THE LIFESPAN OF SOLUTIONS FOR A VISCOELASTIC WAVE EQUATION WITH A STRONG DAMPING AND LOGARITHMIC NONLINEARITY

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Abstract. This paper deals with the following viscoelastic wave equation with a strong damping and logarithmic nonlinearity:

\[ u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}u \ln |u|. \]

A finite time blow-up result is proved for high initial energy. Meanwhile, the lifespan of the weak solution is discussed. The present results in this paper complement and improve the previous work that is obtained by Ha and Park [Adv. Differ. Equ., (2020) 2020: 235].

1. Introduction. In this paper, we are concerned with the following initial-boundary value problem:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}u \ln |u| \quad \text{in } \Omega \times (0, T), \\
u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),
\end{array} \right.
\end{aligned}
\]

where the initial values \( u_0(x) \in H^1_0(\Omega) \) and \( u_1(x) \in L^2(\Omega) \), \( 0 < T \leq \infty \), \( \Omega \subset \mathbb{R}^n \) (\( n \geq 1 \)) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \), and the unknown \( u := u(x, t) \) is a real valued function defined on \( \Omega \times (0, T) \). Throughout this paper, assume that

(A) the exponent \( p \) satisfies

\[ 2 < p < 2^* := \begin{cases} \infty & \text{if } n = 1, 2, \\ \frac{2n}{n-2} & \text{if } n > 2. \end{cases} \]

(B) the relaxation function \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a \( C^1 \) function such that

\[ g(0) > 0, \quad g'(s) \leq 0 \quad \text{and} \quad l := 1 - \int_0^\infty g(s)ds > 0. \]

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Problem (1) has some relevant physical applications as shown in [15]. For instance, problem (1) models the transversal vibrations of a homogeneous string and the longitudinal vibrations of a homogeneous bar, respectively, subject to viscous effects when \( n = 1, 2 \). Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. It is well known that viscoelastic materials show the damping. These damping effects are modeled by integro-differential operators such as
\[
\int_0^t g(t-s) \Delta u(s) ds
\]
from the mathematical point of view. The study on viscoelastic wave equations has attracted many researchers. For example, Guo and Rammaha et al. studied Hadamard well-posedness, blow-up of the solutions for the history value problem of a viscoelastic wave equation which features a fading memory term as well as a supercritical source term and a frictional damping term in [6, 7]. When the logarithmic nonlinearity \(|u|^{p-2} u \ln |u|\) is replaced by the source term \(|u|^{p-2} u\), there exist many related works concerning the blow-up phenomenon of solutions for viscoelastic wave equations, the interested readers can refer to [7, 14, 17], and the reference therein. When the kernel function \( g = 0 \), Ma and Fang [13] considered the following wave equation:
\[
\ddot{u} - \Delta u_t - \Delta u = u \log^2 u.
\]
By constructing a new family of potential wells, together with logarithmic Sobolev inequality and perturbation energy technique, they established sufficient conditions to guarantee the solution exists globally or occurs infinite blow-up under suitable conditions. Di et al. [4] investigated the following strongly damped semilinear wave equations with logarithmic nonlinearity:
\[
\ddot{u} - \Delta u - \Delta u_t = |u|^{p-2} u \ln |u|.
\]
Notice that one cannot directly apply the logarithmic Sobolev inequality [3] in which the diffusion term and the logarithmic nonlinearity must have the same growth order, it is difficult to study the equation above by means of similar methods as in [13]. They derived the finite time blow-up results of weak solutions, and presented the lower and upper bounds for blow-up time by combining the concavity method, perturbation energy method with differential-integral inequality technique. Subsequently, Zu and Guo [19] improved their results by discussing the finite time blow-up for high initial energy and by giving an estimate of the lower bound for blow-up time when \( 1 + \frac{n}{n-2} < p < \frac{2n}{n-2} \).

Until recently, Ha and Park [8] investigated the viscoelastic wave equation with a strong damping and logarithmic nonlinearity, i.e. problem (1). They firstly proved the existence and uniqueness of local weak solutions by using Faedo-Galerkin’s method and contraction mapping principle. Secondly, a finite time blow-up result for the solution with positive initial energy as well as nonpositive initial energy was obtained. However, there exist many unsolved questions, for instance,

1. whether the blow-up phenomenon can happen for high initial energy or not?
2. provided that the blow-up phenomenon can happen for high initial energy, can we give upper and lower bounds of the blow-up time?

