On the derivation and mathematical analysis of some quantum–mechanical models accounting for Fokker–Planck type dissipation: Phase space, Schrödinger and hydrodynamic descriptions

Abstract
This paper is intended to provide the reader with a review of the authors’ latest results dealing with the modeling of quantum dissipation/diffusion effects at the level of Schrödinger systems, in connection with the corresponding phase space and fluid formulations of such kind of phenomena, especially in what concerns the role of the Fokker–Planck mechanism in the description of open quantum systems and the macroscopic dynamics associated with some viscous hydrodynamic models of Euler and Navier–Stokes type.

Keywords
Open quantum system • Wigner-Fokker-Planck equation • Doebner-Goldin equations • nonlinear Schrödinger equations • quantum viscous hydrodynamic equations • quantum Navier-Stokes equations • dissipative quantum mechanics • logarithmic nonlinearity • nonlinear Gauge transformations • Madelung decomposition • osmotic momentum

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1. Introduction
The mathematical modeling of physical processes in which the loss of some observable magnitude becomes relevant constitutes a research field of increasing interest in many disciplines of science, since a great part of the problems dealt with in these areas can be posed in a very realistic way within the theoretical framework of dissipative systems. When only mesoscopic and/or macroscopic scales are involved, the kinetic theory is known to provide quite a powerful tool to represent, simulate (numerically) and analyze (mathematically) a variety of phenomena of this kind, in such a wide scope that they may range from Astronomy to Biology, for instance. On the contrary, whenever smaller scales (say nanoscopic) are present, the quantum effects are shown to become essential, and thus a mathematical description accounting for dissipative processes reveals necessary in this context. Indeed, finding an appropriate formulation that allows to embrace quantum dissipative phenomena does constitute a fundamental task for its great applicability in multiple branches of Physics and Technology such as Brownian dynamics, heavy ion dispersion, quantum optics, energy transfer in photosynthetic complexes or microelectronics, especially concerning the modeling of quantum transport of charged particles in semiconductor devices.

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Our main purpose in this paper is to give an overview of our most recent results as regards quantum dissipative PDE models, particularly from the viewpoint of the wavefunction and the hydrodynamic formulations of quantum mechanics, being our interest mainly focused in the study of what type of nonlinear contributions can take place in a genuinely dissipative Schrödinger equation (as well as in its hydrodynamic counterpart) and the subsequent wellposedness analysis associated with the corresponding initial or initial–boundary value problem. In particular, we do pay special attention to friction and diffusion effects and investigate quantum generalizations of the well–known Langevin equation, commonly used to describe classical Brownian motion.

The problem consisting of how to mathematically describe quantum dissipative phenomena can be tackled from two different (and antagonistic) perspectives. The first one is mainly based on the application of a standard quantization procedure to a classical system. This idea relies upon the existence of a Hamiltonian (or Lagrangian) structure that describes the system dynamics. This is, however, not always possible. An easy example is provided by the Langevin equation

\[ mx''(t) + \eta x'(t) + V'(x) = F(t), \]

that governs the Brownian motion of a classical particle with mass \( m \), immersed in a viscous fluid with friction coefficient \( \eta \), and subject to the action of the potential \( V \) and of the random force \( F \). Indeed, this equation cannot be derived from any system of Hamilton equations. As consequence, it should be concluded that not all phenomena exhibiting dissipation effects can be quantized in the traditional form. In spite of that, various phenomenological models can be found in the literature that are based on nonstandard quantization rules, such as that proposed in 1977 by H. Dekker [46], based on the introduction of complex variables, and subsequently applied to the damped harmonic oscillator.

At variance, the second (obvious) approach consists in starting directly from a quantum system and trying to explain the origin of the observed dissipative behaviour. Of course, the results and intrinsic difficulties of this procedure will depend upon the formulation employed in each case. From the point of view of modeling, the formalism of density matrix operators allows to work with a quantum open system in a rather simple way. This makes it the most appropriate theoretical framework to include dissipation effects in a quantum mechanical process. The main idea standing behind the concept of quantum open system relies on the fact that any dissipative phenomenon observed in the system under study, say \( S \), must be interpreted as the result of the interaction with its surrounding environment \( E \). Then, considering \( S \) and \( E \) as independent quantum systems influencing each other, decomposing the Hamiltonian of the compound system \( S + E \) as the sum of the Hamiltonian of \( S \), that of \( E \), and an interaction \( S–E \) Hamiltonian, and just taking notice of the behaviour of \( S \) (which can be easily done whenever working with density matrix operators, by simply tracing out the degrees of freedom of the environmental system), the well–known 'master equation' approach is obtained, that determines the time evolution law for the reduced (to \( S \)) density matrix operator and does constitute a full description of the quantum dissipative process. The interested reader can find a nice presentation of all these features in the textbooks [39, 133] and the review papers [15, 60, 124].

There exist many models based on a master equation approach that stem from an open system framework and are widely accepted in the physical literature (see for instance [27, 51, 52, 63, 80, 126]). Among the most important ones, it is especially noticeable that introduced in 1983 by A. O. Caldeira and A. J. Leggett [27], where a generalization of the Langevin equation (1) to the quantum case is carried out under the only assumption of high temperatures:

\[ \frac{dR}{dt} = -\frac{i}{\hbar}[H, R] - \frac{i\lambda}{\hbar}[q, \{p, R\}] - \frac{D}{\hbar^2}[q, \{q, R\}]. \]

Here, \( R = R(t) \) is the reduced density matrix operator associated with the system under analysis, \( q \) and \( p \) respectively denote the quantum position and momentum operators, and the symbols \([\ldots]\) and \(\{\ldots\}\) stand for the commutator and the anticommutator operators, respectively. Furthermore, \( H \), denotes the renormalized Hamiltonian [81], \( \hbar \) is the reduced Planck constant and \( i \) is the complex unit. The parameters modeling the interaction of the system with the environment are \( \lambda = \frac{\eta}{m} \) and \( D = \eta k_B T_0 \), where \( \eta > 0 \) and \( T_0 > 0 \) are the coupling constant and the temperature of the environment, respectively, while \( m \) is the effective mass of the particles and \( k_B \) holds for the Boltzmann constant. Eq. (2) is obtained after the interpretation of \( E \) as an infinite set of high–frequency harmonic oscillators in thermodynamic equilibrium (the so–called thermal bath or reservoir), by using the influence functional introduced by R. P. Feynmann and F. L. Vernon in [59] to deal with the interaction effects. Up to now, this is a well accepted dissipative model in quantum mechanics due to its simplicity (as a matter of fact, it is a linear equation that can be solved explicitly) and numerical feasibility and reliability, in spite of some mathematical deficiencies that will be commented on later.
On the other hand, the interpretation of the environment as a thermal bath has given rise to more sophisticated models that generalize that of Caldeira and Leggett (denoted CL from now on). In 1993, L. Diósi explored a Markovian approximation, for the case of medium/high temperatures, of the same (non-Markovian) process previously described by Caldeira and Leggett, obtaining the following master equation:

$$A(R) = -\frac{i\hbar}{\Omega}[q, \{p, R\}] - \frac{1}{\hbar^2} \left( D_{pp}[q, \{p, R\}] + 2D_{pq}[q, \{p, R\}] + D_{qq}[q, \{p, R\}]\right),$$

where $A(R) := \frac{\partial R}{\partial t} + \frac{1}{\hbar}[H, R]$, and where the same notation as for Eq. (2) has been employed. Also $D_{pp}$, $D_{pq}$ and $D_{qq}$ are constants that depend upon the interaction of the system with the thermal bath, given by

$$D_{pp} = \eta k_B T_0, \quad D_{pq} = \frac{\eta\Omega^2}{12\pi m k_B T_0}, \quad D_{qq} = \frac{\eta\hbar^2}{12m^2 k_B T_0},$$

$\Omega$ denoting the cut-off frequency of the reservoir oscillators.

Compared with the CL master equation, the one derived by Diósi incorporates two new terms (that Dekker had already achieved in [46] via its quantization method) that permit Eq. (3) to be written in so-called Kossakowski–Lindblad (KL) form (see §2 and [1, 11, 14, 63, 69, 92, 98] for further information). Equations in KL form are known to enjoy some convenient properties such as dissipativity (in the sense of operators), monotonic variation of entropy under adequate assumptions on the operators $L_j$ (see Eq. (9) below), and conservation of positivity of the density operator (see §2). Several models can be found in the literature that fit this scheme, for instance that derived in [63]. However, the referential CL master equation cannot be written in KL form. What is more, the positivity of the density operator cannot be guaranteed for all times (see [2]). In this sense, Eq. (3) can be understood as an important improvement of this model.

As already pointed out, the density matrix formalism can describe the influence of the environment on a quantum system properly. Nonetheless, it turns out too expensive from a computational point of view (given that it requires $2d$ variables for a $d$–dimensional problem), and too complex from an analytical point of view, since $R$ is an operator typically defined on a Hilbert–Schmidt space, which makes the functional setting to become particularly intricate. This interpretation of quantum mechanics, however, can be reformulated in terms of a probability density function defined in phase space according to the operator transformation $\mathcal{W}$ introduced in 1932 by E. Wigner [134] (see formula (20) below), with the aim of deriving a quantum version of Boltzmann’s distribution. Then, in order to study the motion of a quantum system (either open or isolated) in the $d$–dimensional space, the Wigner function of such a system, $W : \mathbb{R}^{2d} \times [0, \infty) \rightarrow \mathbb{R}$, is defined as $W := \mathcal{W}(R)$, with a rather clear interpretation: $W(t, x, \xi)$ is the probability density associated with the system in the position $x \in \mathbb{R}^d$ and with velocity $\xi \in \mathbb{R}^d$ at time $t \geq 0$. Thus, the Wigner transform settles a canonical representation of a quantum system when seen from a kinetic point of view.

Treating with $\mathcal{W}(R)$ exhibits multiple advantages. As already said before, the Wigner transform of a quantum system is a real function defined in the phase space $\mathbb{R}^{2d}$, which simplifies the interpretation of the calculations associated with the system of interest and allows to establish some analogies with the classical case. As a matter of fact, various dissipative models describing charge transport in semiconductor devices [44, 85, 108], such as the Wigner–BGK and the Wigner–Boltzmann equations, are typically formulated in terms of $W$ (in analogy with the classical situation). It is also significant the fact that the Wigner representation of a quantum system allows to describe many particle systems and conserve the whole information contained in the density matrix operator, making $R$ and $\mathcal{W}(R)$ equivalent. Thus, models derived in the ambit of open quantum systems can be depicted by means of a kinetic formulation that respects all of their physical properties. In particular, Eq. (3), when rewritten in terms of $W$, gives rise to the so-called Wigner–Fokker–Planck (WFP) equation (cf. Eqs. (21)–(22) in §2) that constitutes a widely accepted and well analyzed dissipative model in the quantum–mechanical literature.

In spite of all of the advantages associated with the kinetic description of a quantum process, the Wigner representation also shows some important drawbacks. First of all, the interpretation of $W$ as a probability density is quite a fictitious matter, as it can assume negative values (even in very simple cases). Besides, this representation is rather expensive from a computational point of view since, as for the case of the density operator, the treatment of $W$ in phase space requires doubling the number of variables associated with the problem. This is due to the complexity inherent to the Wigner (or density matrix) picture. This is indeed the reason why in practical applications it is customary to simplify the full model by using the (velocity) moment system associated with the Wigner function, that yields a hydrodynamic description of the quantum system. In effect, multiplying the kinetic equation by the $k$–th power of $\xi$ and then integrating with respect to that variable, the evolution law for the $k$–th moment can be derived, so that a hierarchy of equations is
obtained for the moments of $W$. In particular, a system governing the time evolution of the lower order moments can be constructed (cf. §4).

In the nondissipative case, the hydrodynamic system ruling the time behaviour of the local and current densities, respectively given by

$$n(t, x) = \int_{\mathbb{R}^d} W(t, x, \xi) \, d\xi, \quad nu(t, x) = \int_{\mathbb{R}^d} \xi W(t, x, \xi) \, d\xi,$$

is shown to be closed. Besides, its study turns out equivalent to that of the Wigner equation, so that the physical properties deduced from the former are the same as those deduced from the latter. In this situation, the de Broglie–Bohm formulation of quantum mechanics [19, 40] comes up in terms of $n$ and $u$. Then, one can (formally) prove that $n$ admits a scalar potential $S$ (at least locally), which allows to write down the so-called potential flux hydrodynamic equations. This gives a description of the system in terms of $n$ and $S$.

Nonetheless, when dissipative effects are introduced at the level of $W$, no truncation of the moment system might be expected to be closed, hence it becomes impossible to retain all the physical properties of the Wigner function, and it reveals necessary to overcome this loss of information by imposing certain ad hoc closure relations based on some additional knowledge on the physics underlying the system. This procedure, widely used in classical dissipation theory, permits to derive frictional hydrodynamic models based on $n$ and $nu$. Furthermore, the velocity field $u$ typically generates vorticity (or, in other words, singularities), which makes the existence of a scalar potential impossible (even locally), thus the flux potential formulation cannot be established. On the contrary, by imposing the existence of a function $S$ such that $u = \nabla_S S$ (irrotationality assumption), flux potential hydrodynamic equations can be derived that model friction effects conveniently. Notice that this assumption carries on an additional loss of information, since it leads to the description of the dynamics of a dissipative system through an irrotational representation, which removes the vorticity effects that may stem from dissipative interactions.

