Efficient unitary paths and quantum computational supremacy: A proof of average-case hardness of Random Circuit Sampling

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One-parameter interpolations between any two unitary matrices (e.g., quantum gates) $U_1$ and $U_2$ along efficient paths contained in the unitary group are constructed. Motivated by applications, we propose the continuous unitary path $U(\theta)$ obtained from the QR-factorization

$$U(\theta)R(\theta) = (1-\theta)A + \theta B,$$

where $U_1R_1 = A$ and $U_2R_2 = B$ are the QR-factorizations of $A$ and $B$, and $U(\theta)$ is a unitary for all $\theta$ with $U(0) = U_1$ and $U(1) = U_2$. The QR-algorithm is modified to, instead of $U(\theta)$, output a matrix whose columns are proportional to the corresponding columns of $U(\theta)$ and whose entries are polynomial or rational functions of $\theta$. By an extension of the Berlekamp-Welch algorithm we show that rational functions can be efficiently and exactly interpolated with respect to $\theta$. We then construct probability distributions over unitaries that are arbitrarily close to the Haar measure.

Demonstration of computational advantages of NISQ [15] over classical computers is an imperative near-term goal, especially with the exuberant experimental frontier in academia and industry (e.g., IBM and Google). A candidate for quantum computational supremacy is Random Circuit Sampling (RCS), which is the task of sampling from the output distribution of a random circuit. The aforementioned mathematical results provide a new way of scrambling quantum circuits and are applied to prove that exact RCS is $\#P$-Hard on average, which is a simpler alternative to Bouland et al’s [4]. (Dis)Proving the quantum supremacy conjecture requires approximate average case hardness; this remains an open problem for all quantum supremacy proposals.

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References
I. UNITARY PATHS AND SUMMARY OF RESULTS

A. Paths on the unitary group

Let \( U(N) \) denote the unitary group and suppose \( U_1 \in U(N) \) and \( U_2 \in U(N) \) are two unitary matrices. How can one build a parametrized path \( U(\theta) \) between them such that \( U(\theta) \in U(N) \) for all \( \theta \in [0, 1] \) and \( U(0) = U_1 \) and \( U(1) = U_2 \)? In this subsection we discuss a few different ways of constructing such a \( U(\theta) \).

Let the Hermitian matrix \( H \) be \( H = -i \log(U_1^\dagger U_2) \), then the most natural path is the geodesic:

\[
U(\theta) = U_1 \exp(iH\theta),
\]

which connects \( U_1 = U(0) \) to \( U_2 = U_1 e^{iH} = U(1) \).

An alternative interpolation between the two unitaries is

\[
U(\theta) = U_1^{(1-\theta)} U_2^\theta, \quad \theta \in [0, 1]
\]

where the power is defined, as for any normal matrix, to act only on the eigenvalues.

Motivated by the applications we have in mind we now propose a completely different interpolation scheme that is inspired by numerical linear algebra. In general any \( N \times N \) matrix \( M \) has a QR-decomposition \( UR = M \), where \( U \) is a unitary (or orthogonal if \( M \) is real) and \( R \) is an upper triangular matrix \[17\]. Taking the convention that the diagonal entries of \( R \) are positive, this decomposition is unique so long that the matrix is full rank. Below we refer to both orthogonal and unitary simply as unitary.

Suppose one wants to interpolate between the unitaries in the QR decompositions of the matrices \( A = U_1 R_1 \) and \( B = U_2 R_2 \) on a continuous unitary path. Then, \( U(\theta) \) defined by the QR-decomposition of the interpolation is such a paths:

\[
U(\theta)R(\theta) = (1-\theta) A + \theta B.
\]

By construction \( U(\theta) \) is unitary for all \( \theta \) (See Fig. 1 (left)).

Alternatively, the Euclidean line interval that connects any two unitary matrices \( U_1 \) and \( U_2 \) is the matrix pencil \( (1-\theta)U_1 + \theta U_2 \) for \( \theta \in [0, 1] \). Although this sum is not generally a unitary matrix for \( \theta \in (0, 1) \), the interpolation

\[
(1-\theta)U_1 + \theta U_2
\]
\[ U(\theta)R(\theta) = (1 - \theta) U_1 + \theta U_2, \]  

(3)

provides a unitary path \( U(\theta) \) with the end points \( U(0) = U_1 \) and \( U(1) = U_2 \) (see Fig. (1) (right)).

Since for generic choice of matrices, the sums in Eqs. (2) and (3) are with probability one invertible, \( U(\theta) \) is generically unique for all \( \theta \).

Lastly, an interpolation is achieved by \( U_1 U(\theta) \) and the QR-decomposition

\[ U(\theta)R(\theta) = (1 - \theta) I + \theta U_1^* U_2. \]  

(4)

This decomposition is guaranteed to be unique for all \( \theta \neq 1/2 \) and satisfies \( U(0) = U_1 \) and \( U(1) = U_2 \). Eq. (4) “multiplicatively” interpolates between the unitaries, which is an alternative to the exponential form (Eq. (1)).

Why do we bother with such proposals when we have Eq. (1)? In applications, such as in (quantum) complexity theory (Section (III)), one may access the unitary matrices \( U(\theta_1), U(\theta_2), \ldots, U(\theta_D) \) for some \( D \), and want to determine \( U(\theta) \) for all \( \theta \). That is, the path \( U(\theta) \) needs to be learned by sampling at some number of points. \( U(\theta) \) can be efficiently determined from samples if, for example, the entries of \( U(\theta) \) have a polynomial structure of degree \( D - 1 \). More generally, one wishes for \( U(\theta) \) to have entries whose functional dependence on \( \theta \) are such that they can be inferred efficiently by sampling even if the sample points are noisy.

B. Motivation and summary of the results

The rest of this paper has two main sections: Section (II) is self-contained and may be read by anyone interested in the structure of the unitary group. Starting in Section (III) we apply the mathematical results to a problem in quantum complexity theory.

Section (II) includes an explicit and efficient construction of one-parameter families of unitary matrices that interpolate between any two fixed unitaries. By ‘efficient’ we mean that a modified QR-algorithm is presented (Alg. (1)) that results in unitaries whose entries are proportional to polynomial functions of low degrees in the interpolation parameter. The degrees of the polynomial functions are quantified (Lemma (1) and Corollary (1)). Therefore, we have in our hands an interpolation scheme between any two given unitaries that is continuous, is contained in the unitary group everywhere, and has entries that depend on polynomials of low degrees.

The discovery of Berlekamp-Welch algorithm (BW) [21] was originally motivated by classical error correction schemes such as the Reed-Solomon codes [16]. In these codes, the messages are encoded in the coefficients of polynomials over finite fields. BW takes as input a set of points together with the value of the polynomial at those points. The latter depends on the encoded messages, which may incur errors in the transmission from the sender to the receiver. BW is remarkable in that it can exactly recover such polynomials even if the evaluation of the polynomial at some number of points is erroneous. Here we prove an extension of BW that can efficiently interpolate rational functions (Alg. (2)).

