Fermionic Wigner functional theory

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A Grassmann functional phase space is formulated for the definition of fermionic Wigner functionals. The formulation follows a stepwise process, starting with the identification of suitable fermionic operators that are analogues to bosonic quadrature operators. The Majorana operators do not suffice for this purpose. Instead, a set of fermionic Bogoliubov operators are used. The eigenstates of these operators are shown to provide orthogonal bases, provided that the dual space is defined by augmenting the Hermitian conjugation with a spin transformation. These bases serve as quadrature bases in terms of which the Wigner functionals can be defined analogues to the bosonic case.

I. INTRODUCTION

Quantum field theory [1] has been very successful in formulating theories for most of the fundamental dynamics of nature. The only exception is a quantum description of gravity. The states used for phenomenological calculations in quantum field theory are usually rather simple — tensor products of pure single-particle number states for the different particle fields that are to take part in the interactions. It is not the purpose of quantum field theory to represent exotic quantum states.

On the other hand, formulations of non-relativistic quantum physics that are used for quantum information theory [2], such as those used in quantum optics [3], have been developed to a high level, allowing the description of exotic quantum measurements applied to exotic quantum states. Such descriptions often use phase space representations for such exotic states in terms of ladder operators or quadrature operators [4-6], leading to a quasi-Moyal formalism. The dynamics in these applications are relatively simple compared to the fundamental dynamics of nature as represented in the standard model of particle physics in terms of quantum field theory.

Progress in fundamental physics (specifically with respect to a quantum description of gravity) would require the merger of these two disciplines to obtain a formalism that can successfully represent both the fundamental dynamics and arbitrary complex quantum states. A promising approach is to generalize the Moyal formalism [7,8] to obtain one in which states and operations are represented as distributions on a functional phase space, taking over the role of the path integral domain. The formal resemblance between functional phase space integrals and path integrals implies that the former would allow a similar development to that of quantum field theory based on the latter.

The development of a functional Moyal formalism for bosonic fields has so far been quite successfully based on a generalization of quantum optics [10-13]. The relativistic formulation of a functional phase space for the massless photon field does not present significant challenges. The development of a functional Moyal formalism for fermionic fields is more challenging. In addition to the early work by Cahill and Glauber [14], there are a number of fermion phase space formulations based on the Glauber-Sudarshan P-distribution [15,16] and the Husimi Q-distribution [17] in the context of condensed matter physics. There are also fermion phase space formulations based on the Wigner function [18]. These formulations are often done for finite dimensional systems, leading to phase spaces composed of a finite number of the two-dimensional phase spaces, each representing a one-dimensional system.

In this paper, we consider an infinite dimensional scenario with a functional phase space where the distributions are functionals, leading to a functional Moyal formalism for fermion fields. For this purpose, we choose to work with Wigner functionals. Such a functional Moyal formalism for fermion fields based on Wigner functionals has been presented before [19]. However, it is based on an axiomatic approach that takes the existence of eigenstates of field operators for granted. Here, we show the full development and compute the eigenstates for the pertinent operators explicitly.

The development of a fermionic Moyal formulation with Wigner functionals is frustrated by the fact that the fermionic analogue of the bosonic quadrature operators (i.e., the Majorana operators) do not share the properties of their bosonic equivalents. They do not commute with themselves and their eigenspaces behave differently from those of the bosonic quadrature operators.

Here, we consider a set of fermionic Bogoliubov operators, instead of the Majorana operators. The fermionic Bogoliubov operators consist of linear combinations of ladder operators together with a spin transformation. They are found to behave more closely to the bosonic quadrature operators and therefore enable the development of a fermionic Wigner functional formalism. We obtain their eigenstates and compute the inner products among these eigenstates. The results are surprising in that the dual space of these states do not follow from the usual Hermitian adjoint. Instead, it requires a spin transformation in addition to the Hermitian adjoint, which we call a fermionic adjoint. It follows that the set of fermionic Bogoliubov operators are self-adjoint with respect to this fermionic adjoint.

