Pointwise differentiability of higher order for distributions

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Abstract

For distributions, we build a theory of higher order pointwise differentiability comprising, for order zero, Łojasiewicz’s notion of point value. Results include Borel regularity of differentials, higher order rectifiability of the associated jets, a Rademacher-Stepanov type differentiability theorem, and a Lusin type approximation. A substantial part of this development is new also for zeroth order. Moreover, we establish a Poincaré inequality involving the natural norms of negative order of differentiability. As a corollary, we characterise pointwise differentiability in terms of point values of distributional partial derivatives.

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1 Introduction

In their fundamental paper [CZ61], Calderón and Zygmund created a pointwise differentiability theory for functions in Lebesgue spaces and applied it to the study of strong solutions of systems of linear elliptic equations. Proceeding to weak solutions, one is naturally led to investigate pointwise differentiability theory of distributions first. Our treatment thereof is strongly influenced by the present and future needs of regularity questions in geometric measure theory discussed towards the end of this introduction.

Independent of this motivation, our results also shed new light on the well-established concept, introduced by Łojasiewicz in [Łoj57] and [Łoj58], of point value of a distribution. The latter occurs for instance in the multiplication of distributions (see, e.g., Łojasiewicz [Łoj56] and Itano [Ita66]), in Fourier series (see, e.g., Vindas and Estrada [VE07]), in boundary and initial value problems in partial differential equations (see, e.g., Szmydt [Szm77] and Walter [Wal72]), and in generalised integrals (see Estrada and Vindas [EV12]).

Throughout this introduction, we suppose $k$ is an integer, $0 < \alpha \leq 1$, $k + \alpha \geq 0$, $n$ is a positive integer, $Y$ is a Banach space, and $T \in \mathcal{D}'(\mathbb{R}^n, Y)$. Our chief concern is the case $Y = \mathbb{R}$; hence, separability of $Y$ is hypothesised whenever convenient and separability of $Y^*$ is assumed when it may not be omitted.
1.1 Differentiability theory and Łojasiewicz’s point values

The definition of pointwise differentiability adapts the approach of Rešetnjak, to transform to all objects to the unit ball, from functions to distributions, see \[\text{Res68a}\] p. 294\(^1\). To formulate it, we recall that \(R(\phi)\) is alternatively denoted by \(R_\alpha(\phi(x))\) whenever \(R \in \mathcal{D}'(\mathbb{R}^n, Y)\) and \(\phi \in \mathcal{D}(\mathbb{R}^n, Y)\).

**Definition** (see 2.10). Whenever \(a \in \mathbb{R}^n\), the distribution \(T\) is termed pointwise differentiable of order \(k\) at \(a\) if and only if there exists a polynomial function \(P : \mathbb{R}^n \to Y^*\) of degree at most \(k\) satisfying

\[
\lim_{r \to 0^+} r^{-k-n}(T - S)_r(\phi(r^{-1}(x - a))) = 0 \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y),
\]

where \(S \in \mathcal{D}'(\mathbb{R}^n, Y)\) is defined by \(S(\phi) = \int (\phi, P) d\mathcal{L}^n\) for \(\phi \in \mathcal{D}(\mathbb{R}^n, Y)\); here, by convention, a polynomial function of degree at most \(-1\) is the zero function.

As \(P\) is unique (see 2.9), we may define the \(k\)-th order pointwise differential of \(T\) at \(a\) by \(\text{pt} D^k T(a) = D^k P(a)\) if \(k \geq 0\).

For \(k = 0\), this definition yields Łojasiewicz’s notion of point value (see 2.13). We define pointwise differentiability of order \((k, \alpha)\) by employing the condition

\[
\limsup_{r \to 0^+} r^{-k-\alpha-n}(T - S)_r(\phi(r^{-1}(x - a))) < \infty \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y)
\]

in a similar fashion, see 2.18. This extends Zieleźny’s notion of boundedness at a point which is defined for \((k, \alpha) = (-1, 1)\) and \(n = 1\), see 2.20.

In the author’s view, a theory of higher order pointwise differentiability for a class of objects should consist of at least four results: Borel regularity of the differentials, rectifiability of the family of \(k\) jets, a Rademacher-Stepanov type theorem, and a Lusin type approximation theorem by functions of class \(k\). Such theories (possibly with Borel regularity replaced by appropriate measurability) have been developed for approximate differentiation of functions (successively, by Whitney in [Whi51], Isakov in [Isa87a]\(^2\) and Liu and Tai in [LT94]), for differentiation in Lebesgue spaces with respect to \(\mathcal{L}^n\) (by Calderón and Zygmund in [CZ61]), for pointwise differentiation of sets (by the author in [Men16c]), and for approximate differentiation of sets (by Santilli in [San17]).

For distributions, a pointwise differentiability theory for zeroth order (i.e., \(k + \alpha = 0\)) with almost all four results present was developed in the special case of distributions on the real line (i.e., \(n = 1\)) and \(Y = \mathbb{C}\) by Zieleźny in [Zie60]. As we will elaborate below, it appears rather difficult to extend the method of Zieleźny to general \(n\); in fact, the study of this generalisation was announced for zeroth order in [Zie60] p. 27 but seems not to be available as yet. Employing different methods, we are able to obtain the four indicated key results for general \(n\) and all nonnegative orders in the following Theorems A–D

**Theorem A** (see 2.8 and 4.12). Suppose \(k \geq 0\), \(Y\) is separable, and \(A\) is the set of points at which \(T\) is pointwise differentiable of order \(k\).

Then, \(A\) is a Borel set and, for each \(y \in Y\), the function mapping \(a \in A\) onto the real valued symmetric \(k\) linear map with value

\[
\text{pt} D^k T(a)(v_1, \ldots, v_k)(y) \quad \text{at } (v_1, \ldots, v_k) \in (\mathbb{R}^n)^k,
\]

is a Borel function. In particular, if \(Y^*\) is separable, \(\text{pt} D^k T\) is a Borel function.

\(^1\)The Russian original is [Res68b].

\(^2\)The Russian original is [Isa87b].

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The principal conclusion may alternatively be stated using a natural weak topology (see 2.5) on the space of $Y^*$ valued symmetric $k$ linear maps on $(R^n)^k$. A classic example due to Gelfand (see Gel38, p. 265) shows that, without the separability hypothesis on $Y^*$, the function $pt D^k T$ may be $\mathcal{L}^n \cap A$ nonmeasurable (see 4.14); in particular, that hypothesis may not be omitted.

**Theorem B** (see 4.9). *Suppose $A$ is the set of points at which $T$ is pointwise differentiable of order $(k, \alpha)$. Then, there exists a sequence of compact subsets $C_j$ of $\mathbb{R}^n$ with $A = \bigcup_{j=1}^{\infty} C_j$ and, if $k \geq 0$, also a sequence of functions $f_j : \mathbb{R}^n \to Y^*$ of class $(k, \alpha)$ satisfying

$$pt D^m T(a) = D^m f_j(a) \quad \text{for } a \in C_j \text{ and } m = 0, \ldots, k$$

whenever $j$ is a positive integer.*

As in the case of the differentiability theory of Calderón and Zygmund [CZ61], Theorems 8 and 9, there is no exceptional set in this rectifiability result.

**Theorem C** (see 4.23). *Suppose $Y$ is separable and $A$ is the set of points at which $T$ is pointwise differentiable of order $(k, 1)$. Then, $T$ is pointwise differentiable of order $k + 1$ at $\mathcal{L}^n$ almost all $a \in A$. This theorem in particular proves the existence of point values at $\mathcal{L}^n$ almost all points at which the distribution is bounded in the sense of Zieleńczy provided the latter concept is analogously extended to general $n$, see 2.20 and 4.24.*

**Theorem D** (see 4.25). *Suppose $k \geq 0$, $Y^*$ is separable, and $A$ is the set of points at which $T$ is pointwise differentiable of order $k$. Then, for each $\epsilon > 0$, there exists $g : \mathbb{R}^n \to Y^*$ of class $k$ such that

$$\mathcal{L}^n \left( A \sim (a : pt D^m T(a) = D^m g(a) \text{ for } m = 0, \ldots, k) \right) < \epsilon.$$"

The possible $\mathcal{L}^n \cap A$ nonmeasurability of $pt D^k T$ for nonseparable $Y^*$ shows that one cannot replace $Y^*$ by $Y$ in the separability hypothesis (see 4.26). If $k = 0$, then Theorem [D] follows from Theorem [A] and Lusin’s theorem. If $k \geq 1$, then the weaker statement resulting from replacing $k$ by $k - 1$ in the last line of Theorem [D] follows, at least if $Y = \mathbb{R}$, from Theorems [B] and [C] and Whitney’s Lusin type approximation result for functions of class $(k - 1, 1)$ by functions of class $k$, see [Whi51] Theorem 4. Accordingly, the main additional information contained in Theorem [D] is the equality of the $k$-th derivatives involved.

Beyond the above four basic properties, the pointwise differentiability theory for distributions allows to deduce, for nonnegative integers $l$, differentiability information of order $k + l$ from differentiability information of order $l$ of the $k$-th order distributional derivatives. This property is shared by Calderón and Zygmund’s theory of differentiation in Lebesgue spaces with respect to $\mathcal{L}^n$. However, it fails for approximate differentiation as Kohn’s example in [Koh77] shows. Recalling, for use with $m$-th order partial derivatives, that $\Xi(n, m)$ denotes the set of all $n$-tuples of sequences of nonnegative integers whose sum equals $m$, our next theorem formulates the property in question for distributions.

**Theorem E** (see 3.11(3)). *Suppose $k \geq 1$, $l$ is nonnegative integer, $a \in \mathbb{R}^n$, and $D^k T$ is pointwise differentiable of order $l$ at $a$ for $\xi \in \Xi(n, k)$. Then, $T$ is pointwise differentiable of order $k + l$ at $a$.*

As the converse is elementary (see 2.12), Theorem [E] in particular yields a natural characterisation of pointwise differentiability of order $k$ in terms of the
existence of point values in the sense of Łojasiewicz of the \( k \)-th order partial derivatives (see \( 3.14 \)). For general \( n \) and \( Y = C \), it was previously only known that the existence of point values for the partial derivatives implies the existence of point values for the distribution itself (see Itano \[Ita66\], Lemma 3). In the subcase \( n = 1 \), a related characterisation of point valued by means of “integrable”
distributional derivative was obtained by Kim in \[Kim14\], Proposition 2.

For pointwise differentiability of order \((k + l, \alpha)\), the corresponding theorem reads as follows.

**Theorem E** (see \[3.11\]). Suppose \( k \geq 1 \), \( l \) is an integer, \( l + \alpha \geq 0 \), \( a \in \mathbb{R}^n \), and \( D^k T \) is pointwise differentiable of order \((l, \alpha)\) at \( a \) for \( \xi \in \Xi(n, k) \).

Then, \( T \) is pointwise differentiable of order \((k + l, \alpha)\) at \( a \).

### 1.2 Concept of the proofs and a Poincaré inequality

A short proof of a variant of the four basic properties of a differentiability theory for the simplest case of pointwise (i.e., Peano type) differentiability of functions in the sense of \[Men16c\], 2.6, 2.7 is available in \[Men16c\], 4.6. Here, we mainly focus on the aspects specific to the setting of distributions. For this purpose, we recall that \( B(a, r) \) denotes the closed ball with centre \( a \) and radius \( r \), that

\[ \mathcal{D}_k(R^n, Y) = \mathcal{D}(R^n, Y) \cap \{ \phi : \text{spt} \phi \subset K \} \]

whenever \( K \) is a compact subset of \( \mathbb{R}^n \), and that, for every nonnegative integer \( i \), the norms \( \nu^i_{B(0,1)} \) have value (see \[3.2\])

\[ \sup\{ \| D^m \phi(x) \| : x \in B(0,1), m = 0, \ldots, i \} = \sup \| D^i \phi \| \]

at \( \phi \in \mathcal{D}_{B(0,1)}(R^n, Y) \). Whenever \( \nu \) is a norm defined on a vectorspace containing \( \mathcal{D}_{B(0,1)}(R^n, Y) \), we define (see \[2.14\] and \[2.21\]) the notions of \( \nu \) pointwise differentiability of order \( k \) and \( \nu \) pointwise differentiability of order \((k, \alpha)\) by requiring that the limit conditions in the corresponding definitions without prefix are satisfied uniformly for \( \phi \in \mathcal{D}_{B(0,1)}(R^n, Y) \) with \( \nu(\phi) \leq 1 \).

Then, pointwise differentiability of order \((k, \alpha)\) is equivalent to \( \nu^i_{B(0,1)} \) pointwise differentiability of order \((k, \alpha)\) for some nonnegative integer \( i \), see \[2.22\]. The corresponding statement for pointwise differentiability of order \( k \) holds if and only if \( \dim Y < \infty \), see \[2.22\] and \[2.69\]. With these two facts at hand, Theorems [A] and [B] are derived as in the case of functions; that is, employing basic descriptive set theory for Theorem [A] and Whitney’s extension theorem for Theorem [B].

The key to prove Theorems [C] and [D] is the following theorem which corresponds to \[Men16c\], 4.4 in the case of functions. The pattern of proof however follows \[Men13\], Appendix where the case \( i = 1, k = 0 \), and \( \dim Y < \infty \) was treated by means of the Whitney type partition of unity in \[Fed69\], 3.1.13.

**Theorem F** (see \[4.17\]). Suppose \( i \) is a nonnegative integer, \( Y \) is separable, and \( A \) is the set of points \( a \in \mathbb{R}^n \) at which \( T \) is \( \nu^i_{B(0,1)} \) pointwise differentiable of order \((k - 1, i)\) and \( \text{pt} D^m T(a) = 0 \) for \( m = 0, \ldots, k - 1 \).

Then, \( \mathbb{L}^n \) almost all \( a \in A \) satisfy the following three statements:

(1) The distribution \( T \) is pointwise differentiable of order \( k \) at \( a \).

(2) If \( k > 0 \) or \( Y^* \) is separable, then \( T \) is \( \nu^i_{B(0,1)} \) pointwise differentiable of order \( k \) at \( a \).
If $k > 0$, then $\text{pt } D^k T(a) = 0$.

The hypotheses in the last two items may not be omitted (see 4.19 and 4.20). For separable $Y^*$, Theorem [F] also yields a version of the Rademacher-Stepanov type theorem, Theorem [C] for $\nu_{B(0,1)}^i$ pointwise differentiability (see 4.22(2)).

The key to establish Theorems [E] and $E'$, as well as their versions for $\nu_{B(0,1)}^i$ pointwise differentiability (see 3.11(1)(2)), is the following Poincaré inequality.

