BIFURCATION AND FINAL PATTERNS OF A MODIFIED SWIFT-HOHENBERG EQUATION

YUNCHERL CHOI
Division of General Education
Kwangwoon University, Seoul, 139-701, Korea

TAEYOUNG HA
Division of Computational Sciences
National Institute for Mathematical Sciences, Daejeon, 305-811, Korea

JONGMIN HAN
Department of Mathematics and Research Institute of Basic Sciences
Kyung Hee University, Seoul, 02447, Korea

DOO SEOK LEE
Department of Undergraduate Studies
Daegu Gyeongbuk Institute of Science and Technology, Daegu, 711-873, Korea

(Communicated by Shouhong Wang)

Abstract. In this paper, we study the dynamical bifurcation and final patterns of a modified Swift-Hohenberg equation (MSHE). We prove that the MSHE bifurcates from the trivial solution to an $S^1$-attractor as the control parameter $\alpha$ passes through a critical number $\hat{\alpha}$. Using the center manifold analysis, we study the bifurcated attractor in detail by showing that it consists of finite number of singular points and their connecting orbits. We investigate the stability of those points. We also provide some numerical results supporting our analysis.

1. Introduction. Pattern formation arises from many natural circumstances and has been an important subject in nonequilibrium physics. This happens when the underlying system undergoes phase transitions. Many examples of pattern formation occur in situations that a system is changing from one phase to another, for instance, from a liquid to a geometrically patterned solid, or from a uniform mixture of chemical constituents to a phase-separated pattern of precipitates([12]). It also occurs in a variety of hydrodynamic systems like convection in pure fluids and mixtures, rotating fluids, and chemically reacting fluids.

The concept of instability plays an important role in the understanding of pattern formation. Spatial or temporal patterns emerge when relatively simple systems are

2010 Mathematics Subject Classification. Primary: 37G35; Secondary: 35B32.

Key words and phrases. Modified Swift-Hohenberg equation, dynamical bifurcation, center manifold.

Y. Choi was supported by the Research Grant of Kwangwoon University in 2015. T. Ha was partially supported by the National Institute for Mathematical Sciences (NIMS) grant funded by the Korean government (No. A21300000). J. Han was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2011-0008557).
driven into unstable states during the phase transition. The stable simple (or trivial) system will deform by large amount in response to small perturbation. Such an instability is closely related to a control parameter of the system. As the control parameter moves, solutions can appear or disappear, and change their stability. In particular, the instability usually happens when a control parameter of the system passes through a critical number such that the trivial state loses its stability and turns into a new state leading to a final pattern.

To study spatial or temporal patterns in a system, we do not need the full solutions of realistic equations describing the system. If we are interested in pattern-forming properties of the system, it is sufficient to consider a model equation which shares the long-range effects with the original system \((1)\). For example, the complex Ginzburg-Landau equation is accepted as a model equation describing a variety of phenomena from the nonlinear waves to second-order phase transitions \((1)\). Recently, it is noticed that fourth-order model equations are responsible for lots of phenomena from hydrodynamic instabilities. A canonical form of such fourth order equations is

\[
    u_t = \alpha u - (1 + \partial_{xx})^2 u + f(u, u_x, \mu),
\]

where \(\mu\) is a coupling parameter appearing in a system under consideration. Typical examples of \(f\) are the following: \(f = -u^3\) for the Swift-Hohenberg equation (SHE), \(f = \mu u^2 - u^3\) with \(\mu > 0\) for the generalized Swift-Hohenberg equation (GSHE), and \(f = -uu_x\) for the damped Kuramoto-Sivashinsky equation (DKSE). The SHE is a widely accepted model in the study of the formation of patterns \((2, 15)\). It was derived in \((22)\) as an approximate model for the Rayleigh-Bénard convection describing the pattern formation in layer fluids between horizontal plates. It has attracted a lot of interest in various areas of application regarding pattern formations such as Taylor-Couette flow and lasers \((17)\). The GSHE also provides a useful tool to understand the evolution of hexagonal structure and patterns periodic in one direction \((15)\). The DKSE has emerged as a fundamental tool for understanding the onset and evolution of secondary instabilities in many driven nonequilibrium systems \((10)\). For example, it provides a crude model of directional solidification.

Regarding the stability problem and pattern formation issues for \((1)\), one of the basic approaches is to deal with it in the theory of nonlinear dynamical systems \((12)\). One may consider the model equation as an ODE in a phase space with a control parameter. A general form of this ODE is written as

\[
\begin{align*}
    \frac{du}{dt} &= L_\alpha u + G(u, \mu), \\
    u(0) &= u_0,
\end{align*}
\]

where \(L_\alpha\) is a linear operator in a phase space \(H\), \(G\) is a nonlinear term, and \(\alpha\) is a control parameter of the underlying system. We assume that some of eigenvalues of \(L_\alpha\) change from negative signs to positive signs as \(\alpha\) passes through a critical number. Then, the trivial state of the system loses its stability and bifurcates to some nontrivial attractor \(A_\alpha\). The bifurcated attractor is responsible for the long time behavior of the system and determines the final patterns. Indeed, if \(E_1\) is corresponding eigenspace in the phase space \(H\), the center manifold theory says that there exists a finite dimensional manifold which is locally represented by the graph of a function \(\Phi : E_1 \rightarrow E_1^+\). This manifold, called the center manifold, is locally invariant and tangential to the eigenspace \(E_1\). Moreover, it attracts all flows in a neighborhood \(U\) of the trivial solution in \(H \setminus \Gamma\), where \(\Gamma\) is the stable manifold.
of the trivial solution. The bifurcated attractor is contained in the center manifold so that the long time dynamics of solutions in $U$ is completely determined by the reduced equation of (1.2) on the center manifold. Thus, it is important to verify the structure of the bifurcated attractor, which leads us to the problem of finding the center manifold. Generally, it has been known that it is not easy to obtain the center manifold function since it is defined implicitly. Recently, Ma and Wang derive a useful formula for the approximation of the center manifold functions. This formula provides us the leading terms in certain order and helps us obtain the projected equation of (1.2) onto the center manifold. It will be one of main tools in proving the main theorem. See Theorem 6.1 of [16] for the precise statement.

There have been much efforts on the bifurcation analysis in the above framework as a way of understanding pattern formations for the equation (1.1). See [13, 14, 15, 19, 23, 25] for the SHE, [6, 11] for the GSHE, and [4, 5] for the DKSE. The final patterns for these equations show different behaviors which will be mentioned in detail in Section 2. In this paper, as a variation of the Swift-Hohenberg equation, we are interested in a one dimensional modified Swift-Hohenberg equation (MSHE)

$$u_t = \alpha u - (1 + \partial_{xx})^2 u + \mu u^2 - u^3.$$  

(2.1)

Here, $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, $\alpha \in \mathbb{R}$ is a control parameter related to the driving force of the system and $\mu \in \mathbb{R}$ ([8]). If $\mu = 0$, then (2.1) corresponds to the usual Swift-Hohenberg equation. We consider the MSHE (2.1) under the periodic boundary condition on $\Omega = [-\lambda, \lambda]$, i.e. $u(x, t) = u(x + 2\lambda, t)$ for all $x \in \Omega$ and some $\lambda > 0$. On the other hand, it is easy to see that the MSHE (2.1) is invariant under the evenly periodic condition. In this paper, we deal with the MSHE under the evenly

2. Statement of main Theorem. Let us rewrite the one dimensional MSHE as

$$u_t = \alpha u - (1 + \partial_{xx})^2 u + \mu u^2 - u^3.$$  

(2.1)

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periodic condition. For the functional setting, let

\[ H = \{ u \in L^2(\Omega; \mathbb{R}) : u(-x) = u(x), \ x \in [0, \lambda] \}, \]

\[ H^4_{per}(\Omega; \mathbb{R}) = \{ u \in H^4(\Omega; \mathbb{R}) : \frac{\partial^j}{\partial x^j} u(-\lambda) = \frac{\partial^j}{\partial x^j} u(\lambda) \text{ for } j = 0, 1, 2, 3 \}, \]

\[ H_1 = H^4_{per}(\Omega; \mathbb{R}) \cap H. \]

We formulate (2.1) in an abstract equation

\[
\begin{cases}
\frac{du}{dt} = L_\alpha u + G(u, \mu), \\
u(0) = u_0,
\end{cases}
\]

(2.2)

by setting \( L_\alpha u = -Au + B_\alpha u, \) and

\[ A = \left( \frac{\partial^2}{\partial x^2} + I \right)^2 : H_1 \to H, \]

\[ B_\alpha = \alpha I : H_1 \to H. \]

We also define the nonlinear operator \( G(u, \mu) = G_2(u, u, \mu) + G_3(u, u, u), \) where

\[ G_2(u, v, \mu) = \mu uv, \quad G_3(u, v, w) = -uvw. \]

In what follows, when \( u = v \) or \( u = v = w \), we simply write \( G_2(u, \mu) = G_2(u, u, \mu) \), and \( G_3(u) = -G_3(u, u, u) \).

It is easy to check that \( A, B_\alpha, G : H_1 \to H \) are well defined. The global well-posedness was established in [20]. Moreover, it was proved in [20, 21] that a global attractor exists in the class \( H^k_{per} \) for any \( k \geq 2 \). In particular, a bifurcation analysis with respect to \( \alpha \) was given in [24] for two dimensional MSHE, where the authors characterized the bifurcation by using Lyapunov-Schmidt reduction method on \((0, 2\pi) \times (0, 2\pi)\). In this paper, we carry out the bifurcation analysis of one dimensional problem (2.1) in detail by using center manifold reduction on an interval \((-\lambda, \lambda)\). Our method has an advantage by providing the stability of bifurcated singular points. Also, it is reported from various results (for example, [5, 18, 19]) that the period \( 2\lambda \) of the domain as well as the control parameter \( \alpha \) is another factor causing bifurcation in the equation (1.1). In the following, we will see how the period \( 2\lambda \) is responsible for the bifurcation of the one dimensional MSHE in detail.

