Lattice Perturbation Theory

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Sources of uncertainties in perturbative calculations, tadpole improvement and its role in lattice perturbation theory, and six recent calculations are discussed.

1. INTRODUCTION

Perturbation theory (PT) appears in many important theoretical and practical roles in lattice field theory. Conceptual issues related to the continuum limit can be studied in a perturbative framework, such as the renormalizability of lattice gauge theories \[1\]. Perturbative calculations are used in relating lattice quantities to those defined in continuum schemes and in constructing Symanzik-improved lattice actions and operators. A recent determination of the strong coupling constant \[2\] and a new construction of lattice chiral fermions \[3\] relied on perturbative expansions. PT often helps us to better understand our numerical computations and provides useful checks.

Lattice perturbation theory is a very broad topic. Here, I focus on weak-coupling PT in zero-temperature lattice QCD. First, two important sources of uncertainties in lattice PT calculations are briefly discussed. Next, tadpole improvement and its role in lattice PT are described. I then survey six selected lattice PT papers or series of papers from the recent past. Developments concerning renormalons are omitted, as these will be presented in the talk by Sachrajda \[4\].

2. SOURCES OF UNCERTAINTIES

In this section, I briefly discuss two important sources of uncertainties in any calculation which relies on lattice perturbation theory; namely, the very applicability of PT and the choice of expansion parameter in finite-order approximations. Of course, these issues must also be confronted in continuum perturbation theory.

2.1. Applicability

Because of asymptotic freedom, one generally expects that quantities in QCD dominated by momentum scales \(\mu \gg \Lambda_{\text{QCD}}\) may be reliably determined using perturbation theory. Unfortunately, it is very difficult to anticipate the size of nonperturbative contributions in any calculation. Warnings concerning the enhancement of nonperturbative effects as \(a \to 0\) in the presence of power divergences have been issued \[5\]. The Lepage-Mackenzie \(q^*\) scale (see below) can also serve as a cautionary measure of PT’s reliability. Nevertheless, no harm is done by trying out perturbation theory to see how well it works using as many cross-checks as possible; in fact, much might be learned.

Good agreement between perturbative and Monte Carlo estimates of the critical quark mass for Wilson quarks and various Creutz ratios of Wilson loops was demonstrated in Ref. \[6\]. Estimates of light-quark current renormalizations from boosted PT \[7\] and nonperturbative chiral Ward identities \[8,9\] have been recently compared for the improved clover fermionic action (without tadpole improvement). The results are summarized below (\(Z_V, Z_A, Z_{PS},\) and \(Z_S\) are the vector, axial, pseudoscalar, and scalar current renormalizations, respectively).

| \(\beta\) | Chiral WI | 1-loop BPT |
|---|---|---|
| 6.0 | 0.824(2) | 0.83 |
| 6.2 | 0.817(9) | 0.83 |
| 6.0 | 1.09(3) | 0.97 |
| 6.2 | 1.045(10) | 0.97 |
| 6.0 | 0.60(2) | 0.71 |
| 6.2 | 0.649(9) | 0.72 |

Other nonperturbative determinations of these
currents have also been compared to the above PT results in Ref. [10]. Perturbative estimates of the power-divergent energy shift \( aE_0 \) and heavy-quark mass renormalization \( Z_m \) in NRQCD have been used [11] to determine the \( \Upsilon \) mass \( aM_\Upsilon^{(a)} \) from simulation results \( aE_\Upsilon \) using \( aM_\Upsilon^{(a)} = 2(Z_m aM - aE_0) + aE_\Upsilon \). The results compare very well with the nonperturbative determinations \( aM_\Upsilon^{(b)} \) obtained by fitting to the dispersion \( aE_\Upsilon(p) = aE_\Upsilon + a^2 p^2/(2aM_\Upsilon^{(b)}) \), as shown in the table below.

| \( aM \) | \( aE_\Upsilon \) | \( Z_m \) | \( aE_0 \) | \( aM_\Upsilon^{(a)} \) | \( aM_\Upsilon^{(b)} \) |
|---|---|---|---|---|---|
| 1.71 | 0.453(1) | 1.20 | 0.32 | 3.92 | 3.94(3) |
| 1.80 | 0.451(1) | 1.18 | 0.31 | 4.08 | 4.09(3) |
| 2.00 | 0.444(1) | 1.16 | 0.30 | 4.48 | 4.48(4) |

