LOCAL CHARACTER OF KIM-INDEPENDENCE

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Abstract. We show that NSOP\(_1\) theories are exactly the theories in which Kim-independence satisfies a form of local character. In particular, we show that if \(T\) is NSOP\(_1\), \(M \models T\), and \(p\) is a type over \(M\), then the collection of elementary substructures of size \(|T|\) over which \(p\) does not Kim-fork is a club of \([M]^{|T|}\) and that this characterizes NSOP\(_1\).

We also present a new phenomenon we call dual local-character for Kim-independence in NSOP\(_1\)-theories.

1. Introduction

A well-known theorem of Kim and Pillay characterizes the simple theories as those theories with an independence relation satisfying certain properties and shows that, moreover, any such independence relation must coincide with non-forking independence. As the theory of simple theories was being developed, work of Chatzidakis on \(\omega\)-free PAC fields and Granger on vector spaces with bilinear forms furnished examples of non-simple theories for which there are nonetheless independence relations satisfying many of the fundamental properties of non-forking independence in simple theories. These properties include extension, symmetry, and the independence theorem. Chernikov and the second-named author proved an analogue of one direction of the Kim-Pillay theorem for NSOP\(_1\) theories, showing essentially that the existence of an independence relation with these properties implies that a theory is NSOP\(_1\). To establish the other direction, the first and second-named authors introduced Kim-independence and showed that it is well-behaved in any NSOP\(_1\) theory. The theory of Kim-independence provides an explanation for the simplicity-like phenomena observed in certain non-simple examples and a central issue of research concerning NSOP\(_1\) theories is to determine the extent to which properties of non-forking independence in simple theories carry over to Kim-independence in NSOP\(_1\) theories. This paper addresses the specific issue of local character for Kim-independence.

Simple theories are defined to be the theories in which forking satisfies local character. Local character of non-forking asserts that there is some cardinal \(\kappa (T)\) so that, for any complete type \(p\) over \(A\), there is a set \(B \subseteq A\) with \(|B| < \kappa (T)\) over which \(p\) does not fork. An analogue of local

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character for Kim-independence in NSOP\(_1\) theories was proved by the first- and second-named authors in [KR17, Theorem 4.5]. It was shown there that if \(T\) is NSOP\(_1\) and \(M \models T\), then for any \(p \in S(M)\), there is \(N \prec M\) with \(|N| < \kappa = (2^{|T|})^+\) such that \(p\) does not Kim-fork over \(N\).

However, this result was an unsatisfactory generalization of local character in simple theories for three reasons. First, with respect to non-forking, it follows almost immediately that if \(\kappa(T)\) exists at all, it can be taken to be \(|T|^+\). Given a type \(p \in S(A)\) with no \(B \subseteq A\) of size \(< |T|^+\) over which \(p\) does not fork, one can find a chain of forking types of length \(|T|^+\) and then by the pigeonhole principle, some formula must fork infinitely often with respect to the same disjunction of dividing formulas. This equivalence is no longer immediate when considering Kim-independence, because of the added constraint that the formulas must divide with respect to Morley sequences and it was asked [KR17, Question 4.7] if \((2^{|T|})^+\) can be replaced by \(|T|^+\) in an arbitrary NSOP\(_1\) theory.

Secondly, non-forking independence satisfies base monotonicity, which means that if \(p \in S(A)\) does not fork over \(B\), then \(p\) does not fork over \(B'\) whenever \(A \subseteq B' \subseteq B\). In other words, local character of forking implies that every type does not fork over an entire cone of small subsets of its domain. However, in an NSOP\(_1\) theory \(T\), Kim-independence satisfies base monotonicity if and only if \(T\) is simple. One would like an analogue of local character for NSOP\(_1\) theories that shows that types over models do not Kim-divide over many small submodels. Finally, local character of non-forking independence characterizes simple theories. Many tameness properties of Kim-independence are known to characterize NSOP\(_1\) theories, e.g. symmetry and the independence theorem, so it is natural to ask if local character does as well.

Our main theorem is:

**Theorem 1.1.** Suppose \(T\) is a complete theory with monster model \(M \models T\). The following are equivalent:

1. \(T\) is NSOP\(_1\).
2. There is no continuous increasing sequence of \(|T|^+\)-sized models \(\langle M_i \mid i < |T|^+ \rangle\) with union \(M\) and \(p \in S(M)\) such that \(p \upharpoonright M_{i+1}\) Kim-forks over \(M_i\) for all \(i < |T|^+\).
3. For any \(M \models T\), \(p \in S(M)\), the set of elementary substructures of \(M\) of size \(|T|\) over which \(p\) does not Kim-divide is a stationary subset of \([M]^{|T|}\).
4. For any \(M \models T\), \(p \in S(M)\), the set of elementary substructures of \(M\) of size \(|T|\) over which \(p\) does not Kim-divide contains a club subset of \([M]^{|T|}\).
5. For any \(M \models T\), \(p \in S(M)\), the set of elementary substructures of \(M\) of size \(|T|\) over which \(p\) does not Kim-divide is a club subset of \([M]^{|T|}\).
6. Suppose that \(N \models T\), \(M \prec N\) and \(p \in S(N)\) does not Kim-divide over \(M\). Then the set of elementary substructures of \(M\) of size \(|T|\) over which \(p\) does not Kim-divide is a club subset of \([M]^{|T|}\).
The equivalence of (1) and (2) was noted in [KR17, Corollary 4.6] with \(|T|^+\) replaced by \((2^{\|T\|})^+\), which is considerably weaker than the theorem proved here.

In particular, this theorem implies that if \(T\) is NSOP\(_1\), \(M \models T\), and \(p \in S(M)\), then the set of \(N \prec M\) with \(\|N\| = \|T\|\) such that \(p\) does not Kim-fork over \(N\) is non-empty, answering a question asked by the first and second-named authors [KR17, Question 4.7]. However, by demanding a stronger form of local character, we obtain a new characterization of NSOP\(_1\).

Remark 1.2. In the first draft of this paper, published online on July 2017, we did not yet have (5) or (6) above. Shortly after that draft was available, Pierre Simon have found an easier proof of (1) implies (4), and we thank him for allowing us to include his proof here. Later we found a proof of (6). These proofs uses symmetry of Kim-independence, but are not straightforward as in the proof in simple theories, and our original proof.

Our original proof assumes towards contradiction that local character fails and reaches a contradiction to NSOP\(_1\) as is done in e.g. simple theories. For this approach to work we used stationary logic. This logic expands first-order logic by introducing a quantifier \(\text{aa}\) interpreted so that \(M \models (\text{aa}\, S) \varphi (S)\) if and only if the set of countable subsets \(X \subseteq M\) such that \(M\), when expanded with the predicate \(S\) interpreted as \(X\), satisfies \(\varphi (S)\) contains a club of \([M]^\omega\). This logic was introduced by the third-named author in [She75] and later studied by Mekler and the third-named author [MS86] who showed that the satisfiability of a theory in \(L(\text{aa})\) implies the satisfiability of a theory in a related logic, where the second-order quantifiers range over uncountable sets of a certain size. This theorem, which may be regarded as a version of the upward Lowenheim-Skolem theorem, provides a tool for “stretching” a family of counterexamples to local character in such a way that preserves the cardinality and continuity constraints needed to produce SOP\(_1\).

After further review, we noticed that our original proof gives rise to a new phenomenon, which we call dual local character.

In light of all this, we decided to re-arrange the paper in the following way. After a short preliminaries section, we prove the main theorem. In Section 4 we discuss stationary logic and describe our original proof. In Section 5 we discuss the dual local character.

2. Preliminaries

2.1. NSOP\(_1\) theories, invariant types, and Morley sequences.

Definition 2.1. [DS04, Definition 2.2] A formula \(\varphi (x; y)\) has the 1-strong order property (SOP\(_1\)) if there is a tree of tuples \(\langle a_\eta \mid \eta \in 2^{<\omega}\rangle\) so that

- For all \(\eta \in 2^{<\omega}\), the partial type \(\{\varphi (x; a_\eta \restriction n) \mid n < \omega\}\) is consistent.
- For all \(\nu, \eta \in 2^{<\omega}\), if \(\nu \prec \langle 0 \rangle \leq \eta\) then \(\{\varphi (x; a_\eta), \varphi (x; a_{\nu \cup \langle 1\rangle})\}\) is inconsistent.

A theory \(T\) is NSOP\(_1\) if no formula has SOP\(_1\) modulo \(T\).
Fact 2.2. [KR17] Proposition 2.4] $T$ has NSOP$_1$ if and only if there is a formula $\varphi(x;y)$, $k < \omega$, and a sequence $\langle c_i^1 | i \in I \rangle$ with $c_i^1 = (c_{i,0}, c_{i,1})$ satisfying:

1. For all $i \in I$, $c_{i,0} \equiv_{c_i^1} c_{i,1}$.
2. $\{ \varphi(x;c_{i,0}) | i \in I \}$ is consistent.
3. $\{ \varphi(x;c_{i,1}) | i \in I \}$ is $k$-inconsistent.

