Quantum Heisenberg Chain
with Long-Range Ferromagnetic Interactions
at Low Temperature

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A modified spin-wave theory is applied to the one-dimensional quantum Heisenberg model with long-range ferromagnetic interactions.

\( \mathcal{H} = -J \sum_{i<j}(r_{ij})^{-p} \mathbf{S}_i \cdot \mathbf{S}_j \). Low-temperature properties of this model are investigated. The susceptibility and the specific heat are calculated; the relation between their behaviors and strength of the long-range interactions is obtained. This model includes both the Haldane-Shastry model and the nearest-neighbor Heisenberg model; the corresponding results in this paper are in agreement with the solutions of both the models. It is shown that there exists an ordering transition for \( 1 < p < 2 \) where the model has longer-range interactions than the HS model. The critical temperature is estimated.

KEYWORDS: Heisenberg chain, long-range interactions, Haldane-Shastry model, modified spin-wave theory, susceptibility, specific heat, critical temperature
I. INTRODUCTION

The quantum Heisenberg chain with long-range interactions decaying as $1/r^2$, which is called the Haldane-Shastry model (HS), has been extensively investigated since the exact eigenstates and their eigenenergies are obtained independently by Haldane [1] and Shastry. [2] The thermodynamics of this model was investigated by Haldane. [3] Unfortunately few studies for the general type of $1/r^p$ have been made. What we know are rigorous bounds for the correlation functions in the disordered phase, which are established by Ito. [4]

On the other hand, the model of the classical spins with long-range interactions has been studied for about twenty years. In the case of 1-component spins (Ising model), it was proved by Dyson [5] that the model in the region $1 < p < 2$ has an ordering transition and that the model in the region $p > 2$ doesn’t have it. In the case of 2-component spins (XY model), Šimánek [6] showed that with the low-temperature harmonic approximation there exists a Kosterlitz-Thouless-like transition to a low-temperature phase with infinite susceptibility. In the ferromagnetic case of 3-component spins (Heisenberg model), simulation of the case when $p = 2$ was done by Romano [7] with the Monte Carlo method. The renormalization group approach was made by Fisher et al. [8][9]

In comparison with the model with long-range interactions, the Heisenberg model with a short-range interaction i.e. a nearest-neighbor interaction (NN Heisenberg model) has a longer history. It is well known that for $S = 1/2$ the exact solution is obtained with the Bethe ansatz method. The modified spin-wave approximation was first proposed by one of the authors. [10] The conventional spin-wave theory cannot be applied to the case when the dimensions are less than three. This modification makes the spin-wave theory valid even in one and two dimensions. Results from the modified theory are in good agreement with
those from the Bethe ansatz integral equations. \[11,12\]

In this paper we consider the quantum Heisenberg model with long-range ferromagnetic interactions decaying as $1/r^p$ in one dimension. Its Hamiltonian with a periodic boundary condition is written as follows:

$$
\mathcal{H} = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N-1} J \left( \frac{\pi}{\sin \left( \frac{2\pi m}{N} \right)} \right)^p \mathbf{S}_m \cdot \mathbf{S}_{m+n}
$$

This model in the limit $p \to \infty$ is the NN Heisenberg model and this model for $p = 2$ is the HS model. We discuss a modified spin-wave theory of this model and study the properties at low temperature. In the next section a formulation of the modified spin-wave theory is done. In §3 the terms up to the second order of Bose operators in the transformed Hamiltonian are considered. In the region $p \geq 2$ the temperature-dependence of the susceptibility and of the specific heat are calculated; the critical temperature in the region $1 < p < 2$ is estimated. In §4 the terms up to the forth order of operators are considered. The procedure to obtain physical quantities is shown. The susceptibility and the specific heat are analytically obtained for $p = 2$. In §5 we discuss the results. They are compared to the solutions of the HS model and of the NN Heisenberg model.

