Kinetic evolution and correlation of fluctuations in an expanding quark gluon plasma

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Evolution of spatially anisotropic perturbation created in the system formed after Relativistic Heavy Ion Collisions has been studied. The microscopic evolution of the fluctuations has been examined within the ambit of Boltzmann Transport Equation (BTE) in a hydrodynamically expanding background. The expansion of the background composed of Quark Gluon Plasma (QGP) is treated within the framework of relativistic hydrodynamics. Spatial anisotropic fluctuations with different geometry have been evolved through Boltzmann equation. It is observed that the trace of such fluctuation survive the evolution. Within the relaxation time approximation analytical results have been obtained for the evolution of these anisotropies. Explicit relations between fluctuations and transport coefficients have been derived. The mixing of various Fourier (or $k$) modes of the perturbations during the evolution of the system has been explicitly demonstrated. This study is very useful in understanding the presumption that the measured anisotropies in the data from heavy ion collisions at relativistic energies imitate the initial state effects. The evolution of correlation function for the perturbation in pressure has been studied and shown that the initial correlation between two neighbouring points in real space evolves to a constant value at later time which gives rise to Dirac delta function for the correlation function in Fourier space. The power spectrum of the fluctuation in thermodynamic quantities (like temperature estimated in this work) can be connected to the fluctuation in transverse momentum of the thermal hadrons measured experimentally. The bulk viscous coefficient of the QGP has been estimated by using correlations of pressure fluctuation with the help of Green-Kubo relation. Angular power spectrum of the anisotropies has been estimated in the appendix.

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1. Introduction

The main aim of the heavy ion collision experiments at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) is to create a new state of matter called quark gluon plasma (QGP). Such a state of matter, i.e. QGP\(^1\) may have existed in the early universe after a few microsecond of the Big Bang. One of the motivations to create and study QGP in the laboratory is to understand the state of the universe in the microsecond old era. The fluctuations of physical quantities from their average values can be used to understand several properties of the system i.e. the transport coefficients of the medium, the approach toward equilibrium, etc. The fluctuation becomes extremely important in the neighborhood of nuclear phase transition. The study of temperature ($T$) fluctuation in the cosmic microwave background radiation (CMBR) has provided crucial information about the universe when it was about 300,000 years old. This information has led to tremendous support to the Big Bang model of cosmology. The polarization of the photons resulting from the Thomson scattering at the decoupling surface infected by density fluctuation gets reflected in the quadrupole moment of the phase space distribution of the incident photon. The fluctuation of $T$ in the CMBR is introduced as a perturbation in the phase space distribution of photons. The evolution of this perturbation is studied by using Boltzmann transport equation (BTE\(^2\)) in gravitational field with Thomson
scattering in the collision term. The linear polarization resulted from the scattering is connected with the quadrupole moment of the photon’s phase space distribution.

The fluctuations in the position of nucleons (with finite size) in the colliding nuclei lead to lumpiness in the spatial distribution of initial energy density of the system formed in Relativistic Heavy Ion Collisions Experiments (RHIC-E). The fluctuation in energy density may also originate due to the energy deposition by the propagation of energetic partons produced in the early stage of the RHIC-E. These fluctuations may lead to observable effects similar as temperature fluctuation in CMBR. We have adopted an approach similar to the one followed to study the fluctuation in CMBR. We study the evolution of perturbations by introducing a deviation, $\delta f(x,p,t)$ in the equilibrium distribution function, $f^{(0)}(p)$. The bulk in equilibrium evolves via relativistic hydrodynamics and the evolution of the fluctuations over the equilibrated expanding background, $\delta f$ is treated within the ambit of BTE. In contrast to this the propagation of perturbation has been studied using hydrodynamics in Refs.\textsuperscript{3–7}. $\delta f$ may be used to estimate the fluctuations in various thermodynamical quantities as we will see below. In the present work the equilibrated background is assumed to be quark gluon plasma (QGP) expected to be produced in RHIC-E.

The fluctuations in the thermodynamic quantities (e.g., hot spots created in the initial state of the collisions)\textsuperscript{8} can be related to perturbations in the phase space distributions in the hydrodynamic limit. The evolution of these fluctuations can be analyzed within the ambit of BTE.\textsuperscript{3} Fluctuations in thermodynamic quantities have been proposed as signals of the critical end point in the QCD phase transition.\textsuperscript{9,10} Dissimilar fluctuations in partonic and hadronic phases in the net electric charge and baryon number may shed light on the QCD phase transition in RHIC-E.\textsuperscript{12} Event by event fluctuations in the ratio of positively to negatively charged pions may be used as an indicator of QCD transition as well as for understanding the chemical equilibrium in the system formed in RHIC-E.\textsuperscript{14} Evolution of these fluctuations near the critical end point has been studied by using BTE.\textsuperscript{15} Kinetic theory approach has also been adopted to study fluctuations in particle and energy densities.\textsuperscript{16} In Ref.\textsuperscript{17} it has been argued that the perturbations in hot QGP travel longer distance to reach the border of the medium giving rise to the possibility of detectable signatures of these perturbations.

The study of the fluctuations in the space time structure of the fireball driven by the fluctuations in the position of the nucleons in the colliding nuclei is an important contemporary issue in RHIC-E. Fluctuations in the space-time structure of the system will infect fluctuations in the thermodynamic quantities. How will these fluctuations evolve with time in a hydrodynamically expanding system and how are they connected with the transport coefficients for matter formed in RHIC-E are addressed in this work.

In Ref.\textsuperscript{18} role of non-equilibrium processes on the evolution of QGP was studied within the framework of Parton-Hadron String Dynamics (PHSD) transport approach. It was found that the event-by-event fluctuations on collective variables estimated by the microscopic PHSD model is large due to non-equilibrium processes. However, the ensemble averaged results from these events is close to the results obtained in $(2+1)$-dimensional viscous hydrodynamics. In this context the study of the evolution of perturbations in the microscopic approach is crucial. Therefore, the present study of the perturbations in a hydrodynamically expanding background within the ambit of kinetic theory is appropriate. This is not only expected to achieve better microscopic understanding of the physics but also avoid question of breaking down of hydrodynamic description of fluctuation.\textsuperscript{20} Authors in Ref.\textsuperscript{19} has discussed deposition of energy by the away side mini-jet in the non-equilibrium framework. Corresponding perturbation in the medium has clear anisotropic form in space, as mini-jet deposits energy along its path. Therefore, instead of Gaussian type perturbation consideration of anisotropic perturbation in space will be more appropriate in such cases. In this work we have developed a formalism to find the evolution of fluctuation in hydrodynamically expanding background such that microscopically mode-by-mode analysis can be performed.

The paper is organized as follows. In the next section we discuss the evolution of fluctuations in a non-expanding background and present a relation between energy fluctuation in Fourier space and viscous coefficient. Section 3 is devoted to discuss the progression of fluctuations in a hydrodynamically expanding QGP background. Results are presented in section 4 and section 5 is dedicated to summary and discussions.
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2. Evolution of fluctuations in a non-expanding background

In the following subsections we discuss the connection of $\delta f$, a small deviation of phase space distribution from its equilibrium value with the fluctuations in various thermodynamic variables and its time evolution in a non-expanding background within the framework of BTE (results with expansion will be discussed in section 3). The phase space distribution function, $f(\vec{x}, \vec{p}, t)$ of a system slightly away from equilibrium, at time $t$, position $\vec{x}$, momentum $\vec{p}$ can be written as,

$$f(\vec{x}, \vec{p}, t) = f^0(p)\{1 + \Psi(\vec{x}, \vec{p}, t)\} = f^0(p) + f^0\Psi(\vec{x}, \vec{p}, t)$$

(1)

where $f^0(p)$ is the phase space distribution function in equilibrium and $\Psi(\vec{x}, \vec{p}, t)$ is the fractional deviation from $f^0(p)$. $\Psi(\vec{x}, \vec{p}, t)$ can be used to estimate the fluctuations in various thermodynamic quantities in the system. The evolution of $\Psi$ is governed by BTE, which in turn provides the relation between the dissipative effects and fluctuations in the hydrodynamic limit.

2.1. Fluctuations of various hydrodynamic quantities in Fourier space

A given fluctuations in spatial coordinate can be expressed in terms of various $k$-modes in Fourier space. These $k$-modes or wave number modes can be connected to the wave length ($\lambda = 2\pi/k$) modes, which in turn is related to the angular size ($\vartheta$) of the fluctuations through the relation: $\vartheta = \lambda/d$, where $d$ is the angular diameter as is usually done in analyzing temperature fluctuations in the universe. Fourier analysis is also important to understand what are the $k$-modes of the fluctuations that dissipate during the evolution of the system. For example it is important to determine the viscous horizon in heavy ion collisions.

