Lp-Solvability of Nonlocal Parabolic Equations with Spatial Dependent and Non-Smooth Kernels

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Abstract. In this paper we prove the optimal Lp-solvability of nonlocal parabolic equation with spatial dependent and non-smooth kernels.

1. Introduction

In this paper we are considering the Lp-estimate of the following nonlocal operator:

\[ L^a f = \int_{\mathbb{R}^d} [f(x + y) - f(x) - y^{(\alpha)} \cdot \nabla f(x)]a(x, y)|y|^{-d-\alpha} \, dy, \]

where \( \alpha \in (0, 2) \), \( a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+ \) is a measurable function and 
\[ y^{(\alpha)} := 1_{a \in (1,2)}y + 1_{a=1}y1_{|y| \leq 1}. \]

When \( a(x, y) \) is smooth and 0-homogenous in \( y \), or \( a(x, y) = a(y) \) is independent of \( x \), the Lp-estimates for this type of operators have been studied by Mikulevicius-Pragarauskas [13] and Dong-Kim [8] (see also [19]). However, for nonlinear applications, the smoothness and spatial-independence assumptions are usually not satisfied.

Let us now look at a nonlinear example. Consider the following variational integral appeared in nonlocal image and signal processing [9]:

\[ V(\theta) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(\theta(x) - \theta(y))\kappa(x - y)|y - x|^{-d-\alpha} \, dx \, dy, \quad \alpha \in (0, 2), \]

where \( \phi : \mathbb{R} \to \mathbb{R}^+ \) is an even convex \( C^2 \)-function and \( \kappa(-x) = \kappa(x) \). Assume that \( \phi \) and \( \kappa \) satisfy that for some \( \Lambda > 0 \),

\[ \phi(0) = 0, \quad \Lambda^{-1} \leq \phi''(x) \leq \Lambda, \]

and

\[ \Lambda^{-1} \leq \kappa(x) \leq \Lambda. \]

The Euler-Lagrange equation corresponding to \( V(\theta) \) is given by

\[ \int_{\mathbb{R}^d} \phi'(\theta(t, y) - \theta(t, x))\kappa(y - x)|y - x|^{-d-\alpha} \, dy = 0. \]

In [6], Caffarelli, Chan and Vasseur firstly considered the following time dependence problem:

\[ \partial_t \theta(t, x) = \int_{\mathbb{R}^d} \phi'(\theta(y) - \theta(x))\kappa(y - x)|y - x|^{-d-\alpha} \, dy, \]

and proved that for any \( \theta_0 \in H^{1,2} \), there exists a unique global classical \( C^{1,2} \)-solution to the above equation with \( \theta(0, \cdot) = \theta_0 \) in the \( L^2 \)-sense. The existence of weak solutions with non-increasing

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energy can be deduced by the standard energy argument. To address the regularity problem, they followed the classical idea of De Giorgi and considered the following linearized equation

\[ \partial_t w(t, x) = \int_{\mathbb{R}^d} \phi''(\theta(t, y) - \theta(t, x))(w(t, y) - w(t, x))\kappa(y - x)|y - x|^{-d-\alpha} \, dy, \]

where \( w(t, x) = \nabla \theta(t, x) \). If we set

\[ \hat{k}(t, x, y) = \phi''(\theta(t, y) - \theta(t, x))\kappa(y - x) = \phi'' \left( (y - x) \cdot \int_0^1 w(t, x + s(y - x)) \, ds \right) \kappa(y - x), \]

then, since \( \phi'' \) is an even function, we have

\[ \hat{k}(t, x, y) = \hat{k}(t, y, x), \]

and equation (2) is understood in the weak sense: for all \( \eta \in C_0^\infty(\mathbb{R}^d) \),

\[ \int_{\mathbb{R}^d} \partial_t w(t, x) \eta(x) \, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w(t, y) - w(t, x)) (\eta(y) - \eta(x)) \kappa(t, x, y)|y - x|^{-d-\alpha} \, dy \, dx. \]

Clearly, if we let

\[ a(t, x, y) := \hat{k}(t, x, x + y), \]

then equation (2) becomes

\[ \partial_t w(t, x) = \int_{\mathbb{R}^d} (w(t, x + y) - w(t, x))a(t, x, y)|y|^{-d-\alpha} \, dy. \]

Notice that \( a(t, x, y) \) is usually not smooth apriori in \( x \) and \( y \). This type of equation is our main motivation.

This paper is organized as follows: In Section 2, we give some necessary spaces. In Section 3, we prove some estimates of nonlocal integral operators. In Section 4, the linear nonlocal parabolic equation is studied. In a forthcoming paper, we shall use the result obtained in this paper to study the stochastic differential equations with spatial dependence jump-diffusion coefficients (cf. [18]).

Convention: Throughout this paper, we shall use \( C \) with or without subscripts to denote an unimportant constant.

2. Preliminaries

In this section we introduce some necessary spaces of Dini-type (cf. [15] p.30, (25)). Let \( \mathcal{A}_0 \) be the space of all real bounded measurable functions \( a : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) with finite norm

\[ \|a\|_{\mathcal{A}_0} := \sup_{x, y \in \mathbb{R}^d} |a(x, y)| + \int_0^1 \frac{\omega_a^{(0)}(r)}{r} \, dr < +\infty, \]

where

\[ \omega_a^{(0)}(r) := \sup_{x \in \mathbb{R}^d} \sup_{|y| \leq r} |a(x, y) - a(x, 0)|. \] (3)

Let \( \mathcal{A}_1 \subset \mathcal{A}_0 \) be the subspace with finite norm

\[ \|a\|_{\mathcal{A}_1} := \|a\|_{\mathcal{A}_0} + \int_0^1 \frac{\omega_a^{(1)}(r)}{r} \, dr < +\infty, \]

where

\[ \omega_a^{(1)}(r) := \sup_{|x - x'| \leq r} |a(x, 0) - a(x', 0)|. \] (4)
Let \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For \( p > 1 \) and \( \beta \geq 0 \), let \( \mathbb{H}^\beta_p := (I - \Delta)^{-\frac{\beta}{2}}(L^p) \) be the Bessel potential space with the norm

\[
\|f\|_{\mathbb{H}^\beta_p} := \|(I - \Delta)^{\frac{\beta}{2}}f\|_p \sim \|f\|_p + \|(I - \Delta)^{\frac{\beta}{2}}f\|_p,
\]

and for \( q \in [1, \infty] \), let \( \mathbb{B}^\beta_q \) be the Besov space defined by

\[
\mathbb{B}^\beta_q := (L^p, \mathbb{H}^k_q)_{\frac{\beta}{p}, q}.
\]

where \( k \in \mathbb{N} \) and \( \beta < k \), and \( (\cdot, \cdot)_{p,q} \) stands for the real interpolation space. Let us write

\[
\mathbb{W}^\beta_p := \mathbb{B}^\beta_p.
\]

