Faster Attractor-Based Indexes*

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Abstract

String attractors are a novel combinatorial object encompassing most known compressibility measures for highly-repetitive texts. Recently, the first index building on an attractor of size $\gamma$ of a text $T[1..n]$ was obtained. It uses $O(\gamma \log(n/\gamma))$ space and finds the $occ$ occurrences of a pattern $P[1..m]$ in time $O(m \log n + occ \log^* n)$ for any constant $\epsilon > 0$. We now show how to reduce the search time to $O(m \log n + occ \log \epsilon n)$ within the same space, and ultimately obtain the optimal $O(m + occ)$ time within $O(\gamma \log(n/\gamma) \log n)$ space. Further, we show how to count the number of occurrences of $P$ in time $O(m + \log^3 \epsilon n)$ within $O(\gamma \log(n/\gamma))$ space, or the optimal $O(m)$ time within $O(\gamma \log(n/\gamma) \log n)$ space. These turn out to be the first optimal-time indexes within grammar- and Lempel-Ziv-bounded space. As a byproduct of independent interest, we show how to build, in $O(n \log n)$ expected time and without knowing the size $\gamma$ of the smallest attractor, a run-length context-free grammar of size $O(\gamma \log(n/\gamma))$ generating (only) $T$.

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1 Introduction

Being able to efficiently count and locate the occurrences of a pattern $P$ in a repetitive text $T$ stored in compressed format is becoming an important task due to the constantly-increasing amount of repetitive data that is being generated in domains such as biology, physics, and web repositories (e.g., software or knowledge databases). This has generated, in the last decade, a significant amount of work aimed at designing compressed self-indexes on top of dictionary-compressed text representations. Examples of successful compressors from this family include (but are not limited to) the Lempel-Ziv factorization [19], of size $z$, context-free grammars [18], of size $g$ (e.g., RePair and Greedy), and the run-length Burrows-Wheeler transform [5], of size $\rho$. We refer the reader to an exhaustive review of the state of the art [13]. Recently, Kempa and Prezza [17] showed that all the above-mentioned repetitiveness measures (i.e., $z$, $g$, $\rho$, plus others not mentioned here) are not asymptotically smaller than the size $\gamma$ of a new combinatorial object, the string attractor. This and subsequent works [22, 24] showed that efficient queries can be supported in a space bounded in terms of $\gamma$. By the nature of this new repetitiveness measure, such data structures are universal in the sense that they can be designed on top of any known dictionary-compressed representation of $T$.

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In this paper we obtain the best results on attractor-based indexes, including the first optimal-time searching within space bounded in terms of $\gamma$, $z$, or $g$. We combine and improve upon three recent results:

1. Navarro and Prezza [22] presented the first index that builds on an attractor of size $\gamma$ of a text $T[1..n]$. It uses $O(\gamma \log(n/\gamma))$ space and finds the $occ$ occurrences of a pattern $P[1..m]$ in time $O(m \log n + occ(\log \log(n/\gamma) + \log^e \gamma))$ for any constant $\epsilon > 0$.

2. Christiansen and Ettienne [9] presented an index that builds on the Lempel-Ziv parse of $T$, of $z \geq \gamma$ phrases, which uses $O(z \log(n/z))$ space and searches in time $O(m + \log^e (z \log(n/z)) + occ(\log \log n + \log^e z))$.

3. Navarro [21] presented the first index that builds on the Lempel-Ziv parse of $T$ and counts the number of occurrences of $P$ in $T$ (i.e., computes $occ$) in time $O(m \log^{2+\epsilon}(z \log(n/z)))$.

Our contributions are as follows:

1. We obtain, in space $O(\gamma \log(n/\gamma))$, an index that finds the occurrences of $P$ in $T$ in time $O(m + \log^e \gamma + occ \log^e(\gamma \log(n/\gamma)))$, thereby obtaining the best space and improving the time from previous works [9, 22].

2. We obtain, in space $O(\gamma \log(n/\gamma))$, an index that counts the occurrences of $P$ in $T$ in time $O(m + \log m \log^{2+\epsilon}(\gamma \log(n/\gamma)))$, which outperforms the previous result [21] both in time and space.

3. Using more space, $O(\gamma \log(n/\gamma) \log n)$, we manage to list the occurrences in optimal time $O(m + occ)$ and count them in optimal time $O(m)$. This is the first index with space bounded in terms of $z, g$, or $\gamma$, that searches in optimal time.

To obtain our results, we prove that the locally-consistent and locally-balanced grammar built by Christiansen and Ettienne [9] on Lempel-Ziv can also be built within attractor space. We describe how to do efficient substring extraction and fingerprinting from a run-length context-free grammar (RLCFG), which is assumed but not explained in their paper. We also improve upon their algorithms to find secondary occurrences, adapting a technique [10] that avoids predecessor searches. Further, we use the fact that only $O(\log n)$ partitions of $P$ must be considered for the search [21] to reduce the counting time of Navarro [21], while making its space attractor-bounded as well. As a byproduct of independent interest, we show how to build a RLCFG of size $O(\gamma \log(n/\gamma))$ generating (only) $T$, where $\gamma$ is the size of the smallest attractor, in $O(n \log n)$ expected time and without the need to know the attractor.

## 2 Basic Concepts

We will index a string $T[1..n]$, called the text. Our logarithms are to the base $2$ by default.

Karp-Rabin fingerprinting [16] assigns a string $S[i..\ell]$ the signature $\kappa(S) = (\sum_{i=1}^{\ell} S[i] \cdot c^{i-1}) \mod \mu$ for suitable integers $c > 1$ and prime $\mu$. It is possible to build a function $\kappa$ guaranteeing no collisions between substrings of $T[1..n]$ in $O(n \log n)$ expected time [1].

An attractor $\Gamma = \{p_1, \ldots, p_k\}$ for $T[1..n]$ is a set of positions $p_i \in [1..n]$ such that any substring $T[i..j]$ has at least one copy $T[i'..j']$ that contains some attractor position $p_i$. [17] We make this copy explicit with the function $f[i..j] = [i'..j']$, arbitrarily choosing some copy.

\[1\] Note that in their paper they mistakenly claim a better time of $O(m + \log^e z + occ(\log \log n + \log^e z))$. This can be traced back to the wrong claim that their two-sided range structure, built on $O(z \log(n/z))$ points, answers queries in $O(\log^e z)$ time (the correct time is, instead, $O(\log^e(z \log(n/z)))$).
2.1 Grammar compression

Consider a context-free grammar (CFG) that generates $T$ and only $T$ \[13\]. Each nonterminal must be the left-hand side in exactly one rule, and the size $g$ of the grammar is the sum of the right-hand sides of the rules. The smallest grammar for a text $T$ is NP-complete to compute \[20, 7\], but it is possible to build grammars of size $g = O(z \log(n/z))$ \[13, Lem. 8\].

If we allow, in addition, rules of the form $A \rightarrow B'$, taken to be of size 2 for technical convenience, the result is a run-length context-free grammar (RLCFG) \[23\]. These grammars encompass CFGs and are intrinsically more powerful, for example on the string family $T = a^n$ the smallest CFG is of size $\Theta(\log n)$, whereas a RLCFG of size $O(1)$ can generate it. It is possible to build RLCFGs of size $g = O(z \log(n/z))$ and less \[12\].

