GEODESICS OF POSITIVE LAGRANGIANS FROM SPECIAL LAGRANGIANS WITH BOUNDARY

JAKE P. SOLOMON AND AMITAI M. YUVAL

Abstract. Geodesics in the space of positive Lagrangian submanifolds are solutions of a fully non-linear degenerate elliptic PDE. We show that a geodesic segment in the space of positive Lagrangians corresponds to a one parameter family of special Lagrangian cylinders, called the cylindrical transform. The boundaries of the cylinders are contained in the positive Lagrangians at the ends of the geodesic. The special Lagrangian equation with positive Lagrangian boundary conditions is elliptic and the solution space is a smooth manifold, which is one dimensional in the case of cylinders. A geodesic can be recovered from its cylindrical transform by solving the Dirichlet problem for the Laplace operator on each cylinder.

Using the cylindrical transform, we show the space of pairs of positive Lagrangian spheres connected by a geodesic is open. Thus, we obtain the first examples of strong solutions to the geodesic equation in arbitrary dimension not invariant under isometries. In fact, the solutions we obtain are smooth away from a finite set of points.

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1. Introduction

1.1. Overview. Let \((X, \omega, J, \Omega)\) be a Calabi-Yau manifold. Namely, \(X\) is a Kähler manifold with symplectic form \(\omega\) and complex structure \(J\), and \(\Omega\) is a non-vanishing holomorphic volume form on \(X\). We denote by \(g\) the Kähler metric and by \(n\) the complex dimension.

An oriented Lagrangian submanifold \(\Lambda \subset X\), possibly immersed, is said to be positive if \(\text{Re} \Omega|_\Lambda\) is a positive volume form. A positive Lagrangian submanifold is special if \(\text{Im} \Omega|_\Lambda = 0\). An oriented Lagrangian submanifold is called imaginary special if \(\text{Re} \Omega|_\Lambda = 0\) and \(\text{Im} \Omega|_\Lambda\) is a positive volume form.

Let \(\mathcal{O}\) be a Hamiltonian isotopy class of closed smoothly embedded positive Lagrangians diffeomorphic to a given manifold \(L\). Then \(\mathcal{O}\) is naturally a smooth Fréchet manifold, and for \(\Lambda \in \mathcal{O}\), there is a natural isomorphism

\[
T_\Lambda \mathcal{O} \cong \mathcal{C}^\infty(\Lambda) := \left\{ h \in \mathcal{C}^\infty(\Lambda) \mid \int_\Lambda h \text{Re} \Omega = 0 \right\}.
\]

Following [22], we define a Riemannian metric on \(\mathcal{O}\) by

\[
(h, k) := \int_\Lambda hk \text{Re} \Omega, \quad h, k \in \mathcal{C}^\infty(\Lambda).
\]

It is shown in [23] that the metric \((\cdot, \cdot)\) has a Levi-Civita connection and the associated sectional curvature is non-positive. The Levi-Civita connection, which we describe in detail in Section 2.2, gives rise to the notion of geodesics. If \(\mathcal{O}\) is geodesically connected then there can exist at most one special Lagrangian in \(\mathcal{O}\) [22]. If two Lagrangians \(\Lambda_0, \Lambda_1 \in \mathcal{O}\) are connected by a geodesic, the cardinality of \(\Lambda_0 \cap \Lambda_1\) is bounded below by the number of critical points of a function on \(\Lambda_0\) [18].

The geodesic equation is a fully non-linear partial differential equation. It is shown in [18] that the geodesic equation is degenerate elliptic and the associated boundary value problem in the Euclidean setting has unique weak solutions. In [25] there are examples of smooth geodesics in arbitrary dimension, which are preserved by an isometric action of the orthogonal group \(O(n)\). The group action allows the geodesic equation to be reduced to the one dimensional case, where it becomes an ODE. Further results on geodesics can be found in [8, 27].

An analog to the space \(\mathcal{O}\) under mirror symmetry is the space \(\mathcal{H}\) of almost calibrated \((1, 1)\)-forms on a Kähler manifold. The geodesic equation in \(\mathcal{H}\) is a degenerate form of the deformed Hermitian Yang-Mills equation. The space \(\mathcal{H}\) and its geodesics have been studied in [6, 7].

In the present work, we establish a correspondence between geodesics of positive Lagrangians and one parameter families of imaginary special Lagrangian cylinders. See Theorems 1.1 and 1.5. We call this correspondence the cylindrical transform. By cylinders, we mean manifolds of the form \(N \times [0, 1]\), where \(N\) is a manifold of dimension \(n - 1\). The boundary components of the cylinders corresponding to a geodesic \((\Lambda_t)_{t \in [0, 1]}\) are contained in \(\Lambda_0\) and \(\Lambda_1\) respectively. Positive Lagrangian submanifolds such as \(\Lambda_0\) and \(\Lambda_1\) are an elliptic boundary condition for the imaginary special Lagrangian equation. In Theorem 1.2 we show that the space of imaginary special Lagrangian cylinders with positive Lagrangian boundary conditions is a smooth 1-dimensional manifold.

Using the cylindrical transform and the ellipticity of the imaginary special Lagrangian equation, we establish in Theorem 1.6 a perturbation result for solutions of the geodesic equation. Namely, the space of pairs of positive Lagrangian spheres intersecting transversally at two points that are connected by a geodesic is open. In particular, we show the existence of geodesics connecting positive Lagrangians of arbitrary dimension without any symmetry. The geodesics of Theorem 1.6 are smooth away from a finite number of points. In a sequel [24], we strengthen Theorem 1.6.
so that it produces geodesics of positive Lagrangians that are $C^{1,1}$ submanifolds even at the non-smooth locus.

1.2. Statement of results. To set up the cylindrical transform in its natural generality, we consider geodesics of possibly immersed positive Lagrangians that are smooth away from a finite number of cone point singularities. For the rest of the paper, unless otherwise specified, the term geodesic allows such cone point singularities. We define the critical locus of a geodesic of positive Lagrangians $(A_t)_{t \in [0,1]}$ by

$$\text{Crit}((A_t)_{t}) := \bigcap_{t \in [0,1]} A_t.$$ 

The non-smooth cone points of the geodesics in this paper are contained in their critical loci. The possibly limited regularity at the critical locus is consistent with the result of [18] that the symbol of the linearized geodesic equation has a 1-dimensional kernel except at the critical locus, where the kernel is $(n-1)$-dimensional. A full account of our notion of geodesics is given in Definition 3.38.

Define a positive function $\rho : X \to \mathbb{R}$ by

$$\rho^2 \omega^n/n! = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \Omega \wedge \overline{\Omega}.$$ 

Let $f : L \to X$ be an immersion. Define

$$\Delta_p : C^\infty(L) \to C^\infty(L)$$

by $u \mapsto *d((\rho \circ f) \ast du)$, where $*$ is the Hodge star operator associated to the Riemannian metric $f^*g$. Then $\Delta_p$ is elliptic (see Lemma 4.6). A geodesic $(A_t)_{t \in [0,1]}$ has an associated Hamiltonian, which is the family of functions $h_t \in C^\infty(A_t)$ such that $h_t = \frac{\partial}{\partial t}A_t$. The functions $h_t, t \in [0,1]$, are related by parallel transport of the Levi-Cevita connection on $O$ and, in particular, have the same image and diffeomorphic level sets.

**Theorem 1.1.** Let $(A_t)_{t \in [0,1]}$ be a geodesic of positive Lagrangians and let $(h_t)_{t \in [0,1]}$ denote the associated Hamiltonian. For $c \in \mathbb{R}$, let

$$L_c := \{(p,t) | t \in [0,1], p \in h_t^{-1}(c) \setminus \text{Crit}((A_t)_{t}) \subset A_t\}.$$ 

Then $L_c$ is a smooth immersed submanifold of $X \times [0,1]$ diffeomorphic to the cylindrical manifold $(h_0^{-1}(c) \setminus \text{Crit}((A_t)_{t})) \times [0,1]$, and the map

$$\Phi_c : L_c \to X$$

given by $\Phi_c(p,t) = p$ is an imaginary special Lagrangian immersion mapping the boundary components of $L_c$ to $A_0$ and $A_1$. See Figure 1. Moreover, the map

$$\sigma_c : L_c \to [0,1]$$

given by $\sigma_c(p,t) = t$ satisfies $\Delta_p \sigma_c = 0$.

Let $A_0, A_1 \in O$ and let $N$ be a manifold of dimension $n-1$. We denote by $SLC(N; A_0, A_1)$ the space of imaginary special Lagrangian submanifolds of $X$, perhaps immersed, diffeomorphic to $N \times [0,1]$, such that the boundary corresponding to $N \times \{i\}$ is embedded in $A_i$ for $i = 0, 1$. We denote by $SLC(A_0, A_1)$ the union of the spaces $SLC(N; A_0, A_1)$ as $N$ varies.

**Theorem 1.2.** Given two smooth embedded positive Lagrangians $A_0, A_1$, and a connected closed $(n-1)$-manifold $N$, the space of imaginary special Lagrangian cylinders $SLC(N; A_0, A_1)$ is a smooth 1-dimensional manifold.
Remark 1.3. More generally, positive Lagrangians are natural elliptic boundary conditions for imaginary special Lagrangians with boundary of arbitrary topology, and the associated deformation theory is unobstructed. See Remark 4.9. The deformation theory of closed special Lagrangians was shown to be unobstructed in [16, 19].

The next result is a refinement and partial converse to Theorem 1.1 in the case where $\Lambda_i$, $i = 0, 1$, are smooth embedded spheres intersecting transversally at exactly two points. To formulate the result we need the following definition.

Definition 1.4. The cylindrical transform of a geodesic of positive Lagrangians $(\Lambda_t)_{t \in [0, 1]}$ is the subset of $SLC(\Lambda_0, \Lambda_1)$ parameterized by the family of imaginary special Lagrangian immersions $\Phi_c : L_c \to X$ from Theorem 1.1 for $c \in \mathbb{R}$ such that $L_c \neq \emptyset$.

We refer the reader to Definition 5.13 for the notion of regularity of a connected component in $SLC(S^{n-1}; \Lambda_0, \Lambda_1)$.

Theorem 1.5. Let $\Lambda_0, \Lambda_1 \subset X$ be smooth embedded positive Lagrangian spheres intersecting transversally at exactly two points. The cylindrical transform of a geodesic between $\Lambda_0$ and $\Lambda_1$ is a regular connected component in $SLC(S^{n-1}; \Lambda_0, \Lambda_1)$. Conversely, given a regular connected component $\mathcal{Z} \subset SLC(S^{n-1}; \Lambda_0, \Lambda_1)$, there exists a unique up to reparameterization geodesic between $\Lambda_0$ and $\Lambda_1$ with cylindrical transform $\mathcal{Z}$.

Finally, we apply Theorem 1.5 to prove that geodesics of positive Lagrangian spheres with endpoints intersecting transversally at two points are stable under $C^{2,\alpha}$-small Hamiltonian perturbations. Let $\mathcal{G}_\mathcal{O}$ denote the space of geodesics $(\Lambda_t)_{t \in [0, 1]}$ in $\mathcal{O}$ with endpoints intersecting transversally at two points. We refer the reader to Definition 6.6 for the strong and weak $C^{1,\alpha}$ topologies on $\mathcal{G}_\mathcal{O}$. Roughly speaking, the strong topology controls closeness of all cylinders in the cylindrical transform of a geodesic while the weak topology controls closeness of a single cylinder.

Theorem 1.6. Let $\mathcal{O}$ be a Hamiltonian isotopy class of smooth embedded positive Lagrangian spheres, and let $\Lambda_0, \Lambda_1 \in \mathcal{O}$ intersect transversally at exactly two points. Suppose there exists a geodesic $(\Lambda_t)_{t \in [0, 1]}$ between $\Lambda_0$ and $\Lambda_1$. Let $\alpha \in (0, 1)$. Then, there exists a $C^{2,\alpha}$-open neighborhood $\mathcal{Y}$ of $\Lambda_1$ in $\mathcal{O}$ and a weak $C^{1,\alpha}$-open neighborhood $\mathcal{X}$ of $(\Lambda_t)_{t \in [0, 1]}$ in $\mathcal{G}_\mathcal{O}$ such that for every $\Lambda \in \mathcal{Y}$ there exists a unique geodesic between $\Lambda_0$ and $\Lambda$ in $\mathcal{X}$. This geodesic depends continuously on $\Lambda$ with respect to the $C^{2,\alpha}$ topology on $\mathcal{Y}$ and the strong $C^{1,\alpha}$ topology on $\mathcal{X}$.

In the sequel [24], we strengthen Theorem 1.6 to show that if the geodesic $(\Lambda_t)_t$ is of regularity $C^{1,1}$ at the cone points, so are the geodesics connecting $\Lambda_0$ to any
In [25], there are examples of geodesics of positive Lagrangians of arbitrary dimension, many of which satisfy the conditions of Theorem 1.6. However, they are all preserved by an isometric action of $O(n)$ on the ambient manifold $X$. From Theorem 1.6 we obtain the following.

**Corollary 1.7.** There exist geodesics of positive Lagrangians in arbitrary dimension that are not invariant under any isometries of the ambient manifold.

It should be possible to extend Theorems 1.5 and 1.6 to Lagrangians with more complicated topology and more critical points. Furthermore, it should be possible to extend the techniques of this paper to prove the existence of geodesics a priori. In fact, to show the existence of a geodesic and hence an isotopy between two positive Lagrangians, it may not be necessary to assume the existence of a Hamiltonian isotopy between them, but only an intersection point of Maslov index zero. In the sequel [24], we show that there exists a one parameter family of imaginary special Lagrangian cylinders near any such intersection point. It remains to identify situations in which this family of cylinders can be extended until it terminates at an intersection point of Maslov index $n$. When $n = 2$ and $X$ is hyperkähler, the relation between special Lagrangian submanifolds and holomorphic curves should simplify the analysis. It would also be interesting to study the analogy between the results of this paper and the work relating geodesics in the space of Kähler metrics with families of holomorphic disks [5, 9, 21]. We plan to address these points in future work.

1.3. **Outline.** In Section 2, we collect relevant background material on Lagrangian submanifolds, especially in Calabi-Yau manifolds. Particular attention is paid to a generalization of the Weinstein neighborhood theorem adapted to Lagrangians with boundary in a collection of Lagrangians. Section 3 develops a formalism for differential analysis on immersed submanifolds with cone-points. In Section 4, we study spaces of Lagrangian and imaginary special Lagrangian cylinders and prove Theorem 1.2. We define relative Lagrangian flux and several types of regular families of imaginary special Lagrangian cylinders. Section 5 gives the proofs of Theorems 1.1 and 1.5. Finally, Section 6 gives the proof of Theorem 1.6.

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2. **Background**

2.1. **Weinstein neighborhoods and immersed Lagrangian submanifolds with boundary.** Let $N, M$ be smooth manifolds, $M$ perhaps with boundary. Denote by $\text{Diff}(M)$ the diffeomorphisms of $M$ preserving each boundary component. That is, if $\varphi \in \text{Diff}(M)$ and $B \subset \partial M$ is a component, then $\varphi(B) = B$.

**Definition 2.1.** An immersed (resp. embedded) submanifold of $N$ of type $M$ is an equivalence class of immersions (resp. embeddings),

$$K = \left[f : M \to N\right].$$
where the equivalence is with respect to the Diff(M)-action: The immersions \( f \) and \( f' \) are equivalent if there exists \( \varphi \in \text{Diff}(M) \) such that
\[
f' = f \circ \varphi.
\]
We say that \( K = [f] \) is free if \( f \) has trivial isotropy subgroup. We say that \( K \) has boundary if \( M \) does. In this case, to each boundary component of \( M \) we associate a boundary component of \( K \), which is itself an immersed submanifold.

**Definition 2.2.** Let \( K \) be an immersed submanifold of \( N \) of type \( M \). A point \( p \in K \) is an equivalence class of pairs, \( p = [(f, q)] \), where \( f : M \to N \) represents \( K \) and \( q \in M \). The pairs \( (f, q) \) and \( (f', q') \) are equivalent if there exists \( \varphi \in \text{Diff}(M) \) such that
\[
f' = f \circ \varphi, \quad \varphi(q') = q.
\]
For a point \( p = [f, q] \) of \( K = [f] \), we let \( p_0 \) denote the image of \( p \) in \( N \),
\[
p_0 := f(q).
\]
We say that \( p \) is embedded if \( f^{-1}(p_0) = \{q\} \). By abuse of notation, we abbreviate \( p = f(q) \) when this does not lead to confusion. That is, we may consider \( f(q) \) as a point of \( K \) instead of as a point of \( N \).

We define the **tangent space** at \( p \) by
\[
T_p K := df(T_q M) \subset T_{p_0} N.
\]

The tangent bundle \( TK \) is naturally an immersed submanifold of \( TM \). A **differential form** on \( K \) is an equivalence class of pairs \( \eta = [(f, \tau)] \) where \( f \) is a representative of \( K \) and \( \tau \in \Omega^*(M) \). The pairs \((f, \tau)\) and \((f', \tau')\) are equivalent if there exists \( \varphi \in \text{Diff}(M) \) such that \( f' = f \circ \varphi \) and \( \tau' = \varphi^{*}\tau \). An open subset \( K \) is an immersed submanifold of \( N \) of the form \([f_0]\) where \( f : M \to N \) represents \( K \) and \( U \subset M \) is open. For a differential form \( \eta = [(f, \tau)] \) on \( K \), we may write \( \tau = \varphi^{*}\eta \). When \( \eta \) is a zero form, that is, a function, we also write \( \tau = \eta \circ f \).

Let \( K_i \) be immersed submanifolds of \( N_i \) of type \( M_i \) for \( i = 0, 1 \). A **smooth map** \( g : K \to K' \) is an equivalence class of triples \((f_0, f_1, h)\) where \( h : M_0 \to M_1 \) is smooth and \( f_i : M_i \to N_i \) represents \( K_i \). Two such triples \((f_0, f_1, h)\) and \((f_0', f_1', h')\) are equivalent if there exist \( \varphi_i \in \text{Diff}(M_i) \) such that \( f'_i = f_i \circ \varphi_i \) and \( h' = \varphi_1^{-1} \circ h \circ \varphi_0 \).

We say \( g \) is a diffeomorphism, embedding, and so on, if it is represented by a triple \((f_0, f_1, h)\) where \( h \) is a diffeomorphism, embedding, and so on.

**Remark 2.3.** Every point of an immersed submanifold has a neighborhood that is embedded. An embedded submanifold \( K = [f : M \to N] \) is canonically diffeomorphic to the submanifold of type \( f(M) \) represented by the inclusion \( f(M) \hookrightarrow N \). So, the usual notions of points, maps, and so on apply, and we do not need the added complexity of Definition 2.2.

**Lemma 2.4.** Let \( K \) be a connected immersed submanifold of \( N \) of type \( M \), and suppose \( K \) has an embedded point. Then \( K \) is free.

**Lemma 2.5.** Let \( K \) be an immersed submanifold of \( N \) modeled on \( M \). Suppose that for each connected component \( C \subset M \) there is a boundary component \( B \subset \partial C \) such that the corresponding boundary component of \( K \) has an embedded point. Then \( K \) is free.

**Proof.** Let \( f : M \to N \) be a representative of \( K \) and let \( \varphi \in \text{Diff}(M) \) be such that \( f \circ \varphi = f \). Let \( C \subset M \), and \( B \subset \partial C \) be components. Suppose \( q \in B \) is such that \( p = [(f|_B, q)] \) is an embedded point of the immersed submanifold \([f|_B] \).
As \( \varphi \) preserves each boundary component of \( M \), we have \( \varphi(q) \in B \). Since \( p \) is an embedded point of \([f]_B\), the equality

\[
f(\varphi(q)) = f(q)
\]

implies

\[
\varphi(q) = q.
\]

Now, by local injectivity of \( f \), continuity of \( \varphi \) and the equality \( f \circ \varphi = f \), the subset

\[
U := \{ r \in M \mid \varphi(r) = r \} \subset M
\]

is open. As \( U \) is also closed and \( q \in U \), we deduce \( \varphi|_C = \text{id}_C \) and complete the proof. \( \square \)

Let \((X, \omega)\) be a symplectic manifold of dimension \( 2n \), and let \( L \) be a smooth manifold of dimension \( n \), perhaps with boundary.

**Definition 2.6.** An immersion \( f : L \to X \) is said to be Lagrangian if it satisfies \( f^*\omega = 0 \). Let \( f, f' : L \to X \) be immersions such that, for some \( \varphi \in \text{Diff}(L) \), we have \( f' = f \circ \varphi \). Then \( f \) is Lagrangian if and only if \( f' \) is. We say the associated immersed submanifold \([f] \) is Lagrangian if \( f \) is Lagrangian.

**Notation 2.7.** If \( L \) has no boundary, we let \( \mathcal{L}(X, L) \) denote the space of free immersed Lagrangian submanifolds of \( X \) of type \( L \). Suppose now that \( L \) has \( k \) boundary components \( B_1, \ldots, B_k \), for some \( k \in \mathbb{N} \), and let \( \Lambda_1, \ldots, \Lambda_k \subset X \) be fixed Lagrangians. Let \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) denote the space of free immersed Lagrangian submanifolds \( Z \subset X \) of type \( L \) with boundary components \( \Lambda_1, \ldots, \Lambda_k \), corresponding to the boundary components \( B_1, \ldots, B_k \) of \( L \) such that the following hold.

(a) For \( i = 1, \ldots, k \), the boundary component \( \Lambda_i \) is an immersed submanifold of \( \Lambda_i \).

(b) For \( i = 1, \ldots, k \), and \( p \in \Lambda_i \), we have \( T_p Z \neq T_{nu} \Lambda_i \).

The spaces \( \mathcal{L}(X, L) \) and \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) are Fréchet manifolds locally parameterized by spaces of closed 1-forms. The former is treated in [1], [22], [23] and [25] for embedded Lagrangians, and the generalization to free immersed Lagrangians is performed by means of [4, Theorem 1.5]. In order to study the latter, we formulate the Lagrangian boundary condition in terms of differential forms. This is done below by employing the following version of the Weinstein neighborhood theorem. For a smooth manifold \( M \) and a submanifold \( Q \subset M \) we let \( \nu Q \) denote the conormal bundle of \( Q \) in \( T^* M \).

**Lemma 2.8.** Suppose \( L \) is compact with \( k \) boundary components, \( C_1, \ldots, C_k \). Let \( \Lambda_1, \ldots, \Lambda_k \subset X \) be fixed embedded Lagrangian submanifolds and let \( Z = [f : L \to X] \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \). Then there exist an open neighborhood of the zero section, \( V \subset T^* L \), and a local symplectomorphism \( \varphi : V \to X \), such that the following hold.

(a) Identifying \( L \) with the zero section in \( T^* L \), we have

\[
\varphi|_L = f.
\]

(b) For \( i = 1, \ldots, k \), a point \( p \in C_i \) and a covector \( \xi \in T^*_p L \), we have

\[
\varphi(p, \xi) \in \Lambda_i \iff (p, \xi) \in \nu C_i.
\]

**Proof.** We follow the lines of the argument by Moser presented in [15, Section 3.2], making the necessary adaptations. First we construct a smooth subbundle \( E \subset f^* TX \to L \) satisfying the following.

(1) For \( p \in L \), the fiber \( E_p \subset T_{f(p)} X \) is Lagrangian.
(2) We have \( f^*TX = df(TL) \oplus E \), where \( df : TL \to f^*TX \) is the differential of the immersion \( f \).

(3) For \( p \in C_i, i = 1, \ldots, k \), we have

\[
E_p \cap T_{f(p)} \Lambda_i \neq \{0\}.
\]

It then follows that \( \dim (E_p \cap T_{f(p)} \Lambda_i) = 1 \).

To do so, let \( J \) be an \( \omega \)-compatible almost complex structure. Note that, if the Lagrangian \( Z \) were closed, the bundle \( JTZ \subset f^*TX \) would be a good choice for \( E \). In our case, however, \( JTZ \) does not necessarily satisfy condition (3). We thus perform the following perturbation. For \( p \in L \) and any number \( a \in \mathbb{R} \), the linear space

\[
E_{p,a} := (J + a)df(T_p L) \subset T_{f(p)} X
\]
satisfies conditions (1) and (2). Condition (3) determines \( a \) uniquely for \( p \in \partial L \).

Here, we use condition (b) in the definition of \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \). Hence, constructing \( E \) amounts to extending the smooth assignment \( p \mapsto a(p) \) from \( \partial L \) to all \( L \).

Contraction with \( \omega \) along the immersion \( f \) yields an isomorphism of vector bundles \( E \cong T^*L \). We thus think of \( T^*L \) as a subbundle of \( f^*TX \). Condition (3) in the construction of \( E \) implies that for \( p \in C_i, i = 1, \ldots, k \), the annihilator \( (T_p C_i)^0 \subset T^*_p L \) is identified with a line in \( T_{f(p)} \Lambda_i \).

For \( i = 1, \ldots, k \), let \( \nabla^i \) be a connection on \( TX \), with respect to which the Lagrangian \( \Lambda_i \) is totally geodesic. Let \( (\alpha_i : L \to \mathbb{R})_{i=1}^k \) be a smooth partition of unity with

\[
\alpha_i|_{C_i} \equiv 1, \quad i = 1, \ldots, k.
\]

For a connection \( \nabla \) on \( TX \), let \( \exp^\nabla \) denote the exponential map of \( \nabla \). To each \( p \in L \) we assign a connection \( \nabla^p \) on \( TX \) given by

\[
\nabla^p := \sum_i \alpha_i(p) \nabla^i.
\]

Set

\[
\tilde{V} := \{(p, \xi) \in T^*L \mid \exp^\nabla^p_f(\xi) \text{ exists}\}
\]

and define

\[
\tilde{\varphi} : \tilde{V} \to X, \quad (p, \xi) \mapsto \exp^\nabla^p_f(\xi).
\]

Shrinking \( \tilde{V} \) if necessary, \( \tilde{\varphi} \) is an immersion and one-to-one on each fiber. By construction, we have

\[
(4) \quad \tilde{\varphi} (\nu_{C_i} \cap \tilde{V}) \subset \Lambda_i, \quad i = 1, \ldots, k.
\]

Let \( \omega_0 \) denote the canonical symplectic form on \( \tilde{V} \subset T^*L \). Write

\[
\omega_1 := \tilde{\varphi}^* \omega.
\]

Then \( \omega_0 \) and \( \omega_1 \) coincide on \( T\tilde{V}|_L \). Also, by (4), the conormal bundle \( \nu_{C_i} \) is Lagrangian with respect to both forms for \( i = 1, \ldots, k \). We now construct the desired \( V \subset T^*L \) and an open embedding \( \chi : V \hookrightarrow \tilde{V} \), satisfying the following:

(1) \( \chi^* \omega_1 = \omega_0 \).

