Semianalytical model for planetary resonances

Application to planets around single and binary stars

Tabaré Gallardo¹, Cristián Beaugé², and Cristian A. Giuppone²

¹ Instituto de Física, Facultad de Ciencias, Udelar, Iguá 4225, 11400 Montevideo, Uruguay
² Universidad Nacional de Córdoba, Observatorio Astronómico - IATE, Laprida 854, 5000 Córdoba, Argentina
e-mail: cristian@oa.unc.edu.ar

Received 26 October 2020 / Accepted 21 December 2020

ABSTRACT

Context. Planetary resonances are a common dynamical mechanism acting on planetary systems. However, no general model for describing their properties exists, particularly for commensurabilities of any order and arbitrary eccentricity and inclination values.

Aims. We present a semianalytical model that describes the resonance strength, width, location and stability of fixed points, and periods of small-amplitude librations. The model is valid for any two gravitationally interacting massive bodies, and is thus applicable to planets around single or binary stars.

Methods. Using a theoretical framework in the Poincaré and Jacobi reference system, we developed a semianalytical method that employs a numerical evaluation of the averaged resonant disturbing function. Validations of the model are presented that compare its predictions with dynamical maps for real and fictitious systems.

Results. The model describes many dynamical features of planetary resonances very well. Notwithstanding the good agreement found in all cases, a small deviation is noted in the location of the resonance centers for circumbinary systems. As a consequence of its application to the HD 31527 system, we found that the updated best-fit solution leads to a high-eccentricity stable libration between the middle and outer planets inside the 16/3 mean-motion resonance (MMR). This is the first planetary system whose long-term dynamics appears dominated by such a high-order commensurability. In the case of circumbinary planets, the overlap of N/1 mean-motion resonances coincides very well with the size of the global chaotic region close to the binary, as well as its dependence on the mutual inclination.

Key words. planets and satellites: dynamical evolution and stability – planets and satellites: individual: HD 31527 – planet-star interactions

1. Introduction

There is a fundamental difference between asteroidal and planetary resonances: in the first case, all the dynamics is restricted to the particle, while in the second case, the two massive bodies participate and experience the dynamical effects of the resonance. In the asteroidal (or restricted) case, the planet does not react to the resonance. The asteroid or particle can evolve in a resonance interior to the planet orbit or exterior to it, or it might be coorbital with the planet without perturbing the planet. In planetary resonances we do not distinguish between interior or exterior resonances because the two massive bodies together define the dynamics. The two massive bodies librate with the same period, but it is intuitive that the dynamical effects of the resonance on each planet are inversely proportional to their masses, and when one of them has a negligible mass, the restricted asteroidal case is reproduced.

As first-order resonances play a fundamental role in the structure of planetary systems, most attention has been reserved for this type of resonance. Analytical models describe the resonances between two planets for coplanar orbits very well (Beaugé & Michtchenko 2003; Batygin & Morbidelli 2013; Deck et al. 2013), and several studies have detailed the more complex problem of three-body planetary resonances (e.g., Quillen 2011; Gallardo et al. 2016; Delisle 2017). No model can currently handle arbitrary two-body resonances between massive bodies with arbitrary orbits, however.

To approach the problem, at least three points must be defined: the system of variables, the dimensions of the problem (planar or spatial), and the way in which the resonant disturbing function is computed. An astrocentric system can be adopted, but we cannot obtain a set of canonical variables from it, therefore Poincaré or Jacobi systems should be adopted when we look for a canonical framework. The planar case is simpler and is the rule for analytical treatments. The spatial case adds complexity, and it was only studied by numerical methods (Antoniadou & Voyatzis 2013). The resonant disturbing function is obtained analytically from general classic Laplacian developments (Batygin & Morbidelli 2013) or from developments for the planar case with an improved convergence domain (Beaugé & Michtchenko 2003).

The most important difference of our model with respect to other models is that we numerically calculate the resonant disturbing function without involving expansions. In Sect. 2 we explain the details of this calculation. In order to simplify the system of equations, we also assume that during the resonant motion, both orbits have fixed longitudes of the nodes and periastron. This is well justified when the timescale of the resonant motion is much shorter than the timescale of secular evolution. This approach reduces the involved angular variables to the mean longitudes and the resulting critical angle. With this hypothesis we can numerically calculate the resonant disturbing function and obtain the properties of the resonance: strength, width,
equilibrium points, and period of the small-amplitude oscillations. This approach is a generalization of the model developed by Gallardo (2020) for the restricted case, it reproduces the resonant mode very well. No information can be obtained about the long-term behavior of the system, however. Closely related to this problem is the study of the dynamics of a planet that is in resonance with a star in a binary system (Morais & Giuppone 2012). This case is also considered in our method.

In Sect. 2 we develop the semianalytical model in Poincaré elements, which is suitable for a system with a star and two planets. Then, in Sect. 3 we present the semianalytical model in Jacobi elements, which is more general and in particular suitable for a circumbinary planet. In Sect. 4 we apply the model to describe the resonance between a pair of planets, HD 31527 and HD 74156, while in Sect. 5 we apply the model in Jacobi coordinates to explain chaotic regions for planets around a binary. We conclude the paper in Sect. 6.

2. Resonance model in Poincaré coordinates

Astrocentric or barycentric coordinates are not canonical when applied to a planetary system. In this case the Poincaré coordinates are a valid alternative. According to Laskar (1991); Laskar & Robutel (1995) and Ferraz-Mello et al. (2006), the Hamiltonian for a system of a star with mass $m_0$ and two planets $m_1$ and $m_2$ in Poincaré coordinates is given by

$$H = \frac{\mu_1^2}{2L_1} - \frac{\mu_2^2}{2L_2} - R,$$

where

$$R = k^2 \frac{m_1 m_2}{\Delta} - \frac{m_1 m_2}{m_0} \cdot v_1 \cdot v_2,$$

with $k$ being the Gaussian constant, $\theta_1$ and $\theta_2$ the barycentric velocity vectors, $\mu_i = k^2 (m_0 + m_i)$, $\beta_i = m_0 m_i / (m_0 + m_i)$, $L_i = \beta_i \sqrt{\mu_i} a_i$, $\Delta$ is the mutual distance of the planets, and $a_i$ are the semimajor axes defined in the Poincaré system, which means astrocentric positions and barycentric velocities. The canonical variables are

$$\lambda_i, \quad \ell_i, \quad \sigma_i, \quad \Omega_i, \quad H_i - G_i = G_i (\cos i_i - 1).$$

Now we consider that the planets are close to the resonance $k_2/k_1$ so that $k_1 n_1 \approx k_2 n_2$ (where $n_i = \sqrt{\mu_i / a_i^3}$) and the combination $k_1 \lambda_1 - k_2 \lambda_2$ is a slow varying variable. We note that $k_2 \geq k_1$. We assume that during a libration period the variables $\sigma_i, \Omega_i$ are constants, then $(G_i - L_i)$ and $(H_i - G_i)$ become constant parameters, not variables. We therefore reduce to two degrees of freedom. This is a strong simplification that greatly reduces the complexity of the problem. However, it is an acceptable approximation because in general, the timescale on which the resonant movement occurs is much shorter than the circulation periods of nodes and pericenters. In order to isolate the slowly varying angle, we perform the canonical transformation

$$\theta = k_1 \lambda_1 - k_2 \lambda_2, \quad I_1 = L_1 / k_1, \quad I_2 = L_2 + k_2 L_1 / k_1.$$

The new Hamiltonian then is $H(\theta, I_1, I_2) = H_0(I_1, I_2) - R(\theta, I_2, I_1, I_2)$ with

$$H_0(I_1, I_2) = -\frac{\mu_1^2}{2(k_1 I_1)^2} - \frac{\mu_2^2}{2(k_2 I_2 - k_1 I_1)^2}.$$  \hspace{1cm} (5)\hspace{1cm}

Now we perform a numerical averaging that eliminates $I_2$, and obtain a new Hamiltonian $H(\theta, I_1, I_2)$ with $I_2 = \text{constant}$. In order to perform this averaging, we assume the simplification of Gallardo (2020), where the mean of $R(\theta, I_1, I_2, I_1)$ with respect to $\lambda_2$ is calculated numerically assuming fixed values of $(\theta, I_1, I_2)$. This assumption implies that the orbital elements are fixed during the computation of the numerical averaging that spans over $k_1$ orbital periods of the exterior planet, or equivalently, $k_2$ orbital periods of the interior planet.

