LIMIT THEOREMS FOR THE ‘LAZIEST’ MINIMAL RANDOM WALK MODEL OF ELEPHANT TYPE

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ABSTRACT. We consider a minimal model of one-dimensional discrete-time random walk with step-reinforcement, introduced by Harbola, Kumar, and Lindenberg (2014): The walker can move forward (never backward), or remain at rest. For each \( n = 1, 2, \cdots \), a random time \( U_n \) between 1 and \( n \) is chosen uniformly, and if the walker moved forward [resp. remained at rest] at time \( U_n \), then at time \( n + 1 \) it can move forward with probability \( p \) [resp. \( q \)], or with probability \( 1 - p \) [resp. \( 1 - q \)] it remains at its present position. For the case \( q > 0 \), several limit theorems are obtained by Coletti, Gava, and de Lima (2019). In this paper we prove limit theorems for the case \( q = 0 \), where the walker can exhibit all three forms of asymptotic behavior as \( p \) is varied. As a byproduct, we obtain limit theorems for the cluster size of the root in percolation on uniform random recursive trees.

1. INTRODUCTION

The elephant random walk, introduced by Schütz and Trimper [23], is defined as follows:

- The first step \( Y_1 \) of the walker is +1 with probability \( s \), and –1 with probability \( 1 - s \).
- For each \( n = 1, 2, \cdots \), let \( U_n \) be uniformly distributed on \( \{1, \cdots, n\} \), and

\[
Y_{n+1} = \begin{cases} 
Y_{U_n} & \text{with probability } p, \\
-Y_{U_n} & \text{with probability } 1 - p.
\end{cases}
\]

Each of choices in the above procedure is made independently. The sequence \( \{Y_i\} \) generates a one-dimensional random walk \( \{S_n\} \) by

\[
S_0 := 0, \quad \text{and} \quad S_n := \sum_{i=1}^{n} Y_i \quad \text{for } n = 1, 2, \cdots.
\]

It admits a phase transition from diffusive to superdiffusive behavior at the critical value \( p_c = 3/4 \). Several limit theorems are obtained in [1] [2] [9] [10]. Variations of elephant random walks studied mainly from mathematical viewpoint are found in [3] [4] [5] [7] [12] [13].

Kumar, Harbola, and Lindenberg [20] proposed a random walk model of elephant type, which exhibits asymptotic subdiffusion, normal diffusion, and superdiffusion as a single parameter is swept. An even simpler model of this kind was introduced by Harbola, Kumar, and Lindenberg [16]. Assume that

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$p \in (0, 1)$, $q \in [0, 1)$ and $s \in [0, 1]$. Define a sequence $\{X_i\}$ of $\{0, 1\}$-valued random variables as follows:

- $P(X_1 = 1) = 1 - P(X_1 = 0) = s$.
- For each $n = 1, 2, \ldots$, let $U_n$ be uniformly distributed on $\{1, \ldots, n\}$.
  - If $X_{U_n} = 1$, then $X_{n+1} = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$
  - If $X_{U_n} = 0$, then $X_{n+1} = \begin{cases} 1 & \text{with probability } q, \\ 0 & \text{with probability } 1 - q. \end{cases}$

A random walk $\{H_n\}$ on $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is defined by

$$H_0 := 0, \quad H_n := \sum_{i=1}^{n} X_i \quad \text{for } n = 1, 2, \ldots.$$

Note that for $n = 1, 2, \ldots$, the conditional distribution of $X_{n+1}$ given the history up to time $n$ is

$$P(X_{n+1} = 1 \mid X_1, \ldots, X_n) = 1 - P(X_{n+1} = 0 \mid X_1, \ldots, X_n) = p \cdot \frac{\# \{i = 1, \ldots, n : X_i = 1\}}{n} + q \cdot \frac{\# \{i = 1, \ldots, n : X_i = 0\}}{n}.$$  

Since

$$\# \{i = 1, \ldots, n : X_i = 1\} = H_n \quad \text{and} \quad \# \{i = 1, \ldots, n : X_i = 0\} = n - H_n,$$  

the conditional expectation of $X_{n+1}$ is

$$E[X_{n+1} \mid X_1, \ldots, X_n] = P(X_{n+1} = 1 \mid X_1, \ldots, X_n) = \alpha \cdot \frac{H_n}{n} + q, \quad (1.1)$$

where $n = 1, 2, \ldots$ and $\alpha := p - q \in (-1, 1)$. Solving

$$E[H_{n+1}] = \left(1 + \frac{\alpha}{n}\right) E[H_n] + q,$$

we have

$$E[H_n] = \frac{qn}{1 - \alpha} + \left(s - \frac{q}{1 - \alpha}\right) \cdot \frac{\Gamma(n + \alpha)}{\Gamma(1 + \alpha)\Gamma(n)} \quad (1.2)$$

$$\sim \begin{cases} \frac{qn}{1 - \alpha} & (q > 0), \\ \frac{s}{1 - \alpha} \cdot n^p & (q = 0), \end{cases}$$

where $x_n \sim y_n$ means that $x_n/y_n$ converges to 1 as $n \to \infty$. The walker is ballistic if $q > 0$. On the other hand, in the case $q = 0$, which we call the ‘laziest’ minimal random walk model of elephant type, all three phases of asymptotic behavior are observed as $p \in (0, 1)$ varies (see [16]).

Coletti, Gava, and de Lima [8] proved several limit theorems for this model. The strong law of large numbers holds for any $q \in [0, 1)$:

$$\lim_{n \to \infty} \frac{H_n - E[H_n]}{n} = 0 \quad \text{a.s..}$$

In particular, together with (1.2),

$$\lim_{n \to \infty} \frac{H_n}{n} = \frac{q}{1 - \alpha} \quad \text{a.s..} \quad (1.3)$$
For $q > 0$, further limit theorems are proved in [8]. Let $\rho := q/(1-\alpha) \in (0,1)$ and $\phi(t) := \sqrt{2t \log \log t}$.