Lastly, a subsection is devoted to the Haar measure, and the construction of unitaries whose distributions may be arbitrarily close to the Haar measure in total variational distance (TVD). We discuss the intimate connection of the Haar measure to the QR-factorization of random gaussian matrices and then provide a new construction of distributions over unitary matrices that are \( \theta \)—close in TVD to the Haar measure for any given small \( \theta \).

In Section (III) we apply the above mathematical results to prove a result in quantum complexity theory. Before stating the results, we provide an introduction to the field.
A quantum computation is a rotation in the Hilbert space of the standard basis $e_1 = (1,0,\ldots,0)^T$. The standard notation denotes $e_1$ by $|0\rangle^\otimes n$, which is a $2^n$ dimensional vector; the rotation is achieved by the unitary matrix $U$, whose exact form is fixed by the quantum algorithm being implemented. The final state of the quantum computation, denoted by $|\psi\rangle$, is the vector in the Hilbert space $|\psi\rangle \equiv U (|0\rangle^\otimes n)$. The square of the absolute value of any entry of $|\psi\rangle$ quantifies the probability of occurrence of that particular outcome upon measurement. Good quantum algorithms provide $U$’s that, with high probability, result in outputs that encode the answer to the desired computational task more efficiently than any classical computer.

One then says that the quantum computation involves $n$ quantum bits (qubits), each of which corresponds to $\mathbb{C}^2$, and that the circuit $C$ implements $U$. Constrained by the difficulties in the experimental realizations, $U$ is almost always taken to be a product of many 'local' unitaries (gates) of the form $I_{2^{n-1}} \otimes U_i$ or $I_{2^{n-2}} \otimes U_{i,j}$ where $U_i$ and $U_{i,j}$ are $2 \times 2$ and $4 \times 4$ unitaries respectively. $U_i$ acts on the Hilbert space corresponding to the $i$th qubit and $U_{i,j}$ acts on the joint Hilbert space of the neighboring qubits $i$ and $j$. One says that the circuit $U$ is generic with respect to the architecture $A$ if the local unitaries $U_i$ and $U_{i,j}$ are drawn independently from the Haar measure.

Currently, one of the most active frontiers in quantum computation and information science is centered around “Noisy Intermediate Scale Quantum (NISQ)” computers [15] --noisy because of the absence of quantum error correction. What can a NISQ computer do? Would these offer a quantum advantage over any current classical computer?

Because of a large industrial push (e.g., from IBM and Google), NISQ computers with hundred(s) of qubits are at the brink of existence with the promise of outperforming any classical computer [7]. A milestone is to prove unambiguously a substantial advantage of a NISQ computer over classical ones. This event has been termed quantum supremacy, and we have yet to witness it. It states that there are computational tasks that a NISQ computer can efficiently perform, that any classical computer would find formidable. The two main current proposals to demonstrate this are BosonSampling [1], and Random Circuit Sampling (RCS). Our focus in this paper will be RCS, which roughly speaking, is the task of sampling from the output distribution of a circuit whose local gates are random.

Recently generic aspects of quantum circuits have been foundational. Generic in mathematics means with probability one or almost surely. Generic circuits are therefore circuits whose local gates are random unitaries. Most natural is to draw them from the Haar measure-- drawn independently and uniformly from the space of all unitaries. Such circuits appear in quantum complexity theory [2], the study of black holes [10], holographic models of quantum gravity [19], and quantum supremacy [7, 9]. The latter was initially proposed to demonstrate that quantum computers have capabilities beyond classical. Aaronson and Arkhipov first proposed the BosonSampling problem as a candidate for testing quantum supremacy [1]. Later the Google team proposed that random quantum circuits might demonstrated this supremacy in the near-term quantum devices [3]. We currently know that computing the output probabilities of RCS, even approximately, is $\#P$-Hard in the worst case [6]. This implies that no exact worst-case classical simulation algorithm exists unless the polynomial hierarchy collapses [1, 6]. An advantage of RCS over BosonSampling is that anti-concentration bounds have been proven [5, 8]. This led Bouland et al to prove that the exact average-case RCS is also $\#P$-Hard [4]. What is needed and remains open in all supremacy proposals to date, however, is to prove approximate average-case hardness. The status of the field is summarized in the table below.

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1 $\#P$ is a generalization of NP, which is for decision problems, to counting problems. In particular, an NP-complete problem is: Does a 3-SAT instance have any solutions? The answer is a yes or no (i.e., zero solutions). Whereas, $\#P$ asks: How many solutions does a 3-SAT instance have? Therefore, $\#P$ contains NP.
The application that motivates this paper is to (dis)prove the so called quantum supremacy conjecture, which we pick up in Section III. Although the conjecture remains open, we provide an entirely a new proof of the exact average case hardness of RCS, which is alternative to the recent paper by Bouland et al [4] and does not use the standard techniques. The standard technique for proving average case hardness is via polynomial reduction of the worst case hardness to the average, which is based on Lipton’s technique [11]. This essentially says:

Suppose a problem is known to be \#P-Hard in the worst case. Now suppose the worst case problem, \( A \), can be deformed to \( A(\theta) \) for \( \theta \in [0, 1] \) such that \( A(1) = A \) is the \#P-Hard instance, and \( A(\theta) \) is a polynomial of low degree \( D \) in \( \theta \). If \( \theta \in [0, 1) \) contains at least \( D + 1 \) ‘generic’ instances of the problem, then the problem is generically \#P-Hard. The reason is that if the problem were generically easy, then because of the polynomial structure, the evaluation of \( A(\theta) \) at \( D + 1 \) random points would uniquely determine \( A(\theta) \). One can then read off \( A(1) \) by polynomial extrapolation. But \( A(1) \) is a \#P-Hard problem so the generic instances must have had \#P-Hard instances.

To make use of this polynomial interpolation technique, Aaronson and Arkhipov proposed the BosonSampling, whose underlying computational task is a permanent, which is a polynomial function of the entries [1].

In RCS, the goal to demonstrate quantum supremacy is to compute

\[
p_y(C) \equiv |\langle y|C|0^n\rangle|^2. \tag{5}
\]

Quantum circuits, however, do not have a natural polynomial form. To obtain polynomials corresponding to the generic local circuit, Bouland et al [4] deform the quantum gates towards a Haar distribution and then truncate the Taylor series expansions of the exponential functions that arise in Eq. (1). The truncation results in ‘non-unitarity’ of the local gates and therefore the quantum circuit as a whole. They then justify that the errors due to the truncations are small enough that the average case \#P-Hardness of the non-unitary approximation is still necessary for the approximate average-case hardness conjecture [1] to be true.

