Holographic Entanglement Entropy, Complexity, Fidelity Susceptibility and Hierarchical UV/IR Mixing Problem in $AdS_2$/open strings

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In this paper, we will compute the holographic complexity (dual to a volume in AdS), holographic fidelity susceptibility and the holographic entanglement entropy (dual to an area in AdS) in a two-dimensional version of $AdS$ which is dual to open strings. We will explicitly demonstrate that these quantities are well defined, and then argue that a relation for fidelity susceptibility and time should hold in general due to the $AdS_2$ version of the classical Kepler’s principle. We will demonstrate that it holds for $AdS_2$ solution as well as conformal copies metrics in bulk theory of a prescribed dual conformal invariant quantum mechanics which have been obtained in open string theory. We will also show that hierarchical UV/IR mixing exists in boundary string theory through the holographic bulk picture.

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I. INTRODUCTION

Varied studies done in wide areas of physics have proved that the fundamental laws of physics can be reformulated in terms of relevant information theory quantities \[1, 2\]. Entropy quantifies the amount of information bits that are lost in a certain (un)physical process, and thus, it is supposed to be one of the most important physical quantities related to any such information theoretical process. The entropy has been used to realize several physical phenomena from condensed matter physics (like phase transitions, critical phenomena) to gravitational physics, where entropy looks just like the area of certain surfaces, called horizons. Also, it is believed that the geometry of spacetime can be understood as an emergent object, which emerges due to some types of information theoretical process. A simple reason to believe it is that, in the Jacobson formalism where gravitational field equations are related to thermodynamical quantities, it is always possible to derive the Einstein equations from thermodynamics of horizon by presuming a certain scaling form for the entropy \[3, 4\]. We know that the maximum entropy of a certain region of space scales with the horizon’s area, although this observation has been acquired holding the physics of black holes. This naive relation between area (boundary) and entropy (a quantity relates to the quantum states inside a system) makes the idea of holographic principle \[5, 6\], and the AdS/CFT correspondence, one of the most significant dualities between two regimes (strong/weak) of several physical systems \[7\].

The AdS/CFT correspondence makes it possible to describe quantum entanglement in complex systems in the form of the holographic entanglement entropy (HEE) \[8, 10\].

The HEE of a given quantum field theory in \(d + 1\) dimensions (even non relativistic on) is holographically calculated in terms of the area of a minimal surface defined in the geometry of an asymptotically \(AdS_d\) dual geometry. Let us consider HEE for a given subsystem \(A\) with its complement \(A'\). The Ryu-Takayanagi (RT) expression for the holographic entanglement entropy
where $G$ is the gravitational constant in $d$ dimensions, $\gamma_A$ is the $(d-1)$-minimal surface in the $AdS_d$ geometry. We assumed that the boundary of this surface named as $\mathcal{A}(\gamma_A)$ is the same as the boundary of the quantum entangled system $\partial A$. Because of non renormalizability of Einstein gravity as well as existence of cutoffs, in RT scheme to compute HEE, there are UV divergence terms like $\varepsilon^{-n}, n \geq 1, \ln \varepsilon, \ldots$, etc. Consequently we are required to find a regularization strategy to improve (remove) these divergences. Inspired from quantum field theory, we consider a deformed geometry $D$ and we define the area as follows,

$$\mathcal{A}(\gamma_A) = \mathcal{A}_D(\gamma_A) - \mathcal{A}_{AdS}(\gamma_A),$$  

where $\mathcal{A}_D(\gamma_A)$ is defined in deformed geometry (for example excited states), and $\mathcal{A}_{AdS}(\gamma_A)$ is defined in the background $AdS$ spacetime (ground state). We hope that if one defines the holographic entanglement entropy for a deformed geometry by subtracting the contribution coming from the background $AdS$ spacetime, we are only left with a finite part. We will use this scheme of renormalization through this paper.

