Singlet structure function $g_1$ at small $x$ and small $Q^2$

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Explicit expressions for the singlet $g_1$ at small $x$ and small $Q^2$ are obtained with the total resummation of the leading logarithmic contributions. It is shown that $g_1$ practically does not depend on $x$ in this kinematic region. In contrast, it would be interesting to investigate its dependence on the invariant energy $2pq$ because, being $g_1$ positive at small $2pq$, it can turn negative at greater values of this variable. The position of the turning point is sensitive to the ratio between the initial quark and gluon densities, so its experimental detection would enable to estimate this ratio.

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I. INTRODUCTION

The Standard Approach (SA) for the theoretical description of the spin structure function $g_1$ at large and small $x$ is based on the Altarelli-Parisi or DGLAP $Q^2$-evolution equations \[ \text{[1]} \] complemented with global fits \[ \text{[2]} \] for the initial parton densities. Originally, SA was suggested for describing the region of large $x$ but later it has been applied for investigating the polarized DIS at small $x$ as well. As SA neglects the total resummation of leading $\ln(1/x)$, which becomes necessary at small $x$, the singular ($\sim x^{-\alpha}$) factors are introduced in the fits for the initial parton densities. As it was shown in Refs. \[ \text{[3, 4]} \], such factors act as the leading singularities in the Mellin space. They ensure the steep rise of $g_1$ at $x \ll 1$ and indeed mimic the impact of the total resummation of leading $\ln(1/x)$ terms. Alternatively, when the total resummation is taken into account, those singular factors become unnecessary, so the initial parton densities can be fitted with much simpler expressions. The total resummation of $\ln(1/x)$ contributions to the anomalous dimensions and the coefficient functions of the singlet component of $g_1$ was done in Ref. \[ \text{[5]} \] in the Double-Logarithmic Approximation under the assumption of $\alpha_s$ fixed at an unknown scale. More precise results including the running $\alpha_s$ effects were obtained in Ref. \[ \text{[6]} \].

In the present paper, we extend the results of Ref. \[ \text{[6]} \] to consider the small- $x$ behavior of the singlet $g_1$ in more detail. In particular, we give a special attention to the kinematic region where not only $x$ but also $Q^2$ are small. On one hand, this kinematics has been investigated experimentally by the COMPASS collaboration, see Ref. \[ \text{[7]} \]. On the other hand, the region of small $Q^2$ is clearly beyond the reach of SA. We show that in this kinematics $g_1$ can be practically independent of $x$ even for $x \ll 1$. We obtain that $g_1$, being positive at small values of the invariant energy $2pq$, can turn negative when $2pq$ increases. The position of the turning point is sensitive to the ratio between the initial quark and gluon densities. Then we also show that, in spite of the presence of large factors providing $g_1$ with the Regge behavior at small $x$, the interplay between initial quark and gluon densities might keep $g_1$ at small $x$ close to zero even at small $x$, regardless of values of $Q^2$.

The paper is organized as follows: in Sect. 2 we remind and explain the basic formulae for $g_1$ singlet obtained in Ref. \[ \text{[6]} \]. These formulae include the total resummation of the leading logarithms of $x$. In our approach, the coefficient functions for $g_1$ are expressed through new anomalous dimensions. Explicit expressions for them are presented in Sect. 3. We focus on $g_1$ at small $Q^2$ in Sect. 4. As our approach is perturbative, we are interested in minimizing the influence of non-perturbative contributions. To this aim we introduce an optimal mass scale in Sec. 5. The asymptotics of $g_1$ at small $x$ is considered in Sect. 6. Suggestions for new simple fits for the initial parton densities at small $Q^2$ are briefly discussed in Sec. 7. Sect. 8 contains our numerical results, and finally Sect. 9 is for the concluding remarks.

1 They are simple poles whereas the total resummation leads to the leading singularity as the square root branch point.
II. EXPRESSIONS FOR $g_1$ AT LARGE $Q^2$

The singlet structure function $g_1$ at small $x$ was studied in Ref. [6]. According to it, $g_1$ can be represented in the form of the Mellin integral:

$$g_1(x, Q^2) = \frac{<e_q^2>}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \left( \frac{1}{\sqrt{2\pi}} \right)^{\omega} \left[ (C_q^{(+)} e^{\Omega_{(+)} y} + C_q^{(-)} e^{\Omega_{(-)} y}) \delta q + (C_g^{(+)} e^{\Omega_{(+)} y} + C_g^{(-)} e^{\Omega_{(-)} y}) \delta g \right]$$ (1)

where $<e_q^2>$ stands for the sum of electric charges: $<e_q^2> = 10/9$ for $n_f = 4$, $y = \ln(Q^2/\mu^2)$, with $\mu$ being the starting point of the $Q^2$- evolution, $\delta q$ is the initial averaged quark density: $<e_q^2> \delta q = e_q^2 \delta u + e_q^2 \delta d + \ldots$ whereas $\delta g$ is the initial gluon density.

