Object oriented programming based MATLAB toolbox to solve transient quasi-harmonic equation using finite element methods

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Abstract. Many heat conduction problems encountered in engineering applications are time intensive they are dependent on unsteady state variables. In many situations, temperature varies with respect to time function, for which a solution needs to be found. In this research paper, a 2D transient Quasi-harmonic equation is solved using Finite element method. In engineering applications, many engineering problems are governed by this Quasi-harmonic equation. The main aim of this analysis is to determine the variation of the temperature as a function of time by finding element conductivity matrix using standard Gauss quadrature numerical integration method as it is both accurate and efficient. The obtained results of MATLAB coding are verified with ANSYS with examples. Toolbox is created in MATLAB using Object Oriented Programming language for this analysis. The template of this toolbox is also added in this paper.

1. Introduction

Finite element method is a powerful tool for engineers to solve many engineering problems. The advantages of the finite element method, as compared to other numerical approaches, are numerous. The method is completely general with respect to geometry and material properties [1]. Basically they are of two types one is structural problems and another is heat transfer problems. This paper deals with transient thermal problems that is time dependent heat transfer problems. In some instances, thermal engineering problems are dependent on variables like time and they need to be analysed according to time variations [2]. In situations such as these, finite element method can be used to easily solve such unsteady state problems. As Finite element method is a method of discretization, continuum domain is divided into small divisions called elements. Each element has a property that depends on analysis type. In thermal analysis, for an element conductivity property has to use. So, for obtaining solution, conductivity matrix is necessary to find nodal temperature values and conductivity relationship for using it in finite element method.

After getting the conductivity matrix, post-processing has to be done to get the final solution by imposing proper boundary conditions. Generally, many thermal problems are governed by transient Quasi-harmonic equation [2]. The main aim of thermal analysis is to find temperature values and its
distribution behaviour according to time variation parameter. In this paper, the approach to solve 2D transient Quasi-Harmonic equation using finite element method is given. MATLAB is used as a solving tool and MATLAB code is used for obtaining temperature values and profile of temperature distribution. The obtained results are verified with Standard 2x2 Gauss quadrature numerical integration method.

Conductivity matrix is an inherent material property in thermal analysis. It is the same as stiffness matrix in structural analysis. For heat transfer problems in finite element method, the conductivity matrix contains the geometric and material behaviour like thermal conductivity, such information that indicates certain behaviour when subjected to temperature or any thermal loading. FEM matrix equation which is used to solve the problems is obtained from solving weak form of governing equation and it is given by,

$$[K]\{T\} = \{F\}$$

in which [K] is the conductivity matrix and it is analogous to the stiffness matrix in structural analysis.

Heat conduction equations can offer solutions for problems only if the appropriate boundary and initial conditions are stated in the problem. In thermal analysis, two boundary conditions can be implemented; the first is temperature boundary condition called Dirichlet boundary condition and the second is heat flux boundary condition called Neumann boundary condition [3]. This paper is attempts to implement Dirichlet boundary condition. After providing temperature boundary conditions at certain nodes it is possible to find the temperature on every node.

Using Object Oriented Programming, Toolbox is created in MATLAB GUI. It is easiest and user-friendly computing toolbox that helps to get the required solution directly. The user does not need to do any calculation, only basic input parameters such as geometry description, element data, material model, mesh data etc. are needed. Once this information is given in the toolbox, it will automatically show the results and plots. Physical interference of this toolbox is similar to that of the ANSYS toolbox. This toolbox is based on the algorithm of Standard 2x2 Gauss quadrature numerical integration method. Toolbox proposed in this paper is for 2-Dimensional domain and it can be implemented to 3-dimensional domain also which can be used for transient heat transfer in functionally graded material [4].

2. Quasi-Harmonic equation

Quasi-harmonic equation is one of the partial differentiation equation which represents the temperature and heat transfer behaviour. The determination of temperature distribution in a medium is the main objective of a conduction analysis, i.e. to know the temperature in the medium as a function of space at steady state and as a function of time during the transient state.

