Complete minimal submanifolds with nullity in Euclidean space

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Abstract

In this paper, we investigate minimal submanifolds in Euclidean space with positive index of relative nullity. Let $M^m$ be a complete Riemannian manifold and let $f: M^m \rightarrow \mathbb{R}^n$ be a minimal isometric immersion with index of relative nullity at least $m - 2$ at any point. We show that if the Omori-Yau maximum principle for the Laplacian holds on $M^m$, for instance, if the scalar curvature of $M^m$ does not decrease to $-\infty$ too fast or if the immersion $f$ is proper, then the submanifold must be a cylinder over a minimal surface.

1 Introduction

A frequent theme in submanifold theory is to find geometric conditions for an isometric immersion of a complete Riemannian manifold into Euclidean space $f: M^m \rightarrow \mathbb{R}^n$ with index of relative nullity $\nu \geq k > 0$ at any point to be a $k$-cylinder. This means that the manifold $M^m$ splits as a Riemannian product $M^m = M^{m-k} \times \mathbb{R}^k$ and there is an isometric immersion $g: M^{m-k} \rightarrow \mathbb{R}^{n-k}$ such that $f = g \times id_{\mathbb{R}^k}$.

The index of relative nullity introduced by Chern and Kuiper turned out to be a fundamental concept in the theory of isometric immersions. At a point of $M^m$ the index is just the dimension of the kernel of the second fundamental form of $f: M^m \rightarrow \mathbb{R}^n$ at that point. The kernels form an integrable distribution along any open subset where the index is constant and the images under $f$ of the leaves of the foliation are (part of) affine subspaces in the ambient space. Moreover, if $M^m$ is complete then the leaves are also complete along the open subset where the index reaches its minimum (cf. [4]). Thus, to conclude that $f$ is a cylinder one has to show that the images under $f$ of the leaves of relative nullity are parallel in the ambient space.

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A fundamental result asserting that an isometric immersion \( f: M^m \to \mathbb{R}^n \) with positive index of relative nullity must be a \( k \)-cylinder is Hartman’s theorem [14] that requires the Ricci curvature of \( M^m \) to be nonnegative; see also [19]. A key ingredient for the proof of this result is the famous Cheeger-Gromoll splitting theorem used to conclude that the leaves of minimum relative nullity split intrinsically as a Riemannian factor. Even for hypersurfaces, the same conclusion does not hold if instead we assume that the Ricci curvature is nonpositive. Notice that the latter is always the case if \( f \) is a minimal immersion. Counterexamples easy to construct are the complete irreducible ruled hypersurfaces of any dimension discussed in [7, p. 409].

Some of the many papers containing characterizations of submanifolds as cylinders without the requirement of minimality are [5, 6, 13, 14, 19, 21, 23]. When adding the condition of being minimal we have [1, 8, 11, 12, 13, 15, 26, 28].

In this paper, we extend a result for hypersurfaces due to Savas-Halilaj [24] to the situation of arbitrary codimension.

**Theorem 1** Let \( M^m \) be a complete Riemannian manifold and \( f: M^m \to \mathbb{R}^n \) be a minimal isometric immersion with index of relative nullity \( \nu \geq m - 2 \) at any point of \( M^m \). If the Omori-Yau maximum principle holds on \( M^m \), then \( f \) is a cylinder over a minimal surface.

We recall that the Omori-Yau maximum principle holds on \( M^m \) if for any bounded from above function \( \varphi \in C^\infty(M) \) there exists a sequence of points \( \{x_j\}_{j \in \mathbb{N}} \) such that

\[
\lim \varphi(x_j) = \sup \varphi, \quad \|\nabla \varphi\|(x_j) \leq 1/j \quad \text{and} \quad \Delta \varphi(x_j) \leq 1/j
\]

for each \( j \in \mathbb{N} \).

The category of complete Riemannian manifolds for which the principle is valid is quite large. For instance, it contains the manifolds with Ricci curvature bounded from below. It also contains the class of properly immersed submanifolds in a space form whose norm of the mean curvature vector is bounded (cf. [22, Example 1.14]).

**Corollary 2** Let \( M^m \) be a complete Riemannian manifold and \( f: M^m \to \mathbb{R}^n \) be a minimal isometric immersion with index of relative nullity \( \nu \geq m - 2 \) at any point of \( M^m \). Assume that either the scalar curvature \( \text{scal} \) of \( M^m \) satisfies \( \text{scal} \geq -c(d \log d)^2 \) outside a compact set, where \( c > 0 \) and \( d = d(\cdot, o) \) is the geodesic distance to a reference point \( o \in M^m \), or that \( f \) is proper. Then \( f \) is a cylinder over a minimal surface.

