POPULATION DYNAMICAL BEHAVIOR OF A TWO-PREDATOR ONE-PREY STOCHASTIC MODEL WITH TIME DELAY

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Abstract. In this paper, the convergence of the distributions of the solutions (CDS) of a stochastic two-predator one-prey model with time delay is considered. Some traditional methods that are used to study the CDS of stochastic population models without delay can not be applied to investigate the CDS of stochastic population models with delay. In this paper, we use an asymptotic approach to study the problem. By taking advantage of this approach, we show that under some simple conditions, there exist three numbers \( \rho_1 > \rho_2 > \rho_3 \), which are represented by the coefficients of the model, closely related to the CDS of our model. We prove that if \( \rho_1 < 1 \), then \( \lim_{t \to +\infty} N_i(t) = 0 \) almost surely, \( i = 1, 2, 3 \); If \( \rho_i > 1 > \rho_{i+1} \), \( i = 1, 2 \), then \( \lim_{t \to +\infty} N_j(t) = 0 \) almost surely, \( j = i + 1, ..., 3 \), and the distributions of \( (N_1(t), ..., N_3(t))^T \) converge to a unique ergodic invariant distribution (UEID); If \( \rho_3 > 1 \), then the distributions of \( (N_1(t), N_2(t), N_3(t))^T \) converge to a UEID. We also discuss the effects of stochastic noises on the CDS and introduce several numerical examples to illustrate the theoretical results.

1. Introduction. As one of the most important dynamical properties in population dynamics, the stability of equilibria has attracted rapidly increasing attention \([35]\). Since the nature is full of uncertainty and random phenomena can have critical effects on population dynamics, it is of great significance to study the stability of stochastic population models. What is more, how to reveal the effects of random perturbations on the stability of stochastic population models becomes a crucial problem \([35]\). However, most stochastic population models do not allow for traditional positive equilibrium state, that is to say, the solutions of stochastic

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population models cannot tend to some positive fixed point. As a result, several authors began to consider the convergence of the distributions of the solutions (CDS) of stochastic population models (see, e.g., [9, 17, 33, 44, 45, 50, 52, 53, 55]). By solving the corresponding Fokker-Planck equation, Pasquali [41] investigated the CDS of stochastic logistic equations. Moreover, on the basis of the Markov semigroup theory, CDS of stochastic predator-prey systems were thought over in [9, 44, 45, 52]. In addition, according to the Lyapunov function methods (see e.g. [12]), Ji, Jiang and Shi [17] explored the CDS of a stochastic predator-prey system with modified Leslie-Gower and Holling-type II schemes; Mao [33] studied the CDS of a stochastic Lotka-Volterra facultative mutualism system; Zhao et al. [53] considered the CDS of a stochastic competitive model in a polluted environment.

A great deal of researchers have devoted themselves to single-species or two species stochastic models without time delay. However, on the one hand, it has been recognized that single-species or two species ecological models can’t describe the natural phenomena accurately and many vital behaviors can only be exhibited by systems with three or more species ([13, 40, 43]). On the other hand, as is well known, time delay should not be neglected because the processes of the reproductions of species are not instantaneous and time delay can reflect natural phenomena more authentically ([11, 20, 23]). For instance, when a group of sheep eat some grass, the number of their population will not increase at once owing to the fact that the processes of metabolism, growth and breed all need plenty of time which cannot be neglected. Therefore, it is important and interesting to investigate the CDS of stochastic three-species models with time delay.

As is well known, when it comes to the CDS of stochastic population models, using the explicit solution of the corresponding Fokker-Planck equation is one of the traditional methods (see e.g., [41]). However, for almost all stochastic population models with time delay, it is impossible to obtain the explicit solution of the corresponding Fokker-Planck equation. Besides that, another traditional method is to use the Markov semigroup theory, see e.g. [7, 9, 44, 45, 52]. However, this method cannot be applied to deal with delay population models owing to the fact that the Markov semigroup theory needs to have some standard measures. The phase space of ODEs is a subset of $\mathbb{R}^n$ which has a standard measure—the Lebesgue measure. The phase space of DDEs is some space of functions where it is tough to decide what is a standard measure (communicated with Professor Ryszard Rudnick). The third traditional method to study the CDS of stochastic population models is to use the Lyapunov function method (see, e.g., [17, 33, 55]), but allow for the fact that the Lyapunov function method requires that the solution of a stochastic model must be a Markov process while the solution of a delay stochastic model is not a Markov process. And as a result, this approach cannot be used to study delay stochastic population models either.

In this paper, consider the fact that it is a generally natural phenomenon that several predators compete for a prey, we formulate a two-predator one-prey stochastic population model with time delay in Section 2. In Section 3, we analyze the CDS of the model by using an asymptotic approach ([28]) and discuss the results and their implications from the viewpoint of biology. In Section 4, we present some numerical simulations to illustrate the theoretical findings. In Section 5, we give some concluding remarks, and show that the approach can be applied to study the CDS of other stochastic population models with/without delay. As an example, we establish the sufficient conditions for the CDS of a three-species stochastic delay
mutualism system. We also propose some interesting problems to be explored in the future.

2. Model derivations and main results. At the first place, consider the following Lotka-Volterra delay model with two competing predators and one prey, which has been widely investigated in [1, 10, 13, 19, 36, 47]:

\[
\begin{align*}
\frac{dN_1(t)}{dt} &= N_1(t) \left[ r_1 - a_{11}N_1(t) - a_{12}N_2(t - \tau_{12}) - a_{13}N_3(t - \tau_{13}) \right] dt, \\
\frac{dN_2(t)}{dt} &= N_2(t) \left[ r_2 - a_{21}N_1(t - \tau_{21}) - a_{22}N_2(t) - a_{23}N_3(t - \tau_{23}) \right] dt, \\
\frac{dN_3(t)}{dt} &= N_3(t) \left[ r_3 - a_{31}N_1(t - \tau_{31}) - a_{32}N_2(t - \tau_{32}) - a_{33}N_3(t) \right] dt,
\end{align*}
\]

with initial data

\[
N_i(\theta) = \phi_i(\theta), \quad \theta \in [-\bar{\tau}, 0], \quad \bar{\tau} = \max \{\tau_{ij}\},
\]

where \(N_i(t)\) is the population size of the \(i\)-th species at time \(t\) \((i = 1, 2, 3)\). \(r_1 > 0\) stands for the growth rate of specie 1, \(r_2 < 0\) and \(r_3 < 0\) are the death rates of specie 2 and 3 respectively. \(a_{11} > 0\), \(a_{22} > 0\), \(a_{33} > 0\) mean the inside struggle of the specie \(i\), \(i = 1, 2, 3\). \(a_{12} > 0\) and \(a_{13} > 0\) represent the capture rates. \(a_{21}\) and \(a_{31}\) are negative constants representing the growth from food. \(a_{23} > 0\) and \(a_{32} > 0\) signify the competitions between specie 2 and 3. \(\tau_{ij} \geq 0\) denotes the delay, \(\phi_i(\theta) > 0\) is a continuous function on \([-\bar{\tau}, 0]\) \((i, j = 1, 2, 3)\).

Based on the above model, let us allow for the environmental perturbations in addition. In most cases, the environmental perturbations can be modeled by a colored noise (e.g., [39]). Several researchers (see, e.g., [39]) have claimed that the colored noise could be approximated by a Gaussian white noise if it was not strongly correlated, and the approximation would be quite suitable.

Several approaches have been put forward to introduce a Gaussian white noise into deterministic population models. Here, enlightened by the approach used in [4, 5, 15, 17, 18, 21, 25, 27, 29, 30, 31, 32, 33, 41, 45, 52, 53, 54, 55], we consider the parameter perturbation and suppose that the environmental perturbations mainly affect the growth rates of the populations, with \(r_i \rightarrow r_i + \sigma_i W_i(t)\), where \(\{W_i(t)\}_{t \geq 0} (i = 1, 2, 3)\) stand for standard independent Brownian motions which are defined on a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), \(\sigma_i^2\) is the intensity of the white noise. Then we obtain the following stochastic model with delay:

\[
\begin{align*}
\frac{dN_1(t)}{dt} &= N_1(t) \left[ r_1 - a_{11}N_1(t) - a_{12}N_2(t - \tau_{12}) - a_{13}N_3(t - \tau_{13}) \right] dt + \sigma_1 N_1(t) dW_1(t), \\
\frac{dN_2(t)}{dt} &= N_2(t) \left[ r_2 - a_{21}N_1(t - \tau_{21}) - a_{22}N_2(t) - a_{23}N_3(t - \tau_{23}) \right] dt + \sigma_2 N_2(t) dW_2(t), \\
\frac{dN_3(t)}{dt} &= N_3(t) \left[ r_3 - a_{31}N_1(t - \tau_{31}) - a_{32}N_2(t - \tau_{32}) - a_{33}N_3(t) \right] dt + \sigma_3 N_3(t) dW_3(t),
\end{align*}
\]

with initial data \([1].\) Model (2) is approximate “to age-structured populations, with populations growth taking place in discrete time steps” ([4, 5]), and as a result, we utilize the Itô calculus rather than the Stratonovich calculus.
To make it more convenient, we introduce the following notations. Let \( R^3_+ = \{ a = (a_1, a_2, a_3) \in R^3_+ | a_i > 0, i = 1, 2, 3 \} \) and \( b_i = r_i - \frac{\sigma^2_i}{2} \). Let \( C([-\bar{\tau}, 0], R^3_+) \) stand for all the continuous functions from \([-\bar{\tau}, 0]\) to \( R^3_+ \). Denote \( \alpha_1 = (a_{11}, a_{21}, a_{31})^T \), \( \alpha_2 = (a_{12}, a_{22}, a_{32})^T \), \( \alpha_3 = (a_{13}, a_{23}, a_{33})^T \) and define \( \beta = (r_1, r_2, r_3)^T \), \( \gamma = \frac{1}{2} (\sigma^2_1, \sigma^2_2, \sigma^2_3)^T \). In addition, we define

\[
A = \det(\alpha_1, \alpha_2, \alpha_3), \quad R = \det(\alpha_1, \beta, \gamma), \quad A_1 = \det(\beta, \alpha_2, \alpha_3), \quad \tilde{A}_1 = \det(\gamma, \alpha_2, \alpha_3),
\]
\[
A_2 = \det(\alpha_1, \beta, \gamma), \quad \tilde{A}_2 = \det(\alpha_1, \gamma, \alpha_3), \quad A_3 = \det(\alpha_1, \alpha_2, \beta), \quad \tilde{A}_3 = \det(\alpha_1, \alpha_2, \gamma),
\]
\[
\Delta_1 = a_{22}r_1 - a_{12}r_2, \quad \Delta_2 = a_{11}r_2 - a_{21}r_1, \quad \Delta_3 = a_{11}r_3 - a_{31}r_1,
\]
\[
\tilde{\Delta}_1 = a_{22}\sigma^2_2/2 - a_{12}\sigma^2_2/2, \quad \tilde{\Delta}_2 = a_{11}\sigma^2_2/2 - a_{21}\sigma^2_2/2, \quad \tilde{\Delta}_3 = a_{11}\sigma^2_3/2 - a_{31}\sigma^2_3/2,
\]
\[
\langle f(t) \rangle = t^{-1} \int_0^t f(s)ds, \quad (f)^* = \limsup_{t \to +\infty} t^{-1} \int_0^t f(s)ds, \quad (f)_* = \liminf_{t \to +\infty} t^{-1} \int_0^t f(s)ds.
\]

Before we state our main results, let us introduce some assumptions.

