Weak decays of triply heavy baryons in the light-front approach

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In this work, we investigate the weak decays of ground-state triply heavy baryons. We first obtain the form factors using the light-front quark model in the three-quark picture, and then apply them to arrive at some phenomenological predictions, including the decay widths of semileptonic decays and nonleptonic decays. Our results are expected to be helpful for the experimental search for triply heavy baryons.

I. INTRODUCTION

Since LHCb reported the discovery of the doubly charmed baryon $\Xi_{cc}^{++}$ in the final state $\Lambda_c^+ K^- \pi^+ \pi^+$ [1], the experimentalists have made great progress in the field of doubly heavy baryons (DHBs) – They measured the lifetime of $\Xi_{cc}^{++}$ [2], and then found two other new channels $\Xi_{cc}^{++} \rightarrow \Xi_c^{++} \pi^+$ [3] and $\Xi_{cc}^{++} \rightarrow \Xi_c^{++} \pi^+$ [4]. Surrounded by optimism, people tend to expect that triply heavy baryons (THBs) may be discovered in the near future. The theoretical research on THBs are highly demanded.

Some efforts have been made in this direction. Ref. [5] obtained the masses of the ground-state THBs using QCD sum rules. In Ref. [6], the authors performed an analysis on the weak decays with the help of SU(3) flavor symmetry. In recent works [7, 8], the authors investigated the weak decays of $\Omega_{ccc}^{++}$ and $\Omega_{bbb}^{--}$. More works can be found in Refs. [9–39]. However, there is still a lack of one comprehensive quantitative analysis on the weak decays of THBs. This work and the forthcoming ones aim to fill this gap.

In this work, we only consider S-wave THBs, which can only be spin-3/2 or spin-1/2. We have, for the former, $\Omega_{ccc}^{++}$, $\Omega_{bcc}^{++}$, $\Omega_{bbc}^{0}$ and $\Omega_{bbc}^{--}$, while for the latter, $\Omega_{bcc}^{++}$ and $\Omega_{bbc}^{0}$. Their flavor wavefunctions are given as:

\begin{align}
\Omega_{QQQ} &= QQQ,
\Omega'_{QQQ'} &= \frac{1}{\sqrt{3}}(Q'QQ + QQ'Q + QQQ'),
\Omega_{QQQ'} &= QQQ',
\end{align}

where $Q, Q' = c/b$. It can be seen that, for the spin-3/2 THBs, the three heavy quarks are on an equal footing, while for the spin-1/2 THBs, the two identical heavy quarks are usually considered

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1 Readers may have noticed our naming principle here: keep the symbols of the ground states as simple as possible. The same goes for DHBs.
to form an axial-vector diquark. The spin-3/2 $\Omega'_{QQQ}$ and spin-1/2 $\Omega_{QQQ}'$ have the same quark components, while the latter is usually considered as the ground state. As can be seen in Ref. [40], the flavor mixing between baryons with the same quark components, is ubiquitous. The dynamics mechanism comes from the gluon exchange between quarks inside baryons. However, the mixing is small when a heavy quark serves as a source of color charge. In a word, the mass eigenstates discovered in experiments are the flavor mixing states of $\Omega'_{QQQ}$ and $\Omega_{QQQ}'$, but this mixing can be neglected. The experimentalists should prioritize searching for the ground-state $\Omega^{++}_{ccc}$, $\Omega^{-}_{bbb}$, and $\Omega^{++}_{bcc}$, $Q^{0}_{bbc}$.

Therefore, in this work, we will investigate the weak decays of THBs for the $3/2 \to 1/2$ case and $1/2 \to 1/2$ case, and the method of light-front quark model (LFQM) will be adopted. LFQM has been widely used to study the properties of mesons [41–64], and has achieved great success. In recent years, LFQM is further applied to the baryon sector: in Refs. [65–69], the diquark picture is adopted, while in Refs. [70–78], the three-quark picture is used. In the diquark picture, the two loosely-bound spectator quarks are viewed as a whole to form so-called “diquark”, whose spin-parity can only be $0^+$ or $1^+$ for a ground-state baryon. The diquark picture has some inevitable flaws. Firstly, a diquark is arbitrarily designated for the convenience of research. Secondly, the diquark picture includes parameters such as the diquark mass that are difficult to determine. Lastly, the overlap factor, which is actually the inner product of the flavor-spin wavefunctions of the initial and final baryons, is not easy to obtain in the diquark picture. In this work, we will adopt the three-quark picture.

Specifically, in this work, we will consider the following $3/2 \to 1/2$ processes

\[
\begin{align*}
\Omega^{++}_{ccc} &\rightarrow \Xi^{+}_{cc}(dcc)/\Omega^{+}_{cc}(scc), \\
\Omega^{+}_{bcc}(ccb) &\rightarrow \Xi^{0}_{bc}(dbc)/\Omega^{0}_{bc}(scb), \\
\Omega^{0}_{bcc}(cbb) &\rightarrow \Xi^{-}_{bb}(dbb)/\Omega^{-}_{bb}(sbb), \\
\Omega^{0}_{bce}(bce) &\rightarrow \Xi^{+}_{cc}(ucc), \\
\Omega^{0}_{bce}(bce) &\rightarrow \Xi^{+}_{bc}(ubc), \\
\Omega^{0}_{bce}(bcc) &\rightarrow \Xi^{0}_{bc}(ubb).
\end{align*}
\]

\[\text{(2)}\]

and the following $1/2 \to 1/2$ processes

\[
\begin{align*}
\Omega^{+}_{bec}(ccb) &\rightarrow \Xi^{(0)}_{bc}(dbc)/\Omega^{(0)}_{bc}(scb), \\
\Omega^{0}_{bce}(cbb) &\rightarrow \Xi^{-}_{bb}(dbb)/\Omega^{-}_{bb}(sbb), \\
\Omega^{+}_{bce}(bce) &\rightarrow \Xi^{+}_{cc}(ucc), \\
\Omega^{0}_{bce}(bce) &\rightarrow \Xi^{0}_{bc}(ubc),
\end{align*}
\]

\[\text{(3)}\]

where for the final-state DHBs, we have denoted, for example,

\[\Xi^{0}_{bc} = \frac{1}{2}(bc - cb)d,\]
\[ \Xi^{(i)0}_{bc} = \frac{1}{2}(bc + cb)d. \]

For \( \Xi^{(i)0}_{bc} \), the two heavy quarks are usually considered as a scalar (axial-vector) diquark, while for \( \Xi_{QQ} \), the two identical heavy quarks are considered to form an axial-vector diquark. For more details on the classification of ground-state DHBs, one can refer to Ref. [79]. The \( \frac{1}{2} \to \frac{1}{2} \) processes can be further divided into two categories – according to whether the system of two spectator quarks is a scalar or axial vector. However, for the \( \frac{3}{2} \to \frac{1}{2} \) processes, the system of the two spectator quarks can only be an axial vector. Therefore, we will consider the following three typical processes:

- \( \Omega^{(i)+}_{bcc}(ccb) \to \Omega^{0}_{bc}(scb) \) for the \( \frac{3}{2} \to \frac{1}{2} \) case,
- \( \Omega^{(i)+}_{bcc}(cb) \to \Omega^{0}_{bc}(scb) \) for the \( \frac{1}{2} \to \frac{1}{2} \) case with a scalar spectator diquark,
- \( \Omega^{(i)+}_{bcc}(cb) \to \Omega^{0}_{bc}(scb) \) for the \( \frac{1}{2} \to \frac{1}{2} \) case with an axial-vector spectator diquark.

The rest of this paper is arranged as follows. In Sec. II, the framework of light-front approach under the three quark-picture is briefly introduced. We will discuss the transition matrix elements for three typical processes and extract the corresponding form factors. The non-trivial overlap factors in flavor-spin space are also shown. In Sec. III, the numerical results for the form factors are presented, and then used to give some phenomenological predictions. We conclude this paper in the last section.

II. FORM FACTORS IN THE LIGHT-FRONT APPROACH

A. The light-front approach

In Ref. [77], we reviewed the light-front quark model of baryons under the three-quark picture and provided constructive proofs for the spin wavefunctions of S-wave baryons. Some of the main results are listed below.

In the framework of light-front approach under the three-quark picture, a baryon state is expressed as

\[
|B(P, S, S_z)\rangle = \int \{d^3\tilde{p}_1\} \{d^3\tilde{p}_2\} \{d^3\tilde{p}_3\} \frac{2(2\pi)^3 \delta^3(\tilde{P} - \tilde{p}_1 - \tilde{p}_2 - \tilde{p}_3)}{\sqrt{P^+}} \times \sum_{\lambda_1, \lambda_2, \lambda_3} \Psi^{SSz}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \lambda_1, \lambda_2, \lambda_3) C^{ijk}_{\lambda_1 \lambda_2 \lambda_3} q_i^1(p_1, \lambda_1) q_j^2(p_2, \lambda_2) q_k^3(p_3, \lambda_3),
\]

where \( p_i (\lambda_i) \) is the light-front momentum (helicity) of the quark \( q_i \), the color wavefunction \( C^{ijk} = \epsilon^{ijk}/\sqrt{6} \), and the flavor-spin and momentum wavefunctions are contained in \( \Psi^{SSz} \). The light-front momentum is decomposed into \( p_i = (p_i^-, p_i^+, p_{i\perp}) \) with \( p_i^\pm = p_i^0 \pm p_i^3 \) and \( p_{i\perp} = (p_i^1, p_i^2) \), and the
quarks are assumed to be on mass shell:

\[ p_i^- = \frac{m_i^2 + p_{i\perp}^2}{p_i^+}. \]  

(6)

In Eq. (5),

\[ \hat{p}_i = (p_i^+, p_{i\perp}), \quad \{d^3\hat{p}_i\} = \frac{dp_i^+ d^2p_{i\perp}}{2(2\pi)^3}. \]

(7)