Let us investigate the eigenvalues of the operator \( L_\alpha \) on \( H \). By a simple computation, one can find that \( L_\alpha \) has an eigenvalue sequence

\[ \beta_n(\alpha) = \alpha - \alpha_n, \quad \alpha_n = \left[ 1 - \left( \frac{n\pi}{\lambda} \right)^2 \right]^2, \quad n = 0, 1, 2, \ldots \]

(2.3)

with the corresponding eigenvectors

\[ \phi_0(x) = \frac{1}{\sqrt{2}}, \quad \phi_n(x) = \cos \frac{n\pi x}{\lambda} \]

for \( n \geq 1 \). We also define \( \psi_n(x) = \sin (n\pi x/\lambda) \) for \( n \geq 1 \). We note that the eigenvectors are orthogonal to each other and \( \| \phi_n \|_H^2 = \lambda \) for all \( n \geq 0 \). Since \( \alpha_n \) is a quadratic function of \( (n\pi/\lambda)^2 \), there exists \( N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) such that either

\[ \alpha_n > \alpha_N \quad \forall \ n \neq N, \]

(2.4)

or

\[ \alpha_n > \alpha_N = \alpha_{N+1} \quad \forall \ n \neq N, \ N + 1. \]

(2.5)
In both cases, we denote 
$$\hat{\alpha} = \alpha_N = \inf\{\alpha_n | n \in \mathbb{N}_0\}.$$ 

We note that (2.5) occurs when $$\lambda = \lambda(N)$$ for some $$N \in \mathbb{N}_0$$, where

$$\lambda(N) = \pi \sqrt{\frac{2N^2 + 2N + 1}{2}}. \quad (2.6)$$

The main result of this paper is to verify the dynamic bifurcation of the MSHE defined in $$H$$. For the case (2.4), the MSHE has one dimensional center manifold when $$\alpha$$ is slightly bigger than $$\alpha_N$$. In this case, the bifurcation phenomena of the MSHE for the case (2.4) was well established in [3]. It turns out that the MSHE has a pitchfork bifurcation. Moreover, the bifurcation also depends on the period $$2\lambda$$ but there is no dependence on $$\mu$$.

In this paper, we deal with the second case (2.5). Our result has a big difference with the case (2.4) in that (i) the center manifold has two dimension, (ii) the MSHE bifurcates to an $$S^1$$-attractor, and (iii) the structure of the bifurcated attractor depends on the number $$N$$ and $$\mu$$. We state the main theorems of this paper as follows.

**Theorem 2.1.** Suppose that (2.5) holds true for some $$N \in \mathbb{N}_0$$. Then, as $$\alpha$$ passes $$\hat{\alpha}$$, the MSHE (2.1) defined in $$H$$ bifurcates from the trivial solution to an attractor $$A_{\alpha,N}$$ which is homeomorphic to $$S^1$$. The bifurcated attractor $$A_{\alpha,N}$$ consists singular points and their connecting orbits. The singular points and their stabilities are described in the following.

(i) If $$N = 0$$ and $$\mu > 0$$, there are four singular points:

$$u_0^+ = \pm a_0 \phi_0 + o(\beta_0^{1/2}),$$
$$u_1^+ = a_1 \phi_0 \pm a_2 \phi_1 + o(\beta_0^{3/4}),$$

where $$\beta_0 = \alpha - \alpha_0 = \alpha - 1$$ and

$$a_0 = \sqrt{2} \beta_0^{1/2}, \quad a_1 = \sqrt{\frac{2}{3}} \beta_0^{1/2}, \quad a_2 = \sqrt{\frac{2}{|\mu|}} \left( \frac{\beta_0}{3} \right)^{3/4}.$$ 

The singular points $$u_0^+$$ are asymptotically stable, while $$u_1^+$$ are saddle points.

If $$N = 0$$ and $$\mu < 0$$, there are four singular points $$u_0^+$$ and

$$u_1^+ = a_1 \phi_0 \pm a_2 \phi_1 + o(\beta_0^{3/4}).$$

In this case, $$u_0^+$$ are asymptotically stable and $$u_1^+$$ are saddle points.

(ii) If $$N = 1$$, there are two singular points

$$u^\pm = \pm b_0 \phi_2 + o(\beta_1^{1/2}),$$

where $$\beta_1 = \alpha - \alpha_1 = \alpha - 9/25$$ and

$$b_0 = \sqrt{\frac{180\beta_1}{135 + 16\mu^2}}.$$ 

If $$\mu > 0$$, $$u^-$$ is asymptotically stable and $$u^+$$ is saddle. If $$\mu < 0$$, $$u^-$$ is saddle and $$u^+$$ is a asymptotically stable.
(iii) Let \( 2 \leq N \leq 7 \). Then, there exists a number \( \mu_N > 1 \) satisfying the following. If \( |\mu| < \mu_N \) and \( \mu \neq 0 \), then there are eight singular points

\[
\begin{align*}
    u_{N1}^\pm &= \pm c_1 \phi_N + o(\beta_N^{1/2}), \\
    u_{N2}^\pm &= \pm c_2 \phi_{N+1} + o(\beta_N^{1/2}), \\
    u_{N3}^\pm &= c_3 \phi_N \pm c_4 \phi_{N+1} + o(\beta_N^{1/2}), \\
    u_{N4}^\pm &= -c_3 \phi_N \pm c_4 \phi_{N+1} + o(\beta_N^{1/2}),
\end{align*}
\]  

(2.7)

where \( c_i \) is defined by \( [\text{3.28}] \) for \( i = 1, 2, 3, 4 \). Moreover, \( u_{N1}^\pm \) and \( u_{N2}^\pm \) are asymptotically stable, while \( u_{N3}^\pm \) and \( u_{N4}^\pm \) are saddle points.

If \( |\mu| > \mu_N \), then there are four singular points \( u_{N1}^\pm \) and \( u_{N2}^\pm \). The points \( u_{N1}^\pm \) are asymptotically stable and \( u_{N2}^\pm \) are saddle points. The value \( \mu_N \) is explicitly given in \( [\text{3.32}] \) for each \( 2 \leq N \leq 7 \).

(iv) If \( N \geq 8 \) and \( \mu \neq 0 \), then there are eight singular points in (2.7). The points \( u_{N1}^\pm \) and \( u_{N2}^\pm \) are asymptotically stable. The points \( u_{N3}^\pm \) and \( u_{N4}^\pm \) are saddle.

We will prove Theorem 2.1 in the next section. We give some remarks on our result. Theorem 2.1 shows the final patterns of the solutions of the MSHE if \( \alpha \) is slightly bigger than \( \hat{\alpha} = \alpha_N = \alpha_{N+1} \). Although the bifurcated attractor is homeomorphic to \( S^1 \) (which we call \( S^1 \)-attractor) with the scale \( O(\sqrt{\alpha - \hat{\alpha}}) \) for each \( N \) and \( \mu \neq 0 \), its structure relies on both \( \alpha \) and \( \mu \). If \( N \geq 8 \), the structure is the same for any \( \mu \neq 0 \); there are eight singular points. The structure of the stable singular points is perturbations of each eigenvector \( \phi_N \) and \( \phi_{N+1} \). Whereas the saddle points are perturbations of superpositions of \( \phi_N \) and \( \phi_{N+1} \). If \( 2 \leq N \leq 7 \), there is no dependence on the sign of \( \mu \) but the size of it. For small \( |\mu| \), we have the same structure as the case \( N \geq 8 \). For large \( |\mu| \), we have only four singular points such that two perturbations of \( \phi_N \) are stable and two perturbations of \( \phi_{N+1} \) are saddle. The cases \( N = 0, 1 \) are dependent on the signs of \( \mu \). Furthermore, in the case \( N = 0 \), although the bifurcated attractor is homeomorphic to \( S^1 \), it looks like the perturbation of \( \phi_0 \). In fact, the coefficients \( a_0 \) and \( a_1 \) have a scale \( (\alpha - \alpha_0)^{1/2} \) but \( a_2 \) has a scale \( (\alpha - \alpha_0)^{3/4} \) so that \( a_2 \ll a_0, a_1 \) for all \( \alpha \) close enough to \( \alpha_0 \).

It is quite interesting to compare our result with the dynamical bifurcation of other types of [11]. Although the linear stability analysis is the same, the final patterns can be different due to the nonlinear effect \( f(u, u_x, \mu) \). We compare our result with the dynamical bifurcation phenomena for the SHE \( f = -u^3 \), the GSHE \( f = \mu u^2 - u^3 \), and the DKSE \( f = -u u_x \) under the condition (2.5) as the control parameter \( \alpha \to \hat{\alpha} \).

First, it is known from [11] that the SHE bifurcates from the trivial solution to an \( S^1 \)-attractor \( A_{\alpha, N}^{\text{SHE}} \) in the space of odd periodic functions. The attractor \( A_{\alpha, N}^{\text{SHE}} \) has the same structure for all \( N \geq 1 \). Indeed, it consists of eight static solutions and their connecting orbits for each \( N \geq 1 \). Four of the static solutions are stable points coming from single eigenvectors \( \phi_N \) and \( \phi_{N+1} \). The others are saddle points which are superpositions of \( \phi_N \) and \( \phi_{N+1} \). In the sequel, \( A_{\alpha, N}^{\text{SHE}} \) has the structure exhibited in Theorem 2.1 (iv) for all \( N \geq 1 \).