It is well-known that if \( \beta \) is an integer and \( p > 1 \), an equivalent norm in \( \mathbb{W}^\beta_p = \mathbb{H}^\beta_p \) is given by

\[
\|f\|_{\mathbb{W}^\beta_p} := \sum_{k=0}^\beta \|\nabla^k f\|_p,
\]

where \( \nabla^k \) denotes the \( k \)-order generalized gradient; and if \( 0 < \beta \neq \text{integer} \) and \( p > 1 \), an equivalent norm in \( \mathbb{W}^\beta_p \) is given by

\[
\|f\|_{\mathbb{W}^\beta_p} := \|f\|_p + \sum_{k=0}^{[\beta]} \left( \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\nabla^k f(x) - \nabla^k f(y)|^p}{|x - y|^{d+|\beta|p}} \, dx \, dy \right)^{\frac{1}{p}},
\]

where for a number \( \beta > 0 \), \([\beta]\) denotes the integer part of \( \beta \) and \([\beta] := \beta - [\beta]\). It is also well-known that Riesz’s transform \( \nabla (\Delta)^{-\frac{\beta}{2}} \) is a bounded linear operator in \( L^p \)-space for any \( p > 1 \) (see [15]). Moreover, the following interpolation inequality holds: for any \( \beta \in (0, \gamma) \), \( p > 1 \) and \( f \in \mathbb{H}^\gamma_p \),

\[
\|(-\Delta)^{\frac{\beta}{2}} f\|_p \leq C \|f\|_p^{1-\frac{\beta}{\gamma}} \|(-\Delta)^{\frac{\gamma}{2}} f\|_p^{\frac{\beta}{\gamma}}.
\]

The following lemma is an easy consequence of [11] Lemma 2.1.

**Lemma 2.1.** For any \( \beta \in (0, 1) \), there exits a constant \( C = C(\beta, d) > 0 \) such that for all \( p \geq 1 \) and \( f \in \mathbb{H}^\beta_p \),

\[
\|f(\cdot + y) - f(\cdot)\|_p \leq C|y|^{\beta}\|(-\Delta)^{\frac{\beta}{2}} f\|_p.
\]

For each \( t \in [0, 1] \), write \( \mathbb{V}^\beta_{1,p} := L^p([0, t]; \mathbb{H}^\beta_p) \) with the norm

\[
\|u\|_{\mathbb{V}^\beta_{1,p}} := \left( \int_0^t \|u(s)\|_{\mathbb{H}^\beta_p}^p \, ds \right)^{\frac{1}{p}},
\]

and let \( \mathbb{V}^\beta_{1,p} \) be the completion of all functions \( u \in C^\infty([0, t]; \mathcal{S}(\mathbb{R}^d)) \) with respect to the norm

\[
\|u\|_{\mathbb{V}^\beta_{1,p}} := \sup_{s \in [0, t]} \|u(s)\|_{\mathbb{H}^\beta_{1-p}} + \|u\|_{\mathbb{V}^\beta_{1,p}} + \|\partial_\tau u\|_{\mathbb{V}^\beta_{1-p}}.
\]

It is well-known that (cf. [11] p.180, Theorem III 4.10.2]),

\[
\mathbb{V}^\beta_{1,p} \hookrightarrow C([0, t]; \mathbb{W}^{\frac{1}{p}-\frac{1}{p}, p}).
\]

For simplicity of notation, we also write

\[
\mathbb{V}^\beta_{1,p} := \mathbb{V}^\beta_{1,1}, \quad \mathbb{V}^\beta_{p} := \mathbb{V}^\beta_{p, p}.
\]
3. \( L^p \)-estimate of Nonlocal Operators

Let \( \nu \) be a \( \sigma \)-finite measure on \( \mathbb{R}^d \), which is called a Lévy measure if \( \nu(\{0\}) = 0 \) and
\[
\int_{\mathbb{R}^d} 1 \land |x|^2 \nu(dx) < +\infty.
\]

Let \( \Sigma \) be a finite measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \). For \( \alpha \in (0, 2) \), define
\[
\nu^{(\alpha)}(B) := \int_{S^{d-1}} \left( \int_0^{\infty} \frac{1_B(r\theta)dr}{r^{1+\alpha}} \right) \Sigma(d\theta), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]
(9)

Then \( \nu^{(\alpha)} \) is the Lévy measure corresponding to the \( \alpha \)-stable process.

**Definition 3.1.** (i) Let \( \nu_1 \) and \( \nu_2 \) be two Borel measures on \( \mathbb{R}^d \). We say that \( \nu_1 \) is less than \( \nu_2 \) if \( \nu_1(B) \leq \nu_2(B), \quad B \in \mathcal{B}(\mathbb{R}^d) \), and we simply write \( \nu_1 \leq \nu_2 \) in this case.

(ii) The Lévy measure \( \nu^{(\alpha)} \) defined by (9) is called nondegenerate if
\[
\int \theta_0 \cdot \theta^\alpha \Sigma(d\theta) \neq 0, \quad \forall \theta_0 \in S^{d-1}.
\]
(10)

Throughout this paper we make the following assumption:

\( (H_\nu^{(\alpha)}) \) Let \( \nu \) be a Lévy measure and satisfy that for some \( \alpha \in (0, 2) \),
\[
\nu^{(\alpha)}_1 \leq \nu \leq \nu^{(\alpha)}_2, \quad 1_{\alpha=1} \int_{r \leq |x| < R} \nu(dy) = 0, \quad 0 < r < R < +\infty,
\]
(11)

where \( \nu^{(\alpha)}_i, i = 1, 2 \) are two Lévy measures with the form (9), and \( \nu^{(\alpha)}_1 \) is nondegenerate.

Let us recall the following result from [19] Corollary 4.4.

**Theorem 3.2.** Assume \( (H_\nu^{(\alpha)}) \) with \( \alpha \in (0, 2) \). Then for any \( p \in (1, \infty) \), there exists a constant \( C_0 \in (0, 1) \) such that for all \( f \in \mathbb{H}^{\alpha,p} \),
\[
C_0\|(-\Delta)^{\frac{\alpha}{2}} f\|_p \leq \|L^p f\|_p \leq C_0^{-1}\|(-\Delta)^{\frac{\alpha}{2}} f\|_p.
\]
(12)

Below, for simplicity of notation, we write
\[
\mathcal{J}^{(\alpha)}_f(x, y) := f(x + y) - f(x) - y^{(\alpha)} \cdot \nabla f(x).
\]
(13)

We first prepare the following lemma for later use.