One cannot directly follow the line from [4, 19] because of the presence of the viscoelastic term. The viscoelastic term in fact plays an important role in preventing the occurrence of the blow-up phenomenon. In this paper, we shall answer the two questions above. The organization of this paper is as follows. In Section 2, some notations, definitions and lemmas that will be used in the sequel are introduced. In Section 3, the finite time blow-up of solutions for high initial energy will be considered. An upper bound for the blow-up time is derived by combining the
concaivity method. Moreover, we obtain a lower bound of the blow-up time by making full use of the strong damping $\Delta u_t$.

2. Preliminaries. Throughout this paper, we denote by $\|\cdot\|_p$ and $\|\nabla\cdot\|_2$ the norm on $L^p(\Omega)$ with $1 \leq p \leq \infty$ and $H^1_0(\Omega)$, respectively. Let $\lambda_1$ be the first eigenvalue of the following boundary value problem

$$
\begin{cases}
-\Delta \psi = \lambda \psi & \text{for } x \in \Omega, \\
\psi = 0 & \text{for } x \in \partial \Omega.
\end{cases}
$$

For completeness, firstly let us introduce the previous works on problem (1). The weak solution to problem (1) is as follows:

**Definition 2.1. (Weak solution)** Let $T > 0$. We say that a function $u := u(x,t)$ is a weak solution to problem (1) if

$$
u \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; H^{-1}(\Omega))$$

leads to

$$
\int_{\Omega} (u_t v + \nabla u \nabla v - \int_0^t g(t-s)\nabla u(s)\nabla vds + \nabla u_t \nabla v) dx = \int_\Omega |u|^{p-2}u \ln |u| v dx,
$$

for any with $v \in H^1_0(\Omega)$ and $t \in (0,T)$, and

$$u(x,0) = u_0(x) \quad \text{in } H^1_0(\Omega), \quad u_t(x,0) = u_1(x) \quad \text{in } L^2(\Omega).$$

**Theorem 2.2** ([8]). (Local existence) Let (B) hold. Assume that the exponent $p$ satisfies

$$2 < p < \begin{cases}
\infty & \text{if } n = 1,2, \\
\frac{2(n-1)}{n-2} & \text{if } n > 2.
\end{cases}$$

For the initial data $u_0 \in H^1_0(\Omega), u_1 \in L^2(\Omega)$, there exists a unique weak solution $u$ to problem (1).

If the exponent $p$ satisfies the condition (A), we can directly prove that there exists a unique weak solution for problem (1) by the combination of the theory of monotone operators, nonlinear semigroups with energy methods in [6]. In particular, when $p = 2$, that is, the nonlinear term is $u \ln |u|$, we can follow from the work in [1, 2] to prove the existence of global solutions and blow-up at $+\infty$ under some suitable assumptions.

Define the modified energy functional by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{p} \int_\Omega |u|^p \ln |u| dx + \frac{1}{p^2} \|u\|_{p^2}^p,$$

where $(g \circ u)(t) := \int_0^t (g(t-s)u(t) - u(s))\|\nabla u\|_2^2 ds$. Define

$$J(u) = \frac{1}{2} \left(1 - \int_0^\infty g(s)ds\right) \|\nabla u\|_2^2 - \frac{1}{p} \int_\Omega |u|^p \ln |u| dx + \frac{1}{p^2} \|u\|_{p^2}^p;$$

$$I(u) = \left(1 - \int_0^\infty g(s)ds\right) \|\nabla u\|_2^2 - \int_\Omega |u|^p \ln |u| dx,$$

then

$$J(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(1 - \int_0^\infty g(s)ds\right) \|\nabla u\|_2^2 + \frac{1}{p} I(u) + \frac{1}{p^2} \|u\|_{p^2}^p.$$
Define the potential depth as
\[ d = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\lambda u) = \inf_{u \in N} J(u), \]
where \( N \) is well-known Nehari manifold given by
\[ N = \{ u \in H^1_0(\Omega) \setminus \{0\} | I(u) = 0 \}. \]
Let us follow from [16], it is not hard to verify that \( d \) is positive.

