HODGE CLASSES OF TYPE (2, 2) ON HILBERT SQUARES OF PROJECTIVE K3 SURFACES

SIMONE NOVARIO

Abstract. We give a basis for the vector space generated by rational Hodge classes of type (2, 2) on the Hilbert square of a projective K3 surface general in its rank, which is a subspace of the singular cohomology ring with rational coefficients: we use Nakajima operators and an algebraic model developed by Lehmann and Sorger as main tools. We then obtain a basis of the lattice generated by integral Hodge classes of type (2, 2) on the Hilbert square of a projective K3 surface general in its rank: we exploit lattice theory, a theorem by Qin and Wang and a result by Ellingsrud, Göttsche and Lehmann.

1. Introduction

Irreducible holomorphic symplectic (IHS) manifolds are a generalization to higher dimensions of K3 surfaces. The interest in these varieties has been increasing thanks to the Beauville–Bogomolov decomposition theorem: up to a finite étale cover, any compact Kähler manifold with trivial first Chern class is the product of a complex torus, Calabi–Yau manifolds and IHS manifolds, see [Bea83, Théorème 2]. In this paper we deal with Hilbert schemes of 2-points on a K3 surface, also known as Hilbert squares of K3 surfaces, the first example of IHS manifold other than K3 surfaces to be found, see [Fuj87]: if $S$ is a K3 surface, we denote by $S^{[2]}$ its Hilbert square. There exists a quadratic form on the second cohomology group $H^2(S^{[2]}, \mathbb{Z})$, called Beauville–Bogomolov–Fujiki (BBF) form and denoted by $q_{S^{[2]}}$, so that $(H^2(S^{[2]}, \mathbb{Z}), q_{S^{[2]}})$ is a lattice.

If $S$ is a K3 surface, we denote by $T(S) = NS(S)^{\perp}$ the transcendental lattice of $S$, where $NS(S)$ is the Néron–Severi group of $S$ and the orthogonal is taken with respect to the intersection product. Moreover, $T(S)_\mathbb{Q} = T(S) \otimes \mathbb{Q}$ is a rational Hodge structure of weight 2, and we denote by

$$E_S := \text{Hom}_0(T(S)_\mathbb{Q}, T(S)_\mathbb{Q})$$

the algebra of endomorphisms on $T(S)_\mathbb{Q}$ of weight 0. See [Huy16, §3] for details on Hodge structures. Zarhin showed in [Zar83] that $E_S$ is either $\mathbb{Q}$, or a totally real field or a CM field. We say that $S$ is general in its rank, or simply general, if $E_S \cong \mathbb{Q}$. As remarked in [vG08], see also [EJ16, §1.1], K3 surfaces which are not general gives a locus of positive codimension in the analytic moduli space of K3 surfaces of fixed Picard rank, unless the Picard rank is 20, in which case $E_S$ is always a CM field.

In this paper we study the $\mathbb{Q}$-vector space of rational Hodge classes of type (2, 2) and the lattice of integral Hodge classes of type (2, 2) on the Hilbert square of a general projective K3 surface $S$, defined respectively as

$$H^{2,2}(S^{[2]}, \mathbb{Q}) := H^4(S^{[2]}, \mathbb{Q}) \cap H^{2,2}(S^{[2]}), \quad H^{2,2}(S^{[2]}, \mathbb{Z}) := H^4(S^{[2]}, \mathbb{Z}) \cap H^{2,2}(S^{[2]}).$$

The integral bilinear form considered on $H^{2,2}(S^{[2]}, \mathbb{Z})$ is the cup product. Hodge classes are usually studied in the context of the so-called Hodge conjecture: this states that given a smooth complex projective variety $Y$, the subspace of $H^{2k}(Y, \mathbb{Q})$ generated by algebraic cycles, i.e., classes which are obtained as fundamental cohomological classes $[Z]$ of subvarieties $Z \subset Y$, coincides with $H^{k,k}(Y, \mathbb{Q})$, which is by definition the set $H^{2k}(Y, \mathbb{Q}) \cap H^{k,k}(Y)$ of rational Hodge classes of type $(k,k)$. Similarly one defines integral Hodge classes of type $(k,k)$ as the elements which belong to the set $H^{k,k}(Y, \mathbb{Z}) := H^{2k}(Y, \mathbb{Z}) \cap H^{k,k}(Y)$. We now present the main result of this paper, which gives a basis of the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$, where $S$ is a general projective K3 surface whose Picard group is known, cf. Theorem 8.3.

Theorem. Let $S$ be a general projective K3 surface and let $\{b_1, \ldots, b_r\}$ be a basis of $\text{Pic}(S)$. Then the lattice $H^{2,2}(S^{[2]}, \mathbb{Z})$ is odd. Moreover, $\text{rk}(H^{2,2}(S^{[2]}, \mathbb{Z})) = \frac{(r+1)r}{2} + r + 2$, and a basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$ is the following:

$$\left\{ b_i b_j, \frac{b_i^2 - b_i \delta}{2}, \frac{1}{8} \left( \delta^2 + \frac{1}{3} c_2(S^{[2]}) \right), \delta^2 \right\}_{1 \leq i \leq j \leq r}, \quad (1.1)$$

where $c_2(S^{[2]}) \in H^{2,2}(S^{[2]}, \mathbb{Z})$ is the second Chern class of $S^{[2]}$. 

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Clearly the elements in (1.1) give a basis also for the $\mathbb{Q}$-vector space $H^{2,2}(S^{[2]}, \mathbb{Q})$: a basis in terms of Nakajima operators is described in Theorem 6.2. Topics discussed in this paper are related to [She16], where Shen studies Hyperkähler manifolds of Jacobian type.

The paper is organised as follows. In Section 2 we recall some notions of lattice theory. In Section 3 we give the definition of IHS manifold, together with general properties. In Section 4 we present some useful results which hold for an IHS manifold $X$ of $K3^{[2]}$-type, in particular the relation between the BBF bilinear form and the cup product between elements in $H^4(X, \mathbb{Q})$, and the definition of the dual $q_X^\ast \in H^{2,2}(X, \mathbb{Q})$ of the BBF quadratic form. In Section 5 we introduce Nakajima operators, following [Nak97] and [Leh99], and the algebraic model developed by Lehn and Sorger in [LS03]. In Section 6, using Nakajima operators and the model of Lehn–Sorger, we prove Theorem 6.2, which gives a basis of the $\mathbb{Q}$-vector space of rational Hodge classes of type $(2, 2)$ on the Hilbert square of a general projective $K3$ surface whose Picard group is known: the basis is presented in terms of Nakajima operators. In Section 7, using a result by Ellingsrud, Göttsche and Lehn in [EGL01], we obtain an explicit description of the second Chern class of the Hilbert square of a projective $K3$ surface depending on Nakajima operators, cf. Proposition 7.4. We use this description in Section 8, where we pass to study integral Hodge classes of type $(2, 2)$ on the Hilbert square of a general projective $K3$ surface. The main result of this section is Theorem 8.3, which describes a basis for $H^{2,2}(S^{[2]}, \mathbb{Z})$ for a general projective $K3$ surface $S$: the strategy of the proof is to combine Proposition 7.4 with a result by Qin and Wang in [QW05], together with the property that the lattice $H^2(S, \mathbb{Z})$ is unimodular.

A generic $K3$ surface of degree $2t$ is a general projective $K3$ surface $S_{2t}$ whose Picard group is generated by the class of an ample divisor $H$ with $H^2 = 2t$ with respect to the intersection form. This paper is based on Chapter 3 of the author’s PhD thesis, where Theorem 8.3 is used to show that Hilbert squares of general $K3$ surfaces of degree $2t$, with $t \neq 2$, admitting an ample divisor $D$ with $q_X(D) = 2$ are double EPW sextics, see [O’G06] for the definition, where $q_X$ represents the BBF quadratic form on $X$. Proofs and details that will be omitted in this paper can be found in the author’s PhD thesis, see [Nov].

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2. Lattices

In this section we recall the most important definitions and results of lattice theory that we need. General references are [Nik80] and [CS13].

**Definition 2.1.** A lattice $L$ is a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear form $b : L \times L \to \mathbb{Z}$. We denote by $q : L \to \mathbb{Z}$ the quadratic form defined by $q(x) := b(x, x)$ for every $x \in L$.

If $L$ is a lattice of rank $n$ and $B := \{e_1, \ldots, e_n\}$ is a $\mathbb{Z}$-basis of $L$, we call Gram matrix of $L$ associated to $B$ the following symmetric matrix:

$$
\begin{pmatrix}
    b(e_1, e_1) & \cdots & b(e_1, e_n) \\
    \vdots & \ddots & \vdots \\
    b(e_n, e_1) & \cdots & b(e_n, e_n)
\end{pmatrix}.
$$

A non-degenerate lattice is a lattice $L$ of rank $n$ such that for any non-zero $l \in L$ there exists $l' \in L$ such that $b(l, l') \neq 0$, equivalently, $\det(G) \neq 0$ if $G$ is a Gram matrix of $L$. A lattice $L$ is even if $b(l, l) \in 2\mathbb{Z}$ for every $l \in L$, and odd if it is not even. A sublattice of a lattice $L$ is a free submodule $L' \subseteq L$ with symmetric bilinear form $b' := b|_{L' \times L'}$. A sublattice $L' \subseteq L$ is primitive if $L/L'$ is a free module. The direct sum of two lattices $L_1$ and $L_2$ is by definition the lattice $L_1 \oplus L_2$ whose bilinear form is $b(v_1 + v_2, w_1 + w_2) := b_1(v_1, w_1) + b_2(v_2, w_2)$ for every $v_1, w_1 \in L_1$ and $v_2, w_2 \in L_2$, where $b_1$ and $b_2$ are the bilinear forms of $L_1$ and $L_2$ respectively.

For a lattice $L$ of rank $n$ we write $L_{R} := L \otimes_{\mathbb{Z}} \mathbb{R}$ and we extend $\mathbb{R}$-bilinearly the bilinear form $b$ to $L_{R}$, similarly we extend $q$ to $L_{R}$. If the lattice is non-degenerate, the signature of $L$ is the signature $(l_{(+)}, l_{(-)})$ of the quadratic form on $L_{R}$. A non-degenerate lattice is positive definite if $l_{(-)} = 0$, similarly it is negative definite if $l_{(+)} = 0$, while it is indefinite if $l_{(+)}, l_{(-)} \neq 0$.

The determinant of a lattice $L$ is the determinant of a Gram matrix $G$ of the lattice, and the discriminant of $L$ is $\det(L) := |\det(G)|$. A unimodular lattice is a lattice $L$ such that $\det(L) = 1$: if $L$ is a unimodular lattice, for every
x ∈ L there exists y ∈ L such that b(x, y) = 1. Let k be an integer: the easiest example of lattice is ⟨k⟩, which is the rank one lattice L = ℤe with bilinear form b(e, e) = k. Other two important examples of lattices are the hyperbolic lattice, which is denoted by U, and E₈(−1): the former is the unique unimodular lattice of rank 2 and signature (1, 1), the latter is an even unimodular lattice of signature (0, 8). The following two matrices are Gram matrices respectively for U and E₈(−1):

\[
\begin{pmatrix}
  -2 & 1 \\
  1 & -2 & 1 \\
  1 & -2 & 1 & 1 \\
  1 & -2 & 1 \\
  1 & -2 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
  1 & -2 \\
  1 & -2 \\
  1 & -2 \\
  1 & -2 \\
\end{pmatrix}.
\]

(2.1)

A morphism of lattices φ : L → L’ between two lattices L and L’ whose bilinear forms are respectively b and b’ is a morphism of ℤ-modules such that for every l₁, l₂ ∈ L we have b(l₁, l₂) = b’(φ(l₁), φ(l₂)). Morphisms between two non-degenerate lattices are necessarily injective. We call isometry a bijective morphism of lattices. A lattice L embeds primitively in a lattice L’ if there is a morphism φ : L → L’ such that φ(L) is a primitive sublattice of L’.

3. Generalities on IHS manifolds

In this section we recall basics on irreducible holomorphic symplectic manifolds.

Definition 3.1. An irreducible holomorphic symplectic (IHS) manifold is a simply connected compact complex Kähler manifold X such that \(H^0(X, \Omega^2_X)\) is generated by a non-degenerate holomorphic 2-form, called symplectic form.

The dimension of an IHS manifold is necessarily even as a consequence of the existence of a symplectic form. The Enriques–Kodaira classification of compact complex surfaces shows that the only IHS manifolds of dimension 2 are K3 surfaces. If X is an IHS manifold, then the ℂ-vector space \(H^0(X, \Omega^2_X)\) is zero if p is odd, and it is generated by \(σ^2\) if 0 ≤ p ≤ dim(X) is even, where σ is a symplectic form, see [Bea83, Proposition 3]. The Picard group Pic(X) is isomorphic to the Néron–Severi group NS(X) = \(H^{1,1}(X)_{\mathbb{R}} \cap H^2(X, \mathbb{Z})\), and this embeds in the second cohomology group \(H^2(X, \mathbb{Z})\). By the universal coefficient theorem, the second singular cohomology group \(H^2(X, \mathbb{Z})\) is torsion free. Moreover, by a result due to Beauville, Bogomolov and Fujiki, \(H^2(X, \mathbb{Z})\) can be equipped with a non-degenerate integral quadratic form, denoted by \(q_X\) and called Beauville–Bogomolov–Fujiki (BBF) form, see [Bea83] and [Fuj87] for details: in particular \((H^2(X, \mathbb{Z}), q_X)\) has a structure of even lattice of signature \((3, b_2(X) - 3)\), where \(b_2(X)\) is the second Betti number of X. For K3 surfaces the BBF form coincides with the intersection form. The transcendental lattice of an IHS manifold X is defined as \(T(X) := (\text{NS}(X))^{\perp}\): the orthogonal is taken with respect to the BBF form.