**Lemma 3.3.** Suppose that \( a \in \mathcal{A}_0 \) and \( \nu \leq \nu^{(\alpha)} \) for some \( \alpha \in (0, 2) \). For any \( p > 1 \), there exists a constant \( C = C(\alpha, p, d) > 0 \) such that for all \( f \in \mathbb{H}^{\alpha,p} \) and \( \varepsilon \in (0, 1) \),
\[
\left\| \int_{|y| \leq \varepsilon} \mathcal{J}^{(\alpha)}_f(a(\cdot, y) - a(\cdot, 0))\nu(dy) \right\|_p \leq C\|(-\Delta)^{\frac{\alpha}{2}} f\|_p \int_0^\varepsilon \frac{\omega^0_u(r)}{r}dr,
\]
where \( \omega^0_u \) is defined by (3).

**Proof.** Let us look at the case of \( \alpha \in [1, 2) \). Since \( a \in \mathcal{A}_0 \), by Minkowski’s inequality we have
\[
\left\| \int_{|y| \leq \varepsilon} \left[ f(\cdot + y) - f(\cdot) - y \cdot \nabla f(\cdot) \right](a(\cdot, y) - a(\cdot, 0))\nu(dy) \right\|_p \leq \int_{|y| \leq \varepsilon} |y| \left( \int_0^1 \|\nabla f(\cdot + sy) - \nabla f(\cdot)\|_p ds \right) \omega^0_u(|y|)\nu^{(\alpha)}(dy)
\]
\[
\begin{align*}
\mathcal{L}^\alpha f(x) := & \int_{\mathbb{R}^d} \mathcal{F}_f^{(\alpha)}(x,y) a(x,y) \nu(dy), \\
\text{where } \mathcal{F}_f^{(\alpha)}(x,y) \text{ is given by (13). We now establish the following characterization about the domain of } \mathcal{L}^\alpha.
\end{align*}
\]

Theorem 3.4. Let \( \alpha \in (0,2) \). Assume that (H,\( \alpha \)) holds and \( a \in \mathcal{A}_0 \) satisfies that for some \( 0 < a_0 < a_1 \) and any \( 0 < r < R < \infty \),

\[
a_0 \leq a(x,0) \leq a_1, \quad 1_{\alpha=1} \int_{\nu(x,y) < R} \frac{\nu(x,y)}{y} dy = 0.
\]

Then for any \( p \in (1,\infty) \), there exists a constant \( C_1 \in (0,1) \) depending only on \( a_0, a_1, \nu_1^{(\alpha)}, \nu_2^{(\alpha)} \) and \( \alpha, d, p \) such that for all \( f \in \ell^{\alpha,p} \),

\[
C_1 \| f \|_{\alpha,p} \leq \| \mathcal{L}^\alpha f \|_p + \| f \|_p \leq C_1^{-1} \| f \|_{\alpha,p}.
\]

Proof. We make the following decomposition:

\[
\begin{align*}
\mathcal{L}^\alpha f(x) = & a(x,0) \mathcal{L}^\alpha f(x) + \int_{\|y\| < e} \mathcal{F}_f^{(\alpha)}(x,y) (a(x,y) - a(x,0)) \nu(dy) \\
& + \int_{\|y\| > e} \mathcal{F}_f^{(\alpha)}(x,y) (a(x,y) - a(x,0)) \nu(dy) \\
=: & I_1(x) + I_2(x) + I_3(x).
\end{align*}
\]

For \( I_1(x) \), by Theorem 3.2 and condition (14), we have

\[
a_0 C_0 (-\Delta)^{\alpha/2} f \|_p \leq \| I_1 \|_p \leq a_1 C_0^{-1} (-\Delta)^{\alpha/2} f \|_p.
\]

For \( I_2(x) \), if \( \alpha = 1 \), by (14) we have

\[
\| I_2 \|_p = \left\| \int_{\|y\| > 1} \left[ (\cdot + y) - f(\cdot) \right] \nu(dy) \right\|_p \leq 4 \| f \|_p \| a \|_\infty \nu(B_e);
\]

if \( \alpha \in (0,1) \), we have

\[
\| I_2 \|_p \leq 4 \| f \|_p \| a \|_\infty \nu(B_e^c);
\]

if \( \alpha \in (1,2) \), we have

\[
\| I_2 \|_p \leq 4 \| f \|_p \| a \|_\infty \int_{\|y\| > e} \nu(dy) + 2 \| \nabla f \|_p \| a \|_\infty \int_{\|y\| > e} |y| \nu(dy) \\
\leq C_e \| f \|_p + C_e \| f \|_{\alpha,p} \| f \|_p^{1 - \frac{1}{\alpha}} \leq C e \| f \|_{\alpha,p} + C e \| f \|_p.
\]

For \( I_3(x) \), by Lemma 3.3 we have

\[
\| I_3 \|_p \leq C \gamma(\varepsilon) \| (-\Delta)^{\alpha/2} f \|_p,
\]
where $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Now, combining the above calculations, we obtain the right hand side estimate in (15). Moreover, we also have

$$\|L^\alpha f\|_p \geq \|I_1\|_p^\alpha - \|I_2\|_p^\alpha - \|I_3\|_p^\alpha \geq (a_0 C_0 - \varepsilon - C\gamma(\varepsilon)) \|(-\Delta)^{\alpha/2} f\|_p - C_\varepsilon \|f\|_p.$$  

Letting $\varepsilon$ be small enough, we obtain the left hand side estimate in (15).

\[\square\]

4. Nonlocal linear parabolic equation

In this section we fix a Lévy measure $\nu$ satisfying (H\textsuperscript{(\alpha)})\textsuperscript{3.2}. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative and locally integrable function. Let $N(dt, dx)$ be the Poisson random point measure with intensity measure $\tilde{N}(dt, dx) := \lambda(t) d\nu(dx)$. Let $\tilde{N}(dr, dx) := N(dr, dx) - \tilde{N}(dr, dx)$ be the compensated random martingale measure. Let $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ be a locally integrable function. For $t \geq 0$, define

$$X_t := \int_0^t \theta(r) dr + \int_0^t \int_{B^{(\alpha)}} y \tilde{N}(dr, dy) + \int_0^t \int_{\mathbb{R}^d - B^{(\alpha)}} y N(dr, dy),$$  

(16)

where $B^{(\alpha)} = \{ x : |x| \leq 1 \}$ if $\alpha = 1$; $B^{(\alpha)} = \mathbb{R}^d$ if $\alpha \in (1, 2)$; and $B^{(\alpha)} = \emptyset$ if $\alpha \in (0, 1)$.