### 2.2. Choice of expansion parameter

A major source of ambiguity in perturbative calculations is the choice of expansion parameter. Consider the perturbative prediction for an observable \( O(Q) \) depending on a single external momentum \( Q \) of the generic form

\[
O(Q) = c_0 \left[ 1 + c_1(Q/\mu,RS) \alpha_s(\mu,RS) + c_2(Q/\mu,RS) \alpha_s^2(\mu,RS) + \ldots \right].
\]  

The coefficients \( c_i(Q/\mu,RS) \) and the expansion parameter \( \alpha_s(\mu,RS) \) depend on the choice of scale \( \mu \) and the renormalization scheme (RS), i.e., the definition of \( \alpha_s(\mu,RS) \), in such a way that the observable \( O(Q) \) is independent of both \( \mu \) and the RS. However, in practice, one is forced, either by exhaustion or the asymptotic nature of the series, to truncate the perturbation expansion at some finite order in the coupling. Unfortunately, the resulting finite-order approximants are no longer independent of \( \mu \) and the RS.

For a good choice of expansion parameter, the uncalculated higher-order terms should be small. Since these contributions are unknown, one must ultimately rely on a guess about their sizes. This guess can be based on the sensitivity to changes in the scale and the RS, the apparent rate of convergence of the perturbation series for a variety of quantities, the size of the coefficient of the last term in the truncated series, or some physical criteria. The signal for a poor expansion parameter is the appearance of large higher-order coefficients in the perturbative expansions of numerous observables.

Choosing an expansion parameter involves fixing the RS (defining the coupling), specifying the scale at which to evaluate the coupling (in the case of a running coupling), and determining the numerical value of the coupling at that scale. Various examples of expansion parameter choices are described below.

#### 2.2.1. Coupling definitions

Some couplings which have appeared in the literature include:

- (a) The bare lattice coupling \( \alpha_0 \). This is a poor expansion parameter, yielding first-order corrections for various short-distance quantities which are consistently much too small in comparison to simulation measurements. Large expansion coefficients routinely appear in perturbative series when expressed in terms of this coupling.

- (b) The “boosted” coupling \( \alpha_b = 3\alpha_0/(\langle \text{Tr} U_{\text{plaq}} \rangle) \) defined by rescaling the bare coupling using the mean plaquette. Here, the mean field contributions responsible for the problems in (a) are absorbed into the coupling, resulting in better behaved perturbative series.

- (c) The coupling \( \alpha_{SF}(q) \) defined through the Schrödinger functional using a recursive finite-size scaling technique [12,13]. This coupling runs with the finite system size \( L = 1/q \). An alternative coupling \( \alpha_{TF}(q) \) based on the correlations of Polyakov loops in systems with twisted boundary conditions has also been proposed [14].

- (d) The coupling \( \alpha_{q0}(q) = 1/4r^2F(r) \) for large \( q = 1/r \) defined in terms of the interquark force \( F(r) = dV/dr \) [15].

- (e) The coupling \( \alpha_V(q) \) defined in terms of the short-distance static quark potential \( V(q) \) using \( V(q) = -4\pi C_F \alpha_V(q)/q^2 \), for \( q \) large [16]. By absorbing higher-order contributions to the static-quark potential into the coupling, it is hoped that higher-order contributions to other physical quantities in terms of this coupling will then be small. This definition facilitates scale setting since the running coupling’s argument can be easily related to a gluon momentum. Order-\( \alpha_V \) agreement of perturbative with simulation results for several short-distance quantities has also been
demonstrated.

(f) The coupling \( \alpha_{\text{MS}} \) defined in the familiar modified minimal subtraction scheme.

2.2.2. Scale settings

Various scale setting prescriptions have been devised:

(a) Since \( \alpha_0 \) and \( \alpha_b \) do not run, no scale setting is required for these couplings.

(b) A very simple possibility is to somehow guess the scale. For some quantities, it should be possible to crudely estimate the scale likely to dominate the processes involved.

(c) When the two-loop contribution is known, the scale may be chosen so that the one-loop coefficient vanishes. This procedure works well when relating couplings in different schemes (and is equivalent to choosing the relative scale as the ratio of \( \Lambda \)-parameters) but has not been tested on other quantities.

