On cycle-supermagic labelings of the disconnected graphs

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Abstract. A graph \(G(V,E)\) has an \(H\)-covering if every edge in \(E\) belongs to a subgraph of \(G\) isomorphic to \(H\). Suppose \(G\) admits an \(H\)-covering. An \(H\)-magic labeling is a total labeling \(\lambda\) from \(V(G) \cup E(G)\) onto the integers \(\{1,2,\cdots,|V(G) \cup E(G)|\}\) with the property that, for every subgraph \(A\) of \(G\) isomorphic to \(H\) there is a positive integer \(c\) such that \(\sum_{v \in V(A)} \lambda(v) + \sum_{e \in E(A)} \lambda(e) = c\).

A graph that admits such a labeling is called \(H\)-magic. In addition, if \(\{\lambda(v)\}_{v \in V} = \{1,2,\cdots,|V|\}\), then the graph is called \(H\)-supermagic. In this paper we formulate cycle-supermagic labelings for the disjoint union of isomorphic copies of different families of graphs. We also prove that disjoint union of non isomorphic copies of fans and ladders are cycle-supermagic.

Keywords : \(H\)-supermagic labeling, cycles, disjoint union of graphs.

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1 Introduction

Let \(G = (V,E)\) be a finite, simple, planar, connected and undirected graph, where \(V\) and \(E\) are its vertex-set and edge-set respectively. A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually positive or non-negative integers). Let \(H\) and \(G = (V,E)\) be finite simple graphs with the property that every edge of \(G\) belongs to at least one subgraph isomorphic to \(H\). A bijection \(\lambda : V \cup E \rightarrow \{1,\ldots,|V| + |E|\}\) is an \(H\)-magic labeling of \(G\) if there exist a positive integer \(c\) (called the magic constant), such that for any subgraph \(H'(V',E')\) of \(G\) isomorphic to \(H\), the sum \(\sum_{v \in V'} \lambda(v) + \sum_{e \in E'} \lambda(e)\) is equal to magic constant \(c\). This sum is also known as weight of \(H\). A graph is \(H\)-magic if it admits an
$H$-magic labeling. In addition, if $H$-magic labeling $\lambda$ has the property that $\{\lambda(v)\}_{v \in V} = \{1, \ldots, |V|\}$, then the graph is $H$-supermagic. In our terminology, super edge-magic total labeling is a $K_2$-supermagic labeling. The notion of $H$-magic graph was introduced by A. Gutierrez and A. Llado [11], as an extension of the magic valuation given by Rosa [8], which corresponds to the case $H \cong K_2$.

There are many results for $H$-(super)magic graphs. Llado and Moragas [9] studied cycle-magic labeling of wheels, prisms, books, wind mill graphs and subdivided wheels by using the technique of partitioning sets of integers. Ngurah et al. [13] constructed cycle-supermagic labelings for some families of graphs namely fans, ladders, and books etc. For any connected graph $G$, Maryati et al. [10], proved that disjoint union of $k$ isomorphic copies of $G$ is a $G$-supermagic graph if and only if $|V(G)| + |E(G)|$ is even or $k$ is odd. For a disconnected graph $G$, having at least 2 vertices in its every components, the result was proved again by Maryati et al. [12]. For a positive integer $k$, they proved that (i) If $kG$ is $G$-magic, then $|V(G)| + |E(G)|$ is even or $k$ is odd and (ii) If $|V(G)| + |E(G)|$ is even or $k$ is odd, then $kG$ is $G$-supermagic. For further detail, see the recent survey by Gallian [6].

In this paper, we study the problem that if a connected graph $G$ is cycle-supermagic, then either disjoint union of $m$ isomorphic copies of graph $G$ denoted by $mG$, is cycle-supermagic or not? We study cycle-supermagic labelings for the disjoint union of isomorphic copies of fans, ladders, triangular ladders, wheel graphs, book graph and generalized antiprism graph. Also, we study cycle-supermagic labelings of disjoint union of non-isomorphic copies of ladders and fans.

2 Main Results

Before giving our main results, let us consider the following lemma found in [8] that gives a necessary and sufficient condition for a graph to be super edge-magic.

**Lemma 1.** A graph $G$ with $v$ vertices and $e$ edges is super edge-magic if and only if there exists a bijective function $\lambda: V(G) \rightarrow \{1, 2, \ldots, v\}$ such that the set $S = \{\lambda(x) + \lambda(y) | xy \in E(G)\}$ consists of $e$ consecutive integers. In such a case, $\lambda$ extends to a super edge-magic labeling of $G$ with magic constant $c = v + e + s$, where $s = \min(S)$.

Moreover, while proving our some results, we will use following result to formulate the $(a, 2)$-edge-antimagic vertex labeling for a subgraph isomorphic to $mP_n$. 
Theorem 1. [4] The graph $mP_n$ has a $(m + 2,2)$-edge-antimagic vertex labeling, for every $m \geq 2$ and $n \geq 2$.

2.1 Cycle-supermagic labelings of the disjoint union of isomorphic graphs

In the following theorem, we consider $C_3$-supermagic labeling for disjoint union of isomorphic copies of fans. For $n \geq 3$, the fan $F_n \cong K_1 + P_n$ is a graph with $V(F_n) = \{c\} \cup \{v_i : 1 \leq i \leq n\}$, $E(F_n) = \{cv_i : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$.

Theorem 2. For any positive integer $m \geq 2$, and $n \geq 3$, the graph $G \cong mF_n$ is $C_3$-supermagic.

Proof. Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = m(n + 1)$, $e = m(2n - 1)$. We denote the vertex and edge sets of $G$ as follows: $V(G) = \{c_j : 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$, $E(G) = \{v_i^jv_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{cv_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$. For $1 \leq i \leq n - 1, 1 \leq j \leq m$, let $C_3^{(i,j)}$ be subcycle of graph $G$ with $V(C_3^{(i,j)}) = \{c_j, v_i^j, v_{i+1}^j\}$ and $E(C_3^{(i,j)}) = \{c_jv_i^j, v_i^jv_{i+1}^j, v_{i+1}^jc_j\}$.

