REFINED ALGEBRAIC QUANTIZATION OF CONSTRAINED SYSTEMS WITH STRUCTURE FUNCTIONS

O. Yu. Shvedov

Sub-Dept. of Quantum Statistics and Field Theory, Department of Physics, Moscow State University
Vorobievy gory, Moscow 119899, Russia

Abstract
The method of refined algebraic quantization of constrained systems which is based on modification of the inner product of the theory rather than on imposing constraints on the physical states is generalized to the case of constrained systems with structure functions and open gauge algebras. A new prescription for inner product for the open-algebra systems is suggested. It is illustrated on a simple example. The correspondence between refined algebraic and BRST-BFV quantizations is investigated for the case of nontrivial structure functions.

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1shvedov@qs.phys.msu.su
1 Introduction

Theory of constrained systems is a basis of modern physics: gauge field theories, quantum gravity and supergravity, string and superstring models are examples of systems with constraints. For such theories, one should specify not only an evolution equation but additional requirements (constraints) imposing on initial conditions. For some cases, Hamiltonian is zero, so that all the dynamics is involved to constraints, and the well-known problem of time in reparametrization-invariant theories arises.

In quantum mechanics of constraint systems evolution (if exists) is taken into account in a standard way: evolution transformation is presented as $\exp(-iHt)$ for some Hamiltonian $H$. However, constraints can be taken into account in different ways.

First, one can use the original Dirac idea \[1\] and consider the states to obey the following additional conditions:

$$\hat{\Lambda}^+_a \Psi = 0, \quad a = 1, M, \tag{1}$$

where $\hat{\Lambda}^+_a$ are quantum analogs of constraints. Requirements (1) do not contradict each other, provided that the constraints commute on the constraint surface. For the quantum case, this means that

$$[\hat{\Lambda}^+_a, \hat{\Lambda}^+_b] = i(U^c_{ab})^+ \hat{\Lambda}^+_c \tag{2}$$

for some operators $U^c_{ab}$ called usually as structure functions \[2\]. Relation (1) should also conserve under time evolution, so that constraints should commute with the Hamiltonian for the states obeying eq.(1):

$$[H^+, \hat{\Lambda}^+_a] = i(R^c_a)^+ \hat{\Lambda}^+_c \tag{3}$$

for some operators $R^c_a$.

The most difficult problem of the Dirac approach is construction of the inner product, since $\Psi(q)$ are distributions rather than square-integrable functions. One usually imposes additional gauge conditions \[1, 2, 3\] in such a way that each gauge orbit should be taken into account once. Unfortunately, this approach is gauge-dependent, especially for the case of the Gribov copies problem \[4, 5\].

Therefore, other ways of quantization of constrained systems were invented. The BRST-BFV approach \[6\] is based on extension of the phase space by introducing Lagrange multipliers $\lambda^a$ and momenta $\pi_a$, ghosts and antighosts $C^a$, $\overline{C}^a$, momenta $\Pi_a$, $\Pi^a$. A manifestly covariant operator formulation of nonabelian gauge theories was obtained in this approach \[7\].

An alternative way to develop the quantum theory is to use the refined algebraic quantization technique \[8\] and modify the inner product instead of imposing requirements (1). States are specified by arbitrary functions $\Phi(q)$ called auxiliary state vectors, but their inner product is given by the nontrivial formula

$$\langle \Phi, \eta \Phi \rangle. \tag{4}$$

For the abelian case, $\eta \sim \Pi_a \delta(\hat{\Lambda}_a)$. For the general case, the main requirement for the operator $\eta$ ($\eta = \eta^+$) is the following

$$\eta \hat{\Lambda}_a = 0. \tag{5}$$

The positivity condition is also imposed. Two states are called equivalent if their difference $\Delta \Phi$ has zero norm. This is certainly the case if

$$\Delta \Phi = \hat{\Lambda}_a Y^a$$

because of (3). The classes of equivalence are identified with the Dirac states with the help of formula

$$\Psi = \eta \Phi. \tag{6}$$

The constrained conditions (1) are automatically satisfied then. The inner product for physical states is introduced.
Analogous ideas were used in the projection operator approach \[9, 5\].