The numerical calculation is done as follows. We fix $\theta = 0^\circ$, and varying $\lambda_2$ from 0 to $k_1 2\pi$, we calculate the corresponding $\lambda_1 = (\theta + k_2 \lambda_2) / k_1$. Given both mean longitudes, we calculate the rectangular coordinates and velocities for the two bodies, and using Eq. (2), we calculate $R$. We repeat the calculation of $R$ for 10000-20000 equispaced values of $\lambda_2$ between in the interval $(0, k_1 2\pi)$. Computing the mean of the calculated $R$, we obtain $R(0^\circ)$, which is equivalent to a numerical integration using the rectangle rule. Now we repeat the procedure for $\theta = 1^\circ$ and so on with a series of values $\theta_i$ to obtain a numerical representation $R(\theta)$ with $0^\circ \leq \theta \leq 360^\circ$. The last simplification that we adopt is that $R(\theta, I_1)$ is evaluated at the exact resonance given by $I_1 = I_0$. Then

$$H(\theta, I_1; I_2) \approx H_0(I_1; I_2) - R(\theta, I_1 = I_0; I_2).$$  \hspace{1cm} (6)\hspace{1cm}

Taking into account that $I_1$ in $R$ is evaluated at the exact resonance $(I_1 = I_0)$ and recalling that $I_2$ is constant, we have $H_0$ depending only on the variable $I_1$ and $R$ depending only on the variable $I_2$, remains as a parameter. It is possible to calculate the level curves of this Hamiltonian in the space $(\theta, I_1)$ where the separatrices and equilibrium points can be visualized. The equilibrium points are defined by $dI_1 / dt = d\theta / dt = 0$, which implies $dH_0 / dI_1 = 0, dR / d\theta = 0$. \hspace{1cm} (7)\hspace{1cm}

The first equation leads to the trivial result $n_1 k_1 = n_2 k_2$ (exact resonance), and the second can be computed numerically to yield the values of the equilibrium points $\theta_i$. It is standard to refer to the equilibrium points in terms of the critical angle,

$$\sigma = \theta + (k_2 - k_1) \sigma_i,$$

which has a meaning for near coplanar orbits, but in the spatial case, several combinations of $\Omega_i$ with $\sigma_i$ can be present in $\sigma$. Moreover, in a high-eccentricity or inclination regime, several critical angles could be relevant for a specific resonance. Our model imposes fixed $\Omega_i$ and $\sigma_i$, therefore our results describing the dynamics of the resonance are independent of these angles. We drop here the subindex of $I$ for simplicity, but we recall that we refer to the variable $I_1$. Because $(\theta_i, I_0)$ is an equilibrium point in canonical variables, if we consider some small displacement $(\Delta \theta = s, \Delta I = S)$, a first-order expansion of the Hamiltonian around the equilibrium point can be written as

$$H(\theta, I) = H(\theta_0, I_0) + H_{\theta \theta} + H_{II},$$  \hspace{1cm} (9)\hspace{1cm}

where the subscripts in $H$ mean partial derivatives evaluated at the equilibrium point. Using the canonical equations, we can obtain

$$\frac{dS}{dt} = -H_0 = -H_{\theta} s - H_{I} S$$  \hspace{1cm} (10)\hspace{1cm}
of the type

\[ H = H_I s + H_{II} s, \]  

(11)

with

\[ H_{II} = -\frac{3k_1^2}{\beta \alpha_1^2} - \frac{3k_2^2}{\beta_2 \alpha_2^2}, \]  

(13)

and \( H_{II} = -R_{II} \), which can be calculated numerically, and both \( H_{II}, H_{II} \) are evaluated at the stable equilibrium point.

To obtain the width of the resonance in terms of \( \Delta I \), we follow the same reasoning as in Gallardo (2020). Figure 1 shows that the resonance half-width is equal to the difference between \( I_0 \) and \( I_{sep} \), where \( I_{sep} \) is defined by the separatrix such that

\[ H(\theta_s, I_{sep}) = H(\theta_a, I_0), \]  

(14)

where \( \theta_s \) and \( \theta_a \) are the stable and unstable equilibrium points. When we assume that the domain of the resonance is symmetric with respect to the nominal position, \( I_0 \), the total width is twice \( \Delta I \). In some cases, asymmetries with respect to \( I = I_0 \) are present and a precise examination of the Hamiltonian level curves is required to obtain the exact width. In general, however, because \( \Delta H = H(\theta_s, I_{sep}) - H(\theta_a, I_0) \), we can approximate

\[ \Delta H = \frac{\partial H}{\partial I} \Delta I + \frac{\partial^2 H}{\partial I^2} (\Delta I)^2 + \ldots \]  

(15)

Evaluating the derivatives at the stable equilibrium point (\( \theta_s, I_0 \)) and using Eq. (14), we have

\[ \Delta H = H(\theta_s, I_0) - H(\theta_a, I_0) = \frac{\partial^2 H}{\partial I^2} (\Delta I)^2. \]  

(16)

Using Eq. (6), the left-hand side is

\[ H(I_0) - R(\theta_a) - (H(I_0) - R(\theta_a)) = -(R(\theta_a) - R(\theta_a)) = -\Delta R, \]  

(17)

while \( \frac{\partial H}{\partial I} = H_{II} \) is given by Eq. (13), then

\[ -\Delta R = H_{II} (\Delta I)^2. \]  

(18)

We recall that \( \Delta I \) is in fact \( \Delta I_1 \). Strength is commonly defined as \( SR = (R) - R_{min} \) (Gallardo 2006), which can be associated with the coefficient multiplying \( \cos(\sigma) \) when \( R \) is a well-behaved cosine-like function, but in general, \( R \) could be very different from a simple cosine. In these situations, \( SR \) therefore is a reliable estimate of the sensitivity of \( R \) with \( \sigma \). The strength is a global property of the resonance, as are the libration center and period. It is an indicator of the dynamical relevance of the resonance, and it generates different \( \Delta \) on each planet.

The width in terms of \( \Delta a_1 \) is obtained from the definition of \( I_1 \) in Eq. (4) and \( \Delta a_2 \) is obtained from \( I_2 = \text{constant} \) and is evaluated at the center of the resonance. We obtain the half-widths in au:

\[ \Delta a_1 = \sqrt{a_1} \frac{m_0 + m_1}{m_0 m_1} k_1 \sqrt{2} \sqrt{-\frac{\Delta R}{k^2 H_{II}}}, \]  

(19)

\[ \Delta a_2 = \sqrt{a_2} \frac{m_0 + m_2}{m_0 m_2} k_2 \sqrt{2} \sqrt{-\frac{\Delta R}{k^2 H_{II}}}. \]  

(20)

The higher the planetary mass \( m_i \), the narrower the relative width \( \Delta a_i/a_i \). The resonance is the same dynamical mechanism that affects the two planets, but stronger effects are present in the planet with the lower mass.

It is important to stress that \( \Delta a_i \) are given in terms of Poincaré elements, but for planetary systems, the computed widths \( \Delta a_i \) in Poincaré elements are very similar to the widths in astrocentric elements. Moreover, the widths obtained by the formulae above are correct as long as \( \Delta a_i \ll a_i \). If large widths are obtained using these formulae, it is convenient to recalculate the resonance limits (\( I_{\text{max}}, I_{\text{min}} \)) by a numerical procedure that considers the separatrix obtained from the Hamiltonian given by Eq. (6). In this circumstance, it is possible to observe asymmetries between the two limits of the resonance around the exact (nominal) position of the resonance.

3. Resonance model in Jacobi coordinates

It is also possible to formulate the method in Jacobi coordinates, which are more general than Poincaré coordinates and are particularly well suited for the study of planets evolving in binary stellar systems. Following Lee & Peale (2003), the Hamiltonian in Jacobi coordinates is

\[ H = -k^4 \frac{m_0 m_1}{2(m_0 + m_1)L_i^2} + k^4 \frac{(m_0 + m_1)^3 m_2^3}{2(m_0 + m_1 + m_2)L_i^2} - R, \]  

(21)

where

\[ R = k^2 m_0 m_2 \left( \frac{1}{\Delta_{a_2}} - \frac{1}{r_2} \right) + k^2 m_1 m_2 \left( \frac{1}{\Delta_{a_1}} - \frac{1}{r_2} \right), \]  

(22)

where \( a_i \) are the osculating semimajor axes in the Jacobi system, \( \Delta_{a_2} \) are the mutual distances between bodies \( m_2 \) and \( m_1 \) and \( r_2 \) are the distance of the mass \( m_2 \) to the barycenter of the subsystem \( m_0 \) and \( m_1 \). The canonical variables are defined in analogy to the Poincaré system, but now we have

\[ L_i = \frac{m_0 m_1}{m_0 + m_1} \sqrt{k^2 (m_0 + m_1) a_i}. \]  

(23)
the half-widths of the resonances for the general case. For the
0:1 resonance with
a
3:2 resonance with
composed of a star with
mass
⊙
1
96
0.001
M⊙, mutual inclination of 20°,
0
= 0.5, and
a
2 = 0.9 located in
3:2 resonance with
a
1 = 1 au. The stable equilibrium points are shown with an O and the unstable point is shown with an X. We apply Eq. (28)
and predict a total width of 0.048 au in
a
1.

\[ L_2 = \frac{(m_0 + m_1)m_2}{m_0 + m_1 + m_2} \sqrt{k^2(m_0 + m_1 + m_2)a_2}. \]  

(24)

We perform the same canonical transformation as in Eq. (4),
and then the new Hamiltonian is
\[ H_0(I_1, I_2) = H_0(I_1, I_2) - R(\theta_1, \theta_2, I_2), \]
where

\[ H_0(I_1, I_2) = -k^2 \frac{m_0^2m_1^2}{2(m_0 + m_1)k^2I_1^2} - k^2 \frac{(m_0 + m_1)^3m_2^2}{2(m_0 + m_1 + m_2)(I_2 - k_2I_1)^2}. \]  

(25)

We perform a mean on
a
λ
3
and obtain a Hamiltonian
H(\theta, I_1, I_2)
 analogous to Eq. (6). We can calculate level curves for this
Hamiltonian, and the equilibrium points are determined as well
as the separatrix.

