An Angular Formalism for Spin One Half.

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Understanding spin one half is a crucial issue in the De Broglie Bohm framework. In this paper a concrete relativistic realization of spin one half in terms of angular coordinates is developed. A Lagrange formulation is found, equations of motion are derived, and Lorentz invariance is discussed.

I. INTRODUCTION

A. General Picture

The de Broglie Bohm interpretation of quantum mechanics (dBB) is an attempt to describe quantum phenomena in a deterministic way. It allows to assign physical reality to a system, even at the instances between two measurements. In this sense it goes beyond the Copenhagen interpretation of quantum mechanics. A general introduction to the approach and a discussion of most of the conceptual issues can be found in [1,2]. This formulation has recently attracted some interest due to its potential connection to quantum gravity and unification schemes [3–9]. However, the description of spin one half within this framework proved to be hard and not unique [10–15]. This article is based on the combination of a concrete relativistic realization of spin one half in terms of angular coordinates is developed. A special formulation of the Dirac equation. Those two starting points are introduced in the subsections (I B and I C).

B. Angular representation of Pauli matrices

In [1] a description of non-relativistic spin 1/2 in terms of the Euler-angles of a rigid rotator is given. The algebraic definitions of this formulation are the basis of our work and thus they will be shortly defined in this section. In this formulation the role of two component spinors is taken by scalar functions with additional angular degrees of freedom. The key step is to introduce angular operators $\hat{M}^k$, that act on those functions in such a way that they reproduce the algebra of the Pauli spin matrices $\sigma^k$

\[
\psi_a(x) \quad \text{spinor with two components} \quad \sigma^k
\]

\[
\frac{\psi(x, \alpha_b)}{2M^k} \quad \text{scalar function with angular dependence}
\]

The angles $\alpha_b = \alpha, \beta, \gamma$ are defined with respect to the external space axes and the operators $\hat{M}_k$ are

\[
\hat{M}_1 = i(\cos \beta \partial_\alpha - \sin \beta \cot \alpha \partial_\beta + \sin \beta \sin \gamma \partial_\gamma)
\]

\[
\hat{M}_2 = i(- \sin \beta \partial_\alpha - \cos \beta \cot \alpha \partial_\beta + \cos \beta \sin \gamma \partial_\gamma)
\]

\[
\hat{M}_3 = i\partial_\beta
\]

\[
\hat{M}_1' = i(\cos \gamma \partial_\alpha + \sin \beta \csc \alpha \partial_\beta - \sin \gamma \cot \alpha \partial_\gamma)
\]

\[
\hat{M}_2' = i(\sin \gamma \partial_\alpha - \cos \gamma \csc \alpha \partial_\beta + \cos \gamma \cot \alpha \partial_\gamma)
\]

\[
\hat{M}_3' = i\partial_\gamma
\]

Which reads in a matrix notation

\[
\hat{M}_k = i\hat{\kappa}_{ka} \partial_a \quad \text{(4)}
\]

\[
\hat{M}_k' = i\hat{\kappa}'_{ka} \partial_a \quad \text{(5)}
\]

Those operator-matrices obey the commutation relations

\[
[\hat{\kappa}_{la} \partial_a, \hat{\kappa}_{mb} \partial_b] = \varepsilon^{klm} \hat{\kappa}_{kc} \partial_c \quad \text{(6)}
\]

\[
[\hat{\kappa}'_{la} \partial_a, \hat{\kappa}'_{mb} \partial_b] = -\varepsilon^{klm} \hat{\kappa}'_{kc} \partial_c \quad \text{(7)}
\]

From those relations one sees that the operators $\hat{M}$ follow the same commutation relations as $\sigma_k/2$

\[
[\hat{M}_i, \hat{M}_j] = i\varepsilon^{ijk} \hat{M}_k \quad \text{(8)}
\]

while the operators $\hat{M}'$ follow anomalous commutation relations

\[
[\hat{M}'_i, \hat{M}'_j] = -i\varepsilon^{ijk} \hat{M}'_k \quad \text{(9)}
\]

At this point it is interesting to note that those operators do not fulfill the anti-commutation relations in general. This only happens when the operators are applied to functions with special properties. Holland finds the
following complete set of eigenfunctions for the operator $M^2$ with the eigenvalue $s(s+1) = 3/4$

$$u_i(\alpha_b) = \begin{cases} 
u_+ = \frac{1}{\pi \sqrt{\alpha}} \cos \left( \frac{\alpha}{2} \right) e^{-i(\gamma+\beta)/2} \\ 
u_- = \frac{1}{\pi \sqrt{\alpha}} \sin \left( \frac{\alpha}{2} \right) e^{i(\gamma+\beta)/2} \\ v_+ = \frac{1}{\pi \sqrt{\alpha}} \sin \left( \frac{\alpha}{2} \right) e^{i(\gamma-\beta)/2} \\ v_- = \frac{1}{\pi \sqrt{\alpha}} \cos \left( \frac{\alpha}{2} \right) e^{i(\gamma+\beta)/2} \end{cases}$$

(10)

These functions differ by their eigenvalues $\pm 1/2$ with respect to both, $M_1$ and $M_2$. Now, when acting on this set of functions, the operators $\hat{M}$ fulfill the Clifford algebra of the Pauli matrices $\sigma_k/2$

$\{A_{k\alpha} \partial_{\alpha}, A_{\beta\beta} \partial_{\beta}\} u_i + \frac{1}{2} \delta_{kl} u_i = 0$ ,

(11)

$\{A_{k\alpha} \partial_{\alpha}, A_{\beta\beta} \partial_{\beta}\} u_i + \frac{1}{2} \delta_{kl} u_i = 0$ .

(12)

This technique is typically only applied to non-relativistic spinors with two components [11 16]. But as it will be shown in the following sections it is also sufficient for a description of relativistic two component spinors.

