MAGNETIC FLOWS ON Sol-MANIFOLDS: DYNAMICAL AND SYMPLECTIC ASPECTS

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ABSTRACT. We consider magnetic flows on compact quotients of the 3-dimensional solvable geometry \textbf{Sol} determined by the usual left-invariant metric and the distinguished monopole. We show that these flows have positive Liouville entropy and therefore are never completely integrable. This should be compared with the known fact that the underlying geodesic flow is completely integrable in spite of having positive topological entropy. We also show that for a large class of twisted cotangent bundles of solvable manifolds every compact set is displaceable.

1. INTRODUCTION

The Lie group \textbf{Sol} is the semidirect product associated with the action of $\mathbb{R}$ on $\mathbb{R}^2$ given by

$$u \cdot (y_0, y_1) = (e^u y_0, e^{-u} y_1).$$

The group \textbf{Sol} is diffeomorphic to $\mathbb{R}^3$ and the product is

$$(y_0, y_1, u) \ast (y_0', y_1', u') = (e^u y_0' + y_0, e^{-u} y_1' + y_1, u + u').$$

It is not difficult to see that \textbf{Sol} admits cocompact lattices. Let $A \in SL(2, \mathbb{Z})$ be such that there is $P \in GL(2, \mathbb{R})$ with

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$$

and $\lambda > 1$. There is an injective homomorphism

$$\mathbb{Z}^2 \ltimes_A \mathbb{Z} \hookrightarrow \textbf{Sol}$$

given by $(m, n, l) \mapsto (P(m, n), \log \lambda l)$ which defines a cocompact lattice $\Delta$ in \textbf{Sol}. The closed 3-manifold $\Sigma := \Delta \setminus \textbf{Sol}$ is a 2-torus bundle over the circle with hyperbolic gluing map $A$.

The Riemannian metric

$$ds^2 = e^{-2u} dy_0^2 + e^{2u} dy_1^2 + du^2$$

is left-invariant and descends to a Riemannian metric on $\Sigma$. It is a remarkable fact discovered by A. Bolsinov and I. Taimanov \cite{2} that the geodesic flow of $(\Sigma, ds^2)$ is
completely integrable in the sense of Liouville with the two additional integrals
\[ f = p_{y_0}p_{y_1} \]
\[ F = \exp \left( \frac{-1}{p_{y_0}^2 p_{y_1}^2} \right) \sin \left( 2\pi \frac{\log |p_{y_0}|}{\log \lambda} \right). \]

The geodesic flow has topological entropy \( h_{\text{top}} = 1 \) but Liouville (or metric) entropy \( h_{\mu} = 0 \). It is the simplest example of a geodesic flow on a compact homogeneous space with these properties. Note that the lattice \( \Delta \) has exponential word growth and the entropy is all carried in the minimizing Aubry-Mather sets given by \( p_u = \pm 1, p_{y_0} = p_{y_1} = 0 \). The dynamics on these sets is Anosov and given by the suspension of \( A \). We refer to [1] for a detailed description of the foliation by Liouville tori and for spectral properties of the Laplace-Beltrami operator of \( (\Sigma, ds^2) \).

The manifold \( \Sigma \) has a distinguished monopole, i.e. a closed non-exact 2-form which generates \( H^2(\Sigma, \mathbb{R}) \) given by \( \Omega = dy_0 \wedge dy_1 \). This form is harmonic and Hodge dual to the generator \( du \) of \( H^1(\Sigma, \mathbb{R}) \). The Aubry-Mather sets we mentioned before are calibrated by the closed 1-forms \( \pm du \).

The first goal of this paper is the study of the dynamics of the magnetic flow determined by the metric \( ds^2 \) and the monopole \( \Omega \). We will modulate the intensity of the magnetic field \( \Omega \) with a parameter \( s \in [0, \infty) \) and we will always consider the magnetic flow \( \varphi^s \) running with speed one. The analysis of the flow is carried out in Section 3. One of our findings is that the magnetic flow ceases to be Liouville integrable as soon as the magnetic field is switched on. The reason is that the Liouville entropy becomes positive.

Theorem A. The Liouville entropy of \( \varphi^s \) is given by
\[ h_{\mu}(\varphi^s) = \int_S |\bar{\nu}| \, d\theta \]
where \( \bar{\nu} \) is the average of \( \nu \) over the level sets of the Casimir \( f \). Moreover, \( h_{\mu}(\varphi^s) > 0 \) for all \( s > 0 \) and approaches 1/2 as \( s \to \infty \), while \( h_{\text{top}}(\varphi^s) \equiv 1 \).

This result should be compared with the well-known example of the magnetic flow on a compact hyperbolic surface with magnetic field given by the area form. In this example, as the intensity \( s \) increases the flow becomes “simpler”. Indeed, topological entropy decreases; at \( s = 1 \) we hit the horocycle flow and for \( s > 1 \), the flow has all its orbits closed and becomes integrable. The opposite seems to be happening for our magnetic flow on \( \text{Sol} \). On the other hand, the well-known Rydberg model of a hydrogen atom in a strong magnetic field is believed to exhibit behaviour similar to
that described in Theorem A. We are unaware of any proof, as opposed to evidence, that the Rydberg model has positive Liouville entropy.

The second goal of this paper is to try to explain these drastic changes in the dynamics in terms of changes in the symplectic topology of twisted cotangent bundles.

Let $\Sigma$ be a closed manifold and let $\omega_0$ be the canonical symplectic form of the cotangent bundle $\tau : T^*\Sigma \to \Sigma$. Given a closed 2-form $\sigma$ we let $\omega_\sigma := \omega_0 - \tau^*\sigma$ be the twisted symplectic form determined by $\sigma$. Recall that given a compact set $K$, the displacement energy of $K$ is defined as

$$e(K) := \inf \{ \rho(1, h) : h \in \text{Ham}_c(T^*\Sigma, \omega_\sigma), \ h(K) \cap K = \emptyset \}$$

