Inequalities for quantum channels assisted by limited resources

Vittorio Giovannetti

NEST-INFM & Scuola Normale Superiore, piazza dei Cavalieri 7, I-56126 Pisa, Italy.

The information capacities and “distillability” of a quantum channel are studied in the presence of auxiliary resources. These include prior entanglement shared between the sender and receiver and free classical bits of forward and backward communication. Inequalities and trade-off curves are derived. In particular an alternative proof is given that in the absence of feedback and shared entanglement, forward classical communication does not increase the quantum capacity of a channel.

Any realistic scheme for information transmission must take into account the presence of noise. Given the technological challenge we are facing in controlling decoherence (e.g. [1] and references therein) this is even more important when transmitting quantum information through quantum channels [2]. One way to reduce the effects of noise is to provide the communicating parties with some extra resource that can be used to implement more efficient communication protocols. It is known, for instance, that teleportation [3] and superdense coding [4] can increase both the quantum and classical capacities of a channel by allowing the sender and receiver of the message to share a sufficient amount of prior entanglement [5, 6]. Alternatively, using entanglement distillation protocols [5, 8], the channel performances can be improved by introducing a classical feedback side channel [6] or by allowing the sender and receiver to communicate freely through a classical two-way side channel [3, 4]. Following the suggestion of Refs. [6, 10, 11, 12] in this paper we analyze the relationships between different resources by focusing on the case where resources are limited.

The material is organized as follows. In Sec. I we introduce the notation, define the distillability of a channel in the presence of finite resources, and establish some preliminary results. In Sec. II we give a new proof that, in the absence of prior entanglement and feedback, free forward classical communication does not increase the quantum capacity of a channel. In Sec. III we study the quantum and classical capacities as functions of the resource parameters and we establish some trade-offs and asymptotic limits. In Sec. IV we provide some identities for the distillability: in particular we show that with only free classical forward communication the distillability of a channel cannot be greater than its unassisted quantum capacity. The paper ends in Sec. V with the conclusion.

I. QUANTUM CHANNELS WITH LIMITED RESOURCES

Consider a memoryless quantum channel [2] described by a Completely Positive, Trace preserving (CPT) map \( \mathcal{M} \) defined in a \( d \)-dimensional Hilbert space \( \mathcal{H} \). For the sake of simplicity we will measure the capacities of such channel in “dits” or “qudits” per channel use, where 1 dit stands for \( \log_2 d \) bits of classical information and 1 qudit for \( \log_d 2 d \) qubits of quantum information. Analogously we define 1 “e-dit” the entanglement associated with a maximally entangled state of \( \mathcal{H} \otimes \mathcal{H} \), i.e. \( \log_2 d \) e-bits.

We are interested in the scenario depicted in Fig. 1. At each channel use the sender and the receiver are provided on average with \( x \) dits of backward (i.e. from the receiver to the sender) classical communication, \( y \) dits of forward (i.e. from the sender to the receiver) classical communication, and \( p \) e-dits of shared entanglement.

A rigorous definition of \( x \), \( y \) and \( p \) requires one to consider a limit \( N \to \infty \) in the total number \( N \) of channel uses. That is, if \( X_N \) is the total number of dits of classical feedback available on \( N \) uses of \( \mathcal{M} \), we define \( x \) as \( \lim_{N \to \infty} X_N / N \). Analogous definitions apply for \( y \) and \( p \). Moreover in defining \( y \) and \( p \) we do not assume the classical information transmitted through the side channel to be independent of the classical message transmitted through \( \mathcal{M} \). This is different from the definition adopted in [13] where the communicating parties cannot use the
side channel to directly transfer part of the message (in their case however y is ideally infinite and our definition would produce only trivial results).

Exploiting the resources x, y and p the two communicating parties can improve the performances of the channel $\mathcal{M}$. In particular by means of x and y they can set up purification protocols to augment the number $p$ of shared maximally entangled states. On the other hand they can employ $p$ in teleportation and superdense coding schemes to transfer reliably (i.e. with unit fidelity) quantum or classical information from the sender to the receiver. Alternatively, $p$ and $x$ can be used to create a quantum feedback connection through teleportation. In general the optimal strategy associated with the resources $x$, $y$ and $p$ consists in some complicated composition of all these effects.

