

Nöther symmetries in Bianchi universes

S. Capozziello, G. Marmo, C. Rubano, P. Scudellaro

Dipartimento di Scienze Fisiche, Univ. di Napoli and INFN, Sez. di Napoli
Mostra d’Oltremare, pad. 19, 80125 Napoli. Italy

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Abstract

We use our Nöther Symmetry Approach to study the Einstein equations minimally coupled with a scalar field, in the case of Bianchi universes of class A and B. Possible cases, when such symmetries exist, are found and two examples of exact integration of the equations of motion are given in the cases of Bianchi AI and BV.

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capozziello@na.infn.it; gimarmo@na.infn.it; rubano@na.infn.it; scud@na.infn.it
1 Introduction

Astronomical observations imply that, on a very large scale, our universe can be simply considered as homogeneous and isotropic for cosmological purposes, being thus well described by a Friedman–Robertson–Walker (FRW) metric \([1]\). It was suggested \([2]\) that such a scenario could be nothing else but the result of a different previous history and, therefore, be also consistent with a large amount of primordial anisotropy.

If, moreover, one adopts an inflationary scenario to describe the very early history of the universe, it can be expected that what is actually observed today does not depend on specific initial conditions, including possible initial anisotropies \([3]\). On the contrary, pre–inflationary anisotropies can play a role if primordial friction coefficients, responsible for the coupling between the gravitational field and the inflaton field, are supposed. As a matter of fact, such coefficients are proportional to the rate of expansion of volume, and their increasing makes inflation work better (see for example \([4]\)).

Furthermore, since we generally expect that all homogeneous anisotropies do not increase, it is not strange that we find very small anisotropies in the microwave background \((10^{-7} \div 10^{-4})\) on large and small angular scales \([1], [5], [6]\). Anyway, it is possible to conjecture that anisotropic cosmological models isotropize as time approaches our epoch \([5], [7]\), which makes it meaningful to study situations in which isotropy is not necessary at early times.

On the other side, as we just mentioned before, introduction of a scalar field in cosmology is widely used, to obtain inflationary eras at the very beginning and to treat primordial matter fields in the schemes of Grand Unified Theories.

For all these reasons, it is worthwhile to understand dynamics of anisotropy and to search for anisotropic cosmological models which are exactly solvable.

In this paper, we investigate homogeneous theories of gravity, in which a scalar field is minimally coupled to gravity, searching for point symmetries in the cosmological Lagrangian density which allow to solve exactly the dynamical problem (we begun such a program in \([8]\)).

We start from the action

\[
A = \int d^4x \sqrt{-g} \left( R + \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi^{,\nu} - V(\varphi) \right),
\]

where \(R\) is the Ricci scalar curvature for a spatially homogeneous spacetime, described by the metric

\[
d s^2 = -dt^2 + g_{ab}(t) \omega^a \omega^b,
\]

\(\omega^a\) being 1-forms, not necessarily closed \((a, b = 1, 2, 3)\). By \(\varphi_{,\mu}\) \((\mu = 0, 1, 2, 3)\) we denote the covariant derivative of the scalar field \(\varphi\); \(g\) is the determinant of the metric, and \(V(\varphi)\) is the potential for \(\varphi\).

Brans-Dicke gravitational theories have been recently \([9]\) examined in a wider context. Anyway, a search for point symmetries in vacuum situations, in which \(\varphi\) is nonminimally coupled to gravity as well, has been recently performed in \([10]\) in a form, however, which is not immediately reducible to the minimally coupled ones.
Here we search for Noether symmetries of Lagrangian (1), in absence of ordinary matter, in order to select all Bianchi models which are exactly solvable by the so called Noether Symmetry Approach [8], [11].

The paper is organized as follows. The Bianchi classification and the Lagrangian description are sketched in Sect. 2. In Sect. 3 we apply our procedure to determine the cases where symmetries are present. Sect. 4 is devoted to the analysis of the results, and in Sect. 5 we give two examples of exact integration of the equations of motion. Conclusions are drawn in Sect. 6.