**Theorem 2.3** ([8]). Let \((A)\) and \((B)\) hold. Assume that \( I(u_0) < 0 \) and \( E(0) = \alpha d \), where \( \alpha < 1 \), and the kernel function satisfies
\[ \int_0^\infty g(s)ds \leq \frac{p-2}{p-2+\frac{(1-\hat{\alpha})^2 p^2}{2(1-\hat{\alpha})}}, \]
where \( \hat{\alpha} = \max\{0, \alpha\} \). Moreover, suppose that \( \int_\Omega u_0 u_1 dx > 0 \) when \( E(0) = 0 \). Then the solution \( u \) for problem (1) blows up at finite time \( T^* \) in the sense of
\[ \lim_{t \to T^*} \left( \|u\|_2^2 + \int_0^t \|
abla u\|_2^2 ds \right) = \infty. \]  

**Lemma 2.4** ([8]). The energy functional \( E(t) \) satisfies
\[ E'(t) = \frac{1}{2} \left( g' \circ \nabla u)(t) \right) - \frac{1}{2} g(t) \|
abla u\|_2^2 - \|
abla u_t\|_2^2 \leq 0. \]

**Lemma 2.5** ([10]). Let \( \rho \) be a positive number. Then we have the following inequalities:
\[ \Psi^p \log \Psi \leq \frac{e-1}{p} \Psi^{p+\rho} \text{ for all } \Psi \geq 1, \]
and
\[ \left| \Psi^p \log \Psi \right| \leq (ep)^{-1} \text{ for all } 0 < \Psi < 1. \]

**Lemma 2.6** ([11, 12]). Suppose a positive, twice-differentiable function \( \psi(t) \) satisfies the inequality
\[ \psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0, \]
where \( \theta > 0 \). If \( \psi(0) > 0 \), \( \psi'(0) > 0 \), then \( \psi(t) \to \infty \) as \( t \to t_1 \leq t_2 = \frac{\psi(0)}{\theta \psi'(0)} \).

We are now in a position to give a pivotal lemma which can be used to prove our main results. The crucial lemma is as follows:

**Lemma 2.7.** Let \((A)\) and \((B)\) hold, and \( l > \frac{1}{(p-1)p} \). If the initial data \( u_0 \in H^1_0(\Omega) \) and \( u_1 \in L^2(\Omega) \) such that
\[ 0 < E(0) < \frac{C}{p} \int_\Omega u_0 u_1 dx. \]
Then the weak solution \( u \) to problem (1) satisfies
\[ \int_\Omega u u_t dx - \frac{p}{C} E(t) \geq \left( \int_\Omega u_0 u_1 dx - \frac{p}{C} E(0) \right) e^{Ct} > 0 \]
for any \( t \in [0, T) \), where
\[ C = \min \left\{ 2 + p, \frac{2\lambda_1 [p(p-2) - (p-1)^2 (1-l)]}{2p + \lambda_1} \right\}. \]
Proof. The idea of this proof comes from Lemma 4.1 in [9, 18]. Using directly the first equality for problem (1) and integration by parts, one has

\[ \frac{d}{dt} \int_{\Omega} uu_t dx = \|u_t\|^2 + \int_{\Omega} uu_{tt} dx \]

\[ = \|u_t\|^2 - \|\nabla u\|^2 + \int_0^t g(t-s) \int_{\Omega} \nabla u(s) \nabla u(t) ds \]

\[ - \int_{\Omega} \nabla u \nabla u_t dx + \int_{\Omega} |u|^p \ln |u| dx \]

\[ = \|u_t\|^2 - \left[ 1 - \int_0^t g(t-s) ds \right] \|\nabla u\|^2 \]

\[ + \int_0^t g(t-s) \int_{\Omega} \nabla u(t)(\nabla u(s) - \nabla u(t)) ds \]

\[ - \int_{\Omega} \nabla u \nabla u_t dx + \int_{\Omega} |u|^p \ln |u| dx. \quad (8) \]

Taking full advantage of Young’s inequality, then

\[ \int_0^t g(t-s) \int_{\Omega} \nabla u(t)(\nabla u(s) - \nabla u(t)) ds \]

\[ \geq - \frac{1}{2p} \int_0^t g(s) ds \|\nabla u\|^2 - \frac{p}{2} (g \circ \nabla u)(t), \quad (9) \]

\[ \int_{\Omega} \nabla u \nabla u_t dx \leq \frac{C}{4p} \|\nabla u\|^2 + \frac{p}{C} \|\nabla u_t\|^2. \quad (10) \]

