The uniqueness theorem for entanglement measures

Matthew J. Donald
The Cavendish Laboratory, Madingley Road, Cambridge, CB3 0HE, Britain.
E-mail: matthew.donald@phy.cam.ac.uk

Michał Horodecki
Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland.
E-mail: fizmh@univ.gda.pl

Oliver Rudolph
Quantum Optics & Information Group, Dipartimento di Fisica “A. Volta,” Università di Pavia, Via Bassi 6, I-27100 Pavia, Italy.
E-mail: rudolph@fisicavolta.unipv.it

Abstract We explore and develop the mathematics of the theory of entanglement measures. After a careful review and analysis of definitions, of preliminary results, and of connections between conditions on entanglement measures, we prove a sharpened version of a uniqueness theorem which gives necessary and sufficient conditions for an entanglement measure to coincide with the reduced von Neumann entropy on pure states. We also prove several versions of a theorem on extreme entanglement measures in the case of mixed states. We analyse properties of the asymptotic regularization of entanglement measures proving, for example, convexity for the entanglement cost and for the regularized relative entropy of entanglement.

1 Introduction

Quantifying entanglement [1, 2, 3, 4] is one of the central topics of quantum information theory. Any function that quantifies entanglement is called an entanglement measure. Entanglement is a complex property of a state and, for arbitrary states, there is no unique definitive measure. In general, there are two “regimes” under which entanglement can be quantified: they may be called the “finite” and the “asymptotic” regimes. The first deals with the entanglement of a single copy of a quantum state. In the second, one is interested in how entanglement behaves when one considers tensor products of a large number of identical copies of a given state. It turns out that by studying the asymptotic regime it is possible to obtain a clearer physical understanding of the nature of entanglement. This is seen, for example, in the so-called “uniqueness theorem” [2, 3, 4, 5] which states that, under appropriate conditions, all entanglement measures coincide on pure bipartite states and are equal to the von Neumann entropy of the corresponding reduced density operator. However, this theorem was never rigorously proved under unified assumptions and definitions. Rather, there are various versions of the argument scattered through the literature.

In Ref. [4], the uniqueness theorem was put into a more general perspective. Namely there are two basic
measures of entanglement [1] – entanglement of distillation ($E_D$) and entanglement cost ($E_C$) – having the following dual meanings:

- $E_D(\rho)$ is the maximal number of singlets that can be produced from the state $\rho$ by means of local quantum operations and classical communication (LQCC operations).
- $E_C(\rho)$ is the minimal number of singlets needed to produce the state $\rho$ by LQCC operations.

(more precisely (cf. Definitions [16] and [17]): $E_D(\rho)$ [$E_C(\rho)$] is the maximal [minimal] number of singlets per copy distillable from the state $\rho$ [needed to form $\rho$] by LQCC operations in the asymptotic regime of $n \to \infty$ copies). It is important here that the conversion is not required to be perfect: the transformed state needs to converge to the required state only in the limit of large $n$. Now, in Ref. [4] it was shown that the two basic measures of entanglement are, respectively, a lower and an upper bound for any entanglement measure satisfying appropriate postulates in the asymptotic regime [4]. This suggests the following clear picture: entanglement cost and entanglement of distillation are extreme measures, and provided they coincide on pure states, all other entanglement measures coincide with them on pure states as well. However as mentioned above, the fact that $E_D$ and $E_C$ coincide on pure states was not proven rigorously. Moreover, it turned out that the postulates are too strong. They include convexity, and some additivity and continuity requirements. It is not known whether any measure exists which satisfies all the requirements. $E_D$ and $E_C$ satisfy the additivity requirement, but it is not known whether or not they are continuous in the sense of Ref. [4]. There are also indications that the entanglement of distillation is not convex [8]. On the other hand, two other important measures, the entanglement of formation (denoted by $E_F$) and the relative entropy of entanglement (denoted by $E_R$) are continuous [4] and convex, but there are problems with additivity. The relative entropy of entanglement is certainly not additive [4], and we do not know about the entanglement of formation.

In this situation it is desirable to prove the uniqueness theorem from first principles, and to study to what extent we can relax the assumptions and still get uniqueness of entanglement measures on pure states. In the present paper we have solved the problem completely by providing necessary and sufficient conditions for a measure of entanglement to be equal to the von Neumann entropy of the reduced density operator for pure states. We also show that if we relax the postulate of asymptotic continuity, then any measure of entanglement (not unique any longer) for pure states must lie between the two analogues of $E_D$ and $E_C$ corresponding to perfect fidelity of conversion. These are $\tilde{E}_C(\psi) = S_0(\rho)$ and $E_D(\psi) = S_\infty(\rho)$, where $\rho$ is the reduced density matrix of $|\psi\rangle$, and $S_0$, $S_\infty$ are R\önyi entropies. In [4, 13, 14], one of us has studied entanglement measures based on cross norms and proved an alternative uniqueness theorem for entanglement measures stemming from the Khinchin-Faddeev characterization of Shannon entropy.

The present paper also contains further developments on the problem of extreme measures. We provide two useful new versions of the theorem of Ref. [4]. In one of them, we show that for any (suitably normalized) function $E$ for which the regularization $E^\infty(\rho) = \lim_{n \to \infty} E(\rho^{\otimes n})/n$ exists and which is (i) nonincreasing under local quantum operations and classical communication (LQCC operations) and (ii) asymptotically continuous, the regularization $E^\infty$ must lie between $E_D$ and $E_C$. The theorem and its proof can easily be generalized by replacing the class of LQCC operations by other classes of operations, or by considering conversions between any two states. Moreover, it is valid for multipartite cases. Therefore the result will be an important tool for analysing asymptotic conversion rates between different states. In particular, it follows from our result that to establish irreversibility of conversion between two states (see [13]), one needs to compare regularizations of asymptotically continuous entanglement measures for these states.

In the other new version of the extreme measures theorem, we are able to weaken the postulates of Ref. [4], so that they are at least satisfied by $E_C$. On the other hand, we do not have a proof that $E_D$ is asymptotically continuous for mixed states, although there is strong evidence that this is the case. If it is, then we would finally have a form of the theorem, in which both $E_D$ and $E_C$ could be called extreme measures, not only in the sense provided by the inequalities we prove, but also in the sense that they belong to the set described by the postulates.

To obtain our results we perform a detailed study of possible postulates for entanglement measures in the finite and the asymptotic regime. In particular, we examine which postulates survive the operation of regularization. We show that if a function is convex and subadditive (i.e., $f(\rho \otimes \sigma) \leq f(\rho) + f(\sigma)$), then its
regularization is convex too. Hence, both the regularization of the relative entropy of entanglement \[2\] as well as of the entanglement of formation \[1\] are convex.

It should be emphasized that our results are stated and proved in language accessible for mathematicians or mathematical physicists who have not previously been involved in quantum information theory. This is in contrast to many papers in this field, where many implicit assumptions are obstacles for understanding the meaning of the theorems and their proofs by non-specialists. For this reason, we devote Sections 2 and 3 to careful statements of some essential definitions and results. In Section 4 we present a self-contained and straightforward proof of the difficult implication in Nielsen’s theorem. This is a theorem which we shall use several times. Properties of entanglement measures and relations between them are analysed in Section 5. The most prominent entanglement measures – entanglement of distillation, entanglement cost, entanglement of formation and relative entropy of entanglement – are defined and studied in Section 6. In Section 7 we present our versions of the theorem on extreme measures. Finally, Section 8 contains our version of the uniqueness theorem for entanglement measures, stating necessary and sufficient conditions for a functional to coincide with the reduced von Neumann entropy on pure states.

2 Preliminaries

Throughout this paper, all spaces considered are assumed to be finite dimensional. The set of trace class operators on a Hilbert space \( \mathcal{H} \) is denoted by \( \mathcal{T}(\mathcal{H}) \) and the set of bounded operators on \( \mathcal{H} \) by \( \mathcal{B}(\mathcal{H}) \). A density operator (or state) is a positive trace class operator with trace one. The set of states on \( \mathcal{H} \) is denoted by \( \Sigma(\mathcal{H}) \) and the set of pure states by \( \Sigma_p(\mathcal{H}) \). The trace class norm on \( \mathcal{T}(\mathcal{H}) \) is denoted by \( \| \cdot \|_1 \). For a wavefunction \( |\psi\rangle \in \mathcal{H} \) the corresponding state will be denoted by \( P_\psi \equiv |\psi\rangle\langle\psi| \). The support of a trace class operator is the subspace spanned by its eigenvectors with non-zero eigenvalues.

In the present paper we restrict ourselves mainly to the situation of a composite quantum system consisting of two subsystems with Hilbert space \( \mathcal{H}^A \otimes \mathcal{H}^B \) where \( \mathcal{H}^A \) and \( \mathcal{H}^B \) denote the Hilbert spaces of the subsystems. Often these systems are to be thought of as being spatially separate and accessible to two independent observers, Alice and Bob.

**Definition 1** Let \( \mathcal{H}^A \) and \( \mathcal{H}^B \) be Hilbert spaces. A density operator \( \varrho \) on the tensor product \( \mathcal{H}^A \otimes \mathcal{H}^B \) is called separable or disentangled if there exist a sequence \( (r_i) \) of positive real numbers, a sequence \( (\rho^A_i) \) of density operators on \( \mathcal{H}^A \) and a sequence \( (\rho^B_i) \) of density operators on \( \mathcal{H}^B \) such that

\[
\varrho = \sum_i r_i \rho^A_i \otimes \rho^B_i,
\]

where the sum converges in trace class norm.

The Schmidt decomposition \[4\] is of central importance in the characterization and quantification of entanglement associated with pure states.

**Lemma 2** Let \( \mathcal{H}^A \) and \( \mathcal{H}^B \) be Hilbert spaces and let \( |\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \). Then there exist a sequence of non-negative real numbers \( (p_i)_i \) summing to one and orthonormal bases \( (|a_i\rangle)_i \) and \( (|b_i\rangle)_i \) of \( \mathcal{H}^A \) and \( \mathcal{H}^B \) respectively such that

\[
|\psi\rangle = \sum_i \sqrt{p_i} |a_i \otimes b_i\rangle.
\]

By \( S(\varrho) \) we will denote von Neumann entropy of the state \( \varrho \) given by

\[
S(\varrho) := -\text{tr}_A \log_2 \varrho.
\]

The von Neumann reduced entropy for a pure state \( \sigma \) on a tensor product Hilbert space \( \mathcal{H}^A \otimes \mathcal{H}^B \) is defined as

\[
S_{vN}(\sigma) := -\text{tr}_A((\text{tr}_B \sigma) \log_2 (\text{tr}_B \sigma)),
\]
where \( \mathrm{tr}_A \) and \( \mathrm{tr}_B \) denote the partial traces over \( \mathcal{H}^A \) and \( \mathcal{H}^B \) respectively. For \( \sigma = P_\psi = |\psi \rangle \langle \psi | \), it is a straightforward consequence of Lemma 2 that

\[
-\mathrm{tr}_A((\mathrm{tr}_B P_\psi) \log_2(\mathrm{tr}_B P_\psi)) = -\mathrm{tr}_B((\mathrm{tr}_A P_\psi) \log_2(\mathrm{tr}_A P_\psi)) = -\sum_i p_i \log_2 p_i
\]

where \( (p_i) \) denotes the sequence of Schmidt coefficients of \( |\psi \rangle \). However, for a general mixed state \( \sigma \), \( \mathrm{tr}_A((\mathrm{tr}_B \sigma) \log_2(\mathrm{tr}_B \sigma)) \) may not equal \( \mathrm{tr}_B((\mathrm{tr}_A \sigma) \log_2(\mathrm{tr}_A \sigma)) \).

