When Black Holes Meet Kaluza-Klein Bubbles

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Abstract

We explore the physical consequences of a recently discovered class of exact solutions to five dimensional Kaluza-Klein theory. We find a number of surprising features including: (1) In the presence of a Kaluza-Klein bubble, there are arbitrarily large black holes with topology $S^3$. (2) In the presence of a black hole or a black string, there are expanding bubbles (with de Sitter geometry) which never reach null infinity. (3) A bubble can hold two black holes of arbitrary size in static equilibrium. In particular, two large black holes can be close together without merging to form a single black hole.
1 Introduction

In four dimensions, there is a well known static solution describing two black holes held apart by a strut (conical singularity) running between them \cite{1, 2}. It is natural to ask if there is an analogous solution in higher dimensions. One problem is that if one tries in higher dimensions to construct a one dimensional strut using a line density of matter, it is likely to form a horizon around it and become a black string. A black string connecting two black holes is not likely to be static, but will instead collapse to a single black hole.

In five dimensional Kaluza-Klein (KK) theory, this is one of the questions that has recently been settled by a new class of exact solutions presented by Emparan and Reall \cite{3}. There are, indeed, static solutions involving two black holes which are nonsingular everywhere outside the horizons. The new solutions were found by generalizing an approach first used by Weyl \cite{4} to construct all static, axisymmetric vacuum solutions in four dimensions. In this case, one can choose suitable variables so that Einstein’s equation essentially reduces to a linear equation. Emparan and Reall showed that Weyl’s approach can be generalized to higher dimensional solutions with enough symmetry.

In the static two black hole solution, the role of the strut is played by a Kaluza-Klein “bubble of nothing”. This bubble was first found by Witten \cite{5} almost twenty years ago as the endstate of the decay of the KK vacuum. It can be obtained by a double analytic continuation of the five dimensional Schwarzschild solution. The characteristic feature of the bubble is that the KK circle smoothly shrinks to zero size at a finite radius. The result is a minimal area $S^2$ which is called the bubble, or “bubble of nothing” since there is no space inside. In Witten’s example, the bubble rapidly expands outward and hits null infinity. Its intrinsic geometry is de Sitter space.

The new solutions describe various configurations of black holes and KK bubbles. We explore the physical consequences of some of these solutions and find a number of surprising features\cite{3}. For example, one might have thought that a black hole with $S^3$ topology in KK theory is possible only if its size is smaller than the size of the circle at infinity. A much larger black hole should become a black string with topology $S^2 \times S^1$. Nevertheless, it turns out that in the presence of a KK bubble, there are arbitrarily large black holes with $S^3$ topology. Furthermore, two of these large black

\textsuperscript{1}The special case of just one black hole on a KK bubble was examined in detail in \cite{3} in terms of C-metric coordinates. The possibility of two black holes on a bubble was briefly mentioned in \cite{3}, but the solution was not explicitly constructed.
holes can be held in static equilibrium by just a small piece of bubble! Normally one expects that when two black holes are brought together, they merge into one; another event horizon usually forms which encloses both. However, we find that even when the separation between the black holes is a negligible fraction of their size, the solution remains static and does not form a single black hole. Yet another surprising feature involves the behavior of expanding bubbles in the presence of a black hole or black string. We will see that there are bubbles whose intrinsic geometry is de Sitter space, yet the bubbles never reach null infinity.

These results should be contrasted with related results about black holes and bubbles in Kaluza-Klein theory. It was shown in [2] that arbitrarily large $S^3$ black holes can exist for a fixed size circle at infinity. However, that was for extremally charged black holes in a five dimensional Einstein-Maxwell theory. In that case, the horizon remains a round sphere, and the size of the circle grows as one comes in from infinity. We are considering neutral black holes and five dimensional vacuum solutions. We will see that the circle does not grow, but the horizon can distort itself to fit into the space. It has also been shown that bubbles constructed from the Kerr metric do not reach null infinity [6]. However, our examples appear to be the first de Sitter bubbles with a complete null infinity.

Since KK bubbles can arise from the quantum instability of the vacuum, one would expect to produce not just one, but several expanding bubbles. The question then arises: what happens when two bubbles collide. In a recent paper [7], colliding Kaluza-Klein bubbles are studied in a 3+1-dimensional spacetime. The relevant solution is obtained from the previously mentioned two black hole solution in four dimensions by analytic continuation. The resulting metric has no struts and it describes two colliding $S^1$ Kaluza-Klein bubbles. It is shown that the collision produces a black hole with a Schwarzschild type singularity inside.

It was argued in [7] that a similar bubble collision in 4+1 dimensions should have symmetry $SO(2,1) \times U(1)$, and hence not be a Weyl solution. What happens if one analytically continues the five-dimensional static two black hole solution? We find a new solution describing two $S^2$ KK bubbles stuck on a (pre-existing) black string. The bubbles collide inside the horizon, but this is simply because everything inside the horizon must hit the singularity.

It was suggested in [3] that by continuously changing parameters one could find examples of black holes turning into black strings. This would be extremely interesting, since the nature of the transition between static black holes and black strings is not well understood (see, e.g., [8]). However, when examined more closely we find that this transition occurs in the space of Weyl solutions only with a drastic change
in the boundary conditions at infinity. If one keeps the boundary conditions fixed, continuous changes of the parameters do not change the horizon topology.

We begin with a brief review of the four and five dimensional Weyl solutions describing simple black holes and bubbles. In section 3 we present and analyze the exact solution describing two black holes on a KK bubble. The physical consequences are discussed in section 4, including the demonstration that a small piece of bubble can hold two large black holes apart. In section 5 we consider analytic continuations of the metric of section 3. In section 5.1 we analyze the solution describing two bubbles on a black string and show that there are expanding bubbles with de Sitter geometry which do not reach null infinity. In section 5.2 we present a different analytic continuation describing 3 adjacent Kaluza-Klein bubbles. Finally, we discuss our results in section 6.

2 Review of the Weyl Solutions

We begin by reviewing the Weyl form of simple black hole and bubble solutions in four and five dimensions. These will form the basis for the more general solutions we discuss later.