Quantum observables \( H \) can be introduced in this approach as operators in the space of auxiliary states \( \Phi \). The unitarity property is written as follows

\[
(e^{-iHt})^\dagger \eta e^{-iHt} = \eta. 
\]

This means that

\[
H^\dagger \eta = \eta H. \quad (7)
\]

Let \( \Phi \) be an auxiliary state corresponding to the Dirac state \( \Psi = \eta \Phi \). The observable \( H \) takes it to \( H \Phi \). This corresponds to the Dirac state \( \eta H \Phi = H^\dagger \eta \Phi = H^\dagger \Psi \). Therefore, it is the operator \( H^\dagger \) that corresponds to the observable \( H \) in the Dirac approach, while \( \exp(-iH^\dagger t) \) is an evolution operator.

Thus, an observable in the refined algebraic quantization approach is an operator satisfying properties (3), (7).

An explicit form of the operator \( \eta \) has been obtained only for the case of closed algebra of constraints, where \( U_{ab}^c = \text{const} \). For this case, \( \eta \) is expressed via the integral over gauge group of the representation of the constraint group [10]. It has been stressed in [11] that generalization of this formula to systems with structure functions \( U_{ab}^c \neq \text{const} \) is an interesting open problem. The purpose of this paper is to write down the corresponding prescription for the inner product.

### 2 A proposal for the inner product bilinear form

It is convenient to use another well-developed quantization technique, the BRST-BFV approach [6, 2]. Additional operators \( \lambda^a, \pi_a, C^a, \overline{C}_a, \Pi^a, \overline{\Pi}_a \) obeying the following nontrivial commutation relations

\[
[\lambda^a, \pi_b] = i\delta^a_b, \quad [C^a, \overline{\Pi}_b]_+ = \delta^a_b, \quad [\overline{C}_a, \Pi^b]_+ = \delta^b_a
\]

are introduced. Operators \( \overline{C}_a \) and \( \Pi^b \) are anti-Hermitian. The main object is the B-charge which is a nilpotent Hermitian operator looked for in the following form

\[
\Omega = -i\pi_a \Pi^a + C^a \overline{\Lambda}_a + ... + \Omega^{nb_1...b_{n-1}}_{a_1...a_n} \Pi_{b_1}...\Pi_{b_{n-1}} C^{a_1}...C^{a_n} + ...
\]

(8)

The operators \( \Pi \) and \( C \) are ordered in formula (8) in such a way that ghosts \( C \) are put to the right, while the momenta \( \Pi \) are put to the left. The property \( \Omega^+ = \Omega \) means that in general

\[
\hat{\Lambda}^+ \neq \hat{\Lambda}. \quad (9)
\]

The operator-valued coefficient functions \( \Omega^{nb_1...b_{n-1}}_{a_1...a_n} \) being antisymmetric separately with respect to \( b_1, ..., b_{n-1} \) and separately with respect to \( a_1, ..., a_n \) are constructed in a standard way [2] from recursive relations that are corollaries of the property \( \Omega^2 = 0 \).

Physical states in the BRST-BFV approach satisfy the requirement

\[
\Omega \Upsilon = 0, \quad (10)
\]

while the transformation

\[
\Upsilon \rightarrow \Upsilon + \Omega \Upsilon \quad (11)
\]

is gauge: states \( \Omega \Upsilon \) are equivalent to zero.

