Local uniqueness of semiclassical bounded states for a singularly perturbed fractional Kirchhoff problem

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Abstract. In this paper, we consider the following singularly perturbed fractional Kirchhoff problem
\[
(\varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx) (-\Delta)^s u + V(x) u = |u|^{p-2} u, \quad \text{in } \mathbb{R}^N,
\]
where \(a, b > 0, 2s < N < 4s\) with \(s \in (0, 1)\), \(2 < p < 2^*_s = \frac{2N}{N-2s}\) and \((-\Delta)^s\) is the fractional Laplacian. For \(\varepsilon > 0\) sufficiently small and a bounded continuous function \(V\) of \(x\), we establish a type of local Pohožáev identity by extension technique and then we can obtain the local uniqueness of semiclassical bounded solutions based on our recent results on the uniqueness and non-degeneracy of positive solutions to the limit problem.

Keywords: Fractional Kirchhoff equations; Pohozáev identity; Lyapunov-Schmidt reduction.

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1 Introduction and main results

Let \(H^s(\mathbb{R}^N)(0 < s < 1)\) be the fractional Sobolev space defined by
\[
H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},
\]
edowed with the natural norm
\[
\|u\|^2 = \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy.
\]
We continue to consider the following singularly perturbed fractional Kirchhoff problem
\[
(\varepsilon^{2s} a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx) (-\Delta)^s u + V(x) u = |u|^{p-2} u, \quad \text{in } \mathbb{R}^N, \quad (1.1)
\]
where \(a, b > 0, \varepsilon > 0\) is a parameter, \(V : \mathbb{R}^N \to \mathbb{R}\) is a bounded continuous function and \(p\) satisfies
\[
1 < p < 2^*_s - 1 = \begin{cases} \frac{N+2s}{N-2s}, & 0 < s < \frac{N}{2}, \\ +\infty, & s \geq \frac{N}{2}, \end{cases}
\]
where \(2^*_s\) is the standard fractional Sobolev critical exponent. The fractional Laplacian \((-\Delta)^s\) is the pseudo-differential operator defined by
\[
\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,
\]

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\[ (-\Delta)^s u(x) = -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \]

where
\[ C(N,s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos \xi_1}{|\xi|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N). \]

If \( s = 1 \), Eq. (1.1) reduces to the well-known Kirchhoff type problem, which and their variants have been studied extensively in the literature. The equation that goes under the name of Kirchhoff equation was proposed in [28] as a model for the transverse oscillation of a stretched string in the form
\[ \rho \partial_t^2 u - \left( p_0 + \frac{\xi h}{2L} \int_0^L |\partial_x u|^2 dx \right) \partial_{xx} u = 0, \quad t \geq 0, \quad 0 < x < L, \]

for \( t \geq 0 \) and \( 0 < x < L \), where \( u = u(t,x) \) is the lateral displacement at time \( t \) and at position \( x \), \( \mathcal{E} \) is the energy, \( \rho \) is the mass density, \( h \) is the cross section area, \( L \) the length of the string, \( p_0 \) the initial stress tension.

Problem (1.2) and its variants have been studied extensively in the literature. Bernstein obtains the global stability result in [8], which has been generalized to arbitrary dimension \( N \geq 1 \) by Pohozaev in [39]. We also point out that such problems may describe a process of some biological systems dependent on the average of itself, such as the density of population (see e.g. [7]). After Lions [31] introducing an abstract functional framework to this problem, this type of problem has received much attention. We refer to e.g. [13, 38, 42] and to e.g. [14, 17, 24, 25, 26, 27, 29, 30] for mathematical researches on Kirchhoff type equations on bounded domains and in the whole space, respectively. We also refer to [40] for a recent survey of the results connected to this model.

On the other hand, the interest in generalizing the model introduced by Kirchhoff to the fractional case does not arise only for mathematical purposes. In fact, following the ideas of [9] and the concept of fractional perimeter, Fiscella and Valdinoci proposed in [18] an equation describing the behaviour of a string constrained at the extrema in which appears the fractional length of the rope. Recently, problem similar to (1.1) has been extensively investigated by many authors using different techniques and producing several relevant results (see, e.g. [3, 4, 5, 6, 20, 21, 22, 32, 33, 34, 44]).

From the viewpoint of calculus of variation, the fractional Kirchhoff problem (1.1) is much more complex and difficult than the classical fractional Laplacian equation as the appearance of the term \( b \left( \int_{\mathbb{R}^N} \left| (-\Delta)^s u \right|^2 dx \right) (-\Delta)^s u \), which is of order four. This fact leads to difficulty in obtaining the boundedness of the \((PS)\) sequence for the corresponding energy functional if \( p \leq 3 \). Recently, Rădulescu and Yang [41] established uniqueness and nondegeneracy for positive solutions to Kirchhoff equations with subcritical growth. More precisely, they proved that the following fractional Kirchhoff equation
\[ \left( a + b \int_{\mathbb{R}^N} \left| (-\Delta)^s u \right|^2 dx \right) (-\Delta)^s u + mu = u^p, \quad \text{in } \mathbb{R}^N, \quad (1.3) \]

where \( a, b, m > 0 \), \( \frac{N}{p} < s < 1 \), \( 1 < p < 2^*_s - 1 \), has a unique nondegenerate positive radial solution. One of the main idea is based on the scaling technique which allows us to find a relation between solutions of (1.3) and the following equation
\[ (-\Delta)^s Q + Q = Q^p, \quad \text{in } \mathbb{R}^N \quad (1.4) \]

where \( 0 < s < 1 \) and \( 1 < p < 2^*_s - 1 \). For high dimension and critical case we refer to [23, 45, 46, 47]. We first summarize the main results in [41] for convenience.

**Proposition 1.1** Let \( a, b, m > 0 \) Assume that \( \frac{N}{p} < s < 1 \) and \( 1 < p < 2^*_s - 1 \). Then equation (1.3) has a ground state solution \( U \in H^s(\mathbb{R}^N) \).

\( i \) \( U > 0 \) belongs to \( C^\infty(\mathbb{R}^N) \cap H^{2s+1}(\mathbb{R}^N) \);

\( ii \) there exist some \( x_0 \in \mathbb{R}^N \) such that \( U(\cdot - x_0) \) is radial and strictly decreasing in \( r = |x - x_0| \);
as concentrating solutions since they will concentrate at certain point of the potential function

Now we state the existence result as follows.

Proposition 1.2 Under the assumptions of Proposition 1.1, the ground state solution $U$ of (1.3) is
unique up to translation. Moreover, $U$ is nondegenerate in $H^s(\mathbb{R}^N)$ in the sense that there holds

$$\ker L_+ = \text{span}\{ \partial_{x_1} U, \partial_{x_2} U, \cdots, \partial_{x_N} U \},$$

where $L_+$ is defined as

$$L_+ \varphi = \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^s U|^2 dx \right) (-\Delta)^s \varphi + m \varphi - p u^{p-1} \varphi + 2b \left( \int_{\mathbb{R}^N} (-\Delta)^s U (-\Delta)^s \varphi dx \right) (-\Delta)^s U$$

acting on $L^2(\mathbb{R}^N)$ with domain $H^s(\mathbb{R}^N)$.

By Proposition 1.2, it is now possible that we apply Lyapunov-Schmidt reduction to study the
perturbed fractional Kirchhoff equation (1.1). We want to look for solutions of (1.1) in the Sobolev space

$H^s(\mathbb{R}^N)$ for sufficiently small $\epsilon$, which named semiclassical solutions. We also call such derived solutions as concentrating solutions since they will concentrate at certain point of the potential function $V$. In fact, this method can be traced back to Floer and Weinstein’s work ([19]) on the Schrödinger equation. Later, Oh [36, 37] generalized Floer-Weinstein’s results to higher dimension and obtained the existence of positive multi-bump solutions concentrating at any given set of nondegenerate critical points of $V(x)$ as $\epsilon \to 0$. Also, the existence of a single-peak solution concentrating at the critical point of $V(x)$ which may be degenerate as $\epsilon \to 0$ was obtained by Ambrosetti et al. [2].

Now, it is expected that this approach can deal with problem (1.1) for all $1 < p < 2^*_s - 1$, in a unified
way. To state our following results, we first fix some notations that will be used throughout the paper.
For $\epsilon > 0$ and $y = (y_1, y_2, \cdots, y_N) \in \mathbb{R}^N$, write

$$U_{\epsilon,y}(x) = U \left( \frac{x - y}{\epsilon} \right), \quad x \in \mathbb{R}^N.$$

Assume that $V : \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(V1) $V$ is a bounded continuous function with $\inf_{x \in \mathbb{R}^N} V > 0$;

(V2) There exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$V(x_0) < V(x) \quad \text{for} \quad 0 < |x - x_0| < r_0$$

and $V \in C^\alpha(B_{r_0}(x_0))$ for some $0 < \alpha < \frac{N + 4s}{2}$. That is, $V$ is of $\alpha$-th order Hölder continuity
around $x_0$;

(V3) There exist $m > 1$ and $\delta > 0$ such that

$$\left\{ \begin{array}{l}
V(x) = V(x_0) + \sum_{i=1}^N c_i |x_i - x_{0,i}|^m + O \left( |x - x_0|^{m+1} \right), \quad x \in B_\delta(x_0) \\
\frac{\partial V}{\partial x_i} = m c_i |x_i - x_{0,i}|^{m-2} (x_i - x_{0,i}) + O \left( |x - x_0|^m \right), \quad x \in B_\delta(x_0)
\end{array} \right.$$\]

where $c_i \in \mathbb{R}$ and $c_i \neq 0$ for $i = 1, 2, \cdots, N$.

Without loss of generality, we assume $x_0 = 0$ for simplicity. The assumption (V1) allows us to introduce
the inner products

$$\langle u, v \rangle_\epsilon = \int_{\mathbb{R}^N} \left( c^{2s} a(-\Delta)^s u \cdot (-\Delta)^s v + V(x) uv \right) dx$$

for $u, v \in H^s(\mathbb{R}^N)$. We also write

$$H_\epsilon = \left\{ u \in H^s(\mathbb{R}^N) : ||u||_\epsilon = \langle u, u \rangle_\epsilon^{\frac{1}{2}} < \infty \right\}.$$

Now we state the existence result as follows.
Theorem 1.1 Under the assumptions of Proposition 1.1 and assume that \( V \) satisfies (\( V_1 \)) − (\( V_3 \)). Then there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \),

(i) problem (1.1) has a positive solution \( u_\varepsilon \), which has a global maximum point \( \eta_\varepsilon \in \mathbb{R}^N \) such that
\[
\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V(x_0);
\]

(ii) If \( u_\varepsilon^{(i)}, i = 1, 2, \) are two solutions derived as above, then
\[
u_\varepsilon^{(1)} \equiv u_\varepsilon^{(2)}
\]
holds for \( \varepsilon \) sufficiently small.

Moreover, let
\[
u_\varepsilon = U \left( \frac{x - y_\varepsilon}{\varepsilon} \right) + \varphi_\varepsilon.
\]
be the unique solution, then there hold
\[
|y_\varepsilon| = o(\varepsilon),
\]
\[
\|\varphi_\varepsilon\|_\varepsilon = O \left( \varepsilon^{\frac{N}{2} + m(1 - \tau)} \right),
\]
for some \( 0 < \tau < 1 \) sufficiently small.

As in [41], we only need the conditions (\( V_1 \)) and (\( V_2 \)) to obtain the existence of the semiclassical solutions by Lyapunov-Schmidt reduction. To prove the local uniqueness, we will follow the idea of [11]. More precisely, if \( u_\varepsilon^{(i)}, i = 1, 2, \) are two distinct solutions, then it is clear that the function
\[
\xi_\varepsilon = \frac{u_\varepsilon^{(1)} - u_\varepsilon^{(2)}}{\|u_\varepsilon^{(1)} - u_\varepsilon^{(2)}\|_{L^\infty(\mathbb{R}^N)}}
\]
satisfies \( \|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1 \). We will show, by using the equations satisfied by \( \xi_\varepsilon \), that \( \|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( \varepsilon \to 0 \). This gives a contradiction, and thus follows the uniqueness. To deduce the contradiction, we will need quite delicate estimates on the asymptotic behaviors of solutions and the concentrating point \( y_\varepsilon \). Our results extend the results in [29] to the fractional Kirchhoff problem. But one will see that it is much more difficult to obtain some of the estimates due to the presence of the nonlocal term \( \left( \int_{\mathbb{R}^N} (-\Delta)^s u^2 \right) (-\Delta)^s u \) and the fractional operator.

This paper is organized as follows. We prove the existence of semiclassical solutions in Section 2 and consider their concentration behavior in Section 3. In Section 4, we give the local Pohožaev identity and some basic estimates which will be used later. Finally, we finish the proof of local uniqueness in Theorem 1.1.

Notation. Throughout this paper, we make use of the following notations.

- For any \( R > 0 \) and for any \( x \in \mathbb{R}^N \), \( B_R(x) \) denotes the ball of radius \( R \) centered at \( x \);
- \( \| \cdot \|_q \) denotes the usual norm of the space \( L^q(\mathbb{R}^N) \), \( 1 \leq q \leq \infty \);
- \( o_n(1) \) denotes \( o_n(1) \to 0 \) as \( n \to \infty \);
- \( C \) or \( C_i (i = 1, 2, \cdots) \) are some positive constants may change from line to line.

2 Semiclassical solutions for the fractional Kirchhoff equation

In this section, we mainly prove the existence of semiclassical solution for the fractional Kirchhoff equation (1.1) via Lyapunov-Schmidt reduction method. In fact, all the results in the following has been obtained in [41] under the condition (\( V_1 \)) − (\( V_2 \)). We recall the result and give the outline for the proof for convenience.
Theorem 2.1 Let $a,b > 0$. Assume that $\frac{a}{2} < s < 1$ and $1 < p < 2_s^* - 1$, and assume that $V$ satisfies $(V_1)$ and $(V_2)$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, problem (1.1) has a solution $u_{\varepsilon}$ of the form

$$u_{\varepsilon} = U\left(\frac{x - y_{\varepsilon}}{\varepsilon}\right) + \varphi_{\varepsilon}$$

with $y_{\varepsilon} \in \mathbb{R}^N$, $\varphi_{\varepsilon} \in H_\varepsilon$, satisfying

$$y_{\varepsilon} \to x_0, \quad \|\varphi_{\varepsilon}\|_\varepsilon = o\left(\varepsilon^{\frac{N}{2}}\right),$$

as $\varepsilon \to 0$.

