RATIONAL NUMBERS IN $\times b$-INVARIANT SETS

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Abstract. Let $b \geq 2$ be an integer and $S$ be a finite non-empty set of primes not containing divisors of $b$. For any non-dense set $A \subseteq [0,1)$ such that $A \cap \mathbb{Q}$ is invariant under $\times b$ operation, we prove the finiteness of rational numbers in $A$ whose denominators can only be divided by primes in $S$. A quantitative result on the largest prime divisors of the denominators of rational numbers in $A$ is also obtained.

1. Introduction

Let $C$ be the classical middle-third Cantor set, which consists of real numbers in $[0,1]$ whose ternary expansions do not contain digit 1. In 1984, Mahler [5] asked how close can irrational elements in $C$ be approximated by rational numbers in $C$. A related question is what are the rational numbers in $C$. For any $n \geq 1$, we know that there are exactly $2^{n+1}$ rational numbers of the form $\frac{a}{3^n}$ in $C$ with $a \in \mathbb{Z}$. One may ask what happens if the denominator is a $d$-power for some $d \geq 2$ instead of a 3-power. For $d = 2$, Wall [10] proved that $\frac{1}{4}$ and $\frac{3}{4}$ are the only dyadic rationals in $C$. More generally, let $S$ be a finite set of primes, then the set of $S$-integers $\mathbb{Z}_S$ is defined to be the set of rational numbers whose denominators can only be divided by primes in $S$. Equivalently,

$$\mathbb{Z}_S = \{ \alpha \in \mathbb{Q} : v_p(\alpha) < 0 \text{ implies } p \in S \},$$

where $v_p(\alpha)$ is the unique integer such that $\alpha = p^{v_p(\alpha)}m/n$ for some $m, n \in \mathbb{Z}$ coprime with $p$. One may wonder what does the set $\mathbb{Z}_S \cap C$ look like. When $S = \{2, 5\}$, it was proved by Wall [11] that $\mathbb{Z}_{\{2,5\}} \cap C$ consists of exactly 14 elements. Later, Nagy [6] showed that, if $S = \{p\}$ for some prime $p > 3$, then $C$ contains only finitely many $S$-integers. Recently, based on a heuristic argument as well as numerical evidence, Rahm, Solomon, Trauthwein and Weiss [7] formulated an asymptotic for the number $N^*(T)$ of reduced rational numbers in $C$ with denominators bounded by $T$.

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The distribution of rational numbers could also be studied for generalized Cantor sets. Let \( b \geq 2 \) be an integer and \( D \) be a non-empty subset of \( \{0, 1, \ldots, b - 1\} \), the generalized Cantor set \( C(b, D) \) is the set of real numbers in \([0, 1)\) whose base \( b \) expansions only consist of digits in \( D \). Extending Nagy’s work, Bloshchitsyn [1] proved that for any integer \( b \geq 3 \), \( D \) with cardinality \( b - 1 \) and prime \( p > b^2 \), the set \( \mathbb{Z}_p \cap C(b, D) \) is finite. A very recent work of Schleischitz [8, Corollary 4.4] showed that \( C(b, D) \) contains only finitely many \( S \)-integers if no element of \( S \) divides \( b \) and \( D \) has cardinality at most \( b - 1 \). Shparlinski [9] proved a quantitative strengthen of Schleischitz’s result. To state Shparlinski’s result, we first introduce some notations. For an integer \( d \geq 2 \), denote the largest prime divisor of \( d \) by \( P(d) \) and define the radical of \( d \) by

\[
\text{rad}(d) = \prod_{p|d, \ p \text{ prime}} p.
\]

**Theorem 1.1** ([9]). Let \( b \geq 2 \) be an integer and \( D \subseteq \{0, 1, \ldots, b - 1\} \) be a non-empty set of cardinality at most \( b - 1 \). Then there exists a constant \( c_b > 0 \), depending only on \( b \), such that for any rational number \( \frac{a}{d} \) in \( C(b, D) \) with \( \gcd(ab, d) = 1 \), we have

\[
\text{rad}(d) \geq c_b \log d \quad \text{and} \quad P(d) \geq c_b \sqrt{\log d \log \log d}.
\]

In this article, we investigate \( S \)-integers in \( T_b \)-invariant sets. For any integer \( b \geq 2 \), the transformation \( T_b: [0, 1) \rightarrow [0, 1) \) is defined by

\[
T_b(x) = bx \pmod{1}.
\]

We say that a set \( A \subseteq [0, 1) \) is \( T_b \)-invariant if \( T_b(A) \subseteq A \). Clearly all generalized Cantor sets \( C(b, D) \) are \( T_b \)-invariant. Our finiteness result is as follows.

**Theorem 1.2.** Let \( b \geq 2 \) be an integer, \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \), and \( A \) be a subset of \([0, 1)\). If \( A \) is not dense in \([0, 1]\) and \( T_b(A \cap \mathbb{Q}) \subseteq A \), then \( A \) contains at most finitely many \( S \)-integers.

