ON THE NORM-CONTINUITY FOR EVOLUTION FAMILY ARISING FROM NON-AUTONOMOUS FORMS

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ABSTRACT. We consider evolution equations of the form
\[ \dot{u}(t) + A(t)u(t) = 0, \quad t \in [0, T], \quad u(0) = u_0, \]
where \( A(t), \ t \in [0, T], \) are associated with a non-autonomous sesquilinear form \( a(t, \cdot, \cdot) \) on a Hilbert space \( H \) with constant domain \( V \subset H \). In this note we continue the study of fundamental operator theoretical properties of the solutions. We give a sufficient condition for norm-continuity of evolution families on each spaces \( V, H \) and on the dual space \( V' \) of \( V \). The abstract results are applied to a class of equations governed by time dependent Robin boundary conditions on exterior domains and by Schrödinger operator with time dependent potentials.

INTRODUCTION

Throughout this paper \( H, V \) are two separable Hilbert spaces over \( \mathbb{C} \) such that \( V \) is densely and continuously embedded into \( H \) (we write \( V \hookrightarrow_d H \)). We denote by \( \langle \cdot | \cdot \rangle_V \) the scalar product and \( \| \cdot \|_V \) the norm on \( V \) and by \( \langle \cdot | \cdot \rangle_H, \| \cdot \|_H \) the corresponding quantities in \( H \). Let \( V' \) be the antidual of \( V \) and denote by \( \langle \cdot, \cdot \rangle \) the duality between \( V' \) and \( V \). As usual, by identifying \( H \) with \( H' \) we have \( V \hookrightarrow_d H \cong H' \hookrightarrow_d V' \) see e.g., [5].

Let \( a : [0, T] \times V \times V \to \mathbb{C} \) be a non-autonomous sesquilinear form, i.e., \( a(t; \cdot, \cdot) \) is for each \( t \in [0, T] \) a sesquilinear form,
\[ a(\cdot; u, v) \text{ is measurable for all } u, v \in V, \]
such that
\[ |a(t, u, v)| \leq M\|u\|_V\|v\|_V \quad \text{and} \quad \text{Re } a(t, u, u) \geq \alpha\|u\|_V^2 \quad (t, s \in [0, T], u, v \in V), \tag{2} \]
for some constants \( M, \alpha > 0 \) that are independent of \( t, u, v \). Under these assumptions there exists for each \( t \in [0, T] \) an isomorphism \( A(t) : V \to V' \) such that \( \langle A(t)u, v \rangle = a(t, u, v) \) for all \( u, v \in V \). It is well known that \(-A(t)\), seen as unbounded operator with domain \( V \), generates an analytic \( C_0 \)-semigroup on \( V' \). The operator \( A(t) \) is usually called the operator associated with \( a(t; \cdot, \cdot) \) on \( V' \). Moreover, we associate an operator \( A(t) \) with \( a(t; \cdot, \cdot) \) on \( H \) as follows
\[ D(A(t)) = \{ u \in V | \exists f \in H \text{ such that } a(t; u, v) = \langle f | v \rangle_H \text{ for all } v \in V \}, \]
\[ A(t)u = f. \]

It is not difficult to see that \( A(t) \) is the part of \( A(t) \) in \( H \). In fact, we have \( D(A(t)) = \{ u \in V : A(t)u \in H \} \) and \( A(t)u = A(t)u \). Furthermore, \(-A(t)\) with domain \( D(A(t)) \) generates a holomorphic \( C_0 \)-semigroup on \( H \) which is the restriction to \( H \) of that generated by \(-A(t)\). For all this results see e.g. [22, Chapter 2] or [3, Lecture 7].

We now assume that there exist \( 0 < \gamma < 1 \) and a continuous function \( \omega : [0, T] \to [0, +\infty) \) such that
\[ |a(t, u, v) - a(s, u, v)| \leq \omega(t - s)\|u\|_{V^*}\|v\|_V \quad (t, s \in [0, T], u, v \in V), \tag{3} \]

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Recall that for symmetric forms, i.e., if $V$ where $a(t; u, v) = \overline{a(t; v, u)}$ for all $t, u, v$, then the square root property is satisfied.

