Recovering quantum properties of continuous-variable states in the presence of measurement errors

E. Shchukin and P. van Loock
Johannes-Gutenberg University of Mainz, Institute of Physics, Staudingerweg 7, 55128 Mainz

We present two results which combined enable one to reliably detect multimode, multipartite entanglement in the presence of measurement errors. The first result leads to a method to compute the best (approximated) physical covariance matrix given a measured non-physical one. The other result states that a widely used entanglement condition is a consequence of negativity of partial transposition. Our approach can quickly verify entanglement of experimentally obtained multipartite states, which is demonstrated on several realistic examples. Compared to existing detection schemes, ours is very simple and efficient. In particular, it does not require any complicated optimizations.

PACS numbers: 03.67.Mn, 03.65.Ud, 42.50.Dv
Keywords: continuous variables; multipartite entanglement; bound entanglement

Introduction. Measurement errors are inevitable in real experiments. Not only do they introduce imprecision into the measured data, but the key property of physicality of the object under study may even be violated. This is well known in quantum state tomography, where the reconstructed density matrix may not be positive semidefinite. Techniques like those proposed in Refs. [1, 2] are used to clean up the measured data and produce physical results. Another instance of this nonphysicality problem and one of the main subjects of the present work is related to the covariance matrix (matrix of second-order moments) of a multiparticle or multimode quantum state such as an optical, continuous-variable state: The measured covariance matrix may not satisfy the physicality condition.

There are several approaches to tackle this problem in general. One is to model the experimental setup and the measurement process to deduce the most probable physical set of data obtained in this process. This is the approach taken in Ref. [1]. However, sometimes we do not have the luxury of knowing the measurement process or its model would be exceedingly complicated. In this case we have to recover a physical approximation only from the measured data itself without any reference to the process in which this data was obtained.

The natural question to ask is how good is the obtained physical approximation. To give a reliable answer to this question we need another piece of information in the form of the strength of the measurement errors. If we know the standard deviation, $\sigma$, of the measured quantity from its average value, we can say that our physical approximation is good if it fits into a small $\sigma$ interval centered at the average. It remains to be understood how small this interval should be.

The acceptable size of this interval depends on the standards adopted by the community in different disciplines. For example, in clinical trials a relatively weak $2\sigma$ criterion is used (Refs. [3–5], though the full story of clinical studies is more complicated than expressed by the $2\sigma$ criterion). On the other hand, in particle physics an informal standard refers to results with the significance $3\sigma$ as ”evidence” and as a true discovery if the significance is $5\sigma$ [6]. A recent result of fundamental importance is the discovery of gravitational waves, where the significance level was also reported to be $5\sigma$ [8]. These numbers should not be taken too literally, i.e., that they must be exactly three and exactly five. For example, in Tevatron experiments the observation of the top quark was first reported as ”evidence” with the significance $2.8\sigma$ [9] and later as a real discovery with the significance levels $4.6\sigma–4.8\sigma$ [10, 11]. In general, a significance level of $\sigma$ means that to make the wrong conclusion by chance an event outside of an $\sigma$ interval must be realized in the experiment and the probability of such an event quickly decreases as $s$ increases. In some cases relatively small values of $s$ are acceptable, but for more fundamental results a stronger confidence is needed. We refer to the above results to illustrate that a significance of $5\sigma$ is considered to be sufficient even for the discovery of fundamental properties of Nature, and therefore here we shall not impose stronger criteria for accepting the experimental results.

The results that we present here are twofold. First, we propose an algorithm to obtain the best physical approximation to a measured non-physical covariance matrix. If, in addition, the standard deviations of the individual matrix elements are known then one can also estimate how good this approximation is. The absence of a good physical approximation to the measured matrix is then a signature of inaccuracy of the experiment. Our algorithm is based on semidefinite optimization and there is a very efficient free software for this (and also a much more general) kind of optimization. A low-end desktop PC is enough to perform this algorithm for states with tens of modes.

Another result of our work is to demonstrate that the well-known entanglement condition obtained in Ref. [12] is, in fact, based on negativity of partial transposition (PT). Combined together, the two results give a way to quickly test a multimode quantum state for entanglement even in the presence of measurement errors. This ap-
approach does not require any optimization and it works surprisingly well; sometimes even better than an alternative approach based on genetic optimization proposed in the literature. The simplicity of our method is very attractive in the high-partite case, where otherwise the optimization can take a long time, while taking eigenvectors of matrices, like in our scheme, even with thousands of rows and columns is fast on a common PC.