The equilibrium points verify the same conditions given
by Eq. (7). The first condition gives the location of the
exact resonance
k_1 = k_2 = m_2, where the mean motions are now
n_1 = \sqrt{k^2(m_0 + m_1)/a_1^2} and
n_2 = \sqrt{k^2(m_0 + m_1 + m_2)/a_2^2}. The second condition gives the values of the libration centers \theta_0, and the stable libration periods are given by Eq. (12), where in this case

\[ H_{11} = - \frac{3k_1^2(m_0 + m_1)}{m_0m_1a_1^2} - \frac{3k_2^2(m_0 + m_1 + m_2)}{(m_0 + m_1)m_2a_2^2}, \]  

(26)

and
H_{90} = -R_{90}, which is calculated numerically. H_{11} and H_{90}
are evaluated at the stable equilibrium point. Following an
analogous reasoning as the previous section, we obtain

\[ \Delta a_1 = \frac{\sqrt{m_0 + m_1}m_2}{m_0m_1} k_1 2 \sqrt{\frac{-\Delta R}{k^2H_{11}}}, \]  

(27)

\[ \Delta a_2 = \frac{\sqrt{m_0 + m_1 + m_2}m_2}{(m_0 + m_1)m_2} k_2 2 \sqrt{\frac{-\Delta R}{k^2H_{11}}}. \]  

(28)

We show an example in Fig. 2. The expressions above give us
the half-widths of the resonances for the general case. For the
particular case
m_2/m_1 \sim 0, which corresponds to a planet around
a binary system, we obtain
H_{11} \sim -\frac{m_2^3}{m_1m_2^2}, \Delta a_1 \sim 0 \text{ and }
\Delta a_2 \approx \frac{\sqrt{8/3}}{m_2} \sqrt{\Delta R/m_2}.

(29)

This recovers the resonance width for the restricted case
(Gallardo 2020) in Jacobi coordinates because in the restricted case, m_2 is not present in R. When a planet orbits close to a
stellar binary system, the widths estimated by this method may
exceed the planet semimajor axis. In this case a better result
can be obtained by numerically calculating the borders of
the resonance identifying the separatrix using the expression of
the Hamiltonian given by Eq. (6). No method can compute the
resonance widths for the cases in which the planet is located so close
to the binary that close encounters occur that turn its dynamic
chaotic.

To avoid erroneous results generated by unrealistic \Delta R due
to close encounters, good practice is to monitor the mutual
distance when the integral \( \tilde{R}(\theta) \) is computed and to discard
the calculations that are obtained when close encounters occur.
A reasonable tolerance criterion could be around three mutual Hill
radii, see, for example, Gallardo (2020) and the discussion in
the next section.

We recall that our model assumes fixed pericenters and
nodes. As they evolve over time, the resonance widths change
over time. For a given set of pericenters and nodes, the widths are
maximum, and for others they are minimum. This leads to
the idea of resonance fragility (Gallardo 2020), a parameter
that measures the variability of the widths.

Results obtained with this model are quite good in compar-
ison with the numerical integration of the full equations of
motion. They improve the results obtained with analytical
methods based in truncated expansions. For the system HD 45364 that
was studied by Batygin & Morbidelli (2013), for example, the
dynamics is qualitatively well described by an analytical method,
but it cannot reproduce the libration frequency correctly. Instead,
our model correctly predicts the libration period and provides
a good complement to the analytical theory. We show some
applications of our model in the next sections.

4. Application to planetary systems

The resonant model described in this paper has several differ-
te applications. While it may be employed to study isolated
commensurabilities, giving information about the equilibrium
solutions, libration periods, and widths, it may also be used to
map a region in phase space. It may therefore aid in quantifying
the proximity of the system to resonant configurations and
identify regions that are susceptible to instabilities stemming
from resonance overlap. In certain ways, the model allows us to
obtain a qualitative description of the resonant structure of the phase
plane for a fraction of the computational cost of a dynamical
map (e.g., Megno, Mean Exponential Growth of Nearby Orbits,
Cincotta & Simó 2000).

4.1. System HD 31527

In our first application we analyze the orbital evolution and reso-
nant structure in the vicinity of system HD 31527. Discovered
by Mayor et al. (2011) and recently revisited by Coffinet et al.
(2019), this star hosts three Neptune-size planets orbiting a G0V
star with mass
m_0 = 0.96 M⊙. Even though not all the planetary

Fig. 2. Hamiltonian level curves for a fictitious planetary system
composed of a star with
m_0 = 1 M⊙ and two planets both with mass
0.001 M⊙, mutual inclination of 20°,
0 = 0.5, and
a_2 = 0.9 located in
3:2 resonance with
a_1 = 1 au. The stable equilibrium points are shown with an O and
the unstable point is shown with an X. We apply Eq. (28)
and predict a total width of 0.048 au in
a_1.
parameters are provided, it is possible to obtain a best fit of the radial velocity data based on the Data & Analysis Center for Exoplanets\(^1\) (DACE; see Buchschacher et al. 2015). Keplarian model initial conditions were computed using the formalism described in Delisle et al. (2016). We quadratically added a noise level of 0.75 m/s to the published radial velocity (RV) uncertainties to take stellar jitter and the typical instrumental noise floor seen in HARPS data into account. In addition to the signals fitted, we allowed an RV offset. Night binning reduces from 256 to 244 the in HARPS data, although the orbital fit remained virtually unchanged.

The results are shown in Table 1, where the error bars in the semimajor axes and masses were estimated from the joint posterior distribution of the model parameters using a Markov chain Monte Carlo (MCMC) analysis (Díaz et al. 2014). Computed with previous estimates, the greatest change in the orbital fit lies in the eccentricity of the outer planet. Mayor et al. (2011) found \(e \approx 0.38\), while Coffinet et al. (2019) reported \(e \approx 0.67\). Because neither of these works presented estimates for mean longitudes or longitudes of pericenter, no stability check was carried out. To the best of our knowledge, no dynamical studies have been performed, and the long-term orbital evolution of the planet has yet to be determined. Following usual conventions, for our dynamical analysis we denote the inner planet (HD 31527 b) \(m_1\), the middle body (HD 31527 c) \(m_2\), and the outermost planet (HD 31527 d) \(m_3\).

The top graph of Fig. 3 shows the resonant structure in the \((a_2, e_2)\) plane around the nominal position of the middle planet. The libration width of each commensurability was determined from Eqs. (27)–(28) assuming constant semimajor axes, eccentricities, and longitudes of pericenter. Mutual inclinations were taken equal to zero, and all orbital elements (with the exception of \(a_2\) and \(e_2\)) were assumed equal to their nominal values. Because the resonance widths are determined by the maximum range attained by the orbit-averaged disturbing function (i.e., \(\Delta \omega \propto \sqrt{R} = \sqrt{R_{\text{max}} - R_{\text{min}}}\)), the libration width increases for initial conditions closer to the collision curves, highlighted in gray. For eccentricities beyond this curve, the maximum value of \(R\) can attain very high values that may not be correctly estimated using finite differences. This leads to a resonant structure that appears jagged, irregular, and unreliable. Moreover, very high values of \(R\) are associated with close encounters between the planets and with instabilities. In the absence of a suitable protective mechanism (i.e., mean-motion resonance), all trajectories with \(e_2\) above the collision curves would lead to a disruption of the system.

\(^1\) https://dace.unige.ch

### Table 1. Minimum masses and orbits for system HD 31527.

| Parameter | HD 31527 b | HD 31527 c | HD 31527 d |
|-----------|------------|------------|------------|
| \(K\) [m s\(^{-1}\)] | 2.818 ± 0.092 | 2.661 ± 0.093 | 1.70 ± 0.20 |
| \(P\) [d] | 16.5545 ± 0.0024 | 51.265 ± 0.023 | 272.84 ± 0.78 |
| \(e\) | 0.137 ± 0.033 | 0.030 ± 0.034 | 0.596 ± 0.055 |
| \(\omega\) [deg] | 47 ± 14 | 277 ± 69 | 183.3 ± 6.5 |
| \(T_R\) [d] | 55 499.7379 | 55 466.7627 | 55 444.7471 |
| \(m \sin I\) [\(m_1\)] | 10.8279 ± 0.50 | 15.0399 ± 0.71 | 13.5044 ± 1.20 |
| \(a\) [au] | 0.1254 ± 0.002 | 0.2664 ± 0.005 | 0.8121 ± 0.014 |

**Notes.** The mass of the star is \(m_0 = 0.96 M_\odot\).
The resonant motion and long-term stability of the system, we performed an \( N \)-body simulation of the nominal solution for \( \sim 3 \times 10^6 \) yr. The calculation of the maximum Lyapunov exponent (LCE) is shown in the lower right-hand frame of Fig. 5, yielding a characteristic time of about \( t_{\text{LCE}} \sim 10^5 \) yr. This indicates that the best-fit system is only weakly chaotic, and the departure from a purely regular trajectory should only be noticeable well before the resonances increase monotonically for higher eccentricities, leading to an overlap of adjacent MMRs for initial conditions close to the collision curve. Because the structures beyond this point may not be reliable, they are not plotted.