Theorem[1] is truly global in nature since there are plenty of (noncomplete) examples of minimal submanifolds of any dimension \( m \) with constant index \( \nu = m - 2 \) that are not part of a cylinder on any open subset. They can be all locally parametrically described in terms of a certain class of elliptic surfaces; see Theorem 22 in [5]. In particular, there is a Weierstrass type representation for these submanifolds when the manifold
possesses a Kähler structure; see Theorem 27 in [5]. On the other hand, after the results of this paper what remains as a challenging open problem is the existence of a minimal complete noncylindrical submanifold $f : M^3 \rightarrow \mathbb{R}^n$ with $\nu \geq 1$.

The main difficulty in the proof of Theorem 1 arises from the fact that the index of relative nullity $\nu$ is allowed to vary. Consequently, one has to fully understand the structure of the set of points $A \subset M^m$ where $f$ is totally geodesic in order to conclude that the relative nullity foliation on $M^m \setminus A$ extends smoothly to $A$.

Recently Jost, Yang and Xin [17] proved various Bernstein type results for complete $m$-dimensional minimal graphical submanifolds in Euclidean space with $\nu \geq m - 2$. We observe that from a result in [7] it follows that the submanifolds considered in [17, Theorem 1.1] are cylinders over 3-dimensional complete minimal submanifolds with $\nu \geq 1$. Moreover, from Corollary 2 it follows that the submanifolds considered in [17, Theorem 1.2] are just cylinders over complete minimal surfaces, since entire graphs are proper submanifolds. Thus, to prove a Bernstein theorem for such submanifolds is equivalent to show a Bernstein theorem for entire minimal 2-dimensional graphs in Euclidean space.

2 Preliminaries

In this first section, we recall some basic facts from the theory of isometric immersions that will be used in the proof of Theorem 1.

Let $M^m$ be a Riemannian manifold and $f : M^m \rightarrow \mathbb{R}^n$ be an isometric immersion. As usual, often $M^m$ will be locally identified with its image. The relative nullity subspace $\mathcal{D}(x)$ of $f$ at $x \in M^m$ is the kernel of its second fundamental form $\alpha : T_M \times T_M \rightarrow N_fM$ with values in the normal bundle, that is,

$$\mathcal{D}(x) = \{X \in T_x M : \alpha(X, Y) = 0 \text{ for all } Y \in T_x M\}.$$

Then, the dimension $\nu(x)$ of $\mathcal{D}(x)$ is called the index of relative nullity of $f$ at $x \in M^m$. Let $U \subset M^m$ be an open subset where the index of relative nullity $\nu = s > 0$ is constant. It is a standard fact that the relative nullity distribution $\mathcal{D}$ along $U$ is integrable, that the leaves of relative nullity are totally geodesic submanifolds of $M^m$ and that their images under $f$ are open subsets of affine subspaces in $\mathbb{R}^n$. The following is a well-known result in the theory of isometric immersions (cf. [4, Theorem 5.3]).

**Proposition 3** Let $\gamma : [0, b] \rightarrow M^m$ be a geodesic curve such that $\gamma([0, b])$ is contained in a leaf of relative nullity contained in $U$. Then also $\nu(\gamma(b)) = s$.

The conullity space of $f$ at $x \in M^m$ is the orthogonal complement $\mathcal{D}^\perp(x)$ of $\mathcal{D}(x)$ in the tangent bundle $TM$. We write $X = X^v + X^h$ according to the orthogonal splitting
\(TM = D \oplus D^\perp\) and denote \(\nabla^h_X Y = (\nabla_X Y)^h\). The splitting tensor \(C: D \times D^\perp \to D^\perp\) is given by
\[
C(T, X) = -\nabla^h_X T
\]
for any \(T \in D\) and \(X \in D^\perp\). The following differential equations for the tensor \(C_T = C(T, \cdot)\) are well-known to hold (cf. [4] or [7]):
\[
\nabla_S C_T = C_T C_S + C_{\nabla S T}
\]
(1)
and
\[
(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{\nabla^h_X T}Y - C_{\nabla^h_Y T}X,
\]
(2)
for any \(S, T \in \Gamma(D)\) and \(X, Y \in \Gamma(D^\perp)\).