Measuring covariance matrices. It is well known that a $2n \times 2n$ real symmetric matrix $\gamma = \begin{pmatrix} \gamma_{xx} & \gamma_{xp} \\ \gamma_{xp}^T & \gamma_{pp} \end{pmatrix}$ is a covariance matrix of an $n$-partite quantum state iff it satisfies either of the two equivalent conditions

$$\gamma \pm \frac{i}{2} J \geq 0,$$  

(1)

where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and $E$ is the $n \times n$ identity matrix. The covariance matrices obtained in experiments often violate these conditions, i.e., the matrices on the left-hand side of Eq. (1) have small negative eigenvalues. The natural question to ask is: What is the most probable physical covariance matrix corresponding to the measured slightly non-physical one? Provided that the experiment was performed correctly, the individual measured matrix elements $\gamma_{ij}$ should not differ significantly from the matrix elements $\gamma_{ij}^*$ of the "true" covariance matrix $\gamma^*$. Thus, we can say that the most probable physical matrix is the matrix $\gamma^*$ with $\max_{ij} |\gamma_{ij} - \gamma_{ij}^*|$ as small as possible. This is illustrated by Fig. 1. If we have standard deviations $\sigma_{ij}$ of measurements of the individual matrix elements $\gamma_{ij}$ then the right quantity to minimize is $\max_{1 \leq i, j \leq 2n} |\gamma_{ij} - \gamma_{ij}^*|/\sigma_{ij}$, so that each true matrix element $\gamma_{ij}^*$ lies in an as small sigma interval around the mean value $\gamma_{ij}$ as possible. The solution of this problem can also be used as a test for correctness of the experiment — when the minimized quantity is too large, i.e., there is no physical covariance matrix close to the measured one, then the experiment is likely to have been performed inaccurately.

We can now formulate the optimization problem for the best physical approximation of a non-physical covariance matrix $\gamma^0$:

$$\min_{\gamma} \max_{1 \leq i, j \leq 2n} \frac{|\gamma_{ij} - \gamma_{ij}^0|}{\sigma_{ij}}.$$  

(2)

where the minimization is over all physical covariance matrices $\gamma$. An optimal solution of this problem $\gamma^*$ is the most probable physical covariance matrix corresponding to the measured matrix $\gamma^0$. If $\gamma^0$ happens to be physical from the very beginning then $\gamma^* = \gamma^0$, as it must be. It does not mean that in this case $\gamma^0$ is the true physical matrix of the state, it just means it the best matrix we obtain from this experiment.

Note that if we do not have the $\sigma$ matrix, then we can minimize $\max_{1 \leq i, j \leq 2n} |\gamma_{ij} - \gamma_{ij}^0|$, which formally coincides with the problem (2) where all $\sigma_{ij}$ are set to 1. We denote the solution of this problem by $\gamma^*$. This means that the same approach works both when we have $\sigma$ and when we do not. But in the latter case we will not be able to estimate how good the solution $\gamma^*$ is.

The problem (2) can be solved by formulating it as a semidefinite optimization problem. The conditions (1) can be equivalently written in real form as follows:

$$\begin{pmatrix} \gamma_{xx} & 0 & \gamma_{xp} & \mp \frac{1}{2} E \\ 0 & \gamma_{xx} & \pm \frac{1}{2} E & \gamma_{xp} \\ \gamma_{xp}^T & \pm \frac{1}{2} E & \gamma_{pp} & 0 \\ \mp \frac{1}{2} E & \gamma_{xp}^T & 0 & \gamma_{pp} \end{pmatrix} \geq 0.$$  

(3)

We can also introduce a new real variable, say $s$, which corresponds to the maximum in (2). Then from the condition $\max_{1 \leq i, j \leq 2n} |\gamma_{ij} - \gamma_{ij}^0|/\sigma_{ij} = s$ it follows that the variables of the optimization problem ($s$ and $\gamma_{ij}$) satisfy the following restrictions:

$$s \sigma_{ij} + \gamma_{ij} \geq \gamma_{ij}^0, \quad s \sigma_{ij} - \gamma_{ij} \geq -\gamma_{ij}^0.$$  

(4)

Now the condition $s \geq 0$, Eq. (3) and all the inequalities (4) can be written as a nonnegativity of a single large matrix $M(s, \gamma)$ where $s$ and all individual conditions of Eq. (4) are put on the diagonal in addition to the block of Eq. (3). This matrix is given explicitly in the Appendix. The optimization problem we have to solve reads as

$$\min_{M(s, \gamma) \geq 0} s.$$  

(5)

It is a semidefinite optimization problem with $2n^2 + n + 1$ variables ($s$ and $\gamma$) and the matrix condition of size $4n^2 + 6n + 1$. For a block-diagonal $\gamma_{ij}^0$, $\gamma_{xp} = 0$, the number of variables is $n^2 + n + 1$ and the size of the condition is

FIG. 1. (Color online) Covariance matrix of a bipartite state, two-mode state. The measured elements are shown in black, the most probable (yet unknown) matrix elements are in green. The shape of the probability distribution of each $\gamma_{ij}$ is characterized by the corresponding standard deviation $\sigma_{ij}$, also known from the experiment.
$2n^2 + 6n + 1$. This problem can be easily solved with an appropriate software.