**Assumption 1.** \( A > 0 \), \( A_i > 0 \), \( i = 1, 2, 3 \), which mean that model (2) has a positive equilibrium state if there is no stochastic perturbations. \( \Delta_j > 0, \quad j = 2, 3 \), which mean that both species 1 and species \( j \) can coexist if both stochastic perturbations and the other predator are absent.

**Assumption 2.** \( a_{11} > a_{12} + a_{13} \), \( a_{22} > -a_{21} + a_{23} \), \( a_{33} > -a_{31} + a_{32} \). That is to say, the intra-specific competition rates are stronger than the interaction among different species (see, e.g., [3]).

The following assumption is a technical assumption to make the proof work.

**Assumption 3.** \( A_{12} < 0, \quad A_{13} < 0 \), where \( A_{ij} \) is the symbol of the complement minor of \( a_{ij} \) in the determinant \( A \), \( i, j = 1, 2, 3 \).

We also suppose that \( R > 0 \). By (30) below, we have \( \Delta_2/\tilde{\Delta}_2 > \Delta_3/\tilde{\Delta}_3 \). Note that \( \Delta_i/\tilde{\Delta}_i \) measures the persistence ability of species \( i \) when its competitor is absent, \( i = 2, 3 \). Hence \( R > 0 \) means that the persistence ability of species 2 is greater than that of species 3.

**Theorem 2.1.** For model (2), let Assumptions 1, 3 hold and \( R > 0 \). Set \( \rho_1 = 2r_1/\sigma^2_1, \quad \rho_2 = \Delta_2/\Delta_2, \quad \rho_3 = A_3/A_3 \), then \( \rho_1 > \rho_2 > \rho_3 \) and moreover,

(i) If \( \rho_1 < 1 \), then species \( i, \quad i = 1, 2, 3 \), go to extinction almost surely (a.s.), i.e.,
\[
\lim_{t \to +\infty} N_i(t) = 0, \quad i = 1, 2, 3, \quad a.s.
\]

(ii) If \( \rho_1 > 1 > \rho_2 \), then species 2 and 3 go to extinction a.s., and the distribution of \( N_1(t) \) converges weakly to a unique invariant distribution \( \nu_1 \) which is ergodic:
\[
\lim_{t \to +\infty} \langle N_1(t) \rangle = \int_{R_+} z_1\nu_1(dz_1) = \frac{b_1}{a_{11}}, \quad a.s.
\]

(iii) If \( \rho_2 > 1 > \rho_3 \), then species 3 goes to extinction, and the distribution of \( (N_1(t), N_2(t))^T \) converges weakly to a unique invariant distribution \( \nu_2 \) which is ergodic:
\[
\lim_{t \to +\infty} \langle N_i(t) \rangle = \int_{R_2^2} z_i\nu_2(dz_1, dz_2) = \frac{\Delta_i - \tilde{\Delta}_i}{A_{33}}, \quad i = 1, 2, \quad a.s.
\]
Remark 1. It is useful to point out that if $R < 0$, under Assumptions [13] some similar results can be established by the symmetry of $N_2$ and $N_3$.

Remark 2. Now let us see the biological relevance of the conditions of Theorem 2.1

- $\rho_1 < 1$ means $r_1 < \frac{\sigma_1^2}{2}$. That is to say the intensity of the stochastic noise of the prey exceeds the resist ability of the prey, hence the prey goes to extinction.
- $\rho_1 > 1 > \rho_2$ means $r_1 > \frac{\sigma_1^2}{2}$, $\frac{\sigma_2^2}{2} > r_2 + \left[- \frac{a_{12}}{a_{11}}(r_1 - \frac{\sigma_1^2}{2})\right]$, and $\frac{\sigma_2^2}{2} > r_3 + \left[- \frac{a_{23}}{a_{31}}(r_1 - \frac{\sigma_1^2}{2})\right]$ (because $\Delta_2/\overline{\Delta}_2 > \Delta_3/\overline{\Delta}_3$, see (30) below). The terms $\left[- \frac{a_{23}}{a_{31}}(r_1 - \frac{\sigma_1^2}{2})\right]$ and $\left[- \frac{a_{12}}{a_{11}}(r_1 - \frac{\sigma_1^2}{2})\right]$ measure the “help” from the prey (i.e., the capture). That is to say, the intensity of the stochastic noise of the prey is small and it could resist, but the intensities of the stochastic noises of the predators are large and the “help” from the prey is invalid. Therefore the prey is persistent and the predators go to extinction.
- $\rho_2 > 1 > \rho_3$ means $r_1 > \frac{\sigma_1^2}{2}$, $\frac{\sigma_2^2}{2} < r_2 - \frac{a_{23}}{a_{31}}(r_1 - \frac{\sigma_1^2}{2})$ and

\[
\frac{\sigma_3^2}{2} + \frac{a_{32}}{A_{33}}(\Delta_2 - \overline{\Delta}_2) > r_3 - \frac{a_{31}}{A_{33}}(\Delta_1 - \overline{\Delta}_1). \tag{6}
\]

The second term in the left side of (6) measures the competition form the species 2, and the second term in the right side measures the “help” from the prey. That is to say, for species 2, the intensity of the stochastic noise is small and it could be persistent with the “help” of the prey. However, for species 3, the intensity of the stochastic noise and the competition form species 2 are large, and the “help” from the prey is invalid.

- $\rho_3 > 1$ means $r_1 > \frac{\sigma_1^2}{2}$, $\frac{\sigma_2^2}{2} < r_2 - \frac{a_{23}}{a_{31}}(r_1 - \frac{\sigma_1^2}{2})$ and

\[
\frac{\sigma_3^2}{2} + \frac{a_{32}}{A_{33}}(\Delta_2 - \overline{\Delta}_2) < r_3 - \frac{a_{31}}{A_{33}}(\Delta_1 - \overline{\Delta}_1).
\]

That is to say, for species 3, the intensity of the stochastic noise and the competition from species 2 are small, it could be persistent with the “help” of prey.

Remark 3. The results in Theorem 2.1 have some useful and interesting biological interpretations.

(a) Theorem 2.1 shows that the extinction and convergence of distributions (which means persistence) of $N_3$ only rely on $\rho_3$. Clearly,

\[
d(\rho_3)/d(\sigma_1^2) < 0, \quad d(\rho_3)/d(\sigma_2^2) > 0, \quad d(\rho_3)/d(\sigma_3^2) < 0,
\]

which mean that when $\sigma_1^2$ or $\sigma_2^2$ increases, species 3 tends to extinction but the stochastic noise of species 2 is favorable for the persistence of species 3. These characteristics are reasonable from the biological point of view. When the noise intensity $\sigma_1^2$ increases, the species 1 tends to extinction. Species 3 will be restricted because the food of species 3 become less which gives rise to the abatement of species 3. As a result, the noise intensity $\sigma_2^2$ is unfavorable for
the reproduction of species 3. On the other hand, the species 2 and 3 compete intensely with each other, and note that the noise intensity $\sigma_2^2$ can promote the extinction of $N_2$, which means that species 3 will get more food. Thus the noise intensity $\sigma_3^2$ renews the reproduction of species 3. The effects of $\sigma_1^2$ on the extinction of species 2 can be obtained similar and hence are omitted.

(b) From Theorem 2.1 one can find that $\rho_1$ decides the extinction or not of species 1. It is not difficult to see that

$$d(\rho_1)/d(\sigma_1^2) < 0, \quad d(\rho_1)/d(\sigma_2^2) = 0, \quad d(\rho_1)/d(\sigma_3^2) = 0,$$

which signify that the increment of $\sigma_1^2$ leads to the extinction of $N_1$ while $\sigma_2^2$ and $\sigma_3^2$ have nothing to do with the extinction of species 1. That is to say, only by reducing the noise intensity $\sigma_1^2$ rather than $\sigma_2^2$ or $\sigma_3^2$ can we conserve the prey population under this circumstance, which is a meaningful and useful conclusion.

3. **Proofs.** Before we prove Theorem 2.1 let us recall some definitions.

**Definition 3.1.** A stochastic process, $x(t)$, which represents a population size at time $t$, is said to be stable in time average if $\lim_{t \to +\infty} \langle x(t) \rangle$ is a positive constant, $a.s.$

**Definition 3.2.** Let $N(t) = (N_1(t), N_2(t), N_3(t))^T$ and $M(t) = (M_1(t), M_2(t), M_3(t))^T$ be two arbitrary solutions of model (2) with initial values $N(\theta) \in C([-\tau, 0], R^3_+)$ and $M(\theta) \in C([-\bar{\tau}, 0], R^3_+)$ respectively. If for every $i = 1, 2, 3$, $\lim_{t \to +\infty} E[N_i(t) - M_i(t)] = 0$ $a.s.$, then we say model (2) is globally attractive (21).

**Definition 3.3.** Model (2) is said to be stable in distribution ($\text{34}$) if there exists a unique probability measure $\nu$ such that for any $N(\theta) \in C([-\tau, 0], R^3_+)$, the transition probability $p(t, \phi, \cdot)$ of $N(t)$ converges weakly to $\nu$ as $t \to +\infty$.

The proof of Theorem 2.1 is rather long, so we divide it into three parts.