(d) Another scale setting scheme is the Lepage-Mackenzie \( q^* \) prescription. For a one-loop contribution \( \int d^4q \; \xi(q) \), where \( q \) is the momentum of the exchanged gluon, one chooses \( q^* \) such that \( \alpha_V(q^*) \int d^4q \; \xi(q) = \int d^4q \; \alpha_V(q) \; \xi(q) \). Inserting the lowest-order form for the running coupling into the integral on the right-hand side yields the result \( \ln(q^{*2}) = \int d^4q \ln(q^2) \; \xi(q) / \int d^4q \; \xi(q) \). This procedure is the lattice analogue of the Brodsky-Lepage-Mackenzie prescription in continuum perturbation theory. Difficulties with this procedure can arise when \( \int d^4q \; \xi(q) \approx 0 \), and note that the mean value theorem guarantees that \( q^* \) will satisfy \( 0 \leq q^* \leq 2\pi \) only if \( \xi(q) \geq 0 \) for all \( q \) throughout the region of integration (or \( \leq 0 \) for all such \( q \)).

2.2.3. Value determinations

Numerical values for the chosen coupling may be assigned in various ways:

(a) In a given simulation, determining the values for \( \alpha_0 \) and \( \alpha_b \) is straightforward.

(b) The value of the chosen coupling may be obtained using a perturbative expansion in terms of the bare couplings \( \alpha_0 \) or \( \alpha_b \). For example, in SU(3):

\[
\alpha_V(s/a) = \alpha_0 + (6.71 - 1.75 \ln s) \; \alpha_0^2 + \cdots,
\]

\[
= \alpha_b + (2.52 - 1.75 \ln s) \; \alpha_b^2 + \cdots,
\]

\[
\alpha_{\text{MS}}(s/a) = \alpha_0 + (5.88 - 1.75 \ln s) \; \alpha_0^2
\]

\[
+ (43.41 - 21.89 \ln s + 3.06 \ln^2 s) \; \alpha_0^3 + \cdots,
\]

\[
= \alpha_b + (1.69 - 1.75 \ln s) \; \alpha_b^2
\]

\[
+ (6.31 - 7.23 \ln s + 3.06 \ln^2 s) \; \alpha_b^3 + \cdots,
\]

\[
\alpha_{SF}(s/a) = \alpha_0 + (4.62 - 1.75 \ln s) \; \alpha_0^2 + \cdots,
\]

\[
= \alpha_b + (0.43 - 1.75 \ln s) \; \alpha_b^2 + \cdots.
\]

(c) The coupling strength can also be determined by accurately measuring in a simulation some short-distance quantity whose perturbative expansion is reliable and known to sufficiently high order. In Ref. 17, the logarithm of the mean plaquette was used:

\[-\ln\left(\frac{1}{3} \text{Tr} U_{\text{plaq}}^a\right) = \frac{4\pi}{3} \alpha_V(3.41/a) \right \{1 - 1.185\alpha_V\},\]

where the scale 3.41/a is determined using the \( q^* \) prescription mentioned above. Values of the coupling at other large values of \( q \) are then obtained using the familiar two-loop perturbative evolution equation.

(d) A finite-size scaling procedure can be used to measure \( \alpha_{SF}(q) \) in terms of Sommer’s scale \( r_0 \) without recourse to perturbation theory.

3. TADPOLE IMPROVEMENT

Tadpole improvement (TI) refers to the simple procedure of modifying any lattice gauge field operator by rescaling the link variable \( U_{\mu}(x) \) by a mean field factor: \( U_{\mu}(x) \rightarrow U_{\mu}(x)/u_0 \), where a convenient gauge-invariant choice for the mean field parameter is \( u_0 = \langle \frac{1}{3} \text{Tr} U_{\text{plaq}}^a \rangle^{1/4} \). The purpose of this procedure is to assist in the construction of improved lattice operators, that is, lattice operators with diminished discretization errors and lattice-to-continuum renormalization factors nearer to unity.

