Singularity of \(\{-1\}^m\)-matrices and asymptotics of the number of threshold functions

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Abstract

Two results concerning the number of threshold functions \(P(2, n)\) and the probability \(\mathbb{P}_n\) that a random \(n \times n\) Bernoulli matrix is singular are established. We introduce a supermodular function \(\eta_\star : 2^{\mathbb{R}P^n}_{\text{fin}} \to \mathbb{Z}_{\geq 0}\), defined on finite subsets of \(\mathbb{R}P^n\), that allows us to obtain a lower bound for \(P(2, n)\) in terms of \(\mathbb{P}_{n+1}\). This, together with L. Schlafli’s famous upper bound, give us asymptotics

\[ P(2, n) \sim 2 \left( \frac{2^n - 1}{n} \right), \quad n \to \infty. \]

Also, the validity of the long-standing conjecture concerning \(\mathbb{P}_n\) is proved:

\[ \mathbb{P}_n \sim (n - 1)^2 2^{1-n}, \quad n \to \infty. \]

Keywords. Threshold function, Bernoulli matrices, Möbius function, supermodular function, combinatorial flag.

1 Introduction.

Definition 1 A function \(f : \{-1\}^n \to \{-1\}\) is called a threshold function, if there exist real numbers \(\alpha_0, \alpha_1, \ldots, \alpha_n\), such that

\[ f(x_1, \ldots, x_n) = 1 \iff \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_0 \geq 0. \]

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Denote by $P(2, n)$ the number of threshold functions.

Let us note that

$$f(x_1, \ldots, x_n) = \text{sign}(\bar{\alpha}, (1, \bar{x})),$$

where $(1, \bar{x}) = (1, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $\bar{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$. This observation allows us to correspond a threshold function its $(n+1)$-weight vector $\bar{\alpha}$ as a point in the dual space $(\mathbb{R}^{n+1})^* = \mathbb{R}^{n+1}$.

Let $A^\perp$ be a finite arrangement of hyperplanes all passing through the zero in $\mathbb{R}^{n+1}$ (central arrangement) and denote by $A = \{w_1, \ldots, w_T\}$ the set of their normal vectors. For any $w \in \mathbb{R}^{n+1} \setminus \mathbf{0}$, we consider the linear space $\langle w \rangle$, generated by $w$, as a point of the projective space $\mathbb{RP}^n$. By definition, two hyperplane arrangements $A_1 = \{w_1^1, \ldots, w_T^1\}$ and $A_2 = \{w_1^2, \ldots, w_T^2\}$ are equal, $A_1^\perp \equiv A_2^\perp$, iff subsets $\langle A_1 \rangle \overset{\text{def}}{=} \{\langle w_1^1 \rangle, \ldots, \langle w_T^1 \rangle\} \subset \mathbb{RP}^n$ and $\langle A_2 \rangle \overset{\text{def}}{=} \{\langle w_1^2 \rangle, \ldots, \langle w_T^2 \rangle\} \subset \mathbb{RP}^n$ coinside, $\langle A_1 \rangle = \langle A_2 \rangle$.

It is shown in the paper [23], that $P(2, n)$ can be expressed by the number $C(\langle E_n \rangle)$ of disjoint chambers, obtained as compliment in $\mathbb{R}^{n+1}$ to the arrangement of $2^n$ hyperplanes all passing through the origin with the normal vectors from the set

$$E_n = \{(1, b_1, \ldots, b_n) \mid b_i \in \{\pm 1\}, i = 1, \ldots, n\}. \quad (1)$$

The upper bound of the number $C(\langle H \rangle)$ for any central arrangement of hyperplanes with a set $H$ of normal vectors was established by L. Schlaffi in [18]. For the case $H = E_n$, we have the following upper bound:

$$P(2, n) = C(\langle E_n \rangle) \leq 2 \sum_{i=0}^{n} \binom{2^n - 1}{i}. \quad (2)$$

It should be noted, that in the early 60s of the 20th century the upper bound (2) was obtained by several authors [3], [12], [22]. The detailed information of contribution of above mentioned authors can also be found in [4].

One of the first lower bound of $P(2, n)$ was established by S. Muroga in [16]:

$$P(2, n) \geq 2^{0.33048 n^2}. \quad (3)$$

S. Yajima and T. Ibaraki in [24] improved the order of the logarithm of the lower bound (3) upto $n^2/2$:
\[ P(2, n) \geq 2^{n(n-1)/2 + 8} \text{ for } n \geq 6. \] (4)

Further significant improvements of the bound (4) were obtained basing on the paper [17] of A. M. Odlyzko. In the paper [26], it was noted that from the papers [17], [25] follows:

\[ C(E) = P(2, n) \geq 2^{n^2 - 10n^2/\ln n + O(n \ln n)}. \] (5)

Taking into account the upper bound (2) and inequality (5), it is easy to see that

\[ \lim_{n \to \infty} \frac{\log_2 P(2, n)}{n^2} = 1. \] (6)

In the paper [9], it was suggested an original geometric construction that, in combination with the result from the paper [17], improved the inequality (5) to:

\[ P(2, n) \geq 2^{n^2(1 - 7 \log n)} \cdot P \left( 2, \left[ \frac{7(n - 1)}{\log_2(n - 1)} \right] \right). \] (7)

The generalization of the inequality (7) for the number of threshold \( k \)-logic functions was obtained in [11]. Asymptotics of logarithm of the number of polynomial threshold functions has been recently obtained in [1].

In parallel to finding the asymptotics of the number of threshold functions, studies were conducted to find the asymptotics of the number of singular \( \{\pm 1\} \) (or \( \{0, 1\} \)) \( n \times n \)-matrices.

Let \( M_n = (a_{ij}) \) be a random \( n \times n \) \( \{\pm 1\} \)-matrix, whose entries are independent identically distributed (i.i.d.) Bernoulli random variables:

\[ \Pr(a_{ij} = 1) = \Pr(a_{ij} = -1) = \frac{1}{2}. \]

Many researchers have devoted considerable attention to the old problem of finding the probability

\[ \mathbb{P}_n \overset{\text{def}}{=} \Pr(\det M_n = 0) \]

that a random Bernoulli \( n \times n \) \( \{\pm 1\} \)-matrix \( M_n \) is singular.
In 1963, J. Komlós [14] proved P. Erdös’ conjecture that the probability
that a random Bernoulli $n \times n \{0,1\}$-matrix is singular approaches 0 as $n$
tends to infinity. It is also true for random Bernoulli $\{\pm 1\}$-matrices:

$$P_n = o_n(1).$$  \hfill (8)

In 1977, J. Komlos [15] improved his result by proving that

$$P_n < O\left(\frac{1}{\sqrt{n}}\right).$$  \hfill (9)

The proof of (9) is based on the lemma usually referred to as the Littlewood-
Offord lemma, which was proved by P. Erdős (6).

In 1995, J. Kahn, J. Komlós, and E. Szemerédi established in [13] for
the first time an exponential decay of the upper bound of the singularity
probability of random Bernoulli matrices:

$$P_n \leq (1 - \varepsilon + o_n(1))^n, \quad \text{where } \varepsilon = 0.001.$$  \hfill (10)

In [19], T. Tao and V. Vu improved the result (10) for $\varepsilon = 0.06191$, and
then in [20], they sharpened their technique to prove (10) for $\varepsilon = 0.25$:

$$P_n \leq \left(\frac{3}{4} + o_n(1)\right)^n.$$  \hfill (11)

In 2009, Tao-Vu’s result (11) was further improved by J. Bourgain,
V. H. Vu, and P. M. Wood (see [2]). They proved that

$$P_n \leq \left(\frac{\sqrt{2}}{2} + o_n(1)\right)^n.$$  \hfill (12)

In 2018, K. Tikhomirov finally obtained in [21] that

$$P_n = \left(\frac{1}{2} + o_n(1)\right)^n.$$  \hfill (13)

In this paper, we prove the validity of the long standing conjecture (see
[15], [17], [13]) that dominant sources of singularity are the cases when a
matrix $M_n$ contains two identical (or opposite) rows or two identical (or
opposite) columns.
Theorem 6 Asymptotics of the probability that a random Bernoulli matrix is singular is \((n-1)^{2^{1-n}}\):
\[
P_n \sim (n-1)^{2^{1-n}}, \quad n \to \infty.
\]
We also obtain a new lower bound for the number of threshold functions
\[
P(2, n) \geq 2 \left[1 - \frac{n^2}{2n} (1 + o_n(1))\right] \left(\frac{2^n - 1}{n}\right). \tag{14}
\]
Combining the lower bound (14) with the upper bound (2), we get

Theorem 7 Asymptotics of the number of threshold functions is equal to \(2^{\left(\frac{2^n - 1}{n}\right)}\):
\[
P(2, n) \sim 2^{\left(\frac{2^n - 1}{n}\right)}, \quad n \to \infty.
\]

2 Function \(\eta^\star\) and its properties.

As we mentioned in the previous section, any central hyperplane arrangement \(H^\perp\) with the set of normal vectors \(H = \{w_1, \ldots, w_T\} \in \mathbb{R}^{n+1}_-\), we can identify with the subset \(\langle H \rangle \overset{\text{def}}{=} \{\langle w_1 \rangle, \ldots, \langle w_T \rangle\} \subset \mathbb{P}^n\) of the \(n\)-dimensional projective space. We define a partially ordered set (poset) \(L^H\) in the following way. By definition, any subspace of \(\mathbb{R}^{n+1}_-\) generated by some (possibly empty) subset of \(H\) is an element of the poset \(L^H\). An element \(s \in L^H\) is less than an element \(t \in L^H\) iff the subspace \(t\) contains the subspace \(s\). For any poset \(P\), we can define a simplicial complex \(\Delta_P\) in the following way. The set of vertices of \(\Delta_P\) coincides with the set of elements \(P\) and a set of vertices of \(P\) defines a simplex of \(\Delta_P\) iff this set forms a chain in \(P\). Let us denote by \(\Delta_{L^H}\) the simplicial complex of the poset \(\langle 0_{L^H}, 1_{L^H} \rangle \overset{\text{def}}{=} \{z \in L^H \mid 0_{L^H} < z < 1_{L^H}\}\), where \(0_{L^H}\) and \(1_{L^H}\) are the elements of the poset \(L^H\) corresponding to the zero subspace of \(\mathbb{R}^{n+1}_-\) and the subspace \(\langle w_1, \ldots, w_T \rangle\), respectively. Without loss of generality, we can assume that
\[
dim \langle H \rangle \overset{\text{def}}{=} \dim \text{span } \langle w_1, \ldots, w_T \rangle = n + 1,
\]
i.e.,

\[ \text{span } \langle w_1, \ldots, w_T \rangle = \mathbb{R}^{n+1}. \]

It has been shown in [25] that the number \( C(\langle H \rangle) \) of \((n+1)\)-dimensional regions into which \( \mathbb{R}^{n+1} \) is divided by hyperplanes from the set \( H^+ \) can be found by the formula:

\[ C(\langle H \rangle) = \sum_{t \in L^u_H} |\mu(0_{L^u}, t)|, \quad (15) \]

where \( \mu(s, t) \) is Möbius function of the poset \( L^H \). Möbius function of partially ordered set in Zaslavsky’s formula (15) for calculation of the number of chambers \( C(\langle H \rangle) \) can be interpreted by tools of algebraic topology in the following way. First, we introduce a simplicial complex \( K^H \). The set of vertices of \( K^H \) coincides with the set \( \langle H \rangle \). A subset \( \{\langle w_{i_1}\rangle, \ldots, \langle w_{i_s}\rangle\} \) of \( \langle H \rangle \) forms a simplex of \( K^H \) iff

\[ \text{span } \langle w_{i_1}, \ldots, w_{i_s} \rangle \neq \text{span } \langle w_1, \ldots, w_T \rangle = \mathbb{R}^{n+1}. \]

Taking into account the results of the papers [7], [8], it is possible to show (see [10]) that the absolute value of the Möbius function \( |\mu(0_{L^u}, u)| \) is equal to the dimension of the reduced homology group of the complex \( K^{H \cap u} \) with coefficients in an arbitrary field \( F \):

\[ |\mu(0_{L^u}, u)| = \text{rank } H_{\dim u - 2}(K^{H \cap u}; F). \quad (16) \]

Here, the set \( \langle H \rangle \cap u \) consists of all elements \( \langle H \rangle \) belonging to the subspace \( u \subset \mathbb{R}^{n+1} \) and is considered as a subset of \( \mathbb{R}^{\dim u} \defeq u \).