**Remark 1.3.** Note that in Theorem 1.2 we only require that \( T_b(A \cap \mathbb{Q}) \subseteq A \), which is weaker than that \( A \) is \( T_b \)-invariant.

Indeed, we have obtained a result on the \( \varepsilon \)-dense property of orbits of \( S \)-integers under \( T_b \), which directly gives Theorem 1.2.

**Theorem 1.4.** Let \( b \geq 2 \) be an integer and \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \). For any \( \varepsilon > 0 \), there
exists an effectively computable positive number $D$, such that for any $\frac{a}{d} \in \mathbb{Z}_S \cap [0,1)$ with $(a,d) = 1$ and $d > D$, the orbit of $\frac{a}{d}$ under $T_b$,

$$\text{Orb}_{T_b} \left( \frac{a}{d} \right) := \left\{ T_b^i \left( \frac{a}{d} \right) : i \geq 0 \right\},$$

(1.1)

is $\varepsilon$-dense in $[0,1]$.

**Remark 1.5.** The number $D$ in Theorem 1.4 is given in (2.7).

We also have a quantitative result that strengthens Theorem 1.2.

**Theorem 1.6.** Let $b \geq 2$ be an integer and $A \subseteq [0,1)$ be a set satisfying $T_b(A \cap \mathbb{Q}) \subseteq A$. Suppose $A$ is not dense in $[0,1]$ and let $\varepsilon = \sup\{\text{dist}(x,A) : x \in [0,1]\}$, where $\text{dist}(x,A)$ denotes the distance between $x$ and $A$. Then there exists an absolute constant $K > 0$ such that for any rational number $\frac{a}{d}$ in $A$ with $\gcd(ab,d) = 1$ and $\varepsilon d \geq 3$, we have

$$P(d) \geq \begin{cases} K \sqrt{\frac{1}{\log b} \log (2\varepsilon d) \log \log (2\varepsilon d)} & \text{if } P(d) > b, \\ K \sqrt{\frac{1}{\log b} \log (2\varepsilon d)} & \text{if } P(d) < b. \end{cases}$$

(1.2)

**Remark 1.7.** (i) The absolute constant $K$ in Theorem 1.6 can be effectively computed.

(ii) Note that the assumption that $\gcd(ab,d) = 1$ guarantees that $P(d) \neq b$.

(iii) The assumption that $\varepsilon d \geq 3$ is simply to guarantee that $\log \log (2\varepsilon d)$ is positive. The number 3 can be slightly decreased if needed.

(iv) Theorem 1.6 can be applied to any generalized Cantor set $C(b, \mathcal{D})$, since such set is $T_b$-invariant and $\varepsilon = \frac{m}{2b}$, where $m$ is the largest number of consecutive integers (which may be a single integer) in $\{0,1,\ldots,b-1\} \setminus \mathcal{D}$. Note that our bounds become sharper when $m$ increases, which coincides with the intuition that when $m$ increases there are less rational numbers in $C(b, \mathcal{D})$. This phenomenon is not reflected in Theorem 1.1.

2. **Finiteness of $S$-integers**

Let $b \geq 2$ be an integer and $S$ be a non-empty finite set of primes not containing any prime divisor of $b$. In this section, we prove our $\varepsilon$-dense result Theorem 1.4, and then deduce the finiteness result Theorem 1.2.

We begin with some notations. For any positive integers $b, L$, we use $\overline{b}$ (mod $L$) to denote the coset in $\mathbb{Z}/L\mathbb{Z}$ containing $b$, and when $L$ is clear, we simply write it as $\overline{b}$. Let $(\mathbb{Z}/L\mathbb{Z})^\times$ be the multiplicative
group which consists of \( \ell \) with \( \ell \) relatively prime with \( L \). If \( b \) and \( L \) are coprime, then we denote the order of \( \overline{b} \) by \( \text{ord}(\overline{b}, L) \) and the cyclic subgroup generated by \( \overline{b} \) is \( \langle \overline{b} \rangle \). Equivalently, \( \text{ord}(\overline{b}, L) \) is the smallest positive integer such that \( b^{\text{ord}(\overline{b}, L)} \equiv 1 \pmod{L} \). Let \( G \) be a finite group, the exponent of \( G \), denoted by \( \exp(G) \), is the smallest positive integer such that \( g^{\exp(G)} = 1 \) for all \( g \in G \).

Recall the following basic result on the multiplicative group \((\mathbb{Z}/p^n\mathbb{Z})^\times\).

**Lemma 2.1** ([4, Chapter 4]). For any \( n \geq 3 \), we have

\[
(\mathbb{Z}/2^n\mathbb{Z})^\times \cong \langle -1 \rangle \times \langle 5 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}.
\]

For any odd prime \( p \) and \( n \geq 1 \), we have

\[
(\mathbb{Z}/p^n\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)p^{n-1}\mathbb{Z}.
\]

Next lemma gives relation between orders of elements.