It is known that every solution to Eq. (1.1) is a critical point of the energy functional $I_\varepsilon : H_\varepsilon \to \mathbb{R}$, given by

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b_{\varepsilon}4s-N}{4}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1} dx$$

for $u \in H_\varepsilon$. It is standard to verify that $I_\varepsilon \in C^2(H_\varepsilon)$. So we are left to find a critical point of $I_\varepsilon$. However, due to the presence of the double nonlocal terms $(-\Delta)^s$ and $(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2)$, it requires more careful estimates on the orders of $\varepsilon$ in the procedure. In particular, the nonlocal terms brings new difficulties in the higher order remainder term, which is more complicated than the case of the fractional Schrödinger equation or usual Kirchhoff equation.

Denote $Q$ be the solution of the following equation:

$$\begin{cases} (-\Delta)^s u + u = u^p, \\
 u(0) = \max_{x \in \mathbb{R}^N} u(x), \\
 u > 0, x \in \mathbb{R}^N.
\end{cases}$$

Then, it is easy to see that the function

$$W_\lambda(x) := \lambda^\frac{1}{p} Q\left(\lambda^{\frac{1}{p}} x\right)$$

satisfies the equation

$$(-\Delta)^s u + \lambda u = u^p, \quad x \in \mathbb{R}^N.$$ 

Therefore, for any point $\xi \in \mathbb{R}^N$, taking $\lambda = V(\xi)$, it follows that $V(\xi)^{\frac{1}{p+1}} W\left(V(\xi)^{\frac{1}{p}} x\right)$ satisfies

$$\varepsilon^{2s}(-\Delta)^s u + V(\xi) u = u^p, x \in \mathbb{R}^N.$$ 

By the same idea and the proof of Theorem 1.1, we have known $U(x) = m \frac{\varepsilon_0^s}{|x|^{s-N}} Q\left(m^{\frac{s}{p}} \varepsilon_0^{\frac{1}{p}} x\right)$ is a positive unique solution of (1.3), where $\varepsilon_0 = a + b\|(-\Delta)^{s/2} U\|_2^2$. Moreover, we have the polynomial decay instead of the usual exponential decay of $U$ of and its derivatives (see Section 3). That is,

$$U(x) + |(-\Delta)^{s/2} U(x)| \leq \frac{C}{1 + |x|^{N+2s}}, \quad x \in \mathbb{R}^N \quad (2.1)$$

for some $C > 0$.

Following the idea from Cao and Peng [12] (see also [29]), we will use the unique ground state $U$ of (1.3) with $m = V(x_0)$ to build the solutions of (1.1). Since the $\varepsilon$-scaling makes it concentrate around $\xi$, this function constitutes a good positive approximate solution to (1.1).

For $\delta, \eta > 0$, fixing $y \in B_{\delta}(x_0)$, we define

$$M_{\varepsilon,y} = \{(y, \varphi) : y \in B_\delta(x_0), \varphi \in E_{\varepsilon,y}, \|\varphi\|^2 \leq \eta \varepsilon^N\}$$

where we denote $E_{\varepsilon,y}$ by

$$E_{\varepsilon,y} := \left\{\varphi \in H_\varepsilon : \left(\delta \frac{\partial U_{\varepsilon,y}}{\partial y^i}, \varphi\right) = 0, i = 1, \ldots, N\right\}.$$
We will restrict our argument to the existence of a critical point of \( I_\varepsilon \) that concentrates, as \( \varepsilon \) small enough, near the spheres with radii \( r_0 \). Thus we are looking for a critical point of the form
\[
u_\varepsilon = U_{\varepsilon, y} + \varphi_\varepsilon
\]
where \( \varphi_\varepsilon \in E_{\varepsilon, y} \), and \( \varepsilon y \rightarrow r_0, \| \varphi_\varepsilon \|^2 = o (\varepsilon N) \) as \( \varepsilon \rightarrow 0 \). For this we introduce a new functional \( J_\varepsilon : M_{\varepsilon, y} \rightarrow \mathbb{R} \) defined by
\[
J_\varepsilon (y, \varphi) = I_\varepsilon (U_{\varepsilon, y} + \varphi), \quad \varphi \in E_{\varepsilon, y}.
\]
In fact, we divide the proof of first part of Theorem 1.1 into two steps:

**Step1**: for each \( \varepsilon, \delta \) sufficiently small and for each \( y \in B_\delta (x_0) \), we will find a critical point \( \varphi_{\varepsilon, y} \) for \( J_\varepsilon (y, \cdot) \) (the function \( y \mapsto \varphi_{\varepsilon, y} \) also belongs to the class \( C^1 (H_\varepsilon) \));

**Step2**: for each \( \varepsilon, \delta \) sufficiently small, we will find a critical point \( y_\varepsilon \) for the function \( j_\varepsilon : B_\delta (x_0) \rightarrow \mathbb{R} \) induced by
\[
y \mapsto j_\varepsilon (y) \equiv J (y, \varphi_{\varepsilon, y}). \tag{2.2}
\]

That is, we will find a critical point \( y_\varepsilon \) in the interior of \( B_\delta (x_0) \).

It is standard to verify that \( (y_\varepsilon, \varphi_{\varepsilon, y_\varepsilon}) \) is a critical point of \( J_\varepsilon \) for \( \varepsilon \) sufficiently small by the chain rule. This gives a solution \( u_\varepsilon = U_{\varepsilon, y_\varepsilon} + \varphi_{\varepsilon, y_\varepsilon} \) to Eq. (1.1) for \( \varepsilon \) sufficiently small in virtue of the following lemma.

**Lemma 2.1** There exist \( \varepsilon_0, \eta_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \), \( \eta \in (0, \eta_0) \), and \( (y, \varphi) \in M_{\varepsilon, y} \) the following are equivalent:

(i) \( u_\varepsilon = U_{\varepsilon, y} + \varphi_{\varepsilon, y} \) is a critical point of \( I_\varepsilon \) in \( H_\varepsilon \).

(ii) \( (y, \varphi) \) is a critical point of \( J_\varepsilon \).

Now, in order to realize Step 1, we expand \( J_\varepsilon (y, \cdot) \) near \( \varphi = 0 \) for each fixed \( y \) as follows:
\[
J_\varepsilon (y, \varphi) = J_\varepsilon (y, 0) + l_\varepsilon (\varphi) + \frac{1}{2} \langle L_\varepsilon \varphi, \varphi \rangle + R_\varepsilon (\varphi)
\]
where \( J_\varepsilon (y, 0) = I_\varepsilon (U_{\varepsilon, y}) \), and \( l_\varepsilon, L_\varepsilon \) and \( R_\varepsilon \) are defined for \( \varphi, \psi \in H_\varepsilon \) as follows:
\[
l_\varepsilon (\varphi) = \langle I_\varepsilon' (U_{\varepsilon, y}), \varphi \rangle
\]
\[
= (U_{\varepsilon, y}, \varphi_\varepsilon) + b \varepsilon^{-N} \left( \int \varepsilon \Delta \varphi U_{\varepsilon, y}^2 dx \right) \int (\Delta) \varphi U_{\varepsilon, y} \cdot (\Delta) \varphi dx - \int U_{\varepsilon, y} \varphi dx \tag{2.3}
\]
and \( L_\varepsilon : L^2 (\mathbb{R}^N) \rightarrow L^2 (\mathbb{R}^N) \) is the bilinear form around \( U_{\varepsilon, y} \) defined by
\[
\langle L_\varepsilon \varphi, \psi \rangle = \langle I_\varepsilon' (U_{\varepsilon, y}), [\varphi, \psi] \rangle
\]
\[
= \langle \varphi, \psi \rangle_\varepsilon + b \varepsilon^{-N} \left( \int (\Delta) \varphi U_{\varepsilon, y}^2 dx \right) \int (\Delta) \varphi \cdot (\Delta) \varphi dx
\]
\[
+ 2 \varepsilon^{4-N} \frac{b}{\varepsilon} \left( \int \varphi U_{\varepsilon, y} \cdot (\Delta) \varphi dx \right) \left( \int \varphi U_{\varepsilon, y} \cdot (\Delta) \varphi dx \right) - \int U_{\varepsilon, y} \varphi \psi dx
\]
and \( R_\varepsilon \) denotes the second order reminder term given by
\[
R_\varepsilon (\varphi) = J_\varepsilon (y, \varphi) - J_\varepsilon (y, 0) - l_\varepsilon (\varphi) - \frac{1}{2} \langle L_\varepsilon \varphi, \varphi \rangle. \tag{2.4}
\]

We remark that \( R_\varepsilon \) belongs to \( C^2 (H_\varepsilon) \) since so is every term in the right hand side of (2.4). In the rest of this section, we consider \( l_\varepsilon : H_\varepsilon \rightarrow \mathbb{R} \) and \( R_\varepsilon : H_\varepsilon \rightarrow \mathbb{R} \) and give some elementary estimates.

**Lemma 2.2** Assume that \( V \) satisfies \((V_1)\) and \((V_2)\). Then, there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that for any \( y \in B_1 (0) \), there holds
\[
|l_\varepsilon (\varphi)| \leq C \varepsilon^{\frac{4}{N}} (\varepsilon^\alpha + (|V(y) - V(x_0)|) \| \varphi \|_\varepsilon
\]
for \( \varphi \in H_\varepsilon \). Here \( \alpha \) denotes the order of the Hölder continuity of \( V \) in \( B_{r_0} (x_0) \).
Next we give estimates for $R_\varepsilon$ and its derivatives $R_\varepsilon^{(i)}$ for $i = 1, 2$.

**Lemma 2.3** There exists a constant $C > 0$, independent of $\varepsilon$ and $b$, such that for $i \in \{0, 1, 2\}$, there hold
\[
\left\| R_\varepsilon^{(i)}(\varphi) \right\| \leq C\varepsilon^{-\frac{2(i-1)}{2(i+1)}} \|\varphi\|_{L^p}^{p+1-i} + C(b+1)\varepsilon^{-\frac{1}{4}} \left(1 + \varepsilon^{-\frac{1}{4}} \|\varphi\|_2\right) \|\varphi\|_{L^2}^{N-i}
\]
for all $\varphi \in H_\varepsilon$.

Now we will give the energy expansion for the approximate solutions.

**Lemma 2.4** Assume that $V$ satisfies (V1) and (V2). Then, for $\varepsilon > 0$ sufficiently small, there is a small constant $\tau > 0$ and $C > 0$ such that
\[
I_\varepsilon(U_{\varepsilon,y}) = A\varepsilon^N + B\varepsilon^N ((V(y) - V(x_0)) + O(\varepsilon^{N+\alpha})
\]
where
\[
A = \frac{1}{2} \int_{\mathbb{R}^N} (a|(-\Delta)\tilde{U}|^2 + U^2)\,dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |(-\Delta)\tilde{U}|^2\,dx\right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1}\,dx,
\]
and
\[
B = \frac{1}{2} \int_{\mathbb{R}^N} U^2\,dx.
\]

Now we complete Step 1 for the Lyapunov-Schmidt reduction method as before. We first consider the operator $L_\varepsilon$, 
\[
\langle L_\varepsilon \varphi, \psi \rangle = \langle \varphi, \psi \rangle_\varepsilon + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} U_{\varepsilon,y}|^2\,dx \int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} \varphi \cdot (-\Delta)^{\frac{1}{2}} \psi\,dx + 2\varepsilon^{4s-N} b \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{1}{2}} \varphi\,dx\right) \left(\int_{\mathbb{R}^N} (-\Delta)^{\frac{1}{2}} U_{\varepsilon,y} \cdot (-\Delta)^{\frac{1}{2}} \psi\,dx\right) - \frac{p}{p+1} \int_{\mathbb{R}^N} U_{\varepsilon,y}^{p+1}\,\varphi\psi\,dx
\]
for $\varphi, \psi \in H_\varepsilon$. The following result shows that $L_\varepsilon$ is invertible when restricted on $E_{\varepsilon,y}$

**Lemma 2.5** There exist $\varepsilon_1 > 0, \delta_1 > 0$ and $\rho > 0$ sufficiently small, such that for every $\varepsilon \in (0, \varepsilon_1), \delta \in (0, \delta_1)$, there holds
\[
\| L_\varepsilon \varphi \|_\varepsilon \geq \rho \| \varphi \|_\varepsilon, \quad \forall \varphi \in E_{\varepsilon,y}
\]
uniformly with respect to $y \in B_\delta(x_0)$.

Lemma 2.5 implies that by restricting on $E_{\varepsilon,y}$, the quadratic form $L_\varepsilon : E_{\varepsilon,y} \rightarrow E_{\varepsilon,y}$ has a bounded inverse, with $\| L_\varepsilon^{-1} \| \leq \rho^{-1}$ uniformly with respect to $y \in B_\delta(x_0)$. This further implies the following reduction map.