When constructing any lattice operator \( O[U] \) which is a sum of various basic gauge-field operators \( O_j[U] \), that is, \( O[U] = \sum_j c_j(g) \; O_j[U] \), where each \( O_j \) is a simple gauge-invariant product of link variables, the coefficients \( c_j \) must somehow be determined. Lattice perturbation theory is often called upon to fix these coefficients.
However, the $c_j$’s generally contain large mean-field contributions; to reliably account for these tadpole effects using low-order perturbative expansions, even when using a good expansion parameter, is asking much from perturbation theory. Tadpole improvement offers a better alternative: use mean-field theory instead of PT to treat the tadpole contributions. By tadpole improving the basic operators, $O[U] = \sum_j \hat{c}_j(g) \ O_j[U/u_0]$, the mean-field effects are removed nonperturbatively, resulting in smaller leftover coefficients $\hat{c}_j$ which should be much more reliably estimated by low-order perturbative expansions. Note that $u_0$ is a nonperturbatively measured parameter when simulating, but its perturbative expansion must be used when computing the $\hat{c}_j$ in PT.

Hence, tadpole improvement should be viewed as a simple means of combining mean-field theory with perturbation theory in order to determine the parameters in a lattice operator. For example, consider the Symanzik-improved gluon action of Lüscher and Weisz\cite{18} with the chair coupling set to zero. The ratios of the rectangle $\beta_{rt}$ and parallelogram $\beta_{pl}$ couplings to the plaquette $\beta_{pl}$ coupling change under TI as follows:

$$-20 \frac{\beta_{rt}}{\beta_{pl}} = (1 + 2.02\alpha_s) \ T/L \ u_0^{-2} (1 + 0.48\alpha_s)$$

$$- \frac{\beta_{pg}}{\beta_{pl}} = 0.03\alpha_s \ T/L \ 0.03u_0^{-2}\alpha_s,$$

where $u_0$ is measured in the simulation (the basic operators without TI correction are multiplied by these coefficients). This is not the same as simply using the boosted $\alpha_b$. Note that mean-field corrections are sometimes large; heavy quarkonium spin splittings are dramatically underestimated by a factor of 1/2 using tree-level couplings if TI is not implemented.

4. HIGHLIGHTS FROM RECENT PAST

Six selected lattice perturbation theory papers or series of papers from the recent past are surveyed in this section. For recent developments concerning renormalons, see Ref.\cite{4}.

4.1. On the viability of lattice perturbation theory

Apparent discrepancies between perturbative and Monte Carlo estimates of various short-distance quantities were shown in Ref.\cite{8} to result from the use of the bare lattice coupling $\alpha_0$ as the expansion parameter. An expansion parameter $\alpha_V(q^*)$ defined in terms of the physical static-quark potential was advocated. Studying the expectation value of the trace of a link in Landau gauge, the critical mass for Wilson quarks, and various Creutz ratios of Wilson loops, the authors demonstrated that such discrepancies do not occur when a renormalized coupling such as $\alpha_V(q^*)$ is used.

Also in this paper, the nonlinear relation between the link operator and the gauge field was identified as a source of large mean-field renormalizations which hamper attempts to construct improved lattice operators. A tadpole improvement scheme, as previously discussed, was suggested to remedy this problem.

Lastly, the onset of asymptotic or perturbative scaling in lattice QCD using the standard Wilson action was investigated. The 1P-1S mass splittings in charmonium and bottomonium and the string tension were shown to scale well for $\beta$ values as low as 5.7 when expressed in terms of the scale parameter $\Lambda_V$ associated with $\alpha_V$; scaling is not observed for these quantities when expressed in terms of $\Lambda_0$ associated with the bare lattice coupling. This suggests that lattice spacings used in current simulations are small enough for reliable studies of QCD.

4.2. $\alpha_{\text{MS}}$ in terms of $\alpha_0$

In an impressive series of papers\cite{9}, Lüscher and Weisz have recently extended to two-loop order the perturbative expansion of $\alpha_{\text{MS}}$ in terms of the bare lattice coupling $\alpha_0$ for SU($N$) gauge theories. By matching the perturbative expansions of corresponding correlation functions in the lattice and continuum theories, they found

$$\alpha_{\text{MS}}(s/a) = \alpha_0 + d_1(s) \ \alpha_0^2 + d_2(s) \ \alpha_0^3 + \cdots,$$

where

$$d_1(s) = -\frac{11N}{6\pi} \ln s - \frac{\pi}{2N} + k_1N,$$
\[ d_2(s) = d_1^2(s) - \frac{17N^2}{12\pi^2} \ln s + \frac{3\pi^2}{8N^2} + k_2 + k_3N^2, \]
\[ k_1 = 2.135730074078457(2), \]
\[ k_2 = -2.8626215972(6), \]
\[ k_3 = 1.24911585(3). \]

This calculation is part of an overall strategy for measuring the running coupling using lattice simulations and a nonperturbatively defined coupling \( \alpha_{SE} \) related to the Schrödinger functional.