where $\rho$ is Hofer’s distance and $\text{Ham}_c(T^*\Sigma, \omega_\sigma)$ is the set of compactly supported Hamiltonian diffeomorphisms. Recall also that a compact set $K$ is said to be displaceable if there exists $h \in \text{Ham}_c(T^*\Sigma, \omega_\sigma)$ such that $h(K) \cap K = \emptyset$. Thus $K$ is displaceable iff $e(K)$ is finite. A well known result of M. Gromov [6] asserts that the zero section of $(T^*\Sigma, \omega_0)$ is not displaceable. On the other hand, if $\sigma$ is non-zero and $\Sigma$ has zero Euler characteristic, results of F. Laudenbach and J.-C. Sikorav [8] and L. Polterovich [12] imply that the zero section of $(T^*\Sigma, \omega_\sigma)$ is actually displaceable (if $\sigma$ is non-zero, the zero section of $T^*\Sigma$ ceases to be Lagrangian). Finite displacement energy has important implications. According to a recent result of F. Schlenk [13], if a compact energy level of an autonomous Hamiltonian is displaceable, then it will have finite $\pi_1$-sensitive Hofer-Zehnder capacity which in turn yields almost everywhere existence of contractible closed orbits (i.e. there is a full measure set of values of the energy for which the corresponding energy level has a contractible closed orbit). Let us illustrate this discussion with the following example. Consider a closed hyperbolic 3-manifold and let $\sigma$ be any non-zero closed 2-form. For high values of the energy the magnetic flow will be Anosov, since it can be seen as a perturbation of a geodesic flow on a negatively curved manifold. Thus for high energies, the magnetic flow will have no contractible closed orbits (the magnetic flow will be topologically conjugate to the geodesic flow and it is well known that the latter has no contractible closed geodesics). Schlenk’s result now implies that high energy levels are not displaceable, while low energy levels are by the results of Laudenbach-Sikorav and Polterovich. If we take the closed 3-manifold to have non-zero first Betti number, then it will have non-zero second Betti number and we may choose magnetic fields $\sigma$ with non-zero cohomology classes (monopoles).

Returning to our example on $\text{Sol}$ we note that the geodesic flow of $(\Sigma, ds^2)$ has no contractible closed orbits, but as soon as the magnetic field is switched on, contractible closed orbits appear. These orbits are related to the vanishing of $\tilde{\nu}$, see Remark 3.4 were these observations are proved. It turns out that every compact set in $(T^*(\Delta \setminus \text{Sol}), \omega_0)$ is displaceable. Our last result shows that this is also true for a large class of solvable manifolds.

We say that a Lie group $G$ is completely solvable if it is a closed subgroup of the group of upper triangular matrices with positive diagonal entries. The class of completely solvable groups lies strictly in between nilpotent and solvable groups.
Given a Lie algebra $\mathfrak{g}$, let $L : \Lambda_2(\mathfrak{g}) \to \mathfrak{g}$ be the linear map induced by the Lie bracket, where $\Lambda_2(\mathfrak{g})$ is the second exterior power of $\mathfrak{g}$. Recall that 2-vectors are elements in $\Lambda_2(\mathfrak{g})$ of the form $x \wedge y$ with $x, y \in \mathfrak{g}$.

**Theorem B.** Let $G$ be a simply connected completely solvable group and suppose $\text{Ker} \ L$ is generated by 2-vectors. Let $\Gamma$ be a cocompact lattice and $\Sigma := \Gamma \setminus G$. Then, for any monopole $\sigma$ and any compact set $K \subset (T^*\Sigma, \omega_\sigma)$, $e(K) < \infty$.

Certainly, our example $(T^*(\Delta \setminus \text{Sol}), \omega_\Omega)$ fits the hypotheses of the theorem. For tori, the theorem also follows from the proof of Theorem 3.1 in [5]. It is quite likely that Theorem B holds for any simply connected solvable Lie group with lattice. We do not know of an example of a solvable Lie algebra where $\text{Ker} \ L$ is not generated by 2-vectors. In Section 4 we show how Theorem B applies to compact quotients of some of the standard nilpotent Lie algebras, like the Heisenberg Lie algebra $h_{2n+1}$ and the Lie algebra of upper triangular matrices $u_n$. Finally, in Subsection 4.2 we discuss these results in the context of Aubry-Mather theory and Mañé’s critical values.

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### 2. Preliminaries

Let $\text{Sol}$ be the semidirect product of $\mathbb{R}^2$ with $\mathbb{R}$, with coordinates $(u, y_0, y_1)$ and multiplication

\[(y_0, y_1, u) \ast (y'_0, y'_1, u') = (y_0 + e^u y'_0, y_1 + e^{-u} y'_1, u + u').\]

The map $(y_0, y_1, u) \mapsto u$ is the epimorphism $\text{Sol} \to \mathbb{R}$ whose kernel is the normal subgroup $\mathbb{R}^2$. The group $\text{Sol}$ is isomorphic to the matrix group

\[
\begin{pmatrix}
  e^u & 0 & y_0 \\
  0 & e^{-u} & y_1 \\
  0 & 0 & 1
\end{pmatrix}.
\]

If one denotes by $p_u, p_{y_0}$ and $p_{y_1}$ the momenta that are canonically conjugate to $u$, $y_0$ and $y_1$ respectively, then the functions

\[
\begin{align*}
\alpha_0 &= e^u p_{y_0}, \\
\alpha_1 &= e^{-u} p_{y_1}, \\
\nu &= p_u
\end{align*}
\]

are left-invariant functions on $T^*\text{Sol}$. The closed 2-form

\[(3) \quad \Omega = dy_0 \wedge dy_1\]

is also left-invariant, and consequently,

\[(4) \quad \omega_s = dp_u \wedge du + dp_{y_0} \wedge dy_0 + dp_{y_1} \wedge dy_1 - sdy_0 \wedge dy_1\]

is a left-invariant twisted symplectic form on $T^*\text{Sol}$ for any real number $s$. The Poisson bracket induced by $\omega_s$ is denoted by $\{.,\}_s$. The Poisson brackets of the
coordinate functions are
\begin{align*}
\{\nu, u\}_s &= 1, & \{\alpha_0, \alpha_1\}_s &= s, \\
\{\alpha_0, y_0\}_s &= e^u, & \{\nu, \alpha_0\}_s &= \alpha_0, \\
\{\alpha_1, y_1\}_s &= e^{-u}, & \{\nu, \alpha_1\}_s &= -\alpha_1,
\end{align*}
and all others vanish. Define the Hamiltonian \( H \) on \( T^*\text{Sol} \) by
\begin{equation}
2H = \nu^2 + \alpha_0^2 + \alpha_1^2,
\end{equation}
so that when \( s = 0 \), \( H \) is the Hamiltonian of the left-invariant Riemannian metric mentioned in the Introduction. The equations of the magnetic flow induced by \( H \) are
\begin{equation}
X_H = \begin{cases}
\dot{u} &= \nu, & \dot{\nu} &= -\alpha_0^2 + \alpha_1^2, \\
\dot{y}_0 &= e^u\alpha_0, & \dot{\alpha}_0 &= -\alpha_1s + \nu\alpha_0, \\
\dot{y}_1 &= e^{-u}\alpha_1, & \dot{\alpha}_1 &= \alpha_0s - \nu\alpha_1,
\end{cases}
\end{equation}
or \( X_H(\bullet) = \{H, \bullet\}_s \).

