CONTAINMENT PROBLEM FOR THE QUASI STAR CONFIGURATIONS OF POINTS IN $\mathbb{P}^2$

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Abstract. In this paper, the containment problem for the defining ideal of a special type of zero dimensional subschemes of $\mathbb{P}^2$, so called quasi star configurations, is investigated. Some sharp bounds for the resurgence of these types of ideals are given. As an application of this result, for every real number $0 < \varepsilon < \frac{1}{2}$, we construct an infinite family of homogeneous radical ideals of points in $\mathbb{K}[\mathbb{P}^2]$ such that their resurgences lie in the interval $[2 - \varepsilon, 2)$. Moreover, the Castelnuovo-Mumford regularity of all ordinary powers of defining ideal of quasi star configurations are determined. In particular, it is shown that, all of them have linear resolution.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field and let $R = \mathbb{K}[x_0, \ldots, x_N] = \mathbb{K}[\mathbb{P}^N]$ be the coordinate ring of the projective space $\mathbb{P}^N$. Let $I$ be a nontrivial homogeneous ideal of $R$. For each positive integer $m$, two different kinds of powers of $I$ can be constructed. The first one, and the most algebraic one, is the ordinary power $I^m$ of $I$, generated by all products of $m$ elements of $I$. The second one, is the symbolic power $I^{(m)}$ of $I$, which is defined as follows:

$$I^{(m)} = \bigcap_{\mathbb{P} \in \text{Ass}(I)} \mathbb{P}^m,$$

where $U$ is the multiplicative closed set $R - \bigcup_{\mathbb{P} \in \text{Ass}(I)} \mathbb{P}$. The $m$th symbolic power of $I$, rather than the algebraic nature, has a geometric nature. For example, if $I \subseteq \mathbb{K}[\mathbb{P}^N]$ is the radical ideal of a finite set of points $p_1, \ldots, p_n \in \mathbb{P}^N$, then $I^{(m)}$, which is called fat points ideal, geometrically is defined as the ideal of all homogeneous forms vanishing to order at least $m$ at all points $p_i$, and algebraically is defined as $I^{(m)} = \bigcap_{i} I(p_i)^m$, where $I(p_i)$ is the ideal of polynomials vanishing at the point $p_i$. From the above definitions, immediately follows that $I^m \subseteq I^{(m)}$.

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In recent years, comparing the behavior of symbolic powers and ordinary powers of $I$ has prompted many challengeable problems and conjectures in algebraic geometry and commutative algebra (see [2, 12, 15]). One of these problems, known as containment problem, asks for what pairs of positive integers $(m, r)$ one may has $I^{(m)} \subseteq I^r$. The containment problem has an asymptotic interpretation. Indeed, instead of searching for pair of integers $(m, r)$ such that $I^{(m)} \subseteq I^r$, one can approach to the problem asymptotically. Harbourne and Bocci [3] introduced an asymptotic invariant, known as the resurgence, as follows:

$$\rho(I) = \sup\{\frac{m}{r} \mid I^{(m)} \not\subseteq I^r\},$$

where measures the discrepancies between the symbolic powers of a homogeneous ideal and its ordinary powers. The resurgence of $I$ exists and from its definition immediately follows that

$$(1) \quad I^{(m)} \subseteq I^r \text{ if } \frac{m}{r} > \rho(I).$$

In dealing with the containment problem, Ein, Lazarsfeld, and Smith in [7] as well as, Hochster and Huneke in [13], showed, but with different methods, that for all pairs of positive integers $(m, r)$ such that $m \geq Nr$, the containment $I^{(m)} \subseteq I^r$ holds. In particular this implies $1 \leq \rho(I) \leq N$.

Nevertheless, computing this numerical invariant of a homogeneous ideal $I$ is difficult task and there are only few ideals $I$ for which the exact value of $\rho(I)$ is known (see [3, 6, 2, 4]).

