The running of the cosmological and the Newton constants controlled by the cosmological event horizon

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Abstract
We study the renormalization group running of the cosmological and the Newton constants, where the renormalization scale is given by the inverse of the radius of the cosmological event horizon. In this framework, we discuss the future evolution of the universe, where we find stable de Sitter solutions, but also ‘big crunch’-like and ‘big rip’-like events, depending on the choice of the parameters in the model.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The recently observed [1] accelerated expansion of the universe may have its reason in the existence of dark energy (DE), an energy form with negative pressure, which is so far not understood. The cosmological constant (CC) is theoretically the simplest candidate for DE, because it occurs as a classical parameter in Einstein’s equations, and further it has an origin as vacuum energy in quantum field theory (QFT). On the other hand, it is difficult to explain the tiny value of the CC and the actual coincidence of the energy densities of the CC and non-relativistic matter [2].

There are lots of models which describe DE as a dynamical quantity, e.g. by using scalar fields. Another possibility is the modification of the theory of gravity by introducing extra terms in the equations of the cosmological evolution, or extending our spacetime by additional spacetime dimensions. However, in most of these models the accelerated cosmic expansion is due to new and unknown physics, which often means a large amount of arbitrariness and limited predictability.

In this work, we investigate the CC in the sense that it emerges anyway on a formal level in QFT. There, the zero-point or vacuum energy of a quantized field has the same equation...
of state as the CC occurring in Einstein’s equations. Unfortunately, it is unknown how to
calculate its value in a unique way, because it can be written in the form of a quartically
divergent momentum integral like $\int d^3 p \cdot p$. The naive assumption of an ultra-violet (UV)
cutoff in this integral at some known energy scale usually leads to an unobserved high value of
the CC, which is also called the old CC problem. However, the procedure of renormalization
of (coupling) constants in QFTs can handle infinities, thereby leading to a dependence of the
renormalized constants on some energy scale $\mu$. In many cases, this renormalization scale can
be identified with an external momentum, or at least with some characteristic scale (e.g. the
temperature) of the environment. Studying QFT on curved spacetime \[3\] leads to infinities in
the vacuum expectation values (VEV) of the energy–momentum tensors of the fields. This
can be treated by renormalization to yield a scale-dependent or running CC and a running
Newton constant (NC). The absolute values are still not calculable, but the change with respect
to the renormalization scale can be described by the renormalization group equations (RGE).
Unlike the running coupling constants in the standard model of particles, here, the physical
meaning of the scale is not given by the theory. This becomes obvious since, in the language
of Feynman graphs, the formally infinite value of the CC corresponds to a closed loop without
external legs and hence no distinct energy scale. Connecting this renormalization scale with
physics thus requires an additional theoretical input. Usual choices in the literature are the
Hubble scale, the square root of the Ricci scalar and combinations of similar quantities. The
RG running of the CC and the NC has been studied in several different frameworks and models,
some recent results can be found in \[4–8\].

In our investigation of the RG running of the CC and the NC, we choose the inverse of
the radius of the cosmological event horizon as the renormalization scale. Such a horizon
usually exists in accelerating universes like ours. In addition, the possible relevance of the
horizon scale for DE has often been pointed out \[9–11\]. In section 2, we derive the RGEs and
their dependence on the renormalization scale and the parameters, mainly the field masses.
Since the event horizon in an evolving universe is not constant in time, the CC and the NC are
also time dependent, implying that the usual covariant conservation equations for the energy–
momentum tensor have to be modified (section 3). In section 4, we discuss the properties
of the new evolution equation for the cosmic scale factor and derive some conditions on the
existence and the stability of the final states of the universe. The cosmic fate is the main point
of the discussion, since the characteristic behaviour in the far future depends crucially on the
parameters in the RGEs. In section 5, we illustrate the possible final states of the universe by
showing some numerical solutions and their dependence on the parameters. Finally, section 6
contains our conclusions and some open points of this setup.