Despite the natural significance of a Poincaré inequality, little appears to be known on such inequalities when norms of negative order of differentiability are employed. For our purposes, the following theorem is sufficient. We recall that $\langle y, \upsilon \rangle$ indicates the value $\upsilon(y)$ of the pairing of $y \in Y$ and $\upsilon \in Y^*$ and that $\langle \cdot, \cdot \rangle$ is similarly employed for functions with values in these spaces.

Theorem G (see 3.3). Suppose $i$ is a nonnegative integer, $k \geq 1$, $0 \leq \kappa < \infty$, and $|D^\xi T(\phi)| \leq \kappa \sup \text{im} \|D^i \phi\|$ for $\phi \in D_B(0,2)(\mathbb{R}^n, Y)$, and $o \in \mathfrak{X}(n,k)$. Then, there exists a polynomial function $P: \mathbb{R}^n \to Y^*$ is of degree at most $k - 1$ such that, for $m = 0, \ldots, k - 1$ and $\xi \in \mathfrak{X}(n,m)$, there holds

$$\left|\langle D^\xi T(\theta), D^\xi P \rangle \right| \leq \Gamma \kappa \sup \text{im} \|D^i \theta\| \quad \text{for } \theta \in D_B(0,1)(\mathbb{R}^n, Y),$$

where $0 \leq \Gamma < \infty$ is determined by $i, k$, and $n$.

If $k = 1$, the proof is carried out by comparing $T$ firstly to a convolution of $T$ with a suitable $\Phi \in D(\mathbb{R}^n, \mathbb{R})$ and subsequently to the value at 0 of the function representing that convolution. The cases $k > 1$ then follow inductively.

There seems to be ample room for further development here. For instance, one may wish to study possible subsequent embedding results leading to strengthenings of the estimate in the conclusion. An intriguing example of such an improvement (under supplementary hypotheses not available in the present circumstances) was given by Allard in [All86, §1], see 3.7.

In our present approach to point values, we considerably deviate from the traditional one based on Łojasiewicz’s characterisation thereof (see [Łoj57, 2.3 Théorème] for $n = 1$ and [Łoj58, 4.2 Théorème 1'] for general $n$). The latter yields a local representation of the distribution as high order partial derivative of a function with a certain pointwise differentiability property. Zieleźny’s treatment in [Zie60] is based on this characterisation and the observation that for $n = 1$ the representations are essentially unique. As for general $n$ they are highly nonunique, we instead directly employ the norms dual to $\nu_{B(0,1)}^i D_B(0,1)(\mathbb{R}^n, Y)$.

1.3 Envisaged future developments

Geometric measure theory

The utility to consider the validity of Theorem [D] became apparent during the author’s ongoing investigation of a special case of the varifold regularity problem formulated jointly with Scharrer in [MS17, Question 3]. Furthermore, the present paper is the third in a sequence of studies (initiated by the author in [Men16c] and continued by Santilli in [San17]) that is ultimately directed towards possible higher order pointwise differentiability properties of stationary integral varifolds. For approximate differentiability of second order, the central elliptic partial
differential equation involves an inhomogeneous term precisely a distribution, that is \( \nu_B^{(0,1)} \) pointwise differentiable of order 0 at all points in a set, that is compact and has positive \( \mathcal{L}^n \) measure but does not possess further regularity properties (see [Men13, 4.4(6)]).

**Elliptic partial differential equations**

From the point of view of elliptic partial differential equations, it is natural to aim to replace the norms \( \nu_i^{B(0,1)} \) in the present theory by the norms \( \nu_{i,p} \) defined by

\[
\nu_{i,p}(\phi) = \left( \int \|D^i \phi\|^p d\mathcal{L}^n \right)^{1/p}
\]

whenever \( i \) is a nonnegative integer and \( 1 < p < \infty \). A Rademacher-Stepanov type result for these norms with \( (k, \alpha) = (-1, 1) \), \( i = 1 \), and \( \dim Y < \infty \) was proven by the author in [Men13, 3.13]. Furthermore, initial elements of the corresponding pointwise differentiability theory for weak solutions of (linear and non-linear) elliptic partial differential equations were provided by him in [Men12, 8.4] and [Men13, 3.11, 3.18]. Extending these results to a more complete theory modelled upon that of Calderón and Zygmund [CZ61], but including certain non-linear equations, appears natural not only as development within elliptic partial differential equations but also as case study for the afore-mentioned regularity questions in geometric measure theory.

**Distribution theory**

Following a different line of thought, one might strive to determine optimal conditions on the Banach space \( Y \) in the spirit of the Radon-Nikodým property (see [BL00, §5]) for the validity of the present theory. Finally, Yoshinaga’s reduction (see [Yos67, p.24]) of Łojasiewicz’s concept of fixation of variables (see [Łoj58]) to point values with \( Y = \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) suggests to additionally include certain locally convex spaces \( Y \) in the study of possible extensions.

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**1.5 Notation**

As a rule, our terminology is that of [Fed69, pp.669–676]; in particular, the field involved in vectorspaces and linear maps is considered to be \( \mathbb{R} \) by default and

\[
\phi_{(p)}(f) = \left( \int |f|^p \, d\phi \right)^{1/p}
\]

and

\[
\phi_{(\infty)}(f) = \inf \{ s : s \geq 0, \phi \{ x : |f(x)| > s \} = 0 \}
\]

whenever \( \phi \) measures \( X \), \( Z \) is separable Banach space, \( f \) is a \( \phi \) measurable function mapping \( \phi \) almost all of \( X \) into \( Z \), and \( 1 \leq p \leq \infty \). The only exception to this rule is that we employ the more common locally convex topology on \( \mathcal{D}(\mathbb{R}^n, Y) \) defined for instance in [Men16a, 2.13]. Finally, we additionally employ the term function of class \( (k, \alpha) \) defined for \( k \geq 0 \) in [Men16c, 2.4].
For the convenience of the reader, we next review some of the basic terminology from multilinear algebra related to our treatment of polynomial functions (see [Fed69], §1.9, §1.10). Whenever $Z$ is a normed space and $k \geq 1$, $\odot^k(R^n, Z)$ denotes the normed space of $k$ linear symmetric maps of $(R^n)^k$ into $Z$ with

$$\|\phi\| = \sup \{|\phi(v_1, \ldots, v_k)| : v_m \in R^n, |v_m| \leq 1 \text{ for } m = 1, \ldots, k\}$$

for $\phi \in \odot^k(R^n, Z)$. Moreover, $\odot^0(R^n, Z) = Z$. The symmetric algebra of $R^n$, given by

$$\odot_m R^n = \bigoplus_{m=0}^\infty \odot_m R^n,$$

is a commutative associative graded algebra with unit element $1 \in R = \odot_0 R^n$ and $\odot$ denotes its multiplication. Whenever $e_1, \ldots, e_n$ form a basis of $R^n$, the vectors $e_\xi = (e_1)^{\xi_1} \cdots (e_n)^{\xi_n}$ corresponding to $\xi \in \Xi(n, m)$ form a basis of $\odot_m R^n$. Moreover, the canonical linear isomorphism

$$\text{Hom}(\odot_m R^n, Z) \cong \odot^m(R^n, Z)$$

maps $h \in \text{Hom}(\odot_m R^n, Z)$ onto $\psi \in \odot^m(R^n, Z)$ satisfying

$$\psi(v_1, \ldots, v_m) = h(v_1 \odot \cdots \odot v_m) \quad \text{for } v_1, \ldots, v_m \in R^n.$$  

Accordingly, one alternately denotes $h(\eta)$ by $\langle \eta, \psi \rangle$ for $\eta \in \odot_m R^n$. The interior multiplications $\odot : \odot_m R^n \times \odot_k(R^n, Z) \to \odot^{k-m}(R^n, Z)$ corresponding to $m = 0, \ldots, k$ are characterised by

$$\langle \zeta, \eta \odot \psi \rangle = \langle \zeta \odot \eta, \psi \rangle \quad \text{for } \zeta \in \odot_{k-m} R^n, \eta \in \odot_m R^n, \text{ and } \psi \in \odot^k(R^n, Z).$$

Finally, the norm on $\odot_m R^n$ is defined by

$$\|\eta\| = \sup \{\|\eta, \psi\| : \psi \in \odot^m(R^n, R), \|\psi\| \leq 1\} \quad \text{whenever } \eta \in \odot_m R^n.$$

## 2 Basic properties

In the present section, we firstly collect the necessary functional analytic preliminaries in §2.1. Then, we formally introduce our definitions of pointwise differentiability in §2.9. Finally, we derive basic properties of these concepts along with four examples in §2.24–2.37.

### 2.1 Definition

(see [DS58], p. 420). Suppose $Y$ is a Banach space. Then, the topology on $Y^*$ inherited from $R^Y$ is termed the $Y$ topology.

### 2.2. Suppose $\mu$ measures $X$, $X$ is countably $\mu$ measurable, $Y$ is a separable Banach space, and

$$H : L_1(\mu, Y) \to R$$

is a linear homomorphism for which $M = \sup \{|H(\theta)| : \mu(\{\theta\}) \leq 1\} < \infty$. Then, there exists a $\mu$ almost unique, $Y^*$ valued function $g$ that is $\mu$ measurable with respect to the $Y$ topology and satisfies $\mu(\{g\}) = M$ and

$$H(\theta) = \int \langle \theta, g \rangle \, d\mu \quad \text{for } \theta \in L_1(\mu, Y);$$

in fact, this is a special case of [ITT169, Chapter 7, Section 4].

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$^3$More elementary, one may pass from the case $Y = R$ treated in [Fed69, 2.5.7(ii)] to the general case adapting the method of [Fed69, 2.5.12]; this is carried out in [Men04, 4.6.3.2].
2.3 Definition. Suppose $Y$ and $Z$ are normed spaces.
Then, we define $||\tau||$ for $\tau \in Y \otimes Z$ to be the infimum of the set of numbers
$$\sum_{i=1}^{N} |y_i| |z_i|$$
corresponding to all positive integers $N$ and $y_i \in Y$, $z_i \in Z$ for $i = 1, \ldots, N$ satisfying $\tau = \sum_{i=1}^{N} y_i \otimes z_i$.

2.4 Remark. As in [Rya02, Proposition 2.1, Theorem 2.9], where the slightly more elaborate case of Banach spaces is treated, we see that the function $||\cdot||$ is a norm on $Y \otimes Z$ inducing a linear isometry of $(Y \otimes Z)^*$ with the space of continuous bilinear maps from $Y \times Z$ into $R$.

2.5 Remark. As in the case of vectorspaces (see [Fed69, 1.1.2]), if $Z_1$ and $Z_2$ are normed spaces, then $Y \otimes (Z_1 \oplus Z_2) \simeq (Y \otimes Z_1) \oplus (Y \otimes Z_2)$ as normed spaces.

2.6 Remark. If $Y$ is complete and $\dim Z < \infty$, then $Y \otimes Z$ is complete.

2.7 Remark. The norm $||\cdot||$ on $\bigotimes_{m} \mathbb{R}^n$ renders the canonical linear isomorphism $\text{Hom}(\bigotimes_{m} \mathbb{R}^n, Z) \simeq \bigotimes_{m} (\mathbb{R}^n, Z)$ an isometry whenever $Z$ is a normed space; in fact, this follows as in [2.4] from the characterisation of $||\cdot||$ in [Fed69, 1.10.5].

2.8. Suppose $Y$ is a Banach space and $Z_m = \bigotimes_{n} (\mathbb{R}^n, Y^*)$ whenever $m$ is a nonnegative integer. Then, the normed space $Y \otimes \bigotimes_{m} \mathbb{R}^n$ is complete by 2.6 and we will employ the canonical linear isometry (see 2.4 and 2.7)

$$Z_m \simeq (Y \otimes \bigotimes_{m} \mathbb{R}^n)^*$$

and the $Y \otimes \bigotimes_{m} \mathbb{R}^n$ topology on $Z_m$ defined by requiring this isometry to be a homeomorphism with the $Y \otimes \bigotimes_{m} \mathbb{R}^n$ topology on $(Y \otimes \bigotimes_{m} \mathbb{R}^n)^*$. The compact topology inherited from this topology by $Z_m \cap \{\psi : ||\psi|| \leq 1\}$ is metrisable if and only if $Y$ is separable (see 2.5 and [DS58, V.4.2, V.5.1]). In this case, the class of Borel sets of the $Y \otimes \bigotimes_{m} \mathbb{R}^n$ topology on $Z_m$ is generated by the family consisting of all sets

$$Z_m \cap \{\psi : (y \otimes \eta, \psi) < \kappa\}$$
corresponding to $y \in Y$, $\eta \in \bigotimes_{m} \mathbb{R}^n$, and $\kappa \in \mathbb{R}$; in fact, as $Y \otimes \bigotimes_{m} \mathbb{R}^n$ is spanned by such $y \otimes \eta$, one may apply [Men16a, 2.22]. If $Y^*$ is separable in its norm topology, then so is $Z_m$ and hence the classes of Borel sets in $Z_m$ with respect to the $Y \otimes \bigotimes_{m} \mathbb{R}^n$ topology and the norm topology agree.

2.9 Lemma. Suppose $k$ is a nonnegative integer, $P : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, $\Delta$ is a dense subset of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, $a \in \mathbb{R}^n$, and

$$\lim_{r \to 0^+} r^{-k-n} \int_{(a-x)^k} \phi(r^{-1}(x-a))P(x) \, d\mathbb{Z}^n \, x = 0 \quad \text{for } \phi \in \Delta.$$ 

Then, $D^m P(a) = 0$ for $m = 0, \ldots, k$.

Proof. Possibly replacing $P(x)$ by $\sum_{k=0}^{k} ((x-a)^m \cdot !, D^m P(a))$, we assume degree $P \leq k$. We let $Z$ denote the vectorspace of all polynomial functions $Q : \mathbb{R}^n \to \mathbb{R}$ of degree at most $k$ endowed with the norm whose value at $Q$ equals $\sum_{m=0}^{k} ||D^m Q(0)||$. Since $\dim Z < \infty$, [Bon87, Chapter 1, p. 13, Theorem 2]

*More elementary, one may recall the fifth paragraph of the proof of [Fed69, 2.5.12].*
yields that $Z$ may be homeomorphically embedded into $\mathbb{R}^\Delta$ by associating to $Q \in Z$ the function with value $\int \phi Q \, d\mathcal{L}^n$ at $\phi \in \Delta$. Hence, as

$$r^{-k-n} \int \phi(r^{-1}(x-a))P(x) \, d\mathcal{L}^n \, x = \int \phi(x)P_r(x) \, d\mathcal{L}^n \, x,$$

where $P_r(x) = r^{-k}P(a+rx)$, we conclude $P_r \to 0$ as $r \to 0+$ and $P = 0$. \hfill \Box

2.10 Definition. Suppose $k$ is an integer with $k \geq -1$, $Y$ is a Banach space, $T \in \mathcal{D}(\mathbb{R}^n, Y)$, and $a \in \mathbb{R}^n$.