Combining (9) and (10) with (8), and using the definition of \( E(t) \) and the condition (B), it follows that

\[ \frac{d}{dt} \int_{\Omega} uu_t dx \geq \|u_t\|^2 - \left[ 1 - \int_0^t g(t-s) ds \right] \|\nabla u\|^2 - \frac{1}{2p} \int_0^t g(s) ds \|\nabla u\|^2 \]

\[ - \frac{p}{2} (g \circ \nabla u)(t) - \frac{C}{4p} \|\nabla u\|^2 - \frac{p}{C} \|\nabla u_t\|^2 + \int_{\Omega} |u|^p \ln |u| dx \]

\[ \geq \left( 1 + \frac{p}{2} \right) \|u_t\|^2 + \left[ \left( \frac{p}{2} - 1 \right) + \left( 1 - \frac{1}{2p} - \frac{p}{2} \right) \right] \int_0^t g(s) ds \|\nabla u\|^2 \]

\[ + \frac{1}{p} \|u\|^p - pE(t) - \frac{C}{4p} \|\nabla u\|^2 - \frac{p}{C} \|\nabla u_t\|^2 \]

\[ \geq \left( 1 + \frac{p}{2} \right) \|u_t\|^2 + \left[ \left( \frac{p}{2} - 1 \right) + \left( 1 - \frac{1}{2p} - \frac{p}{2} \right)(1 - l) - \frac{C}{4p} \right] \|\nabla u\|^2 \]

\[ + \frac{1}{p} \|u\|^p - pE(t) - \frac{p}{C} \|\nabla u_t\|^2. \quad (11) \]

Recalling (7), then

\[ C \leq \frac{2\lambda_1 [p(p-2) - (p-1)^2(1-l)]}{2p + \lambda_1} < 2[p(p-2) - (p-1)^2(1-l)]. \]

Therefore, (11) yields that
\[
\frac{d}{dt} \int_{\Omega} uu_t \, dx \geq \left( 1 + \frac{p}{2} \right) \|u_t\|^2_2 + \left[ \left( \frac{p}{2} - 1 \right) + \left( 1 - \frac{1}{2p} - \frac{p}{2} \right) (1 - l) - \frac{C}{4p} \right] \lambda_1 \|u\|^2_2 \\
- pE(t) - \frac{p}{C} \|\nabla u_t\|^2_2,
\]

(12)

where we use \( \lambda_1 \|u\|^2_2 \leq \|\nabla u\|^2_2 \).

Define
\[
F(t) = \int_{\Omega} uu_t \, dx - \frac{p}{C} E(t).
\]

Combining (12) with Lemma 2.4, one has
\[
\frac{d}{dt} F(t) \geq \left( 1 + \frac{p}{2} \right) \|u_t\|^2_2 + \left[ \left( \frac{p}{2} - 1 \right) + \left( 1 - \frac{1}{2p} - \frac{p}{2} \right) (1 - l) - \frac{C}{4p} \right] \lambda_1 \|u\|^2_2 - pE(t) \\
\geq C \left( \frac{1}{2} \|u_t\|^2_2 + \frac{1}{2} \|u\|^2_2 - \frac{p}{C} E(t) \right) \geq CF(t)
\]

(13)

according to (7).

It is noted that \( F(0) = \int_{\Omega} u_0 u_1 \, dx - \frac{p}{C} E(0) > 0 \). By Gronwall’s inequality, thus
\[
F(t) \geq e^{Ct} F(0) > 0.
\]

This proof is complete.

This proof is complete.

\[ \square \]

3. Blow-up for high initial energy. In this section, we are committed to proving the finite time blow-up for high initial energy and to estimating an upper bound of the blow-up time. At first, the finite time blow-up for high initial energy is derived as follows:

**Theorem 3.1. (Finite time blow-up for high initial energy)** Let all the assumptions in Lemma 2.7 be fulfilled. Then the solution \( u \) for problem (1) blows up in finite time.