2 Bianchi classification and Lagrangian formulation

Bianchi spacetimes, in modern geometric language, are $G_3$–principal fiber bundles over $R$. The bundle being trivializable is usually written as $M^4 = R \times G_3$. $G_3$ stays for the three–dimensional group of isometries of the metric on $M^4$. If one is not paying attention to the topology of $G_3$ (i.e. not distinguishing between the simply connected group with respect to the quotient groups obtained by quotienting it with discrete normal subgroups (not distinguishing among Lie groups with the same Lie algebra) we can use a parallelization of $M^4$ with the vector fields $X_0, X_1, X_2, X_3$ with $X_0$ the ”time”–generator and $X_1, X_2, X_3$ the Killing vector fields of the metric closing on the Lie algebra of $G_3$. Of course, when $X_1, X_2, X_3$ are known (and complete) we can integrate them to the action of $G_3$ on $M^4$.

In the framework of this procedure, in 1897 Bianchi worked out the first classification of three–dimensional Lie algebras [13]. Of course, metrics for which corresponding groups are isometries describe cosmological models with specified symmetry properties. Bianchi started with the adjoint representation and his approach has been followed since then. Here, we would like to comment briefly on the classification by using the coadjoint representation, as recently proposed in [14] and [15].

As it is well known a Lie group is locally characterized by its structure constants $c_{bc}^a$. If $(X_a)$ is a basis for the vector space underlying the algebra $\mathcal{G}$ we have $[X_a, X_b] = c_{ab}^c X_c$ and the Jacobi identity $[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$ gives a quadratic constraint on the structure constants.

By identifying the vector space $\mathcal{G}$ with the linear functions on $\mathcal{G}^*$, $\mathcal{G} = \text{Lin}(\mathcal{G}^*, R)$, we can define a Poisson bracket on $\mathcal{G}^*$ by setting $\{x_a, x_b\} = c_{ab}^c x_c$. The classification problem is reduced now to the problem of classifying linear Poisson brackets on a three-dimensional vector space. By using the transformation properties of a Poisson bracket under change of coordinates one shows that with any bracket is associated a bivector field $\Lambda$. In our case $\Lambda = c_{ab}^c x_c \frac{\partial}{\partial x_a} \wedge \frac{\partial}{\partial x_b}$.

In three dimensions, any bivector field is associated with a one-form $\alpha$ by means of the contraction with a volume form $\Omega$, say $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. Thus, we have to consider and classify quadratic one-forms $\alpha = c_{ab}^c x_c e^{ab} dx_r = H^{cr} x_c dx_r$. The Jacobi identity is equivalent to $\alpha \wedge d\alpha = 0$ [14].

The one-form $\alpha$ can be written by decomposing $H^{cr}$ into symmetric and skew-symmetric...
part
\[ \alpha = \frac{1}{2} H^{(cr)}(x_r dx_c + x_c dx_r) + \frac{1}{2} H^{(cr)}(x_r dx_c - x_c dx_r) . \] (3)

By using linear transformations on the three dimensional vector space, we can bring \( \alpha \) to the normal form
\[ \alpha = \frac{1}{2} d(N_1 x_1^2 + N_2 x_2^2 + N_3 x_3^2) + \frac{a}{2} (x_2 dx_3 - x_3 dx_2) . \] (4)

Imposing the Jacobi identity, we find \( a N_1 = 0 \) and, by means of a suitable change of coordinate basis, we may set \( N_i = 0, -1, +1 \). Thus all quadratic one-forms giving rise to three dimensional Lie algebras are classified in two classes

1. Bianchi A, \( \alpha \) exact, i.e. \( a = 0 \)
2. Bianchi B, \( d\alpha \neq 0 \), i.e. \( a \neq 0 \).

In this approach, class A is associated with a foliation of \( \mathbb{R}^3 \) given by the Casimir function \( N_1 x_1^2 + N_2 x_2^2 + N_3 x_3^2 \). The level sets of this function are the coadjoint orbits of the Lie group \( G_3 \) associated with the Lie algebra \( \mathcal{G} \). They give information on the topology of the group. Clearly, if \( N_1 N_2 N_3 \neq 0 \), we have either \( O(3) \) or \( O(2,1) \), according to the signs of \( N_i N_j \) (the second, if one of them is negative). If at least one of the \( N \) is zero, the group is noncompact.

