Adiabatic and entropy perturbations propagation in a bouncing universe.

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Abstract: By studying some bouncing universe models dominated by a specific class of hydrodynamical fluids, we show that the primordial cosmological perturbations may propagate smoothly through a general relativistic bounce. We also find that the purely adiabatic modes, although almost always fruitfully investigated in all other contexts in cosmology, are meaningless in the bounce or null energy condition (NEC) violation cases since the entropy modes can never be neglected in these situations: the adiabatic modes exhibit a fake divergence that is compensated in the total Bardeen gravitational potential by inclusion of the entropy perturbations.
1. Introduction

The singularity problem in cosmology [1], which is not addressed in the framework of inflationary paradigm [2], (note however the recent “emergent universe” proposal [3] in that respect) may be solved either through a Pre-Big-Bang (PBB) type phase [4] or by means of a bounce [5, 6]. Both cases in fact demand that the universe undergoes at least one transition from a collapsing to an expanding epoch, although the PBB situation demands that this happens in the Einstein frame in which general relativity (GR) holds true. It is this kind of bounce that we shall be concerned with here.

A bouncing phase may originate from quantum gravity [7] or quantum cosmology [8], and have been seen to occur in some string-motivated models [9]. In one such model, based on the brane hypothesis in a five dimensional context, namely the ekpyrotic scenario [10, 11], the bounce in fact does not address the singularity problem since it is assumed to be singular. This leads to some difficulties [12] which the model still have to deal with (see Ref. [12]).

It is worth pointing out that in some improvements of the regular bouncing models, not necessarily relying on general relativity at all times, perturbations may even be made scale-invariant [13], thus moving one step towards an alternative model of the Universe, namely one for which there is only a short period of accelerated expansion, which happens during the bouncing era following a contraction phase, in contrast with the relatively long period characteristic of usual inflationary scenarios.
Bouncing models without a long period of inflation afterwards, besides solving the singularity and the horizon problems, can be made to avoid the trans-Planckian issue and may also provide a causal explanation, in terms of a quantum mechanical origin, for the large scale structures in our Universe. However, they yet depend on some theory of initial conditions coming from quantum cosmology and/or string theory in order to address the flatness and isotropization issues, which are solved naturally in usual inflationary scenarios. In short, bouncing models can somehow be viewed either as potential alternatives to inflation, or as complementary to it, providing help in constructing a full consistent cosmological model for our Universe and addressing issues which inflationary models do not.

In the present work, we are interested in pointing out the relevance of entropy fluctuations during the bounce phase, in particular at the time at which the transition from an epoch where the null energy condition (NEC) is valid to an epoch where it ceases to be valid occurs (which will be called, from now on, the NEC transition), which may affect the observable power spectrum of adiabatic perturbations in the present expanding phase of the abovementioned bouncing models, including the PBB case.

In a previous study, concentrating on pure general relativity as the theory for describing gravity, we showed that a bouncing universe described in terms of hydrodynamical fluids was unstable with respect to adiabatic perturbations. More recently, we managed to obtain a model in which all perturbations are finite at all times, including the full Bardeen potential without imposing any adiabatic condition, by considering the case of a two fluids bouncing Universe, one of the fluids described by a scalar field. The case of Ref. is but one particular case of Ref., called case there, with the scale factor behaving near the bounce as an even function of the conformal time, at least up to the fifth order. The bounce itself, i.e. the point at which the scale factor derivative with respect to time vanishes, is stable in this case, in the sense that the perturbations are bounded while passing through it. Instabilities appear however as logarithmic divergences in the second derivative of the adiabatic part of the gauge invariant Bardeen potential, hence also in the Einstein tensor and pressure perturbations, at the NEC transition point. In fact, any model presenting a NEC transition faces the abovementioned divergence in the adiabatic Bardeen potential.

The aim of this paper is to show, in a general framework, that the instability of adiabatic perturbations at the NEC transition point does not stem from a possible ill-defined decomposition of the pressure perturbation into adiabatic and entropy parts, but resides on the fact that entropy fluctuations cannot be neglected there, even for arbitrarily large wavelengths. Note that in the framework of the present article, entropy modes mean modes of mixing entropy between the various fluid constituents of the universe, as opposed to intrinsic entropy modes, which we assume are absent. Hence, any attempt to calculate the power spectrum of perturbations on bouncing models relying only on matching conditions for the adiabatic Bardeen potential through the bounce is not sufficient and may lead to erroneous results. The entropy fluctuations, in particular at the NEC transition point, must be considered because they are not negligible, in fact they are extremely important for the stability of the model, and they may transfer power to the adiabatic perturba-
tions afterwards. We also settle down general conditions in which perturbations can be consistently defined in a bouncing model.

2. Bouncing Background

Let us begin by a review of the main general aspects of bouncing backgrounds. Within our conventions, the FLRW metric reads

\[ ds^2 = a^2(\eta) \left(d\eta^2 - \gamma_{ij}dx^idx^j\right), \]  

(2.1)

with the spatial three-metric \( \gamma_{ij} \) given by

\[ \gamma_{ij} \equiv \left(1 + \frac{K}{4}x^2\right)^{-2}\delta_{ij}, \]  

(2.2)

and \( \eta \) being the conformal time from which one derives the cosmic time \( t \) as the solution of the equation \( ad\eta = dt \), with a given scale factor \( a(\eta) \). In Eq. (2.2), the parameter \( K \), representing the spatial curvature, can be normalized to \( K = 0, \pm 1 \), and \( x^2 \equiv \delta_{ij}x^ix^j \) (units are such that we are setting \( \hbar = c = 1 \)).

The stress energy tensor, source of Einstein field equations, will take the form

\[ T_{\mu}^\nu = (\epsilon + p)u^\mu u_\nu - p\delta_\mu^\nu, \]  

(2.3)

for energy density \( \epsilon \), pressure \( p \) and 4-velocity (fluid tangent vector) \( u \), which can be expressed as \( u^\mu = (1/a)\delta_\mu^0 \), \( i.e., u_\nu = a\delta_{\nu 0} \). Einstein equations for the metric (2.1) and stress energy tensor (2.3) read

\[ H^2 + K = \frac{\ell_p^2}{a^2} \epsilon, \]  

(2.4)

and

\[ \beta \equiv H^2 - H' + K = \frac{3}{2}\ell_p^2 a^2(\epsilon + p), \]  

(2.5)

where \( \ell_p \equiv 8\pi G_N/3 \) is the Planck length (\( G_N \) being Newton constant), \( H \equiv a'/a \) the conformal Hubble parameter, and a prime denotes a derivative with respect to the conformal time \( \eta \). At the point of NEC transition we have \( (\epsilon + p) = 0 \), \( i.e., \beta = 0 \).

We now show that in the case where the energy density can be written as a function of the scale factor \( a(\eta) \), one can prove that this function is even around the bounce, behaving as

\[ a(\eta) = a_0 + b\eta^2 + c\eta^4 + \cdots, \]  

(2.6)

with \( \eta = 0 \) representing the bounce. Note that such a bounce needs a NEC transition point\(^{1}\). Indeed, the scale factor \( a(\eta) \) of a general bouncing model can be expanded around the bounce as follows:

\[ a = a_0 + b\eta^{2n} + d\eta^{2n+1} + e\eta^{2n+2} + f\eta^{2n+3} + \cdots, \]  

(2.7)

\(^{1}\)In fact, a NEC transition point may not be present in such models if the curvature of the spatial sections is positive and if \( 2b \leq a_0 \). However, in order for a realistic model to satisfy these conditions, one needs either to impose a tremendous amount of fine-tuning, or to have an inflationary phase between the bounce and the radiation dominated phase; see Ref. [16] for details.
where $a_0 > 0$, and the integer $n$ satisfies $n \geq 1$. This means we demand that $a(\eta)$ must be at least $C^{2n+3}$ near the bounce. In order that Eq. (2.7) indeed represents a bounce, the otherwise arbitrary parameter $b$ must satisfy $b > 0$.

