A SYSTEM OF THIRD-ORDER DIFFERENTIAL OPERATORS
CONFORMALLY INVARIANT UNDER $\mathfrak{so}(8, \mathbb{C})$

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Abstract. In earlier work, Barchini, Kable, and Zierau constructed a number of conformally invariant systems of differential operators associated to Heisenberg parabolic subalgebras in simple Lie algebras. The construction was systematic, but the existence of such a system was left open in several anomalous cases. Here, a conformally invariant system is shown to exist in the most interesting of these remaining cases. The construction may also be interpreted as giving an explicit homomorphism between generalized Verma modules for the Lie algebra of type $D_4$.

1. Introduction

Conformally invariant systems of differential operators on a manifold $M$ on which a Lie algebra $\mathfrak{g}$ acts by first order differential operators were studied by Barchini, Kable, and Zierau in [1] and [2]. Loosely speaking, a conformally invariant system is a list of differential operators $D_1, \ldots, D_m$ that satisfies the bracket identity

$$[\Pi(X), D_j] = \sum_i C_{ij}(X)D_i,$$

where $\Pi(X)$ is the differential operator corresponding to $X \in \mathfrak{g}$ and $C_{ij}(X)$ are smooth functions on $M$. We shall give the definition of conformally invariant systems more precisely in Section 2. While a general theory of conformally invariant systems is developed in [2], examples of such systems of differential operators associated to the Heisenberg parabolic subalgebras of any complex simple Lie algebras are constructed in [1]. The purpose of this paper is to answer a question, left open in [1], concerning the existence of a certain conformally invariant system of third-order differential operators. This is done by constructing the required system. This result may be interpreted as giving an explicit homomorphism between two generalized Verma modules, one of which is non-scalar. The problem of constructing and classifying homomorphisms between scalar generalized Verma modules has received a lot of attention; for recent work, see, for example, [3]. Much less is known about maps between generalized Verma modules that are not necessarily scalar.

In order to explain our main results in this paper, we briefly review the results of [1] here. To begin with, let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the parabolic subalgebra of Heisenberg type; that is, $\mathfrak{n}$ is a two-step nilpotent algebra with one-dimensional center. We denote by $\gamma$ the highest root of $\mathfrak{g}$. For each root $\alpha$ let $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ be a corresponding $\mathfrak{sl}(2)$-triple, normalized as in Section 2 of [1]. Then $\text{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues $-2, -1, 0, 1, 2$, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus \mathfrak{V}^- \oplus \mathfrak{I} \oplus \mathfrak{V}^+ \oplus \mathfrak{z}(\mathfrak{n}).$$
Let $\mathbb{D}[n]$ be the Weyl algebra of $n$. Then each system constructed in [1] derives from a $\mathbb{C}$-linear map $\Omega_k : \mathfrak{g}(2 - k) \to \mathbb{D}[n]$ with $1 \leq k \leq 4$ and $\mathfrak{g}(2 - k)$ the $2 - k$ eigenspace of $\text{ad}(H_\gamma)$. Let $\Pi_s : \mathfrak{g} \to \mathbb{D}[n]$ be the Lie algebra homomorphism constructed in Section 4 in [1]. Here $s$ is a complex parameter. We say that the $\Omega_k$ system has special value $s_0$ when the system is conformally invariant for $\Pi_{s_0}$.

In [1] the special values of $s$ are determined for the $\Omega_k$ systems with $k = 1, 2, 4$ for all complex simple Lie algebras, but only exceptional cases are considered for the $\Omega_3$ system. A table in Section 8.10 in [1] lists the special values of $s$. The reader may want to notice that the entries in the columns for the systems $\Omega_2^\text{big}$ and $\Omega_2^\text{small}$ for types $B_r$ and $C_r$ should be transposed. Theorem 21 in [2] then shows that the $\Omega_3$ system does not exist for $A_r$ with $r \geq 3$, $B_r$ with $r \geq 3$, and $D_r$ with $r \geq 5$. There remain two open cases, namely, the $\Omega_3$ system for type $A_2$ and the $\Omega_3$ system for type $D_4$. The aim of this paper is to show that the $\Omega_3$ system does exist for type $D_4$ (see Theorem 3.13). In order to achieve the result we use several facts from both [1] and [2]. By using these facts, we significantly reduce the amount of computation to show the existence of the system. In the other remaining case, for the algebra of type $A_2$, the Heisenberg parabolic subalgebra coincides with the Borel subalgebra, and the existence of the $\Omega_3$ system(s) follows from the standard reducibility result for Verma modules (see for instance [3, Theorem 7.6.23]).

There are two differences between our conventions here and those used in [1]. One is that we choose the parabolic $Q_0 = L_0N_0$ for the real flag manifold, while the opposite parabolic $\bar{Q}_0 = L_0\bar{N}_0$ is chosen in [1]. Because of this, our special values of $s$ are of the form $s = -s_0$, where $s_0$ are the special values shown in Section 8.10 in [1]. The other is that we identify $(V^+)^*$ with $V^-$ by using the Killing form, while $(V^+)^*$ in [1] is identified with $V^+$ by using the non-degenerate alternating form $\langle \cdot, \cdot \rangle$ on $V^+$ defined by $[X_1, X_2] = \langle X_1, X_2 \rangle X_\gamma$ for $X_1, X_2 \in V^+$. Because of this difference, the right action $R$, which will be defined in Section 2, will play the role played by $\Omega_1$ in [1]. In addition to these notational differences, there are also some methodological differences between [1] and what we do here. These stem from the fact that we make systematic use of the results of [2] to streamline the process of proving conformal invariance.