In this paper, we study the containment problem of the defining ideal of a special kind of configuration of points in $\mathbb{P}^2$, so called quasi star configuration (see Definition 3.1), which we denote it by $Z_d$. Our main result, provides upper and lower bounds for $\rho(I(Z_d))$ as follows:

**Main Theorem** (Theorem 4.3). Let $I$ be the defining ideal of a quasi star configuration of points $Z_d$ in $\mathbb{P}^2$. If $d \geq 10$ then $2 - \frac{2}{\sqrt{d+1}} \leq \rho(I) \leq 2 - \frac{2}{d+1}$.

This result, with more details, is proved in Example 4.1 and Theorem 4.3. Furthermore, as a corollary to the above theorem, we obtain the following result.

**Theorem** (Corollary 5.1). Let $0 < \varepsilon < 1/2$ be a real number. Then there exists a radical ideal $I_\varepsilon$ in $\mathbb{K}[\mathbb{P}^2]$ such that $\rho(I_\varepsilon) \in [2 - \varepsilon, 2]$.

In [3 Corollary 1.1.1], it is shown that for any homogeneous ideal $I \subset \mathbb{K}[\mathbb{P}^N]$ the containment $I^{(rN)} \subseteq I^r$, where $r$ is a positive integer, is optimal.
Whenever $N = 2$, we can show this optimality can be achieved via quasi star configurations (see Corollary 5.3).

By (1), a necessary condition for the failure of $I^{(2r-1)} \subseteq I^r$, for $r \geq 2$, is that $\rho(I) \geq \frac{2r-1}{r}$, but possibly this is not a sufficient condition for the containment. One of our results, in this paper is to show that, for every integer $r \geq 2$, there exists a radical ideal $I$ such that meets this necessary condition. More generally:

**Theorem** (Corollary 5.2). Let $r \geq 2$ be an integer. Then there exists a configuration of points in $\mathbb{P}^2$ such that its defining ideal $I$ gives the necessary condition for the failure of the containment $I^{(2r-1)} \subseteq I^r$, i.e., $\rho(I) \geq \frac{2r-1}{r}$.

2. Preliminaries

Among the numerical invariants of an arbitrary homogeneous ideal which have been developed to study the containment problem, the Waldschmidt constant, plays as a decisive role. This constant is defined as:

$$\hat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^m)}{m} = \inf_{m \geq 1} \frac{\alpha(I^m)}{m},$$

where $\alpha(I)$ is the least degree of a nonzero polynomial in $I$ and is called the initial degree of $I$. It is shown that this limit exists [3, Lemma 2.3.1].

The containment $I^m \subseteq I^{(m)}$ holds for any positive integer $m$, which implies $\alpha(I^{(m)}) \leq \alpha(I^m) = ma(I)$. Therefore $\hat{\alpha}(I) \leq \alpha(I)$ and consequently $\alpha(I)/\hat{\alpha}(I) \geq 1$. Moreover, see [1, Section 2.1], we have

$$\alpha(I^m)(m+1) \leq \hat{\alpha}(I) \leq \frac{\alpha(I^m)}{m}.$$  \hfill (2)

In general, computing the resurgence of an arbitrary homogeneous ideal $I$ is quite difficult. However, whenever $I$ is the defining ideal of a zero dimensional subscheme in $\mathbb{P}^N$, Bocci and Harbourne [3, Theorem 1.2.1] used another numerical invariants of $I$, i.e., the Castelnuovo-Mumford regularity $\text{reg}(I)$, its initial degree, and the Waldschmidt constant to bound $\rho(I)$ in terms of these invariants as follows:

$$\frac{\alpha(I)}{\hat{\alpha}(I)} \leq \rho(I) \leq \frac{\text{reg}(I)}{\hat{\alpha}(I)}.$$  \hfill (3)

Recall that, if $0 \to F_r \to \cdots \to F_i \to \cdots \to F_0 \to I \to 0$ is a minimal free resolution of $I$ over $R$, then $\text{reg}(I)$ is defined to be $\max\{f_j - j \mid j \geq 0\}$, where $f_j$ is the maximum degree of the generators of the free module $F_j$. In particular, we have

$$\rho(I) = \frac{\alpha(I)}{\hat{\alpha}(I)} \text{ if } \text{reg}(I) = \alpha(I).$$
But it is rare to happen \( \text{reg}(I) = \alpha(I) \), and even if the equality holds, it is a hard task to compute \( \text{reg}(I) \) and \( \hat{\alpha}(I) \). Nevertheless, computing these two invariants may be easier than computing \( \rho(I) \).

