THE SPACE OF RATIONAL MAPS ON $\mathbb{P}^1$

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Abstract. The set of morphisms $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d$ is parametrized by an affine open subset $\text{Rat}_d$ of $\mathbb{P}^{2d+1}$. We consider the action of $\text{SL}_2$ on $\text{Rat}_d$ induced by the conjugation action of $\text{SL}_2$ on rational maps; that is, $f \in \text{SL}_2$ acts on $\phi$ via $\phi^f = f^{-1} \circ \phi \circ f$. The quotient space $M_d = \text{Rat}_d/\text{SL}_2$ arises very naturally in the study of discrete dynamical systems on $\mathbb{P}^1$. We prove that $M_d$ exists as an affine integral scheme over $\mathbb{Z}$, that $M_2$ is isomorphic to $\mathbb{A}^2_\mathbb{Z}$, and that the natural completion of $M_2$ obtained using geometric invariant theory is isomorphic to $\mathbb{P}^2_\mathbb{Z}$. These results, which generalize results of Milnor over $\mathbb{C}$, should be useful for studying the arithmetic properties of dynamical systems.

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§1. Notation and summary of results

A rational map $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree $d$ over a field $K$ is given by a pair of homogeneous polynomials

$\phi = [F_a, F_b] = [a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d, b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d]$ of degree $d$ such that $F_a$ and $F_b$ have no common roots (in $\mathbb{P}^1(\overline{K})$). This last condition is equivalent to the condition that

$\text{Res}(F_a, F_b) \neq 0$.
where the resultant $\text{Res}(F_a, F_b)$ is a certain bihomogeneous polynomial in the coefficients $a_0, a_1, \ldots, a_d, b_0, \ldots, b_d$. We will also frequently write such maps $\phi$ in non-homogeneous form as

$$\phi(z) = \frac{a_0 z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d}{b_0 z^d + b_1 z^{d-1} + \cdots + b_{d-1} z + b_d}.$$

We are interested in studying the space of all rational maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d$. These maps are parametrized by the coefficients of $F_a$ and $F_b$, but notice that these are homogeneous coordinates, since for any non-zero constant $c$ we have $[F_a, F_b] = [cF_a, cF_b]$. Thus the space of rational maps of degree $d$ is the open subset of $\mathbb{P}^{2d+1}$ given by the condition $\text{Res}(F_a, F_b) \neq 0$. Notice that this set is an affine variety, since it is the complement of a hyperplane.

**Definition.** The *space of rational maps of degree* $d$ is the affine open subscheme of $\mathbb{P}^{2d+1} = \text{Proj} \mathbb{Z}[a_0, \ldots, b_d]$ defined by

$$\text{Rat}_d = \mathbb{P}^{2d+1} \setminus \{\text{Res}(F_a, F_b) = 0\}.$$  

To ease notation, we will write

$$A_d = \mathbb{Z}[a_0, a_1, \ldots, a_d, b_0, b_1, \ldots, b_d],$$

$$\rho = \rho(a, b) = \text{Res}(F_a, F_b) \in A_d.$$

Then $\text{Rat}_d = \text{Proj} A_d \setminus \{\rho = 0\}$, so

$$H^2(\text{Rat}_d, \mathcal{O}_{\text{Rat}_d}) = A_d[\rho^{-1}]_{(0)}$$

$$= \mathbb{Z} \left[ \frac{a_0^{i_0} a_1^{i_1} \cdots a_d^{i_d} b_0^{j_0} b_1^{j_1} \cdots b_d^{j_d}}{\rho} \right]_{i_0 + \cdots + i_d + j_0 + \cdots + j_d = 2d},$$

where the “(0)” subscript denotes elements of degree 0 (i.e., rational functions whose numerator and denominator are homogeneous of the same degree).

**Remark.** The space $\text{Rat}_d(\mathbb{C})$ of rational maps over the complex numbers has been studied in some detail. In particular, Segal [12] has studied the topology of $\text{Rat}_d(\mathbb{C})$ intrinsically and as a subset of the space of all continuous maps $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $d$. For example, he proves that the fundamental group $\pi_1(\text{Rat}_d(\mathbb{C}))$ is cyclic of order $2d$ and he gives an explicit description of the universal cover of $\text{Rat}_d(\mathbb{C})$. We will not consider topological questions of this nature in this paper.

The general linear group $\text{GL}_2$ acts on $\mathbb{P}^1$ via linear fractional transformations in the usual way,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : [X, Y] \mapsto [\alpha X + \beta Y, \gamma X + \delta Y].$$

The scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ act trivially, so $\text{GL}_2$ actually acts through its quotient $\text{PGL}_2 = \text{GL}_2 / \mathbb{G}_m$. For various reasons, we will instead consider the action of the special linear group $\text{SL}_2$. There is very little lost in doing this, since over an algebraically closed field, the map $\text{SL}_2 \to \text{PGL}_2$ is surjective with kernel equal to $\{\pm 1\}$. (In general over a field, one has

$$1 \to \mu_n(K) \to \text{SL}_n(K) \to \text{PGL}_n(K) \xrightarrow{\text{det}} \mathbb{K}^* / \mathbb{K}^* \to 1.$$}


The action of $\text{SL}_2$ on $\mathbb{P}^1$ induces several actions on the space of rational functions. The one we will be interested in is the conjugation action given as follows:

For $f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2$ and $\phi = [F_a, F_b] \in \text{Rat}_d$,

\[
\phi^f = f^{-1} \circ \phi \circ f = \begin{bmatrix} \delta F_a(\alpha X + \beta Y, \gamma X + \delta Y) - \beta F_b(\alpha X + \beta Y, \gamma X + \delta Y), \\
- \gamma F_a(\alpha X + \beta Y, \gamma X + \delta Y) + \alpha F_b(\alpha X + \beta Y, \gamma X + \delta Y) \end{bmatrix}.
\]

**Definition.** The *space of conjugacy classes of rational maps of degree* $d$ is the quotient space of $\text{Rat}_d$ by the conjugacy action of $\text{SL}_2$ (in whatever sense this quotient exists). It is denoted by

\[ M_d = \text{Rat}_d / \text{SL}_2. \]

The natural projection map from $\text{Rat}_d$ to $M_d$ will be denoted

\[ \langle \cdot \rangle : \text{Rat}_d \rightarrow M_d. \]

Our principal aim in this paper is to study the extent to which $M_d$ has any sort of nice structure. A priori, about the only thing one can say is that for an algebraically closed field $\Omega$, the quotient $M_d(\Omega) = \text{Rat}_d(\Omega) / \text{SL}_2(\Omega)$ exists as a set.

**Remark.** Over the complex numbers, it seems to be known (but not published?) that $M_d(\mathbb{C})$ has a natural structure as a complex orbifold, and this is made explicit for $M_2(\mathbb{C})$ in [7]. In fact, Milnor shows that $M_2(\mathbb{C}) \cong \mathbb{C}^2$, and this is one of the results we will generalize in this paper. See also [11] for a detailed analysis of various parameter spaces for rational maps of degree two over $\mathbb{C}$ and the loci corresponding to rational maps which have various complex dynamical properties.

Our first result says that the quotient space $M_d$ exists as a geometric quotient scheme over $\text{Spec} \mathbb{Z}$ in the sense of Mumford’s geometric invariant theory. We can further use geometric invariant theory to deduce various properties about $M_d$ and to construct a natural completion.

**Theorem 1.1.** The quotient $M_d = \text{Rat}_d / \text{SL}_2$ exists as a geometric quotient scheme over $\text{Spec} \mathbb{Z}$. It is an affine integral connected scheme whose affine coordinate ring is the ring of invariant functions

\[ H^0(M_d, O_{M_d}) = H^1(\text{Rat}_d, O_{\text{Rat}_d})^{\text{SL}_2} = (A_d[\rho^{-1}]_{(0)})^{\text{SL}_2}. \]

**Remark.** The precise definition of geometric quotient can be found in [10, definition 0.6]. Briefly, in addition to those properties described in the theorem, the quotient scheme $M_d / \mathbb{Z}$ has the following pleasant properties:

1. The following diagram commutes:

\[
\begin{array}{ccc}
\text{SL}_2 \times_{\mathbb{Z}} \text{Rat}_d & \xrightarrow{\text{action of } \text{SL}_2 \text{ on } \text{Rat}_d} & \text{Rat}_d \\
\downarrow \text{proj}_2 & & \downarrow \langle \cdot \rangle \\
\text{Rat}_d & \xrightarrow{\langle \cdot \rangle} & M_d
\end{array}
\]
Intuitively, this says that the action of $SL_2$ on $Rat_d$ descends to the trivial action on $M_d$.

(2) For any algebraically closed field $\Omega$, the natural map $\langle \cdot \rangle : Rat_d(\Omega) \to M_d(\Omega)$ is surjective, and its fibers are the $SL_2(\Omega)$ orbits of points in $Rat_d(\Omega)$.

(3) If $U \subset M_d$ is an open set, then its inverse image in $Rat_d$ is also open.

As remarked above, Milnor [7] proved that $M_2(\mathbb{C}) \cong \mathbb{C}^2$. More precisely, he describes explicitly two functions $\sigma_1, \sigma_2$ on $Rat_2 = \mathbb{P}^5 \setminus \{ \rho = 0 \}$ which are invariant under the action of $SL_2(\mathbb{C})$ and which induce a bijection $(\sigma_1, \sigma_2) : M_2(\mathbb{C}) \to \mathbb{C}^2$.

We will prove the following generalization of Milnor’s result.

**Theorem 1.2.** There are functions

$$\sigma_1, \sigma_2 \in (A_2[\rho^{-1}]_{(0)})^{SL_2}$$

(given explicitly in section 5) which are invariant under the action of $SL_2$ and which induce an isomorphism

$$(\sigma_1, \sigma_2) : M_2 \xrightarrow{\sim} \mathbb{A}^2_\mathbb{Z}$$

of schemes over $\mathbb{Z}$.

Geometric invariant theory also provides the means to embed $M_d$ in larger quotient spaces, as described in the following result.

**Theorem 1.3.** There are open subschemes of $\mathbb{P}^{2d+1}$ (over $\mathbb{Z}$)

$$Rat_d \subset (\mathbb{P}^{2d+1})^s \subset (\mathbb{P}^{2d+1})^{ss}$$

which are invariant under the conjugation action of $SL_2$ and such that the quotients

$$M_d = Rat_d/SL_2, \quad M_d^s = (\mathbb{P}^{2d+1})^s/SL_2, \quad \text{and} \quad M_d^{ss} = (\mathbb{P}^{2d+1})^{ss}/SL_2$$

exist. More precisely, $M_d^s$ is a geometric quotient, $M_d^{ss}$ is a categorical quotient which is proper and of finite type over $\mathbb{Z}$, and $M_d$ sits as a dense open subset of both $M_d^s$ and $M_d^{ss}$.

The spaces $M_d^s$ and $M_d^{ss}$ are called the spaces of stable and semi-stable conjugacy classes of rational maps respectively. Intuitively, the stable locus $(\mathbb{P}^{2d+1})^s$ is the largest set for which the quotient by $SL_2$ satisfies

$$(\mathbb{P}^{2d+1})^s(\Omega)/SL_2(\Omega) \xrightarrow{\sim} M_d^s(\Omega) \quad \text{for all algebraically closed fields } \Omega.$$
Corollary 1.4. The stable and semi-stable loci coincide if and only if \( d \) is even. Hence if \( d \) is even, then \( M^s_d = M^s_0 \) is both a geometric quotient and is proper over \( \text{Spec} \mathbb{Z} \).

Working over \( \mathbb{C} \), Milnor [7] shows that the space \( M^2_2(\mathbb{C}) \sim \mathbb{C}^2 \) has a natural compactification \( \hat{M}^2_2(\mathbb{C}) \sim \mathbb{P}^2(\mathbb{C}) \). As Milnor says, this compactification is natural in the sense that the extra points at infinity “can be thought of very roughly as the limits of quadratic rational maps as they degenerate towards a fractional linear or constant map. However, caution is needed, since such a limit cannot be uniform over” all of \( \mathbb{P}^1(\mathbb{C}) \). From the viewpoint of geometric invariant theory, the “compactification” \( M^s_2 \) of \( M^2_2 \) naturally consists of \( M^2_2 \), an extra affine line \( \mathbb{A}^1 \), and an extra point.

Theorem 1.5. There is a natural isomorphism \( M^s_2 \cong \mathbb{P}^2 \) over \( \mathbb{Z} \) so that the following diagram commutes:

\[
\begin{array}{ccc}
M^2_2 & \sim & \mathbb{A}^2 \\
\downarrow & & \downarrow \\
M^s_2 & \sim & \mathbb{P}^2.
\end{array}
\]

The functions \( \sigma_1, \sigma_2 \) are defined in terms of the multipliers associated to a rational map. We will postpone a complete definition until section 4 and be content here to describe them geometrically. Let \( \Omega \) be an algebraically closed field, and let \( \phi \in \text{Rat}_d(\Omega) \) be a rational map of degree \( d \) defined over \( \Omega \). The fixed points of \( \phi \) are the points

\[
\text{Fix}(\phi) = \{ P \in \mathbb{P}^1(\Omega) : \phi(P) = P \}.
\]

We consider this to be a set with multiplicities. Counted with multiplicity, the set \( \text{Fix}(\phi) \) contains exactly \( d+1 \) points. If \( P \in \text{Fix}(\phi) \), then the derivative \( \phi'(P) \in \Omega \) is well-defined independent of the choice of coordinates on \( \mathbb{P}^1 \); that is, it depends only on the conjugacy class \( \langle \phi \rangle \in M_d(\Omega) \). The number \( \phi'(P) \) is called the multiplier of \( \phi \) at \( P \). A basic identity asserts that

\[
\sum_{P \in \text{Fix}(\phi)} \frac{1}{1 - \phi'(P)} = 1.
\]

(See [7] for an analytic proof. But this formula is essentially algebraic in nature, so the analytic proof implies that it is a formal identity, hence valid over any field.)

