ON \((p, q)\)-STANCU-SZÁSZ-BETA OPERATORS AND THEIR APPROXIMATION PROPERTIES

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Abstract. In the present paper, we have introduced the generalized form of \((p, q)\)-analogue of the Szász-Beta operators with Stancu type parameters. We have studied the local approximation properties of these operators and obtained the convergence rate and weighted approximation.

Keywords: Szász-Beta operators; Stancu type parameters; weighted approximation.

1. Introduction and preliminaries

In the last two decades, the applications of \(q\)-calculus emerged as a new area in the field of approximation theory. The development of \(q\)-calculus has led to the discovery of various modifications of Bernstein polynomials involving \(q\)-integers. The aim of these generalizations is to provide appropriate and powerful tools to application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations.

In 1987, Lupas \[11\] introduced the first \(q\)-analogue of the classical Bernstein operators and investigated its approximating and shape preserving properties. Another \(q\)-generalization of the classical Bernstein polynomial is due to Phillips \[20\]. Several generalization of well known positive linear operators based on \(q\)-integers were introduced and their approximation properties have been studied by several researchers.

Recently, Mursaleen et al introduced the use of \((p, q)\)-calculus in approximation theory and constructed the \((p, q)\)-analogue of Bernstein operators \[13\] and \((p, q)\)-analogue of Bernstein-Stancu operators \[15\]. Most recently, the \((p, q)\)-analogue of

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some more operators have been studied in [1]- [3], [5], [12], [14], [16], [17], [18] and [19].

The \((p, q)\)-integer was introduced to generalize or unify several forms of \(q\)-oscillator algebras well known in the Physics literature related to the representation theory of single parameter quantum algebras. The \((p, q)\)-integer is defined by

\[
\begin{aligned}
\begin{cases}
\frac{p^n - q^n}{p - q} & (p \neq q \neq 1) \\
\frac{1 - q^n}{1 - q} & (p = 1) \\
n & (p = q = 1)
\end{cases}
\end{aligned}
\]

The \((p, q)\)-binomial expansion is

\[
(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{(n-k)(n-k-1)} q^{k(k-1)/2} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} a^{n-k} b^k x^{n-k} y^k,
\]

\[(x + y)^n_{p,q} := (x + y)(px + qy)(p^2 x + q^2 y) \cdots (p^{n-1} x + q^{n-1} y),
\]

\[(1 - x)^n_{p,q} := (1 - x)(p - qx)(p^2 - q^2 x) \cdots (p^{n-1} - q^{n-1} x).
\]

The \((p, q)\)-binomial coefficients are defined by

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!},
\]

The definite integral of a function \(f\) is defined by

\[
\int_0^a f(t) d_{p,q} t = (q - p) a \sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1} a}\right) \frac{p^k}{q^{k+1}}, \quad if \ | \frac{p}{q} | < 1,
\]

\[
\int_0^a f(t) d_{p,q} t = (p - q) a \sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1} a}\right) \frac{q^k}{p^{k+1}}, \quad if \ | \frac{q}{p} | < 1.
\]

There are two \((p, q)\)-analogues of the classical exponential function defined as follows

\[
e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2}}{[n]_{p,q}!} x^n,
\]
and
\[ E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_{p,q}^{p,q}}, \]
which satisfy the equality \( e_{p,q}(x)E_{p,q}(-x) = 1 \). For \( p = 1 \), \( e_{p,q}(x) \) and \( E_{p,q}(x) \) reduce to \( q \)-exponential functions.

For \( m, n \in \mathbb{N} \), the \((p, q)\)-Beta and the \((p, q)\)-Gamma functions are defined by
\[ B_{p,q}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} d_{p,q}x, \]
and
\[ \Gamma_{p,q}(n) = \int_0^\infty p^\frac{n(n-1)}{2} E_{p,q}(-qx)d_{p,q}x, \quad \Gamma_{p,q}(n+1) = [n]_{p,q}^{p,q}. \]
respectively. The two functions are connected through
\[ B_{p,q}(m, n) = q^{\frac{2-m(m-1)}{2}} p^{-\frac{m(m+1)}{2}} \frac{\Gamma_{p,q}(m)\Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}. \]

For \( p = 1 \), all the notions of the \((p, q)\)-calculus reduce to those of \( q \)-calculus.

