CRANK-NICOLSON FINITE ELEMENT APPROXIMATIONS
FOR A LINEAR STOCHASTIC HEAT EQUATION
WITH ADDITIVE SPACE-TIME WHITE NOISE

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ABSTRACT. We formulate an initial- and Dirichlet boundary- value problem for a linear stochastic heat equation, in one space dimension, forced by an additive space-time white noise. First, we approximate the mild solution to the problem by the solution of the regularized second-order linear stochastic parabolic problem with random forcing proposed by Allen, Novosel and Zhang (Stochastics Stochastics Rep., 64, 1998). Then, we construct numerical approximations of the solution to the regularized problem by combining the Crank-Nicolson method in time with a standard Galerkin finite element method in space. We derive strong a priori estimates of the modeling error made in approximating the mild solution to the problem by the solution to the regularized problem, and of the numerical approximation error of the Crank-Nicolson finite element method.

1. INTRODUCTION

1.1. Formulation of the problem. Let $T > 0$, $D = (0, 1)$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Then, we consider the following initial- and Dirichlet boundary- value problem for a linear stochastic heat equation: find a stochastic function $u : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that

$$
u_t = \nu_{xx} + \dot{W} \quad \text{in} \quad (0, T] \times D,$$

$$u(t, \cdot)\big|_{\partial D} = 0 \quad \forall t \in (0, T],$$

$$u(0, x) = 0 \quad \forall x \in D,$$

a.s. in $\Omega$, where $\dot{W}$ denotes a space-time white noise on $[0, T] \times D$ (see, e.g., [19], [9]). The mild solution of the problem above is given by the formula

$$u(t, x) = \int_0^t \int_D G_{t-s}(x, y) dW(s, y),$$

where $G_t(x, y)$ is the space-time Green kernel of the solution to the deterministic parabolic problem: find a deterministic function $v : [0, T] \times \overline{D} \rightarrow \mathbb{R}$ such that

$$v_t = v_{xx} \quad \text{in} \quad (0, T] \times D,$$

$$v(t, \cdot)\big|_{\partial D} = 0 \quad \forall t \in (0, T],$$

$$v(0, x) = v_0(x) \quad \forall x \in D,$$

where $v_0 : \overline{D} \rightarrow \mathbb{R}$ is a deterministic initial condition. In particular, it holds that

$$v(t, x) = \int_D G_t(x, y) v_0(y) dy \quad \forall (t, x) \in (0, T] \times \overline{D}.$$
and
\[
G_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 t} \varepsilon_k(x) \varepsilon_k(y) \quad \forall (t, x) \in (0, T] \times \overline{D},
\]
where \( \lambda_k := k \pi \) for \( k \in \mathbb{N} \), and \( \varepsilon_k(z) := \sqrt{2} \sin(\lambda_k z) \) for \( z \in \overline{D} \) and \( k \in \mathbb{N} \).

The aim of the paper at hand is to analyze the convergence of a Crank-Nicolson finite element method for the approximation of the mild solution \( u \) to the problem (1.1), adapting properly the approach in [21].

1.2. An approximate problem. For \( N_*, J_* \in \mathbb{N} \), we define the mesh-lengths \( \Delta t := \frac{T}{N_*} \) and \( \Delta x := \frac{x}{N_*} \); the nodes \( t_n := n \Delta t \) for \( n = 0, \ldots, N_* \) and \( x_j := j \Delta x \) for \( j = 0, \ldots, J_* \), and the subintervals \( T_n := (t_{n-1}, t_n) \) and \( D_j := (x_{j-1}, x_j) \) for \( j = 1, \ldots, J_* \) and \( n = 1, \ldots, N_* \). Also, for a nonempty set \( A \), we will denote by \( \chi_A \) the indicator function of \( A \).

Then, we consider the approximate stochastic parabolic problem proposed in [1], which is formulated as follows: find stochastic function \( \hat{u} : [0, T] \times \overline{D} \rightarrow \mathbb{R} \) such that
\[
\hat{u}_t = \hat{u}_{xx} + \mathcal{W} \quad \text{in} \ (0, T] \times D,
\]
\[
\hat{u}(t, \cdot)_{|_{\partial D}} = 0 \quad \forall t \in (0, T],
\]
\[
\hat{u}(0, x) = 0 \quad \forall x \in D,
\]
a.e. in \( \Omega \), where \( \mathcal{W} \in L^2((0, T) \times D) \) is a piecewise constant stochastic function given by
\[
\mathcal{W} := \frac{1}{\Delta t \Delta x} \sum_{n=1}^{N_*} \sum_{j=1}^{J_*} \mathcal{R}_n^j \chi_{T_n \times D_j},
\]
where
\[
\mathcal{R}_n^j := \mathcal{W}(T_n \times D_j), \quad n = 1, \ldots, N_*, \quad j = 1, \ldots, J_*.
\]
The standard theory of parabolic problems (see, e.g., [13]), yields
\[
\hat{u}(t, x) = \int_0^t \int_D G_{t-s}(x, y) \mathcal{W}(s, y) \, ds \, dy \quad \forall (t, x) \in [0, T] \times \overline{D},
\]
from which we conclude that: the stochastic function \( \hat{u} \) has a finite noise structure depending on \( N_*, J_* \) random variables (see Remark 1.1), and the space-time regularity of \( \hat{u} \) is higher than that of the mild solution \( u \) to (1.1).

Remark 1.1. If \( S \subset [0, T] \times \overline{D} \) then \( \mathcal{W}(S) \sim N(0, |S|) \). Thus, for \( j = 1, \ldots, J_* \) and \( n = 1, \ldots, N_* \), we have \( \mathcal{W}(T_n \times D_j) \sim N(0, \Delta t \Delta x) \). Also, if \( S_1, S_2 \subset [0, T] \times \overline{D} \) and \( S_1 \cap S_2 = \emptyset \), then \( \mathcal{W}(S_1) \) and \( \mathcal{W}(S_2) \) are independent random variables. Thus, the random variables \( (\mathcal{R}_n^j)_{j=1}^{J_*} \) are independent. For further details we refer the reader to [13] and [9].

1.3. The numerical method. Let \( M \in \mathbb{N} \), \( \Delta r := \frac{T}{M} \), \( \tau_m := m \Delta r \) for \( m = 0, \ldots, M \) and \( \Delta_m := (\tau_{m-1}, \tau_m) \) for \( m = 1, \ldots, M \). Also, let \( \mathcal{W}_h \subset H^1_{0 \epsilon}(D) \) be a finite element space consisting of functions which are piecewise polynomials of degree at most \( r \) over a partition of \( D \) in intervals with maximum mesh-length \( h \). Then, we construct computable fully-discrete approximations of \( \hat{u} \) using the Crank-Nicolson method for time-stepping and the standard Galerkin finite element method for space discretization. The proposed method is as follows: first we set
\[
U_h^0 := 0,
\]
and then, for \( m = 1, \ldots, M \), we are seeking \( U_h^m \in S_h^m \) such that
\[
(U_h^m - U_h^{m-1}, \chi)_{0,D} + \Delta_m (\partial(U_h^m + U_h^{m-1}), \partial \chi)_{0,D} = \int_{\Delta_m} (\mathcal{W}, \chi)_{0,D} \, ds \quad \forall \chi \in S_h^m,
\]
where \((\cdot, \cdot)_{0,D}\) is the usual \( L^2(D) \) inner product.
1.4. References and main results of the paper. The Crank-Nicolson method for stochastic evolution equations has been analyzed in [7] and [8] under the assumption that the additive space-time noise is smooth in space. In [20], the solution of a semilinear stochastic heat equation with multiplicative noise is approximated by a numerical method that combines a θ-time discretization method with a finite element method in space over a uniform mesh. Also, for the proposed family of methods an error analysis has been developed, the outcome of which depends on the value of θ. In particular, for the Backward Euler finite element method (θ = 1), the error estimate obtained is of the form $O(\Delta t^{1/2} + h^{1/2})$, which is in agreement with that in [18] for the stochastic heat equation with multiplicative noise. However, for the Crank-Nicolson finite element method (θ = 1/2), the corresponding error estimate is of the form $O(\Delta t^{1/2} + h^{1/2})$, which means that convergence follows when $\Delta t = o(h^{1/3})$. Taking into account that both methods must have the same order of convergence since the regularity of the solution is low, one is wondering if the difference between the above error estimates is caused by the fact that the Crank-Nicolson method is not strongly stable while the Backward Euler method is. Here, adapting properly the analysis in [21], we show that, in the special case of an additive space time white noise, is that obtained in [18] for the Backward Euler finite element method and thus no mesh conditions are required to ensure its convergence.

The first step in our analysis is to show that the solution $\hat{u}$ to the regularized problem (1.6) converges to the mild solution $u$ to the problem (1.4) when both $\Delta x$ and $\Delta t$ tend freely to zero. This is established by proving in Theorem 3.1 the following $L^{\infty}(L^2_{p}(L^2_{0}))$ modeling error bound:

$$\max_{t \in [0,T]} \left( \mathbb{E} \left[ \| u(t, \cdot) - \hat{u}(t, \cdot) \|_{L^2(D)}^2 \right] \right)^{1/2} \leq C \left( \epsilon^{-1/2} \Delta x^{-1/2} - \epsilon + \Delta t^{1/2} \right)$$

for $\epsilon \in (0, \frac{1}{2}]$, which measures the effect of breaking the infinite noise dimension of $u$ to the finite one of $\hat{u}$.

The second step is the estimation of the discretization error of the Crank-Nicolson finite element method for the approximation of $\hat{u}$ that is formulated in Section 1.3. We achieve the derivation of a discrete in time $L^{\infty}(L^2_{p}(L^2_{0}))$ error estimate of the following form:

$$\max_{0 \leq m \leq h} \left( \mathbb{E} \left[ \| U_h^m - \hat{u}(t_m, \cdot) \|_{L^2(D)}^2 \right] \right)^{1/2} \leq C \left( \epsilon_1^{-1/2} \Delta t^{1/2} - \epsilon_1 + \epsilon_2^{-1/2} h^{1/2} - \epsilon_2 \right)$$

for $\epsilon_1 \in (0, \frac{1}{2}]$ and $\epsilon_2 \in (0, \frac{1}{2}]$, where the constant $C$ is independent of $\Delta t$ and $\Delta x$. To get the latter estimate, we use a discrete Duhamel principle (cf. [18], [2]) that is based on the representation of the numerical approximations via the outcome of a modified Crank-Nicolson method for the deterministic problem. The analysis is moving along the lines of the convergence analysis developed in [21] for a Crank-Nicolson finite element method proposed to approximate the mild solution of a fourth order, linear parabolic problem with additive space time white noise.

Let us give an epigrammatic description of the paper. In Section 2 we introduce notation and recall several known results. In Section 3 we estimate the modeling error. In Section 4 we introduce and analyze the convergence of modified Crank-Nicolson time-discrete and fully-discrete approximations of $v$. Finally, Section 5 contains the error analysis of the Crank-Nicolson finite element approximations of $\hat{u}$ formulated in Section 1.3.
isometry property for stochastic integrals reads
\[ \|g\|_{i,D} \leq C_{RF} \|g\|_{1,D} \quad \forall g \in H^1_0(D). \]

The sequence of pairs \( \{ (\lambda_i^2, \varepsilon_i) \}_{i=1}^\infty \) is a solution to the eigenvalue/eigenfunction problem: find nonzero \( z \in H^2(D) \cap H^1_0(D) \) and \( \lambda \in \mathbb{R} \) such that \(-\lambda^2 = \lambda z \) in \( D \). Since \( (\varepsilon_i)_{i=1}^\infty \) forms a complete orthonormal system in \( (L^2(D), (\cdot, \cdot)_{i,D}) \), for \( s \in \mathbb{R} \), we define the following subspace of \( L^2(D) \)
\[ \mathcal{V}^s(D) := \left\{ g \in L^2(D) : \sum_{i=1}^\infty \lambda_i^{2s} (g, \varepsilon_i)_{i,D}^2 < \infty \right\} \]
provided with the natural norm \( \|g\|_{s,D} := \left( \sum_{i=1}^\infty \lambda_i^{2s} (g, \varepsilon_i)_{i,D}^2 \right)^{1/2} \) for \( g \in \mathcal{V}^s(D) \). For \( s \geq 0 \), the space \( (\mathcal{V}^s(D), \| \cdot \|_{s,D}) \) is a complete subspace of \( L^2(D) \) and we define \( \mathcal{H}^{-s}(D) \) as the completion of \( (\mathcal{V}^s(D), \| \cdot \|_{s,D}) \) or, equivalently, as the dual of \( \mathcal{H}^s(D) \).

