On the decay rate for degenerate gradient flows subject to persistent excitation

Dario Prandi
Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et des systèmes

Joint work with Yacine Chitour and Paolo Mason

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Degenerate gradient flows

Consider

\[ \dot{x}(t) = -c(t)c(t)^\top x(t), \quad x \in \mathbb{R}^n, \quad c : [0, +\infty) \to \mathbb{R}^n. \quad \text{(DGF)} \]

These systems appear in algorithms for, e.g.,

1. Gradient descent with incomplete knowledge of the gradient
2. Identification and model reference adaptive control
3. Consensus in multi-agent systems

**Objective**

Guarantee convergence and stability of (DGF) at the origin, and extract information on the decay rate.
Motivation: Adaptive filters

Consider the scalar output system

\[ z(t) = h^\top c(t). \]

**Problem**

Estimate the parameter \( h \in \mathbb{R}^n \), knowing the input \( c : \mathbb{R}_+ \to \mathbb{R}^n \) and the output \( z : [0, +\infty) \to \mathbb{R} \).

Given an estimate \( \hat{h} : [0, +\infty) \to \mathbb{R}^n \), we let \( \hat{z}(t) = \hat{h}(t)^\top c(t) \). Then let

\[ \frac{d}{dt} \hat{h}(t) = (z(t) - \hat{z}(t))c(t). \]

Then, the misalignement vector \( x(t) = h - \hat{h}(t) \) satisfies (DGF):

\[ \dot{x}(t) = - (z(t) - \hat{z}(t)) c(t) = - (x(t)^\top c(t)) c(t) \]
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\]

Then, the misalignment vector \( x(t) = h - \hat{h}(t) \) satisfies (DGF):
\[
\dot{x}(t) = - (z(t) - \hat{z}(t))c(t) = - (x(t)^\top c(t)) c(t)
\]

Convergence to 0 of (DGF) \( \iff \) Quality of the estimator \( \hat{h} \)
Persistent excitation

We say that \( c \) verifies the *persistent excitation* condition if there exists \( a, b, T > 0 \) such that

\[
\forall t \geq 0, \quad a \mathbf{1} \mathbb{R}^n \leq \int_t^{t+T} c(s)c(s)\top ds \leq b \mathbf{1} \mathbb{R}^n.
\]  

(PE)

**Theorem (cf., Anderson, Narendra, et al.)**

A signal \( c \) verifies (PE) if and only if (DGF) is uniformly globally exponentially stable at 0. That is, there exist \( C, \alpha > 0 \) such that

\[
\|x(t)\| \leq Ce^{-\alpha(t-s)}\|x(s)\|, \quad \forall t > s \geq 0.
\]

- (PE) says that \( c \) “visits all directions of \( \mathbb{R}^n \) during a time window”.
- The upper bound \( b \) is essential. Indeed, by Barabanov et al. (2005), if \( b = +\infty \) it can happen that

\[
x(t) \rightarrow \bar{x} \neq 0 \quad \text{as } t \rightarrow +\infty
\]
Under (PE), the system $\dot{x} = -cc^T x$ is **globally exponentially stable**:

$$\|x(t)\| \leq Ce^{-\alpha t}\|x(0)\|, \quad \forall t \geq 0.$$  \hspace{1cm} (GES)

The **decay rate** for (DGF) is

$$R(c) := \sup\{\alpha > 0 \mid \text{(GES) holds}\} = -\limsup_{t \to +\infty} \frac{\log \|\Phi_c(t, 0)\|}{t},$$

where $\Phi_c(t, 0)$ is the fundamental matrix of (DGF) from 0 to $t$.

**Definition**

The **worst decay rate** is

$$R(a, b, T, n) := \inf\{R(c) \mid c \text{ satisfies (PE) with parameters } a, b, T\}.$$  

$\rightsquigarrow$ **Yields the guaranteed** convergence rate of the system.
Main result

Many lower bounds for $R(a, b, T, n)$ exist in the literature, of the type:

**Theorem (cf., Andersson and Krishnaprasad (2002))**

There exists $C_1 > 0$ such that

$$R(a, b, T, n) \geq \frac{C_1 a}{(1 + nb^2)T}, \quad \forall T > 0, \ a < b, \ n \in \mathbb{N}.$$ 

**Problem:** Are these bounds optimal?
Main result

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**Problem:** Are these bounds optimal?