## 3 Classes of quantum operations

In quantum information theory it is important to distinguish between the class of quantum operations on a composite quantum system which can be realized by separate local actions on the subsystems (i.e. separate actions by “Alice” and by “Bob”) and those which cannot. The class of local quantum operations assisted by classical communication (LQCC) is of central importance in quantum cryptography and the emerging theory of quantum entanglement.

An operation is a positive linear map \( \Lambda : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2) \) such that \( \mathrm{tr}(\Lambda(\sigma)) \leq 1 \) for all \( \sigma \in \Sigma(\mathcal{H}_1) \). Quantum operations are operations which are completely positive. We shall be interested in the trace preserving quantum operations. By the Choi-Kraus representation these are precisely the linear maps \( \Lambda : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2) \) which can be written in the form \( \Lambda(B) = \sum_{i=1}^{n_1} W_i B W_i^\dagger \) for \( B \in \mathcal{T}(\mathcal{H}_1) \) with operators \( W_i : \mathcal{H}_1 \to \mathcal{H}_2 \) satisfying \( \sum_{i=1}^{n_2} W_i^\dagger W_i = 1_1 \), where \( n_1 = \dim \mathcal{H}_3 \), \( n_2 = \dim \mathcal{H}_2 \), and \( 1_1 \) is the identity operator on \( \mathcal{H}_1 \). These can also be characterized as precisely the linear maps which can be composed out of the following elementary operations

(O1) Adding an uncorrelated ancilla:

\[
\Lambda_1 : \mathcal{T}(\mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_1 \otimes \mathcal{K}_1), \Lambda_1(\rho) := \rho \otimes \sigma, \quad \text{where } \mathcal{H}_1 \text{ and } \mathcal{K}_1 \text{ denote the Hilbert spaces of the original quantum system and of the ancilla respectively and where } \sigma \in \Sigma(\mathcal{K}_1);
\]

(O2) Tracing out part of the system:

\[
\Lambda_2 : \mathcal{T}(\mathcal{H}_2 \otimes \mathcal{K}_2) \to \mathcal{T}(\mathcal{H}_2), \Lambda_2(\rho) := \mathrm{tr}_{\mathcal{K}_2}(\rho) \quad \text{where } \mathcal{H}_2 \otimes \mathcal{K}_2 \text{ denote the Hilbert spaces of the full original quantum system and of the dismissed part respectively and where } \mathrm{tr}_{\mathcal{K}_2} \text{ denotes the partial trace over } \mathcal{K}_2;
\]

(O3) Unitary transformations:

\[
\Lambda_3 : \mathcal{T}(\mathcal{H}_3) \to \mathcal{T}(\mathcal{H}_3), \Lambda_3(\rho) = U \rho U^\dagger \quad \text{where } U \text{ is a unitary operator on } \mathcal{H}_3.
\]

A discussion of this material with complete proofs from first principles may be found in the initial archived draft of this paper.

Defining a local operation as quantum operation on a individual subsystem, we now turn to the definition of local operations assisted by classical communication. As always in this paper we consider a quantum system consisting of two (possibly separate) subsystems A and B with (initial) Hilbert spaces \( \mathcal{H}^A \) and \( \mathcal{H}^B \) respectively. There are three cases: the communication between A and B can be unidirectional (in either direction) or bidirectional.

Let us first define the class of local quantum operations (L0) assisted by unidirectional classical communication (operations in this class will be called one-way LQCC operations) with direction from system A (Alice) to system B (Bob). In this case, the operations performed by Bob depend on Alice’s operations, but not conversely.

**Definition 3** A completely positive map \( \Lambda : \mathcal{T}(\mathcal{H}_1^A \otimes \mathcal{H}_1^B) \to \mathcal{T}(\mathcal{H}_2^A \otimes \mathcal{H}_2^B) \) is called a one-way LQCC operation from A to B if it can be written in the form

\[
\Lambda(\sigma) = \sum_{i,j=1}^{K,L} (1^A_i \otimes W^B_{ji})(V^A_i \otimes 1^B_j)\sigma(V^A_i^\dagger \otimes 1^B_j)(1^A_i^\dagger \otimes W^B_{ji}^\dagger)
\]
for all \( \sigma \in \mathcal{T}(\mathcal{H}_i^A \otimes \mathcal{H}^B) \) and some sequences of operators \( (V^A_i : \mathcal{H}_i^A \rightarrow \mathcal{H}_i^A)_i \) and \( (W^B_{ji} : \mathcal{H}_j^B \rightarrow \mathcal{H}_j^B)_{ji} \) with \( \sum_{i=1}^K V^A_i \cdot V^A_i = 1 \) and \( \sum_{j=1}^L W^B_{ji} \cdot W^B_{ji} = 1 \) for each \( i \), where \( 1^A_i, 1^B_i \) and \( 1^A_2 \) are the unit operators acting on the Hilbert spaces \( \mathcal{H}_i^A, \mathcal{H}_i^B \) and \( \mathcal{H}_2^A \), respectively.

Of course, by the Choi-Kraus representation any operation \( \Lambda \) of the form

\[
\Lambda = \Lambda^A \otimes I^B,
\]

where \( \Lambda^A : \mathcal{T}(\mathcal{H}^A) \rightarrow \mathcal{T}(\mathcal{H}^A) \) is a completely positive trace preserving map and \( I^B \) is the identity operator on \( \mathcal{T}(\mathcal{H}_1^B) \), is a one-way LQCC operation from A to B.

Let us now define local quantum operations assisted by bidirectional classical communication (LQCC operations).

**Definition 4** A completely positive map \( \Lambda : \mathcal{T}(\mathcal{H}^A \otimes \mathcal{H}^B) \rightarrow \mathcal{T}(\mathcal{K}^A \otimes \mathcal{K}^B) \) is called an LQCC operation if there exist \( n > 0 \) and sequences of Hilbert spaces \( (\mathcal{H}^A_k)_{k=1}^{n+1} \) and \( (\mathcal{H}^B_k)_{k=1}^{n+1} \) with \( \mathcal{H}^A_k = \mathcal{H}^A(B) \) and \( \mathcal{H}^A_{n+1} = \mathcal{K}^A(B) \), such that \( \Lambda \) can be written in the following form

\[
\Lambda(\sigma) = \sum_{i_1, \ldots, i_{2n} = 1}^{K_{1}, \ldots, K_{2n}} V_{i_1}^{AB} \cdots V_{i_{2n}}^{AB} \sigma V_{i_{2n}}^{AB} \cdots V_{i_1}^{AB} \dagger
\]

for all \( \sigma \in \mathcal{T}(\mathcal{H}^A \otimes \mathcal{H}^B) \) where \( V_{i_1, \ldots, i_{2n}}^{AB} : \mathcal{H}^A \otimes \mathcal{H}^B \rightarrow \mathcal{K}^A \otimes \mathcal{K}^B \) is given by

\[
V_{i_1, \ldots, i_{2n}}^{AB} := (1^A_{n+1} \otimes W_{2n}^{i_{2n-1}, \ldots, i_1})(V_{2n}^{i_{2n-1}, \ldots, i_1} \otimes 1^B_n)(1^A_n \otimes W_{2n-2}^{i_{2n-2}, \ldots, i_1}) \cdots (1^A_2 \otimes W_{2}^{i_2, i_1})(V_{1}^{i_1} \otimes 1^B_1)
\]

with families of operators

\[
(V_{2k-1}^{i_{2k-1}, \ldots, i_1} : \mathcal{H}_k^A \rightarrow \mathcal{H}_{k+1}^A)_{k=1}^{n}, \quad (W_{2k}^{i_{2k}, \ldots, i_1} : \mathcal{H}_k^B \rightarrow \mathcal{H}_{k+1}^B)_{k=1}^{n}
\]

such that for \( k = 0, \ldots, n - 1 \) and each sequence of indices \( (i_{2k}, \ldots, i_1) \)

\[
\sum_{i_{2k+1} = 1}^{K_{2k+1}} (V_{2k+1}^{i_{2k+1}, \ldots, i_1} \dagger V_{2k+1}^{i_{2k+1}, \ldots, i_1}) = 1^A_{k+1}
\]

and for \( k = 1, \ldots, n \) and each sequence of indices \( (i_{2k-1}, \ldots, i_1) \)

\[
\sum_{i_{2k} = 1}^{K_{2k}} (W_{2k}^{i_{2k}, \ldots, i_1} \dagger W_{2k}^{i_{2k}, \ldots, i_1}) = 1^B_k
\]

where for all \( k > 0 \), \( 1^A_k \) and \( 1^B_k \) denote the unit operator on \( \mathcal{H}_k^A \) and \( \mathcal{H}_k^B \) respectively.

Obviously the class of one-way LQCC operations is a subclass of the class of LQCC operations. There is another important class: separable operations. A separable operation is an operation of the form:

\[
\Lambda : \mathcal{T}(\mathcal{H}^A \otimes \mathcal{H}^B) \rightarrow \mathcal{T}(\mathcal{K}^A \otimes \mathcal{K}^B), \quad \Lambda(\sigma) = \sum_{i=1}^{k} (V_i \otimes W_i) \sigma (V_i \otimes W_i)^\dagger
\]

with \( \sum_{i=1}^{k} (V_i \otimes W_i)^\dagger V_i \otimes W_i = 1^{AB} \) where \( 1^{AB} \) denotes the unit operator acting on \( \mathcal{H}^A \otimes \mathcal{H}^B \). The class of separable operations is strictly larger than the LQCC class \([2]\).

One can also consider a small class obtained by taking the convex hull \( \mathcal{C} \) of the set of all maps of the form \( \Lambda^A \otimes \Lambda^B \). Such operations require in general one-way classical communication, but they do not cover the whole class of one-way LQCC operations.
All the classes above are closed under tensor multiplication, convex combinations, and composition. The results of our paper apply in principle to all the classes apart from the last (i.e., apart from the class of all operations in the convex hull $C$ of the set of all maps of the form $\Lambda^A \otimes \Lambda^B$). For definiteness, in the sequel we will use $LQCC$ operations.

Finally, we conclude this section with a useful technical lemma.

**Lemma 5** Let $\Lambda : T(H_1) \rightarrow T(H_2)$ be a positive trace-preserving map and suppose that $B \in T(H_1)$ with $B = B^*$. Then $\|\Lambda(B)\|_1 \leq \|B\|_1$.

**Proof:** Suppose that $B$ has eigenvalue expansion $B = \sum_{i=1}^{n_1} |\beta_i\rangle\langle \beta_i|$. Then

$$\|\Lambda(B)\|_1 \leq \sum_{i=1}^{n_1} |\beta_i| \|\Lambda(|\psi_i\rangle\langle \psi_i|)\|_1 = \|B\|_1$$

as $\|B\|_1 = \sum_{i=1}^{n_1} |\beta_i|$ and $\Lambda(|\psi_i\rangle\langle \psi_i|)$ is a positive trace class operator with unit trace. ■

## 4 Nielsen’s theorem

A beautiful and powerful result of entanglement theory is Nielsen’s theorem [22]. In one direction, the proof is straightforward, and we refer to [22]. The other direction is more difficult. We present here an entirely self-contained, simple, and direct proof. Alternative proofs have previously been given by Hardy [23] and by Jensen and Schack [24].