The parameters \( a, N_i \) are sufficient to characterize Bianchi classification, according to the following table

| Type | \( a \) | \( N_1 \) | \( N_2 \) | \( N_3 \) |
|------|--------|--------|--------|--------|
| I    | 0      | 0      | 0      | 0      |
| II   | 0      | 1      | 0      | 0      |
| III  | 1      | 0      | 1      | -1     |
| IV   | 1      | 0      | 0      | 1      |
| V    | 1      | 0      | 0      | 0      |
| VI\(_o\) | 0    | 1      | -1     | 0      |
| VI\(_h\) | a    | 0      | 1      | -1     |
| VII\(_o\) | 0    | 1      | 1      | 0      |
| VII\(_h\) | a    | 0      | 1      | 1      |
| VIII | 0      | 1      | 1      | -1     |
| IX   | 0      | 1      | 1      | 1      |

A parameter \( h \overset{\text{def}}{=} a^2/(N_2 N_3) \), is also introduced for subclassifying types VI and VII.

Because any Lie group is parallelizable, we can use left-invariant (or right invariant) one-forms in the metric (2), so that it will be invariant under left (or right) action of \( G_3 \) on \( M^4 \). By using a diagonal form for \( g_{ab} \) and selecting (for obvious reasons) a positive
definite one, we can parametrize \( g_{ab} \) by factoring \( \det \| g_{ab} \| \), namely, following Misner, we obtain \[ \| g_{ab} \| = e^{2\lambda(t)} e^{2\beta_{ab}(t)}, \]

where the matrix \( \beta \) is diagonal and traceless, thus depending on two variables only. A widely used choice is \[ \| \beta_{ab} \| = \text{diag} \left( -\frac{1}{2} \beta_1 + \frac{\sqrt{3}}{2} \beta_2 , -\frac{1}{2} \beta_1 - \frac{\sqrt{3}}{2} \beta_2 , \beta_1 \right). \]

Being \( \sqrt{-g} = e^{3\lambda} \), the expansion of volume is wholly determined by \( \lambda \). As to the shear, it is determined only by the \( \beta_i \). This is typical only for the ‘orthogonal’ case, when there are no effects of rotation and tilt. Spatially homogeneous sections are then orthogonal to the fluid flow vector in the universe \[ \| \beta_{ab} \| = \text{diag} \left( -\frac{1}{2} \beta_1 + \frac{\sqrt{3}}{2} \beta_2 , -\frac{1}{2} \beta_1 - \frac{\sqrt{3}}{2} \beta_2 , \beta_1 \right). \]

As shown in \[ \| \beta_{ab} \| = \text{diag} \left( -\frac{1}{2} \beta_1 + \frac{\sqrt{3}}{2} \beta_2 , -\frac{1}{2} \beta_1 - \frac{\sqrt{3}}{2} \beta_2 , \beta_1 \right). \]

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According to a well established procedure \[ \| \beta_{ab} \| = \text{diag} \left( -\frac{1}{2} \beta_1 + \frac{\sqrt{3}}{2} \beta_2 , -\frac{1}{2} \beta_1 - \frac{\sqrt{3}}{2} \beta_2 , \beta_1 \right). \]

we look for Nöther symmetries of the Lagrangian (7). In other words we look for a vector field \( X \) on the configuration space,
such that the Lie derivative with respect to the tangent lift $X^T$ of $X$ on the tangent space vanishes

$$L_{X^T}L = 0.$$  \hfill (11)

The conditions imposed by this equation generally puts restrictions on the class of allowed potentials for $\varphi$.

But let us first observe that there are a number of trivial cases.

