PATHWISE STOCHASTIC INTEGRATION
WITH FINITE VARIATION PROCESSES
UNIFORMLY APPROXIMATING CÀDLÀG PROCESSES

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Abstract. For any real-valued stochastic process $X$ with càdlàg paths we define non-empty family of processes which have locally finite total variation, have jumps of the same order as the process $X$ and uniformly approximate its paths on compacts. The application of the defined class is the definition of stochastic integral with semimartingale integrand and integrator as a limit of pathwise Lebesgue-Stieltjes integrals. This construction leads to the stochastic integral with some correction term (different from the Stratonovich integral). We compare the obtained result with classical results of Wong-Zakai and Bichteler on pathwise stochastic integration. As a "byproduct" we obtain an example of a series of double Skorohod maps of a standard Brownian motion, which is not a semimartingale.

1. Introduction

Let $X = (X_t)_{t \geq 0}$ be a real-valued stochastic process with càdlàg paths and let $T \geq 0$. The total variation of the process $X$ on the interval $[0; T]$ is defined with the following formula

$$TV(X, T) := \sup_n \sup_{0 \leq t_0 < t_1 < \ldots < t_n \leq T} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|.$$  

Unfortunately, many of the most important families of stochastic processes are characterized with a "wild" behavior, demonstrated by their infinite total variation. This fact arguably caused the need of the development of the general theory of stochastic integration. The main idea allowing to overcome the problematic infinite total variation and define stochastic integral with respect to a semimartingale utilizes the fact that the quadratic variation of the semimartingale is still finite. The similar idea may be applied when $p$-variation of the integrator is finite for some $p \in (1; 2)$. This approach utilizes Love-Young inequality and may be used e.g. to define stochastic integral with respect to fractional Brownian motion (cf. [9]). Further developments, where Hölder continuity plays crucial role, led to the rough paths theory developed by T. Lyons and his co-workers (cf. [5]); some other generalization introduces Orlicz norms and may be found in the recent book by Dudley and Norvaiša [4, Chapt. 3]). The approach used in this article is somewhat different. It is similar to the old approach of Wong and Zakai [15] and is based on the simple observation that in the neighborhood (in sup norm) of every càdlàg function defined on compact interval $[0; T]$ one easily finds another function with finite total variation. Thus, for every $c > 0$, the process $X$ may be decomposed as the sum

$$X = X^c + (X - X^c)$$

where $X^c$ is a "nice" process with finite total variation and the difference $X - X^c$ is a process with small amplitude (no greater than $K_T c$) but possibly "wild" behaviour with infinite total variation. More precisely, let $F$ be some fixed, right continuous filtration such that $X$ is adapted to $F$. Now, for every $c > 0$ we introduce (non-empty, as it will be shown in the sequel) family $X^c$ of processes with càdlàg paths, satisfying the following conditions. If $X^c \in X^c$ then

1. the process $X^c$ has locally finite total variation;
2. $X^c$ has càdlàg paths;
3. for every $T \geq 0$ there exists such $K_T < +\infty$ that for every $t \in [0; T]$, $|X_t - X^c_t| \leq K_T c$;
4. for every $T \geq 0$ there exists such $L_T < +\infty$ that for every $t \in [0; T]$, $|\Delta X^c_t| \leq L_T |\Delta X_t|$;
5. the process $X^c$ is adapted to the filtration $F$.

We will prove that if processes $X$ and $Y$ are càdlàg semimartingales on a filtered probability space \( (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, F) \), with a probability measure $\mathbb{P}$, such that usual hypotheses hold (cf. [13] Sect. 1.1), then the sequence of pathwise Lebesgue-Stieltjes integrals

\[ \int_0^T Y_s \, dX^c_s, \quad c > 0, \]
with $X^c \in \mathcal{X}^c$, tends uniformly in probability $\mathbb{P}$ on compacts to $\int_0^T Y_\cdot \, dX + [X^{\text{cont}}, Y^{\text{cont}}]_T$ as $c \downarrow 0$; $\int_0^T Y_\cdot \, dX$ denotes here the (semimartingale) stochastic integral and $X^{\text{cont}}$ and $Y^{\text{cont}}$ denote continuous parts of $X$ and $Y$ respectively. Moreover, for any square summable sequence $(c(n))_{n \geq 1}$ we get a.s., uniform on compacts, convergence of the sequence $\int_0^T Y_\cdot \, dX^{c(n)}$, $n = 1, 2, \ldots$ (cf. Theorem 1).

We shall stress here that for each $c > 0$ and each pair of càdlàg paths $(X(\omega), Y(\omega)), \omega \in \Omega$, the value of $\int_0^T Y_\cdot (\omega) \, dX^c(\omega)$ (and thus the limit, if it exists) is independent of the probability measure $\mathbb{P}$. Thus we obtain a result in the spirit of Wong and Zakai [13], Bichteler (see [2] or [8]) or the recent result of Nutz [13], where operations almost surely leading to the stochastic integral, independent of probability measures and filtrations, are considered. The old approach of Wong and Zakai is very straightforward, since it just replaces stochastic integral with Lebesgue-Stieltjes integral. However, it deals with very limited family of possible integrands and integrators (diffusions driven by a Brownian motion), $x_t = \int_0^t g(s) \, ds + \int_0^t f(s) \, dB_s$, and using appropriate continuous, finite variation approximation of $x$, $x^n$, one gets a.s. in the limit the Stratonovich integral

$$\lim_{n \to \infty} \int_0^T \psi (x^n, t) \, dx^n = \int_0^T \psi (x_t, t) \, dx_t + \frac{1}{2} \int_0^T f^2(t) \frac{\partial \psi}{\partial x} (x_t, t) \, dt.$$  

(Modification of this approach is possible, cf. [9], but it requires introduction of a probability measure on the Skorohod space and rather strong UT (uniform tightness) condition, which is sometimes difficult to verify in practice, and is not satisfied e.g. by the picewise linear approximation of the Wiener process. Moreover, the obtained convergence holds in distribution.)

Bichteler’s remarkable approach allows to integrate any adapted càdlàg process $Y$ with semimartingale integrator $X$, and is based on the approximation

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} \left| Y_0 X_0 + \sum_{i=1}^{\infty} Y_{\tau_{i-1}^n \wedge t} \left( X_{\tau_{i-1}^n \wedge t} - X_{\tau_{i-1}^n -} \right) - \int_0^T Y_\cdot \, dX \right| = 0 \text{ a.s.},$$

where $\tau^n = (\tau^n_i), i = 0, 1, 2, \ldots$, is the following sequence of stopping times: $\tau^n_0 = 0$ and for $i = 1, 2, \ldots,$

$$\tau^n_i = \inf \left\{ t > \tau^n_{i-1} : \left| Y_t - Y_{\tau^n_{i-1}} \right| \geq 2^{-n} \right\}.$$

Remark 1. Following the proof of [8] Theorem 2] it is easy to see that Bichteler’s construction works for any sequence $\tau^n = (\tau^n_i), i = 0, 1, 2, \ldots$, of stopping times, such that $\tau^n_0 = 0$ and

$$\tau^n_i = \inf \left\{ t > \tau^n_{i-1} : \left| Y_t - Y_{\tau^n_{i-1}} \right| \geq c(n) \right\},$$

for $i = 1, 2, \ldots$, given $c(n) > 0$, $\sum_{n=1}^{\infty} c^2(n) < +\infty$.