Then, $T$ is termed pointwise differentiable of order $k$ at $a$ if and only if there exists a polynomial function $P : \mathbb{R}^n \to Y^*$ of degree at most $k$ satisfying

$$\lim_{r \to 0+} r^{-k-n}(T-S)_x(\phi(r^{-1}(x-a))) = 0 \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y),$$

where $S \in \mathcal{D}'(\mathbb{R}^n, Y)$ is defined by $S(\phi) = \int (\phi, P) \, d\mathcal{L}^n$ for $\phi \in \mathcal{D}(\mathbb{R}^n, Y)$; here, by convention, a polynomial function of degree at most $-1$ is the zero function.

As $P$ is unique by 2.9, we term $S$ the $k$-jet of $T$ at $a$ and, if $k \geq 0$, we define the $k$-th order pointwise differential of $T$ at $a$ by $\text{pt}D^k(T)(a) = D^kP(a)$.

2.11 Remark. Whenever $m = 0, \ldots, k$, it follows that $T$ is pointwise differentiable of order $m$ and $\text{pt}D^m(T)(a) = D^mP(a)$.

2.12 Remark. If $T$ is pointwise differentiable of order $k$ at $a$ and $S$ is the $k$-jet of $T$ at $a$, then, for $m = 0, \ldots, k$ and $\xi \in \mathbb{E}(n,m)$, the distribution $D^\xi S$ is pointwise differentiable of order $k-m$ at $a$, and, denoting by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, we have

$$\text{pt}D^m(D^\xi T)(a) = e^\xi \cdot \text{pt}D^{m+\mu}T(a) \quad \text{for } \mu = 0, \ldots, k-m,$$

where $e^\xi = (e_1)^{\xi_1} \cdot \cdots \cdot (e_n)^{\xi_n}$ and the powers are computed in $\bigodot \mathbb{R}^n$.

The converse type of implication will be treated in 3.11 and 3.14.

2.13 Remark. For $k = 0$, $Y = \mathbb{C}$, and $T$ linear, this concept was introduced in [Łoj58 §3]; the subcase $n = 1$ had appeared before in [Łoj57 1.1].

2.14 Definition. Suppose $k$ is a nonnegative integer, $Y$ is a Banach space, $T \in \mathcal{D}'(\mathbb{R}^n, Y)$, $K$ is a compact subset of $\mathbb{R}^n$ containing $0$ in its interior, $\nu$ is a norm on a vector space containing $\mathcal{D}_K(\mathbb{R}^n, Y)$, and $a \in \mathbb{R}^n$.

Then, $T$ is termed $\nu$ pointwise differentiable of order $k$ at $a$ if and only if there exists a polynomial function $P : \mathbb{R}^n \to Y^*$ of degree at most $k$ satisfying

$$\lim_{r \to 0+} \sup \{ r^{-k-n}(T-S)_x(\phi(r^{-1}(x-a))) : \phi \in \mathcal{D}_K(\mathbb{R}^n, Y), \nu(\phi) \leq 1 \} = 0,$$

where $S \in \mathcal{D}'(\mathbb{R}^n, Y)$ is defined by $S(\phi) = \int (\phi, P) \, d\mathcal{L}^n$ for $\phi \in \mathcal{D}(\mathbb{R}^n, Y)$.

2.15 Remark. In this case, $T$ is pointwise differentiable of order $k$ at $a$ and $S$ is the $k$-jet of $T$ at $a$.

2.16 Remark. In the present paper, this notion will usually be employed with $K = B(0,1)$ and $\nu = \nu^{\bullet}_{B(0,1)}$ for some nonnegative integer $i$. Adding the parameter $1 \leq p \leq \infty$, the family of norms $\nu_{i,p}$ on $\mathcal{D}_B(0,1)(\mathbb{R}^n, Y)$ defined by

$$\nu_{i,p}(\phi) = (\mathcal{L}^n)^{i/p}(D^i \phi) \quad \text{whenever } \phi \in \mathcal{D}_B(0,1)(\mathbb{R}^n, Y)$$

could be studied; in fact, for the case $k = 0$, $n = 1$, $Y = \mathbb{C}$, and $C$ linear $T$, the subcase $i = 0$ and $p = 1$ of the concept occurred in [SLO14 Lemma 3] and a basic characterization of the subcase $i = 0$ and $p < \infty$ was obtained in [Kim14 Theorem 10].
2.17 Remark. The case $K = B(0, 1), \nu = \nu_{B(0, 1)}$, $k = 0$, $Y = C$, and $C$ linear $T$ of this concept was introduced as “value of $T$ at $a$ of order not exceeding $n$” in [Lo58] § 8.2; the subcase $n = 1$ had appeared before in [Lo57] § 2.4 Définition, § 4.4 Théorème.

2.18 Definition. Suppose $k$ is an integer, $0 < \alpha \leq 1$, $k + \alpha \geq 0$, $Y$ is a Banach space, $T \in \mathcal{D}'(R^n, Y)$, and $a \in R^n$.

Then, $T$ is termed pointwise differentiable of order $(k, \alpha)$ at $a$ if and only if there exists a polynomial function $P : R^n \to Y^*$ of degree at most $k$ satisfying

$$\limsup_{r \to 0^+} r^{-k-\alpha-n}(T - S)_x(\phi(r^{-1}(x - a))) < \infty \quad \text{for } \phi \in \mathcal{D}(R^n, Y),$$

where $S \in \mathcal{D}'(R^n, Y)$ is defined by $S(\phi) = \int \phi(P) d\mathcal{L}^n$ for $\phi \in \mathcal{D}(R^n, Y)$.

2.19 Remark. In this case, $T$ is pointwise differentiable of order $k$ at $a$, $S$ is the $k$ jet of $T$ at $a$, and, whenever $K$ is a compact subset of $R^n$ and $0 < s < \infty$, there exist a nonnegative integer $i$ and $0 \leq M < \infty$ satisfying

$$|r^{-k-\alpha-n}(T - S)_x(\phi(r^{-1}(x - a)))| \leq M\nu^i_K(\phi) \quad \text{for } \phi \in \mathcal{D}_K(R^n, Y)$$

whenever $0 < r \leq s$; in fact, defining $R_r : \mathcal{D}_K(R^n, Y) \to R$ by

$$R_r(\phi) = r^{-k-\alpha-n}(T - S)_x(\phi(r^{-1}(x - a))) \quad \text{for } 0 < r \leq s \text{ and } \phi \in \mathcal{D}_K(R^n, Y),$$

and noting the continuity of $R_r(\phi)$ in $r$, the last part follows from the principle of uniform boundedness (see [DS58] II.1.11) since $\mathcal{D}_K(R^n, Y)$ is an “$F$-space” in the terminology of [DS58] II.1.10 (see, e.g., [Men16a] 2.14 and [Men16b] 2.4).

2.20 Remark. The case $n = 1, k = -1, \alpha = 1, Y = C$, and $C$ linear $T$ of this concept was introduced in [Zie60] §1.1 Definition.

2.21 Definition. Suppose $k$ is an integer, $0 < \alpha \leq 1$, $k + \alpha \geq 0$, $Y$ is a Banach space, $T \in \mathcal{D}'(R^n, Y)$, $K$ is a compact subset of $R^n$ containing $0$ in its interior, $\nu$ is a norm on a vectorspace containing $\mathcal{D}_K(R^n, Y)$, and $a \in R^n$.

Then, $T$ is termed $\nu$ pointwise differentiable of order $(k, \alpha)$ at $a$ if and only if there exists a polynomial function $P : R^n \to Y^*$ of degree at most $k$ satisfying

$$\limsup_{r \to 0^+} \sup \left\{ r^{-k-\alpha-n}(T - S)_x(\phi(r^{-1}(x - a))) : \phi \in \mathcal{D}_K(R^n, Y), \nu(\phi) \leq 1 \right\} < \infty,$$

where $S \in \mathcal{D}'(R^n, Y)$ is defined by $S(\phi) = \int \phi(P) d\mathcal{L}^n$ for $\phi \in \mathcal{D}(R^n, Y)$.

2.22 Remark. In this case, $T$ is pointwise differentiable of order $(k, \alpha)$ at $a$ and $S$ is the $k$ jet of $T$ at $a$ by 2.19. Hence, 2.19 also yields the following proposition. Whenever $k$ is an integer, $0 < \alpha \leq 1$, $k + \alpha \geq 0$, $Y$ is Banach space, $T \in \mathcal{D}'(R^n, Y)$, and $a \in R^n$, the distribution $T$ is pointwise differentiable of order $(k, \alpha)$ at $a$ if and only if, for some nonnegative integer $i$, it is $\nu_{B(0, 1)}$ pointwise differentiable of order $(k, \alpha)$ at $a$. The case $n = 1, k = -1, \alpha = 1, Y = C$, and $C$ linear $T$ thereof also follows from [Zie60] §1.1 Satz 1.1.

2.23 Remark. According to the characterisation [Zie60] Satz 1.5], the case $n = 1, k = -1, \alpha = 1, Y = C, C$ linear $T$, and $\nu = \nu_{B(0, 1)}$ of our concept is equivalent to the notion of “boundedness of order not exceeding $i$” of [Zie60] §1.3 Definition.
2.24 Remark. Our notions of pointwise differentiability of higher order (see 2.10, 2.14, 2.18 and 2.21) are adaptations of similar concepts for functions in [Res68a, p. 294] to distributions.

2.25 Lemma. Suppose $k$ is an integer, $k \geq -1$, $Y$ is a Banach space, $\Delta$ is a sequentially dense subset of $\mathcal{D}(\mathbb{R}^n, Y)$, $T \in \mathcal{D}(\mathbb{R}^n, Y)$, $a \in \mathbb{R}^n$, $T$ is pointwise differentiable of order $(k, 1)$ at $a$, and $P : \mathbb{R}^n \to Y^*$ is a polynomial function of degree at most $k + 1$ satisfying

$$\lim_{r \to 0^+} r^{-k-1-n}(T - S)_x(\phi(r^{-1}(x - a))) = 0 \quad \text{for } \phi \in \Delta,$$

where $S \in \mathcal{D}'(\mathbb{R}^n, Y)$ is defined by $S(\phi) = \int (\phi, P) \, d\mathcal{L}^n$ for $\phi \in \mathcal{D}(\mathbb{R}^n, Y)$.

Then, $T$ is pointwise differentiable of order $k + 1$ at $a$ and $S$ is the $k + 1$ jet of $T$ at $a$. If, additionally, we have that $\dim Y < \infty$, that $i$ is a nonnegative integer, and that $T$ is $\nu^i_{B(0, 1)}$ pointwise differentiable of order $(k, 1)$ at $a$, then $T$ is $\nu^i_{B(0, 1)}$ pointwise differentiable of order $k + 1$ at $a$.

Proof. Suppose $\theta \in \mathcal{D}(\mathbb{R}^n, Y)$. Taking $\phi_j \in \Delta$ with $\phi_j \to \theta$ as $j \to \infty$, there exists (see, e.g., [Men69 2.14, 2.15]) a number $0 \leq t < \infty$ with $\text{spt } \phi_j \subset K$ for every positive integer $j$, where $K = \mathbb{R}^n \cap B(0, t)$. Applying 2.19 with $\alpha = 1 = s$, we obtain $i$ and $M$. Since $\text{pt } D^n T(a) = \text{pt } D^n S(a)$ for $m = 0, \ldots, k$, by 2.9 and

$$\int_{B(a,r)} |x - a|^{k+1}(k + 1)! \|D^{k+1} P(a)\| d\mathcal{L}^n x = N r^{n+k+1},$$

where $N = (k + 1)! |S^{s-1}(n+k+1)^{-1}r^{n+k+1} \|D^{k+1} P(a)\|$, we conclude

$$|r^{-k-1-n}(T - S)_x(\phi^j(r^{-1}(x - a)))| \leq (M + N) \nu^i_K(\phi)$$

whenever $\phi \in \mathcal{D}_K(\mathbb{R}^n, Y)$ and $0 < r \leq 1$. Consequently, we have

$$\limsup_{r \to 0^+} |r^{-k-1-n}(T - S)_x(\phi^j(r^{-1}(x - a)))| \leq (M + N) \nu^i_K(\theta - \phi_j)$$

whenever $j$ is a positive integer, whence the principal conclusion follows.

Noting 2.11 and 2.22 the hypotheses of the postscript yield $0 \leq M < \infty$ and $s > 0$ that, whenever $0 < r \leq s$, we have

$$|r^{-k-1-n}(T - S)_x(\phi^j(r^{-1}(x - a)))| \leq M \nu^i_{B(0, 1)}(\phi) \quad \text{for } \phi \in \mathcal{D}_{B(0, 1)}(\mathbb{R}^n, Y).$$

As $\dim Y < \infty$ implies, by use of the Ascoli theorem (see [Fed60 2.10.21]), that

$$\mathcal{D}_{B(0, 1)}(\mathbb{R}^n, Y) \cap \{ \phi : \nu^i_{B(0, 1)}(\phi) \leq 1 \}$$

is $\nu^i_{B(0, 1)}$ totally bounded, we readily combine the preceding estimate with the principal conclusion to obtain the postscript. $\square$

2.26 Remark. In view of 2.17 and 2.23 the case $n = 1$, $k = -1$, $Y = \mathbb{C}$, $\Delta = \mathcal{D}(\mathbb{R}, \mathbb{C})$, and $C$ linear $T$, was treated in [Zie60 §1.4 Satz].

2.27 Remark. Combining 2.15, 2.22 and 2.25 we obtain the following proposition: Whenever $k$ is a nonnegative integer, $Y$ is a finite dimensional Banach space, $T \in \mathcal{D}'(\mathbb{R}^n, Y)$, and $a \in \mathbb{R}^n$, the distribution $T$ is pointwise differentiable of order $k$ at $a$ if and only if, for some nonnegative integer $i$, the distribution $T$ is $\nu^i_{B(0, 1)}$ pointwise differentiable of order $k$ at $a$. The case $k = 0$, $Y = \mathbb{C}$, and $C$ linear $T$ of this characterisation also follows from [Loj58 4.2 Théorème Y].
2.28 Example. The image \( \Delta \) of \( \mathcal{D}(\mathbb{R}^n, Y) \) under the canonical monomorphism is sequentially dense in \( \mathcal{D}(\mathbb{R}^n, Y) \) by [Fed69, 3.1].

