Solution of the Kramers’ problem about isothermal sliding of moderately dense gas with accommodation boundary conditions

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Abstract

Half-space boundary Kramers’ problem about isothermal sliding of moderate dense gas with accommodation boundary conditions along a flat firm surface is solving. The new method of the solution of boundary problems of the kinetic theory is applied (see JVMMF, 2012, 52:3, 539-552). The method allows to receive the solution with arbitrary degree of accuracy. The idea of representation of boundary condition on distribution function in the form of source in the kinetic equation serves as the basis for the method mentioned above. By means of Fourier integrals the kinetic equation with a source comes to the Fredholm integral equation of the second kind. The solution has been received in the form of Neumann’s number.

Keywords: Kramers’ problem, moderately dense gas, mirror-diffusion boundary conditions, Neumann series.

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1 Introduction

The Kramers’ problem is one of the major problems in kinetic theory of gases. This problem has great practical importance. The solution of this problem is set forth in such monographies, as [1] and [2].

55 years ago precisely K. M. Case in the well-known work [3] laid the foundation of the analytical solution of boundary problems of the transport theory. The idea of this method included the following: to find the general solution of the non-uniform characteristic equation corresponding to the transport equation, in class of the generalized functions in the form of the sum two generalized functions (an integral principal value and composed, proportional to Dirac delta function).

The first of those composed is the partial solution of non-uniform characteristic equation, and the second is the common solution of corresponding homogeneous equation answering to the non-uniform characteristic equation. As proportionality coefficient in this expression costs so named dispersion function. Zero of dispersion function are connected biunique with partial solution of the initial transport equations.

We come to the characteristic equation after division of variables in the transport equation. By means of spectral parametre we divide spatial and velocity variables in the transport equation.
The general solution of the characteristic equation include as a particular solution singular Cauchy kernel, which denominator is a difference of velocity and spectral variable.

Cauchy kernel provides the use of all powers of methods of the theory functions complex variable, in particular, theory of boundary value Riemann—Hilbert problems.

So, construction of eigen functions of the characteristic equations leads to concept of the dispersion equation, which roots are in biunique conformity with partial (discrete) solutions of the initial transport equation.

The general solution of boundary problems for the transport equation is searched in form of linear combination of discrete solutions with arbitrary coefficients (these coefficients are called as discrete coefficients) and integral on spectral parametre from the eigen functions of the continuous spectrum, multiplicated by unknown function (coefficient of continuous spectrum). Some discrete coefficients are setting and considered as known. Discrete coefficients answer to the discrete spectrum, or, in some cases, answer to spectrum, attached to the continuous.

Substitution of general solution in boundary conditions leads to singular integral equation with Cauchy kernel. The solution of it allows to construct the solution of an initial boundary problem of the transport equations.

Acting in such a way, Carlo Cercignani in 1962 in work [4] constructed the exact solution of Kramers’ problem with pure diffusion boundary conditions about isothermal sliding. Cercignani in [4] gave the exact solution of linear kinetic stationary Boltzmann equation with integral of collisions in form of \( \tau \)–model. Here with there were used diffusion boundary conditions.

Works [3, 4] laid down the foundations of the analytical methods for finding of exact solutions of the modelling kinetic equations.

Generalization of this method on a vector case (system of the kinetic equations) with constant frequency of collisions encounters on the considerable difficulties (see, for example, [5]). Such difficulties have faced authors of works
To overcome these difficulties it was possible in work \cite{7}, in which the problem about temperature jump has been solved.

The problem about temperature jump with frequency of collisions, proportional to the module of molecule velocity, it has been solved by Wiener—Hopf’s method in work \cite{8}. Then in more general statement taking into account weak evaporation (condensation) this problem has been solved by Case’ method in our work \cite{9}. These problems are reduced to the solution of the vector kinetic equations.

The Kramers’ problem has been generalized further on the case of the binary gases \cite{10} and \cite{11}, it has been solved with use of the higher models of Boltzmann equations \cite{12}, it has been generalized on the case accommodation \cite{13} and mirror–diffusion boundary conditions \cite{14}.

Last decade the Kramers’ problem has been formulated and it is analytically solved for quantum gases \cite{15} and \cite{16}.

In our works \cite{17}, \cite{18} and \cite{19} the approximation methods of the solution of boundary problems of kinetic theory have been developed.

In the present work we use a new effective method of the solution of boundary problems with mirror–diffusion boundary conditions \cite{19} for the solution of Kramers’ problem for moderately dense gas.

The question on integral of collisions for dense gases has been studying during last decades \cite{22}. In the solution of this very question there are various approaches, since Enskog’ models \cite{23} and finishing modern decomposition of integral collisions abreast on density degrees. General line, characteristic for all approaches, is nonlocal property of integral collisions for dense gases.

In work \cite{24} the model of integral collisions for moderately dense gas has been offered. Then in works \cite{25} – \cite{28} this model has been used in concrete problems. Besides, in work \cite{29} the Kramers’ problem for moderately dense gas with diffusion boundary condition is has been formulated and analytically solved.

At the heart of an offered method lays the idea to include the boundary
condition on function of distribution in the form of a source in the kinetic equation.

The main point of an offered method include the following. Initially our Kramers’ problem for moderately dense gas about isothermal sliding with mirror–diffusion boundary conditions is formulated in half-space $x > 0$. Then distribution function proceeds in the conjugated (interfaced) half-space $x < 0$ in the even manner on to spatial and on velocity variables. In half-space $x < 0$ also are formulated an initial problem.

After it is received the linear kinetic equation let us break required function (which we will name also distribution function) on two composed: Chapman–Enskog’ distribution function $h_{as}(x, \mu)$ and the second part of distributions function $h_{c}(x, \mu)$, correspond to the continuous spectrum

$$h(x, \mu) = h_{as}(x, \mu) + h_{c}(x, \mu).$$

($as \equiv asymptotic, c \equiv continuous$).

By virtue of the fact that The Chapman–Enskog distribution function is a linear combination of discrete solutions of the initial equation, the function $h_{c}(x, \mu)$ also is the solution of the kinetic equations. Function $h_{c}(x, \mu)$ passes in zero far from wall. On a wall this function satisfies to mirror–diffusion boundary condition.

Further we will transform the kinetic equation for function $h_{c}(x, \mu)$. We include in this equation the boundary condition on to wall for function $h_{c}(x, \mu)$ in the form of a member of type the source laying in a plane $x = 0$. We will underline, that function $h_{c}(x, \mu)$ satisfies to the received kinetic equation in both conjugated half-spaces $x < 0$ and $x > 0$.

We solve this kinetic equation in the second and the fourth quadrants of phase plane $(x, \mu)$ as the linear differential equation of the first order, considering known mass velocity of gas $U_{c}(x)$. From the received solutions it is found the boundary values of unknown function $h^{\pm}(x, \mu)$ at $x = \pm 0$, entering into the kinetic equation.
Now we expand unknown function $h_c(x, \mu)$, unknown mass velocity $U_c(x)$ and delta-Dirac function as Fourier integrals. Boundary values of unknown function $h_c^\pm(0, \mu)$ are thus expressed by the same integral on the spectral density $E(k)$ of mass velocity.