The function $\mathcal{H}(\eta)$ coming from Eq. (2.7) reads

\[
\mathcal{H} = \frac{1}{a_0^2} \left[ 2nb^2a_0\eta^{2n-1} + (2n+1)da_0\eta^{2n} + 2(n+1)e_0\eta^{2n+1} + (2n+3)fa_0\eta^{2n+2} - 2nb^2\eta^{4n-1} - bd(4n+1)\eta^{4n} + \ldots \right],
\]

while $\beta(\eta)$ is

\[
\beta = \frac{1}{a_0^2} \left[ Ka_0^2 - 2n(2n-1)b_0\eta^{2n-2} - 2n(2n+1)da_0\eta^{2n-1} - 2(n+1)(2n+1)e_0\eta^{2n} - 2(2n+3)(n+1)f_0\eta^{2n+1} + 2n(6n-1)b^2\eta^{4n-2} + 8bdn(1+3n)\eta^{4n-1} + \ldots \right].
\]

In the case where the energy density can be written as a function of $a$, $e_0 = e_0(a)$, one can prove that $a(\eta)$ is even and that $n = 1$ in Eq. (2.7). This can be seen as follows: from the energy-momentum conservation equation we obtain $ade/da = -3(\epsilon + p)$, implying that $p$ is also a function of $a$, $p = p(a)$. As a consequence, the right-hand-side of Eq. (2.7) implies that $\beta$ can also be written as a function of $a$, and hence it can be Taylor expanded as

\[
\beta(a) \simeq \beta(a_0) + \beta_a(a_0) \left( b_0^2n + d_0\eta^{2n+1} + \ldots \right) + \frac{\beta_{aa}(a_0)}{2} \left( b_0^2n + d_0\eta^{2n+1} + \ldots \right)^2,
\]

where the “$a$” indices indicate differentiation with respect to the scale factor. Equations (2.8) and (2.10) are, respectively, the left hand side (LHS) and RHS of Eq. (2.7). Comparing powers of the conformal time, and identifying the coefficients, one finds that the term containing $d_0\eta^{2n-1}$ in the LHS has no counterpart in the RHS. Hence, $d$ must vanish. With $d = 0$, the term containing $f_0\eta^{2n+1}$ in Eq. (2.9) also has no corresponding term in Eq. (2.10), so that $f$ must vanish as well. By induction, we can prove that all coefficients of terms containing odd powers of $\eta$ in the expansion Eq. (2.7) must be zero. Hence, $a(\eta)$ must be an even function of the conformal time.\(^2\) Also, if $n > 1$, the term containing $\eta^{2n-2}$ in Eq. (2.9) has no counterpart in Eq. (2.10); this imposes that $n = 1$, and hence a scale factor that behaves quadratically in $\eta$.

A realistic bouncing universe, i.e. one leading smoothly to a radiation dominated phase, cannot be modeled by means of a single barotropic fluid with constant equation of state, since this would imply the single fluid in question to consist in radiation, and it is well-known that radiation alone cannot prevent the occurrence of a singularity forming \(^4\), even for positively curved spatial sections. One could also try to describe such a bounce with just one fluid by allowing a varying equation of state for this fluid, assuming the equation of state parameter $\omega \equiv p/\epsilon$ (with $\epsilon$ the energy density and $p$ the pressure) to depend on time in such a way that $\lim \omega = 1/3$ for large times. This case, somehow implicitly assumed in Ref. \(^1\), will not be treated here since it would demand complete knowledge.

\(^2\)As an alternative argument, let us note that $e(\eta) = e(\eta(a)) = e(a)$ is true if and only if the inverse of $a(\eta)$, $\eta(a)$, exists, which is possible, in a bounce framework (i.e. if $a$ is not a monotonic function of $\eta$), if and only if $a(\eta)$ is even, as $a$ is positive.
of the behavior of the intrinsic entropy of the fluid, which is model-dependent. We shall accordingly in what follows consider the next-to-simple case of $N$ constant equations of state components, later allowing for exchanges in the various modes of perturbations they may produce at a given scale, thereby providing the possibility to introduce mixing entropy modes in a model-independent way.

We want to concentrate on the situation relevant for the rest of the paper for which the matter content is described by an arbitrary number of non interacting hydrodynamical perfect fluids, each with a barotropic equation of state with constant ratio between the energy density and pressure; this is but a special case of the one where the total energy density can be written as a function of the scale factor, and hence with even scale factor, bouncing behavior of Eq (2.6), and a NEC transition point. This means that the components are noninteracting perfect fluids with stress-energy tensors given by

\[ T^\mu_\nu = \sum_{i=1}^{N} T^\mu_\nu_i, \quad T^\mu_\nu_i = (\epsilon_i + p_i)u^\mu u^\nu - p_i g^\mu_\nu, \]  

relations in which $u^\mu u_\mu = 1$ is the same timelike vector for all fluids, in agreement with the symmetry assumptions leading to a FLRW Universe. In the simple case with which we are interested here, we demand that the fluid equations of state be fixed, namely

\[ p_i = \omega_i \epsilon_i, \quad \omega_i = \text{const}. \]  

The total energy density and pressure that enter Einstein equations are $\epsilon = \sum_i \epsilon_i$ and $p = \sum_i p_i$. Energy-momentum conservation for each fluid, $\nabla^\mu T^\mu_\nu = 0$ implies

\[ \epsilon'_i + 3 \mathcal{H} \epsilon_i(1 + \omega_i) = 0 \quad \implies \quad \epsilon_i = c_i a^{-3(1+\omega_i)}, \]  

with $c_i$ arbitrary constants. Note at this point that the energy density in this case is indeed a function of $a$, so that, from the arguments above, $a(\eta)$ must be even with $n = 1$ in the expansion (2.7). This situation corresponds to the so-called case (4) of Ref. [16], which we want to investigate in greater details below.

Specializing to two fluids and upon using Eq. (2.4), one gets

\[ \left( \frac{a'}{a} \right)^2 = \ell^2_p a^{-(1+3\omega_1)} \left[ c_1 + c_2 a^{3(\omega_1-\omega_2)} \right] - \mathcal{K}, \]  

from which it is clear that for non-positive curvature spatial section, i.e. for $\mathcal{K} \leq 0$, it is only possible to have a bounce, as a point in time for which $a' = 0$, provided one of the fluids has negative energy; this means that for $\mathcal{K} \leq 0$, one of the constants $c_1$ or $c_2$ ought to be negative. The case with positive spatial curvature can allow both constants to be positive, but none can vanish if one demands radiation to be present, and therefore that either $\omega_1$ or $\omega_2$ be equal to one third.

As an aside, let us remark that, at the background level, a scalar field $\phi$, with dynamics stemming from the action

\[ S_\phi = \int \left[ \frac{1}{2} (\nabla_\mu \phi) (\nabla^\mu \phi) - V(\phi) \right] \sqrt{-g} d^4x, \]  

(2.15)
is equivalent to a perfect fluid with varying equation of state since one has
\[ p_s = \frac{1}{2a^2}(\phi')^2 - V(\phi), \quad \epsilon_s = \frac{1}{2a^2}(\phi')^2 + V(\phi). \] (2.16)

This is no longer true at the perturbation level, unless the potential \( V \) vanishes, which is then equivalent to a stiff matter fluid \( \omega = 1 \): when \( V = 0 \), one has indeed \( \phi' \propto 1/a^2 \to \epsilon_\phi \propto 1/a^6 \). If the kinetic terms in Eq. (2.16) are negative, and including also a radiation fluid, one recovers the prototypical bouncing model already discussed in Ref. [13], which we shall discuss later on in Sec. [12]. In the case for which the Universe is positively curved, a bounce is possible even if it is dominated by a single scalar field [17, 18, 19].

We now turn to the perturbations in this class of models, which we wish to expand on either a basis of a fluid-by-fluid decomposition, or into adiabatic and entropy modes.

3. General perturbations.

The metric for a perturbed universe with no anisotropic stress contribution can be written, in full generality, in the longitudinal gauge (we use the notations of Ref. [20])
\[ ds^2 = a^2(\eta) \left[ (1 + 2\Phi) d\eta^2 - (1 - 2\Phi) \gamma_{ij} dx^i dx^j \right], \] (3.1)
where \( \Phi \) is the gauge invariant Bardeen potential [21], which is unique since we aim at describing a situation with no anisotropic pressure. The perturbed Einstein equations with this metric then read
\[ \nabla^2 \Phi - 3H \Phi' - 3(\dot{H}^2 - \mathcal{K}) \Phi = 4\pi G a^2 \delta\varepsilon^{\text{(gi)}}, \] (3.2)
\[ \Phi'' + 3H \Phi' + (2\dot{H} + \mathcal{H}^2 - \mathcal{K}) \Phi = 4\pi G a^2 \delta p^{\text{(gi)}}. \] (3.3)
All these quantities are defined in such a way as to be gauge invariant, as emphasized by the superscript “\( \text{(gi)} \)” (see Ref. [20]). We shall for now on take it for granted that all quantities under consideration are gauge invariant, and therefore avoid the use of the notation “\( \text{(gi)} \)”, which will be implicit.