**Proof.** We prove this theorem by contradiction. That is, assume that the solution \( u \) for problem (1) is global.

On the one hand, Hölder’s inequality and Lemma 2.4 indicate that for all \( t \in [0, \infty) \),
\[
\|u\|_2 = \left\| u_0(x) + \int_0^t u_\tau \, d\tau \right\|_2 \leq \|u_0(x)\|_2 + \int_0^t \|u_\tau\|_2 \, d\tau \\
\leq \|u_0(x)\|_2 + \frac{1}{\sqrt{\lambda_1}} \int_0^t \|\nabla u_\tau\|_2 \, d\tau \\
\leq \|u_0(x)\|_2 + \frac{\sqrt{t}}{\sqrt{\lambda_1}} \left( \int_0^t \|\nabla u_\tau\|^2_2 \, d\tau \right)^{\frac{1}{2}} \\
\leq \|u_0(x)\|_2 + \frac{\sqrt{t}}{\sqrt{\lambda_1}} (E(0) - E(t))^{\frac{1}{2}}.
\]

(14)

Since \( u \) is a global solution of problem (1), we have \( E(t) \geq 0 \) for all \( t \in [0, \infty) \). Otherwise, there exists \( t_0 \in [0, \infty) \) such that \( E(t_0) < 0 \). Recalling the definitions of \( E(t), J(u), I(u) \) and (3), we obtain
\[
E(t) \geq \frac{1}{2} \|u_t\|^2_2 + J(u) + \frac{1}{2} (g \circ \nabla u)(t)
\]
\[\begin{align*}
\frac{1}{2} \|\mathbf{u} \|_2^2 + \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \int_0^\infty g(s) ds \right) \| \nabla \mathbf{u} \|_2^2 \\
+ \frac{1}{p} I(u) + \frac{1}{p^2} \| u \|_p^p + \frac{1}{2} (g \circ \nabla u)(t).
\end{align*}\] (15)

Since \( p > 2 \) and the condition \((B)\), we get that \( E(t_0) < 0 \) implies \( I(u(x, t_0)) < 0 \) by (15). Choosing \( u(x, t_0) \) as the new initial data, Theorem 2.3 indicates that \( u \) blows up in finite time, which is a contradiction. Thus, according to Lemma 2.4, we obtain \( 0 \leq E(t) \leq E(0) \). Further, (14) can be rewritten as

\[\| \mathbf{u} \|_2 \leq \| \mathbf{u}_0(x) \|_2 + \frac{\sqrt{t}}{\sqrt{\lambda_1}} (E(0))^\frac{1}{2}\] (16)

for all \( t \in [0, \infty) \).

On the other hand, let us apply (6), then we get

\[\frac{d}{dt} \| \mathbf{u} \|_2^2 = 2 \int_\Omega u \mathbf{u}_t dx \geq 2F(0)e^{Ct} + \frac{2p}{C} E(t) \geq 2F(0)e^{Ct} > 0.\] (17)

Integrating (17) from 0 to \( t \) yields

\[\| \mathbf{u} \|_2^2 = \| \mathbf{u}_0(x) \|_2^2 + 2 \int_0^t \int_\Omega u \mathbf{u}_t dx \, dt \geq \| \mathbf{u}_0(x) \|_2^2 + 2 \int_0^t e^{Ct} F(0) \, dt \]

\[= \| \mathbf{u}_0(x) \|_2^2 + \frac{2}{C} (e^{Ct} - 1) F(0),\]

which contradicts (16) for \( t \) sufficiently large. Thus, the solution \( u \) for problem (1) blows up in finite time. \( \square \)

**Theorem 3.2. (Upper bound of the blow-up time)** Let all the assumptions in Lemma 2.7 be fulfilled. In addition, if

\[E(0) \leq \frac{C}{2p} \| \mathbf{u}_0 \|_2^2,\] (19)

then the solution \( u \) for problem (1) blows up at some finite time \( T^* \) in the sense of (4). Furthermore, the blow-up time \( T^* \) can be estimated from above as follows

\[T^* \leq \frac{2(||\mathbf{u}_0||_2^2 + \gamma \sigma^2)}{(p-2) \left[ \int_\Omega u_0 u_1 dx + \gamma \sigma \right] - 2 \| \nabla \mathbf{u}_0 \|_2^2},\]

where \( C \) is the positive constant shown in (7), \( \gamma = \frac{-2pE(0)+C||\mathbf{u}_0||_2^2}{2p}, \) and \( \sigma > 0 \) is sufficiently large such that

\[ (p-2) \left[ \int_\Omega u_0 u_1 dx + \gamma \sigma \right] - 2 \| \nabla \mathbf{u}_0 \|_2^2 > 0.\] (20)

**Proof.** Obviously, Theorem 3.1 implies that the solution \( u \) for problem (1) blows up in finite time. Suppose that the blow-up time is \( T^* \). Now, we need to estimate an upper of \( T^* \).