We call $Q(x,y,p)$ the quantum capacity achievable in the communication scenario of Fig. 1, while we use $C(x,y,p)$ to indicate the corresponding classical capacity. These two objects represent respectively the maximum number of unknown qudits or dits that can be reliably transmitted through $\mathcal{M}$ per channel use $2$. We also introduce the “distillability” $D(x,y,p)$ which gives, in dits per channel use, the maximum number of maximally entangled states of $\mathcal{H} \otimes \mathcal{H}$ that can be asymptotically shared between the sender and receiver by using $\mathcal{M}$ and the resources $x$, $y$ and $p$. Such a quantity is not a proper capacity of $\mathcal{M}$, as the communicating parties know a priori which states (i.e. the maximally entangled states) they are going to share. Apart from regularization over multiple uses (see Appendix A for details) $D(x,y,p)$ can be expressed as

$$D(x,y,p) = \lim_{n \to \infty} \max_{R} \left\{ \frac{P_{x,y,p}[\mathcal{M} \otimes 1^n \otimes (R)]}{n} \right\},$$

where the maximization is performed over all density matrices $R$ defined in the Hilbert space $\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n$. In this equation $R' \equiv (\mathcal{M} \otimes 1^n \otimes (R))$ is the state we get by sending half of $R$ through $n$ copies of the channel $\mathcal{M}$ and doing nothing (i.e. applying the identity superoperator $1^n$) to the other half. Finally the quantity $P_{x,y,p}[R']$ is the maximum number of e-dits that can be asymptotically extracted from $R'$ through purification protocols which employs, on average, $x$ dits of feedback, $y$ dits of classical forward communication and $p$ e-dits of prior shared entanglement for any use of the channel $\mathcal{M}$. In other words, $D(x,y,p)$ is obtained by maximizing over all possible input $R$ the distillability of the output state $R'$ achievable by using protocols that employ $x$, $y$ and $p$ resources.

The aim of this paper is to study the dependence of $Q(x,y,p)$, $C(x,y,p)$ and $D(x,y,p)$ upon the variables $x$, $y$ and $p$. Such an endeavor is connected with the study of the channel capacity for the simultaneous transmission of quantum and classical information since, for instance, we can interpret the resource $y$ as the classical information transmitted through $\mathcal{M}$ in some previous channel use.

### A. Basic properties

The classical capacity and distillability provide two trivial upper bounds for the quantum capacity of the channel, i.e.

$$C(x,y,p) \geq Q(x,y,p),$$

$$D(x,y,p) \geq Q(x,y,p).$$

In fact, on the one hand, at each channel use we can transmit $Q(x,y,p)$ dits of classical information by encoding them into qudits. On the other hand, at each channel use we can produce $Q(x,y,p)$ e-bits between the sender and the receiver by transmitting $Q(x,y,p)$ halves of maximally entangled states of $\mathcal{H} \otimes \mathcal{H}$. The relation between $D(x,y,p)$ and $C(x,y,p)$ is more complex and even though there are situations in which the later is bigger than the former, we are not able to provide a definitive ordering (see also Sec. IV).

The quantities $Q(x,y,p)$, $C(x,y,p)$ and $D(x,y,p)$ are non-decreasing, jointly-concave functions of their arguments. The first property derives simply from the fact that the sender and receiver can discard part of the resources they are given to exactly simulate communication scenarios with fewer initial resources. The concavity derives instead from the possibility that the communicating parties use their resources within time-sharing strategies (see App. B).

**Unassisted capacities:** For $x, y, p = 0$ the capacities defined above give the unassisted capacities of $\mathcal{M}$, i.e. $Q \equiv Q(0,0,0)$ and $C \equiv C(0,0,0)$. These quantities can be determined by maximizing (over multiple channel uses) the coherent information $14$ and the output Holevo information of the channel $15$, respectively.

**Entanglement assisted capacities:** The functions $Q(0,0,0)$ and $C(0,0,0)$ represent the entanglement assisted capacities of $\mathcal{M}$ in the absence of classical feedback and classical forward communication. Shor recently gave $6$ a procedure to compute the value of $C(0,0,0)$ while a method to calculate $Q(0,0,0)$ is provided by Devetak, Harrow and Winter in Ref. $11$. In the limit of large $p$ it has been shown $7$ that $C_E \equiv C(0,0,\infty)$ can be obtained by maximizing the quantum mutual information $16$ of the channel while, due to teleportation and superdense coding,

$$Q_E = C_E/2.$$

The minimum values $E_Q$ and $E_C$ of $p$ for which $Q(0,0,0)$ and $C(0,0,0)$ achieve respectively $Q_E$ and $C_E$ are known to be of the order of one e-bit per channel use $2,7$. Moreover the following relations have been established $17$:

$$E_C + Q_E \geq E_Q \geq E_C - Q_E,$$

$$E_Q \geq Q_E - Q, \quad E_C \geq C_E - C.$$
classical feedback \( x \) on the quantum and classical capacity of a channel. These results can be summarized in our formalism by the following relations,

\[
C(x, 0, p) = C(0, 0, \infty) \equiv C_E \quad \text{for} \quad p \geq \mathcal{E}_C \\
Q(x, 0, p) = Q(0, 0, \infty) \equiv Q_E \quad \text{for} \quad p \geq \mathcal{E}_Q ,
\]

which imply that, in the absence of forward classical communication \((y = 0)\), feedback cannot be used to increase the capacities above the level achieved with arbitrary shared entanglement.