As mentioned, the entropy measures the amount of the information which we lost during a physical process. It is very common to define the complexity as a quantity which quantifies the difficulty to obtain the information of a system. The complexity has been introduced to investigate different physical systems from gravitational physics to condensed matter physics, and even quantum information theory. This lost information never can be retracted using any possible physical process [11]. Because complexity has only been recently introduced to investigate miscellaneous physical systems, there are different schemes to define the complexity for a CFT. However, recently inspired by RT proposal for holographic entanglement entropy, holographic complexity (HC) has been holographically conjectured as a certain types of volumes in the dual anti-de Sitter (AdS) background [12] - [17]. Moreover, it is adequate to specify a subsystem $A$ with its complement, and define this volume as $V = V(\gamma_A)$, i.e., the volume enclosed by the same minimal surface which was proposed to estimate the HEE [18] is given as follows (called holographic complexity or HC),

$$C_A = \frac{V(\gamma_A)}{8\pi RG_{d+1}},$$  

here $R$ and $V(\gamma_A)$ are the radius of the curvature and the volume in the $AdS_d$ bulk geometry. This volume contains UV divergences, and so we need an appropriate regularization scheme for it. In analogous to the HEE, we define the regularized volume as

$$\Delta V(\gamma_A) = V_D(\gamma_A) - V_{AdS}(\gamma_A).$$
Here $\Delta V_{D}(\gamma_{A})$ denotes the volume in deformed geometry, and $V_{AdS}(\gamma_{A})$ is the volume in the background $AdS$ spacetime. This again improves the divergences and one is again left with a finite part. Several examples for HC have been studied in literature [19]-[38].

A type of duality between dilaton gravity (gravity in 2 dimensions) on $AdS_2$ and open strings was discovered in [39] and it was clearly shown how $AdS_2$ is equivalent to conformal quantum mechanics (CQM) as a non relativistic limit of $CFT_1$ (see [39] and other papers of these authors). This lower dimensional version of $AdS_{d+1}/CFT_d$ conjecture argued that gravity on $AdS_2$ is holographically dual to a one-dimensional conformal field theory on the boundary of $AdS_2$. One reason to believe that such duality exists, is due to the fact that classical two-dimensional (dilaton) gauge theory for gravity has trivial conformal symmetry. It can be possible to reformulate this gauge theory as a nonlinear sigma-model [40] and it was proved that when we take the classical limit of this toy model, we were observed the conformal symmetry. This is a reason to think about gravity on $AdS_2$ as a natural dual to a one-dimensional CFT Entanglement entropy, Holographic Complexity and Fidelity Susceptibility in open strings. This duality is called as $AdS_2$/open string or $AdS_2$/CQM.

The object of interest in this paper is to compute HEE, HC and fidelity susceptibility for a generic open string system using the $AdS_2$/open string duality. The question of interest is how these quantities evolve with time.

The structure of the paper is as follows. In Sec. 2, we fleetingly review the formal frameworks of gravity on $AdS_2$. In Sec. 3, we compute HEE of a string via RT formalism. In Sec. 4, we calculate HC of the string. In Sec. 5, we investigate fidelity susceptibility holographically. In Sec. 6, we summarize our results.

II. GRAVITY IN TWO DIMENSIONS AND $AdS_2$/open string DUALITY

Let us start by two-dimensional dilaton theory with the following action,

$$S = \frac{1}{2\kappa^2} \int d^2 x \sqrt{-g} \left( \varphi R + V(\varphi) \right),$$

where the potential is $V(\varphi) = 2\lambda^2 \varphi$, $\lambda^2$ stands for cosmological constant $\Lambda$, and we redefine $\varphi = e^{-2\varphi}$ where $\varphi$ is dilatonic field. A reason for breaking the conformal symmetry is the existence of a non uniform profile for the dilaton scalar field $\varphi$, i.e., when $\nabla_{\mu} \varphi \neq 0$. It is remarkable to mention here that the Birkhoff’s theorem holds generally, consequently we always can find static blackholes [41]. Let us consider an asymptotically $AdS_2$ solution for dilaton action given in Eq. (5). The trivial symmetry generators are a set of the infinitesimal diffeomorphisms. An equivalent description to these diffeomorphisms is given by the pure gauge.
Note that these equivalence hold only for regions near (almost existed) on the boundary. It was demonstrated that these symmetries are governed by a Virasoro algebra \([42]\). The dilaton gravity action Eq. (5) can be cast in a nonlinear conformal sigma model form. In Ref. \([40]\) the duality between dilaton gravity action (5) on \(AdS_2\) and open strings was proved and a clear dualities exist between the two theories, one in the quantum system on boundary and the other as the geometry of a \(AdS_2\) copy of the AdS spacetime.