The parse tree of a CFG has internal nodes labeled with nonterminals and leaves labeled with terminals. The root is the initial symbol and the concatenation of the leaves yields $T$. If $A \rightarrow A_1 \ldots A_s$, then node $A$ has children $A_1 \ldots A_s$. We obtain the grammar tree by pruning the parse tree: all but the leftmost occurrence of each nonterminal is converted into a leaf and its subtree is pruned. Then the grammar tree has exactly one internal node per distinct nonterminal and the total number of nodes is $g + 1$.

We call $exp(A)$ the expansion of nonterminal $A$, that is, the string it generates (or the concatenation of the leaves under any node $A$ in the parse tree), and $|A| = |exp(A)|$. A grammar is said to be balanced if the parse tree is of height $O(\log n)$, and locally balanced if for some constant $d$ the height of each nonterminal $A$ in the parse tree of $T$ is $\leq d \cdot \log |A|$.

2.2 Locally-consistent locally-contracting parsing

A string $S[1..n]$ can be parsed in a locally consistent way, meaning that equal substrings are largely parsed in the same form. We use a variant due to Mehlhorn et al. \[20\]. Let us define a repetitive area in a string as a maximal substring repeating one symbol, of length 2 or more.

The parsing proceeds in two passes. First, it groups the repetitive areas into blocks, which are seen as single symbols. On the resulting sequence, $S_b[1..n_b]$, it creates a new level of blocks. A permutation $\pi$ chosen uniformly at random is applied to the alphabet of $S_b$, formed by the original symbols of $S$ and the blocks that represent repetitive areas. Then, a local minimum is defined as a position $1 < i < n_b$ such that $\pi(S_b[i-1]) > \pi(S_b[i]) < \pi(S_b[i+1])$.

The parsing of $S_b$ into blocks is then defined as follows: a new block starts at position 1 and at the position following any local minimum. We assume that $n > 1$ and $S$ is terminated with a special symbol $S[n] = \$, for which it always holds $\pi(\$) = 1$; this guarantees that the parsing leaves no isolated symbols.

It is easy to see that this procedure cuts the initial string $S[1..n]$ into at most $\lceil n/2 \rceil$ blocks, and any substring $S[i..j]$ covers or overlaps at most $\lfloor (j - i + 1)/2 \rfloor$ blocks. We say that the parsing is locally contracting. Further, two equal substrings $S[i..j] = S[i'..j']$ that correspond to sequences of complete blocks are parsed identically. Otherwise, two equal substrings are parsed identically except for their two first and two last blocks: the local minima of $S[i..j]$ and $S[i'..j']$ are the same except possibly the first and the last of each, and the blocks within the shared local minima are the same. A difference in the first/last local minima may change the first/last two blocks. We thus say the parsing is locally consistent.

The average length of the blocks produced by the parsing is at most $3$ \[9, Lem. 3\].

2.3 Locally-consistent locally-balanced grammars

Christiansen and Ettienne \[9\] build a RLCFG on a text $T[1..n]$ using the parsing of Section 2.2. They perform the first pass of the parsing, collecting the distinct repetitive areas of the form...
a^\ell$, and create run-length nonterminals of the form $A \rightarrow a^\ell$ to replace the corresponding repetitive areas in $T$. They then perform the second pass on this new text. Each distinct block $a_1 \ldots a_k$ is replaced by a distinct nonterminal of the form $A \rightarrow a_1 \ldots a_k$ (each $a_i$ can be a letter or a run-length nonterminal created in the first pass).

The blocks are then replaced by those created nonterminals $A$. The resulting text, $T'$, is of length $n' \leq \lceil n/2 \rceil$. Note that the last block of $T'$ contains $\$$, and thus it is unique too. Its nonterminal plays the role of $\$$ in $T'$ (i.e., we ensure we assign it $\pi(T'[n']) = 1$).

The process is then repeated again on $T'$, and iterated for at most $\lceil \log n \rceil$ rounds, until a single nonterminal is obtained. This is the initial symbol of the grammar, which contains all the nonterminals created across all rounds. The height of the grammar is at most $\lceil \log n \rceil$.

It is not hard to see that the grammar is locally balanced, because any nonterminal $A$ whose parse tree is of height $h$ holds $|A| \geq 2^h$. Further, the grammar is also locally consistent [9, Sec. 2.3], in the following way: Consider two equal substrings $T[i..j] = T[i'..j']$. These induce $T(i,j)$ and $T(i',j')$, the sets of nodes that are ancestors of the leaves $i \ldots j$ and $i' \ldots j'$, respectively, in the parse tree of $T$. Then, due to the locally-consistent parsing done at each round, $T(i,j)$ and $T(i',j')$ must be equal except, possibly, for the two leftmost and two rightmost nodes generated in each round.

### 2.4 Real-time access to grammars

Gasieniec et al. [14] show how any prefix of length $\ell$ of a nonterminal in a CFG of size $g$ in Chomsky Normal Form can be extracted in time $O(\ell)$, using a data structure of size $O(g)$. This was later extended to general CFGs [10, Sec. 4.3]. We extend their result to RLCFGs.

**Lemma 1** (cf. [14] and [10, Sec. 4.3]). Given a RLCFG of size $g$, there exists a data structure of size $O(g)$ such that any prefix or suffix of length $\ell$ of $\exp(A)$ can be obtained from any nonterminal $A$ in time $O(\ell)$.

**Proof.** We follow the original structure [14, 10] showing that, for rules $X \rightarrow Y^t$, we can simulate the rule $X \rightarrow Y \ldots Y$ with $t$ copies of $Y$. See Appendix A.1 for details.

### 3 Bounding the Size of the Grammar

In this section we show that the RLCFG of Section 2.3 is of size $g = O(\gamma \log(n/\gamma))$, where $\gamma$ is the minimum size of an attractor of $T[1..n]$. The key is to prove that distinct nonterminals are produced only around the attractor elements.

We start by defining bordering blocks. For technical convenience, we add the positions 1 and $n$ to the attractor $\Gamma$, which makes it of size at most $\gamma + 2$.

**Definition 2.** Given an attractor $\Gamma = \{p_1, \ldots, p_{\gamma+2}\}$ of $T[1..n]$ containing the positions 1 and $n$, and a parsing of $T$ into blocks, we say that the blocks covering the positions $p_i$, as well as their preceding and following blocks, are bordering.

To count the number of different blocks produced in the first round of Section 2.3, we will pessimistically assume that all the bordering blocks are different. The number of such blocks is at most $3(\gamma + 2)$. On the other hand, we will show that non-bordering blocks do not need to be considered, because they will be counted somewhere else, when they appear near an attractor element (i.e., as a bordering block).

**Lemma 3.** For each non-bordering block, there is an identical bordering block.
Proof. A non-bordering block, padded with its preceding and following blocks, must have a copy crossing an attractor. At that position, the same block must be formed because it has the same preceding and following context. That copy of the block is bordering. See Appendix A.2 for details.

\[\text{We now show that the first round produces a RLCFG of size } O(\gamma).\]

\[\text{Lemma 4. The first round of parsing produces a grammar of size } O(\gamma).\]

Proof. The expected length of a block is \(O(1)\), and the grammar size is the sum of \(O(\gamma)\) such random variables. The sum is \(O(\gamma)\) in expectation, and can be turned worst-case after trying \(O(1)\) permutations \(\pi\), in expectation. See Appendix A.3 for details.