To give some illustrating examples we refer to Ref. [13], where three measured covariance matrices of four-, six- and ten-partite states were analyzed. We have solved the optimization problem for those matrices and found that the minimal values of $s$ for them are 1.88, 0.17 and 0.37, respectively. The computation times to solve these problems are: nearly instantly for the four-partite case, around 3 sec for the six-partite case and 45 sec for the ten-partite case (on one core of a low-end desktop PC). In the latter case the optimization problem has 111 variables and matrix condition of size 261. In the four-partite case the minimal $s$ is large enough and nine of the twenty elements of the optimal matrix are $\approx 1.88\sigma$ away from the corresponding measured values. One could argue that it might be possible to trade quality of approximations of individual elements for the number of elements with large deviations. For example, it might be possible to find a physical matrix where only one or two elements lie within larger interval, say $3\sigma$, but the rest are much closer to the measured values. In other words, it is the total probability that must be minimized, like the product of probabilities of individual elements, not the individual probabilities themselves. Unfortunately, this argument does not work since for a correctly performed experiment all the elements must be measured accurately; it does not help if only one or two are imprecise but the others are measured perfectly. This is the reason to believe that the experiment with four modes was performed far less accurately than the other two, where the results are quite satisfactory. The results are discussed in more detail in the Appendix.

Let us compare our approach to finding a physical covariance matrix given a measured nonphysical one to the approach used in Ref. [13]. The matrix $\gamma'$ considered there as a physical approximation is given by

$$\gamma' = \gamma + 1.001|\lambda_{\min}|E,$$

where $\lambda_{\min}$ is the most negative eigenvalue of $\gamma + (i/2)J$. As we noted before, we should look for the true physical matrix within small distance of the measured one, so each difference $|\gamma_{ij} - \gamma'_{ij}|$ must be a small multiple of $\sigma_{ij}$. The procedure expressed by Eq. [6] changes only the diagonal elements of the matrix $\gamma$, so computing the ratio $|\gamma_{ii} - \gamma'_{ii}|/\sigma_{ii}$ in the four-partite case for the $\gamma_{xx}$ part, we get the values 16.59, 6.59, 6.29, 9.87. For the $\gamma_{pp}$ part, we get 11.84, 5.30, 3.70, 11.91. From the discussion of probabilities in Ref. [14] (and virtually any textbook on statistics) it follows that the probability of the matrix elements being outside of $11\sigma$ interval is smaller than $10^{-27}$, so it is absolutely unrealistic that the matrix $\gamma'$ given by Eq. [6] was the true covariance matrix in the experiment. Our approach gives a physical covariance matrix with much closer elements, but it is still not good enough so that we believe that the four-partite experiment contains inaccurate data. For the six-partite case the matrix $\gamma'$ gives a better approximation with only one element differing from the corresponding element of $\gamma$ by a quantity larger than $2\sigma$. In the ten-partite case some elements differ by $3\sigma-4\sigma$ and thus are unlikely to be the true values of the corresponding matrix elements. In general, there is no obvious reason to be sure that the expression on the right-hand side of Eq. (6) produces a good approximation and we have just seen that indeed it may not. A much more reliable approach is expressed by the solution of the minimization problem (2).

Detecting entanglement. In Ref. [13] the matrices were also analyzed for entanglement by a complicated optimization algorithm. We now demonstrate that comparable (and sometimes even better) results can be obtained immediately without any optimization at all. For this purpose, first, we demonstrate that a condition of Ref. [14] is equivalent to the positivity of partial-transposition test of the covariance matrix, which is a useful result by itself. For arbitrary real $n$-vectors $h$, $h'$, $g$ and $g'$ let us introduce the operators $\hat{u} = (h, x)$, $\hat{u}' = (h', x)$, $\hat{v} = (g, p)$ and $\hat{v}' = (g', p)$. Then the inequality [3] can be equivalently written as follows:

$$\langle (\Delta \hat{u} + \Delta \hat{u}')^2 + (\Delta \hat{v} + \Delta \hat{v}')^2 \rangle \geq \langle (h, g) - (h', g') \rangle. \quad (7)$$

This condition for arbitrary vectors $h$, $h'$, $g$ and $g'$ is equivalent to Eq. [4] and thus is necessary and sufficient for physicality of the covariance matrix. Setting $h' = g' = 0$ we get a simpler but weaker condition

$$\langle (\Delta \hat{u} + \Delta \hat{v})^2 \rangle \geq \langle (h, g) \rangle. \quad (8)$$

This condition is equivalent to the following ones:

$$\begin{pmatrix} \gamma_{xx} & \pm \frac{1}{2}E \\ \pm \frac{1}{2}E & \gamma_{pp} \end{pmatrix} \geq 0, \quad (9)$$

where the matrix on the left-hand side is the central submatrix of the matrix in Eq. [5].