II. FORMULATION OF MODIFIED SPIN-WAVE THEORY

First the Holstein-Primakoff transformation

$$
S^x_m = S^x_m + iS^y_m = \sqrt{2S} f_m(S) a_m
$$

$$
S^-_m = S^x_m - iS^y_m = \sqrt{2S} a^\dagger_m f_m(S)
$$

$$
S^z_m = S - a^\dagger_m a_m
$$

$$
f_m(S) = \sqrt{1 - \left( \frac{1}{2S} \right) a^\dagger_m a_m} = 1 - \frac{1}{4S} a^\dagger_m a_m + O(S^{-2})
$$

(2.1)
is applied to the Hamiltonian (1.1). Expanded with respect to $1/S$, the Hamiltonian is rewritten as follows:

$$
\mathcal{H} = E_0 + \mathcal{H}_2 + \mathcal{H}_4 + O(S^{-1}),
$$

$$
E_0 = -\frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N-1} JS^2 \left[ \frac{\pi}{\sin(\frac{\pi n}{N})} \right]^p,
$$

$$
\mathcal{H}_2 = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N-1} JS \left[ \frac{\pi}{\sin(\frac{\pi n}{N})} \right]^p (a_{m+n}^\dagger - a_m^\dagger)(a_{m+n} - a_m),
$$

$$
\mathcal{H}_4 = \frac{1}{8} \sum_{m=1}^{N} \sum_{n=1}^{N-1} J S \left[ \frac{\pi}{\sin(\frac{\pi n}{N})} \right]^p \left\{ a_m^\dagger a_{m+n} (a_{m+n} - a_m)^2 
+ (a_m^\dagger - a_{m+n}^\dagger)^2 a_m a_{m+n} \right\}.
$$

Next the site representation is changed to the momentum representation with the Fourier transformation $a_m = (1/\sqrt{N}) \sum_k e^{ikm} a_k$, $a_m^\dagger = (1/\sqrt{N}) \sum_k e^{-ikm} a_k^\dagger$. Then $\mathcal{H}_2$ and $\mathcal{H}_4$ are transformed to the following equations:

$$
\mathcal{H}_2 = \sum_k a_k^\dagger a_k S \{ \eta(0) - \text{Re}[\eta(k)] \}, \quad \eta(k) \equiv J \sum_{n=1}^{N-1} \left[ \frac{\pi}{\sin(\frac{\pi n}{N})} \right]^p e^{ikn}
$$

$$
\mathcal{H}_4 = \frac{1}{8N} \sum_{k_1,k_2,k_3} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_1+k_2-k_3} \left\{ \eta(k_1) + \eta(-k_2) + \eta(-k_3) 
+ \eta(k_1 + k_2 - k_3) - 2\eta(k_1 - k_3) - \eta(k_3 - k_2) - \eta(k_2 - k_3) \right\}.
$$

where $\text{Re}[x]$ stands for the real part of $x$.

We consider the expectation value of $\mathcal{H}$ for the state

$$
|\{n_k\} > = \prod_k (n_k!)^{-\frac{1}{2}} (a_k^\dagger)^{n_k} |0 >
$$

which is an eigenstate of $\mathcal{H}_2$. Then we get

$$
E = E_0 + \langle \mathcal{H}_2 \rangle + \langle \mathcal{H}_4 \rangle,
$$

$$
\langle \mathcal{H}_2 \rangle = \sum_k < n_k > S \{ \eta(0) - \text{Re}[\eta(k)] \},
$$

$$
\langle \mathcal{H}_4 \rangle = \frac{1}{N} \sum_{k_1,k_2,k_3} < n_{k_1} > < n_{k_2} > < n_{k_3} > \left\{ \eta(k_1) + \eta(-k_2) + \eta(-k_3) 
+ \eta(k_1 + k_2 - k_3) - 2\eta(k_1 - k_3) - \eta(k_3 - k_2) - \eta(k_2 - k_3) \right\}.
$$
\[ < \mathcal{H}_4 > = -\frac{1}{2N} \left( \sum_k < n_k^2 > - < n_k > \right) \left( 2\eta(0) - \eta(k) - \eta(-k) \right) \]
\[ + \sum_{k,k'(k \neq k')} < n_k n_{k'} > \left\{ \eta(0) + \eta(k - k') - \eta(k) - \eta(k') \right\} , \]

where \( n_k = a_k^\dagger a_k \). Magnetization in the \( z \)-direction is given by \( SN - \sum_k a_k^\dagger a_k \), so the zero-magnetization condition is

\[ SN - \sum_k a_k^\dagger a_k = 0. \quad (2.5) \]

