In this work the construction of supergravity duals to the noncommutative $\mathcal{N} = 4$ SYM theory in the infinite momentum frame but with constant momentum density is attempted. In the absence of the content of noncommutativity, it has been known for some time that the previous $AdS_5/CFT_4$ correspondence should be replaced by the $K_5/CFT_4$ (with $K_{(p+2)}$ denoting the generalized Kaigorodov spacetime) correspondence with the pp-wave propagating on the BPS brane worldvolume. Interestingly enough, putting together the two contents, i.e., the introduction of noncommutativity and at the same time that of the pp-wave along the brane worldvolume, leads to quite nontrivial consequences such as the emergence of “time-space” noncommutativity in addition to the “space-space” noncommutativity in the manifold on which the dual gauge theory is defined. Taking the gravity decoupling limit, it has been realized that for small $u$, the solutions all reduce to $K_5 \times S^5$ geometry confirming our expectation that the IR dynamics of the dual gauge theory should be unaffected by the noncommutativity while as $u \to \infty$, the solutions start to deviate significantly from $K_5 \times S^5$ limit indicating that the UV dynamics of the dual gauge theory would be heavily distorted by the effect of noncommutativity.
1 Introduction

The celebrated $AdS/CFT$ correspondence \cite{1} is a conjectured (and by now fully tested) equivalence between the two seemingly very different theories; type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in $D = 4$ with gauge group $SU(N)$. Since the advent of this original duality conjecture, quite a few attempts have been made to extend it to a larger context of gauge/gravity duality. The motivation of the present work can be thought of as being along this line as it attempts to extend the original correspondence to the case when the dual CFT is in an infinitely-boosted frame on a noncommutative manifold. To be more concrete, we would like to construct the extremal $D3$-brane solution with a superimposed gravitational pp-wave in the presence of a $NSNS$ $B$-field background that is supposed to be the supergravity dual (in the $AdS/CFT$ sense) to the non-commutative $\mathcal{N} = 4$ super Yang-Mills (NCSYM) in the infinite momentum frame. Thus at this point, it seems relevant to address the issue of $AdS/CFT$ correspondence in cases where there is a pp-wave (for rather rigorous definition of the pp-wave, see appendix B) propagating along a direction longitudinal to a classical $p$-brane worldvolume $\mathcal{F}$. And in this discussion, one has to carefully distinguish between the two cases; the BPS-case and the non-BPS case.

First in the non-BPS case, the effect of including the gravitational pp-wave is locally equivalent to performing a Lorentz boost along the propagation direction of the wave. (This point will be demonstrated explicitly in the appendix A.) Furthermore, if the direction along which the pp-wave propagates is uncompactified, then the equivalence is indeed valid globally, while if the propagation direction is wrapped on a circle, it is valid only locally. For this reason, $p$-branes with superimposed pp-waves propagating on their worldvolumes are often referred to as “boosted” $p$-branes. One, however, should bear in mind that the global structure may not be precisely describable by a Lorentz boost. In the BPS case, on the other hand, the inclusion of the pp-wave and the performance of a Lorentz boost along the propagation direction are not even locally equivalent. This is because in the BPS limit, the Lorentz boost that relates the two metrics, one with the pp-wave and the other without, becomes singular, corresponding to the “infinite boost” with the velocity approaching the speed of light. Thus in the BPS limit, one has essentially two distinct configurations corresponding to the cases with and without the pp-wave which are not even locally equivalent.

Next, an interesting consequence of this observation is the new structure of the near-horizon geometry of the boosted $p$-branes. It is well-known that in the absence of the pp-wave, the typical near-horizon geometries corresponding to the decoupling limit is of type $AdS_{(p+2)} \times S^{D-(p+2)}$ for extremal, BPS $p$-branes such as BPS $M2$, $M5$ and $D3$-branes. On the other hand, in the presence of the pp-wave propagating on a BPS $p$-brane, one can find that the $AdS$-metric of near-horizon geometry is replaced by a new type of metric, which in 4-dimensions was first constructed long ago by Kaigorodov \cite{5} (and turned out to be of type $N$ in the Petrov classification). In other words, generally the near-horizon geometry of a BPS $p$-brane with a pp-wave
propagating on its worldvolume turns out to be of the type $K_{(p+2)} \times S^{D-(p+2)}$ with $K_{(p+2)}$ denoting the $(p+2)$-dimensional generalization of the Kaigorodov metric. Like $AdS$ geometry, the Kaigorodov spacetime is a homogeneous Einstein manifold, but they differ significantly in both their local and global structures. In particular, although the Kaigorodov spacetimes approach $AdS$ locally at infinity, their boundaries are related to those of $AdS$ by an infinity Lorentz boost. This implies that the boundary of the generalized Kaigorodov metric is in an infinite momentum frame and one can moreover show that in the gravity decoupling limit, in order to maintain the structure of the Kaigorodov metric, the momentum per unit $p$-volume must be held fixed. As a consequence, one may conclude that the previous $AdS_{(p+2)}/CFT_{(p+1)}$ correspondence [1], the supergravity-boundary field theory duality that we associated to the extremal BPS $p$-brane solutions should now be replaced by the $K_{(p+2)}/CFT_{(p+1)}$ (with an infinite boost and a constant momentum density) correspondence in the presence of the pp-wave propagating on the $p$-brane solutions. That is, in the spirit of gauge/gravity correspondence, it appears to be natural to conjecture that string theory in $K_{(p+2)} \times S^{D-(p+2)}$ is dual to some CFT in an infinitely-boosted frame, i.e., in the infinite momentum frame. Indeed, this conjecture has been tested and actually confirmed to be true in a recent literature [3]. Next, in the case of non-BPS $p$-branes, the situation is somewhat different. Namely, in the presence of the pp-wave propagating on non-BPS $p$-branes, the near-horizon geometries turn out to be of the type corresponding to the $\{\text{Carter-Novotný-Horský spacetime}\} \otimes S^{D-(p+2)}$. An interesting point, however, in this non-BPS case is that there is locally no distinction between the case where there is a superimposed pp-wave, and the case with no pp-wave. In fact, this can be attributed to the fact that a coordinate transformation (which is, as mentioned earlier, a Lorentz boost along the wave propagation direction) allows the harmonic function associated with the pp-wave to be set to unity. As a consequence, the local form of the Carter-Novotný-Horský (CNH) [2] metrics remains the same whether or not a pp-wave is included in the original $p$-brane solution. And the coordinate transformation (namely, the Lorentz boost) becomes singular in the extremal BPS-limit, which explains why there are two distinct cases in the BPS-situation leading either to the $AdS$ or else to the generalized Kaigorodov metrics after spherical reduction. To summarize, unlike in the extremal case, in the non-extremal case, the pp-wave content can be erased by a coordinate transformation (i.e., via a finite Lorentz boost). And for this reason, we shall in the present work, focus exclusively on the extremal limit in which case there are two clearly distinct correspondences: $AdS/CFT$ and $K/CFT$.

With this observations in mind, we now turn to the explicit construction of the “extremal” $(D1-D3)$ system with a superimposed gravitational pp-wave. From the rules for intersecting branes, it is known [6] that $M$-branes are parallelly intersecting with the gravitational wave, i.e., $M2||W$ and $M5||W$. Thus starting with one of these configurations and applying the well-known duality web, one can deduce a set of rules for putting $W$ (the pp-wave) on various p-branes in $D=10$ type II supergravity (SUGRA). Therefore, following this standard procedure, we shall consider the following duality chain to obtain the $(D1-D3)$ system in the $NSNS$ $B$-field background with a pp-wave.

\[
M2||W \xrightarrow{\text{(KK)}} D2||W \xrightarrow{\text{(BMM-T)}} (D1-D3)||W
\]  

(1)
where \( KK \) denotes the Kaluza-Klein dimensional reduction from \( D = 11 \) to \( D = 10 \) along an isometry direction and BMM-T indicates the procedure suggested by Brekenridge, Michaud and Myers [13] to generally construct a \( D_p - D_{(p+2)} \) bound state via a \( T \)-duality transformation. Note that the endpoint of this duality chain is an “electric” solution charged under electric \( RR \) tensor field. If instead one follows the other duality chain, say,

\[
M5||W \xrightarrow{(KK)} D4||W \xrightarrow{(T)} D3||W \xrightarrow{(T)} D2||W \xrightarrow{(BMM-T)} (D1 - D3)||W,
\]

then one would end up with a “magnetic” \( (D1 - D3) \) system in the \( NSNS \) \( B \)-field background with a pp-wave. And this is because the starting point is the \( M5 \)-brane (with superimposed pp-wave) which is the magnetic dual of the electric \( M2 \)-brane solution of \( D = 11 \) SUGRA.