One can get an entanglement condition based on negativity of PT by using a partially transposed matrix $\gamma^{\text{PT}}$ in the conditions Eq. [3] or Eq. [4]. Let us consider a bipartition $I = \{I, J\}$ and partial transposition corresponding to this bipartition. The covariance matrix of the partially transposed state can be obtained from the original one by changing the sign of its rows and columns with indices that are transposed (the diagonal elements change their sign twice and thus remain positive). Applying the condition [9] to this covariance matrix we get the following inequality:

$$\begin{pmatrix} \gamma_{xx} & \pm \frac{1}{2}E_I \\ \pm \frac{1}{2}E_I & \gamma_{pp} \end{pmatrix} \geq 0, \quad (10)$$

which was introduced in Ref. [14]. The matrix $E_I$ has $\pm 1$ on the main diagonal and the other elements are zero.
The diagonal elements with indices from the same group, \(I\) or \(J\), have the same sign. We stress that the two conditions \([9]\), which differ only by the sign of the off-diagonal block, are equivalent and thus only one of them needs to be tested in practice. In vector form the inequalities \([10]\) read as \(\langle (\Delta \hat{u})^2 + (\Delta \hat{v})^2 \rangle \geq \langle |\langle \mathbf{h}_I, \mathbf{g}_J \rangle | - |\langle \mathbf{h}_J, \mathbf{g}_I \rangle | \rangle\), where \(\mathbf{h}_I = \{h_{i_1}, \ldots, h_{i_k}\}\), \(I = \{i_1, \ldots, i_k\}\) and similar notation is used for \(\mathbf{h}_J, \mathbf{g}_I\) and \(\mathbf{g}_J\). Combining it with Eq. \((6)\) and noting that \(\langle \mathbf{h}, \mathbf{g} \rangle = \langle \mathbf{h}_I, \mathbf{g}_J \rangle + \langle \mathbf{h}_J, \mathbf{g}_I \rangle\) we get
\[
\langle (\Delta \hat{u})^2 + (\Delta \hat{v})^2 \rangle \geq \langle |\langle \mathbf{h}_I, \mathbf{g}_J \rangle | + |\langle \mathbf{h}_J, \mathbf{g}_I \rangle | \rangle,
\]
which is exactly the inequality obtained in Ref. \([12]\). This method can be generalized to produce similar inequalities for partitions of modes into several groups, not only two. We see that the results of that work are just a consequence of the positivity of PT applied to the inequality \([9]\), which is a “part” of the more general inequality \([5]\). Thus, a more general condition could be obtained from the inequality \((7)\).

It is not enough just to test partial transpositions of the recovered physical matrix for positive semidefiniteness. In the presence of the measurement errors we need to verify a stronger statement. Not only should we just violate the condition \([11]\), where the left-hand side is computed with the recovered physical covariance matrix, but we should also violate it with some margin. In other words, we must have
\[
\frac{|\langle \mathbf{h}_I, \mathbf{g}_J \rangle | + |\langle \mathbf{h}_J, \mathbf{g}_I \rangle | - \langle (\Delta \hat{u})^2 + (\Delta \hat{v})^2 \rangle}{\sigma(\mathbf{h}, \mathbf{g})} \geq s_0,
\]
where the denominator of the left-hand side reads as
\[
\sigma(\mathbf{h}, \mathbf{g}) = \left( \sum_{i,j=1}^{n} \sigma_{xx,ij}^2 h_i^2 h_j^2 + \sigma_{pp,ij}^2 g_i^2 g_j^2 \right)^{1/2},
\]
the standard deviation of the random variable (a linear combination of the random variables \(\gamma_{xx,ij}\) and \(\gamma_{pp,ij}\))
\[
\langle (\Delta \hat{u})^2 + (\Delta \hat{v})^2 \rangle = \sum_{i,j=1}^{n} \gamma_{xx,ij} \delta_{xx} h_i h_j + \gamma_{pp,ij} g_i g_j,
\]
and \(s_0\) is the chosen level of confidence, say \(s_0 = 3\). If we can find \(\mathbf{h}\) and \(\mathbf{g}\) with such \(s_0\), than we can be sure that the condition \([11]\) is really violated. It has been noted in Ref. \([14]\) that \(|\langle \mathbf{h}_I, \mathbf{g}_J \rangle | + |\langle \mathbf{h}_J, \mathbf{g}_I \rangle | = \text{Tr} \left( \sqrt{X_I P_I} \sqrt{X_J} + \sqrt{X_J} P_J \sqrt{X_I} \right)\), where \(X = \mathbf{h h}^T\), \(P = \mathbf{g g}^T\) are rank-1 matrices and \(X_I\) is the submatrix of \(X\) whose row and column indices are in \(I\). This means that the quantity \([12]\) is a special case of the one used in Ref. \([13]\) to detect entanglement.