Finally, we extend the reasoning to all the rounds of parsing.

\[\text{Theorem 5. Let } T[1..n] \text{ have an attractor of size } \gamma. \text{ Then there exists a locally-balanced locally-consistent RLCFG of size } g_{rl} = O(\gamma \log(n/\gamma)) \text{ that generates (only) } T.\]

Proof. We perform \(O(\log(n/\gamma))\) rounds of locally-consistent parsing, where the output of each round (a sequence of nonterminals representing blocks) is the input to the next. The length of the string halves in each iteration, and the grammar grows only by \(O(\gamma)\) in each round. See Appendix A.4 for details.

Finally, we note that the grammar can be built without knowing the attractor \(\Gamma\), which yields the following byproduct. This opens the door to building indexes based on the smallest string attractor of \(T\) without having to find that attractor (which is NP-complete [17]).

\[\text{Theorem 6. Let } T[1..n] \text{ have an attractor } \Gamma \text{ of size } \gamma. \text{ Then we can build a locally-balanced locally-consistent RLCFG of size } g_{rl} = O(\gamma \log(n/\gamma)) \text{ that generates (only) } T \text{ in } O(n \log n) \text{ expected time, without knowing } \Gamma.\]

Proof. We carry out \(\log n\) iterations and the grammar does not exceed this size. Instead of requiring that blocks around attractors add up to \(O(1)\) amortized, we check that all the distinct blocks add up to \(O(1)\) amortized. See Appendix A.5.

### 4 Locating Pattern Occurrences

Let \(G\) be a locally-balanced RLCFG of \(r\) rules and size \(g \geq r\) on \(T[1..n]\), formed with the procedure of Section 3, thus \(g = O(\gamma \log(n/\gamma))\). We show how to build an index of size \(O(g)\) that locates the \(occ\) occurrences of a pattern \(P[1..m]\) in time \(O(m + (occ + 1) \log^k n)\).

We make use of the parse and grammar trees of the RLCFG. The parse tree of a RLCFG is defined by interpreting the rules \(A \rightarrow B^t\) as \(A \rightarrow B \ldots B\) (\(t\) copies of \(B\)). For the grammar tree, instead, we treat those rules as \(A \rightarrow BB'\), where \(B'\) is marked with the value \(t - 1\) and it is always a leaf (\(B\) may also be a leaf, if it is not the leftmost occurrence of \(B\)). Since we define the size of \(B'\) as 2, the grammar tree still is of size \(g + 1\), and contains \(r\) internal nodes and \(g + 1 - r\) leaves.

The leaves of the grammar tree induce a partition of \(T\) into \(p = g + 1 - r\) phrases. We say that an occurrence of \(P[1..m]\) at \(T[i..i + m - 1]\) is primary if the lowest grammar tree node deriving a range of \(T\) that contains \(T[i..i + m - 1]\) is internal (or, which is the same, the occurrence crosses the boundary between two phrases); otherwise it is secondary.
4.1 Finding the primary occurrences

To find the primary occurrences, we create a two-dimensional data structure of size $p \times p$. We reverse all the phrase contents, forming a sorted multiset $X$ of $p$ strings. Similarly, the multiset $Y$ contains the suffixes of $exp(A)$, for each nonterminal $A \rightarrow A_1, \ldots, A_s$, that start at positions $|A_1| + 1, |A_1| + |A_2| + 1, \ldots, |A_1| + \cdots + |A_{s-1}| + 1$. Each phrase beginning, except the first, is the starting position of exactly one such suffix, of which there are $g - r$.

We set a point at $(x, y)$ whenever the phrase $X_x$ immediately precedes the suffix $Y_y$ in $T$. It is sufficient, then, to consider every partition $P[1..k] \cdot P[k + 1..m]$ for $1 \leq k < m$, search for $P[1..k]$ reversed, $P[1..k]^{rev}$, in $X$ to find the range $[x_1..x_2]$ of phrases that are suffixed by $P[1..k]$, search for $P[k + 1..m]$ in $Y$ to find the range $[y_1..y_2]$ of phrase-aligned nonterminal suffixes starting with $P[k + 1..m]$, and finally search the two-dimensional structure for all the points in the range $[x_1..x_2] \times [y_1..y_2]$ to retrieve all the primary occurrences whose leftmost cut in $P$ is at position $k$. The first grammar tree node $A_i$ of each suffix $Y_y$ is associated with the corresponding point of the grid. From this node, we can report the position in $T$ of the primary and all the corresponding secondary occurrences.

This arrangement follows previous strategies to index CFGs [10]. To include rules $A \rightarrow B'$, we just index the reverse of the first $B$ and the suffix $B'^{-1}$. It is not necessary to index other split points [9] Sec. 4.1. The main difference, however, is that we need to search only for a few values of $k$.

4.2 Parsing the pattern in linear time

Christiansen and Ettienne [9] show that, thanks to the locally-consistent and locally-contracting parsing, it is sufficient to probe only $\tau = O(\log m)$ positions $k \in [1..m - 1]$. We now show that these positions can be found in time $O(m)$.

We build the parse tree of $P[1..m]$ using the locally consistent RLCFG of Section 3 in time $O(m)$. Since the grammar is locally balanced, $P$ is parsed into $O(\log m)$ levels, where at each level we parse $P$ into a sequence of blocks whose total number is at most one half of the preceding one (even if $P$ does not occur in $T$, as we see soon). Therefore, if we can find the partition into blocks in linear time at any given level, the whole parsing takes time $O(m)$.

To carry out the parsing, we must preserve the permutations $\pi$ of the alphabet used at each of the $\log(n/\gamma)$ rounds of the parsing of $T$, so as to parse $P$ in the same way. The alphabets in each round are disjoint because all the blocks are of length 2 at least. Therefore the total size of these permutations coincides with the total number of terminals and nonterminals in the grammar, and thus can be stored in $O(g)$ space.

Let us describe the first round of the parsing. We first traverse $P$ left-to-right and identify the repetitive areas $a^\ell$. Those are sought in a perfect hash table where we store all the first-round pairs $(a, \ell)$ existing in the text, and replaced by their corresponding nonterminal. We then traverse $P$ again, finding the local minima in $O(m)$ time. The identified blocks are sought in a trie where the first-round blocks of $T$ are stored (the size of this trie is proportional to the right-hand-sides of the rules created during the parsing, so it is $O(g)$ added over all the rounds). If the trie uses perfect hashing to store its children, we can identify all the blocks in time $O(m)$ and proceed to the next round.

Note that, if $P$ does not occur in $T$, its parsing may produce blocks that are not in the trie. Say that the two first and two last blocks in the parse of $P$ are external and the others are internal. As seen in Section 2.2 if we do not find an internal block in the trie, this means that $P$ is not equal to any $T[i..i + m - 1]$ and we finish. On the other hand, the external blocks into which we parse $P$ can differ from those of its occurrences in $T$, and this happens
at any level. Indeed, we do not attempt to find the external blocks in the trie.

For the next level, we take the sequence of nonterminals corresponding to the internal blocks of $P$ we found, and repeat the process. As said, this adds up to $O(m)$ time because the lengths are halved at each iteration.