Let \( m \in \mathbb{N}_0 \). It is well-known (see, e.g., [10]) that
\[ \mathcal{H}^m(D) = \left\{ g \in H^m(D) : \partial^{2i} g \big|_{\partial D} = 0 \quad \text{if} \quad 0 \leq 2i < m \right\} \]
and there exist constants \( C_{m,a} \) and \( C_{m,b} \) such that
\[ C_{m,a} \|g\|_{m,D} \leq \|g\|_{\mathcal{H}^m} \leq C_{m,b} \|g\|_{m,D} \quad \forall g \in \mathcal{H}^m(D). \]

Also, we define on \( L^2(D) \) the negative norm \( \| \cdot \|_{-m,D} \) by
\[ \|g\|_{-m,D} := \sup \left\{ \frac{(g, \varphi)_{m,D}}{\|\varphi\|_{m,D}} : \varphi \in \mathcal{H}^m(D) \quad \text{and} \quad \varphi \neq 0 \right\} \quad \forall g \in L^2(D), \]
for which, using [23,33], it is easy to conclude that there exists a constant \( C_{-m} > 0 \) such that
\[ \|g\|_{-m,D} \leq C_{-m} \|g\|_{\mathcal{H}^m} \quad \forall g \in L^2(D). \]

Let \( L_2 = (L^2(D), (\cdot, \cdot)_{i,D}) \) and \( L^2(D) \) be the space of linear, bounded operators from \( L_2 \) to \( L_2 \). An operator \( \Gamma \in L^2(D) \) is Hilbert-Schmidt, when \( \|\Gamma\|_{HS} := \left( \sum_{i=1}^\infty \|\Gamma \varepsilon_i\|_{i,D}^2 \right)^{1/2} \) is finite, where \( \|\Gamma\|_{HS} \) is the so called Hilbert-Schmidt norm of \( \Gamma \). We note that the quantity \( \|\Gamma\|_{HS} \) does not change when we replace \( (\varepsilon_i)_{i=1}^\infty \) by another complete orthonormal system of \( L_2 \). It is well known (see, e.g., [6], [13]) that an operator \( \Gamma \in L^2(D) \) is Hilbert-Schmidt iff there exists a measurable function \( \gamma : D \times D \to \mathbb{R} \) such that \( \Gamma(v) = \int_D \gamma(s,y) v(y) dy \) for \( v \in L^2(D) \), and then, it holds that
\[ \|\Gamma\|_{HS} = \left( \int_D \int_D \gamma^2(x,y) dxdy \right)^{1/2}. \]

Let \( L_{HS}(L_2) \) be the set of Hilbert Schmidt operators of \( L(L_2) \) and \( \Phi \in L^2([0,T]; L_{HS}(L_2)) \). Also, for a random variable \( X \), let \( \mathbb{E}[X] \) be its expected value, i.e., \( \mathbb{E}[X] := \int_\Omega X dP \). Then, the Itô isometry property for stochastic integrals reads
\[ \mathbb{E} \left[ \left\| \int_0^T \Phi(t) dW \right\|_{i,D}^2 \right] = \int_0^T \|\Phi(t)\|_{HS}^2 dt. \]

We recall that (see Appendix A in [11]): if \( \varepsilon_s > 0 \), then
\[ \sum_{i=1}^\infty \frac{1}{\lambda_i^s + \varepsilon_s} \leq \left( \frac{1 + 2e_s}{e_s \pi} \right)^{\nu-1} \forall \nu \in (0,2], \]
and if \( \mathcal{H}, (\cdot, \cdot)_\mathcal{H} \) is a real inner product space, then
\[ (g - v, g)_\mathcal{H} \geq \frac{1}{2} \left[ (g, g)_\mathcal{H} - (v, v)_\mathcal{H} \right] \forall g, v \in \mathcal{H}. \]

Also, for any \( L \in \mathbb{N} \) and functions \( (v^\ell)_{\ell=0}^L \subset L^2(D) \) we set \( v^{\ell + 1/2} := \frac{1}{\sqrt{2}} (v^\ell + v^{\ell-1}) \) for \( \ell = 1, \ldots, L. \)

Finally, for \( \alpha \in [0,1] \) and for \( n = 0, \ldots, M - 1 \), we set \( \tau_{n+\alpha} := \tau_n + \alpha \Delta \tau. \)
2.1. A projection operator. Let $\mathcal{O} := (0, T) \times D$, $\mathfrak{Q}_{N_*} := \text{span}(\mathcal{X}_{0})_{n=1}^{N_*}$, $\mathfrak{Q}_{J_*} := \text{span}(\mathcal{X}_{0})_{j=1}^{J_*}$ and $\Pi : L^2(\mathcal{O}) \to \mathfrak{Q}_{N_*} \otimes \mathfrak{Q}_{J_*}$ be defined by

$$\Pi g(s, y) := \frac{1}{\sqrt{\Delta t}} \sum_{n=1}^{N_*} \sum_{j=1}^{J_*} \left( \int_{T_n} \int_{D_j} g(t, x) dx dt \right) \mathcal{X}_{0,n}(s) \mathcal{X}_{0,j}(y) \quad \forall (s, y) \in \mathcal{O},$$

which is the usual $L^2(\mathcal{O})-$projection onto the finite dimensional space $\mathfrak{Q}_{N_*} \otimes \mathfrak{Q}_{J_*}$. In the lemma below, we connect the projection operator $\Pi$ to the stochastic load $W$ (cf. Lemma 2.1 in [12, 11]).

**Lemma 2.1.** For $g \in L^2(\mathcal{O})$, it holds that

$$\int_{0}^{T} \int_{D} \Pi g(s, y) \, dW(s, y) = \int_{0}^{T} \int_{D} W(t, x) g(t, x) \, dx dt. \quad (2.10)$$

**Proof.** Using the properties of the stochastic integral (see, e.g., [19]) along with (2.9), (17) and (18), we obtain

$$\int_{0}^{T} \int_{D} \Pi g(s, y) \, dW(s, y) = \frac{1}{\sqrt{\Delta t}} \sum_{n=1}^{N_*} \sum_{j=1}^{J_*} \left( \int_{T_n} \int_{D_j} g(t, x) dx dt \right) W(T_n \times D_j) = \int_{0}^{T} \int_{D} W(t, x) g(t, x) \, dx dt. \quad \square$$

Since it holds that

$$\int_{0}^{T} \int_{D} \Pi g(s, y) \varphi dyds = \int_{0}^{T} \int_{D} g(s, y) \varphi(s, y) dyds \quad \forall \varphi \in \mathfrak{Q}_{N_*} \otimes \mathfrak{Q}_{J_*}, \quad \forall g \in L^2(\mathcal{O}),$$

we, easily, conclude that

$$\int_{0}^{T} \int_{D} (\Pi g(s, y))^2 dyds \leq \int_{0}^{T} \int_{D} (g(s, y))^2 dyds \quad \forall g \in L^2(\mathcal{O}). \quad (2.11)$$

2.2. Linear elliptic and parabolic operators. For given $f \in L^2(D)$, let $T_{f} \in \hat{H}^{2}(D)$ be the solution to the problem: find $v_{E} \in \hat{H}^{2}(D)$ such that

$$v_{E}'' = f \quad \text{in} \quad D, \quad (2.12)$$

i.e. $T_{f} := v_{E}$. It is well-known that

$$\langle v_1, T_{E}v_2 \rangle_{0,D} = -(\partial(T_{E}v_1), \partial(T_{E}v_2))_{0,D} = (T_{E}v_1, v_2)_{0,D} \quad \forall v_1, v_2 \in L^2(D), \quad (2.13)$$

and, for $m \in \mathbb{N}_0$, there exists a constant $C_E^m > 0$ such that

$$\|T_{E}f\|_{m,D} \leq C_E^m \|f\|_{m-2,D}, \quad \forall f \in H^{\max\{0,m-2\}}(D). \quad (2.14)$$

Let $(S(t)v_0)_{t \in [0,T]}$ be the standard semigroup notation for the solution $v$ of (1.3). Then (see, e.g., [16]) for $t \in [0, T]$, $\beta \geq 0$, $p \geq 0$ and $q \in (0, p + 2\ell]$, there exists a constant $C_{p, \beta, \ell} > 0$ such that

$$\|\partial_{t}^{\epsilon} S(t)v_0\|_{H^p} \leq C_{p, \beta, \ell} t^{-\frac{\epsilon}{p-\epsilon}} \|v_0\|_{\bar{H}^q} \quad \forall t > 0, \quad \forall v_0 \in \hat{H}^{q}(D), \quad (2.15)$$

and a constant $C_{\beta} > 0$ such that

$$\int_{0}^{t_b} (s-t_a)^{\beta} \|\partial_{t}^{\epsilon} S(s)v_0\|_{H^p}^2 ds \leq C_{\beta} \|v_0\|_{\bar{H}^{p+2\ell-\beta-1}}^2 \quad (2.16)$$

for all $v_0 \in \hat{H}^{p+2\ell-\beta-1}(D)$ and $t_a, t_b \in [0, T]$ with $t_a < t_b$. 


2.3. Discrete operators. Let \( r \in \mathbb{N} \) and \( Z_h^r \subset H^1_0(D) \) be a finite element space consisting of functions which are piecewise polynomials of degree at most \( r \) over a partition of \( D \) in intervals with maximum mesh-length \( h \). It is well-known (cf., e.g., [4], [3]) that there exists a constant \( C_{FE} > 0 \) such that

\[
\inf_{\chi \in \mathcal{S}_h^r} \| g - \chi \|_{1,D} \leq C_{FE} h \| g \|_{2,D} \quad \forall g \in H^2(D) \cap H^1_0(D).
\]  

Then, we define the discrete Laplacian operator \( \Delta_h : Z_h^r \to Z_h^r \) by \( (\Delta_h \varphi, \chi)_{0,D} = (\partial \varphi, \partial \chi)_{0,D} \), for \( \varphi, \chi \in Z_h^r \), the \( L^2(D) \)–projection operator \( P_h : L^2(D) \to Z_h^r \) by \( (P_h f, \chi)_{0,D} = (f, \chi)_{0,D} \) for \( \chi \in Z_h^r \) and \( f \in L^2(D) \), and the standard Galerkin finite element approximation \( v_{E,h} \in Z_h^r \) of the solution \( v_E \) to (2.12) is specified by requiring

\[
- \Delta_h v_{E,h} = P_h f.
\]

Let \( T_{E,h} : L^2(D) \to Z_h^r \) be the solution operator of the finite element method (2.18), i.e. \( T_{E,h} f := v_{E,h} = -\Delta_h^{-1} P_h f \) for \( f \in L^2(D) \). Then, we can easily conclude that

\[
(f, T_{E,h} g)_{0,D} = -\langle \partial(T_{E,h} f), \partial(T_{E,h} g) \rangle_{0,D} = (g, T_{E,h} f)_{0,D} \quad \forall f, g \in L^2(D),
\]

which, along with (2.17), yields

\[
\| T_{E,h} f \|_{1,D} \leq C \| f \|_{-1,D} \quad \forall f \in L^2(D).
\]

Due to the approximation property (2.17), the theory of the standard Galerkin finite element method for second order elliptic problems (cf., e.g., [1], [3]), yields that

\[
\| T_E f - T_{E,h} f \|_{0,D} \leq C h^2 \| T_E f \|_{2,D}
\]

(2.21) yields

\[
\| T_E f - T_{E,h} f \|_{0,D} \leq C h^2 \| f \|_{0,D} \quad \forall f \in L^2(D).
\]

3. Estimating the Modeling Error

Here, we derive an \( L^\infty_t(L^2_x(L^2_y)) \)–estimate of the modeling error in terms of \( \Delta t \) and \( \Delta x \) (cf., [1], [2], [10], [12]).