3. Linear free resolution of a quasi star configuration

The authors of [5], in the process of classification of all configurations of reduced points in \( \mathbb{P}^2 \) with the Waldschmidt constant less than \( 9/4 \), introduced a special type of configuration of points, which called it quasi star configuration, and is constructed from what is known as star configuration that was introduced in [10]. In the following we recall the definition of quasi star configuration.

**Definition 3.1.** [5, Definition 2.3]. Let \( S_2(2, d) = p_1 + \cdots + p_{\binom{d}{2}} \) be a star configuration of points in \( \mathbb{P}^2 \) which is obtained by pair wise intersections of \( d \geq 3 \) general lines \( L_1, \ldots, L_d \) and let \( T_d = \{q_1, \ldots, q_d\} \) be a set of \( d \) distinct points in \( \mathbb{P}^2 \), such that \( T_d \cap S_2(2, d) = \emptyset \). We say that \( Z_d = T_d + S_2(2, d) \) is a quasi star configuration of points if for each \( i = 1, \ldots, d \), the point \( q_i \), lie on the line \( L_i \). Moreover, these points are not collinear.

In the above definition, the points of \( T_d \) need not to lie on a line. But if all points of \( T_d \) are collinear, then \( Z_d \) would be the star configuration \( S_2(2, d + 1) \).

The quasi star configuration of points \( Z_5 \) is depicted in the figure below.

![Figure 1](image)

Our next goal is to show that the defining ideal of a quasi star configuration \( Z_d \) has a linear minimal free resolution. To do this, the following lemma is needed. Recall that whenever \( J \) is a saturated ideal in \( R = \mathbb{K}[\mathbb{P}^N] \), the multiplicity of \( R/J \), denoted by \( e(R/J) \), is equal to the degree of the closed subscheme associated to \( J \).
Lemma 3.2. Let \( d \) be a positive integer and let \( I \) be the defining ideal of \( t = d(d+1)/2 \) reduced points \( \{p_1, \ldots, p_t \} \) in \( \mathbb{P}^2 \). Let \( J \) be a saturated homogeneous ideal of \( R = \mathbb{K}[\mathbb{P}^2] \) such that \( J \subseteq I \) and let \( J \) has the following minimal free resolution

\[
0 \rightarrow R^{\beta_2}(-d - 1) \rightarrow R^{\beta_1}(-d) \rightarrow J \rightarrow 0.
\]

Then \( J = I \).

Proof. Let \( X(I) \) and \( X(J) \) be the subschemes correspond to the ideals \( I \) and \( J \), respectively. Since \( J \subseteq I \), the support of \( X(I) \), i.e., \( \{p_1, \ldots, p_t \} \), should be contained in the support of the scheme \( X(J) \). Moreover, the minimal free resolution of \( J \) implies that the projective dimension of \( J \) is equal to two and hence \( X(J) \) is a zero-dimensional subscheme of \( \mathbb{P}^2 \). Applying a theorem of Huneke and Miller ([14, Theorem 1.2]) to this minimal free resolution, implies that \( e(R/J) = d(d+1)/2 = t \). On the other hand, since the multiplicity of the coordinate ring of a finite set of reduced points is equal to the number of its points, hence \( e(R/I) = t \). But, since \( I \) and \( J \) are the ideals of points such that \( J \subseteq I \) and \( e(R/I) = e(R/J) \), therefore \( I = J \), as required.

Now we are ready to show \( I(Z_d) \) has a linear minimal free resolution.

Theorem 3.3. Let \( I \subset R = \mathbb{K}[\mathbb{P}^2] \) be the ideal associated to quasi star configuration of points \( Z_d = T_d + S_2(2,d) \) in \( \mathbb{P}^2 \). Then the resolution

\[
0 \rightarrow R^d(-d - 1) \rightarrow R^{d+1}(-d) \rightarrow I \rightarrow 0
\]

is the minimal free resolution of \( I \). In particular, \( \alpha(I) = \text{reg}(I) = d \).

