Entropy fluctuations as a mixedness quantifier

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Abstract. We propose a mixedness quantifier based on entropy fluctuations. Unlike the entropy or the purity parameter, the proposed parameter does not depend on the dimension of the Hilbert space and may be used also for quantized fields.
1. Introduction

Probably the most common tools to assess the degree of mixedness of a given quantum state are entropy \([1,3]\) and the so-called purity parameter \([4,5]\). They define properly the purity of a given state and, because they depend on the dimensionality of the Hilbert space, the degree of mixedness has to be defined for each problem. To clarify this point, let us assume that we are given a measure of entropy, \(S_0\), for a system ‘living’ in an infinite Hilbert space, a quantized field, for instance. We are not be able to say much about how mixed is the state given by the entropy \(S_0\). This will become clear when we define a new mixedness parameter (MP) and apply it to some field density matrices below. We now briefly introduce both the entropy and, its limiting case, the purity parameter.

1.1. Entropy

The quantum mechanical entropy is defined as \([1]\)

\[
S = \langle \hat{S} \rangle = \langle -\ln \hat{\rho} \rangle = -Tr\{\hat{\rho} \ln \hat{\rho}\},
\]

and is also known as the von Neumann entropy. It delivers information about the purity of a given state \(\rho\). It may be seen as the expectation value of the entropy operator \([6]\)

\[
\hat{S} = -\ln \hat{\rho}.
\]

Depending on the density matrix state, we have that for a pure state, \(S = 0\), while if it is in a mixed state, \(S > 0\). This makes \(S\) a good measure of the deviation from pure states. Because the density matrix of the system, \(\rho(t)\), is governed by a unitary time evolution operator, the entropy of a closed system is time independent.

But we usually do not have closed systems, as systems may interact with other systems and/or with an environment, making the entropy to evolve during those kind of interactions. If we consider a system composed by two sub-systems, although the entropy of the whole system does not change in time, we can ask ourselves about the entropy of each subsystem. If we call one sub-system \(A\) and the other \(B\), then the trace of the total density matrix on the \(A\) subsystem basis gives us the density matrix for the \(B\) subsystem

\[
\hat{\rho}_B = Tr_A\{\hat{\rho}\},
\]

and viceversa

\[
\hat{\rho}_A = Tr_B\{\hat{\rho}\}.
\]

The entropies for \(A\) and \(B\) may be defined as

\[
S(\hat{\rho}_{A,B}) = -Tr_{A,B}\{\hat{\rho}_{A,B} \ln \hat{\rho}_{A,B}\}.
\]

The effect of tracing over one of the subsystems variables means that each subsystem is no longer governed by a unitary time evolution, which produces that the entropy of each subsystem becomes time dependent and it may evolve now from pure states to mixed states (or viceversa).
Araki and Lieb \[7\] stated the following inequalities for two interacting subsystems
\[ |S(\hat{\rho}_A) - S(\hat{\rho}_B)| \leq S \leq S(\hat{\rho}_A) + S(\hat{\rho}_B). \] (5)
Therefore, if the two subsystems are initially in a pure state, the whole entropy is zero \((S = 0)\), such that both subsystems will have the same entropy, \(S(\hat{\rho}_A) = S(\hat{\rho}_B)\), in such a way that if the Hilbert space of one subsystems is smaller than the other, it will dictate the maximum entropy of the large one.

1.2. Purity

Another common tool to study the purity of a state is by means of the so-called purity parameter, \(\xi = \langle \hat{\xi} \rangle\),
\[ \xi = \langle (1 - \hat{\rho}) \rangle = 1 - Tr\{\hat{\rho}^2\}. \] (6)
By using the eigenbasis of the density matrix it can be shown that
\[ Tr\{\hat{\rho}^2\} = \sum_n \rho_n^2 \leq \sum_n \rho_n = 1. \] (7)
Because the equality holds only for pure states, the purity parameter, \(\xi\) discriminates uniquely between mixed and pure states. By using the fact that \(1 - \rho_n \leq -\ln \rho_n\) for \(0 < \rho_n \leq 1\) a lower bound for the entropy is found
\[ \xi \leq S. \] (8)

2. Mixedness parameter

We now introduce a MP based on entropy fluctuations
\[ (\Delta S)^2 = \langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2, \] (9)
as follows
\[ Q_S = \exp \left[ -\frac{(\Delta S)^2}{S} \right]. \] (10)
The parameter \(Q_S\) is bounded from zero for pure states to one for completely mixed states, unlike the entropy for which its maximum depends on the Hilbert space dimension and for an infinite Hilbert space, it is not clear what the maximum should be. The MP is independent of the dimension of the Hilbert space.

