A MONOTONE SELECTION PRINCIPLE IN C*-ALGEBRAS

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Abstract. We apply to operator algebra theory a monotone selection principle which apparently escaped attention (of operator algebra theorists) so far. This principle relates to the basic order theoretic characterisation of von Neumann algebras given by Kadison, and the simplified form this result takes in separable Hilbert spaces. In the separable case we need only consider increasing sequences rather than increasing nets. We apply an argument of Klaus Floret to show that, within the realm of commutativity, there exists a general monotone selection principle providing for this simplification. Thereby we obtain a valuable shortcut and a handy tool for related purposes. Actually, a more general selection principle is proved within the framework of vector lattices.

1. Introduction

We first would like to put into perspective the needed elements of operator algebra theory. The following three notions are cornerstones of the order theoretic aspect of operator algebras:
- monotone complete hermitian parts of C*-algebras,
- monotone closed subsets of the above,
- normal positive linear maps between the above.

We shall review these notions and their relevance in the next paragraph.

Let $A$ be a C*-algebra. We shall denote by $A_{sa}$ the hermitian part of $A$. One says that $A_{sa}$ is monotone complete if each non-empty upper bounded upward directed subset of $A_{sa}$ has a supremum in $A_{sa}$, cf. [3, 3.9.2]. For example the hermitian part of a von Neumann algebra is monotone complete. Similarly, if $B$ is a C*-subalgebra of a C*-algebra $A$ with monotone complete hermitian part $A_{sa}$, then $B_{sa}$ is called monotone closed in $A_{sa}$ if $B_{sa}$ contains the supremum in $A_{sa}$ of each non-empty upward directed subset of $B_{sa}$ which is upper bounded in $A_{sa}$. The following theorem of Kadison occupies a central place in the order theoretic part of operator algebra theory. If $H$ is a Hilbert space, and $A$ is a C*-subalgebra of $B(H)$, then $A$ is a von Neumann algebra if and only if $A_{sa}$ is monotone closed in $B(H)_{sa}$. For a proof, see [3, 2.4.4]. Let $\phi : A \to B$ be a positive linear map between two C*-algebras $A$ and $B$. (For example $\phi$ a state or a representation.)

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If $A_{sa}$ and $B_{sa}$ are monotone complete, then $\phi$ is called normal, if for each non-empty upper bounded upward directed subset $J$ of $A_{sa}$ one has $\phi(\sup J) = \sup_{j \in J} \phi(j)$, cf. [3, 2.5.1]. It can be shown that if $\phi : A \to B$ is a normal $*$-algebra homomorphism between C*-algebras $A$ and $B$ with monotone complete hermitian parts, then $\phi(A)_{sa}$ is a monotone closed C*-subalgebra of $B_{sa}$, cf. the proof of [3, 2.5.3]. Thus, if $B$ moreover is a von Neumann algebra, then $\phi(A)$ is a von Neumann algebra.

It has been noted that on a separable Hilbert space, the above conditions can be relaxed. Namely, it is enough to consider increasing sequences instead of increasing nets, cf. [3, 2.4.3]. One then speaks of:
- monotone sequentially complete hermitian parts of C*-algebras,
- monotone sequentially closed subsets of the above,
- sequentially normal positive linear maps between the above.

It has also been noted that for many purposes, the assumption of separability of the Hilbert space can be relaxed to assuming that the von Neumann algebra is $\sigma$-finite. We shall review this notion in the next three paragraphs.

A state $\psi$ on a C*-algebra $A$ is called faithful if $\psi(a) > 0$ for every $a \in A_+ \setminus \{0\}$, cf. [3, 3.7.2]. The following fact is well-known: in a C*-algebra carrying a faithful state every set of pairwise orthogonal projections is countable, cf. [1, 2.5.6]. It is to this well-known fact that we shall adjoin a versatile supplement.

(We open a parenthesis for a proof here because it is so short. Let $\psi$ be a faithful state on a C*-algebra $A$. Let $(\pi_\psi, H_\psi, c_\psi)$ be the GNS representation of $\psi$. Let $P$ be a set of pairwise orthogonal projections in $A$. Then $(\pi_\psi(p))_{p \in P}$ is a family of pairwise orthogonal projections in $H_\psi$. Let $q$ denote its sum. By the Parseval equality it follows that $\|qc_\psi\|^2 = \sum_{p \in P} \|\pi_\psi(p)c_\psi\|^2$. Since the left side of this equality is finite, the sum on the right can only contain countably many non-zero terms. All other terms satisfy $0 = \|\pi_\psi(p)c_\psi\|^2 = \psi(p^*p)$, whence $0 = p^*p = p$ as $\psi$ is faithful.)