### 3.1. Proof of extinction and stability in time average.

**Lemma 3.4.** For any given initial condition $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C([-\bar{\tau}, 0], R^3_+)$, model (2) has a unique global positive solution $N(t)$ on $t \geq 0$ $a.s.$

**Proof.** To begin with, let us consider the following stochastic system

$$\begin{align*}
dx_1(t) &= \left[ r_1 - \frac{1}{2} \sigma_1^2 - a_{11}e^{x_1(t)} - a_{12}e^{x_2(t-\tau_{12})} - a_{13}e^{x_3(t-\tau_{13})} \right] dt + \sigma_1 dW_1(t), \\
dx_2(t) &= \left[ r_2 - \frac{1}{2} \sigma_2^2 - a_{21}e^{x_1(t-\tau_{11})} - a_{22}e^{x_2(t)} - a_{23}e^{x_3(t-\tau_{23})} \right] dt + \sigma_2 dW_2(t), \\
dx_3(t) &= \left[ r_3 - \frac{1}{2} \sigma_3^2 - a_{31}e^{x_1(t-\tau_{31})} - a_{32}e^{x_2(t-\tau_{32})} - a_{33}e^{x_3(t)} \right] dt + \sigma_3 dW_3(t),
\end{align*}$$

with initial condition

$$x_i(\theta) = \ln \phi_i(\theta), \quad \theta \in [-\bar{\tau}, 0], \quad i = 1, 2, 3.$$  

It is not difficult to see that the coefficients of model (7) satisfy the local Lipschitz condition, therefore system (7) has a unique local solution $x(t)$ on $[0, \tau_e)$, where $\tau_e$ represents the explosion time. According to Itô’s formula, we can see that

$$N(t) = (N_1(t) = e^{x_1(t)}, \quad N_2(t) = e^{x_2(t)}, \quad N_3(t) = e^{x_3(t)})^T$$
is the unique positive local solution to system (2). Now let us prove \( \tau_c = +\infty \). To this end, let us introduce the following auxiliary model:

\[
\begin{align*}
\frac{dy_1(t)}{dt} &= y_1(t) \left[ r_1 - a_{11} y_1(t) \right] dt + \sigma_1 y_1(t) dW_1(t), \\
\frac{dy_2(t)}{dt} &= y_2(t) \left[ r_2 - a_{21} y_1(t - \tau_{21}) - a_{22} y_2(t) \right] dt + \sigma_2 y_2(t) dW_2(t), \\
\frac{dy_3(t)}{dt} &= y_3(t) \left[ r_3 - a_{31} y_1(t - \tau_{31}) - a_{33} y_3(t) \right] dt + \sigma_3 y_3(t) dW_3(t),
\end{align*}
\]

with initial data

\[ y_i(\theta) = \phi_i(\theta) > 0, \quad \theta \in [-\tau, 0], \quad i = 1, 2, 3. \]

Taking advantage of the comparison theorem for stochastic equation \([10]\) yields that for \( t \in [0, \tau_c) \),

\[ N_i(t) \leq y_i(t) \text{ a.s., } \quad i = 1, 2, 3. \quad (9) \]

In view of Theorem 4.2 in Jiang and Shi \([18]\), the explicit solution of system (8) is

\[
\begin{align*}
y_1(t) &= \frac{\exp\left(\int_{-\tau}^{0} a_{11} ds + \int_{0}^{t} \sigma_1 W_1(s) ds\right)}{y_1(0) + \int_{0}^{t} \sigma_1 W_1(s) ds}, \\
y_2(t) &= \frac{\exp\left(\int_{-\tau}^{0} a_{21} y_1(s - \tau_{21}) ds + \int_{0}^{t} \sigma_2 W_2(s) ds\right)}{y_2(0) + \int_{0}^{t} \sigma_2 W_2(s) ds}, \\
y_3(t) &= \frac{\exp\left(\int_{-\tau}^{0} a_{31} y_1(s - \tau_{31}) ds + \int_{0}^{t} \sigma_3 W_3(s) ds\right)}{y_3(0) + \int_{0}^{t} \sigma_3 W_3(s) ds}.
\end{align*}
\]

It is easy to see that \( y_1(t), \ y_2(t) \) and \( y_3(t) \) are existent on \( t \geq 0 \), thereby \( \tau_c = +\infty \).

\( \square \)

**Lemma 3.5.** \([31]\) Let \( z(t) \in C[\Omega \times [0, +\infty), R_+] \).

(I) If there exist two positive constants \( T \) and \( \lambda_0 \) such that

\[ \ln z(t) \leq \lambda t - \lambda_0 \int_{0}^{t} z(s) ds + \sum_{i=1}^{n} \sigma_i W_i(t), \]

for all \( t \geq T \), where \( W_i(t) \), \( 1 \leq i \leq n \), are independent standard Brownian motions and \( \sigma_i \), \( 1 \leq i \leq n \), are constants, then

\[
\begin{align*}
\langle z \rangle^* &\leq \lambda / \lambda_0 \text{ a.s., if } \lambda \geq 0; \\
\lim_{t \to +\infty} z(t) &= 0 \text{ a.s., if } \lambda < 0.
\end{align*}
\]

(II) If there exist three positive constants \( T, \lambda \) and \( \lambda_0 \) such that

\[ \ln z(t) \geq \lambda t - \lambda_0 \int_{0}^{t} z(s) ds + \sum_{i=1}^{n} \sigma_i W_i(t) \]

for all \( t \geq T \), then \( \langle z \rangle^* \geq \lambda / \lambda_0 \) a.s.

**Lemma 3.6.** For model (8),

(a) If \( b_1 < 0 \), then \( \lim_{t \to +\infty} y_i(t) = 0 \text{ a.s., } \quad i = 1, 2, 3; \)

(b) \([8]\),

(c) \([8]\),

(d) \([8]\),

(e) \([8]\),

(f) \([8]\),

(g) \([8]\),

(h) \([8]\),

(i) \([8]\),

(j) \([8]\),

(k) \([8]\),

(l) \([8]\),

(m) \([8]\),

(n) \([8]\),

(o) \([8]\),

(p) \([8]\),

(q) \([8]\),

(r) \([8]\),

(s) \([8]\),

(t) \([8]\),

(u) \([8]\),

(v) \([8]\),

(w) \([8]\),

(x) \([8]\),

(y) \([8]\),

(z) \([8]\),
We know that

At the beginning, let us prove (a). Applying Itô’s formula to model (8), we obtain that

\begin{align}
\ln y_1(t) - \ln y_1(0) &= b_1 t - a_{11} \int_0^t y_1(s) \, ds + \sigma_1 W_1(t), \\
\ln y_2(t) - \ln y_2(0) &= b_2 t - a_{21} \int_0^t y_1(s) \, ds - a_{22} \int_0^t y_2(s) \, ds + \sigma_2 W_2(t), \\
\ln y_3(t) - \ln y_3(0) &= b_3 t - a_{31} \int_0^t y_1(s) \, ds - a_{33} \int_0^t y_3(s) \, ds + \sigma_3 W_3(t).
\end{align}

It follows from (10) that

\[ t^{-1} \ln \frac{y_1(t)}{y_1(0)} \leq b_1 + \sigma_1 W_1(t). \]

Now let us prove (b). According to the condition \( b_1 \geq 0 \), applying (I) and (II) in Lemma 3.5 gives

\[ \lim_{t \to +\infty} y_2(t) = 0, \ a.s. \]

We know that

\[ t^{-1} \ln \frac{y_1(t)}{y_1(0)} = b_1 - a_{11} \langle y_1(t) \rangle + t^{-1} \sigma_1 W_1(t), \]
In the similar way, we can get that when $b - a_{21}y_1(t) - a_{22}y_2(t) + t^{-1}\sigma_2 W_2(t)$,
\begin{equation}
t^{-1}\ln \frac{y_2(t)}{y_2(0)} = b_2 - a_{21}\langle y_1(t) \rangle - a_{22}\langle y_2(t) \rangle + t^{-1}\sigma_2 W_2(t), \tag{16}
\end{equation}
and
\begin{equation}
t^{-1}\ln \frac{y_3(t)}{y_3(0)} - t^{-1}a_{31}\left(\int_{t-\tau_{31}}^{t} y_1(s)ds - \int_{0}^{0} y_1(s)ds\right) = b_3 - a_{31}\langle y_1(t) \rangle - a_{33}\langle y_3(t) \rangle + t^{-1}\sigma_3 W_3(t). \tag{17}
\end{equation}
Using (14) in (15) gives that
\begin{equation}
\lim_{t \to +\infty} t^{-1}\ln y_1(t) = 0. \tag{18}
\end{equation}

Multiplying (15) and (16) by $-a_{21}$ and $-a_{11}$ respectively, and adding them, one can derive that
\begin{equation}
t^{-1}a_{11}\ln \frac{y_2(t)}{y_2(0)} - \int_{t-\tau_{21}}^{t} y_1(s)ds - \int_{0}^{0} y_1(s)ds = t^{-1}a_{21}\ln \frac{y_1(t)}{y_1(0)}
\end{equation}
\begin{equation}= b_2a_{11} - a_{21}a_{22}\langle y_2(t) \rangle + t^{-1}a_{11}\sigma_2 W_2(t) - b_1a_{21} - t^{-1}a_{22}\sigma_1 W_1(t). \tag{19}
\end{equation}

An application of (14) gives that
\begin{equation}
\lim_{t \to +\infty} t^{-1}\int_{t-\tau_{21}}^{t} y_1(s)ds = \lim_{t \to +\infty} t^{-1}\int_{0}^{t} y_1(s)ds - \int_{0}^{t-\tau_{21}} y_1(s)ds = 0. \tag{20}
\end{equation}

Consequently, utilizing (18), (19), (20) and Lemma 3.5 yields that
\begin{equation}
\lim_{t \to +\infty}\langle y_2(t) \rangle = \frac{b_2 - a_{21}b_1/a_{11}}{a_{22}}, \text{ a.s.} \tag{21}
\end{equation}

In the similar way, we can get that when $b_3 - a_{31}b_1/a_{11} \geq 0$,
\begin{equation}
\lim_{t \to +\infty}\langle y_3(t) \rangle = \frac{b_3 - a_{31}b_1/a_{11}}{a_{33}}, \text{ a.s.} \tag{22}
\end{equation}

Now we are in the position to prove (c). If $b_2 - a_{21}b_1/a_{11} < 0$, applying (18), (19), (20) and Lemma 3.5 to (16), one can see that
\begin{equation}
\lim_{t \to +\infty} y_2(t) = 0, \text{ a.s.} \tag{23}
\end{equation}

What is more, if $b_3 - a_{31}b_1/a_{11} < 0$, in the same way, we can get that
\begin{equation}
\lim_{t \to +\infty} y_3(t) = 0, \text{ a.s.} \tag{24}
\end{equation}