**Lemma 2.6** There exist $\varepsilon_0 > 0, \delta_0 > 0$ sufficiently small such that for all $\varepsilon \in (0, \varepsilon_0), \delta \in (0, \delta_0)$, there exists a $C^1$ map $\varphi_\varepsilon : B_\delta(x_0) \rightarrow H_\varepsilon$ with $y \mapsto \varphi_{\varepsilon,y} \in E_{\varepsilon,y}$ satisfying
\[
\left\langle \frac{\partial J_\varepsilon (y, \varphi_{\varepsilon,y})}{\partial \varphi}, \psi \right\rangle_\varepsilon = 0, \quad \forall \psi \in E_{\varepsilon,y}.
\]
Moreover, there exists a constant $C > 0$ independent of $\varepsilon$ small enough and $\kappa \in (0, \frac{2}{7})$ such that
\[
\| \varphi_{\varepsilon,y} \|_\varepsilon \leq C\varepsilon^{\frac{7}{7}+\alpha-\kappa} + C\varepsilon^{\frac{7}{7}} \left( V(y) - V(x_0) \right)^{1-\kappa}.
\]

**Proof of Theorem 2.1:** Let $\varepsilon_0$ and $\delta_0$ be defined as in Lemma 2.6 and let $\varepsilon < \varepsilon_0$. Fix $0 < \delta < \delta_0$. Let $y \mapsto \varphi_{\varepsilon,y}$ for $y \in B_\delta(x_0)$ be the map obtained in Lemma 2.6. As aforementioned in Step 2, it is equivalent to find a critical point for the function $j_\varepsilon$ defined as in (2.2) by Lemma 2.1. By the Taylor expansion, we have
\[
j_\varepsilon(y) = J(y, \varphi_{\varepsilon,y}) = I_\varepsilon(U_{\varepsilon,y}) + l_\varepsilon (\varphi_{\varepsilon,y}) + \frac{1}{2} \langle L_\varepsilon \varphi_{\varepsilon,y}, \varphi_{\varepsilon,y} \rangle + R_\varepsilon (\varphi_{\varepsilon,y}).
\]
We analyze the asymptotic behavior of $j_\varepsilon$ with respect to $\varepsilon$ first.
By Lemma 2.2-2.5, we have

\[ j_\varepsilon(y) = I_\varepsilon(U_{\varepsilon,y}) + O \left( \|U_\varepsilon\| \|\varphi_\varepsilon\| + \|\varphi_\varepsilon\|^2 \right) \]
\[ = A\varepsilon^N + B\varepsilon^N (V(y) - V(x_0)) + O(\varepsilon^N) \left( \varepsilon^{\alpha-\kappa} + (V(y) - V(x_0))^{1-\kappa} \right)^2 + O(\varepsilon^{N+\alpha}). \]  

(2.5)

Now consider the minimizing problem

\[ j_\varepsilon(y) \equiv \inf_{y \in B_\delta(x_0)} j_\varepsilon(y). \]

Assume that \( j_\varepsilon \) is achieved by some \( y_\varepsilon \) in \( B_\delta(x_0) \). We will prove that \( y_\varepsilon \) is an interior point of \( B_\delta(x_0) \).

To prove the claim, we apply a comparison argument. Let \( \varepsilon \in \mathbb{R}^N \) with \( |\varepsilon| = 1 \) and \( \eta > 1 \). We will choose \( \eta \) later. Let \( z_\varepsilon = \varepsilon_\eta \varepsilon \in B_\delta(0) \) for a sufficiently large \( \eta > 1 \). By the above asymptotics formula, we have

\[ j_\varepsilon(z_\varepsilon) = A\varepsilon^N + B\varepsilon^N (V(z_\varepsilon) - V(0)) + O(\varepsilon^{N+\alpha}) \]
\[ + O(\varepsilon^N) \left( \varepsilon^{\alpha-\kappa} + (V(z_\varepsilon) - V(0))^{1-\kappa} \right)^2. \]

Applying the Hölder continuity of \( V \), we derive that

\[ j_\varepsilon(z_\varepsilon) = A\varepsilon^N + O(\varepsilon^{N+\alpha}) + O(\varepsilon^{N+\alpha}) \]
\[ + O(\varepsilon^N) \left( \varepsilon^{2(\alpha-\kappa)} + \varepsilon^{2\eta\alpha(1-\kappa)} \right) \]
\[ = A\varepsilon^N + O(\varepsilon^{N+\alpha}). \]

where \( \eta > 1 \) is chosen to be sufficiently large accordingly. Note that we also used the fact that \( \kappa \ll \alpha/2 \).

Thus, by using \( j(y_\varepsilon) \leq j(z_\varepsilon) \) we deduce

\[ B\varepsilon^N (V(y_\varepsilon) - V(0)) + O(\varepsilon^N) \left( \varepsilon^{\alpha-\kappa} + (V(y_\varepsilon) - V(0))^{1-\kappa} \right)^2 \leq O(\varepsilon^{N+\alpha}) \]

That is,

\[ B (V(y_\varepsilon) - V(0)) + O(1) \left( \varepsilon^{\alpha-\kappa} + (V(y_\varepsilon) - V(0))^{1-\kappa} \right)^2 \leq O(\varepsilon^\alpha) \]  

(2.6)

If \( y_\varepsilon \in \partial B_\delta(0) \), then by the assumption \( (V_2) \), we have

\[ V(y_\varepsilon) - V(0) \geq c_0 > 0 \]

for some constant \( 0 < c_0 \ll 1 \) since \( V \) is continuous at \( x = 0 \) and \( \delta \) is sufficiently small. Thus, by noting that \( B > 0 \) from Lemma 2.4 and sending \( \varepsilon \to 0 \), we infer from (2.6) that

\[ c_0 \leq 0. \]

We reach a contradiction. This proves the claim. Thus \( y_\varepsilon \) is a critical point of \( j_\varepsilon \) in \( B_\delta(x_0) \). Then the existence of solutions now follows from the claim and Lemma 2.1.

3 Concentration behavior of solutions

First we explore some properties of the solutions derived as in Section 2. Set

\[ \bar{u}_\varepsilon(x) = u_\varepsilon(x + y_\varepsilon). \]

Then \( \bar{u}_\varepsilon > 0 \) solves

\[ \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^{p/2} \bar{u}_\varepsilon|^2 \, dx \right) (-\Delta)^s \bar{u}_\varepsilon + \bar{V}_\varepsilon(x) \bar{u}_\varepsilon = \bar{u}_\varepsilon^p \quad \text{in } \mathbb{R}^N \]  

(3.1)

with \( \bar{V}_\varepsilon(x) = V(\varepsilon x + \varepsilon y_\varepsilon) \). Then in this section we first show that the solutions concentrate around the minima of \( V \).

8
3.1 $L^\infty$-estimate

In order to study the concentration behavior of the semiclassical solutions obtained above, we first establish the $L^\infty$-estimate. Now we recall the following result for completeness.

**Lemma 3.1** Suppose that $h : \mathbb{R} \to \mathbb{R}$ is convex and Lipschitz continuous with the Lipschitz constant $L$, $h(0) = 0$. Then for each $u \in H^s(\mathbb{R}^N)$, $h(u) \in H^s(\mathbb{R}^N)$ and

$$(-\Delta)^s h(u) \leq h'(u)(-\Delta)^s u$$

in the weak sense.

**Proof:** First, we claim that $h(u) \in H^s(\mathbb{R}^N)$ for $u \in H^s(\mathbb{R}^N)$. In fact

$$[h(u)]_{D^s, 2} = \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|h(u(x)) - h(u(y))|^2}{|x - y|^{N + 2s}} dx dy \right)^{1/2} \leq L \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} dx dy = L \|u\|_{D^{s, 2}},$$

which implies that $h(u) \in D^{s, 2}(\mathbb{R}^N)$. Moreover,

$$\int_{\mathbb{R}^N} |h(u)|^2 dx = \int_{\mathbb{R}^N} |h(u) - h(0)|^2 dx \leq \int_{\mathbb{R}^N} L^2 |u|^2 dx < \infty,$

which yields that $h(u) \in L^2(\mathbb{R}^N)$. Therefore, the claim is true.

Next we show that (3.2) holds. Observe that $h'$ exists a.e. in $\mathbb{R}$ since $h$ is Lipschitz continuous. For $\psi \in C^\infty_0(\mathbb{R}^N, \mathbb{R})$ with $\psi \geq 0$, combining with the convexity of $h$, there holds

$$\int_{\mathbb{R}^N} (-\Delta)^s (h(u))\psi dx = -\frac{1}{2} C(s) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{h(u(x) + y) + h(u(x) - y) - 2h(u(x))}{|y|^{N + 2s}} \psi(x) dy dx,$$

$$\leq -\frac{1}{2} C(s) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|h'(u(x))(u(x) + y) - u(x)) + h'(u(x))(u(x) - y) - u(x))|}{|y|^{N + 2s}} \psi(x) dy dx$$

$$= -\frac{1}{2} C(s) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{h'(u(x))|u(x) + y| + u(x) - 2u(x)}{|y|^{N + 2s}} \psi(x) dy dx$$

This completes the proof.

**Remark 3.1** In fact, from the above arguments, one can see that (3.2) holds for a.e. $x \in \mathbb{R}^N$. Moreover, Lemma 3.1 is true for general dimension $N$.

The following uniform $L^\infty$-estimate plays a fundamental role in the study of behavior of the maximum points of the solutions, whose proof is related to the Moser iterative method [35]. A similar result for the fractional Schrödinger equation can be found in [15] or [48].

**Lemma 3.2** Let $\varepsilon \to 0^+$ and $u_\varepsilon$ be a positive solution of (1.1). Then up to a subsequence, $u_\varepsilon(x) := \frac{u_\varepsilon(\varepsilon x + \varepsilon y_c)}{u_\varepsilon(\varepsilon x + \varepsilon y_c)}$ satisfies that $u_\varepsilon \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C.$$
**Proof:** By standard variational method, we can know that \( \{ \bar{u}_\varepsilon \} \) has a convergent subsequence still denote by \( \{ \bar{u}_\varepsilon \} \). Therefore, there exists some \( C > 0 \) such that

\[
\| \bar{u}_\varepsilon \| \leq C,
\]

and hence

\[
\| \bar{u}_\varepsilon \|_{2^*} \leq C. \tag{3.3}
\]

Let \( T > 0 \), we define

\[
H(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
\beta t^{\beta - 1} + T^\beta, & \text{if } 0 < t < T, \\
\beta T^{\beta - 1} (t - T) + T^\beta, & \text{if } t \geq T,
\end{cases} \tag{3.4}
\]

with \( \beta > 1 \) to be determined later.

Since \( H \) is convex and Lipschitz, by Lemma 3.1,

\[
( -\Delta )^s H(u) \leq H'(u)( -\Delta )^s u. \tag{3.5}
\]

Thus, by (3.1), (3.5) and the Sobolev embedding theorem, we have

\[
\begin{align*}
\| H(\bar{u}_\varepsilon) \|_{L^{2^*_s}(\mathbb{R}^N)}^2 & \leq S^{-1} \int_{\mathbb{R}^N} |( -\Delta )^s H(\bar{u}_\varepsilon) |^2 dx \\
& \leq \frac{1}{sa} \int_{\mathbb{R}^N} (a + b) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{| \bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(y) |^2}{|x - y|^{N + 2s}} dx dy \int_{\mathbb{R}^N} |( -\Delta )^s H(\bar{u}_\varepsilon) |^2 dx \\
& \leq C \int_{\mathbb{R}^N} H(\bar{u}_\varepsilon) H'(\bar{u}_\varepsilon) \left( -\Delta \right)^s \bar{u}_\varepsilon dx \\
& \leq C \int_{\mathbb{R}^N} H(\bar{u}_\varepsilon) H'(\bar{u}_\varepsilon) |-\Delta \varepsilon| \leq C \int_{\mathbb{R}^N} H(\bar{u}_\varepsilon) H'(\bar{u}_\varepsilon) (1 + |\bar{u}_\varepsilon|^{2^*_s - 1}) dx.
\end{align*}
\]

Using the fact \( H(\bar{u}_\varepsilon) H'(\bar{u}_\varepsilon) \leq \beta^2 \bar{u}_\varepsilon^{2^*_s - 1} \) and \( \bar{u}_\varepsilon H'(\bar{u}_\varepsilon) \leq \beta H(\bar{u}_\varepsilon) \), we obtain

\[
\begin{align*}
\| H(\bar{u}_\varepsilon) \|_{L^{2^*_s}(\mathbb{R}^N)}^2 & \leq C \int_{\mathbb{R}^N} \left( \beta^2 \bar{u}_\varepsilon^{2^*_s - 1} + \beta |\bar{u}_\varepsilon|^{2^*_s - 2} H^2(\bar{u}_\varepsilon) \right) dx \\
& \leq C \beta^2 \int_{\mathbb{R}^N} \left( \bar{u}_\varepsilon^{2^*_s - 1} + |\bar{u}_\varepsilon|^{2^*_s - 2} H^2(\bar{u}_\varepsilon) \right) dx, \tag{3.7}
\end{align*}
\]

where \( C \) is independent of \( \beta \). Notice that the last integral in (3.7) is well defined for every \( T \) in the definition of \( H \).

Now we choose \( \beta \) in (3.7) such that \( 2\beta - 1 = 2^*_s \), and denote it as

\[
\beta_1 := \frac{2^*_s + 1}{2}.
\]

Let \( R > 0 \) to be fixed later. For the last integral in (3.7), we apply the Hölder’s inequality with exponents \( r = \frac{2^*_s}{2} \) and \( r' = \frac{2^*_s}{2^*_s - 2} \).

\[
\int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2^*_s - 2} H^2(\bar{u}_\varepsilon) dx = \int_{\{ \bar{u}_\varepsilon \leq R \}} |\bar{u}_\varepsilon|^{2^*_s - 2} H^2(\bar{u}_\varepsilon) dx + \int_{\{ \bar{u}_\varepsilon > R \}} |\bar{u}_\varepsilon|^{2^*_s - 2} H^2(\bar{u}_\varepsilon) dx \\
\leq R^{2^*_s - 1} \int_{\{ \bar{u}_\varepsilon \leq R \}} \frac{H^2(\bar{u}_\varepsilon)}{\bar{u}_\varepsilon} dx + \left( \int_{\{ \bar{u}_\varepsilon > R \}} |\bar{u}_\varepsilon|^{2^*_s - 2} dx \right)^{\frac{2^*_s - 2}{2^*_s - 2}} \tag{3.8}
\]

\[
\times \left( \int_{\mathbb{R}^N} H(\bar{u}_\varepsilon) dx \right)^{\frac{2^*_s - 2}{2^*_s - 2}}. \]
Since \( \{\bar{u}_\varepsilon\} \) is bounded in \( H_\varepsilon \), we can take \( R \) large enough such that
\[
\left( \int_{\{u_\varepsilon > R\}} |\bar{u}_\varepsilon|^{2_s^*} \, dx \right)^{\frac{2}{2_s^* - 2}} \leq \frac{1}{2C\beta_1^2}.
\] (3.9)

From (3.7)-(3.9) we have
\[
\left( \int_{\mathbb{R}^N} |H(\bar{u}_\varepsilon)|^{2_s^*} \, dx \right)^{\frac{2}{2_s^*}} \leq 2C\beta_1^2 \left( \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^*} \, dx + R^{2_s^* - 1} \int_{\mathbb{R}^N} H^2(\bar{u}_\varepsilon) \, dx \right). \tag{3.10}
\]
Using the fact that \( H(\bar{u}_\varepsilon) \leq \bar{u}_\varepsilon^{\beta_1} \) in the right hand side and taking \( T \to \infty \) we obtain
\[
\left( \int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2_s^* \beta_1} \, dx \right)^{\frac{2}{2_s^* \beta_1}} \leq 2C\beta_1^2 \left( \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* - 1} \, dx + R^{2_s^* - 1} \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^*} \, dx \right) < \infty,
\]

hence
\[\bar{u}_\varepsilon \in L^{2_s^* \beta_1}(\mathbb{R}^N)\).