The proof of the inequality \([11]\) suggests a simple way to choose vectors \(\mathbf{h}\) and \(\mathbf{g}\) to have large \(s_0\) in Eq. \([12]\). Suppose that the matrices \([10]\) have negative eigenvalues (which means that partial transposition of the state is negative) and \(\mathbf{z} = (z_1, \ldots, z_n)\) is the eigenvector with the most negative eigenvalue. We can try to use vectors \((z_1, \ldots, z_n)\) and \((\bar{z}_{n+1}, \ldots, \bar{z}_{2n})\) as the vectors \(\mathbf{h}\) and \(\mathbf{g}\).

There is no guarantee that these vectors will always satisfy the inequality \([12]\) with large \(s_0\), but it is a good starting point. If they satisfy that inequality, we are ready, if they do not, we will have to perform a more complicated analysis.

To illustrate the applicability of this approach, let us apply it to the three experimentally measured covariance matrices considered before. We consider only bipartitions, since if the bipartitions pass the test then there is no reason to test other partitions. We have also applied our method to the matrix \(\gamma^*\), Eq. \((6)\), which we know could not be the true matrix in the experiment, but we included it to demonstrate that our approach gives comparable results without performing any optimization. Just to compare the results we give the level of violation for the matrix \(\gamma^*\) obtained as a solution of the optimization problem \([2]\) where the errors \(\sigma\) are not available. All these results are given in the Appendix.

We see that our approach applied to the matrix \(\gamma^\prime\) produces comparable violations which are just marginally smaller than those obtained with optimization, and this difference is completely irrelevant for such large violations. For the ten-partite case the minimal violation was reported to be 1.1 for the bipartition 1,10|23456789. If we apply our method to the matrix \(\gamma^\prime\) we get the violation 2.6. In this case our approach immediately produces a much better result than that obtained with the optimization approach of Ref. \([13]\). For the matrix \(\gamma^*\), which is the most probably true covariance matrix in the experiment, this violation is 3.6. Using the technique of Ref. \([14]\) we are even able to verify that the four-partite state is genuine multipartite entangled with confidence level 3 (see Appendix).

**Conclusion.** In conclusion, we have presented an algorithm to recover the best physical approximation to the experimentally measured covariance matrix and applied it to some realistic measured data. The algorithm is based on semidefinite optimization and can be easily implemented by using free optimization software. In addition, we have proved that a widely used entanglement condition is in fact based on the negativity of PT and can be easily checked by testing the recovered covariance matrix of the partially transposed state for negative eigenvalues. It has been shown that the eigenvector corresponding to the most negative eigenvalue can be a good entanglement witness even in the presence of measurement errors and sometimes it is even better than one obtained with complicated optimization algorithms (like in the ten-partite case considered in Ref. \([13]\)). Applying the technique of our previous work to the recovered covariance matrix of the four-partite state under study we demonstrate that that state is genuine multipartite entangled.
Appendix: Semidefinite optimization problem

Here we give all the details missing in the main part of the work. First, we provide a detailed discussion of the optimization problem (5). The condition $M(s, \gamma) \geq 0$ is equivalent to the condition $M_1(s, \gamma) \leq M_0$, where the matrices $M_0$ and $M_1(s, \gamma)$ in the singlemode case read as

$$M_1(s, \gamma) = \begin{pmatrix}
  s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & \gamma_{xx} & 0 & \gamma_{xp} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \gamma_{xx} & 0 & \gamma_{xp} & 0 & 0 & 0 & 0 \\
  0 & \gamma_{xp} & 0 & \gamma_{pp} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & \gamma_{xp} & 0 & \gamma_{pp} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & s\sigma_{xx} + \gamma_{xx} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & s\sigma_{xx} - \gamma_{xx} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & s\sigma_{xp} + \gamma_{xp} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & s\sigma_{xp} - \gamma_{xp} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\sigma_{pp} + \gamma_{pp} \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\sigma_{pp} - \gamma_{pp}
\end{pmatrix}, \quad (15)$$

$$M_0 = \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & -\gamma^\circ_{xx} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & \gamma^\circ_{xx} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & -\gamma^\circ_{xp} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \gamma^\circ_{xp} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma^\circ_{pp} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma^\circ_{pp}
\end{pmatrix}. \quad (16)$$

The structure of these matrices in the multimode case is the same, but the size is much larger. We stress that the parameters $\gamma^\circ$ and $\sigma$ in these matrices are constants (they
are the data obtained in an experiment) and the variables of the optimization problem are $s$ and the elements of $\gamma$. The optimization problem now reads as

$$\max_{M_1(s,\gamma) \leq M_0} s,$$

(17)

and it is a semidefinite optimization problem. The matrices $M_0$ and $M_1$ are sparse, which allows to compactly represent them even for states with hundreds of modes. This semidefinite optimization problem can be solved with an appropriate software. The software of our choice is cvxopt15.