Finally, we have the following elementary result from the theory of submanifolds.

**Proposition 4** Let \(f: M^m \to \mathbb{R}^n\) be an isometric immersion with constant index of relative nullity \(\nu = s > 0\) and complete leaves of relative nullity. If the splitting tensor \(C\) vanishes, then \(f\) is a \(s\)-cylinder.

**Proof:** That \(C = 0\) is equivalent to \(D\) being parallel in \(M^m\). Consequently, the images via \(f\) of the leaves of \(D\) are also parallel in \(\mathbb{R}^n\). \(\square\)

### 3 The proofs

The possible structures of an isometric immersion \(f: M^m \to \mathbb{R}^n\) when \(M^m\) is complete and the index of relative nullity of \(f\) satisfies \(\nu \geq m - 2\) at any point was completely described in [7]. In particular, if \(f\) is real analytic then it has to be either completely ruled or a cylinder over a 3-dimensional complete submanifold with \(\nu \geq 1\). In the case of minimal submanifolds, it follows from Theorem 16 in [5] that we only have to consider the case of a nontrivial minimal \(f: M^3 \to \mathbb{R}^n\) with \(\nu \geq 1\) at any point of \(M^3\).

Let \(U \subset M^3\) be an open subset where \(\nu = 1\) and the line bundle of relative nullity is trivial. Fix a smooth unit section \(e\) spanning the relative nullity distribution along \(U\) and let \(J\) denote the unique, up to sign, almost complex structure acting on the conullity distribution \(D^\perp = \{e\}^\perp\). For simplicity, we set \(\mathcal{C} = C_e\). Observe that our aim of proving Theorem [1] will be achieved if we show that \(\mathcal{C}\) is identically zero. The following lemma is of crucial importance.

**Lemma 5** There are harmonic functions \(u, v \in C^\infty(U)\) such that
\[
\mathcal{C} = vI - uJ
\]
(3)
where \(I\) stands for the identity map on the conullity distribution.
Proof: We may assume that the immersion $f$ is substantial, that is, it does not reduce codimension. Let $A_\xi$ be the shape operator of $f$ with respect to the normal direction $\xi$, i.e.,

$$\langle A_\xi \cdot, \cdot \rangle = \langle \alpha(\cdot, \cdot), \xi \rangle.$$ 

From the Codazzi equation for $A_\xi|_{D^\perp}$ restricted to $D^\perp$ we have that

$$\nabla e_i A_\xi|_{D^\perp} = A_\xi|_{D^\perp} \circ C + A_{\nabla^\perp \xi}|_{D^\perp}$$

for any normal vector field $\xi \in N_f M$. Thus $A_\xi|_{D^\perp} \circ C$ has to be symmetric, and hence

$$A_\xi|_{D^\perp} \circ C = C^t \circ A_\xi|_{D^\perp}. \quad (4)$$

On the other hand, the minimality condition is equivalent to

$$A_\xi|_{D^\perp} \circ J = J^t \circ A_\xi|_{D^\perp}. \quad (5)$$

First we consider the hypersurface case $n = m + 1$. Take a local orthonormal tangent frame $e_1, e_2, e_3$ that diagonalizes the shape operator of $f$ such that

$$Je_1 = e_2 \quad \text{and} \quad e_3 = e$$

and let $\xi$ be a unit normal along the hypersurface. Set

$$u = \langle \nabla e_2 e_1, e_3 \rangle \quad \text{and} \quad v = \langle \nabla e_1 e_1, e_3 \rangle.$$

From the Codazzi equation

$$\langle \nabla e_i A_\xi \rangle e_3 = \langle \nabla e_3 A_\xi \rangle e_i,$$

where $1 \leq i \leq 2$, we have that $\langle \nabla e_2 e_2, e_3 \rangle = v$. Moreover, from

$$\langle (\nabla e_1 A_\xi) e_2, e_3 \rangle = \langle (\nabla e_2 A_\xi) e_1, e_3 \rangle,$$

we obtain that $\langle \nabla e_1 e_2, e_3 \rangle = -u$. Now we can readily see that (3) holds true.