To specify an inner product for BRST-BFV quantization is also a difficult problem (see, for example, [12, 13]). The most general approach was suggested in [13]. One considers such physical states that

\[
\pi_a \Upsilon = 0, \quad C^a \Upsilon = 0, \quad \overline{C}_a \Upsilon = 0. \quad (12)
\]
Such states are automatically $B$-invariant: condition (10) is satisfied. Since the inner product $(\Upsilon, \Upsilon)$ contains the factor like $\infty \cdot 0$, one uses another prescription

$$(\Upsilon, e^{i[\Omega, \rho]} \Upsilon)$$

being formally equivalent to $(\Upsilon, \Upsilon)$ but without divergences. It is convenient to choose the gauge fermion $\rho$ as

$$\rho = -\lambda^a \Pi_a$$

so that

$$[\Omega, \rho]_+ = -\Pi_a \Pi^a - \lambda^a \hat{\Omega}_a$$

with

$$\hat{\Omega}_a = \Omega_a(\Pi, C) = [\Pi_a, \Omega]_+ = \hat{\Lambda}_a + ... + n\Omega^a_{b_1...b_{n-1}} \Pi_{b_1}...\Pi_{b_{n-1}} C^{a_1}...C^{a_{n-1}} + ...$$

It is convenient to introduce the functional Schrödinger representation. $B$-states $\Upsilon$ are specified as functions $\Upsilon(q, \lambda, \Pi, \bar{\Pi})$. The inner product reads

$$(\Upsilon_1, \Upsilon_2) = \int dq \prod_{a=1}^M d\mu_a d\Pi_a d\Pi^a \Upsilon_1(q, i\mu, \Pi, \bar{\Pi})^* \Upsilon_2(q, -i\mu, \Pi, \bar{\Pi})$$

where $(\Pi^a)^* = -\Pi^a$, $\Pi^a_a = \Pi_a$. The ghost momenta and bosonic coordinates are presented as multipliers $\Pi^a$ and $\bar{\Pi}_a$, while

$$C^a = \frac{\partial}{\partial \Pi_a}, \quad \bar{C}_a = \frac{\partial}{\partial \bar{\Pi}^a}, \quad \pi_a = -i \frac{\partial}{\partial \lambda^a}, \quad p_i = -i \frac{\partial}{\partial q^i}$$

(where the left derivatives are considered). Conditions (12) mean

$$\Upsilon = \Phi(q).$$

It has been argued in [15] that $\Phi(q)$ should be identified with the auxiliary state in the refined algebraic quantization approach. Formulas (13) and (14) give us the following inner product,

$$\int dq \Phi^*(q) \prod_{a=1}^M d\mu_a d\Pi_a d\Pi^a e^{-\bar{\Pi}_a \Pi^a + i\mu_a \hat{\Omega}_a} \Phi(q)$$

with $\hat{\Omega}_a = \Omega_a(\Pi, \partial/\partial \Pi)$. Formula (14) is of the type (3) with

$$\eta = \int \prod_{a=1}^M d\mu_a d\Pi_a d\Pi^a e^{-\bar{\Pi}_a \Pi^a + i\mu_a \hat{\Omega}_a(\Pi, \partial/\partial \Pi)}.$$
3 Properties of the inner product

Let us investigate properties of the operator $\eta$ (17). First of all, check that $\eta^+ = \eta$, so that formula (15) gives us real values. One has

$$(\phi, \eta \phi) = \int \prod_{a=1}^{M} d\mu_a (\phi, \exp[\pi^a \Pi_a + i\mu_a \hat{\Omega}_a] \phi)$$

and

$$(\phi, \eta \phi)^* = \int \prod_{a=1}^{M} d\mu_a (\phi, \exp[\pi^a \Pi_a - i\mu_a \hat{\Omega}_a^+] \phi)$$

After change of variables $\mu_a \to -\mu_a$ and using the property $\hat{\Omega}_a^+ = \hat{\Omega}_a$ being a corollary of the relations $\Pi_a^+ = \Pi_a$ and $\Omega^+ = \Omega$, we find $\eta^+ = \eta$.

