UNIQUENESS/NONUNIQUENESS FOR NONNEGATIVE SOLUTIONS OF THE CAUCHY PROBLEM FOR $u_t = \Delta u - u^p$ IN A PUNCTURED SPACE

ROSS G. PINSKY

Abstract. Consider classical solutions to the following Cauchy problem in a punctured space:

$$u_t = \Delta u - u^p \quad \text{in} \quad (\mathbb{R}^n - \{0\}) \times (0, \infty);$$

(0.1) $$u(x,0) = g(x) \geq 0 \quad \text{in} \quad \mathbb{R}^n - \{0\};$$

$$u \geq 0 \quad \text{in} \quad (\mathbb{R}^n - \{0\}) \times [0, \infty).$$

We prove that if $p \geq \frac{n}{n-2}$, then the solution to (0.1) is unique for each $g$. On the other hand, if $p < \frac{n}{n-2}$, then uniqueness does not hold when $g = 0$; that is, there exists a nontrivial solution with vanishing initial data.

1. Introduction and Statement of Results

The study of uniqueness in the class of all classical solutions for the Cauchy problem

$$u_t = Lu + f(x,u) \quad \text{in} \quad \mathbb{R}^n \times (0, \infty);$$

(1.1) $$u(x,0) = g(x) \geq 0 \quad \text{in} \quad \mathbb{R}^n;$$

$$u \geq 0 \quad \text{in} \quad \mathbb{R}^n \times [0, \infty),$$

where $L$ is a second order elliptic operator, goes back to Brezis [1], where uniqueness was proved in the case that $L = \Delta$ and $f(x,u) = -u^p$ with $p > 1$. In recent papers [3, 7], the dichotomy between uniqueness/nonuniqueness was investigated for general second order elliptic operators $L$ and quite general nonlinearities $f$, which approach $-\infty$ at a superlinear rate as $u \to \infty$. We emphasize that uniqueness here is with regard to the class of all classical solutions.
solutions to (1.1) (with no growth restrictions). For example, let $L = \Delta$ and $f(x,u) = -\gamma(x)u^p$. In [3] it was proved that uniqueness holds for (1.1) if $\gamma(x) \geq c_1 \exp(-c_2|x|^2)$, for some $c_1, c_2 > 0$, while uniqueness does not hold in (1.1) with initial data $g = 0$ if $\gamma(x) \leq c \exp(-|x|^{2+\epsilon})$, for some $c > 0$. On the other hand, if one looks only at mild solutions, then it is well known that uniqueness holds above for all bounded $\gamma$ [6].

In this paper, we study the question of uniqueness for the same semilinear equation $u_t = \Delta u - u^p$ studied by Brezis, but replace the space $\mathbb{R}^n$ by the punctured space $\mathbb{R}^n - \{0\}$, $n \geq 2$, thus allowing for unboundedness of solutions in a neighborhood of 0 at all times $t \geq 0$:

$$u_t = \Delta u - u^p \text{ in } (\mathbb{R}^n - \{0\}) \times (0, \infty);$$

(1.2) $$u(x,0) = g(x) \geq 0 \text{ in } \mathbb{R}^n - \{0\};$$

$$u \geq 0 \text{ in } (\mathbb{R}^n - \{0\}) \times [0, \infty).$$

We assume that $g \in C(\mathbb{R}^n - \{0\})$.

We prove the following theorem.

**Theorem.**

(1) Let $p < \frac{n}{n-2}$. Then there exists a nontrivial solution to (1.2) with initial data $g = 0$.

(2) Let $p \geq \frac{n}{n-2}$. Then for each $g$ there exists a unique solution to (1.2).

**Remark 1.** For the case $p \in (1,2]$, this result has an important interpretation with regard to the theory of super-Brownian motion; see [8] for details.

**Remark 2.** Brezis and Friedman [2] studied the problem $u_t = \Delta u - u|u|^{p-1}$ in $\mathbb{R}^n \times (0, \infty)$, with the initial condition $u(x) = \delta_0(x)$, the Dirac $\delta$-function at 0. They showed that a solution exists if and only if $p < \frac{n+2}{n}$. More recently, Marcus and Veron [5] have shown that for positive solutions of the above equation, the same critical exponent appears when one allows for even more singular initial conditions—namely, not necessarily locally bounded Borel measures. In these papers, the solution is required to be classical at
x = 0, for t > 0, whereas the present paper deals with the situation in which x = 0 is excluded for all times t ≥ 0. As such, it is ‘easier’ to obtain nontrivial solutions in the present case, and this is manifested through the larger critical exponent, \( \frac{n}{n-2} \) as compared to \( \frac{n+2}{n} \).