We now outline the remainder of this paper. In Section 2, we review the setting and results of Section 5 in [2], simultaneously specializing them to the situation considered here. It would be helpful for the reader to be familiar with [2], particularly the concepts of $\mathfrak{g}$-manifold and $\mathfrak{g}$-bundle, at this point; the definitions may be found on pp. 790-791 of [2]. In Section 3, we specialize further by taking $\mathfrak{g}$ to be of type $D_4$. We fix a suitable Chevalley basis and give the definition of the $\hat{\Omega}_3$ system whose conformal invariance is to be established. A remark on notation might be helpful here. In [1], a system $\Omega'_3$ is initially defined. It is then shown to decompose as a sum of a leading term $\bar{\Omega}_3$ and a correction term $C_3$. These two are recombined with different coefficients to give $\Omega_3$, which is finally shown to be conformally invariant for exceptional algebras. For type $D_4$, it emerges that $\Omega_3 = \bar{\Omega}_3$, so that the correction term $C_3$ is discarded completely. For this reason, we do not recapitulate the process. Rather, we simply introduce $\hat{\Omega}_3$ and proceed to show that it is conformally invariant. This is done in Theorem 3.11, which is our main result.
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2. Conformally Invariant Systems

The purpose of this section is to introduce the notion of conformally invariant systems. Let $G_0$ be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. Let $Q_0$ be a parabolic subgroup of $G_0$ and $Q_0 = L_0N_0$ a Levi decomposition of $Q_0$. By the Bruhat decomposition, the subset $N_0Q_0$ of $G_0$ is open and dense in $G_0$, where $N_0$ is the nilpotent subgroup of $G_0$ opposite to $N_0$. Let $\mathfrak{u}$ and $\mathfrak{q}$ be the complexifications of the Lie algebras of $\tilde{N}_0$ and $Q_0$, respectively; we have the direct sum $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{q}$. For $Y \in \mathfrak{g}$, write $Y = Y_\mathfrak{u} + Y_\mathfrak{q}$ for the decomposition of $Y$ in this direct sum. Similarly, write the Bruhat decomposition of $g \in \tilde{N}_0Q_0$ as $g = \tilde{\mathfrak{u}}(g)\mathfrak{q}(g)$ with $\tilde{\mathfrak{u}}(g) \in \tilde{N}_0$ and $\mathfrak{q}(g) \in Q_0$. Note that for $Y \in \mathfrak{g}_0$ we have

$$Y_\mathfrak{u} = \frac{d}{dt} \tilde{\mathfrak{u}}(\exp(tY))|_{t=0},$$

and a similar equality holds for $Y_\mathfrak{q}$.

We consider the homogeneous space $G_0/Q_0$. Let $\mathbb{C}_{\chi^{-s}}$ be the one-dimensional representation of $L_0$ with character $\chi^{-s}$. The representation $\chi^{-s}$ is extended to a representation of $Q_0$ by making it trivial on $N_0$. For any manifold $M$, denote by $C^\infty(M, \mathbb{C}_{\chi^{-s}})$ the smooth functions from $M$ to $\mathbb{C}_{\chi^{-s}}$. The group $G_0$ acts on the space

$$C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) = \{ F \in C^\infty(G_0, \mathbb{C}_{\chi^{-s}}) \mid F(qg) = \chi^{-s}(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0 \}$$

by left translation, and the action $\Pi_s$ of $\mathfrak{g}$ on $C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ arising from this action is given by

$$(\Pi_s(Y) \bullet F)(g) = \frac{d}{dt} F(\exp(-tY)g)|_{t=0}$$

for $Y \in \mathfrak{g}_0$. Here the dot $\bullet$ denotes the action of $\Pi_s(Y)$. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We use the same symbols for the extended actions.

The restriction map $C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \rightarrow C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}})$ is an injection whose image is dense for the smooth topology. We may define the action of $\mathcal{U}(\mathfrak{g})$ on the image of the restriction map by $\Pi_s(u) \bullet f = (\Pi_s(u) \bullet F)|_{\tilde{N}_0}$ for $u \in \mathcal{U}(\mathfrak{g})$ and $F \in C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ with $f = F|_{\tilde{N}_0}$. Define a right action $R$ of $\mathcal{U}(\tilde{\mathfrak{u}})$ on $C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(R(X) \bullet f)(\tilde{\mathfrak{u}}) = \frac{d}{dt} f(\tilde{\mathfrak{u}} \exp(tX))|_{t=0}$$

for $X \in \tilde{\mathfrak{u}}_0$ and $f \in C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}})$. A direct computation shows that

$$(\Pi_s(Y) \bullet f)(\tilde{\mathfrak{u}}) = -sd\chi((\text{Ad}(\tilde{\mathfrak{u}}^{-1})Y)_\mathfrak{q})f(\tilde{\mathfrak{u}}) - (R((\text{Ad}(\tilde{\mathfrak{u}}^{-1})Y)\tilde{\mathfrak{u}}) \bullet f)(\tilde{\mathfrak{u}})$$

for $Y \in \mathfrak{g}$ and $f$ in the image of the restriction map $C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \rightarrow C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}})$. This equation implies that the representation $\Pi_s$ extends to a representation of $\mathcal{U}(\mathfrak{g})$ on the whole space $C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}})$. Note that for all $Y \in \mathfrak{g}$, the linear map $\Pi_s(Y)$ is in $C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \otimes \mathfrak{X}(\tilde{N}_0)$, where
\( \mathcal{X}(\tilde{N}_0) \) is the space of smooth vector fields on \( \tilde{N}_0 \). This property of \( \Pi_s(Y) \) makes \( \tilde{N}_0 \) a \( \mathfrak{g} \)-manifold in the sense of [2] page 790.