The energy momentum tensor, $T_{\mu\nu}$ of the system under study can be written as: $T_{\mu\nu} = T_{\mu\nu}^0 + \Delta T_{\mu\nu}$, where the equilibrium (ideal) part, $T_{\mu\nu}^0$ is determined by $f^0(p)$ and the dissipative part, $\Delta T_{\mu\nu}$ is determined by $\Psi$, i.e.

$$T_{\mu\nu} = \int d^3 p \frac{p_{\mu}p_{\nu}}{p^0} f(\vec{x}, \vec{p}, t)$$

(2)

where $f(\vec{x}, \vec{p}, t)$ is given by Eq. 1. The $T_{\mu\nu}$ of the system in equilibrium can be obtained from Eq. 2 by setting $\Psi = 0$. The metric, $g_{\mu\nu}$, in Minkowski space-time is taken as $g_{\mu\nu} = (-1, 1, 1, 1)$. We assume that the momentum, $\vec{p}$ can be written as $p_i = p n_i$, i.e. $\vec{p} = \hat{p} \hat{n}$ where $\hat{n}$ is a unit vector and $d^3 p = p^2 dp d\Omega$, $d\Omega$ being solid angle associated with $n_i$ which satisfies $\int d\Omega n_in_j = 4\pi \delta_{ij}/3$ and $\int d\Omega n_in_j n_k = 0$.

It is straightforward to obtain various components of $T_{\mu}^0$ from Eq. 2. The deviation of the components of the stress energy tensor, $\Delta T_{\mu\nu}$ from their ideal values can be expressed in terms of the perturbation, $\Psi$ as follows:

$$\Delta T_{\mu\nu}^0 = -\int d^3 p \epsilon f^0(p) \Psi(\vec{x}, \vec{p}, t),$$

$$\Delta T_{\mu\nu}^0 = \int d^3 p \epsilon f^0(p) \Psi(\vec{x}, \vec{p}, t),$$

(3)

$$\Delta T_{\mu\nu}^0 = \int d^3 p \epsilon f^0(p) \Psi(\vec{x}, \vec{p}, t)$$

where $\epsilon = p^0 = -p_0 = \sqrt{p^2 + m^2}$ and $m$ is the mass of the particle. It is to be noted that the integral, $\int d^3 p p n_i f^0(p) = 0$, since in equilibrium the system is isotropic i.e. all directions are equally probable in its rest frame, the phase space average of momentum vector then is zero.

The ideal part of the stress energy tensor in the hydrodynamic limit is given by,

$$T_{\nu}^\mu = (\hat{p} + \hat{P}) U^\mu U_\nu - \hat{P} \delta_{\nu}^\mu,$$

(4)

where $U^\mu = dx^\mu/d\tau = \gamma(1, \vec{v})$ is the four velocity of the fluid and $\tau$ is the proper time. The relations among various components of $\tilde{T}_{\mu\nu}^0$ and the thermodynamic variables are: $\tilde{T}_0^0 = -\hat{P}$, $\tilde{T}_t^i = -\tilde{T}_i^0 = 0$, and $\tilde{T}_j^i = \hat{P} \delta_{ij}$. Therefore, $\hat{P} = \int p^2 dp d\Omega \epsilon f^0(p)$, is the average energy density. Other thermodynamic quantities like pressure, number density etc can be estimated in a similar way. Equilibrium distribution is isotropic, therefore, integration over $d\Omega$ will simply give $4\pi$. If the fluid is slightly away from equilibrium with space
time dependent fluctuations in energy density, pressure, velocity, etc., then the system will evolve toward equilibrium through dissipative processes. In such situation the components of the energy momentum tensors can be explicitly expressed in terms of the thermodynamic variables as:

\[
\begin{align*}
T_{0}^{0}(x_{i}, t) &= -\{\dot{\rho} + \delta \rho(x_{i}, t)\}, \\
T_{i}^{0}(x_{i}, t) &= -T_{0}^{i} = (\dot{\rho} + P)v_{i}, \\
T_{j}^{i}(x_{i}, t) &= \{\dot{P} + \delta P(x_{i}, t)\}\delta_{j}^{i} + \Sigma_{j}^{i}(x_{i}, t),
\end{align*}
\]  
\( \Delta T_{j}^{i} = 0, \)

where \( v_{i} \) is the \( i^{th} \) component of the velocity perturbation. We can choose a frame which is moving with velocity close to the velocity of the fluid, so that the fluid velocity measured from this frame is small. From Eq. 5 we get \( \Sigma_{j}^{i} = T_{j}^{i} - \delta_{j}^{i}T_{k}^{k}/3 \). By using Eqs. 3 and 5 we get,

\[
\begin{align*}
\delta \rho(x_{i}, t) &= -\Delta T_{0}^{0}(x_{i}, t), \\
v_{i}(x_{i}, t) &= \delta T_{0}^{i}(x_{i}, t) / (\dot{\rho} + P), \\
\delta T_{j}^{i}(x_{i}, t) &= \delta P(x_{i}, t)\delta_{j}^{i} + \Sigma_{j}^{i}(x_{i}, t),
\end{align*}
\]

with \( \delta P(x_{i}, t) = \delta T_{j}^{i}(x_{i}, t)/3 \). The shear stress, \( \Sigma_{j}^{i}(x_{i}, t) \) can be expressed in terms of shear viscous coefficient, \( \eta \) as \( \Sigma_{j}^{i}(x_{i}, t) = -\eta(\partial U_{i}/\partial x_{j} + \partial U_{j}/\partial x_{i}) - 2\delta_{j}^{i}\partial U_{k}/\partial x_{k} \) and the thermal conductivity (\( \chi \)) is defined through the relation, \( \Delta T_{0}^{i} = -\chi(\partial T / \partial x_{i}) \).

It is useful to express these quantities, i.e. various components of \( \delta T_{\mu \nu} \) in Fourier or \( k \)-space because expansion of these quantities in terms the spherical harmonics \( (Y_{lm}) \) will enable to connect the angular scales set by \( l \) in terms of \( k \) analogous to the determination of angular scale in CMBR. In \( k \)-space these quantities are marked by tilde (\( \tilde{\cdot} \)) as:

\[
\tilde{\Sigma}_{j}^{i}(k_{i}, t) = -i\eta(\tilde{U}^{i}k_{j} + \tilde{U}_{j}k^{i} - \frac{2}{3}\delta_{j}^{i}k^{r}\tilde{U}_{r})
\]  
\( \tilde{\cdot} \)

and

\[
\tilde{\Delta}T_{0}^{i}(k_{i}, t) = -\chi \left[ ik_{i}\tilde{T}(k_{i}, t) + \tilde{T}(k_{i}, t)\frac{\partial \tilde{U}_{i}}{\partial \theta} \right].
\]

By using Eqs. 3 and 6 the fluctuations in \( k \)-space can be expressed in terms of Fourier mode, \( \Psi \) as:

\[
\begin{align*}
\tilde{\delta} \rho(k_{i}, t) &= \int p^{2} dp d\Omega \epsilon f^{(0)}(p)\tilde{\Psi}(k_{i}, p, n_{i}, t), \\
\tilde{\delta} v_{i}(k_{i}, t) &= -\frac{1}{(\dot{\rho} + P)} \int p^{2} dp d\Omega \, n_{i} f^{(0)}(p)\tilde{\Psi}, \\
\tilde{\delta} \tilde{\Sigma}_{j}^{i}(k_{i}, t) &= \int p^{2} dp d\Omega \frac{p^{2}}{\epsilon} (n_{i}n_{j} - \frac{1}{3}\delta_{j}^{i})f^{(0)}(p)\tilde{\Psi}, \\
\tilde{\delta} \tilde{\dot{P}}(k_{i}, t) &= \frac{1}{3} \int p^{2} dp d\Omega \frac{p^{2}}{\epsilon} f^{(0)}(p)\tilde{\Psi}
\end{align*}
\]

where \( \tilde{\Psi} \) is the Fourier transform of \( \Psi \). Now we take the zenith direction along \( \vec{k} \) and then the angular dependence of \( \tilde{\Psi}(k_{i}, p, n_{i}, t) \) can be expressed in terms of angles between \( \vec{k} \) and \( \vec{n} \). Depending on the symmetries of the problem under consideration \( \tilde{\Psi} \) can be expanded in a series of suitably chosen basis functions e.g. for axial symmetry in terms of Legendre polynomials and in absence of such symmetry it can be expressed in terms of spherical harmonics.

