On a Second Discretization of the ZS-AKNS Spectral Problem: Revisit

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Abstract

In this paper we revisit a discrete spectral problem which was proposed by Ragnisco and Tu in 1989, as a second discretization of the ZS-AKNS spectral problem. We show that the spectral problem corresponds to a bidirectional discretization of the derivative of two wave functions $\phi_{1,x}$ and $\phi_{2,x}$. As a connection with higher dimensional systems, the spectral problem and a related hierarchy can be derived from Lax triads of the differential-difference KP hierarchy via a symmetry constraint. Isospectral and nonisospectral flows derived from the spectral problem compose a Lie algebra. By considering its infinite dimensional subalgebras and continuum limit of recursion operator, three semi-discrete AKNS hierarchies are constructed.

Keywords: ZS-AKNS spectral problem, gauge transformation, symmetry constraint, differential-difference KP equation

MSC (2010): 35Q51, 37K60

1 Introduction

It is well known that the fundamental ZS-AKNS spectral problem\cite{1, 2}

$$\Phi_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \Phi, \quad \Phi = (\phi_1, \phi_2)^T$$

has a discretization given by Ablowitz and Ladik \cite{3, 4}:

$$\Phi_{n+1} = \begin{pmatrix} \lambda & Q_n \\ R_n & 1/\lambda \end{pmatrix} \Phi_n, \quad \Phi_n = (\phi_{1,n}, \phi_{2,n})^T,$$

which bears their names and is called the Ablowitz-Ladik (AL) spectral problem. Here $f_n$ stands for a function $f(n,t)$ defined on $\mathbb{Z} \times \mathbb{C}$. (1.2) leads to a semidiscrete AKNS (sdAKNS) hierarchy through suitably combining the AL flows, and one-field reductions yield the semidiscrete KdV, modified KdV and nonlinear Schrödinger hierarchies (cf.\cite{5–7}). In 1989 Ragnisco and Tu proposed a discrete spectral problem\cite{8}

$$\Theta_{n+1} = \begin{pmatrix} \lambda^2 + Q_n R_n \\ R_n \end{pmatrix} \Theta_n, \quad \Theta_n = (\theta_{1,n}, \theta_{2,n})^T,$$

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which was then studied in [9, 10]. (1.3) leads to a hierarchy of semidiscrete equations which recover the AKNS hierarchy in continuum limit but one-field reduction was not available [10].

In this paper, we revisit the discrete spectral problem (1.3). It will be shown that (1.3) is gauge equivalent to the form

\[
\begin{pmatrix}
\phi_{1,n+1} \\
\phi_{2,n-1}
\end{pmatrix} = \begin{pmatrix}
\lambda & Q_n \\
-R_n & \lambda
\end{pmatrix}
\begin{pmatrix}
\phi_{1,n} \\
\phi_{2,n}
\end{pmatrix}.
\]  

(1.4)

Compared with the AL spectral problem (1.2), the above one is obtained by discretizing the first order derivatives of wave functions \((\phi_1, \phi_2)^T\) in (1.1) in bidirection, i.e.

\[
\phi_{1,x} \sim \frac{\phi_{1,n+1} - \phi_{1,n}}{\epsilon}, \quad \phi_{2,x} \sim \frac{\phi_{2,n} - \phi_{1,n-1}}{\epsilon}.
\]  

(1.5)

Both (1.2) and (1.3) recover the AKNS spectral problem in continuum limit by defining

\[
\Phi(n+j) = \Phi(x+j\epsilon), \quad (Q_n, R_n) = \epsilon(q, r), \quad \lambda = e^{\epsilon\eta},
\]

and taking \(\epsilon \to 0\).

Besides bidirectional discretization of (1.1), the discrete spectral problem (1.3) or (1.4) is interesting in two more aspects. One is that (1.3) is related to a symmetry constraint of the differential-difference Kadomtsev-Petviashvili (KP) equation. This fact demonstrates a link between (1+1)-dimensional and (2+1)-dimensional semidiscrete integrable systems. The other is that, as a spectral problem, (1.3) is a Darboux transformation of the AKNS hierarchy. Note that a Darboux transformation can act as a discrete spectral problem to generate semidiscrete and fully discrete integrable systems. Let us give more details in the following.

It is well known that in continuous case the AKNS hierarchy can be viewed as a symmetry constraint of the Lax pairs of the KP hierarchy [11–14]. The differential-difference KP (D\(^2\)\(\Delta\)KP) equation\(^1\) reads [15]

\[
\Delta \left( \frac{\partial u}{\partial t_2} + 2 \frac{\partial u}{\partial x} - 2u \frac{\partial u}{\partial x} \right) = (2 + \Delta) \frac{\partial^2 u}{\partial x^2},
\]

(1.6)

which is related to the spectral problem [16]

\[
\mathcal{L}\varphi_n = \xi\varphi_n, \quad \mathcal{L} = \Delta + u_{0,n} + u_{1,n}\Delta^{-1} + u_{2,n}\Delta^{-2} + \cdots,
\]  

(1.7)

where \(\Delta = E - 1\), \(E f_n = f_{n+1}\), \(\mathcal{L}\) is called a pseudo-difference operator and in (1.6) \(u = u_{0,n}\). In this paper we will show that by a symmetry constraint the spectral problem (1.7) leads to a spectral problem which is gauge-equivalent to (1.3) and the Lax triads of the D\(^2\)\(\Delta\)KP hierarchy yields a sdAKNS hierarchy. This link will be explained in detail in Sec.3.

It is also well known that a Darboux transformation \(\tilde{\Phi} = D(u, \tilde{u}, \lambda)\Phi\) of a continuous spectral problem \(\Phi_x = M(u, \eta)\Phi\), where \(D(u, \tilde{u}, \lambda)\) is a Darboux matrix with parameter \(\lambda\) and \(\tilde{\Phi}\) and \(\tilde{u}\) stand for new eigenfunction and potential corresponding to \(\lambda\), can act as a discrete spectral problem (by considering \(\tilde{\Phi} = \Phi_{n+1}\) and \(\tilde{u} = u_{n+1}\))

\[
\Phi_{n+1} = D(u_n, u_{n+1}, \lambda)\Phi_n
\]  

(1.8)

\(^1\)D\(^2\)\(\Delta\) indicates 2 continuous and 1 discrete independent variables.
to generate semidiscrete integrable systems as a compatible condition with $\Phi_x = M(u, \eta)\Phi$ \cite{17, 18}. Moreover, Darboux transformations with different parameters, say

\[ \tilde{\Phi} = D(u, \tilde{u}, \lambda_1)\Phi, \quad \bar{\Phi} = D(u, \bar{u}, \lambda_2)\Phi \]

can be used as a Lax pair to generate fully discrete integrable systems. As examples one can refer to \cite{19–22}. In \cite{23} the spectral problem (1.3) was studied as a Darboux transformation of the ZS-AKNS spectral problem (1.1) (with $\lambda^2 = 2(\eta - \gamma)$). It is natural that the semidiscrete equations generated from a Darboux transformation (as a discrete spectral problem) are related via suitable continuum limit to the original continuous spectral problem.