**Lemma 2.2.** [2, Proposition 5, Chapter 2, Section 3] Let \( G \) be a group and \( g \in G \) be an element with finite order \( s \). Then for each integer \( t \geq 1 \), the order of \( g^t \) is \( \frac{s}{\text{gcd}(s, t)} \).

Now we prove a key lemma concerning the order of \( \overline{b} \).

**Lemma 2.3.** Let \( b \geq 2 \) be an integer and \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \). For each \( p \in S \), define

\[
n_p = \begin{cases} 
\max\{3, v_2(b-1), v_2(b+1)\}, & \text{if } p = 2, \\
\max\{1, v_p(b^{p-1}-1)\}, & \text{if } p \neq 2,
\end{cases}
\]

and

\[
N_p = \max\{n_p - v_p(\text{ord}(\overline{b}, p^{n_p})) + v_p(\text{ord}(\overline{b}, q^{n_q})) : q \in S\}.
\]

Then for any integer \( d = \prod_{p \in S} p^{e_p} \), we have

\[
\text{ord}(\overline{b}, d) = \left( \prod_{\substack{p \in S: e_p > N_p}} p^{e_p-N_p} \right) \text{ord} \left( \overline{b}, \prod_{p \in S} p^{\min\{e_p, N_p\}} \right). \tag{2.2}
\]

where \( \prod_{p \in S: e_p > N_p} p^{e_p-N_p} \) is defined to be 1 if \( e_p \leq N_p \) for all \( p \in S \).

**Proof.** We start with the case \( S = \{2\} \). In this case, we have \( N_2 = n_2 \). If \( e_2 \leq N_2 \), then (2.2) is trivial. So it suffices to prove

\[
\text{ord}(\overline{b}, 2^e) = 2^{e-n_2} \text{ord}(\overline{b}, 2^{n_2})\text{ for any } e > n_2.
\]

Since \( e > n_2 \), we have \( b \not\equiv \pm 1 \pmod{2^e} \). So Lemma 2.1 implies that \( \overline{b} \in \langle 5 \rangle \) or \( \langle -5 \rangle \) in \((\mathbb{Z}/2^e\mathbb{Z})^\times\). Let \( \overline{g} = \overline{5} \) or \( -\overline{5} \) such that \( \overline{b} = \overline{g}^t \) for
some \( t \geq 1 \). In the group \((\mathbb{Z}/2^{n_2+1}\mathbb{Z})^\times\), the element \( \overline{g} \) has order \( 2^{n_2-1} \) by Lemma 2.1, so Lemma 2.2 and \( \overline{f} = \overline{g}^t \) imply that
\[
\text{ord}(\overline{b}, 2^{n_2+1}) \mid \gcd(t, 2^{n_2-1}) = 2^{n_2-1}.
\]
Note that \( b \not\equiv 1 \pmod{2^{n_2+1}} \), so \( \text{ord}(\overline{b}, 2^{n_2+1}) \neq 1 \), hence \( \gcd(t, 2^{n_2-1}) \leq 2^{n_2-2} \), and thus \( v_2(t) \leq n_2 - 2 \). Applying the same argument to the groups \((\mathbb{Z}/2^{n_2}\mathbb{Z})^\times\) and \((\mathbb{Z}/2^e\mathbb{Z})^\times\), we have
\[
\text{ord}(\overline{b}, 2^{n_2}) \mid \gcd(t, 2^{n_2-2}) = 2^{n_2-2}, \quad \text{ord}(\overline{b}, 2^e) \mid \gcd(t, 2^{e-2}) = 2^{e-2}.
\]
Now \( v_2(t) \leq n_2 - 2 \) implies that \( \gcd(t, 2^{n_2-2}) = \gcd(t, 2^{e-2}) \), so
\[
\text{ord}(\overline{b}, 2^e) = 2^{e-n_2} \text{ord}(\overline{b}, 2^{n_2}).
\]