- If $V = \text{const.}$, $\mathcal{L}$ does not depend on $\varphi$, which is therefore cyclic. Thus, for all classes and types, we always have the symmetry

$$X = \frac{\partial}{\partial \varphi}.$$  \hfill (12)

- In Bianchi AI and BV, we have that $R^*$ does not depend on $\beta_1, \beta_2$, which are therefore cyclic. We have thus the symmetries

$$X_1 = \frac{\partial}{\partial \beta_1}; \quad X_2 = \frac{\partial}{\partial \beta_2}.$$  \hfill (13)

Moreover, since the term $\dot{\beta}_1^2 + \dot{\beta}_2^2$ is rotationally invariant, we have the symmetry

$$X_3 = \beta_1 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial \beta_1} + \dot{\beta}_1 \frac{\partial}{\partial \beta_2} - \dot{\beta}_2 \frac{\partial}{\partial \beta_1}.$$  \hfill (14)

- In Bianchi AII, $R^*$ depends on $\beta_1$ only, so that we have the symmetry

$$X = \frac{\partial}{\partial \beta_2}.$$  \hfill (15)

In the following, using (11), we investigate the possibility of other symmetries, not so immediately evident.

First of all, we have to say that the configuration space is made of four variables, namely $\lambda, \beta_1, \beta_2$ and $\varphi$. Let us set

$$X = L \frac{\partial}{\partial \lambda} + B_1 \frac{\partial}{\partial \beta_1} + B_2 \frac{\partial}{\partial \beta_2} + F \frac{\partial}{\partial \varphi},$$  \hfill (16)

where $L, B_1, B_2$ and $F$ are unknown functions of $\lambda, \beta_1, \beta_2, \varphi$. The tangent lift will be

$$X^T = X + \frac{dL}{dt} \frac{\partial}{\partial \lambda} + \frac{dB_1}{dt} \frac{\partial}{\partial \beta_1} + \frac{dB_2}{dt} \frac{\partial}{\partial \beta_2} + \frac{dF}{dt} \frac{\partial}{\partial \varphi},$$  \hfill (17)

where $d/dt$ means Lie derivative along the dynamical vector field; for example, it is

$$\frac{dL}{dt} = \frac{\partial L}{\partial \lambda} \dot{\lambda} + \frac{\partial L}{\partial \beta_1} \dot{\beta}_1 + \frac{\partial L}{\partial \beta_2} \dot{\beta}_2 + \frac{\partial L}{\partial \varphi} \dot{\varphi}.$$  \hfill (18)
The expression $L_{\tau T_L}$ is a homogeneous quadratic polynomial in $\dot{\lambda}, \dot{\beta}_1, \dot{\beta}_2, \dot{\varphi}$, plus a term of zeroth degree. Therefore, it vanishes if and only if all the coefficients are zero independently. This leads to a system of eleven partial differential equations

\begin{align}
3L + 2 \frac{\partial L}{\partial \lambda} &= 0 ; \quad 3L + 2 \frac{\partial B_1}{\partial \beta_1} = 0 \\
3L + 2 \frac{\partial B_2}{\partial \beta_1} &= 0 ; \quad 3L + 2 \frac{\partial F}{\partial \varphi} = 0 \\
4 \frac{\partial L}{\partial \beta_1} - \frac{\partial B_1}{\partial \lambda} &= 0 ; \quad 4 \frac{\partial L}{\partial \beta_2} - \frac{\partial B_2}{\partial \lambda} = 0 \\
6 \frac{\partial L}{\partial \varphi} - \frac{\partial F}{\partial \lambda} &= 0 ; \quad \frac{\partial B_1}{\partial \beta_2} + \frac{\partial B_2}{\partial \beta_1} = 0 \\
3 \frac{\partial B_1}{\partial \varphi} + 2 \frac{\partial F}{\partial \beta_1} &= 0 ; \quad 3 \frac{\partial B_2}{\partial \varphi} + 2 \frac{\partial F}{\partial \beta_2} = 0
\end{align}

\[L(R^* + 6V(\varphi) + N_1N_2N_3(1 + N_1N_2N_3)) + B_1 \frac{\partial R^*}{\partial \beta_1} + B_2 \frac{\partial R^*}{\partial \beta_2} + 2FV'' = 0,\] (24)

which are valid for both classes A and B, with relatively different expressions for $R^*$.

We first obtain a general solution for equations (19-23), and then use (24) as a constraint to limit the class of solutions and find a condition on the potential.