**Theorem (Chitour-Mason-Prandi)**

There exists $C_0 > 0$ such that

$$R(a, b, T, n) \leq \frac{C_0 a}{(1 + b^2)T}, \quad \forall T > 0, \, a < b, \, n \in \mathbb{N}.$$  

$\Rightarrow$ We recover the result by Barabanov et al. (2005)
Application I: Generalized persistent excitation

More general condition considered in Barabanov and Ortega (2017), Praly (2017), Efimov et al. (2018):

\[ a_\ell \text{Id}_n \leq \int_{\tau_\ell}^{\tau_{\ell + 1}} c(s)c(s)^T \, ds \leq b_\ell \text{Id}_n, \quad \forall \ell \in \mathbb{N} \tag{GPE} \]

where \( a_\ell, b_\ell > 0 \), and \( (\tau_\ell)_{\ell \in \mathbb{N}} \) is strictly increasing with \( \tau_\ell \to +\infty \).

**Theorem (Praly (2017))**

*Condition (GPE) guarantees global asymptotic stability of (DGF) if*

\[ \sum_{\ell=1}^{\infty} \frac{a_\ell}{(1 + b_\ell)^2} = +\infty. \tag{2} \]
Application I: Generalized persistent excitation

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Theorem (Chitour-Mason-Prandi)

*For every sequence \((a_\ell)_{\ell \in \mathbb{N}}, (b_\ell)_{\ell \in \mathbb{N}} \subset (0, +\infty)\) not satisfying (2), there exists a signal \( c \) satisfying (GPE) for which (DGF) is not globally asymptotically stable.*
Application II: $L^2$-gain for (DGF) systems with linear input

Consider the controlled (DGF) system:

$$\dot{x}(t) = -c(t)c(t)^{\top}x(t) + u(t), \quad u \in L^2([0, +\infty), \mathbb{R}^n).$$

Let $\gamma(c)$ be the $L^2$-gain of the input/output map $u \mapsto x$:

$$\gamma(c) = \sup_{u \in L^2 \setminus \{0\}} \frac{\|x_u\|_2}{\|u\|_2}$$

Rantzer (1999) posed the problem of determining the worst $L^2$ gain:

$$\gamma(a, b, T, n) = \sup\{\gamma(c) \mid c \text{ satisfies (PE)}\}.$$
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Rantzer (1999) posed the problem of determining the worst \( L^2 \) gain:
\[ \gamma(a, b, T, n) = \sup\{ \gamma(c) \mid c \text{ satisfies (PE)} \}. \]

**Theorem (Chitour-Mason-Prandi)**

There exists \( \kappa_0, \kappa_1 > 0 \) such that for all \( T > 0, a \leq b, n \geq 2 \), it holds
\[ \kappa_0 \frac{(1 + b^2)T}{a} \leq \gamma(a, b, T, n) \leq \kappa_1 \frac{(1 + nb^2)T}{a}. \]
### Idea

Connect $R(a, b, T, n) = \inf R(c)$ with an optimal control problem.
Sketch of the proof

Idea

Connect $R(a, b, T, n) = \inf R(c)$ with an optimal control problem.

Recall that

$$R(c) = -\limsup_{t \to +\infty} \frac{1}{t} \sup \left\{ \log \frac{\|x(t)\|}{\|x(0)\|} \mid x(0) \in \mathbb{R}^n \right\}$$

• The dynamics of $\omega$ are independent of $r$.
• The dynamics of $r$ yield:

$$-\log \frac{\|x(T)\|}{\|x(0)\|} = -\log r(T) r(0) = \int_0^T (c^\top \omega)^2 \, ds =: J_T(c, \omega_0).$$
Sketch of the proof

**Idea**

Connect \( R(a, b, T, n) = \inf R(c) \) with an optimal control problem.