Theorem 3.1. Let \( u \) be the mild solution to (1.1), \( \tilde{u} \) be the solution to (1.6) and \( Z(t) := \left( \mathbb{E} \left[ \| u(t, \cdot) - \tilde{u}(t, \cdot) \|_{0,D}^2 \right] \right)^{1/2} \) for \( t \in [0,T] \). Then, there exist a constant \( C_{MR} > 0 \), independent of \( \Delta t \) and \( \Delta x \), such that

\[
\max_{t \in [0,T]} Z(t) \leq C_{MR} \left( \Delta t^\frac{1}{2} + \varepsilon^{-\frac{1}{2}} \Delta x^\frac{1}{2} - \varepsilon \right) \quad \forall \varepsilon \in (0,\frac{1}{2}].
\]

Proof. The proof is moving along the lines of the proof of Theorem 3.1 in [12]. To simplify the notation we set \( S_{n,j} := T_n \times D_j \) for \( n = 1, \ldots, N_x \) and \( j = 1, \ldots, J_x \). Also, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta t \) and \( \Delta x \), and may changes value from the one line to the other.

Let \( t \in (0,T]. \) Using (1.2), (1.3), Lemma 2.1 and Itô isometry, we conclude that

\[
Z(t) = \left( \int_0^T \left( \int_D \int_D |\Psi_{t,x}(s,y)|^2 \, dx \, dy \right) \, ds \right)^{1/2},
\]

where \( \Psi_{t,x}(s,y) := g_{t,x}(s,y) - \Pi g_{t,x}(s,y) \) and \( g_{t,x}(s,y) := \chi(t) \chi_x(s) \chi_y(x,y) \). Now, we introduce the following splitting

\[
Z(t) \leq Z_1(t) + Z_2(t)
\]

where \( Z_\ell(t) := \left( \int_0^T \left( \int_D \int_D |\Psi_{t,x}^{\ell}(s,y)|^2 \, dx \, dy \right) \, ds \right)^{1/2} \) for \( \ell = 1, 2, \) and

\[
\Psi_{t,x}^1(s,y) := \frac{1}{\Delta \Delta x} \int_{s_{n,j}} \chi(t) \chi_x(s,x,y) \left[ G_{t-s}(x,y) - G_{t-s}(x,y') \right] \, dy \, ds',
\]

\[
\Psi_{t,x}^2(s,y) := \frac{1}{\Delta \Delta x} \int_{s_{n,j}} \left[ \chi(t) \chi_x(s,x,y') - \chi(t) \chi_x(s',x,y') \right] \, dy' \, ds'
\]
for \((s, y) \in S_{n, j}\) and for \(n = 1, \ldots, N_*\) and \(j = 1, \ldots, J_*\).

**Estimation of \(Z_1(t)\):** Using \((1.5)\) and the \(L^2(D)\)– orthogonality of \((\varepsilon_k)_{k=1}^\infty\) we have

\[
Z_1^2(t) = \frac{1}{\Delta x^2} \sum_{n=1}^{N_*} \sum_{j=1}^{J_*} \int_D \left( \int_{D_j} \mathcal{X}_{(0,t)}(s) \left[ G_{t-s}(x, y) - G_{t-s}(x, y') \right] dy' \right)^2 dy ds dx
\]

\[
= \frac{1}{\Delta x^2} \sum_{n=1}^{N_*} \sum_{j=1}^{J_*} \int_{S_{n,j}} \mathcal{X}_{(0,t)}(s) \left( \sum_{k=1}^\infty e^{-2\gamma^2 t} \left( \int_{D_j} (\varepsilon_k(y) - \varepsilon_k(y')) dy' \right)^2 \right) dy ds
\]

\[
= \frac{1}{\Delta x^2} \sum_{k=1}^\infty \left( \int_0^t e^{-2\gamma^2 t} ds \right) \left( \int_{D_j} \left( \int_{D_j} (\varepsilon_k(y) - \varepsilon_k(y')) dy' \right)^2 dy \right)
\]

which, along with the Cauchy-Schwarz inequality, yields

\[
Z_1^2(t) \leq \sum_{k=1}^\infty \left( \int_0^t e^{-2\gamma^2 t} ds \right) \left( \frac{1}{\Delta x^2} \sum_{j=1}^{J_*} \int_{D_j \times D_j} (\varepsilon_k(y) - \varepsilon_k(y'))^2 dy dy' \right).
\]

Let \(k \in \mathbb{N}\) and \(j \in \{1, \ldots, J_*\}\). Using the mean value theorem we have

\[
\sup_{y, y' \in D_j} |\varepsilon_k(y) - \varepsilon_k(y')|^2 \leq 2 \min \left\{ 1, \lambda_k^2 \Delta x^2 \right\}
\]

\[
\leq 2 \min \left\{ 1, \lambda_k^2 \Delta x^2 \right\} \gamma
\]

\[
\leq 2 \lambda_k^{2\gamma} \Delta x^{2\gamma} \quad \forall \gamma \in [0, 1],
\]

and

\[
\int_0^t e^{-2\gamma^2 t} dt = \frac{1}{2\lambda_k^2} \left( 1 - e^{-2\lambda_k^2 t} \right)
\]

\[
\leq \frac{1}{2\lambda_k^2}.
\]

Let \(\gamma \in [0, 1/2]\) and \(\epsilon = \frac{1}{2} - \gamma \in (0, \frac{1}{2})\). We combine \((3.2), (3.3), (3.4)\) and \((2.7)\) to obtain

\[
Z_1(t) \leq \frac{\Delta x^2}{\sqrt{(1-\gamma)}} \left( \sum_{k=1}^\infty \frac{1}{k^{1+2\gamma}} \right)^{1/2}
\]

\[
\leq C \Delta x^{1/2 - \epsilon} \epsilon^{-\frac{1}{2}}.
\]

**Estimation of \(Z_2(t)\):** Using again \((1.5)\) and the \(L^2(D)\)– orthogonality of \((\varepsilon_k)_{k=1}^\infty\) we have

\[
Z_2(t) = \left( \sum_{k=1}^\infty \Upsilon_1^k \Upsilon_2^k \right)^{1/2}
\]

where

\[
\Upsilon_1^k := \frac{1}{\Delta x} \sum_{j=1}^{J_*} \left( \int_{D_j} \varepsilon_k(y') dy' \right)^2
\]

\[
\Upsilon_2^k := \frac{1}{\Delta x^2} \sum_{n=1}^N \int_{S_n} \left( \int_{S_n} \mathcal{X}_{(0,t)}(s) e^{-\lambda_k^2 (t-s)} \mathcal{X}_{(0,t)}(s') e^{-\lambda_k^2 (t-s')} ds' \right)^2 ds.
\]
Observing that
\[
\gamma_k^1 \leq \sum_{j=1}^{J^*} \int_{D_j} \varepsilon_k^2(y') \, dy' \\
\leq \int_D \varepsilon_k^2(y') \, dy' \\
\leq 1, \quad \forall k \in \mathbb{N},
\]
and letting \(N(t) \in \{1, \ldots, N_*\}\) such that \(t \in (t_{N(t)-1}, t_{N(t)}]\) we conclude that
\[
(3.7) \quad \mathcal{Z}_2(t) \leq \left( \sum_{k=1}^{N(t)} \sum_{n=1}^{N_*} \Psi_k^1 \right)^{1/2},
\]
where
\[
\Psi_k^1 := \frac{1}{\Delta^2} \int_{T_n} \left( \int_{T_n} \left( \int_s^{s'} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt' \right) \, ds' \right)^2 \, ds.
\]

Let \(k \in \mathbb{N}\) and \(n \in \{1, \ldots, N(t) - 1\}\). Then, we have
\[
\Psi_k^1 = \frac{1}{\Delta^2} \int_{T_n} \left( \int_{T_n} \left( \int_s^{\max\{s', s\}} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt' \right) \, ds' \right)^2 \, ds \\
\leq \frac{1}{\Delta^2} \int_{T_n} \left( \int_{T_n} \left( \int_{t_{n-1}}^{s'} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt' \right) \, ds' \right)^2 \, ds \\
\leq \frac{2}{\Delta^2} \int_{T_n} \left( \int_{T_n} \left( \int_{t_{n-1}}^{s} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt \right) \, ds' \right)^2 \, ds \\
+ \frac{2}{\Delta^2} \int_{T_n} \left( \int_{T_n} \left( \int_{t_{n-1}}^{s} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt \right) \, ds' \right)^2 \, ds,
\]
from which, using the Cauchy-Schwarz inequality, we obtain
\[
\Psi_k^1 \leq 4 \int_{T_n} \left( \int_{t_{n-1}}^{s} \lambda_k^2 e^{-\lambda_k^2 (t-s)} \, dt \right)^2 \, ds \\
\leq 4 \int_{T_n} \left( e^{-\lambda_k^2 (t-s)} - e^{-\lambda_k^2 (t-t_{n-1})} \right)^2 \, ds \\
\leq 4 \int_{T_n} e^{-2\lambda_k^2 (t-s)} \left( 1 - e^{-\lambda_k^2 (s-t_{n-1})} \right)^2 \, ds \\
\leq 4 \left( 1 - e^{-\lambda_k^2 \Delta t} \right)^2 \int_{T_n} e^{-2\lambda_k^2 (t-s)} \, ds \\
\leq 2 \left( 1 - e^{-\lambda_k^2 \Delta t} \right)^2 \frac{e^{-\lambda_k^2 (t-t_{n-1})} - e^{-\lambda_k^2 (t-t_{n-1})}}{\lambda_k^2}.
\]
Thus, by summing with respect to \(n\), we obtain
\[
(3.8) \quad \sum_{n=1}^{N(t)-1} \Psi_k^1 \leq 2 \left( 1 - e^{-\lambda_k^2 \Delta t} \right)^2 \frac{1}{\lambda_k^2}.
\]
Also, we have

\[
\Psi_k^{N(t)} = \frac{1}{\Delta t^2} \int_{t_{N(t)-1}}^t \left( \int_{t_{N(t)-1}}^{t'} \lambda_k^2 e^{-\lambda_k^2 (t-\tau)} d\tau \right) ds' + \int_t^{t_{N(t)}} e^{-\lambda_k^2 (t-s)} ds' \right)^2 ds
\]

\[
+ \frac{1}{\Delta t^2} \int_t^{t_{N(t)}} \left( \int_{t_{N(t)-1}}^{t'} e^{-\lambda_k^2 (t-s')} ds' \right)^2 ds
\]

\[
\leq \frac{1}{\Delta t^2} \int_{t_{N(t)-1}}^t \left( \int_{t_{N(t)-1}}^{t'} \lambda_k^2 e^{-\lambda_k^2 (t-\tau)} d\tau + \Delta t e^{-\lambda_k^2 (t-s)} \right)^2 ds
\]

\[
+ \frac{1}{\Delta t} \left( 1-e^{-\lambda_k^2 (t-t_{N(t)-1})} \right)^2
\]

\[
\leq \frac{2}{\Delta t^2} \int_{t_{N(t)-1}}^t \left( \int_{t_{N(t)-1}}^{t'} \lambda_k^2 e^{-\lambda_k^2 (t-\tau)} d\tau \right) ds'
\]

\[
+ \frac{1}{\Delta t} \left( 1-e^{-\lambda_k^2 (t-t_{N(t)-1})} \right)^2
\]

\[
\leq 8 \int_{t_{N(t)-1}}^t \left( e^{-\lambda_k^2 (t-s)} - e^{-\lambda_k^2 (t-t_{N(t)-1})} \right)^2 ds
\]

\[
\leq 8 \int_{t_{N(t)-1}}^t \left( e^{-\lambda_k^2 (t-s)} - e^{-\lambda_k^2 (t-t_{N(t)-1})} \right)^2 ds \leq \frac{1}{\Delta t} \left( 1-e^{-\lambda_k^2 \Delta t} \right)^2
\]

which finally gives

\[
(3.9) \quad \Psi_k^{N(t)} \leq 5 \left( 1-e^{-2\lambda_k^2 \Delta t} \right)^2 + \frac{1}{\Delta t} \left( 1-e^{-\lambda_k^2 \Delta t} \right)^2.
\]