The proofs of (d) and (e) are similar to that of (b) and (c) respectively, and hence are omitted. □

By Lemma 3.6 for all $\tau \geq 0$, if $\lim_{t \to +\infty} y_j(t) = 0$, then
\begin{equation}
\lim_{t \to +\infty} t^{-1}\int_{t-\tau}^{t} y_j(s)ds = \lim_{t \to +\infty} t^{-1}\left(\int_{0}^{t} y_j(s)ds - \int_{0}^{t-\tau} y_j(s)ds\right) = 0 - 0 = 0, \text{ a.s., } j = 1, 2, 3. \tag{25}
\end{equation}

If $\lim_{t \to +\infty} t^{-1}\int_{0}^{t} y_j(s)ds$ is a constant, then
\begin{equation}
\lim_{t \to +\infty} t^{-1}\int_{t-\tau}^{t} y_j(s)ds = \lim_{t \to +\infty} t^{-1}\left(\int_{0}^{t} y_j(s)ds - \int_{0}^{t-\tau} y_j(s)ds\right) = 0, \text{ a.s., } j = 1, 2, 3. \tag{26}
\end{equation}
Lemma 3.7. In model (8),
\[ \lim_{t \to +\infty} t^{-1} \int_{t-\tau}^t y_j(s) ds = 0 \ a.s., \ j = 1, 2, 3. \] (25)

Lemma 3.8. The solution \( N(t) \) of model (8) obeys
\[ \limsup_{t \to +\infty} \ln N_i(t)/t \leq 0, \ a.s., \ i = 1, 2, 3. \] (26)

Proof. By (9), we only need to prove
\[ \limsup_{t \to +\infty} y_i(t)/t \leq 0, \ a.s., \ i = 1, 2, 3. \] (27)

To get the required assertion, let us analyze the five outcomes in Lemma 3.6 one by one. First of all, in situation (a) we have
\[ \lim_{t \to +\infty} y_i(t) = 0, \ a.s., \ i = 1, 2, 3. \]

Consequently,
\[ \limsup_{t \to +\infty} y_i(t)/t \leq 0, \ a.s. \] (28)

Next, when (14), (20) and (21) are used in (16), we can easily prove that \( \limsup_{t \to +\infty} y_2(t)/t \leq 0, \ a.s. \). In the similar way, we can show \( \limsup_{t \to +\infty} y_3(t)/t \leq 0, \ a.s. \) here. Substituting (14) and (23) into (16), we can attain that
\[ t^{-1} \ln y_2(t)/y_2(0) - t^{-1} a_{21} \left( \int_{t-\tau_{21}}^t y_1(s) ds - \int_{t-\tau_{21}}^0 y_1(s) ds \right) \leq b_2 - a_{21} b_1/a_{11} + \varepsilon + t^{-1} \sigma_2 W_2(t). \] (29)

Combining \( b_2 - a_{21} b_1/a_{11} < 0 \) in situation (c) with (20) and the arbitrariness of \( \varepsilon \), one can observe that \( \limsup_{t \to +\infty} y_2(t)/t \leq 0, \ a.s. \). And in the similar way, we have \( \limsup_{t \to +\infty} y_3(t)/t \leq 0, \ a.s. \).

Finally, the proofs of (d) and (e) are similar to that of (b) and (c) respectively, and thus are omitted.

Lemma 3.9. Consider model (3), let Assumptions (7) and (8) hold. Suppose that \( R > 0 \), then \( \rho_1 > \rho_2 > \rho_3 \) and moreover,

| Condition | Limit |
|-----------|-------|
| (i') \( 1 > \rho_1 \) | \( \lim_{t \to +\infty} N_i(t) = 0, \ i = 1, 2, 3, \ a.s. \) |
| (ii') \( \rho_1 > 1 > \rho_2 \) | \( \lim_{t \to +\infty} N_1(t) = \frac{a_{11}}{A_{11}}, \lim_{t \to +\infty} N_2(t) = \lim_{t \to +\infty} N_3(t) = 0, \ a.s. \) |
| (iii') \( \rho_2 > 1 > \rho_3 \) | \( \lim_{t \to +\infty} N_1(t) = \frac{A_{11} - A_{21}}{A_{21} - A_{11}}, \lim_{t \to +\infty} N_2(t) = \frac{A_{21} - A_{31}}{A_{31}}, \lim_{t \to +\infty} N_3(t) = 0, \ a.s. \) |
| (iv') \( \rho_3 > 1 \) | \( \lim_{t \to +\infty} N_i(t) = \frac{A_{11} - A_{21}}{A_{21}}, \ i = 1, 2, 3, \ a.s. \) |
Applying Itô’s formula to model (2) yields that
\[ \rho \]
Hence,
\[ \rho_1 = 2r_1/\sigma_1^2 > \Delta_2/\Delta_2 = \rho_2. \]
At the same time,
\[ \Delta_2 - \Delta_3 = \frac{a_{11}R}{\Delta_2\Delta_3} > 0, \]
namely that
\[ \Delta_2/\Delta_2 > \Delta_3/\Delta_3. \]
In the similar way, we can gain that \( \Delta_3/\Delta_3 > A_3/A_3. \) Therefore,
\[ \frac{r_1}{\sigma_1^2/2} = \frac{\Delta_2}{\Delta_2} > \frac{A_3}{A_3}, \]
i.e., \( \rho_1 > \rho_2 > \rho_3. \)

Applying Itô’s formula to model (2) yields that
\[ \ln N_1(t) - \ln N_1(0) = b_1 t - a_{11} \int_0^t N_1(s)ds - a_{12} \int_0^t N_2(s - \tau_1)ds - a_{13} \int_0^t N_3(s - \tau_1)ds + \sigma_1 W_1(t), \]
(31)
\[ \ln N_2(t) - \ln N_2(0) = b_2 t - a_{22} \int_0^t N_2(s)ds - a_{21} \int_0^t N_1(s - \tau_2)ds - a_{23} \int_0^t N_3(s - \tau_2)ds + \sigma_2 W_2(t), \]
(32)
\[ \ln N_3(t) - \ln N_3(0) = b_3 t - a_{33} \int_0^t N_3(s)ds - a_{31} \int_0^t N_1(s - \tau_3)ds - a_{32} \int_0^t N_2(s - \tau_3)ds + \sigma_3 W_3(t). \]
(33)

First of all, let us prove (i’). We know that \( a_{11}, a_{12}, a_{13} > 0, \) thus by (31) we obtain
\[ t^{-1} \ln \frac{N_1(t)}{N_1(0)} \leq b_1 + t^{-1} \sigma_1 W_1(t). \]
(34)

Note that \( b_1 = r_1 - 0.5\sigma_1^2 < 0 \) and \( \lim_{t \to +\infty} t^{-1} \sigma_1 W_1(t) = 0. \) Using (I) in Lemma 3.5 we get
\[ \lim_{t \to +\infty} N_1(t) = 0, \text{ a.s.} \]
Substituting this identity into (32), one can observe that for sufficiently large \( t, \)
\[ \ln N_2(t) - \ln N_2(0) = b_2 t - a_{22} \int_0^t N_2(s)ds - a_{21} \int_0^t N_1(s - \tau_2)ds - a_{23} \int_0^t N_3(s - \tau_2)ds + \sigma_2 W_2(t) \leq b_2 t + \varepsilon t - a_{22} \int_0^t N_2(s)ds + \sigma_2 W_2(t), \]
(35)
where $\varepsilon$ is sufficiently small such that $b_2 + \varepsilon < 0$. According to (1) in Lemma 3.5 we obtain
\[
\lim_{t \to +\infty} N_2(t) = 0, \text{ a.s.}
\]
In the same way, we have
\[
\lim_{t \to +\infty} N_3(t) = 0, \text{ a.s.}
\]
Now let us prove $(ii')$. From $\Delta_2/\hat{\Delta}_2 > \Delta_3/\hat{\Delta}_3$ and $\Delta_2/\hat{\Delta}_2 < 1$, we get
\[
\Delta_3/\hat{\Delta}_3 < 1 \text{ i.e., } \frac{a_{11}r_3 - a_{31}r_1}{a_{11}\sigma_3^2/2 - a_{31}\sigma_1^2/2} < 1.
\]
Simplifying the above inequality gives that $a_{11}b_3 < a_{31}b_1$, which means that $b_3 - a_{31}b_1/a_{11} < 0$. What is more, from $\Delta_2/\hat{\Delta}_2 < 1$ we have $b_2 - a_{21}b_1/a_{11} < 0$. According to (c) in Lemma 3.6 one can see that
\[
\lim_{t \to +\infty} y_2(t) = 0, \quad \lim_{t \to +\infty} y_3(t) = 0, \text{ a.s.}
\]
In view of (9), one can observe that
\[
\lim_{t \to +\infty} N_2(t) = 0, \quad \lim_{t \to +\infty} N_3(t) = 0, \text{ a.s.}
\]
Substituting the above two inequalities into (31) and using Lemma 3.5 we can obtain that
\[
\lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{b_1}{a_{11}}, \text{ a.s.}
\]
Now let us turn to $(iii')$. Dividing (31), (32) and (33) by $t$, we can derive the following equations:
\[
t^{-1} \ln \frac{N_1(t)}{N_1(0)} = b_1 - a_{11} \langle N_1(t) \rangle - a_{12} \langle N_2(t) \rangle - a_{13} \langle N_3(t) \rangle
\]
\[+ t^{-1} a_{12} \left( \int_{t - \tau_12}^t N_2(s)ds - \int_{t - \tau_12}^0 N_2(s)ds \right) + t^{-1} a_{13} \left( \int_{t - \tau_13}^t N_3(s)ds - \int_{t - \tau_13}^0 N_3(s)ds \right) + t^{-1} \sigma_1 W_1(t), \tag{36}
\]
\[
t^{-1} \ln \frac{N_2(t)}{N_2(0)} = b_2 - a_{21} \langle N_1(t) \rangle - a_{22} \langle N_2(t) \rangle - a_{23} \langle N_3(t) \rangle
\]
\[+ t^{-1} a_{21} \left( \int_{t - \tau_21}^t N_1(s)ds - \int_{t - \tau_21}^0 N_1(s)ds \right) + t^{-1} a_{23} \left( \int_{t - \tau_23}^t N_3(s)ds - \int_{t - \tau_23}^0 N_3(s)ds \right) + t^{-1} \sigma_2 W_2(t), \tag{37}
\]
\[
t^{-1} \ln \frac{N_3(t)}{N_3(0)} = b_3 - a_{31} \langle N_1(t) \rangle - a_{32} \langle N_2(t) \rangle - a_{33} \langle N_3(t) \rangle
\]
\[+ t^{-1} a_{31} \left( \int_{t - \tau_31}^t N_1(s)ds - \int_{t - \tau_31}^0 N_1(s)ds \right) + t^{-1} a_{32} \left( \int_{t - \tau_32}^t N_2(s)ds - \int_{t - \tau_32}^0 N_2(s)ds \right) + t^{-1} \sigma_3 W_3(t). \tag{38}
\]
Denote $m, n$ as the solution of the following equations:
\[
\begin{cases}
a_{11}m + a_{21}n = a_{31}, \\
a_{12}m + a_{22}n = a_{32}.
\end{cases}
\]
Consequently
\[ m = \frac{-A_{13}}{A_{33}} > 0, \quad n = \frac{A_{23}}{A_{33}} > 0. \]