In Section III, the results of Section II are used to give an entirely a new proof of \#P-hardness of RCS. We prove that the modified QR algorithm results in probabilities (Eq. 5) that are rational functions of the interpolation parameter \( \theta \). These can then be efficiently learned using the new BW algorithm for rational functions. Moreover, we prove that proving \#P-hardness of exact computation of Eq. (5) is necessary for proving the quantum supremacy conjecture. This is ensured by our construction as the local gates will be proved to be \( \theta \)-close to the Haar distribution in TVD.

In addition to being a more direct proof, the advantages of the new proof over [4] include:
1. No truncations are needed and the interpolation is contained in the unitary group everywhere.
2. The construction is explicit in that the degrees and coefficients can be quantified; this might help in proving the quantum supremacy conjecture (see [1] (arXiv version, page 81)).
3. The new mathematical results may be of independent interest.

II. RATIONAL AND POLYNOMIAL PATHS ON THE UNITARY GROUP

A. Standard QR decomposition

The standard QR-decomposition algorithm applied to the columns of a matrix, \( M \), results in an orthonormal set of vectors that comprise the columns of a unitary matrix \( U \), whose linear
combinations dictated by the $R$ matrix result in $M = UR$. It is instructive to quickly recap this algorithm.

Let $A = [a_1, \ldots, a_N]$, $B = [b_1, \ldots, b_N]$ where $a_i$ and $b_i$ are the $i^{th}$ columns of $A$ and $B$ respectively. Let $M(\theta) = (1 - \theta)A + \theta B$ have the QR-decomposition $M(\theta) = U(\theta)R(\theta)$. Let $m_i(\theta)$ be the columns of $M(\theta) = [m_1(\theta), \ldots, m_N(\theta)]$ and denote the columns of $U(\theta)$ by $U(\theta) = [u_1(\theta), \ldots, u_N(\theta)]$.

Comment: Below at times we drop the dependence on $\theta$ for simplicity and denote the columns of $m(\theta)$ simply by $m$ etc. It is obvious that whenever the matrix depends on $\theta$, so do its columns.

Also recall that the projection of the vector $x$ onto $y$ is

$$
\text{proj}_y x = \frac{\langle y, x \rangle}{\|y\|^2} y,
$$

where $\| \cdot \|$ denotes the standard Euclidean 2-norm.

The algorithm starts by $v_1 := m_1 = (1 - \theta)a_1 + \theta b_1$. Let $u_1 = v_1 / \|v_1\|$, then $v_2 = m_2 - \text{proj}_{u_1} m_2$ and $u_2 = v_2 / \|v_2\|$. Generally,

$$
v_k = m_k - \sum_{j=1}^{k-1} \text{proj}_{u_j} m_k, \quad u_k = v_k / \|v_k\|.
$$

It is easy to see that the functional form of the entries of any $u_i$ is a ratio, whose numerator and denominator are in general sums of terms some of which involve square roots of polynomials. For example

$$
u_1 = \frac{(1 - \theta)a_1 + \theta b_1}{\sqrt{(1 - \theta)^2 \|a_1\|^2 + \theta^2 \|b_1\|^2 + \theta(1 - \theta) \langle a_1, b_1 \rangle + \langle b_1, a_1 \rangle}};
$$

$$
v_2 = [(1 - \theta)a_2 + \theta b_2] - \frac{\theta(1 - \theta) \langle a_1, b_2 \rangle + \langle b_2, a_1 \rangle}{(1 - \theta)^2 \|a_1\|^2 + \theta^2 \|b_1\|^2 + \theta(1 - \theta) \langle a_1, b_1 \rangle + \langle b_1, a_1 \rangle},
$$

and $u_2 = v_2 / \|v_2\|$ would involve square roots of polynomials. The standard QR-algorithm, therefore, is inadequate for the applications we have in mind as the entries involve untamable square roots of polynomials that cannot be learned by sampling.

Remark 1. In numerical linear algebra the QR-decomposition is performed by more preferred methods that are more efficient and stable such as Householder transformations, or Givens rotations \cite{[20]}. Although these methods, in the standard form, do not yield any polynomial structure in $U(\theta)$ to be exploited, they can be used to evaluate the numerical values of $U(\theta)$ for any fixed $\theta$.

Suppose we have a matrix $A$ that is $\ell \times k$ with $\ell \geq k$. The computational complexity of QR decomposition depends on the algorithm and is \cite{[20]}:

- Householder transformations: $2\ell k^2 - 2k^3 / 3$.
- Standard QR decomposition procedure: $2\ell k^2$.
- Modified QR decomposition (below) after normalization: $2\ell k^2$. 
B. Modified QR decomposition and rational function Berlekamp-Welch interpolation algorithm

We propose an unnormalized version of the QR-decomposition algorithm, which when applied to pencils of matrices in Eq. (3), results in a matrix whose columns are orthogonal but not normalized. The upshot is that the entries of this matrix can be expressed as polynomial or rational functions of \( \theta \). Below we use the same notion for the columns of the matrices as we did in the standard QR-algorithm above.

**Algorithm 1.** Let \( M(\theta) = (1 - \theta)A + \theta B \), we seek the QR-decomposition of \( U(\theta)R(\theta) = M(\theta) \).

The algorithm first solves for the unnormalized vectors \( z_i \) by performing the following:

1. Let \( z_1 = v_1 = m_1 \), which is linear in \( \theta \).
2. \( v_2 = m_2 - \text{proj}_{z_1} m_2 \), it is instructive to write it explicitly

\[
v_2 = \frac{[(1 - \theta)a_2 + \theta b_2] \|z_1\|^2 - \langle z_1, m_2 \rangle z_1}{\|z_1\|^2}
\]

where

\[
\|z_1\|^2 = (1 - \theta)^2 \|a_1\|^2 + \theta^2 \|b_1\|^2 + \theta(1 - \theta) [\langle a_1, b_1 \rangle + \langle b_1, a_1 \rangle]
\]

\[
\langle z_1, m_2 \rangle z_1 = [(1 - \theta)^2 \langle a_1, a_2 \rangle + \theta^2 \langle b_1, b_2 \rangle + \theta(1 - \theta) (\langle a_1, b_2 \rangle + \langle b_1, a_2 \rangle)] [(1 - \theta)a_1 + \theta b_1]
\]

Therefore, every entry of \( v_2 \) is a rational function with numerator being a polynomial of degree three and denominator a polynomial of degree two. We can now define

\[
z_2 = \|z_1\|^2 v_2
\]

which is a polynomial valued vector, whose entries are polynomials of degree three.

