Estimates of convolution operators of functions from $L^p_{2\pi}$

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Abstract
We generalize and slight improve the result of I. I. Sharapudinov [Mat. Zametki, 1996, Volume 59, Issue 2, 291–302]. Some applications to the de la Vallée Poussin operator will also be given.

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1 Introduction

Let $p = p(x)$ will be a measurable $2\pi$-periodic function, $p_- = \inf \{p(x) : x \in \mathbb{R}\}$, $p^- = \sup \{p(x) : x \in \mathbb{R}\}$, $1 \leq p_- \leq p^- < \infty$ and $L^p_{2\pi}$ will be the space of measurable $2\pi$-periodic functions $f$ such that $\int_{-\pi}^{\pi} |f(x)|^{p(x)} \, dx < \infty$.

Putting
\[
\|f\|_p = \inf \left\{ \alpha > 0 : \int_{-\pi}^{\pi} \left| \frac{f(x)}{\alpha} \right|^{p(x)} \, dx \leq 1 \right\}
\]
we turn $L^p_{2\pi}$ into the Banach space. We write $\Pi_{2\pi}$ for the set of all $2\pi$-periodic variable exponents $p = p(x) \geq 1$ satisfying the condition
\[
|p(x) - p(y)| \ln \frac{2\pi}{|x - y|} = O(1), \ (x, y \in [-\pi, \pi]). \tag{1}
\]

In the paper [2] I. I. Sharapudinov proved the following theorem:

Theorem A. Let $k_\lambda = k_\lambda(x)$ ($1 \leq \lambda < \infty$) be a measurable $2\pi$-periodic essentially bounded function (kernel) satisfying the conditions:

A\(^\circ\) $\int_{-\pi}^{\pi} |k_\lambda(t)| \, dt \leq c_1^\circ$,
\[ B^\circ \sup_t |k_\lambda(t)| \leq c_2^2 \lambda^{\eta}, \]
\[ C^\circ \) |k_\lambda(t)| \leq c_3, \ (\lambda^{-\gamma} \leq |t| < \pi), \]
where \( \gamma, \eta, c_1, c_2, c_3 > 0 \) are independent of \( \lambda \).

If \( f \in L^p_{2\pi} \) with \( p = p(x) \in \Pi_{2\pi} \), then
\[ \|K_\lambda[f]\|_p = O(1) \|f\|_p, \]
where
\[ K_\lambda[f](x) = \int_{-\pi}^{\pi} f(t)k_\lambda(t-x)dt. \]

We generalize and slight improve this result considering the wider family of two parameters convolution operators. Some applications to the de la Vallée Poussin operator will also be given.

2 Main result

Denote by \( k_{m,n} = k_{m,n}(x) \), for every \( 0 \leq m \leq n < \infty \), a measurable \( 2\pi \) -periodic essentially bounded function (kernel). Let define the linear operator
\[ K_{m,n}[f] = k_{m,n}(x) = \int_{-\pi}^{\pi} f(t)k_{m,n}(t-x)dt \]
in space \( L^p_{2\pi} \). We will say that the kernel family \( \{k_{m,n}(x)\}_{0 \leq m \leq \infty} \) satisfies the conditions B) and C), respectively, if the following estimates hold:

B) \( \sup_{t \leq h_m} |k_{m,n}(t)| \leq c_2 (n + 1)^\gamma, \)
C) \( |k_{m,n}(t)| \leq c_3, \ (h_m \leq |t| < \pi), \)
where \( h_m = \frac{\pi}{m+1} \) and \( \gamma, \eta, c_2, c_3 > 0 \) are independent of \( m, n \).

For the operator \( K_{m,n}[f] \) we will prove the following general estimate:

**Theorem 1.** Let \( k_{m,n} = k_{m,n}(x) \) \( (0 \leq m \leq n < \infty) \) satisfy the conditions B) and C). If \( f \in L^p_{2\pi} \) with \( p = p(x) \in \Pi_{2\pi} \), then
\[ \|K_{m,n}[f]\|_p = O(\mu_{m,n}) \|f\|_p, \]
where
\[ \mu_{m,n} = \int_{-h_m}^{h_m} \|k_{m,n}(t)\| dt. \]

**Proof.** Let
\[ x_k = kh_n - \pi, \quad k = 0, \pm 1, \pm 2, \ldots \quad (2) \]
\[ s_k = \min \{p(x) : x_{k-1} \leq x \leq x_{k+2}\}, \quad k = 0, \pm 1, \pm 2, \ldots \quad (3) \]
\[ p_t(x) = s_k \quad (x_{k-1} - t \leq x \leq x_{k+1} - t), \quad k = 0, \pm 1, \pm 2, \ldots \quad (4) \]
whence \( p_t(x) = p_0(x + t) \) is a \( 2\pi \) - periodic step function such that
\[ p_t(x) \leq p(x). \]
Denote by
\[ E_m(x) = (-\pi, \pi) \setminus (x - h_m, x + h_m), \]
when \((x - h_m, x + h_m) \subset (-\pi, \pi)\), but
\[ E_m(x) = (-\pi, \pi) \setminus (-\pi, x + h_m) \]
or
\[ E_m(x) = (-\pi, \pi) \setminus (-\pi, x - h_m), \]
when \(x - h_m < -\pi\) or \(\pi < x + h_m\), respectively.