Many engineering problems are governed by Quasi-harmonic equation containing coefficients which are dependent on any unknown variable or its derivative according to some prescribed law. Basically Quasi-harmonic equation is derived from Fourier’s law which is given by [5],

$$q = - kA \frac{\delta T}{\delta x}$$  

Where,

$q$ = Heat transfer  
$k$ = Thermal conductivity  
$\frac{\delta T}{\delta x}$ = Temperature gradient

In many physical situations the generation of stresses or loads is created due to the diffusion or law of any physical quantity such as heat or mass within the solid. In such case, a scalar variable φ is used to represent the temperature, heat flux or some other physical variables. The balanced equation can be written as,

$$\frac{\delta}{\delta x} (k_x \frac{\delta f}{\delta x}) + \frac{\delta}{\delta y} (k_y \frac{\delta f}{\delta y}) + Q = 0$$
2.1 Weak formulation of transient Quasi-harmonic equation

Transient Quasi-harmonic equation is given by [6],

$$\frac{\delta}{\delta x} (k_x \frac{\delta T}{\delta x}) + \frac{\delta}{\delta y} (k_y \frac{\delta T}{\delta y}) + Q = \rho c_p \frac{\delta T}{\delta t}$$  \hspace{1cm} (4)

Solution for weak form is given by,

$$\sum N_i f(x,y)$$ \hspace{1cm} (5)

Where $N_i$ is a shape function as weight function, and here $f(x,y)$ will be,

$$f(x,y) = \frac{\delta}{\delta x} (k_x \frac{\delta T}{\delta x}) + \frac{\delta}{\delta y} (k_y \frac{\delta T}{\delta y}) + Q - \rho c_p \frac{\delta T}{\delta t}$$ \hspace{1cm} (6)

$$\int N_i \left( \frac{\delta}{\delta x} (k_x \frac{\delta T}{\delta x}) + \frac{\delta}{\delta y} (k_y \frac{\delta T}{\delta y}) + Q - \rho c_p \frac{\delta T}{\delta t} \right)$$ \hspace{1cm} (7)

... (multiplying $N_i$ to function)

$$\int N_i \left( \frac{d}{dx} (k_x \frac{dT}{dx}) + \int N_j \left( \frac{d}{dy} (k_y \frac{dT}{dy}) + \int N_l Q - \int N_i \rho c_p \frac{dT}{dt} \right) \right)$$ \hspace{1cm} (8)

Applying integration by parts, we will get,

$$\int k_x(T) \frac{\delta N_i}{\delta x} \frac{\delta N_j}{\delta x} T_j(t) + k_y(T) \frac{\delta N_i}{\delta y} \frac{\delta N_j}{\delta y} T_j(t) + \int (N_l Q \cdot N_i \cdot \rho c_p \frac{\delta T}{\delta t} T_j(t))$$ \hspace{1cm} (9)

where i, j are nodes

So, FEM equation from a weak form is given by,

$$[C] \left( \frac{\delta(T)}{\delta t} \right) + [K] \{T\} = \{F\}$$ \hspace{1cm} (10)

Where $[C]$ is a capacitance matrix in transient heat transfer.

While solving the transient heat transfer problem, we get an additional matrix, related to the capacity of a material to absorb heat, called capacitance matrix, which is given by,

$$[C]=\rho C_p \int \left[ \left[ [N] \right]^T \left[ [N] \right] \right] dv$$ \hspace{1cm} (11)

Capacitance matrix refers to carrying heat energy capacity. Derivative of temperature with respect to time is multiplied with this capacitance matrix as shown in weak form.

3. Numerical example

Sample Problem - Consider a rectangular plate with dimensions 20×10×1 m. The right face of the plate is maintained at 100°C, 200°C, 300°C, 400°C, 500°C. Consider thermal conductivity to be $k=100$ w/ m°C. Compute all the temperature values at each node in Transient condition. Consider specific heat and density as 1 and time step size as 0.005.
3.1 Boundary conditions
As mentioned in above numerical example, Boundary condition has been applied to right face of the rectangular plate.