Based on \((p, q)\)-calculus, very recently Acar \[1\] defined the \((p, q)\) analogue of Szász operators as
\[ S_{n,p,q}(f; x) = \sum_{k=0}^{[n]_{p,q}} s_{n,k}^{p,q}(x)f\left(\frac{[k]_{p,q}}{q^{k-2}[n]_{p,q}}\right) \]
for \( x \in [0, \infty), 0 < q < p \leq 1 \), where
\[ s_{n,k}^{p,q}(x) = \frac{q^{k(k-1)} x^k}{E_{p,q}([n]_{p,q}x) [k]_{p,q}}. \]

Gupta and Noor \[9\] proposed Szász-Beta operators and obtained some direct results in simultaneous approximation. Gupta and Aral \[8\] extended the studies and they proposed the \( q \)-analogue of Szász-Beta operators. Later on Aral and Gupta \[4\] introduced the \((p, q)\)-analogue of the Szász-Beta operators as follows
\[ D_{n}^{(p,q)}(f; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1+pt)^{n+k+1}} f(p^{k+1}qt) d_{p,q}t. \]
where $s_{n,k}^{p,q}(x)$ is defined in (1.2). In this paper, we have generalized this operator (1.3) with Stancu type parameters. Assuming that $0 \leq \alpha \leq \beta$, for $x \in [0, \infty), 0 < q < p \leq 1$, we define

$$D_{n,p,q}^{\alpha,\beta}(f; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_{0}^{\infty} \frac{t^{k-1}}{(1+pt)^{n+k+1}} f \left( \frac{[n]_{p,q}t^{k+1} + \alpha}{[n]_{p,q} + \beta} \right) dt.$$

(1.4)

2. Auxiliary results

Lemma 2.1. For $x \in [0, \infty), 0 < q < p \leq 1$, we have

(i) $D_{n}^{p,q}(1; x) = 1$,

(ii) $D_{n}^{p,q}(t; x) = x$,

(iii) $D_{n}^{p,q}(t^2; x) = \frac{[2]_{p,q}qx}{p[n-1]_{p,q}} + \frac{pn[n]_{p,q}x^2}{[n-1]_{p,q}}$,

(iv) $D_{n}^{p,q}(t^3; x) = \frac{p^3[n]_{p,q}^2}{q^3[n-1]_{p,q}[n-2]_{p,q}} x^3$

+ $\left( \frac{(p2_{p,q} + p^2)[n]_{p,q}}{p^2 q^4([n-1]_{p,q}[n-2]_{p,q})} + \frac{(p^2 q + 2pq^2)[n]_{p,q}}{q^6([n-1]_{p,q}[n-2]_{p,q})} \right) x^2$

+ $\left( \frac{[2]_{p,q}}{p^3 q^6([n-1]_{p,q}[n-2]_{p,q})} + \frac{(p2_{p,q} + p^2)}{q^3 p^3 5q^5([n-1]_{p,q}[n-2]_{p,q})} \right) x$,

(v) $D_{n}^{p,q}(t^4; x) = \frac{q^{12}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}}{p^n[n]_{p,q}^{13}} x^4$

+ $\frac{[n]_{p,q}^2 (p^5 + 3p^3 q^2 + 2p^2 q^3 + p^3 q^4 + 3q^6)}{q^{11}[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} x^3$

+ $\frac{[n]_{p,q}}{p^3 q^4([n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q})} \left( p^8 + 3p^7 q + 5p^6 q^2 \right.$

+ $5p^5 q^3 + 4p^4 q^4 + p^4 q^4 + 3p^3 q^3 + 2p^2 q^4 + 4pq^5 \right) x^2$

+ $\frac{(p^6 + 2p^5 q + p^4 q^2 + p^3 q^3 + p^3 q^4 + p^3 q^3 + p^3 q^4 + 2pq^4 + 2p^4 q^2 + 2pq^5 + pq^3 + q^6)}{p^n q^n[n-1]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} x.$