Proof. Let the star configuration \( S_2(2,d) = p_1 + \cdots + p_{(d)} \) be obtained by pairwise intersections of \( d \geq 3 \) general lines \( L_1, \ldots, L_d \) and let \( T_d = \{q_1, \ldots, q_d \} \). Let \( L_i' \), where \( 1 \leq i \leq d \), be the line through \( q_i \) that does not pass through any of the other points \( Z_d \). Let \( A \) be the matrix

\[
A = \begin{bmatrix}
L_1 & 0 & \cdots & 0 \\
0 & L_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L_d \\
L_1' & L_2' & \cdots & L_d'
\end{bmatrix} \in \mathcal{M}_{(d+1) \times (d)}(R),
\]

where \( A \) has as the top \( d \times d \) rows a diagonal matrix with \( L_1, L_2, \ldots, L_d \) along the diagonal and as the last row the vector \( [L_1', L_2', \ldots, L_d'] \). Let \( I(A) \) be the ideal generated by the maximal minors of \( A \). Thus we have

\[
I(A) = (L_1L_2 \cdots L_d, L_1'L_2' \cdots L_d', L_1L_2L_3 \cdots L_d, \ldots, L_1L_2 \cdots L_{d-1}L_d').
\]
Since the columns of matrix $A$ is the first syzygy of $I(A)$, then this ideal
has the following free resolution
\[ 0 \longrightarrow R^d(-d-1) \xrightarrow{A} R^{d+1}(-d) \longrightarrow I(A) \longrightarrow 0. \]
In the sequel, we show this resolution is minimal. First, we show $I(A)$ is the
ideal of points. Indeed, since $I(A)$ is not a principal ideal and since there is
not any linear form, for example $L$, such that $L$ divides all $d+1$ elements of
$I(A)$, thus $I(A)$ is the defining ideal of a zero dimensional subscheme in $\mathbb{P}^2$.

Since the coordinate ring of a zero-dimensional subscheme in $\mathbb{P}^2$ is always
Cohen-Macaulay, so $\dim(R/I(A)) = \text{depth}(R/I(A)) = 1$. Thus the projective dimension $R/I(A)$ is always
eq 2, which implies that the above free resolution for $I(A)$ is minimal.

Finally, it is easy to see that $I(A) \subset I$. Since two ideals $I$ and $I(A)$ satisfy
the conditions of Lemma 3.2, thus $I = I(A)$. As a consequence, we have $\alpha(I) = \text{reg}(I) = d$. \hfill \Box

Our next theorem is an extension of [8, Theorem 4.6]. As a special case,
we may also characterize a quasi star configuration of points. To
state it, we need to recall some preliminaries.

Let $X = m_1p_1 + \cdots + m_rp_r$ be a zero dimensional subscheme of $\mathbb{P}^2$ and
let $I = I(X)$ be the corresponding saturated ideal of $X$ in $R = \mathbb{K}[\mathbb{P}^2]$. Recall that the Hilbert function of $X$, denoted by $H(R/I, t)$, is a numerical
invariant of $X$, defined by $H(R/I, t) := \dim_{\mathbb{K}}(R/I)_t$, where $(R/I)_t$ is the $t$-th graded component of $R/I$. Moreover, $X$ is called has a generic Hilbert function if for all nonnegative integers $t$, $H(R/I, t) = \min\{\binom{t+2}{2}, \deg X\}$, where \( \deg X = \sum_i \binom{m_i+1}{2} \). In particular, when $X$ is a finite set of reduced
points, then $\deg X = |X|$.

**Theorem 3.4.** Let $X$ be a finite set of reduced points in $\mathbb{P}^2$ and let $I = I(X)$. Also let $\alpha(I) = \alpha$. Then the following conditions are equivalent:

(i) The ideal $I$ is minimally generated by $\alpha + 1$ generators of degree $\alpha$.

(ii) The scheme $X$ has the generic Hilbert function and $|X| = \binom{\alpha+1}{2}$.