Substitution of Fourier integrals in the kinetic equation and expression of mass velocity leads to the characteristic system of the equations. If to exclude from this system the spectral density $\Phi(k, \mu)$ of function $h_c(x, \mu)$, we will receive Fredholm integral equation of the second kind.

Considering the gradient of mass speed is set, we will expand the unknown velocity of sliding, and also spectral density of mass velocity and distribution functions in series by degrees on diffusions coefficients $q$ (these are Neumann’s numbers). On this way we receive numerable system of the hooked equations on coefficients of series for spectral density. Thus all equations on coefficients of spectral density for mass velocity have singularity (a pole of second order in zero). Excepting these singularities consistently, we will construct all members of series for velocity slidings, and also series for spectral density of the mass velocity and distribution function.

2 Raising an issue

Assuming that moderately dense gas occupies half-space $x > 0$ over firm flat motionless wall. We take the Cartesian system of coordinates with an axis $x$, perpendicular wall, and with plane $(y, z)$, coinciding with a wall, so that the origin of coordinates lays on a wall.

Let us assume, that far from a wall and along an axis $y$ the gradient of mass velocity of the gas which quantity is equal $g_v$ is set. The setting of gradient of mass velocity of gas causes motion of gas along a wall. Let us consider this motion in absence of the tangential gradient of pressure and at to constant temperature. In these conditions the mass velocity of gas will have only one
tangential component \( u_y(x) \), which far from a wall will vary under the linear law. A deviation from linear distribution will be to occur near to a wall in a layer, often named Knudsen’ layer, which thickness has an order of length of the mean free path \( l \). Out of Knudsen’ layer the gas motion is described by the equations of Navier—Stokes. The phenomenon of movement of gas along a surface, caused the gradient of mass velocity set far from a wall, is called isothermal sliding of gas. For the solution of Navier—Stokes equations it is required to put boundary conditions on to wall. As such boundary condition is accepted the extrapolated value of hydrodynamic velocity on a surface (quantity \( u_{sl} \)). We note that a real profile of velocity in Knedsen’ layer it is distinct from the hydrodynamic. For reception quantity \( u_{sl} \) it is required to solve the Boltzmann equation in Knudsen’ layer.

At the small gradients of velocity it is had

\[
u_{sl} = K_v l g_v, \quad g_v = \left( \frac{du_y(x)}{dx} \right)_{x \to +\infty}.
\]

Problem of finding of isothermal velocity sliding \( u_{sl} \) is called as Kramers’ problem (see, for example, [1]). Definition of the quantity \( u_{sl} \) allows, as we will see more low, completely construct the distributionfunction of gas molecules in the given problem. And find the profile of distribution of mass velocity of gas in half-space, and also find the quantity of mass velocity of gas on half-space border.

As the kinetic equation for distribution function let us use the linear kinetic stationary Boltzmann equation with integral of collisions for moderately dense gas

\[
u_x \frac{\partial \varphi(x, \mathbf{C})}{\partial x} = \nu \left[ 2C_y u_y(x) + 2\gamma C_y C_x \frac{du_y(x)}{dx} - \varphi(x, \mathbf{C}) \right]. \tag{2.1}
\]

Here for molecules—hard spheres,

\[
\gamma = \frac{4}{15} \pi n \sigma^3,
\]

where \( \sigma \) is the effective diameter of molecules, \( n \) is the concentration (number density) of gas.
In equation (2.1) \( \nu \) is the effective collision frequency, \( \tau = 1/\nu \) is the characteristic time between two consecutive collisions, \( C \) is the dimensionless velocity of gas molecules, \( C = \sqrt{\beta} v \), function \( \varphi(x, C) \) is connected with absolute Maxwellian equality

\[
f(x, C) = f_0(v)[1 + \varphi(x, C)],
\]

where \( f_0(v) \) is the absolute Maxwellian,

\[
f_0(x, v) = n_0 \left( \frac{\beta}{\pi} \right)^{3/2} \exp \left\{ -\beta [v_x^2 + v_y^2 + v_z^2] \right\}, \quad \beta = \frac{m}{2kT},
\]

\( v = (v_x, v_y, v_z) \) is the velocity of molecule, \( k \) is the Boltzmann constant, \( m \) is the mass of molecule, \( n \) and \( T \) are accordingly numerical density (concentration) and temperature, considered in the given problem are constants.

Let us present the equation (1.1) in the dimensionless form. For this purpose let us enter dimensionless mass velocity in an axis direction \( y U(x) = \sqrt{\beta} u_y(x) \) and coordinate \( x_1 = \nu \sqrt{\beta} x \). Clearly, that dimensionless gradient of mass velocity

\[
G_v = \left( \frac{dU}{dx_1} \right)_{x_1 \to +\infty}
\]

It is connected with gradient \( g_v \) by equality \( G_v = g_v/\nu \).

Now the equation (2.1) registers in the form

\[
C_x \frac{\partial \varphi(x_1, C)}{\partial x_1} + \varphi(x_1, C) =
\]

\[
= \frac{2C_y}{\pi^{3/2}} \int e^{-C' r^2} C'_y \left[ \varphi(x_1, C') + \gamma C_x \frac{d\varphi(x_1, C')}{dx_1} \right] d^3 C'. \quad (2.2)
\]

Let us continue function of distribution to the conjugated half-space in the symmetric manner

\[
f(x_1, C) = f(-x_1, -C_x, C_y, C_z). \quad (2.3)
\]

Continuation according to (2.3) on half-space \( x_1 < 0 \) allow to include boundary conditions in the problem equations.
Such continuation of function of distribution allows to consider actually two problems, one of which the second is defined in "positive" half-space $x_1 > 0$, and in "negative" half-space $x_1 < 0$.

Let us formulate mirror–diffusion boundary conditions for distribution function accordingly for "positive" and for "negative" half-space

\[ f(+0, C) = qf_0(C) + (1 - q)f(+0, -C_x, C_y, C_z), \quad C_x > 0, \quad (2.4) \]

\[ f(-0, C) = qf_0(C) + (1 - q)f(-0, -C_x, C_y, C_z), \quad C_x < 0. \quad (2.5) \]

Here $q$ is the diffusion coefficient, $0 \leq q \leq 1$, $f_0(C)$ is the absolute Maxwellian,

\[ f_0(C) = n\left(\frac{\beta}{\pi}\right)^{3/2} \exp(-C^2). \]

In the equations (2.4) and (2.5) parametre $q$ is the part of the molecules, dissipating diffusion on border, $1-q$ is the part of molecules dissipating mirror.