Next we treat the case \( S = \{p\} \) for some odd prime \( p \). The proof for this case is a simpler version of the \( S = \{2\} \) case. Again, it suffices to prove
\[
\text{ord}(\overline{b}, p^e) = p^{e-n_p} \text{ord}(\overline{b}, p^{n_p}) \quad \text{for any } e > n_p.
\]
Let \( \overline{g} \) be a generator of \((\mathbb{Z}/p^n\mathbb{Z})^\times\), so \( \overline{b} = \overline{g}^t \) for some \( t \geq 1 \). Applying Lemma 2.1 and Lemma 2.2 to the groups \((\mathbb{Z}/p^n\mathbb{Z})^\times\), \((\mathbb{Z}/p^e\mathbb{Z})^\times\) and \((\mathbb{Z}/p^{n_p+1}\mathbb{Z})^\times\), we have
\[
\begin{align*}
\text{ord}(\overline{b}, p^n) \mid \gcd(t, (p - 1)p^{n_p-1}) &= (p - 1)p^{n_p-1}, \\
\text{ord}(\overline{b}, p^e) \mid \gcd(t, (p - 1)p^{e-1}) &= (p - 1)p^{e-1}, \\
\text{ord}(\overline{b}, p^{n_p+1}) \mid \gcd(t, (p - 1)p^{n_p}) &= (p - 1)p^{n_p}.
\end{align*}
\]
Since \( b^{p-1} \not\equiv 1 \pmod{p^{n_p+1}} \), we have \( p - 1 \nmid \text{ord}(\overline{b}, p^{n_p+1}) \). So \( p^{n_p} \nmid \gcd(t, (p - 1)p^{n_p}) \), and thus \( v_p(t) \leq n_p - 1 \). Then \( \gcd(t, (p - 1)p^{n_p-1}) = \gcd(t, (p - 1)p^{e-1}) \) and therefore
\[
\text{ord}(\overline{b}, p^e) = p^{e-n_p} \text{ord}(\overline{b}, p^{n_p}).
\]

Finally we consider the general case. We are going to show that
\[
v_q(\text{ord}(\overline{b}, d)) = v_q \left( \prod_{p \in S, \ e_p > n_p} p^{e_p-n_p} \right) + v_q \left( \text{ord} \left( \overline{b}, \prod_{p \in S} p^{\min\{e_p, n_p\}} \right) \right)
\]
for all primes \( q \), which would imply (2.2).

By the Chinese Reminder Theorem, the map
\[
f : \mathbb{Z}/d\mathbb{Z} \to \prod_{p \in S} \mathbb{Z}/p^{e_p}\mathbb{Z}
\]
\[
\overline{a} \pmod{d} \mapsto (\overline{a} \pmod{p^{e_p}})_{p \in S}
\]
is a ring isomorphism. So it induces a group isomorphism

\[ f : (\mathbb{Z}/d\mathbb{Z})^\times \rightarrow \prod_{p \in S} (\mathbb{Z}/p^{e_p} \mathbb{Z})^\times. \]

Therefore

\[
\text{ord}(\overline{b}, d) = \exp(\overline{b} \pmod{d}) \\
= \exp((\overline{b} \pmod{p^{e_p}})_{p \in S}) \\
= \text{lcm}\{\exp(\overline{b} \pmod{p^{e_p}}) : p \in S\} \\
= \text{lcm}\{\text{ord}(\overline{b}, p^{e_p}) : p \in S\}.
\]

So for any prime \( q \), we have

\[
v_q(\text{ord}(\overline{b}, d)) = v_q(\text{lcm}\{\text{ord}(\overline{b}, p^{e_p}) : p \in S\}) \\
= \max\{v_q(\text{ord}(\overline{b}, p^{e_p})) : p \in S\}. \tag{2.3}
\]

Similarly,

\[
v_q \left( \text{ord} \left( \overline{b}, \prod_{p \in S} p^{\min\{e_p, N_p\}} \right) \right) = \max\{v_q(\text{ord}(\overline{b}, p^{e_p})) : p \in S\}. \tag{2.4}
\]

For any \( p \in S \), since \( N_p \geq n_p \), the cases we have proven imply that

\[
\text{ord}(\overline{b}, p^{e_p}) = p^{\max\{e_p-n_p, 0\}} \text{ord}(\overline{b}, p^{\min\{e_p, n_p\}}) \tag{2.5}
\]

and

\[
\text{ord}(\overline{b}, p^{e_p}) = p^{\max\{e_p-N_p, 0\}} \text{ord}(\overline{b}, p^{\min\{e_p, N_p\}}). \tag{2.6}
\]

If either \( q \notin S \) or \( e_q \leq N_q \), we have

\[
v_q(\text{ord}(\overline{b}, p^{e_p})) = v_q(\text{ord}(\overline{b}, p^{\min\{e_p, N_p\}})).
\]

So (2.3) and (2.4) imply that

\[
v_q(\text{ord}(\overline{b}, d)) = v_q \left( \text{ord} \left( \overline{b}, \prod_{p \in S} p^{\min\{e_p, N_p\}} \right) \right). 
\]

Then (2.2) follows from this since \( v_q(\prod_{p \in S: e_p > N_p} p^{e_p-N_p}) = 0 \).
If \( q \in S \) and \( e_q > N_q \), for any \( p \neq q \), we have
\[
v_q(\text{ord}(\overline{b}, p^{e_p})) = v_q(\text{ord}(\overline{b}, p^{n_p})) \quad \text{(by (2.5))}
\]
\[
\leq N_q - n_p + v_q(\text{ord}(\overline{b}, q^{n_q}))
\]
\[
< e_q - n_p + v_q(\text{ord}(\overline{b}, q^{e_q}))
\]
\[
= v_q(q^{e_p-n_p} \text{ord}(\overline{b}, q^{n_q}))
\]
\[
= v_q(\text{ord}(\overline{b}, q^{e_q})), \quad \text{(by (2.5))}
\]
where the second inequality follows from the definition of \( N_q \) and the third inequality holds since \( N_q < e_q \). So
\[
\max\{v_q(\text{ord}(\overline{b}, p^{e_p})) : p \in S\} = v_q(\text{ord}(\overline{b}, q^{e_q}))
\]
\[
= q^{e_q - N_q} + v_q(\text{ord}(\overline{b}, q^{N_q})).
\]
Combine this with (2.3) and (2.4), we deduce (2.2). \( \square \)