The individual multipliers form an unordered set, so we take the corresponding elementary symmetric functions:

\[
\prod_{P \in \text{Fix}(\phi)} (T + \phi'(P)) = \sum_{i=0}^{d+1} \sigma_i(\phi) T^{d+1-i}.
\]

The \( \sigma_i \)'s depend only on the conjugacy class \( \langle \phi \rangle \), and their definition is clearly algebraic, so they give functions on \( M_d \). More generally, we can use points of period \( n \),

\[
\text{Per}_n(\phi) = \text{Fix}(\phi^n),
\]

and compute the multipliers and symmetric functions of \( \phi^n \) at the points in \( \text{Per}_n(\phi) \). These, too, will give functions on \( M_d \), which we will denote by \( \sigma_i^{(n)} \), \( i = 1, 2, \ldots \).
(Actually, it is more efficient to define these functions using only orbits of formal period $n$. See section 4 for the precise definition of the $\sigma_i^{(n)}$s.)

In [7], Milnor uses his description $M_2(C) \cong C^2$ to show that for maps of degree two, every $\sigma_i^{(n)}$ is a polynomial in $\mathbb{C}[\sigma_1, \sigma_2]$. We can use the above Theorem to strengthen this.

**Corollary 1.6.** Every invariant function on $M_2$, including in particular the $\sigma_i^{(n)}$s, is a polynomial in $\mathbb{Z}[\sigma_1, \sigma_2]$.

§2. The Quotient Spaces $M_d$, $M_d^s$, and $M_d^{ss}$

In this section we will use geometric invariant theory to construct the quotient spaces $M_d$, $M_d^s$, and $M_d^{ss}$. We will follow closely the methods described in [10]. We will try to give complete references to the required results from [10], but in the interest of brevity, we will not take the time to repeat all of the requisite definitions.

The main construction in [10] says that if a reductive group $G$ acts linearly on a variety (or scheme) $X$, then the stable locus $X^s \subset X$ admits a geometric quotient $Y^s = X^s/G$, and the semi-stable locus $X^{ss}$ admits a categorical quotient $Y^{ss} = X^{ss}/G$. Further, the semi-stable quotient $Y^{ss}$ will be proper (complete) over the base in most situations. In addition, various nice properties of $X$ descend to the quotients $Y^s$ and $Y^{ss}$. Applying this general theory to our specific situation yields the following result.

**Theorem 2.1.** We use the notation from section 1.

(a) The space of rational function $\text{Rat}_d \subset \mathbb{P}^{2d+1}$ is an $\text{SL}_2$-invariant dense open subset of the stable locus $(\mathbb{P}^{2d+1})^s$ in $\mathbb{P}^{2d+1}$. Hence the geometric quotient $M_d = \text{Rat}_d/\text{SL}_2$ exists as a scheme over $\mathbb{Z}$.

(b) The geometric quotient $M_d^s = (\mathbb{P}^{2d+1})^s/\text{SL}_2$ and the categorical quotient $M_d^{ss} = (\mathbb{P}^{2d+1})^{ss}/\text{SL}_2$ exist as schemes over $\mathbb{Z}$, and the natural inclusions

$$M_d \subset M_d^s \subset M_d^{ss}$$

exhibit each scheme as a dense open subscheme of the next.

(c) The schemes $M_d$, $M_d^s$, and $M_d^{ss}$ are all connected, integral, normal, and of finite type over $\mathbb{Z}$. Further, $M_d$ is affine and $M_d^{ss}$ is proper over $\mathbb{Z}$.

(d) More precisely, if we let $A_d = \mathbb{Z}[a_0, \ldots, a_d, b_0, \ldots, b_d]$ and $\rho = \text{Res}(F_a, F_b) \in A_d$, then

$$M_d^{ss} \cong \text{Proj} A_d^{\text{SL}_2} \quad \text{and} \quad M_d \cong \text{Spec} A_d[\rho^{-1}]^{\text{SL}_2}_{(0)}.$$  

The indicated rings of invariants $A_d^{\text{SL}_2}$ and $A_d[\rho^{-1}]^{\text{SL}_2}_{(0)}$ are finitely generated over $\mathbb{Z}$.

**Proof.** (a) The fact that $\text{Rat}_d \subset (\mathbb{P}^{2d+1})^s$ can be proven similarly to the proof of [10, proposition 4.2], using the resultant form $\rho(a, b) = \text{Res}(F_a, F_b)$ in place of the discriminant form. Alternatively, the inclusion $\text{Rat}_d \subset (\mathbb{P}^{2d+1})^s$ follows immediately from the numerical criterion (Proposition 2.2) proven below. It is also clear that $\text{Rat}_d$ is an $\text{SL}_2$-invariant subset of $\mathbb{P}^{2d+1}$, since $\text{SL}_2$ fixes the resultant form. Hence $\text{Rat}_d$ is an $\text{SL}_2$-stable and $\text{SL}_2$-invariant scheme, so its geometric quotient exists. Over a field, this is a consequence of Mumford’s construction of quotients [10, chapter 1], and over $\mathbb{Z}$ it follows by essentially the same methods using Seshadri’s theorem that a reductive group scheme is geometrically reductive. See [13] and [10, appendix 1.G].
(b) The existence of the quotients follows from the work of Mumford and Seshadri as cited in (a). The fact that the inclusions are dense open immersions follows from the analogous fact for the inclusions $\text{Rat}_d \subset (\mathbb{P}^{2d+1})^s \subset (\mathbb{P}^{2d+1})^{ss}$.

(c,d) The schemes $\text{Rat}_d$, $(\mathbb{P}^{2d+1})^s$, and $(\mathbb{P}^{2d+1})^{ss}$ are open subschemes of $\mathbb{P}^{2d+1}$, so they are all connected, integral, and normal. It follows from [10, section 2, remark (2)] that the quotients $M_d$, $M^s_d$, and $M^{ss}_d$ have the same properties. The fact that $M_d$ is affine and $M^{ss}_d$ is proper and of finite type over $\mathbb{Z}$ also follows from Seshadri’s work [13] (see also [10, theorem 1.1 and appendix 1.G]), as does the description of $M_d$ and $M^{ss}_d$ via rings of invariants in (d).

Next we use Mumford’s numerical criterion to describe exactly which points in $\mathbb{P}^{2d+1}$ are (semi)-stable for the action of $\text{SL}_2$.

Proposition 2.2. Identifying $\mathbb{P}^{2d+1}$ with pairs of homogeneous polynomials
\[ \phi = [F_a, F_b] = [a_0 X^d + a_1 X^{d-1} Y + \cdots + a_d Y^d, b_0 X^d + b_1 X^{d-1} Y + \cdots + b_d Y^d], \]
we let $\text{SL}_2$ act via conjugation as described in section 1. Also let $\Omega$ be an algebraically closed field.

(a) A point in $\mathbb{P}^{2d+1}(\Omega)$ is unstable if and only if, after an $\text{SL}_2(\Omega)$-conjugation, it satisfies
\[ a_i = 0 \text{ for all } i \leq \frac{d-1}{2} \text{ and } b_i = 0 \text{ for all } i \leq \frac{d+1}{2}. \]

(b) A point in $\mathbb{P}^{2d+1}(\Omega)$ is not stable if and only if, after an $\text{SL}_2(\Omega)$-conjugation, it satisfies
\[ a_i = 0 \text{ for all } i < \frac{d}{2} \text{ and } b_i = 0 \text{ for all } i < \frac{d+1}{2}. \]

As a trivial corollary, we obtain the following useful result.

Corollary 2.3. If $d$ is even, then every semi-stable point is stable, so $M^s_d = M^{ss}_d$.

Remark. Let $K$ be a non-algebraically closed field of characteristic 0. It is an interesting question to ask whether the natural map
\[ \text{Rat}_d(K) \longrightarrow M_d(K) \]
is surjective. This is equivalent to asking whether the field of moduli of a conjugacy class of maps $\langle \phi \rangle$ is also a field of definition. This question was studied in [15], where it is proved that if $d$ is even, then $\text{Rat}_d(K)$ always surjects onto $M_d(K)$; but if $d$ is odd and the Brauer group of $K$ is non-trivial, then it never surjects. It is tempting to speculate that the even/odd dichotomies in [15] and corollary 2.3 are related to one another.

Proof of Proposition 2.2. We will use the numerical criterion described in [10, chapter 2]. For a similar computation, see [10, chapter 4, sections 1 and 2].

Fix a maximal torus $T \subset \text{SL}_2$. After a change of coordinates, the action of $T$ on its canonical representation space $\mathbb{A}^2$ can be diagonalized, so $T$ becomes the group of matrices
\[ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \]
subject to the condition $\alpha \delta = 1$. 

We are identifying the space of pairs \([F_a, F_b]\) with the projective space \(\mathbb{P}^{2d+1}\), and the canonical action of \(\text{SL}_2\) on \(\mathbb{A}^2\) is dual to the action on forms, so the (conjugation) action of an element of \(f = (\alpha \ 0 \ \delta) \in T\) on a point \(\phi = [F_a, F_b] \in \mathbb{P}^{2d+1}\) is given by

\[
\phi^f = [F_a, F_b]^f = [\alpha F_a(\alpha^{-1}X, \delta^{-1}Y), \delta F_b(\alpha^{-1}X, \delta^{-1}Y)].
\]

So if we write \(F_a = \sum a_i X^{d-i}Y^i\) and \(F_b = \sum b_i X^{d-i}Y^i\), then the action on the \((a, b)\)-coordinates is given explicitly as

\[
a_i \mapsto \alpha^{i+d-1} - \delta^{-i}a_i \quad \text{and} \quad b_i \mapsto \alpha^{-i} - \delta^{i-1}b_i.
\]

Now consider a one-parameter subgroup \((1-\text{PS}) \lambda : \mathbb{G}_m \to \text{SL}_2\). Attached to each such \(\lambda\) there is a numerical invariant \(\mu(\phi, \lambda)\) as described in [10, chapter 2]. The numerical criterion of [10, theorem 2.1] says that \(\phi\) is unstable \(\iff\) \(\mu(\phi, \lambda) < 0\) for some \(1-\text{PS} \lambda\), and \(\phi\) is not stable \(\iff\) \(\mu(\phi, \lambda) \leq 0\) for some \(1-\text{PS} \lambda\).

We will now compute this invariant in our situation.

After a change of coordinates, any \(1-\text{PS} \lambda\) can be transformed to lie in a maximal torus and be given by \(\lambda_r(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}\) for some integer \(r \geq 1\).

The action of \(\lambda_r\) on \([F_a, F_b]\) is given by

\[
a_i \mapsto t^{-r(d-1-2i)}a_i \quad \text{and} \quad b_i \mapsto t^{-r(d+1-2i)}b_i.
\]

Then a formula of Mumford [10, proposition 2.3] says that

\[
\mu(\phi, \lambda_r) = \max \{ r(d-1-2i) : a_i \neq 0 \} \cup \{ r(d+1-2i) : b_i \neq 0 \}.
\]

Combining this with the numerical criterion says that \(\phi\) is unstable (respectively not stable) if and only if \(\phi\) is conjugate to a map with

\[
\max \{ r(d-1-2i) : a_i \neq 0 \} \cup \{ r(d+1-2i) : b_i \neq 0 \} < 0 \quad \text{(respectively \(\leq 0\)).}
\]

This is equivalent to the two conditions

\[
a_i \neq 0 \implies d - 1 - 2i < 0 \quad \text{(respectively \(\leq 0\))}
\]

\[
\text{and}
\]

\[
b_i \neq 0 \implies d + 1 - 2i < 0 \quad \text{(respectively \(\leq 0\)),}
\]

which in turn are the same as

\[
a_i = 0 \text{ for all } \frac{d - 1}{2} \geq i \quad \text{(respectively \(> i\))}
\]

\[
\text{and}
\]

\[
b_i = 0 \text{ for all } \frac{d + 1}{2} \geq i \quad \text{(respectively \(> i\)).}
\]

This completes the proof proposition 2.2.

**Proof of Corollary 2.3.** Corollary 2.3 follows immediately from proposition 2.2, since if \(d\) is even, then \((d \pm 1)/2\) is not an integer, so the unstable condition in (a) and the not-stable condition in (b) are equivalent.
In this section we will look at two functors from the category of schemes to the category of sets. We begin by fixing a realization
\[ \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{Z}} = \text{Proj} \mathbb{Z}[X,Y]. \]
Equivalently, we fix a basis \(X, Y\) for the space of global sections \(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))\).