Second, for the GSHE, since there is no invariance of odd or even periodic conditions, we need to treat it under the full periodic condition. It was shown in [3] [11] that the GSHE bifurcates to an \( S^3 \)-attractor \( A_{\alpha, N}^{\text{GSH}} \) for the condition (2.5) for each \( N \geq 1 \). The attractor \( A_{\alpha, N}^{\text{GSH}} \) contains two invariant circles of static solutions and
two dimensional torus consisting of static solutions. Moreover, the direction of the bifurcation depends on the size of $|\mu|$ but not on the sign of $\mu$. Indeed, there is a number $H(N, \lambda, \mu)$ such that if $H(N, \lambda, \mu) > 0$, then the bifurcation is subcritical, i.e., the GSHE bifurcates as $\alpha$ passes through $\hat{\alpha}$ to the right. If $H(N, \lambda, \mu) < 0$, then the bifurcation is supercritical, i.e., the GSHE bifurcates as $\alpha$ passes through $\hat{\alpha}$ to the left. It turns out that the $\mu$-factor in the expression of $H(N, \lambda, \mu)$ contains only $\mu^2$. Consequently, the sign of $\mu$ is not important in the direction of bifurcation but the size of $\mu$ plays a crucial role. In particular, if $|\mu|$ is small enough, then we have a supercritical bifurcation.

Third, regarding the DKSE, it was proved in [4] that the DKSE bifurcates from the trivial solution to an $S^1$-attractor $A_{DKSE}^{\alpha,N}$ in the space of odd periodic functions. However, the structure of the bifurcated attractor depends on the number $N \geq 1$. If $N = 1$, there are two singular points such that one is a stable node and the other is a saddle point. If $2 \leq N \leq 7$, there are four singular points such that two of them are stable nodes and the others are saddle points. Stable nodes come from the eigenvector $\phi_N$ and saddle points come from $\phi_{N+1}$. If $N \geq 8$, there are eight singular points such that $A_{DKSE}^{\alpha,N}$ has the same structure as $A_{SHE}^{\alpha,N}$.

Finally, we observe that the GSHE, the DKSE, and the MSHE have quadratic nonlinear terms coupled with a parameter $\mu$. These terms give different structures of the bifurcated attractors when the critical frequency $N$ is low, precisely $N \leq 7$. However, if $N \geq 8$, then the bifurcated attractors have the same structure as in the case of the SHE: it consists of the eight singular points and their connecting orbits. There are two single modes $\phi_N$ and $\phi_{N+1}$ which provide the four stable points. The superposition of $\phi_N$ and $\phi_{N+1}$ become saddle points. For low numbers $N \leq 7$, the structures are different from each other. For example, in the MSHE, if $2 \leq N \leq 7$, then single modes are stable and the superpositions are saddle (Theorem 2.1 (iii)). However, in the DKSE, if $2 \leq N \leq 7$, then there are only single modes such that two of them are stable nodes and the others are saddle points.

3. Proof of Theorem 2.1 This section is devoted to the proof of Theorem 2.1. We will apply the attractor bifurcation Theorem of [16, 17]. The main point is the center manifold analysis. We will use some formula to calculate the center manifold function derived in [16]. We assume that $\alpha$ is slightly bigger than $\hat{\alpha} = \alpha_N = \alpha_{N+1}$. We set

$$\beta(\alpha) := \beta_N(\alpha) = \beta_{N+1}(\alpha) = \alpha - \hat{\alpha}. \quad (3.1)$$

We note that

$$\beta(\alpha) = \begin{cases} < 0, & \text{if } \alpha < \hat{\alpha} \\
0, & \text{if } \alpha = \hat{\alpha} \\
> 0, & \text{if } \alpha > \hat{\alpha} \end{cases} \quad (3.2)$$

and

$$\operatorname{Re}\beta_n(\hat{\alpha}) < 0, \quad \forall n \neq N, N + 1. \quad (3.3)$$

Lemma 3.1. For $\alpha \leq \hat{\alpha}$, the trivial solution $u = 0$ is locally asymptotically stable in $H$.

For the proof this lemma, we need the center manifold analysis for the case $\alpha = \hat{\alpha}$ given in this section. So, we postpone the proof of this lemma in Section 4. Lemma 3.1 together with (3.1) satisfies the assumptions of the attractor bifurcation Theorem 6.1 of [16]. Hence, (2.1) bifurcates from the trivial solution to an attractor $A_{\alpha,N}$ as $\alpha$ passes through $\hat{\alpha}$ to the right. Furthermore, by the Poincare-Bendixon
Theorem, the bifurcated attractor $A_{\alpha, N}$ is homeomorphic to $S^1$ and consists of singular points and their connecting orbits. In the following, we study the stability of singular points, which gives us an information of the final patterns of $A_{\alpha, N}$. This depends on the behavior of solutions on the center manifold since the center manifold at the trivial solution is locally attractive.

Let $E_1 = \text{span} \{ \phi_N, \phi_{N+1} \}$ and $E_2 = E_1^\perp$ in $H$. For $j = 1, 2$, let $P_j : H \to E_j$ be the canonical projections and $\mathcal{L}^2_j = L_{\alpha} |_{E_j}$. If $\Phi(\cdot, \alpha, \mu) : E_1 \to E_2$ is a center manifold function and $v = P_1 u = y_N \phi_N + y_{N+1} \phi_{N+1}$, then the reduced equation of (2.1) on the center manifold is

$$
\frac{dv}{dt} = \mathcal{L}^2_1 v + P_1 G(v + \Phi(v, \alpha, \mu), \mu).
$$

(3.4)

By taking the inner product of (3.4) with $\phi_N$ and $\phi_{N+1}$, we obtain

$$
\begin{cases}
\frac{dy_N}{dt} = \beta y_N + F_1(y_N, y_{N+1}), \\
\frac{dy_{N+1}}{dt} = \beta y_{N+1} + F_2(y_N, y_{N+1})
\end{cases}
$$

(3.5)

where

$$
F_1(y_N, y_{N+1}) = \frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \mu), \phi_N \rangle + \frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_N \rangle,
$$

$$
F_2(y_N, y_{N+1}) = \frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \mu), \phi_{N+1} \rangle + \frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu), \phi_{N+1} \rangle.
$$

In the following, we compute $F_1$ and $F_2$. We deduce from Theorem 3.8 of [16] that the center manifold function $\Phi$ can be expressed as

$$
\Phi(v, \alpha, \mu) = (-\mathcal{L}^2_1)^{-1} P_2 G_2(v, \mu) + O(\beta(\alpha) \cdot (y_N^2 + y_{N+1}^2)) + o(y_N^2 + y_{N+1}^2)
$$

(3.6)

as $\alpha \searrow \hat{\alpha}$, where the last equality comes from (3.1). The computation is split into three cases according to the value $N$.

Before proceeding further, we derive some useful identities for eigenvectors which will be used in the following calculations. We use elementary formula of trigonometric functions such as the half-angle formula. Here is the list:

(i) $(\phi_n)_x = -\frac{n\pi}{\lambda} \psi_n$, $(\psi_n)_x = \frac{n\pi}{\lambda} \phi_n$ for $n \geq 1$,

(ii) $\phi_n^2 = \frac{\sqrt{2} \phi_0 + \phi_2 n}{2}$, $\psi_n^2 = \frac{\sqrt{2} \phi_0 - \phi_2 n}{2}$ for $n \geq 1$,

(iii) $\phi_n \psi_m = \frac{\phi_{n-m} + \phi_{n+m}}{2}$, $\psi_n \psi_m = \frac{\phi_{n-m} - \phi_{n+m}}{2}$ for $n > m \geq 1$.

In particular, we obtain that for $n \geq m \geq 1$ and $k \geq 1$,

$$
\langle \psi_n \psi_m, \phi_k \rangle = \begin{cases}
\frac{\lambda}{2} & \text{for } n - m = k, \\
-\frac{\lambda}{2} & \text{for } n + m = k.
\end{cases}
$$

(3.7)
We also note that
\[
\begin{align*}
\int_{-\lambda}^{\lambda} \phi_N^4 \, dx &= \int_{-\lambda}^{\lambda} \phi_{N+1}^4 \, dx = \frac{3}{4} \lambda, \\
\int_{-\lambda}^{\lambda} \phi_N^2 \phi_{N+1}^2 \, dx &= \frac{1}{2} \lambda, \\
\int_{-\lambda}^{\lambda} \phi_N^3 \phi_{N+1}^3 \, dx &= \int_{-\lambda}^{\lambda} \phi_N \phi_{N+1}^3 \, dx = 0.
\end{align*}
\]

(3.8)

3.1. The Case \( N = 0 \). In this case, \( \alpha_0 = \alpha_1 = 1 \) such that \( v = y_0 \phi_0 + y_1 \phi_1 \). Moreover, it comes from (2.6) and (3.1) that
\[
\lambda = \frac{\pi}{\sqrt{2}}, \quad \beta(\alpha) = \alpha - 1.
\]

(3.9)

By direct computation, we obtain
\[
G_2(v, \mu) = \mu \left[ (y_0 \phi_0 + y_1 \phi_1) \right]^2 = \frac{\mu \pi^2}{\lambda^2} y_1^2 \psi_1^2 = \sqrt{2} \mu y_1^2 \phi_0 - \mu y_1^2 \phi_2.
\]

To use (3.6), let
\[
(-L_2^0)^{-1} P_2 G_2(v, \mu) = \sum_{n=2}^{\infty} r_n \phi_n.
\]

Then,
\[
-\mu y_1^2 \phi_2 = P_2 G_2(v, \mu) = -L_2^0 \left( \sum_{n=2}^{\infty} r_n \phi_n \right) = -\sum_{n=2}^{\infty} r_n \beta_n \phi_n
\]

Hence, \( r_2 = \mu y_1^2 / \beta_2 \) and \( r_n = 0 \) for all \( n \geq 3 \). This tells us from (3.6) that
\[
\Phi(v, \alpha, \mu) = \mu y_1^2 \phi_2 / \beta_2 + o(y_0^2 + y_1^2).
\]

Using this expression for the center manifold function, we can compute \( F_1 \) and \( F_2 \).