The action given by (5) basically has two solutions given by pure \(AdS_2\) (ground state) and Schwarzschild-AdS (SAdS) with mass \(M = m_{bh}\) which corresponds to the excited state in dual quantum theory. We write these solutions as following:

**AdS**

\[
\begin{align*}
    ds_1^2 &= \frac{1}{\lambda^2 x^2} (-dt^2 + dx^2), \quad \varphi = \varphi_0 \\
    \varphi &= \varphi_0 \lambda \tau.
\end{align*}
\]

**SAdS**

\[
\begin{align*}
    ds_2^2 &= \left(\frac{a}{\sinh(a \lambda \tau)}\right)^2 (-d\tau^2 + d\sigma^2), \quad a = \frac{2m_{bh}}{\lambda \varphi_0}, \quad \varphi = \varphi_0 \lambda \sigma.
\end{align*}
\]

The spacetime is the gravitational dual of an open string with a conformal symmetry near its quantum critical point.

Note that the metric of pure AdS can be obtained as a limit of \(a \to 0\) \(ds_2^2\), and the two metrics (7), (6) are related by the change of coordinates. However, this transformation can not cover all the spacetime manifold and is singular, so this transformation is formal and we won’t use it in our investigation.

### III. HOLOGRAPHIC ENTANGLEMENT ENTROPY FOR STRING

Using celebrated RT proposal\([8]-[10]\), we need to calculate minimal area for two regions of entangled systems. We use time dependant formalism to calculate holographic entanglement entropy and holographic complexity \([43, 44]\). To compute entanglement entropy and complexity at the boundary using bulk we need to specify the boundary entangled region. The entangling region in the boundary is taken to be a string with width \(L\) such that

**Pure AdS**

\[\text{Pure AdS} = A_1 = \{ t = t(x), \quad -L \leq x \leq L \},\]  

**SAdS**

\[\text{SAdS} = A_2 = \{ \tau = \tau(\sigma), \quad -L \leq \sigma \leq L \},\]  

where \(L\) is the extent of the subsystem in the time direction. Since the string has time translational invariance, one can describe the profile of the extremal surface by \(t = t(x)\) for pure AdS and \(\tau = \tau(\sigma)\) for blackhole. With this set up in place, we can now proceed to compute the RT area enclosed by the minimal surface extending from the boundary into the bulk.
A. Pure AdS

In this case, this is given by

\[ A_1(\gamma) = \int_{-L}^{L} \frac{dx}{|\lambda x|} \sqrt{1 - \dot{t}^2}, \]  

(10)

where "dot" denotes derivative with respect to \( x \). The minimization of this area functional determines the function \( t(x) \) which reads,

\[ \dot{t} = \frac{|\lambda px|}{\sqrt{1 + (\lambda px)^2}}. \]  

(11)

Boundary conditions are given by,

\[ t(x = 0) = t^*, \quad \dot{t}(x = 0) = 0, \quad t(x = L) = T - \varepsilon. \]  

(12)

where \( T \) denotes total time elapsed in system and \( \varepsilon \) is smallness parameter. An Exact solution for (11) is given as follows,

\[ t(x) = \text{sgn}(\lambda px) \frac{\lambda p}{\sqrt{1 + (\lambda px)^2}} + C_1. \]  