For $\varphi \in C^2_b(\mathbb{R}^d)$, by Itô’s formula we have

$$\mathbb{E} \varphi(x + X_t - X_s) = \varphi(x) + \mathbb{E} \int_s^t \theta(r) \cdot \nabla \varphi(x + X_r - X_s) dr$$

$$+ \mathbb{E} \int_s^t \int_{\mathbb{R}^d} [\varphi(x + X_r - X_s + y) - \varphi(x + X_r - X_s) - y^{(\alpha)} \cdot \nabla \varphi(x + X_r - X_s)] \tilde{N}(dr, dy).$$

Thus, if we let

$$T_{t,s} \varphi(x) := T_{t,s}^{x_{\lambda_0} \theta} \varphi(x) := \mathbb{E} \varphi(x + X_t - X_s),$$  

(17)

then one sees that

$$\partial_t T_{t,s} \varphi = L^{\lambda(t)_0 \theta} T_{t,s} \varphi + \theta(t) \cdot \nabla T_{t,s} \varphi.$$  

The following result is a slight extension of [19, Theorem 4.2].

**Theorem 4.1.** Assume (H\textsuperscript{(\alpha)}) with $\alpha \in (0, 2)$. Let $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ be a locally integrable function and $\lambda : \mathbb{R}_+ \to [\lambda_0, \infty)$ be a measurable function, where $\lambda_0 > 0$. Let $T_{t,s}^{x_{\lambda_0} \theta}$ be defined by (17).

Then for any $p \in (1, \infty)$, there exists a constant $C = C(\lambda_0, \lambda_1, \lambda_2, \alpha, p, d) > 0$ such that for any $T > 0$ and $f \in L^p((0, T) \times \mathbb{R}^d)$,

$$\int_0^T \left\| L^\alpha \int_0^T \mathcal{M}_{t,s} f(s, \cdot) ds \right\|_p^p dt \leq C \int_0^T \left\| f(t) \right\|_p^p dt.$$  

(18)

**Proof.** Let $N^{(1)}(dt, dx)$ and $N^{(2)}(dt, dx)$ be two independent Poisson random point measures with intensity measures $\tilde{N}^{(1)}(dt, dx) := (\lambda(t) - \lambda_0) d\nu(dx)$ and $\tilde{N}^{(2)}(dt, dx) := \lambda_0 d\nu(dx)$ respectively. Let $X_t^{(1)}$ be defined by (16) in terms of $N^{(1)}$, and $X_t^{(2)}$ be defined by

$$X_t^{(2)} := \int_0^t \int_{B^{(\alpha)}} y \tilde{N}^{(2)}(dr, dy) + \int_0^t \int_{\mathbb{R}^d - B^{(\alpha)}} y N^{(2)}(dr, dy).$$

In fact, $X_t^{(2)}$ is the Lévy process corresponding to the Lévy measure $\lambda_0 \nu(dx)$. By Itô’s formula, we have

$$T_{t,s}^{x_{\lambda_0} \theta} f(x) = \mathbb{E} f(x + X_t^{(1)} - X_s^{(1)} + X_t^{(2)} - X_s^{(2)}) = \mathbb{E} T_{t,s}^{x_{\lambda_0} \theta} f(x + X_t^{(1)} - X_s^{(1)}).$$  

(19)
Thus, by \([19, \text{Theorem 4.2}]\), we have
\[
\int_0^T \left\| \mathcal{L} \int_0^t \mathcal{T}_{t,s} f(s, \cdot) ds \right\|_p^p \, dt \leq \mathbb{E} \int_0^T \left\| \mathcal{L} \int_0^t \mathcal{T}_{t,s} f(s, \cdot + X_s^{(1)} - X_s^{(1)}) ds \right\|_p^p \, dt
= \mathbb{E} \int_0^T \left\| \mathcal{L} \int_0^t \mathcal{T}_{t,s} f(s, \cdot) ds \right\|_p^p \, dt
\leq C \mathbb{E} \int_0^T \left\| f(s, \cdot - X_s^{(1)}) \right\|_p^p \, ds = C \int_0^T \left\| f(s) \right\|_p^p \, ds.
\]
The proof is complete. \(\square\)

Consider the following time-dependent linear nonlocal parabolic system:
\[
\partial_t u = \mathcal{L}^{ad_{(y)}} u + b^{(a)} \cdot \nabla u + f, \quad u(0) = \varphi, \quad (20)
\]
where \(u, f : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^n\), \(a : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\) and \(b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) are Borel measurable functions, and
\[
b^{(a)}(t, x) = 1_{a \in [1,2]} b(t, x). \quad (21)
\]
We make the following assumptions on \(a\) and \(b\):

(H\(_a\)) For each \(t \geq 0\), \(a(t) \in \mathcal{A}_1\) satisfies
\[
\sup_{t \in [0,1]} \|a(t)\|_{\mathcal{A}_1} < +\infty, \quad a_0 \leq a(t, x, 0) \leq a_1,
\]
where \(a_0, a_1 > 0\), and for all \(0 < r < R < +\infty\),
\[
1_{a=1} \int_{r \leq |x| < R} y a(t, x, y) \nu(dy) = 0. \quad (22)
\]

(H\(_b\)) For all \(t \geq 0\) and \(x, y \in \mathbb{R}^d\),
\[
|b^{(a)}(t, x) - b^{(a)}(t, y)| \leq 1_{a=1} \omega_b(|x - y|) + 1_{a \in (1,2)} C_b,
\]
where \(\omega_b : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is an increasing function with \(\lim_{s \rightarrow 0} \omega_b(s) = 0\).

Let us first prove the following apriori estimate by the method of freezing the coefficients (cf. \([19, \text{Lemma 5.1}]\)).

**Lemma 4.2.** Suppose that \(a(t, x, y) = a(t, x)\) is independent of \(y\) and satisfies (H\(_a\)), and \(b\) satisfies (H\(_b\)). Let \(p > 1\) and not equal to \(\frac{\alpha}{\alpha-1}\) if \(\alpha \in (1,2)\), and let \(f \in \mathbb{Y}^{0,p}\) and \(u \in \mathbb{Y}^{\alpha,p}\) satisfy (20). Then for all \(t \in [0,1]\),
\[
\|u\|_{\mathbb{Y}^{\alpha,p}} \leq C \left( \|u(0)\|_{\mathbb{Y}_{-1}^{\alpha,p}} + \|f\|_{\mathbb{Y}_{-1}^{\alpha,p}} \right), \quad (23)
\]
where \(C\) depends only on \(a_0, a_1, \|a\|_{\mathcal{A}_1}, \|b\|_{\infty}, d, p, \alpha \) and \(\omega_b\).

