Quantum State Detection via Elimination

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Abstract

We present the view of quantum algorithms as a search-theoretic problem. We show that the Fourier transform, used to solve the Abelian hidden subgroup problem, is an example of an efficient elimination observable which eliminates a constant fraction of the candidate secret states with high probability. Finally, we show that elimination observables do not always exist by considering the geometry of the hidden subgroup states of the dihedral group $D_N$.

1 Introduction

In the classic game of “Twenty Questions”, Player 1 thinks of a secret number between 1 and $N$. Player 2 tries to guess the number in as few tries as possible by asking questions of the form, “Is the secret number less than or equal to $x$?” It is well known that if Player 1 always answers correctly then $\lceil \log N \rceil$ questions are necessary and sufficient to determine the number. Questions like this are studied in combinatorial search theory [2, 1]. More formally, a search problem is a pair $(S, F)$ where $S$ is a set called the search space and $F$ is a set of functions defined on $S$, called the set of allowable questions. There are two players, and the problem is for Player 2 to determine a secret element $x_0 \in S$ initially only known by Player 1. To learn this secret, Player 2 can ask questions $f \in F$ to which Player 1 must answer with elements $x \in S$ for which $f(x) = f(x_0)$. Another way of phrasing this is to say that the allowable questions form a subset of the partition lattice of

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and the answer discloses the block of the chosen partition question which contains the secret element \( x_0 \).

In the present paper we cast known quantum algorithms in a similar light. We present a quantum search-theoretic game in which Player 1 chooses a secret quantum state \( \rho \) from among a set \( S \) of possible states and supplies Player 2 with multiple copies of the secret state. Player 2 applies a sequence of observables for the purpose of discovering the secret state. The observables are the quantum analogues of the partition questions of traditional search theory. Certain sets \( S \) have a geometric quality that permits the construction of a POVM called an \textit{efficient elimination observable} which eliminates any nonsecret state with high probability. In this case the secret state is efficiently discoverable, i.e., in \( O(\log^{O(1)}(|S|)) \) observables, with high probability. This is the quantum analogue of Twenty Questions. We show that the Fourier Transform, the well-known solution of the hidden subgroup problem, is such an observable. This permits an understanding of the hidden subgroup solution independent of harmonic analysis. We then use the hidden subgroup states of the dihedral group \( D_N \) as an example of a set \( S \) which does not possess the geometric property permitting the construction of an elimination observable. This example shows the essential nature of the probabilistic data discussed in \[3\].

2 Elimination Observables

We consider the following quantum state detection game played between two players. There is a set of possible \textit{secret states} \( S = \{\rho_1, \ldots, \rho_N\} \) known to both players. Player 1 (who may be “nature”) chooses a secret state \( \rho_0 \in S \) and provides multiple copies of the secret state when requested to do so by Player 2. The task for Player 2 is to design an observable (POVM) or sequence of observables which allows him to guess the secret state in as few requests to Player 1 as possible. Similar scenarios have been considered \[5\].

\textbf{Definition 1} Let \( S = \{\rho_1, \ldots, \rho_N\} \) be a set of states and \( A = \{A_1, \ldots, A_m\} \) a POVM. For each \( \rho \in S \) we define the elimination set of \( \rho \) with respect to \( A \) as \( E_A(\rho) = \{A_i \in A : \text{tr}(\rho A_i) = 0\} \) and the elimination operator of \( \rho \) with respect to \( A \) as \( A_\rho = \sum_{A_i \in E_A(\rho)} A_i \). \( A \) is called an \textit{efficient elimination observable} for \( S \) if there exists a constant \( c \in (0,1] \) such that for all \( \rho, \rho' \in S \) we have \( \text{tr}(\rho A_\rho) \geq c \) or \( \text{tr}(\rho' A_\rho) \geq c \). An efficient elimination POVM \( A \) is called \textit{optimal} if for all \( A_i \in A, \rho \in S \), and positive operators \( B \), if \( \text{support}(B) \subseteq \text{support}(A_i) \) then \( \text{tr}(\rho B) = 0 \) implies \( \text{tr}(\rho A_i) = 0 \).
Intuitively, an efficient elimination observable allows us to detect a secret state in a polynomial number of experiments with high probability. Suppose the secret state is $\rho = \rho_0$. If $A$ is an efficient elimination observable then for any other $\rho' \in S$, by measuring $A$ on $\rho$ we either eliminate the possibility that $\rho'$ is the secret state with probability at least $c$ or the probability that the secret state is $\rho'$ drops exponentially. This suffices to eliminate $\rho'$ in a polynomial number of experiments with exponentially high probability. Thus in a polynomial number of measurements we eliminate everything except the secret state $\rho_0$.