We assume here that \( n_k \) is the Bose distribution; the entropy and the free energy are respectively written as follows:

\[(\text{entropy}) = \sum_k \left\{ (1 + n_k) \ln(1 + n_k) - n_k \ln n_k \right\} , \quad (2.6)\]

\[ F = E - T \times (\text{entropy}) . \quad (2.7) \]

Moreover \( < n_k^2 > \) which expresses the expectation value of \( n_k^2 \) is \( 2\tilde{n}_k^2 + \tilde{n}_k \) in terms of \( \tilde{n}_k \) (\( = < n_k > \)). We want to know \( \tilde{n}_k \) which minimizes the free energy under the constraint condition of zero magnetization, so we introduce the Lagrange multiplier \( \mu \) and minimize the following quantity \( W \):

\[ W = F - \mu (\sum_k \tilde{n}_k - SN) . \quad (2.8) \]

From \( \partial W / \partial \tilde{n}_k = 0 \) the Bose-Einstein distribution

\[ \tilde{n}_k = \frac{1}{e^{\beta (\varepsilon(k) - \mu)} - 1} \quad (2.9) \]

is reproduced where \( \varepsilon(k) = \partial E / \partial \tilde{n}_k \) and \( \beta = T^{-1} \). From \( \partial W / \partial \mu = 0 \) we have the self-consistent condition
\[ S = \frac{1}{N} \sum_k \tilde{n}_k \]  

which determines the chemical potential \( \mu \).

Using the rotational averaging, we obtain the static susceptibility;

\[ \chi = \frac{\beta}{3N} \sum_k (\tilde{n}_k^2 + \tilde{n}_k). \]  

(2.11)

### III. QUADRATIC THEORY

In this section we consider the first two terms up to the quadratic term of operators in the Holstein-Primakoff transformed Hamiltonian (2.2). Then the dispersion relation is written as follows:

\[ \varepsilon(k) = JS \sum_{n=1}^{N-1} \left[ \frac{\pi}{N \sin(\pi n/N)} \right]^p \{1 - \cos(kn)\}. \]  

(3.1)

Here we take the thermodynamic limit \( N \to \infty \); the dispersion relation is rewritten to

\[ \varepsilon(k) = 2JS \sum_{n=1}^{\infty} \frac{1 - \cos(kn)}{n^p}. \]  

(3.2)

The Bose-Einstein integral function gives us the dominant term of the dispersion for any positive and small \( k \) for arbitrary \( p \ (> 1) \).

\[ \varepsilon(k) \simeq \begin{cases} 
JS \zeta(p-2)k^2 & (p > 3) \\
-JSk^2 \ln k & (p = 3) \\
JS\omega(p)k^{p-1} & (1 < p < 3) 
\end{cases} \]

\[ \omega(p) \equiv \frac{\pi}{\Gamma(p) \cos[\pi(p-2)/2]} \]  

(3.3)

\[ 6 \]
In the region \( p \geq 2 \), we can determine the chemical potential \( \mu \) from the self-consistent condition (2.10). We have no Bose condensation which breaks this condition (2.10). The satisfaction of this condition means that the system has no ordering transition. We use the determined \( \mu \) to calculate the susceptibility and the specific heat at low temperature for \( p \geq 2 \). The continuum approximation of the state density is valid in the region \( p \geq 2 \); then the self-consistent condition (2.10) is rewritten to

\[
S = \frac{1}{\pi} \int_0^{\pi} \frac{dk}{e^{\beta \varepsilon(k) + v} - 1} = \frac{1}{\pi} \int_0^{\pi} \frac{d\varepsilon}{d\varepsilon e^{\beta \varepsilon + v} - 1}
\]

(3.4)

where \( v = -\beta \mu \).

