Exact quasinormal modes for a special class of black holes

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Abstract

Analytic exact expressions for the quasinormal modes of scalar and electromagnetic perturbations around a special class of black holes are found in $d \geq 3$ dimensions. It is shown that, the size of the black hole provides a bound for the angular momentum of the perturbation. Quasinormal modes appear when this bound is fulfilled, otherwise the excitations become purely damped.
I. INTRODUCTION

Small perturbations of the geometry or matter fields drained by a black hole give rise to the so-called quasinormal modes. Since the spectrum is independent of the initial conditions, and it is characterized only on the parameters of the black hole and on the fundamental constants of the system, it contains relevant information about the intrinsic properties of the black hole. Furthermore, by virtue of the AdS/CFT correspondence, quasinormal modes determine the relaxation time scale of the thermal states in the dual theory \cite{1}. Exact results about quasinormal modes and frequencies certainly help to have a deeper understanding of this phenomenon. Nevertheless, only few analytic results are known \cite{2}-\cite{19}. Our purpose is to report that there is a special class of black holes in $d \geq 3$ dimensions, for which exact analytic expressions for the quasinormal modes of scalar and electromagnetic perturbations can be obtained. One can see that the size of the black hole provides a bound for the angular momentum of the perturbation, such that quasinormal modes appear when this bound is fulfilled; otherwise the excitations become purely damped.

Let us consider the following spacetime in $d$-dimensions

$$ds^2 = -\frac{1}{l^2} \left( r^2 - r_+^2 \right) dt^2 + \frac{l^2 dr^2}{r^2 - r_+^2} + r^2 d\Sigma_{d-2}^2,$$  \hspace{1cm} (1)

where $l$ is the AdS radius and $d\Sigma_{d-2}^2$ stands for the metric of a smooth "base" manifold $\Sigma_{d-2}$ of $d - 2$ dimensions which can be assumed to be compact and orientable. This metric describes an asymptotically locally AdS black hole whose event horizon is located at $r = r_+$. In $d = 3$ dimensions this metric corresponds to the static BTZ black hole \cite{20}, while in $d = 4$ it solves the field equations of conformal gravity provided the base manifold is of constant curvature \cite{21}. This is also so for conformal gravity in even dimensions \cite{22}. In odd dimensions, for an arbitrary base manifold $\Sigma_{d-2}$, the black hole (1) provides a solution for a special case of Lovelock gravity \cite{23}. This case is such that the coefficients are fixed so as the theory admits a unique maximally symmetric AdS vacuum \cite{24} and the Lagrangian can be expressed as a Chern-Simons form \cite{25}. The case of $\Sigma_{d-2}$ of constant curvature was previously analyzed in \cite{26}, \cite{27} and for spherical symmetry Eq. (1) reduces to the solution found in \cite{28}. In five dimensions the metric (1) still solves the field equations even in the presence of a nontrivial fully antisymmetric torsion \cite{29}. The black hole (1) also provides a solution for the Lovelock theory in $d = 8$ compactified to five dimensions \cite{30}.
II. FREE MASSIVE SCALAR FIELD

Here it is shown that the Klein-Gordon equation

\[(\Box - m^2) \phi = 0 , \tag{2}\]

admits an analytic solution when it is solved on the background metric given by (1), which allows to find an exact expression for the quasinormal modes. This can be seen as follows:

The metric (1) can be expressed as

\[ds^2 = - \left( \frac{\dot{r}^2}{l^2} - 1 \right) dt^2 + \frac{dr^2}{\hat{r}^2 - 1} + \hat{r}^2 d\hat{\Sigma}_{d-2}^2 , \tag{3}\]

where the time and radial coordinates have been rescaled as \(\hat{r} = \frac{r}{r_+}, \hat{t} = \frac{r_+}{r} t,\) and the new base manifold \(\hat{\Sigma}_{d-2}\) is related to \(\Sigma_{d-2}\) by means of a global conformal transformation given by \(d\hat{\Sigma}_{d-2}^2 = \frac{r_+^2}{r^2} d\Sigma_{d-2}^2.\) Thus, since the quasinormal modes for a free scalar field propagating on the metric (3) have been already found in [4] requiring the scalar field to be purely ingoing at the horizon and the vanishing of the energy flux at infinity, the result we are looking for can be obtained by means of a simple rescaling. Indeed, as for each mode the scalar field propagating on (3) acquires the form