The new result of Nutz goes even further, since it does not assume càdlàg property of the integrand, but to prove his result one needs the existence of Mokobodżki’s medial limits (cf. [12]), which one can not prove under standard Zermelo–Fraenkel set theory with the axiom of choice.

The results of this paper seem to indicate that Bichteler’s approach is the most flexible (under standard Zermelo–Fraenkel set theory with the axiom of choice) since we will prove that even in the case when the integrator is a standard Brownian motion, our construction [11] can not be extended to arbitrary adapted, bounded by a constant, continuous integrand $Y$. Moreover, similar to the Wong - Zakai, but a more general construction,

$$\int_0^T Z^c \, dX^c, \ c > 0,$$

can not be extended to arbitrary continuous semimartingale integrand $Z$ and semimartingale integrator $X$. The construction of the appropriate $Y$ and $Z$, adapted to the natural filtration of $B$ and leading to divergent series of integrals $\int_0^T Y \, dB^{(n)}, \int_0^T Z^{(n)} \, dB^{(n)},$ with $B^{(n)}, Z^{(n)}, B^{(n)}$ satisfying conditions (1)-(5) for some semimartingales $Z, B$, with $\gamma(n), \delta(n) \downarrow 0$ as $n \uparrow +\infty$, will utilise the recent findings of Bednorz, Lochowski and Milos on truncated variation (see [1], [11]) and its relation with the double Skorohod map on $[-c; c]$ (cf. [3]).

Let us shortly comment on the organization of the paper. In the next section we prove, for any $c > 0$, the existence of non-empty family of processes $X^c$. In the third section we deal with the limit of pathwise, Lebesgue-Stieltjes integrals $\int_0^T Y_\cdot \, dX^c$ as $c \downarrow 0$. The fourth section is devoted to the construction of counterexamples. Last section - Appendix - summarizes the necessary facts on the relation between the truncated variation and double Skorohod map on $[-c; c]$.

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2. Existence of the sequence \((X^c_{c > 0})\)

In this section we will prove that for every \(c > 0\) the family of processes \(X^c\), satisfying the conditions (1)-(5) of Section 1 is non-empty. For given \(c > 0\) we will simply construct a process \(X^c\) satisfying all these conditions. We start with few definitions.

For fixed \(c > 0\) we define two stopping times

\[
T_{u,k}^{2c} = \inf \left\{ s \geq 0 : \sup_{t \in [0,s]} X_t - X_0 > c \right\},
\]

\[
T_{d,k}^{2c} = \inf \left\{ s \geq 0 : X_0 - \inf_{t \in [0,s]} X_t > c \right\}.
\]

Assume that \(T_{d,k}^{2c} \geq T_{u,k}^{2c}\), i.e. the first upward jump of the process \(X\) from \(X_0\) of size \(c\) appears before the first downward jump of the same size \(c\) or both times are infinite (there is no upward or downward jump of size \(c\)). Note that in the case \(T_{d,k}^{2c} < T_{u,k}^{2c}\) we may simply consider the process \(-X\). Now we define sequences \((T_{d,k}^{2c})_{k=1}^{\infty}, (T_{u,k}^{2c})_{k=1}^{\infty}\) in the following way: \(T_{u,0}^{2c} = T_{u,k}^{2c}\) and for \(k = 0,1,2,\ldots\)

\[
T_{d,k}^{2c} = \begin{cases} 
\inf \left\{ s \geq T_{u,k}^{2c} : \sup_{t \in [T_{u,k}^{2c},s]} X_t - X_s > 2c \right\} & \text{if } T_{u,k}^{2c} < +\infty, \\
+\infty & \text{otherwise},
\end{cases}
\]

\[
T_{u,k+1}^{2c} = \begin{cases} 
\inf \left\{ s \geq T_{d,k}^{2c} : X_s - \inf_{t \in [T_{d,k}^{2c},s]} X_t > 2c \right\} & \text{if } T_{d,k}^{2c} < +\infty, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Remark 2. Note that for any \(s > 0\) there exists such \(K < \infty\) that \(T_{u,k}^{2c} > s\) or \(T_{d,k}^{2c} > s\). Otherwise we would obtain two infinite sequences \((s_k)_{k=1}^{\infty}\), \((S_k)_{k=1}^{\infty}\) such that \(0 \leq s(1) < S(1) < s(2) < S(2) < \ldots \leq s\) and \(X_{S(k)} - X_{S(k)} \geq c\). But this is a contradiction since \(X\) is a càdlàg process and for any sequence such that \(0 \leq s(1) < S(1) < s(2) < S(2) < \ldots \leq s\) sequences \((X_{S(k)})_{k=1}^{\infty}, (X_{s(k)})_{k=1}^{\infty}\) have a common limit.

Now we define, for the given process \(X\), the process \(X^c\) with the formulas

\[
X^c_s = \begin{cases} 
X_0 & \text{if } s \in [0; T_{u,0}^{2c}), \\
\sup_{t \in [T_{u,k}^{2c}; T_{d,k}^{2c}]} X_t - c & \text{if } s \in [T_{u,k}^{2c}; T_{d,k}^{2c}], k = 0,1,2,\ldots, \\
\inf_{t \in [T_{d,k}^{2c}; T_{u,k+1}^{2c}]} X_t + c & \text{if } s \in [T_{d,k}^{2c}; T_{u,k+1}^{2c}], k = 0,1,2,\ldots.
\end{cases}
\]

Remark 3. Note that due to Remark 2, \(s\) belongs to one of the intervals \([0; T_{u,0}^{2c}), [T_{u,k}^{2c}; T_{d,k}^{2c})\) or \([T_{d,k}^{2c}; T_{u,k+1}^{2c})\) for some \(k = 0,1,2,\ldots\) and the process \(X^c_s\) is defined for every \(s \geq 0\).

Now we are to prove that \(X^c\) satisfies conditions (1)-(5).

Proof. (1) The process \(X^c\) has finite total on compact intervals, since it is monotonic on intervals of the form \([T_{u,k}^{2c}; T_{d,k}^{2c})\), \([T_{d,k}^{2c}; T_{u,k+1}^{2c})\) which sum up to the whole half-line \([0; +\infty)\).

(2) From formula (2.1) it follows that \(X^c\) is also càdlàg.

