Recent developments of biharmonic conjecture and modified biharmonic conjectures

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Abstract

A submanifold $M$ of a Euclidean $m$-space $E^m$ is said to be biharmonic if $\Delta H = 0$ holds identically, where $H$ is the mean curvature vector field and $\Delta$ is the Laplacian on $M$. In 1991, the author conjectured in [14] that every biharmonic submanifold of a Euclidean space is minimal. The study of biharmonic submanifolds is nowadays a very active research subject. In particular, since 2000 biharmonic submanifolds have been receiving a growing attention and have become a popular subject of study with many progresses.

In this article, we provide a brief survey on recent developments concerning my original conjecture and generalized biharmonic conjectures. At the end of this article, I present two modified conjectures related with original biharmonic conjecture.

1 Introduction

Let $x : M \to E^m$ be an isometric immersion from a Riemannian $n$-manifold into a Euclidean $m$-space. Denote by $\Delta$, $x$ and $H$ the Laplacian, the position vector and the mean curvature vector of $M$, respectively. Then $M$ is called a biharmonic submanifold if $\Delta H = 0$. Due to the well-known Beltrami’s formula, $\Delta x = -nH$, it is obvious that every minimal submanifold of $E^m$ is a biharmonic submanifold.

The study of biharmonic submanifolds was initiated by the author in the middle of 1980s in his program of understanding the finite type submanifolds in Euclidean spaces; also independently by G. Y. Jiang [29] for his study of Euler-Lagrange’s equation of bienergy functional in the sense of Eells and Lemaire.

The author showed in 1985 that biharmonic surfaces in $E^3$ are minimal (unpublished then, also independently by Jiang [29]). This result was the starting point of I. Dimitric’s work on his doctoral thesis at Michigan State University (cf. [27]). In particular, Dimitric extended author’s unpublished result to biharmonic hypersurfaces of $E^m$ with at most two distinct principal curvatures [27]. In his thesis, Dimitric also proved that every biharmonic submanifold of finite type in $E^m$ is minimal. Another extension of this result on biharmonic surfaces was given by T. Hasanis and T. Vlachos in [30] (see also [26]). They proved that biharmonic hypersurfaces of $E^4$ are minimal.

Formally, the author made in [14] the following.

Biharmonic Conjecture: The only biharmonic submanifolds of Euclidean spaces are the minimal ones.
A biharmonic map is a map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds that is a critical point of the bienergy functional:

\[
E^2(\phi, D) = \frac{1}{2} \int_D \|\tau_\phi\|^2 \, \ast 1
\]

for every compact subset \( D \) of \( M \), where \( \tau_\phi = \text{trace}_g \nabla d\phi \) is the tension field \( \phi \). The Euler-Lagrange equation of this functional gives the biharmonic map equation (see [29])

\[
\tau_\phi^2 := \text{trace}_g (\nabla^\phi \nabla^\phi - \nabla^\phi^M) \tau_\phi - \text{trace}_g R^N(d\phi, \tau_\phi) d\phi = 0,
\]

where \( R^N \) is the curvature tensor of \((N, h)\). Equation (2) states that \( \phi \) is a biharmonic map if and only if its bi-tension field \( \tau_\phi^2 \) vanishes.

Let \( M \) be an \( n \)-dimensional submanifold of a Euclidean \( m \)-space \( \mathbb{E}^m \). If we denote by \( \iota : M \to \mathbb{E}^m \) the inclusion map of the submanifold, then the tension field of the inclusion map is given by \( \tau_\iota = -\Delta_\iota = -nH \) according to Beltrami’s formula. Thus \( M \) is a biharmonic submanifold if and only if

\[
n \Delta H = -\Delta^2 \iota = -\tau_\iota^2 = 0,
\]

i.e., the inclusion map \( \iota \) is a biharmonic map.

Caddeo, Montaldo and Oniciuc [10] proved that every biharmonic surface in the hyperbolic 3-space \( H^3(-1) \) of constant curvature \(-1\) is minimal. They also proved that biharmonic hypersurfaces of \( H^n(-1) \) with at most two distinct principal curvatures are minimal [9].