It follows from the definition of Möbius function that

\[ \sum_{0_{L^u} \leq u < 1_{L^H}} |\mu(0_{L^u}, u)| \geq - \sum_{0_{L^u} \leq u < 1_{L^H}} \mu(0_{L^u}, u) = |\mu(0_{L^u}, 1_{L^H})|. \]

Hence,

\[ C(\langle H \rangle) = |\mu(0_{L^u}, 1_{L^H})| + \sum_{0_{L^u} \leq u < 1_{L^H}} |\mu(0_{L^u}, u)| \geq 2|\mu(0_{L^u}, 1_{L^H})|. \quad (17) \]

From (16) and (17), we have:

\[ C(\langle H \rangle) \geq 2 \text{ rank } H_{n-1}(K^H; F). \quad (18) \]
As a consequence of (18) for the case $H = E_n$, we have:

$$P(2, n) = C((E_n)) \geq 2 \text{rank } H_{n-1}(K^E_n; F).$$  \tag{19}$$

Let us fix an arbitrary order on the set $\langle H \rangle$:

$$\pi : [T] \to \langle H \rangle \subset \mathbb{R}P^n, \quad |\langle H \rangle| = T, \quad \langle w_i \rangle \overset{\text{def}}{=} \pi(i), \quad 1 \leq i \leq T.$$  \tag{20}$$

Let us denote by $\langle H \rangle^{\times s}$, $s = 1, \ldots, T$, the set of ordered collections $(\langle w_{i_1} \rangle, \ldots, \langle w_{i_s} \rangle)$ of different $s$ elements from $\langle H \rangle$ and let $\langle H \rangle^{\times s}_{\neq 0} \subset \langle H \rangle^{\times s}$ and $\langle H \rangle^{\times s}_0 \subset \langle H \rangle^{\times s}$ be the subsets

$$\langle H \rangle^{\times s}_{\neq 0} \overset{\text{def}}{=} \{ (\langle w_{i_1} \rangle, \ldots, \langle w_{i_s} \rangle) \in \langle H \rangle^{\times s} \mid \dim \text{span } \langle w_{i_1}, \ldots, w_{i_s} \rangle = s \}. \tag{21}$$

$$\langle H \rangle^{\times s}_0 \overset{\text{def}}{=} \{ (\langle w_{i_1} \rangle, \ldots, \langle w_{i_s} \rangle) \in \langle H \rangle^{\times s} \mid \dim \text{span } \langle w_{i_1}, \ldots, w_{i_s} \rangle < s \}. \tag{22}$$

\textbf{Definition 2} We say that an ordered collection of different elements $(\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle) \in \langle H \rangle^{\times n}$ satisfies to $\eta_n^\pi(\langle H \rangle)$ condition iff the following requirements are fulfilled:

1. $2 \leq i_1 < i_2 < \cdots < i_n \leq T$;

2. $\forall l, 1 \leq l \leq n$, the element $\langle w_{i_l} \rangle$ is minimal in order $\pi$ among all points from the set $\langle H \rangle \cap \text{span } (\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle)$.

It follows from the definition [2] that if a collection $(\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle)$ satisfies to $\eta_n^\pi(\langle H \rangle)$ condition, then for all $l = 1, \ldots, n$, we have

$$\dim \text{span } \langle w_{i_1}, \ldots, w_{i_n} \rangle = n - l + 1,$$

i.e.,

$$(\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle) \in \langle H \rangle^{\times n}_{\neq 0},$$

and

$$\text{span } \langle w_1, w_{i_1}, \ldots, w_{i_n} \rangle = \mathbb{R}^{n+1}. \tag{23}$$

Denote by $B^\pi(\langle H \rangle)$ the set

$$B^\pi(\langle H \rangle) \overset{\text{def}}{=} \{ W \in \langle H \rangle^{\times n}_{\neq 0} \mid W \text{satisfies to } \eta_n^\pi(\langle H \rangle) \text{ condition} \}. \tag{24}$$

The theorem 7 of [10] is also true for any finite subset $\langle H \rangle \subset \mathbb{R}P^n$. It asserts that the number of collections $(\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle)$ satisfying to $\eta_n^\pi(\langle H \rangle)$
condition is equal to the rank $H_{n-1}(K;F)$. Hence, the number of collection satisfying to $\eta_n(\langle H \rangle)$ condition doesn’t depend on the order $\pi$ on the set $\langle H \rangle$. Let us denote this number by $\eta_\pi^\star(\langle H \rangle)$. Thus on the set $2^{\text{RP}_n}$ of finite subsets of $\text{RP}_n$, the function $\eta_\pi^\star : 2^{\text{RP}_n} \rightarrow \mathbb{Z}_{\geq 0}$ satisfies to the formula:

$$\eta_\pi^\star(\langle H \rangle) = \text{rank } H_{n-1}(K;F), \quad \langle H \rangle \subset \text{RP}_n.$$ 

(25)

**Proposition 1** $\eta_\pi^\star$ is a supermodular function on $2^{\text{RP}_n}$.

**Proof.** It is necessary to demonstrate that for any finite subset $\langle H \rangle \subset \text{RP}_n$, $|\langle H \rangle| = T$, and any two different elements $\langle u \rangle, \langle v \rangle \in \text{RP}_n \setminus \langle H \rangle$ the following inequality

$$\eta_\pi^\star(\langle H \rangle \cup \{\langle u \rangle\}) - \eta_\pi^\star(\langle H \rangle) \leq \eta_\pi^\star(\langle H \rangle \cup \{\langle u \rangle, \langle v \rangle\}) - \eta_\pi^\star(\langle H \rangle \cup \{\langle v \rangle\})$$

(26)

is fullfilled.

For any order $\pi : [T] \rightarrow \langle H \rangle$, we define orders $\pi^{u,v} : [T + 2] \rightarrow \langle H \rangle \cup \{\langle u \rangle, \langle v \rangle\}$, $\pi^u : [T + 1] \rightarrow \langle H \rangle \cup \{\langle u \rangle\}$, and $\pi^v : [T + 1] \rightarrow \langle H \rangle \cup \{\langle v \rangle\}$ such that

$$\pi^{u,v}(i) = \pi^u(i) = \pi^v(i) = \pi(i), \quad \forall i = 1, \ldots, T;$$

$$\pi^{u,v}(T + 1) = \langle u \rangle, \quad \pi^{u,v}(T + 2) = \langle v \rangle;$$

$$\pi^u(T + 1) = \langle u \rangle, \quad \pi^v(T + 1) = \langle v \rangle.$$ 

(27)

Then the expression in the left part of the inequality (26) equals to the number of collections $((w_1), \ldots, (w_{n-1}), \langle u \rangle)$ satisfying to $\eta_\pi^\star(\langle H \rangle \cup \{\langle u \rangle\})$ condition. Due to (27), these collections also satisfy to $\eta_\pi^{u,v}(\langle H \rangle \cup \{\langle u \rangle, \langle v \rangle\})$ condition. The expression in the right side of the inequality (26) equals to cardinality of the set consisting of collections of the form $((w_1), \ldots, (w_{n-1}), \langle u \rangle)$ and $((w_{i_1}), \ldots, (w_{i_{n-2}}), \langle u \rangle, \langle v \rangle)$ satisfying to $\eta_\pi^{u,v}(\langle H \rangle \cup \{\langle u \rangle, \langle v \rangle\})$ condition. Hence, the inequality (26) is proved.

Q.E.D.

Denote by $P_{R^{n+1}}^{(w)}$ the orthogonal projector along the linear subspace $\langle w \rangle \subset \text{R}^{n+1}$ onto its $n$-dimensional orthogonal compliment $\langle w \rangle^\perp \subset \text{R}^{n+1}$, and denote by $v^{\perp w}$ the image of a vector $v \in \text{R}^{n+1}$:

$$v^{\perp w} \stackrel{\text{def}}{=} P_{R^{n+1}}^{(w)}(v).$$

(28)

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For $\langle H \rangle = \{\langle w_1 \rangle, \ldots, \langle w_T \rangle \} \subset \mathbb{RP}^n$ and $\langle w \rangle \notin \langle H \rangle$, we denote by $\langle H \rangle^\perp w$ the set:

$$\langle H \rangle^\perp w \overset{\text{def}}{=} P_{R^{n+1}}^w(\langle H \rangle) = \{\langle w_1^\perp w \rangle, \ldots, \langle w_T^\perp w \rangle \} \subset \mathbb{RP}^{n-1}.$$  \hspace{1cm} (29)

**Theorem 1** For any finite subset $\langle H \rangle \subset \mathbb{RP}^n$ and element $\langle u \rangle \in \mathbb{RP}^n \setminus \langle H \rangle$, we have:

$$\eta_n^* (\langle H \rangle \cup \{\langle u \rangle \}) = \eta_n^* (\langle H \rangle) + \eta_{n-1}^* (\langle H \rangle^\perp u).$$  \hspace{1cm} (30)

**Proof.** Let $\pi : [T + 1] \to \langle H \rangle \cup \{\langle u \rangle \}$ be an order on $\langle H \rangle \cup \{\langle u \rangle \} \subset \mathbb{RP}^n$ such that $\pi(T + 1) = \langle u \rangle$. For any $\langle w^\perp u \rangle \in \langle H \rangle^\perp u$, let

$$T^\pi(w^\perp u) \overset{\text{def}}{=} \{i \in [T] \mid \langle w_i^\perp u \rangle = \langle w^\perp u \rangle \}$$

$$m(w^\perp u) \overset{\text{def}}{=} \min \{i \in T^\pi(w^\perp u)\}.$$ \hspace{1cm} (31)

For $\langle x^\perp u \rangle, \langle y^\perp u \rangle \in \langle H \rangle^\perp u$, we say that $\langle x^\perp u \rangle <_{\pi^\perp} \langle y^\perp u \rangle \iff m(x^\perp u) < m(y^\perp u).$ \hspace{1cm} (32)

Let $|\langle H \rangle^\perp u| = T'$. Then we define the order $\pi|_{u^\perp} : [T'] \to \langle H \rangle^\perp u$ as the unique map preserving the linear orders:

$$i < j \iff \pi|_{u^\perp}(i) <_{\pi|_{u^\perp}} \pi|_{u^\perp}(j).$$ \hspace{1cm} (33)

In the proof of the Proposition [11] we have shown that the cardinality of the set

$$B_n^\pi(\langle H \rangle \cup \{\langle u \rangle \}) \overset{\text{def}}{=} \left\{ W \in B^\pi(\langle H \rangle \cup \{\langle u \rangle \}) \mid W = (W', \langle u \rangle), \ W' \in \langle H \rangle^\times (n-1)\right\}$$

is equal to the number $\eta_n^* (\langle H \rangle \cup \{\langle u \rangle \}) - \eta_n^* (\langle H \rangle)$:

$$\eta_n^* (\langle H \rangle \cup \{\langle u \rangle \}) - \eta_n^* (\langle H \rangle) = |B_n^\pi(\langle H \rangle \cup \{\langle u \rangle \})|. \hspace{1cm} (34)$$

For any $W = (\langle w_1 \rangle, \ldots, \langle w_{n-1} \rangle, \langle u \rangle) \in B_n^\pi(\langle H \rangle \cup \{\langle u \rangle \})$, we assert that

$$W^\perp u \overset{\text{def}}{=} (\langle w_1^\perp u \rangle, \ldots, \langle w_{n-1}^\perp u \rangle) \in B_{n-1}^\pi(\langle H \rangle^\perp u).$$ \hspace{1cm} (35)
First of all, we note that for \( W \in B_n^\pi(\langle H \rangle \cup \{ \langle u \rangle \}) \), we have
\[
i_l = m(i_l) \overset{\text{def}}{=} m(w_i^\perp u), \quad \forall l = 1, \ldots, n - 1. \tag{36}
\]
Indeed, if the condition (36) is not fulfilled for some \( l, 1 \leq l \leq n - 1 \), then \( i_l > m(i_l) \). Taking into account that
\[
w_i^\perp u = w_i - \beta_i u, \quad \beta_i = \frac{\langle w_i, u \rangle}{\langle u, u \rangle}, \quad i = 1, \ldots, n - 1, \tag{37}
\]
we have
\[
\langle w_{m(i_l)} \rangle \in \text{span} \langle \langle w_i \rangle, \langle u \rangle \rangle \subset \text{span} \langle \langle w_i \rangle, \ldots, \langle w_n - 1 \rangle, \langle u \rangle \rangle. \tag{38}
\]
The inclusion (38) contradicts to our choice \( W \in B_n^\pi(\langle H \rangle \cup \{ \langle u \rangle \}) \).