Now assume that $f$ is not an hypersurface. Consider the space

$$N_f^1(x) = \text{span}\{\alpha(X, Y) : \text{for all } X, Y \in T_x M\}.$$ 

Notice that the dimension of $N_f^1(x)$ is at most two due to minimality. Suppose that there is an open subset $V \subset M^3$ where $\dim N_f^1 = 1$. A simple argument using the Codazzi equation [4, Corollary 4.7] shows that $N_f^1$ is parallel in the normal bundle along $V$, and thus the map $f|_V$ reduces codimension to an hypersurface. But due to real analyticity, the same would hold globally, and that is a contradiction. Hence, there is an open dense subset $W$ of $M^3$ where $\dim N_f^1 = 2$. We conclude from (4) and (5) that $C \in \text{span}\{I, J\}$ on $U \cap W$. By continuity, we then get that $C \in \text{span}\{I, J\}$ on $U$. Therefore, also in this case there are functions $u, v \in C^\infty(U)$ such that (3) holds.
It remains to show that $u, v$ are harmonic. From (1) and (2) we have
\[ \nabla^b e^C = C^2 \] (6)
and
\[ (\nabla^b e^C) Y \equiv (\nabla^b e^C) X \] (7)
for any $X, Y \in \Gamma(D^\perp)$. For a local orthonormal tangent frame $e_1, e_2, e_3$ such that $Je_1 = e_2$ and $e_3 = e$, it follows from (3) that
\[ v = \langle \nabla e_1 e_1, e_3 \rangle = \langle \nabla e_2 e_2, e_3 \rangle \] (8)
and
\[ u = -\langle \nabla e_1 e_2, e_3 \rangle = \langle \nabla e_2 e_1, e_3 \rangle. \] (9)
It is easily seen that (6) is equivalent to
\[ e_3(v) = v^2 - u^2 \quad \text{and} \quad e_3(u) = 2uv \] (10)
whereas (7) to
\[ e_1(u) = e_2(v) \quad \text{and} \quad e_2(u) = -e_1(v). \] (11)
The Laplacian of $v$ is given by
\[ \Delta v = \sum_{j=1}^{3} e_j e_j(v) + \omega_{12}(e_2) e_1(v) - \omega_{12}(e_1) e_2(v) - (\omega_{13}(e_1) + \omega_{23}(e_2)) e_3(v) \] (12)
where
\[ \omega_{ij}(e_k) = \langle \nabla e_k e_i, e_j \rangle, \]
where $1 \leq i, j, k \leq 3$. Using (2) and (11), we have that
\[ e_1(e_1(v) + e_2 e_2(v)) = -e_1 e_2(u) + e_2 e_1(u) = [e_2, e_1](u) \]
\[ = \nabla e_2 e_1(u) - \nabla e_1 e_2(u) \]
\[ = \omega_{12}(e_1) e_1(u) + \omega_{12}(e_2) e_2(u) + (\omega_{13}(e_1) - \omega_{23}(e_1)) e_3(u) \]
\[ = \omega_{12}(e_1) e_2(v) - \omega_{12}(e_2) e_1(v) + 2ue_3(u). \]
Inserting the last equality into (12) and using (8) and (10) yields
\[ \Delta v = e_3 e_3(v) + 2ue_3(u) - 2ve_3(v) = 0. \]
That also $u$ is harmonic is proved in a similar manner.

Let us focus in the 3-dimensional case, i.e., let $f: M^3 \to \mathbb{R}^n$ be a minimal isometric immersion of a complete Riemannian manifold with index of relative nullity $\nu(x) \geq 1$ at any point $x \in M^3$, that is, the index is either 1 or 3. Let $\mathcal{A}$ denote the set of totally
geodesic points of $f$. From Proposition 3 the relative nullity foliation $\mathcal{D}$ is a line bundle on $M^3 \setminus \mathcal{A}$. Due to the real analyticity of the submanifold, the square of the norm of the second fundamental form is a real analytic function. It follows that $\mathcal{A}$ is a real analytic set. According to Lojasewicz’s structure theorem [18, Theorem 6.3.3] the set $\mathcal{A}$ locally decomposes as

$$\mathcal{A} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3,$$

where each $\mathcal{V}^d$, $0 \leq d \leq 3$, is either empty or a disjoint finite union of $d$-dimensional real analytic subvarieties. A point $x_0 \in \mathcal{A}$ is called a regular point of dimension $d$ if there is a neighborhood $\Omega$ of $x_0$ such that $\Omega \cap \mathcal{A}$ is a $d$-dimensional real analytic submanifold of $\Omega$. If otherwise $x_0$ is said to be a singular point. The set of singular points is locally a finite union of submanifolds.