Let's search for distribution function in the form

\[ \varphi(x_1, C) = 2C_y h(x_1, \mu). \]

According (2.3) we have

\[ h(x_1, \mu) = h(-x_1, -\mu), \quad \mu > 0. \]

The equation (2.2) transforms following manner

\[ C_x \frac{\partial h}{\partial x_1} + h(x_1, \mu) = 2U(x_1) + 2\gamma \frac{dU(x_1)}{dx_1}. \]

We obtain the formula for finding of mass velocity

\[ U(x_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-C^2_x) h(x_1, C_x) dC_x. \]

Therefore, for function $h(x_1, \mu)$ we obtain the equation

\[ \mu \frac{\partial h}{\partial x_1} + h(x_1, \mu) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \left[ h(x_1, t) + \gamma \mu \frac{\partial h(x_1, t)}{\partial x_1} \right] dt. \quad (2.6) \]
Boundary condition (2.4) and (2.5) transforms into following boundary condition

\[ h(+0, \mu) = (1 - q)(+0, -\mu) = (1 - q)(-0, \mu), \quad \mu > 0, \quad (2.4') \]
\[ h(-0, \mu) = (1 - q)(-0, -\mu) = (1 - q)(+0, \mu), \quad \mu < 0. \quad (2.5') \]

The right part of the equation (2.6) is the sum of mass velocity of gas

\[ U(x_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} h(x_1, t) dt \]

and its derivative on coordinate.

Let us present function \( h(x_1, \mu) \) in the form

\[ h(x_1, \mu) = h_{\text{as}}^\pm(x_1, \mu) + h_c(x_1, \mu), \quad \text{если} \quad \pm x_1 > 0, \]

where asymptotic part of distribution function (so-called Chapman—Enskog distribution function)

\[ h_{\text{as}}^\pm(x_1, \mu) = U_{\text{sl}}(q) \pm G_v[x_1 - (1 - \gamma)\mu], \quad \text{если} \quad \pm x_1 > 0, \quad (2.7) \]

also is the solution of the kinetic equation (2.6).

Here \( U_{\text{sl}}(q) \) is the required velocity (dimensionless) of the isothermal slidings.

As far from the wall \((x_1 \to \pm\infty)\) distribution function \( h(x_1, \mu) \) passes into Chapman—Enskog \( h_{\text{as}}^\pm(x_1, \mu) \), for function \( h_c(x_1, \mu) \), corresponding to continuous spectrum, we receive the following boundary condition

\[ h_c(\pm\infty, \mu) = 0. \]

From here for mass velocity of gas it is received

\[ U_c(\pm\infty) = 0. \quad (2.8) \]

Let us notice, that in equality (2.7) sign on a gradient in "negative" half-space varies on the opposite. Therefore the condition (2.8) it is carried out automatically for functions \( h_{\text{as}}^\pm(x_1, \mu) \).
Then boundary conditions (2.4') and (2.5') pass in the following

\[ h_c(+0, \mu) = \]
\[ = -h_{as}^+(+0, \mu) + (1 - q)h_{as}^+(+0, -\mu) + (1 - q)h_c(+0, -\mu), \quad \mu > 0, \]
\[ h_c(-0, \mu) = \]
\[ = -h_{as}^-(0, \mu) + (1 - q)h_{as}^-(0, -\mu) + (1 - q)h_c(-0, -\mu), \quad \mu < 0. \]

We denote

\[ h_0^\pm(\mu) = -h_{as}^\pm(0, \mu) + (1 - q)h_{as}^\pm(0, -\mu) = \]
\[ = -qU_{sl}(q) + (2 - q)(1 - \gamma)G_v|\mu|. \]

Let us rewrite the previous boundary conditions in the form

\[ h_c(+0, \mu) = h_0^+(\mu) + (1 - q)h_c(+0, -\mu), \quad \mu > 0, \]
\[ h_c(-0, \mu) = h_0^-(\mu) + (1 - q)h_c(-0, -\mu), \quad \mu < 0, \]
where

\[ h_0^\pm(\mu) = -h_{as}^\pm(0, \mu) + (1 - q)h_{as}^\pm(0, -\mu) = \]
\[ = -qU_{sl}(q) + (2 - q)(1 - \gamma)G_v|\mu|. \]

Considering symmetric continuation of function of distribution, we have

\[ h_c(-0, -\mu) = h_c(+0, +\mu), \quad h_c(+0, -\mu) = h_c(-0, +\mu). \]

Hence, boundary conditions will be rewritten in the form

\[ h_c(+0, \mu) = h_0^+(\mu) + (1 - q)h_c(-0, \mu), \quad \mu > 0, \quad (2.9) \]
\[ h_c(-0, \mu) = h_0^-(\mu) + (1 - q)h_c(+0, -\mu), \quad \mu < 0. \quad (2.10) \]

Let us include boundary conditions (2.9) and (2.10) in the kinetic equation as follows

\[ \mu \frac{\partial h_c}{\partial x_1} + h_c(x_1, \mu) = 2U_c(x_1) + 2\gamma \mu \frac{dU_c(x_1)}{dx_1} + \]
\[ + |\mu| \left[ h_0^\pm(\mu) - qh_c(\mp0, \mu) \right] \delta(x_1), \quad (2.11) \]
where $U_c(x_1)$ is the part of mass velocity corresponding to the continuous spectrum,

$$U_c(x_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} h_c(x_1, t) \, dt. \quad (2.12)$$

Really, let, for example, $\mu > 0$. Let us integrate both parts of the equation (2.11) on $x_1$ from $-\varepsilon$ to $+\varepsilon$. It is as result we receive equality

$$h_c(+\varepsilon, \mu) - h_c(-\varepsilon, \mu) = h_0^+(\mu) - q h_c(-\varepsilon, \mu),$$

whence at $\varepsilon \to 0$ in accuracy it is received the boundary condition (2.9).

On the basis of definition of mass velocity (2.12) it is concluded, that for it the condition (2.8) is satisfied

$$U_c(+\infty) = 0.$$

Hence, in half-space $x_1 > 0$ profile of mass velocity of gas it is calculated under the formula

$$U(x_1) = U_{as}(x_1) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} h_c(x_1, t) \, dt,$$

and far from a wall has following linear distribution

$$U_{as}(x_1) = U_{sl}(q) + G_v x_1, \quad x_1 \to +\infty.$$

3 Kinetic equation in the second and the fourth quadrants of phase plane

Solving the equation (2.11) at $x_1 > 0$, $\mu < 0$, considering set the mass velocity $U(x_1)$, we receive, satisfying boundary conditions (1.10), the following solution

$$h_c^+(x_1, \mu) = -\frac{1}{\mu} \exp\left(-\frac{x_1}{\mu}\right) \int_{-\infty}^{+\infty} \exp\left(+\frac{t}{\mu}\right) 2 \left[U_c(t) + \gamma \mu \frac{dU_c(t)}{dt}\right] dt. \quad (3.1)$$
Similarly at $x_1 < 0$, $\mu > 0$ it is found

$$h_c^-(x_1, \mu) = \frac{1}{\mu} \exp\left(-\frac{x_1}{\mu}\right) \int_{-\infty}^{x_1} \exp\left(+\frac{t}{\mu}\right) 2 \left[ U_c(t) + \gamma \mu \frac{dU_c(t)}{dt} \right] dt. \quad (3.2)$$