3. In general,

\[
v_k = m_k - \sum_{j=1}^{k-1} \text{proj}_{z_j} m_k.
\]

\[
\begin{align*}
z_k &= \left( \prod_{j=1}^{k-1} \|z_j\|^2 \right) \left\{ m_k - \sum_{j=1}^{k-1} \frac{\langle z_j, m_k \rangle z_j}{\|z_j\|^2} \right\} \\
&= \left( \prod_{j=1}^{k-1} \|z_j\|^2 \right) \left\{ m_k - \sum_{j=1}^{k-1} \frac{\langle z_j, m_k \rangle z_j}{\|z_j\|^2} \right\}
\end{align*}
\] (7)

The set of vectors \( v_1, \ldots, v_N \) are orthogonal but unnormalized. We shall keep them unnormalized to retain the rational function dependence of the entries on \( \theta \) (see Lemma[1] and Corollary[1]). Similarly, the set of vectors \( z_1, \ldots, z_N \) are orthogonal but unnormalized and have entries that are polynomials in \( \theta \). They are obtained by simply multiplying through by the common denominator of all \( v_j \)'s. For example, \( z_1 = m_1, z_2 = v_2 \|m_1\|^2 \) etc..

4. (Optional) Given \( z_1, \ldots, z_N \), the columns of \( U(\theta) \) are obtained by normalizing \( z_1, \ldots, z_N \). Each entry of \( U(\theta) \) will be a ratio of a polynomial function in \( \theta \) with the square root of a polynomial function in \( \theta \):

\[
u_k(\theta) = z_k(\theta) / \sqrt{\langle z_k(\theta), z_k(\theta) \rangle}, \quad 1 \leq k \leq N.
\] (8)

This is sufficient for our purposes as we will need the norm-square of such quantities which are exactly rational functions.
Definition 1. Let \( p_\ell(\theta) \) be the set of polynomial vectors of degree \( \ell \) in \( \theta \); that is, every entry of the vector \( v \in p_\ell(\theta) \) is a polynomial of degree \( \ell \) in \( \theta \). Let \( q_\ell(\theta) \) be the set of polynomial functions of degree \( \ell \) in \( \theta \). The algebra is such that for \( v \in p_\ell(\theta), u \in p_r(\theta), f \in q_\ell(\theta), \) and \( g \in q_r(\theta), \) we have \( fg \in q_{\ell+r}(\theta), \langle u, v \rangle \in q_{\ell+r}(\theta), \) and \( g \circ v \in p_{\ell+r}(\theta). \)

Definition 2. A vector \( v(\theta) \) whose entries are rational functions of \( \theta \) is denoted by \( v(\theta) \in p_\ell(\theta)/q_\ell(\theta) \) if its entries have numerators that are polynomials of degree \( \ell \) and denominators are polynomials of degree \( r \).

Lemma 1. \( v_k(\theta) \) given by Eq. (6) in the modified QR algorithm satisfies \( v_k(\theta) \in p_{Dk+1}(\theta)/q_{Dk}(\theta), \) with \( D_k = (3^{k-1} - 1). \)

Proof. We first prove that \( v_k(\theta) \in p_{Dk+1}(\theta)/q_{Dk}(\theta) \) for some positive integer \( D_k \). Note that \( v_1(\theta) = z_1(\theta) = m_1(\theta) \in p_1(\theta)/q_0(\theta), \) suppose \( v_k-1(\theta) \in p_{Dk-1+1}(\theta)/q_{Dk-1}(\theta), \) then \( z_{k-1} \in p_{Dk-1+1}(\theta) \) and

\[
v_k = \left\{ m_k - \sum_{j=1}^{k-2} \text{proj}_{j} m_k \right\} - \text{proj}_{k-1} m_k
\]

and the sum of the terms in the braces are of the same type as \( v_{k-1}(\theta) \) by construction. Hence, by the induction hypothesis they are of the type \( p_{Dk-1+1}(\theta)/q_{Dk-1}(\theta). \) Further, \( \text{proj}_{k-1} m_k = \langle z_{k-1}, m_k \rangle z_{k-1}/\|z_{k-1}\|^2 \in p_{2Dk-1+3}(\theta)/q_{2Dk-1+2}(\theta). \) So we have

\[
v_k \in \frac{p_{Dk-1+1}(\theta)}{q_{Dk-1}(\theta)} - \frac{p_{2Dk-1+3}(\theta)}{q_{2Dk-1+2}(\theta)} = \frac{p_{3Dk-1+3}(\theta)}{q_{3Dk-1+2}(\theta)}.
\]

Letting \( D_k = 3D_{k-1} + 2 \) we arrive at the desired result. Lastly, \( D_k = (3^{k-1} - 1) \) is the solution of the recursion relation \( D_{k+1} = 3D_k + 2 \) with the initial condition \( D_1 = 0. \)

Corollary 1. Let \( z_k = q_{Dk}(\theta) v_k \), then \( z_k \in p_{Dk+1}(\theta). \) That is, each entry of \( z_k \) in Eq. (7) is a polynomial function of degree \( D_k + 1 = 3^{k-1}. \)

Remark 2. Because the local gates in a quantum circuit that implements a quantum computation are represented by \( 2 \times 2 \) or \( 4 \times 4 \) (unitary) matrices, we note that \( D_2 + 1 = 4 \) and \( D_4 + 1 = 28. \)

We now turn to the issue of uniquely determining a rational function by efficient sampling.

Lemma 2. Any rational function of degree \( (k_1, k_2) \) in one variable \( \theta \) has the general form

\[
F(\theta) = \frac{a_{k_1} \theta^{k_1} + a_{k_2} \theta^{k_2} + \cdots + a_0}{b_{k_1} \theta^{k_1} + b_{k_2} \theta^{k_2} + \cdots + b_0}
\]

and is uniquely determined by \( k_1 + k_2 + 1 \) points provided that \( F(\theta_i) = f_i, \) \( \theta_i < \infty \) for \( i \in [k_1 + k_2 + 1] \) are independent conditions and the numerator and denominator are relatively prime.

Proof. Since a rational function is determined up to a constant multiple of numerator and denominator, we can factor out \( a_{k_1}/b_{k_2}, \) and use homogeneous coordinates with \( k_1 + k_2 + 1 \) unknowns. By multiplying both sides by the denominator and then evaluating \( F(\theta) \) at \( k_1 + k_2 + 1 \) points \( F(\theta_i) = f_i, \) the coefficients become the solution of the linear system of equations in \( (k_1 + k_2 + 1) \) variables. Given that the \( f_i \) are independent, the coefficients are uniquely determined (unique point of intersection of hyperplanes). Lastly, \( f_i < \infty \) is to emphasize that we discard any \( \theta_i \) that is a root of the denominator; such \( \theta \)’s are of measure zero anyway.
In coding theory, and especially in Reed-Solomon codes [16], the messages $a_1, \ldots, a_k$ may be encoded into a polynomial $a_1 + a_2 \theta + \cdots + a_k \theta^k$, which then is evaluated at $n > k + t$ points. Then, the decoding procedure recovers the polynomial and hence the message exactly despite $t$ errors. The decoding procedure relies on Berlekamp-Welch (BW) algorithm for polynomial interpolation [18, 21]. BW can be extended to interpolate rational functions. The proof follows Sudan’s and is a generalization of it from polynomial to rational functions [18].