(i) If $0 \leq \alpha < 1/2$, then
$$\frac{H_n - E[H_n]}{\sqrt{\frac{\rho(1-\rho)}{1-2\alpha}n}} \xrightarrow{d} N(0,1), \quad \text{and} \quad \limsup_{n \to \infty} \frac{H_n - E[H_n]}{\phi\left(\frac{\rho(1-\rho)}{1-2\alpha}n\right)} = 1 \ \text{a.s.}$$

(ii) If $\alpha = 1/2$, then
$$\frac{H_n - E[H_n]}{\sqrt{\rho(1-\rho)n \log n}} \xrightarrow{d} N(0,1), \quad \text{and} \quad \limsup_{n \to \infty} \frac{H_n - E[H_n]}{\phi(\rho(1-\rho)n \log n)} = 1 \ \text{a.s.}$$

(iii) If $1/2 < \alpha < 1$, then there exists a random variable $W$ with positive variance such that
$$\lim_{n \to \infty} \frac{H_n - E[H_n]}{a_n} = W \ \text{a.s. and in } L^2,$$
where $a_n := \frac{\Gamma(n+\alpha)}{\Gamma(1+\alpha)\Gamma(n)}$. Moreover, essentially the same calculation as in [19] gives that
$$\frac{H_n - E[H_n] - W \cdot a_n}{\sqrt{\frac{\rho(1-\rho)}{2\alpha-1}n}} \xrightarrow{d} N(0,1),$$
and
$$\limsup_{n \to \infty} \frac{H_n - E[H_n] - W \cdot a_n}{\phi\left(\frac{\rho(1-\rho)}{2\alpha-1}n\right)} = 1 \ \text{a.s.}$$

Remark 1.1. If $s = \rho \in (0,1)$, then the minimal random walk model is equivalent to the “correlated Bernoulli process” introduced by Drezner and Farnum [11]. In the latter context, the central limit theorem was proved by Heyde [13].

Although the $q = 0$ case is the most interesting, only partial results are obtained in Coletti, Gava, and de Lima [8]. The aim of this article is to prove several limit theorems for this case.

2. Results

In this section we consider the ‘laziest’ minimal random walk model of elephant type $(q = 0)$. Note that if $X_1 = 0$, then $X_n = 0$ for all $n$. Hereafter we assume that $s = 1$ in addition. The dynamics can be summarized as follows: Let $p \in (0,1)$.

- The first step $X_1$ of the walker is 1 with probability one.
- For each $n = 1,2,\cdots$, let $U_n$ be uniformly distributed on $\{1,\cdots,n\}$, and

$$X_{n+1} = \begin{cases} X_U \ & \text{with probability } p, \\ 0 \ & \text{with probability } 1 - p. \end{cases}$$
The equation (1.1) becomes
\[ E[X_{n+1} \mid \mathcal{F}_n] = P(X_{n+1} = 1 \mid \mathcal{F}_n) = p \cdot \frac{H_n}{n}, \tag{2.1} \]
where \( \mathcal{F}_n \) is the \( \sigma \)-algebra generated by \( X_1, \cdots, X_n \). Noting that
\[ E[H_{n+1} \mid \mathcal{F}_n] = \left(1 + \frac{p}{n}\right) H_n, \]
we introduce
\[ a_0 := 1, \quad \text{and} \quad a_n := \prod_{k=1}^{n-1} \left(1 + \frac{p}{k}\right) = \frac{\Gamma(n + p)}{\Gamma(n) \Gamma(1 + p)} \quad \text{for} \quad n = 1, 2, \cdots, \tag{2.2} \]
and set
\[ \hat{M}_n := \frac{H_n}{a_n} \quad \text{for} \quad n = 0, 1, 2, \cdots. \]
Then \( \{\hat{M}_n\} \) satisfies a martingale property \( E[\hat{M}_{n+1} \mid \mathcal{F}_n] = \hat{M}_n \). Since \( \hat{M}_n \) is nonnegative, Doob’s convergence theorem implies that
\[ \lim_{n \to \infty} M_n = \lim_{n \to \infty} \frac{H_n}{a_n} = \hat{W} \quad \text{a.s.}. \tag{2.3} \]
In Corollary 2.2 below, we show that \( \hat{W} \) is positive with probability one.

2.1. Moments of the position. For \( k = 1, 2, \cdots, \) let
\[ a_n^{(k)} := \frac{\Gamma(n + kp)}{\Gamma(n) \Gamma(1 + kp)}. \]
Note that \( a_n^{(1)} = a_n \). The moments of the position \( H_n \) up to the fourth are calculated in section 4.6 of Coletti, Gava, and de Lima [8]:
\begin{align*}
E[H_n] &= a_n^{(1)}, \\
E[(H_n)^2] &= 2a_n^{(2)} - a_n^{(1)}, \\
E[(H_n)^3] &= 6a_n^{(3)} - 6a_n^{(2)} + a_n^{(1)}, \\
E[(H_n)^4] &= 24a_n^{(4)} - 36a_n^{(3)} + 14a_n^{(2)} - a_n^{(1)}. \tag{2.4}
\end{align*}
We could not find a simple way to describe the coefficients. Let \( (x)_1 := x \) and \( (x)_k := x(x - 1) \cdots (x - k + 1) \) for \( k = 2, 3, \cdots \). The \( k \)-th factorial moment of a random variable \( X \) is defined by \( E[(X)_k] \). Using (2.4), we can see that
\begin{align*}
E[(H_n)_2] &= E[(H_n)^2] - E[H_n] = 2(a_n^{(2)} - a_n^{(1)}), \\
E[(H_n)_3] &= E[(H_n)^3] - 3E[(H_n)^2] + 2E[H_n] = 6(a_n^{(3)} - 2a_n^{(2)} + a_n^{(1)}), \\
E[(H_n)_4] &= E[(H_n)^4] - 6E[(H_n)^3] + 11E[(H_n)^2] - 6E[H_n] \\
&= 24(a_n^{(4)} - 3a_n^{(3)} + 3a_n^{(2)} - a_n^{(1)}). \end{align*}
The following theorem, which will be proved in section 3 gives the general solution for \( E[(H_n)_k] \).
Theorem 2.1. Assume that \( p \in (0,1), q = 0, \) and \( s = 1. \) For any \( k = 1, 2, \ldots \) and \( n = 1, 2, \ldots, \)

\[
E[(H_n)_k] = k! \cdot \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} a_n^{(i)}. \tag{2.5}
\]