**Definition.** Let \(d \geq 1\) be an integer. The functor \(\text{Rat}_d\) of rational maps (really morphisms) of degree \(d\) on \(\mathbb{P}^1\) is the functor
\[ \text{Rat}_d : \text{Sch} \rightarrow \text{Sets} \]
defined by
\[ \text{Rat}_d(S) = \{ S\text{-morphisms } \phi : \mathbb{P}^1_S \rightarrow \mathbb{P}^1_S \text{ satisfying } \phi^* \mathcal{O}_{\mathbb{P}^1_S}(1) \cong \mathcal{O}_{\mathbb{P}^1_S}(d) \}. \]

Of course, we still write \(\text{Rat}_d\) for the scheme defined in section 1. That is, \(\text{Rat}_d\) is the affine scheme
\[ \text{Rat}_d = \text{Spec} \mathbb{Z}\left[\frac{a_i a_j \cdots b_i b_j}{\rho} \right]_{i_0 + \cdots + i_d + j_0 + \cdots + j_d = 2d}, \]
where we will write as usual
\[ \rho = \rho(a,b) = \text{Res}(F_a, F_b) \]
for the resultant polynomial. Over \(\text{Rat}_d\) we have a universal morphism \(\phi^{\text{univ}}\) of degree \(d\),
\[ \begin{array}{ccc} \mathbb{P}^1_{\text{Rat}_d} & \xrightarrow{\phi^{\text{univ}}} & \mathbb{P}^1_{\text{Rat}_d} \\ [X,Y] \rightarrow & [F_a(X,Y), F_b(X,Y)] \end{array} \]

**Definition.** For any scheme \(S\), we define an equivalence relation on the set \(\text{Rat}_d(S)\) as follows. Two \(S\)-morphisms \(\phi, \psi \in \text{Rat}_d(S)\) are equivalent, denoted \(\phi \sim \psi\), if there is an \(S\)-isomorphism \(f : \mathbb{P}^1_S \sim \mathbb{P}^1_S\) such that \(\phi \circ f = f \circ \psi\) (i.e., \(\phi^f = \psi\)). We then define the functor \(\text{M}_d\) to be the quotient of \(\text{Rat}_d\) by this equivalence relation:
\[ \text{M}_d : \text{Sch} \rightarrow \text{Sets}, \quad S \rightarrow \text{Rat}_d(S)/\sim. \]

Our first result says that the functor \(\text{Rat}_d\) is representable.
**Theorem 3.1.** The scheme $\text{Rat}_d$ represents the functor $\text{Rat}_d$, and in fact the universal construction described above makes $\text{Rat}_d$ into a fine moduli space for $\text{Rat}_d$.

**Proof.** Given any $S$-valued point of $\text{Rat}_d$, say $\sigma : S \to \text{Rat}_d$, we can use $\sigma$ to base extend the universal map $\phi_{\text{univ}}$ and obtain a morphism $\phi_{\sigma} \in \text{Rat}_d(S)$ defined by the following diagram:

$$
\begin{array}{ccc}
\mathbb{P}^1_{\text{Rat}_d} \times \text{Rat}_d S & \xrightarrow{\phi_{\text{univ}} \times 1_S} & \mathbb{P}^1_{\text{Rat}_d} \times \text{Rat}_d S \\
\downarrow & & \downarrow \\
\mathbb{P}^1_S & \xrightarrow{\phi_{\sigma}} & \mathbb{P}^1_S
\end{array}
$$

This gives a map from $\text{Rat}_d(S) = \text{Hom}(S, \text{Rat}_d)$ to $\text{Rat}_d(S)$ for every scheme $S$.

Next suppose that we start with an element $\phi \in \text{Rat}_d(S)$. We will further suppose that $S$ is affine, $S = \text{Spec } B$. We have assumed that we have fixed a $\mathbb{Z}$-basis $X, Y$ for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, so $X, Y$ will certainly be a $B$-basis for $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(1))$, and similarly $X^d, X^{d-1}Y, \ldots, Y^d$ is a (canonical, given the initial choice of $X$ and $Y$) $B$-basis for $H^0(\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(d))$. Further, the definition of $\text{Rat}_d$ implies that $\phi^* \mathcal{O}_{\mathbb{P}^1_S}(1) \cong \mathcal{O}_{\mathbb{P}^1}(d)$, so taking global sections we can write

$$
\phi^* X = \alpha_0 X^d + \alpha_1 X^{d-1}Y + \cdots + \alpha_d Y^d = F_\alpha,
\phi^* Y = \beta_0 X^d + \beta_1 X^{d-1}Y + \cdots + \beta_d Y^d = F_\beta,
$$

where $\alpha_0, \ldots, \beta_d \in B$ are uniquely determined by $\phi$ (and our choice of $X, Y$).

We are going to prove that $\rho(\alpha, \beta) = \text{Res}(F_\alpha, F_\beta)$ is a unit in $B$. Assuming this, we see that $(\alpha, \beta)$ defines a point $\tau = \tau(\phi)$ in $\text{Rat}_d(B)$, and then it is clear that $\phi_{\tau} \in \text{Rat}_d(S)$ is just the original map $\phi$. This means that the maps

$$
\begin{array}{ccc}
\text{Rat}_d(S) & \longrightarrow & \text{Rat}_d(S) \\
\sigma & \longmapsto & \phi_{\sigma} \\
\phi & \longmapsto & \tau(\phi) = (F_\alpha(\phi), F_\beta(\phi))
\end{array}
$$

are inverses, at least on affine schemes $S$. However, the uniqueness of the $(F_\alpha, F_\beta)$ associated to a given $\phi \in \text{Rat}_d(S)$ means that we can glue to get the same result for arbitrary schemes $S$.

It remains to show that $\rho(\alpha, \beta) \in B^*$. We know that $\phi : \mathbb{P}^1_S \to \mathbb{P}^1_S$ is a morphism, so in particular $\phi^* X, \phi^* Y$ generate the sheaf $\mathcal{O}_{\mathbb{P}^1_S}(1)$. Hence for any (closed) point $P \in \mathbb{P}^1_S$, at least one of the sections $(\phi^* X)_P$ and $(\phi^* Y)_P$ must be non-zero. In other words, for every (maximal) ideal $\mathfrak{p} \in \text{Spec } B$, the forms

$$
\phi^* X = F_\alpha(X, Y) \mod \mathfrak{p} \quad \text{and} \quad \phi^* Y = F_\beta(X, Y) \mod \mathfrak{p}
$$

define the trivial locus in $\mathbb{P}^1_{B/\mathfrak{p}}$. This implies that $\text{Res}(F_\alpha, F_\beta) \notin \mathfrak{p}$, and since this is true for all (maximal) ideals, we conclude as desired that $\text{Res}(F_\alpha, F_\beta) \in B^*$.

Next we consider the functor $M_d$.

**Theorem 3.2.** There is a natural map of functors

$$
M_d \longrightarrow \text{Hom}(\cdot, M_d)
$$
with the property that $\mathcal{M}_d(\Omega) \cong M_d(\Omega)$ for every algebraically closed field $\Omega$.

Proof. Let $\xi \in \mathcal{M}_d(S)$. Within the equivalence class $\xi$ we choose an element $\phi \in \text{Rat}_d(S)$. From theorem 3.1, we may regard $\phi$ as an element of $\text{Rat}_d(S)$, and then the construction of $M_d$ as a quotient (theorem 2.1) gives us a point $\lambda = \lambda(\xi, \phi) \in M_d(S)$. We claim that $\lambda$ is independent of the choice of $\phi$, and so gives a well-defined map $\mathcal{M}_d(S) \to M_d(S)$. To verify this, let $\psi = \phi f$ be another element of $\xi$, where $f : P^1_S \to P^1_S$ is an $S$-isomorphism. Then $\phi$ and $\psi$ are $S$-valued points of $\text{Rat}_d$, and we want to show that the compositions

$$S \xrightarrow{\psi} \text{Rat}_d \to M_d$$

give the same map $S \to M_d$. Covering $S$ by affine open sets, we may assume that $S = \text{Spec } B$. The $S$-isomorphism $f$ satisfies $f^* O_{P^1_S}(1) \cong O_{P^1_S}(1)$, so

$$f^* X = \alpha X + \beta Y \quad \text{and} \quad f^* Y = \gamma X + \delta Y$$

for some $\alpha, \beta, \gamma, \delta \in B$. Further, the fact that $f$ has an inverse means that $\det f = \alpha \delta - \beta \gamma \in B^*$. Let $B' = B[\sqrt{\alpha \delta - \beta \gamma}]$ and $S' = \text{Spec } B'$. Notice that $B'/B$ is a finite extension, so $S' \to S$ is surjective. This allows us to replace $S$ by $S'$, and then we may replace $f$ with the map $f'$ determined by the conditions

$$f'^* X = \frac{\alpha}{\sqrt{\alpha \delta - \beta \gamma}} X + \frac{\beta}{\sqrt{\alpha \delta - \beta \gamma}} Y \quad \text{and} \quad f'^* Y = \frac{\gamma}{\sqrt{\alpha \delta - \beta \gamma}} X + \frac{\delta}{\sqrt{\alpha \delta - \beta \gamma}} Y.$$  

It is still true that $\phi \circ f' = f' \circ \psi$, and now $\det f' = 1$.

Thus $\phi$ and $\psi$ are $\text{SL}_2(B')$-equivalent, so any function in the ring of invariants

$$H^0(M_d, \mathcal{O}_{M_d}) = H^0(\text{Rat}_d, \mathcal{O}_{\text{Rat}_d})^{\text{SL}_2}$$

will take the same value at $\phi$ and $\psi$. Since $M_d$ is the spectrum of this ring, it follows that $\phi$ and $\psi$ give the same $S$-valued point of $M_d$. This completes the proof that the map $\mathcal{M}_d(S) \to M_d(S)$ defined above is indeed well defined, independent of the choice of a representative in $\text{Rat}_d(S)$.

We also need to show that $\mathcal{M}_d(S)$ is isomorphic to $M_d(S)$ on geometric points $S = \text{Spec } \Omega$ (i.e., where $\Omega$ is an algebraically closed field). But this is clear, since over an algebraically closed field we have

$$\mathcal{M}_d(\Omega) = \text{Rat}_d(\Omega)/\text{PGL}_2(\Omega) \quad \text{and} \quad M_d(\Omega) = \text{Rat}_d(\Omega)/\text{SL}_2(\Omega),$$

and the map $\text{SL}_2(\Omega) \to \text{PGL}_2(\Omega)$ is surjective, so the quotients are the same.

4. Fixed points, periodic points and multiplier systems

In this section we will construct functions on the quotient space $M_d$ by associating to each $\phi \in M_d$ the system of multipliers of its fixed (or more generally periodic) points. To motivate this construction, we begin by describing the situation over an algebraically closed field $k$. 

Thus let $\phi \in \text{Rat}_d(k)$, so $\phi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a rational map of degree $d$. Such a map has exactly $d + 1$ fixed points (counted with multiplicity), say $\xi_1, \ldots, \xi_{d+1}$. For each $\xi = \xi_i$, the map $\phi$ induces a $k$-linear map $\phi^*$ from $\Omega_{\mathbb{P}^1, \xi}$ to itself, where $\Omega_{\mathbb{P}^1, \xi}$ denotes the space of germs of differential 1-forms at $\xi$. This vector space has dimension 1 over $k$, so if we take any non-zero differential form $\omega \in \Omega_{\mathbb{P}^1, \xi}$, then $\phi^*(\omega)$ is a multiple of $\omega$, say $\phi^*(\omega) = \phi'(\xi)\omega$. The number $\phi'(\xi) \in k$ is called the multiplier of $\phi$ at the fixed point $\xi$, and the set $\{\phi'(\xi_1), \phi'(\xi_2), \ldots, \phi'(\xi_{d+1})\}$ is the associated multiplier system.

The fixed points depend algebraically, but not rationally, on the coefficients of $\phi$, and in any case they only form an unordered set. We define quantities $\sigma_i = \sigma_i(\phi)$ by taking the symmetric functions of the multipliers:

$$\prod_{i=1}^{d+1} (T + \phi'(\xi_i)) = \sum_{i=0}^{d+1} \sigma_i T^{d+1-i}. $$

The $\sigma_i$'s are symmetric functions of the $\phi'(\xi_i)$'s, and each $\phi'(\xi)$ is a rational function of $\xi$, so the $\sigma_i$'s are actually rational functions of the coefficients of $\phi$. In other words, the $\sigma_i$'s are rational (in fact, regular) functions on $\text{Rat}_d/k$.

Now let $f \in \text{PGL}_2(k)$ be an automorphism of $\mathbb{P}^1_k$. Then $f^{-1}(\xi_1), \ldots, f^{-1}(\xi_{d+1})$ are the fixed points of $\phi^f$, and the chain rule tells us that

$$\begin{align*}
(\phi^f)'(f^{-1}(\xi)) &= (f^{-1} \circ \phi \circ f)'(f^{-1}(\xi)) \\
&= (f^{-1})'((\phi(\xi))) \cdot \phi'(\xi) \cdot f'(f^{-1}(\xi)) \\
&= (f^{-1})'(\xi) \cdot \phi'(\xi) \cdot f'(f^{-1}(\xi)) \\
&= \phi'(\xi).
\end{align*}$$

Thus the multiplier system of $\phi$ is $\text{PGL}_2(k)$-invariant, so the $\sigma_i$'s descend to give regular functions on $\text{Rat}_d$.

We now explain how this construction generalizes over $\mathbb{Z}$.

**Theorem 4.1.** Let $\phi = \phi^{\text{univ}} : \mathbb{P}^1_{\text{Rat}_d} \to \mathbb{P}^1_{\text{Rat}_d}$ be the universal morphism of degree $d$ as described in Section 3. There exists a unique reduced closed subscheme $\text{Fix} \subset \mathbb{P}^1_{\text{Rat}_d}$ having the following two properties:

(i) $\phi|_{\text{Fix}} = 1_{\text{Fix}}$, i.e., $\phi$ induces the identity map on $\text{Fix}$.

(ii) If $Z \subset \mathbb{P}^1_{\text{Rat}_d}$ is a reduced closed subscheme with the property that $\phi|_Z = 1_Z$, then $Z \subset \text{Fix}$.