Lemma 2.2. Let $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$. For $x \in [0, \infty), 0 < q < p \leq 1, 0 \leq \alpha \leq \beta$, we have

(i) $D_{n,p,q}^{\alpha,\beta}(e_0; x) = 1$,

(ii) $D_{n,p,q}^{\alpha,\beta}(e_1; x) = \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta}$. 
(iii) \( D_{n,p,q}^{(\alpha,\beta)} (e_2; x) \) = \[
\frac{p[n\beta_d^3_{p,q}]}{[n-1]_{p,q}([n\beta_d^3_{p,q} + \beta])^2} x^2 + \frac{[n]_{p,q}(q(p + q)[n\beta_d^3_{p,q} + 2\alpha p[n-1]_{p,q})}{p([n\beta_d^3_{p,q} + \beta])^2[n-1]_{p,q}} x^2 + \frac{([n\beta_d^3_{p,q} + \beta])^2}{\alpha^2} \]

(iv) \( D_{n,p,q}^{(\alpha,\beta)} (e_3; x) \) = \[
\frac{p^3[n\beta_d^3_{p,q}]}{q^3([n\beta_d^3_{p,q} + \beta])^3[n-1]_{p,q}([n-2]_{p,q})} x^3 + \frac{[n]_{p,q}(p^3 q + 2p^2 q^2 + 2p + q) + 3p^2 q^3}{pq^6([n\beta_d^3_{p,q} + \beta])^3[n-1]_{p,q}([n-2]_{p,q})} x^2 + \frac{([n\beta_d^3_{p,q} + \beta])^3}{\alpha^3} \]

(v) \( D_{n,p,q}^{(\alpha,\beta)} (e_4; x) \) = \[
\frac{p^6[n\beta_d^3_{p,q}]}{q^4([n\beta_d^3_{p,q} + \beta])^3[n-1]_{p,q}([n-2]_{p,q}([n-3]_{p,q})} x^4 + \frac{[n]_{p,q}(p^5 + 3p^3 q^2 + 2p^2 q^3 + 2p q^4 + q^5) + 4p^3 q^5}{q^6([n\beta_d^3_{p,q} + \beta])^3[n-1]_{p,q}([n-2]_{p,q}([n-3]_{p,q})} x^3 + \frac{([n\beta_d^3_{p,q} + \beta])^3}{\alpha^4} \]

\[
+ \frac{5p^5 q^3 + 2p^4 q^4 + p^3 q^4 + 2p^3 q^3 + 2p^2 q^4 + pq^5}{n\beta_d^3_{p,q}([n-2]_{p,q}([n-3]_{p,q})} x^2 + \frac{4\alpha [n\beta_d^3_{p,q}([n-3]_{p,q})]}{6\alpha^2 p^6 q^9[n-2]_{p,q}([n-3]_{p,q})} x + \frac{\alpha^3[n-1]_{p,q}([n-2]_{p,q}([n-3]_{p,q})^6}{([n\beta_d^3_{p,q} + \beta])^4} \]

Proof. Using Lemma 2.1, we can easily say, (i) \( D_{n,p,q}^{(\alpha,\beta)} (e_0; x) = 1 \). Moreover

(ii) \( D_{n,p,q}^{(\alpha,\beta)} (e_1; x) \) = \[
\sum_{k=0}^{\infty} \frac{s_{n,k}^{p,q}(x)}{B_{p,q}(k,n+1)} \frac{1}{(1+pt)^{n+k+1}} \left( \frac{[n]_{p,q} p^{k+1} q t + \alpha}{[n\beta_d^3_{p,q} + \beta]} \right) d_{p,q} t \]

= \[
\frac{[n]_{p,q}}{[n\beta_d^3_{p,q} + \beta]} \sum_{k=0}^{\infty} \frac{s_{n,k}^{p,q}(x)}{B_{p,q}(k,n+1)} \frac{p^{k+1} q t}{(1+pt)^{n+k+1}} d_{p,q} t \]