We will define a total labeling $\lambda : V(G) \cup E(G) \rightarrow \{1,2,\ldots,3mn\}$ as follows:

Let $1 \leq i \leq n, 1 \leq j \leq m$.

Step 1. At this step, we will only label the vertices of graph $G$ as follows:

$\lambda(c_j) = m(n + 1) - j + 1$

For $i$-odd

$\lambda(v_i^j) = j + m\left(\frac{i - 1}{2}\right)$

For $i$-even

$\lambda(v_i^j) = \frac{2n + 2i + 3 + (-1)^{n+1}m + 4j}{4}$

After labeling the vertices of graph $G$, we can see that the set of all weights of subcycle $C_3^{(i,j)}$ under labeling $\lambda$ consists of consecutive integers $\frac{3mn}{2} + m + 2, \frac{3m(n+1)}{2} + m + 3, \ldots, \frac{5mn}{2} + 1$, for even $n$ and $\frac{3m(n+1)}{2} + 2, \frac{3m(n+1)}{2} + 3, \ldots, \frac{m(5n+1)}{2} + 1$, for odd $n$. 
Step 2. Now, we will label the edges \( c_jv_i^j \) as follow:

\[
\lambda(c_jv_i^j) = 3mn - m(i - 1) - j + 1
\]

After labeling the vertices and edges \( c_jv_i^j \) of graph \( G \), we can see that the set of all weights of subcycle \( C_{3}^{(i,j)} \) consists of consecutive integers \( \frac{m(26n+5+(-1)^{n+1})}{4} + 3, \frac{m(26n+5+(-1)^{n+1})}{4} + 4, ..., \frac{m(26n+5+(-1)^{n+1})}{4} + m(n - 1) + 2 \), for any \( n \).

Step 3. Now we are left with the labeling of edges \( v_i^jv_{i+1}^j : 1 \leq i \leq n - 1 \), with consecutive integers \( mn + m + 1, mn + m + 2, ..., 2mn \). Here, we can easily label these edges by using Lemma 1 to have \( C_3 \)-supermagic labeling of graph \( G \) as follows:

\[
\lambda(v_i^jv_{i+1}^j) = m(n + i) + j.
\]

After combining all three steps, we can see that for every subcycle \( C_3^{(i,j)} \) of \( G \), the sum \( \lambda(c_{ij}) + \lambda(v_i^j) + \lambda(v_{i+1}^j) + \lambda(c_jv_i^j) + \lambda(c_jv_{i+1}^j) + \lambda(v_i^jv_{i+1}^j) \) is \( \frac{mn}{2}[34n + 5 + (-1)^{n+1}] + 3 \). Hence \( mF_n \) is \( C_3 \)-supermagic. □

In the next two theorems, we prove that disjoint union of isomorphic copies of ladder graphs, namely ladder graphs and triangular ladders admit cycle-supermagic labelings.

Let \( L_n \cong P_n \times P_2 \), \( n \geq 2 \), be a ladder graph with \( V(L_n) = \{u_i, v_i : 1 \leq i \leq n\} \), \( E(L_n) = \{u_iu_{i+1} : 1 \leq i \leq n \} \cup \{u_iu_{i+1} : 1 \leq i \leq n - 1 \} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 1 \} \).

**Theorem 3.** For every \( m \geq 2 \), \( n \geq 2 \), the graph \( G \cong mL_n \) is \( C_4 \)-supermagic.

**Proof.**

Let \( n = |V(G)| = 2mn \), and \( e = |E(G)| = 3mn - 2m \). We denote the vertex and edge sets of \( G \) as follows:

\[
V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\};
\]

\[
E(G) = \{u_i^jv_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^jv_{i+1}^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{v_i^jv_{i+1}^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\}.
\]

Define a total labeling \( \lambda : V(G) \cup E(G) \to \{1, 2, ..., 5mn - 2m\} \) as follows:

Step 1. For \( 1 \leq i \leq n, 1 \leq j \leq m \)

\[
\lambda(u_i^j) = j + m(i - 1)
\]
\[ \lambda(v_i^j) = v - m(i - 1) - j + 1 \]

After labeling the vertices of graph \( G \), we can see that weight of every subcycle \( C_4^{(i,j)} \) under labeling \( \lambda \) is \( 2v + 2 \).

Step 2. Now, we will use Theorem 1 to label the edges \( u_i^j v_i^j \), by assuming that these edges are vertices of subgraph isomorphic to \( mP_v \) and labeling is as follows:

\[ \lambda(u_i^j v_i^j) = v + m(i - 1) + j \]

At this stage, set of all weights of \( C_4^{(i,j)} \) under labeling \( \lambda \) forms an arithmetic sequence with initial term \( 4v + m + 2 \) and the difference is \( 2 \).

Step 3. For \( 1 \leq i \leq n - 1, 1 \leq j \leq m \)

\[ \lambda(u_i^j v_{i+1}^j) = v + e - m(n - 1) - m(i - 1) - j + 1 \]

\[ \lambda(v_i^j v_{i+1}^j) = v + e - m(i - 1) - j + 1 \]

Let \( C_4^{(i,j)} : 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq m \), be the subcycle of graph \( G \) with \( V(C_4^{(i,j)}) = \{u_i^j, u_{i+1}^j, v_j^i, v_{i+1}^j\} \) and \( E(C_4^{(i,j)}) = \{u_i^j u_{i+1}^j, v_j^i v_{i+1}^j, u_i^j v_{i+1}^j, u_{i+1}^j v_i^j\} \).

After combining all three steps, we can verify that \( \sum C_4^{(i,j)} = \lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_j^i) + \lambda(v_{i+1}^j) + \lambda(u_i^j v_{i+1}^j) + \lambda(u_{i+1}^j v_i^j) = m(17n - 2) + 4 \). Hence \( mL_n \) is \( C_4 \)-supermagic.