The intrinsic variables \((x_i, k_{i\perp})\) are introduced through

\[ p_i^+ = x_i P^+, \quad p_{i\perp} = x_i P_{\perp} + k_{i\perp}, \]

\[ \sum_{i=1}^3 x_i = 1, \quad \sum_{i=1}^3 k_{i\perp} = 0, \]

(8)

where \(x_i\) is the light-front momentum fraction constrained by \(0 \leq x_i \leq 1\). Define \(\bar{P}^\mu \equiv p_1^+ + p_2^+ + p_3^+\) and \(M_0^2 \equiv \bar{P}^2\), and it can be shown that

\[ M_0^2 = \frac{k_{1\perp}^2 + m_1^2}{x_1} + \frac{k_{2\perp}^2 + m_2^2}{x_2} + \frac{k_{3\perp}^2 + m_3^2}{x_3}. \]

(9)

The internal momenta are defined by

\[ k_i = (k_i^-, k_i^+, k_{i\perp}) = (e_i - k_{i\perp}, e_i + k_{i\perp}, k_{i\perp}) = \left( \frac{m_i^2 + k_{i\perp}^2}{x_i M_0}, x_i M_0, k_{i\perp} \right), \]

(10)

then it is easy to verify:

\[ e_i = \frac{x_i M_0}{2} + \frac{m_i^2 + k_{i\perp}^2}{2x_i M_0}, \]

\[ k_{i\perp} = \frac{x_i M_0}{2} - \frac{m_i^2 + k_{i\perp}^2}{2x_i M_0}, \]

(11)

where \(e_i\) is the energy of the quark \(q_i\) in the rest frame of \(\bar{P}\).

For S-wave baryons, three typical wavefunctions are given as follows:

- **ΛQ-type:**

\[ \Psi^{S=\frac{1}{2}, S_z}(p_i, \lambda_i) = A_0 \bar{u}(p_3, \lambda_3)(\bar{P} + M_0)(-\gamma_5)C\bar{u}^T(p_2, \lambda_2) \]

\[ \times \bar{u}(p_1, \lambda_1)u(\bar{P}, S_z)\Phi(x_i, k_{i\perp}), \]

(12)

- **ΣQ-type:**

\[ \Psi^{S=\frac{1}{2}, S_z}(p_i, \lambda_i) = A_1 \bar{u}(p_3, \lambda_3)(\bar{P} + M_0)(\gamma^\mu - \nu^\mu)C\bar{u}^T(p_2, \lambda_2) \]

\[ \times \bar{u}(p_1, \lambda_1)\left( \frac{1}{\sqrt{3}} \gamma_\mu \gamma_5 \right)u(\bar{P}, S_z)\Phi(x_i, k_{i\perp}), \]

(13)
• $\Sigma_Q$-type:

$$\Psi_1^{S=\frac{1}{2},S_z}(\vec{p}_i, \lambda_i) = A_1^i \bar{u}(p_3, \lambda_3)(\vec{P} + M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2, \lambda_2) \times \bar{u}(p_1, \lambda_1)u_\mu(\vec{P}, S_z)\Phi(x_i, k_{i\perp}),$$

where quark 1 is $Q = b/c$ while quarks 2, 3 are $u, d$ or $d, u$, $v^\mu \equiv \vec{P}^\mu/M_0$, $\Phi$ is the momentum wavefunction, and the normalization factors

$$A_0 = A_1 = A_1^i = \frac{1}{4\sqrt{M_0^2(e_1 + m_1)(e_2 + m_2)(e_3 + m_3)}}.$$ (15)

More discussion is needed regarding the three types of spin wavefunctions. Respectively denote the spin wavefunctions of $\Lambda_Q, \Sigma_Q$ and $\Sigma_Q^*$ as

$$\psi_0(321) \equiv \bar{u}(p_3, \lambda_3)(\vec{P} + M_0)(-\gamma_5)C\bar{u}^T(p_2, \lambda_2)\bar{u}(p_1, \lambda_1)u(\vec{P}, S_z),$$

$$\psi_1(321) \equiv \bar{u}(p_3, \lambda_3)(\vec{P} + M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2, \lambda_2)\bar{u}(p_1, \lambda_1)(\frac{1}{\sqrt{3}}\gamma_\mu\gamma_5)u(\vec{P}, S_z),$$

$$\psi_{1\mu}(321) \equiv \bar{u}(p_3, \lambda_3)(\vec{P} + M_0)(\gamma^\mu - v^\mu)C\bar{u}^T(p_2, \lambda_2)\bar{u}(p_1, \lambda_1)u_\mu(\vec{P}, S_z).$$ (16)

In $\psi_0(321)$, quarks 3 and 2 are usually considered to form a scalar diquark, while in $\psi_1(321)$ and $\psi_{1\mu}(321)$, they are considered to form an axial-vector diquark. It can be shown that $\psi_{0,1,1\mu}(321)$ have the same normalization factor $A_0 = A_1 = A_1^i$. As expected, a scalar diquark and an axis-vector diquark are orthogonal, so $\psi_0(321)$ is orthogonal to $\psi_1(321)$ and $\psi_{1\mu}(321)$. Therefore, under the valence quark approximation, the amplitude of $\Lambda_b \rightarrow \Sigma_c$ is zero. One can easily check that,

$$\psi_0(231) = -\psi_0(321),$$

$$\psi_1(231) = +\psi_1(321),$$ (17)

and at least in the sense of spin coupling

$$\begin{pmatrix} \psi_0(312) \\ \psi_1(312) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_0(321) \\ \psi_1(321) \end{pmatrix}. $$ (18)

Also at least in the sense of spin coupling, it can be checked that

$$\psi_{1\mu}(321) = \psi_{1\mu}(213) = \psi_{1\mu}(132) = \cdots,$$ (19)

which is as expected because the three quarks in $\psi_{1\mu}(321)$ are on an equal footing.

For the momentum wavefunction, we will adopt the following form

$$\Phi(x_i, k_{i\perp}) = \sqrt{\frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0}} \times 3^{3/4} \times 16 \left( \frac{\pi}{3} \right)^{3/2} \exp \left( -\frac{k_1^2 + k_2^2 + k_3^2}{2\beta^2} \right),$$ (20)

where $k_i \equiv (k_{i\perp}, k_{i\parallel})$, $\beta$ is the shape parameter that characterizes the momentum distribution inside a baryon. In Eq. (20), we have put the three momenta $\vec{k}_{1,2,3}$ on an equal footing, and
assumed that their distributions all follow a Gaussian distribution. A similar assumption has also been adopted in Ref. [78]. The verification of normalization for Eq. (20) is included in Appendix A.

Some comments on the momentum wavefunction are in order. In the literature, the following momentum wavefunction is often used (see, for example, Refs. [8, 70, 77]):

\[
\Phi(x_i, k_{i\perp}) = \sqrt{\frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0}} \varphi(\vec{k}_1, \beta_1) \varphi(\frac{\vec{k}_2 - \vec{k}_3}{2}, \beta_{23})
\]  

(21)

where

\[
\varphi(\vec{k}, \beta) = 4 \left(\frac{\pi}{\beta^2}\right)^{3/4} \exp\left(-\frac{\vec{k}^2}{2\beta^2}\right),
\]  

(22)

and the shape parameters \(\beta_1\) and \(\beta_{23}\) respectively characterize the momentum distribution between quark 1 and the system of quark 2 and quark 3, and the momentum distribution between quark 2 and quark 3. One can see from Eq. (21) that, the three quarks are not equally treated, while its form should be suitable for discussing the internal structure of excited states of baryons.

The shape parameter \(\beta\) in Eq. (20) can be viewed as the most important parameter in the light-front approach. In Ref. [77], we proposed using the pole residue to determine this parameter. In the following, we provide some details for determining the shape parameter for a spin-3/2 baryon. For the case of a spin-1/2 baryon, one can refer to our previous work [77].

**B. The shape parameter**

To be specific, we will take the spin-3/2 \(\Omega^+_{bc}\) as an example to illustrate how to determine the shape parameter.

On the one hand, the pole residue of a spin-3/2 baryon is defined as:

\[
\langle 0|J^\mu_{bc}(P, S_z)\rangle_{\Omega^+_{bc}} = \lambda_+ u_\mu(P, S_z),
\]  

(23)

where the interpolating current of \(\Omega^+_{bc}\) can be given by [5]

\[
J^\mu_{bc} = \epsilon_{ijk}(c_i^T C \gamma_\mu c_j) b_k
\]  

(24)

with \(i, j, k\) the color indices.

On the other hand, the matrix element \(\langle 0|J^\mu_{bc}|\Omega^+_{bc}\rangle\) can be calculated in LFQM

\[
\langle 0|J^\mu_{bc}||\Omega^+_{bc}\rangle = \int \frac{dx_2 d^2 k_{2\perp}}{2(2\pi)^3} \frac{dx_3 d^2 k_{3\perp}}{2(2\pi)^3} \frac{A}{\sqrt{x_1 x_2 x_3}} \Phi(x_i, k_{i\perp}) \times \sqrt{6} \times \frac{1}{\sqrt{2}} \times 2
\]

\[
\times \text{Tr}[C \gamma_\mu (\bar{\psi}_3 + m_3)(\bar{\psi} + M_0)(\gamma^\rho - v^\rho) C (\bar{\psi}_2 + m_2)^T]
\]

\[
\times (\bar{\psi}_1 + m_1) u_\rho(P, S_z),
\]  

(25)
where $\sqrt{6}$ comes from the color space, $1/\sqrt{2}$ comes from the normalization of the baryon state, 2 comes from two equivalent contractions, and the normalization factor $A$ can be found in Eq. \[(15)\].

Multiplying Eqs. \[(23)\] and \[(25)\] with $\sum_S \bar{u}^\mu(P, S_z)\gamma^+ \mu$ from the left, also noting that $\gamma^+ u_\mu(P) = \gamma^+ u_\mu(P)$ \[(80)\], one can obtain the expression of the pole residue in LFQM. Then, use the pole residue obtained from other theoretical methods, such as QCD sum rules, as input to determine the shape parameter.