Now consider the case when the condition (35) is not fulfilled, i.e., there exist \( l, k \), \( 1 \leq l \leq n - 1 \), and \( k < i_l \) such that
\[
\langle w_k^\perp u \rangle \in \text{span} \langle \langle w_i^\perp u \rangle, \ldots, \langle w_{n - 1}^\perp u \rangle \rangle.
\]
Hence, there exist \( \alpha_l, \ldots, \alpha_{n - 1} \in \mathbb{R} \) such that
\[
w_k^\perp u = \alpha_l w_i^\perp u + \cdots + \alpha_{n - 1} w_{i_{n - 1}}^\perp u. \tag{39}
\]
From (37) and (39) we have
\[
w_k = \alpha_l w_i + \cdots + \alpha_{n - 1} w_{i_{n - 1}} - (\alpha_l \beta_i + \cdots + \alpha_{n - 1} \beta_{i_{n - 1}} - \beta_k) u. \tag{40}
\]
The last equation means that \( \langle w_k \rangle \in \text{span} \langle \langle w_i \rangle, \ldots, \langle w_{n - 1} \rangle, \langle u \rangle \rangle \), i.e., contradicts to our requirement \( W \in B_n^\pi(\langle H \rangle \cup \{ \langle u \rangle \}) \).

Thus we can define the map
\[
\psi_u^\pi : B_n^\pi(\langle H \rangle \cup \{ \langle u \rangle \}) \to B_{n - 1}^\pi(\langle H \rangle^\perp u)
\]
by the rule
\[
\psi_u^\pi(W) \overset{\text{def}}{=} W^\perp u. \tag{41}
\]
We assert that \( \psi_u^\pi \) is injective. If we assume the opposite, then there exist \( W_1, W_2 \in B_n^\pi(\langle H \rangle \cup \{ \langle u \rangle \}) \), \( W_1 \neq W_2 \), \( W_1 = (\langle w_i \rangle, \ldots, \langle w_{i_{n - 1}} \rangle, \langle u \rangle) \), \( W_2 = (\langle w_j \rangle, \ldots, \langle w_{j_{n - 1}} \rangle, \langle u \rangle) \) such that
\[
\psi_u^\pi(W_1) = (\langle w_i^\perp u \rangle, \ldots, \langle w_{i_{n - 1}}^\perp u \rangle) = (\langle w_j^\perp u \rangle, \ldots, \langle w_{j_{n - 1}}^\perp u \rangle) = \psi_u^\pi(W_2). \tag{42}
\]
Let \( l, 1 \leq l \leq n - 1 \), be the maximal number such that \( w_i \neq w_{ji} \). Without loss of generality, we can assume that \( i_l < j_l \). It follows from (36) that \( m(i_l) = i_l < j_l = m(j_l) \). Then
\[
\langle w_{i_l}^{\perp u} \rangle = \langle w_{m(i_l)}^{\perp u} \rangle \neq \langle w_{m(j_l)}^{\perp u} \rangle = \langle w_{j_l}^{\perp u} \rangle.
\]
(43)
The inequality (43) contradicts to our assumption (42) that \( \langle w_i^{\perp u} \rangle = \langle w_j^{\perp u} \rangle \), \( l = 1, \ldots, n - 1 \).

We assert that \( \psi_u^\pi \) is surjective. Let \( X_u^{\perp} = (\langle x_1 \rangle, \ldots, \langle x_{n-1} \rangle) \in B_u^{\pi} (\langle H \rangle^{\perp u}) \). Let us put
\[
(\psi_u^\pi)^{-1} (X_u^{\perp}) \overset{\text{def}}{=} (\langle w_{m(x_1)} \rangle, \ldots, \langle w_{m(x_{n-1})} \rangle, \langle u \rangle).
\]
(44)
It is necessary to demonstrate that
\[
(\psi_u^\pi)^{-1} (X_u^{\perp}) \in B_u^{\pi} (\langle H \rangle \cup \{ \langle u \rangle \}).
\]
(45)
If we assume that the inclusion (45) isn’t true, then there exist \( l, k, 1 \leq l \leq n - 1 \), and \( k < m(x_l) \) such that
\[
\langle w_k \rangle \in \text{span} (\langle w_{m(x_1)} \rangle, \ldots, \langle w_{m(x_{n-1})} \rangle, \langle u \rangle).
\]
Hence,
\[
\langle w_k^{\perp u} \rangle \in \text{span} (\langle w_{m(x_1)}^{\perp u} \rangle, \ldots, \langle w_{m(x_{n-1})}^{\perp u} \rangle).
\]
(46)
Since \( k < m(x_l) \), then
\[
m(w_k^{\perp u}) < m(x_l) < \cdots < m(x_{n-1}).
\]
(47)
From inclusion (46) and inequalities (47) we get a contradiction to \( X_u^{\perp} \in B_u^{\pi} (\langle H \rangle^{\perp u}) \).

Thus we have demonstrated that
\[
|B_u^{\pi} (\langle H \rangle \cup \{ \langle u \rangle \})| = |B_u^{\pi} (\langle H \rangle^{\perp u})|.
\]
(48)
Since
\[
\eta_{n-1}^* (\langle H \rangle^{\perp u}) = |B_u^{\pi} (\langle H \rangle^{\perp u})|,
\]
(49)
our Theorem follows from the equalities (34), (48), and (49).

Q.E.D.
For any finite subset $\langle H \rangle \subset \mathbb{RP}^n$ and element $\langle w \rangle \in \mathbb{RP}^n$, we denote by $\left( \begin{array}{c} \langle H \rangle \\ \eta_n^* \end{array} \right)^{(w)}$ the following sum:

$$\left( \begin{array}{c} \langle H \rangle \\ \eta_n^* \end{array} \right)^{(w)} \overset{\text{def}}{=} \sum_{\{\langle w_i \rangle, \ldots, \langle w_n \rangle\} \subset \langle H \rangle} \eta_n^* \{\{\langle w \rangle, \langle w_i \rangle, \ldots, \langle w_n \rangle\}\}. \quad (50)$$

**Theorem 2** For any $n \geq 1$, finite subset $\langle H \rangle \subset \mathbb{RP}^n$, and element $\langle w \rangle \in \mathbb{RP}^n$, we have:

$$\eta_n^*(\langle H \rangle \cup \{\langle w \rangle\}) \leq \left( \begin{array}{c} \langle H \rangle \\ \eta_n^* \end{array} \right)^{(w)}. \quad (51)$$

**Proof.** Let $\pi : [T] \to \langle H \rangle \cup \{\langle w \rangle\}$ be an order on $\langle H \rangle \cup \{\langle w \rangle\} \subset \mathbb{RP}^n$, $T = |\langle H \rangle \cup \{\langle w \rangle\}|$, such that $\pi(1) = \langle w \rangle$. It is easy to see that

$$\eta_n^*(\{\langle w \rangle, \langle w_i \rangle, \ldots, \langle w_n \rangle\}) = 1 \iff \iff \text{span } \langle w, w_i, \ldots, w_n \rangle = \mathbb{R}^{n+1}.$$ 

It follows from definition 2 that if a collection $(\langle w_i \rangle, \ldots, \langle w_n \rangle) \in B^\pi(\langle H \rangle \cup \{\langle w \rangle\})$, then (see (23))

$$\text{span } \langle w_1, w_{i_1}, \ldots, w_{i_n} \rangle = \text{span } \langle w, w_{i_1}, \ldots, w_{i_n} \rangle = \mathbb{R}^{n+1}.$$

Now the Theorem follows from the equality $\eta_n^*(\langle H \rangle \cup \{\langle w \rangle\}) = |B^\pi(\langle H \rangle \cup \{\langle w \rangle\})|$.

Q.E.D.

3 A formula for $\eta_n^*$ in terms of combinatorial flags on a central hyperplane arrangement.

For any $W = (\langle w_1 \rangle, \ldots, \langle w_{i_n} \rangle) \in \langle H \rangle_{\neq 0} \times^n$ and $l = 1, \ldots, n$, let

$$q_l^W \overset{\text{def}}{=} |L_l(W) \cap \langle H \rangle| \overset{\text{def}}{=} |\text{span } \langle \langle w_{i_{n-l+1}} \rangle, \ldots, \langle w_{i_n} \rangle \rangle \cap \langle H \rangle|. \quad (52)$$
Definition 3 For any $W \in \langle H \rangle^{\times n}$, the ordered set of numbers

$$W(\langle H \rangle) \overset{\text{def}}{=} (q_1^W, q_{n-1}^W, \ldots, q_n^W) \quad (53)$$

is called a combinatorial flag on $\langle H \rangle \subset \mathbb{RP}^n$ of the ordered set $W$.

If $W \in \langle H \rangle^{\times n}_{\neq 0}$, then $W(\langle H \rangle)$ is called a full combinatorial flag of $W$.

For the sake of simplicity, we will use the following notation:

$$W[H] \overset{\text{def}}{=} q_n^W \cdot q_{n-1}^W \cdot \cdots \cdot q_1^W. \quad (54)$$

To define the set $B^\pi(\langle H \rangle)$ (see (24)), we fixed an order $\pi : [T] \to \langle H \rangle \subset \mathbb{RP}^n$ (see (20)) that allowed us to compare elements of $\langle H \rangle$:

$$\langle w_i \rangle <_\pi \langle w_j \rangle \iff i < j. \quad (55)$$

Thus $\Gamma$ can be identified with the symmetric group $\text{Sym}([T])$, and any permutation $\sigma : [T] \to [T]$ defines the basis of the homology group $H_{n-1}(K^H; \mathbb{F})$, considered as a vector space over an fixed field $\mathbb{F}$, say $\mathbb{Z}_2$, as the subset of collections of $n$ elements from $\langle H \rangle$

$$B^\pi\sigma(\langle H \rangle) \subset \langle H \rangle^{\times n}_{\neq 0}$$

obeying to $\eta^\pi\sigma(\langle H \rangle)$ condition.

Theorem 3 For any probability distribution $p = (p_1, \ldots, p_T)$ on a subset $\langle H \rangle \subset \mathbb{RP}^n$, span $\langle H \rangle = \mathbb{R}^{n+1}$, the following equality is true:

$$\eta^*_n(\langle H \rangle) = \sum_{W \in \langle H \rangle^{\times n}_{\neq 0}} \frac{1 - p_{i_1} - p_{i_2} - \cdots - p_{i_m}}{W[H]} W[H]. \quad (56)$$

Here, the indices used in the numerator correspond to elements from

$$L_n(W) \cap \langle H \rangle = \left\{ \langle w_{i_1} \rangle, \ldots , \langle w_{i_m} \rangle, \ldots , \langle w_{i_{m'}} \rangle \right\}.$$
Proof. We define the probability distribution \( \tilde{p} \) on the set \( \Gamma \cong Sym([T]) \) by the formula:

\[
\tilde{p}(\gamma) = p_{\gamma(1)} \frac{1}{(T-1)!}, \quad \gamma \in Sym([T]).
\]

For any collection \( W = (\langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle) \in H^x \neq 0 \), we define the random function \( I_W: \Gamma \to \mathbb{R} \) by the formula:

\[
I_W(\gamma) \overset{\text{def}}{=} \begin{cases} 
1, & \text{if } W \text{ satisfies to } \eta^\ast_n(\langle H \rangle) \text{ condition;} \\
0, & \text{in all other cases.}
\end{cases}
\]

Let

\[
I \overset{\text{def}}{=} \sum_{W \in \langle H \rangle^x \neq 0} I_W: \Gamma \to \mathbb{R}.
\]

Then for any \( \gamma \in \Gamma \),

\[
I(\gamma) = \text{const} = |B^\pi(\langle H \rangle)| = \text{rank } H_{n-1} (K^H; F) = \eta^\ast_n(\langle H \rangle).
\]

Hence, the expectation of \( I \) is equal to \( \eta^\ast_n(\langle H \rangle) \):

\[
\mathbb{E}[I] = \eta^\ast_n(\langle H \rangle). \tag{57}
\]

Additivity of expectation reduces the problem of calculation \( \mathbb{E}[I] \) to counting the probability \( \Pr(I_W = 1) \):

\[
\mathbb{E}[I] = \sum_{W \in \langle H \rangle^x \neq 0} \mathbb{E}[I_W] = \sum_{W \in \langle H \rangle^x \neq 0} \Pr(I_W = 1). \tag{58}
\]