Now we give the details about the three cases considered in the main part. The measured covariance matrix in the four-partite case is

$$\gamma_{xx}^o = \begin{pmatrix}
1.09921 & 0.16092 & -0.17608 & -0.84831 \\
0.16092 & 0.40938 & -0.1606 & -0.18963 \\
-0.17608 & -0.1606 & 0.46060 & 0.04318 \\
-0.84831 & -0.18963 & 0.04318 & 1.064185
\end{pmatrix},$$

(18)

and $\gamma_{xp}^o = 0$. The standard deviations are given by the matrices

$$\sigma_{xx} = \begin{pmatrix}
0.00326 & 0.01041 & 0.00893 & 0.00646 \\
0.01041 & 0.00822 & 0.01847 & 0.01899 \\
0.00893 & 0.01847 & 0.00861 & 0.01345 \\
0.00646 & 0.01899 & 0.01345 & 0.00549
\end{pmatrix},$$

$$\sigma_{pp} = \begin{pmatrix}
0.00457 & 0.01009 & 0.02767 & 0.04288 \\
0.01009 & 0.01022 & 0.02100 & 0.02085 \\
0.02767 & 0.02100 & 0.01465 & 0.01955 \\
0.04288 & 0.02085 & 0.01955 & 0.00455
\end{pmatrix}.$$

(19)

The solution of the problem (17) for this data is

$$\gamma_{xx}^* = \begin{pmatrix}
1.10535 & 0.14133 & -0.16983 & -0.84598 \\
0.14133 & 0.42485 & -0.19067 & -0.19569 \\
-0.16983 & -0.19067 & 0.46261 & 0.04967 \\
-0.84598 & -0.19569 & 0.04967 & 1.06482
\end{pmatrix},$$

$$\gamma_{pp}^* = \begin{pmatrix}
1.10782 & 0.33634 & 0.41646 & 0.89981 \\
0.33634 & 0.94206 & 0.53487 & 0.40398 \\
0.41646 & 0.53487 & 1.07097 & 0.36679 \\
0.89981 & 0.40398 & 0.36679 & 1.06472
\end{pmatrix}.$$

(20)

We have computed the eigenvector corresponding to the most negative eigenvalue for three different matrices — $\gamma^i$ used in Ref. [13], the optimal solution $\gamma^*$ given by Eq. (20), and the optimal solution $\hat{\gamma}^*$ obtained without taking $\sigma$ into account — and checked the significance $s_0$ these vectors provide in Eq. (12). The results are shown in Table I. It can be seen that these vectors give results comparable to those obtained with sophisticated optimization, and the difference is absolutely irrelevant for such high $s_0$.

| Bipartition | Ref. [13] | Violation | $\gamma^i$ | $\gamma^*$ | $\gamma^*$ |
|------------|-----------|-----------|------------|------------|------------|
| 1/234      | 20.93     | 16.18     | 19.09      | 18.46      |
| 2/134      | 13.17     | 13.04     | 16.52      | 15.93      |
| 3/124      | 11.21     | 11.18     | 15.42      | 14.52      |
| 4/123      | 21.06     | 16.24     | 19.27      | 18.96      |
| 12/34      | 24.34     | 18.14     | 21.69      | 20.77      |
| 13/24      | 23.52     | 15.97     | 18.87      | 18.44      |
| 14/23      | 4.66      | 4.29      | 8.48       | 7.57       |

TABLE I. Comparison of the confidence level $s_0$, Eq. (12), for the four-partite case.

A similar comparison for the six-partite case is given in Table II. Here we see the same behavior — our eigenvector approach gives slightly smaller but comparable violations. The violations are even larger than in the previous four-partite case, and this difference plays no role, especially taken into account that computing eigenvalues of such a small matrix costs nothing. For the ten-partite case we cannot give full comparison, because the results are not provided in Ref. [13], but it has been mentioned in the main part that for at least one bipartition our approach gives substantially stronger violation than the optimization algorithms implemented in that work. We stress that taking eigenvalues is a very cheap operation and can be efficiently performed for matrices with thousands rows and columns even on a low-end desktop PC. So, our approach is a very simple yet effective method that can immediately detect entanglement of states with huge number of parts even in the case of imperfect measurements. In rare cases where this approach does not work one can utilize a more expensive optimization technique.