As we mentioned, our primary concern in this work is the construction of supergravity duals to the noncommutative \( \mathcal{N} = 4 \) SYM theory in the infinite momentum frame (but with constant momentum density)\(^1\). Indeed, the construction of supergravity duals to the commutative \( \mathcal{N} = 4 \) SYM theory in the infinite momentum frame has been discussed by Cvetic, Lu and Pope [2] and the supergravity duals to the noncommutative \( \mathcal{N} = 4 \) SYM theory has been constructed by Hashimoto and Itzhaki [7] and by Maldacena and Russo [8] some time ago. Thus it may seem that the present work is a natural extension of these earlier works containing rather straightforwardly enlarged results. Interestingly enough, however, we shall see in a moment that putting the two contents, i.e., the introduction of noncommutativity and at the same time that of the pp-wave along the brane worldvolume, together leads to quite non-trivial consequences such as the emergence of “time-space” noncommutativity in addition to the “space-space” noncommutativity in the manifold on which the dual gauge theory is defined.

Lastly, perhaps it would be appropriate to distinguish the motivation and nature of the present work from those of the recent development in the string theory on maximally supersymmetric pp-wave in association with the \( AdS/CFT \) correspondence. Penrose [9] has long ago pointed out that, in the neighborhood of a null geodesic, all spacetimes locally have a plane wave as a limit. Indeed, plane waves are known examples of exact classical string vacuum. Thus by taking the Penrose limit, any exact classical string vacuum can be related to the plane waves. Then the great recent interest in the string/M-theory in the plane wave background resulted from the realization that the maximally supersymmetric plane wave solution of type IIB SUGRA can thus be obtained from the Penrose limit of the string vacuum solution, \( AdS_5 \times S^5 \)[10] and that the superstring theory (particularly in the Green-Schwarz formalism with the choice of light-cone gauge) is exactly soluble in this plane wave background [11]. Thus the motivation and the nature of this programme is to extend the \( AdS/CFT \) correspondence to the regime of massive string states [12] whereas those of the present work is to extend the original \( AdS/CFT \) correspondence still for the massless string spectrum to \( K/CFT \) correspondence with the boundary gauge theory being defined in the noncommutative manifold and

\(^1\)We were informed that some of the issues related to the ones addressed in the present work also has been discussed in [4].
moreover in the infinite momentum frame with a constant momentum density. We hope that the distinction between the two is now clear.

2 Near-horizon geometries of extremal/non-extremal M2-brane with a superimposed pp-wave

We begin with the non-extremal M2-brane solution with a superimposed gravitational pp-wave (see appendix A for a derivation)

\[ ds^2_{11} = H^{-2/3} \left[ -K^{-1} f dt^2 + K \{ dx_1 + \coth \mu_1 (K^{-1} - 1) dt \}^2 + dx_2^2 \right] + H^{1/3} \left[ f^{-1} dr^2 + r^2 d\Omega_7^2 \right], \]

\[ A_{[3]} = \coth \mu_1 (H^{-1} - 1) (dt \wedge dx_1 \wedge dx_2) \quad \text{with} \]

\[ H(r) = 1 + \frac{Q_1}{r^6}, \quad K(r) = 1 + \frac{Q_2}{r^6}, \quad f(r) = 1 - \frac{\mu}{r^6}, \]

\[ Q_1 = \mu \sinh^2 \mu_1, \quad Q_2 = \mu \sinh^2 \mu_2, \quad \mu = k \kappa_1^{4/3} \]

where \( Q_1 \) is the usual (electric) RR charge and \( Q_2 \) is a new parameter representing the momentum along the \( x_1 \)-direction, i.e., \( \mu_2 \) parameterizes the Lorentz boost factor as \( \gamma = (1 - \beta^2)^{-1/2} = \cosh \mu_2 \) with \( \beta = \tanh \mu_2 \). And \( \kappa_1 \) denotes the 11-dimensional gravitational constant. Then the associated extremal solution amounts to the limiting case

\[ \mu \to 0, \quad \mu_1, \mu_2 \to \infty \quad \text{with} \quad Q_1 = \mu \sinh^2 \mu_1, \quad Q_2 = \mu \sinh^2 \mu_2 \quad \text{kept fixed} \]

when the solution above becomes

\[ ds^2_{11} = H^{-2/3} \left[ -K^{-1} dt^2 + K \{ dx_1 + (K^{-1} - 1) dt \}^2 + dx_2^2 \right] + H^{1/3} \left[ dr^2 + r^2 d\Omega_7^2 \right], \]

\[ A_{[3]} = (H^{-1} - 1) (dt \wedge dx_1 \wedge dx_2). \]

In this section, we would like to explore the nature of the near-horizon geometries of both extremal and non-extremal M2-brane solutions with the superimposed pp-wave in some detail following [2].

2.1 Extremal solution

We begin with the extremal case. Note first that under the coordinate transformation

\[ t \to \frac{3}{2} t - \frac{1}{2} x_1, \quad x_1 \to \frac{1}{2} t + \frac{1}{2} x_1, \]

it follows \( K \to (K - 1) = Q_2/r^6 \). This implies that the constant term “1” may be dropped from \( K(r) \) when it is more convenient to do so as it can always be removed via the coordinate transformation given in eq.(6). Next, the near-horizon region is defined to be \( r \to 0 \) where
Thus the metric of the near-horizon geometry of the extremal $M_2$-brane with superimposed pp-wave becomes
\[
\begin{align*}
  ds_{11}^2 &= Q_1^{-2/3} r^4 \left[-K^{-1} dt^2 + K \{dx_1 + (K^{-1} - 1) dt\}^2 + dx_2^2\right] + Q_1^{1/3} \frac{dt^2}{r^2} + Q_1^{1/3} d\Omega_7^2.
\end{align*}
\] (7)

Thus spacetime represented by this metric is a product $M_4 \times S^7$ and particularly, since the coefficient of the $S^7$ metric $d\Omega_7^2$ is a constant, $M_4$ here must be an Einstein manifold with its metric being a solution of $D = 4$ gravity with a pure cosmological constant term
\[
S_4 = \int d^4x \sqrt{|g_4|} R_4 - 2\Lambda
\] (8)
where $\Lambda = -12Q_1^{-1/3}$. Now, upon the $S^7$ reduction and writing $r = e^\rho$, the metric of the near-horizon geometry given above takes the form
\[
\begin{align*}
  ds_4^2 &= Q_1^{1/3} \left[-e^{10\rho} dt^2 + e^{-2\rho}(dx_1 + e^{6\rho} dt)^2 + e^{4\rho} dx_2^2 + d\rho^2\right]
\end{align*}
\] (9)
where the charge parameters have been partly absorbed by rescaling the worldvolume coordinates. It is straightforward to see that this is an homogeneous Einstein metric and indeed it can be identified with the metric discovered first by Kaigorodov. We shall henceforth denote it by $K_4$ and its generalization to arbitrary dimensions by $K_n$. Note also that the $K_4$-metric has a 5-dim. isometry group (i.e., it possesses 5-Killing vectors) and it preserves $1/4$ of the supersymmetry owned by the Minkowsi metric. Lastly, it might be of some interest to compare this Kaigorodov metric with that of $AdS$ generally in $D = (n + 3)$ dimensions. Consider the following family of metrics
\[
\begin{align*}
  ds_D^2 &= -e^{2a\rho} dt^2 + e^{2b\rho}[dx + e^{(a-b)\rho} dt]^2 + e^{2c\rho} dy^i dy^i + d\rho^2
\end{align*}
\] (10)
with $1 \leq i \leq n$. Then there are actually two inequivalent solutions corresponding to the metric having this ansatz that solves the Einstein equation $R_{\mu\nu} = \Lambda g_{\mu\nu}$ for a Einstein manifold and they are
\[
[AdS_{n+3}] \quad a = b = c = 2L,
\]
\[
[K_{n+3}] \quad a = (n + 4)L, \quad b = -nL, \quad c = 2L
\]
with $L \equiv \frac{1}{2} \sqrt{\frac{-\Lambda}{(n + 2)}}$. ($\Lambda < 0$)