Let S be a K3 surface: we denote by \(S^{[n]}\) the Hilbert scheme of \(n\) points on S, which is the scheme which parametrizes zero-dimensional closed subschemes of length \(n\) on S. By [Bea83, Théorème 3] the variety \(S^{[n]}\) is an IHS manifold of dimension \(2n\). If \(S^{[n]}\) is the quotient of \(S^n = S \times \cdots \times S\) by the symmetric group of \(n\) elements, the morphism \(\rho : S^{[n]} \to S^n\) defined by \(\rho([x]) = \sum x_i \cdot \mathcal{O}_{S^n}(x)\) is called Hilbert–Chow morphism, see for instance [Ive06]. In particular \(\rho\) is a desingularisation of \(S^{[n]}\), whose singular locus is the so-called diagonal, i.e., the set of cycles \(p_1 + \cdots + p_n\) such that there exist \(i\) and \(j\) with \(i \neq j\) and \(p_i = p_j\). The pre-image of the diagonal in \(S^{[n]}\) is an irreducible divisor \(E\) on \(S^{[n]}\), and there exists a primitive class \(δ \in \text{Pic}(S^{[n]})\) such that \(2δ = [E]\). An important fact is the following: there exists a primitive embedding of lattices

\[i : H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[n]}, \mathbb{Z})\]

such that \(H^2(S^{[n]}, \mathbb{Z}) = i(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}δ\), and \(q_{S^{[n]}}(δ) = -2(n-1)\). For a K3 surface S there exists an isometry of lattices \(H^2(S, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}\), see for instance [BHPVdV15, §VII.3], in particular \(H^2(S, \mathbb{Z})\) is an even unimodular lattice of signature \((3, 19)\). Hence there is an isometry of lattices

\[H^2(S^{[n]}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1)\rangle,\]

similarly Pic(S^{[n]}) = i(Pic(S)) \oplus \mathbb{Z}δ: see [Bea83, §6] for details. Moreover, the singular cohomology ring \(H^*(S^{[n]}, \mathbb{Z})\) for a K3 surface S and \(n \geq 1\) is torsion free by [Mar07, Theorem 1]. When \(n = 2\), we call \(S^{[2]}\) the Hilbert square of S. An IHS manifold which is deformation equivalent to the Hilbert square of a K3 surface is said to be of \(K3^{[2]}\)-type.
The other known examples of IHS manifolds up to deformation equivalence are generalised Kummer varieties, see [Bea83, §7], an isolated example of dimension 10 and second Betti number $b_2 = 24$, see [O’G99], and an isolated example of dimension 6 and second Betti number $b_2 = 8$, see [O’G08]. We do not discuss details on these examples since in this paper we deal only with Hilbert squares of K3 surfaces.

4. Generalities on IHS manifolds of $K^3$-[2]-type

Let $X$ be an IHS manifold of dimension 4 of $K^3$-[2]-type. In this section we recall some useful properties from [O’G08, §2], in particular the link between the intersection pairing on $H^4(X, \mathbb{Q})$ and the BBF form on $H^2(X, \mathbb{Q})$.

First of all, we state the following corollary of Verbitsky’s results in [Ver96], obtained by Guan in [Gua01], see also [O’G10, Corollary 2.5]. We denote by $b_i(X)$ the $i$-th Betti number of $X$.

**Proposition 4.1.** Let $X$ be an IHS manifold of dimension 4. Then $b_2(X) \leq 23$. If equality holds then $b_3(X) = 0$ and the map

$$\text{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q})$$

induced by the cup product is an isomorphism. In particular this happens when $X$ is an IHS fourfold of $K^3$-[2]-type.

We denote by $\langle \cdot, \cdot \rangle$ the intersection pairing on the singular cohomology group $H^4(X, \mathbb{Z})$ induced by the cup product and defined by $\langle \alpha, \beta \rangle := \int_X \alpha \beta$. We use the same notation $\langle \cdot, \cdot \rangle$ for the $\mathbb{Q}$-bilinear extension of the intersection pairing above to $H^4(X, \mathbb{Q})$, obtaining a $\mathbb{Q}$-valued intersection pairing on $\text{Sym}^2 H^2(X, \mathbb{Q})$. Let $X$ be an IHS manifold of $K^3$-[2]-type: then the intersection pairing $\langle \cdot, \cdot \rangle$ can be described in terms of the $\mathbb{Q}$-extension of the BBF bilinear form on $H^2(X, \mathbb{Q})$ thanks to the following Proposition, see [O’G08, Remark 2.1].

**Proposition 4.2** (O’Grady). Let $X$ be an IHS fourfold of $K^3$-[2]-type. The intersection pairing $\langle \cdot, \cdot \rangle$ defined above coincides with the bilinear form on $\text{Sym}^2 H^2(X, \mathbb{Q})$ given by

$$\langle \alpha_1 \alpha_2, \alpha_3 \alpha_4 \rangle = \langle \alpha_1, \alpha_2 \rangle \langle \alpha_3, \alpha_4 \rangle + \langle \alpha_1, \alpha_3 \rangle \langle \alpha_2, \alpha_4 \rangle + \langle \alpha_1, \alpha_4 \rangle \langle \alpha_2, \alpha_3 \rangle$$

for every $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^2(X, \mathbb{Q})$, where $\langle \cdot, \cdot \rangle$ denotes the BBF bilinear form on $H^2(X, \mathbb{Q})$.

Let $q_X$ be the BBF quadratic form on $X$ and $\{e_1, \ldots, e_{23}\}$ be a basis of $H^2(X, \mathbb{Q})$. Let $\{e_1^\vee, \ldots, e_{23}^\vee\}$ be the dual basis in $H^2(X, \mathbb{Q})^\vee$, i.e., $e_i^\vee(e_j) = \delta_{i,j}$. Then we have

$$q_X = \sum_{i,j} g_{i,j} e_i^\vee \otimes e_j^\vee, \quad q_X^\vee = \sum_{i,j} m_{i,j} e_i e_j,$$

where $g_{i,j} := (e_i, e_j)$, the matrix $(g_{i,j})$ is symmetric and $(m_{i,j}) = (g_{i,j})^{-1}$. As shown in [O’G08, Proposition 2.2], the products $\langle q_X^\vee, \alpha \rangle$ for every $\alpha \in H^4(X, \mathbb{Q})$ can be computed exploiting the BBF bilinear form on $H^2(X, \mathbb{Q})$.

**Proposition 4.3** (O’Grady). Let $X$ be an IHS fourfold of $K^3$-[2]-type. Let $\langle \cdot, \cdot \rangle$ be the bilinear form described in Proposition 4.2. Then $\langle \cdot, \cdot \rangle$ is non-degenerate and

$$\langle q_X^\vee, \alpha \beta \rangle = 25 \langle \alpha, \beta \rangle \quad \text{for all } \alpha, \beta \in H^2(X, \mathbb{Q}),$$

$$\langle q_X^\vee, q_X^\vee \rangle = 23 \cdot 25.$$

O’Grady has shown in [O’G08, §3] that $q_X^\vee$ is a rational multiple of $c_2(X)$, the second Chern class of the tangent bundle of $X$, in particular it is an element of $H^{2,2}(X, \mathbb{Q})$, i.e., it is a rational Hodge class of type $(2, 2)$.

**Proposition 4.4** (O’Grady). Let $X$ be an IHS fourfold of $K^3$-[2]-type. Then $q_X^\vee \in H^{2,2}(X, \mathbb{Q})$, i.e., $q_X^\vee$ is a rational Hodge class of $X$ of type $(2, 2)$, and

$$\frac{6}{5} q_X^\vee = c_2(X) \in H^{2,2}(X, \mathbb{Z}).$$

Moreover, $\frac{2}{5} q_X^\vee \in H^{2,2}(X, \mathbb{Z})$ is an integral Hodge class of $X$ of type $(2, 2)$.
5. Nakajima operators and the Lehn–Sorger model

In this section we introduce Nakajima operators following [Nak97] and [Leh99], and the algebraic model developed by Lehn and Sorger in [LS03].

Let $S$ be a smooth complex projective surface and let $S^{[n]}$ be the Hilbert scheme of $n$ points on $S$ for any integer $n > 0$. We define

$$H^S_n := \bigoplus_{n=0}^{4n} H^1(S^{[n]}, \mathbb{Q}), \quad H^n := \bigoplus_{n \geq 0} H^S_n.$$  

The unit of the $\mathbb{Q}$-algebra, with cup product, $H^S_n \otimes \mathbb{Q}$ is called the vacuum vector and it is denoted by $|0\rangle$. If $f : S^{[n]}$ denotes the unit for the cup product in $H^S_n$ for every $n$, the unit in $H^n$ for the cup product is given by $|1\rangle := \sum_{n \geq 0} 1_{S^{[n]}}$. The space $H^n$ is double graded by $(n, i)$: we say that $n$ is the conformal weight and $i$ is the cohomological degree, denoted by $| \cdot |$. Let $f \in \text{End}(H^n)$ be a linear operator. We say that $f$ is homogeneous of bidegree $(\nu, \iota)$ if for any $n$ and $i$ we have $f(H^1(S^{[n]}, \mathbb{Q})) \subset H^{i+n}(S^{[n+k]}, \mathbb{Q})$. The commutator of two homogeneous operators $f, g \in \text{End}(H^n)$ is defined by

$$[f, g] := f \circ g - (-1)^{|f| \cdot |g|} g \circ f.$$  

We now define an intersection pairing $\langle \cdot, \cdot \rangle$ on $H^n$. First of all, fix an integer $n \geq 0$ and let $\alpha, \beta \in H^n$: we set

$$\langle \alpha, \beta \rangle := \int_{S^{[n]}} \alpha \beta \quad (5.1)$$

Note that $\langle \alpha, \beta \rangle = 0$ if $|\alpha| + |\beta| \neq 4n$. Then $\langle \cdot, \cdot \rangle$ extends naturally to a non-degenerate graded symmetric bilinear form on $H^n$, which we denote again by $\langle \cdot, \cdot \rangle$. If $f \in \text{End}(H^n)$ is a homogeneous operator, we define the adjoint operator $f^!$ as the homogeneous operator characterised by the relation

$$\langle f^!(\alpha), \beta \rangle = (-1)^{|f| \cdot |\beta|} \langle \alpha, f(\beta) \rangle.$$  

Following [Leh99, §1.2], we define an irreducible subvariety $S^{[n,n+k]}$ of $S^{[n]} \times S \times S^{[n+k]}$ for any integers $n \geq 0, k > 0$. Let $X^{[n,n+k]} \subset S^{[n]} \times S^{[n+k]}$ be the uniquely determined closed subscheme with the property that any morphism $f = (f_1, f_2) : T \rightarrow S^{[n]} \times S^{[n+k]}$ from an arbitrary variety $T$ factors through $X^{[n,n+k]}$ if and only if the following holds:

$$(f_1 \times \text{id}_S)^{-1}(\Xi^S_n) \subset (f_2 \times \text{id}_S)^{-1}(\Xi^S_{n+k}),$$

where $\Xi^S_n \subset S^{[n]} \times S$ is the universal family of subschemes parametrised by $S^{[n]}$. Closed points in $X^{[n,n+k]}$ correspond to pairs $(\xi, \xi')$ of subschemes with $\xi \subseteq \xi'$. Then one obtains a morphism similar to the Hilbert–Chow morphism:

$$\rho : X^{[n,n+k]} \rightarrow \text{Sym}^k S.$$  

We set $X^{[n,n+k]}_0 := \rho^{-1}(\Delta)$, where $\Delta \subset \text{Sym}^k S$ is the small diagonal, and we consider on $X^{[n,n+k]}_0$ the reduced induced subscheme structure. Set-theoretically we can identify $X^{[n,n+k]}_0$ with the following subset of $S^{[n]} \times S \times S^{[n+k]}$:

$$X^{[n,n+k]}_0 := \{ (\xi, x, \xi') \mid \xi \subseteq \xi' \text{ and } \text{Supp}(\mathcal{I}_\xi/\mathcal{I}_{\xi'}) = x \},$$

where $\mathcal{I}_\xi$ is the ideal sheaf of $\xi$. We define $S^{[n,n+k]}$ as

$$S^{[n,n+k]} := \{ (\xi, x, \xi') \in X^{[n,n+k]}_0 \mid \text{Proj}(\mathcal{I}_\xi/\mathcal{I}_{\xi'}) = x \},$$

where the closure is the Zariski closure. Then $S^{[n,n+k]}$ is an irreducible subvariety of $S^{[n]} \times S \times S^{[n+k]}$ of complex dimension $2n + k + 1$ by [Leh99, Lemma 1.1]. Consider the following diagram

$$\begin{array}{ccc}
S^{[n]} & \xrightarrow{\varphi} & S^{[n,n+k]} \\
\downarrow \rho \downarrow & & \downarrow \psi \\
S & \xrightarrow{\psi} & S^{[n+k]} \\
\end{array}$$

where $\varphi, \rho$ and $\psi$ are the projections respectively on $S^{[n]}$, $S$ and $S^{[n+k]}$.  

**Definition 5.1.** Let $S$ be a smooth complex projective surface. The Nakajima creation operators, known also as Heisenberg operators, are defined as follows: for $\alpha \in H^*(S, \mathbb{Q})$ and $x \in H^*(S^{[n]}, \mathbb{Q})$ we set for $k > 0$

$$q_k : H^*(S, \mathbb{Q}) \rightarrow \text{End}(H^S), \quad q_k(\alpha)(x) := \psi_* \left( PD^{-1} \left[ S^{[n,n+k]} \right] \cdot \varphi^*(x) \cdot \rho^!(\alpha) \right),$$  

where $[S^{[n,n+k]}] \in H_{4n+2k+2}(S^{[n]} \times S \times S^{[n+k]}, \mathbb{Q})$ is the fundamental class of $S^{[n,n+k]}$, we denote by $PD$ the Poincaré duality, the dot is the cup product, and $\psi_*$ is the Gysin homomorphism, i.e., $\psi_* = PD^{-1} \psi_* PD$, where $\psi_*$ in the
right-hand side is the pushforward in homology. By convention \( q_0 = 0 \), and the Nakajima annihilation operators are defined as

\[
q_k(\alpha) := (-1)^k q_{-k}(\alpha)^\dagger \quad \text{for all } k < 0.
\]

Note that the unit \( |1\rangle \in \mathbb{H}^S \) can be represented as \( |1\rangle = \sum_{n \geq 0} 1_{S^n} = \exp(q_1(1_S))|0\rangle \), which gives

\[
1_{S^n} = \frac{1}{n!} q_1(1^n)|0\rangle.
\] (5.4)

The following commutation formula was obtained by Nakajima in [Nak97].

**Theorem 5.2** (Nakajima). The operators \( q_i \) satisfy the following commutation formula:

\[
[q_i(\alpha), q_j(\beta)] = i \cdot \delta_{i+j,0} \cdot \int_S \alpha \beta \cdot \text{id}_{\mathbb{H}^S},
\]

where \( \delta \) is the Kronecker delta.