Before we state the theorem we need the following definition.

**Definition 6** Let $(p_i)_{i=1}^{m_1}$ and $(q_i)_{i=1}^{m_2}$ be two probability distributions with probabilities arranged in decreasing order, i.e., $p_1 \geq p_2 \geq \cdots \geq p_{m_1}$ and similarly for $(q_i)$. Then we will say that $(q_i)$ majorizes $(p_i)$, (in symbols $(q_i) \succ (p_i)$) if for all $k \leq \min\{m_1, m_2\}$ we have

$$\sum_{i=1}^{k} q_i \geq \sum_{i=1}^{k} p_i.$$

(10)

**Theorem 7 (Nielsen)** Let $H^A$ and $H^B$ be Hilbert spaces and let $\{|\chi_i\rangle\rangle_{m=1}^{M}$ and $\{|\kappa_i\rangle\rangle_{m=1}^{M}$ be orthonormal bases for $H^A$ and $H^B$ respectively. Let $|\Psi\rangle = \sum_{m=1}^{M} \sqrt{p_m}|\chi_m\kappa_m\rangle$ and $|\Phi\rangle = \sum_{m=1}^{M} \sqrt{q_m}|\chi_m\kappa_m\rangle$ be Schmidt decompositions of normalized vectors $|\Psi\rangle$ and $|\Phi\rangle$ in $H^A \otimes H^B$ with $p_1 \geq p_2 \geq \cdots \geq p_M$ and $q_1 \geq q_2 \geq \cdots \geq q_M$. Then $|\Psi\rangle\langle \Psi|$ can be converted into $|\Phi\rangle\langle \Phi|$ by $LQCC$ operations if and only if $(q_i)$ majorises $(p_i)$.

**Proof:** (One direction only.) Suppose that $(q_i)$ majorises $(p_i)$. Set $\rho \equiv |\Psi\rangle\langle \Psi|$ and $\sigma \equiv |\Phi\rangle\langle \Phi|$. We shall prove that there is a sequence $(\Lambda_n)_{n=1}^{N}$ with $N < M$ of completely positive maps on $T(H^A \otimes H^B)$ of the form

$$\Lambda_n(\omega) = (C_n \otimes U_n)\omega(C_n \otimes U_n) + (D_n \otimes V_n)\omega(D_n \otimes V_n)$$

(11)

where $U_n, V_n \in B(H^B)$ are unitary and $C_n, D_n \in B(H^A)$ satisfy $C_n^\dagger C_n + D_n^\dagger D_n = 1^A$ such that $\Lambda_1 \circ \Lambda_2 \circ \cdots \circ \Lambda_N(\rho) = \sigma$. Note that all the $\Lambda_n$ are one-way $LQCC$ operations from $A$ to $B$ and hence their composition also is. As the Schmidt decomposition is symmetrical between $A$ and $B$, we could also use one-way $LQCC$ operations from $B$ to $A$.

Let $N = N(|\Psi\rangle, |\Phi\rangle)$ be the number of non-zero $\delta_k$. We shall prove the result by induction on $N$. $|\Psi\rangle = |\Phi\rangle$ if and only if $\delta_1 = \delta_2 = \cdots = \delta_{M-1} = 0$. In this case $N(|\Psi\rangle, |\Phi\rangle) = 0$, $\rho = \sigma$, and the result is certainly true.

Suppose that the result holds for all pairs $(|\Psi\rangle, |\Phi\rangle)$ satisfying the conditions of the proposition with $N(|\Psi\rangle, |\Phi\rangle) = 0, \ldots, L$ and that $(|\Psi\rangle, |\Phi\rangle)$ is a pair with $N(|\Psi\rangle, |\Phi\rangle) = L + 1$. Then there exists $J \geq 1$ such that $\delta_1 = \delta_2 = \cdots = \delta_{J-1} = 0$ and $\delta_{J} > 0$. Setting $\delta_0 := 0$, we have $q_j - p_j = \delta_{j-1} + q_j - p_j = \delta_j$ for $j = 1, \cdots, J$. This implies that $p_j = q_j$ for $j = 1, \cdots, J - 1$ and that $q_J > p_J$. Suppose that $\delta_k > 0$ for
Define \((r_m)_{m=1}^M\) by \(r_m := p_m\) for \(m \neq J, K\) and by \(r_j := p_j + \delta\), \(r_K := p_K - \delta\) where \(\delta := \min\{\delta_k : k = J, \ldots, K-1\}\). By construction \(\delta > 0\). Now \(\delta \leq \delta_j\) implies \(q_j \geq r_j \geq p_j\) and \(\delta \leq \delta_K-1\) implies \(p_K \geq r_K \geq q_K\). This in turn implies that \(r_1 \geq r_2 \geq \cdots \geq r_M\). Thus for \(k = 1, \ldots, J - 1\) and for \(k = K, \ldots, M\),

\[
\sum_{m=1}^k r_m = \sum_{m=1}^k p_m \leq \sum_{m=1}^k q_m.
\]

For \(k = J, \ldots, K - 1\), \(\sum_{m=1}^k r_m = \sum_{m=1}^k p_m + \delta\) and so, as \(0 < \delta \leq \delta_k\),

\[
\sum_{m=1}^k p_m < \sum_{m=1}^k r_m \leq \sum_{m=1}^k q_m.
\]

Define \(|\Xi\rangle := \sum_{m=1}^M \sqrt{r_m}|\chi_m\kappa_m\rangle\). Then \(N(|\Xi\rangle, |\Phi\rangle) \leq L\) so that, by the inductive hypothesis, there is a sequence \((\Lambda_n)_{n=1}^N\) of maps of the required form with \(N = N(|\Xi\rangle, |\Phi\rangle)\) such that

\[
\Lambda_1 \circ \Lambda_2 \circ \cdots \circ \Lambda_N(|\Xi\rangle) = \sigma.
\]

Thus to complete the proof, we need only find a completely positive map \(\Lambda\) of the required form such that

\[
\Lambda(|\Psi\rangle) = |\Xi\rangle\langle\Xi|.
\]

To this end set \(P := \sum_{m \neq J, K} |\chi_m\rangle\langle\chi_m|\). Set

\[
C := \sqrt{\frac{r_J p_J - r_K p_K}{r_J - r_K}} \left( P + \sqrt{\frac{r_J}{p_J}} |\chi_J\rangle \langle\chi_J| + \sqrt{\frac{r_K}{p_K}} |\chi_K\rangle \langle\chi_K| \right)
\]

and \(U := 1^B\). Set

\[
D := \sqrt{\frac{r_J p_K - r_K p_J}{r_J - r_K}} \left( P + \sqrt{\frac{r_K}{p_J}} |\chi_K\rangle \langle\chi_K| + \sqrt{\frac{r_J}{p_K}} |\chi_J\rangle \langle\chi_J| \right)
\]

and \(V := |\kappa_K\rangle \langle\kappa_J| + |\kappa_J\rangle \langle\kappa_K| + \sum_{m \neq J, K} |\kappa_m\rangle \langle\kappa_m|\).

Note that \(p_J \geq p_K > q_K \geq 0\), that \(r_J > r_K\), that \(r_J p_J > r_K p_K\), and that \(r_J p_K - r_K p_J = (p_J + \delta)p_K - (p_K - \delta)p_J = \delta(p_K + p_J) > 0\). Note also that \(r_J^2 - r_K^2 = (r_J - r_K)(r_J + r_K) = (r_J - r_K)(p_J + p_K)\) so that

\[
\frac{r_J p_J - r_K p_K}{r_J - r_K} + \frac{r_J p_K - r_K p_J}{r_J - r_K} = 1.
\]

With these definitions and notes, the completion of the proof is straightforward.

\[
\square
\]

5 Entanglement measures

**Definition 8** Consider a bipartite composite quantum system with Hilbert space of the form \(\mathcal{H}^A \otimes \mathcal{H}^B\) where \(\mathcal{H}^A \equiv \mathcal{H}^B \equiv \mathbb{C}^d\). Assume that isomorphisms between \(\mathbb{C}^d\), \(\mathcal{H}^A\), and \(\mathcal{H}^B\) are chosen. For a chosen orthonormal basis \((|\psi_i\rangle)_{i=1}^d\) of \(\mathbb{C}^d\), we let

\[
|\Psi_+ (\mathbb{C}^d)\rangle \equiv \sum_{i=1}^d \frac{1}{\sqrt{d}} |\psi_i \otimes \psi_i\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B.
\]

|\Psi_+ (\mathbb{C}^d)\rangle is a maximally entangled wavefunction. All other maximally entangled wavefunctions in \(\mathcal{H}^A \otimes \mathcal{H}^B\) can be obtained by applying a unitary operator of the form \(1^A \otimes U_B\) to \(|\Psi_+ (\mathbb{C}^d)\rangle\) where \(U_B\) is a unitary operator on \(\mathcal{H}^B\). The pure state corresponding to \(|\Psi_+ (\mathbb{C}^d)\rangle\) will be denoted by \(P_+ (\mathbb{C}^d) \equiv |\Psi_+ (\mathbb{C}^d)\rangle \langle\Psi_+ (\mathbb{C}^d)|\).

In an arbitrary bipartite composite system, we shall refer to any wavefunction with the same Schmidt coefficients as \(|\Psi_+ (\mathbb{C}^d)\rangle\) as a representative of \(|\Psi_+ (\mathbb{C}^d)\rangle\) and to the corresponding state as a representative of \(P_+ (\mathbb{C}^d)\).
5.1 Conditions on mixed states

The degree of entanglement of a density operator on the Hilbert space of a bipartite composite quantum system can be expressed by an “entanglement measure.” This a non-negative real-valued functional $E$ defined on $\Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ for all finite-dimensional Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$. Any of the following conditions might be imposed on $E$:[1][2][3][4][5]

(E0) If $\sigma$ is separable, then $E(\sigma) = 0$.

(E1) (Normalization.) If $P_+^d$ is any representative of $P_+(\mathbb{C}^d)$, then $E(P_+^d) = \log_2 d$ for $d = 1, 2, \ldots$.

A weaker condition is:

(E1') $E(P_+(\mathbb{C}^2)) = 1$.

(E2) (LQCC Monotonicity.) Entanglement cannot increase under procedures consisting of local operations on the two quantum systems and classical communication. If $\Lambda$ is an LQCC operation, then

$$E(\Lambda(\sigma)) \leq E(\sigma)$$

for all $\sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$.

