THE MODULI SPACE OF STABLE RANK 2 PARABOLIC BUNDLES OVER AN ELLIPTIC CURVE WITH 3 MARKED POINTS

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Abstract. We explicitly describe the moduli space $M^s(X, 3)$ of stable rank 2 parabolic bundles over an elliptic curve $X$ with trivial determinant bundle and 3 marked points. Specifically, we exhibit $M^s(X, 3)$ as a blow-up of an embedded elliptic curve in $(\mathbb{C}P^1)^3$. The moduli space $M^s(X, 3)$ can also be interpreted as the $SU(2)$ character variety of the 3-punctured torus. Our description of $M^s(X, 3)$ reproduces the known Poincaré polynomial for this space.

1. Introduction

Given a curve $C$, one can define a moduli space $M^s(C, n)$ of stable rank 2 parabolic bundles over $C$ with trivial determinant bundle and $n$ marked points. The space $M^s(C, n)$ has the structure of a smooth complex manifold of dimension $3(g-1)+n$, where $g$ is the genus of the curve $C$. In general, the space $M^s(C, n)$ depends on a positive real parameter $\mu$ known as the weight. For $\mu$ sufficiently small ($\mu < 1/n$ will suffice), the space $M^s(C, n)$ is independent of $\mu$, but as $\mu$ increases it may cross critical values at which $M^s(C, n)$ undergoes certain birational transformations [2, 13]. The moduli space $M^s(C, n)$ can also be interpreted as an $SU(2)$-character variety, which is defined as the space of conjugacy classes of $SU(2)$-representations of the fundamental group of $C$ with $n$ punctures, where loops around the punctures are required to correspond to $SU(2)$-matrices conjugate to $\text{diag}(e^{2\pi i \mu}, e^{-2\pi i \mu})$. Moduli spaces of parabolic bundles on curves are natural objects of study in algebraic geometry, and also play an important role in low-dimensional topology. In particular, these spaces have a canonical symplectic structure and can be used to define Floer homology theories of links [4, 5, 6, 7].

Explicit descriptions of $M^s(C, n)$ are known for small values of $n$ and $C$ a rational or elliptic curve. For rational curves, it is well-known that for small weight we have

\begin{align*}
M^s(\mathbb{C}P^1, 0) &= M^s(\mathbb{C}P^1, 1) = M^s(\mathbb{C}P^1, 2) = \emptyset, \\
M^s(\mathbb{C}P^1, 3) &= \{pt\}, \\
M^s(\mathbb{C}P^1, 4) &= \mathbb{C}P^1 - \{3 \text{ points}\}, \\
M^s(\mathbb{C}P^1, 5) &= \mathbb{C}P^2 \# 4\mathbb{C}P^2.
\end{align*}

The structure of $M^s(\mathbb{C}P^1, 6)$ for $\mu = 1/4$, corresponding to the traceless character variety, was recently described by Kirk [9]. For an elliptic curve $X$, it is straightforward to show that for small weight we have

\begin{align*}
M^s(X, 0) &= \emptyset, & M^s(X, 1) &= \mathbb{C}P^1.
\end{align*}

It was recently shown by Vargas [15] that $M^s(X, 2)$ is the complement of an embedded elliptic curve in $(\mathbb{C}P^1)^2$.

Our goal in this paper is to explicitly describe the structure of $M^s(X, 3)$. We prove the following result:

Theorem 1.1. For small weight, the moduli space $M^s(X, 3)$ for an elliptic curve $X$ is a blow-up of an embedded elliptic curve in $(\mathbb{C}P^1)^3$.

To prove Theorem 1.1, we make use of an explicit description of Hecke modifications of rank 2 holomorphic vector bundles on elliptic curves that is derived in [3]. Roughly speaking, a Hecke modification is a way of locally modifying a vector bundle near a point to obtain a new vector bundle. The moduli space $M^s(X, 3)$ plays an important role in a conjectural Floer homology theory for links in lens spaces discussed in [3], and Theorem 1.1 was motivated by this application.