In this paper, as new results we mainly achieve the following:

- find that the discrete spectral problem (1.3) is (gauge) equivalent to (1.4) which is a bidirectional discretization of the ZS-AKNS spectral problem (1.1);

- build connection between the spectral problem (1.3) and pseudo-difference operator spectral problem (1.7) via a symmetry constraint of the $D^2\Delta KP$ equation, as well as a connection between a sdAKNS hierarchy and the Lax triads of the $D^2\Delta KP$ hierarchy;

- obtain three sdAKNS hierarchies that are different from those derived from the AL spectral problem (1.2).

The paper is organized as follows. Sec.2 contains necessary notions and notations. In Sec.3 we show connections between (1.3), (1.4) and (1.7), and in Sec.4 we derive (1.3) and a sdAKNS hierarchy from the Lax triads of the $D^2\Delta KP$ hierarchy. In Sec.5 we discuss possible sdAKNS hierarchies related to (1.3). Sec.6 is for conclusions and discussions. There is one Appendix which, as a comparison, gives the sdAKNS hierarchies derived from the AL spectral problem.

2 Basic notions

Let us shortly describe some notions and notations that we will use in the paper. (We mainly follow \cite{24, 25}).

For functions $Q_n$ and $R_n$ defined on $\mathbb{Z}$ and vanishing rapidly as $n \to \pm\infty$, let $U_n = (Q_n, R_n)^T$. Consider a differential-difference evolution equation

\[ U_{n,t} = K(U_n), \quad U_n \in \mathcal{M}, \]

(2.1)

where by $\mathcal{M}$ we denote the infinite dimensional linear manifold of functions $U_n$. The solution $U_n = U(n, t)$ is usually asked to depend in a $C^\infty$-way on the time parameter $t$. Let $S$ be the fiber of the tangent bundle $T\mathcal{M}$ at any point $U_n \in \mathcal{M}$. In principle there is an identification between the linear spaces $\mathcal{M}$ and $S$, but it is convenient to regard them as different objects for a better geometrical understanding (i.e. $\mathcal{M}$ is the manifold under examination, $S$ is the tangent space at any point $U_n \in \mathcal{M}$). Let $S^*$ be the dual space of $S$ w.r.t. the bilinear form $\langle \cdot, \cdot \rangle: S^* \times S \to \mathbb{R}$ defined as

\[ \langle f_n, g_n \rangle = \sum_{n=-\infty}^{+\infty} f_n g_n, \quad f_n \in S^*, g_n \in S. \]

(2.2)
The Gâteaux derivative of a function (or an operator or a functional) $F(U_n)$ on $\mathcal{M}$ in the direction $g_n \in S$ is defined as

$$F'[g_n] = \left. \frac{\partial}{\partial \varepsilon} F(U_n + \varepsilon g_n) \right|_{\varepsilon=0}, \quad U_n \in \mathcal{M}, g_n \in S.$$  

The above definition is valid as well for the case $F = F(U_n, t)$, where $F$ depends explicitly on the time parameter $t$ and we treat $U_n$ and $t$ as independent variables (cf. [25]). For the sake of a more generic sense, in the following definitions are given for the time-dependent cases. They are valid as well when we remove the independent time variable $t$.

For two vector fields $F(U_n, t), G(U_n, t) : \mathcal{M} \times \mathbb{R} \rightarrow S$, their standard commutator is defined as

$$[F, G] = F'[G] - G'[F].$$  

Vector field $G(U_n, t) : \mathcal{M} \times \mathbb{R} \rightarrow S$ is called a symmetry of equation (2.1) if

$$\partial_t G(U_n, t) + [G(U_n, t), K(U_n)] = 0$$  

holds everywhere in $\mathcal{M} \times \mathbb{R}$.

A linear operator $L(U_n, t) : S \rightarrow S$ is called a strong symmetry operator of equation (2.1) if

$$\partial_t L + L'[K] = [K', L]$$  

holds everywhere on $\mathcal{M}$, where $[A, B] = AB - BA$. A linear operator $L(U_n, t) : S \rightarrow S$ is called to be hereditary (or a hereditary operator) if

$$L'[LF]G - L'[LG]F = L(L'[F]G - L'[G]F), \quad \forall F, G \in S.$$  

If $L$ is a hereditary operator, so is $L^{-1}$. If $L$ is a hereditary operator and is a strong symmetry of equation (2.1), then $L$ is also a strong symmetry of equation $U_{n,t} = L K(U_n)$.

### 3 Gauge equivalent forms of (1.3)

In this section we will list out some spectral problems which are gauge equivalent to the spectral problem (1.3).

In addition to (1.3) and (1.4), we list the following spectral problems

$$\Phi_{n+1} = \left( \lambda \frac{Q_n}{R_{n+1}} (1 + Q_n R_{n+1}) / \lambda \right) \Phi_n, \quad \Phi_n = (\phi_{1,n}, \phi_{2,n})^T,$$  

$$\Psi_{n+1} = \left( \lambda^2 \frac{1}{R_{n+1}^2} (1 + Q_n R_{n+1}) \right) \Psi_n, \quad \Psi_n = (\psi_{1,n}, \psi_{2,n})^T,$$  

$$\Sigma_{n+1} = \left( \frac{1}{R_{n+1}^2} \lambda^2 (1 + Q_n R_{n+1}) \lambda^2 \right) \Sigma_n, \quad \Sigma_n = (\sigma_{1,n}, \sigma_{2,n})^T,$$  

$$\Pi_{n+1} = \left( \lambda^2 \frac{\lambda Q_n}{R_{n+1}^2} (1 + Q_n R_{n+1}) / \lambda^2 \right) \Pi_n, \quad \Pi_n = (\pi_{1,n}, \pi_{2,n})^T,$$  

$$\begin{pmatrix} \psi_{1,n+1} \\ \psi_{2,n-1} \end{pmatrix} = \begin{pmatrix} \lambda^2 & Q_n \\ -R_n & 1 \end{pmatrix} \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix},$$  

(3.5)
\[ \mathcal{L} \varphi_n = \xi \varphi_n, \quad \mathcal{L} = \Delta - Q_n R_n - Q_n \Delta^{-1} R_n. \] \tag{3.6}

They are related to each other as in the following diagram.

\[ \text{(1.4)} \xrightarrow{\text{GT}_1} \text{(3.5)} \xrightarrow{\text{GT}_2} \text{(1.3)} \]
\[ \begin{array}{c}
\text{(3.1)} \xrightarrow{\text{GT}_1} \text{(3.2)} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{(3.3)} \quad \text{GT}_3 \quad \text{GT}_4 \\
\text{(3.4)} \end{array} \]

Fig.1 Relations of eight spectral problems

Here are the gauge transformations

\[ \text{GT}_1 : \quad \Phi_n = T_1 \Psi_n, \quad T_1 = \lambda^{-n} \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix}, \tag{3.7a} \]
\[ \text{GT}_2 : \quad \Psi_n = T_2 \Theta_n, \quad T_2 = \begin{pmatrix} 1 & 0 \\ R_n & 1 \end{pmatrix}, \tag{3.7b} \]
\[ \text{GT}_3 : \quad \Phi_n = T_3 \Sigma_n, \quad T_3 = T_1^{-1}, \tag{3.7c} \]
\[ \text{GT}_4 : \quad \Psi_n = T_4 \Pi_n, \quad T_4 = \begin{pmatrix} 1/\lambda & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.7d} \]

**Theorem 3.1.** The spectral problem (1.3) is (gauge) equivalent to (1.4) which is a bidirectional discretization of the ZS-AKNS spectral problem (1.1).