The vector component \( n_{i} \) and tensor components \( (n_{i}n_{j} - \frac{1}{3}\delta_{j}^{i}) \), appearing in the expressions for \( v_{i}(k_{i}, t) \) and \( \Sigma_{j}^{i}(k_{i}, t) \) respectively, can be converted into functions of \( \theta \) (angle between \( \vec{k} \) and \( \vec{n} \)), by taking contraction with suitable tensors made out of the components of \( \vec{k} \). If we contract \( n_{i} \) with \( k_{i} \) then we get \( k\vec{k} \cdot \vec{n} = k \cos \theta = kP_{1}(\vec{k} \cdot \vec{n}) \) and by contracting \( (n_{i}n_{j} - \frac{1}{3}\delta_{j}^{i}) \) with \( (\vec{k}_{i}\vec{k}_{j} - \frac{1}{3}\delta_{j}^{i}) \) we get \( \frac{k^{2}}{2} (3(\vec{k} \cdot \vec{n})^{2} - 1) = \frac{2}{3} (3 \cos^{2} \theta - 1) = \frac{2}{3} P_{2}(\vec{k} \cdot \vec{n}) \), where \( P_{l}s \) are Legendre polynomials. For axial symmetric distribution of \( \vec{n} \) it helps to connect
different co-efficient of expansion of $\tilde{\Psi}$ in terms of Legendre polynomials with corresponding scalar quantities obtained from $v_i(k_i, t)$ and $\Sigma^j_i(k_i, t)$ due to orthogonality relation satisfied by $P_l$’s. We define the scalar quantities like $\Delta$, $\theta$ and $\sigma$ as\footnote{as used in the appendix to find the angular correlations} which, as will be seen later allow us to get evolution equation for the Fourier modes. The fluctuation in energy density in Fourier space is given by,

$$\Delta(k_i, t) = \frac{\delta \tilde{\rho}(k_i, t)}{\tilde{\rho}} = -\frac{\delta \tilde{T}^0(k_i, t)}{\tilde{\rho}}.$$  \hspace{1cm} (10)

Similarly we define energy flux,

$$\theta(k_i, t) = i k^{i} \tilde{v}_i = \frac{i k^{i} \delta \tilde{T}^0(k_i, t)}{(\tilde{\rho} + \tilde{P})},$$  \hspace{1cm} (11)

and the shear stress as,

$$(\tilde{\rho} + \tilde{P}) \sigma(k_i, t) = -(\tilde{\epsilon}^{i}_{\ j} - \frac{1}{3} \delta^{i}_{\ j}) \Sigma^{j}_i(k_i, t),$$  \hspace{1cm} (12)

The quantity, $\theta(k_i, t) = i k^{i} v_j$ originates from the velocity gradient. $\theta$ and $\sigma$ can be expressed in terms of the shear viscous coefficient ($\eta$) and thermal conductivity ($\chi$) as follows:

$$\sigma(k, t) = \frac{4}{3 \tilde{\rho} + \tilde{P}} i k^{i} \tilde{v}_i(k, t)$$  \hspace{1cm} (13)

and

$$\theta(k, t) = \frac{\chi}{\tilde{\rho} + \tilde{P}} (k^2 \tilde{T}(k, t) - i k^{i} \tilde{T}(k, t) \tilde{v}_i(k, t)).$$  \hspace{1cm} (14)

The left hand side of both the equation above can be estimated from the solution BTE. Fourier transformation of these equations in frequency space will lead to dispersion relation. This relation can be used to determine those $k$ (wave number) values which will dissipate due to viscous effects.

Now Eqs. 9, 10, 11 and 12 can be used to obtain the fluctuations in the energy density, pressure and velocity in $k$-space as:

$$\Delta(k_i, t) = \frac{1}{4\pi} \int d\Omega \int \frac{p^2 dp \epsilon f^{(0)}(p) \tilde{\Psi}(k_i, p, n_i, t)}{\int p^2 dp \epsilon f^{(0)}(p)},$$

$$\delta \tilde{P}(k_i, t) = \frac{1}{4\pi} \int d\Omega \int \frac{p^2 dp (p^2 / \epsilon) f^{(0)}(p) \tilde{\Psi}(k_i, p, n_i, t)}{\int p^2 dp (p^2 / \epsilon) f^{(0)}(p)},$$

$$\theta(k_i, t) = \frac{i k}{4\pi} \int d\Omega (\tilde{k}, \tilde{n}) \int \frac{p^2 dp f^{(0)}(p) \tilde{\Psi}(k_i, p, n_i, t)}{\int p^2 dp (\epsilon + p^2 / 3 \epsilon) f^{(0)}(p)},$$

$$\sigma(k_i, t) = -\frac{1}{4\pi} \int d\Omega (\tilde{k}, \tilde{n})^2 \int \frac{p^2 dp f^{(0)}(p) \tilde{\Psi}(k_i, p, n_i, t)}{\int p^2 dp (\epsilon + p^2 / 3 \epsilon) f^{(0)}(p)}.$$  \hspace{1cm} (15)

To understand the angular scale determined by the multipole number $l$ (as used in the appendix to find the angular correlations) we expand $\tilde{\Psi}$ in terms of Legendre polynomials for an axially symmetric distribution as:

$$\tilde{\Psi}(\tilde{k}, \tilde{n}, p, t) = \sum_{l=0}^{\infty} (-i)^l (2l + 1) \Psi_l(\tilde{k}, \tilde{n}, t) P_l(\tilde{k}, \tilde{n}),$$  \hspace{1cm} (16)

where the factor $(-i)^l (2l + 1)$ is used to simplify the expansion of a plane wave form of $\tilde{\Psi}$. The $l$ is related to angular resolution of the anisotropies, i.e. smaller angular scale will require larger $l$ and vice versa. The temperature fluctuations ($\Delta T$) may be obtained from Eq. 16 by using the relation\footnote{which, as will be seen later allow us to get evolution equation for the Fourier modes.}

$$\frac{\Delta T}{\tilde{T}} = -(\partial \ln f^{(0)} / \partial \rho p) \frac{1}{T} \Psi$$  \hspace{1cm} (17)

For simplicity we will consider the massless limit, $m = 0$ which gives the relation $\epsilon = p$. The energy density is a scalar quantity whereas the velocity and the shear tensor are vector and tensor respectively and these aspects are also bound to reflect in the corresponding fluctuations. Therefore, the orthogonality of $P_l$’s ensure
that the fluctuations in scalar, vector and tensor quantities are dictated by the coefficients $\Psi_0$, $\Psi_1$ and $\Psi_2$ which are obtained by substituting $\Psi$ from Eq. 16 in Eq. 17 and performing the angular integration as,

$$\Delta(k_i, t) = \delta \rho(k_i, t)/\bar{\rho} = \int p^2 dp pf^{(0)}(p)\Psi_0(k_i, p, t)/\bar{\rho},$$

$$\delta P(k_i, t)/\bar{P} = \int p^2 dp pf^{(0)}(p)\Psi_0(k_i, p, t)/\bar{\rho}$$

$$\theta(k_i, t) = 3ik\delta T^0/(4\bar{\rho}) = \frac{3}{4}k \int p^2 dp pf^{(0)}(p)\Psi_1(k_i, p, t)/\bar{\rho}$$

$$\sigma(k_i, t) = \frac{1}{2} \int p^2 dp pf^{(0)}(p)\Psi_2(k_i, p, t)/\bar{\rho}$$

where $\bar{\rho} = \int p^2 dp pf^{(0)}(p)$. The above set of equations can be written in a more compact form through the expansion of the function $F(\vec{k}, \hat{n}, t)$ which is obtained by integrating $\delta f$ over the magnitude of momentum, $\vec{p}$.

$$F(\vec{k}, \hat{n}, t) = \int p^2 dp pf^{(0)}(p)\Psi(\vec{k}, p, n_i, t)/\bar{\rho}$$

therefore, $F$ has the angular dependence of $\Psi$ and consequently $F$ can be expressed as:

$$F(\vec{k}, \hat{n}, t) = \sum_{l=0}^{\infty} (-i)^l(2l + 1)F_l(\vec{k}, t)P_l(\hat{k} \cdot \hat{n}).$$

with

$$F_l(\vec{k}, t) = \int p^2 dp pf^{(0)}(p)\Psi_l(\vec{k}, p, t)/\epsilon_0$$

The fluctuations in terms of $F_l$’s are now given by

$$\Delta(\vec{k}, t) = F_0(\vec{k}, t), \quad \delta P(\vec{k}, t)/\bar{P} = F_0(\vec{k}, t), \quad \theta(\vec{k}, t) = 3/4k F_1(\vec{k}, t), \quad \sigma(\vec{k}, t) = 1/2 F_2(\vec{k}, t).$$