As a first step, in the subsection below, we show that if the matter content of a background bouncing model can be described by an arbitrary number of non interacting fluids with constant equations of state, then the full Bardeen potential and all its derivatives are completely regular at all times. Then, in the following subsection, we define adiabatic and entropy fluctuation modes. We recover the general results of Ref. [16] that there are always divergences in the gauge invariant adiabatic potential around the bounce, except in the case with bouncing behavior of Eq. (2.4), and around the NEC transition, but using a different method: instead of approximating the equations near the singular points and solving them afterwards, we adopt the more accurate and rigorous method of solving the exact equation and then expanding the corresponding solution and its derivatives near the singular point (note that both methods are expected to give similar results near regular points [22]). In particular, the logarithmic divergence in the second derivative of the adiabatic Bardeen potential at the NEC transition is reobtained for any model in which
this transition occurs. We then rephrase, in more general terms, the result that there is a class of models with NEC transition in which the full Bardeen potential and all its derivatives are completely regular at all times, but whose second derivative of the adiabatic perturbation at the NEC transition is divergent. Finally, we complete this section by examining the properties of the curvature perturbation \[21, 23, 24\] on uniform density hypersurfaces associated to adiabatic perturbations.

3.1 The case for a bounded full Bardeen potential: Fluid by Fluid decomposition

We now prove that if the matter content of the model is described by an arbitrary number \(N\) of non interacting fluids with constant equations of state, then the Bardeen potential and all its derivatives are regular at all times. Using the linearity of Eqs. (3.2) and (3.3), we can decompose the total gravitational perturbation \(\Phi\) as the sum \(\Phi = \sum_i \Phi_i\), and construct the \(N\) sets of decoupled equations

\[
\nabla^2 \Phi_i - 3H \Phi_i' - 3(H^2 - K) \Phi_i = 4\pi G a^2 \delta\epsilon_i, \\
\Phi_i'' + 3H \Phi_i' + (2H' + H^2 - K) \Phi_i = 4\pi G a^2 \delta p_i,
\]

for each value of \(i = 1, \cdots, N\), i.e. for each fluid. Note at this point that the decomposition of \(\Phi\) as the sum of functions \(\Phi_i\) is merely a mathematical tool, without any particular physical meaning, to prove the regularity of the the Bardeen potential. The functions \(\Phi_i\) prove convenient as they encode all the information necessary to describe the dynamical system as a whole.

Substituting \(\delta p_i = \omega_i \delta\epsilon_i\) in Eq. (3.5), and inserting it into Eq. (3.4), one then obtains the following decoupled equations for each \(\Phi_i\):

\[
\Phi_i'' + 3H(1 + \omega_i)\Phi_i' - \omega_i \nabla^2 \Phi_i + [2H' + (H^2 - K)(1 + 3\omega_i)] \Phi_i = 0,
\]

Hence, \(2N\) initial conditions are necessary to obtain the full Bardeen potential; this can also be seen by counting the number of degrees of freedom and constraint equations \[21\].

Models of this type, as proven above, are symmetric around the bounce. Since \(H\) and \(H'\) are regular everywhere, and as the \(\omega_i\) are constants, then the coefficients appearing in Eqs. (3.6) are completely regular at all times, in particular around the bounce and around a possible point of NEC transition. Then, by Fuchs property \[22\], \(\Phi_i\), and all their derivatives are also regular at all points. Consequently, the same applies true for the full Bardeen potential \(\Phi = \sum_i \Phi_i\).

Note that Eqs. (3.4) and (3.5) can be cast in the form of Eqs. (3.6) if and only if the fluid perturbations satisfy constant equations of state, \(\delta p_i = \omega_i \delta\epsilon_i\), with \(\omega_i = \text{const.}\), i.e. if the fluids have vanishing intrinsic entropy perturbations; this most crucial assumption is not always emphasized. In the case where the equations of state contain \(\omega_i\) which are not constants, Eqs. (3.6) cannot be obtained in this way, the relationship between \(\delta p_i\) and \(\delta\epsilon_i\) may contain divergent coefficients, as we will see below, and the proof of the regularity of the Bardeen potential, if possible at all, would be, at least, much more involved. Note also that if the parameters \(\omega_i\) are allowed to vary, one cannot guarantee that \(a(\eta)\) is even, and that the expansion \(a(\eta) = a_0 + b\eta^2 + c\eta^4 + \cdots\) is valid near the bounce.
3.2 Unboundedness of adiabatic perturbations

We now investigate the more usual, although alternative in the bouncing context, description of the perturbations based on an expansion into adiabatic and entropy modes. Such an expansion works perfectly well in all other known situation encountered in cosmology [20], and it is therefore of interest to understand whether it can be rendered meaningful during a bouncing phase.

Let us assume the condition, known to be valid for most fluids, that the pressure depends on two parameters only, namely the energy density $\epsilon$, and the entropy $S$. This allows the expansion

$$\delta p = c_s^2 \delta \epsilon + \tau \delta S,$$  

(3.7)

where

$$c_s^2 \equiv \left( \frac{\partial p}{\partial \epsilon} \right)_S = \frac{p'}{\epsilon'} = -\frac{1}{3} \left( 1 + \frac{\beta'}{H\beta} \right)$$  

(3.8)

is the “sound velocity” (see Ref. [16] for a discussion of this quantity, named $\Upsilon$ in that reference), and $\tau \equiv (\partial p/\partial S)_{\epsilon}$. Plugging Eq. (3.3) into (3.2), and making use of the expansion (3.7), one recovers the usual Bardeen equation

$$\Phi'' + 3H(1 + c_s^2)\Phi' + \left[ 2H' + (H^2 - K) \left( 1 + 3c_s^2 \right) - c_s^2 k^2 \right] \Phi = \frac{3}{2} \ell_p^2 a^2 (\bar{\tau} \delta S).$$  

(3.9)

From now on, we shall Fourier decompose the Bardeen potential and any other relevant space-dependent quantities on the basis of the eigenfunctions of the Laplace-Beltrami operator as,

$$\Phi = \sum_k u_k(x) \Phi_k(\eta), \text{ with } (\nabla^2 + k^2) u_k = 0,\quad (3.10)$$

where the eigenvalues $k$ depend on the spatial curvature $K$ [24]. In what follows, we shall assume that such an expansion has been done for all the quantities involved, which will then be subsequently identified with their Fourier modes, i.e. we shall not write the index $k$, which will be implicit, and make the replacement $\nabla^2 \to -k^2$ throughout.

The evolution equation for the adiabatic modes is given by Eq. (3.9) in which one sets $\delta S = 0$ and $\Phi \to \Phi_{ad}$, namely

$$\Phi_{ad}'' + 3H(1 + c_s^2)\Phi_{ad}' + \left[ c_s^2 k^2 + 2H' + (H^2 - K) \left( 1 + 3c_s^2 \right) \right] \Phi_{ad} = 0.$$  

(3.11)

The presence of $c_s^2$, coming from the expansion (3.7), in both Eq. (3.9) and its equivalent for the purely adiabatic modes Eq. (3.11), together with the fact, as we will see below, that this quantity may diverge both at the bounce and at the NEC transition point, indicate some possible bad behavior of $\Phi$ around these points. As we will see, this is true for Eq. (3.11) but not for Eq. (3.9).

It is well known that Eq. (3.11) can be solved by means of the following usual change of variable [23]

$$\Phi_{ad} = \frac{3\ell_p^2 H u}{2a^2 \theta}, \quad \theta \equiv \frac{H}{a} \sqrt{\frac{3}{2\beta}},$$  

(3.12)
which transforms the original equation into the parametric oscillator equation

$$u'' + \left( c_s^2 k^2 - \frac{\theta''}{\theta} \right) u = 0. \tag{3.13}$$

The general solution of this last equation can be constructed iteratively in the regime for which $c_s^2 k^2 \ll V_u \equiv \theta''/\theta$ to yield

$$\frac{u}{\theta} = B_1 \left[ 1 - k^2 \int \frac{d\tau}{\theta^2} \int^{\tau} d\sigma (c_s \theta)^2 + k^4 \int \frac{d\tau}{\theta^2} \int^{\tau} d\sigma (c_s \theta)^2 \int \frac{d\rho}{\theta^2} \int^{\rho} d\zeta (c_s \theta)^2 \right]$$

$$+ B_2 \int \frac{d\tau}{\theta^2} \left[ 1 - k^2 \int d\sigma (c_s \theta)^2 \int \frac{d\rho}{\theta^2} + k^4 \int d\sigma (c_s \theta)^2 \int \frac{d\rho}{\theta^2} \int d\zeta (c_s \theta)^2 \int \frac{d\varphi}{\theta^2} \right]$$

$$+ \cdots, \tag{3.14}$$

where $B_1$ and $B_2$ are constants, although usually depending on the scale $k$ (they are in principle calculated through a matching with the region where $c_s^2 k^2 \gg \theta''/\theta$), and the remaining terms represented by the dots are of order $O(k^6)$ compared with those indicated.

We will now reobtain the divergences we got in Ref. [10], this time using expansion (3.14).