Let \( \mathcal{L}_{-s} \) be the trivial bundle of \( \tilde{N}_0 \) with fiber \( \mathbb{C}_{\chi^{-s}} \). Then the space of smooth sections of \( \mathcal{L}_{-s} \) is identified with \( C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \). An operator \( D : C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \rightarrow C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \) is said to be a differential operator if it is of the form

\[
D = \sum_{|\alpha| \leq k} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha},
\]

where \( a_\alpha \in C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \), \( k \in \mathbb{Z}_{\geq 0} \), and multi-index notation is being used.

Denote the space of differential operators by \( \mathbb{D}(\mathcal{L}_{-s}) \). The elements of \( C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \) may be regarded as differential operators by identifying them with the multiplication operator they induce. A computation shows that in \( \mathbb{D}(\mathcal{L}_{-s}) \),

\[
([\Pi_s(Y), f])(\bar{n}) = -\big(R((\text{Ad}(\bar{n}^{-1})Y)\bar{n}) \cdot f\big)(\bar{n})
\]

for \( Y \in \mathfrak{g} \) and \( f \in C^\infty(\tilde{N}_0) \). This verifies that \( \Pi_s \) gives \( \mathcal{L}_{-s} \) the structure of a \( \mathfrak{g} \)-bundle in the sense of [2] page 791.

**Definition 2.2.** Let \( \Pi_s \) and \( \mathcal{L}_{-s} \) be as above. A conformally invariant system on \( \mathcal{L}_{-s} \) with respect to \( \Pi_s \) is a list of differential operators \( D_1, \ldots, D_m \in \mathbb{D}(\mathcal{L}_{-s}) \) so that the following two conditions are satisfied:

- (C1) The list \( D_1, \ldots, D_m \) is linearly independent at each point of \( \tilde{N}_0 \).
- (C2) For each \( Y \in \mathfrak{g} \) there is an \( m \times m \) matrix \( C(Y) \) of smooth functions on \( \tilde{N}_0 \) so that, in \( \mathbb{D}(\mathcal{L}_{-s}) \),

\[
[[\Pi_s(Y), D_j]] = \sum_i C_{ij}(Y)D_i.
\]

The map \( \mathfrak{C} : \mathfrak{g} \rightarrow M_{m \times m}(C^\infty(\tilde{N}_0)) \) is called the structure operator.

Now we define

\[
\mathbb{D}(\mathcal{L}_{-s})^\bar{n} = \{ D \in \mathbb{D}(\mathcal{L}_{-s}) \mid [[\Pi_s(X), D]] = 0 \text{ for all } X \in \bar{n}\}.
\]

**Proposition 2.3.** [2] Proposition 13] Let \( D_1, \ldots, D_m \) be a list of operators in \( \mathbb{D}(\mathcal{L}_{-s})^\bar{n} \). Suppose that the list is linearly independent at \( e \) and that there is a map \( b : \mathfrak{g} \rightarrow \mathfrak{gl}(m, \mathbb{C}) \) such that

\[
([\Pi_s(Y), D_i] \cdot f)(e) = \sum_{j=1}^{m} b(Y)_{ji}(D_j \cdot f)(e)
\]

for all \( Y \in \mathfrak{g} \), \( f \in C^\infty(\tilde{N}_0, \mathbb{C}_{\chi^{-s}}) \), and \( 1 \leq i \leq m \). Then \( D_1, \ldots, D_m \) is a conformally invariant system on \( \mathcal{L}_{-s} \). The structure operator of the system is given by \( \mathfrak{C}(Y)(\bar{n}) = b(\text{Ad}(\bar{n}^{-1})Y) \) for all \( \bar{n} \in \tilde{N}_0 \) and \( Y \in \mathfrak{g} \).
As shown on p.802 in [2] the differential operators in $\mathbb{D}(\mathcal{L}_{-})\bar{n}$ can be described in terms of elements of the generalized Verma module

$$M(\mathbb{C}_{sd\chi}) = \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(q) \mathbb{C}_{sd\chi},$$

where $\mathbb{C}_{sd\chi}$ is the $\mathfrak{q}$-module derived from the $Q_0$-representation $(\chi^s, \mathbb{C})$. By identifying $M(\mathbb{C}_{sd\chi})$ as $\mathcal{U}(\bar{n}) \otimes \mathbb{C}_{sd\chi}$, the map $M(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{n})$ given by $u \otimes 1 \mapsto u$ is an isomorphism. The composition

$$(2.4) \quad M(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{n}) \to \mathbb{D}(\mathcal{L}_{-})\bar{n}$$

is then a vector-space isomorphism, where the map $\mathcal{U}(\bar{n}) \to \mathbb{D}(\mathcal{L}_{-})\bar{n}$ is given by $u \mapsto R(u)$.