Note that, in each round, there are at most four external blocks, and thus might be different from those in its occurrences (or may even not occur in $T$). These add up to $O(\log m)$ blocks that can be parsed differently. Christiansen and Ettienne [9, Sec. 4.3] show that the limits $k$ of these blocks inside $P$ are all that need to be considered when partitioning $P = P[1..k] \cdot P[k+1..m]$. We can easily obtain the limiting positions of those blocks by storing the size $|A|$ of every nonterminal $A$ in our grammar.

### 4.3 Searching for the pattern prefixes and suffixes

We can search for $\tau$ pattern prefixes and suffixes in $X$ or $Y$ in time $O(m + \tau \log^2 m)$ by building on the following results.

▶ **Lemma 7** (cf. [1] [11] [13]). Let $S$ be a set of strings and assume we have a data structure supporting extraction of any length-$\ell$ prefix of strings in $S$ in time $f_e(\ell)$ and computation of a given Karp-Rabin fingerprint $\kappa$ of any length-$\ell$ prefix of strings in $S$ in time $f_h(\ell)$. We can then build a data structure of $O(|S|)$ words such that, later, we can solve the following problem in $O(m + \tau(f_h(m) + \log m) + f_e(m))$ time: given a pattern $P[1..m]$ and $\tau > 0$ suffixes $Q_1, \ldots, Q_\tau$ of $P$, find the ranges of strings in (the lexicographically-sorted) $S$ prefixed by $Q_1, \ldots, Q_\tau$.

**Proof.** The proof follows closely that of Gagie et al. [13, Lem 5.2]. An inspection of that proof shows that the fingerprints and substrings that need to be extracted are always prefixes of length at most $m$ of strings in $S$. See Appendix A.6 for details. ◀

▶ **Lemma 8.** With our data structure, we can compute Karp-Rabin fingerprints of prefixes of length $\ell$ of strings in $X$ or $Y$ in time $f_h(\ell) = O(\log^2 \ell)$.

**Proof.** Analogously as for extraction (Lemma 1), we consider the $O(\log \ell)$ levels of the grammar subtree containing the desired prefix. For each level, we find in $O(\log \ell)$ time the prefix/suffix of the rule contained in the desired prefix. Fingerprints of those prefixes/suffixes of rules are precomputed. See Appendix A.7 for details. ◀

For extraction, as shown in the proof of Lemma 7, all we need is to extract a prefix or a suffix of a phrase, that is, of a nonterminal or a sequence of consecutive children of a nonterminal. We can therefore use the structure of Lemma 1 directly, obtaining $f_e(\ell) = O(\ell)$.

### 4.4 Reporting secondary occurrences

Our data structure maintains the grammar tree, where every node representing a nonterminal $B$ also points to (1) its nearest ancestor that is the root or represents a nonterminal $A$ that appears more than once in the grammar tree, (2) its offset inside $\text{exp}(A)$, and (3) the next appearance of $B$ to the right in the grammar tree (those next appearances must be leaves). If $A \rightarrow B'$, then the first child $B$ of $A$ points to the second child, $B'$, as its next appearance, and $B'$ points to the next appearance of $B$ (recall the beginning of Section 4).

Given a primary occurrence of $P$ contained by the minimal node $v$, labeled $B$, we recursively report the corresponding occurrence contained in the next appearance (3) of $B$ to the right, and continue to the ancestor (1) of $v$. The special symbols $B'$ marked with $t-1$
are treated as particular cases, each copy of \( B \) recursively calling the next one until they are exhausted, and then going to the next appearance of \( B \) in the tree.

Whenever we reach the root, we report the final position. A different occurrence of \( P \) is reported each time. Because every time we go upwards (1) in the path to the root, we reach the root or a node whose nonterminal has other occurrences to the right (3), the total time amortizes to \( O(1) \) per occurrence \([10]\): the recursion can be regarded as a binary tree following edges (1) and (3), with a distinct occurrence per leaf. Along the way, we must maintain the offset of the occurrence of \( P \) inside the current nonterminal. For this purpose, as we move upwards, we add up the offsets stored (2). All these structures require \( O(g) \) space.

### 4.5 Short patterns

The search for pattern prefixes and suffixes in \( \mathcal{X} \) and \( \mathcal{Y} \) takes time \( O(m + \tau \log^2 m) = O(m + \log^2 m) \subseteq O(m) \), and the data structures use \( O(|\mathcal{X}| + |\mathcal{Y}| + |T_G| + |T_C|) = O(g) \) space. A geometric data structure using \( O(p) = O(g) \) space can perform each range search in time \( O(\log^g g) \) plus \( O(\log^\gamma g) \) per primary occurrence found \([6]\). The secondary occurrences are reported in time \( O(1) \) each, on a data structure whose size is proportional to that of the grammar tree, \( O(g) \).

This yields time \( O(m + \log m \log^\gamma g + \text{occ} \log^\epsilon g) \) to find the \( \text{occ} \) occurrences of \( P[1..m] \). Next we show how to remove the additive term \( O(\log m \log^\epsilon g) \) by dealing separately with short patterns. This uses \( O(g) \) further space and leaves only an additive \( O(\log^g g) \)-time term needed for finding short patterns that do not occur in \( T \); this term is then slightly reduced.

This cost comes from the \( O(\log m) \) geometric searches, each having a component \( O(\log^p p) \) that cannot be charged to the primary occurrences found \([6]\). The resulting cost, \( O(\log m \log^\gamma g) \), may be \( \omega(m) \) only if \( m = O(\log^\epsilon g \log \log g) \subseteq O(\log^\epsilon g) \) for any \( \epsilon' > \epsilon \). Analogously to the idea of Christiansen and Ettienne \([9, \text{Sec. 6}]\), since there are at most \( \gamma \ell \) distinct substrings of length \( \ell \) in \( T \), there are at most \( \gamma \log^{2\epsilon'} g \) distinct substrings of length up to \( \log^\epsilon g \). We will store them all in a perfect hash table, recording for each the \( O(\log(\log^\epsilon g)) = O(\log \log g) \) partitions that are relevant for the search and are nonempty in the range search data structure. Since each partition position requires \( O(\log \log g) \) bits, we encode all this information in \( O(\gamma (\log^{2\epsilon'} g)(\log \log g)^2) \) bits, which is \( O(\gamma) \) space for any \( \epsilon' < 1/2 \). Avoiding the partitions that do not produce any result effectively removes the \( O(\log m \log^\epsilon g) \) additive term on the short patterns, because that cost can be charged to the first primary occurrence found.

We must, however, discard short patterns that do not occur in \( T \) but could collide with some short string in the perfect hash table. To do this, we proceed as follows. If the first partition returned by the hash table turns out to be empty, then this was due to a collision with another pattern, and we can immediately return that \( P \) does not occur in \( T \). If, on the other hand, the first partition does return some occurrence, we extract the text around the first one in order to verify that the pattern is actually \( P \). If it is not, then this is also due to a collision and we return that \( P \) does not occur in \( T \). In total, the cost to detect that a short pattern \( P \) does not occur in \( T \) is \( O(m + \log^\gamma g) \). Since \( \log^\gamma g \in O(\log^\gamma g + \log(n/\gamma)) \), we can further reduce the cost to \( O(m + \log^\gamma g) \) by storing a perfect hash function with all the patterns shorter than \( \log \log(n/\gamma) \). This time we can afford to store a grammar tree position where each pattern occurs in \( T \), as that takes \( O(\gamma (\log \log(n/\gamma))^2) \subseteq O(\gamma \log(n/\gamma)) \) space. Any search for a pattern of length up to \( \log \log(n/\gamma) \) that does not occur in \( T \) can then be handled in \( O(m) \) time.

\[ \textbf{Theorem 9.} \text{Let } T[1..n] \text{ have an attractor of size } \gamma. \text{ Then, there exists a data structure of} \]
size \( g = O(\gamma \log(n/\gamma)) \) that can find the \( occ \) occurrences of any pattern \( P[1..m] \) in \( T \) in time \( O(m + \log^2 \gamma + occ \log^2 g) \) \( \subseteq O(m + (occ + 1) \log^2 n) \) for any constant \( \epsilon > 0 \).

