CORRIGENDUM: ON THE ABEL DIFFERENTIAL EQUATIONS OF THIRD KIND

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Abstract. In this paper, using the Poincaré compactification technique we classify the topological phase portraits of a special kind of quadratic differential system, the Abel quadratic equations of third kind. In [1] where such investigation was presented for the first time some phase portraits were not correct and some were missing. Here we provide the complete list of non equivalent phase portraits that the Abel quadratic equations of third kind can exhibit and the bifurcation diagram of a 3-parametric subfamily of it.

1. Statement of the main results. For more details about the Abel quadratic differential polynomial of third kind see [1]. The Abel quadratic differential polynomial systems of third kind are of the form

\[
\begin{align*}
\dot{x} &= y^2, \\
\dot{y} &= a_0 + a_1 x + a_2 x^2 + (b_0 + b_1 x)y,
\end{align*}
\] (1)

where \(a_0, a_1, a_2, b_0, b_1, b_2\) are real parameters with \(a_0^2 + a_1^2 + a_2^2 \neq 0\).

The main result of the paper is the following one.

Theorem 1.1. The Abel quadratic polynomial differential equations (1) after a linear change of variables and a rescaling of its independent variable can be written as one of the following systems

\[
\begin{align*}
\dot{x} &= y^2, & \dot{y} &= k_0 + k_1 y + x^2 + k_2 xy & k_0, k_1 \in \mathbb{R} \text{ and } k_2 \in \{0, 1\}, & (i) \\
\dot{x} &= y^2, & \dot{y} &= x + k_1 y + k_2 xy & k_1 \in \mathbb{R} \text{ and } k_2 \in \{-1, 0, 1\}, & (ii) \\
\dot{x} &= y^2, & \dot{y} &= 1 + k_2 xy & k_2 \in \{-1, 1\}, & (iii) \\
\dot{x} &= y^2, & \dot{y} &= 1 + k_1 y & k_1 \in \{0, 1\}. & (iv)
\end{align*}
\]

Moreover,

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(a) system (i) is a three-parametric system, it admits 40 non-equivalent phase portraits on the Poincaré disk, see Figures 1 and 2. The bifurcation diagram of system (i) is given in Figure 3. More precisely, from the 40 non-equivalent phase portraits of this family 3 phase portraits have not finite singular points; 6 phase portraits admit one finite singular point and 31 have two finite singular points; 3 phase portraits have a saddle-node as the unique finite singular point; 5 phase portraits admit one cusp and one saddle-node and 1 phase portrait admits two cusps as finite singular points; 24 phase portraits admit a pair of saddle-nodes as finite singular points; 10 phase portraits admit a separatrix connection, namely $S_4, S_5, S_6, S_{18}, L_2, L_6$ have a separatrix connection between the two finite saddle-nodes and $S_9, S_{10}, L_4, L_5$ have a separatrix connection between one of the finite saddle-nodes and an infinite singular point (saddle or saddle-node).

(b) system (ii) is a two-parametric system with a unique finite singular point (a saddle-node or a cusp). This system admits 6 non-equivalent phase portraits on the Poincaré disk, given in Figure 4. The phase portraits $V_{13}$ and $V_{15}$ of system (ii) are topologically equivalent to $S_5$ and $S_2$ of system (i), respectively. The phase portrait $S_{21}$ has a separatrix connection between the finite saddle-node and the infinite saddle and,

- when $k_1 \neq 0$ and $k_2 \in \{-1, 0\}$ the phase portrait is $V_{13}$;
- when $k_1 = 0$ and $k_2 \in \{-1, 0\}$ the phase portrait is $S_{19}$;
- when $k_1 = 0$ and $k_2 = 1$ the phase portrait is $S_{20}$;
- when $k_1 \neq 0$ and $k_2 = 1$ the phase portrait is $V_{14}$ for $0 < |k_1| < 0.6$, $S_{21}$ for $k_1 \approx 0.6$ and $S_{15}$ for $|k_1| > 0.6$.

(c) system (iii) is a one-parametric system, it admits two non-equivalent phase portraits, which are topologically equivalent to $V_1$ and $V_2$ of Figure 1. More precisely,

- when $k_2 = -1$ the phase portrait is $V_1$ of Figure 1;
- when $k_2 = 1$ the phase portrait is $V_2$ of Figure 1.

(d) system (iv) admits a unique phase portrait topologically equivalent to $V_1$ of Figure 1.