Suppose that $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ and $l \in L_0$. Then we define an action of $L_0$ on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(l \cdot f)(\bar{n}) = \chi^{-s}(l)f(l^{-1}\bar{n}l).$$

This action agrees with the action of $L_0$ by left translation on the image of the restriction map $C^\infty(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. In terms of this action we define an action of $L_0$ on $\mathbb{D}(\mathcal{L}_{-})$ by

$$(l \cdot D)\bullet f = l \cdot (D\bullet(l^{-1} \cdot f)).$$

One can check that we have $l \cdot R(u) = R(\text{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{n})$; in particular this $L_0$-action stabilizes the subspace $\mathbb{D}(\mathcal{L}_{-})\bar{n}$. Also $L_0$ acts on $M(\mathbb{C}_{sd\chi})$ by $l \cdot (u \otimes z) = \text{Ad}(l)u \otimes z$, and with these actions, the isomorphism (2.4) is $L_0$-equivariant. For $D \in \mathbb{D}(\mathcal{L}_{-})$, we denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D\bullet f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. The following result is the specialization of Theorem 15 in [2] to the present situation.

**Theorem 2.5.** Suppose that $F$ is a finite-dimensional $\mathfrak{q}$-submodule of the generalized Verma module $M(\mathbb{C}_{sd\chi})$. Let $f_1, \ldots, f_k$ be a basis of $F$ and define constants $a_{ri}(Y)$ by

$$Yf_i = \sum_{r=1}^k a_{ri}(Y)f_r$$

for $1 \leq i \leq k$ and $Y \in \mathfrak{q}$. Let $D_1, \ldots, D_k \in \mathbb{D}(\mathcal{L}_{-})\bar{n}$ correspond to the elements $f_1, \ldots, f_k \in F$. Then

$$[\Pi_q(Y), D_i]_{\bar{n}} = \sum_{r=1}^k a_{ri}((\text{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}})(D_r)_{\bar{n}} - sd\chi((\text{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}})(D_i)_{\bar{n}}$$

for all $Y \in \mathfrak{g}$, $1 \leq i \leq k$, and $\bar{n} \in \bar{N}_0$.

3. **The $\Omega_3$ System on $\mathfrak{so}(8, \mathbb{C})$**

In this section, we specialize to the situation where $G_0$ is a real form of the group $SO(8, \mathbb{C})$ that contains a real parabolic subgroup of Heisenberg type. In this setting, we construct a system of differential operators on the bundle $\mathcal{L}_1$ and show that it is conformally invariant. We first introduce some notation.

Let $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\Delta$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix $\Delta^+$ a positive system and denote by $S$ the corresponding set of simple roots. We denote the highest root by $\gamma$. Let $B_{\mathfrak{g}}$ denote a positive multiple of the Killing form on $\mathfrak{g}$ and denote by $(\cdot, \cdot)$ the corresponding inner product induced on $\mathfrak{h}^*$. The normalization of $B_{\mathfrak{g}}$ will be specified.
Let us write $||\alpha||^2 = (\alpha, \alpha)$ for any $\alpha \in \Delta$. For $\alpha \in \Delta$, we let $g_\alpha$ be the root space of $g$ corresponding to $\alpha$. For any $\text{ad}(h)$-invariant subspace $V \subset g$, we denote by $\Delta(V)$ the set of roots $\alpha$ so that $g_\alpha \subset V$.

It is known that we can choose $X_\alpha \in g_\alpha$ and $H_\alpha \in h$ for each $\alpha \in \Delta$ in such a way that the following conditions hold. The reader may want to note that our normalizations are special cases of those used in [1].

(C1) For each $\alpha \in \Delta^+$, $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ is an sl(2)-triple. In particular, $[X_\alpha, X_{-\alpha}] = H_\alpha$.

(C2) For each $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$.

(C3) For $\alpha \in \Delta$ we have $B_{g_\alpha}(X_\alpha, X_{-\alpha}) = 1$; in particular, $(\alpha, \alpha) = 2$.

(C4) For $\alpha, \beta \in \Delta$ we have $\beta(H_\alpha) = (\beta, \alpha)$.

Let $q$ be the parabolic subalgebra of $g$ of Heisenberg type; that is, the parabolic subalgebra corresponding to the subset $\{\alpha \in S \mid (\alpha, \gamma) = 0\}$. Denote by $l$ the Levi factor of $q$ and by $n$ the nilpotent radical of $q$. Then the action of $\text{ad}(H_\gamma)$ on $g$ has eigenvalues $-2, -1, 0, 1, 2$, and the corresponding eigenvalue decomposition of $g$ is denoted by $g = z(\bar{n}) \oplus V^- \oplus l \oplus V^+ \oplus z(n)$.

Note that $V^+$ and $V^-$ are irreducible $l$-modules, since the Heisenberg parabolic $q$ is maximal (see [4] Exercise 5, page 638 for instance).

Let $D_\gamma(g, h)$ be the deleted Dynkin diagram associated to the Heisenberg parabolic $q$; that is, the subdiagram of the Dynkin diagram of $(g, h)$ obtained by deleting the node corresponding to the simple root that is not orthogonal to $\gamma$, and the edges that involve it.