We now introduce the boundary operator. Fix a positive integer \( n > 0 \) and let \( \lambda = \{ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0 \} \) be a partition of \( n \), i.e., an \( s \)-uple of ordered positive integers such that \( \lambda_1 + \lambda_2 + \cdots + \lambda_s = n \). Consider the Hilbert–Chow morphism \( \rho : S^{[n]} \to \text{Sym}^n S \). We set

\[
\text{Sym}^n S := \{ \alpha \in \text{Sym}^n S \mid \alpha = \sum_{1 \leq i \leq s} \lambda_i x_i, \ x_i \in S \text{ pairwise distinct} \}
\]

for a fixed partition \( \lambda \). By a theorem of Briançon \( S_0^{[n]} := \rho^{-1}(\text{Sym}^n S) \) is irreducible of dimension \( n + s \), see [Bri77]. For example, \( S_0^{[1,1,1,1]} \) is the open subset of \( S_0^{[n]} \) which corresponds to the configuration space of unordered \( n \)-tuples of pairwise distinct points: it is the only stratum which is open. The only stratum of codimension 1 is \( S_0^{[2,1,1,1]} \).

**Definition 5.3.** The boundary of \( S^{[n]} \) for \( n \geq 2 \) is the irreducible divisor

\[
\partial S^{[n]} := \bigcup_{\lambda \neq (1,1,\ldots,1)} S^{[n]}_{\lambda} = S_0^{[2,1,1,1]},
\]

**Remark 5.4.** For \( n = 1 \) the boundary \( \partial S^{[n]} \) is the empty set.

There is a description of the divisor \( \partial S^{[n]} \) in sheaf theoretic terms. Consider the universal family of subschemes \( \Xi_n \subset S^{[n]} \times S \). Let \( p : \Xi_n \to S^{[n]} \) be the projection on the first factor. Since \( p \) is a flat morphism of finite degree \( n \), we have that \( \mathcal{O}_{\Xi_n} := p_*(\mathcal{O}_{\Xi_n}) \) is a locally free sheaf of rank \( n \). Then \( \partial S^{[n]} = -2c_1(\mathcal{O}_{\Xi_n}^{[n]}) \) by [Leh99, Lemma 3.7].

We can now define the boundary operator and the derivative of a linear operator \( f \in \text{End}(\mathbb{H}^S) \).

**Definition 5.5.** The boundary operator \( \partial : \mathbb{H}^S \to \mathbb{H}^S \) is the homogeneous linear map of bidegree \((0,2)\) given by

\[
\partial(x) := c_1(\mathcal{O}_{S}^{[n]}) \cdot x = -\frac{1}{2} (\partial S^{[n]}) \cdot x \quad \text{for all } x \in H^*(S^{[n]}, \mathbb{Q}).
\]

For any endomorphism \( f \in \text{End}(\mathbb{H}^S) \) we define the derivative of \( f \) as

\[
f' := [\partial, f] = \partial \circ f - f \circ \partial.
\]

We denote by \( f^{(n)} \) the higher derivatives.

The following relation was obtained by Lehn, see [Leh99, Theorem 3.10, Proposition 3.12].

**Theorem 5.6** (Lehn). Let \( S \) be a smooth complex projective surface. Then for any integers \( n, m \) such that \( n + m \neq 0 \) and cohomology classes \( \alpha, \beta \in H^*(S, \mathbb{Q}) \) we have

\[
[q_n(\alpha), q_m(\beta)] = -nm \cdot q_{n+m}(\alpha \beta).
\]

We now state the Ellingsrud–Göttsche–Lehn formula. Let \( \Xi_n \subset S^{[n]} \times S \) be the universal family of subschemes parametrised by \( S^{[n]} \). By [Leh99, Theorem 1.9] we have that \( S^{[n,n+1]} \) is a smooth irreducible variety isomorphic to
the blow-up of $S^{[n]} \times S$ along the universal family, i.e., $S^{[n,n+1]} \cong \text{Bl}_S(S^{[n]} \times S)$. We get the following diagram

$$
\begin{array}{c}
S^{[n,n+1]} \xrightarrow{\psi} S^{[n+1]} \\
\downarrow \sigma \\
S^{[n]} \xrightarrow{\rho} S, \\
\end{array}
\begin{array}{c}
\uparrow \rho \\
\uparrow \sigma \\
S^n \xrightarrow{p} \Xi_n, \\
\end{array}
$$

(5.5)

where $\sigma$ is the blow-up of $S^{[n]} \times S$ in $\Xi_n$, the morphism $\iota$ is the inclusion and the other maps are the projections. If $N$ is the exceptional divisor of $\sigma$, let $L := \mathcal{O}_{S^{[n,n+1]}}(-N)$. Let $T_n$ be the tangent bundle $TS^{[n]}$ and $\omega_S$ be the canonical bundle of $S$. Given a smooth projective variety $X$, we define the dual $F^\vee$ as

$$
F^\vee := \sum_i (-1)^i \text{Ext}^i(F, \mathcal{O}_X).
$$

(5.6)

Let $K(X)$ be the Grothendieck group of vector bundles on $X$. We denote by $f^!$ the pullback between the Grothendieck groups of a morphism $f$ of smooth projective varieties, see [Ful13, §15.1] for more details. We can now state the Ellingsrud–Göttsche–Lehn formula, see [EGL01, Proposition 2.3].

**Proposition 5.7** (EGL formula). Keep notation as above. The following relation holds in $K(S^{[n,n+1]}):$

$$
\psi^!T_{n+1} = \varphi^!T_n + L - \varphi^!\sigma^!(\mathcal{O}_{\Xi_n}) + \varphi^!\rho^!\omega_S^\vee - \omega_S^\vee - \rho^!(\mathcal{O}_S - T_S + \omega_S^\vee).
$$

(5.7)

Let now $S$ be a projective K3 surface. The following theorem by Qin and Wang gives integral basis of $H^2(S^{[2]}, \mathbb{Z})$ and $H^4(S^{[2]}, \mathbb{Z})$ in terms of Nakajima operators, see [BNWS13, p.17]. See [QW05, Theorem 5.4, Remark 5.6] for a more general statement.

**Theorem 5.8** (Qin–Wang). Let $S$ be a projective K3 surface and $X = S^{[2]}$ be its Hilbert square. Let $\{\alpha_i\}_{i=1, \ldots, 22}$ be an integral basis of $H^2(S, \mathbb{Z})$. Denote by $1 \in H^0(S, \mathbb{Z})$ the unit and by $x \in H^4(S, \mathbb{Z})$ the class of a point.

(i) An integral basis of $H^2(X, \mathbb{Z})$ in terms of Nakajima operators is given by

$$
\left\{ \frac{1}{2} q_2(1)[0], q_1(1)[0], q_1(\alpha_i)[0] \right\}_{i=1, \ldots, 22}.
$$

(ii) An integral basis of $H^4(X, \mathbb{Z})$ in terms of Nakajima operators is given by the following elements:

$$
q_1(1)[0], \quad q_1(\alpha_i)[0], \quad q_1(\alpha_i)[0] \quad \text{with} \quad i < j,
$$

$$
m_{1,1}(\alpha_i)[0] := \frac{1}{2} (q_1(\alpha_i)^2 - q_2(\alpha_i))[0],
$$

with $i, j = 1, \ldots, 22$.

**Remark 5.9.** By Definition 5.1 with $k = 2$, $n = 0$, we have $\frac{1}{2} q_2(1)[0] = \delta$, where $\delta \in H^2(S^{[2]}, \mathbb{Z})$ is such that $2\delta$ is the class of the exceptional divisor of the Hilbert–Chow morphism. Moreover, if $\alpha \in H^2(S, \mathbb{Z})$, then by Definition 5.1 with $n = k = 1$ we obtain that $\alpha$ seen as an element of $H^2(S^{[2]}, \mathbb{Z})$ is represented by $q_1(1)[0]$.

Consider the diagonal embedding $\tau_2 : S \to S \times S$ of the K3 surface $S$. We denote the Gysin homomorphism followed by the Küneth isomorphism by $\tau_{2*} : H^*(S, \mathbb{Z}) \to H^*(S, \mathbb{Z}) \otimes H^*(S, \mathbb{Z})$. We take the following basis $\{\alpha_1, \ldots, \alpha_8\}$ for the lattice $H^2(S, \mathbb{Z})$: let $\{\alpha_1, \ldots, \alpha_8\}$ and $\{\alpha_9, \ldots, \alpha_{16}\}$ be the basis of the two copies of $E_6(-1)$ whose Gram matrix is the second matrix of Example 2.1 and $\{\alpha_{17}, \alpha_{18}\}, \{\alpha_{19}, \alpha_{20}\}, \{\alpha_{21}, \alpha_{22}\}$ be the basis of the three copies of $U$ whose Gram matrix is the first matrix in Example 2.1. The following Lemma will be useful, see [BNWS13, p.18].

**Lemma 5.10.** Let $X = S^{[2]}$ be the Hilbert square of a projective K3 surface $S$. Assume that $\{\alpha_1, \ldots, \alpha_{22}\}$ is the basis of the lattice $H^2(S, \mathbb{Z})$ constructed above. Then

$$
\tau_{2*} : = \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1,
$$

where the $\mu_{i,j}$’s are represented in Table 1 (we write down only the $\mu_{i,j}$’s which are non zero and such that $i \leq j$):
Proof. We can write
\[
\tau_{2*}1 = \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + a(1 \otimes x) + b(x \otimes 1)
\]
for some \( \mu_{i,j} \in \mathbb{Z} \) such that \( \mu_{i,j} = \mu_{j,i} \) with \( i, j \in \{1, \ldots, 22\} \), and some \( a, b \in \mathbb{Z} \). Let \( \langle \cdot, \cdot \rangle \) be the intersection pairing of \( \mathbb{H}^S \) in (5.1). Since \( \tau_{2*} \) is the adjoint of the cup product, we have the relation \( \langle \tau_{2*}1, \alpha_k \otimes \alpha_l \rangle = \int_S \alpha_k \alpha_l \), which gives, together with \( \tau_{2*}1 = \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + a(1 \otimes x) + b(x \otimes 1) \), the following system:

\[
\sum_{i,j} \mu_{i,j} \int_S \alpha_i \alpha_k \int_S \alpha_j \alpha_l = \int_S \alpha_k \alpha_l. \tag{5.7}
\]

From (5.7), with the help of a computer, we can compute the coefficients \( \mu_{i,j} \). The solution of the system is given in Table 1 (we have written down only the \( \mu_{i,j} \)'s which are non zero and such that \( i \leq j \), since \( \mu_{i,j} = \mu_{j,i} \)). Similarly, from the relations
\[
\langle \tau_{2*}1, 1 \otimes x \rangle = \int_S x = 1, \quad \langle \tau_{2*}1, x \otimes 1 \rangle = \int_S x = 1,
\]
we obtain \( a = b = 1 \).

We now introduce the algebraic model developed by Lehn and Sorger in [LS03]. Recall that a graded Frobenius algebra of degree \( d \) is a finite dimensional graded vector space \( A = \bigoplus_{l=-d}^{d} A^l \) endowed with a graded commutative and associative multiplication \( A \otimes A \to A \) of degree \( d \), i.e., \( \deg(ab) = \deg(a) + \deg(b) + d \), and unit element 1, necessarily of degree \( -d \), together with a linear form \( T : A \to \mathbb{Q} \) of degree \( -d \) such that the induced symmetric bilinear form \( \langle a, b \rangle := T(ab) \) is non-degenerate and of degree 0. Note that \( d \) must be an even number since \( 1 \cdot 1 = 1 \). An example of graded Frobenius algebra is the shifted cohomology ring \( H^*(X, \mathbb{Q})[d] \) of a compact complex manifold \( X \) of complex dimension \( d \). From now on we consider \( A := H^*(S, \mathbb{Q})[2] \), where \( S \) is a projective K3 surface. The linear form \( T : A \to \mathbb{Q} \) of \( T(\alpha) := -\int_S \alpha, \) and the induced bilinear form is \( \langle \alpha, \beta \rangle = T(\alpha \beta) = -\int_S \alpha \beta \), which is the intersection pairing changed of sign. The graded Frobenius algebra structure of \( A \) induces a structure of graded Frobenius algebra on \( A^{\otimes n} \): as remarked in [HIT12], since \( S \) is a K3 surface, \( A \) has only graded pieces of even degree, so the general construction of Lehn-Sorger simplifies. In this case, the multiplication induced on \( A^{\otimes n} \) is
\[
(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = (a_1 b_1) \otimes \cdots \otimes (a_n b_n),
\]
and the linear form is
\[
T : A^{\otimes n} \to \mathbb{Q}, \quad a_1 \otimes \cdots \otimes a_n \mapsto T(a_1)T(a_2)\ldots T(a_n).
\]

The symmetric group \( S_n \) of order \( n \) acts on \( A^{\otimes n} \) as
\[
\pi(a_1 \otimes \cdots \otimes a_n) := a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(n)}.
\]

Given a partition \( n = n_1 + \cdots + n_k \) with \( n_1, \ldots, n_k \in \mathbb{Z}_{>0} \), there is a generalised multiplication map \( A^{\otimes n} \to A^{\otimes k} \) defined by
\[
a_1 \otimes \cdots \otimes a_n \mapsto (a_1 \ldots a_{n_1}) \otimes \cdots \otimes (a_{n_1+\ldots+n_{k-1}+1} \ldots a_{n_1+\ldots+n_k}).
\]
Given a finite set $I \subset \{1, \ldots, n\}$, let $A^{\otimes I}$ denote the tensor power with factors indexed by elements of $I$. Let $n$ be a fixed positive integer, $\pi \in S_n$ be a permutation and $\langle \pi \rangle \subset S_n$ be the subgroup generated by $\pi$. If $[n] := \{1, \ldots, n\}$, denote by $\langle \pi \rangle \setminus [n]$ the set of orbits. Then we set

$$A\{S_n\} := \bigoplus_{\pi \in S_n} A^{\otimes (\pi) \setminus [n]} \cdot \pi.$$

**Example 5.11.** If $n = 2$, we have $S_2 = \{\text{id}, (12)\}$. Moreover, $(\text{id}) \setminus [2] = \{\{1\}, \{2\}\}$ and $(\langle 12 \rangle) \setminus [2] = \{\{1, 2\}\}$, hence we obtain $A\{S_2\} = A^{\otimes 2}\text{id} \oplus A(12)$. Similarly for $n = 3$ we have

$$A\{S_3\} = A^{\otimes 3}\text{id} \oplus A^{\otimes 2}(1 2) \oplus A^{\otimes 2}(1 3) \oplus A^{\otimes 2}(2 3) \oplus A(1 2 3) \oplus A(1 3 2).$$

Let $\sigma \in S_n$. There is a bijection

$$\sigma : \langle \pi \rangle \setminus [n] \rightarrow \langle \sigma \pi \sigma^{-1} \rangle \setminus [n], \quad x \mapsto \sigma x$$

and an isomorphism

$$\tilde{\sigma} : A\{S_n\} \rightarrow A\{S_n\}, \quad a\pi \mapsto (a\sigma)\sigma \pi \sigma^{-1}.$$

Thus we obtain an action of the symmetric group $S_n$ on $A\{S_n\}$. We denote by

$$A^{[n]} := (A\{S_n\})^{S_n}$$

the subspace of invariants. We can now state the main theorem of [LS03].