Feedback and forward communication: For \( p = 0 \) and arbitrary backward and forward classical communication \( Q(x, y, p) \) gives the two-way quantum capacity, \( Q_{2\text{-way}} \equiv Q(\infty, \infty, 0) \). There is no simple recipe to compute \( Q_{2\text{-way}} \), but using teleportation one can show that this capacity coincides with the maximum amount of maximally entangled state that can be shared between the sender and receiver using arbitrary two-way purification protocols, i.e.

\[
Q(\infty, \infty, 0) = D(\infty, \infty, 0) ,
\]

with \( D(\infty, \infty, 0) \) the distillability of Eq. (11) evaluated for \( x = y = \infty \) and \( p = 0 \).

The relations between the quantities defined above have not yet completely understood. In the case of quantum capacities, we know for instance that \( Q_E \) and \( Q_{2\text{-way}} \) are always greater than or equal to \( Q \) and recently it has been proved \[^{12} \] that \( Q_E \geq Q_{2\text{-way}} \). Equation (6) establishes that \( Q_E \) is greater than or equal to the quantum capacity of the channel in the presence of arbitrary amount of feedback \( Q_{FB} \equiv Q(\infty, 0, 0) \), but it is unclear \[^{12} \] if this last quantity is strictly smaller than \( Q_{2\text{-way}} \).

In the following sections we will try to characterize the relations among communication scenarios assisted by different initial resources. We begin in Sec. II by deriving two simple identities which involve forward classical communication.

II. CAPACITIES ASSISTED BY FORWARD COMMUNICATION

In this section we analyze the role of free forward classical communication and show that for any \( x, y \) and \( p \),

\[
C(x, y, p) = y + C(x, 0, p) ,
\]

\[
Q(0, y, 0) = Q(0, 0, 0) .
\]

The first expression states that if the sender is provided with \( y \) classical dits of forward communication per channel use, the classical capacity of the channel cannot be increased of more than \( y \) dits per channel use. This result can be interpreted as an application of the additivity property of the entanglement breaking channels \[^{18} \] to the case \( x, p \neq 0 \). We provide an explicit proof of Eq. (12) in Sec. II A. Equation (13) is a little more subtle: it implies that, for \( x = p = 0 \), free forward classical communication cannot be used to boost the quantum capacity of a channel. This fact was first pointed out in Ref. \[^{18} \] and successively in Ref. \[^{19} \] by explicitly proving that from any protocol that uses classical forward communication one can derive another protocol which does not use such resource but that achieves, asymptotically, the same communication rate. In Sec. II B we give an alternative derivation of this result by direct calculation of the capacity \( Q(0, y, 0) \). In Sec. II C we will prove a stronger version of this identity by showing that \( D(0, y, 0) = Q(0, 0, 0) \).

A. Classical capacity assisted by forward classical communication

To derive Eq. (12) we notice that the right-hand side of this equation is a lower bound for the capacity \( C(x, y, p) \). In fact, for each use of channel \( \mathcal{M} \), the sender can exploit the forward communication resource to directly transmit \((in average)\) \( y \) dits. Moreover, by using the channel \( \mathcal{M} \) with \( x \) dits of feedback and \( p \) e-dits of share entanglement, she/he can still communicate at a rate \( C(x, 0, p) \). Hence to prove the identity in Eq. (12) we only need to show that

\[
C(x, y, p) \leq y + C(x, 0, p) .
\]

This can be accomplished, for instance, by considering the capacity \( C(x, 0, p) \) in the absence of free forward communication. By definition, using the channel \( N \) times the sender cannot transmit more than \( NC(x, 0, p) \) dits of classical communication. Suppose now that she/he decides to use the classical dits transmitted in a fraction \( \gamma \in [0, 1] \) of the \( N \) channels as a resource to boost the capacity of the remaining \((1 - \gamma)N\) channel uses. The total number of dits transmitted in the first part of the protocol is \( \gamma NC(x, 0, p) \), which provides in average

\[
y \equiv \gamma C(x, 0, p)/(1 - \gamma)
\]

dits per channel use of forward communication available as a resource for the remaining \((1 - \gamma)N\) uses. In this way, when using these channels she/he can achieve a capacity \( C(x, y, p) \geq C(x, 0, p) \) and hence a total of \((1 - \gamma)NC(x, y, p)\) dits of classical communication transmitted. Since this number cannot exceed \( NC(x, 0, p) \), we have

\[
C(x, 0, p) \geq (1 - \gamma)C(x, y, p) ,
\]

which yields Eq. (12) by solving for \( \gamma \) in terms of \( y \) through Eq. (11).