\textbf{2.2 Non-extremal solution}

Next, we turn to the study of the near-horizon geometry of the non-extremal $M_2$-brane with a superimposed pp-wave. In the near-horizon region, where $H(r) \sim Q_1/r^6 = (k\kappa_1^{4/3}) \sinh^2 \mu_1/r^6$, the metric of non-extremal $M_2$-brane with a superimposed pp-wave becomes
\[
\begin{align*}
  ds_{11}^2 &= Q_1^{-2/3} r^4 \left[-K^{-1} f dt^2 + K \{dx_1 + \coth \mu_2(K^{-1} - 1) dt\}^2 + dx_2^2\right] + Q_1^{1/3} \frac{dt^2}{r^2} + Q_1^{1/3} d\Omega_7^2.
\end{align*}
\] (11)
which is again of the metric form for $M_4 \times S^7$ with $M_4$ metric being a solution of $D = 4$ pure gravity with only a cosmological constant term represented by the action

$$S_4 = \int d^4x \sqrt{g_4}[R_4 - 2\Lambda]$$

(12)

where $\Lambda = -12Q_1^{-1/3} = -12(k\kappa_{11}^{4/3} \sinh^2 \mu_1)^{-1/3}$. As before, we take $S^7$ as the “internal” sphere having the metric $ds^2 = Q_1^{1/3} d\Omega_7^2 = (k\kappa_{11}^{4/3} \sinh^2 \mu_1)^{1/3} d\Omega_7^2$ or equivalently having the radius $R_7 = Q_1^{1/6} = (k\kappa_{11}^{4/3} \sinh^2 \mu_1)^{1/6}$. Then upon the $S^7$-reduction and writing $r = e^\rho$, the metric of the near-horizon geometry given above takes the form

$$ds^2 = Q_1^{2/3} e^{4\rho} \left[-K^{-1} f dt^2 + K \{dx_1 + \coth \mu_2(K^{-1} - 1)dt\}^2 + dx_2^2\right] + Q_1^{1/3} f^{-1} d\rho^2$$

(13)

with $K(\rho) = 1 + k \sinh^2 \mu_2 e^{-6\rho}$, $f(\rho) = 1 - k e^{-6\rho}$. (Here, we have set the gravitational constant $\kappa_{11}$ to unity for convenience.) This is indeed a metric for an Einstein manifold found by Carter and by Novotný and Horský (CNH). In the asymptotic region where $r \to \infty$, we have $f(r) \to 1$ and hence this CNH metric goes over to the Kaigorodov metric discussed earlier. Lastly, we comment on the generalization of this (originally 4-dimensional) CNH metric to arbitrary dimensions for later use. First, the generalized CNH metric that typically arises in the spherical $S^{D-1}$-reduction of the non-extremal $p$-brane with a superimposed pp-wave is given by

$$ds^2 = c_1 e^{4\rho} \left[-K^{-1} f dt^2 + K \{dx_1 + \coth \mu_2(K^{-1} - 1)dt\}^2 + dy^i dy^i\right] + c_2 f^{-1} d\rho^2$$

(14)

with $K(\rho) = 1 + k \sinh^2 \mu_2 e^{-6\rho}$, $f(\rho) = 1 - ke^{-6\rho}$ and $d = (p + 1)$, $(\tilde{d} + 2) = D - d$. Or more generally, the solution to the Einstein equation $R_{\mu \nu} = \Lambda g_{\mu \nu}$ in $D = (n + 3)$ that represents the generalization of the above CNH metric is given by

$$ds^2_D = -e^{2a \rho} f dt^2 + e^{2b \rho} [dx + e^{(a-b)\rho} dt]^2 + e^{2c \rho} dy^i dy^i + f^{-1} d\rho^2$$

(15)

where $1 \leq i \leq n$, $f(\rho) = 1 - ke^{-(a-b)\rho}$ and

$$a = (n + 4)L, \quad b = -nL, \quad c = 2L, \quad \text{with} \quad L = \frac{1}{2} \sqrt{-\Lambda \over (n+2)} \quad (\Lambda < 0).$$

(16)

Note also that this generalized CNH metric can also be put in the form

$$ds^2_D = e^{-2nL \rho} dx^2 + e^{4L \rho} (2 dx dt + k dt^2 + dy^i dy^i) + [1 - ke^{-2(n+2)L \rho}]^{-1} d\rho^2.$$  

(17)

Both of these expressions for the generalized CNH metric given in eqs.(15) and (17) reduce to that the generalized Kaigorodov metric given in eq.(10) for $k = 0$. This completes the study of near-horizon geometries of $M2$-brane with a superimposed pp-wave. In the following section, we get back to our main task of constructing the extremal $(D1 - D3)$ system with a superimposed pp-wave in the presence of the NSNS $B$-field.
3 Construction of \((D1 - D3)\) system with a superimposed pp-wave

3.1 Construction of the solution

We now describe our strategy briefly. From the rules for intersecting branes, it is known [6] that \(M\)-branes are parallely intersecting with the \(M\)-wave, i.e., \((1|M2, W)\) (or \(M2||W\) for short) and \((1|M5, W)\) (or \(M5||W\) for short). Thus (I) we shall start with the extremal \(M2\)-brane solution with a superimposed gravitational pp-wave, \(M2||W\) and then perform a Kaluza-Klein (KK) dimensional reduction to \(D = 10\) along a U(1)-isometry direction which is chosen to be a coordinate transverse to the \(M2\)-brane to get \(D2||W\) in IIA theory. Then next, (II) we shall proceed with the procedure suggested by Breckenridge, Michaud and Myers (BMM) [13] to finally obtain \((D1 - D3)||W\) via a T-dual transformation:

(I) \(M2||W \overset{\text{KK}}{\longrightarrow} D2||W\)

Consider the extremal \(M2\)-brane solution with a superimposed pp-wave and its KK reduction,

\[
ds_{11}^2 = H^{-2/3} \left[ -K^{-1} dt^2 + K \left\{ dx_1 + (K^{-1} - 1)dt \right\}^2 + dx_2^2 \right] + H^{1/3} \left[ \sum_{m=3}^{10} dx_m^2 \right] \\
A_{[3]} = \left( H^{-1} - 1 \right) dt \wedge dx_1 \wedge dx_2 \\
A_{[3]} = \frac{1}{3!} A_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda + \frac{1}{2!} B_{\mu\nu} dx^\mu \wedge dx^\nu \wedge dy
\]

where \(A_{[3]}\) and \(B_{[2]}\) denote respectively a 3-form RR and a 2-form NSNS potential and \(A^{(1)}\) is the KK gauge field. \(H(r)\) and \(K(r)\) are as given before. Reduction to \(D = 10\) along a direction transverse to the \(M2\)-brane amounts to choosing, say, \(y = x_3\). This then implies that we should identify \(e^{4/3\phi} = H^{1/3}\) which, in turn, yields

\[
e^{2\phi} = g_s^2 H^{1/2}, \quad A_{[1]}^{(1)} = 0, \quad B_{[2]}^{(2)} = 0, \quad A_{\mu\nu\lambda}^{(3)} = \{ A_{\mu\nu\lambda}^{(3)} = (H^{-1} - 1) \}.
\]

Thus the result is a \(D2\)-brane solution in \(D = 10\) type IIA SUGRA with a superimposed pp-wave given by

\[
ds_{10}^2 = H^{-1/2} \left[ -K^{-1} dt^2 + K \left\{ dx_1 + (K^{-1} - 1)dt \right\}^2 + dx_2^2 \right] + H^{1/2} \left[ \sum_{m=3}^9 dx_m^2 \right], \\
A_{[3]} = \frac{1}{g_s^2} (H^{-1} - 1) dt \wedge dx_1 \wedge dx_2, \\
e^{2\phi} = g_s^2 H^{1/2}
\]
where \( g_s \) denotes the string coupling representing \( g_s = e^{\phi_{\infty}} \).

\[
(II) \quad D2||W \xrightarrow{(\text{BMM} - \text{T})} (D1 - D3)||W
\]

Finally, following the procedure suggested by Breckenridge, Michaud and Myers (BMM)\(^{[13]}\), we now construct the bound state of extremal \( D1 - D3 \) system with a superimposed pp-wave. For the case at hand, the suggested procedure of BMM to construct a \( D1 - D3 \) system in the \( NSNS \) B-field background consists of the following 3-steps.