Once the shape parameter is determined, one can proceed to calculate the weak decay form factors.

**C. Form factors of the $3/2 \rightarrow 1/2$ case**

In this subsection, we will take $\Omega_{bc}^{l+} \rightarrow \Omega_{bc}^0$ as an example to illustrate how to extract the form factors of the $3/2 \rightarrow 1/2$ case. This is a $c \rightarrow s$ process, and the $b$ quark and one $c$ quark are spectators. For the initial state, whether in flavor space or spin space, the three quarks are on an equal footing. Especially, one can consider the two spectator quarks as an axial-vector diquark. For the final state, the two spectator quarks can only be an axial-vector diquark, which has been mentioned above.

One the one hand, the weak decay matrix element can be parameterized in terms of form factors

$$
\langle \Omega_{bc}^0(P', S_z')|\bar{s}\gamma^\mu c|\Omega_{bc}^{l+}(P, S_z)\rangle = \bar{u}(P', S_z') \left[ \gamma^\mu \frac{P^{\alpha}}{M'} f_1^V(q^2) + \frac{P^\mu P^{\alpha}}{M'M'} f_2^V(q^2) \right] \gamma_5 u_\alpha(P, S_z),
$$

\[(26)\]

$$
\langle \Omega_{bc}^0(P', S_z')|\bar{s}\gamma^\mu c|\Omega_{bc}^{l+}(P, S_z)\rangle = \bar{u}(P', S_z') \left[ \gamma^\mu \frac{P^{\alpha}}{M'} f_1^A(q^2) + \frac{P^\mu P^{\alpha}}{M'M'} f_2^A(q^2) \right] u_\alpha(P, S_z),
$$

\[(27)\]

where $q = P - P'$, $f_i^{V,A}$ are the form factors, and $M^{(i)}$ is the mass of the initial (final) baryon.

On the other hand, the matrix element can also be calculated in LFQM

$$
\langle \Omega_{bc}^0(P', S_z')|\bar{s}\gamma^\mu (1 - \gamma_5)c|\Omega_{bc}^{l+}(P, S_z)\rangle = \int \left\{ d^3\vec{p}_2 \right\} \left\{ d^3\vec{p}_3 \right\} \frac{A' A}{\sqrt{p_1^+ p_2^+ P^+ P^+}} \Phi^*(x'_i, k'_i) \Phi(x_i, k_i) \times \frac{1}{\sqrt{2}} \times 2 \times \text{Tr}[(\bar{P} + M_0)(\gamma^\mu - v^\sigma)(\bar{p}_2 + m_2)\gamma^\nu C(\gamma^\sigma - \gamma^\nu)(\bar{P}' + M'_0)(\bar{p}_3 + m_3)] \times \bar{u}(P', S_z') \left( \frac{1}{\sqrt{3}} \gamma_\sigma \gamma_5 (\bar{p}_1 + m_1)^\nu (1 - \gamma_5)(\bar{p}_1 + m_1) u_\mu(P, S_z),
$$

\[(28)\]

where $v^\gamma \equiv \bar{P}^\gamma / M_0$, $v^\sigma \equiv \bar{P}^\sigma / M'_0$, and the two factors $1/\sqrt{2}$ and 2 respectively come from the normalization of the initial state and two equivalent contractions.
The form factors \( f_i^V \) can be extracted in the following way. Respectively multiply Eq. (26) by \( \sum_{s_z} \bar{u}(P, S_z)(\Gamma_5)p_\beta u(P', S'_z) \) with \( (\Gamma_5)p_\beta \in \{ \gamma_\mu P_\mu', P_\mu P'_\beta, P'_\mu P_\mu, g_{\mu\beta} \} \gamma_5 \) to arrive at one set of expressions. Do the same thing for the vector-current part of Eq. (28) to arrive at the other set of expressions. Equate the two sets of expressions to extract the form factors \( f_i^V \). Similarly, one can also extract the form factors \( f_i^A \).

D. Form factors of the \( 1/2 \to 1/2 \) case with a scalar spectator diquark

In this subsection, we will take \( \Omega_{bc}^+ \to \Omega_{bc}^0 \) as an example to illustrate how to extract the form factors of the \( 1/2 \to 1/2 \) case with a scalar spectator diquark. This is also a \( c \to s \) process, and the \( b \) quark and one \( c \) quark are spectators. For the initial state, the two charm quarks are considered to form an axial-vector diquark, while for the final state, the two heavy quarks are considered to form a scalar diquark.

One the one hand, the weak decay matrix element can be parameterized in terms of form factors

\[
\langle \Omega_{bc}^0(P', S'_z) | \bar{s} \gamma^\mu c | \Omega_{bc}^+(P, S_z) \rangle = \bar{u}(P', S'_z) \left[ \gamma^\mu f_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{M} f_2(q^2) + \frac{q^\mu}{M} f_3(q^2) \right] u(P, S_z),
\]

and

\[
\langle \Omega_{bc}^0(P', S'_z) | \bar{s} \gamma^\mu \gamma_5 \gamma_5 c | \Omega_{bc}^+(P, S_z) \rangle = \bar{u}(P', S'_z) \left[ \gamma^\mu g_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{M} g_2(q^2) + \frac{q^\mu}{M} g_3(q^2) \right] \gamma_5 u(P, S_z),
\]

where \( q = P - P', f_i, g_i \) are the form factors, and \( M \) is the mass of the initial baryon.

On the other hand, the matrix element can also be calculated in LFQM in two different ways.

\( 1 \) The direct way

Doing the contraction directly, one can obtain

\[
\langle \Omega_{bc}^0(P', S'_z) | \bar{s} \gamma^\mu (1 - \gamma_5) c | \Omega_{bc}^+(P, S_z) \rangle = \int \left\{ d^3 \bar{p}_2 \right\} \left\{ d^3 p_3 \right\} A^A \frac{A^A}{\sqrt{p_{1z}^+ p_{1z}^0 p_{1z}^+ P_+ P^+}} \Phi(x_i', k_i') \Phi(x_i, k_i) \times (-1) \times \frac{1}{\sqrt{2}} \times 2 \times \bar{u}(\tilde{P}', S'_z)(p_1' + m_1) \gamma^\mu (1 - \gamma_5)(p_1 + m_1)(\tilde{P} + M_0)(\gamma^\rho - \nu^\rho) \times C(p_3 + m_3)^T C \gamma_5 (\tilde{P}' + M_0)(p_2 + m_2)(\frac{1}{\sqrt{3}} \gamma_\rho \gamma_5) u(P, S_z),
\]

where the four quarks can be numbered as \( (1, 1', 2, 3) = (c^1, s, b, c^2) \) with \( c^i \) the \( i \)-th charm quark, and the factor \(-1\) comes from the exchange of the \( b \) quark and the \( c \) quark in the final state.

\( 2 \) The indirect way

The spin wavefunction of the initial state can be written as \( \psi_1(cbc) \), where the two charm quarks are considered as an axial-vector diquark. Expand \( \psi_1(cbc) \) in terms of a new diquark basis \( \psi_{0,1}(cbc) \)

\[
\psi_1(cbc) = \left( -\frac{\sqrt{3}}{2} \right) \psi_0(cbc) + \left( -\frac{1}{2} \right) \psi_1(cbc),
\]
see Eq. (18), while the spin wavefunction of the final state is
\[ \psi_0(cbs), \]
then one can read the factor \(-\sqrt{3}/2\). It turns out that
\[
\langle \Omega^+_{bc}(P', S'_z) | \bar{s} \gamma^\mu (1 - \gamma_5) c | \Omega^0_{bcc}(P, S_z) \rangle
\]
\[
= \int \{ d^3 \tilde{p}_2 \} \{ d^3 \tilde{p}_3 \} \frac{A' A}{\sqrt{p_1^+ p_1^+ p^+ P}} \Phi'(x'_i, k'_i \perp) \Phi(x_i, k_i \perp) \times \frac{-\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \times 2 \times \frac{1}{\sqrt{2}} \times 2 \times \frac{-1}{\sqrt{2}} \times 2
\]
\[
\times \bar{u}(P', S'_z) (p_1' + m_1') \gamma^\mu (1 - \gamma_5) (\bar{p}_1 + m_1) u(P, S_z). \tag{34}
\]

One can check that, the expressions in Eqs. (31) and (34) are equivalent. In the actual calculation below, we will adopt the latter expression as it has a clearer meaning and closely corresponds to the calculation under the diquark picture, see, for example, Ref. [79].

Some comments are in order. In Ref. [79], the so-called “overlap factor” is introduced, which is actually the inner product of the flavor-spin wavefunctions of the initial and final states. However, in the diquark picture, the unclear definition of diquark makes it difficult to determine this factor. It is obvious that this factor is very important because it appears directly as a multiplication factor in the calculation of matrix elements. This factor is also present in the three-quark picture. For example, in Eq. (31), this factor is \((-1) \times 1/\sqrt{2} \times 2\), while in Eq. (34), it is \((-\sqrt{3}/2) \times 1/\sqrt{2} \times 2\). It can be seen that if different calculation schemes are used, this factor can also be different. Readers who want to use this method must be careful enough. More discussion can be found in Subsec. II F.

The form factors \( f_i \) can be extracted in the following way. Respectively multiply Eq. (29) by \( \sum_{S_z} \bar{u}(P, S_z) \Gamma_\mu u(P', S'_z) \) with \( \Gamma_\mu \in \{ \gamma_\mu, P_\mu, P'_\mu \} \) to arrive at one set of expressions. Do the same thing for the vector-current part of Eq. (31), but at this point, take the approximation \( P^{(i)} \to \bar{P}^{(i)} \) within the integral. Equate the two sets of expressions to extract the form factors \( f_i \). Similarly, one can also extract the form factors \( g_i \).