Since \( a_n^{(k)} \sim \frac{n^{kp}}{\Gamma(1+kp)} \) as \( n \to \infty, \) we have

\[
\lim_{n \to \infty} \frac{E[(H_n)_k]}{a_n^{(k)}} = \lim_{n \to \infty} \frac{E[(H_n)_k]}{a_n^{(k)}} = k! \text{ for any } k = 1, 2, \ldots.
\]

The following corollary is a much more precise result than (1.3) for \( q = 0. \)

Corollary 2.2. For any \( k = 1, 2, \ldots, \)

\[
\lim_{n \to \infty} E \left[ \frac{H_n}{\Gamma(1+np)} \right]^k = \frac{k!}{\Gamma(1+kp)}.
\]

Thus the martingale \( \{\hat{M}_n\} \) is \( L^k \)-bounded for any \( k, \) and the almost sure limit

\[
\mathcal{W} := \lim_{n \to \infty} \frac{H_n}{\Gamma(1+np)} = \frac{\hat{W}}{\Gamma(1+1)}
\]

has a Mittag–Leffler distribution with parameter \( p \) (see Appendix C). In particular, \( P(\mathcal{W} > 0) = 1. \)

2.2. Limit theorems for the laziest case. By Corollary 2.2, the limit \( \hat{W} \) in (2.3) satisfies \( P(\mathcal{W} > 0) = 1. \) Based on this fact, we obtain central limit theorems in the following form.

Theorem 2.3. Assume that \( q = 0 \) and \( s = 1. \) For \( 0 < p < 1, \)

\[
\frac{H_n - \hat{W} \cdot a_n}{\sqrt{\hat{W} \cdot a_n}} \overset{d}{\to} N(0,1),
\]

and

\[
\frac{H_n - \hat{W} \cdot a_n}{\sqrt{\hat{W} \cdot a_n}} \overset{d}{\to} \sqrt{\mathcal{W}'} \cdot Z,
\]

where \( Z \) is distributed as \( N(0,1), \) and \( \mathcal{W}' \) is independent of \( Z \) and has the same distribution as \( \mathcal{W}. \)

This situation is quite different from \( q > 0: \) The central limit theorem holds for whole regions of parameter space, with random centering and random norming.

We also prove the law of the iterated logarithm.

Theorem 2.4. Assume that \( q = 0 \) and \( s = 1. \) For \( 0 < p < 1, \)

\[
\limsup_{n \to \infty} \frac{H_n - \hat{W} \cdot a_n}{\phi \left( \hat{W} \cdot a_n \right)} = 1 \text{ a.s.,}
\]

where \( \phi(t) := \sqrt{2t \log \log t}. \)

Theorems 2.3 and 2.4 will be proved in section 4.
Applications to percolation on random recursive trees. Kürsten [21] found important connections between (several variations of) elephant random walks and percolation on random recursive trees. Consider the following procedure for obtaining a sequence \( \{T_i\} \) of recursive trees: The first graph \( T_1 \) consists of a single vertex labeled 1. For each \( i = 2, 3, \ldots \), the graph \( T_i \) is evolved from \( T_{i-1} \) by joining a new vertex labeled \( i \) to a uniformly chosen vertex labeled \( u_{i-1} \) from \( T_{i-1} \). We perform Bernoulli bond percolation on \( T_n \): Each edge of \( T_n \) is independently removed with probability \( 1 - p \), and otherwise retained. Then we can see that the size \( \#C_{1,n} \) of the cluster containing the vertex labeled 1 has the same distribution as the position \( H_n \) of the laziest minimal random walk model of elephant type (this relation is implicitly mentioned before eq. (43) in [21]).

Thus all of our results described above have counterparts for \( \#C_{1,n} \). To our best knowledge, Theorems 2.1, 2.3 and 2.4 are new also in this context. We remark that the \( \#C_{1,n} \)-version of Corollary 2.2 is Lemma 3 in Businger [7], where it plays a crucial role in analyzing the shark random swim, and is proved using a connection with the Yule process. Our proof of Corollary 2.2 based on Theorem 2.1 is a short alternative.

In Appendix A we give a precise description of the relation between the minimal random walk model with \( 0 \leq q < p < 1 \) and percolation on random recursive trees. As an easy and useful application of this, we derive new expressions of the expectation and the variance of \( H_n \), in terms of size of percolation clusters.

3. Factorial moments of the position

To prove Theorem 2.1 we use the probability generating functions: Let

\[
 f_n(x) := E[x^{H_n}] \quad \text{for } n = 1, 2, \ldots.
\]

Recall that \( f_n^{(k)}(1) = E[(H_n)_k] \), where \( f_n^{(k)}(1) \) denotes the \( k \)-th derivative of \( f_n(x) \) at \( x = 1 \). By (2.1), we can see

\[
 E[x^{H_{n+1}} | \mathcal{F}_n] = x^{H_n} \cdot \left\{ x^1 \cdot p \cdot \frac{H_n}{n} + x^0 \cdot \left( 1 - p \cdot \frac{H_n}{n} \right) \right\}
 = x^{H_n} \cdot \left\{ \frac{p(x - 1)}{n} \cdot H_n + 1 \right\},
\]

and

\[
 f_{n+1}(x) = E \left[ x^{H_n} \cdot \left\{ \frac{p(x - 1)}{n} \cdot H_n + 1 \right\} \right]
 = \frac{p(x - 1)}{n} \cdot E[H_n x^{H_n}] + E[x^{H_n}] = \frac{p x(x - 1)}{n} \cdot f_n'(x) + f_n(x).
\]