Remark 3. Note that for n = 2, nonuniqueness prevails for the problem in this paper for all p > 1.

We give a very simple proof of part (1) of the theorem by exploiting a recent result in [7]. For the proof of part (2), we construct appropriate supersolutions.

2. Proof of Theorem

We begin by noting that existence follows by standard methods; see [4] or [3] (this latter reference treats the case that the domain is \( \mathbb{R}^n \), but the same techniques work in the punctured space). Thus, it remains to consider uniqueness.

Proof of part (1). Since the problem is radially symmetric, it suffices to show that uniqueness fails for the radially symmetric equation

\[
\begin{align*}
    u_t &= u_{rr} + \frac{n-1}{r} u_r - u^p, \quad r \in (0, \infty), \ t > 0; \\
    u(r, 0) &= 0, \quad r \in (0, \infty); \\
    u &\geq 0, \quad r \in (0, \infty), \ t \geq 0.
\end{align*}
\]

(2.1)

By assumption, we have \( p < \frac{n}{n-2} \), or equivalently, \( n < \frac{2p}{p+1} \). Thus, the function \( W(x) = C x^{-\frac{p+1}{p-1}} \), where \( C^{p-1} = \frac{2}{p-1} (\frac{2p}{p+1} - n) \), is a positive, stationary solution of the parabolic equation \( u_t = u_{rr} + \frac{n-1}{r} u_r - u^p \) in \((0, \infty)\). By [7, Theorem 2-ii], the fact that there exists a nontrivial positive, stationary solution guarantees that uniqueness does not hold for the corresponding parabolic equation with initial data 0; that is, uniqueness does not hold for (2.1). Actually, the result in [7] is for equations with domain \( \mathbb{R}^n \), \( n \geq 1 \), whereas the domain here is \((0, \infty)\). One can check that the proof also holds
in a half space, but more simply, one can make the change of variables $z = \frac{1}{2} - x$, which converts the problem to all of $R$.

**Proof of part (2).** We will prove uniqueness for (1.2) in the case of vanishing initial data. This is enough because by [3, Proposition 3], uniqueness for arbitrary $g \geq 0$ follows from uniqueness for the case $g = 0$. We write the condition $p \geq \frac{n}{n-2}$ in the form $n \geq \frac{2p}{p-1}$. For technical reasons, it will be necessary to treat the cases $n > \frac{2p}{p-1}$ and $n = \frac{2p}{p-1}$ separately.

We first consider the case $n > \frac{2p}{p-1}$. For $\epsilon$ and $R$ satisfying $0 < \epsilon < 1$ and $R > 1$, and some $l \in (0, 1]$, define

$$\phi_{R, \epsilon}(x) = ((|x| - \epsilon)(R - |x|))^{-\frac{2}{p-1}}(1 + |x|)^{\frac{2}{p-1}}(1 + \frac{\epsilon^l}{|x|^l}R^{\frac{2}{p-1}}).$$

Also, for $R$ and $\epsilon$ as above, and some $\gamma > 0$, define

$$\psi_{R, \epsilon}(x, t) = \phi_{R, \epsilon}(x) \exp(\gamma(t + 1)).$$

Note that $\psi_{R, \epsilon}(x, 0) > 0$, for $|x| \in (\epsilon, R)$, and $\psi_{R, \epsilon}(x, t) = \infty$, for $|x| = \epsilon$ or $|x| = R$. We will show that for all sufficiently large $R$ and all sufficiently small $\epsilon$, and for $\gamma$ sufficiently large and $l$ sufficiently small, independent of those $R$ and $\epsilon$, one has

$$\Delta \psi_{R, \epsilon} - \psi_{R, \epsilon}^p - (\psi_{R, \epsilon})_t \leq 0, \text{ for } \epsilon < |x| < R \text{ and } t > 0.$$ 

It then follows from the maximum principle for semi-linear equations that every solution $u(x, t)$ to (1.2) satisfies

$$u(x, t) \leq \psi_{R, \epsilon}(x, t), \text{ for } \epsilon < |x| < R \text{ and } t \in [0, \infty).$$