Now the equations (2.11) and (2.12) can be copied, having replaced the second member into square bracket from (2.11) according to (3.1) and (3.2), in the form

$$\frac{\partial h_c}{\partial x_1} + h_c(x_1, \mu) =$$

$$= 2U_c(x_1) + 2\gamma \mu \frac{dU_c(x_1)}{dx_1} + |\mu| \left[ h_0^+(\mu) - q h_0^+(0, \mu) \right] \delta(x_1), \quad (3.3)$$

$$U_c(x_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} h_c(x_1, t) dt. \quad (3.4)$$

In equalities (3.3) boundary values $h_c^\pm(0, \mu)$ are expressed through the component of the mass velocity corresponding to continuous spectrum

$$h_c^\pm(0, \mu) = -\frac{1}{\mu} e^{-x_1/\mu} \int_0^{\pm\infty} e^{t/\mu} 2 \left[ U_c(t) + \gamma \mu \frac{dU_c(t)}{dt} \right] dt.$$

For the solution of the equations (3.4) and (3.3) we search in the form of Fourier integrals

$$U_c(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} E(k) \, dk, \quad \delta(x_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \, dk, \quad (3.5)$$

$$h_c(x_1, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \Phi(k, \mu) \, dk. \quad (3.6)$$

Thus distribution function $h_c^+(x_1, \mu)$ is expressed through spectral density $E(k)$ of mass velocity as follows

$$h_c^+(x_1, \mu) = -\frac{1}{\mu} \exp\left(-\frac{x_1}{\mu}\right) \int_{-\infty}^{+\infty} \exp\left(+\frac{t}{\mu}\right) dt \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikt} E(k) (1 + i\gamma \mu k) \, dk =$$
\[
\int_{-\infty}^{\infty} e^{ikx_1} \frac{1 + i\gamma k\mu}{1 + ik\mu} E(k) \, dk = \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{1 - i(1 - \gamma)k\mu + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk.
\]

Similarly arguing, we receive, that for function \( h_c^-(x_1, \mu) \) precisely same formula takes place

\[
h_c^-(x_1, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{1 + i\gamma k\mu}{1 + ik\mu} E(k) \, dk = \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{1 - i(1 - \gamma)k\mu + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk.
\]

Therefore

\[
h_c^\pm(x_1, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{1 + i\gamma k\mu}{1 + ik\mu} E(k) \, dk = \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{1 - i(1 - \gamma)k\mu + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk.
\]

Using parity of function \( E(k) \) further it is received

\[
h_c^\pm(0, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - i(1 - \gamma)k\mu + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk = \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{1 + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk.
\] (3.7)

4 Characteristic system and Fredholm integral equation

Now we will substitute Fourier integrals (3.6) and (3.5), and also equality (3.7) in the equations (3.3) and (3.4). We receive characteristic system of the
equations
\[ \Phi(k, \mu)(1 + ik\mu) = E(k)(1 + i\gamma k\mu) + \]
\[ = |\mu| \left[ -2qU_{sl}(q) + 2(1 - \gamma)(2 - q)G_v|\mu| - \frac{q}{\pi} \int_0^\infty \frac{1 + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk \right], \quad (4.1) \]
\[ E(k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \Phi(k, t) \, dt. \quad (4.2) \]
From equation (4.1) we receive
\[ \Phi(k, \mu) = \frac{1 + i\gamma k\mu}{1 + ik\mu} E(k) + \]
\[ + \frac{|\mu|}{1 + ik\mu} \left[ -2qU_{sl}(q) + 2(1 - \gamma)(2 - q)G_v|\mu| - \right. \]
\[ \left. - \frac{q}{\pi} \int_0^\infty \frac{1 + \gamma(k\mu)^2}{1 + (k\mu)^2} E(k) \, dk \right], \quad (4.3) \]
Let us substitute \( \Phi(k, \mu) \), defined by equality (4.3), in (4.2). We receive, that:
\[ E(k)L(k) = -2qU_{sl}(q)T_1(k) + 2(1 - \gamma)(2 - q)G_vT_2(k) - \]
\[ - \frac{q}{\pi^{3/2}} \int_0^\infty E(k_1)dk_1 \int_{-\infty}^{\infty} \frac{e^{-t^2}|t|(1 - ikt)(1 + \gamma k^2t^2)dt}{(1 + k^2t^2)(1 + k_1^2t^2)}. \quad (4.4) \]
Here
\[ T_n(k) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n dt, \quad n = 0, 1, 2, \ldots, \]
\( L(k) \) is the dispersion function,
\[ L(k) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{1 - i(1 - \gamma)kt + \gamma(kt)^2}{1 + (kt)^2} dt = \]
\[ = 1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \frac{1 + \gamma(kt)^2}{1 + (kt)^2} dt = \]
\[ = 1 - T_0(k) - \gamma k^2 T_2(k) = (1 - \gamma)k^2 T_2(k). \]
It is easy to see, that
\[
1 - T_0(k) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{1 + k^2 t^2} = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2} dt}{1 + k^2 t^2} = k^2 T_2(k).
\]

Besides, internal integral in (4.4) we will transform also we will designate as follows
\[
J(k, k_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} |t|(1 - ikt)(1 + \gamma k^2 t^2) dt}{(1 + k^2 t^2)(1 + k_1^2 t^2)} =
\]
\[
= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2} t(1 + \gamma k_1^2 t^2) dt}{(1 + k^2 t^2)(1 + k_1^2 t^2)}.
\]

Further more general integrals will be necessary for us
\[
J^{(m)}(k, k_1) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-t^2} t^m(1 + \gamma k_1^2 t^2) dt}{(1 + k^2 t^2)(1 + k_1^2 t^2)}, \quad m = 1, 2, \ldots,
\]

besides
\[
J^{(1)}(k, k_1) = J(k, k_1).
\]
Let us copy now the equation (4.4) by means of the previous equality in the following form

\[
E(k)L(k) + \frac{q}{\pi} \int_0^{\infty} J(k, k_1) E(k_1) \, dk_1 =
\]

\[
= -2qU_{sl}(q)T_1(k) + 2(1 - \gamma)(2 - q)G_vT_2(k).
\]

(4.5)

The equation (4.5) is integral equation Фредгольма of the second kind. The kernel of this integral equation is the sum

\[
J(k, k_1) = J_1(k, k_1) + \gamma k^2 J_3(k, k_1),
\]

where

\[
J_n(k, k_1) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t^2} t^n \, dt}{(1 + k^2 t^2)(1 + k_1^2 t^2)},
\]

besides

\[
J_1(0, k_1) = T_1(k_1), \quad J_1(k, 0) = T_1(k).
\]
We note that

\[ T_n(k) = T_n(0) - k^2 T_{n+2}(k), \quad T_n(0) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n dt, \]

besides

\[ T_0(0) = 1, \quad T_1(0) = \frac{1}{\sqrt{\pi}}, \quad T_2(0) = \frac{1}{2}, \quad T_3(0) = \frac{1}{\sqrt{\pi}}, \]

\[ T_4(0) = \frac{3}{4}, \quad T_5(0) = \frac{2}{\sqrt{\pi}}, \quad \cdots. \]

5 Neumann series and infinitely system of equations

Further we also search for the solution in the form of series. These series are Neumann’s series. Members of these series are expressed by multiple integrals. Multiplicity of integrals grows with growth of number of the member of the series.