(2) For \( q \in V \) and \( i = 1, \ldots, k \), we have

\[
\chi(q) \in \nu_{C_i} \iff q \in \nu_{C_i}.
\]

(3) For \( q \in L \subset V \) we have \( \chi(q) = q \).
This will complete the proof since we can take \( \varphi := \tilde{\varphi} \circ \chi \). The embedding \( \chi \) is obtained as the flow of a time-dependent vector field as follows. Let \( H : \tilde{V} \times [0,1] \to \tilde{V}, \quad ((p, \xi), t) \mapsto (p, t\xi) \) and \( \pi : \tilde{V} \times [0,1] \to \tilde{V}, \quad ((p, \xi), t) \mapsto (p, \xi) \).

Let \( \eta \) be the 1-form on \( \tilde{V} \) given by
\[
\eta := \pi_* H^*(\omega_1 - \omega_0),
\]
where \( \pi_* \) denotes pushing forward by integration along the fibers of \( \pi \). For a discussion of the properties of integration along the fiber, see [14, Section 3.1]. Then \( \eta \) satisfies
\[
d\eta = \omega_1 - \omega_0, \quad \eta|_{\mathcal{I}_L \tilde{V}} = 0, \quad \eta|_{\nu_{C_i}} = 0, \quad i = 1, \ldots, k.
\]

Shrinking \( \tilde{V} \) again, we may assume that, for \( t \in [0,1] \), the closed 2-form \( \omega_t := t\omega_1 + (1-t)\omega_0 \) is non-degenerate. For \( t \in [0,1] \), define a vector field \( u_t \) on \( \tilde{V} \) by
\[
i_{u_t} \omega_t = -\eta.
\]

Then \( u_t \) vanishes on \( L \) and is tangent to \( \nu_{C_i} \) for \( i = 1, \ldots, k \). Let \( \chi_t, \ t \in [0,1] \), denote the flow of \( u_t \). That is,
\[
\chi_0 = \text{id}, \quad \frac{d}{dt} \chi_t = u_t \circ \chi_t.
\]

Let \( V \) be the domain of \( \chi := \chi_1 \). By the Cartan formula we have
\[
\frac{d}{dt} \chi_t^* \omega_t = \chi_t^* \left( \frac{d}{dt} \omega_t + i_{u_t} d\omega_t + di_{u_t} \omega_t \right)
= \chi_t^* ((\omega_1 - \omega_0) - d\eta)
= 0.
\]

Hence we have \( \chi^* \omega_1 = \omega_0 \), as desired. \( \square \)

**Definition 2.9.** We call the pair \((V, \varphi)\) of Lemma 2.8 an immersed Weinstein neighborhood of \( Z \) compatible with \( \Lambda_1, \ldots, \Lambda_k \).

We now show the space \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) is a Fréchet manifold. For a manifold \( M \) with boundary, we let \( \Omega^1(M) \) denote the space of smooth 1-forms on \( M \) and \( \Omega^1_\mathbb{B}(M) \subset \Omega^1(M) \) the Fréchet subspace consisting of closed forms annihilating the boundary.

**Corollary 2.10.** Suppose \( L \) is compact with \( k \) boundary components and let \( \Lambda_i \subset X, \ i = 1, \ldots, k, \) be fixed embedded Lagrangians. Then \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) is a Fréchet manifold. In fact, for \( Z \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) an immersed Weinstein neighborhood of \( Z \) compatible with \( \Lambda_1, \ldots, \Lambda_k \), gives rise to a local parameterization
\[
X : \mathcal{U} \subset \Omega^1_\mathbb{B}(L) \to \tilde{\mathcal{U}} \subset \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k).
\]

**Proof.** Let \( Z \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) and choose a Lagrangian immersion \( f : L \to X \) representing \( Z \). Let \((V, \varphi)\) be a compatible immersed Weinstein neighborhood. Set
\[
\mathcal{U} := \{ \alpha \in \Omega^1_\mathbb{B}(L) \mid \text{Graph}(\alpha) \subset V \}.
\]

Then \( \mathcal{U} \) is open in \( \Omega^1_\mathbb{B}(L) \). For \( \alpha \in \mathcal{U} \), write
\[
\mathcal{G}_\alpha := \varphi(\text{Graph}(\alpha)) \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k).
\]

One verifies that the map \( X \) given by \( \alpha \mapsto \mathcal{G}_\alpha \) is one-to-one and onto an open subset in \( \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k) \) equipped with the quotient topology (compare with [4, Theorem 1.5]). Here we rely on freeness of \( Z \). Repeating the above procedure
for a Lagrangian $Z' \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ close to $Z$, the induced transition map $U \to U' \subset \Omega_B^1(L)$ is smooth.

Lemma 2.11 and Remark 2.12 below describe tangent vectors in spaces of Lagrangian submanifolds explicitly. Lemma 2.11 (a) is stated and proved in [1], while part (b) is the analogue for Lagrangians with boundary. Remark 2.12 provides a hands-on approach.

**Lemma 2.11.**

(a) Suppose $L$ is closed. For $Z \in \mathcal{L}(X, L)$, the tangent space $T_Z \mathcal{L}(X, L)$ is canonically isomorphic to the space of closed $1$-forms on $Z$.

(b) Suppose $L$ has $k$ boundary components. For $Z \in \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$, the tangent space $T_Z \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ is canonically isomorphic to the space $\Omega_B^1(Z)$.

**Remark 2.12.** The canonical isomorphism of Lemma 2.11 can be understood as follows. Let $(Z_t)_{t \in (-\epsilon, \epsilon)}$ be a smooth path of Lagrangians diffeomorphic to $L$, which may or may not have boundary. Let $\Psi_t : L \to X$ for $t \in (-\epsilon, \epsilon)$ be a smooth family of immersions such that $\Psi_t$ represents $Z_t$. Then, recalling Definition 2.2, we have

$$
\frac{d}{dt} Z_t = \left[ (\Psi_t, i_{\frac{d}{dt} \Psi_t} \omega) \right] \in \Omega_B^1(Z_t).
$$

**Definition 2.13.**

(a) Suppose $L$ is closed. Let $\Lambda = (\Lambda_t)_{t \in [0, 1]} \subset \mathcal{L}(X, L)$ be a smooth path. We say $\Lambda$ is exact if $\frac{d}{dt} \Lambda_t$ is exact for $t \in [0, 1]$.

(b) Suppose $L$ has $k$ boundary components. Let $Z = (Z_t)_{t \in [0, 1]} \subset \mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ be a smooth path. We say $Z$ is exact relative to the boundary if $\frac{d}{dt} Z_t$ is exact relative to the boundary for $t \in [0, 1]$, that is, if there exists a function $h_t \in C^\infty(Z_t)$ vanishing on $\partial Z_t$ such that $\frac{d}{dt} Z_t = d h_t$.

**Remark 2.14.** Recall that a smooth path of embedded closed Lagrangians is exact if and only if it is induced by a Hamiltonian flow on $X$ [1].

2.2. Calabi-Yau manifolds, special Lagrangians and the space of positive Lagrangians.

**Definition 2.15.** A Calabi-Yau manifold is a quadruple $(X, \omega, J, \Omega)$, where $(X, \omega)$ is a symplectic manifold, $J$ is an $\omega$-compatible integrable complex structure, and $\Omega$ is a nowhere-vanishing holomorphic $(n, 0)$ form (with respect to $J$). In particular, $X$ is a Kähler manifold with the metric $g = g_J = \omega(\cdot, J\cdot)$.

**Remark 2.16.** In the literature, various definitions of Calabi-Yau manifolds can be found. In previous work by the authors, the notion used here would be called an almost Calabi-Yau manifold. A more restrictive definition adds the requirement $\rho \equiv 1$, where $\rho$ is the function defined in (3).

In what follows, we fix a Calabi-Yau manifold $(X, \omega, J, \Omega)$. Recall the following observation of [13].

**Lemma 2.17.** Let $p \in X$ and let $S \subset T_p X$ be an oriented Lagrangian subspace. Then $\Omega$ does not vanish on $S$. In fact, for an oriented basis $v_1, \ldots, v_n \in S$, we have

$$
|\Omega(v_1, \ldots, v_n)| = \rho \text{vol}(v_1, \ldots, v_n),
$$

where vol denotes the Riemannian volume form.
In particular, it follows from Lemma 2.17 that if $\text{Im} \Omega|_{\Lambda} = 0$, then $\text{Re} \Omega|_{\Lambda}$ is non-vanishing. If $\text{Re} \Omega|_{\Lambda} = 0$, then $\text{Im} \Omega|_{\Lambda}$ is non-vanishing.

**Definition 2.18.** Let $p \in X$ and let $S \subset T_p X$ be an oriented Lagrangian subspace. The *phase* $\theta_S \in S^1$ of $S$ is the argument of $\Omega(v_1, \ldots, v_n)$ for $v_1, \ldots, v_n \in S$ an oriented basis. We say that $S$ is positive if $\theta_S \in (-\frac{\pi}{2}, \frac{\pi}{2})$. An oriented Lagrangian submanifold $\Lambda \subset X$ has a *phase function* $\theta_\Lambda : \Lambda \to S^1$ given by $\theta_\Lambda(p) = \theta_{\pi^*\Lambda}$. The Lagrangian $\Lambda$ is positive if all its tangent spaces are positive.

Let $O$ be a Hamiltonian isotopy class of closed embedded positive Lagrangians in $X$ of type $L$, and let $\Lambda \in O$. By Remark 2.14, the tangent space $T_\Lambda O$ consists of exact 1-forms on $\Lambda$. Positivity of $\Lambda$ yields the isomorphism (1), which in turn gives rise to the Riemannian metric $(\cdot, \cdot)$ defined in (2). As shown in [23], the metric $(\cdot, \cdot)$ has a Levi-Civita connection.

**Definition 2.19.** Let $\Lambda = \{\Lambda_t\}_{t \in [0,1]}$ be a smooth path in $O$. A lifting of $\Lambda$ is a smooth path of embeddings $\Psi_t : L \to X$, $t \in [0,1]$, such that $\Psi_t$ represents $\Lambda_t$. A lifting $(\Psi_t)$ is *horizontal* if it satisfies

$$i_{\Psi_t^*} \text{Re} \Omega = 0, \quad t \in [0,1].$$

It is shown in [22] that, for a path $\Lambda$ as above, every embedding representing $\Lambda_0$ extends uniquely to a horizontal lifting. This allows us to describe the Levi-Civita connection of $(\cdot, \cdot)$ as follows. Let $u_t$, $t \in [0,1]$, be a vector field along $\Lambda$. That is,

$$u_t \in \mathcal{C}^\infty(\Lambda_t), \quad t \in [0,1].$$

Let $(\Psi_t)$ be a horizontal lifting of $\Lambda$. Then the covariant derivative of $u_t$ is given by

$$(D \frac{dt}{dt}u_t) \circ \Psi_t = \frac{d}{dt}(u_t \circ \Psi_t).$$

Thus, the path $\Lambda$ is a *geodesic* if and only if it admits a lifting $(\Psi_t)$ and a family of functions $h_t : \Lambda_t \to \mathbb{R}$ satisfying the equations

$$(6) \quad i_{\Psi_t^*} \omega = d(h_t \circ \Psi_t), \quad i_{\Psi_t^*} \text{Re} \Omega = 0, \quad \frac{d}{dt}(h_t \circ \Psi_t) = 0.$$  

This equivalent definition allows us to extend the notion of geodesics to a larger class of Lagrangian paths. Indeed, it makes sense for paths of immersed Lagrangians which are not necessarily embedded or closed. In the present article we also consider geodesics of non-smooth Lagrangians with singular locus consisting of finitely many cone points. We provide an explicit definition of geodesics in a more general form after setting up the necessary theory of submanifolds with cone points.

3. **LAGRANGIANS WITH CONE POINTS**

For the purposes of this article it is natural to consider Lagrangians with *cone points* and paths thereof. In Section 3.1 below we discuss differentiability on general submanifolds with cone points, where no additional geometric structure is assumed. In Section 3.2 we discuss spaces of positive Lagrangians with cone points in a Calabi-Yau manifold.

3.1. **Oriented blowups and cone-immersed submanifolds.**

**Definition 3.1.**

(1) Let $n \in \mathbb{N}$ and let $V$ be a real vector space of dimension $n$. For $0 \neq v \in V$, the *ray* spanned by $v$ is the subset $\{\lambda v \mid \lambda \geq 0\} \subset V$. The *oriented projective space* $\mathbb{P}^+(V)$ is defined to be the set of rays in $V$. As a set, $\mathbb{P}^+(V)$ is naturally identified with the quotient $(V \setminus \{0\})/\mathbb{R}^+$. We equip $\mathbb{P}^+(V)$ with the smooth
structure making this identification a diffeomorphism. In particular, $\mathbb{P}^+(V)$ is diffeomorphic to the sphere $S^{n-1}$. The oriented blowup of $V$ is defined by
\[
\tilde{V} := \{(q,r) \in V \times \mathbb{P}^+(V) \mid q \in r\}.
\]
The blowup projection $\pi : \tilde{V} \to V$ is given by $(q,r) \mapsto q$. We call $E = \pi^{-1}(0)$ the exceptional sphere. Then $\tilde{V}$ is a smooth manifold with boundary $E$. The restricted blowup projection $\pi|_{\tilde{V}\setminus E} : \tilde{V} \setminus E \to V \setminus \{0\}$ is a diffeomorphism.

(2) Let $M$ be an $n$-dimensional smooth manifold and let $p \in M$. Let $p \in U \subset M$ be open with a diffeomorphism $\varphi : U \to \mathbb{R}^n$ carrying $p$ to 0. Let $\tilde{U}$ denote the oriented blowup of $U$ with respect to the vector space structure induced by $\varphi$, let $\pi_U$ denote the blowup projection and let $E$ denote the exceptional sphere. We define the oriented blowup of $M$ at $p$ by gluing $\tilde{U}$ and $M \setminus \{p\}$ along $\pi_U$:
\[
\tilde{M}_p := \tilde{U} \cup (M \setminus \{p\}) / (q,r) \sim q, (q,r) \in \tilde{U} \setminus E.
\]
The projection $\pi_U$ then extends to the global blowup projection $\pi : \tilde{M}_p \to M$. We call $E_p = \pi^{-1}(p)$ the exceptional sphere over $p$. One verifies that everything defined here is independent of $U$ and $\varphi$, and the exceptional sphere $E_p$ is naturally identified with $\mathbb{P}^+(T_pM)$. Once again, the blowup projection $\pi$ restricts to a diffeomorphism
\[
\pi|_{\tilde{M}_p \setminus E} : \tilde{M}_p \setminus E \to M \setminus \{p\}.
\]
(3) Let $M$ be as in (2), and let $S \subset M$ be finite. For $p \in S$, let $\tilde{M}_p$ denote the oriented blowup of $M$ at $p$. The oriented blowup of $M$ at $S$, denoted by $\tilde{M}_S$, is obtained by gluing all the oriented blowups $\tilde{M}_p \setminus \{S \setminus \{p\}\}$, $p \in S$, in the obvious manner. For $p \in S$, the exceptional sphere over $p$ is naturally identified with $\mathbb{P}^+(T_pM)$. The oriented blowup $\tilde{M}_S$ is a smooth manifold with boundary equal to the disjoint union of the exceptional spheres. It comes with a blowup projection $\pi : \tilde{M}_S \to M$ as before. We write
\[
\tilde{M}_S^2 = \tilde{M}_S \setminus \partial \tilde{M}_S
\]
for the interior of $\tilde{M}_S$.

(4) Let $M$ and $S$ be as in (3). Let $p \in S$ and let $\tilde{p} \in E_p$. Cone coordinates at $\tilde{p}$ are a triple $(U,X,\alpha)$ where $\tilde{p} \in U \subset E_p$ is open,
\[
X = (x^1, \ldots, x^m) : U \to \mathbb{R}^m
\]
are local coordinates, and $\alpha : U \times [0,\epsilon) \to \tilde{M}_S$ is a smooth open embedding such that $\alpha(\tilde{q},0) = \tilde{q}$ for $\tilde{q} \in U$. We generally denote the coordinate on $[0,\epsilon)$ by $s$. Given a map $f : M \to N$, we abbreviate
\[
f \circ \pi \circ \alpha(\tilde{q},s) = f(\tilde{q},s) = f(x_1, \ldots, x_m, s), \quad (x_1, \ldots, x_m) = X(\tilde{q}).
\]

Remark 3.2. Let $V$ be an $n$-dimensional real vector space. Given a smooth section $\sigma$ of the quotient map $V \setminus \{0\} \to \mathbb{P}^+(V)$, we identify
\[
[0,\infty) \to \tilde{V}
\]
by $(r,s) \mapsto (s\sigma(r), r)$. Since $\mathbb{P}^+(V) \cong S^{n-1}$, we identify $S^{n-1} \times [0,\infty) \cong \tilde{V}$.

Definition 3.3. Let $M$ be a smooth manifold, let $p \in M$. Let $\sigma$ be a smooth section of the quotient map $T_pM \setminus \{0\} \to \mathbb{P}^+(T_pM) \cong S^{n-1}$. A polar coordinate map centered at $p$ associated with $\sigma$ is a smooth map
\[
\kappa : S^{n-1} \times [0,\epsilon) \to M
\]
satisfying the following.

1. \( \kappa|_{S^{n-1} \times \{0\}} \) is the constant map to \( p \).
2. \( \kappa|_{S^{n-1} \times (0,\epsilon)} \) is an open embedding.
3. For \( r \in S^{n-1} \),
   \[ \frac{\partial \kappa}{\partial s}(r,0) = \sigma(r). \]

An elementary argument shows that for every \( \sigma \) there exist many polar coordinate maps. We may sometimes speak of a polar-coordinate map \( \kappa \) without mentioning \( \sigma \) explicitly. In this case, we refer to the section \( \sigma \) determined by condition (3) as the section associated with \( \kappa \).

The oriented blowup gives rise to the following notions associated with differentiability, which are weaker than the usual ones.

**Definition 3.4.** Let \( M \) and \( N \) be smooth manifolds, let \( p \in M \), and let \( \Psi : M \to N \) be continuous. Let \( \tilde{M}_p, E_p \) and \( \pi : \tilde{M}_p \to M \) as in Definition 3.1.

1. The map \( \Psi \) is said to be cone-smooth at \( p \) if there exists an open \( E_p \subset \tilde{U} \subset \tilde{M}_p \) such that the composition \( \Psi \circ \pi|_U : \tilde{U} \to N \) is smooth.
2. Suppose \( \Psi \) is cone-smooth at \( p \). The cone-derivative of \( \Psi \) at \( p \) is the unique map
   \[ d\Psi_p : T_pM \to T_{\Psi(p)}N \]
   satisfying the equality
   \[ d(\Psi \circ \pi)_\tilde{p} = d\Psi_p \circ d\pi_\tilde{p} \]
   for \( \tilde{p} \in E_p \). One verifies that the cone-derivative is well-defined and homogeneous of degree 1. Also, the restricted map \( d\Psi_p|_{T_pM \setminus \{0\}} \) is smooth. Nevertheless, the cone-derivative is not linear in general.
3. Suppose \( \Psi \) is cone-smooth at \( p \). We say \( \Psi \) is cone-immersive at \( p \) if the restricted map
   \[ d\Psi_p|_{T_pM \setminus \{0\}} : T_pM \setminus \{0\} \to T_{\Psi(p)}N \]
   is a smooth immersion. In particular, in this case we have \( d\Psi_p(v) \neq 0 \) for \( 0 \neq v \in T_pM \).

Recall that the Euler vector field on a real vector space is the radial vector field that integrates to rescaling by \( e^t \). We let \( \varepsilon \) denote the Euler vector field.

**Lemma 3.5.** Let \( \Psi : M \to N \) be cone-smooth at \( p \) and let \( \kappa : S^{n-1} \times [0,\epsilon) \to M \) be a polar-coordinate map at \( p \). Then \( \Psi \) is cone-immersive at \( p \) if and only if

\[ \frac{\partial}{\partial s}(\Psi \circ \kappa)|_{s=0} : S^{n-1} \to T_{\Psi(p)}N \]

is an immersion nowhere tangent to the Euler vector field.

**Proof.** Let \( \sigma \) be the section associated with \( \kappa \). Let \( r \in S^{n-1} \). Observe that

\[ \frac{\partial}{\partial s}(\Psi \circ \kappa)(r,0) = d\Psi_p(\sigma(r)). \]

So, by the chain rule,

\[ d(d\Psi_p)_{\sigma(r)} \circ d\sigma_r = \frac{\partial}{\partial s}(\Psi \circ \kappa)|_{s=0}(r,0). \]

Also,

\[ d(d\Psi_p)_{\sigma(r)}(\varepsilon(\sigma(r))) = \varepsilon(d\Psi_p(\sigma(r))). \]

Finally, observe that

\[ T_{\sigma(r)}T_pM = \mathbb{R}\langle \varepsilon(\sigma(r)) \rangle \oplus d\sigma_r(T_rS^{n-1}). \]
The lemma follows. \hfill \Box

**Lemma 3.6.** In the setting of Definition 3.4, suppose $\Psi$ is cone-immersive at $p$.

(a) There exists an open neighborhood $p \in V \subset M$ such that, for $p \neq q \in V$, we have $\Psi(q) \neq \Psi(p)$.

(b) Let $p \in V \subset M$ as in (a) and let $\tilde{V} := \pi^{-1}(V) \subset \tilde{M}_p$. Then there exists a unique continuous $\tilde{\Psi} : \tilde{V} \rightarrow \tilde{N}_{\Psi(p)}$ satisfying

$$\pi_N \circ \tilde{\Psi} = \Psi \circ \pi|_{\tilde{V}},$$

where $\tilde{N}_{\Psi(p)}$ denotes the oriented blowup of $N$ at $\Psi(p)$ and $\pi_N$ is the associated blowup projection. Moreover, for some open subset $E_p \subset \tilde{V}' \subset \tilde{V}$, the restricted map $\tilde{\Psi}|_{\tilde{V}'}$, is a smooth immersion carrying $E_p$ into $E_{\Psi(p)}$ and satisfying, for $\tilde{p} \in E_p$ and $v \in T_{\tilde{p}}\tilde{M}_p$,

$$d\tilde{\Psi}_p(v) \in T_{\tilde{\Psi}(\tilde{p})}E_{\Psi(p)} \Leftrightarrow v \in T_{\tilde{p}}E_p.$$

**Proof.** As the lemma is local, we may assume $M = \mathbb{R}^{m+1}$ and $N = \mathbb{R}^{n+1}$. Keeping in mind Remark 3.2, we have

$$\Psi \circ \pi : S^m \times [0, \infty) \rightarrow \mathbb{R}^{n+1}, \quad \Psi \circ \pi|_{S^m \times \{0\}} = 0.$$

Let $s$ denote the $[0, \infty)$-coordinate. Since $\Psi$ is cone-immersive, for $r \in S^m$, the one-sided directional derivative $\partial(\Psi \circ \pi)|_{(r, 0)}$ is non-vanishing. So, there exist $\epsilon > 0$, an open half space $H \subset \mathbb{R}^{n+1}$, and a neighborhood $r \in U \subset S^m$, such that $\partial(\Psi \circ \pi)|_{(r', s')} \in H$ for $(r', s') \in U \times [0, \epsilon)$. By the mean-value theorem, $\Psi \circ \pi(r', s') \neq 0$ for $(r', s') \in U \times (0, \epsilon)$. Part (a) now follows from compactness of $S^m$.

Again by Remark 3.2 we have $\tilde{N}_{\Psi(p)} = \mathbb{P}^+ \left( \mathbb{R}^{n+1} \right) \times \{0\}$. Let

$$\mathbb{P}^+(d\Psi_0) : \mathbb{P}^+ \left( \mathbb{R}^{m+1} \right) \rightarrow \mathbb{P}^+ \left( \mathbb{R}^{n+1} \right)$$

denote the oriented projectivization of $d\Psi_0$. By assumption, $\mathbb{P}^+(d\Psi_0)$ is a well-defined smooth immersion. The equality (7) determines the desired $\tilde{\Psi}$ of part (b) uniquely away from the exceptional sphere $\mathbb{P}^+ \left( \mathbb{R}^{m+1} \right) \times \{0\}$. We set

$$\tilde{\Psi}(c, 0) := (\mathbb{P}^+(d\Psi_0)(c), 0), \quad c \in \mathbb{P}^+ \left( \mathbb{R}^{m+1} \right).$$

One verifies by a straightforward computation that this is indeed a smooth continuation of $\tilde{\Psi}|_{\mathbb{P}^+ \left( \mathbb{R}^{m+1} \right) \times \{0\}}$. Also, the derivative $d\tilde{\Psi}(c, 0)$ is one-to-one for $c \in \mathbb{P}^+ \left( \mathbb{R}^{m+1} \right)$. Since being an immersion is an open property, we can find $\tilde{V}'$ as desired. One direction of implication (8) is trivial. To see the other direction, suppose $v \in T_{\tilde{p}}\tilde{M}_p \setminus T_{\tilde{p}}E_p$. Then, $d\pi(v) \neq 0$, so $d\Psi_p \circ d\pi(v) \neq 0$. It follows that $d\tilde{\Psi}_p(v) \notin T_{\tilde{\Psi}(\tilde{p})}E_{\Psi(p)}$ as desired. \hfill \Box

**Definition 3.7.** The map $\tilde{\Psi}$ of Lemma 3.6 (b) is called the **strict transform** of $\Psi|_{\tilde{V}}$.

**Corollary 3.8.** Let $M, N$, be smooth manifolds of equal dimension, let $S \subset M$ be finite and let $(\Psi : M \rightarrow N, S)$ be a cone-immersion. Then $\Psi$ is an open map.

**Proof.** It suffices to prove that for $p \in S$ and every open $p \in V \subset M$ as in Lemma 3.6 (a), the image $\Psi(V)$ contains an open neighborhood of $\Psi(p)$. Indeed, consider the strict transform $\tilde{\Psi} : \tilde{V} \rightarrow \tilde{N}_{\Psi(p)}$ as in Lemma 3.6 (b). Then, $\tilde{\Psi}(E_p) \subset E_{\Psi(p)}$ is open and closed, so $\tilde{\Psi}(E_p) = E_{\Psi(p)}$. By the inverse function theorem, there is an open $E_p \subset U$ with $E_{\Psi(p)} \subset \tilde{\Psi}(U)$ and $\tilde{\Psi}(U)$ open. So, the image under the blowup projection $\pi_N \left( \tilde{\Psi}(U) \right)$ is open, which implies the claim. \hfill \Box
Definition 3.9. Let $M$ and $N$ be smooth manifolds. Let $S \subset M$ be a finite subset and $\Psi : M \to N$ a continuous map.

(a) We say the pair $(\Psi, S)$ is cone-smooth if $\Psi$ is smooth away from $S$ and cone-smooth at every element of $S$.

(b) We say the pair $(\Psi, S)$ is a cone-immersion from $(M, S)$ to $N$ if $\Psi$ is a smooth immersion away from $S$ and cone-immersive at every element of $S$.

(c) Suppose $N = M$ and $\Psi(p) = p$ for all $p \in S$. The pair $(\Psi, S)$ is a cone-diffeomorphism of $(M, S)$ if $\Psi$ is a smooth diffeomorphism away from $S$ and for $p \in S$, the cone derivative $d\Psi|_{T_pM \setminus \{0\}} : T_pM \setminus \{0\} \to T_pM$ is a diffeomorphism onto $T_pM \setminus \{0\}$. We let $\text{Diff}(M, S)$ denote the group of cone-diffeomorphisms of $(M, S)$ that act trivially on the set of connected components.

