Jost-Lehmann-Dyson Representation
and Froissart-Martin Bound
in Quantum Field Theory
on Noncommutative Space-Time *

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Abstract In the framework of quantum field theory (QFT) on noncommutative (NC) space-time with $SO(1,1) \times SO(2)$ symmetry, which is the feature arising when one has only space-space noncommutativity ($\theta_{0i} = 0$), we prove that the Jost-Lehmann-Dyson representation, based on the causality condition usually taken in connection with this symmetry, leads to the mere impossibility of drawing any conclusion on the analyticity of the $2 \rightarrow 2$-scattering amplitude in $\cos \Theta$, $\Theta$ being the scattering angle. A physical choice of the causality condition rescues the situation and as a result an analog of Lehmann’s ellipse as domain of analyticity in $\cos \Theta$ is obtained. However, the enlargement of this analyticity domain to Martin’s ellipse and the derivation of the Froissart bound for the total cross-section in NC QFT is possi-

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ble only in the special case when the incoming momentum is orthogonal to the NC plane. This is the first example of a nonlocal theory in which the cross-sections are subject to a high-energy bound. For the general configuration of the direction of the incoming particle, although the scattering amplitude is still analytic in the Lehmann ellipse, no bound on the total cross-section has been derived. This is due to the lack of a simple unitarity constraint on the partial-wave amplitudes, which could be used in this case. High-energy upper bounds on the total cross-section, among others, are also obtained for an arbitrary flat (noncompact) dimension of NC space-time.

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1 Introduction

The development of QFT on NC space-time, especially after the seminal work of Seiberg and Witten [1], which showed that the NC QFT arises from string theory, has triggered lately the interest also towards the formulation of an axiomatic approach to the subject. Consequently, the analytical properties of scattering amplitude in energy $E$ and forward dispersion relations have been considered [2, 3], Wightman functions have been introduced and the CPT theorem has been proven [4, 5], and as well attempts towards a proof of the spin-statistics theorem have been made [5].

In the axiomatic approach to commutative QFT, one of the fundamental results consisted of the rigorous proof of the Froissart bound on the high-energy behaviour of the scattering

*In the context of the Lagrangian approach to NC QFT, the CPT and spin-statistics theorems have been proven in general in [6]; for CPT invariance in NC QED, see [7, 8], and in NC Standard Model [9].
amplitude, based on its analyticity properties [10, 11]. In this paper we aim at obtaining
the analog of this bound when the space-time is noncommutative. Such an achievement,
besides being topical in itself, will also prove fruitful in the conceptual understanding of
subtle issues, such as causality, in nonlocal theories to which the NC QFT’s belong.

In the following we shall consider NC QFT on a space-time with the commutation relation

$$[x_\mu, x_\nu] = i\theta_{\mu\nu},$$  \hspace{1cm} (1.1)

where $\theta_{\mu\nu}$ is an antisymmetric constant matrix (for a review, see, e.g., [12, 13]). Such NC
theories violate Lorentz invariance, while translational invariance still holds. We can always
choose the system of coordinates, such that $\theta_{13} = \theta_{23} = 0$ and $\theta_{12} = -\theta_{21} \equiv \theta$. Then, for
the particular case of space-space noncommutativity, i.e. $\theta_{0i} = 0$, the theory is invariant
under the subgroup $SO(1,1) \times SO(2)$ of the Lorentz group. The requirement that time be
commutative ($\theta_{0i} = 0$) discards the well-known problems with the unitarity [14] of the NC
theories and with causality [15, 16]. As well, the $\theta_{0i} = 0$ case allows a proper definition of
the $S$-matrix [3].

In the conventional (commutative) QFT, the Froissart bound was first obtained [10] using
the conjectured Mandelstam representation (double dispersion relation) [17], which assumes
analyticity in the entire $E$ and $\cos \Theta$ complex planes. The Froissart bound,

$$\sigma_{tot}(E) \leq c \ln^2 \frac{E}{E_0},$$  \hspace{1cm} (1.2)

expresses the upper limit of the total cross-section $\sigma_{tot}$ as a function of the CMS energy $E$,
when $E \to \infty$. However, such an analyticity or equivalently the double dispersion relation
has not been proven, while smaller domains of analyticity in $\cos \Theta$ were already known [18].
One of the main ingredients in rigorously obtaining the Froissart bound is the Jost-Lehmann-Dyson representation [19, 20] of the Fourier transform of the matrix element of the commutator of currents, which is based on the causality as well as the spectral conditions (for an overall review, see [21]). Based on this integral representation, one obtains the domain of analyticity of the scattering amplitude in \( \cos \Theta \). This domain proves to be an ellipse—the so-called Lehmann’s ellipse [18]. Using this domain of analyticity, Greenberg and Low [23] found a weaker bound on the high-energy behaviour of the total cross-section:

\[
\sigma_{\text{tot}}(E) \leq c \ E^2 \ln^2 \frac{E}{E_0}.
\]  

However, the domain of analyticity in \( \cos \Theta \) can be enlarged to the so-called Martin’s ellipse by using the dispersion relations satisfied by the scattering amplitude and the unitarity constraint on the partial-wave amplitudes. Using this larger domain of analyticity, the Froissart bound (1.2) was rigorously proven [11] (for a review, see [22]).

Further on, the analog of the Froissart-Martin bound was rigorously obtained for the \( 2 \to 2 \)-particle scattering in a space-time of arbitrary dimension \( D \) [24, 25].