By [9], [27] and Lemma 3.8, we have
\[ \limsup_{t \to +\infty} t^{-1} \ln N_i(t) \leq 0, \quad i = 1, 2, 3, \quad (39) \]
\[ \lim_{t \to +\infty} t^{-1} \int_{t-\tau_{ij}}^t N_j(s)ds = 0, \quad i \neq j \text{ and } i, j = 1, 2, 3. \quad (40) \]

According to Lemma 3.8, for arbitrarily given \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that when \( t > T_1 \)
\[ t^{-1} \left( m \ln \frac{N_1(t)}{N_1(0)} + n \ln \frac{N_2(t)}{N_2(0)} \right) \leq \varepsilon. \]

Multiplying (36), (37), and (38) by \(-m, -n, \) and 1 respectively and adding them, one can observe that for sufficiently large \( t \) such that \( t > T_1 \),
\[ t^{-1} \ln \frac{N_3(t)}{N_3(0)} - t^{-1} \left( m \ln \frac{N_1(t)}{N_1(0)} + n \ln \frac{N_2(t)}{N_2(0)} \right) = A_3 - \tilde{A}_3 = \frac{A}{A_{33}} (N_3(t)) - m t^{-1} a_{12} \left( \int_{t-\tau_{12}}^t N_2(s)ds - \int_{t-\tau_{12}}^0 N_2(s)ds \right) 
- n t^{-1} a_{13} \left( \int_{t-\tau_{13}}^t N_3(s)ds - \int_{t-\tau_{13}}^0 N_3(s)ds \right) - n t^{-1} a_{21} \left( \int_{t-\tau_{21}}^t N_1(s)ds 
- \int_{t-\tau_{21}}^0 N_1(s)ds \right) - n t^{-1} a_{23} \left( \int_{t-\tau_{23}}^t N_3(s)ds - \int_{t-\tau_{23}}^0 N_3(s)ds \right) + t^{-1} a_{31} \left( \int_{t-\tau_{31}}^t N_1(s)ds - \int_{t-\tau_{31}}^0 N_1(s)ds \right) + t^{-1} a_{32} \left( \int_{t-\tau_{32}}^t N_2(s)ds - \int_{t-\tau_{32}}^0 N_2(s)ds \right) \quad (41) \]

Using (27) in (41) yields that
\[ t^{-1} \ln \frac{N_3(t)}{N_3(0)} \leq \frac{A_3 - \tilde{A}_3}{A_{33}} (N_3(t)) + t^{-1} (m \sigma_1 W_1(t) + n \sigma_2 W_2(t) - \sigma_3 W_3(t)). \quad (42) \]

Since \( \rho_3 = A_3/\tilde{A}_3 < 0 \), we can choose \( \varepsilon > 0 \) be sufficiently small such that \((A_3 - \tilde{A}_3)/A_{33} + 2 \varepsilon < 0 \). Making use of the arbitrariness of \( \varepsilon \) and Lemma 3.5 gives that
\[ \lim_{t \to +\infty} N_3(t) = 0, \quad a.s. \]

Consequently, model (2) reduces to the following predator-prey model:
\[ \begin{align*}
\frac{dN_1(t)}{dt} = & N_1(t) \left[ r_1 - a_{11} N_1(t) - a_{12} N_2(t - \tau_{12}) \right] dt + \sigma_1 N_1(t) dW_1(t), \\
\frac{dN_2(t)}{dt} = & N_2(t) \left[ r_2 - a_{21} N_1(t - \tau_{21}) - a_{22} N_2(t) \right] dt + \sigma_2 N_2(t) dW_2(t),
\end{align*} \quad (43) \]

which has already been investigated in [30]. Then similar to the proof of Theorem 1 in [30], the following identities can be derived:
\[ \lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{\Delta_1 - \tilde{\Delta}_1}{A_{33}}, \quad \lim_{t \to +\infty} \langle N_2(t) \rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{A_{33}}, \quad a.s. \]
Now we are in the position to prove (iv'). By (42), since \( \rho_3 = A_3/\tilde{A}_3 > 1 \), we know from the arbitrariness of \( \varepsilon \) and Lemma 3.3 that

\[
(N_3(t))^* \leq \frac{A_3 - \tilde{A}_3}{A}, \quad \text{a.s.} \tag{44}
\]

Denote \( p, q \) as the solution of the following equations:

\[
\begin{align*}
ap_{11} + a_{31}q & = a_{21}, \\
ap_{13} + a_{33}q & = a_{23}.
\end{align*}
\]

Then we have

\[
p = -\frac{A_{12}}{A_{22}} > 0, \quad q = \frac{A_{32}}{A_{22}} > 0.
\]

According to Lemma 3.8, for arbitrarily given \( \varepsilon > 0 \), there exists \( T_2 > 0 \) such that

\[
t^{-1} \left( p \ln \frac{N_1(0)}{N_1(0)} + q \ln \frac{N_3(0)}{N_3(0)} \right) \leq \varepsilon.
\]

Multiplying (36), (37) and (38) by \( -p, 1 \) and \( -q \) respectively and adding them, we can obtain that for \( t > T_2 \)

\[
t^{-1} \ln \frac{N_2(t)}{N_2(0)} - t^{-1} \left( p \ln \frac{N_1(t)}{N_1(0)} + q \ln \frac{N_3(t)}{N_3(0)} \right)
\]

\[
= \frac{A_2 - \tilde{A}_2}{A_{22}} \left( N_2(t) - pt^{-1}a_{12} \left( \int_{t-T_1}^{t} N_2(s)ds - \int_{t-T_2}^{0} N_2(s)ds \right) \right)
\]

\[
- t^{-1}a_{13} \left( \int_{t-T_1}^{t} N_3(s)ds - \int_{t-T_2}^{0} N_3(s)ds \right) + t^{-1}a_{23} \left( \int_{t-T_1}^{t} N_1(s)ds - \int_{t-T_2}^{0} N_1(s)ds \right)
\]

\[
- \int_{t-T_3}^{0} N_1(s)ds + t^{-1}a_{23} \left( \int_{t-T_3}^{t} N_2(s)ds - \int_{t-T_3}^{0} N_2(s)ds \right)
\]

\[
- t^{-1}a_{31} \left( \int_{t-T_3}^{t} N_3(s)ds - \int_{t-T_3}^{0} N_3(s)ds \right) + t^{-1}a_{32} \left( \int_{t-T_3}^{t} N_2(s)ds - \int_{t-T_3}^{0} N_2(s)ds \right)
\]

\[
\leq 2\varepsilon - \frac{A_2 - \tilde{A}_2}{A_{22}} (N_2(t)) + t^{-1} (p\sigma_1W_1(t) - \sigma_2W_2(t) + q\sigma_3W_3(t)).
\]

Using (27) in (45) gives that

\[
t^{-1} \ln \frac{N_2(t)}{N_2(0)} \leq \frac{A_2 - \tilde{A}_2}{A_{22}} + 2\varepsilon - \frac{A_2}{A_{22}} (N_2(t)) + t^{-1} (p\sigma_1W_1(t) - \sigma_2W_2(t) + q\sigma_3W_3(t)).
\]

Note that \( A_2/\tilde{A}_2 > A_3/\tilde{A}_3 > 1 \). According to the arbitrariness of \( \varepsilon \) and Lemma 3.5, we have

\[
(N_2(t))^* \leq \frac{A_2 - \tilde{A}_2}{A}, \quad \text{a.s.} \tag{46}
\]

It follows that for any sufficiently small \( \varepsilon \), there exist \( T_3 \) and \( T_4 \) such that

\[
\begin{align*}
a_{12}(N_2(t)) \leq a_{12}(N_2(t))^* + \varepsilon \leq \frac{a_{12}(A_2 - \tilde{A}_2)}{A} + \varepsilon, \quad t > T_3, \\
a_{13}(N_3(t)) \leq a_{13}(N_3(t))^* + \varepsilon \leq \frac{a_{13}(A_3 - \tilde{A}_3)}{A} + \varepsilon, \quad t > T_4.
\end{align*}
\]
Substituting (47) into (36) results in that for sufficiently large $t$,
\[
\begin{aligned}
t^{-1} \ln \frac{N_1(t)}{N_1(0)} \geq & b_1 - a_{11}(N_1(t)) - \frac{a_{12}(A_2 - \hat{A}_2)}{A} - \frac{a_{13}(A_3 - \hat{A}_3)}{A} - 3\varepsilon + t^{-1}\sigma_1 W_1(t) \\
= & \frac{a_{11}(A_1 - \hat{A}_1)}{A} - a_{11}(N_1(t)) - 3\varepsilon + t^{-1}\sigma_1 W_1(t).
\end{aligned}
\]

According to the arbitrariness of $\varepsilon$ and Lemma 3.5, we have
\[
\langle N_1(t) \rangle_* \geq \frac{A_1 - \hat{A}_1}{A}, \text{ a.s.} \tag{48}
\]

By $a_{31} < 0$, $a_{32} > 0$, one can observe that for every sufficiently small $\varepsilon$, there exist $T_5$ and $T_6$ such that
\[
\begin{aligned}
a_{31} \langle N_1(t) \rangle \leq & a_{31} \langle N_1(t) \rangle_* + \varepsilon \leq \frac{a_{31}(A_1 - \hat{A}_1)}{A} + \varepsilon, \quad t > T_5, \\
a_{32} \langle N_2(t) \rangle \leq & a_{32} \langle N_2(t) \rangle_* + \varepsilon \leq \frac{a_{32}(A_2 - \hat{A}_2)}{A} + \varepsilon, \quad t > T_6.
\end{aligned}
\tag{49}
\]