Using the relation $\sigma(\vec{k}, t) = -4\eta k^2 v_j(\vec{k}, t)/3(\bar{\rho} + \bar{P})$, and writing $ik\vec{k}v_j(\vec{k}, t) = \Theta(\vec{k}, t)$ we get an important relation which connects the fluctuation ($F_2$) with the transport coefficient ($\eta$),

$$F_2(\vec{k}, t) = -\frac{8\eta}{3(\bar{\rho} + \bar{P})}\Theta(\vec{k}, t) = -\frac{8\eta}{3\bar{T}_s}\Theta(\vec{k}, t)$$

where the thermodynamic relation, $\bar{h} = \bar{\rho} + \bar{P} = s\bar{T}$, among enthalpy density ($\bar{h}$), entropy density ($s$) and temperature ($\bar{T}$) has been used. The $\eta$ appears as a coefficient of 2nd rank tensor involving gradient in the $i$th direction of the $j$th component of velocity, as a result the $l = 2$ term appears in the expression for $\eta$ in Eq. 23. In a similar way, using Eq. 22 the bulk viscosity $\zeta$ can be related to the fluctuation in pressure ($F_0$) as: $\delta \bar{P} = -ik_0 v^i \zeta$.

### 2.2. Fluctuations in Fourier space and transport coefficients in relaxation time approximation

The temperature fluctuations, $\Delta T(\theta, \phi)$ in CMBR is generally expanded in Laplace series in terms of spherical harmonics, $Y_{lm}(\theta, \phi)$. The maximum value of $l$ is determined by the angular resolution of the detector which can be connected to the wave number ($k$) corresponding to the Fourier transform of the spatial anisotropy. Therefore, in analogy with fluctuations in the CMBR the spatial anisotropy is studied here in Fourier space. But first we briefly discuss it in coordinate space.

The BTE, $p^\mu \partial_\mu f = C[f]$ in absence of external force and in the relaxation time approximation (similar approximation were used e.g. in Refs. 24 and 27) reduces to

$$\frac{\partial \Psi}{\partial t} + \frac{p^i}{\epsilon} \frac{\partial \Psi}{\partial x^i} = -\frac{\Psi(\vec{x}, \vec{p}, t)}{\tau_R}.$$

for $\Psi$. In Eq. 24 $\tau_R$ is the relaxation time. In this work we assume that the system is close to the local equilibrium and the collisions between the particles bring the system back to the equilibrium within a time
scale $\tau_R$. For the present work the relaxation time can be estimated as the inverse of the reaction rate of the quarks and gluons using pQCD cross sections and Hard Thermal Loop Approximations. The solution of Eq. \ref{eq:24} for a given initial (at time $t_0$) distribution, $\Psi_{i\alpha}(\tilde{x}, \tilde{p}, t_0)$ is

\[
\Psi(\tilde{x}, \tilde{p}, t - t_0) = \Psi_{i\alpha}(\tilde{x} - \frac{\tilde{p}}{p_0}(t - t_0)), \tilde{p}) \exp \left[ -\frac{(t - t_0)}{\tau_R} \right]
\]

Knowing $\Psi$ it is straightforward to estimate the fluctuation in energy density from the following expression:

\[
\Delta(\tilde{x}, t - t_0) = \frac{\int p^2 dp d\Omega \epsilon f^{(0)}(p) \Psi(\tilde{x}, \tilde{p}, t - t_0)}{\int p^2 dp d\Omega \epsilon f^{(0)}(p)}.
\]

The solution of Eq. \ref{eq:24} given by Eq. \ref{eq:25} is useful to study the time evolution of spatial anisotropy of the matter.

Now we would like to derive a relation between the fluctuation in energy density and transport coefficients. To facilitate this we write Eq. \ref{eq:24} for massless particles (as the case may be for partonic plasma produced in RHIC-E) in $k$-space:

\[
\frac{\partial \Psi}{\partial t} + i\hat{k} \hat{n} \Psi = -\frac{\Psi(\tilde{k}, \tilde{p}, t)}{\tau}.
\]

With the help of Eq. \ref{eq:26} Eq. \ref{eq:27} can be reduced to an equation describing the time evolution of $F$ as

\[
\frac{\partial F}{\partial t} + i\hat{k} \hat{n} F = -\frac{F(\tilde{k}, t)}{\tau}.
\]

This equation has the following solution,

\[
F(\tilde{k}, \hat{n}, t) = F(\tilde{k}, \hat{n}, t_0) \exp \left[ -\frac{1}{\tau} + i k \mu (t - t_0) \right]
\]

where $\hat{k}, \hat{n} = \mu$. The value of $F(\tilde{k}, \hat{n}, t)$ can be obtained from its value at initial time, $t_0$. Eq. \ref{eq:29} is a general expression for the fluctuations in the sense that all the quantities, e.g. $\Delta, \theta, \sigma$, discussed above at time $t$ can be obtained from this expression if their corresponding initial values are supplied. Expanding $F(\tilde{k}, \hat{n}, t)$ as in Eq. \ref{eq:25} and using the orthogonality relations of $P(\mu)$'s we get,

\[
F_l(\tilde{k}, t) = \frac{1}{2} e^{-\frac{(t-t_0)}{\tau_0}} \sum_{s=0}^{\infty} (-i)^{s-l}(2s+1)F_s(\tilde{k}, t_0) \int_{-1}^{1} d\mu P_l(\mu) P_s(\mu) e^{-ik\mu(t-t_0)}.
\]

For $l = 0, 1, 2$ we have,

\[
\begin{pmatrix}
F_0(k, t) \\
F_1(k, t) \\
F_2(k, t)
\end{pmatrix} = \begin{pmatrix}
I_0 & (-i)3I_1 & (-5)\frac{1}{2}(3I_2 - I_0) \\
I_1 & (-i)3I_2 & (-5)\frac{1}{2}(3I_3 - I_1) \\
\frac{1}{2}(3I_2 - I_0) & (-i)\frac{1}{2}(I_3 - I_1) & (-5)\frac{1}{2}(3I_4 - 6I_2 + I_0)
\end{pmatrix} \begin{pmatrix}
F_0(k, t_0) \\
F_1(k, t_0) \\
F_2(k, t_0)
\end{pmatrix},
\]

where $I_n \equiv I_n(k, t)$ and $\alpha = k(t - t_0)$

\[
I_n(\alpha) = \int_{-1}^{1} d\mu \mu^{n} e^{-i\mu \alpha},
\]

for $n = 0$ $I_0$ is given by

\[
I_0(\alpha) = (-i) \frac{2 \sin \alpha}{\alpha}
\]

The following relations may be used to obtain $I_j(\alpha)$ for $j = 1, 2, ..., n = 0$

\[
I_{n+k}(\alpha) = \frac{1}{(-i)^k} \frac{d^k I_n}{d^2 \alpha}.
\]

Therefore, the energy density fluctuation at time $t$ is given by:

\[
\Delta(\tilde{k}, t) = \frac{1}{2} e^{-\frac{(t-t_0)}{\tau_0}} \sum_{s=0}^{\infty} (-i)^{s}(2s+1)F_s(\tilde{k}, t_0) \int_{-1}^{1} d\mu P_s(\mu) e^{-ik\mu(t-t_0)}.
\]
Taking terms up to $s = 2$ in the expression for $\Delta(\vec{k}, t)$ we get,

$$\Delta(\vec{k}, t) = \frac{1}{2} e^{-\frac{1}{2}(t-t_0)} \sum_{n=0}^{2} (-i)^n (2s + 1) F_n(\vec{k}, t_0) \int_{-1}^{+1} d\mu P_n(\mu) e^{-ik\mu(t-t_0)}. \quad (32)$$

Performing the integration over $\mu$ we get,

$$\Delta(\vec{k}, t) = e^{-\frac{1}{2}(t-t_0)} \left\{ F_0(\vec{k}, t_0) \frac{\sin k(t-t_0)}{k(t-t_0)} \right\} + 3 F_1(\vec{k}, t_0) \left\{ \frac{\cos k(t-t_0)}{k(t-t_0)} - \frac{\sin k(t-t_0)}{(k(t-t_0))^2} \right\}
- 5 F_2(\vec{k}, t_0) \left\{ \frac{\sin k(t-t_0)}{k(t-t_0)} + \frac{3 \cos k(t-t_0)}{(k(t-t_0))^2} - \frac{3 \sin k(t-t_0)}{(k(t-t_0))^3} \right\}. \quad (33)$$