From Eq. (19a), we immediately get

\[L = e^{-\frac{2}{3} \lambda} \bar{L}(\beta_1, \beta_2, \varphi),\] (25)

so that from (21)

\[B_{1,2} = -\frac{8}{3} \frac{\partial \bar{L}}{\partial \beta_{1,2}} e^{-\frac{2}{3} \lambda} + \bar{B}_{1,2}(\beta_1, \beta_2, \varphi),\] (26)

and from (22a)

\[F = -4 \frac{\partial \bar{L}}{\partial \varphi} e^{-\frac{2}{3} \lambda} + \bar{F}(\beta_1, \beta_2, \varphi).\] (27)

Using (22b), we get then

\[-\frac{8}{3} \frac{\partial^2 \bar{L}}{\partial \beta_1 \partial \beta_2} e^{-\frac{2}{3} \lambda} + \frac{\partial \bar{B}_1}{\partial \beta_2} = \frac{8}{3} \frac{\partial^2 \bar{L}}{\partial \beta_1 \partial \beta_2} e^{-\frac{2}{3} \lambda} - \frac{\partial \bar{B}_2}{\partial \beta_1}.\] (28)

Since $\bar{B}_{1,2}$ do not depend on $\lambda$, it must be

\[\frac{\partial^2 \bar{L}}{\partial \beta_1 \partial \beta_2} = 0,\] (29)
so that
\[ \bar{L} = \bar{L}_1(\beta_1, \varphi) + \bar{L}_2(\beta_2, \varphi) ; \] (30)

moreover (22b) gives
\[ \frac{\partial \bar{B}_1}{\partial \beta_2} = -\frac{\partial \bar{B}_2}{\partial \beta_1} . \] (31)

Substituting into (19b), we get
\[ 3e^{-3/2\lambda} \left( \bar{L}_1 + \bar{L}_2 - \frac{16 \partial^2 \bar{L}_1}{9} \right) + \frac{\partial \bar{B}_1}{\partial \beta_1} = 0 . \] (32)

Now, since only \( \bar{L}_2 \) and \( \bar{B}_1 \) depend on \( \beta_2 \), and \( \bar{B}_1 \) does not depend on \( \lambda \), it must be
\[ \bar{L}_2 = 0 ; \frac{\partial \bar{B}_1}{\partial \beta_1} = 0 . \] (33)

Substituting into (20a) we get analogously
\[ \bar{L}_1 = 0 ; \frac{\partial \bar{B}_2}{\partial \beta_2} = 0 . \] (34)

Thus, we find that \( L = 0 \) and \( B_1 = B_1(\beta_2, \varphi) \), \( B_2 = B_2(\beta_1, \varphi) \). From (22b) we obtain now
\[ B_1 = f(\varphi) \beta_2 + g_1(\varphi) ; \quad B_2 = -f(\varphi) \beta_1 + g_2(\varphi) , \] (35)

while from (20b) and (22a) we get \( F = F(\beta_1, \beta_2) \), and from (23a)
\[ 3f' \beta_2 + 3g_1' = -2 \frac{\partial F}{\partial \beta_1} . \] (36)

The left hand side does not depend on \( \beta_1 \), thus
\[ F = -\frac{3}{2} \left( g_1' - f' \beta_2 \right) \beta_1 + \tilde{F}(\beta_2) , \] (37)

and, analogously, from (23b)
\[ F = -\frac{3}{2} \left( g_2' - f' \beta_1 \right) \beta_2 + \tilde{F}(\beta_1) . \] (38)

By comparison we get
\[ f' = 0 ; \quad g_1'' = g_2'' = 0 , \] (39)
\[ -\frac{3}{2} g_1' \beta_1 - \tilde{F} \beta_1 = -F_0 ; \quad \frac{3}{2} g_2' \beta_2 + \tilde{F} \beta_2 = F_0 , \] (40)

where \( F_0 \) is an arbitrary constant. The general solution is thus
\[ L = 0 \quad ; \quad B_1 = c \beta_2 + c_1 \varphi + c_0 \] (41)
\[ B_2 = -c \beta_1 + c_2 \varphi + c'_0 \quad ; \quad F = F_0 - \frac{3}{2} (c_1 \beta_1 + c_2 \beta_2) , \] (42)
with \( c, c_1, c_2, c_0, c'_0, F_0 \) arbitrary constants.