A condition which, as we shall confirm below (Lemma[6]), is weaker than (E2), is

(E2') $E(\Lambda(\sigma)) = E(\sigma)$ whenever $\sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ and $\Lambda$ is a strictly local operation which is either unitary or which adds extraneous dimensions. On Alice’s side, these local operations take the form, either of $\Lambda_1(\varrho) = (U^A \otimes I^B)\varrho(U^A \otimes I^B)^\dagger$ where $U^A : \mathcal{H}^A \to \mathcal{H}^A$ is unitary, or of $\Lambda_2(\varrho) = (W^A \otimes I^B)\varrho(W^A \otimes I^B)^\dagger$ where $\mathcal{H}^A \subset \mathcal{K}^A$ and $W^A : \mathcal{H}^A \to \mathcal{K}^A$ is the inclusion map. There are equivalent local operations on Bob’s side.

(E2'') $E(\Lambda(\sigma)) = E(\sigma)$ whenever $\sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ and $\Lambda$ is a strictly local unitary operation.

Without further remark, we shall always assume that all our measures satisfy (E2'').

A weaker condition deals only with approximations to pure states:

(E3') Same as (E3) but with $\varrho_n \in \Sigma_p(\mathcal{H}^A_n \otimes \mathcal{H}^B_n)$ for all $n$.

Sometimes we are interested in entanglement measures which satisfy an additivity property:

(E4) (Additivity.) For all $n \geq 1$ and all $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$

$$\frac{E(\varrho^{\otimes n})}{n} = E(\varrho).$$

Here $\varrho^{\otimes n}$ denotes the $n$-fold tensor product of $\varrho$ by itself which acts on the tensor product $(\mathcal{H}^A)^{\otimes n} \otimes (\mathcal{H}^B)^{\otimes n}$.

An apparently weaker property, which as we shall see in Lemma[7] is actually equivalent to (E4), is

(E4') (Asymptotic Additivity.) Given $\varepsilon > 0$ and $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$, there exists an integer $N > 0$ such that $n \geq N$ implies

$$\frac{E(\varrho^{\otimes n})}{n} - \varepsilon \leq E(\varrho) \leq \frac{E(\varrho^{\otimes n})}{n} + \varepsilon.$$
(E5) (Subadditivity.) For all $\varrho, \sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$,
\[ E(\varrho \otimes \sigma) \leq E(\varrho) + E(\sigma). \]

(E5') For all $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ and $m, n \geq 1$,
\[ E\left(\varrho^{\otimes (m+n)}\right) \leq E\left(\varrho^{\otimes m}\right) + E\left(\varrho^{\otimes n}\right). \]

(E5'') (Existence of a regularization.) For all $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$, the limit
\[ E^\infty(\varrho) \equiv \lim_{n \to \infty} \frac{E(\varrho^{\otimes n})}{n} \]
exists.

In Lemma 12 we shall prove the well-known result that (E5') is a sufficient condition for (E5''). When (E5'') holds, we shall refer to $E^\infty$ as the regularization of $E$. We shall discuss some general properties of $E^\infty$ in Proposition 13.

(E6) (Convexity.) Mixing of states does not increase entanglement.
\[ E(\lambda \varrho + (1 - \lambda)\sigma) \leq \lambda E(\varrho) + (1 - \lambda)E(\sigma) \]
for all $0 \leq \lambda \leq 1$ and all $\varrho, \sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$.

(E6') For any state $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ and any decomposition $\varrho = \sum_i p_i \left| \psi_i \right\rangle \left\langle \psi_i \right|$ with $\left| \psi_i \right\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$, $p_i \geq 0$ for all $i$ and $\sum_i p_i = 1$, we require
\[ E(\varrho) \leq \sum_i p_i E(P_{\psi_i}). \]

5.2 Conditions on pure states

The conditions imposed on an entanglement measure can be weakened by requiring that they only apply for pure states. Indeed, it might not even be required that the measure is defined except on pure states. Recall that $\Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$ denotes the set of pure states on the composite space.

(P0) If $\sigma \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$ is separable, then $E(\sigma) = 0$.

(P1) = (E1) (Normalization.) If $P_+^d$ is any representative of $P_+(\mathbb{C}^d)$, then $E(P_+^d) = \log_2 d$ for $d = 1, 2, \ldots$.

(P1') = (E1') $E(P_+(\mathbb{C}^2)) = 1$.

(P2) Let $\Lambda$ be an operation which can be realized by means of local operations and classical communications. If $\sigma \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$ is such that $\Lambda(\sigma)$ is also pure, then
\[ E(\Lambda(\sigma)) \leq E(\sigma). \]

(P2') For $\sigma \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$, $E(\sigma)$ depends only on the non-zero coefficients of a Schmidt decomposition of $\sigma$. 

9
By Nielsen’s theorem and the proof of Lemma 3 below, (P2) is equivalent to assuming (P2’) and that if the Schmidt coefficients of $\varrho$ majorize those of $\sigma$ then $E(\varrho) \leq E(\sigma)$. Our proof of the theorem shows that, given (P2’), only local operations and operations of the specific form of Equation (11) need be considered for (P2) (cf. (2)).

Below we will in particular be interested in entanglement measures satisfying the following additional conditions:

(P3) Let $(\mathcal{H}_n^A)_{n \in \mathbb{N}}$ and $(\mathcal{H}_n^B)_{n \in \mathbb{N}}$ be sequences of Hilbert spaces and let $\mathcal{H}_n \equiv \mathcal{H}_n^A \otimes \mathcal{H}_n^B$ for all $n$. For all sequences $(\varrho_n)_{n \in \mathbb{N}}$ and $(\sigma_n)_{n \in \mathbb{N}}$ of states with $\varrho_n, \sigma_n \in \Sigma_p(\mathcal{H}_n^A \otimes \mathcal{H}_n^B)$, such that $\|\varrho_n - \sigma_n\|_1 \to 0$, we require that

$$E(\varrho_n) - E(\sigma_n) \over 1 + \log_2 \dim \mathcal{H}_n \to 0.$$  

(P4) For all $n \geq 1$ and all $\varrho \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$,

$$E(\varrho^{\otimes n}) \over n = E(\varrho).$$

Of course, when $\varrho$ is pure, so is $\varrho^{\otimes n}$.

(P4’) Given $\epsilon > 0$ and $\varrho \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$, there exists an integer $N > 0$ such that $n \geq N$ implies

$$E(\varrho^{\otimes n}) \over n - \epsilon \leq E(\varrho) \leq E(\varrho^{\otimes n}) \over n + \epsilon.$$

(P5”) (Existence of a regularization on pure states.) For all $\varrho \in \Sigma_p(\mathcal{H}^A \otimes \mathcal{H}^B)$, the limit

$$E^\infty(\varrho) \equiv \lim_{n \to \infty} E(\varrho^{\otimes n}) \over n$$

exists.

5.3 Some connections between the conditions

Lemma 9 (E2’) is implied by (E2).

Proof: By Equation (4), the operations considered in (E2’) are LQCC. To see this for $\Lambda_2$, note that $W^{A_1}W^{A} = 1_A$. Thus (E2) implies $E(\Lambda_i(\sigma)) \leq E(\sigma)$ for $i = 1, 2$. Unitary maps are invertible and so $E(\Lambda_1(\sigma)) \geq E(\sigma)$. On the other hand, if $\mathcal{H}^A \subset \mathcal{K}^A$ and $P^A$ is the projection onto $\mathcal{H}^A$, then, for any $\tau^A \in \Sigma_p(\mathcal{H}^A)$, the map $\Lambda_3^A : \Sigma(\mathcal{K}^A) \to \Sigma(\mathcal{H}^A)$ defined by $\Lambda_3^A(\varrho) := P^A \varrho P^A + \text{tr}(\varrho (1 - P^A)) \tau^A$ is completely positive and trace preserving, so by Equation (4), the map on $\Sigma(\mathcal{K}^A \otimes \mathcal{H}^A)$ defined by $\Lambda_3 = \Lambda_3^A \otimes I^B$ is LQCC.

$$\Lambda_3(\Lambda_2(\sigma)) = \sigma$$ and hence (E2) implies $E(\sigma) \leq E(\Lambda_2(\sigma))$.

Lemma 10 (E4’) is equivalent to (E4) and (P4’) is equivalent to (P4).

Proof: That (E4) implies (E4’) is immediate. Suppose (E4’) and choose $m$, $\varrho$, and $\epsilon$.

By (E4’), there exists $N$ such that $n \geq N$ implies $|E(\varrho) - E(\varrho^{\otimes n})/n| \leq \epsilon$ and $|E(\varrho^{\otimes m}) - (E(\varrho^{\otimes n})^{\otimes n})/n| \leq \epsilon$. But, by definition, $(\varrho^{\otimes m})^{\otimes n} = \varrho^{\otimes mn}$ where the equality relates equivalent density matrices on products of isomorphic local spaces. Thus $n \geq N$ implies

$$|E(\varrho) - E(\varrho^{\otimes m})/m| \leq |E(\varrho) - E(\varrho^{\otimes mn})/mn| + |E(\varrho^{\otimes mn})/mn - E(\varrho^{\otimes m})/m| \leq 2\epsilon.$$  

(E4) follows. The same proof shows the equivalence of (P4’) and (P4).
Lemma 11 Let $E$ be an entanglement measure which satisfies (P1'), (P2), and (P4). Then $E$ satisfies (P0) and (P1). Moreover, if $E$ is defined on mixed states and satisfies either (E2) or (E6'), then (E0) is satisfied.

Proof: First we deal with separable states.

Choose $\epsilon > 0$. Any pair of separable pure states are interconvertible by local unitary operators. If $\sigma$ is such a state, then so is $\sigma^{\otimes n}$, and so, by (P2), $E(\sigma) = E(\sigma^{\otimes n})$. But (P4) implies that $E(\sigma) = E(\sigma^{\otimes n})/n$ and hence $E(\sigma) = 0$. This gives (P0) and the $d = 1$ case of (P1).

Now let $\rho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ be a mixed separable state. Expanding the states $\rho_i^A$ and $\rho_i^B$ of Equation 11 into pure components shows that $\sigma$ is a convex combination of pure separable states: $\sigma = \sum_i p_i \sigma_i$.

Thus (E6') is sufficient to go from (P0) to (E0). But (E2) is also sufficient, because if $\Lambda_i : \mathcal{T}(\mathcal{H}^A \otimes \mathcal{H}^B) \rightarrow \mathcal{T}(\mathcal{H}^A \otimes \mathcal{H}^B)$ is a local operation such that $\Lambda_i(\sigma_1) = \sigma_i$, then $\Lambda := \sum_i p_i \Lambda_i$ is an LQCC operation such that $\Lambda(\sigma_1) = \sigma$ and so (E2) and (P0) yield $E(\sigma) \leq E(\sigma) = 0$.

Now we turn to showing that, for $d \geq 2$, $E(P^d_+) = \log_2 d$ follows from (P1'), (P2), and (P4). By (P2'), $E(P^d_+)$ is independent of the representative of $P_+(\mathcal{C}^d)$ considered.

Choose $\epsilon > 0$ and $d \geq 2$. Choose $N > 1/\epsilon$. Set $w(n) = E(P^d_n)$.

By Nielsen's theorem, (P2) implies that $w(d_1) \leq w(d_2)$ whenever $d_1 \leq d_2$.

Up to local isomorphisms, $(P^d_n)^{\otimes n} = P^d_n$, so that, by (P4), $w(d) = w(d^n)/n$ for all $n$ and, by (P1'), $w(2) = w(2^n)/n = 1$.

