Matrix $\sigma$-models for Multi D-brane Dynamics

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Abstract

We describe a dynamical worldsheet origin for the Lagrangian describing the low-energy dynamics of a system of parallel D-branes. We show how matrix-valued collective coordinate fields for the D-branes naturally arise as couplings of a worldsheet $\sigma$-model, and that the quantum dynamics require that these couplings be mutually noncommutative. We show that the low-energy effective action for the $\sigma$-model couplings describes the propagation of an open string in the background of the multiple D-brane configuration, in which all string interactions between the constituent branes are integrated out and the genus expansion is taken into account, with a matrix-valued coupling. The effective field theory is governed by the non-abelian Born-Infeld target space action which leads to the standard one for D-brane field theory.

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1. Introduction

With the advent of Dirichlet-brane field theory, introduced by Witten [1] and elucidated on by Taylor [2], there has been a lot of activity in describing the short-distance spacetime structure of string theory (see for example [3]). The D-brane action is obtained from dimensional reduction, to the world-volume of the D-brane, of ten-dimensional supersymmetric $U(N)$ Yang-Mills theory which describes the low-energy dynamics of open superstrings with the usual Neumann boundary conditions. The target space coordinate fields are now $N \times N$ matrices and hence describe some sort of noncommutative spacetime geometry, giving an explicit realization of the idea that at small distances the conventional notion of spacetime must be abandoned.

The supersymmetric $N \times N$ matrix quantum mechanics for D-particles with action

$$S_M = \frac{1}{4\pi \alpha'} \int dt \text{tr} \left( \left( D_t Y^i \right)^2 - \frac{g_s}{2} [Y^i, Y^j]^2 + 2\psi^T D_t \psi - 2\sqrt{g_s} \psi^T \gamma_i \left[ \psi, Y^i \right] \right)$$

(1.1)

has been of particular interest recently, in light of the conjecture [4] that in the large-$N$ limit it provides a Hamiltonian formalism for the low-energy dynamics of M Theory in the infinite momentum frame of the 11-dimensional spacetime. The action (1.1) is the reduction of 10-dimensional supersymmetric Yang-Mills theory down to $(0 + 1)$-dimensions, with $D_t = \partial_t - i[A_0, \cdot]$. The fields $Y^i(t)$, $i = 1, \ldots, 9$, are $N \times N$ Hermitian matrices in the adjoint representation of $U(N)$ describing the collective coordinates of a system of $N$ parallel D0-branes (with infinitesimal separation), and $\psi$ are their superpartners. The trace in (1.1) is taken in the fundamental representation of $U(N)$, $1/2\pi \alpha'$ is the string tension, and $g_s$ is the (closed) string coupling constant. The $N$ D-particles play the role of partons in the light-cone theory and the large-$N$ limit naturally incorporates the quantum field theoretical Fock space into the model. Among other things, the gauge symmetry group $U(N)$ contains the permutation symmetry $S_N$ of $N$ identical particles as its Weyl subgroup, and so the action (1.1) naturally describes a second quantized theory from the onset.

The D-brane field theory is canonically obtained from the target space low-energy dynamics of open superstrings, and the string coupling $g_s$ originates from the ten-dimensional Yang-Mills coupling constant. In this letter we will describe how to derive (1.1) from a worldsheet $\sigma$-model for a system of $N$ D-branes interacting via the exchange of fundamental strings. We show how the matrix-valued D-brane configuration fields $Y^i$ arise as couplings in such a $\sigma$-model. Starting with a configuration where the D-branes lie on top of each other, we demonstrate that the spacetime coordinates are necessarily described by matrices. We then demonstrate that the quantum dynamics of the $N$ D-brane system imply that the D-branes must acquire an infinitesimal separation. The topological expansion of the $\sigma$-model induces independent off-diagonal degrees of freedom describing this separation, and it also provides a geometrical and dynamical origin for the string coupling constant in the low-energy effective target space action. This latter action, which
is shown to be equivalent to (1.1) in the strong tension limit, describes the propagation (in $\sigma$-model coupling constant space) of a single fundamental string in the background of a “fat brane”, i.e. the multi D-brane system with all string interactions between the constituent branes integrated out and with a matrix-valued coupling to the open string. These results exhibit a worldsheet origin for the appearance of matrices as coordinate fields at short distance scales, and it may have implications for the recent attempts [5, 6] at a geometrical interpretation of (1.1) as a sum over random surfaces appropriate to the perturbation expansion of string theory.

2. Worldsheet $\sigma$-models and the Emergence of Matrices

We start by showing how a worldsheet $\sigma$-model construction yields a natural realization of one of the central ideas implied by Matrix Theory [1, 4] – that the spacetime induced by coincident D-branes is described by noncommuting matrix-valued coordinates. Let us first describe the action for a string in the background of a single D0-brane within the $\sigma$-model approach. The open string is described by a worldsheet $\Sigma$ which at tree-level we usually take to have the topology of a disc. The endpoints of the string are fixed at the same point in spacetime, the coordinates of the D0-brane. The collective excitations of the D0-brane are described by the worldsheet embedding fields $x^i(z, \bar{z}) : \Sigma \rightarrow \mathbb{R}^9$, $i = 1, \ldots, 9$, obeying Dirichlet boundary conditions on $\partial \Sigma$ while the target space temporal coordinate $x^0(z, \bar{z})$ satisfies the standard Neumann boundary conditions. The spatial embedding fields of $\Sigma$ are therefore constant on its boundary while the temporal fields vary along $\partial \Sigma$,