First we consider the region \( p > 3 \). From the dispersion relation and the self-consistent condition (3.4):

\[
S = \frac{1}{2\pi \sqrt{JS\zeta(p-2)}} \int_0^{\varepsilon(p)} \frac{\varepsilon^{-\frac{1}{2}} d\varepsilon}{e^{\beta \varepsilon + v} - 1}
\]

\[
\approx \frac{1}{2\pi \sqrt{\beta JS \zeta(p-2)}} \int_0^{\infty} \frac{x^{-\frac{1}{2}} dx}{e^{x + v} - 1}
\]

\[
= \frac{1}{2\pi \sqrt{\beta JS \zeta(p-2)}} \Gamma\left(\frac{1}{2}\right) \left(\Gamma\left(\frac{1}{2}\right) v^{-\frac{1}{2}} + \zeta\left(\frac{1}{2}\right) + \cdots\right),
\]

(3.5)

we obtain \( v \) as follows:

\[
v^{-1} = 4\zeta(p-2)S^3 \beta J
\]

(3.6)

where the Bose-Einstein integral function is used. From eq. (2.11) the susceptibility in this region is calculated as follows:

\[
\chi \approx \frac{2S^4 \zeta(p-2)}{3} \beta^2 J
\]

(3.7)

In the same way, we have
\( v^{-1} \approx \begin{cases} 
(S^p[p(p)]^{p-1}\omega(p)\beta J)^{\frac{1}{p-2}} & (2 < p < 3) \\
\exp(\beta JS^2\pi^2) & (p = 2) 
\end{cases} \)

where \( \kappa(p) \equiv (p - 1)\sin[\pi/(p - 1)] \); then we have

\[
\chi = \begin{cases} 
\frac{p-2}{3(J(p-1)}[\omega(p)]^{\frac{1}{p-2}} \{S^2\kappa(p)\beta J\}^{\frac{1}{p-2}} & (2 < p < 3) \\
\frac{1}{3JS^2\pi^2} \exp(\beta JS^2\pi^2) & (p = 2) 
\end{cases}
\]

In this section the free energy per site is given as follows:

\[
f \equiv \frac{F}{N} = \frac{E_0}{N} + \frac{1}{N} \sum_k \bar{n}_k \varepsilon(k) - \frac{T}{N} \sum_k \{(1 + \bar{n}_k) \ln(1 + \bar{n}_k) - \bar{n}_k \ln \bar{n}_k \}
\] (3.8)

In the limit \( N \to \infty \), we have

\[
f = e_0 + S\mu + \frac{T}{\pi} \int_{0}^{\varepsilon(\pi)} \frac{dk}{d\varepsilon} \ln(1 - e^{-\beta\varepsilon-v})d\varepsilon
\] (3.9)

where \( E_0/N \to e_0 \ ( = -JS\zeta(p) ) \) as \( N \to \infty \). From the dispersion relation the free energy is obtained as follows:

\[
\frac{f - e_0}{T} \approx \begin{cases} 
-\frac{\zeta(3/2)}{\sqrt{2\pi}(p-2)} \sqrt{\frac{T}{J}} & (p > 3) \\
-\pi^{-1} \lambda(p)[S\omega(p)]^{\frac{1}{p-2}}(\frac{T}{J})^{\frac{1}{p-2}} & (1 < p < 3) 
\end{cases}
\]

where \( \lambda(p) \equiv \Gamma\left(1/(p-1)\right)\zeta\left(p/(p-1)\right)/(p-1) \). Then we can calculate the specific heat per site; we have

\[
c \approx \begin{cases} 
\frac{3\zeta(3/2)}{4\sqrt{2\pi}(p-2)} \sqrt{\frac{T}{J}} & (p > 3) \\
\frac{p}{\pi(p-1)^2} \lambda(p)[\omega(p)/2]^{\frac{1}{p-2}}(\frac{T}{J})^{\frac{1}{p-2}} & (1 < p < 3) 
\end{cases}
\]

for \( S = 1/2 \).