Let us consider, that the gradient of mass velocity in the equation (4.5) is set. We will expand solutions of the characteristic systems (4.3) and (4.5) abreast on degrees diffusion coefficient \( q \)

\[ E(k) = G_v (1 - \gamma) (2 - q) \left[ E_0(k) + q E_1(k) + q^2 E_2(k) + \cdots \right], \]  

\[ \Phi(k, \mu) = G_v (1 - \gamma) (2 - q) \left[ \Phi_0(k, \mu) + q \Phi_1(k, \mu) + q^2 \Phi_2(k, \mu) + \cdots \right]. \]  

For velocity of sliding \( U_{sl}(q) \) we will search thus in the form

\[ U_{sl}(q) = G_v (1 - \gamma) \frac{2-q}{q} \left[ U_0 + U_1 q + U_2 q^2 + \cdots + U_n q^n + \cdots \right]. \]  

Expansion (5.1) and (5.2) according to (3.5) and (3.6) mean, that mass velocity and the function of distribution corresponding to continuous spectrum, are in the form of expansion

\[ U_c(x_1) = G_v (1 - \gamma) (2 - q) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \left[ E_0(k) + q E_1(k) + q^2 E_2(k) + \cdots \right] dk, \]
\[ h_c(x_1, \mu) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \times \]
\[ \times \int_{-\infty}^{\infty} e^{ikx_1} \left[ \Phi_0(k, \mu) + q\Phi_1(k, \mu) + q^2\Phi_2(k, \mu) + \cdots \right] dk. \]

The offered method allows to consider expansion in series (5.1) - (5.3) which are assumed converging, at all values of coefficient \( q : 0 \leq q \leq 1 \). Besides, it is necessary to notice, that a developed method it is possible to apply not only to problems with mirror–diffusion boundary conditions. The method suits and those classes of problems, for example, with diffusion boundary conditions \( (q = 1) \), which it is impossible to solve analytically.

Let us substitute numbers (5.1) – (5.3) in the equations (4.3) and (4.5). We receive system of the equations

\[ \left[ E_0(k) + qE_1(k) + q^2E_2(k) + \cdots \right] L(k) = \]
\[ = T_2(k) - \left[ U_0 + qU_1 + q^2U_2 + \cdots \right] T_1(k) - \]
\[ -q \cdot \frac{1}{\pi} \int_{0}^{\infty} J(k, k_1) \left[ E_0(k_1) + qE_1(k_1) + q^2E_2(k_1) + \cdots \right] dk_1 \]

and

\[ \Phi_0(k, \mu) + q\Phi_1(k, \mu) + q^2\Phi_2(k, \mu) + \cdots = \]
\[ = \frac{1 + i\gamma k\mu}{1 + ik\mu} \left[ E_0(k) + qE_1(k) + q^2E_2(k) + \cdots \right] + \]
\[ + \frac{|\mu|}{1 + ik\mu} \left[ \mu| - U_0 + qU_1 + q^2U_2 + \cdots \right] - \]
\[ -q \cdot \frac{1}{\pi} \int_{0}^{\infty} \frac{1 + i\gamma k_1\mu}{1 + ik_1\mu} \left[ E_0(k_1) + qE_1(k_1) + q^2E_2(k_1) + \cdots \right] dk_1. \]

Now these integral equations of break up to equivalent infinite system of the equations. In zero approach it is received the following system of the equations

\[ E_0(k)L(k) = T_2(k) - U_0T_1(k), \quad (5.4) \]
20

\[ \Phi_0(k, \mu) = \frac{1 + i\gamma k \mu}{1 + ik \mu} E_0(k) + \frac{|\mu|}{1 + ik \mu} [ |\mu| - U_0 ], \quad (5.5) \]

In first approximation we have

\[ E_1(k) L(k) = -U_1 T_1(k) - \frac{1}{\pi} \int_0^\infty J(k, k_1) E_0(k_1) dk_1, \quad (5.6) \]

\[ \Phi_1(k, \mu) = \frac{1 + i\gamma k \mu}{1 + ik \mu} E_1(k) + \]

\[ + \frac{|\mu|}{1 + ik \mu} \left[ -U_1 - \frac{1}{\pi} \int_0^\infty \frac{1 + \gamma(k_1 \mu)^2}{1 + k_1^2 \mu^2} E_0(k_1) dk_1 \right]. \quad (5.7) \]

In second approximation we have

\[ E_2(k) L(k) = -U_2 T_1(k) - \frac{1}{\pi} \int_0^\infty J(k, k_1) E_1(k_1) dk_1, \quad (5.8) \]

\[ \Phi_2(k, \mu) = \frac{1 + i\gamma k \mu}{1 + ik \mu} E_2(k) + \]

\[ + \frac{|\mu|}{1 + ik \mu} \left[ -U_2 - \frac{1}{\pi} \int_0^\infty \frac{1 + \gamma(k_1 \mu)^2}{1 + k_1^2 \mu^2} E_1(k_1) dk_1 \right]. \quad (5.9) \]

In \( n \)-th approximation we have

\[ E_n(k) L(k) = -U_n T_1(k) - \frac{1}{\pi} \int_0^\infty J(k, k_1) E_{n-1}(k_1) dk_1, \quad (5.10) \]

\[ \Phi_n(k, \mu) = \frac{1 + i\gamma k \mu}{1 + ik \mu} E_n(k) + \]

\[ + \frac{|\mu|}{1 + ik \mu} \left[ -U_n - \frac{1}{\pi} \int_0^\infty \frac{1 + \gamma(k_1 \mu)^2}{1 + k_1^2 \mu^2} E_{n-1}(k_1) dk_1 \right], \quad n = 1, 2, 3, \cdots \quad (5.11) \]
6 Zero approximation

From the formula (5.4) for zero approximation it is found

$$E_0(k) = \frac{T_2(k) - U_0T_1(k)}{L(k)}.$$  

(6.1)

Zero approximation of mass velocity on the basis of (6.1) is equal

$$U_{c(0)}(x_1) = G_v(1 - \gamma) \frac{2 - q}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} E_0(k) \, dk =$$

$$= G_v(1 - \gamma) \frac{2 - q}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{-U_0T_1(k) + T_2(k)}{L(k)} \, dk.$$  