Let \( H \cong TL_n \) be a triangular ladder graph with \( V(H) = \{u_i, v_i : 1 \leq i \leq n\} \), \( E(H) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i : 1 \leq i \leq n - 1\} \).

**Theorem 4.** The graph \( G \cong mTL_n \) admits \( C_3 \)-supermagic labelings for \( m \geq 2 \) and \( n \geq 3 \).

**Proof.**

Let \( v = |V(G)| \) and \( e = |E(G)| \). Then \( v = 2mn, e = m(4n - 3) \). We denote the vertex and edge sets of \( G \) as follows:

\[
V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\},
\]

\[
E(G) = \{u_i^j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{u_{i+1}^j v_i^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\}.
\]

Define a total labeling \( \lambda : V(G) \cup E(G) \to \{1, 2, \ldots, 6mn - 3m\} \) as follows:

For \( 1 \leq i \leq n, 1 \leq j \leq m \),

\[ \lambda(u_i^j) = j + 2m(i - 1) \]
Let $\lambda$ define a total labeling and positive integer $G$ the vertex and edge sets of $G$.

For $1 \leq i \leq n - 1, 1 \leq j \leq m$

$$\lambda(u_i^j u_{i+1}^j) = m(6n - 3) - 2m(i - 1) - j + 1$$

It is easy to verify that for every subcycle $C_3^{k,j} : 1 \leq i \leq 2n - 2$ and $1 \leq j \leq m$, $\lambda(u_i^j) + \lambda(u_{i+1}^j) + \lambda(v_i^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(v_i^j v_{i+1}^j) = 14mn - 3m + 3$. Hence $mTL_n$ is $C_3$-supermagic.

Next we consider $C_3$-supermagic labeling of disjoint union of isomorphic copies of odd wheel graphs. Let $W_n = K_1 + C_n$ be a wheel graph with $V(W_n) = \{v_i : 1 \leq i \leq n\}$, and $E(W_n) = \{cv_i : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n\}$.

**Theorem 5.** The graph $G \cong mW_n$ is $C_3$-supermagic, for any odd $n \geq 3$ and positive integer $m \geq 2$.

**Proof.**

Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = m(n + 1)$, $e = 2mn$. We denote the vertex and edge sets of $G$ as follows:

$V(G) = \{c_j : 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$, and $E(G) = \{v_i^j v_{i+1}^j : 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{c_j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Define a total labeling $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(3n + 1)\}$ as follows:

For $1 \leq i \leq n, 1 \leq j \leq m$

$$\lambda(c_j) = j$$

$$\lambda(c_j v_i^j) = m(2n - i + 1) - j + 1$$

$$\lambda(c_j v_n) = m(2n + 1) - j + 1$$

$$\lambda(v_n v_1) = m(2n + 3) - j + 1$$

$$\lambda(v_i^j v_{i+1}^j) = m(2n + i + 3) - j + 1$$

$$\lambda(v_i^j) = \begin{cases} 
    m\left(\frac{i+1}{2}\right) + j, & \text{if } i \text{ odd} \\
    m\left(\frac{i+2}{2}\right) + j, & \text{if } i \text{ even}
\end{cases}$$
It is easy to verify that for every subcycle $C^{(i,j)}_3 : 1 \leq i \leq n, 1 \leq j \leq m$ of $mW_n$, the weight of every subcycle $C^{(i,j)}_3$ of $G$ is $\frac{m}{2}(13n + 11) + 3$. Hence $mW_n$ is $C_3$-supermagic.

\[\square\]

**Theorem 6.** For even $n \geq 4$, and $m \geq 2$, the graph $G \cong mW_n$ is $C_3$-supermagic.

**Proof.**

Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = m(n+1)$, $e = 2mn$. We denote the vertex and edge sets of $G$ as follows:

$V(G) = \{v_j^i \colon 1 \leq j \leq m\} \cup \{v_j^i \colon 1 \leq i \leq n, 1 \leq j \leq m\}$, and

$E(G) = \{v_j^iv_{j+1}^i \colon 1 \leq i \leq n - 1, 1 \leq j \leq m\} \cup \{v_0^jv_j^i \colon 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_n^jv_j^i \colon 1 \leq j \leq m\}$.

Define a total labeling $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(3n + 1)\}$ as follows:

Throughout the following labeling we will consider $1 \leq j \leq m$,

**Case 1:** $\frac{n}{2}$-even

\[
\lambda(v_0^i) = m\left(\frac{n}{4} + \left\lfloor \frac{n-1}{2} \right\rfloor + 1\right) + j - m
\]

\[
\lambda(v_j^i) = \begin{cases} 
  m\left\lfloor \frac{i}{2} \right\rfloor + j, & \text{if } i - \text{odd}, 1 \leq i \leq n \\
  \frac{mn}{2}(i + n) + j - m, & \text{if } i - \text{even}, 1 \leq i \leq \frac{n}{2} \\
  \frac{mn}{2}(i + n) + j, & \text{if } i - \text{even}, \frac{n}{2} + 1 \leq i \leq n 
\end{cases}
\]

\[
\lambda(v_0^jv_1^i) = \begin{cases} 
  m(2n + 2 - i) - (j - 1), & \text{if } 1 \leq i \leq \frac{n}{2} \\
  m(2n + 1 - i) - (j - 1), & \text{if } \frac{n}{2} + 1 \leq i \leq n - 1 \\
  m(\frac{2n}{2} + 1) - (j - 1), & \text{if } i = n 
\end{cases}
\]

\[
\lambda(v_i^jv_{i+1}^j) = \begin{cases} 
  m(2n + 2 + i) - (j - 1), & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \\
  m(2n + 3 + i) - (j - 1), & \text{if } \frac{n}{2} \leq i \leq n - 2 \\
  m(\frac{2n}{2} + 2) - (j - 1), & \text{if } i = n - 1 
\end{cases}
\]

$\lambda(v_0^jv_1^i) = 2m(n + 1) - (j - 1)$
It is easy to verify that weight of every subcycle $C_3^{(i,j)} : 1 \leq i \leq n, 1 \leq j \leq m$ of $mW_n$ is $m\lceil \frac{n-1}{2} \rceil + \frac{m}{4}(27n+16) + 3$. Hence $mW_n$ is $C_3$-supermagic.