Choose $n_1, n_2 > N$ such that $2^{n_2} + 1 \geq d n_1 \geq 2^{n_2}$. Then $\log_2 d \geq n_2/n_1$, $|n_2/n_1 - \log_2 d| \leq 1/n_1 < \epsilon$ and, using (P4),

$$|w(d) - \log_2 d| = |w(d^n) - n_2|/n_1 + |n_2/n_1 - \log_2 d| \leq |w(d^n) - n_2 w(2)|/n_1 + \epsilon$$

and

$$|w(d^n) - n_2 w(2)|/n_1 = |w(d^n) - w(2^n)|/n_1 \leq |w(2^{n_2} + 1) - w(2^{n_2})|/n_1 = 1/n_1 \leq \epsilon.$$ 

It follows that $w(d)$ is arbitrarily close to $\log_2 d$.

Lemma 12 (E5') implies (E5'). Indeed, (E5') implies that $\frac{E(\rho^{\otimes m})}{m} \to \inf \left\{ \frac{E(\rho^{\otimes m})}{m} : m \geq 1 \right\}$.

Proof: (see [2] Theorem 4.9). Fix $k > 0$. Every $m \geq 1$ can be written $m = nk + r$ with $0 \leq r < k$. Then for all $m > 0$ set $f(m) := E(\rho^{\otimes m})$. (E5') implies that

$$\frac{f(m)}{m} \leq \frac{n f(k) + f(r)}{nk + r} \leq \frac{n f(k)}{nk} + \frac{f(r)}{nk} = \frac{f(k)}{k} + \frac{f(r)}{nk}.$$

As $m \to \infty$ then $n \to \infty$ so $\limsup_{m \to \infty} \frac{f(m)}{m} \leq \frac{f(k)}{k}$ and thus $\limsup_{m \to \infty} \frac{f(m)}{m} \leq \inf_{k \geq 1} \frac{f(k)}{k}$. Now $\inf_{k \geq 1} \frac{f(k)}{k} = \liminf_{m \to \infty} \frac{f(m)}{m}$ shows that $\lim_{m \to \infty} \frac{f(m)}{m}$ exists and equals $\inf_{k \geq 1} \frac{f(k)}{k}$.

Proposition 13 Let $E$ be an entanglement measure which satisfies (E5'). Then

1. $E^\infty$ satisfies (E4).
2. If $E$ satisfies (E0), then so does $E^\infty$.
3. If $E$ satisfies (E1), then so does $E^\infty$.
4. If $E$ satisfies (E2), then so does $E^\infty$.
5. If $E$ satisfies (E5), then so does $E^\infty$.
6. If $E$ satisfies (E5) and (E6), then so does $E^\infty$.
Proof:
1) For all \( m \) and \( \varrho \),
\[
\frac{E^\infty(\varrho^{\otimes m})}{m} = \lim_{n \to \infty} \frac{E(\varrho^{\otimes n})}{nm} = E^\infty(\varrho).
\]
2) If \( \sigma \) is separable, then so is \( \sigma^{\otimes n} \) for all \( n \).
3) If \( P^\varrho_+ \) is a representative of \( P_+(\mathbb{C}^d) \), then \( (P^\varrho_+)^{\otimes n} \) is a representative of \( P_+(\mathbb{C}^d)^n \).
4) If \( \Lambda \) is \( \mathbb{L}_\mathbb{Q} \), then so is \( \Lambda^{\otimes n} \) and \( \Lambda(\sigma)^{\otimes n} = \Lambda^{\otimes n}(\sigma^{\otimes n}) \).
5) For all \( \varrho, \sigma \) and \( k \geq 1 \), (E5) implies that
\[
\frac{E((\varrho \otimes \sigma)^{\otimes k})}{k} \leq \frac{E(\varrho^{\otimes k})}{k} + \frac{E(\sigma^{\otimes k})}{k}.
\]
6) Suppose that \( E \) satisfies (E5) and (E6). Let \( \varrho, \sigma \in \Sigma(H^A \otimes H^B) \) and choose \( x_1, x_2 \in [0, 1] \) with \( x_1 + x_2 = 1 \). Let \( \omega = x_1 \varrho + x_2 \sigma \). Expanding \( \omega^{\otimes n} \) as a sum of products, using convexity of \( E \), and then using local isomorphisms to re-order the terms in each product, gives
\[
E(\omega^{\otimes n}) \leq \sum_{k=0}^{\infty} \binom{n}{k} x_1^k x_2^{n-k} E(\varrho^{\otimes k}) \leq \sum_{k=0}^{\infty} \binom{n}{k} x_1^k x_2^{n-k} (E(\varrho^{\otimes k}) + E(\sigma^{\otimes (n-k)}))
\]
where the second inequality is a consequence of (E5). To complete the proof, we need the following lemma:

**Lemma 14** As \( n \to \infty \), \( \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} x_1^k x_2^{n-k} E(\varrho^{\otimes k}) \to x_1E^\infty(\varrho) \) and \( \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} x_1^k x_2^{n-k} E(\sigma^{\otimes (n-k)}) \to x_2E^\infty(\sigma) \).

**Proof:** It is sufficient to prove the first limit. Set \( g(n) = E(\varrho^{\otimes n})/n \) and \( L = E^\infty(\varrho) \). Choose \( \epsilon > 0 \). By Lemma 12 there exists \( K \) such that \( k \geq K \) implies \( |g(k) - L| < \epsilon/2 \) and there is a constant \( C > 0 \) such that \( |g(k) - L| < C \) for all \( k \). \( N > K \) implies that
\[
\frac{1}{N} \sum_{k=0}^{K} \binom{N}{k} x_1^k x_2^{N-k} \leq \frac{K}{N} \sum_{k=0}^{N} \binom{N}{k} x_1^k x_2^{N-k} = \frac{K}{N}.
\]
Set \( h(x) = (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \). \( xh'(x) = nx(x + y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^k y^{n-k} \). Thus \( x_1 + x_2 = 1 \) implies that \( \sum_{k=0}^{n} \binom{n}{k} kx_1^k x_2^{n-k} = nx_1 \).

Choose \( N_0 > K \) such that \( KC/N_0 < \epsilon/2 \). Then \( N > N_0 \) implies
\[
\left| \frac{1}{N} \sum_{k=0}^{N} \binom{N}{k} x_1^k x_2^{N-k} E(\varrho^{\otimes k}) - x_1E^\infty(\varrho) \right| = \left| \frac{1}{N} \sum_{k=0}^{N} \binom{N}{k} kx_1^k x_2^{N-k} g(k) - x_1L \right|
\]
\[
= \left| \frac{1}{N} \sum_{k=0}^{N} \binom{N}{k} kx_1^k x_2^{N-k} g(k) - L \right|
\]
\[
\leq \frac{1}{N} \sum_{k=0}^{K} \binom{N}{k} kx_1^k x_2^{N-k} C
\]
\[
+ \frac{1}{N} \sum_{k=K+1}^{N} \binom{N}{k} kx_1^k x_2^{N-k} (g(k) - L)
\]
\[
\leq KC/N + \epsilon/2 \sum_{k=K+1}^{N} \binom{N}{k} x_1^k x_2^{N-k}
\]
\[
\leq \epsilon.
\]
Continuity (E3) is not mentioned in Proposition 13, although we could use Lemma 12 to deduce upper-semicontinuity from (E3) and (E5'), as the infimum of a family of real continuous functions is upper-semicontinuous. For an example which may be relevant, consider the sequence of functions on [0, 1] defined by \( f_n(x) = nx^2 \). Clearly \( f_{m+n}(x) \leq f_m(x) + f_n(x) \). \( g_n(x) = x^n \) converges (pointwise) as \( n \to \infty \) to a discontinuous, but upper-semicontinuous, function.

6 Examples of important entanglement measures

In this section we will present some important entanglement measures and check which of the postulates from Section 5 they satisfy.

6.1 Operational measures

Here we shall describe two entanglement measures, entanglement of distillation and entanglement cost [1], which are defined in terms of specific state conversions.

Lemma 15 Let \( \varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \) with \( \mathcal{H}^A \equiv \mathcal{H}^B \equiv \mathcal{H} \) and \( \dim \mathcal{H} = d \). Let \( |\phi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \) be a separable wavefunction and \( P_+^d \) be a representative of \( P_+(\mathbb{C}^d) \) on \( \mathcal{H}^A \otimes \mathcal{H}^B \). Then there exist LQCC operations \( \Lambda_1 \) and \( \Lambda_2 \) such that \( \Lambda_1(\varrho) = |\phi\rangle\langle\phi| \) and \( \Lambda_2(P_+^d) = \varrho \).

Proof: Let \( (\psi_i^A)_{i=1}^d \) (resp. \( (\psi_i^B)_{i=1}^d \)) be an orthonormal basis for \( \mathcal{H}^A \) (resp. \( \mathcal{H}^B \)) and define \( \Lambda_1 \) by

\[
\Lambda_1(\sigma) = \sum_{i,j=1}^d (|\phi^A\rangle \otimes |\phi^B\rangle \langle\psi_i^A| \otimes 1_B) \sum_{i=1}^d (|\phi^A\rangle \langle\psi_i^A| \otimes 1_B) \sigma (|\phi^A\rangle \langle\phi^A| \otimes 1_B) = |\phi\rangle \text{tr}(\sigma) \langle\phi| = |\phi\rangle \langle\phi|
\]

for all \( \sigma \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \).

For \( \Lambda_2 \), we note that if \( |\Psi\rangle\langle\Psi| \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \) is any pure state, then, by Nielsen’s theorem, there exists an LQCC operation mapping \( P_+^d \) to \( |\Psi\rangle\langle\Psi| \) because the distribution \( (\frac{1}{d})_{i=1}^d \) is majorized by any probability distribution on \( \{1, \ldots, d\} \). Now, as in the proof of Lemma 11, we can construct \( \Lambda_2 \) as a convex combination of operations mapping \( P_+^d \) to pure components of \( \varrho \). \( \blacksquare \)

Given a state \( \varrho \) on \( \mathcal{H}^A \otimes \mathcal{H}^B \), consider a sequence of LQCC operations \( (\Lambda_n) \) with \( \Lambda_n : \mathcal{T}((\mathcal{H}^A)^{\otimes n} \otimes (\mathcal{H}^B)^{\otimes n}) \to \mathcal{T}((\mathcal{H}^A)^{\otimes n} \otimes (\mathcal{H}^B)^{\otimes n}) \). Suppose, that \( \sigma_n \equiv \Lambda_n(\varrho^{\otimes n}) \) satisfies

\[
\|P_+^d - \sigma_n\|_1 \to 0
\]

for some representative \( P_+^d \) of \( P_+(\mathbb{C}^d) \) on \( (\mathcal{H}^A)^{\otimes n} \otimes (\mathcal{H}^B)^{\otimes n} \). We call such a sequence \( (\Lambda_n) \) an LQCC distillation protocol. The asymptotic ratio attainable via this protocol is then defined by

\[
E_D((\Lambda_n), \varrho) \equiv \limsup_{n \to \infty} \frac{\log_2 d_n}{n}.
\]

(14)

Lemma 13 shows that, for any state, a distillation protocol always exists with \( d_n \equiv 1 \).

