THE CONSERVATION OF MASS-MOMENT PARAMETERS

DAN COMĂNESCU

ABSTRACT. In this paper we study a concept of mass-moment parameter which is the generalization of the mass and the moments of inertia of a continuous media. We shall present some interesting kinematical results in the hypothesis that a set of mass-moment parameters are conserved in a motion of a continuous media.

Mathematics Subject Classification: 74A05; 70B10; 70S10

Keywords: continuum mechanics; kinematics; mass-moment parameter.

1. INTRODUCTION

A mass-moment parameter of a continuous media is a kinematical parameter which is used to describe the distribution of the matter in the continuous media. The most common mass-moment parameters are the mass and the moments of inertia. The mass-moment parameters, with the exception of the mass, are used especially in the theory of rigid bodies. In a motion of a rigid body the moments of inertia are conserved with respect to a frame of reference rigidly connected to the body.

The objective of our study are:

• to present the concept of mass-moment parameter which is used in this paper;
• to introduce the conservation of a mass-moment parameter for a motion of a continuous media;
• to deduce the local laws in the cases in which one or two mass-moment parameters are conserved in a motion;
• to study the conservation of some moments of inertia.

2. THE MASS-MOMENT PARAMETERS

In this paper we consider the mass-moment parameters of the form (with respect to a frame of reference $\mathcal{R}$):

\begin{equation}
    P(\mathcal{P}, t) = \int_{P_t} \rho(\vec{x}, t)p(\vec{x}, t)dv
\end{equation}

where $t$ is a time-moment, $\mathcal{P}$ is a part of the continuous media, $P_t$ is the image of $\mathcal{P}$ (at time $t$), $\vec{x}$ is the position vector of a particle (at time $t$), with respect to frame of reference $\mathcal{R}$, $\rho$ is the mass density and $p$ is the reduced density of the mass moment parameter $P$.

The mass-moment parameter $P$ is defined by a Lebesgue integral with respect to the volume measure. The functions $\rho$ and $p$ are supposed to be continuous.
For the reduced density \( p \equiv 1 \) on obtain the mass of the continuous media:

\[
M(\mathcal{P}, t) = \int_{\mathcal{P}} \rho(\vec{x}, t) dv
\]

The moment of inertia with respect to the point \( Q \) is defined by the integral:

\[
I_Q(\mathcal{P}, t) = \int_{\mathcal{P}} \rho(\vec{x}, t)d^2(\vec{x}, \vec{x}_Q) dv
\]

where \( d(\vec{x}, \vec{x}_Q) \) is the distance between an arbitrary point of \( \mathcal{P} \) and \( Q \). The moment of inertia with respect to the plane \( \Pi \) is defined by the integral:

\[
I_\Pi(\mathcal{P}, t) = \int_{\mathcal{P}} \rho(\vec{x}, t)d^2(\vec{x}, \Pi) dv
\]

where \( d(\vec{x}, \Pi) \) is the distance between an arbitrary point of \( \mathcal{P} \) and the plane \( \Pi \). The moment of inertia with respect to the \( \Delta \)-axis is defined by the integral:

\[
I_\Delta(\mathcal{P}, t) = \int_{\mathcal{P}} \rho(\vec{x}, t)d^2(\vec{x}, \Delta) dv
\]

where \( d(\vec{x}, \Delta) \) is the distance between an arbitrary point of \( \mathcal{P} \) and the line \( \Delta \).

3. The conservation of a set of mass-moment parameters

In this section we present some results concerning the conservation of one or two mass-moment parameters in a motion of a continuous media.

First, we deduce the local laws for the conservation of a mass-moment parameter. Second, two conservation type results are deduced.

**Definition 3.1.** A mass-moment parameter \( P \) is conserved in a motion of the continuous media if we have:

\[
P(\mathcal{P}, t_1) = P(\mathcal{P}, t_2)
\]

for all parts \( \mathcal{P} \) of the continuous media and for all two time-moments \( t_1 \) and \( t_2 \).