In NC QFT with \( \theta_{0i} = 0 \) we shall follow the same path for the derivation of the high-energy bound on the scattering amplitude, starting from the Jost-Lehmann-Dyson representation and adapting the derivation to the new symmetry \( SO(1, 1) \times SO(2) \) and to the nonlocality of the NC theory. In Section 2 we derive the Jost-Lehmann-Dyson representation satisfying the light-wedge (instead of light-cone) causality condition, which has been used so far, being inspired by the above symmetry. In Section 3 we show that no analyticity of the scattering amplitude in \( \cos \Theta \) can be obtained in such a case. However, with a newly introduced causality condition, based on physical arguments, we obtain from the
Jost-Lehmann-Dyson representation a domain of analyticity in cos Θ, which coincides with the Lehmann ellipse. In Section 4 we show that the extension of this analyticity domain to Martin’s ellipse is possible in the case of the incoming particle momentum orthogonal to the NC plane (x_1, x_2), which eventually enables us to derive rigorously the analog of the Froissart-Martin bound (1.2) for the total cross-section. The general configuration of incoming particle momentum is also discussed, together with the problems which arise. In Section 5 we consider higher-dimensional NC theories and obtain the high-energy bounds for a particular setting, in the case of even dimension D; we also discuss the peculiarities of the NC theories on space-time of higher odd dimension. Section 6 is devoted to conclusion and discussions.

2 Jost-Lehmann-Dyson representation

The Jost-Lehmann-Dyson representation [19, 20] is the integral representation for the Fourier transform of the matrix element of the commutator of currents:

\[ f(q) = \int d^4 x e^{iqx} f(x) , \]  

where

\[ f(x) = \langle p' | [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] | p \rangle , \]  

satisfying the causality and spectral conditions. The process considered is the 2 → 2 scalar particles scattering, \( k + p \rightarrow k' + p' \), and \( j_1 \) and \( j_2 \) are the scalar currents corresponding to the incoming and outgoing particles with momenta \( k \) and \( k' \) (see also [21, 26]).

For NC QFT with \( SO(1, 1) \times SO(2) \) symmetry, in [27] a new causality condition was pro-
posed, involving (instead of the light-cone) the light-wedge corresponding to the coordinates 
\( x_0 \) and \( x_3 \), which form a two-dimensional space with the \( SO(1,1) \) symmetry. Accordingly we shall require the vanishing of the commutator of two currents (in general, observables) at space-like separations in the sense of \( SO(1,1) \) as:

\[
[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, \quad \text{for} \quad x^2 \equiv x_0^2 - x_3^2 < 0.
\] (2.3)

The spectral condition compatible with (2.3) would require now that the physical momenta be in the forward light-wedge:

\[
\tilde{p}_2^2 \equiv p_0^2 - p_3^2 > 0 \quad \text{and} \quad p_0 > 0.
\] (2.4)

The spectral condition (2.4) will impose restrictions on \( f(q) \). Using the translational invariance in (2.2), one can express the matrix element of the commutator of currents, \( f(x) \), in the form:

\[
f(x) = \int dq e^{-iqx+ip+p'}\frac{q}{2} G_1(q) - \int dq e^{iqx-ipy}\frac{q}{2} G_2(q)
\]

\[
= \int dq e^{-iqx} \left[ G_1 \left( q + \frac{1}{2}(p+p') \right) - G_2 \left( -q + \frac{1}{2}(p+p') \right) \right],
\] (2.5)

where

\[
G_1(q) = \langle p'|j_1(0)|q \rangle \langle q|j_2(0)|p \rangle,
\]

\[
G_2(q) = \langle p'|j_2(0)|q \rangle \langle q|j_1(0)|p \rangle.
\] (2.6)

Comparing (2.5) with the inverse Fourier transformation\(^\dagger\), \( f(x) = \int dq e^{-iqx} f(q) \), it follows that

\[
f(q) = f_1(q) - f_2(q) = G_1 \left( q + \frac{1}{2}(p+p') \right) - G_2 \left( -q + \frac{1}{2}(p+p') \right).
\] (2.7)

\(^\dagger\)Throughout the paper we omit all the inessential factors of \((2\pi)^n\), which are irrelevant for the analyticity considerations.
Given the way the functions $G_1$ and $G_2$ are defined in (2.6), one finds that $f(q) = 0$ in the region where the momenta $q + \frac{1}{2}(p + p')$ and $-q + \frac{1}{2}(p + p')$ are simultaneously nonphysical, i.e. when they are out of the future light-wedge (2.4).

In order to express the condition for $f(q) = 0$, we shall define the $SO(1,1)$-invariant \[ \tilde{m}^2 = k_0^2 - k_3^2 = f(m^2, k_1^2 + k_2^2), \] where $k$ is the momentum of an arbitrary state and $m$ is its mass. However, we have to point out that $\tilde{m}$ is only a kinematical variable, invariant with respect to $SO(1,1)$ (but not the mass).

For the physical states with momentum $q + \frac{1}{2}(p + p')$, we take $\tilde{m}_1$ to be the minimal value of the $SO(1,1)$-invariant quantity above. Then, in the Breit frame, where $\frac{1}{2}(p + p') = (p_0, 0, 0, 0)$, one finds that $f_1(q) \neq 0$ for all the $q$ values, satisfying the spectral condition $q_0 + p_0 \geq 0$ and $(q_0 - p_0)^2 - q_3^2 \geq 0$. In other words, $f_1(q) = 0$ for $q_0 < -p_0 + \sqrt{q_3^2 + \tilde{m}_1^2}$.

Similarly one finds that $f_2(q) = 0$ for $p_0 - \sqrt{q_3^2 + \tilde{m}_2^2} < q_0$ (where $\tilde{m}_2$ has a meaning analogous to that of $\tilde{m}_1$, but for the states with the momentum $-q + \frac{1}{2}(p + p')$).

As a result, due to the spectral condition (2.4), $f(q) = 0$ in the region outside the hyperbola

\[ p_0 - \sqrt{q_3^2 + \tilde{m}_2^2} < q_0 < -p_0 + \sqrt{q_3^2 + \tilde{m}_1^2}. \] (2.8)

To derive the Jost-Lehmann-Dyson representation, further we consider the 6-dimensional space-time with the Minkowskian metric $\left(+,-,-,-,-,-\right)$. On this space, we define the vector $z = (x_0, x_1, x_2, x_3, y_1, y_2)$. For practical purposes we introduce also the notations for the 2-dimensional vector $\tilde{x} = (x_0, x_3)$ and the 4-dimensional vector $\tilde{z} = (z_0, z_3, z_4, z_5) \equiv (x_0, x_3, y_1, y_2)$. On the 6-dimensional space we define the function

\[ F(z) = f(x)\delta(\tilde{x}^2 - y^2) = f(x)\delta(\tilde{z}^2), \] (2.9)
depending on all 6 coordinates.