As on p.789 in [1] the operator $\Omega_2$ is given in terms of $R$ by

$$\Omega_2(Z) = -\frac{1}{2} \sum_{\alpha, \beta \in \Delta(V^+)} N_{\beta, \gamma'} M_{\alpha, \beta'}(Z) R(X_{-\alpha}) R(X_{-\beta})$$

for $Z \in l$. It follows from Theorem 5.2 of [1] and the data tabulated in Section 8.10 of [1] that each $\Omega_2$ system associated to a singleton component of $D_\gamma(g, h)$ is conformally invariant on the line bundle $L_1$. The reader may want to note here that the special values of our $\Omega_2$ system are of the form $-s_0$ with $s_0$ the special values of the $\Omega_2$ system given in [1], because the parabolic $q$ is chosen in this paper, while the opposite parabolic $\bar{q}$ is chosen in [1]. One can also check that we have $\Omega_2(\text{Ad}(l)Z) = \chi(l) \cdot \Omega_2(Z)$ for all $l \in L_0$. Note that this is different from the $\text{Ad}(l)$ transformation law that appears in [1], for the same reason. We extend the $C$-linear maps $d\chi$, $R$, and $\Omega_2$ to be left $C^\infty(\bar{N}_0)$-linear so that certain relationships can be expressed more easily.

In the rest of this paper our line bundle is assumed to be $L_1$ and for simplicity we denote $\Pi_1$ by $\Pi$. Now we define an operator $\bar{\Omega}_3$ on $C^\infty(\bar{N}_0, \mathbb{C}_\chi)$ by
\[ \tilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) \Omega_2([X_\epsilon, Y]) \]

for \( Y \in V^- \).

**Lemma 3.1.** Let \( W_1, \ldots, W_m \) be a basis for \( V^+ \) and \( W_1^*, \ldots, W_m^* \) be the \( B_g \)-dual basis of \( V^- \). Then

\[
\tilde{\Omega}_3(Y) = \sum_{i=1}^m R(W_i^*) \Omega_2([W_i, Y]).
\]

**Proof.** Suppose that \( \Delta(V^+) = \{\epsilon_1, \ldots, \epsilon_m\} \). Each \( W_i \) then may be expressed by

\[
W_i = \sum_{j=1}^m a_{ij} X_{\epsilon_j}
\]

for \( a_{ij} \in \mathbb{C} \). Let \([a_{ij}]\) be the change of basis matrix and set \([b_{ij}] = [a_{ij}]^{-1}\). Then define

\[
W_i^* = \sum_{k=1}^m b_{ki} X_{-\epsilon_k}
\]

for \( i = 1, \ldots, m \). Since \( B_g(X_{\epsilon_i}, X_{-\epsilon_j}) = \delta_{ij} \) with \( \delta_{ij} \) the Kronecker delta, it follows that \( B_g(W_i, W_j^*) = \delta_{ij} \).

Thus \( \{W_1^*, \ldots, W_m^*\} \) is the dual basis of \( \{W_1, \ldots, W_m\} \). Note that we have \( \sum_{i=1}^m b_{ki} a_{ij} = \delta_{kj} \) since \([b_{ij}][a_{ij}] = I\). Then a direct computation shows that

\[
\sum_{i=1}^m R(W_i^*) \Omega_2([W_i, Y]) = R \sum_{k=1}^m \left( \sum_{i=1}^m b_{ki} a_{ij} \right) R(X_{-\epsilon_k}) \Omega_2([X_{\epsilon_j}, Y])
\]

\[
= \sum_{j=1}^m R(X_{-\epsilon_j}) \Omega_2([X_{\epsilon_j}, Y]).
\]

This completes the proof. \( \square \)

**Lemma 3.2.** For all \( l \in L_0, Z \in l, \) and \( Y \in V^- \), we have

\[
\tilde{\Omega}_3(\text{Ad}(l)Y) = \chi(l) l \cdot \tilde{\Omega}_3(Y)
\]

and

\[
[\Pi(Z), \tilde{\Omega}_3(Y)] = \tilde{\Omega}_3([Z, Y]) - d\chi(Z) \tilde{\Omega}_3(Y).
\]

**Proof.** Recall that \( l \cdot R(u) = R(\text{Ad}(l)u) \) for \( l \in L_0 \) and \( u \in \mathcal{U}(\bar{n}) \). Since we have \( \Omega_2(\text{Ad}(l)W) = \chi(l) l \cdot \Omega_2(W) \) for \( l \in L_0 \) and \( W \in l \), it follows that

\[
\chi(l) l \cdot \tilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(\text{Ad}(l)X_{-\epsilon}) \Omega_2([\text{Ad}(l)X_\epsilon, \text{Ad}(l)Y]).
\]

By Lemma 3.1, the value of \( \tilde{\Omega}_3(Y) \) is independent from a choice of a basis for \( V^+ \). Therefore the right hand side of (3.4) is equal to the sum \( \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) \Omega_2([X_\epsilon, \text{Ad}(l)Y]) \), which is \( \tilde{\Omega}_3(\text{Ad}(l)Y) \).

The second equality is obtained by differentiating the first. \( \square \)
Lemma 3.6. For all \( X \in \mathfrak{g}, Y \in V^- \), and \( \bar{n} \in \bar{N}_0 \).