+ \[
\frac{\alpha}{[n\beta_d^3_{p,q} + \beta]} \sum_{k=0}^{\infty} \frac{s_{n,k}^{p,q}(x)}{B_{p,q}(k,n+1)} \frac{1}{(1+pt)^{n+k+1}} d_{p,q} t \]
\[
D_n^{p,q}(e_1; x) = \frac{[n]_{p,q}}{[n]_{p,q} + \beta} D_n^{p,q}(e_1; x) + \frac{\alpha}{[n]_{p,q} + \beta} D_n^{p,q}(e_0; x)
\]

\[
= \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta}.
\]

\[
(iii) D_{n,p,q}^{(\alpha,\beta)}(e_2; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} \left( \frac{[n]_{p,q}p^{k+1} + \alpha}{[n]_{p,q} + \beta} \right)^2 d_p q t
\]

\[
= \frac{[n]_{p,q}^2}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^k}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ 2\alpha [n]_{p,q} \frac{[n]_{p,q}^2}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^k}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ \frac{[n]_{p,q}^2}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_2; x) + \frac{2\alpha [n]_{p,q}}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_1; x)
\]

\[
+ \frac{\alpha^2}{([n]_{p,q} + \beta)^2} D_n^{p,q}(e_0; x)
\]

\[
(iv) D_{n,p,q}^{(\alpha,\beta)}(e_3; x) = \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} \left( \frac{[n]_{p,q}p^{k+1} + \alpha}{[n]_{p,q} + \beta} \right)^3 d_p q t
\]

\[
= \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{3k+3} q^3}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k+2}}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ 3\alpha [n]_{p,q}^2 \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{2k+2} q^2}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k+1}}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ 3\alpha^2 [n]_{p,q} \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{p^{k+1} q}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^k}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ \frac{\alpha^3}{([n]_{p,q} + \beta)^3} \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n+1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} d_p q t
\]

\[
+ \frac{[n]_{p,q}^3}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_3; x) + \frac{3\alpha [n]_{p,q}^2}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_2; x)
\]

\[
+ \frac{\alpha^3}{([n]_{p,q} + \beta)^3} D_n^{p,q}(e_1; x)
\]
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\[
\begin{align*}
+ 3\alpha^2 [p, q, \alpha, \beta] D_n^{p, q}(e_1; x) & \quad + \frac{\alpha^3}{(n, p, q + \beta)^3} D_n^{p, q}(e_0; x) \\
+ \frac{p^3 [n, q, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}{q^3 [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
+ \frac{[n, p, q] [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}{p q^3 [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} & \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \sum_{k=0}^{\infty} \frac{[n, p, q] [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]

\[
\begin{align*}
+ & \frac{1}{2} \int_0^x (n, q, p, q + \beta)^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha)
+ \frac{3 q \alpha [2, p, q [n, p, q + \beta]^3 [n - 1, p, q [n - 2, p, q] [n - 2, p, q] \alpha}
\end{align*}
\]
We readily obtain the following lemma.

**Lemma 2.3.** For \( x \in [0, \infty) \), \( 0 < q < p \leq 1 \), \( 0 \leq \alpha \leq \beta \), we have

\[
(i) D_{n,p,q}^{\alpha,\beta}((t - x)^2) = \left( \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^2 - [n]_{p,q} + 1} \right) x^2 + \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \\
(ii) D_{n,p,q}^{\alpha,\beta}((t - x)^4) = \left( \frac{p^6 [n]_{p,q}^7}{q^4 ([n]_{p,q} + \beta)^4 [n - 1]_{p,q} [n - 2]_{p,q} [n - 3]_{p,q}} \right) x^4 - \frac{4p^3 [n]_{p,q}^5}{q^6 ([n]_{p,q} + \beta)^3 [n - 1]_{p,q} [n - 2]_{p,q}} + \frac{4[11]([n]_{p,q} + \beta)}{([n]_{p,q} + \beta)^4 [n - 1]_{p,q} [n - 2]_{p,q}} + 1 \right) x^4 \\
+ \left( \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^2 [n - 1]_{p,q}} - \frac{4[n]_{p,q}}{([n]_{p,q} + \beta)} + 1 \right) x^4 \\
+ \left( \frac{[n]_{p,q}}{([n]_{p,q} + \beta)^4 [n - 1]_{p,q} [n - 2]_{p,q} [n - 3]_{p,q}} \right) x^4
\]
In this section, we present local approximation theorem for operators \( D_{\alpha,\beta}^{n,p,q} \). By \( C_B[0, \infty) \), we denote the space of all real-valued continuous and bounded functions \( f \) defined on the interval \( [0, \infty) \). The norm \( \| \cdot \| \) on the space \( C_B[0, \infty) \) is given by