(3) In order to prove condition (3) we consider 3 possibilities.

\[
\bullet \text{ } s \in [0; T_{u,0}^{2c}) \text{. In this case, since } 0 \leq s < T_{u,0}^{2c} X \leq T_{u,0}^{2c} X, \text{ by definition of } T_{u,0}^{2c} X \text{ and } T_{d,0}^{2c} X,
\]

\[
X_s - X^c_s = X_s - X_0 \in [-c; c].
\]

\[
\bullet \text{ } s \in [T_{u,k}^{2c}; T_{d,k}^{2c}) \text{, for some } k = 0,1,2,\ldots \text{. In this case, by definition of } T_{d,k}^{2c}, \sup_{t \in [T_{u,k}^{2c},s]} X_t - X_s \text{ belongs to the interval } [0; 2c]; \text{ hence}
\]

\[
X_s - X^c_s = X_s - \sup_{t \in [T_{u,k}^{2c},s]} X_t + c \in [-c; c].
\]

\[
\bullet \text{ } s \in [T_{d,k}^{2c}; T_{u,k+1}^{2c}) \text{ for some } k = 0,1,2,\ldots \text{. In this case } X_s - \inf_{t \in [T_{d,k}^{2c},s]} X_t \text{ belongs to the interval } [0; 2c], \text{ hence}
\]

\[
X_s - X^c_s = X_s - \inf_{t \in [T_{d,k}^{2c},s]} X_t - c \in [-c; c].
\]
(4) We will prove stronger fact than (4), namely that for every \( s > 0 \),
\[
|\Delta X^c_s| \leq |\Delta X_s|.
\]
Indeed, from formula \[(2.1)\] it follows that for any \( s \neq \{T^2_{u,k}; T^2_{d,k}\} \), \[(2.2)\] holds, hence let us assume that \( s \in \{T^2_{u,k}; T^2_{d,k}\} \). We consider several possibilities. If \( s = T^2_{u,0} \) then, by the definition of \( T^2_{u,0} \),
\[
X_s - X^c_s = X_s - c - X_0 \geq 0 \quad \text{and} \quad X^c_s - X_s = X_s - X_0 - c \leq X_s - X^c_s.
\]
If \( s = T^2_{u,k}, k = 1,2,\ldots \), then, by the definition of \( T^2_{u,k} \),
\[
X^c_s - X_s = X_s - c - \left( \inf_{t \in [T^2_{d,k} - 1; s]} X_t + c \right) = X_s - \inf_{t \in [T^2_{d,k} - 1; s]} X_t - 2c \geq 0
\]
and, on the other hand,
\[
X^c_s - X_s = X_s - \inf_{t \in [T^2_{d,k} - 1; s]} X_t - 2c \leq X_s - X^c_s.
\]
Similar arguments may be applied for \( s = T^2_{d,k}, k = 0,1,\ldots \).

(5) The process \( X^c \) is adapted to the filtration \( F \) since it is adapted to any right continuous filtration containing the natural filtration of the process \( X \).

\[ \square \]

**Remark 4.** It is possible to define the process \( X^c \) in many different ways. For example, defining
\[
X^c = X_0 + \text{UTV}^c(X, \cdot) - \text{DTV}^c(X, \cdot)
\]
we obtain a process satisfying all conditions (1)-(5) and having (on the intervals of the form \([0; T], T > 0\) the smallest possible total variation among all processes, increments of which differ from the increments of the process \( X \) by no more than \( c \). \( \text{UTV}^c(X, \cdot) \) and \( \text{DTV}^c(X, \cdot) \) denote here upward and downward truncated variation processes, defined as
\[
\text{UTV}^c(X,t) := \sup_n \sup_{0 \leq t_1 < t_2 < \ldots < t_n \leq t} \sum_{i=1}^n \max \left\{ X_{t_i} - X_{t_{i-1}} - c, 0 \right\},
\]
\[
\text{DTV}^c(X,t) := \sup_n \sup_{0 \leq t_1 < t_2 < \ldots < t_n \leq t} \sum_{i=1}^n \max \left\{ X_{t_{i-1}} - X_{t_i} - c, 0 \right\}.
\]
Moreover, for any \( T > 0 \) we have
\[
\text{TV}(X^c; T) = \text{UTV}^c(X,T) + \text{DTV}^c(X,T) = \sup_n \sup_{0 \leq t_1 < t_2 < \ldots < t_n \leq T} \sum_{i=1}^n \max \left\{ |X_{t_i} - X_{t_{i-1}}| - c, 0 \right\} =: \text{TV}^c(X; T).
\]
We will call \( TV^c \) truncated variation. For more on truncated variation, upward truncated variation and downward truncated variation see [10] or [11].

Some other construction may be done with the Skorohod map on \([-\alpha; \beta] \) (cf. [3]) where \( \alpha, \beta : [0; +\infty) \rightarrow (0; +\infty) \) are (possibly time-dependent) continuous boundaries such that \( \sup_{0 \leq t \leq T} \alpha^c(t) \leq K_T \), \( \sup_{0 \leq t \leq T} \beta^c(t) \leq K_T \) and \( \inf_{0 \leq t} (\beta^c(t) + \alpha^c(t)) > 0 \). The Skorohod map on \([-\alpha; \beta] \) allows to construct such a locally finite variation càdlàg process \(-X^c\) that
\[
X + (-X^c) \in [-\alpha^c; \beta^c].
\]
From further properties of this map it follows that \( X^c \) satisfies all conditions (1)-(5). In fact, construction \[(2.1)\] of \( X^c \) is based on a Skorohod map on the interval \([-c; c]\). In the Appendix we will prove this as well as other interesting properties of this map.

3. Pathwise Lebesgue-Stieltjes integration with respect to the process \( X^c \)

Let us now consider a measurable space \((\Omega, \mathcal{F})\) equipped with a right-continuous filtration \( F \) and two processes \( X \) and \( Y \) with càdlàg paths, adapted to \( F \). For \( T > 0 \) and for a sequence of processes \((X^c)_{c > 0}\) with \( X^c \in X^c \) let us consider the sequence
\[
\int_0^T Y \cdot dX^c.
\]
The integral in (3.1) is understood in the pathwise, Lebesgue-Stieltjes sense (recall that for any \( c > 0 \), \( X^c \) has bounded variation). We have

**Theorem 5.** Assume that \( \mathbb{P} \) is a probability measure on \((\Omega, \mathcal{F})\) such that \( X \) and \( Y \) are semimartingales with respect to this measure and filtration \( F \), which is complete under \( \mathbb{P} \), then

\[
\int_0^T Y \cdot dX^c \to^{ucp} \int_0^T Y \cdot dX + [X^{cont}, Y^{cont}]_T \quad \text{as } c \downarrow 0,
\]

where "\( \rightarrow^{ucp} \)" denotes uniform convergence on compacts in probability \( \mathbb{P} \) and \([X^{cont}, Y^{cont}]_T \) denotes quadratic covariation of continuous parts \( X^{cont}, Y^{cont} \) of \( X \) and \( Y \) respectively.