2. Renormalization group equations

To formulate the RGEs for the CC and the NC $G$, we consider free quantum fields on a curved spacetime, namely a Friedmann–Robertson–Walker (FRW) universe with a positive CC $\lambda$.
For one fermionic and one bosonic degree of freedom with masses $m_F$ and $m_B$, respectively,
the 1-loop effective action can be written in the form \[3\]

$$S_{\text{eff}} = \int d^4 x \sqrt{-g} \left[ \frac{\text{Ric}}{16\pi G} - \Lambda + \left( D + \ln \frac{m_F/\mu}{m_B/\mu} \right) \times \frac{m_F^4 - m_B^4}{32\pi^2} - \frac{\text{Ric}}{16\pi^2} \left[ \left( \xi - \frac{1}{6} \right) m_B^2 - \frac{1}{12} m_F^2 \right] \right] + C, \quad (1)$$
where \( D = \frac{1}{2} \gamma_{\text{Euler}} + \lim_{n \to 4} (n - 4)^{-1} \) is a divergent term, which does not depend on the renormalization scale \( \mu \). Furthermore, \( \xi \) is a coupling constant\(^1\), and the variable \( C \) represents all further terms in the effective action, that are neither proportional to the Ricci scalar \( \text{Ric} \) nor to the vacuum energy density \( \Lambda := \lambda/(8\pi G) \).

The relevant \( \beta \)-functions in the \( \overline{\text{MS}} \)-scheme for the vacuum energy density \( \Lambda \) and the NC \( G \) are obtained by the requirement that the effective action \( S_{\text{eff}} \) must not depend on the renormalization scale \( \mu \),

\[
\mu \frac{dS_{\text{eff}}}{d\mu} = 0.
\]

Because of this condition, \( \Lambda \) and \( G \) have to be treated as \( \mu \)-dependent functions in equation (1), which have to obey the RGEs given by

\[
\mu \frac{d\Lambda}{d\mu} = - \frac{m_F^4 - m_B^4}{32\pi^2}, \quad \mu \frac{dG}{d\mu} = - \frac{1}{\pi} \left[ (\xi - \frac{1}{6}) m_B^2 - \frac{1}{12} m_F^2 \right].
\]

Note, that the divergent term \( D \) has dropped out, leaving over just the masses \( m_F/B \) and \( \xi \).

Assuming constant masses, the RGEs can be integrated. Hence, the equation for the vacuum energy density reads

\[
\Lambda(\mu) = \Lambda_0 \left( 1 - q_1 \ln \frac{\mu}{\mu_0} \right), \quad \Lambda_0 := \Lambda(\mu_0),
\]

where the sign of the parameter

\[
q_1 := \frac{1}{32\pi^2 \Lambda_0} \left( m_F^4 - m_B^4 \right)
\]

depends on whether bosons or fermions dominate. In this context, a real scalar field counts as one bosonic degree of freedom, and a Dirac field as four fermionic ones. The generalization to more than one quantum field in the RGE can be achieved by summing over the fourth powers of their masses. For the NC \( G \) we obtain the RGE in the integrated form

\[
G(\mu) = \frac{G_0}{1 - q_2 \ln \frac{\mu}{\mu_0}}, \quad G_0 := G(\mu_0).
\]

Again, we omit the generalization to more fields, that follows from summing over the squared masses of the fields. For one bosonic and one fermionic degree of freedom the mass parameter \( q_2 \) is given by

\[
q_2 := \frac{G_0}{\pi} \left[ (\xi - \frac{1}{6}) m_B^2 - \frac{1}{12} m_F^2 \right].
\]

Finally, we remark that equation (2) for the running vacuum energy density \( \Lambda(\mu) \) was derived in a renormalization scheme, which is usually associated with the high energy regime. Unfortunately, the corresponding covariantly derived equations for the low energy sector are not known yet [12]. Therefore, we prefer to work with the above RGEs, which were derived in a covariant way, and study the consequences and the constraints on the mass parameters \( q_1 \) and \( q_2 \).