The Lie algebra of left-invariant functions on \( T^*\text{Sol} \) has a non-trivial centre generated by the Casimir
\begin{equation}
f = s\nu + \alpha_0\alpha_1.
\end{equation}

**Remark 2.1.** The 2-form \( \Omega \) defines a central extension of \( \text{Sol}: \mathbb{R} \hookrightarrow G \twoheadrightarrow \text{Sol} \). The Lie algebra \( \mathfrak{g} \) of \( G \) is isomorphic to the Lie algebra with basis \( s, \nu, \alpha_0, \alpha_1 \) and Lie bracket \( \{\cdot, \cdot\}_s \). The equations of the magnetic Hamiltonian \( H \) (equation 7) may be viewed as the symplectic reduction of a Kaluza-Klein metric Hamiltonian on \( T^*G \) at a non-zero level of momentum. From this point of view, \( f \) and \( s \) are Casimirs of the Poisson bracket on \( \mathfrak{g}^* \).

Actually, the group \( G \) may be identified with one of the solvable 4-dimensional geometries, namely \( \text{Sol}^3_1 \) [15]. It has a matrix representation
\[
\begin{pmatrix}
1 & y & z \\
0 & e^t & x \\
0 & 0 & 1
\end{pmatrix},
\]
where \( x, y, z, t \in \mathbb{R} \). Via the Kaluza-Klein metric, Theorem A could be reinterpreted as follows: the geodesic flow on compact quotients of \( \text{Sol}^3_1 \) has positive Liouville entropy and is not completely integrable.

### 3. Analysis of the Magnetic Flow

Since the Hamiltonian vector field \( X_H \) (equation 7) is left-invariant, the vector field factors onto a vector field \( E_h \) on \( \mathfrak{s}^* \) through the projection map \( T^*\text{Sol} \twoheadrightarrow \mathfrak{s}^* \) induced by the left-framing of \( T^*\text{Sol} \). The *Euler* vector field \( E_h \) is a Hamiltonian vector field on \( \mathfrak{s}^* \) equipped with the Lie bracket \( \{\cdot, \cdot\}_{\mathfrak{s}} \). The Hamiltonian \( h: \mathfrak{s}^* \to \mathbb{R} \) is the Hamiltonian which induces \( H \). It is clear the dynamics of \( X_H \) can be reconstructed from the dynamics of \( E_h \).
Let $S = h^{-1}(\frac{1}{2})$ be the unit sphere in $\mathfrak{s}^*$; the unit-sphere bundle $H^{-1}(\frac{1}{2})$ is naturally diffeomorphic to $\text{Sol} \times S$. The functions $\nu, \alpha_0, \alpha_1$ will be regarded as coordinate functions on $\mathfrak{s}^*$. Define the standard smooth measure $\theta$ on $S$ by
\begin{equation}
4\pi \times \theta = \nu d\alpha_0 \wedge d\alpha_1 + \alpha_0 d\alpha_1 \wedge d\nu + \alpha_1 d\nu \wedge d\alpha_0|_S.
\end{equation}
The measure $\theta$ may be decomposed as $\theta = \mathfrak{m} \wedge \mathfrak{m}_f$. The measure $\mathfrak{m}$ is defined so that for each connected component of $f^{-1}(c) \cap S$, call it $f_c$, $\mathfrak{m}$ induces a smooth probability measure on $f_c$ that is $E_h$-invariant. Let $\bar{\nu} : S \to \mathbb{R}$ be defined by
\begin{equation}
\bar{\nu}(\mu) := \int_{f(\mu)} \nu \ d\mathfrak{m} \quad \forall \mu \in S,
\end{equation}
that is, $\bar{\nu}(\mu)$ is the mean value of $\nu$ along the connected component of the level set of $f|S$ containing $\mu$.

Here is a more prosaic definition of $\mathfrak{m}$. Because the vector field $E_h$ preserves the volume form $d\nu \wedge d\alpha_0 \wedge d\alpha_1$ on $\mathfrak{s}^*$, and $E_h$ is tangent to the unit sphere $S$, the vector field $E_h|S$ is Hamiltonian with respect to the symplectic form $\theta$ (the Hamiltonian is $g = 4\pi \times f$). Therefore, if $c$ is a non-trivial regular value of the integral $f$, then a neighbourhood of $f_c$ in $S$ admits action-angle coordinates $(I, \phi \mod 1)$ such that $g = g(I)$,
\begin{equation}
E_h = \begin{cases}
\dot{\phi} = \frac{\partial g(I)}{\partial I}, \\
\dot{I} = 0,
\end{cases}
\end{equation}
and $\theta = d\phi \wedge dI$. The measure $\mathfrak{m}$ in these coordinates is
\begin{equation}
\mathfrak{m} = d\phi,
\end{equation}
while
\begin{equation}
\bar{\nu} = \int_0^1 \nu(\phi, I) \ d\phi.
\end{equation}

**Proposition 3.1.** For $s \neq 0$, $\bar{\nu} : S \to \mathbb{R}$ is a continuous, $\psi^s$-invariant function which is real-analytic off the set of non-elliptic singular levels of $f|S$.

**Proof.** The real-analyticity of $\bar{\nu}$ on the regular-point set follows from the fact that $\bar{\nu}$ and $f$ are real-analytic and the action-angle coordinates are real-analytic.

**Case 1, $|s| \neq 0, 1$:** When $|s| < 1$, $f$ has a pair of peaks (resp. pits) at $\alpha_0 = \alpha_1 = \pm \alpha, \nu = s$ (resp. $\alpha_0 = -\alpha_1 = \pm \alpha, \nu = -s$) where $\alpha = \sqrt{\frac{1}{2}(1 - s^2)}$. When $|s| \geq 1$, $f$ has a single peak (resp. pit) at $\alpha_0 = \alpha_1 = 0, \nu = 1$ (resp. $\alpha_0 = \alpha_1 = 0, \nu = -1$). These critical points are all non-degenerate for $|s| \neq 0, 1$.

**Case 1a, elliptic singularity:** Let $p \in S$ be a peak or pit for $f|S$, hence an elliptic singularity of $E_h$ on $S$. There is a canonical system of coordinates $(x, y)$ defined on a neighbourhood of $p$ such that the Hamiltonian $g$ of $E_h|S$ is in Birkhoff normal form:
\begin{equation}
g(x, y) = g_1 I + g_2 I^2 + \cdots, \quad I = \frac{1}{2} (x^2 + y^2), \quad x + iy = \sqrt{2} e^{2\pi i \phi}.
\end{equation}
FIGURE 1. S seen from the point of view of f, 0 < |s| < 1.