Note that after the early work [9, 10], the spectral problem (1.3) has been reinvestigated in different forms (for example, (3.1) in [26, 27], (3.2) in [28], (3.3) in [29], (3.4) in [30–32]), (3.6) in [33], etc.). However, since in these gauge transformations GT only eigenfunctions are involved (without any changes of potentials), the evolution equations derived from all the spectral problems listed in Fig.1 are the same (up to some combinations of flows). In fact, for two evolution equations which are derived respectively as compatibilities of linear problems

\[ \Phi_{n+1} = M_n(u_n, \lambda) \Phi_n, \quad \Phi_{n,t} = N_n \Phi_n, \tag{3.8} \]

and

\[ \Psi_{n+1} = U_n(u_n, \lambda) \Psi_n, \quad \Psi_{n,t} = V_n \Psi_n, \tag{3.9} \]

if they are gauge equivalent via transformation \( \Phi_n = T_n \Psi_n \), then there are relations

\[ M_n = T_{n+1} U_n T_n^{-1}, \quad N_n = T_{n,t} T_n^{-1} + T_n V_n T_n^{-1}, \]

and consequently their compatibilities are related by

\[ M_{n,t} - N_{n+1} M_n + M_n N_n = T_{n+1}(U_{n,t} - V_n U_n + U_n V_n) T_n^{-1}, \]

which means the two equations derived from (3.8) and (3.9) as compatibilities are same.

### 4 A symmetry constraint of the D\(^2\)\(\Delta\)KP equation

In this section we investigate in detail a symmetry constraint of the D\(^2\)\(\Delta\)KP equation (1.6), by which we reduce (1.7) to (3.6) and generate a sdAKNS hierarchy as well.
4.1 Spectral problem (3.6) as a symmetry constraint of (1.7)

Spectral problem (3.6) is connected with (1.7) through a symmetry constraint of the $D^2\Delta KP$ equation (1.6). To explain this, let us start from Lax triad of the $D^2\Delta KP$ equation [34]. Consider the pseudo-difference operator $\mathcal{L}$ defined in (1.7), i.e.,

$$\mathcal{L} = \Delta + u_{0,n} + u_{1,n}\Delta^{-1} + u_{2,n}\Delta^{-2} + \cdots ,$$

(4.1)

where $u_{i,n} = u_i(n,x,t)$ and $t = (t_1,t_2,\cdots)$. The difference operator $\Delta$ obeys the discrete Leibniz rule

$$\Delta^s g(n) = \sum_{i=0}^{\infty} C_s^i (\Delta^i g(n+s-i))\Delta^{s-i}, \quad s \in \mathbb{Z},$$

(4.2)

where

$$C_s^i = \frac{s(s-1)(s-2)\cdots(s-i+1)}{i!}, \quad C_0^0 = 1.$$  

(4.3)

The one-field $D^2\Delta KP$ hierarchy can be derived from the following Lax triad [34],

$$\mathcal{L}\varphi_n = \xi\varphi_n,$$

(4.4a)

$$\varphi_{n,x} = A_1\varphi_n, \quad A_1 = \Delta + u_{0,n},$$

(4.4b)

$$\varphi_{n,t_j} = A_j\varphi_n, \quad (j = 1,2,\cdots),$$

(4.4c)

where $A_j = (\mathcal{L}^j)_+$ denotes the difference part of $\mathcal{L}^j$, the first two of which are

$$A_1 = \Delta + u_{0,n},$$

(4.5a)

$$A_2 = \Delta^2 + ((\Delta u_{0,n}) + 2u_{0,n})\Delta + (\Delta u_{0,n}) + u_{0,n}^2 + (\Delta u_{1,n}) + 2u_{1,n}.$$  

(4.5b)

Compatibility of (4.4) reads

$$\mathcal{L}_x = [A_1, \mathcal{L}],$$

(4.6a)

$$\mathcal{L}_{t_j} = [A_j, \mathcal{L}], \quad (j = 1,2,\cdots),$$

(4.6b)

$$A_{1,t} - A_{j,x} + [A_1,A_j] = 0, \quad (j = 1,2,\cdots),$$

(4.6c)

where $[A,B] = AB - BA$. Among (4.6), the first equation (4.6a) provides expressions of $u_{j,n}$ in terms of $u_{0,n}$, which are

$$\Delta u_{1,n} = u_{0,n,x},$$

(4.7a)

$$\Delta u_{k+1,n} = u_{k,n,x} - \Delta u_{k,n} - u_{0,n}u_{k,n} + \sum_{j=0}^{k-1}(-1)^j C^j_{k-1} u_{k-j,n} \Delta^j u_{0,n-k}, \quad k \geq 1.$$  

(4.7b)

Besides, (4.6c), written as

$$u_{0,n,t_j} = K_j = A_{j,x} - [A_1,A_j], \quad (j = 1,2,\cdots),$$

(4.8)

provides zero curvature representations of a hierarchy of the one-field $D^2\Delta KP$ equation if substituting $u_{j,n}$ with $u_{0,n}$ by using (4.7). Particularly, when $j = 2$ one gets the $D^2\Delta KP$ equation (1.6).

As in continuous case, the $D^2\Delta KP$ equation (1.6) has a symmetry composed of eigenfunction $\varphi_n$ and its adjoint function $\bar{\varphi}_n$. 

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Lemma 4.1. \[16\] \((\varphi_n \tilde{\varphi}_n)_x\) is a symmetry of the \(D^2\Delta KP\) equation (1.6) provided

\[ \varphi_{n,x} = A_1 \varphi_n, \quad \tilde{\varphi}_{n,x} = -A_1^* \tilde{\varphi}_n, \quad (4.9) \]

and

\[ \varphi_{n,t_2} = A_2 \varphi_n, \quad \tilde{\varphi}_{n,t_2} = -A_2^* \tilde{\varphi}_n. \quad (4.10) \]

Here \(A_1^*\) stands for the formal adjoint operator of \(A_i\) w.r.t. the bilinear form (2.2). Explicit forms of them are

\[
A_1^* = -\Delta E^{-1} + u_{0,n}, \quad A_2^* = \Delta^2 E^{-2} - \Delta E^{-1}(\Delta u_{0,n}) + 2u_{0,n} + u_{0,n}^2 + u_{0,n,x} + 2(\Delta^{-1}u_{0,n,x}),
\]

where we have replaced \(u_{1,n}\) with \(\Delta^{-1}u_{0,n,x}\).