The Leibniz rule yields

\[
 f_{n+1}^{(k)}(x) = \frac{p x(x - 1)}{n} \cdot f_n^{(k+1)}(x) + \binom{k}{1} \cdot \frac{2 p x - p}{n} \cdot f_n^{(k)}(x)
 + \binom{k}{2} \cdot \frac{2 p}{n} \cdot f_n^{(k-1)}(x) + f_n^{(k)}(x).
\]
Thus we have

\[ E[(H_{n+1})_k] = f_{n+1}^{(k)}(1) = \frac{kp}{n} \cdot f_n^{(k)}(1) + \frac{k(k-1)p}{n} \cdot f_n^{(k-1)}(1) + f_n^{(k)}(1) \]

\[ = \frac{n + kp}{n} \cdot E[(H_n)_k] + \frac{k(k-1)p}{n} \cdot E[(H_n)_{k-1}]. \]

We prove \((2.5)\) by induction. For each \(n = 1, 2, \ldots\), \(E[H_n] = a_n^{(1)}\) in \((2.4)\) clearly satisfies \((2.5)\). On the other hand,

\[ E[H_1] = \frac{n + kp}{n} \cdot a_n^{(1)} + \frac{k(k-1)p}{n} \cdot a_{n-1}^{(1)}. \]

satisfy \((2.5)\), as \(a_n^{(1)} = 1\). Assume that \(k > 2\), and that \(E[(H_n)_{k-1}]\) and \(E[(H_n)_k]\) satisfy \((2.5)\). We can see that

\[ E[(H_{n+1})_k] = \frac{n + kp}{n} \cdot k! \cdot \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} a_n^{(i)} \]

\[ + \frac{k(k-1)p}{n} \cdot (k-1)! \cdot \sum_{i=1}^{k-1} (-1)^{k-1-i} \binom{k-2}{i-1} a_n^{(i)} \]

\[ = \frac{n + kp}{n} \cdot k! \cdot a_n^{(k)} \]

\[ + k! \cdot \sum_{i=1}^{k-1} (-1)^{k-i} \binom{k-1}{i-1} a_n^{(i)} \left( \frac{n + kp}{n} - \frac{(k-1)p}{n} \cdot \frac{k-i}{k-1} \right) \]

\[ = k! \cdot a_{n+1}^{(k)} + k! \cdot \sum_{i=1}^{k-1} (-1)^{k-i} \binom{k-1}{i-1} a_n^{(i)} \cdot \frac{n + ip}{n} \]

\[ = k! \cdot \sum_{i=1}^{k} (-1)^{k-i} \binom{k-1}{i-1} a_{n+1}^{(i)}. \]

This completes the proof of Theorem 2.1

4. Limit theorems

The structure of our proof of Theorems 2.3 and 2.4 is similar to that of [19]. For \(n = 0, 1, 2, \ldots\), we set

\[ M_n := \frac{H_n - E[H_n]}{a_n}(= \overline{M}_n - 1), \]

where \(a_n\) is defined in \((2.2)\). Clearly \(\{M_n\}\) is a martingale with mean zero. As in the proof of Lemma 1 of [19], we can obtain

\[ M_{n+1} - M_n = \frac{X_{n+1} - E[X_{n+1} | \mathcal{F}_n]}{a_{n+1}} \quad (4.1)\]
for each $n = 1, 2, \cdots$. Noting that $X^2_n = X_n$, we have
\[ E[(M_{n+1} - M_n)^2 | \mathcal{F}_n] = \frac{E[(X_{n+1} - E[X_{n+1} | \mathcal{F}_n])^2 | \mathcal{F}_n]}{(a_{n+1})^2} \]
\[ = \frac{E[X^2_{n+1} | \mathcal{F}_n] - (E[X_{n+1} | \mathcal{F}_n])^2}{(a_{n+1})^2} \]
\[ = \frac{E[X^2_{n+1} | \mathcal{F}_n] - (1 - E[X_{n+1} | \mathcal{F}_n])}{(a_{n+1})^2}. \tag{4.2} \]

Note that (4.1) and (4.2) hold also for $n = 0$, where $\mathcal{F}_0$ is the trivial $\sigma$-algebra.

For $k = 1, 2, \cdots$, let
\[ d_k := M_k - M_{k-1} = \frac{X_k - E[X_k | \mathcal{F}_{k-1}]}{a_k}. \]

Note that $|d_k| \leq \frac{1}{a_k} \leq 1$. Using (4.2), (1.3), and (2.3),
\[ E[(d_k)^2 | \mathcal{F}_{k-1}] = \frac{E[X_k | \mathcal{F}_{k-1}] \cdot (1 - E[X_k | \mathcal{F}_{k-1}])}{(a_k)^2} \]
\[ = p \cdot \frac{H_{k-1}}{k-1} \cdot \left(1 - p \cdot \frac{H_{k-1}}{k-1}\right) \cdot \frac{1}{(a_k)^2} \]
\[ = p \cdot \frac{H_{k-1}}{a_{k-1}} \cdot \frac{a_{k-1}}{k-1} \cdot \left(1 - p \cdot \frac{H_{k-1}}{k-1}\right) \cdot \frac{1}{(a_k)^2} \]
\[ \sim p \cdot \frac{W}{ka_k} \quad \text{as } k \to \infty \text{ a.s.} \]

As $|d_k| \leq 1$, the bounded convergence theorem yields that
\[ E[(d_k)^2] \sim \frac{p \cdot E[W]}{ka_k} = \frac{p}{ka_k} \quad \text{as } k \to \infty. \]

Thus we have
\[ V^2_n := \sum_{k=n}^{\infty} E[(d_k)^2 | \mathcal{F}_{k-1}] \]
\[ \sim p \cdot \frac{W \cdot \Gamma(1 + p)}{n^{1+p}} \sum_{k=n}^{\infty} \frac{1}{k^{1+p}} \]
\[ \sim p \cdot \frac{W \cdot \Gamma(1 + p)}{np} = \frac{W \cdot \Gamma(1 + p)}{n^{1+p}} \sim \frac{W}{a_n} \quad \text{as } n \to \infty \text{ a.s.,} \]
and
\[ s^2_n := \sum_{k=n}^{\infty} E[(d_k)^2] \sim \frac{1}{a_n} \quad \text{as } n \to \infty, \]

which imply that
\[ \lim_{n \to \infty} \frac{V^2_n}{s^2_n} = W \quad \text{a.s..} \tag{4.3} \]
Proof of Theorems 2.3 and 2.4. We check the conditions of Theorem B.2(ii) and (iii) in Appendix B are satisfied.