Combining (3.7), (3.8) and (3.9) we obtain

\[
(3.10) \quad Z_2(t) \leq \left( 7 \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \frac{1}{\Delta t} \right)^{1/2}.
\]

The last step in the proof is to bound the series above in terms of \( \Delta t \). For the first series, we proceed as follows

\[
\sum_{k=1}^{\infty} \frac{\left( 1-e^{-2\lambda_k^2 \Delta t} \right)^2}{\lambda_k^2} \leq \frac{1}{\Delta t^2} \int_1^{\infty} \frac{\left( 1-e^{-2x^2\pi^2 \Delta t} \right)^2}{x^2} dx
\]

\[
\leq \frac{2}{\Delta t^2} \left( 1-e^{-2\pi^2 \Delta t} \right)^2 + 8 \Delta t \int_1^{\infty} \left( 1-e^{-2x^2\pi^2 \Delta t} \right) e^{-2x^2\pi^2 \Delta t} dx
\]

\[
\leq \frac{2}{\Delta t^2} \left( \int_0^1 \left( -e^{-2x^2\pi^2 \Delta t} \right)' dx \right)^2 + \frac{8}{\Delta t} \sqrt{\Delta t} \int_0^\infty e^{-2y^2} dy
\]

\[
\leq \frac{2}{\Delta t^2} \left( 2\pi^2 \Delta t \int_0^1 e^{-2x^2\pi^2 \Delta t} dx \right)^2 + \frac{8}{\Delta t} \sqrt{\Delta t} \left( \int_0^1 e^{-2y^2} dy + \int_1^{\infty} e^{-2y^2} dy \right)
\]

\[
\leq 8\pi^2 \Delta t^2 + \frac{8}{\pi} \sqrt{\Delta t} \left( 1 + \frac{1}{2} \int_1^{\infty} \frac{1}{y^2} dy \right)
\]
from which we obtain
\begin{equation}
\sum_{k=1}^{\infty} \frac{(1-e^{-\lambda_k^2\Delta t})^2}{\lambda_k^2} \leq C \sqrt{\Delta t}.
\end{equation}

We treat the second series, in a similar manner, as follows
\begin{align*}
\sum_{k=1}^{\infty} \frac{(1-e^{-\lambda_k^2\Delta t})^2}{\lambda_k^2} & \leq \frac{(1-e^{-\pi^2\Delta t})^2}{\pi^4} + \int_1^{\infty} \frac{(1-e^{-\pi^2\Delta t})e^{-\pi^2\Delta t}}{x^2} dx \\
& \leq \frac{4}{3\pi^4} \left(1 - e^{-\pi^2\Delta t}\right)^2 + \frac{4\Delta t}{3\pi^4} \int_1^{\infty} \frac{1-e^{-\pi^2\Delta t}e^{-\pi^2\Delta t}}{x^2} dx \\
& \leq \frac{4}{3\pi^4} \left(\pi^2 \Delta t \int_0^{\infty} e^{-\pi^2\Delta t} dx\right)^2 \\
& \quad + \frac{4}{3\pi^4} \Delta t \left(1 - e^{-\pi^2\Delta t} + 2\pi^2 \Delta t \int_1^{\infty} e^{-\pi^2\Delta t} dx\right) \\
& \leq \frac{8}{3} \Delta t^2 + \frac{8}{3} \Delta t^2 \int_0^{\infty} e^{-y^2} dy \\
& \leq \frac{8}{3} \Delta t^2 + \frac{8}{3} \Delta t^2 \left(\int_0^{1} e^{-y^2} dy + \int_1^{\infty} e^{-y^2} dy\right) \\
& \leq \frac{8}{3} \Delta t^2 + \frac{8}{3} \Delta t^2 \left(1 + \int_1^{\infty} \frac{1}{y} dy\right),
\end{align*}

from which we arrive at
\begin{equation}
\frac{1}{\Delta t} \sum_{k=1}^{\infty} \frac{(1-e^{-\lambda_k^2\Delta t})^2}{\lambda_k^2} \leq C \sqrt{\Delta t}.
\end{equation}

Using the bounds (3.10), (3.11) and (3.12) we arrive at
\begin{equation}
Z_2(t) \leq C \Delta t^{\frac{3}{4}}.
\end{equation}

Since \(Z(0) = 0\), the error bound (3.1) follows easily from (3.2), (3.6) and (3.13).

\begin{remark}
In [1] and [2] is obtained an \(O(\Delta t^{\frac{3}{4}} + \Delta x \Delta t^{-\frac{3}{4}})\) a priori estimate of the modelling error measured in the \(L_2^p(L_2^p(L_2^p))\) norm, which introduces the need to assume a CFL condition in order to conclude a rate of convergence, when \(\Delta t, \Delta x \to 0\). The \(L_2^p(L_2^p(L_2^p))\)–modelling error estimate derived in Theorem 3.1 is valid, without requiring any mesh condition between \(\Delta t\) and \(\Delta x\).
\end{remark}

4. A Modified Crank-Nicolson Method for the Deterministic Problem

Following [21], we introduce and analyze modified Crank-Nicolson time-discrete and fully-discrete approximations of the solution to the deterministic problem (1.3), which are necessary to carry out the convergence analysis of the Crank-Nicolson finite element method defined in Section 1.3

4.1. Time-Discrete approximations. The modified Crank-Nicolson time-discrete approximations of the solution \(v\) to (1.3) follow, first, by setting
\begin{equation}
V^0 := v_0
\end{equation}
and by finding \(V^1 \in \hat{H}^2(D)\) such that
\begin{equation}
V^1 - V^0 = \Delta t^2 \partial^2 V^1,
\end{equation}

\[10\]

\[10\]
and then, for $m = 2, \ldots, M$, by specifying $V^m \in \tilde{H}^2(D)$ satisfying
\begin{equation}
V^m - V^{m-1} = \Delta \tau \, \partial^2 V^{m-1}.
\end{equation}

The first convergence result we provide, is a discrete in time $L^2(D)$ estimate of time-averages of the nodal error.

**Proposition 4.1.** Let $(V^m)_{m=0}^M$ be the time-discrete approximations of the solution $v$ to the problem defined by (4.1) (4.3). Then, there exists a constant $C > 0$, independent of $\Delta \tau$, such that
\begin{equation}
\left( \Delta \tau \sum_{m=1}^M \| V^{m-1} - V^{m-2} \|_{0,D}^2 \right)^{1/2} \leq C \Delta \tau^\theta \| v_0 \|_{\tilde{H}^{2-\theta}} \quad \forall \theta \in [0,1], \; \forall v_0 \in \tilde{H}^1(D),
\end{equation}
where $v^\ell(\cdot) := v(\tau_\ell, \cdot)$ for $\ell = 0, \ldots, M$.

**Proof.** In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta \tau$ and may changes value from the one line to the other.

Let $E^2 := (\tau_{\ell+1}, \cdot) - V^\ell$ and $E^m := V^m - V^m$ for $m = 0, \ldots, M$. Then, using (4.3) and (4.4), we conclude that
\begin{equation}
T_E(E^m - E^{m-1}) = \Delta \tau \, E^{m-1} + \rho_m, \; \; m = 2, \ldots, M,
\end{equation}
where $\rho_m(\cdot) := \int_{\Delta_m} [v(s, \cdot) - v^{m-1}(\cdot)] \, ds$. Taking the $L^2(D)$–inner product of both sides of (4.5) with $E^{m-1}$, using (4.3) and then summing with respect to $m$ from 2 up to $M$, we obtain
\begin{equation}
|T_EE^m|_{1,D}^2 - |T_EE^1|_{1,D}^2 + 2 \Delta \tau \sum_{m=2}^M \| E^{m-1} \|_{0,D}^2 = -2 \sum_{m=2}^M \langle \rho_m, E^{m-1} \rangle_{\tilde{H}^1},
\end{equation}
which, after applying the Cauchy-Schwarz inequality and the geometric mean inequality, yields
\begin{equation}
\Delta \tau \sum_{m=2}^M \| E^{m-1} \|_{0,D}^2 \leq |T_EE^1|_{1,D}^2 + \Delta \tau^{-1} \sum_{m=2}^M \| \rho_m \|_{0,D}^2.
\end{equation}

Next, we bound the residual functions $(\rho_m)_{m=2}^M$ as follows:
\begin{equation}
\| \rho_m \|_{0,D}^2 = \frac{1}{4} \int_D \left( - \int_{\Delta_m} \int_0^{\tau_m} \partial_x v(s, x) \, dsd\tau + \int_{\Delta_m} \int_0^{\tau_m} \partial_x v(s, x) \, dsd\tau \right)^2 \, dx
\end{equation}
\begin{equation}
\leq \int_D \left( \int_{\Delta_m} \int_{\Delta_m} |\partial_x v(s, \cdot)\|^2_{0,D} \, dsd\tau \right)^2 \, dx
\leq \Delta \tau^3 \int_{\Delta_m} \| \partial_x v(s, \cdot) \|_{0,D}^2 \, ds, \; \; m = 2, \ldots, M.
\end{equation}

Also, observing that $E^0 = 0$ and combining (4.6), (4.7) and (2.16) (with $\beta = 0$, $\ell = 1$, $p = 0$), we obtain
\begin{equation}
\Delta \tau \sum_{m=1}^M \| E^{m-1} \|_{0,D}^2 \leq \Delta \tau \| E^1 \|_{0,D}^2 + |T_EE^1|_{1,D}^2 + \Delta \tau^2 \int_0^{\tau} \| \partial_x v(s, \cdot) \|_{0,D}^2 \, ds
\end{equation}
\begin{equation}
\leq \Delta \tau \| E^1 \|_{0,D}^2 + |T_EE^1|_{1,D}^2 + C \Delta \tau^2 \| v_0 \|_{\tilde{H}^1}^2.
\end{equation}