The calculation exploits the background field technique of de Wit. In order to place their calculation on firm theoretical grounds, the authors first show in Ref. [21] that lattice gauge theory with a background gauge field is renormalizable to all orders in perturbation theory. The proof is based on the BRS, background gauge, and background shift symmetries of the lattice functional integral. They find that no new counterterms are required in addition to those already needed in the absence of the background field.

An important advantage in using the background field technique is the fact that the relation between \( \alpha_{MS} \) and \( \alpha_0 \) can be extracted solely from the background field 3-point function: the 3-point vertex function need not be considered. This dramatically reduces the number of Feynman diagrams which must be computed. Also, diagrams with two external legs are much simpler to evaluate than those with three external legs.

The calculation is described in detail in Ref. [22] and involves 4 one-loop diagrams and 31 two-loop diagrams (7 factorizable, 7 ring, 7 tadpole, 3 diamond, 3 eye, and 4 bigmac diagrams). All vertex factors were generated using algebraic manipulation programs written in MAPLE. The Feynman loop-integrals were evaluated using a new, innovative position-space method described in Ref. [20]. This method is based on efficiently evaluating the free massless propagator in coordinate space using a recursion relation which expresses the propagator in terms of its values close to the origin. Convergence of lattice sums is accelerated using known asymptotic forms of the propagator. The method is even useful for evaluating Feynman diagrams with non-zero external momenta.

### 4.3. Tadpole-improved heavy-light lattice operators

In Ref. [23], the lattice-to-continuum renormalizations of the temporal components of the point \( (A_\mu) \) and point-split \( (A_{ps}^\mu) \) axial currents in the heavy-quark effective theory (HQET) were computed to one-loop order. The standard Wilson fermionic action with massless light quarks was used with several values for the Wilson \( r \) parameter. This was the first HQET calculation to implement tadpole improvement and to use \( \alpha_V(q^*) \) as the expansion parameter. Writing the continuum current \( A_c \) in terms of the point and point-split lattice currents as \( A_c(\mu) \approx Z(\mu)A_L(\mu)/2 \approx Z_{ps}(\mu)A_{ps}^\mu(\mu)/(2u_0) \), the renormalizations for \( r = 1 \) were found to be

\[ Z(\mu) = 1 + (-1.48 + 0.318 \ln \mu) \alpha_V(2.18/\mu), \]
\[ Z_{ps}(\mu) = 1 + (-0.76 + 0.318 \ln \mu) \alpha_V(2.13/\mu). \]

These results may be compared to those from boosted PT in which one writes \( A_c(\mu) \approx \sqrt{2\kappa_{bc}Z(\mu)}A_L(\mu) \approx \sqrt{2\kappa_{bc}Z_{ps}(\mu)A_{ps}^\mu(\mu)} \), where \( \kappa_{bc} = u_0\kappa_c \) is the boosted critical hopping parameter, \( \alpha_b = \alpha_0/u_0^4 \), and, for \( r = 1 \),

\[ Z(\mu) = 1 + (-1.64 + 0.318 \ln \mu) \alpha_b, \]
\[ Z_{ps}(\mu) = 1 + (0.12 + 0.318 \ln \mu) \alpha_b. \]

Comparisons for a few values of \( \beta \) are shown in the table below:

| \( \beta \) | \( \mu \) | \( Z/2 \sqrt{2\kappa_{bc}Z} \) | \( Z_{ps}/2u_0 \sqrt{2\kappa_{bc}Z_{ps}} \) |
|------------|--------|-------------------|-------------------|
| 5.7        | 1      | 0.34              | 0.48              |
| 5.7        | 2      | 0.37              | 0.41              |
| 6.2        | 4      | 0.38              | 0.41              |
| 6.2        | 5      | 0.39              | 0.43              |
| 6.2        | 6      | 0.40              | 0.43              |

One sees that differences between the one-loop results from tadpole-improved renormalized PT and boosted PT can be as large as 10%.