Note that \(u_{0,n,x}\) is also a symmetry of the \(D^2\Delta KP\) equation (1.6) (cf.[34]). Consider a symmetry \(\sigma = u_{0,n,x} + (\varphi_n \tilde{\varphi}_n)_x\). Taking \(\sigma = 0\) leads to a group invariant solution to (1.6). On the other hand, \(\sigma = 0\) provides a symmetry constraint \(u_{0,n} = -\varphi_n \tilde{\varphi}_n\). For convenience, we write \(\varphi_n = Q_n, \tilde{\varphi}_n = R_n\), and then we have

\[ u_{0,n} = -Q_n R_n \quad (4.11) \]

and (4.9) reads

\[ Q_{n,x} = Q_{n+1} - Q_n - Q_n^2 R_n, \quad R_{n,x} = R_n - R_{n-1} + Q_n R_n^2. \quad (4.12) \]

If replacing \(R_n\) with \(R_{n+1}\), (4.12) provides a Bäcklund transformation for the nonlinear Schrödinger equations (cf.[23]), and through suitable continuum limit it yields the nonlinear Schrödinger equations (cf.[35]). Next we investigate the change of \(L\) under constraint (4.11) and (4.12).

Lemma 4.2. If \(u_{0,n}\) is given by (4.11) where \(Q_n\) and \(R_n\) obey (4.12), then \(u_{k,n}\) defined in (4.7) can be expressed as

\[ u_{k+1,n} = (-1)^{k+1} Q_n \Delta^k R_{n-k-1}, \quad k = 0, 1, 2, \ldots. \quad (4.13) \]

Proof. From (4.7), (4.11) and (4.12) it is not difficult to find

\[ u_{1,n} = -Q_n R_{n-1}, \quad u_{2,n} = Q_n \Delta R_{n-2}, \quad u_{3,n} = -Q_n \Delta^2 R_{n-3}. \]

Now we suppose (4.13) is valid up to \(u_{k+1,n}\). Then for \(u_{k+2,n}\), from (4.7) we have

\[ \Delta u_{k+2,n} = u_{k+1,n,x} - \Delta u_{k+1,n} - u_{0,n} u_{k+1,n} + \sum_{j=0}^{k} (-1)^j C_k^j u_{k+1-j,n} \Delta^j u_{0,n-k-1}. \quad (4.14) \]

For the first three terms on the right hand side, substituting (4.13) into them and making use of (4.11) and (4.12), we find

\[ u_{k+1,n,x} - \Delta u_{k+1,n} - u_{0,n} u_{k+1,n} = \Delta((-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}) + (-1)^{k+1} Q_n \Delta^k (Q_{n-k-1} R_{n-k-1}). \]
Using formula (4.2), the last term on the right hand side of the above equation yields
\[
(-1)^{k+1} Q_n \Delta^k(Q_{n-k-1} R_{n-k-1}^2) \\
=(-1)^{k+2} Q_n \Delta^k(R_{n-k-1} u_{0,n-k-1}) \\
=(-1)^{k+2} \sum_{j=0}^{k} C_{k}^{j} Q_n (\Delta^{k-j} R_{n-k-j-1})(\Delta^{j} u_{0,n-k-1}) \\
= - \sum_{j=0}^{k} (-1)^{j} C_{k}^{j} u_{k+1-j,n} \Delta^{j} u_{0,n-k-1},
\]
which is just canceled by the last term on the right hand side of (4.14). Thus we reach
\[
\Delta u_{k+2,n} = \Delta((-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}),
\]
i.e.,
\[
u_{k+2,n} = (-1)^{k+2} Q_n \Delta^{k+1} R_{n-k-2}.
\]
Based on mathematical induction, we complete the proof. \qed

With Lemma 4.2 in hand and making use of formula (4.2) (for \( s = -1 \)), we immediately find
\[
(\mathcal{L})_{-} = \sum_{j=1}^{\infty} u_{j,n} \Delta^{-j} = -Q_n \Delta^{-1} R_n, \tag{4.15}
\]
As a result we reach the following theorem.

**Theorem 4.1.** Under symmetry constraint (4.11) where \( Q_n \) and \( R_n \) obey (4.12), the spectral problem (4.4a) is written as (3.6).

Note that in [36] (4.15) is called constrained discrete KP hierarchy, which is from here actually a result of the symmetry constraint (4.11) together with (4.12).

### 4.2 The sdAKNS hierarchy from symmetry constraint

There is a sdAKNS hierarchy coming from (4.4c) and its adjoint form under the constraint (4.11). This agrees with the continuum limit of the \( D^2 \Delta KP \) hierarchy and symmetry constraint of the continuous KP hierarchy (cf.[12, 14, 37])

In the following we prove

**Theorem 4.2.** For the pseudo-difference operator
\[
\mathcal{L} = \Delta - Q_n R_n - Q_n \Delta^{-1} R_n, \tag{4.16}
\]
define \( A_m = (\Sigma^m)_+ \) and \( A_m^* \) to be the adjoint operator of \( A_m \) w.r.t. the bilinear form (2.2). Then
\[
Q_{n,t_m} = A_m Q_n, \tag{4.17a}
\]
\[
R_{n,t_m} = -A_m^* R_n \tag{4.17b}
\]
provide a recursive relation
\[
\begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_{m+1}} = \mathcal{L}^{(+)} \begin{pmatrix} Q_n \\ R_n \end{pmatrix}_{t_{m}}, \tag{4.18}
\]

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where
\[ \mathcal{L}^{(+)} = \begin{pmatrix} \Delta - Q_n(E + 1)\Delta^{-1}R_n - Q_nR_n & -Q_n(E + 1)\Delta^{-1}Q_n \\ R_n(E + 1)\Delta^{-1}R_n & -\Delta E^{-1} + R_n(E + 1)\Delta^{-1}Q_n - Q_nR_n \end{pmatrix}. \]
(4.19) generates a sdAKNS hierarchy (see (5.20)).

To prove the theorem we first give two lemmas.