The other ingredients of the integrand in Eq. (1) are expressed in terms of the anomalous dimensions $H_{ik}$, with $i, k = q, g$. The exponents $\Omega_{(\pm)}$ and coefficient functions $C_g^{(\pm)}$ are:

$$\Omega_{(\pm)} = \frac{1}{2} [H_{gg} + H_{qq} \pm R],$$ (2)

$$C_q^{(+)} = \frac{\omega}{RT} \left[ (H_{qq} - \Omega_{(-)}) (\omega - H_{gg}) + H_{qq} H_{gg} + H_{gg} (\omega - \Omega_{(-)}) \right],$$ (3)

$$C_q^{(-)} = \frac{\omega}{RT} \left[ (\Omega_{(+)} - H_{qq}) (\omega - H_{gg}) - H_{qq} H_{gg} + H_{gg} (\Omega_{(+)} - \omega) \right],$$

$$C_g^{(+)} = \frac{\omega}{RT} \left[ (H_{gg} - \Omega_{(-)}) (\omega - H_{qq}) + H_{gg} H_{gg} + H_{gg} (\omega - \Omega_{(-)}) \right] \left( -\frac{A'}{2\pi \omega^2} \right),$$

$$C_g^{(-)} = \frac{\omega}{RT} \left[ (\Omega_{(+)} - H_{gg}) (\omega - H_{qq}) - H_{gg} H_{gg} + H_{gg} (\Omega_{(+)} - \omega) \right] \left( -\frac{A'}{2\pi \omega^2} \right).$$

Here

$$R = \sqrt{(H_{qq} - H_{gg})^2 + 4H_{qq} H_{gg}}, \quad T = \omega^2 - \omega (H_{gg} + H_{qq}) + (H_{gg} H_{qq} - H_{gg} H_{gg})$$ (4)

and

$$A'(\omega) = \frac{1}{b} \left[ \frac{1}{\eta} - \int_0^\infty \frac{d\rho e^{-\omega \rho}}{(\rho + \eta)^2} \right]$$ (5)

with $\eta = \ln(\mu^2/\Lambda^2_{QCD})$ and $b = (33 - 2n_f)/(12\pi)$. The additional factor $\left(-\frac{A'}{2\pi \omega^2}\right)$ in the coefficients $C_g^{(\pm)}$ is the small-$\omega$ estimate for the quark box which relates the initial gluons to the electromagnetic current. $A'(\omega)$ stands for the QCD coupling $\alpha_s$ in the box in the Mellin space.

III. ANOMALOUS DIMENSIONS

The anomalous dimensions $H_{ik}$ obey the following system of equations:

$$\omega H_{qq} = b_{qq} + H_{gg} H_{gg} + H_{qq}^2$$ (6)

$$\omega H_{gg} = b_{gg} + H_{gg} H_{gg} + H_{gg}^2$$

$$\omega H_{gg} = b_{gg} + H_{gg} H_{gg} + H_{gg}^2$$

$$\omega H_{gg} = b_{gg} + H_{gg} H_{gg} + H_{gg}^2$$

where

$$b_{ik} = a_{ik} + V_{ik},$$ (7)

with the Born contributions $a_{ik}$ defined as follows:

$$a_{qq} = \frac{A(\omega) C_F}{2\pi}, \quad a_{gg} = \frac{A'(\omega) C_F}{\pi}, \quad a_{qg} = \frac{n_f A'(\omega)}{2\pi}, \quad a_{gg} = \frac{4NA(\omega)}{2\pi}.$$ (8)
is given by the Eq. (5) and

$$A'(\omega) = \frac{1}{b} \left( \frac{\eta}{b \eta^2 + \pi^2} - \frac{1}{2} \int_0^\infty \frac{d\rho e^{-\omega \rho}}{(\rho + \eta)^2 + \pi^2} \right)$$  \hspace{1cm} (9)$$