Case 2: $\frac{n}{2}$-odd

$$\lambda(v_0^i) = \frac{3m(n+2)}{4} + j - m$$

$$\lambda(v_i^j) = \begin{cases} 
    m\lfloor \frac{i}{2} \rfloor + j, & \text{if } i \text{ odd, } 1 \leq i \leq n \\
    m\lfloor \frac{i}{2} \rfloor + j, & \text{if } i \text{ even, } 1 \leq i \leq \frac{n}{2} \\
    m\lfloor \frac{i}{2} \rfloor + j, & \text{if } i \text{ even, } \frac{n}{2} + 1 \leq i \leq n-1 \\
    m(\frac{n}{2}) + j, & \text{if } i = n \\
\end{cases}$$

$$\lambda(v_0^i v_j^i) = \begin{cases} 
    2m(n+1) - (j-1), & \text{if } i = 1 \\
    m(2n+2-i) - (j-1), & \text{if } 2 \leq i \leq \frac{n}{2} \\
    m(2n-i) - (j-1), & \text{if } i \text{ even, } \frac{n}{2} + 1 \leq i \leq n-1 \\
    m(2n+2-i) - (j-1), & \text{if } i \text{ odd, } \frac{n}{2} + 1 \leq i \leq n \\
    m(\frac{3n}{2} + 1) - (j-1), & \text{if } i = n \\
\end{cases}$$

$$\lambda(v_i^j v_j^{i+1}) = \begin{cases} 
    m(2n+1) - (j-1), & \text{if } i = 1 \\
    m(2n+1+i) - (j-1), & \text{if } 2 \leq i \leq \frac{n}{2} - 1 \\
    m(2n+2+i) - (j-1), & \text{if } \frac{n}{2} \leq i \leq n-1 \\
\end{cases}$$

$$\lambda(v_n^i v_1^j) = m\left(\frac{5n}{2} + 1\right) - (j-1)$$

The weight of every subcycle $C_3^{(i,j)} : 1 \leq i \leq n, 1 \leq j \leq m$ of $mW_n$, under the labeling $\lambda$ is $m\left(\frac{5n}{2} + 1\right) + 3$. Hence $mW_n$ is $C_3$-supermagic. □

The book graph $B_n \cong K_{1,n} \times K_2$ has the vertex set $V(G) = \{u_1, u_2, v_1, w_1\}$ and the edge set $E(G) = \{u_1, u_2, u_1w_1, u_2v_1, v_1w_1\}$ for $1 \leq i \leq n$. 
Theorem 7. For $m \geq 2$, $n \geq 2$ the graph $G \cong mB_n$ admits $C_4$-supermagic labelings.

Proof. Let $G$ be a graph and let $v = |V(G)|$ and $e = |E(G)|$. Then $v = 2m(n+1)$, $e = m(3n+1)$. We denote the vertex and edge sets of $G$ as follows:

$V(G) = \{w_j^1, w_j^2 : 1 \leq j \leq m\} \cup \{v_i^1, w_i^1 : 1 \leq i \leq n, 1 \leq j \leq m\}$

$E(G) = \{w_j^1w_j^2 : 1 \leq j \leq m\} \cup \{w_i^1w_i^2, w_i^1v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^1w_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Throughout the following labeling, we will consider $1 \leq i \leq n, 1 \leq j \leq m$.

$\lambda(u_j^1) = m(i-1) + j, i = 1, 2$

$\lambda(v_i^j) = m(i+1) + j$

$\lambda(w_i^j) = m(2n - i) + j$.

Further, we will split the labelings into two cases.

Case 1: $n$ is even.

$\lambda(u_1^1u_1^2) = m\left(\frac{5n}{2} + 3\right) - j + 1$

$\lambda(u_2^1v_i^j) = \begin{cases} 
    m(2n + 2 + i) - j + 1, & \text{if } 1 \leq i \leq \frac{n}{2} \\
    m(2n + 3 + i) - j + 1, & \text{if } \frac{n}{2} + 1 \leq i \leq n 
\end{cases}$

$\lambda(u_1^1w_i^j) = \begin{cases} 
    m(5n + 5 - 2i) - j + 1, & \text{if } 1 \leq i \leq \frac{n}{2} \\
    m(6n + 4 - 2i) - j + 1, & \text{if } \frac{n}{2} + 1 \leq i \leq n 
\end{cases}$

$\lambda(v_i^jw_i^j) = \begin{cases} 
    \left(\frac{m}{2}\right)(7n + 6 + 2i) - j + 1, & \text{if } 1 \leq i \leq \frac{n}{2} \\
    \left(\frac{m}{2}\right)(5n + 6 + 2i) - j + 1, & \text{if } \frac{n}{2} + 1 \leq i \leq n 
\end{cases}$

It is easy to verify that for every subcycle $C_4^{(i)} : 1 \leq i \leq n$ and $1 \leq j \leq m$, the weight is $\lambda(u_1^1) + \lambda(u_2^1) + \lambda(v_i^j) + \lambda(w_i^j) + \lambda(u_1^1u_1^2) + \lambda(u_1^1v_i^j) + \lambda(u_1^1w_i^j) + \lambda(v_i^jw_i^j) + \lambda(v_i^jw_i^{j+1}) + \lambda(w_i^jv_i^{j+1}) + \lambda(v_i^{j+1}w_i^j) + \lambda(v_i^{j+1}w_i^{j+2}) + \lambda(w_i^{j+2}v_i^{j+1}) + \lambda(v_i^{j+1}v_i^{j+2}) + \lambda(w_i^{j+2}w_i^{j+1})$
Case 2: $n$ is odd.