This is the fluctuations in energy density, from which the fluctuations in temperature can be obtained by using the relation: $\delta \rho/\rho = 4 \delta T/T$ for $\rho \sim T^4$. We use Eqs. 22, 23 and 33 to obtain the energy density fluctuation in terms of transport coefficients as:

$$\Delta(\vec{k}, t) = e^{-\frac{1}{2}(t-t_0)/\tau} \left\{ \frac{\sin k(t-t_0)}{k(t-t_0)} \right\} + \frac{4}{3} \sin \frac{k \tau}{T} \left\{ \frac{\cos k(t-t_0)}{k(t-t_0)} - \frac{\sin k(t-t_0)}{(k(t-t_0))^2} \right\}
+ \frac{40}{3} \frac{\eta}{\tau} \sin \left\{ \frac{k \tau}{T} \right\} + \frac{3 \cos k(t-t_0)}{(k(t-t_0))^2} - \frac{3 \sin k(t-t_0)}{(k(t-t_0))^3} \right\}. \quad (34)$$

Eq. 34 provides the connection of the fluctuation in energy density in Fourier space with various transport coefficients e.g. thermal conductivity ($\chi$) and viscosity ($\eta$). It may be easily checked that the above solution satisfies the condition, $\Delta(\vec{k}, t) \rightarrow \Delta(\vec{k}, t_0)$ in the limit $t \rightarrow t_0$. It is interesting to note that the $k \sim 0$ mode (or large wave length mode) which is insensitive to spatial gradient is damped by the exponential time dependence only. $\Delta(\vec{k}, t_0)$ represents the mode of the initial perturbation that takes the whole system (as in $k \rightarrow 0$, length scale of inhomogeneity $\lambda \rightarrow \infty$) slightly away from its equilibrium value. However, the non-zero $k$ modes, in addition to the exponential decay, are damped out also due to spatial gradient which is signalled by the presence of terms involving shear viscosity and thermal conductivity in Eq. 34.

The fluctuation in energy density in position space can be obtained by taking Fourier transformation of Eq. 34 as,

$$\frac{\delta \rho}{\rho}(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \Delta(\vec{k}, t) \exp(i\vec{k} \cdot \vec{x}) \quad (35)$$

If the initial ($t = t_0$) energy density fluctuation, gradient of velocity, viscosity to entropy ratio and temperature of the system in equilibrium are known then Eqs. 34 and 35 can be used to get fluctuations at any time, $t > t_0$. Angular correlation function for these fluctuation has been estimated in the appendix.

### 3. Evolution of fluctuation in a hydrodynamically expanding QGP background

In the previous section we have considered the evolution of the fluctuations in a non-expanding background. However, in a realistic scenario in RHIC-E the system expands due to high internal pressure. Therefore, in this section we include the effects of the expansion on the spatial anisotropy through the solutions of relativistic hydrodynamics. The fluid velocity and all the thermodynamic quantities become function of space time coordinates to be determined by the solution of the hydrodynamical equations.

The space time variation of quantities such as energy density and flow velocity is governed by relativistic hydrodynamics. Therefore, we solve the equation:

$$\partial_\mu \mathcal{T}^{\mu \nu} = 0 \quad (36)$$

with the assumption that net baryon number density ($n_B$) is zero at the central rapidity region, hence we need not consider the equation $\partial_\mu (n_B u^\mu) = 0$. We also assume boost invariance along the longitudinal direction and solve the Eq. 36 numerically with equation of state $\mathcal{P} = \tilde{\rho}/3$ for initial condition taken from optical Glauber at the highest RHIC energy ($\sqrt{s_{NN}} = 200$ GeV) for Au+Au collision. The hydrodynamic solution for flow velocity and temperature ($\bar{T} = [30\tilde{\rho}/(g\pi^2)]^{1/4}$) have been used to study the space time evolution of the fluctuations in an expanding background.
The evolution of $\delta f = f_0(\psi)$ is governed by the equation, $p^\mu \partial_\mu f = (p \cdot u)C[f]$ which reduces to the following for an expanding system in the relaxation time approximation:

$$\frac{\partial}{\partial t} + \frac{\vec{p}}{p^0} \cdot \frac{\partial}{\partial \vec{x}} + \frac{(p^0 u_0 - \vec{p} \cdot \vec{u})}{p^0 \tau_R(x)} \delta f(x, p) = - \left( \frac{\partial}{\partial t} + \frac{\vec{p}}{p^0} \cdot \frac{\partial}{\partial \vec{x}} \right) f_0(x, p) \quad (37)$$

The solution of Eq. $37$ is given by $31$

$$\delta f(x, p) = D(t, t_0) \left[ \delta f_{\text{in}}(p, \vec{x} - \frac{\vec{p}}{p^0}(t - t_0)) + \int_{t_0}^{t} B(\vec{x} - \frac{\vec{p}}{p^0}(t' - t_0), t') D(t_0, t') dt' \right] \quad (38)$$

where

$$D(t_2, t_1) = \exp \left[ - \int_{t_1}^{t_2} dt' A(p, \vec{x} - \frac{\vec{p}}{p^0}(t' - t_0), t') \right] \quad (39)$$

with

$$A(p, \vec{x}, t) = \frac{u_0(x) - \vec{p} \cdot \vec{u}(x)/p_0}{\tau_R(x)} \quad (40)$$

and

$$B(\vec{x}, t) = - \left( \frac{\partial}{\partial t} + \frac{\vec{p}}{p^0} \cdot \frac{\partial}{\partial \vec{x}} \right) f_0(x, p) \quad (41)$$

For

$$f_0(x, p) = f_{e_2} = \frac{1}{e^{\beta(x)(u^\mu p_\mu)} - 1} \quad (42)$$

The expression for $B$ reduces to:

$$B(\vec{x}, t) = -f_{e_2}(1 + f_{e_2}) \frac{p^\mu}{p^0} \partial_\mu \left[ \beta(x) u^\mu p_\mu \right] \quad (43)$$

We took the relaxation time as $\tau_R^{-1}(x) = 1.1 \alpha_s T(x)^23$ (we have taken constant value of $\alpha_s = 0.2$ here), $\beta = 1/T(x)$, $u^\mu(x) = (\gamma, \gamma \vec{v})$ is the four velocity of the fluid and $\gamma(x) = u^0(x) = (1 - v(x)^2)^{-1/2}$. The interaction of the expanding background with the fluctuation is affected through the relaxation time which depends on $T$ and the space-time variation of the temperature and velocity fields are determined by the solution of the relativistic hydrodynamic equations. Eq. $38$ provides the space-time evolution of fluctuation in phase space distribution for an expanding QGP background. This equation may be used to estimate various auto-correlations and fluctuations in thermodynamic quantities which can be measured experimentally. For example, the fluctuation in temperature may be estimated by using Eqs. 36 and 15 which may be connected to fluctuations in the transverse momentum (Ref. 30) measured experimentally.

To simulate initial spatial anisotropy with different geometry, we choose,

$$\delta f(p, \vec{x}, t_0) = A_0 \exp \left[ -r(1 + a_n \cos n\phi) \right] \quad (44)$$

We have taken $n = 2, 3, 4, 5$ and 11 to simulate different initial anisotropy. $A_0$ is set to unity for numerical results discussed below. From the solution in coordinate space one can get Fourier modes of fluctuations using Fourier Transformation which will give evolution of different Fourier modes of fluctuations. We take $a_n = 0.3$ for $n = 2, 3, 4, 5$ and 11.

### 4. Results

In this work we have used BTE to study the anisotropic fluctuations. The description of the evolution of anisotropy induced fluctuations within the ambit of kinetic theory approach helps in getting better microscopic insight on the evolution. Moreover, kinetic theory approach has validity over a wider range of phase space compared to hydrodynamical descriptions. In the two subsections below we present results on the evolution of fluctuations in a static and subsequently for a realistic scenario of expanding backgrounds respectively.
Fig. 1. Evolution of the fluctuation in energy density with $r$ at different $t$ for a non-expanding QGP background.