2.1 Four Dimensions

Static, axisymmetric vacuum solutions in four dimensions can be written in the form

\[ ds^2 = -e^{2U} dt^2 + e^{-2U} r^2 d\phi^2 + e^{2\nu} (dr^2 + dz^2), \]

where \( U \) is an axisymmetric solution to the Laplace equation in the flat metric \( dr^2 + r^2 d\phi^2 + dz^2 \), and \( \nu \) is determined by \( U \).

If \( U \) is the Newtonian potential of a rod of mass \( M \) and length \( 2M \), the metric (2.1) describes a Schwarzschild black hole of mass \( M \). Note that even though the effective source for \( U \) is only axisymmetric, the resulting spacetime is spherically symmetric. If one analytically continues \( t \rightarrow i\chi \) one obtains the euclidean black hole. Regularity requires that \( \chi \) be periodically identified so one has Kaluza-Klein boundary conditions. If one further analytically continues \( \phi \rightarrow i\tau \), one obtains the four dimensional Kaluza-Klein bubble. The bubble is defined to be the surface where \( g_{\chi\chi} = 0 \), but this is just the analytic continuation of the \( S^2 \) that was the black hole horizon. This means that the induced metric on the bubble is two dimensional de Sitter space. Thus, even though the metric looks static, the bubble is really expanding.
The form of the metric one obtains from (2.1) corresponds to writing de Sitter space in static coordinates.

If $U$ is chosen to be the Newtonian potential of two non-intersecting rods with masses $M_i$ and lengths $2M_i$, $i = 1, 2$, the metric (2.1) describes two black holes. The black holes are held apart by forces due to struts that arise from conical singularities on the axis between the two black holes. The configuration requires a single strut between the two black holes or alternatively two struts extending from each black hole to infinity along the axis of symmetry. This construction generalizes to solutions of $n$ collinear black holes: the lengths of the rods determine the sizes of the black holes, and in general there will be struts between the black holes.

Starting with two equal size static black holes and analytically continuing $t \rightarrow i\chi$ and $\phi \rightarrow i\tau$, the metric (2.1) describes two expanding $S^1$ Kaluza-Klein bubbles. This solution, which is completely free of conical deficits, was analyzed in [7].

### 2.2 Five Dimensions

As shown in [3], the Weyl ansatz can be extended to higher dimensions. Static, five dimensional vacuum solutions with two additional commuting isometries can be written in the form

$$ds^2 = -e^{2U_1}dt^2 + e^{2U_2}d\phi^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2),$$

(2.2)

where for $i = 1, 2, 3$, the functions $U_i$ are axisymmetric solutions to Laplace’s equation in three-dimensional flat space such that $U_1 + U_2 + U_3 = \log r + \text{constant}$. The function $\nu$ is determined by the $U_i$‘s. It is convenient to again take the $U_i$ to be the potentials of rods placed along the $z$ axis with mass per unit length $1/2$. (Other choices tend to produce naked singularities.) For each $i$, $U_i = \log r$ near a rod, so the corresponding metric function always vanishes at the rods. This produces event horizons when the associated coordinate is timelike, or axes for rotational symmetry when the associated coordinate is spacelike. If $U_1$ has no source, and $U_2$ has a rod of length $2M$, one obtains the trivial product of time and the euclidean Schwarzschild solution. This is the static $S^2$ Kaluza-Klein bubble.

Following [3], let $U_1$ be the potential of a rod with mass per length $1/2$ placed on the $z$-axis for $-\mu < z < \mu$, and let $U_2$ and $U_3$ be the potentials of semi-infinite rods with $z > \mu$ and $z < -\mu$ respectively, and mass per length $1/2$. The solution (2.2) then describes a five-dimensional Schwarzschild black hole with Schwarzschild radius $2\sqrt{\mu}$. By analytically continuing $t \rightarrow i\chi$ and $\phi \rightarrow i\tau$ the solution becomes that of an expanding Kaluza-Klein bubble which describes the decay [5] of the Kaluza-Klein
In analogy to the four-dimensional case, there are static solutions involving two five-dimensional black holes. As we shall see in the next section, in five dimensions the role of a strut providing the force to hold the two black holes apart is played by a Kaluza-Klein bubble; however, no conical deficits are associated with the bubble and even for black holes of different sizes we find that the solution is free of conical singularities.

3 Two Black Holes on a Kaluza-Klein Bubble

In section 3.1 we construct the solution describing two black holes on a Kaluza-Klein bubble. We identify the horizons and the bubble in section 3.2, and we analyze the asymptotic behavior in section 3.3.

3.1 The Solution

We study the metric (2.2) where \( U_i, i = 1, 2, 3 \), are the Newtonian potentials of line-masses with densities \( 1/2 \) such that for positive real numbers \( a, b, c \) with \( a, c > b \) we have sources (see Fig. 1)

\( U_1 \): Two finite rods positioned on the \( z \)-axis for \(-c < z < -b \) and \( b < z < a \);

\( U_2 \): One finite rod positioned on the \( z \)-axis for \(-b < z < b \);

\( U_3 \): Two semi-infinite rods positioned on the \( z \)-axis for \( z < -c \) and \( z > a \).
Define as in [3]

\[ \zeta_1 = z - a ; \quad \zeta_2 = z - b ; \quad \zeta_3 = z + b ; \quad \zeta_4 = z + c, \] (3.1)

and for \( i = 1, \ldots, 4 \)

\[ R_i = \sqrt{r^2 + \zeta_i^2} \] (3.2)

\[ Y_{ij} = R_i R_j + \zeta_i \zeta_j + r^2. \] (3.3)

We can then write the first three metric components as

\[ g_{tt} = -e^{2U_1} = -\frac{(R_2 - \zeta_2)(R_4 - \zeta_4)}{(R_1 - \zeta_1)(R_3 - \zeta_3)} \] (3.4)

\[ g_{\phi\phi} = e^{2U_2} = \frac{R_3 - \zeta_3}{R_2 - \zeta_2} \] (3.5)

\[ g_{\psi\psi} = e^{2U_3} = (R_1 - \zeta_1)(R_4 + \zeta_4), \] (3.6)

and solving for \( \nu \) using the methods described in [3], we find

\[ g_{rr} = g_{zz} = e^{2\nu} = \frac{Y_{14}Y_{23}}{4R_1 R_2 R_3 R_4} \sqrt{\frac{Y_{12}Y_{34}}{Y_{13}Y_{24}}} \frac{(R_1 - \zeta_1)}{(R_4 - \zeta_4)}. \] (3.7)