2.29 Example. There exists a separable Banach space \( Y \) and \( f : \mathbb{R} \to Y^* \), continuous with respect to the \( Y \) topology on \( Y^* \), such that \( \| f(x) - f(a) \| = 1 \) whenever \( a, x \in \mathbb{R} \) and \( a \neq x \); in fact, noting 2.2 and [Men16b, 2.1, 2.6], we may take \( Y \simeq L_1(\mathcal{L}^1, \mathbb{R}) \), and \( f(x) \) to be associated to the characteristic function of the interval \( \{ t x : 0 \leq t \leq 1 \} \). Evidently, such a function \( f \) must be \( \mathcal{L}^1 \) nonmeasurable with respect to the norm topology on \( Y^* \) by [Fed69, 2.2.4].

2.30 Remark. The type of function considered originates from [Gel38, p.265].

2.31 Lemma. Suppose \( k \) is a nonnegative integer, \( Y \) is a separable Banach space, \( f \) is a \( Y^* \) valued function, that is \( \mathcal{L}^n \) measurable with respect to the \( Y \) topology on \( Y^* \) and satisfies

\[
\int_K \| f \| \, d\mathcal{L}^n < \infty \quad \text{whenever } K \text{ is a compact subset of } \mathbb{R}^n,
\]

and \( P : \mathbb{R}^n \to Y^* \) is a polynomial function of degree at most \( k \).

Then, a distribution \( S \in \mathcal{D}'(\mathbb{R}^n, Y) \) may be defined by

\[
S(\phi) = \int \langle \phi, f \rangle \, d\mathcal{L}^n \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y)
\]

and the following five statements hold:

1. If \( a \in \mathbb{R}^n \) satisfies

\[
\limsup_{r \to 0^+} r^{-n-k} \int_{B(a,r)} \| f - P \| \, d\mathcal{L}^n < \infty,
\]

then \( S \) is pointwise differentiable of order \( k \) at \( a \) and \( \lim_{r \to 0^+} r^{-n-k} \int_{B(a,r)} \| (y, f(x) - P(x)) \| \, d\mathcal{L}^n x = 0 \) for \( y \in Y \),

2. For \( \mathcal{L}^n \) almost all \( a \), the distribution \( S \) is pointwise differentiable of order 0 at \( a \) and \( \lim_{r \to 0^+} r^{-n-k} \int_{B(a,r)} \| f - P \| \, d\mathcal{L}^n = 0 \).

3. Whenever \( a \in \mathbb{R}^n \) and \( 0 < r < \infty \), we have

\[
\int_{B(a,r)} \| f \| \, d\mathcal{L}^n = \sup \{ S(\phi) : \phi \in \mathcal{D}_{B(a,r)}(\mathbb{R}^n, Y) \text{ and } \nu_{B(a,r)}^0(\phi) \leq 1 \}.
\]

4. For \( a \in \mathbb{R}^n \), the distribution \( S \) is \( \nu_{B(0,1)}^0 \) pointwise differentiable of order \( k \) at \( a \) with \( \lim_{r \to 0^+} r^{-n-k} \int_{B(a,r)} \| f - P \| \, d\mathcal{L}^n = 0 \) if and only if

\[
\lim_{r \to 0^+} r^{-n-k} \int_{B(a,r)} \| f - P \| \, d\mathcal{L}^n = 0.
\]

5. If \( Y^* \) is separable in its norm topology, then \( S \) is \( \nu_{B(0,1)}^0 \) pointwise differentiable of order 0 at \( \mathcal{L}^n \) almost all \( a \).

Proof. The legitimacy of the definition of \( S \) is noted in [Fed69, 4.1.1]. [1] is a consequence of 2.25 and 2.28. Moreover, employing a countable dense subset of \( Y \) and [Fed69, 2.8.18, 2.9.9], [2] may be deduced from [1]. Observing (see [Men04, 4.6.1]) that

\[
\int_{B(a,r)} \| f \| \, d\mathcal{L}^n = \sup \{ \int_{B(a,r)} \langle \theta, f \rangle \, d\mathcal{L}^n : \theta \in \mathcal{L}_\infty(\mathcal{L}^n, Y), (\mathcal{L}^n)_{(\infty)}(\theta) \leq 1 \}
\]

for \( a \in \mathbb{R}^n \) and \( 0 < r < \infty \), [3] follows by suitably approximating such \( \theta \). Finally, [3] implies [1] and, by [Fed69, 2.8.18, 2.9.9], [1] and 2.8 yield [5].  

\[\square\]
2.32 Remark. Under the hypothesis of [5], the separability hypothesis on $Y$ is redundant by [DS58] II.3.16.

2.33 Remark. In view of [4], 2.29 shows that the separability hypothesis on $Y^*$ in [5] may not be omitted even if $f$ is continuous with respect to the $Y^*$ topology. For such $f$, the distribution $S$ is $\nu_{\mathcal{B}(0,1)}^0$ pointwise differentiable of order $(-1,1)$ at each $a \in \mathbb{R}^n$ by [3], since $\|f\|$ is locally bounded by [DS58] II.3.21.

2.34 Remark. If $f$ is “$w^*$-differentiable with $w^*$-differential $\psi$ at $a$” in the sense of [AK09] 3.4 and $P(x) = f(a) + \langle x - a, \psi \rangle$ for $x \in \mathbb{R}^n$, then $S$ is pointwise differentiable of order 1 at $a$ and pt $D^1 S(a) = \psi$ by [1].

2.35 Example. Whenever $Y$ is a separable Hilbert space with $\dim Y = \infty$, there exists a distribution $S$ associated to some $f : \mathbb{R} \to Y$ as in 2.24 such that $S$ is $\nu^0_{\mathcal{B}(0,1)}$ pointwise differentiable of order $(-1,1)$ at $0$ and $S$ is pointwise differentiable of order 0 at 0 but, for every nonnegative integer $i$, $S$ is not $\nu^i_{\mathcal{B}(0,1)}$ pointwise differentiable of order 0 at 0; in fact, in view of [2.31](1), [3], choosing an orthonormal sequence $y_j$ in $Y$, we may take

$$f(x) = y_j \quad \text{if } x \in \mathcal{B}(0, 2^{1-j}) \sim U(0, 2^{-j})$$

for some positive integer $j$ and $f(x) = 0$ if $x = 0$ or $x \in \mathbb{R} \sim \mathcal{B}(0, 1)$.

2.36 Remark. The preceding example shows that, neither from the postscript of 2.25 nor from 2.27, the hypothesis $\dim Y < \infty$ may be omitted.

2.37 Example. Suppose $Y$ is a separable Banach space, $S \in \mathcal{D}'(\mathbb{R}^n, Y)$ is representable by integration — that is, see [Fed69] 4.1.5. $\|S\|$ is a Radon measure and there exists a $Y^* \cap \{\psi : \|\psi\| = 1\}$ valued function $g$, that is $\|S\|$ measurable with respect to the $Y$ topology on $Y^*$ and satisfies $S(\phi) = \int \langle \phi, g \rangle \, d\|S\|$ for $\phi \in \mathcal{D}(\mathbb{R}^n, Y)^-$, and $V = \{(a, B(a, r)) : a \in \mathbb{R}^n, 0 < r < \infty\}$ is the standard $\mathcal{L}^n$ Vitali relation (see [Fed69] 2.8.18). Then, for $\mathcal{L}^n$ almost all $a$, the distribution $S$ is pointwise — in case $Y^*$ is separable, also $\nu^0_{\mathcal{B}(0,1)}$ pointwise — differentiable of order 0 at $a$ with

$$\text{pt } D^0 S(a) = \mathbf{D}(\|S\|, \mathcal{L}^n, V, a) g(a) \text{ if } a \in \text{dmm } g, \quad \text{pt } D^0 S(a) = 0 \text{ else;}$$

in fact, employing [Fed69] 2.9.2, 2.9.7 to express the absolutely continuous part $\|S\|_{\mathcal{L}^n}$, of $\|S\|$ with respect to $\mathcal{L}^n$ via its $V$ derivative $\mathbf{D}(\|S\|, \mathcal{L}^n, V, \cdot)$ and reducing to the case that $g$ is a Borel function with respect to the $Y$ topology and that $\text{dmm } g = \mathbb{R}^n$ by [2.8] and [Fed69] 2.3.6, we use [Fed69] 2.4.10 to obtain

$$S(\phi) = \int \langle \phi(x), g(x) \rangle \mathbf{D}(\|S\|, \mathcal{L}^n, V, x) \, d\mathcal{L}^n x + \int \langle \phi, g \rangle \, d(\|S\| - \|S\|_{\mathcal{L}^n})$$

for $\phi \in \mathcal{D}(\mathbb{R}^n, Y)$, whence we infer the inclusion by [2.31](2, [4]) as we are assured that $\mathbf{D}(\|S\| - \|S\|_{\mathcal{L}^n}, \mathcal{L}^n, V, x) = 0$ for $\mathcal{L}^n$ almost all $x$ by [Fed69] 2.9.10.

3 Poincaré inequality

The main purpose of this section is to establish the indicated Poincaré inequality in [5.1] [5.7]. Furthermore, we include its applications to the study of the relation of pointwise differentiability and distributional derivatives in [3.8] [3.19].
3.1. Suppose \( i \) is a nonnegative integer, \( K \) is a compact subset of \( \mathbb{R}^n \), \( Y \) is a separable Banach space, \( T \in \mathcal{D}'(\mathbb{R}^n,Y) \), and \( 0 \leq \kappa < \infty \). Then, we will verify (for use in 3.7 only) the equivalence of the following two conditions.

(1) For \( \phi \in \mathcal{D}_K(\mathbb{R}^n,Y) \), we have \( |T(\phi)| \leq \kappa \sup \text{im} \| D^j \phi \| \).

(2) There exists \( S \in \mathcal{D}'(\mathbb{R}^n,\mathcal{O}^j(\mathbb{R}^n,Y)) \) satisfying \( \text{spt} S \subseteq K \), \( \| S \|(\mathbb{R}^n) \leq \kappa \), and
\[
T(\phi) = S(D^j \phi) \quad \text{for} \quad \phi \in \mathcal{D}_K(\mathbb{R}^n,Y).
\]

In fact, proceeding as in [Fed69, 4.1.12], if \( 1 \) holds, \( V = \mathcal{D}(\mathbb{R}^n,\mathcal{O}^j(\mathbb{R}^n,Y)) \), \( \sigma \) is the seminorm on \( V \) defined by \( \sigma(v) = \kappa \sup \{ |v(x)| : x \in K \} \) for \( v \in V \), and \( Q : \mathcal{D}_K(\mathbb{R}^n,Y) \to V \) is the linear monomorphism induced by \( D^j \), then we have \( T \circ Q^{-1} \leq \sigma |Q| \) and \( 2 \) follows by extending \( T \circ Q^{-1} \) to a linear map \( S : V \to \mathbb{R} \) with \( S \leq \sigma \) by means of the Hahn-Banach theorem [Fed69, 2.4.12].

3.2. Suppose \( i \) is a nonnegative integer, \( a \in \mathbb{R}^n \), \( 0 < r < \infty \), and \( Y \) is a Banach space. Then, there holds \( \sup \| D^m \phi \| \leq r^{i-m} \sup \| D^j \phi \| \) for \( m = 0, \ldots, i \) and \( \phi \in \mathcal{D}_B(a,r)(\mathbb{R}^n,Y) \) by [Fed69, 2.2.7, 3.1.1, 3.1.11]; in particular, we have
\[
\nu^j_{B(0,1)}(\phi) = \sup \| D^j \phi \| \quad \text{for} \quad \phi \in \mathcal{D}_B(0,1)(\mathbb{R}^n,Y).
\]

3.3 Theorem. Suppose \( i \) is a nonnegative integer, \( k \) is a positive integer, \( Y \) is a Banach space, \( T \in \mathcal{D}'(\mathbb{R}^n,Y) \), \( 0 < r < \infty \), \( \Phi \in \mathcal{D}_B(0,r)(\mathbb{R}^n,\mathbb{R}) \),
\[
\Phi \ast Q = Q
\]
whenever \( Q : \mathbb{R}^n \to \mathbb{R} \) is a polynomial function of degree at most \( k - 1 \), \( 0 \leq \kappa < \infty \), \( C \) is a compact convex subset of \( \mathbb{R}^n \), \( K = \bigcup \{ B(c, kr) : c \in C \} \),
\[
|\langle D^k T(\phi) \rangle| \leq \kappa \sup \| D^i \phi \| \quad \text{for} \quad \phi \in \mathcal{D}_K(\mathbb{R}^n,Y) \quad \text{and} \quad a \in \mathcal{E}(n,k),
\]
a \( C \), and \( P : \mathbb{R}^n \to Y^* \) is the polynomial function of degree at most \( k - 1 \) satisfying
\[
\langle y, D^k P(a) \rangle = \langle D^k T \rangle_k(\Phi(b - a)y)
\]
whenever \( m = 0, \ldots, k - 1 \), \( \xi \in \mathcal{E}(n,m) \), and \( y \in Y \).
Then, for \( m = 0, \ldots, k - 1 \) and \( \xi \in \mathcal{E}(n,m) \), there holds
\[
|\langle D^k T(\theta) \rangle - \langle \theta, D^k P \rangle | \leq \Gamma_m \kappa \sup \| D^i \theta \| \quad \text{for} \quad \theta \in \mathcal{D}_C(\mathbb{R}^n,Y),
\]
where \( \Gamma_m = (n \alpha(n) \sup \| D^i \Phi \|)^{k-m} \prod_{\mu=1}^{k-m} (2\mu r + \text{diam } C)^{1+n+i} \).