It is well-known that g has a formal Birkhoff normal form; Zung has proven that the formal Birkhoff normal form converges when g is completely integrable [16]. Inspection of equations (11–13) shows that \( \bar{\nu} \) may be written as

\[
\bar{\nu}(\mu) = \frac{1}{T} \times \int_{0}^{T} \nu \circ \psi^{s}_{t}(\mu) \, dt, \quad \forall \mu \in S,
\]

where \( \psi^{s} \) is the Euler flow of \( E_{h}|S \) and T is the period of the orbit through \( \mu \). In an action-angle chart \( T = \frac{\partial I}{\partial g} \), and one sees that T extends over the critical point at \( I = 0 \) as a real-analytic function. Therefore, define

\[
t \cdot \mu = \psi^{s}_{tT(\mu)}(\mu), \quad \forall t \in S^{1} = \mathbb{R}/\mathbb{Z}.
\]

This defines a real-analytic action of \( S^{1} \) on a neighbourhood of the critical point \( p \). In angle-action coordinates, this action is just \( t \cdot (\phi, I) = (\phi + t \mod 1, I) \). The integral in equation (15) is then

\[
\bar{\nu}(\mu) = \int_{0}^{1} \nu(t \cdot \mu) \, dt, \quad \forall \mu \in S
\]

i.e. \( \bar{\nu} \) is the average of \( \nu \) under the real-analytic action of \( S^{1} \). This shows that \( \bar{\nu} \) is real-analytic in a neighbourhood of the elliptic critical point \( p \).

Case 1b, hyperbolic singularity: In this case, it is known that there are canonical co-ordinates \((x, y)\) which send the hyperbolic fixed point to \((0, 0)\), its stable and unstable manifolds to the \( x \)- and \( y \)-axes respectively, and in which the hamiltonian is of the form

\[
g = g_{1}\tau + g_{2}\tau^{2} + \cdots, \quad \text{where } \tau = xy.
\]

In this coordinate system, the flow is simply

\[
\psi^{s}_{t}(x, y) = (xe^{-t\omega(\tau)}, ye^{t\omega(\tau)})
\]

where \( \omega = \frac{\partial g(x)}{\partial x} \). Without loss of generality, one may assume that the coordinate system is defined on a square centred on the origin, as in figure 2. For a point \( p \) along the right-hand face of the square above the \( x \)-axis, let \( q \) be the corresponding point along the orbit which intersects the top face, with the convention that when \( p = P \) lies on the stable manifold, the corresponding point is \( q = Q \) on the unstable
manifold. The orbit consists of two segments: the segment $\overline{pq}$ inside the box, and the segment $\overline{qp}$ lying in the complement of the box. The period $T = T(p)$ of this orbit is the sum of the time $T_0(p)$ that the orbit spends on the segment $\overline{pq}$ plus the time $T_1(p)$ that the orbit spends on the segment $\overline{qp}$. The time $T_1(p)$ is a real-analytic function that approaches the finite limit $T_1(P)$ as $p \to P$; $T_0(p)$ is also real-analytic and approaches $+\infty$ as $p \to P$.

From equation (15), one has the equation

$$\tilde{\nu}(p) = \frac{T_0}{T^2} \times \int_0^{T_0} \nu \circ \psi_t(p) \, dt + \frac{T_1}{T^2} \times \int_{T_0}^T \nu \circ \psi_t(p) \, dt.$$  

The second term is bounded by a constant times $\frac{T_1}{T}$, which converges to 0 as $p \to P$. The first term converges to $\nu(0) = \tilde{\nu}(0) = \tilde{\nu}(P)$ as $p \to P$.

A similar, but slightly more involved, argument shows that if $p$ lies in the right-hand face of the square below the $x$-axis, then $\tilde{\nu}(p)$ converges to $\tilde{\nu}(P)$, also. By symmetry and invariance of $\tilde{\nu}$ under $\psi_t$, this proves that $\tilde{\nu}$ is a continuous function in a neighbourhood of the hyperbolic singularity and its stable and unstable manifold.

The reader may verify by direct computation that, if $\nu = y$ in the coordinate box, then $\frac{\partial \nu}{\partial y}$ diverges to $+\infty$ as $p \to P$ ($y \to 0$).

Case 2, $|s| = 1$: In this case, $f|S$ has two critical points - at $\alpha_0 = \alpha_1 = 0, \nu = \pm 1$ - that are both degenerate. The argument of case 1a may be adapted to show that $\tilde{\nu}$ is a continuous function at each of these critical points. \[\square\]
Here are some further properties of \( \bar{\nu} \). Since \( \bar{\nu} \) is \( \psi^s \)-invariant, one may view it as a function defined on the image of \( f|S \). In this case, it makes sense to say that \( \bar{\nu} \) is monotone increasing.

**Proposition 3.2.** If \( s > 1 \) (resp. \( s < -1 \)), then \( \bar{\nu} \) is a monotone increasing (resp. decreasing) function that vanishes only on the zero level of \( f|S \).

**Proof.** The symmetry of \( f \) and the symplectic form \( \theta \) dictate that \( \bar{\nu}(c) \) be an odd function of \( c \). Therefore \( \bar{\nu} \) always vanishes on the zero level of \( f \).

Let us suppose that \( s > 0 \); the case where \( s < 0 \) is analogous. From the previous proposition, it suffices to prove that \( \bar{\nu} \) is monotone increasing on the regular levels of \( f|S \). From equation (13), one sees that if \( \nu_2(\phi, I) > \nu_1(\phi, I) \) for all \( \phi, I \), then \( \bar{\nu}_2(I) > \bar{\nu}_1(I) \) for all \( I \). For our purposes, let \( \nu_1 = \nu \) and let \( \nu_2 = \nu \circ \gamma_\tau \) where \( \gamma \) is a gradient-like flow for \( f|S \) – that takes the form \( \gamma_\tau(\phi, I) = (\phi, I + \tau) \) in angle-action coordinates – and \( \tau > 0 \) is a small positive number. That is, if the derivative of \( \nu \) in the direction of the gradient-like flow \( \gamma \) is positive, then \( \bar{\nu} \) is a monotone increasing function. Let us remark that to test the positivity of this directional derivative, it suffices to use any gradient-like vector field; in particular, it suffices to compute the directional derivative of \( \nu \) with respect to the standard gradient vector field of \( f|S \).

One computes that

\[
\langle d\nu, \nabla(f|S) \rangle = \begin{bmatrix} \alpha_0 & \alpha_1 \\ s & \nu \\ \nu & s \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}.
\]

The symmetric matrix is positive definite if \( s > 0 \) and \( s^2 > \nu^2 \). If \( s > 1 \), then the matrix is always positive definite, whence the right-hand side vanishes only at \( \alpha_0 = \alpha_1 = 0, \nu = \pm 1 \). This proves the proposition.