Further we calculate the number of permutations \( \gamma \) such that \( I_W(\gamma) = 1 \). Since \( \langle w_{\gamma(1)} \rangle \notin L_n(W) \), then \( q_n^W \) elements from \( L_n(W) \cap \langle H \rangle \) can be located in any places except the first one, i.e., \( \gamma^{-1}(j) \neq 1 \) for any \( j \in [T] \) such that \( \langle w_j \rangle \in L_n(W) \cap \langle H \rangle \). The arrangement of the remaining elements from \( \langle H \rangle \setminus \{\langle w_{\gamma(1)} \rangle \cup \{L_n(W) \cap \langle H \rangle\}\} \) does not affect the fulfillment of \( \eta^\pi_n(\langle H \rangle) \) condition. By \( \eta^\pi_n(\langle H \rangle) \) condition, the element \( \langle w_{i_1} \rangle \) has to be in the first place among the selected \( q_n^W \) positions for arrangement of the set \( L_n(W) \cap \langle H \rangle \), while \( q_n^{W-1} \) elements from \( L_{n-1}(W) \cap \langle H \rangle \) can be located in any of the remained \( q_n^W - 1 \) places. The arrangement of the elements from \( \{L_n(W) \cap \langle H \rangle\} \setminus \{\langle w_{i_1} \rangle \cup \{L_{n-1}(W) \cap \langle H \rangle\}\} \) in \( q_n^W - q_n^{W-1} - 1 \) places, left after choosing
\(q_{n-1}^W + 1\) places for arrangement of the set \(L_{n-1}(W) \cap \langle H \rangle\) and \(\langle w_{i_1} \rangle\), doesn’t affect the fulfillment of \(\eta_n^{\pi \circ \gamma}(\langle H \rangle)\) condition. Continuing the same way, we get that in the first place among \(q_l^W\) positions selected for the elements from \(L_l(W) \cap \langle H \rangle\) has to be located the element \(\langle w_{i_{n-l+1}} \rangle\), while \(q_{l-1}^W\) elements from \(L_{l-1}(W) \cap \langle H \rangle\) can be located in any of the remained \(q_l^W - q_{l-1}^W - 1\) places, and the positions of the elements from \(\{L_l(W) \cap \langle H \rangle\} \setminus \{(w_{i_{n-l+1}}) \cup \{L_{l-1}(W) \cap \langle H \rangle\}\}\) in \(q_l^W - q_{l-1}^W - 1\) places, left after choosing \(q_{l-1}^W + 1\) places for arrangement of the set \(L_{l-1}(W) \cap \langle H \rangle\) and \(w_{i_{n-l+1}}\), doesn’t affect the fulfillment of \(\eta_n^{\pi \circ \gamma}(\langle H \rangle)\) condition.

Denote by \(N(\gamma(1) = i)\) the number of permutations \(\gamma\) with fixed value \(\gamma(1) = i\) such that \(\langle w_{i} \rangle \notin L_n(W)\). Then
\[
N(\gamma(1) = i) = \left(\frac{T-1}{q_i^W}\right) (T - q_i^W)! \cdot \left(\frac{q_{i-1}^W}{q_i^W}\right) (q_i^W - q_{i-1}^W - 1)! \cdot \ldots \cdot \left(\frac{q_{n-2}^W}{q_{n-1}^W}\right) (q_{n-1}^W - q_{n-2}^W - 1)! = \frac{(T-1)!}{q_i^W} \cdot \frac{(q_{i-1}^W)!}{q_i^W} \cdot \ldots \cdot \frac{(q_{n-2}^W)!}{q_i^W} \cdot \frac{(q_{n-1}^W)!}{q_i^W} = \frac{(T-1)!}{q_i^W q_{i-1}^W \cdots q_{n-2}^W q_{n-1}^W} = \frac{(T-1)!}{W[H]} \quad \text{(since } q_1^W = 1).\]

Then we have
\[
\Pr(I_W = 1) = \sum_{i \in [T] \text{ s.t. } \langle w_i \rangle \notin L_n(W)} p_i \frac{1}{(T - 1)!} \frac{(T - 1)!}{W[H]} = \frac{1 - p_1 - \ldots - p_{q_W^W}}{W[H]},
\]
where \(L_n(W) \cap \langle H \rangle = \{(w_{i_1}) \ldots, \langle w_{i_n} \rangle \ldots, \langle w_{q_W^W} \rangle\}\). Now the Theorem follows from (57) and (58).

**Q.E.D.**

**Remark 1** Since the right side of equation of Theorem 3 is expressed by a polynomial of degree 1, then the Theorem 3 is true for any \(p_i \in \mathbb{R}, i = 1, \ldots, T\), such that \(\sum_{i=1}^T p_i = 1\).
4 A lower bound for \( \eta^\star \).

Next, we are going to get an upper bound for the rank \( H_n \left( K^H, K^H_{n-1}; F \right) \),
where \( \langle H \rangle = \{ \langle w_1 \rangle, \ldots, \langle w_T \rangle \} \subset \mathbb{R}P^n \), and \( K^H_{n-1} \) is the \((n-1)\)-skeleton of \( K^H \). The nonzero part of the homology exact sequence of the pair \( (K^H, K^H_{n-1}) \)
has the following form (see (11) of [10]):

\[
0 \to H_n \left( K^H, K^H_{n-1}; F \right) \to H_{n-1} \left( K^H_{n-1}; F \right) \to H_{n-1} \left( K^H; F \right) \to 0.
\] (59)

For any \( \Delta = (\langle w_{i_1} \rangle, \ldots, \langle w_{i_{n+1}} \rangle) \in \langle H \rangle_{0}^{(n+1)} \), \( i_1 < \ldots < i_{n+1} \), let put

\[
n(\Delta) \overset{\text{def}}{=} \{ t \in [n] \mid w_{i_t} \in \text{span} \langle w_{i_{t+1}}, \ldots, w_{i_{n+1}} \rangle \},
\] (60)

\[
\Delta(H) \overset{\text{def}}{=} \{ \langle w_p \rangle \in \langle H \rangle \mid \exists t \in [n] \text{ s.t. } p < i_{t+1} \text{, and } w_p \in \text{span} \langle w_{i_{t+1}}, \ldots, w_{i_{n+1}} \rangle \},
\] (61)

\[
t(\Delta) \overset{\text{def}}{=} \max_{t \in n(\Delta)} t,
\] (62)

\[
\langle w(\Delta) \rangle \overset{\text{def}}{=} \max_{\langle w \rangle \in \Delta(H)} \langle w \rangle.
\] (63)

Let

\[
C^\pi_n (H) \overset{\text{def}}{=} \{ \Delta = (\langle w_{i_1} \rangle, \ldots, \langle w_{i_{n+1}} \rangle) \in \langle H \rangle_{0}^{(n+1)} \mid 1 < i_1 < \ldots < i_{n+1}, \langle w_{i_{t(\Delta)}} \rangle = \langle w(\Delta) \rangle \},
\] (64)

It was shown in the paper [10] (see Lemma 5 and Lemma 6) that the set of homology classes of \( H_n \left( K^H, K^H_{n-1}; F \right) \) corresponded to the set of simplices \( C^\pi_n (H) \) forms a basis of \( H_n \left( K^H, K^H_{n-1}; F \right) \). Hence, the cardinality \( |C^\pi_n (H)| \) doesn’t depend on the choice of order \( \pi \) on \( \langle H \rangle \).

Let us fix any element \( \langle w \rangle \in \langle H \rangle \). Since \( |C^\pi_n (H)| \) doesn’t depend on the choice of \( \pi \), we can assume that \( \langle w \rangle \) is the minimal element in \( \pi \), i.e., \( \pi(1) = \langle w \rangle \). Let put

\[
D^\pi_n (H; w) \overset{\text{def}}{=} \{ \Delta = (\langle w \rangle, \langle w_{i_1} \rangle, \ldots, \langle w_{i_n} \rangle) \in \langle H \rangle_{0}^{(n+1)} \mid 1 < i_1 < \ldots < i_n \}.
\] (65)
It is clear that
\[ D_n(H; w) \overset{\text{def}}{=} |D_n^\pi(H; w)| \] (66)
doesn’t depend on the choice of \( \pi \).

Remark 2  From our definition, we have
\[ D_n(H; w) = \left( T - 1 \right) \left( \frac{\langle H \rangle}{\eta} \right)^{\langle w \rangle}. \] (67)

The map \( \hat{t} : C_n^\pi(H) \to \langle H \rangle^n \) defined on \( \Delta = (\langle w_1 \rangle, \ldots, \langle w_{i+1} \rangle) \in C_n^\pi(H) \) by the formula
\[ \hat{t}(\Delta) = (\langle w_1 \rangle, \ldots, \langle w_{i_{\Delta}-1} \rangle, \langle w_{i_{\Delta}} \rangle, \langle w_{i_{\Delta}+1} \rangle, \ldots, \langle w_{i+1} \rangle), \]
is a monomorphism. Note that for all \( \Delta \) from the set
\[ C_n^{\pi; \neq 0}(H; w) \overset{\text{def}}{=} \{ \Delta = (\langle w_1 \rangle, \ldots, \langle w_{i_{+1}} \rangle) \in C_n^\pi(H) | \text{span} \langle w_1, \ldots, w_{i_{+1}} \rangle = n, \text{and} \langle w \rangle \notin \text{span} \langle w_1, \ldots, w_{i_{+1}} \rangle \}, \] (68)
we have
\[ \hat{t}(\Delta) \in \langle H \rangle^n_{\neq 0}. \]

Lemma 1  The set of homology classes of \( H_n(K^H, K^{H_{n-1}}; \mathbb{Z}_2) \) corresponded to the set of simplices \( C_n^{\pi; \neq 0}(H; w) \cup D_n^\pi(H; w) \) generates \( H_n(K^H, K^{H_{n-1}}; \mathbb{Z}_2) \).

Proof.  Since the set of homology classes corresponded to the set of simplices \( C_n^\pi(H) \) forms a basis of \( H_n(K^H, K^{H_{n-1}}; \mathbb{Z}_2) \), it is sufficient to show that any simplex \( \Delta = (\langle w_1 \rangle, \ldots, \langle w_{i_{+1}} \rangle) \in C_n^\pi(H) \), such that
\[ \dim \text{span} \langle w_1, \ldots, w_{i_{+1}} \rangle < n, \text{and} \langle w \rangle \notin \{ \langle w_1 \rangle, \ldots, \langle w_{i_{+1}} \rangle \}; \]
or
\[ \dim \text{span} \langle w_1, \ldots, w_{i_{+1}} \rangle = n, \text{and} \langle w \rangle \in \text{span} \langle w_1, \ldots, w_{i_{+1}} \rangle, \]
but \( \langle w \rangle \notin \{ \langle w_1 \rangle, \ldots, \langle w_{i_{+1}} \rangle \}, \]
as a chain belongs to \( \text{span}(C_n^{\pi; \neq 0}(H; w), D_n^\pi(H; w), \delta_{n+1}(C_{n+1}(K^H; \mathbb{Z}_2))) \). In both cases, we have
\[ \Delta_{n+1} \overset{\text{def}}{=} (\langle w \rangle, \langle w_1 \rangle, \ldots, \langle w_{i_{+1}} \rangle) \in C_{n+1}(K^H; \mathbb{Z}_2), \]

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and

$$\delta_{n+1}(\Delta_{n+1}) = \Delta + \sum_{k=1}^{n+1} (\langle w \rangle, \langle w_{i_1} \rangle, \ldots, \langle w_{i_k} \rangle, \ldots, \langle w_{i_{n+1}} \rangle).$$

Hence, $$\Delta \in \text{span} \langle D^n_\pi(H; w), \delta_{n+1}(C_{n+1}(K^H; \mathbb{Z}_2)) \rangle.$$ Q.E.D.