Define the auxiliary function as in [8]

\[M(t) = \| \mathbf{u} \|_2^2 + \int_0^t \| \nabla \mathbf{u} \|_2^2 dx + (T^* - t) \| \nabla \mathbf{u}_0 \|_2^2 + \gamma (t + \sigma)^2 \text{ for } t \in [0, T^*).\]
By a direct computation, one has
\[ M'(t) = 2 \int_{\Omega} uu_t dx + \| \nabla u \|^2 - \| \nabla u_0 \|^2 + 2\gamma(t + \sigma) = 2 \int_{\Omega} uu_t dx + 2 \int_0^t \int_{\Omega} \nabla u \nabla u_t dx d\tau + 2\gamma(t + \sigma) \quad \text{for } t \in [0, T^*). \]

From the equality above and problem (1), it is obtained that
\[ M''(t) = 2\| u_t \|^2 + 2 \int_{\Omega} uu_t dx + 2 \int_{\Omega} \nabla u u_t dx + 2\gamma \]
\[ = 2\| u_t \|^2 - 2\| \nabla u \|^2 + 2 \int_0^t g(t-s) \int_{\Omega} \nabla u(s) \nabla u(t) dx ds + 2 \int_{\Omega} |u|^p \ln |u| dx + 2\gamma \]
\[ = 2\| u_t \|^2 - 2 \left( 1 - \int_0^t g(s)ds \right) \| \nabla u \|^2 - 2 \int_0^t g(t-s) \int_{\Omega} \nabla u(t)(\nabla u(t) - \nabla u(s)) dx ds \]
\[ + 2 \int_{\Omega} |u|^p \ln |u| dx + 2\gamma \]
for \( t \in [0, T^*) \). Applying Cauchy-Schwarz inequality and Young’s inequality, one has
\[ \xi(t) := \left[ \| u_t \|^2 + \int_0^t \| \nabla u \|^2 d\tau + \gamma(t + \sigma)^2 \right] \left[ \| u_t \|^2 + \int_0^t \| \nabla u_t \|^2 d\tau + \gamma \right] \]
\[ - \left[ \int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} \nabla u \nabla u_t dx d\tau + \gamma(t + \sigma) \right]^2 \geq 0 \quad \text{for } t \in [0, T^*). \]

Therefore,
\[ M(t)M''(t) - \frac{p+2}{4} (M'(t))^2 \]
\[ = M(t)M''(t) - \frac{p+2}{4} \left[ 2 \int_{\Omega} uu_t dx + 2 \int_{\Omega} \nabla u \nabla u_t dx d\tau + 2\gamma(t + \sigma) \right]^2 \]
\[ = M(t)M''(t) - (p+2)M(t) \left( \| u_t \|^2 + \int_0^t \| \nabla u_t \|^2 d\tau + \gamma \right) + (p+2)\xi(t) \]  \hfill (21)
\[ + (p+2)(T^*-t)\| u_0 \|^2 \left( \| u_t \|^2 + \int_0^t \| \nabla u_t \|^2 d\tau + \gamma \right) \geq M(t)\eta(t) \quad \text{for } t \in [0, T^*), \]
where
\[ \eta(t) = M''(t) - (p+2) \left( \| u_t \|^2 + \int_0^t \| \nabla u_t \|^2 d\tau + \gamma \right) \]
\[ = -p\| u_t \|^2 - 2 \left( 1 - \int_0^t g(s)ds \right) \| \nabla u \|^2 \]
\[ - 2 \int_0^t g(t-s) \int_{\Omega} \nabla u(t)(\nabla u(t) - \nabla u(s)) dx ds \]
\[ + 2 \int_{\Omega} |u|^p \ln |u| dx - (p+2) \int_0^t \| \nabla u_t \|^2 d\tau - p\gamma. \]
Using (2), Lemma 2.4 and Young’s inequality with $\beta > 0$, we obtain