**Proof.** Let \((\rho_\varepsilon)_{\varepsilon \in (0,1)}\) be a family of mollifiers in \(\mathbb{R}^d\), i.e., \(\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x)\), where \(\rho \in C_0^\infty(\mathbb{R}^d)\) with \(\int \rho = 1\) is nonnegative. Define
\[
u(t) := u(t) * \rho_\varepsilon, \quad a_\varepsilon(t) := a(t) * \rho_\varepsilon, \quad b_\varepsilon(t) := b(t) * \rho_\varepsilon, \quad f_\varepsilon(t) := f(t) * \rho_\varepsilon,
\]
where * stands for the convolution. Taking convolutions for both sides of (20), we have
\[
\partial_t u_\varepsilon = \mathcal{L}^{ad_{(y)}} u_\varepsilon + b^{(a)} \cdot \nabla u_\varepsilon + h_\varepsilon, \quad (24)
\]
where
\[
h_\varepsilon := f_\varepsilon + (\mathcal{L}^{ad_{(y)}} u) * \rho_\varepsilon - \mathcal{L}^{ad_{(y)}} u_\varepsilon * (b^{(a)} \cdot \nabla u) * \rho_\varepsilon - b^{(a)} \cdot \nabla u_\varepsilon.
\]
By the assumption, it is easy to see that for all \( \varepsilon \in (0, 1) \) and \( t \in [0, 1] \) and \( x, y \in \mathbb{R}^d \),
\[
|a_\varepsilon(t, x) - a(t, y)| \leq \omega_\varepsilon^+(|x - y|), \quad |b_\varepsilon(t, x) - b(t, y)| \leq 1_{a(1, 2)}(\omega_\varepsilon(|x - y|)) + 1_{a(1, 2)}C_b,
\]
and
\[
|a_\varepsilon(t, x) - a(t, x)| \leq \omega_\varepsilon^+(\varepsilon), \quad |b_\varepsilon(t, x) - b(t, x)| \leq 1_{a(1, 2)}\omega_b(\varepsilon) + 1_{a(1, 2)}C_b.
\]
Moreover, by the property of convolutions, we also have
\[
\lim_{\varepsilon \to 0} \int_0^1 \|h_\varepsilon(t) - f(t)\|_p dt = 0.
\]
Below, we use the method of freezing the coefficients to prove that for all \( t \in [0, 1] \),
\[
\|u_\varepsilon\|_{L^p_t} \leq C_{1, p} \left(\|u_\varepsilon(0)\|_{W^{1,p} - \frac{d}{p}} + \|h_\varepsilon\|_{L^p_t}\right),
\]
where the constant \( C \) is independent of \( \varepsilon \). After proving this estimate, (26) immediately follows by taking limits for (26).

For simplicity of notation, we drop the subscript \( \varepsilon \) below. Fix \( \delta > 0 \) being small enough, whose value will be determined below. Let \( \zeta \) be a smooth function with support in \( B_\delta \) and \( \|\zeta\|_p = 1 \). For \( z \in \mathbb{R}^d \), set
\[
\zeta_\varepsilon(z) := \zeta(z - z), \quad \lambda_\varepsilon^a(z) := a(t, z), \quad \theta_\varepsilon^b(z) := 1_{a=1}b(t, z).
\]
Multiplying both sides of (24) by \( \zeta_\varepsilon \), we have
\[
\partial_t(u_\varepsilon \zeta_\varepsilon) = \lambda_\varepsilon^a \mathcal{L}^\nu(u_\varepsilon \zeta_\varepsilon) + \theta_\varepsilon^b \cdot \nabla(u_\varepsilon \zeta_\varepsilon) + g_\varepsilon^\nu,
\]
where
\[
g_\varepsilon^\nu := (a - \lambda_\varepsilon^a) \mathcal{L}^\nu u_\varepsilon \zeta_\varepsilon + \lambda_\varepsilon^a \left(\mathcal{L}^\nu u_\varepsilon \zeta_\varepsilon - \mathcal{L}^\nu (u_\varepsilon \zeta_\varepsilon)\right) + \left(b^{(a)} - \theta_\varepsilon^a\right) \cdot \nabla(u_\varepsilon \zeta_\varepsilon) - ub^{(a)} \cdot \nabla \zeta_\varepsilon + h_\varepsilon \zeta_\varepsilon.
\]
Let \( T \) be defined by (17) in terms of \( \nu = \lambda_\varepsilon^a \nu \) and \( \theta_\varepsilon^b \). By Duhamel’s formula, \( u_\varepsilon \zeta_\varepsilon \) can be written as
\[
u
u
\]
\[
\int_0^T \nu \int_0^T \|\mathcal{L}^\nu (u_\varepsilon \zeta_\varepsilon)(t)\|_p dt \leq 2^{p-1} \left(\int_0^T \|\mathcal{L}^\nu T_{t, 0}^{\lambda_\varepsilon^a, \theta_\varepsilon^b}(u(0) \zeta_\varepsilon)(t)\|_p dt + \int_0^T \|\mathcal{L}^\nu T_{t, s}^{\lambda_\varepsilon^a, \theta_\varepsilon^b} g_\varepsilon^\nu(s, x) ds\|_p dt\right)
\]
\[
= : 2^{p-1}(I_1(T, z) + I_2(T, z)).
\]
For \( I_1(T, z) \), by (19) and \( \|\mathcal{L}^\nu f(\cdot + z)\|_p = \|\mathcal{L}^\nu f\|_p \), we have
\[
\int_0^T \|\mathcal{L}^\nu T_{t, 0}^{\lambda_\varepsilon^a, \theta_\varepsilon^b}(u(0) \zeta_\varepsilon)(t)\|_p dt \leq \int_0^T \|\mathcal{L}^\nu T_{t, 0}^{\lambda_\varepsilon^a, \theta_\varepsilon^b}(u(0) \zeta_\varepsilon)(t)\|_p dt \leq \mathcal{C}\|u(0) \zeta_\varepsilon\|_p^{\nu}, \quad \mathcal{T}, \frac{1}{2}, p
\]
where the last step is due to (17, p.96 Theorem 1.14.5] and [19, Corollary 4.5]. Thus, by definition (5), it is easy to see that
\[
\int_{\mathbb{R}^d} I_1(T, z) dz \leq C \int_{\mathbb{R}^d} \|u(0) \zeta_\varepsilon\|_p^{\nu} dz \leq C\left(\|u(0)\|_{W^{1,p} - \frac{d}{p}}^{\nu} + \|u(0)\|_p^{\nu}\right).
\]
For \( I_2(T, z) \), by Theorem 4.11 we have
\[
I_2(T, z) \leq C \int_0^T \|g_\varepsilon^\nu(s)\|_p ds \leq C \int_0^T \mathcal{G}((a - \lambda_\varepsilon^a)(\mathcal{L}^\nu u \zeta_\varepsilon))(s)\|_p ds
\]
Choosing where the last step is due to the interpolation inequality, Young’s inequalities and Theorem 3.2. Moreover, it is easy to see that