E. The 1/2 \( \to \) 1/2 case with an axial-vector spectator diquark

In this subsection, we will take \( \Omega^+_{bcc} \to \Omega^0_{bc} \) as an example to illustrate how to extract the form factors of the 1/2 \( \to \) 1/2 case with an axial-vector spectator diquark. This is also a \( c \to s \) process, and the \( b \) quark and one \( c \) quark are spectators. For the initial state, the two charm quarks are considered to form an axial-vector diquark, while for the final state, the two heavy quarks are considered to form an axial-vector diquark.

One the one hand, the weak decay matrix element can be parameterized in terms of form factors, see Eqs. (29) and (30). On the other hand, the matrix element can also be calculated in LFQM in two different ways.
(1) The direct way  Doing the contraction directly, one can obtain

\[ \langle \Omega_0^{bc}(P', S_z') | \bar{s} \gamma_{\mu} (1 - \gamma_5) c | \Omega_{b\bar{c}c}(P, S_z) \rangle = \int \{ d^3 \tilde{p}_2 \} \{ d^3 \tilde{p}_3 \} \frac{A' A}{\sqrt{P_1^+ \tilde{p}_1^+ (P + P')}} \Phi^{\nu*}(x'_i, k'_i, k_{i\perp}) \Phi(x_i, k_{i\perp}) \times \frac{1}{\sqrt{2}} \times 2 \times \bar{u}(\bar{P}', S_z')(\psi_{3} + m_3)^T C(\gamma_{\sigma} - v_{\sigma}')(\tilde{P}' + M_0') (\psi_{1} + m_1) \]  

(35)

where the four quarks can also be numbered as \((1, 1', 2, 3) = (c^1, s, b, c^2)\) with \(c^i\) the \(i\)-th charm quark.

(2) The indirect way  The spin wavefunction of the initial state has been expanded in terms of a new diquark basis in Eq. (32), while the spin wavefunction of the final state is \(\psi_1(cbs)\), then one can read the factor \(-\frac{1}{2}\). It turns out that

\[ \langle \Omega_0^{bc}(P', S_z') | \bar{s} \gamma_{\mu} (1 - \gamma_5) c | \Omega_{b\bar{c}c}(P, S_z) \rangle = \int \{ d^3 \tilde{p}_2 \} \{ d^3 \tilde{p}_3 \} \frac{A' A}{\sqrt{P_1^+ \tilde{p}_1^+ (P + P')}} \Phi^{\nu*}(x'_i, k'_i, k_{i\perp}) \Phi(x_i, k_{i\perp}) \times (-\frac{1}{2}) \times \frac{1}{\sqrt{2}} \times 2 \times \bar{u}(\bar{P}', S_z')(\psi_{3} + m_3)^T C(\gamma_{\sigma} - v_{\sigma}')(\tilde{P}' + M_0') (\psi_{1} + m_1) \]  

(36)

One can check that, the expressions in Eqs. (35) and (36) are equivalent. In the actual calculation below, we will adopt the latter expression.

The method for extracting the form factors is the same as that in Subsec. IID.

F. The overlap factor

As mentioned above, the overlap factor is so important that it is worth further discussing in a new subsection. It must be emphasized again that the overlap factor may be different if different calculation schemes are used. For the form factors of the \(1/2 \rightarrow 1/2\) case, regardless of whether the spectator diquark is a scalar or an axial-vector, we all adopt the indirect calculation schemes.

In the three-quark picture, the overlap factor consists of the following parts:

- inner product in spin space,
- the normalization factor of initial state,
- the normalization factor of final state,
- contraction factor.
The overlap factors of other processes can be found in Table I.

| 3/2 → 1/2 transition | Overlap factor | 1/2 → 1/2 transition | Overlap factor |
|-----------------------|---------------|-----------------------|---------------|
| $\Omega_{c\bar{c}}^{++}(ccc) \rightarrow \Xi_{cc}^{++}(d\bar{c})/\Omega_{c\bar{c}}^{0}(scc)$ | $1 \times \frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{2}} \times 6$ | $\Omega_{bc}^{0}(cc) \rightarrow \Xi_{bc}^{0}(dcb)/\Omega_{bc}^{0}(sbc)$ | $(\frac{1}{\sqrt{3}})^2 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ |
| $\Omega_{c\bar{c}}^{0}(ccb) \rightarrow \Xi_{c\bar{c}}^{0}(dbc)/\Omega_{c\bar{c}}^{0}(scc)$ | $1 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ | $\Omega_{b\bar{c}}^{0}(bcc) \rightarrow \Xi_{b\bar{c}}^{0}(dbb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $(\frac{1}{\sqrt{3}})^2 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ |
| $\Omega_{b\bar{c}}^{0}(cbb) \rightarrow \Xi_{b\bar{c}}^{0}(dbb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $1 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times 2$ | $\Omega_{b\bar{c}}^{0}(bcc) \rightarrow \Xi_{b\bar{c}}^{0}(dbb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $(\frac{1}{\sqrt{3}})^2 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ |
| $\Omega_{b\bar{c}}^{0}(bcc) \rightarrow \Xi_{b\bar{c}}^{0}(cccc)/\Omega_{b\bar{c}}^{0}(sbb)$ | $1 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times 2$ | $\Omega_{b\bar{c}}^{0}(bcc) \rightarrow \Xi_{b\bar{c}}^{0}(dbb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $(\frac{1}{\sqrt{3}})^2 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ |
| $\Omega_{b\bar{c}}^{0}(bbc) \rightarrow \Xi_{b\bar{c}}^{0}(bcb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $1 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times 2$ | $\Omega_{b\bar{c}}^{0}(bcc) \rightarrow \Xi_{b\bar{c}}^{0}(dbb)/\Omega_{b\bar{c}}^{0}(sbb)$ | $(\frac{1}{\sqrt{3}})^2 \times \frac{1}{\sqrt{2}} \times 1 \times 2$ |

Of course, the latter three items originate from the symmetry in flavor space. The overlap factors of the three typical processes have been given above. Here, we only present another example, where the initial state contains more identical quarks – $\Omega_{c\bar{c}}^{++}(ccc) \rightarrow \Omega_{c\bar{c}}^{0}(scc)$:

- For the initial state, whether in flavor space or spin space, the three charm quarks are on an equal footing. Especially, one can consider the two spectator charm quarks as an axial-vector diquark. For the final state, the two spectator charm quarks form an axial-vector diquark. Therefore, the inner product in spin space is 1.

- The normalization factor of initial state is $1/\sqrt{6}$. This is because when calculating $\langle \Omega_{c\bar{c}}^{0}(\Omega_{c\bar{c}}^{++})$, a contraction factor 6 appears, therefore, in order to normalize the flavor wavefunction of $|\Omega_{c\bar{c}}^{++}\rangle$, an additional factor $1/\sqrt{6}$ should be added.

- The normalization factor of final state is $1/\sqrt{2}$. Similar to the last item.

- The contraction factor of $\langle \Omega_{c\bar{c}}^{0}|\gamma^\alpha(1-\gamma_5)c|\Omega_{c\bar{c}}^{++}\rangle$ is 6.

The overlap factors of other processes can be found in Table I.

### III. NUMERICAL RESULTS AND PHENOMENOLOGICAL APPLICATIONS

In this section, we will first present the numerical results of the form factors. Subsequently, these form factors will be applied to arrive at some phenomenological predictions, including semileptonic decays and nonleptonic decays. For the latter, we are constrained to consider only the factorable W-emission diagram, i.e., the diagram with a W boson emitting outward and decaying into a meson. It should be noted that, nonleptonic decays contain additional non-perturbative contributions, our predictions can only be considered as rough estimates. However, considering nonleptonic decays have practical significance for the experimental search for THBs, we still believe that our estimates here are valuable.
TABLE II: Masses and pole residues of THBs and DHBs, as well as shape parameters determined by pole residues.

| $\Omega_{ccc}^{++}$ | $\Omega_{bc}^{++}$ | $\Omega_{bc}^{0}$ | $\Omega_{bbb}^{-}$ | $\Omega_{bcc}^{+}$ | $\Omega_{bcc}^{0}$ |
|---------------------|---------------------|-------------------|-------------------|-------------------|-------------------|
| $m$/GeV             | 4.81                | 8.03              | 11.23             | 14.43             | 8.02              | 11.22             |
| $\lambda$/GeV$^3$   | 0.208 ± 0.031       | 0.225 ± 0.025     | 0.324 ± 0.046     | 0.942 ± 0.139     | 0.430 ± 0.047     | 0.565 ± 0.081     |
| $\beta$/GeV         | 0.704 ± 0.036       | 0.869 ± 0.032     | 0.970 ± 0.045     | 1.134 ± 0.054     | 0.890 ± 0.033     | 0.966 ± 0.046     |
| $m$/GeV             | 3.622               | 3.738             | 6.943             | 6.998             | 10.143            | 10.273            |
| $\lambda$/GeV$^3$   | 0.109 ± 0.021       | 0.123 ± 0.024     | 0.176 ± 0.040     | 0.188 ± 0.041     | 0.281 ± 0.071     | 0.347 ± 0.083     |
| $\beta$/GeV         | 0.583 ± 0.037       | 0.600 ± 0.039     | 0.687 ± 0.051     | 0.694 ± 0.050     | 0.803 ± 0.065     | 0.853 ± 0.065     |

A. Inputs

The masses of the initial THBs and the final DHBs are collected in Table II. The former come from the QCD sum rules calculation [5], while the latter come from the experimental measurement [1] and the lattice QCD calculation [81]. In addition, we will take $m_{\Xi_{bc}} = m_{\Xi'_{bc}}$ and $m_{\Omega_{bc}} = m_{\Omega'_{bc}}$.

In Table II, we also list the pole residues of THBs and DHBs, which are respectively calculated in Refs. [5] and [82] in QCD sum rules. With the help of pole residues, we determine the shape parameters, which are also listed in Table II. One can see from Table II that, although the pole residues of spin-3/2 $\Omega_{bcc}^{+}$ ($\Omega_{bcc}^{0}$) and spin-1/2 $\Omega_{bcc}^{+}$ ($\Omega_{bcc}^{0}$) differ greatly, the shape parameters determined by them are almost equal, which is as expected. In this sense, taking $\beta_{\Xi_{bc}} = \beta_{\Xi'_{bc}}$ and $\beta_{\Omega_{bc}} = \beta_{\Omega'_{bc}}$ is reasonable.