The notion of an optimal efficient elimination observable captures the notion that it does not help to refine any of the outcomes $A_i \in A$ because to do so would not provide any increased information in the form of knowing more eliminated states.

For any set of states $S$ we may attempt to construct an elimination measurement in the following natural way. For each $\rho \in S$ define $\rho^\perp = \ker(\rho) = \text{support}(\rho)^\perp$ to be the orthogonal complement of the support of $\rho$ in $\mathcal{H}$. We now take intersections of various subsets $\{\rho_1^\perp, \ldots, \rho_m^\perp\}$ in the hope of finding subspaces of $\mathcal{H}$ which lie in many of the $\rho^\perp$s. The idea is that if $L \leq \rho_{i_1}^\perp \cap \cdots \cap \rho_{i_k}^\perp$ is such an elimination subspace and if $L$ is the outcome of a measurement, then we may eliminate $\rho_{i_1}, \ldots, \rho_{i_k}$ as possibilities for the secret state. This method does not always yield an efficient elimination observable as we will see below in the case of hidden subgroups of the dihedral group $D_N$. To obtain an efficient elimination observable we need to find enough of these elimination subspaces to span $\mathcal{H}$. Sometimes the geometry of $S$ does not permit this. However in the case of hidden subgroups of finite, Abelian groups this technique does work and, interestingly enough, yields the same observable which has been used in previous quantum algorithms, namely the Fourier observable.

### 3 Elimination and Hidden Subgroup States

In the hidden subgroup problem we are given a finite group $G$ and an oracle function on $G$ which is promised to be constant and distinct on cosets of some subgroup $H \leq G$. Let $K$ be a transversal for $H$ in $G$ and let $C = K^m$. Using the oracle we may easily prepare multicoset states of order $m$, a tensor product of $m$ coset states of $H$, i.e.

$$|\psi(H, c)\rangle = |c_1 H\rangle \otimes \cdots \otimes |c_m H\rangle,$$
where for any non-empty subset $Y \subseteq G$, 
\[ |Y \rangle = \frac{1}{\sqrt{|Y|}} \sum_{y \in Y} |y \rangle \]
and $c = (c_1, \ldots, c_m) \in C$. Define the mixed state 
\[ \rho_H = \left( \frac{|H|}{|G|} \right)^m \sum_{c \in C} |\psi(H, c)\rangle \langle |\psi(H, c)|. \]

The state $\rho_H$ is equal to the mixed state we obtain by applying the oracle function $m$ times and tracing out the $m$ registers holding the function-values.

Let $S_G = \{ \rho_H : H \leq G \}$ be the set of possible mixed states. Note that $S_G$ also implicitly depends on $m$. In [4] it shown that these mixed states are distinguishable for $m$ being on the order of $\log |G|$. In other words, hidden subgroup states are distinguishable in a small number of oracle calls. A number of problems are reducible to hidden subgroup problems including discrete logarithm, graph isomorphism, code equivalence, and various equivalent problems thought to be strictly harder than graph isomorphism [4]. As an example of this last category we mention restricted graph automorphism, where given a graph $\Gamma$ on $n$ vertices and a subgroup $J$ of $S_n$ given by generators one should find a set of generators for the subgroup $\text{Aut}(\Gamma) \cap J$.