The paper is organized as follows. In Sections 2 and 3, we review the background material we will need on parabolic bundles and vector bundles on elliptic curves. In Section 4, we use Hecke-modification methods to explicitly describe $M^s(X, 3)$. In Section 5, we relate $M^s(X, 3)$ to $M^{\ast}(X, 2)$, the moduli space of $S$-equivalence classes of rank 2 semistable parabolic bundles with trivial determinant bundle and 2 marked points. In Section 6, we use our description of $M^s(X, 3)$ to reproduce the known Poincaré polynomial for this space.
2. PARABOLIC BUNDLES

The concept of a parabolic bundle was introduced in [10]. We will not need this concept in its full generality; rather, we will consider only parabolic bundles of a certain restricted form, which is discussed at greater length in [3, Appendix B]. For our purposes here, a rank 2 parabolic bundle over a curve \( C \) consists of a rank 2 holomorphic vector bundle \( E \) over \( C \) with trivial determinant bundle, distinct marked points \( p_1, \cdots, p_n \in C \), a line \( \ell_p \in \mathbb{P}(E_p) \) in the fiber \( E_p \) over each marked point \( p_i \), and a positive real parameter \( \mu \) known as the weight. For simplicity, we will suppress the curve \( C \) and weight \( \mu \) in the notation and denote a parabolic bundle as \( (E, \ell_{p_1}, \cdots, \ell_{p_n}) \).

In order to describe the stability properties of parabolic bundles, it is helpful to introduce some additional terminology. Recall that the degree of the proper subbundles of a vector bundle \( E \) in the fiber \( E_p \) is bounded above. Given a rank 2 holomorphic vector bundle \( E \), we say that a line \( \ell_p \in \mathbb{P}(E_p) \) is bad if there is a line subbundle \( L \) of \( E \) of maximal degree such that \( \ell_p = L_p \), and good otherwise. We say that lines \( \ell_{p_1}, \ell_{p_2}, \ell_{p_3} \in \mathbb{P}(E_{p_1}), \ell_{p_1} \in \mathbb{P}(E_{p_2}) \) are bad in the same direction if there is a line subbundle \( L \) of \( E \) of maximal degree such that \( \ell_p = L_p \) for \( i = 1, \cdots, n \).

Consider a parabolic bundle \( \mathcal{E} = (E, \ell_{p_1}, \cdots, \ell_{p_n}) \). Let \( m \) denote the maximum number of lines of \( \mathcal{E} \) that are bad in the same direction. For sufficiently small weight \( \mu < 1/n \) will suffice, we can characterize the stability and semistability of \( \mathcal{E} \) as follows. If \( E \) is unstable, then \( \mathcal{E} \) is unstable. If \( E \) is semistable, then \( \mathcal{E} \) is stable if \( m < n/2 \), semistable if \( m \leq n/2 \), and unstable if \( m > n/2 \). Note that if \( n \) is odd then stability and semistability are equivalent.

We define a moduli space \( M^s(C, n) \) of isomorphism classes of stable parabolic bundles with \( n \) marked points. As with vector bundles, one can define a notion of \( S \)-equivalent semistable parabolic bundles, and we define a moduli space \( M^{ss}(C, n) \) of \( S \)-equivalence classes of semistable parabolic bundles. For odd \( n \) we have that \( M^s(C, n) = M^{ss}(C, n) \). For even \( n \) we have that \( M^s(C, n) \) is an open subset of \( M^{ss}(C, n) \).

3. VECTOR BUNDLES ON ELLIPTIC CURVES

Vector bundles on elliptic curves were classified by Atiyah [1], and are well understood. Here we briefly summarize the results regarding vector bundles on elliptic curves that we will need; these results are either well-known (see for example [1, 8, 12, 14]) or derived in [3, Section 5].

3.1. Line bundles. Isomorphism classes of degree 0 line bundles on an elliptic curve \( X \) are parameterized by the Jacobian \( \text{Jac}(X) \). Given a basepoint \( e \in X \), we define the Abel-Jacobi isomorphism \( X \to \text{Jac}(X) \), \( p \mapsto [\mathcal{O}(p-e)] \). Given points \( p, e \in X \), we define a translation map \( \tau_{p,e} : \text{Jac}(X) \to \text{Jac}(X) \), \( [L] \mapsto [L \otimes \mathcal{O}(p-e)] \). We say that a line bundle \( L \) is 2-torsion if \( L^2 = \mathcal{O} \). There are four 2-torsion line bundles, which we denote \( L_i \) for \( i = 1, 2, 3, 4 \).

3.2. Semistable rank 2 vector bundles. For our purposes here, we need only consider semistable rank 2 vector bundles on an elliptic curve \( X \) with trivial determinant bundle. There are three classes of such bundles.