**Definition 3.** (Error polynomial) Suppose $f = (f_1, \ldots, f_n)$ is a vector. Let $F(\theta)$ be a rational function of degree $(k_1, k_2)$. We define the error polynomial $E(\theta)$ as one that satisfies

$$E(\theta_i) = 0 \quad \text{if} \quad F(\theta_i) \neq f_i, \quad \deg(E(\theta)) \leq t.$$  

**Algorithm 2.** (Berlekamp-Welch for Rational Functions) Given $(\theta_1, f_1), (\theta_2, f_2), \ldots, (\theta_n, f_n)$, find a rational function $F(\theta)$ of degree $(k_1, k_2)$ exactly by evaluating it at $n > k_1 + k_2 + 2t$ points despite $t$ errors in the evaluation points:

$$|\{i \in [n] | F(\theta_i) \neq f_i\}| \leq t.$$  

**Proof.** The error polynomial by Def. (3) satisfies

$$E(\theta_i) F(\theta_i) = E(\theta_i) f_i. \quad (9)$$

Let $W(\theta_i) \equiv E(\theta_i) f_i$, which implies that $f_i = W(\theta_i) / E(\theta_i)$. Since $W(\theta) = E(\theta) F(\theta)$ is a $(k_1 + t, k_2)$ rational function, by Eq. (9), $f_i$ is a $(k_1 + t, k_2 + t)$ rational function of $\theta$. By Lemma (2), the linear system defined by Eq. (9), has a solution as long as $n > k_1 + k_2 + 2t$. If $W(\theta)/E(\theta)$ results in a rational function of degree $(k_1, k_2)$ we are done and we simply output it as $F(\theta)$, otherwise we decide that there were too many errors.

Can the algorithm find distinct $(E_1(\theta), W_1(\theta))$ and $(E_2(\theta), W_2(\theta))$? We now show that $W_1(\theta)/E_1(\theta)$ and $W_2(\theta)/E_2(\theta)$ are equal, which means that $F(\theta)$ is learned uniquely even if there are multiple solutions. We have

$$\frac{W_1(\theta)}{E_1(\theta)} = \frac{W_2(\theta)}{E_2(\theta)} \iff E_1(\theta) W_2(\theta) = E_2(\theta) W_1(\theta).$$

Recall that both sides are bounded degree rational functions (of degree $(k_1 + 2t, k_2)$). So by evaluating them at enough points we can determine them uniquely (Lemma (2)). Since at every $\theta_i$

$$E_1(\theta_i) f_i = W_1(\theta_i), \quad E_2(\theta_i) f_i = W_2(\theta_i),$$

solving for $f_i$, we have $E_1(\theta_i) W_2(\theta_i) f_i = f_i E_2(\theta_i) W_1(\theta_i)$ at every $\theta_i$. If $f_i = 0$, then by Eq. (9) $W(\theta_i) = 0$, otherwise we just cancel the $f_i$. This proves the claim that $F(\theta) = E_1(\theta)/W_1(\theta) = E_2(\theta)/W_2(\theta)$. Since $n > k_1 + k_2 + 2t$, we are guaranteed to have enough points. \hfill \Box

**Remark 3.** Here we gave an algorithm in which the number of errors $t < (n - k_1 - k_2)/2$, but as in standard BW algorithm, it is entirely possible to find algorithms that can handle more errors. The above is sufficient for our purposes so we leave finding such algorithms for future work.

**Remark 4.** As in the standard BW algorithm, this algorithm works just as well over finite fields, reals or complex numbers. As far as we know, the performance with respect to approximate $f_i$’s is not known.
C. The Haar measure and distributions close to it in total variational distance (TVD)

Let $O(N)$, $U(N)$, and $\text{SP}(2N)$ denote the set of orthogonal, unitary, and symplectic matrices respectively. The entries of these matrices are drawn from real ($\beta = 1$), complex ($\beta = 2$), and ($\beta = 4$) respectively. In what follows we ignore $\beta = 4$ case. The set of such matrices with determinant equal to one are, respectively, denoted by $SO(N)$ and $SU(N)$. Moreover, $O(N)$, $SO(N)$ form subgroups of the set of $N \times N$ real matrices. Similarly, $U(N)$ and $SU(N)$ form subgroups of the set of $N \times N$ complex matrices. If $G$ is any one of the matrix groups defined above, then a uniform random element of $G$ is a matrix $V \in G$ whose distribution is translation invariant, which means for any fixed $M \in G$,

$$VM \overset{d}{=} MV \overset{d}{=} V,$$

where $\overset{d}{=}$ denotes equality in the distribution sense. Alternatively, the the distribution of a uniform random element of $G$ is a translation invariant probability measure. Below we denote both “orthogonal” and “unitary” simply by unitary. A standard theorem of compact classical matrix groups is the following (See E. Meckes’ excellent notes [12]).

**Theorem 1.** Let $G$ be any of $O(N)$, $SO(N)$, $U(N)$, or $SU(N)$. Then there is a unique translation-invariant probability measure on $G$, which is called the Haar measure.

To explicitly and geometrically construct the Haar measure, one starts by considering an $N \times N$ matrix $X$ whose entries are standard gaussians. The joint probability density of the entries is:

$$d\mu_\beta(X) = (2\pi)^{-\beta N^2/2} \prod_{i,j=1}^{N} dx_{i,j} \exp \left(-|x_{i,j}|^2/2\right), \quad (10)$$

where $\beta = 1$ corresponds to real, $\beta = 2$ to complex and $\beta = 4$ to quaternion entries. It is easy to check that this measure is invariant under a left-multiplication by a unitary matrix $V$. That is, by the change of variables $y_{i,j} = [VX]_{i,j}$, one exactly obtains the foregoing equations with $x_{i,j}$ replaced by $y_{i,j}$ everywhere.

For compact Lie groups, left translation invariance implies right translation invariance and therefore the Haar measure.

Although we have a measure that is invariant under left and right multiplications by any unitary, the space of matrices being described is not unitary. The standard way to proceed is to orthonormalize the columns of $X$ by a QR-factorization $X = UR$, where $U$ is the desired Haar measure unitary. $U$ is exactly the unitary matrix that results by performing a Gram-Schmidt process on the columns of $X$.

**Remark 5.** Although Gram-Schmidt results in unitaries from Haar measure, numerically stable QR algorithms such as Householder, do not perform Gram-Schmidt and can result in distributions different than Haar. To fix this, one must constrain the diagonal entries of $R$ to be positive and then use whatever blackbox QR algorithm available [13].

It is easy to see that the translation invariance of the space of such unitary matrices is preserved as the Gram-Schmidt process (equivalently the QR-decomposition) commutes with left multiplication by a fixed unitary matrix as we now show.