(13)

here \( \text{sgn}(x) = \frac{x}{|x|} \). Using (12) we obtain the ultimate form of solution for the extremal surfaces in \( A_1 \), as following,

\[ t(x) = \text{sgn}(\lambda px) \frac{\lambda p}{\sqrt{1 + (\lambda px)^2}} + t^* \mp \frac{1}{|\lambda p|}. \]  

(14)

Substituting the above expression for \( t(x) \) in eq.(10) and putting a cut-off \( p^{-1} \) for the \( x \) integral, we have the following expression for HEE ,

\[ S_{\text{HEE}}^{\text{AdS}} = \ln(\lambda p L/2) \frac{\lambda}{\lambda} + \lim_{\sigma \to 0} \ln(\lambda p \sigma/2) \frac{\lambda}{\lambda} + O(p^2). \]  

(15)

We will see when we substract (15) from the HEE for SAdS the divergence term will be cancelled.

B. SAdS

Now we calculate HEE for (7). Area functional for (6) is given by

\[ A_2(\sigma) = \int_{-L}^{L} \frac{d\sigma}{|\sinh(\lambda a\sigma)|} \sqrt{1 - \dot{\tau}^2}, \]  

(16)

Euler-Lagrange Eq, is obtained as follows:

\[ \dot{\tau} = \frac{\eta \sinh(\lambda a\sigma)}{\sqrt{1 + (\eta \sinh(\lambda a\sigma))^2}}. \]  

(17)
Here $\eta \ll 1$ is a conserved charge of system like $p$ in the last section. Boundary conditions are given by,

$$\tau(\sigma = 0) = \tau^*, \quad \dot{\tau}(\sigma = 0) = 0, \quad \tau(\sigma = L) = T^* - \varepsilon.$$  \hspace{1cm} (18)

$T^*$ denotes the total time elapsed in system and $\varepsilon$ is smallness parameter. Exact solution for (17) is given by follows,

$$\tau(\sigma) = \tau^* - \frac{\eta}{\alpha\lambda|\eta|} \ln(1 + \eta) + \frac{\eta \sinh(\alpha\lambda\sigma)}{\alpha\lambda|\eta| \sinh(\alpha\lambda\sigma)} \times \left( \eta \cosh(\alpha\lambda\sigma) + \sqrt{1 + \eta^2 \sinh^2(\alpha\lambda\sigma)} \right).$$  \hspace{1cm} (19)

and $\eta \to 0$. Expanding $\tau(\sigma)$ in a Laurent series in $\eta$, we find the following expression for HEE,

$$S_{\text{HEE}}^{SAdS} = \tanh^{-1} \left( \frac{1}{2} \frac{1 + (\cosh(\alpha\lambda L))^2}{\cosh(\alpha\lambda L)} \right) + \lim_{\eta \to 0} \frac{4}{\alpha\lambda\eta} + O(\eta^2).$$  \hspace{1cm} (21)

So, the total, finite HEE is given by Eq. \textit{(21)}-(15),

$$S_{\text{HEE}}^{Net} = S_{\text{HEE}}^{SAdS} - S_{\text{HEE}}^{AdS} = \frac{1}{4G} \left( \tanh^{-1} \left( \frac{1}{2} \frac{1 + (\cosh(\alpha\lambda L))^2}{\cosh(\alpha\lambda L)} \right) \right) - \ln(\lambda pL/2) \frac{1}{4G} + \lim_{\eta \to 0} \left( \frac{4}{\alpha\lambda\eta} - \frac{\ln(\eta/2)}{\lambda} \right).$$  \hspace{1cm} (22)

The leading divergent term of the HEE of an extremal co-dimension one hypersurface in the $AdS_2$ geometry is,

$$S_{\text{HEE}}^{\text{div}} \sim \frac{1}{G\lambda} \left( \frac{1}{\alpha\eta} - \ln\left( \frac{\lambda^2 Lp^2\eta}{4} \right) \right).$$  \hspace{1cm} (23)

where $\eta \to 0, p \to \infty$. If we define $b \equiv \eta p < \infty$, the divergent term is rewritten as follows:

$$S_{\text{HEE}}^{\text{div}} \sim \frac{1}{G\lambda} \left( \frac{1}{\alpha\eta} + \ln(\eta) + \text{finite term} \right).$$  \hspace{1cm} (24)