(i) Delocalize (or smear out) the \( D2 \)-brane (oriented in \((x_1 - x_2)\) plane) in one of the transverse (say, \( x_3 \)) directions.

(ii) Perform a rotation on the delocalized \( D2 \)-brane in \((x_2 - x_3)\) plane namely, we make it to be ‘tilted’ in \((x_2 - x_3)\) plane.

(iii) Apply \( T \)-duality on (rotated) \( \tilde{x}_3 \)-direction.

In other words, we begin with the extremal \( D2 \)-brane plus wave given above but oriented instead at an angle in \((x_2 - x_3)\) plane and then apply \( T \)-duality on \( x_3 \) to find a solution describing the bound state of an extremal \( D1 \) and \( D3 \)-brane with a pp-wave. Thus we start by rewriting the \( D2||W \) solution given above as

\[
\begin{align*}
ds_{10}^2 &= H^{1/2} \left[ -K^{-1} dt^2 + K \{ dx_1 + (K^{-1} - 1) dt \}^2 + \sum_{m=3}^{9} dx_m^2 + \sum_{m=3}^{9} dx_m^2 \right], \\
A_{[3]} &= \frac{1}{g_s} (H^{-1} - 1) dt \wedge dx_1 \wedge dx_2, \\
e^{2\phi} &= g_s^2 H^{1/2} = g_s^2 \left[ 1 + \frac{Q_1}{r^5} \right]^{1/2}.
\end{align*}
\]

Recall that \( H(r) \) here is a harmonic function in the transverse coordinates which solves the Poisson’s equation with some delta function source. According to the suggested prescription of BMM, one needs a slightly different harmonic function \( H(r) \) (and \( K(r) \) as well for the case at hand where we consider the superposition of a pp-wave) in order to ‘delocalize’ the present extremal \( D2 \)-brane (with a pp-wave) in one of the transverse directions, say, \( x_3 \). And then they pointed out that this can be done in at least two different ways. Firstly, the delta function source can be chosen so that \( \partial_i \partial^i H = -5 Q_1 A_6 \prod_{i=3}^{9} \delta(x_i) \) (where \( A_n \) denotes the area of a unit \( n \)-sphere) and the delocalization of the \( D2 \)-brane can be achieved by following the so-called ‘vertical reduction’ approach. Namely, one adds an infinite number of identical sources in a periodic array along the \( x_3 \)-axis. Then a smeared solution may be extracted from the long-range
fields, for which the $x_3$-dependence is exponentially suppressed. The second approach, which may be termed, ‘vertical oxidation’, consists in simply replacing the above 7-dimensional delta function source by that of a line source extending along $x_3$, i.e., $\partial_i \partial^i H = -4Q_1 A_5 \prod_{i=4}^9 \delta(x_i)$. Whichever method one may employ, the number of dimensions transverse to the “smeared-out” $D2$-brane becomes ‘effectively’ only 6 rather than 7, i.e.,

$$ds_{10}^2 = H^{1/2} \left[ \frac{-K^{-1} dt^2 + K \{dx_1 + (K^{-1} - 1)dt\}^2 + dx_2^2 + dx_3^2 + \sum_{m=4}^9 dx_m^2}{H} \right],$$

$$H(r) = 1 + \frac{Q_1}{r^4}, \quad K(r) = 1 + \frac{Q_2}{r^4} \quad \text{and} \quad r^2 = \sum_{i=4}^9 x_i^2. \quad (22)$$

Then the form of the antisymmetric $RR$ tensor potential $A_{[3]} = \frac{1}{g_s} (H^{-1} - 1)dt \wedge dx_1 \wedge dx_2$ reveals that we now have a $D2$-brane oriented along $(x_1 - x_2)$ plane and smeared out in $x_3$-direction. We now consider performing a rotation on our delocalized $D2$-brane. Note, however, that since our $D2$-brane was originally extended in $(x_1 - x_2)$ plane, we may have two inequivalent options:

(A) the rotation in $(x_2 - x_3)$ plane with $x_2$ being the ‘spectator’ direction (with respect to the superimposed pp-wave) or

(B) the rotation in $(x_1 - x_3)$ plane now with $x_1$ being the ‘boost’ (or wave propagation) direction.

Obviously, the option (A) would not distort the pp-wave propagating on the brane world-volume and we discuss this case first and then the option (B) later on. Namely the rotation

$$\begin{cases} 
    dx_3 = \cos \varphi d\tilde{x}_3 - \sin \varphi d\tilde{x}_2, \\
    dx_2 = \sin \varphi d\tilde{x}_3 + \cos \varphi d\tilde{x}_2
\end{cases} \quad (23)$$

with $\varphi$ being the angle between $\tilde{x}_2$-axis and $x_2$-axis, takes the $D2$-brane plus the pp-wave solution given above to

$$ds_{10}^2 = H^{1/2} \left[ \frac{-K^{-1} dt^2 + K \{dx_1 + (K^{-1} - 1)dt\}^2 + dx_2^2 + dx_3^2 + \sum_{m=4}^9 dx_m^2}{H} \right] + \left( \frac{\cos^2 \varphi}{H} + \sin^2 \varphi \right) d\tilde{x}_2^2$$

$$+ \left( \frac{\sin^2 \varphi}{H} + \cos^2 \varphi \right) d\tilde{x}_3^2 + 2 \cos \varphi \sin \varphi \left( \frac{1}{H} - 1 \right) d\tilde{x}_2 d\tilde{x}_3 + \sum_{m=4}^9 dx_m^2 \right],$$

$$A_{[3]} = \frac{1}{g_s} (H^{-1} - 1)dt \wedge dx_1 \wedge (\cos \varphi d\tilde{x}_2 + \sin \varphi d\tilde{x}_3), \quad (24)$$

$$e^{2\phi} = g_s^2 H^{1/2} = g_s^2 \left[ 1 + \frac{Q_1}{r^4} \right]^{1/2}. \quad (25)$$
Lastly, applying the generalized Buscher’s $T$-duality\textsuperscript{2} on $\tilde{x}_3$, we end up with

\[
\begin{align*}
 ds_{10}^2 &= H^{1/2} \left[ -K^{-1}dt^2 + K\{dx_1 + (K^{-1} - 1)dt\}^2 + \frac{d\tilde{x}_2^2 + d\tilde{x}_3^2}{1 + (H - 1) \cos^2 \varphi} + \sum_{m=4}^{9} dx_m^2 \right],
 B_{[2]} &= \frac{1}{g_s} \frac{(H - 1) \cos \varphi \sin \varphi}{1 + (H - 1) \cos^2 \varphi}(d\tilde{x}_2 \wedge d\tilde{x}_3),
 A_{[2]} &= \frac{1}{g_s} \frac{(H - 1) \sin \varphi (dt \wedge dx_1)},
 A_{[4]} &= \frac{1}{g_s} \frac{(H - 1) \cos \varphi}{1 + (H - 1) \cos^2 \varphi}(dt \wedge dx_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3),
 e^{2\varphi} &= \frac{g_s^2}{H} \frac{1}{1 + (H - 1) \cos^2 \varphi}
\end{align*}
\]

with $H(r)$, $K(r)$ as given before. Note that it is evident from the emergence of $RR$ potentials $A_{[2]}$ and $A_{[4]}$ that we indeed have a bound state of a $D1$ and $D3$-branes. Moreover, this solution manifests itself as representing a $D1$-brane “dissolved” in the $D3$-brane and having a superimposed pp-wave propagating on its worldvolume.

As has been noted in the introduction, one can obtain a “magnetic” ($D1 - D3$) system in the $NSNS$ B-field background with a pp-wave as well by starting with the $M5$-brane (with superimposed pp-wave) and following the duality chain given in eq.(2). Then the magnetically $RR$-charged solution turns out to be the same as the electrically-charged solution given above except that now the $RR$ tensor field strengths are given, instead of $F_{[3]}^e = dA_{[2]}$ and $F_{[5]}^e = dA_{[4]}$, by

\[
\begin{align*}
 F_{[7]}^m &= * F_{[3]}^e = * (dA_{[2]}) = -\frac{4Q_1 \sin \varphi}{g_s r^6[1 + (H - 1) \cos^2 \varphi]} \times \\
 & \left[ x_4(dx_2 \wedge dx_3 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) - x_5(dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) \\
 & + x_6(dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_7 \wedge dx_8 \wedge dx_9) - x_7(dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_8 \wedge dx_9) \\
 & + x_8(dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_9) - x_9(dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8) \right], \\
 F_{[5]}^m &= * F_{[3]}^e = * (dA_{[4]}) = \frac{4Q_1 H \cos \varphi}{g_s r^6[1 + (H - 1) \cos^2 \varphi]} \times \\
 & \left[ x_4(dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) - x_5(dx_4 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) \\
 & + x_6(dx_4 \wedge dx_5 \wedge dx_7 \wedge dx_8 \wedge dx_9) - x_7(dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_8 \wedge dx_9) \\
 & + x_8(dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_9) - x_9(dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8) \right]
\end{align*}
\]

where the Hodge dual is taken with respect to the metric solution given in eq.(25) above. Note that $F_{[5]}^m \neq F_{[5]}^e$ and hence $F_{[5]}^e \neq * F_{[5]}^e$, namely the 5-form $RR$ field strength is not self-dual.