First, we have vector bundles of the form \( E = L \oplus L^{-1} \), where \( L \) is a degree 0 line bundle such that \( L^2 \neq \mathcal{O} \). There are two bad lines \( L_p, (L^{-1})_p \in \mathbb{P}(E_p) \) in the fiber \( E_p \) over any point \( p \in X \), and all other lines in \( \mathbb{P}(E_p) \) are good. The automorphism group \( \text{Aut}(E) \) of \( E \) consists of \( GL(2, \mathbb{C}) \) matrices of the form

\[
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}.
\]

Each bad line \( L_p, (L^{-1})_p \in \mathbb{P}(E_p) \) is fixed by the automorphisms of \( E \), and there is a unique (up to rescaling by a constant) automorphism carrying any good line \( \ell_p \in \mathbb{P}(E_p) \) to any other good line \( \ell'_p \in \mathbb{P}(E_p) \).

Second, we have four vector bundles of the form \( E = L_i \oplus L_i \), where \( L_i \) is a 2-torsion line bundle. All lines \( \ell_p \in \mathbb{P}(E_p) \) in the fiber \( E_p \) over any point \( p \in X \) are bad. The automorphism group \( \text{Aut}(E) \) of \( E \) is \( GL(2, \mathbb{C}) \), and there is a unique (up to rescaling by a constant) automorphism carrying any triple of lines \( (\ell_{p_1}, \ell_{p_2}, \ell_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3}) \) such that no two lines are bad in the same direction to any other triple of lines \( (\ell'_{p_1}, \ell'_{p_2}, \ell'_{p_3}) \in \mathbb{P}(E_{p_1}) \times \mathbb{P}(E_{p_2}) \times \mathbb{P}(E_{p_3}) \) such that no two lines are bad in the same direction.

Third, we have four vector bundles of the form \( E = F_2 \oplus L_i \), where \( L_i \) is a 2-torsion line bundle and \( F_2 \) is the unique non-split extension of \( \mathcal{O} \) by \( \mathcal{O} \):

\[
0 \longrightarrow \mathcal{O} \longrightarrow F_2 \longrightarrow \mathcal{O} \longrightarrow 0.
\]

There is a unique bad \( (L_i)_p \in \mathbb{P}(E_p) \) in the fiber \( E_p \) over any point \( p \in X \), and all other lines in \( \mathbb{P}(E_p) \) are good. The automorphism group \( \text{Aut}(E) \) of \( E \) consists of \( GL(2, \mathbb{C}) \) matrices of the form

\[
\begin{pmatrix}
A & B \\
0 & A
\end{pmatrix}.
\]

The bad line \( (L_i)_p \in \mathbb{P}(E_p) \) is fixed by the automorphisms of \( E \), and there is a unique (up to rescaling by a constant) automorphism carrying any good line \( \ell_p \in \mathbb{P}(E_p) \) to any other good line \( \ell'_p \in \mathbb{P}(E_p) \).
We define \( M^{ss}(X) \) to be the moduli space of \( S \)-equivalence classes of semistable rank 2 vector bundles on \( X \) with trivial determinant bundle. In [14] it is shown that \( M^{ss}(X) \) is isomorphic to \( \mathbb{CP}^1 \), as can be understood as follows. The above classification shows that we can parameterize semistable rank 2 vector bundles with trivial determinant bundle as \( L \oplus L^{-1} \) for \( [L] \in \text{Jac}(X) \), together with the four bundles \( F_2 \otimes L_i \). The bundles \( L \oplus L^{-1} \) and \( L^{-1} \oplus L \) are isomorphic, hence \( S \)-equivalent, and are thus identified in \( M^{ss}(X) \). One can show that the bundles \( F_2 \otimes L_i \) and \( L_i \oplus L_i \) are \( S \)-equivalent, and are thus identified in \( M^{ss}(X) \). It follows that \( M^s(X) \) is the quotient of \( \text{Jac}(X) \) by the involution \( [L] \mapsto [L^{-1}] \), which yields a space known as the pillowcase that is isomorphic to \( \mathbb{CP}^1 \). We define a map \( p : \text{Jac}(X) \rightarrow M^{ss}(X) \), \( [L] \mapsto [L \oplus L^{-1}] \), which is a branched double-cover with four branch points \( p([L_i]) \in M^{ss}(X) \) corresponding to four ramification points \( [L_i] \in \text{Jac}(X) \) that are fixed by the involution \( [L] \mapsto [L^{-1}] \).