Under the assumptions (1)-(5) it is known that for each $x_0 \in V$ the non-autonomous homogeneous Cauchy problem
\begin{align}
\dot{u}(t) + A(t)u(t) &= 0 \quad \text{a.e. on } [0, T], \\
u(0) &= x_0,
\end{align}
has a unique solution $u \in MR(V, H) := L^2(0, T; V) \cap H^1(0, T; H)$ such that $u \in C([0, T]; V)$. This result has been proved by Arendt and Monniaux [4] (see also [10]) when the form $a$ satisfies the weaker condition
\begin{align}
|a(t, u, v) - a(s, u, v)| \leq \omega(|t - s|)||u||_V||v||_V, \quad (t, s \in [0, T], u, v \in V).
\end{align}

In this paper we continue to investigate further regularity of the solution of (6). For this it is necessary to associate to the Cauchy problem (6) an evolution family
\begin{align*}
\mathcal{U} := \left\{ U(t, s) : 0 \leq s \leq t \leq T \right\} \subset \mathcal{L}(H)
\end{align*}
which means that:
\begin{enumerate}
\item $U(t, t) = I$ and $U(t, s) = U(t, r)U(r, s)$ for every $0 \leq r \leq s \leq t \leq T$,
\item for every $x \in X$ the function $(t, s) \mapsto U(t, s)x$ is continuous into $H$ for $0 \leq s \leq t \leq T$
\item for each $x_0 \in H, U(.)x_0$ is the unique solution of (6).
\end{enumerate}

**Definition 0.1.** Let $Y \subseteq H$ be a subspace. An evolution family $\mathcal{U} \subset \mathcal{L}(H)$ is said to be **norm continuous in $Y$** if $\mathcal{U} \subset \mathcal{L}(Y)$ and the map $(t, s) \mapsto U(t, s)$ is norm continuous with value in $\mathcal{L}(Y)$ for $0 \leq s < t \leq T$.

If the non-autonomous form $a$ satisfies the weaker condition (7) then it is known that (6) is governed by an evolution family which is norm continuous in $V$ [15, Theorem 2.6], and norm continuous in $H$ if in addition $V \hookrightarrow H$ is compact [15, Theorem 3.4]. However, for many boundary value problem the compactness assumption fails.

In this paper we prove that the compactness assumption can be omitted provided $a$ satisfies (3) instead of (7). This will allow us to consider a large class of examples of applications. One of the main ingredient used here is the non-autonomous **returned adjoint form** $a^*_{\gamma} : [0, T] \times V \times V \to \mathbb{C}$ defined by
\begin{align}
a^*_{\gamma}(t, u, v) := a(T - t, v, u) \quad (t, s \in [0, T], u, v \in V).
\end{align}

The concept of returned adjoint forms appeared in the work of D. Daners [7] but for different interest. Furthermore, [15, Theorem 2.6] cited above will be also needed to prove our main result.

We note that the study of regularity properties of the evolution family with respect to $(t, s)$ in general Banach spaces has been investigated in the case of constant domains by Komatsu [14] and Lunardi [17], and by Acquistapace [1] for time-dependent domains.

We illustrate our abstract results by two relevant examples. The first one concerns the Laplacian with non-autonomous Robin boundary conditions on an unbounded Lipschitz domain. The second one traits a class of Schrödinger operators with time dependent potential.
1. Preliminary results

Let \( a : [0, T] \times V \times V \to \mathbb{C} \) a non-autonomous sesquilinear form satisfying (1) and (2). Then the following well known result regarding \( L^2 \)-maximal regularity in \( V' \) is due to J. L. Lions

**Theorem 1.1** (Lions, 1961). For each given \( s \in [0, T] \) and \( x_0 \in H \) the homogeneous Cauchy problems

\[
\begin{aligned}
&\dot{u}(t) + A(t)u(t) = 0 \quad \text{a.e. on } [s, T], \\
u(s) = x,
\end{aligned}
\]

(9)

has a unique solution \( u \in MR(V, V') := MR(s, T; V, V') := L^2(s, T; V) \cap H^1(s, T; V') \).

Recall that the maximal regularity space \( MR(V, V') \) is continuously embedded into \( C([s, T], H) \) [21, page 106]. A proof of Theorem 1.1 using a representation theorem of linear functionals, known in the literature as Lions’s representation Theorem can be found in [21, page 112] or [8, XVIII, Chapter 3, page 513].