The constituent quark masses are adopted as (in units of GeV) [50–58]

$$m_u = m_d = 0.25, \quad m_s = 0.37, \quad m_c = 1.4, \quad m_b = 4.8.$$  \hspace{1cm} (37)

The decay constants of mesons in the final states of nonleptonic decays can be found in Refs. [45, 58, 83, 84]. The Wilson coefficient $a_1(\mu) \equiv C_1(\mu) + C_2(\mu)/3$ is taken as $a_1(\mu_b) = 1.03$ for the bottom quark decay and $a_1(\mu_c) = 1.10$ for the charmed quark decay [85]. All the other inputs can be found in PDG [86].

B. Form factors

Our predictions for the form factors are given in Tables III, IV, and V where the values of the form factors at $q^2 = 0$ are shown. At this point, the SU(3) flavor symmetry and heavy quark symmetry, as well as their breaking degrees can be seen most clearly. Some comments are in order.

- The $c \to d$ and $c \to s$ processes are related by SU(3) flavor symmetry. One can see that the SU(3) symmetry breaking is roughly 10-20%.
For a discussion on the validity of this assumption, see Ref. [58].

For example, $\Omega_{cc}^{++} \to \Xi_{cc}^{++}$ and $\Omega_{bb}^{++} \to \Xi_{bb}^{++}$ are related by heavy quark symmetry. One can see that this symmetry is bad, which should be mainly attributed to the fact that the charm quark is not heavy enough.

Comparing Table IV and Table V we find that the form factors involving a scalar spectator diquark are larger than the corresponding ones involving an axial-vector spectator diquark.

To access the $q^2$ dependence in the kinematic region, we adopt the following single pole assumption:

$$F(q^2) = \frac{F(0)}{1 - q^2/m_{pole}^2}.$$  \hspace{1cm} (38)

For the $c \to d/s$ and $b \to u$ decays, $m_{pole}$ is respectively taken as the mass of $D$, $D_s$, and $B$ meson. For a discussion on the validity of this assumption, see Ref. [58].

### Table IV: Form factors for the $3/2 \to 1/2$ processes with a scalar spectator diquark.

| Transition  | $f_1(0)$ | $f_2(0)$ | $f_3(0)$ | $f_4(0)$ | $f_5(0)$ | $f_6(0)$ | $g_1(0)$ | $g_2(0)$ | $g_3(0)$ |
|-------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\Omega_{bcc}^+ \to \Xi_{bc}^0$ | -0.582 | 2.932 | -1.106 | -0.396 | -2.141 | 29.448 |
| $\Omega_{bcc}^0 \to \Xi_{bc}^+$ | -0.630 | 3.269 | -1.500 | -0.445 | -2.479 | 31.279 |

| Transition  | $f_1(0)$ | $f_2(0)$ | $f_3(0)$ | $f_4(0)$ | $f_5(0)$ | $f_6(0)$ | $g_1(0)$ | $g_2(0)$ | $g_3(0)$ |
|-------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\Omega_{bcc}^+ \to \Xi_{bc}^0$ | -0.582 | 2.932 | -1.106 | -0.396 | -2.141 | 29.448 |
| $\Omega_{bcc}^0 \to \Xi_{bc}^+$ | -0.630 | 3.269 | -1.500 | -0.445 | -2.479 | 31.279 |

### Table V: Form factors for the $1/2 \to 1/2$ processes with an axial-vector spectator diquark.

| Transition  | $f_1(0)$ | $f_2(0)$ | $f_3(0)$ | $g_1(0)$ | $g_2(0)$ | $g_3(0)$ |
|-------------|----------|----------|----------|----------|----------|----------|
| $\Omega_{bcc}^+ \to \Xi_{bc}^0$ | -0.335 | 0.237 | -0.339 | 0.076 | 0.411 | -5.648 |
| $\Omega_{bcc}^0 \to \Xi_{bc}^-$ | 0.526 | -1.804 | 1.649 | -0.112 | -1.054 | 19.543 |
| $\Omega_{bcc}^+ \to \Omega_{bc}^0$ | -0.363 | 0.361 | -0.769 | 0.086 | 0.476 | -6.001 |
| $\Omega_{bcc}^0 \to \Omega_{bc}^-$ | 0.610 | -2.710 | 1.947 | -0.143 | -1.010 | 17.581 |
| $\Omega_{bcc}^+ \to \Xi_{cc}^{++}$ | 0.035 | 0.036 | -0.023 | -0.009 | -0.007 | 0.036 |
| $\Omega_{bcc}^0 \to \Xi_{cc}^{'+}$ | -0.012 | -0.010 | 0.007 | 0.003 | 0.004 | -0.034 |
C. Semileptonic decays

1. The $3/2^+ \rightarrow 1/2^+$ case

Define the helicity amplitudes as

$$H_{\lambda',\lambda W}^{V(A)} \equiv \langle B'(\lambda')|\bar{q}\gamma^\mu(q_0)Q|B(\lambda)\rangle\epsilon^*_{W\mu}(\lambda W),$$

where $\lambda = \lambda_W - \lambda'$. These amplitudes can be written in terms of form factors:

$$H_{\frac{3}{2},1}^{V,A} = \mp i\sqrt{\frac{2}{3}}\sqrt{2MM'(\omega + 1)}\left[2(\omega + 1)f_1^{V,A} + f_4^{V,A}\right],$$

$$H_{\frac{3}{2},-1}^{V,A} = \mp i\sqrt{2MM'(\omega + 1)}f_4^{V,A},$$

$$H_{\frac{3}{2},0}^{V,A} = \pm i\sqrt{\frac{2}{3}}\sqrt{2MM'(\omega + 1)}\left[(\omega + 1)\omega + 1)f_1^{V,A} + r(\omega^2 - 1)f_2^{V,A} + (\omega^2 - 1)f_3^{V,A} + (r\omega - 1)f_4^{V,A}\right],$$

$$H_{\frac{3}{2},t}^{V,A} = \mp i\sqrt{\frac{2}{3}}(\omega + 1)\sqrt{2MM'(\omega + 1)}\left[(\omega + 1)f_1^{V,A} + (r - \omega)f_3^{V,A} + r f_4^{V,A}\right],$$

where the upper (lower) sign corresponds to $V(A)$, $\omega \equiv v \cdot v' = (P \cdot P')/(MM')$, $r \equiv M'/M$, and $\hat{q} \equiv \sqrt{q^2}/M$. The other helicity amplitudes can be obtained by

$$H_{-\lambda',-\lambda W}^{V,A} = \mp H_{\lambda',\lambda W}^{V,A}.$$  (41)

The polarized differential decay widths for $B(3/2^+) \rightarrow B'(1/2^+)l\nu$ are

$$\frac{d\Gamma_L}{dq^2} = \frac{G_F^2|V_{CKM}|^2|\vec{P}'|q^2(1 - \hat{m}_l^2)^2}{768\pi^3M^2}\left[(2 + \hat{m}_l^2)(|H_{-\frac{1}{2},0}|^2 + |H_{\frac{1}{2},0}|^2) + 3\hat{m}_l^2(|H_{-\frac{1}{2},-1}|^2 + |H_{\frac{1}{2},-1}|^2)\right],$$

$$\frac{d\Gamma_T}{dq^2} = \frac{G_F^2|V_{CKM}|^2|\vec{P}'|q^2(1 - \hat{m}_l^2)^2(2 + \hat{m}_l^2)}{768\pi^3M^2}\left[|H_{-\frac{1}{2},0}|^2 + |H_{\frac{1}{2},0}|^2 + |H_{-\frac{1}{2},-1}|^2 + |H_{\frac{1}{2},-1}|^2\right],$$  (42)

where $H_{\lambda',\lambda W} \equiv H_{\lambda',\lambda W}^{V,A} - H_{\lambda',\lambda W}^{A}$, $\hat{m}_l \equiv m_l/\sqrt{q^2}$, and $|\vec{P}'|$ is the magnitude of 3-momentum of $B'$ in the rest frame of $B$.

Our predictions for the semileptonic decays of the $3/2 \rightarrow 1/2$ processes are given in Table VII. One can see that, the decay widths of the $c \rightarrow d$ processes are almost exactly one order of magnitude smaller than the corresponding ones of the $c \rightarrow s$ processes, due to $|V_{cd}/V_{cs}|^2 \approx 0.054$.