When the causality condition (2.3) is fulfilled, i.e. for the physical region, \(f(x)\) and \(F(z)\) determine each other, since

\[
\int dy_1dy_2 F(z) = f(x)\theta(\bar{x}^2) = \begin{cases} f(x) & \text{for } \bar{x}^2 > 0 , \\ 0 & \text{for } \bar{x}^2 < 0 . \end{cases} \tag{2.10}
\]

The Fourier transform of \(F(z)\),

\[
F(r) = \int d^6ze^{izr} F(z) , \tag{2.11}
\]

can be expressed, using (2.9) and (2.10), as

\[
F(r) = \int d^4qD_1(r - \hat{q})f(q) . \tag{2.12}
\]

Denoting the remaining 4-dimensional vector \(\tilde{r} = (r_0, r_3, r_4, r_5)\), we have

\[
D_1(r) = \int d^6ze^{izr}\delta(\bar{x}^2) = \frac{\delta(r_1)\delta(r_2)}{\bar{r}^2} = \delta(r_1)\delta(r_2)D_1(\tilde{r}) , \tag{2.13}
\]

with \(D_1(\tilde{r}) = \frac{1}{\bar{r}^2}\).

We define now the ”subvector” of a 6-dimensional vector as \(\hat{q} = (q_0, q_1, q_2, q_3, 0, 0)\) and we find the relation between \(F(\hat{q})\) and \(f(q)\) in view of the causality condition (2.3):

\[
F(\hat{q}) = \int d^4xf(x)\theta(\bar{x}^2)e^{iqx} = f(q) . \tag{2.14}
\]

\(D_1(\tilde{r})\) satisfies the 4-dimensional wave-equation:

\[
\Box_4D_1(\tilde{r}) = 0 , \tag{2.15}
\]

where the d’Alembertian is defined with respect to the coordinates \(r_0, r_3, r_4, r_5\). Then, due to (2.12), it follows that \(F(r)\) satisfies the same equation,

\[
\Box_4F(r) = 0 . \tag{2.16}
\]
It is crucial to note that \( F(r) \) depends on all 6 variables \( r_0, \ldots r_5 \):

\[
F(r) = \int d^4q f(q) D_1(\tilde{r} - \tilde{q}) \delta(r_1 - q_1) \delta(r_2 - q_2),
\]

where \( \tilde{q} = (q_0, q_3, 0, 0) \).

The solution of (2.16) can be written in the form [30]:

\[
F(r') = \int d^3\Sigma_\alpha \int \int dr_1 dr_2 \left[ F(r) \frac{\partial D(\tilde{r} - \tilde{r}')}{\partial \tilde{r}_\alpha} - D(\tilde{r} - \tilde{r}') \frac{\partial F(r)}{\partial \tilde{r}_\alpha} \right] \delta(r_1) \delta(r_2),
\]

where \( D(\tilde{r}) \) satisfies the homogeneous differential equation \( \Box_4 D(\tilde{r}) = 0 \), with the initial conditions

\[
D(\tilde{r})|_{r_0=0} = 0 \quad \text{and} \quad \frac{\partial D}{\partial r_0}(\tilde{r})|_{r_0=0} = \prod_{i=1}^3 \delta(r_i).
\]

The first condition implies that \( D(\tilde{r}) \) is an odd function, with the result that:

\[
D(\tilde{r}) = \int d^4ze^{-iz\tilde{r}} (z_0) \delta(z^2) = \epsilon(r_0) \delta(\tilde{r}^2). \tag{2.17}
\]

We note here that the surface \( \Sigma \) is 3-dimensional and not 5-dimensional as it is in the commutative case with light-cone causality condition. Now we can express \( f(q) \) using (2.14) as:

\[
f(q) = F(\tilde{q}) = \int dr_1 dr_2 \delta(r_1 - q_1) \delta(r_2 - q_2)
\times \int d^3\Sigma_\alpha [F(r)\frac{\partial D(\tilde{r} - \tilde{q})}{\partial \tilde{r}_\alpha} - D(\tilde{r} - \tilde{q})\frac{\partial F(r)}{\partial \tilde{r}_\alpha}]. \tag{2.18}
\]

Due to the arbitrariness of the surface \( \Sigma \), one can reduce the integration over \( r_4 \) and \( r_5 \), using the cylindrical symmetry, to the integral over \( \kappa^2 = r_4^2 + r_5^2 \). Subsequently we change the notation of variables \( r_i \) to \( u_i \) and use the explicit form of \( D(\tilde{r}) \) from (2.17) to obtain:

\[
f(q) = \int du_1 du_2 \delta(u_1 - q_1) \delta(u_2 - q_2) \int d^3\Sigma_j d\kappa^2
\]
\[
\times \left\{ F(u, \kappa^2) \frac{\partial}{\partial \tilde{u}_j} \left[ \epsilon(u_0 - \tilde{q}_0)\delta((\tilde{u} - \tilde{q})^2 - \kappa^2) \right] - \epsilon(u_0 - q_0)\delta((\tilde{u} - \tilde{q})^2 - \kappa^2) \frac{\partial F(u, \kappa^2)}{\partial \tilde{u}_j} \right\}.
\]

(2.19)

Using the standard mathematical procedure [30] for performing the integration in (2.19), we obtain the Jost-Lehmann-Dyson representation in NC QFT, satisfying the light-wedge causality condition (2.3):

\[
f(q) = \int d^4 u d\kappa^2 \epsilon(q_0 - u_0)\delta((q_0 - u_0)^2 - (q_3 - u_3)^2 - \kappa^2) \times \delta(q_1 - u_1)\delta(q_2 - u_2)\phi(u, \kappa^2),
\]

(2.20)

where \(\phi(u, \kappa^2) = -\frac{\partial F(u, \kappa^2)}{\partial \tilde{u}_0}\).