Substituting (2.2) and (2.3) in (2.5), letting $\epsilon \to 0$, and then letting $R \to \infty$, one concludes that $u(x, t) \equiv 0$. Thus, it remains to show (2.4).
From now on we will use radial coordinates, writing \( \phi(r) \) for \( \phi(x) \) with \(|x| = r\), and similarly for \( \psi \). We have

\[
(2.6) \quad \exp(-\gamma(t+1)) (\psi_{R,\epsilon})_r =
\]

\[
- \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r))^{\frac{2}{p-1}-1} (R+\epsilon-2r) \left( 1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}} \right)
\]

\[
+ \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r))^{\frac{2}{p-1}-1} (1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}})
\]

\[
- l((r-\epsilon)(R-r))^{\frac{2}{p-1}} (1 + r)^{\frac{2}{p-1}} \frac{\epsilon l}{r^{l+1} R^{\frac{2}{p-1}}},
\]

and

\[
(2.7) \quad \exp(-\gamma(t+1)) ((r-\epsilon)(R-r))^{-\frac{2}{p-1}} (\psi_{R,\epsilon})_{rr} =
\]

\[
- \frac{2}{p-1} \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r))^{\frac{2}{p-1}-1} (R+\epsilon-2r) (1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}})
\]

\[
+ 2 \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r))^{\frac{2}{p-1}-1} \left( 1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}} \right)
\]

\[
- 2 \left( \frac{2}{p-1} \right)^2 ((r-\epsilon)(R-r)) (R+\epsilon-2r) \left( 1 + r \right)^{\frac{2}{p-1}-1} \left( 1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}} \right)
\]

\[
+ 2l \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r)) (R+\epsilon-2r) (1 + r)^{\frac{2}{p-1}-1} \frac{\epsilon l}{r^{l+1} R^{\frac{2}{p-1}}}
\]

\[
+ \left( \frac{2}{p-1} \right) \left( \frac{2}{p-1} - 1 \right) ((r-\epsilon)(R-r))^2 (1 + r)^{\frac{2}{p-1}-2} \left( 1 + \frac{\epsilon l}{r} R^{\frac{2}{p-1}} \right)
\]

\[
- 2l \left( \frac{2}{p-1} \right) ((r-\epsilon)(R-r))^2 (1 + r)^{\frac{2}{p-1}-1} \frac{\epsilon l}{r^{l+1} R^{\frac{2}{p-1}}}
\]

\[
+ l(l+1) ((r-\epsilon)(R-r))^2 (1 + r)^{\frac{2}{p-1}} \frac{\epsilon l}{r^{l+2} R^{\frac{2}{p-1}}},
\]
Using (2.2), (2.3), (2.6) and the fact that $\frac{2}{p-1} + 2 = \frac{2p}{p-1}$, we have

(2.8)

$$\exp(-\gamma(t+1))(r-\epsilon)(R-r)^{-\frac{2}{p-1} - 2\left(\frac{n-1}{r}(\psi_{R,\epsilon})_r - \psi_{R,\epsilon}^p - (\psi_{R,\epsilon})_t\right)} =$$

$$-\left(\frac{2}{p-1}\right)(\frac{n-1}{r})(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon}{r}R^{\frac{2}{p-1}})$$

$$+\left(\frac{2}{p-1}\right)((r-\epsilon)(R-r))^{2}(1+r)^{\frac{2}{p-1} - 1}(1+\frac{\epsilon}{r}R^{\frac{2}{p-1}})$$

$$-\gamma((r-\epsilon)(R-r))^{2}(1+r)^{\frac{2}{p-1}}(1+\frac{\epsilon}{r}R^{\frac{2}{p-1}})$$

$$- (1+r)^{\frac{2p}{p-1}}(1+\frac{\epsilon}{r}R^{\frac{2}{p-1}})^p \exp((p-1)\gamma(t+1)).$$

We will show that for all sufficiently large $R$ and sufficiently small $\epsilon$, and for $\gamma$ sufficiently large and $l$ sufficiently small, independent of those $R$ and $\epsilon$, the sum of the right hand sides of (2.7) and (2.8) is non-positive. This will prove (2.4).