\[\phi = e^{-i\omega \hat{t}/l} f(\hat{r}) Y(\hat{\Sigma}) , \tag{4}\]

where \(Y(\hat{\Sigma})\) is an eigenfunction of the Laplace operator on \(\hat{\Sigma}\) with eigenvalue \(-\hat{Q},\) i.e. \(\nabla_{\hat{\Sigma}}^2 Y(\hat{\Sigma}) = -\hat{Q} Y(\hat{\Sigma}),\) the scalar field on the black hole (1), is given by

\[\phi = e^{-i\omega t/l} f(r) Y(\Sigma) , \tag{5}\]

with

\[\omega = \frac{r_+}{l} \hat{\omega} , \quad Q = \frac{r_+^2}{l^2} \hat{Q} . \tag{6}\]

Hence, as \(\hat{\omega}\) was already found in [4], the quasinormal frequencies of the black hole (1) turn out to be given by

\[\omega = -\sqrt{Q - \left( \frac{d-3}{2} \right)^2 \frac{r_+^2}{l^2} - i \frac{r_+}{l} \left( 2n + 1 + \sqrt{\left( \frac{d-1}{2} \right)^2 + m^2 l^2} \right) , \tag{8}\]
where $Q$ is the eigenvalue of the Laplace operator on $\Sigma_{d-2}$, and $n = 0, 1, 2, \ldots$.

The radial function can be expressed in terms of $z = 1 - r_+^2 / r^2$, and it reads

$$f(z) = z^\alpha (1 - z)^\beta F(a, b, c, z),$$

where

$$\alpha = -\frac{il\omega}{2r_+},$$

$$\beta = \beta_\pm = \frac{d - 1}{4} \pm \frac{1}{2} \sqrt{\left(\frac{d - 1}{2}\right)^2 + m^2 l^2},$$

and $F$ is the hypergeometric function with parameters defined by

$$a = -\left(\frac{d - 3}{4}\right) + \alpha + \beta_\pm + \frac{i}{2} \sqrt{\frac{l^2}{r_+^2} Q - \left(\frac{d - 3}{2}\right)^2},$$

$$b = -\left(\frac{d - 3}{4}\right) + \alpha + \beta_\pm - \frac{i}{2} \sqrt{\frac{l^2}{r_+^2} Q - \left(\frac{d - 3}{2}\right)^2},$$

$$c = 1 + 2\alpha.$$  

(10)  

(11)  

(12)  

(13)  

(14)

Note that, as it occurs for AdS spacetime, stability is guaranteed provided the Breitenlohner-Freedman bound is fulfilled [31]

$$m^2 l^2 \geq -\left(\frac{d - 1}{2}\right)^2.$$  

(15)

From Eq. (8), one can see that ringing modes exist provided the following bound on the eigenvalue of the Laplace operator on $\Sigma_{d-2}$ is fulfilled

$$Q > \left(\frac{d - 3}{2}\right)^2 \frac{r_+^2}{l^2},$$

(16)

otherwise the excitations are purely damped.

In the case of spherical symmetry, i.e. for $\Sigma_{d-2} = S^{d-2}$, since $Q = L (L + d - 3)$ the equation (16) provides a bound for the angular momentum $L$ of the perturbation. It is natural then to expect that this bound should be related with the impact parameter that a geodesic has to possess in order to avoid being swallowed by the black hole.

Note also that, when (16) is fulfilled, the damping time scale is independent of $Q$, unlike what occurs for the Schwarzschild-AdS black hole, for which the damping time scale increases with the angular momentum of the mode [1].
Using the results found in this section, it has been recently argued that the mass and area spectrum of these black holes have a strong dependence on the base manifold, and they are not evenly spaced \[32\].

III. NONMINIMAL COUPLING

Let us consider the following massive scalar field nonminimally coupled with the scalar curvature

\[ (\Box - m^2 + \xi R) \phi = 0, \]

where the conformal coupling is recovered for \( \xi = -\frac{1}{d-2}. \) Here \( R \) is the Ricci scalar of the background metric, which for the black hole (1) is given by

\[ R = -\frac{d(d-1)}{l^2} + \frac{l^2 R_{\Sigma} + (d-2)(d-3)r_+^2}{l^2 r^2} := A + \frac{B}{r^2}, \]

where \( R_{\Sigma} \) is the Ricci scalar of the base manifold \( \Sigma_{d-2} \), which hereafter is assumed to be constant in order to ensure the separability of equation (17). Unlike the case of AdS spacetime, the Ricci scalar of the black hole (18) is not constant, and hence the nonminimal coupling contributes now to the field equation with more than just a shift in the mass. Nevertheless, remarkably, the effect of the nonminimal coupling amounts to a shift in \( Q \), compared with the previous case. This can be seen as follows:

Using separation of variables as in Eq. (5), the equation for the radial function reads

\[ \frac{d^2 f}{dr^2} + \left[ \frac{d-2}{r^2} + \frac{2r}{r^2 - r_+^2} \right] \frac{df}{dr} + \left[ \omega^2 l^4 \left( \frac{r^2 - r_+^2}{r^2} \right)^2 - \frac{l^2 (Q - \xi B)}{r^2} r_+^2 - \frac{l^2 (m^2 - \xi A)}{r^2} \right] f = 0, \]

Note that one obtains the same equation as in the case of minimal coupling \( (\xi = 0) \), which has already been solved in the previous section, but with an effective mass and "angular momentum" given by

\[ Q_{\text{eff}} = Q - \left( R_{\Sigma} + (d-2)(d-3)\frac{r_+^2}{l^2} \right) \xi, \]

\[ m^2_{\text{eff}} = m^2 + \frac{d(d-1)}{l^2} \xi. \]

Therefore, the solution of Eq. (19) can be written as in (5) with (9), replacing \( Q \) and \( m^2 \) by \( Q_{\text{eff}} \) and \( m^2_{\text{eff}} \), respectively. The quasinormal frequencies are then given by (8) with the
same replacements. The presence of a nonminimal coupling changes the boundary condition on the vanishing of the energy flux at infinity, such that it can be compatible with scalar fields possessing slow fall-off. If the mass and the coupling constant $\xi$ satisfy the relation

$$\xi + \beta + 4\xi\beta = 0,$$

with

$$\beta := \frac{d-1}{4} \pm \frac{1}{2} \sqrt{\left(\frac{d-1}{2}\right)^2 + m_{eff}^2 l^2},$$

then for the range of effective masses given by

$$-\left(\frac{d-1}{2}\right)^2 < m_{eff}^2 l^2 < 1 - \left(\frac{d-1}{2}\right)^2,$$

there is a second set of modes for which the frequencies can be obtained from Ref. [4].

Thus, using the corresponding scalings and shifts explained in this section, the second set of quasinormal frequencies turns out to be

$$\omega = -\sqrt{Q_{eff} - \left(\frac{d-3}{2}\right)^2 \frac{r_+^2}{l^2} - \frac{r_+}{l} \left(2n + 1 - \sqrt{\left(\frac{d-1}{2}\right)^2 + m_{eff}^2 l^2}\right)}.$$

**IV. ELECTROMAGNETIC FIELD**

The quasinormal modes for an electromagnetic perturbation that propagates on the black hole (1) can be obtained following the same strategy as the one for the scalar field. Indeed, the quasinormal modes for the Maxwell field propagating on the massless topological black hole (3) have been found by López-Ortega in Ref. [33], who showed that the problem can be reduced to the case of the scalar field solved in Ref. [4], since scalar and vector modes of the electromagnetic perturbation are equivalent to scalar field perturbations but with different precise masses. Therefore by virtue of Eqs. (6) and (7), the quasinormal frequencies for the scalar and vector modes of the electromagnetic perturbation on the black hole (1) are respectively given by

$$d = 4 : \omega_s = -\sqrt{Q_s - \frac{1}{4} \frac{r_+^2}{l^2} - 2i \frac{r_+}{l} \left(n + \frac{3}{4}\right)},$$

$$d \geq 5 : \omega_s = -\sqrt{Q_s - \left(\frac{d-3}{2}\right)^2 \frac{r_+^2}{l^2} - 2i \frac{r_+}{l} \left(n + \frac{3}{4}\right)}.$$
and
\[ d \geq 4 : \omega_v = -\sqrt{Q_v - \left[1 + \left(\frac{d - 3}{2}\right)^2\right] \frac{r_+^2}{l^2} - 2i\frac{r_+}{l}\left(n + \frac{d - 1}{4}\right)}, \tag{28} \]

where \( Q_s \) and \( Q_v \) are the eigenvalues of the Laplacian on the base manifold \( \Sigma_{d-2} \) for scalar and vector harmonics, respectively.

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