In a second calculation, the libration widths were calculated limiting the value of \( R_{\text{max}} \) to configurations where the distance between the planets was larger than a lower bound \( \Delta_{\text{min}} \). With this approach we aimed to reduce the reach of the commensurability to include only those initial conditions that avoid close encounters and are therefore indicative of stable motion.

Without detailed stability criteria, the necessary value of \( \Delta_{\text{min}} \) can only be estimated qualitatively. For the restricted three-body problem, Gallardo (2020) showed that good agreement with \( N \)-body simulations was found using \( \Delta_{\text{min}} \sim 3R_{\text{Hill}} \), where \( R_{\text{Hill}} \) is the Hill radius of the perturber. Following the same idea, in the planetary case involving bodies \( m_i \) and \( m_j \), we adopted \( \Delta_{\text{min}} = 2\sqrt[3]{R_{\text{Hill}}} \), where

\[
R_{\text{Hill}}^{(i,j)} = \frac{a_i + a_j}{2} \left[ \frac{m_i + m_j}{3m_0} \right]^{1/3}
\]

is the so-called mutual Hill radius of the planets (Marzari & Weidenschilling 2002). This expression for \( \Delta_{\text{min}} \) is equal to the minimum distance between planets in circular orbits leading to Hill stability, as deduced originally by Marchal & Saari (1975) and Marchal & Bozis (1982), and later made commonly known by Gladman (1993). Although this choice for \( \Delta_{\text{min}} \) does not constitute a credible stability limit for multiplanet systems in eccentric orbits, we found it a simple expression leading to reasonable agreement with \( N \)-body simulations. A more rigorous analysis is left for future studies.

The libration widths incorporating \( \Delta_{\text{min}} \) as a minimum distance between the planets are shown with continuous white lines in Fig. 4. The effect of the cutoff starts being noticeable well below the collision curve and reduces the effective size of the stable resonance domain. For the 5/1, 16/3, and 11/2 MMRs the analytical results agree very well with the dynamical map, even though the predicted regions are overestimated. The modified model also predicts finite libration widths for higher-order and weaker commensurabilities; but no corresponding structure is observed in the \( N \)-body map.

The nominal position of the outer planet is shown with a large filled yellow circle. The error bars are very evident in both the semimajor axis and eccentricity plane of the outer body. Only MMRs with \( m_2 \) are plotted. Even though all commensurabilities are of a high order, the high eccentricity of the planet increases their reach, generating a significant widening of the libration region for values of \( e_1 \) close to the nominal value. The best-fit solution lies close to the collision curve, and the top of the error bar almost reaches this limit. As in the previous frame, the nominal semimajor axis almost coincides with that of the 16/3 MMR, whose width in this plane is substantially larger than previously seen in \((a_3, e_3)\). Some effect on the dynamical evolution of the system might be possible after all.

Figure 4 shows a Megno (Cincotta & Simé 2000) dynamical map, again in the \((a_3, e_3)\) plane, but now focusing on a smaller region around the outer planet. A total of \( 300 \times 300 \) initial conditions were integrated for \( 10^4 \) yr. All other orbital elements, including the initial mean anomalies, were set to those given in Table 1. Initial conditions leading to escapes or collisions during the simulation are shown in red. Regular motion (at least within the integration span) is shown with dark colors, and increasingly chaotic trajectories are identified with lighter tones. Only the central region of the stronger MMRs proves to be a protective mechanism that is sufficiently effective to avoid disruptive perturbations and close encounters. These safe zones appear as green stalagmites above the collision curve, shown in gray.

The separatrices of several mean-motion resonances are shown with white lines, the most important of which are identified with rounded labels. To estimate the libration widths, we employed two different approximations. The dashed lines show results obtained without specifying any upper bounds in \( R_{\text{max}} \). They yield results that are analogous to those in Fig. 3. As before, the resonant widths increase monotonically for higher eccentricities, leading to an overlap of adjacent MMRs for initial conditions close to the collision curve. Because the structures beyond this point may not be reliable, they are not plotted.

Fig. 4. Megno map for the HD 31527 system, as obtained from the numerical integration of a grid of \( 300 \times 300 \) initial conditions in the \((a_3, e_3)\) plane. All other orbital elements were set to the nominal values. Integration time was set to \( 10^4 \) yr, and escapes during this time span are highlighted in red. The color code shows values of \( (\langle Y \rangle - 2) \). Dark blue corresponds to more regular orbits, and lighter tones of green indicate increasingly chaotic motion. The location of the outer planet is shown with a filled yellow circle, and the collision curve is plotted in gray. The libration widths of the most relevant MMRs determined analytically are plotted as white lines. See text for details.
aid the visualization of the resonant and secular periods, only the first 10^5 yr of the integration are plotted. The solution appears to be completely quasi-periodic, and we found no trace of a secular or long-term increase in the amplitudes. Finally, the bottom left-hand graph shows the behavior of the secular angle \( \Delta \varpi(3,2) = \varpi_3 - \varpi_2 \) in green, and the red curve corresponds to the resonant angle \( \sigma_2 = 3\lambda_2 - 16\lambda_3 + 13\varpi_3 \).

From these results, it appears that the middle and outer planets of the HD 31527 system lie inside the 16/3 MMR, displaying a moderate-amplitude libration around \( \sigma_2 = \pi \). To the best of our knowledge, this is the first time that an exoplanetary system is found in such a high-order and high-degree mean-motion commensurability, and whose resonant motion acts as an effective protection mechanism. The chaoticity of the surrounding regions is so strong that it is difficult to imagine a formation scenario for this system involving planetary migration, even if this process were responsible for the excitation of the eccentricity. Two stronger MMRs (5/1 and 11/2) are found close by and to either side of the present-day position; it is hard to imagine how these could have been avoided while capturing the system in the (weaker) 16/3 resonance.

The libration period may also be estimated with the semi-analytical model and yields a value of \( T_{\text{lib}} \approx 22 \text{ yr} \), while a Fourier decomposition of the N-body simulation yields a value close to \( \sim 19 \text{ yr} \). We believe the agreement is very satisfactory, particularly considering that the analytical estimate corresponds to zero-amplitude librations, while the numerical solution has a considerable oscillation around the fixed point. This calculation additionally shows that the model may be used to obtain qualitative information on the resonant dynamics, regardless of the eccentricity or mutual inclination of the bodies.

### 4.2. System HD 74156

In the early 2000s it was suggested that most planetary systems should be dynamically packed (e.g., Barnes & Quinn 2001; Barnes & Raymond 2004), with little room for additional bodies between adjacent planets (e.g., Giuppone et al. 2013). Based on this idea, several works analyzed the phase space of known systems searching for stable regions that may host undetected planets. One of these systems was HD 74156.

Discovered by Naef et al. (2004), the system hosts two currently confirmed planets that orbit a \( m_0 = 1.24 M_\odot \) star. One of the bodies lies close to the star with an orbital period of \( P_b \approx 60 \) days, and the second body is found significantly farther out (\( P_c \approx 5.5 \) yr). After a series of N-body simulations, Raymond & Barnes (2005) found that a Saturn-type planet could well exist between the two known bodies and that the resulting system would be stable for timescales on the order of the age of the star.
Table 2. Masses and orbital parameters of the three-planet HD 74156 system proposed by Barnes et al. (2008).

| Parameter     | HD 74156 b | HD 74156 d | HD 74156 c |
|---------------|------------|------------|------------|
| $m$ [M$_\odot$] | 1.847      | 0.412      | 7.995      |
| $a$ [au]      | 0.292      | 1.023      | 3.848      |
| $e$           | 0.629      | 0.227      | 0.426      |
| $\omega$ [deg] | 176.45     | 191.81     | 262.17     |
| $M$ [deg]     | 211.64     | 67.51      | 340.20     |

| Planet #     | 1          | 2          | 3          |

Notes. The mass of the star is $m_0 = 1.24 M_\odot$.

As new observations became available, Barnes et al. (2008) extended the numerical study and further constrained the mass and orbit of the hypothetical third planet to the new radial velocity data. They found that the region between the two confirmed planets contains a rich dynamical structure defined by both isolated and interacting mean-motion resonances with the two bodies. The most promising system configuration is reproduced in Table 2.