Together with (3.3), we have
\[\|\bar{u}_\varepsilon\|_{2_s^* \beta_1} \leq C, \tag{3.11}\]

uniformly in \( \varepsilon \). Now we suppose \( \beta > \beta_1 \). Thus, using that \( H(\bar{u}_\varepsilon) \leq \bar{u}_\varepsilon^{\beta_1} \) in the right hand side of (3.7) and taking \( T \to \infty \), we get
\[
\left( \int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2_s^* \beta} \, dx \right)^{\frac{2}{2_s^* \beta}} \leq C\beta^2 \left( \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* - 1} \, dx + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx \right) < \infty. \tag{3.12}\]

Set \( a_1 := \frac{2_s^* (2_s^* - 1)}{2_s^* - 2} \) and \( b_1 := 2_s^* - 1 \), \( a_1 \) and \( b_1 > 0 \). Thus, by using Young’s inequality with exponents \( r = \frac{2_s^*}{a_1} \) and \( r' = \frac{2_s^*}{2_s^* - a_1} \), we have
\[
\int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* - 1} \, dx \leq \frac{a_1}{2_s^*} \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^*} \, dx + \frac{2_s^*}{2_s^* - a_1} \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx
\leq \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^*} \, dx + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx
\leq C \left( 1 + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx \right). \tag{3.13}\]

Combining (3.12) and (3.13), we conclude that
\[
\left( \int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2_s^* \beta} \, dx \right)^{\frac{2}{2_s^* \beta}} \leq C\beta^2 \left( 1 + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx \right), \tag{3.14}\]

where \( C \) remains independently of \( \beta \). Therefore,
\[
\left( 1 + \int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2_s^* \beta} \, dx \right)^{\frac{2}{2_s^* (\beta + 1)}} \leq (C\beta^2)^{\frac{2}{2_s^* (\beta - 1)}} \left( 1 + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* + 2_s^* - 2} \, dx \right)^{\frac{1}{2_s^* (\beta - 1)}}. \tag{3.15}\]

Repeating this argument we will define a sequence \( \beta_i \), \( i \geq 1 \) such that
\[2_s^* \beta_{i+1} + 2_s^* - 2 = 2_s^* \beta_i. \]

Thus,
\[\beta_{i+1} - 1 = \left( \frac{2_s^*}{2_s^*} \right)^i (\beta_1 - 1). \]

Replacing it in (3.15) one has
\[
\left( 1 + \int_{\mathbb{R}^N} |\bar{u}_\varepsilon|^{2_s^* \beta_{i+1}} \, dx \right)^{\frac{2}{2_s^* \beta_{i+1} + 1}} \leq (C\beta_{i+1}^2)^{\frac{1}{2_s^* (\beta_{i+1} + 1)}} \left( 1 + \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{2_s^* \beta_{i+1} + 2_s^* - 2} \, dx \right)^{\frac{1}{2_s^* (\beta_{i+1} + 1)}}. \tag{3.16}\]
Denoting $C_{i+1} = C/\beta_{i+1}^2$ and

$$K_i := \left(1 + \int_{\mathbb{R}^N} u_i^{2\beta_i + 2^{*} - 2} dx\right)^{\frac{1}{\beta_i + 1}}.$$ 

So we can rewrite (3.16) as

$$K_{i+1} \leq C_{i+1}^{\frac{1}{\beta_{i+1} + 1}} K_i,$$

and hence we conclude that there exists a constant $D > 0$ independent of $i$, such that

$$K_{i+1} \leq \prod_{j=2}^{i+1} C_j^{\frac{1}{\beta_j + 1}} K_2 \leq DK_1.$$

Therefore,

$$\bar{u}_\varepsilon \in L^\infty(\mathbb{R}^N), \ \forall \varepsilon.$$

Jointly with (3.11),

$$\|\bar{u}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq C,$$

uniformly in $\varepsilon$. This finishes the proof of Lemma 3.2. ■

### 3.2 Concentration behavior of solutions

Now we are ready to give the proof of concentration behavior of solutions obtained in Theorem 1.1. Let $u_{\varepsilon}$ be a positive solution of (1.1) obtained in Theorem 1.1, then $u_{\varepsilon}(x) := u_{\varepsilon}(\varepsilon x + \varepsilon y_\varepsilon)$ is a solution of the problem

$$\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u_{\varepsilon}|^2 dx\right)(-\Delta) u_{\varepsilon} + V_{\varepsilon}(x)u_{\varepsilon} = \bar{u}_\varepsilon^p \quad \text{in } \mathbb{R}^N, \quad (3.17)$$

with $\{y_\varepsilon\} \subset \mathbb{R}^N$, $V_\varepsilon(x) := V(\varepsilon x + \varepsilon y_\varepsilon)$. From [48, Lemma 4.6], we have

$$\lim_{\varepsilon \to 0} I_\varepsilon(u_{\varepsilon}) = \lim_{\varepsilon \to 0} J_\varepsilon(\bar{u}_\varepsilon) = c_V(x_0),$$

where

$$J_\varepsilon(\bar{u}_\varepsilon) = \frac{a}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} \bar{u}_\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x)\bar{u}_\varepsilon^2 dx + \frac{b}{4} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} \bar{u}_\varepsilon|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} \bar{u}_\varepsilon^{p+1} dx$$

is the corresponding energy functional of (3.17) and we define the ground energy corresponding to (3.17) by

$$e_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon,$$

with the Nehari manifold associated to $J_\varepsilon$ defined as

$$\mathcal{N}_\varepsilon := \{u \in H\setminus\{0\} : \langle J_\varepsilon'(u), u \rangle = 0\}.$$

Similar to [48, Lemma 4.8], after extracting a subsequence, we have

$$\bar{u}_\varepsilon \to u, \ \text{in } H^s(\mathbb{R}^N), \quad (3.18)$$

and

$$\bar{y}_\varepsilon := \varepsilon y_\varepsilon \to y \in \Lambda := \{y \in \mathbb{R}^N : V(y) = V(x_0)\}. \quad (3.19)$$

**Claim 1:** $u$ is a positive ground state solution to the limit equation

$$\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} u|^{2} dx\right)(-\Delta) u(x) + V(x_0)u = u^p \quad \text{in } \mathbb{R}^N. \quad (3.20)$$

In fact, for any $\varphi \in H^s(\mathbb{R}^N)$, observe that

$$\left(a + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{1}{2}} \bar{u}_\varepsilon|^2 dx\right)\int_{\mathbb{R}^N} (-\Delta)\bar{u}_\varepsilon \varphi dx + \int_{\mathbb{R}^N} \bar{V}_\varepsilon(x)\bar{u}_\varepsilon \varphi dx = \int_{\mathbb{R}^N} \bar{u}_\varepsilon^p \varphi dx.$$

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and by (3.18) (3.19) and the fact that \( V \) is uniformly continuous,

\[
\left| \int_{\mathbb{R}^N} \nabla V(x) \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^N} V(x_0) \nabla \varphi \, dx \right| \leq \int_{\mathbb{R}^N} \left| \nabla V(x) \right| \left| \nabla (u_\varepsilon \varphi - u \varphi) \right| \, dx + \int_{\mathbb{R}^N} \left| (\nabla V(x) - \nabla V(x_0)) u \varphi \right| \, dx \to 0.
\]

By (3.18), (3.19) and (3.21), it is easy to check that

\[
\left( a + b \int_{\mathbb{R}^N} (\nabla^2 u \varphi)^2 \, dx \right) \int_{\mathbb{R}^N} (-\Delta) u \varphi \, dx + V(x_0) \int_{\mathbb{R}^N} u \varphi \, dx = \int_{\mathbb{R}^N} u^p \varphi \, dx, \ \forall \varphi \in H^s(\mathbb{R}^N),
\]

which yields that \( u \) is a solution to (3.20).

On the other hand, by Fatou’s Lemma, we have

\[
c_{V(x_0)} \leq \frac{a}{2} \int_{\mathbb{R}^N} (\nabla^2 u)^2 \, dx + \frac{V(x_0)}{4} \int_{\mathbb{R}^N} u^2 \, dx + b \left( \int_{\mathbb{R}^N} (\nabla^2 u)^2 \, dx \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \, dx
\]

\[
\leq \liminf_{n \to +\infty} \left( \frac{a}{4} \int_{\mathbb{R}^N} (\nabla^2 u_\varepsilon)^2 \, dx + \frac{V(x_0)}{4} \int_{\mathbb{R}^N} (\nabla^2 u_\varepsilon)^2 \, dx \right)
\]

\[
\leq \liminf_{n \to +\infty} \left( \mathcal{J}_{\varepsilon n}(u_\varepsilon) - \frac{1}{4} \mathcal{J}_{\varepsilon n}^{\prime}(\bar{u}_\varepsilon, \bar{u}_\varepsilon) \right)
\]

\[
= c_{V(x_0)}.
\]

Thus, \( u \) is a positive ground state solution to (3.20).

**Claim 2:** \( \bar{u}_\varepsilon(x) \to 0 \) as \( |x| \to \infty \), uniformly in \( \varepsilon \).

Indeed, we rewrite (3.17) as

\[
(-\Delta)^s \bar{u}_\varepsilon + \bar{u}_\varepsilon = \Upsilon_\varepsilon, \text{ in } \mathbb{R}^N.
\]

where

\[
\Upsilon_\varepsilon(x) = (a + b \int_{\mathbb{R}^N} (\nabla^2 u_\varepsilon)^2 \, dx)^{-1} \left[ u_\varepsilon(x) - \bar{u}_\varepsilon(x) \right] - \bar{u}_\varepsilon(x).
\]

Then we know from [16] that

\[
\bar{u}_\varepsilon = \mathcal{K} * \Upsilon_\varepsilon = \int_{\mathbb{R}^N} \mathcal{K}(x-y) \Upsilon_\varepsilon(y) \, dy,
\]

where \( \mathcal{K} \) is the Bessel kernel. Moreover, \( \mathcal{K} \) has the following properties:

- \( \mathcal{K} \) is positive, radially symmetric and smooth in \( \mathbb{R}^N \setminus \{0\} \),
- there is \( C > 0 \) such that \( \mathcal{K}(x) \leq \frac{C}{|x|^{N-2s}} \),
- \( \mathcal{K} \in L^q(\mathbb{R}^N), \forall q \in [1, \frac{N}{N-2s}) \).

By (3.18), Lemma 3.2 and its proof and the interpolation on the \( L^p \)-spaces,

\[
\bar{u}_\varepsilon \to u, \text{ in } L^p(\mathbb{R}^N), \forall p \in (2, +\infty).
\]

Set

\[
\Upsilon(x) = (a + b \int_{\mathbb{R}^N} (\nabla^2 u)^2 \, dx)^{-1} [u(x) - V(x)u(x)] + u(x).
\]

It follows from (3.18) and (3.22) that

\[
\Upsilon_\varepsilon \to \Upsilon, \text{ in } L^p(\mathbb{R}^N), \forall p \in (2, +\infty)
\]

and

\[
\| \Upsilon_\varepsilon \|_{L^\infty(\mathbb{R}^N)} \leq C
\]

for some \( C > 0 \) and all \( \varepsilon \).
From (3.22)-(3.24), repeating the proof of [1, Lemma 2.6] with small modifications, we conclude that
\[ \bar{u}_\varepsilon(x) \to 0 \text{ as } |x| \to \infty, \]
uniformly in \( \varepsilon \).