Proof. Let \( F \) be the subspace of \( \mathcal{M}(\mathbb{C}_{-d\chi}) \) spanned by \( X_{-\alpha} \otimes 1 \) and \( 1 \otimes 1 \) with \( \alpha \in \Delta(V^+) \). A direct computation shows that \( F \) is a \( q \)-submodule of \( \mathcal{M}(\mathbb{C}_{-d\chi}) \) and that for \( Z \in \mathfrak{I} \) and \( U \in \mathfrak{n} \) we have

\[
Z(X_{-\alpha} \otimes 1) = [Z, X_{-\alpha}] \otimes 1 - d\chi(Z)X_{-\alpha} \otimes 1
\]

and

\[
U(X_{-\alpha} \otimes 1) = -d\chi([U, X_{-\alpha}]_1) \otimes 1.
\]

Then it follows from Theorem \( 2.5 \) that if \( X \in \mathfrak{g} \) and \( (\Ad(\bar{n}^{-1})X)_q = Z + U \) with \( Z \in \mathfrak{I} \) and \( U \in \mathfrak{n} \) then for \( Y \in V^- \),

\[
[\Pi(X), R(Y)]_{\bar{n}} = R([Z, Y])_{\bar{n}} - d\chi([U, Y]).
\]

Since \( [Z, Y] = [\Ad(\bar{n}^{-1})X, Y]_V \) and \( [U, X_{-\alpha}]_1 = [\Ad(\bar{n}^{-1})X, Y]_1 \), this completes the proof.

Proposition 3.5. We have

\[
[\Pi(X), R(Y)]_{\bar{n}} = R([\Ad(\bar{n}^{-1})X, Y]_V)_{\bar{n}} - d\chi([\Ad(\bar{n}^{-1})X, Y])_{\bar{n}}
\]

for all \( X \in \mathfrak{g}, Y \in V^- \), and \( \bar{n} \in \bar{N}_0 \).

Proof. Let \( \omega_2(X) \) denote the element in \( U(\bar{n}) \otimes \mathbb{C}_{-d\chi} \) that corresponds to \( \Omega_2(X) \) under \( R \).

Lemma 3.6. For \( W, Z \in \mathfrak{I} \), we have

\[
\omega_2([Z, W]) = Z \omega_2(W) + 2d\chi(Z) \omega_2(W).
\]

Proof. Since \( \Omega_2(\Ad(l)W) = \chi(l)I \cdot \Omega_2(W) \) for \( l \in \L_0 \), we have \( \omega_2(\Ad(l)W) = \chi(l) \Ad(l) \omega_2(W) \) by Lemma 18 in [2]. Then the formula is obtained by replacing \( l \) by \( \exp(tZ) \) with \( Z \in \mathfrak{I}_0 \), differentiating, and setting at \( t = 0 \).

Proposition 3.7. We have

\[
[\Pi(X), \Omega_2(W)]_{\bar{n}} = \Omega_2([\Ad(\bar{n}^{-1})X, W]_1)_{\bar{n}} - d\chi((\Ad(\bar{n}^{-1})X)_1) \Omega_2(W)_{\bar{n}}
\]

for all \( X \in \mathfrak{g}, W \in \mathfrak{I}, \) and \( \bar{n} \in \bar{N}_0 \).

Proof. Recall that the \( \Omega_2 \) system is conformally invariant on the line bundle \( \mathcal{L}_1 \). Therefore \( F \equiv \text{span}_{\mathbb{C}}\{\omega_2(W) \mid W \in \mathfrak{I}\} \) is a \( q \)-submodule of \( \mathcal{M}(\mathbb{C}_{-d\chi}) \). By applying Lemma 3.6 with \( Z = H_{\gamma} \), we obtain \( H_{\gamma} \omega_2(W) = -4\omega_2(W) \) for all \( W \in \mathfrak{I} \). For \( U \in V^+ \) we have \( H_{\gamma} U \omega_2(W) = -3U \omega_2(W) \), and \( H_{\gamma} X_{\epsilon} \omega_2(W) = -2X_{\epsilon} \omega_2(W) \) for all \( W \in \mathfrak{I} \). Therefore if \( U \in \mathfrak{n} \) then \( U \omega_2(W) = 0 \) for all \( W \in \mathfrak{I} \), because otherwise \( U \omega_2(W) \) would have the wrong \( H_{\gamma} \)-eigenvalue to lie in \( F \). Since Lemma 3.6 shows that

\[
Z \omega_2(W) = \omega_2([Z, W]) - 2d\chi(Z) \omega_2(W)
\]

for \( Z, W \in \mathfrak{I} \), the proposed formula now follows from Theorem \( 2.5 \).