Unfortunately, subsequent orbital fits including stellar jitter (Baluc 2009) did not confirm planet d, arguing that its signal could be due to annual systematic errors from the High Resolution Spectrograph (HRS) of the Hobby-Eberly Telescope. Wittenmyer et al. (2009), Meschiari et al. (2011), and Feng et al. (2015) later updated the system with additional observations and also failed to detect the proposed middle planet.

However, the resulting dynamical structure of the three-planet system proposed by Barnes et al. (2008) proved sufficiently interesting to be a good test for our model. Even if the third planet may not exist, the hypothetical extended system constitutes a challenging benchmark for the predictive power of the model, as well as for its capacity of identifying underlying features in the dynamical maps.

As before, we numbered the planets according to their distance to the central star, as identified in the lower row in Table 2. The confirmed planets are thus $m_1$ and $m_3$, and the proposed unconfirmed body is labeled $m_2$. Because we study a hypothetical system, we need not be tied to specific values for the angles. We therefore modified the orbital elements given in Table 2 such that

$$M_1 = M_2 = M_3 = 0; \quad \sigma_1 = \sigma_2 = \sigma_3 = \pi.$$  \hfill (32)

In other words, all planets are assumed to lie on their peri-centers, and the difference in the initial apsidal orientation between adjacent bodies is taken to be $\Delta \sigma_{1(2,1)} = \sigma_2 - \sigma_1 = 0$ and $\Delta \sigma_{2(3,2)} = \sigma_3 - \sigma_2 = \pi$. This choice helps preserve the symmetry in the libration regions in the dynamical map and in some cases maximizes their width.

For the representative plane, we chose the $(a_2, e_2)$ plane, more or less centered on the proposed position of the middle planet, and with limits similar to those used in Fig. 2 of Barnes et al. (2008). The left-hand plots of Fig. 6 show color plots of $\Delta \epsilon$ for a grid of $900 \times 900$ initial conditions varying the values of the semimajor axis $a_2$ and eccentricity $e_2$ while maintaining all other orbital elements fixed. Each initial condition was then integrated for $10^3$ yr, keeping track of any ejections or collisions during this time. The color code corresponds to the maximum change in eccentricity of $m_2$ during the integration (i.e., $\Delta \epsilon = \max(e_2) - \min(e_2)$), and has proven to be a powerful indicator of secular and resonant dynamics (e.g., Dvorak et al. 2004; Ramos et al. 2015).

While the vertical light-toned plumes are generated by the action of mean-motion resonances, the dark diagonal stripe in the upper left-hand plot represents the so-called Mode I secular mode (Michtchenko & Malhotra 2004) between the inner and middle planet and represents the locus of initial conditions leading to zero-amplitude oscillation of the secular angle $\Delta \sigma_{2(1)}$ around zero. In the plot the Mode I secular mode may be seen as a diagonal curve running from $(a_2, e_2) \approx (0.8, 0.2)$ down to $(a_2, e_2) \approx (0.95, 0.1)$. An analogous but less evident stripe may also be seen in the lower left-hand plot for higher values of the semimajor axis. Because the longitude of pericenter of the outer planet was chosen such that $\Delta \sigma_{3(2)} = \pi$, no indication of a Mode I with the outer planet is observed.

For the right-hand plots, we reduced the grid to a set of $300 \times 300$ initial conditions, but extended the integration time to $10^4$ yr. The color code now shows the Mengo values at the end of the simulation. As in the dynamical map constructed for the previous planetary system, dark blue corresponds to regular orbits, while lighter tones represent increasingly chaotic trajectories. In both sets of graphs, the semimajor axis domain was split into two sections: the region with $a_2 \leq 0.95$ au is shown in the top frames, and the bottom graphs show results for $a_2 \geq 0.95$ au. The position of the proposed planet is highlighted with filled white circles.

The maximum widths of the relevant MMRs as obtained with the model are shown with red lines (for commensurabilities with the inner planet) and blue lines (for resonances with the outer planet). They are identified in the top part of the graphs using text with the corresponding color. In most cases we note very good agreement between the predicted resonant structure and the dynamical maps. For example, the hour-glass figure centered at $a_2 \approx 0.86$ au is caused by the 5/1 MMR with the inner planet. For eccentricities $e_2 \lesssim 0.13$, the stable equilibrium solution occurs for $\sigma = l_1 - 5l_2 + 4e_2 = \pi$, resulting in chaotic trajectories inside the separatrix. Conversely, for higher eccentricities, the stable solution occurs for values of the resonant angle $\sigma = 0$ and is thus coincidental with the angles used to define the representative plane. The central region of the resonance is therefore more regular and admits librational solutions.

These hour-glass structures observed for the 5/1 MMR are not new and were detected in studies of circumbinary planets by Zoppetti et al. (2018). The center of the X shape appears to coincide with the forced eccentricity of the planet (Zoppetti et al. 2019), a correlation that also appears in the plots shown here.

Another interesting feature is the large chaotic region located at $a_2 \approx 0.96$ au and visible in the lower plot. The resonance analysis shows that it is generated by the interaction between two distinct commensurabilities, the 6/1 between planets $m_1$ and $m_2$, and the 8/1 resonance between $m_2$ and $m_3$. A similar structure occurs at $a_2 \approx 1.06$ au between two 7/1 MMRs, but the overlap does not occur for near-circular orbits this time, which leads to a wedge-shaped region of regular motion at low eccentricities.

For a more general analysis of the phase plane, we recall that an $n_1/n_2 = k_2/k_1$ MMR has two independent resonant angles that we may denote by

$$\sigma_1 = k_1 l_1 - k_2 l_2 + (k_2 - k_1) \sigma_j,$$
$$\sigma_2 = k_1 l_1 - k_2 l_2 + (k_2 - k_1) \sigma_j.$$  \hfill (33)

All other possible critical angles may be written as linear combinations of $\sigma_1$, $\sigma_2$ and the difference in longitudes of pericenter.
Our model analyzes the behavior of the resonances based on the study of the combination $k_1 \lambda_i - k_2 \lambda_j$, independent of the definition of $\sigma$. We can calculate the location of the equilibrium points corresponding to $\sigma_1$, for example, and we can analyze how the predictions of this model correlate with the dynamical structures found in the Megno map. Nevertheless, the model can also be used to separate the mean-motion resonances into three distinct groups, according to the number of stable solutions of $\sigma = \sigma_1$ and their equilibrium values. These are detailed below.

The first group corresponds to two-planet commensurabilities, between the middle planet $m_2$ and one of its neighbors, in which a the stable equilibrium solution corresponds to a value of the resonant angle $\sigma = 0$. Examples are dc-10/1, bd-11/2, bd-6/1, dc-17/2, and bd-13/2. Following the notation used in the figure, the letters indicate the planets involved in the MMR. These solutions are found to be stable for all the values of $e_2$ up to the edge of the collision curve. In some cases, these solutions are stable only for a certain range of eccentricities. These include bd-5/1 (for $e_2 \gtrsim 0.13$), dc-8/1 ($e_2 \gtrsim 0.01$), and bd-7/1 ($e_2 \gtrsim 0.1$).

The second group is similar to the previous case, but these resonances are characterized by a single stable solution located at $\sigma = \pi$. An example is dc-9/1. Most of these solutions are found for a limited interval of the eccentricity $e_2$, however. These include bd-5/1 (for $e_2 \lesssim 0.13$), bd-7/1 ($e_2 \lesssim 0.1$), and dc-7/1 ($e_2 \gtrsim 0.05$). In these cases the solutions found for other eccentricities usually correspond to $\sigma = 0$ librations.

Finally, the third group corresponds to the case of asymmetric stable solutions that harbors fixed points with values of $\sigma$ different from 0 or $\pi$. Examples were found only in resonances with the outer planet and include dc-7/1 (for $e_2 \lesssim 0.05$) and dc-8/1 (for $e_2 \lesssim 0.01$). For higher eccentricities, these solutions reduce to symmetric librations, either with $\sigma = 0$ or $\sigma = \pi$.

In general, the behavior of resonant initial conditions in the dynamical map agrees very well with the behavior expected from the model. Thus, resonances with stable solutions in $\sigma = k_2 \pi$ are expected to show regular solution inside the libration domain for the chosen representative plane. This behavior is clearly observed in both the dc-10/1 and dc-9/1 commensurabilities and to lesser degree in the bd-5/1 resonance.

Conversely, resonances with stable solutions in $\sigma = \pi$ should exhibit more chaotic motion in the representative plane. This prediction is observed in the case of dc-7/1 and, for example, low-eccentricity initial conditions in the bd-5/1 resonance. It is important to recall, however, that there is no unique resonant angle, particularly for high-order MMRs, and the most dominant combination between the mean longitudes and longitudes of pericenter depends on the eccentricities and relative masses of the planets involved in the commensurability. Even so, the
information that the model yields still serves as an initial analysis of the resonance, and serves to identify characteristics or interesting features that deserve further study.