After considering these CKM matrix elements, we find that, the SU(3) symmetry breaking between $\Omega_{ccc}^{++} \rightarrow \Xi_{cc}^{++}e^+\nu_e$ and $\Gamma_{ccc}^{++} \rightarrow \Omega_{cd}^{++}e^+\nu_e$ is about 30%, that between $\Omega_{bcc}^{++} \rightarrow \Xi_{bc}^{++}e^+\nu_e$ and $\Omega_{bcc}^{++} \rightarrow \Omega_{cc}^{++}e^+\nu_e$ is about 10%, and that between $\Omega_{bbc}^{0} \rightarrow \Xi_{bc}^{0}e^+\nu_e$ and $\Omega_{bbc}^{0} \rightarrow \Omega_{bc}^{0}e^+\nu_e$ is about 40%.
where \( H_{d, d} \) and \( d \). Similarly define the helicity amplitudes as in Eq. (39). These amplitudes can also be written in terms of form factors:

\[
H_{V}^{+} = -i \sqrt{2Q} \left( f_1 - \frac{M + M'}{M} f_2 \right),

H_{V}^{-} = -i \sqrt{2Q} \left( f_1 + \frac{q^2}{M} f_3 \right),

H_{A}^{+} = -i \sqrt{2Q} \left( g_1 + \frac{M - M'}{M} g_2 \right),

H_{A}^{-} = -i \sqrt{2Q} \left( g_1 - \frac{q^2}{M} g_3 \right),
\]

(43)

where \( Q_{\pm} = (M \pm M')^2 - q^2 \) and \( M^{(\mu)} \) is the mass of the initial (final) baryon. The other helicity amplitudes can be obtained by

\[
H_{V,A}^{+, \lambda_W} = \pm H_{V,A}^{-, \lambda_W}.
\]

(44)

The polarized differential decay widths for \( B(1/2^+) \to B'(1/2^+) l \nu \) are

\[
\frac{d\Gamma_L}{dq^2} = \frac{G_{\mu}^2 |V_{CKM}|^2 |\vec{P}|^2 q^2 (1 - m_l^2)}{384 \pi^3 M^2} \left[ (2 + \hat{m}_l^2)(|H_{\frac{1}{2},0}^+|^2 + |H_{\frac{1}{2},0}^-|^2) + 3\hat{m}_l^2(|H_{\frac{1}{2},0}^+|^2 + |H_{\frac{1}{2},0}^-|^2) \right],
\]

\[
\frac{d\Gamma_T}{dq^2} = \frac{G_{\mu}^2 |V_{CKM}|^2 |\vec{P}|^2 q^2 (1 - m_l^2)^2}{384 \pi^3 M^2} \left[ |H_{\frac{1}{2},-1}^+|^2 + |H_{\frac{1}{2},-1}^-|^2 \right],
\]

(45)

where \( H_{\lambda_W}^{+, \lambda_W} \equiv H_{\lambda_W}^{+, \lambda_W} - H_{\lambda_W}^{+, \lambda_W}, \hat{m}_l \equiv m_l / \sqrt{q^2} \), and \( |\vec{P}| \) is the magnitude of 3-momentum of \( B' \) in the rest frame of \( B \).

Our predictions for the semileptonic decays of the \( 1/2 \to 1/2 \) processes are given in Tables VII and VIII. Some comments are in order.

- As expected, the decay widths of the \( c \to d \) processes are almost exactly one order of magnitude smaller than the corresponding ones of the \( c \to s \) processes.
TABLE VII: Decay widths and $\Gamma_L/\Gamma_T$ for the semileptonic decays of the $1/2 \to 1/2$ processes with a scalar spectator diquark.

| Channel | $\Gamma$/ GeV | $\Gamma_L/\Gamma_T$ |
|---------|---------------|-------------------|
| $\Omega_{bc}^+ \to \Xi_{bc}^0 e^+ \nu_e$ | $9.67 \times 10^{-15}$ | 0.96 |
| $\Omega_{bc}^0 \to \Omega_{bc}^0 e^+ \nu_e$ | $1.57 \times 10^{-13}$ | 0.98 |
| $\Omega_{bc}^0 \to \Xi_{bc}^+ e^- \bar{\nu}_e$ | $3.34 \times 10^{-18}$ | 0.89 |

TABLE VIII: Decay widths and $\Gamma_L/\Gamma_T$ for the semileptonic decays of the $1/2 \to 1/2$ processes with an axial-vector spectator diquark.

| Channel | $\Gamma$/ GeV | $\Gamma_L/\Gamma_T$ |
|---------|---------------|-------------------|
| $\Omega_{bc}^+ \to \Xi_{bc}^0 e^+ \nu_e$ | $8.91 \times 10^{-16}$ | 4.29 |
| $\Omega_{bb}^0 \to \Xi_{bb}^0 e^+ \nu_e$ | $2.49 \times 10^{-15}$ | 3.22 |
| $\Omega_{bc}^+ \to \Omega_{bc}^0 e^+ \nu_e$ | $1.45 \times 10^{-14}$ | 4.19 |
| $\Omega_{bb}^0 \to \Omega_{bb}^0 e^+ \nu_e$ | $2.98 \times 10^{-14}$ | 3.59 |
| $\Omega_{bc}^+ \to \Xi_{bc}^+ e^- \bar{\nu}_e$ | $1.83 \times 10^{-18}$ | 3.15 |
| $\Omega_{bb}^0 \to \Xi_{bb}^+ e^- \bar{\nu}_e$ | $2.57 \times 10^{-19}$ | 3.38 |

• The decay widths involving a scalar spectator diquark are almost exactly one order of magnitude larger than the corresponding ones involving an axial-vector spectator diquark.

• Our predictions on $\Gamma_L/\Gamma_T$ are quite interesting. For the decay widths involving a scalar spectator diquark, $\Gamma_L \approx \Gamma_T$, while for those involving an axial-vector spectator diquark, $\Gamma_L \gg \Gamma_T$. Specifically, take the two decays $\Omega_{bc}^+ \to \Xi_{bc}^0 e^+ \nu_e$ as an example. Both of these two decays are the $1/2^+ \to 1/2^+$ semileptonic decays, and the only difference between them is that the two final states have different flavor-spin wavefunctions. It seems that $\Gamma_L/\Gamma_T$ can tell us some information about the internal structures of $\Xi_{bc}^0$.

D. Nonleptonic decays

In this subsection, we will only consider the contribution from the factorable W-emission diagram, in which the W boson is emitted outward and decays into a pseudoscalar or vector meson.

1. The $3/2^+ \to 1/2^+$ case

The corresponding decay widths are

$$\Gamma(B \to B'P) = \frac{|\lambda|^2 f^2 m^2 |\vec{P}|}{32\pi M^2} \left( |H_{-\frac{1}{2}}|^2 + |H_{\frac{1}{2}}|^2 \right),$$

$$\Gamma(B \to B'V) = \frac{|\lambda|^2 f^2 m^2 |\vec{P}|}{32\pi M^2} \left( |H_{-\frac{3}{2},-1}|^2 + |H_{\frac{3}{2},-1}|^2 + |H_{-\frac{1}{2},1}|^2 + |H_{\frac{1}{2},1}|^2 \right)$$
+ |H_{-\frac{1}{2},0}|^2 + |H_{\frac{1}{2},0}|^2),
\end{align}

where
\begin{equation}
\lambda \equiv \frac{G_F}{\sqrt{2}} \xi a_1
\end{equation}

with $\xi$ the product of the corresponding CKM matrix elements and $a_1(\mu) \equiv C_1(\mu) + C_2(\mu)/3$, $|\vec{P}'|$ is the magnitude of 3-momentum of $B'$ in the rest frame of $B$, and $m$ is the mass of the meson.

Our predictions for the nonleptonic decay widths of the $3/2 \to 1/2$ processes are given in Table IX. Some comments are in order.

- For the $c \to d/s$ processes, the decay width of $B \to B'V$ is one order of magnitude larger than the corresponding decay width of $B \to B'P$, while for the $b \to u$ processes, the ratio of the two is only around 2-3.

- Some words about so-called golden channels. Roughly speaking, the decay mode $B \to B'\pi$ is a candidate. This situation is similar to that of $\Xi^{++}_{cc}$, for which the two channels $\Xi^{++}_{cc} \to \Xi^{(t)+}_{cc} \pi^+$ have been established experimentally. However, if we choose, for example, the decay $\Omega^{++}_{ccc} \to \Xi^{++}_{cc} \pi^+$ as a golden channel to search for $\Omega^{++}_{ccc}$, we should discover $\Xi^{++}_{cc}$ first. Therefore, one may want to know the decay width of $\Omega^{++}_{ccc} \to \Xi^{++}_{cc} \pi^+$, where the spin-3/2 $\Xi^{++}_{cc}$ further decays into other charged final states. These $3/2 \to 3/2$ processes are left to our future works.

2. **The 1/2+ → 1/2+ case**

The corresponding decay widths are
\begin{align}
\Gamma(B \to B'P) &= \frac{|\lambda|^2 f_P^2 m^2 |\vec{P}'|}{16\pi M^2} (|H_{-\frac{1}{2},1}|^2 + |H_{\frac{1}{2},1}|^2),
\Gamma(B \to B'V/A) &= \frac{|\lambda|^2 f_{V/A}^2 m^2 |\vec{P}'|}{16\pi M^2} (|H_{-\frac{1}{2},0}|^2 + |H_{\frac{1}{2},0}|^2 + |H_{-\frac{1}{2},1}|^2 + |H_{\frac{1}{2},1}|^2).
\end{align}

Our predictions for the nonleptonic decay widths of the 1/2 → 1/2 processes are given in Tables X and XI. Some comments are in order.

- We usually consider the spin-1/2 $\Omega^{++}_{bcc}$ ($\Omega^{0}_{bcc}$) as the ground state and the spin-3/2 $\Omega^{++}_{bcc}$ ($\Omega^{0}_{bcc}$) as an excited state. In experiments, the experimentalists should prioritize searching for the former.