Using (49) in (48), we can get that when $t$ is large enough,
\[
\begin{aligned}
t^{-1} \ln \frac{N_3(t)}{N_3(0)} \geq & b_3 - a_{33}(N_3(t)) - \frac{a_{31}(A_1 - \hat{A}_1)}{A} - \frac{a_{32}(A_2 - \hat{A}_2)}{A} - 3\varepsilon + t^{-1}\sigma_3 W_3(t) \\
= & \frac{a_{33}(A_3 - \hat{A}_3)}{A} - a_{33}(N_3(t)) - 3\varepsilon + t^{-1}\sigma_3 W_3(t).
\end{aligned}
\]

According to the arbitrariness of $\varepsilon$ and Lemma 3.5, we can obtain that
\[
\langle N_3(t) \rangle_* \geq \frac{A_3 - \hat{A}_3}{A}, \text{ a.s.} \tag{50}
\]

Combining (44) with (50), one can observe that
\[
\lim_{t \to +\infty} \langle N_3(t) \rangle = \frac{A_3 - \hat{A}_3}{A}, \text{ a.s.} \tag{51}
\]

In the similar way, using (44), (48) in (57) and then combining them with (46) yield that
\[
\lim_{t \to +\infty} \langle N_2(t) \rangle = \frac{A_2 - \hat{A}_2}{A}, \text{ a.s.} \tag{52}
\]

Subsequently, substituting (51), (52) into (36) and combining them with (48), we can attain that
\[
\lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{A_1 - \hat{A}_1}{A}, \text{ a.s.}
\]

This completes the proof. \qed

3.2. Proof of global attractivity.

**Lemma 3.10.** For any $p > 1$, there exists a constant $K = K(p)$ which makes the solution $N(t)$ of model (2) satisfy the property that
\[
\limsup_{t \to +\infty} E[N_i^p(t)] \leq K, \quad i = 1, 2, 3. \tag{53}
\]

**Proof.** The proof is rather standard and hence is omitted (see e.g. [15]). \qed

From Lemma 3.10 there is a $T > 0$ satisfying that for $t \geq T$, $E[N_i^p(t)] \leq 2K$. Note that $E[N_i(t)]$ is continuous, thus there is a constant $K_1 > 0$ such that $E[N_i^p(t)] < K_1$ when $-\tau \leq t < T$. Denote $L = \max\{2K, K_1\}$, then we have
\[
E[N_i^p(t)] \leq L = L(p), \quad t \geq \tau, \quad p > 0, \quad i = 1, 2, 3. \tag{54}
\]
Lemma 3.11. Model \( \square \) is globally attractive if Assumption \( \square \) holds.

Proof. Denote
\[
\hat{A} = \begin{pmatrix}
a_{11} & -a_{12} & -a_{13} \\
a_{21} & a_{22} & -a_{23} \\
a_{31} & -a_{32} & a_{33}
\end{pmatrix}, \quad \hat{L} = \begin{pmatrix}
a_{12} & a_{13} & -a_{12} \\
-a_{21} & a_{21} & -a_{23} \\
-a_{31} & a_{32} & a_{31} - a_{32}
\end{pmatrix}.
\]

Let \( k_i \) be the cofactor of the \( i \)-th diagonal element of \( \hat{L} \hat{A} \). Then according to Kirchhoff’s Matrix Tree Theorem (see, e.g., [37]), we obtain that \( k_i > 0, \ i = 1, 2, 3 \). Define
\[
V_i(t) = k_i|\ln N_i(t) - \ln M_i(t)| + \sum_{j \neq i, j=1} g(a_{ij}) \int_{t-\tau_{ij}}^{t} |N_j(s) - M_j(s)|ds, \ i = 1, 2, 3,
\]

Calculating the right differential \( d^+ V(t) \), we can attain that
\[
d^+ V(t) = \sum_{i=1}^{3} \left( k_i \text{sgn}(N_i(t) - M_i(t)) \right.
\]
\[
\left. - \sum_{j \neq i, j=1}^{3} a_{ij}(N_j(t) - M_j(t)) \right) dt
\]
\[
\left. + \sum_{j \neq i, j=1}^{3} k_i|a_{ij}|N_j(t) - M_j(t)|dt - \sum_{j \neq i, j=1}^{3} k_i|a_{ij}|N_j(t) - M_j(t)|dt \right)
\]
\[
\leq \sum_{i=1}^{3} \left( -k_i a_{ii} |N_i(t) - M_i(t)|dt + \sum_{j \neq i, j=1}^{3} k_i|a_{ij}|N_j(t) - M_j(t)|dt \right)
\]
\[
= - \sum_{i=1}^{3} \sum_{j=1}^{3} \left( k_i c_{ij} |N_j(t) - M_j(t)| \right) dt,
\]

where \( (c_{ij})_{3 \times 3} = \hat{A} \). By Theorem 2.3 in Li and Shuai [22],
\[
\sum_{i=1}^{3} \sum_{j=1}^{3} \left( k_i c_{ij} |N_j(t) - M_j(t)| \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} (k_i c_{ij} |N_i(t) - M_i(t)|).
\]

Then we have
\[
d^+ V(t) \leq - \sum_{i=1}^{3} \sum_{j=1}^{3} \left( k_i c_{ij} |N_i(t) - M_i(t)| \right) dt
\]
\[
= - \sum_{i=1}^{3} k_i \left[ a_{ii} - \sum_{j \neq i, j=1}^{3} |a_{ij}| \right] |N_i(t) - M_i(t)| dt.
\]

Namely that
\[
\mathbb{E}(V(t)) \leq V(0) - \int_{0}^{t} \sum_{i=1}^{3} k_i \left[ a_{ii} - \sum_{j \neq i, j=1}^{3} |a_{ij}| \right] \mathbb{E}|N_i(s) - M_i(s)| ds.
\]
Subsequently
\[ E(V(t)) + \int_0^t \frac{3}{\sum_{i=1}^3 k_i} \left[ a_{ii} - \sum_{j \neq i, j=1}^3 |a_{ij}| \right] E|N_i(s) - M_i(s)|ds \leq V(0) < \infty. \]

Note that \( V(t) \geq 0 \), then \( E|N_i(t) - M_i(t)| \) is integrable on \([0, +\infty)\).

Now we are in the position to prove that \( E|N_i(t) - M_i(t)| \) is uniformly continuous. In fact, it follows from (2) that
\[
\begin{align*}
\mathbb{E}N_1(t) &= N_1(0) \\
&\quad + \int_0^t \mathbb{E} \left[ r_1 N_1(s) - a_{11} N_1^2(s) - a_{12} N_1(s) N_2(s - \tau_{12}) - a_{13} N_1(s) N_3(s - \tau_{13}) \right] ds,
\end{align*}
\]
\[
\begin{align*}
\mathbb{E}N_2(t) &= N_2(0) \\
&\quad + \int_0^t \mathbb{E} \left[ r_2 N_2(s) - a_{21} N_1(s) N_1(s - \tau_{21}) - a_{22} N_2^2(s) - a_{23} N_2(s) N_3(s - \tau_{23}) \right] ds,
\end{align*}
\]
\[
\begin{align*}
\mathbb{E}N_3(t) &= N_3(0) \\
&\quad + \int_0^t \mathbb{E} \left[ r_3 N_3(s) - a_{31} N_1(s) N_2(s - \tau_{31}) - a_{32} N_3(s) N_2(s - \tau_{32}) - a_{33} N_3^2(s) \right] ds.
\end{align*}
\]

That is to say, \( \mathbb{E}(N_i(t)), i = 1, 2, 3 \), are continuously differentiable functions with respect of \( t \). On the other hand, in view of (54),
\[
\frac{d(\mathbb{E}N_1(t))}{dt} \leq r_1 \mathbb{E}(N_1(t)) - a_{11} \mathbb{E}(N_1^2(t)) \leq L_1,
\]
\[
\frac{d(\mathbb{E}N_2(t))}{dt} \leq a_{21} \mathbb{E}(N_2(t)) \mathbb{E}(N_1(t - \tau_{21})) \leq \frac{a_{21}}{2} \mathbb{E} |N_1^2(t - \tau_{21}) + N_2^2(t)| \leq L_1,
\]
and,
\[
\frac{d(\mathbb{E}N_3(t))}{dt} \leq a_{31} \mathbb{E}(N_3(t)) \mathbb{E}(N_1(t - \tau_{31})) \leq \frac{a_{31}}{2} \mathbb{E} |N_1^2(t - \tau_{31}) + N_3^2(t)| \leq L_1,
\]
where \( L_1 > 0 \) is a constant. Hence \( \mathbb{E}(N_i(t)), i = 1, 2, 3 \), are uniformly continuous functions. According to the well known Barbalat’s conclusion (3), we obtain
\[
\lim_{t \to +\infty} \mathbb{E}(N_i(t) - M_i(t)) = 0, \ i = 1, 2, 3.
\]

3.3. Proof of stability in distribution.

**Lemma 3.12.** Model (2) is stable in distribution if Assumption 2 holds.

**Proof.** Our proof is motivated by [34]. Set \( P(t, \phi, B) \) for the probability of event \( N(t; \phi) \in B \) with initial data \( \phi(\theta) \in C([-\bar{\tau}, 0]; \mathbb{R}_+^3) \). Based on Lemma 3.10 and Chebyshev’s inequality, we can see that the family of \( p(t, \phi, \cdot) \) is tight.