Furthermore, we consider the non-autonomous adjoint form \( a^* : [0, T] \times V \times V \to \mathbb{C} \) of \( a \) defined by

\[
a^*(t; u, v) := \overline{a(t; v, u)}
\]

for all \( t \in [0, T] \) and \( u, v \in V \). Finally, we will need to consider the returned adjoint form \( a^*_r : [0, T] \times V \times V \to \mathbb{C} \) given by

\[
a^*_r(t, u, v) := a^*(T - t, u, v).
\]

Clearly, the adjoint form is a non-autonomous sesquilinear form and satisfies (1) and (2) with the same constant \( M, \alpha \). Moreover, the adjoint operators \( A^*(t), t \in [0, T] \) of \( A(t), t \in [0, T] \) coincide with the operators associated with \( a^* \) on \( H \). Thus applying Theorem 1.1 to the returned adjoint form we obtain that the Cauchy problem associated with \( A_r^*(t) := A^*(T - t) \)

\[
\begin{aligned}
&\dot{v}(t) + A_r^*(t)v(t) = 0 \quad \text{a.e. on } [s, T], \\
v(s) = x,
\end{aligned}
\]

(10)

has for each \( x \in H \) a unique solution \( v \in MR(V, V') \). Accordingly, for every \((t, s) \in \Delta := \{(t, s) \in [0, T]^2 : t \leq s\} \) and every \( x \in H \) we can define the following family of linear operators

\[
U(t, s)x := u(t) \quad \text{and} \quad U_r^*(t, s)x := v(t),
\]

where \( u \) and \( v \) are the unique solutions in \( MR(V, V') \) respectively of (9) and (10). Thus each family

\[
\{U(t, s) : (t, s) \in \Delta\} \quad \text{and} \quad \{U_r^*(t, s) : (t, s) \in \Delta\}
\]

yields a contractive, strongly continuous evolution family on \( H \) [15, Proposition].

In the autonomous case, i.e., if \( a(t, \cdot, \cdot) = a(\cdot, \cdot) \) for all \( t \in [0, T] \), then one knows that \(-A_0\), the operator associated with \( a_0 \) in \( H \), generates a \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) in \( H \). In this case \( U(t, s) := T(t - s) \) yields a strongly continuous evolution family on \( H \). Moreover, we have

\[
(12) \quad U(t, s) = T(t - s) = U^*(t, s) = U_r^*(t, s).
\]

Here, \( T(\cdot) \) denote the adjoint of \( T(\cdot) \) which coincides with the \( C_0 \)-semigroup \((T^*(t))_{t \geq 0}\) associated with the adjoint form \( a^* \). In the non-autonomous setting however, (12) fails in general even in the finite dimensional case, see [7, Remark 2.7]. Nevertheless, Proposition 1.2 below shows that the evolution families \( U \) and \( U_r^* \) can be related in a similar way. This formula appeared in [7, Theorem 2.6].

**Proposition 1.2.** Let \( U \) and \( U_r^* \) be given by (11). Then we have

\[
(13) \quad [U_r^*(t, s)]' x = U(T - s, T - t)x
\]

for all \( x \in H \) and \((t, s) \in \Delta\).

The equality (13) will play a crucial role in the proof of our main result. We include here a new proof for the sake of completeness.
**Proof.** (of Proposition 1.2) Let \( \Lambda = (0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n+1} = T) \) be a subdivision of \([0, T]\). Let \( a_k : V \times V \to \mathbb{C} \) for \( k = 0, 1, \ldots, n \) be given by

\[
a_k(u, v) := a_{k, \Lambda}(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(r; u, v) \, dr \quad \text{for} \ u, v \in V.
\]

All these forms satisfy (2) with the same constants \( \alpha, M \). The associated operators in \( V' \) are denoted by \( A_k \in \mathcal{L}(V, V') \) and are given for all \( u \in V \) and \( k = 0, 1, \ldots, n \) by

\[
A_k u := A_{k, \Lambda} := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} A(r) u \, dr.
\]

Consider the non-autonomous form \( a_\Lambda : [0, T] \times V \times V \to \mathbb{C} \) defined by

\[
a_\Lambda(t; \cdot, \cdot) := \begin{cases} a_k(\cdot, \cdot) & \text{if } t \in [\lambda_k, \lambda_{k+1}) \\
a_n(\cdot, \cdot) & \text{if } t = T. \end{cases}
\]

Its associated time dependent operator \( A_\Lambda(\cdot) : [0, T] \subset \mathcal{L}(V, V') \) is given by

\[
A_\Lambda(t) := \begin{cases} A_k & \text{if } t \in [\lambda_k, \lambda_{k+1}) \\
A_n & \text{if } t = T. \end{cases}
\]