As sheds from the previous discussion, both the kinetic and the hydrodynamic formulations constitute powerful tools to describe dissipative properties of quantum–mechanical systems. Concerning the wavefunction formulation, it is noticeable the fact that the introduction of effects that may generate gain or loss of some observable magnitude, must give rise to either an effective time–dependent Hamiltonian or to the presence of nonlinear terms (to be understood as dissipation operators) in the Schrödinger equation. Both consequences are in principle unpleasant, as they may lead to features that cannot be framed in the standard theory, such as the violation of the superposition principle. A deep knowledge as well as a right interpretation of the nonlinear terms that prove useful to model dissipative effects at the Schrödinger level are both of wide interest, as a wave description is unquestionably fundamental in quantum mechanics (see §4).

Up to now, there has been a big lack regarding the correct understanding of how to model dissipative interactions in the Schrödinger picture. However, there exist several approaches to this problem that rely on different interpretations. In 1981, N. Gisin introduced a family of dissipative Schrödinger equations starting from a very general open quantum system [68] and using a projection operator technique that allowed to recover a single state in the system under study. Although theoretically interesting because of its generality in interpreting the Hamiltonian of the system as well as that of the interaction, the model is constructed in a rather unrealistic context, since it is known that the time evolution of a quantum system composed of two subsystems generates entanglement between them, which leads to quantum decoherence and makes the materialization of the Gisin state hardly possible, as in fact occurs with any representation of an open system (hence of any system that undergoes dissipation) through a single wavefunction.

Therefore, we may conclude that any representation of a dissipative process by means of a Schrödinger equation is linked with an intrinsic loss of information. However, as seen before, there also exist hydrodynamic descriptions that do not contain all the physical properties of the system under study, but even though give good approximations. For instance, it can be simply shown that the expectation values of the position and the velocity associated with the system 

$$\langle x \rangle_{\text{hyd}} := \int_{\mathbb{R}^d} x n(t, x) \, dx, \quad \langle v \rangle_{\text{hyd}} := \int_{\mathbb{R}^d} nu(t, x) \, dx,$$

coincide with their counterparts calculated directly from the WFP system:

$$\langle x \rangle_{W} := \int_{\mathbb{R}^{2d}} x W(t, x, \xi) \, d\xi \, dx, \quad \langle v \rangle_{W} := \int_{\mathbb{R}^{2d}} \xi W(t, x, \xi) \, d\xi \, dx.$$
This property is also true for the systems dealt with in [75, 102], among others.

Under these considerations, one can search for quantum wave equations that describe observable dissipation processes in a phenomenological way. M. D. Kostin already exploited this idea in [93] to depict Langevin dynamics in terms of a nonlinear Schrödinger model that enjoys some suitable properties, such as energy dissipation [94], as well as gives rise to numerical approximations that reproduce the Langevin dynamics appropriately [120]. Besides, this equation is equivalent to the CL master equation (except for the loss of information described above). This means that starting gives rise to numerical approximations that reproduce the Langevin dynamics appropriately [120].

Along with the Kostin equation, another well contrasted class of Schrödinger equations accounting for dissipative and diffusive effects is that derived by H. D. Doebner and G. Goldin in [53]. The nonlinearities participating in the Doebner–Goldin (DG) class were derived in the algebraic frame of group theory, more precisely from the representation analysis of the quantum kinematical group of diffeomorphisms Diff(R^3), and proved to be physically motivated in many different situations comprising diffusive quantum mechanics.

In the picture of the balance-of-mass and balance-of-momentum equations within the DG framework (see §3.2), the overall motion of the system under study can be thought of as the motion of a quantum fluid having local density \( n \) and mean velocity \( u \), subject to the influence of an external potential augmented by the quantum Bohm potential (see the definition given in (66) below), which represents current arising as a result of density gradient effects, and by several other viscosity and dissipation forces that will be reported on later.

The paper is structured as follows: In Section 2 we summarize the main aspects of the WFP equation as the prototype kinetic model to describe quantum–mechanical dissipation in phase space. Section 3 concerns the fluid equations associated with some WFP and DG type interaction models. Section 4 deals with the Schrödinger description of such phenomenology, making especial emphasis on some nonlinear models of logarithmic type derived from the WFP equation as well as on the nonlinear Schrödinger–Langevin equation. Finally, in an Appendix we include a graphic scheme intended to relate the different families of models dealt with in this paper (of Wigner, hydrodynamic and Schrödinger type) among them, emphasizing their mutual interconnections.

2. Phase space description: From the density matrix approach to the WFP equation

Our starting point in this section is Eq. (3), which might be also written in the following way (see for example [27], Eq. (3.20) in [126], Eq. (9) in [63] and Eq. (1.4) in [80], which is reduced to that of [126] in the case of a purely Ohmic environment):

\[
\partial_t \rho = -\frac{i}{\hbar} (H_\lambda - H_0) \rho - \lambda (x - y) \cdot (\nabla_x - \nabla_y) \rho \\
+ \left( D_{pp} |\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2} |x - y|^2 - \frac{2i}{\hbar} D_{pq} (x - y) \cdot (\nabla_x + \nabla_y) \right) \rho .
\]

Here,

\[
H_0 = -\frac{\hbar^2}{2m} \Delta_x + V
\]

is the electron Hamiltonian (\( H_\lambda, H_0 \) standing for copies of \( H_0 \) acting on the \( x \) and \( y \) variables, respectively) and \( V \) holds for the electric potential. The positive constants \( \lambda, D_{pp}, D_{pq}, D_{qq} \) are as stated in (4) (cf. [51]). The first term in the right-hand side of Eq. (6) corresponds to the Liouvillian evolution induced by (7), the one proportional to \( \lambda \) is a dissipative term, and all of the remaining terms are diffusive. Note that Eq. (6) was derived in [51] as the Markovian approximation (that is, elimination of memory effects via perturbative analysis, see [39]) of the originally non Markovian evolution of the electron in the oscillator bath.
The density matrix operator of the electron (which is a linear, nonnegative and selfadjoint trace class operator) adopts the following form:

\[ (R(t)) (x) = \int_{\mathbb{R}^d} I(y) \rho(t, x, y) \, dy, \quad (8) \]

where the integral kernel \( \rho(t) \) is a function that typically belongs to \( L^2(\mathbb{R}^d \times \mathbb{R}^d) \). A great deal of properties of open quantum systems can be shown to hold if the Markovian evolution equation for the operator given in (8) belongs to the KL class, as commented before. An equation of the type

\[ \frac{dR}{dt} = -\frac{i}{\hbar} [H, R] + A(R) \]

is said to be in KL form if \( H \) is a selfadjoint operator (Hamiltonian) and there exists a family of countably many linear operators \( \{L_j\}_{j \in \mathbb{N}} \) such that \( A(R) \) can be expressed as

\[ A(R) = \sum_{j \in \mathbb{N}} L_j R L_j^* - \frac{1}{2} (LR + RL), \quad \text{with} \quad L = \sum_{j \in \mathbb{N}} L_j^* L_j, \quad (9) \]

where \( A^* \) denotes the adjoint operator of \( A \). In addition, some other technical hypothesis on \( H \) and \( A \) are also necessary, for instance that \( -\frac{i}{\hbar} [H, \cdot] + A(\cdot) \) generates a strongly continuous semigroup (the interested reader may wish to consult [116]) on the space of trace class operators on \( L^2(\mathbb{R}^d) \). We also refer to [34] for details.

The main properties of evolution equations in KL form are the following:

(a) Conservation of positivity (cf. [34, 98]): \( R(0) \geq 0 \Rightarrow R(t) \geq 0 \) for all \( t > 0 \) (in the sense of positive definite operators). In fact, the KL form even entails complete positivity of the evolution semigroup \([15, 98, 129]\), a stronger property than just positivity related to the positivity preservation for all possible statistical couplings of the system of interest with finite-dimensional remote and inert systems (so-called ancillas). Note that the total charge of the electron ensemble, given by \( \text{tr}(R(t)) \), is left invariant by the evolution: \( \text{tr}(R(t)) = \text{tr}(R(0)) \) for all \( t > 0 \), so that

\[ q = \text{tr}(R(0)) = \int_{\mathbb{R}^{2d}} W(t, x, \xi) \, d\xi \, dx \geq 0. \quad (10) \]

(b) Dissipativity (in the space of trace class operators): The inequality

\[ \langle A(R), \text{sgn}(R) \rangle_{HS} \leq 0 \]

holds, where \( \langle A, B \rangle_{HS} = \text{tr}(AB^*) \) is the usual scalar product on the Hilbert space of Hilbert–Schmidt operators on \( L^2(\mathbb{R}^d) \). \( A(R) \), given as in (9), is then a dissipative operator in the sense of the semigroup generator.

(c) Entropy growth: If

\[ \sum_{j=1}^{\infty} L_j^* L_j \leq \sum_{j=1}^{\infty} L_j^* L_j, \quad (11) \]

then the von Neumann entropy \( S(R) := -\text{tr}(R \log R) \) satisfies

\[ \frac{d}{dt} S(R(t)) \geq 0 \quad (12) \]

for all operators \( R(t) \).
Obviously, the properties (b) and (c) refer to the irreversibility of the evolution equation generated by Eq. (3) (in nontrivial cases).

To write Eq. (6) in Lindblad form we can set

\[ L_j = r x_j + \delta \partial x_j, \quad r, \delta \in \mathbb{C}, \quad j = 1, \ldots, d, \quad (13) \]

where \( x_j \) stands for the operator representing multiplication by the \( j \)-th position coordinate. Clearly, \( L_j^* = r x_j - \delta \partial x_j \).

Also, we set

\[ L_{d+j} = w x_j, \quad L_{2d+j} = \varphi \partial x_j, \quad w, \varphi \in \mathbb{C}, \quad j = 1, \ldots, d. \quad (14) \]

Moreover, we define the adjusted Hamiltonian

\[ H_{adj} = H - i \frac{\hbar \mu}{2} \sum_{j=1}^d \{ x_j, \partial x_j \}, \quad \mu \in \mathbb{R}, \quad (15) \]

where \( \{ A, B \} = AB + BA \) denotes the anticommutator of the operators \( A \) and \( B \). Obviously, \( H_{adj} \) is (formally) selfadjoint. Lengthy but simple calculations give the integral kernel \( a = a(x, y) \) of

\[ \frac{-i}{\hbar} [H_{adj}, R] + \sum_{j=1}^{3d} L_j R L_j^* - \frac{1}{2} \left( \sum_{j=1}^{3d} L_j^* L_j + R \sum_{j=1}^{3d} L_j L_j^* \right). \]

which has the form

\[ a(x, y) = \frac{\mu}{\hbar} (H_x - H_y) \rho + d (\text{Re} \sigma - \mu) \rho - \frac{1}{2} ([r^2 + |w|^2] |x - y|^2 \rho + \frac{1}{2} (|\delta|^2 + |\varphi|^2) |\nabla_x + \nabla y|^2 \rho + \left( \kappa x \cdot \nabla_x - \sigma y \cdot \nabla_x - \overline{x} x \cdot \nabla y + \overline{y} y \cdot \nabla y \right) \rho, \quad (16) \]

with \( \sigma = \delta \tau \) and \( \kappa = \mu + ilm(\sigma) \). A comparison shows that we can choose the parameters \( r, w, \delta, \varphi \in \mathbb{C} \) and \( \mu \in \mathbb{R} \) such that the right-hand side of Eq. (6) comes out if and only if

\[ |(r, w)|^2 = \frac{2D_{pq}}{\hbar^2}, \quad |(\delta, \varphi)|^2 = 2D_{qq}, \quad \mu = \text{Re} \sigma = \lambda, \quad \text{Im} \sigma = -\frac{2D_{pq}}{\hbar}. \quad (17) \]

Then, we conclude that \( r, w, \delta, \varphi, \mu \) can be found satisfying these equations if and only if the reservoir parameters are such that the matrix

\[ \begin{pmatrix} D_{qq} & D_{pq} + \frac{1}{2} \hbar \lambda \\ D_{pq} - \frac{1}{2} \hbar \lambda & D_{pp} \end{pmatrix} \]

is positive definite, i.e.

\[ D_{pp} D_{qq} - D_{pq}^2 \geq \frac{\hbar^2 \lambda^2}{4}. \quad (18) \]

In terms of the original thermal bath constants, this condition reads

\[ \frac{\hbar \Omega}{k_B} \eta \leq \sqrt{3} \pi \quad (19) \]

or \( \eta = 0 \) (no coupling to the thermal bath). The constraint (19) is satisfied for medium/high temperatures. We remark that (18), (19) can be found in [47, 52].

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Under condition (18), one possible choice of the KL operators is given by

\[ r = \frac{\sqrt{2D_{pq}}}{\hbar}, \quad \delta = \frac{\hbar \lambda - 2iD_{pq}}{\sqrt{2D_{pp}}}, \quad w = 0, \quad \varphi^2 = 2D_{qq} - |\delta|^2. \]

This implies \( \sum_{j=1}^{3d}(L^*_j L_j - L_j L^*_j) = -2d \lambda \), thus entropy growth for all initial density matrices can only be concluded in the frictionless case \( \lambda = 0 \) (see (11), (12)).