Equivalently, denoting \(\tilde{u} = (u_0, u_3)\), (2.20) can be written as:

\[
f(q) = \int d^2 \tilde{u} d\kappa^2 \epsilon(q_0 - u_0)\delta((\tilde{q} - \tilde{u})^2 - \kappa^2)\phi(\tilde{u}, q_1, q_2, \kappa^2).
\]

(2.21)

The function \(\phi(\tilde{u}, q_1, q_2, \kappa^2)\) is an arbitrary function, except that the requirement of spectral condition determines a domain in which \(\phi(\tilde{u}, q_1, q_2, \kappa^2) = 0\). This domain is outside the region where the \(\delta\) function in (2.21) vanishes, i.e.

\[
(\tilde{q} - \tilde{u})^2 - \kappa^2 = 0,
\]

(2.22)

but with \(\tilde{q}\) in the region given by (2.8), where \(f(q) = 0\). Putting together (2.22) and (2.8), we obtain the domain out of which \(\phi(\tilde{u}, q_1, q_2, \kappa^2) = 0\):

\begin{align*}
& a) \quad \frac{1}{2}(\tilde{p} + \tilde{p}') \pm \tilde{u} \text{ are in the forward light-wedge (cf. (2.4));} \\
& b) \quad \kappa \geq \max \left\{ 0, m_1 - \sqrt{\left( \frac{\tilde{p} + \tilde{p}'}{2} + \tilde{u} \right)^2}, m_2 - \sqrt{\left( \frac{\tilde{p} + \tilde{p}'}{2} - \tilde{u} \right)^2} \right\}.
\end{align*}

(2.23)
For the purpose of expressing the scattering amplitude, we actually need the Fourier transform $f_R(q)$ of the retarded commutator,

$$f_R(x) = \theta(x_0) f(x) = \langle p' | \theta(x_0) [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] | p \rangle .$$  \hspace{1cm} (2.24)

Using (2.24) and the Fourier transformation $f(x) = \int dq' e^{-iq'x} f(q')$, we can express $f_R(q)$ as follows:

$$f_R(q) = \int dx e^{iqx} f_R(x) = \int dx e^{iqx} \theta(x_0) f(x) = \int dq' f(q') \int dx e^{i(q-q')x} \theta(x_0) .$$  \hspace{1cm} (2.25)

Taking into account that

$$\int dx_0 e^{i(q-q')x} \theta(x_0) = -i e^{i(\vec{q}-\vec{q})\vec{x}} \frac{q_0 - q'_0}{q_0 - q'},$$

eq. (2.25) becomes:

$$f_R(q) = i \int dq'_0 \frac{f(q'_0, \vec{q})}{q'_0 - q_0}.$$  

Now in the above formula we introduce the Jost-Lehmann-Dyson representation (2.21), with the result:

$$f_R(q) = i \int dq'_0 \frac{f(q'_0, \vec{q})}{q'_0 - q_0}.$$  

In (2.26) one can integrate over $q'_0$, using the known formula of integration with a $\delta$-function,

$$\int G(x) \delta(g(x)) dx = \sum_i \frac{G(x_{0i})}{g'(x_{0i})},$$

where $x_{0i}$ are the simple roots of the function $g(x)$. We identify in (2.26) $G(q'_0) = \frac{e(q'_0 - u_0)}{q'_0 - q_0}$ and $g(q'_0) = (q'_0 - u_0)^2 - (q_3 - u_3) - \kappa^2$ (with the roots $q'_0 = u_0 \pm [(q_3 - u_3)^2 + \kappa^2]^{1/2}$).
With these considerations, from (2.26) we obtain the NC version of the Jost-Lehmann-Dyson representation for the retarded commutator:

\[ f_R(q) = \int d^2 \tilde{u} \kappa^2 \frac{\phi(\tilde{u}, q_1, q_2, \kappa^2)}{(q_0 - u_0)^2 - (q_3 - u_3)^2 - \kappa^2} . \]  

(2.27)

Compared to the usual Jost-Lehmann-Dyson representation,

\[ f^\text{comm}_R(q) = \int d^4 u \kappa^2 \frac{\phi(u, \kappa^2)}{(q_0 - u_0)^2 - (\vec{q} - \vec{u})^2 - \kappa^2} , \]

(2.28)

the expression (2.27) is essentially different in the sense that the arbitrary function \( \phi \) now depends on \( q_1 \) and \( q_2 \). This feature will have further crucial implications in the discussion of analyticity of the scattering amplitude in \( \cos \Theta \).

3 Analyticity of the scattering amplitude in \( \cos \Theta \).

Lehmann’s ellipse

In the center-of-mass system (CMS) and in a set in which the incoming particles are along the vector \( \vec{\beta} = (0, 0, \theta) \), the scattering amplitude in NC QFT depends still on only two variables, the CM energy \( E \) and the cosine of the scattering angle, \( \cos \Theta \) (for a discussion about the number of variables in the scattering amplitude for a general type of noncommutativity, see [28]).

In terms of the Jost-Lehmann-Dyson representation, the scattering amplitude is written

\[ f^\text{comm}_R(q) = \int d^4 u \kappa^2 \frac{\phi(u, \kappa^2)}{(q_0 - u_0)^2 - (\vec{q} - \vec{u})^2 - \kappa^2} , \]

(2.28)

The ‘magnetic’ vector \( \vec{\beta} \) is defined as \( \beta_i = \frac{1}{2} \epsilon_{ijk} \theta_j \). The terminology stems from the antisymmetric background field \( B_{\mu \nu} \) (analogous to \( F_{\mu \nu} \) in QED), which gives rise to noncommutativity in string theory, with \( \theta_{\mu \nu} \) essentially proportional to \( B_{\mu \nu} \) (see, e.g., [1]).
as (cf. [21] for commutative case):

\[
M(E, \cos \Theta) = i \int d^2 \tilde{u} d\kappa^2 \frac{\phi(\tilde{u}, \kappa^2, k + p, (k' - p')_1, 2)}{\left[\frac{1}{2}(\tilde{k}' - \tilde{p}') + \tilde{u}\right]^2 - \kappa^2},
\]

(3.29)

where \(\phi(\tilde{u}, \kappa^2,...)\) is a function of its \(SO(1, 1)\)- and \(SO(2)\)-invariant variables: \(u^2_0 - u^2_3\), \((k_0 + p_0)^2 - (k_3 - p_3)^2\), \((k_1 + p_1)^2 + (k_2 + p_2)^2\), \((k'_1 - p'_1)^2 + (k'_2 - p'_2)^2\), ... The function \(\phi\) is zero in a certain domain, determined by the causal and spectral conditions, but otherwise arbitrary.