\( \square \)

**Remark 3.3.** When \( |s| < 1 \), the function \( \bar{\nu} \) cannot be monotone increasing. As one can see in figure (1), \( \bar{\nu} \) attains its maximum value of unity at the hyperbolic fixed point \( \alpha_0 = \alpha_1 = 0, \nu = 1 \); at the same point \( f = s \). On the other hand, at the elliptic critical points \( \alpha_0 = \alpha_1 = \pm \sqrt{\frac{1}{2}(1 - s^2)}, \nu = s \), \( \bar{\nu} \) attains a value of \( s \) while \( f = \frac{1}{2}(1 + s^2) \). Thus: \( s < \frac{1}{2}(1 + s^2) \) while \( 1 = \bar{\nu}(s) > \bar{\nu}(\frac{1}{2}(1 + s^2)) = s \). Numerical calculations do suggest that \( \bar{\nu} \) is monotone increasing on \([-s, s]\) and decreasing on the two complementary subintervals (see figure 3).

A related issue concerns the monotone nature of the function \( s \mapsto h_\mu(\varphi^s) \). In figure (1) we give evidence from numerical computations that this function is a monotone function on \( (-\infty, 0) \) and \( [0, \infty) \).

The function \( \bar{\nu} \) is approximated by integrating the Euler equations in the almost canonical variables \( \nu, \phi \) (see the discussion around equation (33)) using the Runge-Kutta 4-step method and averaging \( \nu \) over a numerically computed period. The function \( h_\mu(\varphi^s) \) is approximated by numerically integrating \( \bar{\nu} \) over a grid using Simpson’s rule. Data and source code is available from [here](#).
Figure 3. The function $\bar{\nu}$ as a function of $f$ for selected values of $s$. Note the loss of differentiability at the hyperbolic critical level $f = s$ and the lack of monotonicity.

Figure 4. The function $h_\mu(\varphi^s)$ as a function of $s$. Inset (left): on the interval $[0, 5 \times 10^{-3}]$; Inset (right): on the interval $[0, 1 \times 10^{-4}]$.

Remark 3.4. Consider an orbit of the magnetic flow on $\text{Sol}$ that projects onto a closed orbit of $E_h$. From equation (7) it is clear that $u$ is a periodic function of time if and only if $\bar{\nu} = 0$. Left-invariance—or an easy check using (7)—gives that the functions $p_{y_0} + sy_1$ and $p_{y_1} - sy_0$ are first integrals in $\text{Sol}$. Since $\alpha_0$ and $\alpha_1$ are periodic, we
conclude that \( p_{y_0} = e^{-u_\alpha_0} \) and \( p_{y_1} = e^{u_\alpha_1} \) are periodic if \( u \) is periodic. Thus, if \( s > 0 \) and \( \bar{\nu} = 0 \), the orbit of the magnetic flow on \( \text{Sol} \) is periodic. Since there are always closed orbits of \( E_h \) with \( \bar{\nu} = 0 \) we conclude that for \( s > 0 \) the magnetic flow on \( \Delta \setminus \text{Sol} \) always has contractible closed orbits.

Observe that for the geodesic flow \((s = 0)\) no closed orbit is contractible, since if \( u \) is periodic, \( y_0 \) and \( y_1 \) must diverge linearly.

3.1. Cocompact subgroups of \( \text{Sol} \). To compute the metric entropy of the magnetic flow, it is useful to view the lattice subgroup \( \Delta \) of \( \text{Sol} \), especially the diagonalizing transformation \( P \) described in the introduction, intrinsically.

Given a lattice subgroup \( \Delta \) of \( \text{Sol} \), there is an exact sequence \( \mathbb{Z}^2 \hookrightarrow \Delta \to \mathbb{Z} \) induced by the exact sequence \( \mathbb{R}^2 \hookrightarrow \text{Sol} \to \mathbb{R} \). The quotient group \( \mathbb{Z} \) acts on \( \mathbb{Z}^2 \) via a representation \( \rho : \mathbb{Z} \to SL(2, \mathbb{Z}) \). The generator \( \rho(1) = A \) from the introduction is a hyperbolic matrix with eigenvalues \( \lambda \pm 1, |\lambda| > 1 \).

In terms of the coordinate system (equation (1)), the group \( \Delta \) can be described as follows. Let \( F = \mathbb{Q}(\lambda) \) be the quadratic number field obtained by adjoining \( \lambda \) to the rationals. The integers of \( F \), \( \mathcal{O} \), is isomorphic to \( \mathbb{Z}^2 \) as an abelian group, and the unit group of \( \mathcal{O} \), \( U \), acts as an automorphism group. The group \( \Delta \) is naturally isomorphic to a finite-index subgroup of the semi-direct product \( U \rtimes \mathcal{O} \). We shall henceforth identify \( \Delta \) with a subgroup of \( U \rtimes \mathcal{O} \).

The volume of \( \Delta \triangleleft U \rtimes \mathcal{O} \) can be defined to be

\[
\text{vol} \Delta := \log |\lambda| \times \det \begin{bmatrix} a_0^{(0)} & a_0^{(1)} \\ a_1^{(0)} & a_1^{(1)} \end{bmatrix},
\]

where \( a_0, a_1 \) generate \( \Delta \cap \mathcal{O} \) and \( a^{(j)} \) is the \( j \)-th conjugate of \( a \). One can see that \( \text{vol} \Delta \) is the determinant of the injection of \( \mathbb{Z}^2 \hookrightarrow \Delta \) into \( \text{Sol} \) defined in the introduction; indeed, the matrix \( P \) introduced there is effectively the matrix on the right-hand side of equation (22). It is clear that \( \text{vol} \Delta \) is the volume of a fundamental region for \( \Delta \) in \( \text{Sol} \) relative to the volume form \( du \wedge dy_0 \wedge dy_1 \). That is

\[
\text{vol}(\Delta \setminus \text{Sol}) = \text{vol} \Delta.
\]

Let \( \Sigma = \Delta \setminus \text{Sol} \) and let \( \mu \) be the \( X_H \)-invariant probability measure on \( \Sigma \times S \) induced by \( \omega_s^\Sigma = -du \wedge dy_0 \wedge dy_1 \wedge d\nu \wedge d\alpha_0 \wedge d\alpha_1 \), i.e.

\[
\mu = \frac{1}{\text{vol} \Delta} \times du \wedge dy_0 \wedge dy_1 \wedge \theta.
\]