is the Mellin representation of the QCD running coupling $\alpha_s$ involved in the quark-gluon ladder, with the proper account of its analytic properties. In Eq. (5) we use the standard notations for $C_F = (N^2 - 1)/(2N) = 4/3$ and $N = 3$. Finally,

$$V_{sk} = \frac{m_{sk}}{\sqrt{2}} D(\omega) ,$$  \hspace{1cm} (10)$$

where

$$m_{qq} = \frac{C_F}{2N} , \quad m_{gg} = -2N^2 , \quad m_{qg} = n_f \frac{N}{2} , \quad m_{qg} = -NC_F ,$$  \hspace{1cm} (11)$$

and

$$D(\omega) = \frac{1}{2b^2} \int_0^\infty d\rho e^{-\omega \rho} \ln \left( (\rho + \eta)/\eta \right) \left[ \frac{\rho + \eta}{(\rho + \eta)^2 + \pi^2} + \frac{1}{\rho + \eta} \right]$$  \hspace{1cm} (12)$$

is the factor that accounts for non-ladder diagrams.

The solution to Eq. (13) is

$$H_{qq} = \frac{1}{2} \left[ \omega + Z - \frac{b_{qq} - b_{gg}}{Z} \right] , \quad H_{gg} = \frac{b_{gg}}{Z} ,$$  \hspace{1cm} (13)$$

$$H_{qg} = \frac{1}{2} \left[ \omega + Z - \frac{b_{qq} - b_{gg}}{Z} \right] , \quad H_{qq} = \frac{b_{qq}}{Z} .$$

where

$$Z = \frac{1}{\sqrt{2}} \sqrt{\omega^2 - 2(b_{qq} + b_{gg})} + \sqrt{(\omega^2 - 2(b_{qq} + b_{gg})^2 - 4(b_{qq} - b_{gg})^2 - 16b_{gg}^2} .$$  \hspace{1cm} (14)$$

IV. EXPRESSIONS FOR THE SINGLET $g_1$ AT SMALL $Q^2$

Eq. (11) states that $g_1$ does not depend on $Q^2$ when $Q^2 \sim \mu^2$. In this case $g_1$ depends only on $z \equiv \mu^2/(2pq)$:

$$g_1(z) = \frac{<\gamma^2>}{2} \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left( \frac{1}{z} \right)^\omega \left[ \omega - H_{gg} + H_{gg}^\prime \delta q + \omega - H_{gg} + H_{gg}^\prime \left( -\frac{A'}{2\pi \omega^2} \right) \delta q \right] .$$  \hspace{1cm} (15)$$

Eq. (11) was obtained for $Q^2 \gg \mu^2$ and cannot be used for studying the $Q^2$ -dependence of $g_1$ at $Q^2 < \mu^2$. However, it would be interesting to extend our approach to this region. In order to do so, we suggest to modify Eq. (11), replacing $Q^2$ by $(Q^2 + \mu^2)$. Although such a shift takes us out of the logarithmic accuracy we have always kept in our approach, it looks quite reasonable and natural. Indeed, let us in the first place consider a contribution of a ladder Feynman graph at $x \ll 1$ with the DL accuracy. The graph includes the quark and gluon rungs. Integrations of the quark rungs are infrared-stable, being regulated with the quark mass $m_q$. On the contrary, integrations of the gluon rungs are IR-divergent, so they must be regulated. The standard way of the IR-regulating in QED and QCD is providing gluons with a mass $\mu$ which acts as an IR cut-off. It is also convenient to choose $\mu \gg m_q$ and replace $m_q$ by $\mu$ in the quark propagators as was first suggested in Ref. [3]. After that $m_q$ can be dropped. Now both gluon and quark rungs of the ladder are IR-stable and $\mu$ -dependent. The simplification of the spin structure can be done with the standard means (see e.g. the review [4]). It is appropriate to use the standard Sudakov variables for integrations over momenta $k_i$ of ladder virtual quarks and gluons: $k_i = \alpha_i(p - (m_q^2/(2p_q))) + \beta_i(q + xp) + k_{\perp}$. After that the DL contribution of a ladder graph with $n$ rungs is proportional to the integral $J_n$:

$$J_n = \int \frac{dk_{n+1}^2}{2\pi} d\alpha_n d\beta_n \delta(w \beta_n - Q^2 - \mu^2 + w \alpha_n \beta_n - k_{n+1}^2)$$  \hspace{1cm} (16)$$