**Claim 3:** There exist \( C > 0 \) such that
\[ \bar{u}_\varepsilon(x) \leq C \frac{1}{1 + |x|^{N+2s}}, \forall x \in \mathbb{R}^N. \]

In fact, according to [16, Lemma 4.2], there exists a continuous function \( \bar{\omega} \) such that
\[ 0 < \bar{\omega}(x) \leq C \frac{1}{1 + |x|^{N+2s}}, \quad (3.25) \]
and
\[ (-\Delta)^s \bar{\omega} + \frac{V(x_0)}{2(a+2b[u]_{D^{s},2})^2} \bar{\omega} = 0, \text{ in } \mathbb{R}^N \setminus B_R(0) \quad (3.26) \]
for some suitable \( R > 0 \). From (3.18), \( \bar{u}_\varepsilon \to u \) in \( L^2(\mathbb{R}^N) \), and hence
\[ [\bar{u}_\varepsilon]_{D^{s},2} \to [u]_{D^{s},2}. \]
Since \( \bar{u}_\varepsilon \) solves (3.17) and \( \bar{u}_\varepsilon(x) \to 0 \) as \( |x| \to \infty \) uniformly in \( \varepsilon \), then, for some large \( R_1 > 0 \), we obtain
\[ (-\Delta)^s \bar{u}_\varepsilon + \frac{V(x_0)}{2(a+2b[u]_{D^{s},2})^2} \bar{u}_\varepsilon = \frac{\bar{u}_\varepsilon^p - \bar{V}(x) \bar{u}_\varepsilon}{a + b[\bar{u}_\varepsilon]_{D^{s},2}^2} + \frac{V(x_0)}{2(a+2b[u]_{D^{s},2})^2} \bar{u}_\varepsilon \]
\[ = \frac{\bar{u}_\varepsilon^p - \bar{V}(x) \bar{u}_\varepsilon + (a + b[\bar{u}_\varepsilon]_{D^{s},2}^2) \frac{V(x_0)}{2(a+2b[u]_{D^{s},2})^2} \bar{u}_\varepsilon}{a + b[\bar{u}_\varepsilon]_{D^{s},2}^2} \]
\[ \leq \frac{\bar{u}_\varepsilon^p - \bar{V}(x) \bar{u}_\varepsilon + \frac{V(x_0)}{2} \bar{u}_\varepsilon}{a + b[\bar{u}_\varepsilon]_{D^{s},2}^2} \]
\[ \leq \frac{\bar{u}_\varepsilon^p - \bar{V}(x) \bar{u}_\varepsilon + \frac{V(x_0)}{2} \bar{u}_\varepsilon}{a + b[\bar{u}_\varepsilon]_{D^{s},2}^2} \]
\[ \leq 0, \]
for \( x \in \mathbb{R}^N \setminus B_{R_1}(0) \). Now we take \( R_2 := \max \{ R, R_1 \} \) and set
\[ z_\varepsilon := (\alpha + 1) \bar{\omega} - \beta \bar{u}_\varepsilon, \quad (3.28) \]
where \( \alpha := \sup_{\mathbb{R}^N} \| \bar{u}_\varepsilon \|_\infty < \infty \) and \( \beta := \min_{B_{R_2}(0)} \bar{\omega} > 0 \). We next show that \( z_\varepsilon \geq 0 \) in \( \mathbb{R}^N \). For this we suppose by contradiction that, there is a sequence \( \{ x_j^\varepsilon \} \) such that
\[ \inf_{x \in \mathbb{R}^N} z_\varepsilon(x) = \lim_{j \to \infty} z_\varepsilon(x_j^\varepsilon) < 0. \]
(3.29)

Observe that
\[ \lim_{|x| \to \infty} \bar{\omega}(x) = 0. \]
Combing with \( \bar{u}_\varepsilon(x) \to 0 \) as \( |x| \to \infty \) uniformly in \( \varepsilon \), we obtain
\[ \lim_{|x| \to \infty} z_\varepsilon(x) = 0, \]
uniformly in \( \varepsilon \). Consequently, the sequence \( \{ x_j^\varepsilon \} \) is bounded and therefore, up to a subsequence, we may assume that \( x_j^\varepsilon \to x_\varepsilon^* \) as \( j \to \infty \) for some \( x_\varepsilon^* \in \mathbb{R}^N \). Hence (3.29) becomes
\[ z_\varepsilon(x_\varepsilon^*) = \inf_{x \in \mathbb{R}^N} z_\varepsilon(x) < 0. \]
(3.30)

From (3.30), we have
\[ (-\Delta)^s z_\varepsilon(x_\varepsilon^*) = -\frac{C(s)}{2} \int_{\mathbb{R}^N} \frac{z_\varepsilon(x_\varepsilon^* + y) + z_\varepsilon(x_\varepsilon^* - y) - 2z_\varepsilon(x_\varepsilon^*)}{|y|^{N+2s}} dy \leq 0. \]
(3.31)
By (3.28), we get
\[ z_\varepsilon(x) \geq \alpha \beta + \bar{\omega} - \alpha \beta > 0, \text{ in } B_{R_2}(0). \]
Therefore, combining this with (3.30), we see that
\[ x_\varepsilon^* \in \mathbb{R}^N \setminus B_{R_2}(0). \]
(3.32)
From (3.26)-(3.27), we conclude that
\[ (-\Delta)^{s/2} z_\varepsilon + \frac{V(x_0)}{2(a + 2b|u(z_\varepsilon)|^2)} z_\varepsilon \geq 0, \text{ in } \mathbb{R}^N \setminus B_{R_2}(0). \]
(3.33)
Thanks to (3.32), we can evaluate (3.33) at the point \( x_\varepsilon^* \), and recall (3.30),(3.31), we conclude that
\[ 0 \leq (-\Delta)^{s/2} z_\varepsilon(x_\varepsilon^*) + \frac{V(x_0)}{2(a + 2b|u(z_\varepsilon^*)|^2)} z_\varepsilon(x_\varepsilon^*) < 0, \]
this is a contradiction, so \( z_\varepsilon(x) \geq 0 \) in \( \mathbb{R}^N \). That is to say, \( \bar{u}_\varepsilon \leq (\alpha + 1)^{-1} \bar{\omega} \), which together with (3.25), implies that
\[ \bar{u}_\varepsilon(x) \leq \frac{C}{1 + |x|^N}, \forall x \in \mathbb{R}^N. \]
Now, we end the proof of concentration behavior of semiclassical solutions. Using [43, Proposition 2.9] again, we see that \( \bar{u}_\varepsilon \in C^{1,\alpha}(\mathbb{R}^N) \) for any \( \alpha < 2s-1 \). Considering \( \eta_\varepsilon \) the global maximum point of \( \bar{u}_\varepsilon \), by Lemma 3.2 and Claim 2, we see that \( \eta_\varepsilon \in B_R(x_0) \) for some \( R > 0 \). Thus, the global maximum point of \( u_\varepsilon \) given by \( z_\varepsilon = \eta_\varepsilon + y_\varepsilon \) satisfies \( \varepsilon z_\varepsilon = \varepsilon \eta_\varepsilon + \varepsilon y_\varepsilon \). Since \( \{\eta_\varepsilon\} \) is bounded, it follows that \( \varepsilon z_\varepsilon \rightarrow y \), thus \( V(\varepsilon z_\varepsilon) \rightarrow V(x_0) \) as \( \varepsilon \rightarrow \infty \). Moreover, by Claim 3, we have the following decay estimate
\[ u(x/\varepsilon) = \tilde{u}_\varepsilon(x/\varepsilon - y_\varepsilon) \leq \frac{C}{1 + |x/\varepsilon - y_\varepsilon|^N}, \forall x \in \mathbb{R}^N. \]
Now setting \( \tilde{v}_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \) we can easily see that \( \tilde{v}_\varepsilon(x) \) has the desired properties.

4 Local Pohozaev identity

In this section, we derive a local Pohozaev type identity which plays an important role in the proof of Theorem 1.1.

Lemma 4.1 Let \( u \) be a positive solution of (1.1) obtained as above. Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \). Then, for each \( i = 1, 2, \ldots, N \), there hold
\[
\int_\Omega \frac{\partial V}{\partial x_i} u^2 dx = \left( \varepsilon^a + \varepsilon^{4s-N} b \right) \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx \int_{\partial \Omega} \left( |(-\Delta)^{s/2} u|^2 \nu_i - 2 \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_i} \right) d\sigma
\]
\[+ \int_{\partial \Omega} Vu^2 \nu_i d\sigma - \frac{2}{p+1} \int_{\partial \Omega} u^{p+1} \nu_i d\sigma. \]
Here \( \nu = (\nu_1, \nu_2, \ldots, \nu_N) \) is the unit outward normal of \( \partial \Omega \).

Proof: We use the same ideas in [11, 22]. Indeed, the definition of nonlocal operator cause some techniques developed for local case can not be adapted immediately to nonlocal case. To overcome these difficulties, we will use an approach due to Caffarelli and Silvestre [10], that is, we will apply the s-harmonic extension technique to transform a nonlocal problem to a local one.

For this, we will denote by \( \mathbb{R}^{N+1}_+ := \mathbb{R}^N \times (0, +\infty) \). Also, for a point \( z \in \mathbb{R}^{N+1}_+ \), we will use the notation \( z = (x, y) \), with \( x \in \mathbb{R}^N \) and \( y > 0 \). For any \( u \in H^s(\mathbb{R}^N) \), we define that \( w = E_s(u) \) is its s-harmonic extension to the upper half-space \( \mathbb{R}^{N+1}_+ \), if \( w \) is a solution of the problem
\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+ \setminus \mathbb{R}^N \times \{y = 0\}, \\
w = u & \text{on } \mathbb{R}^N \times \{y = 0\}.
\end{cases}
\]
Moreover, we define the space $X^s(\mathbb{R}^{N+1}_+)$ and $\dot{H}^s(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^{N+1}_+)$ and $C_0^\infty(\mathbb{R}^N)$ under the norms
\[
\|w\|_{X^s}^2 := \int_{\mathbb{R}^{N+1}_+} \kappa_s y^1-2s|\nabla w|^2 dx dy,
\]
\[
\|w\|_{\dot{H}^s}^2 := \int_{\mathbb{R}^N} |\nabla w|^2 dx,
\]
where $\kappa_s > 0$ is a normalization constant.

Now we may reformulate the nonlocal Kirchhoff equation (1.1) in a local way, that is
\[
- \left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \right) \int_{\Omega^+} \kappa_s y^1-2s \nabla w(z) \frac{\partial w}{\partial x_i} dz = -\int_{\Omega^+} V(z) w(z) \frac{\partial w}{\partial x_i} dz + \int_{\Omega^+} |w(z)|^p \frac{\partial w}{\partial x_i} dz.
\]
(4.3)

Note that

LHS of (4.3) = \left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \right) \left( \frac{1}{2} \int_{\Omega^+} \kappa_s y^1-2s |\nabla w|^2 \nu_i d\sigma - \int_{\partial \Omega^+} \kappa_s y^1-2s \frac{\partial w}{\partial \nu} \frac{\partial w}{\partial y_i} d\sigma \right).

On the other hand, by Green’s formula, we have

LHS of (4.3) = -\frac{1}{2} \int_{\partial \Omega^+} u^2(x) V(z) \nu_i(z) d\sigma + \frac{1}{2} \int_{\Omega^+} u^2(z) \frac{\partial V(z)}{\partial x_i} d\sigma + \frac{1}{p+1} \int_{\partial \Omega^+} |w(z)|^{p+1} \nu_i(z) d\sigma.

Combining with them, we obtain

\[
\left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 dx \right) \left( \frac{1}{2} \int_{\partial \Omega^+} \kappa_s y^1-2s |\nabla w|^2 \nu_i d\sigma - \int_{\partial \Omega^+} \kappa_s y^1-2s \frac{\partial w}{\partial \nu} \frac{\partial w}{\partial y_i} d\sigma \right) = \frac{1}{2} \int_{\Omega^+} V(x) u^2 \nu_i d\sigma - \frac{1}{2} \int_{\Omega^+} \frac{\partial V}{\partial x_i} u^2 d\sigma - \frac{1}{p+1} \int_{\partial \Omega^+} |w(x)|^{p+1} \nu_i d\sigma,
\]
which means (4.1).

Now, let $u_\varepsilon = U_{\varepsilon,y_\varepsilon} + \varphi_{\varepsilon,y_\varepsilon}$ be an arbitrary solution of (1.1) derived as in Section 2. We know $y_\varepsilon = o(1)$ as $\varepsilon \to 0$. We will improve this asymptotics estimate by assuming that $V$ satisfies the additional assumption (V3), and by means of the above Pochozâev type identity. We first recall some useful estimates.

Lemma 4.2 Suppose that $V(x)$ satisfies (V3), then we have
\[
\int_{\mathbb{R}^N} (V(x_0) - V(x)) U_{\varepsilon,y_\varepsilon}(x) u(x) dx = O\left( \varepsilon^{\frac{N}{2}+m} + \varepsilon^{\frac{N}{2}} |y_\varepsilon - x_0|^m \right) \|u\|_\varepsilon,
\]
(4.4)
and
\[
\int_{B_d(y_\varepsilon)} \frac{\partial V(x)}{\partial x_i} U_{\varepsilon,y_\varepsilon}(x) u(x) dx = O\left( \varepsilon^{\frac{N}{2}+m-1} + \varepsilon^{\frac{N}{2}} |y_\varepsilon - x_0|^{m-1} \right) \|u\|_\varepsilon,
\]
for any $d \in (0, \delta)$, where $u(x) \in H_\varepsilon$. 

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\textbf{Proof:} First, from (V) and Hölder’s inequality, for a small constant \(d\), we have

\[
\left| \int_{B_{d}(y)} (V(x_0) - V(x)) U_{\varepsilon, y_{0}}(x) u(x) dx \right| \\
\leq C \int_{B_{d}(y)} |x - x_{0}|^{m} U_{\varepsilon, y_{0}}(x)|u(x)|dx \\
\leq C \left( \int_{B_{d}(y)} |x - x_{0}|^{2m} U_{\varepsilon, y_{0}}^{2}(x) dx \right)^{\frac{1}{2}} \left( \int_{B_{d}(y)} u^{2}(x) dx \right)^{\frac{1}{2}} \\
\leq C\varepsilon^{\gamma} (\varepsilon^{m} + |y_{\varepsilon} - x_{0}|^{m}) \|u\|_{\varepsilon}.
\] (4.5)

Also, by the polynomial decay of \(U_{\varepsilon, y_{0}}(x)\) in \(\mathbb{R}^{N} \setminus B_{d} (y_{\varepsilon})\), we can deduce that, for any \(\gamma > 0\),

\[
\left| \int_{\mathbb{R}^{N} \setminus B_{d}(y_{0})} (V(x_0) - V(x)) U_{\varepsilon, y_{0}}(x) u(x) dx \right| \leq C\varepsilon^{\gamma} \|u\|_{\varepsilon}.
\] (4.6)

Then, taking suitable \(\gamma > 0\), from (4.5) and (4.6), we get (4.4).