Proof. We assume \( a = 0 \) and abbreviate \( \delta = \text{diam } C \) and \( \lambda = \sup \| D^i \Phi \| \).
Replacing \( T \) and \( k \) by \( D^k T \) and \( k - m \), it suffices to show the assertion
\[
|T(\theta) - \langle \theta, P \rangle | \leq \Gamma_0 \kappa \sup \| D^i \theta \| \quad \text{for} \quad \theta \in \mathcal{D}_C(\mathbb{R}^n,Y).
\]

Firstly, the special case \( k = 1 \) thereof will be proven. Suppose \( \theta \in \mathcal{D}_C(\mathbb{R}^n,Y) \) with \( \sup \| D^i \theta \| \leq 1 \). Using the coordinate functions \( X_j : \mathbb{R}^n \to \mathbb{R} \) given by \( X_j(x) = x_j \) whenever \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we define \( \phi_j : \mathbb{R}^n \to Y^* \) by
\[
\phi_j(x) = \int_0^1 \Phi(b) X_j(b) \theta(x - tb) \, d\mathcal{L}^1 t \, d\mathcal{L}^n b
\]
whenever $x \in \mathbb{R}^n$ and $j = 1, \ldots, n$. We note $\phi_j \in \mathcal{D}_K(\mathbb{R}^n, Y)$ with

$$D^m \phi_j(x) = \int_0^1 \Phi(b) X_j(b) D^m \theta(x - tb) \, d\mathcal{L}^1 \, t \, d\mathcal{L}^n b$$

for $m = 0, 1, 2, \ldots$ and $x \in \mathbb{R}^n$ and infer

$$\sup \text{im} \|D^j \phi_j\| \leq r(\mathcal{L}^n)_{(1)}(\Phi) \leq \alpha(n)r^{1+n+i}$$

for $j = 1, \ldots, n$ by [3.2] and also $\theta - \Phi \ast \theta = \sum_{j=1}^n D_j \phi_j$ since

$$\theta(x) - (\Phi \ast \theta)(x) = \int \Phi(b)(\theta(x) - \theta(x - b)) \, d\mathcal{L}^n b = \sum_{j=1}^n D_j \phi_j(x) \quad \text{for} \ x \in \mathbb{R}^n.$$ 

Consequently, we obtain $|T(\theta - \Phi \ast \theta)| \leq n\alpha(n)r^{1+n+i}$. In view of [Fed69, 4.1.2], we may define $f \in \mathcal{D}'(\mathbb{R}^n, Y^*)$ by requiring

$$\langle y, f(x) \rangle = T_b(\Phi(b - x))y$$

for $x \in \mathbb{R}^n$ and $y \in Y$. Since $\langle y, D_j f(x) \rangle = (D_j T)_b(\Phi(b - x))y$ for $y \in Y$, we infer

$$\|D_j f(x)\| \leq \kappa \lambda$$

for $x \in C$ and $j = 1, \ldots, n$.

Recalling $T(\Phi \ast \theta) = \int (\theta(f) \, d\mathcal{L}^n$ from [Fed69, 4.1.2] and noting $\sup \text{im} |\theta| \leq \delta^i$, where $\delta^0 = 1$, by [3.2] the preceding estimate implies

$$|T(\Phi \ast \theta) - \int \langle \theta(x), f(0) \rangle \, d\mathcal{L}^n x| \leq n\alpha(n)\delta^{1+n+i} \kappa \lambda$$

and the conclusion follows in the present case.

Proceeding inductively, we now establish that the validity of the assertion for some $k$ implies its validity for $k + 1$. For this purpose, we observe that $D = \bigcup \{B(e, kr) : c \in C\}$ is a convex set and define $S \in \mathcal{D}'(\mathbb{R}^n, Y)$ by

$$S(\theta) = T(\theta) - \int \langle \theta(x), \langle x^k/k!, D^k P(0) \rangle \rangle \, d\mathcal{L}^n x \quad \text{for} \ \theta \in \mathcal{D}(\mathbb{R}^n, Y).$$

For $\theta \in \mathcal{D}(\mathbb{R}^n, Y)$ and $\xi \in \mathfrak{N}(n, k)$, noting

$$(D^\xi S)(\theta) = (D^\xi T)(\theta) - \int \langle \theta(x), D^\xi P(0) \rangle \, d\mathcal{L}^n x,$$

we apply the special case with $T$ and $C$ replaced by $D^\xi T$ and $D$ to conclude

$$\|D^\xi S(\theta)\| \leq n\alpha(n)\lambda(2(k + 1)r + \delta)^{1+n+i} \kappa \sup \text{im} \|D^\xi \theta\|.$$
3.4 Remark. The estimate of \( T(\theta - \Phi \ast \theta) \) is adapted from the use of the smoothing homotopy formulae for currents in \[\text{Fed69} 4.1.18\].

3.5 Remark. From \(\text{CZ61} \) Lemma 2.6], we recall the following proposition. If \( k \) is a positive integer, then there exists \( \Phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) satisfying \( \Phi \ast Q = Q \) whenever \( 0 < r < \infty \) and \( Q : \mathbb{R}^n \to \mathbb{R} \) is a polynomial function of degree at most \( k - 1 \), where \( \Phi_n(x) = r^{-n} \Phi(r^{-1}x) \) for \( x \in \mathbb{R}^n \).

3.6 Remark. If, for some positive integer \( k \), a function \( \Phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) is such that \( \Phi \ast Q = Q \) for every polynomial function \( Q : \mathbb{R}^n \to \mathbb{R} \) of degree at most \( k - 1 \), then the functions \( \phi_o \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \), defined by \( \phi_o(x) = (o!)^{-1} D^o \Phi(-x) \) for \( x \in \mathbb{R}^n \), \( m = 0, \ldots, k - 1 \), and \( o \in \Xi(n,m) \), satisfy the conditions

\[
\int \phi_o(x) x^\xi \mathcal{L}^n x = 1 \text{ if } o = \xi, \quad \int \phi_o(x) x^\xi \mathcal{L}^n x = 0 \text{ if } o \neq \xi
\]

whenever \( m = 0, \ldots, k - 1 \) and \( \xi \in \Xi(n,m) \), where \( x^\xi = \prod_{j=1}^n (x_j)^{\xi_j} \), and \( 0^0 = 1 \). Such a family of functions was employed in \[\text{Res68a} \] p. 297 to construct, for a purpose very similar to ours, a “projection operator” of \( \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) onto the subspace of distributions corresponding to a polynomial function of degree at most \( k - 1 \). We also note that, for \( o \in \Xi(n,0) \), the condition on \( \phi_o \) is equivalent to the invariance property in question.

3.7 Remark. It is illustrative to compare the preceding theorem (for \( i = 1 = k \) and \( Y = \mathbb{R} \)) to the strong constancy lemma of \[\text{All86} \] §1 which, in view of 3.1, may be restated as follows. Whenever \( n \) is a positive integer, \( 0 < \lambda \leq 1 \), and \( \epsilon > 0 \), there exists \( 0 \leq \Gamma < \infty \) with the following property: If \( a \in \mathbb{R}^n \), \( 0 < r < \infty \), \( T \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \), \( T(\phi) \geq 0 \) for \( 0 \leq \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \), \( 0 \leq \kappa < \infty \), and

\[
\| (D_j T)(\phi) \| \leq \kappa \sup \mathcal{L} |D \phi| \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \text{ and } j = 1, \ldots, n,
\]

then there exists \( 0 \leq c < \infty \) such that

\[
|T(\theta) - c \int \theta \mathcal{L}^n| \leq (\epsilon \| T \| \mathcal{B}(a,r) + \Gamma \kappa) \sup \mathcal{L} \theta
\]

for \( \theta \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \). In our case, \( T \) need not correspond to a monotone Daniell integral and the summand \( \| T \| \mathcal{B}(a,r) \) does not occur; accordingly, a stronger norm is employed for \( \theta \).

3.8 Lemma. Suppose \( k \) is a nonnegative integer, \( n \) is a positive integer, and \( \nu \) is a norm on \( \mathcal{D}_B(1)(\mathbb{R}^n, \mathbb{R}), v \in \mathbb{S}^{n-1}, K = B(v,1) \cap B(0,1), \) and \( P : \mathbb{R}^n \to \mathbb{R} \) is a polynomial function of degree at most \( k \).

Then, for some \( 0 \leq \Gamma < \infty \) determined by \( k, n, \) and \( \nu \), there holds

\[
\sup \{ \| D^m P(x) \| : x \in B(0,1), m = 0, \ldots, k \}
\]

\[
\leq \Gamma \sup \{ \int \phi P \mathcal{L}^n : \phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}), \nu(\phi) \leq 1 \}.
\]

Proof. Using rotations, it is sufficient to consider a fixed vector \( v \). Then, we notice (see \[\text{Fed69} 2.6.5\]) that both sides represent norms on the finite dimensional space of real valued polynomial functions on \( \mathbb{R}^n \) of degree at most \( k \). \( \square \)

3.9 Remark. The particular form of the set \( K \) will be employed in the proof of 3.8. For the present section, \( K = B(0,1) \) would be sufficient.
3.10 Remark. Whenever \(i\) is a nonnegative integer, we may take \(\nu = \nu^i_{B(0,1)}\) and employ (2.2) to infer, for \(a \in \mathbb{R}^n\), \(0 < s < \infty\), and \(m = 0, \ldots, k\), that
\[
s^m \| D^m P(a) \| \leq \Gamma s^{-n-i} \sup \left\{ \| \phi P \|_{\mathscr{L}^n_\nu} : \phi \in \mathcal{D}(R^n, R), \sup \| D^\alpha \phi \| \leq 1 \right\}.
\]

3.11 Theorem. Suppose \(i, k, \) and \(l\) are integers, \(i \geq 0, k \geq 1, 0 < \alpha \leq 1, l + \alpha \geq 0\), \(Y\) is a Banach space, \(T \in \mathcal{D}^l(R^n, Y)\), and \(a \in \mathbb{R}^n\).

Then, the following four statements hold.

(1) If \(l \geq 0\) and, for \(o \in \mathfrak{E}(n,n)\), the distribution \(D^o T\) is \(\nu^i_{B(0,1)}\) pointwise differentiable of order \(l\) at \(a\), then, for \(m = 0, \ldots, k - 1\) and \(\xi \in \mathfrak{E}(n,m)\), the distribution \(D^o T\) is \(\nu^i_{B(0,1)}\) pointwise differentiable of order \(k + l - m\) at \(a\).

(2) If, for \(o \in \mathfrak{E}(n,n)\), the distribution \(D^o T\) is \(\nu^i_{B(0,1)}\) pointwise differentiable of order \((l, \alpha)\) at \(a\), then, for \(m = 0, \ldots, k - 1\) and \(\xi \in \mathfrak{E}(n,m)\), the distribution \(D^o T\) is \(\nu^i_{B(0,1)}\) pointwise differentiable of order \((k + l - m, \alpha)\) at \(a\).

(3) If \(l \geq 0\) and, for \(o \in \mathfrak{E}(n,n)\), the distribution \(D^o T\) is pointwise differentiable of order \(l\) at \(a\), then, for \(m = 0, \ldots, k - 1\) and \(\xi \in \mathfrak{E}(n,m)\), the distribution \(D^o T\) is pointwise differentiable of order \(k + l - m\) at \(a\).

(4) If, for \(o \in \mathfrak{E}(n,n)\), the distribution \(D^o T\) is pointwise differentiable of order \((l, \alpha)\) at \(a\), then, for \(m = 0, \ldots, k - 1\) and \(\xi \in \mathfrak{E}(n,m)\), the distribution \(D^o T\) is pointwise differentiable of order \((k + l - m, \alpha)\) at \(a\).

Moreover, under the hypotheses of any of these statements, there holds (see (2.12))
\[
e^{o - \xi} \cdot \text{pt} D^{k-m+\mu}(D^o T)(a) = \text{pt} D^\mu(D^o T)(a)
\]
whenever \(m = 0, \ldots, k - 1, \xi \in \mathfrak{E}(n,m), \mu = 0, \ldots, l, o \in \mathfrak{E}(n,k),\) and \(\xi \leq o\).

Proof. We will reduce the problem by adding the condition \(\text{pt} D^\mu(D^o T)(a) = 0\) for \(\mu = 0, \ldots, l\) and \(o \in \mathfrak{E}(n,k)\) to the hypotheses of the four statements. Indeed, noting (2.11) the hypotheses of any of the four statements allow to apply (2.12) with \(T, k, m, \xi,\) and \(\mu\) replaced by \(D^o T, \mu, \mu, \pi,\) and \(0\) to conclude that \(D^{o+\pi} T = D^o D^\pi T\) is pointwise differentiable of order \(0\) at \(a\) with
\[
\text{pt} D^{n+\pi}(D^{o+\pi} T)(a) = \langle e^\pi, \text{pt} D^\mu(D^o T)(a) \rangle
\]
for \(\mu = 0, \ldots, l, \pi \in \mathfrak{E}(n,\mu),\) and \(o \in \mathfrak{E}(n,k)\). Defining \(\psi_\mu \in \mathcal{O}^{k+\mu}(R^n, Y^*)\) by
\[
\langle e^\pi, \psi_\mu \rangle = \text{pt} D^\mu(D^o T)(a)\]
for \(\mu = 0, \ldots, l\) and \(o \in \mathfrak{E}(n,k + \mu)\) and \(Q : R^\mu \to Y^*\) by
\[
Q(x) = \sum_{\mu = 0}^{k+\mu} ((x - a)^{k+\mu}/(k + \mu)!), \psi_\mu\]
for \(x \in R^n\), we then infer
\[
D^\mu D^o Q(a) = e^{\pi} \cdot \psi_\mu = \text{pt} D^\mu(D^o T)(a)\]
for \(\mu = 0, \ldots, l\) and \(o \in \mathfrak{E}(n,k)\), whence the indicated reduction follows by replacing \(T(\theta)\) by \(T(\theta) - \int \langle \theta, Q \rangle d\mathcal{L}^n\) for \(\theta \in \mathcal{D}(R^n, Y)\).