C. Lagrangian Formulation for Relativistic Spin 1/2

In order to have a direct applicability of the concepts in the previous section we seek a quadratic and two component formulation of spin one half. Such a formulation was given in 1958 by Laurie Brown [17], who found a two component fermion theory with a Lagrangian which seemingly can not be found for the a single Feynman-Gell-Mann equation [18,19]. One defines

$p^+ = p_0 + \vec{\sigma} \vec{p} \equiv \sigma^\mu p_\mu$ ,

(13)

$p^- = p_0 - \vec{\sigma} \vec{p} \equiv \sigma^\mu p_\mu$ .

(14)

Due to the Clifford algebra of the Pauli matrices one has

$\sigma_\mu \sigma_\nu = g_{\mu\nu} + h_{\mu\nu}$

(15)

$\sigma_\mu \sigma_\nu = g_{\mu\nu} + h'_{\mu\nu}$

with

$h_{0i} = -h_{i0} = -h'_{0i} = h'_{i0} = \sigma_i$

(16)

$h_{ij} = h'_{ij} = -\frac{1}{2}[\sigma_i, \sigma_j]$

Further one finds from $D_\mu = p_\mu - A_\mu$

$D^+ D^- = D_\mu D^\mu - \frac{i}{2} h_{\mu\nu} F^{\mu\nu} = D^2 - i\vec{\sigma}(\vec{E} + i\vec{B})$

(17)

and

$D^+ D^- = D_\mu D^\mu - \frac{i}{2} h'_{\mu\nu} F^{\mu\nu}$ .

(18)

A Lagrangian is defined by introducing an auxiliary two component spinor $\Omega$

$\mathcal{L}_{LB} = m^{-1}(D^+ \Omega)^\dagger(D^- \psi) - m\Omega^\dagger \psi + c.c.$ .

(19)

After a partial integration this Lagrangian can be written in the form

$\mathcal{L}_{LB} = (D_\mu \Omega)^\dagger D^\mu \psi - \frac{i}{2} \rho^\dagger H_{\mu\nu}^{1/2} F^{\mu\nu} \psi - m m^2 \Omega^\dagger \psi + c.c.$ ,

(20)

where

$H_{\mu\nu}^{1/2} = -H_{i\alpha}^{1/2} = \sigma_i, \quad H_{ij}^{1/2} = -\frac{1}{2} [\sigma_i, \sigma_j]$ .

(21)

By varying with respect to the bi-spinor fields one obtains two equations of motion which are apart from one term the Feynman-Gell-Mann equations

$m^2 \psi = D^+ D^- \psi = (D^\mu D_\mu - \frac{i}{2} h_{\mu\nu} F^{\mu\nu}) \psi$

(22)

$m^2 \Omega = D^+ D^- \Omega = (D^\mu D_\mu - \frac{i}{2} h'_{\mu\nu} F^{\mu\nu}) \Omega$ .

(23)

The equations contain positive and negative energy solutions. One can project onto the positive solutions by putting [22]

$iD^+ \Omega = m \psi, \quad iD^- \psi = m \Omega$ .

(24)

Under the discrete symmetries C and P the equations [22] and [23] transform into each other (where C additionally has $\vec{E} \rightarrow -\vec{E}$). Those are the properties of the left handed and right handed Weyl spinors.

The two equations in [24] can be combined to a single equation for a four component spinor

$(D_\mu - \alpha_i D_i - \beta m) \begin{pmatrix} \Omega \\ \psi \end{pmatrix} = 0$ ,

(25)

with

$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

(26)

By multiplying this with $\beta$ from the left hand side one obtains

$(\gamma^\mu D_\mu - m) \begin{pmatrix} \Omega \\ \psi \end{pmatrix} = 0$

(27)

where $\gamma_\mu = (\beta, \beta \alpha_i)$. Equation [27] is the Dirac equation in Weyl representation. This shows that at the level of equations of motion, the action [19] is equivalent to Dirac’s formulations of spin one half.
II. AN ACTION FOR ANGLES AND SPIN 1/2

In this section it will investigated whether the formulation that was given in subsection [13] can be directly and uniquely adapted to the formulation of spin 1/2 that was given in subsection [14]. As first approach we propose the following action

\[ S = \int d^4x \int d\omega \left( \mathcal{L}_0 + \Lambda^{1/2} \right) + c.c \quad , \]  

(28)

where \( \omega = \sin(\alpha)d\alpha d\beta d\gamma \) and the angular integrals are over \( 0 \leq \alpha \leq \pi, 0 \leq \beta \leq 2\pi, 0 \leq \gamma \leq 4\pi \). The integral contains a main part

\[ \mathcal{L}_0 = m^{-1}(\hat{D}^+ \Omega) (\hat{D}^- \psi) - m\Omega^1 \psi \quad , \]

(29)

the Lagrange multiplier

\[ \Lambda^{1/2} = \lambda \cdot \Omega^1 (\hat{M}^2 - \hbar^2 \frac{3}{2}) \psi \quad , \]

(30)

and the operators

\[ \hat{D}^\pm = D^\pm |_{[\sigma_1 \rightarrow 2i\hbar \partial_\sigma, \sigma_0 = t \rightarrow 1]} \quad . \]

(31)

This is a straight forward guess, motivated by the form of the spinor action [19]. The two Lagrange multipliers ensure that the functions \( \psi \) and \( \Omega \) are eigenfunctions of the total spin operator \( \hat{M}^2 \) with the eigenvalue \( s(s+1) = 3/4 \). Having this constraint, the most general wave functions can be expressed as linear combinations of the \( \hat{M}^2 \) eigenfunctions given in [10]. With this representations of the functions \( \psi, \Omega \) checks the following relations:

\[ \int d\omega \ u^*_i u_j = \delta_{ij} \quad . \]

(32)

One further finds two sets of Pauli matrices

\[ 2i \int d\omega \ u^*_i \hat{A}_{ka} \partial_a u_j = \left( \begin{array}{cc} \sigma_k & 0 \\ 0 & \sigma_k \end{array} \right)_{ij} \quad , \]

(33)

one for \( u_+ \) and one for \( v_\pm \). Integrating this relation by parts and using

\[ 2i \int d\omega \ u^*_i (\partial_a \hat{A}_{ka}) u_j = 0 \quad \]

(34)

one finds that

\[ 2i \int d\omega \ (\partial_a u^*_i) \hat{A}_{ka} u_j = - \left( \begin{array}{cc} \sigma_k & 0 \\ 0 & \sigma_k \end{array} \right)_{ij} \quad . \]

(35)

It is possible to write down many different types of integrals, containing two derivative operators \( \partial_a \). A selection of such integrals is The explicit results for those integrals are given in the appendix VI A. The remaining integrals can be obtained from [74] by complex conjugation or partial integration.