The expected construction time is \( O(n \log n) \), dominated by the rounds of parsing and by the verification that the Karp-Rabin signatures of Section 4.3 are collision-free \([1]\).

### 4.6 Using more space

By using \( O(\gamma \log n) \) space, we can maintain in the latter hash table all the \( O(\gamma \log^2 \epsilon g) \) distinct short strings of length up to \( \log^\epsilon g \), with their grammar tree position, to allow verifying them. This reduces the time to \( O(m + occ \log^\epsilon g) \).

By using \( O(p \log \log p) \) space for the grid, the range queries run in time \( O(\log \log p) \) per query and per returned item \([1]\). With the same techniques for short patterns, this yields an index using \( O(\gamma \log(n/\gamma) \log \log n) \) space and searching in time \( O(m + (occ + 1) \log \log n) \). Further, using \( O(\gamma (\log \log n)^2) \) further space, we obtain \( O(m + occ \log \log n) \) search time.

Yet another geometric structure \([6]\) uses \( O(p \log p) \) space and reports in \( O(\log \log p) \) time per query and \( O(1) \) per result. This yields \( O(\gamma \log(n/\gamma) \log n) \) space and \( O(m + \log m \log \log n + occ) \) time. To remove the second term, we can index all the patterns of length \( m < (\log \log n)^2 \), of which there are \( O(\gamma (\log \log n)^2) \). The trick of giving the nonempty partitions is not sufficient this time, because we cannot afford even the \( O(\log \log p) \) time of the nonempty areas. However, we only care about the short patterns that, in addition, occur less than \( (\log \log n)^2 \) times, since otherwise the third term, \( O(occ) \), absorbs the second.

Storing all the occurrences of such patterns requires \( O(\gamma (\log \log n)^4) \) space: we use a compact trie with all the \( O(\gamma (\log \log n)^2) \) patterns of length \( (\log \log n)^2 \). The trie uses perfect hashing to store the children of the nodes. Each node has a pointer to the grammar tree to allow verifying the pattern sought in optimal time. Each node also records whether it occurs less than \( (\log \log n)^2 \) times. Only the leaves (i.e., the patterns of length exactly \( (\log \log n)^2 \)) store their (at most \( (\log \log n)^2 \)) occurrences. Shorter patterns correspond to internal trie nodes, and for them we must traverse all the descendant leaves to collect their occurrences.

**Theorem 10.** Let \( T[1..n] \) have an attractor of size \( \gamma \). Then, there exists a data structure of size \( O(\gamma \log(n/\gamma) \log n) \) that can find the \( occ \) occurrences of any pattern \( P[1..m] \) in \( T \) in time \( O(m + occ) \).

Such an optimal search time has been obtained only by using \( O(p (\log(n/p)) \) space, where \( p \) is the number of runs of the Burrows-Wheeler Transform of \( T \) \([13]\) or using \( O(\epsilon) \) space, where \( \epsilon \) is the size of the smallest automaton recognizing \( T \) \([2]\). None of those measures have useful known upper bounds in terms of \( \gamma \). Optimal time had not been obtained within space bounded by \( z \geq \gamma \), the size of the Lempel-Ziv parse of \( T \), or using grammars.

### 5 Counting Pattern Occurrences

Navarro \([21]\) shows how an index like the one we describe in Section 4 can be used for counting the number \( occ \) of times \( P[1..m] \) occurs in \( T \). First, he uses the result \([8]\) that a \( p \times p \) grid can be enhanced by associating elements of any algebraic semigroup to the points, so that later we can operate all the elements inside a rectangular area in time \( O(\log^{2+\epsilon} p) \), for any constant \( \epsilon > 0 \), with a structure using \( O(p) \) space (assuming the semigroup elements fit in a word). Then he shows that, in a binary grammar, one can associate the number of secondary occurrences triggered by each point in a grid analogous to that of Section 4.1 so that their sums can be computed as described.
We now improve upon the space and time using our grammar of Section 4. Although it is not binary, each point in the grid corresponds to a distinct primary occurrence of \( P \), and the secondary occurrences triggered by each primary occurrence are all distinct \( [10] \) (note that the same point may be contained in various ranges for different partitions of \( P \), but each time it corresponds to a different offset of \( P \)). Therefore, we can associate with each point in the grid the number of occurrences that its primary occurrence will trigger. Then, counting the number of occurrences of a partition \( P = P[1..k] \cdot P[k+1..m] \) corresponds to summing up the number of occurrences of the points that lie in the corresponding range of the grid.

As seen in Section 4.3 with our particular grammar there are only \( O(\log n) \) partitions of \( P \) that must be tried in order to recover all its occurrences; the others can be shown to yield no points \( [9] \). Therefore, we find the \( O(\log n) \) relevant ranges as in Section 4.3 and then count the number of occurrences in each such range in time \( O(\log^{2+\epsilon} n) \subseteq O(\log^{2+\epsilon} g) \).

This improves the previous result \( [21] \) in space (as it built the grammar on a Lempel-Ziv parse instead of on attractors) and in time (as it considered all the \( m-1 \) partitions of \( P \)).

**Theorem 11.** Let \( T[1..n] \) have an attractor of size \( \gamma \). Then, there exists a data structure of size \( g = O(\gamma \log(n/\gamma)) \) that can count the number of occurrences of any pattern \( P[1..m] \) in \( T \) in time \( O(m + m \log^{2+\epsilon} g) \subseteq O(m + m \log^{3+\epsilon} n) \) for any constant \( \epsilon > 0 \).

If we are willing to use \( O(p \log p) \) space, the cost to compute the sum over a range decreases to \( O(\log p) \) \( [26] \). Therefore, using \( O(\gamma \log(n/\gamma) \log n) \) space, we could count in time \( O(m + m \log g) \subseteq O(m + m \log \log n) \).

To reduce this time to \( O(m) \), note that, if \( \log(n/\gamma) \leq \log \log n \), then \( \gamma \geq n/\log n \), and thus a suffix tree using space \( O(n) \) = \( O(\gamma \log n) \) can count in optimal time \( O(m) \). Thus, assume \( \log(n/\gamma) > \log \log n \). Further, assume that \( m \leq \log \log n \), since otherwise the counting time is already \( O(m) \). Since \( m \leq \log \log n \leq \log^2 n \), we have \( n \leq 2 \log n \log \log n \leq 2 \log n \log(n/\gamma) \). Therefore, we can index in a compact trie all the text substrings of length up to \( 2 \log n \log(n/\gamma) \) and directly store their number of occurrences. Since there are \( \gamma \ell \) distinct substrings of length \( \ell \), this requires \( O(\gamma \log n \log(n/\gamma)) \) space (as before, we use perfect hashing for the children of trie nodes, plus a pointer to the grammar tree to allow verifying the pattern sought). Those short patterns are then counted in time \( O(m) \).