We denote the seven terms on the right hand side of (2.7) by $J_1 - J_7$, and the five terms on the right hand side of (2.8) by $I_1 - I_5$. Note that the terms that are positive are $J_1, J_2, J_4, J_5, J_7$ and $I_2$. In what follows, $M$ will denote a positive number that can be made as large as one desires by choosing $\gamma$ sufficiently large. Consider first those $r$ satisfying $r \geq cR$, where $c$ is a fixed positive number. For $r$ in this range, we have $|I_5| \geq MR^{\frac{2}{p-1}+2}(1+\epsilon R^{\frac{2}{p-1}})$.

It is easy to see that for $M$ sufficiently large, $|I_5|$ dominates each of the positive terms, uniformly over large $R$ and small $\epsilon$, and thus (since $M$ can be made arbitrarily large) also the sum of all of the positive terms. Now consider those $r$ for which $\delta_0 \leq r \leq C$, for some constants $0 < \delta_0 < C$. For $r$ in this range and $\epsilon$ sufficiently small, we have $|I_4| \geq MR^2(1+\epsilon R^{\frac{2}{p-1}})$, and it is easy to see that for $M$ sufficiently large, $|I_4|$ dominates each of the positive terms, uniformly over large $R$ and small $\epsilon$, and thus, also the sum of all of the positive terms. One can also show that the transition from $r$ of order unity to $r$ of order $R$ causes no problem. Thus, we conclude that
for any fixed $\delta_0 > 0$ and $\gamma$ sufficiently large, the sum of the right hand sides of (2.7) and (2.8) is negative for all large $R$ and small $\epsilon$. Note that all this holds uniformly over $l \in (0, 1]$. The parameter $l$ has not been needed yet.

We now turn to the delicate situation—when $\epsilon \leq r \leq \delta_0$. For later use, we remind the reader that $\delta_0$ may be chosen as small as one likes. (Note that at $r = \epsilon$, all the terms vanish except $J_1$ and $I_5$. Using the fact that $\frac{2^p}{p-1} + 2 = \frac{2^p}{p-1}$, it is easy to see that for sufficiently large $\gamma$, $|I_5(\epsilon)|$ dominates $J_1(\epsilon)$, uniformly over all large $R$ and small $\epsilon$. However, when $r$ is small, but on an order larger than $\epsilon$, the analysis becomes a lot more involved.) In the sequel, whenever we say that a condition holds for $\gamma$ or $M$ sufficiently large, or for $l$ sufficiently small, we mean independent of $R$ and $\epsilon$.

We first take care of the easy terms. Clearly, $J_5 \leq |I_4|$ if $\gamma$ is sufficiently large. Also $J_7 = \frac{k+1}{n} |I_3| \leq |I_3|$, if $l$ is chosen sufficiently small. (This last inequality holds since by assumption, $n > \frac{2p}{p-1}$; thus, $n > 2$ for all choices of $p$.)

We now show that for $\gamma$ sufficiently large, $J_2 \leq |I_4| + |I_5|$, for $\epsilon \leq r \leq \delta_0$. (We are reusing $|I_4|$ here. Later we will reuse $|I_5|$. This is permissible because $\gamma$ can be chosen as large as we like.) To show this inequality, it suffices to show that for $M$ sufficiently large,

\begin{equation}
(r - \epsilon)R \leq M(r - \epsilon)^2 R^2 + M(1 + \frac{\epsilon^l}{R^{p-1}}) R^{p-1}, \quad \text{for } r \in [\epsilon, \delta_0]
\end{equation}

A trivial calculation shows that the left hand side of (2.9) is less than the first term on the right hand side if $r \geq \epsilon + \frac{1}{RM}$. If $r \in [\epsilon, \epsilon + \frac{1}{RM}]$, then the left hand side of (2.9) is less than or equal to $\frac{1}{M}$ while the second term on the right hand side is greater than $M$. We conclude that (2.9) holds with $M \geq 1$.

It remains to consider $J_1, J_4$ and $I_2$. We will show that for $\gamma$ sufficiently large,

\begin{equation}
J_1 + J_4 + I_2 + I_1 + I_5 \leq 0, \quad \text{for } r \in [\epsilon, \delta_0].
\end{equation}
Since $I_2$ has the factor $(r - \epsilon)^2$, while $I_1$ has the factor $(r - \epsilon)$, and since \( \frac{R-r}{R+\epsilon-2r} \) can be made arbitrarily close to 1 by choosing $R$ sufficiently large, it follows that for any $\eta > 0$, we have $I_2 \leq \eta |I_1|$, for $r \in [\epsilon, \delta_0]$, if we choose $\delta_0$ sufficiently small and $R$ sufficiently large. Note that $J_4 \leq \frac{2l}{n-1} |I_1|$. Thus,