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With the assumed fermionic quadrature bases, we proceed to formulate the fermionic Wigner functional theory. It is shown that the Weyl transformation for the Wigner function of fermions allows transformations between fermionic density operators and Grassmann Wigner functionals. The Moyal star product is also derived allowing quantum operations to be represented purely in terms of fermionic Wigner functionals.

II. BACKGROUND

A. Bosonic operators

Since we intend to develop the fermionic phase space representation in analogy to the phase space associated with bosonic Wigner distributions, it helps to review bosonic Wigner functions. We start by considering only the particle-number degrees of freedom, ignoring all other degrees of freedom. In such a case, the pair of ladder operators — the annihilation operator $\hat{a}$ and its Hermitian adjoint, the creation operator $\hat{a}^\dagger$ — obey the simple commutation relations $[\hat{a}, \hat{a}^\dagger] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$ and $[\hat{a}, \hat{a}^\dagger] = 1$. The quadrature operators are defined as the sum and difference of the ladder operators

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = -\frac{i}{\sqrt{2}}(\hat{a} - \hat{a}^\dagger).$$

(1)

They are Hermitian operators, obeying another set of simple commutation relations $[\hat{q}, \hat{q}] = [\hat{p}, \hat{p}] = 0$ and $[\hat{q}, \hat{p}] = i$, which follow unambiguously from those of the ladder operators. Their eigenstates, which are solutions of the eigen-equations

$$\hat{q}|q\rangle = |q\rangle q, \quad \hat{p}|p\rangle = |p\rangle p,$$

(2)

with nondegenerate real-valued eigenvalues $q$ and $p$, represent orthogonal bases. The eigen-bases of the respective quadrature operators are mutually unbiased — the inner product $(q|p) = \exp(\text{i}qp)$ implies that $|(q|p)|^2 = 1$.

The Wigner function of a state is given in terms of the quadrature eigenstates by

$$W_\rho(q, p) = \int \langle q + \frac{1}{2}x|\hat{\rho}|q - \frac{1}{2}x\rangle \exp(-\text{i}xp) \, dx,$$

(3)

where $\hat{\rho}$ is the density operator of the state. Such a Wigner function can be computed for any operator. The properties of the quadrature operators and their eigenbases are crucial for the definition of the Wigner function.

Bogoliubov operators are also produced as linear combinations of ladder operators,

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger, \quad \hat{b}^\dagger = v^*\hat{a} + u^*\hat{a}^\dagger.$$

(4)

Unlike the quadrature operators, they are not Hermitian. The defining property of the Bogoliubov operators is that they obey the same commutation relations as the ladder operators from which they are constructed:

$$[\hat{b}, \hat{b}^\dagger] = [\hat{b}^\dagger, \hat{b}^\dagger] = 0 \quad \text{and} \quad [\hat{b}, \hat{b}^\dagger] = 1. \quad \text{It thus follows that} \quad |u|^2 - |v|^2 = 1. \quad \text{For} \quad u = 1 \quad \text{and} \quad v = 0, \quad \text{the Bogoliubov operators are equal to the original ladder operators.}$$

Incorporating the other degrees of freedom (spin and spatiotemporal degrees of freedom), represented by a spin index $s$ and a dependence on the wave vector $k$, the eigen-equations for the quadrature operators become

$$\hat{q}_s(k)|\rangle q\rangle = |q\rangle q_s(k), \quad \hat{p}_s(k)|\rangle p\rangle = |p\rangle p_s(k).$$

(5)

The eigenvalue now becomes a spectral eigenvalue function. Each such function is associated with a unique eigenstate. Hence, the phase space becomes a functional phase space, and the Wigner function becomes a Wigner functional. It is represented by

$$W[q, p] = \int \exp\left[ -i \sum_s \int p_s(k)x_s(k) \frac{q^3 k^2}{(2\pi)^3 \omega} \right] \times \langle q + \frac{1}{2}x|\hat{\rho}|q - \frac{1}{2}x\rangle \, D[x],$$

(6)

where $D[x]$ represents a functional integration measure over the space of all finite energy functions $x_s(k)$. Apart from the complexities introduced by the functional nature of the resulting formalism, all the properties of the operators and the states remain essentially the same, thus leading to a successful formulation for the states and operations of massless bosonic fields.