A general spin one half expansion in terms of \( M^2 \) eigenfunctions is given by

\[ \psi(x, \alpha) = \sum_{a=1}^4 \psi_a(x) u^a(\alpha) \quad . \]

(36)

For the case of spin half it is sufficient to stick to the eigenfunctions of \( u_+, u_- \)

\[ \psi(x, \alpha_b) = \sum_{a=1}^2 \psi_a(x) u^a(\alpha_b) \quad . \]

(37)

Having the relations [32-84] together with [5] and [11] allows to show that for the spin one half functions [37] allows to show that for the spin one half functions

\[ \int d\omega \mathcal{L}_0|_{s=1/2} = \mathcal{L}_{LB} \quad . \]

(38)

This means that after integration over the angular coordinates, the Lagrangian [28] with the replacements [31] is exactly identical to the Lagrangian [19]. After this integration the expansion coefficients \( \psi_a(x) \) from [37] become the spinor components in [19]. The same result without any mixing terms is obtained if the eigenfunctions \( v_+, v_- \) are used. Thus, with the action [28] an explicit Lagrange formulation of relativistic spin one half in terms of the angles \( \alpha, \beta, \gamma \) has been found. However, the identities [11] and [84] allow to write this angular Lagrangian in many different forms. Further ambiguities could in principle arise when doing the replacement \( \sigma^k \rightarrow 2i\hbar \partial_\sigma \partial_a \) within the action. It is a priori not clear whether the derivative \( \partial_a \) should act to the right, or to the left, whether it should act on the functions \( \psi \) and \( \Omega \) only, or whether derivatives of the kind \( \partial_a \hat{A}_{kb} \) should also be allowed. A careful analysis of the integrals [33-74] shows that defining the derivatives as ones that always act to the right gives the right result. When using other definitions, one has to take possible sign changes like in [33 and 35] into account.

III. THE EQUATIONS OF MOTION

The equations of motion for the spinors \( \psi \) and \( \Omega \) were given in [22] and [23]. Now we will turn to the angular dependent Lagrangian [28]. Using the Clifford algebra [11], the main part of [28] can be written in the form

\[ \mathcal{L}_0 = m^{-1} \left( (D^- \Omega)^\dagger D^- \psi - 2\Omega^1 F_{\mu\nu} \mathcal{C}_{\nu k}^\mu \hat{M}^k \psi \right) - m\Omega^1 \psi \quad . \]

(39)

where \( \mathcal{C}_{\nu k}^\mu \) is defined as

\[ \mathcal{C}_{00}^k = 0, \quad \mathcal{C}_{0j}^k = -\mathcal{C}_{j0}^k = i\delta_k^j, \quad \mathcal{C}_{ij}^k = \epsilon_{ij}^k \quad . \]

(40)

We will use this Lagrangian in order to derive angular equations of motion for spin one half. When doing this
one has to take into account that the Lagrangian contains explicit dependence on angular degrees of freedom. The Lagrangian further contains derivatives of order one in the angles. Note that this Lagrangian is only valid if the constraints \[ \text{[Equation]} \] are applied, since only in this case the Clifford algebra \[ \text{[Equation]} \] can be used. Using this, the equation of motion for the auxiliary field \( \Omega(x, \alpha_b) \) reads

\[
\frac{\partial L}{\partial \dot{\Omega}^\dagger} - \frac{D}{Dx^k} \left( \frac{\partial L}{\partial \Omega^\dagger_{,k}} \right) = 0 ,
\]

where the total derivative as is given in the appendix \[ \text{[Equation]} \] is defined as

\[
\frac{D}{Dx^k} = \left( \frac{\partial}{\partial x^k} + \frac{\partial \psi_a}{\partial \dot{x}^k} \frac{\partial}{\partial \psi_a} + \frac{\partial \psi_{a,\sigma}}{\partial \dot{x}^k} \frac{\partial}{\partial \psi_{a,\sigma}} \right) + \frac{\partial \Omega_{,\dagger}}{\partial \dot{x}^k} \frac{\partial}{\partial \Omega_{,\dagger}} \left( \frac{\partial}{\partial \dot{x}^k} \right) .
\]

For the case of \[ \text{[Equation]} \] only the first line of \[ \text{[Equation]} \] contributes, while the second line of \[ \text{[Equation]} \] gives zero. First, one evaluates derivatives of the Lagrangian:

\[
\frac{\partial L}{\partial \dot{\Omega}^\dagger} = m^{-1} \left( -ieA_\mu D^\mu \psi - 2ieA_\mu \psi \Omega^\dagger \right) - \partial \psi \Omega^\dagger (x, \alpha_b)
\]

\[
\frac{\partial L}{\partial \Omega^\dagger_{,\rho}} = m^{-1} \left( D^\rho \psi - 2i\psi \partial \Omega^\dagger_{,\rho} \right) .
\]