$$= \int \frac{dk_{n-1}^2}{2\pi} d\alpha_{n-1} d\beta_{n-1} \delta(w \alpha_{n-1} \beta_{n-1} - \mu^2 - (k_{n-1}^2 + k_{n-1}^2)$$  \hspace{1cm} (16)$$

$$\cdots$$

$$\int \frac{dk_1^2}{2\pi} d\alpha_1 d\beta_1 \delta(-w \alpha_1 - \mu^2 + w \alpha_1 \beta_1 - k_1^2)$$  \hspace{1cm} (16)$$
where the rungs are numbered from the bottom to the top of the ladder. We have used the notation \( w \equiv 2pq \). As we consider \( x \ll 1 \), we neglected the term \(- wx_{\text{cut}}\) coming from the representation \( 2q_{\text{cut}} = w\beta_n - wx_{\text{cut}} \) in the argument of the first \( \delta \)-function and similar terms in the other \( \delta \)-functions. Eq. (16) manifests that the \( Q^2 \)-dependence in \( J_n \) at \( x \ll 1 \) is given by the term \( Q^2 + \mu^2 \) in the first \( \delta \)-function only. Neither accounting for non-ladder graphs nor accounting for single logarithms, including the running \( \alpha_s \) effects, change this situation. So, the replacement

\[
Q^2 \to \tilde{Q}^2 \equiv Q^2 + \mu^2 ,
\]

is confirmed by the analysis of the structure of the Feynman graphs, though as the replacement is beyond the logarithmic accuracy, it can be called model-dependent.

Further, DGLAP exploits the \( Q^2 \)-evolution, with \( Q^2 \) being the upper limit of integrations over \( k_r^2 \) \((r = 1, 2, ..)\), so the first loop DGLAP double-logarithmic evolution \( J^{\text{DGLAP}}_1 \equiv \ln(Q^2/\mu^2) \). This contribution is large only when \( Q^2 \gg \mu^2 \). In contrast, we use the evolution with respect to \( \mu^2 \), the integrations over \( k_r^2 \) run up to \( w \) instead of \( Q^2 \), so our approach is not restricted by the region of large \( Q^2 \). For example, when \( n = 1 \), Eq. (16) yields

\[
J_1 = \ln(w/\mu^2) ,
\]

so \( J_1 \) does not depend on \( Q^2 \) at all. The \( Q^2 \)-dependence appears in the next loops. In particular, when \( n = 2 \),

\[
J_2 = -(1/2) \ln^2(w/\mu^2) \ln(\tilde{Q}^2/w) + (1/6) \ln^3(\tilde{Q}^2/w) ,
\]

so it depends on \( Q^2 \) through \( Q^2 + \mu^2 \). It agrees with Eq. (17). In contrast to DGLAP, double logarithms in Eqs. (18,19) do not disappear when \( Q^2 \to 0 \).

Using Eq. (17) makes possible to rewrite Eq. (1) in the form convenient equally for large and small \( Q^2 \):

\[
g_1(x, Q^2) = \frac{e^2}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \left( \frac{1}{z + x} \right)^{\omega} \left[ \left( C_q^{(+)}/(\omega) \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega_{(+)}} + C_{g}^{(-)}(\omega) \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega_{(-)}} \right) \delta q \right] - \frac{A'}{2\pi \omega^2} \left[ C_q^{(+)}/(\omega) \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega_{(+)}} + C_{g}^{(-)}(\omega) \left( \frac{Q^2 + \mu^2}{\mu^2} \right)^{\Omega_{(-)}} \right] \delta q .
\]