Lemma 3.8. For \( X \in V^+ \) and \( Y \in V^- \), we have

\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2([\{X, X_{-\epsilon}\}, [X_{-\epsilon}, Y]]) = 2\Omega_2([X, Y]).
\]
Proof. Since we have \(|e|^2 = 2\) for all \(e \in \Delta(V^+)\), it follows from Proposition 2.2 of [1] that
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2([[X, X_{-\epsilon}], [X_\epsilon, Y]]) = \frac{1}{2} \sum_{\epsilon} p(D_4, \epsilon) \Omega_2(\text{pr}_e([[X, Y]])),
\]
where \(\mathcal{C}\) are the connected components of \(\mathcal{D}_r(g, \mathfrak{h})\) as in [1] and \(\text{pr}_e([[X, Y]])\) is the projection of \([X, Y]\) onto \(l(\mathcal{C})\), the ideal of \([l, l]\) corresponding to \(\mathcal{C}\). (See Section 2 of [1] for further discussion.) One can find in Section 8.4 of [1] that \(p(D_4, \epsilon) = 4\) for all the components \(\mathcal{C}\). Then the fact that \(\Omega_2(H_\gamma) = 0\) shows that we obtain
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2([[X, X_{-\epsilon}], [X_\epsilon, Y]]) = 2\Omega_2([[X, Y]]),
\]
which is the proposed formula. \(\square\)

Now with the above lemmas and propositions we are ready to show the following key theorem.

**Theorem 3.9.** We have \(\|\Pi(X), \tilde{\Omega}_3(Y)\|_\epsilon = 0\) for all \(X \in V^+\) and all \(Y \in V^-\).

**Proof.** The commutator \([\Pi(X), \tilde{\Omega}_3(Y)]\) is a sum of two terms. One of them is given by
\[
\sum_{\epsilon \in \Delta(V^+)} [\Pi(X), R(X_{-\epsilon})] \Omega_2([[X_\epsilon, Y]]) = \sum_{\epsilon \in \Delta(V^+)} R((\text{Ad}(-1)X, X_{-\epsilon})_{V^-}) \Omega_2([[X_\epsilon, Y]]) - \sum_{\epsilon \in \Delta(V^+)} d\chi([[\text{Ad}(-1)X, X_{-\epsilon}]]) \Omega_2([[X_\epsilon, Y]]),
\]
by Proposition 3.8. At \(\epsilon\), the first term is zero, since \([X, X_{-\epsilon}]_{V^-} = 0\) for all \(\epsilon \in \Delta(V^+)\). By writing out \(X\) as a linear combination of \(X_\alpha\) with \(\alpha \in \Delta(V^+)\), one can see that at the identity the second term in (3.10) evaluates to
\[
- \sum_{\epsilon \in \Delta(V^+)} d\chi([[X, X_{-\epsilon}]]_\epsilon) \Omega_2([[X_\epsilon, Y]]) = -\Omega_2([[X, Y]])_\epsilon
\]
since \(d\chi(H_\alpha) = 1\) for \(\alpha \in \Delta(V^+)\). The other term is given by
\[
\sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) [\Pi(X), \Omega_2([[X_\epsilon, Y]])] = \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) \Omega_2([[\text{Ad}(-1)X, [X_\epsilon, Y]]]) - \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) d\chi([[\text{Ad}(-1)X]]_\epsilon) \Omega_2([[X_\epsilon, Y]]),
\]
by Proposition 3.7. To further evaluate this expression, we make use of a simple general observation. Namely, if \(D\) is a first order differential operator, \(\phi\) and \(\psi\) are smooth functions, and \(\phi(0) = 0\) then \(D_e(\phi \psi) = D_e(\phi) \psi(0)\). Notice that \(\tilde{n} \mapsto \text{ad}(\text{Ad}(\tilde{n}^{-1})X)\) is a smooth function on \(\tilde{N}_0\). It follows from the left \(C^\infty(\tilde{N}_0)\)-linear extension of \(\Omega_2\) that the first term of the right hand side of (3.11) can be expressed as
\[
\sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon}) (\text{ad}(\text{Ad}(-1)X)_\epsilon \cdot \Omega_2([[X_\epsilon, Y]])),
\]
where \(\text{ad}(\text{Ad}(-1)X)_\epsilon\) denotes the map \(Z \mapsto [\text{Ad}(-1)X, Z]_\epsilon\) for \(Z \in \mathfrak{g}\). Since we have
\[
(R(X_{-\epsilon}) \cdot (\text{Ad}(-1)X))(\epsilon) = [X, X_{-\epsilon}],
\]
For \( [X, [X, Y]]_t = 0 \), and \( X_t = 0 \), the right hand side of (3.11) then evaluates at the identity to
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2(([X, X_{-\epsilon}], [X_{\epsilon}, Y])]_e) \omega_2([X, X_{-\epsilon}], [X_{\epsilon}, Y])]_e - \sum_{\epsilon \in \Delta(V^+)} d\chi([X, X_{-\epsilon}]_e) \Omega_2([X_{\epsilon}, Y])_e,
\]
which is equivalent to
\[
\sum_{\epsilon \in \Delta(V^+)} \Omega_2(([X, X_{-\epsilon}], [X_{\epsilon}, Y])]_e) - \Omega_2([X, Y])_e.
\]
Therefore we obtain
\[
[[\Pi(X), \tilde{\Omega}_3(Y)]_e = \sum_{\epsilon \in \Delta(V^+)} \Omega_2(([X, X_{-\epsilon}], [X_{\epsilon}, Y])]_e) - 2\Omega_2([X, Y])_e.
\]
Now it follows from Lemma 3.8 that \([\Pi(X), \tilde{\Omega}_3(Y)]_e = 0\).