$$x^i(z, \bar{z})|_{\partial \Sigma} = y^i(x^0), \quad i = 1, \ldots, 9; \quad \partial_\perp x^0(z, \bar{z})|_{\partial \Sigma} = 0 \quad (2.1)$$

where $y^i(x^0)$ are the collective coordinates of the D0-brane and $\partial_\perp$ denotes the normal derivative to the boundary of $\Sigma$. So the surface $\Sigma$ is effectively a sphere with a marked point on it since the field $x$ is constant on the boundary. This means that an open string with both of its ends resting on a single 0-brane is effectively a closed string forced to pass through a fixed point in spacetime.

The action for the string propagation is defined by the $\sigma$-model

$$S_1 = \frac{1}{4\pi \alpha'} \int_\Sigma \eta_{\mu\nu} \partial x^\mu \bar{\partial} x^\nu + \oint_{\partial \Sigma} y^i(x^0) \partial_\perp x^i \quad (2.2)$$

where $\eta_{\mu\nu}$ is a (critical) flat Minkowski spacetime metric. The worldsheet boundary operator in (2.2) describes the excitation of the D-particle and it exploits the fact [1, 4]

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1In this letter we shall, for definiteness, discuss the case of D-particles, but the analysis and results trivially generalize to D-instantons and extended D-branes as well.

2In the following we will use Latin letters to denote spatial indices and Greek letters for spacetime indices in the target space, which we assume for simplicity is a flat Minkowski spacetime.
that the D-brane configurations \( y^i \) couple to the boundary vertex operator \( \partial_\perp x^i|_{\partial \Sigma} \). In this action formalism, appropriate for the computation of vacuum amplitudes that we shall present, the Dirichlet boundary conditions are \( x^i(z, \bar{z})|_{\partial \Sigma} = 0 \) so that the boundary couplings \( y^i \) depend only on \( x^0 \). Using the alternative approach of implementing the non-zero boundary conditions in (2.1) would set the \( y^i \)-coupling in (2.2) to 0 [8]. The two approaches are related in a simple fashion at tree-level of the semi-classical expansion. With the boundary conditions (2.1), the shift \( x^i \rightarrow x^i + y^i \) sets \( x^i|_{\partial \Sigma} = 0 \) and the leading order expansion of the bulk part of the action (2.2) induces the boundary term there. In general the transformation is more complicated and involves a redefinition of the background fields [8]. For the time being, we suppress, for ease of notation, the explicit boundary operator in (2.2) which corresponds to the Neumann boundary condition on \( x^0 \) in (2.1).

For a heavy non-relativistic D-particle we can write

\[
y^i(x^0) = y^i(0) + u^i x^0
\]

(2.3)

where \( u^i \) is the constant velocity of the D-brane. Upon T-dualization (see section 4), the configurations \( y^i(x^0) = -2\pi \alpha' A^i(x^0) \) form the components of a dimensionally-reduced background \( U(1) \) gauge field \( A \), and integrating out the embedding fields \( x^\mu \) gives, to lowest order in the derivative expansion, the Born-Infeld effective action [9, 10]

\[
\Gamma_{BI} = \frac{1}{g_s} \int dt \sqrt{1 - (2\pi \alpha' F_0^0)^2} = \frac{1}{g_s} \int dt \sqrt{1 - (\dot{y}^i)^2}
\]

(2.4)

with \( t \) the constant mode of the temporal embedding field \( x^0 \). The action (2.4) is just the relativistic free particle action for the D0-brane. The beta-function equations for the renormalization group flow associated with the worldsheet ultraviolet scale can be interpreted as the classical equations of motion of the D-brane [8]. The quantization of (2.4) is obtained by summing over all worldsheet topologies in the pinched approximation as described in [11]. In the following we shall derive the generalization of (2.4) to the case of a multi D-particle system.

In the presence of two D-particles at positions \( y^{(1)i} \) and \( y^{(2)i} \) there are excitations of the strings whose two endpoints are connected to a single D-brane. These degrees of freedom are described by the fields \( x^{(a)\mu} \), with \( a = 1, 2 \), defined on two (identical) worldsheets \( \Sigma^{(a)} \). In addition there are excitations corresponding to strings stretching between the two D-branes, which are described by a field \( x^{(12)\mu} \) defined on a worldsheet which is an annulus \( \Sigma^{(12)} \). The fields \( x^{(12)i} \) have constant values \( y^{(1)i} \) and \( y^{(2)i} \) on the two boundaries \( \partial \Sigma^{(1)} \) and \( \partial \Sigma^{(2)} \) of \( \Sigma^{(12)} \), respectively. The worldsheet \( \sigma \)-model action is a generalization of (2.2)

\[
S_{12} = \frac{1}{4\pi \alpha'} \sum_{a=1,2} \int_{\Sigma^{(a)}} \eta_{\mu\nu} \partial x^{(a)\mu} \partial x^{(a)\nu} + \frac{1}{4\pi \alpha'} \int_{\Sigma^{(12)}} \eta_{\mu\nu} \partial x^{(12)\mu} \partial x^{(12)\nu} + \sum_{a=1,2} \left( \oint_{\partial \Sigma^{(a)}} y^{(a)(x^0)} \partial_\perp x^{(a)i} + \oint_{\partial \Sigma^{(12)}} y^{(a)(x^0)} \partial_\perp x^{(12)i} \right)
\]