**Theorem 5.12 (Theorem 3.2 in [LS03]).** Let $S$ be a projective $K3$ surface. Then there is a canonical isomorphism of graded rings

$$(H^\ast (S, \mathbb{Q})[2])^{[n]} \simeq H^\ast (S^{[n]}, \mathbb{Q})[2n].$$

The structure of graded Frobenius algebra of $H^\ast (S^{[n]}, \mathbb{Q})[2n]$ is obtained by setting

$$T(a) := (-1)^n \int_{S^{[n]}} a \quad \text{for all } a \in H^\ast (S^{[n]}, \mathbb{Q}).$$

We sketch the description of the isomorphism of Theorem 5.12. Let $\mathcal{V}(A) := \text{Sym}^\ast (A \otimes t^{-1} \mathbb{Q}[t^{-1}])$, which is bigraded by degree and weight: an element $a \otimes t^{-m} \in A \otimes t^{-m}$ has degree $|a|$ and weight $m$. The component of $\mathcal{V}(A)$ of constant weight $n$ is the graded vector space

$$\mathcal{V}(A)_n \cong \bigoplus_{|\alpha| = n} \otimes \text{Sym}^\alpha A,$$

where the direct sum is taken over all the possible partitions $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ of $n$, and $|\alpha| := \alpha_1 \cdot 1 + \alpha_2 \cdot 2 + \cdots$.

We now fix $\pi \in S_n$. Let $f : \{1, 2, \ldots, N\} \rightarrow \langle \pi \rangle \setminus [n]$ be an enumeration of the orbits of $\pi \in S_n$. We denote by $l_i$ the length of the $i$-th orbit, i.e., $l_i := |f(i)|$. We define $\tilde{\Phi} : A^{\otimes N} \rightarrow \mathcal{V}(A)$ as

$$a_1 \otimes \cdots \otimes a_N \xrightarrow{\Phi'} \frac{1}{n!} \left( a_1 \otimes t^{-l_1} \cdots (a_N \otimes t^{-l_N}) \right).$$

Let $\Phi : \bigoplus_{n \geq 0} A\{S_n\} \rightarrow \mathcal{V}(A)$ be defined on each summand $A^{\otimes (\pi) \setminus [n]} \cdot \pi$ by the composition

$$A^{\otimes (\pi) \setminus [n]} \cdot \pi \xrightarrow{f^{-1}} A^{\otimes N} \xrightarrow{\Phi} \mathcal{V}(A),$$

where $f^{-1}$ denotes the identification of $A^{\otimes (\pi) \setminus [n]} \cdot \pi$ with $A^{\otimes N}$ through the enumeration $f$. Then $\Phi$ induces an isomorphism of graded vector spaces $A^{[n]} \cong \mathcal{V}(A)_n$ by [LS03, Proposition 2.11]. Moreover, there is an isomorphism of graded vector spaces $\Psi : \mathcal{V}(A) \rightarrow \bigoplus_{n \geq 0} H^\ast (S^{[n]}, \mathbb{Q})[2n]$ given by

$$(a_1 t^{-l_1} \cdots a_N t^{-l_N}) \xrightarrow{\Psi} q_{n_1}(a_1) \cdots q_{n_8}(a_8)[0],$$

see [LS03, Theorem 3.6]. Combining (5.8) and (5.9) we can obtain a basis of $H^\ast (S^{[n]}, \mathbb{Q})[2n]$ from a basis of $H^\ast (S, \mathbb{Q})[2]$. In order to use correctly the isomorphism of Theorem 5.12, if $A := H^\ast (S, \mathbb{Q})[2]$, we have to work with elements of $A\{S_n\}$ which are invariant for the action of $S_n$. The basis of $H^\ast (S^{[n]}, \mathbb{Q})[2n]$ obtained is clearly also a basis of $H^\ast (S^{[n]}, \mathbb{Q})$ with the standard grading. For instance, let $\{\alpha_1, \ldots, \alpha_{22}\}$ be a basis of $H^2(S, \mathbb{Z})$, denote by $1 \in H^0(S, \mathbb{Z})$ the unit and by $x \in H^4(S, \mathbb{Z})$ the class of a point: one can show that $\{\frac{1}{4} q_2(1)[0], q_1(1) q_1(\alpha_1)[0]\}_{i=1, \ldots, 22}$ is a basis of $H^2(S^{[2]}, \mathbb{Q})$ and $\{\frac{1}{4} q_2(\alpha_1)[0], q_1(1) q_1(\alpha)(0), q_1(\alpha_1) q_1(\alpha_1)[0]\}_{1 \leq i, j \leq 22}$ is a basis of $H^4(S^{[2]}, \mathbb{Q})$.

We omit the definition of the product on $(H^\ast (S, \mathbb{Q})[2])^{[n]}$ defined in [LS03] to give the structure of ring. We state the following result which describes the cup product between elements in $H^2(S^{[2]}, \mathbb{Z})$, see [BNWS13, p.13].
Lemma 5.13. Let $X = S^{[2]}$ be the Hilbert square of a projective K3 surface $S$. Let $\{\alpha_1, \ldots, \alpha_{22}\}$ be the basis of the lattice $H^2(S, \mathbb{Z})$ used in Lemma 5.10. Then the following equalities hold in $H^4(X, \mathbb{Z})$.

(i) For every $\alpha \in H^2(S, \mathbb{Z})$ we have
\[ \frac{1}{2}q_2(1)|0) \cup q_1(1)q_1(\alpha)|0) = q_2(\alpha)|0). \]

(ii) For every $\alpha, \beta \in H^2(S, \mathbb{Z})$ we have
\[ q_1(1)q_1(\alpha)|0) \cup q_1(1)q_1(\beta)|0) = \left( \int_S \alpha \beta \right) q_1(1)q_1(x)|0) + q_1(1)q_1(x)|0). \]

(iii) If $\mu_{i,j}$, with $i, j = 1, \ldots, 22$, are the coefficients computed in Lemma 5.10, then
\[ \frac{1}{2}q_2(1)|0) \cup \frac{1}{2}q_2(1)|0) = - \sum_{i<j} \mu_{i,j}q_1(\alpha_i)q_1(\alpha_j)|0) - \frac{1}{2} \sum_i \mu_{i,i}q_1(\alpha_i^2)|0) - q_1(1)(x)|0). \]

Remark 5.14. The result for the product $\frac{1}{2}q_2(1)|0) \cup \frac{1}{2}q_2(1)|0)$ given in [BNWS13, p.18] is not correct: a change of sign is needed in the right-hand side. The map $\Delta$, in [BNWS13, p.18] corresponds to $r_2$, in our notation: in that article the cohomology ring taken is $H^*(S, \mathbb{Q})$, without the shifting used in the Lehn–Sorger model, which gives the change of sign of the intersection pairing on $H^*(S, \mathbb{Q})[2]$, as seen above. See also [HHT12, Remark 3.1].

6. Rational Hodge classes of type (2, 2) on Hilbert squares of K3 surfaces

Let $S$ be a projective K3 surface. In this section we use Theorem 5.12 to compute a basis of the $\mathbb{Q}$-vector space $H^{2,2}(S^{[2]}, \mathbb{Q})$ of rational Hodge classes of type $(2, 2)$ on $S^{[2]}$.

Recall that the rational cohomology groups $H^{2i}(S, \mathbb{Q})$ and $H^{2j}(S^{[n]}, \mathbb{Q})$, where $i, j \in \mathbb{Z}_{\geq 0}$, are Hodge structures of weight $2i$ and $2j$ respectively, and the shifted cohomology groups $H^{2i}(S, \mathbb{Q})[2]$ and $H^{2j}(S^{[n]}, \mathbb{Q})[2n]$ are Hodge structures of weight $2i - 2$ and $2j - 2n$ respectively, with the following Hodge decompositions:
\[ H^{2i}(S, \mathbb{C})[2] = \bigoplus_{p+q=2i-2} H^{p,q}(S)[2], \quad H^{2j}(S^{[n]}, \mathbb{Q})[2n] = \bigoplus_{r+s=2j-2n} H^{r,s}(S^{[n]}[2n], \mathbb{Q}), \]

where $p, q \in \{-1, 0, \ldots, i-1\}$ and $r, s \in \{-n, 1-n, \ldots, j-n\}$, and
\[ H^{p,q}(S)[2] = H^{p+1,q+1}(S), \quad H^{r,s}(S^{[n]}[2n]) = H^{r+n,s+n}(S^{[n]}). \]

For details on Hodge structures, see [Huy16, §3]. If $A := H^*(S, \mathbb{Q})[2]$, recall that $A^{[n]} := \bigoplus_{|\alpha|=n} \bigotimes_i \operatorname{Sym}^{\alpha_i} A$. We denote by $(A^{[n]})^{2i}$ the component of degree $2i - 2n$ of $A^{[n]}$. Then the Hodge structures $H^{2j}(S, \mathbb{Q})[2]$ give rise to a Hodge structure on $(A^{[n]})^{2i}$. Since the weights of the Hodge structures considered depend only on the (shifted) cohomological degrees, we have that $(A^{[n]})^{2i}$ is a Hodge structure of weight $2i - 2n$. Note that $H^{2i}(S^{[n]}, \mathbb{Q})[2n]$ is a Hodge structure of weight $2i - 2n$ and $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]}, \mathbb{Q})[2n]$ by Theorem 5.12. Recall that, given two Hodge structures $V$ and $W$, a morphism of weight $k$ is a linear map $f : V \to W$ such that its $\mathbb{C}$-linear extension satisfies $f(V^{p,q}) \subseteq W^{p+k,q+k}$. The pullback and the Gysin homomorphism induced by a morphism $f : X \to Y$ between two complex projective manifolds are morphisms of Hodge structures of weight respectively 0 and $r = \dim_{\mathbb{C}}(Y) - \dim_{\mathbb{C}}(X)$, see [Voi02, §7.3.2]. Using this property, Definition 5.1 and the isomorphisms (5.8) and (5.9) one can show the following result, which is well known to experts and implicitly given in [LS03].

Theorem 6.1. Let $S$ be a projective K3 surface. Let $A := H^*(S, \mathbb{Q})[2]$ and consider $(A^{[n]})^{2i} \cong H^{2i}(S^{[n]}, \mathbb{Q})[2n]$, the isomorphism induced by Theorem 5.12 on the components of (shifted) cohomological degree $2i - 2n$. Take on $(A^{[n]})^{2i}$ the Hodge structure of weight $2i - 2n$ described above and on $H^{2i}(S^{[n]}, \mathbb{Q})[2n]$ the Hodge structure of weight $2i - 2n$ induced by shifted cohomology. Then
\[ (A^{[n]})^{2i} \cong H^{2i}(S^{[n]}, \mathbb{Q})[2n] \]
is an isomorphism of Hodge structures of weight 0.

Let $S$ be a K3 surface and $T(S)$ be its transcendental lattice, which is a Hodge structure of weight 2. Denote by $E_S := \operatorname{Hom}_0(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}})$ the algebra of endomorphisms on $T(S)_{\mathbb{Q}}$ of weight 0. Recall that a K3 surface $S$ is general if $E_S \cong \mathbb{Q}$. We can now give a basis of the $\mathbb{Q}$-vector space $H^{2,2}(S^{[2]}, \mathbb{Q})$ of rational Hodge classes of type $(2, 2)$ on the Hilbert square of a general projective K3 surface.
Theorem 6.2. Let $S$ be a general projective $K3$ surface and let \{${b_1, \ldots, b_r}$\} be a basis of $\text{Pic}(S)$. Then:

(i) $\dim(H^{2,2}(S^{[2]}, \mathbb{Q})) = \frac{(r+1)r}{2} + r + 2$.

(ii) A basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ is given by the following elements:

- $\frac{1}{2}q_2(b_i)(0)$ for $i = 1, \ldots, r$.
- $q_1(1)q_1(x)(0)$, where $1 \in H^0(S, \mathbb{Q})$ is the unit and $x \in H^4(S, \mathbb{Q})$ is the class of a point.
- $\frac{1}{2}q_1(b_i)^2(0)$ for $i = 1, \ldots, r$.
- $q_1(b_i)q_1(b_j)(0)$ for $1 \leq i < j \leq r$.

where $\{\alpha_1, \ldots, \alpha_22\}$ is the basis of $H^2(S, \mathbb{Z})$ used in Lemma 5.10 and the $\alpha_{i,j}$'s are given in Table 1.

Proof. By Theorem 5.12 and Theorem 6.1 the following is an isomorphism of Hodge structures of weight 0:

$$H^4(S^{[2]}, \mathbb{Q}) \cong H^2(S, \mathbb{Q}) \oplus (H^0(S, \mathbb{Q}) \otimes H^4(S, \mathbb{Q})) \oplus \text{Sym}^2(H^2(S, \mathbb{Q})), \quad (6.1)$$

we omit the shiftings of the cohomology groups. The Hodge classes of $S^{[2]}$ of bidegree $(2, 2)$ have bidegree $(0, 0)$ in the shifted cohomology, so we look for the components of (shifted) bidegree $(0, 0)$ in the right-hand side of (6.1). If $\text{NS}(S)$ is the Néron–Severi group of $S$ and $T(S) = (\text{NS}(S))^\perp$ is the transcendental lattice, $H^2(S, \mathbb{Q})$ can be decomposed as

$$H^2(S, \mathbb{Q}) \cong \text{NS}(S)_\mathbb{Q} \oplus T(S)_\mathbb{Q}, \quad (6.2)$$

where $\text{NS}(S)_\mathbb{Q} = \text{NS}(S) \otimes \mathbb{Q}$ and $T(S)_\mathbb{Q} = T(S) \otimes \mathbb{Q}$.

The first summand of (6.1) has the $\mathbb{Q}$-vector space $\text{NS}(S)_\mathbb{Q}[2]$ as component of bidegree $(0, 0)$. Since $\text{NS}(S) \cong \text{Pic}(S)$ and $\text{rk}(\text{Pic}(S)) = r$ by assumption, the component of bidegree $(0, 0)$ of $H^{2}(S^{[2]}, \mathbb{Q})$ has dimension $r$. By (5.8) and (5.9) we have $b_i \xrightarrow{\psi_0\phi'} \frac{1}{2}q_2(b_i)(0)$, so $\frac{1}{2}q_2(b_i)(0)$ is in a basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$ for $i = 1, \ldots, r$.