(6.2)

According to (6.2) we will impose on zero approximation of mass velocity the requirement: $U_{c(+\infty)} = 0$. This condition leads to that subintegral expression from Fourier integral (6.2) in a point $k = 0$ finite. Hence, we should eliminate the pole of the second order in a point $k = 0$ at function $E_0(k)$. Noticing, that

$$T_2(0) = \frac{1}{2}, \quad T_1(0) = \frac{1}{\sqrt{\pi}},$$

we seek the zero approximation $U_0$:

$$U_0 = \frac{T_2(0)}{T_1(0)} = \frac{\sqrt{\pi}}{2} \approx 0.8862.$$  

Let us find expression numerator (6.1)

$$T_2(k) - U_0T_1(k) = k^2 \int_{0}^{\infty} \frac{e^{-t^2}t^3}{1 + k^2t^2} \left(1 - \frac{2t}{\sqrt{\pi}}\right) \, dt = k^2 \left[\frac{\sqrt{\pi}}{2} T_3(k) - T_4(k)\right].$$

Therefore we have

$$T_2(k) - \frac{\sqrt{\pi}}{2} T_1(k) = k^2 \varphi_0(k),$$

where

$$\varphi_0(k) = \int_{0}^{\infty} \left(1 - \frac{2t}{\sqrt{\pi}}\right) \frac{e^{-t^2}t^3}{1 + k^2t^2} \, dt = \frac{\sqrt{\pi}}{2} T_3(k) - T_4(k).$$
According to (6.1) and (5.5) we have

\[ E_0(k) = \frac{\varphi_0(k)}{(1 - \gamma)T_2(k)} \]

and

\[ \Phi_0(k, \mu) = 1 + \frac{i\gamma k \mu}{1 + i k \mu} E_0(k) + \frac{|\mu|}{1 + i k \mu} (|\mu| - U_0), \]

and, hence, in zero approximation mass velocity, corresponding to the continuous spectrum, it is equal

\[ U^{(0)}(x_1) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \frac{\varphi_0(k)}{T_2(k)} dk. \]

Corresponding function of distribution is equal

\[ h_c^{(0)}(x_1, \mu) = G_v 2(2 - q) \times \]

\[ \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1 + i\gamma k \mu \varphi_0(k)}{1 + i k \mu T_2(k)} + (1 - \gamma) \frac{\mu^2 - U_0 |\mu|}{1 + i k \mu} \right] e^{ikx_1} dk. \]

7 First approximation

Let us pass to the first approximation. As a first approximation from equations (5.6), (5.7) we find amendments of spectral density mass velocity and distribution function

\[ E_1(k) = -\frac{1}{L(k)} \left( U_1 T_1(k) + \frac{1}{1 - \gamma} \cdot \frac{1}{\pi} \int_{0}^{\infty} \frac{J(k, k_1)}{T_2(k_1)} \varphi_0(k_1) dk_1 \right) \quad (7.1) \]

and

\[ \Phi_1(k, \mu) = \frac{1 + i\gamma k \mu}{1 + i k \mu} E_1(k) - \]

\[ - \frac{|\mu|}{1 + i k \mu} \left[ U_1 + \frac{1}{\pi} \int_{0}^{\infty} \frac{1 + i\gamma k_1 \mu}{1 + i k_1 \mu} E_0(k_1) dk_1 \right]. \quad (7.2) \]
Under these amendments we will construct mass velocity and function of distribution as the first approximation

$$U_c(x_1) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \left[ E_0(k) + qE_1(k) \right] dk$$

and

$$h_c(x_1) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} \left[ \Phi_0(k, \mu) + q\Phi_1(k, \mu) \right] dk.$$

The first amendment to mass velocity looks like

$$U_c^{(1)}(x_1) = G_v(1 - \gamma) \frac{2 - q}{2\pi} \int_{-\infty}^{\infty} e^{ikx_1} E_1(k) dk.$$

The requirement $U_c(+\infty) = 0$ leads to the finiteness requirement subintegral expression in the previous Fourier integral. In expression (7.1) we will substitute decomposition

$$T_1(k) = \frac{1}{\sqrt{\pi}} - k^2T_3(k)$$

and

$$J^{(1)}(k, k_1) = J^{(1)}(0, k_1) - k^2J^{(3)}(k, k_1).$$

It is as a result received, that

$$E_1(k) = -\frac{1}{(1 - \gamma)k^2T_2(k)} \left[ U_1 \frac{1}{\sqrt{\pi}} + \frac{1}{1 - \gamma} \frac{1}{\pi} \int_{0}^{\infty} J^{(1)}(0, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1 - \right.$$  

$$-k^2 \left( U_1 T_3(k) + \frac{1}{1 - \gamma} \frac{1}{\pi} \int_{0}^{\infty} J^{(3)}(k, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1 \right) \right].$$

From this expression it is visible, that for elimination of the pole of the second order in zero, it is necessary to demand, that

$$U_1 = -\frac{1}{1 - \gamma} \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} J^{(1)}(0, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1 =$$
\[ E_1(k) = \frac{1}{(1 - \gamma)T_2(k)} \left[ U_1T_3(k) + \frac{1}{1 - \gamma} \frac{1}{\pi} \int_0^\infty J^{(3)}(k, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1 \right]. \]

Let us transform expression in a square bracket from last expression

\[ U_1T_3(k) + \frac{1}{1 - \gamma} \frac{1}{\pi} \int_0^\infty J^{(3)}(k, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1 = \]

\[ = \frac{1}{1 - \gamma} \frac{1}{\pi} \int_0^\infty S(k, k_1) \frac{\varphi_0(k_1)}{T_2(k_1)} dk_1. \]  

(7.3)

Here

\[ S(k, k_1) = J^{(3)}(k, k_1) - \sqrt{\pi}T_3(k)J^{(1)}(0, k_1) = \]

\[ = J_3(k, k_1) - \sqrt{\pi}T_3(k)T_1(k_1) + \gamma k_1^2 [J_5(k, k_1) - \sqrt{\pi}T_3(k)T_3(k_1)] = \]

\[ = S_1(k, k_1) + \gamma k_1^2 S_2(k, k_1), \]

where

\[ S_1(k, k_1) = J_3(k, k_1) - \sqrt{\pi}T_3(k)T_1(k_1), \]

\[ S_2(k, k_1) = J_5(k, k_1) - \sqrt{\pi}T_3(k)T_3(k_1). \]

Now the spectral density is equal

\[ E_1(k_1) = \frac{\varphi_1(k_1)}{(1 - \gamma)^2 T_2(k_1)}. \]

Here the designation is entered

\[ \varphi_1(k_1) = \frac{1}{\pi} \int_0^\infty S(k_1, k_2) \frac{\varphi_0(k_2)}{T_2(k_2)} dk_2. \]
Now spectral density (7.1) and (7.2) completely are constructed. Velocity of sliding is as the first approximation equal

\[ U_{sl}(q) = G_v(1 - \gamma)^2 q \left[ 0.8862 + \frac{0.1405 + 0.2009 \gamma}{1 - \gamma} \right] = \]

\[ = G_v \frac{2 - q}{q} \left[ 0.8862(1 - \gamma) + (0.1405 + 0.2009\gamma)q \right]. \]

8 Second approximation

Let us pass to the second approximation of the problem. We take the equations (5.8) and (5.9).