The periodicity of \( \psi \) and \( \phi \) are fixed by the requirement of regularity on the associated axes of rotation. As mentioned above, these axes correspond to the location of the rods for \( U_2 \) and \( U_3 \), since \( U_i \to -\infty \) near the rod causing the corresponding metric component to vanish. Thus for \( z < -c \) and \( z > a \), the orbit of \( \psi \) vanishes in the limit \( r \to 0 \). Regularity requires that in these \( z \)-regions

\[ \lim_{r \to 0} \frac{\sqrt{g_{\psi\psi}} \Delta \psi}{\int_0^r g_{rr} dr} = 2\pi, \] (3.8)

where \( \Delta \psi \) denotes the period of \( \psi \). For both cases, \( z < -c \) and \( z > a \), we find that (3.8) is satisfied by taking \( \Delta \psi = 2\pi \).

For \( |z| < b \), the orbit of \( \phi \) vanishes as \( r \to 0 \). Taking \( \phi \) to have period

\[ \Delta \phi = \frac{8\pi b(a + c)}{\sqrt{(a + b)(b + c)}}, \] (3.9)

the solution is regular and free of conical deficits.
3.2 Event Horizons and Bubbles

The metric component \( g_{tt} \) vanishes when \( r = 0 \) and \( b < z < a \) or when \( r = 0 \) and \( -c < z < -b \), corresponding to the two black hole horizons. For the first horizon the constant-\( t \) metric with \( r = 0 \) is given by

\[
\begin{align*}
&d\tilde{s}_{bh1}^2 = \frac{z - b}{z + b} d\phi^2 + 4(a - z)(z + c) d\psi^2 + \frac{(a + c)^2(a - b)(z + b)}{(a + b)(z - b)(a - z)(z + c)} dz^2
\end{align*}
\]

where \( b < z < a \), and for the second horizon the metric is

\[
\begin{align*}
&d\tilde{s}_{bh2}^2 = \frac{z + b}{z - b} d\phi^2 + 4(a - z)(z + c) d\psi^2 + \frac{(a + c)^2(c - b)(z - b)}{(b + c)(z + b)(a - z)(z + c)} dz^2
\end{align*}
\]

where \( -c < z < -b \). Since the \( \phi \) circles shrink to zero size at \( z = b \) and the \( \psi \) circles shrink to zero size at \( z = a \) or \( z = -c \), we see that topologically the horizons are \( S^3 \). The area of the first horizon is

\[
A_{bh1} = \frac{32\pi^2 b (a + c)^2 (a - b)^{3/2}}{(a + b)(b + c)^{1/2}}
\]

and the area of the second horizon is obtained by interchanging \( a \) and \( c \) in (3.12).

When we checked regularity we observed that for \( |z| < b \), the orbit of \( \phi \) vanished for \( r = 0 \). This means that between the two black holes sits a Kaluza-Klein “bubble of nothing”. The metric on this minimal bubble, for constant \( t \) and \( r = 0 \), is given by

\[
\begin{align*}
&d\tilde{s}_{\text{bubble}}^2 = 4(a - z)(c + z) d\psi^2 + \frac{4b^2(a + c)^2}{(a + b)(b + c) b^2 - z^2} dz^2
\end{align*}
\]

Note that the \( \psi \)-orbit does not close off at \( z = \pm b \), so that the “bubble” is actually a cylinder rather than an \( S^2 \); yet we shall continue to refer to it as a bubble.

The proper distance between the two black holes is

\[
s = \frac{2\pi b(a + c)}{\sqrt{(a + b)(b + c)}}
\]

along a curve of constant \( \psi \).
3.3 Asymptotic Behavior and Total Mass

Consider to leading order the asymptotic behavior of the metric. As \( \rho = \sqrt{r^2 + z^2} \to \infty \), we find to order \( O(\rho^{-2}) \)

\[
\begin{align*}
g_{tt} &= -\left(1 - \frac{a + c - 2b}{\rho}\right) \\
g_{\phi\phi} &= 1 - \frac{2b}{\rho} \\
g_{\psi\psi} &= r^2 \left(1 + \frac{a + c}{\rho}\right) \\
g_{rr} = g_{zz} &= 1 + \frac{a + c}{\rho}.
\end{align*}
\]

The leading order metric is

\[
ds^2_{\rho \to \infty} \sim -dt^2 + d\phi^2 + r^2d\psi^2 + dr^2 + dz^2,
\]

so asymptotically the space is \( M^{3,1} \times S^1 \). The \( S^1 \) at infinity is parametrized by \( \phi \) and its size is given by (3.9). Notice that the size of the circle at infinity is precisely four times the distance between the two black holes (3.14). Note also that \( M^{3,1} \) has a complete null infinity; the asymptotic flat metric is not cut-off. Both of these facts will be important in the next section.

Using the next to leading order metric \( h_{\mu\nu} \), we compute the total mass of the configuration. The ADM mass is given by

\[
M = \frac{1}{16\pi} \lim_{\rho \to \infty} \int_{S^2 \times S^1} (\partial_i h_{ij} - \partial_j h_{ii}) N^i dV,
\]

where the surface \( S^2 \times S^1 \) is at constant \( \rho \); here \( N^i \) is the unit normal vector of this surface and indices \( i \) and \( j \) label coordinates \( \phi, x, y, \) and \( z \) with \( x = r \cos \psi \) and \( y = r \sin \psi \). We find

\[
M = \frac{4\pi b(a + c - b)(a + c)}{\sqrt{(a + b)(b + c)}}.
\]

It is clear that the total mass is always positive.

4 Physical Consequences

In this section we analyze the physical behavior of the solution. We consider first the limit of two small black holes on a KK bubble (section 4.1). We then let the black
holes be much larger than their separation, and also study what happens when the separation goes to zero (section 4.2). It turns out that big black holes on a KK bubble resemble fat black strings, so in section 4.3 we compare the entropy of a black string with that of the two black hole solution.

