A Domain Wall Solution by Perturbation of the Kasner Spacetime

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Abstract

Plane symmetric perturbations are applied to an axially symmetric Kasner spacetime which leads to no momentum flow orthogonal to the planes of symmetry. This flow appears laminar and the structure can be interpreted as a domain wall. We further extend consideration to the class of Bianchi Type I spacetimes and obtain corresponding results.

*Subject headings:* general relativity, Kasner metric, Bianchi Type I, domain wall, perturbation
1. Finding New Solutions

There are three principal exact solutions to the Einstein Field Equations (EFE) that are most relevant for the description of astrophysical phenomena. They are the Schwarzschild (internal and external), Kerr and Friedman-LeMaître-Robertson-Walker (FLRW) models. These solutions are all highly idealized and involve the introduction of simplifying assumptions such as symmetry.

The technique we use which has led to a solution of the EFE involves the perturbation of an already symmetric solution (Wilson and Dyer 2007). Whereas standard perturbation techniques involve specifying the forms of the energy-momentum tensor to determine the form of the metric, we begin by defining a new metric with an applied perturbation:

\[ g_{ab} = \tilde{g}_{ab} + h_{ab} \]  

where \( \tilde{g}_{ab} \) is a known symmetric metric and \( h_{ab} \) is the applied perturbation. We then use this new metric to find the components of the Einstein tensor, \( G_{ab} \), in terms of the metric components. Next, we invert the EFE, where now \( T_{ab} = G_{ab}/\kappa \), to ascertain whether or not components of the energy-momentum tensor, \( T_{ab} \), exist to satisfy the perturbation. If such an energy-momentum tensor is physically acceptable, we then try to limit the behaviour of our perturbation by imposing conditions to model particular astronomical phenomenon. For example, if we were modelling a galaxy-like structure, we could choose our density to scale as \( \rho \sim r^{-2} \) from the galactic centre (Wilson and Dyer 2007). This makes the technique flexible and relevant in that we can model a physically viable structure. With these real physical constraints applied, we have a new metric. We now re-compute the energy-momentum components of this new metric and investigate any implications that may arise from our new exact solution to the EFE.
2. The Base Solution

The metric we choose to perturb is the Kasner spacetime \cite{Kasner1921}. This metric describes an anisotropic universe that is a vacuum solution to the EFE classified as a Bianchi Type I universe. It is characterized by a spacetime that is anisotropically expanding (or contracting) in two directions while contracting (or expanding) in the third with space-like slices that are spatially flat and with a singularity occurring at $t = 0$. Hence, the Kasner spacetime is of special interest in cosmology since the standard cosmological solutions to the EFE near the cosmological singularity, such as the aforementioned FLRW model, can be described as a succession of Kasner epochs \cite{Lifshitz1963}. Furthermore, by breaking a homogeneity along one direction, Bianchi Type solutions can easily be generalized to spacetimes with $G_2$ isotropy groups \cite{Harvey1990}.

The Kasner spacetime in Cartesian coordinates with metric signature $(+,-,\ldots,-)$ takes the form (in spacetime dimension $D$):

$$ds^2 = dt^2 - \sum_{j=1}^{D-1} t^{2P_j} [dx_j]^2$$

where $P_j$ are the Kasner exponents which satisfy:

$$\sum_{j=1}^{D-1} P_j = 1 \quad \text{and} \quad \sum_{j=1}^{D-1} P_j^2 = 1 \quad (3)$$

The first condition in (3) describes a plane whereas the second condition describes a sphere of dimension $D - 1$. Thus, the Kasner exponents lie on a sphere of dimension $D - 2$. For $D = 4$, at each $t = constant$ hypersurface, there exists a flat 3-dimensional space, whose worldlines of constant $x$, $y$ and $z$ are time-like geodesics along which galaxies, or other test particles, can be imagined to move. Since the solution represents an anisotropically expanding (or contracting) universe, a volume element $dV$ increases (or decreases) in time as $\sqrt{|g|} \, d^3x = td^3x$, where $g = det|g_{ab}|$. 


The Kasner spacetime in Cartesian coordinates in a $4-D$ spacetime takes the form:

$$ds^2 = dt^2 - t^{2P_1} dx^2 - t^{2P_2} dy^2 - t^{2P_3} dz^2$$  \hspace{1cm} (4)$$

with the same restrictions imposed by the Kasner exponents as given in (3).