**Theorem 12.** Let \( T[1..n] \) have an attractor of size \( \gamma \). Then, there exists a data structure of size \( O(\gamma \log(n/\gamma) \log n) \) that can count the number of occurrences of any pattern \( P[1..m] \) in \( T \) in time \( O(m) \).

6 Final Remarks

We have managed to obtain state-of-the-art counting and locating times within attractor-bounded space. Other indexing problems can also be addressed with our approach. For example, document listing is the problem of, given a collection of texts of total size \( N \), tell in which documents a pattern appears. Navarro \( [21] \) Thm. 2 recently obtained the first document listing index with provable performance for repetitive text collections, listing the \( ndoc \) documents where \( P[1..m] \) appears in time \( O(m \log^{1+\epsilon} N \cdot ndoc) \). This complexity, unfortunately, multiplies the input by the output sizes. In the full version we will show that his grammar can be changed to a locally-consistent one so that there are only \( O(\log m) \) pattern splits to try. We can then reduce their document listing time to \( O(m + m \log m \log^{1+\epsilon} N \cdot ndoc) \).
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Faster Attractor-Based Indexes

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A Omitted Proofs

A.1 Proof of Theorem

Let us first consider prefixes. Define a forest of tries \( T_G \) with one node per distinct nonterminal or terminal symbol. Let us identify symbols with nodes of \( T_G \). Terminal symbols are trie roots, and \( B \) is the parent of \( A \) in \( T_G \) iff \( B \) is the leftmost symbol in the rule that defines \( A \), that is, \( A \to B \ldots \). For the rules \( A \to B^l \), we just let \( B \) be the parent of \( A \). We augment \( T_G \) to support constant-time level ancestor queries \cite{navarro2010suffix}. To extract \( \ell \) symbols of \( \text{exp}(A) \), we start with the node \( A \) of \( T_G \) and immediately return the terminal \( a \) associated with its trie root (found with a level ancestor query). We now find the ancestor of \( \gamma \). Similarly, \( k \) is mapped to \( \pi \) with the node \( A \) of \( T_G \) and let \( \text{exp} \) be made of \( T \) nonterminals, and let \( O \) can be made of \( T \) rules to form the block of \( \ell \) symbols is of size \( k \). We hold that \( \text{exp}(A) \) at depth 3. Let \( C \) be this node, with \( C \to BC_2 \ldots C_r \), and so on. When we have to obtain the next symbols from a nonterminal \( X \to Y^t \), we treat it exactly as \( X \to Y \ldots Y \) of size \( t \), that is, we extract \( \text{exp}(Y) \) \( t - 1 \) further times. Overall, we output each next symbol in \( O(1) \) time. Suffixes are analogous, and can be obtained in real-time in reverse order by defining a similar tree \( T'_G \) where \( B \) is the parent of \( A \) iff \( B \) is the rightmost symbol in the rule that defines \( A \), \( A \to \ldots B \).

A.2 Proof of Lemma

Let \( \Gamma = \{ p_1, \ldots, p_s \} \) be an attractor of \( T \). Recalling Section \( \Gamma \), we define a function \( f \) such that, for any interval \( [i..j] \), \( f[i..j] \) includes some \( p_i \) and \( T[i..j] = T[f[i..j]] \).

Now consider a non-bordering block of the form \( T[p..p + \ell - 1] = a_1^{\ell_1} a_2^{\ell_2} \ldots a_k^{\ell_k} \), with \( k \geq 2 \), all \( \ell_i \geq 1 \), and all \( a_i \neq a_{i+1} \). This is formed by first creating blocks for all the repetitive areas \( a_i^{\ell_i} \) with \( \ell_i > 1 \) (creating the rule \( A_i \to a_i^{\ell_i} \)), and then collapsing a block with the \( k \) resulting symbols, \( a_i \) if \( \ell_i = 1 \) or \( A_i \) otherwise. Consider the extended area \( T[p..p + \ell - 1] \) formed by \( T[p..p + \ell - 1] \) and the preceding and following blocks. Since \( T[p..p + \ell - 1] \) is not bordering, \( T[p..p + \ell + y] \) does not contain any position \( p_i \), and therefore it holds that \( f[p..p + \ell + y] = [q..q + y] \), where \( q \neq p \), \([q..q + y] \) includes a position \( p_i \), \( T[q..q + y] = T[p..p + \ell + y] \), and the block \( T[p..p + \ell - 1] \) is mapped to \( T[q.q + \ell - 1] \) inside \( T[q..q + y] \). Since the blocks contain at least 2 distinct symbols and a block starts at \( T[p] \), \( T[p..p - 1] \) has a suffix of the form \( a_0^{\ell_0} \) with \( a_0 \neq a_1 \) and \( \pi(a_0) > \pi(a_1) \) (i.e., \( a_0 \) is a local minimum), and therefore a block starts at \( T[q] \) as well. Similarly, \( T[q + \ell] = T[q + \ell] = a_{k+1} \), where \( a_k \neq a_{k+1} \) and \( \pi(a_{k+1}) > \pi(a_k) \) (i.e., \( a_k \) is a local minimum), and therefore a block also ends at \( T[q + \ell] \). Further, there are no local minima inside \( T[p..p + \ell - 2] = T[q..q + \ell - 2] \). Therefore, a block is also formed with exactly \( T[q..q + \ell - 1] \). This block is bordering because \( T[q..q + \ell + y] \) includes an attractor position \( p_i \), so we do not need to count the distinct non-bordering block \( T[p..p + \ell - 1] \).

A.3 Proof of Lemma

Lemma shows that we produce at most \( 3(\gamma + 2) \) distinct blocks. Each block \( a_1^{\ell_1} a_2^{\ell_2} \ldots a_k^{\ell_k} \), however, adds up to \( 3k \) to the size of the RLCFG (the rules \( A_i \to a_i^{\ell_i} \) are of size 2, and the rule to form the block of \( k \) symbols is of size \( k \)). We will show that their total contribution can be made \( O(\gamma) \), though. Consider the text \( T_b \) after collapsing repetitive areas into nonterminals, and let \( p_i' \) be the positions to which the attractor elements \( p_i \) are mapped in \( T_b \). Let \( p_i'^{(2)} < p_i'^{(1)} < p_i' \leq p_i^{(1)} < p_i^{(2)} \) be the 4 local minima surrounding \( p_i' \), and
define the 4 random variables $X_i^{-2} = p_i^{(-1)} - p_i^{(-2)}$ (length of the block $T[p_i^{(-2)} + 1..p_i^{(-1)}]$), $X_i^{-1} = (p_i^{+1}) - p_i^{-1}$ (length of the block prefix $T[p_i^{(-1)} + 1..p_i^{+1}]$ plus 2), $X_i^1 = p_i^{+1} - (p_i^{(-2)}$ (length of the block suffix $T[p_i^{+1}]$ plus 1), and $X_i^2 = p_i^{+2} - p_i^{+1}$ (length of the block $T[p_i^{+1}]$). Then the 3 bordering blocks around $p_i$ add up to length $k = p_i^{+2} - p_i^{(-2)}$, which is $X_i^{-2} + X_i^{-1} + X_i^1 + X_i^2 - 3$, and they increase the RLCFG size by at most $3k = 3(X_i^{-2} + X_i^{-1} + X_i^1 + X_i^2 - 3)$. Added over the $\gamma + 2$ attractor elements, the increase in the grammar size is bounded by the random variable $Y = 3 \sum_{i=1}^{k} (X_i^{-2} + X_i^{-1} + X_i^1 + X_i^2 - 3)$. Each variable $X_i^1$ counts the trials, starting afresh from a fixed position, until finding a local minimum: $X_i^1$ moving forward from $p_i^{+1}$, $X_i^2$ moving forward from $X_i^{-1}$, moving backwards from $p_i^{+2}$, and $X_i^2$ moving backwards from $X_i^{-2}$. The expectation of all those variables is $E(X_i^1) \leq 3 \gamma$ by [2] Lem. 3, and therefore $E(X_i^{-2} + X_i^{-1} + X_i^1 + X_i^2 - 3) = 9$ and $E(Y) \leq 27(\gamma + 2)$, even if the variables $X_i^1$ are dependent on each other. This means that, for at least half of the permutations $\pi$ it holds that $Y \leq 54(\gamma + 2)$. Thus, we can try out random values of $\pi$ until $Y \leq 54(\gamma + 2)$ and we will succeed with 2 trials on average. Therefore, we can guarantee a RLCFG of size $O(\gamma)$ with linear expected-time construction.