Based on these, Caddeo, Montaldo and Oniciuc made in [9] the following.

The generalized Chen’s conjecture: Any biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.

The study of biharmonic submanifolds is nowadays a very active research subject. In particular, since 2000 biharmonic submanifolds have been receiving a growing attention and have become a popular subject of study with many progresses.

In this article, we provide a brief survey on recent developments concerning my original conjecture and generalized biharmonic conjectures. At the end of this article, I present two modified conjectures related with original biharmonic conjecture.

2 Recent developments on Chen’s original biharmonic conjecture

Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a Riemannian \( n \)-manifold \( M \) into a Euclidean \( m \)-space \( \mathbb{E}^m \). Then \( M \) is biharmonic if and only if it satisfies the following fourth order strongly elliptic semi-linear PDE system (see, for instance, [13, 16, 20])

\[
\begin{align*}
\Delta^D H + \sum_{i=1}^n \sigma(A_H e_i, e_i) &= 0, \\
n \nabla \langle H, H \rangle + 4 \text{trace} A_DH &= 0,
\end{align*}
\]

where \( \Delta^D \) is the Laplace operator associated with the normal connection \( D \), \( \sigma \) the second fundamental form, \( A \) the shape operator, \( \nabla \langle H, H \rangle \) the gradient of the squared mean curvature, and \( \{e_1, \ldots, e_n\} \) an orthonormal frame of \( M \).

An immersed submanifold \( M \) in a Riemannian manifold \( N \) is said to be properly immersed if the immersion is a proper map, i.e., the preimage of each compact set in \( N \) is compact in \( M \).

The total mean curvature of a submanifold \( M \) in a Riemannian manifold is given by \( \int_M |H|^2 \, dv \).
Denote by $K(\pi)$ the sectional curvature of a given Riemannian $n$-manifold $M$ associated with a plane section $\pi \subset T_p M$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_p M$, the scalar curvature $\tau$ at $p$ is defined to be $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$.

Let $L$ be a subspace of $T_p M$ of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. The scalar curvature $\tau(L)$ of $L$ is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r.$$  

For an integer $r \in [2, n-1]$, the $\delta$-invariant $\delta(r)$ of $M$ is defined by (cf. [17, 20])

$$\delta(r)(p) = \tau(p) - \inf\{\tau(L)\}, \quad (3)$$

where $L$ run over all $r$-dimensional linear subspaces of $T_p M$.

For any $n$-dimensional submanifold $M$ in $\mathbb{E}^m$ and any integer $r \in [2, n-1]$, the author proved the following general sharp inequality (cf. [17, 20]):

$$\delta(r) \leq \frac{n^2(n-r)}{2(n-r+1)} |\nabla^2 f|^2. \quad (4)$$

A submanifold in $\mathbb{E}^m$ is called $\delta(r)$-ideal if it satisfies the equality case of (4) identically. Roughly speaking ideal submanifolds are submanifolds which receive the least possible tension from its ambient space (cf. [17, 20]).

A hypersurface of a Euclidean space is called weakly convex if it has non-negative principle curvatures.

It follows immediately from the definition of biharmonic submanifolds and Hopf’s lemma that every biharmonic submanifold in a Euclidean space is non-compact.

The following provides an overview of some affirmative partial solutions to my original biharmonic conjecture.

- Biharmonic surfaces in $\mathbb{E}^3$ (B.-Y. Chen [14] 20 and G. Y. Jiang [29]).
- Biharmonic curves (I. Dimitric [27, 28]).
- Biharmonic hypersurfaces in $\mathbb{E}^4$ (T. Hasanis and T. Vlachosin [30]) (a different proof by F. Defever [26]).
- Spherical submanifolds (B.-Y. Chen [16]).
- Biharmonic hypersurfaces with at most 2 distinct principle curvatures (I. Dimitric [27]).
- Biharmonic submanifolds of finite type (I. Dimitric [27, 28]).
- Pseudo-umbilical biharmonic submanifolds (I. Dimitric [28]).
- Biharmonic submanifolds which are complete and proper (Akutagawa and Maeta [1]).
- Biharmonic properly immersed submanifolds (S. Maeta [36]).
- Biharmonic submanifolds satisfying the decay condition at infinity

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^2 dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$ (G. Wheeler [15]).
In [33], Y.-L. Ou constructed examples to show that my original biharmonic conjecture cannot be generalized to the case of biharmonic conformal submanifolds in Euclidean spaces.