It is well known [7] that when $G$ is Abelian one can efficiently find a hidden subgroup $H$ using only a single coset state at a time, i.e. working in the Hilbert space $\mathbb{C}[G]$, utilizing the quantum Fourier transform. The Fourier transform is a change of basis transformation from the point mass basis to the basis of characters of $G$. Let $\hat{G} = \{ \chi_1, \ldots, \chi_{|G|} \}$ be the group of characters of $G$ and let $|\chi \rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi(g) |g \rangle$. We may alternatively refer to the Fourier observable $F(G)$ which is the self-adjoint operator defined by the character basis:
\[ F(G) = \sum_{i=1}^{|G|} i |\chi_i \rangle \langle \chi_i|. \]

The following result casts this well known fact in the state distinguishability paradigm.

**Theorem 1** If $S_G$ is the set of hidden subgroup states of an Abelian group $G$ then the Fourier observable is a refinement of the unique optimal elimination POVM for $S_G$. 

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This result implies that we may rederive the Fourier observable from strictly geometric considerations. It is our hope that this geometric perspective may yield insight into the value of new observables for similar state distinguishability problems.

The theorem follows immediately from the following two lemmas. Although the proofs of the lemmas rely on results from harmonic analysis for brevity, we emphasize that one may derive them without recourse to these results. Admittedly this may involve difficult calculations but the point is that this basic quantum algorithm may be understood without any knowledge of Fourier analysis.

The first lemma is basic and describes the elimination subspaces of hidden subgroup states. Let \( H \leq G \). Define the orthogonal group to \( H \) as \( H^\perp = \{ \chi : \chi(h) = 1 \text{ for all } h \in H \} \).

**Lemma 2** For any \( H \leq G \), \( \rho_H^\perp = \langle |\chi\rangle : \chi \notin H^\perp \rangle \).

**Proof** Notice \( \rho_H \) corresponds to a random choice of a coset state \( |cH\rangle \). By basic results in Fourier analysis on Abelian groups \( \mathbb{G} \) we have \( |H^\perp| = \frac{|G|}{|H|} \) and we may write the coset state as 

\[
|cH\rangle = \sqrt{|H|/|G|} \sum_{\chi \in H^\perp} \chi(c)|\chi\rangle.
\]

Therefore for all \( \chi \notin H^\perp \) we have \( \langle \chi |cH\rangle = 0 \). This means \( \chi \in \rho_H^\perp \) and therefore \( \langle \chi | : \chi \notin H^\perp \rangle \subseteq \rho_H^\perp \). For the other inclusion, notice that for \( c, d \in G \) with \( c \notin |dH\rangle \), we have \( \langle cH |dH\rangle = 0 \). This means that \( \dim(\rho_H^\perp) = \dim(H) - \frac{|G|}{|H|} \). But \( |\{ \chi : \chi \notin H^\perp \}| = |G| - \frac{|G|}{|H|} = \dim(H) - \frac{|G|}{|H|} \). The lemma follows. \( \square \)

Before stating the second lemma we require some definitions which describe the optimal elimination observable.

Define an equivalence relation on \( \hat{G} \) by \( \chi_1 \equiv \chi_2 \) if and only if \( \chi_1 \in \langle \chi_2 \rangle \) and \( \chi_2 \in \langle \chi_1 \rangle \). Let the equivalence classes be \( \{[\chi_1], \ldots, [\chi_s]\} \). For each class let \( \mathcal{H}_i = \langle |\chi\rangle : \chi \in [\chi_i] \rangle \) be the subspace of \( \mathcal{H} = \mathbb{C}[G] \) spanned by the members of \( [\chi_i] \). Let \( P_i \) be the projection onto \( \mathcal{H}_i \) and define the self-adjoint operator \( A(G) = \sum_{i=1}^s iP_i \).

**Lemma 3** If \( S_G \) is the set of hidden subgroup states of \( G \) then the unique optimal elimination POVM for \( S_G \) is \( A(G) \).
Proof We first prove $A(G)$ is an efficient elimination observable. The argument is actually the standard argument that the quantum algorithm efficiently finds a hidden subgroup but phrased in the present terminology. Suppose the hidden subgroup is $H$, i.e. $\rho = \rho_H$, and let $J \leq G$. If $J \leq H$ then $H^\perp \leq J^\perp$, and further, if $J$ is strictly contained in $H$, then $\frac{|J^\perp|}{|H^\perp|} \leq \frac{1}{2}$. Using the formula above we see $\text{tr}(\rho_J A(G)_{\rho_H}) \geq \frac{1}{2}$. If $J \not\leq H$ then $|H^\perp| |J^\perp \cap H^\perp| \leq \frac{1}{2}$ and thus $\text{tr}(\rho_H A(G)_{\rho_H}) \geq \frac{1}{2}$.