$$
\eta(t) = -2pE(t) + (p-2)\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_p^2 + p(g \circ \nabla u)(t) + \frac{2}{p}\|u\|_p^p
- 2\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(t) - \nabla u(s))dxds - (p+2)\int_0^t \|\nabla u_r\|_2^2dt - p\gamma
= -2pE(0) + (p-2)\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_p^2 + p(g \circ \nabla u)(t) + \frac{2}{p}\|u\|_p^p
- 2\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(t) - \nabla u(s))dxds - (p+2)\int_0^t \|\nabla u_r\|_2^2dt - p\gamma
\geq -2pE(0) + \left[(p-2)\left(1 - \int_0^t g(s)ds\right) - 2\beta\int_0^t g(s)ds\right]\|\nabla u\|_2^2
+ \left[p - \frac{1}{2\beta}\right](g \circ \nabla u)(t) + \frac{2}{p}\|u\|_p^p + (p-2)\int_0^t \|\nabla u_r\|_2^2dt - p\gamma
$$

for $t \in [0, T^*)$. By letting $\beta = \frac{1}{2p}$, and recalling $t > \frac{1}{(p-1)^2}$, then for $t \in [0, T^*)$,

$$
\eta(t) \geq -2pE(0) + \left[(p-2) - \left(1 - \frac{1}{p}\right)\lambda_1\|u_0\|_2^2 - p\gamma\right]
\geq -2pE(0) + \left[(p-2) - \left(1 - \frac{1}{p}\right)\lambda_1\|u_0\|_2^2 - p\gamma\right]
$$

Notice that (17) implies

$$
\|u\|_2^2 \geq \|u_0\|_2^2 \quad \text{for } t \in [0, T^*).
$$

Thus, it follows by combining (21) with (23) (24) and (19) that

$$
M(t)M'(t) - \frac{p+2}{4}(M'(t))^2
\geq M(t)\left\{-2pE(0) + \left[(p-2) - \left(1 - \frac{1}{p}\right)\lambda_1\|u_0\|_2^2 - p\gamma\right]\right\}
\geq M(t)\left\{-2pE(0) + C\|u_0\|_2^2 - p\gamma\right\} \geq 0 \quad \text{for } t \in [0, T^*),
$$

here we use $\gamma = \frac{-2pE(0)+C\|u_0\|_2^2}{2p}$ and (19). Notice that

$$
M(0) = \|u_0\|_2^2 + T^*\|\nabla u_0\|_2^2 + \gamma\sigma^2 > 0,
$$

$$
M'(0) = 2\int_\Omega u_0u_1dx + 2\gamma\sigma > 0,
$$

thus, making use of Lemma 2.6 yields that $M(t) \to \infty$ as $t \to T^*$ with

$$
T^* \leq \frac{4M(0)}{(p-2)M'(0)} = \frac{2(\|u_0\|_2^2 + T^*\|\nabla u_0\|_2^2 + \gamma\sigma^2)}{(p-2)\left[\int_\Omega u_0u_1dx + \gamma\sigma\right]}
$$

It follows from (20) that

$$
T^* \leq \frac{2(\|u_0\|_2^2 + \gamma\sigma^2)}{(p-2)\left[\int_\Omega u_0u_1dx + \gamma\sigma\right] - 2\|\nabla u_0\|_2^2}
$$

This proof is complete. \qed
Remark 1. As a consequence of Theorem 3.2, it is obvious to prove that for any \( m > d \), there exist initial data \( u_0 \) and \( u_1 \) such that \( E(0) > m \) as well as \( u_0 \) and \( u_1 \) satisfy all the assumptions in Theorem 3.2 by applying the similar proof of Theorem 3.13 in [5]. Then the weak solution of problem (1) blows up at finite time for arbitrarily high energy initial data.

**Theorem 3.3. (Lower bound for the blow-up time)** Let \( n \geq 3 \). If all the conditions of Theorem 3.2 are satisfied, then the blow-up time \( T^* \) can be estimated from below as follows

\[
\int_{R(0)}^{+\infty} \frac{1}{C_1 y^{p-1+\mu} + y + C_2} dy \leq T^*,
\]

where \( \mu > 0 \) is constant, the constants \( C_1, C_2 \) and the initial data \( R(0) \) are defined in (33), (34) and (36), respectively.