The subscheme $\text{Fix}$ also satisfies:

(iii) $\text{Fix}$ is integral (i.e., reduced and irreducible).

(iv) The projection $\text{Fix} \to \text{Rat}_d$ is a finite morphism of degree $d + 1$.

**Proof.** To ease notation, we will write $\phi = \phi^{\text{univ}} = [F_a, F_b]$. If $\text{Fix}$ exists, its uniqueness is clear from (i) and (ii), so we just need to find a subscheme with properties (i) and (ii). We set

$$\text{Fix} = V(YF_a(X, Y) - XF_b(X, Y)) \subset \mathbb{P}^1_{\text{Rat}_d}. $$

(2)
In other words, we start with the hypersurface of type \((d + 1, 1)\) in \(\mathbb{P}^1 \times \mathbb{P}^{2d+1}\) defined by the bihomogeneous form \(YF_a - XF_b\), and then we take its intersection with \(\mathbb{P}^1 \times \text{Rat}_d\). It is clear that \(\varphi\) fixes \(\text{Fix}\), since \(\text{Fix}\) is the subscheme defined by the “condition” \(\varphi([X, Y]) = [X, Y]\).

Next let \(Z \subset \mathbb{P}^1_{\text{Rat}_d}\) be fixed by \(\varphi\). If \(\sigma : \text{Spec}\, k \rightarrow Z\) is any geometric point of \(Z\), say \(\sigma(k) = [\alpha, \beta] \in \mathbb{P}^1(k)\), then \(\varphi\) fixes \(\sigma(k)\), so \([\alpha, \beta] = \varphi([\alpha, \beta]) = [F_a(\alpha, \beta), F_b(\alpha, \beta)]\).

Hence \(\sigma(k) \in \text{Fix}(k)\), which shows that every geometric point of \(Z\) lies in \(\text{Fix}\). It follows that \(Z\) is a subscheme of \(\text{Fix}\) (this is where we use the assumption that \(Z\) is reduced). This completes the proof that the subscheme \(\text{Fix}\) defined by (2) satisfies (i) and (ii).

Next we observe that the bihomogeneous form \(YF_a - XF_b\) is clearly irreducible, since it has degree 1 in \((a, b)\). More precisely, if it were to factor in \(\mathbb{Z}[a, b][X, Y]\), then one of the factors would have to lie in \(\mathbb{Z}[X, Y]\), and it is clear that \(YF_a - XF_b\) has no such factors. Hence \(\text{Fix}\) is irreducible, and it is reduced by assumption, which verifies (iii).

To check (iv), we observe from (2) that \(\text{Fix}\) is clearly quasi-finite over \(\text{Rat}_d\), and similarly it is proper (even projective) over \(\text{Rat}_d\). Therefore \(\text{Fix}\) is finite over \(\text{Rat}_d\) (see [3, exercise 11.2] or [6, chapter I, proposition 1.10]). The degree of the map is then clear, since \(YF_a - XF_b\) is homogeneous of degree \(d + 1\) in \((X, Y)\).

It follows from Theorem 4.1(i) that \(\varphi\) induces an \(\mathcal{O}_{\text{Fix}}\)-linear map \(\varphi^*\) from

\[
\Omega^1_{\mathbb{P}^1_{\text{Rat}_d}/\text{Rat}_d} \otimes_{\mathcal{O}_{\text{Rat}_d}} \mathcal{O}_{\text{Fix}}
\]

to itself. This sheaf is (locally) free of rank 1 over \(\mathcal{O}_{\text{Fix}}\), so from (iii) it is (locally) free of rank \(d + 1\) over \(\mathcal{O}_{\text{Rat}_d}\). Thus \(\varphi^*\) defines an \(\mathcal{O}_{\text{Rat}_d}\)-linear map of a (locally) free sheaf of rank \(d + 1\), so we can compute its characteristic polynomial

\[
\det(T + \varphi^*) = \sum_{i=0}^{d+1} \sigma_i T^{d+1-i}
\]

to obtain (local) sections \(\sigma_1, \ldots, \sigma_{d+1}\) of \(\mathcal{O}_{\text{Rat}_d}\).

**Proposition 4.2.** The functions \(\sigma_1, \ldots, \sigma_{d+1}\) described above are global sections of \(\mathcal{O}_{\text{Rat}_d}\). Further, they are invariant under the conjugation action of \(\text{SL}_2\), and hence they descend to give global sections of \(\mathcal{O}_{\mathbb{M}_d}\).

**Proof.** The scheme \(\text{Rat}_d\) is affine, say \(\text{Rat}_d = \text{Spec}\, B\); and \(\text{Fix}\) is finite over \(\text{Rat}_d\), so it too is affine, say \(\text{Fix} = \text{Spec}\, B'\), where \(B'/B\) is a finite extension of degree \(d + 1\). Then

\[
\Omega^1_{\mathbb{P}^1_{\text{Rat}_d}/\text{Rat}_d} \otimes_{\mathcal{O}_{\text{Rat}_d}} \mathcal{O}_{\text{Fix}} = \mathcal{O}_{\mathbb{P}^1_{B'}/B} \otimes_{\mathcal{O}_{\mathbb{P}^1_{B}}} B'
\]

is a free \(B'\)-module of rank 1 generated by \(dz\), where \(z\) is a uniformizer on \(\mathbb{P}^1_{B'}\) at \(\text{Fix}\). (We can think of \(\text{Fix}\), an integral subscheme of \(\mathbb{P}^1_{B'}\), as a point of \(\mathbb{P}^1_{B'}\).) The map \(\varphi\) induces an endomorphism \(\varphi^*\) of this module, so

\[
\varphi^*(dz) = \varphi' \cdot dz
\]

for some element \(\varphi' \in B'\).
Now $B'$ is a free $B$-module of rank $d + 1$, so multiplication by $\phi'$ gives an $B$-linear endomorphism of $B'^{d + 1}$. The characteristic polynomial of this endomorphism is well-defined independent of the choice of a basis, which gives

$$\det(T + \phi') = \sum_{i=0}^{d+1} \sigma_i T^{d+1-i} \quad \text{for elements } \sigma_i \in B.$$ 

In other words, $\sigma_1, \ldots, \sigma_{d+1}$ are global sections of $\mathcal{O}_{\text{Rat}_d}$.

In order to show that the $\sigma_i$’s descend to $M_d$, we must show that they are $\text{SL}_2$-invariant. Since $M_d$ is reduced, it suffices to check invariance on geometric points, so let $k$ be an algebraically closed field, let $\phi \in \text{Rat}_d(k)$, and let $f \in \text{SL}_2(k)$. Then the equality $\sigma_i(\phi) = \sigma_i(\phi')$ follows from the chain-rule calculation (1). Hence the $\sigma_i$’s are global sections of $(\mathcal{O}_{\text{Rat}_d})_{\text{SL}_2} = \mathcal{O}_{M_d}$.

We continue to let $\phi = \phi^\text{univ}$ be the universal morphism of degree $d$ over $\text{Rat}_d$. Theorem 4.1 above describes the fixed subscheme of $\phi$. More generally, for any $n \geq 1$, we can consider the periodic subscheme of period $n$, as described in the following theorem.

**Theorem 4.3.** For every $n \geq 1$, there exists a unique reduced closed subscheme

$$\text{Per}_n \subset \mathbb{P}^1_{\text{Rat}_d}$$

having the following two properties:

1. $\phi^n |_{\text{Per}_n} = 1_{\text{Per}_n}$, i.e., $\phi^n$ induces the identity map on $\text{Per}_n$.
2. If $Z \subset \mathbb{P}^1_{\text{Rat}_d}$ is a reduced closed subscheme with the property that $\phi^n |_{Z} = 1_{Z}$, then $Z \subset \text{Per}_n$.

The scheme $\text{Per}_n$ is called the scheme of periodic points of period $n$. The subscheme $\text{Per}_n$ also satisfies:

(iii) The projection $\text{Per}_n \to \text{Rat}_d$ is a finite morphism of degree $d^n + 1$.

**Proof.** Most of the proof is very similar to the proof of Theorem 4.1, so we just briefly sketch. We can write $\phi^n = [F_a^{(n)}, F_b^{(n)}]$, where $F_a^{(n)}$ and $F_b^{(n)}$ are bihomogeneous polynomials in $\mathbb{Z}[a, b][X, Y]$ of bidegree $((d^n - 1)/(d - 1), d^n)$. Consider the closed subscheme of $\mathbb{P}^1_{\text{Rat}_d}$ defined by the equation

$$Y F_a^{(n)} - X F_b^{(n)}, \quad (3)$$

and let $\text{Per}_n$ be this subscheme with the induced reduced subscheme structure. It is clear that $\phi^n$ induces the identity map on (3), hence also on $\text{Per}_n$, and then (i) and (ii) and the fact that $\text{Per}_n$ is finite over $\text{Rat}_d$ are proven in the same way as Theorem 4.1. It remains to show that the degree of $\text{Per}_n$ over $\text{Rat}_d$ is exactly $d^n + 1$.

Since $\text{Per}_n$ is the subscheme of $\mathbb{P}^1_{\text{Rat}_d}$ given by (3) with the induced reduced subscheme structure, and (3) has degree $d^n + 1$ in the variables $(X, Y)$, we must show that the polynomial $Y F_a^{(n)} - X F_b^{(n)}$ has no repeated factors when factored in $\mathbb{Z}[a, b][X, Y]$. For this, it suffices to show that it has no repeated factors when we specialize $(a, b)$. Consider the rational map $\phi = [X^d, Y^d] \in \text{Rat}_d(\mathbb{C})$. For this map we have $\phi^n = [X^{d^n}, Y^{d^n}]$, so (3) becomes

$$Y F_a^{(n)} - X F_b^{(n)} = XY(X^{d^n-1} - Y^{d^n-1}).$$
which is a polynomial with distinct roots in \( \mathbb{P}^1(\mathbb{C}) \). This proves that (3) has no repeated factors in \( \mathbb{Z}[a, b][X, Y] \) (and in fact, no repeated factors in \( \mathbb{F}_p[a, b][X, Y] \) provided \( d^n \not\equiv 1 \pmod{p} \)).

Theorem 4.1 included the assertion that \( \text{Fix} \) is irreducible, but the analogous statement for \( \text{Per}_n \) was omitted in Theorem 4.3. In fact, \( \text{Per}_n \) is always reducible for \( n \geq 2 \), since in particular we always have \( \text{Fix} \subset \text{Per}_n \). In terms of polynomials, it is easy to check that the equation (3) defining \( \text{Per}_n \) is divisible by \( YF_a - XF_b \).

More generally, we can decompose \( \text{Per}_n \) into pieces as described in the following theorem.

**Theorem 4.4.** With notation as in Theorems 4.1 and 4.3, there are unique reduced closed subschemes 
\[
\text{Per}_m^* \subset \mathbb{P}^1_{\text{Rat}_d},
\]
one for each \( m \geq 1 \), with the following properties:
(i) \( \text{Per}_1^* = \text{Fix} \).
(ii) \( \text{Per}_n = \bigcup_{m|n} \text{Per}_m^* \) for every \( n \geq 1 \).

The scheme \( \text{Per}_m^* \) is called the scheme of periodic points of formal\(^1\) period \( m \).

In addition:
(iii) \( \phi^m_{|\text{Per}_m^*} = 1_{\text{Per}_m^*} \).
(iv) \( \text{Per}_m^* \) is finite over \( \text{Rat}_d \), and if we let \( \nu_m \) be the degree of \( \text{Per}_m^* \) over \( \text{Rat}_d \), then
\[
d^n + 1 = \sum_{m|n} \nu_m \quad \text{and} \quad \nu_m = \sum_{r|m} \mu(m/r)(d^r + 1),
\]
where \( \mu \) is the Möbius function.

**Proof.** The proof is by induction on \( m \). We have \( \text{Per}_1^* = \text{Fix} \) from (i), so we are okay for \( m = 1 \). Now suppose that we know the theorem for all \( m < n \). Consider the scheme \( \text{Per}_n \) from Theorem 4.3. For any \( m|n \) with \( m < n \), we know that \( \phi^m \) fixes \( \text{Per}_m^* \), and so \( \phi^n = (\phi^m)^{(n/m)} \) also fixes \( \text{Per}_m^* \). It follows from Theorem 4.3 that \( \text{Per}_m^* \) is a subscheme of \( \text{Per}_n \). Since \( \text{Per}_n \) and the \( \text{Per}_m^* \)'s are finite over the (affine irreducible) scheme \( \text{Rat}_d \), it follows that
\[
\text{Per}_n = \left( \bigcup_{m|n, m<n} \text{Per}_m^* \right) \cup \text{Per}_n^*,
\]
where \( \text{Per}_n^* \) is a union of irreducible components of \( \text{Per}_n \). Further, since \( \phi^n \) induces the identity map on \( \text{Per}_n \), it clearly induces the identity map on \( \text{Per}_n^* \); and since \( \text{Per}_n \) is finite over \( \text{Rat}_d \), the same is true of \( \text{Per}_n^* \). Finally, the degree of \( \text{Per}_n^* \) over \( \text{Rat}_d \) satisfies
\[
\nu_n = \deg(\text{Per}_n^* \to \text{Rat}_d) = \deg(\text{Per}_n \to \text{Rat}_d) - \sum_{m|n, m<n} \deg(\text{Per}_m^* \to \text{Rat}_d)
= (d^n + 1) - \sum_{m|n, m<n} \nu_m.
\]

\(^1\)In [9], these were called points of “essential” period \( m \) and were denoted \( Z^*_m \); but we feel that Milnor’s “formal” [7] is a better terminology.
This gives the first part of (iv), and the second part is just Möbius inversion.