On the other hand, we cannot determine the chemical potential \( \mu \) in the region \( 1 < p < 2 \) as can do in the region \( p \geq 2 \). This is because the zero-magnetization condition is
broken by the Bose condensation. Instead we can know the critical temperature $T_c (=1/\beta_c)$ of spontaneous magnetization by the estimation of the critical temperature of the Bose condensation. The critical temperature in the region where $p$ is less than 2 and where $p$ is in the vicinity of 2 can be obtained from the constant terms of the Bose-Einstein integral function. We should use $\varepsilon(k) = JS\{\omega(p)k^{p-1} + \zeta(p-2)k^2\}$ as the dispersion relation to obtain a better estimation of $T_c$ near $p = 2$; then we have

$$T_c = JS\omega(p)[\tau(p)]^{p-1}$$

$$\tau(p) = \tau_0(p) + \frac{(4 - p)\Gamma(\frac{4-p}{p-1})\zeta(\frac{4-p}{p-1})\zeta(p-2)}{(p-1)\omega(p)\Gamma(\frac{1}{p-1})\zeta(\frac{1}{p-1})}(\tau_0(p))^{4-p}$$

$$\tau_0(p) = \frac{\pi S(p-1)}{\Gamma(\frac{1}{p-1})\zeta(\frac{1}{p-1})}.$$  

We note that $\varepsilon(k) \to \varepsilon(\pi)$ for any $k > 2$ as $p \to 1+$. The critical temperature for $p$ which is greater than 1 and which is in the vicinity of 1 can be estimated as follows;

$$T_c = \frac{(2p - 1)\zeta(p)}{2^{p-1}\ln 3}J$$

for $S = 1/2$.

**IV. QUARTIC THEORY**

In this section we consider the first three terms up to the quartic term of operators in the Hamiltonian (2.2). Because the dispersion relation is given by $\varepsilon(k) = \partial E/\partial \tilde{n}_k$, we have

$$\varepsilon(k) = \sum_{n=1}^{N-1} J\left[\frac{\pi}{\sin(\frac{\pi n}{N})}\right]^p \{1 - \cos(kn)\}[S - \frac{1}{N} \sum_{k'} \tilde{n}_{k'} + \frac{1}{N} \sum_{k'} \tilde{n}_{k'} \cos(k'n)].$$

The even function $\tilde{n}_k$ in respect of $k$ is expanded into the following Fourier series:
\[ \tilde{n}_k = \frac{f(0)}{2} + \sum_{m=1}^{\infty} f(m) \cos(km), \quad (4.2) \]

\[ f(m) = \int_{-\pi}^{\pi} \frac{dk}{\pi} \tilde{n}_k \cos(km), \quad (m = 0, 1, 2, \ldots). \quad (4.3) \]

The dispersion \( \varepsilon(k) \) in the thermodynamic limit \( N \to \infty \) is expressed by

\[ \varepsilon(k) = J \sum_{n=1}^{\infty} \frac{1 - \cos(kn)}{n^p} f(n) \quad (4.4) \]

where we use the self-consistent condition of zero magnetization (2.10) in this limit i.e. \( S = f(0)/2 \).

So the problem of calculating physical quantities is reduced to obtaining the dispersion and the distribution which satisfy the three eqs. (2.9), (4.3) and (4.4) under the self-consistent condition of zero magnetization. For arbitrary \( p (\geq 2) \) we can obtain \( \varepsilon(k) \) and \( v \) by the following equations:

\[ \varepsilon(k) = \frac{2J}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\pi} dq \frac{(1 - \cos(kn)) \cos(qn)}{n^p(e^{\beta \varepsilon(q)+v} - 1)} \quad (4.5) \]

\[ S\pi = \int_{0}^{\pi} \frac{dk}{e^{\beta \varepsilon(k)+v} - 1} \quad (4.6) \]

The dispersion and chemical potential which are obtained from the iteration of eqs. (4.5) and (4.6) give us physical quantities.

Fortunately we can make an analytical treatment for \( p = 2 \); it is shown in the rest of this section. Equation (1.4) for \( p = 2 \) is differentiated twice; we have

\[ \frac{d^2 \varepsilon}{dk^2} = J\tilde{n}_k - JS. \quad (4.7) \]

The introduction of a function \( g(k) = d\varepsilon/dk \) and the integration of the differentiating equation (4.7) give us the following equation:
\[ g^2 = \frac{2J}{\beta} \ln \left| \frac{1 - e^{-\beta \varepsilon - v}}{1 - e^{-v}} \right| - 2JS \varepsilon \] (4.8)

where the initial conditions \( g = \varepsilon = 0 \) at \( k = 0 \) are used. From eq. (4.8), we use the equation:

\[ dk = \frac{d\varepsilon}{\sqrt{\frac{2J}{\beta} \ln \left| \frac{1 - e^{-\beta \varepsilon - v}}{1 - e^{-v}} \right| - 2JS \varepsilon}} \] (4.9)

to change an integral variable from \( k \) to \( \varepsilon \) in the integrations.