### 3.2.1 The bounce

For the bounce itself, we use expansion (2.7) without the term $f \eta^{2n+3}$, which is unnecessary here. All the following behaviors are written up to first order.

The possible cases are:

1. $n > 1$ and $K \neq 0$. In this case, we find the following behaviors (for details, see Ref. [10]),

$$c_s^2 = \frac{2(2n-1)(n-1)}{3K \eta^2}, \tag{3.15}$$

$$\mathcal{H} = \frac{2nb}{a_0} \eta^{2n-1}, \tag{3.16}$$

$$\beta = K, \tag{3.17}$$

$$z \propto \frac{1}{\eta^{2n-2}}, \tag{3.18}$$

$$\theta \propto \eta^{2n-1}, \tag{3.19}$$

$$V_u \propto \frac{1}{\eta^2}. \tag{3.20}$$

One can see that $c_s^2$ diverges with the same power as $V_u$. Hence, as long as $k^2 \ll 1$, the expansion (3.14) can be applied as close to the bounce as we want. Inserting Eqs. (3.15), (3.19) and (3.16) into (3.14) and (3.12), one can easily find that

$$\Phi_{ad} \propto \eta^{2-2n}, \tag{3.21}$$

which diverges at the bounce, as stated in Ref. [10]. This divergence already appears in the term of order $k^0$, and it is independent of $k$. 
(2) $n > 1$ and $\mathcal{K} = 0$. For the special case of a flat background, the various quantities needed to describe perturbations are modified as

\[
\psi_s^2 = -\frac{a_0(n - 1)}{3b n \eta^{2n}},
\]

\[
\mathcal{H} = \frac{2nb}{a_0} \eta^{2n-1},
\]

\[
\beta = -\frac{b}{a_0} 2n(2n - 1) \eta^{2n-2},
\]

\[
z \propto \text{const.}
\]

\[
\theta \propto \eta^n,
\]

\[
V_u \propto \frac{1}{\eta^2}.
\]

(3.22) (3.23) (3.24) (3.25) (3.26) (3.27)

Here, $\psi_s^2$ diverges faster than $V_u$. Hence, expansion (3.14) cannot be applied as we approach the bounce. In this situation, one is obliged to approximate the equation and find its solution. However, in this case, a divergence already in the adiabatic perturbation itself is found in Ref. [16],

\[
\Phi_{ad} \propto \eta^{(3n-2)/2} e^{\alpha |\alpha|},
\]

where

\[
\alpha \equiv k \sqrt{a_0/[3nb(n - 1)]} \eta^{1-n}.
\]

Note there is no divergence for $k = 0$.

(3) $n = 1$, $d \neq 0$, $\forall \mathcal{K}$. This is the case where the second derivative of $a(\eta)$ is non vanishing and $a(\eta)$ is not even. The relevant quantities are

\[
\psi_s^2 = -\frac{a_0 d}{b(2b - \mathcal{K} a_0) \eta},
\]

\[
\mathcal{H} = \frac{2b}{a_0} \eta,
\]

\[
\beta = \mathcal{K} - \frac{2b}{a_0},
\]

\[
z \propto \frac{1}{\sqrt{\eta}}
\]

\[
\theta \propto \eta,
\]

\[
V_u \propto \frac{1}{\eta}.
\]

(3.29) (3.30) (3.31) (3.32) (3.33) (3.34)

As in the first case, $\psi_s^2$ diverges with the same power as $V_u$. Again, as long as $k^2 \ll 1$, the expansion (3.14) can be applied as close to the bounce as we want. Inserting Eqs. (3.15), (3.19) and (3.16) into (3.14) and (3.12) one can easily find that the adiabatic perturbation is finite but its first derivative diverges logarithmically, $\Phi_{ad}' \propto B_2 k^2 \ln(\eta)$, and its second derivative diverges as $\Phi_{ad}'' \propto B_2 k^2 / \eta$, exactly as stated in Ref. [16]. There is no divergence for $k = 0$.

(4) $n = 1$, $\forall \mathcal{K}$, and $d = 0$. 

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This is the case we will examine in detail in this paper. The relevant quantities are

\[ c_s^2 = \frac{8b^2 + (Kb - 12c)a_0}{3b(2b - Ka_0)}, \quad (3.35) \]
\[ \mathcal{H} = \frac{2b}{a_0} \eta, \quad (3.36) \]
\[ \beta = K - \frac{2b}{a_0}, \quad (3.37) \]
\[ z \propto \frac{1}{\eta}, \quad (3.38) \]
\[ \theta \propto \eta, \quad (3.39) \]
\[ V_u = \text{const.} \quad (3.40) \]

In this case, neither \( c_s^2 \) nor \( V_u \) diverge at the bounce. Hence, Eq. (3.11) is regular around the bounce, and so are all its solutions.

### 3.2.2 The NEC transition

In what follows, we concentrate on the point where \( \beta = 0 \), so that we shift the origin of time: for the rest of this section, \( \eta = 0 \) when \( \beta = 0 \), and we denote by an index 0 quantities evaluated at this point.

We now assume that the scale factor around \( \eta = 0 \) is, again, differentiable at least up to third order, so that the following expansion

\[ a(\eta) = a_0 \left[ 1 + \mathcal{H}_0 \eta + \frac{1}{2} (2\mathcal{H}_0^2 + K) \eta^2 + \frac{1}{3!} a_3 \eta^3 + \cdots \right], \quad (3.41) \]

holds.\(^4\) In this relation, \( a_3 \equiv a'''(0)/a_0 \), and use has been made of \( a''(0)/a_0 = 2\mathcal{H}_0^2 + K \), which is a simple rewriting of \( \beta = 0 \).

Using the expansion (3.41), we find that

\[ \mathcal{H} = \mathcal{H}_0 + \eta(\mathcal{H}_0^2 + K) + \frac{1}{2} \left[ a_3 - \mathcal{H}_0 (4\mathcal{H}_0^2 + 3K) \right] \eta^2 + \mathcal{O}(\eta^3), \quad (3.42) \]

leading to

\[ \beta \simeq \left[ \mathcal{H}_0 (6\mathcal{H}_0^2 + 5K) - a_3 \right] \eta + \mathcal{O}(\eta^2), \quad (3.43) \]

while the sound velocity takes the form

\[ c_s^2 = -\frac{1}{3\mathcal{H}_0 \eta} + \mathcal{O}(\eta^0), \quad (3.44) \]

which manifestly diverges at the NEC violating point.

\(^3\)Note that, in all these discussions, we are not examining situations where the constants appearing in the relevant quantities exhibited above cancel out exactly, as this requires some fine tuning.

\(^4\)Apart from the fine tuned case having \( 2b = a_0 \), the situation for which the point of NEC transition coincides with the bounce [i.e. no linear term in the expansion (3.41)] is nothing but the case (2) of the previous subsection, which is thus already treated.
We will now use Eqs. (3.12) and (3.14) to evaluate the divergences in the adiabatic perturbation. The relevant quantities in these equations are, to leading order

\[ \theta^{-2} = \frac{2a_0^2}{3H_0^2} \left( H_0 \left( 6H_0^2 + 5K \right) - a_3 \right) \eta + O(\eta^2), \]  

whose behavior, combined with the divergence in \( c_s^2 \), yields

\[ c_s^2 \theta^2 \approx \frac{H_0 \eta^{-2} + K \eta^{-1}}{2a_0^2 \left[ a_3 - H_0 \left( 6H_0^2 + 5K \right) \right]} = b_1 \eta^{-2} + b_2 \eta^{-1} + O(\eta^0), \]  

and finally

\[ \frac{\mathcal{H}}{a^2} = \frac{H_0}{a_0^2} + \frac{(K - H_0^2)}{a_0^2} \eta + \left[ a_3 - 3H_0 \left( 2H_0^2 + 3K \right) \right] \eta^2, \]  

plus terms of order \( O(\eta^3) \).

Inserting these expressions into Eqs. (3.12) and (3.14), one obtains, in the terms with coefficient \( B_1 \), the quantities \( k^2 b_2 c_1 \eta^2 \ln(\eta)/2 \) and \( -k^4 b_1^2 c_1^2 \eta^2 \ln(\eta)/2 \), whose second derivative diverges as \( \ln(\eta) \). This is exactly the type of divergence obtained in Ref. [16] by another method. Here, however, the \( k \)-dependence of the divergences is obtained more precisely. There is no divergence for \( k = 0 \). Note that if \( K = 0 \), the coefficient \( b_2 \) vanishes, so that the divergence appears only at order \( k^4 \). We will return to this point in the last section.

The divergence in the adiabatic perturbation presented above, which is present in any model with NEC transition points (including the class of the precedent subsection), suggests that adiabatic perturbations cannot be defined in such models. We will turn to this point in details in Sec. 4 by concentrating on the simplest situation involving only two fluids. In the meantime, let us end up the setting of the general formalism for \( N \) fluids by looking at the curvature perturbation.