### 4.4. Couplings in NRQCD

Nonrelativistic lattice QCD (NRQCD) is an effective field theory designed for studying hadrons containing heavy quarks. The NRQCD action includes interactions which systematically correct for relativity and finite-lattice-spacing errors.
4.5. Structure functions

The renormalization constants and mixing coefficients for the lowest-twist lattice operators appearing in the Wilson expansion of the product of two hadronic currents have recently been calculated to one-loop order by two groups in Refs. [24–28]. These quantities are needed to extract from simulations the hadronic matrix elements relevant for determining moments of the quark and gluon distributions inside hadrons, i.e., the deep inelastic structure functions. The lattice operators considered were of the form

\[
\begin{align*}
Q_{\tau_1,\tau_n}^J &= 2^{-n}\overline{\psi} \gamma_{\tau_1} \overleftrightarrow{D}_{\tau_2} \cdots \overleftrightarrow{D}_{\tau_n} \lambda^f \psi, \\
Q_{\tau_1,\tau_n}^5 &= 2^{-n}\overline{\psi} \gamma_{\tau_1} \gamma_5 \overleftrightarrow{D}_{\tau_2} \cdots \overleftrightarrow{D}_{\tau_n} \lambda^f \psi, \\
G_{\tau_1,\tau_n} &= \sum_{\rho} \text{Tr}(F_{\tau_1,\rho} \overleftrightarrow{D}_{\tau_2} \cdots \overleftrightarrow{D}_{\tau_{n-1}} F_{\rho,\tau_n}),
\end{align*}
\]

ignoring trace terms, where \(\psi\) is a quark field, \(D_\mu\) is a covariant lattice derivative, \(F_{\mu,\nu}\) is the cloverleaf gluon field strength tensor, and \(\lambda^f\) are flavour matrices. The rank-two flavour-singlet operators \(Q_{\mu,\nu}\) and \(G_{\mu,\nu}\) mix and are related to the first moments of the quark and gluon distributions, respectively; the rank-three \(Q_{\mu,\nu,\tau}\) is related to the second moment of the quark distribution. The quantities calculated were the renormalization factors

\[
Z_{kl}(\mu a) = \delta_{kl} - \frac{\alpha_0}{4\pi} C_F (\gamma_{kl} \ln \mu a + B_{kl})
\]

relating the bare lattice operators to a set of finite operators \(\hat{O}_k(\mu) = Z_{kl}(\mu a)O(l)\) renormalized by requiring that the matrix elements of \(\hat{O}\) in external massless quark and/or gluon states having momentum \(p^2 = \mu^2\) are identical to the tree-level matrix elements of the bare lattice operators, for \(\mu a \ll 1\) to minimize discretization effects. Only operators which cannot mix with lower dimensional operators due to symmetry considerations were studied.

In Refs. [24,27], results were obtained for both the Wilson and the improved clover actions in the chiral limit; tadpole improvement was not implemented. For the rank two operators, internal quark loops were taken into account, and flavour singlet and non-singlet renormalizations were studied; only the quenched theory was used for the rank three quark operators. The calculations relied heavily on the use of the algebraic manipulation languages FORM and SCHOONSCHIP, with modifications to properly treat the Dirac...
matrices. The following operators were considered: $O_1 = Q_{(14)}$, $O_2 = G_{(14)}$, $O_3 = Q_{NS}^{NS}$, $O_A = Q_{114} - \frac{1}{2}(Q_{22} + Q_{133})$, $O_B = Q_{114} + Q_{114} - \frac{1}{2}(Q_{22} + Q_{224} + Q_{333} + Q_{334})$, and $O_C = Q_{(123)}$, where \{\cdots\} denotes symmetrization and the superscript $NS$ denotes non-singlet (operators are flavour singlets unless otherwise indicated). Some selected results for $r = 1$ (quenched) are given in the table below (results for the improved fermion action are indicated by the superscript $I$):

| $k, l$ | $\gamma_{kl}$ | $B_{kl}$ | $B_{kl}^I$ |
|-------|---------------|---------|-----------|
| 1, 1  | 16/3          | -3.165  | -15.816   |
| $A, A$ | 13/3 | -18.824  | -27.389   |
| $A, B$ | 2    | -0.924   | -3.603    |
| $B, A$ | 4    | -2.955   | -11.803   |
| $B, B$ | 19/3 | -17.540  | -18.538   |
| $C, C$ | 25/3 | -19.005  | -29.815   |

The rank two renormalizations using the Wilson action were compared with previous determinations: some discrepancies were found.