(d) Let the diffeomorphism group $\text{Diff}(M, S)$ act on cone-immersions from $(M, S)$ to $N$ by composition. A cone-immersed submanifold of $N$ of type $(\Psi, S)$ is an orbit of the $\text{Diff}(M, S)$-action.

(e) Suppose $M$ is orientable and let $\text{Diff}^+(M, S) \circ \text{Diff}(M, S)$ denote the normal subgroup of orientation preserving cone-smooth diffeomorphisms. An orientation on a cone-immersed submanifold $K$ of type $(M, S)$ is an equivalence class of pairs $(O, C)$ where $O$ is an orientation of $M$ and $C$ is a $\text{Diff}^+(M, S)$ orbit inside the $\text{Diff}(M, S)$ orbit $K$. There is a natural $\text{Diff}(M, S)/\text{Diff}^+(M, S)$ action on such pairs and this gives rise to the desired equivalence relation.

Remark 3.10. It follows from Lemma 3.6 that a map $\Psi : (M, S) \to (M, S)$ is a cone-diffeomorphism if and only if it lifts to a diffeomorphism $\Psi : \tilde{M}_S \to \tilde{M}_S$ such that $\tilde{\Psi}(E_p) = E_p$ for $p \in S$.

Remark 3.11. In this article, all cone-immersions are assumed to have embedded cone locus. That is, for a cone-immersion $(\Psi, S)$ and points $c \neq c' \in S$ we have $\Psi(c) \neq \Psi(c')$.

Definition 3.12. Let $K = [(\Psi : M \to N, S)]$ be a cone-immersed submanifold of type $(M, S)$. As in Definition 2.2, a point $p$ in $K$ is an equivalence class of pairs $((\chi, S), q)$, where $(\chi : M \to N, S)$ is a representative of $K$ and $q \in M$. We let $p_0$ denote the image of $p$ in $N$. That is, $p_0 = \chi(q)$ for $((\chi, S), q)$ a representative of $p$. The cone-immersed submanifold $K$ thus has a well-defined cone locus

$$K^C := \{[(\Psi, S), c] | c \in S\}.$$

A cone point is an element of the cone locus. We define the tangent cone of $K$ at a point $p = [(\Psi, S), q]$ to be the cone-immersed submanifold

$$TC_pK := [(d\Psi)_c : T_cM \to T_{p_0}N, \{0\}]$$

of $T_{p_0}N$. The tangent cone $TC_pK$ is indeed a cone, that is, invariant under scalar multiplication. Moreover, it is independent of the choice of $\Psi$. If $\Psi$ is smooth at $q$, then $TC_pK$ is smoothly embedded and recovers the usual notion of tangent space. We define the projective tangent cone of $K$ at $p$ by

$$\mathbb{P}^+(TC_pK) := \mathbb{P}^+(d\Psi)_c : \mathbb{P}^+(T_cM) \to \mathbb{P}^+(T_{p_0}N).$$

This is a smooth immersed sphere in $\mathbb{P}^+(T_{p_0}N)$.

A function on $K$ is an equivalence class of pairs $((\chi, S), f)$, where $(\chi, S)$ is a representative of $K$ and $f$ is a function on $M$. We say the function $h = [(\Psi, S), f]$ is cone-smooth at the point $p = [(\Psi, S), q]$ if $f$ is cone-smooth at $q$. In this case $h$ has
a well-defined cone-derivative \( dh_0 : TC_p K \to \mathbb{R} \), which is a degree-1 homogeneous function.

**Lemma 3.13.** Let \( M, S, \pi : \tilde{M}_S \to M, p \) and \( \tilde{p} \) be as in Definition 3.1(3) and (4). Let \( (U, X, \alpha) \) be cone coordinates at \( \tilde{p} \) and abbreviate \( V = \operatorname{Im}(\alpha) \subset \tilde{M}_S \). Define sections \( e_i \) of the vector bundle \( \pi^*TM|_V \) by

\[
e_i(q, s) := \begin{cases} \frac{1}{2} \partial \tilde{\pi} (q, s), & s \neq 0, \\ \nabla_s \frac{\partial \tilde{\pi}}{\partial s} (q, s), & s = 0, \end{cases} \quad i = 1, \ldots, m,
\]

where \( \nabla \) is an arbitrary connection. Define \( e_0 := \frac{\partial \tilde{\pi}}{\partial s} \). Then \( e_0, \ldots, e_m \), are independent of the choice of \( \nabla \), smooth, and everywhere linearly independent.

The proof of Lemma 3.13 relies on the following elementary observation (compare with [17, Lemma 2.1]).

**Lemma 3.14.** Let \( f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R} \) be smooth with \( f(x, 0) = 0, x \in \mathbb{R}^k \). Then we have \( f(x,s) = s \cdot g(x,s) \) for \( (x,s) \in \mathbb{R}^k \times \mathbb{R} \), where \( g : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R} \) is smooth and satisfies

\[
g(x,0) = \frac{\partial f}{\partial s}(x,0), \quad \frac{\partial g}{\partial s}(x,0) = \frac{1}{2} \frac{\partial^2 f}{\partial s^2}(x,0), \quad x \in \mathbb{R}^k.
\]

**Proof of Lemma 3.13.** As the blowup projection \( \pi \) maps the exceptional sphere \( E_p \) to the point \( p \), we have

\[
\frac{\partial \pi}{\partial \tilde{\pi}} (\tilde{q}, 0) = 0, \quad \tilde{q} \in U, \quad i = 1, \ldots, m.
\]

The sections \( e_i, \quad i = 1, \ldots, m \), are thus well-defined on \( U \) independently of the choice of connection \( \nabla \). They are smooth by Lemma 3.14. One shows the sections \( e_0, \ldots, e_m \), are everywhere linearly independent by using the definition of the blowup projection \( \pi \). \( \square \)

Let \( (\Psi : M \to N, S) \) be a cone-smooth map and let \( p \in S \). Recall that the cone derivative \( d\Psi_p : T_p M \to T_{\Psi(p)} N \) is homogeneous of degree 1 and \( d\Psi_p|_{T_p M \setminus \{0\}} \) is smooth. It follows that for \( v \in T_p M \) and \( \lambda > 0 \), we have \( d(\Psi_p)_v = d(\Psi_p)_v \lambda v \) under the canonical identification \( T_v T_p M \cong T_{\Psi(p)} M \cong T_{\lambda v} T_p M \). Thus, for \( \tilde{p} = [v] \in P^*(T_p M) \), we define

\[
d(\Psi_p)_{\tilde{p}} := d(\Psi_p)_v : T_p M \to T_{\Psi(p)} N.
\]

**Lemma 3.15.** Let \( (\Psi : M \to N, S) \) be a cone-smooth map. Consider the map \( d (\Psi|_{M \setminus S}) : TM|_{M \setminus S} \to \Psi^*TN|_{M \setminus S} \). Pulling back by \( \pi \) gives a map

\[
\pi^* d (\Psi|_{M \setminus S}) : \pi^*TM|_{\tilde{M}_S} \to \pi^*\Psi^*TN|_{\tilde{M}_S}.
\]

This map extends uniquely to a map of bundles \( \tilde{\Psi} : \pi^*TM \to \pi^*\Psi^*TN \). Moreover, for \( p \in S \) and \( \tilde{p} \in E_p \), we have

\[
\tilde{\Psi}_{\tilde{p}} = d(\Psi_p)_{\tilde{p}}.
\]

In particular, if \( \Psi \) is a cone-immersion, then \( \tilde{\Psi} \) is an injective map of vector bundles.

**Proof.** By Lemma 3.13 and using the notation therein, it suffices for the first claim to show that the sections \( \pi^* d\Psi(e_i)|_{s > 0} \) extend to smooth sections of \( \pi^*\Psi^*TN \) for \( i = 0, \ldots, m \) and any choice of cone coordinates. Linearity of \( d\tilde{\Psi} \) follows by continuity. Consider the differential of the blowup projection \( d\pi|_{\tilde{M}_S \setminus E_p} : T\tilde{M}_S|_{\tilde{M}_S} \to \pi^*TM|_{\tilde{M}_S} \). We have

\[
\pi^* d (\Psi|_{M \setminus S}) = \pi^* d (\Psi|_{M \setminus S}) \circ d\pi \circ d\pi^{-1}|_{\tilde{M}_S} = d(\Psi \circ \pi) \circ d\pi^{-1}|_{\tilde{M}_S}.
\]
By definition of cone-smooth, \( d(\Psi \circ \pi) : T\tilde{M}_S \to \Psi^*TN \) is a smooth bundle map. For \( p \in S \), since \( \Psi \circ \pi \) maps \( E_p \) to a point, it follows that \( d(\Psi \circ \pi)|_{\{s=0\}} = 0 \). On the other hand, \( d\pi^{-1}(e_i|_{s>0}) = \frac{\partial}{\partial x^i} \) for \( i = 1, \ldots, m \), and \( d\pi^{-1}(e_0|_{s>0}) = \frac{\partial}{\partial s} \). So, it follows from Lemma 3.14 that \( \pi^* (\Psi(e_i|_{s>0})) \) extends to a smooth section of \( \pi^*\Psi^*TN \), which is \( \tilde{\Psi}(e_i) \). Thus, the map \( \tilde{\Psi} \) is well-defined.

To prove equation (9), we claim that for \( \tilde{q} \in U \),
\[
\tilde{\Psi}(e_i)(\tilde{q}, 0) = \frac{\partial^2 \Psi}{\partial x^i \partial s}(\tilde{q}, 0), \quad i = 1, \ldots, m,
\]
Indeed,
\[
\frac{\partial^2 \Psi}{\partial x^i \partial s}(\tilde{q}, 0) = \frac{\partial^2 \Psi}{\partial s \partial x^i}(\tilde{q}, 0) = \frac{\partial}{\partial s} \left( \tilde{\Psi} \left( \frac{\partial \pi}{\partial x^i} \right) \right)(\tilde{q}, 0) = \tilde{\Psi}(e_i)(\tilde{q}, 0).
\]
Similarly,
\[
\frac{\partial \Psi}{\partial s}(\tilde{q}, 0) = \tilde{\Psi} \left( \frac{\partial \pi}{\partial s} \right)(\tilde{q}, 0) = \tilde{\Psi}(e_0)(\tilde{q}, 0).
\]
On the other hand, writing \( v(\tilde{q}) = e_0(\tilde{q}, 0) \in T_pM \), recalling the definition of the cone derivative \( d\Psi_p \), and identifying \( T_pT_pM \simeq T_pM \), we have
\[
\frac{\partial^2 \Psi}{\partial x^i \partial s}(\tilde{q}, 0) = \frac{\partial}{\partial x^i} d\Psi_p(v(\tilde{q})) = d(d\Psi_p)_v(e_i(\tilde{q}, 0)), \quad i = 1, \ldots, m,
\]
\[
\frac{\partial \Psi}{\partial s}(\tilde{q}, 0) = d\Psi_p(v) = d(d\Psi_p)_v(e_0(\tilde{q}, 0)).
\]
Equation (9) follows. \(\square\)

**Definition 3.16.** Let \( S \subset M \) be a finite subset. The **blowup tangent bundle of** \( (M, S) \) is the bundle \( \tilde{T}\tilde{M}_S := \pi^*TM \to \tilde{M}_S \). When clear from the context, we may omit the subscript \( S \). Let \( (\Psi : M \to N, S) \) be a cone-smooth map. The **blowup differential of** \( \Psi \) is the map
\[
\tilde{\Psi} : \tilde{TM}_S \to \pi^*\Psi^*TN
\]
given by Lemma 3.15.

Let \( M \) be a smooth manifold and let \( S \subset M \) be a finite subset and let \( \pi : \tilde{M}_S \to M \) denote the blowup projection. Given a differential form \( \alpha \) on \( M \) we can pull-back \( \alpha \) as a section of \( \Lambda^*(T^*M) \) to obtain a section of \( \pi^*\Lambda^*(T^*M) \simeq \Lambda^*(\tilde{T}\tilde{M}_S^*) \). We denote this pull-back by \( \pi^{-1}\alpha \). Observe that this pull-back is different from the pull-back of \( \alpha \) as a differential form, \( \pi^*\alpha \), which would be a section of \( \Lambda^*(T^*\tilde{M}_S) \). A similar distinction applies to pull-backs of metrics.

**Definition 3.17.** A **cone-smooth differential form** on \( (M, S) \) is a smooth differential form \( \alpha \) on \( M \setminus S \) such that \( \pi^{-1}\alpha \) extends to a smooth section \( \tilde{\alpha} \) of \( \Lambda^*(\tilde{T}\tilde{M}_S^*) \). We call \( \tilde{\alpha} \) the **blowup form**. We say that \( \alpha \) and \( \tilde{\alpha} \) are closed if \( \alpha \) is closed as a differential form on \( M \setminus S \).

A **cone-smooth Riemannian metric** on \( (M, S) \) is a smooth Riemannian metric \( g \) on \( M \setminus S \) such that \( \pi^{-1}g \) extends to a smooth metric \( \tilde{g} \) on the blowup tangent bundle \( \tilde{T}\tilde{M}_S \). We call \( \tilde{g} \) the **blowup metric**.

Let \( (\Psi : M \to N, S) \) be a cone-smooth map. A **cone-smooth vector field along** \( \Psi \) is a smooth section \( \xi \) of \( (\Psi|_{M \setminus S})^*TN \) such that \( (\pi|_{\tilde{M}_S})^* \xi \) extends to a smooth section \( \tilde{\xi} \) of \( (\Psi \circ \pi)^*TN \). We call \( \tilde{\xi} \) the **blowup vector field**.
Lemma 3.20. Let $(\Psi : M \to N, S)$ be a cone-smooth map, let $\alpha$ be a smooth differential form on $N$ and let $\xi$ be a cone-smooth vector field along $\Psi$. Then $(\Psi|_{M\setminus S})^* \alpha$ and $i_\xi \alpha$ are cone-smooth differential forms on $(M, S)$. If the blowup vector field $\xi$ vanishes on $\partial \tilde{M}_S$, then so does the blowup form $i_\xi \alpha$. If $(\Psi, S)$ is a cone-immersion, and $g$ is a Riemannian metric on $N$, then $(\Psi|_{M\setminus S})^* g$ is a cone-smooth Riemannian metric on $(M, S)$.

Proof. In light of the preceding lemma, we abbreviate
\begin{equation}
(10) \Psi^* := (\Psi|_{M\setminus S})^* \alpha, \quad \Psi^* g := (\Psi|_{M\setminus S})^* g.
\end{equation}

Lemma 3.21. Let $\alpha$ be a cone-smooth differential form on $(M, S)$. Then, the pullback differential form $(\pi|_{\tilde{M}_S})^* \alpha$ extends to a smooth differential form on $\tilde{M}_S$. If the blowup form $\tilde{\alpha}$ vanishes on $\partial \tilde{M}_S$, then so does the extension of $(\pi|_{\tilde{M}_S})^* \alpha$ considered as a section of $\Lambda^* (T^* \tilde{M}_S)|_{\partial \tilde{M}_S}$.

Proof. The dual of the differential of $\pi$ gives a map of vector bundles
\[ d\pi^* : T^* \tilde{M}_S = \pi^* T^* M \to T^* \tilde{M}_S, \]
which induces a map $\Lambda^* (d\pi^*) : \Lambda^* (T^* \tilde{M}_S) \to \Lambda^* (T^* \tilde{M}_S)$. Since
\[ (\pi|_{\tilde{M}_S})^* \alpha = \Lambda^* (d\pi^*) \circ \tilde{\alpha}|_{\tilde{M}_S}, \]
the required extension is given by $\Lambda^* (d\pi^*) \circ \tilde{\alpha}$. The vanishing claim is immediate. \hfill $\square$

We write $\pi^* \alpha$ for the extension of $(\pi|_{\tilde{M}_S})^* \alpha$ given by the preceding lemma.

Lemma 3.22. Let $\alpha$ be a cone-smooth differential 1-form on $(M, S)$ such that there exists a smooth function $f^0 : M \setminus S \to \mathbb{R}$ with $\alpha|_{M\setminus S} = df^0$ and the blowup form $\tilde{\alpha}$ vanishes on $\partial \tilde{M}_S$. Then $f^0$ extends to a cone-smooth function $f$ on $(M, S)$, and $S$ is contained in the critical locus of $f$.

Proof. Since $d(\alpha|_{M\setminus S}) = d(df^0) = 0$, it follows that $d(\pi^* \alpha) = 0$. Lemma 3.21 gives $\pi^* \alpha|_{\partial \tilde{M}_S} = 0$, so $\pi^* \alpha$ is exact. Let $\tilde{f} : \tilde{M}_S \to \mathbb{R}$ be smooth with $d\tilde{f} = \pi^* \alpha$. After possibly adding a constant to $\tilde{f}$, we may assume that $\tilde{f}|_{\tilde{M}_S} = f^0 \circ \pi|_{\tilde{M}_S}$. Again invoking the vanishing of $\pi^* \alpha|_{\tilde{M}_S}$, it follows that $\tilde{f}|_{\partial \tilde{M}_S}$ is locally constant. So, we
take \( f : M \to \mathbb{R} \) to be the unique function such that \( f \circ \pi = \tilde{f} \), and \( f \) is cone-smooth by definition. The vanishing of the cone-derivative of \( f \) at \( S \) follows from the vanishing of \( \pi^*\alpha|_{\partial \tilde{M}_S} \) as a section of \( T^*\tilde{M}_S|_{\partial \tilde{M}_S} \), given by Lemma 3.21. \( \Box \)

**Definition 3.23.** Suppose \( M \) is oriented and let \( \alpha \) be a cone-smooth differential form on \((M, S)\) such that \( \pi^*\alpha \) has compact support. We define the integral of \( \alpha \) by

\[
\int_M \alpha := \int_{\tilde{M}_S} \pi^*\alpha.
\]

**Lemma 3.24.** Let \( \xi \) be a cone-smooth vector field on \((M, S)\) with blowup vector field \( \tilde{\xi} \) vanishing on \( \partial \tilde{M}_S \). Then, there exists a unique smooth vector field \( \tilde{\xi} \) on \( \tilde{M}_S \) such that \( d\pi(\tilde{\xi}|_{\tilde{M}_S}) = \xi \). Moreover, \( \tilde{\xi} \) is tangent to \( \partial \tilde{M}_S \).

Proof. Since \( \pi|_{\tilde{M}_S} \) is a diffeomorphism, there exists a unique vector field \( \tilde{\xi} \) on \( \tilde{M}_S \) such that

\[
(11) \quad d\pi(\tilde{\xi}) = \xi.
\]

To prove the lemma, it suffices to show that \( \tilde{\xi} \) extends smoothly to a vector field \( \tilde{\xi} \) on \( M \) that is tangent to \( \partial M \). To this end, we use Lemma 3.13 and the notation therein. Abbreviate \( V^c = V \cap \tilde{M}_S \). It suffices to show that \( \tilde{\xi}|_{V^c} \) extends to \( V \). Write \( \tilde{\xi}|_{V} = \sum_{i=0}^m \tilde{\xi}_i e_i \). So,

\[
\tilde{\xi}|_{V^c} = \tilde{\xi}_0 V + \sum_{i=1}^m \tilde{\xi}_i \frac{\partial}{\partial x^i}.
\]

Furthermore, write

\[
\tilde{\xi}|_{V^c} = \tilde{\xi}_0 \frac{\partial}{\partial s} + \sum_{i=1}^m \tilde{\xi}_i \frac{\partial}{\partial x^i}.
\]

Equation (11) gives

\[
\tilde{\xi}_0 = \tilde{\xi}_0|_{V^c}, \quad \tilde{\xi}_i = \frac{1}{s} \tilde{\xi}_i|_{V^c}, \quad i = 1, \ldots, m.
\]

Since \( \tilde{\xi} \) vanishes on \( \partial \tilde{M}_S \), it follows that the functions \( \tilde{\xi}_i \) vanish on \( \partial \tilde{M}_S \). By Lemma 3.14, the functions \( \tilde{\xi}_i \) extend smoothly to functions \( \hat{\xi}_i \) on \( V \). Take \( \hat{\xi}_0 = \tilde{\xi}_0 \). Define \( \hat{\xi} \) on \( V \) by

\[
\hat{\xi}|_V = \hat{\xi}_0 \frac{\partial}{\partial s} + \sum_{i=1}^m \hat{\xi}_i \frac{\partial}{\partial x^i}.
\]

Since \( \hat{\xi}_0 = \tilde{\xi}_0 \) vanishes on \( \partial \tilde{M}_S \), it follows that \( \hat{\xi} \) is tangent to \( \partial \tilde{M}_S \). \( \Box \)

**Remark 3.25.** Let \( \Psi \in \text{Diff}(M, S) \). Then Remark 3.10 and the fact that

\[
\pi^*\Psi^*TM = \Psi^*\pi^*TM = \Psi^*\tilde{TM}_S
\]

imply that the blowup differential gives an isomorphism of vector bundles

\[
\hat{d}\Psi : \tilde{TM}_S \cong \Psi^*\tilde{TM}_S.
\]

In particular, cone-smooth diffeomorphisms act by pull-back on cone-smooth differential forms.

**Definition 3.26.** Let \( K = [\Psi : M \to N, S] \) be a cone-immersed submanifold of type \((M, S)\). A cone-smooth differential form on \( K \) is an equivalence class \( \tau = [(\chi, S), \alpha] \) where \((\chi, S)\) represents \( K \) and \( \alpha \) is a cone-smooth differential form on \((M, S)\). Two pairs are equivalent if they belong to the same orbit of the \( \text{Diff}(M, S) \).
action given by Remark 3.25. We may write \( \alpha = \Psi^* \tau \). Given a smooth form \( \eta \) on \( N \), the restriction to \( K \) is the cone-smooth form given by

\[
\eta_{|K} := \left[ (\Psi, S), \Psi^* \eta \right].
\]

We say that \( \tau \) is closed if \( \alpha \) is. We say that \( \tau \) vanishes at the cone locus if \( \bar{\alpha} \) vanishes on \( \partial M_S \). Given an orientation on \( K \), if \( \pi^* \alpha \) has compact support, we define

\[
\int_K \tau := \int_M \alpha.
\]

**Definition 3.27.** Let \( K = [(\Psi : M \to N, S)] \) be a cone-immersed submanifold, let \( p = [(\Psi, S), q] \) be a cone point, and let \( \bar{p} = [\Psi^* (d\Psi_q), \bar{q}] \in \mathbb{P}^+ (TC_p K) \). The tangent space of \( K \) at \( \bar{p} \) is defined by

\[
T_{\bar{p}} K := d\Psi_{\bar{q}} \left( \overline{T M_\bar{q}} \right) \subset T_p N,
\]

which is independent of the choice of representatives. At a smooth point \( p \) of \( K \), we define the tangent space \( T_p K \) as in Definition 2.2.

Let \( h = [(\Psi, S), f] \) be a cone-smooth function on \( K \). Then the differential of \( h \) at \( \bar{p} \) is defined by

\[
dh_{\bar{p}} := df_{\bar{p}} \circ d\Psi_{\bar{q}}^{-1} : T_{\bar{p}} K \to \mathbb{R},
\]

which is independent of choices of representatives.

**Remark 3.28.** An orientation on a cone immersed submanifold \( K \) as in Definition 3.9 (e) is equivalent to a continuously varying orientation on its tangent spaces.

**Definition 3.29.**

1. Let \((f : M \to \mathbb{R}, S)\) be a cone-smooth function, and let \( p \in S \). The point \( p \) is said to be a critical point of \( f \) if the cone-derivative \( df_p : T_p M \to \mathbb{R} \) vanishes identically.
2. Let \( K = [(\Psi : M \to N, S)] \) be a cone-immersed submanifold, let \( h = [(\Psi, S), f] \) be a cone-smooth function on \( K \), and let \( p = [(\Psi, S), q] \) be a cone point. The point \( p \) is a critical point of \( h \) if \( q \) is a critical point of \( f \).

**Remark 3.30.** It follows from Lemma 3.15 that in the situation of part (1) of the preceding definition, if \( p \) is a critical point of \( f \), then \( df_p = 0 \) for all \( \bar{p} \in E_p \). The analogous statement holds in the situation of part (2).

**Definition 3.31.** Let \((f : M \to \mathbb{R}, S)\) be a cone-smooth function, and let \( p \in S \) be a critical point of \( f \).

1. The cone-Hessian of \( f \) at \( p \) is the map

\[
\nabla df : T_p M \to T_p M^* ,
\]

smooth away from 0 and homogeneous of degree 1, defined as follows. By Remark 3.30, the blowup differential \( \tilde{df} \) vanishes on the exceptional sphere \( E_p \subset \overline{M}_S \). So, the restriction of the second covariant derivative \( \nabla \tilde{df} \in \text{Hom} \left( \overline{T M}_S, \overline{T M}_S^* \right) \) to \( E_p \) is independent of the choice of connection. Moreover, \( \nabla \tilde{df} \) vanishes on \( TE_p \subset \overline{T M}_S |_{E_p} \). Recall that a vector \( 0 \neq v \in T_p M \) gives rise to a point \([v] \in \mathbb{P}^+ (T_p M) \simeq E_p \). For \( v \in T_p M \), and \( \bar{v} \in \overline{T [v]} \overline{M}_S \) such that \( d\pi_{[v]} (\bar{v}) = v \), we define

\[
\nabla_v df := \nabla \bar{v} \tilde{df} \in \left( \overline{T M}_S^* \right)_{[v]} = T_p^* M.
\]

2. The critical point \( p \) is said to be degenerate if there exists a tangent vector \( 0 \neq v \in T_p M \) with \( \nabla_v df = 0 \).
Lemma 3.32. Continue with the notation of Definition 3.31. Let \( 0 \neq v \in T_pM \) and write \( \tilde{p} = [v] \). Take local cone coordinates with \( \frac{\partial s}{\partial x^i}(\tilde{p},0) = v \) and let \( e_0, \ldots, e_m \), denote the induced local frame of the blowup tangent bundle as in Lemma 3.13. The cone-Hessian of \( h \) at \( \tilde{p} \) is given by

\[
(\nabla_v df)(e_0(\tilde{p},0)) = \frac{\partial^2 f}{\partial s^2}(\tilde{p},0), \quad (\nabla_v df)(e_i(\tilde{p},0)) = \frac{1}{2} \frac{\partial^3 f}{\partial s^2 \partial x^i}(\tilde{p},0).
\]

Proof. Since \( \partial \tilde{s} \partial x^i(\tilde{q},0) = 0 \), \( \tilde{q} \in U \), Lemma 3.14 gives \( \frac{\partial f}{\partial x^i} = g_i \), where \( g_i \) is smooth and

\[
\frac{\partial g_i}{\partial s}(\tilde{q},0) = \frac{1}{2} \frac{\partial^3 f}{\partial s^2 \partial x^i}(\tilde{q},0).
\]

Observe that for \( s > 0 \), we have

\[
df(e_i) = \frac{1}{s} \frac{\partial f}{\partial x^i} = g_i.
\]

By continuity, we have \( df(e_i) = g_i \) everywhere. Use the connection on \( \hat{T}M_S \) with respect to which the frame \( e_0, \ldots, e_m \), is parallel. Then, for \( i = 1, \ldots, m \),

\[
(\nabla_v df)(e_i(\tilde{p},0)) = \frac{\partial}{\partial s} df(e_i) \bigg|_{(\tilde{p},0)} = \frac{\partial g_i}{\partial s}(\tilde{p},0) = \frac{1}{2} \frac{\partial^3 f}{\partial s^2 \partial x^i}(\tilde{p},0).
\]

The proof of the left-hand equality is similar but easier.

\[
\square
\]

Definition 3.33. Let \( K = [(\Psi : M \to N,S)] \) be a cone-immersed submanifold, let \( h = [(\Psi, S, f)] \) be a cone-smooth function and let \( p = [(\Psi, S, q)] \) be a critical cone point. Let \( \hat{TC}_pK = TC_pK \setminus \{0\} \) denote the punctured tangent cone. The cone-Hessian of \( h \) at \( p \) is the section \( \nabla dh \) of the cotangent bundle of the punctured tangent cone

\[
T^*\hat{TC}_pK \to \hat{TC}_pK
\]

defined as follows. Let \( v = [(df_q, w)] \) be a point of the punctured tangent cone. Observe that \( T_v\hat{TC}_pK \simeq T_vK \) by equation (9). Recall that \( (TM_S)[w] \simeq T_qM \) and by definition \( T_vK = d\Psi[w]((TM_S)[w]) \). We define

\[
\nabla_v dh := \nabla_w df \circ d\Psi^{-1}_w : T_vK \to \mathbb{R}.
\]

The critical cone point \( p \) is degenerate if \( q \) is a degenerate critical point of \( f \).