Next, we choose the renormalization scale \( \mu \) to be the inverse of the radius \( R \) of the cosmological event horizon. In the FRW universe the (radial) horizon radius \( R \) at the cosmic time \( t \) is given by

\[
\mu^{-1} = R(t) := a(t) \cdot \int_t^\infty \frac{dt'}{a(t')},
\]

\(^1\) In the action \( S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + \frac{1}{2} \phi \partial \mu \phi - \frac{1}{4} [m^2 + \xi \text{Ric}] \phi^2 \right] \) of a scalar field \( \phi \) on a curved spacetime, the constant \( \xi \) occurs in the coupling term \( \xi \cdot \text{Ric} \cdot \phi^2 \) between the scalar field and the Ricci scalar \( \text{Ric} \).
where \( a(t) \) is the cosmic scale factor, corresponding to the line element:

\[
ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right).
\]

For universes, which end within finite time, the upper limit of the integral in equation (6) should be replaced by the time when the universe ends.

The choice of the scale \( \mu = R^{-1} \) can be motivated by the thermodynamical properties of the cosmological event horizon. This horizon emits radiation, whose temperature is given by the Gibbons–Hawking [13] temperature \( T_{\text{GH}} = (2\pi R)^{-1} \), that is proportional to our renormalization scale \( \mu \). For a comoving observer in a de Sitter universe the only cosmological energy scale is given by this temperature. We have to admit that this is no proof of the rightness of this choice. On the other hand, the investigation of the cosmological evolution with this specific renormalization scale is the main point of this work, and the resulting solutions are quite interesting. From equation (6) we see that in an evolving universe the event horizon radius \( R \) and thus the scale \( \mu \) are usually not constant in time. Therefore, the vacuum energy density \( \Lambda \) and the NC \( G \) will be time dependent, too. This requires a generalization of the covariant conservation conditions, and complicates the solutions of Friedmann’s equations.

### 3. Evolution equation for the scale factor

In this section, we derive the evolution equation for the cosmic scale factor \( a(t) \) in the framework of the spatially isotropic and homogeneous FRW universe with a time-dependent CC and NC. On this background, radiation and pressureless matter (dust) can both be described by a perfect fluid with the energy density \( \rho \) and the pressure \( p = \omega \rho \), where the constant \( \omega \) characterizes the equation of state. The corresponding energy–momentum tensor for these energy forms reads

\[
T_{\alpha \beta} = (\rho + p)u^\alpha u^\beta - pg_{\alpha \beta},
\]

with \( u^\alpha \) being the 4-velocity vector field of the fluid. With our choice of the renormalization scale, \( G \) and \( \Lambda \) depend only on the cosmic time \( t \). From Einstein’s equations

\[
G_{\alpha \beta} = 8\pi G(\Lambda g_{\alpha \beta} + T_{\alpha \beta})
\]

and from the contracted Bianchi identities \( G_{\alpha \beta ; \beta} = 0 \) for the Einstein tensor \( G_{\alpha \beta} \), we obtain the generalized conservation equations:

\[
0 = [G\Lambda g_{\alpha \beta} + GT_{\alpha \beta}]_{,\beta} = 0 \quad \text{for} \quad G_{\alpha \beta}.
\]

Note that the simple scaling rule \( \rho \propto a^{-3(1+\omega)} \) for the matter content is not valid anymore, because it is now possible to transfer energy between the matter and the vacuum, in addition to \( G \neq 0 \). Therefore, we have to combine the Friedmann equations for the Hubble scale \( H := \frac{\dot{a}}{a} \) and the acceleration \( \frac{\ddot{a}}{a} \),

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} G(t)(\Lambda(t) + \rho(t)), \quad \frac{\ddot{a}}{a} = \frac{8\pi}{3} G(t)(\Lambda(t) - Q\rho(t)), \quad Q := \frac{1}{2}(1 + 3\omega),
\]

### Notes

1. The constant \( k \) fixes the spatial curvature of the universe.
2. Dust: \( \omega = 0 \); radiation: \( \omega = \frac{1}{3} \).
3. We do not assume \( T_{\alpha \beta ; \beta} = 0 \).
to eliminate the matter energy density $\rho$. The left-hand side of the result is abbreviated by $F(t)$:

$$F(t) := \dot{a} + Q \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] = \frac{8\pi}{3} G \Lambda \cdot (1 + Q).$$