\[
\lambda(u_i^j w_i^j) = m(2n + 3) - j + 1
\]

\[
\lambda(u_i^j v_i^j) = m(2n + 3 + i) - j + 1
\]

\[
\lambda(w_i^j) = \begin{cases} 
  m(5n + 5 - 2i) - j + 1, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  m(6n + 5 - 2i) - j + 1, & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n
\end{cases}
\]

\[
\lambda(v_i^j w_i^j) = \begin{cases} 
  \left(\frac{m}{2}\right)(7n + 5 + 2i) - j + 1, & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  \left(\frac{m}{2}\right)(5n + 5 + 2i) - j + 1, & \text{if } \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n
\end{cases}
\]

It is easy to verify that for every subcycle $C_{4}^{(i)} : 1 \leq i \leq n, 1 \leq j \leq m$, the weigh is $\lambda(u_i^j) + \lambda(u_i^j) = \lambda(u_i^j) + \lambda(u_i^j) + \lambda(u_i^j v_i^j) + \lambda(u_i^j w_i^j) + \lambda(v_i^j w_i^j) = \frac{m}{2}(29n + 35) + 4$. Hence $mB_n$ for $n$-odd is $C_4$-supermagic. \hfill \Box

The prism is a graph $G \cong C_m \times P_n$ with vertex-set $V(G) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge-set $E(G) = \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n - 1\} \cup \{v_{i,j}v_{i+1,j}, 1 \leq i \leq m, 1 \leq j \leq n\}$.

A generalized antiprism $A_m^n$ is a graph obtained by completing the prism graph $C_m \times P_n$ by adding the edges $v_{i,j+1}v_{i+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n - 1$.

**Theorem 8.** For $l \geq 2$ and $m, n \geq 3$, the generalized antiprism $A_m^n$ is $C_3$-supermagic with magic constant $lm(9n - 4) + 3$.

**Proof.**

Let $v = |V(A_m^n)|$ and $e = |E(A_m^n)|$. Then $v = lmn$, $e = lm(3n - 2)$. We denote the vertex and edge-sets of $G$ as follows:

\[
V(A_m^n) = \{v_i^k : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l\},
\]

\[
E(A_m^n) = \{v_i^k v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n - 1, 1 \leq k \leq l\} \cup \{v_{i,j} v_{i+1,j}^k : 1 \leq i \leq m, 1 \leq j \leq n - 1, 1 \leq k \leq l\}.
\]

Let $B_{i,j}^k$ be the 3-cycle $v_{i,j}^k v_{i+1,j}^k v_{i,j+1}^k$ and $C_{i,j}^k$ be the 3-cycle $v_{i,j+1}^k v_{i+1,j+1}^k v_{i,j}^k$. 

Define a total labeling $\lambda : V(A^n_m) \cup E(A^n_m) \rightarrow \{1, 2, ..., 4lmn - 2lm\}$ as follows:

**Case 1: $m$-odd**

$$\lambda(v_{i,j}^k) = \begin{cases} l[m(j-1) + \frac{m-j+2}{2}] + k - l, & \text{if } i \text{ - odd, } j \text{ - odd} \\ l[m(j-1) + \frac{2m-j+2}{2}] + k - l, & \text{if } i \text{ - even, } j \text{ - odd} \\ l[m(j-1) + \frac{4+1}{2}] - k + 1, & \text{if } i \text{ - odd, } j \text{ - even} \\ l[m(j-1) + \frac{m+j+1}{2}] - k + 1, & \text{if } i \text{ - even, } j \text{ - even} \end{cases}$$

$$\lambda(v_{i,j}^k v_{i,j+1}^k) = \begin{cases} l[m(4n-j-2) + \frac{j+1}{2}] - k + 1, & \text{if } i \text{ - odd, } j \text{ - odd} \\ l[m(4n-j-2) + \frac{m+j+1}{2}] - k + 1, & \text{if } i \text{ - even, } j \text{ - odd} \\ l[m(4n-j-2) + \frac{m-i}{2}] + k - l, & \text{if } i \text{ - odd, } j \text{ - even, } i \neq m \\ lm(4n-j-1) + k - l, & \text{if } i = m, j \text{ - even} \\ l[m(4n-j-2) + \frac{2m-i}{2}] + k - l, & \text{if } i \text{ - even, } j \text{ - even} \end{cases}$$

For $n$-even

$$\lambda(v_{i,j}^k v_{i,j+1}^k) = \begin{cases} l[m(2n-j-1) + \frac{m+1}{2}] + k - l, & \text{if } i \text{ - odd, } j \text{ - odd} \\ l[m(2n-j-1) + \frac{i}{2}] + k - l, & \text{if } i \text{ - even, } j \text{ - odd} \\ l[m(2n-j-1) + \frac{2m-i+1}{2}] + k - l, & \text{if } i \text{ - odd, } j \text{ - even} \\ l[m(2n-j-1) + \frac{m-i+1}{2}] + k - l, & \text{if } i \text{ - even, } j \text{ - even} \end{cases}$$

For $n$-odd

$$\lambda(v_{i,j}^k v_{i,j+1}^k) = \begin{cases} l[m(2n-j-1) + \frac{m+1}{2}] - k + 1, & \text{if } i \text{ - odd, } j \text{ - odd} \\ l[m(2n-j-1) + \frac{i}{2}] - k + 1, & \text{if } i \text{ - even, } j \text{ - odd} \\ l[m(2n-j-1) + \frac{2m-i+1}{2}] - k + 1, & \text{if } i \text{ - odd, } j \text{ - even} \\ l[m(2n-j-1) + \frac{m-i+1}{2}] - k + 1, & \text{if } i \text{ - even, } j \text{ - even} \end{cases}$$
For $n$-even

\[
\lambda(v_{i,j}^k v_{i+1,j}^k) = \begin{cases} 
  l[m(3n - j - 2) + \frac{2m - i + 1}{2}] - k + 1, & \text{if } i - \text{odd}, j - \text{odd} \\
  l[m(3n - j - 2) + \frac{m + i + 1}{2}] - k + 1, & \text{if } i - \text{even}, j - \text{odd} \\
  l[m(3n - j - 2) + \frac{m + i}{2}] - k + 1, & \text{if } i - \text{odd}, j - \text{even} \\
  l[m(3n - j - 2) + \frac{1}{2}] - k + 1, & \text{if } i - \text{even}, j - \text{even}
\end{cases}
\]