The following lemma is well-known for extrema of smooth functions. We show it is true also for cone-smooth functions.

Lemma 3.34. Let \( M \) be a smooth manifold, let \( p \in M \), let \( 0 \neq v \in T_pM \) and let \( h : M \to \mathbb{R} \) be cone-smooth at \( p \) with \( dh_p = 0 \). Assume further that \( p \) is an extremum point of \( h \). If the equality \( (\nabla_v dh)(v) = 0 \) holds, then we have \( \nabla_v dh = 0 \). In particular, \( p \) is a degenerate critical point of \( h \) in this case.

Proof. Without loss of generality we assume \( p \) is a minimum of \( h \). Let \( \tilde{p} := [v] \in \mathbb{P}^+(T_pM). \) Let \((U, X, \alpha)\) be local cone coordinates with \( \frac{\partial s}{\partial x^i}(\tilde{p},0) = v \) and let \( e_0, \ldots, e_m \), denote the induced local frame of the blowup tangent bundle as in Lemma 3.13. By the assumption and Lemma 3.32, we have

\[
\frac{\partial^2 h}{\partial s^2}(\tilde{p},0) = 0.
\]

Since \( p \) is a critical point where \( h \) attains a minimum, we have

\[
\frac{\partial^2 h}{\partial s^2}(\tilde{q},0) \geq 0, \quad \tilde{q} \in U.
\]
For $i = 1, \ldots, m$, Lemma 3.32 and equality (12) together with inequality (13) yield

\begin{equation}
(\nabla_v dh) (e_i (\tilde{\rho}, 0)) = \frac{1}{2} \frac{\partial^3 h}{\partial s^2 \partial x^2} \tilde{\rho}, 0) \\
= \frac{1}{2} \frac{\partial^3 h}{\partial x^2 \partial s^2} (\tilde{\rho}, 0) \\
= 0.
\end{equation}

Since $e_0 (\tilde{\rho}, 0) = v$, the assumption and (14) give $\nabla_v dh = 0$, as desired. $\square$

Recall the meaning of a polar coordinate map from Definition 3.3.

**Lemma 3.35.** Let $M$ be a smooth manifold of dimension $m + 1$ and let $p \in M$. Let $h: M \to \mathbb{R}$ be cone-smooth at $p$ such that $h$ is a non-degenerate critical point and an extremum point of $h$. Then there exist a positive $\epsilon$ and a polar coordinate map $\kappa : S^m \times [0, \epsilon) \to M$ centered at $p$ such that for each $s \in (0, \epsilon)$ the restricted map $\kappa|_{S^m \times \{s\}}$ parameterizes a level set of $h$.

**Proof.** Without loss of generality we suppose $h(p) = 0$ is a minimum. Let $\pi : \overline{M}_p \to M$ denote the blowup projection. For simplicity, we write $h$ instead of $h \circ \pi$ and think of $h$ as a function on $\overline{M}_p$. Identify a neighborhood $E_p \subset V \subset \overline{M}_p$ with $\mathbb{P}^+(T_p M) \times [0, \epsilon)$ and let $r$ denote the $[0, \epsilon)$-coordinate. Then we have

$$\frac{\partial h}{\partial r} (\tilde{\rho}, 0) = 0, \quad \tilde{p} \in \mathbb{P}^+(T_p M).$$

By Lemmas 3.32 and 3.34, we have

$$\frac{\partial^2 h}{\partial r^2} (\tilde{\rho}, 0) > 0, \quad \tilde{p} \in \mathbb{P}^+(T_p M).$$

Applying Lemma 3.14 twice and diminishing $\epsilon$ if necessary, we write

$$h (\tilde{p}, r) = r^2 f (\tilde{p}, r),$$

where $f : \mathbb{P}^+(T_p M) \times [0, \epsilon) \to \mathbb{R}$ is smooth and positive. It follows that the function $\sqrt{h} : \mathbb{P}^+(T_p M) \times [0, \epsilon) \to \mathbb{R}$ is smooth. Diminishing $\epsilon$ again if necessary, $\sqrt{h}$ has no critical points. By the implicit function theorem, there exists a diffeomorphism

$$\tilde{\kappa} : S^m \times [0, \epsilon) \to \mathbb{P}^+(T_p M) \times [0, \epsilon)$$

satisfying

$$\sqrt{h} (\tilde{\kappa}(q, s)) = s, \quad (q, s) \in S^m \times [0, \epsilon).$$

We claim that the map $\kappa := \pi \circ \tilde{\kappa}$ has all the desired properties. Let

$$\varpi : T_p M \setminus \{0\} \to \mathbb{P}^+(T_p M)$$

denote the projection. To show that $\kappa$ satisfies property (3) of a polar coordinate map, it suffices to show that $\frac{\partial \varpi}{\partial s} (q, 0) \neq 0$ for $q \in S^m$ and the composition $\varpi \circ \frac{\partial \kappa}{\partial s} (\cdot, 0) : S^m \to \mathbb{P}^+(T_p M)$ is a diffeomorphism. Indeed,

$$\frac{\partial \kappa}{\partial s} (q, 0) = d\varpi \circ \frac{\partial \tilde{\kappa}}{\partial s} (q, 0).$$

Since $\tilde{\kappa}$ is a diffeomorphism, $\frac{\partial \tilde{\kappa}}{\partial s} (q, 0)$ is not tangent to $E_p$. So, $\frac{\partial \kappa}{\partial s} (q, 0) \neq 0$. Finally, $\varpi \circ \frac{\partial \tilde{\kappa}}{\partial s} (\cdot, 0) = \tilde{\kappa}(\cdot, 0)$, which is a diffeomorphism by construction. The remaining properties of $\kappa$ are immediate. $\square$
3.2. Cone-immersed Lagrangians. Let \((X, \omega)\) be a \(2n\)-dimensional symplectic manifold and let \(L\) be a connected \(n\)-dimensional smooth manifold. A cone-immersion \((\Psi : L \to X, S)\) is said to be Lagrangian if it is Lagrangian away from its cone locus. The cone-immersed submanifold represented by a Lagrangian cone-immersion is also said to be Lagrangian. Suppose \(\Lambda = [(\Psi : L \to X, S)]\) is Lagrangian, let \(p \in \Lambda\) be a cone point, and let \(\tilde{p} \in \mathbb{P}^+(TC_p\Lambda)\). As the Lagrangian Grassmannian bundle of \(X\) is a closed subset of the \(n\)-Grassmannian bundle, it follows that \(T_{\tilde{p}}\Lambda\) is a Lagrangian subspace of \(T_{\tilde{p}O}X\). Thus, we have a well-defined phase function \(\theta_{\Psi} : \tilde{L}_S \to S^1\) given by \(\theta_{\Psi}(q) = \theta_{\frac{d\Psi_t}{dt}((T\tilde{L}_S)_{\Psi_t})}\).

We wish to study paths of cone-immersed Lagrangians with static cone locus. For a finite subset \(C_0 \subset X\), we let \(\mathcal{L}(X, L; S, C_0)\) denote the space of oriented cone-immersed Lagrangians in \(X\) of type \((L, S)\) with cone locus image equal to \(C_0\). For a path \(\Lambda = (\Lambda_t)_{t \in [0, 1]}\) in \(\mathcal{L}(X, L; S, C_0)\), a lifting of \(\Lambda\) is a family of cone-immersions, \([(\Psi_t : L \to X, S)_{t \in [0, 1]}\), such that \((\Psi_t, S)\) represents \(\Lambda_t\) for \(t \in [0, 1]\). The path \(\Lambda\) is smooth if it admits a smooth lifting, that is, if the family of maps \(\Psi_t \circ \pi\) is smooth, where \(\pi : \tilde{L}_S \to L\) is the blowup projection. Given a smooth path \(\Lambda = (\Lambda_t)_{t \in [0, 1]}\), we define a family of 1-forms \(\sigma_t\) on \(\Lambda_t\) as follows. Let \([(\Psi_{t, S} : L \to X, S)_{t \in [0, 1]}\) be a smooth lifting. We abbreviate
\[
\frac{d}{dt}\Psi_t := \frac{d}{dt}\Psi_t|_{L \setminus S},
\]
which is a cone-smooth vector field along \(\Psi_t\). Moreover, the blowup vector field \(\frac{d}{dt}\Psi_t\) vanishes on \(\partial\tilde{L}_S\). We define
\[
\sigma_t := \left[\left(\Psi_t, S\right), i_{\frac{d\Psi_t}{dt}}\omega\right].
\]

Lemma 3.36. The form \(\sigma_t\) is independent of the choice of \(\Psi_t\), closed, and vanishes at the cone locus of \(\Lambda_t\).

Proof. The proof that \(\sigma_t\) is independent of \(\Psi_t\) and closed is analogous to the proof of [1, Lemma 2.1]. Lemma 2.30 implies that \(\sigma_t\) vanishes at the cone locus of \(\Lambda_t\). 

We call \(\sigma_t\) the time derivative and write
\[
\frac{d}{dt}\Lambda_t := \sigma_t.
\]
A path of cone-immersed Lagrangians is said to be exact if its time-derivative is the differential of a cone-smooth function \(\frac{d}{dt}\Lambda_t = dh_t\). In this case, it follows that every cone point is a critical point of \(h_t\).

Suppose \((X, \omega, J, \Omega)\) is Calabi-Yau. An oriented cone-immersed Lagrangian \(\Lambda \in \mathcal{L}(X, L; S, C_0)\) is positive if the tangent space \(T_p\Lambda\) is positive for each smooth point \(p\), and for each cone point \(p\) and \(\tilde{p} \in \mathbb{P}^+(TC_p\Lambda)\) the tangent space \(T_{\tilde{p}}\Lambda\) is positive. Note that this is stronger than positivity at the smooth locus. Assume now that \(L\) is closed. Let \(\mathcal{O} \subset \mathcal{L}(X, L; S, C_0)\) be an exact isotopy class of positive cone-immersed Lagrangians and let \(\Lambda = [(\Psi : L \to X, S)] \in \mathcal{O}\). Recall Definition 3.26. As the blowup \(\tilde{L}_S\) is compact, the volume form \(\text{Re} \Omega|_\Lambda\) is integrable. Let \(C^\infty(\Lambda)\) denote the space of cone-smooth functions on \(\Lambda\). Set
\[
C^\infty(\Lambda) := \left\{h \in C^\infty(\Lambda) \left| \int_\Lambda h \text{Re} \Omega = 0, \forall c \in \Lambda^c dh_c = 0 \right. \right\}.
\]
Then the isomorphism (1) and Riemannian metric (2) make sense as in the smooth case. Let \((\Lambda_t)\) be a smooth path in \(\mathcal{O}\) and let \((\Psi_t)_{t \in [0, 1]}\) be a smooth lifting. We say \((\Psi_t)_{t \in [0, 1]}\) is horizontal if it satisfies \(i_{\frac{d\Psi_t}{dt}}\text{Re} \Omega = 0\). It is shown in [22] that every compactly supported path of smooth positive Lagrangians admits horizontal liftings. We show the same for cone-immersed positive Lagrangians.
Lemma 3.37. Let \((\Lambda_t)_{t \in [0,1]}\) be a smooth path in \(\mathcal{O}\), and let \((\Psi, S)\) be a representative of \(\Lambda_0\). Then \((\Psi, S)\) extends uniquely to a horizontal lifting of \((\Lambda_t)_t\).

Proof. We imitate the argument presented in [22]. Let \((\langle \Psi_t, S \rangle)_t\) be a smooth lifting of \((\Lambda_t)_t\) with \(\Psi_0 = \Psi\). For \(t \in [0,1]\), let \(w_t\) denote the unique vector field on \(L \setminus S\) satisfying

\[
i_{w_t} \Psi_t^* \Re \Omega = -i_{\hat{\Psi}_t} \Re \Omega.
\]

We claim that \(w_t\) is cone-smooth and the blowup vector field \(\tilde{w}_t\) vanishes on \(\partial \tilde{L}_S\). Indeed, Lemma 3.20 implies that the differential forms \(\Psi_t^* \Re \Omega\) and \(i_{\hat{\Psi}_t} \Re \Omega\) are cone-smooth and the blowup differential form \(i_{\tilde{\Psi}_t} \Re \Omega\) vanishes on \(\partial \tilde{L}_S\). Moreover, since \(\Lambda_t\) is positive, it follows from Lemma 3.15 that the blowup form \(\Psi_t^* \Re \Omega\) is non-vanishing. Let \(\tilde{w}_t\) be the unique section of \(\tilde{T}_L\) such that

\[
i_{\tilde{w}_t} \Psi_t^* \Re \Omega = -i_{\hat{\Psi}_t} \Re \Omega.
\]

Then, \(\tilde{w}_t|_{\tilde{L}_S} = \pi|_{\tilde{L}_S} w_t\). So, \(w_t\) is cone-smooth and \(\tilde{w}_t\) is the blowup vector field. Furthermore, \(\tilde{w}_t\) vanishes on \(\partial \tilde{L}_S\) since \(i_{\tilde{\Psi}_t} \Re \Omega\) vanishes on \(\partial \tilde{L}_S\).

By Lemma 3.24 there exists a unique vector field \(\tilde{w}_t\) on \(\tilde{L}_S\) such that

\[
d\pi \left( \tilde{w}_t|_{\tilde{L}_S} \right) = w_t.
\]

Moreover, \(\tilde{w}_t\) is tangent to \(\partial \tilde{L}_S\). Let \((\tilde{\varphi}_t)_t\) denote the flow of \((\tilde{w}_t)_t\). Then, \(\tilde{\varphi}_t(\partial \tilde{L}_S) = \partial \tilde{L}_S\), and \(\tilde{\varphi}_t\) descends to a map \(\tilde{\varphi}_t : M \to M\). Remark 3.10 implies that \(\varphi_t\) is cone-smooth. Thus, the family of compositions \((\Psi_t \circ \varphi_t)_t\) gives the desired horizontal lifting.

By virtue of Lemma 3.37, the Levi-Civita connection described in Section 2.2 extends naturally to exact isotopy classes of cone-immersed positive Lagrangians. We use horizontal lifts to define geodesics in such classes. The definition below is more general than that used in previous works, as it allows the Lagrangians in question to be non-closed or non-smooth. Note, however, that it is equivalent to the old definition when the Lagrangians in question are smoothly embedded and closed.

Definition 3.38. Let \((X, \omega, J, \Omega)\) be a Calabi-Yau manifold, let \(L\) be a connected smooth manifold, not necessarily closed, and let \(S \subset L\) be a finite subset. Let \(C_0 \subset X\) be finite, let \(\mathcal{O} \subset \mathcal{L}(X, L; S, C_0)\) be an exact isotopy class of cone-immersed Lagrangians, and let \((\Lambda_t)_{t \in [0,1]}\) be a path in \(\mathcal{O}\). The path \((\Lambda_t)_t\) is a geodesic if it admits a horizontal lifting \((\langle \Psi_t, S \rangle)_{t \in [0,1]}\) and a family of functions \(h_t \in C^\infty(\Lambda_t)\) satisfying

\[
\frac{d}{dt} \Lambda_t = dh_t,
\]

We call the family \((h_t)_{t \in [0,1]}\) the Hamiltonian of the geodesic. We also call the time independent function \(h = h_t \circ \Psi_t : L \to \mathbb{R}\) the Hamiltonian with respect to the horizontal lifting \((\Psi_t)_t\). Observe that \(h_t = [\langle \Psi_t, h \rangle]_t\). If \(L\) is not compact, the Hamiltonian is only well-defined up to a time independent constant. If \(C_0\) is empty, we say \((\Lambda_t)_t\) is a smooth geodesic or geodesic of smooth Lagrangians.

From now on, unless otherwise specified, the term geodesic will be used in the sense of the preceding definition.

Lemma 3.39. Let \((\Lambda_t)_t\) be a path in \(\mathcal{L}(X, L; S, C_0)\) and let \((\Lambda^\circ_t)_t\) be the path in \(\mathcal{L}(X, L \setminus S)\) obtained by removing the cone points. If \((\Lambda^\circ_t)_t\) is a geodesic, then so is \((\Lambda_t)_t\).

Proof. This follows from Lemma 3.22 and Lemma 3.36.
4. LAGRANGIAN AND SPECIAL LAGRANGIAN CYLINDERS

4.1. The space of Lagrangian cylinders. We start this section with the definition of its main objects.

Definition 4.1.

(a) Let $(X, \omega)$ be a symplectic manifold, and let $\Lambda_0, \Lambda_1 \subset X$ be smooth embedded Lagrangians. A Lagrangian cylinder between $\Lambda_0$ and $\Lambda_1$ is a smooth immersed Lagrangian submanifold with boundary, $Z = [f : L \to X] \in \mathcal{L}(X, L; \Lambda_0, \Lambda_1)$, where $L = N \times [0, 1]$ for some smooth manifold $N$, and the restricted immersion $f|_{N \times \{i\}}$ is an embedding into $\Lambda_i$ for $i = 0, 1$. We let $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ denote the space of Lagrangian cylinders between $\Lambda_0$ and $\Lambda_1$ of type $N \times [0, 1]$. We let $\mathcal{LC}(\Lambda_0, \Lambda_1)$ denote the space of Lagrangian cylinders between $\Lambda_0$ and $\Lambda_1$ of general topological type.

(b) If $X$ is Calabi-Yau and $\Lambda_0$ and $\Lambda_1$ are positive, we let $\mathcal{SLC}(N; \Lambda_0, \Lambda_1) \subset \mathcal{LC}(N; \Lambda_0, \Lambda_1)$ denote the subspace consisting of imaginary special Lagrangian cylinders. The space $\mathcal{SLC}(\Lambda_0, \Lambda_1)$ is defined analogously.

Remark 4.2.

(1) Let $Z = [f : N \times [0, 1] \to X]$ as in Definition 4.1 (a). By Lemma 2.5 the requirement that $f|_{N \times \{i\}}$ is an embedding implies that $f$ is free as required in Notation 2.7.

(2) Let $X, \Lambda_0, \Lambda_1$, be as in Definition 4.1 (b). Let $Z = [f : N \times [0, 1] \to X]$ be an immersed imaginary special Lagrangian with the boundary component corresponding to $N \times \{i\}$ in $\Lambda_i$. Then $Z$ automatically satisfies condition (b) in Notation 2.7.

It is well-known that special Lagrangian submanifolds can be described as solutions to an elliptic PDE (see [2, 13] and the references therein or [16, 19] for a geometric approach). We show below that any pair of positive Lagrangian submanifolds, $\Lambda_0$ and $\Lambda_1$, provides an elliptic boundary condition to the imaginary special Lagrangian equation. In particular, we shall see in Proposition 4.7 that, if $N$ is compact, $\mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ is a smooth 1-dimensional submanifold of the Fréchet manifold $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$. Our approach is similar to that of McLean [16] with some necessary adaptations.

Fix two Lagrangian submanifolds, $\Lambda_0, \Lambda_1 \subset X$, and a compact $n-1$-dimensional manifold $N$. By Corollary 2.10, the space $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$ is a Fréchet manifold modeled locally on $\Omega^1_B(N \times [0, 1])$, the space of closed 1-forms on $N \times [0, 1]$ annihilating the boundary. Moreover, by Lemma 2.11 (b), for $Z \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$ we have a canonical isomorphism

$$T_Z\mathcal{LC}(N; \Lambda_0, \Lambda_1) \cong \Omega^1_B(Z).$$

The following observation allows us to replace the spaces $\Omega^1_B(N \times [0, 1])$ and $\Omega^1_B(Z)$ with spaces of functions.

Lemma 4.3. Let $N$ be a smooth manifold without boundary, and let $\sigma \in \Omega^1(N \times [0, 1])$ be closed with pullback to the boundary component $N \times \{0\}$ zero. Then $\sigma$ is exact.

Proof. Let $\pi_0 : N \times [0, 1] \to N \times [0, 1]$ be given by $(p, t) \mapsto (p, 0)$. Let $\gamma : S^1 \to N \times [0, 1]$ be a smooth loop, and write $\gamma_0 := \pi_0 \circ \gamma$. Then $\gamma_0$ is homotopic to $\gamma$. As $\sigma$ is closed and annihilates the boundary component $N \times \{0\}$, we have

$$\int_{S^1} \gamma^* \sigma = \int_{S^1} \gamma_0^* \sigma = 0,$$

and the lemma follows. \qed

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In the rest of this section, we assume $N$ is connected. By Lemma 4.3, every differential form in $\Omega^1_B(N \times [0, 1])$ is exact with primitive constant on each boundary component. So, by Corollary 2.10 we obtain the following lemma. For a smooth differential form in $\Omega^k_X$, let $\Lambda$ and $Z$ component. So, by Corollary 2.10 we obtain the following lemma. For a smooth differential form in $\Omega^k_X$, let $\Lambda$ and $Z$

Moreover, we have a canonical isomorphism

$x \in U \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$. Then any immersed Weinstein neighborhood of $Z$ compatible with $\Lambda_0$ and $\Lambda_1$ gives rise to a local parameterization

$$\mathbf{X} : U \subset \mathcal{C}^{\infty}_{\text{COB}}(Z) \to \tilde{U} \subset \mathcal{LC}(N; \Lambda_0, \Lambda_1).$$

Moreover, we have a canonical isomorphism

$$T_{\mathbf{X}} \mathcal{LC}(N; \Lambda_0, \Lambda_1) \cong \mathcal{C}^{\infty}_{\text{COB}}(Z).$$

Continuing with the same setting, we now assume further that $(X, \omega, J, \Omega)$ is Calabi-Yau, $\Lambda_0$ and $\Lambda_1$ are positive and $Z \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$.Abbreviate $\mathcal{L} = N \times [0, 1]$. Fix an immersion $f : L \to X$ representing $Z$. By Lemma 2.8, choose an immersed Weinstein neighborhood $(V, \varphi)$ of $Z$ compatible with $\Lambda_0$ and $\Lambda_1$, where $V \subset T^*L$ and $\varphi : V \to X$ with $\varphi|_L = f$. Let $\pi_L : T^*L \to L$ denote the projection. For $u \in \mathcal{C}^{\infty}_{\text{COB}}(L)$, let $\text{Graph}(du) \subset T^*L$ denote the graph. Write

$$U := \{ u \in \mathcal{C}^{\infty}_{\text{COB}}(L) | \text{Graph}(du) \subset V \}.$$

For $u \in U$, let $j_u : L \to X$ be given by

$$j_u = \varphi \circ (\pi_L|_{\text{Graph}(du)})^{-1}.$$  

Define

$$F : U \to C^{\infty}(L), \quad u \mapsto *j_u^* \text{Re} \Omega,$$

where $*$ denotes the Hodge star operator of $f^*g$. Then, for $u \in U$, the cylinder $[j_u]$ is imaginary special Lagrangian if and only if the function $u$ satisfies $F(u) = 0$. We have thus established a local characterization of $\mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ as the zero set of a differential operator. Since $Z \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$, we have $F(0) = 0$. We compute the linearization of $F$ in Lemma 4.5 below. By Lemma 2.17 we have

$$f^* \text{Im} \Omega = \rho_f \text{vol}_f,$$

where $\text{vol}_f$ denotes the Riemannian volume form of $f^*g$ and $\rho_f := \rho \circ f$ with $\rho$ the positive function defined in (3).

**Lemma 4.5.** Consider the above setting.

(a) Let $f_t : L \to X$, $t \in (-\epsilon, \epsilon)$, be a smooth family of Lagrangian immersions with $f_0 = f$. Write $v := \frac{d}{dt}|_{t=0} f_t$ and suppose we have $i_\mathbf{X} \omega = du$ for some $u \in C^{\infty}(L)$. Then

$$\frac{d}{dt}|_{t=0} (*f_t^* \text{Re} \Omega) = *d(\rho_f * du).$$

(b) The linearization of the operator $F$ of (15) at $0 \in U$ is given by

$$dF_0(u) = *d(\rho_f * du).$$
Proof. We prove part (a). Decompose $v$ into $v = v^T + v^\perp$, where $v^T$ is tangent to $Z$ and $v^\perp$ is orthogonal to $Z$ with respect to the Kähler metric $g$. As $Z$ is Lagrangian, we have

$$i_{v^\perp} \omega = i_v \omega = du,$$

which implies

(17) $v^\perp = -J df(\nabla u)$,

where $\nabla$ denotes the gradient with respect to $f^*g$. By (16) and (17), as $Z$ is imaginary special Lagrangian and $\Omega$ is complex-linear, we have

(18) $i_v \Re \Omega = i_{v^\perp} \Re \Omega = i_{\nabla u} f^* \Im \Omega = \rho f i_{\nabla u} \vol f = \rho f^* du$.

Finally, as $\Omega$ is closed, from the Cartan formula and (18) we deduce

$$\ast \left( \frac{d}{dt} \bigg|_{t=0} f^* \Re \Omega \right) = \ast di_v \Re \Omega = \ast d(\rho f^* du),$$

as desired. Part (b) is a particular case of (a). \hfill \Box

In light of Lemma 4.5 we define the linear second-order operator

(19) $\Delta_\rho : C^\infty_{\text{COB}}(L) \to C^\infty(L)$, $u \mapsto \ast d(\rho f^* du)$.