Next result concerns the orbits of \( S \)-integers under \( T_b \), it is the key of this article that leads to the proofs of our main theorems.

**Theorem 2.4.** Let \( b \geq 2 \) be an integer, \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \), and \( \frac{a}{d} \) be an \( S \)-integer with \( (a, d) = 1 \) and \( d = \prod_{p \in S} p^{e_p} \). Let \( N_p \) be as in (2.1), and set \( d_0 = \prod_{p \in S: e_p > N_p} p^{e_p - N_p}, d_1 = \prod_{p \in S} p^{\min\{e_p, N_p\}} \). Define
\[
A_1 = \left\{ T_b^i \left( \frac{a}{d} \right) : 0 \leq i \leq \text{ord}(\overline{b}, d) - 1 \right\},
\]
\[
A_2 = \left\{ \frac{1}{d_0} T_b^i \left( \frac{a}{d_1} \right) + \frac{j}{d_0} : 0 \leq i \leq \text{ord}(\overline{b}, d_1) - 1, 0 \leq j \leq d_0 - 1 \right\}.
\]
Then \( A_1 = A_2 \).

**Proof.** First we prove \( A_1 \subseteq A_2 \). For any \( 0 \leq i \leq \text{ord}(\overline{b}, d) - 1 \), by the definition of order, there exists \( 0 \leq i' \leq \text{ord}(\overline{b}, d_1) - 1 \) such that \( b^i \equiv b^{i'} \) (mod \( d_1 \)). Let \( j = \lfloor d_0 T_b^i \left( \frac{a}{d} \right) \rfloor \), it is an integer between 0 and \( d_0 - 1 \) since \( T_b^i \left( \frac{a}{d} \right) \in [0, 1) \). Then
\[
d_0 T_b^i \left( \frac{a}{d} \right) = T_b^i \left( \frac{d_0 a}{d} \right) + j
\]
\[
= T_b^i \left( \frac{a}{d_1} \right) + j \quad \text{(since } d = d_0 d_1)\]
\[
= T_b^{i'} \left( \frac{a}{d_1} \right) + j.
\]
So
\[ T_b^i \left( \frac{a}{d} \right) = \frac{1}{d_0} T_b^{i'} \left( \frac{a}{d_1} \right) + \frac{j}{d_0} \in A_2. \]

Since \( i \) is arbitrary, we have \( A_1 \subseteq A_2 \).

Now we compute the cardinality of \( A_1 \). For any \( 0 \leq i_1, i_2 \leq \text{ord}(\overline{b}, d) - 1 \), if \( T_b^{i_1} \left( \frac{a}{d} \right) = T_b^{i_2} \left( \frac{a}{d} \right) \), then \( ab^{i_1} \equiv ab^{i_2} \pmod{d} \). Without loss of generality, we assume \( i_1 \leq i_2 \). Then since both \( a, b \) are coprime with \( d \), we have \( b^{i_2 - i_1} - 1 \equiv 0 \pmod{d} \) and thus \( \text{ord}(\overline{b}, d) \mid (i_2 - i_1) \), which only happens when \( i_1 = i_2 \) as \( 0 \leq i_2 - i_1 \leq \text{ord}(\overline{b}, d) - 1 \). Therefore the cardinality of \( A_1 \) equals \( \text{ord}(\overline{b}, d) \).

Next we compute the cardinality of \( A_2 \). Suppose that \( \frac{1}{d_0} T_b^{i_1} \left( \frac{a}{d_1} \right) + \frac{j_1}{d_0} = \frac{1}{d_0} T_b^{i_2} \left( \frac{a}{d_1} \right) + \frac{j_2}{d_0} \)

for some \( 0 \leq i_1, i_2 \leq \text{ord}(\overline{b}, d_1) - 1 \) and \( 0 \leq j_1, j_2 \leq d_0 - 1 \). Multiply both sides by \( d_0 \), we have
\[ T_b^{i_1} \left( \frac{a}{d_1} \right) + j_1 = T_b^{i_2} \left( \frac{a}{d_1} \right) + j_2. \]
Comparing the integer parts of both sides, we deduce that \( j_1 = j_2 \). Then \( T_b^{i_1} \left( \frac{a}{d_1} \right) = T_b^{i_2} \left( \frac{a}{d_1} \right) \). By a similar argument as in the previous paragraph, we have \( i_1 = i_2 \). Therefore the cardinality of \( A_1 \) equals \( d_0 \text{ord}(\overline{b}, d_1) \).