4.1 Small Black Holes

To gain physical intuition about this solution, we start with the case $a = c = b + \epsilon$ where $\epsilon \ll b$. For $\epsilon = 0$, it is easy to see that $g_{tt} = -1$ and the solution reduces to the product of time and the euclidean Schwarzschild solution. This is the static Kaluza-Klein bubble with topology $S^2$, radius $2b$, and circle at infinity of length $8\pi b$. For $\epsilon \neq 0$, one adds two small black holes on opposite sides of this bubble. The geometry of the black holes (3.10), (3.11) simplify considerably in the limit of small $\epsilon$. Since the horizon corresponds to $b \leq |z| \leq b + \epsilon$, $g_{\phi\phi} = O(\epsilon)$. So to leading order in $\epsilon$, we can ignore the change in the periodicity of $\phi$. Let $\phi = 4b\varphi$ where $\varphi$ has period $2\pi$. Then to leading order in $\epsilon$, the horizon geometry (for $z > 0$) becomes

$$ds^2_{bh1} = 8b(z-b)d\varphi^2 + 8b(b+\epsilon-z)d\psi^2 + \frac{2b\epsilon}{(z-b)(b+\epsilon-z)}dz^2$$ (4.1)

Letting $z = b + \epsilon \sin^2 \theta$, with $0 \leq \theta \leq \pi/2$, this becomes

$$ds^2_{bh1} = 8b\epsilon(d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2)$$ (4.2)

The second black hole is similar. So the black holes are round three spheres with radius $\sqrt{8b\epsilon}$. Normally a small black hole in Kaluza-Klein theory would be localized on the $S^1$, but that is not what is happening here. The solution retains the rotational symmetry around the circle. A small spherical black hole is possible because the size of the compact direction shrinks to zero on the KK bubble.

At first sight, it is surprising that one can add black holes to a static bubble and keep the solution static. One might expect that adding some mass would cause the bubble to collapse. One can understand what is happening as follows. For a fixed size circle at infinity, one can construct a one parameter family of initial data describing bubbles of varying radii [9]. The initial data is time symmetric and the metric takes the form

$$ds^2 = F(\rho)d\chi^2 + F^{-1}(\rho)d\rho^2 + \rho^2d\Omega$$ (4.3)

where $F(\rho) = 1 - 4M/\rho - c/\rho^2$. The two free parameters are the total mass $M$, and an arbitrary constant $c$. The bubble is located at the positive zero of $F$, $\rho = \rho_+ \equiv$
$2M + \sqrt{4M^2 + c}$. Regularity requires that $\chi$ be periodic with period

$$L = \frac{2\pi \rho_+^2}{\rho_+ - 2M}.$$  \hspace{1cm} (4.4)

This is the length of the circle at infinity. Although the full evolution of this initial data is not known, Corley and Jacobson \[10\] showed that the second derivative of the bubble area with respect to proper time along the bubble is given by

$$\frac{d^2A}{d\tau^2} = 8\pi \left(1 - \frac{4M}{\rho_+}\right)$$  \hspace{1cm} (4.5)

Using (4.4) this can be rewritten as

$$\frac{d^2A}{d\tau^2} = 8\pi \left(\frac{4\pi \rho_+}{L} - 1\right)$$  \hspace{1cm} (4.6)

This clearly shows that there is a linear relationship between the bubble size and its initial acceleration.

Now consider the metric on the bubble (3.13). If one starts with the static bubble $a = c = b$, and increases $a$ and $c$ to add the two black holes, it is easy to see that the bubble size increases. So the interpretation is that the new bubble would normally accelerate outward, but is kept in place by the black holes. Conversely, the attraction between the black holes is exactly canceled by the natural expansion of the bubble. Since the general static solution has $a \neq c$, it is clear that the bubble can adjust itself to support unequal mass black holes. The fact that the solution remains static even with only one black hole on the bubble ($c = b, a > b$) indicates that the black hole can attract the bubble itself and stop it from expanding. We will see a clear example of this in section (5.1).

Now let us continue to increase the size of the black holes. One might expect that the black holes will merge when they become larger than the size of the bubble. However, it is easy to see that this does not happen. We have already remarked that the separation between the black holes is always one fourth of the size of the circle at infinity. So keeping our asymptotic metric fixed, the black holes never touch. To see what happens we now turn to the opposite limit of black holes much larger than the size of the bubble.

### 4.2 Big Black Holes

We now study the limit of two big black holes stabilized by the bubble between them. Taking $a, c \gg b$ the bubble metric (3.13) reduces to

$$ds^2_{\text{bubble}} = 4ac d\psi^2 + \frac{4b^2(a + c)^2}{ac} d\eta^2$$  \hspace{1cm} (4.7)
Figure 2: Geometry of a thin cylindrical Kaluza-Klein bubble separating two large $S^3$ black holes. One can view this as starting with a black string $S^2 \times S^1$ (with the $S^2$ radius much larger than the circle), cutting the $S^2$ into two pieces, separating the pieces, and gluing in a KK bubble.

where $z = -b \cos \eta$ with $0 < \eta < \pi$. The bubble is a flat cylinder with large radius $2\sqrt{ac}$ and small height $2\pi b(a + c)/\sqrt{ac}$.

Away from the bubble, the metric of each black hole horizon is approximately

$$ds^2_{bh1,2} = d\phi^2 + 4(a - z)(z + c)\,d\psi^2 + \frac{(a + c)^2}{(a - z)(z + c)}\,dz^2$$

where $b \ll z < a$ for horizon 1 and $c < z \ll -b$ for horizon 2. This can be further simplified by defining $z = [(a - c) + (a + c)\cos \theta]/2$. Then the horizon metric becomes

$$ds^2_{bh1,2} = d\phi^2 + (a + c)^2[d\theta^2 + \sin^2 \theta\,d\psi^2]$$

This is just the product of a circle and (part of) a round two-sphere of radius $a + c$. The first black hole corresponds to roughly $0 \leq \theta < \theta_0$ where $\cos \theta_0 \approx (c - a)/(a + c)$, while the second is approximately $\theta_0 < \theta \leq \pi$. The estimate for $\theta_0$ is only approximate since when $|z|$ approaches $b$, the horizon no longer takes the form (4.8). In this region, the radius of the $\phi$ circles becomes $z$ dependent and shrinks to zero at $|z| = b$. This is how one obtains arbitrarily large black holes with $S^3$ topology in a space with one direction compactified. One starts with a large four dimensional black hole, which in Kaluza-Klein theory is really a $S^2 \times S^1$ black string. One then adds a KK bubble which pinches off the horizon changing its topology to $S^3$.