We have now to plug solutions (42) into (24) and check for compatibility. Being \( L = 0 \) we get

\[
B_1 \frac{\partial R^*}{\partial \beta_1} + B_2 \frac{\partial R^*}{\partial \beta_2} = -2FV'.
\]  
(43)

It is easy to see that the two sides must be zero independently, so that we have two subcases: \( V = \text{const.} \) and \( F = 0 \). In both of them (43) is satisfied for types I and V. Examining the other types, we find that for Class A Eq. (43) is verified only for type II and \( B_1 = 0 \), while for Class B Eq. (43) is verified for type VI \( h \neq 0 \) iff

\[
c = 0 \quad ; \quad c_2 = -\frac{1}{a\sqrt{3}}c_1 \quad ; \quad c'_0 = -\frac{1}{a\sqrt{3}}c_0.
\]  
(44)

The other types are excluded from the beginning, as said above.

4 Results

Case 1 \(- V = \text{const.} \)

The situation for \( V = \text{const.} \) is summarized as follows

1a) Bianchi AI

The most general symmetry is written (on \( Q \)) as

\[
X = (c\beta_2 + c_1\varphi + c_0) \frac{\partial}{\partial \beta_1} + (-c\beta_1 + c_2\varphi + c'_0) \frac{\partial}{\partial \beta_2} + (F_0 - \frac{3}{2}(c_1\beta_1 + c_2\beta_2)) \frac{\partial}{\partial \varphi},
\]  
(45)

with obvious lift to \( TQ \). A basis of symmetries on \( TQ \) is given by

\[
X_1 = \frac{\partial}{\partial \beta_1} \quad ; \quad X_2 = \frac{\partial}{\partial \beta_2} \quad ; \quad X_3 = \frac{\partial}{\partial \varphi}
\]  
(46)

\[
X_4 = \varphi \frac{\partial}{\partial \beta_1} - \frac{3}{2} \beta_1 \frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial \beta_1} - \frac{3}{2} \beta_1 \frac{\partial}{\partial \dot{\varphi}}
\]  
(47)

\[
X_5 = \varphi \frac{\partial}{\partial \beta_2} - \frac{3}{2} \beta_2 \frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial \beta_2} - \frac{3}{2} \beta_2 \frac{\partial}{\partial \dot{\varphi}}
\]  
(48)

\[
X_6 = \beta_1 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial b} + \beta_1 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial \dot{b}}.
\]  
(49)

It turns out that only five of them are independent. For instance, we can take the first five of them. With these symmetries, we associate the following constants of the motion

\[
K_1 = e^{3\lambda} \beta_1 \quad ; \quad K_2 = e^{3\lambda} \beta_2 \quad ; \quad K_3 = e^{3\lambda} \dot{\varphi}
\]  
(50)

\[
K_4 = e^{3\lambda} \left( \varphi \beta_1 - \frac{3}{2} \beta_1 \varphi \right) \quad ; \quad K_5 = e^{3\lambda} \left( \varphi \beta_2 - \frac{3}{2} \beta_2 \varphi \right) \quad ; \quad K_6 = e^{3\lambda} (\beta_1 \beta_2 - \dot{\beta}_1 \beta_2).
\]  
(51)
These symmetries close on the following Lie algebra

\[ [X_i, X_j] = 0 \ , \ i, j = 1, 2, 3 \ ; \ [X_1, X_4] = -\frac{3}{2} X_3 \ ; \ [X_2, X_4] = 0 \]  

(52)

\[ [X_1, X_5] = 0 \ ; \ [X_2, X_5] = -\frac{3}{2} X_3 \ ; \ [X_3, X_4] = X_1 \ ; \ [X_3, X_5] = X_2 \]  

(53)

\[ [X_4, X_5] = \frac{3}{2} X_6 \ ; \ [X_1, X_6] = X_2 \ ; \ [X_2, X_6] = -X_1 \]  

(54)

\[ [X_3, X_6] = 0 \ ; \ [X_4, X_6] = X_5 \ ; \ [X_5, X_6] = X_4 \ , \]  

(55)

and the \( K_i \) close on the same algebra in terms of Poisson brackets.