Let \( \mathcal{P}(C([-\bar{\tau}, 0]; \mathbb{R}_+^3)) \) be all the probability measures on \( C([-\bar{\tau}, 0]; \mathbb{R}_+^3) \) and for any two measures \( P_1, P_2 \in \mathcal{P} \), we denote the following metric
\[
d_G(P_1, P_2) = \sup_{g \in G} \left| \int_{\mathbb{R}_+^3} g(N)P_1(dN) - \int_{\mathbb{R}_+^3} g(N)P_2(dN) \right|,
\]
where
\[
G = \left\{ g : C([-\bar{\tau}, 0]; \mathbb{R}_+^3) \to \mathbb{R} \mid |g(\delta) - g(\beta)| \leq ||\delta - \beta||, \ |g(\cdot)| \leq 1 \right\}.
\]
Subsequently, for any \( g \in G \) and \( t, s > 0 \), we have

\[
\left| \mathbb{E}_g(N(t + s; \phi)) - \mathbb{E}_g(N(t; \phi)) \right| \\
= \mathbb{E} \left[ \mathbb{E} \left( g(N(t + s; \phi)) | \mathcal{F}_s \right) \right] - \mathbb{E}_g(N(t; \phi)) \\
\leq \int_{\mathbb{R}_+^3} \left| \mathbb{E}_g(N(t; \xi)) - \mathbb{E}_g(N(t; \phi)) \right| p(s, \phi, d\xi).
\]

By virtue of \textbf{Lemma 3.11}, there must exist a \( T > 0 \) such that for \( t \geq T \),

\[
\sup_{g \in G} \left| \mathbb{E}_g(N(t; \xi)) - \mathbb{E}_g(N(t; \phi)) \right| \leq \varepsilon,
\]

namely that, \( \left| \mathbb{E}_g(N(t + s; \phi)) - \mathbb{E}_g(N(t; \phi)) \right| \leq \varepsilon \). Due to the arbitrariness of \( g \), we have

\[
\sup_{g \in G} \left| \mathbb{E}_g(N(t + s; \phi)) - \mathbb{E}_g(N(t; \phi)) \right| \leq \varepsilon,
\]

which means that

\[
d_G(p(t + s, \phi, \cdot), p(t, \phi, \cdot)) \leq \varepsilon, \; \forall t \geq T, \; s > 0.
\]

Therefore for any \( \phi \in C([-\tau, 0]; \mathbb{R}_+^3) \), \( \{p(t, \phi, \cdot) : t \geq 0\} \) is Cauchy in \( \mathcal{P} \) with metric \( d_G \). Choose \( \phi(\theta) \equiv 0.1 \). Hence there exists a unique \( v(\cdot) \in \mathcal{P}(C([-\tau, 0]; \mathbb{R}_+^3)) \) such that

\[
\lim_{t \to +\infty} d_G(p(t, 0.1, \cdot), v(\cdot)) = 0.
\]

According to \textbf{Lemma 3.11}

\[
\lim_{t \to +\infty} d_G(p(t, \varphi, \cdot), p(t, 0.1, \cdot)) = 0.
\]

As a result,

\[
\lim_{t \to +\infty} d_G(p(t, \varphi, \cdot), v(\cdot)) \\
\leq \lim_{t \to +\infty} d_G(p(t, \varphi, \cdot), p(t, 0.1, \cdot)) + \lim_{t \to +\infty} d_G(p(t, 0.1, \cdot), v(\cdot)) = 0.
\]

This completes the proof. \( \square \)

\textbf{Proof of Theorem 2.1} To begin with, let us show \( (iv) \). By \( (iv') \) in \textbf{Lemma 3.9} all the species are stable in time average:

\[
\lim_{t \to +\infty} \langle N_i(t) \rangle = \frac{A_i - \bar{A}_i}{A} > 0, \; i = 1, 2, 3, \; a.s. \tag{55}
\]

On the other hand, note that model \( \{2\} \) is stable in distribution, then the transition probability \( p(t, \phi, \cdot) \) of \( N(t) \) converges weakly to a unique probability measure which is denoted by \( \nu_3 \). By Kolmogorov-Chapman equation, \( \nu_3 \) is an invariant measure. According to Corollary 3.4.3 in \[12\], we can obtain that \( \nu_3 \) is strong mixing. Thanks to Theorem 3.2.6 in \[12\], one can see that \( \nu_3 \) is ergodic. By virtue of (3.3.2) in \[12\] and (55), one gets

\[
\lim_{t \to +\infty} \langle N_i(t) \rangle = \int_{\mathbb{R}_+^3} z_i \nu_3(dz_1, dz_2, dz_3) = \frac{A_i - \bar{A}_i}{A}, \; i = 1, 2, 3, \; a.s.
\]
We are in the position to prove (iii). By \((iii')\) in Lemma 3.9
\[
\lim_{t \to +\infty} N_3(t) = 0, \quad \lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{\Delta_1 - \tilde{\Delta}_1}{A_{33}}, \quad \lim_{t \to +\infty} \langle N_2(t) \rangle = \frac{\Delta_2 - \tilde{\Delta}_2}{A_{33}}, \text{ a.s.}
\]
Consider the following predator-prey model:
\[
\begin{align*}
\begin{cases}
\mathrm{d}N_1(t) &= \tilde{N}_1(t) \left[ r_1 - a_{11} \tilde{N}_1(t) - a_{12} \tilde{N}_2(t - \tau_{12}) \right] \mathrm{d}t + \sigma_1 \tilde{N}_1(t) \mathrm{d}W_1(t), \\
\mathrm{d}N_2(t) &= \tilde{N}_2(t) \left[ r_2 - a_{21} \tilde{N}_1(t - \tau_{21}) - a_{22} \tilde{N}_2(t - \tau_{22}) \right] \mathrm{d}t + \sigma_2 \tilde{N}_2(t) \mathrm{d}W_2(t),
\end{cases}
\end{align*}
\]
with initial data \(\tilde{N}_i(\theta) = N_i(\theta), \; i = 1, 2.\) Since \(\lim_{t \to +\infty} N_3(t) = 0\) a.s., the CDS of \((N_1(t), N_2(t))^T\) is the same with that of \((\tilde{N}_1(t), \tilde{N}_2(t))^T\). Similar to the proof of Lemma 3.12 model (56) is stable in distribution. Then similar to the proof of \((iv)\), we can show that the transition probability of \((\tilde{N}_1(t), \tilde{N}_2(t))^T\) converges weakly to a unique invariant ergodic measure which is denoted by \(\nu_2\), and
\[
\lim_{t \to +\infty} \langle N_i(t) \rangle = \int_{R^2_+} \nu_2(\mathrm{d}z_1, \mathrm{d}z_2) = \frac{\Delta_i - \tilde{\Delta}_i}{A_{33}}, \; i = 1, 2.
\]

The proof of \((ii)\) is similar to that of \((iii)\) and hence is omitted. \((i)\) follows from \((i')\) in Lemma 3.9, and this completes the proof. \(\square\)

4. Numerical simulations. In this section, we give some numerical simulations by using the Milstein method mentioned in Higham [14] to illustrate our main results. Consider the following discretization equations:
\[
\begin{align*}
\begin{cases}
N_1^{(k+1)} &= N_1^{(k)} \left[ r_1 - a_{11} N_1^{(k)} - a_{12} N_2^{(k-\tau_{12}/\Delta t)} - a_{13} N_3^{(k-\tau_{13}/\Delta t)} \right] \Delta t \\
&+ \sigma_1 N_1^{(k)} \sqrt{\Delta t} \zeta_1^{(k)} + \sigma_2^2 N_1^{(k)} ((\zeta_1^{(k)})^2 - 1) \Delta t,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
N_2^{(k+1)} &= N_2^{(k)} \left[ r_2 - a_{21} N_1^{(k-\tau_{21}/\Delta t)} - a_{22} N_2^{(k-\tau_{22}/\Delta t)} - a_{23} N_3^{(k-\tau_{23}/\Delta t)} \right] \Delta t \\
&+ \sigma_2 N_2^{(k)} \sqrt{\Delta t} \zeta_2^{(k)} + \sigma_2^2 N_2^{(k)} ((\zeta_2^{(k)})^2 - 1) \Delta t,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
N_3^{(k+1)} &= N_3^{(k)} \left[ r_3 - a_{31} N_1^{(k-\tau_{31}/\Delta t)} - a_{32} N_2^{(k-\tau_{32}/\Delta t)} - a_{33} N_3^{(k-\tau_{33}/\Delta t)} \right] \Delta t \\
&+ \sigma_3 N_3^{(k)} \sqrt{\Delta t} \zeta_3^{(k)} + \sigma_3^2 N_3^{(k)} ((\zeta_3^{(k)})^2 - 1) \Delta t,
\end{cases}
\end{align*}
\]
where \(\zeta_i^{(k)}, \; i = 1, 2, 3, \; k = 1, ..., n,\) are independent Gaussian random variables \(N(0, 1).\)
In this section, we choose \(r_1 = 1.2, \; r_2 = -0.15, \; r_3 = -0.01, \; a_{11} = 1.6, \; a_{12} = 1.2, \; a_{13} = 0.3, \; a_{21} = -0.85, \; a_{22} = 1.9, \; a_{23} = 0.6, \; a_{31} = -0.4, \; a_{32} = 1, \; a_{33} = 2.1, \; \tau_{12} = 3, \; \tau_{13} = 7, \; \tau_{21} = 1, \; \tau_{23} = 5, \; \tau_{31} = 4, \; \tau_{32} = 10, \; N_1(\theta) = 0.5 + 0.1 \sin \theta, \; N_2(\theta) = 0.1 + 0.05 \sin \theta, \; N_3(\theta) = 0.05 + 0.03 \sin \theta.\) Then \(A = 7.2510, \; A_1 = 4.3905, \; A_2 = 1.3442, \; A_3 = 0.1634, \; \Delta_1 = 2.4600, \; \Delta_2 = 0.7800, \; \Delta_3 = 0.4640, \; A_{11} = 3.3900, \; A_{12} = -1.5450, \; A_{13} = -0.0900, \; A_{21} = 2.2200, \; A_{31} = 0.1500.\) Clearly, Assumptions 1 and 2 hold.
(I) To begin with let us set $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.05$, $\sigma_3^2/2 = 0.05$, then $\tilde{A}_1 = 0.9135$, $\tilde{A}_2 = 0.5767$, $\tilde{A}_3 = 0.0720$, $R = 0.000035 > 0$. Then Assumption 3 holds and $\rho_3 = 2.2694 > 1$. According to (iv) in Theorem 2.1

$$\lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{A_1 - \tilde{A}_1}{A} = 0.4808,$$

$$\lim_{t \to +\infty} \langle N_2(t) \rangle = \frac{A_2 - \tilde{A}_2}{A} = 0.1058,$$

$$\lim_{t \to +\infty} \langle N_3(t) \rangle = \frac{A_3 - \tilde{A}_3}{A} = 0.0126.$$

See Fig.1(a). By using the numerical method mentioned in [7], we present the probability density functions of $N_1(t)$, $N_2(t)$ and $N_3(t)$ in Fig.1(b), which are obtained from 1000 simulation at $t = 2000$.

(II) Now let us choose $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.05$, $\sigma_3^2/2 = 0.05$, $r_1 = 1.2$, $r_2 = -0.15$, $r_3 = -0.01$, $a_{11} = 1.6$, $a_{12} = 1.2$, $a_{13} = 0.3$, $a_{21} = -0.85$, $a_{22} = 1.9$, $a_{23} = 0.6$, $a_{31} = -0.4$, $a_{32} = 1$, $a_{33} = 2.1$, $\tau_{12} = 3$, $\tau_{13} = 7$, $\tau_{21} = 1$, $\tau_{23} = 5$, $\tau_{31} = 4$, $\tau_{32} = 10$, $N_1(\theta) = 0.5 + 0.1 \sin \theta$, $N_2(\theta) = 0.1 + 0.05 \sin \theta$, $N_3(\theta) = 0.05 + 0.03 \sin \theta$.