For the benefit of the subsequent discussion, we summarize the properties that we associate with bosonic quadrature operators and bosonic Bogoliubov operators. For bosonic quadrature operators they are:

- **bQ1**: Quadrature operators are Hermitian; their eigenstates exist, are orthogonal and the associated eigenvalues are real.

- **bQ2**: The pair of quadrature operators do not commute; their commutation relation differs from that of the ladder operator.

- **bQ3**: Quadrature operators commute with themselves.

- **bQ4**: The eigenbases of the respective quadrature operators are mutually unbiased.

The properties of Bogoliubov operators are:

- **bB1**: Bogoliubov operators are not Hermitian; they are pairwise Hermitian adjoints of each other.

- **bB2**: A pair of Bogoliubov operators do not commute; their commutation relations have the same form as those of the ladder operators from which they are constructed.

- **bB3**: Bogoliubov operators commute with themselves.
The third property of the respective sets follows trivially from the commutation relations of the ladder operators. We state these trivial properties explicitly because they become nontrivial in the fermionic case when we replace “commute” by “anti-commute.”

### B. Majorana operators

Now we leave the bosonic case behind and hence only consider fermionic operators, unless explicitly stated otherwise. We use the same notation for the fermionic case. Since anti-fermionic ladder operators anti-commute with fermionic ladder operators, we do not consider the anti-fermionic fields here. Without all but the particle-number degrees of freedom, the fermionic ladder operators obey the simple anti-commutation relations

\[ \{ \hat{a}, \hat{a}^\dagger \} = \{ \hat{a}^\dagger, \hat{a} \} = 0, \quad \{ \hat{a}, \hat{a}^\dagger \} = 1. \]  

(7)

where \( \hat{a} \) and its Hermitian adjoint \( \hat{a}^\dagger \) represent the fermionic annihilation and creation operators, respectively. Without any other degrees of freedom, the fermionic Hilbert space is two-dimensional, consisting of linear combinations of the vacuum \( |\text{vac}\rangle \) and the single-particle state \( |1\rangle \). There are only four basic operations

\[ \hat{a} |\text{vac}\rangle = 0, \quad \hat{a} |1\rangle = |\text{vac}\rangle, \]

\[ \hat{a}^\dagger |1\rangle = 0, \quad \hat{a}^\dagger |\text{vac}\rangle = |1\rangle. \]  

(8)

We are seeking fermionic operators produced as linear combinations of the fermionic ladder operators, such that they behave in a way that is analogous to the bosonic quadrature operators. The expected properties for such fermionic quadrature operators are:

- **fQ1**: Fermionic quadrature operators are Hermitian; their eigenstates exist, are orthogonal and the associated eigenvalues are real.

- **fQ2**: Pairs of fermionic quadrature operators do not anti-commute; their anti-commutation relation differs from that of the ladder operators.

- **fQ3**: Each fermionic quadrature operator anti-commutes with itself.

- **fQ4**: A pair of fermionic quadrature operators have eigen-bases that are mutually unbiased.

To this end, we consider the Hermitian operators defined in terms of the sum and difference of the fermionic ladder operators

\[ \hat{m} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{n} = \frac{i}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger). \]  

(9)

They are the **Majorana operators**. However, we find that they produce the commutation relations

\[ \{ \hat{m}, \hat{m}^\dagger \} = \{ \hat{n}, \hat{n}^\dagger \} = 1, \quad \{ \hat{m}, \hat{n} \} = 0. \]  

(10)

Therefore, the Majorana operators do not satisfy conditions fQ2 and fQ3.