- Dirichlet boundary condition
  1. On node 2, $\phi(x, t) = 500$
  2. On node 7, $\phi(x, t) = 400$
  3. On node 8, $\phi(x, t) = 300$
  4. On node 9, $\phi(x, t) = 200$
  5. On node 6, $\phi(x, t) = 100$

Whereas, $\phi$ is a temperature function

- Time boundary condition
  1. Initial time $t = 0$ to $t = 1$ ($0 < t < 1$)

3.2 Results and Discussion

Table 1. Comparison between obtained temperature values from MATLAB coding and ANSYS at two different time instances

| Node No. | Temperature values using MATLAB coding when time t=1 | Temperature values using ANSYS when time t=1 | Temperature values using MATLAB coding when time t=0.75 | Temperature values using ANSYS when time t=0.75 |
|----------|------------------------------------------------------|---------------------------------------------|--------------------------------------------------------|---------------------------------------------|
| 1        | 93.275                                               | 95.860                                      | 58.6900                                                | 64.180                                      |
| 2        | 500.0000                                             | 500.00                                      | 500.0000                                               | 500.00                                      |
| 3        | 109.0301                                             | 111.12                                      | 76.4462                                                | 80.771                                      |
| 4        | 158.4097                                             | 159.47                                      | 132.3048                                               | 134.2                                       |
| 5        | 241.3373                                             | 241.61                                      | 226.5782                                               | 226.82                                      |
| 6        | 100.0000                                             | 100.00                                      | 100.0000                                               | 100.00                                      |
| 7        | 400.0000                                             | 400.00                                      | 400.0000                                               | 400.00                                      |
| 8        | 300.0000                                             | 300.00                                      | 300.0000                                               | 300.00                                      |
|   |          |          |          |          |
|---|----------|----------|----------|----------|
| 9 | 200.0000 | 200.00   | 200.0000 | 200.00   |
|10 |  92.7153 |  95.300  |  58.1375 |  63.627  |
|11 | 199.4189 | 199.70   | 184.6632 | 184.91   |
|12 | 147.9656 | 149.03   | 121.8666 | 123.77   |
|13 | 108.1399 | 110.23   |  75.5633 |  79.889  |
|14 |  92.9098 |  95.495  |  58.3309 |  63.820  |
|15 |  92.9953 |  95.580  |  58.4138 |  63.903  |
|16 |  93.0808 |  95.665  |  58.4966 |  63.986  |
|17 | 109.2035 | 111.30   |  76.6207 |  80.946  |
|18 | 108.5850 | 110.68   |  76.0048 |  80.330  |
|19 | 107.9665 | 110.06   |  75.3888 |  79.715  |
|20 | 155.3518 | 156.41   | 129.2478 | 131.15   |
|21 | 153.1876 | 154.25   | 127.0857 | 128.99   |
|22 | 151.0235 | 152.08   | 124.9237 | 126.83   |
|23 | 243.1502 | 243.43   | 228.3915 | 228.63   |
|24 | 220.3781 | 220.66   | 205.6207 | 205.86   |
|25 | 197.6061 | 197.88   | 182.8499 | 183.09   |

**Figure 2.** Temperature distribution using MATLAB coding
Figure 3. Temperature distribution using ANSYS coding

3.3 Comparison between MATLAB inbuilt PDE toolbox and proposed toolbox based on object oriented programming

Figure 4. MATLAB inbuilt toolbox
Figure 5. Proposed toolbox

Inbuilt PDE toolbox is based on mathematics and statistics. Its conventional toolbox is neither simple nor user-friendly as the user should give appropriate partial differentiation equation and Boundary conditions manually. Only users with good knowledge of mathematics and statistics, can use this toolbox. On the other hand, the toolbox which is proposed in this paper is based on Finite element analysis and has the advantage of being very user-friendly. It works on geometrical description and inputs for this toolbox are element type, material type and mesh data. In this toolbox, geometry data can be directly given from an Excel file and the graph of displacement and stress will automatically be plotted depending on whether the problem is either structural or thermal.

This toolbox is also designed for functionally graded materials that is materials which have variable properties. As this toolbox is user-friendly, it is possible for anyone who has basic knowledge about FEA to easily use this toolbox and obtain the needed solution.

3.4 Flowchart for toolbox
4. Conclusion

Transient heat transfer analysis is studied in this paper. Temperature distribution of the plate has been analysed under thermal loading. The 2D transient Quasi-harmonic equation is solved using finite element method. The solution of the Quasi-harmonic equation gives the main aim of thermal analysis that is temperature values and temperature distribution over the given region. Solving this equation using the suggested finite element approach is easy and accurate results can be obtained. MATLAB’s inbuilt partial differential equation toolbox is studied. The toolbox which is presented in this paper has the advantage of being user-friendly and gives the answer directly. It is similar to that of the popular ANSYS toolbox. The results obtained from MATLAB coding are verified with ANSYS for homogeneous material. It is noted that the suggested method works well for structural as well as thermal analysis.

5. References

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