### A.4 Proof of Theorem 5

Lemmas [3] and [4] show that, in the first round, we produce at most $3(\gamma + 2)$ distinct blocks, and can find a RLCFG of size $O(\gamma)$ to represent them. For the second round, we create a reduced sequence $T'$ from $T$ by replacing all the blocks by their corresponding nonterminals. The new sequence is of length $n' \leq n/2$, as seen in Section 2.2.

We define a new attractor $\Gamma'$ of size $\gamma' \leq 3(\gamma + 2)$ on $T'$, containing all the positions $\{p_i^{+1}, p^-_i, p_i^{+1} + 1\}$, where $p_i^{+1}$ is the position in $T'$ of the nonterminal covering $T[p_i^{+1}]$. That is, all the bordering blocks of $T$ become attractor elements in $T'$. To see that $\Gamma'$ is an attractor of $T'$, consider any string $T'[i',j']$ not including a position of $\Gamma'$. Then $T'[i',j']$ comes from a string $T[i..j]$ formed by a sequence of non-bordering blocks. As in Lemma 3 we extend $T[i..j] \to T[i-x..j+y]$ so as to include one more block in each direction. Since $T[i-x..j+y]$ does not include an attractor position, $f[i-x..j+y] = [p-x..q+y]$ includes a position $p_i$ and $T[p-x..q+y] = T[i-x..j+y]$, so $T[p..q]$ is parsed in the same way as $T[i..j]$, as a sequence of whole blocks. Let $T[p..q]$ be mapped to $T'[i'..j']$. Therefore, $T'[i'..j']$ includes a position in $\Gamma'$. We can then start the second round on $T'$, and so on.

Let us call $T_r$ the sequence $T$ after $r$ iterations (so $T = T_0$ and $T' = T_1$) and $N_r$ the number of distinct blocks created when converting $T_r$ into $T_{r+1}$. In the first iteration, since there may be up to 3 bordering blocks around each attractor position (one covering it, and its preceding and following blocks), we may create $N_1 \leq 3(\gamma + 2)$ distinct blocks. Those blocks become new positions in the attractor $\Gamma_1$ of $T_1$. Note that those attractor positions in $\Gamma_1$ are grouped into $\gamma + 2$ regions of up to 3 consecutive elements. In each new iteration, $T_r$ is parsed into blocks again. Lemma 3 shows that non-bordering blocks are not distinct, so we can focus on the number of new blocks produced when parsing each of the $\gamma + 2$ regions, plus a preceding and a following block in each region. The parsing of 3 consecutive elements produces at most 2 blocks, and thus 4 bordering blocks, so $|\Gamma_2| \leq 4(\gamma + 2)$. The parsing of 4 consecutive elements produces at most 3 blocks, and thus 5 bordering blocks, so $|\Gamma_3| \leq 5(\gamma + 2)$. The parsing of 5 consecutive elements produces at most 3 blocks, and thus 5 bordering blocks again, that is, $|\Gamma_r| \leq 5(\gamma + 2)$ for all $r$. Therefore, we have at most 5 bordering blocks, and produce at most 5 distinct blocks, around each of the $\gamma + 2$ areas.

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2 The constant 54 can be reduced to $27 + \epsilon$ for any constant $\epsilon$ while requiring $O(1/\epsilon)$ trials on average.
After $r$ rounds, the sequence is of length at most $n/2^r$ and we have generated at most $5(\gamma + 2)r$ distinct blocks. Therefore, if we perform $r = \log(n/\gamma)$ rounds, the sequence will be of length at most $\gamma$ and the number of distinct blocks will be $O(\gamma \log(n/\gamma))$.

Since we can make each block induce an increase of $O(1)$ in the grammar size, the final RLCFG can be made of size $O(\gamma \log(n/\gamma))$, which includes the final rule of size $\gamma$ that derives from a new initial symbol, all the symbols remaining after the $r$ rounds.

A.5 Proof of Theorem 6

If we do not know $\gamma$, we can still run our construction along $\log n$ rounds, and have a grammar of height $\log n$. After the round $\log(n/\gamma)$, the grammar will be of size $O(\gamma \log(n/\gamma))$, and the next rounds will add at most size $O(\gamma)$ to the grammar. The other place where we use the knowledge of $\Gamma$ is when checking that the bordering blocks produced add up to length $2 \cdot 9(\gamma + 2)$. But since these are all the distinct blocks produced in the round, we can use an alternative check: If the parsing produces $b$ distinct blocks, the grammar will grow proportionally to their total length, so we rather check that their lengths add up to $O(b)$.

Our arguments on the strings $X_i$ around attractor positions (even if now we do not know them) shows that these strings have expected length up to $3$, and thus we can ensure that their sum is at most $6b$ for half of the permutations. Thus with $O(1)$ attempts we find a suitable permutation. This adds up to $O(n \log n)$ expected time along $\log n$ rounds.

A.6 Proof of Lemma 7

First, we require a Karp-Rabin function $\kappa$ that is collision-free between equal-length text substrings whose length is a power of two. We can find such a function at index construction time in $O(n \log n)$ expected time and $O(n)$ space [4]. We extend the collision-free property to pairs of equal-letter strings of arbitrary length by switching to the hash function $\kappa'$ defined as $\kappa'(T[i..i+\ell-1]) = \langle \kappa(T[i..i+2^\lceil \log \ell \rceil - 1]), \kappa(T[i+\ell-2^\lceil \log \ell \rceil..i+\ell-1]) \rangle$.

Z-fast tries [11] Sec. H.2] solve the weak part of the lemma in $O(m \log(\sigma)/w + \tau \log m)$ time. They have the same topology of a compact trie on $S$, but use function $\kappa'$ to find a candidate node for $Q_i$ in time $O(\log |Q_i|) = O(\log m)$. We compute the $\kappa'$-signatures of all pattern suffixes $Q_1, \ldots, Q_r$ in $O(m)$ time, and then search the z-fast trie for the $\tau$ suffixes $Q_i$ in time $O(\tau \log m)$.