Fig. 2. Evolution of the spatial anisotropy of the perturbation with initial elliptic geometry at time $\tau_0 = 0.6$ fm/c (upper panel). The lower panel shows the geometry after a time 4 fm/c has elapsed. Hydrodynamic expansion of the QGP background has been taken into account. Here the boundary of the background has an elliptic shape with the dimension of major and minor axes are approximately 6 fm and 4 fm respectively. The colours from red to violet represent highest to lowest values of the perturbations.
4.1. Evolution of fluctuation in energy density for a non-expanding system

We display the spatial variation of the fluctuation in Fig. 1 at different times for a non-expanding background. We substitute Eq. 25 in Eq. 26 with $\Psi_{in}$ as a Gaussian in space at the initial time and evaluate the evolution of the fluctuation. For the sake of illustration we take $T = 400$ MeV and $\tau \sim 1$ fm/c. The results indicate a rapid dissipation and displacement of the peak of the fluctuations with increase in time. The displacement of the peak of the initial fluctuation given by Eq. 44 centered at $r = |\vec{x}| = 0$ is governed by the factor, $\vec{x} - \vec{p}(t - t_0)/p_0$ appearing in the solution for $\Psi$ (Eq. 25) and the dissipation is controlled by the relaxation time, $\tau_R$ involves in the exponential factor in the same equation. The dissipation will slow down in an expanding medium because the relaxation time will increase with decreasing temperature due to expansion.

4.2. Evolution of fluctuation in an expanding QGP background

In this section we would like to do some case study of how a given spatial anisotropy characterized by some geometric shape will evolve with space and time in an expanding QGP medium governed by relativistic hydrodynamics. This will give us some idea on the evolution of elliptical or triangular anisotropic perturbations created in the collisions.

We present the results now for a realistic scenario where the background QGP is expanding hydrodynamically as described in section 3. Taking the value of the temperature dependent relaxation time relevant for QCD plasma the evolution of initial spatial anisotropies introduced through $\Psi_{in}$ (or $\delta f$) have been studied. It is to be noted that due to expansion the temperature decreases and hence the relaxation time increases which slows down the dissipation. Therefore, the dissipation of the perturbation gets slower with
the expansion of the system. The effects of perturbation has better chance of survivability in the direction of lesser extent because the expansion is faster along that direction due to larger pressure gradient. It implies that systems with same energy density the perturbations has larger chances to survive in systems with smaller size. Then it is expected that the presence of perturbations will be dominant in relatively smaller size systems between events of same class. In Fig. 2 the evolution of the initial elliptic spatial anisotropy of the perturbation (upper panel), realized by taking \( n = 2 \) in Eq. 44 is depicted. The initial thermalization time is taken as \( \tau_0 = 0.6 \text{ fm/c} \). The evolution is studied up to \( \tau = 4 \text{ fm/c} \). The red to violet colours used in the figures for distinct visibility, represent correspondingly the highest to lowest values of the perturbations. For \( n = 2 \) the anisotropy has an elliptic shape having stronger gradient along \( x \)-axis resulting in faster expansion along \( x \) compared to \( y \) axis. Therefore, the propagation of the perturbation generates a pattern similar to the one generated in water waves by the impact of a stick on the still water surface. This kind of pattern is clearly observed in the lower panel of Fig. 2 such type of fluctuation may be created by the propagation of jets through the QGP. It may be observed that the solution of BTE given in Eq. 38 is subjected to two different kinds of mechanism - (a) dissipation of the fluctuations and (b) hydrodynamic expansion of the background. The expansion velocity will be larger along \( x \)-axis than along \( y \)-axis due to different pressure gradient imposed by the initial geometry of the fluctuation. This results in the splitting of the fluctuation as observed in the lower panel of Fig. 2. By switching off the dissipation (appearing through the exponential term in \( D(t, t_0) \)) we have noticed that the fluctuation still splits in two parts but the peak of the fluctuation does not reduce significantly. It is also important to note that the two oppositely propagating perturbations are correlated which may have interesting observation effects.
Moreover, if the perturbation is created near the boundary of the system then the wave propagating outward will dissipate less than the one moving inward. The results in Fig. 2 indicate a rapid dissipation of the peak. The peak has been reduced by more than 90% at a time $\tau = 4$ fm/c. The expansion of the QGP background is governed by the equation of state i.e. by the velocity sound in the QGP (the maximum displacement is determined by the sound horizon: i.e. the distance travelled by the sound wave: $\int_{\tau_0}^{\tau} d\tau c_s(\tau) d\tau$.

We have taken the sound velocity, $c_s = 1/\sqrt{3}$, independent of $\tau$ for the expanding QGP background). The displacement of the fluctuation (primarily $\delta f$) is regulated by the factor: $\vec{x} - \vec{p}(t - t_0)/p_0$ appearing in $\Psi$ (Eq. 25). Therefore, the net displacement is determined by the combination of these two factors. The dissipation of the fluctuation is dictated by the relaxation time which is a function of space-time coordinate through the relation: $\tau_R^{-1} \sim T(t, \vec{x})$. Therefore, the amplitude of the displacement becomes a space-time dependent quantity which is evident from the results displayed in Fig. 2.

The spatial anisotropic structure of the system formed in RHIC-E can be understood with the help of Fourier analysis in terms of its various coefficients. Work on the space-time evolution of the angular power spectrum for more realistic initial condition for the hydrodynamical solution derived from Glauber Monte-Carlo techniques is under progress.\cite{38}

In Figs. 3 - 5 the evolutions of the spatial anisotropic perturbations with different initial geometry like triangular, quadrangular and pentagonal for $n = 3, 4$ and 5 respectively have been depicted. We would like to see how these anisotropies dissipate. The perturbations introduce pressure gradient in the system. The magnitude of the perturbation gets reduced by the force arising due to pressure imbalance. It is observed that the spatial anisotropies of such perturbations dissipate fast. The propagation of these anisotropic perturbations are affected primarily by the velocity of sound in the QGP background as well as by the velocity of
the fluctuation appearing in $\Psi$ as $\vec{x} - \vec{p}(t - t_0)/p_0$, and hence, on the thermal mass of the degrees of freedom that constitute the perturbation. The splitting of the peaks are resulted from the expanding background with different magnitude of velocities due to different pressure gradient imposed by the initial geometry of the fluctuation. The propagating waves for the perturbation take shape analogous to water waves created on the calm surface if perturbed initially with similar geometric shape. Theoretical analysis of the angular power spectrum of the anisotropies arising from such perturbations in the evolving stage will shed light on possibility of selecting out the signatures of the early stage of the evolving matter.

In Fig. 6 we display an initial perturbation (introduced at $r = 0$) with smaller angular dimension implemented through a hendecagonal ($n = 11$) geometric shape to check whether such perturbations survive the evolution (upper panel). The fate of the perturbation after space-time evolution is depicted in the lower panel of Fig. 6. We observe that the perturbations of small angular size dissipate substantially. In fact, the perturbation with size corresponding to $n = 5$ and $n = 11$ look similar at a time $4 \text{ fm/c}$ after the initial time. We introduce the initial perturbation at distance $2.5 \text{ fm}$ away from the origin along the positive $x$-axis (upper panel, Fig. 7). It is clear from the results displayed in Fig. 7 (lower panel) that the perturbation moving outward (away from the centre) has suffered less dissipation compared to the one propagating inward and hence has a better chance to carry detectable signature.

An elliptic perturbation is imparted near the boundary (Fig. 8 upper panel), $3 \text{ fm}$ away from the centre along the (positive) $x$-axis. The fate of the perturbation after $2 \text{ fm/c}$ and $4 \text{ fm/c}$ are shown in the middle and lower panels of Fig. 8 respectively. It is interesting to note that the perturbation propagating away from the center dissipates less and the one moving toward the centre of the background QGP decay fast. Therefore, if any perturbation is created near the boundary the possibility of getting it detected is more.
The evolution of fluctuation, $\delta n(k_x, k_y, t)$ obtained by integrating $\delta f$ over $p$ in transverse $k$-space is depicted in Fig. 9 for the initial shape at time 0.6 fm/c realized with $n = 2$ (Eq. 44). We observe that the pattern of the perturbation changes substantially from its initial distribution (upper panel) due to the mixing of various $k$-modes (lower panel) at a later time (4 fm/c). The perturbation is propagating over a hydrodynamically expanding background which makes all the variables like, temperature ($\bar{T}$), flow velocity ($v$), pressure ($\bar{P}$), energy density ($\epsilon$), etc explicit functions of time and space. The interaction of the perturbation ($\delta f$) with the background is incorporated through the relaxation time which is a function of temperature and hence space-time coordinates. Therefore, it is expected that various modes of the perturbations in the Fourier space will get mixed during its propagation over the expanding background as clearly visible in Fig. 9 (lower panel). The peak of the fluctuation has reduced significantly due the exponential factor determined by the relaxation time.