Our goal now is to show that $\mathcal{A} = \mathcal{V}^1$, unless $f$ is just an affine subspace in $\mathbb{R}^n$ in which case Theorem 1 trivially holds. After excluding the latter trivial case, we have from the real analyticity of $f$ that $\mathcal{V}^3$ is empty.

**Lemma 6** The set $\mathcal{V}^2$ is empty.

**Proof:** We only have to show that there is no regular point in $\mathcal{V}^2$. Suppose to the contrary that such a point do exist. Let $\Omega \subset M^3$ be an open neighborhood of a smooth point $x_0 \in \mathcal{V}^2$ such that $L^2 = \Omega \cap \mathcal{A}$ is an embedded surface. Let $e_1, e_2, e_3, \xi_1, \ldots, \xi_{n-3}$ be an orthonormal frame adopted to $M^3$ along $\Omega$ near $x_0$. The coefficients of the second fundamental form are

$$h^a_{ij} = \langle \alpha(e_i, e_j), \xi_a \rangle$$

where from now on $1 \leq i, j, k \leq 3$ and $1 \leq a, b \leq n - 3$.

The Gauss map $\gamma : M^3 \to Gr(3, n)$ of $f$ as a map into the Grassmannian of oriented 3-dimensional subspaces in $\mathbb{R}^n$ is defined by $\gamma(x) = T_xM^3 \subset \mathbb{R}^n$, up to parallel translation in $\mathbb{R}^n$ to the origin. Regarding $Gr(3, n)$ as a submanifold in $\wedge^3 \mathbb{R}^n$ via the map for the Plücker embedding, we have that $\gamma = e_1 \wedge e_2 \wedge e_3$. Then

$$\gamma^* e_i = \sum_{j,a} h^a_{ij} e_{ja}$$

(13)

where $e_{ja}$ is obtained by replacing $e_j$ with $\xi_a$ in $e_1 \wedge e_2 \wedge e_3$. Then

$$\sum_i (\gamma^* e_i, \gamma^* e_i) = \sum_{i,j,a} (h^a_{ij})^2 = \|\alpha\|^2$$

where the inner product of two simple 3-vectors in $\wedge^3 \mathbb{R}^n$ is defined by

$$\langle a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3 \rangle = \det \left( \langle a_i, b_j \rangle \right).$$

For a fixed simple 3-vector $A = a_1 \wedge a_2 \wedge a_3$ let $w_A : M^3 \to \mathbb{R}$ be the function defined by

$$w_A = \langle \gamma, A \rangle.$$
Note that $w_A$ is a kind of height function. Because the immersion $f$ is minimal, the function $w_A$ satisfies
\[
\Delta w_A = -\|\alpha\|^2 w_A + \sum_{i,a \neq b,j \neq k} h_{ij}^a h_{ik}^b \langle e_{ja,kb}, A \rangle
\]
where $e_{ja,kb}$ is obtained by replacing $e_j$ with $\xi_a$ and $e_k$ with $\xi_b$ in $e_1 \wedge e_2 \wedge e_3$ (cf. [27, p. 36]). Let $\varepsilon_1, \ldots, \varepsilon_n$ be an orthonormal basis of $\mathbb{R}^n$. The set
\[
\{\varepsilon_{j_1} \wedge \varepsilon_{j_2} \wedge \varepsilon_{j_3} : 1 \leq j_1 < j_2 < j_3 \leq n\}
\]
of 3-vectors is an orthonormal basis of $\wedge^3 \mathbb{R}^n$ by means of which identify $\wedge^3 \mathbb{R}^n$ with $\mathbb{R}^3$. Denoting by $\{A_J\}_{J \in \{1, \ldots, N\}}$ the corresponding base in $\mathbb{R}^N$, we have
\[
\gamma = \sum_{J=1}^N w_J A_J \text{ where } w_J = \langle \gamma, A_J \rangle.
\]
From $h_{ij}^a = \langle \gamma_a e_i, e_{ja} \rangle$, we obtain
\[
h_{ij}^a = \sum_J \langle e_{ja}, A_J \rangle e_i(w_J).
\]  \hfill (14)
Moreover, for any $J \in \{1, \ldots, N\}$, it holds
\[
\Delta w_J = -\|\alpha\|^2 w_J + \sum_{i,a \neq b,j \neq k} h_{ij}^a h_{ik}^b \langle e_{ja,kb}, A_J \rangle.
\]  \hfill (15)
Take a local chart $\phi: U \to \mathbb{R}^3$ of coordinates $x = (x_1, x_2, x_3)$ on an open subset $U$ of $\Omega$ and set
\[
e_i = \sum_j \mu_{ij} \partial x_j.
\]  \hfill (16)
Setting $\theta_J = w_J \circ \phi^{-1}$, we obtain the map $\theta = : \phi(U) \subset \mathbb{R}^3 \to \mathbb{R}^N$ given by
\[
\theta = \sum_J \theta_J A_J = (\theta_1, \ldots, \theta_N).
\]
Note that $\theta = \gamma \circ \phi^{-1}$, i.e., $\theta$ is just the representation of the Gauss map with respect to the above mentioned charts. From (14) and (16) we have
\[
h_{ij}^a = \sum_{k,J} \mu_{ik} \langle e_{ja}, A_J \rangle \langle \theta_J \rangle_{x_k}
\]  \hfill (17)
and
\[
\|\alpha\|^2 = \sum_{i,j,a} \left( \sum_{k,J} \mu_{ik} \langle e_{ja}, A_J \rangle \langle \theta_J \rangle_{x_k} \right)^2.
\]  \hfill (18)
The Laplacian of $M^3$ is given by