Eq. (20) coincides with Eq. (1) when \( Q^2 \gg \mu^2 \) and also reproduces Eq. (16) when \( Q^2 = 0 \). Eq. (20) shows that the \( x \)- and \( Q^2 \)-dependence of \( g_1 \) are getting weaker with decreasing \( Q^2 \) so that \( g_1 \) at \( Q^2 \ll \mu^2 \) depends rather on \( z = \mu^2/(2pq) \) than on \( x \) or \( Q^2 \). Eqs. (1,12) describe the leading \( Q^2 \)-dependence of \( g_1 \) at \( Q^2 \gg \mu^2 \). Similarly, Eq. (20) describes the leading \( Q^2 \)-dependence at \( Q^2 \ll \mu^2 \): although logarithms of \( (Q^2 + \mu^2)/\mu^2 \) are small here, they are multiplied by leading, double logarithms of \( 1/z \) contrary to other, unaccounted \( Q^2 \)-terms. In what follows we will not discuss the \( Q^2 \)-dependence of \( g_1 \) at small \( Q^2 \) in detail. Instead, we focus on investigating \( g_1 \) at \( Q^2 \to 0 \) where \( g_1 \) is given by any of Eqs. (15,20). According to Eqs. (1,15,20), the total resummation of double-logarithms for \( g_1 \) makes it depend on the value and the way of introducing the cut-off \( \mu \). This dependence will vanish when the probabilities to find a polarized quark and gluon are calculated and used in expressions for \( g_1 \) instead of the phenomenological initial densities \( \delta q \) and \( \delta g \).

V. OPTIMAL SCALE FOR \( \mu \)

In order to estimate \( \mu \), we discuss below the restrictions for it. From

\[
\alpha_s(k^2) = \frac{1}{\beta \ln(k^2/\Lambda^2_{QCD})} ,
\]

as \( k^2 \gg \Lambda^2_{QCD} \) and \( k^2 > \mu^2 \), we obtain the obvious restriction for the value of \( \mu \):

\[
\mu^2 \gg \Lambda^2_{QCD} .
\]

Then, the DL contributions from ladder quark rungs are infrared-stable, with logarithms there containing masses \( m_q \) of the involved quarks in denominators. In order to calculate ladder fermion graphs with Infa-Red Evolution Equations these logs should be regulated with the infrared cut-off \( \mu \). It brings the second restriction for \( \mu \):

\[
\mu > m_q .
\]
Basically, there are no other restrictions for $\mu$. However, some additional information of $\mu$ comes from the small-$x$ asymptotics of $g_1$. In Ref. [10] it was shown that

$$g_1 \sim (1/x)^{\omega_0}$$  
(24)

when $x \to 0$. It turned out that $\omega_0$ depends on $\eta = \ln(\mu^2/\Lambda_{QCD}^2)$, in such a way that $\omega_0$ is maximal at $\eta = \eta_S \approx 7.5$ with

$$\omega_0(\eta_S) \equiv \Delta_S \approx 0.86.$$  
(25)

We have called $\Delta_S$ the intercept of the singlet $g_1$. This value is in agreement with the analysis of experimental data [10]. Assuming $\Lambda_{QCD} = 0.1$ GeV leads to the estimate

$$\mu_S = 5 \Lambda_{QCD}^{3.75} \approx 5.5 \text{ GeV}. \quad (26)$$

On the other hand, from physical considerations, the intercept $\Delta_S$ should be a constant and should not depend on $\mu$. This dependence is the artefact of our approach: we account for perturbative contributions to the asymptotics and leave out a possible impact of non-perturbative ones. Taken together, the perturbative and non-perturbative contributions would make $\omega_0$ to be $\mu$-independent. We suggested in Ref. [6] that non-perturbative contributions to $\omega_0$ happened to be minimal at $\eta = \eta_S$ so in order to minimize the impact of (the basically unknown) non-perturbative contributions on $g_1$, we should fix $\mu = \mu_S = 5.5$ GeV. We call $\mu_S$ the Optimal Scale for the singlet $g_1$. We expect that choosing this scale for $\mu$ would bring a better agreement between experimental data and our formulae than other values of $\mu$. We also suggest that the initial parton densities can be fitted mostly simply when $\mu$ is fixed at the Optimal Scale. It is worth of mentioning that in Refs. [11] the Optimal Scale $\mu_{NS}$ for the non-singlet component of $g_1$ is 5 times smaller: $\mu_{NS} = 1$ GeV.