**Lemma 4.3.** Suppose that \( p_{-1,n}^{(m)} = \text{Res}_\Delta \mathcal{L}^m \), i.e. \( p_{-1,n}^{(m)} \) is the coefficient of \( \Delta^{-1} \) term in \( \mathcal{L}^m \). Then we have
\[ \Delta p_{-1,n}^{(m)} = -(Q_nR_n)t_m. \]
(4.20)

**Proof.** Comparing the constant terms of the left and right hand sides of the Lax equation
\[ \mathcal{L}_{t_m} = [A_m, \mathcal{L}] = -(\mathcal{L}^m)_-, \]
one immediately obtains (4.20). Here \((\mathcal{L}^m)_- = \mathcal{L}^m - A_m\). \( \square \)

**Lemma 4.4.** The following relations hold,
\[ (Q_n\Delta^{-1}R_n A_m)_+ = Q_n\Delta^{-1}R_n A_m - Q_n\Delta^{-1}(A^*_m R_n), \]
(4.21)
\[ (A_m Q_n\Delta^{-1}R_n)_+ = A_m Q_n\Delta^{-1}R_n - (A_m Q_n)\Delta^{-1}R_n. \]
(4.22)

**Proof.** We prove them one by one. For (4.21) we only need to prove
\[ (Q_n\Delta^{-1}R_n A_m)_- = Q_n\Delta^{-1}(A^*_m R_n). \]
(4.23)
In fact, supposing that \( A_m = \sum_{j=0}^{m} a_{j,n}\Delta^{m-j} \) and noting that \( R_n \to 0 \) as \( |n| \to \infty \), we have
\[
(Q_n\Delta^{-1}R_n A_m)_- = \left( Q_n\Delta^{-1}R_n \sum_{j=0}^{m} a_{j,n}\Delta^{m-j} \right)_-
= \left[ Q_n \sum_{j=0}^{m} \sum_{s=1}^{\infty} (-1)^{s-1}(\Delta^{s-1}E^{-s}R_n a_{j,n})\Delta^{m-j-s} \right]_-
= \left[ Q_n \sum_{l=1}^{\infty} \left( (-1)^{l-1}\Delta^{l-1}E^{-l} \sum_{j=0}^{m} (-1)^{m-j}(\Delta^{m-j}E^{-(m-j)}a_{j,n}R_n) \right) \Delta^{-l} \right]_-
= Q_n\Delta^{-1}\Delta \sum_{l=1}^{\infty} \left[ (-1)^{l-1}\Delta^{l-1}E^{-l}(A^*_m R_n) \right] \Delta^{-l}
= Q_n\Delta^{-1} \sum_{l=1}^{\infty} \left( [(\Delta^*)^{l-1}(A^*_m R_n)]\Delta^{-l+1} - [(\Delta^*)^l(A^*_m R_n)]\Delta^{-l} \right)
= Q_n\Delta^{-1}(A^*_m R_n),
\]
i.e. (4.21).

Next, we prove (4.22). To calculate \((A_m Q_n\Delta^{-1}R_n)_-\) we rewrite the operator \( A_m Q_n \) as a form of pure difference operator \( A_m Q_n = \sum_{j=0}^{m} b_{j,n}\Delta^{m-j} \) in which only the constant term \( b_{m,n} \) makes sense in \((A_m Q_n\Delta^{-1}R_n)_-\). Since \( b_{m,n} = (A_m Q_n) \) we immediately find
\[ (A_m Q_n\Delta^{-1}R_n)_- = (A_m Q_n)\Delta^{-1}R_n, \]
which leads to the relation (4.22). \( \square \)
Thus, \( A_{m+1} = \left[ (\Delta - Q_n R_n - Q_n \Delta^{-1} R_n) (A_m + p_{-1,n}^{(m)} \Delta^{-1}) \right]_+ \)
\[ = \Delta A_m - Q_n R_n A_m - [E \Delta^{-1} (Q_n R_{n,tm} + R_n Q_{n,tm})] - (Q_n \Delta^{-1} R_n A_m)_+ , \]
where we have made use of Lemma 4.3. Substituting (4.21) into the above we find
\[ A_{m+1} = \Delta A_m - Q_n R_n A_m - Q_n \Delta^{-1} R_n A_m - Q_n \Delta^{-1} R_{n,tm} - [E \Delta^{-1} (Q_n R_{n,tm} + R_n Q_{n,tm})] . \]
(4.24)

Note that the last term is a scalar.

Next, we calculate \( A_{m+1} \) in another way:
\[ A_{m+1} = \left[ (A_m + p_{-1,n}^{(m)} \Delta^{-1}) (\Delta - Q_n R_n - Q_n \Delta^{-1} R_n) \right]_+ \]
\[ = A_m \Delta - A_m Q_n R_n + p_{-1,n}^{(m)} - (A_m Q_n \Delta^{-1} R_n)_+ \]
\[ = A_m \Delta - A_m Q_n R_n - A_m Q_n \Delta^{-1} R_n + Q_{n,tm} \Delta^{-1} R_n - [\Delta^{-1} (Q_n R_{n,tm} + R_n Q_{n,tm})] , \]
where we have made use of Lemma 4.3 and (4.22). Its adjoint form reads
\[ A_{m+1}^* = \Delta^* A_m^* - Q_n R_n A_m^* + R_n E \Delta^{-1} Q_n A_m^* - R_n E \Delta^{-1} Q_{n,tm} - [\Delta^{-1} (Q_n R_{n,tm} + R_n Q_{n,tm})] . \]
(4.25)

Now, imposing (4.24) on \( Q_n \) and (4.25) on \( R_n \), respectively, and making use of (4.17), we arrive at the recursive relation (4.18). Thus we complete the proof.

Here we remark that it is possible to prove that \( (\varphi_n \bar{\varphi}_n)_x \) is a symmetry of the whole \( D^2 \Delta KP \) hierarchy. Following the idea in [39] on additional symmetry
\[ \mathfrak{L}_z = -[\varphi_n \Delta^{-1} \bar{\varphi}_n, \mathfrak{L}] , \]
(4.26)
where \( \mathfrak{L} \) is the pseudo-difference operator (4.1), \( \varphi_n \) and \( \bar{\varphi}_n \) respectively satisfy (4.4c) and its adjoint form \( \bar{\varphi}_{n,t} = -A_j^* \bar{\varphi}_n \), and we additionally request \( \varphi_n \) and \( \bar{\varphi}_n \) satisfy (4.9) as well due to the Lax triad (4.4), equation (4.26) yields for \( \Delta^0 \) term that \( u_{0,n,z}^{n+1} = \varphi_{n+1}^{n+1} - \varphi_{n}^{n} \bar{\varphi}_{n-1} \), which is \( u_{0,n,z}^{n} = (\varphi_n \bar{\varphi}_n)_x \) under (4.9). It has been proved that [40] \( [\partial_{t_j}, \partial_z] \mathfrak{L} = 0 \), which means \( (\varphi_n \bar{\varphi}_n)_x \) and the \( D^2 \Delta KP \) flows \( K_j \) defined in (4.8) commute, i.e. \( [K_j, (\varphi_n \bar{\varphi}_n)_x] = 0 \). Thus, \( (\varphi_n \bar{\varphi}_n)_x \) is a symmetry of the whole \( D^2 \Delta KP \) hierarchy (4.8). Such a result can also be proved as in [41] for continuous case from another approach. With this symmetry, the extended \( D^2 \Delta KP \) hierarchy [38] can be interpreted as a kind of symmetry constraints and then the sources provided by \( (\varphi_n \bar{\varphi}_n)_x \) are automatically self-consistent. We also remark that, as we can see, the additional condition (4.9), coming from the item in Lax triad for the independent variable \( x \), plays a significant role. Anyway, by means of symmetry constraint (4.11), one may study integrability properties of the sdAKNS hierarchy from a viewpoint of the \( D^2 \Delta KP \) hierarchy. This will be considered elsewhere.
5 The sdAKNS hierarchies related to (1.3)

In this section we will first list isospectral and nonisospectral flows together with their Lie algebra derived from (1.3) in [9]. As new results, from Lie algebraic structures of these flows we construct new symmetries of nonisospectral equations. In addition, by considering infinite dimensional subalgebras and continuum limit of recursion operator, we construct three types of isospectral and nonisospectral sdAKNS hierarchies.