Let us check relation (5). One has

$$\eta^* \hat{\Lambda}_b Y^b(q) = \int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi_a \exp[\pi^a \Pi_a + i\mu_a \hat{\Omega}_a] \Omega \Pi_b Y^b(q).$$

(18)

Since $\Omega^2 = \Omega$, the operators $\Omega$ and $[\Omega, \rho]_+$ commute:

$$\Omega[\Omega, \rho]_+ = \Omega \rho \Omega = [\Omega, \rho]_+ \Omega,$$

so that

$$e^{\pi^a \Pi_a + i\mu_a \hat{\Omega}_a} \Omega = \Omega e^{\pi^a \Pi_a + i\mu_a \hat{\Omega}_a}.$$

Formula (18) transforms then to

$$\eta^* \hat{\Lambda}_b Y^b(q) = \int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi_a \exp[\pi^a \Pi_a + i\mu_a \hat{\Omega}_a] \Pi_b Y^b(q)$$

(20)

since $\Omega = \Omega^+$. The operator $\Omega^+$ can be presented in representation (15) as

$$\Omega^+ = \frac{1}{i} \frac{\partial}{\partial \mu_a} \Pi_a^+ + \frac{\partial}{\partial \Pi_a} \hat{\Lambda}_a^+ + ... + \Omega_{a_1...a_n}^{b_1...b_{n-1}} \frac{\partial}{\partial \Pi_{a_1}} \frac{\partial}{\partial \Pi_{a_2}} ... \frac{\partial}{\partial \Pi_{a_n}} \Pi_{b_{n-1}} ... \Pi_{b_1} + ...$$

(21)

Integral (20) then vanishes as an integral of full derivative. Formula (3) is checked. Thus, formula (17) obeys the desired properties of the operator $\eta$ entering to the inner product. However, the problem of positive definiteness of the inner product remains to be investigated.

4 Correspondence between BRST-BFV, Dirac and refined algebraic quantization approaches

Let us show that correspondence between BFV, auxiliary and Dirac states found in [15] for the Lie-algebra case remains valid for the case of nontrivial structure functions.

Let $\Upsilon$ be an arbitrary BFV state obeying eq.(10) but not satisfying in general eq.(12). For this BFV state, consider the function

$$\Phi(q) = \Upsilon(q, 0, 0, 0)$$

(22)

It occurs to play a role of an auxiliary wave function in the refined algebraic quantization approach. If the conditions (12) are valid, relation (22) is obvious. It was advocated in [13] that any physical state can be taken to the gauge (12) by transformation (11). However, the auxiliary state

$$(\Omega \Upsilon)(q, 0, 0, 0) = \hat{\Lambda}_a \frac{\partial}{\partial \Pi_a} |_{\Pi=0,\lambda=0} Y$$
is equivalent to zero (has zero norm) because of (5). Therefore, for any \( \Upsilon \) definition (22) is valid since equivalent auxiliary states are identified.

Eq.(3) for the Dirac wave function can be rewritten as

\[
\Psi(q) = \int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a (e^{[\Omega, \rho]} \Phi)(q, -i\mu, \Pi, \Pi).
\]

Let \( \Upsilon \) be an arbitrary BFV state that is equivalent to \( \Phi \) (and, therefore, to \( e^{[\Omega, \rho]} \Phi \)). Let us check that

\[
\Psi(q) = \int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a \Upsilon(q, -i\mu, \Pi, \Pi).
\]

(23)

It is sufficient to justify that equivalent BFV states (11) give equal Dirac wave functions (23). However, it follows directly from (21) that

\[
\int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a (\Omega^+ \Upsilon)(q, -i\mu, \Pi, \Pi) = 0
\]

since integrals of full derivatives vanish. Formula (23) is obtained.