Next, we establish (1) and (2) and the validity of the formula in the postscript under the hypotheses of any of these two statements. For this purpose, we define \(\gamma = l\) in case of (1) and \(\gamma = l + \alpha\) in case of (2), hence \(\gamma \geq 0\). We choose \(\Phi_{\gamma}\) and
\(\Phi_r\) as in [3.3]. For \(0 < r < \infty\), we also define polynomial functions \(P_r : \mathbb{R}^n \to Y^*\) of degree at most \(k - 1\) characterised by

\[\langle y, D^k P_r(a) \rangle = (D^k T)a(\Phi_r(b - a)y)\]

whenever \(m = 0, \ldots, k - 1, \xi \in \mathcal{E}(n, m),\) and \(y \in Y\), abbreviate \(C(r) = B(a, r)\) and \(K(r) = B(a, (k + 1)r)\), and let \(\kappa(r)\) denote the supremum of the set of all numbers

\[s^{-n-\gamma-i}(D^n T)(\phi)\]

corresponding to \(0 < s \leq r, a \in \mathcal{E}(n, k),\) and \(\phi \in \mathcal{D}_{K(r)}(\mathbb{R}^n, Y)\) satisfying \(\|D^k \phi\| \leq 1.\) By [3.2] the hypotheses of [1] and [2] yield

\[\lim_{r \to 0^+} \kappa(r) = 0\] in case of [1], \(\limsup_{r \to 0^+} \kappa(r) < \infty\) in case of [2];

in particular, there exists \(0 < \delta < \infty\) with \(\kappa(\delta) < \infty.\) For \(0 < r \leq \delta,\) applying [3.3] with \(\Phi, \kappa,\) and \(C\) replaced by \(\Phi_r, r^{n+\gamma+i}\kappa(r),\) and \(C(r),\) we obtain

\[|\langle D^k T\theta - \langle \theta, D^k P_r \rangle, d\mathcal{L}^n \rangle| \leq \Delta_1 r^{-k+m+n+\gamma+i} \kappa(r),\]

where \(\Delta_1 = \left(\sup_{n, \alpha(n)} \|D^k \Phi\| \right) + 1 \left(\|(k + 1)!2^k\right)^{1+n+i},\)

for \(m = 0, \ldots, k - 1, \xi \in \mathcal{E}(n, m),\) and \(\theta \in \mathcal{D}_{C(r)}(\mathbb{R}^n, Y)\) with \(\|D^k \theta\| \leq 1.\) Moreover, whenever \(0 < \theta / 2 \leq s \leq r \leq \delta,\) noting

\[|\langle \theta, P_r - P_s \rangle, d\mathcal{L}^n \rangle| \leq 2\Delta_1 r^{k+n+\gamma+i} \kappa(r) \sup \|D^k \theta\| \quad \text{for} \quad \theta \in \mathcal{D}_{C(r)}(\mathbb{R}^n, Y),\]

implies

\[\|D^m (P_r - P_s)(a)\| \leq \Delta_2 r^{-k+m+\gamma+i} \kappa(r) \quad \text{for} \quad m = 0, \ldots, k - 1,\]

where \(0 \leq \Delta_2 < \infty\) is determined by \(i, k, n,\) and \(\Phi.\) Therefore, as \(k - m + \gamma \geq 1\), we may define \(P : \mathbb{R}^n \to Y^*\) by \(P(x) = \sum_{m=0}^{k-1} (x - a)^m / m!\), \(\lim_{r \to 0^+} \|D^k P_r(a)\|\) for \(x \in \mathbb{R}^n\) and estimate

\[\|D^m (P_r - P_s)(a)\| \leq \Delta_2 r^{-k+m+\gamma+i} \kappa(r) \quad \text{for} \quad m = 0, \ldots, k - 1 \quad \text{and} \quad 0 < r \leq \delta,\]

both using the geometric series. Employing [3.2] for \(\theta\) and Taylor’s formula (see [Fed69] 10.4, 3.1.11]) to bound \(\|D^k (P_r - P)(x)\|\) for \(x \in C(r),\) we conclude

\[|\langle \theta, D^k (P_r - P) \rangle, d\mathcal{L}^n \rangle| \leq 6\alpha(n)\Delta_2 r^{-k+m+n+\gamma+i} \kappa(r),\]

\[|\langle D^k \theta \rangle - \langle \theta, D^k P \rangle, d\mathcal{L}^n \rangle| \leq (\Delta_1 + 6\alpha(n)\Delta_2) r^{-k+m+n+\gamma+i} \kappa(r)\]

for \(m = 0, \ldots, k - 1, \xi \in \mathcal{E}(n, m),\) and \(\theta \in \mathcal{D}_{C(r)}(\mathbb{R}^n, Y)\) with \(\|D^k \theta\| \leq 1.\) Whence we infer [1] and [2] and the corresponding part of the postscript.

Combining [2] and [2.2], we obtain [4] and its part of the postscript.

To establish [3], we firstly notice that the case \(\dim Y < \infty\) of [3], including its postscript, follows from [1] and [2.27]. To treat the general case of [3], we define \(Y^* \in \mathcal{D}'(\mathbb{R}^n, \mathbb{R})\) by \(Y^*(\zeta) = T_{\zeta(x)}(y)\) for \(y \in Y\) and \(\zeta \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}).\) From the case \(\dim Y < \infty\) of [3] and its postscript, we conclude that \(Y^*\) is pointwise differentiable of order \(k + l\) at \(a\) and \(ptD^{k+l} Y^*(a) = 0\) for \(\mu = 0, \ldots, l.\) By [4],
$T$ is pointwise differentiable of order $(k + l - 1, 1)$ at $a$. Defining $P : \mathbb{R}^n \to Y^*$ by
\[ P(x) = \sum_{m=0}^{k-1} \langle (x - a)^m / m!, \text{pt} D^m T(a) \rangle \quad \text{for } x \in \mathbb{R}^n, \]
we readily verify
\[ \langle \eta, \text{pt} D^\nu T^\nu(a) \rangle = \langle y, \langle \eta, D^\nu P(a) \rangle \rangle \]
whenever $\nu = 0, \ldots, k + l$, $\eta \in \mathcal{D}$, $\mathbb{R}^n$, and $y \in Y$. In combination with 2.28 we then apply 2.25 with $k$ replaced by $k + l - 1$ to infer that $T$ is pointwise differentiable of order $k + l$ at $a$ and that $P$ corresponds to the $k + l$ jet of $T$ at $a$. Finally, 2.12 with $k$ and $\mu$ replaced by $k + l$ and $k - m + \mu$ yields the remaining cases of [3] and the postscript.

3.12 Remark. The preceding theorem is partly analogous to [CZ61, Theorem 11], where differentiability in Lebesgue spaces with respect to $\mathcal{L}^n$ is treated for $k = 1$ in such a manner as to include embedding results.

3.13 Remark. By 2.31[1] and 2.37, taking $i = 0 = l$ in [1] yields the differentiability result for real valued functions on $\mathbb{R}^n$ whose $k$-th order distributional partial derivatives are representable by integration contained in [Reš68a, Theorem 1].

3.14 Corollary. $T$ is pointwise differentiable of order $k$ at $a$ if and only if, for $a \in \mathfrak{S}(n, k)$, the distribution $D^\nu T$ is pointwise differentiable of order $0$ at $a$.

Proof. Combine 2.12 and 3.11[3].

3.15 Remark. For the case $k = 1 = n$, $Y = \mathcal{C}$, and $\mathcal{C}$ linear $T$, a related characterisation was noted in [Kim14, Proposition 2]. The weaker statement that pointwise differentiability of order 0 at $a$ of the distributions $D^\nu T$ for $a \in \mathfrak{S}(n, k)$ implies pointwise differentiability of $T$ of order 0 at $a$ had been proven for the same case in [Loj57, 3.2 Théorème] and follows for general $n$ from [Ita66, Lemma 3].

### 4 Differentiability theory

In this section, we firstly carry out the necessary adaptations of various known results to cover the case of infinite dimensional target spaces in 4.1–4.25. Then, we establish the main theorems of the differentiability theory in 4.25–4.25.

4.1 Lemma. Suppose $k$ is a positive integer, $Y$ is a separable Banach space, $f : \mathbb{R}^n \to Y^*$ is of class $k - 1$, $M = \text{Lip} D^{k-1} f < \infty$, and $Z_k$ is defined and endowed with the $Y \otimes \mathcal{O}_k \mathbb{R}^n$ topology as in 2.8.

Then, there exists an $\mathcal{L}^n$ almost unique, $Z_k$ valued $\mathcal{L}^n$ measurable function $g$ satisfying $\mathcal{L}^n(\langle \|g\| \rangle) \leq M$ and
\[ \int \langle \eta, D^k \zeta(x) \rangle (y, f(x)) \, d\mathcal{L}^n x = (-1)^k \int \zeta(x) \langle y, \langle \eta, g(x) \rangle \rangle \, d\mathcal{L}^n x \]
whenever $\zeta \in \mathcal{D}(\mathbb{R}^n, \mathbb{R})$, $\eta \in \mathcal{O}_k \mathbb{R}^n$, and $y \in Y$.

Proof. If $f$ is of class $k$, then we may take $g = D^k f$ and, by [Fed69, 2.2.7, 3.1.1, 3.1.11], notice that $M = \sup \text{in} \| D^k f \|$. In the general case, the image of $f$ is contained in a separable subspace of $Y^*$, hence convolution as described.
Whenever a corresponding to 4.4 Remark. Suppose 4.2 Lemma. By [BL00, 5.12(i)],
Proof. In view of 4.1, one may apply 2.31(2)(3)(4) with
−∞ < a < b < ∞ by [Men16b, 2.1] and 2.2 both with
By 2.8, that case also follows from the fact
\[ \left(\frac{\langle \theta, D f \rangle}{d \mathcal{L}^n} = \int \langle \theta, g \rangle d \mathcal{L}^n \quad \text{for} \quad \theta \in L^1(\mathcal{L}^n, Y \otimes \mathcal{O}_k \mathbb{R}^n). \]

In view of the special case, the conclusion now readily follows.

4.2 Lemma. Suppose k is a positive integer, Y is a separable Banach space, \( f : \mathbb{R}^n \to Y^* \) is of class \( (k-1,1) \), and \( S \in \mathcal{D}(\mathbb{R}^n, Y) \) satisfies

\[ S(\phi) = \int \langle \phi, f \rangle d \mathcal{L}^n \quad \text{for} \quad \phi \in \mathcal{D}(\mathbb{R}^n, Y). \]

Then, the following three statements hold.

(1) For \( a \in \mathbb{R}^n \), S is \( \nu^0_{B(0,1)} \) pointwise differentiable of order \( (k-1,1) \) at \( a \) and \( pt D^n S(a) = pt D^m f(a) \) for \( m = 0, \ldots, k-1 \).

(2) For \( \mathcal{L}^n \) almost all \( a \), S is pointwise differentiable of order \( k \) at \( a \).

(3) If \( Y^* \) is separable, then S is \( \nu^0_{B(0,1)} \) pointwise differentiable of order \( k \) at \( \mathcal{L}^n \) almost all \( a \).

Proof. In view of 4.1, one may apply [2.31][2][3][1] with S replaced by \( D^k T \) whenever \( \xi \in \Xi(n, k) \). Therefore, the conclusion follows from [3.11][1][2][3].

4.3 Theorem (Gelfand’s Rademacher theorem). Suppose Y is a Banach space, \( Y^* \) is separable, and \( f : \mathbb{R}^n \to Y^* \) is locally Lipschitzian.

Then, \( f \) is differentiable at \( \mathcal{L}^n \) almost all \( a \).

Proof. By [BL00, 5.12(i)], \( Y^* \) has the Radon-Nikodým property in the sense of [BL00, 5.4]. Therefore, the conclusion follows from [BL00, 4.3, 6.41].

4.4 Remark. The main case \( n = 1 \) originates from [Gel38, §9, Satz 2, Hilfssatz 5]. By 2.8, that case also follows from the fact \( f(b) - f(a) = \int_0^1 (1, g(x)) d \mathcal{L}^1 x \) for \( -\infty < a < b < \infty \), where \( g \) is as in [4.1] with \( k = 1 \), via [Fed69, 2.8.18, 2.9.9].

4.5 Theorem (Whitney’s extension theorem). Suppose Y is a normed vector-space, \( k \) is a nonnegative integer, \( 0 < \alpha \leq 1 \), A is a closed subset of \( \mathbb{R}^n \), and to each \( a \in A \) corresponds a polynomial function

\[ P_a : \mathbb{R}^n \to Y \text{ with degree } P_a \leq k. \]

Whenever \( C \subset A \) and \( \delta > 0 \), let \( g(C, \delta) \) be the supremum of the set of all numbers

\[ \| D^m P_a(b) - D^m P_b(b) \| \cdot |a - b|^{m-k} \cdot (k - m)! \]

corresponding to \( m = 0, \ldots, k \) and \( a, b \in C \) with \( 0 < |a - b| \leq \delta \).

If \( \limsup_{\delta \to 0+} \delta^{-n} g(C, \delta) < \infty \) for each compact subset \( C \) of A, then there exists a map \( g : \mathbb{R}^n \to Y \text{ of class } (k, \alpha) \) such that

\[ D^m g(a) = D^m P_a(a) \quad \text{for} \quad m = 0, \ldots, k \text{ and } a \in A. \]
Proof. Assuming $A \neq \emptyset$, we proceed as in [Fed69, 3.1.14]; hence, $U = \mathbb{R}^n \sim A$ and there is a map $g : \mathbb{R}^n \to Y$ of class $k$ such that $g|U$ is of class $\infty$ and

$$D^m g(a) = D^m P_a(a) \quad \text{for } m = 0, \ldots, k \text{ and } a \in A.$$  

Noting that lines 19–23 of [Fed69, p. 226] remain valid if $i = k + 1$, we record, from the proof of [Fed69, 3.1.14], the two estimates

$$\|D^{k+1} g(x)\| \leq M_{k+1} \text{dist}(x, A)^{-1} \rho(C, 6 \text{dist}(x, A)) \quad \text{if } x \in U,$$

$$\|D^k g(x) - D^k g(a)\| \leq (M_k + 1) \rho(C, 6|x - a|)$$

whenever $a \in A, x \in B(a, 1/3)$, and $C = A \cap B(a, 2)$, where $M_k$ and $M_{k+1}$ are real numbers determined by $n$ and $k$.

Suppose $a \in A, C = A \cap B(a, 3)$, notice that there exists $0 \leq \kappa < \infty$ with $g(C, \delta) \leq \kappa \delta$ for $0 < \delta \leq 2$, and let $y, z \in B(a, 1/9)$. If

$$\text{dist}(y + t(z - y), A) \geq |y - z|/2 \quad \text{for } 0 \leq t \leq 1,$$

we obtain from the first estimate that

$$\|D^k g(y) - D^k g(z)\| \leq 12\kappa M_{k+1}|y - z|^\alpha$$

using [Fed69, 2.2.7, 3.1.1] and $\|D^k g\| = \|D^{k+1} g\|$. If

$$\text{dist}(y + t(z - y), A) < |y - z|/2 \quad \text{for some } 0 \leq t \leq 1,$$

then, we take $c \in A$ with $|y + t(z - y) - c| < |y - z|/2$, notice

$$\sup\{|y - c|, |z - c|\} \leq 3|y - z|/2 \leq 1/3, \quad |c - a| \leq |c - y| + |y - a| \leq 1,$$

and obtain from the second estimate (with $a$ replaced by $c$)

$$\|D^k g(z) - D^k g(y)\| \leq \|D^k g(z) - D^k g(c)\| + \|D^k g(c) - D^k g(y)\|$$

$$\leq 6\kappa(M_k + 1)(|z - c|^\alpha + |y - c|^\alpha) \leq 18\kappa(M_k + 1)|z - y|^\alpha.$$

Accordingly, the map $(D^k g)|B(a, 1/9)$ is Hölder continuous with exponent $\alpha$. $\square$

4.6 Remark. The last paragraph of the proof is adapted from [Ste70, VI.2.2.2]; see also [Ste70, VI.2.3.1–VI.2.3.3].

4.7 Theorem. Suppose $k$ is a nonnegative integer, $Z$ is a separable Banach space, $f : \mathbb{R}^n \to Z$ is of class $k$, and $A = \text{dimn} D^k f$.