In order to bound the first two terms in the right hand side of (4.8), we introduce the following splittings
\begin{equation}
|T_EE^1|_{1,D}^2 \leq 2 \left( |T_E(v(\tau_1, \cdot) - v(\tau_{\ell+1}, \cdot))|_{1,D}^2 + |T_EE^\ell|_{1,D}^2 \right),
\end{equation}
\begin{equation}
\Delta \tau \| E^1 \|_{0,D}^2 \leq 2 \Delta \tau \left( \| v(\tau_1, \cdot) - v(\tau_{\ell+1}, \cdot) \|^2_{0,D} + \| E^\ell \|_{0,D}^2 \right).
\end{equation}
We continue by estimating the terms in the right hand side of (4.9) and (4.10). First, we observe that \( \|v(\tau_1, \cdot) - v(\tau_2, \cdot)\|^2_{\partial_t, \theta, \cdot, 0, \cdot} \leq \frac{\Delta \tau}{2} \int_{\tau_2}^{\tau_1} \|\partial_t v(\tau, \cdot)\|^2_{\partial_t, \theta, \cdot, 0, \cdot} d\tau \), which, along with (2.16) (with \( \ell = 1, p = 0, \beta = 0 \)), yields
\[
(4.11) \quad \|v(\tau_1, \cdot) - v(\tau_2, \cdot)\|^2_{\partial_t, \theta, \cdot, 0, \cdot} \leq C \Delta \tau \|v_0\|^2_{\mathcal{H}^1}.
\]
Next, we use (2.14) and (2.4), to get
\[
|T_E(v(\tau_1, \cdot) - v(\tau_2, \cdot))|^2_{0, \cdot, 0, \cdot} \leq C \Delta \tau \int_{\tau_2}^{\tau_1} \|\partial_t v(\tau, \cdot)\|^2_{\partial_t, \theta, \cdot, 0, \cdot} d\tau \leq C \Delta \tau \int_{\tau_2}^{\tau_1} \|\partial_t v(\tau, \cdot)\|^2_{\partial_t, \theta, \cdot, 0, \cdot} d\tau.
\]

Finally, using (2.13) and (4.2) we have
\[
(4.13) \quad T_E(\mathcal{E}^{1, \cdot}_x - \mathcal{E}^{0}) = \frac{\Delta \tau}{2} \mathcal{E}^{1, \cdot}_x + \rho^{1, \cdot}_x
\]
with \( \rho^{1, \cdot}_x := \int_0^{T^*} \{v(s, \cdot) - v(\tau, \cdot)\} ds \). Since \( \mathcal{E}^{0} = 0 \), after taking the \( L^2(D) \)-inner product of both sides of (4.13) with \( \mathcal{E}^{1, \cdot}_x \) and using (2.13) and the Cauchy-Schwarz inequality along with the arithmetic mean inequality, we obtain
\[
(4.14) \quad |T_E \mathcal{E}^{1, \cdot}_x|^2_{0, \cdot, 0, \cdot} + \frac{\Delta \tau}{2} \|\mathcal{E}^{1, \cdot}_x\|^2_{0, \cdot, 0, \cdot} \leq \frac{1}{\Delta \tau} \|\rho^{1, \cdot}_x\|^2_{0, \cdot, 0, \cdot} + \frac{\Delta \tau}{2} \|\mathcal{E}^{1, \cdot}_x\|^2_{0, \cdot, 0, \cdot}.
\]
Now, using (2.16) (with \( \beta = 0, \ell = 1, p = 0 \)) we obtain
\[
|T_E \mathcal{E}^{1, \cdot}_x|^2_{0, \cdot, 0, \cdot} + \frac{\Delta \tau}{2} \|\mathcal{E}^{1, \cdot}_x\|^2_{0, \cdot, 0, \cdot} \leq C \Delta \tau \|v_0\|^2_{\mathcal{H}^1},
\]
which, along with (4.14), yields
\[
(4.15) \quad |T_E \mathcal{E}^{1, \cdot}_x|^2_{0, \cdot, 0, \cdot} + \frac{\Delta \tau}{2} \|\mathcal{E}^{1, \cdot}_x\|^2_{0, \cdot, 0, \cdot} \leq C \Delta \tau \|v_0\|^2_{\mathcal{H}^1}.
\]
Thus, from (4.8), (4.9), (4.10), (4.11), (4.12) and (4.15), we conclude that (4.4) holds for \( \theta = 1 \).

We continue, by observing that (4.2) and (3.3) are equivalent to
\[
(4.16) \quad T_E(V^1 - V^0) = \frac{\Delta \tau}{2} V^1,
\]
\[
(4.17) \quad T_E(V^m - V^{m-1}) = \Delta \tau V^{m-\frac{3}{2}}, \quad m = 2, \ldots, M.
\]
Next, we take the $L^2(D)$–inner product of both sides of (4.17) with $\mathcal{V}^{m-\frac{1}{2}}$, use (2.13) and sum with respect to $m$ from 2 up to $M$, to obtain

$$
\Delta \tau \|V\|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \|\mathcal{V}^{m-\frac{1}{2}}\|_{0,D}^2 \leq \Delta \tau \|V\|_{0,D}^2 + |T_E V|_{1,D}^2.
$$

Now, we take the $L^2(D)$–inner product of both sides of (4.16) with $V^1$, use (2.13) along with (2.11) to get

$$
|T_E V|_{1,D}^2 + \Delta \tau \|V\|_{0,D}^2 \leq |T_E v^0|_{1,D}^2.
$$

Combining (4.18) and (4.19) and then using (2.14) and (2.21), we obtain

$$
\Delta \tau \|V\|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \|\mathcal{V}^{m-\frac{1}{2}}\|_{0,D}^2 \leq C \|v_0\|_{-1,D}^2
$$

and

$$
\Delta \tau \|V\|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \|\mathcal{V}^{m-\frac{1}{2}}\|_{0,D}^2 \leq 2 \Delta \tau \sum_{m=1}^{M} \|V^m\|_{0,D}^2,
$$

and

$$
2 \Delta \tau \sum_{m=1}^{M} \|V^m\|_{0,D}^2 \leq 2 \Delta \tau - 1 \sum_{m=1}^{M} \int_D \left( \int_{\tau_{m-1}}^{\tau_m} \partial_x \left[ (\tau - \tau_{m-1}) v(\tau, x) \right] d\tau \right)^2 dx
$$

$$
\leq 2 \Delta \tau - 1 \sum_{m=1}^{M} \int_D \left( \int_{\tau_{m-1}}^{\tau_m} [v(\tau, x) + (\tau - \tau_{m-1}) v_x(\tau, x)] d\tau \right)^2 dx
$$

$$
\leq 4 \sum_{m=1}^{M} \int_{\tau_{m-1}}^{\tau_m} \left[ \|v(\tau, \cdot)\|_{0,D}^2 + (\tau - \tau_{m-1})^2 \|v_x(\tau, \cdot)\|_{0,D}^2 \right] d\tau
$$

$$
\leq 4 \int_{\tau_{m-1}}^{\tau_m} \left( \|v(\tau, \cdot)\|_{0,D}^2 + (\tau - \tau_{m-1})^2 \|v_x(\tau, \cdot)\|_{0,D}^2 \right) d\tau,
$$

which, along with (2.10) (taking $(\beta, \ell, p) = (0, 0, 0)$ and $(\beta, \ell, p) = (2, 1, 0)$), yields

$$
2 \Delta \tau \sum_{m=1}^{M} \|V^m\|_{0,D}^2 \leq C \|v_0\|_{-1,D}^2.
$$

Observing that $\mathcal{E}^{\frac{1}{2}} = \frac{1}{2} \mathcal{E}^1$, we have

$$
\Delta \tau \sum_{m=1}^{M} \|\mathcal{E}^{m-\frac{1}{2}}\|_{0,D}^2 \leq 2 \left( \Delta \tau \|V^1\|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \|\mathcal{V}^{m-\frac{1}{2}}\|_{0,D}^2 \right)
$$

$$
+ 2 \left( \Delta \tau \|v^1\|_{0,D}^2 + \Delta \tau \sum_{m=2}^{M} \|v^{m-\frac{1}{2}}\|_{0,D}^2 \right),
$$

which, after using (4.20), (4.21) and (4.22), yields that (4.4) holds for $\theta = 0$.

Hence, the estimate (4.4) follows by interpolation.

Next, we establish a discrete in time $L^2(D)$–inner product of both sides of (4.17) with $\mathcal{V}^{m-\frac{1}{2}}$, use (2.13) and sum with respect to $m$ from 2 up to $M$, to obtain

$$
(4.23) \quad \left( \Delta \tau \sum_{m=1}^{M} \|V^m - v^m\|_{0,D}^2 \right)^{1/2} \leq C \Delta \tau^{\frac{1}{2}} \|v_0\|_{H^{\ell-1}} \quad \forall \delta \in [0, 1], \quad \forall v_0 \in \mathcal{H}^1(D),
$$

Proposition 4.2. Let $(\mathcal{V}^m)_{m=0}^M$ be the modified Crank-Nicolson time-discrete approximations of the solution $v$ to the problem (1.13) defined by (4.11–4.13). Then, there exists a constant $C > 0$, independent of $\Delta \tau$, such that

$$
(4.23) \quad \left( \Delta \tau \sum_{m=1}^{M} \|V^m - v^m\|_{0,D}^2 \right)^{1/2} \leq C \Delta \tau^{\frac{1}{2}} \|v_0\|_{H^{\ell-1}} \quad \forall \delta \in [0, 1], \quad \forall v_0 \in \mathcal{H}^1(D),
$$

13
where \( v^\ell := v(\tau_\ell, \cdot) \) for \( \ell = 0, \ldots, M \).

**Proof.** We will arrive at the error bound (4.23) by interpolation after proving it for \( \delta = 1 \) and \( \delta = 0 \) (cf. Proposition 4.1). In both cases, the error estimation is based on the following bound

\[
(\Delta \tau M \sum_{m=1}^M \|V^m - v^m\|^2_{0,D})^{1/2} \leq S_1 + S_2 + S_3
\]

where

\[
S_1 := (\Delta \tau M \sum_{m=2}^M \|V^m - V^{m-\frac{1}{2}}\|^2_{0,D})^{1/2},
\]

\[
S_2 := (\Delta \tau \|V^1 - v^1\|^2_{0,D} + \Delta \tau M \sum_{m=2}^M \|V^{m-\frac{1}{2}} - v^{m-\frac{1}{2}}\|^2_{0,D})^{1/2},
\]

\[
S_3 := (\Delta \tau M \sum_{m=2}^M \|v^{m-\frac{1}{2}} - v^m\|^2_{0,D})^{1/2}.
\]

In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta \tau \) and may change value from one line to the other.

Taking the \( L^2(D) \)–inner product of both sides of (4.25) with \( (V^m - V^{m-1}) \) and then integrating by parts, we easily arrive at

\[
\|V^m - V^{m-1}\|^2_{0,D} + \Delta \tau \frac{1}{2} \left( |V^m|^2_{1,D} - |V^{m-1}|^2_{1,D} \right) = 0, \quad m = 2, \ldots, M.
\]

After summing both sides of (4.25) with respect to \( m \) from 2 up to \( M \), we obtain

\[
\Delta \tau \sum_{m=2}^M \|V^m - V^{m-1}\|^2_{0,D} + \Delta \tau \frac{1}{2} \left( |V^M|^2_{1,D} - |V^1|^2_{1,D} \right) = 0,
\]

which yields

\[
(\Delta \tau \sum_{m=2}^M \|V^m - V^{m-1}\|^2_{0,D}) \leq \Delta \tau \frac{1}{2} |V^1|^2_{1,D}.
\]

Taking the \( L^2(D) \)–inner product of both sides of (4.2) with \( V^1 \), and then integrating by parts and using (2.8), we obtain: \( \|V^1\|^2_{0,D} - \|V^0\|^2_{0,D} + \Delta \tau |V^1|^2_{1,D} \leq 0 \), which yields

\[
\Delta \tau |V^1|^2_{1,D} \leq \|v_0\|^2_{0,D}.
\]

Thus, combining (4.20) and (4.21), we get

\[
S_1 = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^M \|V^m - V^{m-1}\|^2_{0,D} \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\Delta \tau}} \Delta \tau |V^1|_{1,D} \leq \sqrt{\Delta \tau} \|v_0\|_{0,D}.
\]

Also, we observe that the estimate (4.4), for \( \theta = \frac{1}{2} \), yields

\[
S_2 \leq C \sqrt{\Delta \tau} \|v_0\|_{0,D}.
\]
Finally, using (2.15) (with $\ell = 1, p = 0, q = 0$), we obtain
\[
S_3 \leq \left( \Delta \tau \sum_{m=2}^{M} \left\| \int_{\Delta_{m}} \partial_{\tau} v(\tau, \cdot) d\tau \right\|_{0,D}^2 \right)^{\frac{1}{2}} \\
\leq \left( \Delta \tau^2 \int_{\Delta} \| \partial_{\tau} v(\tau, \cdot) \|_{0,D}^2 d\tau \right)^{\frac{1}{2}} \\
\leq C \left( \Delta \tau^2 \int_{\Delta} \tau^{-2} \| v_0 \|_{0,D}^2 d\tau \right)^{\frac{1}{2}} \\
\leq C \Delta \tau \| v_0 \|_{0,D} \left( \frac{1}{\Delta \tau} - \frac{1}{\tau} \right)^{\frac{1}{2}} \\
\leq C \sqrt{\Delta \tau} \| v_0 \|_{0,D}.
\] (4.30)

Thus, from (4.24), (4.28), (4.29) and (4.30) we conclude (4.23) for $\delta = 1$.