B. Quantum capacity assisted by forward classical communication

To prove Eq. (13) it is sufficient to show that the right-hand side term is greater than or equal to the left-hand side term. In fact by definition we have that
$Q(0,y,0) \geq Q(0,0,0)$ for all $y$. We remind the reader that the unassisted capacity $Q = Q(0,0,0)$ of a channel $\mathcal{M}$ can be calculated as the sup over $n$ successive uses of the channel,

$$Q = \sup_n Q_n/n,$$  \hspace{1cm} (13)

of the maximum coherent information\cite{14,20},

$$Q_n \equiv \max_{\rho \in \mathcal{H}^\otimes n} \left\{ S(\mathcal{M}^\otimes n(\rho)) - S((\mathcal{M}^\otimes n \otimes \mathbb{1}_{\text{anc}})(\Phi_p)) \right\},$$ \hspace{1cm} (14)

achievable over the set of the input density matrices $\rho$ of $\mathcal{H}^\otimes n$. In Eq. (14), $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy expressed in dits, and $\Phi_p$ is a purification \cite{14} of $\rho$ defined in the extended space obtained by adding an ancillary space $\mathcal{H}_{\text{anc}}$ to $\mathcal{H}^\otimes n$.

An expression analogous to Eq. (14) can be used to compute the capacity $Q(0,y,0)$ of the channel $\mathcal{M}$ in the presence of $y$ dits of classical forward communication. In fact, consider a Hilbert space $\mathcal{H}'$ of dimension $\Delta \geq d^y$ and define $T$ the CPT map which describes the complete decoherence of the system in the orthonormal basis $\{|\omega\rangle\}$ of $\mathcal{H}'$, i.e.

$$T(|\omega\rangle\langle\omega|) \equiv \delta_{\omega,\omega'} |\omega\rangle\langle\omega|.$$ \hspace{1cm} (15)

This map is an entanglement breaking channel\cite{15} when used in the absence of any external resource, it cannot transfer quantum information but can reliably transmit (at least) $y$ dits of classical information by encoding them into the occupation numbers of the basis $\{|\omega\rangle\}$. The super-operator $\bar{\mathcal{M}} \equiv \mathcal{M} \otimes T$ defines a quantum channel which acts on the input Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. Clearly the quantum capacity $\bar{Q}(0,y,0)$ of this channel is at least as big as the capacity $Q(0,y,0)$ of the original channel $\mathcal{M}$: by employing $T$ to transmit $\log_2 \Delta$ dits of classical information at each use of $\bar{\mathcal{M}}$, we can simulate the performance of the channel $\mathcal{M}$ when it is assisted by $y$ dits of forward communication. The identity \cite{9} can be hence proved by showing that

$$\bar{Q}(0,0,0) \leq Q(0,0,0).$$ \hspace{1cm} (16)

In the following we will do that by expressing $\bar{Q}(0,0,0)$ in terms of the coherent information of $\mathcal{M}$ as in Eq. (14).

Given the $n$-elements vector $\bar{\omega} \equiv (\omega_1, \cdots, \omega_n)$, define $|\bar{\omega}\rangle \equiv \otimes_{j=1}^n |\omega_j\rangle$ the orthonormal basis of $\mathcal{H}^\otimes n$ obtained by taking $n$ copies of the basis $\{|\omega_j\rangle\}$ of $\mathcal{H}'$. The map $\bar{\mathcal{M}}^\otimes n$ transforms any density matrix $R$ of $(\mathcal{H} \otimes \mathcal{H}')^\otimes n$ according to

$$\bar{\mathcal{M}}^\otimes n(R) \equiv \sum_\omega \lambda_\omega \mathcal{M}^\otimes n(\rho_\omega) \otimes |\bar{\omega}\rangle\langle\bar{\omega}|,$$ \hspace{1cm} (17)

where

$$\lambda_\omega \rho_\omega \equiv \langle\bar{\omega}|R|\bar{\omega}\rangle,$$ \hspace{1cm} (18)

is the unnormalized density matrix of $\mathcal{H}^\otimes n$ obtained by projecting $R$ into $|\bar{\omega}\rangle$, $\lambda_\omega$ being the probability associated with such a projection. From the orthogonality of $\{|\bar{\omega}\rangle\}$ we can thus express the von Neumann entropy of $\bar{\mathcal{M}}^\otimes n(R)$ as \cite{14}

$$S(\bar{\mathcal{M}}^\otimes n(R)) = H(\lambda_\omega) + \sum_\omega \lambda_\omega S(\mathcal{M}^\otimes n(\rho_\omega)),$$ \hspace{1cm} (19)

with

$$H(\lambda_\omega) \equiv -\sum_\omega \lambda_\omega \log_2 \lambda_\omega,$$ \hspace{1cm} (20)

the Shannon entropy associated to the probabilities $\lambda_\omega$.