To see this, let $X_1, \ldots, X_N$ be the columns of a matrix $X$. The Gram-Schmidt process orthonormalizes these columns by systematically removing the projections of the others. For example, starting from $X_1$, the second column $X_2$ is replaced by $X_2 - \langle X_1, X_2 \rangle X_2$. Now suppose we left
multiply the unitary matrix $V$, whereby the columns become $VX_1, VX_2, \ldots, VX_N$. The resulting second column, by the unitarity of $V$, becomes

$$ VX_2 - \langle VX_1, VX_2 \rangle VX_1 = VX_2 - \langle X_1, X_2 \rangle VX_1. $$

The right hand side is equivalent to first doing Gram-Schmidt and then multiplying the resulting vectors by $V$. In other words, one can first apply a rotation to all the columns and then orthonormalize or first orthonormalize and then apply the rotation.

In conclusion, if we start from the space of Gaussian matrices we have a unitary invariant measure, which then through QR factorization results in a translation invariant measure over the space of unitary matrices (i.e., Haar measure).

We now show that we can achieve measures arbitrarily close the Haar using pencils of matrices proposed in the interpolation in Eq. (2):

**Lemma 3.** Consider the QR-decomposition $U(\theta) R(\theta) = (1 - \theta) X + \theta I$, where the entries of $X$ are independently and identically distributed gaussians $x_{i,j} \sim N_\beta(0,1)$. Then for $\theta \ll 1$, the distribution over $U(\theta)$ is $O(\theta)$-close to the Haar measure.

**Proof.** Let us denote by $Z(X; \theta) = (1 - \theta) X + \theta I$. Clearly, $Z$ is a linear transformation of a gaussian random matrix $X$, whose distribution is given in Eq. (10). Therefore, the distribution over $Z(X; \theta)$ denoted by $dv_\beta(Z(X))$ is:

$$ dv_\beta(Z(X)) = (2\pi(1-\theta)^2)^{-\beta N^2/2} \left( \prod_{i,j=1}^{N} dx_{i,j} \right) \exp \left[ -\frac{1}{2(1-\theta)^2} \left( \sum_{i \neq j} |x_{i,j}|^2 + \sum_{i} |x_{i,i} - \theta|^2 \right) \right] $$

$$ = d\mu_\beta(X) \left\{ 1 + \theta \left[ \beta N^2 - \sum_{i} (|x_{i,i}|^2 - \text{Re}(x_{i,i})) \right] + O(\theta^2) \right\}, \tag{11} $$

where we assume $\theta \ll 1$. Since the finite moments of a finite gaussian matrix are bounded, the total variation distance between $\mu_\beta(X)$ and $v_\beta(Z)$ obeys

$$ \| \mu_\beta(X) - v_\beta(Z(X)) \|_{TV} = \max_{A \subset \mathbb{R}} |\mu(A) - v(A)| \leq O(\beta N^2 \theta). $$

To prove that $U$ and $U(\theta)$ that result from a QR decomposition have a TVD that is at most $O(\beta N^2 \theta)$, we set up a coupling argument. Suppose $Y$ is distributed like $Z$ and is independent of $X$. We set up the coupling $(X, Y)$, whose marginals are just $X$ and $Y$. Applying the QR algorithm to the coupling results in the independent unitaries $(X', Y')$ and we need to argue that $TVD(X', Y') \leq TVD(X, Y) \equiv Pr(X \neq Y)$. If we start with $X = Y$, the algorithm results in $X' = Y'$ by the fact that $X$ and $Y$ are both with probability one invertible and the essential uniqueness of QR for invertible matrices. However, if $X \neq Y$, the algorithm can result in $X' \neq Y'$ or $X' = Y'$. Therefore, $Pr(X' \neq Y') \leq Pr(X \neq Y) = O(\beta N^2 \theta)$. This proves that the TVD over the resulting unitaries is at most $O(\beta N^2 \theta)$.

**Remark 6.** Intuitively, the output of any function of the random matrices $Z$ over the distribution $v_\beta(Z(X))$, maps to a distribution whose distance is at most $O(\beta N^2 \theta)$ from what would have been obtained if the input were from the distribution $\mu_\beta(X)$. Similarly, any algorithm whose input is from $v_\beta(Z)$, in particular the distribution over $U(\theta)$ resulting from Gram-Schmidt process, is also $O(\beta N^2 \theta)$ close in distribution to the space of unitary matrices from the Haar measure. So we conclude that the total variation distance between the distribution over $U(\theta)$ and the Haar measure is $O(\beta N^2 \theta)$. 


III. EXACT AVERAGE-CASE \#P-HARDNESS OF RANDOM CIRCUIT SAMPLING (RCS)

A. Proof structure (informal)

We now apply the foregoing mathematical results to a problem in quantum computational complexity. It is known that there exist local circuits with \( n \) qubits whose probability amplitudes are \#P-Hard to estimate to within \( 1/\text{poly}(n) \) multiplicative error [1, 6]. By a quantum circuit we have in mind a specific architecture \( A \) that implements a specific unitary transformation of \( |0\rangle^\otimes n \). As stated in Subsection (I B) the unitary matrix \( U \) that encodes the quantum computation is made up of a product of unitary matrices with one unitary per layer of the circuit. The unitary in each layer, in turn, is made up of a tensor product of 1–or 2–qubits unitaries (i.e., gates). Therefore, if \( C \) is the quantum circuit, it is a product of many gates \( C = C_mC_{m-1} \ldots C_2C_1 \), where each \( C_i \) may be equal to a one qubit gate that implements \( I \otimes U_i \) or a 2-qubit gate that implements \( I \otimes U_{ij} \).

Definition 4. Below by random circuit sampling problem, or simply RCS, we mean sampling from the output distribution of a random circuit.

The goal is to prove the following conjecture:

Conjecture 1. (Informal supremacy conjecture) Approximating to \( 1/\text{poly}(n) \) multiplicative error of most amplitudes of most quantum circuits is a \#P hard problem.

The statement we prove in this section informally reads:

Theorem 2. (Informal statement of the theorem) Exact calculation of most amplitudes of most quantum circuits is a \#P Hard problem.

The key words that distinguish our results from the stated conjecture are ‘approximating’ vs. ‘exact calculation’.

The proof structure is as follows. We start with an arbitrary circuit, which may be the worst case circuit \( C \), and whose exact specification is irrelevant for our purposes. We just assume that it has an architecture \( A \) and as usual is made up of local gates.

Our goal is to show that if the local gates were implementing local unitaries independently drawn from the Haar measure, then the problem remains \#P-Hard to compute on average over this distribution. The way we do this is that we \( \theta \)–deform the worst case circuit \( C \) to be sufficiently close to such a random circuit denoted by \( C(\theta) \) such that \( C(1) = C \). The proposed deformation is rational function valued in \( \theta \) for the quantity whose \#P-Hardness we want to show. But in the previous section we provided an efficient BW algorithm that can efficiently interpolate rational functions. Therefore, we can invoke a Lipton-like argument (see italic text in Section (I B)) to prove that the average case should also be \#P-Hard, because otherwise \( C(1) \) would be easy to compute by interpolation.