(iii) The ideal $I$ has a linear minimal free resolution as follows:
\[ 0 \longrightarrow R^{\alpha}(-\alpha-1) \longrightarrow R^{\alpha+1}(-\alpha) \longrightarrow I \longrightarrow 0. \]

(iv) For the ideal $I$, we have $\text{reg}(I) = \alpha(I)$.

(v) For all $m \geq 1$, we have $\text{reg}(I^m) = m\text{reg}(I) = m\alpha(I)$.

(vi) The ideal $I^2$ is minimally generated by $\binom{\alpha+2}{2}$ generators of degree $2\alpha$.

(vii) The ideal $I^2$ has a linear minimal free resolution as follows:
\[ 0 \rightarrow R^{\binom{\alpha}{2}}(-2\alpha+2) \rightarrow R^{\binom{\alpha+1}{2}}(-2\alpha+1) \rightarrow R^{\binom{\alpha+2}{2}}(-2\alpha) \rightarrow I^2 \rightarrow 0. \]
Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from [8, Theorem 4.6] and its proof. Moreover, (iii) $\Rightarrow$ (iv) is immediate.

To show (iv) implies (v), let $d(I)$ denote the maximum degree of elements of a minimal set of generators of $I$. By definition of the regularity, it is clear that $d(I^{m}) = md(I) \leq \reg(I^{m})$. Moreover, since Krull dimension of $R/I$ is one, by [9, Theorem 1.1], $\reg(I^{m}) \leq m\reg(I)$. Since $\alpha(I) = \reg(I)$, thus $d(I) = \reg(I)$. Therefore, these two latter inequalities imply $\reg(I^{m}) = m\reg(I)$.

We show (v) implies (i). Since $\alpha(I) = \reg(I)$, so the degree of all elements of a minimal set of generators of $I$ are equal. Let $c$ be the number of a minimal set of generators of $I$ of degree $\alpha$. Now we have to show $c = \alpha + 1$. Since $I$ is generated by $c$ generators of degree $\alpha$, hence $H(R/I, \alpha) = \binom{\alpha+2}{2} - c$ and also $H(R/I, t) = \binom{t+2}{2}$, for $t \leq \alpha - 1$, in particular $H(R/I, \alpha - 1) = \binom{\alpha+1}{2}$. Since $X$ is a zero dimensional subscheme in $\mathbb{P}^2$, the Hilbert function of $X$ would be constant for $t \geq \reg(I) - 1 = \alpha - 1$, which is equal to $\deg X$ (see [3, Section 1.2]). Thus $H(R/I, \alpha - 1) = H(R/I, \alpha)$. In particular, $(\binom{\alpha+1}{2}) = (\binom{\alpha+2}{2}) - c$, which yields $c = \alpha + 1$.

By [8, Lemma 4.2], (i) implies (vi). To show (vi) implies (i), let $\mathcal{G}$ be a set of minimal generators of $I$ and let $d = d(I)$ denote the maximum degree of elements of $\mathcal{G}$. We claim that $d = \alpha$. On the contrary, let $d > \alpha$, so there exists two polynomials $F$ and $G$ in $\mathcal{G}$ such that $\deg F = \alpha$ and $\deg G = d$. By [8, Theorem 2.3], the polynomial $FG$ is of degree $(d+\alpha)$ and is an element of a minimal set of generators of $I^2$. But $(d+\alpha) > 2\alpha$, which contradicts our assumption. Now, let $c$ be the number of elements of degree $\alpha$ in $\mathcal{G}$. By [8, Lemma 4.2], the ideal $I^2$ has $\binom{\alpha+1}{2}$ minimal generators of degree $2\alpha$. Therefore, $\binom{c+1}{2} = \binom{\alpha+2}{2}$. In particular, $c = \alpha + 1$.

By Theorem 2.3 of [8], (iii) implies (vii). At last, the minimality of free resolution of $I^2$, immediately gives the implication (vii) $\Rightarrow$ (vi).