(2.5)
Let us now consider what happens when the two D-branes coincide, i.e. \( y^{(1)} = y^{(2)} \). In this case the values of the field \( x^{(12)} \) on the two boundaries of \( \Sigma^{(12)} \) coincide,\(^\dagger\)

\[
x^{(12)}|_{\partial \Sigma^{(12)}_1} = \zeta x^{(12)}|_{\partial \Sigma^{(12)}_2}
\]

where \( \zeta = +1 \) for unoriented and \( \zeta = -1 \) for oriented open strings. It is then possible to express (2.3) as the action for the collective excitations of a single brane but with fields which are matrix valued. We define the \( 2 \times 2 \) symmetric matrix field

\[
X^\mu = \begin{pmatrix} x^{(1)\mu} & x^{(12)\mu} \\ x^{(12)\mu} & x^{(2)\mu} \end{pmatrix}
\]

The action (2.3) then becomes formally equivalent to (2.2),

\[
S_{12} = \frac{1}{4\pi\alpha'} \int_\Sigma \text{tr} \eta_{\mu\nu} \partial X^\mu \overline{\partial} X^\nu + \oint_{\partial \Sigma} \text{tr} Y^i(x^0) \partial_{\perp} X^i
\]

where the \( 2 \times 2 \) matrix \( Y^i \) is the boundary value of the matrix \( X^i \),

\[
Y^i(x^0) = y^i(x^0) \begin{pmatrix} 1 & \frac{1}{\zeta} \\ \frac{1}{\zeta} & 1 \end{pmatrix}
\]

and the worldsheet \( \Sigma \) is a disc (equivalently a sphere with a single marked point).

Thus the above worldsheet \( \sigma \)-model point of view leads to a rather simple interpretation of the fact that, in the presence of coincident D0-branes, spacetime becomes matrix-valued. These arguments generalize in the obvious way to an arbitrary number \( N \) of coincident D0-branes. The worldsheet \( \sigma \)-model action is constructed from \( N \) embedding fields \( x^{(a)i} \), \( a = 1, \ldots, N \), describing the the collective excitations of the \( N \) D0-branes themselves, and from \( \frac{1}{2} N(N-1) \) fields \( x^{(ab)i} \), with \( 1 \leq a < b \leq N \), describing the fundamental strings exchanged between the various D0-branes. The matrix action (2.8) is then defined in terms of \( N \times N \) symmetric (for \( \zeta = +1 \)) or antisymmetric (for \( \zeta = -1 \)) matrices. It possesses a global \( O(N) \) symmetry defined by simultaneous rotations of the matrices \( X^i \) and \( Y^i \).

3. Quantum Dynamics of D-branes

The mutually commuting matrix fields (2.9) describe strings with their ends rigidly attached at the same point in space (coincident D-branes). To describe the collective quantum dynamics of the system of D-branes we must perturb \( y^{(1)} = y^{(2)} + \varepsilon \) by a parameter \( \varepsilon \rightarrow 0 \) which provides an infinitesimal separation between the D-particles. This configuration arises, as we now demonstrate, from quantum fluctuations of the system around the fixed configurations above, due to, for instance, recoil effects from the scattering of closed string states off the D-brane background \([11] - [13]\) or from the emission of closed or open string states representing interactions between the membranes. Likewise,
the endpoints of the fundamental string exchanged between the two branes should be considered to differ from \( y^{(a)j} \) by infinitesimal amounts, but to maintain the closed nature \((2.0)\) only one more degree of freedom is added to describe the boundary of the exchanged propagating string.

We consider the propagation of an open string in a single D-brane background, corresponding, for example, to the collective position coordinates \( y_i^{(1)} \). In such a process the recoil of the D-brane background due to scattering by the open string can be described by operators of the form \([12]\)

\[
C(x^{(1)}; y^{(1)}) \equiv \epsilon y_i^{(1)}(0) \Theta(x^0) \partial_{x^{(1)i}}, \quad D(x^{(1)}; y^{(1)}) \equiv u_i^{(1)} x^0 \Theta(x^0) \partial_{x^{(1)i}} \quad (3.1)
\]

where \( u_i^{(1)} \) is the recoil velocity of the brane, due to the string emission or scattering, and the infinitesimal parameter \( \epsilon \) regularizes the step function \( \Theta(x^0) \). It can be shown \([12]\) that \( \epsilon^2 \) is related to logarithmic divergences on the worldsheet via \( \log \Lambda \sim \epsilon^{-2} \), with \( \Lambda \) the (ultraviolet) renormalization group scale on \( \Sigma^{(1)} \).