Next denote by \( T_k \) the \( C_0 \)-semigroup associated with \( a_k \) in \( H \) for all \( k = 0, 1 \ldots n \). Then applying Theorem 1.1 to the form \( a_\Lambda \) we obtain that in this case the associated evolution family \( U_\Lambda(t, s) \) is given explicitly for \( \lambda_{m-1} \leq s < \lambda_m < \ldots < \lambda_{l-1} \leq t < \lambda_l \) by

\[
U_\Lambda(t, s) := T_{l-1}(t - \lambda_{l-1}) T_{l-2}(\lambda_{l-1} - \lambda_{l-2}) \ldots T_{m}(\lambda_{m+1} - \lambda_m) T_{m-1}(\lambda_m - s),
\]

and for \( \lambda_{l-1} \leq a \leq b < \lambda_l \) by

\[
U_\Lambda(t, s) := T_{l-1}(t - s).
\]

By [20, Theorem 3.2] we know that \( (U_\Lambda)_\Lambda \) converges weakly in \( MR(V, V') \) as \( |\Lambda| \to 0 \) and

\[
\lim_{|\Lambda| \to 0} \|U_\Lambda - U\|_{MR(V, V')} = 0
\]

The continuous embedding of \( MR(V, V') \) into \( C([0, T]; H) \) implies that \( \lim_{|\Lambda| \to 0} U_\Lambda = U \) in the weak operator topology of \( \mathcal{L}(H) \).

Now, let \((t, s) \in \overline{\Delta} \) with \( \lambda_{m-1} \leq s < \lambda_m < \ldots < \lambda_{l-1} \leq t < \lambda_l \) be fixed. Applying the above approximation argument to \( a^*_k \) one obtains that

\[
U^*_{a, r}(t, s) = T_{l-1, r}(t - \lambda_{l-1}) T_{l-2, r}(\lambda_{l-1} - \lambda_{l-2}) \ldots T_{m, r}(\lambda_{m+1} - \lambda_m) T_{m-1, r}(\lambda_m - s),
\]

where \( T_{k, r} \) and \( T^*_{k, r} \) are the \( C_0 \)-semigroups associated with

\[
a_{k, r}(u, v) := \frac{1}{\lambda_{k+1} - \lambda_k} \int_{\lambda_k}^{\lambda_{k+1}} a(T - r; u, v) \, dr = \frac{1}{\lambda_{k+1} - \lambda_k} \int_{T - \lambda_{k+1}}^{T - \lambda_k} a(r; u, v) \, dr
\]

and its adjoint \( a^*_{k, r} \), respectively. Recall that \( T^*_{k, r} = T^r_{k, r} \).

On the other hand, the last equality in (21) implies that \( T_{k, r} \) coincides with the semigroup associated with \( a_{k, A_T} \) where \( A_T \) is the subdivision \( \Lambda_T := (0 = T - \lambda_{n+1} < T - \lambda_n < \ldots < T - \lambda_1 < T - \lambda_0 = T) \). It follows from (17)-(18) and (19)-(20) that

\[
\left[U^*_{a, r}(t, s)\right] = \left[T_{m-1, r}(\lambda_m - s) T_{m, r}(\lambda_{m+1} - \lambda_m) \ldots T_{l-2, r}(\lambda_{l-1} - \lambda_{l-2}) T_{l-1, r}(t - \lambda_{l-1}) \right]
\]

\[
= T_{m-1} \left((T - s) - (T - \lambda_m)\right) T_m \left((T - \lambda_m) - (T - \lambda_{m+1})\right) \ldots T_{l-1} \left((T - \lambda_{l-1}) - (T - t)\right)
\]

\[
= U_{A_T}(T - s, T - t)
\]

Finally, the desired equality (13) follows by passing to the limit as \( |\Lambda| = |A_T| \to 0 \). \( \square \)
Remark 1.3. The coerciveness assumption in (2) may be replaced with
\[ \text{Re } a(t, u, u) + \omega \| u \|_H^2 \geq \alpha \| u \|_V^2 \quad (t \in [0, T], u \in V) \]
for some $\omega \in \mathbb{R}$. In fact, $a$ satisfies (22) if and only if the form $a_\omega$ given by $a_\omega(t, \cdot, \cdot) := a(t, \cdot, \cdot) + \omega(\cdot, \cdot)$ satisfies the second inequality in (2). Moreover, if $u \in MR(V, V')$ and $v := e^{-\omega}u$, then $v \in MR(V, V')$ and $u$ satisfies (9) if and only if $v$ satisfies
\[ \dot{v}(t) + (\omega + A(t))v(t) = 0 \quad t\text{-a.e. on } [s, T], \quad v(s) = x. \]