We also use following notation. Write $a \downarrow^M B$ for $tp(a/MB)$ is finitely satisfiable in $M$, in other words it is a coheir of its restriction to $M$. A type $p \in S(M)$ is an heir of its restriction to $N \prec M$ if for every formula $\varphi(x;y) \in L(N)$ and every $b \in M$, if $\varphi(x;b) \in p$ then $\varphi(x;b') \in p$ for some $b' \in N$. We denote this by $c \downarrow^h_N M$. This is equivalent to saying that $M \downarrow^h_N c$.

Definition 2.3. A global type $q \in S(M)$ is called $A$-invariant if $b \equiv_A b'$ implies $\varphi(x;b) \in q$ if and only if $\varphi(x;b') \in q$. A global type $q$ is invariant if there is some small set $A$ such that $q$ is $A$-invariant. If $q(x)$ and $r(y)$ are $A$-invariant global types, then the type $(q \otimes r)(x,y)$ is defined to be $tp(a,b/M)$ for any $b \models r$ and $a \models q_{|Mb}$. It is also $A$-invariant. We define $q^\otimes n(x_0,\ldots,x_{n-1})$ by induction: $q^\otimes 1 = q$ and $q^\otimes n+1 = q(x_n) \otimes q^\otimes n(x_0,\ldots,x_{n-1})$.

Fact 2.4. [She90] Lemma 4.1] If $T$ is any complete theory, $M \models T$, and $p \in S(M)$, then there is a complete global type $q$ extending $p$ which is, moreover, finitely satisfiable in $M$. In particular, $q$ is $M$-invariant.

Definition 2.5. Suppose $q$ is an $A$-invariant global type and $I$ is a linearly ordered set. By a Morley sequence in $q$ over $A$ of order type $I$, we mean a sequence $\langle b_\alpha | \alpha \in I \rangle$ such that for each $\alpha \in I$, $b_\alpha \models q_{|Ab_{<\alpha}}$ where $b_{<\alpha} = \{ b_\beta | \beta < \alpha \}$. Given a linear order $I$, we will write $q^\otimes I$ for the $A$-invariant type in variables $\langle x_\alpha | \alpha < I \rangle$ so that for any $B \supseteq A$, if $b \models q^\otimes I | B$ then $b_\alpha \models q_{|Bb_{<\alpha}}$ for all $\alpha \in I$. If $q$ is, moreover, finitely satisfiable in $A$, then we refer to a Morley sequence in $q$ over $A$ as a coheir sequence over $A$.

The above definition of $q^\otimes I$ generalizes the finite tensor product $q^\otimes n$ – given any global $A$-invariant type $q$ and linearly ordered set $I$, one may easily show that $q^\otimes I$ exists and is $A$-invariant by compactness.

Definition 2.6. Suppose $M$ is a model.

1. Given a formula $\varphi(x;b)$ and a global $M$-invariant type $q \supseteq tp(b/M)$, we say that $\varphi(x;b)$ $k$-Kim-divides over $M$ via $q$ if, whenever $\langle b_i | i < \omega \rangle$ is a Morley sequence over $M$ in $q$, then $\{ \varphi(x;b_i) | i < \omega \}$ is $k$-inconsistent.
2. If $q$ is a global $M$-invariant type with $q \supseteq tp(b/M)$, we say $\varphi(x;b)$ Kim-divides over $M$ via $q$ if $\varphi(x;b)$ $k$-Kim-divides over $M$ via $q$ for some $k < \omega$.
3. We say $\varphi(x;b)$ Kim-divides over $M$ if $\varphi(x;b)$ Kim-divides over $M$ via $q$ for some global $M$-invariant $q \supseteq tp(b/M)$. 
(4) We say that $\varphi(x;b)$ Kim-forks over $M$ if it implies a finite disjunction of formulas, each Kim-dividing over $M$.

(5) We write $a \downarrow^{K}_M B$ for $\text{tp}(a/MB)$ does not Kim-fork (or Kim-independent) over $M$.

Note that if $a \downarrow^{w}_M B$ then $a \downarrow^{f}_M B$ (i.e. $\text{tp}(a/BM)$ does not fork over $M$) which implies $a \downarrow^{K}_M B$.

**Fact 2.7.** [KR17 Theorem 3.15] The following are equivalent for the complete theory $T$:

1. $T$ is NSOP$_1$.
2. (Kim’s lemma for Kim-dividing) Given any model $M \models T$ and formula $\varphi(x;b)$, $\varphi(x;b)$ Kim-divides via $q$ for some global $M$-invariant $q \supseteq \text{tp}(b/M)$ if and only if $\varphi(x;b)$ Kim-divides via $q$ for all global $M$-invariant $q \supseteq \text{tp}(b/M)$.

From this it easily follows that Kim-forking is equal to Kim-dividing [KR17 Proposition 3.19]. The notion of Kim independence, denoted by $\perp^K$, satisfies many nice properties which turn out to be equivalent to NSOP$_1$.

**Fact 2.8.** [KR17 Theorem 8.1] The following are equivalent for the complete theory $T$:

1. $T$ is NSOP$_1$.
2. Symmetry of Kim independence over models: $a \perp^K_M b$ iff $b \perp^K_M a$ for any $M \models T$.
3. Independence theorem over models: if $A \perp_M B$, $c \perp_M A$, $c' \perp_M B$ and $c \equiv_M c'$ then there is some $c'' \perp_M AB$ such that $c'' \equiv_M c$ and $c'' \equiv c'M_B$.

**Fact 2.9.** [KR17 Lemma 7.6] Suppose that $T$ is NSOP$_1$ and that $\langle a_i \mid i < \omega \rangle$ is an $\perp^K$-Morley sequence over $M$ in the sense that $a_i \perp^K_M a_i$ and the sequence is indiscernible. Then if $\varphi(x,a_0)$ does not Kim-divide over $M$, then $\{ \varphi(x,a_i) \mid i < \omega \}$ does not Kim-divide over $M$, and in particular it is consistent.

### 2.2. The generalized club filter

**Definition 2.10.** Let $\kappa$ be a cardinal and $X$ a set with $|X| \geq \kappa$. We write $[X]^\kappa$ to denote $\{ Y \subseteq X \mid |Y| = \kappa \}$.

1. A set $C \subseteq [X]^\kappa$ is **unbounded** if for every $Y \in [X]^\kappa$, there is some $Z \in C$ with $Y \subseteq Z$.
2. A set $C \subseteq [X]^\kappa$ is **closed** if, whenever $\langle Y_i \mid i < \alpha \leq \kappa \rangle$ is a chain in $C$, i.e. each $Y_i \in C$ and $i < j < \alpha$ implies $Y_i \subseteq Y_j$, then $\bigcup_{i<\alpha} Y_i \in C$.
3. A set $C \subseteq [X]^\kappa$ is **club** if it is closed and unbounded.
4. A set $S \subseteq [X]^\kappa$ is **stationary** if $S \cap C \neq \emptyset$ for every club $C \subseteq [X]^\kappa$.

The **club filter** on $[X]^\kappa$ is the filter generated by the clubs. If $|X| = \kappa$, then the club filter on $[X]^\kappa$ is the principal ultrafilter consisting of subsets of $[X]^\kappa$ containing $X$. 
Example 2.11. If $M$ is an $L$-structure of size $\geq \kappa \geq |L|$, then the collection of elementary substructures of $M$ of size $\kappa$ is a club in $[M]^{\kappa}$.

Remark 2.12. In the literature, e.g. [Jec13 Definition 8.21], the above definitions are given instead for subsets of $\mathcal{P}_{\kappa^+}(X) = \{Y \subseteq X \mid |Y| < \kappa^+\}$ but note that $[X]^{\kappa}$ is a club subset of $\mathcal{P}_{\kappa^+}(X)$, hence all definitions relativize to this set in the natural way.

Fact 2.13. Let $\kappa$ be a cardinal and $X$ a set with $|X| \geq \kappa^+$.

1. The club filter on $[X]^{\kappa}$ is $\kappa^+$-complete [Jec13 Theorem 8.22].
2. For every club $C \subseteq [X]^{\kappa}$, there is a collection of finitary functions $\mathcal{T} = \{f_i \mid i < \kappa\}$ with $f_i : X^{n_i} \rightarrow X$ such that $C_{\mathcal{T}} := \{Y \in [X]^{\kappa} \mid f_i(Y^{n_i}) \subseteq Y \text{ for all } i < \kappa\}$ is contained in $C$. Equivalently, there is a function $F : X^{<\omega} \rightarrow [X]^{\kappa}$ such that $C_F \subseteq C$ [Jec13 Lemma 8.26].
3. Conversely, given a collection of finitary functions $\mathcal{T} = \{f_i \mid i < \kappa\}$ with $f_i : X^{n_i} \rightarrow X$, the set $C_\mathcal{T}$ is club in $[X]^{\kappa}$.
4. When $\kappa = \omega$, for any club $C \subseteq [X]^{\kappa}$, there is a function $F : X^{<\omega} \rightarrow X$ such that $C_F \subseteq C$ [Jec13 Theorem 8.28].

We leave the proof of the next lemma to the reader.