5.1 Flows related to (1.3) and their Lie algebra

In Ref.[9], from spectral problem (1.3), the following isospectral hierarchy $U_{n,t_s} = K_s$ and nonisospectral hierarchy $U_{n,t_s} = \sigma_s$ were derived:

$$U_{n,t_s} = K_s = L^sK_0, \quad K_0 = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \quad s \in \mathbb{Z}, \quad (5.1)$$

$$U_{n,t_s} = \sigma_s = L^s\sigma_0, \quad \sigma_0 = \begin{pmatrix} (n+\frac{1}{2})Q_n \\ -(n-\frac{1}{2})R_n \end{pmatrix}, \quad s \in \mathbb{Z}, \quad (5.2)$$

where $U_n = (Q_n, R_n)^T$, $L$ is a recursion operator

$$L = \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} - \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}(E + 1)\Delta^{-1}(R_n, Q_n) - Q_nR_n, \quad (5.3)$$

and its inverse $L^{-1}$ is

$$L^{-1} = \begin{pmatrix} E^{-1}\mu_n^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} E^{-1}Q_n\mu_n^{-1} & 0 \\ 0 & -R_n\mu_n^{-1} \end{pmatrix}(E + 1)\Delta^{-1}(\mu_n^{-1}R_{n+1}, \mu_n^{-1}Q_nE) \quad (5.4)$$

with $\mu_n = 1 + Q_nR_{n+1}$. $L$ is a hereditary operator. The simplest ispospectral flows and nonisospectral flows are

$$K_{-1} = \begin{pmatrix} Q_n^{-1/2} & 0 \\ 0 & \mu_n^{-1/2} \end{pmatrix}, \quad K_0 = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \quad K_1 = \begin{pmatrix} Q_{n+1} - Q_n^2R_n \\ -R_{n+1} + R_n^2Q_n \end{pmatrix}, \quad (5.5a)$$

$$K_2 = \begin{pmatrix} Q_{n+2} - Q_n^2Q_{n+1} - Q_n^2R_{n+1} - 2Q_nQ_{n+1}R_n + Q_n^3R_n^2 \\ -R_n^2 + R_n^2Q_{n+1} + R_n^2Q_n + 2R_nR_{n+1}Q_n - R_n^3Q_n^2 \end{pmatrix}, \quad (5.5b)$$

and

$$\sigma_{-1} = \begin{pmatrix} (n-1/2)Q_n^{-1/2} & 0 \\ 0 & \mu_n^{-1/2} \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} (n+1/2)Q_n \\ -(n-1/2)R_n \end{pmatrix}, \quad (5.6a)$$

$$\sigma_1 = \begin{pmatrix} (n+3/2)Q_{n+1} - (n+3/2)Q_n^2R_n - 2Q_nQ_n^2R_n^{-1}Q_n^2 \\ -(n-3/2)R_{n+1} + (n+1/2)R_n^2Q_n + 2R_nR_{n+1}Q_n \end{pmatrix}, \quad (5.6b)$$

The flows $K_s$ and $\sigma_s$ defined in (5.1) and (5.2) constitute a centerless Virasoro algebra [9],

$$[K_m, K_s] = 0, \quad (5.7a)$$

$$[K_m, \sigma_s] = mK_{m+s}, \quad (5.7b)$$

$$[\sigma_m, \sigma_s] = (m-s)\sigma_{m+s}, \quad m, s \in \mathbb{Z}. \quad (5.7c)$$
5.2 New symmetries of nonisospectral equations (5.2)

Note that algebraic structure (5.7) is the same as the one generated by the AL flows (see [42]). This means the hierarchies (5.1) and (5.2) and the AL hierarchies can share those results obtained from the algebraic structure (5.7). One remarkable result is that the nonisospectral hierarchy (5.2) possesses symmetries. In fact, it is very rare for a nonisospectral equation to have infinitely many symmetries. However, making use of minus indices in the structure (5.7), infinitely many symmetries for the nonisospectral hierarchy (5.2) can be constructed.

Theorem 5.1. Any given member $U_{n,tm} = \sigma_m$ in the nonisospectral hierarchy (5.2) possesses two sets of symmetries,

$$\eta^{(m)}_s = \sum_{j=0}^{s} C^j_s (mt_m)^{s-j} \sigma_{m-jm}, \quad (s = 0, 1, 2 \cdots),$$

(5.8a)

$$\gamma^{(m)}_s = \sum_{j=0}^{s} C^j_s (mt_m)^{s-j} K_{-jm}, \quad (s = 0, 1, 2 \cdots),$$

(5.8b)

and these symmetries form a centerless Virasoro algebra with structure

$$[[\eta^{(m)}_l, \gamma^{(m)}_s]] = 0,$$

(5.9a)

$$[[\eta^{(m)}_l, \eta^{(m)}_s]] = -ml\eta^{(m)}_{l+s-1},$$

(5.9b)

$$[[\eta^{(m)}_l, \gamma^{(m)}_s]] = -m(l-s)\eta^{(m)}_{l+s-1}.$$  

(5.9c)

Here the suffix (m) corresponds to equation $U_{n,tm} = \sigma_m$.

We skip the proof, for which one can refer to Proposition 5.1 in [42].

For an isospectral equation $U_{n,tm} = K_m$ in the isospectral hierarchy (5.1), it also has two sets of symmetries,

$$\{K_s\} \text{ and } \{\tau^{(m)}_s = m t_m K_{m+s} + \sigma_s\}, \quad s \in \mathbb{Z}.$$ 

and they form a centerless Virasoro algebra as well, with structure

$$[K_l, K_s] = 0,$$

$$[K_l, \tau^{(m)}_s] = lK_{l+s},$$

$$[\tau^{(m)}_l, \tau^{(m)}_s] = (l-s)\tau^{(m)}_{l+s}. $$

5.3 Infinite dimensional subalgebras and new sdAKNS hierarchies

Equations (5.1) and (5.2) are not the sdAKNS hierarchies. It is interesting to consider infinite dimensional subalgebras of the algebra (5.7). These subalgebras, together with continuum limits of the recursion operator (5.3), can be used to construct and identify sdAKNS hierarchies.
5.3.1 Infinite dimensional subalgebras

Define

\[
\begin{align*}
\tilde{K}_s^{(+)} &= (L^{(+)})^s K_0, \quad \tilde{\sigma}_s^{(+)} = (L^{(+)})^s \sigma_0, \quad s = 0, 1, 2, \ldots, \\
\tilde{K}_s^{(-)} &= (L^{(-)})^s K_0, \quad \tilde{\sigma}_s^{(-)} = (L^{(-)})^s \sigma_0, \quad s = 0, 1, 2, \ldots,
\end{align*}
\]

and

\[
\begin{align*}
\tilde{K}_{2s+1}^{(+)} &= L^s (K_1 - K_{-1})/2, \quad \tilde{K}_{2s}^{(+)0} = L^s K_0, \quad s = 0, 1, 2, \ldots, \\
\tilde{\sigma}_{2s+1}^{(+)} &= L^s (\sigma_1 - \sigma_{-1})/2, \quad \tilde{\sigma}_{2s}^{(+)0} = L^s \sigma_0, \quad s = 0, 1, 2, \ldots,
\end{align*}
\]

where

\[
\begin{align*}
L &= L - 2I + L^{-1}, \quad L^{(+)} = L - I, \quad L^{(-)} = I - L^{-1}
\end{align*}
\]

and \(I\) is the \(2 \times 2\) unit matrix.