To prove conditions a) and a’ with \( \eta^2 = \hat{W} \) are satisfied, we will show

\[
\lim_{n \to \infty} \frac{1}{s_k^2} \sum_{k=n}^{\infty} \{(d_k)^2 - E[(d_k)^2 \mid \mathcal{F}_{k-1}]\} = 0 \quad \text{a.s.}
\]

By the tail version of Kronecker’s lemma (see Lemma 1 (ii) in Heyde [17]), it is sufficient to show

\[
\sum_{k=1}^{\infty} \frac{1}{s_k^2} \{(d_k)^2 - E[(d_k)^2 \mid \mathcal{F}_{k-1}]\} < +\infty \quad \text{a.s.. (4.4)}
\]

Let \( \tilde{d}_k \) denote the summand. Note that

\[
\sum_{k=1}^{\infty} E[(d_k)^2 \mid \mathcal{F}_{k-1}] \leq \sum_{k=1}^{\infty} \frac{1}{s_k^4} E[(d_k)^4 \mid \mathcal{F}_{k-1}].
\]

Since

\[
E[(d_k)^4 \mid \mathcal{F}_{k-1}] \leq \frac{1}{(a_k)^2} \cdot E[(d_k)^2 \mid \mathcal{F}_{k-1}]
\]

\[
\sim s_k^4 \cdot \frac{p \cdot \hat{W}}{k a_k} \sim s_k^4 \cdot \frac{p \cdot \hat{W} \cdot \Gamma(1 + p)}{k^{1+p}} \quad \text{as} \ k \to \infty,
\]

the series in the right hand side converges a.s.. Theorem B.1 implies (4.4).

For \( \varepsilon > 0 \), noting that

\[
E[(d_k)^2 : |d_k| > \varepsilon s_n] \leq \frac{1}{\varepsilon^2 s_n^2} E[(d_k)^4],
\]

and

\[
E[(d_k)^4] \leq \frac{1}{(a_k)^2} \cdot E[(d_k)^2] \sim \frac{p \cdot \Gamma(1 + p)^2}{k^{1+3p}} \quad \text{as} \ k \to \infty,
\]

we have

\[
\frac{1}{s_n^2} \sum_{k=n}^{\infty} E[(d_k)^2 : |d_k| > \varepsilon s_n] \leq \frac{1}{\varepsilon^2 s_n^2} \sum_{k=n}^{\infty} E[(d_k)^4]
\]

\[
\leq C_1 n^{2p} \cdot \frac{1}{n^{3p}} = C_1 \frac{n^p}{n^p} \to 0 \quad \text{as} \ n \to \infty.
\]

In view of Remark B.3, condition b) is also satisfied.

Similarly, for \( \varepsilon > 0 \) we have

\[
\frac{1}{s_k} E[|d_k| : |d_k| > \varepsilon s_k] \leq \frac{1}{s_k} \cdot \frac{1}{s_k} E[(d_k)^4]
\]

\[
\leq C_2 k^{2p} \cdot \frac{1}{k^{1+3p}} = C_2 \frac{k^{1+p}}{k^{1+p}},
\]

which implies that condition c) holds.

Condition d) is implied by

\[
\sum_{n=1}^{\infty} \frac{1}{s_n^4} E[(d_n)^4] < +\infty.
\]
To deduce the conclusions of Theorems 2.3 and 2.4 from Theorem B.2 (ii) and (iii), note that

\[
W - M_n = a_n \cdot W - (H_n - E[H_n]) = a_n \cdot \hat{W} - H_n,
\]

and

\[
a_n \cdot \hat{\phi} \left( \frac{\hat{W}}{a_n} \right) \sim \hat{\phi} \left( a_n \cdot \hat{W} \right) \quad \text{as } n \to \infty,
\]

where \( \hat{\phi}(t) = \sqrt{2t \log |\log t|} \).

\[\square\]

**Appendix A. The minimal random walk model and percolation on random recursive trees**

We explore a relation between the minimal random walk model by Hambola, Kumar, and Lindenberg [16], explained in the Introduction, and percolation on random recursive trees. Throughout this section we assume that \(0 \leq q < p < 1\), and set \(\alpha = p - q\) and \(\rho = q/(1 - \alpha)\).

The sequence \(\{T_i\}\) of random recursive trees is defined in section 2.3. Consider bond percolation on \(T_n\) with parameter \(\alpha\). The expectation regarding this model is denoted by \(E_\alpha[\cdot]\). There are at most \(n\) clusters, which are denoted by \(C_1, C_2, \ldots, C_n\) (for convenience we regard \(C_j = \emptyset\) if \(j\) is larger than the number of clusters). We quote some of results in Kürsten [21].

**Lemma A.1.** For bond percolation on \(T_n\) with parameter \(\alpha \in (0,1)\),

\[
E_\alpha[\#C_1, n] = \frac{\Gamma(n + \alpha)}{\Gamma(1 + \alpha) \Gamma(n)}, \quad (A.1)
\]

\[
E_\alpha[\#C_1^2, n] = \frac{2\Gamma(n + 2\alpha)}{\Gamma(1 + 2\alpha) \Gamma(n)} - \frac{\Gamma(n + \alpha)}{\Gamma(1 + \alpha) \Gamma(n)}, \quad (A.2)
\]

\[
\sum_{j=1}^{n} E_\alpha[\#C_{j, n}^2] = \begin{cases} 
\frac{1}{1 - 2\alpha} \cdot n + \frac{1}{2\alpha - 1} \cdot \frac{\Gamma(n + 2\alpha)}{\Gamma(2\alpha) \Gamma(n)} & (\alpha \neq 1/2), \\
\sum_{\ell=1}^{n} \frac{1}{\ell} & (\alpha = 1/2).
\end{cases} \quad (A.3)
\]

**Remark A.2.** In [21], (A.1) and (A.3) are derived from basic results on the original elephant random walk, found in [23]. In view of the connection with the laziest minimal random walk model explained in section 2.3, (A.1) and (A.2) are paraphrases of (2.4). Those are obtained by solving relatively easy difference equations.