**Proof.** Fixing \( c > 0 \) and using integration by parts formula (cf. [7] formula (1), page 519]) we get

\[
Y_T X_T^c - Y_0 X_0^c = \int_0^T Y_t \cdot dX_t^c + \int_0^T X_t^c \cdot dY_t + [Y, X^c]_T
\]

(the above equality and subsequent equalities in the proof hold \( \mathbb{P} \) a.s.). By the uniform convergence, \( X_t^c \to X_t \) as \( c \downarrow 0 \) (note that the bound \(|X^c| \leq |X| + K_T c\) and a.s. pointwise convergence \( X_t^c \to X_t \) as \( c \downarrow 0 \) are sufficient) we get

\[
\int_0^T X_t^c \cdot dY_t \to^{ucp} \int_0^T X_t \cdot dY_t.
\]

Since \( X^c \) has locally finite variation, we have (cf. [7] Theorem 26.6 (viii)),

\[
[Y, X^c]_T = \sum_{0 < s \leq T} \Delta Y_s \Delta X_s^c.
\]

We calculate the (pathwise) limit

\[
\lim_{c \downarrow 0} [Y, X^c]_T = \lim_{c \downarrow 0} \sum_{0 < s \leq T} \Delta Y_s \Delta X_s^c = \sum_{0 < s \leq T} \Delta Y_s \Delta X_s
\]

(notice that for any \( 0 \leq s \leq T \), \(|\Delta X_s^c| \leq L_T |\Delta X_s|\), thus the above sum is convergent by dominated convergence) and finally obtain

\[
\int_0^T Y_t \cdot dX_t^c \to^{ucp} \left\{ Y_T X_T^c - Y_0 X_0^c - \int_0^T X_t^c \cdot dY_t - [Y, X^c]_T \right\}
\]

(3.2)

On the other hand, again by the integration by parts formula, we obtain

(3.3)

\[
\int_0^T X_t \cdot dY_t = Y_T X_T - Y_0 X_0 - \int_0^T Y_t \cdot dX_t - [Y, X]_T.
\]

Finally, comparing (3.2) and (3.3), and using [7] Corollary 26.15, we obtain

\[
\int_0^T Y_t \cdot dX_t^c \to^{ucp} \int_0^T Y_t \cdot dX_t + [Y, X]_T - \sum_{0 < s \leq T} \Delta Y_s \Delta X_s \quad \text{as } c \downarrow 0
\]

\[
= \int_0^T Y_t \cdot dX_t + [X^{cont}, Y^{cont}]_T.
\]

□

Note that to prove Theorem 5 we did not need the pathwise uniform convergence of the processes \( X^c \) to the process \( X \); we might simply use local boundedness and a.s. pointwise convergence \( X_t^c \to X_t \) as \( c \downarrow 0 \). Using the pathwise uniform convergence of the sequence \( (X^c)_{c>0} \) we are able to prove a bit stronger result. We have

**Theorem 6.** Assume that \( \mathbb{P} \) is a probability measure on \((\Omega, \mathcal{F})\) such that \( X \) and \( Y \) are semimartingales with respect to this measure and filtration \( F \), which is complete under \( \mathbb{P} \), then for any \( T > 0 \) and any sequence \((c(n))_{n \geq 1}\) such that \( c(n) > 0 \), \( \sum_{n=1}^{\infty} c(n)^2 < +\infty \) we have

\[
\lim_{n \to +\infty} \sup_{0 \leq t \leq T} \left| \int_0^t Y \cdot dX^{c(n)} - \int_0^t Y \cdot dX + [X^{cont}, Y^{cont}]_t \right| = 0 \quad \mathbb{P} \text{ a.s.}
\]
Thus we obtain that $\Omega$ and a process $\Delta Y_s \Delta (X^c_s - X_s) - \int_0^t (X^c - X) \, dY$.

\[
\sum_{0 < s \leq t} \Delta Y_s \Delta (X^c_s - X_s) \leq K_{TC} (|Y_0| + |Y_t|) + \sum_{0 < s \leq t} \Delta Y_s \Delta (X^c_s - X_s) + \int_0^t (X^c - X) \, dY.
\]

Thus we get
\[
\sup_{0 \leq t \leq T} \left| \int_0^t Y_- \, dX^c - \int_0^t Y_- \, dX - [X^\text{cont}, Y^\text{cont}]_t \right| \leq K_{TC} (|Y_0| + \sup_{0 \leq t \leq T} |Y_t|) + \sum_{0 < s \leq t} \Delta Y_s \Delta (X^c_s - X_s) + \sup_{0 \leq t \leq T} \left| \int_0^t (X^c - X) \, dY \right|.
\]

Since $Y$ has càdlàg paths, it is locally bounded and hence $K_{TC} (|Y_0| + \sup_{0 \leq t \leq T} |Y_t|) \to 0$ a.s. as $c \downarrow 0$.

Since for every $t \in [0; T]$, $|X^c_t - X_t| \leq K_{TC}$ (condition (3)), for $s \in [0; t]$ we have $|\Delta (X^c_s - X_s)| \leq 2 K_{TC}$. Similarly, by condition (4),
\[
|\Delta (X^c_s - X_s)| \leq |\Delta X^c_s| + |\Delta X_s| \leq (L_T + 1) |\Delta X_s|.
\]

Thus we obtain that
\[
|\Delta (X^c_s - X_s)| \leq \min \{2 K_{TC}, (L_T + 1) |\Delta X_s|\} \leq (2 K_T + L_T + 1) \min \{c, |\Delta X_s|\}
\]

and using this, we estimate
\[
\sup_{0 \leq t \leq T} \left| \sum_{0 < s \leq t} \Delta Y_s \Delta (X^c_s - X_s) \right| \leq \sup_{0 \leq t \leq T} \sqrt{\sum_{0 < s \leq t} |\Delta Y_s|^2} \sqrt{\sum_{0 < s \leq t} |\Delta (X^c_s - X_s)|^2} \leq \sqrt{[\Delta Y]_T (2 K_T + L_T + 1) \sum_{0 < s \leq t} \min \{c^2, |\Delta X_s|^2\}} \to 0 \text{ a.s. as } c \downarrow 0.
\]