Consider now a purification $\Phi_R \equiv |\Phi_R\rangle\langle\Phi_R|$ of $R$. Its projection into the state $|\bar{\omega}\rangle$ of $\mathcal{H}^\otimes n$ gives

$$\langle\bar{\omega}|\Phi_R\rangle \equiv \sqrt{\lambda_\omega} |\Phi_{\rho_\omega}\rangle,$$ \hspace{1cm} (21)

where $|\Phi_{\rho_\omega}\rangle \in \mathcal{H}^\otimes n \otimes \mathcal{H}_{\text{anc}}$ is a purification of the density matrix $\rho_\omega$ of Eq. (18). From Eq. (17) derives thus

$$\bar{\mathcal{M}}^\otimes n(\Phi_R) = \sum_\omega \lambda_\omega \mathcal{M}^\otimes n(\Phi_{\rho_\omega}) \otimes |\bar{\omega}\rangle\langle\bar{\omega}|,$$ \hspace{1cm} (22)

with $\Phi_{\rho_\omega} \equiv |\Phi_{\rho_\omega}\rangle\langle\Phi_{\rho_\omega}|$, and hence

$$S(\mathcal{M}^\otimes n(\Phi_R)) = H(\lambda_\omega) + \sum_\omega \lambda_\omega S(\mathcal{M}^\otimes n(\Phi_{\rho_\omega}))).$$ \hspace{1cm} (23)

From Eqs. (19) and (23) we finally obtain

$$S((\bar{\mathcal{M}}^\otimes n)(R)) - S((\bar{\mathcal{M}}^\otimes n \otimes \mathbb{1}_{\text{anc}})(\Phi_R)) = \sum_\omega \lambda_\omega [S(\mathcal{M}^\otimes n(\rho_\omega)) - S((\mathcal{M}^\otimes n \otimes \mathbb{1}_{\text{anc}})(\Phi_{\rho_\omega})))],$$ \hspace{1cm} (24)

which shows that the coherent information of the map $\bar{\mathcal{M}}^\otimes n$ relative to the input state $R$ of $(\mathcal{H} \otimes \mathcal{H}')^\otimes n$ can be expressed as a convex combination of the coherent informations of the map $\mathcal{M}^\otimes n$. According to \cite{14} the quantum capacity $\bar{Q}(0,0,0)$ of $\bar{\mathcal{M}}$ is obtained by maximizing over $R$ the left-hand side term of Eq. (24) and then by taking the sup over $n$. The inequality (16) finally derives by noticing that $Q_n$ of Eq. (14) is greater than or equal to the right-hand side of Eq. (24) for all $R$ and for all $n$ integer.

III. ASYMPTOTIC LIMITS AND TRADE-OFF CURVES FOR CAPACITIES

In this section we analyze in detail the relations between the capacities associated with different resources. We will focus mostly on the properties of $Q(x, y, p)$. For the sake of simplicity part of the material relative to the no-feedback case $(x = 0)$ has been postponed in Appendix C.
A. Quantum capacity

The behavior of $Q(x, y, p)$ as a function of the parameter $p$ is sketched in Fig. 2. We begin by showing that for all $x, y$ and $p$ one has

$$pQ(x, y, 0)/D(x, y, 0) + Q(x, y, 0) \geq Q(x, y, p),$$

where $D(x, y, 0)$ is the distillability of $\mathcal{M}$ defined in Eq. (11). This relation essentially states that $p$ e-dits cannot produce more than $p$ qubits of quantum information. To prove it we proceed analogously to Sec. II A and consider $N >> 1$ uses of the channel $\mathcal{M}$ in the presence of $x$ qubits of feedback, $y$ qubits of classical forward information and no shared entanglement. By definition the sender cannot transfer more than $NQ(x, y, 0)$ qubits. Suppose now that she/he decides to use a fraction $\gamma$ of the channels to share with the receiver some maximally entangled state that will then be employed as a resource for the remaining $(1-\gamma)N$ uses. Since the total number of e-dits transmitted on $\gamma N$ channels is at most $\gamma N D(x, y, 0)$, they obtain on average

$$p \equiv \gamma D(x, y, 0)/(1-\gamma)$$

e-dits per channel as a resource for the $(1-\gamma)N$ remaining channels. In the second part of the protocol the communicating parties can thus achieve a quantum capacity $Q(x, y, p)$ which is greater than the initial $Q(x, y, 0)$. The total number of qubits transmitted in this way is hence equal to $(1-\gamma)N Q(x, y, p)$. Equation (25) then follows by requiring this quantity to be smaller than the maximum number of qubits transmittable, i.e. $N Q(x, y, 0)$. 