Let \(\|f\|_p \leq 1\).

It is clear that for \(p = \max\{p(x) : -\pi \leq x \leq \pi\}\)
\[
\left( \int_{-\pi}^{\pi} |K_{m,n}[f](x)|^{p(x)} \, dx \right)^{1/p}
\]
\[
= \left( \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} f(t)k_{m,n}(t-x) \, dt \right|^{p(x)} \, dx \right)^{1/p}
\]
\[
= \left( \int_{-\pi}^{\pi} \left( \int_{E_m(x)} + \int_{x-h_m}^{x+h_m} \right) f(t)k_{m,n}(t-x) \, dt \right|^{p(x)} \, dx \right)^{1/p}
\]
\[
\leq \left( \int_{-\pi}^{\pi} \int_{E_m(x)} f(t)k_{m,n}(t-x) \, dt \right|^{p(x)} \, dx \right)^{1/p}
+ \left( \int_{-\pi}^{\pi} \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x) \, dt \right|^{p(x)} \, dx \right)^{1/p}
\]
\[
= J_m^{1/p} + J_x^{1/p}.
\]

Since
\[ \|f\|_q \leq (2\pi + 1) \|f\|_p \]
with \(q(x) \leq p(x)\) for \(-\pi \leq x \leq \pi\) (cf. [1]), by condition C),
\[
\left| \int_{E_m(x)} f(t)k_{m,n}(t-x) \, dt \right|
\]
\[
= O(1) \int_{E_m(x)} |f(t)| \, dt = O(1) \int_{-\pi}^{\pi} |f(t)| \, dt = O(1) \|f\|_p = O(1)
\]
and therefore
\[ J_m = O(1). \]
In case of integral $J_x$, using (1), (2) and (3), for $x_k \leq x \leq x_{k+1}$, we obtain that
\[ |p(x) - s_k| = O\left(\frac{1}{\ln 2(n+1)\gamma}\right), \]
and therefore by condition B)
\[ \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt = O\left(\frac{1}{\ln 2(n+1)\gamma}\right) \]
Next,
\[ J_x = \sum_{k=0}^{2^{|n|}-1} \int_{x_k}^{x_{k+1}} \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \]
Thus
\[ J_x = O(1) \sum_{k=0}^{2^{|n|}-1} (\mu_{m,n})^s_k \int_{x_k}^{x_{k+1}} \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \]
By the Jensen inequality,
\[ J_x = O(1) \sum_{k=0}^{2^{|n|}-1} (\mu_{m,n})^s_k \int_{x_k}^{x_{k+1}} \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \int_{x-h_m}^{x+h_m} f(t)k_{m,n}(t-x)dt \int_{x-h_m}^{x+h_m} |f(t)|^p dt \]
Similar to [2] (17) p.295 we have
\[ \int_{-\pi}^\pi |f(x)|^{p(x)} dx \leq (2\pi + 1)^\gamma. \]
Hence
\[ J_1^{1/\mathcal{P}} = O(\mu_{m,n}) \]
and our result follows. \hfill \Box

**Corollary 2.** If we put \( m = n = \lambda \) in the assumptions of Theorem 1, then for \( f \in L_{2\pi}^p \) with \( p = p(x) \in \Pi_{2\pi} \) the following estimate
\[ \|K_\lambda[f]\|_p = O(\mu_\lambda \|f\|_p) \]
holds, where \( \mu_\lambda = \int_{-h_\lambda}^{h_\lambda} |\kappa_\lambda(t)| \, dt \) and \( k_\lambda = k_{\lambda,\lambda} \).

**Remark 3.** If we additionally assume that \( \mu_\lambda = O(1) \), then Corollary 2 gives Theorem A with the result \([2]\) of I. I. Sharapudinov, under the slight weaker conditions.

