The origin of regular Newtonian potential in infinite derivative gravity

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Abstract

It turns out that the infinite derivative gravity (IDG) is ghost-free and renormalizable when one chooses the exponential of an entire function. For this IDG case, the corresponding Newtonian potential generated from the delta function is non-singular at the origin. However, we will explicitly show that the source generating this non-singular potential is given not by the delta-function due to the point-like source of mass, but by the Gaussian mass distribution. This explains clearly why the IDG with the exponential of an entire function yields the finite potential at the origin.

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1 Introduction

There was a conjecture that renormalizable higher-derivative gravity has a finite Newtonian potential at the origin \[1, 2, 3, 4\]. This relation was first mentioned in Stelle’s seminal work \[5\] which showed that the fourth-derivative gravity is renormalizable and has a finite potential at origin. Unfortunately, this gravity suffers from ghost problem because it has a massive pole with negative residue which could be interpreted either as a state of negative norm or a state of negative energy.

It turned out that the infinite derivative gravity (IDG) can resolve the problem of massive ghost as well as it may avoid the singularity of the Newtonian potential at the origin, when one chooses the exponential of an entire function. We would like to note that this model is also named super-renormalizable quantum gravity \[6\]. However, one does not understand fully how this nonlocal gravity could provide a regular potential. It was argued that the cancellation of the singularity at the origin is an effect of an infinite amount of hidden ghost-like complex poles \[2\].

On the other hand, Tseytlin \[7\] showed already in string theory that the corrected Poisson equation \[e^{-\alpha'\Delta} \delta^N = -\mu \delta^N(x)\] with \[\Delta = \delta_{ij} \partial_i \partial_j\] regularizes effectively the delta-function source at the scale of \[\sqrt{\alpha'}\]. In other words, replacing the source \[\delta^N(x)\) by \[\delta^N(x) = e^{\alpha'\Delta} \delta^N(x)\], one can interpret “\[\delta^N = -\mu \delta^N(x)\]” as a Poisson equation with smearing of the delta-function source at the string scale \[\sqrt{\alpha'}\]. Furthermore, the source of \[\delta^N(x)\] removes Newtonian singularity \[8\], similar to the simpler fourth-derivative gravity \[5\].

Especially, we would like to point out that the Gaussian mass distribution gives rise to a regular Newtonian potential in the Einstein gravity \[12\].

In this work, we will show that the source generating the regular Newtonian potential is originated not from the delta-function due to the point-like mass source but from the Gaussian mass distribution. This indicates clearly why the IDG with the exponential of an entire function yields the finite potential at the origin. Transforming the delta-function source (Einstein theory) to the Gaussian distribution (IDG) makes the Newtonian potential regular.

2 Propagator for IDG

In this section, we wish to mention briefly a process for obtaining a non-singular Newtonian potential from the propagator of the IDG. Starting from the general version of higher-derivative gravitational action up to the second
order in curvature without restricting the number of derivative

\[ S = \frac{1}{4\kappa^2} \int d^4x \sqrt{-g} \left[ -2R + RF_1(\Box_\sigma)R + R_{\mu\nu}F_2(\Box_\sigma)R^{\mu\nu} + R_{\mu\nu\rho\sigma}F_3(\Box_\sigma)R^{\mu\nu\rho\sigma} \right], \]

(1)

where \( \sigma \) is the length scale at which the nonlocal modifications become important. Here \( F_i \)'s are infinite-derivative functions of \( \Box_\sigma \) \[ F_i(\Box_\sigma) = \sum_{n=0}^\infty c_n \Box_\sigma^n \]. One needs a specific constraint of \( 2F_1 + F_2 + 2F_3 = 0 \) \[ 10, 11 \] when expanding around a Minkowski spacetime \( (g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}) \) with \( \eta_{\mu\nu} = \text{diag}(+ - - -) \) so that the bilinear action can provide a ghost-free (massive) tensor propagator. The simplest condition could be achieved if one chooses \( F_i \)'s as

\[ F_1(\Box_\sigma) = a(\Box_\sigma) - 1, \quad F_2(\Box_\sigma) = -2F_1(\Box_\sigma), \quad F_3(\Box_\sigma) = 0, \]

(2)

which corresponds to the relation of \( a(\Box_\sigma) = c(\Box_\sigma) \) with

\[ a(\Box_\sigma) = 1 - \frac{\Box_\sigma F_2(\Box_\sigma)}{2}, \quad c(\Box_\sigma) = 1 + \Box_\sigma(F_2(\Box_\sigma) + 3F_1(\Box_\sigma)) \].