Taking again the $L^2(D)$—inner product of both sides of (4.2) with $V^1$ and then integrating by parts and using (2.1), (2.4) and (2.3) along with the arithmetic mean inequality, we obtain
\[
\| V^1 \|_{0,D}^2 + \frac{\Delta \tau}{2} | V^1 |_{1,D}^2 = (v_0, V^1)_{0,D} \\
\leq \| v_0 \|_{-1,D} \| V^1 \|_{1,D} \\
\leq C \| v_0 \|_{H^{-1}} \| V^1 \|_{1,D} \\
\leq C \Delta \tau^{-1} \| v_0 \|_{H^{-1}}^2 + \frac{\Delta \tau}{2} | V^1 |_{1,D}^2,
\]
which yields that
\[
\Delta \tau | V^1 |_{1,D} \leq C \| v_0 \|_{H^{-1}}.
\] (4.31)

Thus, combining (4.20) and (4.31), we conclude that
\[
S_1 = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^{M} \| V^m - V^{m-1} \|_{0,D}^2 \right)^{\frac{1}{2}} \\
\leq C \| v_0 \|_{H^{-1}}.
\] (4.32)

Also, the estimate (4.4), for $\theta = 0$, yields
\[
S_2 \leq C \| v_0 \|_{H^{-1}}.
\] (4.33)

Using the Cauchy-Schwarz inequality and (4.22), we have
\[
S_3 = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^{M} \| v^m - v^{m-1} \|_{0,D}^2 \right)^{\frac{1}{2}} \\
\leq \frac{\sqrt{\tau}}{2} \left( 2 \Delta \tau \sum_{m=1}^{M} \| v^m \|_{0,D}^2 \right)^{\frac{1}{2}} \\
\leq C \| v_0 \|_{H^{-1}}.
\] (4.34)

Thus, from (4.24), (4.32), (4.33) and (4.34) we conclude (4.23) for $\delta = 0$.

4.2. Fully-Discrete Approximations. In this section we construct and analyze finite element approximations, $\left( V^m_h \right)_{m=0}^M$, of the modified Crank-Nicolson time-discrete approximations defined in Section 4.1. The method begins by setting
\[
V^0_h := P_h v_0
\]
and finding $V^1_h \in Z_h$ such that
\[
V^1_h - V^0_h + \frac{\Delta \tau}{2} \Delta_h V^1_h = 0.
\] (4.35)
Then, for \( m = 2, \ldots, M \), it specifies \( \mathcal{V}^m_h \in \mathbb{Z}_h^m \) such that
\begin{equation}
\mathcal{V}^m_h - \mathcal{V}^{m-1}_h + \Delta \tau \Delta a \mathcal{V}^{m-\frac{1}{2}}_h = 0.
\end{equation}

First, we show a discrete in time \( L^2(\mu^t) \) a priori estimate of time averages of the nodal error between the modified Crank-Nicolson time-discrete approximations presented in the previous section and the modified Crank-Nicolson fully-discrete approximations defined above.

**Proposition 4.3.** Let \( (\mathcal{V}^m_h)_{m=0}^M \) be the Crank-Nicolson time-discrete approximations defined by (4.11)-(4.13) and \( (\mathcal{V}^m_h)_{m=0}^M \) be the modified Crank-Nicolson fully-discrete approximations defined by (4.35)-(4.37). Then, there exists a constant \( C > 0 \), independent of \( h \) and \( \Delta \tau \), such that
\begin{equation}
\left( \Delta \tau \| V^1_h - V^0_h \|^2_{1,D} + \Delta \tau \sum_{m=2}^M \| \mathcal{V}^m_{1/2} - \mathcal{V}^{m-\frac{1}{2}}_h \|^2_{0,D} \right)^{1/2} \leq C h^{2\theta} \| v_0 \|_{H^{2\theta-1}}
\end{equation}
for all \( \theta \in [0, 1] \) and \( v_0 \in \dot{H}^1(D) \).

**Proof.** We will get the error estimate (4.38) by interpolation after proving it for \( \theta = 1 \) and \( \theta = 0 \) (cf. [12]). In the sequel, we will use the symbol \( C \) to denote a generic constant that is independent of \( \Delta \tau \) and \( h \), and may change value from one line to the other.

Letting \( \Theta^\ell := V^\ell - V^\ell_h \) for \( \ell = 0, \ldots, M \), we use (1.2), (4.36), (4.3) and (4.37), to arrive at the following error equations:
\begin{align}
T_{E,h}(\Theta^1 - \Theta^0) &= \frac{\Delta \tau}{2} \Theta^1 + \frac{\Delta \tau}{2} \xi_1, \\
T_{E,h}(\Theta^m - \Theta^{m-1}) &= \Delta \tau \Theta^{m-\frac{1}{2}} + \Delta \tau \xi_m, \quad m = 2, \ldots, M,
\end{align}
where
\begin{align}
\xi_1 := & (T_{E,h} - T_E) \partial^2 V^1, \\
\xi_\ell := & (T_{E,h} - T_E) \partial^2 V^{\ell-\frac{1}{2}}, \quad \ell = 2, \ldots, M.
\end{align}

Taking the \( L^2(D) \)–inner product of both sides of (4.40) with \( \Theta^{m-\frac{1}{2}} \) and then using (2.19), the Cauchy-Schwarz inequality along with the arithmetic mean inequality, we obtain
\begin{equation}
| T_{E,h} \Theta^m_{1,D} - T_{B,h} \Theta^{m-1}_{1,D} |^2_{1,D} + \Delta \tau \| \Theta^{m-\frac{1}{2}} \|^2_{0,D} \leq \Delta \tau \| \xi_m \|^2_{0,D}, \quad m = 2, \ldots, M.
\end{equation}

After summing with respect to \( m \) from 2 up to \( M \), the relation above yields
\begin{equation}
\Delta \tau \| \Theta^1 \|^2_{1,D} + \Delta \tau \sum_{m=2}^M \| \Theta^{m-\frac{1}{2}} \|^2_{0,D} \leq | T_{E,h} \Theta^1_{1,D} + \Delta \tau \| \Theta^1 \|^2_{0,D} + \Delta \tau \sum_{m=2}^M \| \xi_m \|^2_{0,D}.
\end{equation}

Observing that \( T_{E,h} \Theta^0 = 0 \), we take the \( L^2(D) \)–inner product of both sides of (4.40) with \( \Theta^1 \) and then use the Cauchy-Schwarz inequality along with the arithmetic mean inequality, to get
\begin{equation}
| T_{E,h} \Theta^1_{1,D} + \frac{\Delta \tau}{2} \| \Theta^1 \|^2_{0,D} \leq \frac{\Delta \tau}{2} \| \xi_1 \|^2_{0,D}.
\end{equation}

Thus, using (4.43), (4.44), (4.41), (4.42) and (2.21), we easily conclude that
\begin{equation}
\Delta \tau \| \Theta^1 \|^2_{1,D} + \Delta \tau \sum_{m=2}^M \| \Theta^{m-\frac{1}{2}} \|^2_{0,D} \leq \Delta \tau \| \xi_1 \|^2_{0,D} + \Delta \tau \sum_{m=2}^M \| \xi_m \|^2_{0,D}
\end{equation}
\begin{equation}
\leq C h^4 \left( \Delta \tau | V^1 |^2_{1,2,D} + \Delta \tau \sum_{m=2}^M | \mathcal{V}^{m-\frac{1}{2}} |^2_{1,2,D} \right).
\end{equation}

Taking the \( L^2(D) \)–inner product of (4.3) with \( \partial^2 \mathcal{V}^{m-\frac{1}{2}} \), and then integrating by parts and summing with respect to \( m \), from 2 up to \( M \), it follows that
\begin{equation}
| V^m |^2_{1,D} - | V^1 |^2_{1,D} + 2 \Delta \tau \sum_{m=2}^M | \mathcal{V}^{m-\frac{1}{2}} |^2_{1,2,D} = 0.
\end{equation}
which yields

$$\Delta \tau |V^1_{1,D}|^2 + \sum_{m=2}^{M} \Delta \tau |V^{m-\frac{1}{2}}_{1,D}|^2 \leq \frac{1}{2} |V^1_{1,D}|^2 + \Delta \tau |V^1_{1,D}|^2.$$  \hspace{1cm} (4.46)

Now, take the $L^2(D)$–inner product of (4.2) with $\partial^2 V^1$, and then integrate by parts and use (2.8) to get

$$|V^1_{1,D} - |V^0_{1,D}| + \Delta \tau |V^1_{1,D}| \leq 0,$$

which, along with (2.3), yields

$$|V^1_{1,D} - |V^0_{1,D}| + \Delta \tau |V^1_{1,D}| \leq \|v_0\|^2_{H^1}.$$  \hspace{1cm} (4.47)

Thus, combining (4.45), (4.46) and (4.47), we obtain (4.38) for all $\theta$.

From (4.36) and (4.37), it follows that

$$-T_{E,h}(V^1_h - V^0_h) + \Delta \tau \frac{\partial^2}{\partial t} V^1_h = 0,$$

which yields

$$-T_{E,h}(V^m_h - V^{m-1}_h) + \Delta \tau V^{m-\frac{1}{2}}_h = 0, \quad m = 2, \ldots, M.$$  \hspace{1cm} (4.49)

Taking the $L^2(D)$–inner product of (4.49) with $V^{m-\frac{1}{2}}_h$ and using (2.8), we have

$$|T_{E,h} V^0_h|_{1,D}^2 - |T_{E,h} V^{m-1}_h|_{1,D}^2 + 2 \Delta \tau \|V^{m-\frac{1}{2}}_h\|^2_{0,D} = 0, \quad m = 2, \ldots, M,$$

which, after summing with respect to $m$ from 2 up to $M$, yields

$$\Delta \tau \|V^1_h\|^2_{0,D} + \Delta \tau \sum_{m=2}^{M} |V^{m-\frac{1}{2}}_h|^2_{0,D} \leq \frac{1}{2} |T_{E,h} V^1_h|_{1,D}^2 + \Delta \tau \|V^1_h\|^2_{0,D}.$$  \hspace{1cm} (4.50)

Now, take the $L^2(D)$–inner product of (4.48) with $V^1_h$ and use (2.8) and (4.38), to have

$$|T_{E,h} V^1_h|_{1,D}^2 + \Delta \tau \|V^1_h\|^2_{0,D} \leq |T_{E,h} P^1_h v_0|_{1,D}^2 \leq |T_{E,h} v_0|^2_{1,D}.$$  \hspace{1cm} (4.51)

Combining (4.50), (4.51), (2.20) and (2.8), we obtain

$$\left( \Delta \tau \|V^1_h\|^2_{0,D} + \Delta \tau \sum_{m=2}^{M} |V^{m-\frac{1}{2}}_h|^2_{0,D} \right)^{1/2} \leq C \|v_0\|_{H^{-1}}.$$  \hspace{1cm} (4.52)

Finally, combine (4.52) with (4.20) to get (4.38) for $\theta = 0$. \hspace{1cm} \[\square\]

Next, we derive a discrete in time $L^2(L^2)$ a priori estimate of the nodal error between the modified Crank-Nicolson time-discrete approximations and the modified Crank-Nicolson fully-discrete approximations.