The second summand of (6.1) is $H^0(S, \mathbb{Q})[2] \otimes H^4(S, \mathbb{Q})[2]$, which is a vector space over $\mathbb{Q}$ of dimension 1. This is generated by $1 \otimes x$, which is an element of bidegree $(0, 0)$. Since $1 \otimes x + x \otimes 1 \xrightarrow{\psi_0\phi'} q_1(1)q_1(x)(0)$, the element $q_1(1)q_1(x)(0)$ is in a basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$.

Consider $\text{Sym}^2(H^2(S, \mathbb{Q})[2])$, the third summand of (6.1). Using (6.2), we can decompose it as

$$\text{Sym}^2(H^2(S, \mathbb{Q})[2]) \cong \text{Sym}^2(\text{Pic}(S)_\mathbb{Q}[2]) \oplus \text{Sym}^2(T(S)_\mathbb{Q}[2]) \oplus (\text{Pic}(S)_\mathbb{Q}[2] \otimes T(S)_\mathbb{Q}[2]). \quad (6.3)$$

By assumption $\text{rk}(\text{Pic}(S)) = r$, so $\text{Sym}^2(\text{Pic}(S)_\mathbb{Q}[2])$, whose elements have all bidegree $(0, 0)$, has dimension $\frac{(r+1)r}{2}$ as $\mathbb{Q}$-vector space. By (5.8) and (5.9) we have:

$$b_i \otimes b_i \xrightarrow{\psi_0\phi'} \frac{1}{2}q_1(b_i)^2(0), \quad b_i \otimes b_j + b_j \otimes b_i \xrightarrow{\psi_0\phi'} q_1(b_i)q_1(b_j)(0)$$

for $i, j \in \{1, \ldots, r\}$ and $i < j$. Then the elements $\frac{1}{2}q_1(b_i)^2(0)$, for $i = 1, \ldots, r$, and $q_1(b_i)q_1(b_j)(0)$, for $1 \leq i < j \leq r$, are in a basis of $H^{2,2}(S^{[2]}, \mathbb{Q})$. Note that $\text{Pic}(S)_\mathbb{Q}[2] \otimes T(S)_\mathbb{Q}[2]$ does not contain any element of bidegree $(0, 0)$. It remains to determine the elements of bidegree $(0, 0)$ in $\text{Sym}^2(T(S)_\mathbb{Q}[2])$, i.e.,

$$\left(\text{Sym}^2(T(S)_\mathbb{Q}[2])\right)^{0,0} \cap \left(\text{Sym}^2(T(S)_\mathbb{Q}[2])\right).$$

Consider $T(S)_\mathbb{Q} \otimes T(S)_\mathbb{Q}$ with the standard grading. Then $T(S)_\mathbb{Q}$ is the minimal sub-Hodge structure of $H^2(S, \mathbb{Q})$ with $H^{0,0}(S) = T(S)_\mathbb{Q}[2]$, see [Huy16, Definition 3.2.5, Lemma 3.3.1], in particular $T(S)_\mathbb{Q}$ is a Hodge structure of weight 2. By [Huy16, Example 3.1.3, (iii)] the dual $T(S)_\mathbb{Q}^* = \text{Hom}(T(S)_\mathbb{Q}, \mathbb{Q})$ is a Hodge structure of weight $-2$, and there is an isomorphism of Hodge structures of weight $-2$ from $T(S)_\mathbb{Q}$ to $T(S)_\mathbb{Q}^*$. This implies that

$$T(S)_\mathbb{Q} \otimes T(S)_\mathbb{Q} \cong T(S)_\mathbb{Q} \otimes T(S)_\mathbb{Q} \cong T(S)_\mathbb{Q}^* \otimes T(S)_\mathbb{Q}^{*0,0},$$

and by [Huy16, Example 3.1.4, (iv)] we have

$$E_S = \text{Hom}_0(T(S)_\mathbb{Q}, T(S)_\mathbb{Q}) \cong (T(S)_\mathbb{Q} \otimes T(S)_\mathbb{Q}) \cap (T(S)_\mathbb{Q} \otimes T(S)_\mathbb{Q})^{0,0},$$

where $\text{Hom}_0(T(S)_\mathbb{Q}, T(S)_\mathbb{Q})$ denotes the space of Hodge endomorphisms on $T(S)_\mathbb{Q}$ of weight 0. Since by assumption $S$ is a general K3 surface we have $\text{Hom}_0(T(S)_\mathbb{Q}, T(S)_\mathbb{Q}) \cong \mathbb{Q} \cdot \text{id}$. Passing to the shifted cohomology groups, this implies that the $\mathbb{Q}$-vector space $(\text{Sym}^2(T(S)_\mathbb{Q}[2]))^{0,0} \cap (\text{Sym}^2(T(S)_\mathbb{Q}[2]))$ has dimension 1. We now describe the element induced by its generator on $H^4(S^{[2]}, \mathbb{Q})$. Let $\{b_{r+1}, \ldots, b_{22}\}$ be an orthogonal basis of $T(S)_\mathbb{Q}$ with respect to the
intersection form, and let \( \{\beta_{r+1}, \ldots, \beta_{22}\} \) be the basis of \( T(S)^*_Q \) given by \( \beta_i := (\beta_i, \cdot) \in \text{Hom}(T(S)\_Q, Q) \cong T(S)\_Q \) for \( i \in \{r + 1, \ldots, 22\} \). Then

\[
\text{id} = \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i^\vee \otimes \beta_i \in T(S)^*_Q \otimes T(S)_Q
\]

since for every \( k \in \{r + 1, \ldots, 22\} \) we have

\[
\left( \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i^\vee \otimes \beta_i \right) (\beta_k) = \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} (\beta_i, \beta_k) \cdot \beta_i = \beta_k.
\]

Note that \((\beta_i, \beta_i) \neq 0\) since \( \{\beta_{r+1}, \ldots, \beta_{22}\} \) is an orthogonal basis and the intersection form on \( H^2(S, Q) \) is non-degenerate. Since \( T(S)_Q \cong T(S)^*_Q \) by the map \( \beta_i \mapsto \beta_i^\vee \), the identity, seen as element in \( T(S)_Q \otimes T(S)_Q \), is

\[
\text{id} = \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} \beta_i \otimes \beta_i \in T(S)_Q \otimes T(S)_Q.
\] (6.4)

We see that (6.4) is invariant for the action of the symmetric group \( S_2 \) on \( A\{S_2\} \), where \( A := H^*(S, Q)[2] \), so we obtain the following element of \( H^4(S^{[2]}, Q)[4] \):

\[
\text{id} \overset{\Psi, \Phi^\vee}{=} \frac{1}{2} \sum_{i=r+1}^{22} \frac{1}{(\beta_i, \beta_i)} q_1(\beta_i)^2 |0\rangle.
\] (6.5)

Hence the last element of the basis of \( H^{2,2}(S^{[2]}, Q) \) is (6.5). After some tedious computations, one can show that the element in (6.5) can be substituted in the basis of \( H^{2,2}(S^{[2]}, Q) \) obtained by the following element:

\[
\delta^2 = -\sum_{i<j} \mu_{i,j} q_1(\alpha_i) q_1(\alpha_j) |0\rangle - \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2 |0\rangle - q_1(1) q_1(x) |0\rangle.
\]

We omit this part of the proof. We conclude that \( \dim(H^{2,2}(S^{[2]}, Q)) = \frac{(r+1)r}{2} + r + 2 \) and a basis of the \( Q \)-vector space \( H^{2,2}(S^{[2]}, Q) \) is the one presented in the statement of the theorem. \( \square \)

**Remark 6.3.** Note that the assumption that \( S \) is a **general** K3 surface, i.e., \( E_S \cong Q \), is necessary in the statement of Theorem 6.2. If \( S \) is general, as seen in the proof, the component \( \text{Sym}^2(T(S)_Q[2]) \) in (6.3) gives only the element (6.4) in a basis of \( H^{2,2}(S^{[2]}, Q) \) (we have substituted it with \( \delta^2 \) in the end of the proof). If \( S \) is not general, \( E_S \) is bigger than \( Q \), hence the component \( \text{Sym}^2(T(S)_Q[2]) \) in (6.3) gives not only (6.4), but also other elements in a basis of \( H^{2,2}(S^{[2]}, Q) \).

One can combine Theorem 6.2 with Theorem 5.8 to find a basis, in terms of Nakajima operators, of the lattice \( H^{2,2}(S^{[2]}, Z) \) of integral Hodge classes of type \((2, 2)\) of a **generic** K3 surface \( S \), which is by definition a general projective K3 surface with Picard group of rank 1. The proof of this result is very similar to the one of Theorem 8.3, which generalises it to the Hilbert square of any general projective K3 surface. For this reason, we only sketch the proof: details can be found in the author’s PhD thesis, see [Nov, Theorem 3.3.17]. While Theorem 8.3, as we will see, needs Theorem 8.1 to be proven, Theorem 6.4 can be shown without using it.

**Theorem 6.4.** Let \( S \) be a generic K3 surface and \( h \in \text{Pic}(S) \) be the ample generator of \( \text{Pic}(S) \). Then a basis of the lattice \( H^{2,2}(S^{[2]}, Z) \) of integral Hodge classes of type \((2, 2)\) is given by the following elements:

\[
q_2(h) |0\rangle, \quad q_1(1) q_1(x) |0\rangle, \quad \frac{1}{2} (q_1(h)^2 - q_2(h)) |0\rangle, \quad -\sum_{i<j} \mu_{i,j} q_1(\alpha_i) q_1(\alpha_j) |0\rangle - \frac{1}{2} \mu_{i,i} q_1(\alpha_i)^2 |0\rangle - q_1(1) q_1(x) |0\rangle,
\] (6.6)

where \( x \in H^4(S, Z) \) is the class of a point, \( 1 \in H^0(S, Z) \) is the unit, \( \{\alpha_1, \ldots, \alpha_{22}\} \) is the basis of \( H^2(S, Z) \) used in Lemma 5.10 and the \( \mu_{i,j} \)'s are the integers given by Table 1.

**Sketch of the proof.** After a slight modification of the basis given in Theorem 6.2, we see that (6.6) is a basis of \( H^{2,2}(S^{[2]}, Q) \). In order to prove the theorem, we give a basis of the lattice \( H^4(S^{[2]}, Z) \) which contains the elements in (6.6). The Picard group \( \text{Pic}(S) \cong Z h \) can be primitively embedded in a unique way up to isometries in \( H^2(S, Z) \), see [Huy16, Theorem 14.1.12]; we can identify \( h \) with \( a_{17} + ta_{18} \). After some computations, using Theorem 5.8 and the
fact that $\mu_{21,22} = 1$ by Table 1 (compare with the proof of Theorem 8.3), it is possible to show that the following is a basis of $H^4(S^{[2]}, \mathbb{Z})$:  

\begin{align*}
q_1(1)q_1(x)|0),
q_2(\beta_i)|0 \text{ for } i = 1, \ldots, 22, 
q_1(\beta_i)q_1(\beta_j)|0) \text{ for } 1 \leq i < j \leq 22 \text{ and } (i,j) \neq (21,22), 
\frac{1}{2} (q_1(\beta_i)^2 - q_2(\beta_i))|0 \text{ for } i = 1, \ldots, 22, 
- \sum_{i<j} \mu_{ij} q_1(\alpha_i)q_1(\alpha_j)|0) - \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0) - q_1(1)q_1(x)|0),
\end{align*}

(6.7)

where $\beta_i := \alpha_i$ for $i \neq 17$ and $\beta_{17} := \alpha_{17} + t\alpha_{18}$. Since the elements in (6.6) are contained in (6.7) and $\beta_{17} = h$, we conclude that (6.6) is a basis of $H^{2,2}(S^{[2]}, \mathbb{Z})$. \qed

7. SECOND CHERN CLASS OF THE HILBERT SQUARE OF A K3 SURFACE

Let $S$ be a projective K3 surface and $X := S^{[2]}$ be its Hilbert square. In this section we look for a representation of the second Chern class $c_2(X)$ of $X$ in terms of Nakajima operators: this will be fundamental to find a basis of the lattice $H^{2,2}(X, \mathbb{Z})$ of integral Hodge classes of type $(2,2)$ on the Hilbert square of a general projective K3 surface. Let $q_X^\vee \in H^{2,2}(X,\mathbb{Q})$ be the dual of the BBF-form. As always we denote by $\delta \in \text{Pic}(X)$ the class such that $2\delta$ is the class of the exceptional divisor of the Hilbert–Chow morphism. First of all, the following proposition gives a basis of $H^{2,2}(X, \mathbb{Q})$ which does not depend on Nakajima operators, where $X$ is the Hilbert square of a generic K3 surface: we omit the proof.

Proposition 7.1. Let $X := S^{[2]}$ be the Hilbert square of a generic K3 surface and let $h \in \text{Pic}(X)$ be the class induced by the ample generator of $\text{Pic}(S)$. Then $\{h^2, h\delta, \delta^2, \frac{3}{2}q_X^\vee\}$ is a basis of the $\mathbb{Q}$-vector space $H^{2,2}(X, \mathbb{Q})$.