3.3. Hecke modifications of rank 2 vector bundles. Given a rank 2 vector bundle \( E \) over a curve \( C \), distinct points \( p_1, \ldots, p_n \in C \), and lines \( \ell_p \in \mathbb{P}(E_{p_i}) \) for each point \( p_i \), one can perform a Hecke modification of \( E \) at each point \( p_i \) using data provided by the line \( \ell_p \), so as to obtain a new vector bundle that we will denote \( H(E, \ell_{p_1}, \ldots, \ell_{p_n}) \). One way to describe \( H(E, \ell_{p_1}, \ldots, \ell_{p_n}) \) is as follows. Let \( \mathcal{E} \) denote the sheaf of sections of \( E \), and define a subsheaf \( \mathcal{F} \) of \( \mathcal{E} \) whose set of sections over an open subset \( U \) of \( X \) is given by

\[
\mathcal{F}(U) = \{ s \in \mathcal{E}(U) \mid p_i \in U \implies s(p_i) \in \ell_{p_i} \text{ for } i = 1, \ldots, n \}.
\]

We define \( \mathcal{H}(E, \ell_{p_1}, \ldots, \ell_{p_n}) \) to be the vector bundle whose sheaf of sections is \( \mathcal{F} \).

Hecke modifications of rank 2 vector bundles on elliptic curves are described explicitly in [3]. For our purposes here, all the results we will need can be described in terms of a few properties of a certain map \( h_e \), which we define as follows. Given a semistable rank 2 vector bundle \( E \) with trivial determinant bundle over an elliptic curve \( C \), distinct points \( p, q, e \in X \) such that \( p + q = 2e \), and lines \( \ell_p \in \mathbb{P}(E_p) \) and \( \ell_q \in \mathbb{P}(E_q) \) that are not bad in the same direction, we define \( h_e(E, \ell_p, \ell_q) \in M^{ss}(X) \) as

\[
h_e(E, \ell_p, \ell_q) = [H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e)].
\]

One can show that if the lines \( \ell_p \) and \( \ell_q \) are not bad in the same direction, then \( H(E, \ell_p, \ell_q) \otimes \mathcal{O}(e) \) is in fact a semistable vector bundle with trivial determinant bundle and thus represents a point in \( M^{ss}(X) \). It is clear that the order of the lines doesn’t matter in the definition of \( H(E, \ell_p, \ell_q) \), so

\[
h_e(E, \ell_p, \ell_q) = h_e(E, \ell_p, \ell_q).
\]

If \( (E, \ell_p, \ell_q) \) and \( (E', \ell'_p, \ell'_q) \) are isomorphic parabolic bundles, one can show that

\[
h_e(E, \ell_p, \ell_q) = h_e(E', \ell'_p, \ell'_q).
\]

In particular, for \( \phi \in \text{Aut}(E) \) we have

\[
h_e(E, \ell_p, \ell_q) = h_e(E, \phi(\ell_p), \phi(\ell_q)).
\]

If the line \( \ell_p \) is good and \( E \neq L \oplus L_i \), then we have the following result:

**Theorem 3.1** ([3, Lemma 5.26]). If \( E = L \oplus L^{-1} \) for \( L \neq \mathcal{O} \) or \( E = F_2 \otimes L_i \), and \( \ell_p \in \mathbb{P}(E_p) \) is a good line, then the map \( \mathbb{P}(E_p) \rightarrow M^{ss}(X) \), \( \ell_q \rightarrow h_e(E, \ell_p, \ell_q) \) is an isomorphism.