The solution analyzed here provides a way of deforming a black string with horizon $S^2 \times S^1$ into a configuration of two large $S^3$ black holes, separated by a cylindrical KK bubble (see Fig. 2). If the radius of the black string $S^2$ is much larger than the radius
of the $S^1$, we can cut the $S^2$ along any latitude and insert a KK bubble between the two parts such that the $\phi$-orbit on the horizon shrinks smoothly to zero where the bubble intersects the horizon. In section 4.3, we compare the entropy of the original black string to the entropy of the black holes on the bubble.

It seems quite remarkable that a small piece of Kaluza-Klein bubble can hold two large black holes apart. Not only does it compensate for the expected gravitational attraction, but it also prevents the black holes from merging. In 3 + 1 dimensions, if one brings two black holes close together, a third horizon forms surrounding the two. (This can be clearly seen in terms of initial data.) In these Kaluza-Klein solutions, this does not happen even when the separation between the black holes is negligible compared to their size. We do not have an intuitive explanation for this.

It is easy to see that adding the bubble does not change the total mass significantly: From (3.21) and (3.9), one sees that the total mass divided by the length of the circle at infinity is $M/L = (a + c - b)/2 \approx (a + c)/2$. This agrees with the mass per unit length of a black string with Schwarzschild radius $a + c$ and length $L$.

One possibly relevant fact is that if one includes the time direction, the metric on the bubble (4.7) becomes

$$ds^2_{\text{bubble}} = 4ac d\psi^2 + \frac{4b^2(a + c)^2}{ac} \left[ d\eta^2 - \frac{\sin^2 \eta}{4(a + c)^2} dt^2 \right], \quad (4.10)$$

Thus the bubble is the product of a circle, and a static patch of two dimensional de Sitter spacetime. We have seen that the coordinate $t$ is a unit time translation at infinity. The form of $g_{tt}$ means that there is a redshift between the bubble and infinity which can be interpreted as arising from the proximity of the black holes. The de Sitter geometry indicates that the bubble is expanding, but only a static patch appears outside the black holes. So the distance between the black holes remains constant. The de Sitter horizons coincide with the black hole horizons.

What happens to the bubble inside the black holes? The bubble is defined by $g_{\phi\phi} = 0$. The black hole horizon is a three sphere, and $g_{\phi\phi} = 0$ just selects a circle on this sphere. The usual Penrose diagram for a black hole suppresses the entire sphere, so it corresponds to fixing one point on the sphere. If one fixes this point to lie on the line $g_{\phi\phi} = 0$, then the bubble extends throughout the black hole in a Penrose diagram. On the other side of the black hole throat is another region of spacetime which is identical to the one on this side. So there is another static patch of a de Sitter bubble on the other side which joins onto the first. The result is shown in Fig. 3.\[\] The global topology of the bubble is now $T^2$ since the direction through the

\[\text{We have made the simplest assumption that the spacetime on the other side of the two black}\]
black holes provides a second circle in addition to the one parameterized by $\psi$. This qualitative discussion can be made more precise using the coordinates introduced in section 5, which extend the metric inside the event horizons and beyond the black hole throat.

Most other two dimensional slices through the geometry have a more conventional description. For instance, if we fix $\phi$, $\psi$ and $z$, the geometry on the $r,t$ plane depends on the value of $z$. If $z > a$ or $z < -c$, then the spacetime is causally the same as two dimensional Minkowski space. If $b < z < a$ or $-c < z < -b$, the causal structure is the same as Schwarzschild, with its two asymptotically flat regions. If $-b < z < b$, and we now include the $\phi$ circles, the geometry looks like a static bubble: the radius of the $\phi$ circles goes from a constant at infinity to zero on the bubble. There is a redshift between the bubble and infinity which increases as $|z|$ approaches $b$.

As we discussed in section 2, the Weyl solutions can be characterized by the rods which act like sources for the functions $U_i$. From a comparison of the configurations of rods for the two black holes on a Kaluza-Klein bubble with that of a four-dimensional Schwarzschild times a flat direction — i.e. a black string — it is tempting to conclude that the limit $b \to 0$ corresponds to a transition between black holes and a black string. This would be very interesting since one expects (in the absence of KK bubbles) that a small $S^3$ black hole in Kaluza-Klein theory becomes a black string when its size increases, yet exactly how this transition occurs is not understood. Unfortunately, the rods are misleading. It is true that in the limit $b \to 0$, the length of Kaluza-Klein bubble goes to zero and the two black holes approach each other. However
the size of the circle at infinity also vanishes. In fact, in this limit, the two horizons match up to form a perfect two-dimensional sphere and the solution becomes that of a four-dimensional Schwarzschild black hole with a four-dimensional mass \((a + c)/2\). However, this limit is very singular, since e.g. the curvature on the Kaluza-Klein bubble diverges as its size goes to zero. It appears one cannot use these Weyl metrics to study the black hole–black string transition.

### 4.3 Entropy

In a spacetime which asymptotically is \(M^{3,1} \times S^1\) we have studied the exact solution describing two black holes sitting on a Kaluza-Klein bubble. In terms of the parameters \(a\), \(b\), and \(c\) the circle at infinity has length

\[
L = \frac{8\pi b(a + c)}{\sqrt{(a + b)(b + c)}},
\]

the ADM mass of the configuration is

\[
M = \frac{4\pi b(a + c - b)(a + c)}{\sqrt{(a + b)(b + c)}},
\]

and the total area of the two black holes is

\[
A_{2BH} = A_{bh1} + A_{bh2} = \frac{32\pi^2 b(a + c)^2}{(a + b)^{1/2} (b + c)^{1/2}} \frac{(a - b)^{3/2} + (c - b)^{3/2}}{(a + b)^{1/2} + (c + b)^{1/2}}
\]

(4.13)