Theorem 6.4 gives us a basis of the lattice $H^{2,2}(S^{[2]} , \mathbb{Z})$, where $S$ is a generic K3 surface, in terms of Nakajima operators. It is natural to wonder how to describe this basis without Nakajima operators, in particular we want to obtain a basis of $H^{2,2}(S^{[2]} , \mathbb{Z})$ in terms of some rational linear combination of $h^2, h\delta, \delta^2, \frac{3}{2}q_X^\vee \in H^{2,2}(S^{[2]} , \mathbb{Q})$ of Proposition 7.1. Note that using Lemma 5.13 we can represent $h^2, h\delta$ and $\delta^2$ in terms of Nakajima operators, while we do not know how to write $q_1(1)q_1(x)|0)$ of Theorem 6.4 in terms of $h^2, h\delta, \delta^2, \frac{3}{2}q_X^\vee$. Hence we need to express $\frac{3}{2}q_X^\vee \in H^{2,2}(X, \mathbb{Z})$ in terms of Nakajima operators. Recall that $\frac{3}{2}q_X^\vee = c_2(X)$ by Proposition 4.4, so we look for a representation of $c_2(X) \in H^{2,2}(X, \mathbb{Z})$ in terms of Nakajima operators. The tool that we use is the EGL formula of Proposition 5.7

Let $S$ be a projective K3 surface. We denote by $T_2 := T_{S^{[2]}}$ the tangent bundle of $S^{[2]}$. From now on, we will denote by $\cdot$ the cup product. We define the following operator on the cohomology ring $H^*(S^{[2]}, \mathbb{Q})$:

\[ \text{ch}(T_2) : H^*(S^{[2]}, \mathbb{Q}) \to H^*(S^{[2]}, \mathbb{Q}), \quad x \mapsto \text{ch}(T_2) \cdot x. \]

By the general construction of $S^{[n,n+k]}$ seen in Section 5, if $\Delta \subset S \times S$ is the diagonal, we have $S^{[1,2]} \cong \text{Bl}_\Delta(S^2)$. Diagram (5.5) for $n = 1$ gives

\[ \begin{array}{c}
S^{[1,2]} \xrightarrow{\psi} S^{[2]} \\
\downarrow \sigma \\
S \times S \xrightarrow{\rho} S,
\end{array} \]

(7.1)

Note that the morphisms $\varphi, \rho$ and $\psi$ appearing in diagram (7.1) correspond to the morphisms $\varphi, \rho$ and $\psi$ of diagram (5.2), with $n = k = 1$, precomposed with the inclusion of $S^{[1,2]}$ in $S \times S \times S^{[2]}$. With this notation, the definition of the Nakajima operator $q_1$ is the same of (5.3) without the component $PD^{-1}[S^{[1,2]}]$, i.e., for $\alpha, x \in H^*(S)$ we have

\[ q_1(\alpha)(x) = \psi_*(\varphi^*(x) \cdot \rho^*(\alpha)). \]
By properties of cup and cap product, the latter denoted by ∩, we have \( \varphi^* (x) \cdot \rho^* (\alpha) = PD^{-1} ([S^{[1,2]}] \cap \varphi^* (x) \cdot \rho^* (\alpha)) \) in \( H^* (S^{[1,2]}) \) for every \( \alpha, x \in H^* (S) \). Then we get

\[
\text{ch}(T_2) \cdot q_1 (\alpha) (x) = \text{ch}(T_2) \cdot \psi_* (\varphi^* (x) \cdot \rho^* (\alpha)) = \text{ch}(T_2) \cdot PD^{-1} \psi_* ([S^{[1,2]}] \cap \varphi^* (x) \cdot \rho^* (\alpha)) = PD^{-1} \psi_* ([S^{[1,2]}] \cap \psi^*(\text{ch}(T_2)) \cdot \varphi^* (x) \cdot \rho^* (\alpha)) = PD^{-1} \psi_* ([S^{[1,2]}] \cap \text{ch}(\psi^T T_2) \cdot \varphi^* (x) \cdot \rho^* (\alpha)),
\]

where \( \psi_* \) in the first equality is the Gysin homomorphism, while in the other equalities is the pushforward in homology, and the third equality comes from the projection formula. Applying Proposition 5.7 with \( n = 1 \) and \( \omega_S \) trivial we get:

\[
\text{ch}(T_2) \cdot q_1 (\alpha) (x) = PD^{-1} \psi_* ([S^{[1,2]}] \cap \varphi^*(\text{ch}(T_S) \cdot x) \cdot \rho^* (\alpha)) + PD^{-1} \psi_* ([S^{[1,2]}] \cap \text{ch}(L) \cdot \varphi^* (x) \cdot \rho^* (\alpha)) - PD^{-1} \psi_* ([S^{[1,2]}] \cap \text{ch}(L) \cdot \sigma^*(\text{ch}(O_{\Delta}^c)) \cdot \varphi^* (x) \cdot \rho^* (\alpha)) + PD^{-1} \psi_* ([S^{[1,2]}] \cap \text{ch}(L^c) \cdot \varphi^* (x) \cdot \rho^* (\alpha)) - PD^{-1} \psi_* ([S^{[1,2]}] \cap \text{ch}(L^c) \cdot \sigma^*(\text{ch}(O_{\Delta})) \cdot \varphi^* (x) \cdot \rho^* (\alpha)) - PD^{-1} \psi_* ([S^{[1,2]}] \cap \varphi^* (x) \cdot \rho^* (\text{ch}(2O_S - T_S))) ,
\]

where \( L := O_S (-N) \) and \( N \) is the exceptional divisor of the blowing up \( \sigma : \text{Bl}_{\Delta} (S^2) \to S^2 \). If we set \( x := q_1 (1)[0] \) and \( \alpha := 1 \), we can use formula (7.2) to compute \( c_2 (S^{[2]}) \) in terms of Nakajima operators. Recall that here the dual is defined by (5.6), while we denote by \( F^* := \text{Hom}(F, O_X) \) the classical dual of a coherent sheaf \( F \). We now study the duals appearing in the right-hand side of (7.2).

**Lemma 7.2.** Keep notation as above. Then:

(i) The dual \( L^c \) is isomorphic to the dual \( L^* = \text{Hom}(L, O_S^{[1,2]}) \).

(ii) \( O_{\Delta} = O_{\Delta} \).

**Proof.**

(i) By [Har13, Proposition III.6.7] we have \( \mathcal{E}xt^i (L, O_S^{[1,2]}) \cong \mathcal{E}xt^i (O_S^{[1,2]}, O_S^{[1,2]}) \otimes L^* \). Moreover, by [Har13, Proposition III.6.3] we have that \( \mathcal{E}xt^i (O_S^{[1,2]}, O_S^{[1,2]}) \) is 0 for \( i > 0 \) and it is equal to \( O_S^{[1,2]} \) if \( i = 0 \). We conclude that \( \sum_i (-1)^i \mathcal{E}xt^i (L, O_S^{[1,2]}) \cong L^* \).

(ii) We apply [Sch, Lemma 1] with \( X = S \times S, Z = \Delta \) and \( L = O_{S \times S} \). We have \( \mathcal{E}xt^i (O_{\Delta}, O_{S \times S}) = 0 \) if \( i \neq 2 \) and \( \mathcal{E}xt^2 (O_{\Delta}, O_{S \times S}) = \text{det} N_{\Delta|S \times S} \). Moreover, \( N_{\Delta|S \times S} = T_S \) and \( T_S \cong O_S \) since \( S \) is a K3 surface. We conclude that \( \sum_i (-1)^i \mathcal{E}xt^i (O_{\Delta}, O_{S \times S}) = O_{\Delta} \), as we wanted.

We recall the computation of the Chern character of \( O_{\Delta} := i_* O_{\Delta} \), where \( i : \Delta \hookrightarrow S \times S \) is the inclusion.

**Lemma 7.3.** Let \( S \) be a K3 surface and \( \Delta \subset S \times S \) be the diagonal. Denote by \( [\Delta] \in H^4 (S \times S, \mathbb{Z}) \) the fundamental cohomological class of \( \Delta \) in \( S \times S \). Then \( \text{ch}(O_{\Delta}) = [\Delta] - 2y \), where \( y \in H^8 (S \times S, \mathbb{Q}) \) is the class of a point in \( S \times S \).

**Proof.** Let \( i : \Delta \hookrightarrow S \times S \) be the inclusion. By Grothendieck–Riemann–Roch Theorem, see [AH59, Theorem 1] and [HA62], we have

\[
\text{ch}(O_{\Delta}) = i_* (\text{ch}(O_{\Delta} \cdot \text{td}(S)) \cdot \text{td}(S \times S)^{-1}) = i_* (\text{td}(S) \cdot i^* \text{td}(S \times S)^{-1}) = i_* (\text{td}(S)^{-1}) ,
\]

where \( i_* \) is the Gysin homomorphism and \( O_{\Delta} \) in the left-hand side is \( i_* O_{\Delta} \). Let \( x \in H^4 (S, \mathbb{Q}) \) be the class of a point of \( S \). The Todd class of a K3 surface is \( \text{td}(S) = 1 + 2x \), hence we obtain \( \text{ch}(O_{\Delta}) = [\Delta] - 2y \).
Consider formula (7.2). We introduce the following notation:

$L1 := \text{ch}(\mathcal{T}_2) \cdot q_1(\alpha)(x)$,
$R1 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \varphi^\ast(\text{ch}(\mathcal{T}_S) \cdot x) \cdot \rho^\ast(\alpha))$
$R2 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \text{ch}(\mathcal{L}) \cdot \varphi^\ast(x) \cdot \rho^\ast(\alpha))$
$R3 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \text{ch}(\mathcal{L}^\vee) \cdot \varphi^\ast(x) \cdot \rho^\ast(\alpha))$
$R4 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \text{ch}(\mathcal{L}^\vee) \cdot \sigma^\ast(\text{ch}(\mathcal{O}_\Delta)) \cdot \varphi^\ast(x) \cdot \rho^\ast(\alpha))$
$R5 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \text{ch}(\mathcal{L}^\vee) \cdot \sigma^\ast(\text{ch}(\mathcal{O}_\Delta)) \cdot \varphi^\ast(x) \cdot \rho^\ast(\alpha))$
$R6 := PD^{-1} \psi_\ast ([S^{[1,2]}] \cap \varphi^\ast(x) \cdot \rho^\ast(\text{ch}(2\mathcal{O}_S - \mathcal{T}_S) \cdot x))$.

Recall that we have taken $x = q_1(1)|0$ and $\alpha = 1$. We can now compute $c_2(S^{[2]})$ in terms of Nakajima operators.

**Proposition 7.4.** Let $S$ be a projective K3 surface. Then the second Chern class $c_2(S^{[2]}) \in H^{2,2}(S^{[2]}, \mathbb{Z})$ of $S^{[2]}$ in terms of Nakajima operators is

$$c_2(S^{[2]}) = 27q_1(1)q_1(x)|0 + 3 \sum_{i<j} \mu_{i,j}q_1(\alpha_i)q_1(\alpha_j)|0 + \frac{3}{2} \sum_i \mu_{i,i}q_1(\alpha_i)^2|0,$$

where $1 \in H^0(S, \mathbb{Z})$ is the unit and $x \in H^4(S, \mathbb{Z})$ is the class of a point, $\{\alpha_1, \ldots, \alpha_{22}\}$ is the basis of the lattice $H^2(S, \mathbb{Z})$ used in Lemma 5.10 and the $\mu_{i,j}$’s are given in Table 1.

Note that by Table 1 the integers $\mu_{i,j}$ are all even, so (7.3) is an element of $H^{2,2}(S^{[2]}, \mathbb{Z})$.

**Proof.** We make some computations on $L1, R1, \ldots, R6$ introduced above.

- We have $L1 = \text{ch}(\mathcal{T}_2) \cdot q_1(\alpha)(x) = \text{ch}(\mathcal{T}_2) \cdot q_1(1)q_1(1)|0$. By the definition of exponential Chern character and by (5.4), which gives $q_1(1)q_1(1)|0 = 2 \cdot 1_{S^{[2]}}$, we obtain

$$L1 = 8 \cdot 1_{S^{[2]}} - 2c_2(S^{[2]}) + \frac{1}{6} c_2(S^{[2]})^2 - \frac{1}{3} c_4(S^{[2]}).$$

- Since $S$ is a K3 surface we have $c_1(S) = 0$ and $c_2(S) = 24x$, where $x \in H^4(S, \mathbb{Z})$ is the class of a point on $S$. Hence $\text{ch}(\mathcal{T}_S) = 2 - 24x$ and we obtain

$$R1 = 2q_1(1)q_1(1)|0 - 24q_1(1)q_1(x)|0 = 4 \cdot 1_{S^{[2]}} - 24q_1(1)q_1(x)|0,$$

where the second equality comes from (5.4).

- Let $d := c_1(\mathcal{L})$. Then we have $\text{ch}(\mathcal{L}) = \sum_{\nu \geq 0} \frac{1}{\nu!}d^\nu$. Now, [Leh99, Lemma 3.9] implies that the cycle $[S^{[1,2]}] \cap d^\nu$ induces the operator $q_1(\nu)$, as observed in the proof of [Leh99, Lemma 4.2], hence we obtain

$$R2 = \sum_{\nu \geq 0} \frac{1}{\nu!}q_1(\nu)(\alpha) \cdot x = \sum_{\nu \geq 0} \frac{1}{\nu!}q_1(\nu)(1)q_1(1)|0.$$ We now compute $q_1(\nu)(1)q_1(1)|0$ for every $\nu \geq 0$. If $\nu = 0$, we have

$$q_1(0)(1)q_1(1)|0 = q_1(1)q_1(1)|0 = 2 \cdot 1_{S^{[2]}} \in H^0(S^{[2]}, \mathbb{Z}).$$

If $\nu = 1$, by Theorem 5.6 we have

$$q_1(1)q_1(1)|0 = -q_2(1)|0 \in H^2(S^{[2]}, \mathbb{Z}).$$

(7.4)

If $\nu = 2$, we have

$$q_1(2)(1)q_1(1)|0 = (\partial q_1' - q_1' \partial)(1)q_1(1)|0 = \partial q_1(1)q_1(1)|0 - q_1' \partial q_1(1)|0.$$ The boundary of $S$ is empty by Remark 5.4, so $q_1(\partial q_1(1)|0 = 0$. Moreover, using (7.4), we get $\partial q_1(1)q_1(1)|0 = -q_2(1)|0$, and by Definition 5.5 we obtain

$$q_1(2)(1)q_1(1)|0 = \frac{1}{2} q_2(1)|0 \cdot q_2(1)|0 \in H^4(S^{[2]}, \mathbb{Z}).$$
Similarly for \( \nu = 3 \) and \( \nu = 4 \) we obtain the following:

\[
\begin{align*}
q_1^{(3)}(1)q_1(1)|0\rangle &= -\frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot q_2(1)|0\rangle \in H^6(S^{[2]}, \mathbb{Z}), \\
q_1^{(4)}(1)q_1(1)|0\rangle &= \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot q_2(1)|0\rangle \in H^8(S^{[2]}, \mathbb{Z}).
\end{align*}
\]

If \( \nu \geq 5 \), we obtain an element in \( H^{2\nu}(S^{[2]}, \mathbb{Z}) = 0 \). We conclude that

\[
R2 = 2 \cdot 1_{S^{[2]}} - q_2(1)|0\rangle + \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \\
+ \frac{1}{3} \left( \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \right).
\]

By Lemma 7.2 we have \( \text{ch}(\mathcal{O}_\Delta) = \text{ch}(\mathcal{O}_\Delta^\vee) \), and by Lemma 7.3 we have \( \text{ch}(\mathcal{O}_\Delta) = |\Delta| - 2y \), where \( y \in H^8(S \times S, \mathbb{Z}) \) is the class of a point. Moreover, \( |\Delta| = \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1 \) by Lemma 5.10, where \( \{\alpha_1, \ldots, \alpha_{22}\} \) is the basis of \( H^2(S, \mathbb{Z}) \) used in Lemma 5.10 and the \( \mu_{i,j} \)'s are given in Table 1. Recall the notation of diagram (7.1). By the Künneth theorem we have \( y = x \otimes x \). Then

\[
\sigma^*(\text{ch}(\mathcal{O}_\Delta)) = \sigma^* \left( \sum_{i,j} \mu_{i,j} \alpha_i \otimes \alpha_j + 1 \otimes x + x \otimes 1 - 2(x \otimes x) \right)
\]

\[
= \sum_{i,j} \mu_{i,j} \varphi^*(\alpha_i) \cdot \varphi^*(\alpha_j) + \varphi^*(1) \cdot \rho^*(x) + \varphi^*(x) \cdot \rho^*(1) - 2 \varphi^*(x) \cdot \rho^*(x).
\]