To check relation (1), use the property

\[
\int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a (\Omega^+ \Pi_a \Upsilon)(q, -i\mu, \Pi, \Pi) = 0.
\]

Since \( \Omega^+ \Upsilon = \Omega \Upsilon = 0 \), one can rewrite it as follows,

\[
\int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a ([\Omega^+, \Pi_a] + \Upsilon)(q, -i\mu, \Pi, \Pi) = 0.
\]

(24)

The anticommutator has the form

\[
[\Omega^+, \Pi_a] = \hat{\Lambda}_a^+ + \ldots + n(\Omega_{n a_1 \ldots a_{n-1} a})^+ \frac{\partial}{\partial \Pi_{a_{n-1}}} \ldots \frac{\partial}{\partial \Pi_{a_1}} \Pi_{b_{n-1}} \ldots \Pi_{b_1} + \ldots
\]

It contains full derivatives, except for the term \( \hat{\Lambda}_a^+ \). Thus, eq.(24) can be presented as

\[
\hat{\Lambda}_a^+ \int \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a \Upsilon(q, -i\mu, \Pi, \Pi) = 0.
\]

Eq.(1) is obtained.

Thus, the formal relationship between refined algebraic quantization, Dirac and BFV states is found. However, the topology problems which may lead to integration over some domain in (23) are to be investigated in future.

5 Quantum observables

Let us consider the properties of quantum observables. In the BRST-BFV approach, observables are viewed as series

\[
H_B = H + \ldots + H_{n b_1 \ldots b_{n-1} a_1 \ldots a_n} \Pi_{b_1} \ldots \Pi_{b_n} C_{a_1} \ldots C_{a_n} + \ldots
\]
The operator coefficient functions $H^{n_{b_1} \ldots b_n}_{a_1 \ldots a_n}(\hat{p}, \hat{q})$ are chosen in such a way that

$$H^+_B = H_B, \quad [\Omega, H_B] = 0. \quad (25)$$

These properties provide that physical states (10) are taken by the operator $H$ to physical, while equivalent states are taken to equivalent.

Since $(H_B \Upsilon)(q,0,0,0) = H \Upsilon(q,0,0,0)$, the operator coefficient $H$ is an observable in the refined algebraic quantization approach. One also has

$$\int M d\mu a d\Pi a d\Pi^a H^+_B \Upsilon(q, -i\mu, \Pi, \Pi) = H^+_B \int M d\mu a d\Pi a d\Pi^a \Upsilon(q, -i\mu, \Pi, \Pi)$$

because integral of full derivative vanishes. Therefore, $H^+_B$ is a Hamiltonian in the Dirac approach.

To check property (3), consider the expressions

$$\Omega H_B \Pi_c \Phi(q)|_{\Pi=\Pi=0} = (\hat{\Lambda}_c H + \hat{\Lambda}_b H^1_c)\Phi(q)$$

and

$$H_B \Omega \Pi_c \Phi(q)|_{\Pi=\Pi=0} = H \hat{\Lambda}_c \Phi(q).$$

They should be equal because of (25), so that $[\Omega; H_B] = \hat{\Lambda}_b H^1_c$ and eq.(3) is satisfied if $-iR^c_a = H^1_a$.

Let us verify formula (7). One has

$$\eta H \Phi(q) = \int M d\mu a d\Pi a d\Pi^a \exp[\Pi^a \Pi_a + i\mu_a \hat{\Omega}_a] H_B \Phi(q), \quad (26)$$

while

$$H^+ \eta \Phi(q) = \int M d\mu a d\Pi a d\Pi^a H^+_B \exp[\Pi^a \Pi_a + i\mu_a \hat{\Omega}_a] \Phi(q). \quad (27)$$

Here we have taken into account that $C^a \Phi(q) = 0$ and that the integral of full derivative vanishes. Consider the difference of eqs.(26), (27). Let us make use of the following relation,

$$H^+_B e^{[\Omega; \rho]} - e^{[\Omega; \rho]} H_B = \int_0^1 d\tau e^{\tau [\Omega, \rho]} [\Omega, \rho] e^{(1-\tau)[\Omega, \rho]}, \quad (28)$$

since $H^+_B = H_B$. Moreover, $[[\Omega, \rho], H_B] = [\Omega, [H_B, \rho]] +$. It follows from eq.(19) that

$$H^+_B e^{[\Omega; \rho]} - e^{[\Omega; \rho]} H_B = [\Omega; A] +$$

with

$$A = \int_0^1 d\tau e^{\tau [\Omega, \rho]} [H_B; \rho] e^{(1-\tau)[\Omega, \rho]}, \quad (29)$$