The sum over pinched worldsheet genera, which are the dominant configurations in the random surface sum, leads to quantum fluctuations of the \( \sigma \)-model couplings \([13]\), in analogy with higher-dimensional wormhole calculus \([14]\). Consider the partition function of an open string \( \sigma \)-model in a recoiling D-brane background

\[
Z^{(1)} = \sum_g \int [dx^{(1)}] e^{-S_0[x^{(1)}] - \int_{\partial \Sigma^{(1)}(1)} C(x^{(1)}; y^{(1)}) - \int_{\partial \Sigma^{(1)}(1)} D(x^{(1)}; y^{(1)})} \quad (3.2)
\]

where \( S_0 \) is the free \( \sigma \)-model action and the sum is over all worldsheet genera \( g \). In the case of weakly-coupled strings the genus expansion can be truncated to a resummation over annulus graphs. Then the dominant configurations in the sum are pinched worldsheet annuli, with vanishing pinching size \( \delta \rightarrow 0 \). Such insertions \( \Delta S \) in the path integral \((3.2)\) are expressed in terms of bilocal operators \( O_i O_j \) on \( \Sigma^{(1)} \), and exponentiate to give

\[
\sum_g^{(p)} \Delta S_g = e^{\Delta S} = e^{\int_{\Sigma^{(1)}} O_i O_j g^{ij}} = \int \prod_k d\alpha_k e^{-\frac{1}{2} \alpha_i \alpha_j g^{ij}} e^{\alpha_i f_{\Sigma^{(1)}} O_i} \quad (3.3)
\]

where \( \alpha_j \) are worldsheet wormhole parameters \([14]\), and \( g^{ij} \) is an appropriate metric in parameter space. The Gaussian weight in \((3.3)\) can be thought of as a wormhole parameter space distribution function. When \((3.3)\) is inserted into the path integral \((3.2)\) one obtains a quantization \([13]\) of the \( \sigma \)-model couplings \( g^i \) of the deformation operators \( g^i f_{\Sigma^{(1)}} O_i \),

\[
g^i \rightarrow \hat{g}^i = g^i + \alpha^i \quad (3.4)
\]

and the \( \sigma \)-model coupling constants thus become operators in the target space.

In the case of the boundary conformal field theory for the recoil operators \((3.1)\), these operators carry zero conformal dimension \([15]\) which leads to \( \log \delta \) (modular) divergences in the string propagator across thin worldsheet strips. The analysis of divergences made in \([11]\) reveals that the leading \( \log \delta \) divergences can be cancelled by momentum conservation,
while subleading divergences can be absorbed by renormalizing the widths of the wormhole
distribution functions for the $y_i$ and $u_i$ couplings

$$
P_{u_i} = e^{-(u_i^{(1)} - u_i^{(1)})^2/(2\Delta u_i^{(1)})^2}, \quad P_{y_i} = e^{-(y_i^{(1)} - y_i^{(1)})^2/(2\Delta y_i^{(1)})^2} \quad (3.5)$$

The widths of the distributions are the respective quantum uncertainties

$$
(\Delta u_i^{(1)})^2 \sim g_s^2 \epsilon^2 \log \delta, \quad (\Delta y_i^{(1)})^2 \sim g_s^2 \epsilon^{-2} \log \delta \quad (3.6)
$$

One can absorb the $\delta \to 0$ infinities into a renormalization $g_s^2 = (g_s^{\text{ren}})^2 / \log \delta$ of the string coupling constant, thereby obtaining an uncertainty in the collective coordinates of the D-brane of order $(\Delta y_i^{(1)})^2 \sim (g_s^{\text{ren}})^2 \epsilon^{-2}$. For heavy branes, their BPS mass $M_D \sim 1/g_s^{\text{ren}} \to \infty$ and one may assume that the string coupling is weak enough so that $g_s^{\text{ren}} \ll \epsilon^2$ implying

$$
\Delta y_i^{(1)} \sim \epsilon \equiv g_s^{\text{ren}} \epsilon^{-1} \ll 1 \quad (3.7)
$$

Such infinitesimal fluctuations imply that $y_i^{(1)} - y_i^{(2)} \sim \epsilon$. This quantum uncertainty destroys the classical rigidity of the two overlapping branes, and promotes the distance between two quantum D-branes to a dynamical degree of freedom. This property can be extended to strongly-coupled strings (for which the above perturbative computation of the uncertainties is not strictly valid) by invoking a strong-weak coupling duality.

Thus to properly account for the sum over worldsheet topologies the boundary matrix is represented in the generic symmetric or antisymmetric form

$$
Y^i = \begin{pmatrix} y_{(1)i} & y_{(2)i} \\ \zeta y_{(12)i} & y_{(2)i} \end{pmatrix} \quad (3.8)
$$

4. T-dual Formalism

The discussion of the previous section shows that to describe the quantum dynamics of the multi D-brane system one needs to “glue” two discs $\Sigma^{(a)}$ representing the D-particles to the ends of the “worldtube” $\Sigma^{(12)}$ which is a cylinder representing the fundamental string between them. Alternatively, we glue a handle $\Sigma^{(12)}$ to the surface of a sphere with two marked points. We must then allow the two ends of the tube to fluctuate in spacetime. This can be described within a T-dual formalism \[(16)\] for the Dirichlet action in (2.5) for the exchanged string with worldsheet cylinder $\Sigma^{(12)}$,

$$
S_D[x^{(12)}] = \frac{1}{4\pi\alpha'} \int_{\Sigma^{(12)}} \eta_{\mu\nu} \partial x^{(12)\mu} \partial x^{(12)\nu} + \sum_{a=1,2} \oint_{\partial\Sigma^{(12)}} y_a^{(12)i}(x^0) \partial_\perp x^{(12)i} \quad (4.1)
$$