Expanding the fields according to \[ \text{[Equation]} \], this can be rewritten giving

\[
\frac{\partial L}{\partial \dot{\Omega}^\dagger} = m^{-1} \left( -ieA_\mu D^\mu \psi \right) - \psi \Omega^\dagger - \partial \psi \Omega^\dagger (x, \alpha_b)
\]

\[
\frac{\partial L}{\partial \Omega^\dagger_{,\rho}} = m^{-1} \left( D^\rho \psi \right) - \psi \partial \Omega^\dagger_{,\rho} (x, \alpha_b)
\]

Joining \[ \text{[Equation]} \] - \[ \text{[Equation]} \] with the corresponding terms from the Lagrange multipliers one obtains the equation of motion from varying with respect to \( \Omega^\dagger \):

\[
D_\rho D^{\rho \prime} \psi(x_\mu, \alpha_b) - eF_{\mu \nu} C_k^{\mu \nu} \hat{M}_k \psi(x_\mu, \alpha_b)
\]

\[
m^2 \psi(x_\mu, \alpha_b) = m \lambda (\hat{M}^2 - R^2 \frac{3}{2}) \psi(x_\mu, \alpha_b).
\]

The corresponding constraint equation from \( \lambda \) is

\[
\Omega^\dagger (\hat{M}^2 - R^2 \frac{3}{2}) \psi(x_\mu, \alpha_b) = 0 .
\]

Expanding the functions \( \Omega, \psi \) in terms of a full set of \( \hat{M}^2 \) eigenfunctions one finds that due to equation \[ \text{[Equation]} \] only the functions \[ \text{[Equation]} \] survive. Thus the equation of motion \[ \text{[Equation]} \] subject to the constraint \[ \text{[Equation]} \] reads

\[
D_\rho D^{\rho \prime} \psi_a(x_\mu, \alpha_b) - eF_{\mu \nu} C_k^{\mu \nu} \hat{M}_k \psi_a(x_\mu, \alpha_b)
\]

\[
m^2 \psi_a(x_\mu, \alpha_b) = m \lambda (\hat{M}^2 - R^2 \frac{3}{2}) \psi(x_\mu, \alpha_b).
\]

Similarly, the equation of motion for \( \Omega \) can be obtained by varying the action \[ \text{[Equation]} \] with respect to \( \psi \) and the constraint \( \lambda^* \). This calculation can be largely simplified by performing a partial integration with respect to all angular coordinates \( \partial_\theta \) before evaluating the equations of motion. One finds that the analog equation of motion for \( \Omega \)

\[
D_\rho D^{\rho \prime} \Omega_a(x_\mu, \alpha_b) - eF_{\mu \nu} C_k^{\mu \nu} \hat{M}_k \psi_a(x_\mu, \alpha_b)
\]

\[
m^2 \Omega_a(x_\mu, \alpha_b) = 0 .
\]

In any case both \( \psi(x_\mu, \alpha_b^* \) and \( \Omega(x_\mu, \alpha_b^* \) are just scalar functions and as such they can only transform due to

IV. LORENTZ INVARIANCE

A. Lorentz transformations as coordinate transformation in angular space

Knowing that spinors in the standard picture transform under a certain representation of Lorentz group, one has to find the corresponding transformation in the angular picture. The spinors in the Lagrangian \[ \text{[Equation]} \] transform under a Lorentz transformation \( x'_\mu = \Lambda^\mu_\mu x_\mu \), according to

\[
\psi(x') = S_\psi(\Lambda) \psi(x) = e^{\frac{1}{2} (\theta_\xi - i \xi_\xi) \sigma_3} \psi(x)
\]

\[
\Omega(x') = S_\Omega(\Lambda) \psi(x) = e^{\frac{1}{2} (\theta_\xi + i \xi_\xi) \sigma_3} \psi(x)
\]

where \( \theta_\xi \) correspond to rotations and \( \xi_\xi \) correspond to boosts and the \( S_\psi(\Lambda), S_\Omega(\Lambda) \) are matrices in the spinor space. Following the scheme \[ \text{[Equation]} \] this suggests in the angular formulation

\[
\psi(x', \alpha_b) = e^{i(\theta - i \xi_\xi) \hat{M}_k} \psi(x, \alpha_b)
\]

\[
\Omega(x', \alpha_b^*) = e^{i(\theta + i \xi_\xi) \hat{M}_k^*} \Omega(x, \alpha_b^*)
\]

One can expand the operators in \[ \text{[Equation]} \] and \[ \text{[Equation]} \] for infinitesimal transformations

\[
e^{i(\theta - i \xi_\xi) \hat{M}_k} = N \approx 1 - \theta_\xi A_{kao} \partial_a + i \xi_\xi A_{kao} \partial_a
\]

\[
e^{i(\theta + i \xi_\xi) \hat{M}_k^*} = \bar{N} \approx 1 - \theta_\xi A_{kao} \partial_a - i \xi_\xi A_{kao} \partial_a
\]

In any case both \( \psi(x, \alpha_b^* \) and \( \Omega(x, \alpha_b^* \) are just scalar functions and as such they can only transform due to
their coordinates. Thus, every non trivial transformation like the one for spinors in \cite{51,52} has to originate from a transformation of the angular coordinates

\begin{align}
\psi'(x',\alpha_b) &= \psi(x,\alpha_b') \\
\Omega'(x',\alpha_b') &= \Omega(x,\alpha_b') .
\end{align}

Here \( \alpha'_b = \alpha_b + \delta \alpha_b \) are the angular coordinates that are transformed by the action of a certain Lorentz group element. This implies a unique transformation of the additional angular coordinates

\begin{align}
\psi'(\alpha_b) &= \exp \left( i(\theta_k - i\xi_k)\hat{M}_k \right) \psi(\alpha_b) \\
&\approx \psi(\alpha_b) - (\theta_k - i\xi_k)\hat{M}_k \partial_a \psi(\alpha_b) \\
&\approx \psi(\alpha_b) + \partial_a \psi(\alpha_b) \delta \alpha_a \implies \\
\delta \alpha_b &= -(\theta_k - i\xi_k)\hat{M}_k \partial_a \alpha_b .
\end{align}