In order to estimate
\[
I^c(T) := \sup_{0 \leq t \leq T} \left| \int_0^t (X^c - X_-) \, dY \right|
\]

let us decompose the semimartingale $Y$ into a local martingale $M$ with bounded jumps (hence a local $L^2$ martingale) and a process $A$ with locally finite variation (this is possible due to [17, Lemma 26.5] but the decomposition may depend on the measure $P$). $Y = M + A$. Let $(\tau(k))_{k \geq 1}$ be a sequence of stopping times increasing to $+\infty$ such that $(M_{t \wedge \tau(k)})_{t \geq 0}$ is a square integrable martingale. We will use elementary estimate $(a + b)^2 \leq 2a^2 + 2b^2$, the Burkholder inequality and localization. On the set $\Omega_N = \{\omega \in \Omega : TV (A, T) \leq N\}$ we have
\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau(k)} \left| \int_0^t (X^c - X_-) \, dY \right|^2 ; \Omega_N \right] \leq 2E \sup_{0 \leq t \leq T \wedge \tau(k)} \left| \int_0^t (X^c - X_-) \, dM \right|^2 + 2 \left[ E \left| \int_0^T (X^c - X_-) \, dA \right|^2 ; \Omega_N \right] \leq 2 (4K_T^2 c^2 E [M, M]_{T \wedge \tau(k)} + K_T^2 c^2 N^2) \leq 8 E [M, M]_{T \wedge \tau(k)} + N^2) K_T^2 c^2.
\]
Let now \((c(n))_{n \geq 1}\) be such a sequence that \(\sum_{n=1}^{\infty} c(n)^2 < +\infty\). We have
\[
\mathbb{E} \left[ \sum_{n=1}^{\infty} \sup_{0 \leq t \leq T \land \tau(\cdot)} \left| \int_0^t \left( X_{c(n)}^e - X_\tau^e \right) dY \right|^2 ; \Omega_N \right]
\]
\[
= \sum_{n=1}^{\infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T \land \tau(\cdot)} \left| \int_0^t \left( X_{c(n)}^e - X_\tau^e \right) dY \right|^2 ; \Omega_N \right]
\]
\[
\leq 8 \left( \mathbb{E} \left[ M, M \right]_{T \land \tau(\cdot)} + N^2 \right) K_T^2 \sum_{n=1}^{\infty} c(n)^2 < +\infty.
\]
Hence, the sequence \(I^{c(n)}(T \land \tau(k)), n = 1, 2, \ldots\), converges to 0 on the set \(\Omega_N\). Since \(\Omega = \bigcup_{N \geq 1} \Omega_N\), we get that \(I^{c(n)}(T \land \tau(k))\) converges \(\mathbb{P}\) a.s. to 0. Finally, since \(\tau(k) \to +\infty\) a.s. we get that \(I^{c(n)}(T)\) converges \(\mathbb{P}\) a.s. to 0.

\[\square\]

4. \textbf{Counterexamples}

In this section, using further properties of the sequence \(X^e\) defined in Section 2 which we prove in the Appendix, we will show that even for the integrator \(X = B\) being a standard Brownian motion Theorem 5 can not be extended to the case when \(Y\) is not a semimartingale. To prove this we start with few definitions. First, we define sequence \(\beta(n), n = 1, 2, \ldots\) in the following way \(\beta(1) = 1\) and for \(n = 2, 3, \ldots\):
\[
\beta(n) = n^2 \beta(n - 1)^6.
\]
Now we define \(\alpha(n) := \beta(n)^{1/2}\), \(\gamma(n) := \beta(n)^{-1}\) and
\[
Y := \sum_{n=2}^{\infty} \alpha(n) \left( B - B^{\gamma(n)} \right),
\]
where \(B\) is a standard Brownian motion and for any \(c > 0\), \(B^c\) is defined as in Section 2 (with formulas (2.1) or symmetric). Notice that \(Y\) is well defined, since
\[
\left| \alpha(n) \left( B - B^{\gamma(n)} \right) \right| \leq \alpha(n) \gamma(n) = \gamma(n)^{1/2}
\]
and for \(n = 2, 3, \ldots\),
\[
\gamma(n)^{1/2} = \beta(n)^{-1/2} = n^{-1} \beta(n - 1)^{-3}
\]
\[
\leq 2^{-1} \beta(n - 1)^{-1/2} = 2^{-1} \gamma(n - 1)^{1/2}.
\]
Hence the series
\[
\sum_{n=2}^{\infty} \alpha(n) \left( B - B^{\gamma(n)} \right)
\]
is uniformly convergent to a bounded, continuous process, adapted to the natural filtration of \(B\). We will use facts proved in Appendix as well as [11, Theorem 1], stating that for any continuous semimartingale \(X\)
\[
\lim_{c \downarrow 0} c \cdot TV^c (X, 1) = \langle X \rangle_1
\]
(where \(TV^c (X, T)\) was defined in Remark 1), from which follows that
\begin{equation}
\lim_{c \downarrow 0} c \cdot TV^c (B, 1) = 1.
\end{equation}
We will also use the Gaussian concentration of \(TV^c (B, T)\) (see [11, Remark 6]), from which follows that for \(c \in (0; 1)\) and \(k = 1, 2, \ldots\),
\begin{equation}
\mathbb{E} TV^c (B, 1)^k \leq C_k c^{-k},
\end{equation}
where \(C_k\) is a constant depending on \(k\) only.

We have

\textbf{Fact 7.} The sequence of integrals
\[
\int_0^1 Y \cdot dB^{\gamma(n)}
\]
diverges.
Proof. Let us fix $n = 2, 3, 4, \ldots$ and split $\int_0^1 Y_\cdot dB^{\gamma(n)}$ into two summands, $\int_0^1 Y_\cdot dB^{\gamma(n)} = I + II$, where
\[
I = \sum_{m=2}^{n-1} \alpha(m) \int_0^1 \left( B - B^{\gamma(m)} \right) dB^{\gamma(n)}
\]
and
\[
II = \int_0^1 \left\{ \alpha(n) \left( B - B^{\gamma(n)} \right) + \sum_{m=n+1}^\infty \alpha(m) \left( B - B^{\gamma(m)} \right) \right\} dB^{\gamma(n)}.
\]

Firstly, we consider the second summand, $II$. Let us notice that for $m \geq 3$, $\gamma(m)^{1/2} \leq 3^{1/2} \gamma(m-1)^{1/2}$ which implies
\[
\left| \sum_{m=n+1}^\infty \alpha(m) \left( B - B^{\gamma(m)} \right) \right| \leq \sum_{m=n+1}^\infty \alpha(m) \gamma(m) = \sum_{m=n+1}^\infty \gamma(m)^{1/2}
\leq \gamma(n)^{1/2} \sum_{i=1}^\infty 3^{-i} = \frac{1}{2} \gamma(n)^{1/2}.
\]

Hence
\[
\left| \int_0^1 \sum_{m=n+1}^\infty \alpha(m) \left( B - B^{\gamma(m)} \right) dB^{\gamma(n)} \right| \leq \frac{1}{2} \gamma(n)^{1/2} \int_0^1 \left| dB^{\gamma(n)} \right| = \frac{1}{2} \gamma(n)^{1/2} \cdot TV \left( B^{\gamma(n)}, 1 \right).
\]
By the equality (5.2) (see the Appendix),
\[
\alpha(n) \int_0^1 \left( B - B^{\gamma(n)} \right) dB^{\gamma(n)} = \gamma(n)^{1/2} TV \left( B^{\gamma(n)}, 1 \right)
\]
and by two last estimates we get
(4.3)
\[
II \geq \frac{1}{2} \gamma(n)^{1/2} TV \left( B^{\gamma(n)}, 1 \right) \geq \frac{1}{2} \gamma(n)^{1/2} TV 2^{\gamma(n)} (B, 1),
\]
where the last estimate follows from $TV \left( B^{\gamma(n)}, 1 \right) \geq TV 2^{\gamma(n)} (B, 1)$ (see (5.1) in the Appendix).