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_{x_i} \left( \sqrt{g} g^{ij} \partial_{x_j} \right)$$

where $g_{ij}$ are the components of the metric of $M^3$ and $g = \text{det}(g_{ij})$. Using (17) and (18) we see that (15) is of the form

$$\sum_{i,j} g_{ij} (\theta J)_{x_i x_j} + C_J (x, \theta, \theta x_1, \theta x_2, \theta x_3) = 0,$$

where $C_J : \phi(U) \times \mathbb{R}^{4N} \to \mathbb{R}$ is given by

$$C_J(x, y, z_1, z_2, z_3) = \frac{1}{\sqrt{g}} \sum_{i,j} (\sqrt{g} g^{ij})_{x_i z_j} + y_J \sum_{i,j,a} \left( \sum_{k,l} \mu_i \langle e_{ja}, A_l \rangle z_{kI} \right)^2$$

$$- \sum_{I,K} \sum_{i,j,m \neq k} \mu_i m \langle e_{ja,bk}, A_I \rangle \langle e_{ja}, A_K \rangle \langle e_{kb}, A_I \rangle z_{mI} z_{lK}$$

where $y = (y_1, \ldots, y_N)$, $z_i = (z_{i1}, \ldots, z_{iN})$, $i, m, l \in \{1, 2, 3\}$ and $I, J, K \in \{1, \ldots, N\}$. Therefore, we have that the vector valued map $\theta = (\theta_1, \ldots, \theta_N)$ satisfies the elliptic equation

$$L\theta = \sum_{i,j} A_{ij}(x) \theta_{x_i x_j} + C(x, \theta, \theta x_1, \theta x_2, \theta x_3) = 0$$

where $A_{ij} = g^{ij} I_N$, $I_N$ being the identity $N \times N$ matrix and $C = (C_1, \ldots, C_N)$. Moreover, we have from (13) that $\theta$ is constant on $\phi(L^2)$ and $\bar{n}(\theta) = 0$ on $\phi(L^2)$ where $\bar{n}$ is a unit normal field to the surface $\phi(L^2)$ in $\mathbb{R}^3$.

Consider the Cauchy problem $L\theta = 0$ with the following initial conditions: $\theta$ is constant on $\phi(L^2)$ and $\bar{n}(\theta) = 0$ on $\phi(L^2)$. According to the Cauchy-Kowalewsky theorem (cf. [25]) the problem has a unique solution if the surface $\phi(L^2)$ is noncharacteristic. This latter is satisfied if $Q(\bar{n}) \neq 0$, where $Q$ is the characteristic form given by

$$Q(\zeta) = \text{det}(\Lambda(\zeta))$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ and

$$\Lambda(\zeta) = \sum_{i,j} g^{ij} \zeta_i \zeta_j I_N$$

is the symbol of the differential operator $L$. That the surface $\phi(L^2)$ is noncharacteristic follows from

$$Q(\zeta) = \left( \sum_{i,j} g^{ij} \zeta_i \zeta_j \right)^N.$$

Because $C(x, y, 0, 0, 0) = 0$ the constant maps are solutions to the Cauchy problem. From the uniqueness part of the Cauchy-Kowalewsky theorem we conclude that the Gauss map $\gamma$ is constant on an open subset of $M^3$, and that is not possible.
Lemma 7  The set $\mathcal{V}^0$ is empty.