Then, for each $\varepsilon > 0$, there exists a map $g : \mathbb{R}^n \to Z$ of class $k + 1$ such that

$$\mathcal{L}^n (A \sim \{a : D^m f(a) = D^m g(a) \text{ for } m = 0, \ldots, k\}) = 0.$$  

Proof. We may proceed as in [Fed69, 3.1.15]; in fact, it is sufficient to replace “Assured . . . measurable”, $m$, and $\mathbb{R}^n$ in its proof by “Assured by [Fed69, 3.1.1] that $A$ is a Borel set and $D^k f$ is a Borel function”, $n$, and $Z$, respectively. $\square$

4.8 Remark. In contrast to [Fed69, 3.1.15], one may not replace $A$ by $\mathbb{R}^n \cap \{a : \limsup_{x \to a} |f(x) - f(a)|/|x - a| < \infty\}$; in fact, in view of [Men16b, 2.1], we obtain a separable Banach space $Z$ and a Lipschitzian function $f : \mathbb{R} \to Z$ with $\mathcal{L}^1 (\mathbb{R} \sim \text{dimn} D f) > 0$ from [Fed69, p. 265], observe $\text{dimn} D f = \text{dimn} \text{ap} D f$ by [Fed69, 3.1.5], and note that, for $g : \mathbb{R} \to Z$ of class 1, [Fed69, 2.8.18, 2.9.11] yields $\mathcal{L}^1 (\{a : f(a) = g(a)\} \sim \text{ap} D f) = 0$.  

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4.9 Theorem. Suppose $k$ is an integer, $0 < \alpha \leq 1$, $k + \alpha \geq 0$, $Y$ is a Banach space, $T \in \mathcal{D}'(\mathbb{R}^n, Y)$, and $A$ is the set of points at which $T$ is pointwise differentiable of order $(k, \alpha)$.

Then, there exists a sequence of compact subsets $C_j$ of $\mathbb{R}^n$ with $A = \bigcup_{j=1}^{\infty} C_j$ and, if $k \geq 0$, also a sequence of functions $f_j : \mathbb{R}^n \to Y^*$ of class $(k, \alpha)$ satisfying

$$\text{pt} \mathcal{D}^m T(a) = \mathcal{D}^m f_j(a) \quad \text{for } a \in C_j \text{ and } m = 0, \ldots, k$$

whenever $j$ is a positive integer.

Proof. Suppose $Z_m$ are defined and endowed with the $Y \otimes \bigoplus_m \mathbb{R}^n$ topology as in 4.8. Whenever $j$ is a positive integer, we abbreviate $\nu_j = \nu_{\mathcal{B}(0,1)}$ and define compact sets $L_j$ to consist of those $(a, \psi) \in \mathbb{R}^n \times \bigoplus_{m=0}^{k} Z_m$ satisfying

$$|a| \leq j, \quad \|\psi_m\| \leq j \quad \text{for } m = 0, \ldots, k,$$

$$\left| T_x \left( \phi \left( \frac{x-a}{r} \right) \right) \right| = \sum_{m=0}^{k} \left| \int \phi \left( \frac{x-a}{m} \right) \, d\mathcal{L}^n x, \psi_m \right| \leq jr^{k+\alpha+n}$$

whenever $0 < r \leq j^{-1}$, $\phi \in \mathcal{D}_{\mathcal{B}(0,1)}(\mathbb{R}^n, Y)$, and $\nu_j(\phi) \leq 1$.

Then, we have

$$\left( A \times \bigoplus_{m=0}^{k} Z_m \right) \cap \{(a, \psi) : \psi_m = \text{pt} \mathcal{D}^m T(a) \text{ for } m = 0, \ldots, k\} = \bigcup_{j=1}^{\infty} L_j$$

by 2.22. We define compact sets $C_j = \{(a, \psi) \in L_j \text{ for some } \psi\}$, whenever $j$ is a positive integer, as well as polynomial functions $P_a : \mathbb{R}^n \to Y^*$ by letting

$$P_a(x) = \sum_{m=0}^{k} (x-a)^m / m! \cdot \text{pt} \mathcal{D}^m T(a) \quad \text{for } a \in A \text{ and } x \in \mathbb{R}^n.$$

To apply 4.5, we suppose $j$ is a positive integer and $a, b \in C_j$, $0 < |a-b| \leq 1/j$. 

Abbreviating $r = |a-b|$, $v = r^{-1}(a-b)$, and $K = \mathcal{B}(v, 1) \cap \mathcal{B}(0, 1)$, we notice $r^{-1}(x-a) = r^{-1}(x-b) + v$ for $x \in \mathbb{R}^n$ and infer

$$\left| \int \phi \left( \frac{x-a}{r} \right), (P_a - P_b)(x) \right| \, d\mathcal{L}^n x \leq \left| \int \phi \left( \frac{x-a}{r} \right), P_a(x) \right| \, d\mathcal{L}^n x - T_x \left( \phi \left( \frac{x-a}{r} \right) \right) \right|$$

$$+ \left| T_x \left( \phi \left( \frac{x-a}{r} \right) \right) - \int \left( \phi \left( \frac{x-a}{r} \right), P_a(x) \right) \, d\mathcal{L}^n x \right|$$

$$\leq 2j r^{k+\alpha+n}$$

whenever $\phi \in \mathcal{D}_K(\mathbb{R}^n, Y)$ and $\nu_j(\phi) \leq 1$, where $\theta \in \mathcal{D}_{\mathcal{B}(0,1)}(\mathbb{R}^n, Y)$ satisfies $\theta(x) = \phi(x+v)$ for $x \in \mathbb{R}^n$. Consequently, applying 4.8 with $\nu$ and $P(x)$ replaced by $\nu_j$ and $(y, (P_a - P_b)(a + r x))$, whenever $y \in Y$, we conclude

$$\sup \{ r^m \| \mathcal{D}^m(P_a - P_b)(x) \| : x \in \mathcal{B}(a, r), m = 0, \ldots, k \} \leq 2j r^{k+\alpha+n},$$

where $0 \leq \kappa < \infty$ is determined by $j$, $k$, and $n$; hence, 4.5 yields the conclusion.

4.10 Corollary. The set $A$ is a Borel subset of $\mathbb{R}^n$ and the functions mapping $a \in A$ onto $\mathcal{D}^m T(a) \in \bigotimes_m \mathcal{D}(\mathbb{R}^n, Y^*)$ are Borel functions for $m = 0, \ldots, k$.\footnote{Concerning the case $k = -1$, recall that the direct sum of an empty family of vectorspaces is the zero vectorspace and that the sum over the empty set equals zero.}
4.11 Remark. In case \( n = 1, k = -1, \alpha = 1, Y = C, \) and \( C \) linear \( T \), the fact that \( A \) is the union of a countable family of compact sets was already obtained in [Zie60] §3.1 Folgerung 2.

4.12 Theorem. Suppose \( k \) is a nonnegative integer, \( Y \) is a separable Banach space, \( Z_k \) is defined and endowed with the \( Y \otimes \bigotimes_k \mathbb{R}^n \) topology as in 2.8. \( T \in \mathcal{D}'(\mathbb{R}^n, Y) \), and \( A \) is the set of points at which \( T \) is pointwise differentiable of order \( k \).

Then, \( A \) is a Borel set and the function mapping \( a \in A \) onto \( \text{pt} D^k T(a) \in Z_k \) is a Borel function.

Proof. Suppose \( 0 \leq M < \infty \). It is sufficient to prove the assertion with \( A \) replaced by

\[
A' = A \cap \{ a : \| \text{pt} D^m T(a) \| \leq M \text{ for } m = 0, \ldots, k \}.
\]

For this purpose, let \( B \) be the Borel set (see 4.9) of points at which \( T \) is pointwise differentiable of order \((k - 1, 1)\). Moreover, whenever \( i \) and \( j \) are positive integers and \( \phi \in \mathcal{D}(\mathbb{R}^n, Y) \), we define the closed sets \( L(i, j, \phi) \) to consist of those \( (a, \psi) \in \mathbb{R}^n \times \bigoplus_{m=0}^k Z_m \), satisfying

\[
\| \psi_m(t) \| \leq M \text{ for } m = 0, \ldots, k,
\]

\[
\left| T_k(\phi(x_a^t)) - \int \phi(x_a^t) \sum_{m=0}^k \langle (x - a)^m / m!, \psi_m \rangle d\mathcal{L}^n x \right| \leq r^{k+n} / i
\]

for \( 0 < r < 1 / j \), where \( Z_m \) are defined and endowed with the \( Y \otimes \bigotimes_m \mathbb{R}^n \) topology as in 2.8. Employing [Men16a, 2.2, 2.24] to choose a countable sequentially dense subset \( \Delta \) of \( \mathcal{D}(\mathbb{R}^n, Y) \), we infer that

\[
G = \left( B \times \bigoplus_{m=0}^k Z_m \right) \cap \bigcap_{\phi \in \Delta} \bigcup_{i,j,\phi} L(i, j, \phi)
\]

is a Borel set. Noting \( (a, \psi) \in G \) if and only if \( a \in A' \) and \( \psi_m = \text{pt} D^m T(a) \) for \( m = 0, \ldots, k \) by 2.25, we apply [Men16c, 4.1] with \( \mathbb{R}^n \) and \( (\bigoplus_{m=0}^k Z_m) \cap \{ \psi : \| \psi_m \| \leq M \text{ for } m = 0, \ldots, k \} \) to infer the conclusion.

\[ \square \]

4.13 Remark. In view of 2.17 and 2.27 in case \( n = 1, k = 0, Y = C, \) and \( C \) linear \( T \), slightly more precise information on \( A \) may be obtained from [Zie60] §3.3 Folgerung 1.

4.14 Remark. From 2.29 and 2.31 \( \square \) we infer that, with respect to the norm topology on \( Z_k \), the function \( \text{pt} D^k T \) may be \( \mathcal{L}^n \)-\( A \) nonmeasurable.

4.15 Lemma. Suppose \( n \) is a positive integer, \( A \) is a closed subset of \( \mathbb{R}^n \), \( Y \) is a Banach space, \( T \in \mathcal{D}'(\mathbb{R}^n, Y) \), \( A \cap \text{spt} T = \emptyset \), \( 0 < r < \infty \), \( 0 \leq \kappa < \infty \), \( 0 \leq \lambda < \infty \), \( i \) is a nonnegative integer, and

\[
|T(\phi)| \leq \kappa s^{n+\lambda+i} \sup \| D^i \phi \|
\]

whenever \( b \in A \), \( 0 < s \leq 3r \), and \( \phi \in \mathcal{D}_b(b) \). Then, for some \( 0 \leq \Gamma < \infty \) determined by \( n, \lambda, \) and \( i \), there holds

\[
|T(\phi)| \leq \Gamma r^{\lambda+i} \mathcal{L}^n(B(a, 3r) \sim A) \sup \| D^i \phi \|
\]

whenever \( a \in A \) and \( \phi \in \mathcal{D}(B(a, 3r)) \).
Proof. We assume \( r < 1/2 \) and suppose \( a \in A \) and \( \phi \in \mathcal{D}(\mathbb{R}^n, Y) \).
We define \( h : \mathbb{R}^n \to \mathbb{R} \) by
\[
h(x) = \frac{1}{20} \inf \{1, \inf \{a, \inf \{\phi(x)\} \} \} \quad \text{for } x \in \mathbb{R}^n \sim A.
\]
Applying \[Fed69\] 3.1.13 with \( m \) and \( \Phi \) replaced by \( n \) and \( \{\mathbb{R}^n \sim A\} \), we obtain a countable subset \( C \) of \( \mathbb{R}^n \sim A \), nonnegative functions \( \zeta_c \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) for \( c \in C \), and a sequence, determined by \( n \), of numbers \( 0 \leq V_m < \infty \) for every nonnegative integer \( n \) such that the family \( \{B(c, h(c)) : c \in C\} \) is disjointed, such that
\[
h(x) \geq h(c)/3 \quad \text{for } x \in B(c, 10h(c)),
\]
\[\text{card}(C \cap \{c : B(c, 10h(c)) \cap B(x, 10h(x)) \neq \emptyset\}) < \infty \quad \text{for } x \in \mathbb{R}^n \sim A,
\]
\[
\|D^m \zeta_c(x)\| \leq V_m h(x)^{-m} \quad \text{for } x \in \mathbb{R}^n \sim A, \quad m = 0, 1, 2, \ldots
\]
for \( c \in C \), and such that \( \sum_{c \in C} \zeta_c(x) = 1 \) for \( x \in \mathbb{R}^n \sim A \). We let
\[
C' = C \cap \{c : B(c, 10h(c)) \cap B(a, r) \cap \text{spt } T \neq \emptyset\}.
\]
Since \( B(a, r) \cap \text{spt } T \) is a compact subset of \( \mathbb{R}^n \sim A \), the set \( C' \) is finite and \( B(a, r) \cap \text{spt } T \) is contained in the interior of \( \{x : \sum_{c \in C'} \zeta_c(x) = 1\} \), whence it follows
\[
T(\phi) = \sum_{c \in C'} T(\zeta_c \phi).
\]
Next, we choose \( \xi : C' \to A \) with \( |c - \xi(c)| = \inf(c, A) \) and verify
\[
10h(c) \leq r, \quad B(c, 10h(c)) \subset B(\xi(c), 30h(c)), \quad B(c, h(c)) \subset B(a, 3r) \sim A
\]
for \( c \in C' \); in fact, we have
\[
20h(c) \leq |c - \xi(c)| \leq |c - a| \leq 10h(c) + r \leq 2r < 1, \quad 20h(c) = |c - \xi(c)|.
\]
Whenever \( c \in C \), considering both \( \zeta_c \) and \( \theta \) as maps into the normed algebra \( \mathcal{O}, Y \), the general Leibniz formula in \[Fed69\] 3.1.11 yields
\[
D^i(\zeta_c \phi)(x) = \sum_{m=0}^{i} D^m \zeta_c(x) \odot D^{i-m} \phi(x) \quad \text{for } x \in \mathbb{R}^n.
\]
Using \[Fed69\] 1.10.5 and \[3.2\] we therefore estimate
\[
\sup \text{im } \|D^i(\zeta_c \theta)\| \leq \sum_{m=0}^{i} \binom{i}{m} V_m 3^n h(c)^{-m} r^m \sup \text{im } \|D^i \phi\|
\]
\[
\leq \Delta h(c)^{-i} r^i \sup \text{im } \|D^i \phi\|
\]
whenever \( c \in C' \), where \( \Delta = \sum_{m=0}^{i} \binom{i}{m} V_m 3^n \). Accordingly, abbreviating \( \Gamma = \Delta(30)^{n+1} \gamma \alpha(n)^{-1} \), we obtain
\[
|T(\zeta_c \phi)| \leq \kappa(30h(c))^{n+\lambda+1} \sup \text{im } \|D^i(\zeta_c \phi)\|
\]
\[
\leq \Gamma \kappa r^{\lambda+1} \sum_{m=0}^{i} B(c, h(c)) \sup \text{im } \|D^i \phi\|
\]
for \( c \in C' \), whence \( |T(\phi)| \leq \Gamma \kappa r^{\lambda+1} \sum_{m=0}^{i} (B(a, 3r) \sim A) \sup \text{im } \|D^i \phi\| \) follows. \( \square \)
4.16 Remark. The preceding lemma extends [Men13, A.1] where the case \( \lambda = 0 \), \( i = 1 \), and \( Y \) a finite dimensional Euclidean space is treated.