By Lemma 2.3, we have \( \text{ord}(\overline{b}, d) = d_0 \text{ord}(\overline{b}, d_1) \). So \( A_1 \) and \( A_2 \) have the same cardinality and thus \( A_1 = A_2 \).

**Proof of Theorem 1.4.** Let
\[ D = \frac{1}{2\varepsilon} \prod_{p \in S} p^{N_p}. \] (2.7)

For any \( i \geq 0 \), write \( i = k \text{ord}(\overline{b}, d) + i' \) where \( k \in \mathbb{N} \) and \( 0 \leq i' \leq \text{ord}(\overline{b}, d) - 1 \). Since \( b^{\text{ord}(\overline{b}, d)} \equiv 1 \pmod{d} \), we have
\[ T_b^i \left( \frac{a}{d} \right) = T_b^{i'} T_b^{k \text{ord}(\overline{b}, d)} \left( \frac{a}{d} \right) = T_b^{i'} \left( \frac{a}{d} \right). \]
Hence \( \text{Orb}_{T_b} \left( \frac{a}{d} \right) = A_1 \), and thus \( \text{Orb}_{T_b} \left( \frac{a}{d} \right) = A_2 \) by Theorem 2.4. Note that
\[ A_2 \ni \left\{ \frac{1}{d_0} \cdot \frac{a}{d_1} + \frac{j}{d_0} : 0 \leq j \leq d_0 - 1 \right\}. \]
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and the distance of any two consecutive elements in the later set is $\frac{1}{d_0}$. By the definition of $d_0$, we have

$$\frac{1}{d_0} \leq \prod_{p \in S} p^{N_p - e_p} = \frac{1}{d} \prod_{p \in S} p^{N_p} < \frac{1}{D} \prod_{p \in S} p^{N_p} = 2\varepsilon.$$  

Therefore $\text{Orb}_{T_b} \left( \frac{a}{d} \right)$ is $\varepsilon$-dense in $[0, 1]$. □

We next deduce Theorem 1.2 from Theorem 1.4.

**Proof of Theorem 1.2.** Since $A$ is not dense in $[0, 1]$, there exists an interval $I \subseteq [0, 1] \setminus A$ with positive length $|I|$. Let $\varepsilon = \frac{|I|}{2}$, so $A$ is not $\varepsilon$-dense in $[0, 1]$. Let $\frac{a}{d} \in A \cap \mathbb{Q}$ with $\gcd(a, d) = 1$. Since $T_b(A \cap \mathbb{Q}) \subseteq A$, we have $\text{Orb}_{T_b} \left( \frac{a}{d} \right) \subseteq A$, and so $\text{Orb}_{T_b} \left( \frac{a}{d} \right)$ is also not $\varepsilon$-dense in $[0, 1]$. Therefore Theorem 1.4 implies that $d < D$ for some positive number $D$. Clearly there are only finitely many rational numbers $\frac{a}{d} \in [0, 1)$ with $d < D$, hence $A$ contains at most finitely many $S$-integers. □

3. LARGEST PRIME DIVISOR

In this section, we use Theorem 1.4 to prove Theorem 1.6. Since we do not intend to compute the exact value of the absolute constant in Theorem 1.6, we use the notations $\alpha \gg \beta$ and $\beta \ll \alpha$ to mean that $|\alpha| \geq c|\beta|$ for some absolute constant $c$, which can be effectively computed.

We start with upper bounds on $n_p$ and $N_p$.

**Lemma 3.1.** Keep the notations of Lemma 2.3. Let $P$ be the largest prime in $S$. Then for any $p \in S$, we have

$$n_p \ll \frac{p \log b}{\log p} \quad \text{and} \quad N_p \ll \frac{\log b}{\log p} (p + P).$$

**Proof.** The bound of $n_p$ is deduced from its definition and the trivial inequality

$$v_p(x) \leq \frac{\log x}{\log p} \quad \text{for any} \quad x > 0.$$

For any $q \in S$, note that $\text{ord}(\overline{b}, q^{n_q})$ cannot be bigger than the order of $(\mathbb{Z}/q^{n_q}\mathbb{Z})^\times$, which equals $(q - 1)q^{n_q - 1}$, so

$$v_p(\text{ord}(\overline{b}, q^{n_q})) \leq \frac{\log (q - 1)q^{n_q - 1}}{\log p} \leq \frac{n_q \log q}{\log p} \ll \frac{q \log b}{\log p}.$$