We shall now argue that the condition (18) is also necessary for Eq. (6) to be of KL form. The above representation in KL form is (as usual) not unique. However, only the mixed \( \lambda \)-terms in Eq. (6) are relevant for the validity of the KL form. They arise from the operators \( L, R L_j^* \). Since these operators are positive, the cancellation of different KL operators is not possible. From the structure of Eqs. (16) and (6) we readily see that only KL operators of the form (13)–(14) can be used to represent the right–hand side of Eq. (6). Assume now that Eq. (6) can be represented by two operators of the form (13) (the other cases are trivial). Using then the (obvious generalization of) the relations (17) we estimate

\[ \left| \lambda + \frac{2i}{\hbar} D_{pq} \right| = | \delta_1 r_1 + \delta_2 r_2 |^2 \leq (| \delta_1 |^2 + | \delta_2 |^2 )( | r_1 |^2 + | r_2 |^2 ) \leq \frac{4}{\hbar^2} D_{qq} D_{pp} \]

and (18) follows.

Note that the CL master equation \( (D_{pq} = D_{qq} = 0) \) is not of KL form, and as commented previously the conservation of positivity of \( R(t) \) cannot be guaranteed. The very high temperature model \( (\lambda = D_{pq} = D_{qq} = 0) \) analyzed in [105], however, is of KL form.

The WFP equation can be understood in terms of a (quasi)probability distribution function \( W(t, x, \xi) \) defined on the position–momentum phase space, which arises after applying the well–known Wigner transform to the function \( \rho \) (see [67, 99]), namely

\[ W(t, x, \xi) := W(\rho)(t, x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \rho\left( t, x - \frac{\hbar}{2m} y, x + \frac{\hbar}{2m} y \right) e^{-iy \cdot \xi} dy. \]  

(20)

This transformation yields the following quantum–kinetic Markovian equation of Fokker–Planck type

\[ \partial_t W + (\xi \cdot \nabla_x) W + \theta_h[V]W = \mathcal{L}_{WF}[W], \]

(21)

with

\[ \mathcal{L}_{WF}[W] = \frac{D_{pq}}{m^2} \Delta_x W + 2\lambda \text{div}_x(\xi W) + \frac{2D_{pq}}{m} \text{div}_x(\nabla_x W) + D_{qq} \Delta_x W, \]

(22)

where \( \theta_h[V]W \) denotes the (eventually) nonlinear pseudo–differential term (for instance, it is quadratic in the case of Hartree–type interactions) given by

\[ \theta_h[V]W(t, x, \xi) = \frac{i}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{1}{\hbar} \left[ V\left( t, x + \frac{\hbar}{2m} y \right) - V\left( t, x - \frac{\hbar}{2m} y \right) \right] W(t, x, \xi') e^{-i(\xi - \xi') y} d\xi' dy. \]

(23)

A lot of attention has been paid in last years to the WFP equation, especially from the point of view of its mathematical properties. We should mention here the pieces of work carried out in [6–11, 28, 29, 105, 117, 123, 135], where different aspects regarding the derivation of the model, the wellposedness of the associated initial value problem, the long time behaviour and the numerical representation of its solutions were taken into consideration. Particularly, the following existence plus uniqueness results were proved by some of us in the case of physically admissible initial data \( W_0 \) (i.e. such that the density matrix operator corresponding to \( W_0 \) is nonnegative) and 3D Poisson coupling, that is

\[ V(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{n(t, y)}{|x - y|} dy, \]

giving rise to the so–called Wigner–Poisson–Fokker–Planck (WPFP) equation.
Theorem 1 ([28]). Let \( W_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}_+^3) \cap L^1(\mathbb{R}^3; L^2(\mathbb{R}_+^3)) \) be such that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}_+^3} |\xi|^2 |W_0(x, \xi)| d\xi dx < \infty.
\]

Then, the WPFP equation (with \( D_{pq} = 0 \)) admits a unique global mild solution

\[
W \in C\left([0, \infty); L^1(\mathbb{R}^3 \times \mathbb{R}_+^3) \right) \cap C\left([0, \infty); L^1(\mathbb{R}^3; L^2(\mathbb{R}_+^3)) \right).
\]

Moreover, \( W \in C\left(0, \infty); W^{1, \infty}(\mathbb{R}^3 \times \mathbb{R}_+^3) \right) \). Also, the local density and the electric potential satisfy the following Hölder–regularity properties: for all \( t > 0 \),

\[
n(t, \cdot) \in C^{0, \alpha}(\mathbb{R}_+^3) \text{ with } 0 < \alpha < \frac{1}{2}, \quad V(t, \cdot) \in C^{1, \beta}(\mathbb{R}_+^3) \text{ with } 0 < \beta < \frac{1}{3}.
\]

As pointed out in [28], the same proof applies to the general WPFP case (with nonvanishing \( D_{pq} \)-term) with slight modifications in the expression of the fundamental solution.

Theorem 2 ([105]). Let \( W_0 \) belong to \( L^1(\mathbb{R}^3 \times \mathbb{R}_+^3) \cap L^1(\mathbb{R}^3; L^2(\mathbb{R}_+^3)) \). Then, there exists \( 0 < T_{\max} \leq \infty \) such that the WPFP equation (with \( \lambda = D_{pq} = D_{qq} = 0 \)) admits a unique mild solution

\[
W \in C\left(0, T_{\max}; L^1(\mathbb{R}^3 \times \mathbb{R}_+^3) \cap L^1(\mathbb{R}^3; L^2(\mathbb{R}_+^3)) \right)
\]

when solved along with a given initial data \( W_0 \) in the conditions above.

Besides, if

\[
|||\xi|^2 W_0|||_{L^1(\mathbb{R}^3 \times \mathbb{R}_+^3)} + |||\nabla_x W_0|||_{L^1(\mathbb{R}^3 \times \mathbb{R}_+^3)} + |||W_0|||_{L^1(\mathbb{R}^3; L^p(\mathbb{R}_+^3))} < \infty
\]

for some \( p > 3 \), then \( T_{\max} = \infty \).

3. Fluid description: On the viscous hydrodynamic system, the quantum Navier–Stokes equations and some of their variants.

This section aims to interpret some quantum dissipative models in terms of their associated viscous hydrodynamic systems. As already settled down, these models describe a quantum system that comes into contact with a reservoir which absorbs the energy lost by the system. That is the case, for example, of active regions and contacts in semiconductor devices [5]. These fluid descriptions are typically based on master equations in KL form [85] for the density matrix function, collisional Wigner equations [5, 85], or suitable Schrödinger models (DG system [53, 104], Schrödinger–Langevin equation [93, 103]), which usually entail high complexity and singularities. In this manner, quantum hydrodynamic models arise as macroscopic formulations that prove especially useful for numerical simulations.

In order to derive the hydrodynamic system associated with a master equation (for the Wigner or the density matrix functions, which are connected by the transformation \( \mathcal{W} \) introduced in §2 and its inverse \( \mathcal{W}^{-1} \)), we start from its general kinetic formulation

\[
\partial_t W + (\xi \cdot \nabla_x) W + \theta_n [V] W = Q[W].
\]

The operator in the right–hand side of Eq. (24) describes collisional events. By formal integration of the Wigner equation over the momentum space, evolution equations for the particle density \( n \), moment density \( nu \), and energy density \( ne \), recalled here to be defined by \( (n, nu, ne) = \int_{\mathbb{R}^3} (1, \xi, \frac{1}{2} |\xi|^2) W d\xi \), can be derived. However, higher order moments appear which cannot be expressed in terms of the moments \( (n, nu, ne) \). This problem is typically recognized...
as the closure problem, and can be solved by assuming that the Wigner function in the higher order moments can be approximated by the quantum equilibrium distribution $M[W]$. According to P. Degond and C. Ringhofer [45], given a Wigner function $W$ and its corresponding moments, the quantum equilibrium is defined (if it exists) as the maximizer of the quantum free energy (or quantum entropy) subject to the constraints $(n, nu, ne) = \int_{\mathbb{R}}(1, \xi, \frac{1}{2}|\xi|^2)M[W]d\xi$:

$$M[W](t, x, \xi) = \text{Exp}\left\{A(t, x) - \frac{|\xi - U(t, x)|^2}{2T(t, x)}\right\}.$$  \hspace{1cm} (25)

The functions $A$, $U$, and $T$ are the Lagrange multipliers of the constrained extremal problem, and $\text{Exp}\{W\} = \mathcal{W}\{\exp\{W^{-1}(W)\}\}$ is the quantum exponential defined in [45]. The multiplier $U$ is linked to the velocity $u$ by the relation $U = u + O(\hbar^2)$, and it holds $U = u$ for irrotational flows (see [83, p. 295]). The rigorous solvability of the constrained maximization problem is a delicate issue. In that direction, a unique maximizer of the quantum entropy in the 1D setting subject to a given local particle density was proven recently in [109].

In this way, quantum hydrodynamic equations were derived (see [58, 65, 86, 87]) whose numerical solution is less demanding than for Wigner models. Also, viscous corrections were obtained when a Chapman–Enskog expansion around the quantum equilibrium is applied, $W = M[W] + \beta g$, \hspace{1cm} (26)

where $g$ is some first order correction and $\beta > 0$ is some parameter. This procedure was recently applied by S. Brull and F. Méhats [23] to Eq. (24) with $Q[W] = \mathcal{L}_{BGK}[W]$, where

$$\mathcal{L}_{BGK}[W] = \frac{1}{\beta}(M[W] - W),$$

$\beta > 0$ being the relaxation time. The Wigner–BGK model leads to nonlocal quantum Navier–Stokes (QNS) equations, consisting of the mass conservation equation and the momentum balance equation. Local equations are realized by approximating the quantum equilibrium up to order $O(\hbar^4)$, yielding a density–dependent viscosity in the stress tensor. The full model, including the energy equation, was derived in [89]. In fact, integrating the Wigner–BGK equation over the momentum variable, closing the moment equations by $M[W]$, performing the Chapman–Enskog expansion (26), and expanding $M[W]$ in powers of $\hbar^2$, the (local) QNS equations up to $O(\hbar^4)$

\begin{align*}
\partial_t n + \text{div}_x(nu) &= 0, \hspace{1cm} (27) \\
\partial_t(nu) + \text{div}_x((nu \otimes u + P) + n\nabla_x V) &= \text{div}_x S, \hspace{1cm} (28) \\
\partial_t(ne) + \text{div}_x((P + ne\beta)u + q) + nu \cdot \nabla_x V &= \text{div}_x(Su), \hspace{1cm} (29)
\end{align*}

are obtained, where $I$ is the identity matrix in $\mathbb{R}^{2d}$,

$$ne = \frac{d}{2}nT + \frac{1}{2}n|u|^2 - \frac{\hbar^2}{24}n\Delta_x\log(n), \hspace{1cm} P = nT - \frac{\hbar^2}{12}n\nabla_x \otimes \nabla_x \log(n).$$  \hspace{1cm} (30)

are the energy density and quantum stress tensor, respectively, and where

\begin{align*}
q &= -\frac{5}{2}\beta nT\nabla_x u - \frac{\hbar^2}{24}n(\Delta_x u + 2\nabla_x \text{div}_x u), \hspace{1cm} (31) \\
S &= 2\beta nT \left(\text{Sym}(\nabla_x u) - \frac{1}{3}(\text{div}_x u)I\right), \hspace{1cm} (32)
\end{align*}

are the quantum heat flux and viscous stress tensor, respectively. Here, $\text{Sym}(T)$ stands for the symmetric part of the tensor $T$. It is worthy remarkable that the structure of (27)–(29) is analogous to its classical Navier–Stokes counterpart. Indeed, as noticed by R. Harvey for the first time in [78], this system contains the quantum generalization of the classical, fluid–mechanical viscosity term, with nonconstant (density–dependent) diffusion coefficients. We also remark that there
exists a class of QNS models derived from Lagrangian mechanics of the space of probability measures [62] such that this model class goes beyond the VQH equations.

On the other hand, when integrating Eq. (24) with $Q[W] = \mathcal{L}_{WFP}[W]$ (cf. (22)) over the momentum variable, closing the moment equations by $\mathcal{M}[W]$, and expanding the quantum equilibrium in powers of $\hbar^2$, the viscous quantum hydrodynamic (VQH) equations up to $O(\hbar^4)$

$$
\partial_t n + \text{div}_x (nu) = D_{q\hbar} \Delta_x n,
$$

$$
\partial_t (nu) + \text{div}_x (nu \otimes u + P + 2D_{q\hbar} nI) + n \nabla_x V = -2\lambda nu + D_{q\hbar} \Delta_x (nu),
$$

$$
\partial_t (ne) + \text{div}_x \{ (P + ne \otimes u + 2D_{q\hbar} nI)u \} + nu \cdot \nabla_x V = dD_{q\hbar} n - 4\lambda ne + D_{q\hbar} \Delta_x (ne),
$$

are derived (see for instance [71, 72]).