Next, we turn to the option (B) in which the $D2$-brane (delocozalized in the $x_3$-direction) is to be rotated in $(x_1 - x_3)$ plane with $x_1$ being the boost (i.e., wave propagation) direction.

\textsuperscript{2}The explicit generalized Bucher’s $T$-duality rules employed in this work can be found for instance in [14].
Namely, upon the rotation

\[
\begin{aligned}
&dx_3 = \cos \varphi d\tilde{x}_3 - \sin \varphi d\tilde{x}_1, \\
&dx_1 = \sin \varphi d\tilde{x}_3 + \cos \varphi d\tilde{x}_1
\end{aligned}
\]  

(28)

with this time \( \varphi \) being the angle between \( \tilde{x}_1 \)-axis and \( x_1 \)-axis, the \( D2 \)-brane plus the pp-wave solution given above becomes

\[
d s_{10}^2 = H^{1/2} \left[ \frac{1}{H} \{- (2 - K) dt^2 + 2(1 - K) dt (\cos \varphi d\tilde{x}_1 + \sin \varphi d\tilde{x}_3) \} + \frac{dx_2^2}{H} \\
+ \left( \frac{K}{H} \cos^2 \varphi + \sin^2 \varphi \right) d\tilde{x}_1^2 + \left( \frac{K}{H} \sin^2 \varphi + \cos^2 \varphi \right) d\tilde{x}_3^2 \\
+ 2 \cos \varphi \sin \varphi \left( \frac{K}{H} - 1 \right) d\tilde{x}_1 d\tilde{x}_3 + \sum_{m=4}^9 dx_m^2 \right],
\]

\[
A_{[3]} = \frac{1}{g_s} (H^{-1} - 1) \{ dt \wedge (\cos \varphi d\tilde{x}_1 + \sin \varphi d\tilde{x}_3) \wedge dx_2 \},
\]

(29)

\[
e^{2\varphi} = g_s^2 H^{1/2} = g_s^2 \left[ 1 + \frac{Q_1}{r^4} \right]^{1/2}.
\]

Lastly, applying the generalized Buscher’s \( T \)-duality on \( \tilde{x}_3 \), we are left with

\[
d s_{10}^2 = H^{1/2} \left[ - \frac{K^{-1} dt^2}{H} + \frac{K \{ d\tilde{x}_1 + \cos \varphi (K^{-1} - 1) dt \}^2}{K + (H - K) \cos^2 \varphi} + \frac{dx_2^2}{H} \\
+ \frac{d\tilde{x}_3^2}{K + (H - K) \cos^2 \varphi} + \sum_{m=4}^9 dx_m^2 \right],
\]

\[
B_{[2]} = \frac{1}{g_s} \frac{(K - 1) \sin \varphi}{K + (H - K) \cos^2 \varphi} (dt \wedge d\tilde{x}_3) + \frac{1}{g_s} \frac{(H - K) \cos \varphi \sin \varphi}{K + (H - K) \cos^2 \varphi} (d\tilde{x}_1 \wedge d\tilde{x}_3),
\]

\[
A_{[2]} = \frac{1}{g_s} (H^{-1} - 1) \sin \varphi (dt \wedge dx_2),
\]

(30)

\[
A_{[4]} = \frac{1}{g_s} \frac{(H - 1) \cos \varphi}{K + (H - K) \cos^2 \varphi} (dt \wedge d\tilde{x}_1 \wedge dx_2 \wedge d\tilde{x}_3),
\]

\[
e^{2\varphi} = g_s^2 \frac{H}{K + (H - K) \cos^2 \varphi}
\]

with again \( H(r), K(r) \) as given before.

Next, the associated magnetically \( RR \)-charged solution again turns out to be the same as the electrically-charged solution given above except that the \( RR \) tensor field strengths are given, instead of \( F^e_{[3]} = dA_{[2]} \) and \( F^e_{[5]} = dA_{[4]} \), by

\[
F^m_{[7]} = * F^e_{[3]} = * (dA_{[2]}) = - \frac{4Q_1 \sin \varphi}{g_s r^6 [K + (H - K) \cos^2 \varphi]} \times
\]

\[
\left[ x_4 (dx_3 \wedge dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) - x_5 (dx_3 \wedge dx_4 \wedge dx_6 \wedge dx_7 \wedge dx_8 \wedge dx_9) \right]
\]

(31)
that the Hodge dual is taken with respect to the metric solution given in eq. (30). Note again that \( F^{m}_{[5]} \neq F^{e}_{[5]} \) and hence \( F^{e}_{[5]} \neq * F^{e}_{[5]} \), namely the 5-form \( RR \) field strength is not self-dual.

### 3.2 Nature of the solution

First of all, again it is evident from the emergence of \( RR \) potentials \( A_{[2]} \) and \( A_{[4]} \) that we are left with a bound state of a \( D1 \) and \( D3 \)-branes. Moreover, this solution still appears to represent a \( D1 \)-brane “dissolved” in a \( D3 \)-brane with a superimposed pp-wave propagating in \( \tilde{x}_1 \)-direction which is tilted from the original \( x_1 \)-direction by an angle \( \varphi \). Particularly, a remarkable feature of the solution corresponding to option (B) that can be contrasted from that of our previous solution corresponding to option (A) is that now we have the non-vanishing \( NSNS \) \( B \)-field component \( B_{t\tilde{x}_3} \) as well as the usually expected component \( B_{\tilde{x}_1\tilde{x}_3} \). Note that we shall eventually propose that these solutions are the dual supergravity description of noncommutative SYM at large coupling and in the infinite-momentum-frame particularly if we consider the gravity decoupling limits of their extremal versions. And of course this interpretation is based on the \( K_{(p+2)}/CFT_{(p+1)} \) (in the infinite-momentum-frame but with a constant momentum density) correspondence we discussed earlier in the introduction. In this spirit, the emergence of non-vanishing components \( B_{t\tilde{x}_3} \) and \( B_{\tilde{x}_1\tilde{x}_3} \) in the dual supergravity solution corresponding to option (B) implies that its dual SYM theory at large coupling should be defined on a manifold consisting of the two noncommutative hypersurfaces \( (t - \tilde{x}_3) \) and \( (\tilde{x}_1 - \tilde{x}_3) \) planes. Namely, we now ended up with both “time-space” and “space-space” noncommutativity in option (B) in contrast to option (A) where one was left with just “space-space” noncommutativity. Although it may, at first sight, seem quite a surprise, it, on second thought, was rather an expected result. That is to say, first notice that it is the “tilting” procedure of the delocalized \( D2 \)-brane that essentially generates the \( NSNS \) \( B \)-field components upon performing the \( T \)-duality. Then in option (A), the rotation is done in \( (x_2 - x_3) \) plane with \( x_2 \) being a spectator direction with respect to the propagating pp-wave on the brane. Thus the \( T \)-duality can at most generates the component \( B_{\tilde{x}_2\tilde{x}_3} \). In option (B), on the other hand, the rotation is performed in \( (\tilde{x}_1 - \tilde{x}_3) \) plane instead with \( x_1 \) now being the wave propagation direction. As a result, due to the non-vanishing metric component \( g_{t\tilde{x}_1} \) this time, the \( T \)-duality turns out to generate non-zero component \( B_{t\tilde{x}_3} \) as well as \( B_{\tilde{x}_1\tilde{x}_3} \). And this is why we have both time-space
and space-space noncommutativity in its dual SYM theory for option (B). Namely, it is the non-trivial role played by the superimposed pp-wave that leads to the full noncommutativity in its dual SYM theory. Lastly, we also note that if \( Q_2 = 0 \) (and hence \( K(r) = 1 \)), namely in the absence of the superimposed pp-wave, both of these \( (D1 - D3) \) bound state solutions given above correctly reduce to that of Hashimoto and Itzhaki [7] or of Maldacena and Russo [8] with two (spectator) longitudinal directions \( x_1, x_2 \) which can now be freely interchanged.