Lemma 2.14. Suppose $\lambda$ is a cardinal, $X$ is a set with $|X| = \lambda^+$, and $\{Y_\alpha \mid \alpha < \lambda^+\}$ is an increasing continuous sequence of sets of cardinality $\lambda$ with union $X$. Then $\{Y_\alpha \mid \alpha < \lambda^+\}$ is a club of $[X]^\lambda$. In particular, if $X = \lambda^+$ and $C \subseteq \lambda^+ \setminus \lambda$ is a club of $\lambda^+$, then $C$ is a club of $[X]^\lambda$.

3. Proof of Theorem 1.1

3.1. A short proof of (1) implies (4) in Theorem 1.1 using heirs. Here we give a short proof of (1) implies (4) in Theorem 1.1 due to Pierre Simon. We thank him for allowing us to include this proof.

Lemma 3.1. Suppose $p(x) \in S(M)$, $M \models T$. Then the set of $N \prec M$ such that $|N| = |T|$ and $p$ is an heir of $p|_N$ is a club subset of $[M]^{\|T\|}$.  

Proof. It is easy to verify that this set is closed under increasing unions, so it is enough to show that it contains a club.

Consider the $L_p$-structure $M_p$ expanding $M$ by forcing $p$ to be definable — i.e. for every $L$-formula $\varphi(x; y)$ add a relation $R_\varphi(y)$ interpreted as $\{b \in M^{[b]} \mid \varphi(x, b) \in p\}$. Note that $|L_p| = |L|$. Then if $N' \prec M_p$ then its $L$-reduct $N$ is such that $p$ is an heir of $p|_N$. Thus we are done by Example 2.11.  

Theorem 3.2. Suppose $T$ is NSOP$_1$. If $M \models T$ and $p \in S(M)$, then the set of elementary substructures $N \prec M$ with $|N| = |T|$ such that $p$ does not Kim-divide over $N$ contains a club.
Proof. By Lemma 3.3, it suffices to show that if \( p \) is an heir of \( p \restriction N \), then \( p \) does not Kim-divide over \( N \). But if \( p \) is an heir of \( p \restriction N \), then, given \( c \models p, M \downarrow N c \), hence \( M \downarrow N c \) by symmetry of Kim-independence (in fact one needs only a weak version of symmetry, see [KR17, Proposition 3.22]) which implies \( c \downarrow N M \). This shows that \( p \) does not Kim-divide over \( N \).

\[ \square \]

3.2. A proof of (1) implies (6) in Theorem 1.1

Lemma 3.3. Suppose \( T \) is an arbitrary theory and \( M \models T \) with \( |M| \geq |T| = \kappa \). Given any global \( M \)-finitely satisfiable type \( q \), let \( C_q \) denote the set of \( N \prec M \) with \( |N| = \kappa \) such that \( q^\omega|_N = r^\omega|_N \) for some global \( N \)-finitely satisfiable type \( r \). Then:

1. \( C_q \) is in the club filter on \( [M]^\kappa \).
2. Given any set \( A \), there is some \( N \prec M \) of size \( \leq |T| + |A| \) such that \( A \subseteq N \) and \( q^\omega|_N \) is a type of a Morley sequence generated by some global type \( r \) finitely satisfiable in \( N \) and if \( \varphi(x,c) \) Kim-divides over \( M \) via \( q \) then \( \varphi(x,c) \) Kim-divides over \( N \) via \( r \).

Proof. One proof of (1) essentially follows from the proof of [KR17, Lemma 4.4], so we also give an alternative one. Let \( \bar{a} = (a_i \mid i < \omega) \) be a coheir sequence generated by \( q \) over \( M \). Then, \( N \in C_q \) iff \( N \prec M \) and \( \bar{a} \) is a coheir sequence over \( N \) in the sense that \( \text{tp}(a_i/a_{<i}N) \) if finitely satisfiable in \( N \). Thus it is easy to see that \( C_q \) is closed under unions.

Note that if \( N \prec M \) is such that \( \text{tp}(\bar{a}/M) \) is an heir extension of its restriction to \( N \), then \( N \in C_q \); if \( \varphi(a_i, a_{<i}) \) holds when \( \varphi(x,y) \) is some formula over \( N \), then for some \( c \in M \), \( \varphi(c, a_{<i}) \) holds, and by choice of \( N \), we may assume that \( c \in N \). Now Lemma 3.1 finishes the proof.

(2) is immediate from (1), applied to the theory \( T(A) \) obtained from \( T \) by adding constants for the elements of \( A \).

\[ \square \]

Theorem 3.4. Suppose \( T \) is \( \text{NSOP}_1 \) with \( |T| = \kappa \) and \( M \models T \). Then for a finite tuple \( b \) and any set \( A \), the following are equivalent:

1. \( A \downarrow M b \).
2. There is a club \( C \subseteq [M]^\kappa \) of elementary substructures of \( M \) such that \( A \downarrow M b \) for all \( N \in C \).
3. There is a stationary set \( S \subseteq [M]^\kappa \) of elementary substructures of \( M \) such that \( A \downarrow M b \) for all \( N \in S \).

Proof. (1) \( \implies \) (2) Suppose that \( A \downarrow M b \). Let \( q \supseteq \text{tp}(b/M) \) be a global \( M \)-finitely satisfiable type and choose \( \langle b_i \mid i < \omega \rangle \models q^\omega|_M \) with \( b_0 = b \). By Lemma 3.3, there is a club \( C_q \) of elementary substructures \( N \prec M \) with \( |N| = |T| \) so that \( q^\omega|_N = r^\omega|_N \) for some global \( N \)-finitely satisfiable type \( r \). Fix \( N \in C, a \) a finite tuple from \( A \) and \( \varphi(x;b,n) \in \text{tp}(a/Nb) \). As \( a \downarrow M b \), we know \( \{\varphi(x;b_i,n) \mid i < \omega\} \) is consistent. As \( \langle b_i \mid i < \omega \rangle \) is also a Morley sequence over \( N \) in a global \( N \)-finitely satisfiable type, it follows from Kim’s lemma for Kim-dividing (Fact 2.7) that \( \varphi(x;b,n) \)
does not Kim-divide over $N$. As $\varphi (x;b,n)$ was arbitrary, we conclude $a \perp^K_N b$. Since this was true for any $a$, we have that $A \perp^K_N b$.

(2) $\implies$ (3) is immediate.

(3) $\implies$ (1) Suppose $a \perp^K_M b$ for some finite tuple $a$ from $A$. Let $\varphi (x;b,m) \in \text{tp}(a/Mb)$ be a formula witnessing this. Fix $q \supseteq \text{tp}(b/M)$ a global $M$-finitely satisfiable type and $\langle b_i | i < \omega \rangle \models q^{\omega^\omega}|_M$. Let $C' = \{ N \prec M | |N| = |T| \text{ and } m \in N \}$. The set $C'$ is clearly club so the intersection $C'' = C_q \cap C'$ is in the club filter on $|M|^\kappa$. If $N \in C''$ and $q^{\omega^\omega}|_N = r^{\omega^\omega}|_N$ for some global type $r$ finitely satisfiable in $N$, then $\varphi (x;b,m) \in \text{tp}(a/Nb)$ and $\langle b_i | i < \omega \rangle$ realizes $r^{\omega^\omega}|_N$. As $\{ \varphi (x;b_i,m) | i < \omega \}$ is inconsistent, we have $a \perp^K_N b$. As $S$ is stationary, it must intersect $C''$, so we get a contradiction.

\begin{corollary}
Suppose $T$ is NSOP$_1$ with $|T| = \kappa$ and $M \models T$. Then for a finite tuple $a$ and any set $B$, the following are equivalent:

1. $a \perp^K_M B$.
2. There is a club $C \subseteq [M]^\kappa$ of elementary substructures of $M$ such that $a \perp^K_N B$ for all $N \in C$.
3. There is a stationary set $S \subseteq [M]^\kappa$ of elementary substructures of $M$ such that $a \perp^K_N B$ for all $N \in S$.

\begin{proof}
Follows immediately from symmetry of Kim-independence and Theorem 3.3. \hfill \Box
\end{proof}

\begin{lemma}
Suppose $T$ is NSOP$_1$. Assume $M \prec N$. Suppose that $a \perp^K_M N$ and $\varphi (x,a)$ Kim-divides over $N$ for $\varphi (x,y) \in L(M)$. Then $\varphi (x,a)$ Kim-divides over $M$.

\begin{proof}
Let $\langle a_i | i < \omega \rangle$ be an indiscernible sequence over $N$ starting with $a_0 = a$ such that $a_i \perp^h_N a_{<i}$ and $\{ \varphi (x,a_i) | i < \omega \}$ is inconsistent (to construct it, let $\langle b_i | i \in \mathbb{Z} \rangle$ be a coheir sequence in the type of $\text{tp}(a/N)$, so in particular $b_i \perp^u_N b_{<i}$ for $i < 0$, hence $b_{>i} \perp^u_N b_{>i}$ by transitivity of $\perp^u$, and let $a_i = b_{-i}$ for $i < \omega$).