The form of the Kasner spacetime that we consider is the axisymmetric case with $G_2$ isometry that occurs when two of the exponents are equal, which by (3), requires that $P_1 = P_2 = 2/3$ and $P_3 = -1/3$. With this restriction, the metric in cylindrical coordinates becomes:

$$ds^2 = dt^2 - t^{4/3} (dr^2 + r^2 d\phi^2) - t^{-2/3} dz^2$$ \hspace{1cm} (5)$$

3. The Perturbations

The set of perturbations we implement are only on the $z$-component of the Kasner spacetime represented in cylindrical coordinates (5). Clearly, the results we obtain in cylindrical coordinates will be similar to those in Cartesian coordinates since $r^2 = x^2 + y^2$. For simplicity, we define the Kasner spacetime in cylindrical coordinates as:

$$ds^2 = dt^2 - t^{4/3} (dr^2 + r^2 d\phi^2) - g_{zz} dz^2$$ \hspace{1cm} (6)$$

where $g_{zz} = \tilde{g}_{zz} + h_{ab}$ is our perturbed metric. We consider the three Cases shown in Table 1.

| Cases | $g_{zz} = \tilde{g}_{zz} + h_{ab}$ |
|-------|----------------------------------|
| I     | $[t^{-2/3} + H]$                |
| II    | $[t^{-1/3} + H]^2$              |
| III   | $[t^{-2/3} + Z(z)T(t)]$         |

Table 1: Implemented perturbations.
In Cases I and II, we first investigate $H$ as a differentiable function of $z$. We then investigate Cases I and II again, but where $H$ is now made a differentiable function of $t$, $r$ and $z$. In Case III, we investigate the result of a perturbation with a product separable differentiable solution.

To aid in these computations, we use the REDUCE Computer Algebra System \cite{Hearn2009} with the REDTEN Tensor Analysis Package \cite{Harper1994}.

4. Results of the Perturbations

Both Cases I and II produce a $G_{ab}$ with non-zero components. When the perturbation is $H(z)$, the result produces non-zero terms for $G_{tt}$, $G_{rr}$ and $G_{\phi\phi}$. All other terms, including $G_{zz}$, are zero. When the perturbation is $H(t, r, z)$, a similar result is obtained for all cases, but with the addition of cross-components, $G_{ab}$ for $a \neq b$. These cross terms can be made zero if we choose the perturbation to contain only first-order terms in $H$ or by making $\partial H/\partial r = \partial H/\partial \phi = 0$. Also, in cylindrical coordinates for all the cases reviewed, $G_{rr} = r^2G_{\phi\phi}$, which is expected. The product separable perturbation of Case III reveals the same results as Case I and II; we arrive at non-zero terms for $G_{tt}$, $G_{rr}$ and $G_{\phi\phi}$ and a $G_{zz} = 0$. Thus all perturbations that involved only the $g_{zz}$ term of the Kasner metric reveals that we will always obtain $G_{zz} = 0$ when considering axisymmetric or plane symmetric Kasner spacetimes, when $P1 = P2$.

Relating these results to the energy-momentum tensor $T_{ab}$, both the $G_{zz}$ and $G_{za} = G_{az}$ (for $a = \{0, 1, 2\}$) components of the Einstein tensor being zero reveals that there is no momentum-flux across the $z = constant$ surface regardless of the type of perturbation applied. Each of the applied perturbations resulted in the metric collapsing (or expanding) in the $z$-direction while expanding (or collapsing) in the $x$- and $y$-directions, as is expected.
of the Kasner metric. But, as a result of the $G_{zz}$ term being zero, there is no interaction between the stratified layers of the matter above or below the $xy$-plane. This is suggestive of laminar flow.

Inverting the EFE to get $T_{ab} = G_{ab}/\kappa$, leads to $T_{ab}$ being diagonal but with no $T_{zz}$ component. The energy-momentum tensor for an infinite, static plane-symmetric domain wall as suggested by Campanelli et al is:

$$T_{ab} = \delta(z)\text{diag}(\rho, -p, -p, 0)$$

(7)

where $\rho$ is the energy-density and $p$ is the pressure. This energy-momentum tensor (Campanelli et al. 2003) corresponds to an infinite, static plane-symmetric domain wall (Kibble 1976) lying in the $xy$-plane. Domain walls correspond to a particular class of topological defects whereby the energy-density is trapped, and from a cosmological point of view, these over-dense regions could lead to structure formation (Brandenberger 1997). Furthermore, it has been suggested (Friedland et al. 2003) that the Universe may be dominated by a network of domain walls and these domain wall structures could represent an alternative view of dark energy theories.

5. Energy Conditions

In order to rule out any non-physical solutions to the EFE, energy conditions are applied to the state of matter content for gravitational and non-gravitational fields. These energy conditions consist of the Weak, Null, Strong and Dominant energy conditions, which are coordinate-invariant constraints on the energy-momentum tensor.

For the cases in which our Kasner spacetime was perturbed with $H(z)$, we apply the least stringent of these conditions, the Null Energy Condition (NEC). This condition states
that for all future-pointing null vectors, $k^a$:

$$T_{ab}k^a k^b \geq 0$$  \hfill (8)

This restriction implies that $\rho + p \geq 0$, whereby the energy-density may be negative as long as there is a compensating pressure. If this condition is violated in any of the perturbed Kasner spacetimes, it would then indicate that our solutions are unstable.