The upper bound of Eq. (25) is not always tight and for large values of $p$ is replaced by

$$y/2 + Q_E \geq Q(x, y, p).$$

To prove this inequality consider the scenario where the communicating parties are provided with $x$ qubits of feedback, $y = 0$ forward classical communication, and $p$ e-dits of shared entanglement. In this case, by using the channel $N$ times the sender can transfer at most $N Q(x, y, p)$ qubits. Compare this with the number of qubits that can be transmitted when a fraction of the channels are employed to produce some qubits of classical communication as a resource for the remaining channels. In the limit $N >> 1$ we get

$$y/2 + Q_E \geq Q(x, y, 0).$$

which for $p \to \infty$ becomes (see Eqs. (8) and (10))

$$y/2 + Q_E \geq Q(x, y, \infty).$$

The inequality (27) derives now from the monotonicity of $Q(x, y, p)$ with respect to $p$.

Simple lower bounds for $Q(x, y, p)$ are obtained by exploiting teleportation. In fact if $p \geq y/2$, the sender can use the $y$ qubits of forward communication and $y$ e-dits of shared entanglement to teleport $y/2$ qubits to the receiver. At this point she/he can still use the channel $\mathcal{M}$ to transmit at a rate equal to $Q(x, 0, p - y/2)$. i.e.

$$Q(x, y, p) \geq y/2 + Q(x, 0, p - y/2) \quad \text{for } p \geq y/2.$$ (30)

Analogously one can show that

$$Q(x, y, p) \geq p + Q(x, y - 2p, 0) \quad \text{for } p \leq y/2.$$ (31)

Consider now the case $p \geq y/2 + \mathcal{E}_Q$, with $\mathcal{E}_Q$ defined as in Sec. I. In this limit, Eq. (6) implies $Q(x, 0, p - y/2) = Q_E$, and by confronting Eq. (27) with Eq. (30) we obtain

$$Q(x, y, p) = y/2 + Q_E \quad \text{for } p \geq y/2 + \mathcal{E}_Q.$$ (32)

The right-hand side of this expression is thus an asymptote for the capacity (see Fig. 2). It is achieved for a critical value of $p$ which is smaller than $y/2 + \mathcal{E}_Q$ and greater than the intercept between the asymptote and the upper bound of Eq. (25) [Notice that by imposing $y/2 + \mathcal{E}_Q$ to be greater than this point one gets $\mathcal{E}_Q \geq Q(x, y, 0)$ which gives Eq. (11) for $x = y = 0$. From these considerations and from the concavity of $Q(x, y, p)$, we can now establish the linear lower bound of plot a) of Fig. 2, i.e.

$$Q(x, y, p) \geq p y/2 + Q_E - Q(x, y, 0) / y/2 + \mathcal{E}_Q + Q(x, y, 0).$$ (33)
for \( p \leq y/2 + \mathcal{E}_Q \). For \( x = 0 \) this inequality can be improved by means of Eqs. (31) and (32). In fact in this limit Eq. (31) yields
\[
Q(0, y, p) \geq p + Q(0, y - 2p, 0) = p + Q(0, 0, 0) ,
\]
for \( p \leq y/2 \), which by comparison with Eqs. (25) and (2) implies
\[
Q(0, y, p) = p + Q(0, 0, 0) \quad \text{for} \quad p \leq y/2 .
\]
This equation generalizes the identity (3) to the case of shared entanglement. In Appendix B we provide a more detailed analysis of the no-feedback case, by analyzing a conjecture proposed by Bowen [17].