Then $\langle a_i | i < \omega \rangle$ is an $\perp^K$-Morley sequence over $M$ in the sense that $a_i \perp^K_M a_{<i}$. To see this, suppose not, i.e., by symmetry suppose that $a_{<i} \perp^K_M a_i$. Then for some formula $\psi (z,x)$ over $M$, $\psi (a_{<i},a_i)$ holds and $\psi (z,a_i)$ Kim-divides over $M$. Since $a_{<i} \perp^u_N a_i$, for some $n \in N$, $\psi (n,a_i)$ holds. However, since $a_i \equiv_N a_i$ by symmetry $N \perp^K_M a_i$ — contradiction.

Suppose that $\varphi (x,a)$ does not Kim-divide over $M$. Then by Fact 2.29 $\{ \varphi (x,a_i) | i < \omega \}$ is consistent — contradiction. \hfill \Box
\end{proof}

\begin{lemma}
Suppose $T$ is NSOP$_1$. Suppose that $\langle M_i | i \leq \alpha \rangle$ is an increasing sequence of elementary substructures of a model $N$, that $M_\alpha = \bigcup \{ M_i | i < \alpha \}$ and that $p \in S(N)$. Assume that $p$ does not Kim-fork over $M_i$ for all $i < \alpha$. Then $p$ does not Kim-fork over $M_\alpha$.
\end{lemma}
Theorem 3.8. Suppose that $|M_{\alpha}|_M a$. Suppose not. Then there is some formula $\varphi(x,y)$ in $L(M_{\alpha})$ and some $b \in N$ such that $\varphi(b,a)$ holds and $\varphi(x,a)$ Kim-divides over $M_{\alpha}$. Let $i < \alpha$ be such that $\varphi(x,y) \in L(M_i)$. Since $M_{\alpha} \subseteq N$ and $a \downarrow K_{M_{\alpha}} N$ by assumption, $a \downarrow K_{M_i} a$. Hence by Lemma 3.6, $\varphi(x,a)$ Kim-divides over $M_i$. Hence $b \notin K_{M_i} a$. But this is a contradiction since $a \downarrow K_{M_i} N$ so by symmetry $b \downarrow K_{M_i} a$. \hfill \Box

We can now prove (1) $\implies$ (6) from Theorem 1.1.

**Theorem 3.8.** Suppose that $T$ is NSOP$_1$. Suppose that $a$ is a finite tuple, $a \downarrow K_{M_i} N$ and $M \prec N$. Then the set $E$ of $M' \in [M]^{|T|}$ such that $M' \prec M$ and $a \downarrow K_{M_i} N$ is a club.

Proof. The family $E$ is closed under unions by Lemma 3.7. Hence to finish we only need to show that $E$ contains a club, and this follows from Corollary 3.8 (1) $\implies$ (2). \hfill \Box

3.3. The equivalence (1)–(6). We finish the proof of Theorem 1.1 with the following.

**Theorem 3.9.** Suppose $T$ is a complete theory. The following are equivalent:

1. $T$ is NSOP$_1$.
2. There is no continuous increasing sequence of $|T|$-sized models $\langle M_i \mid i < |T|^+ \rangle$ with union $M$ and $p \in S(M)$ such that $p \upharpoonright M_{i+1}$ Kim-forks over $M_i$ for all $i < |T|^+$.
3. For any $M \models T$, $p \in S(M)$, the set of elementary substructures of $M$ of size $|T|$ over which $p$ does not Kim-divide is a stationary subset of $[M]^{|T|}$.
4. For any $M \models T$, $p \in S(M)$, the set of elementary substructures of $M$ of size $|T|$ over which $p$ does not Kim-divide contains a club subset of $[M]^{|T|}$.
5. For any $M \models T$, $p \in S(M)$, the set of elementary substructures of $M$ of size $|T|$ over which $p$ does not Kim-divide is a club subset of $[M]^{|T|}$.
6. Suppose that $N \models T$, $M \prec N$ and $p \in S(N)$ does not Kim-divide over $M$. Then the set of elementary substructures of $M$ of size $|T|$ over which $p$ does not Kim-divide is a club subset of $[M]^{|T|}$.

Proof. (1) $\implies$ (6) is Theorem 3.8.

(6) $\implies$ (5) $\implies$ (4) $\implies$ (3) is trivial (for (6) implies (5), note that for $p \in S(M)$, $p$ does not Kim-divide over $M$ trivially).

(3) $\implies$ (2) By Lemma 2.14, $C = \{M_i \mid i < |T|^+ \}$ is a club of $[M]^{|T|}$. As $T$ is NSOP$_1$, there is a stationary set $S \subseteq [M]^{|T|}$ such that $N \in S$ implies $p$ does not Kim-fork over $N$. Choose any $M_i \in C \cap S$ to obtain a contradiction.

(2) $\implies$ (1). Suppose $T$ has SOP$_1$ as witnessed by some formula $\varphi(x,y)$. Let $T^{sk}$ be a Skolemized expansion of $T$. Then $T^{sk}$ also has SOP$_1$ as witnessed by $\varphi(x,y)$. Thus by Proposition 2.2 we can find a formula $\varphi(x,y)$ and an array $\langle c_{i,j} \mid i < \omega, j < 2 \rangle$ such that $c_{i,0} \models \varphi(c_{i,0}, c_{i,1})$ for all
$i < \omega$, \{\varphi(x,c_i,0) \mid i < \omega\}$ is consistent and \{\varphi(x,c_i,1) \mid i < \omega\} is 2-inconsistent (all in $M^{\kappa}$). By Ramsey and compactness we may assume that $(\varphi_i \mid i < \omega)$ is indiscernible (with respect to $M^{\kappa}$) and extend this sequence to length $|T|^+$. For $i \leq |T|^+$, let $N_i = \text{dcl}(\varphi_i)$ (in $M^{\kappa}$). Then for every limit ordinal $\delta < |T|^+$, $\varphi(x,c_{\delta,1})$ Kim-divides over $N_i$ as the sequence $\langle c_{\delta,1} \mid \delta \leq j < |T|^+ \rangle$ is indiscernible and for all $\delta \leq j$, $\varphi_j \downarrow_{N_i} \varphi_{j+}$.

As $c_{\delta,1} \equiv \varphi_{\delta,0}$, it follows that $c_{\delta,1} \equiv N_a c_{\delta,0}$, and hence $\varphi(x,c_{\delta,0})$ also Kim-divides. Let $p \in S\left(N_{|T|^+}\right)$ be any complete type containing $\{\varphi(x,c_{\delta,0}) \mid \delta < \kappa\}$, which is possible as this partial type is consistent. The sequence $\langle N_\delta \mid \delta \in \text{lim}\left(\rangle\left|T|^+\right)\rangle$ is an increasing and continuous sequence of elementary substructures of $N_{|T|^+}$ of size $|T|$ with union $N_{|T|^+}$ witnessing that (2) fails. \hfill $\square$

**Corollary 3.10.** Suppose $T$ is NSOP$_1$, $M \models T$, $M \prec N$, and $p \in S(N)$. Then $p$ does not Kim-fork over $M$ iff for every $\kappa$ with $|T| \leq \kappa \leq |M|$, the set of elementary substructures of $M$ of size $\kappa$ over which which $p$ does not Kim-divide is a club subset of $[M]^{\kappa}$.

**Proof.** Suppose that $p$ does not Kim-fork over $M$. Let $A \subseteq M$ be any subset of $M$ of size $\kappa$ and apply Theorem 3.1 to the theory $T(A)$ obtained from $T$ by adding new constant symbols for the elements of $A$.

For the other direction, apply the left hand side with $\kappa = |T|$ and use Corollary 3.10. \hfill $\square$

**Corollary 3.11.** Suppose $T$ is NSOP$_1$ and $M \models T$. Then given any set $A$, there is a club $E \subseteq [M]^{|T|+|A|}$ such that $N \in E$ iff $A \downarrow_N^K M$.

**Proof.** Let $\kappa = |A| + |T|$. By Corollary 3.10 we know for each finite tuple $a$ from $A$, there is a club $E_a \subseteq [M]^{\kappa}$ so that $N \in E_a$ iff $a \downarrow_N^K M$. Let $E = \bigcap_{a \in A} E_a$. As $|A| \leq \kappa$ and the club filter on $[M]^{\kappa}$ is $\kappa^+$-complete (Fact 2.13(i)), $E$ is a club of $[M]^{\kappa}$. By the strong finite character of Kim-independence, we have $A \downarrow_N^K M$ iff $N \in E$. \hfill $\square$

3.4. A sample application.

**Proposition 3.12.** Suppose $T$ is NSOP$_1$ and $A \models T$. Given any set $C$, there is some $C' \supseteq C$ with $|C'| = |C| + |T|$ such that $C' \cap A$ is a model and $C' \downarrow_{A \cap C'}^K A$.

**Proof.** Let $\kappa = |C| + |T|$. Let $C_0 = C$ and, by Corollary 3.11 we may let $E_0 \subseteq [A]^{\kappa}$ be a club of elementary substructures of $A$ such that $N \in E_0$ implies $C_0 \downarrow_N^K A$. By induction, we will choose sets $C_i$, clubs $E_i \subseteq [A]^{\kappa}$, and models $X_i \prec A$ such that

1. $X_i \in \bigcap_{j \leq i} E_i$ and $C_i \cap A \subseteq X_i$,
2. $C_{i+1} = C_i \cup X_i$,
3. For all $N \in E_i$, we have $C_i \downarrow_N^K A$. 