From the equation (5.8) it is found

\[ E_2(k) = -\frac{1}{L(k)} \left[ U_2 T_1(k) + \frac{1}{\pi} \int_0^\infty J(k, k_1) E_1(k_1) dk_1 \right]. \]

The second amendment to mass speed looks like

\[ U_c^{(2)}(x_1) = G_v(1 - \gamma)^2 \frac{2 - q}{2\pi} \int_{-\infty}^\infty e^{ikx_1} E_2(k) dk. \]

The condition \( U_c(+\infty) = 0 \) leads to the requirement of limitation of function \( E_2(k) \) in a point \( k = 0 \). Let us eliminate the pole of the second order in the point \( k = 0 \) in the right part of equalities for \( E_2(k) \). For this purpose we will present expression in a square bracket from (8.1) in the form

\[ E_2(k) = -\frac{1}{(1 - \gamma)k^2 T_2(k)} \left[ U_2 \left( \frac{1}{\sqrt{\pi}} - k^2 T_3(k) \right) + \right] + \frac{1}{(1 - \gamma)^2} \frac{1}{\pi} \int_0^\infty \left[ J^{(1)}(0, k_1) - k^2 J^{(3)}(k, k_1) \right] \frac{\varphi_1(k_1)}{T_2(k_1)} \right]. \]

From this equality it is visible, that for pole elimination in zero it is necessary to demand, that

\[ U_2 = -\frac{1}{(1 - \gamma)^2} \frac{1}{\sqrt{\pi}} \int_0^\infty J^{(1)}(0, k_1) \frac{\varphi_1(k_1)}{T_2(k_1)} dk_1. \]
Let us present expression (8.2) in the form convenient for calculations

\[
U_2 = -\frac{1}{(1-\gamma)^2} \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \left\{ T_1(k_1)S_1(k_1, k_2) + \gamma \left[ k_1^2 T_3(k_1) S_1(k_1, k_2) + k_2^2 T_1(k_1) S_2(k_1, k_2) \right] + \gamma^2 k_1^2 k_2^2 T_3(k_1) S_2(k_1, k_2) \right\} \frac{\varphi_0(k_2)dk_1dk_2}{T_2(k_1)T_2(k_2)} = \\
= - \frac{J_0 + \gamma J_1 + \gamma^2 J_2}{(1-\gamma)^2}.
\]

Here

\[
J_0 = \frac{1}{\pi^{3/2}} \int_0^\infty \int_0^\infty T_1(k_1)S_1(k_1, k_2) \frac{\varphi_0(k_2)dk_1dk_2}{T_2(k_1)T_2(k_2)} = 0.0116,
\]

\[
J_1 = \frac{1}{\pi^{3/2}} \int_0^\infty \int_0^\infty \left[ k_1^2 T_3(k_1) S_1(k_1, k_2) + k_2^2 T_1(k_1) S_2(k_1, k_2) \right] \times \\
\times \frac{\varphi_0(k_2)dk_1dk_2}{T_2(k_1)T_2(k_2)} = 0.0125,
\]

\[
J_2 = \frac{1}{\pi^{3/2}} \int_0^\infty \int_0^\infty k_1^2 k_2^2 T_3(k_1) S_2(k_1, k_2) \frac{\varphi_0(k_2)dk_1dk_2}{T_2(k_1)T_2(k_2)} = -0.0306.
\]

Thus, in the second approximation it is found, that

\[
U_2 = - \frac{J_0 + \gamma J_1 + \gamma^2 J_2}{(1-\gamma)^2} = - \frac{0.0116 + 0.0125\gamma - 0.0306\gamma^2}{(1-\gamma)^2}.
\]

On the basis of (8.2) from the previous it is found

\[
E_2(k) = -\frac{1}{(1-\gamma)T_2(k)} \left[ U_2 T_2(k) + \frac{1}{(1-\gamma)^2} \frac{1}{\pi} \int_0^\infty J^{(3)}(k, k_1) \frac{\varphi_1(k_1)}{T_2(k_1)} dk_1 \right] = \\
= \frac{1}{(1-\gamma)^3 T_2(k)} \frac{1}{\pi} \int_0^\infty \left[ J^{(3)}(k, k_1) - \sqrt{\pi} T_3(k) J^{(1)}(0, k_1) \right] \frac{\varphi_1(k_1)}{T_2(k_1)} dk_1 = \\
= \frac{\varphi_2(k)}{(1-\gamma)^3 T_2(k)}.
\]
Here
\[ \varphi_2(k) = \frac{1}{\pi} \int_0^\infty S(k, k_1) \frac{\varphi_1(k_1)}{T_2(k_1)} \, dk_1, \]
where
\[ S(k, k_1) = J^{(3)}(k, k_1) - \sqrt{\pi} T_3(k) J^{(1)}(0, k_1). \]

Under the found amendments of the second order it is possible to construct the mass velocity and distribution function in the second approximation
\[ U_c(x_1) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx_1} \left[ E_0(k) + q E_1(k) + q^2 E_2(k) \right] \, dk \]
and
\[ h_c(x_1) = G_v(1 - \gamma)(2 - q) \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx_1} \left[ \Phi_0(k, \mu) + q \Phi_1(k, \mu) + q^2 \Phi_2(k, \mu) \right] \, dk. \]

The amendment \( \Phi_2(k, \mu) \) we seek from the equation (5.9) with use of the constructed function \( E_2(k) \) and constants \( U_2 \).

Velocity of sliding is equal in the second approximation
\[ U_{sl}(q) = G_v(1 - \gamma) \frac{2 - q}{q} \left[ U_0 + U_1 q + U_2 q^2 \right]. \]

The formula (8.2) we will transform through calculation of the double integral of the following form
\[ U_2 = -\frac{1}{(1 - \gamma)^2} \frac{1}{\pi^{3/2}} \int_0^\infty \int_0^\infty \frac{J^{(1)}(0, k_1) S(k_1, k_2)}{T_2(k_1) T_2(k_2)} \varphi_0(k_2) \, dk_1 \, dk_2. \]

Also we will transform the formula for calculation \( \varphi_2(k) \)
\[ \varphi_2(k) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \frac{S(k, k_1) S(k_1, k_2)}{T_2(k_1) T_2(k_2)} \varphi_0(k_2) \, dk_1 \, dk_2. \]