**Proposition 4.4.** Let $\left(V^m\right)_{m=0}^M$ be the modified Crank-Nicolson time-discrete approximations defined by (4.1)–(4.3), and $\left(V^m_h\right)_{m=0}^M$ be the modified Crank-Nicolson finite element approximations specified by (4.35)–(4.37). Then, there exists a constant $C > 0$, independent of $h$ and $\Delta \tau$, such that

$$\left( \Delta \tau \sum_{m=1}^{M} \|V^m - V^m_h\|^2_{0,D} \right)^{1/2} \leq C \left( \Delta \tau \frac{4}{\theta} \|v_0\|_{H^{\frac{1}{2}} - 1} + h^{2\theta} \|v_0\|_{H^{2\theta - 1}} \right)$$  \hspace{1cm} (4.53)

for all $\delta$, $\theta \in [0, 1]$ and $v_0 \in \tilde{H}^2(D)$.

**Proof.** The proof is based on the estimation of the terms in the right hand side of the following triangle inequality:

$$\left( \Delta \tau \sum_{m=1}^{M} \|V^m - V^m_h\|^2_{0,D} \right)^{1/2} \leq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3$$  \hspace{1cm} (4.54)
In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta \tau$ and may change value from one line to the other.

Taking the $L^2(D)$–inner product of both sides of (4.37) with $(V^m_h - V^{m-1}_h)$, we have

\[
\|V^m_h - V^{m-1}_h\|^2_{0,D} + \Delta \tau \|V^m - V^{m-1}_h\|^2_{0,D} + \sum_{m=2}^M \|V^m - V^{m-1}_h\|^2_{0,D} = 0, \quad m = 2, \ldots, M.
\]

After summing both sides of (4.55) with respect to $m$ from 2 up to $M$, we obtain

\[
\Delta \tau \sum_{m=2}^M \|V^m_h - V^{m-1}_h\|^2_{0,D} + \Delta \tau^2 \sum_{m=2}^M \|V^m - V^{m-1}_h\|^2_{0,D} = 0,
\]

which yields

\[
\Delta \tau \sum_{m=2}^M \|V^m_h - V^{m-1}_h\|^2_{0,D} \leq \Delta \tau^2 \sum_{m=2}^M \|V^m - V^{m-1}_h\|^2_{0,D}.
\]

Taking the $L^2(D)$–inner product of both sides of (4.36) with $V^1_h$ and then using (2.8), we obtain

\[
\|V^1_h\|^2_{0,D} - \|V^0_h\|^2_{0,D} + \Delta \tau \|V^1_h\|^2_{1,D} \leq 0,
\]

from which we conclude that

\[
\Delta \tau \|V^1_h\|^2_{1,D} \leq \|v_0\|^2_{0,D}.
\]

Thus, combining (4.50) and (4.57) we have

\[
S_3 = \frac{1}{2} \left( \Delta \tau \sum_{m=2}^M \|V^m_h - V^{m-1}_h\|^2_{0,D} \right)^{1/2}
\leq \frac{1}{2} \Delta \tau \|V^1_h\|_{1,D}
\leq \Delta \tau \frac{1}{2} \|v_0\|_{0,D}.
\]

Taking again the $L^2(D)$–inner product of both sides of (4.30) with $V^1_h$ and then using (2.4) and (2.1) along with the arithmetic mean inequality, we obtain

\[
\|V^1_h\|^2_{0,D} + \Delta \tau \|V^1_h\|^2_{1,D} = (P_h v_0, V^1_h)_{0,D}
= (v_0, V^1_h)_{0,D}
\leq \|v_0\|_{-1,D} \|V^1_h\|_{1,D}
\leq C \|v_0\|_{H-1} \|V^1_h\|_{1,D}
\leq C \Delta \tau^{-1} \|v_0\|^2_{H-1} + \Delta \tau \|V^1_h\|^2_{1,D},
\]

which yields that

\[
\Delta \tau \frac{1}{2} \|V^1_h\|^2_{1,D} \leq C \|v_0\|^2_{H-1}.
\]
Thus, combining (4.59) and (4.50), we conclude that
\[
S_3 = \frac{1}{2} \left( |\Delta \tau| \sum_{m=2}^{M} \| V^m_h - V^{m-1}_h \|_{0,D}^2 \right)^{1/2}
\]
(4.60)
\[
\leq \frac{1}{2\tau} |\Delta \tau| |V_h^0|_{0,D}
\]
\[
\leq C \|v_0\|_{H^{1-}}.
\]

Also, from (4.28) and (4.32), we have
\[
S_1 \leq C |\Delta \tau\|_{H^{1-}}.
\]
(4.61)
\[
S_1 \leq C \|v_0\|_{H^{1-}}.
\]
and
\[
S_1 \leq C \|v_0\|_{H^{1-}}.
\]
(4.62)
By interpolation, from (4.58), (4.61), (4.60) and (4.62), we conclude that
\[
S_2 + S_3 \leq C |\Delta \tau\|_{H^{1-}} \quad \forall \delta \in [0, 1].
\]
(4.63)
Finally, the estimate (4.38) reads
\[
S_2 \leq C h^2 |\Delta \tau\|_{H^{1-}} \quad \forall \theta \in [0, 1].
\]
(4.64)

Thus, (4.53) follows as a simple consequence of (4.54), (4.63) and (4.64).

5. Convergence Analysis of the Crank-Nicolson finite element method

In this section, we focus on the derivation of an estimate of the approximation error of the Crank-Nicolson finite element method introduced in Section 1.3. For that, we will use, as a comparison tool (cf. [17], [18], [2], [12], [21]), the corresponding Crank-Nicolson time-discrete approximations of \( \hat{u} \), which are defined first by setting
\[
U^0 := 0
\]
and then, for \( m = 1, \ldots, M \), by specifying
\[
U^m = \hat{H}^2(D)
\]
\[
U^m - U^{m-1} = |\Delta \tau| \partial^2 U^{m-1} + \int_{\Delta m} \omega \ ds \quad \text{a.s.}
\]
Thus, we split the discretization error of the Crank-Nicolson finite element method as follows
\[
\max_{0 \leq m \leq M} \left( \mathbb{E} \left[ \| \hat{u}^m - U^m_h \|_{0,D}^2 \right] \right)^{1/2} \leq \max_{1 \leq m \leq M} E_{\text{tor}}^m + \max_{1 \leq m \leq M} E_{\text{sdr}}^m
\]
(5.2)
where \( \hat{u}^m := \hat{u}(\tau_m, \cdot) \), \( E_{\text{tor}}^m := \left( \mathbb{E} \left[ \| \hat{u}^m - U^m_h \|_{0,D}^2 \right] \right)^{1/2} \) is the time discretization error at \( \tau_m \), and \( E_{\text{sdr}}^m := \left( \mathbb{E} \left[ \| U^m - U^m_h \|_{0,D}^2 \right] \right)^{1/2} \) is the space discretization error at \( \tau_m \).

In the sequel, we estimate the above defined type of error using a discrete Duhamel principle technique along with the convergence results for the modified Crank-Nicolson method obtained in Section 4 (cf. [17], [18], [2], [12], [21]).

5.1. Estimating the time discretization error.

Proposition 5.1. Let \( (U^m)^{m=0}_m \) be the Crank-Nicolson time discrete approximations specified by (6.1)-(6.2). Then, there exists a constant \( C_{\text{tor}} \geq 0 \), independent of \( \Delta t, \Delta x \) and \( \Delta \tau \), such that
\[
\max_{1 \leq m \leq M} E_{\text{tor}}^m \leq C_{\text{tor}} \epsilon^{4} |\Delta \tau|^{4-\epsilon} \quad \forall \epsilon \in \left( 0, \frac{1}{2} \right).
\]
(5.3)

Proof. Let \( I : L^2(D) \rightarrow L^2(D) \) be the identity operator, \( Y : H^2(D) \rightarrow L^2(D) \) be defined by
\[
Y := I + \frac{\Delta \tau}{2 \sigma^2} \partial^2
\]
and \( \Lambda : L^2(D) \rightarrow \hat{H}^2(D) \) be the inverse elliptic operator \( \Lambda := \left( 1 - \frac{\Delta \tau}{2 \sigma^2} \partial^2 \right)^{-1} \). Then, for \( m = 1, \ldots, M \), we define an operator \( Q^m : L^2(D) \rightarrow \hat{H}^2(D) \) by
\[
Q^m := (\Lambda \circ Y)^{m-1} \circ \Lambda,
\]
which has a Green’s function \( G_{Q^m} \) given by \( G_{Q^m}(x, y) := \sum_{k=1}^{\infty} \frac{1 - \Delta \tau}{(1 + \frac{\Delta \tau}{2 \sigma^2} \lambda^2_m)^{m-1}} \epsilon_{\lambda}(x) \epsilon_{\lambda}(y) \quad \forall x, y \in \bar{D} \). Also, for given \( v_0 \in \hat{H}^1(D) \), let \( (S_{\Delta t}^m(v_0))^{M}_{m=0} \) be the time-discrete approximations of the
Next, we introduce the following splitting \((2.10), (2.6), (2.5)\) and \((2.11)\), we obtain a generic constant that is independent of \(\Delta t\). Let \(A\) be the argument, we conclude that a solution to the deterministic problem \((1.3)\), defined by \((4.1)–(4.3)\). Then, using a simple induction argument applied to \((5.2)\), yields

\[
E^m_{\text{tdr}} = \mathbb{E} \left[ \left( \int_0^T \left( \int_D (K_m - G_m)(\tau; x, y) W(\tau, y) d\tau \right)^2 d\tau \right)^{1/2} \right],
\]

with \(K^m(\tau; x, y) := \sum_{\ell=1}^m X_{\ell} \mathbb{E}_{\mathbb{G}_{\tau_{\ell-1}}} (x, y)\) for \(x, y \in \overline{D}, \tau \in [0, T]\). Then, using \((5.5), (1.9), (2.10), (2.6), (2.5)\) and \((2.11)\), we obtain

\[
\max_{1 \leq m \leq M} E^m_{\text{tdr}} \leq \max_{1 \leq m \leq M} A_m + \max_{1 \leq m \leq M} B_m,
\]

with

\[
A_m := \left( \sum_{\ell=1}^m \int_{\Delta \tau} \|S^m_{\Delta \tau} - S(\tau_{\ell-1})\|_{H^1}^2 d\tau \right)^{1/2},
\]

\[
B_m := \left( \sum_{\ell=1}^m \int_{\Delta \tau} \|S(\tau_{\ell-1}) - S(\tau_{\ell-1})\|_{H^1}^2 d\tau \right)^{1/2}.
\]