5. Application to circumbinary planets

The *Kepler* mission revolutionized our knowledge of the existence of circumbinary planets (CBPs), with the discovery of 13 transiting CBPs orbiting 11 Kepler eclipsing binaries. Recently, the first circumbinary planet was reported using the data from TESS mission (TOI 1338, Kostov et al. 2020). These planets bring new challenges for formation theories, and their formation mechanism is still debated because most of them lie near their instability boundary at about three to five binary separations. Recent studies have argued that the occurrence rate of giant Kepler-like CBPs is comparable to that of giant planets in single-star systems (see Martin et al. 2019, and references therein).

In this sample, all the transiting binaries that host planets have short periods (<30 days) (Schwarz et al. 2016) with eccentricities ranging from quasi-circular orbits (like Kepler-47, Orosz et al. 2012b) to highly eccentric (like Kepler-34, Welsh et al. 2012). Typical planets detected around binary stellar systems have a radius of about 10 Earth radii and orbital periods of about 160 days on almost circular coplanar orbits (i.e., $a_s \sim 0.35 \, \text{au}$, Martin (2018).

Transiting inclined CBPs are more difficult to detect because the selection bias. However, planets do form in circumbinary disks (CBDs), and highly noncoplanar CBDs such as 99 Herculis (Kennedy et al. 2012), IRS 43 (Brinch et al. 2016), GG Tau (Cazzoletti et al. 2017; Aly et al. 2018), HD 142527 (Avenhaus et al. 2017), and HD 98800 (Kennedy et al. 2019) were detected.

Previous analysis of the stability of inclined massless particles was reported by Doolin & Blundell (2011), who considered different binary eccentricities and mass ratios between its components. More recently, Cuello & Giuppone (2019) and Giuppone & Cuello (2019) studied the evolution of CBDs and the stability of inclined Jupiter-like planets around binaries. Recently, circumbinary retrograde coplanar stability was addressed by Hong & van Putten (2021) for circular and eccentric planets. Furthermore, there is some evidence that the N/1 MMRs are related to the planetary parking location and their evolution in the disk. With a simplified model, Zoppetti et al. (2018) suggested capture in 5/1 MMR and subsequent tidal evolution of the planet, whereas some effort with fine-tuning hydrodynamical simulations tried to place some constraints on the protoplanetary disks that allow migration of CB planets, but do not show capture at MMRs (see, e.g., Penzlin et al. 2020, and references therein).

In this section we study the N/1 and N/2 mean-motion resonances (identified as $k_2/k_1$ in the model). Our aim is to correlate unstable regions near the binary with the MMR overlap.

We used the model described in Sect. 3 to map a region in phase space, identifying regions susceptible to instabilities stemming from resonance overlap. Regular motion and chaos of a coplanar CBP in the plane ($\alpha, e$) are presented in Sect. 5.1 (with $\alpha = a_1/a_2$), and we study an inclined CBP in the plane $(\alpha, i)$ that is initially placed on a circular orbit in Sect. 5.2. We consider different types of binaries such those detected by *Kepler* and TESS. We do not expect effects on the binary orbit because the mass of the planet is much lower than that of the binary companion. To describe the orbit of the binary, we use orbital elements with subindex $B$, and orbital elements without subindex for the planet.

![Fig. 7. Megno map for a circumbinary system in the ($\alpha, e$) plane. The color code shows values of $Y' = \log_{10}(Y) - 2$. Dark blue corresponds to more regular orbits, and lighter tones of green indicate increasingly chaotic motion. The libration widths of the most relevant MMRs determined with the model are plotted as white lines, setting $\sigma = 180^\circ$. The top scale indicates the positions of N/1 resonances with respect to the mean motion of the binary. See text for details.](image-url)

Without loss of generality, we set initial conditions $\{\alpha_B = 0^\circ, \varpi_B = 0^\circ, \Omega_B = 0^\circ\}$. Our fiducial binary has $a_B = 0.1 \, \text{au}$, and the sum of the component masses is equal to 1. The angles of a given planet are measured from the direction of the binary pericenter, and Jacobi orbital elements of the circumbinary planet are given in the model. Our fiducial binary has $a_B = 0.1 \, \text{au}$, and the sum of the component masses is equal to 1. The angles of a given planet are measured from the direction of the binary pericenter, and Jacobi orbital elements of the circumbinary planet are given in the model. Our fiducial binary has $a_B = 0.1 \, \text{au}$, and the sum of the component masses is equal to 1. The angles of a given planet are measured from the direction of the binary pericenter, and Jacobi orbital elements of the circumbinary planet are given in the model.

5.1. Coplanar circumbinary planets

In this subsection we start with coplanar orbits, and we only show the location of the MMR in the plane $(\alpha, e)$ for a binary with a mass ratio $q = 0.2$ and eccentricity $e_B = 0.1$, defining $\alpha = a_1/a_2$. Figure 7 shows a Megno dynamical map for a planet, focusing on a region between $1.5 < \alpha < 4.5$. A total of 250 $\times$ 100 initial conditions were integrated for $10^5$ binary periods. All angular elements were set to zero. Initial conditions leading to escapes or collisions during the simulation are shown in brown.

The semianalytical model predicts that all the N/1 MMRs have one stable center of resonance around $0^\circ$, in the region plotted in Fig. 7. However, maybe because of the fast precession of $\sigma$ induced by the binary (see Farago & Laskar 2010; Li et al. 2014), closer resonances are unstable (3/1 and 4/1). In contrast to the planetary case, we identify that regions inside of the stronger N/1 MMRs are chaotic (mainly observed in MMR 3/1 and 4/1). Overlap of 3/1, 4/1, and 5/1 delimits the unstable regions in the plane $(\alpha, e)$, showing a perfect agreement of the model with numerical integrations. The V-shape of each resonance is evident and the resonant-width decreases as we increase $N$; it is almost zero for circular planets.

Additionally, we plot in Fig. 7 the position of some similar CB systems with the orbital parameters listed in Table 3. We indicate with $\alpha_c$ the minimum stable coplanar-prograde semimajor axis, using the criteria of Holman & Wiegert (1999). This empirical criterion, which is widely used, is based in a polynomial fit that considers quadratical exponents of $ab$, $e_B$, and $q$. According to Fig. 7, this limit roughly corresponds to the closest region to the binary where secular motion is allowed (at the correct position of MMR 4/1).
Table 3. Masses and orbital parameters of CB planets.

| System      | q    | α   | e_B  | m [m_{Jup}] | e   |
|-------------|------|-----|------|-------------|-----|
| Kepler-38   | 0.26 | 3.16| 0.10 | 0.38        | 0.03|
| Kepler-453  | 0.21 | 4.26| 0.05 | 0.05        | 0.04|
| TOI-1338    | 0.28 | 3.49| 0.16 | 0.10        | 0.09|

References.  
(1)Orosz et al. (2012a), (2)Welsh et al. (2015), (3)Kostov et al. (2020)

We verified that the dynamical maps do not show a difference in the closer regions when we chose initial conditions with \( \Delta \sigma = 0^\circ \) or with \( \Delta \sigma = 180^\circ \). We must note that in regions farther away from the binary, differences in structures appear inside the resonance, as shown in Fig. 3 of Zoppetti et al. (2018).

5.2. Inclined circumbinary planets

In the view of variety of CBDs around binaries and to test the model for arbitrary inclination, we studied the dynamics of inclined planets around binaries (P-type planets). We must note that for inclined particles, the nodal libration (the regions in which inclination and node librates) is determined by the separatrix \( F \), simplifying the expression of Farago & Laskar (2010),

\[
F = \sqrt{\frac{1 - e_B^2}{1 - 5 e_B^2 \cos^2 \Omega + 4 e_B^2}}.  
\]

(34)

Then, libration of the node \( \Omega \) and inclination \( i \) of the CB planet around \( 90^\circ \) is expected when

\[
\arcsin F < i < \pi - \arcsin F.  
\]

(35)

It is important also to remark that outside the separatrix region, \( \Omega \) circulates (prograde or retrograde direction). We recall that librating orbits and stationary angles change when brown dwarf companions are considered (Chen et al. 2019), that is, for masses that are not compatible with CB detected planets.

We explored the MMR location and widths around a wide variety of circumbinary inclined orbits for different binary mass ratios \( q = [0.2, 0.5, 1] \). However, we chose to center our attention on three particular configurations for a mass ratio \( q = 0.2 \). We used Eq. (28) of the semianalytical model to calculate the width of the resonances, although we can numerically search for the exact extension of each separatrix to estimate their interior and exterior limit. We found that 2/1 MMR alone exhibits appreciable differences between Eq. (28) and the exact extension deduced from the level curves of the Hamiltonian. Although our planet was initially placed on circular orbits, the mean value of the planet eccentricity in regular orbits is roughly \( e \sim 0.05 \) due to the excitation induced by the binary, and we used this value to calculate the width of the resonances.