If the line \( \ell_p \) is bad, then \( h_e(E, \ell_p, \ell_q) \) is uniquely determined by \( E \) and the points \( p \) and \( e \):

**Theorem 3.2** ([3, Lemma 5.27]). If \( \ell_p \in \mathbb{P}(E_p) \) is a bad line and \( \ell_q \in \mathbb{P}(E_q) \) is not bad in the same direction as \( \ell_p \), then \( h_e(E, \ell_p, \ell_q) \) is given by

\[
\begin{align*}
&h_e(L \oplus L^{-1}, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L]), \\
&h_e(F_2 \otimes L_i, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L_i]).
\end{align*}
\]

Note that \( (p \circ \tau_{p-e})([L]) = (p \circ \tau_{e-p})([L]) \). Note also that in Theorem 3.2 the line \( \ell_q \in \mathbb{P}(E_q) \) is allowed to be bad, not just not bad in the same direction as \( \ell_p \in \mathbb{P}(E_p) \). For example, we have

\[
\begin{align*}
&h_e(L \oplus L^{-1}, \ell_p, \ell_q) = (p \circ \tau_{p-e})([L]) = (p \circ \tau_{e-q})([L]).
\end{align*}
\]

4. Description of \( M^s(X, 3) \)

We consider here the moduli space \( M^s(X, 3) \) of stable rank 2 parabolic bundles over an elliptic curve \( X \) with trivial determinant bundle and 3 marked points \( p_1, p_2, p_3 \in X \). If \( [E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3) \), then \( E \) is semistable and no two of the lines \( \ell_{p_1}, \ell_{p_2}, \ell_{p_3} \) are bad in the same direction. It follows that \( E \) has one of the three forms described in Section 3.2; that is, \( E = L \oplus L^{-1} \) for \( L \neq \mathcal{O} \), \( E = L_i \oplus L_i \), or \( E = F_2 \otimes L_i \). Choose points \( e_1, e_2, e_3 \in X \) such that

\[
p_1 + p_2 = 2e_3, \quad p_3 + p_1 = 2e_2, \quad p_2 + p_3 = 2e_1.
\]
Define a map \( \pi = (\pi_1, \pi_2, \pi_3) : M^s(X, 3) \to (M^{ss}(X))^3 \) by
\[
\pi([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = ([E], h_{e_2}(E, \ell_{p_1}, \ell_{p_3}), h_{e_1}(E, \ell_{p_2}, \ell_{p_3})).
\]
Note that since \( E \) is semistable and no two of the lines \( \ell_{p_1}, \ell_{p_2}, \ell_{p_3} \) are bad in the same direction, we can in fact define \( h_{e_2}(E, \ell_{p_1}, \ell_{p_3}) \) and \( h_{e_1}(E, \ell_{p_2}, \ell_{p_3}) \). It is useful to decompose \( M^s(X, 3) \) as
\[
M^s(X, 3) = \{ \ell_{p_3} \text{ good} \} \cup \{ \ell_{p_3} \text{ bad} \},
\]
where the open submanifold \( \{ \ell_{p_3} \text{ good} \} \) and the closed submanifold \( \{ \ell_{p_3} \text{ bad} \} \) consist of points \([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in M^s(X, 3)\) for which \( \ell_{p_3} \) is a good and bad line, respectively. The open submanifold \( \{ \ell_{p_3} \text{ good} \} \) is described by the following result:

**Theorem 4.1.** The restriction of \( \pi : M^s(X, 3) \to (M^{ss}(X))^3 \) to \( \{ \ell_{p_3} \text{ good} \} \to \pi(\{ \ell_{p_3} \text{ good} \}) \) is an isomorphism.

**Proof.** If \([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{ \ell_{p_3} \text{ good} \}\), then \( E \) must have good lines, hence \( E = L \oplus L^{-1} \) for \( L^2 \neq 0 \) or \( E = F_2 \otimes L_i \). For each point \([E] \in M^{ss}(X)\), choose a representative \( E \) of \([E]\) and a good line \( \ell'_{p_3} \in \mathbb{P}(E_{p_3}) \). We can define a map
\[
\pi^{-1}_1([E]) \cap \{ \ell_{p_3} \text{ good} \} \to \mathbb{P}(E_{p_3}) \times \mathbb{P}(E_{p_2}),
\]
\[
[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto (\phi(\ell_{p_1}), \phi(\ell_{p_2}))
\]
where \( \phi \) is the unique (up to rescaling by a constant) automorphism of \( E \) such that \( \phi(\ell_{p_3}) = \ell'_{p_3} \). This map is an isomorphism onto its image, hence by Theorem 3.1 the map \( (\pi_2, \pi_3) : \pi^{-1}_1([E]) \cap \{ \ell_{p_3} \text{ good} \} \to (M^{ss}(X))^2 \) is an isomorphism onto its image. \( \square \)