4 Decoupling Limits

The metric sector of the extremal \( (D1 - D3) \) bound state solutions with a superimposed pp-wave given above all asymptote to the 10-dimensional flat spacetime as \( r \to \infty \). Very near the horizon at \( r = 0 \), on the other hand, they nearly look like \( K_5 \times S^5 \) with \( K_5 \) being the 5-dimensional generalisation of the “Kaigorodov” metric we discussed in some detail earlier. And the throat connecting these two asymptotic regions contains non-zero NSNS and RR fields. Thus on the boundary, we would have the \( N = 4 \) SYM theory in the infinitely-boosted frame but with constant and finite momentum density in the spirit of \( K(p+2)/CFT(p+1) \) correspondence in the presence of the pp-wave propagating on the extremal \( p \)-brane worldvolume. Therefore, we now elaborate on this point. In order eventually to have noncommutative SYM theory in the infinitely-boosted frame (living in the worldvolume of \( (D1 - D3) \) system placed on the boundary near the horizon) on the boundary of \( K_5 \), we take the following ordinary field theory decoupling limit :

\[
\alpha' \to 0, \quad \tan \varphi = \frac{\hat{b}}{\alpha'},
\]

\[
r = \alpha' R^2 u, \quad g_s = \frac{\alpha'}{\hat{b}} \hat{g}_s
\]

where \( R^4 = 4\pi g_s N \) and \( x_{0,1} = \tilde{x}_{0,1}, x_{2,3} = \frac{\alpha'}{\hat{b}} \tilde{x}_{2,3} \) for the solution corresponding to option (A) and \( x_{0,2} = \tilde{x}_{0,2}, x_{1,3} = \frac{\alpha'}{\hat{b}} \tilde{x}_{1,3} \) for the solution corresponding to option (B), with \( u, \hat{g}_s, \tilde{x}_\mu, \hat{b} \) being kept fixed. Here the factor of \( \hat{b} \) has been introduced for later convenience. And \( \hat{g}_s \) here is the value of string coupling in the IR regime. Now in this decoupling limit, the extremal solution corresponding to option (A) becomes

\[
d_{10}^2 = \alpha' R^2 \left[ u^2 \{ -K^{-1} d\tilde{t}^2 + K(d\tilde{x}_1 + (K^{-1} - 1)d\tilde{t})^2 \} + u^2 \hat{h}(d\tilde{x}_2^2 + d\tilde{x}_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right],
\]

\[
B_{[2]} = B\frac{(\alpha'^2 R^4 u^4 - 1)}{(1 + \alpha'^2 u^4)} (d\tilde{x}_2 \wedge d\tilde{x}_3),
\]

\[
A_{[2]} = -\frac{1}{g_s} \left( \frac{\hat{b}}{\alpha'} \right) (\alpha'^2 R^4 u^4 - 1) (d\tilde{t} \wedge d\tilde{x}_1),
\]

\[
A_{[4]} = \frac{\hat{h}}{g_s} (\alpha'^2 R^4 u^4 - 1) (d\tilde{t} \wedge d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3),
\]
\[ e^{2\phi} = \hat{g}_s^2 \hat{h} \]

where \( K = Q_2/(\alpha' R^2 u)^4 \), \( \hat{h} = 1/(1 + a^4 u^4) \) and \( B_{\infty} = \alpha'/\hat{b} = \alpha'(R^2/a^2) \) with \( a = R\sqrt{b} \).

Next, the decoupling limit of the extremal solution corresponding to option (B) reads

\[
\begin{align*}
 ds^2_{10} &= \alpha' R^2 \left[ u^2 \left( -K^{-1} d\tilde{t}^2 + d\tilde{x}_2^2 \right) + u^2 \hat{h} \left\{ K [d\tilde{x}_1 + (K^{-1} - 1)d\tilde{t}^2 + d\tilde{x}_3^2] + \frac{du^2}{u^2} + d\Omega^2_{\infty} \right\} \right], \\
 B_{[2]} &= B_{\infty} \frac{(1 - K) a^4 u^4}{(1 + K a^4 u^4)} (d\tilde{t} \wedge d\tilde{x}_3) + B_{\infty} \frac{K (\alpha'/\hat{b}) a^4 u^4 - 1}{(1 + K a^4 u^4)} (d\tilde{x}_1 \wedge d\tilde{x}_3), \\
 A_{[2]} &= \frac{1}{g_s} \left( \frac{\hat{b}}{\alpha'} \right) (\alpha'^2 R^4 u^4 - 1)(d\tilde{t} \wedge d\tilde{x}_2), \\
 A_{[4]} &= \frac{\hat{h}}{g_s} (\alpha'^2 R^4 u^4 - 1)(d\tilde{t} \wedge d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3), \\
 e^{2\phi} &= \hat{g}_s^2 \hat{h}
\end{align*}
\]

where \( K \) and \( B_{\infty} \) are as given above but now \( \hat{h} = 1/(1 + K a^4 u^4) \).

The decoupling limit of these extremal (\( D1 - D3 \)) bound state solutions with a superimposed pp-wave given above is the main result we would like to report in this work. Namely, in the spirit of \( K_{(p+2)}/\text{CFT}_{(p+1)} \) correspondence that we discussed earlier in the introduction, we propose that the decoupling limit of the extremal solutions given above constitute the dual supergravity description of the SYM theory in the infinitely-boosted frame on noncommutative 4-dimensional manifold. And in the same spirit, we expect that the decoupling limit of the non-extremal solutions should be the dual supergravity description of the noncommutative SYM theory in the infinitely-boosted frame at finite temperature. At this point, it seems relevant to point out the interesting role played by the presence of the pp-wave parallely-intersecting with the (\( D1 - D3 \)) bound state. To do so, recall first that the solution construction corresponding to option (A) involves, when obtaining the (\( D1 - D3 \)) bound state from the (\( D2 \))-brane solution via the so-called BMM T-duality prescription, the rotation in a plane containing a spectator direction (with respect to the superimposed pp-wave), while the one corresponding to option (B) involves the rotation in the other plane containing the boost (i.e., wave propagation) direction. These rather technically-looking choices corresponding to the two inequivalent options in the solution construction procedure, however, turn out to lead to physically interesting consequences. Namely, the decoupling limit of the solution corresponding to option (A) is expected to be the dual gravity description of the SYM theory in the infinitely-boosted frame on a manifold with one noncommutative hypersurface, \((\tilde{x}_2 - \tilde{x}_3)\) plane. In contrast, that corresponding to option (B) is supposed to be the dual gravity description of the same gauge theory, this time on a manifold with two noncommutative hypersurfaces, \((\tilde{t} - \tilde{x}_3), (\tilde{x}_1 - \tilde{x}_3)\) planes. In other words, one ends up with both “time-space” and “space-space” noncommutativity in option (B) in contrast to option (A) where only “space-space” noncommutativity is present. The essential reason that underlies this emergence of “time-space” noncommutativity already has been discussed earlier. Here, we stress that on purely technical side, intersecting the (\( D1 - D3 \)) bound state parallely with a gravitational pp-wave turns out to provide yet another way of generating
the “time-space” noncommutativity in its dual SYM theory different from those suggested in
the literature in the absence of the pp-wave.
Now, other comments concerning the decoupling limit of these extremal \((D1-D3)\) bound state
with a superimposed pp-wave are in order:

(i) These extremal solutions all approach \(K_5 \times S^5\) (with \(K_5\) denoting the 5-dimensional
generalisation of the “Kaigorodov” metric) for small \(u\), which corresponds to the IR regime of the
dual SYM theory in the infinitely-boosted frame since “\(u\)” plays the role of energy scale on the
gauge theory side. This is indeed what one would naturally expect since the noncommutative
SYM theory should reduce to the ordinary (commutative) SYM theory at long distances. And
the solutions start to deviate from the \(K_5 \times S^5\) solution roughly at \(u \sim 1/a\) (with \(a = R\sqrt{\beta}\)
carrying the dimension of the length), namely at a distance scale of order \(a = R\sqrt{\beta}\). As Maldacena
and Russo [8] pointed out, for large ’tHooft coupling, i.e., for \(\lambda = g_{YM}^2 N = 4\pi g_s N = R^4 \sim \text{large},\)
this clearly is greater than the naively expected distance scale of \(L \sim \sqrt{\beta}\).