Therefore, the difference between formulas (26) and (27) reads

$$(H^+ \eta - \eta H) \Phi(q) = \int M d\mu a d\Pi a d\Pi^a [\Omega + A\Omega] \Phi(q).$$

This integral vanishes since $\Omega \Phi(q) = 0$ and an integral of full derivative is zero. Thus, relation (7) is satisfied.
6 A simple example

Consider a simple example of a system with structure functions. Investigate the model with 3 degrees of freedom \((p_i, q^i), i = 1, 3\) and 2 classical constraints

\[
\Lambda_1 = a(q^2, q^3)p_1, \quad \Lambda_2 = p_2.
\] (28)

Since \(\{\Lambda_1, \Lambda_2\} = \partial_2 \log a(q^2, q^3)\Lambda_1\), the constraints forms an algebra with structure functions. Let us look for the B-charge in the form (8). In classical theory, it should be written as

\[
\Omega = -i\pi_1 \Pi^1 - i\pi_2 \Pi^2 + p_1 a \Pi^1 + p_2 a \Pi^2 + (\alpha_1 \Pi_1 + \alpha_2 \Pi_2)C_1 C^2
\] (29)

for some functions \(\alpha_a(p, q)\). The property \(\{\Omega, \Omega\} = 0\) means that

\[
p_1 a \alpha_1 + p_2 a \alpha_2 = [p_1 a; p_2],
\]

so that

\[
\alpha_1 = -i\partial_2 \log a; \quad \alpha_2 = 0.
\]

We see that classically

\[
\Omega = -i\pi_1 \Pi^1 - i\pi_2 \Pi^2 + p_1 a \Pi^1 + p_2 a \Pi^2 - i\partial_2 \log a \Pi_1 C_1 C^2.
\]

To quantize the B-charge, one should choose the operator ordering. If the \(\Pi\)-operators were put to the left with respect to \(C\)-operators, the quantum B-charge would be not Hermitian. To obey the condition \(\Omega^+ = \Omega\), let us use the Weyl quantization

\[
\Omega = -i\pi_1 \Pi^1 - i\pi_2 \Pi^2 + p_1 a \Pi^1 + (p_2 - i\Pi_1 \partial_2 \log a \Pi^1 + \frac{i}{2} \partial_2 \log a)C^2
\] (30)

It is remarkable that in quantum theory the constraint \(\hat{\Lambda}_2\) should be modified with respect to the classical theory (28); it follows from eq. (8) that

\[
\hat{\Lambda}_2 = p_2 + \frac{i}{2} \partial_2 \log a,
\]

so that the operator \(\hat{\Lambda}_2\) becomes formally non-Hermitian. This feature of quantum constraints is known from the theory of constrained systems with nonunimodular closed algebra [16, 11].

Let us evaluate the inner product (16). Consider the wave function

\[
\Upsilon^t(q, \Pi, \Pi) = e^{-i\Pi_a \Pi_a + it\mu_a \Omega_a \phi(q)}.
\] (31)

Since

\[
\hat{\Omega}_1 = p_1 a + i\Pi_1 \partial_2 \log a \Pi^2, \quad \hat{\Omega}_2 = p_2 - i\Pi_1 \partial_2 \log a \Pi^1 + \frac{i}{2} \partial_2 \log a,
\]

the state (31) obeys the following Cauchy problem

\[
\frac{\partial}{\partial t} \Upsilon^t = [-\Pi_1 \Pi^1 - \Pi_2 \Pi^2 + a \mu_1 \partial_1 + \mu_2 \partial_2 - \frac{\mu_2}{2} \partial_2 \log a - \mu_1 \Pi_1 \partial_2 \log a \frac{\partial}{\partial \Pi_2} + \mu_2 \Pi_1 \partial_2 \log a \frac{\partial}{\partial \Pi_1}] \Upsilon^t, \quad \Upsilon^0 = \Phi(q)
\] (32)