Next, from (V) and Hölder’s inequality, for any \(d \in (0, \bar{d}]\), we have

\[
\left| \int_{B_{d}(y_{0})} \frac{\partial V(x)}{\partial x_{i}} U_{\varepsilon, y_{0}}(x) u(x) dx \right| \\
\leq C \int_{B_{d}(y_{0})} |x - x_{0}|^{m-1} U_{\varepsilon, y_{0}}(x)|u(x)|dx \\
\leq C\varepsilon^{\frac{\gamma}{2}} (\varepsilon^{m-1} + |y_{\varepsilon} - x_{0}|^{m-1}) \|u\|_{\varepsilon}.
\]

\[
\int_{B_{r}(x_{0})} |u_{i}(x)| dx < +\infty, \quad i = 1, \ldots, l.
\]

Then for any \(x_{0}\), there exist a small constant \(d\) and another constant \(C\) such that

\[
\int_{\partial B_{d}(x_{0})} |u_{i}(x)| d\sigma \leq C \int_{\mathbb{R}^{N}} |u_{i}(x)| dx, \quad \text{for all} \ i = 1, \ldots, l.
\] (4.7)

\textbf{Proof:} Let \(M_{i} = \int_{\mathbb{R}^{N}} |u_{i}(x)| dx\), for \(i = 1, \ldots, l\). Then for a fixed small \(r_{0} > 0\),

\[
\int_{B_{r_{0}}(x_{0})} \left( \sum_{i=1}^{l} |u_{i}(x)| \right) dx \leq \sum_{i=1}^{l} M_{i}, \quad \text{for all} \ i = 1, \ldots, l.
\] (4.8)

On the other hand,

\[
\int_{B_{r_{0}}(x_{0})} \left( \sum_{i=1}^{l} |u_{i}(x)| \right) dx \geq \int_{0}^{r_{0}} \int_{\partial B_{r}(x_{0})} \left( \sum_{i=1}^{l} |u_{i}(x)| \right) d\sigma dr.
\] (4.9)

Then (4.8) and (4.9) imply that there exists a constant \(d < r_{0}\) such that

\[
\int_{\partial B_{r}(x_{0})} |u_{i}(x)| d\sigma \leq \int_{\partial B_{d}(x_{0})} \left( \sum_{i=1}^{l} |u_{i}(x)| \right) dx \leq \sum_{i=1}^{l} M_{i}, \quad \text{for all} \ i = 1, \ldots, l.
\] (4.10)

So taking \(C = \max_{1 \leq i \leq l} \sum_{i=1}^{l} \frac{M_{i}}{r_{0} M_{i}}\), we can obtain (4.7) from (4.10).

Applying Lemma 4.3 to \(\varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} \varphi_{\varepsilon}|^{2} + \varphi_{\varepsilon}^{2}\), there exists a constant \(d = d_{\varepsilon} \in (1, 2)\) such that

\[
\int_{\partial B_{d}(y_{0})} \varepsilon^{2s} |(-\Delta)^{\frac{s}{2}} \varphi_{\varepsilon}|^{2} + \varphi_{\varepsilon}^{2} d\sigma \leq \|\varphi_{\varepsilon}\|_{\varepsilon}^{2}
\] (4.11)
By an elementary inequality, we have
\[ \int_{\partial B_d(y_{\epsilon})} |(-\Delta)^{s/2} \varphi_{\epsilon}|^2 \, ds \leq 2 \int_{\partial B_d(y_{\epsilon})} |(-\Delta)^{s/2} U_{\epsilon,y_{\epsilon}}|^2 \, ds + 2 \int_{\partial B_d(y_{\epsilon})} |(-\Delta)^{s/2} \varphi_{\epsilon}|^2 \, ds. \]

By the proof in Section 3.2 we can know that there exists a small constant \( d_1 \), such that for any \( \gamma > 0 \) and \( 0 < d < d_1 \), we have
\[ U_{\epsilon,y_{\epsilon}} + |(-\Delta)^{s/2} U_{\epsilon,y_{\epsilon}}| = O(\epsilon^\gamma), \quad \text{for} \quad x \in B_d(x), \quad (4.12) \]
and
\[ U_{\epsilon,y_{\epsilon}} + |(-\Delta)^{s/2} U_{\epsilon,y_{\epsilon}}| = o(\epsilon^\gamma), \quad \text{for} \quad x \in \partial B_d(x). \quad (4.13) \]
Hence, for the constant \( d \) chosen as above, we deduce
\[ \epsilon^{2s} \int_{\partial B_d(y_{\epsilon})} |(-\Delta)^{s/2} \varphi_{\epsilon}|^2 \, ds = O\left( \|\varphi_{\epsilon}\|_2^2 + \epsilon^\gamma \right). \quad (4.14) \]
In particular, it follows from (4.12) and (4.13) that for any \( \gamma > 0 \), it holds
\[ \int_{\mathbb{R}^N} U_{\epsilon,y_{\epsilon}}^{q_1} U_{\epsilon,y_{\epsilon}}^{q_2} \, dx = O(\epsilon^\gamma), \quad (4.15) \]
and
\[ \int_{\mathbb{R}^N} \epsilon^{2s} (-\Delta)^{s/2} U_{\epsilon,y_{\epsilon}} (-\Delta)^{s/2} U_{\epsilon,y_{\epsilon}} \, dx = O(\epsilon^\gamma), \quad (4.16) \]
where \( q_1, q_2 > 0 \).

Now we can improve the estimate for the asymptotic behavior of \( y_{\epsilon} \) with respect to \( \epsilon \).

**Lemma 4.4** Assume that \( V \) satisfies (V1)–(V3). Let \( u_{\epsilon} = U_{\epsilon,y_{\epsilon}} + \varphi_{\epsilon} \) be a solution derived as in Theorem 2.1. Then
\[ |y_{\epsilon}| = o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \]

**Proof:** The proof can be found in [11, 29] for the case \( s = 1 \), we give the nonlocal version due to the presence of the nonlocal term \( \int_{\mathbb{R}^N} (-\Delta)^{s/2} u \, (-\Delta)^{s} u \) and the fractional operator. To analyze the asymptotic behavior of \( y_{\epsilon} \) with respect to \( \epsilon \), we apply the Pohozaev-type identity (4.1) to \( u = u_{\epsilon} \) with \( \Omega = B_d(y_{\epsilon}) \), where \( d \in (1,2) \) is chosen as in (4.11). Note that \( d \) is possibly dependent on \( \epsilon \). We get
\[ \int_{B_d(y_{\epsilon})} \frac{\partial V}{\partial x_i} (U_{\epsilon,y_{\epsilon}} + \varphi_{\epsilon})^2 \, dx = \sum_{i=1}^{3} I_i \quad (4.17) \]
with
\[ I_1 = \left( \epsilon^{2s} + \epsilon^{4s-N} \right) \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \varphi_{\epsilon} \right|^2 \, dx \int_{\partial B_d(y_{\epsilon})} \left| (-\Delta)^{s/2} \varphi_{\epsilon} \right|^2 \nu_i - 2 \frac{\partial u_{\epsilon}}{\partial \nu} \frac{\partial u}{\partial x_i} \, \sigma, \]
\[ I_2 = \int_{\partial B_d(y_{\epsilon})} V(x) u_{\epsilon}^2(x) \nu_i \, d\sigma, \]
and
\[ I_3 = -\frac{2}{p+1} \int_{\partial B_d(y_{\epsilon})} u_{\epsilon}^{p+1}(x) \nu_i \, d\sigma. \]
It follows from Theorem 2.1 that
\[ \epsilon^{2s} a + \epsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} \varphi_{\epsilon} \right|^2 \, dx = O(\epsilon^{2s}). \]

Thus, from (4.14) we deduce \( I_1 = O\left( \|\varphi_{\epsilon}\|_2^2 + \epsilon^\gamma \right) \). Using similar arguments and choosing a suitable \( d \) if necessary, we also get \( I_2 = O\left( \|\varphi_{\epsilon}\|_2^2 + \epsilon^\gamma \right) \). For \( I_3 \), by Lemma 4.3 we have
\[ I_3 \leq C \left( \int_{\partial B_d(y_{\epsilon})} |\varphi_{\epsilon}(x)|^{p+1} \, d\sigma + \epsilon^\gamma \right) \leq C \left( \int_{\mathbb{R}^N} |\varphi_{\epsilon}(x)|^{p+1} \, dx + \epsilon^\gamma \right) \leq C(\|\varphi_{\epsilon}\|_2^2 + \epsilon^\gamma). \]
Hence
\[ \sum_{i=1}^{3} I_i = O \left( (\| \varphi_\varepsilon \|_2^2 + \varepsilon^7) \right). \] (4.18)

To estimate the left hand side of (4.17), notice that from Lemma 4.2
\[
\int_{B_\delta(y_\varepsilon)} \frac{\partial V(x)}{\partial x_i} (U_{\varepsilon,y_\varepsilon} + \varphi_\varepsilon)^2 \, dx \\
= \int_{B_\delta(y_\varepsilon)} \frac{\partial V(x)}{\partial x_i} (U_{\varepsilon,y_\varepsilon}^2 + \varphi_\varepsilon^2) \, dx + 2 \int_{B_\delta(y_\varepsilon)} \frac{\partial V(x)}{\partial x_i} U_{\varepsilon,y_\varepsilon} \varphi_\varepsilon(x) \, dx \\
= \int_{B_\delta(y_\varepsilon)} \frac{\partial V(x)}{\partial x_i} U_{\varepsilon,y_\varepsilon}^2 \, dx + O \left( \| \varphi_\varepsilon \|_2^2 + \varepsilon^{N+2m-2} + \varepsilon^N |y_\varepsilon|^{2m-2} \right). \] (4.19)

By the assumption \((V_3)\), we deduce, for each \(i = 1, 2, \cdots, N\),
\[
\int_{B_\delta(y_\varepsilon)} \frac{\partial V}{\partial x_i} U_{\varepsilon,y_\varepsilon}^2 \, dx = m_c \int_{B_d(y_\varepsilon)} |x_i|^{m-2} x_i U_{\varepsilon,y_\varepsilon}^2 \, dx + O \left( \int_{B_d(y_\varepsilon)} |x|^m U_{\varepsilon,y_\varepsilon}^2 \right) \\
= m_c \varepsilon^N \int_{B_{\varepsilon^N}(0)} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) U^2(z) \, dz + O \left( \varepsilon^N (\varepsilon^m + |y_\varepsilon|^m) \right) \\
= m_c \varepsilon^N \int_{\mathbb{R}^N} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) U^2(z) \, dz + O \left( \varepsilon^N (\varepsilon^m + |y_\varepsilon|^m) \right).
\]

Which gives
\[
\int_{B_\delta(y_\varepsilon)} \frac{\partial V}{\partial x_i} (U_{\varepsilon,y_\varepsilon} + \varphi_\varepsilon)^2 \, dx = m_c \varepsilon^N \int_{\mathbb{R}^N} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) U^2 \, dx \\
+ O \left( \varepsilon^{\frac{N}{2}} \| \varphi_\varepsilon \|_2 + \| \varphi_\varepsilon \|_2^2 + \varepsilon^N (\varepsilon^m + |y_\varepsilon|^m) \right). \] (4.20)

Since \(c_i \neq 0\) by assumption \((V_3)\), combining (4.17)-(4.20) we deduce
\[
\varepsilon^N \int_{\mathbb{R}^N} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) U^2 \, dx = O \left( \varepsilon^{\frac{N}{2}} \| \varphi_\varepsilon \|_2 + \| \varphi_\varepsilon \|_2^2 + \varepsilon^N (\varepsilon^m + |y_\varepsilon|^m) \right).
\]

By Lemma 2.6 and \((V_3)\),
\[
\| \varphi_\varepsilon \|_2 = O \left( \varepsilon^{\frac{N}{2}} (\varepsilon^m + |y_\varepsilon|^{m(1-\tau)}) \right). \]

Thus,
\[
\int_{\mathbb{R}^N} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) U^2 \, dx = O \left( \varepsilon^{m-\tau} + |y_\varepsilon|^{m(1-\tau)} \right). \] (4.21)

On the other hand, let \(m^* = \min(m, 2)\). We have
\[
|y_{\varepsilon,i}|^m \leq |\varepsilon z_i + y_{\varepsilon,i}|^m - m |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) \varepsilon z_i \\
+ C \left( |\varepsilon z_i + y_{\varepsilon,i}|^{m-m^*} |\varepsilon z_i|^{m^*} + |\varepsilon z_i|^m \right) \\
\leq m |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) y_{\varepsilon,i} + C \left( |\varepsilon z_i|^m + |y_{\varepsilon,i}|^{m-m^*} |\varepsilon z_i|^m \right) \] (4.22)

by the following elementary inequality: for any \(e, f \in \mathbb{R}\) and \(m > 1\), there holds
\[
|e + f|^m - |e|^m - m |e|^{m-2} e f \leq C \left( |e|^{m-m^*} |f|^{m^*} + |f|^m \right)
\]
for some \(C > 0\) depending only on \(m\). So, multiplying (4.22) by \(U^2\) on both sides and integrate over \(\mathbb{R}^N\). We get
\[
|y_{\varepsilon,i}|^m \int_{\mathbb{R}^N} U^2 \, dx \leq m \int_{\mathbb{R}^N} |\varepsilon z_i + y_{\varepsilon,i}|^{m-2} (\varepsilon z_i + y_{\varepsilon,i}) y_{\varepsilon,i} U^2 \, dx + O \left( \varepsilon^m + |y_\varepsilon|^{m-m^*} \varepsilon^{m^*} \right)
\]
for each \(i\). Applying (4.21) to the above estimate yields
\[
|y_\varepsilon|^m = O \left( (\varepsilon^{m-\tau} + |y_\varepsilon|^{m(1-\tau)}) |y_\varepsilon| + \varepsilon^m + |y_\varepsilon|^{m-m^*} \varepsilon^{m^*} \right).
\]
Recall that $m\tau < 1$. Using $\varepsilon$-Young inequality
\[
XY \leq \delta X^m + \frac{m}{m-1} Y^{m-1}, \quad \forall \delta, X, Y > 0
\]
we deduce
\[
|y_{\varepsilon}| = O(\varepsilon).
\]

We have to prove that $|y_{\varepsilon}| = o(\varepsilon)$. Assume, on the contrary, that there exist $\varepsilon_k \to 0$ and $y_{\varepsilon_k} \to 0$ such that $y_{\varepsilon_k}/\varepsilon_k \to A \in \mathbb{R}^N$ with $A = (A_1, A_2, \ldots, A_N) \neq 0$. Then (4.21) gives
\[
\int_{\mathbb{R}^N} \left| z_i + \frac{y_{\varepsilon_k}}{\varepsilon_k} \right|^{m-2} \left( z_i + \frac{y_{\varepsilon_k}}{\varepsilon_k} \right) U^2 dx = O(\varepsilon^{m-\tau}).
\]
Taking limit in the above gives
\[
\int_{\mathbb{R}^N} \left| z_i + A_i \right|^{m-2} \left( z_i + A_i \right) U^2(z) dz = 0.
\]
However, since $U = U(|z|)$ is strictly decreasing with respect to $|z|$, we infer that $A = 0$. We reach a contradiction. The proof is complete.