The substitution of \( \pi \) for \( k \) in eqs. (4.4) and (4.8) gives

\[ \varepsilon(\pi) = \frac{JS \pi^2}{2} + O(v \ln \frac{1}{v}), \] (4.10)

\[ e^v = 1 - \frac{1 - e^{-\beta \varepsilon(\pi)}}{1 - e^{Sg \beta \varepsilon(\pi)}} \] (4.11)

respectively. At low temperature, then we have

\[ v \simeq \exp(-\frac{\beta JS^2 \pi^2}{2}). \] (4.12)

From eqs. (4.3) and (2.11) the susceptibility at low temperature of the HS model is calculated as \( \chi \simeq (\beta/6)\sqrt{2/\beta J \pi} \exp(\beta JS^2 \pi^2/2) \). For \( S = 1/2 \) the susceptibility is obtained as follows:

\[ \chi \simeq \frac{\beta}{6} \sqrt{\frac{2}{\beta J \pi}} \exp(\frac{\beta J \pi^2}{8}). \] (4.13)

The free energy per site in this section is given as follows:

\[ f = \frac{1}{4} \sum_{n=1}^{\infty} \frac{J}{n^p} \left\{ (f(n))^2 - JS \sum_{n=1}^{\infty} \frac{f(n)}{n^p} + \mu S \right\}
+ \frac{1}{\beta \pi} \int_0^{\varepsilon(\pi)} dk \frac{d\varepsilon}{d\varepsilon} \ln(1 - e^{-\beta \varepsilon - v}) d\varepsilon. \] (4.14)

Then we obtain the following dominant term of the specific heat per site:

\[ c \simeq \frac{2}{3} \left( \frac{T}{J} \right) \] (4.15)

for \( p = 2 \) and \( S = 1/2 \).
V. DISCUSSION

In this paper we have investigated the one-dimensional quantum Heisenberg model with long-range ferromagnetic interactions by the modified spin-wave approximation. This approximation makes it possible to treat the cases not only of the special values of \( p \).

We can apply the modified spin-wave theory to the NN Heisenberg model; its validity has already been checked. The limit \( p \to \infty \) shifts both the quadratic and quartic theory in this paper straightforward to the cases of the NN Heisenberg model.

The susceptibility and the entropy density of the ferromagnetic HS model are obtained by Haldane as follows:

\[
\chi = \frac{\beta}{2 \pi} \int_0^{\pi/2} dv \exp(-2 \beta J h(v)) \approx \frac{\beta}{4} \sqrt{\frac{2}{\beta J \pi}} \exp\left(\frac{\beta J \pi^2}{8}\right), \tag{5.1}
\]

\[
s = \frac{2}{\pi} \int_0^{\pi/2} dv \left\{ \ln[2 \cosh(\beta J h(v))] - \beta J h \tanh(\beta J h(v)) \right\} \tag{5.2}
\]

respectively, where \( h(v) = \frac{v^2 - (\pi/2)^2}{4} \). From eq. (5.2) the specific heat per site is obtained; we have

\[
c = \frac{2}{3} \left( \frac{T}{J} \right) + \cdots. \tag{5.3}
\]

Both the results of the specific heat in the quadratic theory and that in the quartic theory are the same as this specific heat (5.3). Our result of the susceptibility from the quadratic theory is not the same as the expression (5.1) but is in agreement with it from the point of view that both have exponential divergence at low temperature. So it is reasonable that the quadratic theory is qualitatively valid.