This situation was examined in \[8\] (with different variables). In that paper, it was shown how to use the symmetries in order to obtain exact integration of the dynamics. We give an example below.

1b) Bianchi AII

The general form of \( X \) (on \( Q \)) is

\[ X = (c_2 \varphi + c_0) \frac{\partial}{\partial \beta_2} + \left( F_0 - \frac{3}{2} c_2 \beta_2 \right) \frac{\partial}{\partial \varphi} , \]  

(56)

from which we get the independent fields (on \( TQ \))

\[ X_1 = \frac{\partial}{\partial \beta_2} \ ; \ X_2 = \frac{\partial}{\partial \varphi} \ ; \ X_3 = \varphi \frac{\partial}{\partial \beta_2} - \frac{3}{2} \beta_2 \frac{\partial}{\partial \varphi} + \dot{\varphi} \frac{\partial}{\partial \beta_2} - \frac{3}{2} \dot{\beta_2} \frac{\partial}{\partial \varphi} \]  

(57)

with Lie algebra

\[ [X_1, X_2] = 0 \ ; \ [X_1, X_3] = -\frac{3}{2} X_2 \ ; \ [X_2, X_3] = X_1 \ . \]  

(58)

The associated constants of the motion are a subset of the above ones.

In spite of the existence of three symmetries, the exact integration is not straightforward in this case, so that we shall not study the situation here.

1c) Bianchi BV

The situation is quite similar to Bianchi AI, with the same symmetries and constants of the motion. The equations of course are different. An example of exact integration is given below.

1d) Bianchi BVI \( h \neq 0 \)

We have

\[ X_1 = \frac{\partial}{\partial \varphi} \ ; \ X_2 = \frac{\partial}{\partial \beta_1} - \frac{1}{a \sqrt{3}} \frac{\partial}{\partial \beta_2} \]  

(59)

\[ X_3 = \varphi \frac{\partial}{\partial \beta_1} - \frac{1}{a \sqrt{3}} \varphi \frac{\partial}{\partial \beta_2} - \frac{3}{2} \left( \beta_1 \frac{3}{2} \beta_2 \right) \frac{\partial}{\partial \varphi} + \dot{\varphi} \frac{\partial}{\partial \beta_1} - \frac{1}{a \sqrt{3}} \dot{\varphi} \frac{\partial}{\partial \beta_2} - \frac{3}{2} \left( \dot{\beta}_1 \frac{3}{2} \dot{\beta}_2 \right) \frac{\partial}{\partial \varphi} \]  

(60)
with algebra

\[ [X_1, X_2] = 0 \quad [X_1, X_3] = X_2 \quad [X_2, X_3] = \left( \frac{1}{3a^2} - \frac{3}{2} \right) X_2 \]  

(61)

and constants of the motion

\[ K_1 = e^{3\lambda} \dot{\beta}_1 \quad K_2 = e^{3\lambda} \left( \dot{\beta}_1 - \frac{3}{2} \dot{\beta}_2 \right) \]  

(62)

\[ K_3 = e^{3\lambda} \left( -3\varphi \beta_1 + \frac{1}{a^2} \varphi \dot{\beta}_2 + 3\beta_1 \dot{\phi} - \frac{3}{a\sqrt{3}} \right) . \]  

(63)

**Case 2 – \( V \neq \text{const.} \)**

2a) Bianchi AI

We obtain the following independent symmetries

\[ X_1 = \frac{\partial}{\partial \beta_1} \quad X_2 = \frac{\partial}{\partial \beta_2} \quad X_3 = \beta_1 \frac{\partial}{\partial \beta_2} - \beta_2 \frac{\partial}{\partial \beta_1} + \dot{\beta}_1 \frac{\partial}{\partial \dot{\beta}_2} \frac{\partial}{\partial \beta_1} \]  

(64)

with algebra

\[ [X_1, X_2] = 0 \quad [X_1, X_3] = -X_2 \quad [X_2, X_3] = X_1 \]  

(65)

and constants of the motion

\[ K_1 = e^{3\lambda} \dot{\beta}_1 \quad K_2 = e^{3\lambda} \dot{\beta}_2 \quad K_3 = e^{3\lambda} (\beta_2 \dot{\beta}_1 - \beta_1 \dot{\beta}_2) . \]  

(66)

This case was examined already in \[8\] and, although it was not possible to achieve exact integration, qualitative consideration on the behaviour of the solutions were made possible.