Remark 2.1. My original biharmonic conjecture is still open.

Remark 2.2. My biharmonic conjecture is false if the ambient Euclidean space were replaced by a pseudo-Euclidean space. The simplest examples are constructed by Chen and Ishikawa in [21]. For instance, we have the following.

Example. Let \( f(u, v) \) be a proper biharmonic function, i.e. \( \Delta f \neq 0 \) and \( \Delta^2 f = 0 \). Then

\[
x(u, v) = (f(u, v), f(u, v), u, v)
\]

defines a biharmonic, marginally trapped surface in the Minkowski 4-space \( \mathbb{E}^4_1 \) endowed with the Lorentzian metric \( g_0 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \).

Here, by a marginally trapped surface, we mean a space-like surface in \( \mathbb{E}^4_1 \) with light-like mean curvature vector field.

It was proved in [21] that the biharmonic surfaces defined by (5) are the only biharmonic, marginally trapped surfaces in \( \mathbb{E}^4_1 \).

3 Recent developments on Caddeo-Montaldo-Oniciuc’s Generalized Chen’s biharmonic conjecture

Let \( M \) be a submanifold of a Riemannian manifold with inner product \( \langle , \rangle \), then \( M \) is called \( \epsilon \)-superbiharmonic if

\[
\langle \Delta H, H \rangle \geq (\epsilon - 1) |\nabla H|^2,
\]

where \( \epsilon \in [0, 1] \) is a constant. For a complete Riemannian manifold \( (N, h) \) and \( \alpha \geq 0 \), if the sectional curvature \( K^N \) of \( N \) satisfies

\[
K^N \geq -L(1 + \text{dist}_N(\cdot, q_0)^2)^\alpha
\]

for some \( L > 0 \) and \( q_0 \in N \), then we call that \( K^N \) has a polynomial growth bound of order \( \alpha \) from below.

There are also many affirmative partial answers to the generalized Chen’s biharmonic conjecture. The following provides a brief overview of the affirmative partial answers to this generalized conjecture.

• Biharmonic hypersurfaces in the hyperbolic 3-space \( \mathbb{H}^3(-1) \) (Caddeo, Montaldo and Oniciuc [9]).

• Biharmonic hypersurfaces in \( \mathbb{H}^4(-1) \) (Balmuş, Montaldo and Oniciuc [7]).

• Pseudo-umbilical biharmonic submanifolds of \( \mathbb{H}^m(-1) \) (Caddeo, Montaldo and Oniciuc [9]).

• Biharmonic hypersurfaces of \( \mathbb{H}^{n+1}(-1) \) with at most two distinct principal curvatures (Balmuş, Montaldo and Oniciuc [8]).
• Totally umbilical biharmonic hypersurfaces in Einstein spaces (Y.-L. Ou [44]).

• Biharmonic hypersurfaces with finite total mean curvature in a Riemannian manifold of non-positive Ricci curvature (Nakauchi and Urakawa [40]).

• Biharmonic submanifolds with finite total mean curvature in a Riemannian manifold of non-positive sectional curvature (Nakauchi and Urakawa [41]).

• Complete biharmonic hypersurfaces $M$ in a Riemannian manifold of non-positive Ricci curvature whose mean curvature vector satisfies $\int_M |H|^\alpha \, dv < \infty$ for some $\epsilon > 0$ with $1 + \epsilon \leq \alpha < \infty$ (S. Maeta [38]).