4.3. Correlation in pressure fluctuation and bulk viscosity

The auto-correlation function for fluctuation in pressure arising from perturbations is defined as:

$$C_{\Delta P}(r, t, \Delta \phi) = \int d\phi \delta P(r, \phi, t) \delta P(r, \phi + \Delta \phi, t)$$

(45)

We evaluate $C_{\Delta P}$ at fixed $r (= 3$ fm here) as a function of $t$ and $\Delta \phi$. The variation of $C_{\Delta P}$ with the angular separation $\Delta \phi$ is plotted in Fig. 10 for perturbation with elliptic geometry ($n = 2$) at different times as indicated. For $t = 0.6$ fm/c the correlation function decreases with $\Delta \phi$ attains a dip at $\Delta \phi \sim \pi/2$ and again increases to produce a symmetric behaviour about the dip. At $t = 1$ fm/c the $\Delta \phi$ variation of $C_{\Delta P}$ is similar.
Fig. 8. Same as Fig. 2 but the perturbation is given at a distance of 3 fm away from the origin along x-axis. The middle (lower) panel shows the results after a time 2 fm/c (4 fm/c) has elapsed.

to earlier time with an overall reduction in the magnitude. At a later time, $t = 2.5$ fm/c the $C_{\Delta P}$ evolves to a plateau. This indicates that the power spectrum, $|\delta P(k)|^2$ is a Dirac delta function. It is also interesting to note that the $C_{\Delta P}$ at a given $r$ and $\Delta \phi$ decreases monotonically with time i.e. the correlation becomes weaker in real space as the perturbation reduces and the system approaches toward equilibrium with the progress of time. The evolution of correlation in the pressure fluctuation is crucial for the study of flow harmonics in RHIC-E. The angular differential pressure will give rise to various non-zero flow coefficients like, elliptic, triangular and higher orders. The measured anisotropy can be extrapolated backward in time through theoretical model to characterize the early state of the matter formed in RHIC-E.
Fig. 9. (upper panel). The fluctuations in $k$-space at time $\tau = 0.6$ fm/c (upper panel). The lower panel shows the results after a time 4 fm/c has elapsed. The mixing of $k$-modes is visible in the lower panel.

Fig. 10. The angular auto-correlation of pressure is shown at $r = 3$ fm as a function of $\Delta \phi$. The $\Delta \phi$ variation of $C_{\delta \phi}$ is displayed at different times such as 0.6 fm/c (blue), 1 fm/c (orange), 2.5 fm/c (green), 3.5 fm/c (red) and 4 fm/c (black line).

The bulk viscosity of matter created in RHIC-E is a field of high contemporary interest. We use the current formalism to estimate the bulk viscous coefficient ($\zeta$). The fluctuations in thermodynamic quantities can be used to estimate various transport coefficients. For example, the fluctuations in pressure ($\delta P \ll \bar{P}$) determined by the $\delta f$ (Eq. 25) can be employed to calculate the bulk viscous coefficient (several methods...
Fig. 11. The variation of bulk viscosity to entropy ratio ($\zeta/s$) as a function of temperature.

have been employed in the literature to estimate bulk viscosity of QGP some of these are discussed in Ref.\r of the quarks with thermal mass by using Green-Kubo relation in the domain of linear response. The bulk viscosity ($\zeta$) is related to the correlation of time dependent pressure fluctuation as follows:

$$\zeta = \frac{1}{T} \int_0^\infty dt \langle \delta P(t) \delta P(0) \rangle$$

We estimate the $\zeta$ by using this relation and compare the bulk viscosity to entropy density ($s$) ratio as a function of temperature to the results obtained in Ref.\r in the strong coupling limit with two flavour NJL model (Fig 11). We observe that the behaviour of $\zeta/s$ in the high $T (> 225 \text{ MeV})$ regime is similar to that obtained in Ref.\r. This is reasonable because the relaxation time used in the present work has been estimated for weakly coupled QGP which may be realized at the high $T$ regime. The $\zeta/s$ calculated in Ref.\r rises very fast with lowering of $T$ (for $T < 225 \text{ MeV}$) due to multi-loop contributions, inclusion of such contributions is beyond the scope of the present work. However, it has been verified that the $\zeta/s$ obtained here is similar to the $\zeta/s$ reported in Ref.\r with single loop contribution which may be a good approximation for weakly coupled system.

4.4. **Temperature fluctuation**

The solution of the BTE, $\delta f$ can be used to estimate the fluctuation in useful thermodynamic quantities in QGP. For example, the fluctuation in temperature, $\Delta T$ in different azimuthal bins ($\Delta \phi$) can be calculated as follows. Since the average transverse momentum ($\langle p_T \rangle$) is directly proportional to the temperature of the QGP the fluctuation in temperature in a bin $\Delta \phi$ is given by the relation,

$$\Delta T \sim \int_{\phi_1}^{\phi_2} d\phi \int_0^\infty p_T dp_T \int d^3x \delta f$$

Therefore, the $\Delta \phi$ variation of temperature fluctuation in little bang i.e. for the system formed in RHIC-E can be estimated analogous to the temperature fluctuation in the universe in the recombination era. These issues along with the power spectrum similar to the CMBR will be addressed in Ref.\r.

5. **Summary and Discussion**

The evolution of fluctuations have been studied in Ref.\r using relativistic hydrodynamical model. In contrast we use a more microscopic approach to investigate the evolution of fluctuations within the framework of BTE in a relativistically expanding QGP background. The background of the spatial fluctuations has been assumed as a thermalized expanding QGP. The expansion of the background has been dealt with the (2+1) dimensional relativistic hydrodynamical model. The evolution of initial spatial anisotropic perturbations with different geometry have been studied and analytical results have been obtained. It is found that
the perturbations dissipate during its propagation, however, the creation of such anisotropic perturbations near the boundary of the plasma may lead to detectable effects. Theoretical analysis of these anisotropies will help in understanding the early stage of the matter. We have provided an explicit relation between the fluctuations and transport coefficients. The mixing of various $k$-modes of the perturbations during the course of evolution has been demonstrated. The evolution of correlations of perturbation in pressure has been studied and shown that the correlation between two points in real space reaches a plateau at later time. We have used the calculated correlation in pressure fluctuation to estimate the bulk viscous coefficient. The results presented here may be used to estimate correlations of multiplicity fluctuation at the freeze-out surface which may be measured in experiments to extract transport coefficients like shear viscosity as shown in Ref. [33].

The power spectrum of temperature fluctuations (Eq. 17) can be estimated and compared with the power spectrum of angular momentum reported in Ref. [32]. This will provide insight into the fate of the anisotropies created initially in the system produced in RHIC-E. A comment on the present study may be made here. Although in the present work we have discussed the evolution spatial correlations in the following way,

$$\langle \Delta(\hat{n}_1)\Delta(\hat{n}_2) \rangle = \sum_{l,m} a_{lm}(\hat{x},t) Y_{lm}(\hat{n}), \quad (48)$$

with $a_{lm} = (-i)^l 4\pi \int d^3\hat{n} Y^{*}_{lm}(\hat{n}) \Delta(\hat{x},\hat{n},t)$ and $\langle a_{lm} a^{*}_{l'm'} \rangle = C_l(\hat{x},t) \delta_{ll'} \delta_{mm'}$. Using Eq. 48 we can find the correlations in the following way,

$$\langle \Delta(\hat{x},\hat{n}_1,t) \Delta(\hat{x},\hat{n}_2,t) \rangle = \sum_{l,m,l',m'} \langle a_{lm} a^{*}_{l'm'} \rangle Y_{lm}(\hat{n}_1) Y^{*}_{l'm'}(\hat{n}_2),$$

$$= \sum_{l,m,l',m'} C_l(\hat{x},t) \delta_{ll'} \delta_{mm'} Y_{lm}(\hat{n}_1) Y^{*}_{l'm'}(\hat{n}_2), \quad (49)$$

which leads to

$$\langle \Delta(\hat{x},\hat{n}_1,t) \Delta(\hat{x},\hat{n}_2,t) \rangle = \frac{1}{4\pi} \sum_l (2l + 1) C_l(\hat{x},t) P_l(\hat{n}_1 \cdot \hat{n}_2),$$

since, $P_l(\hat{n}_1 \cdot \hat{n}_2) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{n}_1) Y^{*}_{lm}(\hat{n}_2)$. Eq. 49 defines the correlations of fluctuations observed from two different directions in terms of co-efficient $C_l$s. $C_l$s are the angular power spectrum which contains the information of the anisotropies. Work is under progress to estimate these coefficients by solving the hydrodynamical equations with the initial conditions taken from Glauber Monte-Carlo method. Similarly, one can define these co-efficient corresponding to $k$-space presentation of fluctuations. Time evolution of these co-efficients can be obtained from evolution of energy density fluctuation, $\Delta(\vec{k},\hat{n},t)$ given by Eq. 34. Therefore, we have

$$\langle \Delta(\vec{x},\hat{n}_1,t) \Delta(\vec{x},\hat{n}_2,t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} \langle \Delta(\vec{k},\hat{n}_1,t) \Delta(\vec{k}',\hat{n}_2,t) \rangle$$

$$= \sum_l (2l+1) C_l(\hat{x},t) P_l(\hat{n}_1 \cdot \hat{n}_2), \quad (50)$$

The results presented here may be used to estimate correlations of multiplicity fluctuation at the freeze-out surface which may be measured in experiments to extract transport coefficients like shear viscosity as shown in Ref. [33]. The power spectrum of temperature fluctuations (Eq. 17) can be estimated and compared with the power spectrum of transverse momentum fluctuation of thermal hadrons measured experimentally. A detail analysis of the power spectrum with hydrodynamically expanding background is underway [33] the pattern of which looks similar to the power spectrum of angular momentum reported in Ref. [32]. This will provide insight into the fate of the anisotropies created initially in the system produced in RHIC-E. A comment on the present study may be made here. Although in the present work we have discussed the evolution spatial anisotropy in an expanding QGP background, the formalism discussed is relevant for studying space-time evolution of fluctuations in any relativistically expanding background.