\[ \int_{\mathbb{R}^d} \int_0^T \left| \frac{d}{ds} \mathcal{L}^\alpha(s) \cdot \nabla \mathcal{L}^\alpha(s) \right| dz = C \int_0^T \left| \frac{d}{ds} \mathcal{L}^\alpha(s) \cdot \nabla \mathcal{L}^\alpha(s) \right| dz. \]

For \( I_{21}(T, z) \), by \( (25) \) and \( \|\xi\|_p = 1 \), we have
\[ \int_{\mathbb{R}^d} I_{21}(T, z)dz \leq C \omega_d^{(1)}(\delta)^p \int_0^T \left| \mathcal{L}^\alpha(s) \cdot \nabla \mathcal{L}^\alpha(s) \right| dz = C \omega_d^{(1)}(\delta)^p \int_0^T \left| \mathcal{L}^\alpha(s) \right| dz. \]

For \( I_{22}(T, z) \), using \( (7) \) and as in the proof of \([19, \text{Lemma 2.5}]\), for any \( \beta \in (0 \lor (\alpha - 1), \alpha) \), we have
\[ \int_{\mathbb{R}^d} I_{22}(T, z)dz \leq C a_1 \int_0^T \int_{\mathbb{R}^d} \left| \mathcal{L}^\alpha(s) - \mathcal{L}^\alpha(s) \right| dzds \]
\[ \leq C \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds + C \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds \]
\[ \leq C \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds + \frac{1}{4p} \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds, \]

where the last step is due to the interpolation inequality, Young’s inequalities and Theorem 3.2.

For \( I_{23}(T, z) \), as above we have
\[ \int_{\mathbb{R}^d} I_{23}(T, z)dz \leq C 1 \omega_b(\delta)^p \left( \int_0^T \left| \nabla \mathcal{L}^\alpha(s) \right| ds + \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds \right). \]

Moreover, it is easy to see that
\[ \int_{\mathbb{R}^d} I_{24}(T, z)dz \leq C \|b\|_p \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds, \]
\[ \int_{\mathbb{R}^d} I_{25}(T, z)dz \leq C \int_0^T \left| h(s) \right| ds. \]

Combining the above calculations, we get
\[ \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds = \int_0^T \int_{\mathbb{R}^d} \left| \mathcal{L}^\alpha(s) \cdot \nabla \mathcal{L}^\alpha(s) \right| dzds \leq 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \left| \mathcal{L}^\alpha(s) \right| dzds \]
\[ + 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \left| \mathcal{L}^\alpha(s) - \mathcal{L}^\alpha(s) \right| dzds \]
\[ \leq C \|u(0)\|_{\mathcal{W}^{\alpha,p}_d} + \left( \frac{1}{4} + C (\omega_1^{(1)}(\delta)^p + \omega_b(\delta)^p) \right) \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds \]
\[ + C \int_0^T \left| \mathcal{L}^\alpha(s) \right| ds + C \int_0^T \left| h(s) \right| ds. \]

Choosing \( \delta_0 > 0 \) being small enough so that
\[ C (\omega_1^{(1)}(\delta_0)^p + \omega_b(\delta_0)^p) \leq \frac{1}{4}, \]
we obtain that for all $T \in [0, 1]$,
\[
\int_0^T \|\mathcal{L}^u(s)\|_p^p \, ds \leq C\|u(0)\|_{\frac{p}{p-\frac{\alpha}{\alpha-1}}}^p + C \int_0^T \|u(s)\|_{\frac{p}{p-\frac{\alpha}{\alpha-1}}}^p \, ds + C \int_0^T \|h(s)\|_p^p \, ds.
\] (28)

On the other hand, by (24), it is easy to see that for all $t \in [0, 1]$,
\[
\|u(t)\|_p^p \leq C\|u(0)\|_p^p + C1_{\alpha \in [1, 2]} \int_0^t \|\nabla u(s)\|_p^p \, ds + C \int_0^t \|h(s)\|_p^p \, ds,
\]
which together with (28) and Gronwall’s inequality yields that for all $t \in [0, 1]$,
\[
\sup_{s \in [0, t]} \|u(s)\|_p^p + \int_0^t \|\mathcal{L}^u(s)\|_p^p \, ds \leq C \left(\|u(0)\|_{\frac{p}{p-\frac{\alpha}{\alpha-1}}}^p + \int_0^t \|h(s)\|_p^p \, ds\right).
\] (29)

From equation (24), we also have
\[
\int_0^t \|\partial_s u(s)\|_p^p \, ds \leq C \left(\|a\|_\infty \int_0^t \|\mathcal{L}^u(s)\|_p^p \, ds + \|b^{(a)}\|_\infty \int_0^t \|\nabla u(s)\|_p^p \, ds + \int_0^t \|h(s)\|_p^p \, ds\right),
\]
which together with (29) and (12) gives (26), and therefore (23).

We now prove the following main result of this paper.

**Theorem 4.3.** Suppose $(\mathbf{H}_a^{(\alpha)})$, $(\mathbf{H}_b^{(\alpha)})$ and $(\mathbf{H}^p)$ and for some $k \in \mathbb{N} \cup \{0\}$,
\[
|\nabla^j a(x, y)| + |\nabla^j b(x, y)| \leq C_j, \quad j = 0, \ldots, k.
\]

For given $p \in (1, \infty)$ not equal to $\frac{\alpha}{\alpha-1}$ when $\alpha \in (1, 2)$, and for $\varphi \in \mathbb{X}^{k+\alpha-\frac{\alpha}{\alpha-1},p}$, there exists a unique $u \in \mathbb{X}^{k+\alpha, p}$ satisfying equation (20). Moreover, for all $t \in [0, 1]$,
\[
\|u\|_{\mathbb{X}^{k+\alpha, p}} \leq C_{k,p} \left(\|\varphi\|_{\mathbb{X}^{k+\alpha-\frac{\alpha}{\alpha-1},p}} + \|f\|_{\mathbb{X}^{k, p}}\right),
\] (30)
where $C_{0,p}$ depends only on $a_0, a_1, \|a\|_{\mathcal{L}_1}, \|b\|_{\infty}, d, p, \alpha$ and $\omega_b$.