(9)

Now the RGEs for $\Lambda$ and $G$ from equations (2) and (4) are inserted in equation (9) to yield, with the specific choice of our renormalization scale $\mu_0 = 1/R$, the main equation of this work

$$\mu(t) = \frac{\mu_0}{\mu_0} = \exp \left[ \frac{K_0 F(t) - 1}{q_1 - q_2 K_0 F(t)} \right].$$

(10)

In this equation the constant $K_0$ is defined as

$$K_0 := \frac{3}{8\pi G_0 \Lambda_0 (Q + 1)} = \frac{H_0^{-2}}{\Omega_{\Lambda 0} (Q + 1)},$$

(11)

where $\Omega_{\Lambda 0} = 8\pi G_0 \Lambda_0 / (3H_0^2)$ is the relative vacuum energy density and $H_0$ is the Hubble scale at the time $t = t_0$.

Note that for a dominant matter energy density $\rho \gg \Lambda$ and flat spatial curvature ($k = 0$), the acceleration quantity $q := \frac{\ddot{a}}{a} \frac{\dot{a}}{a^2}$ is given by the negative value of the (new) equation of state parameter $Q = (1 + 3\omega)/2$, which we have introduced in equation (8).

4. Discussion

This section is devoted to the discussion of the properties of our main equation (10), thereby placing special interest in the late-time behaviour of the scale factor $a(t)$. Solving equation (10) for $F$ leads to

$$K_0 F = \frac{1 + q_1 \ln \frac{R}{R_e}}{1 + q_2 \ln \frac{R}{R_e}},$$

(12)

which is plotted as a function of the horizon radius $R$ in figures 4, 5, 6 and 7 for four cases, depending on the signs of $q_1$ and $q_2$. For reasonable field masses $m$, the magnitude of the mass parameter $q_1$ is always greater than that of $q_2$. In the definition of the parameter $q_2$ the field masses $m$ are divided by today’s Planck mass $M_{\text{Planck,}0} = 1/\sqrt{G_0}$, therefore, $|q_2|$ is a tiny quantity today. This implies a very weak running of the NC $G$, which agrees with the strong bounds on the time variation of $G$ [14]. Additionally, this has the advantage that today we are far away from the Landau pole of $G(\mu)$, where the function $F$ diverges. Very high values of $|F|$ could render our calculations invalid, since we do not implement higher powers of the curvature scalar in the gravitational action. These contributions are probably relevant in regimes with large $|F|$. Whether they are able to prevent the singular behaviour in the scale factor, that occurs in the numerical solutions for some cases, requires further investigation and is not treated in this paper.

Much less problematic are final states of the universe that are de Sitter like. In this case, the scale factor acquires the form $a(t) \propto \exp(H_e t)$ for large $t$, where $H_e$ denotes the constant Hubble scale, and the radius of the event horizon is given by $R_e = 1/H_e$. Plugging this asymptotic form for $a(t)$ into equation (12), one arrives at

$$\frac{K_0 (1 + Q)}{R_e^2} = q_3 x^{-2} = \frac{1 + q_1 \ln x}{1 + q_2 \ln x},$$

where
where the variables $x := R_e/R_0$ and $q_3 := K_0(1 + Q)/R_0^2 > 0$ have been introduced. Neglecting the running of the NC, we set $q_2 = 0$ and therefore have to solve $q_3 x^{-2} = 1 + q_1 \ln x$ for $x$. The results are given by

$$x = \frac{R_e}{R_0} = \sqrt{\frac{2q_3}{q_1 \cdot W_u(\frac{2q_3}{q_1} e^{2/q_1})}},$$

involving Lambert’s $W$-function $W_u(z)$, which is the inverse function of $z = x e^x$. The index $u$ denotes the different branches of this function. Only for $u = 0, -1$ does it take on real values for real arguments $z > -e^{-1}$. Additionally, $W_{-1}(z)$ is not real valued for $z \geq 0$, as can be seen in Figure 1, where both branches are plotted. From these properties of the $W$ function we get for negative $q_1$ the constraint $q_3 \leq -\frac{2}{q_1} \exp\left(-\frac{2}{q_1} - 1\right)$, which implies a lower bound for the initial value $R_0$ of the horizon radius:

$$R_0 \geq R_{0\text{min}} := \sqrt{-\frac{2}{q_1 \Omega_{\Lambda 0} H_0^2} \exp\left(\frac{2}{q_1} + 1\right)}.$$  