The dynamics of CB inclined planets at these distances leads to circulation of \( \sigma \) because of the interaction with the binary. Moreover, outside the nodal resonances, the node \( \Omega \) circulates (Farago & Laskar 2010; Doolin & Blundell 2011; Li et al. 2014). Our model calculates the width of the resonance for a fixed value of \( \sigma \) and \( \Omega \), and we analyzed the dependence of widths for different values from \( 0^\circ \) to \( 180^\circ \). We show the results in Fig. 8, setting a binary with \( q = 0.2 \) and \( a_B = 0.1 \) au. At first sight, we observe common features in all the panels, where the width of the resonances decreases with increasing \( N \) and the prograde resonances are wider than retrograde resonances. The changes in the width of the separatrices due to different values of \( \sigma \) (left panels) are higher than when we change initial values of \( \Omega \) (right panels). Furthermore, for the cases when we set initial \( \Omega = 90^\circ \) the separatrices show a wobbling shape at high inclinations. Additionally, the effect of change the value of \( \Omega \) have also profound effects in separatrices of retrograde orbits, blurring the regions around 2/1, 3/1 and 4/1 MMR (see middle frames at Fig. 8).

In each left-hand panel we indicate with \( \alpha_e \) the minimum stable coplanar-prograde semimajor axis using the criterion of Holman & Wiegert (1999). According to Fig. 8, this limit roughly correspond to the closest region to the binary where secular motion is allowed (outside the resonance overlap region).

The difference between the top and middle frames of Fig. 8 is the eccentricity of the binary. For a low eccentricity of the binary (\( e_B = 0.1 \) at top frames), we observe the region of superposition in coplanar prograde orbits between 2/1 and 3/1 MMR when we set different \( \sigma \) values, while the other resonances almost remain constant. The change in \( \Omega \) in the top right frame does not affect the coplanar prograde or retrograde resonances. When we increase the eccentricity of the binary (\( e_B = 0.5 \), center frames), the separatrix at prograde coplanar resonances shows huge changes: they overlap up to the 6/1 MMR. Retrograde resonances are less affected, and the location of the 4/1 MMR is associated with the most stable region near the binary. Islands of stable motion are possible, where the gray lines do not cross at polar inclinations \( i \sim 90^\circ \).

Varying \( \Omega \) (center right-frame) does not affect the width of the low-inclined prograde resonances, but enhances the width of the highly inclined retrograde resonances. In the bottom left frame of Fig. 8 we plot the resonance widths for a fixed value of \( \Omega = 0^\circ \) and varying \( \sigma \), where the initial conditions are outside the libration region of \( \Omega \). The separatrix location encompasses wider regions in coplanar and highly inclined systems, while for retrograde coplanar orbits, the separatrices shrink and leave favorable conditions for regular orbits closer to the binary. The bottom right frame shows a separatrix extension for \( \sigma = 90^\circ \) and different fixed values of \( \Omega \), where inclined orbits are more affected by this variation.

Some works showed dynamical maps around binaries in the plane \((a, i)\) using ejection times (see, e.g., Doolin & Blundell 2011) or the Megno indicator (see, e.g., Cuello & Giuppone 2019; Giuppone & Cuello 2019) but did not convincingly explain the regular spatial motion. To compare with the model, we ran numerical integrations and show the results in Fig. 9 using the color scale \( Y = \log_{10}(\langle Y \rangle - 2) \), where \( \langle Y \rangle \) is the Megno indicator. We also overplot the \( N/1 \) resonances that we constructed for the Fig. 8. The planet has initial \( \sigma = 0^\circ \), \( e = 0 \), and the binary mass ratio is \( q = 0.2 \). Each frame is constructed from the numerical integration of a grid of \( 300 \times 300 \) initial conditions in the \((a, i)\) plane. All other initial orbital elements were set to the nominal values. Integration time was set to \( 3 \times 10^5 \) yr (\( \sim 10^4 \) orbital periods of the binary).

The top frame of Fig. 9 shows \( e_B = 0.1 \) (almost like Kepler-38, Kepler-453, or TOI-1338), where unstable regions coincide with the overlap of the \( N/1 \) resonances. While the prograde MMRs have considerable widths, the MMR widths shrink with increasing inclination. The reported stability island around polar orbits (see, e.g., Doolin & Blundell 2011; Cuello & Giuppone 2019; Giuppone & Cuello 2019) are easily explained with the resonance width of the MMR at highly inclined orbits (\( i \sim 90^\circ \)). Moreover, as the inclination increases, the resonance width decreased; retrograde resonances are more compact.
Fig. 8. Widths and location of $N/1$ resonances setting different values of $\sigma$ (left frames) and $\Omega$ (right frames) for a planet around a binary with a mass ratio $q = 0.2$ and $a_B = 0.1$ in the plane $(a, i)$. The top scale of each frame indicates the positions of $N/1$ resonances with respect to the mean motion of the binary. We calculate each resonance with the semianalytical model for 20 different values of $\sigma$ in the left-hand frames ($\Omega$ in the right-hand frames) from $0^\circ$ to $180^\circ$ and plot each one with gray lines. Black lines denote the value for $\sigma = 180^\circ$ (left frames) or $\Omega = 180^\circ$ (right frames). Horizontal red lines indicate the separatrix location for nodal libration, i.e., regions in which the dynamics only allow libration of $\Omega$ around $90^\circ$. See text for more detail. Bottom panels: do not present horizontal lines because the nodal resonance is absent, then $\Omega$ circulates independently of the inclination $i$.

When we increase the binary eccentricity $e_B = 0.5$ (center and bottom frames of Fig. 9, similar to Kepler-34), more chaotic regions near the binary at coplanar prograde and retrograde orbits appear as result of resonance overlap (see the center frames of Fig. 8). Regular orbits closer to the binary survive at $i \sim 90^\circ$. Finally, in the bottom frame of Fig. 9 we show the case were $e_B = 0.5$ but dynamical maps with initial $\Omega = 0^\circ$ (i.e., $\Omega$ does not fall inside the nodal libration region for any inclination and always circulates). Unstable regions of low-inclination prograde orbits appear that agree with the $N/1$ plotted resonances.

We recall that the circulation of $\sigma$ causes the overlap that is observed in the bottom frame of Fig. 8.

A shift in semimajor axis for the location of the nominal MMR is evident in the dynamical maps. This shift is higher when the mass ratio $q$ or the binary eccentricity $e_B$ are increased (only by about 1% in the case of a binary with $q = 0.2$ and $e_B = 0.1$). This shift is also present for mean values of the semimajor axis for the dynamical maps.

Finally, in Fig. 10 we compare the width of MMR $N/1$ and $N/2$ around a binary with eccentricity $e_B = 0.5$ in the plane $(a, i)$.
for different mass ratios \( q = 0.2 \) and \( q = 1 \). We set fixed values of \( \Omega = 90^\circ \) and \( \varpi = 0^\circ \). The mass ratio of the binary components considerably modifies the width of the MMRs in prograde orbits (see, e.g., 2/1, 5/2, 3/1, and 4/1). In prograde orbits, the 2/1 resonance is most important for determining instability regions because it overlaps regions of the 4/1 resonance (or the 5/1 resonance, depending on the mass ratio).

Retrograde highly inclined resonances hardly change their width with the mass ratio. For initial values of \( \Omega = 90^\circ \), the wobbling structure of the resonances changes with \( q \) and \( i \); although

Fig. 9. Megno maps of inclined Jupiter planets around a binary with \((e_b, \Omega) = (0.1, 90^\circ)\), \((e_b, \Omega) = (0.5, 90^\circ)\), and \((e_b, \Omega) = (0.5, 0^\circ)\) from top to bottom panels respectively. As before, top scale of each frame indicates the positions of N/1 MMR and horizontal red lines delimit the nodal separatrix using Eq. (35). White lines are the separatrices of each N/1 resonances using the semianalytical model, varying the values of \( \Omega \) and \( \varpi \) (equivalent to gray lines in Fig. 8). See text for details.

Fig. 10. Resonance N/1 and N/2 widths setting a different mass ratio between components \( q = [0.2, 1] \) in the plane \((a, i)\) calculated with the semianalytical model. For \( q = 0.2 \) we use green (N/1) and magenta (N/2), and for \( q = 1 \) we use orange (N/1) and blue (N/2) resonances. Horizontal lines indicate the separatrix location for nodal libration. See text for more detail.

6. Conclusions

We presented a semianalytical model for planetary mean-motion resonances around a single star or binary stars, yielding the location and stability of fixed points of the averaged system, the libration period of small-amplitude oscillations, and the widths of the libration region in the semimajor axis domain. It is a natural extension of the model developed by Gallardo (2020) for the restricted three-body problem.

Applications to the HD 31527 and HD 74156 planetary systems, as well as to fictitious and real circumbinary planets show very good agreement between the model predictions and dynamical features deduced from pure N-body simulations, even for very high eccentricities. These dynamical characteristics include the resonance widths, the libration periods, and the stability of the fixed points. Together, these features allow a more detailed interpretation of the dynamical maps and help estimate regions corresponding to long-term orbital stability.