For $n$-odd

\[
\lambda(v_{i,j}^k v_{i+1,j}^k) = \begin{cases} 
  l[m(3n - j - 2) + \frac{2m - i + 1}{2}] + k - l, & \text{if } i - \text{odd}, j - \text{odd} \\
  l[m(3n - j - 2) + \frac{m + i + 1}{2}] + k - l, & \text{if } i - \text{even}, j - \text{odd} \\
  l[m(3n - j - 2) + \frac{m + i}{2}] + k - l, & \text{if } i - \text{odd}, j - \text{even} \\
  l[m(3n - j - 2) + \frac{1}{2}] + k - l, & \text{if } i - \text{even}, j - \text{even}
\end{cases}
\]

For $1 \leq i \leq m$, $1 \leq j \leq n - 1$ and $1 \leq k \leq l$,

\[
\sum (B_{i,j}^k) = \lambda(v_{i,j}^k) + \lambda(v_{i+1,j}^k) + \lambda(v_{i,j}^k v_{i+1,j}^k) + \lambda(v_{i,j}^k v_{i+1,j}^k v_{i+1,j}^k)
\]

where $i$ is odd and $j$ is odd,

\[
\sum (B_{i,j}^k) = l[m(j - 1) + \frac{m - i + 2}{2}] + k - l + l[m(j - 1) + \frac{2m - i + 2}{2}] + k - l + l[m(j - 1) + \frac{i + 1}{2}] - k + 1 + l[m(4n - j - 2) + \frac{i + 1}{2}] - k + 1 + l[m(2n - j - 1) + \frac{m + i}{2}] - k + 1 + l[m(3n - j - 2) + \frac{2m - i + 1}{2}] + k - l
\]

where $i$ is even and $j$ is even,

\[
\sum (B_{i,j}^k) = (m - 1) + \frac{m + i + 1}{2} - k + 1 + l[m(j - 1) + \frac{m + i + 1}{2} - k + 1 + l[m(j - 1) + \frac{m + i + 1}{2}]
\]

\[
\sum (B_{i,j}^k) = l[m(j - 1) + \frac{m + i + 1}{2} - k + 1 + l[m(j - 1) + \frac{m + i + 1}{2}]
\]
On cycle-supermagic labelings of the disconnected graphs

\[ \frac{i + 2}{2} - k + 1 + l[m(j - 1) + \frac{2m - i + 2}{2}] + k - l + \]
\[ + l[m(4n - j - 2) + \frac{2m - i}{2}] + k - l + l[m(2n - j - 1) + \]
\[ + \frac{m - i + 1}{2}] \]
\[ \sum (B_{i,j}^k) = lm[9n - 4] + 3 \]

Thus, it can be easily verified that \( \sum (B_{i,j}^k) = lm(9n - 4) + 3 \) for the remaining cases of \( i \) and \( j \), where \( 1 \leq i \leq m, 1 \leq j \leq n - 1 \) and \( 1 \leq k \leq l \).

Next we prove \( \sum (C_{i,j}^k) = lm(9n - 4) + 3 \), where \( 1 \leq i \leq m, 1 \leq j \leq n - 1 \) and \( 1 \leq k \leq l \).

\[ \sum (C_{i,j}^k) = \lambda(v_{i+1,j}^k) + \lambda(v_{i+1,j+1}^k) + \]
\[ + \lambda(v_{i,j+1}^k v_{i+1,j+1}^k) + \lambda(v_{i,j}^k v_{i+1,j}^k v_{i+1,j+1}^k) \]

where \( i \) is odd and \( j \) is odd, and \( i \neq m \),

\[ \sum (C_{i,j}^k) = l[mj + \frac{m + i + 1}{2}] - k + 1 + l[m(j - 1) + \]
\[ + \frac{2m - i + 1}{2}] + k - l + l[mj + \frac{m + i + 2}{2}] - k + 1 + \]
\[ + l[m(4n - j - 3) + \frac{m - i}{2}] + k - l + l[m(2n - j - 1) + \]
\[ + \frac{i + 1}{2}] - k + 1 + l[m(3n - j - 2) + \frac{2m - i + 1}{2}] + k - l \]
\[ \sum (C_{i,j}^k) = lm[9n - 4] + 3 \]

where \( i \) is even and \( j \) is even,

\[ \sum (C_{i,j}^k) = l[mj + \frac{2m - i + 2}{2}] + k - l + l[m(j - 1) + \]
\[ + \frac{i + 2}{2}] - k + 1 + l[mj + \frac{m - i + 1}{2}] + k - l + \]
\[ + l[m(4n - j - 3) + \frac{m + i + 1}{2}] - k + 1 + l[m(2n - j - 1) + \]
\[ + \frac{2m - i}{2}] + k - l + l[m(3n - j - 2) + \frac{i + 1}{2}] - k + 1 \]
\[ \sum (C_{i,j}^k) = lm[9n - 4] + 3 \]
Similarly for the other cases of $i$ and $j$, $\sum (B_{i,j}^k) = \sum (C_{i,j}^k) = lm[9n-4] + 3$, where $1 \leq i \leq m$, $1 \leq j \leq n-1$ and $1 \leq k \leq l$. Thus we get a $C_3$-supermagic labeling of $A_m^n$ with supermagic constant $c = lm[9n-4] + 3$. Hence $A_m^n$ is $C_3$-supermagic.