**Proof.** The strategy is to prove the apriori estimate (30) and then use the continuity method (cf. [12, 20]).

(Step 1) Let us first rewrite equation (20) as
\[
\partial_t u(t, x) = a(t, x, 0)\mathcal{L}^u u(t, x) + b^{(a)}(t, x) \cdot \nabla u(t, x) + \tilde{f}(t, x),
\]
where
\[
\tilde{f}(t, x) := f(t, x) + \int_{\mathbb{R}^d} \mathcal{F}^{(a)}_{u(t, \cdot)}(x, y)(a(t, x, y) - a(t, x, 0)) \nu(dy)
\]
and
\[
\mathcal{F}^{(a)}_{u(t, \cdot)}(x, y) := u(t, x + y) - u(t, x) - y \cdot \nabla u(t, x).
\]
Notice that by Lemma 3.3
\[
\|\tilde{f}(t)\|_p \leq \|f(t)\|_p + \left\|\int_{|y|<r} \mathcal{F}^{(a)}_{u(t, \cdot)}(\cdot, y)(a(t, \cdot, y) - a(t, \cdot, 0)) \nu(dy)\right\|_p
\]
\begin{align*}
&+ \left\|\int_{|y|\geq r} \mathcal{F}^{(a)}_{u(t, \cdot)}(\cdot, y)(a(t, \cdot, y) - a(t, \cdot, 0)) \nu(dy)\right\|_p \\
&\leq \|f(t)\|_p + 2a_1 \left(\|u(t)\|_p \nu(B_r^c) + 1_{a \in (1, 2)} \|\nabla u(t)\|_p\right)
\end{align*}
\begin{align*}
&+ C_{k,p}\|u(t)\|_p + C\gamma_0(\varepsilon)\|(-\Delta)^{\frac{\alpha}{2}} u(t)\|_p \\
&\leq \|f(t)\|_p + C_\omega\|u(t)\|_p + \gamma_1(\varepsilon)\|(-\Delta)^{\frac{\alpha}{2}} u(t)\|_p,
\end{align*}
where the last step is due to the interpolation inequality and Young’s inequalities, and
\[\gamma_0(\varepsilon) := \int_0^\varepsilon \frac{\omega_0^{(0)}(r)}{r} \, dr, \quad \gamma_1(\varepsilon) := \varepsilon + C\gamma_0(\varepsilon).\]

By Lemma 4.2, we have
\[\|u\|_{\mathcal{E}^p} \leq C_1\|\varphi\|_{\mathcal{V}^{p-\frac{n}{p}}} + C_2\|f\|_{\mathcal{X}^p}.\]

In particular, for all \(t \in [0, 1]\),
\[\sup_{s \in [0, t]}\|u(s)\|_p + \int_0^t \|(-\Delta)^{\frac{n}{2}} u(s)\|_p \, ds \leq C_1\|\varphi\|_{\mathcal{V}^{p-\frac{n}{p}}} + \gamma_1(\varepsilon) \int_0^t \|(-\Delta)^{\frac{n}{2}} u(s)\|_p \, ds + C_2 \int_0^t \|u(s)\|_p \, ds + C_2 \int_0^t \|f(s)\|_p \, ds.

Letting \(\varepsilon\) be small enough and using Gronwall’s inequality, we obtain (30) with \(k = 0\).

(Step 2) We now estimate the higher order derivatives. Write
\[w^{(n)}(t, x) := \nabla^n u(t, x).

By the chain rule, one can see that
\[\partial_t w^{(n)} = L^{(n)} w^{(n)} + b^{(a)} \cdot \nabla w^{(n)} + g^{(n)},\]

where
\[g^{(n)} := \nabla^n f + \sum_{j=1}^n \frac{n!}{(n-j)!j!} \left( L^{(n-j)} (\nabla^{n-j} u) + \nabla (b^{(a)} \cdot \nabla^{n-j} u) \right)\]

and
\[L^{(n-j)}(\nabla^{n-j} u)(t, x) = \int_{\mathbb{R}^d} \mathcal{F}^{(a)}_{\nabla^{n-j} u} (x, y) \nabla_j a(t, x, y) \nu(dy).\]

By Step 1, we know that
\[\|w^{(n)}\|_{\mathcal{E}^{p,n}} \leq C \left( \|w^{(n)}(0)\|_{\mathcal{V}^{p-\frac{n}{p}}} + \|g^{(n)}\|_{\mathcal{X}^{p,n}} \right). \tag{31}\]

By Minkowski’s inequality, we have
\[\|L^{(n-j)}(\nabla^{n-j} u)(t)\|_p \leq C_j \int_{\mathbb{R}^d} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot) - y^{(a)} \cdot \nabla \nabla^{n-j} u(t, \cdot)\|_p \nu(dy)\]

\[\leq C_j \int_{|b| > 1} \left( 2\|\nabla^{n-j} u(t)\|_p + 1_{\alpha \in (0, 1)} |b| \cdot \|\nabla^{n-j} u(t)\|_p \right) \nu(dy) + C_j 1_{\alpha \in (0, 1)} \int_{|b| < 1} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot)\|_p \nu(dy)\]

\[+ C_j 1_{\alpha \in [1, 2]} \int_{|b| < 1} \|\nabla^{n-j} u(t, \cdot + y) - \nabla^{n-j} u(t, \cdot) - y \cdot \nabla^{n-j} u(t, \cdot)\|_p \nu(dy)\]

\[\leq 2C_j \nu(B^1_1) \|\nabla^{n-j} u(t)\|_p + C_j \|\nabla^{n-j} u(t)\|_p \times \left( \int_{B^1_1} |b| 1_{\alpha \in (0, 1)} \nu(dy) + \int_{B_1} |b| 1_{\alpha \in (1, 2)} \nu(dy) \right)\]

\[+ C_j 1_{\alpha \in [1, 2]} \int_{|b| < 1} |b| \left( \int_0^1 \|\nabla^{n-j} u(t, \cdot + sy) - \nabla^{n-j} u(t, \cdot)\|_p \, ds \right) \nu(dy)\]

\[\leq C \|\nabla^{n-j} u(t)\|_p + C \|\nabla^{n-j} u(t)\|_p + C_1 \|\nabla^{n-j} u(t)\|_p \int_{B_1} |b|^{1+\beta} 1_{\alpha \in [1, 2]} \nu(dy),\]
where $\beta \in ((\alpha - 1) \vee 0, 1)$ and the last step is due to (7).