Next we consider the closed submanifold \( \{ \ell_{p_3} \text{ bad} \} \). Elements of \( \{ \ell_{p_3} \text{ bad} \} \) have one of three forms:
\[
[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}], \quad [F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}], \quad [L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}],
\]
where \( L^2 \neq 0 \) and \( L_i \) is a 2-torsion line bundle. Note that elements of the form \([L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]\) can be converted into the first of the three listed forms by applying the isomorphism \( \phi : L \oplus L^{-1} \to L^{-1} \oplus L \):
\[
[L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}] = [L^{-1} \oplus L, \phi(\ell_{p_1}), \phi(\ell_{p_2}), (L_i)_{p_3}] = [M \oplus M^{-1}, \phi(\ell_{p_1}), \phi(\ell_{p_2}), M_{p_3}],
\]
where we have defined \( M = L^{-1} \) and used the fact that \( \phi((L_i)_{p_3}) = (L_i)_{p_3} \).

Recall that we defined a map \( \pi_1 : M^s(X, 3) \to M^{ss}(X), [E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto [E] \). We can lift \( \pi_1 : \{ \ell_{p_3} \text{ bad} \} \to M^{ss}(X) \) to the branched double-cover \( p : \text{Jac}(X) \to M^{ss}(X) \) by using the bad line \( \ell_{p_3} \) to distinguish between distinct vector bundles \( L \oplus L^{-1} \) and \( L^{-1} \oplus L \) that are identified in \( M^{ss}(X) \):

\[
\xymatrix{ \text{Jac}(X) \ar[rd]^p \ar[rr]^{\pi_1} & & \{ \ell_{p_3} \text{ bad} \} \ar[ll]_{\tilde{\pi}_1} \ar[rr]^\pi & & (M^{ss}(X))^3, }
\]
where \( \tilde{\pi}_1 : \{ \ell_{p_3} \text{ bad} \} \to \text{Jac}(X) \) is defined such that
\[
\tilde{\pi}_1([L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = [L], \quad \tilde{\pi}_1([F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \pi_1([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = [L_i].
\]
Define a map \( f : \text{Jac}(X) \to (M^{ss}(X))^3, f = (p, p \circ \tau_{p_3-e_2}, p \circ \tau_{p_3-e_1}) \).

**Theorem 4.2.** We have a commutative diagram
\[
\xymatrix{ \{ \ell_{p_3} \text{ bad} \} \ar[rr]_{\tilde{\pi}_1} & & \text{Jac}(X) \ar[rr]^f & & (M^{ss}(X))^3. }
\]

**Proof.** The fiber of \( \tilde{\pi}_1 \) over a point \([L] \in \text{Jac}(X)\) such that \( L^2 \neq 0 \) is
\[
\pi^{-1}_1([L]) = \{ [L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}] \}.
\]
From Theorem 3.2 it follows that \( \pi(\pi^{-1}_1([L])) = f([L]). \) The fiber of \( \tilde{\pi}_1 \) over the point \([L_i] \in \text{Jac}(X)\) is
\[
\tilde{\pi}_1^{-1}([L_i]) = \{ [F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}] \} \cup \{ [L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \}.
\]
From Theorem 3.2 it follows that
\[
\pi([F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, (L_i)_{p_3}]) = \pi([L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = f([L_i]).
\]
Thus \( \pi(\tilde{\pi}_1^{-1}([L_i])) = f([L_i]). \) \( \square \)
Theorem 4.3. The map $f : \text{Jac}(X) \to (M^{ss}(X))^3$ is injective.

Proof. Take $[L], [M] \in \text{Jac}(X)$ such that $f([L]) = f([M])$. Projecting onto the first factor of $(M^{ss}(X))^3$, we find that either $[M] = [L]$ or $[M] = [L^{-1}]$. If $[M] = [L^{-1}]$, then projecting onto the second factor of $(M^{ss}(X))^3$ gives either $[L \otimes \mathcal{O}(p_3 - e_2)] = [L^{-1} \otimes \mathcal{O}(p_3 - e_2)]$ or $[L \otimes \mathcal{O}(p_3 - e_2)] = [L \otimes \mathcal{O}(e_2 - p_3)]$. In the first case $[L] = [L^{-1}]$. The second case cannot actually occur, since otherwise $2p_3 = 2e_2 = p_3 + p_1$ and thus $p_1 = p_3$, contradiction. □

Given a point $[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \in \{\ell_{p_3} \text{ bad}\}$, we have that $E$ is semistable and $\ell_{p_1}$ and $\ell_{p_2}$ cannot be bad in the same direction, hence we can define a map $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$,

$$h([E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}]) = h_{e_3}(E, \ell_{p_1}, \ell_{p_2}).$$

Theorem 4.4. The map $(\tilde{\pi}_1, h) : \{\ell_{p_3} \text{ bad}\} \to \text{Jac}(X) \times M^{ss}(X)$ is an isomorphism.