- It is interesting to compare the decay widths of these three decays: $\Omega^{++}_{bcc} \to \Omega^{0}_{bcc} \pi^+$, $\Omega^{++}_{bcc} \to \Omega^{+}_{bcc} \pi^+$ and $\Omega^{+}_{bcc} \to \Omega^{0}_{bcc} \pi^+$, and we find: $\Gamma(\Omega^{++}_{bcc} \to \Omega^{0}_{bcc} \pi^+) < \Gamma(\Omega^{++}_{bcc} \to \Omega^{+}_{bcc} \pi^+) < \Gamma(\Omega^{+}_{bcc} \to \Omega^{0}_{bcc} \pi^+)$.
TABLE IX: Decay widths for the nonleptonic decays of the $3/2 \to 1/2$ processes.

| Channel | $\Gamma$ / GeV | Channel | $\Gamma$ / GeV |
|---------|----------------|---------|----------------|
| $\Omega^{+}_{ccc} \to \Xi^{+}_{cc} \pi^{+}$ | $1.39 \times 10^{-15}$ | $\Omega^{+}_{ccc} \to \Xi^{+}_{cc} \rho^{+}$ | $2.46 \times 10^{-14}$ |
| $\Omega^{+}_{ccc} \to \Xi^{+}_{cc} K^{+}$ | $1.41 \times 10^{-16}$ | $\Omega^{+}_{ccc} \to \Xi^{+}_{cc} K^{*+}$ | $1.65 \times 10^{-15}$ |
| $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} \pi^{+}$ | $1.05 \times 10^{-15}$ | $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} \rho^{+}$ | $1.66 \times 10^{-14}$ |
| $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} K^{+}$ | $9.71 \times 10^{-17}$ | $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} K^{*+}$ | $1.04 \times 10^{-15}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bb} \pi^{+}$ | $6.22 \times 10^{-16}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bb} \rho^{+}$ | $1.15 \times 10^{-14}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bb} K^{+}$ | $5.98 \times 10^{-17}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bb} K^{*+}$ | $7.27 \times 10^{-16}$ |
| $\Omega^{+}_{ccc} \to \Omega^{+}_{cc} \pi^{+}$ | $2.91 \times 10^{-14}$ | $\Omega^{+}_{ccc} \to \Omega^{+}_{cc} \rho^{+}$ | $4.46 \times 10^{-13}$ |
| $\Omega^{+}_{ccc} \to \Omega^{+}_{cc} K^{+}$ | $2.62 \times 10^{-15}$ | $\Omega^{+}_{ccc} \to \Omega^{+}_{cc} K^{*+}$ | $2.70 \times 10^{-14}$ |
| $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} \pi^{+}$ | $2.10 \times 10^{-14}$ | $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} \rho^{+}$ | $3.27 \times 10^{-13}$ |
| $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} K^{+}$ | $1.84 \times 10^{-15}$ | $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} K^{*+}$ | $1.88 \times 10^{-14}$ |
| $\Omega^{0}_{bcc} \to \Omega^{0}_{bb} \pi^{+}$ | $1.27 \times 10^{-14}$ | $\Omega^{0}_{bcc} \to \Omega^{0}_{bb} \rho^{+}$ | $1.72 \times 10^{-13}$ |
| $\Omega^{0}_{bcc} \to \Omega^{0}_{bb} K^{+}$ | $9.87 \times 10^{-16}$ | $\Omega^{0}_{bcc} \to \Omega^{0}_{bb} K^{*+}$ | $7.89 \times 10^{-15}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{cc} \pi^{+}$ | $3.18 \times 10^{-20}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{cc} \rho^{+}$ | $1.03 \times 10^{-19}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{cc} K^{+}$ | $2.64 \times 10^{-21}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{cc} K^{*+}$ | $5.55 \times 10^{-21}$ |
| $\Omega^{+}_{bcc} \to \Xi^{+}_{bc} D^{+}$ | $5.15 \times 10^{-21}$ | $\Omega^{+}_{bcc} \to \Xi^{+}_{bc} D^{*-}$ | $1.14 \times 10^{-20}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{+}$ | $1.38 \times 10^{-19}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{*-}$ | $2.86 \times 10^{-19}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} \pi^{+}$ | $1.70 \times 10^{-20}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} \rho^{+}$ | $5.73 \times 10^{-20}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} K^{+}$ | $1.42 \times 10^{-21}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} K^{*+}$ | $3.13 \times 10^{-21}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} D^{+}$ | $3.06 \times 10^{-21}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} D^{*-}$ | $7.17 \times 10^{-21}$ |
| $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} D^{+}$ | $8.26 \times 10^{-20}$ | $\Omega^{0}_{bcc} \to \Xi^{0}_{bc} D^{*-}$ | $1.81 \times 10^{-19}$ |
| $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} \pi^{+}$ | $1.15 \times 10^{-19}$ | $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} \rho^{+}$ | $3.92 \times 10^{-19}$ |
| $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} K^{+}$ | $9.60 \times 10^{-21}$ | $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} K^{*+}$ | $2.15 \times 10^{-20}$ |
| $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} D^{+}$ | $2.14 \times 10^{-20}$ | $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} D^{*-}$ | $5.15 \times 10^{-20}$ |
| $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} D^{+}$ | $5.79 \times 10^{-19}$ | $\Omega^{0}_{bbc} \to \Xi^{0}_{bb} D^{*-}$ | $1.30 \times 10^{-18}$ |

TABLE X: Decay widths for the nonleptonic decays of the $1/2 \to 1/2$ processes with a scalar spectator diquark.

| Channel | $\Gamma$ / GeV | Channel | $\Gamma$ / GeV |
|---------|----------------|---------|----------------|
| $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} \pi^{+}$ | $5.48 \times 10^{-15}$ | $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} \rho^{+}$ | $3.17 \times 10^{-14}$ |
| $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} K^{+}$ | $4.95 \times 10^{-16}$ | $\Omega^{+}_{bcc} \to \Xi^{0}_{bc} K^{*+}$ | $1.71 \times 10^{-15}$ |
| $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} \pi^{+}$ | $1.05 \times 10^{-13}$ | $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} \rho^{+}$ | $5.86 \times 10^{-13}$ |
| $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} K^{+}$ | $9.32 \times 10^{-15}$ | $\Omega^{+}_{bcc} \to \Omega^{0}_{bc} K^{*+}$ | $2.89 \times 10^{-14}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} \pi^{+}$ | $6.49 \times 10^{-20}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} \rho^{+}$ | $1.96 \times 10^{-19}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} K^{+}$ | $5.38 \times 10^{-21}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} K^{*+}$ | $1.04 \times 10^{-20}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{+}$ | $1.07 \times 10^{-20}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{*-}$ | $1.74 \times 10^{-20}$ |
| $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{+}$ | $2.89 \times 10^{-19}$ | $\Omega^{0}_{bcc} \to \Xi^{+}_{bc} D^{*-}$ | $4.27 \times 10^{-19}$ |
TABLE XI: Decay widths for the nonleptonic decays of the $1/2 \rightarrow 1/2$ processes with an axial-vector spectator diquark.

| Channel                        | $\Gamma$/ GeV | Channel                        | $\Gamma$/ GeV |
|--------------------------------|---------------|--------------------------------|---------------|
| $\Omega_{bcc}^+ \rightarrow \Xi_{bc}^0 \pi^+$ | $1.31 \times 10^{-15}$ | $\Omega_{bcc}^+ \rightarrow \Xi_{bc}^0 \rho^+$ | $2.58 \times 10^{-15}$ |
| $\Omega_{bcc}^+ \rightarrow \Xi_{bc}^0 K^+$    | $1.18 \times 10^{-16}$ | $\Omega_{bcc}^+ \rightarrow \Xi_{bc}^0 K^{*+}$ | $1.05 \times 10^{-16}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi_{bc}^0 \pi^+$ | $3.40 \times 10^{-15}$ | $\Omega_{bcc}^0 \rightarrow \Xi_{bc}^0 \rho^+$ | $7.40 \times 10^{-15}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi_{bc}^0 K^+$   | $3.18 \times 10^{-16}$ | $\Omega_{bcc}^0 \rightarrow \Xi_{bc}^0 K^{*+}$ | $3.05 \times 10^{-16}$ |
| $\Omega_{bcc}^+ \rightarrow \Omega_{bcc}^0 \pi^+$ | $2.46 \times 10^{-14}$ | $\Omega_{bcc}^+ \rightarrow \Omega_{bcc}^0 \rho^+$ | $4.30 \times 10^{-14}$ |
| $\Omega_{bcc}^+ \rightarrow \Omega_{bcc}^0 K^+$ | $2.23 \times 10^{-15}$ | $\Omega_{bcc}^+ \rightarrow \Omega_{bcc}^0 K^{*+}$ | $1.56 \times 10^{-15}$ |
| $\Omega_{bcc}^0 \rightarrow \Omega_{bc}^0 \pi^+$ | $5.92 \times 10^{-14}$ | $\Omega_{bcc}^0 \rightarrow \Omega_{bc}^0 \rho^+$ | $8.71 \times 10^{-14}$ |
| $\Omega_{bcc}^0 \rightarrow \Omega_{bc}^0 K^+$ | $5.31 \times 10^{-15}$ | $\Omega_{bcc}^0 \rightarrow \Omega_{bc}^0 K^{*+}$ | $2.09 \times 10^{-15}$ |
| $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} \pi^+$ | $1.05 \times 10^{-19}$ | $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} \rho^+$ | $2.87 \times 10^{-19}$ |
| $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} K^+$ | $8.61 \times 10^{-21}$ | $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} K^{*+}$ | $1.47 \times 10^{-20}$ |
| $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} D^-$ | $1.45 \times 10^{-20}$ | $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} D^{*-}$ | $1.57 \times 10^{-20}$ |
| $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} D^*_-$ | $3.82 \times 10^{-19}$ | $\Omega_{bcc}^+ \rightarrow \Xi^{++}_{cc} D^{*-}$ | $3.67 \times 10^{-19}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} \pi^+$ | $1.49 \times 10^{-20}$ | $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} \rho^+$ | $4.08 \times 10^{-20}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} K^+$ | $1.22 \times 10^{-21}$ | $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} K^{*+}$ | $2.10 \times 10^{-21}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} D^-$ | $2.17 \times 10^{-21}$ | $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} D^{*-}$ | $2.30 \times 10^{-21}$ |
| $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} D^*_-$ | $5.75 \times 10^{-20}$ | $\Omega_{bcc}^0 \rightarrow \Xi^{0}_{bc} D^{*-}$ | $5.40 \times 10^{-20}$ |

E. Uncertainties

The shape parameter is one of the most important parameters in the light-front approach, and in this work we use the pole residue to determine this parameter. The pole residue is taken from the calculation of QCD sum rules, which includes a given error. The error of the pole residue is transmitted to the shape parameter, the errors in the shape parameters of the initial and final baryons are then transmitted to the form factors, and then to the semileptonic and nonleptonic decay widths.