Then

$$N_p \leq n_p + \max_{q \in S} v_p(\text{ord}(\overline{b}, q^{n_q})) \ll \frac{p \log b}{\log p} + \frac{P \log b}{\log p}. \quad \square$$
Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Suppose \( S \) is the set of all prime divisors of \( d \) and the prime decomposition of \( d \) is \( d = \prod_{p \in S} p^{e_p} \). By Theorem 1.4 and the choice of \( D \) in its proof (cf. (2.7)), we have

\[
\log d \leq \log D = -\log(2\varepsilon) + \sum_{p \in S} N_p \log p.
\]

So Lemma 3.1 implies

\[
\log (2\varepsilon d) \ll \sum_{p \in S} ((p + P) \log b) \leq (\log b) \sum_{p \in S} 2P = 2P\#S \log b,
\]

where \( \#S \) denotes the cardinality of \( S \). The prime number theorem says \( \#S \ll \frac{P}{\log P} \), hence we have

\[
\log (2\varepsilon d) \ll \frac{P^2}{\log P} \log b. \tag{3.1}
\]

If \( P > b \), then \( \log \log (2\varepsilon d) \leq 2 \log P \), and so

\[
\log (2\varepsilon d) \log \log (2\varepsilon d) \ll P^2 \log b. \tag{3.2}
\]

If \( P < b \), then it follows from (3.1) and the trivial bound \( \frac{1}{\log P} \ll 1 \) that

\[
\log (2\varepsilon d) \ll P^2 \log b. \tag{3.3}
\]

Now (3.2) and (3.3) yield the desired inequity (1.2). \( \square \)

4. Further discussion

In this section, we present some corollaries of our main results, discuss the \( T_b \)-invariant condition, and raise questions for future research.

4.1. Rational numbers of more general form. In Theorem 1.2, we concerns rational numbers whose denominators do not share prime divisors with \( b \). The following corollary says that if we restrict to rational numbers of the form \( \frac{a}{d^n} \), then it is fine to have \( \gcd(b, d) > 1 \), as long as \( d \) has a prime divisor not dividing \( b \).

**Corollary 4.1.** Let \( b \geq 2 \) be an integer, \( d \geq 2 \) be another integer such that there exists at least one prime \( p \mid d \) such that \( p \nmid b \), and \( A \) be a subset of \([0, 1)\). If \( A \) is not dense in \([0, 1]\) and \( T_{b}(A \cap \mathbb{Q}) \subseteq A \), then \( A \) contains at most finitely many rational numbers of the form \( \frac{a}{d^n} \), \( n \in \mathbb{N} \).

**Proof.** Let \( \tilde{d} \) be the largest divisor of \( d \) satisfying \( \gcd(\tilde{d}, b) = 1 \), that is, \( \tilde{d} = \prod_{p \mid b} p^{v_p(\tilde{d})} \). If \( \frac{a}{d^n} \in A \), we apply \( T_b \) to it enough times to make the denominator not containing prime divisors of \( b \) anymore. In other
words, we choose a big enough integer \( m \) such that \( mv_p(b) \geq nv_p(d) \) for all \( p \mid b \), and then

\[
T_b^m \left( \frac{a}{d^n} \right) = \frac{\tilde{a}}{d^n},
\]

for some integer \( \tilde{a} \). Now Theorem 1.2 says that \( \tilde{d}^n \) is bounded, and hence \( n \) is bounded. Therefore \( A \) contains at most finitely many rational numbers of the form \( \frac{a}{d} \).

In Theorem 1.4, the same conclusion holds if we replace \( \frac{a}{d} \) in (1.1) by a rational multiple \( \frac{aa'}{dd'} \), where \( a' \leq d' \in \mathbb{N} \) with \( (a', d') = 1 \). More precisely, we have the following.

**Corollary 4.2.** Let \( b \geq 2 \) be an integer and \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \). Let \( \frac{a'}{d'} \) be a rational number in \( [0, 1) \) with \( (a', d') = 1 \). Then for any \( \varepsilon > 0 \), there exists an effectively computable positive number \( D \), such that for any \( \frac{aa'}{dd'} \in \mathbb{Z}_S \cap [0, 1) \) with \( (aa', dd') = 1 \) and \( d > D \), the orbit

\[
\text{Orb}_{T_b} \left( \frac{aa'}{dd'} \right) = \left\{ T_b^i \left( \frac{aa'}{dd'} \right) : i \geq 0 \right\},
\]

is \( \varepsilon \)-dense in \( [0, 1] \).