From our results in the quadratic theory we can divide the region of \( p \) which determines the strength of long-range interactions into two parts. One is the region where temperature-dependence except for coefficients of physical quantities is the same as the case of the NN
Heisenberg model at low temperature. The other is the region where at low temperature it is different from the case of the NN Heisenberg model. We will call the former region the effectively short-range region (ESRR) and we will call the latter region the essentially long-range region (ELRR). In the region $p > 3$ the susceptibility has the same power of $T$ as that of the NN Heisenberg model in spite that the model in this region has long-range interactions. The specific heat also has the same power of $T$. In the region $2 < p < 3$ the susceptibility and the specific heat have behaviors of power of $T$ as temperature goes to zero. The powers of temperature in the expressions of the susceptibility and of the specific heat, however, are changed as $p$ goes from 3 to 2. Especially the power of the susceptibility goes to infinity as $p \to 2+$; the susceptibility for $p = 2$ has exponential divergence at low temperature. Then the region $p > 3$ is the ESRR and the region $p < 3$ is the ELRR. The authors believe that the model for $p = 3$ is in the ELRR. This is because there is a possibility that the susceptibility has the divergence of $1/T^2$ with a factor of correction $\ln(1/T)$. The conclusion of the quadratic theory is summarized in Fig. 1.

Until now no approaches to the estimation of the critical temperature in the region $1 < p < 2$ have been known. So we cannot compare our results with others. Our estimation of $T_c$, however, is convincing from the following two points of view. One is that $T_c$ of the expression (3.10) vanishes as $p \to 2$. The other is that $T_c$ of the expression (3.11) is divergent as $p \to 1$. This has no contradiction to the fact that it takes an infinite amount of energy to make one-magnon state from the vacuum.

Our result (4.13) of the susceptibility in the quartic theory are in good agreement with the expression (5.1); the only difference is the constant factor $2/3$. This constant factor comes from the point that the modified spin-wave theory is not rotationally invariant. This constant factor reminds us of the difference between the susceptibility from the modified
spin-wave theory and that from the Schwinger boson mean field theory. Results from
the Schwinger boson mean field theory will be published elsewhere. There it will be shown
that almost all the same discussion in this paper is made again. The only difference will be
the improvement of this constant factor in the susceptibility.

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APPENDIX A

The Bose-Einstein integral function

\[ F(\alpha, v) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{x^{\alpha-1}dx}{e^x+v-1} = \frac{e^{-v}}{1^\alpha} + \frac{e^{-2v}}{2^\alpha} + \cdots, \tag{A.1} \]

is often used in this paper. Analytical property of this function near \( v = 0 \) is known; we
have

\[ F(\alpha, v) = \Gamma(1 - \alpha)v^{\alpha-1} + \sum_{n=0}^\infty \frac{\zeta(\alpha - n)}{n!}(-v)^n \quad (\alpha \notin \mathbb{N}), \]

\[ F(\alpha, v) = \frac{(-v)^{\alpha-1}}{(\alpha - 1)!} \left\{ \sum_{r=1}^{\alpha-1} \frac{1}{r} - \ln v \right\} + \sum_{n=\alpha-1}^\infty \frac{\zeta(\alpha - n)}{n!}(-v)^n \quad (\alpha \in \mathbb{N}), \]

where \( \zeta(\alpha) \) is Riemann’s zeta function. We need the summation

\[ \sum_{n=1}^\infty \frac{1 - \cos(kn)}{n^\alpha}, \tag{A.2} \]
to obtain the dispersion relation. We can calculate the dominant term of this summation for small $k$ using $F(\alpha, v)$; (A.2) is expressed by $\zeta(\alpha) - \text{Re}[F(\alpha, -ik)]$. The following integrations are very useful to obtain the susceptibility and the free energy:

$$\int_0^\infty \frac{x^{\alpha-1}e^{x+v}}{(e^{x+v}-1)^2} \, dx = \Gamma(\alpha)F(\alpha-1, v),$$

(A.3)

$$\int_0^\infty x^{\alpha-1} \ln(1 - e^{-x-v}) \, dx = -\Gamma(\alpha)F(\alpha+1, v)$$

(A.4)

respectively.

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Figure Captions

Fig.1  The critical temperature in the region $1 < p < 2$ and the exponent of the susceptibility in the region $p \geq 2$. The solid line stands for the exponent. The exponent for $p = 3$ isn’t obtained in this paper. The critical temperature is expressed by the broken line. Numerical results link the analytical results (3.10) and (3.11).