\[
\| f \| = \sup_{0 \leq x < \infty} | f(x) | .
\]

Further, let us consider the following \( K \)-functional:

\[
K_2(f, \delta) = \inf_{g \in W^2} \left\{ \| f - g \| + \delta \| g'' \| \right\}
\]

where \( \delta > 0 \) and \( W^2 = \{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \} \). By Theorem 2.4 of [6], there exists an absolute constant \( C > 0 \) such that

\[
(3.1) \quad K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta})
\]

where

\[
\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} \left| f(x + 2h) - 2f(x + h) + f(x) \right|
\]
is the second order modulus of smoothness of \( f \in C_B[0, \infty) \). The usual modulus of continuity of \( f \in C_B[0, \infty) \) is defined by
\[
\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} | f(x + h) - f(x) | .
\]

**Theorem 3.1.** Let \( f \in C_B[0, \infty) \) and \( 0 < q < p \leq 1 \), \( 0 \leq \alpha, \beta \leq 1 \). Then for all \( n \in \mathbb{N} \), there exists an absolute constant \( C > 0 \) such that
\[
| D_{n,p,q}^{\alpha,\beta}(f; x) - f(x) | \leq C \omega_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)),
\]
where
\[
\delta_n(x) = \sqrt{D_{n,p,q}^{\alpha,\beta}((t-x)^2; x) + (\alpha_n(x))^2}, \quad \alpha_n(x) = \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta} - x.
\]

**Proof.** For \( x \in [0, \infty) \), we consider the auxiliary operators \( \bar{D}^*_n \) defined by
\[
\bar{D}^*_n(f; x) = D_{n,p,q}^{\alpha,\beta}(f; x) + f(x) - f \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta} \right).
\]
From Lemma 2.2 (i), (ii) and Lemma 2.3 (i), we observe that the operators \( \bar{D}^*_n(f; x) \) are linear and reproduce the linear functions. Hence
\[
\bar{D}^*_n(1; x) = D_{n,p,q}^{\alpha,\beta}(1; x) + 1 - 1 = 1,
\]
\[
\bar{D}^*_n(t; x) = D_{n,p,q}^{\alpha,\beta}(t; x) + x - \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta} \right) = x,
\]
\[
\bar{D}^*_n((t-x); x) = \bar{D}^*_n(t; x) - x \bar{D}^*_n(1; x) = 0.
\]
Let \( x \in [0, \infty) \) and \( g \in W^2 \). Using the Taylor’s formula
\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.
\]
Applying \( \bar{D}^*_n \) to both sides of the above equation, we have
\[
\bar{D}^*_n(g; x) - g(x) = g'(x)\bar{D}^*_n((t-x); x) + \bar{D}^*_n\left( \int_x^t (t-u)g''(u)du; x \right)
\]
\[
= D_{n,p,q}^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du; x \right)
\]
\[
- \int_x^t \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta} \left( \frac{[n]_{p,q}}{[n]_{p,q} + \beta} x + \frac{\alpha}{[n]_{p,q} + \beta} - u \right) g''(u)du.
\]
On the other hand, since
\[
| \int_x^t (t-u)g''(u)du | \leq \int_x^t | t-u | g''(u) | du \leq \| g'' \| \int_x^t | t-u | | du | \leq (t-x)^2 \| g'' \| .
\]
and

\[
\left| \int_{x}^{\lfloor n \rfloor_{p,q} + \frac{x}{\lfloor n \rfloor_{p,q} + \beta}} + \int_{x}^{\lfloor n \rfloor_{p,q} + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta}} \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} - u \right) g''(u) \, du \right| \\
\leq \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} - x \right)^2 \| g'' \|.
\]