In the following, we will continue to consider these three typical processes: $\Omega_{bcc}^+ \rightarrow \Omega_{bcc}^0$, $\Omega_{bcc}^+ \rightarrow \Omega_{bc}^0$, and $\Omega_{bcc}^+ \rightarrow \Omega_{bc}^0$. We tune the shape parameters of initial and final baryons to the upper limits, thereby obtaining the upper limits of the form factors and decay widths in Table XII. As a comparison, we also copy the corresponding central values in this table. It can be seen that, the uncertainties of the shape parameters of the initial and final baryons result in about 15% uncertainties for the semileptonic decay widths and about 10% uncertainties for the nonleptonic decay widths.
TABLE XII: Central values and upper limits for the form factors and decay widths (in units of $10^{-14}$ GeV) of the three typical processes.

| Decay width | $\Gamma(\Omega_{bc}^+ \to \Omega_{bc}^0 e^+ \nu_e)$ | $\Gamma(\Omega_{bc}^+ \to \Omega_{bc}^0 e^+ \nu_e)$ | $\Gamma(\Omega_{bc}^+ \to \Omega_{bc}^0 e^+ \nu_e)$ |
|-------------|---------------------------------|---------------------------------|---------------------------------|
| Central value | 8.44                           | 15.7                            | 1.45                            |
| Upper limit  | 9.66                           | 18.0                            | 1.66                            |

F. Comparison

Recently, Refs. [7] and [8] investigated the weak decays of $\Omega_{cc}^{++}$ and $\Omega_{bb}^{-}$ in the light-front approach, where the diquark picture and three-quark picture are respectively adopted. In Table XIII, our predicted decay widths are compared with those in these two works. It can be seen that, there exist large differences. Some comments are in order.

- The key parameters – the shape parameters, in this work, are determined using the corresponding pole residues of baryons, while Refs. [7] and [8] selected these parameters based on experience. The uncertainties in the shape parameters are the main source of error.

- For $\Gamma(\Omega_{cc}^{++} \to \Xi_{cc}^0 e^+ \nu_e)$ and $\Gamma(\Omega_{cc}^{++} \to \Omega_{cc}^0 e^+ \nu_e)$, our results are several times larger than those in Refs. [7] and [8]. This is due to the fact that we have adopted larger shape parameters in this work. For example, in Ref. [7], the following values are adopted for the shape parameters

$$\beta_{e(cc)} = 0.553, \quad \beta_{d(cc)} = 0.370, \quad \beta_{s(cc)} = 0.435,$$ (49)

while in this work, they are respectively determined as

$$\beta_{\Omega_{cc}} = 0.704, \quad \beta_{\Xi_{cc}} = 0.583, \quad \beta_{\Omega_{cc}} = 0.600.$$ (50)

- For $\Gamma(\Omega_{bb}^{-} \to \Xi_{bb}^0 e^- \bar{\nu}_e)$ and $\Gamma(\Omega_{bb}^{-} \to \Omega_{bb}^0 \pi^-)$, our results are much smaller than those in Ref. [8]. This is likely due to the significant difference in the bottom quark mass used in Ref.
TABLE XIII: Our predicted decay widths (in units of $10^{-14}$ GeV) are compared with those in the literature.

| Decay                                                                 | This work | Ref. [7] | Ref. [8], Case I | Ref. [8], Case II |
|----------------------------------------------------------------------|-----------|----------|------------------|------------------|
| $\Gamma(\Omega_{ccc}^{+++} \to \Xi_{ccc}^{++} e^+ \nu_e)$           | 1.11      | 0.32     | 0.307            | 0.359            |
| $\Gamma(\Omega_{ccc}^{+++} \to \Omega_{ccc}^{++} e^+ \nu_e)$        | 13.1      | 4.95     | 4.78             | 5.51             |
| $\Gamma(\Omega_{bb}^{--} \to \Xi_{bb}^{0} e^- \bar{\nu}_e)$         | $1.41 \times 10^{-3}$ | -        | $6.03 \times 10^{-3}$ | $10.1 \times 10^{-3}$ |
| $\Gamma(\Omega_{ccc}^{+++} \to \Xi_{ccc}^{++} \pi^+)$               | 0.139     | 0.160    | 0.280            | 0.362            |
| $\Gamma(\Omega_{ccc}^{+++} \to \Omega_{ccc}^{++} \pi^+)$            | 2.91      | 7.32     | 6.40             | 8.03             |
| $\Gamma(\Omega_{bb}^{--} \to \Xi_{bb}^{0} \pi^-)$                   | $1.15 \times 10^{-5}$ | -        | $12.9 \times 10^{-5}$ | $17.8 \times 10^{-5}$ |

and that adopted in this work. In Ref. [8], $m_b = 4.4$ GeV, while in this work, $m_b = 4.8$ GeV. In addition, Ref. [8] adopted larger shape parameters.

IV. CONCLUSIONS

The recent progress in the study of doubly heavy baryons gives us confidence to believe that we may not be far from the discovery of triply heavy baryons. However, considering that there is still a lack of one comprehensive quantitative analysis on the weak decays of triply heavy baryons, this work and the forthcoming ones aim to fill this gap.

Of course, we should first search for the ground-state triply heavy baryons in experiments, which include: the spin-$3/2$ $\Omega_{ccc}^{++}$ and $\Omega_{bb}^{-}$, and the spin-$1/2$ $\Omega_{bcc}^{+}$ and $\Omega_{bbc}^{0}$. Therefore, in this work, we investigate the weak decays of triply heavy baryons for the $3/2 \to 1/2$ case and $1/2 \to 1/2$ case. We first obtain the form factors using the light-front quark model in the three-quark picture, and then apply them to arrive at some phenomenological predictions, including the decay widths of semileptonic decays and nonleptonic decays. Our results are expected to be helpful for the experimental search for triply heavy baryons.

The road to searching for triply heavy baryons is long and arduous, but the journey of a thousand miles begins with a single step. As far as we are concerned, the following efforts can be made:

- As mentioned above, the $3/2 \to 3/2$ processes may have larger decay widths for the spin-$3/2$ $\Omega_{ccc}^{++}$ and $\Omega_{bb}^{-}$. In addition, for the sake of theoretical completeness, the $1/2 \to 3/2$ processes are also worth studying. In Ref. [8], we investigated the weak decays of doubly heavy baryons for the $1/2 \to 3/2$ case, and found that the decay widths are approximately one order of magnitude smaller than those of the $1/2 \to 1/2$ case. One may expect a similar pattern to hold for the case of triply heavy baryons.

- There is another type of processes that we deliberately avoid because they do not have priority in experiments, but are also worth studying for the sake of theoretical completeness, that is, the $b \to c$ processes.
• As can be seen in Subsec. III F, despite all adopting the light-front approach, different literatures still yield significantly different results. Other methods, such as QCD sum rules and lattice QCD, are needed to clarify these things.

• In order to search for triply heavy baryons, lifetime is another important reference. Ref. [31] roughly estimated the lifetimes of triply heavy baryons, however, the contribution of the four-quark operators may play a significant role. We intend to make some efforts in this direction in the future.

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Appendix A: The normalization of the momentum wavefunction

In this appendix, we will demonstrate the normalization of the momentum wavefunction, which requires:

\[
\int \frac{dx_1 d^2 k_1}{2(2\pi)^3} \frac{dx_2 d^2 k_2}{2(2\pi)^3} \frac{dx_3 d^2 k_3}{2(2\pi)^3} \delta(1 - \sum x_i) \delta^2(\sum k_i) \frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0} \varphi_1 \varphi_2^2 = 1, \tag{A1}
\]

where we have denoted

\[
\varphi_1 = \varphi(\vec{k}_1, \beta_1), \quad \varphi_{23} = \varphi(\frac{\vec{k}_2 - \vec{k}_3}{2}, \beta_{23}), \tag{A2}
\]

and the definition of \( \varphi \) can be found in Eq. (21).

We rewrite the left-hand side of Eq. (A1):

\[
\text{LHS} = \int \frac{dx_2 d^2 k_2}{2(2\pi)^3} \frac{dx_3 d^2 k_3}{2(2\pi)^3} \frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0} \varphi_1^2 \varphi_{23} \\
= \int \frac{dk_2 d^2 k_2}{2(2\pi)^3} \frac{dk_3 d^2 k_3}{2(2\pi)^3} \varphi^2(\vec{k}_1, \beta_1) \varphi^2(\frac{\vec{k}_2 - \vec{k}_3}{2}, \beta_{23}) \\
= \int \frac{d^3 k_2}{2(2\pi)^3} \frac{d^3 k_3}{2(2\pi)^3} \varphi^2(\vec{k}_2 + \vec{k}_3, \beta_1) \varphi^2(\frac{\vec{k}_2 - \vec{k}_3}{2}, \beta_{23}) \\
= \int \frac{d^3 k_{23}}{2(2\pi)^3} \frac{d^3 l_{23}}{2(2\pi)^3} \varphi^2(\vec{k}_{23}, \beta_1) \varphi^2(\vec{l}_{23}, \beta_{23}) \\
= 1 = \text{RHS}. \tag{A3}
\]

In the second step, we have used

\[
\frac{\partial k_{23}}{\partial x_2} \frac{\partial k_{23}}{\partial x_2} = \frac{e_1 e_2 e_3}{x_1 x_2 x_3 M_0}, \tag{A4}
\]
in the fourth step, we define $\vec{k}_{23} \equiv \vec{k}_2 + \vec{k}_3$ and $\vec{l}_{23} \equiv \vec{k}_2 - \vec{k}_3/2$, and note that $\int d^3k_2 d^3k_3 = \int d^3k_{23} d^3l_{23}$, and in the last step,

$$\int \frac{d^3k}{(2\pi)^3} \varphi^2(\vec{k}, \beta) = 1. \tag{A5}$$

In Eq. (A1), if we replace $\varphi_1^2 \varphi_2^2$ with $\varphi_{123}^2$, where

$$\varphi_{123} \sim \exp \left(-\frac{\vec{k}_1^2 + \vec{k}_2^2 + \vec{k}_3^2}{2\beta^2}\right) = \exp \left(-\frac{\vec{k}_{23}^2 + \vec{l}_{23}^2}{2\beta^2}\right), \tag{A6}\right.$$

then one can easily obtain the normalization factor of $\varphi_{123}$, see Eq. (20).

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