Note that according Lie’s theorem, it is enough to consider the infinitesimal transformations, which lead after iterations to any finite transformation. Following this logic one finds the finite transformations of the angles

\begin{align}
\alpha'_b &= \exp \left( -(\theta_k - i\xi_k)\hat{M}_k \right) \alpha_b
\end{align}

where \( \alpha_b \) are the angles and \( \theta_k, \xi_k \) are the parameters of the Lorentz transformation. This expansion shows clearly that rotations change the real values of the angles while boosts actually introduce an imaginary part to the angles. Thus, after a boost the distinction between \( \alpha_a \) and \( \alpha'_a \) becomes important. Those complex parameters could cause problems, since it would be ambiguous to have within a part of the integral functions and operators that depend on both \( \alpha_a \) and \( \alpha'_a \). However, using the definition \cite{52} where \( \Omega = \Omega(x,\alpha'_a) \), one exactly avoids this type of problem. With reference to the eigenfunctions \cite{10} one sees that \( (\Omega(x,\alpha'_a))^* \) is effectively a function of the \( \alpha_a \) and not of the \( \alpha'_a \). Noting that all parts of the action combine \( \Omega^* \) and \( \psi \) (or the complex conjugation of both) one finds that no such ambiguity exists. This defines for a given Lorentz transformation the transformed (possibly complex) angles. It is important to note that the integral \( \int d\omega \) over those (possibly complex) angles is still effectively three dimensional, since it is simply a transformed version of the three dimensional real integral.

Please note that for rotations \( \theta_k \) the angular transformation behavior \cite{61} is just an application of a rotation in the Euler angles. For boosts \( \xi_k \) however the transformation \cite{61} generates an imaginary part for the angles and it also distorts their real part. This behavior is exemplified in figure \( \ref{fig:1} \) where the real and the imaginary part of an angle is plotted versus its value before the boost. In figure \( \ref{fig:2} \) it is shown how those final values evolve as function of the boost parameter \( \xi \). One observes that on the one hand the real part of the internal parameters \( \alpha_b \) changes as one would expect from a boosted coordinate system. The imaginary part, on the other hand, reflects a rescaling of the eigenfunctions \cite{10} and their normality conditions.

### B. Lorentz invariance of the angular Lagrangian

From the expansion \cite{55} one finds after a partial integration in the angular coordinate that the mass term stays invariant under such an infinitesimal transformation

\begin{align}
\langle \Omega^* \psi' \rangle &= \langle \Omega^* \psi \rangle + \mathcal{O}(\theta^2, \xi^2, \theta_k \xi_k) .
\end{align}

The same procedure works for the kinetic term of the Lagrangian \cite{39}

\begin{align}
\langle (D_\mu \Omega)^* D^\mu \psi' \rangle &= \langle (D_\mu \Omega)^* D^\mu \psi \rangle + \mathcal{O}(\theta^2, \xi^2, \theta_k \xi_k) ,
\end{align}

where the vector and spinor transformations cancel independently. Before going to the interaction term, it is useful to discuss the fundamental correspondence between \( \sigma^k \) and \( \hat{M}^k(\alpha_a) \). Given the fact that the matrices \( \sigma^k \) do not transform at all it seems puzzling that the operators \( \hat{M}^k(\alpha_a) \), who contain functions of the angles, actually do transform due to the transformation of
those angles. This puzzle persists even after an integration over \( d\omega \). The solution to this lies in the fact that in this picture the transformation of spatial and angular coordinates are connected. Thus, one can consider the possibility that the spatial index \( k \) of the operators also transforms, given that \( \hat{M}_k(\alpha_0) \) is not a constant matrix any more. In that case an infinitesimal Lorentz transformation of this operator can be written as

\[
\hat{M}'_k(\alpha_0') = \hat{M}_k(\alpha_0) + \hat{\omega}_k^\dagger \hat{M}_k(\alpha_0) + (\partial_b \hat{M}_k(\alpha_0))\partial \alpha_b .
\]  
(64)

On the right hand side, \( \hat{\omega}_k^\dagger \) is an infinitesimal transformation of the external index \( k \) and the \( \partial \alpha_b \) term corresponds to the transformation of the angular parameter that was found in (60). Allowing for this possibility one can find a \( \hat{\omega}_k^\dagger \) such that both transformation terms cancel and the angular operator is actually constant under Lorentz transformation, just as the usual Pauli matrices

\[
\hat{M}'_k(\alpha_0') = \hat{M}_k(\alpha_0) .
\]  
(65)

This condition implies

\[
\hat{\omega}_k^\dagger = \frac{i}{2}(\theta_n - i\partial_n)\epsilon^{nkl} .
\]  
(66)

Thus, using the above definition all operators \( \hat{M}_k(\alpha_0) \) can be treated as constants under Lorentz transformations. It is straightforward to show that the commutation and anti-commutation relations (8) and (11) are not affected by this kind of index and or angular transformation. This identity turns out to be useful for the interaction term, where one further has to make use of the second order angular integrals \( I_{k_1}^{\dagger} \), \( I_{k_1} \) and \( I_{k_1}^b \). One finds that the resulting terms cancel exactly with the terms from the transformation of the external electromagnetic field \( F^{\mu\nu} = \Lambda_\mu^\rho \Lambda_\nu^\sigma F^{\rho\sigma} \)

\[
\langle \Omega^* F_{\mu\nu} \bar{C}_k^\dagger M^k(\alpha_b) | \psi \rangle = \langle \Omega^* F_{\mu\nu} \bar{C}_k^\dagger \hat{M}_k^\dagger | \psi \rangle + O(\theta, \xi)^2 .
\]  
(67)