Definition 16 The distillable entanglement or entanglement of distillation \( E_D \) is defined as the supremum of Equation (14) over all possible LQCC distillation protocols:

\[
E_D(\varrho) \equiv \sup_{(\Lambda_n)} E_D((\Lambda_n), \varrho).
\]

(15)
By construction $E_D$ satisfies the properties (E2) and (E4) of entanglement measures. The proof is analogous to the proof of Lemma 1 in [29]. It is not known whether $E_D$ satisfies (E3) or (E6). (Indeed, as already mentioned, there is evidence that (E6) may not be satisfied [8]). We shall confirm in Lemma 24 that (E0) and (E1) are satisfied.

The so-called entanglement cost $E_C$ is defined in a complementary way. Given a state $\varrho$ consider a sequence of LQCC operations $\Lambda_n : T(\mathbb{C}^{d_n^A} \otimes \mathbb{C}^{d_n^B}) \to T((\mathcal{H}^A)^{\otimes n} \otimes (\mathcal{H}^B)^{\otimes n})$ transforming a representative of $P_+(\mathbb{C}^{d_n})$ into a state $\sigma_n$ such that

$$\|\sigma_n - \varrho^{\otimes n}\|_1 \to 0.$$  

The asymptotic ratio attainable via this formation-protocol is then given by

$$E_C((\Lambda_n) , \varrho) \equiv \liminf_{n \to \infty} \frac{\log_2 d_n}{n}.$$  

(16)

Once again Lemma 15 shows that, for any state, a formation protocol always exists with $d_n \equiv d_n$ where $d = \max\{\dim \mathcal{H}^A, \dim \mathcal{H}^B\}$.

**Definition 17** The entanglement cost $E_C$ is defined as the infimum of Equation (16) over all possible LQCC formation protocols:

$$E_C(\varrho) \equiv \inf_{\{\Lambda_n\}} E_C((\Lambda_n) , \varrho).$$  

(17)

By construction $E_C$ satisfies property (E2). As we shall discuss in the next section, by [29] and Proposition 13, it also satisfies (E0), (E1), (E2), (E4), (E5), and (E6). It is not known whether it satisfies (E3). We shall also prove below that for pure states both $E_D$ and $E_C$ are equal to the reduced von Neumann entropy given by Equation (3). (This was first realized in [25] and a rigorous proof was sketched in [22].)

6.2 Abstract measures

The entanglement measures discussed in this subsection quantify entanglement mathematically but their definitions do not admit a direct operational interpretation in terms of entanglement manipulations. The first one is the so-called entanglement of formation [1] which is defined as follows:

**Definition 18** Let $\mathcal{H}^A$ and $\mathcal{H}^B$ be finite dimensional Hilbert spaces and let $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$, then the entanglement of formation is defined for pure states as

$$E_F(\psi) := S_{vN}(\psi),$$  

(18-a)

where $S_{vN}(\psi)$ (defined in Equation (3)) is the von Neumann entropy of either of the reduced density matrices of $|\psi\rangle$. For mixed states $\varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B)$ we define

$$E_F(\varrho) := \inf \sum_i p_i E_F(\psi_i),$$  

(18-b)

where the infimum is taken over all possible decompositions of $\varrho$ of the form $\varrho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ with $p_i \geq 0$ for all $i$ and $\sum_i p_i = 1$.

The entanglement of formation satisfies (E0) – (E3), (E5), and (E6). In particular, (E2) was shown in Ref. [1], (E3) in Ref. [1], and (E0), (E1), (E5), and (E6) follow directly from the definition of $E_F$.

The entanglement of formation $E_F$ is believed but not known to be equal to the entanglement cost $E_C$. However, it is known that the regularized entanglement of formation $E_F^\infty$ (which exists by (E5′)), is equal to the entanglement cost [29]. This allows us to apply Proposition 13 to $E_C$.

Let us now present another important measure, namely, the relative entropy of entanglement [26, 2]. It is defined as follows

$$E_R(\varrho) \equiv \inf_{\sigma} S_{rel}(\varrho|\sigma),$$  

(19)
where \( S_{\text{rel}}(\rho|\sigma) \equiv \text{tr} \log_2 \rho - \text{tr} \log_2 \sigma \) is the quantum relative entropy, and where the infimum is taken over all separable states \( \sigma \). One can consider variations of the above measure, by changing the set of states over which the infimum is taken (this set should be closed under LQCC operations though). Like the entanglement of formation, \( E_R \) satisfies (E0)–(E3), (E5), and (E6). In particular, (E1) and (E2) were shown in Ref. [26], (E3) in Ref. [3], (E0) follows immediately and (E5) almost immediately from the definition of \( E_R \), (E6) follows from the convexity of the quantum relative entropy \( S_{\text{rel}} \).

The properties of \( E_R \) and Proposition 13 show that the regularized relative entropy of entanglement \( E^\infty_R \) exists and satisfies (E0), (E1), (E2), (E4), (E5), and (E6). It is shown in [10] that \( E_R \) does not satisfy (E4).

This implies, of course, that \( E_R \) and \( E^\infty_R \) are not always equal (cf. [34]).

Finally, let us note that for pure states both the entanglement of formation (by definition) and the relative entropy of entanglement (as shown in [2], [35]) are equal to the reduced von Neumann entropy \( S_{\text{vN}} \) (defined in Equation (3) above). An immediate consequence of the additivity of \( S_{\text{vN}} \) is that \( E^\infty_F = E^\infty_C \) and \( E^\infty_R \) are also equal to \( S_{\text{vN}} \) on pure states (see also Theorem 23).

### 7 Entanglement of distillation and entanglement cost as extreme measures

In this section we improve the theorem of Ref. [4] by giving precise conditions under which \( E_D \) and \( E_C \) are lower and upper bounds for entanglement measures. We propose three versions of the theorem.

**Proposition 19** Suppose that \( E \) is an entanglement measure defined on mixed states which satisfies (E1)–(E4). Then for all states \( \rho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \)

\[
E_D(\rho) \leq E(\rho) \leq E_C(\rho).
\] (20)

**Proof:** Choose \( \epsilon > 0 \). We shall prove the result in three steps:

I. First we prove that, having if necessary passed to a subsequence, there exists an integer \( N_1 > 0 \) such that \( n \geq N_1 \) implies

\[
\frac{E(\rho^\otimes n)}{n} \geq E_D(\rho) - \epsilon.
\] (21)

Consider a near-optimal LQCC protocol \( (\Lambda_n)_n \). By the definition of distillable entanglement, there exists a LQCC protocol \( (\Lambda_n)_n \) such that, after possibly passing to a subsequence,

\[
\left\| P^d_n - \Lambda_n(\rho^\otimes n) \right\|_1 \to 0
\] (22-a)

and

\[
\left| E_D(\rho) - \frac{\log_2 d_n}{n} \right| \leq \epsilon/2
\] (22-b)

for all \( n \geq N_1' \). (E3) implies that

\[
\left| \frac{E(\Lambda_n(\rho^\otimes n)) - E\left( P^d_n^+ \right)}{1 + n \log_2 d} \right| \to 0
\] (23)

as \( n \to \infty \) where \( d = \text{dim} \mathcal{H}^A \otimes \mathcal{H}^B \). It follows that we can choose \( N_1'' > 0 \) such that \( n \geq N_1'' \) implies

\[
\left| \frac{E(\Lambda_n(\rho^\otimes n))}{n} - \frac{E\left( P^d_n^+ \right)}{n} \right| \leq \epsilon/2
\] (24)
and so, using (E2), for \( n \geq N_1 = \max\{N_1', N_1''\}, \)
\[
\frac{E(\rho^{\otimes n})}{n} \geq \frac{E(\Lambda_n(\rho^{\otimes n}))}{n} \geq \frac{E\left( P^d_n \right)}{n} - \epsilon/2 = \frac{\log_2 d_n}{n} - \epsilon/2 \geq E_D(\rho) - \epsilon. \tag{25}
\]

**II.** As a second step, we prove that, having if necessary passed to another (perhaps disjoint) subsequence, there exists an integer \( N_2 \geq N_1 \) such that \( n \geq N_2 \) implies
\[
\frac{E(\rho^{\otimes n})}{n} \leq E_C(\rho) + \epsilon. \tag{26}
\]
This is similar to the first step. Consider a near-optimal protocol \((\Lambda_n)_n\) for \( \rho \). We have (after possibly passing to a suitable subsequence of \((\Lambda_n)_n\), for all sufficiently large \( n \),
\[
\frac{E(\rho^{\otimes n})}{n} \leq \frac{E\left( \Lambda_n \left( P^d_n \right) \right)}{n} + \epsilon/2 \leq \frac{E\left( P^d_n \right)}{n} + \epsilon/2 = \frac{\log_2 d_n}{n} + \epsilon/2 \leq E_C(\rho) + \epsilon. \tag{27}
\]

**III.** The final step is to invoke (E4) to give
\[
E_D(\rho) - \epsilon \leq E(\rho) = \frac{E(\rho^{\otimes n})}{n} \leq E_C(\rho) + \epsilon. \tag{28}
\]

Unfortunately, as we do not at present know of any function for which we can prove that postulates (E1)–(E4) hold for all states, it is possible that Proposition 19 may be empty. Nevertheless, by modifying the final step of the proof, we can obtain the following:

**Proposition 20** Let \( E \) be an entanglement measure defined on mixed states and satisfying (E1), (E2), (E3), and (E3'). Then for all states \( \rho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \),
\[
E_D(\rho) \leq E^\infty(\rho) \leq E_C(\rho). \tag{29}
\]

**Proof:** Without using condition (E4) or any properties of \( E^\infty \) except its existence, we can maintain the structure of the previous proof, simply by replacing \( E(\rho) \) in (28) by \( E^\infty(\rho) \).

Proposition 20 is certainly non-empty. Indeed, as mentioned in the previous section, both the entanglement of formation and the relative entropy of entanglement satisfy all assumptions of the proposition. We obtain

**Corollary 21** The entanglement of distillation \( E_D \) is less than or equal to the entanglement cost \( E_C \) for all states.

Although, in physical terms, Corollary 21 seems almost necessary, a rigorous proof requires some control both over changes in state and over changes in dimension.

Let us now consider yet another version, where we weaken the assumptions in the theorem on extreme measures of Ref. 4. We impose the condition (E3') which is stronger than (P3) but weaker than (E3).

One mechanism for deriving condition (E3') which is stronger than (P3) but weaker than (E3). This follows immediately from two facts:

(i) Fannes inequality \[31, 32\]
\[
|S(\sigma) - S(\rho)| \leq \|\sigma - \rho\|_1 \log_2 \dim \mathcal{H} + \eta(\|\sigma - \rho\|_1) \tag{31}
\]
which holds for any two states \( \sigma \) and \( \rho \) acting on the Hilbert space \( \mathcal{H} \) and satisfying \( \|\sigma - \rho\|_1 \leq \frac{1}{3} \); here \( \eta(s) \equiv -s \log s \) and \( S \) denotes the standard von Neumann entropy as above;

16
\( \| \sigma_A - \varrho_A \|_1 \leq \| \sigma - \varrho \|_1 \) where \( \sigma_A \) and \( \varrho_A \) are the reduced density operators of \( \sigma \) and \( \varrho \) respectively.