3.2. Metric Entropy of the Magnetic Flow.

Theorem A. Let \( s \neq 0 \) and \( \varphi^s : \mathbb{R} \times \Sigma \times S \to \Sigma \times S \) be the magnetic flow with infinitesimal generator \( X_H \). The metric entropy of the time-1 map \( \varphi_1^s \) is

\[
h_\mu(\varphi_1^s) = \int_S |\bar{\nu}| \, d\theta.
\]

\[\text{[The field } \mathbb{Q}(\lambda) \text{ is a quadratic number field and so equals } \mathbb{Q}(\sqrt{d}) \text{ for some positive, square-free integer } d. \text{ The map } \sqrt{d} \mapsto -\sqrt{d} \text{ induces a field automorphism, and the image of } a = a^{(0)} \text{ under this automorphism is referred to as a conjugate of } a \text{ and denoted by } a^{(1)}.\]
Therefore, since \( \nu \) is non-zero on a positive measure set, \( h_\mu(\varphi_1^t) > 0 \). Moreover, \( h_\mu(\varphi_1^s) \) approaches \( 1/2 \) as \( s \to \infty \) and \( h_{\text{top}}(\varphi_1^t) \equiv 1 \).

**Remark 3.5.** As is proven in Proposition 3.1, \( \nu \) is a continuous function that is real-analytic on the complement of the critical levels of \( f|S \). Therefore \( \nu \) is non-zero on a set of full measure. It is almost certain that \( \nu \) vanishes only on one level of \( f|S \); Proposition 3.2 proves this when \( |s| > 1 \).

**Proof.** First, consider a flow \( \varphi : \mathbb{R} \times \text{Sol} \times T^1 \to \text{Sol} \times T^1 \) which is a skew product over a translation

\[
\varphi_t(g, \phi) = (\gamma(t, g, \phi), \phi + at \mod 1) \quad \forall g \in \text{Sol}, \phi \in T^1.
\]

If \( \varphi \) is assumed to be left-invariant, then the cocycle \( \gamma \) satisfies \( \gamma(t, g, \phi) = g \gamma(t, 1, \phi) \).

Therefore, if \( T = 1/a \), then

\[
\varphi_T(g, \phi) = (g \gamma(1, \phi), \phi \mod 1) \quad \forall g \in \text{Sol}, \phi \in T^1,
\]

where \( \gamma(\phi) = \gamma(T, 1, \phi) \). Therefore, for all \( n \in \mathbb{Z} \),

\[
\varphi_{nT}(\Delta g, \phi) = (\Delta g \gamma(\phi)^n, \phi \mod 1) \quad \forall g \in \text{Sol}, \phi \in T^1,
\]

The cocycle \( \gamma(\phi) \in \text{Sol} \) either takes values in the non-hyperbolic subgroup \( \mathbb{R}^2 \) or it has a non-trivial projection to \( \mathbb{R} \). In the former case, \( \varphi_T \) has zero entropy. In the latter case, \( T\text{Sol} \) splits into 3 complementary, left-invariant line-bundles \( E^+, E^- \) and \( E^0 \). These line bundles are determined by their value at the identity of \( \text{Sol} \). If we identify \( T\text{Sol} \) as the Lie algebra \( \mathfrak{s} \), then \( E^+ \) is the unstable subspace, \( E^- \) is the stable subspace and \( E^0 \) is the centralizer of \( \text{Ad}g \), respectively. Let \( \lambda_+(g) \) be the log of the largest eigenvalue of \( \text{Ad}g \), \( g \in \text{Sol} \). One sees using Pesin’s formula that

\[
h_{\mu_c}(\varphi_1) = \frac{1}{T} \times \int_0^1 \lambda_+(\gamma(\phi)) \, d\phi,
\]

where \( \mu_c = \frac{1}{\text{vol} \Delta} \times du \wedge dy_0 \wedge dy_1 \wedge d\phi \) is a \( \varphi \)-invariant probability measure on \( \Sigma \times T^1 \).

A simple computation shows that \( \lambda_+(g) \) is the projection \( g \mapsto |u(g)| \) induced by \( \text{Sol} \xrightarrow{u} \mathbb{R} \). In addition, if we observe that \( \gamma(\phi) = \gamma(0, 1, \phi)^{-1} \gamma(T, 1, \phi) \) and use the fact that \( u \) is a group homomorphism, then

\[
|\Delta u| = |u(\gamma(\phi))| = \lambda_+(\gamma(\phi)),
\]

where \( \Delta u \) is the change in \( u \) over the time interval \([0, T]\).

Let us turn to the magnetic flow: Let \( c \) be a regular value of \( f|S \) and introduce action-angle variables in a neighbourhood of \( f_c \subset S \). The flow, \( \varphi^s \), of \( X_H \) restricted to \( \Sigma \times f_c \) is of the form described by equation (25), with \( a = \frac{\partial g}{\partial t} \), see equation (12). The Liouville measure \( \mu \) on \( \Sigma \times S \) induces the invariant conditional probability measure \( \mu_c \) on \( \Sigma \times f_c \). Inspection of equation (7) shows that over the period \( T \), \( u \) changes by

\[
\Delta u = \int_0^T \nu(t) \, dt.
\]

Since \( \nu \) is a periodic function, the integral for \( \Delta u \) is independent of the angle variable \( \phi \). Equations (28, 29) therefore show that \( |\Delta u|/T \) is the metric entropy of \( \varphi^s|\Sigma \times f_c \)
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with respect to the conditional probability measure \( \mu_c \). Using equation (11) in action-angle coordinates, one obtains

\[
\frac{\Delta u}{T} = \frac{1}{T} \times \int_0^1 \nu(\phi, I) \, d\phi \times \frac{\partial I}{\partial g} = \int_0^1 \nu(\phi, I) \, d\phi = \bar{\nu},
\]

since \( T = \frac{\partial I}{\partial g} \). Therefore, we can integrate to obtain the metric entropy of the magnetic flow on the unit-sphere bundle

\[
h_{\mu}(\varphi^s) = \int_{\Sigma \times S} |\Delta u| \, d\mu = \int_S |\bar{\nu}| \, d\theta.
\]

This proves equation (24). Let us prove the remaining two points in Theorem A.

Topological entropy. We note that the arguments above also imply that the sum of the non-negative Liapunov exponents of \( \varphi^s \) is given by \( |\bar{\nu}| \leq 1 \). Thus by Ruelle’s inequality and the variational principle for topological entropy we see that \( h_{\text{top}}(\varphi^s) \leq 1 \). Since the flow on the set \( p_u = \pm 1, p_{y_0} = p_{y_1} = 0 \) is the same for all \( s \) and carries entropy 1 we conclude that \( h_{\text{top}}(\varphi^s) \equiv 1 \).