Proof. The fiber of $\tilde{\pi}_1$ over a point $[L] \in \text{Jac}(X)$ such that $L^2 \neq \mathcal{O}$ is

$$\tilde{\pi}_1^{-1}([L]) = \{(L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3})\} \to \mathbb{C}P^1,$$

We will argue that $\tilde{\pi}_1^{-1}([L])$ is isomorphic to $\mathbb{C}P^1$. Choose a local trivialization of $E := L \oplus L^{-1}$ over an open set containing $p_1$ and $p_2$ so as to obtain identifications $\psi_i : \mathbb{P}(E_{p_i}) \to \mathbb{C}P^1$ for $i = 1, 2$. We can choose the local trivialization such that $\psi_i(L_{p_i}) = \infty$ and $\psi_i((L^{-1})_{p_i}) = 0$. Define $z_i = \psi_i(\ell_{p_i})$ and note that $(z_1, z_2) \in \mathbb{C}^2 - \{(0, 0)\}$. An automorphism of $E$ induces the transformation $(z_1, z_2) \mapsto a(z_1, z_2)$ for $a \in \mathbb{C}^*$, hence we have an isomorphism

$$\tilde{\pi}_1^{-1}([L]) = \{(L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3})\} \to \mathbb{C}P^1,$$

A canonical version of this statement is that the restriction of $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to $\tilde{\pi}_1^{-1}([L])$ gives an isomorphism $\tilde{\pi}_1^{-1}([L]) \to M^{ss}(X)$. In particular, from Theorems 3.1 and 3.2 it follows that

$$h([L \oplus L^{-1}, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = \{(p \circ \tau_{e_3-p_1})([L]), (p \circ \tau_{e_3-p_2})([L])\},$$

The fiber of $\tilde{\pi}_1$ over the point $[L_i] \in \text{Jac}(X)$ is

$$\tilde{\pi}_1^{-1}([L_i]) = \{(F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}) \cup [L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}]\}. $$

Note that $\{(L_i + L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3})\}$ consists of a single point, since there is a unique (up to rescaling by a constant) automorphism of $L_i \oplus L_i$ that induces an isomorphism of any pair of stable parabolic bundles $(L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3})$ and $(L_i \oplus L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3})$. We will argue that $\{(F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3})\}$ is isomorphic to $\mathbb{C}$. Choose a local trivialization of $E := F_2 \otimes L_i$ over an open set containing $p_1$ and $p_2$ so as to obtain identifications $\psi_i : \mathbb{P}(E_{p_i}) \to \mathbb{C}P^1$ for $i = 1, 2$. We can choose the local trivialization such that $\psi_i(L_{p_i}) = \infty$ and $\psi_i((L^{-1})_{p_i}) = 0$. Define $z_i = \psi_i(\ell_{p_i})$ and note that $(z_1, z_2) \in \mathbb{C}^2$. An automorphism of $E$ induces the transformation $(z_1, z_2) \mapsto (z_1 + b, z_2 + b)$ for $b \in \mathbb{C}$, hence we have an isomorphism

$$\tilde{\pi}_1^{-1}([L_i]) \to \mathbb{C}, $$

A canonical version of these results is that the restriction of $h : \{\ell_{p_3} \text{ bad}\} \to M^{ss}(X)$ to $\tilde{\pi}_1^{-1}([L_i])$ gives an isomorphism $\tilde{\pi}_1^{-1}([L_i]) \to M^{ss}(X)$. In particular, from Theorems 3.1 and 3.2 it follows that

$$h([F_2 \otimes L_i, \ell_{p_1}, \ell_{p_2}, L_{p_3}]) = \{(p \circ \tau_{p_1-e_3})([L_i])\},$$

Note that $(p \circ \tau_{p_1-e_3})([L_i]) = (p \circ \tau_{e_3-p_1})([L_i]) = (p \circ \tau_{e_3-p_2})([L_i]). □$

Theorems 4.1–4.4 prove Theorem 1.1 from the Introduction.