**Corollary 1** We have

$$|C^n_\pi(H)| \leq |C^n_{\pi; \neq 0}(H; w)| + D_n(H; w). \quad (69)$$

We fix any $$\langle w \rangle \in \langle H \rangle,$$ and for any $$\langle u \rangle \in \langle H \rangle,$$ and order $$\pi : [T] \to \langle H \rangle,$$ such that $$\pi(1) = \langle w \rangle,$$ we put

$$\langle H^\pi_{<u} \rangle \overset{\text{def}}{=} \{ \langle h \rangle \in \langle H \rangle \mid \langle h \rangle <_\pi \langle u \rangle \}.$$

For any subset $$\langle U \rangle = \{ \langle u_{i_{n-k}} \rangle, \ldots, \langle u_{i_1} \rangle \} \subset \langle H \rangle \setminus \langle w \rangle, 0 \leq k \leq n,$$ such that

$$\langle u_{i_{n-k}} \rangle <_\pi \cdots <_\pi \langle u_{i_1} \rangle,$$

$$\dim \text{span} \langle U \rangle \overset{\text{def}}{=} \dim \text{span} \langle u_{i_{n-k}}, \ldots, u_{i_1} \rangle = n - k,$$

and $$\langle w \rangle \notin \text{span} \langle U \rangle,$$

we define the set $$C_{n-k}^\pi(H^\pi_{U}; U) \subset \widehat{t}(C_{n; \neq 0}^\pi(H; w)),$$ where $$\langle H^\pi_{U} \rangle \overset{\text{def}}{=} \langle H^\pi_{<u_{i_{n-k}}} \rangle,$$ in the following way

$$C_{n-k}^\pi(H^\pi_{U}; U) \overset{\text{def}}{=} \{ \gamma \in \langle H \rangle^{\times n} \mid \exists \Delta = (\langle w_{j_1} \rangle, \ldots, \langle w_{j_{k+1}} \rangle, \langle u_{i_{n-k}} \rangle, \ldots, \langle u_{i_1} \rangle) \in C_{n; \neq 0}^\pi(H; w) \text{ s.t. } \widehat{t}(\Delta) = \gamma \}. \quad (71)$$

**Remark 3** Note that from the conditions $$\Delta \in C_{n; \neq 0}^\pi(H; w)$$ and $$\dim \text{span} \langle U \rangle = n - k,$$ follows that $$\langle w(\Delta) \rangle <_\pi \langle u_{i_{n-k}} \rangle <_\pi \cdots <_\pi \langle u_{i_1} \rangle.$$

**Definition 4** Let a subset $$\langle U \rangle \subset \langle H \rangle \setminus \langle w \rangle$$ satisfy the conditions (70) and

$$\langle U^s \rangle \overset{\text{def}}{=} \{ \langle u_{i_s} \rangle, \ldots, \langle u_{i_1} \rangle \}, 1 \leq s \leq n - k.$$ We say that $$\langle U \rangle$$ is a $$\pi$$-closed $$(n - k)$$-subset of $$\langle H \rangle$$ iff

$$\text{span} \langle U^s \rangle \cap \langle H^\pi_{U^s} \rangle = \emptyset, \quad 1 \leq s \leq n - k. \quad (72)$$
Definition 5  We say that \( \langle U \rangle \) is a \( \pi \)-boundary \( (n - k) \)-subset of \( \langle H \rangle \) iff
\[
\text{span}\langle U^\pi_s \rangle \cap \langle H^\pi_{U^\pi_s} \rangle = \mathcal{O}, \quad 1 \leq s \leq n - k - 1, \tag{73}
\]
\[
\text{span}\langle U \rangle \cap \langle H^\pi_{\pi} \rangle \neq \mathcal{O}. \tag{74}
\]

Let \( B(\langle H \rangle; n - k; \pi) \) denotes the set of all \( \pi \)-boundary \( (n - k) \)-subsets of \( \langle H \rangle \), \( 0 \leq k \leq n - 2 \). Note that for \( k = n - 1 \), \( B(\langle H \rangle; 1; \pi) = \mathcal{O} \).

Lemma 2  The set \( \hat{\gamma}(C_{n; \neq 0}^\pi(H; w)) \subset \langle H \rangle_{\neq 0}^\pi \) can be expressed as disjoint union of \( C_{n-k}^\pi(H^\pi_U; U) \) as follows:
\[
\hat{\gamma}(C_{n; \neq 0}^\pi(H; w)) = \bigcup_{k=0}^{n-2} \bigcup_{(U) \in B(\langle H \rangle; n-k; \pi)} C_{n-k}^\pi(H^\pi_U; U). \tag{75}
\]

Proof. Let us assume that there exists \( \gamma \in C_{n-k}^\pi(H^\pi_U; U) \cap C_{n-l}^\pi(H^\pi_V; V) \subset \langle H \rangle_{\neq 0}^\pi \) for some two different \( \pi \)-boundary subsets \( \langle U \rangle \) and \( \langle V \rangle \). Then
\[
\hat{\gamma}^{-1}(\gamma) = \Delta = \\
\langle (w_{j1}), \ldots, w_{jk}, \ldots, w_{j1} \rangle, \langle u_{i_{n-k}}, \ldots, u_{i_{1}} \rangle = \langle (w_{r1}), \ldots, w_{r_{1}}, \ldots, (v_{i_{n-l}}, \ldots, v_{i_{1}}) \rangle.
\]
Since \( \langle U \rangle \) and \( \langle V \rangle \) are different, then \( k \neq l \). Let \( k < l \). Then \( \langle V \rangle \subset \langle U \rangle \), \( \langle v_{i_{l}} = \langle u_{i_{1}}, \ldots, v_{i_{n-l}} = \langle u_{i_{n-l}} \rangle \). Since \( \langle U \rangle \) and \( \langle V \rangle \) are \( \pi \)-boundary subsets, then
\[
\mathcal{O} \neq \text{span}\langle V \rangle \cap \langle H^\pi_{\pi} \rangle = \text{span}\langle U^\pi_{n-l} \rangle \cap \langle H^\pi_{U_{n-l}}^\pi \rangle = \mathcal{O}.
\]
This contradicts to our assumption. Hence, for any two different \( \pi \)-boundary subsets \( \langle U \rangle \) and \( \langle V \rangle \), we have
\[
C_{n-k}^\pi(H^\pi_U; U) \cap C_{n-l}^\pi(H^\pi_V; V) = \mathcal{O}.
\]
For any \( \Delta \in C_{n; \neq 0}^\pi(H; w) \), \( \Delta = \langle (w_{i_{1}}, \ldots, w_{i_{n}}) \rangle \), let
\[
M(\Delta) \overset{\text{def}}{=} \{ t \in [n] | \text{span}\langle w_{it+1}, \ldots, w_{i_{n+1}} \rangle \cap \langle H^\pi_{w_{it+1}} \rangle \neq \mathcal{O} \},
\]
and
\[
m(\Delta) \overset{\text{def}}{=} \max_{t \in M(\Delta)} t.
\]
Then the set \( \langle U(\Delta) \rangle \overset{\text{def}}{=} \{ \langle w_{i_{m(\Delta)+1}}, \ldots, w_{i_{n+1}} \rangle \} \) is a \( \pi \)-boundary \( (n - m(\Delta) + 1) \)-subset. Since \( t(\Delta) \leq m(\Delta) \), then
\[
\hat{\gamma}(\Delta) \in C_{n-m(\Delta)+1}^\pi(H^\pi_{U(\Delta)}; U(\Delta)).
\]
Let denote by $\langle H; w \not\in \rangle_{\neq 0}^{x}$ the set

$$\langle H; w \not\in \rangle_{\neq 0}^{x} \overset{\text{def}}{=} \{ U \in \langle H \rangle_{\neq 0}^{x} | w \not\in \text{span}(U) \}.$$  \hspace{1cm} (76)

**Theorem 4** Let $\langle H \rangle \subset \mathbb{RP}^{n}$ be a finite subset, $|\langle H \rangle| = T$. Then for any $\langle w \rangle \in \langle H \rangle$, the following inequality is true

$$\text{rank} H_{n} (K^{H}, K_{n-1}^{H}; F) \leq \frac{1}{(T-1)!} \sum_{k=0}^{n-2} \sum_{U \in \langle H; w \not\in \rangle_{\neq 0}^{x}} \frac{1}{q_{n-k-1}^{U} \cdots q_{1}^{U}} \times$$

$$\cdot \sum_{d=k+3}^{T-q_{n-k-1}^{U}} \frac{(T-d)!(T-q_{n-k-1}^{U} - 1)!}{(T-q_{n-k-1}^{U} - 1)!} \cdot A_{n-k}(U; d) \left( \begin{array}{c} d - 2 \\ k \end{array} \right) + D_{n}(H; w),$$

where

$$A_{n-k}(U; d) \overset{\text{def}}{=} \left[ \left( \frac{T-q_{n-k-1}^{U} - 2}{q_{n-k-1}^{U} - q_{n-k-1}^{U} - 1} \right) - \left( \frac{T-q_{n-k-1}^{U} - d}{q_{n-k-1}^{U} - q_{n-k-1}^{U} - 1} \right) \right] \times$$

$$\times (q_{n-k-1}^{U} - q_{n-k-1}^{U} - 1)!. \hspace{1cm} (77)$$

**Proof.** Let denote by $\Gamma$ the set of all orders on the set $\langle H \rangle$ such that $\tilde{\gamma}(1) = \langle w \rangle$, $\forall \tilde{\gamma} \in \Gamma$. For a fixed order $\pi$ (see (20)) such that $\pi(1) = \langle w \rangle$, any $\tilde{\gamma} \in \Gamma$ can be expressed as composition

$$\tilde{\gamma} = \pi \circ \gamma : [T] \to [T] \to \langle H \rangle$$

$\pi$ with a permutation $\gamma \in \text{Sym}([T]; 1) \subset \text{Sym}([T])$ such that $\gamma(1) = 1$. Let $p$ be the uniform probability distribution on $\Gamma \cong \text{Sym}([T]; 1)$:

$$p(\gamma) = \frac{1}{(T-1)!}, \hspace{0.5cm} \gamma \in \text{Sym}([T]; 1).$$

For any collection $U = (\langle u_{n-k} \rangle, \ldots, \langle u_{1} \rangle) \in H_{\neq 0}^{x(n-k)}$ such that $\langle w \rangle \not\in \text{span}(U)$, we define the random function $I_{n-k}^{U} : \text{Sym}([T]; 1) \to \mathbb{R}$ by the
Formula:

\[ I_{n-k}^U(\gamma) \overset{\text{def}}{=} \begin{cases} 
(\pi \circ \gamma)^{-1}(\langle u_{i_{n-k}} \rangle) - 2, & \text{if } U \in B(\langle H \rangle; n-k; \pi \circ \gamma); \\
0 & \text{in all other cases.}
\end{cases} \quad (78) \]

From definition (71) follows that for any \((n-k)\)-subset \(\langle U \rangle \subset \langle H \rangle \setminus \langle w \rangle\) and order \(\pi : [T] \to \langle H \rangle\) satisfying to (70), we have:

\[ |C_{n-k}^\pi(H_U; U)| < \binom{|H_U|}{k}. \quad (79) \]

Let

\[ I_{n-k} \overset{\text{def}}{=} \sum_{U \in \langle H; w \rangle \setminus \langle n-k \rangle} I_n^U : \text{Sym}([T]; 1) \to \mathbb{R}. \quad (80) \]

Then from Corollary 1, (79), and Lemma 2 for any \(\gamma \in \text{Sym}([T]; 1)\), we have

\[ \text{rank } H_n \left( K^H, K_{n-1}^H; F \right) = |C_{n-k}^\pi(H_U; U)| \leq \sum_{k=0}^{n-2} I_{n-k}(\gamma) + D_n(H; w). \quad (81) \]

Hence, the inequality (81) holds if we change the right hand side of (81) by its expectation:

\[ \text{rank } H_n \left( K^H, K_{n-1}^H; F \right) \leq \sum_{k=0}^{n-2} \mathbb{E}[I_{n-k}] + D_n(H; w). \quad (82) \]

Let \(U = (\langle u_{i_{n-k}} \rangle, \ldots, \langle u_{i_1} \rangle) \in H_{\neq 0}^{(n-k)}\) such that \(\langle w \rangle \notin \text{span}(U)\) and \(q_{n-k}^U > q_{n-k-1}^U + 1\). We don’t need to consider the case \(q_{n-k}^U = q_{n-k-1}^U + 1\), because for any \(\gamma \in \text{Sym}([T]; 1)\) such \(U\) cannot be a \(\pi \circ \gamma\)-boundary subset of \(\langle H \rangle\).