For the discussion of analyticity of \(M(E, \cos \Theta)\) in \(\cos \Theta\), it is of crucial importance that all dependence on \(\cos \Theta\) be contained in the denominator of (3.29). But, since the arbitrary function \(\phi\) depends now on \((k' - p')_1, 2\), it also depends on \(\cos \Theta\). This makes impossible the mere consideration of any analyticity property of the scattering amplitude in \(\cos \Theta\).

One might wonder now whether in the above derivation there is any condition which could be subject to challenge. In that case there might also appear the possibility that an analyticity domain can be obtained, which might lead to a high-energy upper bound on the scattering amplitude, analogous to the Froissart-Martin bound. In this connection, we would like to mention that all perturbative calculations performed in NC QFT show that the scattering amplitude respects the Froissart bound when \(\theta_{0i} = 0\) (see, e.g., [28]).

Since the Jost-Lehmann-Dyson representation reflects the effect of the causal and spectral axioms, we notice that the hypotheses (2.3) and (2.4) used for the present derivation of JLD representation are too weak, in the sense of their physical implications, since they allow for a much larger physical region, by not at all taking into account the effect of the NC coordinates \(x_1\) and \(x_2\). This remark is also in accord with the result of [2], in which it was shown that the forward dispersion relation cannot be obtained in NC QFT with the causality condition.
(2.3), except for the case of incoming particle momentum being orthogonal to the NC plane, i.e. in the direction of the $\vec{\beta}$-vector.

### 3.1 Causality in NC QFT

In the following, we shall challenge the causality condition (2.3)

$$[j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right)] = 0, \quad \text{for } x^2 \equiv x_0^2 - x_3^2 < 0,$$

(3.30)

which takes into account only the variables connected with the $SO(1, 1)$ symmetry.

This causality condition would be suitable in the case when nonlocality in NC variables $x_1$ and $x_2$ is infinite, which is not the case on a space with the commutation relation $[x_1, x_2] = i\theta$, which implies $\Delta x_1 \Delta x_2 \geq \frac{\theta}{2}$. The fact that in the causality condition (3.30) the coordinates $x_1$ and $x_2$ do not enter means that the propagation of a signal in this plane is instantaneous: no matter how far apart two events are in the noncommutative coordinates, the allowed region for correlation is given by only the condition $x_0^2 - x_3^2 > 0$, which involves the propagation of a signal only in the $x_3$-direction, while the time for the propagation along $x_1$- and $x_2$-directions is totally ignored.

The uncertainty relation $\Delta x_1 \Delta x_2 \sim \theta$, which follows from the considered commutation relation of the coordinates, puts a lower bound on localization. Admitting that the scale of nonlocality in $x_1$ and $x_2$ is $l \sim \sqrt{\theta}$, then the propagation of interaction in the noncommutative coordinates is instantaneous only within this distance $l$. It follows then that two events are correlated, i.e. $[j_1\left(\frac{\delta}{2}\right), j_2\left(-\frac{\delta}{2}\right)] \neq 0$, when $x_1^2 + x_2^2 \leq l^2$ (where $x_1^2 + x_2^2$ is the distance in the NC plane with $SO(2)$ symmetry), provided also that $x_0^2 - x_3^2 \geq 0$ (the events are time-like...
separated in the sense of $SO(1,1)$). Adding the two conditions, we obtain that

$$[j_1(x/2), j_2(-x/2)] \neq 0, \text{ for } x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) \geq 0.$$  \hspace{1cm} (3.31)

The negation of condition (3.31) leads to the conclusion that the locality condition should indeed be given by:

$$[j_1(x/2), j_2(-x/2)] = 0, \text{ for } \bar{x}^2 - (x_1^2 + x_2^2 - l^2) \equiv x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0,$$

or, equivalently,

$$[j_1(x/2), j_2(-x/2)] = 0, \text{ for } x_0^2 - x_3^2 - (x_1^2 + x_2^2) < -l^2,$$  \hspace{1cm} (3.32)

where $l^2$ is a constant proportional to NC parameter $\theta$. When $l^2 \to 0$, (3.32) becomes the usual locality condition.

When $x_1^2 + x_2^2 > l^2$, for the propagation of a signal only the difference $x_1^2 + x_2^2 - l^2$ is time-consuming and thus in the locality condition it is the quantity $x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2)$ which will occur. Therefore, we shall have again the locality condition in the form:

$$[j_1(x/2), j_2(-x/2)] = 0, \text{ for } x_0^2 - x_3^2 - (x_1^2 + x_2^2 - l^2) < 0,$$

which is equivalent to (3.32).

The new causality condition (3.32) is strongly supported by calculations performed in the first paper dealing with the causality in NC QFT, in Lagrangian approach, [15]. There it was shown, through the study of a scattering process, that space-space NC $\phi^4$ in 2+1 dimensions is causal at macroscopical level\(^5\). Moreover, the same study [15] has shown that the scattered wave appears to originate from a position shifted by $\frac{1}{2} \theta p$, where $p$ is the momentum of the

\(^5\)We would like to emphasize that, if one admits the causality condition in the form (3.30) for 3+1-
incoming wave packet. The physical interpretation given in [15] is that the incident particles should be viewed as extended rigid rods, of the size $\theta p$, perpendicular to their momentum. In other words, the noncommutativity introduces a scale $\theta$ of the spatial nonlocality. The effect calculated in [15] is actually an amplification at macroscopic scale of the (micro)causality condition (3.32).