### 3.3 Curvature perturbation in the adiabatic case

Another relevant function that can be useful for calculating the primordial spectrum of cosmological perturbation was introduced in Ref. [23]. It is the curvature perturbation \( \zeta \) on uniform density hypersurfaces (or its generalization for non-flat background \( \zeta_{\text{BST}} [25] \))

\[ \zeta = \frac{2}{3} \left( \frac{H^{-1} \Phi' + \Phi}{1 + \omega} \right) + \Phi = \left( \frac{H\Phi}{\beta} \right)' + 2\Phi, \]  

where \( \omega \) is the total equation of state which can be evaluated by means of Eqs. (2.4) and (2.5) through

\[ 1 + \omega = \frac{\epsilon + p}{\epsilon} = \frac{2}{3} \left( \frac{\beta}{H^2 + K} \right) \]  

and the second equality of Eq. (3.48) stems from this relation together with the explicit assumption \( K = 0 \).

This variable is useful in particular when it comes to describing ordinary transitions such as the radiation to matter domination, or the reheating at the end of inflation: on
large (super-Hubble, often misleadingly called superhorizon \([19]\) scales, \(\zeta\) is approximately constant. Indeed, in this case, it is found that \([23]\)

\[
\zeta' \simeq -\frac{\mathcal{H}}{\epsilon + p} \delta p_{\text{nad}},
\]

(3.50)

where \(\delta p_{\text{nad}} = \delta p - c_s^2 \delta \epsilon\) is the nonadiabatic part of the pressure perturbation, so that \(\zeta\) is expected to be conserved for adiabatic perturbations. In this latter case, to which we restrict our attention in this section, it therefore suffices to evaluate it before the transition to obtain its value after the transition, without prior knowledge of the detailed structure of the transition itself. It is immediately clear however from Eq. (3.50) that this will no longer hold whenever a nonnegligible amount of entropy perturbation is present, or if the NEC violation occurs at some stage. As we have seen above, both these conditions generically take place in a bouncing scenario, so that care must be taken in examining this particular case. In a fashion similar to Eq. (3.12), one can define

\[
\zeta_{\text{ad}} = -\sqrt{\frac{3}{2}} \ell_p \frac{v}{z},
\]

(3.51)

where \(v\) gives yet another way of obtaining the gravitational potential \(\Phi\) through \([20]\)

\[
\Phi_{\text{ad}} = \sqrt{\frac{3}{2}} \ell_p \beta^{1/2} \frac{1}{ac_s k^2} \left( \frac{v}{z} \right)',
\]

(3.52)

with

\[
z \equiv \frac{a \beta^{1/2}}{\mathcal{H}c_s} = \sqrt{\frac{3}{2} \frac{1}{c_s \theta}},
\]

(3.53)

where, again, care must be taken when \(\beta\) and \(c_s^2\) change sign, which occurs at the NEC transition. Note that Eq. (3.51) is only valid provided one considers adiabatic perturbations; if entropy perturbations were present at a nonnegligible level, then Eq. (3.51) would have to be modified by inclusion of an extra term, proportional to the entropy perturbation in question (and to \(z\)), in order to define the full \(\zeta\).

It is worth noting at this point that the variable \(v\) draws its importance in the theory of cosmological perturbations from the fact that, in the case of a single fluid (or a scalar field) dominating the universe, it allows to write the total (gravitational and fluid) action as that of a simple scalar field with varying mass, which can then be easily quantized \([20]\). It is from this variable, by assuming a Bunch-Davies \([26]\) vacuum state for its relevant modes, that one sets initial conditions for the cosmological perturbation.

The effective action derivable for \(v\) permits to cast its equation of motion in the same form as Eq. (3.12), namely

\[
v'' + \left( c_s^2 k^2 - \frac{z''}{z} \right) v = 0,
\]

(3.54)

where now the effective potential reads

\[
V_v \equiv \frac{z''}{z},
\]

(3.55)
and the RHS of Eq. (3.54), although vanishing in the adiabatic case to which we restrict our attention here, contains in principle a source term proportional to the entropy perturbation. The perturbation $v$, being commonly used in the literature as a quantum scalar field, may be argued to possess some amount of physical significance, although it is not directly observable. Note also that $v$ cannot account for all the degrees of freedom if more than one fluid are acting on comparable levels; in this latter case, the variable $v$ merely encodes the information on the adiabatic part of the full Bardeen potential.

The solution can be similarly expanded as in Eq. (3.14) for the variable $u$, provided the replacements $u \rightarrow v$ and $\theta \rightarrow z$ are done. Using Eq. (3.52), this gives the curvature perturbation directly as

$$
\zeta_{\text{ad}} \propto \frac{v}{z} \simeq C_1 \left[ 1 - k^2 \int_\eta^\tau d\tau (c_s \theta)^2 \int_\tau^\sigma \frac{d\sigma}{\theta^2} \right] + C_2 \int_\eta^\tau d\tau (c_s \theta)^2 \left[ 1 - k^2 \int_\tau^\sigma \frac{d\sigma}{\theta^2} \int_\sigma^\rho \frac{d\rho (c_s \theta)^2}{\theta^2} \right],
$$

where $C_1$ and $C_2$ are again constants depending on scale and we have dropped higher order terms.

In principle, using either $u$ or $v$ to propagate adiabatic perturbations through a given period should lead to the same gravitational adiabatic potential $\Phi_{\text{ad}}$, and hence to the same primordial spectrum, i.e. the same physical predictions. In practice however, because of the existence of poles in the effective potentials and “sound velocity”, there are instances leading to discrepancies (see Refs. [19, 27] for a more thorough discussion).

The results obtained in this section suggest that the very definition of adiabatic modes may be unattainable in such bouncing models. In fact, as explained in Sec. 2, one needs at least two fluids in order to construct a realistic bouncing model. This, in turn, demands that the perturbations cannot be purely adiabatic: even if one considers fluids with no intrinsic entropy perturbation, the relative entropy contribution must be present. The question is if the entropy fluctuations, although necessarily present, can be considered to be irrelevant all along in order for the adiabatic perturbations to be sufficient to accurately describe the evolution of the overall perturbations, as usually assumed to be the case on large scales. In order to answer this question, we examine in details in the following section (Sec. 4) a prototypical example of a two non interacting fluids bouncing model, both fluids having barotropic equations of state with constant ratio.

4. Bouncing with two fluids

In what follows, we establish the differential equations governing the entropy perturbations and the Bardeen potential dynamical evolution, and show that entropy fluctuations cannot be neglected at the NEC transition point, and hence that purely adiabatic perturbations have no meaning there. To conclude this section, i.e. in 4.2, we restrict ourselves to the particular case of a flat universe filled with a negative energy stiff matter and radiation, or, in other words, the situation equivalent to the case presented in Ref. [13]. There we show explicitly that, although presenting the abovementioned divergences in the adiabatic fluctuations, the equations governing the Bardeen potential and entropy fluctuations, when
taken together without any adiabatic condition, yield a perfectly regular fourth order equation for the full Bardeen potential, which is equal to the one obtained in Ref. [13], whose solutions and derivatives must be regular at any time, including at the NEC transition. This reinforces the idea that there is no problem in defining entropy fluctuations in such bouncing models, but that we cannot neglect them around the NEC transition point. We also study, in this particular example, the behavior of curvature perturbations.

4.1 Adiabatic and entropy modes

We now specialize to the $N = 2$ case for which many calculations, in particular the evolution of entropy perturbations, can be done explicitly. We now have $\delta \epsilon = \delta \epsilon_1 + \delta \epsilon_2$ and $\delta p = \delta p_1 + \delta p_2$, from which we can derive the constraint

$$ (a \Phi)_i' = 4\pi G a^2 \left[ \epsilon_1 (1 + \omega_1) \delta u_{(1)i} + \epsilon_2 (1 + \omega_2) \delta u_{(2)i} \right], $$

which will be useful later. The sound velocity now takes the simple form

$$ c_s^2 = \frac{\omega_1 (1 + \omega_1) \epsilon_1 + \omega_2 (1 + \omega_2) \epsilon_2}{(1 + \omega_1) \epsilon_1 + (1 + \omega_2) \epsilon_2}, $$

which diverges at the points of NEC violation $\epsilon + p = 0$. With these explicit relations, one can calculate the entropy contribution to the pressure fluctuation. This is

$$ \bar{\tau} \delta S = (\omega_1 - \omega_2) \frac{(1 + \omega_1)(1 + \omega_2) \epsilon_1 \epsilon_2}{(1 + \omega_1) \epsilon_1 + (1 + \omega_2) \epsilon_2} \times \left[ \frac{\delta \epsilon_1}{(1 + \omega_1) \epsilon_1} - \frac{\delta \epsilon_2}{(1 + \omega_2) \epsilon_2} \right], $$

for which we now seek a time evolution equation.