We choose the null vector field $k^a$, in the Kasner spacetime to be,

$$k^a = \left( \sqrt{1 - n^2 g_{zz}}, 0, 0, n^2 \right)$$  \hfill (9)

where $n \in \mathbb{R}$. This class of null vectors lies in the $T_{zz}$ plane of interest. A summary of the energy-density restrictions is shown in Table 2, where $\dot{T} = \partial T / \partial t$. For all Cases I to III, $T_{ab}k^a k^b$ is positive definite for positive $H$.

| Cases | $g_{zz} = \tilde{g}_{zz} + h_{ab}$ | $T_{ab}k^a k^b$ |
|-------|---------------------------------|----------------|
| I     | $[t^{-2/3} + H(z)]$             | $\frac{4t^{1/3}(t^{1/3}n^2 + t)H}{9t^{4/3}(6t^{2/3}H + 1)}$ |
| II    | $[t^{-1/3} + H(z)]^2$           | $\frac{4(t^{1/3}n^2 + t)H}{9t^{4/3}(t^{1/3}H + 1)}$ |
| III   | $t^{-2/3} + H(z)T(t)$           | $\frac{2t^{1/3}(3Tn^2t + 3\tilde{T}t^2 + 2t^{1/3}Tn^2 + 2Tt)H}{9t^{4/3}(6t^{2/3}HT + 1)}$ |

Table 2: Applied NEC results

6. Bianchi Type I Metrics

As mentioned in Section 2, the Kasner spacetime is a special class of Bianchi Type I spacetime being a homogeneous and anisotropic vacuum solution to the EFE. Therefore, it is appropriate to apply the perturbations to the more generalized Bianchi Type I to
investigate its dynamics and compare the results to the perturbed Kasner models. The Bianchi Type I (Stephani 2003) metric has the form:

\[ ds^2 = c^2 dt^2 - g_{11} dx^2 - g_{22} dy^2 - g_{33} dz^2 \]  

(10)

where,

\[ g_{\alpha\alpha} = (-g)^{1/3} \left[ ct / (\dot{M} ct + A) \right]^{2P_\alpha - 2/3} \]

(11)

such that \( \alpha = 1, 2, 3 \), with no sum on \( \alpha \). The quantity \( \dot{M} = \kappa \mu c^2 \sqrt{-g} \) is a constant and \( \sqrt{-g} = 3ct(\dot{M} ct + A)/4 \), where \( A \) is an integration constant. Using these relationships and an appropriate constant rescaling of coordinates, we can re-write (10) as:

\[ ds^2 = dt^2 - t^{4/3} dx^2 - t^{4/3} dy^2 - t^{-2/3}(1 + \epsilon t)^2 dz^2 \]

(12)

where \( \epsilon = \dot{M}/A \). As before, since we wish to compare this result to the axisymmetric perturbed Kasner spacetime, we impose \( P_1 = P_2 = 2/3 \) and \( P_3 = -1/3 \).

There are two limiting cases of the Bianchi Type I metric in (12). If we choose \( \epsilon t \ll 1 \), the initial singularity is approached at early times and we regain the original Kasner spacetime. For late times, if we choose \( \epsilon t \gg 1 \), the metric isotropizes to the well-known Einstein-de Sitter dust metric, which is a sub-set of the FLRW metric. Indeed, the Kasner solution is a past asymptotic state as mentioned in Section 2.

7. Perturbations on Bianchi Type I

The perturbations applied to the Bianchi Type I metric are summarized in Table 3.
| Cases | $g_{zz} = \tilde{g}_{zz} + h_{ab}$ |
|-------|----------------------------------|
| I     | $t^{-2/3}[(1 + \epsilon t)^2 + H(z)]$ |
| II    | $t^{-2/3}[(1 + \epsilon t)^2 + H(z)T(t)]$ |

Table 3: Implemented Perturbations on Bianchi Type I

Investigating $G_{ab}$ for the perturbations of Cases I and II, we arrive at the same conclusion as was found for our perturbed Kasner spacetimes; we obtain $G_{zz} = 0$ for the axisymmetric Bianchi Type I spacetimes.

8. Symmetries

The nonlinear nature of the EFE makes it difficult to find exact solutions. All of the known solutions have admitted simplifying symmetries in order to attain a solution. The technique we apply involves the perturbation of an already highly symmetric spacetime in hopes of producing a new spacetime solution. We now wish to determine to what extent any symmetries that were inherent in the original spacetime metric still retain (or break) any symmetries.

To investigate symmetries, we will consider conformal Killing vector fields, $\xi^a$, that satisfy the conformal Killing equations:

$$\xi_{[a||b]} + \xi_{b||a} = \phi g_{ab}$$  \hspace{1cm} (13)

where $\phi = \frac{k_i}{4} \xi^c_{||c}$ and $||$ denotes covariant differentiation. Utilizing the REDUCE/REDTEN computer algebra system, we computed the conformal Killing equations as 10 symmetric rank-2 tensors for each of the perturbation cases investigated. A Killing vector from the original metric was then calculated along with its covariant derivative and substituted in the conformal Killing equation (13). For the perturbations applied to the two Cases, no
such Killing vectors or conformal Killing vectors were found to remain; the original Killing vector is no longer a Killing vector.

9. Conclusion

Starting from two known, highly symmetric, solutions to the Einstein Field Equations we have applied plane symmetric perturbations and have shown that the resulting perturbed spacetimes exhibit the existence of structure that can be interpreted as a domain wall. It was demonstrated that these solutions do not violate the Null Energy Condition and thus would permit us to apply an appropriate energy-momentum tensor in future investigations.

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