The limit of metric entropy. To compute \( \lim_{s \to \infty} h_{\mu}(\varphi^s) \), note that \( \frac{1}{s} \times f = \nu + \alpha_0 \alpha_1 \times \frac{1}{s} \). Therefore, as \( s \to \infty \), the regular level sets of \( f|S \) converge uniformly in the \( C^1 \) topology to the level sets of \( \nu \), i.e. the regular level sets of \( f|S \) converge to circles at a constant height off the \( \alpha_0 - \alpha_1 \) plane (and \( f|S \) has only the points \( \alpha_0 = \alpha_1 = 0, \nu = \pm 1 \) as critical points).

Let us coordinatize \( S - \{(0, 0, \pm 1)\} \) by spherical coordinates

\[
\alpha_0 = \cos(2\pi \xi) \sin(\eta), \alpha_1 = \sin(2\pi \xi) \sin(\eta), \nu = \cos(\eta) \quad 0 < \eta < \pi, 0 \leq \xi < 1.
\]

The angle \( \xi \) is the normalized longitudinal angle which vanishes along \( \{\alpha_1 = 0, \alpha_0 > 0\} \) and has \( \frac{\partial \xi}{\partial \alpha_1} > 0 \) along the same privileged longitude. In spherical coordinates, the normalized area form is

\[
\theta = \frac{1}{2} \times \sin(\eta) \, d\eta \wedge d\xi.
\]

For \( s > 1 \), we normalize the action-angle coordinates \( (I, \phi) = (I_s, \phi_s) \) on \( S - \{(0, 0, \pm 1)\} \) as follows: first, \( \phi_s = 0 \) and \( \frac{\partial \phi_s}{\partial \alpha_1} > 0 \) along the privileged longitude \( \{\alpha_1 = 0, \alpha_0 > 0\} \); second, \( I_s(\mu) \) is defined to be the area of the sublevel set \( \{f \leq f(\mu)\} \) in \( S \). The above paragraph shows that as \( s \to \infty \), \( I_s \) converges to the function \( I_\infty \) which gives the area of the region in \( S \) below height \( \nu \). A computation shows that \( I_\infty = \frac{1 + \nu}{2} \). On the other hand, \( \phi_s \) converges to the normalized longitudinal angle \( \xi \).

Inspection of equations (12,13) shows that the mean value of \( \nu \) averaged with respect to the measure \( d\phi_s \) converges to \( \nu \) as \( s \to \infty \). This convergence is in the uniform \( C^0 \) topology. Therefore

\[
\lim_{s \to \infty} h_{\mu}(\varphi^s_1) = \int_0^1 \int_0^\pi |\cos(\eta) \sin(\eta)| \, d\eta \, d\xi \times \frac{1}{2} = \frac{1}{2},
\]

as asserted. □
3.3. A variation. There is an interesting variation of the previous example. Consider the group $G = \text{Sol} \times \mathbb{R}$ and the left-invariant 2-form given by $\Omega := du \wedge dt$, where $t$ denotes the variable on the $\mathbb{R}$-factor. We consider on $G$ the left-invariant metric given by $ds^2 + dt^2$ and the cocompact lattice $\Delta \times \mathbb{Z}$. The magnetic flow $\varphi^s$ on the compact quotient thus obtained has the following remarkable properties for $s \neq 0$ (as before $s$ is the intensity):

- $h_{top}(\varphi^s) = 0$ for $s \neq 0$. This shows that topological entropy may be discontinuous when a twist in the symplectic structure is introduced;
- $\varphi^s$ is completely integrable with real analytic integrals. If we let $\tau := pt$, then the integrals are $\alpha_0 \alpha_1$, $\alpha_0 e^{-\tau/s}$ (the two Casimirs) and $\tau - su$, which can be made invariant under the lattice just by composing with a suitable periodic function.

We leave the details of the proofs of these claims to the reader, but they do follow in a straightforward fashion from an analysis quite similar to the one done in this section.

4. Proof of Theorem B

We first prove the following easy lemma.

**Lemma 4.1.** Let $\mathfrak{g}$ be a Lie algebra such that $\text{Ker} \ L$ is generated by 2-vectors. Let $\Omega$ be an antisymmetric bilinear form on $\mathfrak{g}$ such that $\Omega(x, y) = 0$ for all $x, y$ with $[x, y] = 0$. Then $\Omega$ is exact, that is, there exists $b \in \mathfrak{g}^*$ such that $\Omega(x, y) = b([x, y])$ for all $x, y \in \mathfrak{g}$.

**Proof.** Let $L^* : \mathfrak{g}^* \rightarrow (\Lambda_2(\mathfrak{g}))^*$ be the dual of $L : \Lambda_2(\mathfrak{g}) \rightarrow \mathfrak{g}$. It suffices to show that $\Omega$ is in the image of $L^*$. But the image of $L^*$ coincides with the annihilator of $\text{Ker} \ L$, so it suffices to check that $\Omega(q) = 0$ for all $q \in \text{Ker} \ L$. But if $\text{Ker} \ L$ is generated by 2-vectors we may write $q = \sum_i x_i \wedge y_i$ with $x_i \wedge y_i \in \text{Ker} \ L$. Thus $\Omega(q) = \sum_i \Omega(x_i, y_i)$. But since $[x_i, y_i] = 0$, $\Omega(x_i, y_i) = 0$ by hypothesis and $\Omega(q) = 0$.

We now break the proof of Theorem B into a few simple steps.

1. Let $\sigma$ be a closed 2-form in $\Sigma$ with non-zero cohomology class. By a theorem of A. Hattori [7] (which in turn is a generalization of a theorem of K. Nomizu for nilmanifolds [10]), there exists a left-invariant closed 2-form $\Omega$ cohomologous to $\sigma$. This is the only part of the proof in which we use that $G$ is completely solvable. We denote by the same symbol $\Omega$ the 2-form on $G$ or on $\Sigma$.

   Write $\sigma = \Omega + d\theta$ for some smooth 1-form $\theta$. The fibrewise shift $(x, p) \mapsto (x, p - \theta)$ takes compact sets to compact sets and is a symplectomorphism between $(T^*\Sigma, \omega_\sigma)$ and $(T^*\Sigma, \omega_\Omega)$. Hence, from now on we may suppose that the monopole is given by a closed left-invariant 2-form $\Omega$.