Let us calculate the number of permutations \(\gamma \in \text{Sym}([T]; 1)\) such that

\[ I_{n-k}^U(\gamma) = \binom{d-2}{k}, \quad (83) \]

for some \(d, k + 3 \leq d \leq T - q_{n-k-1}^U\). From (78), we have

\[ (\pi \circ \gamma)^{-1}(\langle u_{i_{n-k}} \rangle) = d. \]
From the condition (73) follows that positions of elements from \( L_{n-k-1}(U) \cap \langle H \rangle \) in the order \( \pi \circ \gamma \) have to be chosen from the set \([d + 1, T] = [d + 1, d + 2, \ldots, T]\), and if they are fixed, we have \( A_{n-k}(U; d) \) possibilities for arrangement of elements from the set

\[
\{ L_{n-k}(U) \cap \langle H \rangle \} \setminus \{ \{ \langle u_{n-k} \rangle \cup L_{n-k-1}(U) \} \cap \langle H \rangle \}
\]

to fulfill the \( \pi \circ \gamma \)-boundary condition (74). The arrangement of the remaining elements from

\[
\langle H \rangle \setminus \{ \{ \langle w \rangle \cup L_{n-k}(U) \} \cap \langle H \rangle \}
\]

will not affect the fact that \( U \) is a \( \pi \circ \gamma \)-boundary subset of \( \langle H \rangle \). Since the collection \( U_{n-k-1} \) defined \( \langle \langle u_{i_{n-k-1}} \rangle, \ldots, \langle u_{1} \rangle \rangle \) has to be a \( \pi \circ \gamma \)-closed subset of \( \langle H \rangle \), then the element \( \langle u_{i_{n-k-1}} \rangle \) has to be in the first place among any \( q_{n-k-1}^{U} \) positions selected from the set \([d + 1, T]\) for arrangement of the set \( L_{n-k-1}(U) \cap \langle H \rangle \), while \( q_{n-k-2}^{U} \) elements from \( L_{n-k-2}(U) \cap \langle H \rangle \) can be located in any of the remained \( q_{n-k-1}^{U} - 1 \) places. The arrangement of the elements from

\[
\{ L_{n-k-1}(U) \cap \langle H \rangle \} \setminus \{ \{ \langle u_{i_{n-k-1}} \rangle \cup L_{n-k-2}(U) \} \cap \langle H \rangle \}
\]

in \( q_{n-k-1}^{U} - q_{n-k-2}^{U} - 1 \) places, left after choosing \( q_{n-k-2}^{U} + 1 \) places for arrangement of the set \( L_{n-k-2}(U) \cap \langle H \rangle \) and \( \langle u_{i_{n-k-1}} \rangle \), doesn’t affect the fulfillment of \( \pi \circ \gamma \)-closeness condition for \( U_{n-k-1} \).

Continuing the same way, we get that the number \( N(U; d) \) of permutations \( \gamma \in Sym([T]; 1) \), such that \(( \pi \circ \gamma)^{-1}(\langle u_{n-k} \rangle) = d\) and \( U \) is a \( \pi \circ \gamma \)-boundary subset of \( \langle H \rangle \), is

\[
N(U; d) = (T - q_{n-k-1}^{U} - 1)!A_{n-k}(U; d) \left( \begin{array}{c} T - d \\ q_{n-k-1}^{U} \end{array} \right) \times
\]

\[
\times \left( \begin{array}{c} q_{n-k-1}^{U} - 1 \\ q_{n-k-2}^{U} \end{array} \right) \left( q_{n-k-1}^{U} - q_{n-k-2}^{U} - 1 \right)! \times \cdots \times \left( \begin{array}{c} q_{1}^{U} - 1 \\ q_{1}^{U} - 1 \end{array} \right) \left( q_{1}^{U} - q_{1}^{U} - 1 \right)! \times
\]

\[
\cdots \times \left( \begin{array}{c} q_{1}^{U} - 1 \\ q_{1}^{U} \end{array} \right) \left( q_{1}^{U} - q_{1}^{U} - 1 \right)! = \frac{(T - q_{n-k-1}^{U} - 1)!A_{n-k}(U; d)(T - d)!}{(T - q_{n-k-1}^{U} - d)!} \frac{1}{q_{n-k-1}^{U} \cdots q_{n-k-2}^{U} \cdots q_{1}^{U}}.
\]

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Hence,

$$E[T_{n-k}^U] = \frac{1}{(T-1)!} \frac{1}{q_{n-k-1}^U \cdots q_1^U} \times$$

$$\sum_{d=k+3}^{T-q_{n-k-1}^U} \frac{(T-d)!(T-q_{n-k-1}^U - 1)!}{(T-q_{n-k-1}^U - d)!} A_{n-k}(U; d) \binom{d-2}{k}$$

(84)

The Theorem follows from (82), (84), and additivity of expectation.

Q.E.D.

Taking into account the inequality (19) and definition (25), we return to elaborating of $\eta^\ast((\langle H \rangle))$ for $\langle H \rangle = \langle E_n \rangle \subset \mathbb{R}P^n$ (see (1)).

We define $\delta_{n,k}$, $k = 1, \ldots, n + 1$, as

$$\delta_{n,k} \overset{def}{=} \frac{|\langle E_n \rangle^k \setminus \langle E_n \rangle^k_{\neq \emptyset}|}{|\langle E_n \rangle^k|}, \quad k = 1, \ldots, n + 1.$$  

(85)

We choose subspaces

$$V_k \subset \mathbb{R}^{n+1}, \quad \text{dim } V_k = k, \quad k = 1, \ldots, n + 1,$$

such that the orthogonal projectors

$$P_k : \mathbb{R}^{n+1} = V_k^\perp \oplus V_k \rightarrow V_k^\perp \oplus V_k = \mathbb{R}^{n+1},$$

$$P_k(v) = v_2, \quad \forall v = v_1 + v_2 \in V_k^\perp \oplus V_k, \quad k = 1, \ldots, n + 1,$$

satisfy the following conditions:

for any $k$ linear independent vectors $w_{i_1}, \ldots, w_{i_k} \in E_n$, the vectors

$$w_{i_s}^k \overset{def}{=} P_k(w_{i_s}), \quad s = 1, \ldots, k,$$

are linear independent as well.

Let $E_{n,k}$ denote the set

$$E_{n,k} \overset{def}{=} P_k(E_n).$$  

(86)
For $W^{k+1} = (\langle w^{k+1}_{i_1} \rangle, \ldots, \langle w^{k+1}_{i_k} \rangle) \in \langle E_{n,k+1} \rangle^k \neq 0$, we use the following notations:

\[
L(W^{k+1}) \overset{def}{=} \text{span} \langle w^{k+1}_{i_1}, \ldots, w^{k+1}_{i_k} \rangle \subset V_{k+1} = \mathbb{R}^{k+1};
q_k^{W^{k+1}} \overset{def}{=} |L(W^{k+1}) \cap E_{n,k+1}|;
E_{n,k+1}^m \overset{def}{=} \left\{ W^{k+1} \in \langle E_{n,k+1} \rangle^k \neq 0 \mid q_k^{W^{k+1}} = k + m \right\};
\gamma^m_{k+1} \overset{def}{=} \frac{|E_{n,k+1}^m|}{|E_{n,k+1} \times (k+1)|};
B_{k+1} \overset{def}{=} \left\{ (\langle w^{k+1}_{i_1} \rangle, \ldots, \langle w^{k+1}_{i_k} \rangle) \in \langle E_{n,k+1} \rangle^{k+1} \setminus \langle E_{n,k+1} \rangle^k \neq 0 \mid \langle w^{k+1}_{i_1} \rangle, \ldots, \langle w^{k+1}_{i_k} \rangle \in \langle E_{n,k+1} \rangle^k \neq 0 \right\};
\epsilon_{k+1} = \frac{|B_{k+1}|}{|E_{n,k+1} \times (k+1)|}.
\] (87)

Note that

\[
\delta_{n,s} = \frac{|\langle E_{n,k+1} \rangle^s \setminus \langle E_{n,k+1} \rangle^s \neq 0|}{|\langle E_{n,k+1} \rangle^s|}, \quad s = 1, \ldots, k + 1.
\] (88)

Since

\[
\langle E_{n,k+1} \rangle^{(k+1)} \setminus \langle E_{n,k+1} \rangle^{(k+1)} =
\left\{ (\langle E_{n,k+1} \rangle^{(k+1)} \setminus \langle E_{n,k+1} \rangle^{(k+1)} \setminus B_{k+1}) \right\} \cup B_{k+1},
\]
then

\[
\delta_{n,k+1} = \delta_{n,k} + \epsilon_{k+1}.
\] (89)

From definition (87) we have

\[
|B_{k+1}| = \sum_{m=1}^{2^{k-1}-k} |E_{n,k+1}^m|m,
\] (90)

\[
|E_{n,k+1}^m| = \gamma^m_{k+1}|\langle E_{n,k+1} \rangle^k|,
\] (91)

and from (88)

\[
|\langle E_{n,k+1} \rangle^k| = (1 - \delta_{n,k})|\langle E_{n,k+1} \rangle^k|.
\] (92)

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Hence, from (90), (91), (92) we get
\[ \epsilon_{k+1} = (1 - \delta_{n,k}) \sum_{m=1}^{2^{k-1} - k} \gamma_{k+1}^m \frac{m}{2^n - k}. \] (93)

**Definition 6** We say that a vector \( w \in \mathbb{R}^{n+1} \) is in general position to the sets \( E_n = P_{n+1}(E_n), P_n(E_n), \ldots, P_2(E_n) \) iff for any \( k, k = 2, \ldots, n+1, \) and vectors \( w_{i_1}, \ldots, w_{i_{k-1}} \in E_n \subset \mathbb{R}^{n+1}, \) the vector \( w_k \overset{\text{def}}{=} P_k(w) \) doesn't belong to the linear span of vectors \( w_{i_1}^k, \ldots, w_{i_{k-1}}^k: \)
\[ w_k \notin \text{span}(w_{i_1}^k, \ldots, w_{i_{k-1}}^k). \]

Next, we are going to apply the Theorem 4 to the sets
\[ \langle Z_k \rangle \overset{\text{def}}{=} \langle E_{n,k} \rangle \cup \langle w_k \rangle \subset \mathbb{RP}^{k-1}, \; k = 2, \ldots, n+1, \]
where \( w \in \mathbb{R}^{n+1} \) is in general position to the sets \( P_k(E_n), k, k = 2, \ldots, n+1. \) We express the right hand side of the inequality (77) as the sum of four summands
\[ \text{rank} H_{k-1} \left( K^{Z_k}, K^{Z_{k-2}}_{k-2}; F \right) \leq S^k_0 + S^k_\leq + S^k_\geq + D_{k-1}(Z_k; w^k), \] (94)
where
\[ S^k_0 \overset{\text{def}}{=} \frac{1}{(2^n)!} \sum_{U \in \langle Z_k; w^k \rangle \times (k-1)} \frac{1}{q_{k-2}^U \cdots q_1^U} \times \] (95)
\[ \times \sum_{d=3}^{2^n+1-q_k^U-2} \frac{(2^n + 1 - d)!}{(2^n+1-q_k^U-d)!} \cdot \frac{(2^n - q_k^U)!}{(2^n + 1 - q_k^U - d)!} \cdot A_{k-1}(U; d); \]
\[ S^k_\leq \overset{\text{def}}{=} \frac{1}{(2^n)!} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_k; w^k \rangle \times (k-t-1)} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \times \] (96)
\[ \times \sum_{d=t+3}^{q_k^U-t+2} \frac{(2^n + 1 - d)!}{(2^n+1-q_k^U-d)!} \cdot \frac{(2^n - q_k^U-t-1)!}{(2^n - q_k^U-t-2)!} \cdot A_{k-t-1}(U; d) \left( \begin{array}{c} d - 2 \\ t \end{array} \right); \]
\[ S^k_\geq = \frac{1}{(2^n)!} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_k; \omega^k, \xi \rangle_{\not=0}^{(k-t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \times \]

\[ \sum_{d=\max(q_{k-t-2}^U + 2, t+3)}^{2^n+1-q_{k-t-2}^U} \frac{(2^n + 1 - d)! (2^n - q_{k-t-1}^U)!}{(2^n + 1 - q_{k-t-2}^U - d)!} \cdot A_{k-t-1}(U; d) \left( \binom{d-2}{t} \right), \]

where

\[ A_{k-t-1}(U; d) = \left[ \left( \frac{2^n - q_{k-t-2}^U - 1}{q_{k-t-1}^U - q_{k-t-2}^U - 1} \right) - \left( \frac{2^n - q_{k-t-2}^U - d + 1}{q_{k-t-1}^U - q_{k-t-2}^U - 1} \right) \right] \times \]

\[ \times (q_{k-t-1}^U - q_{k-t-2}^U - 1)!. \]