Let \(\xi_1, \xi_2, \ldots, \xi_n\) be a sequence of independent random variables, which is also independent from bond percolation, satisfying

\[
P_{p,q,s}(\xi_1 = 1) = 1 - P_{p,q,s}(\xi_1 = 0) = s, \quad \text{and}
\]

\[
P_{p,q,s}(\xi_j = 1) = 1 - P_{p,q,s}(\xi_j = 0) = \rho \quad \text{for } j > 1.
\]

The transition probability in (1.1) can be interpreted as follows: For each step \(n = 2, 3, \ldots\),
• with probability $\alpha$, the walker repeats the behavior at a uniformly chosen time, and
• with probability $1 - \alpha$, the walker moves forward with probability $\rho$, or remains at rest otherwise.

Similarly to [21], we can see that $H_n$ has the same distribution as

$$\sum_{j=1}^{n} \xi_j \cdot (\#C_{j,n}).$$  \hspace{1cm} (A.4)

In the case $q > 0$, computation of the second moment (and the variance) of $H_n$ by solving difference equations is straightforward but quite tedious, as is imagined from very complicated equations (8), (9) and (10) in [16]. Using the above connection with percolation, we can easily obtain concise formulae described in terms of the moments of the size of open clusters.

**Theorem A.3.** Let $E_{p,q,s}$ and $V_{p,q,s}$ denote the expectation and the variance for the minimal random walk model. Assume that $0 \leq q < p < 1$, and set $\alpha = p - q$ and $\rho = q/(1 - \alpha)$. Then we have the following.

$$E_{p,q,s}[H_n] = \rho n + (s - \rho)E_{\alpha}[\#C_{1,n}],$$  \hspace{1cm} (A.5)

$$V_{p,q,s}[H_n] = \rho(1 - \rho) \sum_{j=1}^{n} E_{\alpha}[\#C_{j,n}^2] + (1 - 2\rho)(s - \rho)E_{\alpha}[\#C_{1,n}^2] - (s - \rho)^2(E_{\alpha}[\#C_{1,n}])^2.$$  \hspace{1cm} (A.6)

**Proof.** Since $E_{p,q,s}[\xi_1] = s$ and $E_{p,q,s}[\xi_j] = \rho$ for $j > 1$, we have

$$E_{p,q,s}[H_n] = \sum_{j=1}^{n} E_{p,q,s}[\xi_j] \cdot E_{\alpha}[\#C_{j,n}] = sE_{\alpha}[\#C_{1,n}] + \rho \sum_{j=2}^{n} E_{\alpha}[\#C_{j,n}]$$

$$= \rho \sum_{j=1}^{n} E_{\alpha}[\#C_{j,n}] + (s - \rho)E_{\alpha}[\#C_{1,n}] = \rho n + (s - \rho)E_{\alpha}[\#C_{1,n}].$$

Turning to the mean square displacement, similarly as eq. (17) in [21],

$$E_{p,q,s}[(H_n)^2]$$

$$= \sum_{j=1}^{n} E_{p,q,s}[(\xi_j)^2] \cdot E_{\alpha}[\#C_{j,n}^2]$$

$$+ 2 \sum_{1 \leq j < k \leq n} E_{p,q,s}[\xi_j] \cdot E_{p,q,s}[\xi_k] \cdot E_{\alpha}[\#C_{j,n} \cdot \#C_{k,n}]$$

$$= \rho \sum_{j=1}^{n} E_{\alpha}[\#C_{j,n}^2] + 2\rho^2 \sum_{1 \leq j < k \leq n} E_{\alpha}[\#C_{j,n} \cdot \#C_{k,n}]$$

$$+ (s - \rho)E_{\alpha}[\#C_{1,n}^2] + 2(s - \rho)\rho \sum_{1 < k \leq n} E_{\alpha}[\#C_{1,n} \cdot \#C_{k,n}].$$  \hspace{1cm} (A.7)
The first two terms in (A.7) are

\[ \rho^2 E_\alpha \left[ \left( \sum_{j=1}^{n} \# \mathcal{C}_{j,n} \right)^2 \right] + (\rho - \rho^2) \sum_{j=1}^{n} E_\alpha[(\# \mathcal{C}_{j,n})^2] = \rho^2 n^2 + \rho(1 - \rho) \sum_{j=1}^{n} E_\alpha[(\# \mathcal{C}_{j,n})^2]. \]

Noting that

\[ \sum_{1 < k \leq n} E_\alpha[(\# \mathcal{C}_{1,n}) \cdot (\# \mathcal{C}_{k,n})] = E_\alpha \left[ (\# \mathcal{C}_{1,n}) \cdot \sum_{1 < k \leq n} (\# \mathcal{C}_{k,n}) \right] = E_\alpha[(\# \mathcal{C}_{1,n}) \cdot (n - \# \mathcal{C}_{1,n})], \]

the other two terms in (A.7) are

\[ (s - \rho) E_\alpha[(\# \mathcal{C}_{1,n})^2] + 2(s - \rho) \rho n E_\alpha[\# \mathcal{C}_{1,n}] = (1 - 2\rho)(s - \rho) E_\alpha[(\# \mathcal{C}_{1,n})^2] + 2(s - \rho) \rho n E_\alpha[\# \mathcal{C}_{1,n}]. \]

Using

\[ (E_{p,q,s}[H_n])^2 = \rho^2 n^2 + 2(s - \rho) \rho n E_\alpha[\# \mathcal{C}_{1,n}] + (s - \rho)^2 (E_\alpha[\# \mathcal{C}_{1,n}])^2, \]

we have the conclusion. \( \Box \)

Combining (A.6) with Lemma (A.1) we can obtain the asymptotics of the variance.