Hence, by the assumptions, we obtain
\[
\|g^{(n)}\|_{\psi_{2,p}}^p \leqslant \|f\|_{\psi_{2,p}}^p + C\|u\|_{\psi_{2,p}}^p + C\|u\|_{\psi_{2,p}}^p 1_{\alpha \in [1,2]}.
\]

Summing over $n$ from 0 to $k$ for (31) yields
\[
\|u(t)\|_{\psi_{2,p}}^p + \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds \leqslant C\|\varphi\|_{\psi_{2,p}}^p + C1_{\alpha \in [1,2]} \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds
\]
\[
+ C \int_0^t \|f(s)\|_{\psi_{2,p}}^p ds + C \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds
\]
\[
\leqslant C\|\varphi\|_{\psi_{2,p}}^p + C1_{\alpha \in [1,2]} \int_0^t \|u(s)\|_{\psi_{2,p}}^p \|u(s)\|_{\psi_{2,p}}^p(1 - 1/\alpha) ds
\]
\[
+ C \int_0^t \|f(s)\|_{\psi_{2,p}}^p ds + C \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds
\]
\[
\leqslant C\|\varphi\|_{\psi_{2,p}}^p + \frac{1}{2}1_{\alpha \in [1,2]} \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds
\]
\[
+ C \int_0^t \|f(s)\|_{\psi_{2,p}}^p ds + C \int_0^t \|u(s)\|_{\psi_{2,p}}^p ds,
\]

which then gives (30) by Gronwall’s inequality.

(Step 3) For $\lambda \in [0, 1]$, define an operator
\[
U_\lambda := \partial_t - \lambda \mathcal{L}^\alpha - \lambda b^{(\alpha)} \cdot \nabla - (1 - \lambda)\mathcal{L}^\nu.
\]

By (15), it is easy to see that
\[
U_\lambda : \mathcal{X}_{k+\alpha}^{k,p} \rightarrow \mathcal{X}_{k+\alpha}^{k,p}.
\]

For given $\varphi \in \mathcal{X}_{k+\alpha}^{k,p}$, let $\mathcal{X}_{k+\alpha}^{k,p}$ be the space of all functions $u \in \mathcal{X}_{k+\alpha}^{k,p}$ with $u(0) = \varphi$. It is clear that $\mathcal{X}_{k+\alpha}^{k,p}$ is a complete metric space with respect to the metric $\| \cdot \|_{\mathcal{X}_{k+\alpha}^{k,p}}$. For $\lambda = 0$ and $f \in \mathcal{X}_{k,p}$, it is well-known that there is a unique $u \in \mathcal{X}_{k+\alpha}^{k,p}$ such that
\[
U_{0}u = \partial_t u - \mathcal{L}^\nu u = f.
\]

In fact, by Duhamel’s formula, the unique solution can be represented by
\[
u(t, x) = \mathcal{T}_{t,0}^{\nu,0} \varphi(x) + \int_0^t \mathcal{T}_{t,s}^{\nu,0} f(s, x) ds,
\]

where $\mathcal{T}_{t,s}^{\nu,0}$ is defined by (17). Suppose now that for some $\lambda_0 \in [0, 1)$, and for any $f \in \mathcal{X}_{k,p}$, the equation
\[
U_{\lambda_0}u = f
\]

admits a unique solution $u \in \mathcal{X}_{k+\alpha}^{k,p}$. Thus, for fixed $f \in \mathcal{X}_{k,p}$ and $\lambda \in [\lambda_0, 1]$, and for any $u \in \mathcal{X}_{k+\alpha}^{k,p}$, by (32), the equation
\[
U_{\lambda}w = f + (U_{\lambda_0} - U_{\lambda})u
\]

admits a unique solution $w \in \mathcal{X}_{k+\alpha}^{k,p}$. Introduce an operator
\[
w = Q_{\lambda}u.
\]
We now use the apriori estimate (30) to show that there exists an \( \varepsilon > 0 \) independent of \( \lambda_0 \) such that for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \),

\[
Q_\lambda : \mathcal{X}^{k+\alpha,p}_\varphi \to \mathcal{X}^{k+\alpha,p}_\varphi
\]

is a contraction operator.

Let \( u_1, u_2 \in \mathcal{X}^{k+\alpha,p}_\varphi \) and \( w_i = Q_\lambda u_i \), \( i = 1, 2 \). By equation (33), we have

\[
U_{\lambda_0}(w_1 - w_2) = (U_{\lambda_0} - U_{\lambda})(u_1 - u_2) = (\lambda_0 - \lambda)(\mathcal{L}^{(a-1)p} + b^{(\alpha)} \cdot \nabla)(u_1 - u_2).
\]

By (30) and (15), it is not hard to see that

\[
\|Q_\lambda u_1 - Q_\lambda u_2\|_{\mathcal{X}^{k+\alpha,p}} \leq C_{k,p}|\lambda_0 - \lambda| \cdot \|\mathcal{L}^{(a-1)p} + b^{(\alpha)} \cdot \nabla)(u_1 - u_2)\|_{\mathcal{X}^{k,p}}
\]

where \( C_0 \) is independent of \( \lambda, \lambda_0 \) and \( u_1, u_2 \). Taking \( \varepsilon = 1/(2C_0) \), one sees that

\[
Q_\lambda : \mathcal{X}^{k+\alpha,p}_\varphi \to \mathcal{X}^{k+\alpha,p}_\varphi
\]

is a 1/2-contraction operator. By the fixed point theorem, for each \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \), there exists a unique \( u \in \mathcal{X}^{k+\alpha,p}_\varphi \) such that

\[
Q_\lambda u = u,
\]

which means that

\[
U_{\lambda} u = f.
\]

Now starting from \( \lambda = 0 \), after repeating the above construction \( \lfloor 1/\varepsilon \rfloor + 1 \)-steps, one obtains that for any \( f \in \mathcal{X}^{k,p}_\varphi \),

\[
U_{\lambda} u = f
\]

admits a unique solution \( u \in \mathcal{X}^{k+\alpha,p}_\varphi \). \( \square \)

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