We denote by \( \vec{X} \) and \( \vec{x} \) the position occupied by a particle at the initial moment and in the configuration at \( t \). The law (3.1) has the form:

\[
\int_{\mathcal{P}_1} \rho(\vec{x}, t)p(\vec{x}, t) dv = \int_{\mathcal{P}_0} \rho(\vec{X}, 0)p(\vec{X}, 0) dV
\]

**Remark 3.1.** The problem of conservation of a mass-moment parameter is connected with the mathematical theory of integral invariants.

**Theorem 3.1. (local law in material variables)** Let the \( C^1 \)-map \( \vec{x} = \chi(\vec{X}, t) \) of the motion and \( P \) a mass-moment parameter with \( p \) the reduced density. The proposition are equivalents:

(i) \( P \) is conserved in the motion;

(ii) for all time-moments \( t \) and \( \vec{X} \in D_0 \) we have:

\[
\rho(\vec{x}, t)p(\vec{x}, t)J(\vec{X}, t) = \rho(\vec{X}, 0)p(\vec{X}, 0)
\]
where $D_0$ is the image of the continuous media at the initial moment and $J$ the determinant of the deformation gradient; i.e.:

$$J(\vec{X}, t) = \det(\frac{\partial x_i}{\partial X_j}(\vec{X}, t))_{i,j\in\{1,2,3\}}$$

Proof. We need the following result:

**Lemma 3.2.** (see [8] pp. 47) If $f: D \subset \mathbb{R}^3 \to \mathbb{R}$ is a continuous function on the domain $D$ such that for all domains $D \subset D$ is satisfied $\int_D f dv = 0$ then $f \equiv 0$.

Using the relation (3.2) and the transformation of the integrals in the material variables we deduce that for all domains $P_0 \subset D_0$ and for all times $t$:

$$\int_{P_0} \rho(\vec{x}, t)p(\vec{x}, t)J(\vec{X}, t) - \rho(\vec{X}, 0)p(\vec{X}, 0)dV = 0.$$

The theorem follows from (3.4) and Lemma 3.2. \qed

**Theorem 3.3.** *(local law in spatial variables)* Let the $C^2$-map $\vec{x} = \chi(\vec{X}, t)$ of the motion and $P$ a mass-moment parameter with the reduced density $p$ a $C^1$-function. The proposition are equivalents:

(i) $P$ is conserved in the motion;

(ii) for all time-moments $t$ and $\vec{x} \in D_t$ we have:

$$\frac{d}{dt}dJ(\vec{X}, t) = J(\vec{X}, t)\text{div}_x \vec{v}(\vec{x}, t) = 0$$

where $\vec{v}$ is the velocity and $\text{div}_x \vec{v}$ is the divergence of $\vec{v}$ with respect to the spatial variables.

Proof. We need the Euler’s lemma (see [8], pp. 36).

**Lemma 3.4.** *(Euler)* For a $C^1$-motion $\vec{x} = \chi(\vec{X}, t)$ of a continuous media is satisfied the relation:

$$\frac{d}{dt}J(\vec{X}, t) = J(\vec{X}, t)\text{div}_x \vec{v}(\vec{x}, t).$$

We have the sequence of equivalences:

$$(i) \iff \frac{d}{dt}\int_{P_t} \rho p dv = 0 \ \forall P \iff \int_{P_0} \frac{d}{dt} \rho p + \frac{d}{dt} \rho p \ dV = 0 \ \forall P$$

Using Euler’s lemma and Lemma 3.2 we deduce:

$$(i) \iff \int_{P_t} \frac{d}{dt} \rho p + (\rho p)\text{div}_x (\vec{v}, t) dv = 0 \ \forall P \iff (ii)$$