(a) is the paths of $N_1(t)$, $N_2(t)$ and $N_3(t)$ and their time average; (b) is the probability density functions of $N_1(t)$, $N_2(t)$ and $N_3(t)$.

See Fig.2(a). Fig.2(b) is the probability density functions of $N_1(t)$ and $N_2(t)$ at $t = 2000$. 

**Figure 1.** Model 2 with $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.05$, $\sigma_3^2/2 = 0.05$, $r_1 = 1.2$, $r_2 = -0.15$, $r_3 = -0.01$, $a_{11} = 1.6$, $a_{12} = 1.2$, $a_{13} = 0.3$, $a_{21} = -0.85$, $a_{22} = 1.9$, $a_{23} = 0.6$, $a_{31} = -0.4$, $a_{32} = 1$, $a_{33} = 2.1$, $\tau_{12} = 3$, $\tau_{13} = 7$, $\tau_{21} = 1$, $\tau_{23} = 5$, $\tau_{31} = 4$, $\tau_{32} = 10$, $N_1(\theta) = 0.5 + 0.1 \sin \theta$, $N_2(\theta) = 0.1 + 0.05 \sin \theta$, $N_3(\theta) = 0.05 + 0.03 \sin \theta$.
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Figure 2. Model (2) with $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.05$, $\sigma_3^2/2 = 0.5$, other parameters are taken as Fig.1. (a) is the paths of $N_1(t)$, $N_2(t)$ and $N_3(t)$ and the time average of $N_1(t)$ and $N_2(t)$; (b) is the probability density functions of $N_1(t)$ and $N_2(t)$.

(III) Let $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.47$, $\sigma_3^2/2 = 0.5$, then we can get that $\tilde{A}_1 = 0.0486$, $\tilde{A}_2 = 1.4916$, $\tilde{A}_3 = 1.0254$, $R = 0.1565 > 0$ and $\rho_1 = 4$, $\rho_2 = 0.7746$. Clearly, Assumption 3 holds and $\rho_1 > 1 > \rho_2$. By (ii) in Theorem 2.1 we can observe that species 2 and 3 are extinct and

$$
\lim_{t \to +\infty} \langle N_1(t) \rangle = \frac{b_1}{a_{11}} = 0.5625.
$$

See Fig.3(a). Fig.3(b) presents the probability density function of $N_1(t)$ at $t = 2000$.

(IV) Finally, choose $\sigma_1^2/2 = 2$, $\sigma_2^2/2 = 0.47$, $\sigma_3^2/2 = 0.5$, then we can see that $\tilde{A}_1 = 5.8116$, $\tilde{A}_2 = 4.1181$, $\tilde{A}_3 = 0.8724$, $R = 0.0689$ and $\rho_1 = 0.6000$. It is easy to see that Assumption 3 holds and $1 > \rho_1$. According to (i) in Theorem 2.1 all the species go to extinction. See Fig.4.

Figure 3. Model (2) with $\sigma_1^2/2 = 0.3$, $\sigma_2^2/2 = 0.47$, $\sigma_3^2/2 = 0.5$, other parameters are taken as Fig.1. (a) is the paths of $N_1(t)$, $N_2(t)$ and $N_3(t)$ and the time average of $N_1(t)$; (b) is the probability density functions of $N_1(t)$. 
5. Concluding remarks. Understanding the influences of environmental perturbations on the coexistence and extinction of species are important problems in population biology. Mathematical and ecological results have shown that time delay should not be neglected ([11, 20]) and population models with three-species are more appropriate than single-species models and two-species models ([13, 40]), therefore we pay attention to a two-predator one-prey stochastic model with time delay in this paper. Our main result is Theorem 2.1 which establishes the sharp sufficient conditions for the convergence of distributions for each \( N_i(t), i = 1, 2, 3, \) and reveals the influences of environmental perturbations on the coexistence and extinction of the species.

Traditional approaches to investigate the CDS of stochastic population models are to solve the corresponding Fokker-Planck equations ([11]), or to use the Markov semigroup theory ([83, 84, 52]), or to use the Lyapunov function method ([17, 33, 55]). However, for most stochastic delay models, these three approaches can not be used. In this paper, we use an asymptotic method ([28]) to study the CDS problems, which can avoid the difficulties of these three traditional approaches. The algorithm of this method can be described as follows: consider the CDS problem of the following stochastic Lotka-Volterra model with time delay:

\[
dN_i(t) = N_i(t) \left[ r_i - a_{ii} N_i(t) - \sum_{j=1,j \neq i}^{n} a_{ij} N_j(t - \tau_{ij}(t)) \right] dt + \sigma_i N_i(t) dW_i(t), \ 1 \leq i \leq n, \tag{57}
\]

where \( N_i(\theta) = \phi_i(\theta), \ \theta \in [-\bar{\tau}, 0]. \)

**Step 1.** Establish sufficient conditions for stability in time average of all species in model (57). For most two-species models, these sufficient conditions have been obtained in the literature (see e.g., [26, 30, 48]). However, for general \( n \) species model (57), the sufficient conditions are difficult to establish. One possible way to obtain the sufficient conditions is to apply Itô’s formula to \( \ln N_i \) and then using Lemma 3.5 and the comparison theorem for stochastic equation to estimate \( \liminf_{t \to +\infty} (\langle N_i(t) \rangle) \) and \( \limsup_{t \to +\infty} (\langle N_i(t) \rangle). \)
Step 2. Prove that model (57) is globally attractive. There are two steps to show this. First, to give some conditions under which model (57) satisfies (53) (Liu [24] (Lemma 2.2) established some sufficient conditions (not necessary) for this, the author showed that if \( a_{ii} > \frac{4}{3} \sum_{j=1,j\neq i}^{3} |a_{ij}| \), then model (57) satisfies (53)). The second step is to apply Itô's formula to \( V(t) = \sum_{i=1}^{n} V_i(t) \), where

\[
V_i(t) = k_i \ln N_i(t) - \ln M_i(t) + \sum_{j \neq i,j=1}^{n} k_i|a_{ij}| \int_{t-\tau_{ij}}^{t} |N_j(s) - M_j(s)| ds,
\]

and \( k_i \) is the cofactor of the \( i \)-th diagonal element of the corresponding matrix (see the proof of Lemma 3.11), and then use (53) to complete the proof of global attractivity.

Step 3. Prove that model (57) is stability in distribution under the conditions in Step 2. The proof is a slight modification of that in Lemma 3.12 (i.e., \( R^3_+ \) changes to \( R^n_+ \) in the proof).

Step 4. Use the results in Step 3 and [42] to show that the asymptotic measure is ergodic, and then use the results in Step 1 to give the ergodic identity (see (5)), this complete the whole proof.

Now, as an example, let us study the CDS of the following three-species stochastic delay mutualism system (38) by using the above algorithm:

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left[ r_1 - a_{11}x_1(t) - a_{12}x_2(t - \tau_{12}) - a_{13}x_3(t - \tau_{13}) \right] dt + \sigma_1 x_1(t) dW_1(t), \\
\dot{x}_2(t) &= x_2(t) \left[ r_2 - a_{21}x_1(t - \tau_{21}) - a_{22}x_2(t) - a_{23}x_3(t - \tau_{23}) \right] dt + \sigma_2 x_2(t) dW_2(t), \\
\dot{x}_3(t) &= x_3(t) \left[ r_3 - a_{31}x_1(t - \tau_{31}) - a_{32}x_2(t - \tau_{32}) - a_{33}x_3(t) \right] dt + \sigma_3 x_3(t) dW_3(t),
\end{align*}
\]

(58)

where \( a_{ii} > 0 \) and \( a_{ij} < 0 \) for \( i, j = 1, 2, 3, j \neq i \), \( x_i(\theta) = \phi_i(\theta), \theta \in [-\bar{T}, 0] \). For model (58), Xia et al. [48] have shown that if \( a_{ii} > -\sum_{j=1,j\neq i}^{3} a_{ij}, \ i = 1, 2, 3, \) and

\[
r_i > \frac{\sigma^2}{2}, \ i = 1, 2, 3,
\]

(59)

then model (58) is stability in time average:

\[
\lim_{t \to +\infty} \langle x_i(t) \rangle = \frac{A_i - \hat{A}_i}{A} \text{ a.s., } \ i = 1, 2, 3.
\]

(60)

Now let us turn to Step 2. In fact, according to the methods in Step 2, we can show that if

\[
a_{ii} > -\frac{4}{3} \sum_{j=1,j\neq i}^{3} a_{ij}, \ i = 1, 2, 3
\]

(61)

then model (58) is globally attractive, and hence by the results in Step 3, model (58) is stability in distribution. Finally, by Step 4 and (60), we obtain the following results:

**Proposition 1.** If (59) and (61) hold, then the distribution of \( (x_1(t), x_2(t), x_3(t))^T \) in model (58) converges weakly to a unique invariant distribution \( \nu_4 \) which is ergodic:

\[
\lim_{t \to +\infty} \langle x_i(t) \rangle = \int_{\mathbb{R}^3_+} z_i \nu_4(dz_1, dz_2, dz_3) = \frac{A_i - \hat{A}_i}{A}, \ i = 1, 2, 3, \text{ a.s.}
\]
Figure 5. Solutions of model (2) with $a_{12} = 1.32$, other parameters are taken as Fig.1(a), initial values $N_1(\theta) = 0.5 + 0.1 \sin \theta$, $N_2(\theta) = 0.1 + 0.05 \sin \theta$, $N_3(\theta) = 0.05 + 0.03 \sin \theta$, $M_1(\theta) = 0.4 + 0.1 \sin \theta$, $M_2(\theta) = 0.3 + 0.05 \sin \theta$, $M_3(\theta) = 0.1 + 0.05 \sin \theta$.

Some interesting topics deserve further investigation. An interesting question is to consider what happens if Assumptions 2 and 3 do not hold. It is useful to point out that Assumptions 2 and 3 are just sufficient. For example, Assumption 2 is used to prove the global attractiveness of model (2). In Fig.5, Assumption 2 does not hold. However, Fig.5 shows that model (2) still is globally attractive. Another question is to take other random noises into account, for example, regime switching (see e.g., [32, 54, 49]) or Lévy noises (see e.g., [2, 51]). We leave these questions for future work.

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