Remark. The scheme $\text{Per}_n \subset \text{Rat}_d$ is given by the vanishing of the homogeneous polynomial

$$\Phi_n \stackrel{\text{def}}{=} YF_a^{(n)} - XF_b^{(n)}.$$

The defining property of $\text{Per}_n$ (or a direct calculation) shows that if $m|n$, then $\Phi_m|\Phi_n$ in $\mathbb{Z}[a,b][X,Y]$. Then the fact that $\Phi_n$ is reduced (i.e., has no repeated factors) and a simple inclusion-exclusion argument shows that the product

$$\Phi^*_n \stackrel{\text{def}}{=} \prod_{m|n} (\Phi_m)^{\nu(n/m)}.$$

is in $\mathbb{Z}[a,b][X,Y]$. Looking at the defining properties of $\text{Per}^*_n$ in Theorem 4.4, we see that $\text{Per}^*_n$ is given by the equation $\Phi^*_n = 0$.

It is almost certainly the case that $\Phi^*_n$, and thus $\text{Per}^*_n$, are irreducible, but this has not yet been proven. A similar problem for the space of (monic) polynomial maps is treated by Morton in [8].

Remark. The scheme $\text{Per}^*_n$ is finite over $\text{Rat}_d$, and its degree $\nu_n$ gives the number of periodic points of formal period $n$ for a rational map of degree $d$. The following table gives the value of $\nu_n$ for small values of $d$ and $n$.

| $d$ | $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|-----|---|---|---|---|---|---|---|---|
| 2   | 2   | 3 | 6 | 12| 30| 54| 126|240|
| 3   | 4   | 6 | 24| 72|240|696|2184|6480|
| 4   | 5   | 12| 60|240|1020|4020|16380|65280|
| 5   | 6   | 20|120|600|3120|15480|78120|390000|
| 6   | 7   | 30|210|1260|7770|46410|279930|1678320|
| 7   | 8   |42 |336|2352|16800|117264|823536|5762400|
| 8   | 9   |56 |504|4032|32760|261576|2097144|16773120|

The degree of $\text{Per}^*_n$ over $\text{Rat}_d$ is a (locally) free sheaf of rank 1 over $\mathcal{O}_{\text{Rat}_d}$, hence (locally) free of rank $\nu_n$ over $\mathcal{O}_{\text{Rat}_d}$. The map $\phi$ induces a linear endomorphism of this sheaf, and we compute the characteristic polynomial

$$\det(T + \phi^*) = \sum_{i=0}^{\nu_n} \sigma_i^{(n)} T^{\nu_n-i}$$

for certain sections $\sigma_i^{(n)}$ of $\mathcal{O}_{\text{Rat}_d}$. The following result generalizes Proposition 4.2.

**Theorem 4.5.** The functions $\sigma_i^{(n)}$, for all $n \geq 1$ and $1 \leq i \leq \nu_n$, as described above, are global sections of $\mathcal{O}_{\text{Rat}_d}$. Further, they are invariant under the conjugation action of $\text{SL}_2$, and hence they descend to give global sections of $\mathcal{O}_{\text{M}_d}$.

**Proof.** The proof is the same, mutatis mutandis, as the proof of Proposition 4.2.
§5. The space $M_2$ is isomorphic to $\mathbb{A}^2$

In this section we will prove the following theorem.

Theorem 5.1. The natural map

$$M_2 \rightarrow \text{Spec} \, \mathbb{Z}[\sigma_1, \sigma_2] \cong \mathbb{A}^2_{\mathbb{Z}}$$

is an isomorphism of schemes over $\mathbb{Z}$.

Remark. Theorem 5.1 may be compared with Milnor’s result [7, lemma 3.1] which says that there is an algebraic bijection between $M_2(\mathbb{C})$ and $\mathbb{C}^2$. Milnor uses his result to deduce [7, lemma D.1] that the higher order invariants $\sigma_i^{(n)}$ are in $\mathbb{C}[\sigma_1, \sigma_2]$. He illustrates this corollary with the examples

$$\begin{align*}
\sigma_1^{(2)} &= 2\sigma_1 + \sigma_2, \\
\sigma_1^{(3)} &= \sigma_1(2\sigma_1 + \sigma_2) + 3\sigma_1 + 3, \\
\sigma_2^{(3)} &= (\sigma_1 + \sigma_2)^2(2\sigma_1 + \sigma_2) - \sigma_1(\sigma_1 + 2\sigma_2) + 12\sigma_1 + 28.
\end{align*}$$

Using Theorem 5.1, we can strengthen Milnor’s result by showing that the $\sigma_i^{(n)}$’s are always polynomials in $\sigma_1$ and $\sigma_2$ with integer coefficients.

Corollary 5.2. The ring of $SL_2$-invariant functions on $\text{Rat}_2$ is exactly $\mathbb{Z}[\sigma_1, \sigma_2]$. In particular, all of the higher order invariants $\sigma_i^{(n)}$ are in $\mathbb{Z}[\sigma_1, \sigma_2]$.

Remark. If we write

$$\phi(z) = \frac{a_0z^2 + a_1z + a_2}{b_0z^2 + b_1z + b_2} = \frac{F_a(z)}{F_b(z)},$$

then the corresponding resultant is

$$\rho = \rho(a, b) = \text{Res}(F_a, F_b)$$

$$= a_2^2b_0^2 - a_1a_2b_0b_1 + a_0a_2b_1^2 + a_2b_1b_2 - 2a_0a_2b_0b_2 - a_0a_1b_1b_2 + a_0^2b_2^2.$$

The space of rational functions $\text{Rat}_2$ is the subset of $\mathbb{P}^5 = \text{Proj} \, \mathbb{Z}[a_0, a_1, a_2, b_0, b_1, b_2]$ given by the non-vanishing condition $\rho(a, b) \neq 0$, so $\text{Rat}_2$ is the affine scheme

$$\text{Rat}_2 = \text{Spec} \, \mathbb{A}_2[\rho^{-1}(0)] = \text{Spec} \, \mathbb{Z} \left[ \frac{a_0a_1a_2b_0b_1b_2}{\rho(a, b)} \right]_{i_0 + i_1 + i_2 + j_0 + j_1 + j_2 = 4}.$$

The action of $SL_2$ on $\text{Rat}_2$ is given by its action on the $a_i$’s and $b_i$’s corresponding to the rule $\phi' = f^{-1} \circ \phi \circ f$. We will omit giving the action explicitly, but we note from Theorem 2.1 that $M_2$ is the affine scheme whose affine coordinate ring is the ring of invariants of this $SL_2$-action. According to Theorem 5.1, this ring of invariants is exactly $\mathbb{Z}[\sigma_1, \sigma_2]$, so it seems worthwhile to write down $\sigma_1$ and $\sigma_2$ explicitly in terms of the $a_i$’s and $b_i$’s.

$$\rho(a, b)\sigma_1(\phi) = a_1^3b_0 - 4a_0a_1a_2b_0 - 6a_0^2b_0^2 - 4a_1a_2^2b_0^2 + 4a_1^2a_2^2b_0 + 4a_1a_2b_0b_1$$

$$- 2a_0a_2b_1^2 + 4a_2b_1^3 - 2a_0^2b_0b_2 + 4a_0a_2b_0b_2 - 4a_2b_0b_1b_2 - a_1b_1^2b_2$$

$$+ 2a_0^2b_2^2 + 4a_1b_0b_2^2;$$

$$\rho(a, b)\sigma_2(\phi) = -a_0^2a_1^2 + 4a_0a_2^3 - 2a_1^3b_0 + 10a_0a_1a_2b_0 + 12a_0^2b_0^2 - 4a_0^2a_2b_1$$

$$- 7a_1a_2b_0b_1 - a_1^3b_2^2 + 5a_0a_2b_1^2 - 2a_2b_1^3 + 4a_0a_1b_1 + 5b_0^2b_2b_2$$

$$- 4a_0b_2b_0b_2 - a_0a_1b_1b_2 + 10a_2b_0b_1b_2 - 4a_1b_0b_2^2 + 2a_0b_1b_2^2 - b_1^2b_2^2$$

$$+ 4b_0b_2^3.$$
These formulas make the map $\text{Rat}_2 \to M_2$ completely explicit using the identifications $\text{Rat}_2 \subset \mathbb{P}^5$ and $M_2 \cong \text{Spec} \mathbb{Z}[\sigma_1, \sigma_2]$.

We will prove Theorem 5.1 in a number of steps. One of the tools we will use is a set of normal forms for rational maps of degree two modulo $\text{SL}_2$-conjugation. Normal forms are typically created by moving fixed, periodic, and/or critical points into specified locations, see for example [7, appendix C]. We will take the same approach, but some care is needed because ultimately we will be working over rings and fields which may have finite characteristic, including characteristic 2. So for example, we will not want to use, either implicitly or explicitly, the “fact” that a rational map of degree 2 has exactly two critical points, since in characteristic 2 a map of degree two either has one critical point, or else it is inseparable and every point is critical.

Remark. The relation $\sigma_1 = \sigma_3 + 2$, that is $\mu_1 + \mu_2 + \mu_3 = \mu_1 \mu_2 \mu_3 + 2$, implies the formal identities

$$(\mu_1 - 1)^2 = (\mu_1 \mu_2 - 1)(\mu_1 \mu_3 - 1) \quad \text{and} \quad (\mu_2 - 1)^2 = (\mu_2 \mu_1 - 1)(\mu_2 \mu_3 - 1).$$

In particular, over any field (or even over any reduced ring), the condition $\mu_1 \mu_2 = 1$ is equivalent to $\mu_1 = \mu_2 = 1$.

Normal Forms Lemma 5.3. Let $\Omega$ be an algebraically closed field of characteristic $p$, and let $\xi \in M_2(\Omega)$ have multipliers $\mu_1, \mu_2, \mu_3$. For any $\phi(z) = F(z)/G(z)$ with $F, G \in \Omega[z]$, we will write $\rho$ for the resultant $\text{Res}(F, G)$.

(i) If $\mu_1 \mu_2 \neq 1$, then

$$\phi(z) = \frac{z^2 + \mu_1 z}{\mu_2 z + 1} \in \xi$$

with $\rho = 1 - \mu_1 \mu_2$.

(ii) If $\mu_1 \neq 0$, then there is a $\beta \in \Omega$ satisfying

$$\beta^2 = \left(1 - \frac{2}{\mu_1}\right)^2 - \mu_2 \mu_3$$

such that

$$\phi(z) = \frac{1}{\mu_1} \left( z + \frac{1}{z} \right) + \beta \in \xi$$

and $\rho = \mu_1^2$.

Proof. We start with any rational map

$$\phi(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2} \in \xi$$

and make coordinate changes to put $\phi$ into the desired form.

(i) As noted above, the condition $\mu_1 \mu_2 \neq 1$ implies that $\mu_1 \neq 1$ and $\mu_2 \neq 1$. The associated fixed points thus have multiplicity 1, so they must be distinct. Making a change of variables, we can move them to 0 and $\infty$ respectively. This means that $\phi(z) = (a_0 z^2 + a_1 z)/(b_1 z + b_2) \in \xi$ with $a_0 b_2 \neq 0$, so we can dehomogenize by setting $a_0 = 1$. Taking derivatives, we find that

$$\phi'(0) = \frac{a_1}{b_2} = \mu_1 \quad \text{and} \quad \phi'(\infty) = b_1 = \mu_2,$$
so $\phi$ has the form $\phi(z) = (z^2 + b_2 \mu_1 z)/(\mu_2 z + b_2)$. Finally, $b_2^{-1} \phi(b_2 z)$ puts $\phi$ into the desired form, and one easily verifies that the resultant is $1 - \mu_1 \mu_2$.

(ii) The assumption that $\mu_1 \neq 0$ means that the associated fixed point is not critical. We move this fixed point to $\infty$, which forces $a_0 = 0$, and then $a_0 \neq 0$, so we can dehomogenize by setting $a_0 = 1$. Further, $\phi'(\infty) = b_1 = \mu_1$. Next we observe that $\phi^{-1}(\infty)$ consists of $\infty$ and one other point. This follows from the fact that the multiplier at $\infty$ is non-zero, or we can just note that $\phi(-b_2/\mu_1) = \infty$. In any case, we use the change of variables $z \mapsto z - b_2/\mu_1$ to move this point to 0, which puts $\phi$ in the form

$$\phi(z) = \frac{z^2 + a_1 z + a_2}{\mu_1 z}.$$  

Note that $a_2 \neq 0$, so the final variable change $z \mapsto \sqrt{a_2} z$ puts $\phi$ into the desired form with $\beta = a_1/\mu_1 \sqrt{a_2}$. The resultant is easily computed to equal $\rho = \mu_1^2$, and with a bit more effort one computes the multiplier

$$\sigma_3 = \mu_1 \mu_2 \mu_3 = \mu_2 - \mu_2 \beta^2 - 4 + 4/\mu_1.$$  

Solving for $\beta^2$ completes the proof of the lemma.

Using these normal forms, it is not hard to show that the map $M_2 \to \mathbb{A}^2$ is bijective on geometric points. This may be compared with [7, lemma 3.1], where the same result is proven over $\mathbb{C}$ in essentially the same way.

**Lemma 5.4.** Let $\Omega$ be an algebraically closed field. Then the map

$$(\sigma_1, \sigma_2) : M_2(\Omega) \longrightarrow \Omega^2$$

is a bijection (of sets).