**Lemma 3** For sufficiently large \( n \), the following inequality holds:

\[ S^k_\geq + S^k_\leq \leq \left( 1 + o \left( \frac{n^2}{2^n} \right) \right) \frac{k-3}{2^n} \left( \binom{2^n}{k-1} \right). \]
Proof. From (97) and the inequality $q_{t-2}^k \geq k - t - 2$ we have

$$S^k_\geq \leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_{k, w^k} \rangle_{\neq 0}^{(k-t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \times$$

$$\times \sum_{d=\max(q_{k-t-2}^{U}, t+3)}^{2^n+1-q_{k-t-2}^{U}} \frac{(2^n - d + 1) \cdots (2^n - d - q_{k-t-2}^U + 2)}{(2^n - 1) \cdot (2^n - q_{k-t-2}^U)} \frac{(d - 2)}{t} \leq$$

$$\leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_{k, w^k} \rangle_{\neq 0}^{(k-t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \frac{(2^n - 1) \cdots (2^n - t)}{t!} \times$$

$$\times \sum_{d=\max(q_{k-t-2}^{U}, t+3)}^{2^n+1-q_{k-t-2}^{U}} (1 - \alpha_d)^{q_{k-t-2}^U} \alpha_d^t \leq$$

$$\leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_{k, w^k} \rangle_{\neq 0}^{(k-t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \frac{(2^n - 1) \cdots (2^n - t)}{t!} \times$$

$$\times \sum_{d=\max(q_{k-t-2}^{U}, t+3)}^{2^n+1-q_{k-t-2}^{U}} (1 - \alpha_d)^{k-t-2} \alpha_d^t.$$
From (96) we have

\[
S_k^k \leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_k, w^k \rangle^{(k-t-1)}} q_{k-t-2}^U \cdots q_1^U \times
\]

\[
\sum_{d=t+3}^{q_{k-t-2}+2} (2^n - q_{k-t-2}^U - 1) \cdots (2^n - q_{k-t-2}^U - (d - 2)) \left( \frac{d - 2}{t} \right) \leq
\]

\[
\leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_k, w^k \rangle^{(k-t-1)}} q_{k-t-2}^U \cdots q_1^U \times
\]

\[
\frac{1}{t!} \left( \frac{2^n - 1}{2^n - 1} \right) \cdots \left( \frac{2^n - t}{2^n - t} \right) \times
\]

\[
\times \sum_{d=t+3}^{q_{k-t-2}+2} \left( 1 - \frac{q_{k-t-2}^U}{2^n - 1} \right) \alpha_d^t \leq
\]

\[
\leq \frac{1}{2^n} \sum_{t=1}^{k-3} \sum_{U \in \langle Z_k, w^k \rangle^{(k-t-1)}} q_{k-t-2}^U \cdots q_1^U \times
\]

\[
\frac{1}{t!} \left( \frac{2^n - 1}{2^n - 1} \right) \cdots \left( \frac{2^n - t}{2^n - t} \right) \times
\]

\[
\times \sum_{d=t+3}^{q_{k-t-2}+2} (1 - \alpha_d)^{k-t-2} \alpha_d^t.
\]

The last inequality in (100) follows from the inequality

\[
\left( 1 - \frac{q}{2^n - 1} \right)^{(2^n-1)x} \leq (1 - x)^{k-t-2}
\]

that holds for \( x \in [0, \frac{q}{2^n-1}] \) and \( q \geq k - t - 2 \).
From (99) and (100) we get

\[ S_k^\geq + S_k^\leq \leq \frac{1}{2^n} \sum_{t=1}^{k-3} \frac{(2^n - t)}{t!} \sum_{d=2}^{2n+1} (1 - \alpha_d)^{k-t-2} \alpha_d^t \]

\[ \times \sum_{U \in (Z_k; w^k \notin \varnothing)^{k-(t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \leq \]

\[ \leq \frac{1}{2^n} \sum_{t=1}^{k-3} \frac{(2^n - 1) \cdots (2^n - t)}{t!} \left( \frac{\Gamma(k-t-1)\Gamma(t+1)}{\Gamma(k)} + \frac{n^2}{12(2^n - 1)^2} \right) \times \]

\[ \times \frac{1}{(k-t-2)!} \sum_{U \in (Z_k; w^k \notin \varnothing)^{k-(t-1)}} 1 \leq \left( 1 + o\left( \frac{n^3}{2^n} \right) \right) \frac{k-3}{2^n} \left( \frac{2^n}{k-1} \right). \]

Here we used the formula 853.21 from [3] that holds for \( t \geq 1 \):

\[ \int_0^1 (1-x)^{k-t-2}x^t \, dx = \frac{\Gamma(k-t-1)\Gamma(t+1)}{\Gamma(k)} = \frac{(k-t-2)!t!}{(k-1)!} \]

and the midpoint rule for sum estimation by integral.

Q.E.D.

**Lemma 4** For any \( n \geq 64 \), the following inequality holds:

\[ S_k^0 \leq \left( \frac{1}{2^n} \left( 1 + o\left( \frac{n^3}{2^n} \right) \right) + (\delta_{n,k} - \delta_{n,k-1}) \left( 1 - \frac{k-1}{2^n} \right) \right) \left( \frac{2^n}{k-1} \right). \]

(101)

**Proof.** From (95) we have

\[ S_k^0 \leq \frac{1}{2^n} \sum_{U \in (Z_k; w^k \notin \varnothing)^{k-(t-1)}} \frac{1}{q_{k-t-2}^U \cdots q_1^U} \left( J(U, k; >) + J(U, k; \leq) \right), \]

(102)

where

\[ J(U, k; >) \overset{\text{def}}{=} \sum_{d=q_{k-t-2}^U+3}^{2^n+1} \frac{(2^n - d + 1) \cdots (2^n - d - q_{k-t-2}^U + 2)}{(2^n - 1) \cdots (2^n - q_{k-t-2}^U)} \]

(103)
and

\[
J(U, k; \leq) \overset{\text{def}}{=} \sum_{d=3}^{q_{k-2}+2} \left( \frac{(2^n - q_{k-2}^U - 1) \cdots (2^n - q_{k-2}^U - (d - 2))}{(2^n - 1) \cdots (2^n - (d - 2))} - \frac{(2^n - q_{k-1}^U) \cdots (2^n - q_{k-1}^U - (d - 3))}{(2^n - 1) \cdots (2^n - (d - 2))} \right).
\]

(104)

We have

\[
\frac{1}{q_{k-2}^U \cdots q_1^U} J(U, k; >) \leq \frac{1}{q_{k-2}^U \cdots q_1^U} \sum_{d=q_{k-2}^U+3}^{2^n+1-q_{k-2}^U} (1 - \alpha_d) q_{k-2}^U \leq \frac{k-2}{(k-2)! q_{k-2}^U} \left( \frac{1}{q_{k-2}^U + 1} + \frac{(q_{k-2}^U)^2}{24(2^n - 1)^2} \right) \leq \frac{1}{(k-1)!} \left( 1 + \frac{n^2 q_{k-2}^U}{(2^n - 1)^2} \right) = \frac{1}{(k-1)!} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right).
\]

(105)

Here \( \alpha_d = \frac{d-2}{2^n-1} \).

We assert that for any \( U \in \langle Z_k; w^k, \phi \rangle \times (k-1) \) for \( n \geq 64 \), the following inequality holds:

\[
\frac{1}{q_{k-2}^U \cdots q_1^U} J(U, k; \leq) \leq \frac{q_{k-1}^U - (k-1)}{(k-1)!}.
\]

(106)
For \( U \in \langle Z_k; w^k \varphi_0 \rangle \times (k-1) \) such that \( q^U_{k-1} \geq n^3 \), we have

\[
\frac{1}{q^U_{k-1} - (k-1)} \frac{q^U_{k-1} - (k-1)}{q^U_{k-2} \cdots q^U_1} J(U, k; \leq) \leq
\]

\[
\leq \frac{1}{q^U_{k-1} - (k-1)} \frac{1}{q^U_{k-2}} \frac{q^U_{k-1} - (k-1)}{(k-3)!} q^U_{k-2} \leq
\]

\[
(q^U_{k-1} \geq n^3)
\]

\[
\leq \frac{q^U_{k-1} - (k-1)}{(k-1)!}.
\]  

(107)

For \( U \in \langle Z_k; w^k \varphi_0 \rangle \times (k-1) \) such that \( q^U_{k-1} < n^3 \), it can be proven by mathematical induction that for any \( d, 3 \leq d \leq q^U_{k-2} + 2 \), and \( n \geq 64 \) (for \( n \geq 64 \), we have \( n^{10} < 2^n - 1 \)), the following inequality holds:

\[
\frac{(2^n - q^U_{k-2} - 1) \cdots (2^n - q^U_{k-2} - (d - 2))}{(2^n - 1) \cdots (2^n - (d - 2))} - \frac{(2^n - q^U_{k-1}) \cdots (2^n - q^U_{k-1} - (d - 3))}{(2^n - 1) \cdots (2^n - (d - 2))} \leq \frac{d - 2}{n (q^U_{k-2})^2}.
\]  

(108)

Hence, for \( U \in \langle Z_k; w^k \varphi_0 \rangle \times (k-1) \) such that \( q^U_{k-1} < n^3 \), we have:

\[
\frac{1}{q^U_{k-1} - (k-1)} \frac{q^U_{k-1} - (k-1)}{q^U_{k-2} \cdots q^U_1} J(U, k; \leq) \leq
\]

\[
\leq 1 \cdot \frac{q^U_{k-1} - (k-1)}{(k-2)!} \frac{1}{n (q^U_{k-2})^2} \sum_{d=3}^{q^U_{k-2} + 2} (d - 2) \leq \frac{q^U_{k-1} - (k-1)}{(k-1)!}.
\]  

(109)

The assertion (106) follows from (107) and (109).
It follows from (102), (105), and (106) that for any $n \geq 64$, the following inequality holds:

$$S_0^k \leq \frac{1}{2^n} \left( \frac{1 + o \left( \frac{n^3}{2^n} \right)}{(k - 1)!} \sum_{U \in (Z_k; w_k) \setminus \mathcal{Z}^{(k-1)}} 1 + \right.$$  

$$+ \frac{1}{2^n (k - 1)!} \sum_{U \in (Z_k; w_k) \setminus \mathcal{Z}^{(k-1)}} (q_U^{k-1} - (k - 1)).$$

Taking into account (89), we note that

$$\left( q_U^{k-1} - (k - 1) \right) = \frac{1}{2^n} |B_k| =$$

$$= \frac{\epsilon_k}{2^n} 2^n (2^n - 1) \cdots (2^n - k + 1) = (\delta_{n,k} - \delta_{n,k-1}) \left( \frac{2^n - 1}{k - 1} \right) (k - 1)!. \tag{111}$$

The Lemma follows from (110) and (111).

Q.E.D.

**Theorem 5** For any $n \geq 64$, the following inequality holds for $k = 1, \ldots, n$:

$$\eta^*_k (\langle Z_{k+1} \rangle) \geq \left[ 1 - \delta_{n,k} - \frac{k - 1}{2^n} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) - \left( 1 - \frac{k}{2^n} \right)(\delta_{n,k+1} - \delta_{n,k}) \right] \left( \frac{2^n}{k} \right) =$$

$$= \left( \frac{E_n}{\eta^*_k} \right)^{\langle w \rangle} - \left[ \frac{k - 1}{2^n} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) + \left( 1 - \frac{k}{2^n} \right)(\delta_{n,k+1} - \delta_{n,k}) \right] \left( \frac{2^n}{k} \right). \tag{112}$$

**Proof.** From the exact sequence (59), we get:

$$\eta^*_k (\langle Z_{k+1} \rangle) = \left( \frac{2^n}{k} \right) - \text{rank} H_k \left( K^{Z_k+1}, K^{Z_{k+1}}; F \right). \tag{113}$$
From inequality (94), Lemma 3, and Lemma 4, for any $n \geq 64$, we get for $k = 1, \ldots, n$:

$$
\text{rank} H_k \left( K^{Z_{k+1}}, K^{Z_{k+1}}_{k-1}; F \right) \leq \\
\leq \left[ \delta_{n,k} + \frac{k-1}{2n} \left( 1 + o \left( \frac{n^3}{2n} \right) \right) + \left( 1 - \frac{k}{2n} \right) (\delta_{n,k+1} - \delta_{n,k}) \right] \left( \frac{2^n}{k} \right)
$$

(114)

The Theorem follows from (113), (114), and Remark 2.

Q.E.D.