3. Degree of mixedness for several states

3.1. Two-level system

Consider the state of a two-level system given by the density matrix
\[ \hat{\rho} = \cos^2 \phi \langle e \rangle \langle e \rangle + \sin^2 \phi \langle g \rangle \langle g \rangle, \] (11)
it is direct to show that the entropy is
\[ S = -\cos^2 \phi \ln (\cos^2 \phi) - \sin^2 \phi \ln (\sin^2 \phi), \] (12)
while entropy fluctuations may be easily calculated as
\[ \Delta S = \frac{1}{2} \left| \sin 2\phi \ln(\cot^2 \phi) \right| . \]  \hspace{1cm} (13)

In Figure 1 we plot the normalized entropy, \( S/\ln 2 \), and the quantifier we are introducing to measure mixedness, the MP. It may be observed a similar behaviour.

![Figure 1](image-url)

**Figure 1.** We plot the normalized entropy, \( S/\ln 2 \), and the MP \( e^{-(\Delta S)^2/S} \) for the two-states density matrix

\[ \hat{\rho} = \cos^2 \phi \vert e \rangle \langle e \vert + \sin^2 \phi \vert g \rangle \langle g \vert \]  \hspace{1cm} (14)

as a function of \( \phi \).

### 3.2. Infinite systems

In order to show that this parameter may be applied to infinite systems, we apply it to three cases of a quantized field: statistical mixtures of two and three coherent states and to a thermal field.

**I)** An statistical mixture of two coherent states is written as
\[ \hat{\rho} = \frac{1}{2} \vert \alpha \rangle \langle \alpha \vert + \frac{1}{2} \vert -\alpha \rangle \langle -\alpha \vert , \]  \hspace{1cm} (14)

where \( \vert \alpha \rangle \) is a coherent state, may be purified to the wave function
\[ \vert \Psi \rangle = \vert 1 \rangle \vert \psi_1 \rangle + \vert 2 \rangle \vert \psi_2 \rangle , \]  \hspace{1cm} (15)

with the unnormalized wavefunctions \( \vert \psi_1 \rangle = \frac{1}{\sqrt{2}} \vert \alpha \rangle \) and \( \vert \psi_2 \rangle = \frac{1}{\sqrt{2}} \vert -\alpha \rangle \). The above expression allows to find the entropy of the state in a simply way
\[ S = -\lambda_1 \ln \lambda_1 - \lambda_2 \ln \lambda_2 , \]  \hspace{1cm} (16)

as well as its entropy fluctuations
\[ \Delta S = \sqrt{\lambda_1 \lambda_2} \left| \ln \left( \frac{\lambda_1}{\lambda_2} \right) \right| , \]  \hspace{1cm} (17)
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where $\lambda_1$ and $\lambda_2$ are the eigenvalues of the matrix

$$
\begin{pmatrix}
\langle \psi_1 | \psi_1 \rangle & \langle \psi_1 | \psi_2 \rangle^* \\
\langle \psi_1 | \psi_2 \rangle & \langle \psi_2 | \psi_2 \rangle
\end{pmatrix}.
$$

(18)

Equations (16) and (17) may be written explicitly as

$$
S = -\left(\frac{1 + e^{-2|\alpha|^2}}{2}\right) \ln \left(\frac{1 + e^{-2|\alpha|^2}}{2}\right) - \left(\frac{1 - e^{-2|\alpha|^2}}{2}\right) \ln \left(\frac{1 - e^{-2|\alpha|^2}}{2}\right),
$$

(19)

and

$$
\Delta S = \frac{\sqrt{1 - e^{-4|\alpha|^2}}}{2} \ln \left(\frac{1 + e^{-2|\alpha|^2}}{1 - e^{-2|\alpha|^2}}\right).
$$

(20)

In Figure 2 we plot the normalized entropy and the mixedness parameter based on entropy fluctuations. Again they show a similar behaviour. However, note that in order to normalize the entropy in this (infinite) case, we had to have knowledge that the field was an statistical mixture of two coherent states.

II) If the statistical mixture (14) is substituted by a mixture of three coherent states

$$
\hat{\rho} = \frac{1}{3} |\alpha\rangle \langle \alpha | + \frac{1}{3} |\alpha\rangle \langle -\alpha | + \frac{1}{3} |2\alpha\rangle \langle 2\alpha |,
$$

the entropy and its fluctuations may be written, respectively as

$$
S = -\lambda_1 \ln \lambda_1 - \lambda_2 \ln \lambda_2 - \lambda_3 \ln \lambda_3 ,
$$

$$
\Delta S = \sqrt{\lambda_1 (\ln \lambda_1)^2 + \lambda_2 (\ln \lambda_2)^2 + \lambda_3 (\ln \lambda_3)^2 - S^2},
$$

(22)

Figure 2. We plot the normalized entropy, $S/\ln 2 - S$, and $e^{-(\Delta S)^2/S}$ for the field density operator $\hat{\rho} = (|\alpha\rangle \langle \alpha | + |\alpha\rangle \langle -\alpha | + |\alpha\rangle \langle 2\alpha |)/2$ as a function of $|\alpha|$. 