Proof: Suppose that $x_0 \in \mathcal{V}^0$ and let $\Omega$ be an open neighborhood around $x_0$ such that $\nu = 1$ on $\Omega \setminus \{x_0\}$. Let $\{x_j\}_{j \in \mathbb{N}}$ be a sequence in $\Omega \setminus \{x_0\}$ converging to $x_0$. Let $e_j = e(x_j) \in T_{x_j}M$ be the sequence of unit vectors contained in the relative nullity distribution of $f$. By passing to a subsequence, if necessary, there is a unit vector $e_0 \in T_{x_0}M$ such that $\lim e_j = e_0$. By continuity, the geodesic tangent to $e_0$ at $x_0$ is a leaf of relative nullity outside $x_0$. But this is a contradiction in view of Proposition 3.

Lemma 8  The foliation $\mathcal{F}$ of the nullity distribution extends analytically over the regular points of $\mathcal{A}$.

Proof: First observe that the relative nullity distribution extends continuously over the smooth points of $\mathcal{A}$. In fact, by the previous lemmas it remains to consider the case when $\Omega$ is an open subset of $M^3$ such that $\Omega \cap \mathcal{A}$ is a open segment in a straight line in the ambient space. But in this situation the result follows by a argument of continuity similar than in the proof of Lemma 7.

Let $\Omega$ be an open subset of $M^3 \setminus \mathcal{A}$ and let $e_1, e_2, e_3$ be a local frame on $\Omega$ as in the proof of Lemma 5. Consider the map $F: \Omega \to S^{n-1}$ into the unit sphere given by $F = f_* e_3$. A straightforward computation using (8), (9) and (11) gives that its tension field

$$\tau(F) = \sum_{j=1}^{3} \left( \nabla_{e_j} F_* e_j - F_* \nabla_{e_j} e_j \right)$$

vanishes. Here $\nabla$ denotes the Levi-Civita connection of $S^{n-1}$. Hence $F$ is a harmonic map. Because $\mathcal{A} = \mathcal{V}^1$ its 2-capacity $\text{cap}_2(\mathcal{A})$ must be zero (cf. [10, Theorem 3]). Since the map $F$ is continuous on $M^3$, it follows from a theorem of Meier [20, Theorem 1]) that $F$ is of class $C^2$ on $M^3$. But then $F$ is real analytic by a result due to Eells-Sampson [9, Proposition p. 117].

Lemma 9  The set $\mathcal{A}$ has no singular points.

Proof: According to Lemmas 6 and 7 the set $\mathcal{A}$ only contains subvarieties of dimension one with possible isolated singular points. Thus, by Lemma 8 the set of smooth points of $\mathcal{A}$ just contains segments of straight lines. Hence, if there is a singular point in $\mathcal{A}$ it must be the intersection of such geodesic lines, and that is clearly not possible.

The proof of our main result relies heavily on the following consequence or the Omori-Yau maximum principle; see [3, Theorem 28] or [16, Lemma 4.1].

Lemma 10  Let $M^m$ be a complete Riemannian manifold for which the Omori-Yau maximum principle holds. If $\varphi \in C^\infty(M)$ satisfies $\Delta \varphi \geq 2\varphi^2$ and $\varphi \geq 0$, then $\varphi = 0$. 

10
Proof of Theorem 1: Without loss of generality we may assume that $M^3$ is oriented by passing to the oriented double cover if necessary. It follows from Lemmas 8 and 9 that $J$ is globally defined and that $\|C\|^2 = u^2 + v^2$ is real analytic on $M^3$. From Lemma 5 and (10) it follows that
\[
\Delta(u^2 + v^2) = 2\|\nabla u\|^2 + 2\|\nabla v\|^2 \geq 2(u^2 + v^2)^2.
\]
We deduce from Lemma 10 that $C = 0$, and by Proposition 4 this implies the desired splitting result. □

Proof of Corollary 2: The Omori-Yau maximum principle holds on $M^m$ under the assumption on the scalar curvature (see [2] or [3, Theorem 2.4]) or if the immersion $f$ is proper (see [3, Theorem 2.5]). □

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