For this, we consider the path integral

$$
Z^{(12)} \equiv \int [dx^{(12)\mu}] \ e^{-S_D[x^{(12)}]}
$$

$$
= \int [dx^{(12)\mu}] \ [dh^i] \ [d\lambda^i] \ \exp \left\{ -\frac{1}{\pi\alpha'} \int_{\Sigma^{(12)}} \left( h_i^2 h_i^2 + h_i^2 \partial x^{(12)i} - h_i^2 \partial x^{(12)i} \right) + \frac{1}{4\pi\alpha'} \int_{\Sigma^{(12)}} \partial x^{(12)i} \partial x^{(12)i} + \sum_{a=1,2} \oint_{\partial\Sigma^{(12)}} \lambda_a^i \left( x^{(12)i} - y_a^{(12)i} \right) \right\} \quad (4.2)
$$
The fields $h^{i}_{x,z}$ implement a functional Gaussian integral transform of the spatial part of the kinetic term in (4.2). We have also exploited the fact that the string propagation is free, apart from the Dirichlet boundary conditions at its endpoints which are implemented by Lagrange multiplier fields $\lambda^{i}_{x}$. Integrating by parts over the worldsheet yields

$$Z^{(12)} = \int [dx^{(12)\mu}] [dh^{i}] [d\lambda^{i}] \exp \left\{ -\frac{1}{\pi \alpha'} \int_{\Sigma^{(12)}} \left( -\frac{i}{4} \partial x^{0} \partial x^{0} + h^{i}_{x} h^{i}_{z} - x^{(12)\mu} \left[ \partial h^{i}_{x} - \partial h^{i}_{z} \right] \right) \right\}$$

$$+ \sum_{a=1,2} \left( i \oint_{\partial \Sigma^{(12)}_{a}} \lambda^{i}_{a} \left( x^{(12)\mu} - y^{(12)\mu}_{a} \right) + \frac{1}{\pi \alpha'} \int_{0}^{1} h^{i}_{a} x^{(12)\mu} ds_{a} \right) \right\}$$

(4.3)

where $s_{a} \in [0, 1]$ parametrize the circles $\partial \Sigma^{(12)}_{a}$.

Integrating out $x^{(12)\mu}$ in the interior and $\lambda^{i}$ on the boundary of $\Sigma^{(12)}$ in (4.3) gives

$$Z^{(12)} = \int [dx^{(12)0}] [dh^{i}] \delta[\partial h^{i}_{x} - \partial h^{i}_{z}]$$

$$\times \exp \left\{ -\frac{1}{\pi \alpha'} \int_{\Sigma^{(12)}} \left( -\frac{i}{4} \partial x^{0} \partial x^{0} + h^{i}_{x} h^{i}_{z} \right) + \frac{1}{\pi \alpha'} \sum_{a=1,2} \int_{0}^{1} y^{(12)\mu}_{a} h^{i}_{a} ds_{a} \right\}$$

(4.4)

The delta-function constraint on $h^{i}$ in (4.4) implies that its Hodge decomposition over the worldsheet has the form (neglecting harmonic modes)

$$h^{i}_{z}(z, \bar{z}) = \frac{1}{2} \partial x^{(12)\mu}(z, \bar{z}) + \bar{\partial} \xi^{i}(z, \bar{z}), \quad h^{i}_{x}(z, \bar{z}) = \frac{1}{2} \partial x^{(12)\mu}(z, \bar{z}) - \partial \xi^{i}(z, \bar{z})$$

(4.5)

with $\nabla^{2} \xi^{i} = 0$. Substituting (4.3) into (4.4), we see that the transverse part $\xi^{i}$ of $h^{i}$ only contributes boundary terms to the path integral, yielding some overall constants (as do the harmonic modes of $h^{i}$). The effective action in (4.4) thus depends only on the longitudinal part of (4.3), and defining $\bar{x}^{(12)0} \equiv x^{(12)0}$ it can be written as

$$S_{N}[\bar{x}^{(12)}] = \frac{1}{4 \pi \alpha'} \int_{\Sigma^{(12)}} \eta_{\mu \nu} \partial \bar{x}^{(12)\mu} \partial \bar{x}^{(12)\nu} + \sum_{a=1,2} \int_{0}^{1} C^{i}_{a}(\bar{x}^{0}) \frac{\partial \bar{x}^{(12)\mu}_{a}}{\partial s_{a}} ds_{a}$$

(4.6)

where

$$C^{i}_{a}(\bar{x}^{0}) = -\frac{1}{2 \pi \alpha'} y^{(12)\mu}_{a}(\bar{x}^{0})$$

(4.7)

The action (4.6) is the worldsheet $\sigma$-model action for an open string with the usual Neumann boundary conditions $\partial_{\perp} \bar{x}^{(12)\mu}_{a}(\partial \Sigma^{(12)}_{a}) = 0$ at its two endpoints which allow the fields $\bar{x}^{(12)\mu}_{a}(z, \bar{z})$ to vary along the boundary of the worldsheet. Generally, the path integral manipulation above implements T-duality as a functional canonical transformation of the open string theory (16), mapping the spatial embedding fields $x^{i}$ to their duals $\bar{x}^{i}$ and converting the Dirichlet boundary conditions to Neumann ones. Note that the temporal direction is not T-dualized ($x^{0} = \bar{x}^{0}$).