1. Quantum capacity with forward classical communication

The dependence of \( Q(x, y, p) \) with respect to the resource \( y \) has been plotted in Fig. 3. The asymptote
\[
Q(x, \infty, p) = p + Q(x, \infty, 0) ,
\]
is derived by considering Eqs. (25) and (31) in the limit \( y \gg 2p \). For \( x \neq 0 \), we do not have a method to characterize the critical \( y \) for which this asymptotic regime is achieved. However, Eq. (35) shows that in the case of no-feedback (\( x = 0 \)) this quantity is smaller than \( 2p \). The upper bound given in the plot \( a) \) of Fig. 3 is provided by Eq. (28). For \( p \geq \mathcal{E}_Q \) and \( y \leq (p - \mathcal{E}_Q)/2 \) the value of \( Q(x, y, p) \) is determined by Eq. (32) while Eq. (27) gives a better lower bound for \( Q(x, y, p) \) than (36) (see plot \( b) \) of Fig. 3.

B. Classical capacity

Equation (33) allows us to focus only on the case \( y = 0 \). A lower bound for \( C(x, 0, p) \) derives from the concavity with respect to \( p \) and from the definition of \( \mathcal{E}_C \) given in Eq. (3), i.e.
\[
C(x, 0, p) \geq p \frac{C_E - C}{\mathcal{E}_C} + C .
\]
As in the case of Eq. (28) we can establish the following inequality,
\[
p C(x, 0, 0) D(x, 0, 0) + C(x, 0, 0) \geq C(x, 0, p) ,
\]
with \( D(x, 0, 0) \) the distillability of the channel \( \mathcal{M} \) achieved by using on average \( x \) dits of classical feedback. Equation (38) can be derived by comparing the capacity \( C(x, 0, 0) \) with the number of bits transmissible with a protocol where the sender and receiver employ part of the channel uses to share maximally entangled pairs.

IV. IDENTITIES FOR DISTILLABILITY

From the definition of \( D(x, y, p) \) of Eq. (11) one can prove that for any \( x, y \) and \( p \) the following identities hold
\[
D(x, \infty, p) = Q(x, \infty, p) ,
\]
\[
D(x, y, p) = p + D(x, y, 0) ,
\]
\[
D(0, 0, 0) = D(0, 0, 0) = Q(0, 0, 0) .
\]
The first identity is a trivial generalization of Eq. (7). It derives by noticing that, with infinite free forward classical communication, each of the maximally entangled state distilled from the channel can be used to teleport one qudit of quantum information. This implies that \( Q(x, \infty, p) \geq D(x, \infty, p) \) which together with Eq. (2) gives the relation (39). The identity (40) is the analog of Eq. (3) in the context of the distillability of a channel. It states that adding \( p \) e-dits of shared entanglement per channel use to the resources, the distillability of \( \mathcal{M} \) cannot be increased by more than \( p \) e-dits per channel use.

A less trivial identity is Eq. (41) which can be seen as a stronger version of Eq. (3). For the special case of generalized depolarizing channels it was first proved in...
Ref. 7. It implies that free forward communication is not sufficient to extract maximally entangled states at a rate higher than the unassisted capacity of a channel. To prove it, consider the inequality (29) for \( x = 0 \) and \( p \leq y/2 \). Using the properties (9) and (10) we get

\[
Q(0, 0, 0) \geq D(0, y, 0),
\]

which, according to Eq. (2) gives the identity (31).

Equations (10) shows that for \( p \gg \mathcal{E}_Q, \mathcal{E}_C \) and finite values of \( y \) the distillability \( D(x, y, p) \) is bigger than the corresponding capacities \( Q(x, y, p) \) and \( C(x, y, p) \) (these two quantities saturate respectively to \( y/2 + Q_E \) and \( y + C_E \)). However, according to this same equation, \( p \) cannot be considered as a proper resource for distillation protocols. The interesting cases are hence those where the distillability coincides with the quantum capacity of the channel and can be thus strictly lower than the classical capacity.

V. CONCLUSION

In this paper we studied the performance of a quantum channel \( \mathcal{M} \) in the presence of external resources. In particular we focused on the dependence of its capacities with respect to the resource parameters, deriving some inequalities and trade-offs. We have also introduced the concept of distillability of a quantum channel, by maximizing the distillable entanglement one can get at the input and output ports of \( \mathcal{M} \) over all purification protocols that exploit only finite amount of resource per channel use.

APPENDIX A: DISTILLABILITY

A purification protocol \( \mathcal{P} \) acting on \( k \) copies of \( R' \) produces \( m(k) \) copies of a given maximally entangled state \( |\Psi\rangle \) of \( \mathcal{H} \otimes \mathcal{H} \) with fidelity \( F(k) \) which approaches unity in the limit of large \( k \). Here we are interested only on those protocols \( \mathcal{P} \) that operate on \( R' \otimes k \) by employing in total \( knx \) dits of classical feedback, \( kny \) dits of free classical forward communication, and \( knp \) e-dits of shared entanglement. The quantity \( P_{x,y,p}[R'] \) is then defined by the ratio \( m(k)/k \) by optimizing \( m(k) \) over all \( \mathcal{P} \) and by considering the limit \( k \rightarrow \infty \). Thus besides the sup over \( n \) of Eq. (11), the computation of \( D(x, y, p) \) also requires a regularization over the parameter \( k \) defined above.