We now show optimality and uniqueness. Let $\chi \equiv \chi'$. Then note that for any $H$, $\chi \in H^\perp$ if and only if $\chi' \in H^\perp$. So by the first lemma $\chi \in \rho_H^\perp$ if and only if $\chi' \in \rho_H^\perp$. This means that $A(G)$ is a refinement of any elimination observable. Unique optimality follows from this. \qed

4 Subgroup States of $D_N$ Cannot be Eliminated

In [3] the hidden subgroup problem over $D_N$ is considered. The quantum algorithm given in that paper results in probabilistic data which information-theoretically determines the hidden subgroup but for which there is no known processing technique which enables the hidden subgroup to be found efficiently. Obviously, it is therefore reasonable to seek another quantum algorithm based on an elimination observable for which any necessary post-processing may be performed efficiently. Unfortunately, there does not exist any such efficient elimination observable for the hidden subgroup states of $D_N$, which we now show.

For the sake of clarity let us consider $D_P$ where $P$ is a prime. Similar arguments hold when $N$ is composite but are cluttered by irrelevant details. Notationally we write elements of $D_P$ as ordered pairs $(a, b) \in \mathbb{Z}_P \times \mathbb{Z}_2$. Without loss of generality we may assume the hidden subgroup has the form $H = \{(0, 0), (k, 1)\}$ [3] and thus refer to hidden reflections. We try to construct an elimination observable in the generic way outlined in the first section. This method fails however because the intersection of any two elimination spaces always results in the same one-dimensional space. We now make this precise.

Let $\mathcal{H}_1$ be the elimination subspace of the hidden reflection $(k_1, 1)$. This means that all vectors in $\mathcal{H}_1$ are orthogonal to all coset states $|a, 0\rangle + |(a + k_1, 1)\rangle$. It is easy to see that $\mathcal{H}_1$ consists of the subspace of $(k_1, 1)$-
antiperiodic vectors, i.e., vectors of the form

$$\sum_{i=0}^{P-1} \lambda_i \langle i, 0 \rangle - \sum_{i=0}^{P-1} \lambda_i \langle (i + k_1) \mod P, 1 \rangle,$$

where $\lambda_0, \ldots, \lambda_{P-1} \in \mathbb{C}$ are complex numbers. Similarly the elimination space $\mathcal{H}_2$ of the hidden reflection $(k_2, 1)$ contains precisely the $(k_2, 1)$-antiperiodic vectors. So $\mathcal{H}_1 \cap \mathcal{H}_2$ contains those vectors which are both $(k_1, 1)$-antiperiodic and $(k_2, 1)$-antiperiodic. But because $P$ is prime the only vectors of this form lie in the one-dimensional subspace spanned by the vector

$$v = \sum_{i=0}^{P-1} \langle i, 0 \rangle - \sum_{i=0}^{P-1} \langle i, 1 \rangle.$$

Therefore the intersection of any two (or more) elimination spaces is the space spanned by $v$. Clearly $\mathbb{C}[D_P]$ is not spanned by subspaces which satisfy the elimination observable criterion. This shows the nonexistence of an elimination observable for $D_P$.

5 Conclusion

We have shown that the quantum algorithm for finding hidden subgroups of finite Abelian groups is an example of a quantum state distinguishability game. This game is a search-theoretic quantum analogue of “Twenty Questions”. The Fourier transform is the unique optimal elimination observable corresponding to the set of hidden subgroup states. We have also shown an example of a search space with a geometry that prevents the construction of an elimination observable, the hidden subgroup states of $D_N$. We mention several possibilities for future work. The classical search-theoretic literature seems to be concerned exclusively with elimination search. What types of probabilistic search are possible and can they be adapted to situations like $D_N$ where elimination is impossible? Finally, are there other, natural problems that may be phrased as state distinguishability problems and solved via elimination observables?

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