(3)

This implies, for non-linear case, that a simplest model of the IDG is given by \[ 2 \]

\[ S_{\text{IDG}} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R + G_{\mu\nu} \frac{a(\Box_\sigma)}{\Box_\sigma} R^{\mu\nu} \right] \]

(4)

with \( \kappa^2 = 8\pi G \). Bilinearizing the Lagrangian of Eq.\( (4) \) together with imposing the de Donder gauge, one obtains \( \mathcal{L}_{\text{IDG}}^{\text{bil}} = h^{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h^{\alpha\beta}/2 \). The inverse operator \( (1/\mathcal{O}) \) corresponds to the propagator. The propagator of the action \( (1) \) takes the form with \( k^2 = k_0^2 - k^2(k^2 = k \cdot k = |k|^2) \)

\[ \mathcal{P}_{\mu\nu,\rho\sigma}^{\text{IDG}}(k) = \frac{1}{a(-k_0^2)} \frac{1}{k^2} \left[ P_{\mu\nu,\rho\sigma}^{(2)} - P_{\mu\nu,\rho\sigma}^{(0-s)} \right], \]

(5)

where \( P^{(2)} \) and \( P^{(0-s)} \) are the Barnes-Rivers spin projection operators. If one chooses \( a(\Box_\sigma) = e^{\sigma^2\Box/4}[a(-k_0^2) = e^{-\sigma^2k^2/4}] \) which is the exponential of an entire function, there is no room to introduce ghost poles at the perturbative level for the IDG. This is because \( a(\Box_\sigma) \) has no zeros or no poles. In this case, the only dynamical pole resides at \( k^2 = 0 \) which corresponds to a massless pole of the spin-2 propagator. Given this entire function, furthermore, the propagator becomes exponentially suppressed in the UV and the vertex factors are exponentially enhanced so that this theory can be renormalizable. We note that the other choice of \( a(\Box_\sigma) = e^{-\sigma^2\Box/4}[a(-k_0^2) = e^{\sigma^2k^2/4}] \) with \( \eta_{\mu\nu} = \text{diag}(+ - - -) \) does not leads to the Gaussian distribution in momentum space.

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Let us pay attention to the relation between static propagator of $D_{\mu\nu,\rho\sigma}^{IDG}(k)|_{k_0=0}$ and classical potential generated by a point-like mass source of $T_{\mu\nu} = \rho_{\mu\nu}h_{\nu\nu}$ with $\rho = M\delta(x)$,

$$V(r) = \frac{k^2M}{(2\pi)^3} \int d^3k e^{ikr}D_{00,00}(k), \quad D_{00,00}^{IDG}(k) = -\frac{e^{-\sigma^2|k|^2/4}}{2|k|^2}. \quad (6)$$

Then, we obtain the Newtonian potential

$$V^{IDG}(r) = -\frac{GM}{r}\text{Erf}\left(\frac{r}{\sigma}\right). \quad (7)$$

Since the error function Erf$(x)$ takes a series form around $x = 0$ as

$$\text{Erf}(x) \simeq \frac{2x}{\sqrt{\pi}} - \frac{x^3}{\sqrt{\pi}} + \frac{11x^5}{20\sqrt{\pi}} - \frac{241x^7}{840\sqrt{\pi}} + \cdots, \quad (8)$$

the regular potential is obtained from Eq.(7) as

$$V^{IDG}(r) \simeq \frac{2GM}{\sqrt{\pi}\sigma}\left[-1 + \frac{r^2}{2\sigma^2} - \cdots\right]. \quad (9)$$

As a result, we find that around $r = 0$, the singularity disappears due to the non-locality and the effect depends on a specific form of $a(\Box r)$. Hence, the action (4) provides a ghost-free and singularity-free gravity which is also renormalizable.