2. Norm continuous evolution family

In this section we assume that the non-autonomous form $a$ satisfies (2)-(5). Thus as mentioned in the introduction, under these assumptions the Cauchy problem (9) has $L^2$-maximal regularity in $H$. Thus for each $x \in V$,
\[ U(\cdot, s)x \in MR(V, H) := MR(s, T; V, H) := L^2(s, T; V) \cap H^1(s, T; H). \]
Moreover, $U(\cdot, s)x \in C[s, T; V]$ by [4, Theorem 4.2]. From [15, Theorem 2.7] we known that the restriction of $U$ to $V$ defines an evolution family which norm continuous. The same is also true for the Cauchy problem (10) and the associated evolution family $U^\ast_r$ since the returned adjoint form $a^\ast_r$ inherits all properties of $a$. In the following we establish that $U$ can be extended to a strongly continuous evolution family on $V'$.

Proposition 2.1. Let $a$ be a non-autonomous sesquilinear form satisfying (2)-(5). Then $U$ can be extended to a strongly continuous evolution family on $V'$, which we still denote $U$.

Proof. Let $x \in H$ and $(t, s) \in \Delta$. Then using Proposition 1.2 and the fact that $U$ and $U^\ast_r$ define both strongly continuous evolution families on $V$ and $H$ we obtain that
\[ \|U(t, s)x\|_V = \sup_{\|v\|_V = 1} |< U(t, s)x, v > | = \sup_{\|v\|_V = 1} |(U(t, s)x)v)_H| = \sup_{\|v\|_V = 1} |(x(U(t, s))v)_H| = \sup_{\|v\|_V = 1} |(x)U^\ast_r(T - s, T - t)v)_H| \]
\[ = \sup_{\|v\|_V = 1} |< x, U^\ast_r(T - s, T - t)v > | \leq \|x\|_V \|U^\ast_r(T - s, T - t)\|_{\mathcal{L}(V)} \leq c\|x\|_V, \]
where $c > 0$ is such that $\sup_{t, s \in \Delta} \|U^\ast_r(t, s)\|_{\mathcal{L}(V)} \leq c$. Thus, the claim follows since $H$ is dense in $V'$. \qed

Let $\Delta := \{(t, s) \in \Delta | t \geq s\}$. The following theorem is the main result of this paper.

Theorem 2.2. Let $a$ be a non-autonomous sesquilinear form satisfying (2)-(5). Let $\{U(t, s) : (t, s) \in \Delta\}$ given by (11). Then the function $(t, s) \mapsto U(t, s)$ is norm continuous on $\Delta$ into $\mathcal{L}(X)$ for $X = V, H$ and $V'$.

Proof. The norm continuity for $U$ in the case where $X = V$ follows from [15, Theorem 2.7]. On the other hand, applying [15, Theorem 2.7] to $a^\ast_r$ we obtain that $U^\ast_r$ is also norm continuous on $\Delta$ with values in $\mathcal{L}(V)$. Using Proposition 1.2, we obtain by similar arguments as in the proof of Lemma 2.1
\[ \|U(t, s) - U(t', s')\|_{\mathcal{L}(V)} \leq \|U^\ast_r(T - s, T - t)x - U^\ast_r(T - s', T - t')x\|_{\mathcal{L}(V)} \]
for all $x \in V'$ and $(t, s), (t', s') \in \Delta$. This implies that $U$ is norm continuous on $\Delta$ with values in $\mathcal{L}(V')$. Finally, the norm continuity in $H$ follows then by interpolation. \qed
3. Examples

This section is devoted to some relevant examples illustrating the theory developed in the previous sections. We refer to [4] and [19] and the references therein for further examples.

(i) **Laplacian with time dependent Robin boundary conditions on exterior domain** Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary $\Gamma$. Denote by $\sigma$ the $(d-1)$-dimensional Hausdorff measure on $\Gamma$. Let $\Omega_{ext}$ denote the exterior domain of $\Omega$, i.e., $\Omega_{ext} := \mathbb{R}^d \setminus \Omega$. Let $T > 0$ and $\alpha > 1/4$. Let $\beta : [0, T] \times \Gamma \to \mathbb{R}$ be a bounded measurable function such that

$$|\beta(t, \xi) - \beta(t, \eta)| \leq c|t - s|^{\alpha}$$

for some constant $c > 0$ and every $t, s \in [0, T], \xi, \eta \in \Gamma$. We consider the from $a : [0, T] \times H^1(\Omega_{ext}) \times H^1(\Omega_{ext}) \to \mathbb{C}$ defined by