We conclude that

\[
\left| \bar{D}_n^* (g; x) - g(x) \right| \leq \left| D_{n,p,q}^{\alpha,\beta} \left( \int_{x}^{t} (t - u) g''(u) \, du; x \right) \\
- \int_{x}^{\lfloor n \rfloor_{p,q} + \frac{x}{\lfloor n \rfloor_{p,q} + \beta}} + \int_{x}^{\lfloor n \rfloor_{p,q} + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta}} \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} - u \right) g''(u) \, du \right| \\
\leq \| g'' \| \left| D_{n,p,q}^{\alpha,\beta} ((t - x)^2; x) + \| g'' \| \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} - x \right)^2 \right| \\
= \| g'' \| \delta_n^2(x).
\]

Now, taking into account boundedness of \( \bar{D}_n^* \), we have

\[
\left| \bar{D}_n^* (f; x) \right| \leq \left| D_{n,p,q}^{\alpha,\beta} (f; x) \right| + 2 \| f \| \leq 3 \| f \|.
\]

Therefore

\[
\left| D_{n,p,q}^{\alpha,\beta} (f; x) - f(x) \right| \leq \left| \bar{D}_n^* (f - g; x) - (f - g)(x) \right| + \left| f \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} \right) - f(x) \right| \\
+ \left| \bar{D}_n^* (g; x) - g(x) \right| \\
\leq \left| \bar{D}_n^* (f - g; x) \right| + \left| (f - g)(x) \right| + \left| f \left( \frac{\lfloor n \rfloor_{p,q}}{\lfloor n \rfloor_{p,q} + \beta} x + \frac{\alpha}{\lfloor n \rfloor_{p,q} + \beta} \right) - f(x) \right| \\
+ \left| \bar{D}_n^* (g; x) - g(x) \right| \\
\leq 4 \| f - g \| + \omega(f, \alpha_n(x)) + \delta_n^2(x) \| g'' \|.
\]

Hence, taking the infimum on the right-hand side over all \( g \in W^2 \), we have the following result

\[
\left| D_{n,p,q}^{\alpha,\beta} (f; x) - f(x) \right| \leq 4 K_2(f, \delta_n^2(x)) + \omega(f, \alpha_n(x)).
\]

In view of the property of \( K \)-functional, we get

\[
\left| D_{n,p,q}^{\alpha,\beta} (f; x) - f(x) \right| \leq C_2(f, \delta_n(x)) + \omega(f, \alpha_n(x)).
\]

This completes the proof of the theorem. \( \square \)
4. Approximation properties in weighted spaces

Let \( B_\rho[0, \infty) \) be the space of all real valued functions on \([0, \infty)\) satisfying the condition \(|f(x)| \leq M_f \rho(x)\), where \( M_f \) is a constant depending only on \( f \) and \( \rho(x) \) is a weight function.

Let \( C_\rho[0, \infty) \) be the space of all continuous functions in \( B_\rho[0, \infty) \) with the norm

\[
\|f\|_\rho = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\rho(x)}
\]

and

\[
C_\rho^0 = \left\{ f \in C_\rho[0, \infty) : \lim_{x \to \infty} \frac{|f(x)|}{\rho(x)} < \infty \right\}.
\]

In what follows, we assume the weight function as \( \rho(x) = 1 + x^2 \).