Despite the fact that there is no a priori relation between the BGK relaxation parameter $\beta$ and the WFP coefficients, there is a surprising structural connection between the above models (27)–(29) and (33)–(35) by means of the change of variables

$$
v = u - \beta \nabla_x \log(n),
$$

when $\beta = D_{q\hbar}$ is chosen, which make them (formally) equivalent. An analogous transformation at the level of the quantum momentum operator might be applied to the WFP equation, giving rise to a QNS type model that has no need for a Chapman–Enskog expansion along its derivation. The following section is devoted to develop this idea. Further details can be consulted in [90].

Concerning the viscous quantum hydrodynamic models based on (nonlinear) Schrödinger equations, we observe that the macroscopic moments associated with the wavefunction $\psi$ are obtained in the usual manner:

$$
n := |\psi|^2, \quad nu := J = \frac{\hbar}{m} \text{Im} (\overline{\psi} \nabla_x \psi),
$$

so that the corresponding hydrodynamic system can be derived by just performing straightforward calculations. It is noticeable that this kind of models are closed at the level of $n$ and $J$, in such a way that an additional equation for the energy density becomes superfluous. Section §3.2 is devoted to the fluid system associated with the already mentioned DG class. There, we sketch the ideas developed in [104]. An analogous transformation to (36) will be also useful in order to describe the associated dynamics.

### 3.1. From WFP to QNS: The role of the osmotic momentum operator

Later on we present an alternative derivation of the QNS model, starting from the WFP equation and then applying a (conveniently modified) moment method.

---

**Fig 1.** Macroscopic quantum models derived from the Wigner–BGK or WFP equations. The derivation expressed by Arrow 4 is that we are interested in.
When applied to the Wigner–BGK equation, the standard moment method gives viscous corrections if combined with a (first order) Chapman–Enskog expansion around the equilibrium (Arrow 1 in Figure 1). However, viscous corrections to the quantum hydrodynamic equations can be directly obtained by applying the moment method to the WFP equation, doing in this case without any asymptotic expansion of the Wigner function (Arrow 2 in Figure 1). According to the new velocity defined in (36), the VQH system can be formulated as the QNS system if \( h > \beta = D_{qg} \) [84] (Arrow 3 in Figure 1). We may then adapt the idea underlying Arrow 3, based on the so–called osmotic velocity

\[
\eta = \xi - D_{qg} \nabla_x.
\]

This definition is also related to the deformed momentum of R. Mosna, I. Hamilton, and L. Delle Site [48, Formula (8)]. As a matter of fact, we show in [90] that (39) originates from a local gauge transformation in the Schrödinger picture.

The first advantage of our approach is that, contrary to the techniques governing the procedure underlying Arrow 1, the Chapman–Enskog expansion can now be avoided, which simplifies significantly the derivation. The second advantage is that the viscosity coefficient can be identified with the quantum diffusion coefficient of the operator \( \mathcal{L}_{WFP}[W] \), which is proportional to the de Broglie wavelength \( \lambda_{dB} = h/\sqrt{4m\hbar^2} \) (cf. the definition of \( D_{qg} \) in formula (4)), and whose numerical value can be easily determined. On the other hand, the viscosity coefficient in [23, 89] equals the scaled relaxation time in the BGK operator and it may be less easier to determine its numerical value.

We finally believe that a viscous quantum hydrodynamic model can be derived from the Wigner–BGK equation using a similar transformation (Arrow 5 in Figure 1), but this still remains an open problem.

After these considerations, our main result is the following.

**Theorem 3 ([90]).**

Let \( W_\beta \) be a (smooth) solution to Eq. (24) with \( Q[W] = \mathcal{L}_{BGK}[W] + \mathcal{L}_{WFP}[W] \). Then, the limit function \( W = \lim_{\beta \to 0} W_\beta \) solves, up to terms of order \( O(h^4) \), the QNS equations

\[
\begin{align*}
\partial_t n + \text{div}_x(nu_\eta) &= 0, \\
\partial_t (nu_\eta) + \nabla_x \cdot (nu_\eta \otimes u_\eta + P_\eta) + n \nabla_x V &= -2\lambda nu_\eta + \text{div}_x S_\eta, \\
\partial_t (ne_\eta) + \text{div}_x \left( (P_\eta + ne_\eta) u_\eta + q_\eta \right) - D_{qg} n \Delta_x V + nu_\eta \cdot \nabla_x V &= dD_{qg} n - 4\lambda ne_\eta + \text{div}_x (S_\eta u_\eta),
\end{align*}
\]

where \( (n, nu_\eta) = \int_{\mathbb{R}^d}(1, \eta) W \, d\xi \), and where we have assumed that \( R(u_\eta) = \frac{1}{2}(\nabla u_\eta - \nabla u_\eta^T) = O(h^2) \) and \( \nabla_x \log(T) = O(h^2) \), with \( T \) being defined in (44) below. Furthermore, the modified quantum stress tensor \( P_\eta \), the quantum heat flux
the viscous stress tensor $S_\eta$ and the energy density $n e_\eta$ are given by

\[
\begin{align*}
    P_\eta &= n \left( T + 2 D_{pq} + 2 \lambda D_{qq} \right) - \frac{\hbar^2}{12} n \nabla_s \otimes \nabla_s \log(n), \\
    q_\eta &= \frac{\hbar^2}{24} n \left( \Delta_s u_\eta + 2 \nabla_s \cdot \nabla s u_\eta \right), \quad S_\eta = 2 D_{qq} n \text{Sym}(\nabla_s u_\eta), \\
    n e_\eta &= \frac{d}{2} n T + \frac{1}{2} n |u_\eta|^2 - \frac{\hbar^2}{24} n \Delta_s \log(n) - D_{qq} n \nabla_s \cdot \nabla_s u_\eta. \tag{44}
\end{align*}
\]

This model can be interpreted as follows. Eqs. (41)-(43) are the balance equations of the particle, momentum, and energy densities. The stress tensor $P_\eta$ consists of the pressure $n T$, which expresses the Boyle law for ideal gases, the increase $2 D_{pq} + 2 \lambda D_{qq}$ to the temperature due to quantum diffusion, and the quantum tensor $-\frac{\hbar^2}{12} n \nabla_s \otimes \nabla_s \log(n)$. Beside the diffusive term $(2 D_{pq} + 2 \lambda D_{qq}) n$, the quantum stress tensor $P_\eta$ exactly coincides with the expression derived in other quantum fluid models [83].

Compared to the QNS model of [89] (Eqs. (27)-(29) above), there are some differences. First, the general expression of the viscous stress tensor $S$ can be formulated as

\[
S = 2 \mu \text{Sym}(\nabla_s u) + \left( \zeta - \frac{2}{3} \mu \right) (\nabla_s u)^2,
\]

where $\mu$ is the shear viscosity and $\zeta$ the bulk viscosity. In the model of [89], the bulk viscosity vanishes. At variance, in our model we have $\zeta = \frac{1}{3} \mu$. Second, the quantum heat flux in [89] contains the stabilizing Fourier term $-\frac{\hbar}{2} B n T \nabla_s T$ (see formulae (31)-(32)), which comes up from the Chapman–Enskog expansion and is missing in the above model. Third, the pressure part in the stress tensor $P_\eta$ contains a larger temperature expression than that in [89]. The difference is due to the diffusion parameters $D_{pq}$ and $D_{qq}$. Finally, the energy density contains the term $D_{qq} n \nabla_s \cdot \nabla_s u_\eta$, which is not present in [89].

The energy density $n e_\eta$ is the sum of the thermal energy $\frac{1}{2} n T$, the kinetic energy $\frac{1}{2} n |u_\eta|^2$, and two quantum contributions. When integrated over space, the energy becomes (after an integration by parts)

\[
\int_{\mathbb{R}^d} n e_\eta \, dx = \int_{\mathbb{R}^d} \left( \frac{d}{2} n T + \frac{1}{2} n |u_\eta|^2 \right) \, dx + \frac{\hbar^2}{6} \int_{\mathbb{R}^d} |\nabla_s \sqrt{n}|^2 \, dx - D_{qq} \int_{\mathbb{R}^d} n \nabla_s u_\eta \, dx.
\]

The second term in the right-hand side can be interpreted as the Fisher information, whereas the last term describes the work due to compression and vanishes for incompressible fluids.

### 3.2. Hydrodynamic formulation of DG models

This section is devoted to the derivation of the quantum hydrodynamic system associated with some wavefunction-based dissipative models. A large number of the nonlinear Schrödinger equations proposed in the literature involve several (real and complex) nonlinearities describing different phenomenologies in astrophysics, quantum gravity or condensed matter physics. In particular, the family of DG equations

\[
\begin{align*}
    i \hbar \partial_t \psi &= H_0 \psi + \frac{i \hbar D}{2} \left( \frac{\Delta_s n}{n} \right) \psi + \frac{m^2 D'}{\hbar} \left( \frac{\nabla_s f}{n^2} \psi \right) + m D' \left( \mu_i \frac{\nabla_s f}{n^2} \right) \psi \psi \cdot \nabla_s \left( \mu_0 \frac{\Delta_s n}{n} \right) \psi + \hbar D' \left( \mu_2 \frac{\Delta_s n^2}{n} + \mu_3 \frac{\Delta_s n}{n^2} \right) \psi \psi, \tag{45}
\end{align*}
\]

was introduced in [53] as the most general class of Schrödinger type equations compatible with the Fokker–Planck continuity equation for the probability density. Here, $D$ and $D'$ represent diffusion coefficients while the parameters $\mu_1, \ldots, \mu_3$ are adimensional real constants. Some of the most important properties of this family of equations are collected in the following list (see [54, 55] and references therein, where several studies concerning symmetries and explicit time-dependent solutions can be found):
(i) The total charge \( \int_{\mathbb{R}^d} n(t, x) \, dx \) is preserved along the time evolution for all values of the coefficients \( \mu_i \).

(ii) If \( \psi \) is a solution to Eq. (45), so is \( \lambda \psi \) for any \( \lambda \in \mathbb{C} \).

(iii) For initially uncorrelated \( N \)-particle systems, noninteracting subsystems remain uncorrelated.

(iv) Plane waves \( \phi(t, x) = A_0 e^{i k \cdot x - \omega t} \) always solve Eq. (45) in the free–particle regime \( (V \equiv 0) \) in \( H_0 \).

(v) Solutions to Eq. (45) are time–translation and Euclidean invariant.

(vi) If the external potential \( V \) is chosen to be stationary, then there exist stationary state solutions to Eq. (45) that typically ‘tend’ to the corresponding stationary solutions of the linear Schrödinger equation as \( D \) and \( D' \) approach zero.

The macroscopic fluid system associated with the DG class includes a broad range of observable behaviours that are described in the following general result.

**Theorem 4 ([104]).**

Let \( \psi \) be a (smooth) solution to the general DG equation (45). Also, let \( n > 0 \) and \( nu \) be defined as in (37). Then, the couple \( (n, nu) \) satisfies the following hydrodynamic system:

\[
\begin{align*}
\partial_t n + \text{div}_x (nu) &= D \Delta_x n, \\
\partial_t (nu) + \text{div}_x (nu \otimes \nabla_x + P_{DG}) &= -\frac{1}{m} n \nabla_x V + D(\Delta_n)u + \alpha_1 \text{div}_x (n \text{Sym}(\nabla_x u)) \\
&\quad+ n \nabla_x \otimes \nabla_x \log(n) \left( \alpha_4 u + \alpha_5 \nabla_x \log(n) \right),
\end{align*}
\]

where

\[
P_{DG} = -\frac{\hbar^2}{4m^2} \left( 1 - \frac{4mD'\mu_3}{\hbar} \right) n \nabla_x \otimes \nabla_x \log(n)
\]

is the quantum stress tensor and where

\[
\alpha_1 = -D' \mu_1, \quad \alpha_2 = -\frac{2mD'}{\hbar} \mu_1, \quad \alpha_3 = -D' \mu_4, \quad \alpha_4 = -D'(\mu_1 + \mu_4), \quad \alpha_5 = -\frac{\hbar D'}{m} (\mu_2 + 2\mu_3).
\]

It is worth noticing at this point that the fluid–mechanical vorticity (or rotational) tensor \( R(u) := \frac{i}{2} \left( \nabla_x u - \nabla_x u^\top \right) \) vanishes in this situation, as when associated with a wavefunction \( \psi \) the fluid mean velocity \( u = \frac{\hbar}{\pi} \text{Im} \left( \nabla_x \psi \right) \) is an irrotational field (in virtue of Schwartz’s lemma). Indeed, if \( \psi = \sqrt{\pi} \exp \left\{ \frac{i}{\hbar} \sigma \right\} \) one has \( u = \frac{\hbar}{\pi} \nabla_x \sigma \), so that the tensor \( \nabla_x u \) is clearly symmetric.

The hydrodynamic system (46)–(47) can be also recast in conservative form by means of the introduction of the effective velocity

\[
v := u - D \nabla_x \log(n).
\]

Indeed, in terms of this new magnitude the continuity equation becomes the usual conservation law of probability density \( \partial_t n + \text{div}_x (nu) = 0 \). Besides, Eq. (49) is analogous to (36) (and also to (39)) with \( D \) instead of \( \beta \). It is also noticeable the fact that the new velocity \( v \) becomes a fully nontrivial ‘object’ when the DG equation (45) retains the diffusive contribution stemming from the complex–valued potential (i.e. when \( D \neq 0 \)). The interested reader can find a wide discussion upon the fundamental role played by the osmotic velocity in deriving the Schrödinger equation from Nelsonian mechanics in [61].