Since eq. (32) is a first-order partial differential equation, it can be solved by the characteristic method. The solution is looked for in the following form

\[
\Upsilon^t(Q^t, \Pi^t, \Pi) = \exp[\int_0^t dt [-\Pi_1^t \Pi^1 - \Pi_2^t \Pi^2 - \frac{\mu^2}{2} \partial_2 \log a (Q^r)]] \Upsilon^0(Q^0, \Pi^0, \Pi),
\]

The solution is given by

\[
\Upsilon^t(Q^t, \Pi^t, \Pi) = \Phi(Q^0, \Pi^0, \Pi) \exp[\int_0^t dt [-\Pi_1^t \Pi^1 - \Pi_2^t \Pi^2 - \frac{\mu^2}{2} \partial_2 \log a (Q^r)]].
\]
where the functions \( Q^t, \tilde{\Pi}^t \) satisfy the following ordinary differential equations,

\[
\begin{align*}
\dot{Q}^t_1 &= -a(Q_2, Q_3)\mu_1; \\
\dot{Q}^t_2 &= -\mu_2; \\
\dot{Q}^t_3 &= 0,
\end{align*}
\]

\[
\frac{d}{dt}\tilde{\Pi}^t_2 = \mu_1\tilde{\Pi}_1\partial_2 \log a(Q_2, Q_3), \\
\frac{d}{dt}\tilde{\Pi}^t_1 = -\mu_2\tilde{\Pi}_1\partial_2 \log a(Q_2, Q_3),
\]

so that the classical characteristic trajectory is

\[
\begin{align*}
Q^t_3 &= Q^0_3, \\
Q^t_2 &= Q^0_2 - \mu_2 t, \\
Q^t_1 &= Q^0_1 - \int_0^t d\tau a(Q^0_2 - \mu_2 \tau, Q^0_3)\mu_1,
\end{align*}
\]

\[
\begin{align*}
\tilde{\Pi}^t_1 &= \frac{a(Q^0_2 - \mu_2 t, Q^0_3)}{a(Q^0_2, Q^0_3)}\tilde{\Pi}^0_1, \\
\tilde{\Pi}^t_2 &= \tilde{\Pi}^0_2 + \frac{1}{a(Q^0_2, Q^0_3)}\mu_1\tilde{\Pi}^0_1 \int_0^t d\tau \partial_2 a(Q^0_2 - \mu_2 \tau, Q^0_3).
\end{align*}
\]

Combining all factors, one finds the solution the Cauchy problem \( \text{(32)} \),

\[
\gamma^t(x, \Pi, \Pi) = \sqrt{\frac{a(x_2, x_3)}{a(x_2 + \mu_2 t, x_3)}} \exp\left[-\int_0^t d\tau \frac{a(x_2 + \mu_2 t, x_3)}{a(x_2, x_3)}\Pi_1 \Pi^t - t\Pi_2 \Pi^2\right] \exp\left[\int_0^t d\tau \mu_1 \frac{\partial_2 \log a(x_2 + \mu_2 \tau, x_3)}{a(x_2, x_3)}\Pi_1 \Pi^t \right] \Phi(x_1 + \int_0^t d\tau a(x_2 + \mu_2 \tau, x_3)\mu_1, x_2 + \mu_2 t, x_3) \tag{33}
\]

One can also check by the direct computations that expression \( \text{(33)} \) really satisfies the Cauchy problem \( \text{(32)} \). The inner product \( \text{(16)} \) reads

\[
\int dx\Phi^*(x) \prod_{a=1}^{M} d\mu_a d\Pi_a d\Pi^a \gamma^t(x, \Pi, \Pi). \tag{34}
\]

Integration over ghost variables gives us the multiplier

\[
t\int_0^t d\tau a(x_2 + \mu_2 \tau, x_3)\frac{1}{a(x_2, x_3)}.
\]