### 3 Origin of regular potential

At this stage, let us remind that the Newtonian potential $V^{IDG}(r)$ was obtained from the dressed propagator (5) with the point-like source $\rho = M\delta^3(r)$. It was argued that the cancellation of singularity may be seen as an effect of an infinite amount of hidden ghost-like complex poles [2]. However, we point out that the dressed propagator has only a simple pole at $k^2 = 0$ because $a(-k^2)$ is the exponential of an entire function. Thus, this could not explain an origin of the regular potential (7). This is so because one derived the potential from the propagator by assuming the point-like source.

Since the regular potential implies the renormalizable gravity, we must explore the origin of the regular potential by investigating the mass source. For this purpose, we derive the linearized equation from (4) with the external source $T_{\mu\nu}$ [13]

$$a(\Box_r)\left[\Box h_{\mu\nu} - (\partial_\mu\partial_\alpha h^\alpha_{\nu} + \partial_\alpha\partial_\nu h^\alpha_{\mu}) + (\gamma_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + \partial_\mu\partial_\nu h) - \eta_{\mu\nu}\Box h\right] = -2\kappa^2 T_{\mu\nu}. \quad (10)$$
Considering the Newtonian approximation of $\partial_0 h_{\mu\nu} = 0$, the trace and the 00-component of (10) take the forms

$$2a(\Box_\sigma) \left[ -\Box h + \partial_\alpha \partial_\beta h^{\alpha\beta} \right] = -2\kappa^2 \rho, \quad (11)$$
$$a(\Box_\sigma) \left[ \Box h_{00} + \partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h \right] = -2\kappa^2 \rho. \quad (12)$$

Choosing the perturbation as the Newtonian and Bardeen potentials as

$$h_{\mu\nu} = \text{diag}\{2\Phi, 2\Psi, 2\Psi, 2\Psi\}, \quad (13)$$

the trace and the 00-component equations are given by

$$2a(\Delta_\sigma) \left[ 2\Delta(\Phi - 3\Psi) + 2\Delta \Psi \right] = -2\kappa^2 \rho, \quad (14)$$
$$4a(\Delta_\sigma) \Delta \Psi = 2\kappa^2 \rho. \quad (15)$$

By comparing (14) with (15), we find that two potentials $\Phi$ and $\Psi$ satisfy the same equation

$$2a(\Delta_\sigma) \Delta \{\Phi, \Psi\} = \kappa^2 M \delta^3(r). \quad (16)$$

We wish to find the solution to (16) by going into the momentum space and then, going back to the coordinate space. Fourier transforming (16) leads to

$$-2a(k^2)k^2 \Phi(k) = \kappa^2 M, \quad a(k^2) = e^{\sigma^2 |k|^2 / 4} \quad (17)$$

whose potential is determined by inverse-Fourier transforming as

$$\Phi(r) = -\frac{\kappa^2 M}{2} \int \frac{d^3k}{(2\pi)^3} e^{ikr} a(k^2) = -\frac{\kappa^2 M}{(2\pi)^2} \frac{1}{r} \int_0^\infty d|k| \frac{e^{-\sigma^2 |k|^2 / 4} \sin(|k|r)}{|k|}. \quad (18)$$

Noting

$$\int_0^\infty d|k| \frac{e^{-|k|^2 / 4} \sin(|k|r)}{|k|} = \frac{\pi}{2} \text{Erf} \left( \frac{r}{\sigma} \right), \quad (19)$$

the Newtonian potential is given by

$$\Phi_{\text{IDG}}(r) = -\frac{GM}{r} \text{Erf} \left( \frac{r}{\sigma} \right), \quad (20)$$

which is exactly the same form as in Eq. (17).