A closer look at to the proof of the implication (iv) $\Rightarrow$ (v) in Theorem 3.4 shows that for each positive integer $m$, the ideal $I^m$ has linear resolution. This has the worth to be mentioned as a corollary.

Corollary 3.5. Let $I$ be the ideal of a configuration of points in $\mathbb{P}^2$ such that $\reg(I) = \alpha(I)$. Then all ordinary powers of $I$ has linear free resolution.

Remark 3.6. In addition to the quasi star configurations, two distinct classes of configurations of points in $\mathbb{P}^2$ can be named, such that their defining ideals meet the conditions of Theorem 3.4. These are:

(a) the star configuration $S_2(2, d+1)$, for which $\reg(I(S_2(2, d+1))) = \alpha(I(S_2(2, d+1))) = d$, by [8, Lemma 2.4.2];
(b) for any integer \( d \geq 1 \), the configuration \( X \) consists of \( n = \frac{(d-1)+2}{2} \)
generic points, in \( \mathbb{P}^2 \), for which \( \text{reg}(I(X)) = \alpha(I(X)) = d \) (see [3, Section 1.3]).

An interesting problem that may arise in this regard, is to classify all configurations of points \( Z \) in \( \mathbb{P}^2 \) such that \( I(Z) \) satisfies in one of the equivalent conditions of Theorem 3.4.

The following example shows that the above three mentioned classes of configurations of points may have different features that distinguish them from each other.

Example 3.7. Let \( X \) be a configuration of 6 generic points in \( \mathbb{P}^2 \), \( Y = S_2(2,4) \), be a star configuration generated by 4 general lines in \( \mathbb{P}^2 \) and finally let \( W = Z_3 = T_3 + S_2(2,3) \) be the quasi star configuration generated by 3 generic lines in \( \mathbb{P}^2 \). Then by our assumptions, the number of points in \( Y \) and \( W \) is the same as the number of points in \( X \), while their resurgences are different.

(1) By [3, Corollary 1.3.1], \( \rho(X) = 5/4 \);
(2) By [3, Theorem 2.4.3], \( \rho(Y) = 3/2 \);
(3) By Example 4.1, \( \rho(W) = 4/3 \).

4. Proof of the main Theorem

By Theorem 3.3, for a quasi star configuration \( Z_d \), we have \( \alpha(I(Z_d)) = \text{reg}(I(Z_d)) = d \). Thus by [3], computing \( \rho(I(Z_d)) \) depends on the computing of \( \hat{\alpha}(I(Z_d)) \). For the initial case of \( d \), i.e., \( d = 3 \), we can compute the exact value of \( \rho(I(Z_d)) \). In fact

Example 4.1. Let \( Z_3 \) be the quasi star configuration of 6 points. Theorem 3.3 implies \( \alpha(I(Z_3)) = \text{reg}(I(Z_3)) = 3 \) and by [5, Proposition 3.1], we have \( \hat{\alpha}(I(Z_3)) = 9/4 \). Then [3] yields \( \rho(I(Z_3)) = 4/3 \).

However, finding the exact value of \( \hat{\alpha}(I(Z_d)) \), whenever \( d \geq 4 \), seems to be a hard problem. Hence, it is reasonable to look for good bounds for \( \hat{\alpha}(I(Z_d)) \).

In the sequel, our goal is to establish an upper bound for the Waldschmidt constant of the defining ideal of a quasi star configuration \( I(Z_d) \).

Recall that for a real number \( a \), \( \lfloor a \rfloor \) denotes the least integer greater than or equal to \( a \), and \( \lceil a \rceil \) denotes the greatest integer less than or equal to \( a \).