After rescaling of variables \( \mu \)

\[
\xi_1 = \int_0^t d\tau a(x_2 + \mu_2 \tau, x_3)\mu_1, \quad \xi_2 = t\mu_2
\]

one finds that the integral \( \text{(34)} \) takes a simple form

\[
\int dx_1 dx_2 dx_3 d\xi_1 d\xi_2 \Phi^*(x_1, x_2, x_3) \frac{1}{\sqrt{a(x_2 + \xi_2, x_3)a(x_2, x_3)}} \Phi(x_1 + \xi_1, x_2 + \xi_2, x_3)
\]

We see that the bilinear form \( \eta \) can be defined as

\[
(\Phi, \eta \Phi) = \int dx_3|\int dx_1 dx_2 \frac{\Phi(x_1, x_2, x_3)}{\sqrt{a(x_2, x_3)}}|^2,
\]

so that the correspondence between the Dirac wave function \( \Psi \) and the auxiliary state \( \Phi \) is

\[
\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{a(x_2, x_3)}} \int dy_1 dy_2 \frac{\Phi(y_1, y_2, x_3)}{\sqrt{a(y_2, x_3)}}.
\]
It obeys the constraints

\[ a(x_2, x_3)p_1 \Psi \equiv \hat{\Lambda}_1^+ \Psi = 0, \quad \frac{1}{\sqrt{a(x_2, x_3)}} p_2 \sqrt{a(x_2, x_3)} \Psi \equiv \hat{\Lambda}_2^+ \Psi = 0. \]

while the gauge transformation of \( \Phi \) is

\[ \Phi \rightarrow \Phi + \sqrt{a(x_2, x_3)} p_2 \frac{1}{\sqrt{a(x_2, x_3)}} \Psi \equiv U \Psi. \]

for some functions \( Y^1 \) and \( Y^2 \). We see that all properties of \( \eta \) (including positive definiteness) are indeed satisfied in this example.

### 7 Discussion

Thus, the refined algebraic quantization formula (4) is generalized to the case of structure functions. A simple exactly solvable example is investigated; an explicit formula for the inner product is obtained.

A wide class of such examples of systems with structure functions can be constructed as follows. Consider the Lie-algebra constrained system: \( \hat{\Lambda}_a = L_a - \frac{i}{2} f^{bc}_a, U^a_{bc} = f^a_{bc} = \text{const} \) such that \( L_a \) are linear in momenta, \( L_a = \alpha_{aj}(x)p_j + \beta_a(x) \). The B-charge is

\[ \Omega_0 = C^a L_a - \frac{i}{2} f^{bc}_a \Pi_a C^b C^c - \frac{i}{2} f^{ab}_a C^b - i\pi_a \Pi^a. \]

Consider the unitary transformation being an exponent of the operator quadratic with respect to ghost variables,

\[ U = \exp[\Pi_a A^a(x) C^b - \frac{1}{2} A^a(x)] \]

It generates a linear canonical transformation of ghosts. The transformed B-charge \( U^{-1} \Omega_0 U = \Omega \) is Hermitian and nilpotent. It contains terms \( \Omega^1 \) and \( \Omega^2 \) only and corresponds to the new system with classical constraints

\[ \Lambda_{a'} = L_a (\exp A)^a_{a'}. \]  \hspace{1cm} (35)

with quantum corrections. Generally, they form an algebra with nontrivial structure functions. Since \( \Omega^n = 0, n \geq 3 \), while the constraints are linear in momenta, the Cauchy problem analogous to (32) still corresponds to the first-order partial differential equation and can be solved exactly, so that it is also possible to perform an integration in eq.(17) explicitly.

We see that the system with classical constraints (35) which was mentioned in [11] can be exactly investigated by the approach proposed in this paper.

The case of an open gauge algebra corresponding to nontrivial coefficient functions \( \Omega^n, n \geq 3 \) is much more complicated for the exact calculations. However, the integral formula (17) is still valid, so that one can use it for numerical calculations or for application of asymptotic methods such as perturbation theory or semiclassical approximation.

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