**Proposition 3.12.** For \( Y \in V^- \), we have \([\Pi(X), \tilde{\Omega}_3(Y)]_e = 0\).

**Proof.** Since \( \mathfrak{z}(\mathfrak{n}) = [V^+, V^+] \), it suffices to show that \([\Pi([X_1, X_2]), \tilde{\Omega}_3(Y)]_e = 0 \) for \( X_1, X_2 \in V^+ \). Note that we have \( \Pi([X_1, X_2]) = \Pi(X_1) \Pi(X_2) \), so it follows from the Jacobi identity that \([\Pi([X_1, X_2]), \tilde{\Omega}_3(Y)] \) may be expressed as a sum of two terms. The first is
\[
[\Pi(X_1), [\Pi(X_2), \tilde{\Omega}_3(Y)]] = \Pi(X_1)[\Pi(X_2), \tilde{\Omega}_3(Y)] - [\Pi(X_2), \tilde{\Omega}_3(Y)]\Pi(X_1).
\]
By (2.11), we have \( \Pi(X)_e = 0 \) for all \( X \in \mathfrak{n} \). Using this fact and Theorem 3.9 it is obtained that \([\Pi(X_1), [\Pi(X_2), \tilde{\Omega}_3(Y)]_e = 0 \) since \( (D_1D_2)_e = (D_1)_eD_2 \) for \( D_1, D_2 \in \mathfrak{d}(L_1) \). The second term is
\[
[\Pi(X_2), [\tilde{\Omega}_3(Y), \Pi(X_1)]_e = \Pi(X_2)[\tilde{\Omega}_3(Y), \Pi(X_1)] - [\tilde{\Omega}_3(Y), \Pi(X_1)]_e \Pi(X_2),
\]
It follows from the same argument for the first term that we have \([\Pi(X_2), [\tilde{\Omega}_3(Y), \Pi(X_1)]_e = 0 \). This concludes that the proposition.

**Theorem 3.13.** Let \( \mathfrak{g} \) be the complex simple Lie algebra of type \( D_4 \), and \( \mathfrak{q} \) be the parabolic subalgebra of Heisenberg type. Then the \( \tilde{\Omega}_3 \) system is conformally invariant on the line bundle \( L_1 \).

**Proof.** For \( Y \in V^- \), it follows from Lemma 3.2 that
\[
[\Pi(Z), \tilde{\Omega}_3(Y)]_e = \tilde{\Omega}_3([Z, Y])_e - d\chi(Z)\tilde{\Omega}_3(Y)_e
\]
for all \( Z \in \mathfrak{l} \). Also Theorem 3.9 and Proposition 3.12 show that \([\Pi(U), \tilde{\Omega}_3(Y)]_e = 0 \) for all \( U \in \mathfrak{n} \). By the definition of \( \tilde{\Omega}_3(Y) \), it is clear that \([\Pi(U), \tilde{\Omega}_3(Y)]_e = 0 \) for all \( \tilde{U} \in \mathfrak{n} \). Now by applying Proposition 2.3 we conclude that the \( \tilde{\Omega}_3 \) system is conformally invariant on \( L_1 \).

Let \( \omega_3(Y) \) denote the element in \( \mathcal{U}(\mathfrak{n}) \otimes \mathbb{C}_{-d\chi} \) that corresponds to \( \tilde{\Omega}_3(Y) \) under \( R \). Theorem 3.13 then implies that \( E \equiv \text{span}_\mathbb{C}\{\omega_3(Y) \mid Y \in V^-\} \) is a \( q \)-submodule of \( \mathcal{M}(\mathbb{C}_{-d\chi}) \). Note that it follows from (3.3) that we have \( \omega_3(\text{Ad}(l)Y) = \chi(l)\text{Ad}(l)\omega_3(Y) \) for \( l \in L_0 \). By using the \( \text{Ad}(l) \) transformation law, one can check that a map \( Y \otimes 1 \mapsto \omega_3(Y) \) from \( V^- \otimes \mathbb{C}_{-d\chi} \) to \( E \) is \( L_0 \)-equivariant with the standard action of \( L_0 \) on \( V^- \otimes \mathbb{C}_{-d\chi} \). In particular, \( E \) is an irreducible \( L_0 \)-module, because \( V^- \) is \( L_0 \)-irreducible. Since \( \omega_3 \) has the same \( \text{Ad}(l) \) transformation law as \( \omega_2 \), we have
\[
\omega_3([Z, Y]) = Z\omega_3(Y) + 2d\chi(Z)\omega_3(Y) \quad (3.14)
\]
for $Y \in V^-$ and $Z \in \mathfrak{l}$. The same argument in the proof of Proposition 3.7 then shows that $\mathfrak{n}$ acts on $E$ trivially. Hence, $E$ is a leading $\mathfrak{l}$-type in the sense of [2, page 808].

Now there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism from a generalized Verma module $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{q}) E$ to $\mathcal{M}(\mathbb{C} - d\chi)$, that is given by

\[ u \otimes \omega_3(Y) \mapsto u \cdot \omega_3(Y). \]

It follows from (3.14) that $H_\gamma$ acts on $E$ by $-5$, while it acts on $\mathbb{C} - d\chi$ by $-2$; in particular, $E$ is not equivalent to $\mathbb{C} - d\chi$. We now conclude the following corollary.

**Corollary 3.15.** Let $\mathfrak{g}$ be the complex simple Lie algebra of type $D_4$, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the generalized Verma module $\mathcal{M}(\mathbb{C} - d\chi)$ is reducible.

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