The new orbital fit of the radial velocity data of HD 31527 we presented shows that the middle and outer planets are probably locked in a 16/3 mean-motion resonance and a high-eccentricity moderate-amplitude libration of the resonant angle \( 16 \lambda_3 - 3 \lambda_5 - 13 \varpi_2 \). This configuration appears to be stable for timescales of at least about \( 10^7 \) yr. Because no significant secular changes in the semimajor axes or eccentricities are observed within this time frame, it is possible that the orbital fit is even stable for timescales comparable to the age of the star.

Differences between predicted and observed widths, as well as the location of the resonance centers, can be attributed to long-term variations in the secular angles (i.e., longitudes of pericenter and ascending nodes) that are assumed constant in the model. These effects are almost negligible in MMR among
planetary bodies, but they become more visible in binary systems, especially for high mass-ratios between the stellar components. Notwithstanding this effect, the general features of the resonant dynamics are well reproduced by the model.

Using the semi-analytical model, we convincingly demonstrated that for P-type orbits (circumbinary planets), the dynamical maps in planes the regions closer to the binary. We compared our results with MMR overlap is directly associated with the instability limit in the semimajor axis domain. While this issue is rarely observable in planetary systems around single stars, it becomes more evident for polar inclinations. As we approach the binary, the proximity causes circulation of \( \Omega \) and \( \bar{\sigma} \) is on short timescales, thus the separatrix of each resonance moves and creates a wider effective separatrix.

When we analyzed the MMR experienced by an inclined planet around a binary, we found that prograde resonances are wider than retrograde resonances, and the N/1 resonance overlap is the main cause of instabilities near the binary. The N/2 resonances are considerably weaker than the N/1 resonance, but the former contribute to the overlap between consecutive N/1 resonances. We found that N/1 and N/2 resonances delimit the islands of stability for polar configurations and that the resonance shape in the plane \((a,i)\) strongly depends on the initial value of \( \Omega \), either 0° or 90° for polar inclinations. As we approach the binary, the proximity causes circulation of \( \Omega \) and \( \bar{\sigma} \) is on short timescales, thus the separatrix of each resonance moves and creates a wider effective separatrix.

The model assumes that the center of the libration domain coincides with the nominal position of the resonance. Although this approximation has been proved to be very reliable in most cases, some values of the system parameters lead to a noticeable offset that appears as a shift of the resonant structure in the semimajor axis domain. While this issue is rarely observable in planetary systems around single stars, it becomes more evident for circumbinary planets with high values of the mass ratio \( q \) or binary eccentricity \( e_B \). A more detailed study of the resonance offset, as well as its description, is left for a forthcoming work.

Codes for the computation of the resonances can be found online\(^2\).

Acknowledgements. We thank the anonymous reviewer who, through a rigorous examination of the manuscript, suggested substantial improvements to this final version. N-body computations were performed at Mulatona Cluster from CCAD-UNC, which is part of SNiCAD-MinCyT, Argentina. T.G. acknowledges support from PEDECIBA. C.G. acknowledges the training acquired in the 2020 Sagan Exoplanet Summer Virtual Workshop.

References

Aly, H., Lodato, G., & Cazzoletti, P. 2018, MNRAS, 480, 4738
Antoniadou, K. L., & Voyatzis, G. 2013, Celest. Mech. Dyn. Astron., 115, 161
Avenhaus, H., Quanz, S. P., Schmid, H. M., et al. 2017, AJ, 154, 33
Baluve, R. V. 2009, MNRAS, 393, 969
Barnes, R., & Raymond, S. N. 2004, ApJ, 617, 569
Barnes, R., Goździewski, K., & Raymond, S. N. 2008, ApJ, 680, L57
Batygin, K., & Morbidelli, A. 2013, A&A, 556, A28
Beaugé, C., &ichitchenko, T. A. 2003, MNRAS, 341, 760
Birnstiel, T., & Lodato, G. 2017, A&A, 599, A102
Cazzoletti, P., Ricci, L., Birnstiel, T., & Lodato, G. 2017, A&A, 599, A102
Chen, C., Franchini, A., Lubow, S. H., & Martin, R. G. 2018, MNRAS, 490, 5634

Cincotta, P. M., & Simó, C. 2000, A&AS, 147, 205
Coffinet, A., Lovis, C., Dumusque, X., & Pepe, F. 2019, A&A, 629, A27
Cuello, N., & Giuppone, C. A. 2019, A&A, 628, A119
Deck, K. M., Payne, M., & Holman, M. J. 2013, ApJ, 774, 129
Delisle, J. B., Ségransan, D., Buchschacher, N., & Alesina, F. 2016, A&A, 590, A134
Díaz, R. F., Almenara, J. M., Santerne, A., et al. 2014, MNRAS, 441, 983
Doolin, S., & Blundell, K. M. 2011, MNRAS, 418, 2656
Dvorak, R., Pilat-Lohinger, E., Schwarz, R., & Freistetter, F. 2004, A&A, 426, L37
Farago, F., & Laskar, J. 2010, MNRAS, 401, 1189
Feng, Y. K., Wright, J. T., Nelson, B., et al. 2015, ApJ, 800, 22
Ferraz-Mello, S., Michtchenko, T. A., & Beaugé, C. 2006, Chaotic Worlds: from Order to Disorder in Gravitational N-body Dynamical Systems, eds. B. A. Steves, A. J. Maciejewski, & M. Hendry (Dordrecht: Springer), 255
Gallardo, T., 2006, Icarus, 184, 29
Gallardo, T., 2020, Celest. Mech. Dyn. Astron., 132, 9
Gallardo, T., Costa, L., & Badano, L. 2016, Icarus, 274, 83
Giuppone, C. A., & Cuello, N. 2019, J. Phys. Conf. Ser., 1365, 012023
Giuppone, C. A., Mora, M. H. M., & Correia, A. C. M. 2013, MNRAS, 436, 3547
Gladman, B. 1993, Icarus, 106, 247
Gomez, J. M., & Wieget, P. A. 1999, AJ, 117, 621
Hong, C., & van Putten, M. H. P. M. 2021, New Astron., 84, 101516
Kennedy, G. M., Wyatt, M. C., Sibthorpe, B., et al. 2012, MNRAS, 421, 2264
Kennedy, G. M., Marín, L., Facchini, S., et al. 2019, Nat. Astron., 189
Kostov, V. B., Oros, J. A., Feinstein, A. D., et al. 2020, AJ, 159, 253
Laskar, J. 1991, Series B NATO Advanced Science Institutes (ASI), 271, 93
Laskar, J., & Robutel, P. 1995, Celest. Mech. Dyn. Astron., 62, 193
Lee, M. H., & Peale, S. J. 2003, ApJ, 592, 1201
Li, D., Zhou, J.-L., & Zhang, H. 2014, MNRAS, 437, 3832
Marchal, C., & Bozis, G. 1982, Celest. Mech., 26, 311
Marchal, C., & Saar, D. G. 1975, Celest. Mech., 12, 115
Martin, D. V. 2018, Populations of Planets in Multiple Star Systems (Berlin: Springer), 156
Martín, D. V., Triandaf, A. H. M. J., Udry, S., et al. 2019, A&A 624, A68
Marzari, F., & Weidenschilling, S. J. 2002, Icarus, 156, 570
Mayor, M., Marmier, M., Lovis, C., et al. 2011, ArXiv e-prints [arXiv:1109.2497]
Meschiarri, S., Laughlin, G., Vogt, S. S., et al. 2011, ApJ, 727, 117
Michtchenko, T. A., & Malhotra, R. 2004, Icarus, 168, 237
Mora, M. H. M., & Giuppone, C. A. 2012, MNRAS, 424, 52
Naef, D., Mayor, M., Beuzit, J. L., et al. 2004, A&A, 414, 351
Orosz, J. A., Welsh, W. F., Carter, J. A., et al. 2012a, ApJ, 758, 87
Orosz, J. A., Welsh, W. F., Carter, J. A., et al. 2012b, Science, 337, 1511
Penzlin, A. B. T., Kley, W., & Nelson, R. P. 2020, A&A, 645, A68
Quillen, A. C. 2011, MNRAS, 418, 1043
Ramos, X. S., Correa-Otto, J. A., & Beaugé, C. 2015, Celest. Mech. Dyn. Astron., 123, 453
Raymond, S. N., & Barnes, R. 2005, ApJ, 619, 549
Schwarz, R., Funk, B., Zechner, R., & Baßos, Á. 2016, MNRAS, 460, 3598
Welsh, W. F., Orosz, J. A., Carter, J. A., et al. 2012, Nature, 481, 475
Welsh, W. F., Orosz, J. A., Short, D. R., et al. 2015, ApJ, 809, 26
Wittenmyer, R. A., Endl, M., Cochran, W. D., Levison, H. F., & Henry, G. W. 2009, ApJS, 182, 97
Zoppetti, F. A., Beaugé, C., & Leiva, A. M. 2018, MNRAS, 477, 5301
Zoppetti, F. A., Beaugé, C., & Leiva, A. M. 2019, J. Phys. Conf. Ser., 1365, 012029

\(^2\) www.fisica.edu.uy/~gallardo/atlas/plares.html