2b) Bianchi AII

Only one symmetry is left, namely

\[ X = \frac{\partial}{\partial \varphi} . \]  

(67)

2c) Bianchi BV

Again the situation is essentially the same as for Bianchi AI.

2d) Bianchi BVI\[0\]

We have only one symmetry

\[ X = \frac{\partial}{\partial \beta_1} - \frac{3}{2} \frac{\partial}{\partial \beta_2} \]  

(68)

with the corresponding constant of the motion

\[ K = e^{3\lambda} \left( \dot{\beta}_1 - \frac{1}{a\sqrt{3}} \dot{\beta}_2 \right) . \]  

(69)
5 Examples of exact integration

In this Section, as an example, we obtain exact integration in the Bianchi AI and BV cases, when the potential \( V(\varphi) \) is zero.

As said earlier, the first order Einstein equation is a first integral of the second order equations derived from the Lagrangian (7). It turns out that it is always equivalent to

\[
E_L \overset{\text{def}}{=} \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L = 0 .
\]  

(70)

It is in fact this equation that is left after reduction. Substituting the constants of the motion found above we get

\[
6\ddot{\lambda}^2 - K^2 e^{-6\lambda} - \alpha^2 e^{-2\lambda} = 0 ,
\]  

(71)

where \( K^2 = 3(K_1^2 + K_2^2)/2 + K_3^2 \) and \( \alpha^2 = 6a^2 e^{\frac{2}{3a^2}} \).

In Bianchi AI, i.e. when \( \alpha = 0 \), we obtain immediate integration

\[
\lambda = \frac{1}{3} \log \left[ \sqrt{\frac{3}{2}} K(t - t_0) \right] ,
\]  

(72)

which coincides with the one found in [8] and therefore, we avoid further comments on it. In Bianchi BV, setting \( x = e^{2\lambda} \), we obtain the equation

\[
\frac{3}{2} x^2 - K^2 x - \alpha^2 = 0 ,
\]  

(73)

which integrates to

\[
\sqrt{\frac{2}{3}} t = \sqrt{x(1 + x)} - \log(\sqrt{x} + \sqrt{1 + x}) ,
\]  

(74)

where we have set \( K = 1 \) (that is \( x(0) = 0 \)), and \( \alpha = 1 \) (that is, a rescaling of \( x \) and \( t \)).

This relation cannot be inverted but can be used for a qualitative analysis. Indeed, it is easy to see that, for small \( t \), and hence small \( x \), we have that the mean scale factor \( e^\lambda \) is proportional to \( t^{1/2} \), while for large \( t \) it is proportional to \( t^{1/3} \). It is also possible to see that power law-inflation never occurs.

As for the behaviour of \( \varphi \), it is immediately derivable from the expression of \( K_3 \). In the first case we get

\[
\varphi = \sqrt{\frac{2}{3}} \frac{K_3}{K} \log(t - t_0) ,
\]  

(75)

while in the second we have \( \varphi \propto t^{-1/2} \) for small \( t \) and \( \varphi \propto \log t \) for large \( t \).
6 Conclusions

We have found all Noether symmetries for the Bianchi universes for which a Lagrangian function, of the type given in [14], is known. This allowed immediate exact integration in simple cases. In the other cases we have seen that the number of symmetries is often sufficiently high to permit a good reduction of the configuration space. These situations are of great physical interest and a complete analysis shall be done elsewhere.

It is important to observe that the Lagrangian (7), in the case when we want to add ordinary matter (in form of dust decoupled from the scalar field), changes only by an additive constant. This is not irrelevant, because this constant changes the equation $E_L = 0$, but of course the second order equations and the structure of the symmetries are exactly the same. This means that all the discussion above is valid for this case, except for the integration, which is now more complicate.

Other types of ordinary matter only give a contribution in the coefficient of $L$ in Eq. (24). Since $L = 0$, we see that again the structure of symmetries is the same and the problems for integration may still arise from the energy condition.

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