Spending similar reasonings, for \( n \)-th approximation \( (n = 1, 2, \cdots) \) according to (5.10) and (5.11) it is received
\[ U_n = -\frac{1}{(1 - \gamma)^n} \frac{1}{\sqrt{\pi}} \int_0^\infty J^{(1)}(0, k_1) \frac{\varphi_n(k_1)}{T_2(k_1)}, \]
besides
\[ E_n(k) = \frac{1}{(1 - \gamma)^{n+1}} \frac{\varphi_n(k)}{T_2(k)}, \]

where
\[ \varphi_n(k) = -\frac{1}{\pi} \int_0^\infty S(k, k_1) \frac{\varphi_{n-1}(k_1)}{T_2(k_1)} dk_1, \quad n = 1, 2, \ldots. \]

Let us write out \( n \)th approximation \( U_n \) and \( \varphi_n(k) \), expressed through multiple integrals and zero approximation of spectral density of mass velocity \( E_0(k) \). We have
\[
U_n = -\frac{1}{(1 - \gamma)^n \pi^{n-1/2}} \int_0^\infty \cdots \int_0^\infty J^{(1)}(0, k_1) \frac{S(k, k_1) S(k_1, k_2) \cdots S(k_{n-1}, k_n)}{T_2(k_1) \cdots T_2(k_n)} \times \\
\times \varphi_0(k_n) dk_1 \cdots dk_n,
\]
\[
E_n(k) = \frac{1}{(1 - \gamma)^{n+1} \pi^{n} T_2(k)} \int_0^\infty \cdots \int_0^\infty \frac{S(k, k_1) S(k_1, k_2) \cdots S(k_{n-1}, k_n)}{T_2(k_1) \cdots T_2(k_n)} \times \\
\times \varphi_0(k_n) dk_1 \cdots dk_n,
\]
\[
\varphi_n(k) = \frac{1}{\pi^n} \int_0^\infty \cdots \int_0^\infty \frac{S(k, k_1) S(k_1, k_2) \cdots S(k_{n-1}, k_n)}{T_2(k_1) \cdots T_2(k_n)} \times \\
\times \varphi_0(k_n) dk_1 \cdots dk_n, \quad n = 1, 2, \ldots.
\]

9 Comparison with previous results

Let us compare zero, the first and the second approximation of velocity of sliding with the exact solution of the equation (2.11). It has been above found, that zero, first and second amendments (coefficients of expansion) of velocity of sliding are accordingly equal
\[ U_0 = \frac{\sqrt{\pi}}{2} = 0.8862, \]
\[
U_1 = \frac{0.1405 + 0.2009\gamma}{1 - \gamma},
\]
\[
U_2 = -\frac{0.0116 + 0.0125\gamma - 0.0306\gamma^2}{(1 - \gamma)^2}.
\]

On the basis of these equalities we will write out dimensionless velocity of sliding of moderately dense gas in the second approach

\[
U_{sl}(q) = G_v(1 - \gamma)\left(2 - \frac{q}{q}\right)\left[0.8862 + \frac{0.1405 + 0.2009\gamma}{1 - \gamma}q - \frac{0.0116 + 0.0125\gamma - 0.0306\gamma^2}{(1 - \gamma)^2}q^2\right].
\] (9.1)

We will consider the first limiting case, when \( q \to 0 \), i.e. we consider the Kramers’ problem with the pure diffusion boundary conditions. From equality (9.1) considering, that parametre \( \gamma \) satisfies to an inequality: \( \gamma \ll 1 \), we find, that

\[
U_{sl}(q) = G_v(1.0152 - 0.6862\gamma).
\] (9.2)

Из результатов работы [19] вытекает, что точное выражение безразмерной скорости скольжения равно:

\[
U_{sl}(q) = G_v(V_1 - \frac{1}{\sqrt{2}}) = G_v(1.0162 - 0.7071\gamma).
\] (9.3)

From comparison of formulas (9.2) and (9.3) follows, that for the rarefied gases (\( \gamma = 0 \)) exact velocity of sliding differs on 0.1% from the second approximation of this velocity, and the parametres proportional to density coefficients \( \gamma \), differ on 2% in the same approximation.

Let us consider the second limiting case when moderately dense gas passes in rarefied, i.e. the case \( \gamma \to 0 \). In this case according to the reasonings giving above velocity of sliding it is equal

\[
U_{sl}(q) = G_v\frac{2 - q}{q}\left[U_0 + U_1q + U_2q^2\right],
\]
or, according to equality (9.1)

\[
U_{sl}(q) = G_v\frac{2 - q}{q}\left[0.8862 + 0.1405q - 0.0116q^2\right].
\] (9.2)
According to results from [19] velocity of sliding in the second approximation in accuracy coincides with expression (9.2). If to take here \( q = 1 \), how it was already specified, velocity of sliding is equal \( U_{sl}(q) = 1.0152G_v \), i.e. velocity slidings in the second approximation differs from the exact \( U_{sl}(q) = V_1G_v = 1.0162G_v \) на 0.1%.

Let us reduce the formula for velocity of sliding (4.3) in a dimensional form. Let us write down this formula in the form

\[
 u_{sl} = \frac{2 - q}{q} U(q) \frac{lg_v}{l\sqrt{\beta\nu}},
\]

where

\[
 U(q) = U_0 + U_1q + U_2q^n + \cdots.
\]

Let us choose length of free mean path \( l \) agree Cercignani [4]: \( l = \eta \sqrt{\pi \beta / \rho} \), where \( \eta \) is the dynamic viscosity of gas, \( \rho \) is the its density. For the given problems \( \eta = \rho / (2\nu \beta) \). Hence, in the dimensional form velocity of isothermal sliding it is equal

\[
 u_{sl} = K_v(q)l \left( \frac{du_y(x)}{dx} \right)_{x \to +\infty},
\]

where \( K_v(q) \) is the coefficient of isothermal sliding,

\[
 K_v(q) = \frac{2 - q}{q} U(q) \frac{2}{\sqrt{\pi}}.
\]

### 10 Conclusion

In this very work developed earlier (see ЖВММФ, 2012, 52:3, 539-552) the new method of solution half-spatial boundary problems of the kinetic theory is applied for the solution of the classical problem of the kinetic theory (Kramers’ problem) about isothermal sliding of moderately dense gas along the flat firm surface. The general mirror–diffusion boundary conditions (Maxwell conditions) are used. The kinetic equation with non local integral of collisions is applied. At the heart of this method the idea to continue
distribution function in the conjugated half-space \( x_1 < 0 \) lays. And to include in the kinetic equation boundary condition in the form of a member of type of a source on the function of distribution corresponding to the continuous spectrum. With the help of the Fourier transformation the kinetic equation it is reduced to Fredholm integral equation of the second kind which we solve by method of consecutive approximations. An offered method possesses the quite high efficiency. So, comparison with the exact solution shows, that in the second approximation an error in finding one coefficient does not surpass 0.1\%, and the second does not surpass 2\%.

This method has been successfully applied [20] and [21] in the solution of some such challenges of the kinetic theory, which do not presume the analytical solution.

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