Since the black holes resemble a black string everywhere away from the bubble, we compare this area to the area of a five-dimensional black string with the same mass (4.12) and size of circle at infinity (4.11). The black string metric can be written

\[
ds_{BS}^2 = -(1 - \frac{R_0}{R}) dt^2 + \left(1 - \frac{R_0}{R}\right)^{-1} dR^2 + R^2 d\Omega_2^2 + dz^2
\]

(4.14)

with \(z \sim z + L\). The ADM mass of the black string is \(M = R_0 L/2\) and the horizon area is \(A_{BS} = 4\pi R_0^2 L\). Thus in terms of the mass, the area is \(A_{BS} = 16\pi M^2/L\). Inserting the values (4.11) and (4.12) we find, using dimensionless scalings \(x, y > 1\) defined by \(a = xb\) and \(c = yb\), that the ratio of the areas is given by

\[
\frac{A_{2BH}}{A_{BS}} = \frac{x + y}{(x + y - 1)^2} \left[\frac{(x - 1)^{3/2}}{(x + 1)^{1/2}} + \frac{(y - 1)^{3/2}}{(y + 1)^{1/2}}\right].
\]

(4.15)

Analyzing this function we find that for all \(x, y > 1\),

\[
A_{2BH} < A_{BS}
\]

(4.16)
so that the simple black string is always entropically favored over the configuration of two black holes on a Kaluza-Klein bubble. Hence we should not expect a black string to spontaneously generate a KK bubble that splits the black string horizon $S^1 \times S^2$ into two black hole $S^3$ horizons connected by the bubble.

5 Analytic Continuation

In this section we consider double analytic continuations of the metric describing two black holes on a Kaluza-Klein bubble. We shall do this in two ways, leading to two distinct solutions: one describing two $S^2$ KK bubbles on a black string (section 5.1), the other describing three adjacent KK bubbles (section 5.2). The first solution has the standard Kaluza-Klein boundary conditions, but the second does not — it approaches $M^3 \times S^1 \times S^1$ asymptotically. For this reason, we concentrate on the first solution and discuss the second only briefly.

5.1 Two bubbles on a black string

We analytically continue the solution from section 3 by taking $t \to i\chi$ and $\phi \to i\tau$, so that the metric is now

$$ds^2 = e^{2U_1}d\chi^2 - e^{2U_2}d\tau^2 + e^{2U_3}d\psi^2 + e^{2\nu}(dr^2 + dz^2),$$  \hspace{1cm} (5.1)

with $U_i$ and $\nu$ given by equations (3.4)-(3.7). To avoid conical singularities, we must set $a = c$ and make $\chi$ periodic with period

$$\Delta \chi = 8\pi a \sqrt{\frac{a - b}{a + b}}. \hspace{1cm} (5.2)$$

The Kaluza-Klein circle at infinity is now parameterized by $\chi$.

When $r \to 0$, $g_{\tau\tau}$ vanishes for $|z| < b$, so at $r = 0$ and $|z| < b$ we have a horizon. This horizon has topology $S^2 \times S^1$, so it is a black string. The $S^2$ is parameterized by $z$ and $\chi$, and the $S^1$ is parameterized by $\psi$. The constant-$\tau$ metric of the horizon is

$$ds^2_{\text{horizon}} = \frac{b^2 - z^2}{a^2 - z^2}d\chi^2 + 4(a^2 - z^2)d\psi^2 + \frac{16a^2 b^2}{(a + b)^2 b^2 - z^2}dz^2,$$

with $|z| < b$. The area of the horizon is

$$A_{\text{horizon}} = \frac{(16\pi ab)^2}{a + b} \sqrt{\frac{a - b}{a + b}}. \hspace{1cm} (5.4)$$
For $b < |z| < a$, the orbit of $\chi$ vanishes as $r \to 0$. This means that at $r = 0$ we have two KK bubbles with $b < z < a$ and $-a < z < -b$, respectively. The metric of the first bubble is (constant $\tau$ and $r = 0$)

$$ds^2_{\text{bubble}1} = 4 (a^2 - z^2) d\psi^2 + \frac{4 a^2 (a-b)}{a+b} \frac{(z+b)}{(z-b)} \frac{dz^2}{a^2 - z^2}$$

with $b < z < a$. Topologically, this describes a disk not an $S^2$ since the orbit of $\psi$ does not close off at $z = b$. However, these coordinates do not cover the entire constant-$\tau$ surface. Consider a geodesic with constant $\tau, \chi, \psi$ as it approaches $r = 0$. If either $g_{\chi\chi}$ or $g_{\psi\psi}$ vanishes at $r = 0$, then this is just the axis of a rotational symmetry, and the geodesic continues to positive values of $r$ with $\chi$ or $\psi$ shifted by half its period. This is the case for $|z| \geq b$. However, for $|z| < b$, both $g_{\chi\chi}$ and $g_{\psi\psi}$ remain nonzero and one can continue the spacetime past $r = 0$. Since the metric only depends on $r^2$, the natural extension is to let $r$ become negative. This yields another copy of the geometry which is exactly analogous to the region on the other side of the Schwarzschild throat in the maximally extended Schwarzschild geometry. The spacetime is clearly invariant under $r \to -r$ which implies that the black string horizon at $r = 0, |z| \leq b$ is a minimal $S^2 \times S^1$.

Returning to the bubble, we now see that there is another copy of the disk (5.3) on the negative $r$ side and the two disks smoothly join at $z = b$ to make an $S^2$. A similar argument applies to the bubble at $z = -b$. The net result is that the configuration on a constant $\tau$ slice describes two $S^2$ bubbles on opposite sides of a $S^2 \times S^1$ horizon. The spacetime appears static, but since the coordinates do not cover the entire spacetime, this is misleading. We now show that under evolution the two bubbles collide at the black string singularity.

When one considers the evolution of this spacetime, one immediately faces an apparent contradiction. Consider the limit when $a$ is close to $b$, so the bubbles are small. We saw in the previous section, that before the analytic continuation, this spacetime described two small $S^3$ black holes on a KK bubble. Since the black holes are round spheres, after the analytic continuation, the bubble geometry must be three dimensional de Sitter space $dS_3$. In other words, these are expanding bubbles. It was shown in [6] that de Sitter bubbles expand out and hit null infinity. However, it is clear from the asymptotic form of the metric that null infinity is complete. So the bubble never reaches null infinity. The resolution is that the bubbles are held in by the black string. Most of each bubble lies inside the black string, and only the static patch extends outside the horizon (see Fig. 4).