Since they are Hermitian, we expect the Majorana operators to have eigenstates that are orthogonal with real eigenvalues. Due to the two-dimensional Hilbert space, these eigenstates are linear combinations of \( |\text{vac}\rangle \) and \( |1\rangle \). For \( \hat{m} \), they are

\[ |m_s\rangle = \frac{1}{\sqrt{2}} (|\text{vac}\rangle \pm |1\rangle), \]  

(11)

with eigenvalues \( \pm 1/\sqrt{2} \), respectively, and for \( \hat{n} \),

\[ |n_s\rangle = \frac{1}{\sqrt{2}} (|\text{vac}\rangle \mp i|1\rangle), \]  

(12)

with eigenvalues \( \pm 1/\sqrt{2} \), respectively. The eigenstates of each Majorana operator are orthogonal and the two sets are mutually unbiased. Therefore, these operators satisfy fQ1 and fQ4. We can use either of the two sets of eigenstates as an orthogonal basis for the Hilbert space.

We can increase the dimension of the Hilbert spaces by including the other degrees of freedom. For instance, if we include spin, the Hilbert space becomes four-dimensional, with basis elements \( |\text{vac}\rangle, |1\rangle, |1\rangle, \) and \( |1\rangle |1\rangle |1\rangle |1\rangle \), where the subscripts represent the two spins. The ladder operators are represented as \( \hat{a}_s \) and \( \hat{a}_s^\dagger \), where \( s \) is the spin index. Operators associated with different degrees of freedom anti-commute. Therefore, the anti-commutation relations become

\[ \{ \hat{a}_r, \hat{a}_s \} = \{ \hat{a}_r^\dagger, \hat{a}_s^\dagger \} = 0, \quad \{ \hat{a}_r, \hat{a}_s^\dagger \} = \delta_{r,s}. \]  

(13)

The Majorana operators now include spin

\[ \hat{m}_s = \frac{1}{\sqrt{2}} (\hat{a}_s + \hat{a}_s^\dagger), \]

\[ \hat{n}_s = \frac{-i}{\sqrt{2}} (\hat{a}_s - \hat{a}_s^\dagger). \]  

(14)

They still do not satisfy conditions fQ2 and fQ3.

To obtain the eigenstates, we can consider the different spins separately. These eigenstates are

\[ |m_{s}, \rangle = \frac{1}{\sqrt{2}} (|\text{vac}\rangle \pm |1\rangle \), \]  

(15)

\[ |n_{s}, \rangle = \frac{1}{\sqrt{2}} (|\text{vac}\rangle \mp i|1\rangle \),

with eigenvalues \( \pm 1/\sqrt{2} \) in both cases. A basis for the full Hilbert space is obtained from all the tensor products of these eigenstates, leading to complete orthogonal bases for \( \hat{m}_s \) and \( \hat{n}_s \), respectively. These two bases are mutually unbiased. As a result, they satisfy fQ1 and fQ4.

The inclusion of the spatiotemporal degrees of freedom does not resolve the issues. If the spatiotemporal degrees of freedom are treated as a finite number of discrete modes, their inclusion follows the same pattern as the inclusion of spin with a larger dimension. The discrete basis is constructed as tensor products of all the
eigenstates for the individual modes. If the number of modes is increased to infinity, the resulting bases become similar to those associated with the Fock states. Such bases that are constructed as tensor products of infinitely many eigenstates of discrete modes are often cumbersome to work with and not suitable for a Moyal formalism.