Let \(\delta \in \left[0, \frac{1}{2}\right]\). Then, using \((4.23)\) and \((4.27)\), we have

\[
A_m = \left[ \sum_{k=1}^\infty \left( \Delta \tau \sum_{\ell=1}^m \|S^m_{\Delta \tau} - S(\tau_{\ell-1})\|_{H^1}^2 \right)^{1/2} \right]^{1/2}
\]

\[
= \left[ \sum_{k=1}^\infty \left( \Delta \tau \sum_{\ell=1}^m \|S^m_{\Delta \tau} - S(\tau_{\ell})\|_{H^1}^2 \right)^{1/2} \right]^{1/2}
\]

\[
\leq C \Delta \tau^{1/2} \left( \sum_{k=1}^\infty \|\varepsilon_k\|_{H^{\delta-1}}^2 \right)^{1/2}
\]

\[
\leq C \Delta \tau^{1/2} \left( \sum_{k=1}^\infty \|\varepsilon_k\|_{H^{\delta-1}}^2 \right)^{1/2}
\]

\[
\leq C \Delta \tau^{1/2} \left( \sum_{k=1}^\infty \|\varepsilon_k\|_{H^{\delta-1}}^2 \right)^{1/2}
\]

\[
\leq C \left( \frac{1}{\Delta \tau} \right)^{1/2} \Delta \tau^{1/2}, \quad m = 1, \ldots, M,
\]
which, after setting $\epsilon = \frac{1}{4} - \frac{\delta}{2} \in (0, \frac{1}{4}]$, yields

$$\max_{1 \leq m \leq M} A_m \leq C \varepsilon^{-\frac{\gamma}{4}} \Delta \tau^{\frac{\gamma}{4}}. \quad (5.7)$$

Now, observing that $S(t)\varepsilon = e^{-\lambda_t^2 t} \varepsilon$ for $t \geq 0$ and $\kappa \in \mathbb{N}$, we obtain

$$B_m = \left( \sum_{\kappa=1}^{\infty} \sum_{\ell=1}^{m} \left( \int_{\Delta_t} \left[ e^{-\lambda_t^2 (\tau_{m-\ell-1})} - e^{-\lambda_t^2 (\tau_{m-\ell})} \right]^2 \varepsilon^2 (x) \, dx \right) \, d\tau \right)^{1/2}$$

$$= \left( \sum_{\kappa=1}^{\infty} \sum_{\ell=1}^{m} e^{-2\lambda_t^2 (\tau_{m-\ell})} \left( 1 - e^{-\lambda_t^2 (\tau_{m-\ell})} \right)^2 \, d\tau \right)^{1/2}$$

$$\leq \left( \sum_{\kappa=1}^{\infty} (1 - e^{-\lambda_t^2 \Delta \tau})^2 \int_{0}^{\tau_m} e^{2\lambda_t^2 (\tau - \tau_m)} \, d\tau \right)^{1/2}$$

$$\leq \left( \sum_{\kappa=1}^{\infty} (1 - e^{-2\lambda_t^2 \Delta \tau})^2 \right)^{1/2}, \quad m = 1, \ldots, M,$$

from which, applying (5.11), we obtain

$$\max_{1 \leq m \leq M} B_m \leq \Delta \tau^{\frac{\gamma}{2}}. \quad (5.8)$$

Finally, the estimate (5.14) follows easily combining (5.6), (5.7) and (5.8). \qed

5.2. Estimating the space discretization error.

**Lemma 5.1.** Let $f \in L^2(D)$ and $g_h, \psi_h, z_h \in Z_h$, such that

$$\psi_h + \frac{\Delta h}{2} \Delta_h \psi_h = P_h f \quad \text{and} \quad z_h = g_h - \frac{\Delta h}{2} \Delta_h g_h.$$

Then there exist functions $G_h, \tilde{G}_h \in C(\bar{D} \times \bar{D})$ such that

$$\psi_h(x) = \int_{D} G_h(x, y) f(y) \, dy \quad \forall \, x \in \bar{D}, \quad (5.10)$$

$$z_h(x) = \int_{D} \tilde{G}_h(x, y) g_h(y) \, dy \quad \forall \, x \in \bar{D}. \quad (5.11)$$

**Proof.** Let $\nu_h := \dim(Z_h^\perp)$ and $\gamma : Z_h^\perp \times Z_h^\perp \rightarrow \mathbb{R}$ be an inner product on $Z_h^\perp$ defined by $\gamma(\chi, \varphi) := \langle \partial \chi, \partial \varphi \rangle_{\partial \bar{D}}$ for $\chi, \varphi \in Z_h^\perp$. Then, we can construct a basis $\{\varphi_j\}_{j=1}^{\nu_h}$ of $Z_h^\perp$ which is $L^2(D)$—orthogonal, i.e., $\langle \varphi_i, \varphi_j \rangle_{\partial \bar{D}} = \delta_{ij}$ for $i, j = 1, \ldots, \nu_h$, and $\gamma$—orthogonal, i.e. there exist positive $\{\varepsilon_{h,j}\}_{j=1}^{\nu_h}$ such that $\gamma(\varphi_i, \varphi_j) = \varepsilon_{h,i} \delta_{ij}$ for $i, j = 1, \ldots, \nu_h$ (see Sect. 8.7 in [5]). Thus, there exist real numbers $\{\mu_j\}_{j=1}^{\nu_h}$ and $\{\mu_j\}_{j=1}^{\nu_h}$ such that $\psi_h = \sum_{j=1}^{\nu_h} \mu_j \varphi_j$ and $z_h = \sum_{j=1}^{\nu_h} \mu_j \varphi_j$. Then, (5.9) yields $\mu_\ell = \frac{1}{1 + \frac{h}{2} \varepsilon_{h,\ell}} (f, \varphi_\ell)_{\partial \bar{D}}$ and $\bar{\mu}_\ell = (1 - \frac{1}{4} \epsilon_{\varepsilon_{h,\ell}}) (g_h, \varphi_\ell)_{\partial \bar{D}}$ for $\ell = 1, \ldots, \nu_h$. Thus, we conclude (5.10) and (5.11) with $G_h(x, y) = \sum_{j=1}^{\nu_h} \frac{1}{1 + \frac{h}{2} \varepsilon_{h,j}} \varphi_j(x, y)$ and $\tilde{G}_h(x, y) = \sum_{j=1}^{\nu_h} (1 - \frac{1}{4} \epsilon_{\varepsilon_{h,j}}) \varphi_j(x, y)$ for $x, y \in \bar{D}$. \qed

**Proposition 5.2.** Let $(U_h^m)^M_{m=0}$ be the fully discrete approximations defined by (1.10), (1.11) and $(U^m)^M_{m=0}$ be the time discrete approximations defined by (5.1) and (5.2). Then, there exists a constant $C_{\text{HOR}} > 0$, independent of $\Delta t$, $\Delta x$, $\Delta \tau$, and $h$, such that

$$\max_{1 \leq m \leq M} E_h^m \leq C_{\text{HOR}} \left( \epsilon_1^{\frac{\gamma}{4}} h^{\frac{\gamma}{4} - \epsilon_1} + \epsilon_2^{\frac{\gamma}{4}} \Delta \tau^{\frac{\gamma}{4} - \epsilon_2} \right) \quad \forall \epsilon_1 \in (0, \frac{1}{2}], \quad \forall \epsilon_2 \in (0, \frac{1}{2}]. \quad (5.12)$$

**Proof.** In the sequel, we will use the symbol $C$ to denote a generic constant that is independent of $\Delta t$, $\Delta x$, $\Delta \tau$, and $h$, and may changes value from one line to the other.

Let $L^2(D) \rightarrow L^2(D)$ be the identity operator, $Y_h : Z_h^\perp \rightarrow Z_h^\perp$ be defined by $Y_h := I - \frac{\Delta h}{2} \Delta_h$ and $\Lambda_h : L^2(D) \rightarrow Z_h^\perp$ be the inverse discrete elliptic operator given by $\Lambda_h := (I + \frac{\Delta h}{2} \Delta_h)^{-1} \circ P_h$. Also, for $m = 1, \ldots, M$, we define a discrete operator $Q_h^m : L^2(D) \rightarrow Z_h^\perp$ by $Q_h^m := (\Lambda_h \circ Y_h)^{m-1} \circ \Lambda_h$, where
which has a Green’s function \( G_{Q^m} \) (see Lemma 5.1). Also, for given \( v_0 \in \dot{H}^1(D) \), let \( (S^m_h(v_0))_{m=0}^M \) be fully discrete discrete approximations of the solution to the deterministic problem (1.3), defined by (4.35)–(4.37). Then, using a simple induction argument, we conclude that \( S^m_h(v_0) = Q^m_h(v_0) \) for \( m = 1, \ldots, M \).

Applied an induction argument on (1.11), we obtain

\[
U^m_h(x) = \sum_{\ell=1}^m \int_\mathbb{D} Q^m_{h-\ell+1} W(\tau, x) \, d\tau
\]

(5.13)

where \( K^m_h(\tau; x, y) := \sum_{\ell=1}^m K^m_{h-\ell+1}(x, y) \) for all \( \tau \in [0, T] \) and \( x, y \in \mathbb{D} \). Using (5.13), (5.9), (2.10), (2.5) and (2.11), we get

\[
E^m_{SDF} \leq \left( \int_0^T \left( \int_D \int_D (K^m_h(\tau; x, y) - K^m_h(\tau; x, y))^2 \, dy \, dx \right) \, d\tau \right)^{1/2} \leq \left( \Delta \tau \sum_{\ell=1}^m \| S^\ell_{\Delta \tau} - S^\ell_{h} \|_{H^1}^2 \right)^{1/2}.
\]

Let \( \delta \in [0, \frac{1}{2}] \) and \( \theta \in [0, \frac{1}{4}] \). Using (4.53), we obtain

\[
(E^m_{SDF})^2 \leq \sum_{k=1}^M \left( \Delta \tau \sum_{\ell=1}^m \| S^\ell_{\Delta \tau}(\varepsilon_k) - S^\ell_{h}(\varepsilon_k) \|_{H^1}^2 \right) \leq C \left( h^{4\theta} \sum_{k=1}^M \| \varepsilon_k \|_{H^{2\theta-1}}^2 + \Delta \tau^\delta \sum_{k=1}^M \| \varepsilon_k \|_{H^{1-\delta}}^2 \right) \leq C \left( h^{4\theta} \sum_{k=1}^M \lambda_k^{\frac{1}{1+2(1-2\theta)}} + \Delta \tau^\delta \sum_{k=1}^M \lambda_k^{\frac{1}{1+4(1-\theta)}} \right),
\]

which, after using (2.7), yields

\[
E^m_{SDF} \leq C \left( h^{4\theta} \left( \frac{1}{2} - 2\theta \right)^{-\frac{1}{2}} + \Delta \tau^\delta \left( \frac{1}{4} - \delta \right)^{-\frac{1}{2}} \right).
\]

(5.14)

and setting \( \epsilon_1 = \frac{1}{2} - 2\theta \in (0, \frac{1}{2}] \) and \( \epsilon_2 = \frac{1}{4} - \delta \in (0, \frac{1}{4}] \) we, easily, arrive at (5.12). \( \square \)

5.3. Estimating the discretization error.

**Theorem 5.3.** Let \( \hat{u} \) be the solution to the problem (1.6), and \( (U^m_h)_{m=0}^M \) be the Crank-Nicolson fully-discrete approximations of \( \hat{u} \) defined by (1.10)–(1.11). Then, there exists a constant \( C > 0 \), independent of \( \Delta x, \Delta t, h, \) and \( \Delta \tau \), such that

\[
\max_{0 \leq m \leq M} \mathbb{E} \left[ \left\| U^m_h - \hat{u}(\tau_m, \cdot) \right\|_{H^{1\theta}}^2 \right] \leq C \left( \epsilon_1^{\frac{1}{2}} h^{\frac{1}{2} - \epsilon_1} + \epsilon_2^{\frac{1}{2}} \Delta \tau^{\frac{1}{4} - \epsilon_2} \right)
\]

(5.15)

for all \( \epsilon_1 \in (0, \frac{1}{2}] \) and \( \epsilon_2 \in (0, \frac{1}{4}] \).

**Proof.** The error estimate (5.15) is a simple consequence of (5.3), (5.4) and (5.12). \( \square \)

**Remark 5.1.** For an optimal, logarithmic-type choice of the parameter \( \epsilon \) in (3.1) and of the parameters \( \epsilon_1 \) and \( \epsilon_2 \) in (5.15), we refer the reader to the discussion in Remark 3 of [13].
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