First, we note that
\[
G_2(v + \Phi(v, \alpha, \mu), \mu) = \mu \left\{ -y_1 \pi \psi_1 - \frac{\mu y_1^2}{\beta_2} 2\pi \psi_2 + o(|y_0|^3 + |y_1|^3) \right\}^2 = \frac{\mu \pi^2}{\lambda^2} \left( y_1^2 \psi_1^2 + \frac{4 \mu y_1^2}{\beta_2} \psi_1 \psi_2 \right) + o(|y_0|^3 + |y_1|^3))
\]

(3.10)

\[
= \sqrt{2} \mu y_1^2 \phi_0 + \frac{4 \mu^2 y_1^3}{\beta_2} \phi_1 - \mu y_1^2 \phi_2 - \frac{4 \mu^2 y_1^3}{\beta_2} \phi_3 + o(|y_0|^3 + |y_1|^3))
\]

As a consequence, we are led to
\[
\begin{align*}
\frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \mu), \phi_0 \rangle &= \sqrt{2} \mu y_1^2 + o(|y_0|^3 + |y_1|^3)), \\
\frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \mu), \phi_1 \rangle &= \frac{4 \mu^2 y_1^3}{\beta_2} + o(|y_0|^3 + |y_1|^3)).
\end{align*}
\]

On the other hand, since \( \Phi(v, \alpha, \mu) = O(y_0^2 + y_1^2) \), we obtain
\[
G_3(v + \Phi(v, \alpha, \mu)) = - \left\{ y_0 \phi_0 + y_1 \phi_1 + O(y_0^2 + y_1^2) \right\}^3
\]

(3.11)

\[
= - y_0^3 \phi_0^3 - 3 y_0^2 y_1 \phi_0^2 \phi_1 + 3 y_0 y_1^2 \phi_0 \phi_1^2 - y_1^3 \phi_1^3 + o(|y_0|^3 + |y_1|^3)
\]

\[
= - \left( \frac{y_0^3}{2} + \frac{3 y_0 y_1^2}{2} \right) \phi_0 - \left( \frac{3 y_0^2 y_1}{2} + \frac{3}{4} y_1^3 \right) \phi_1 - \frac{3 y_0 y_1^2}{2 \sqrt{2}} \phi_2 - \frac{y_1^3}{4} \phi_3 + o(|y_0|^3 + |y_1|^3)).
\]
Thus, we have
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_0 \rangle = -\frac{1}{2}(y_0^3 + 3y_0y_1^2) + o(|y_0|^3 + |y_1|^3),
\]
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_1 \rangle = -\frac{3}{4}(2y_0^3y_1 + y_1^3) + o(|y_0|^3 + |y_1|^3).
\]

As a consequence, we obtain
\[
\begin{align*}
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_0 \rangle &= -\frac{1}{2}(y_0^3 + 3y_0y_1^2) + o(|y_0|^3 + |y_1|^3), \\
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_1 \rangle &= -\frac{3}{4}(2y_0^3y_1 + y_1^3) + o(|y_0|^3 + |y_1|^3).
\end{align*}
\]

Let us find nontrivial singular points of the truncated system of (3.5):
\[
\begin{align*}
h_1(y_0, y_1) &= \beta y_0 + \sqrt{2}\mu y_1^2 - \frac{1}{2}(y_0^3 + 3y_0y_1^2) = 0, \\
h_2(y_0, y_1) &= \beta y_1 + \frac{4\mu^2}{\beta_2} y_1^3 - \frac{3}{4}(2y_0^2y_1 + y_1^3) = 0.
\end{align*}
\]

Figure 1 shows the graphs of (3.11) where dotted curves are \( h_1(y_0, y_1) = 0 \) and the others are \( h_2(y_0, y_1) = 0 \).

\[
\begin{align*}
F_1(y_0, y_1) &= \sqrt{2}\mu y_1^2 - \frac{1}{2}(y_0^3 + 3y_0y_1^2) + o(|y_0|^3 + |y_1|^3), \\
F_2(y_0, y_1) &= \frac{4\mu^2}{\beta_2} y_1^3 - \frac{3}{4}(2y_0^2y_1 + y_1^3) + o(|y_0|^3 + |y_1|^3).
\end{align*}
\]

(3.10)

(3.11)

Figure 1. Truncated system for \( N = 0 \) with \( \beta = 0.0001 \).

It is easy to check that \( (y_0, y_1) = (\pm a_0, 0) \) are solutions of (3.11), where \( a_0 > 0 \) and \( a_0^2 = 2\beta \). Suppose that \( y_1 \neq 0 \). Then, by the second equation of (3.11)

\[
\frac{3}{2} y_0^2 + \left( \frac{3}{4} - \frac{4\mu^2}{\beta_2} \right) y_1^2 = \beta.
\]

(3.12)

Since \( \beta_2 < 0 \) by (3.3), we see that \( y_0^2 + y_1^2 = O(\beta) \) with \( 0 < \beta = \alpha - 1 \ll 1 \). So, we can write the first equation of (3.11) as

\[
\sqrt{2}\mu y_1^2 = O(\beta^{3/2}), \quad \text{namely,} \quad y_1^2 = O(\beta^{3/2}).
\]

(3.13)

Combining (3.12) and (3.13), we deduce

\[
y_0^2 = \frac{2}{3} \beta + O(\beta^{3/2}).
\]

(3.14)

Multiplying the first equation of (3.11) by \( y_0 \), we get

\[
\frac{4}{9} \beta^2 + \sqrt{2}\mu y_0 y_1^2 + O(\beta^{5/2}) = 0.
\]

(3.15)
Suppose that \( \mu > 0 \). Since \( y_0y_1^2 = O(\beta^2) \), we conclude that \( y_0 < 0 \), i.e., \( y_0 = -a_1 \) where by (3.14)
\[
a_1 = \sqrt{\frac{2\beta}{3} + O(\beta^{3/4})}. \tag{3.16}
\]
Similarly, if \( \mu < 0 \), then \( y_0 = a_1 < 0 \). In both cases, by plugging \( y_0 = \pm a_1 \) in (3.15), we obtain
\[
y_1^2 = -\frac{4\beta^2}{9\sqrt{2}\mu y_0} + O(\beta^2) = \frac{2}{|\mu|} \left( \frac{\beta}{3} \right)^{3/2} + O(\beta^2).
\]
Hence, \( y_1 = \pm a_2 \) where
\[
a_2 = \sqrt{\frac{2}{|\mu|} \cdot \left( \frac{\beta}{3} \right)^{3/4} \cdot \sqrt{1 + O(\beta^{1/2})}} = \sqrt{\frac{2}{|\mu|} \cdot \left( \frac{\beta}{3} \right)^{3/4} + O(\beta^{3/4})} \tag{3.17}
\]
Now, we study the stability of singular points. Let \( h = (h_1, h_2) \) such that
\[
Dh(y_0, y_1) = \begin{pmatrix} \beta - \frac{3}{2} y_0^2 - \frac{3}{2} y_1^2 & 2\sqrt{2}\mu y_1 - 3y_0 y_1 \\ -3y_0 y_1 & \beta + \frac{12\mu^2 y_1^2}{\beta_2} - \frac{3}{2} y_1^2 - \frac{9}{4} y_1^2 \end{pmatrix},
\]
Since \( Dh(\pm a_0, 0) \) has negative eigenvalues \(-2\beta\), \((\pm a_0, 0)\) are nondegenerate asymptotically stable point of (3.11). Next, suppose \( \mu > 0 \). We have
\[
Dh(-a_1, \pm a_2) = \begin{pmatrix} \zeta_{11}^\pm & \zeta_{12}^\pm \\ \zeta_{21}^\pm & \zeta_{22}^\pm \end{pmatrix},
\]
where
\[
\zeta_{11}^+ = \zeta_{11}^- = \beta - \frac{3}{2} a_1^2 - \frac{3}{2} a_2^2 = O(\beta^{3/2}),
\]
\[
\zeta_{12}^+ = -\zeta_{12}^- = 2\sqrt{2}|\mu| a_2 + 3a_1 a_2,
\]
\[
\zeta_{21}^+ = -\zeta_{21}^- = 3a_1 a_2,
\]
\[
\zeta_{22}^+ = \zeta_{22}^- = \beta + \frac{12\mu^2 a_2^2}{\beta_2} - \frac{3}{2} a_1^2 - \frac{9}{4} a_2^2 = O(\beta^{3/2}).
\]
We note that
\[
\zeta_{12} \zeta_{21} = 6\sqrt{2}|\mu| a_1 a_2 + 9a_1^2 a_2^2 = \frac{8}{3} \beta^2 + O(\beta^{5/2}).
\]
Thus, if \( 0 < \beta = \alpha - \hat{\alpha} \ll 1 \),
\[
\zeta_{11} \zeta_{22} - \zeta_{12} \zeta_{21} = -\frac{8}{3} \beta^2 + O(\beta^{5/2}) < 0,
\]
which implies that \( Dh(-a_1, \pm a_2) \) have a positive eigenvalue and a negative eigenvalue. So, \((-a_1, \pm a_2)\) are saddle points. By a similar argument we conclude that \((a_1, \pm a_2)\) are saddle points for \( \mu < 0 \). See Figure 2.
By direct calculation, the Case: 