Given \( \langle C_i, X_i, E_i \mid i \leq n \rangle \), let \( C_{n+1} = C_n \cup X_n \). By Corollary 3.11 we may let \( E_{n+1} \subseteq [A]^n \) be a club such that \( N \in E_{n+1} \) implies \( C_{n+1} \downarrow ^K_N A \). As
\[
\{ X \in [A]^n \mid C_{n+1} \cap A \subseteq X \}
\]
is a club of \([A]^n\), we may choose \( X_{n+1} \in \bigcap_{i \leq n+1} E_i \) containing \( C_{n+1} \cap A \). This completes the induction.

Let \( C_\omega = \bigcup_{i<\omega} C_i \). By construction, \( C_\omega \cap A = \bigcup_{i<\omega} X_i \). As \( i < j \) implies \( X_i \subseteq X_j \), and \( i \geq n \) implies \( X_i \in E_n \), it follows that \( C_\omega \cap A = \bigcup_{i \geq n} X_i \in E_n \) for all \( n \), as \( E_n \) is club. Also as each \( X_i \) is a model, this additionally shows that \( C_\omega \cap A \) is a model. Moreover, if \( c \in C_\omega \) is a finite tuple, there is some \( n \) so that \( c \in C_n \), hence \( c \downarrow ^K_{C_\omega \cap A} A \), by the choice of \( E_n \). Setting \( C' = C_\omega \), we finish. \( \square \)

3.5. Open questions.

**Question 3.13.** Is the dual of Lemma 3.6 also true? Namely, suppose that \( a \downarrow ^M N \) and \( \varphi(x,a) \) Kim-divides over \( M \) for \( \varphi(x,y) \in L(M) \). Then is it true that \( \varphi(x,a) \) Kim-divides over \( N \)?

If the answer to Question 3.13 is “yes”, then we have the following weak form of transitivity (note that a full version of transitivity does not hold, see [KR17, Section 9.2]).

**Claim 3.14.** (Weak form of transitivity) Suppose the answer is “yes”. Let \( M \prec N \). Suppose that \( a \downarrow ^M N \) and \( a \downarrow ^N B \). Then \( a \downarrow ^M B \).

**Proof.** Suppose not. Then by symmetry there is a formula \( \varphi(x,y) \) over \( M \) such that \( \varphi(b,a) \) holds for some \( b \in B \) and \( \varphi(x,a) \) Kim-divides over \( M \). However, since \( b \downarrow ^K_N A \), \( \varphi(x,a) \) does not Kim-divide over \( N \). By assumption we arrive at a contradiction. \( \square \)

**Question 3.15.** Does the weak form of transitivity hold in NSOP\(_1\) theories?

**Question 3.16.** The proof of (1) implies (6) in Theorem 1.1 relied heavily on symmetry of Kim-independence, whose proof assumes that the whole theory is NSOP\(_1\). However, a closer look at the proof of (1) implies (4) given in Section 3, or observing the proof using stationary logic given below, we see that for (1) implies (4), we only need that a particular formula \( \varphi(x,y) \) does not have an SOP\(_1\) array as in Fact 2.2. Can the same be said for (1) implies (6)?

**Question 3.17.** Is there a local counterpart to Lemma 3.7. Namely, under NSOP\(_1\), assume that \( \varphi(x,a) \) does not Kim-fork over \( M_i \) for \( i < \alpha \) an increasing union. Is it true that \( \varphi(x,a) \) does not Kim-fork over \( \bigcup_{i<\alpha} M_i \)?

**Question 3.18.** Is it true that \( T \) is NSOP\(_1\) if and only if for every \( M \models T \) and complete type \( p \in S(M) \), there is some \( N \prec M \) of cardinality \( \leq |T| \) such that \( p \) does not Kim-fork over \( N \)?
4. A proof of (1) implies (4) in Theorem 4.1 using stationary logic

4.1. More on clubs.

**Definition 4.1.** Suppose $\kappa$ is a cardinal and $A \subseteq B$, $S \subseteq [A]^\kappa$, and $T \subseteq [B]^\kappa$. We define $S^B \in [A]^\kappa$ and $T \upharpoonright A \in [A]^\kappa$ by

$$S^B = \{ Y \in [B]^\kappa \mid Y \cap A \in S \}$$

$$T \upharpoonright A = \{ X \in [A]^\kappa \mid \text{there is } Y \in T \text{ such that } X = Y \cap A \}.$$

**Fact 4.2.** [Jec13 Theorem 8.27] Suppose $\kappa$ is a cardinal, $A \subseteq B$, $S \subseteq [A]^\kappa$, and $T \subseteq [B]^\kappa$.

1. If $S$ is stationary in $[A]^\kappa$, then $S^B$ is stationary in $[B]^\kappa$.
2. If $T$ is stationary in $[B]^\kappa$, then $T \upharpoonright A$ is stationary in $[A]^\kappa$.

**Lemma 4.3.** Suppose $X$ is a set and $\lambda$ and $\kappa$ are cardinals with $\lambda \leq \kappa < |X|$. Suppose, moreover, we are given a stationary subset $S \subseteq [X]^\kappa$ and, for every $Y \in S$, a stationary subset $S_Y \subseteq [Y]^\lambda$. Then $S' = \bigcup_{Y \in S} S_Y$ is a stationary subset of $[X]^\lambda$.

**Proof.** Suppose $D \subseteq [X]^\lambda$ is a club. We must show $S' \cap D \neq \emptyset$. By Fact 2.13(3), there is a sequence of finitary functions $\overrightarrow{f} = \langle f_i \mid i < \lambda \rangle$ where for all $i < \lambda$, $f_i : X^{n_i} \to X$ and the set $C_{\overrightarrow{f}} \subseteq [X]^\lambda$ of $\lambda$-sized subsets of $X$ closed under $\overrightarrow{f}$ is a club with $C_{\overrightarrow{f}} \subseteq D$. The subsets of $X$ of size $\kappa$ closed under $\overrightarrow{f}$ form a club $C_{\overrightarrow{f}}^\kappa \subseteq [X]^\kappa$, hence $C_{\overrightarrow{f}}^\kappa \cap S \neq \emptyset$. Fix $Y \in C_{\overrightarrow{f}}^\kappa \cap S$. Define a sequence of functions $\overrightarrow{g} = \langle g_i \mid i < \lambda \rangle$ by $g_i = f_i \upharpoonright Y^{n_i}$ for all $i < \lambda$. This definition makes sense as $Y$ is closed under the functions $f_i$, so that $C_{\overrightarrow{f}} \cap [Y]^\lambda = C_{\overrightarrow{g}}$, the subsets of $Y$ closed under $\overrightarrow{g}$, hence is a club of $[Y]^\lambda$. Therefore $C_{\overrightarrow{g}} \cap [Y]^\lambda \cap S_Y \neq \emptyset$. In particular, this shows $D \cap S' \neq \emptyset$, which completes the proof. \qed

The club filter on $[X]^\omega$ was characterized by Kueker in terms of games of length $\omega$ [Kue72]. The natural analogue for games of length $\lambda$ determines a filter on $\mathcal{P}_{\lambda^+}(X)$, which, in general, differs from the club filter. In generalizing stationary logic to quantification over sets of some uncountable size $\lambda$, it turns out that this filter provides a more useful analogue to the club filter on $[X]^\omega$ than the club filter on $[X]^\lambda$.

**Definition 4.4.** Suppose $X$ is a set and $\lambda$ is a regular cardinal. Given a subset $F \subseteq \mathcal{P}_{\lambda^+}(X)$, we define the game $G(F)$, to be the game of length $\lambda$ where Players I and II alternate playing an increasing $\lambda$ sequence of elements of $\mathcal{P}_{\lambda^+}(X)$. In this game, Player II wins if and only if the union of the sets played is in $F$. The filter $D_\lambda(X)$ is defined to be the filter generated by the sets $F \subseteq \mathcal{P}_{\lambda^+}(X)$ in which Player II has a winning strategy in $G(F)$. We say $Y \subseteq \mathcal{P}_{\lambda^+}(X)$ is $D_\lambda(X)$-stationary if $Y$ intersects every set in $D_\lambda(X)$.

It is easy to check that every club $C \subseteq [X]^\lambda$ is an element of $D_\lambda(X)$ and, therefore, that every $S \subseteq [X]^\lambda$ that is $D_\lambda(X)$-stationary is also stationary with respect to the usual club filter on $[X]^\lambda$. 
It was remarked in [MS86] that if \( \lambda = \lambda^{<\lambda} \), then \( D_\lambda (\lambda^+) \) is just the filter generated by the clubs of \( \lambda^+ \) intersected with the set of ordinals of cofinality \( \lambda \) (considered as initial segments of \( \lambda^+ \)). More precisely, we have the following fact. (We omit its proof since it is not necessary for the rest.)

**Fact 4.5.** Suppose \( \lambda \) is an infinite cardinal and write \( S^\lambda_\lambda \) for the stationary set \( \{ \alpha < \lambda^+ \mid \text{cf}(\alpha) = \lambda \} \).