Proceeding as for \( R2 \) we get

\[
R3 = \sum_{i,j} \mu_{i,j} \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(x)q_1(1)|0\rangle + \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(x)q_1(1)|0\rangle
\]

\[
+ \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(1)q_1(x)|0\rangle - 2 \sum_{\nu \geq 0} \frac{1}{\nu!} q_1^{(\nu)}(x)q_1(1)|0\rangle.
\]

We call \( R3_{\nu=1} \) the component of \( R3 \) obtained by putting \( \nu = i \) in (7.5) for \( i \geq 0 \). Using the commutativity rule given by Theorem 5.2 we have

\[
R3_{\nu=0} = \sum_{i,j} \mu_{i,j} q_1(1)|0\rangle q_1(1)|0\rangle + 2q_1(1)q_1(x)|0\rangle - 2q_1(x)q_1(1)|0\rangle.
\]

Note that \( q_1(1)|0\rangle q_1(1)|0\rangle \in H^4(S^{[2]}, \mathbb{Z}) \) and \( q_1(1)|0\rangle q_1(x)|0\rangle \in H^8(S^{[2]}, \mathbb{Z}) \). If \( \nu \geq 3 \) we obtain elements in \( H^1(S^{[2]}, \mathbb{Q}) \) with \( i \geq 10 \), so these are equal to zero. We do not compute explicitly \( R3_{\nu=1} \): we will see that this is not necessary. If \( \nu = 2 \), using Definition 5.5 we obtain, after some computations,

\[
R3_{\nu=2} = \sum_{i,j} \mu_{i,j} \frac{1}{2} \left( \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \right) \cdot q_1(1)|0\rangle q_1(1)|0\rangle + \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot q_1(1)|0\rangle q_1(x)|0\rangle,
\]

which is an element of \( H^8(S^{[2]}, \mathbb{Z}) \). We conclude that

\[
R3 = \sum_{i,j} \mu_{i,j} q_1(1)|0\rangle q_1(1)|0\rangle + 2q_1(1)q_1(x)|0\rangle - 2q_1(x)q_1(1)|0\rangle + R3_{\nu=1}
\]

\[
+ \frac{1}{2} \sum_{i,j} \mu_{i,j} \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot q_1(1)|0\rangle q_1(1)|0\rangle
\]

\[
+ \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot q_1(1)|0\rangle q_1(1)|0\rangle.
\]

By Lemma 7.2 we have \( \mathcal{L}^\vee \cong \text{Hom}(\mathcal{L}, \mathcal{O}_{S^{[1]}, 2}) \). Hence if \( d := c_1(\mathcal{L}) \) we have \( \text{ch}(\mathcal{L}^\vee) = \sum_{\nu \geq 0} \frac{(-1)^\nu}{\nu!} d^\nu \), so \( R4 \) is computed in the same way as \( R2 \), with a change of sign for the components obtained when \( \nu = 1 \) and \( \nu = 3 \). We obtain

\[
R4 = 2 \cdot 1_{S^{[2]}} + q_2(1)|0\rangle + \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \\
+ \frac{1}{3} \left( \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \right).
\]

\[
+ \frac{1}{17} \left( \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \cdot \frac{1}{2}q_2(1)|0\rangle \right).
\]}
• By Lemma 7.2 we have \( L' \cong \text{Hom}(L, \mathcal{O}_{S[12]}) \) and \( \mathcal{O}_{\Delta}^\vee = \mathcal{O}_\Delta \), so \( R5 \) is computed in the same way as \( R3 \), with a change of sign for the component \( R3_{\nu=1} \), so we obtain

\[
R5 = \sum_{i,j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0| + 2q_1(1)q_1(x)|0| - 2q_1(x)q_1(|0|) - R3_{\nu=1}
\]

\[
\begin{align*}
+ \frac{1}{2} \sum_{i,j} \mu_{i,j} \frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| \cdot q_1(\alpha_i)q_1(\alpha_j)|0| \\
+ \frac{1}{8} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| \cdot q_1(1)q_1(x)|0| .
\end{align*}
\]

• Since \( S \) is a K3 surface, we have \( \text{ch}(2O_S - T_S) = 24x \), where \( x \in H^4(S, \mathbb{Z}) \) is the class of a point. Then

\[ R6 = 24q_1(1)q_1(x)|0|, \]

where we have used \( q_1(x)q_1(1)|0| = q_1(1)q_1(x)|0| \) from Theorem 5.2.

Thus formula (7.2) with \( x = q_1(1)|0| \) and \( \alpha = 1 \) gives

\[
L1 = 8 \cdot 1_{S[2]} - 52q_1(1)q_1(x)|0| + 2(\frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0|) - 2 \sum_{i,j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0|
\]

\[
+ \frac{1}{2} \left( \frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| \right) \\
+ 4q_1(x)q_1(1)|0|
\]

\[
= \sum_{i,j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0| + 2q_1(1)q_1(x)|0| - 2q_1(x)q_1(|0|) - R3_{\nu=1}
\]

(7.6)

where

\[
L1 = 8 \cdot 1_{S[2]} - 2c_2(S^{[2]}) + \frac{1}{6} c_2(S^{[2]})^2 - \frac{1}{3} c_4(S^{[2]}).
\]

(7.7)

We now impose equalities between elements belonging to \( H^4(S^{[2]}, \mathbb{Z}) \) in the right-hand side of (7.6) and (7.7). We obtain

\[
c_2(S^{[2]}) = 26q_1(1)q_1(x)|0| - \frac{1}{2} q_2(1)|0| \cdot \frac{1}{2} q_2(1)|0| + \sum_{i,j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0|.
\]

Using the commutativity rule of Theorem 5.2 and Lemma 5.13.(iii), we get

\[
c_2(S^{[2]}) = 27q_1(1)q_1(x)|0| + 3 \sum_{i<j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0| + \frac{3}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0|,
\]

and we are done. \( \square \)

8. Integral Hodge classes of type \((2, 2)\) on Hilbert squares of any K3 surface

In this section we compute a basis for the lattice \( H^{2,2}(S^{[2]}, \mathbb{Z}) \) of integral Hodge classes of type \((2, 2)\) on the Hilbert square of a general projective K3 surface \( S \) whose Picard group is known. Proposition 7.4 implies the following.

Theorem 8.1. Let \( S \) be a projective K3 surface and \( X = S^{[2]} \) be its Hilbert square. Consider \( q_X^\vee \in H^{2,2}(X, \mathbb{Q}) \), the dual of the BBF quadratic form. Then

\[
\frac{2}{3} q_X^\vee = 9q_1(1)q_1(x)|0| + \sum_{i,j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0| + \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0| \in H^{2,2}(X, \mathbb{Z}),
\]

(8.1)

where \( 1 \in H^0(S, \mathbb{Z}) \) is the unit, \( x \in H^4(S, \mathbb{Z}) \) is the class of a point, \( \{\alpha_1, \ldots, \alpha_{22}\} \) is the basis of \( H^2(S, \mathbb{Z}) \) used in Lemma 5.10 and the \( \mu_{i,j} \)'s are the integers given in Table 1. Moreover, \( \frac{2}{3} q_X^\vee \) is indivisible in \( H^{2,2}(X, \mathbb{Z}) \) and

\[
\frac{1}{8} \left( s^2 + \frac{2}{3} q_X^\vee \right) = q_1(1)q_1(x)|0| \in H^{2,2}(X, \mathbb{Z}).
\]

Proof. By Proposition 4.4 we have \( \frac{2}{3} q_X^\vee = c_2(X) \), so Proposition 7.4 implies (8.1). Moreover, taking the basis of \( H^4(X, \mathbb{Z}) \) given by Theorem 5.8, we see that \( \frac{2}{3} q_X^\vee \) is indivisible in \( H^4(X, \mathbb{Z}) \), i.e., there is no \( \alpha \in H^4(X, \mathbb{Z}) \) such that \( n\alpha = \frac{2}{3} q_X^\vee \) for some integer \( n \in \mathbb{Z}_{>1} \): by Theorem 5.8 it suffices to find some \( \mu_{i,j} \) which are coprime with 9, the
coefficient of \( q_1(1)q_1(x)|0 \) in (8.1), in Table 1. This implies that \( \frac{2}{5}q_X^\vee \) is indivisible also in \( H^{2,2}(X, \mathbb{Z}) \). Recall that by Lemma 5.13 we have
\[
\delta^2 = -\sum_{i<j} \mu_{i,j}q_1(\alpha_i)q_1(\alpha_j)|0| - \frac{1}{2} \sum_i \mu_{i,i}q_1(\alpha_i)^2|0| - q_1(1)q_1(x)|0|,
\]
thus from (8.1) and (8.2) we obtain \( \delta^2 + \frac{2}{5}q_X^\vee = 8q_1(1)q_1(x)|0| \in H^{2,2}(X, \mathbb{Z}) \), which implies
\[
\frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right) = q_1(1)q_1(x)|0| \in H^{2,2}(X, \mathbb{Z}).
\]

Note that the element \( \frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right) \) appears also in [She16, Lemma 4.3]. It is possible to verify that Theorem 8.1 is consistent with the computation of \( \text{ch}(S[2]) \) given by the MADE program in [BNW07, §11], based on results obtained by Boissière in [Boi05], see [Nov, §3.4.3] for details.

**Remark 8.2.** If \( S \) is a projective K3 surface, the lattice \( H^{2,2}(S[2], \mathbb{Z}) \) is always an odd lattice: this follows from the product \( \left( \frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right), \frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right) \right) = 1 \).

We can now pass to the main result of this paper. Let \( S \) be a general projective K3 surface. In Theorem 6.2 we have given a basis of the vector space \( H^{2,2}(S[2], \mathbb{Q}) \) for a general projective K3 surface \( S \). Then in Theorem 6.4 we have described a basis of the lattice \( H^{2,2}(S[2], \mathbb{Z}) \) when \( S \) is generic, i.e., general with Picard group of rank \( r = 1 \). We now present a basis of the lattice \( H^{2,2}(S[2], \mathbb{Z}) \) for any general projective K3 surface \( S \) with Picard group of rank \( r \), where \( 1 \leq r \leq 19 \), since \( h^{1,1}(S) = 20 \) for a K3 surface and, as remarked in Section 1, a K3 surface of Picard rank 20 is not general. We will give both a basis in terms of Nakajima operators and a basis which does not depend on Nakajima operators. In the particular case of general K3 surfaces, the next theorem will prove the conjecture in Section 7. We will use results obtained in Section 7 and Theorem 8.1.

**Theorem 8.3.** Let \( S \) be a general projective K3 surface and let \( \{b_1, \ldots, b_r\} \) be a basis of \( \text{Pic}(S) \). Then:
\[ (i) \quad \text{rk} \left( H^{2,2}(S[2], \mathbb{Z}) \right) = \frac{(r+1)r}{2} + r + 2. \]
\[ (ii) \quad \text{The lattice } H^{2,2}(S[2], \mathbb{Z}) \text{ is odd and a basis is given by the following elements:} \]
- \( q_2(b_i)|0| \), for \( i = 1, \ldots, r \),
- \( q_1(1)q_1(x)|0| \), where \( 1 \in H^0(S, \mathbb{Z}) \) is the unit and \( x \in H^4(S, \mathbb{Z}) \) is the class of a point.
- \( \frac{1}{2} \left( q_1(b_i)^2 - q_2(b_i) \right)|0| \), for \( i = 1, \ldots, r \),
- \( q_1(b_i)q_1(b_j)|0| \), for \( 1 \leq i < j \leq r \),
- \( -\sum_{i<j} \mu_{i,j}q_1(\alpha_i)q_1(\alpha_j)|0| - \frac{1}{2} \sum_i \mu_{i,i}q_1(\alpha_i)^2|0| - q_1(1)q_1(x)|0| \), where \( \{\alpha_1, \ldots, \alpha_{22}\} \) is the basis of \( H^2(S, \mathbb{Z}) \)
used in Lemma 5.10 and the \( \mu_{i,j} \)'s are given in Table 1.