\[
(2.11) \quad J_1 + J_4 + I_2 + I_1 \leq J_1 + (1 - \frac{2l}{n-1} - \eta)I_1 = J_1 + (1 - \kappa)I_1,
\]

where $\kappa = \frac{2l}{n-1} + \eta$. Also note that since we are free to choose $l$ and $\eta$ as small as we like, the same holds for $\kappa$. We have

\[
(2.12) \quad J_1 + (1 - \kappa)I_1 = (1 + r)^{2(p-1)}(R + \epsilon - 2r)(1 + \frac{e^l}{r^l}R^{2p-1}) \times
\]

\[
\left( \frac{2(p+1)}{(p-1)^2}(R + \epsilon - 2r) - (1 - \kappa) \frac{2(n-1)}{p-1} \frac{r - \epsilon}{r} (R - r) \right).
\]

From the assumption that $n > \frac{2p}{p-1}$, it follows that for $\kappa$ sufficiently small and $R$ sufficiently large,

\[
(2.13) \quad \left( \frac{2(p+1)}{(p-1)^2}(R + \epsilon - 2r) - (1 - \kappa) \frac{2(n-1)}{p-1} \frac{r - \epsilon}{r} (R - r) \right) \leq C \frac{e^l}{r} R, \quad r \in [\epsilon, \delta_0],
\]

for some $C > 0$. From (2.11)-(2.13), we obtain

\[
(2.14) \quad J_1 + J_4 + I_2 + I_1 \leq C \frac{e^l}{r} R(1 + r)^{2(p-1)}(R + \epsilon - 2r)(1 + \frac{e^l}{r^l}R^{2p-1}),
\]

for $r \in [\epsilon, \delta_0]$.

In light of (2.14), in order to prove (2.10), it suffices to show that

\[
(2.15) \quad \frac{e^l}{r} R^2 \leq M(1 + \frac{e^l}{r^l} R^{2p-1})^{p-1}, \quad r \in [\epsilon, \delta_0],
\]

for sufficiently large $M$. Choose $l$ sufficiently small so that $l(p - 1) \leq 1$. Then the right hand side of (2.15) is greater or equal to $M \frac{e^l}{r} R^2$.

We now turn to the case $n = \frac{2p}{p-1}$. For $\epsilon$ and $R$ satisfying $0 < \epsilon < 1$ and $R > 1$, and some $c \geq 2$, define

\[
(2.16) \quad \phi_{R,\epsilon}(x) = (((|x| - \epsilon)(R - |x|))^{\frac{2p-1}{p-1}}(1 + |x|)^{\frac{2p}{p-1}}(1 + \frac{R^2}{\log \frac{c|x|}{\epsilon}})^{\frac{1}{p-1}}).
\]
Note that the only difference between $\phi_{R,\epsilon}$ here and $\phi_{R,\epsilon}$ in the previous case is that the term $\frac{\epsilon}{|x|}$ has been changed to $(\frac{1}{\log \frac{R}{\epsilon}})^{\frac{1}{p-1}}$. As before, we define

$$
\psi_{R,\epsilon}(x, t) = \phi_{R,\epsilon}(x) \exp(\gamma(t + 1)),
$$

and convert to radial coordinates, with $|x| = r$. Note that $\frac{1}{p-1} + 1 = \frac{p}{p-1}$ and $\frac{1}{p-1} + 2 = \frac{2p-1}{p-1}$. In place of (2.7) and (2.8), we have

$$
(2.17)
\exp(-\gamma(t + 1))((r - \epsilon)(R - r))^{-\frac{2}{p-1}}(\psi_{R,\epsilon})_{rr} =
$$

$$
\frac{2}{p-1} \left( \frac{2}{p-1} + 1 \right) (R + \epsilon - 2r)^2 (1 + r)^{\frac{2}{p-1}} \left( 1 + \left( \frac{R^2}{\log \frac{R}{\epsilon}} \right)^{\frac{1}{p-1}} \right)
$$

$$
+ 2 \left( \frac{2}{p-1} \right) (r - \epsilon)(R - r) \left( 1 + r \right)^{\frac{2}{p-1}} \left( 1 + \left( \frac{R^2}{\log \frac{R}{\epsilon}} \right)^{\frac{1}{p-1}} \right)
$$