As a consequence, by (3.7) we are led to 

Hence, by (3.6) the center manifold function is can be written as 

By direct calculation, 

Hence, by [3.6] the center manifold function is can be written as 

So, we obtain that 

As a consequence, by (3.7) we are led to 

Meanwhile, since \( \Phi(v, \alpha, \mu) = O(y_0^2 + y_1^2) \), we obtain 

\[ G_3(v + \Phi(v, \alpha, \mu)) = -y_1^3 \phi_1^3 - 3y_1^2 y_2 \phi_1^2 \phi_2 - 3y_1 y_2^2 \phi_1 \phi_2^2 - y_2^3 \phi_2^3 + o(|y_1|^3 + |y_2|^3), \]
which yields from (3.8) that
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_1 \rangle = -\frac{3y_1^3 + 6y_1y_2^2}{4} + o(|y_1|^3 + |y_2|^3),
\]
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_2 \rangle = -\frac{6y_1^2y_2 + 3y_2^3}{4} + o(|y_1|^3 + |y_2|^3).
\]

In the sequel,
\[
\begin{align*}
F_1(y_1, y_2) &= \frac{4\mu y_1 y_2}{5} + \frac{48\mu^2 y_1 y_2^2}{25\beta_3} - \frac{3y_1^3 + 6y_1y_2^2}{4} + o(|y_1|^3 + |y_2|^3), \\
F_2(y_1, y_2) &= -\frac{\mu y_2^2}{5} + \frac{24\mu^2 y_1^2 y_2}{25\beta_3} + \frac{64\mu^2 y_3^3}{25\beta_4} - \frac{6y_1^2y_2 + 3y_2^3}{4} + o(|y_1|^3 + |y_2|^3).
\end{align*}
\]

As before, we want to find nontrivial singular points of the truncated system of (3.5):
\[
\begin{align*}
\begin{cases}
h_1(y_1, y_2) := \beta y_1 + \frac{4\mu y_1 y_2}{5} + \frac{48\mu^2 y_1 y_2^2}{25\beta_3} - \frac{3y_1^3 + 6y_1y_2^2}{4} = 0, \\
h_2(y_1, y_2) := \beta y_2 - \frac{\mu y_2^2}{5} + \frac{24\mu^2 y_1^2 y_2}{25\beta_3} + \frac{64\mu^2 y_3^3}{25\beta_4} - \frac{6y_1^2y_2 + 3y_2^3}{4} = 0.
\end{cases}
\end{align*}
\]

The Figure 3 shows the graphs of (3.20) where dotted curves are \( h_1(y_1, y_2) = 0 \) and the others are \( h_2(y_1, y_2) = 0 \).

![Figure 3. Truncated system for \( N = 1 \) with \( \beta = 0.0001 \) and \( \mu = 1 \).](image)

First, it is easy to see that \((y_1, y_2) = (0, \pm b_0)\) are solutions of the system (3.20) if \( 0 < \beta \ll 1 \). Here, \( b_0 > 0 \) and
\[
b_0^2 = \beta \left( \frac{3}{4} - \frac{64\mu^2}{25\beta_4} \right)^{-1} = \frac{180\beta}{135 + 16\mu^2} + O(\beta^2),
\]
where we used the following formula
\[
\beta_4 = \beta + (\alpha_1 - \alpha_4) = \beta - \frac{720}{25}.
\]

If we set \( h = (h_1, h_2) \), then
\[
Dh(0, \pm b_0) = \begin{pmatrix} \zeta_1^{\pm} & 0 \\ 0 & \zeta_2 \end{pmatrix}.
\]
where

\[
\zeta_1^\pm = \beta \mp \frac{4\mu}{5}b_0 + \left(\frac{48\mu^2}{25\beta_3} - \frac{3}{2}\right)b_0^2 = \pm \frac{4\mu}{5}b_0 + O(\beta),
\]

\[
\zeta_2 = \beta + \left(\frac{192\mu^2}{25\beta_4} - \frac{9}{4}\right)b_0^2 = -2\beta + O(\beta^2) < 0.
\]

Here, we used

\[
\beta_3 = \beta + (\alpha_1 - \alpha_3) = \beta - \frac{160}{25}.
\]

Consequently, if \(\mu > 0\), then \((0, -b_0)\) is a stable point and \((0, b_0)\) is a saddle point. If \(\mu < 0\), then \((0, -b_0)\) is a saddle point and \((0, b_0)\) is a stable point (See Figure 4).

Next, we show that \((y_1, y_2) = (0, \pm b_0)\) are the only nontrivial solutions of the system (3.20). If \((y_1, y_2)\) is a nontrivial solution of (3.20) with \(y_1 \neq 0\), then \(y_2 \neq 0\) by the second equation of (3.20), where \(\mu \neq 0\). Multiplying the first equation of (3.20) by \(y_1\) and the second equation by \(4y_2\) and adding them, we obtain

\[
\beta y_1^2 + 4\beta y_2^2 = \frac{3}{4}y_1^4 + Py_1^2y_2^2 + Qy_2^4,
\]

(3.21)

where

\[
P = \frac{15}{2} - \frac{144\mu^2}{25\beta_3} > 0, \quad Q = 3 - \frac{256\mu^2}{25\beta_4} > 0.
\]

Hence, \(y_1^2 + y_2^2 = O(\beta)\). Then, by the first equation of (3.20)

\[
y_1^2 = \frac{4}{3}\beta + \frac{16\mu y_2}{15} + \left(\frac{64\mu^2}{25\beta_3} - 2\right)y_2^2.
\]

(3.22)

Inserting this into the second equation of (3.20), we have

\[
\left(\frac{16\mu^2}{75} + O(\beta^{1/2})\right)y_2 = -\frac{4\mu\beta}{15}.
\]

Hence

\[
y_2 = -\frac{5\beta}{4\mu} + O(\beta^{3/2}).
\]

(3.23)

Plugging (3.23) in (3.22), we get \(y_1^2 = O(\beta^{3/2})\). In particular, \(y_2^2/y_1^2 = O(\beta^{1/2})\). Dividing (3.21) by \(\beta y_1^2\), we are led to

\[
1 + O(\beta^{1/2}) = \frac{1}{\beta y_1^2} \left(\frac{3}{4}y_1^4 + Py_1^2y_2^2 + Qy_2^4\right) = O(\beta^{1/2}).
\]

This yields a contradiction since \(\beta \ll 1\).

3.3. The Case: \(N \geq 2\). In this case, by (2.6),

\[
\begin{align*}
\pi^2 \lambda^2 &= \frac{2}{2N^2 + 2N + 1}, \\
\bar{\alpha} &= \alpha_N = \alpha_{N+1} = \left(\frac{2N + 1}{2N^2 + 2N + 1}\right)^2, \\
\beta(\alpha) &= \alpha - \bar{\alpha}.
\end{align*}
\]

(3.24)

Moreover,

\[
\alpha_n = \left(\frac{2N^2 + 2N + 1 - 2n^2}{2N^2 + 2N + 1}\right)^2, \quad n \in \mathbb{N}_0.
\]

(3.25)
Figure 4. Structure of the bifurcated attractor for $N = 1$.

For simplicity, let $z_1 = y_N$ and $z_2 = y_{N+1}$, and set $z = (z_1, z_2)$. Then for $v = z_1 \phi_N + z_2 \phi_{N+1}$, we have

$$G_2(v, \mu) = \frac{\mu \pi^2}{\lambda^2} \left\{ N^2 \frac{\phi_0^2}{\phi_1} + N(N + 1) z_1 z_2 \psi_N \psi + (N + 1)^2 \frac{\phi_0^2}{\phi_1} \right\}$$

$$= \frac{\mu \pi^2}{\lambda^2} \left\{ \frac{\sqrt{2}(N^2 \frac{\phi_0^2}{\phi_1} + (N + 1)^2 \frac{\phi_0^2}{\phi_1})}{2} \phi_0 + N(N + 1) z_1 z_2 \phi_1 \right\}$$

$$- N(N + 1) z_1 z_2 \phi_{N+1} - \frac{(N + 1)^2 \frac{\phi_0^2}{\phi_1}}{2} \phi_{N+2} + o(|z|^2).$$

Hence, we derive from (3.6) that

$$\Phi(v, \alpha, \mu) = - \frac{\mu \pi^2}{\lambda^2} \left\{ \frac{\sqrt{2}(N^2 \frac{\phi_0^2}{\phi_1} + (N + 1)^2 \frac{\phi_0^2}{\phi_1})}{2} \phi_0 + N(N + 1) z_1 z_2 \phi_1 \right\}$$

$$- N(N + 1) z_1 z_2 \phi_{N+1} - \frac{(N + 1)^2 \frac{\phi_0^2}{\phi_1}}{2} \phi_{N+2} + o(|z|^2).$$

So, we obtain that

$$G_2(v + \Phi(v, \alpha, \mu), \mu) = \frac{\mu \pi^2}{\lambda^2} \left\{ \frac{\phi_0^2}{\phi_1} + N(N + 1) z_1 z_2 \psi_N \psi + (N + 1)^2 \frac{\phi_0^2}{\phi_1} \right\}$$

$$- 2 \frac{\mu \pi^2}{\lambda^2} \left\{ \frac{\phi_0^2}{\phi_1} + N(N + 1) z_1 z_2 \psi_N \psi + (N + 1)^2 \frac{\phi_0^2}{\phi_1} \right\}$$

$$- \frac{N(N + 1) \phi_0^2}{\phi_1 \phi_{N+1}} \phi_{N+1} \psi_N - \frac{N(N + 1)^2 \phi_0^2}{\phi_1 \phi_{N+2}} \phi_{N+2} \psi_N$$

$$+ \frac{N(N + 1)^2 z_1 z_2 \psi_N \psi + (N + 1)^2 z_1 z_2 \psi_{N+1}}{\phi_1 \phi_{N+1} \phi_{N+2}} + \phi_{N+2} \psi_N \psi_{N+1}$$

$$- \frac{N(N + 1)^2 z_1 z_2 \psi_N \psi + (N + 1)^2 z_1 z_2 \psi_{N+1}}{\phi_1 \phi_{N+1} \phi_{N+2}} + \phi_{N+2} \psi_N \psi_{N+1}$$