**Case 2: $m$-even**

$$\lambda(v_{i,j}^k) = \begin{cases} 
  l[m(j - 1) + i] + k - l, & \text{if } j \text{ odd} \\
  l[mj + i + 1] - k + 1, & \text{if } j \text{ even}
\end{cases}$$

$$\lambda(v_{i+1,j}^k) = \begin{cases} 
  l[m(4n - j - 1) - i] - k + 1, & \text{if } j \text{ odd, } i \neq m \\
  lm(4n - j - 1) - k + 1, & \text{if } j \text{ odd, } i = m \\
  l[m(4n - j - 1) + i + 1] + k - l, & \text{if } j \text{ even, } i \neq m \\
  l[m(4n - j - 2) + 1] + k - l, & \text{if } j \text{ even, } i = m
\end{cases}$$

$$\lambda(v_{i,j+1}^k) = \begin{cases} 
  l[m(2n - j) - i + 1] + k - l, & \text{if } j \text{ odd, } n \text{ even} \\
  l[m(2n - j) - i + 1] - k + 1, & \text{if } j \text{ odd, } n \text{ odd} \\
  l[m(2n - j - 1) + i] - k + 1, & \text{if } j \text{ even, } n \text{ odd} \\
  l[m(2n - j - 1) + i] + k - l, & \text{if } j \text{ even, } n \text{ even}
\end{cases}$$

$$\lambda(v_{i+1,j+1}^k) = \begin{cases} 
  l[m(3n - j - 2) + i] - k + 1, & \text{if } j \text{ odd, } n \text{ even} \\
  l[m(3n - j - 2) + i] + k - l, & \text{if } j \text{ odd, } n \text{ odd} \\
  l[m(3n - j - 1) - i + 1] - k + 1, & \text{if } j \text{ even, } n \text{ even} \\
  l[m(3n - j - 1) - i + 1] + k - l, & \text{if } j \text{ even, } n \text{ odd}
\end{cases}$$

For $1 \leq i \leq m$, $1 \leq j \leq n-1$ and $1 \leq k \leq l$,

$$\sum (B_{i,j}^k) = \lambda(v_{i,j}^k) + \lambda(v_{i+1,j}^k) + \lambda(v_{i,j+1}^k) + \lambda(v_{i+1,j+1}^k)$$
where $j$ is odd and $i \neq m$,

\[
\sum (B_{i,j}^k) = l[m(j-1) + i] + k - l + l[m(j-1) + i + 1] + k - l + l[m(j+1) - i + 1] - k + 1 + l[m(4n-j-1) - i] - k + 1 + l[m(2n-j) - i + 1] + k - l + l[m(3n-j-2) + i] - k + 1
\]

where $j$ is even and $i = m$,

\[
\sum (B_{i,j}^k) = \lambda(v_{m,j}^k) + \lambda(v_{i,j}^k) + \lambda(v_{m,j+1}^k) + \lambda(v_{m,j+1}^{k+1}) + \lambda(v_{m,j}^{k+1}) + \lambda(v_{m,j+1}^{k+1})
\]

Similarly for the remaining cases of $i$ and $j$, it can be easily verified that $\sum (B_{i,j}^k) = ln(9n-4) + 3$ for $1 \leq i \leq m, 1 \leq j \leq n-1$ and $1 \leq k \leq l$.

Next we prove $\sum (C_{i,j}^k) = ln(9n-4) + 3$, where $1 \leq i \leq m, 1 \leq j \leq n-1$ and $1 \leq k \leq l$.

where $j$ is odd and $i \neq m$,

\[
\sum (C_{i,j}^k) = l[m(j+1) - i + 1] - k + 1 + l[m(j-1) - i + 1] + k - l + l[m(j+1) - i + 1] - k + 1 + l[m(3n-j-2) + i] - k + 1 + l[m(4n-j-3) + i + 1] + k - l
\]

where $j$ is even and $i = m$,

\[
\sum (C_{m,j}^k) = \lambda(v_{m,j+1}^k) + \lambda(v_{1,j}^k) + \lambda(v_{m,j+1}^{k+1}) + \lambda(v_{m,j+1}^{k+1}) + \lambda(v_{m,j}^{k+1}) + \lambda(v_{m,j+1}^{k+1})
\]

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\[
\sum (C_{m,j}^k) = l[mj + m] + k - l + lmj - k + 1 \\
+ l[mj + 1] + k - l + l[m(3n - j - 1) + 1 - m] + k - l + \\
+ l[m(2n - j - 1) + 1] - k + 1 + l[m(4n - j - 2) - m + 1] - k + 1 \\
\sum (C_{m,j}^k) = lm[9n - 4] + 3
\]

Similarly for the other cases of \(i\) and \(j\), \(\sum(B_{i,j}^k) = \sum(C_{i,j}^k) = lm[9n - 4] + 3\), where \(1 \leq i \leq m, 1 \leq j \leq n - 1\) and \(1 \leq k \leq l\). Thus we get a \(C_3\)-supermagic labeling of \(A_n^m\) with supermagic constant \(c = lm[9n - 4] + 3\). Hence \(A_n^m\) is \(C_3\)-supermagic.

\[\square\]

In the following section, we study the cycle-supermagic labeling for the disjoint union of non isomorphic copies of fans and ladders.

### 3 Cycle-supermagic labelings of the disjoint union of non isomorphic graphs

Let \(G \cong sF_n^{s+1} \cup kF_n\), \(s, k \geq 1\), we denote the vertex and edge sets of \(G\) as follows:

\[\begin{align*}
V(G) &= \{v_j : 1 \leq j \leq s + k\} \cup \{v_j' : 1 \leq i \leq b, 1 \leq j \leq s + k\}, \\
E(G) &= \{v_i v_{i+1}' : 1 \leq i \leq b, 1 \leq j \leq s + k\} \cup \{c_j v_j' : 1 \leq i \leq b, 1 \leq j \leq s + k\},
\end{align*}\]

where

\[
b = \begin{cases} 
    n & \text{if } 1 \leq j \leq s, \\
    n - 1 & \text{if } s + 1 \leq j \leq s + k.
\end{cases}
\]

**Theorem 9.** For any positive integer \(s, k\) and \(n \geq 3\), the graph \(G \cong sF_n^{s+1} \cup kF_n\) is \(C_3\)-supermagic.