The perturbed fluid tangent vector, for each fluid, has components given by $\delta u_0 = -a^2 \delta u^0 = a \Phi$ and $\delta u_{(1,2)i} = -a^2 \gamma_{ij} \delta u_{j(1,2)} = -a \partial_i v_{(1,2)}$, thus defining the potentials $v_1$ and $v_2$. Expanding the energy-momentum conservation $\nabla_\mu T^\mu_\nu = 0$ for both fluid to first order yields

$$ \delta \epsilon_i' + 3 \mathcal{H} (\delta \epsilon_i + \delta p_i) + (\epsilon_i + p_i) (\nabla^2 v_i - 3 \Phi') = 0, $$

for the time component, and

$$ [ (\epsilon_i + p_i) \partial_j v_i ]' + 4 \mathcal{H} (\epsilon_i + p_i) \partial_j v_i + \partial_j \delta p_i + (\epsilon_i + p_i) \partial_j \Phi = 0, $$

for a spatial component. Defining the density contrasts through

$$ \delta_i \equiv \frac{\delta \epsilon_i}{\epsilon_i}, $$

and using the background energy momentum conservation, which implies $\delta_i' = [\delta \epsilon_i' + 3 \mathcal{H} (1 + \omega_i) \delta \epsilon_i]/\epsilon_i$, one transforms Eq. [13] into

$$ \delta_i' = (1 + \omega_i) (3 \Phi' - \nabla^2 v_i). $$

Let us emphasize at this stage the well-known fact that the adiabaticity condition

$$ \delta S = 0 \iff s = \frac{1}{\omega_1 - \omega_2} \left( \frac{\delta_1}{1 + \omega_1} - \frac{\delta_2}{1 + \omega_2} \right) = 0 $$

for
is conserved in time in the long wavelength limit \( k \to 0 \) since \( (\omega_1 - \omega_2)s' = k^2(v_1 - v_2) \). This is one of the reasons for considering adiabatic modes, the other being that the observed spectrum of primordial fluctuations can be reconstructed, e.g. from CMBFAST \[28\], with initial conditions deep in the radiation era satisfying \( s = 0 \), whereas isocurvature modes lead to significant disagreement \[29\] with the data \[30\], although some mixture is still acceptable \[31\] (the situation is similar to that of having a small component of the perturbations in the form of topological defects \[22\]).

Projecting Eq. (4.5) along the vector \( k_j \) leads to the following dynamical equation for the velocity potentials \( v_i \):

\[
v_i' + \mathcal{H}(1 - 3\omega_i)v_i + \omega_i \frac{\delta_i}{1 + \omega_i} + \Phi = 0. \tag{4.9}
\]

With the definition (4.8) and Eqs. (4.7) and (4.9), upon using the constraint equation written as

\[
(3\mathcal{K} - k^2)\Phi = \frac{3}{2} \beta^2 \ell^2 a^2 \sum_i \{ \epsilon_i \delta_i - 3\mathcal{H} [\epsilon_i (1 + \omega_i)v_i] \}, \tag{4.10}
\]

which is nothing but a rewriting in a convenient way of Eq. (3.2) using Eq. (4.1) and the background Einstein equation (2.5), the dynamical equation for the variable \( s \) follows directly. This is:

\[
s'' + \mathcal{H}(1 - 3c_s^2)s' + k^2c_s^2 s = \frac{k^2}{\beta} (k^2 - 3\mathcal{K})\Phi, \tag{4.11}
\]

where the definition \[33\]

\[
c_s^2 \equiv \frac{\omega_2 \epsilon_1 (1 + \omega_1) + \omega_1 \epsilon_2 (1 + \omega_2)}{(1 + \omega_1) \epsilon_1 + (1 + \omega_2) \epsilon_2} \tag{4.12}
\]

has been used. It should be noted that this function, just as the “sound velocity” \( c_s^2 \), is also singular at the NEC violating point, for which \( \epsilon + p \to 0 \).

Eq. (4.11) is the main result of this section. Note that its source term diverges at the NEC transition point. It means that we cannot neglect \( s \) at this point for any small but finite \( k \). Hence, adiabatic perturbations cannot be defined there. Note also that the divergent source is of order \( k^2 \) if \( \mathcal{K} \neq 0 \), and \( k^4 \) if \( \mathcal{K} = 0 \), exactly the orders in which the divergences in the adiabatic potential appear as calculated in section 3. Finally, we want to emphasize from Eq. (4.11) that the adiabatic case is, in most of the usual situations not involving NEC violation or bounces, i.e. in standard inflationary models, the only one that is tractable self consistently, in particular in the flat \( \mathcal{K} = 0 \) situation. This is because \( s \approx 0 \) solves Eq. (4.11) at leading order in \( k^2 \). The symmetric situation, with “purely entropic modes”, having \( \Phi = 0 \), is not self-consistent because \( s \) [or \( \delta S \) in Eq. (3.9)] then sources the gravitational potential at the same order.

Eqs. (3.3) and (4.11) form a closed system for the perturbation variables \( s \) and \( \Phi \) depending only on the background functions. In fact, one can express the entropy and sound “velocities” \( c_z \) and \( c_s \) simply as

\[
c_s^2 = \frac{3}{2} \frac{(1 + \omega_1 + \omega_2 + \omega_1 \omega_2)}{(1 + \omega_1) \epsilon_1 + (1 + \omega_2) \epsilon_2} \mathcal{H}^2 + \frac{\mathcal{K}}{\beta} - 1, \tag{4.13}
\]
\[ c_s^2 = -\frac{1}{3} \left( 1 + \frac{\beta'}{\mathcal{H}} \right), \] (4.14)

whereas the left hand side of Eq. (3.9) can be given the form
\[ \frac{3}{2} \ell_r^2 a^2 (\ddot{r} \delta S) = -\beta \left( c_s^2 \right)' s. \] (4.15)

All these relations permit to gather the overall system into a much simplified form involving only the scale factor \( a(\eta) \), system which is therefore particularly well suited for a more specific investigation. Let us accordingly now apply these relations to the particular case of Ref. [13], which is, as mentioned above, equivalent to a two fluids model consisting of a negative energy stiff matter and radiation. We want to verify whether, without any adiabatic condition, Eqs (3.9) and (4.11) are consistent with the general result that the full Bardeen potential and all its derivatives are bounded at all times.

### 4.2 A worked-out example

We now consider the particular case of a bounce occurring for a flat \((\mathcal{K} = 0)\) universe filled with radiation \((\omega_1 = 1/3)\) and some negative energy stiff matter having \(p_s = \epsilon_s < 0\) (i.e. \(\omega_2 = 1\)). Energy conservation is valid separately for both fluids, yielding \(\epsilon_r = c_r/a^4\) and \(\epsilon_s = -c_s/a^6\), with \(c_r\) and \(c_s\) two positive constants. Note that this stiff matter can also be modeled by a negative kinetic energy free massless scalar field, whose background dynamics and perturbations were studied in Ref. [13].

The background FLRW metric has a scale factor that takes the form \[ a = a_0 \sqrt{1 + \left( \frac{\eta}{\eta_0} \right)^2}, \] (4.16)

where \(a_0^2 = c_s/c_r\) and \(\eta_0^2 = c_s/(c_r^2 \ell_r^2)\). Note that this form satisfies the symmetry requirement \(\eta \to -\eta\) proven in section [3].

The relevant quantities to be calculated, namely \(\mathcal{H}, \mathcal{H}', \beta, c_s^2, \) and \(c_s^2\) read, in this example,
\[ \mathcal{H}(x) \equiv \frac{x}{\eta_0 (1 + x^2)}, \quad \mathcal{H}'(x) \equiv \frac{1 - x^2}{\eta_0^2 (1 + x^2)^2}, \] (4.17)
\[ c_s^2(x) \equiv \frac{2x^2 - 7}{3(2x^2 - 1)}, \quad c_z^2(x) \equiv \frac{2x^2 + 1}{2x^2 - 1}, \] (4.18)
\[ \beta(x) \equiv \frac{2x^2 - 1}{\eta_0^2 (x^2 + 1)^2}, \] (4.19)

where we have set for further convenience \(x \equiv \eta/\eta_0\).