If $R_0$ is smaller than this minimal value, then equation (13) has no positive solutions and a final de Sitter state does not exist. For $R_0 = R_{0\text{min}}$ there is exactly one solution $x = \exp\left(-\frac{1}{q_1} - \frac{1}{2}\right)$, for higher values $R_0$ there are two solutions. In the case of a positive value of the parameter $q_1$, the initial value $R_0$ must be smaller than $\frac{1}{\sqrt{H_0^2 \Omega_{\Lambda 0}}}$. Otherwise the final horizon radius $R_e$ is smaller than the initial one, $x < 1$. Both cases are plotted in Figure 2.

Since we have found several de Sitter solutions, we have to study the stability of these final states. Therefore, we write $K_0 F$ as a function of $K_0 F = K_0(H + (Q + 1)H^2)$,

$$K_0 F = q_1 \left[H - \frac{1}{R_0} \exp\left(\frac{1 - K_0 F}{q_1}\right)\right].$$
The running of the cosmological and the Newton constants

\[ R_0H_0 \Omega^{1/2}_\Lambda \]

where we used \( \dot{R} = RH - 1 \). In the final de Sitter state we have \( K_0 \dot{F} = 0 \) and \( R = R_e = 1/H_e = \text{const.} \). Near this point we can neglect \( \dot{H} \) in the function \( F \) and replace \( H \) by \( \sqrt{K_0(Q + 1)} \). For a stable solution it is required that

\[
\frac{d(K_0 F)}{d(K_0 F)} < 0
\]

at the final point, where \( K_0 F = K_0(Q + 1)H_e^2 = q_3 x^{-2} \). With \( q_3 \) and \( x \) from above, this yields the stability condition

\[
\left[ W_u \left( \frac{2q_1 e^{2/q_1}}{q_1} \right) \right]^{-1} < -1,
\]

implying that there are no stable de Sitter solutions for positive values of \( q_1 \), because the \( W \) function is positive. For negative \( q_1 \) we get the condition

\[
W_u \left( \frac{2q_1 e^{2/q_1}}{q_1} \right) > -1,
\]

which means that only the solution with \( u = 0 \) is stable. This renders the final event horizon radius \( R_e = R_0x \) unique.

Finally, we take a closer look at the ratio \( R_e/R_0 \) as a function of the mass parameter \( q_1 \). For initial values \( R_0 < 1/(H_0\sqrt{\Omega_{\Lambda}/\Omega_0}) \), which means \( q_3 > 1 \), there is a certain range of values of \( q_1 \) where no solutions for \( R_e \) exist. This range is again given by the requirement that the argument of the \( W \) function must be greater than or equal to \(-e^{-3}\), leading to the conditions

\[
q_1 \leq \frac{2}{W_0(-e^{q_1})} \quad \text{or} \quad q_1 \geq \frac{2}{W_{-1}(-e^{q_1})}.
\]

In figure 3 the exclusion range for \( q_1 \) is obvious for \( q_3 > 1 \). In the case that \( q_1 \) lies above this range, the unstable solution for \( R_e \) is reached first during the future cosmic evolution. For \( q_1 \) below this range, the stable solution is nearer to the initial value \( R_0 \) than the unstable one.
Figure 3. This figure shows the ratio \( R_e/R_0 \) between the final and the initial event horizon radius as a function of \( q_1 \). We plotted four cases for different values of \( q_3 = 1/(H_0^2 \Omega_{\Lambda_0} \Omega_{\Lambda_0}) \), where the choice \( q_3 = 0, 0.8; 0.9 \) corresponds to an initial radius \( R_0 > 1/(H_0 \sqrt{\Omega_{\Lambda_0}}) \), and \( q_3 = 1.1; 1.2 \) to \( R_0 < 1/(H_0 \sqrt{\Omega_{\Lambda_0}}) \). In the latter case there are no solutions for a certain range of values of \( q_1 \) as described by equation (14). The thick lines show the \((u = 0)\)-branch of \( R_e/R_0 \), and the thin lines the \((u = -1)\)-branch, respectively.