Our technique is not limited to detect only simple kind of entanglement, we are also able to detect genuine multipartite entanglement in some cases. This is substantially more complicated problem and our approach works for small number of parts. The goal is to violate all the equations (12) simultaneously for all bipartitions. Using the method of Ref. [14] we have found the following pair of matrices:

$$X = \begin{pmatrix}
0.29331 & 0.03784 & 0.22823 & 0.23107 \\
0.03784 & 0.58693 & 0.17803 & 0.17187 \\
0.22823 & 0.17803 & 0.38153 & 0.19831 \\
0.23107 & 0.17187 & 0.19831 & 0.28106
\end{pmatrix},$$

$$P = \begin{pmatrix}
0.20468 & -0.00241 & -0.08516 & -0.10549 \\
-0.00241 & 0.3480 & -0.15864 & -0.13005 \\
-0.08516 & -0.15864 & 0.23421 & 0.01436 \\
-0.10549 & -0.13005 & 0.01436 & 0.21535
\end{pmatrix}.$$
| Bipartition | Violation $\gamma^*$ | $\gamma^*$ | $\gamma^*$ |
|------------|---------------------|------------|------------|
| 1|24|35| 40.09 | 38.47 | 39.05 | 38.81 |
| 2|34|56| 36.18 | 35.48 | 36.45 | 36.23 |
| 3|25|46| 20.27 | 19.32 | 19.93 | 19.81 |
| 4|12|35| 20.01 | 18.69 | 19.14 | 19.06 |
| 5|12|43| 27.15 | 26.66 | 27.42 | 27.28 |
| 6|12|34| 49.22 | 45.70 | 46.34 | 46.09 |
| 12|34|56| 53.54 | 48.79 | 49.50 | 49.23 |
| 13|24|56| 45.57 | 42.28 | 42.81 | 42.56 |
| 14|23|46| 44.79 | 39.85 | 40.44 | 40.24 |
| 15|23|45| 45.28 | 38.01 | 38.51 | 38.34 |
| 16|23|45| 31.18 | 29.99 | 30.87 | 30.68 |
| 23|14|56| 40.16 | 38.78 | 39.68 | 39.44 |
| 24|13|56| 37.70 | 35.89 | 36.84 | 36.60 |
| 25|13|46| 35.26 | 31.18 | 32.07 | 31.85 |
| 26|13|45| 47.02 | 40.27 | 40.80 | 40.63 |
| 34|12|56| 24.83 | 21.49 | 22.14 | 22.01 |
| 35|12|46| 28.79 | 25.05 | 25.74 | 25.69 |
| 36|12|45| 50.19 | 45.09 | 45.73 | 45.51 |
| 45|13|26| 30.63 | 28.26 | 28.93 | 28.81 |
| 46|12|23| 51.50 | 47.23 | 47.82 | 47.58 |
| 56|12|34| 56.08 | 51.54 | 52.24 | 51.95 |
| 123|456| 56.66 | 52.09 | 52.76 | 52.49 |
| 124|356| 54.40 | 49.40 | 50.12 | 49.86 |
| 125|346| 50.65 | 45.55 | 46.17 | 45.95 |
| 126|345| 29.96 | 25.71 | 26.47 | 26.39 |
| 134|256| 47.68 | 42.95 | 43.53 | 43.29 |
| 135|246| 47.28 | 40.51 | 40.99 | 40.82 |
| 136|245| 34.24 | 29.69 | 30.60 | 30.38 |
| 145|236| 47.33 | 38.89 | 39.40 | 39.27 |
| 146|235| 35.34 | 33.31 | 34.14 | 33.94 |
| 156|234| 39.49 | 38.47 | 39.39 | 39.13 |

TABLE II. Comparison of the confidence level $s_0$, Eq. (12), for the six-partite case.

These matrices are a genuine entanglement witness for the four-partite state under study or, more precisely, for the state with the covariance matrix given by Eq. (20). In fact, for the bipartition 1|234 the maximum is attained at

$$X' = \begin{pmatrix} 0.29331 & 0.11794 & 0.14763 & 0.11827 \\ 0.11794 & 0.58693 & 0.17803 & 0.17187 \\ 0.14763 & 0.17803 & 0.38153 & 0.19831 \\ 0.11827 & 0.17187 & 0.19831 & 0.28106 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.20468 & 0.01396 & 0.05572 & 0.04133 \\ 0.01396 & 0.36480 & -0.15863 & -0.13005 \\ 0.05572 & -0.15863 & 0.23421 & 0.01436 \\ 0.04133 & -0.13005 & 0.01436 & 0.21535 \end{pmatrix}$$

and the violation is equal to

$$\frac{B_{1|234}(X,P) - G(X,P)}{\sigma(X,P)} \approx 5.57.$$ (22)