Theorem 4.2. Let \( d \geq 4 \) be an integer and let \( Z_d = T_d + S_2(2,d) \) be a quasi star configuration of points in \( \mathbb{P}^2 \). Let \( I = I(Z_d) \). Then
(a) if \( d \leq 9 \) then \( \hat{\alpha}(I) \leq \frac{d+c_d}{2} \), where \( c_d = 2, 2, \frac{12}{5}, \frac{21}{8}, \frac{48}{17} \), and 3 for \( d = 4, \ldots, 9 \), respectively;
(b) if \( d \geq 10 \) then \( \hat{\alpha}(I) \leq \frac{d+\sqrt{d}}{2} \).
Proof. (a) Let $W_d = \{p_1, \ldots, p_d\}$ be a set of $4 \leq d \leq 9$ general points in $\mathbb{P}^2$. It is known that $\alpha(I(W_d)^{(m)}) = \lceil c_d m \rceil$ (see for example [11]). Since the points of $W_d$ are in general position then $\alpha(I(T_d)^{(m)}) \leq \alpha(I(W_d)^{(m)}) = \lceil c_d m \rceil$. Let now $c_d = \frac{a}{b}$, where $a$ and $b$ are two positive integers and let $D$ be the polynomial $D = (L_1 L_2 \ldots L_d)^{bm}$. It is clear that $D \in I(S_2(2,d)^{(2bm)})$ and $D \in I(T_d)^{(bm)}$. Since $\alpha(I(T_d)^{(bm)}) \leq \lceil c_d bm \rceil = \lceil \frac{ab}{b} m \rceil = am$, thus there exists a polynomial of degree $am$, for example $F$, vanishes to order $bm$ along $T_d$. Thus $FD \in I((2bm))$. Therefore

$$\hat{\alpha}(I) \leq \frac{\alpha(I)^{(2bm)}}{2bm} \leq \frac{\deg F + \deg D}{2bm} = \frac{am + bmd}{2bm} = \frac{\frac{d}{2} + d}{2} = \frac{d + c_d}{2}.$$

(b) To prove the assertion, we use the same method as in the proof of (a). Let $d \geq 10$ be an integer and let $W_d$ be a set of $d$ points in $\mathbb{P}^2$. It is always true that $\hat{\alpha}(I(W_d)) \leq \sqrt{d}$ ([?, Proposition 3.4]). By inequality (2), for all $m \geq 1$, we have $\frac{\alpha(I(T_d)^{(m)})}{m+1} \leq \hat{\alpha}(I(T_d))$. Therefore $\alpha(I(T_d)^{(m)}) \leq (m+1)\sqrt{d}$. It means that there exists a polynomial of degree $\lceil (m+1)\sqrt{d} \rceil$, for example $F'$, vanishes to order $m$ along $T_d$. Let now $D'$ be the polynomial $D' = (L_1 L_2 \ldots L_d)^m$. It is easy to see that $F'D'$ is an element of $I((2m))$ and is of degree $md + \lceil (m+1)\sqrt{d} \rceil \leq m(d + \sqrt{d}) + \sqrt{d}$. If allowing $m$ tends to infinity, then $\hat{\alpha}(I) \leq (d + \sqrt{d})/2$. Hence the claim stabilizes.

Now we can use Theorems 3.3 and 4.2 to bound the resurgence $\rho(I(Z_d))$ of defining ideal of quasi star configuration $Z_d$ as follows:

**Theorem 4.3.** Let $I$ be the ideal associated to quasi star configuration of points $Z_d$ in $\mathbb{P}^2$.

(a) If $4 \leq d \leq 9$ then $2 - \frac{2c_d}{d + c_d} \leq \rho(I) \leq 2 - \frac{2}{d+1}$,

(b) If $d \geq 10$ then $2 - \frac{2}{\sqrt{d+1}} \leq \rho(I) \leq 2 - \frac{2}{d+1}$,

where $c_d$ is the one which defined in Theorem 4.2.

**Proof.** By Theorem 3.3, we have $\text{reg}(I) = \alpha(I) = d$. Thus by [3], $\rho(I) = \frac{\alpha(I)}{\alpha(I)}$. Hence the statement depends on bounds for $\hat{\alpha}(I)$. By [12] Proposition 3.1], for every finite set of points $W$ in $\mathbb{P}^2$, we have $\frac{\alpha(I(W))}{d+1} \leq \hat{\alpha}(I(W))$. In particular, $\frac{d+1}{2} \leq \hat{\alpha}(I)$, which yields $\rho(I) = \frac{\alpha(I)}{\alpha(I)} \leq \frac{2d}{d+1} = 2 - \frac{2}{d+1}$. By Theorem 4.2 for $d \geq 10$, we have $\hat{\alpha}(I) \leq \frac{d + \sqrt{d}}{2}$, which yields $\rho(I) = \frac{\alpha(I)}{\alpha(I)} \geq \frac{2d}{d + \sqrt{d}} = 2 - \frac{2\sqrt{d}}{d+1}$. For $4 \leq d \leq 9$ with a similar argument, we can deduce $\rho(I) \geq 2 - \frac{2c_d}{d + c_d}$, as required.