**Lemma 5.1.** The flows

\[
\begin{align*}
(I) : \{\tilde{K}_s^{(+)}, \tilde{\sigma}_s^{(+)}\}, \quad (II) : \{\tilde{K}_s^{(-)}, \tilde{\sigma}_s^{(-)}\}, \quad (III) : \{\tilde{K}_{2m+j}, \tilde{\sigma}_{2s+k}\},
\end{align*}
\]

generate three infinite dimensional subalgebras, respectively, with structures

\[
\begin{align*}
(I) : & [\tilde{K}_s^{(+)}, \tilde{K}_l^{(+)}] = 0, \\
& [\tilde{K}_s^{(+)}, \tilde{\sigma}_l^{(+)}] = s(\tilde{K}_{s+l}^{(+)} + \tilde{K}_{s+l-1}^{(+)}), \\
& [\tilde{\sigma}_s^{(+)}, \tilde{\sigma}_l^{(+)}] = (s - l)(\tilde{\sigma}_{s+l}^{(+)} + \tilde{\sigma}_{s+l-1}^{(+)}),
\end{align*}
\]

\[
\begin{align*}
(II) : & [\tilde{K}_s^{(-)}, \tilde{K}_l^{(-)}] = 0, \\
& [\tilde{K}_s^{(-)}, \tilde{\sigma}_l^{(-)}] = -s(\tilde{K}_{s+l}^{(-)} - \tilde{K}_{s+l-1}^{(-)}), \\
& [\tilde{\sigma}_s^{(-)}, \tilde{\sigma}_l^{(-)}] = -(s - l)(\tilde{\sigma}_{s+l}^{(-)} - \tilde{\sigma}_{s+l-1}^{(-)}),
\end{align*}
\]

\[
\begin{align*}
(III) : & [\tilde{K}_{2m+j}, \tilde{K}_{2s+k}] = 0, \\
& [\tilde{K}_{2m}, \tilde{\sigma}_{2s}] = 2m \tilde{K}_{2(m+s)-1}, \\
& [\tilde{K}_{2m}, \tilde{\sigma}_{2s+1}] = 2m \tilde{K}_{2(m+s)} + \frac{m}{2} \tilde{K}_{2(m+s+1)}, \\
& [\tilde{K}_{2m+1}, \tilde{\sigma}_{2s+k}] = (2m + 1) \tilde{K}_{2(m+s)+k} + \frac{m+1}{2} \tilde{K}_{2(m+s+1)+k}, \\
& [\tilde{\sigma}_{2m}, \tilde{\sigma}_{2s}] = 2(m - s) \tilde{\sigma}_{2(m+s)-1}, \\
& [\tilde{\sigma}_{2m+j}, \tilde{\sigma}_{2s+k}] = [2(m - s) - 1 + j] \tilde{\sigma}_{2(m+s)+j} + \frac{m - s - 1 + j}{2} \tilde{\sigma}_{2(m+s+1)+j},
\end{align*}
\]

where \(j, k \in \{0, 1\}\), \(m, s \geq 0\) and we define \(\tilde{K}_s^{(\pm)} = \tilde{\sigma}_s^{(\pm)} = \tilde{K}_{-1} = \tilde{\sigma}_{-1} = 0\). Obviously, the set (III) has a subalgebra \(\{\tilde{K}_{2m+1}, \tilde{\sigma}_{2s+1}\}\).

This lemma can be verified directly. The structure of set (III) has been proved in [6]. Note that operator \(L - L^{-1}\) does not generate a subalgebra of (5.7).
5.3.2 Three sdAKNS hierarchies

Let us construct the sdAKNS hierarchies through considering possible combinations of the flows \( \{K_s\} \) and \( \{\sigma_s\} \) defined in (5.1) and (5.2). The criteria is that these combined flows should be closed as a subalgebra of (5.7) and they must yield their counterparts in the continuous AKNS flows in reasonable continuum limits. For this purpose we investigate continuum limits of initial flows, \( L \) and \( L^{-1} \) under a uniform scheme\(^2\)

\[
U_{n+j} = \epsilon(q(x + je, t), r(x + je, t))^T, \quad n\epsilon = x, (n \to \infty, \epsilon \to 0). \quad (5.15)
\]

We find

\[
K_0 = \epsilon(q, -r)^T, \quad (K_1 - K_{-1})/2 = \epsilon^2(q, r)^T_x + O(\epsilon^3),
\]

\[
\sigma_0 = x(q, -r)^T + O(\epsilon), \quad (\sigma_1 - \sigma_{-1})/2 = \epsilon(qx_x + q, xr_x + r)^T + O(\epsilon^2),
\]

and

\[
L = I + \epsilon L_{AKNS} + \frac{\epsilon^2}{2} L_{NLS} + O(\epsilon^3),
\]

\[
L^{-1} = I - \epsilon L_{AKNS} + \frac{\epsilon^2}{2} (2L_{AKNS} - L_{NLS}) + O(\epsilon^3),
\]

where

\[
L_{AKNS} = \begin{pmatrix}
\partial_x - 2q\partial_{x}^{-1}r & -2q\partial_{x}^{-1}q \\
2r\partial_{x}^{-1}x & -\partial_x + 2r\partial_{x}^{-1}x
\end{pmatrix}
\]

is the recursion operator of the continuous AKNS hierarchy, and

\[
L_{NLS} = \begin{pmatrix}
\partial_x^2 & 0 \\
0 & \partial_x^2
\end{pmatrix} - 2qr,
\]

which yields the nonlinear Schrödinger system by acting on \((q, -r)^T\). (5.17) indicates

\[
\mathcal{L}^{(+)} = L - I = \epsilon L_{AKNS} + O(\epsilon^2),
\]

\[
\mathcal{L}^{(-)} = I - L^{-1} = \epsilon L_{AKNS} + O(\epsilon^2),
\]

\[
\mathcal{L} = L - 2I + L^{-1} = \epsilon^2 L_{AKNS} + O(\epsilon^3).
\]

Thus, based on the continuum limits of initial flows in (5.16), the definition of the flows (5.10,5.11,5.12) and Lemma 5.1, we obtain three sets of sdAKNS hierarchies.