**Theorem 4.1.** Let \( 0 < q = q_n < p = p_n \leq 1 \) such that \( q_n \to 1, p_n \to 1 \), as \( n \to \infty \). For each \( f \in C_\rho^0 \), we have

\[
\lim_{n \to \infty} \|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)\|_\rho = 0.
\]

**Proof.** With elementary calculations, it can be easily followed that \( \lim_{n \to \infty} \|D_{\alpha, \beta}^{n, p_n, q_n}(e_i; \cdot) - e_i\|_\rho = 0 \), where \( e_i(x) = x^i, i = 0, 1, 2 \). By weighted Korovkin theorem given in [7], we get the required result.

Next we give the following theorem to approximate all functions in \( C_\rho^0 \). This type of result is discussed in [10] for locally integrable functions.

**Theorem 4.2.** Let \( 0 < q = q_n < p = p_n \leq 1 \) such that \( q_n \to 1, p_n \to 1, q_n^a \to 1, p_n^a \to 1 \) as \( n \to \infty \). For each \( f \in C_\rho^0 \) and \( a > 0 \), we have

\[
\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)|}{(1 + x^2)^{1+a}} = 0.
\]

**Proof.** For any fixed \( x_0 > 0 \),

\[
\sup_{x \in [0, \infty)} \frac{|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)|}{(1 + x^2)^{1+a}} \leq \sup_{x \leq x_0} \frac{|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)|}{(1 + x^2)^{1+a}} + \sup_{x \geq x_0} \frac{|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)|}{(1 + x^2)^{1+a}}
\]

\[
\leq \|D_{\alpha, \beta}^{n, p_n, q_n}(f; x) - f(x)\|_{C[0, x_0]} + \|f\|_\rho \sup_{x \geq x_0} \frac{|D_{\alpha, \beta}^{n, p_n, q_n}(1 + t^2; x)|}{(1 + x^2)^{1+a}}
\]

\[
+ \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+a}}
\]
\begin{equation}
  = I_1 + I_2 + I_3.
\end{equation}

Since \(|f(x)| \leq \|f\|_\rho (1 + x^2)\), we have

\[
I_3 = \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+a}} \leq \sup_{x \geq x_0} \frac{\|f\|_\rho}{(1 + x^2)^{1+a}} \leq \frac{\|f\|_\rho}{(1 + x_0^2)^{1+a}}
\]

Let \(\epsilon > 0\) be arbitrary. There exists \(n_1 \in \mathbb{N}\) such that

\[
\|f\|_\rho \sup_{x \geq x_0} \frac{|D^{\alpha,\beta}_{n,p,q_n}(1 + t^2; x)|}{(1 + x^2)^{1+a}} < \frac{1}{(1 + x_2^2)^{1+a}} \|f\|_\rho \left(1 + \frac{\epsilon}{3}\right), \quad \forall n \geq n_1
\]

\begin{equation}
< \frac{\|f\|_\rho}{(1 + x^2)^{1+a}} + \frac{\epsilon}{3}, \quad \forall n \geq n_1.
\end{equation}

Hence

\[
\|f\|_\rho \sup_{x \geq x_0} \frac{|D^{\alpha,\beta}_{n,p,q_n}(1 + t^2; x)|}{(1 + x^2)^{1+a}} < \frac{\|f\|_\rho}{(1 + x_0^2)^{1+a}} + \frac{\epsilon}{3}, \quad \forall n \geq n_1.
\]

Thus

\[
I_2 + I_3 < \frac{2\|f\|_\rho}{(1 + x_0^2)^{1+a}} + \frac{\epsilon}{3}, \quad \forall n \geq n_1.
\]

Now, let us choose \(x_0\) to be so large that \(\frac{\|f\|_\rho}{(1 + x_2^2)^{1+a}} < \frac{\epsilon}{6}\).