For the bipartition 2|134 the maximum is attained at

$$X' = \begin{pmatrix} 0.29331 & 0.12608 & 0.22823 & 0.23107 \\ 0.12608 & 0.58693 & 0.08149 & 0.08280 \\ 0.22823 & 0.08149 & 0.38153 & 0.19831 \\ 0.23107 & 0.08280 & 0.19831 & 0.28106 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.20468 & 0.06250 & -0.08516 & -0.10549 \\ 0.06250 & 0.36480 & 0.01133 & 0.00155 \\ -0.08516 & 0.01133 & 0.23421 & 0.01436 \\ -0.10549 & 0.00155 & 0.01436 & 0.21535 \end{pmatrix}$$

and the violation is equal to

$$\frac{B_{2|134}(X,P) - G(X,P)}{\sigma(X,P)} \approx 3.04.$$ (23)

For the bipartition 3|124 the maximum is attained at

$$X' = \begin{pmatrix} 0.29331 & 0.03784 & 0.17402 & 0.23107 \\ 0.03784 & 0.58693 & 0.09827 & 0.17187 \\ 0.17402 & 0.09827 & 0.38153 & 0.29962 \\ 0.23107 & 0.17187 & 0.22906 & 0.28106 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.20468 & -0.00241 & 0.01222 & -0.10549 \\ -0.00241 & 0.36480 & -0.00151 & -0.13005 \\ 0.01222 & -0.00151 & 0.23421 & 0.11148 \\ -0.10549 & -0.13005 & 0.11148 & 0.21535 \end{pmatrix}$$

and the violation is equal to

$$\frac{B_{3|124}(X,P) - G(X,P)}{\sigma(X,P)} \approx 3.05.$$ (24)

For the bipartition 4|123 the maximum is attained at

$$X' = \begin{pmatrix} 0.29331 & 0.03784 & 0.22823 & 0.13781 \\ 0.03784 & 0.58693 & 0.17803 & 0.10005 \\ 0.22823 & 0.17803 & 0.38153 & 0.29978 \\ 0.13781 & 0.10005 & 0.22978 & 0.28106 \end{pmatrix}$$

$$P' = \begin{pmatrix} 0.20468 & -0.00241 & -0.08516 & 0.01806 \\ -0.00241 & 0.36480 & -0.15864 & -0.10017 \\ -0.08516 & -0.15864 & 0.23421 & 0.10894 \\ 0.01806 & -0.00107 & 0.10894 & 0.21535 \end{pmatrix}$$

and the violation is equal to

$$\frac{B_{4|123}(X,P) - G(X,P)}{\sigma(X,P)} \approx 5.03.$$ (25)

For the bipartition 12|34 the maximum is attained at
and the violation is equal to
\[ \frac{B_{12|34}(X,P) - G(X,P)}{\sigma(X,P)} \approx 15.28. \] (26)

For the bipartition 13|24 the maximum is attained at
\[
X' = \begin{pmatrix}
0.29331 & 0.10141 & 0.22823 & 0.14157 \\
0.10141 & 0.58693 & 0.09542 & 0.17187 \\
0.22823 & 0.09542 & 0.38153 & 0.21426 \\
0.14157 & 0.17187 & 0.21426 & 0.28106
\end{pmatrix}
\]
\[ P' = \begin{pmatrix}
0.20468 & 0.03763 & -0.08516 & 0.00885 \\
0.03763 & 0.36480 & -0.01959 & -0.13005 \\
-0.08516 & -0.01959 & 0.23421 & 0.11690 \\
0.00885 & -0.13005 & 0.11690 & 0.21535
\end{pmatrix}
\]
and the violation is equal to
\[ \frac{B_{13|24}(X,P) - G(X,P)}{\sigma(X,P)} \approx 6.66. \] (27)

For the bipartition 14|23 the maximum is attained at
\[
X' = \begin{pmatrix}
0.29331 & 0.10225 & 0.17399 & 0.23107 \\
0.10225 & 0.58693 & 0.17803 & 0.09153 \\
0.17399 & 0.17803 & 0.38153 & 0.21520 \\
0.23107 & 0.09153 & 0.21520 & 0.28106
\end{pmatrix}
\]
\[ P' = \begin{pmatrix}
0.20468 & 0.03253 & 0.00714 & -0.10549 \\
0.03253 & 0.36480 & -0.15864 & -0.02570 \\
0.00714 & -0.15864 & 0.23421 & 0.10579 \\
-0.10549 & -0.02570 & 0.10579 & 0.21535
\end{pmatrix}
\]
and the violation is equal to
\[ \frac{B_{14|23}(X,P) - G(X,P)}{\sigma(X,P)} \approx 3.04. \] (28)

We thus have that for all bipartitions \( I \) the inequality
\[ \frac{B_I(X,P) - G(X,P)}{\sigma(X,P)} > 3 \] (29)
is satisfied, so we can safely claim that the four-partite state under study is genuine entangled (provided that the experiment has been performed correctly).