1. If \( C \subseteq \lambda^+ \) is a club, then \( C \cap S^\lambda_\lambda \in D_\lambda (\lambda^+) \).
2. Suppose \( \lambda = \lambda^{<\lambda} \). Then \( D_\lambda (\lambda^+) \) is generated by sets of the form \( C \cap S^\lambda_\lambda \), where \( C \subseteq \lambda^+ \) is a club.

**Lemma 4.6.** Suppose \( X \) is a set of size \( \lambda^+ \), and \( \{ \lambda \mid \alpha < \lambda^+ \} \) is an increasing and continuous sequence from \( P_{\lambda^+} (X) \) with union \( X \). Suppose \( S \subseteq P_{\lambda^+} (X) \) is \( D_\lambda (X) \)-stationary. Then the set \( S_* = \{ \alpha < \lambda^+ \mid \text{cf}(\alpha) = \lambda, X_\alpha \subseteq S \} \) is a stationary subset of \( \lambda^+ \).

**Proof.** As \( |X| = \lambda^+ \), we may assume \( X = \lambda^+ \). Let \( C \subseteq \lambda^+ \) consist of the ordinals \( \alpha < \lambda^+ \) such that \( X_\alpha = \alpha \). This set is easily seen to be a club.

Let \( C_* \subseteq \lambda^+ \) be a club. We must show \( C_* \cap S_* \neq \emptyset \). By Fact 4.5(1), \( C_* \cap C \cap S^\lambda_\lambda \in D_\lambda (X) \), hence \( (S^\lambda_\lambda \cap C \cap C_*) \cap S \neq \emptyset \). Pick \( Y \) in this intersection. Then by definition of \( C \), \( Y = X_\alpha = \alpha \) for some \( \alpha \in S^\lambda \). As \( X_\alpha \subseteq S \), we have \( \alpha \in S_* \). This shows \( S_* \cap C_* \neq \emptyset \). \( \Box \)

**Lemma 4.7.** Suppose \( A \subseteq B \) and \( S \subseteq P_{\lambda^+} (B) \) is \( D_\lambda (B) \)-stationary. Then the set \( S \mid A = \{ X \cap A \mid X \in S \} \) is \( D_\lambda (A) \)-stationary.

**Proof.** It is enough to show that if \( F \in D_\lambda (A) \) then \( F^B = \{ X \in P_{\lambda^+} (B) \mid X \cap A \subseteq F \} \in D_\lambda (B) \).

We may assume that there is some winning strategy \( f \) for Player II in the game \( G (F) \), since \( F \in D_\lambda (A) \). That is, the function \( f \) is defined so that if, at stage \( i \), Player I has played \( \langle A_j \mid j \leq i \rangle \) then \( f (\langle A_j \mid j \leq i \rangle) \) outputs the play for Player II.

Now we will define a winning strategy for Player II in the game \( G (F^B) \). At stage \( i \), if Player I has played \( \langle A_j \mid j \leq i \rangle \), Player II plays \( B_i = A_i \cup f (\langle A_j \cap A \mid j \leq i \rangle) \). As the rules of the game require that the sets are increasing, we have

\[
A_i \cap A \subseteq f (\langle A_j \cap A \mid j \leq i \rangle) \subseteq A,
\]

hence \( B_i \cap A = f (\langle A_j \cap A \mid j \leq i \rangle) \). It follows that

\[
\begin{array}{c|ccc}
& A_0 \cap A & A_1 \cap A & \cdots \\
I & A_0 \cap A & A_1 \cap A & \cdots \\
II & B_0 \cap A & B_1 \cap A & \cdots \\
\end{array}
\]

is a play according to \( f \) in \( G (F) \). Therefore,

\[
\left( \bigcup_{i < \lambda} A_i \cup B_i \right) \cap A = \bigcup_{i < \lambda} (A_i \cap A) \cup (B_i \cap A) \in F,
\]
which shows \( \bigcup_{i<\lambda} A_i \cup B_i \in F^B \). We have shown that Player II has a winning strategy in \( G(F^B) \) so \( F^B \in D_\lambda(B) \). ☐

4.2. **Stationary logic.** The stationary logic \( L(aa) \) was introduced in [She75] (where it was called \( L(Q^*_n) \)). The logic is defined as follows: given a first-order language \( L \), expand the language with countably many new unary predicates \( \{ S_i \mid i < \omega \} \) and a new quantifier \( aa \). The formulas of \( L \) in \( L(aa) \) are the the smallest class containing the first-order formulas of \( L \), closed under the usual first-order formation rules together with the rule that if \( \varphi \) is a formula, then \( (aaS_i) \varphi \) is also a formula, for any new unary predicate \( S_i \). Satisfaction is defined as usual, together with the rule that \( M \models (aaS) \varphi(S) \) if and only if \( M \models \varphi(S) \) when \( S^M = X \) for “almost all” \( X \in [M]^\omega \)—that is, \( \{ X \in [M]^\omega \mid S^M = X \text{ then } M \models \varphi(S) \} \) contains a club of \([M]^\omega\). We define the quantifier \( \text{stat} \) dually: \( M \models (\text{stat}S) \varphi(S) \) if and only if \( M \models \neg(aaS) \neg \varphi(S) \). Note that \( M \models (\text{stat}S) \varphi(S) \) if and only if \( \{ X \in [M]^\omega \mid S^M = X \text{ then } M \models \varphi(S) \} \) is stationary. Given an \( L \)-structure \( M \), we write \( \text{Th}_{aa}(M) \) for the set of \( L(aa) \)-sentences satisfied by \( M \). We refer the reader to [BKM78] Section 1 for a detailed treatment of stationary logic.

Later work by Mekler and the third-named author extended stationary logic, which quantifies over countable sets, to a logic that permits quantification over sets of higher cardinality [MSS86]. For \( \lambda \) a regular cardinal, the logic \( L(aa^\lambda) \) is defined analogously to \( L(aa) \), with semantics defined so that \( M \models (aa^\lambda S) \varphi(S) \) if and only if \( \{ X \in [M]^\lambda \mid S^M = X \text{ then } M \models \varphi(S) \} \in D_\lambda(M) \). The quantifier \( \text{stat}^\lambda \) is also understood dually: \( M \models (\text{stat}^\lambda S) \varphi(S) \) if and only if \( M \models \neg(aa^\lambda S) \neg \varphi(S) \). If \( T \) is an \( L(aa) \)-theory, one obtains an \( L(aa^\lambda) \)-theory by replacing the quantifier \( aa \) with \( aa^\lambda \).

We call this theory the \( \lambda \)-interpretation of \( T \). By working with \( D_\lambda(M) \) instead of the full club filter on \([M]^\lambda\), one is able to relate satisfiability of an \( L(aa) \)-theory to the satisfiability of its \( \lambda \)-interpretation. Below, the “moreover” clause about \( \lambda \)-saturation is not stated in [MSS86], but is immediate from the proof.

Fact 4.8. [MSS86] Theorem 1.3 | Suppose \( \lambda = \lambda^<\lambda \) and \( T \) is a consistent \( L(aa) \)-theory of size at most \( \lambda \). Then the \( \lambda \)-interpretation of \( T \) has a model of size at most \( \lambda^+ \). In fact, there is such a model which is, moreover, \( \lambda \)-saturated.

The following easy observation is also useful:

Lemma 4.9. Suppose \( \varphi \) is a first-order formula, possibly with parameters from \( M \) and \( |\varphi(M)| > \aleph_0 \). Then if \( M' \models \text{Th}_{aa}(M) \) in the \( \lambda \)-interpretation, then \( |\varphi(M')| > \lambda \).

Proof. Suppose not. Then \( \{ S \in [M']^\lambda \mid \varphi(M') \subseteq S \} \) is a club of \([M']^\lambda\) hence an element of \( D_\lambda(M') \). Therefore \( M' \models (aa^\lambda S) \forall x (\varphi(x) \rightarrow S(x)) \). As \( M' \models \text{Th}_{aa}(M') \) in the \( \lambda \)-interpretation, \( M \models (aaS) \forall x (\varphi(x) \rightarrow S(x)) \), so \( \varphi(M) \) is countable, a contradiction. ☐
4.3. Reduction to a countable language.

Remark 4.10. Suppose that $T$ is an $NSOP_1$ theory in the language $L$. Suppose that $M \models T$ and $\varphi(x,y)$ is any formula. Then for any language $L' \subseteq L$ containing $\varphi$, and any $b \in M$, $\varphi(x,b)$ Kim-divides over $M$ in $L$ iff $\varphi(x,b)$ Kim-divides over $M' := M \upharpoonright L'$ (in the sense of $T \upharpoonright L'$).

Indeed, this follows from Kim’s lemma for Kim-dividing (Fact 2.7) and the fact that if $\bar{b}$ is a coheir sequence in $L$ over $M$ starting with $b$, then it is also in $L'$.

Lemma 4.11. Suppose $T$ is an $NSOP_1$ theory in the language $L$, $M \models T$ and for some $p \in S(M)$, the set

$$S = \{N \prec M \mid |N| = |T|, p \text{ Kim-divides over } N\}$$

is stationary in $[M]^{|T|}$. Then there is a countable sublanguage $L' \subseteq L$ and a stationary set $S' \subseteq [M]^{\omega}$ so that, setting $p' = p \upharpoonright L$, we have that for all $N' \in S'$, $p'$ Kim-divides over $N'$.