$$a(t; u, v) := \int_{\Omega_{ext}} \nabla u \cdot \nabla v \, d\xi + \int_{\Omega_{ext}} \beta(t, \cdot) u \overline{v} \, d\sigma$$

where $u \to u|_{\Gamma} : H^1(\Omega_{ext}) \to L^2(\Gamma, \sigma)$ is the trace operator which is bounded [2, Theorem 5.36]. The operator $A(t)$ associated with $a(t; \cdot, \cdot)$ on $H := L^2(\Omega_{ext})$ is minus the Laplacian with time dependent Robin boundary conditions

$$\partial_\nu u(t) + \beta(t, \cdot) u = 0 \quad \text{on } \Gamma.$$ 

Here $\partial_\nu$ is the weak normal derivative. Thus the domain of $A(t)$ is the set

$$D(A(t)) = \left\{ u \in H^1(\Omega_{ext}) \mid \Delta u \in L^2(\Omega_{ext}), \partial_\nu u(t) + \beta(t, \cdot) u|_{\Gamma} = 0 \right\}$$

and for $u \in D(A(t))$, $A(t) u := -\Delta u$. Thus similarly as in [4, Section 5] one obtains that $a$ satisfies (2)-(5) with $\gamma := r_0 + 1/2$ and $\omega(t) = t^\beta$ where $r_0 \in (0, 1/2)$ such that $r_0 + 1/2 < 2\alpha$. We note that [4, Section 5] the authors considered the Robin Laplacian on the bounded Lipschitz domain $\Omega$. The main ingredient used there is that the trace operators are bounded from $H^{s}(\Omega)$ with value in $H^{s-1/2}(\Gamma, \sigma)$ for all $1/2 < s < 3/4$. This boundary trace embedding theorem holds also for unbounded Lipschitz domain, and thus for $\Omega_{ext}$, see [18, Theorem 3.38] or [6, Lemma 3.6].

Thus applying [4, Theorem 4.1] and Theorem 2.2 we obtain that the non-autonomous Cauchy problem

$$\begin{cases}
\dot{u}(t) - \Delta u(t) &= 0, \quad u(0) = x \in H^1(\Omega_{ext}) \\
\partial_\nu u(t) + \beta(t, \cdot) u &= 0 \quad \text{on } \Gamma
\end{cases}$$

has $L^2$-maximal regularity in $L^2(\Omega_{ext})$ and its solution is governed by an evolution family $U(\cdot, \cdot)$ that is norm continuous on each space $V, L^2(\Omega_{ext})$ and $V'$.

3.1. **Non-autonomous Schrödinger operators.** Let $m_0, m_1 \in L^1_{Loc}(\mathbb{R}^d)$ and $m : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that there exist positive constants $\alpha_1, \alpha_2$ and $\kappa$ such that

$$\alpha_1 m_0(\xi) \leq m(t, \xi) \leq \alpha_2 m_0(\xi), \quad |m(t, \xi) - m(s, \xi)| \leq \kappa |t - s|m_1(\xi)$$

for almost every $\xi \in \mathbb{R}^d$ and every $t, s \in [0, T]$. Assume moreover that there exist a constant $c > 0$ and $s \in [0, 1]$ such that for $u \in C^\infty_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} m_1(\xi)|u(\xi)|^2 \, d\xi \leq c\|u\|_{H^s(\mathbb{R}^d)}.$$

Consider the non-autonomous Cauchy problem

$$\begin{cases}
\dot{u}(t) - \Delta u(t) + m(t, \cdot) u(t) &= 0, \\
u(0) &= x \in V.
\end{cases}$$

Here $A(t) = -\Delta + m(t, \cdot)$ is associated with the non-autonomous form $a : [0, T] \times V \times V \to \mathbb{C}$ given by

$$V := \left\{ u \in H^1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} m_0(\xi)|u(\xi)|^2 \, d\xi < \infty \right\}$$
and
\[ a(t; u, v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v d\xi + \int_{\mathbb{R}^d} m(t, \xi) |u(\xi)|^2 d\xi. \]

The form \( a \) satisfies also (2)-(5) with \( \gamma := s \) and \( \omega(t) = t^a \) for \( \alpha > \frac{s}{2} \) and \( s \in [0, 1] \).

This example is taken from [19, Example 3.1]. Using our Theorem 2.2 we have that the solution of Cauchy problem (26) is governed by a norm continuous evolution family on \( L^2(\mathbb{R}^d), V \) and \( V' \).

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