For $Z \in SLC(N; \Lambda_0, \Lambda_1)$, and $f : L \to X$ an immersion with $Z = [f]$, we have a canonical identification $C^\infty(Z) = C^\infty(L)$, so we obtain an operator

$\Delta_\rho : C^\infty_{\text{COB}}(Z) \to C^\infty(Z)$.

This operator does not depend on the choice of $f$. The operator $\Delta_\rho$ is similar to the usual Riemannian Laplacian in a manner made precise in Lemma 4.6 below.

For $k \geq 0$ and $\alpha \in (0, 1)$ we let $C^{k,\alpha}_{\text{COB}}(L)$ denote the completion of $C^\infty_{\text{COB}}(L)$ with respect to the Hölder $C^{k,\alpha}$-norm. We let $C^{k,\alpha}(L; \partial L)$ denote the space of functions on $L$ of regularity $C^{k,\alpha}$ which vanish on the boundary. For $P, Q$, smooth manifolds, we let $C^{k,\alpha}(P, Q)$ denote the smooth Banach manifold of maps $P \to Q$ of regularity $C^{k,\alpha}$.

Lemma 4.6. Let $k$ be a non-negative integer and let $\alpha \in (0, 1)$. Then the naturally extended linear operator $\Delta_\rho : C^{k+2,\alpha}_{\text{COB}}(L) \to C^{k,\alpha}(L)$ is surjective with a 1-dimensional kernel consisting of $C^\infty$ functions.

Proof. We note first that the principal symbol of $\Delta_\rho$ differs from that of the usual Riemannian Laplacian by the positive function $\rho f$. Hence $\Delta_\rho$ is elliptic. Also, $\Delta_\rho$ annihilates constants. Equivalently, expressing $\Delta_\rho$ in local coordinates as

$$\Delta_\rho u = a^{ij} u_{ij} + b^i u_i + cu,$$

the coefficient $c$ vanishes. As $L$ is compact, it now follows from standard arguments (see, for example, [12, Chapter 6] and [26, Section 5.1]) that

$\Delta_\rho : C^{k+2,\alpha}(L; \partial L) \to C^{k,\alpha}(L)$

is one-to-one and surjective. Finally, for $a \in \mathbb{R}$, there exists a unique $u \in C^{k+2,\alpha}_{\text{COB}}(L)$ satisfying

$$\Delta_\rho u = 0, \quad u|_{N \times \{1\}} = a,$$
which is in fact $C^\infty$. The lemma follows.

Proposition 4.7. Let $\Lambda_0, \Lambda_1 \subset X$ be smooth embedded positive Lagrangians, let $N$ be a connected closed smooth manifold, and abbreviate $L = N \times [0, 1]$. Then the space $\mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ is a smoothly embedded 1-dimensional submanifold of $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$. Moreover, for $Z = [f : L \to X] \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$, we have

$$T_Z \mathcal{SLC}(N; \Lambda_0, \Lambda_1) = \ker \Delta_p,$$

where $\Delta_p : C^\infty_{\text{COB}}(L) \to C^\infty(L)$ is defined as in (19).

Proof. Let $Z = [f : L \to X] \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$. Recall the definition of the operator $F : U \to C^\infty(L)$ from (15). Pick $\alpha \in (0, 1)$ and let $U^{2, \alpha} \subset C^2_{\text{COB}}(L)$ be an open set with $U^{2, \alpha} \cap C^\infty_{\text{COB}}(L) = U$. Extend the operator $F$ to an operator $F : U^{2, \alpha} \to C^\alpha(L)$.

Since $\Omega$ is smooth, and the map $U^{2, \alpha} \to C^{1, \alpha}(L, X)$, $u \mapsto j_u$, is smooth, it follows that the extended operator $F$ is smooth. Since $Z$ is imaginary special Lagrangian, $F(0) = 0$. By Lemmas 4.5 and 4.6, the linearization

$$dF_0 = \Delta_p : C^2_{\text{COB}}(L) \to C^\alpha(L)$$

is onto with a 1-dimensional kernel. By the implicit function theorem, there exist $\epsilon > 0$, an open $0 \in W \subset C^2_{\text{COB}}(L)$ and a smooth embedding $\gamma : (-\epsilon, \epsilon) \to W$ with $\gamma(0) = 0$, such that for $f \in W$, we have $F(f) = 0$ if and only if $f = \gamma(t)$ for some $t \in (-\epsilon, \epsilon)$. The path $\gamma$ satisfies

$$\Delta_p(\gamma(0)) = \frac{d}{dt} \bigg|_{t=0} F(\gamma(t)) = 0.$$

By elliptic regularity (e.g. [12, Chapter 17]), for $t \in (-\epsilon, \epsilon)$ the function $\gamma(t)$ is in fact in $C^\infty_{\text{COB}}(L)$. Moreover, for every $k \geq 2$, the above argument with 2 replaced by $k$ shows that $\gamma$ is smooth as an embedding into the space $C^k_{\text{COB}}(L)$. It follows that $\gamma$ is smooth as a map into $C^\infty_{\text{COB}}(L)$.

Proof of Theorem 1.2. Follows from Proposition 4.7.

For a Lagrangian cylinder $Z \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$, let $Z^H \subset \mathcal{LC}(N; \Lambda_0, \Lambda_1)$ denote the space of Lagrangian cylinders exact isotopic to $Z$ relative to the boundary. Then we have

$$T_Z Z^H = C^\infty(Z; \partial Z) \subset C^\infty_{\text{COB}}(Z),$$

the space of smooth functions vanishing on the boundary. As we have

$$C^\infty_{\text{COB}}(Z) = C^\infty(Z; \partial Z) \oplus \ker \Delta_p,$$

the following is a consequence of Proposition 4.7.

Corollary 4.8. Let $\Lambda_0, \Lambda_1 \subset X$ be smooth embedded positive Lagrangians and let $Z \in \mathcal{LC}(N; \Lambda_0, \Lambda_1)$. Then the intersection $Z^H \cap \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ is a discrete subset of $\mathcal{LC}(N; \Lambda_0, \Lambda_1)$.

Remark 4.9. Using the above technique, one can generalize Proposition 4.7 to spaces of imaginary special Lagrangians modeled by an arbitrary manifold $L$ with any number of boundary components. Namely, recalling Notation 2.7, for $\Lambda_1, \ldots, \Lambda_k \subset X$ positive Lagrangians, let

$$\mathcal{SL}(X, L; \Lambda_1, \ldots, \Lambda_k) \subset \mathcal{LC}(X, L; \Lambda_1, \ldots, \Lambda_k)$$

denote the subspace of imaginary special Lagrangians. As Lemma 4.3 fails to hold for arbitrary manifolds $L$, one considers the space $\Omega^B_p(L)$ as a local model.
of $\mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ as in Corollary 2.10. Special Lagrangian submanifolds near $Z \in \mathcal{S}\mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ are then parameterized by $\Delta_\rho$-harmonic 1-forms on $L$, where $\Delta_\rho$ in this case is a modified Hodge Laplacian. It follows from the Hodge decomposition for manifolds with boundary (see [20], [26, Section 5.9] and the references therein) that $\mathcal{S}\mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ has the dimension of the real relative cohomology space $H^1(L, \partial L)$. Similarly to Corollary 4.8, the intersection $Z^H \cap \mathcal{S}\mathcal{L}(X, L; \Lambda_1, \ldots, \Lambda_k)$ is discrete.

4.2. Relative Lagrangian flux. Let $(X, \omega)$ be a symplectic manifold. Recall the notion of Lagrangian flux [3, 11, 22]. Roughly speaking, given a path of closed Lagrangian submanifolds, $\Lambda_t \subset X$, $t \in [0, 1]$, its Lagrangian flux is a linear functional on $H_1(\Lambda_0)$ which measures the path’s deviation from being exact. Suppose now that $\Lambda_0, \Lambda_1 \subset X$ are fixed Lagrangians and $Z_s$, $s \in [0, 1]$, is a path in $\mathcal{L}\mathcal{C}(N; \Lambda_0, \Lambda_1)$. Then the relative Lagrangian flux of the path is a linear functional on $H_1(Z_0, \partial Z_0)$ which measures the deviation of the path from being exact relative to the boundary. As the relative homology group $H_1(Z_0, \partial Z_0)$ is generated by a single element, one can think of the relative Lagrangian flux of a path of cylinders as a number. The precise definition is as follows.

Let $N$ be a closed connected $n-1$-dimensional smooth manifold and let $Z_s$, $s \in [s_0, s_1]$, be a smooth path in $\mathcal{L}\mathcal{C}(N; \Lambda_0, \Lambda_1)$. Let $\Phi : N \times [0, 1] \times [s_0, s_1] \to X$ be a smooth parameterization of the path $(Z_s)$. Namely, for $s \in [s_0, s_1]$, the restricted map $\Phi_s := \Phi|_{N \times [0, 1] \times \{s\}}$ is a parameterization of the cylinder $Z_s$, and for $(p, s) \in N \times [s_0, s_1]$ we have

$$\Phi(p, i, s) \in A_i, \quad i = 0, 1.$$ 

For $s \in [s_0, s_1]$, write

$$h_s := \frac{d}{ds} Z_s \in C^\infty_{\text{Cob}}(Z_s).$$

For $i = 0, 1$, let $C_{i,s}$ denote the boundary component of the cylinder $Z_s$ corresponding to $N \times \{i\}$, let $A_s \in \mathbb{R}$ satisfy

$$h_s|_{C_{i,s}} \equiv A_s.$$ 

Let $\gamma : [0, 1] \to N \times [0, 1]$ be a smooth path representing the fundamental class in the relative homology group $H_1(N \times [0, 1], \partial(N \times [0, 1]))$. Namely, $\gamma$ satisfies

$$\gamma(i) \in N \times \{i\}, \quad i = 0, 1.$$ 

Set

$$\tau : [0, 1] \times [s_0, s_1] \to X, \quad (t, s) \mapsto \Phi(\gamma(t), s).$$

Lemma 4.10. In the above setting we have

$$\int_{[0,1] \times [s_0, s_1]} \tau^* \omega = -\int_{s_0}^{s_1} A_s ds. \tag{20}$$

Proof. For $s \in [s_0, s_1]$, let

$$Y_s := \frac{d}{ds} (\Phi_s \circ \gamma) \in \Gamma([0, 1], (\Phi_s \circ \gamma)^*TX).$$

By definition of the derivative of a Lagrangian path (see Remark 2.12), we have

$$i_{Y_s} \omega = d(h_s \circ \Phi_s \circ \gamma).$$

By Fubini’s theorem,

$$\int_{[0,1] \times [s_0, s_1]} \tau^* \omega = -\int_{s_0}^{s_1} \left( \int_{[0,1]} i_{Y_s} \omega \right) ds$$

$$= -\int_{s_0}^{s_1} A_s ds,$$
as desired. \qed

**Definition 4.11.** The quantity in equality (20) is called the relative Lagrangian flux of the path \((Z_s)_{s \in [s_0, s_1]}\). We let \(\text{RelFlux} ((Z_s)_{s \in [s_0, s_1]})\) denote the relative Lagrangian flux. More generally, if \(I \subset \mathbb{R}\) is an interval, possibly open or half open, with endpoints \(a < b\), and \((Z_s)_{s \in I}\) is a path in \(\mathcal{LC}(N; \Lambda_0, \Lambda_1)\), we write

\[
\text{RelFlux} ((Z_s)_{s \in I}) = \lim_{s_0 \to a} \lim_{s_1 \to b} \text{RelFlux} ((Z_s)_{s \in [s_0, s_1]})
\]

whenever the limit exists.

**Remark 4.12.** A straightforward modification of the arguments in [22] shows that \(\text{RelFlux} ((Z_s)_{s \in I})\) is independent of the choices of the parameterization \(\Phi\) and the path \(\gamma\). Moreover, when \(I\) is a closed interval, \(\text{RelFlux} ((Z_s)_{s \in I})\) depends only on the homotopy class of the path \((Z_s)\) relative to its endpoints.

### 4.3. Regular families of special Lagrangian cylinders

Let \((X, \omega, J, \Omega)\) be a Calabi-Yau manifold of real dimension \(2n\), let \(\Lambda_0, \Lambda_1 \subset X\) be smooth embedded positive Lagrangians, and let \(N\) be a closed smooth manifold of dimension \(n - 1\).

**Definition 4.13.** Let \(Z \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)\) and let \(C_i, i = 0, 1\), denote the boundary component of \(Z\) corresponding to \(N \times \{i\}\). The **fundamental harmonic** on \(Z\) is the unique function \(\sigma \in C^\infty_{\text{COB}}(Z)\) satisfying

\[
\Delta_p(\sigma) = 0, \quad \sigma|_{C_i} \equiv 1.
\]

If \(\sigma\) has no critical points, we say \(Z\) has regular harmonics.

For the remainder of this section, we assume that \(N\) is connected. Let

\[
\gamma: (-\epsilon, \epsilon) \to \mathcal{SLC}(N; \Lambda_0, \Lambda_1)
\]

be a smooth path. A smooth lifting of \(\gamma\) is a smooth map \(\Phi: N \times [0, 1] \times (-\epsilon, \epsilon) \to X\) such that for \(s \in (-\epsilon, \epsilon)\) the restriction

\[
\Phi_s := \Phi|_{N \times [0, 1] \times \{s\}} : N \times [0, 1] \to X
\]

represents the immersed submanifold \(\gamma(s) \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)\).

**Lemma 4.14.** Let \(Z \in \mathcal{SLC}(N; \Lambda_0, \Lambda_1)\). Then the following are equivalent.

1. \(Z\) has regular harmonics.
2. For every embedded curve \(\gamma: (-\epsilon, \epsilon) \to \mathcal{SLC}(N; \Lambda_0, \Lambda_1)\) with \(\gamma(0) = Z\), and for every smooth lifting of \(\gamma\),

\[
\Phi: N \times [0, 1] \times (-\epsilon, \epsilon) \to X,
\]

after possibly diminishing \(\epsilon\), the map \(\Phi\) is an immersion.
3. For one embedded curve \(\gamma: (-\epsilon, \epsilon) \to \mathcal{SLC}(N; \Lambda_0, \Lambda_1)\) with \(\gamma(0) = Z\), and one smooth lifting of \(\gamma\),

\[
\Phi: N \times [0, 1] \times (-\epsilon, \epsilon) \to X,
\]

the map \(\Phi\) is an immersion.

**Proof.** First, we prove that condition (3) implies condition (1). Let \(\gamma\) and \(\Phi\) be as in condition (3). Let \(\sigma\) denote the fundamental harmonic on \(Z\). By Proposition 4.7, we have

\[
\dot{\gamma}(0) = a\sigma
\]

for some \(a \in \mathbb{R}\). Let

\[
Y := \frac{d}{ds}\bigg|_{s=0} \Phi(\cdot, \cdot, s) \in \Gamma(N \times [0, 1], \Phi_0^*TX).
\]
By Remark 2.12 we have
\[ (22) \quad d(a\sigma \circ \Phi_0) = iy\omega. \]
As \( \Phi \) is an immersion, the section \( Y \) is nowhere tangent to \( Z \). As \( Z \) is Lagrangian, the right-hand side of (22) is nowhere vanishing. It follows that \( a\sigma \) does not have critical points and neither does \( \sigma \).

Next, we prove that condition (1) implies condition (2). Indeed, suppose \( Z \) has regular harmonics. Let \( \gamma \) and \( \Phi \) be as in condition (2). By Proposition 4.7 we have
\[ \gamma(0) = a\sigma \]
for some \( 0 \neq a \in \mathbb{R} \). Defining \( Y \) as in (21), the equality (22) continues to hold true. By assumption, the left-hand side of (22) is non-vanishing. It follows that \( Y \) is nowhere tangent to \( Z \), which implies that, diminishing \( \epsilon \) if necessary, \( \Phi \) is an immersion.

It is immediate that condition (2) implies condition (3). \( \square \)

The following lemma is, in fact, a particular case of Lemma 4.14. We provide an independent proof for clarity.

**Lemma 4.15.** Equip \( \mathbb{C}^n \) with the standard Calabi-Yau structure, let \( \Lambda_0, \Lambda_1 \subset \mathbb{C}^n \) be positive Lagrangian linear subspaces, and let \( Z \in \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \). Then \( Z \) has regular harmonics if and only if \( Z \) is nowhere tangent to the Euler vector field.

**Proof.** Let \( \Phi_1 : S^{n-1} \times [0,1] \rightarrow \mathbb{C}^n \) be a representative of \( Z \), and set
\[ \Phi : S^{n-1} \times [0,1] \times (0,\infty) \rightarrow \mathbb{C}^n, \quad (p,t,s) \mapsto s \cdot \Phi_1(p,t). \]
Then for \( s \in (0,\infty) \), the map \( \Phi_s : \Phi_1|_{S^{n-1} \times [0,1] \times \{s\}} \) is an immersion representing an element in \( \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \). Hence, we have
\[ \frac{d}{ds} \bigg|_{s=1} [\Phi_s] = a\sigma \]
for some \( 0 \neq a \in \mathbb{R} \), where \( \sigma \) denotes the fundamental harmonic on \( Z \). By Remark 2.12, a point \( q \in Z \) is a critical point of \( \sigma \) if and only if the Euler vector at \( q \) is tangent to \( Z \). The lemma follows. \( \square \)

The following lemma, which is in fact an elementary observation in functional analysis, shows that the fundamental harmonic depends smoothly on the geometry of a cylinder. In particular, the property of having regular harmonics is stable under small perturbations.

**Lemma 4.16.** Let \( N \) be a closed connected smooth manifold, and write \( Z := N \times [0,1] \). Let \( C_i := N \times \{i\}, i = 0,1 \), denote the boundary components of \( Z \). Let \( \alpha \in (0,1) \), let \( k \in \mathbb{N} \), and let \( L : C^{k+2,\alpha}_{\text{COB}}(Z) \rightarrow C^{k,\alpha}(Z) \) be a bounded linear operator, such that \( L_0 := L|_{C^{k+2,\alpha}_{\text{COB}}(Z,\partial Z)} : C^{k+2,\alpha}(Z,\partial Z) \rightarrow C^{k,\alpha}(Z) \) is an isomorphism. Let \( u \in C^{k+2,\alpha}_{\text{COB}}(Z) \) be the unique function satisfying
\[ Lu = 0, \quad u|_{C_1} \equiv 1. \]

Then, letting \( \mathcal{B} \left( C^{k+2,\alpha}_{\text{COB}}(Z), C^{k,\alpha}(Z) \right) \) denote the space of bounded linear operators \( C^{k+2,\alpha}_{\text{COB}}(Z) \rightarrow C^{k,\alpha}(Z) \) endowed with the operator norm, there is an open neighborhood
\[ L \in \mathcal{V} \subset \mathcal{B} \left( C^{k+2,\alpha}_{\text{COB}}(Z), C^{k,\alpha}(Z) \right) \]
such that for \( \tilde{L} \in \mathcal{V} \), there exists a unique \( \tilde{u} = \tilde{u}_L \) satisfying
\[ \tilde{L}\tilde{u} = 0, \quad \tilde{u}|_{C_1} \equiv 1. \]
Moreover, the assignment \( \tilde{L} \mapsto \tilde{u}_L \) is smooth.
Proof. Let $L \in \mathcal{V} \subset \mathcal{B} \left( C^{k+2,\alpha}_{\text{COb}}(Z), C^{k,\alpha}(Z) \right)$ be open such that, for $\tilde{L} \in \mathcal{V}$, the restricted operator $\tilde{L}_0 := \tilde{L}|_{C^{k+2,\alpha}(Z,\mathcal{B}Z)}$ is an isomorphism. The assignment $\tilde{L} \mapsto \tilde{L}_0^{-1}$ is smooth in $\mathcal{V}$. For $\tilde{L} \in \mathcal{V}$, the unique $\tilde{u}$ with the desired properties is given by

$$\tilde{u}_L = u - \tilde{L}_0^{-1}L_0u.$$ 

The lemma follows. \hfill \Box

Next, we define interior regularity and regular convergence for families of special Lagrangian cylinders. As a preliminary, we recall the notion of ends (see [10]). Let $A$ be a topological space. The set of ends $\mathcal{E}(A)$ is given by

$$\mathcal{E}(A) := \lim_{\K \subset A} \pi_0(A \setminus K).$$

Thus, an end $E \in \mathcal{E}(A)$ determines for every compact $K \subset A$ a connected component $E(K)$ of the complement $A \setminus K$, such that for two compact subsets $K \subset K'$ we have $E(K') \subset E(K)$. For our purposes the space $A$ will always be a connected component of $\mathcal{SLC} \left(S^{n-1}; \Lambda_0, \Lambda_1 \right)$, which is 1-dimensional. We thus use the following terminology, which is somewhat more elementary and equivalent in the case at hand. Let $C$ be a connected non-compact 1-dimensional manifold. That is, $C$ is a curve diffeomorphic to the real line. A ray in $C$ is a connected open proper subset $U \subseteq C$ with non-compact closure. Two rays, $U, V \subset C$ are said to be equivalent if $U \subset V$ or $V \subset U$. Finally, an end is an equivalence class of rays. Every curve $C$ as above has exactly two ends.

Definition 4.17.

1. Let $U \subset \mathcal{SLC} \left(N; \Lambda_0, \Lambda_1 \right)$ be open and connected. An interior-regular parameterization of $U$ is a smooth immersion $\Phi : N \times [0,1] \times (a,b) \to X$ satisfying the following conditions:
   (a) The restriction of $\Phi$ to a boundary component $\Phi|_{N \times \{i\} \times (a,b)}$ is an embedding for $i = 0, 1$.
   (b) For $s \in (a, b)$, the restricted immersion $\Phi_s := \Phi|_{N \times [0,1] \times \{s\}}$ represents an element of $U$.
   (c) The map $\chi : (a,b) \to U$, $s \mapsto [\Phi_s]$, is a diffeomorphism.
   The subset $U$ is said to be interior-regular if it admits an interior-regular parameterization.

2. Let $U$ be as in (1) and let $\Phi$ be an interior-regular parameterization of $U$. For $s \in (a, b)$, write $Z_s := [\Phi_s]$ and let $\sigma_s$ denote the fundamental harmonic on $Z_s$. The parameterization $\Phi$ is compatible with the harmonics of $U$ if the following conditions hold:
   (a) For $(p, t, s) \in N \times [0,1] \times (a, b)$ we have
   $$\sigma_s(\Phi(p, t, s)) = t.$$
   (b) For $(p, t, s) \in N \times [0,1] \times (a, b)$, the derivative $\frac{\partial}{\partial s} \Phi(p, t, s)$ is orthogonal to the $t$-level set of $\sigma_s$.

3. Let $Z \subset \mathcal{SLC} \left(S^{n-1}; \Lambda_0, \Lambda_1 \right)$ be a connected component, let $E$ be an end of $Z$, and let $q \in \Lambda_0 \cap \Lambda_1$. A regular parameterization of $E$ about $q$ is a smooth map $\Phi : S^{n-1} \times [0,1] \times [0, e] \to X$ satisfying the following conditions:
   (a) For $(p, t) \in S^{n-1} \times [0,1]$ we have $\Phi(p, t, 0) = q$.
   (b) The restricted map $\Phi|_{S^{n-1} \times [0,1] \times \{0, e\}}$ is an interior-regular parameterization of $U$, for some ray $U \subset Z$ representing $E$.
   (c) The derivative
   $$\frac{\partial}{\partial s} \big|_{s=0} \Phi(\cdot, \cdot, s) : S^{n-1} \times [0,1] \to T_q X \quad \text{satisfying}$$
is an immersion and the restriction \( \frac{\partial}{\partial s} |_{s=0} \Phi(\cdot, s)|_{S^{n-1} \times \{1\}} \) is an embedding for \( i = 0, 1 \).

(d) The Euler vector field on \( T_q X \) is nowhere tangent to the immersion \( \frac{\partial}{\partial s} |_{s=0} \Phi(\cdot, s) \).

In this case, we also say that \( \Phi \) is a regular parameterization of \( U \) about \( q \).

A regular parameterization \( \Phi \) as above is said to be compatible with the harmonics of \( E \) if the interior-regular restricted map \( \Phi|_{S^{n-1} \times [0,1] \times (0,\epsilon)} \) is.

We say the end \( E \) converges regularly to the intersection point \( q \) if it admits a regular parameterization about \( q \). We may use a half-open interval with arbitrary endpoints, open either from below or above, in place of the half-open interval \([0, \epsilon)\).

**Remark 4.18.** It follows from Lemma 4.14 that if a connected open subset \( U \subset S\mathcal{LC}(N; \Lambda_0, \Lambda_1) \) is interior-regular, every element of \( U \) has regular harmonics.

**Remark 4.19.** Let \( Z, E \) and \( q \) be as in Definition 4.17 (3), and suppose \( \Phi \) is a regular parameterization of \( E \) about \( q \). Recalling Definition 3.3, let

\[ \kappa : S^{n-1} \times [0, \epsilon) \rightarrow \mathbb{R}^n \]

be a polar coordinate map centered at zero with image \( U \). For a fixed \( t \in [0, 1] \), denote by \( \Psi_t : U \rightarrow X \) the unique map such that

\[ \Phi|_{S^{n-1} \times \{t\} \times [0, \epsilon)} = \Psi_t \circ \kappa. \]

Then, Lemma 3.5 together with properties (c) and (d) of a regular parameterization shows that \( \Psi_t \) is cone immersive at zero. It follows from property (b) that \( (\Psi_t, 0) \) is a cone immersion.

**Lemma 4.20.**

(a) Let \( U \subset S\mathcal{LC}(N; \Lambda_0, \Lambda_1) \) be interior-regular and let

\[ \Phi : N \times [0, 1] \times (0, 1) \rightarrow X \]

be an interior regular parameterization. Then \( U \) admits a unique interior-regular parameterization \( \hat{\Phi} : N \times [0, 1] \times (0, 1) \rightarrow X \) compatible with its harmonics such that \( \hat{\Phi}|_{N \times \{0\} \times (0, 1)} = \Phi|_{N \times \{0\} \times (0, 1)} \).

(b) Let \( Z \subset S\mathcal{LC}(S^{n-1}; \Lambda_0, \Lambda_1) \) be a connected component and let \( E \) be an end of \( Z \). Let \( q \in \Lambda_0 \cap \Lambda_1 \) be a transverse intersection point, suppose \( E \) converges regularly to \( q \) and suppose \( \Phi : S^{n-1} \times [0,1] \times [0,\epsilon) \rightarrow X \) is a regular parameterization of \( E \) about \( q \). Then \( E \) admits a unique regular parameterization \( \hat{\Phi} : S^{n-1} \times [0,1] \times [0,\epsilon) \rightarrow X \) about \( q \) which is compatible with the harmonics of \( E \) such that \( \hat{\Phi}|_{N \times \{0\} \times [0,\epsilon)} = \hat{\Phi}|_{N \times \{0\} \times [0,\epsilon)} \).