\qed

We shall study now a case in which two mass-moment parameters are conserved.
Theorem 3.5. Let a $C^1$-motion of a continuous media $\vec{x} = \chi(\vec{X}, t)$. We consider two mass-media parameters $P_1$ and $P_2$ with reduced densities $p_1$ and $p_2$ which are conserved in the motion. Then, for all particles and time-moments it is satisfied:

\[(3.6) \quad p_1(\vec{X}, 0)p_2(\vec{x}, t) = p_1(\vec{x}, t)p_2(\vec{X}, 0)\]

Proof. Following Theorem 3.1 one obtains:

\[
\rho(\vec{x}, t)p_1(\vec{x}, t)J(\vec{X}, t) = \rho(\vec{X}, 0)p_1(\vec{X}, 0)
\]

\[
\rho(\vec{x}, t)p_2(\vec{x}, t)J(\vec{X}, t) = \rho(\vec{X}, 0)p_2(\vec{X}, 0)
\]

Using the properties $\rho > 0$ and $J \neq 0$ we deduce our result. \qed

Theorem 3.6. If, moreover, $p_2 \neq 0$ we deduce:

\[(3.7) \quad \frac{p_1(\vec{x}, t)}{p_2} = \frac{p_1(\vec{X}, 0)}{p_2(\vec{X}, 0)}\]

Theorem 3.7. Let a $C^1$-motion of a continuous media $\vec{x} = \chi(\vec{X}, t)$ and $P$ a mass-moment parameter with $p$ the reduced density.

If the mass $M$ (see (2.2)) and $P$ are conserved in the motion then for all particles and time-moments we have:

\[(3.8) \quad p(\vec{x}, t) = p(\vec{X}, 0)\]

4. Applications

In this section are presented some applications of the results of section 3.

4.1. Conservation of mass. If the mass $M$ (see (2.2)) is conserved in a $C^1$-motion $\vec{x} = \chi(\vec{X}, t)$ of a continuous media then, using Theorem 3.1, is obtained the material equation of continuity (Euler 1762):

\[(4.1) \quad \rho(\vec{X}, 0) = \rho(\vec{x}, t)J(\vec{X}, t)\]

If the motion is a $C^2$-function and $\rho$ is a $C^1$-function a consequence of conservation of mass is (see Theorem 3.3) the spatial equation of continuity (Euler 1757):

\[(4.2) \quad \frac{dp}{dt}(\vec{x}, t) + \rho(\vec{x}, t)\text{div}_{\vec{x}} \vec{v}(\vec{x}, t) = 0\]

4.2. Conservation of mass and of an other mass-moment parameter. In this paragraph we suppose that a continuous media has a $C^1$-motion such that the mass $M$ (see (2.2)) is conserved.

Using Theorem 3.7 is easy to obtain the following results.

Theorem 4.1. Let $O$ a fixed point. The following are equivalents:

(i) the moment of inertia with respect to the point $O$ is conserved in the motion;
(ii) the motion of an arbitrary particle is on a sphere with the center $O$. 

Theorem 4.2. Let $\Pi$ a fixed plane. The following are equivalents:
(i) the moment of inertia with respect to the plane $\Pi$ is conserved in the motion;
(ii) the motion of an arbitrary particle is in a plane parallel with $\Pi$.

Theorem 4.3. Let $\Delta$ a fixed axis. The following are equivalents:
(i) the moment of inertia with respect to the $\Delta$-axis is conserved in the motion;
(ii) the motion of an arbitrary particle is on a circular cylinder with $\Delta$ as the axis of symmetry.

4.3. A condition for the equilibrium of a continuous media. Let a $C^1$-motion of a continuous media with respect to a spatial frame of reference $Ox_1x_2x_3$. We suppose that the mass $M$ is conserved in the motion. We denote by $I_O$ the moment of inertia with respect to the point $O$, $I_{x_i}$ ($i \in \{1, 2, 3\}$) the moment of inertia with respect to the $Ox_i$-axis, $I_{Ox_i x_j}$ ($i, j \in \{1, 2, 3\}$) the moment of inertia with respect to the plane $Ox_i x_j$. We have the relations (see [14] pp. 612-613):