We have found that HEE is scaled as follows:

$$S_{\text{HEE}}^{AdS_2,\text{div}} \sim \ln(\eta)$$  \hspace{1cm} (25)

$$S_{\text{HEE}}^{AdS_2,\text{div}} \sim \frac{1}{\eta}$$  \hspace{1cm} (26)

\textbf{IV. COMPUTATION OF HOLOGRAPHIC COMPLEXITY}

In this section we calculate holographic complexity for metrics (6),7) using the RT volume enclosed by the minimal surface extending from the boundary into the bulk [43].
A. Pure AdS

For metric (6) the RT volume enclosed by the minimal surface is given by the following integral,

\[ V_1 = \int_0^L \frac{dx}{\lambda^2 x^2} \left[ t(x) - t^* \right] \]  

(27)

Using solution in Eq. (13) and by change of variables: \( y = \lambda p x \) and \( y_0 = \lambda p L \), we obtain:

\[ V_1 = \frac{1}{\lambda^2} \left( \sinh^{-1} y_0 + y_0 \sqrt{1 + y_0^2} - \frac{(1 + y_0^2)^{3/2}}{y_0} + \frac{1}{y_0 |\lambda p|} \right) \]  

(28)

\[ -\frac{1}{\lambda^2} \lim_{\varepsilon \to 0} \left( \frac{1}{|\lambda p|} - 1 \right) \frac{1}{\varepsilon} \]

Note that we used taylor’s series for integral near \( y = \varepsilon \to 0 \). The divergent term of HC is rewritten as following:

\[ C_{AdS}^2 \sim \frac{1}{\varepsilon}. \]  

(29)

B. SAdS

For SAdS, using metric (7) and by solution given in Eq. (20) we obtain:

\[ V_2 = \Phi(z_0, \varepsilon) + \frac{1}{\lambda^2} \frac{\eta \ln(1 + |\eta|) \cosh z_0}{|\eta|} - \frac{1}{\lambda^2} \lim_{\varepsilon \to 0} \frac{\eta \ln(1 + |\eta|) \frac{1}{\varepsilon}}{\cosh(z_0)} \]  

(30)

where \( z_0 = \lambda a L \) and we define an auxiliary function

\[ \Phi(z_0, \varepsilon) = -\frac{\ln(1 + \frac{\eta}{|\eta|})}{z_0} + \frac{\ln(1 + \frac{\eta}{|\eta|})}{\varepsilon} + O(z_0) \]  

(31)

where the subleading terms start at order \( O(\frac{1}{\varepsilon}) \). Finally if we substract (28) from (30) we obtain a finite value for holographic complexity,

\[ \Delta V = -\frac{\ln(1 + \frac{\eta}{|\eta|})}{\lambda^2 a L} + \frac{1}{\lambda^2} \frac{\eta \ln(1 + |\eta|) \cosh(\lambda a L)}{|\eta|} \cosh(\lambda a L) \]  

(32)

\[ -\frac{1}{\lambda^2} \left( \sinh^{-1} \lambda p L + \lambda p L \sqrt{1 + (\lambda p L)^2} - \frac{(1 + (\lambda p L)^2)^{3/2}}{\lambda p L} + \frac{1}{\lambda p L |\lambda p|} \right) \]

The divergent term of HC is rewritten as follows:

\[ C_{SAdS}^2 \sim \frac{1}{\varepsilon}. \]  

(33)

where the divergent parts is removed safely if different conserved charges \( (p, \eta) \) satisfies the following (resembling an IR-UV cut off relation),

\[ \left( 1 + \frac{\eta}{|\eta|} \right) \left( 1 + |\eta| \right)^{-\frac{\eta}{|\eta|}} = \exp(1 - \frac{1}{|\lambda p|}). \]  

(34)
To further justify this UV-IR relation Eq. (34), it is constructive to extend our discussions for the case where the boundary has a massless scalar field $\psi$ and a massive scalar field $\Psi$ with mass $M$. Obviously if we keep the energy level $p \ll M$, we won’t see the massive field, $p$ is the IR cutoff. If we now take the energy to be $\eta \gg M$ we will see both fields $(\Psi, \psi)$, but now almost all of their energy will be momentum and their mass will be negligible. They will be behaving as two massless fields. This is an example of hierarchical UV/IR mixing [46].