**Proof.** We prove (b). The proof of (a) is similar and less involved.

We identify a neighborhood of \( q \) in \( X \) with a ball \( V \subset \mathbb{C}^n \) via Darboux coordinates carrying \( q \) to the origin and \( \Lambda_i, i = 0, 1 \), to linear subspaces. For \( s \in [0, 1) \), write

\[ V_s := \{ z \in \mathbb{C}^n | sz \in V \}. \]

For \( s \in (0, 1) \) let \( M_s : V_s \rightarrow V \) denote rescaling by \( s \). Define a family of complex structures and \( n \)-forms on \( V_s \) by

\[ J_s := M_s^* J, \quad \Omega_s := s^{-n} M_s^* \Omega, \quad s \in (0, 1), \]

where \( J \) and \( \Omega \) are the complex structure and Calabi-Yau form of \( X \), respectively. Then as \( s \searrow 0 \), \( J_s \) and \( \Omega_s \) converge in the \( C^\infty \)-topology on compact subsets to a constant complex structure and a constant \( n \)-form on \( V_0 = \mathbb{C}^n \) denoted by \( J_0 \) and \( \Omega_0 \) respectively. For \( s \in [0, 1) \), the quadruple \((V_s, \omega, J_s, \Omega_s)\) is a Calabi-Yau manifold. We let \( g_s \) denote the associated Kähler metric. We define \( \rho_s \) as in (3).
Let $\Phi : S^{n-1} \times [0,1] \times [0,1] \rightarrow V$ be a regular parameterization of $E$ about the origin $q$. By Lemma 3.14, we have

$$\Phi(p,t,s) = s \cdot \chi(p,t,s), \quad (p,t,s) \in S^{n-1} \times [0,1] \times [0,1],$$

where $\chi : S^{n-1} \times [0,1] \times [0,1] \rightarrow \mathbb{C}^n$ is smooth with

$$\chi(p,t,0) = \frac{\partial \Phi}{\partial s}(p,t,0).$$

For $s \in [0,1]$, write $\chi_s := \chi|_{S^{n-1} \times [0,1] \times \{s\}}$. Then $\chi_s$ is an immersion representing an $\Omega_s$-imaginary special Lagrangian cylinder with boundary components in $A_i$, $i = 0,1$. Define a smooth family of elliptic linear differential operators

$$\Delta_s : C^\infty \left(S^{n-1} \times [0,1]\right) \rightarrow C^\infty \left(S^{n-1} \times [0,1]\right), \quad u \mapsto *_s d((\rho_s \circ \chi_s) *_s du),$$

where $*_s$ denotes the Hodge star operator of the metric $\chi_s^* g_s$. Let $\tilde{\sigma}_s$ denote the fundamental harmonic on $S^{n-1} \times [0,1]$ with respect to $\Delta_s$. Namely, $\tilde{\sigma}_s : S^{n-1} \times [0,1] \rightarrow \mathbb{R}$ is the unique function satisfying

$$\Delta_s(\tilde{\sigma}_s) = 0, \quad \tilde{\sigma}_s|_{S^{n-1} \times \{i\}} = i, \quad i = 0,1.$$

Note that all the cylinders $[\chi_s]$ have regular harmonics. Indeed, for $s \in (0,1)$, the cylinder $[\chi_s]$ is merely a rescaling of $[\Phi_s]$, which has regular harmonics by Remark 4.18, whereas $[\chi_0]$ has regular harmonics by Lemma 4.15 and property (d) of Definition 4.17 (3). Hence, all the functions $\tilde{\sigma}_s$ have no critical points. By Lemma 4.16, the function $\tilde{\sigma}_s$ depends smoothly on $s$. Define a smooth family of vector fields on $S^{n-1} \times [0,1]$ by

$$Y_s := \frac{\nabla \tilde{\sigma}_s}{|\nabla \tilde{\sigma}_s|^2}, \quad s \in [0,1],$$

where the gradient and modulus are taken with respect to $\chi_s^* g_s$. Then $Y_s$ is $\chi_s^* g_s$-orthogonal to the level sets of $\tilde{\sigma}_s$ and satisfies $Y_s(\tilde{\sigma}_s) \equiv 1$. Let

$$\varphi_s : W_s \subset \mathbb{R} \times \left(S^{n-1} \times [0,1]\right) \rightarrow S^{n-1} \times [0,1]$$

denote the flow of $Y_s$. Note that for $(p,t,s) \in S^{n-1} \times [0,1] \times [0,1]$ we have

$$(t, (p,0)) \in W_s, \quad \tilde{\sigma}_s(\varphi_s(t, (p,0))) = t.$$

Finally, we set

$$\tilde{\chi} : S^{n-1} \times [0,1] \times [0,1] \rightarrow V, \quad (p,t,s) \mapsto \chi_s(\varphi_s(t, (p,0)))$$

and

$$\tilde{\Phi} : S^{n-1} \times [0,1] \times [0,1] \rightarrow V, \quad (p,t,s) \mapsto s \cdot \tilde{\chi}(p,t,s).$$

By construction, $\tilde{\Phi}$ is a regular parameterization of $E$ about $q$ compatible with the harmonics and $\tilde{\Phi}|_{N \times \{0\} \times [0,\varepsilon]} = \Phi|_{N \times \{0\} \times [0,\varepsilon]}$.

Suppose $\tilde{\Phi}'$ is another such regular parameterization. Since

$$(24) \quad \tilde{\Phi}'|_{N \times \{0\} \times [0,\varepsilon]} = \Phi|_{N \times \{0\} \times [0,\varepsilon]} = \tilde{\Phi}'|_{N \times \{0\} \times [0,\varepsilon]},$$

it follows from the definition of regular parameterization that $[\tilde{\Phi}'_s] = [\tilde{\Phi}_s]$ for $s \in (0,\varepsilon)$. So, there exists a diffeomorphism $\psi_s : N \times [0,1] \rightarrow N \times [0,1]$ such that $\tilde{\Phi}'_s = \tilde{\Phi}_s \circ \psi_s$. By (24), we have also $\psi_s|_{N \times \{0\}} = \text{id}_N$. By the chain rule,

$$\frac{d\tilde{\Phi}_s}{dt}(\frac{d\psi_s}{dt}) = \frac{d\tilde{\Phi}'_s}{dt}.$$
Since $d\hat{\Phi}_s$ is injective, it follows that

$$\frac{\partial \hat{\psi}_s}{\partial t} = \frac{\partial}{\partial t} \circ \psi_s.$$  

Consequently, $\psi_s = \text{id}_{N \times [0,1]}$. \hfill \Box

Let $\text{Diff}(N \times (0,1) \to (0,1)) \subset \text{Diff}(N \times (0,1))$ denote the subgroup consisting of diffeomorphisms that carry fibers of the projection $N \times (0,1) \to (0,1)$ to other such fibers. For $\Phi : N \times [0,1] \times (0,1) \to X$ an interior regular parameterization of $U$ and $\phi \in \text{Diff}(N \times (0,1) \to (0,1))$, write

$$\Phi_s = \Phi \circ (\phi \times \text{id}_{[0,1]}): N \times [0,1] \times (0,1) \to X.$$  

Corollary 4.21. Let $U \subset S\text{LC}(N;\Lambda_0,\Lambda_1)$ be interior-regular. Then, the map $(\Phi,\phi) \mapsto \Phi_s$ gives a free transitive action of the group $\text{Diff}(N \times (0,1) \to (0,1))$ on the set of interior regular parameterizations of $U$ compatible with the harmonics.

Proof. It follows from the definitions that the action preserves regular parameterizations compatible with the harmonics. Freeness is immediate. Transitivity follows from the uniqueness claim of Lemma 4.20 (a). \hfill \Box

5. The Relation Between Cylinders and Geodesics

In this section we establish the relation between geodesics of positive Lagrangians and families of imaginary special Lagrangian cylinders. We fix an ambient Calabi-Yau manifold $(X,\omega,J,\Omega)$, a smooth manifold $L$, a finite subset $S \subset L$ and a finite subset $C_0 \subset S$. In the following, all geodesics are geodesics of cone-immersed Lagrangians of type $(L,S)$ with cone locus image $C_0$ unless otherwise mentioned.

Lemma 5.1. Let $(\Lambda_t)_{t\in[0,1]}$ be a geodesic of positive Lagrangians of type $(L,S)$, and let $\Psi_t : L \to \Lambda_t$, $t \in [0,1]$, be a horizontal lifting of $(\Lambda_t)$. Let $(h_t)_t$ be the Hamiltonian of $(\Lambda_t)$, and let $h = h_t \circ \Psi_t : L \to \mathbb{R}$ be the Hamiltonian of $(\Lambda_t)$, with respect to the lifting $(\Psi_t)_t$. For $c$ in the image of $h$, define

$$\Phi_c : (h^{-1}(c) \setminus \text{Crit}(h)) \times [0,1] \to X, \quad (p,t) \mapsto \Psi_t(p).$$

Then $\Phi_c$ is an imaginary special Lagrangian immersion.

Proof. Fix $t_0 \in [0,1]$ and $p \in h^{-1}(c) \setminus \text{Crit}(h)$. Let $u_1, \ldots, u_{n-1} \in T_p h^{-1}(c)$ be a basis and write

$$w_i := d(\Psi_t)_p(u_i) \in T_{\Psi_{t_0}(p)}\Lambda_{t_0}, \quad i = 1, \ldots, n-1.$$  

The tangent vectors $w_1, \ldots, w_{n-1}$ are linearly independent as $\Psi_{t_0}$ is a smooth immersion away from critical points of $h$. Let $v \in T_{\Psi_{t_0}(p)}X$ be the $t$-derivative of $\Phi_c$. That is,

$$v := \frac{\partial \Phi_c}{\partial t}(p,t_0) = \frac{d}{dt} \Psi_t(p).$$  

Since $p$ is a regular point of $h$, we have $v \not \in T_{\Psi_{t_0}(p)}\Lambda_{t_0}$. It follows that the tangent vectors $v, w_1, \ldots, w_{n-1}$ are linearly independent and $\Phi_c$ is indeed an immersion.

As the immersion $\Psi_{t_0}$ is Lagrangian, we have

$$\omega(w_i, w_j) = 0, \quad i,j = 1, \ldots, n-1.$$  

By definition of the derivative of a Lagrangian path (see Remark 2.12), we have

$$\omega(v, w_i) = dh_p(u_i) = 0, \quad i = 1, \ldots, n-1.$$  

The immersion $\Phi_c$ is thus Lagrangian. Finally, by horizontality of the lifting $(\Psi_t)$, we have

$$\text{Re} \Omega(v, w_1, \ldots, w_{n-1}) = 0,$$

and $\Phi_c$ is indeed an imaginary special Lagrangian immersion. \hfill \Box
**Definition 5.2.** In the setting of Lemma 5.1, for \( c \) in the image of the Hamiltonian \( h \), we call the immersed special Lagrangian cylinder \([\Phi_c]\) the **associated cylinder of \( c \)** level sets. Note that \([\Phi_c]\) is independent of the horizontal lifting \((\Psi_t)_t\). It depends only on the choice of additive constant in the intrinsic Hamiltonian \((h_t)_t\).

**Lemma 5.3.** Let \( (\Lambda_t)_t, (\Psi_t)_t, h \) and \((\Phi_c)_c\) be as in Lemma 5.1. For \( c \) in the image of \( h \), let \( Z_c \) denote the associated cylinder \([\Phi_c]\) and define a function

\[
\sigma_c : Z_c \to [0, 1],
\]

by

\[
\sigma_c(\Phi_c(q, t)) = t.
\]

(a) Let \( N \) be an \((n-1)\)-dimensional smooth manifold, let \( c_0 < c_1 \in \mathbb{R} \) and let \( \beta : N \times (c_0, c_1) \to L \) be a smooth embedding with

\[
h(\beta(q, c)) = c, \quad (q, c) \in N \times (c_0, c_1).
\]

For \( c \in (c_0, c_1) \) define a smooth immersion

\[
\phi_c : N \times [0, 1] \to X, \quad (q, t) \mapsto \Phi_c(\beta(q, c), t) = \Psi_t(\beta(q, c)),
\]

and let

\[
v = \frac{d}{dc}\phi_c.
\]

Then,

\[
i_v \omega = -d(\sigma_c \circ \phi_c).
\]

(b) For \( c \) in the image of \( h \) we have \( \Delta_{\psi}(\sigma_c) = 0 \).

**Proof.** Let \( N \) and \( \beta \) be as in (a). Let \( t \) denote the \([0, 1]\)-coordinate on \( N \times [0, 1] \). By Remark 2.12 and the definition of a geodesic, we have

\[
i_v \omega \left( \frac{\partial}{\partial t} \right) = -\omega \left( \frac{d}{dt} \Psi_t, \frac{d}{dc}(\Psi_t \circ \beta) \right)
\]

\[
= -\frac{d}{dc}(h \circ \beta)
\]

\[
= -1
\]

Let \( w \) be a vector field on \( N \times [0, 1] \) tangent to \( N \). Then, since \( \Psi_t \) is a Lagrangian immersion,

\[
i_v \omega (w) = -\omega \left( d\Psi_t(w), \frac{d}{dc}(\Psi_t \circ \beta) \right)
\]

\[
= 0
\]

\[
= -w(\sigma_c \circ \phi_c).
\]

Part (a) now follows from (26) and (27).

Let \( p \in L \) be a regular point of \( h \) and let \( c_0 < c_1 \) with \( h(p) \in (c_0, c_1) \). Let \( B \) denote the standard open ball of dimension \( n-1 \). Let \( p \in W \subset L \) be a ball containing only regular points of \( h \), and let \( \beta : B \times (c_0, c_1) \to W \) be a diffeomorphism with

\[
h \circ \beta(q, c) = c, \quad (q, c) \in B \times (c_0, c_1).
\]

For \( c \in (c_0, c_1) \), define \( \phi_c \) as in (a). Part (b) now follows from (a), Lemma 4.5 (a) and Lemma 5.1. \( \square \)
Remark 5.4. Let $(\Lambda_t)_{t \in [0,1]}$, $(h_t)_t$, $(Z_c)_c$, and $(\sigma_c)_c$, be as in Lemma 5.3. Let $c$ be such that $Z_c$ is compact. In particular, $c$ must be a regular value of $(h_t)_t$. It follows from Lemma 5.3 (b) that $\sigma_c$ is the fundamental harmonic on $Z_c$. Also, by its definition, the function $\sigma_c$ has no critical points. The cylinder $Z_c$ thus has regular harmonics.

Proof of Theorem 1.1. Follows from Lemmas 5.1 and 5.3. \qed

Let $(\Lambda_t)_t$ be a geodesic of positive Lagrangians with Hamiltonian $(h_t : \Lambda_t \rightarrow \mathbb{R})_t$. For $c$ in the image of $h_t$, let $Z_c$ denote the associated cylinder of $c$ level sets. Recall from Definition 1.4 that the family of cylinders $(Z_c)_c$ is called the cylindrical transform of the geodesic $(\Lambda_t)_t$. The subset $\{Z_c \mid c \text{ is a regular value of } h\}$ is the non-singular cylindrical transform. We show that every component in the non-singular cylindrical transform is interior-regular with relative Lagrangian flux given by the Hamiltonian of the geodesic.

Lemma 5.5. Let $(\Lambda_t)_t$ be a geodesic with Hamiltonian $(h_t)_t$ and assume the functions $h_t$ are proper. For $c$ in the image of $h_t$, let $Z_c$ denote the associated cylinder of $c$ level sets. Since the functions $h_t$ are proper, the cylinder $Z_c$ is compact when $c$ is a regular value of $(h_t)_t$. Let $c_0 < c_1 \in \mathbb{R}$ be such that the interval $(c_0, c_1)$ consists of regular values of $(h_t)_t$.

(a) The family of cylinders $U := \{Z_c \mid c \in (c_0, c_1)\}$ is interior-regular.

(b) For $b_0 < b_1 \in (c_0, c_1)$ we have

$$\text{RelFlux}\left((Z_c)_{c \in [b_0, b_1]}\right) = b_1 - b_0.$$ 

Proof. Let $(\Psi_t : L \rightarrow X)_t$ be a horizontal lifting of $(\Lambda_t)_t$, and let $h : L \rightarrow \mathbb{R}$ be the corresponding Hamiltonian,

$$h_t = ([\Psi_t, h]).$$

By assumption there exist an $(n-1)$-dimensional smooth manifold $N$ and a diffeomorphism

$$\beta : N \times (c_0, c_1) \rightarrow h^{-1}(c_0, c_1)$$

with

$$h \circ \beta(q, c) = c, \quad (q, c) \in N \times (c_0, c_1).$$

Define

$$\Phi : N \times [0, 1] \times (c_0, c_1) \rightarrow X, \quad (q, t, c) \mapsto \Psi_t(\beta(q, c)).$$

Then $\Phi$ is an interior-regular parameterization of $U$ verifying (a). Part (b) follows from Lemma 5.3 (a) and Definition 4.11. \qed

Proposition 5.7 below is a converse to Lemma 5.5 (a) showing that an interior-regular family of special Lagrangian cylinders gives rise to a geodesic of positive Lagrangians. We use the following notion.

Definition 5.6. Let $\Lambda_0, \Lambda_1 \subset X$ be smooth Lagrangians and let $N$ be a closed connected smooth manifold of dimension $n-1$. Let $U \subset \mathcal{SLC}(N; \Lambda_0, \Lambda_1)$ be open, connected and interior-regular. Let $\Phi : N \times [0, 1] \times (0,1) \rightarrow X$ be an interior regular parameterization of $U$. For $i = 0, 1$, the submanifold of $\Lambda_i$ swept by $U$ is the image of the embedding $\Phi|_{N \times \{i\} \times (0,1)}$. This is independent of $\Phi$. Similarly, suppose $U \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_1)$ is a ray and $\Phi : S^{n-1} \times [0, 1] \times [0, c) \rightarrow X$ is a regular parameterization of $U$ about an intersection point $q \in \Lambda_0 \cap \Lambda_1$. For $i = 0, 1$, the unpunctured submanifold of $\Lambda_i$ swept by $U$ is the image of the embedding $\Phi|_{S^{n-1} \times \{i\} \times [0, c)}$. 37
Proposition 5.7. Let \( \Lambda_0, \Lambda_1 \subset X \) be smooth embedded positive Lagrangians, let \( N \) be a closed connected smooth manifold of dimension \( n - 1 \) and let
\[
U \subset \mathcal{SLC}(N; \Lambda_0, \Lambda_1)
\]
be open, connected and interior regular. For \( i = 0, 1 \), let \( \Lambda_i^U \) denote the submanifold of \( \Lambda_i \) swept by \( U \). Then, there exists a geodesic of positive Lagrangians between \( \Lambda_0^U \) and \( \Lambda_1^U \) with cylindrical transform \( U \). This geodesic is unique up to reparameterization and has empty critical locus.

Proof. Let \( \Phi : N \times [0, 1] \times (0, 1) \rightarrow X \) be an interior-regular parameterization of \( U \). Since the points of \( U \) are imaginary special Lagrangians, \( N \) must be orientable. Orient \( N \) so that the open embedding
\[
\Phi_{|N \times \{0\} \times (0, 1)} : N \times \{0\} \times (0, 1) \rightarrow \Lambda_0
\]
is orientation preserving. By Lemma 4.20, we may assume that \( \Phi \) is compatible with the harmonics of \( U \). For \( s \in (0, 1) \) write \( \Phi_s := \Phi_{|N \times \{s\} \times \{0\}} \) and \( Z_s := [\Phi_s] \). Let \( \sigma_s \) denote the fundamental harmonic on \( Z_s \). Let \( \xi \) denote the \( t \)-derivative of \( \Phi \),
\[
\xi = \xi(p, t, s) = \frac{\partial}{\partial t}\Phi(p, t, s).
\]
For \( t \in [0, 1] \) write
\[
\Psi_t := \Phi_{|N \times \{t\} \times (0, 1)}.
\]
Then \( \Psi_t \) is a smooth immersion by the definition of an interior regular parameterization. Let \( \Lambda_i^U := [\Psi_t] \) with orientation given by the orientation on \( N \times (0, 1) \). We show in three steps that \( (\Lambda_i^U)_{t \in [0, 1]} \) is a geodesic of positive Lagrangians.

**Step 1:** For \( t \in [0, 1] \), the immersed submanifold \( \Lambda_i^U \) is positive Lagrangian. To show this, pick \( (p, t, s) \in N \times [0, 1] \times (0, 1) \). As \( \Phi \) is compatible with the harmonics of \( U \), the immersed submanifold
\[
K_{t,s} := [\Phi_{|N \times \{t\} \times \{s\}}]
\]
is the \( t \)-level set of \( \sigma_s \) in the Lagrangian cylinder \( Z_s \). In particular, \( K_{t,s} \) is \( \omega \)-isotropic. By Proposition 4.7, there exists \( a(s) \in \mathbb{R} \) such that
\[
\frac{d}{ds}Z_s = a(s)\sigma_s.
\]
Hence, by Remark 2.12, for \( v \in T_{\Psi_t(p, s)}K_{t,s} \) we have
\[
\omega\left( \frac{\partial}{\partial s}\Psi_t(p, s), v \right) = a(s)d(\sigma_s)_{\Psi_t(p, s)}(v) = 0,
\]
showing that \( \Lambda_i^U \) is Lagrangian.

It remains to establish positivity. Fix a basis \( v_1, \ldots, v_{n-1} \in T_{\Psi_t(p, s)}K_{t,s} \). Let \( 0 \neq w \in T_{\Psi_t(p, s)}\Lambda_i^U \) be orthogonal to \( K_{t,s} \). Define \( E_{p, t, s} \subset T_{\Psi_t(p, s)}X \) by
\[
E_{p, t, s} := (T\Psi_t(p, s)K_{t,s})^\perp \cap (T\Psi_t(p, s)K_{t,s})^\perp = \{ u \in T_{\Psi_t(p, s)}X \mid g(u, v_i) = 0 = \omega(u, v_i), \ i = 1, \ldots, n - 1 \}.
\]
Then, \( E_{p, t, s} \) is a \( J \)-invariant linear subspace of real dimension 2. It follows from Definition 4.17(2)(b) and the fact that \( Z_s \) is Lagrangian that \( \xi(p, t, s) \in E_{p, t, s} \). Since \( \Lambda_i^U \) is Lagrangian, it follows that \( w \in E_{p, t, s} \). Since \( \Phi \) is an immersion, \( \xi \neq 0 \), so
\[
w = \alpha \xi + \beta J\xi
\]
for some $\alpha, \beta \in \mathbb{R}$. Also, as $\Phi$ is an immersion, we have $\beta \neq 0$. By Lemma 2.17 and since $Z_s$ is imaginary special Lagrangian, we have

$$0 \neq \Omega(v_1, \ldots, v_{n-1}, \xi) \in \sqrt{-1}\mathbb{R}.$$  

Finally, as $\Omega$ is of type $(n, 0)$, we have

$$\text{Re } \Omega(v_1, \ldots, v_{n-1}, w) = \text{Re} \left( (\alpha + \sqrt{-1}\beta) \Omega(v_1, \ldots, v_{n-1}, \xi) \right)$$

$$= \sqrt{-1}\beta \Omega(v_1, \ldots, v_{n-1}, \xi)$$

$$\neq 0.$$  

Since $\Lambda^U_t$ is positive by assumption, the positivity of $\Lambda^U_t$ follows by continuity.

**Step 2:** *The family of immersions $(\Psi_t)_{t \in [0, 1]}$ is horizontal.* To prove this, we need to verify

$$\text{Re } \Omega(\xi, v_1, \ldots, v_{n-1}) = 0$$  

and

$$\text{Re } \Omega(\xi, v_1, \ldots, \tilde{v}_1, \ldots, v_{n-1}, w) = 0, \quad i = 1, \ldots, n - 1,$$  

where $v_1, \ldots, v_{n-1}$ and $w$ are as above. Equality (30) holds true as all the cylinders in $U$ are imaginary special Lagrangian. Equality (31) holds true by virtue of equality (29), as $\Omega$ is of type $(n, 0)$. This completes the proof of Step 2.

**Step 3:** *Pick $s_0 \in (0, 1)$. Define a family of functions $h^U_t \in C^\infty(\Lambda_s)$ by

$$h^U_t \circ \Psi_t(p, s_1) = \text{RelFlux} \left( (Z_s)_{s \in [s_0, s_1]} \right), \quad s_1 \in (0, 1).$$

In particular, $h^U_t \circ \Psi_t(p, s_1)$ is independent of $t$ and the point $p \in N$. Then,

$$\frac{d}{dt} \Lambda^U_t = dh^U_t.$$  

**Step 3:** To show this, recall the function $a : (0, 1) \to \mathbb{R}$ defined in equality (28). By Definition 4.11 we have

$$\frac{d}{ds_1} \text{RelFlux} \left( (Z_s)_{s \in [s_0, s_1]} \right) = -a(s_1).$$

So, recalling from Definition 4.17(2)(a) the meaning of $\Phi$ being compatible with the harmonics of $U$,

$$\frac{\partial}{\partial s} h^U_t \circ \Psi_t(p, s) = -a(s)$$

$$= -a(s) d(\sigma_t)_{\Psi_t(p, s)}(\xi(p, t, s))$$

$$= -\omega \left( \frac{\partial}{\partial s} \Psi_t(p, s), \xi(p, t, s) \right)$$

$$= \omega \left( \xi(p, t, s), \frac{\partial}{\partial s} \Psi_t(p, s) \right),$$

On the other hand, for $v$ a vector field on $N \times [0, 1]$ tangent to $N$, since $h^U_t \circ \Psi_t(p, s)$ is independent of $p$, we have

$$v \left( h^U_t \circ \Psi_t \right) = 0 = \omega(\xi, d\Psi_t(v)).$$

By Remark 2.12, it follows that $\frac{d}{dt} \Lambda^U_t = dh^U_t$. This completes the proof of Step 3.

Since $h^U_t \circ \Psi_t$ is independent of $t$, it follows that $(\Lambda^U_t)_{t \in [0, 1]}$ is a geodesic. By construction, the cylindrical transform of this geodesic is $U$. Finally, since by Definition 4.17 (1) the parameterization $\Phi$ is an immersion, the time derivative $\frac{d}{dt} \Psi_t = \frac{d}{dt} \Phi = \xi$ is nowhere vanishing, implying that $(\Lambda^U_t)_{t}$ has empty critical locus.
It remains to prove uniqueness. Let \((\Lambda_t^U)_{t \in [0,1]}\) be another geodesic with cylindrical transform \(U\). Let \((\Psi_t : t \in [0,1])\) be a horizontal lifting with \(\Psi_0 = \Psi_0\), and let \(h' : N \times \{0,1\} \to \mathbb{R}\) denote the associated Hamiltonian. Let \(h\) denote the Hamiltonian of \((\Lambda_t^U)_{t}\) with respect to the horizontal lifting \((\Psi_t)\). By Lemma 5.5 (b), after possibly adding a constant to \(h\), we have \(h' = h\). Let \(t_0 \in [0,1]\), let \(c\) be a value of \(h\), and let \(s \in (0,1)\) be such that \(h^{-1}(c) = N \times \{s\}\). Since the cylindrical transforms of \((\Lambda_t^U)_{t}\) and \((\Lambda_t''^U)_{t}\) coincide, the cylinder of \(c\) level sets \(Z_c\) associated with \((\Lambda_t^U)_{t}\) coincides with the cylinder of \(c\) level sets \(Z'_c\) associated with \((\Lambda_t''^U)_{t}\). By Remark 5.4, the immersed submanifolds \([\Psi_{t_0}|_{N \times \{s\}}]\) and \([\Psi_{t_0}|_{N \times \{s\}}]\) both coincide with the \(t_0\) level set of the fundamental harmonic of the cylinder \(Z_c = Z'_c\).