$$
- 2 \left( \frac{2}{p-1} \right)^2 (r - \epsilon)(R - r)(R + \epsilon - 2r) \left( 1 + r \right)^{\frac{2}{p-1}} \left( 1 + \left( \frac{R^2}{\log \frac{R}{\epsilon}} \right)^{\frac{1}{p-1}} \right)
$$

$$
+ \left( \frac{2}{p-1} \right)^2 (r - \epsilon)(R - r)(R + \epsilon - 2r) \left( 1 + r \right)^{\frac{2}{p-1}} \frac{1}{r \left( \log \frac{R}{\epsilon} \right)^{\frac{1}{p-1}}} R^{p-1}
$$

$$
+ \left( \frac{2}{p-1} \right) \left( \frac{2}{p-1} - 1 \right) ((r - \epsilon)(R - r))^2 \left( 1 + r \right)^{\frac{2}{p-1}} \left( 1 + \left( \frac{R^2}{\log \frac{R}{\epsilon}} \right)^{\frac{1}{p-1}} \right)
$$

$$
- \left( \frac{2}{p-1} \right)^2 ((r - \epsilon)(R - r))^2 \left( 1 + r \right)^{\frac{2}{p-1}} \frac{1}{r \left( \log \frac{R}{\epsilon} \right)^{\frac{1}{p-1}}} R^{p-1}
$$

$$
+ \left( \frac{1}{p-1} \right) ((r - \epsilon)(R - r))^2 \left( 1 + r \right)^{\frac{2}{p-1}} \left( \frac{1}{r^2 \left( \log \frac{R}{\epsilon} \right)^{\frac{1}{p-1}}} + \frac{p}{(p-1)r^2 \left( \log \frac{R}{\epsilon} \right)^{\frac{p}{p-1}}} \right) R^{\frac{2}{p-1}}
$$
and

\[(2.18)\]
\[
\exp(-\gamma(t+1))((r-\epsilon)(R-r))^{\frac{p}{p-1}-2} \left(\frac{n-1}{r}(\psi_{R,\epsilon})_r - \psi^p_{R,\epsilon} - (\psi_{R,\epsilon})_r\right) = \\
- \frac{2}{p-1}(n-1)(r-\epsilon)(R-r)(R+\epsilon-2r)(1+r)^{\frac{2}{p-1}1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}} \\
+ \frac{2}{p-1}(n-1)(r-\epsilon)(R-r)2(1+r)^{\frac{2}{p-1}} \left(1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}\right) \\
- \frac{1}{p-1}(n-1)(r-\epsilon)(R-r)2(1+r)^{\frac{2}{p-1}} \frac{1}{r(\log \frac{c}{\epsilon})^{\frac{1}{p-1}}} R^\frac{2}{p-1} \\
- \gamma((r-\epsilon)(R-r))^2(1+r)^{\frac{2}{p-1}} \left(1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}\right) \\
- (1+r)^\frac{2p}{p-1} \left(1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}\right)^p \exp((p-1)\gamma(t+1)).
\]

As before, we denote the terms in \[(2.17)\] and \[(2.18)\] by \(J_1 - J_7\) and \(I_1 - I_5\) respectively. It’s easy to see that the analysis in the previous case carries over to the present case when \(r\) satisfies \(r \geq \delta_0\), where, as above, \(\delta_0\) is an arbitrary positive constant. It remains to consider \(r \in [\epsilon, \delta_0]\).

Exactly as in the previous case, we have \(J_5 \leq |I_4|\) and \(J_2 \leq |I_4| + |I_5|\), and similar to the previous case, it is easy to see that if \(c\) is chosen sufficiently large, then \(J_7 \leq |I_3|\). (For this last inequality, we use the fact that the condition \(n = \frac{2p}{p-1}\) guarantees that \(n > 2\).) We now consider the term \(I_2\).