The new velocity variable \( v(t, x) \) evolves as described in the following
Theorem 5 (104).
Under the same hypotheses of Theorem 4, the couple \((n, nv)\) satisfies the following hydrodynamic system:

\[
\begin{align*}
\partial_t n + \text{div}_x(nv) &= 0, \\
\partial_t (nv) + \text{div}_x \{ n v \otimes v + P_{\text{DC}}^{\text{es}} \} &= -\frac{1}{m} n \nabla_x V + \beta_1 \text{div}_x (n \text{Sym}(\nabla_x v)) + n \text{Sym}(\nabla_x v)(\beta_2 v + \beta_3 \nabla_x \log(n)) \\
&\quad+ n \text{div}_x (\nabla_x \log(n)(\beta_4 v + \beta_5 \nabla_x \log(n))),
\end{align*}
\]

(51)

where

\[
P_{\text{DC}}^{\text{es}} = P_{\text{DC}} + D\mu_1 n \nabla_x \otimes \nabla_x \log(n) = \left\{-\frac{\hbar^2}{4m^2} + D' \left(D\mu_1 + \frac{\hbar}{m} \mu_2\right)\right\} n \nabla_x \otimes \nabla_x \log(n)
\]

(52)

is the quantum stress tensor in the new variables, and where

\[
\begin{align*}
\beta_1 &= D - D' \mu_1, & \beta_2 &= -\frac{2mD'}{\hbar} \mu_3, & \beta_3 &= -D \left(1 + \frac{2mD'}{\hbar} \mu_1\right) - D' \mu_4, & \beta_4 &= -D' \left(\mu_1 + \frac{2mD}{\hbar} \mu_3 + \mu_4\right), \\
\beta_5 &= -D^2 \left(1 + \frac{2mD'}{\hbar} \mu_1\right) - D' \left(D(\mu_1 + 2\mu_3) + \frac{\hbar}{m}(\mu_2 + 2\mu_5)\right).
\end{align*}
\]

(53)

Systems (46)–(48) and (50)–(52) must be interpreted as dissipative corrections of the local hydrodynamic system for a viscous, vorticity–free, compressible quantum fluid.

On one hand, concerning the continuity equation some partial results on the quantum–mechanical consistence (in the sense of local–in–time preservation of the positivity of the particle density) of models of this nature have been given in the preceding literature under suitable technical assumptions (see for example [88]). Such an analysis for a particular DG model was recently carried out in [77]. Dealing with the DG equation in full generality constitutes a much harder problem, whose study is intended to be the subject of future work. On the other hand, the balance–of–momentum equations contain the distinctive term of Navier–Stokes systems, namely \(\text{div}_x(n \text{Sym}(\nabla_x u))\), that models viscosity phenomena whenever \(D > D' \mu_1\), corrected by a list of nonconservative vector contributions including

\[
n \text{Sym}(\nabla_x w)w, \ n \text{Sym}(\nabla_x u_o)u_o, \ n(\nabla_x u_o)u_o, \ n(\nabla_x u_o)u_o, \ \text{with} \ w = u, v,
\]

(53)

which represent different friction effects due to the velocity densities \(u\) or \(v\), and to the osmotic velocity \(u_o = D\nabla_x \log(n)\). Concretely, the first and last terms in (53) have dissipative character since they represent gain or loss of momentum depending upon the sign of \(\alpha_2\) and \(\alpha_5\) (respectively, \(\beta_2\) and \(\beta_5\)), provided that

\[
\int_{\mathbb{R}^d} n \text{Sym}(\nabla_x w) w \, dx = \frac{1}{2} \int_{\mathbb{R}^d} n |w|^2 \, dx, \quad \int_{\mathbb{R}^d} n(\nabla_x u_o) u_o \, dx = \frac{1}{2} \int_{\mathbb{R}^d} n |u_o|^2 \, dx.
\]

The remaining terms in (53) can be rewritten as

\[
n \text{Sym}(\nabla_x w) u_o = n(u_o \cdot \nabla_x) w, \quad n(\nabla_x u_o) w = n(w \cdot \nabla_x) u_o,
\]

that express convection. Here, \(w\) has been indistinctively used to denote \(u\) or \(v\).

These contributions may however identically vanish under some well motivated physical assumptions on the parameters \(D, D'\), and \(\mu_j\), \(1 \leq j \leq 5\). We also point out that the systems (46)–(48) and (50)–(52) are equivalent in the sense that they exhibit identical observable behaviour. Nonetheless, consideration of the effective velocity \(v\) instead of \(u\) leads to some structural differences between both systems (for nonvanishing \(D\)). Eq. (46) is of Fokker–Planck type, while Eq. (50) adopts a conservative form. Likewise, the diffusive term \(D\Delta_t (nu)\) in Eq. (47) is not present in Eq. (51).
which contributes to largely simplify the latter with respect to the former. We further remark that the stress tensor $P_{DC}^{\text{ex}}$ contains an extra diffusive term $DD\rho_0 n \nabla_x (n \log(n))$ as compared to $P_{DC}$ (cf. Eq. (52)).

As for the hydrodynamic equations derived from a Chapman–Enskog expansion of the phase space quasi–probability distribution around the quantum Maxwellian, the system (46)–(48) (as well as (50)–(52)) holds to give a compressible description of a viscous quantum fluid, where viscosity is taken into account from a non Newtonian approach, i.e. the viscosity coefficients are typically density–dependent. On the other hand, two main differences must be stressed between our exact models and those derived from asymptotic expansion techniques. The first one is related to the vorticity tensor $\text{R}$ our exact models and those derived from asymptotic expansion techniques. The second main difference concerns the coefficient of the quantum stress tensor, that (up to diffusive corrections) equals $-\frac{\hbar^2}{4m}$ when derived from the wavefunction picture (for both pure and mixed state cases) and $-\frac{\hbar^2}{12m'}$ when the distribution function is expanded around the quantum thermal equilibrium. A similar observation was already made by M. G. Ancona in [3] in the context of density gradient theories.

Some particular cases of physical relevance were analyzed in [104], including the Galilean invariant and the linearizable DG subfamilies [54, 56], the modular Schrödinger class [12, 64], the integrable DG equations [113] and the special DG equations [127, 128]. We summarize here the results concerning the logarithmic DG models stemming from hydrodynamic and Nelsonian approaches, as derived in §4.1 and §4.2 below.

4. Schrödinger description: The logarithmic equation and some of its variants

As commented previously, there exist several approaches that are intended to go into the right interpretation of nonlinearities at the Schrödinger level in greater depth, particularly into those that prove useful to model dissipative effects. We will focus our attention here on various models related to the logarithmic Schrödinger equation (written in units in which $\hbar = m = 1$)

$$i\partial_t \psi = -\frac{1}{2}\Delta \psi + \gamma \log(n) \psi,$$

(54)

$\gamma \in \mathbb{R}\setminus\{0\}$ representing the strength of the (attractive or repulsive) nonlinear interaction and $n = |\psi|^2$ denoting the local density. This equation was proposed by I. Bialynicki–Birula and J. Mycielski [17] in 1976 as the only nonlinear wave equation of Schrödinger type accounting for some fundamental properties of quantum mechanics, such as the following (see also [16]):

- Separability of noninteracting subsystems, i.e. a solution of the nonlinear equation for the overall system can be constructed by taking the product of two arbitrary solutions of the nonlinear equations for the subsystems.
- Additivity of the total energy for noninteracting subsystems: $E[\psi_1 \psi_2] = E[\psi_1] + E[\psi_2]$.
- Boundedness from below of $E[\psi]$ for a free particle in any number of dimensions.
- Planck’s relation for all stationary states $\psi(t, x) = \sqrt{n(t, x)} \exp\{ -i\omega t \}$, that is, $E[\psi] = \hbar \omega$.
- All solutions satisfy the Ehrenfest relations: $\frac{d}{dt} \langle q \rangle = \frac{1}{m} \langle p \rangle$, $\frac{d}{dt} \langle p \rangle = 0$, where the averaging symbol $\langle \rangle$ means the expectation value calculated with respect to $\psi$.
- Invariance under the transformation $\psi \mapsto a \psi \exp\{ -i\sigma t \log(|\sigma|^2) \}$.

Nevertheless, they only addressed the case $\gamma < 0$. A derivation of this equation from Nelson’s stochastic quantum mechanics [114] was also given by Lemos in [96] (see [111] too).

The single sign choice for the logarithmic term first made in [17] and later continued in [30–32] was owing to the fact that the opposite sign leads to an energy functional not bounded from below. However, the positive sign for the logarithmic nonlinearity was physically justified by M. P. Davidson in [37] as representing a diffusion force within the context of stochastic quantum mechanics. By considering slowly varying profiles (almost constant solutions) in the absence of external forces, one can easily observe that the kinetic contribution $\int_{\mathbb{R}^d} |\nabla_x \psi|^2 \, dx$ is negligible, so that the effective energy operator may be written as

$$E[\psi] = \int_{\mathbb{R}^d} |\psi|^2 \log(n) \, dx \quad (\text{with } \gamma = 1).$$

(55)
This expression is not bounded from below. However, provided \( \psi \) has its support over a domain \( \Omega \) with finite measure in configuration space, it is a simple matter to check that the energy functional expressed by (55) admits a minimizer. If, in addition, the normalization constraint \( \int_{\Omega} n \, dx = 1 \) is assumed to hold, then minimization of the functional

\[
E_\mu[\psi] = \int_{\Omega} \left( n \log(n) + \mu \left( n - \frac{1}{|\Omega|} \right) \right) \, dx
\]

leads to a minimizer verifying \( n \equiv |\Omega|^{-1} \), where we used the Lagrange multiplier \( \mu \). Therefore, the bound \( E[\psi] \geq -\log(|\Omega|) \) is deduced.

Various meaningful physical interpretations have been given to the presence of the logarithmic potential in the Schrödinger equation. Indeed, it can be understood as the effect of statistical uncertainty or as the potential energy associated with the information encoded in the matter distribution described by the probability density \( n \). Also, \( \log(n) \) can be understood as an approximation of \( V \) up to \( O(h^2) \) terms when \( V \) is assumed to be the Hartree electrostatic potential solving the Poisson equation \( \nabla^2 V = n \) [65]. Recently, the model equation (54) has proved useful for the description of several nonlinear phenomena in the ambits of capillary fluids [43], Bose liquids [13], geophysical applications of magma transport [21, 42, 93] and nuclear physics [79]. Other nonlinear Schrödinger models of logarithmic type have proved interesting in the literature, for instance the Schrödinger–Landau–Lifshitz equation for the description of nonconservative quantum systems [93] (see §4.2), or the logarithmic-type and modular corrections introduced by P. C. Sabatier in [118] and more recently dealt with by P. Garbaczewski in [65]. Besides, one of its most relevant potential applications nowadays concerns the modeling of quantum dissipative interactions between a particle ensemble and a thermal reservoir of phonons when a Fokker–Planck scattering mechanism comes into play [75, 101, 102]. In that context, the logarithmic nonlinearity corresponds to a linear velocity–dependent frictional force caused by the interaction of the particle ensemble with a dissipative environment. We shall explain such models in the two following sections.

From a mathematical point of view, the logarithmic Schrödinger equation has been treated by T. Cazenave in [30, 31], who showed the existence of stable, localized nonspreading profiles of Gaussian shape (Gaussons, see also [18]), T. Cazenave and A. Haraux in [32, 33], and by C. Cid and J. Dolbeault in [35], who established some dispersion and asymptotic stability properties via rescaling techniques. Concerning wellposedness, the global–in–time existence of solutions was studied in [30, §9.3] on a suitable subspace of \( H^1_0(\Omega) \) consisting of regular wavefunctions with finite energy, \( \Omega \) being an arbitrary open domain. Also A. Jüngel, M. C. Mariani and D. Rial dealt in [88] with the local wellposedness of the Schrödinger equation equipped with a general family of \( |\psi|^2 \)-dependent nonlinearities, among them that of logarithmic type, in spaces of nonvanishing functions over bounded domains. In [76], the local wellposedness of Eq. (54) was achieved in the whole 3D space. The main result is as follows.

**Theorem 6 ([74, 76]).**

There exists a unique function

\[
\psi \in L^\infty([0, \infty); H^1(\mathbb{R}^3)) \cap C([0, \infty); L^2(\mathbb{R}^3))
\]

that solves the initial value problem

\[
\frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \gamma \log(n) \psi , \quad (t, x) \in [0, \infty) \times \mathbb{R}^3 ,
\]

\[
\psi(0, x) = \psi_0(x), \quad \psi_0 \in H^1(\mathbb{R}^3), \quad |x| \, |\psi_0|^2 \in L^1(\mathbb{R}^3),
\]

in a mild sense, i.e.

\[
\psi(t, x) = U(t)[\psi_0] - i \gamma \int_0^t U(t - s) \left[ \log(n(s)) \psi(s) \right] \, ds ,
\]

where \( U(t)[\psi_0] \) is the solution of the linear Schrödinger equation.