However, both solutions for \( R_e \) lie below \( R_0 \). Initial values \( R_0 > 1/(H_0 \sqrt{\Omega_{\Lambda_0}}) \) (i.e. \( q_3 < 1 \)) lead to stable final states with \( R_e > R_0 \) for all negative values of \( q_1 \).

For initial values \( R_0 \) and mass parameters \( q_1 \), that do not satisfy the above existence and stability conditions, the fate of the universe will be ‘big crunch’ like or ‘big rip’ like\(^5\), respectively. With this notation we mean that the scale factor \( a(t) \) or one of its derivatives \( H, q \) becomes singular in a finite time in the future. There is one exception, that may occur when \( q_1 \) and \( q_2 \) are both positive, where the function \( F \) and thus the Hubble scale \( H \) approach constant values, while the horizon radius \( R \) goes to infinity, see figure 7. Another property of the cosmic evolution for negative \( q_1 \) is that no big crunch may occur. This can be seen from the time derivative of the function \( K_0 F \),

\[
K_0 \dot{F} = q_1 \frac{\dot{R}}{R} = q_1 \left( H - \frac{1}{R} \right), \quad q_1 < 0,
\]

which gets positive for \( H < 0 \) (big crunch), thus preventing a further decrease of \( K_0 F \) and the final collapse of the scale factor.

\(^5\) Recent analyses of future singularities and their properties can be found in [15].
5. Numerical solutions

Up to this section, we analysed the evolution equation (10) for the scale factor analytically. Unfortunately, finding explicit solutions seems to be rather difficult because of the strongly nonlinear form of the equation. Therefore, we solve it numerically, thereby realizing that the form given by equation (10) is not directly suitable for a numerical study because of the integral over the time \( t \) in the radius function \( R(t) \). This integral can be removed by differentiating with respect to \( t \), leading to an ordinary differential equation of the order three,

\[
\frac{\dot{a}}{a} + \frac{(q_2 - q_1)K_0 F}{(q_1 - q_2)K_0 F^2} \exp \left[ \frac{K_0 F - 1}{q_1 - q_2 K_0 F} \right] - \frac{1}{R_0} = 0,
\]

which can be solved for the scale factor \( a(t) \) numerically. Note that one has to check whether the numerical solutions are also solutions of the original equation. This is not guaranteed since differentiating equation (10) possibly changes its set of solutions. Indeed, we encounter such ‘false’ solutions in some cases.

Regarding recent observations, we get acceptable solutions only for \(|q_1| \) of the order 1, which means that the relevant mass scale \( m \) should be near \( \Lambda_0^{1/4} \approx 10^{-3} \text{ eV} \). Actually, the only known particles with such a low mass are neutrinos. This indicates that the influence of higher mass fields is suppressed, or these fields have decoupled, respectively. Unfortunately, the simple form of the RGEs (2) and (4) cannot account for a decoupling mechanism. Therefore, we assume in this work that the mass scale \( m \) is low enough today, so that the solutions are compatible with observations. However, note that at earlier cosmic times high-mass fields \( m \) should be relevant.

Concerning the differential equation, we can fix several initial conditions by using observational results. These are today’s value of the Hubble scale \( H_0 = \frac{2}{3} (t_0) \) and the relative vacuum energy density \( \Omega_{\lambda_0} \). Neglecting the spatial curvature (\( \Omega_k = -k/\dot{a}^2 = 0 \)) and considering only dust (with an equation of state parameter \( Q = 0.5 \)) and the CC as relevant energy forms in the present-day universe, the acceleration parameter \( q_0 = \frac{\dot{a}}{a} (t_0) \) is determined by equation (9): \( q = \Omega_{\lambda} (1 + Q) + Q (\Omega_k - 1) \). Today’s value of the horizon radius \( R_0 \) is unknown, so we have to estimate it. Since it should be the largest physical length scale and the universe seems to be almost de Sitter like, we assume the horizon radius to be around \( R_0 \approx 1.2 H_0^{-1} \).