This equivalent, T-dual picture of the multi D-brane system thus shows how to allow the string endpoints to fluctuate by replacing the Dirichlet boundary conditions with
Neumann ones on \( \partial \Sigma \), and at the same time it relates the D-brane field theory to a ten-dimensional gauge theory [3, 17] which describes the low-energy sector of open superstring theory. Now that the endpoints of the string which stretches between the two D-particles are allowed to propagate in spacetime, it is possible to “twist” the boundary condition (2.6) describing coincident D-branes. For this, we augment each \( y^{(12)i} = |y^{(12)i}| e^{i\theta_i} \) to generic complex valued fields, such that their conjugates \( (y^{(12)i})^* \) measure the changes of orientation between the two boundaries. Then \( |y^{(12)}| \sim |y^{(1)} - y^{(2)}| \) effectively measures the relative distance between the two D-particles and the angular variables \( 2\theta_i \in (0, 2\pi] \) measure their relative orientation along each direction in spacetime. For the canonical boundary values (2.6), we have \( \theta_i = \pi \) for \( \zeta = +1 \) (oriented strings) while \( \theta_i = \frac{\pi}{2} \) for \( \zeta = -1 \) (unoriented strings). The diagonal components of (3.8) remain real-valued since they correspond to definite D-particle positions. Thus we replace (3.8) by the 2 \( \times \) 2 Hermitian matrix field

\[
Y^i(x^0) = \begin{pmatrix} y^{(1)i}(x^0) & y^{(12)i}(x^0) \\ y^{(12)*i}(x^0) & y^{(2)i}(x^0) \end{pmatrix}
\]

Thus, in the general case of \( N \) D-branes, the effective action for the \( \sigma \)-model couplings will possess a generic global \( U(N) \) symmetry \( Y_i \rightarrow U Y_i U^\dagger \) corresponding to rotations of the D-branes relative to one another. This yields a dynamical worldsheet realization of the fact that at very short distances, spacetime is described by mutually noncommutative matrix fields with global \( U(N) \) symmetry.

5. Open Strings in Fat Brane Backgrounds

The dynamics of the system described in the previous sections, as we have so far set it up, do not take into account the interactions between the constituent D-branes and fundamental strings. Promoting the string embedding fields to matrices as described in section 2 and T-dualizing, we have the \( U(N) \)-symmetric action

\[
S_N = \frac{1}{4\pi\alpha'} \int_{\Sigma} \eta_{\mu\nu} \partial \tilde{X}^\mu \partial \tilde{X}^\nu - \frac{1}{2\pi\alpha'} \oint_{\partial \Sigma} \text{tr} \ Y^i(x^0) d\tilde{X}^i(s) + \oint_{\partial \Sigma} \text{tr} \ A^0(x^0) d\tilde{X}^0(s) \\
= \sum_{a=1}^{N} \left( \frac{1}{4\pi\alpha'} \int_{D^2} \eta_{\mu\nu} \partial \tilde{x}^{(a)\mu} \partial \tilde{x}^{(a)\nu} - \frac{1}{2\pi\alpha'} \oint_{S^1} y^{(a)i}(x^0) d\tilde{x}^{(a)i}(s) + \oint_{S^1} A^0_{aa}(x^0) dx^0(s) \right) \\
+ \sum_{a<b} \left( \frac{1}{2\pi\alpha'} \int_{A^2} \eta_{\mu\nu} \partial \tilde{x}^{(ab)\mu} \partial \tilde{x}^{(ab)\nu} - \frac{1}{\pi\alpha'} \oint_{S^1} \text{Re} \ y^{(ab)i}(x^0)^* d\tilde{x}^{(ab)i}(s) \\
+ 2 \oint_{S^1} \text{Re} \ A^0_{ab}(x^0)^* dx^0(s) \right)
\]

where \( \Sigma \) is a disc \( D^2 \) with a set of marked points on it, \( A^2 \) is an annulus, and we have explicitly written the boundary term with \( A^0 \) which parametrizes the original Neumann boundary conditions on the temporal components, \( \partial_s x^0|_{\partial \Sigma} = 0 \). The action (5.1) is a
sum of $N^2$ actions for the independent fields $x^{(a)\mu}, x^{(ab)\mu}$ which describes the assembly of D-brane configurations and fundamental strings. It is therefore missing the information about the interactions between the various D-branes via exchanges of the fundamental strings. Equivalently, the relabelling of the worldsheets between the first and second lines of (5.1) is not quite correct, i.e. the matrix version of the multi D-brane action (describing a single composite D-particle with matrix-valued coordinates as in section 2) does not properly account for the different worldsheet topologies. The proper incorporation of these topologies in fact breaks the $U(N)$ invariance group of the matrix action.