In the sequel, we will pay special attention to the logarithmic models introduced above.
4.1. On the hydrodynamic wavefunction approach to the WFP equation

The model we are dealing with in this section was originally studied in [101] in the 1D case, and in [102] in the multidimensional setting. It aims to describe the full WFP dynamics (Eqs. (21)–(22)) in terms of an appropriate wavefunction. The fundamental idea of the derivation consists in closing adequately the $(n, nu)$–hydrodynamic system associated with the WFP equation, and then constructing such wavefunction through its Madelung form. To this purpose we proceed, as usual, to multiply Eqs. (21)–(22) by $1$ and $\xi$ and integrate against $\xi$. The obtained system is a nonclosed version of Eqs. (33)–(35) and, written in terms of $n$ and $u$, gives rise to

\begin{align}
\partial_{\tau}n + \text{div}_{x}(nu) &= D_{\rho\rho}\Delta n , \\
\partial_{\tau}u + (u \cdot \nabla_{x})u &= \frac{1}{m} \nabla_{x} V - \frac{1}{n} \text{div}_{x} P_{\rho} - 2\lambda u - \frac{2D_{\rho\rho}}{m} \nabla_{x} \log(n) + D_{\rho\rho} \left[ 2 \left( \nabla_{x} \log(n) \cdot \nabla_{x} \right) u + \Delta_{x} u \right] ,
\end{align}

where $P_{\rho} = \int_{\mathbb{R}^{d}} \xi \otimes \xi W \, d\xi - nu \otimes u$ denotes the stress tensor. In order to close this system, our strategy consists in taking advantage of the osmotic correction to the mean velocity variable, namely $\nu = u - u_{o}$ (cf. (36), (38)), in such a way that the stress tensor is easily dealt with. Indeed, system (56)–(57), expressed in terms of $n$ and $\nu$, adopts the following form:

\begin{align}
\partial_{\tau}n + \text{div}_{x}(n\nu) &= 0 , \\
\partial_{\tau}\nu + (\nu \cdot \nabla_{x})\nu &= \frac{1}{m} \nabla_{x} V - \frac{1}{n} \text{div}_{x} P_{\nu} - 2\lambda \nu - \frac{2D_{\rho\rho}}{m} \nabla_{x} \log(n) \\
&+ D_{\rho\rho} \frac{1}{n} \left[ \nabla_{x} (\text{div}_{x}(n\nu)) + \Delta_{x}(n\nu) - 2\lambda \nabla_{x} n \right] + D_{\rho\rho} \frac{1}{n} \nabla_{x}(\Delta_{x} n) ,
\end{align}

where the corrected quantum stress tensor is defined by $P_{\nu} = \int_{\mathbb{R}^{d}} \xi \otimes \xi W \, d\xi - \nu \otimes \nu$.

If the Wigner function is associated with a mixed quantum state $\{ \psi_{k} \}_{k \in \mathbb{N}}$, it may be expressed as

\begin{equation}
W(t, x, \xi) = \frac{1}{(2\pi)^{d}} \sum_{k \in \mathbb{N}} \lambda_{k} \int_{\mathbb{R}^{d}} \psi_{k}(t, y) \psi^{*}_{k}(t, x + \frac{\hbar}{2m} y) e^{-iy \cdot \xi} \, dy ,
\end{equation}

where $\{ \lambda_{k} \}_{k \in \mathbb{N}}$, denoting the occupation probabilities, are such that $\lambda_{k} \geq 0$ for all $k \in \mathbb{N}$ and $\sum_{k} \lambda_{k} = 1$. Then, considering the following Madelung (modulus–argument) decomposition of the complex functions $\psi_{k}$, given by

\begin{equation}
\psi_{k}(t, x) = A_{k}(t, x) \exp \left\{ \frac{i}{\alpha} S_{k}(t, x) \right\} , \quad \alpha = 2mD_{\rho\rho} ,
\end{equation}

using that the velocity associated with the $k$–th quantum state reads as

\begin{equation}
v_{k} = \frac{1}{m} \nabla_{x} S_{k} ,
\end{equation}

and performing straightforward computations, we are led to the decomposition $P_{\nu} = P_{\nu}^{c} + P_{\nu}^{q}$, where

\begin{equation}
P_{\nu}^{c} = \frac{\hbar^{2}}{\alpha^{2}} \alpha \left( \langle \nu \otimes \nu \rangle_{A} - \langle \nu \rangle_{A} \otimes \langle \nu \rangle_{A} \right)
\end{equation}

denotes the classical part of the pressure tensor, in the sense that it describes the standard fluid–dynamic pressure, and $P_{\nu}^{q}$ stands for the quantum part of the pressure tensor, as it retains the quantum–mechanical effects through positive powers of the Planck constant, which become negligible as $\hbar \to 0$ (see formula (63) below). Observe that we denoted $\langle X \rangle_{A} = \frac{1}{n} \sum_{k} \lambda_{k} A_{k}^{2} X_{k}$ the average value of $X$ with respect to the weights $\lambda_{k} A_{k}^{2}$. 

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In order to deal with a consistent system of equations from a mathematical point of view we shall impose the following closure relations \[ \tag{62} P^q_\nu = P^q_\nu(n), \quad A_k = A = \sqrt{n} \; \forall k \in \mathbb{N}. \]

Of course, these assumptions immediately imply that the hierarchy of moment equations established above is not exact any longer. In fact, they give rise to an approximate system of moments which has the advantage to be closed, that is, it can be solved selfconsistently. In particular, the second hypothesis in (62) allows for an expression of the quantum component of the stress tensor which only depends upon macroscopic variables such as the average velocity \( \langle v \rangle_A \) and the local density \( n \):

\[
P^q_\nu = \frac{\hbar}{m} \text{Sym}\left( \langle v \rangle_A \otimes \nabla_s n \right) - \frac{\hbar^2}{4m^2} n \nabla_s \otimes \nabla_s \log(n) - D^2_{qq} \nabla_s n \otimes \nabla_s \log(n). \]  

These new variables describe the (already closed) hydrodynamic system we are searching for:

\[
\begin{align*}
\partial_t n + \frac{\hbar}{\alpha} \text{div}_s \left( n \langle v \rangle_A \right) &= D_{qq} \Delta_s n, \\
\partial_t \langle v \rangle_A + \frac{\hbar}{\alpha} \left( \langle v \rangle_A \cdot \nabla_s \right) \langle v \rangle_A &= D_{qq} \left\{ 2 \left( \nabla_s \log(n) \cdot \nabla_s \right) \langle v \rangle_A + \Delta_s \langle v \rangle_A \right\} \\
&- 2 \lambda \langle v \rangle_A - \frac{\alpha}{\hbar} \left( \frac{1}{m} \nabla_s \left( V + Q \right) + \frac{1}{n} \nabla_s P^\nu + \frac{2D_{pq}}{m} \nabla_s \log(n) \right), 
\end{align*}
\]

where \( Q(t, x) \) is the quantum potential of Bohm defined by

\[
Q := -\frac{\hbar^2}{2m} \Delta \sqrt{n} = -\frac{\hbar^2}{4m} \left( \frac{\Delta_n n}{n} - \frac{[\nabla_s n]^2}{2n^2} \right). \]

Now, if we assume that there exists a function \( S(t, x) \) verifying

\[
\langle v \rangle_A = \frac{1}{m} \nabla_s S, \tag{63}
\]

then we can construct the wavefunction

\[
\Psi(t, x) = \sqrt{n(t, x)} \exp \left\{ \frac{i}{\alpha} S(t, x) \right\}. \tag{68}
\]

If in addition there are two scalar fields \( U \) and \( F \) such that

\[
\frac{1}{n} \text{div}_s P^\nu = \nabla_s U, \quad \left( \nabla_s \log(n) \cdot \nabla_s \right) \nabla_s S = \nabla_s F, \tag{69}
\]

then Eq. (65) can be written in potential–flow form, and thus the logarithmic Schrödinger equation satisfied by \( \Psi \) shows up.

**Theorem 7 (\cite{102, 110}).**

Let \( W \) be a solution to the WFP equation (21)–(22) and assume that the relations in (62) are fulfilled and that there exist scalar fields \( S, U \) and \( F \) satisfying (67) and (69). Then, the wavefunction defined in (68) solves the following dissipative, nonlinear logarithmic Schrödinger equation:

\[
\begin{align*}
\imath \hbar \partial_t \Psi &= -\frac{\hbar^2}{2m} \Delta_s \Psi + \left( V + mU + \frac{2\hbar \lambda}{\alpha} S + 2D_{pq} \log(n) \right) \Psi + \frac{\imath \hbar D_{qq}}{2} \left( \frac{\Delta_n n}{n} \right) \Psi - \frac{\hbar}{m} F \Psi - mD_{pq} \text{div}_s \left( \frac{J}{n} \right) \Psi. \tag{70}
\end{align*}
\]
The following remarks are now in order.

- $n$ and $J$, given as in (37), coincide with the corresponding magnitudes for the Wigner function under the hypotheses of Theorem 7. This highlights the fact that Eq. (70) represents properly the WFP observable dynamics.

- (70) is a strongly nonlinear equation that incorporates friction and diffusion effects, the former modeled by $\frac{2\alpha}{m} S \Psi$ and the latter by the logarithmic term (crossed diffusion) and by $\{ i \hbar \frac{\partial^n_\mathcal{G} }{n} \Delta_n \Psi - \frac{\hbar}{m} F - m D_{qp} \nabla_s \cdot \left( \frac{1}{n} \right) \} \Psi$ (standard diffusion). Nevertheless, the $D_{qp}$-term in the WFP equation, responsible for the process of decoherence (see [82]), does not contribute to the final form of Eq. (70). This is due to the fact that the moment system is truncated at the level of the current equation, while the $D_{qp}$-contribution is only ‘visible’ at the next level, i.e., that of the kinetic energy equation.

- $\frac{2\alpha}{m} S \Psi$ can be easily checked to be an authentic friction term given that $\hbar$ and $\alpha$ have the same units, thus $\frac{\alpha}{m}$ is an adimensional factor. The role played by $\alpha$ as the new action unit in this sort of derivations is discussed in more detail in [75].

In comparing the closure relations leading to Eqs. (64)–(65) with those stemming from an application of the entropy principle (see §3), what one observes is, on one hand, that the assumption on the ‘$h^2$–size’ of the vorticity tensor in Theorem 3 is now replaced by just considering a vanishing vorticity. On the other hand, the assumption of being–near–the–Maxwellian–equilibrium in the context of Theorem 3 is here substituted for the uniformity condition about the amplitudes of the wavefunctions taking part of the Wigner function (see the second formula in (62)). Moreover, the hypothesis imposed on the classical part of the stress tensor, $P^c_r = P^c_r(n)$, is straightforwardly shown to be fulfilled for a wide range of well accepted and broadly used models. For example, one may postulate a diagonal tensor $P^c_r(t, x) = p(t, x) I$, $I$ denoting the identity matrix, with the pressure function $p(t, x)$ adopting the potential form $p = n^b$ for some appropriate exponent $b$ (perfect fluid condition). It is obvious that this example also satisfies the first relation in (69).

Concretely, in the 3D lowest order isotropic approximation, it is well–known that the pressure of the electron gas is proportional to $n^2$. Then, if we assume $P^c_r = \frac{2C}{3} n^2 I$, with $C \in \mathbb{R}$, it is easily followed that $\nabla \cdot P^c_r = \frac{2C}{3} n^2 \nabla \cdot n$, hence $\frac{1}{2} \nabla \cdot P^c_r = C \nabla \cdot \left( \frac{n^2}{2} \right)$ and finally $U$ can be thought of, for example, as the potential nonlinearity of Slater type $C|\Psi|^2$. Other choices of $b$ may produce different potential nonlinearities in the Schrödinger equation, which can be modulated with the presence of Hartree type (attractive or repulsive) potentials giving rise to Schrödinger–Poisson–$X^\alpha$ models which have been widely studied in the framework of semiconductor devices (see for example [20, 119]).

We finally discuss the physical plausibility of the second condition in (69). The identity $G := (\nabla \log(n) \cdot \nabla_s) \nabla_s S = \nabla_s F$ can be equivalently understood as the irrerotationality of the field $G$. Using that $u = \frac{1}{n}$ and the fact that $G = \frac{m}{2} \left( \nabla_s \log(n) \cdot \nabla_s \right) u$, as follows from the relations $\nu = \frac{\hbar}{\alpha} \langle \Psi | - D_{qq} \nabla_s \log(n) \rangle$ and $\langle \nu | = \frac{1}{\alpha} \nabla_s S \nabla_s$, we get

$$\text{rot} \left[ (\nabla_s \log(n) \cdot \nabla_s) u \right] = \frac{1}{n} \text{rot} \left[ (\nabla_s \cdot \nabla_s) u \right] = \frac{1}{n^2} \nabla_s \nabla_s \times (\nabla_s \cdot \nabla_s) u,$$

which (by elementary vector calculus) is proved to be zero if and only if the matrices $\nabla_s u$ and $\nabla_s u_s$ commute (cf. (38)). From a mathematical point of view, this would mean that there exists a system of coordinates with respect to which both velocity fields, $u$ and $u_s$, can be simultaneously diagonalized.