With the above choices for \( f \) and \( g \) one can show that \( E_F \) and \( E_R \) satisfy the inequalities in (30) see [1, 2, 33, 35]. Then, \( E_R^\infty \) and \( E_F^\infty \) also satisfy inequalities (30), because the additivity of the von Neumann entropy implies that both \( f \) and \( g \) satisfy (E4). \( E_D \) also satisfies the inequality \( E_D \leq g \) but we do not know whether or not it satisfies the second inequality. However, a stronger inequality (the so-called hashing inequality), which would have many interesting implications, was conjectured in Ref. [36]. Strong evidence for this conjecture was collected there.

We shall also use the weak form of convexity (E6').

**Proposition 22** Let \( E \) be an entanglement measure defined on mixed states and satisfying (E1), (E2), (E3'), and (E6'). Then for all states \( \varrho \in \Sigma(\mathcal{H}^A \otimes \mathcal{H}^B) \) we have

\[
E_D(\varrho) \leq E(\varrho) \leq E_C(\varrho)
\]

if (E4) holds and

\[
E_D(\varrho) \leq E^\infty(\varrho) \leq E_C(\varrho)
\]

if (E5) holds.

**Proof:** Step I of the proof of Proposition 19 goes through with (E3') replacing (E3) in inequality (24).

To replace step II, we use the estimate \( E_C \geq E_F^\infty \). This follows from Proposition 20 (but also of course from Ref. [29] where it was shown that \( E_C = E_F^\infty \)). For any state \( \varrho \) consider its finite decompositions into pure states

\[
\varrho = \sum_i p_i |\psi_i\rangle \langle \psi_i|
\]

for which

\[
E_F(\varrho) = \sum_i p_i S_{vN}(P_{\psi_i}).
\]

In Ref. [30] it was shown that such a decomposition exists.

As \( (E1) \Rightarrow (P1') \), \( (E2) \Rightarrow (P2) \), and \( (E3') \Rightarrow (P3) \), we can apply Theorem 23 below to show that \( E(P_{\psi_i}) = S_{vN}(P_{\psi_i}) \) if \( E \) satisfies (E4) and \( E^\infty(P_{\psi_i}) = S_{vN}(P_{\psi_i}) \) if \( E \) satisfies (E5).

Now (E6') implies, in the first case, that \( E(\varrho) \leq E_F(\varrho) \) (cf. [30]) and hence

\[
E(\varrho) = \frac{E(\varrho^{\otimes n})}{n} \leq \frac{E_F(\varrho^{\otimes n})}{n}
\]

which yields the required upper bound when \( n \to \infty \). For the second case, we can use the proof of part (6) of Proposition 13 to show that (E6') holds for \( E^\infty \). This yields \( E^\infty(\varrho) \leq E_F(\varrho) \) and

\[
E^\infty(\varrho) = \frac{E^\infty(\varrho^{\otimes n})}{n} \leq \frac{E_F(\varrho^{\otimes n})}{n}.
\]

Again the required bound follows on taking \( n \to \infty \).

8 The uniqueness theorem for entanglement measures

**Theorem 23** Let \( E \) be a functional on pure states. Then the following are equivalent

(1) \( E \) satisfies (P1'), (P2), (P3), and (P4').

(2) \( E \) satisfies (P0), (P1), (P2), (P3), and (P4).

17
(3) $E$ coincides with the reduced von Neumann entropy $E = S_N$.

On the other hand, if $E$ satisfies (P0), (P1), (P2), and (P3), then $E$ satisfies (P3') and, on pure states, $E^\infty = S_N$.

**Proof:** The equivalence of (1) and (2) is proved in Lemmas [10] and [11].

It is clear that the reduced von Neumann entropy satisfies (P0), (P1) and (P4). (P3) follows from the facts (i) and (ii) of the previous section. Finally (P2) is a consequence of Nielsen's Theorem and the fact that the von Neumann entropy is a Schur-concave function [37]. Indeed, with the inductive decomposition operations introduced in our proof of Nielsen’s theorem, we can prove (P2) just by showing, in the notation of Equation (12), that $S_{vN}(\Lambda(|\Psi\rangle\langle\Psi|)) \leq S_{vN}(|\Psi\rangle\langle\Psi|)$. This amounts to proving that, for $p_J \geq p_K$ and suitable $\delta$,

$$-(p_J + \delta) \log_2(p_J + \delta) - (p_K - \delta) \log_2(p_K - \delta) \leq -p_J \log_2 p_J - p_K \log_2 p_K$$

and this is easily confirmed by differentiating with respect to $\delta$.

Now suppose that $E$ satisfies (P0), (P1), (P2), and (P3). Using (P2'), we may assume that $\mathcal{H}^A \equiv \mathcal{H}^B \equiv \mathcal{H}$. Suppose that $\dim \mathcal{H} = d$ and let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$. Write $S \equiv S_{vN}(|\psi\rangle\langle\psi|)$ for the von Neumann entropy of the reduced density matrix of $|\psi\rangle$. Consider $n$ copies of the wavefunction $|\psi\rangle$: $|\psi^\otimes n\rangle \in \mathcal{H}_{tot} \equiv \mathcal{H}^{\otimes n} \otimes \mathcal{H}^{\otimes n}$. Let $\{q_j : j = 1, \ldots, d\}$ be the set of eigenvalues of the reduced density matrix of $|\psi\rangle$ and $\{p_i : i = 1, \ldots, d^{2n}\}$ be the set of eigenvalues of the reduced density matrix of $|\psi^\otimes n\rangle$. Again using (P2'), we may adjust $d$ so that $q_j > 0$ for $j = 1, \ldots, d$. In view of (P0), we may also assume that $S > 0$. Considered as a probability distribution, $\{p_i\}$ is the distribution for $n$ independent trials each with distribution $\{q_j\}$. Choose bases $(e_i) \subset \mathcal{H}^{\otimes n}$ and $(f_i) \subset \mathcal{H}^{\otimes n}$ such that

$$|\psi^\otimes n\rangle = \sum_i \sqrt{p_i} |e_i\rangle \otimes |f_i\rangle.$$

Choose $\epsilon > 0$. By the asymptotic equipartition theorem ([38] Theorem 3.1.2), there exists an integer $N \equiv N(\epsilon)$ such that, for all $n \geq N$, one can find a subset $\text{TYP} \equiv \text{TYP}(n, \epsilon)$ of the set of indices $\{i\}_{i=1}^{d^{2n}}$ with the following properties:

$$2^{-n(3+\epsilon)} \leq p_i \leq 2^{-n(3-\epsilon)}, \quad \text{for } i \in \text{TYP},$$

$$p \equiv \sum_{i \in \text{TYP}} p_i \geq 1 - \epsilon,$$

$$\#\text{TYP} \leq 2^{n(3+\epsilon)}.$$  \(34\text{-c})

Here $\#\text{TYP}$ denotes the number of elements in $\text{TYP}$.

Introduce another wavefunction $|\phi_n\rangle \in \mathcal{H}_{tot}$ given by

$$|\phi_n\rangle \equiv \frac{1}{\sqrt{p}} \sum_{i \in \text{TYP}} \sqrt{p_i} |e_i\rangle \otimes |f_i\rangle.$$

This wavefunction satisfies

$$|\langle \psi^\otimes n |\phi_n\rangle|^2 = p \geq 1 - \epsilon$$

and so

$$|||\psi^\otimes n\rangle\langle\psi^\otimes n| - |\phi_n\rangle\langle\phi_n||_1| = 2\sqrt{(1 - |\langle \psi^\otimes n |\phi_n\rangle|^2)} \leq 2\sqrt{\epsilon}.$$  \(36\)

Now, the crucial observation (cf. [22]) is that for $\epsilon < \min\{\frac{1}{2}S, \frac{1}{2}\}$ and $n$ sufficiently large, there exist completely positive maps $\Lambda_n$ and $\Lambda'_n$ such that

$$\Lambda_n(|\phi_n\rangle \langle \phi_n|) = P^a$$  \(37\text{-a})
for $P^+_n$ a representative of $P_+ (\mathbb{C}^n)$ in $\mathcal{H}_{\text{tot}}$ with $\left| \frac{\log_2 a}{n} - S \right| < \epsilon + \frac{2}{n}$ and

$$\Lambda'_n (P^+_n) = |\phi_n \rangle \langle \phi_n|$$

for $P^+_n$ a representative of $P_+ (\mathbb{C}^n)$ in $\mathcal{H}_{\text{tot}}$ with $\left| \frac{\log_2 b}{n} - S \right| < \epsilon + \frac{1}{n}$. Indeed, to see Equation (37-a), set $a = |p2^{n(\frac{\epsilon}{p})}|$, i.e., $a$ is the largest integer smaller than or equal to $p2^{n(\frac{\epsilon}{p})}$. Then $a \leq p2^{n(\frac{\epsilon}{p})} \leq p/p_i$ and we see that the distribution $\{ \frac{1}{p} \}_{p \in \mathbb{N}}$ is majorized by $\{ \frac{1}{a} \}_{a \in \mathbb{N}}$, hence Equation (37-a) follows from Nielsen’s Theorem. Equation (37-b) follows by a similar argument when we take $b = |p2^{n(\frac{\epsilon}{p})}|$, i.e., $b$ is the smallest integer larger than or equal to $p2^{n(\frac{\epsilon}{p})}$. The conditions on $\epsilon$ and $n$ are sufficient to go from $a = \lfloor p2^{n(\frac{\epsilon}{p})} \rfloor$ to $\left| \frac{\log_2 a}{n} - S \right| < \epsilon + \frac{2}{n}$ and from $b = \lfloor p2^{n(\frac{\epsilon}{p})} \rfloor$ to $\left| \frac{\log_2 b}{n} - S \right| < \epsilon + \frac{1}{n}$, ensuring, for example, that $a \neq 0$.

Now choose a sequence $(\epsilon_j)_{j \in \mathbb{N}}$ of positive numbers such that $\epsilon_j \to 0$ for $j \to \infty$. Suppose that $(n_k)_{k \in \mathbb{N}}$ is a sequence of integers such that $n_k \to \infty$ and $E((\psi \otimes nk) (\psi \otimes nk)) \to L$ for some $L$.

For each $j$, choose $n_{kj} \geq \max\{N(\epsilon_j), 1/\epsilon_j\}$. We can apply the postulates (P0)–(P3) to obtain the following estimates:

$$\frac{E((\psi \otimes nk_j)(\psi \otimes nk_j))}{n_{kj}} = \frac{E((\psi \otimes nk_j)(\psi \otimes nk_j)) - E(|\phi_n_k_j \rangle \langle \phi_n_k_j|)}{n_{kj}} + \frac{E(|\phi_n_k_j \rangle \langle \phi_n_k_j|)}{n_{kj}}$$

As $j \to \infty$, the first term vanishes due to (P3) and the second approaches $S_{\infty}(P_\psi)$ (cf. Ref. [1]). This implies that $L \geq S_{\infty}(|\psi \rangle \langle \psi|)$. The proof of the inequality $L \leq S_{\infty}(|\psi \rangle \langle \psi|)$ is similar:

$$\frac{E((\psi \otimes nk_j)(\psi \otimes nk_j))}{n_{kj}} = \frac{E(|\phi_n_k_j \rangle \langle \phi_n_k_j|)}{n_{kj}} + \frac{E(|\phi_n_k_j \rangle \langle \phi_n_k_j|)}{n_{kj}}$$

We have now shown that every limit point of the sequence $\frac{E((\psi \otimes nk)(\psi \otimes nk))}{n}$ has the value $L = S_{\infty}(|\psi \rangle \langle \psi|)$.