Inspired by Tseytlin’s work [7], we transform (16) into the poisson-like equation

$$\Delta \{\Phi, \Psi\} \equiv 4\pi G \rho(r) \quad (21)$$
whose Fourier-transform is given by \[\Phi(|k|) = -4\pi G \frac{\tilde{\rho}(|k|)}{k^2}.\] (22)

Here \[\tilde{\rho}(|k|) = 4\pi \int_0^\infty r^2 dr \frac{\sin|kr|}{|kr|} \rho(r).\] (23)

For example, given a point-like source of mass \(M\)
\[\rho_p(r) = M \delta^3(r) = \frac{M}{4\pi r^2} \delta(r),\] (24)
plugging (24) into (23) leads to the density in momentum space
\[\tilde{\rho}_p(|k|) = M \int_0^\infty dr \frac{\sin|kr|}{|kr|} \delta(r) = M.\] (25)

Performing inverse-Fourier transformation, one obtains the usual Newtonian potential as
\[\Phi(r)_p = -\frac{2G}{\pi} \int_0^\infty d|k| |k| \frac{\sin|kr|}{|kr|} \tilde{\rho}_p(|k|) = -\frac{2GM}{\pi r} \int_0^\infty dz \frac{\sin[z]}{z} = -\frac{GM}{r}.\] (26)

On the other hand, let us look for the IDG-mass distribution whose momentum distribution is initially given by \([17]\) as
\[\tilde{\rho}_{\text{IDG}}(|k|) = M e^{-\sigma^2|k|^2/4}.\] (27)

It is given by the inverse-Fourier transformation \([12]\)
\[\rho_{\text{IDG}}(r) = \int_0^\infty \frac{|k|^2 d|k| |k| \sin|kr|}{2\pi^2 |k|r} \tilde{\rho}_{\text{IDG}}(|k|) = \frac{M}{\pi^{3/2}} \frac{e^{-r^2/\sigma^2}}{\sigma^3},\] (28)
which is obviously the Gaussian mass distribution (regular matter density) with the width \(\sigma\) when comparing to the point-like source \([24]\). We check that the total mass is given by
\[M = 4\pi \int_0^\infty r^2 dr \rho_{\text{IDG}}(r).\] (29)

Fig. 1 depicts the Gaussian mass density \(\rho_{\text{IDG}}(r)\), the IDG regular potential \(\Phi_{\text{IDG}}(r)\), and the singular Newtonian potential \(\Phi_p(r)\) as functions of \(r\). We observe that the density \(\rho_{\text{IDG}}(r)\) is essentially zero for \(r \geq 3\sigma\), which allows us to identify \(\sigma\) as the length scale of the nonlocality. Here, one finds that \(\Phi_p(r) \simeq \Phi_{\text{IDG}}\) for \(r \geq 3\sigma\).
Figure 1: For Gaussian mass distribution with $\sigma = 1$ and $M = 1$ (top curve), its regular Newtonian potential appears as middle curve. The bottom curve denotes singular Newtonian potential for point-like source of mass $M = 1$ with $G = 1$.

4 Discussions

Even though the nonlocal gravity of (4) with $a(\Box_{\sigma}) = e^{\sigma^2\Box/4}$ provides the regular Newtonian potential (9), one did not know explicitly what makes the potential finite at the origin. On the other hand, the regular potential becomes an important issue because it implies the renormalizable quantum gravity.

In this work, we have shown that Einstein gravity (Poisson equation) provides the singular potential as shown in the process of $\rho_p(r) = M\delta^3(r) \to \tilde{\rho}_p(k) = M \to \Phi_p(r) = -GM/r$, whereas the infinite derivative gravity (IDG) with $a(\Box_{\sigma}) = e^{\sigma^2\Box/4}$ (Poisson-like equation) gives us the regular potential as shown in the process of $\tilde{\rho}_{IDG}(k) = Me^{-\sigma^2|k|^2/4} \to \rho_{IDG}(r) = (M/\pi^{3/2}\sigma^3)e^{-r^2/\sigma^2} \to \Phi_{IDG}(r) = -GMErf(r/\sigma)/r$.

Importantly, we found the Gaussian mass distribution $\rho_{IDG}(r)$ from the Gaussian distribution $\tilde{\rho}_{IDG}(k)$ in momentum space. It implies that the origin of the regular potential in the IDG theory is the smearing of the point-like mass source for the nonlocal scale of $r < 3\sigma$, corresponding to the Gaussian mass distribution. For $r \geq 3\sigma$, one finds that two potentials are nearly the same like as $\Phi_p(r) \simeq \Phi_{IDG}$. This explains clearly why the IDG with the exponential of an entire function yields the finite potential at the origin.
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