Compared with the general DG equation (45), it is clear that the relations $D = D_{qq}$, $D' \mu_1 = -D_{qq}$, $\mu_1 + \mu_4 = 0$ and $\mu_2 = \mu_3 = \mu_5 = 0$ must be satisfied among the parameters in order to fix the second line of Eq. (70). In addition, the resulting DG kernel

$$\Theta[n, J] = -m D_{qq} \frac{\nabla_s \cdot \nabla_s}{n} + m D_{qq} \frac{J \cdot \nabla_s}{n^2} + \frac{i \hbar D_{qq}}{2} \Delta_n \frac{n}{n}$$

is here augmented by the nonlinearities

$$\left( \frac{\lambda}{m D_{qq}} S + \frac{2 D_{qq}}{m} \log(n) - \frac{1}{m} F \right) \Psi.$$

Concerning $F$, needless to say that in general it is not a DG term. Nonetheless, the important fact from a macroscopic viewpoint is that it induces viscous hydrodynamics ($\langle (\Delta_s u - n \Delta_s u, to be precise) peculiar to the DG family.
Applying Theorem 4 and after straightforward calculations involving the terms in (72), the momentum fluid system associated with Eq. (70) reads

\begin{align*}
\partial_t n + \text{div}_v(nu) &= D_{qq}\Delta_n, \\
\partial_t(nu) + \text{div}_v\left(nu \otimes u + P_{\text{th}}\right) &= D_{qq}\left(\Delta_n u + \text{div}_v\left(n\text{Sym}(\nabla_v u)\right)\right) + D_{qq} \text{Sym}(\nabla_v u)\nabla_v n - 2\lambda nu,
\end{align*}

with the quantum stress tensor given by

\[ P_{\text{th}} = -\frac{\hbar^2}{4m^2} n \nabla_v \otimes \nabla_v \log(n) + \frac{2D_{pq} n^3}{m}. \]

After an adequate rewriting, this system exactly fits the VQH equation stated in (34). Indeed, \( P_{\text{th}} \) is as the stress tensor appearing in Eq. (34), up to the prefactor in front of \( n \nabla_v \otimes \nabla_v \log(n) \) and, of course, up to the temperature contribution stemming from the expansion around the quantum Maxwellian.

Now, the gauged equation for \( \nu = u - D_{qq} \nabla_v \log(n) \) adopts the following shape:

\[ \partial_t(n\nu) + \text{div}_v\left(n\nu \otimes \nu + P_{\text{th}}^{\nu}\right) = 2D_{qq}\text{div}_v\left(n\text{Sym}(\nabla_v \nu)\right) - 2\lambda \nu, \]

with

\[ P_{\text{th}}^{\nu} = -\left(\frac{\hbar^2}{4m^2} + D_{qq}^2\right) n \nabla_v \otimes \nabla_v \log(n) + \frac{2\Lambda}{m} n I, \]

which makes it to become a frictional Navier–Stokes system in the quantum–mechanical sense (cf. (42)).

### 4.2. On the Nelsonian wavefunction approach to the WFP equation

A different approach for describing quantum Fokker–Planck hydrodynamics in the Schrödinger picture was first provided in [101] (1D case) and then in [75] (multidimensional case). The main idea underlying our derivation consists of admitting a ‘classical’ interpretation of the continuity equation in terms of Nelsonian stochastic mechanics [36, 114]. This theory, initiated in 1952 by I. Fényes [57], is intended to give a description of quantum mechanics by means of classical probability densities for particles undergoing Brownian motion with diffusive interactions. In this framework, the evolution of a particle subject to nondissipative Brownian motion is shown to be equivalent (in the sense of its probability and current density) to that described by the Schrödinger equation [114].

We start from the \( (n, u) \)-hydrodynamic system associated with the WFP equation, stated in (56)–(57), and assume Brownian motion as produced by the dissipative interaction between the quantum gas and the thermal environment, the particles thus being subject to the action of forward and backward velocities \( u_+ \) and \( u_- \), respectively. Performing a microscopic time inversion of such velocities (see [73]), taking their mean value \( \nu := \frac{1}{2}(u_+ + u_-) \) and assuming it to evolve with time as the WFP mean velocity, we obtain a closure relation for (56)–(57). Therefore, if we suppose that the velocity \( u \) is a irrotational field, we get

\[ u = \frac{1}{m} \nabla_v S \quad (75) \]

for certain scalar field \( S \). Now we are in conditions to state our main result.

**Theorem 8 ([74, 75]).**

Let \( W \) be a solution to the WFP equation (21)–(22) and assume that there exists a scalar field \( S \) satisfying (75). Then, the wavefunction defined in (68) solves the following dissipative, nonlinear logarithmic Schrödinger equation:

\[ i\alpha \partial_t \Psi = H_e \Psi + \frac{\alpha^2}{2m^2} Q \Psi + \Lambda \log(n) \Psi + D_{qq} \left\{ \frac{i\alpha}{n} \frac{\Delta n}{n} + m \text{div}_v \left( \frac{f}{n} \right) \right\} \Psi, \quad (76) \]

where \( H_e = -\frac{\alpha^2}{2m} \Delta + V \) is the electron Hamiltonian, \( Q \) denotes the quantum potential introduced in (66), and where we denoted \( \Lambda := 2D_{pq} + 2m\alpha D_{qq} \).
In this picture, the probability density associated with $\Psi$ coincides with that of the WFP equation (cf. (5)), while the current densities are connected by the relation $J = \frac{h}{2} nu$. As for the hydrodynamic approach, the $D_{pp}$--term does not contribute to the final form of Eq. (76) either. On the other hand, the position diffusion $D_{qq}$--term in the WFP equation plays an essential role in the structure of our model. First of all, it gives rise to a set of nonlinearities belonging to the DG family [53]. Furthermore, it should be noticed that the diffusion governed by $D_{qq}$ turns the standard Schrödinger Hamiltonian $H_0 = -\hbar^2/2m \Delta + V$ into the effective Hamiltonian $H_0 = -\hbar^2/2m \Delta + V$. The parameter $\alpha = 2mD_{qq}$ has the dimensions of an action but it is not a universal constant, as it hinges on the particular system under study. Thus, though $\alpha \neq \hbar$ in general, it plays the role of $\hbar$ in some sense, conferring quantum–mechanical meaning to our wavefunction. At this point, we remark that under the original assumptions on the parameters (cf. §2) one is straightforwardly led to $\alpha \ll \hbar$, which means that the quantum potential effects are however drastically relaxed due to the spatial diffusion introduced by the WFP equation. As consequence, the $D_{qq}$--term confers some sort of classical behaviour to Eq. (76). Finally, we observe that the linear terms due to the $\lambda$–damping and the crossed $D_{pq}$--diffusion in the WFP equation are described in the wavefunction picture by the logarithmic nonlinearity (see Eq. (76) and the definition of $\Lambda$ in Theorem 8). This kind of selfinteraction becomes the most relevant contribution of Eq. (76). In fact, we can take advantage of the following nonlinear gauge transformation [91, 112, 121]

$$G : \Psi \mapsto \Phi = \Psi \exp \left\{ -\frac{i}{2} \log(n) \right\}, \quad (77)$$

which makes Eq. (76) equivalent to the (simpler) purely logarithmic Schrödinger equation

$$i\alpha \partial_t \Phi = H_0 \Phi + \Lambda \log(n) \Phi. \quad (78)$$

In this way, the DG diffusive terms present in Eq. (76) might be neglected only by turning the current density $J_\mu$ into $J_\mu = J_\mu - \hbar \alpha n \nabla \nabla \cdot n$, which does not alter the observable velocity. $G$ is also revealed to enjoy sufficiently good properties (indeed, it preserves the local density and the Ehrenfest equations) so as to study various aspects of Eq. (76) via the logarithmic Schrödinger Equation (78) (see [77]). Since we are mainly interested in the local densities associated with $\Psi$ and $\Phi$, which are identical to each other and also to that stemming from the Wigner function, we are called to focus further analysis on Eq. (76) and its gauge reduction to the purely logarithmic Schrödinger equation established in (78). Anyway, both Eqs. (76) and (78) retain the dissipative effects introduced by the WFP equation.

Eq. (76) belongs (up to the logarithmic nonlinearity) to the DG class of equations (45) with the following choices of the parameters:

$$D' \mu_1 = D = D_{qq}, \quad \mu_1 + \mu_4 = 0, \quad \mu_2 + 2\mu_5 = 0, \quad \mu_3 = 0.$$

As a matter of fact, Eq. (76) is easily seen to satisfy the canonical linearization conditions for the DG family (see [104] and references therein)

$$D' \mu_1 = D = -D' \mu_4, \quad \mu_2 + 2\mu_5 = 0, \quad \mu_3 = 0.$$

The contribution of the logarithmic term to the associated hydrodynamic equations is shown to result in no new contribution to the continuity equation, and in the presence of the term $-\hbar^2/2m \nabla \cdot n$ in the right–hand side of the current density equation, all of which is straightforwardly translated into the following system for the couple $\{n, nu\}$:

$$\partial_t n + \nabla \cdot (nu) = D_{qq} \Delta n, \quad (79)$$

$$\partial_t (nu) + \nabla \cdot (nu \nabla \cdot u + P_{LS}) = D_{qq} \Delta (nu) - 2D_{qq} \nabla \cdot (n \text{Sym}(\nabla \cdot u)), \quad (80)$$

with the modified quantum stress tensor

$$P_{LS} = -2D_{qq}^2 \hbar \nabla \cdot \nabla \log(n) + \frac{2\Lambda}{m} n \mathbb{I},$$

which includes a thermodynamic correction stemming from the logarithmic term, which might be interpreted as connected to the temperature of the electron ensemble within the quantum open system in terms of the (constant) temperature of
By means of the gauge transformation Eq. (78) on an appropriate subset of $H$ wellposed in some Sobolev spaces. To that purpose, our strategy consisted in developing a fixed–point argument for $-\mathbb{W}$. We note that the osmotic change of variables stated in (49) does linearize Eq. (76), given that the condition $-4m^2\frac{D_\psi^2}{\hbar^2} < 0$ is clearly fulfilled.

Concerning more theoretical aspects, the Cauchy problem associated with Eq. (76) was shown in [77] to be wellposed in some Sobolev spaces. To that purpose, our strategy consisted in developing a fixed–point argument for Eq. (78) on an appropriate subset of $H^2(\Omega)$, consisting of those wavefunctions living far from vacuum, in order to obtain regular solutions to the initial–boundary value problem determined by

$$\phi(0, x) = \phi_0(x), \quad x \in \Omega, \quad \phi(t, x) = \phi_B(x), \quad x \in \partial \Omega, \quad t \in [0, T], \quad (79)$$

thus to the original problem subject to

$$\psi(0, x) = \psi_0(x), \quad x \in \Omega, \quad \psi(t, x) = \psi_B(x), \quad x \in \partial \Omega, \quad t \in [0, T], \quad (80)$$

by means of the gauge transformation $G$. This analytical treatment does require an accurate definition of the argument of $\psi$ due to the multivaluedness of the complex logarithm. The required assumptions for our 3D analysis are the following:

(H1) $\Omega \subset \mathbb{R}^3$ is a simply–connected, $C^2$ bounded domain.

(H2) $\psi_0 \in H^2(\Omega)$, $\psi_B \in H^2(\partial \Omega)$, and $\psi_0 = \psi_B$ in $\partial \Omega$.

(H3) There exists $\delta > 0$ such that $\text{ess–inf}\{|\psi_0(x)| : x \in \Omega\} > \delta$, $\text{ess–inf}\{|\psi_B(x)| : x \in \partial \Omega\} > \delta$.

Our main notational conventions are the following: Denote $H$ a subset of $L^2(\Omega)$, $\delta$ a positive constant, and

$$H_\delta = \left\{ \varphi \in H : |\varphi| > \delta \text{ a.e. in } \Omega \right\}, \quad \text{and} \quad X^T = C([0, T]; H^2(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

(i) There exists $T = T(\delta, \psi_0, \psi_0, \Omega) > 0$ such that the nonlinear Schrödinger initial–boundary value problem determined by Eq. (76) (coupled with (81)–(82)) admits a unique strong solution $\psi \in X^T_\delta$.

(ii) The dynamics underlying Eq. (76), when coupled with (81)–(82), is (topologically) equivalent to that associated with the initial–boundary value problem consisting of Eq. (78) subject to the conditions (79)–(80), in the sense that there exists a suitable homeomorphism in $C([0, T]; H^2_\delta)$ that carries strong solutions of (76) (with (81)–(82)) into strong solutions of (78) (with (79)–(80)).