We want to describe the $\sigma$-model of a background of $N$ interacting branes, where the interactions are taken care of by open and closed strings exchanged between them. For this, we shall use the composite configuration described by (5.1) to properly incorporate the dynamics of the D-particles. We call this composite a “fat brane” (Fig. 1). Namely, a fat brane is the above configuration of $N$ parallel D-branes with all string interactions (including the sums over genera) among them integrated out. If we consider the $\sigma$-model with worldsheet that has the topology of a disc, then we integrate out the marked points associated with the various D-brane boundary conditions $y^{(a)}, y^{(ab)}$, leaving a low-energy effective theory. This takes care of the topological configurations associated with the worldsheet dynamics, at the cost of introducing matrix-valued couplings $Y^i$. In the limit where the separations between the $N$ constituent D-branes becomes large, the string interactions among them are negligible and the field theory is the free $U(1)^N$ theory which is described by $N$ copies of the single D-particle action discussed in section 2. When the relative separations vanish, the branes fall on top of each other (giving a usual “thin brane”) and the field theory is described by the composite D-brane $\sigma$-model of section 2 with matrix-valued fields and enhanced $U(N)$ symmetry. This presents an alternative dynamical derivation of the short-distance noncommutativity of spacetime, in which massless excitations in the spectrum of the quantum $U(N)$ gauge theory yield bound states of D-branes with broken supersymmetry (i.e. $[Y^i, Y^j] \neq 0$ for $i \neq j$) [1].

The effective matrix $\sigma$-model describes the target space (equivalently coupling constant space) dynamics of a single fundamental string propagating in the background of the fat brane. The string couples to the fat brane background via a matrix-valued coupling constant $Y^i(x^0)$. By $U(N)$ invariance, the non-trivial matrix-valued interactions are described by the expectation value, in a free $\sigma$-model, of the path-ordered $U(N)$ Wilson loop operator $W[C]$ along the boundary of the worldsheet,

$$\sum_y \int [d\tilde{X}] \int_\Sigma \prod_{a,b=1}^N d^2z_{ab} \ e^{-SN[\tilde{X};A]} \approx \langle \langle W[C = \partial{\Sigma}] \rangle \rangle \equiv \int [d\tilde{x}] \ e^{-N^2S_0[\tilde{z}]} \ tr \ P \ exp \left( i \int_{\partial{\Sigma}} A_\mu(\tilde{x}^0(s)) \ d\tilde{x}\mu(s) \right)$$

(5.2)

where $d\tilde{X}$ is the invariant Haar measure for integration over the Lie algebra of $N \times N$ Hermitian matrices, $z_{ab}$ are marked points on the disc $\Sigma$, and $A_\mu = (A^0, -\frac{1}{2\pi\alpha'}Y^i)$ can be
interpreted as a ten-dimensional $U(N)$ isospin gauge field dimensionally reduced to the worldline of the D-particle. The partition function (5.2) describes the effective dynamics of an open string, with Neumann boundary conditions, propagating in the fat brane background, the latter of which is described by the matrix-valued coupling constants $Y^i(x^0)$ and incorporated by the insertion of the Wilson loop in the free $\sigma$-model action of (5.2). The addition of this term can be thought of as either a modification of the path integral measure which takes the boundary conditions into account, or as the only non-trivial $U(N)$-invariant possible addition to the action. Each matrix element $Y_{ab}$ represents the coupling of the fundamental string to either a constituent D-brane (for $a = b$) or to an exchanged string between the D-particles with a particular orientation (for $a \neq b$).

Notice that the coordinates $\tilde{x}^i$ of the open string are scalars, as is the common ‘time’ $x^0$ for propagation of the fat brane background and the open string. The time-dependence of $Y^i$ can be represented schematically as

$$Y^i_{ab}(x^0) = Y^i_{ab}(x^0 = 0) + U^i_{ab} x^0$$

in the simple case of non-relativistic heavy D-particles. Here $Y^i(x^0 = 0)$ is the matrix-valued coupling constant (4.8). The velocity matrix $U^i$ describes the velocities of the constituent D-branes in the fat brane. We can typically assume that the entire, composite fat brane moves with a single (center of mass) velocity $u$, in which case $U_{ab} = u\delta_{ab}$.

The Neumann boundary conditions for $\tilde{x}^\mu$ in this case are parametrized by the quantities

$$C^\mu(\tilde{x}_0; s) = \bar{\rho}^a(s) A^\mu_{ab}(\tilde{x}_0) \rho^b(s)$$

where $\bar{\rho}^a, \rho^b$ are Chan-Paton isospin factors at the endpoints of the open string. The D-brane configurations can thus be identified with the components of the gauge field $A_\mu$.
restricted to the worldline of the D-particle as

\[ y^i(\tilde{x}^0) = -2\pi\alpha' \tilde{\rho}^i(s) A^i_{ab}(\tilde{x}^0) \rho^b(s) \]  

(5.5)

This relation can be used to map correlators of vertex operators of the open string theory into expectation values of Wilson loop observables of the \( U(N) \) gauge theory \[1, 17, 18\]. For the non-abelian Chan-Paton factors (5.4), the worldsheet renormalization group equations have been derived in \[15\] and the (non-abelian) effective action in \[20\]. The expression (5.2) no longer has the form of a functional integral over the exponential of a local action. However, the path integral for the Neumann \( \sigma \)-model action with the non-abelian Chan-Paton factors (5.4) does \[16\]. Note that when \( \tilde{\rho}^a = \delta^{ac} \) and \( \rho^b = \delta^{bd} \), (5.5) reduces to the \( \sigma \)-model couplings \( Y^i_{ab}(\tilde{x}^0) \).