![Figure 2](image-url)
where $\lambda_1$, $\lambda_2$ and $\lambda_3$ are the eigenvalues of the matrix

$$
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} e^{-2|\alpha|^2} & \frac{1}{3} e^{-|\alpha|^2/2} \\
\frac{1}{3} e^{-2|\alpha|^2} & \frac{1}{3} & \frac{1}{3} e^{-3|\alpha|^2} \\
\frac{1}{3} e^{-|\alpha|^2/2} & \frac{1}{3} e^{-3|\alpha|^2} & \frac{1}{3}
\end{pmatrix}.
$$

(23)

In Figure 3 we plot the normalized entropy, $S/\ln 3$ and the MP $e^{-(\Delta S)^2/S}$ for the statistical mixture $\hat{\rho} = (|\alpha\rangle \langle \alpha| + |\!\!-\alpha\!\rangle \langle \!\!-\alpha\!| + |2\alpha\rangle \langle 2\alpha|)/3$ as a function of $\alpha$.

In this case it is not clear how the entropy may be normalized. In Figure 4 we plot both quantities as a function of the average number of photons. It may be clearly seen how the MP is bounded while the entropy is not.
Figure 4. We plot the entropy, $S$, and the MP $e^{-\langle \Delta S \rangle^2 / S}$ for the thermal distribution as a function of $\bar{n}$.

4. MP in the Jaynes-Cummings model

The interaction Hamiltonian for the atom-field interaction in the rotating wave approximation, i.e., for the Jaynes–Cummings model is given by

$$\hat{H}_I = \lambda (\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+) ,$$

(26)

where we have considered the two-level atom transition frequency equal to the quantized field frequency, i.e. the resonant interaction. Here $\hat{a}$ and $a^\dagger$ are the field annihilation and creation operators, respectively, and $\sigma_+ = |e\rangle \langle g|$ and $\sigma_- = |g\rangle \langle e|$ are the rising and lowering operators for the atom. The kets $|e\rangle$ and $|g\rangle$ represent the excited and ground states of the atom, respectively. The parameter $\lambda$ is the coupling strength, also called the Rabi frequency. It is direct to obtain the evolution operator for the Hamiltonian above [12], which may be written as

$$\hat{U}_I = \begin{pmatrix} \cos \left(\lambda t \sqrt{n} \hat{a}^\dagger \hat{a} \right) & -i \hat{V} \sin \left(\lambda t \sqrt{n} \hat{a}^\dagger \hat{a} \right) \\ -i \hat{V}^\dagger \sin \left(\lambda t \sqrt{n} \hat{a}^\dagger \hat{a} \right) & \cos \left(\lambda t \sqrt{n} \hat{a}^\dagger \hat{a} \right) \end{pmatrix} ,$$

(27)

where $\hat{V} = \frac{1}{\sqrt{\hat{a}^\dagger \hat{a}}} \hat{a}$ and $\hat{V}^\dagger = \hat{a}^\dagger \frac{1}{\sqrt{\hat{a}^\dagger \hat{a}}}$ are the London operators also known as Susskind-Glogower operators. The evolved wavefunction for an initial state given by the initial quantized field in coherent state [8], $|\alpha\rangle$, and the atom in its excited state is given by

$$|\psi\rangle = |\psi_1\rangle |e\rangle + |\psi_2\rangle |g\rangle ,$$

(28)

with the unnormalized wavefunctions

$$|\psi_1\rangle = \cos \left(\lambda t \sqrt{n + 1} \right) |\alpha\rangle , \quad |\psi_2\rangle = -i \hat{V} \sin \left(\lambda t \sqrt{n} \hat{a}^\dagger \right) |\alpha\rangle .$$

(29)
From these equations we may find the atomic and field density matrices as

\[ \hat{\rho}_A = \begin{pmatrix} \langle \psi_1 | \psi_1 \rangle & \langle \psi_1 | \psi_2 \rangle^* \\ \langle \psi_1 | \psi_2 \rangle & \langle \psi_2 | \psi_2 \rangle \end{pmatrix}, \]

(30)

and

\[ \hat{\rho}_F = |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|. \]

(31)

Equation (1) tells us that the entropy for the field and the atom are equal because the total entropy for the state (28) is zero. Then we find that the atomic entropy and its fluctuations are given by

\[ S_A = -\Lambda_1 \ln \Lambda_1 - \Lambda_2 \ln \Lambda_2 = S_F, \]

\[ \Delta S_A = \sqrt{\Lambda_1 \ln^2 \Lambda_1 + \Lambda_1 \ln^2 \Lambda_2 - S_A^2} = \Delta S_F, \]

(32)

where \( \Lambda_1 \) and \( \Lambda_2 \) are the eigenvalues of the atomic density matrix (30).

In Figure 5 we plot the normalized entropy and the MP for the atom. Again, as in former cases, both follow similar behaviour, showing maximums at exactly the same interaction times. We should stress again that we needed a priori information in order to know how to normalize the entropy.

5. Conclusion

We have introduced a new quantifier for mixedness of a given state based on entropy fluctuations. This parameter has the property that it is bounded from zero, for pure states, to one for completely mixed states. As we have shown, for quantized fields,
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entropy requires to have a *priori* knowledge about the state to properly normalize the entropy while for the MP it is not necessary such knowledge. We could not properly normalize the entropy in the example given for the thermal distribution and therefore could not bound it to one.

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