B. Formal results

By inserting a complete set of basis between each \( C_i \) and \( C_{i+1} \), one can write the circuit down in what is at times called “Feynman path integral” form [4]. The amplitude corresponding to the initial state \( |y_0\rangle \) and final state \( |y_m\rangle \) is

\[
\langle y_m|C|y_0\rangle = \sum_{y_1,y_2,\ldots,y_{m-1} \in \{0,1\}^n} \langle y_m|C_m|y_{m-1}\rangle \langle y_{m-1}|C_{m-1}|y_{m-2}\rangle \cdots \langle y_1|C_1|y_0\rangle. \tag{12}
\]

We now define some basic notation following [4].
Definition 5. (Haar random circuit distribution) Let $\mathcal{A}$ be an architecture over circuits and let $\mathcal{H}_A$ be the distribution over circuits in $\mathcal{A}$ whose gates are independently drawn from the Haar measure.

The random circuit sampling is then:

Definition 6. (Random Circuit Sampling (RCS) [4]) Random circuit sampling over a fixed architecture $\mathcal{A}$ is the following task: given a description of a random circuit $C$ from $\mathcal{H}_A$, and a description of an error parameter $\epsilon > 0$, sample from the probability distribution induced by $C$. That is draw $y \in \{0, 1\}^n$ with probability $\Pr(y) = |\langle y | C | 0^n \rangle|^2$ up to a total variation distance $\epsilon$ in time $\text{poly}(n, 1/\epsilon)$.

In RCS one seeks estimations of $|\langle y | C | 0^n \rangle|^2$ but any bit string $|y\rangle$ is simple to obtain by applying Pauli $X$ matrices to positions in $|0^n\rangle$ that correspond to 1’s. By this, so called ‘hiding property’ [1], it is sufficient to prove the hardness of computing

$$p_0(C) \equiv |\langle 0^n | C | 0^n \rangle|^2. \quad (13)$$

Conjecture 2. (Quantum Supremacy Conjecture [1, 4]) There is no classical randomized algorithm that performs RCS to inverse polynomial total variation distance error.

Remark 7. Conjecture [1] implies Conjecture [2] assuming Polynomial Hierarchy is infinite [1].

In order to ultimately prove this, some intermediate steps have been taken. It is known that the estimation of the exact amplitudes of the worst case circuit are $\#P$-Hard. More recently, Bouland et al proved that the exact average case RCS is also $\#P$-Hard [4]. What remains open is to prove that the approximate average RCS is also $\#P$-Hard. This is what is meant by proving “robustness”.

An application of the developments of the previous sections is a new proof of the exact $\#P$-hardness of RCS, which among other things is free of some issues presented in [4]. For example, the truncation of the Taylor series is unsatisfactory from a mathematical perspective. In particular, BW requires exact evaluation of the polynomial at some of the many points, but the truncation introduces errors at all points. One then has to do extra work to justify the validity of the proof. Therefore, we now proceed with the construction of the new reduction of the exact-to-average case hardness based on constructing a path on the unitary group described above.

Definition 7. ($\theta$-deformed Haar) Let $\mathcal{A}$ be an architecture over circuits, $\theta \in [0, 1]$, and $G_1, \ldots, G_m$ be the gates in the architecture. Define the distribution $\mathcal{H}_{A,\theta}$ over circuits in $\mathcal{A}$ by setting each
Figure 3: Schematics for Def. 8: The scrambling of the circuit C to C(θ).

gate in the circuit to be the unitary defined by the QR decomposition \( G_j(\theta) R_j(\theta) = (1 - \theta)X + \theta I \), where \( X \) is a standard complex gaussian matrix with independent entries \( x_{ij} \sim \mathcal{N}(0, 1) \).

Comment: \( G_j(0) \) is a (small \( 2 \times 2 \) or \( 4 \times 4 \)) unitary distributed according to the Haar measure (Sec. (II C)) and \( G_j(1) = I \). This interpolation is exact and no truncations are needed.

**Definition 8.** To reduce the complexity of the worst case circuit \( C = C_m C_{m-1} \ldots C_2 C_1 \) to the average case, denote the latter by \( C(\theta) = C_m(\theta) C_{m-1}(\theta) \ldots C_2(\theta) C_1(\theta) \), in which the \( \theta \)–deformed Haar gates are defined by \( C_j(\theta) \equiv C_j G_j(\theta) \) and \( G_j(\theta) \) is defined in Def. 7. Clearly (see Fig. 3),

\[
\theta = 0 : C_j G(0) \implies C(0) \in \mathcal{H}_A \\
\theta = 1 : C_j G(1) = C_j(1) = C_j I \implies C(1) = C \quad \text{worst case circuit,}
\]

the first relation follows from the translation invariance property of the Haar measure that was discussed in Sec. (II C) (see Fig. 2). Let us define the deformation of Eq. (13) by

\[
p_0(C(\theta)) \equiv |\langle 0^n | C(\theta) | 0^n \rangle|^2, \tag{14}
\]

which gives \( p_0(C(1)) = p_0(C) \equiv |\langle 0^n | C | 0^n \rangle|^2 \).

We are now in a position to prove that the total variation distance between circuits whose local gates are Haar random unitaries and \( C(\theta) = C_1 G_1(\theta) C_2 G_2(\theta) \ldots C_m G_m(\theta) \) is small. We emphasize that the scrambling takes place locally, which preserves the underlying architecture \( A \).

**Lemma 4.** Let \( A \) be an architecture on circuits with \( m \) gates. Then \( C \cdot \mathcal{H}_A \) is distributed according to \( \mathcal{H}_A \), and the total variation distance between \( C \cdot \mathcal{H}_A \) and \( C \cdot \mathcal{H}_{A,\theta} \) is \( O(\theta) \) for \( \theta \ll 1 \).

**Proof.** Recall that Haar measure implies translational invariance with respect to multiplications by fixed unitaries (see Subsection (II C)). Therefore, if \( H_j \) is distributed according to the Haar
measures then so is $C_jH_j$. Moreover the $\ell_1$ norm that defines total variation distance is invariant under unitary multiplication. So it suffices to compare the measures over $\mathcal{H}_{A,B}$ and $\mathcal{H}_A$, which by Lemma (3) have TVD of $O(\theta)$ over a single local gate. By the additivity of TVD, the distribution induced by $C(\theta)$ which is denoted by $\mathcal{H}_{A,B}$ has a TVD from $\mathcal{H}_A$ that is $O(m\theta)$. \hfill \Box

Remark 8. We prove $O(\theta)$ TVD without requiring a priori that $\theta = 1/\text{poly}(n)$. In an $n$-qubit circuit, $m = \text{poly}(n)$ and is often $m = O(n^2)$. Therefore, if we take $\theta = 1/\text{poly}(n)$ we are guaranteed for the TVD between $C \cdot \mathcal{H}_A$ and $C(\theta) \in C \cdot \mathcal{H}_{A,B}$ to be $O(1/\text{poly}(n))$.