The T-dualization of this model using the background field method of the previous section has been carried out in \[11\], yielding the usual D-brane \( \sigma \)-model action with a delta-function constraint for the boundary conditions onto the abelian projection (5.5) of the gauge group. It is crucial though to define the action in (5.2) using Neumann boundary conditions in order to properly incorporate the coupling of the fundamental string to the fat brane. The T-dual picture then shows that the resulting \( \sigma \)-model action is in effect one for a single D-brane with abelian projection coordinates (5.5), as is anticipated from the analysis of section 2. This is the essence behind the fat brane picture that we described above and is the reason for the equivalence of the matrix \( \sigma \)-model action with that for the collective quantum dynamics of multi D-brane configurations.

6. Low-energy Effective Action

The effective action for Wilson loop correlators of the sort (5.2) has been derived recently by Tseytlin \[20\], who showed that it yields the natural non-abelian generalization of the Born-Infeld Lagrangian of \[3\] for open strings in the background of non-abelian Chan-Paton factors. To lowest order in the derivative expansion, the effective action depends only on the field strength \( F_{\mu\nu} = [D_\mu, D_\nu] \), with \( D_\mu = \partial_\mu - i[A_\mu, \cdot] \), and not on its gauge-covariant derivatives. The result in the present case is then \( \langle \langle W[\partial \Sigma] \rangle \rangle \simeq e^{-\Gamma_{NBI}} \) with

\[ \Gamma_{NBI} = c_0 \int dt \text{ tr } \text{Sym} \sqrt{\det_{\mu,\nu} \left[ \eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} \right]} \]  

(6.1)

where \( c_0 \) is a constant, the trace is taken in the fundamental representation of \( U(N) \), and \( \text{Sym} \) denotes the symmetrized product \( \text{Sym}(A_1 \cdots A_n) = \frac{1}{n!} \sum_{\pi \in S_n} A_{\pi_1} \cdots A_{\pi_n} \).

The important technical point in the derivation of (6.1) by the conventional background field method is that covariant derivatives of the field strength tensor \( F_{\mu\nu} \) have been effectively set to zero, i.e. \( D_\mu F_{\nu\rho} = 0 \) \[20\]. In the case at hand the components of the field strength are given by

\[ 2\pi\alpha' F_{0i} = D_0 Y_i \equiv \dot{Y}_i - i[A_0, Y_i] \quad , \quad (2\pi\alpha')^2 F_{ij} = [Y_i, Y_j] \]  

(6.2)
since the coordinates $Y_{ab}$ of the fat brane do not depend on $x^i$, but only on the Neumann time coordinate $x^0$ of the open string propagation in the fat brane background. This implies that the condition for the vanishing of the covariant derivative reads (in the gauge $A_0 = 0$)

$$2\pi\alpha'\bar{\partial}_0 F_{0i} = \bar{Y}_i = 0,$$

$$\bar{\partial}_0[Y_i, Y_j] = 0,$$

$$[Y_j, [Y_i, Y_k]] = 0$$

(6.3)

For the simple boost (5.3) in a non-accelerating fat brane background, the conditions (6.3) are indeed satisfied. Note also that the non-trivial solitonic D-brane configurations correspond to

$$[Y_i, Y_j] = f_{ij}I_N$$

(6.4)

where $I_N$ is the $N \times N$ identity matrix, and hence they also satisfy (6.3).

The matrix model effective action (1.1) now follows, after appropriate rescalings of the fields, from the expansion of (6.1) in powers of the Regge slope $\alpha'$. The linear terms vanish, while the first non-zero term is the quadratic one which yields the bosonic part of (1.1). Thus the (bosonic part of the) matrix model action can be derived as the target space effective action of the non-local $\sigma$-model (5.2) describing the motion of an open string in a fat D-brane background. The crucial aspect of this fat brane description is the matrix-valued couplings $Y^i$ of the $\sigma$-model. For non-diagonal coupling matrices, the coefficient of the commutator term $[Y^i, Y^j]^2$ is determined dynamically from the quantum uncertainties (3.7) arising from the quantum dynamics of the multi D-brane system which effectively renders the matrices $Y^i$ mutually noncommutative. This picture therefore yields a geometrical origin for the appearance of the string coupling constant $g_s$ (here in its renormalized form) in the matrix model action. Furthermore, these same uncertainties naturally determine the solitonic BPS configurations (6.4) from a quantum mechanical origin.

We have shown that the matrix model for M Theory describes the propagation of open strings in a background of $N$ interacting D-branes, and a worldsheet $\sigma$-model description of this process is a (low-energy) effective theory in which the string interactions of the multiple D-brane configurations have been integrated out. In this sense, the open string propagates in the background of a fat brane with a matrix-valued coupling constant (4.8). One important aspect that is not clear from this construction is how to implement target space supersymmetry at the level of the worldsheet $\sigma$-model. A supersymmetric extension of the abelian Born-Infeld effective Lagrangian for a single D-brane has been derived recently in [21], although the $\sigma$-model origins of this construction are not evident. One could mimick the fat brane construction of section 5 by replacing (5.2) with a supersymmetric Wilson loop correlator in the Green-Schwarz light-cone formalism [18]. However, the T-dualization of the resulting $\sigma$-model action to one with Dirichlet boundary conditions appears to be highly non-trivial, even in the single D-particle case.

\footnote{An explicit appearence of $g_s$ also occurs in the scattering of strings off a single fluctuating quantum D-brane background [13].}
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