$$+ o(|z|^3).$$
Applying the formula (3.7), we deduce that
\[
\frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \phi_N) \rangle = \frac{\mu^2 \pi^4}{\lambda^4} \left[ \frac{N^4 z_1^3}{\beta_{2N}} + N(N + 1)^2 \left( \frac{2N + 1}{\beta_{2N+1}} - \frac{1}{\beta_1} \right) z_1 z_2 + o(|z|^3) \right],
\]
and
\[
\frac{1}{\lambda} \langle G_2(v + \Phi(v, \alpha, \mu), \phi_{N+1}) \rangle
\]
\[
= \frac{\mu^2 \pi^4}{\lambda^4} \left[ N^2(N + 1) \left( \frac{2N + 1}{\beta_{2N+1}} + \frac{1}{\beta_1} \right) z_2^2 + (N + 1)^4 z_2^3 + o(|z|^3) \right].
\]
On the other hand, we get
\[
G_3(v + \Phi(v, \alpha, \mu)) = -z_1^3 \phi_N - 3z_1^2 z_2 \phi_N \phi_{N+1} - 3z_1 z_2^2 \phi_N \phi_{N+1} - z_3^2 \phi_{N+1} + o(|z|^3).
\]
Using (3.8), we infer that
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_N \rangle = -3 \frac{3}{4} (z_1^3 + 2z_1 z_2^2) + o(|z|^3),
\]
and
\[
\frac{1}{\lambda} \langle G_3(v + \Phi(v, \alpha, \mu)), \phi_{N+1} \rangle = -3 \frac{3}{4} (2z_1 z_2 + z_2^3) + o(|z|^3).
\]
Gathering the above results, we are led to
\[
\begin{cases}
F_1(z) = -d_{11} z_1^3 - d_{12} z_1 z_2^2, \\
F_2(z) = -d_{21} z_1 z_2 - d_{22} z_2^3,
\end{cases}
\]
where
\[
\begin{align*}
\alpha_1 &= \frac{3}{4} - \frac{\mu^2 \pi^4 N^4}{\lambda^4 \beta_{2N}}, \\
\beta_1 &= \alpha_1 + \beta = \frac{4(N - 1)N(N + 1)N(N + 2)}{2N^2 + 2N + 1)^2} + \beta.
\end{align*}
\]
Similarly,
\[
\begin{align*}
\beta_{2N} &= -\frac{12N^2(N - 1)(3N + 1)}{(2N^2 + 2N + 1)^2} + \beta, \\
\beta_{2N+1} &= -\frac{4N(N + 1)(3N + 1)(3N + 2)}{(2N^2 + 2N + 1)^2} + \beta, \\
\beta_{2N+2} &= -\frac{12(N + 1)^2(N + 2)(3N + 2)}{(2N^2 + 2N + 1)^2} + \beta.
\end{align*}
\]
Using the above identities and \((3.24)\), we have a precise expression for \(d_{ij}\):

\[
\begin{align*}
\frac{d_1}{
\frac{d_{11}(\alpha, N)}{\Delta} &= \frac{3}{4} + \frac{\mu^2 N^2}{3(N - 1)(3N + 1)} + O(\beta), \\
\frac{d_2}{\Delta} &= \frac{3}{2} + \frac{\mu^2 (N + 1)^2 (N^2 - 4N - 2)}{(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\beta), \\
\frac{d_3}{\Delta} &= \frac{3}{2} + \frac{\mu^2 N^2 (N^2 + 6N + 3)}{(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\beta), \\
\frac{d_4}{\Delta} &= \frac{3}{4} + \frac{\mu^2 (N + 1)^2}{3(N + 2)(3N + 2)} + O(\beta).
\end{align*}
\]

Let us find nontrivial singular points of the truncated system of \((3.5)\):

\[
\begin{align*}
\frac{h_1(\alpha)}{\Delta} &= \beta z_1 + F_1(\alpha) = 0, \\
\frac{h_2(\alpha)}{\Delta} &= \beta z_2 + F_2(\alpha) = 0.
\end{align*}
\]

We have eight solutions of \((3.27)\): \((\pm c_1, 0)\), \((0, \pm c_2)\), \((c_3, \pm c_4)\), and \((-c_3, \pm c_4)\). Here, \(c_1 > 0\) and

\[
c_1 = \frac{\beta}{d_{11}}, \quad c_2 = \frac{\beta}{d_{22}}, \quad c_3 = \frac{(d_{22} - d_{12})\beta}{d_{11}d_{22} - d_{12}d_{21}}, \quad c_4 = \frac{(d_{11} - d_{21})\beta}{d_{11}d_{22} - d_{12}d_{21}}. \tag{3.28}
\]

Since \(d_{11}, d_{22} > c_1\) and \(c_2\) are well-defined. We need to show that \(c_3\) and \(c_4\) are also well-defined. By direct calculation, we have

\[
\begin{align*}
d_{11} - d_{21} &= \frac{3}{4} + \frac{\mu^2 N^2 (3N^2 + 28N + 14)}{3(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\beta), \\
d_{22} - d_{12} &= \frac{3}{4} + \frac{\mu^2 (N + 1)^2 (3N^2 - 22N - 11)}{3(N - 1)(N + 2)(3N + 1)(3N + 2)} + O(\beta), \\
d_{11}d_{22} - d_{12}d_{21} &= \frac{27}{16} - \frac{\mu^2 (18N^4 + 36N^3 - 74N^2 - 92N - 23)}{4(N - 1)(N + 2)(3N + 1)(3N + 2)} \\
&\quad - \frac{\mu^4 N^2 (N + 1)^2 (27N^4 + 54N^3 - 821N^2 - 848N - 212)}{9(N + 2)^2 (3N + 2)^2 (3N + 1)^2 (N - 1)^2} + O(\beta).
\end{align*}
\]

We note that \(d_{11} - d_{21} < 0\) for all \(N \geq 2\) and \(\mu \neq 0\). This means that \(d_{11}d_{22} - d_{12}d_{21} < 0\) for the existence of \(c_1\), which in turn implies that \(d_{22} - d_{12} < 0\) for the existence of \(c_3\). We note that

\[
\begin{align*}
3N^2 - 22N - 11 > 0, & \quad \forall N \geq 8, \\
d_{11}d_{22} - d_{12}d_{21} < 0, & \quad \forall N \geq 2, \\
d_{11}d_{22} - d_{12}d_{21} < 0, & \quad \forall N \geq 6.
\end{align*}
\]

Hence, for any \(\mu \neq 0\),

\[
\begin{align*}
d_{11} - d_{21} < 0, & \quad \forall N \geq 2, \\
d_{22} - d_{12} < 0, & \quad \forall N \geq 8, \\
d_{11}d_{22} - d_{12}d_{21} < 0 & \quad \forall N \geq 6.
\end{align*}
\]

So, if \(N \geq 8\), \(c_3\) and \(c_4\) exist for any \(\mu \neq 0\). For other values of \(N\), we need careful analysis. In the following, we study the stability of singular points for each \(N \geq 2\) in detail. Before proceeding further, we notice that the singular points \((\pm c_1, 0)\) are
always asymptotically stable. Indeed, if we set \( h = (h_1, h_2) \), then
\[
D h(z) = \begin{pmatrix} \beta - 3d_{11}z_1^2 - d_{12}z_2^2 & -2d_{12}z_1z_2 \\ -2d_{21}z_1z_2 & \beta - d_{21}z_1^2 - 3d_{22}z_2^2 \end{pmatrix}.
\]

It is not difficult to see that \( D h(\pm c_1, 0) \) have two eigenvalues
\[-2\beta, \ (d_{11} - d_{21})\beta/d_{11}\]
which are negative by (3.29). Hence, \((\pm c_1, 0)\) are asymptotically stable. For existence and stability of other singular points depends on \( N \) and \( \mu \). In the following, we study these subjects by dividing the value \( N \) into two cases.

First suppose that \( N \geq 8 \). In this case, we have eight singular points. We note that
\[
D h(0, \pm c_2) \text{ have two eigenvalues } -2\beta \text{ and } (d_{22} - d_{12})\beta/d_{22}, \tag{3.30}
\]
which are both negative by (3.29). So, \((0, \pm c_2)\) are asymptotically stable. Next, we see that
\[
D h(\pm (c_3, \pm c_4)) = \begin{pmatrix} \beta - 3d_{11}c_3^2 - d_{12}c_4^2 & -2d_{12}c_3c_4 \\ -2d_{21}c_3c_4 & \beta - d_{21}c_3^2 - 3d_{22}c_4^2 \end{pmatrix}.
\]
Then,
\[
\det D h(\pm (c_3, \pm c_4)) = \frac{4\beta^2(d_{22} - d_{12})(d_{11} - d_{21})}{d_{11}d_{22} - d_{12}d_{21}} < 0, \tag{3.31}
\]
which implies that \( D h(\pm (c_3, \pm c_4)) \) have a positive eigenvalue and a negative eigenvalue. Therefore, \((c_3, \pm c_4)\) are all saddle points. See Figure 5.