• Biharmonic properly immersed submanifolds in a complete Riemannian manifold with non-positive sectional curvature whose sectional curvature has polynomial growth bound of order less than 2 from below (S. Maeta [37]).

• Complete biharmonic submanifolds with finite bi-energy and energy in a non-positively curved Riemannian manifold (N. Nakauchi, H. Urakawa and S. Guðmundsson [42]).

• Complete oriented biharmonic submanifolds in a Riemannian manifold with sectional curvature $H \in L^2(M)$ in a Riemannian manifold with non-positive Ricci tensor (Álías, García-Martínez and Rigoli [2]).

• Compact biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature (G.-Y. Jiang [29] and S. Maeta [38]).

• $\epsilon$-superbiharmonic submanifolds in a complete Riemannian manifolds satisfying the decay condition at infinity

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^2 \, dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$ (G. Wheeler [48]).

• Complete biharmonic submanifolds (resp., hypersurfaces) $M$ in a Riemannian manifold of non-positive sectional (resp., Ricci) curvature whose mean curvature vector satisfies $\int_M |H|^p \, dv < \infty$ for some $p > 0$ (Y. Luo [33]).

• Complete biharmonic submanifolds (resp., hypersurfaces) in a Riemannian manifold whose sectional curvature (resp., Ricci curvature) is non-positive with at most polynomial volume growth (Y. Luo [33]).

• Complete biharmonic submanifolds (resp., hypersurfaces) in a negatively curved Riemannian manifold whose sectional curvature (resp., Ricci curvature) is smaller than $-\epsilon$ for some $\epsilon > 0$ (Y. Luo [33]).

• Proper $\epsilon$-superharmonic submanifolds $M$ with $\epsilon > 0$ in a complete Riemannian manifold $N$ whose mean curvature vector satisfying the growth condition

$$\lim_{\rho \to \infty} \frac{1}{\rho^2} \int_{f^{-1}(B_\rho)} |H|^{2+a} \, dv = 0,$$

where $f$ is the immersion, $B_\rho$ is a geodesic ball of $N$ with radius $\rho$, and $a \geq 0$ (Luo [33]).

On the other hand, it was proved by Y.-L. Ou and L. Tang in [46] that the generalized Chen’s biharmonic conjecture is false in general by constructing foliations of proper biharmonic hyperplanes in a 5-dimensional conformally flat space with negative sectional curvature. Further counter-examples were constructed in [31] by T. Liang and Y.-L. Ou.
4 Maeta’s Generalized Chen’s conjecture

A submanifold of a Euclidean space is called $k$-harmonic if its mean curvature vector satisfies $\Delta^{k-1}H = 0$. It follows from Hopf’s lemma that such submanifolds are always non-compact. Some relationships between $k$-harmonic and harmonic maps of Riemannian manifolds into Euclidean $m$-space $\mathbb{E}^m$ have been obtained by the author in [18].

Recently, S. Maeta [35] found some relations between $k$-harmonic and harmonic maps of Riemannian manifolds into non-flat real space forms.

It follows from [18] Proposition 3.1 that every $k$-harmonic submanifold of $\mathbb{E}^m$ is either minimal or of infinite type (in the sense of [13]). On the other hand, it is also well-known that all $k$-harmonic curves in $\mathbb{E}^m$ are of finite type (see [24] Proposition 4.1). Consequently, every $k$-harmonic curve in $\mathbb{E}^m$ is an open portion of line (this known fact was rediscovered recently in [35] Theorem 5.5).

Based on this fact, S. Maeta [35] made another generalized Chen’s conjecture; namely,

“The only $k$-harmonic submanifolds of a Euclidean space are the minimal ones.”

5 Two related biharmonic conjectures

Finally, I present two biharmonic conjectures related to my original biharmonic conjecture.

Biharmonic Conjecture for Hypersurfaces: Every biharmonic hypersurface of Euclidean spaces is minimal.

The global version of my original biharmonic conjecture can be found, for instance, in [1, 38].

Global Version of Chen’s biharmonic Conjecture: Every complete biharmonic submanifold of a Euclidean space is minimal.

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