**Proof.** Let $\xi, \xi' \in M_2(\Omega)$ have the same image in $\Omega^2$. The set of multipliers is determined by the values of $\sigma_1$ and $\sigma_2$, since the multipliers (with multiplicity) are the roots of the polynomial

$$T^3 - \sigma_1 T^2 + \sigma_2 T - (\sigma_1 - 2).$$

Hence $\xi, \xi'$ have the same multiplier systems, say $\{\mu_1, \mu_2, \mu_3\}$. We consider two cases.

First, suppose that $\mu_1 \mu_2 \neq 1$. Then the Normal Forms Lemma 5.3(i) tells us that the rational map

$$\phi(z) = \frac{z^2 + \mu_1 z}{\mu_2 z + 1}$$

is in both $\xi$ and $\xi'$, so $\xi = \xi'$.

Second, suppose that $\mu_1 \mu_2 = 1$. The Normal Forms Lemma 5.3(ii) says that $\xi$ and $\xi'$ each contain a map

$$\phi(z) = \frac{1}{\mu_1} \left( z + \frac{1}{z} \right) + \beta \in \xi$$

for some $\beta \in \Omega$ satisfying

$$\beta^2 = \left( 1 - \frac{2}{\mu_1} \right)^2 - \mu_2 \mu_3.$$
However, by an earlier remark, the condition \( \mu_1 \mu_2 = 1 \) actually implies that \( \mu_1 = \mu_2 = 1 \), and then \( \sigma_1 = \sigma_3 + 2 \) shows that also \( \mu_3 = 1 \). Hence \( \beta^2 = 0 \), so \( \beta = 0 \).

Thus \( \xi \) and \( \xi' \) both contain \( z + 1/z \), so \( \xi = \xi' \). This completes the verification that the map \( M_2(\Omega) \to \Omega^2 \) is injective.

To see that the map is surjective, we take any \((\alpha_1, \alpha_2) \in \Omega^2\), and we let \( \mu_1, \mu_2, \mu_3 \in \Omega \) be the three roots (with multiplicity) of the polynomial

\[
T^3 - \alpha_1 T^2 + \alpha_2 T - (\alpha_1 - 2).
\]

If any \( \mu_1 \mu_2 \neq 1 \), then the rational map

\[
\phi(z) = \frac{z^2 + \mu_1 z}{\mu_2 z + 1}
\]

has degree two, multipliers \( \mu_1, \mu_2, \mu_3 \), and hence invariants \( \sigma_1(\phi) = \alpha_1 \) and \( \sigma_2(\phi) = \alpha_2 \). Similarly, if \( \mu_1 \mu_2 = 1 \), then it follows as usual that \( \mu_1 = \mu_2 = \mu_3 = 1 \), so we need merely observe that the map \( z + 1/z \) has multipliers \( \{1, 1, 1\} \) and invariants \( \sigma_1 = 3 = \alpha_1 \) and \( \sigma_2 = 3 = \alpha_2 \). This proves that the map \( M_2(\Omega) \to \Omega^2 \) is surjective, which completes the proof of the lemma.

Before proceeding further, we want to note that the mere fact that \( M_2 \to A_2 \) is bijective on geometric points (i.e., \( M_2(\Omega) = A_2(\Omega) \) for algebraically closed fields \( \Omega \)) does not imply that the map \( M_2 \to A_2 \) is an isomorphism. The are two possible problems. First, if \( \Omega \) has positive characteristic, then an inseparable map may be bijective on points, yet not be an isomorphism. Second, even in characteristic 0, there are morphisms which are bijective on geometric points, yet have no inverse. A simple example is the map of \( A_1 \) onto the the cuspidal cubic \( y^2 = x^3 \) via the map \( t \mapsto (t^2, t^3) \). The next step in the proof of Theorem 5.1 will be to show that the map \( M_2 \to A_2 \) is proper.

**Lemma 5.5.** The map \( M_2 \to \text{Spec } \mathbb{Z}[\sigma_1, \sigma_2] = A_2^2 \) is a proper morphism.

**Proof.** Let \( F : M^2 \to A_2^2 \) be the given map. We know from general principles that \( M_2 \) and \( A_2^2 \) are separable over \( \mathbb{Z} \), so \( F \) is separable. Further, \( M_2 \) is of finite type of \( \mathbb{Z} \), so \( M_2 \) is Noetherian and \( F \) is of finite type. Hence we may use the valuative criterion [3, II.4.7] to check that \( F \) is proper.

Let

\[
R = \text{a discrete valuation ring}, \quad T = \text{Spec } (R), \quad K = \text{the fraction field of } R, \quad U = \text{Spec } (K),
\]

and suppose we are given a commutative diagram

\[
\begin{array}{ccc}
U & \longrightarrow & M_2 \\
\downarrow & & \downarrow F \\
T & \longrightarrow & A_2^2.
\end{array}
\]

We need to find a map \( T \to M_2 \) making the diagram commute.

Let \( \overline{M}_2 = M_2^a \) be the proper scheme containing \( M_2 \) as described in Theorem 2.1. Since we are working with maps of degree 2, Corollary 2.3 says that \( M_2^a = M_2 \), so \( \overline{M}_2 \) is actually the \( SL_2 \)-quotient of the stable points in \( \mathbb{P}^5 \), but for our purposes it
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suffices to know that there is a certain SL$_2$-stable subset $(\mathbb{P}^5)^{ss}$ of $\mathbb{P}^5$ which contains $\text{Rat}_d$ and whose SL$_2$-quotient $\overline{M}_2$ exists and is proper over $\mathbb{Z}$. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
\text{Rat}_2 & \longrightarrow & (\mathbb{P}^5)^{ss} \\
\downarrow & & \downarrow \\
U & \longrightarrow & M_2 \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathbb{A}^2 \longrightarrow \text{Spec} (\mathbb{Z}).
\end{array}
$$

The map $\overline{M}_2 \to \text{Spec} (\mathbb{Z})$ is proper, so the valuative criterion implies that there is a map $T \to M_2$ making the diagram commute. So we just need to show that the image of this map lies in $M_2$, since this will give a map $T \to M_2$, and then the separability of $\mathbb{A}^2$ over $\mathbb{Z}$ will imply that $T \to M_2$ commutes with the maps in the left-hand square.

(To verify this last assertion, we label some of the maps in the above diagram as

$$
\begin{array}{ccc}
U & \longrightarrow & M_2 \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathbb{A}^2 \longrightarrow \text{Spec} (\mathbb{Z}).
\end{array}
$$

Now suppose that we have constructed a map $\gamma : T \to M_2$ with $\gamma \circ i = \alpha$, but that we only know that $\pi \circ \sigma \circ \gamma = \pi \circ \beta$. We want to show that $\sigma \circ \gamma = \beta$. Of course, we do know that $\sigma \circ \alpha = \beta \circ i$. Consider the commutative square

$$
\begin{array}{ccc}
U & \longrightarrow & \mathbb{A}^2 \\
\downarrow & & \downarrow \\
T & \longrightarrow & \text{Spec} (\mathbb{Z}).
\end{array}
$$

Notice that both of the maps $\sigma \circ \gamma : T \to \mathbb{A}^2$ and $\beta : T \to \mathbb{A}^2$ commute with this square. Hence the separability of $\pi : \mathbb{A}^2 \to \text{Spec} (\mathbb{Z})$ implies the desired equality $\sigma \circ \gamma = \beta$.)

We observe that we are free to replace $K$ by a finite extension $K'$ and $R$ with its integral closure $R'$ in $K'$. This is true because if we can prove that the map $T' = \text{Spec} (R') \to \overline{M}_2$ has image in $M_2$, then the same will be true for $T \to \overline{M}_2$, since $T' \to T$ is surjective.

The given map $\xi : U \to M_2$ is a $K$-valued point $\xi \in M_2(K)$. This SL$_2$-equivalence class of rational maps has invariants $\sigma_1, \sigma_2, \sigma_3 \in K$ and multipliers $\mu_1, \mu_2, \mu_3 \in \overline{K}$ as usual. Replacing $K$ by a finite extension, we will assume that $\mu_1, \mu_2, \mu_3 \in K$. Further, the commutativity of the diagram (4) tells us that $\sigma_1$ and $\sigma_2$ are in $R$. Hence $\mu_1, \mu_2, \mu_3$ are also in $R$, since they are roots of the monic polynomial with coefficients in $R$,

$$
T^3 - \sigma_1 T^2 + \sigma_2 T - (\sigma_1 - 2),
$$
and $R$ is integrally closed. Let $\mathfrak{m}$ be the maximal ideal in the valuation ring $R$. We now consider two cases.

First, suppose that $\mu_1\mu_2 \not\equiv 1 \pmod{\mathfrak{m}}$. Then certainly $\mu_1\mu_2 \not\equiv 1$, so the Normal Forms Lemma 5.3(i) tells us that (after another finite extension of $K$) we can find a map

$$\phi(z) = \frac{z^2 + \mu_1z}{\mu_2z + 1}$$

in the equivalence class of maps $\xi$. In other words, $\phi$ is a point in $\text{Rat}_2(K)$ lifting $\xi$. Recall that $\text{Rat}_2$ is the affine open subset of $\mathbb{P}^5$ given by

$$\{[a_0, a_1, a_2, b_0, b_1, b_2] \in \mathbb{P}^5 : \text{Res}(a_0X^2 + a_1XY + a_2Y^2, b_0X^2 + b_1XY + b_2Y^2) \neq 0\}.$$ 

Thus a point $\psi \in \text{Rat}_2(K) \hookrightarrow \mathbb{P}^5(R)$ will lie in $\text{Rat}_2(R)$ if and only if it has the form

$$\psi(z) = \frac{a_0z^2 + a_1z + a_2}{b_0z^2 + b_1z + b_2} \quad \text{with } a_0, a_1, a_2, b_0, b_1, b_2 \in R \text{ and}$$

$$\text{Res}(a_0X^2 + a_1XY + a_2Y^2, b_0X^2 + b_1XY + b_2Y^2) \equiv 0 \pmod{\mathfrak{m}}.$$ 

The map $\phi(z)$ listed above corresponds to the point $[1, \mu_1, 0, 0, \mu_2, 1] \in \mathbb{P}^5(R)$, and its resultant is $\mu_1\mu_2 - 1$. By assumption, $\mu_1\mu_2 - 1 \not\equiv 0 \pmod{\mathfrak{m}}$, so $\phi$ lies in $\text{Rat}_2(R)$, and hence $\xi$ lies in $M_2(R)$.

For the second case, we suppose that $\mu_1\mu_2 \equiv 1 \pmod{\mathfrak{m}}$. In particular, $\mu_1 \not\equiv 0$, so the Normal Forms Lemma 5.3(ii) says that (after a finite extension of $K$) there is a map

$$\phi(z) = \frac{1}{\mu_1} \left( z + \frac{1}{z} \right) + \beta \in \xi,$$

where $\beta$ satisfies

$$\beta^2 = \left(1 - \frac{2}{\mu_1}\right)^2 - \mu_2\mu_3.$$ 

Again extending $K$, we may assume that $\beta \in K$. Further, the assumption that $\mu_1\mu_2 \equiv 1 \pmod{\mathfrak{m}}$ means that $\mu_1$ is a unit (i.e., $\mu_1 \in R^\times$), so we see that $\beta \in R$. As above, this map $\phi$ corresponds to the point $[1, \beta\mu_1, 1, 0, \mu_1, 1] \in \mathbb{P}^2(R)$ having resultant $\mu_1^2$. We know that $\mu_1$ is a unit, so $\mu_1^2 \not\equiv 0 \pmod{\mathfrak{m}}$. Hence $\phi$ lies in $\text{Rat}_2(R)$, which proves that $\xi$ lies in $M_2(R)$.

This completes the proof that there is a map $T = \text{Spec}(R) \to M_2$ making the diagram (4) commute. By the valuative criterion for properness [3, II.4.7], we conclude that the map $(\sigma_1, \sigma_2) : M_2 \to \mathbb{A}_R^2$ is proper, which completes the proof of Lemma 5.5.

We now know that the map $M_2 \to \mathbb{A}_R^2$ is proper. However, both $M_2$ and $\mathbb{A}_R^2$ are affine varieties. Intuitively, the (geometric) fibers of the map are both affine and complete, which should imply they they consist of a finite set of points. The following generalization of [3, exer. II.4.6] makes this intuition precise. It will be used to show that the map $M_2 \to \mathbb{A}_R^2$ is finite.
Lemma 5.6. Let $X$ and $Y$ be affine integral schemes, and let $F : X \to Y$ be a dominant proper morphism of finite type. Then $F$ is a finite morphism.

Proof. Let $X = \text{Spec} (A)$ and $Y = \text{Spec} (B)$. Since $X$ and $Y$ are integral schemes, $A$ and $B$ are integral domains. We let $K_A$ and $K_B$ be the fraction fields of $A$ and $B$ respectively. Then $F : X \to Y$ induces a homomorphism $B \to A$, and the fact that $F$ is dominant means that this homomorphism is injective. So we also get an injection $K_B \to K_A$.