Even more, we do prove that strong solutions of Eq. (76) (and even of more general Schrödinger equations) can be rigorously decomposed in Madelung’s form in an unique way (up to a change of constant global phase) in terms of a regular argument.
**Theorem 10 ([77, 110]).**

Let \( \Omega \subset \mathbb{R}^3 \) be a simply-connected, \( C^1 \) bounded domain. Let also \( \delta > 0, T > 0, \) and \( \psi \in X_T^\delta \) be a strong solution of the following model Schrödinger equation

\[
i \partial_t \psi = -\frac{\alpha}{2m} \Delta_x \psi + \Theta[n, J] \psi, \tag{83}
\]

where \( \Theta : H^2_{\delta} \times \{ H^1(\Omega) \} \rightarrow L^2(\Omega) \) is a continuous (complex) operator. Then, there exists a unique countable set \( S(\psi) \subset X^T_T \) such that the Madelung decomposition

\[
\psi(t, x) = \sqrt{n(t, x)} \exp \left\{ \frac{i}{\alpha} S(t, x) \right\} \tag{84}
\]

holds almost everywhere in \( \Omega \) for any \( S \in S(\psi) \) and \( t \in [0, T] \). In particular, the following properties are accomplished.

(i) For any \( S \in S(\psi) \) and fixed \( t \in [0, T] \), the equation \( \nabla_x S(t) = J(t) \) holds almost everywhere in \( \Omega \) and

\[
\psi(0, x) = \sqrt{n(0, x)} \exp \left\{ \frac{i}{\alpha} S(0, x) \right\}. \tag{85}
\]

(ii) For any \( S \in S(\psi) \), the evolution equation

\[
\partial_t S = -\frac{\alpha^2}{\hbar^2} Q - \frac{m}{2} \frac{|J|^2}{n^2} - \alpha \Re(\Theta[n, J]) \tag{86}
\]

holds strongly in \([0, T]\).

In the framework of the semiclassical limit of the wavefunction, several authors have proved the existence of solutions with the same structure as (84) for various Schrödinger equations of nonlinear type (Schrödinger–Poisson [100], nonlinear Schrödinger [66, 70], derivative Schrödinger [50], and general modified nonlinear Schrödinger [49] systems).

### 4.3. On the nonlinear Schrödinger–Langevin equation

In this section we deal with a quantum version of the stochastic Langevin equation in the Schrödinger picture. Apart from those models based on a master equation approach that were reported on in §1 (see for instance [27, 51, 52]), there also exist in the literature other linear descriptions of frictional quantum Brownian motion. Regarding the wavefunction, we must lay stress on the generalized Schrödinger–Langevin picture introduced by N. G. van Kampen in [131], namely

\[
i \partial_t \psi = \left\{ H - i\lambda(U + i\mathcal{F}(t)) \right\} \psi, \tag{87}
\]

where \( H \) denotes the system Hamiltonian, \( \lambda \) is the coupling parameter, \( U \) is a linear Hermitian operator representing dissipation, and where \( \mathcal{F}(t) \) holds for a zero–mean random operator accounting for fluctuating contributions. Eq. (87) has been dealt with from a stochastic point of view in [24–26] (see also references therein), where the authors showed that in an adequate approach it is possible to go beyond the Markovian description under a completely positive evolution determined by a quantum dynamical semigroup. Nevertheless, we are mainly interested in nonlinear Schrödinger equations, provided that the expectation values associated with the latter satisfy the classical frictional equations (Eq. (1) with \( F = 0 \)) , a feature that is not guaranteed to be fulfilled by the former. Concretely, our equation of interest is the following (in a unit system in which \( \hbar = m = 1 \))

\[
i \partial_t \psi = \left\{ -\frac{\Delta_x}{2} + V \right\} \psi + \lambda V_i \psi + \Theta[n, J] \psi, \tag{88}
\]

\[
V_i(\psi) = S - \langle S \rangle, \tag{89}
\]
where $V = V(t, x)$ is an external potential and $\lambda V$ is the nonlinear frictional term, which is proportional to $S - \langle S \rangle$. Here, $S$ denotes any of the infinitely many arguments associated with $\psi$ satisfying the Madelung relation

$$\psi(t, x) = \sqrt{n(t, x)} \exp\{iS(t, x)\}, \quad (90)$$

and $\langle S \rangle(t) := \int_{\Omega} S(t, x)n(t, x)\, dx$ describes the quantum expectation value of $S$. Finally, $\Theta$ stands for an arbitrary selfconsistent interaction (Poissonian or thermodynamical, for example) depending upon the wavefunction through its associated observables, say the probability and current densities.

The system (88)–(89), first derived by M. D. Kostin in [93], turns out to preserve the total probability and dissipate energy (see also [94]). Likewise, the technique of stochastic quantization introduced by E. Nelson in [114] can be applied to obtain Eq. (88)–(89) from a hydrodynamic point of view (see [122, 136]). At this point, it should be highlighted that the analysis of (88)–(89) cannot be compared to that of Eq. (87) because of the deep differences underlying these two Langevin approaches for the wavefunction. Indeed, van Kampen’s scheme is stochastic in essence, whereas Kostin’s equation is interpreted in a deterministic manner. Anyhow, the nonlinear character exhibited by Eq. (88)–(89) (which sets the most important separation with respect to other approaches) requires, at variance with Eq. (87), the specific use of genuine mathematical tools (nonlinear Sobolev estimates, fixed–point arguments based on the concept of mild solution, elliptic regularization, trace properties on the boundary, etc.) for the treatment of nonlinear potential operators, say $V_i$.

In what concerns the mathematical analysis of (88)–(89), it is customary that the expectation value of $S$, which in principle does not contain any physical information, be removed from $V_i$ by means of the change of phase $\psi \mapsto \psi\exp\{i\nu(t)\}$, with $\nu(t) = - \int_{0}^{t} e^{-\int_{t}^{\tau} S(\tau)\, d\tau} (91)$ This is the way in which the simplest form of Kostin’s equation, given by

$$i\partial_t \psi = \left(-\frac{1}{2} \Delta + V \right)\psi + \lambda S \psi, \quad (91)$$

comes up. Nevertheless, this simplification seems to be not fully satisfactory. The main difficulty in writing Eq. (91) is, as pointed out before, the ambiguity induced by the multivalued nature of the function $S$. The questions that immediately arise are: What is the mathematical sense of Eq. (91)? How its analysis can be made rigorous? What is the appropriate functional setting in order that all quantities are well defined? In [88] this ambiguity was overcome, in the framework of a smooth and bounded domain $\Omega$, by solving a proper boundary value problem for $S$ subject to a fixed datum $S_B$ on $\partial\Omega$. Under this interpretation of the argument, Eq. (91) is shown to have a unique solution ‘separated from zero’. Nonetheless, this criterion still proves somehow unsatisfactory given that Eq. (91) turns $S_B$–dependent. On the other hand, along with the difficulties in the rigorous treatment of Eq. (91) already discussed, another inconvenience makes this model inappropriate from a physical viewpoint. It does consist in the fact that if $\psi$ is a solution of Eq. (91), there does not exist $0 \neq \nu \in \mathbb{R}$ such that $\phi = e^{i\nu}\psi$ be also a solution. This is in contradiction with the standard quantum–mechanical invariance by changes of global phase (i.e. no matter the chosen $S$ be, the physics of the system remains unaltered).

In [103] all of these issues were elucidated. To that aim, we took advantage of the following simple property: Given a wavefunction $\psi$ such that both $S$ and $S'$ satisfy the Madelung relation (90), then $S - \langle S \rangle = S' - \langle S' \rangle$ and thus the mapping $\psi \mapsto V_i$ is uniquely determined. Furthermore, the formulation (88) (with $V_i$ instead of just $S$) does enjoy the invariance by change of phase since $V_i(\psi) = V_i(\psi e^{i\nu})$ for all $\nu \in \mathbb{R}$. The mathematical approach to $V_i$ is based on the Madelung decomposition of $\psi$ (cf. Eq. (90)), which implies $\nabla_x S = \text{Im}\{\nabla_x \psi/\psi\}$. This equality along with Eq. (89) lead us to $V_i$ through solving the following (wellposed) problem:

$$\nabla_x V_i = \text{Im}\{\nabla_x \psi/\psi\}, \quad \langle V_i \rangle = 0 \quad (92)$$

(see [103] and references therein). Here, $X^T$ reveals as a suitable functional space. In addition to hypotheses (H1)–(H3) above, an assumption on the operator $\Theta$ must be also imposed for our purposes:

$$(H4) \quad \Theta : H^2_0(\Omega) \to H^2(\Omega) \text{ is locally Lipschitz continuous and there exists} \quad \Theta_B \in H^2(\partial\Omega) \text{ such that } \Theta(\psi)|_{\partial\Omega} = \Theta_B \text{ when } \psi|_{\partial\Omega} = \psi_B.$$
Some authors have analyzed different aspects of the (nonlinear) Schrödinger–Langevin equation, mainly from a formal point of view in the perspective of PDEs. We remark [22], in which the global existence of Gaussian solutions in the harmonic oscillator framework was established for the particular case of Kostin's equation, as well as the absence of \( L^2 \) solitary waves in the free–particle regime; [97], where the semiclassical limit of the Kostin–Poisson system is performed in the whole space; [130], where the stability of stationary solutions was studied; and [41], where it is shown how the ground state is reached. Also, in [4] and [88] the Schrödinger–Langevin equation is dealt with as an auxiliary problem in analyzing the wellposedness of the associated quantum hydrodynamic system. In spite of that, there still remained an important lack of mathematical sense in the very core of the formulation of the problem, finally clarified through Theorem 11.

Regarding the study of Eq. (88)–(89) in whole space, there are still a lot of questions left unresolved. Unfortunately, the desirable belonging of its solutions to \( L^2(\mathbb{R}^d) \) is not compatible anymore with working in spaces of wavefunctions separated from zero, hence vacuum density regions cannot be avoided in principle. Instead, the wellposedness analysis of (88)–(89) should emerge from an adequate definition of \( V_2(\psi) \) in domains where the wavefunction may vanish. Indeed, a more detailed knowledge of the problem shows that as long as \( z = \text{Im}(\nabla\psi/\psi) \) belongs to \( L^1_{\text{loc}}(\mathbb{R}^d) \), \( V_1 \) can be constructed (at least locally) as a scalar potential of \( z \) with vanishing expectation value. This definition, however, is not completely satisfactory given that the choice of \( V_1 \) instead of \( S \) does not fully solve the problem of the multivalued character of the friction term. It can be easily found, for instance, two different choices of \( V_1 \) for a given wavefunction \( \psi \) such that \( \text{supp}(\psi) \) is not connected. Therefore, another condition on \( V_1 \) is required to carry out this approach successfully for wavefunctions with vacuum regions.

A different strategy consists of searching for solutions to Eqs. (88), (92) with strictly positive modulus. In [125] it is suggested that the profile \( \psi = \exp\{v\} \), with \( v \in L^p_{\text{loc}}(\mathbb{R}^d) \) and \( \nabla v \) in \( L^p(\mathbb{R}^d) \) for certain \( p > d \), allows to find a nonvanishing \( L^2 \)-solution to this problem. Nevertheless, imposing this ansatz to our equation drives to a new system of coupled derivative Schrödinger equations whose analysis turns out to be even more complicated.

Another idea consists of finding global solutions to Eqs. (88), (92) around a strictly positive profile. In this situation we are called to take advantage of the dissipative character of the system and of the fact that \( \langle V_1 \rangle = 0 \), and work in a proper neighborhood of the ground state (that is, the eigenfunction of the Hamiltonian associated with the smallest eigenvalue) for a fixed external potential \( V \).

Finally, a quantum diffusion term (as those in Eq. (53)) might be added to Eq. (88) as a viscosity contribution in order to get a continuity equation of Fokker–Planck type. Such an equation would allow, on one hand, to apply a minimum principle that guarantees the strict positivity of \( |\psi| \) and does provide the proper definition of \( V_1 \), but on the other hand would oblige us to work with derivative Schrödinger equations. The introduction of an appropriate gauge transformation in the spirit of [115] might be the key point to avoid this difficulty. Anyway, we postpone these approaches to future work.

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**Appendix: Connecting models**

We end up with a schematic representation of the different levels of description of an open quantum system reported on throughout this paper (phase space in the Wigner function approach, hydrodynamical and Schrödinger type), as well as of their mutual interconnections. The captions and formulae accompanying the different (one or two–side) arrows indicate the (forward or reversible) way to achieve one stage from the others. The same notational conventions introduced within the text are followed here, up to the projection operator \( P_\psi := |\psi\rangle\langle\psi| \) over the pure state \( \psi \) (not defined previously), that allows to directly get a density matrix approach starting from the wavefunction representation and vice versa.
On the derivation and mathematical analysis

Density operator $R$

Von Neumann equation

Wigner function $W$

Wigner equation

$n := \int W d\xi$

$n u := \int \xi W d\xi$

Local density $n$ and current density $nu$

Nonlocal drift–diffusion system

Density $n$ and mean velocity $u$

Nonlocal hydrodynamic system

Closure relations for the moment system:
zero temperature or isothermal model, among others

$R = P_\psi$

Local density $n$ and current density $nu$

Local drift–diffusion system

Local density $n$ and mean velocity $u$

Local hydrodynamic system

$u = \frac{1}{m} \nabla_x S$

Wavefunction $\psi$

Schrödinger equation

$\psi := \sqrt{n} \exp \left\{ \frac{i}{\alpha} S \right\}$

$\psi \neq 0$

Density $n$ and phase $S$

Hydrodynamic system (flux potential)

$n := |\psi|^2$, $\nabla_x S = \text{Im} \left( \frac{\nabla_x \psi}{\psi} \right)$, $\psi \neq 0$
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