APPENDIX B: CONCAVITY

In this Appendix we show the joint concavity of \( Q(x, y, p) \), \( C(x, y, p) \) and \( D(x, y, p) \).

For the sake of simplicity write \( Q(x_i) \equiv Q(x, y, p) \) where for \( i = 1, 2, 3, x_1 \equiv x, x_2 \equiv y \) and \( x_3 \equiv p \). Consider the case where the two communicating parties have access to \( N \gg 1 \) uses of the channel \( \mathcal{M} \) and hence to \( X_N(i) \equiv Nx_i \) units of the \( i \)th resource. By definition, the sender cannot transfer more than \( NQ(x_i) \) qubits to the receiver. Suppose now that they divide the set of \( N \) channels into two groups: the group \( A \) with \( N_A \) channels and the group \( B \) with \( N_B \equiv N - N_A \) channels. Moreover, when operating the channels of \( A \), the sender and receiver decide to employ only \( X_A(i) \leq X_N(i) \) units of the \( i \)th resource, corresponding to an average of \( x_A^i \equiv X_A(i)/N_A \) for this set. The remaining \( X_B(i) \equiv X_N(i) - X_A(i) \) units of the \( i \)th resource are instead used when operating the channels of \( B \) (which gives an average of \( x_B^i \equiv X_B(i)/N_B \) units per channel use for this set). In the limit of very large \( N \) the quantum capacity associated with the channels of the set \( A \) is thus given by \( Q(x_A^i) \): the maximum number of qubits that can be transmitted using these channels is thus \( N_AQ(x_A^i) \). Analogously for the set \( B \) we have a maximum number of \( N_BQ(x_B^i) \) qubits transmitted. The sum of these two quantities cannot exceed the optimal value \( NQ(x_i) \) and we obtain the inequality

\[
Q(x_i) \geq \frac{N_A}{N}Q(x_A^i) + \frac{N_B}{N}Q(x_B^i),
\]

which, since \( x_i = x_A^iN_A/N + x_B^iN_B/N \), proves the joint concavity of \( Q(x_i) \). The same procedure can be applied in the case of \( C(x, y, p) \) and \( D(x, y, p) \). Notice that joint concavity with respect to \( x, y \) and \( p \) implies concavity in each of these variables (see for instance [1]).

APPENDIX C: NO-FEEDBACK CASE AND BOWEN CONJECTURE

In this Appendix we discuss a conjecture proposed in [17] showing that in the no-feedback regime \((x = 0)\) it allows one to solve exactly the value of \( Q(0, y, p) \).

The Bowen conjecture implies that for any channel \( \mathcal{M} \) the value of \( \mathcal{E}_Q \) of Eq. (15) is given by

\[
\mathcal{E}_Q = Q_E - Q,
\]

where \( Q_E \) and \( Q \) are, respectively, the entanglement assisted capacity \( Q(0, 0, \infty) \) and the unassisted capacity \( Q(0, 0, 0) \) of the channel. On the one hand the validity of this conjecture was challenged recently by the results of [11] which seem to indicate that this relation does not hold for generic \( \mathcal{M} \). On the other hand we know that there are examples of channels (e.g., dephasing and erasure channels [17, 21]) which satisfy Eq. (C1). In any case, whether or not the Bowen conjecture is a true statement for all CPT map \( \mathcal{M} \), it is worth studying its consequences on the quantum capacity.

If Eq. (C1) is true we can use the inequalities derived in Sec. (11) to verify that the following identity applies

\[
Q(0, 0, p) = p + Q(0, 0, 0) \quad \text{for} \quad p \leq \mathcal{E}_Q,
\]

(this follows for instance by noticing that, for \( x = 0 \), Eq. (C1) implies that the gray region in the plot \( b \) of
Fig. 4: Plot of $Q(0, y, p)$ under the conjecture of Eq. (C4). In this case the capacity is determined by the concavity and the upper bound (25); its value is given by Eq. (C4). Compare this plot with plot b) of Fig. 3 where the conjecture (C1) was not taken into account.

For all $y > 0$, Eq. (35) shows that Eq. (C4) is verified at least for $p \leq y/2$. On the other hand, for $y = 0$ and $C(0,0,0) \neq 0$, one can show that Eq. (28) implies that Eq. (C4) applies at least for $p$ sufficiently small.

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