The equivalent of Eqs (3.6) now read [recall that for the sake of simplicity, and to avoid an unnecessary proliferation of indices, we note simply \(\Phi_r\) and \(\Phi_s\) the \(k\)–modes of \(\Phi\) in the following equations, i.e. e.g. \(\Phi_r \equiv \Phi_r(k, \eta)\)]
\[ \frac{d^2 \Phi_r}{dx^2} + \frac{4x}{1 + x^2} \frac{d \Phi_r}{dx} + \left( \tilde{k}^2 + \frac{2}{(1 + x^2)^2} \right) \Phi_r = 0, \] (4.20)
and
\[
\frac{d^2 \Phi_s}{dx^2} + \frac{6x}{1 + x^2} \frac{d \Phi_s}{dx} + \left( k^2 + \frac{2}{1 + x^2} \right) \Phi_s = 0, \tag{4.21}
\]
where \( k \equiv k \eta_0 \), with \( \Phi = \Phi_r + \Phi_s \), the indices \( r \) and \( s \) corresponding to radiation and stiff matter, respectively. As one can see, all coefficients in these equations are completely regular, even at the point of NEC transition, \( x_{\text{NEC}}^2 = 1/2 \). The full Bardeen potential is therefore regular everywhere.

Let us now examine the evolution of the perturbations from the more usual alternative point of view of Eqs. (3.9) and (4.11). From Eqs. (4.15), (4.17), (4.18), and (4.19), one can turn Eqs. (3.9) and (4.11) into
\[
\frac{d^2 \Phi}{dx^2} + \frac{2x(4x^2 - 5)}{(1 + x^2)(2x^2 - 1)} \frac{d \Phi}{dx} + \left[ \frac{2x^2 - 7}{3(2x^2 - 1)} k^2 - \frac{2}{(1 + x^2)(2x^2 - 1)} \right] \Phi = \frac{8}{3} \frac{s}{(1 + x^2)(2x^2 - 1)}, \tag{4.22}
\]
and
\[
\frac{d^2 s}{dx^2} = \frac{4x}{2x^2 - 1} \frac{ds}{dx} + \frac{2x^2 + 1}{2x^2 - 1} k^2 s = \frac{s}{2x^2 - 1} \left( \frac{x^2 + 1}{2x^2 - 1} \right) k^2 \Phi. \tag{4.23}
\]

The coefficients of \( \Phi, d\Phi/dx, s \) and \( ds/dx \) in these equations, as well as their source terms, diverge as \( 1/(2x^2 - 1) \) at the NEC transition points \( x_{\text{NEC}} \). This may imply divergences in the solutions of these equations near the NEC transition. However, taking into account the entropy modes (which was not possible at all in the framework of Ref. [16]) without any adiabatic assumption allows a more precise analysis since it permits to obtain the fourth order equation satisfied by all the modes of \( \Phi \), i.e. not only the adiabatic ones (whose meaning near NEC transition is in question). The relevant equation, derived from Eqs. (4.22) and (4.23) is
\[
\frac{d^4 \Phi}{dx^4} + \frac{10x}{1 + x^2} \frac{d^3 \Phi}{dx^3} + \left( \frac{4}{3} k^2 + \frac{20}{1 + x^2} \right) \frac{d^2 \Phi}{dx^2} + \frac{6xk^2}{1 + x^2} \frac{d \Phi}{dx} + \frac{4}{3} \left( k^2 + \frac{4}{x^2 + 1} \right) k^2 \Phi = 0. \tag{4.24}
\]
All the coefficients of the above equation, as could have been expected from the analysis in terms of Eqs. (4.20) and (4.21), are regular everywhere. Hence, using again Fuchs property, \( \Phi \) and all its derivatives must be regular at all points, i.e. there cannot be any divergence nowhere, no unbounded growth of the perturbation, in accordance with the alternative analysis based on Eqs. (4.20) and (4.21). Using Eq. (4.17), one can show that Eq. (4.24), as expected, is the same as the one obtained in Ref. [13] [Eq. (35) thereof], where the negative energy stiff matter is described in terms of the negative energy free massless scalar field, namely
\[
\Phi^{(IV)} + 10 \mathcal{H} \Phi'' + \left[ \frac{4}{3} k^2 + 20 \left( \mathcal{H}' + 2 \mathcal{H}^2 \right) \right] \Phi'' + 6 \mathcal{H} k^2 \Phi' + \frac{1}{3} k^2 \left[ k^2 + 4 \left( \mathcal{H}' + 2 \mathcal{H}^2 \right) \right] \Phi = 0. \tag{4.25}
\]

In the same way one can obtain, for the sake of completeness, a similarly decoupled fourth order equation for the entropy \( s \), which reads
\[
s^{(IV)} - 2 \mathcal{H} s''' + \left[ \frac{4}{3} k^2 + 2 \left( \mathcal{H}' + 2 \mathcal{H}^2 \right) \right] s'' - \frac{2}{3} \mathcal{H} k^2 s' + k^2 \left[ \frac{1}{3} k^2 - 2 \left( \mathcal{H}' + 2 \mathcal{H}^2 \right) \right] s = 0. \tag{4.26}
\]
Hence, as $H'$ and $H$ are regular functions, the entropy fluctuations are also well behaved all along.

It should be kept in mind that Eqs. (4.25) and (4.26) do in fact represent the same physical system, and the second does not provide any additional information not already contained in the first. The number of initial conditions, which is obtained from only one of them, is four, as one should expect in the particular case of two fluids [see the discussion below Eq. (3.5)].

Let us now investigate the pure adiabatic $(s = 0)$ perturbations modes. Their equation is given by Eq. (3.11), which we solve by means of Eqs. (3.12) and (3.14). Note that although the term $c_s^2\theta^2$ presents a (simple) pole at $x = x_{\text{NEC}}$, the approximation (3.14) makes perfect sense even around this point since the effective potential for $u$ reads

$$V_u \equiv \frac{\theta''}{\theta} = \frac{8x^6 - 2x^4 + 20x^2 + 3}{[\eta_0 (1 + x^2)(2x^2 - 1)]^2},$$

and hence presents two second order poles: the approximation actually becomes better as one gets closer to the NEC transition points.

The relevant quantities are, to leading order,

$$c_s^2\theta^2 = \frac{x^2(2x^2 - 7)}{2a_0^2(1 + x^2)(2x^2 - 1)^2} \simeq -\frac{1}{8a_0^2} \left[ \frac{1}{(x - x_{\text{NEC}})^2} - \frac{13}{6} \right],$$

the following term being of order $O(x - x_{\text{NEC}})$, and

$$\theta^{-2} = \frac{2a_0^2}{3x^2} \left[ (1 + x^2)(2x^2 - 1) \right] \simeq 4a_0^2 \left[ \sqrt{2} (x - x_{\text{NEC}}) - \frac{5}{3} (x - x_{\text{NEC}})^2 \right],$$

where now the remaining term behaves as $O((x - x_{\text{NEC}})^3)$. These relations are precisely of the form expected from Eqs. (3.46) and (3.45) for $K = 0$, i.e. with $b_2 = 0$.

The integrals in Eq. (3.14) can be evaluated explicitly, and read

$$\int^\eta \frac{d\tau}{\theta^2} = \frac{2}{3} a_0^2 \eta_0 \left( \frac{1}{x} + x + \frac{2}{3} x^3 \right),$$

and

$$\int^\eta c_s^2 \theta^2 d\tau = \frac{\eta_0}{2a_0^2} \left( \frac{x}{2x^2 - 1} + \arctan x \right),$$

which in turn yield

$$\int^\eta \frac{d\tau}{\theta^2} \int^\tau c_s^2 \theta^2 d\sigma = \frac{\eta_0^2}{18x} \left\{ x^3 + 2 \left[ 2x^4 + 3(x^2 + 1) \right] \arctan x + 2x \ln(x^2 + 1) \right\},$$

and

$$\int^\eta c_s^2 \theta^2 d\tau \int^\tau \frac{d\sigma}{\theta^2} = \frac{\eta_0^2}{18(2x^2 - 1)} \left[ 10 - x^2 + 2x^4 + 2(1 - 2x^2) \ln(x^2 + 1) \right].$$

The divergence in the second derivative appears in the term

$$\Phi''_{\text{ad,dom}} \sim B_1 k^4 \left[ c_s^2 \int^\eta \frac{d\tau}{\theta^2} \int^\tau c_s^2 \theta^2 d\sigma + (\theta^{-2})' \int^\eta c_s^2 \theta^2 d\tau \int^\tau \frac{d\sigma}{\theta^2} \int^\sigma c_s^2 \theta^2 d\rho \right],$$

(4.34)
which, after inserting Eqs (4.28), (4.29), (4.32) and (4.18) expressed as Taylor series around $x - x_{\text{NEC}}$ yields

$$\Phi''_{\text{ad,dom}} \propto B_1 k^4 \ln(x - x_{\text{NEC}}),$$

(4.35)

as expected from the results of section 3.