Correspondingly, the spectral condition will read as

$$
\begin{align*}
  p_0^2 - p_1^2 - p_2^2 - p_3^2 &\geq 0, \\
  p_0 &> 0.
\end{align*}
$$

(3.33)

This is the case since a NC QFT with $\theta_{\mu\nu}$ a constant matrix has the twisted Poincaré symmetry. The latter, however, has the representation content identical to the usual Poincaré symmetry and thus the particles dispersion law and the spin structure are the same as the usual (commutative) QFT case [35]. Thus both the causality condition (3.32) and the spectral condition (3.33) are given in terms of the invariants of the theory.

In fact, the consideration of nonlocal theories of the type (3.32) was initiated by Wightman [29], who asked the concrete question whether the vanishing of the commutator of fields (or observables), i.e. $[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0$, for $x_0^2 - x_1^2 - x_2^2 - x_3^2 < -l^2$ would imply its vanishing for $x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0$. It was proven later [30, 31, 32] (see also [33]) that, indeed, in a quantum field theory which satisfies the translational invariance and the spectral axiom (3.33), the nonlocal commutativity

$$
[j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, 
$$

for $x_0^2 - x_1^2 - x_2^2 - x_3^2 < -l^2$

dimensional NC QFT, then for the 2+1-dimensional theory treated in [15] one simply could not write any (micro)causality condition, since all (two) spatial coordinates are noncommutative and the signal should propagate instantaneously in all directions.
implies the local commutativity

\[ [j_1(\frac{x}{2}), j_2(-\frac{x}{2})] = 0, \quad \text{for} \quad x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0. \quad (3.34) \]

This powerful theorem, which does not require Lorentz invariance, can be applied in the noncommutative case, since the hypotheses are fulfilled, with the conclusion that the causality properties of a QFT with space-space noncommutativity are physically identical to those of the corresponding commutative QFT.

It is then obvious that the Jost-Lehmann-Dyson representation (2.28) obtained in the commutative case holds also on the NC space. Consequently, the NC two-particle→two-particle scattering amplitude will have the same form as in the commutative case:

\[ M(E, \cos \Theta) = i \int d^4ud\kappa_2 \frac{\phi(u, \kappa_2^2, k + p)}{\frac{1}{2}(k' - p') + u} - \kappa_2^2. \quad (3.35) \]

This leads to the analyticity of the NC scattering amplitude in $\cos \Theta$ in the analog of the Lehmann ellipse, which behaves at high energies $E$ the same way as in the commutative case, i.e. with the semi-major axis as

\[ y_L = (\cos \Theta)_{\text{max}} = 1 + \frac{\text{const}}{E^4}. \quad (3.36) \]

4 Enlargement of the domain of analyticity in $\cos \Theta$ and use of unitarity. Martin’s ellipse

Two more ingredients are needed in order to enlarge the domain of analyticity in $\cos \Theta$ to the Martin’s ellipse and to obtain the Froissart-Martin bound: the dispersion relations and the unitarity constraint on the partial-wave amplitudes [22].
We recall the conclusion of [2] that, when using the causality condition (2.3), the forward dispersion relation cannot be obtained in NC theory with general direction of the $\vec{\beta}$-vector. However, the conclusion to which we arrived by imposing a physical nonlocal commutativity condition (3.32) and reducing it to the local commutativity (3.34), by using the theorem due to Wightman, Vladimirov and Petrina, leads straightforwardly to the usual forward dispersion relation also in the NC case with a general $\vec{\beta}$ direction. We have to point out that now the derivations of both the Lehmann ellipse and of the forward dispersion relation do not depend on the orientation of the incoming momentum $\vec{p}$ with respect to the NC plane, or equivalently to the $\vec{\beta}$-vector.

As for the unitarity constraint on the partial wave amplitudes, the problem has been dealt with in [28], for a general case of noncommutativity $\theta_{\mu\nu}$, $\theta_{0i} \neq 0$. For space-space noncommutativity ($\theta_{0i} = 0$), the scattering amplitude depends, besides the center-of-mass energy, $E$, on three angular variables. In a system were we take the incoming momentum $\vec{p}$ in the $z$-direction, these variables are the polar angles of the outgoing particle momentum, $\Theta$ and $\phi$, and the angle $\alpha$ between the vector $\vec{\beta}$ and the incoming momentum. The partial-wave expansion in this case reads:

$$A(E, \Theta, \phi, \alpha) = \sum_{l,l',m} (2l' + 1) a_{ll'm}(E) Y_{lm}(\Theta, \phi) P_{l'}(\cos \alpha) ,$$  

(4.37)

where $Y_{lm}$ are the spherical harmonics and $P_{l'}$ are the Legendre polynomials.

Imposing the unitarity condition directly on (4.37) or using the general formulas given in [28], it can be shown that a simple unitarity constraint which involves single partial-wave amplitudes one at a time cannot be obtained in general, but only in a setting where the incoming momentum is orthogonal to the NC plane (equivalently it is parallel to the vector
\( \vec{\beta} \). In this case the amplitude depends only on one angle, \( \Theta \), and the unitarity constraint is reduced to the well-known one of the commutative case, i.e.

\[
Im \ a_1(E) \geq |a_1(E)|^2.
\] (4.38)

For this particular setting, \( \vec{p} \parallel \vec{\beta} \), it is then straightforward, following the prescription developed for commutative QFT, to enlarge the analyticity domain of scattering amplitude to Martin’s ellipse with the semi-major axis at high energies as

\[
y_M = 1 + \frac{\text{const}}{E^2},
\] (4.39)

and subsequently obtain the NC analog of the Froissart-Martin bound on the total cross-section, in the CMS and for \( \vec{p} \parallel \vec{\beta} \):

\[
\sigma_{\text{tot}}(E) \leq c \ln^2 \frac{E}{E_0}.
\] (4.40)

In the same manner, rigorous high-energy bounds on the nonforward scattering amplitude,

\[
|A(E, \cos \Theta)| < \text{const} \ E^{3/2} \ln^{3/2} \left( \frac{E}{E_0} \right) \frac{1}{\sin \Theta^{1/2}},
\] (4.41)

as well as on the differential cross-section,

\[
\left( \frac{d\sigma}{d\Omega} \right) |_{\Theta=0} < \text{const} \ E^2 \ln^4 \frac{E}{E_0},
\] (4.42)

are obtained in the NC case with \( \vec{p} \parallel \vec{\beta} \). The high-energy behaviour of these bounds is identical to the one in the ordinary QFT.