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6. Appendix: Correlations

In this appendix, we evaluate the correlation, $\langle \Delta(\hat{n}_1)\Delta(\hat{n}_2) \rangle$. The fluctuations, $\Delta(\vec{x},\hat{n},t)$ can be written as:

$$\Delta(\vec{x},\hat{n},t) = \sum_{l,m} a_{lm}(\vec{x},t) Y_{lm}(\hat{n}), \quad (48)$$

with $a_{lm} = (-i)^l 4\pi \int d^3\hat{n} Y^{*}_{lm}(\hat{n}) \Delta(\vec{x},\hat{n},t)$ and $\langle a_{lm} a^{*}_{l'm'} \rangle = C_l(\vec{x},t) \delta_{ll'} \delta_{mm'}$. Using Eq. 48 we can find the correlations in the following way,

$$\langle \Delta(\vec{x},\hat{n}_1,t) \Delta(\vec{x},\hat{n}_2,t) \rangle = \sum_{l,m,l',m'} \langle a_{lm} a^{*}_{l'm'} \rangle Y_{lm}(\hat{n}_1) Y^{*}_{l'm'}(\hat{n}_2),$$

$$= \sum_{l,m,l',m'} C_l(\vec{x},t) \delta_{ll'} \delta_{mm'} Y_{lm}(\hat{n}_1) Y^{*}_{l'm'}(\hat{n}_2), \quad (49)$$

which leads to

$$\langle \Delta(\vec{x},\hat{n}_1,t) \Delta(\vec{x},\hat{n}_2,t) \rangle = \frac{1}{4\pi} \sum_l (2l + 1) C_l(\vec{x},t) P_l(\hat{n}_1 \cdot \hat{n}_2),$$

since, $P_l(\hat{n}_1 \cdot \hat{n}_2) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{n}_1) Y^{*}_{lm}(\hat{n}_2)$. Eq. 49 defines the correlations of fluctuations observed from two different directions in terms of co-efficient $C_l$s. $C_l$s are the angular power spectrum which contains the information of the anisotropies. Work is under progress to estimate these coefficients by solving the hydrodynamical equations with the initial conditions taken from Glauber Monte-Carlo method. Similarly, one can define these co-efficients corresponding to $k$-space presentation of fluctuations. Time evolution of these co-efficients can be obtained from evolution of energy density fluctuation, $\Delta(\vec{k},\hat{n},t)$ given by Eq. 34. Therefore, we have

$$\langle \Delta(\vec{x},\hat{n}_1,t) \Delta(\vec{x},\hat{n}_2,t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} \langle \Delta(\vec{k},\hat{n}_1,t) \Delta(\vec{k}',\hat{n}_2,t) \rangle$$

$$= \sum_l (2l+1) C_l(\vec{x},t) P_l(\hat{n}_1 \cdot \hat{n}_2), \quad (50)$$
Now,
\[
\langle \Delta (\vec{k}, \hat{n}_1, t) \Delta (\vec{k}', \hat{n}_2, t) \rangle = \langle \Delta (\vec{x}, \hat{n}_1, t_0) \Delta (\vec{x}, \hat{n}_2, t_0) \rangle L(\vec{k}, t_0; \vec{k}', t_0) \\
+ \langle \Theta (\vec{x}, \hat{n}_1, t_0) \Theta (\vec{x}, \hat{n}_2, t_0) \rangle M(\vec{k}, t_0; \vec{k}', t_0) \\
+ \langle \Delta (\vec{x}, \hat{n}_1, t_0) \Theta (\vec{x}, \hat{n}_2, t_0) \rangle N(\vec{k}, t_0; \vec{k}', t_0),
\]
where,
\[
L(\vec{k}, t, \vec{k}', t_0) = e^{-\frac{2(t-t_0)}{s^2}} \left\{ \frac{\sin k(t-t_0)}{k(t-t_0)} \right\} \left\{ \frac{\sin k'(t-t_0)}{k'(t-t_0)} \right\},
\]
\[
M(\vec{k}, t, \vec{k}', t_0) = e^{-\frac{2(t-t_0)}{s^2}} \left( \frac{40 \, \eta}{3 \, s^2} \right)^2 \left\{ \frac{\sin k(t-t_0)}{k(t-t_0)} \right\} \left\{ \frac{3 \cos k(t-t_0)}{(k(t-t_0))^2} - \frac{3 \sin k(t-t_0)}{(k(t-t_0))^3} \right\} \\
\times \left\{ \frac{\sin k'(t-t_0)}{k'(t-t_0)} + \frac{3 \cos k'(t-t_0)}{(k'(t-t_0))^2} - \frac{3 \sin k'(t-t_0)}{(k'(t-t_0))^3} \right\},
\]
\[
N(\vec{k}, t, \vec{k}', t_0) = e^{-\frac{2(t-t_0)}{s^2}} \left( \frac{40 \, \eta}{3 \, s^2} \right)^2 \left\{ \frac{\sin k(t-t_0)}{k(t-t_0)} \right\} \\
\times \left\{ \frac{\sin k'(t-t_0)}{k'(t-t_0)} + \frac{3 \cos k'(t-t_0)}{(k'(t-t_0))^2} - \frac{3 \sin k'(t-t_0)}{(k'(t-t_0))^3} \right\},
\]
For two functions \( \Delta \) and \( \Theta \), defining the correlation as: \( \langle \Delta (\vec{k}, \hat{n}_1, t) \Theta (\vec{k}', \hat{n}_2, t) \rangle = \langle 2\pi \rangle^3 \delta (\vec{k} - \vec{k}') \delta_{\Delta \Theta} \langle \Delta (\vec{k}, \hat{n}_1, t) \Theta (\vec{k}, \hat{n}_2, t) \rangle \) and \( \Delta (\vec{k}, \hat{n}, t) = \sum_{l,m} a_{lm}(\vec{k}, t) Y_l^m(\hat{n}) \), \( \langle a_{lm} a_{l'm'}^{\ast} \rangle = C_{l \Delta \Delta}(\vec{k}, t) \delta_{ll'} \delta_{mm'} \), we get
\[
\langle \Delta (\vec{k}, \hat{n}_1, t) \Delta (\vec{k}', \hat{n}_2, t) \rangle = (2\pi)^3 \delta (\vec{k} - \vec{k}') \sum_l \frac{2l+1}{4\pi} \langle \hat{n}_1 \cdot \hat{n}_2 \rangle L(l, k, k, t_0) + C_{l \Delta \Delta}(k, t_0) M(k, k, t, t_0).
\]
Using \( \langle \Delta (\vec{k}, \hat{n}_1, t) \Delta (\vec{k}', \hat{n}_2, t) \rangle = (2\pi)^3 \delta (\vec{k} - \vec{k}') \sum_l \frac{2l+1}{4\pi} C_{l \Delta \Delta}(k, t) P_l(\hat{n}_1 \cdot \hat{n}_2) \), in Eq. 54 we get,
\[
C_{l \Delta \Delta}(\vec{k}, t) = C_{l \Delta \Delta}(k, t_0) L(k, k, t, t_0) + C_{l \Theta \Theta}(k, t_0) M(k, k, t, t_0).
\]
Using Eq. 49 and 54 in Eq. 50 we get
\[
C_{l \Delta \Delta}(\vec{x}, t) = \int d^3k \{ C_{l \Delta \Delta}(k, t_0) L(k, k, t, t_0) + C_{l \Theta \Theta}(k, t_0) M(k, k, t, t_0) \}
\]
This provides the correlations at different angular scales at time \( t \) for a given correlations at initial time, \( t_0 (t > t_0) \).

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