### 3 De la Vallée Poussin operator

Let \( f \in L_{2\pi}^1 \) and consider the trigonometric Fourier series
\[ Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^\infty (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \]
with the partial sums \( S_k f \). For \( 0 \leq m \leq n, m, n = 0, 1, 2, \ldots \) denote by
\[ V_{n,m}[f](x) = \frac{1}{m+1} \sum_{k=n-m}^n S_k[f](x) = \frac{1}{\pi(m+1)} \int_{-\pi}^{\pi} f(t) \Phi_{n,m}(t-x) \, dt \]
the de la Vallée Poussin means of the series \( Sf \), where
\[ \Phi_{n,m}(t) = \frac{1}{m+1} \sin \frac{(m+1)t}{2} \sin \frac{(2n-m+1)t}{2} \cdot \frac{2 \sin^2 \frac{t}{2}}{\pi}. \]

It is clear that the kernel family \( \{\Phi_{m,n}(x)\}_{1 \leq m \leq n < \infty} \) satisfies the conditions B) with \( \eta = 1 \) and C) with \( \gamma = \frac{1}{2} \). By the following calculation
\[
\begin{align*}
\int_0^{h_m} & \left| \frac{\sin \frac{(m+1)t}{2} \sin \frac{(2n-m+1)t}{2}}{2 \sin^2 \frac{t}{2}} \right| \, dt \\
= & \ \frac{1}{m+1} \left( \int_0^{\frac{\pi}{m+1}} + \int_{\frac{\pi}{m+1}}^{\frac{2n-m+1}{m+1}} + \int_{\frac{2n-m+1}{m+1}}^{h_m} \right) \left| \sin \frac{(m+1)t}{2} \sin \frac{(2n-m+1)t}{2} \right| \cdot \frac{2 \sin^2 \frac{t}{2}}{\pi} \, dt \\
\leq & \ \int_0^{\frac{\pi}{m+1}} \frac{(2n-m+1) \, dt}{2} + \int_{\frac{\pi}{m+1}}^{\frac{2n-m+1}{m+1}} \frac{dt}{2 \frac{2n-m+1}{m+1}} + \frac{1}{m+1} \int_{\frac{2n-m+1}{m+1}}^{h_m} \frac{dt}{2 \left( \frac{2n-m+1}{m+1} \right)^2} \\
\leq & \ \pi + \frac{\pi}{2} \ln \frac{n+1}{m+1} + \frac{\pi}{2}
\end{align*}
\]
we obtain
\[ \mu (\Phi_{m,n}) = \int_{-h_m}^{h_m} |\Phi_{m,n}(t)| \, dt \leq \pi \left( 3 + \ln \frac{n + 1}{m + 1} \right). \]

Hence, by Theorem 1, we have:

**Theorem 4.** If \( L^p_{2\pi} \) with \( p = p(x) \in \Pi_{2\pi} \), then
\[ \|V_{n,m}[f]\|_p = O \left( 1 + \ln \frac{n + 1}{m + 1} \right) \|f\|_p \quad (0 \leq m \leq n, \ m, n = 0, 1, 2, \ldots). \]

From Theorem 4 we get the following corollary:

**Corollary 5.** Let \( f \in L^p_{2\pi} \) with \( p = p(x) \in \Pi_{2\pi} \). If \( m = O(n) \), then
\[ \|V_{n,m}[f]\|_p = O(1) \|f\|_p \]
hold.

In the special case we can consider the following Fourier operator:
\[ S_n[f](x) = V_{n,0}[f](x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \Phi_{n,0}(t - x) \, dt, \]
where
\[ \Phi_{n,0}(t) = \frac{\sin \left( \frac{(2n+1)t}{2} \right)}{2 \sin \frac{t}{2}}. \]

For this operator we have:

**Corollary 6.** Let \( f \in L^p_{2\pi} \) with \( p = p(x) \in \Pi_{2\pi} \). If we put \( m = 0 \) in Theorem 4, then
\[ \|S_n[f]\|_p = O \left( 1 + \ln (n + 1) \right) \|f\|_p \quad (n = 0, 1, 2, \ldots). \]

**Remark 7.** In the case \( p \equiv 1 \) the results of this section we can find in the monograph of A. Zygmund [A, Ch. II, p.70, Ch. III, p.90] (see e.g. [B, Ch. II, p.117-8].

**References**

[1] Sharapudinov, I. I.: The basis property of the Haar system in the space \( L^{p(x)}([0; 1]) \) and the principle of localization in the mean, Math. Sb., Vol. 130(172), No 2(6). 275-283 (1986).

[2] Sharapudinov, I. I.: Uniform boundedness in \( L^p(p = p(x)) \) of some families of convolution operators, Mat. Zametki, Volume 59, Issue 2, 291-302 (1996).

[3] Sharapudinov, I. I.: On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces, Azerbaijan Journal of Mathematics V. 4, No 1, January (2014).
[4] Sharapudinov, I. I.: Approximation of functions in $L^p(x)$ by trigonometric polynomials, Izv. RAN. Ser. Mat., Volume 77, Issue 2, 197-224 (2013).

[5] Zygmund, A.: Trigonometric series, Cambridge, London, New York, Melbourne, (2002).

[6] Zhuk, V. V.: Approximation of periodic functions (Russian), Leningrad, (1982).