Thus, the unitarity constraint on the partial-wave amplitudes distinguishes a particular setting (\( \vec{p} \parallel \vec{\beta} \)) in which the Lehmann’s ellipse can be enlarged to the Martin’s ellipse and
Froissart-Martin bound can be obtained. However, since the Lehmann’s ellipse is the domain of analyticity in \( \cos \Theta \) irrespective of the direction of \( \vec{p} \), one might attempt to obtain at least a NC analog of the Greenberg-Low bound (1.3) for \( \vec{p} \parallel \vec{\beta} \). However, according to the orthodox procedure for the derivation of this bound [23], besides analyticity in Lehmann’s ellipse and polynomial boundedness, a suitable unitarity constraint on the partial-wave amplitudes is still needed, which we have not succeeded to derive. Nevertheless, this does not exclude the possibility of obtaining a rigorous high-energy bound on the cross-section for \( \vec{p} \parallel \vec{\beta} \), and the issue deserves further investigation.

5 Generalization to the flat (noncompact) higher-dimensional NC space-time

In this section we shall derive rigorous high-energy bounds on the total cross-section, on the two-particle\( \rightarrow \)two-particle nonforward scattering amplitude and on the differential cross-section for the case of NC space-time with an arbitrary number of noncompact dimensions \( D \). For the usual (commutative) case of higher-dimensional space-time, such high-energy bounds have been previously derived [24, 25].

Due to the recent activity in the NC version of higher-dimensional theories (see, e.g., [34] and references therein), the generalization of high-energy bounds derived in the previous section to higher dimensions is of interest, especially since in this case there appear distinct features between the theories in which the space-time dimension \( D \) is odd or even. Specifically, for the case of even \( D \) we shall be able to derive high-energy bounds, analogous to
the bounds in usual commutative QFT, for the special setting when the incoming particle momentum is in the direction of the 'magnetic' vector $\vec{\beta}$ (to be defined below).

Consider QFT on NC space-time of arbitrary dimension $D$, with

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad \mu, \nu = 0, 1, ..., D - 1 \quad (5.43)$$

and $\theta_{\mu\nu} = -\theta_{\nu\mu}$ arbitrary real constants. To avoid the known problems in the case $D = 4$ with unitarity [14] and causality [15, 16], which will appear automatically in the general case of $D$ dimensions, we should have that the noncommutativity is only between spacial components, i.e. $\theta_{0i} = 0$, $i = 1, ..., D - 1$ (or equivalently there is no 'electric' vector $\vec{\epsilon}$, $\epsilon_i \equiv \theta_{0i}$.)

We also mention a crucial point that for a two-particle→two-particle scattering, the amplitude still depends on only two kinematical variables, $E$ and $\cos \Theta$, for any dimension $D$ of ordinary (commutative) space-time [24]. In this case the symmetry is $SO(1, D - 1)$ and the partial-wave expansion is in terms of the Gegenbauer (instead of Legendre) polynomials, for any $D$, and there is no distinction in the final results between odd and even $D$.

Consider now the case of a space-time of even dimension $D$, with commutation relation (5.43) and $\theta_{0i} = 0$. Any $D \times D$ antisymmetric matrix $\theta_{\mu\nu}$ possesses paired eigenvalues $\pm \lambda_\alpha$, with $\alpha = 1, ..., \frac{D}{2}$. Due to the conditions $\theta_{0i} = 0$, $i = 1, ..., D - 1$, there is one pair of light-like vectors corresponding to $\lambda = 0$. We choose this pair of vectors to span the plane $(0, D - 1)$.

With this choice, the 'magnetic' vector $\vec{\beta}$, defined as

$$\beta_i = \epsilon_{i_{i1}i_2...i_{D-2}}\theta_{i_{1}i_{2}...}\theta_{i_{D-3}i_{D-2}} \quad (5.44)$$

*For terminology, see footnote on the definition of vector $\vec{\beta}$ in Section 3.
becomes oriented to the \((D - 1)\)-direction, i.e. \(\beta_1 = \beta_2 = \ldots = \beta_{D-2} = 0\) and \(\beta_{D-1} = det\theta'\) is the determinant of the \((D-2)\times(D-2)\) matrix \(\theta_{i'j'}\), with \(i', j' = 1, \ldots, D-2\). We shall consider the case \(det\theta' \neq 0\), i.e. when there are no more zero eigenvalues \(\lambda_\alpha\). For the generic case with distinct, nonzero eigenvalues of \(\theta_{i'j'}\), the space-time symmetry is \(SO(1,1) \times (D-2)/2 \prod_{l=1}^{(D-2)/2} SO(l)(2)\) (when the eigenvalues are degenerate, the symmetry is enlarged).

For the general relative direction of incoming particle momentum \(\vec{p}\) and the ‘magnetic’ vector \(\vec{\beta}\), a partial-wave expansion of the amplitude can be written down analogous to the one given for \(D = 4\) in [28]. However, for the special configuration where the incoming particle momentum \(\vec{p}\) is parallel to \(\vec{\beta}\), the number of kinematical variables entering the scattering amplitude becomes two, \(E\) and \(\cos \Theta\), and in this case the partial-wave expansion of \(D\)-even dimensional scattering amplitude is analogous to the one treated in [24, 25].