Now let us consider the first summand, $I$. For $m = 2, \ldots, n - 1$, using integration by parts we calculate
\[
\int_0^1 \left( B - B^{\gamma(m)} \right) dB^{\gamma(n)} = \int_0^1 B dB^{\gamma(n)} - \int_0^1 B^{\gamma(m)} dB^{\gamma(n)}
= \left( B_1 - B_1^{\gamma(m)} \right) B_1^{\gamma(n)} + \int_0^1 B^{\gamma(n)} dB^{\gamma(m)} - \int_0^1 B^{\gamma(n)} dB.
\]

By this, the inequality $(a + b + c)^2 \leq 3 \left( a^2 + b^2 + c^2 \right)$ and the Itô isometry we estimate
\[
E \left( \int_0^1 \left( B - B^{\gamma(m)} \right) dB^{\gamma(n)} \right)^2 \leq 3 \gamma(m)^2 E \left( B_1^{\gamma(n)} \right)^2
+ 3 \left\{ \sup_{0 \leq s \leq 1} \left( B_s^{\gamma(n)} \right)^2 TV \left( B^{\gamma(n)}, 1 \right) \right\}
+ 3 \int_0^1 E \left( B_s^{\gamma(n)} \right)^2 ds.
\]

Further, from $a^2 b^2 \leq \frac{1}{2} a^4 + \frac{1}{2} b^4$ and then $\left| B_s^{\gamma(n)} \right| \leq |B_s| + \gamma(n)$, $TV \left( B^{\gamma(m)}, 1 \right) \leq TV 2^{\gamma(m)} (B, 1) + 2 \gamma(m)$ (this follows from the estimate (5.1) and $(a + b)^4 \leq 8 \left( a^4 + b^4 \right)$,
\[
E \left\{ \sup_{0 \leq s \leq 1} \left( B_s^{\gamma(n)} \right)^2 TV \left( B^{\gamma(m)}, 1 \right) \right\} \leq \frac{1}{2} E \sup_{0 \leq s \leq 1} \left( B_s^{\gamma(n)} \right)^4 + \frac{1}{2} E TV \left( B^{\gamma(m)}, 1 \right)^4
\leq \frac{1}{2} 8 E \sup_{0 \leq s \leq 1} \left( B_s^{\gamma(n)} \right)^4 + \frac{1}{2} 8 E TV 2^{\gamma(m)} (B, 1)^4 + 2^4 \gamma(n)^4
\leq 4 E \sup_{0 \leq s \leq 1} B_s^4 + 4 E \sup_{0 \leq s \leq 1} TV 2^{\gamma(m)} (B, 1)^4 + 1.
\]

Similarly, by $\left| B_s^{\gamma(n)} \right| \leq |B_s| + \gamma(n)$ and $(a + b)^2 \leq 2 \left( a^2 + b^2 \right)$ we calculate
\[
E \left( B_1^{\gamma(n)} \right)^2 \leq 2 E \left( B_1^2 + \gamma(n)^2 \right) \leq 3
\]
and

\[ \int_0^1 \mathbb{E}\left( B_s^{(n)} \right)^2 \, ds \leq 3. \]

Hence, by (4.4) and last three estimates,

\[
\mathbb{E}\left( \sum_{m=2}^{n-1} \alpha(m) \int_0^1 (B - B_{\gamma(m)}) \, dB_{\gamma(n)} \right)^2 \leq n \sum_{m=2}^{n-1} \alpha(m)^2 \mathbb{E}\left( \int_0^1 (B - B_{\gamma(m)}) \, dB_{\gamma(n)} \right)^2 
\leq n \sum_{m=2}^{n-1} \alpha(m)^2 \left( 3 \gamma(m)^2 + 4 \mathbb{E} \sup_{0 \leq s \leq 1} B_s^4 + 4 \mathbb{E} TV^{2\gamma(m)}(B, 1)^4 \right) 
\leq n^2 \alpha(n-1)^2 \left( 7 + 4 \mathbb{E} \sup_{0 \leq s \leq 1} B_s^4 + 4 \mathbb{E} TV^{2\gamma(n-1)}(B, 1)^4 \right).
\]

(4.5)

By the Gaussian concentration properties of \( \sup_{0 \leq s \leq 1} B_s \) and \( TV^{2\gamma(n-1)}(B, 1) \) (estimate (4.2)), there exists universal constants \( \tilde{C}, C \) such that

\[ \mathbb{E} TV^{2\gamma(n-1)}(B, 1)^4 \leq \tilde{C} \gamma(n-1)^{-4} \]

and

\[ 3 \left( 7 + 4 \mathbb{E} \sup_{0 \leq s \leq 1} B_s^4 + 4 \mathbb{E} TV^{2\gamma(n-1)}(B, 1)^4 \right) \leq C \gamma(n-1)^{-4} = C \beta(n-1)^4. \]

By (4.5) and (4.6),

\[ \mathbb{E}\left( \sum_{m=2}^{n-1} \alpha(m) \int_0^1 (B - B_{\gamma(m)}) \, dB_{\gamma(n)} \right)^2 \leq n^2 \alpha(n-1)^2 \beta(n-1)^4 = C n^2 \beta(n-1)^5. \]

Now, by (4.4) and the Chebyshev inequality we get

\[ \mathbb{P}\left( |I| \geq \sqrt{3Cn\beta(n-1)^{5/2}} \right) \leq \frac{1}{3}. \]

Thus, for the set \( A_n := \left\{ |I| \leq \sqrt{3Cn\beta(n-1)^{5/2}} \right\} \) we have \( \mathbb{P}(A_n) \geq 2/3 \), and by (4.3) on \( A_n \) we have

\[ \int_0^1 Y_- \, dB_{\gamma(n)} = I + II \geq \frac{1}{2} \gamma(n)^{1/2} TV^z(B_{\gamma(n)}, 1) - \sqrt{2Cn\beta(n-1)^{5/2}} \]

\[ \geq \frac{1}{2} \gamma(n)^{-1/2} \gamma(n) TV^{2\gamma(n)}(B, 1) - \sqrt{2Cn\beta(n-1)^{5/2}} \]

\[ = \frac{1}{2} \beta(n)^{1/2} \gamma(n) TV^{2\gamma(n)}(B, 1) - \sqrt{2Cn\beta(n-1)^{5/2}}. \]