**Proof.** Let

\[
S' = S \cup \{ p : p \text{ is prime with } p \mid d', p \nmid b \}.
\]

Then there exists \( k \geq 0 \) such that for any \( \frac{a'}{d'} \in \mathbb{Z}_S \), \( T_b^k \left( \frac{aa'}{dd'} \right) \) is an \( S' \)-integer of the form \( \frac{\tilde{a}}{d'} \). Then by Theorem 1.4 (in which we take \( S = S' \)), we see that for any \( \varepsilon > 0 \), there exists a positive number \( D \) such that for any \( \frac{a'}{d'} \in \mathbb{Z}_S \) with \( (aa', dd') = 1 \) and \( d > D \), the orbit \( \text{Orb}_{T_b} \left( \frac{\tilde{a}}{d'} \right) \) is \( \varepsilon \)-dense in \( [0, 1] \). It then follows that \( \text{Orb}_{T_b} \left( \frac{aa'}{dd'} \right) \) is also \( \varepsilon \)-dense in \( [0, 1] \) as it contains \( \text{Orb}_{T_b} \left( \frac{\tilde{a}}{d'} \right) \). \( \square \)

4.2. **An application on \( S \)-integers.** Theorem 1.2 yields the following property of \( S \)-integers, which says that the orbit of any infinite set of \( S \)-integers under \( T_b \) is dense in \( [0, 1] \).

**Corollary 4.3.** Let \( b \geq 2 \) be an integer, \( S \) be a non-empty finite set of primes not containing any prime divisor of \( b \). Let \( X \subseteq \mathbb{Z}_S \cap [0, 1) \) be an infinite subset of \( S \)-integers. Then the set

\[
\text{Orb}_{T_b}(X) := \left\{ b^k x \pmod{1} : x \in X, k \geq 0 \right\}
\]
is dense in $[0, 1]$.

Proof. Observe that
\[
\text{Orb}_{T_b}(X) = \bigcup_{k=0}^{\infty} T_b^k X
\]
and thus $\text{Orb}_{T_b}(X)$ is $T_b$-invariant. If $\text{Orb}_{T_b}(X)$ is not dense in $[0, 1]$, then Theorem 1.2 implies that $\text{Orb}_{T_b}(X)$ contains at most finitely many $S$-integers, which contradicts that $\text{Orb}_{T_b}(X) \supseteq X$ and $X$ is an infinite subset of $S$-integers. Hence $\text{Orb}_{T_b}(X)$ is dense in $[0, 1]$.

4.3. The $T_b$-invariant condition. In Theorem 1.2 and Theorem 1.6, we require that $T_b(A \cap \mathbb{Q}) \subseteq A$, which is weaker than $A$ being $T_b$-invariant. If we know $A$ is actually $T_b$-invariant, then all of our results can apply to $S$-integers in $\overline{A}$, the closure of $A$, by noting that $A$ is $T_b$-invariant implies $\overline{A}$ is also $T_b$-invariant.

On one hand, clearly there exist $T_b$-invariant sets which are not generalized Cantor sets. On the other hand, Wu [12] showed that every closed $T_b$-invariant set can be covered by a generalized Cantor set with similar Hausdorff dimension (see [3] for definition).

Theorem 4.4. [12, Proposition 9.3] Let $A \subseteq [0, 1)$ be a $T_b$-invariant set. Then for any $\epsilon > 0$, there exist $k \in \mathbb{N}$ and a generalized Cantor set $C(b^k, D)$ such that $\overline{A} \subseteq C(b^k, D)$ and $\dim_H(\overline{A}) \geq \dim_H(C(b^k, D)) - \epsilon$.

This result, combining with Theorem 1.1, leads to another proof of the finiteness of $S$-integers in $T_b$-invariant sets, which we now briefly sketch. If $A$ is $T_b$-invariant and not dense, then $\overline{A}$ is $T_b$-invariant and $\overline{A} \neq [0, 1]$. From the proof of [12, Proposition 9.3] we see that $\overline{A}$ is contained in a generalized Cantor set $C(b^k, D)$ with $D \subseteq \{0, 1, \ldots, b^k\}$ and $\#D < b^k$. Then we can apply Theorem 1.1 to deduce that $\mathbb{Z}_S \cap C(b^k, D)$ is finite. Therefore the finiteness of $\mathbb{Z}_S \cap \overline{A}$ follows since $\mathbb{Z}_S \cap \overline{A} \subseteq \mathbb{Z}_S \cap C(b^k, D)$.

4.4. Algebraic numbers. Now we have a decent understanding of rational numbers in $T_b$-invariant sets, one may wonder what happens for algebraic numbers. Given an algebraic number $\delta$ with degree at least 2 and a $T_b$-invariant set $A \subseteq [0, 1)$, we ask if the intersection $A \cap \{\frac{a}{\delta^n} \in (0, 1): a, n \in \mathbb{N}\}$ is finite. When $A = C$ is the middle-third Cantor set, Mahler [5] conjectured that all algebraic numbers in $C$ are rational numbers, so $A \cap \{\frac{a}{\delta^n} \in (0, 1): a, n \in \mathbb{N}\}$ is actually the empty set. As solving Mahler’s conjecture seems out of reach at the moment, determine the finiteness of $A \cap \{\frac{a}{\delta^n} \in (0, 1): a, n \in \mathbb{N}\}$ for arbitrary $\delta$ could also be very hard. We wonder if the problem becomes solvable if
\[ \delta \] is restricted to Pisot numbers, which are very close to integers when raising to high powers.

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