Theorem 3. Let $A$ be an architecture such that $p_0(C)$ is $\#P-$Hard to calculate in the worst case. Then it is $\#P$-hard to compute $3/4 + 1/\text{poly}(n)$ of the probabilities over $\mathcal{H}_A$.

Proof. Consider $p_0(C(\theta))$ over the choice of $C(\theta)$ from the distribution induced by the $\theta-$deformed Haar measure $\mathcal{H}_{A,\theta}$ over $G_i(\theta)$’s and for $\theta = 1/\text{poly}(n)$.

Let $\{G_i\}$ be a collection of independent Haar random gates as defined in Def. (7). Recall from Def. (8) that $C(\theta)$ is the scrambled worst-case circuit, which coincides with the worst case circuit at $\theta = 1$. By the modified QR-Algorithm each of the entries of $C(\theta)$ is a ratio of a polynomial with the square root of a polynomial (see Eq. (5) in Alg. (1)).

We now prove that $p_0(\theta) = |\langle 0^n|C(\theta)|0^n\rangle|^2$ is a rational function of degree at most $(2mD_4, 2mD_4)$, which is low degree for $m = \text{poly}(n)$. This follows from Corollary (1) and the following arguments. Before normalizing each $\langle y_j|C_i(\theta)|y_{j-1}\rangle$ is a polynomial of degree at most $D_4$. The product of such functions in Eq. (12) is simply a polynomial of degree at most $mD_4$ because there are $m$ terms in the product, and these polynomials are closed under addition. Therefore, $\langle 0^n|C(\theta)|0^n\rangle$ is a polynomial of degree at most $mD_4$ over the square root of a polynomial of degree at most $2mD_4$. But we are interested in the probability $p_0(\theta) = |\langle 0^n|C(\theta)|0^n\rangle|^2$, which because of taking the absolute value and squaring becomes exactly a rational function of degree at most $(2mD_4, 2mD_4)$. Recall that $D_4 = 27$ is just a constant.

Similar to the argument in (4), a simple counting shows that if there exists an oracle $\mathcal{O}$ that can compute $p_0(\theta)$ for $3/4 + 1/\text{poly}(n)$ of $C(\theta)$, then given that it correctly evaluates $p_0(\theta)$ over $1/2 + 1/\text{poly}(n)$ of the choices of $\{U_i\}$, it must succeed over $1/2 + 1/\text{poly}(n)$ of $\theta$’s as well. This sets the tolerance to errors for interpolating $p_0(\theta)$ based on sampling using the BW algorithm for rational functions (Alg. (2)).

We are seeking an interpolation of a $(2mD_4, 2mD_4)$ rational function. Since it is expected that $1/2 + 1/\text{poly}(n)$ of $\theta$’s may be erroneous, the least expected number of independent evaluation points ($\theta$’s) to exactly interpolate $F(\theta)$ is $2mD_4\text{poly}(n)$, where the $\text{poly}(n)$ factor is exactly the one that appears in the fraction of successful $\theta$’s. This exactly interpolates $p_0(\theta)$ and we can now interpolate $p_0(1)$, which is the worst case probability $p_0(C)$. Since the latter is assumed to be a $\#P$-Hard problem, it would be a contradiction to have an efficient oracle that evaluates $p_0(\theta)$ for generic $\theta$. \hfill \Box

Remark 9. Since the rational BW algorithm above is explicit and exact, it might help characterize $p_0(\theta) = |\langle 0^n|C(\theta)|0^n\rangle|^2$ sufficiently well as to help proving Conjecture (2).

We now prove that Theorem (3) is necessary for the quantum supremacy conjecture. This follows from closeness in total variational distance of $\mathcal{H}_{A,B}$ to the Haar measure local circuit $\mathcal{H}_A$ that was essentially in Lemma (3) and Theorem (3). Namely, we show that any algorithm $\mathcal{O}$ that works on average over circuits drawn from the distribution $\mathcal{H}_{A,B}$ can be used to get an algorithm that works on average over circuits drawn from $\mathcal{H}_A$.

Corollary 2. Exact average case hardness (Theorem (3)) is necessary for proving the quantum supremacy conjecture (conjecture (2)).
Proof. We prove that conjecture (2) implies Theorem (3). Suppose there exists an algorithm $O$ whose input is a circuit $C(0)$ with local gates being Haar distributed (i.e., $H_A$ distributed) and outputs $p_0(C(0))$. Then the same algorithm run on the circuit $C(\theta)$ whose local gates are distributed according to $H_{A,\theta}$ in Lemma (3) results in the output $p_0(C(\theta))$, whose distribution is $O(\theta)$ close to the distribution of $p_0(C(0))$. This is because the total variation distance is the supremum over the positive difference of all functions between the two distributions. Since the algorithm $O$ is such an example, on inputs $H_A$ and $H_{A,\theta}$, it results in output distributions that are at most $O(\theta)$ in total variation distance. Now if there exists an algorithm that exactly computes probabilities on average (with probability $3/4 + 1/poly(n)$), then the algorithm approximately computes the probabilities over $H_A$. Hence if the latter is $\#P$-hard then the former must be too.

Remark 10. We proved the exact average case hardness of sampling from the output distribution of RCS. The previous work of Bouland et al proves a small robustness of $2^{-\text{poly}(n)}$ for the truncated polynomial series expansion [4]. Since the denominators of the rational functions being considered here can be proved to be not too small, we expect similar arguments based on Patrui’s lemma [14] would prove exponentially small robustness in our case as well. However, we leave a more thorough characterization of the robustness based on the ideas herein for future work.

IV. DISCUSSION AND CONCLUSIONS

Here we constructed a general way of interpolating between unitaries (e.g., quantum gates or circuits), which may find independent use in other contexts such as those involving scrambling. The interpolation is efficient in that certain functions of it can be learned by a polynomial number of samples. This follows from the polynomial and rational function dependence of the entries on the interpolation parameter (Alg. (1)). We also showed that the unitary matrices that result may be controlled to have a distribution close to Haar measure in total variation distance.

The mathematical results were then applied to give a fresh new proof of the average case hardness of random circuit sampling. There are other known facts about RCS, such as the equivalence with sampling and anti-concentration results. We refer the reader to Appendices A5-A7 of [4] for a summary and other complexity theoretical background on this problem.

It is important to note that in the definition of RCS, one wants to allow some errors, partly because the errors are experimentally inevitable. The complexity of RCS is an open problem. To prove the quantum supremacy conjecture the robustness needs to be improved. However, the quantum supremacy conjecture may be false or unrealistic to realize experimentally.

Acknowledgements

I thank Adam Bouland, Bill Fefferman, Karthikeyan Shanmugam, Scott Aaronson, Madhu Sudan for discussions. I acknowledge the support of the Frontiers Foundation and the MIT-IBM collaborative grant.

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