As we proceed with the analysis of the collision of the two bubbles, we shall follow
Figure 4: The evolution of an $S^2$ bubble sitting on an $S^2 \times S^1$ black string. The bubble geometry is de Sitter, but does not reach null infinity. There is another bubble on the opposite side of the $S^2 \times S^1$ black string which is not shown.

the analysis of [7] closely. In order to describe the spacetime to the future of the black string horizon, we analytically continue $r$, taking $\tilde{r} = i r$. Then $\tilde{r}$ is the time coordinate and $\tau$ becomes spacelike. We introduce double null coordinates

$$u = \tilde{r} + z,$$

$$v = \tilde{r} - z$$

(5.6)

and just as in [7] we find that the metric in terms of $u$ and $v$ has coordinate singularities on each of the null hypersurfaces $u = b$ or $v = b$. In terms of a better set of null coordinates

$$U = -\sqrt{b - u}$$

and

$$V = -\sqrt{b - v}$$

(5.7)

the metric is no longer singular at $u = b$ or $v = b$. However, along

$$U^2 + V^2 = 2b$$

(5.8)

some metric components diverge. We now show that this is just a coordinate singularity along the two timelike segments, $UV < 0$, corresponding to the motion of the bubbles. However, along the future spacelike segment, $U, V > 0$, there is a real curvature singularity. The past spacelike segment, $U, V < 0$, is simply the black string horizon encountered above.

We now establish these claims. We expand the metric near the time-like segment of (5.8) along the null hypersurface $V = V_0 < 0$, taking

$$U = \sqrt{2b - V_0^2} - \epsilon,$$

(5.9)

In this analytic continuation, $e^{2\nu}$ must be continuous, and hence stay positive.
where \( \epsilon > 0 \) is assumed to be small. To leading order the metric behaves as

\[
g_{XX} \sim O(\epsilon^2), \quad g_{\tau\tau} \sim O(1), \quad g_{\psi\psi} \sim O(1), \quad g_{UV} \sim O(1). \tag{5.10}
\]

This corresponds to a flat space four-dimensional geometry times a circle parametrized by \( \psi \). Thus the metric remains nonsingular while the bubbles approach each other.

For the future spacelike segment, \( V = V_0 > 0 \) and \( U \) again given by (5.9), we find

\[
g_{XX} \sim O(\epsilon^4), \quad g_{\tau\tau} \sim O(\epsilon^{-2}), \quad g_{\psi\psi} \sim O(1), \quad g_{UV} \sim O(\epsilon^4). \tag{5.11}
\]

This is like a black string singularity. To see this, note that the metric for the five-dimensional Schwarzschild black string near the singularity can be written

\[
ds_{BS}^2 \approx \frac{2m}{r} dt^2 - \frac{r}{2m} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\chi^2) + d\psi^2
\]

Setting \( r = \epsilon^2 \) and introducing double null coordinates \([7]\), \( U = \sqrt{2/m \epsilon + \theta} \) and \( V = \sqrt{2/m \epsilon - \theta} \), we obtain the same behavior as in (5.11).

We conclude that the bubbles evolve from the \( r = 0 \) surface, and as they approach each other the metric remains non-singular. The bubbles collide at the null points \((U,V) = (\sqrt{2b},0)\) and \((U,V) = (0,\sqrt{2b})\) corresponding to the black string curvature singularity. These points are at null proper distance because \( g_{UV} \to 0 \).

The question remains whether the black string is formed by the collision of the two bubbles (as in \([\text{7}]\)) or whether the bubbles are just sitting on opposite poles of the \( S^2 \) on the \( S^2 \times S^1 \) horizon and brought together simply because the entire \( S^2 \) shrinks to a point at the singularity. The correct interpretation is the latter. A simple collision of two KK bubbles should have symmetry \( SO(2,1) \times U(1) \), since a single bubble has symmetry \( SO(3,1) \times U(1) \), and a second bubble will break the Lorentz group down to the subgroup acting orthogonal to the direction between the bubbles. This symmetry is incompatible with the Weyl ansatz. By adding the black string, one replaces the \( SO(2,1) \) symmetry by a time translation and axisymmetry. This can be seen by looking at the induced metric at \( z = 0 \). This is the surface exactly half-way between the bubbles. In the simple colliding bubble solution, the \( \psi \) circles should shrink to zero size, and the metric should have \( SO(2,1) \) symmetry. The fact that the \( \psi \) orbit is nonzero on the horizon indicates that there is a pre-existing black string.

### 5.2 Other Configurations

Consider first \( c = b \). Before analytic continuation the metric describes a single black hole sitting on a Kaluza-Klein bubble. Making a double analytic continuation of
this metric, taking $t \to i\chi$ and $\phi \to i\tau$, interchanges the roles of the black hole and bubble. If the original configuration described a small black hole sitting on a big bubble ($a - b \ll b$; as in section 4.1, but with $a \neq c$ and $c = b$), the solution obtained by analytic continuation describes a big black hole on a tiny bubble ($a - b \gg b$; as in section 4.2 with $c = b$). The coordinates do not describe the full constant-$\tau$ surface: there is another asymptotic region described by letting $r$ take negative values. In this region, we find (by symmetry in $r \to -r$) the other "half" of the Kaluza-Klein bubble, so that in the full space the bubble is topologically an $S^2$. The solution of a black hole on a KK bubble was studied in [3] in terms of C-metric coordinates.

Now let $b \neq c$. As in section 5.1, we analytically continue the metric with $a = c$ by taking $t \to i\chi$, but this time we take $\psi \to i\tau$. Again regularity requires $\chi$ to be periodic with period (5.2). This configuration describes three adjacent KK bubbles at $r = 0$: for $|z| < b$ we have an $S^2$ bubble parametrized by $z$ and $\chi$, and for $b < |z| < a$ we find two bubbles parametrized by $z$ and $\phi$. For $r = 0$ and $|z| > a$ we encounter Rindler horizons. Extending the $r$-coordinate to run over all real values, we find that the two latter bubbles are also topologically $S^2$. The solution has two $S^1$’s at infinity, so asymptotically it is $M^3 \times S^1 \times S^1$ and not the Kaluza-Klein vacuum. Although we have not studied this solution in detail, it appears that at least two of the bubbles expand outward and hit null infinity. It is thus remarkable that the bubbles appear never to collide and the spacetime remains nonsingular.