**Proof.** Define an auxiliary function

\[
R(t) = \frac{1}{2} \| u_t \|_2^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \| \nabla u \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) = E(t) + \frac{1}{p} \int_\Omega |u|^p \ln |u| dx - \frac{1}{p^*} \| u \|_{p^*}^p.
\]  

(26)

Recalling Lemma 2.4, we have

\[
R'(t) = E'(t) + \int_\Omega |u|^{p-2} u u_t \ln |u| dx
\]

\[
\leq -\| \nabla u_t \|_2^2 + \int_\Omega |u|^{p-2} u u_t \ln |u| dx.
\]  

(27)

On the other hand, from (2), we get

\[
\frac{1}{2} \| u_t \|_2^2 + \frac{1}{2} \| \nabla u \|_2^2
\]

\[
\leq \frac{1}{2} \| u_t \|_2^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \| \nabla u \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) = R(t).
\]  

(28)

Therefore, the conclusion of Theorem 3.2 and (28) indicate

\[
\lim_{t \to T^*} R(t) = \infty.
\]  

(29)

Let us divide \( \Omega \) into two parts as follows:

\[
\Omega_1 = \{ x \in \Omega : |u| \geq 1 \} \quad \text{and} \quad \Omega_2 = \{ x \in \Omega : |u| < 1 \}.
\]

Applying Lemma 2.5, Hölder’s inequality, Young’s inequality with \( \varepsilon > 0 \) and the embedding \( H^1_0(\Omega) \to L^{2^*}(\Omega) \) with \( 2^* = \frac{2n}{n-2} \), it follows that

\[
\int_\Omega |u|^{p-2} u u_t \ln |u| dx = \int_{\Omega_1} |u|^{p-2} u u_t \ln |u| dx + \int_{\Omega_2} |u|^{p-2} u u_t \ln |u| dx
\]

\[
\leq (e \mu)^{-1} \int_{\Omega_1} |u|^{p-1+\mu} u_t dx + (e(p-1))^{-1} \int_{\Omega_2} |u_t| dx
\]

\[
\leq \left( e \mu \right)^{-1} \left\| u \right\|_{p-1+\mu}^{\frac{2n}{n+2}} \| u_t \|_2^2 + (e(p-1))^{-1} |\Omega_2|^\frac{1}{2} \| u_t \|_2.
\]
Using again the embedding $H^1(\Omega) \hookrightarrow L^{2(n+2)/(n+2)}(\Omega)$ and (28), then
\[
\int_{\Omega} |u|^{p-2} u u_t \log |u| dx \leq \frac{B^2(e\mu)^{-2}}{2\varepsilon} B^2(\mu - 1) \| \nabla u \|^2 (p-1+\mu) + \frac{\varepsilon}{2} \| \nabla u_t \|^2 + \frac{1}{2} (e(p-1))^{-2} |\Omega_2| + \frac{1}{2} |u_t|^2
\]
(30)

Notice that $\frac{2n(p-1)}{n+2} < 2^*$, then we could choose $\mu > 0$ such that $\frac{2n(p-1+\mu)}{n+2} \leq 2^*$.

Using again the embedding $H^1_0(\Omega) \hookrightarrow L^{2(n+2)/(n+2)}(\Omega)$ and (28), then
\[
\int_{\Omega} |u|^{p-2} u u_t \log |u| dx \leq \frac{B^2(e\mu)^{-2}}{2\varepsilon} B^2(\mu - 1) \| \nabla u \|^2 (p-1+\mu) + \frac{\varepsilon}{2} \| \nabla u_t \|^2 + \frac{1}{2} (e(p-1))^{-2} |\Omega_2| + R(t).
\]
(31)

Let us choose $\varepsilon = 2$, and then combine (27) with (31), we obtain
\[
R'(t) \leq C_1 R(t)^{p-1+\mu} + R(t) + C_2
\]
(32)

with
\[
C_1 = \frac{B^2(e\mu)^{-2}}{2\varepsilon} B^2(\mu - 1) \left( \frac{2}{7} \right)^{p-1+\mu},
\]
(33)

\[
C_2 = \frac{1}{2} (e(p-1))^{-2} |\Omega_2|.
\]
(34)

Integrating (32) over $[0, t]$, one has
\[
\int_0^t \frac{R'(s)}{C_1 R(s)^{p-1+\mu} + R(s) + C_2} ds \leq t.
\]
(35)

Let $t \to T^*$ in (35) and recall (29), then
\[
\int_{R(0)}^{+\infty} \frac{1}{C_1 y^{p-1+\mu} + y + C_2} dy \leq T^*
\]
with
\[
R(0) = \frac{1}{2} |u_1|^2 + \frac{1}{2} |\nabla u_0|^2.
\]
(36)

This completes the proof of this Theorem. \qed

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