A von Neumann algebra $\mathcal{M}$ is called $\sigma$-finite if every set of pairwise orthogonal projections in $\mathcal{M}$ is countable, cf. [1, 2.5.1], or [3, 3.8.3]. Obviously a Hilbert space $H$ is separable if and only if the von Neumann algebra of all bounded operators on $H$ is $\sigma$-finite. Also, a von Neumann algebra is $\sigma$-finite if and only if it admits a faithful state. This follows from the above proof together with [1, 2.5.6]. Note that $\sigma$-finiteness of a von Neumann algebra $\mathcal{M}$ is of a commutative nature in the sense that it pertains to commutative C*-subalgebras of $\mathcal{M}$ only.

We give a related selection principle which apparently escaped attention so far. The argument is due to Klaus Floret, cf. his proof [2] p. 228 of the order completeness of the $L^p$ spaces for $1 \leq p < \infty$. A core statement goes as follows. (For a more precise version with proof, see the next section.)
Theorem. Let $A$ be a commutative $C^*$-algebra carrying a faithful state $\psi$. If $A_{sa}$ is monotone sequentially complete, then it is is monotone complete, and the following monotone selection principle holds. For every non-empty upper bounded upward directed subset $J$ of $A_{sa}$ there exists an increasing sequence $(j_n)$ in $J$ with the same supremum as $J$. Every increasing sequence $(j_n)$ in $J$ with $\sup_n \psi(j_n) = \sup_{j \in J} \psi(j)$ does the job.

We feel that this principle is rather smooth in operation compared to the clumsy notion of $\sigma$-finiteness. We have the following consequences.

Corollary. Let $\phi : A \to B$ be a positive linear map between two $C^*$-algebras $A$ and $B$ with monotone complete hermitian parts. Assume that $A$ is commutative and admits some faithful state. If $\phi$ is sequentially normal, then $\phi$ is normal.

Corollary. Let $A$ be a commutative $C^*$-algebra carrying a faithful state $\psi$. If $A_{sa}$ is monotone sequentially complete, and if $\psi$ is sequentially normal, then $A_{sa}$ is monotone complete, and $\psi$ is normal. In this case, the GNS representation $\pi_\psi$ is a normal isomorphism of $C^*$-algebras from $A$ onto a von Neumann algebra.

Proof. The proof of the statement concerning the GNS representation follows from the proofs of [3, 3.3.9] and [3, 2.5.3].

In the next section, a more general selection principle is formulated and proved in the abstract framework of vector lattices. This makes that commutative $C^*$-algebras belong within the reach of the result. Indeed, it is well-known that the hermitian part of a commutative $C^*$-algebra is a vector lattice. (This follows for example from the commutative Gelfand-Naimark theorem.) Conversely, the hermitian part of a non-commutative $C^*$-algebra is never a vector lattice. (This follows from the remark [3, 1.4.9].) However, our results are of immediate interest to non-commutative $C^*$-algebras as well, namely via maximal commutative $C^*$-subalgebras. We mention in this respect the notion of $AW^*$-algebras and the notion of completely additive positive linear maps between $AW^*$-algebras, cf. [3, 3.9.2], and [3, 3.9.6]. Note again that $\sigma$-finiteness is of a commutative nature as well.

2. A MORE PRECISE SELECTION PRINCIPLE AND PROOF

In the following definition we formalise and clarify completeness and closedness with respect to monotone nets and sequences. Also, the notion of a faithful positive linear functional on an ordered vector space is introduced.

Definition. Let $W$ be an ordered vector space, and let $V$ be a vector subspace of $W$. (Typically $W$ the hermitian part of a von Neumann algebra.)
We shall say that $W$ is monotone complete, if each non-empty upper bounded upward directed subset of $W$ has a supremum in $W$. We shall say that $W$ is monotone sequentially complete, if each upper bounded increasing sequence in $W$ has a supremum in $W$.

If $W$ is monotone complete, then $V$ shall be called monotone closed in $W$, if $V$ contains the supremum $\sup J$ in $W$ of each non-empty upward directed subset $J$ of $V$ that is upper bounded in $W$. (It is clear that then $\sup J$ also is the supremum in $V$ of $J$, and that $V$ then is monotone complete.) If $W$ is monotone sequentially complete, then $V$ shall be called monotone sequentially closed in $W$, if $V$ contains the supremum $\sup_n j_n$ in $W$ of each increasing sequence $(j_n)$ in $V$ that is upper bounded in $W$. (It is clear that then $\sup_n j_n$ also is the supremum in $V$ of $(j_n)$, and that $V$ then is monotone sequentially complete.)