By weak we mean that the returned answer for each suffix $Q_i$ is not guaranteed to be correct if $Q_i$ does not prefix any string in $S$: we could therefore have false positives among the answers, though false negatives cannot occur. A procedure for discarding false positives [11] requires extracting substrings and their fingerprints from $S$. We describe this strategy in detail in order to analyze its time complexity in our scenario.

Let $Q_1, \ldots, Q_j$ be the pattern suffixes for which the z-fast trie found a candidate node. Order the pattern suffixes so that $|Q_1| < \cdots < |Q_j|$, that is, $Q_i$ is a suffix of $Q_{i'}$ whenever $i < i'$. In addition, let $v_1, \ldots, v_j$ be the candidate nodes (explicit or implicit) of the z-fast trie such that all substrings below them are prefixed by $Q_1, \ldots, Q_j$ (modulo false positives), respectively, and let $t_i = \text{string}(v_i)$ be the substring read from the root of the trie to $v_i$. Our goal is to discard all nodes $v_j$ such that $t_k \neq Q_k$.

We proceed in rounds. At the beginning, let $a = 1$ and $b = 2$. At each round, we perform the following checks:

1. If $\kappa'(Q_a) \neq \kappa'(t_a)$: discard $v_a$ and set $b \leftarrow b + 1$ and $a$ to the next integer $a'$ such that $v_{w'}$ has not been discarded.
If $|\kappa'(Q_a)| = |\kappa'(t_a)|$: let $R$ be the length-$|t_a|$ suffix of $t_b$, \( R = t_b[|t_b| - |t_a| + 1..|t_b|] \). We have two sub-cases:

a. $\kappa'(Q_a) = \kappa'(R)$. Then, we set $b \leftarrow b + 1$ and $a$ to the next integer $a'$ such that $v_{a'}$ has not been discarded.

b. $\kappa'(Q_a) \neq \kappa'(R)$. Then, discard $v_b$ and set $b \leftarrow b + 1$.

3. If $b = j + 1$: let $v_f$ be the last node that was not discarded. Note that $Q_f$ is the longest pattern suffix that was not discarded; other non-discarded pattern suffixes are suffixes of $Q_f$. We extract $t_f$. Let $s$ be the length of the longest common suffix between $Q_f$ and $t_f$. We report as a true match all nodes $v_i$ that were not discarded in the above procedure and such that $|Q_i| \leq s$.

Intuitively, the above procedure is correct because we check that text substrings read from the root to the candidate nodes form a monotonically increasing sequence according to the suffix relation: $t_i \leq_{\text{suffix}} t_{i'}$ for $i < i'$ (if the relation fails at some step, we discard the failing node). Comparisons to the pattern are relegated to the last step, where we explicitly compare the longest matching pattern suffix with $t_f$. For a full proof, see Gagie et al. [11].

To analyze the running time, note that we compute $\kappa'$-fingerprints of strings ($t_a$, $t_b$, and $R$) that are always suffixes of prefixes of length at most $m$ of strings in $S$ (because our candidate nodes $v_1, \ldots, v_j$ are always at depth at most $m$). By definition, to retrieve $\kappa'(t_a)$ we need to compute the two $\kappa'$-signatures of the length-$2^\epsilon$ prefix and suffix of $t_a$, for some $\epsilon \leq \log |t_a| \leq \log m$, $1 \leq i \leq j$. Computing the needed $\kappa'$-signatures reduces therefore to the problem of computing $\kappa$-signatures of suffixes of prefixes of length at most $m$ of strings in $S$. Let $R' = t_b[|t_b| - s + 1..|t_b|]$ be such a length-$s$ string of which we need to compute $\kappa(R')$. Then, $\kappa(R') = \kappa(t_b) - \kappa(t_b[1..|t_b| - s]) \cdot c^s \mod \mu$. Both signatures on the right-hand side are prefixes of suffixes of length at most $m$ of strings in $S$. The value $c^s \mod \mu$ can moreover be computed in $O(\log m)$ time using the fast exponentiation algorithm. It follows that, overall, computing the needed $\kappa'$-signatures takes $O(f_b(m) + \log m)$ time per candidate node. For the last candidate, we extract the prefix $t_f$ of length at most $m$ ($O(f_e(m))$ time) of one of the strings in $S$ and compare it with the longest candidate pattern suffix ($O(m)$ time). There are at most $\tau$ candidates, so the verification process takes $O(m + \tau \cdot (f_b(m) + \log m) + f_e(m))$ time. Added to the time spent to find the candidates in the z-fast trie, we obtain the claimed bounds.

### A.7 Proof of Lemma 8

Strings in $X$ are reversed expansions of nonterminals. Let every nonterminal $X$ store the fingerprints of the reverses of all the suffixes of $\exp(X)$ that start at $X$’s children. That is, if $X \rightarrow X_1 \ldots X_s$, store the fingerprints of $(\exp(X_1) \ldots \exp(X_s))^{rev}$ for all $i$. We use the trie $T'_G$ of the proof of Lemma 4 where each trie node is a grammar nonterminal and its parent is the rightmost symbol of its defining rule. To extract the fingerprint of the reversed prefix of length $\ell$ of a nonterminal $X$, we go to the node of $X$ in $T'_G$ and run an exponential search over its ancestors, so as to find in time $O(\log \ell)$ the lowest one whose expansion length is $\leq \ell$. Let $B$ be that nonterminal, then $B$ is the first node in the rightmost path of the parse tree from $X$ with $|B| \leq \ell$. Note that the height of $B$ is $O(\log \ell)$ because the grammar is locally balanced, and moreover the parent $A \rightarrow B_1 \ldots B_i B$ of $B$ holds $|A| > \ell$. We then exponentially search the preceding siblings of $B$ until we find the largest $i$ such that $|B_1| + \ldots + |B_i| > \ell$ (we must store these cumulative expansion lengths for each $B_i$). This takes $O(\log \ell)$ time. We collect the stored signature of $(\exp(B_{i+1}) \ldots \exp(B))^{rev}$; this is part of the signature we will assemble. Now we repeat the process from $B_i$, collecting the signature from the remaining
part of the desired suffix. It is easy to see that this costs \( O(\log^2 \ell) \) time. The case of \( Y \) is similar, now using the trie \( T_G \) of Lemma \ref{lem:trie} and computing prefixes of signatures. The only difference is that we start from a given child \( Y_i \) of a nonterminal \( Y \to Y_1 \ldots Y_t \) and the signature may span up to the end of \( Y \). So we start with the exponential search for the first \( Y_j \) such that \( |Y_i| + \ldots + |Y_j| > \ell \); the rest of the process is similar.

When we have rules of the form \( A \to B^t \), we find in constant time the desired copy \( B_i \), from \( \ell \) and \( |B| \). Similarly, we can compute the fingerprint \( \kappa(exp(B_{i+1}) \ldots exp(B)) = \kappa(exp(B)^{r-i+1}) = (\kappa(B) \cdot \frac{e^{r-i+1}}{e^{r-i+1}}) \mod \mu: e^{|B|} \mod \mu \) and \( (e^{|B|} - 1)^{-1} \mod \mu \) can be stored with \( B \), and the exponentiation can be computed in \( O(\log(t-i)) \subseteq O(\log \ell) \) time.