Now let $B' \subset K_A$ be any valuation ring of $K_A$ containing the image of $B$. By definition of valuation ring, every $x \in K_A$ satisfies either $x \in B'$ or $x^{-1} \in B'$. (See [4, XII §4].) In particular, $K_A$ is the fraction field of $B'$. Now consider the commutative diagrams

\[
\begin{array}{ccc}
A & \longrightarrow & K_A \\
\uparrow & & \uparrow \\
B & \longrightarrow & B'
\end{array}
\quad\quad\quad
\begin{array}{ccc}
X = \text{Spec} (A) & \longleftarrow & \text{Spec} (K_A) \\
F & \downarrow & \\
\text{Spec} (B) & \longleftarrow & \text{Spec} (B').
\end{array}
\]

We are given that $F$ is proper, so the valuable criterion of properness [3, II.4.7] tells us that there is a unique map $\text{Spec} (B') \to \text{Spec} (A)$ making the right-hand diagram commute. Equivalently, there is a unique homomorphism $A \to B'$ making the left-hand diagram commute. This proves that every valuation ring of $K_A$ containing the image of $B$ will also contain $A$. It follows from [4, XII §4, prop. 4.9] or [3, II.4.11A] that $A$ is integral over $B$. (That is, every element of $A$ is the root of a monic polynomial in $B[T]$.) But we are also given that $F$ is of finite type, which means that $A$ is of finite type over $B$. Thus $A = B[a_1, \ldots, a_r]$ with each $a_i$ integral over $B$, so $A$ is a finitely generated $B$-module. Therefore $F$ is finite.

Combining Lemmas 5.4, 5.5, and 5.6 shows that the map $M_2 \to \mathbb{A}_Z^2$ is a finite map which is bijective on geometric points. One would expect that this should imply that the map is an isomorphism, but there is still some work to do. In characteristic $p$, the Frobenius map is finite and bijective on geometric points, yet is not an isomorphism; and the same is true of the map $t \to (t^2, t^3)$ of $\mathbb{A}^1$ onto the twisted cubic. We will need to use the fact that $M_2 \to \mathbb{A}_Z^2$ is a morphism of schemes over $\mathbb{Z}$ and the fact that the image $\mathbb{A}_Z^2$ is non-singular. The following general lemma is what we will need to complete the proof of Theorem 5.1.

Lemma 5.7. Let $F : X \to Y$ be a morphism of schemes over $\mathbb{Z}$. Suppose that the following four conditions are true.

(i) $X$ is an integral scheme.
(ii) $Y$ is an integral normal scheme which is dominant over $\mathbb{Z}$.
(iii) $F$ is a finite morphism.
(iv) $F$ induces a bijection on geometric points.

Then $F$ is an isomorphism.

Proof. Before beginning the proof, we remind the reader what the conditions (i)–(iv) really mean. A scheme $X$ is integral if and only if it is reduced and irreducible [3, II.3.1]. This means that the local ring of the generic point is a field, equal to the fraction field of $A$ for any affine open subset $\text{Spec} (A) \subset X$ [3, II.3.6]. The fact that $Y$ is normal means that its local rings are integrally closed domains [3, exer. II.3.8]. The map $F$ induces a natural map on $S$-valued points,
$X(S) \to Y(S)$, for any scheme $S$. Thus if $f : S \to X$ is in $X(S)$, then $F \circ f \in Y(S)$.

Condition (iv) says that this map is a bijection whenever $S = \text{Spec} (\Omega)$ for an algebraically closed field $\Omega$.

We now begin the proof of Lemma 5.7. Geometric points are dense in $Y$, so (iv) implies that $F$ is dominant (i.e., $F(X)$ is dense in $Y$). Hence $F$ induces a map of function fields $F^* : K(Y) \to K(X)$. (In fact, since $F$ is finite from (iii), it follows that $F$ is closed [3, ex. II.3.5(b)], so $F(X)$ is dense and closed, so $F(X) = Y$. But we won’t need to know this stronger fact.)

Take any affine open subset $\text{Spec} (B) \subset Y$. Using the fact (iii) that $F$ is finite, we find that $F^{-1}(\text{Spec} (B)) = \text{Spec} (A)$, where $A$ is a finitely generated $B$-module [3, exer. II.3.4]. Replacing $X$ and $Y$ by $\text{Spec} (A)$ and $\text{Spec} (B)$, we may assume that $X$ and $Y$ are affine. Note that $A$ and $B$ are integral domains, since $X$ and $Y$ are integral schemes from (i) and (ii). Let $K_A$ and $K_B$ be the fraction fields of $A$ and $B$ respectively. We have commutative diagrams

$$
\begin{array}{c@{\quad}c@{\quad}c}
X = \text{Spec} (A) & \Longleftarrow & \text{Spec} (K_A) \\
\downarrow F & & \downarrow F \\
Y = \text{Spec} (B) & \Longleftarrow & \text{Spec} (K_B)
\end{array}
$$

$$
\begin{array}{c@{\quad}c@{\quad}c}
A & \longrightarrow & K_A \\
\uparrow F^* & & \uparrow F^* \\
B & \longrightarrow & K_B.
\end{array}
$$

Notice that if $\Omega$ is any (algebraically closed) field, then

$$
X(\Omega) = \text{Mor}(\text{Spec} \Omega, X) = \text{Mor}(\text{Spec} \Omega, \text{Spec} A) = \text{Hom}(A, \Omega) = \text{Hom}(K_A, \Omega),
$$

where the last equality is true because $K_A$ is the fraction field of the integral domain $A$, and similarly $Y(\Omega) = \text{Hom}(K_B, \Omega)$. Then the map $X(\Omega) \to Y(\Omega)$ induced by $F$ is given by

$$
\text{Hom}(K_A, \Omega) \longrightarrow \text{Hom}(K_B, \Omega), \quad f \longmapsto f \circ F^*.
$$

Now it is a standard fact from the theory of fields that if $\Omega$ is algebraically closed, then any $g \in \text{Hom}(K_B, \Omega)$ can be lifted to an element of $\text{Hom}(K_A, \Omega)$ in exactly $[K_A : F^* K_B]_s$ ways, where the subscript $s$ denotes the separable degree. (See 4, VII §4.) Condition (iv) tells us that $X(\Omega) \to Y(\Omega)$ is bijective, so we conclude that $[K_A : F^* K_B]_s = 1$. However, the assumption (ii) that $Y$ is dominant over $\text{Spec} (\mathbb{Z})$ implies that $B$, and hence also $K_B$, have characteristic 0. So the separable degree is the actual degree, $[K_A : F^* K_B] = 1$, and hence $F^* : K_B \to K_A$ is an isomorphism.

We now have the commutative diagram

$$
\begin{array}{c@{\quad}c}
A & \longrightarrow & K_A \\
\uparrow F^* & & \uparrow F^* \\
B & \longrightarrow & K_B.
\end{array}
$$

Further, $A$ is integral over $B$ from (iii), and $B$ is integrally closed in $K_B$ from (ii). But $K_A = F^*(K_B)$, so $F^*(B)$ is integrally closed in $K_A$. This gives the inclusions

$$
B \overset{F^*}{\longrightarrow} A \subset (\text{integral closure of } B \text{ in } K_A) = F^*(B).
$$
Hence \( F^*: B \to A \) is an isomorphism, which completes the proof that \( F \) is an isomorphism.

We now have all of the pieces to complete the proof of Theorem 5.1.

**Proof of Theorem 5.1.** We will denote by \( \sigma : M_2 \to \mathbb{A}^2_\mathbb{Z} \) the morphism induced by the inclusion of \( \mathbb{Z}[\sigma_1, \sigma_2] \) into the affine coordinate ring of \( M_2 \). Both \( M_2 \) and \( \mathbb{A}^2_\mathbb{Z} \) are affine integral schemes, the former from Theorem 2.1, and the latter trivially. The map \( \sigma \) is bijective on geometric points from Lemma 5.4, so it must be dominant. The map \( \sigma \) is of finite type, since it is a \( \mathbb{Z} \)-morphism, and \( M_2 \) is actually of finite type over \( \mathbb{Z} \) from Theorem 2.1. Finally, \( \sigma \) is a proper morphism from Lemma 5.5. It follows from Lemma 5.6 that \( \sigma \) is a finite morphism.

We want to apply Lemma 5.7 to \( \sigma \), so we have to check the four conditions in Lemma 5.7. First, \( M_2 \) is an integral scheme from Theorem 2.1. Second, it is easy to see that \( \mathbb{A}^2_\mathbb{Z} \) is an integral normal scheme and is dominant over \( \mathbb{Z} \). Third, \( \sigma \) is a finite morphism from the previous paragraph. Fourth, \( \sigma \) induces a bijection on geometric points from Lemma 5.4. Hence we can apply Lemma 5.7 to conclude that \( \sigma \) is an isomorphism.

**Proof of Corollary 5.2.** In general, the affine coordinate ring of \( M_d \) is the ring of \( \text{SL}_2 \)-invariant functions on \( \text{Rat}_d \) (see Theorem 2.1), while the affine coordinate ring of \( \mathbb{A}^2_\mathbb{Z} = \text{Spec} \mathbb{Z}[\sigma_1, \sigma_2] \) is precisely \( \mathbb{Z}[\sigma_1, \sigma_2] \). Now the isomorphism \( M_2 \cong \mathbb{A}^2_\mathbb{Z} \) from Theorem 5.1 shows that \( M_2 \) and \( \mathbb{A}^2_\mathbb{Z} \) have the same affine coordinate rings, which gives the first part of the corollary. The second part is immediate, since the higher order invariants \( \sigma_1^{(n)} \) are in the affine coordinate ring of \( M_2 \).

**§6. The Completion \( M^*_2 \) of \( M_2 \)**

In this section we will prove that the stable completion \( M^*_2 \) of \( M_2 \) has a very simple structure as described in the following theorem.

**Theorem 6.1.** The isomorphism \( (\sigma_1, \sigma_2) : M_2 \cong \mathbb{A}^2 \) extends to an isomorphism \( \sigma : M^*_2 \cong \mathbb{P}^2 \) of schemes over \( \mathbb{Z} \).

**Remark.** Milnor [7, section 4] uses the identification \( (\sigma_1, \sigma_2) : M_2(\mathbb{C}) \cong \mathbb{C}^2 \) to study the completion \( \mathbb{P}^2(\mathbb{C}) \) of \( \mathbb{C}^2 \). He shows that the extra points at infinity in \( \mathbb{P}^2(\mathbb{C}) \) correspond to linear and constant maps which can be thought of as degenerate quadratic maps. This provides a natural completion of \( M_2(\mathbb{C}) \) which is isomorphic to \( \mathbb{P}^2(\mathbb{C}) \), but unfortunately it does not immediately imply that our completion \( M^*_2(\mathbb{C}) \) is isomorphic to \( \mathbb{P}^2(\mathbb{C}) \). The difficulty is that Milnor implicitly defines a degenerating family of maps \( \phi_\epsilon \in \text{Rat}_d \) to be a family for which (at least) one of \( \sigma_1(\phi_\epsilon) \) or \( \sigma_2(\phi_\epsilon) \) tends to infinity as \( t \to t_0 \), but there is no a priori reason that the \( \sigma_i(\phi_\epsilon) \)'s might not approach some indeterminate form \( \frac{0}{0} \). For example, the family of maps

\[
\phi_{(t_1, t_2)}(z) = \frac{t_1 z^2 + 2z}{t_2 z + 1}
\]

over the \((t_1, t_2)\) plane satisfies

\[
\sigma_1(\phi_{(t_1, t_2)}) = \frac{2(t_1^2 - 2t_1 t_2 - t_2^2)}{t_1(t_1 - 2t_2)} \quad \text{and} \quad \sigma_2(\phi_{(t_1, t_2)}) = \frac{-5t_2^2}{t_1(t_1 - 2t_2)}.
\]
so neither of the limits
\[ \lim_{(t_1,t_2) \to (0,0)} \sigma_1(\phi(t_1,t_2)) \quad \text{or} \quad \lim_{(t_1,t_2) \to (0,0)} \sigma_2(\phi(t_1,t_2)) \]
exists. Milnor’s completion corresponds to adding maps which satisfy what one might call a \((\sigma_1,\sigma_2)\)-stability condition. During the proof of Theorem 6.1, we will verify that \((\sigma_1,\sigma_2)\)-stability is the same as the stability criterion from geometric invariant theory used to define the stable sets \((\mathbb{P}^5)^s\) and \(M_2^s\).

**Lemma 6.2.** Let \(\Omega\) be an algebraically closed field, and let \(\phi \in (\mathbb{P}^5)^s \setminus \text{Rat}_2(\Omega)\). That is, \(\phi = [F_a, F_b] \) is in the stable locus, but \(\phi\) is not in \(\text{Rat}_2\) because the resultant \(\text{Res}(F_a, F_b)\) vanishes. Then there exists an \(f \in \text{SL}_2(\Omega)\) so that
\[ \phi^f(X,Y) = [AXY, XY + BY^2] \quad \text{for some} \ [A,B] \in \mathbb{P}^1(\Omega). \]
Further, the homogeneous pair \([A,B]\) is uniquely determined by the conjugacy class \(\langle \phi \rangle\) up to reversing the roles of \(A\) and \(B\).

In other words, there is a well-defined bijection
\[ \mathbb{P}^1(\Omega)/\text{Sym}_2 \longrightarrow (M_2^s \setminus M_2)(\Omega), \quad \text{induced by} \quad [A,B] \longmapsto [AXY, XY + BY^2], \]
where the symmetric group on two letters \(\text{Sym}_2\) acts on \(\mathbb{P}^1\) by interchanging the coordinates.

**Proof.** The assumption that \(\phi = [F_a, F_b] \) is not in \(\text{Rat}_2(\Omega)\) means that \(F_a\) and \(F_b\) have a common root in \(\mathbb{P}^1(\Omega)\). Making an appropriate conjugation, we may move the common root to \([1,0]\), so \(\phi\) looks like
\[ \phi = [a_1XY + a_2Y^2, b_1XY + b_2Y^2]. \]
According to Proposition 2.2, the stability of \(\phi\) (i.e., \(\phi \in (\mathbb{P}^5)^s\)) implies that \(b_1 \neq 0\).