Equivalently, the following is a basis of \( H^{2,2}(S[2], \mathbb{Z}) \):
\[
\left\{ b_i b_j, \frac{b_i^2 - b_j \delta}{2}, \frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right), \delta^2 \right\}_{1 \leq i \leq j \leq r}.
\]

In particular, if \( S = S_{2t} \) is a generic K3 surface of degree \( 2t \), and \( h \in \text{Pic}(S_{2t}) \) is the class induced by the ample generator of \( \text{Pic}(S_{2t}) \), then
\[
H^{2,2}(S_{2t}[2], \mathbb{Z}) = \mathbb{Z} h^2 \oplus \mathbb{Z} \frac{h^2 - h \delta}{2} \oplus \mathbb{Z} \frac{1}{8} \left( \delta^2 + \frac{2}{5}q_X^\vee \right) \oplus \mathbb{Z} \delta^2.
\]

Moreover, \( \text{disc}(H^{2,2}(S_{2t}[2], \mathbb{Z})) = 84t^3 \) and the Gram matrix in the basis given above is the following:
\[
\begin{pmatrix}
12t^2 & 6t^2 & 2t & -4t \\
6t^2 & t(3t - 1) & t & -2t \\
2t & t & 1 & -1 \\
-4t & -2t & -1 & 12
\end{pmatrix}
\]

Proof. By Theorem 6.2 we have \( \text{dim}(H^{2,2}(S[2], \mathbb{Q})) = \frac{(r+1)r}{2} + r + 2 \). Since by [Mar07, Theorem 1] the cohomology groups \( H^i(S[2], \mathbb{Z}) \) are torsion free for \( i \geq 0 \), we obtain \( \text{rk}(H^{2,2}(S[2], \mathbb{Z})) = \frac{(r+1)r}{2} + r + 2 \). Remark 8.2 shows that
$H^{2,2}(S^{[2]},\mathbb{Z})$ is an odd lattice. After a slight modification of the basis given in Theorem 6.2, we have that the following is a basis of $H^{2,2}(S^{[2]},\mathbb{Q})$:

$$q_2(b_i)|0\rangle\text{ for }i=1,\ldots, r$$

$$q_1(1)q_1(x)|0\rangle$$

$$\frac{1}{2} (q_1(b_i)^2 - q_2(b_i))|0\rangle\text{ for }i=1,\ldots, r$$

$$q_1(b_i)q_1(b_j)|0\rangle\text{ for }1\leq i < j \leq r$$

$$- \sum_{i<j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0\rangle - \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0\rangle - q_1(1)q_1(x)|0\rangle.$$

(8.5)

The strategy of the proof is the following: we look for a sublattice $L$ of $H^4(S^{[2]},\mathbb{Z})$ of maximal rank such that $L\cap H^{2,2}(S^{[2]},\mathbb{Q}) = H^{2,2}(S^{[2]},\mathbb{Z})$ and such that a basis of $L$ contains the elements in (8.5). Since the Picard group $\text{Pic}(S)$ of $S$ can be primitively embedded in $H^2(S,\mathbb{Z})$, there exists a basis of $H^2(S,\mathbb{Z})$ of the form \{$b_1,\ldots, b_r, b_{r+1},\ldots, b_{22}$\} for some $b_{r+1},\ldots, b_{22} \in H^2(S,\mathbb{Z})$. By Theorem 5.8 the following is a basis of $H^4(S^{[2]},\mathbb{Z})$:

$$\left\{ q_1(1)q_1(x)|0\rangle, q_2(b_i)|0\rangle, q_1(b_i)q_1(b_j)|0\rangle, \frac{1}{2} (q_1(b_i)^2 - q_2(b_i))|0\rangle \right\},$$

(8.6)

where $i,j \in \{1,\ldots, 22\}$ and $i < j$. Recall that by Lemma 5.13 we have

$$\delta^2 = - \sum_{i<j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0\rangle - \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0\rangle - q_1(1)q_1(x)|0\rangle,$$

(8.7)

where \{\alpha_1,\ldots, \alpha_{22}\} is the basis of the lattice $H^2(S,\mathbb{Z})$ used in Lemma 5.10 and the $\mu_{i,j}$’s are the integers given in Table 1. Using the same procedure of Lemma 5.10 and Lemma 5.13 with the basis \{$b_1,\ldots, b_{22}$\}, we obtain

$$\delta^2 = - \sum_{i<j} \sigma_{i,j} q_1(b_i)q_1(b_j)|0\rangle - \frac{1}{2} \sum_i \sigma_{i,i} q_1(b_i)^2|0\rangle - q_1(1)q_1(x)|0\rangle$$

(8.8)

for some integers $\sigma_{i,j}$. Then the $\mu_{i,j}$’s are associated to the basis \{\alpha_1,\ldots, \alpha_{22}\} by the description of $\delta^2$ in (8.7), and the $\sigma_{i,j}$’s are associated to the basis \{$b_1,\ldots, b_{22}$\} by the description of $\delta^2$ in (8.8).

We show that there exist positive integers $l$ and $k$ with $l < k$ and $k \geq r + 1$ such that $\sigma_{l,k} \neq 0$. Suppose by contradiction that $\sigma_{i,j} = 0$ for every $(i,j)$ such that $i < j$ and $j \geq r + 1$. Then (8.8) becomes

$$\delta^2 = - \sum_{1\leq i<j\leq r} \sigma_{i,j} q_1(b_i)q_1(b_j)|0\rangle - \frac{1}{2} \sum_{i=1}^{22} \sigma_{i,i} q_1(b_i)^2|0\rangle - q_1(1)q_1(x)|0\rangle.$$

(8.9)

Consider the transcendental lattice $T(S) \cong \text{NS}(S)^\perp$. Let $x \in T(S)$. Since $H^2(S,\mathbb{Z})$ is a unimodular lattice, there exists $y \in H^2(S,\mathbb{Z})$ such that $\int_S xy = 1$. By Proposition 4.2 we have

$$\langle \delta^2, xy \rangle = -2.$$

(8.10)

By (8.9), Lemma 5.13 and Theorem 8.1 we have

$$\langle \delta^2, xy \rangle = - \sum_{1\leq i<j\leq r} \sigma_{i,j} \left[ b_ib_j - \left( \int_S b_ib_j \right) \left( \frac{1}{8} \delta^2 + \frac{1}{20} q_1^2 \right) \right]$$

$$- \frac{1}{2} \sum_{i=1}^{22} \sigma_{i,i} \left[ b_i^2 - \left( \int_S b_i^2 \right) \left( \frac{1}{8} \delta^2 + \frac{1}{20} q_1^2 \right) \right]$$

$$- \left( \frac{1}{8} \delta^2 + \frac{1}{20} q_1^2 \right) \langle x, y \rangle.$$

(8.11)

Since $x \in T(S)$ we have $\int_S b_ix = 0$ for $i = 1,\ldots, r$, hence the right-hand side of (8.11) is equal to

$$- \sum_{1\leq i<j\leq r} \sigma_{i,j} \left[ \int_S b_ib_j + \frac{1}{4} \int_S b_ib_j - \frac{5}{4} \int_S b_ib_j \right]$$

$$- \frac{1}{2} \sum_{i=1}^{r} \sigma_{i,i} \left[ \int_S b_i^2 + \frac{1}{4} \int_S b_i^2 - \frac{5}{4} \int_S b_i^2 \right]$$

$$- \frac{1}{2} \sum_{i=r+1}^{22} \sigma_{i,i} \left[ \int_S b_i^2 + 2 \int_S b_ix \int_S b_iy + \frac{1}{4} \int_S b_i^2 - \frac{5}{4} \int_S b_i^2 \right].$$
Note that \( \int_S b_i b_j + \frac{1}{4} \int_S b_i b_j - \frac{5}{4} \int_S b_i b_j = 0 \) and \( \int_S b_i^2 + \frac{1}{4} \int_S b_i^2 - \frac{5}{4} \int_S b_i^2 = 0 \), hence we finally obtain

\[
\langle \delta^2, xy \rangle = - \sum_{i=r+1}^{22} \sigma_{i,i} \int_S b_i x \int_S b_i y - 1. \tag{8.12}
\]

Thus (8.10) and (8.12) imply

\[
\sum_{i=r+1}^{22} \sigma_{i,i} \int_S b_i x \int_S b_i y = 1. \tag{8.13}
\]

The \( \sigma_{i,i} \)'s are all even, otherwise by Theorem 5.8 the element \( \delta^2 \) in (8.8) would not be integral, since a basis of \( H^1(S^{[2]}, \mathbb{Z}) \) is given by (8.6). Hence the left-hand side of (8.13) is even, so it cannot be equal to 1. We get a contradiction, so there exist \( l, k \) positive integers with \( l < k \) and \( k \geq r + 1 \) such that \( \sigma_{l,k} \neq 0 \). Let now \( L \) be the sublattice of \( H^4(S^{[2]}, \mathbb{Z}) \) with the following basis:

(i) \( q_1(1)q_1(x)|0\),

(ii) \( q_1(b_i)q_1(b_j)|0 \) with \( 1 \leq i < j \leq 22 \) and \( (i,j) \neq (l,k) \),

(iii) \( \delta^2 = - \sum_{i<j} \sigma_{i,j} q_1(b_i)q_1(b_j)|0 - \frac{1}{2} \sum_i \sigma_{i,i} q_1(b_i)^2|0 - q_1(1)q_1(x)|0 \),

(iv) \( q_2(b_i)|0 \) for \( i = 1, \ldots, 22 \),

(v) \( \frac{1}{2} (q_1(b_i)^2 - q_2(b_i))|0 \) for \( i = 1, \ldots, 22 \).

Recall that by (8.8) the element in (iii) is also equal to

\[
- \sum_{i<j} \mu_{i,j} q_1(\alpha_i)q_1(\alpha_j)|0 - \frac{1}{2} \sum_i \mu_{i,i} q_1(\alpha_i)^2|0 - q_1(1)q_1(x)|0.
\]

Since \( \sigma_{l,k} \neq 0 \), the elements in (8.14) give a basis for \( H^4(S^{[2]}, \mathbb{Q}) \), thus \( L \) is a sublattice of \( H^4(S^{[2]}, \mathbb{Z}) \) of maximal rank. If \( \sigma_{l,k} = \pm 1 \), then \( q_1(b_i)q_1(b_k)|0 \) can be obtained as an integral linear combination of elements in (8.14), hence every element in the basis (8.6) of \( H^4(S^{[2]}, \mathbb{Z}) \) is in \( L \), so \( L = H^4(S^{[2]}, \mathbb{Z}) \) and (8.5) is a basis of \( H^{2,2}(S^{[2]}, \mathbb{Z}) \): that is what happens in the proof of Theorem 6.4. If \( \sigma_{l,k} \neq \pm 1 \), then \( L \neq H^4(S^{[2]}, \mathbb{Z}) \). More precisely, we have

\[
\frac{H^4(S^{[2]}, \mathbb{Z})}{L} \cong \frac{\mathbb{Z}}{\sigma_{l,k} \mathbb{Z}} \tag{8.15}
\]

generated by \( q_1(b_i)q_1(b_k)|0 \). We show that

\( L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) = H^{2,2}(S^{[2]}, \mathbb{Z}) \).

The inclusion \( L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) \subseteq H^{2,2}(S^{[2]}, \mathbb{Z}) \) is clear. We now prove the inclusion \( L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) \supseteq H^{2,2}(S^{[2]}, \mathbb{Z}) \) by showing that if \( z \notin L \cap H^{2,2}(S^{[2]}, \mathbb{Q}) \) then \( z \notin H^{2,2}(S^{[2]}, \mathbb{Z}) \). If \( z \notin H^{2,2}(S^{[2]}, \mathbb{Q}) \) we are done. Suppose now that \( z \notin L \). Clearly we have \( H^4(S^{[2]}, \mathbb{Z}) \cap H^{2,2}(S^{[2]}, \mathbb{Q}) = H^{2,2}(S^{[2]}, \mathbb{Z}) \). Since the quotient in (8.15) is generated by \( q_1(b_i)q_1(b_k)|0 \), it suffices to show that \( q_1(1)q_1(b_k)|0 \notin H^{2,2}(S^{[2]}, \mathbb{Q}) \) to get the inclusion. Suppose by contradiction that \( q_1(1)q_1(b_k)|0 \in H^{2,2}(S^{[2]}, \mathbb{Q}) \). Hence \( q_1(b_i)q_1(b_k)|0 \) is a rational linear combination of elements in (8.5). Recall that the last element in (8.5) can be written as

\[
\delta^2 = - \sum_{i<j} \sigma_{i,j} q_1(b_i)q_1(b_j)|0 - \frac{1}{2} \sum_i \sigma_{i,i} q_1(b_i)^2|0 - q_1(1)q_1(x)|0.
\]

Since \( q_1(b_i)q_1(b_k)|0 \) appears only in \( \delta^2 \) among the elements in (8.5), we see that \( q_1(b_i)q_1(b_k)|0 \in H^{2,2}(S^{[2]}, \mathbb{Q}) \) only if \( \sigma_{i,j} = 0 \) for \( i \leq j, j \geq r + 1 \) and \( (i,j) \neq (l,k) \).

Let again \( x \in T(S) \) and \( y \in H^2(S, \mathbb{Z}) \) such that \( \int_S xy = 1 \). Similarly to (8.12) we have

\[
\langle \delta^2, xy \rangle = - \sum_{j \geq r+1} \sigma_{i,j} \left( \int_S b_i x \int_S b_j y + \int_S b_j x \int_S b_i y \right) - \frac{1}{2} \sum_{i=r+1}^{22} \sigma_{i,i} \left( 2 \int_S b_i x \int_S b_i y \right) - 1, \tag{8.17}
\]

which implies by (8.10) the following:

\[
\sum_{i<j} \sigma_{i,j} \left( \int_S b_i x \int_S b_j y + \int_S b_j x \int_S b_i y \right) + \sum_{i=r+1}^{22} \sigma_{i,i} \left( \int_S b_i x \int_S b_i y \right) = 1.
\]
Thus we see that (8.16) is not true, otherwise (8.17) becomes $\sigma_{k,l}(\int_S b_l x \int_S b_k y + \int_S b_k x \int_S b_l y ) = 1$, which is false since by assumption $\sigma_{k,l} \neq \pm 1$. Hence there exists $(\alpha, \beta) \neq (l, k)$ with $\alpha \leq \beta$ and $\beta \geq r+1$ such that $\sigma_{\alpha, \beta} \neq 0$. As remarked above, this shows that $q_1(b_i)q_1(b_k)(0) \not\in H^{2,2}(S[2], \mathbb{Q})$. Then

$$L \cap H^{2,2}(S[2], \mathbb{Q}) = H^{2,2}(S[2], \mathbb{Z}).$$

We conclude that (8.5) is a basis of $H^{2,2}(S[2], \mathbb{Z})$. Using Remark 5.9, Lemma 5.13 and Theorem 8.1, the elements in (8.5) can be written as:

$$b_i \delta$$ for $i = 1, \ldots, r$,

$$\frac{1}{8} \left( \delta^2 + \frac{2}{5} q_X^2 \gamma \right),$$

$$\frac{1}{2} \left( b_i^2 - \int_S b_i^2 \cdot \frac{1}{8} \left( \delta^2 + \frac{2}{5} q_X^2 \right) - b_i \delta \right)$$ for $i = 1, \ldots, r$,

$$b_i b_j - \int_S b_i b_j \cdot \frac{1}{8} \left( \delta^2 + \frac{2}{5} q_X^2 \right)$$ for $1 \leq i < j \leq r$.

We write $b_i$ both for the element in Pic(S) and for the line bundle that this induces on Pic(S[2]). By Remark 5.9 we have $b_i = q_1(1)q_1(b_i)(0)$ and $\delta = \frac{1}{2}q_2(1)(0)$ in $H^4(S[2], \mathbb{Z})$. Then, starting from the basis of $H^{2,2}(S[2], \mathbb{Z})$ found before, using Lemma 5.13 one can show that (8.3) is a basis of $H^{2,2}(S[2], \mathbb{Z})$. In particular when $S = S_{2t}$ is a generic K3 surface of degree $2t$ we have (8.4): in this case the discriminant of $H^{2,2}(S_{2t}[2], \mathbb{Z})$ and the Gram matrix can be computed using Proposition 4.3.

\[\square\]

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*Università degli Studi di Milano, Dipartimento di Matematica “F. Enriques”, Via Cesare Saldini 50, 20133 Milano, Italy.

*Université de Poitiers, Laboratoire de Mathématiques et Applications, Téléport 2, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil, France.*

*Email address: simone.novario@unimi.it, simone.novario@math.univ-poitiers.fr*