**Proof.**

Let \(v = |V(G)|\) and \(e = |E(G)|\). Then \(v = s(n + 1) + nk, e = s(2n - 1) + k(2n - 3)\).

Define a total labeling \(\lambda : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, n(3s + 2k) - 3k\}\) as follows:

Throughout the following labeling, we will consider \(1 \leq i \leq b, 1 \leq j \leq s + k\).

\[
\lambda(c_j) = s(n + 1) + nk - j + 1
\]

For \(i\)-odd
\[ \lambda(v^i_j) = j + \left(\frac{s+k}{2}\right)(i-1) \]

For \( i \)-even

\[ \lambda(v^i_j) = j + \left(\frac{s+k}{2}\right)(i-2) + \left(\frac{s+k}{2}\right)n, \]

\[ \lambda(c_jv^i_j) = s(3n-i+1) + k(3n-i-2) - j + 1, \]

\[ \lambda(v^i_jv^j_{i+1}) = s(n+i) + k(n+i-1) + j, \quad 1 \leq i \leq b-1. \]

It is easy to verify that for every subcycle \( C^{(i)}_3 : 1 \leq i \leq (s+k)(n-1)-k \) of \( sF_{n+1} \cup kF_n \), the weight of \( C^{(i)}_3 \) is \( \frac{s+k}{2}(17n) + s - 7k + 3 \). Hence \( sF_{n+1} \cup kF_n \) is \( C_3 \)-supermagic. \( \square \)

**Theorem 10.** The graph \( G \cong sL_{n+1} \cup kL_n \) is \( C_4 \)-supermagic, where \( s, k \geq 1 \) and \( n \geq 2 \).

**Proof.**

Let \( G \) be a graph and let \( v = |V(G)| \) and \( e = |E(G)| \). Then \( v = 2|sL_{n+1} + kL_n| \), \( e = s(3n-2) + k(3n-5) \). We denote the vertex and edge sets of \( G \) as follows:

\[ V(G) = \{u^i_j : 1 \leq i \leq n, 1 \leq j \leq s\} \cup \{v^i_j : 1 \leq i \leq n, 1 \leq j \leq s\} \cup \{a^i_j : 1 \leq i \leq n, 1 \leq j \leq s\} \cup \{b^i_j : 1 \leq i \leq n, 1 \leq j \leq s\}. \]

\[ E(G) = \{u^i_jv^i_j : 1 \leq i \leq n, 1 \leq j \leq s\} \cup \{u^i_ju^i_{j+1} : 1 \leq i \leq n, 1 \leq j \leq s\} \cup \{v^i_jv^{i+1}_{j} : 1 \leq i \leq n-1, 1 \leq j \leq s\} \cup \{a^i_jb^i_j : 1 \leq i \leq n, 1 \leq t \leq k\} \cup \{b^i_jb^{i+1}_j : 1 \leq i \leq n-1, 1 \leq t \leq k\}. \]

Define a total labeling \( \lambda : V(G) \cup E(G) \to \{1, 2, \ldots, 5n(s+k) - 7k - 2s\} \) as follows:

For \( 1 \leq i \leq n, 1 \leq j \leq s, 1 \leq t \leq k \).

\[ \lambda(u^i_j) = i + n(j-1) \]

\[ \lambda(a^i_j) = sn + i + (n-1)(t-1) \]

\[ \lambda(v^i_j) = 2sn + 2k(n-1) - n(j-1) - i + 1 \]

\[ \lambda(b^i_j) = 2sn + 2k(n-1) + n(1-s) + t(1-n) - i \]

\[ \lambda(u^i_jv^{i+1}_j) = 2sn + 2k(n-1) + (n-i)(s+k) + j \]

\[ \lambda(a^i_jb^i_j) = 2sn + 2k(n-1) + (n-i-1)(s+k) + s + t \]
For $1 \leq i \leq n - 1$, $1 \leq j \leq s, 1 \leq t \leq k$.

\[
\lambda(u_i^j u_{i+1}^j) = s(5n - 2) + k(5n - 7) - (s + k)(n - i - 1) - j + 1
\]
\[
\lambda(a_i^j a_{i+1}^j) = s(5n - 2) + k(5n - 7) - (s + k)(n - i - 2) - s - t + 1
\]
\[
\lambda(v_i^j v_{i+1}^j) = 4sn + 4k(n - 1) - (s + k)(n - i) - j + 1
\]
\[
\lambda(b_i^j b_{i+1}^j) = 4sn + 4k(n - 1) - (s + k)(n - i - 1) - s - t + 1
\]

To show that $\lambda$ is a $C_4$-supermagic labeling of $G$, let $C_4^{(i)}, 1 \leq i \leq (s + k)(n - 1) - k$, be the subcycles of $G \cong sL_n + 1 \cup kL_n$. Since, the weight of every $C_4^{(i)}$ is $4 + 17(n + k) - 19k - 2s$, therefore, $G$ is a $C_4$-supermagic

\section{Conclusion}

In this paper we studied the problem that if a graph $G$ has a cycle-supermagic labeling then either disjoint union of isomorphic and non isomorphic copies of $G$ will have a cycle-supermagic labeling or not? We have studied the case only for the disjoint union of $m \geq 2$ isomorphic copies of fans, wheels, ladders graphs. We also proved that disjoint union of non isomorphic copies of fans and ladders are cycle-supermagic. Moreover, we believe that if a graph has a cycle-(super)magic labeling, then disjoint union of that graph also has a cycle-(super)magic labeling. Therefore, we propose the following open problem.

**Open Problem 1** If a graph $G$ is a cycle-(super)magic, determine whether there is a cycle-(super)magic labeling for $mG, m \geq 2$.

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