**Theorem 5.2.** The flows defined in (5.10,5.11,5.12) yield three sets of sdAKNS hierarchies

(I): \( U_{t_s} = \bar{K}_s^{(+)} \), \( U_{t_s} = \bar{\sigma}_s^{(+)} \), \quad (5.20)

(II): \( U_{t_s} = \bar{K}_s^{(-)} \), \( U_{t_s} = \bar{\sigma}_s^{(-)} \), \quad (5.21)

(III): \( U_{t_s} = \bar{K}_s \), \( U_{t_s} = \bar{\sigma}_s \), \quad (5.22)

where \( s = 0,1,\cdots \). They all correspond to the continuous isospectral and nonisospectral AKNS hierarchies under the continuum limit (5.15)\(^3\)

Here we remark that \( \{U_{t_s} = \bar{K}_s^{(+)}\} \) was already found [10], which is just (4.18), the result of symmetry constraint of the D^2\DeltaKP hierarchy. Besides, equation \( U_{t_2} = \bar{K}_2 \) was also mentioned in [10] as a discretization of the 2nd order AKNS equations.

\(^2\)The correspondence between \( x \) and \( n \) is \( x = x_0 + n\epsilon \) where \( \epsilon \) is viewed as a spacing parameter. Here we take \( x_0 = 0 \) for convenience.

\(^3\)In principle we need to suitably rescale \( t_s \) by \( \epsilon^s t_s \). For example, for \( U_{t_s} = \bar{K}_s \), rescale \( t_s \) by \( \epsilon^{-s} t_s \).
6 Conclusions

The spectral problem (1.3) can generate a sdAKNS hierarchy. It is also a Darboux transformation of the ZS-AKNS spectral problem (1.1). By revisiting it, we have shown that it is gauge equivalent to (1.4) which provides a bidirectional discretization of (1.1), while the AL spectral problem (1.2) comes from a monodirectional discretization of (1.1). As a relation with higher dimensional systems, we proved that (1.3) and a related sdAKNS hierarchy can be obtained from the Lax triads of the D²∆KP hierarchy via the symmetry constraint (4.11). This fact, on one side, coincides with the continuous case [12, 14]. On the other side, it exhibits a new aspect of discrete systems: there are two discrete spectral problems, (1.2) and (1.3) which can generate sdAKNS hierarchy, but only (1.3) that is a bidirectional discretisation of the ZS-AKNS spectral problem is related to the symmetry constraint of the D²∆KP hierarchy. In addition to the above results, three sdAKNS hierarchies (5.20), (5.21) and (5.22) are obtained with a criteria that the corresponding flows are closed w.r.t. Lie product (2.4) and in continuum limit they approach to the continuous AKNS hierarchy. Among these sdAKNS hierarchies, \{U_{n,t_j} = \tilde{K}_j^{(+)}\} is the one derived from Lax triad of the D²∆KP hierarchy via the symmetry constraint.

With regard to the symmetry-constrained spectral problem (3.6), in [43] a spectral problem
\[ \hat{\mathcal{L}} \phi_n = \xi \phi_n, \quad \hat{\mathcal{L}} = \Delta + Q_n \Delta^{-1} R_n \] (6.1)
was given as a constrain of \[ \hat{\mathcal{L}} \phi_n = \xi \phi_n, \quad \hat{\mathcal{L}} = \Delta + u_{1,n} \Delta^{-1} + u_{2,n} \Delta^{-1} + u_{2,n} \Delta^{-1} + \cdots \], and the related equations were investigated (e.g. [44]). However, our results are different from them.

There are several interesting problems that could be followed. Apart from continuous correspondence of integrability properties of the new sdAKNS hierarchies there would be many interactions between the (1+1) and (2+1)-dimensional systems based on symmetry constraints, such as solutions and integrability characteristics (e.g. [12, 13, 39, 45] in continuous case). Some known results related to the pseudo-difference operator (4.1) (e.g. [46, 47]) could also be used to investigate the sdAKNS hierarchies.

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A The sdAKNS hierarchies from the AL spectral problem

From the AL spectral problem (1.2) one can derive the AL hierarchy (cf. [7]):
\[ U_{ts} = K_s = \tilde{L}^s K_0, \quad s \in \mathbb{Z}, \] (A.1)
where \( K_0 \) and the first few flows are
\[ K_0 = \begin{pmatrix} Q_n \\ R_n \end{pmatrix}, \quad K_1 = \tilde{\mu}_n \begin{pmatrix} Q_{n+1} \\ -R_{n-1} \end{pmatrix}, \quad K_{-1} = \tilde{\mu}_n \begin{pmatrix} Q_{n-1} \\ -R_{n+1} \end{pmatrix}, \]
the recursion operator reads
\[
\overline{L} = \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} + \left( \begin{pmatrix} -Q_n E \\ R_n \end{pmatrix} \right) \Delta^{-1}(R_n, Q_n) \frac{1}{\mu_n},
\]
with its inverse
\[
\overline{L}^{-1} = \begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix} + \left( \begin{pmatrix} Q_n \\ -R_n E \end{pmatrix} \right) \Delta^{-1}(R_n^{-1}, Q_n E) \overline{L} - 1
\]
\[
+ \overline{\mu}_n \left( \begin{pmatrix} Q_n^{-1} \\ -ER_n \end{pmatrix} \right) \Delta^{-1}(R_n, Q_n) \frac{1}{\mu_n},
\]
and here \(\overline{\mu}_n = 1 - Q_n R_n\). Under the continuum limit scheme (5.15), one can find
\[
\overline{L} = I + \epsilon L_{AKNS} + \frac{\epsilon^2}{2} L_{AKNS}^2 + O(\epsilon^3),
\]
where \(L_{AKNS}\) is given in (5.18).

There are also three sdAKNS hierarchies related to the AL spectral problem:
\[
I : \{ U_{n,t_s} = \bar{K}_s^{(+)} \}, \quad II : \{ U_{n,t_s} = \bar{K}_s^{(-)} \}, \quad III : \{ U_{n,t_s} = \bar{K}_s \},
\]
where
\[
\bar{K}_s^{(+)} = (\bar{L}^{(+)})^s K_0, 
\bar{K}_s^{(-)} = (\bar{L}^{(-)})^s K_0, 
\bar{K}_{2s+1} = \bar{L}^s(K_1-K_{-1})/2, \quad \bar{K}_{2s} = \bar{L}^s K_0, \quad s = 0,1,2,\ldots,
\]
\[
\bar{L} = \bar{L} - 2I + \bar{L}^{-1}, \quad \bar{L}^{(+)} = \bar{L} - I, \quad \bar{L}^{(-)} = I - \bar{L}^{-1}.
\]
The third sdAKNS hierarchy has been well studied and it admits one-field reduction to get sdKdV, sdmKdV and sdNLS hierarchies. As a review one can refer to [7].

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