Let us choose such \( N \) that for any \( n \geq N \),

\[ \mathbb{P}\left( \gamma(n) TV^{2\gamma(n)}(B, 1) \geq \frac{1}{4} \right) \geq \frac{2}{3}. \]

(this is possible by (4.1)). By the definition of \( \beta(n) \), on the set \( A_n \cap D_n \), where

\[ D_n := \left\{ \gamma(n) TV^{2\gamma(n)}(B, 1) \geq \frac{1}{4} \right\}, \]

we get

\[ \frac{1}{2} \beta(n)^{1/2} \gamma(n) TV^{2\gamma(n)}(B, 1) - \sqrt{3Cn\beta(n-1)^{5/2}} \geq \frac{1}{8} \beta(n-1)^3 - \sqrt{3Cn\beta(n-1)^{5/2}}. \]

Since

\[ \frac{1}{8} \beta(n-1)^3 - \sqrt{3Cn\beta(n-1)^{5/2}} \to +\infty \]

as \( n \to +\infty \) and

\[ \mathbb{P}(A_n \cap D_n) \geq \frac{1}{3}, \]

we get that the sequence of integrals

\[ \int_0^1 Y_- \, dB_{\gamma(n)} \]

is divergent. \( \square \)
Remark 8. From Theorem 3 and just proved Fact 2 it follows that the bounded, continuous process
\[ Y = \sum_{n=2}^{\infty} \alpha(n) \left( B - B^{\gamma(n)} \right), \]
adapted to the natural filtration of \( B \), can not be a semimartingale.

The construction of sequences \( Z^{\delta(n)}, \tilde{B}^{\delta(n)}, n = 1, 2, \ldots \) such that the sequence of integrals \( \int_0^1 Z^{\delta(n)} \, d\tilde{B}^{\delta(n)}, n = 1, 2, \ldots \), is divergent as \( n \uparrow +\infty \) and \( Z^{\delta(n)}, \tilde{B}^{\delta(n)} \) satisfy conditions (1)-(5) for some semimartingales \( Z, \tilde{B} \). This is much easier. We set \( \delta(n) = 1/n \), \( Z^{\delta(n)} = 2B^{1/n^2} + n \left( B^{1/(2n^2)} - B^{1/n^2} \right), \tilde{B}^{\delta(n)} = B^{1/n^2} \). We easily check that \( Z^{\delta(n)} \) satisfies (1)-(5) for \( Z = 2B \) and trivially \( \tilde{B}^{\delta(n)} \) satisfies (1)-(5) for \( \tilde{B} = B \). Since for any \( c > 0 \), on the set \( B^c = B - c, dB^c \geq 0 \), and on the set \( B^c = B + c, dB^c \leq 0 \) (see Lemma 10 in the Appendix), and \(-c/2 \leq B - B^{c/2} \leq c/2 \), we get \( B^{c/2} - B^c \geq c/2 \) on the set \( dB^c > 0 \) and \( B^{c/2} - B^c \leq -c/2 \) on the set \( dB^c < 0 \). Thus
\[
\int_0^1 Z^{\delta(n)} \, d\tilde{B}^{\delta(n)} - \int_0^1 2B^{1/n^2} \, dB^{1/n^2} = \int_0^1 n \left( B^{1/(2n^2)} - B^{1/n^2} \right) \, dB^{1/n^2} \\
\geq n/2 \int_0^1 |dB^{1/n^2}| = n/2 \, n^{-2} TV \left( B^{1/n^2}, 1 \right) \\
\geq n/2 \, n^{-2} TV^{1/n^2}(B, 1).
\]
Now, by the usual Lebesque-Stieltjes integration, \( \int_0^1 2B^{1/n^2} \, dB^{1/n^2} = \left( B^{1/n^2} \right)^2 \), and by the just obtained estimate and 4.11 we see that
\[
\int_0^1 Z^{\delta(n)} \, d\tilde{B}^{\delta(n)} \to +\infty.
\]

5. Appendix

In this Appendix we will prove estimates used in Section 4, conceming the process \( X^c \), constructed in Section 2. Before we proceed, let us recall the definitions of truncated variation, upward truncated variation and downward truncated variation from Remark 4. Let us notice that for \( c = 0 \) we simply get that \( TV^0 \) is the (finite or infinite) total variation and \( UTV = UTV^0 \) and \( DTV = DTV^0 \) are positive and negative parts of the total variation. Moreover, we have the Hahn-Jordan decomposition, \( TV = UTV + DTV \).

Lemma 9. For the total variation of the process \( X^c \), constructed in Section 2, one has the following estimates
\[
TV^{2c}(X, T) \leq TV(X^c, T) \leq TV^{2c}(X, T) + 2c.
\]
Proof. The lower bound in (5.1) follows directly from the estimate
\[
|X_t^c - X_s^c| \geq \max \{|X_t - X_s| - 2c, 0\},
\]
valid for any \( 0 \leq s < t \leq T \), which follows directly from inequalities \( |X_t^c - X_s| \leq c, |X_t^c - X_s^c| \leq c \) and the triangle inequality.

To prove the opposite inequality, let us assume that \( T_{d,k}^c X \geq T_{u,k}^c X \) and denote 
\[
M^{2c}_k = \sup_{t \in [T_{d,k}^c, T_{u,k+1}^c]} X_t, k = 0, 1, \ldots,
\]
and consider three possibilities:
- \( T \in [0; T_{u,0}^c] \). In this case \( TV(X^c, T) = UTV(X^c, T) = DTV(X^c, T) = 0 \).
- \( T \in \left[ T_{u,0}^c; T_{d,0}^c \right] \). In this case
\[
UTV(X^c, T) = \sup_{t \in [T_{u,0}^c, T_{d,0}^c]} X_t - c - X_0, DTV(X^c, T) = 0.
\]
and
\[
TV(X^c, T) = UTV(X^c, T) + DTV(X^c, T).
\]
Now, by the definition of \( TV^{2c} \) it is not difficult to see that
\[
TV^{2c}(X, T) \geq \max \left\{ \sup_{t \in [T_{u,0}^c, T]} X_t - X_0 - 3c, 0 \right\} \geq TV(X^c, T) - 2c.
\]
Lemma 10. (3, Theorem 2.6 and Corollary 2.4) and we will write \( \phi \) and the mutually singular measures \( \eta, \nu \). We have

\[
UTV(X^c, T) = (M_0^{2c} - c - X_0) + \sum_{i=1}^{k-1} (M_i^{2c} - m_i^{2c} - 2c) + \sup_{t \in [T^{2c}_{u,k}; T^{2c}_{d,k}]} X_t - m_{k-1}^{2c} - 2c \geq UTV(X^c, T) - 2c,
\]

and

\[
DTV(X^c, T) = \sum_{i=0}^{k-1} (M_i^{2c} - m_i^{2c} - 2c) = DTV(X^c, T)
\]

and

\[
TV(X^c, T) = UTV(X^c, T) + DTV(X^c, T).
\]