Since \(c\) was arbitrary, it follows that \(\Lambda_{t_0}^U = \Lambda_{t_0}'^U\). Since \(t_0\) was arbitrary, the claim follows.

**Lemma 5.8.** Let \(\Lambda_0, \Lambda_1 \subset X\) be smooth embedded positive Lagrangians intersecting transversely at a point \(q\). Suppose there exists a connected component \(Z \subset SLC(S^{n-1}; \Lambda_0, \Lambda_1)\) with an end \(E\) converging regularly to \(q\). Let \(U \subset Z\) be a ray representing \(E\) admitting a regular parameterization about \(q\). For \(i = 0,1\), let \(\Lambda_i^U\) denote the unpunctured submanifold of \(\Lambda_i\) swept by \(U\). Then, there exists a geodesic of positive Lagrangians between \(\Lambda_0^U\) and \(\Lambda_1^U\) with cylindrical transform \(U\). This geodesic is unique up to reparameterization and has critical locus \(\{q\}\).

**Proof.** Let \(\Phi : S^{n-1} \times [0,1] \times (0,\epsilon) \to X\) be a regular parameterization of \(U\) about \(q\) compatible with the harmonics. For \(t \in [0,1]\), define

\[
\Psi_t : S^{n-1} \times (0,\epsilon) \to X, \quad (c,s) \mapsto \Phi(c,t,s).
\]

Recalling Definition 3.3, let

\[
\kappa : S^{n-1} \times (0,\epsilon) \to \mathbb{R}^n
\]

be a polar coordinate map centered at zero with image \(W\). For \(t \in [0,1]\), let

\[
\Psi_t : W \to X
\]

be the unique map such that

\[
\Psi_t = \Psi_t \circ \kappa.
\]

By Remark 4.19, the pair \((\Psi_t,0)\) is a cone-immersion for \(t \in [0,1]\). Write

\[
\Psi_t^0 := \Psi_t|_{W\setminus\{0\}}, \quad \Lambda_t^U := [(\Psi_t,0)], \quad \Lambda_t''^U := [\Psi_t^0].
\]

By Proposition 5.7, the family \((\Lambda_t''^U)_{t \in [0,1]}\) is a geodesic with empty critical locus and cylindrical transform \(U\). Recalling the meaning of a regular parameterization about \(q\) from Definition 4.17(3)(a), it follows that \(\Psi_t(0) = q\) for \(t \in [0,1]\). Thus \(\Lambda_t^U \in \mathcal{L}(X,W;\{0\},\{q\})\) as in Definition 3.38. It follows from Lemma 3.39 that \((\Lambda_t^U)_{t}\) is a geodesic with critical locus \(\{q\}\). The uniqueness of \((\Lambda_t^U)_{t}\) follows from the uniqueness of \((\Lambda_t''^U)_{t \in [0,1]}\) given by Proposition 5.7.

The following lemmas are preliminary to the proof of Theorem 1.5. We first establish relevant notation. Let \((\Lambda_t)_{t \in [0,1]} \in \mathcal{L}(X,L;\{0\})\) be a geodesic. Let \((h_t : \Lambda_t \to \mathbb{R})_{t \in [0,1]}\) denote the Hamiltonian of \((\Lambda_t)_{t}\). Let \(\pi : \tilde{S} \to L\) denote the blowup projection. Let \((((\Psi_t : L \to X,S))_{t \in [0,1]}\) be a horizontal lifting of \((\Lambda_t)_{t}\), and let \(h : L \to \mathbb{R}\) be the Hamiltonian of \((\Lambda_t)_{t}\) with respect to \(((\Psi_t, S))_{t}\). That is,

\[
h_t = [((\Psi_t, S), h)].
\]

Then \((h,S)\) is cone-smooth and \(S\) is contained in its critical locus by Lemma 3.36. Recalling Definition 3.17, Remark 3.18, Lemma 3.20 and notation (10), let \(\nabla'^{h}h\)}
denote the gradient of $h$ with respect to the pullback metric $\Psi^* g$. So, $\nabla^t h$ is a cone-smooth vector field on $(L, S)$. Composing with the differential $d\Psi_t$, we consider $\nabla^t h$ as a cone-smooth vector field along $\Psi_t$. Recall that we denote by $\nabla^t h$ the blowup vector field, which is a section of $\pi^* \Psi^*_t TX$. Let $\theta_t = \theta_{\Psi_t} : \tilde{L}_S \to S^1$ denote the phase function.

**Lemma 5.9.** Let $p_0 \in S$, let $0 \neq v \in T_{p_0} L$ and let $\tilde{p} := [v] \in E_{p_0}$. Then,
\[
\frac{d}{dt} d\Psi_t(v) = -J\nabla_v \nabla^t h - \tan(\theta_t(\tilde{p})) \nabla_v \nabla^t h,
\]
where $\nabla$ is any connection on $TX$.

**Proof.** By [22, Remark 5.6], we have
\[
(32) \quad \frac{d}{dt} \Psi_t \circ \pi(p) = -J\nabla^t h(p) - \tan(\theta_t(p)) \nabla^t h(p), \quad p \in \tilde{L}_S, t \in [0, 1].
\]
Let $s, x^1, \ldots, x^{n-1}$, be local cone-coordinates on $L$ around $\tilde{p}$ with $\frac{dp}{dt} |_\tilde{p} = v$. As $p_0$ is a critical point of $h$ and by Remark 3.30, we have
\[
\nabla^t h(\tilde{p}) = 0, \quad t \in [0, 1].
\]
Hence, covariantly differentiating (32) at $\tilde{p}$ in the direction of $v$, we obtain
\[
\frac{d}{dt} d\Psi_t(v) = \frac{D}{dt} \frac{\partial \Psi_{\tilde{p}}}{\partial s}(\tilde{p}) = \frac{D}{\partial s} \frac{\partial \Psi_{\tilde{p}}}{\partial t}(\tilde{p}) = \nabla_v \left(-J\nabla^t h - \tan(\theta_t(\tilde{p})) \nabla^t h\right) = -J\nabla_v \nabla^t h - \tan(\theta_t(\tilde{p})) \nabla_v \nabla^t h.
\]

\[\Box\]

**Lemma 5.10.** Suppose $\Lambda_0$ and $\Lambda_1$ intersect transversally at $q \in C_0$. Then, $q$ is a non-degenerate critical point of $h_t$, $t \in [0, 1]$.

**Proof.** Let $p \in S$ with $\Psi_t(p) = q$ for $t \in [0, 1]$. By way of contradiction, suppose that $p$ is a degenerate critical point of $h$ and let $0 \neq v \in T_p L$ with $\nabla_v dh = 0$. It follows from Lemma 5.9 that
\[
d\Psi_0(v) = d\Psi_1(v)
\]
contradicting the transversality of $\Lambda_0$ and $\Lambda_1$.

\[\Box\]

**Lemma 5.11.** Suppose $\Lambda_0$ and $\Lambda_1$ intersect transversally at a single point, so $C_0 = \{q\}$. Moreover, for $t \in [0, 1]$ assume $q$ is an absolute minimum or maximum of $h_t$ and all the regular level sets of $h_t$ are $(n-1)$-dimensional spheres. Let $Z$ denote the cylindrical transform of $(\Lambda_t)_t$. Then, one end of $Z$ converges regularly to $q$.

**Proof.** Write $S = \{p\}$, so $\Psi_t(p) = q$. By Lemma 5.10, $p$ is a non-degenerate critical point of $h$. By Lemma 3.35, there exist a positive $\epsilon$ and a polar coordinate map $\kappa : S^{n-1} \times [0, \epsilon) \to L$ centered at $p$ such that for each $s \in (0, \epsilon)$ the restricted map $\kappa|_{S^{n-1} \times \{s\}}$ parameterizes a level set of $h$.

Define a map
\[
\Phi : S^{n-1} \times [0, 1] \times [0, \epsilon) \to X, \quad (c, t, s) \mapsto \Psi_t(\kappa(c, s)).
\]
We show that $\Phi$ is a regular parameterization of one end of $Z$ about the intersection point $q$. We verify the conditions of Definition 4.17 (3) one by one. Condition (a)
is immediate. By Lemma 5.5 (a), $\Phi|_{S^{n-1} \times [0,1] \times (0,\epsilon)}$ is interior regular verifying condition (b). To verify conditions (c) and (d), we show that
$$\frac{\partial \Phi}{\partial s} : S^{n-1} \times [0,1] \times \{0\} \to T_qX$$
is an immersion nowhere tangent to the Euler vector field. Indeed, by Lemma 3.5, for $t \in [0,1]$ the restriction
$$\frac{\partial \Phi}{\partial s} \bigg|_{S^{n-1} \times \{t\} \times \{0\}} : S^{n-1} \times \{t\} \times \{0\} \to T_qX$$
is an immersion nowhere tangent to the Euler vector field. Let $(c,t) \in S^{n-1} \times [0,1]$, and write
$$v = \frac{\partial \kappa}{\partial s}(c,0) \in T_pL, \quad \tilde{\rho} = [v] \in \mathbb{P}^+(T_pL).$$
Let
$$\tilde{q} = [(\mathbb{P}^+((d\Psi_t)_{\tilde{\rho}}), \tilde{\rho})] \in \mathbb{P}^+(TC_q\Lambda_t).$$
By Lemma 5.9 we have
$$(33) \quad \frac{\partial}{\partial s} d \Psi_t(c,t,0) = \frac{\partial}{\partial t} d \Psi_t(c,t,0).$$
Write $w = \nabla_v \tilde{\nabla}h$. As $p$ is a critical point of $h$, Remark 3.30 gives $\tilde{\nabla}h(\tilde{\rho}) = 0$, and it follows that $w \in T_q\Lambda_t \subset T_qX$. Moreover, $w \neq 0$ since $p$ is a non-degenerate critical point of $h$. Write
$$\Upsilon := d \left( \frac{\partial \Psi_t \circ \kappa}{\partial s} \bigg|_{s=0} \right) (T_cS^{n-1}) = d \left( \frac{\partial \Phi}{\partial s} \bigg|_{S^{n-1} \times \{t\} \times \{0\}} \right) (T_cS^{n-1}) \subset T_qX.$$Recalling Definition 3.3, let $\sigma = \frac{\partial \kappa}{\partial s} \big|_{s=0} : \mathbb{P}^+(T_pL) \to T_pL \setminus \{0\}$ denote the section associated to $\kappa$. Let $\tilde{p} \subset T_pL$ and $\tilde{\tilde{q}} \subset T_qX$ denote the 1-dimensional subspaces generated by the rays $\tilde{\rho}, \tilde{\tilde{q}}$, respectively. Then,
$$\tilde{T}L_S|_{\tilde{p}} = T_pL = \tilde{\rho} \oplus d\sigma (T_p\mathbb{P}^+(T_pL)).$$So, we obtain
$$T_{\tilde{q}}\Lambda_t = d\tilde{\Psi}_t \left( \left( T\tilde{L}_S \right)_{\tilde{p}} \right) = \tilde{\tilde{q}} \oplus \Upsilon.$$Since $T_{\tilde{q}}\Lambda_t$ is Lagrangian, we have $T_{\tilde{q}}\Lambda_t \cap JT_{\tilde{q}}\Lambda_t = \{0\}$. It follows from equation (33) that
$$\frac{\partial}{\partial t} \frac{\partial \Phi}{\partial s}(c,t,0) \notin T_{\tilde{q}}\Lambda_t.$$So, the linear subspace
$$\mathbb{R} \left( \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial s}(c,t,0) \right) \oplus T_{\tilde{q}}\Lambda_t =$$
$$= \mathbb{R} \left( \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial s}(c,t,0) \right) \oplus \tilde{\tilde{q}} \oplus d \left( \frac{\partial \Phi}{\partial s} \bigg|_{S^{n-1} \times \{t\} \times \{0\}} \right) (T_cS^{n-1}) \subset T_qX$$
is $n+1$ dimensional. Since the Euler vector field $\varepsilon$ on $T_qX$ satisfies
$$0 \neq \varepsilon \left( \frac{\partial \Phi}{\partial s}(c,t,0) \right) \in \tilde{\tilde{q}},$$the lemma follows. \hfill \Box

We henceforth restrict the discussion to the setting of Theorem 1.5. Namely, we assume $\Lambda_0, \Lambda_1 \subset X$ are smooth embedded positive Lagrangian spheres intersecting transversally at the points $q_0$ and $q_1$ and nowhere else.
Remark 5.12. Note that, under the above assumption, if there exists a geodesic connecting \( \Lambda_0 \) and \( \Lambda_1 \), its critical locus necessarily consists of exactly two points. In particular, letting \( h \) denote the Hamiltonian of the geodesic and \([m,M]\) the image of \( h \), every \( c \in (m,M) \) is a regular value.

We define regularity for a connected component in \( \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \).

**Definition 5.13.** Let \( Z \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) be a connected component. A regular parameterization of \( Z \) is a smooth map \( \Phi : S^{n-1} \times [0,1] \times [0,1] \to X \) satisfying the following conditions:

1. The restricted map \( \Phi|_{S^{n-1} \times [0,1] \times (0,1)} \) is an interior-regular parameterization of \( Z \).
2. The restricted maps \( \Phi|_{S^{n-1} \times [0,1] \times [0,1/2]} \) and \( \Phi|_{S^{n-1} \times [0,1] \times (1/2,1]} \) are regular parameterizations of the two ends of \( Z \) about the intersection points \( q_0 \) and \( q_1 \), respectively.

We say \( Z \) is regular if it admits a regular parameterization.

**Remark 5.14.**

(a) A connected component \( Z \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) is regular if and only if each \( Z \subset \mathcal{SLC} \) has an interior-regular neighborhood and the ends of \( Z \) converge regularly to the intersection points \( q_0 \) and \( q_1 \) respectively. This follows from Lemma 4.20 and Corollary 4.21. Indeed, we use the compactness of \([0,1]\) and the uniqueness claim to glue together parameterizations of the ends of \( Z \) with finitely many interior regular parameterizations of open intervals in \( Z \).

(b) Suppose \( Z \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) is a regular connected component. Then \( Z \) admits a regular parameterization compatible with the harmonics. This again follows from Lemma 4.20.

**Proof of Theorem 1.5.** Suppose there exists a geodesic \( (\Lambda_t)_{t \in [0,1]} \) between the positive Lagrangian spheres \( \Lambda_0 \) and \( \Lambda_1 \). Let \( Z \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) denote the cylindrical transform. Then Lemma 5.5, Lemma 5.11 and Remark 5.14(a), imply that \( Z \) is regular.

Conversely, let \( Z \subset \mathcal{SLC} (S^{n-1}; \Lambda_0, \Lambda_1) \) be a regular connected component. Let \( \Phi : S^{n-1} \times [0,1] \times [0,1] \) be a regular parameterization of \( Z \). It follows from Remark 4.19 and Corollary 3.8 that the image of \( \Phi|_{S^{n-1} \times [0,1] \times (1/2,1]} \subset \mathcal{SLC} \) is open. Since \( S^{n-1} \times [0,1] \) is compact, the image is also closed. Thus, the submanifolds swept by \( Z \) are \( \Lambda_i \setminus \{q_0, q_1\} =: \Lambda_i^r \), \( i = 0,1 \). By Proposition 5.7, there exists a geodesic \( (\Lambda_i^r)_{t \in [0,1]} \) between \( \Lambda_0^r \) and \( \Lambda_1^r \) with cylindrical transform \( \Phi \). For \( j = 0,1 \), let \( U_j \) be a ray representing the end of \( Z \) that converges regularly to \( q_j \). Let \( \Lambda_i^U_j \) denote the unpunctured submanifold of \( \Lambda_i \) swept by \( U_j \). By Lemma 5.8, there exists a geodesic \( (\Lambda_i^U_j)_{t \in [0,1]} \) between \( \Lambda_0^U_j \) and \( \Lambda_1^U_j \) with cylindrical transform \( U_j \). By the uniqueness claim of Proposition 5.7, the three above geodesics glue together to the desired geodesic between \( \Lambda_0 \) and \( \Lambda_1 \).

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6. **Proof of the perturbation theorem**

In this section we prove Theorem 1.6. We assume the setting of the theorem. That is, \( \mathcal{O} \) is a Hamiltonian isotopy class of positive Lagrangian spheres in \( X \), the spheres \( \Lambda_0, \Lambda_1 \in \mathcal{O} \) intersect transversally at \( \Lambda_0 \cap \Lambda_1 = \{q_0, q_1\} \), and \( (\Lambda_t)_{t \in [0,1]} \) is a geodesic between \( \Lambda_0 \) and \( \Lambda_1 \). Let \( \mathcal{Z} \) denote the cylindrical transform of \( (\Lambda_t)_{t \in [0,1]} \). By Theorem 1.5 the cylindrical transform \( \mathcal{Z} \) is a regular connected component of the
1-dimensional manifold $\mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_1)$. Let $\Phi : S^{n-1} \times [0, 1] \times [0, 1] \to X$ be a regular parameterization of $Z$. For $s \in (0, 1)$, we write
$$\Phi_s := \Phi|_{S^{n-1} \times [0, 1] \times \{s\}}, \quad Z_s := [\Phi_s] \in Z.$$ Fix a Weinstein neighborhood $\Lambda_1 \subset W \subset X$ with projection $\pi : W \to \Lambda_1$ satisfying
$$W \cap \Lambda_0 = \pi^{-1}(\{q_0, q_1\}).$$ In particular, spheres in $\Omega$ that are close to $\Lambda_1$ are identified with exact 1-forms on $\Lambda_1$. For a function $h \in C^\infty(\Lambda_1)$ small enough in the $C^1$-sense, we let $\Lambda_{1,h}$ denote the element of $O$ obtained by this identification as the graph of $dh$. We write $\Lambda_0 \cap \Lambda_{1,h} = \{q_0, h, q_1, h\}$. Thus a sufficiently small $C^{k+1,\alpha}$ open set $0 \in W \subset C^\infty(\Lambda_1)$ corresponds to a $C^{k,\alpha}$ open neighborhood of $\Lambda_1$ in $O$.

**Notation 6.1.** Let $\chi : X \to [0, 1]$ be smooth with compact support contained in $W$, and let $h \in C^{k,\alpha}(\Lambda_1)$ for some $k \geq 2$ and $\alpha \in [0, 1]$. Then the function $\chi \cdot h \circ \pi$ is well-defined and of regularity $C^{k,\alpha}$ on $X$ with compact support. We let $\varphi_{h,\chi}$ denote the time-1 Hamiltonian flow of $\chi \cdot h \circ \pi$.

**Remark 6.2.** Let $\chi$ and $h$ be as in Notation 6.1. Then the map $\varphi_{h,\chi} : X \to X$ is a $C^{k+1,\alpha}$ symplectomorphism of $X$ restricting to the identity on $\Lambda_0$.

In the proofs below, we will use the following well-known technical lemma. Let $\Omega_{C^{k,\alpha}}$ denote differential forms of regularity $C^{k,\alpha}$.

**Lemma 6.3.** Let $P, Q, W$ be smooth manifolds. For $r, s, l \geq 1$, the map
$$C^{r,\alpha}(P, Q) \times \Omega_{C^{s,\alpha}}(Q) \to \Omega_{C^\min(r-1,s-1),\alpha}(P), \quad (g, \eta) \mapsto g^*\eta,$$ is of regularity $C^l$.

Abbreviate $L = S^{n-1} \times [0, 1]$.

**Proposition 6.4.** Fix $\alpha \in (0, 1)$. Let $s_0 \in (0, 1)$. Then there exist a positive $\epsilon$, a $C^{3,\alpha}$-open $0 \in W \subset C^\infty(\Lambda_1)$ and a family of smooth immersions
$$f_{s,h} : L \to X, \quad (s, h) \in (s_0 - \epsilon, s_0 + \epsilon) \times W,$$ smooth in $s$ and continuous with respect to the $C^{3,\alpha}$ topology on $W$ and the $C^{1,\alpha}$ topology on $C^\infty(L, X)$ with the following properties:

1. For $s$ and $h$ as above the immersion $f_{s,h}$ represents an imaginary special Lagrangian cylinder $Z_{s,h} \in \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h})$.
2. We have $f_{s,h} = \Phi_s$ for $s \in (s_0 - \epsilon, s_0 + \epsilon)$.
3. For $h \in W$ the family of cylinders $(Z_{s,h})$ is interior regular. Moreover, there exists a $C^{1,\alpha}$ open set $V \subset C^\infty(L, X)$ with
$$f_{s,h} \in V, \quad (s, h) \in (s_0 - \epsilon, s_0 + \epsilon) \times W,$$ such that if $f \in V$ represents an imaginary special Lagrangian cylinder
$$[f] \in \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h})$$ with $h \in W$, then $[f] = [f_{s,h}]$ for some $s \in (s_0 - \epsilon, s_0 + \epsilon)$.

**Proof.** By Lemma 2.8, choose an immersed Weinstein neighborhood $(Y, \psi)$ of $Z_{s_0}$ compatible with $\Lambda_0$ and $\Lambda_1$, where $Y \subset T^*L$ and $\psi : Y \to X$ with $\psi|_L = \Phi_{s_0}$. Let $\pi_L : T^*L \to L$ denote the projection. For $u \in C^2(\Omega_{C^{\infty}}(L))$, let $\text{Graph}(du) \subset T^*L$ denote the graph. For $u$ small enough that $\text{Graph}(du) \subset Y$, let $j_u : L \to X$ be given by
$$j_u = \psi \circ (\pi_L|_{\text{Graph}(du)})^{-1}.$$ Let $U \subset X$ be an open set such that
$$\overline{U} \subset W \setminus \Lambda_0, \quad \Phi_{s_0}(S^{n-1} \times \{1\}) \subset U.$$
Let $\chi : X \to [0, 1]$ be a smooth function such that $\chi|_U \equiv 1$ and $\text{supp}(\chi) \subset W \setminus \Lambda_0$. Choose an open set $\Phi_{x_0}(S^{n-1} \times \{1\}) \subset U'' \subset U$ and an open set $0 \in A \subset C^{3,\alpha}(A_1)$ such that for $h \in A$ we have

$$\varphi_{h,\chi}(U'' \cap A_1) \subset A_{1,h}.$$ 

Let $0 \in U \subset C^{2,\alpha}_{\text{COB}}(L)$ be open such that for $u \in U$, we have $j_u(S^{n-1} \times \{1\}) \subset U'$. Consider the differential operator

$$F : U \times A \to C^\alpha(L), \quad (u, h) \mapsto j_u^*\varphi_{h,\chi}^*\Re \Omega.$$ 

By Lemma 4.5 and Lemma 4.6, the restriction of $\varphi_{h,\chi}$ to $j_u$ represents a Lagrangian cylinder in $\mathcal{L}C(S^{n-1}; \Lambda_0, A_{1,h})$, and this cylinder is imaginary special Lagrangian if and only if $F(u, h) = 0$.

We claim $F$ is continuously differentiable, and for a fixed $h \in A \cap C^\infty(A_1)$, the map $u \mapsto F(u, h)$ is smooth. Indeed, recalling Remark 6.2, since the map

$$A \to C^{2,\alpha}(X, X), \quad h \mapsto \varphi_{h,\chi},$$

is continuously differentiable, it follows from Lemma 6.3 that the map

$$A \to \Omega_{C^{1,\alpha}}(X), \quad h \mapsto \varphi_{h,\chi}^*\Re \Omega,$$

is continuously differentiable. Similarly, since the map

$$U \to C^{1,\alpha}(L, X), \quad u \mapsto j_u,$$

is smooth, it follows from Lemma 6.3 that the map

$$U \times \Omega_{C^{1,\alpha}}(X) \to \Omega_{C^\alpha}(L), \quad (u, \eta) \mapsto j_u^*\eta,$$

is of regularity $C^\alpha$. So, the map $F$ is continuously differentiable as the composition of two continuously differentiable maps. Moreover, for fixed $h \in A \cap C^\infty(A_1)$, we have $\varphi_{h,\chi}^*\Re \Omega \in \Omega_{C^\infty}(X)$, so the map $u \mapsto F(u, h)$ is smooth.

Consider the linearization of $F$,

$$dF_{(0,0)} : C^{2,\alpha}_{\text{COB}}(L) \times C^{3,\alpha}(A_1) \to C^\alpha(L).$$

By Lemma 4.5 and Lemma 4.6, the restriction of $dF_{(0,0)}$ to the subspace

$$C^{2,\alpha}(L; \partial L) \times \{0\} \subset C^{2,\alpha}_{\text{COB}}(L) \times C^{3,\alpha}(A_1)$$

is an isomorphism onto $C^\alpha(L)$. Abbreviate $V := C^{2,\alpha}(L; \partial L)$. Recall that

$$\text{codim} \left( V \subset C^{2,\alpha}_{\text{COB}}(L) \right) = 1.$$ 

Let $\ell \subset C^{2,\alpha}_{\text{COB}}(L)$ be a line consisting of smooth functions such that

$$C^{2,\alpha}_{\text{COB}}(L) = V \oplus \ell.$$ 

By the implicit function theorem, there exist open neighborhoods

$$0 \in V_0 \subset V, \quad 0 \in \ell_0 \subset \ell, \quad 0 \in A_0 \subset A,$$

such that for $(l, h) \in \ell_0 \times A_0$ there exists a unique $v = v(l, h) \in V_0$ with

$$F(v + l, h) = 0.$$ 

By elliptic regularity (e.g. [12, Chapter 17]), the function $v(l, h)$ is smooth if $h$ is. Write

$$W := C^\infty(A_1) \cap A_0.$$ 

Define a family of immersions by

$$\tilde{f}_{l,h} := \varphi_{h,\chi} \circ j_v(l,h)+l, \quad (l, h) \in \ell_0 \times W.$$ (35)
By construction, the immersions $\tilde{f}_{l,h}$ represent special Lagrangian cylinders in $SLC(S^{n-1}; \Lambda_0, \Lambda_1, h)$ for $(l, h) \in \ell_0 \times W$. Moreover, the map 
$$\ell_0 \times W \to C^\infty(L, X), \quad (l, h) \mapsto \tilde{f}_{l,h},$$

is continuous with respect to the $C^{3, \alpha}$ topology on $W$ and the $C^{1, \alpha}$ topology on $C^\infty(L, X)$. Also, for fixed $h \in W$, the map $l \mapsto \tilde{f}_{l,h}$ is smooth.

For a small $\epsilon > 0$, there is a unique open embedding $i : (s_0 - \epsilon, s_0 + \epsilon) \hookrightarrow \ell_0$ such that
$$\left[\tilde{f}_{i(s),0}\right] = Z_s, \quad s \in (s_0 - \epsilon, s_0 + \epsilon).$$

For $s \in (s_0 - \epsilon, s_0 + \epsilon)$, let $\zeta_s \in \text{Diff}(L)$ be the diffeomorphism such that $\tilde{f}_{i(s),0} \circ \zeta_s = \Phi_s$.