The proof of Theorem 1.1 follows from Proposition 1 of [1] and the study presented in Sections 4-6 of [1]. More precisely, in the proof of Theorem 1.2 of [1] we presented the local type of the singular points (finite and infinite) and the possible local bifurcations and invariant lines. However, when we used this information to draw the possible phase portraits we made mistakes, we lost some phase portraits and drawn incorrectly some of them. These mistakes implied that the classification made in [1] was incomplete and incorrect. In this corrigendum all the phase portraits are corrected and well drawn. In particular we list the information about the corrections done in the phase portraits in [1] compared with the corrigendum version.

1. The phase portraits 17 and 18 in Figure 1 of [1] were topologically equivalent;
2. In [1] we distinguished orbits from separatrices by drawing them with different width, however we misdrew some of them. For example, in phase portrait 11 there is a connection of separatrices but we kept a separatrix from the phase portrait 10. This produced that the count about separatrices (S) and canonical regions (R) was wrong in [1]; For phase portrait 11 we have $S = 11$ and $R = 4$. These kind of mistakes happened in other phase portraits.
Figure 1. Global phase portraits in the Poincaré disk of system (i).
3. Phase portraits 6 and 31 in [1] were impossible;
4. Phase portrait 36 was not well drawn.
5. In Theorem 1.2, statements 19)–21) and 22)–26), the condition the condition
   \( k_1 + \sqrt{-k_0k_2} > 0 \) was irrelevant. Also for the cases 27)–34) with \( k_2 > 3/4^{1/3} \).

Concerning the details about the main result in [1] (Theorem 1.2), we re-state
the second part of Theorem 1.2 as the items (a)-(d) of Theorem (1.1) in this cor-
rigendum. Although the main result is the same, taking into account the possible
normal forms that a Abel quadratic system of third kind could have and their non-
equivalent phase portraits in the Poincaré disc, in this version, we present some
additional details that, in our point of view, help the reader to understand the
investigation described here.

When we were working on this corrigendum, specially on the correction of
the phase portraits of system (i) we decided to present the bifurcation diagrams of this
system, that is a 3-parametric system, for which the investigation can be reduced to
the study of two families with two parameters. These bifurcation diagrams are not

Figure 2. Global phase portraits in the Poincaré disk of system (i).
Figure 3. Bifurcation diagram of system (i) for $k_1 = 1$ and $k_1 = 0$. The phase portrait $P_2$ is topologically equivalent to $L_8$. In this diagram, the phase portraits in the surfaces $S_i$, for $i = 4, 5, 6, 9, 10, 18$ and in the curves $L_j$, for $j = 2, 4, 5, 6$ admit a separatrix connection and although such surfaces and curves do not need to be algebraic the surfaces $S_9$ and $S_{10}$ are contained in one component of the algebraic surface given by $-64k_0^3 - k_0^4 + 16k_0^3k_1^2k_2 - 3k_0k_1^4k_2^2 + 16k_0^3k_1^2k_2 - 3k_0^2k_1^2k_2^2 - k_0^3k_2^2 = 0$. 
Figure 4. Global phase portraits in the Poincaré disk of systems (ii). The phase portraits $V_{13}$ and $V_{15}$ of system (ii) are topologically equivalent to the phase portraits $S_3$ and $S_2$ of system (i), respectively. Here we remark that in the phase portraits $S_3$ and $S_2$ of system (i), such system has two complex points and a saddle–node as finite singular points but in system (ii) the phase portraits $V_{13}$ and $V_{15}$ have a saddle–node as the unique finite singular point. The phase portrait $S_{21}$ has a separatrix connection between the finite saddle-node and the infinite saddle.

presented in [1] and they correspond to Figure 3 in this corrigendum. These bifurcation diagrams make clear the classification of the phase portraits in investigation, they help the reader to understand the changes that occurs in the phase portraits of the studied system and avoid missing cases as happened in [1]. The bifurcation diagram in 2 dimensions of case (i) also helps to understand the bifurcations and the coherence of the phase portraits. Moreover using the bifurcation diagram we have had been able to detect the non algebraic bifurcations not presented in [1].

It is important to observe that although the number of non-equivalent phase portraits for the Abel system of third order in this corrigendum and in [1] are the same, in [1], as we have mentioned before, there were mistakes. Some phase portraits in this corrigendum are not in the previous version since they correspond to the missing phase portraits that can be found easily using the bifurcation diagram now available in this version.

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