Of course, we are still free to conjugate by elements of \(\text{SL}_2(\Omega)\) which fix \([1,0]\). First we will conjugate by the matrix \(f = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\). This gives
\[ \phi^f = [(a_1 - b_1\beta)XY - (b_1\beta^2 + b_2\beta - a_1\beta - a_2)Y^2, b_1XY + (b_2 + b_1\beta)Y^2]. \]
We know that \(b_1 \neq 0\), so we can take \(\beta\) to be either of the roots of
\[ b_1\beta^2 + b_2\beta - a_1\beta - a_2 = 0 \]
and dehomogenize by setting \(b_1 = 1\) to obtain
\[ \phi^f = [AXY, XY + BY^2] \quad \text{with} \ A,B \in \Omega. \]
If either \(A\) or \(B\) is non-zero, this is the desired form. But if \(A = B = 0\), so \(\phi^f = [0, XY]\), then conjugation by \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) would give the form \([XY, 0]\), and from Proposition 2.2 this would contradict the stability of \(\phi\). This shows that after conjugation, we can always put \(\phi\) into the desired form.

It remains to determine to what extent the form \([AXY, XY + BY^2]\) is unique. Some algebra and a case-by-case analysis shows that conjugation by the matrix
\( \left( \begin{array}{cc} \alpha & \delta \\ \gamma & \delta \end{array} \right) \) preserves this form in exactly the following cases, where \( u \) denotes an arbitrary element \( u \in \Omega^* \):

\[
\begin{align*}
\text{f} &= \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}, \\
\phi & = [u^2AXY, XY + u^2BXY] \quad \text{(any } A, B) \\
\text{f} &= \begin{pmatrix} u^{-1} & u(A - B) \\ 0 & u \end{pmatrix}, \\
\phi & = [u^2BXY, XY + u^2AXY] \quad \text{(any } A, B) \\
\text{f} &= \begin{pmatrix} u^{-1} & uB \\ -(uB)^{-1} & 0 \end{pmatrix}, \\
\phi & = [-u^2BXY, XY] \quad \text{(if } A = 0, B \neq 0) \\
\text{f} &= \begin{pmatrix} 0 & uA \\ -(uA)^{-1} & 0 \end{pmatrix}, \\
\phi & = [0, XY - u^2BY^2] \quad \text{(if } A = 0, B \neq 0) \\
\text{f} &= \begin{pmatrix} 0 & uA \\ -(uA)^{-1} & u \end{pmatrix}, \\
\phi & = [0, XY - u^2AY^2] \quad \text{(if } B = 0, A \neq 0) \\
\text{f} &= \begin{pmatrix} 0 & uA \\ -(uA)^{-1} & 0 \end{pmatrix}, \\
\phi & = [0, XY - u^2AY^2] \quad \text{(if } B = 0, A \neq 0) 
\end{align*}
\]

It follows that two forms \([AXY, XY + BY^2]\) and \([A'XY, XY + B'Y^2]\) are \( \text{SL}_2(\Omega) \)-equivalent if and only if there is a \( \lambda \in \Omega \) such that either

\((A', B') = (\lambda A, \lambda B) \quad \text{or} \quad (A', B') = (\lambda B, \lambda A)\).

So the set of forms, up to \( \text{SL}_2(\Omega) \)-equivalence, is in one-to-one correspondence with the quotient space \( P^1(\Omega) / \text{Sym}_2 \).

**Lemma 6.3.** (a) Let \( R \) be a discrete valuation ring with fraction field \( K \) and residue field \( k \). Let \( \psi : \text{Spec } R \to M_2^k \) be a morphism. Then the point

\[
[1, \sigma_1(\psi), \sigma_2(\psi)] \sim \in \mathbb{P}^2(k)
\]

depends only on the image of the special fiber \( \psi_k \). (The tilde indicates the natural reduction map \( \mathbb{P}^2(K) \to \mathbb{P}^2(k) \).)

(b) The map \( (\sigma_1, \sigma_2) : M_2 \to k^2 \) induces a birational \( \mathbb{Z} \)-morphism

\[
\sigma = [1, \sigma_1, \sigma_2] : M_2^k \to \mathbb{P}^2.
\]

**Proof.** (a) Note that it is crucial that the special fiber \( \psi_k \) of the family is assumed to be stable, since the example described in the remark above shows that the result is false without the stability assumption.

Our first step will be to lift the map from \( M_2^k \) to \( (\mathbb{P}^5)^s \). To do this, we first replace \( R \) by its strict Henselization. This means that the residue field \( k \) is separably closed and that \( R \) satisfies Hensel’s lemma. (See [14, IV §6] or [2] for information about Henselizations.)

Next we observe that the map \( (\mathbb{P}^5)^s \to M_2^k \) is a smooth morphism. Intuitively, this is true because it is a geometric quotient map whose fibers are isomorphic to the smooth scheme \( \text{SL}_2 \). More precisely, we begin by using [1, corollary VII.1.9].

This says that it suffices to check that the map over each point of \( \text{Spec } \mathbb{Z} \) is smooth. In other words, we need to check that the maps

\[
(\mathbb{P}^5)^s \times \text{Spec } F \to M_2^k \times \text{Spec } F
\]
are smooth, where \( \mathbb{F} \) is either \( \mathbb{Z}/p\mathbb{Z} \) or \( \mathbb{Q} \). This reduces the problem to morphisms over a field. Next we apply [3, III.10.2], which says that it suffices to check that the fiber over each point of \( M_2^+(\mathbb{F}) \) is regular. But as noted above, each such fiber is isomorphic to \( \text{SL}_2(\mathbb{F}) \). This completes the verification that the morphism \( (\mathbb{P}^5)^* \to M_2^+ \) is smooth. Now the lifting property of Henselian rings [2] says that the map \( (\mathbb{P}^5)^*(R) \to M_2^+(R) \) on \( R \)-points is surjective, so we can lift \( \psi \). By abuse of notation, we will also denote the lift by \( \psi : \text{Spec} \; R \to (\mathbb{P}^5)^* \).

If \( \psi_k \in \text{Rat}_2(k) \), then \( \sigma_1(\psi) \) and \( \sigma_2(\psi) \) are already in \( R \), so the desired result follows from the trivial computation

\[
[1, \sigma_1(\psi), \sigma_2(\psi)] = [1, \tilde{\sigma}_1(\psi), \tilde{\sigma}_2(\psi)] = [1, \sigma_1(\psi_k), \sigma_2(\psi_k)].
\]

Suppose now that \( \psi_k \notin \text{Rat}_2(k) \). Of course, by assumption we do know that \( \psi_k \in (\mathbb{P}^5)^*(k) \), so Lemma 6.2 says that after conjugation, we may assume that \( \psi_k \) has the form

\[
\psi_k = [\hat{A}_1 XY, XY + \hat{B}_2 Y^2]
\]

for some \( [\hat{A}_1, \hat{B}_2] \in \mathbb{P}^1(k) \). Lifting \( \hat{A}_1 \) and \( \hat{B}_2 \) to elements \( A_1, B_2 \in R \), this means that we can write \( \psi \) in the form

\[
\psi = [A_0 \pi X^2 + A_1 XY + A_2 \pi Y^2, B_0 \pi X^2 + (1 + B_1 \pi)XY + B_2 Y^2],
\]

where \( A_0, A_1, A_2, B_0, B_1, B_2 \in R \), at least one of \( A_1, B_2 \) is in \( R^* \), and \( \pi \in R \) is a uniformizer. Let

\[
\rho = \rho(\psi) = \text{Res}(A_0 \pi X^2 + A_1 XY + A_2 \pi Y^2, B_0 \pi X^2 + (1 + B_1 \pi)XY + B_2 Y^2).
\]

Notice that \( \hat{\rho} = 0 \). We claim that \( \rho(\psi)\sigma_1(\psi) \) and \( \rho(\psi)\sigma_2(\psi) \) are both in \( R \) and that at least one of them is in \( R^* \). To see this, we use the explicit formulas for \( \sigma_1 \) and \( \sigma_2 \) in section 5. Substituting into these formulas and reducing modulo \( \pi \), we find that

\[
\rho(\tilde{\psi})\tilde{\sigma}_1(\psi) = -A_1 \hat{B}_2 \quad \text{and} \quad \rho(\tilde{\psi})\tilde{\sigma}_2(\psi) = -A_1^2 - B_2^2.
\]

Hence

\[
[1, \sigma_1(\psi), \sigma_2(\psi)] = [\rho(\psi), \rho(\psi)\sigma_1(\psi), \rho(\psi)\sigma_2(\psi)] = [\rho(\psi), \rho(\psi)\tilde{\sigma}_1(\psi), \rho(\psi)\tilde{\sigma}_2(\psi)] = [0, A_1 \hat{B}_2, A_1^2 + B_2^2] \in \mathbb{P}^2(k).
\]

Notice that this is a well-defined point in \( \mathbb{P}^2(k) \), since \( A_1 \) and \( B_2 \) are in \( R \) and at least one of them is a unit. Further, the point clearly depends only on the special fiber \( \psi_k = [A_1 XY, XY + B_2 Y^2] \). This completes the proof of Lemma 6.3(a).

(b) The map \( (\sigma_1, \sigma_2) : M_2 \to \mathbb{A}^2 \) is an isomorphism from Theorem 5.1, so it certainly induces a birational map \( M_2^+ \to \mathbb{P}^2 \). We want to show that this map extends to a morphism. This follows from (a) and general principles. We briefly sketch. If \( F : X \to Y \) is a birational map with the property in (a), and if \( x \in X \) is a closed point, we define \( F(x) \) as follows. Take any discrete valuation ring \( R \) with fraction field \( K \) and residue field \( k(x) \) and any map \( \psi : \text{Spec} \; R \to X \) with
ψ(Spec \(k(x)\)) = x. Since \(F\) is birational, we can also assume that \(ψ(Spec \(K\))\) lies in the domain of \(F\). Then \(F \circ ψ : Spec \(R\) → \(Y\) extends to a morphism (assuming \(Y\) is regular), so we can define \(F(x) = (F \circ ψ)(Spec \(k(x)\))\). The key here is that the property described in (a) says that the value of \(F(x)\) depends only on \(x\), independent of \(ψ\), so \(F(x)\) is well-defined.

We now have the tools needed to complete the proof of Theorem 6.1.

**Proof of Theorem 6.1.** Lemma 6.3(b) says that there is a birational morphism

\[
σ = [1, σ_1, σ_2] : M_s^2 \longrightarrow \mathbb{P}^2.
\]

We claim that \(σ\) is a bijection on geometric points. Theorem 5.1 tells us that \(σ\) is an isomorphism \(M_s^2 \rightarrow \mathbb{A}^2\), so we just need to check the boundary. Let \(Ω\) be an algebraically closed field, and for any \(A,B \in Ω\), let \(φ_{A,B} = [AXY, XY + BY^2] \in \mathbb{P}^5(Ω)\). Then Lemma 6.2 says that the map \([A,B] \rightarrow φ_{A,B}\) induces a bijection \(\mathbb{P}^1(Ω)/\text{Sym}_2 \rightarrow (M_s^2 \setminus M_2)(Ω)\), so we need to show that the map \(\mathbb{P}^1(Ω)/\text{Sym}_2 \rightarrow (\mathbb{P}^2 \setminus \mathbb{A}^2)(Ω), \quad [A,B] \mapsto σ(φ_{A,B})\),

is a bijection. Note that we cannot compute \(σ(φ_{A,B})\) directly, since \(σ_1(φ_{A,B})\) and \(σ_2(φ_{A,B})\) do not exist. However, if we let \(ρ\) denote the resultant form of two polynomials, then \(ρσ_1\) and \(ρσ_2\) will be defined at the point \(φ_{A,B}\). More precisely, using the explicit formulas for \(ρσ_1\) and \(ρσ_2\) given in section 5, we find that

\[
ρ(φ_{A,B}) = 0, \quad (ρσ_1)(φ_{A,B}) = -AB, \quad (ρσ_2)(φ_{A,B}) = -A^2 - B^2,
\]

so we are reduced to showing that the map

\[
\mathbb{P}^1(Ω)/\text{Sym}_2 \longrightarrow (\mathbb{P}^2 \setminus \mathbb{A}^2)(Ω), \quad [A,B] \mapsto [0, AB, A^2 + B^2]
\]

is bijective. This is an exercise which we leave for the reader.

We now know that \(σ : M_s^2 \rightarrow \mathbb{P}^2\) is a birational morphism which is bijective on geometric points. The fact that it is bijective on geometric points certainly implies that it is quasi-finite (i.e., the inverse image of any point is a finite set of points). Further, \(M_s^2\) and \(\mathbb{P}^2\) are both proper over \(\mathbb{Z}\), so it follows from [3, II.4.8(e)] that \(σ\) is a proper morphism. Thus \(σ\) is quasi-finite and proper, so [6, chapter I, proposition 1.10] tells us that \(σ\) is finite.

To complete the proof of Theorem 6.1, we merely need to observe that we now know that \(σ : M_s^2 \rightarrow \mathbb{P}^2\) satisfies the four conditions in Lemma 5.7, and hence \(σ\) is an isomorphism. (We remark that rather than using Lemma 5.7, we could instead give a direct proof that a finite birational morphism \(F : X \rightarrow Y\) of integral schemes with \(Y\) normal is an isomorphism. To do this, we can replace \(X\) and \(Y\) by affines \(\text{Spec} A\) and \(\text{Spec} B\). Then \(A\) is integral over \(B\), the fraction fields of \(A\) and \(B\) coincide, and \(B\) is integrally closed, so \(A = B\).)

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