Then,

\begin{equation}
I_2 + I_3 < \frac{2\epsilon}{3}, \quad \forall n \geq n_1.
\end{equation}

\begin{equation}
I_1 = \|D^{\alpha,\beta}_{n,p,q_n}(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \quad \forall n \geq n_2.
\end{equation}

Let \(n_0 = \max(n_1, n_2)\). Then, combining (4.1)-(4.4), we get

\[
\sup_{x \in [0, \infty)} \frac{|D^{\alpha,\beta}_{n,p,q_n}(f; x) - f(x)|}{(1 + x^2)^{1+a}} < \epsilon, \quad \forall n \geq n_0.
\]

This completes the proof. \(\square\)

Now we present ordinary approximation in terms of Lipschitz constant defined by

\begin{equation}
lip_M(\gamma) = \left\{ f \in C_B[0, \infty) : |f(t) - f(x)| \leq M \frac{|t-x|^{\gamma}}{(t+x)^{\beta}} \right\},
\end{equation}

where \(M\) is a positive constant and \(0 < \gamma \leq 1\).
Theorem 4.3. Let be \( f \in C_B[0, \infty), \ 0 < q < p \leq 1, \ 0 \leq \alpha \leq \beta, \) then for any \( x \in (0, \infty), \) the following inequality holds:

\[
|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq M \left( \frac{\varphi_{n,p,q}(x)}{x} \right)^{\frac{2}{\gamma}},
\]

where \( \varphi_{n,p,q}(x) = D_{n,p,q}^{e_1}(e_1 - x)^2; x). \)

Proof. First, we prove that the result is true for \( \gamma = 1. \) Then, for \( f \in \text{lip}_M(\gamma), \) we obtain

\[
|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq \sum_{k=0}^{\infty} s_{n,k}(x) \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}}
\]

\[
\times f \left( \frac{[n]_{p,q}^{k+1} qt \alpha}{[n]_{p,q} + \beta} \right) - f(x) \right| dt.
\]

Using \( \sqrt{x} < \sqrt{\frac{[n]_{p,q}^{k+1} qt \alpha}{[n]_{p,q} + \beta}} + x \) and the Cauchy-Schwarz inequality, we get

\[
|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq \frac{M}{\sqrt{2}} \sum_{k=0}^{\infty} s_{n,k}(x) \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}}
\]

\[
\times \left| \frac{[n]_{p,q}^{k+1} qt \alpha}{[n]_{p,q} + \beta} - x \right| dt.
\]

Therefore, the result is true for \( \gamma = 1. \) We prove that the result is true for \( 0 < \gamma \leq 1, \)

applying Hölder’s inequality with \( p = \frac{2}{\gamma}, \ q = \frac{1}{\gamma}, \)

\[
|D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq \sum_{k=0}^{\infty} s_{n,k}(x) \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}}
\]

\[
\times f \left( \frac{[n]_{p,q}^{k+1} qt \alpha}{[n]_{p,q} + \beta} \right) - f(x) \right| dt.
\]

\[
\leq \sum_{k=0}^{\infty} s_{n,k}(x) \left( \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}}
\]

\[
\times f \left( \frac{[n]_{p,q}^{k+1} qt \alpha}{[n]_{p,q} + \beta} \right) - f(x) \right| dt.
\]
\[ \left| \frac{[n]_{p,q}B^{k+1}q_t + \alpha}{[n]_{p,q} + \beta} - f(x) \right| d_{p,q}t \right\}^2 \]

\[ \leq \left\{ \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} \right\}^{2z/n} \]

Since \( f \in \text{lip}_M(\gamma) \), we have

\[ |D_{n,p,q}^{\alpha,\beta}(f; x) - f(x)| \leq \frac{M}{x^{\frac{2z}{n}}} \left( \sum_{k=0}^{\infty} s_{n,k}^{p,q}(x) \frac{1}{B_{p,q}(k, n + 1)} \int_0^\infty \frac{t^{k-1}}{(1 + pt)^{n+k+1}} \right) \]

\[ \times \left( \frac{[n]_{p,q}B^{k+1}q_t + \alpha}{[n]_{p,q} + \beta} - x \right)^2 d_{p,q}t \right\}^{\frac{2z}{n}} \]

\[ = \frac{M}{x^{\frac{2z}{n}}} \left( D_{n,p,q}^{\alpha,\beta}((e_1 - x)^2; x) \right)^{\frac{2z}{n}} \leq M \left( \frac{\varphi_{n,p,q}(x)}{x} \right)^{\gamma}. \]

Therefore, the proof is completed. \( \square \)

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