4.17 Theorem. Suppose \( i \) and \( k \) are nonnegative integers, \( Y \) is a separable Banach space, \( R \in \mathcal{D}(\mathbb{R}^n, Y) \), and \( A \) is the set of all \( a \in \mathbb{R}^n \) such that \( R \) is \( \nu_{B(0,1)} \) pointwise differentiable of order \( (k-1,1) \) at \( a \) and \( \text{pt} \ D^m R(a) = 0 \) for \( m = 0, \ldots, k-1 \).

Then, \( \mathcal{L}^n \) almost all \( a \in A \) satisfy the following three statements:

1. The distribution \( R \) is pointwise differentiable of order \( k \) at \( a \).
2. If \( k > 0 \) or \( Y^* \) is separable, then \( R \) is \( \nu_{B(0,1)} \) pointwise differentiable of order \( k \) at \( a \).
3. If \( k > 0 \), then \( \text{pt} \ D^k R(a) = 0 \).

Proof. Whenever \( j \) is a positive integer, we define \( A_j \) to be the set of all \( b \in \mathbb{R}^n \) satisfying

\[
|R(\phi)| \leq j s^{n+k+j} \sup \|D^j \phi\| \quad \text{for } 0 < s \leq 6/j \text{ and } \phi \in \mathcal{D}_{B(b,6)}(\mathbb{R}^n, Y).
\]

By [Fed69, 4.1.2], a point \( b \) belongs to \( A_j \) if and only if

\[
|R(\phi(\frac{x-b}{s}))| \leq j s^{n+k} \nu_{B(0,1)}(\phi) \quad \text{for } 0 < s \leq 6/j \text{ and } \phi \in \mathcal{D}_{B(0,1)}(\mathbb{R}^n, Y);
\]

in particular, the sets \( A_j \) are closed and \( A = \bigcup_{j=1}^{\infty} A_j \).

Supposing \( j \) to be a positive integer, the conclusion will be shown to hold at \( \mathcal{L}^n \) almost all \( a \in A_j \). We choose \( \Phi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}) \) satisfying \( \int \Phi \, d\mathcal{L}^n = 1 \) and \( \text{spt} \Phi \subset B(0,1) \) and abbreviate \( \Delta = \sup \|D^i \Phi\| \).

In this paragraph, we study various objects whenever \( 0 < \epsilon \leq 3/j \). With \( \Phi_\epsilon(x) = e^{\epsilon^2} \Phi(\epsilon^{-1} x) \) for \( x \in \mathbb{R}^n \), we define \( R_\epsilon \in \mathcal{D}(\mathbb{R}^n, Y) \) by

\[
R_\epsilon(\phi) = R(\Phi_\epsilon * \phi) \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y).
\]

From [Fed69, 4.1.2], we infer the existence of \( f_\epsilon \in \mathcal{C}(\mathbb{R}^n, Y^*) \) satisfying

\[
R_\epsilon(\phi) = \int_{[x : \text{dist}(x, A_j) \leq \epsilon]} \langle \phi, f_\epsilon \rangle \, d\mathcal{L}^n \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y),
\]

\[
(y, f_\epsilon(x)) = (R)_b(\Phi_\epsilon(b-x)y) \quad \text{whenever } x \in \mathbb{R}^n \text{ and } y \in Y.
\]

Clearly, we have \( R_\epsilon \to R \) as \( \epsilon \to 0+ \) and

\[
\|f_\epsilon(x)\| \leq j 2^{n+k+1} \Delta \epsilon \quad \text{whenever } \text{dist}(x, A_j) \leq \epsilon.
\]

We define \( S_\epsilon \in \mathcal{D}(\mathbb{R}^n, Y) \) by

\[
S_\epsilon(\phi) = \int_{[x : \text{dist}(x, A_j) \leq \epsilon]} \langle \phi, f_\epsilon \rangle \, d\mathcal{L}^n \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y)
\]

and abbreviate \( T_\epsilon = R_\epsilon - S_\epsilon \). For \( b \in A_j, 0 < s \leq 3/j \), and \( \phi \in \mathcal{D}_{B(b,6)}(\mathbb{R}^n, Y) \) with \( \sup \|D^i \phi\| \leq 1 \), we estimate

\[
\text{spt}(\Phi_\epsilon * \phi) \subset B(b, \epsilon + s), \quad |R_\epsilon(\phi)| \leq j 2^{n+k+1} s^{n+k+1} \text{ if } \epsilon \leq s,
\]

\[
[T_\epsilon(\phi)] \cap [x : \text{dist}(x, A_j) \leq \epsilon] = \emptyset, \quad T_\epsilon(\phi) = 0 \text{ if } \epsilon > s,
\]

\[
|S_\epsilon(\phi)| \leq (\mathcal{L}^n, \mathbb{I}_{\{x : \text{dist}(x, A_j) \leq \epsilon\}})(\mathcal{L}^n)(1) \leq j 2^{n+k+1} \Delta \epsilon \alpha(n) s^{n+k+1},
\]

\[
|T_\epsilon(\phi)| \leq \kappa s^{n+k+1}, \quad \text{where } \kappa = j 2^{n+k+1}(1 + \Delta \alpha(n)).
\]
Therefore, applying \ref{1.15} with \( A, T, \) and \( \lambda \) replaced by \( A_j, T_c, \) and \( k, \) we infer

\[
|T_\epsilon(\phi)| \leq \Gamma \kappa r^{k+1} \mathcal{L}^n(B(a,r) \sim A_j) \sup \|D^k \phi\|
\]

whenever \( a \in A_j, 0 < r \leq 1/j, \) and \( \phi \in \mathcal{D}(B(a,r))(\mathbb{R}^n, Y). \)

Since \( (\mathcal{L}^n, \mathcal{D}(\mathbb{R}^n, \mathcal{D})) = \{ x : \text{dist}(x, A_j) \leq 1/j \}, \) we may use \ref{1} and \ref{2.2} both with \( \mu = \mathcal{L}^n \) and \ref{2.8} with \( Y \) and \( m \) replaced by \( L_1(\mathcal{L}^n, Y) \) and \( 0 \) to construct \( S \in \mathcal{D}(\mathbb{R}^n, Y) \) as a subsequential limit of \( S_k \) as \( \epsilon \to 0^+ \) and a \( Y^* \) valued function that is \( \mathcal{L}^n \) measurable with respect to the \( Y \) topology on \( Y^* \) and that satisfies

\[
(\mathcal{L}^n)(\|f\|) \leq j2^{m+1} \Delta \text{ if } k = 0, \quad f = 0 \text{ if } k > 0,
\]

\[
S(\phi) = \int (\phi, f) \, d\mathcal{L}^n \quad \text{for } \phi \in \mathcal{D}(\mathbb{R}^n, Y).
\]

Defining \( T = R - S, \) the final estimate of the preceding paragraph implies

\[
|T(\phi)| \leq \Gamma \kappa r^{k+1} \mathcal{L}^n(B(a,r) \sim A_j) \sup \|D^k \phi\|
\]

for \( a \in A_j, 0 < r \leq 1/j, \) and \( \phi \in \mathcal{D}(B(a,r))(\mathbb{R}^n, Y). \) Hence, \( T \) is \( \nu^{i}_{B(0,1)} \) pointwise differentiable of order \( k \) at \( a \) with \( \text{pt} D^k T(a) = 0 \) at \( \mathcal{L}^n \) almost all \( a \in A_j \) by \ref{Fed69} 2.8.18, 2.9.11. In view of \ref{2.31}, \ref{2.32}, \ref{2.33}, the conclusion now follows. \( \square \)

\textbf{4.18 Remark.} The preceding theorem generalises \ref{Men13} A.3 where the case \( i = 1, k = 0, \) and \( Y \) a finite dimensional Euclidean space is treated.

\textbf{4.19 Remark.} The hypothesis in \( \ref{2} \) may not be omitted by \ref{2.33}.

\textbf{4.20 Remark.} Considering the distribution associated to any nonzero continuous function \( f : \mathbb{R} \to \mathbb{R}, \) the hypothesis in \ref{3} may not be omitted by \ref{2.31}.

\textbf{4.21 Corollary.} Suppose \( k \) is a positive integer, \( Y \) is a separable Banach space, \( R \in \mathcal{D}(\mathbb{R}^n, Y), \) and \( A \) is the set of all \( a \in \mathbb{R}^n \) such that \( R \) is pointwise differentiable of order \( k - 1, 1 \) at \( a \) and \( \text{pt} D^n R(a) = 0 \) for \( m = 0, \ldots, k - 1. \)

Then, there holds \( \text{pt} D^k R(a) = 0 \) for \( \mathcal{L}^n \) almost all \( a \in A. \)

\textbf{Proof.} We combine \ref{2.22} with \( k \) and \( \alpha \) replaced by \( k - 1 \) and \ref{4.17} \( \square \)

\textbf{4.22 Theorem.} Suppose \( i \) and \( k \) are nonnegative integer, \( Y \) is a separable Banach space, \( T \in \mathcal{D}(\mathbb{R}^n, Y), \) and \( A \) is the set of points at which the distribution \( T \) is \( \nu^{i}_{B(0,1)} \) pointwise differentiable of order \( k - 1, 1. \)

Then, \( \mathcal{L}^n \) almost all \( a \in A \) satisfy the following two statements:

\( 1 \) The distribution \( T \) is pointwise differentiable of order \( k \) at \( a. \)

\( 2 \) If \( Y^* \) is separable, then \( T \) is \( \nu^{i}_{B(0,1)} \) pointwise differentiable of order \( k \) at \( a. \)

\textbf{Proof.} In view of \ref{4.17} \ref{2} \ref{4}, we may assume \( k \geq 1. \) Then, we obtain sequences \( C_j \) and \( f_j \) from \ref{1.2} with \( k \) and \( \alpha \) replaced by \( k - 1 \) and \( 1. \) Associating \( S_j \) with \( f_j \) as in \ref{4.12} and defining \( R_j = T - S_j, \) the distributions \( R_j \) are \( \nu^{i}_{B(0,1)} \) pointwise differentiable of order \( k - 1, 1 \) at \( a \) with \( \text{pt} D^n R_j(a) = 0 \) whenever \( a \in A \cap C_j, m = 0, \ldots, k - 1, \) and \( j \) is a positive integer by \ref{4.9} \ref{1.2} and \ref{4.17} \ref{4.1}. The conclusion now follows from \ref{1.2} \ref{2} \ref{3} and \ref{4.17} \ref{2} \ref{4} \( \square \)
4.23 Corollary. Suppose \( k \) is a nonnegative integer, \( Y \) is a separable Banach space, \( T \in \mathcal{D}'(\mathbb{R}^n,Y) \), and \( A \) is the set of points at which \( T \) is pointwise differentiable of order \((k-1,1)\).

Then, \( T \) is pointwise differentiable of order \( k \) at \( \mathcal{L}^n \) almost all \( a \in A \).

Proof. We combine 4.22 and 4.22(1).

\[ \text{4.24 Remark.} \quad \text{The case } n=1, k=0, Y=\mathbb{C}, \text{ and } C \text{ linear } T \text{ was established in [Zie60, §2.1 Satz 2.1].} \]

\[ \text{4.25 Theorem. Suppose } k \text{ is a nonnegative integer, } Y \text{ is a Banach space, } Y^* \text{ is separable, } T \in \mathcal{D}'(\mathbb{R}^n,Y), \text{ and } A \text{ is the set of points at which } T \text{ is pointwise differentiable of order } k. \]

Then, for each \( \epsilon > 0 \), there exists \( g : \mathbb{R}^n \to Y^* \) of class \( k \) such that

\[ \mathcal{L}^n \left( A \sim \{ a : \text{pt } D^m T(a) = D^m g(a) \text{ for } m = 0, \ldots, k \} \right) < \epsilon. \]

Proof. By [DS58, II.3.16], \( Y \) is separable. In view of 4.28, we infer from 4.12 that \( \text{d}m^k T \) is a Borel set and \( \text{pt } D^m T \) is a Borel function with respect to the norm topology on \( \bigoplus^m \mathbb{R}^n \) for \( m = 0, \ldots, k \). If \( k = 0 \), we may accordingly combine [Fed69, 2.3.5, 3.1.14] to obtain the conclusion.

Suppose now \( k > 0 \). Then, we employ 4.9 with \( k \) and \( \alpha \) replaced by \( k-1 \) and 1 to construct \( f : \mathbb{R}^n \to Y^* \) of class \((k-1,1)\) such that

\[ \mathcal{L}^n \left( A \sim \{ a : \text{pt } D^m T(a) = \text{pt } D^m f(a) \text{ for } m = 0, \ldots, k-1 \} \right) < \epsilon. \]

Noting 4.3 with \( Y \) and \( f \) replaced by \( Y \otimes \bigoplus_{k-1} \mathbb{R}^n \) and \( D^{k-1} f \), we infer that \( D^{k-1} f \) is differentiable at \( \mathcal{L}^n \) almost all \( a \). Therefore, by 4.7 with \( k \) and \( Z \) replaced by \( k-1 \) and \( Y^* \), there exists \( g : \mathbb{R}^n \to Y^* \) of class \( k \) such that

\[ \mathcal{L}^n \left( A \sim \{ a : \text{pt } D^m T(a) = \text{pt } D^m g(a) \text{ for } m = 0, \ldots, k-1 \} \right) < \epsilon. \]

The conclusion now follows from 4.21 applied with \( R \in \mathcal{D}'(\mathbb{R}^n,Y) \) defined by

\[ R(\phi) = T(\phi) - \int \langle \phi, g \rangle \, d\mathcal{L}^n \text{ for } \phi \in \mathcal{D}(\mathbb{R}^n,Y). \]

\[ \text{4.26 Remark. One cannot replace } Y^* \text{ by } Y \text{ in the separability hypothesis by 4.14.} \]

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