Observe that the map
$$(s_0 - \epsilon, s_0 + \epsilon) \to C^\infty(L, L), \quad s \mapsto \zeta_s$$
is smooth. Take $$f_{s,h} := \tilde{f}_{i(s),h} \circ \zeta_s.$$ Since the maps $(l, h) \mapsto f_{l,h}$ and $s \mapsto \zeta_s$ are continuous in the topologies specified above, it follows that the map $(s, h) \mapsto f_{s,h}$ is continuous as desired. Moreover, for fixed $h \in W$, the map $s \mapsto f_{s,h}$ is smooth.

Write $Z_{s,h} := [f_{s,h}]$. Properties (1) and (2) claimed in the proposition are immediate from the construction. To establish property (3), we claim that after possibly shrinking $W$, for $h \in W$, the map $$\Phi^h : L \times (s_0 - \epsilon, s_0 + \epsilon) \to X,$$given by $$\Phi^h(p, s) = f_{s,h}(p)$$is a regular parameterization of the family $(Z_{s,h})_{s \in (s_0 - \epsilon, s_0 + \epsilon)}$. Indeed, the map $\Phi^h$ is smooth because the map $s \mapsto f_{s,h}$ is smooth. Conditions (b) and (c) of Definition 4.17 (1) hold by construction. The remaining conditions in Definition 4.17 (1) are open, and $\Phi^0 = \Phi|_{S^{n-1} \times [0,1] \times (s_0 - \epsilon, s_0 + \epsilon)}$, which is interior regular. So, possibly after shrinking $W$ and $\epsilon$, the map $\Phi^h$ is also an interior regular parameterization for $h \in W$.

Finally, we construct the open set $V$. Given $Q \subset W$ and $P \subset C^\infty(L, X)$, let $$\mathcal{B}(Q, P) := \{(h, f) \in Q \times P \mid f \text{ represents a cylinder } [f] \in \mathcal{L}(S^{n-1}; \Lambda_0, \Lambda_1, h)\}.$$ First, we claim that perhaps after shrinking $W$, there exists a $C^{1, \alpha}$ open set $V_1 \subset C^\infty(L, X)$ such that $$\Phi_s \in V_1, \quad s \in (s_0 - \epsilon, s_0 + \epsilon),$$and if $(h, f) \in \mathcal{B}(W, V_1)$, then $$\varphi^{-1}_{h,\chi} \circ f(L) \subset \psi(Y).$$ Here, the set $\psi(Y)$ is not open in $X$ because $Y$ is a manifold with boundary. Indeed, $Y$ is an open neighborhood of the zero section of $T^*L$, and $L$ is a manifold with boundary. Nonetheless, for $i = 0, 1$, we have $\varphi^{-1}_{h,\chi} \circ f(S^{n-1} \times \{i\}) \subset \Lambda_i$. Moreover, $\varphi^{-1}_{h,\chi} \circ f$ is close in the $C^{1, \alpha}$ topology to $\Phi_s$ for some $s \in (s_0 - \epsilon, s_0 + \epsilon)$, and $\Phi_s(L) \subset \psi(Y)$. So, the claim follows.

For $(h, f) \in \mathcal{B}(W, V_1)$, let $$\kappa_{h,f} := \pi_L \circ \psi^{-1} \circ \varphi^{-1}_{h,\chi} \circ f : L \to L.$$ The map $$\mathcal{B}(W, V_1) \to C^\infty(L, L), \quad (h, f) \mapsto \kappa_{h,f},$$
is continuous with respect to the $C^{3,\alpha}$ topology on $W$ and the $C^{1,\alpha}$ topologies on $V_1$ and $C^\infty(L,L)$. Moreover, diffeomorphisms are open in $C^\infty(L,L)$ in the $C^{1,\alpha}$ topology and $\kappa_{0,h} = \zeta_s$ is a diffeomorphism for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. So, possibly shrinking $W$, we choose an open $V_2 \subset V_1$ such that $\Phi_s \in V_2$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ and if $(h,f) \in B(W,V_2)$, then $\kappa_{h,f}$ is a diffeomorphism.

For $(h,f) \in B(W,V_2)$, by Lemma 4.4 there exists $u_{h,f} \in C^\infty_{C_{OB}}(L)$ such that

$$j_{u_{h,f}} = \kappa_{h,f}^{-1} \circ f \circ \kappa_{h,f}^{-1}.$$  

The map

$$B(W,V_2) \to C^\infty_{C_{OB}}(L), \quad (h,f) \mapsto u_{h,f},$$

is continuous with respect to the $C^{3,\alpha}$ topology on $W$, the $C^{1,\alpha}$ topology on $V_2$ and the $C^{2,\alpha}$ topology on $C^\infty_{C_{OB}}(L)$. Hence, possibly shrinking $W$, we choose an open $V \subset V_2$ such that $\Phi_s \in V$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ and if $(h,f) \in B(W,V)$, then

$$u_{h,f} \in V_0 + \ell_0.$$  

Fix a family of smooth maps

$$f_{s,h} : L \to X, \quad (s,h) \in (0,\varepsilon) \times W,$

smooth in $s$ and continuous with respect to the $C^{3,\alpha}$ topology on $W$ and the $C^{1,\alpha}$ topology on $C^\infty(L,X)$ with the following properties:

(1) For $(s,h) \in (0,\varepsilon) \times W$ the map $f_{s,h}$ is an immersion representing an imaginary special Lagrangian cylinder $Z_{s,h} \in \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h})$.

(2) We have $f_{s,0} = \Phi_s$ for $s \in [0,\varepsilon)$.

(3) For $h \in W$, the map

$$\Phi^h : L \times [0,\varepsilon) \to X,$$

given by

$$\Phi^h(p,s) = f_{s,h}(p)$$

is a regular parameterization about $q_{0,h}$ of the family of imaginary special Lagrangian cylinders $(Z_{s,h})_{s \in (0,\varepsilon)}$.

(4) The map $W \to C^\infty(L,tx)$ given by

$$h \mapsto \frac{\partial f_{s,h}}{\partial s} \bigg|_{s=0}$$

is continuous with respect to the $C^{3,\alpha}$ topology on $W$ and the $C^{1,\alpha}$ topology on $C^\infty(L,tx)$.

A similar family of smooth immersions $f_{s,h}$ exists for $s \in (1 - \varepsilon, 1]$.

**Proof.** We prove the proposition for $s \in (0,\varepsilon)$. Let $\chi : X \to [0,\varepsilon]$ be smooth with compact support in $W$ and equal to $1$ in a neighborhood of the intersection point $q_0$. Then there exist open sets $q_0 \in U \subset X$ and $0 \in A_1 \subset C^1(\Lambda_1)$ such that for $h \in A_1$ we have

$$\varphi_{h,\chi}(U \cap \Lambda_1) \subset \Lambda_{1,h}, \quad \varphi_{h,\chi}(U \cap \Lambda_0) \subset \Lambda_0.$$  

Let $\delta > 0$ such that for $s \in (0,\delta)$ we have $Z_s \subset U$. Thus, for $s \in (0,\delta)$ and $h \in A_1$, $[\varphi_{h,\chi} \circ \Phi_s] \in \mathcal{LC}(S^{n-1}; \Lambda_0, \Lambda_{1,h})$.  

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After possibly shrinking $U$, identify $U$ with a ball $V \subset \mathbb{C}^n$ via a Darboux parameterization

$$X : V \to U$$

such that $X^{-1}(\Lambda_0)$ and $X^{-1}(\Lambda_1)$ are contained in real linear subspaces. For $s \geq 0$, let $M_s : \mathbb{C}^n \to \mathbb{C}^n$ denote multiplication by $s$, and write

$$V_s := M_s^{-1}(V).$$

Let $\mathcal{A} := \mathcal{A}_1 \cap C^{3,\alpha}(\Lambda_1)$. For $h \in \mathcal{A}$, define a complex structure and an $n$-form on $V$ by

$$J_h = J_{h,1} := X^* \varphi_{h,\alpha}^* J, \quad \Omega_h = \Omega_{h,1} := X^* \varphi_{h,\alpha}^* \Omega.$$  

For $h \in \mathcal{A}$ and $s \in (0, \delta)$, define a complex structure and an $n$-form on $V_s$ by

$$J_{h,s} := M_s^* J_h, \quad \Omega_{h,s} := s^{-n} M_s^* \Omega_h.$$  

The complex structures and $n$-forms defined in this manner are of regularity $C^{1,\alpha}$. For $h \in \mathcal{A}$ we have

$$J_{h,s} \xrightarrow{s \searrow 0} J_{h,0}, \quad \Omega_{h,s} \xrightarrow{s \searrow 0} \Omega_{h,0},$$

where $J_{h,0}$ and $\Omega_{h,0}$ are a constant complex structure and a constant $n$-form on $V_0 = \mathbb{C}^n$, and the convergence is with respect to the $C^{1,\alpha}$ topology on compact subsets. Moreover, writing $\Omega_{C^{\infty}}$ for differential forms of regularity $C^{\infty}$, for $s_0 \in (0, \delta)$ the map

$$[0, s_0) \to \Omega_{C^{\infty}}(V_{s_0}), \quad s \mapsto \Omega_{h,s}|V_{s_0},$$

is continuously differentiable.

Recall the regular parameterization $\Phi : S^{n-1} \times [0, 1] \times [0, 1] \to X$. By the choice of $\delta$, we have $\Phi(S^{n-1} \times [0, 1] \times [0, \delta)) \subset U$. By Lemma 3.14, we have

$$X^{-1} \circ \Phi(p, t, s) = s \cdot \Psi(p, t, s), \quad (p, t, s) \in S^{n-1} \times [0, 1] \times [0, \delta),$$

where $\Psi : S^{n-1} \times [0, 1] \times [0, \delta) \to \mathbb{C}^n$ is smooth with

$$\Psi(p, t, 0) = \frac{\partial(X^{-1} \circ \Phi)}{\partial s}(p, t, 0), \quad (p, t) \in S^{n-1} \times [0, 1].$$

For $s \in [0, \delta)$, write

$$\Psi_s := \Psi|_{S^{n-1} \times [0, 1] \times \{s\}}.$$  

For $s \in (0, \delta)$, the map $\Psi_s$ is an immersion representing an $\Omega_{h,s}$-imaginary special Lagrangian cylinder. As $\Phi$ is regular, it follows from Definition 5.13 and Definition 4.17 (3) that the map $\Psi_0$ is an immersion nowhere tangent to the Euler vector field. Thus,

$$Z_{0,0}' := [\Psi_0]$$

is an $\Omega_{0,0}$-special Lagrangian cylinder.

By Lemma 2.8, choose an immersed Weinstein neighborhood $(Y, \psi)$ of $Z_{0,0}'$ compatible with $X^{-1}(\Lambda_0)$ and $X^{-1}(\Lambda_1)$, where $Y \subset T^*L$ and $\psi : Y \to \mathbb{C}^n$ with $\psi|L = \Psi_0$. Let $\pi_L : T^*L \to L$ denote the projection. For $u \in C^{2,\alpha}_{C^{\infty}}(L)$, let $\text{Graph}(du) \subset T^*L$ denote the graph. Let $0 \in U \subset C^{2,\alpha}_{C^{\infty}}(L)$ be open such that for $u \in U$ we have $\text{Graph}(du) \subset Y$. For $u \in U$, let $j_u : L \to X$ be given by

$$j_u = \psi \circ (\pi_L|_{\text{Graph}(du)})^{-1}.$$  

If necessary, diminish $\delta$ so that $\psi(Y) \subset V_s$ for $s \in [0, \delta)$. Define a differential operator

$$\mathcal{F} : U \times \mathcal{A} \times [0, \delta) \to C^{\infty}(L), \quad (u, h, s) \mapsto j_u^* \Re \Omega_{h,s}.$$  

For $(u, h, s) \in U \times \mathcal{A} \times (0, \delta)$, the immersion $\varphi_{h,\chi} \circ X \circ M_s \circ j_u$ represents a Lagrangian cylinder in $\mathcal{C}(S^{n-1}; \Lambda_0, \Lambda_1, h)$, and this cylinder is imaginary special Lagrangian if and only if $\mathcal{F}(u, h, s) = 0$.  

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We claim \( \mathcal{F} \) is continuously differentiable, and for a fixed \( h \in A \cap C^\infty(\Lambda_1) \), the map \( (u, s) \mapsto \mathcal{F}(u, h, s) \) is smooth. Indeed, recalling Remark 6.2, since the map
\[
A \to C^{2,\alpha}(V, X), \quad h \mapsto \varphi_{h, X} \circ X,
\]
is continuously differentiable, it follows from Lemma 6.3 that the map
\[
C \quad \text{is of regularity}
\]
is continuously differentiable. Similarly, since the map
\[
\Omega \quad \text{is smooth}, \quad \text{it follows from Lemma 6.3 that the map}
\]
is smooth, it follows from Lemma 6.3 that the map
\[
U \times [0, \delta) \to C^{4,\alpha}(L, V), \quad (u, s) \mapsto M_s \circ j_u,
\]
is smooth, and for a fixed \( h \in A \cap C^\infty(\Lambda_1) \), we have
\[
\Omega_h = X^* \varphi_{h, X}^* \Re \Omega \in \Omega^*_{C^\infty}(V), \quad \text{so the map } (u, s, \eta) \mapsto \mathcal{F}(u, h, s) \text{ is smooth}.
\]

Consider the linearization of \( \mathcal{F} \),
\[
d\mathcal{F}_{(0,0,0)} : C^{2,\alpha}_{COB}(L) \times C^{3,\alpha}(A_1) \times \mathbb{R} \to C^\alpha(L).
\]
By Lemma 4.5 and Lemma 4.6, the restriction of \( d\mathcal{F}_{(0,0,0)} \) to the subspace
\[
C^{2,\alpha}(L; \partial L) \times \{0\} \subset C^{2,\alpha}_{COB}(L) \times C^{3,\alpha}(A_1) \times \mathbb{R}
\]
is an isomorphism onto \( C^\alpha(L) \). Abbreviate \( \mathcal{V} := C^{2,\alpha}(L; \partial L) \). Let \( \ell \subset C^{2,\alpha}_{COB}(L) \) be a one dimensional subspace consisting of smooth functions such that
\[
C^{2,\alpha}_{COB}(L) = \mathcal{V} \oplus \ell.
\]
By the implicit function theorem, there exist open neighborhoods
\[
0 \in \mathcal{V}_0 \subset \mathcal{V}, \quad 0 \in \ell_0 \subset \ell, \quad 0 \in A_0 \subset A,
\]
and \( \epsilon \leq \delta \) such that for \( (l, h, s) \in \ell_0 \times A_0 \times [0, \epsilon) \) there exists a unique
\[
v = v(l, h, s) \in \mathcal{V}_0
\]
with
\[
\mathcal{F}(v + l, h, s) = 0.
\]
By elliptic regularity (e.g. [12, Chapter 17]), the function \( v(l, h, s) \) is smooth if \( h \) is.

Write
\[
\mathcal{W} := C^\infty(\Lambda_1) \cap A_0.
\]
Since \( \mathcal{F} \) is continuously differentiable, it follows that
\[
v : \ell_0 \times A_0 \times [0, \epsilon) \to \mathcal{V}_0
\]
is continuously differentiable. Moreover, since for fixed \( h \in C^\infty(\Lambda_1) \cap A \) the map 
\((u, s) \mapsto F(u, h, s)\) is smooth, it follows that for fixed \( h \in W \) the map 
\[ t_0 \times [0, \epsilon) \to V_0, \quad (l, s) \mapsto v(l, h, s), \]
is smooth.

Since \( \Psi_0 = j_0 \) and \((\Psi_s)_{s \in [0, \delta)}\) is a smooth family of immersions, after possibly shrinking \( \epsilon \), for each \( s \in [0, \epsilon) \) there exists a unique \( u_s \in V_0 + t_0 \) such that \( j_{u_s} \) and \( \Psi_s \) represent the same immersed \( \Omega_{0, s} \)-imaginary special Lagrangian cylinder. In particular, \( F(u_s, 0, s) = 0 \). Decompose 
\[ u_s = v_s + l_s, \quad v_s \in V_0, \quad l_s \in t_0. \]
Since \( v = v(l_s, 0, s) \) is the unique solution to \( F(v + l_s, 0, s) = 0 \), we conclude that 
\[ v(l_s, 0, s) = v_s. \]

Define a family of smooth maps 
\[ \tilde{f}_{s,h} := \varphi_{h,X} \circ X \circ M_s \circ j_{u_s + v(l_s, h, s)}, \quad (s, h) \in [0, \epsilon) \times W; \]
For \((s, h) \in (0, \epsilon) \times W\), the maps \( f_{s,h} \) are immersions representing imaginary special Lagrangian cylinders in \( S\mathcal{L}_C(S^{n-1}; \Lambda_0, \Lambda_{1, h}) \). Moreover, the map 
\[ [0, \epsilon) \times W \to C^\infty(L, X), \quad (s, h) \mapsto \tilde{f}_{s,h}, \]
is continuous with respect to the \( C^{3, \alpha} \) topology on \( W \) and the \( C^{1, \alpha} \) topology on \( C^\infty(L, X) \). Also, for fixed \( h \in W \), the map \( s \mapsto f_{s,h} \) is smooth.

Recall that for \( s \in [0, \epsilon) \), the immersion \( j_{u_s} \) represents the same immersed cylinder as the immersion \( \Psi_s \). Let \( \zeta_s \in \text{Diff}(L) \) be the diffeomorphism such that 
\[ j_{u_s} \circ \zeta_s = \Psi_s. \]
Observe that the map 
\[ [0, \epsilon) \to C^\infty(L, L), \quad s \mapsto \zeta_s, \]
is smooth. Moreover, since \( \Psi_0 = j_0 \), we have 
\[ \zeta_0 = \text{id}_L. \]
By equation (40) we have \( u_s = l_s + v(l_s, 0, s) \), so it follows from equation (41) that 
\[ \tilde{f}_{s,h} \circ \zeta_s = \Phi_s. \]
Take 
\[ f_{s,h} := \tilde{f}_{s,h} \circ \zeta_s. \]
Since the maps \((s, h) \mapsto \tilde{f}_{s,h} \) and \( s \mapsto \zeta_s \) are continuous in the topologies specified above, it follows that the map \((s, h) \mapsto f_{s,h} \) is continuous as desired. Moreover, for fixed \( h \in W \), the map \( s \mapsto f_{s,h} \) is smooth.

Write \( Z_{s,h} := [f_{s,h}] \) for \((s, h) \in (0, \epsilon) \times W\). Properties (1) and (2) claimed in the proposition are immediate from the construction. We proceed with the proof of property (4). Indeed, by equation (42), we have 
\[ \frac{\partial f_{s,h}}{\partial s} \bigg|_{s=0} = d(\varphi_{h,X} \circ X) \circ j_{0,0 + v(l_0, h, 0)}. \]
So, the continuity of the map (36) follows from the continuity of the maps (37) and (39).

To establish property (3) claimed in the proposition, we argue as follows. The map \( \Phi^h \) is smooth because the map \( s \mapsto f_{s,h} \) is smooth. Condition (a) of Definition 4.17 (3) is a consequence of the fact that \( \varphi_{h,X} \circ X \circ M_0 \) is the constant map
with image \( g_{0,h} \). Conditions (c) and (d) of Definition 4.17 (3) hold after possibly shrinking \( W \) by the following argument. Observe that

\[
\frac{\partial \Phi^0}{\partial s} \bigg|_{s=0} = \frac{\partial \Phi}{\partial s} \bigg|_{s=0},
\]

which satisfies Conditions (c) and (d) of Definition 4.17 (3) by assumption. Since immersions, embeddings and transverse maps, are open in the \( C^1 \) topology, it suffices to show that the map

\[
(43) \quad W \to C^{1,\alpha}(L, TX), \quad h \mapsto \frac{\partial \Phi^h}{\partial s} \bigg|_{s=0},
\]

is continuous. Since \( \Phi(p, s) = f_{s,h}(p) \), this is equivalent to property (4) of the proposition. Condition (b) of Definition 4.17 (3) requires that \( \Phi^h|_{L \times (0, \epsilon)} \) be an interior regular parameterization, which we prove as follows. Conditions (b) and (c) of Definition 4.17 (1) hold by construction. It remains to show that \( \Phi^h|_{L \times (0, \epsilon)} \) is an immersion and \( \Phi^h|_{\partial L \times (0, \epsilon)} \) is an embedding. After possibly shrinking \( \epsilon \), this follows from conditions (c) and (d) of Definition 4.17 (3), which we have already proved.

**Definition 6.6.** Let \( \mathcal{O} \) be a Hamiltonian isotopy class of positive Lagrangian spheres. For \( \Lambda_0, \Lambda_1 \in \mathcal{O} \), we write \( \Lambda_0 \not\equiv \Lambda_1 \) if \( \Lambda_0 \) and \( \Lambda_1 \) intersect transversally at exactly two points. Let

\[
\mathcal{Z}_\mathcal{O} := \left\{ (\Lambda_0, \Lambda_1, Z) \mid \begin{array}{l}
\forall \Lambda_i \in \mathcal{O}, \; i = 0, 1, \; \Lambda_0 \not\equiv \Lambda_1,
\exists \; Z \subset \mathcal{SLC}(\Lambda_0, \Lambda_1) \text{ a regular component}
\end{array} \right\}.
\]

We define the strong and weak \( C^{k,\alpha} \) topologies on \( \mathcal{Z}_\mathcal{O} \) as follows. For

\[
\mathcal{V} \subset C^\infty(S^{n-1} \times [0, 1], X), \quad \mathcal{U} \subset C^\infty(S^{n-1} \times [0, 1], TX),
\]

open subsets in the \( C^{k,\alpha} \) topology, write

\[
\mathcal{T}_{\mathcal{U}, \mathcal{V}} := \left\{ (\Lambda_0, \Lambda_1, Z) \in \mathcal{Z}_\mathcal{O} \mid \begin{array}{l}
\forall Z \in \mathcal{V}, \; \exists f: S^{n-1} \times [0, 1] \to X \text{ representing } Z
\end{array} \right\}
\]

and

\[
\mathcal{X}_\mathcal{V} = \left\{ (\Lambda_0, \Lambda_1, Z) \in \mathcal{Z}_\mathcal{O} \mid \exists Z \in \mathcal{V}, \; \exists f: S^{n-1} \times [0, 1] \to X \text{ representing } Z
\end{array} \right\}.
\]

Then, a basis for the strong \( C^{k,\alpha} \) topology on \( \mathcal{Z}_\mathcal{O} \) is given by sets of the form \( \mathcal{T}_{\mathcal{U}, \mathcal{V}} \) and a sub-basis for the weak \( C^{k,\alpha} \) topology on \( \mathcal{Z}_\mathcal{O} \) is given by sets of the form \( \mathcal{X}_\mathcal{V} \). Let

\[
\mathfrak{O}_\mathcal{O} := \{(\Lambda_t)_{t \in [0, 1]} \mid (\Lambda_t)_{t \in [0, 1]} \text{ is a geodesic in } \mathcal{O}, \; \Lambda_0 \not\equiv \Lambda_1 \}
\]

denote the space of geodesics in \( \mathcal{O} \) with endpoints intersecting transversally at two points. By Theorem 1.5, the cylindrical transform gives a bijection

\[
\mathfrak{O}_\mathcal{O} \simeq \mathcal{Z}_\mathcal{O}.
\]

So, the strong and weak \( C^{k,\alpha} \) topologies on \( \mathcal{Z}_\mathcal{O} \) give rise to topologies on \( \mathfrak{O}_\mathcal{O} \), which we also call the strong and weak \( C^{k,\alpha} \) topologies respectively.

**Proof of Theorem 1.6.** By Propositions 6.4 and 6.5 and the compactness of \([0, 1]\), we find a finite cover of \([0, 1]\) by relatively open intervals \( I_j, j = 0, \ldots, N \), subsets \( 0 \in W^j \subset C^\infty(\Lambda_1) \) open in the \( C^{1,\alpha} \) topology, and families of smooth immersions

\[
f^j_{s,h} : L \to X, \quad (s, h) \in I_j \times W^j,
\]

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It follows from implication (44) that for \( h \) be the interval consisting of the cylinders \( Z \). Thus, the sets \( \{W \} \) possibly shrinking and relabeling the intervals \( I_j \), we can assume that \( 0 \in I_0, 1 \in I_N, \) and \( I_j \cap I_k = \emptyset \) unless \( k = j \pm 1 \). Moreover, we can assume that \( N \geq 2 \).

Let \( W = \cap_{j=0}^N W_j \). For \( h \in W \), let

\[ U^h_j \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h}) \]

be the interval consisting of the cylinders \( Z^h_j = [f^h_j, a] \) for \( s \in I_j \). Choose

\[ s_j \in I_j \cap I_{j+1}, \quad j = 0, \ldots, N - 1. \]

Possibly shrinking \( W \), we may assume by continuity that

\[ f^j_{s_j,h} \in V^{j+1}, \quad h \in W, \quad j = 0, \ldots, N - 2, \]

and

\[ f^N_{s_{N-1},h} \in V^{N-1}, \quad h \in W. \]

It follows from implication (44) that for \( h \in W \), we have

\[ U^h_j \cap U^h_{j+1} \neq \emptyset, \quad j = 0, \ldots, N - 1. \]

Thus, the sets \( \{U^h_j\}_{j=0}^N \) cover an open interval \( Z^h \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h}) \). For \( j = 1, \ldots, N - 1 \), the interval \( U^h_j \) is interior regular by property (3) of Proposition 6.4. For \( j = 0 \) (resp. \( N \)) the interval \( U^h_j \) converges regularly to \( q_{0,h} \) (resp. \( q_{1,h} \)) by property (3) of Proposition 6.5. So, the interval \( Z^h \) is a regular connected component by Remark 5.14(a). Take \( \mathcal{V} \) the \( C^{2,\alpha} \) open neighborhood of \( \Lambda_1 \) in \( \mathcal{O} \) corresponding to \( W \) and take \( X := X_{\mathcal{V}} \). Let \( (\Lambda^t_l)_{t \in [0,1]} \in X \). For \( h \in W \) suppose \( (\Lambda^t_l)_{t \in [0,1]} \) is a geodesic in \( X \) with \( \Lambda^0_l = \Lambda_0 \) and \( \Lambda^1_l = \Lambda_{1,h} \). Let \( Z^t \subset \mathcal{SLC}(S^{n-1}; \Lambda_0, \Lambda_{1,h}) \) denote its cylindrical transform. It follows from implication (44) that \( Z^t \cap Z^h \neq \emptyset \) and thus \( Z^t = Z^h \). So, Theorem 1.5 gives \( (\Lambda^t_l)_t = (\Lambda^1_l)_t \). We have proven the existence and uniqueness part of Theorem 1.6. The continuity claim follows from the continuity of the families \( f^j_{s,h} \) and property (4) of Proposition 6.5.

\[ \square \]

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Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem, 91904, Israel
E-mail address: jake@math.huji.ac.il

Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem, 91904, Israel
E-mail address: amitai.yuval@mail.huji.ac.il

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