In this calculation the infinitesimal transformation \( \Lambda_\mu^\rho \approx \delta_\mu^\rho + \omega_\mu^\nu \delta_\nu^\rho \) with

\[
\omega_\mu^\nu = \begin{pmatrix}
0 & \xi_x & \xi_y & \xi_z \\
\xi_x & 0 & -\theta_y & \xi_z \\
\xi_y & \theta_y & 0 & \xi_z \\
\xi_z & -\theta_z & -\xi_y & 0
\end{pmatrix}
\]  
(68)

was used. Please note that the same proof, term by term, can be done for the other version of the Lagrangian that was previously defined in (28-31). Thus, from (62, 63, 67) one finds that the angular Lagrangian is invariant under the operational transformations (62, 63, 64) after integrating out the angular degrees of freedom. Those integrations involve a volume element of the angular degrees of freedom. Thus, it remains to be shown that also this volume element \( d\omega(\alpha, \beta, \gamma) = \sin \alpha d\phi d\beta d\gamma \) is invariant under a Lorentz transformation

\[
d\omega(\alpha', \beta', \gamma') = d\omega(\alpha, \beta, \gamma) .
\]  
(69)

The above identity is best shown infinitesimally where \( \alpha' \approx \alpha + \delta \alpha \) and

\[
d\omega(\alpha', \beta', \gamma') \approx \sin(\alpha + \delta \alpha) \left| \frac{\partial \alpha'}{\partial \alpha} \right| \sin \theta d\beta d\gamma . \tag{70}
\]

With the angular transformation (61) one finds to first order in \( \eta_k = \theta_k - i\partial_k \) that the Jacobian is given by

\[
\left| \frac{\partial \alpha'}{\partial \alpha} \right| \approx 1 + \eta_1 \cot \alpha \cos \beta - \eta_2 \cot \sin \beta + O(\eta^2) . \tag{71}
\]

For the same transformation the \( \sin(\alpha + \delta \alpha) \) function reads

\[
\sin \alpha' \approx 1 - \eta_1 \cot \alpha \cos \beta - \eta_2 \cot \sin \beta + O(\eta^2) . \tag{72}
\]

Joining (70, 71, and 72) one sees that the relation (69) holds.

After proving the invariance statements (62, 63, 67 and 69) by brute force computation it is instructive to reflect, on what they actually mean in terms of the coordinate transformation \( \alpha_b \rightarrow \alpha'_b \). In this context the relations (62) and (63) are actually trivial since no free index is involved and any arbitrary substitution of the angular coordinates would have left the integrals invariant. What is unique about the angular transformations (61) is that they leave the volume element \( d\omega \) invariant and that they compensate the space-time transformation of \( F_{\mu\nu} \) in (67). This compensation in (67) is possible due to the simultaneous transformation of the index \( k \) and the angles \( \alpha_b \), which allowed to use the identity (65).

Thus, in the context of Lorentz transformations three things have been shown:

(i) Event though the angles \( \alpha_b \) are transformed, Pauli matrices and angular operators can be treated analogue in the sense the identity (65).

(ii) The Lagrangian (19) is Lorentz invariant, in both formulations

(iii) The transformation laws for the left handed and right handed spinors \( \psi_\alpha(x_\mu) \), \( \Omega_\alpha(x_\mu) \) correspond to the same possibly complex transformation of the angular coordinates \( \alpha_b \) in the scalar functions \( \psi(x_\mu, \alpha_b) \) and \( \Omega(x_\mu, \alpha^*_b) \).

Further checks concerning Lorentz invariance are performed in (VTC).

V. CONCLUSIONS

In this paper we presented a new formulation of relativistic spin one half. The corresponding Lagrangian has three additional angular coordinates in terms of the Euler angles (\( \alpha, \beta, \) and \( \gamma \)). In the construction scalar functions with angular dependence play the role of spinors and angular operators play the role of Pauli matrices.
It is shown that, after integrating out the angular degrees for freedom, the bi-spinorial action of Brown is obtained \( [38] \). Since this bi-spinor action is known to be equivalent to the Dirac equation \( [27] \), the main objective of finding a new de Broglie Bohm formulation equivalent to the Dirac equation is achieved. Further, a equation of motion \( [49] \) for the extended space (containing space-time and angular degrees of freedom) is derived. This equation of motion should allow to calculate the Bohmian trajectories in the extended space. Finally, Lorentz invariance is shown and it is demonstrated how boosts and rotations act on the scalar functions \( \psi(x_\mu, \alpha_k) \) and \( \Omega(x_\mu, \alpha_k) \). The beauty of this formulation lies in the fact that Lorentz transformations for spinors simply correspond to a change of the parameters \( \alpha \) as indicated in \( [61] \).

Future investigation will be on explicit solutions of the equation of motion \( [49] \) and on the applicability of the method to systems with spin \( \neq 1/2 \).

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VI. APPENDIX

A. List of angular integrals

This part of the appendix contains the explicit form of angular integrals with two derivatives

\[
\iint d\omega \ u_i^a(\partial_h A_{kk}) u_j = 0 ,
\]

Evaluating those integrals one finds

\[
(I^0_{kl})_{ij} = \frac{1}{4} \begin{pmatrix}
\sigma_j \sigma_k & 0 \\
0 & \sigma_k \sigma_j
\end{pmatrix},
\]

\[
(I^1_{kl})_{ij} = 0,
\]

\[
(I^2_{11})_{ij} = -\frac{\delta_{ij}}{2} \lim_{\varepsilon \to 0} (\cos(\varepsilon) + \log(\tan(\frac{\varepsilon}{2}))),
\]

\[
(I^2_{22})_{ij} = (I^2_{11})_{ij} , \quad (I^2_{\text{rest}})_{ij} = 0,
\]

\[
(I^3_{11})_{ij} = \frac{1}{2} \delta_{ij} , \quad (I^3_{\text{rest}})_{ij} = 0,
\]