The Wilson action results above were confirmed in Ref. \[28\] (except the last row which was not computed). However, these authors advocate the use of the operators $\tilde{O}_A = O_A + O_B$ and $\tilde{O}_B = O_B - 2O_A$ in order to diagonalize the anomalous dimension matrix. They also considered the following operators: $O_{1b} = Q_{2} - \frac{1}{4}(Q_{11} + Q_{22} + Q_{33})$, $O_4 = Q_2^2$, $O_5 = Q_{(12)}^2$, and $O_6 = Q_{(21)}^2$, where \{\cdots\} denotes antisymmetrization. The renormalizations using the $r = 1$ Wilson action are given below:

| $k$ | $\gamma_{kk}$ | $B_{kk}$ |
|-----|---------------|---------|
| 1b  | 16/3          | -1.892(06) |
| 4   | 0             | 15.795(03) |
| 5   | 25/3          | -19.560(10) |
| 6   | 7/3           | -15.680(10) |

MATHEMATICA and MAPLE were used to perform the calculations.

4.6. Stochastic perturbation theory

Recently, an innovative numerical technique \[29\] for obtaining weak-coupling perturbative expansions of local observables in lattice QCD was proposed. An exciting aspect of this method is that it allows one to obtain much longer expansions than presently possible using conventional diagrammatic approaches.

The method is based on Parisi-Wu stochastic quantization in which the gauge field is viewed as a random variable which evolves according to the Langevin equation. One step $t \rightarrow t + \epsilon$ in the discrete Langevin equation consists of a sweep through the lattice, updating links using $U_\mu(x; t + \epsilon) = \exp[-F_\mu(x; t)]U_\mu(x; t)$, where the driving function $F_\mu$ depends on the link variables in all plaquettes containing the link between sites $x$ and $x + a\hat{\mu}$, the Langevin time step $\epsilon$, and a noise matrix. One then writes $U_\mu(x; t) = \exp[A_\mu(x; t)/\sqrt{\beta}]$, rescales the time step $\epsilon = \tau/\beta$, expands $A_\mu(x; t) = \sum_{k \geq 0} \beta^{-k/2} A_\mu^{(k)}(x; t)$ and $F_\mu$ as power series in $1/\sqrt{\beta}$, and truncates to some order to transform the Langevin equation into a system of coupled stochastic finite-difference equations. The coefficients of the perturbative expansion of any local observable $W$ are then given by expectation values of composite operators of the $A_\mu^{(k)}$ which are obtained by averaging over the Langevin history:

$$W = \sum_n \beta^{-n/2} \langle O_n \rangle,$$

$$\langle O_n \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} O_n(t).$$

The discrete Langevin equation has $O(\epsilon)$ systematic errors, making it necessary to extrapolate the results to $\epsilon \rightarrow 0$. Stochastic gauge fixing is necessary for an acceptable signal-to-noise ratio and finite size errors must be controlled.

In Ref. \[31\], the mean plaquette in SU(3),

$$P = 1 - \frac{1}{3} \langle \text{Tr } U_p \rangle = \sum_{n=1}^{\infty} c_n \beta^{-n},$$

was computed to eight-loop order. Results were obtained on an $8^4$ lattice with periodic boundary conditions for step sizes $\tau = 0.01, 0.015, 0.02$, then extrapolated to $\tau \rightarrow 0$. The first three coefficients were found to agree with known values obtained analytically in Ref. \[31\] using the standard diagrammatic approach, as shown in the table below.
Langevin Analytic

\[
\begin{align*}
    c_1 & = 1.998(1) & 2 \\
    c_2 & = 1.218(1) & 1.212(7) \\
    c_3 & = 2.940(5) & 2.9605 \\
    c_4 & = 9.28(2) \\
    c_5 & = 34.0(2) \\
    c_6 & = 134.9(9) \\
    c_7 & = 563(5) \\
    c_8 & = 2488(29)
\end{align*}
\]

5. CONCLUDING REMARKS

The evaluation of Feynman diagrams in lattice PT is difficult because the Feynman integrands are usually complicated functions of the loop and external momenta. Standard tools, such as Feynman parameters and partial integration methods, are not very helpful on the lattice. Progress is being made, however, with the development of new techniques, such as those proposed in Refs. [19]-[22], an increasing reliance on and expertise in using analytical computer programs, such as MAPLE and MATHEMATICA, and the introduction of exciting new stochastic methods. Lattice perturbation theory continues to evolve and to play an important role in lattice field theory.

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