**Corollary A.4.** When \( q > 0, \)

\[ V_{p,q,s}[H_n] \sim \begin{cases} 
\frac{\rho(1 - \rho)}{1 - 2\alpha} n & (\alpha < 1/2), \\
\rho(1 - \rho)n \log n & (\alpha = 1/2), \\
\left[ \frac{\rho(1 - \rho)}{2(2\alpha - 1)\Gamma(2\alpha)} + \frac{(1 - 2\rho)(s - \rho)}{\Gamma(1 + 2\alpha)} - \frac{(s - \rho)^2}{\Gamma(1 + \alpha)^2} \right] n^{2\alpha} & (\alpha > 1/2)
\end{cases} \]
as \( n \to \infty. \)

To close this section, we give a remark on phase transition of the biased elephant random walk \( \{S_n\} \) on \( \mathbb{Z} \):

- With probability \( \alpha \), the walker repeats one of previous steps.
- With probability \( 1 - \alpha \), the walker performs like a simple random walk, which jumps to the right with probability \( \rho \), or to the left with probability \( 1 - \rho \) (The unbiased case \( \rho = 1/2 \) is the original elephant random walk explained in the Introduction, where \( p \geq 1/2 \) and \( \alpha = 2p - 1 \)).

This is obtained from the minimal random walk model as follows: Let

\[ Y_i := 2X_i - 1 \quad \text{and} \quad S_n = \sum_{i=1}^{n} Y_i = 2H_n - n. \]

Then \( P(Y_1 = +1) = 1 - P(Y_1 = -1) = s, \) and by (1.1),

\[ P(Y_{n+1} = \pm 1 \mid \mathcal{F}_n) = \alpha \cdot \frac{\# \{ i = 1, \ldots, n : Y_i = \pm 1 \}}{n} + (1 - \alpha) \cdot \rho. \]
By (1.3), we have
\[
\lim_{n \to \infty} \frac{S_n}{n} = 2\rho - 1 \quad \text{a.s..} \tag{A.8}
\]

Consider bond percolation on \( T_n \) with parameter \( \alpha \), and assign ‘spin’ \( m_j := 2\xi_j - 1 \in \{+1, -1\} \) to each of percolation clusters \( C_{j,n} \), independently for different clusters. By (A.4), \( S_n \) has the same distribution as
\[
\sum_{j=1}^{n} m_j \cdot (\# C_{j,n}).
\]
The above procedure is essentially the same as the “Divide and Color” model introduced by Häggström [14]. When \( s = \rho = 1/2 \), the resulting model resembles the Ising model with zero external field, and increasing \( \alpha \) corresponds to lowering the temperature. The parameter \( \varepsilon := 2\rho - 1 \) plays a similar role to the external field in the Ising model. By (A.8), when \( \varepsilon \neq 0 \), the asymptotic speed of the walker remains unchanged regardless of the value of \( \alpha \). On the other hand, when \( \varepsilon = 0 \), the walker admits a phase transition from diffusive to superdiffusive behavior. This is reminiscent of the fact that the Ising model admits a phase transition only when the external field is zero.

**Appendix B. Martingale limit theorems**

**Theorem B.1** (Hall and Heyde [15], Theorem 2.15). Suppose that \( \{M_n\} \) is a square-integrable martingale with mean 0. Let \( d_k = M_k - M_{k-1} \) for \( k = 1, 2, \cdots \), where \( M_0 = 0 \). On the event
\[
\left\{ \sum_{k=1}^{\infty} E[(d_k)^2] \mid \mathcal{F}_{k-1} \right\} < +\infty,
\]
\( \{M_n\} \) converges a.s..

**Theorem B.2** (Heyde [17], Theorem 1 (b)). Suppose that \( \{M_n\} \) is a square-integrable martingale with mean 0. Let \( d_k = M_k - M_{k-1} \) for \( k = 1, 2, \cdots \), where \( M_0 = 0 \). If
\[
\sum_{k=1}^{\infty} E[(d_k)^2] < +\infty
\]
holds in addition, then we have the following: Let
\[
W_n^2 := \sum_{k=n}^{\infty} (d_k)^2 \quad \text{and} \quad s_n^2 := \sum_{k=n}^{\infty} E[(d_k)^2].
\]

(i) The limit \( M_\infty := \sum_{k=1}^{\infty} d_k \) exists a.s., and \( M_n \stackrel{L^2}{\to} M_\infty \).

(ii) Assume that
\[
a) \quad \frac{W_n^2}{s_n^2} \to \eta^2 \quad \text{as} \ n \to \infty \quad \text{in probability, and}
\]
\[
b) \quad \lim_{n \to \infty} \frac{1}{\eta^2} E \left[ \sup_{k \geq n} (d_k)^2 \right] = 0,
\]
where \( \eta^2 \) is some a.s. finite and non-zero random variable. Then we have

\[
\frac{M_{\infty} - M_n}{W_{n+1}} = \sum_{k=n+1}^{\infty} \frac{d_k}{W_{n+1}} \xrightarrow{d} Z, \quad \text{and}
\]

\[
\frac{M_{\infty} - M_n}{s_{n+1}} = \sum_{k=n+1}^{\infty} \frac{d_k}{s_{n+1}} \xrightarrow{d} \tilde{\eta} \cdot Z,
\]

where \( Z \) is distributed as \( N(0,1) \), and \( \tilde{\eta} \) is independent of \( Z \) and distributed as \( \eta \).