5. Some Applications

In this section we give some applications of Theorem 4.3. As a first consequence of this theorem, we have:
Corollary 5.1. Let $0 < \varepsilon < \frac{1}{2}$ be a real number. Then there exists a radical ideal of points $I_\varepsilon$ in $\mathbb{K}[\mathbb{P}^2]$ such that $\rho(I_\varepsilon) \in [2 - \varepsilon, 2)$.

Proof. Let $d \geq (\frac{2}{\varepsilon} - 1)^2$ be an integer and let $Z_d$ be the quasi star configuration associated to $d$. Since $\varepsilon < \frac{1}{2}$ and since $d \geq (\frac{2}{\varepsilon} - 1)^2$, one can see that $d \geq 10$ and $2 - \varepsilon \leq 2 - \frac{2}{\sqrt{d+1}}$. Now by Theorem 4.3, we have

$$2 - \varepsilon \leq 2 - \frac{2}{\sqrt{d+1}} \leq \rho(I(Z_d)) \leq 2 - \frac{2}{\sqrt{d+1}} < 2.$$ 

\[ \square \]

Known radical ideals of points $I$ in $\mathbb{K}[\mathbb{P}^2]$ for which the containment $I^{(3)} \subseteq I^2$ fails, are rare, and even it is not known for which positive integers $r \geq 3$ the containment $I^{(2r-1)} \subseteq I^r$ fails. As a corollary of Theorem 4.3 we can construct ideals of points such that they may be a candidate for the failure of $I^{(2r-1)} \subseteq I^r$.

Corollary 5.2. Let $r \geq 2$ be an integer. Then there exists a radical ideal of points $I$ such that it has the necessary condition for the failure $I^{(2r-1)} \subseteq I^r$, i.e., $\rho(I) \geq \frac{2r-1}{r}$.

Proof. Let $d \geq (2r-1)^2$ be an integer and let $Z_d = \sum_{i=1}^{d} q_i + S_2(2, d)$ be the quasi star configuration of points in $\mathbb{P}^2$. Since $d \geq (2r-1)^2$, one can see that $\frac{2r-1}{r} \leq 2 - \frac{2}{\sqrt{d+1}}$. Now, by Theorem 4.3 we have $\rho(I) \geq 2 - \frac{2}{\sqrt{d+1}} \geq \frac{2r-1}{r}$.

\[ \square \]

Harbourne and Bocci in [3], showed that for a homogeneous ideal $I$ of $\mathbb{K}[\mathbb{P}^N]$ the containment $I^{(m)} \subseteq I^r$ for the bound $\frac{m}{r} \geq N$ is optimal. As a consequence of Theorem 4.3 when $N = 2$, we can show that the quasi star configurations meet this optimality too.

Corollary 5.3. Let $I$ be a homogeneous ideal of $\mathbb{K}[\mathbb{P}^2]$ and let $m$ and $r$ be two positive integers. Then for $m/r \geq 2$, the containment $I^{(m)} \subseteq I^r$ is optimal.

Proof. On the contrary, let there exists a real number $c < 2$ such that for any two positive integers $m, r$ with $m/r \geq c$ the containment $I^{(m)} \subseteq I^r$ holds. Then we have $\rho(I) \leq c$. Now let $Z_d$ be a quasi star configuration of points in $\mathbb{P}^2$ with defining ideal $I(Z_d)$. Let $d$ tends to infinity. Then by Theorem 4.3 $2 = \rho(I(Z_d)) \leq c < 2$, which is a contradiction.

\[ \square \]

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