6 Discussion

Higher dimensional gravity plays an important role in recent discussions of string theory, M-theory, and brane worlds. We have seen that it has many unexpected properties. Using the methods of [3] we have constructed and studied a three parameter family of exact solutions describing the interaction of black holes and Kaluza-Klein bubbles. One of the most surprising results is that a small piece of bubble can support two enormous black holes in static equilibrium. The gravitational attraction of the black holes is apparently balanced by the tendency of the bubble to expand. In the spirit of the principle of maximum tension recently discussed by Gibbons [12], we find it interesting that we have a static solution with arbitrarily close black holes supported by a KK bubble. However, in this limit it is difficult to define the force between the black holes, so we cannot discuss whether there is a corresponding bound on a maximum repulsive force.

One might hope that these solutions allow one to study the transition between horizon topologies, for example two black holes merging to form a black string in KK
theory. As we have seen, the Weyl solutions we have studied here are not suitable for this purpose. The fact that the proper distance \( \frac{3.14}{3.9} \) between the black holes is one fourth of the size of the circle at infinity \( \frac{3.9}{3.9} \) implies that we cannot let the black holes approach each other without shrinking the circle at infinity. In section 4.2 we saw that in the limit \( b \to 0 \), the circle at infinity vanishes and the black hole horizons join up to form the round \( S^2 \) horizon of a four-dimensional Schwarzschild black hole. However, the curvature blows up on the KK bubble, so the limit is singular.

The relation between the black hole separation and the size of the circle at infinity is a consequence of the requirement of regularity for the \( \phi \)-coordinate in the region \(|z| < b\) when \( r \to 0\). We can ease up on this requirement at the cost of introducing conical singularities, and it is natural to ask how this will influence the black hole merger. Now instead of fixing the period of \( \phi \) by \( \frac{3.9}{3.9} \) as required by regularity, let us set \( \Delta \phi = 2\pi k \) for some positive constant \( k \). This introduces a conical singularity at \( r = 0 \) for \(|z| < b\), and including the \( \psi \)-direction we find that the conical singularity is actually spread over the surface of the KK bubble. The deficit angle associated with the conical singularity is

\[
\delta = 2\pi \left(1 - \frac{k\sqrt{(a+b)(b+c)}}{4b(a+c)}\right).
\]

Note that for given \( a, b, \) and \( c \) the angle \( \delta \) can be positive or negative, depending on the choice of \( k \). If \( k < \frac{4b(a+c)}{\sqrt{(a+b)(b+c)}} \), the angle is a deficit angle so the strut provides a pull. However, since the period of \( \phi \) is smaller than in \( \frac{3.9}{3.9} \), the black holes are now smaller than in the configuration without the strut. So the bubble now balances the smaller black holes as well as the pull of the strut. If \( k > \frac{4b(a+c)}{\sqrt{(a+b)(b+c)}} \), the angle is an excess angle. The combined efforts of the bubble and the strut can now balance two bigger black holes.

Now let the black holes approach each other by taking \( b \to 0 \). If \( k \propto b \), then \( \Delta \phi \to 0 \) as \( b \to 0 \), so the circle at infinity vanishes. Alternatively, if \( k \) approaches some nonzero constant as \( b \to 0 \), then for sufficiently small \( b \) the angle \( \delta \) is an excess angle and in the limit \( b \to 0 \) this excess angle diverges. We conclude, as in section 4.2, that even in the presence of conical singularities the rod picture is deceptive: when \( b \to 0 \), the rod picture suggests that the two black holes merge to form a four-dimensional Schwarzschild black hole times a flat direction — a black string — however, when taking regularity and struts into account we have shown that this limit is singular.

In section 4.2 we showed how a fat black string can be deformed by cutting the \( S^2 \) along any latitude, inserting a piece of KK bubble, and requiring the \( S^1 \) to smoothly
shrink to zero size where the bubble intersects the horizons. The result is two black holes separated by a KK bubble. Away from the bubble, the horizon of a big black hole again looks like a black string and we can repeat the cutting and gluing process to obtain a solution describing two black holes and one black string held apart by KK bubbles. This is illustrated in Fig. 5. If we analytically continue $t \rightarrow i\chi$ and $\phi \rightarrow i\tau$ we obtain a solution with two long thin black strings separated by a KK bubble, and on each black string sits an $S^2$ KK bubble. Generalizing the above, we find solutions describing $n$ collinear black strings separated by KK bubbles; at each end we either have a black hole with an $S^3$ horizon or an $S^2$ KK bubble. Thus solutions with multiple black strings and black holes combine the configurations studied in sections 4 and 5.

We have not investigated the stability of the solutions. On the one hand, one might argue that the solutions describing two static black holes on a bubble should be unstable since they represent a delicate balance between the attraction of the black holes and expansion of the bubble. Intuitively, if a perturbation increases the distance between the black holes, their attraction will decrease, but the expansion may increase, causing the black holes to accelerate apart. Conversely, a perturbation which brings the black holes slightly closer may cause the system to collapse into a single black hole. On the other hand, this Newtonian argument might fail for large black holes supported by a bubble. In this case, the bubble geometry is de Sitter space and is expected to be stable [6]. For the solution in 5.1 describing two KK
bubbles on a black string, in the limit $a \gg b$ the black string takes the form of a small round $S^2$ times a long $S^1$. This is precisely the regime in which one usually expects the black string to be unstable [13]. Can the large KK bubbles stabilize the black string?

We may at this point speculate about generalizations of our solution to higher dimensions. These cannot be achieved as a Weyl solution: the $d$-dimensional generalized Weyl solutions have $d - 2$ commuting Killing vector fields, which are not permitted by $d > 5$-dimensional Schwarzschild solutions. So the interaction of higher dimensional black holes and bubbles will require a new class of solutions.

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