Now it is not difficult to see that

\[
UTV^{2c}(X, T) \geq \max \{ M_0^{2c} - X_0 - 3c, 0 \} + \sum_{i=1}^{k-1} (M_i^{2c} - m_i^{2c} - 2c) + \sup_{t \in [T^{2c}_{u,k}; T^{2c}_{d,k}]} X_t - m_{k-1}^{2c} - 2c \geq UTV(X^c, T) - 2c,
\]

and

\[
TV^{2c}(X, T) = UTV^{2c}(X, T) + DTV^{2c}(X, T) \geq TV(X^c, T) - 2c.
\]

\( s \in [T_{d,k}; T_{u,k+1}] \), for some \( k = 0, 1, 2, \ldots \). The proof follows similarly as in the previous case.

Now we will prove that the construction of \( X^c \) in Section 2 is based on a Skorohod map on the interval \([-c; c]\). Let us recall the definition of the Skorohod problem on the interval \([-c; c]\). Let \( D[0; +\infty) \) denote the set of real-valued càdlàg functions and \( BV[0; +\infty) \), \( BV(0; +\infty) \) denote subspaces of \( D[0; +\infty) \) consisting of nondecreasing functions and functions of bounded variation, respectively. We have

Definition 1. A pair of functions \((\phi, \eta) \in D[0; +\infty) \times BV[0; +\infty)\) is said to be a solution of the Skorohod problem on \([-c, c]\) for \( \psi \) if the following conditions are satisfied:

1. for every \( t \geq 0 \), \( \phi(t) = \psi(t) + \eta(t) \in [-c, c]; \)
2. \( \eta = \eta - \nu_a \), where \( \eta, \nu_a \in BV^+[0; +\infty) \) and the corresponding measures \( d\eta, d\nu_a \) are carried by \( \{ t \geq 0 : \phi(t) = -c \} \) and \( \{ t \geq 0 : \phi(t) = c \} \), respectively.

It is possible to prove that for every \( c > 0 \) there exist a unique solution to the Skorohod problem on \([-c; c]\) (cf. 3 Theorem 2.6 and Corollary 2.4) and we will write \( \phi^c = \Gamma^c(\psi) \) to denote the associated map, called the Skorohod map on \([-c, c]\). Now we will prove

Lemma 10. The process \( X^c \), constructed in Section 2 and the Skorohad map on \([-c; c]\) are related via the equality

\[
X^c = X - \Gamma^c(X)
\]

and the mutually singular measures \( dUTV(X^c, \cdot) \) and \( dDTV(X^c, \cdot) \) are carried by \( \{ t \geq 0 : X_t - X^c_t = c \} \) and \( \{ t \geq 0 : X_t - X^c_t = -c \} \) respectively. Thus, on these sets we have

\[
dUTV(X^c, \cdot) = dX^c, \quad dDTV(X^c, \cdot) = -dX^c
\]

respectively.

Proof. Denote \( V = X - X^c \). We have \( V \in [-c; c] \), i.e. condition 1. in the Definition holds, and to finish the proof it is enough to prove that for the finite variation process \( -X^c \) the corresponding measures \( dUTV(-X^c, \cdot) = dDTV(-X^c, \cdot) \) and \( dDTV(-X^c, \cdot) = dUTV(-X^c, \cdot) \) are carried by \( \{ t \geq 0 : V_t = -c \} \) and \( \{ t \geq 0 : V_t = c \} \) respectively. Notice that by the formula (2.1) the process \( -X^c \) is nonincreasing on the intervals \([T^{2c}_{u,k}; T^{2c}_{d,k}]\) and nondecreasing on the intervals \([T^{2c}_{d,k}; T^{2c}_{u,k+1}]\), \( k = 0, 1, 2, \ldots \). Thus \( d(-X^c_t) = dDTV(X^c, s) = -d\inf_{t \leq s} X_t \) and \( d(-X^c_t) = -dUTV(X^c, s) = -d\sup_{t \leq s} X_t \) on the intervals \([T^{2c}_{d,k}; T^{2c}_{u,k+1}]\) and \([T^{2c}_{u,k}; T^{2c}_{d,k}]\), \( k = 0, 1, 2, \ldots \), respectively.
Now, notice that the only points of increase of the measure $d\text{UTV}^c(X^c, \cdot)$ from the intervals $(T^c_{u,k}; T^c_{d,k}), k = 0, 1, 2, \ldots$ are the points where the process $X$ attains new suprema. But in every such point $s$ we have
\[ X^c_s = \sup_{t \in [T^c_{u,k}; s]} X_t - c = X_s - c \]
and hence $V^c_s = X_s - X^c_s = c$. Similar assertion holds for $d\text{DTV}^c(X^c, \cdot)$.

Next, notice that at the point $s = T^c_{u,0}$ one has $X^c_s = X_s - c \geq X_0 = X_{-}\cdots$, and since for $T^c_{u,k+1} < +\infty, k = 0, 1, \ldots$, one has
\[ T^c_{u,k+1} = \inf \left\{ s \geq T^c_{d,k} : X_s - \inf_{t \in [T^c_{d,k}; s]} X_t > 2c \right\}, \]
then for $s = T^c_{u,k+1} < +\infty, k = 0, 1, \ldots, \inf_{t \in [T^c_{d,k}; s]} X_t = \inf_{t \in [T^c_{d,k}; s]} X_t$ and
\[ X^c_s = X_s - c \geq \inf_{t \in [T^c_{d,k}; s]} X_t + c \]
\[ = \inf_{t \in [T^c_{d,k}; s]} X_t + c = X^c_{-}\cdots. \]
Thus, at the points $s = T^c_{u,k}, k = 0, 1, \ldots$ we have $d\text{DTV}^c(X^c, \cdot) = 0$, $d\text{UTV}^c(X^c, \cdot) \geq 0$ and $V^c_s = c$.

In a similar way one proves that the measure $d\text{DTV}^c(X^c, \cdot)$ is carried by $\{ t \geq 0 : V^c_t = -c \}$.

The last assertion follows from the fact that $\text{UTV}$ and $\text{DTV}$ are positive and negative parts of $dX^c$. \hfill $\square$

The direct consequence of Lemma 10 is the equality
\[ \int_0^T (X - X^c) dX^c = c \cdot \int_0^T |dX^c| = c \cdot \text{TV}(X^c, T), \]
which holds for any $c$, $T > 0$.

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