As a consequence of Lemma 4.4 and the assumption $(V_3)$, we infer that
\[
\|\varphi_{\varepsilon}\|_{\varepsilon} = O\left(\varepsilon^{\frac{N}{2} + m(1-\tau)}\right).
\]
Here we can take $\tau$ so small that $m(1-\tau) > 1$ since $m > 1$.

5  Uniqueness of semiclassical bounded states

In this section we prove the local uniqueness of semiclassical bounded states obtained before. We use a contradiction argument as that of [11, 29]. Assume $u_{\varepsilon}^{(1)} = U_{\varepsilon} z(\varepsilon y_{\varepsilon}) + \varphi_{\varepsilon}^{(1)}, i = 1, 2$, are two distinct solutions derived as in Section 2. By the argument in Section 3, $u_{\varepsilon}^{(1)}$ are bounded functions in $\mathbb{R}^N, i = 1, 2$. Set
\[
\xi_{\varepsilon} = \frac{u_{\varepsilon}^{(1)} - u_{\varepsilon}^{(2)}}{\|u_{\varepsilon}^{(1)} - u_{\varepsilon}^{(2)}\|_{L^\infty(\mathbb{R}^N)}}
\]
and set
\[
\tilde{\xi}_{\varepsilon}(x) = \xi_{\varepsilon} \left( \varepsilon x + y_{\varepsilon}^{(1)} \right).
\]
It is clear that
\[
\|\tilde{\xi}_{\varepsilon}\|_{L^\infty(\mathbb{R}^N)} = 1.
\]
Moreover, by the Claim 3 in Section 3, there holds
\[
\tilde{\xi}_{\varepsilon}(x) \to 0 \quad \text{as } |x| \to \infty 
\]
uniformly with respect to sufficiently small $\varepsilon > 0$. We will reach a contradiction by showing that $\|\tilde{\xi}_{\varepsilon}\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $\varepsilon \to 0$. In view of (5.1), it suffices to show that for any fixed $R > 0$,
\[
\|\tilde{\xi}_{\varepsilon}\|_{L^\infty(B_R(0))} \to 0 \quad \text{as } \varepsilon \to 0.
\]
First we have

Lemma 5.1  There holds
\[
\|\tilde{\xi}_{\varepsilon}\|_{\varepsilon} = O\left(\varepsilon^{\frac{N}{2}}\right).
\]
Proof: Recall the inner product on $H^s(\mathbb{R}^N)$, we can compute that
\[
\int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} u|^2 - |(\Delta)^{\frac{s}{2}} v|^2 \, dx
= \int_{\mathbb{R}^N} ((\Delta)^{\frac{s}{2}} u + (\Delta)^{\frac{s}{2}} v) ((\Delta)^{\frac{s}{2}} u - (\Delta)^{\frac{s}{2}} v) \, dx
= \int_{\mathbb{R}^N} (\Delta)^{\frac{s}{2}} (u + v)(-\Delta)^{\frac{s}{2}} (u - v) \, dx.
\]

Then assume that $u_i^{(i)}$, $i = 1, 2$, are two solutions to (1.1), we obtain that
\[
\left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} u_i^{(1)}|^2 \, dx \right) (\Delta)^{s} \xi + V \xi
= \varepsilon^{4s-N} b \int_{\mathbb{R}^N} (\Delta)^{\frac{s}{2}} (u_i^{(1)} + u_i^{(2)})(-\Delta)^{\frac{s}{2}} \xi \, dx
+ C_\epsilon(x) \xi
\]
and that
\[
\left( \varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} u_i^{(2)}|^2 \, dx \right) (\Delta)^{s} \xi + V \xi
= \varepsilon^{4s-N} b \int_{\mathbb{R}^N} (\Delta)^{\frac{s}{2}} (u_i^{(1)} + u_i^{(2)})(-\Delta)^{\frac{s}{2}} \xi \, dx
+ C_\epsilon(x) \xi
\]
where
\[C_\epsilon(x) = p \int_0^1 \left( tu_i^{(1)}(x) + (1-t)u_i^{(2)}(x) \right)^{p-1} \, dt.
\]
Adding (5.3) and (5.4) together gives
\[
\left( 2\varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} u_i^{(1)}|^2 + |(\Delta)^{\frac{s}{2}} u_i^{(2)}|^2 \, dx \right) (\Delta)^{s} \xi + 2V \xi
= \varepsilon^{4s-N} b \int_{\mathbb{R}^N} (\Delta)^{\frac{s}{2}} (u_i^{(1)} + u_i^{(2)})(-\Delta)^{\frac{s}{2}} \xi \, dx
+ (\Delta)^{s} \left( u_i^{(1)} + u_i^{(2)} \right) + 2C_\epsilon(x) \xi.
\]

Multiply $\xi$ on both sides of (5.5) and integrate over $\mathbb{R}^N$. By throwing away the terms containing $b$, we get
\[
\| \xi \|^2 \leq \int_{\mathbb{R}^N} C_\epsilon \xi^2 \, dx.
\]
On the other hand, letting $\varphi_i^{(i)}$ be the small perturbation term corresponding to $u_i^{(i)}$, then we have
\[
|C_\epsilon(x)| \leq C \left( U_{\varepsilon N}^{p-1} + U_{\varepsilon N}^{p-1} \varphi_i^{(i)}(x) \right)^{p-1} + \left( \varphi_i^{(2)}(x) \right)^{p-1}.
\]
Since $|\xi(x)| \leq 1$, for $i = 1, 2$, we have
\[
\int_{\mathbb{R}^N} U_{\varepsilon N}^{p-1} \varphi_i^{(i)}(x) \, dx \leq C \varepsilon^N,
\]
and
\[
\int_{\mathbb{R}^N} \varphi_i^{(i)}(x)^{p-1} \xi \, dx \leq C \left( \int_{\mathbb{R}^N} (\varphi_i^{(i)})^{2s} \right)^{\frac{2s}{2s+1}} \left( \int_{\mathbb{R}^N} (\xi^2)^{\frac{2s+1-p}{2s}} \right)^{\frac{2s}{2s+1-p}} \, dx
\leq C \sum_{i=1}^2 \left( |(\Delta)^{\frac{s}{2}} \varphi_i^{(i)}|_{L^2(\mathbb{R}^N)} \right)^{p-1} \left( \int_{\mathbb{R}^N} (\xi^2)^{\frac{2s+1-p}{2s}} \right)^{\frac{2s}{2s+1-p}} \, dx
= O \left( \varepsilon^{\frac{p-1}{2s}} \| \xi \|_{L^2} \right)^{\frac{2s}{2s+1-p}}.
\]
In the last inequality we have used the fact that $\| \varphi_i^{(i)} \|_{L^2} = O(\varepsilon^{\frac{s}{2s}})$. Therefore,
\[
\| \xi \|_2^2 = O \left( \varepsilon^{N} + \varepsilon^{\frac{p-1}{2s}} \| \xi \|_{L^2} \right)^{\frac{2s}{2s+1-p}},
\]
which implies the desired estimate. The proof is complete.

Next we study the asymptotic behavior of $\xi$.
Lemma 5.2 Let $\xi_\epsilon = \xi_\epsilon (\varepsilon x + y_\epsilon^{(1)})$. There exist $d_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, such that (up to a subsequence)
\[
\xi_\epsilon \to \sum_{i=1}^{N} d_i \partial_x U \quad \text{in } C^1_{\text{loc}} (\mathbb{R}^N)
\]
as $\varepsilon \to 0$.

**Proof:** It is straightforward to deduce from (5.3) that $\xi_\epsilon$ solves
\[
(a + \varepsilon^{-4s}b \int_{\mathbb{R}^N} |(-\Delta)^{s} u_\epsilon^{(1)}|^2 \, dx) (-\Delta)^s \xi_\epsilon + V(\varepsilon x + y_\epsilon^{(1)}) \xi_\epsilon
\]
\[
= \varepsilon^{-4s}b \left( \int_{\mathbb{R}^N} (-\Delta)^s (u_\epsilon^{(1)} + u_\epsilon^{(2)})(-\Delta)^s \xi_\epsilon \, dx \right) (-\Delta)^s \left( u_\epsilon^{(2)} \left( \varepsilon x + y_\epsilon^{(1)} \right) \right) + C_{\varepsilon} \left( \varepsilon x + y_\epsilon^{(1)} \right) \xi_\epsilon.
\]

(5.6)

For convenience, we introduce
\[
u_\epsilon^{(i)}(x) = u_\epsilon^{(i)} (\varepsilon x + y_\epsilon^{(1)}) \quad \text{and} \quad \varphi_\epsilon^{(i)} = \varphi_\epsilon^{(i)} (\varepsilon x + y_\epsilon^{(1)})
\]
for $i = 1, 2$. Then, we have
\[
\varepsilon^{-4s} \int_{\mathbb{R}^N} |(-\Delta)^s u_\epsilon^{(1)}|^2 \, dx = \int_{\mathbb{R}^N} |(-\Delta)^s \nu_\epsilon^{(1)}|^2 \, dx
\]
and
\[
\varepsilon^{-4s}b \left( \int_{\mathbb{R}^N} |(-\Delta)^s \nu_\epsilon^{(1)}|^2 + |(-\Delta)^s \nu_\epsilon^{(2)}|^2 \, dx \right) = b \left( \int_{\mathbb{R}^N} |(-\Delta)^s \nu_\epsilon^{(1)}|^2 + |(-\Delta)^s \nu_\epsilon^{(2)}|^2 \, dx \right)
\]
which are uniformly bounded. Moreover, we have
\[
\int_{\mathbb{R}^N} |(-\Delta)^s \varphi_\epsilon^{(1)}|^2 \, dx = \varepsilon^{-N} O \left( \left\| \varphi_\epsilon^{(1)} \right\|_{L^2(\mathbb{R}^N)}^2 \right) = O \left( \varepsilon^{2m(1-\tau)} \right)
\]
by (4.23), and
\[
\int_{\mathbb{R}^N} |(-\Delta)^s \xi_\epsilon|^2 \, dx = \varepsilon^{-4s} \int_{\mathbb{R}^N} |(-\Delta)^s \xi_\epsilon|^2 \, dx = O(1)
\]
by Lemma 5.1.

Thus, in view of $\left\| \xi_\epsilon \right\|_{L^\infty(\mathbb{R}^N)} = 1$ and estimates in the above, the elliptic regularity theory implies that $\xi_\epsilon$ is locally uniformly bounded with respect to $\varepsilon$ in $C^1_{\text{loc}} (\mathbb{R}^N)$ for some $\beta \in (0, 1)$. As a consequence, we assume (up to a subsequence) that
\[
\xi_\epsilon \to \bar{\xi} \quad \text{in } C^1_{\text{loc}} (\mathbb{R}^N)
\]
We claim that $\bar{\xi} \in \text{Ker} \mathcal{L}$, that is,
\[
(a + b \int_{\mathbb{R}^N} |(-\Delta)^s U|^2 \, dx) (-\Delta)^s \bar{\xi} + 2b \left( \int_{\mathbb{R}^N} (-\Delta)^s U \cdot (-\Delta)^s \bar{\xi} \, dx \right) (-\Delta)^s U + \bar{\xi} = p U^{p-1} \bar{\xi},
\]
which can be seen as the limiting equation of (5.6). It follows from (5.7) and (5.9) that
\[
\varepsilon^{-4s}b \int_{\mathbb{R}^N} |(-\Delta)^s u_\epsilon^{(1)}|^2 \, dx - b \int_{\mathbb{R}^N} |(-\Delta)^s U|^2 \, dx = b \int_{\mathbb{R}^N} \left( |(-\Delta)^s u_\epsilon^{(1)}|^2 - |(-\Delta)^s U|^2 \right) \, dx
\]
\[
= b \int_{\mathbb{R}^N} \left( |(-\Delta)^s u + (-\Delta)^s \varphi_\epsilon^{(1)}|^2 - |(-\Delta)^s U|^2 \right) \, dx
\]
\[
= O \left( \varepsilon^{m(1-\tau)} \right)
\]
\[
\to 0.
\]
(5.12)
Similarly, we deduce from (5.8)-(5.10) that
\[
\int_{\mathbb{R}^N} (-\Delta)^s \left( \bar{u}^{(1)} \xi + \bar{u}^{(2)} \right) \cdot (-\Delta)^s \xi dx = \int_{\mathbb{R}^N} (-\Delta)^s \left( U \left( x + \left( y^{(1)} - y^{(2)} \right) / \varepsilon \right) - U \right) \cdot (-\Delta)^s \xi dx \\
+ \int_{\mathbb{R}^N} (-\Delta)^s \left( \varphi^{(1)} + \varphi^{(2)} \right) \cdot (-\Delta)^s \xi dx \\
\to 0.
\]

It follows from Lemma 4.4 that, for any \( \Phi \in C_0^\infty (\mathbb{R}^N) \),
\[
\int_{\mathbb{R}^N} (-\Delta)^s \left( \bar{u}^{(2)} - U \right) \cdot (-\Delta)^s \Phi dx = \int_{\mathbb{R}^N} (-\Delta)^s \left( U \left( x + \left( y^{(1)} - y^{(2)} \right) / \varepsilon \right) - U \right) \cdot (-\Delta)^s \Phi dx \\
+ \int_{\mathbb{R}^N} (-\Delta)^s \varphi^{(2)} \cdot (-\Delta)^s \Phi dx \\
\to 0.
\]

Combining the above formulas and \( \xi \to \xi \) in \( C_0^1 \text{loc} \ (\mathbb{R}^N) \), we conclude that
\[
\frac{b}{\varepsilon} \left( \int_{\mathbb{R}^N} (-\Delta)^s \left( u^{(1)} + u^{(2)} \right) \cdot (-\Delta)^s \xi dx \right) (-\Delta)^s \left( \varepsilon x + y^{(1)} \right) \\
\to 2b \left( \int_{\mathbb{R}^N} (-\Delta)^s U \cdot (-\Delta)^s \xi dx \right) (-\Delta)^s U
\]
in \( H^{-s} (\mathbb{R}^N) \).