If $W_+$ denotes the set of positive elements of $W$, then a positive linear functional $\psi$ on $W$ shall be called faithful, if $\psi(a) > 0$ for each $a \in W_+ \setminus \{0\}$.

**Theorem.** Let $W$ be a vector lattice carrying a faithful positive linear functional $\psi$. Assume that $W$ is monotone sequentially complete, and that $V$ is a monotone sequentially closed vector subspace of $W$. Then $W$ is monotone complete, and $V$ is monotone closed in $W$.

Indeed, which is more, the following monotone selection principle holds. Whenever $J$ is a non-empty upward directed subset of $V$ that is upper bounded in $W$, there exists an increasing sequence $(j_n)$ in $J$ with

$$\sup_n j_n = \sup J \text{ in } W.$$  

Every increasing sequence $(j_n)$ in $J$ with

$$\sup_n \psi(j_n) = \sup \psi(j)$$

does the job. From the result that $V$ is monotone closed in $W$, one also has

$$\sup_n j_n = \sup J \text{ in } V.$$  

**Proof.** Let $J$ be a non-empty upward directed subset of $V$ that is upper bounded in $W$. Let $(j_n)_{n \geq 1}$ be any increasing sequence in $J$ with

$$(*)\quad \sup_{n \geq 1} \psi(j_n) = \sup \psi(j).$$

We define

$$j_0 = \sup_{n \geq 1} j_n \text{ in } W,$$

which is possible by the assumption that $W$ is monotone sequentially complete. One has that $j_0$ belongs to $V$ by the assumption that $V$ is monotone sequentially closed in $W$. To prove the theorem, it is enough to show that $j_0 = \sup J$ in $W$. (This is so because if we choose $V := W$, we obtain that $W$ is monotone complete. We clearly then also have
that $V$ is monotone closed in $W$.) To prove that $j_0 = \sup J$ in $W$, it is sufficient to show that $j_0$ is an upper bound of $J$. (Indeed, if $j_0$ is an upper bound of $J$, and $h$ is any other upper bound of $J$ in $W$, then $h$ also is an upper bound of $(j_n)_{n \geq 1}$. Since $j_0$ is the least upper bound of $(j_n)_{n \geq 1}$ in $W$, it follows that $h \geq j_0$. This says that $j_0 = \sup J$ in $W$.)

Let $j \in J$ be arbitrary. We have to prove that $j_0 \geq j$. For arbitrary $n \geq 1$, there exists an element $g_n$ of $J$ with $g_n \geq j_n, g_n \geq j$ by the upward direction of $J$. One then has

$$j - j_0 \leq g_n - j_0 \leq g_n - j_n.$$ 

Using that $W$ is a vector lattice, we have

$$\psi((j - j_0)_+) \leq \psi(g_n - j_n) = \psi(g_n) - \psi(j_n) \leq \sup_{j \in J} \psi(j) - \psi(j_n).$$

Since $n \geq 1$ is arbitrary, it follows with (*) that

$$\psi((j - j_0)_+) \leq \sup_{j \in J} \psi(j) - \sup_n \psi(j_n) = 0,$$

whence $(j - j_0)_+ = 0$ as $\psi$ is faithful. This implies $j - j_0 \leq 0$, or $j_0 \geq j$. □

**Definition.** Let $\phi : W_1 \to W_2$ be a positive linear map between ordered vector spaces $W_1, W_2$. If $W_1, W_2$ are monotone complete, then $\phi$ is called normal, if for each non-empty upper bounded upward directed subset $J$ of $W_1$, one has $\phi(\sup J) = \sup_{j \in J} \phi(j)$. If $W_1, W_2$ are monotone sequentially complete, then $\phi$ is called sequentially normal, if for each upper bounded increasing sequence $(j_n)$ in $W_1$, one has $\phi(\sup_n j_n) = \sup_n \phi(j_n)$.

**Corollary.** Let $\phi : A \to B$ be a positive linear map between monotone complete ordered vector spaces $A$ and $B$. Assume that $A$ is a vector lattice admitting some faithful positive linear functional. If $\phi$ is sequentially normal, then $\phi$ is normal.

**References**

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