5. Relationship between $M^s(X, 3)$ and $M^{ss}(X, 2)$

In [15] it is shown that $M^{ss}(X, 2)$ is isomorphic to $(\mathbb{C}P^1)^2$. From our perspective, we can describe this result by defining a map $M^{ss}(X, 2) \to (M^{ss}(X))^2$,

$$[E, \ell_{p_1}, \ell_{p_2}] \mapsto ([E], h_{e_3}(E, \ell_{p_1}, \ell_{p_2})).$$

One can show that this map is an isomorphism.

We can relate the closed subset $\{\ell_{p_3} \text{ bad}\}$ of $M^s(X, 3)$ to the moduli space $M^{ss}(X, 2)$ as follows. Define a map $\{\ell_{p_3} \text{ bad}\} \to M^{ss}(X, 2)$,

$$[E, \ell_{p_1}, \ell_{p_2}, \ell_{p_3}] \mapsto [E, \ell_{p_1}, \ell_{p_2}].$$
We have a commutative diagram

\[
\begin{array}{c}
\ell_{p_3} \text{ bad} \\
\downarrow \tilde{\pi}_1 \\
\text{Jac}(X) \xrightarrow{p} M^{ss}(X),
\end{array}
\]

where we have defined a map $M^{ss}(X, 2) \to M^{ss}(X)$, $[E, \ell_{p_1}, \ell_{p_2}] \mapsto [E]$.

6. Poincaré polynomial of $M^s(X, 3)$

The Poincaré polynomial of $M^s(C, n)$ is given in [11, Theorem 3.8] for the case $\mu = 1/4$, corresponding to the traceless character variety, and $n$ odd:

\[
P_t(M^s(C, n)) = \frac{(1+t^2)^n(1+t^3)^{2g} - 2^{n-1}t^{2g+n-1}(1+t)^{2g}(1+t^2)}{(1-t^2)(1-t^4)},
\]

where $g$ is the genus of $C$. In fact, the results of [11] are stated for parabolic bundles with fixed determinant bundle of odd degree, but since $\mu = 1/4$, corresponding to a traceless character variety, the results also hold for the moduli space $M^s(C, n)$ for which the determinant bundle of the parabolic bundles is trivial. For an elliptic curve $X$ with $3$ marked points, equation (1) gives

\[
P_t(M^s(X, 3)) = 1 + 4t^2 + 2t^3 + 4t^4 + t^6.
\]

We can reproduce equation (2) using our explicit description of $M^s(X, 3)$. Since $\pi : M^s(X, 3) \to (M^{ss}(X))^3$ restricts to an isomorphism $M^s(X, 3) \setminus \{\ell_{p_3} \text{ bad}\} \to (M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})$, we obtain the following equation for the Poincaré polynomials for cohomology with compact supports:

\[
P_t(M^s(X, 3) \setminus \{\ell_{p_3} \text{ bad}\}) = P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})).
\]

From the long exact sequence for cohomology with compact supports, we have

\[
P_t(M^s(X, 3) \setminus \{\ell_{p_3} \text{ bad}\}) = P_t(M^s(X, 3)) - P_t(\{\ell_{p_3} \text{ bad}\}),
\]

\[
P_t((M^{ss}(X))^3 - \pi(\{\ell_{p_3} \text{ bad}\})) = P_t((M^{ss}(X))^3) - P_t(\pi(\{\ell_{p_3} \text{ bad}\})).
\]

We have that $M^{ss}(X)$ is isomorphic to $\mathbb{C}\mathbb{P}^1$, $\pi(\{\ell_{p_3} \text{ bad}\})$ isomorphic to $	ext{Jac}(X)$, and $\{\ell_{p_3} \text{ bad}\}$ is isomorphic to $\text{Jac}(X) \times M^{ss}(X)$, so

\[
P_t(M^{ss}(X))^3 = (1 + t^2)^3, \quad P_t(\pi(\{\ell_{p_3} \text{ bad}\})) = 1 + 2t + t^2, \quad P_t(\{\ell_{p_3} \text{ bad}\}) = (1 + 2t + t^2)(1 + t^2).
\]

Combining equations (3)–(6), we reproduce equation (2) for $P_t(M^s(X, 3))$.

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