5 Asymptotics of the number of singular $\{\pm 1\}$-matrices.

Lemma 5 For $n \geq 64$ and $k = 1, \ldots, n$, we have

$$
\delta_{n,k+1} - \delta_{n,k} \leq \frac{k-1}{2n} \left( 1 + o \left( \frac{n^3}{2n} \right) \right).
$$

(115)

Proof. We define the probability distribution $p : \langle Z_{k+1} \rangle = \langle E_{n,k+1} \rangle \cup \{ \langle w^{k+1} \rangle \} \rightarrow [0, 1]$ by the rule:

$$
p \left( \langle w^{k+1} \rangle \right) = 1, \quad p \left( \langle w_{i}^{k+1} \rangle \right) = 0, \quad i = 1, \ldots, 2^n.
$$

Then from Theorem 3 we have

$$
\eta_k^* (\langle Z_{k+1} \rangle) = \sum_{W^{k+1} \in \langle E_{n,k+1} \rangle \neq 0} \frac{1}{W^{k+1} [E_{n,k+1}]}.
$$

(116)

For any permutation $\sigma \in Sym[k]$, we define the map $\sigma^* : \langle E_{n,k+1} \rangle^{\times k} \neq 0 \rightarrow \langle E_{n,k+1} \rangle^{\times k}$ by the formula

$$
\sigma^* \left( W^{k+1} \right) = W^{k+1}_\sigma \quad \text{def} \quad (\langle w^{k+1}_{i_1(1)} \rangle, \ldots, \langle w^{k+1}_{i_k(1)} \rangle) \in \langle E_{n,k+1} \rangle^{\times k},
$$

$$
\forall W^{k+1} = (\langle w^{k+1}_{i_1} \rangle, \ldots, \langle w^{k+1}_{i_k} \rangle) \in \langle E_{n,k+1} \rangle^{\times k},
$$

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Note that for the symmetrization of a combinatorial flag, defined by the formula
\[ Sym(W^{k+1}) \overset{def}{=} \sum_{\sigma \in Sym[k]} \frac{1}{W^{k+1}_\sigma [E_{n,k+1}]} , \]
for \( W^{k+1} \in E_{n,k+1}^m \), we have
\[ Sym(W^{k+1}) \leq \frac{k}{k+m}. \] (117)
Hence,
\[ \sum_{W^{k+1} \in E_{n,k+1}^m} \frac{1}{W^{k+1}[E_{n,k+1}]} \leq \gamma_{k+1}^m (1 - \delta_{n,k}) \frac{k}{k+m} \binom{2^n}{k} \quad \text{and} \quad \eta_k^*((Z_{k+1})) \leq (1 - \delta_{n,k}) \binom{2^n}{k} \sum_{m=0}^{2k-1-k} \gamma_{k+1}^m \frac{k}{k+m}. \] (118)
Combining (118) with inequality (112), for \( n \geq 64 \), we get
\[ (1 - \delta_{n,k}) \binom{2^n}{k} \sum_{m=0}^{2k-1-k} \gamma_{k+1}^m \frac{k}{k+m} \geq \eta_k^*((Z_{k+1})) \geq 1 - \gamma_0^0 k^{k+1} - \sum_{m=1}^{2k-1-k} \gamma_{k+1}^m \frac{k}{k+m}. \] (119)
or
\[ \frac{1}{1 - \delta_{n,k}} \left[ \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \binom{k - 1}{2^n} + \left( 1 - \frac{k}{2^n} \right) (\delta_{n,k+1} - \delta_{n,k}) \right] \geq 1 - \gamma_0^0 k^{k+1} - \sum_{m=1}^{2k-1-k} \gamma_{k+1}^m \frac{k}{k+m}. \] (120)
Taking into account the identity
\[ 1 - \gamma_0^0 k^{k+1} = \sum_{m=1}^{2k-1-k} \gamma_{k+1}^m, \]
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the inequality (120) may be expressed as follows
\[
\frac{1}{1 - \delta_{n,k}} \left[ \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \frac{k - 1}{2^n} + \left( 1 - \frac{k}{2^n} \right) (\delta_{n,k+1} - \delta_{n,k}) \right] \geq 2^{k-1-k} \sum_{m=1}^{2^{k-1-k}} \frac{\gamma_{m+1}^m m}{k + m}. \tag{121}
\]

For \(1 \leq k \leq n - 1\), the inequality
\[
\frac{m}{k + m} \geq \frac{2m}{2^n - k} \tag{122}
\]
is fulfilled for all \(m\), \(1 \leq m \leq 2^{k-1-k}\).

For \(k = n\), the inequality (122) is true for all \(m \leq 2^{n-1} - \frac{3}{2} n\). It follows from Littlewood-Offord lemma in the form proven by P. Erdős [6] that for \(n \geq 4\),
\[
\gamma_{m+1}^m = 0, \quad \forall \ m, \text{ such that } 2^{n-1} - \frac{3}{2} n < m < 2^{n-1} - n. \tag{123}
\]

From (121), (122), (123), (89), and (93), we have
\[
\frac{1}{1 - \delta_{n,k}} \left[ \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \frac{k - 1}{2^n} + \left( 1 - \frac{k}{2^n} \right) (\delta_{n,k+1} - \delta_{n,k}) \right] \geq 2^{k-1-k} \sum_{m=1}^{2^{k-1-k}} \frac{\gamma_{m+1}^m m}{2^n - k} = \frac{2(\delta_{n,k+1} - \delta_{n,k})}{1 - \delta_{n,k}} \tag{124}
\]

Hence,
\[
\left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \frac{k - 1}{2^n} \geq \left( 1 + \frac{k}{2^n} \right) (\delta_{n,k+1} - \delta_{n,k}). \tag{125}
\]
The inequality (115) of Lemma 5 follows from (125).

Q.E.D.

For ease of use of established terminology, we formulate an estimate for the cardinality of the set of singular \(\{\pm 1\}\)-matrices in terms of the probability \(\mathbb{P}_n\) of singularity of random Bernoulli matrices. Also, we can identify \(\langle E_n \rangle\) with \(E_n\).
Theorem 6. For $n \to \infty$, we have

$$P_n \sim \frac{(n - 1)^2}{2^{n - 1}}.$$ \hspace{1cm} (126)

Proof. By definition, we have

$$P_{n+1} = \left| \{E_n\}^{n+1} \setminus [E_n]^{(n+1)} \right| \cup \left| [E_n]^{(n+1)} \setminus [E_n_{\neq n}]^{(n+1)} \right| \frac{2^{n(n+1)}}{2^{n+1}},$$ \hspace{1cm} (127)

i.e.,

$$P_{n+1} = \left| \{E_n\}^{n+1} \setminus [E_n]^{(n+1)} \right| + \delta_{n,n+1} \frac{2^{n(n+1)}}{2^{n+1}},$$

where \(\{E_n\}^{n+1} = E_n \times \cdots \times E_{n+1}\).

Cardinality of the subset of matrices containing exactly two equal rows asymptotically plays the main role for estimation of \(\left| \{E_n\}^{n+1} \setminus [E_n]^{(n+1)} \right|\), i.e.,

$$\left| \{E_n\}^{n+1} \setminus [E_n]^{(n+1)} \right| = \frac{n(n + 1)}{2^{n+1}} (1 + o_n(1)).$$ \hspace{1cm} (128)

From Lemma 5 we have

$$\delta_{n,n+1} \leq \delta_{n,n} + \left(1 + o\left(\frac{n^3}{2^n}\right)\right) \frac{n - 1}{2^n} \leq \delta_{n,n-1} + \left(1 + o\left(\frac{n^3}{2^n}\right)\right) \frac{n - 2}{2^n} + \frac{n - 1}{2^n} \leq \ldots$$ \hspace{1cm} (129)

$$\leq \left(1 + o\left(\frac{n^3}{2^n}\right)\right) \sum_{k=1}^{n-1} \frac{k}{2^n} = \left(1 + o\left(\frac{n^3}{2^n}\right)\right) \frac{n(n - 1)}{2^{n+1}}.$$

We need to show that

$$\delta_{n,n+1} \geq \frac{n(n - 1)}{2^{n+1}} (1 + o_n(1)).$$ \hspace{1cm} (130)

Let \(R_{n-1}^{n+1} \subset [E_{n-1}]^{(n+1)}\) be the subset of ordered collections \(W = (w_{i_1}, \ldots, w_{i_{n+1}}) \in [E_{n-1}]^{(n+1)}\) such that the columns \(\bar{1}, Y_2, \ldots, Y_n\) of the
matrix \( M(W) \) with rows \( w_{i1}, \ldots, w_{i_{n+1}} \)

\[
M(W) = \begin{pmatrix} w_{i1} \\ \vdots \\ w_{i_{n+1}} \end{pmatrix} = (\bar{1}, Y_2, \ldots, Y_n)
\]

are not collinear, i.e., \( Y_i \neq \pm Y_j, \forall i \neq j \).

We can construct an ordered collection \( W' \in [E_n]^{(n+1)} \) by placing a column \( \pm Y_i, \) \( i = 2, \ldots, n \), in one of \( n \) positions:

\[
M(W') = (\bar{1}, Y_2, \ldots, Y_i, \ldots, \pm Y_i, \ldots, Y_n).
\]

Then the total number of \( W' \in [E_n]^{(n+1)} \) such that the matrix \( M(W') \) has exactly two equal up to sign columns is not less than

\[
\frac{2(n-1)n}{2} |R_{n-1}^{n+1}| = n(n-1) |R_{n-1}^{n+1}| . \quad (131)
\]

Since

\[
\frac{|[E_{n-1}]^{(n+1)}|}{|[E_n]^{(n+1)}|} = \frac{1}{2^{n+1}} (1 + o_n(1)) \quad \text{and}
\]

\[
|E_{n-1}|^{(n+1)} = (1 + o_n(1)) |R_{n-1}^{n+1}| , \quad (132)
\]

then (130) follows from (131) and (132). The Theorem follows from (127), (128), (129), and (130).

Q.E.D.

6 Asymptotics of the number of threshold functions.

In this section we use notations from the previous section.

**Theorem 7** Asymptotics of the number of threshold functions is equal to

\[
P(2, n) \sim 2^\left( 2^n - 1 \right), \quad n \to \infty. \quad (133)
\]
Proof. We write the inequality (112) of the Theorem 5 for $k = n$ taking into account the inequality (115) of Lemma 5:

$$\eta_n^\star (\langle E_n \rangle \cup \{\langle w \rangle \}) \geq \left[ 1 - \frac{n-1}{2n-1} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \right] \left( \frac{2^n - 1}{n} \right) \geq$$

$$\geq \left[ 1 - \frac{(n-1)^2}{2^n} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \right] \left( \frac{2^n - 1}{n} \right) +$$

$$+ \left[ 1 - \delta_{n,n} - \frac{n-1}{2n-1} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \right] \left( \frac{2^n - 1}{n} \right).$$

(134)

From Theorem 1 we have:

$$\eta_n^\star (\langle E_n \rangle) = \eta_n^\star (\langle E_n \rangle \cup \{\langle w \rangle \}) - \eta_{n-1}^\star (\langle E_n \rangle)^{\perp w}.$$ (135)

From Theorem 2 we have:

$$\eta_{n-1}^\star (\langle E_n \rangle)^{\perp w} \leq \left( \langle E_n \setminus \{u\} \rangle^{\perp w} \right)^{\perp w}.$$ (136)

where $u = \bar{I} \in E_n$.

A summand $\eta_{n-1}^\star (\{\langle u^{\perp w}, \langle w_i^{\perp w} \rangle, \ldots, \langle w_i^{\perp w} \rangle \})$ from the right side of (136) is equal to 1 iff

$$\text{span} \langle u^{\perp w}, w_i^{\perp w}, \ldots, w_i^{\perp w} \rangle = \langle w \rangle^{\perp} = \mathbb{R}^n,$$

or

$$\dim \text{span} \langle u, w_i, \ldots, w_i \rangle = n.$$ (137)

It follows from symmetry of $E_n$, (137), and definition of $\delta_{n,n}$ that the right side of (136) is equal to $(1 - \delta_{n,n}) \left( \frac{2^n - 1}{n-1} \right)$ and

$$\eta_{n-1}^\star (\langle E_n \rangle)^{\perp w} \leq (1 - \delta_{n,n}) \left( \frac{2^n - 1}{n-1} \right).$$ (138)

Taking into account (19), (25), (134), (135), and (138), we get a lower bound for $P(2, n)$:

$$P(2, n) \geq 2 \left[ 1 - \frac{n^2}{2^n} \left( 1 + o \left( \frac{n^3}{2^n} \right) \right) \right] \left( \frac{2^n - 1}{n} \right).$$ (139)

The Theorem follows from the upper bound (2) and the lower bound (139).

Q.E.D.
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