\[
(I^4_{kk})_{ij} = -\frac{1}{4} \delta_{ij} , \quad (I^4_{12})_{ij} = (I^4_{21})_{ij} = 0
\]

\[
(I^4_{13})_{ij} = (I^4_{31})_{ij} = \frac{1}{4} \begin{pmatrix}
\sigma_3 \sigma_1 & 0 \\
0 & \sigma_3 \sigma_1
\end{pmatrix},
\]

\[
(I^4_{23})_{ij} = (I^4_{32})_{ij} = \frac{1}{4} \begin{pmatrix}
\sigma_3 \sigma_2 & 0 \\
0 & \sigma_3 \sigma_2
\end{pmatrix},
\]

\[
(I^5_{kl})_{ij} = 0.
\]

\[
(I^6_{kk})_{ij} = (I^6_{23})_{ij} = (I^6_{13})_{ij} = 0,
\]

\[
(I^6_{12})_{ij} = -\frac{1}{4} \begin{pmatrix}
\sigma_3 & 0 \\
0 & \sigma_3
\end{pmatrix},
\]

\[
(I^6_{32})_{ij} = -\frac{1}{2} \begin{pmatrix}
\sigma_3 \sigma_2 & 0 \\
0 & \sigma_3 \sigma_2
\end{pmatrix},
\]

\[
(I^6_{31})_{ij} = -\frac{1}{2} \begin{pmatrix}
\sigma_3 \sigma_1 & 0 \\
0 & \sigma_3 \sigma_1
\end{pmatrix},
\]

\[
(I^7_{11})_{ij} = -\frac{\delta_{ij}}{2} \lim_{\varepsilon \to 0} (\cos(\varepsilon) + \log(\cot(\frac{\varepsilon}{2}))),
\]

\[
(I^7_{22})_{ij} = (I^7_{11})_{ij} , \quad (I^7_{\text{rest}})_{ij} = 0,
\]

\[
(I^7_{kl})_{ij} = \frac{1}{4} \begin{pmatrix}
\sigma_k \sigma_l & 0 \\
0 & \sigma_k \sigma_l
\end{pmatrix}.
\]

For the spin one half eigenfunctions one finds further the useful relations

\[
4(I^8_{kl})_{ij} = -4 \left( (I^4_{kl})_{ij} + (I^6_{kl})_{ij} \right) = \frac{\sigma_k \cdot \sigma_l}{0 \sigma_k \cdot \sigma_l}.
\]

\[
(I^5_{kl})_{ij} = \frac{1}{2} \left( (I^2_{kl})_{ij} + (I^7_{kl})_{ij} \right) /2 + (I^3_{kl})_{ij} = 0.
\]

When using partial integration one has to take care, since in many cases the boundary terms do not vanish. The integrals \( (I^5_{kl})_{ij} \) and \( (I^7_{kl})_{ij} \) do not converge at the boundaries of \( \alpha \). Therefore the asymptotic result for \( \int_0^{D-\epsilon} d\alpha \ldots \) was given.
B. EOM's from a Lagrangian with Explicit Coordinate Dependence

In this appendix the equations of motion for a physical system that is defined in a region \( R \) and whose coordinates are \( x_k \) \((k = 1..n)\) is discussed. The fields are functions of these variables and they are labeled as \( \psi^\alpha \), where \( \alpha \), runs over all fields defined in the system. The discussion follows basically [20]. The Lagrange density depends explicitly on \( x_k, \psi^\alpha, \) and \( \partial_k \psi^\alpha \). The action is defined as:

\[
\mathcal{A} = \int d(x) \mathcal{L} [x_k, \psi^\alpha(x), \partial_k \psi^\alpha(x)] . \tag{85}
\]

When performing a variation one has to take into account the region \( R \) and the coordinates \( x_k \rightarrow x_k + \delta x_k \). Thus, in general a field if varied according to

\[
\delta \psi^\alpha(x') := \psi^\alpha(x + \delta x) + \delta \psi^\alpha = \psi^\alpha(x) + \partial_k \psi^\alpha(x) \delta x^k + \delta \psi^\alpha \Rightarrow \delta \psi^\alpha(x) := \psi^\alpha(x') - \psi^\alpha(x) = \partial_k \psi^\alpha(x) \delta x^k + \delta \psi^\alpha \tag{86}
\]

\[
\Rightarrow \delta \partial_k \psi^\alpha(x) = \partial_k \partial_k \psi^\alpha(x) \delta x^k + \partial_k \psi^\alpha . \tag{87}
\]

Performing the total variation in the action, one gets

\[
\delta \mathcal{A} = \int d(x + \delta x) \mathcal{L} [x_k + \delta x_k, \psi^\alpha(x) + \delta \psi^\alpha(x), \partial_k \psi^\alpha(x) + \delta \partial_k \psi^\alpha(x)] - \int d(x) \mathcal{L} [x_k, \psi^\alpha(x), \partial_k \psi^\alpha(x)] \Rightarrow \delta \mathcal{A} = \int d(x) \left( \frac{D}{Dx_k} \left[ \mathcal{L} \delta x_k + \frac{\partial \mathcal{L}}{\partial \psi^\alpha_k} \delta \psi^\alpha(x) - \psi^\alpha_k(x) \delta x^k \right] \right)
\]

\[
\text{here, the derivative } \frac{D}{Dx_k} \text{ means the total derivative respect to the coordinates}
\]

\[
\frac{D}{Dx_k} = \partial_x^k + \psi^\alpha_k \partial_x^{\psi^\alpha} + \psi^\alpha_k \psi^{\psi^\alpha_k} \partial_x^{\psi^\alpha_k} . \tag{89}
\]

The first total derivative in (89) is a boundary term, which can be neglected. If one demands that the region \( R \) is left invariant one can set \( \delta x_k = 0 \). The remaining variation of the action is then

\[
\delta \mathcal{A} = \int d(x) \left( \left( \frac{\partial \mathcal{L}}{\partial \psi^\alpha} - \frac{D}{Dx_k} \frac{\partial \mathcal{L}}{\partial \psi^\alpha_k} \right) \delta \psi^\alpha(x) \right) . \tag{90}
\]

Hamilton’s principle requires that the above expression must vanish for every choice of the region of integration and for every choice of the variational functions \( \delta \psi^\alpha(x) \). Thus the resulting equations of motion are

\[
\frac{\partial \mathcal{L}}{\partial \psi^\alpha} - \frac{D}{Dx_k} \frac{\partial \mathcal{L}}{\partial \psi^\alpha_k} = 0 . \tag{91}
\]

The total derivative \([89]\) was defined in such a way that the equations of motion have a form that is similar to the one that is normally used in physics. The equations here are, however, more general since they include explicit coordinate dependence.