Note that in order to get Eq. (4.35), it is necessary to keep the subleading part in the second term of Eq. (4.34) since the leading orders of both terms are $\propto (x - x_{\text{NEC}})^{-1}$ and exactly cancel each others. This result, together with the regular equations (4.24) and (4.26) obtained above, show that the Bardeen potential and the entropy perturbations given as solutions of Eqs. (4.22) and (4.23) make sense at the NEC transition, but that the adiabatic modes in themselves do not. However, since the divergence itself arises at the fourth order in $k$, and in the second derivative of the adiabatic part of the Bardeen potential, it can be argued that, in the flat case $\mathcal{K} = 0$ at least, this adiabatic part of the Bardeen potential and its first derivative could be used in a consistent way to produce matching conditions in a bouncing scenario.

To be complete, let us now compute the curvature perturbation as discussed in Sec. 3.3. We first evaluate the background function $z$ in the background (4.16). This gives

$$z = \frac{a_0}{x} (2x^2 - 1) \left[ \frac{3 (1 + x^2)}{2x^2 - 7} \right]^{1/2},$$

(4.36)

which in turns yields the following complicated form for the potential $V_v$

$$V_v = \frac{14x^6 + 108x^4 + 273x^2 + 98}{[\eta_0 (x^2 + 1) (2x^2 - 7)]^2},$$

(4.37)

and thus exhibits two second-order poles whose origin can be traced back to the time at which the “sound velocity” $c_s$ vanishes, and another pole at the bounce itself. As is apparent from the solution (3.56), however, the points at which $c_s \to 0$ do not generate any divergences, and are in fact perfectly regular as far as the solution is concerned; therefore, we shall not consider them further.

The solution given by Eq. (3.56) involves essentially the same integrals as those in Eq. (3.14), except for the last one (the triple integral), which we could not perform explicitly. It turns out, however, that, contrary to the Bardeen potential, and because of the reverse order in which the integrands appear in the integrals of Eq. (3.56) with respect to Eq. (3.14), the curvature perturbation diverges at the NEC transition, and is thus not a proper quantity to propagate through this point. Since $\Phi_{\text{ad}}$, the expansion rate $\mathcal{H}$, and their first derivatives are all well-behaved at the NEC violating time, this fact was to be expected from the very definition of $\zeta_{\text{ad}}$, see Eq. (3.48).

One point of interest to be mentioned here is the fact that, if one restricts attention to adiabatic perturbations, then both variables $u$ and $v$ exhibit divergences while passing through the bounce. However, those present in $u$ stem from the NEC violation, necessary in this case, and can be understood as mere computational artifacts as they are exactly compensated in the calculation of $\Phi_{\text{ad}}$, which thus appears to be the regular variable to consider in this instance (remember that the divergence in $\Phi_{\text{ad}}$ appears only in its second
derivative, at order $k^4$ in the above example). In contrast, $v$ is actually unbounded also at the bounce itself, as well as at the points for which the sound velocity vanishes, while $\zeta_{\text{ad}}$ exhibits a pole around the NEC transition.

This result merely means that the comoving hypersurfaces are not defined at the NEC violating points, which is hardly surprising if one recalls the definition of these hypersurfaces [21]: they are given by the requirement that the total stress-energy tensor satisfy $T^0_i = 0$. Setting $\delta p_{\text{com}}$ the pressure perturbation with respect to these hypersurfaces, the comoving coordinates are then completed by the comoving time $t_{\text{com}}$ related to the cosmic time $t$ through the relation

$$dt = ad\eta = \left(1 - \frac{\delta p_{\text{com}}}{\epsilon + p}\right) dt_{\text{com}}. \quad (4.38)$$

It is clear that the divergence observed in the curvature perturbation can then be interpreted as a bad choice of coordinates, the transformation to the comoving coordinates being singular at the NEC transition, and hence forbidden. It seems therefore that, at least in this example, propagating $u$, i.e. $\Phi_{\text{ad}}$, through the bounce and NEC transition makes more sense than propagating $v$, i.e. $\zeta_{\text{ad}}$.

In the case one tries to match the collapsing phase to the expanding phase directly, without taking into account the detailed structure of the bounce, then neither $\Phi$ nor $\zeta$ is actually continuous through that particular bounce; in fact, it is the combination $\beta (\zeta - \Phi)/H$ which passes continuously in this case [13]. This bears close resemblance with another situation in which general relativity alone is used to perform the bounce [18, 19]; these results provide new example for the general framework set in Ref. [27].

5. Conclusions

In this paper, we have examined with greater details one particular case previously discussed in Ref. [16] in which adiabatic perturbations in the hydrodynamical framework present divergences in its second derivative around the NEC transition, see Eq. (4.35), even though, as inspection of Eqs. (4.20) and (4.21) reveals, the full Bardeen potential and all its derivatives are perfectly regular. Eqs. (4.20) and (4.21) constitute a set of equations which describe the Bardeen potential, hence the scalar part of the metric fluctuations, completely. An alternative set of equations is the one based on the definition of entropy fluctuations (essential for the definition of an adiabatic perturbation) given in Eq. (3.7), namely, Eqs. (4.23) and (4.22). These equations present divergent coefficients essentially because Eq. (3.7) contains $c_s^2$, which has a pole at the NEC transition, see Eq. (4.18). This fact suggests that the system (4.23)–(4.22) may not be appropriate to describe perturbations around the NEC transition. However, by combining these two equations, we have obtained a completely regular fourth order equation for the full Bardeen potential, where no divergences in their solutions appear at any order, even though the equations from which it was obtained presented coefficients with singular points at the NEC transition. Hence, the hydrodynamical treatment of the example based on Eqs. (1.23) and (1.22) is completely equivalent to the one in terms of fields worked out in Ref. [13], where the same fourth order equation was obtained.
Nevertheless, inspection of Eq. (4.23) shows that entropy fluctuations cannot be neglected, even for arbitrarily small but non-vanishing values of the wavelength $k$, at the NEC transition time as long as its source term diverges there. Hence, adiabatic perturbations cannot be defined at this point and the divergence detected in Ref. [14] is not physically meaningful. That may be understood in much the same way as undamped resonances or shock waves: if one neglects the friction terms thanks to which the energy of the modes can be evacuated away, which is what we do by considering only adiabatic perturbation, one introduces artificial discontinuities (in the case of shocks) or even divergences. The adiabatic perturbation approximation, which does not take into account possible exchanges between the various fluids involved, and therefore does not count correctly the relevant degrees of freedom, can only make sense if one of the fluids clearly dominates over the others, or if there is only one fluid with complicated equation of state, as was considered in Ref. [16].

The case studied in section 4.2 is but an example for which the scale factor near the bounce can be expanded as a symmetric power series in the conformal time $\eta$, namely as $a(\eta) = a_0 + b\eta^2 + e\eta^4 + \cdots$, where $a_0$, $b$ and $e$ are constants (it can be shown that the coefficients up to the fourth derivative of the scale factor are necessary to describe the passing of the perturbations through the bounce [19]). We have shown in section 2 that this is the only possible behavior of the scale factor in a model where the energy density can be written as a function of $a$. This situation comprehends the case of many non-interacting fluids with constant equations of state, where we have proven that [see Eq. (3.6)] the full Bardeen potential and all its derivatives are completely regular at all times.

We have also shown in section 3 that for models with other behaviors near the bounce, the conclusions of Ref. [16] hold correct, namely, that, besides divergences at the NEC transition, there are divergences in the adiabatic perturbation in the bounce itself. As the matter content of these types of bounce cannot be modeled with many non-interacting fluids with constant equations of state, one cannot prove in the way we did that the full Bardeen potential is finite in such cases, if it is. For case (1) of section 3, as the divergence in the adiabatic Bardeen potential comes in zero order in $k$, we can only conjecture that they are genuine divergences of the full Bardeen potential. For the other cases, as they come in at least second order in $k$, it may be that a process analogous to the one described in section 4.2 occurs, and the adiabatic treatment alone may not be meaningful also at the bounce itself. In order to see if this is the case, specific examples of this type must be constructed. In fact, nonsymmetric bounces may be physically more interesting than the symmetric ones. Indeed, some string/brane motivated models [10] have been proposed in the literature, which, although mathematically inconsistent for various reasons [11], suggest that the turning point, i.e. the bounce itself, could be the time at which entropy is produced, in the form of radiation say, not in a smooth way. Seen from a four dimensional point of view, such an entropy production can be modeled by a nonsymmetric bouncing scenario, since the total energy density is then no longer a function of the scale factor. It is clear however that, according to the present analysis, the fate of linear perturbation theory in general relativity bouncing models cannot be guaranteed by examining adiabatic perturbations alone. Also, in the case linear perturbation theory really breaks down, it is possible that such models do have important second order contributions, which means
possibly strong backreaction effects. All these questions deserves further investigation.

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