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Polar coordinates: Letting $x = r\omega$ for $r > 0$ and $\omega \in \mathbb{S}^{n-1}$, (DGF) reads

$$\begin{aligned}
\dot{r} &= -(c^T \omega)^2 r, \\
\dot{\omega} &= -c^T \omega (c - (c^T \omega)\omega), \quad r_0 = \|x(0)\|, \quad \omega_0 = \frac{x(0)}{\|x(0)\|}.
\end{aligned}$$

- The dynamics of $\omega$ are independent of $r$.
- The dynamics of $r$ yield:

$$-\log \frac{\|x(T)\|}{\|x(0)\|} = -\log \frac{r(T)}{r(0)} = \int_0^T (c^T \omega)^2 \, ds =: J_T(c, \omega_0).$$
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Optimal control problem:

\[
\mu(a, b, T, n) := \min J_T(c, \omega_0) = \min \int_0^T (c^T \omega)^2 \, ds
\]

Here, \( c : [0, T] \to \mathbb{R}^n \) runs over all signals satisfying

\[
a \mathbf{1}d_n \leq \int_0^T c(s)c(s)^T \, ds \leq b \mathbf{1}d_n,
\]

and \( \omega \) is a solution to (Pol) with \( \omega(0) = \omega_0 \in \mathbb{S}^{n-1} \).
Sketch of the proof II

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Steps:

1. Prove that

\[ R(a, b, T, n) \leq 2 \frac{\mu(a/2, b/2, T, n)}{T} \]

Show that \( \mu(a/2, b/2, T, n) \) is realized by an optimal control \( c_* : [0, T] \to \mathbb{R}^n \), which extends to a 2T-periodic (PE) signal \( c_* : \mathbb{R}_+ \to \mathbb{R}^n \)
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2. Show that \( \mu(a, b, T, n) \leq \mu(a, b, T, 2) \);
Sketch of the proof II

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\( \leadsto \) Show that \( \mu(a/2, b/2, T, n) \) is realized by an optimal control \( c_* : [0, T] \to \mathbb{R}^n \),

which extends to a 2T-periodic (PE) signal \( c_* : \mathbb{R}_+ \to \mathbb{R}^n \)

2. Show that \( \mu(a, b, T, n) \leq \mu(a, b, T, 2) \);

3. Precisely estimate \( \mu(a, b, T, 2) \).
Sketch of the proof II

Optimal control problem:

\[ \mu(a, b, T, n) := \min J_T(c, \omega_0) = \min \int_0^T (c^T \omega)^2 \, ds \]

Here, \( c : [0, T] \to \mathbb{R}^n \) runs over all signals satisfying

\[ a \text{Id}_n \leq \int_0^T c(s)c(s)^T \, ds \leq b \text{Id}_n, \]

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Steps:

1. Prove that

\[ R(a, b, T, n) \leq 2 \frac{\mu(a/2, b/2, T, n)}{T} \]

\( \leadsto \) Show that \( \mu(a/2, b/2, T, n) \) is realized by an optimal control \( c^* : [0, T] \to \mathbb{R}^n \), which extends to a 2T-periodic (PE) signal \( c^* : \mathbb{R}_+ \to \mathbb{R}^n \).

2. Show that \( \mu(a, b, T, n) \leq \mu(a, b, T, 2) \);
3. Precisely estimate \( \mu(a, b, T, 2) \).

PMP
More general systems

We obtain the same result for the worst rate of decay for the more general system

\[
\dot{x}(t) = -S(t)x(t)
\]

were \( S(t) \in \mathbb{R}^{n \times n} \) is such that \( S(t) \geq 0 \) and for \( a, b, T > 0 \)

\[
a \mathbf{I}d_n \leq \int_{t}^{t+T} S \, ds \leq b \mathbf{I}d_n
\]

\( \hookrightarrow \) The family of signals \( S \) is obtained as the convexification of the family \( cc^\top \) where \( c : [0, T] \to \mathbb{R}^n \) satisfies (PE)

\( \hookrightarrow \) the worst rate of decay is realized by (DGF), e.g., \( S = cc^\top \)
Open question

For $a, b, T$ fixed, what dependence on the dimension?

$$
\frac{C_1}{n} \leq \lim_{b \to \infty} R(a, b, T, n) \left( \frac{1 + b^2 T}{a} \right) \leq C_0.
$$

- The technique used in the proof yields also the lower bound

  $$
  R(a, b, T, n) \geq \frac{\mu(a, b, T, n)}{T}.
  $$

- At the moment we cannot directly access $\mu(a, b, T, n)$ for $n \neq 2$.  

Thank you for your attention!

Y. Chitour, P. Mason, D. Prandi

*Worst Exponential Decay Rate for Degenerate Gradient flows subject to persistent excitation*

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