(ii) We now turn to the behavior of these solutions on the other asymptotic boundary at
\(u \to \infty\). Unlike the case in which the NSNS B-field is absent, the solutions exhibit some
peculiar features. For instance, as this boundary is approached, the physical (proper) size
of the noncommutative directions (i.e., \(\tilde{x}_2 - \tilde{x}_3\) directions) shrink (in string frames), since
\(\tilde{h} = 1/(1 + a^4 u^4) \sim 1/u^4\) as \(u \to \infty\) for the solution corresponding to option (A). Interestingly,
however, this is not the case for the solution corresponding to option (B) since there \(\tilde{h} = 1/(1 + K a^4 u^4) = [1 + Q_2(a/\alpha'R^2)^4]^{-1}\) and hence it is independent of \(u\) in the decoupling
limit. Namely for the solution corresponding to option (B), the physical size of both the
commuting and noncommuting directions exhibits essentially the same (growing) behavior as
the \(u \to \infty\) boundary is approached. And for the solution corresponding to option (A), this
shrinking behavior of the physical size of the noncommuting directions may lead to the danger
of encountering the curvature singularity (in string metric) as \(u \to \infty\) since the type of scaling
isometry near this boundary that exists in the absence of the pp-wave content,

\[
\tilde{x}_{0,1} \to \lambda^{-1} \tilde{x}_{0,1}, \quad \tilde{x}_{2,3} \to \lambda \tilde{x}_{2,3}, \quad u \to \lambda u
\]  

noticed by Maldacena and Russo [8] simply does not exist for the case at hand when the pp-
wave content is present.

(iii) We now briefly comment on the nature of supersymmetry and some duality owned
by these solutions. The transverse 5-sphere is still round and hence possesses \(SO(6)\)-isometry
which corresponds on the dual gauge theory side to the \(SU(4)\) \(R\)-symmetry of the \(N = 4\)
SUSY algebra. And the fact that this \(SO(6)\)-isometry is not contaminated even under the
introduction of the noncommutativity implies that the SUSY is not further broken by the non-
commutativity either. Next, the electric 5-form \(RR\) field strength of the electric solution is
apparently not the same in form as the magnetic 5-form \(RR\) field strength of the magnetic
solution, i.e., \( F^c_{[5]} \neq F^m_{[5]} = \ast F^c_{[5]} \) in eqs.(27) and (32). This is due to the presence of the NSNS B-field (leading to the \((D1 - D3)\) bound state) and the gravitational pp-wave propagating on the brane and implies that this particular type IIB supergravity solution is not self-dual under S-duality.

5 Summary and Discussion

To summarize, in the present work, we attempted to explore the mechanism of non-locality in the noncommutative SYM theory in the infinitely-boosted frame especially at strong coupling from the dual supergravity description in terms of the extremal \((D1 - D3)\) bound state solution with a superimposed pp-wave. One may naturally expect that when the effect of non-locality of order, say, \( a = R\sqrt{b} = (g_{YM}^2 N b^2)^{1/4} \) is turned on, the dynamics at length scales larger than \( a \) would be unaffected while that at length scales smaller than \( a \) would be drastically changed. From the gravity decoupling limit of the dual supergravity solution, we have actually confirmed this intuitive expectation. That is, for small \( u \), the solutions all reduce to \( K_5 \times S^5 \) geometry confirming our expectation that the IR dynamics of the dual gauge theory should be unaffected by the noncommutativity while as \( u \to \infty \), the solutions start to deviate significantly from \( K_5 \times S^5 \) limit indicating that the UV dynamics of the dual gauge theory would be heavily distorted by the effect of noncommutativity.

Nevertheless, aside from our attempt to study it using the \( K_{(p+2)}/CFT_{(p+1)} \) correspondence, the noncommutative SYM theory in the infinitely-boosted frame itself does not seem to have been studied in great detail. Thus it might be challenging to work in this direction as well. The commutative boundary CFT in the infinitely-boosted frame, on the other hand, has been examined by Brecher, Chamblin and Reall [3] in some detail. Thus it seems worth summarizing the results of their study here. They also started by noting that in the spirit of gauge/gravity correspondence, it is natural to conjecture that string theory in the Kaigorodov spacetime is dual to some CFT in the infinitely-boosted frame. Since the momentum density was held fixed in the gravity decoupling limit, however, there is a non-zero background momentum density present. And this background momentum density, in turn, breaks the conformal symmetry group of the boundary field theory down to some smaller group. They showed that actually the isometries of the Kaigorodov spacetime have a natural interpretation as this subgroup of the conformal group that leaves the background momentum density invariant. They then attempted the computation of 2-point functions of field operators in the boundary theory. As is well-known, when conformal symmetry is exact, the 2-point functions of CFT operators are completely determined. For the case under consideration when the conformal invariance is partly broken, the dilatation symmetry still persists and it allows to constrain the form of 2-point functions. As a result, they demonstrated that this surviving symmetry determines the scalar 2-point function up to an arbitrary function of one variable. Moreover, this 2-point
function turned out to be independent of the background momentum density and this point has been attributed to a large $N$ effect. In association with the context of the present work in which the gravity duals to the noncommutative boundary CFT in the infinitely-boosted frame has been developed, then, one might wish to add the noncommutativity content to the type of analysis performed in [3] to eventually study the corresponding dual CFT. Technically, it can be achieved by properly combining the works [8] and [3]. And this will be left for a serious future study.

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**A M2-brane in $D = 11$ SUGRA with a superimposed pp-wave**

Consider the (bosonic sector of) $D = 11$ SUGRA with action

$$S_{11} = \int d^{11}x \sqrt{g} \left[ R - \frac{1}{2 \times 4!} F_{[4]}^2 \right] - \frac{1}{6} \int A_{[3]} \wedge F_{[4]} \wedge F_{[4]},$$

$$A_{[3]} = \frac{1}{3!} A_{MNP} dx^M \wedge dx^N \wedge dx^P, \quad F_{[4]} = dA_{[3]} \quad (37)$$

and the associated classical field equations

$$R_{MN} = \frac{1}{2 \times 3!} \left[ F_{MPQR} F_{N}^{PQR} - \frac{1}{12} g_{MN} F_{[4]}^2 \right],$$

$$\frac{1}{\sqrt{g}} \partial_M \left[ \sqrt{g} F_{MNPQR} \right] = 0. \quad (38)$$

Then the non-extremal $M2$-brane solution to this classical field equations is given by

$$ds_{11}^2 = H^{-2/3} \left[ -f dt^2 + dx_1^2 + dx_2^2 \right] + H^{1/3} \left[ f^{-1} dr^2 + r^2 d\Omega_7^2 \right],$$

$$A_{[3]} = \text{coth} \mu_1 (H^{-1} - 1) (dt \wedge dx_1 \wedge dx_2) \quad \text{with}$$

$$H(r) = 1 + \frac{Q_1}{r^6}, \quad f(r) = 1 - \frac{\mu}{r^6}, \quad Q_1 = \mu \sinh^2 \mu_1 = (2^5 \pi^2 N) l_p^6, \quad \mu = k \kappa_{11}^{4/3} = 2m$$

where $Q_1$ is the usual (electric) $RR$ charge and $\kappa_{11}$ and $l_p$ denotes the 11-dimensional gravitational constant and Planck length respectively and $N$ is the number of coincident branes and Lastly $m$ denotes the (ADM) mass density of the brane. And the horizon of this black $M2$-brane is located at $r_+ = \mu^{1/6} = k^{1/6} \kappa_{11}^{2/9}$ along the transverse radial direction.
Now, note that in the presence, say, of an electric charge (and its field), one way of generating the electromagnetic wave is to go to an infinitely-boosted Lorentz frame, i.e., a Lorentz frame moving at the speed of light with respect to the charge. In a similar manner, in the presence of (some form of) a mass, like $M_2$-brane itself given above, one way of generating the gravitational pp-wave would be to make a transit to an infinitely-boosted Lorentz frame. Thus in order to construct a non-extremal $M_2$-brane solution with a superimposed gravitational pp-wave, we consider performing a Lorentz boost on the non-extremal $M_2$-brane solution given above in the $(t, x_1)$ plane