VI. SMALL- $x$ ASYMPTOTICS OF $g_1$

Before performing numerical analysis of Eq. (24), it could be instructive to consider its small-$x$ asymptotics. This asymptotics is different for small and large $Q^2$ and when $x + z \to 0$, we obtain

$$g_1 \sim \left( \frac{1}{x + z} \right)^{\Delta_S} \frac{K}{\ln^{3/2}(1/(x + z))} \frac{Q^2 + \mu^2}{\mu^2} \Delta_S^{\Delta_S/2} \left( \frac{2}{\Delta_S} + \ln \frac{Q^2 + \mu^2}{\mu^2} \right) \left[ C_q^{\text{as}} \delta q + C_g^{\text{as}} \delta g \right],$$  
(27)

where the intercept $\Delta_S$ is given by Eq. (26) are taken at the intercept point $\omega = \Delta_S$, $\Delta_S \approx 0.86$,

$$K = \sqrt{\frac{\Delta_S}{8\pi}}, \quad \tilde{\Delta}_S = \Delta_S - \partial[(bg_g + b_{qg}) - r]/\partial \omega, \quad r = \sqrt{(b_{gg} - b_{qg})^2 + 4b_{gg}b_{qg}},$$  
(28)

and

$$C_q^{\text{as}} = 1 + \frac{b_{qq} - b_{gg} + 2b_{gg}}{r}, \quad C_g^{\text{as}} = \left(1 + \frac{b_{gg} - b_{qg} + 2b_{gg}}{r}\right)(-\frac{A'(\Delta_S)}{2\pi^2 \Delta_S}),$$  
(29)

with all $b_{ij}$ and their $\omega$-derivatives in Eq. (28) are taken at the intercept point $\omega = \Delta_S$. The initial parton densities $\delta q(\omega)$ and $\delta g(\omega)$ are also fixed at $\omega = \Delta_S$.

When $z \to 0$ and $Q^2 \ll \mu^2$, Eq. (27) turns (we drop here the unessential overall factor) to

$$g_1 \sim S(\Delta_S) \delta q(\Delta_S) \left(\frac{1}{x}\right)^{\Delta_S} / \ln^{3/2}(1/z)$$  
(30)

with

$$S(\Delta_S) = -1 - 0.064 \delta g(\Delta_S)/\delta q(\Delta_S).$$  
(31)

Eqs. (30, 31) show that the asymptotics of $g_1$ does not depend on $x$ and $Q^2$ in the small-$Q^2$ region and the sign of $g_1$ is determined by $S(\Delta_S)$. There can be three options:
A. Large and negative $\delta g$: \[ S(\Delta S) > 0, \]  
When the initial gluon density is negative and large so that \[ \delta g < -15.64\delta q, \]  
the asymptotics of $g_1$ is positive. It is known that $g_1 > 0$ at large $z$ where it is given by its Born expression. Therefore, if $\delta q$ and $\delta g$ are related by Eq. (32), $g_1$ is positive in the whole range of $z$.

B. Positive or small and negative $\delta g$: \[ S(\Delta S) < 0. \]  
On the contrary, when \[ \delta g > -15.64\delta q, \]  
g_1, being positive at large $x$, should pass through the zero value and changes its sign at asymptotically small $z$.

C. Fine tuning: \[ S(\Delta S) = 0 \]  
Finally, there might be a strong correlation between $\delta q$ and $\delta g$: \[ \delta g = 15.64\delta q \]  
when $z \to 0$. In this case $g_1$ is positive at large $x$ and then $g_1 \to 0$ in spite of the large power-like factor $\left(\frac{1}{z}\right)^{\Delta S}$ in Eq. (30).

VII. FITS FOR INITIAL PARTON DENSITIES

In the Standard Approach, the initial parton densities $\delta q(x)$, $\delta g(x)$ are fitted from experimental data at $x \sim 1$ and $Q^2 = \mu^2 \approx 1$ GeV$^2$. Then they are evolved with the anomalous dimensions into the region of large $Q^2$ ; $Q^2 \gg \mu^2$ and finally evolved with the coefficient functions into the region of $x \ll 1$. As the coefficient functions in the SA do not include the total resummation of $\ln x$ and therefore cannot provide $g_1$ with the steep rise at small $x$, this role is assigned, in the standard fits[2], to the singular factors $x^{-\alpha}$ which mimic the resummation. In other words, the impact of the NLO terms in the DGLAP coefficient functions on the small-$x$ behavior of $g_1$ is actually negligibly small compared to the impact of the fit. When the resummation is accounted for, the singular factors can be dropped out and the fits can be simplified down to expressions \[ \sim N_{q,g}(1 + a_{q,g}(z + x)). \]  
Obviously, the straightforward evolution of the fits backwards, to the region of $Q^2 \ll \mu^2$ is beyond SA. We suggest that the analyses of the large $Q^2$- and small $Q^2$- experimental data would be more consistent when the argument $x$ in the new fits is replaced by $(z + x)$. This argument $\approx x$ at large $Q^2$ and $\approx z$ at small $Q^2$. It means that at small $Q^2$ the fits should depend on $2pq$ only. We suggest that the fits for $\delta q(z + x)$, $\delta g(z + x)$ can be chosen as linear forms \[ \sim N_{q,g}(1 + a_{q,g}(z + x)), \]  
with parameters $N_{q,g}, a_{q,g}$ fitted from experimental data either at small or large $Q^2$. However, in the present paper we do not plan to study new fits in detail. Instead, we assume that after the total resummations of $\ln x$ has been accounted for, one can approximate the initial parton densities by constants:

\[ \delta q = N_q, \quad \delta g = N_g. \]  
The analysis of the data on the non-singlet $g_1$ shows that $N_q$ should be positive whereas the sign of $N_g$ is basically unknown.

VIII. NUMERICAL RESULTS FOR $g_1$ AT $Q^2 = 0$

Both Eq. (20) and the asymptotic expression Eq. (30) imply that the $x$- dependence of $g_1$ at small $Q^2$ should be weak even for very small $x$. As a matter of fact, $g_1$ in this region depends on $2pq$ only. so a plot of $g_1$ vs $x$ should reveal a very flat behavior, with the magnitude, $g_1(z)$, depending on the interplay between $\delta q$ and $\delta g$ at different
FIG. 1: $G_1$ evolution with decreasing $z = \mu^2/(2pq)$ for different values of ratio $r = \delta g/\delta q$: curve 1 - for $r = 0$, curve 2 - for $r = -5$, curve 3 - for $r = -8$ and curve 4 - for $r = -15$.

$z = \mu^2/(2pq)$. Presuming that $\delta q > 0$ and approximating $\delta q$ and $\delta g$ by the constants $N_{q,g}$, we rewrite Eq. (20) at $Q^2 = 0$ as

$$g_1(z) = \langle \epsilon_e^2 > /2)N_qG_1(z)$$

and calculate $G_1$ numerically. The results for different values of the ratio $r = N_g/N_q$, $G_1$ are plotted in Fig. 1. When the gluon density is neglected, i.e. $N_g = 0$ (curve 1), $G_1$ being positive at $x \sim 1$, is getting negative very soon, at $z < 0.5$ and falls fast with decreasing $z$. When $N_g/N_q = -5$ (curve 2), $G_1$ remains positive and not large until $z \sim 10^{-1}$, turns negative at $z \sim 0.03$ and falls afterwards rapidly with decreasing $z$. This turning point where $G_1$ changes its sign is very sensitive to the magnitude of the ratio $r$. For instance, at $N_g/N_q = -8$ (curve 3), $G_1$ passes through zero at $z \sim 10^{-3}$. When $N_g/N_q < -10$, $G_1$ is positive at any experimentally reachable $z$ (curve 4). Therefore, the experimental measurement of the turning point would allow to draw conclusions on the interplay between the initial quark and gluon densities.

IX. CONCLUSION

We have shown that the study of $g_1$ at small-$Q^2$ could be as interesting as in the large-$Q^2$ region. Eq. (20) describes the singlet $g_1$ at small $x$ and arbitrary values of $Q^2$, generalizing both Standard Approach and our previous results. It accounts for the total resummation of the leading logarithms of $x$ and $z$ ($z = \mu^2/(2pq)$). It makes possible to simplify the fits for the initial parton densities. In general, $g_1$ includes both perturbative and non-perturbative contributions. In order to minimize the impact of the latter, we have introduced an Optimal Scale for $\mu$. In the small- $Q^2$ region, Eq. (20) predicts that $g_1$ essentially depends on $2pq$ only and practically does not depend on $Q^2$ and $x$ even at $x \ll 1$, making the investigation of the $x$-dependence uninteresting. On the contrary, the study of the $z$-dependence of $g_1$ at small $Q^2$ would be useful. Indeed, the sign of $g_1$ is positive at $z$ close to 1 and can remain positive or become negative at smaller $z$, depending on the ratio between $\delta g$ and $\delta q$. Our numerical results are plotted in Fig. 1. The position of this point is sensitive to the ratio $\delta g/\delta q$, so the experimental measurement of this point would enable to estimate the impact of $\delta g$.

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\footnote{We remind that Eq. (20) is becoming unreliable when $z \sim 1$ and should be modified according to the prescription of Ref. \cite{6} in order to describe this region.}
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