C. Lorentz group and its explicit form

A covering group of the orthochronous Lorentz group \( SL(2, \mathbb{C}) \) is defined by \( 2 \times 2 \) matrices, with null trace and determinant 1. This group is specified by six real valued parameters. As candidates to satisfy all these requirements, one may use:

\[
\left\{ L_k, K_k \right\} \Leftrightarrow \left\{ \frac{1}{2} \sigma_k, -i \frac{1}{2} \sigma_k \right\} .
\]

Which correspond to rotations \( (L_k) \) and boosts \( (K_k) \). They are the generators of Lorentz transformations in this space of representation of \( \sigma \) matrices. The above implies that an arbitrary element of \( SL(2, \mathbb{C}) \) can be written as

\[
\Lambda(\theta, \xi) \equiv \exp i \left( \frac{1}{2} \theta \sigma_k + i \frac{1}{2} \xi \sigma_k \right) = \exp \left( i \frac{1}{2} (\theta_k - i \xi_k) \sigma_k \right) \tag{92}
\]

This corresponds to one representation of the Lorentz group \( SL(2, \mathbb{C}) \), namely \((1/2, 0)\). If one conjugates the previous expression one obtains another representation, namely \((0, 1/2)\), which can be expressed as

\[
\Lambda^*(\theta, \xi) = \sigma_2 \left[ \exp \left( i \frac{1}{2} (\theta_k + i \xi_k) \sigma_k \right) \right] \sigma_2 . \tag{93}
\]

Individually, both act in a two component spinor space (or a Weyl spinor). In this way, a vector in the representation space \((1/2, 0) \otimes (0, 1/2)\), transforms:

\[
\Psi' = \begin{pmatrix} \Lambda(\alpha, \beta) & 0 \\ 0 & \sigma_2 \Lambda^*(\alpha, \beta) \sigma_2 \end{pmatrix} \Psi . \tag{94}
\]

Those are Dirac spinors. For our purposes the formulation in [17] is considering Weyl spinors, as it can be seen from [27]. This means that by choosing a representation for \( \psi \), for instance \((1/2, 0)\), one immediately fixed the representation of \( \Omega \) to be the other one \((0, 1/2)\). With this, Lorentz invariance of the Lagrangian in subsection [17B] is preserved. Remember that \( \Omega \) is connected to \( \psi \) via the equation [24].
A remaining question in our discussion is whether the transformation behavior given in (92 and 93) goes through to the angular formulation. In order to see this one may use the expectation values of the $\hat{M}$ spin operators in the basis of eigenfunctions of $M_3$, $M'_3$ and $M^2$. This can be done since both $\Omega$ (living in $(0,1/2)$) and $\psi$ (living in $(1/2,0)$) are expressed with the same basis functions $u_\alpha$. One has to show that the relation

$$\langle u_\alpha | e^{i(\theta-i\xi)\hat{M}^k} | u_j \rangle = \left( e^{\frac{i}{2}(\theta-i\xi)\sigma^k} \right)_{ij} . \quad (95)$$

holds for those expectation values. This proof involves excessive use of the Baker Campbell Hausdorff relation. It will be given separately for pure rotations and pure boosts. First, we consider $\xi_k = 0$:

$$\langle e^{i\theta_\xi \hat{M}^k} \rangle_{ab} = \langle \mathbb{I} + \frac{1}{2!} \left( i\theta_\xi \hat{M}^k \right) + \frac{1}{2!} \left( i\theta_\xi \hat{M}^k \right)^2 + \frac{1}{3!} \left( i\theta_\xi \hat{M}^k \right)^3 + \ldots \rangle_{ab} \quad \Leftrightarrow \quad \langle \mathbb{I} \rangle_{ab} = \delta_{ab}$$

$$\langle i\theta_\xi \hat{M}^k \rangle_{ab} = \frac{1}{2} i \theta_\xi \sigma^k_{ab} \quad (96)$$

Applying the base of eigenfunctions one can replace the anti-commutator by a Kronecker delta (see 11). This gives

$$\langle (i\theta_\xi \hat{M}^k)^2 \rangle_{ab} = -i \left( \frac{1}{2} \right)^3 \theta^2 \theta^k \sigma^k_{ab} . \quad (99)$$

The remaining terms can be solved in the same way, with the same tricks. So, by induction, one can also finish the entire table of expectation values of powers of $\hat{M}$ operators. Finally:

$$\langle e^{i\theta_\xi \hat{M}^k} \rangle = e^{\frac{i}{2} \xi_k \sigma^k} . \quad (100)$$

Using an identical procedure for pure boosts one finds

$$\langle (i\theta_\xi \hat{M}^k)^3 \rangle_{ab} = -i \left( \frac{1}{2} \right)^2 \theta_\xi \sigma^k \delta_{ab}$$

$$\langle e^{i\theta_\xi \hat{M}^k} \rangle = e^{\frac{i}{2} \xi_k \sigma^k} . \quad (101)$$

Thus, it has been shown that (95) holds for rotations and boosts and that the Lorentz transformations of Weyl spinors can be translated directly to the scalar functions with angular dependence.
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