Second, we consider the case \( 2 \leq N \leq 7 \). By direct computation, we deduce that
\[
d_{22} - d_{12} < 0 \iff -\mu_N < \mu < \mu_N, \quad d_{11}d_{22} - d_{12}d_{21} < 0 \iff -\nu_N < \mu < \nu_N,
\]
where
\[
\begin{align*}
\mu_2 &= \frac{2\sqrt{609}}{43} = 1.2 \cdots, & \nu_2 &= \frac{\sqrt{552902 + 37870\sqrt{75001}}}{2164} = 1.5 \cdots, \\
\mu_3 &= \frac{3\sqrt{22}}{8} = 1.7 \cdots, & \nu_3 &= \frac{\sqrt{83798 + 2002\sqrt{10921}}}{208} = 2.6 \cdots, \\
\mu_4 &= \frac{3\sqrt{461}}{85} = 2.4 \cdots, & \nu_4 &= \frac{3\sqrt{19102902 + 32214\sqrt{729249}}}{4720} = 4.3 \cdots, \\
\mu_5 &= \frac{\sqrt{5474}}{23} = 3.2 \cdots, & \nu_5 &= \frac{3\sqrt{11596623 + 1309\sqrt{160409}}}{130} = 13.8 \cdots, \\
\mu_6 &= \frac{6\sqrt{1330}}{49} = 4.4 \cdots, & \nu_6 &= \infty, \\
\mu_7 &= \frac{3\sqrt{1518}}{16} = 7.3 \cdots, & \nu_7 &= \infty.
\end{align*}
\]
(3.32)
Since \( \mu_N < \nu_N \), \( c_3 \) and \( c_4 \) exist for all \( \mu \in (\mu_N, \mu_N) \). Thus, if \( -\mu_N < \mu < \mu_N \ (\mu \neq 0) \), it follows from (3.30) and (3.31) that \((0, \pm c_2)\) are asymptotically stable and \((c_3, \pm c_4)\) are saddle points. Meanwhile, if \( |\mu| > \mu_N \), then \( c_3 \) and \( c_4 \) does not exist and \((0, \pm c_2)\) are saddle points by (3.30). See Figure 5. Now the proof of Theorem 2.1 is complete.
In this section, we prove Lemma 3.1. We recall \( \alpha_N = \alpha_{N+1} \).

**Case 1.** \( \alpha < \hat{\alpha} \)

We choose small \( \varepsilon > 0 \) such that
\[
\begin{align*}
\hat{\alpha} - (1 - 2\varepsilon)\alpha_n &< 0, \quad \forall \ n \neq N, N + 1, \\
\varepsilon \tau &< -\frac{\alpha - \hat{\alpha}}{2},
\end{align*}
\]
where
\[
\tau = \sup_{\eta_n > 0} \eta_n < \infty \quad \text{and} \quad \eta_n = -\left(\frac{n\pi}{\lambda}\right)^4 + 5\left(\frac{n\pi}{\lambda}\right)^2 - 2.
\]
Suppose that \( \|u_0\|^2 < \delta \), where \( \delta \in (0, 1) \) is to be determined below. We expand the solution of (2.2) as
\[
u(x, t) = \sum_{n=0}^{\infty} a_n(t) \phi_n(x).
\]
Since \( u \in C([0, \infty), H) \), there exists \( T > 0 \) such that
\[
\delta > \|u(\cdot, t)\|^2 = \lambda \sum_{n=0}^{\infty} |a_n(t)|^2
\]
for all \( t \in [0, T] \). We note that \( \lambda |a_n(t)|^2 < \delta \) for all \( t \in [0, T] \) and \( n \geq 0 \). Multiplying (2.2) by \( u \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 = \sum_{n, k=0}^{\infty} \langle \beta_n a_n(t) \phi_n, a_k(t) \phi_k \rangle - \frac{\mu}{2} \int_{-\lambda}^{\lambda} u^2 u_{xx} dx - \|u\|^4
\]
\[
\leq \lambda \sum_{n=0}^{\infty} |\beta_n a_n(t)|^2 + \varepsilon \|u_{xx}\|^2 + C_{\varepsilon, \mu} \|u\|^4
\]
\[
= \lambda \sum_{n=0}^{\infty} \left\{ \beta_n + \varepsilon \left(\frac{n\pi}{\lambda}\right)^4 \right\} |a_n(t)|^2 + C_{\varepsilon, \mu} \|u\|^4,
\]
where $C_{\varepsilon,\mu}$ is a positive constant depending only on $\varepsilon$ and $\mu$. By the one dimensional Gagliardo-Nirenberg inequality, we have

$$\|u\|^4 \leq C\|u_x\| \|u\|^3 \leq \frac{\varepsilon}{C_{\varepsilon,\mu}} \|u_x\|^2 + D_{\varepsilon,\mu} \|u\|^6 \leq \frac{\varepsilon}{C_{\varepsilon,\mu}} \|u_x\|^2 + \delta^2 D_{\varepsilon,\mu} \|u_x\|^2,$$

where $D_{\varepsilon,\mu}$ relies only on $\varepsilon$ and $\mu$. In the sequel,

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \leq \lambda \sum_{n=0}^{\infty} \left\{ \beta_n + \varepsilon \left(\frac{n\pi}{\lambda}\right)^4 + \varepsilon \left(\frac{n\pi}{\lambda}\right)^2 + \delta^2 C_{\varepsilon,\mu} D_{\varepsilon,\mu} \right\} |a_n(t)|^2.$$

We notice that by the choice of $\varepsilon$

$$\beta_n + \varepsilon \left(\frac{n\pi}{\lambda}\right)^4 + \varepsilon \left(\frac{n\pi}{\lambda}\right)^2 = \alpha - \alpha_N + (1 - 2\varepsilon) \alpha_n + \varepsilon \eta_n < \frac{\alpha - \hat{\alpha}}{2}.$$

As a consequence,

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 < \left\{ \frac{\alpha - \hat{\alpha}}{2} + \delta^2 C_{\varepsilon,\mu} D_{\varepsilon,\mu} \right\} \|u\|^2.$$

Thus, if we choose $\delta > 0$ so small such that

$$\frac{\alpha - \alpha_N}{2} + \delta^2 C_{\varepsilon,\mu} D_{\varepsilon,\mu} < 0,$$

then $\|u(\cdot,t)\|^2$ is decreasing for all $t \in [0,T]$. Applying continuation argument, we can conclude that the trivial solution is asymptotically stable in the ball $B_\delta(0) \subset H$.

**Case 2.** $\alpha = \hat{\alpha}$

In this case, since $\beta_N(\alpha) = \beta_{N+1}(\alpha) = 0$, we employ the center manifold reduction. Since the center manifold is locally attractive, it suffices to consider the locally asymptotic stability of the trivial solution on the center manifold. We rewrite the truncated form of the reduction of (3.5) as

$$\begin{cases} 
\frac{dy_N}{dt} = \hat{F}_1(y_N, y_{N+1}), \\
\frac{dy_{N+1}}{dt} = \hat{F}_2(y_N, y_{N+1}),
\end{cases}$$

where $\hat{F}_i$ is the truncation of $F_i$. In the following, we show that $(y_N, y_{N+1}) = (0,0)$ is locally asymptotically stable for this truncated system.

First, suppose that $N = 0$. By (3.10), we have

$$\begin{cases}
\frac{dy_0}{dt} = \sqrt{2\mu} y_1^2 - \frac{1}{2} (y_0^3 + 3y_0 y_1^2), \\
\frac{dy_1}{dt} = \frac{4\mu y_1^3}{\beta_2} - \frac{3}{4} (2y_0 y_1 + y_1^3).
\end{cases}$$

Since $\beta_2 < 0$, if $y_0^2 + y_1^2 \ll 1$, then

$$\frac{d}{dt} \left( \frac{1}{4} y_0^2 + \frac{1}{2} y_1^2 \right) = \frac{\sqrt{2}}{2} \mu y_0^2 y_1^2 - \frac{1}{2} (y_0^6 + 3y_0^4 y_1^2) + \frac{4\mu^2 y_1^4}{\beta_2} - \frac{3}{4} (2y_0 y_1^2 + y_1^4) < y_0^2 y_1^2 \left( \frac{\sqrt{2}}{2} y_0 - \frac{3}{2} \right) < 0.$$

Hence, $u \equiv 0$ is locally asymptotically stable.
Next, let $N = 1$. We note that $\beta_3, \beta_4 < 0$. Hence, it follows from (3.19) that if $y_0^2 + y_1^2 \ll 1$, then
\[
\frac{d}{dt} \left( \frac{1}{2} y_2^2 + 2y_3^2 \right) = 48\mu^2 y_1^2 y_2^2 \frac{y_1^4 + 6y_1^2 y_2^2}{4} + 24 \left( \frac{25\mu^2 y_1^2 y_2^2}{25\beta_3} + \frac{64\mu^2 y_2^4}{25\beta_4} - \frac{6y_2^2 y_2^2 + 3y_2^2}{4} \right) < 0,
\]
which implies that $u \equiv 0$ is locally asymptotically stable.

Finally, we assume $N \geq 2$. Since $\beta_{2N}, \beta_{2N+1}, \beta_{2N+2} < 0$, we deduce from (3.20) that
\[
\frac{1}{2} \frac{d}{dt} (y_N^2 + y_{N+1}^2) = -d_{11} z_1^4 - d_{22} z_2^4 - d_{12} + d_{21}) z_1^2 z_2^2 < 0,
\]
where we use that fact that $d_{11}, d_{22} > 0$ and
\[
d_{12} + d_{21} = 3 - 2\mu^2 \pi^2 N (N + 1)^2 \frac{\lambda^4}{\beta_{2N+1}} > 0.
\]
So, $u \equiv 0$ is locally asymptotically stable. This completes the proof of Lemma 3.1.