Proof. For each $N \in S$, choose some $\varphi_N(x; b_N) \in p$ such that $\varphi(x; b_N)$ Kim-divides over $N$. By Fact 2.13(1), the club filter on $[M]^{|T|}$ is $|T|^+$-complete, so for any a partition of a stationary set into $|T|$ many pieces, we may find some piece which is stationary. Therefore we may assume there is some formula $\varphi$ so that $\varphi_N(x; b_N) = \varphi(x; b_N)$ for all $N \in S$. Let $L'$ be any countable sublanguage of $L$ containing $\varphi$. By Remark 4.10 and (the proof of) Theorem 3.4, for each $N \in S$, there is a club $C_N \subseteq [N]^{\omega}$ of countable $L'$-elementary substructures over which $\varphi(x; b_N)$ Kim-divides. By Lemma 4.13, $S' = \bigcup_{N \in S} C_N$ is a stationary subset of $[M]^{\omega}$. By definition of $S'$, if $N' \in S'$, then there is some $\varphi(x; b_{N'}) \in p$ such that $\varphi(x; b_{N'})$ Kim-divides over $N'$.

4.4. Stretching.

Lemma 4.12. Suppose $T$ is $NSOP_1$, $|T| = \aleph_0$, $M \models T$, and there is $p \in S(M)$ so that the set

$$S_0 = \{N \prec M \mid |N| = \aleph_0 \text{ and } p \text{ Kim-divides over } N\}$$

is stationary. Then, given any regular uncountable cardinal $\lambda = \lambda^{<\lambda}$, there is a model $M' \models T$, $|M'| = \lambda^+$, a formula $\varphi(x;y)$, and a type $p_*$ over $M'$ so that

$$S'_0 = \{N' \prec M' \mid |N'| = \lambda, \text{ there is } \varphi(x; a'_N) \in p_* \text{ that Kim-divides over } N'\}$$

is $D_\lambda(M')$-stationary.

Proof. As no type Kim-divides over its domain, it follows that $M$ is uncountable. For each $N \in S_0$, there is some formula $\varphi_N(x; a_N) \in p$ and $k_N < \omega$ so that $\varphi(x; a_N)$ $k_N$-Kim-divides over $N$ via a Morley sequence in some global $N$-finitely satisfiable type. As the club filter on $[M]^{\omega}$ is $\aleph_1$-complete, Fact 2.13(1), there are $\varphi$ and $k$ so that for some stationary $S \subseteq S_0$, we have $N \in S$ implies $\varphi_N(x; a_N) = \varphi(x; a_N)$ and $k_N = k$. 
Let \( l = |a_N| \) for all \( N \in S \) and let \( \hat{M} \) be an \( \aleph_1 \)-saturated elementary extension of \( M \). Let \( \chi \) be a sufficiently large regular cardinal so that all objects of interest are contained in \( H(\chi) \). In particular, we may choose \( \chi \) so that \( \hat{M}, \omega, \hat{M}, T, L, \) and \( p \) are contained in \( H(\chi) \), together with a bijection to \( \omega \) witnessing the countable cardinality of \( L \), and we consider the structure

\[
\mathcal{H} = \left( H(\chi), \in, M, \hat{M}, L, T, p \right).
\]

By Fact 4.2(1), the set \( S_* = \{ X \in [\mathcal{H}]^\omega \mid X \cap M \in S \} \) is a stationary subset of \([\mathcal{H}]^\omega\).

Let \( \Phi(X) \) be the formula in the language of \( \mathcal{H} \) together with a new predicate \( X \) that naturally asserts: there exists \( c \in M^l \), such that \( \varphi(x;c) \in p \) and such that there exists \( f \in \omega(\hat{M}^l) \) such that:

- \( X \cap M \) is an elementary substructure of \( M \).
- \( f = (f_i \mid i < \omega) \) is an \( (X \cap M) \)-indiscernible sequence such that \( \text{tp}(f_i / (X \cap M) f_{<i}) \) is finitely satisfiable in \( (X \cap M) \).
- \( f(0) = c \).
- \( \{ \varphi(x;f_i) \mid i < \omega \} \) is \( k \)-inconsistent.

We first show the following:

**Claim.** \( \mathcal{H} \models (\text{stat}X) \Phi(X) \).

Proof of claim. As \( S_* \) is stationary, it suffices to show that if \( X \in S_* \) and \( S^\mathcal{H} = S_* \), then \( \mathcal{H} \models \Phi(S) \).

Recall that if \( X \in S_* \), then \( X \cap M \in S \) so \( X \cap M \) is a countable elementary substructure of \( M \), and \( \varphi(x;a_{X \cap M}) \) is a formula in \( p \) that \( k \)-divides over \( X \cap M \). As \( \hat{M} \) is \( \aleph_1 \)-saturated, there is a coheir sequence \( (a_i \mid i < \omega) \) over \( X \cap M \) in \( \hat{M} \) with \( a_0 = a_{X \cap M} \) and \( \{ \varphi(x;a_i) \mid i < \omega \} \) \( k \)-inconsistent. Put \( c = a_0 \) and let \( f \in [\omega(\hat{M}^l)] \) be defined by \( f_i = a_i \), we easily have (1)-(4) satisfied, proving the claim. \( \square \)

By Fact 4.3 there is \( \mathcal{H}' \) which is a model of the \( \lambda \)-interpretation of \( \text{Th}_{\text{ea}}(\mathcal{H}) \) with \( |\mathcal{H}'| = \lambda^+ \), \( \mathcal{H}' = (\mathcal{H}', \epsilon', M', \hat{M}', L', p') \). As \( L \) and \( T \) are coded by natural numbers, the language \( L \) is contained in \( L' \) and thus the definable set \( \{ x \in \mathcal{H}' \mid x \in \hat{M}' \} \) may be regarded as the domain of an \( L' \)-structure whose reduct to \( L \) is a model of \( T \) and likewise for \( M \). Moreover \( M' \prec_L \hat{M}' \) and \( |M'| = \lambda^+ \), by Lemma 4.9. As \( \mathcal{H}' \models (\text{stat}^\lambda X) \Phi(X) \), there is a \( D_\lambda(\mathcal{H}') \)-stationary set \( S'_0 \subseteq [\mathcal{H}']^\lambda \) witnessing this. Let \( S' = S'_0 \upharpoonright M' \)—i.e. \( S' = \{ X \cap M' \mid X \in S'_0 \} \). By Lemma 4.7 \( S' \) is \( D_\lambda(M') \)-stationary. Let \( p_* = p' \upharpoonright L \). To conclude the proof, it suffices to establish the following:

**Claim.** \( p_* \) is a type over \( M' \) and if \( N \in S' \), then \( p_* \) \( k \)-Kim-divides over \( N \) via some \( \varphi(x;a'_{N}) \in p_* \).

Proof of claim. It is clear that \( p_* \) is a consistent type over \( M' \). Now fix \( N \in S' \). By definition of \( S' \), \( N = X \cap M' \) for some \( X \in [\mathcal{H}']^\lambda \) such that \( \mathcal{H}' \models \Phi(S) \) when \( S^{\mathcal{H}'} = X \). It follows that for some \( b \in M' \), there is an \( N \)-indiscernible sequence \( \langle b_i \mid i \in I \rangle \) with \( b_0 = b \), such that \( \text{tp}(b_i/Nb_{<i}) \)
is finitely satisfiable in $N$, $\varphi(x;b_0) \in p'$ and $\{\varphi(x;b_i) \mid i \in I\}$ is $k$-inconsistent, where $I$ denotes the (possibly non-standard) natural numbers of $\mathcal{H}'$. By indiscernibility, $(b_i \mid i \in I)$ is a Morley sequence over $N$ in a global $N$-finitely satisfiable type, which shows $\varphi(x;b_0)$ $k$-Kim-divides over $N$. This completes the proof. 

4.5. The main lemma.

**Lemma 4.13.** (Main Lemma) Suppose $T$ is a complete theory, $M \models T$ is a model with $|M| \geq |T|$, and for some $p \in S(M)$, the set

$$S_0 = \left\{ N \in [M]^{|T|} \mid N \prec M, p \text{ Kim-divides over } N \right\}$$

is stationary. Then $T$ has SOP$_1$.

*Proof.* Towards contradiction suppose $T$ is NSOP$_1$. By Lemma 4.11 there is a countable sub-language $L' \subseteq L$ and a stationary set $S'_0 \subseteq [M]^{\omega}$ such that if $p' = p \upharpoonright L'$ then for all $N \in S$, $N \prec_{L'} M$ and $p'$ Kim-divides over $N$. Therefore, we may assume for the rest of the proof that $T$ is countable.