It is natural to wonder whether the conditions in Theorem 23 can be weakened, and, in particular, whether (P3) is necessary. That it has been noted by Vidal [1]. Consider the entanglement measures defined on pure states by $S_{\infty}(\sigma) = -\log_2 p_1(\sigma)$ where $p_1(\sigma)$ is the largest coefficient in a Schmidt decomposition of $\sigma$ and by $S_0(\sigma) = \log d(\sigma)$ where $d$ is the number of non-zero coefficients. $S_0$ and $S_{\infty}$ both satisfy (P0), (P1), (P2) (by Nielsen’s theorem), and (P4). $S_{\infty}$ is even trace norm continuous on Hilbert spaces of fixed dimension. (P3) however does not hold for either. This is, of course, a consequence of Theorem 23. An explicit example of the failure of (P3) for $S_{\infty}$ is provided by the states $\sigma_n \equiv |\Psi_n \rangle \langle \Psi_n|$, $\phi_n \equiv |\Phi_n \rangle \langle \Phi_n|$. 

with Schmidt decompositions $|\Psi_n\rangle \equiv \sqrt{\frac{1}{2^n}}|\psi_1\rangle + \sum_{i=2}^{4^n-2^n+1} \frac{1}{2^n}|\psi_i\rangle$ and $|\Phi_n\rangle \equiv \sum_{i=1}^{4^n} \frac{1}{2^n}|\psi_i\rangle$ for some orthonormal family $\{|\psi_i\rangle\}$ of wavefunctions. In fact, any entanglement measure $E$ defined on pure states and satisfying (P0), (P1), (P2), and (P4), will satisfy $S_\infty(\sigma) \leq E(\sigma) \leq S_0(\sigma)$ for all pure $\sigma$. The upper bound here is a consequence of Lemma 17 while, for the lower bound, we modify the proof of Theorem 23 using the fact that $|\psi^{\otimes n}\rangle \langle \psi^{\otimes n}|$ can always be converted without approximation into $P_+^c$ where $c$ is the largest integer smaller than or equal to $1/p_1$.

An example of a measure on pure states satisfying (P0), (P1), (P2), (P3), but not (P4), is given by $E(\sigma) = 2(1 - p_1(\sigma))S_\infty(\sigma)$ for $p_1(\sigma) \geq \frac{1}{2}$, $E(\sigma) = S_0(\sigma)$ for $p_1(\sigma) \leq \frac{1}{2}$.

Finally, let us consider entanglement of distillation and entanglement cost in the above context. Using the maps constructed in Theorem 23, we show that they are equal to $S_N$. We have already noted that for $E_C$ this also follows from Lemma 24.

**Lemma 24** The entanglement of distillation $E_D$ and the entanglement cost $E_C$ both coincide on pure states with the von Neumann reduced entropy $E_D(\rho_\psi) = E_C(\rho_\psi) = S_N(\rho_\psi)$ for all $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$.

**Proof:** From Section 2 we know that $E_D \leq E_C$. It suffices to show that on pure states $E_D \geq S_N$ and $E_C \leq S_N$. We will continue to use the notation from the proof of Theorem 23.

That $E_C(\rho_\psi) \leq S_N(\rho_\psi)$ follows directly from the definition of $E_C$, using the operations defined by the $A'_\psi$, which satisfy Equation (37-a) and estimate (36).

To show that $E_D(\rho_\psi) \geq S_N(\rho_\psi)$, let us apply the map $A_{n_j}$ from Equation (37-a) to the state $|\psi^{\otimes n_j}\rangle \langle \psi^{\otimes n_j}|$. We only need check that the resulting state $A_{n_j}(|\psi^{\otimes n_j}\rangle \langle \psi^{\otimes n_j}|)$ approaches $P_+^n$ as $j \to \infty$. But, by Lemma 4,

$$\|A_{n_j}(|\psi^{\otimes n_j}\rangle \langle \psi^{\otimes n_j}|) - P^n\|_1 = \|A_{n_j}(|\psi^{\otimes n_j}\rangle \langle \psi^{\otimes n_j}|) - A_{n_j}(|\phi_{n_j}\rangle \langle \phi_{n_j}|)\|_1 \leq \|\psi^{\otimes n_j}\rangle \langle \psi^{\otimes n_j}| - |\phi_{n_j}\rangle \langle \phi_{n_j}|\|_1$$

and once again estimate (36) is sufficient.

With the results obtained in this paper, we can now prove that $E_D$ is convex on pure decompositions, i.e.,

**Lemma 25**

$$E_D \left( \sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \leq \sum_i p_i E_D (|\psi_i\rangle \langle \psi_i|),$$

where $p_i \geq 0$ for all $i$ and $\sum_i p_i = 1$.

**Proof:** We have seen that $E_C$ is convex and satisfies $E_D \leq E_C$. Using Lemma 24 gives

$$E_D \left( \sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \leq E_C \left( \sum_i p_i |\psi_i\rangle \langle \psi_i| \right) \leq \sum_i p_i E_C (|\psi_i\rangle \langle \psi_i|) = \sum_i p_i S_N(\rho_\psi) = \sum_i p_i E_D (|\psi_i\rangle \langle \psi_i|).$$

**References**

[1] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A 54, 3824 (1996), quant-ph/9604024.

[2] V. Vedral and M.B. Plenio, Entanglement measures and purification procedures, Phys. Rev. A 57, 1619 (1998), quant-ph/9707032.

[3] G. Vidal, Entanglement monotones, Journ. Mod. Opt. 47, 355 (2000), quant-ph/9807077.
[4] M. Horodecki, P. Horodecki and R. Horodecki, Limits for entanglement measures, Phys. Rev. Lett. 84, 2014 (2000), quant-ph/9908065.
[5] S. Popescu, D. Rohrlich, Thermodynamics and the measure of entanglement, Phys. Rev. A 56, R3319 (1997), quant-ph/961004.
[6] M.A. Nielsen, Continuity bounds for entanglement, Phys. Rev. A 61, 64301 (2000), quant-ph/9908086.
[7] An analogous result for the finite regime was obtained in Ref. [3].
[8] P.W. Shor, J.A. Smolin and B.M. Terhal, Nonadditivity of bipartite distillable entanglement follows from conjecture on bound entangled Werner states, Phys. Rev. Lett. 86, 2681 (2001), quant-ph/0010054.
[9] M.J. Donald and M. Horodecki, Continuity of relative entropy of entanglement, Phys. Lett. A 264, 257 (1999), quant-ph/9910002.
[10] K.G.H. Vollbrecht and R.F. Werner, Entanglement measures under symmetry, Phys. Rev. A 64, 062307 (2001), quant-ph/0010095.
[11] N. Linden, S. Popescu, B. Schumacher and M. W. Westmoreland, Reversibility of local transformations of multiparticle entanglement, quant-ph/9912039.
[12] O. Rudolph, A uniqueness theorem for entanglement measures, J. Math. Phys. 42, 2507 (2001), quant-ph/0105014.
[13] O. Rudolph, A new class of entanglement measures, J. Math. Phys. 42, 5306 (2001), math-ph/0005011.
[14] E. Schmidt, Zur Theorie der linearen und nichtlinearen Integralgleichungen. I. Teil: Entwicklung willkürlicher Funktionen nach Systemen vorgeschriebener, Math. Annalen 63, 433 (1907).
[15] K. Kraus, General state changes in quantum theory, Ann. Phys. (N.Y.) 64, 311 (1971).
[16] E.B. Davies, Quantum Theory of Open Systems, Academic, London, 1976.
[17] K. Kraus, States, Effects and Operations, Springer-Verlag, Berlin, 1983.
[18] M.-D. Choi, Completely positive linear maps on complex matrices, Lin. Alg. Appl. 10, 285 (1975).
[19] M.J. Donald, M. Horodecki, and O. Rudolph, The uniqueness theorem for entanglement measures, quant-ph/0105017 v1.
[20] E.M. Rains, An improved bound on distillable entanglement, Phys. Rev. A 60, 179 (1999), quant-ph/9809082.
[21] C.H. Bennett, D.P. DiVincenzo, C.A. Fuchs, T. Mor, E. Rains, P.W. Shor, J.A. Smolin and W.K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A 59, 1070 (1999), quant-ph/9804053.
[22] M.A. Nielsen, Conditions for a class of entanglement transformations, Phys. Rev. Lett. 83, 436 (1999), quant-ph/9811053.
[23] L. Hardy, Method of areas for manipulating the entanglement properties of one copy of a two-particle pure entangled state, Phys. Rev. A 60, 1912 (1999), quant-ph/9903001.
[24] J.G. Jensen, R. Schack, Simple algorithm for local conversion of pure states, Phys. Rev. A 63, 062303 (2001), quant-ph/0006043.
[25] C.H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A 53, 2046 (1996), quant-ph/9511030.
[26] V. Vedral, M.B. Plenio, M.A. Rippin and P.L. Knight, Quantifying entanglement, Phys. Rev. Lett. 78, 2275 (1997), quant-ph/9702027.
[27] D. Jonathan and M. Plenio, Minimal conditions for local pure-state entanglement manipulation, Phys. Rev. Lett. 83, 1455 (1999), quant-ph/9903054.

[28] P. Walters, An Introduction to Ergodic Theory, Springer-Verlag, New York, 1982.

[29] P. Hayden, M. Horodecki and B. Terhal, The asymptotic entanglement cost of preparing a quantum state, J. Phys. A 34, 6891 (2001), quant-ph/0008134.

[30] A. Uhlmann, Entropy and optimal decompositions of states relative to a maximal commutative subalgebra, Open Sys. & Inf. Dyn. 5, 209 (1998), quant-ph/9704017.

[31] M. Fannes, A continuity property of the entropy density for spin lattice systems, Commun. Math. Phys. 31, 291 (1973).

[32] M. Ohya and D. Petz, Quantum Entropy and Its Use, Springer-Verlag, New York, 1993, p. 22.

[33] P. Horodecki, M. Horodecki and R. Horodecki, Entanglement and thermodynamical analogies, Acta Phys. Slovaca 48, 141 (1998), quant-ph/9805072.

[34] K. Audenaert, J. Eisert, E. Jané, M.B. Plenio, S. Virmani, and B. De Moor, Asymptotic relative entropy of entanglement, Phys. Rev. Lett. 87, 217902 (2001), quant-ph/0103096.

[35] M.B. Plenio, S. Virmani and P. Papadopoulos, Operator monotones, the reduction criterion and the relative entropy, J. Phys. A 33, L193 (2000), quant-ph/0002073.

[36] M. Horodecki, P. Horodecki and R. Horodecki, Unified approach to quantum capacities: towards quantum noisy coding theorem, Phys. Rev. Lett. 85, 433 (2000), quant-ph/0003040.

[37] M.A. Nielsen, Characterizing mixing and measurement in quantum mechanics, Phys. Rev. A 63, 022114 (2001), quant-ph/0008073.

[38] T.M. Cover and J.A. Thomas, Elements of Information Theory, John Wiley, New York, 1991.