V. HOLOGRAPHIC FIDELITY SUSCEPTIBILITY

In previous section, we proved that quantum complexity of an open string theory can also be obtained holographically, as the holographic complexity is dual to a volume in $AdS_2$ space time. Furthermore, it has been demonstrated that the holographic complexity of a field theory can be proportional to the fidelity susceptibility of the boundary field theory. Consequently, the fidelity susceptibility of an open string theory can be holographically calculated using a maximal volume $V_{\text{max}}$ in the $AdS_2$ which ends on a covariant zero mean curvature slicing of the time-dependent bulk geometry [45]. We can use this maximal volume in the AdS geometry to define the holographic complexity in such a geometry as

$$F = \frac{V_{\text{max}}}{8\pi R G}.$$  \hspace{1cm} (35)

Now the $AdS_2$/open string holography is a limiting case of usual holography, and it is known that there are divergence terms associated with such volumes. Thus, we need to regularize this volume, before we can define the fidelity susceptibility for $AdS_2$ geometries. This will be done by subtracting the background AdS geometry $V_{\text{max}}^{AdS}$ from the deformed SAdS geometry $V_{\text{max}}^{SAdS}$. So, we can define a regularized maximal volume

$$\Delta V_{\text{max}} = V_{\text{max}}^{SAdS} - V_{\text{max}}^{AdS}. $$ \hspace{1cm} (36)

Now using this regularized maximal volume in the $AdS_2$ geometry, we can define the regularized holographic fidelity susceptibility of an open string boundary theory as

$$\Xi_F = F_{SAdS} - F_{AdS} = \frac{\Delta V_{\text{max}}}{8\pi R G}$$ \hspace{1cm} (37)

where $R$ is the radius of the curvature of this $AdS_2$ geometry. This regularized holographic fidelity susceptibility is equal to fidelity susceptibility of the boundary open string theory, and so the fidelity susceptibility of the open string field theory can be holographically estimated from holographic complexity. It may be noted that recently in Ref. [47], the author demonstrated that the holographic fidelity susceptibility for two quantum states is given by the difference of
the volume of an extremal surface in the fully back reacted dual geometry. Now we can use the Eq. (37) to evaluate fidelity susceptibility as

$$\Xi_F = \frac{1}{8\pi RG} \left( \frac{T - t^*}{L} - \frac{a(T^* - \tau^*)}{\lambda} \frac{\cosh(a\lambda L)}{\sinh(a\lambda L)} \right).$$

(38)

where the divergence parts are removed.

To conclude one may propose that the renormalized fidelity susceptibility of the maximal volume due to the effect of a time interval cancelation provide a holographic Kepler’s second law for fidelity susceptibility in the dual field theory

$$T^* - \tau^* = T - t^*.$$

(39)

Conjecture: An imaginary line connecting two points in an open string in dual AdS$_2$, sweeps out an equal fidelity susceptibility of quantum system in equal amounts of time.

VI. SUMMARY

In this letter, we calculate holographic quantum complexity and the holographic entanglement entropy for string holographically. We propose that a hierarchical UV/IR mixing exists in AdS$_2$/open strings. It can be considered as a holographic version of Kepler’s second law argues that an imaginary line connecting two points in an open string in dual AdS$_2$, sweeps out an equal fidelity susceptibility of quantum system in equal amounts of time. Furthermore, in analogy with the usual Kepler’s second law, the regions analysed were assumed to exist as connected points in string. We argued that such a conjuncture should hold in general, as it is based on the AdS version of the Kepler’s second law.

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