By forcing with the LÃ©vy collapse $\text{Coll}(\lambda^+, 2\lambda)$ for a sufficiently large cardinal, we may assume there is some an uncountable cardinal $\kappa = \kappa^<\kappa$, namely $\kappa = \lambda^+$, while preserving the situation. By Lemma 4.12 there is a model $M' \models T$ with $|M'| = \kappa^+$ and a type $p'$ over $M'$ so that

$$S''_0 = \left\{ N \in [M']^\kappa \mid N \prec M' \text{ and some } \varphi(x;c_N) \in p' \text{ Kim-divides over } N \right\}$$

is $D_\kappa(M')$-stationary. Let $\langle M_\alpha \mid \alpha < \kappa^+ \rangle$ be a continuous and increasing sequence of $\kappa$-sized elementary substructures of $M'$ with union $M'$. The set $S = \left\{ \alpha < \kappa^+ \mid \text{cf}(\alpha) = \kappa, M_\alpha \in S''_0 \right\}$ is a stationary subset of $\kappa^+$ by Lemma 4.15. By intersecting with a club, we may also assume that for all $\alpha \in S$, $M_\alpha$ contains $c_{M_\beta}$ for all $\beta \in \alpha \cap S$.

From here, the proof closely follows the proof of [KR17, Theorem 4.5]. For each $\alpha \in S$, let $c_\alpha$ denote $c_{M_\alpha}$ and let $r_\alpha$ be a global $M_\alpha$-finitely satisfiable type extending $tp(c_\alpha/M_\alpha)$. By reducing $S$, we may assume that there is some $k < \omega$ such that $r$ witnesses that $\varphi(x;c_\alpha)$ $k$-Kim-divides over $M_\alpha$. For each $\alpha \in S$, apply Lemma 4.18(2) to choose a countable $N_\alpha \prec M_\alpha$ such that $r^{o\omega}_{\alpha}|N_\alpha$ is the type of a Morley sequence in some global $N_\alpha$-finitely satisfiable type and hence such that $\varphi(x;c_\alpha)$ $k$-Kim-divides over $N_\alpha$. Define $\rho : S \to \kappa^+$ by $\rho(\alpha) = \min \{ \beta < \alpha \mid N_\alpha \subseteq M_\beta \}$. This is well-defined and pressing down on $S$ as $\kappa$ is regular and uncountable. By Fodor’s lemma, there is $S' \subseteq S$ such that $\rho$ is constant on $S'$, say with constant value $\beta_0$. As $|M_{\beta_0}| = \kappa$, there are $\leq \kappa^{|\beta_0|} = \kappa$ many choices for $N_\alpha \subseteq M_{\beta_0}$ so there is a stationary $S'' \subseteq S'$ and $N'_0$ so that $N_\alpha = N'_0$ for all $\alpha \in S''$. As there are $\leq 2^{\kappa_0} \leq \kappa$ choices for $r^{o\omega}_{\alpha}|N'_0$, there is a stationary $S_* \subseteq S''$ such
that \( r_{\omega}^{\omega}\mid_{N_0'} \) is constant, with value \( s_0^{\omega}\mid_{N_0'} \) for some global coheir \( s_0 \) over \( N_0' \). Let \( \delta_0 = \min S_0, e_0 = c_{\delta_0} \).

Repeating this process \( \omega \) many times, we find an increasing sequence \( \langle \delta_i \mid i < \omega \rangle \) of ordinals in \( \kappa^+ \), an increasing sequence of models \( \langle N_i' \mid i < \omega \rangle \), \( e_i \in M' \) for \( i < \omega \) and global \( N_i' \)-finitely satisfiable types \( s_i \) such that:

\( N_i' \) contains \( e_{<i}, \varphi(x; e_j) \) is \( k \)-Kim-dividing over \( N_i' \) for every \( i \leq j \), \( s_i \) is a global coheir over \( N_i' \) extending \( \text{tp}(e_i/N_i') \) and \( e_j \equiv_{N_i'} e_i \) for all \( j \geq i \). In addition, \( s_j^{\omega}\mid_{N_i'} = s_i^{\omega}\mid_{N_i'} \) for all \( j \geq i \).

Denote \( \overline{e} = \langle e_i : i < \omega \rangle \). Note that \( \{ \varphi(x; e_i) : i < \omega \} \) is a subset of \( p' \), hence consistent. Now, exactly as in the claim in the proof of \([\text{KR17, Theorem 4.5}]\), we can show that if \( i_0 < \ldots < i_{n-1} < \omega \) and for each \( j < n, f_j \models s_j \models \langle N_i' \mid \overline{e}_{<j} \rangle_{f_{<j}} \) then \( e_{i_j} \equiv_{c_{<j} f_{<j}} f_j \) for all \( j < n \) and \( \{ \varphi(x; f_j) : j < n \} \) is \( k \)-inconsistent. By compactness, we can find an array \( \langle (c_{i,0}, c_{i,1}) : i < \omega \rangle \) so that \( \{ \varphi(x, c_{i,0}) : i < \omega \} \) is consistent, \( \{ \varphi(x, c_{i,1}) : i < \omega \} \) is \( k \)-inconsistent, and \( c_{i,0} \equiv_{c_{<i}, c_{i,1}} \) for all \( i < \omega \). By Fact \([2.2] \) we obtain SOP\( _1 \), a contradiction.

\[ \square \]

**Corollary 4.14.** Theorem \([1.1] (1) \Rightarrow (4) \) holds.

5. **Dual local character**

**Definition 5.1.** \( (T \) any theory) Say that a formula \( \varphi(x, a) \) strongly Kim-divides over a model \( M \) if for every global \( M \)-invariant type \( q \supseteq \text{tp}(a/M) \), \( \varphi(x, a) \) Kim-divides over \( M \) via \( q \).

**Remark 5.2.** By Fact \([2.7] \) strong Kim-dividing = Kim-dividing iff \( T \) is NSOP\( _1 \).

**Definition 5.3.** A dual type \( (\) over \( A \)\) in \( x \) is a set \( F \) of \( (A-)\)definable sets in \( x \) such that for some \( k < \omega \), it is k-inconsistent. Say that \( F \) dually divides over a model \( N \), if every \( X \in F \) which is not definable over \( N \) divides over \( N \). Similarly define when \( F \) dually Kim-divides over \( N \) and when \( F \) strongly dually Kim-divides over \( N \).

**Theorem 5.4.** The following are equivalent for a complete theory \( T \).

1. \( T \) is NSOP\( _1 \).
2. There is no continuous increasing sequence of \( |T| \)-sized models \( \langle M_i \mid i < |T|^+ \rangle \) with union \( M \) and a dual type \( F \) over \( M \) such that \( F \upharpoonright M_{i+1} \) does not strongly dually Kim-divide over \( M_i \) for all \( i < |T|^+ \).
3. Assume that \( M \models T \) and \( F \) a dual type over \( M \). Then there is a stationary subset \( S \) of \( [M \upharpoonright |T|] \) such that if \( N \in S \) then \( N < M \) and \( F \) strongly dually Kim-divides over \( N \).
4. (Dual local character) Same as (3) but \( S \) is a club.

**Proof.** The proof is essentially dualizing or inverting the proof (using stationary logic) of Theorem \([1.1] (1) \Rightarrow (4) \), but we go into some details.
(1) $\implies$ (4). We follow the proof of “(1) implies (4)” of Theorem 1.1 as described in Section 4. Namely, assume that (2) fails. This means that there is a stationary subset $S$ of $[M]^{|T|}$ such that if $N \in S$ then $N \prec M$ and there is some $X \in F$ which is not definable over $N$ but still does not Kim-divide over $N$. Using the same proof as in Lemma 4.11 we may assume that the language $L$ is countable and that there is a single formula $\varphi(x,y)$ with $|x| = n$ such that if $N \in S$ then for some $b \in M \setminus N$, $\varphi(x,b)$ does not Kim-divide over $N$ (and $\varphi(x,b)$ is not $N$-definable). Now we repeat the same procedure as in Lemma 4.12. Thus, for a regular uncountable cardinal $\lambda = \lambda^{<\lambda}$, we get a model $M' \models T$, $|M'| = \lambda^+$, a formula $\varphi(x,y)$, and a $k$-inconsistent family $F_\ast$ of definable formulas over $M'$ so that

$$S'_0 = \{ N' \prec M' \mid |N'| = \lambda, \exists \varphi(x,a'_N) \in F_\ast \text{ not } N'-\text{definable and does not Kim-divide over } N' \}$$

is $D_\lambda(M')$-stationary. Now we repeat the proof of Lemma 4.13. The contradiction we will arrive at the end will be the same contradiction, but the roles of the sequences $e_i$ and $f_j$ are reversed. Now $\{ \varphi(x,e_i) \mid i < \omega \}$ is $k$-inconsistent (note that the formulas $\varphi(x,e_i)$ must define distinct definable sets from $F_i$) and $\langle \varphi(x,f_j) \mid j < n \rangle$ is consistent.

(4) $\implies$ (3) $\implies$ (2) is exactly as in the proof of Lemma 3.9. The proof of (2) $\implies$ (1) is just dualizing the proof of “(2) implies (1)” in Theorem 3.9 in the sense that the sequences $\langle c_{i,0} \mid i < \omega \rangle$ and $\langle c_{i,1} \mid i < \omega \rangle$ exchange places. □

Question 5.5. Is there a proof of the dual local character which does not use stationary logic? Such a proof may reveal some new properties of Kim-dividing.

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