Using the fact that \(n - 1 = \frac{p+1}{p-1}\), and replacing \(\frac{r-\epsilon}{r}\) by 1, we have

\[
I_2 \leq \frac{2(p+1)}{(p-1)^2}(r-\epsilon)(R-r)^2(1+r)^{\frac{2}{p-1}1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}},
\]

whereas

\[
|J_3| = 2\left(\frac{2}{p-1}\right)^\frac{R+\epsilon-2r}{R-r}(r-\epsilon)(R-r)^2(1+r)^{\frac{2}{p-1}1 + \left(\frac{R^2}{\log \frac{c}{\epsilon}}\right)^{\frac{1}{p-1}}}. 
\]

Since \(\frac{R+\epsilon-2r}{R-r}\) can be made arbitrarily close to 1 by choosing \(R\) sufficiently large, we have \(I_2 \leq |J_3|\). (Notice that this argument does not work in the case that \(n > \frac{2p}{p-1}\) if \(n\) is chosen sufficiently large. On the other hand, the method of dealing with \(I_2\) that was used above in the case \(n > \frac{2p}{p-1}\)—namely,
treat it together with \( I_1 \)—does not work in the present case that \( n = \frac{2p}{p-1} \). It is because of this that it has been necessary to split the proof into two cases.

Now consider the term \( J_4 \). In the case that \( n > \frac{2p}{p-1} \), \( J_4 \) was treated together with \( I_1 \); in the present borderline case, this will not work. It is here that the amended form of \( \phi_{R,\epsilon} \) is needed. We have

\[
J_4 \leq CR^{\frac{p}{p-1}}(\log \frac{CR}{\epsilon})^{-\frac{p}{p-1}}, \text{ for } r \in [\epsilon, \delta_0],
\]

for some \( C > 0 \). On the other hand,

\[
|I_5| \geq MR^{\frac{2p}{p-1}}(\log \frac{CR}{\epsilon})^{-\frac{p}{p-1}}, \text{ for } r \in [\epsilon, \delta_0],
\]

where \( M \) can be chosen as large as one wants by choosing \( \gamma \) sufficiently large. Thus, by choosing \( \gamma \) sufficiently large, we have \( J_4 \leq |I_5| \).

Finally, the term \( J_1 \) is treated as it was in the previous case, but without the addition of \( J_4 \) and \( I_2 \). Using the fact that \( n = \frac{2p}{p-1} \), the analysis in (2.12)-(2.14) gives

\[
J_1 + I_1 \leq C^\epsilon R (1 + r)^{\frac{2}{p-1}} (R + \epsilon - 2r) \left( 1 + \left( \frac{R^2}{\log \frac{CR}{\epsilon}} \right)^{\frac{1}{p-1}} \right).
\]

Comparing the right hand side of (2.19) with \( |I_5| \), one sees that the inequality

\[
J_1 + I_1 + I_5 \leq 0, \text{ for } r \in [\epsilon, \delta_0],
\]

will hold with \( \gamma \) chosen sufficiently large if

\[
\frac{\epsilon}{r} R^2 \leq M \left( 1 + \left( \frac{R^2}{\log \frac{CR}{\epsilon}} \right)^{\frac{1}{p-1}} \right)^{p-1}, \text{ for } r \in [\epsilon, \delta_0],
\]

holds with \( M \) chosen sufficiently large. The right hand side of (2.20) is larger than \( MR^2(\log \frac{CR}{\epsilon})^{-1} \); thus, (2.20) holds since \( \frac{\epsilon}{r}(\log \frac{CR}{\epsilon})^{-1} \) is bounded for \( r \in [\epsilon, \delta_0] \), uniformly over small \( \epsilon \). This completes the proof of part (2). \( \square \)
References

[1] Brezis, H. Semilinear equations in $\mathbb{R}^N$ without condition at infinity, Appl. Math. Optim. 12 (1984), 271-282.

[2] Brezis, H. and Friedman, A. Nonlinear parabolic equations involving measures as initial conditions, J. Math. Pures et Appl. 62 (1983), 73-97.

[3] Engla¨ nder, J. and Pinsky, R. Uniqueness/nonuniqueness for nonnegative solutions of second-order parabolic equations of the form $u_t = Lu + Vu - \gamma u^p$ in $\mathbb{R}^n$, J. Differential Equations 192 (2003), 396–428.

[4] Lieberman, G. M. Second Order Parabolic Differential Operators, World Scientific Publishing Co., Singapore, 1996.

[5] Marcus, M. and Veron, L. Initial trace of positive solutions of some nonlinear parabolic equations, Comm. in Partial Diff. Equas. 24 (1999), 1445-1499.

[6] Pazy, A. Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[7] Pinsky, R. Positive Solutions of Reaction diffusion equations with super-linear absorption: universal bounds, uniqueness for the Cauchy problem, boundedness of stationary solutions, submitted.

[8] Pinsky, R. The compact support property for measure-valued diffusions, submitted

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel.

E-mail address: pinsky@math.technion.ac.il