\begin{align}
  t' &= (\cosh \mu_2) t + (\sinh \mu_2)x_1, \\
  x_1' &= (\sinh \mu_2)t + (\cosh \mu_2)x_1
\end{align}

where $\gamma = (1 - \beta^2)^{-1/2} = \cosh \mu_2$, $\beta \gamma = \sinh \mu_2$, and thus $\beta = \sinh \mu_2/\gamma = \tanh \mu_2$. Note, as mentioned earlier in the introduction, that this Lorentz boost becomes singular, i.e., becomes an infinite boost in the extremal limit where $\mu_2 \to \infty$. Upon this Lorentz boost, then, the part of the $M_2$-brane worldvolume metric becomes

\begin{align}
  \left[-f(r)dt^2 + dx_1^2 + dx_2^2\right] \\
  &= -f(r)(\cosh \mu_2 dt' - \sinh \mu_2 dx_1')^2 + (\cosh \mu_2 dx_1' - \sinh \mu_2 dt')^2 + dx_2^2 \\
  &= -K^{-1}f(r)dt'^2 + K[dx_1' + \coth \mu_2 (K^{-1} - 1) dt']^2 + dx_2^2 \\
  &\text{with } K(r) = 1 + \frac{Q_2^2}{r^6}, \quad Q_2 = \mu \sinh^2 \mu_2
\end{align}

and where $Q_2$ is a new parameter representing the momentum along $x_1'$-direction. Next, it is straightforward to see that under this Lorentz boost, the $RR$ tensor field (and hence its field strength) remains the same, namely

\begin{align}
  A_{t'x_1'x_2} &= \left(\frac{\partial t}{\partial t'}\frac{\partial x_1}{\partial x_1'} - \frac{\partial x_1}{\partial t'}\frac{\partial t}{\partial x_1'}\right) A_{tx_1x_2} \\
  &= (\cosh^2 \mu_2 - \sinh^2 \mu_2) A_{tx_1x_2} = A_{t_1x_1x_2} \quad \text{and}
\end{align}

\begin{align}
  dt' \wedge dx_1' \wedge dx_2 &= (\cosh^2 \mu_2 - \sinh^2 \mu_2) dt \wedge dx_1 \wedge dx_2 = dt \wedge dx_1 \wedge dx_2 \\
  A_{[3]} &= A_{tx_1x_2} dt \wedge dx_1 \wedge dx_2 = A_{t'x_1'x_2} dt' \wedge dx_1' \wedge dx_2.
\end{align}

Thus putting these results altogether, one can conclude that upon the Lorentz boost in the $(t, x_1)$ plane, the non-extremal $M_2$-brane solution goes over to the non-extremal $M_2$-brane solution with a superimposed gravitational pp-wave given by (henceforth, we shall drop the primes on $t$ and $x_1$ coordinates)

\begin{align}
  ds_1^2 &= H^{-2/3} \left[-K^{-1}f dt^2 + K\{dx_1 + \coth \mu_2 (K^{-1} - 1) dt\}^2 + dx_2^2\right] + H^{1/3} \left[f^{-1}dr^2 + r^2 d\Omega_7^2\right], \\
  A_{[3]} &= \coth \mu_1 (H^{-1} - 1)(dt \wedge dx_1 \wedge dx_2) \quad \text{with}
\end{align}

\begin{align}
  H(r) &= 1 + \frac{Q_1}{r^6}, \quad K(r) = 1 + \frac{Q_2}{r^6}, \quad f(r) = 1 - \frac{\mu}{r^6}, \\
  Q_1 &= \mu \sinh^2 \mu_1, \quad Q_2 = \mu \sinh^2 \mu_2, \quad \mu = k\kappa_{11}^{4/3}.
\end{align}
Next, the extremal $M2$-brane with a superimposed gravitational pp-wave amounts to the limiting case when
\[ \mu \to 0, \quad \mu_1, \mu_2 \to \infty \quad \text{with} \quad Q_1 = \mu \sinh^2 \mu_1, \quad Q_2 = \mu \sinh^2 \mu_2 \quad \text{kept fixed} \]
then the solution above becomes
\[
ds_{11}^2 = H^{-2/3} \left[ -K^{-1} dt^2 + K \{ dx_1 + (K^{-1} - 1) dt \}^2 + dx_2^2 \right] + H^{1/3} \left[ dr^2 + r^2 d\Omega_7^2 \right],
\]
\[A[3] = (H^{-1} - 1)(dt \wedge dx_1 \wedge dx_2). \quad \text{(45)}\]
Finally, upon introducing the retarded ($u$) and the advanced ($v$) null coordinates
\[ u = x_1 - t, \quad v = x_1 + t \quad \text{(46)}\]
the extremal solution takes the form
\[
ds_{11}^2 = H^{-2/3}(r)[dudv + (K - 1)du^2 + dx_2^2] + H^{1/3}(r)[dr^2 + r^2 d\Omega_7^2]. \quad \text{(47)}\]
Obviously, in this extreme limit which corresponds to the infinite Lorentz boost case, the metric on the $M2$-brane worldvolume for $r = \text{const.}$
\[ [dudv + (K - 1)du^2 + dx_2^2] \quad \text{with} \quad (K - 1) = \frac{Q_2}{r^6} \quad \text{(48)}\]
does indeed represent a gravitational wave propagating in $x_1$-direction.

**B What is the gravitational “pp-wave”?**

By definition, a vacuum spacetime is a *plane-fronted gravitational waves* provided it contains a “shear-free” congruence of null geodesics (with tangent $k^\alpha$) and provided it admits “plane wave surfaces” (i.e., spacelike 2-surfaces orthogonal to $k^\alpha$). And because of the existence of plane wave surfaces, the expansion and twist (rotation) must vanish as well. The best-known subclass of these waves are *plane-fronted gravitational waves with parallel rays* ("pp-waves") which are defined by the condition that the null vector $k^\alpha$ is covariantly constant, $\nabla^\beta k_\alpha = 0$.

Generally, for the null vector $k^\alpha$ tangent to null geodesic congruence, $\nabla^\beta k_\alpha$ can be decomposed as
\[
\nabla^\beta k_\alpha = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3} \theta h_{\alpha\beta} - a_\alpha k_\beta \quad \text{(49)}
\]
where $h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha k_\beta$ is the metric induced on the hypersurfaces $\Sigma$ orthogonal to $k^\alpha$ and
\[
\theta \equiv h^{\alpha\beta} \nabla^\beta k_\alpha = \nabla^\alpha k_\alpha,
\]
\[
\sigma_{\alpha\beta} \equiv \frac{1}{2} (h^\mu_{\beta} \nabla_\mu k_\alpha + h^\mu_{\alpha} \nabla_\mu k_\beta) - \frac{1}{3} \theta h_{\alpha\beta} = \nabla_{(\alpha} k_{\beta)} - \frac{1}{3} \theta h_{\alpha\beta},
\]
\[
\omega_{\alpha\beta} \equiv \frac{1}{2} (h^\mu_{\beta} \nabla_\mu k_\alpha - h^\mu_{\alpha} \nabla_\mu k_\beta) = \nabla_{[\alpha} k_{\beta]},
\]
\[
a_\alpha \equiv k^\beta \nabla_\beta k_\alpha.
\]
are the expansion, shear, twist and acceleration, respectively, of the null geodesic congruence. Thus if the expansion, shear, twist and acceleration all vanish, then $\nabla_\beta k_\alpha = 0$. In suitable null coordinates $(u, v, \xi, \bar{\xi})$ such that

$$k_\alpha = \partial_\alpha u, \quad k^\alpha = (\partial/\partial v)^\alpha \quad (51)$$

the metric representing the gravitational pp-wave is given by

$$ds^2 = dudv + H(u, \xi, \bar{\xi})du^2 + (dx_2^2 + dx_3^2)$$

$$= dudv + H(u, \xi, \bar{\xi})du^2 + d\xi d\bar{\xi} \quad (52)$$

where $H$ is a real function of $u$ and $\xi$ which spans the wave 2-surfaces $u = \text{const.}, v = \text{const.}$. The vacuum Einstein field equations imply $2H = f(u, \xi) + \bar{f}(u, \bar{\xi})$ with $f$ being an arbitrary function of $u$ analytic in $\xi$. In general, the gravitational pp-waves have only the single isometry generated by the Killing vector $k^\alpha = (\partial/\partial v)^\alpha$.

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