With the same physical arguments on the finiteness of the instantaneous nonlocal interaction in the NC hyperplane, as presented in Section 3 and leading to the causality condition (3.32), we shall have now the causality condition as:

\[
[j_1(\frac{x_2}{2}), j_2(-\frac{x_2}{2})] = 0 , \text{ for } x_0^2 - x_{D-1}^2 - (x_1^2 + \ldots + x_{D-2}^2 - l^2) < 0 . \quad (5.45)
\]

Using now the extension of the Wightman-Vladimirov-Petrina theorem [30, 31, 32] to an arbitrary dimension \(D\), one arrives to the conclusion that the nonlocality condition of the type (5.45) implies the ordinary causality condition:

\[
[j_1(\frac{x_2}{2}), j_2(-\frac{x_2}{2})] = 0 , \text{ for } x_0^2 - x_{D-1}^2 - (x_1^2 + \ldots + x_{D-2}^2) < 0 . \quad (5.46)
\]

As a final result, for an even-\(D\)-dimensional NC space-time, for the special kinematical configuration of \(\vec{p} \parallel \vec{\beta}\), we obtain the rigorous high-energy bounds on the total cross-section,
nonforward scattering amplitude and on the differential cross-section:

\[ \sigma_{\text{tot}}(E) < c \left( \ln \frac{E}{E_0} \right)^{D-2}, \quad (5.47) \]

\[ |A(E, \cos \Theta)| < \text{const} \ E^{(7-D)/2} \left( \ln \frac{E}{E_0} \right)^{(D-1)/2} \frac{1}{|\sin \Theta|^{(D-3)/2}}, \quad (5.48) \]

\[ \left( \frac{d\sigma}{d\Omega} \right) |_{\Theta=0} < \text{const} \ E^2 \left( \ln \frac{E}{E_0} \right)^{2(D-2)}. \quad (5.49) \]

The bounds (5.47), (5.48) and (5.49) are the analogs of the Froissart and other bounds obtained previously for the ordinary (commutative) QFT in \( D \) dimensions [24, 25], identical in their high-energy behaviour.

We mention that for the case of an odd dimension \( D \) of space-time, one cannot construct a 'magnetic' vector \( \vec{\beta} \) such as (5.44). Instead, there exists a 2-dimensional plane, orthogonal to the \((D-3)\)-dimensional NC hyperplane and thus no kinematical configuration for the incoming particle momentum is possible, in order that no extra angular variables would appear. As a result, no simple unitarity constraint, which would involve single partial-wave amplitudes one at a time, can be derived. Thus, although there exists the Lehmann ellipse for the analyticity domain in \( \cos \Theta \) of the two-particle→two-particle scattering amplitude, no extension to Martin’s ellipse is possible. Consequently, not only the derivation of a Froissart-Martin bound in \( D \) odd NC case is impossible, there seems no way to rigorously obtain even a weaker bound analogous to the Greenberg-Low bound.

### 6 Conclusion and discussions

In this paper we have tackled the problem of high energy bounds on the two-particle→two-particle scattering amplitude in NC QFT with \( SO(1,1) \times SO(2) \) symmetry. We have found
that, using the causal and spectral conditions (2.3) and (2.4) proposed in [27] for NC theories with such symmetry, a new form of the Jost-Lehmann-Dyson representation (2.27) is obtained, which does not permit to draw any conclusion about the analyticity of the scattering amplitude (3.29) in \( \cos \Theta \). However, the physical observation that nonlocality in the noncommuting coordinates is not infinite brought us to imposing a new causality condition (3.32), which accounts for the finitness of the range of nonlocality and prevents the instantaneous propagation of signals in the entire noncommutative plane \((x_1, x_2)\). We proved that the new causality condition is formally identical to the one corresponding to the commutative case (3.34), using the Wightman-Vladimirov-Petrina theorem [30, 31, 32]. We would like to mention that there the NC QFT with noncommutativity given by (1.1) possesses the twisted Poincaré symmetry which preserves the commutation relation (1.1) between the coordinates, while the representation content and the invariants are identical to the ones of ordinary Poincaré algebra [35]. In this sense, the causality and the spectral conditions in the forms (3.32) and (3.33) are described by invariants of the theory, as it should be.

Thus, the scattering amplitude in NC QFT is proved to be analytical in \( \cos \Theta \) in the Lehmann ellipse, just as in the commutative case; moreover, dispersion relations can be written on the same basis as in commutative QFT.

Finally, based on the unitarity constraint on the partial-wave amplitudes in NC QFT, we can conclude that, for theories with space-space noncommutativity \( (\theta_{0i} = 0) \), the total cross-section is subject to an upper bound (4.40) identical to the Froissart-Martin bound in its high-energy behaviour, when the incoming particle momentum \( \vec{p} \) is orthogonal to the NC plane. In the same configuration, similar bounds on nonforward scattering amplitude
(4.41) and on the differential cross-section (4.42) are obtained. This is the first example of a nonlocal theory, in which cross-sections do have an upper high-energy bound.

Furthermore, our considerations have been generalized to the case of NC space-time of an arbitrary dimension $D$. For the case of even dimension $D$, in the specific configuration in which the incoming particle momentum $\vec{p}$ is parallel to the 'magnetic’ vector $\vec{\beta}$, the generalization of the bounds on the cross-sections (5.47), the nonforward scattering amplitude (5.48) and on the differential cross-section (5.49) are obtained. In the case of an odd dimension $D$ of NC space-time, although there exists the Lehmann ellipse as analyticity domain of the scattering amplitude in $\cos \Theta$, there seems to be no way to derive even weaker bounds, such as the Greenberg-Low bound [23]. This is due to the appearance of additional angular variables, thus leading to a complicated form of the unitarity constraint on the partial-wave amplitudes, which seemingly makes its utilization impossible, at least in its present form.

There remain several interesting questions to be studied. Among them is the question whether any bound can in principle be obtained for the case when the incoming particle momentum is not orthogonal to the NC plane, or, equivalently, is not parallel to the 'magnetic’ vector. Similar questions appear also in the case of NC QFT in a space-time of arbitrary dimension $D$, concerning the distinct features between even and odd $D$, as considered in Section 5 of the paper.

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