For the numerical treatment every dimensionful quantity is expressed in terms of today’s Hubble scale \( H_0 \) (Hubble units). In a \( \Lambda CDM \) universe with constant \( \Lambda \) and \( G \), the cosmic age is denoted by \( t_0 \). In our calculations we used the following numbers from [16]:

\[
H_0 = 1.5 \times 10^{-42} \text{ GeV}, \quad \Omega_{\lambda_0} = 0.73, \quad t_0 = 13.7 \text{ Gyr} = 0.99 H_0^{-1},
\]

\[
\Lambda_0 = 2.98 \times 10^{-47} \text{ GeV}^4, \quad \Lambda_0^{1/4} \approx 2.34 \times 10^{-3} \text{ eV}, \quad G_0 = (1.22 \times 10^{19} \text{ GeV})^{-2}.
\]

The first observation from the numerical solutions is, that for a positive value of \( q_1 \) the cosmic age decreases with respect to the age \( t_0 \) of the standard \( \Lambda CDM \) universe, whereas for a negative \( q_1 \) the age increases. Furthermore, the usually small value of \( q_2 \) leads to a negligible time variation of Newton’s constant \( G(t) \).

To show the characteristic future cosmic evolution, we investigate four cases in more detail, which result from the parameter choices \( q_1 = \pm 2 \) and \( q_2 = \pm 0.1 \). Note that due to the suppression by the Planck scale, the realistic value of \( q_2 \) should be much lower than \( \pm 0.1 \). Here, we used a large value for \( q_2 \) to show the differences due to the sign of \( q_2 \). Figures 4–7 show the numerical results for different values of the initial radius \( R_0 \) of the event horizon.
The graphs in each of the four figures illustrate the scale factor $a(t)$, the Hubble scale $H(t)$, the acceleration $q(t)$, the event horizon radius $R(t)$ and $F(t)$ as functions of the cosmic time $t$, respectively. The last graph displays $K_0F$ as a function of the radius $R/R_0$. 

Figure 4. The cosmological evolution for the parameter choice $q_1 = -2$ and $q_2 = -0.1$ and different values of the initial event horizon radius $R_0$. The fate of this universe is either a stable de Sitter state when choosing $R_0 = 1.20; 1.30$, or a big rip (BR) in the case $R_0 = 1.10; 1.15$. $K_0F$ is bounded from above. Nomenclature: scale factor $a$, Hubble scale $H = \frac{\dot{a}}{a}$, acceleration $q = \frac{\ddot{a}}{a}$, event horizon radius $R$ and its initial value $R_0$. For the function $K_0F$ see equations (12), (9) and (11), for the mass parameters $q_1, q_2$ see equations (3) and (5).
Figure 5. For the parameter choice $q_1 = -2$ and $q_2 = +0.1$ the cosmic evolution is not very different from the case $q_2 = -0.1$ (figure 4). In the future, there is either a stable de Sitter state for $R_0 = 1.20; 1.30$, or a big rip (BR) when $R_0 = 1.10; 1.15$. $K_0 F$ is bounded from below.

Nomenclature: scale factor $a$, Hubble scale $H = \dot{a}/a$, acceleration $q = \ddot{a}/\dot{a}^2$, event horizon radius $R$ and its initial value $R_0$. For the function $K_0 F$ see equations (12), (9) and (11), for the mass parameters $q_1, q_2$ see equations (3) and (5).