Now, we estimate \( C_\varepsilon \left( \varepsilon x + y^{(1)} \right) \)
\[
U \left( \frac{x - y^{(1)}}{\varepsilon} \right) - U \left( \frac{x - y^{(2)}}{\varepsilon} \right) = O \left( \frac{y^{(1)} - y^{(2)}}{\varepsilon} \nabla U \left( \frac{x - y^{(1)} + \theta \left( y^{(1)} - y^{(2)} \right)}{\varepsilon} \right) \right),
\]
where \( 0 < \theta < 1 \). Then
\[
u^{(1)}_\varepsilon(x) - u^{(2)}_\varepsilon(x) = o(1) \nabla U \left( \frac{x - y^{(1)} + \theta \left( y^{(1)} - y^{(2)} \right)}{\varepsilon} \right) + O \left( \left| \varphi^{(1)}_\varepsilon(x) \right| + \left| \varphi^{(2)}_\varepsilon(x) \right| \right).
\]
So, from (4.12), for any \( \gamma > 0 \), we have
\[
C_\varepsilon (x) = o(1) \nabla U \left( \frac{x - y^{(1)} + \theta \left( y^{(1)} - y^{(2)} \right)}{\varepsilon} \right) + O \left( \left| \varphi^{(1)}_\varepsilon(x) \right| + \left| \varphi^{(2)}_\varepsilon(x) \right| \right)^{p-1} \\
+ pU^{p-1} \left( \frac{x - y^{(1)}}{\varepsilon} \right) + o(\varepsilon^\gamma), \quad x \in B_d(y^{(1)}).
\]
Then using Lemma 4.4, we can obtain
\[
C_\varepsilon (\varepsilon x + y^{(1)}_\varepsilon) = pU^{p-1}(x) + O \left( \left| \varphi^{(1)}_\varepsilon \left( \varepsilon x + y^{(1)}_\varepsilon \right) \right| + \left| \varphi^{(2)}_\varepsilon \left( \varepsilon x + y^{(1)}_\varepsilon \right) \right| \right)^{p-2} + o(1), \quad x \in B_d(0).
\]

Also, for \( i = 1, 2 \), we know
\[
\int_{\mathbb{R}^N} \left| \varphi^{(i)}_\varepsilon \left( \varepsilon x + y^{(1)}_\varepsilon \right) \right|^p \left| \Phi(x) \right| dx \leq C \left( \int_{\mathbb{R}^N} \left| \varphi^{(i)}_\varepsilon \left( \varepsilon x + y^{(1)}_\varepsilon \right) \right|^{2^*_p} dx \right)^{\frac{p-1}{2^*_p}} \left\| \Phi \right\|_{2^*_p + p} \\
\leq C \left( 2^*_p \left\| \varphi^{(i)}_\varepsilon \right\|_{\varepsilon} \right)^{\frac{p-1}{2^*_p}} \left\| \Phi \right\|_{2^*_p + p} \\
\leq C \left( 2^*_p \left\| \varphi^{(i)}_\varepsilon \right\|_{\varepsilon} \right)^{\frac{p-1}{2^*_p}} \left\| \Phi \right\|_{H^1(\mathbb{R}^N)}
\]
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Then for any $\Phi \in C_0^\infty(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} C_\varepsilon \left( \varepsilon x + y^{(1)} \right) \tilde{\xi}_\varepsilon \Phi - p \int_{\mathbb{R}^N} U^{p-1} \tilde{\xi}_\varepsilon \Phi = o(1) \|\Phi\|_{L^\infty(\mathbb{R}^N)}.
\]

Therefore, we obtain (5.11). Then $\tilde{\xi} = \sum_{i=1}^N d_i \partial_{x_i} U$ follows from Proposition 1.2 for some $d_i \in \mathbb{R}(i = 1, 2, \cdots , N)$, and thus the Lemma is proved.

Now we prove (5.2) by showing the following lemma.

**Lemma 5.3** Let $d_i$ be defined as in Lemma 5.2. Then
\[
d_i = 0 \quad \text{for } i = 1, 2, \cdots , N.
\]

**Proof:** We use the Pohozaev-type identity (4.1) to prove this lemma. Apply (4.1) to $u^{(1)}_\varepsilon$ and $u^{(2)}_\varepsilon$ with $\Omega = B_d \left( y^{(1)} \right)$, where $d$ is chosen in the same way as that of Lemma 5.2. We obtain
\[
\int_{B_d(y^{(1)})} \frac{\partial V}{\partial x_i} \left( (u^{(1)}_\varepsilon)^2 - (u^{(2)}_\varepsilon)^2 \right) dx = \int_{\partial B_d(y^{(1)})} \left( (\Delta) \tilde{\xi}_\varepsilon \right) \left( (\Delta) \tilde{u}^{(1)}_\varepsilon \right) \nu_i - 2 \int_{\partial B_d(y^{(1)})} \left( (\Delta) \tilde{u}^{(1)}_\varepsilon \right) \frac{\partial u^{(1)}_\varepsilon}{\partial x_i} \frac{\partial u^{(1)}_\varepsilon}{\partial x_j} d\sigma,
\]

In terms of $\xi_\varepsilon$, we get
\[
\int_{B_d(y^{(1)})} \frac{\partial V}{\partial x_i} \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_\varepsilon dx = \sum_{i=1}^4 I_i
\]
with
\[
I_1 = \left( \varepsilon^2 a + \varepsilon^{4s-N} b \right) \int_{\mathbb{R}^N} \left| (\Delta) \tilde{\xi}_\varepsilon \right| dx \int_{\partial B_d(y^{(1)})} \left( (\Delta) \tilde{u}^{(1)}_\varepsilon \right) \left( (\Delta) \tilde{u}^{(1)}_\varepsilon \right) \nu_i d\sigma,
\]

\[
I_2 = -2 \left( \varepsilon^2 a + \varepsilon^{4s-N} b \right) \int_{\mathbb{R}^N} \left| (\Delta) \tilde{\xi}_\varepsilon \right| dx \int_{\partial B_d(y^{(1)})} \left( \frac{\partial u^{(1)}_\varepsilon}{\partial x_i} \frac{\partial u^{(1)}_\varepsilon}{\partial x_j} - \frac{\partial u^{(2)}_\varepsilon}{\partial x_i} \frac{\partial u^{(2)}_\varepsilon}{\partial x_j} \right) d\sigma,
\]

\[
I_3 = \varepsilon^{4s-N} b \int_{\partial B_d(y^{(1)})} \left( (\Delta) \tilde{u}^{(2)}_\varepsilon \right)^2 dx \nu_i - 2 \int_{\partial B_d(y^{(1)})} \left( \frac{\partial u^{(2)}_\varepsilon}{\partial x_i} \frac{\partial u^{(2)}_\varepsilon}{\partial x_j} \right) d\sigma \int_{\mathbb{R}^N} \left( (\Delta) \tilde{u}^{(1)}_\varepsilon \right) \left( (\Delta) \tilde{u}^{(2)}_\varepsilon \right) \nu_i d\sigma,
\]

\[
I_4 = \int_{\partial B_d(y^{(1)})} V \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_\varepsilon \nu_i d\sigma - 2 \int_{\partial B_d(y^{(1)})} A_i \xi_\varepsilon \nu_i d\sigma,
\]
where $A_i(x) = \int_0^1 \left( t u^{(1)}_\varepsilon(x) + (1-t) u^{(2)}_\varepsilon(x) \right) dt$.

We estimate (5.14) term by term. Note that
\[
\varepsilon^2 a + \varepsilon^{4s-N} b \int_{\mathbb{R}^N} \left| (\Delta) \tilde{\xi}_\varepsilon \right| dx = O \left( \varepsilon^{2s} \right)
\]
for each $i = 1, 2$. Moreover, by similar arguments as that of Lemma 4.4, we have
\[
\int_{\partial B_d(y^{(1)})} \left( (\Delta) \tilde{u}^{(2)}_\varepsilon \right)^2 d\sigma = O \left( \left\| (\Delta) \tilde{u}^{(2)}_\varepsilon \right\|_{L^2(\mathbb{R}^N)}^2 \right).
\]
Thus, by (5.2) and Lemma 4.2-4.4, we deduce
\[
\sum_{i=1}^{3} I_i = \sum_{i=1}^{2} O \left( \varepsilon^2 \left\| (-\Delta)^{\frac{1}{2}} \varphi_{\varepsilon}^{(i)} \right\|_{L^2(\Omega)} \right) = O \left( \varepsilon^{N+2m(1-\tau)} \right)
\]
and
\[
\int_{\partial B_{\varepsilon}(y_{\varepsilon}^{(i)})} V(x) \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon} \nu_{\varepsilon} d\sigma = O \left( \varepsilon^{N+m(1-\tau)} \right)
\]
and
\[
\int_{\partial B_{\varepsilon}(y_{\varepsilon}^{(i)})} A_{\varepsilon} \xi_{\varepsilon} \nu_{\varepsilon} d\sigma = O \left( \varepsilon^{N+m(1-\tau)} \right).
\]
Hence we conclude that
\[
\text{the RHS of (5.14) } = O \left( \varepsilon^{N+m(1-\tau)} \right). \tag{5.15}
\]

Next we estimate the left hand side of (5.14). We have
\[
\int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} \frac{\partial V}{\partial x_i} \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon}(x) dx
\]
\[
= mc_{i} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^{m-2} x_i \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon}(x) dx + O \left( \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^m \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon}(x) dx \right).
\]
Observe that
\[
mc_{i} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^{m-2} x_i \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon} dx
\]
\[
= mc_{i} \varepsilon^{N} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |\varepsilon y_i + y_{\varepsilon}^{(1)}|^{m-2} \left( \varepsilon y_i + y_{\varepsilon}^{(1)} \right) \left( U(y) + U \left( y + \frac{y_{\varepsilon}^{(1)} - x_{\varepsilon}^{(1)}}{\varepsilon} \right) \right) \xi_{\varepsilon} dx
\]
\[
+ mc_{i} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^{m-2} x_i \left( \varphi_{\varepsilon}^{(1)} + \varphi_{\varepsilon}^{(2)} \right) \xi_{\varepsilon} dx.
\]
Since \( U \) decays at infinity (see Section 3) and \( y_{\varepsilon}^{(i)} = o(\varepsilon) \), using Lemma 5.2 we deduce
\[
mc_{i} \varepsilon^{N} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |\varepsilon y_i + y_{\varepsilon}^{(1)}|^{m-2} \left( \varepsilon y_i + y_{\varepsilon}^{(1)} \right) \left( U(y) + U \left( y + \frac{y_{\varepsilon}^{(1)} - y_{\varepsilon}^{(2)}}{\varepsilon} \right) \right) \xi_{\varepsilon} dx
\]
\[
= 2mc_{i} \varepsilon^{m+2} \sum_{j=1}^{N} d_j \int_{\mathbb{R}^N} |y_i|^{m-2} y_i U(y) \partial_{x_j} U dx + o(\varepsilon^{m+2}) \tag{5.16}
\]
\[
= D_i d_{i} \varepsilon^{m+2} + o(\varepsilon^{m+2})
\]
where
\[
D_i = 2mc_{i} \int_{\mathbb{R}^N} |y_i|^{m-2} y_i U(y) \partial_{x_j} U \neq 0.
\]
On the other hand, by Hölder’s inequality, (4.23) and Lemma 5.1, we have
\[
mc_{i} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^{m-2} x_i \left( \varphi_{\varepsilon}^{(1)} + \varphi_{\varepsilon}^{(2)} \right) \xi_{\varepsilon} dx = \sum_{i=1}^{2} O \left( \int_{\mathbb{R}^N} \left\| \varphi_{\varepsilon}^{(i)} \right\|_{L^2(\Omega)} \right)
\]
\[
= \sum_{i=1}^{2} O \left( \left\| \varphi_{\varepsilon}^{(i)} \right\|_{L^2(\Omega)} \right)
\]
\[
= O \left( \varepsilon^{N+m(1-\tau)} \right).
\] (5.17)

Therefore, from (5.16) and (5.17), we deduce
\[
mc_{i} \int_{B_{\varepsilon}(y_{\varepsilon}^{(i)})} |x_i|^{m-2} x_i \left( u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)} \right) \xi_{\varepsilon} dx = D_i d_{i} \varepsilon^{N+m-1} + o(\varepsilon^{N+m-1}). \tag{5.18}
\]
Similar arguments give
\[
O \left( \int_{B_A(y_i^{(1)})} |x|^{m} \left( u^{(1)}_\varepsilon + u^{(2)}_\varepsilon \right) \xi_\varepsilon dx \right) = O \left( \varepsilon^{N+m} \right).
\] (5.19)

Hence, combining (5.18) and (5.19), we obtain
\[
\text{the LHS of (5.14) = } D_i \varepsilon^{N+m-1} + o \left( \varepsilon^{N+m-1} \right).
\] (5.20)

So (5.15) and (5.20) imply that \( d_i = 0 \).

The proof is complete.  

**Proof of Theorem 1.1(ii):** If there exist two distinct solutions \( u^{(i)}_\varepsilon, i = 1, 2 \), then by setting \( \xi_\varepsilon \) and \( \bar{\xi}_\varepsilon \) as above, we find that
\[
\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 1
\]
by assumption, and that
\[
\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = o(1) \quad \text{as } \varepsilon \to 0
\]
by (5.1) and (5.2). We reach a contradiction. The proof is complete.

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