5. Numerical study. In this section, we will validate Theorem 2.1 using the numerical scheme. Let us consider the one dimensional MSHE (2.1) for $0 \leq x \leq \lambda(N)$ and $t > 0$, where $\lambda(N)$ is defined by (2.6) for given $N \in \mathbb{N}_0$. For numerical simulation, we employ the Crank-Nicholson method. To this end, we assume the space is discretized by the equal mesh spacing $\Delta x = \lambda/M$ and $x_j = j \Delta x, j = 0, 1, 2, \cdots, M$. And $T_{\text{max}}$ is enough time so that the numerical solution is close to steady state solution and $\Delta t = T_{\text{max}}/K$ and $t_n = n \Delta t, n = 0, 1, 2, \cdots, K$. Let $u^n$ or $u^k(x, t)$ be an approximate solution to the solution $u(x_j, t_n)$ at the discrete grid point $(x_j, t_n)$. Then, by applying Crank-Nicholson method to (2.1), we obtain
\[
\frac{u^n_j - u^{n-1}_j}{\Delta t} = \gamma L_\alpha(u^n_j) + (1 - \gamma) L_\alpha(u^{n-1}_j) + G(u^{n-1}_j, \mu),
\]
where $0 \leq \gamma \leq 1$. In our numerical simulation, $\gamma = 1/2$ is chosen. Here, we use the same notations $L_\alpha$ and $G(\cdot, \mu)$ as in Section 2. Finally, the functions for initial condition are chosen by three types as shown in Figure 6
\[
u_0(x) = u(x, 0) = 1 + \phi_1(x),
\]
\[
u_1(x) = u(x, 0) = 1 + 2\phi_1(x) + \phi_2(x),
\]
\[
u_2(x) = 0.1\phi_8(x) + \phi_9(x).
\]
Most numerical simulations of this section use $u_0(x)$ and $u_1(x)$. But, $u_2(x)$ is adopted to the test for the case $N = 8$, We also choose $\alpha = 1 + 10^{-4}$ for all numerical tests.

Recalling the final patterns of solutions described in Theorem 2.1, we decompose $u(\cdot, t)$ as
\[
u(\cdot, t) = v(\cdot, t) + o(\beta^{-1/2}_N).
\]
For example, if $N = 0$ and $\mu > 0$, then $v = \pm a_0 \phi_0$ as in Theorem 2.1(i). To verify the results of Theorem 2.1 numerically, we show the status that the numerical solution goes to $v(\cdot, t)$ as time goes to infinity. The following Figures illustrate that the dotted curves are the numerical $u_h(\cdot, t)$ solutions and the red solid curve is the leading term $v(\cdot, t)$ of the steady state solution $u(\cdot, t)$ obtained in Theorem 2.1.
Figure 7 show the numerical results in the case $N = 0$ and $N = 1$ with $\mu = 1$. From Figure 7a that $u_h(x,t)$ goes to $v(x) = a_0\phi_0(x)$ with the initial condition $u_0(x)$. This confirms the structure shown at Figure 2a. And, Figure 7b, we know that $u_h(x,t)$ goes to $v(x) = -b_0\phi_2(x)$ for initial condition $u_0(x)$. This confirms to structure shown at Figure 4a.

Let us consider the case $2 \leq N \leq 7$. We recall from Theorem 2.1 that the bifurcated attractor shows different shapes according to whether $\|\mu\| > \mu_N$ or $\|\mu\| < \mu_N$. To capture this phenomena, we choose $N = 2$ and set $\mu = 1/2$ or 2. We note from (3.32) that $\mu_2 = 1.2 \cdots$. Then, Theorem 2.1 (iii) says that if $\mu = 1/2 < \mu_2$, both of the single modes $\phi_2$ and $\phi_3$ are asymptotically stable. If $\mu = 2 > \mu_2$, only the single mode $\phi_2$ is asymptotically stable. Figures 8 illustrate these facts. Figure 8a and Figure 8b show that if $\mu = 1/2$, $u_h(x,t)$ goes to $c_1\phi_2(x)$.
for the initial condition \( u_0(x) \), and \( u_h(x,t) \) goes to \( c_2\phi_3(x) \) for the initial condition \( u_1(x) \), respectively. Figure 8c shows that if \( \mu = 2 \), \( u_h(x,t) \) goes to \( c_1\phi_2(x) \) for the initial conditions \( u_1(x) \). For the initial condition \( u_0(x) \), we can obtain the same result in this case. Therefore structures shown at Figure 5a and 5b are verified.

![Figure 8](image)

**Figure 8.** Tests for \( N = 2 \) given by (a) the initial value \( u_0(x) \) and \( \mu = 1/2 \), (b) the initial value \( u_1(x) \) and \( \mu = 1/2 \) and (c) the initial value \( u_1(x) \) and \( \mu = 2 \)

Last, for the test in the case \( N \geq 8 \), we choose \( N = 8 \) and set \( \mu = 1 \). By Theorem 2.1 (iv), the perturbations of the single modes \( \phi_8 \) and \( \phi_9 \) are asymptotically stable. Figure 9a shows that \( u_h(x,t) \) goes to \( v(x) = c_1\phi_8(x) \) for the initial conditions \( u_1(x) \). For the initial condition \( u_0(x) \), we can obtain the same result in this case. On the other hand, if we choose the initial condition \( u_2(x) = 0.1\phi_8(x) + \phi_9(x) \), \( u_h(x,t) \) goes to \( v(x) = c_2\phi_9(x) \) as shown in Figure 9b. This also confirms the structure shown at Figure 5a for the case \( N \geq 8 \).

**REFERENCES**

[1] I. Aranson and L. Kramer, *The world of the complex Ginzburg-Landau equation*, Rev. Mod. Phys., 74 (2002), 99–143.

[2] M. Bestehorn and H. Haken, *Transient patterns of the convection instability: A model-calculation*, Z. Phys. B, 57 (1984), 329–333.

[3] Y. Choi, *Dynamical bifurcation of one dimensional modified Swift-Hohenberg equation*, Bull. Korean Math. Soc., 52 (2015), 1241–1252.

[4] Y. Choi and J. Han, *Dynamical bifurcation of the damped Kuramoto-Sivashinsky equation*, J. Math. Anal. Appl., 421 (2015), 383–398.
Figure 9. Tests for $N = 8$ given by (a) the initial value $u_1(x)$ and $\mu = 1$ and (b) the initial value $u_2(x)$ and $\mu = 1$. 

[5] Y. Choi, J. Han and C. H. Hsia, Bifurcation analysis of the damped Kuramoto-Sivashinsky equation with respect to the period, *Discr. Cont. Dyn. Syst. B.*, 20 (2015), 1933–1957.

[6] Y. Choi, J. Han and J. Park, Dynamical bifurcation of the generalized Swift-Hohenberg equation, *Intern. J. Bifur. Chaos*, 25 (2015), 1550095, 16 pp.

[7] M. Cross and P. Hohenberg, Pattern formation outside equilibrium, *Rev. Mod. Phys.*, 65 (1993), 851–1112.

[8] A. Doelman, B. Sandstede, A. Scheel and G. Schneider, Propagation of hexagonal patterns near onset, *Euro. J. Appl. Math.*, 14 (2003), 85–110.

[9] N. Duan and W. Gao, Optimal control of a modified Swift-Hohenberg equation, *Electr. J. Diff. Eqns.*, 2012 (2012), 1–12.

[10] K. R. Edler, J. D. Gunton and N. Goldenfeld, Transition to spatiotemporal chaos in the damped Kuramoto-Sivashinsky equation, *Phys. Rev. E*, 56 (1997), 1631–1634.

[11] H. Gao and Q. Xiao, Bifurcation analysis of the 1D and 2D generalized Swift-Hohenberg equation, *Intern. J. Bifur. Chaos*, 20 (2010), 619–643.

[12] J. P. Gollub and J. S. Langer, Pattern formation in nonequilibrium physics, *More Things in Heaven and Earth*, (1999), 665–676.

[13] J. Han and C.-H. Hsia, Dynamical bifurcation of the two dimensional Swift-Hohenberg equation with odd periodic condition, *Discr. Cont. Dyn. Syst. B.*, 17 (2012), 2431–2449.

[14] J. Han and M. Yari, Dynamic bifurcation of the periodic Swift-Hohenberg equation, *Bull. Korean Math. Soc.*, 49 (2012), 923–937.

[15] M. Hilalì, S. Métemis, P. Borckmans and G. Dewel, Pattern selection in the generalized Swift-Hohenberg model, *Phys. Rev. E*, 51 (1995), 2046–2052.

[16] T. Ma and S. Wang, *Bifurcation Theory and Applications*, World Scientific, 2005.

[17] T. Ma and S. Wang, *Phase Transition Dynamics in Nonlinear Sciences*, Springer, 2014.

[18] L. A. Peletier and V. Rottschäfer, Pattern selection of solutions of the Swift-Hohenberg equation, *Physica D*, 194 (2004), 95–126.

[19] L. Peletier and J. Williams, Some canonical bifurcations in the Swift-Hohenberg equation, *SIAM J. Appl. Dyn. Sys.*, 6 (2007), 208–235.

[20] M. Polat, Global attractor for a modified Swift-Hohenberg equation, *Computers Math. Appl.*, 57 (2009), 62–66.

[21] L. Song, Y. Zhang and T. Ma, Global attractor for a modified Swift-Hohenberg equation in $H^k$ spaces, *Nonlin. Anal.*, 72 (2010), 183–191.

[22] J. Swift and P. Hohenberg, Hydrodynamic fluctuations at the convective instability, *Phys. Rev. A*, 15 (1977), 319–328.

[23] Q. Xiao and H. Gao, Bifurcation analysis of the Swift-Hohenberg equation with quintic nonlinearity, *Intern. J. Bifur. Chaos*, 19 (2009), 2927–2937.

[24] Q. Xiao and H. Gao, Bifurcation analysis of a modified Swift-Hohenberg equation, *Nonlin. Anal. Real World Appl.*, 11 (2010), 4451–4464.

[25] M. Yari, Attractor bifurcation and final patterns of the $N$-dimensional and generalized Swift-Hohenberg equations, *Discr. Cont. Dyn. Sys. B.*, 7 (2007), 441–456.
X. Zhao, B. Liu, P. Zhang, W. Zhang and F. Liu, Fourier spectral method for the modified Swift-Hohenberg equation, *Adv. Difference Equ.*, **2013** (2013), 1–19.

Received July 2016; revised October 2016.

E-mail address: yuncherl@kw.ac.kr
E-mail address: tha@nims.re.kr
E-mail address: jmhan@khu.ac.kr
E-mail address: dslee@dgist.ac.kr