In section 4 we discussed the evolution equation analytically, and found several conditions for the existence of stable de Sitter final states. These properties are also shown by the
Figure 6. The cosmological evolution for different values of today's horizon radius $R_0$ for the case $q_1 = +2$ and $q_2 = -0.1$. The solutions for $R_0 = 1.09; 1.10$ exhibit a big crunch (BC), whereas the initial conditions $R_0 = 1.11; 1.10$ lead to a big rip (BR). $K_0 F$ is bounded from below. The numerical solutions marked by $(\times)$ are not compatible with the main equation (10), see section 5 for further details. Nomenclature: scale factor $a$, Hubble scale $H = \frac{\dot{a}}{a}$, acceleration $q = \frac{\ddot{a}}{\dot{a}}$, event horizon radius $R$ and its initial value $R_0$. For the function $K_0 F$ see equations (12), (9) and (11), for the mass parameters $q_1, q_2$ see equations (3) and (5).

numerical results. For negative values of $q_1$, we observe only ‘big rip’-like solutions and de Sitter final states, whereas for positive $q_1$, a big crunch may also occur, and all de Sitter
Figure 7. The cosmological evolution for different values of today’s horizon radius $R_0$. Here, we choose $q_1 = +2$ and $q_2 = +0.1$. The solutions for $R_0 = 1.09; 1.10$ exhibit a big crunch (BC), where $K_0 F$ is unbounded from below. For the initial conditions $R_0 = 1.11; 1.10$ the function $F$ and the Hubble scale $H$ approach a finite value, where the horizon radius $R$ diverges. The numerical solutions marked by $(\times)$ are not compatible with the main equation (10), see section 5 for further details. Nomenclature: scale factor $a$, Hubble scale $H = \dot{a}/a$, acceleration $q = \ddot{a}/a^2$, event horizon radius $R$ and its initial value $R_0$. For the function $K_0 F$ see equations (12), (9) and (11), for the mass parameters $q_1, q_2$ see equations (3) and (5).
states are unstable. Note that the catastrophic events, the ‘big rip’ and the ‘big crunch’, usually involve a high gravitational strength, implying that our calculations may not be reliable near these singular points. For positive values of $q_1$ and $q_2$ (see figure 7), we have not observed any ‘big rip’-like solutions. Then the final state may be either a ‘big crunch’ or a forever expanding universe, where the Hubble scale approaches a finite positive value, but the event horizon radius $R$ goes to infinity. This is a contradiction, because an asymptotically constant Hubble scale $H > 0$ implies a finite event horizon radius $R \approx H^{-1}$ in the far future, which is not the case here. Obviously, this numerical solution is not a solution of the original equation (10). For $q_1 > 0$ and $q_2 < 0$ (see figure 6) the ‘big rip’-like events in the numerical solutions occur at a finite and large value of the horizon radius $R$. Again, this behaviour is not compatible with the vanishing of the horizon radius at such an event. Therefore, we can reject these numerical solutions, too.

6. Conclusions

We investigated a cosmological model, where the CC and the NC are determined by renormalization group equations, which emerge from QFT on curved spacetime. By choosing the inverse radius of the cosmological event horizon as the renormalization scale, we get time-dependent constants. Because of this, the evolution equation for the cosmic scale factor becomes more complicated than in standard $\Lambda$CDM-cosmology, leading to cosmological solutions with several very different future final states of the universe. We found ‘big crunch’-like and ‘big rip’-like solutions, but also stable de Sitter final states. Which of these states will be realized depends crucially on the field masses in the renormalization group equations, and on the initial value of today’s radius of the event horizon. In this context, we derived some conditions on the existence of stable de Sitter states. Furthermore, the cosmic evolution was analysed numerically for different values of the field masses. For a realistic cosmic behaviour, we have to require that the masses in the RGEs should be quite low, implying the need for some suppression or decoupling mechanism for high-mass fields. However, such a mechanism for the CC and the NC has not been found yet [12]. This also restricts the main focus of this work to the future behaviour of the universe, because for the cosmic evolution at early times quantum fields with high masses should be taken into account. Moreover, the regimes with a high gravitational strength need a deeper investigation, because such conditions are given not only at early times, but also at the singularities in the future. Finally, we conclude that the specific choice of the renormalization scale in this work leads to cosmological solutions that may become singular in a finite time without introducing exotic forms of matter.

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