(iii) Assume that the following three conditions hold:

\begin{itemize}
  \item[a')] \( \frac{W_n^2}{s_n^2} \to \eta^2 \) as \( n \to \infty \) a.s.,
  \item[c)] \( \sum_{k=1}^{\infty} \frac{1}{s_k^2} E[|d_k| : |d_k| > \varepsilon s_k] < +\infty \) for any \( \varepsilon > 0 \), and
  \item[d)] \( \sum_{k=1}^{\infty} \frac{1}{s_k^4} E[|d_k|^4 : |d_k| \leq \delta s_k] < +\infty \) for some \( \delta > 0 \).
\end{itemize}

Then \( \limsup_{n \to \infty} \frac{M_{\infty} - M_n}{\phi(W_{n+1}^2)} = 1 \) a.s., where \( \phi(t) := \sqrt{2\log |\log t|} \).

Remark B.3. A sufficient condition for b) in Theorem B.2 is that

\[
\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{k=n}^{\infty} E[|d_k|^2 : |d_k| > \varepsilon s_n] = 0
\]

for any \( \varepsilon > 0 \). (See the proof of Corollary 1 in Heyde [17].)

Appendix C. The Mittag–Leffler distribution

The Mittag–Leffler function is defined by

\[
E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+\alpha)} \quad (\alpha, z \in \mathbb{C}).
\]

Note that \( E_1(z) = e^z \) (See e.g. [9], p. 315).

The random variable \( X \) is Mittag–Leffler distributed with parameter \( p \in [0,1] \) if

\[
E[e^{\lambda X}] = E_p(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(1+kp)} \quad \text{for } \lambda \in \mathbb{R}.
\]

Thus the \( k \)-th moment of \( X \) is \( \frac{k!}{\Gamma(1+kp)} \), and this distribution is determined by moments (see [9], p. 329 and p. 391). If \( p = 0 \) (resp. \( p = 1 \)), then \( X \) has the exponential distribution with mean one (resp. \( X \) concentrates on \( \{1\} \)). For \( p \in (0,1) \), the probability density function \( f_p(x) \) of \( X \) is

\[
f_p(x) = \frac{\rho_p(x^{-1/p})}{px^{1+1/p}} \quad \text{for } x > 0,
\]

where \( \rho_p(x) \) is the density function of the one-sided stable\((p)\) distribution. See [22] for details. In particular, \( f_{1/2} \) is the density function of the standard
half-normal distribution:

\[ f_{1/2}(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} \quad \text{for } x > 0. \]

References

[1] Baur, E. and Bertoin, J. (2016). Elephant random walks and their connection to Pólya-type urns, *Phys. Rev. E*, 94, 052134.

[2] Bercu, B. (2018). A martingale approach for the elephant random walk, *J. Phys. A: Math. Theor.*, 51, 015201.

[3] Bercu, B. and Laulin, L. (2019). On the multi-dimensional elephant random walk, *J. Statist. Phys.*, 175, 1146–1163.

[4] Bertoin, J. (2018). Noise reinforcement for Lévy processes, to appear in *Ann. Inst. Henri Poincaré Probab. Stat.* [arXiv:1810.08364]

[5] Bertoin, J. (2020). Universality of noise reinforced Brownian motions, [arXiv:2002.09166](https://arxiv.org/abs/2002.09166)

[6] Bingham, N. H., Goldie, C. M., and Teugels, J. L. (1989). Regular variation, *Encyclopedia of Mathematics and its Applications*, 27, Cambridge University Press.

[7] Businger, S. (2018). The shark random swim (Lévy flight with memory), *J. Statist. Phys.*, 172, 701–717. (See also [arXiv:1710.05671v3](https://arxiv.org/abs/1710.05671v3))

[8] Coletti, C. F., Gava, R. J., and de Lima, L. R. (2019). Limit theorems for a minimal random walk model, *J. Stat. Mech.*, 083206.

[9] Coletti, C. F., Gava, R. J., and Schütz, G. M. (2017a). Central limit theorem for the elephant random walk, *J. Math. Phys.*, 58, 053303.

[10] Coletti, C. F., Gava, R. J., and Schütz, G. M. (2017b). A strong invariance principle for the elephant random walk, *J. Stat. Mech.*, 123207.

[11] Drezner, Z. and Farnum, N. (1993). A generalized binomial distribution, *Comm. Statist. Theory Methods*, 22, 3051–3063.

[12] Gut, A. and Stadtmüller, U. (2018). Variations of the elephant random walk, [arXiv:1812.01915](https://arxiv.org/abs/1812.01915)

[13] Gut, A. and Stadtmüller, U. (2019). Elephant random walks with delays, [arXiv:1906.04930](https://arxiv.org/abs/1906.04930)

[14] Häggström, O. (2001). Coloring percolation clusters at random, *Stoch. Proc. Appl.*, 96, 213–242.

[15] Hall, P. and Heyde, C. C. (1980). Martingale limit theory and its application, *Probability and Mathematical Statistics*, Academic Press.

[16] Harbola, U., Kumar, N., and Lindenberg, K. (2014). Memory-induced anomalous dynamics in a minimal random walk model, *Phys. Rev. E*, 90, 022136

[17] Heyde, C. C. (1977). On central limit and iterated logarithm supplements to the martingale convergence theorem, *J. Appl. Probab.*, 14, 758–775.

[18] Heyde, C. C. (2004). Asymptotics and criticality for a correlated Bernoulli process, *Aust. N. Z. J. Stat.*, 46, 53–57.

[19] Kubota, N. and Takei, M. (2019). Gaussian fluctuation for superdiffusive elephant random walks, *J. Statist. Phys.*, 177, 1157–1171.

[20] Kumar, N., Harbola, U., and Lindenberg, K. (2010). Memory-induced anomalous dynamics: Emergence of diffusion, *Phys. Rev. E*, 82, 021101

[21] Kürsten, R. (2016). Random recursive trees and the elephant random walk, *Phys. Rev. E*, 93, 032111

[22] Pollard, H. (1948). The completely monotonic character of the Mittag–Leffler function $E_a(-z)$, *Bull. Amer. Math. Soc.*, 54, 1115–1116.

[23] Schütz, G. M. and Trimper, S. (2004). Elephants can always remember: Exact long-range memory effects in a non-Markovian random walk, *Phys. Rev. E*, 70, 045101
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