2. We identify $T^*G$ with $G \times \mathfrak{g}^*$ using left translations. Smooth left-invariant functions on $T^*G$ are then identified with $C^\infty(\mathfrak{g}^*)$. The twisted symplectic structure $\omega_\Omega$ determines a Poisson bracket $\{ \cdot, \cdot \}_{\Omega}$. Given $f, g \in C^\infty(\mathfrak{g}^*)$ we
have

\[ \{f, g\}_\Omega(m) = m([d_m f, d_m g]) + \Omega(d_m f, d_m g) \]

for every \( m \in g^* \) where \( d_m f, d_m g \in g \) using the canonical isomorphism \((g^*)^* = g\). This formula is a simple consequence of the definition of the twisted symplectic form on \( T^*G \) plus left-invariance.

(3) If \( f, g \in C^\infty(g^*) \) then, they induce functions on \( T^*\Sigma = \Sigma \times g^* \), which only depend on the \( g^* \)-variables and their Poisson brackets is computed, of course, also using (35).

(4) Since the cohomology class of \( \Omega \) is not zero, we now invoke Lemma 4.1 to obtain two vectors \( x, y \in g \) such that \( [x, y] = 0 \) but \( \Omega(x, y) \neq 0 \).

(5) The vectors \( x, y \) are obviously linearly independent. Consider a basis \( \{e_1 = x, e_2 = y, e_3, \ldots, e_n\} \) of \( g \) and let \( \{e_1^*, \ldots, e_n^*\} \) be its dual basis. Given \( m \in g^* \) write \( m = \sum_i m_i e_i^* \) and let \( f_i(m) := m_i \). Using (35) we have

\[ \{f_1, f_2\}_\Omega(m) = \Omega(x, y) \neq 0. \]

(6) Consider \( f_1 \) as Hamiltonian on \( T^*\Sigma \). Along the Hamiltonian flow of \( f_1 \) we have

\[ \dot{m}_2 = \Omega(x, y) \neq 0 \]

which readily implies that any compact set in \( T^*\Sigma \) may be displaced using the Hamiltonian flow of a suitable cut-off of \( f_1 \). This finishes the proof of Theorem B.

4.1. Examples. Consider the Heisenberg Lie algebra \( h_{2n+1} \) with basis

\[ \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\} \]

and non-zero brackets \([x_i, y_i] = z \) for \( i = 1, \ldots, n \). The image of \( L : \Lambda_2(h_{2n+1}) \to h_{2n+1} \) is obviously one dimensional and generated by \( z \). All the vectors \( x_i \wedge x_j, y_i \wedge y_j, x_i \wedge z, y_i \wedge z \) and \( x_i \wedge y_j \) for \( i \neq j \) are in the Kernel of \( L \). Additional \( n - 1 \), 2-vectors in the Kernel of \( L \) are given by

\[ (x_i + y_i) \wedge (x_1 + y_1) \]

for \( i = 2, \ldots, n \). Thus \( \text{Ker} L \) is generated by 2-vectors.

Another well known nilpotent Lie algebra \( g_{2n+1} \) is given by a basis

\[ \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\} \]

and non-zero brackets \([z, x_i] = y_i \) for \( i = 1, \ldots, n \). Here the kernel of \( L \) is generated by \( x_i \wedge x_j, y_i \wedge y_j, x_i \wedge y_j \) and \( z \wedge y_i \).

Finally consider the nilpotent Lie algebra \( u_n \) of upper triangular \( n \times n \) matrices (with zeros along the diagonal). If \( e_{ij} \) denotes the matrix which has a 1 in its \((i, j)\)-entry and zero everywhere else, then the non-zero brackets are \([e_{ij}, e_{jl}] = e_{il} \) where \( i < j < l \). As in the case of the Heisenberg Lie algebra it is easy to check that \( \text{Ker} L \) is generated by 2-vectors and we leave this to the reader.
The simply connected nilpotent Lie groups associated with these Lie algebras admit cocompact lattices and monopoles. The corresponding second Betti numbers are:

\[ b_2(\mathfrak{h}_{2n+1}) = 2n^2 - n - 1, \quad n \geq 2; \]
\[ b_2(\mathfrak{g}_{2n+1}) = n(n+1); \]
\[ b_2(\mathfrak{u}_n) = \frac{(n-2)(n+1)}{2}. \]

To all of them Theorem B applies.

4.2. Relation with Mañé’s critical value. Let \( M \) be a closed manifold and \( \sigma \) a non-zero closed 2-form. We say that a compact set \( K \subset (T^*M, \omega_\sigma) \) is stably displaceable if \( K \times S^1 \) is displaceable in \( (T^*M \times T^*S^1, \omega_\sigma \oplus \omega_0) \). Let \( g \) be a Riemannian metric on \( M \). Following Schlenk in [13] we define \( d(g, \sigma) \) as the supremum of the values of \( k \in \mathbb{R} \) such that the set of \( (x, p) \in T^*M \) with \( |p|^2_x \leq 2k \) is stably displaceable.

The results of Laudenbach-Sikorav [8] and Polterovich [12] that we mentioned in the Introduction imply that \( d(g, \sigma) > 0 \). We have introduced stable displacement to include the case in which the Euler characteristic of \( M \) is different from zero. Note that this was unnecessary before because all the manifolds we discussed had vanishing Euler characteristic.

Suppose now that \( \sigma \) is weakly exact, that is, its lift \( \tilde{\sigma} \) to the universal covering \( \tilde{M} \) of \( M \) is exact. Mañé’s critical value \( c(g, \sigma) \) is defined as [3]:

\[ c(g, \sigma) := \inf_{u \in C^\infty(\tilde{M}, \mathbb{R})} \sup_{x \in \tilde{M}} \frac{1}{2} |d_x u + \theta_x|^2, \]

where \( \theta \) is any primitive of \( \tilde{\sigma} \). As \( u \) ranges over \( C^\infty(\tilde{M}, \mathbb{R}) \) the form \( \theta + du \) ranges over all primitives of \( \tilde{\sigma} \), because any two primitives differ by a closed 1-form which must be exact since \( \tilde{M} \) is simply connected. The critical value \( c(g, \sigma) < \infty \) if and only if \( \tilde{\sigma} \) has bounded primitives.

**Question.** Is \( d(g, \sigma) = c(g, \sigma) \) always?

As far as we are aware, there are no counterexamples to this equality which is motivated by the desire to relate Aubry-Mather theory with Symplectic Topology. A full motivation for this question together with more examples where equality holds maybe found in [4]. Suppose that \( \pi_1(M) \) is amenable. Then (see [11] Corollary 5.4) \( c(g, \sigma) = \infty \) if and only if \( [\sigma] \neq 0 \). Thus, to test the Question when \( \pi_1(M) \) is amenable and \( \sigma \) is a monopole, we must show that \( d(g, \sigma) = \infty \). This is exactly the content of Theorem B which could then be interpreted as evidence of a positive answer to the Question (recall that solvable groups are amenable).

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