\begin{align}
I_O &= \frac{1}{2}(I_{x_1} + I_{x_2} + I_{x_3}) \\
I_O &= I_{Ox_1 x_2} + I_{Ox_2 x_3} + I_{Ox_3 x_1} \\
I_O &= I_{x_i} + I_{Ox_j x_k}, \ {i, j, k} = \{1, 2, 3\} \\
I_{x_i} &= I_{Ox_i x_j} + I_{Ox_i x_k}, \ {i, j, k} = \{1, 2, 3\} \\
I_{Ox_i x_j} &= \frac{1}{2}(I_{x_i} + I_{x_j} - I_{x_k}), \ {i, j, k} = \{1, 2, 3\}
\end{align}

Theorem 4.4. If three of seven mass-moment parameters from the set $I_O, I_{x_1}, I_{x_2}, I_{x_3}, I_{Ox_1 x_2}, I_{Ox_2 x_3}, I_{Ox_3 x_1}$ are conserved in the motion then the continuous media is in an equilibrium state.

Proof. Using the relations (4.3)-(4.7) we deduce that all the seven mass-moment parameters are conserved in the motion of the continuous media. Applying the Theorem 3.6 for the mass-moment parameters $I_{Ox_1 x_2}, I_{Ox_2 x_3}, I_{Ox_3 x_1}$ and we obtain:
\[
\begin{align*}
x_1^2(\vec{X}, t) &= X_1^2 \\
x_2^2(\vec{X}, t) &= X_2^2 \\
x_3^2(\vec{X}, t) &= X_3^2
\end{align*}
\]

The functions $t \rightarrow x_i(\vec{X}, t)$ are continuous with the initial conditions $x_i(\vec{x}, t) = X_i$. The result is straightforward. $\square$
References

[1] cite.Arnold741V. I. Arnold, *Mathematical methods of classical mechanics*, Nauka, Moscow, 1974.

[2] cite.Balint982St. Balint, *Lecții de mecanică teoretică. Mecanica solidului rigid*, Tip. Univ. de Vest, Timișoara, 1998.

[3] cite.Balint963Lecții de mecanică teoretică. Mecanica mediilor continue, Tip. Univ. de Vest, Timișoara, 1996.

[4] cite.Camenschi004G. Camenschi, *Introducere în mecanica mediilor continue deformabile*, Ed. Univ. București, București, 2000.

[5] cite.Comanescu045D. Comănescu, *Modele și metode în mecanica punctului material*, Mirton, Timișoara, 2004.

[6] cite.Dragoș036L. Dragoș, *Mathematical methods in Aerodynamics*, Kluwer Academic Pub. and Ed. Academiei Române, București, 2003.

[7] cite.Dragoș97M. Dragoș, *Mecanica fluidelor, vol. I. Teoria generală. Fluidul ideal incomprimibil*, Ed. Academiei Române, București, 1999.

[8] cite.Dragoș838Principiile mecanicii mediilor continue, Ed. Tehnică, București, 1983.

[9] cite.Dragoș769Principiile mecanicii analitice, Ed. Tehnică, București, 1976.

[10] cite.Iacob89I. Iacob, *Mecanică teoretică și mecanică, Ed. Academiei R.S.R.*, București, 1989.

[11] cite.Iacob81I. Iacob, *Mecanică teoretică*, Ed. Didactică și Pedagogică, București, 1980.

[12] cite.Soos832E. Soós and C. Teodosiu, *Calcul tensorial cu aplicații în mecanica solidelor*, Ed. Științifică și Enciclopedică, București, 1983.

[13] cite.Truedsell74C. Truesdell, *Introduction a la Mecanique des Milieux Continus*, Mason, Paris, 1974.

[14] cite.Valcovici68V. Vălcovici, St Bălan, and R. Voinea, *Mecanică teoretică, ed. a III-a*, Ed. Tehnică, București, 1968.

Address of author: Department of Mathematics, West University of Timișoara, Bd. Pârvan nr. 4, Timișoara, Romania
E-mail address: comanescu@math.uvt.ro