It is therefore preferable to treat the (infinite dimensional) spatiotemporal degrees of freedom as continuous degrees of freedom. The ladder operators are represented by \( \hat{a}(u) \) and \( \hat{a}^\dagger(u) \), where \( u \) is a generic continuous variable representing the spatiotemporal degrees of freedom. The anti-commutation relations are now given by

\[
\begin{align*}
\{ \hat{a}(v), \hat{a}(u) \} &= \{ \hat{a}^\dagger(v), \hat{a}^\dagger(u) \} = 0, \\
\{ \hat{a}(v), \hat{a}^\dagger(u) \} &= \delta(u - v).
\end{align*}
\]  

The Hilbert space is spanned by an infinite dimensional basis, consisting of anti-symmetric tensor products of single-particle states \(|u\rangle\) and the vacuum \(|\text{vac}\rangle\). The states can have arbitrary numbers of particles, provided that no two of them share the same argument. Operating on the single-particle states, the operators produce

\[
\begin{align*}
\hat{a}(u)|v\rangle &= |\text{vac}\rangle \delta(u - v), \\
\hat{a}^\dagger(u)|v\rangle &= \begin{cases} |u\rangle|v\rangle = -|v\rangle|u\rangle & \text{for } u \neq v, \\
0 & \text{for } u = v.
\end{cases}
\end{align*}
\]  

The Majorana operators are now represented by

\[
\begin{align*}
\hat{m}(u) &= \frac{1}{\sqrt{2}}[\hat{a}(u) + \hat{a}^\dagger(u)], \\
\hat{n}(u) &= \frac{i}{\sqrt{2}}[\hat{a}(u) - \hat{a}^\dagger(u)].
\end{align*}
\]  

A general eigenstate of these Majorana operators can be expressed as an expansion of the form

\[
|\psi\rangle = |\text{vac}\rangle \mu_0 + \int |u\rangle \mu_1(u) du + \int |u\rangle |v\rangle \mu_2(u,v) du dv + \int |u\rangle |v\rangle |w\rangle \mu_3(u,v,w) du dv dw + \ldots.
\]  

where \( \mu_n(\ldots) \) are coefficient functions that are antisymmetric with respect to the interchange of any two variables. To determine the coefficient functions, we substitute the expansion into the eigen-equation

\[
\hat{m}(t)|\psi\rangle = |\psi\rangle \lambda(t),
\]  

where \( t \) is the continuous variable. Then we separate the result into different equations for the different numbers of particles. For the vacuum, we get

\[
\mu_1(t) = \mu_0 \lambda(t),
\]  

which gives \( \mu_1 \) in terms of \( \mu_0 \). Next, we consider terms with single-fermion states, for which we get

\[
2 \int |v\rangle \mu_2(t,v) dv + |t\rangle \mu_0 = \int |u\rangle \mu_1(u) \lambda(t) du.
\]  

When we use the expression for \( \mu_1 \) in Eq. (21), the resulting expression for \( \mu_2 \) becomes

\[
\mu_2(t,v) = \frac{1}{2} \mu_0 \lambda(v) \lambda(t) - \frac{1}{2} \delta(v - t) \mu_0.
\]  

However, there is a problem with the symmetry of the expression. The first term can be made antisymmetric by assuming the eigenvalue function \( \lambda(v) \) is a Grassmann function, but the last term cannot be made antisymmetric and would therefore have to be dropped. Without such a term, the eigen-equation cannot be solved. Similar issues with the symmetry appear in all the expressions for larger numbers of particles.

It may seem that this issue can be avoided by including both discrete degrees of freedom, such as spin, and continuous degrees of freedom. The same procedure then produces an expression for \( \mu_2 \) where the second term has the same symmetric form multiplied by \( \delta_{r,s} \). Hence, the symmetry problem persists. Even in such a general case, the Majorana operators do not have eigenstates.

We conclude that the Majorana operators with continuous variables do not have a full set of eigenstates for all the degrees of freedom. As a result, none of the conditions \( fQ1, fQ2, fQ3 \) and \( fQ4 \), are satisfied. The Majorana operators are therefore not suitable equivalents of the quadrature operators found in the bosonic case.

### III. FERMIONIC BOGOLIUBOV OPERATORS

If we compare the analysis for the eigenstates of the Majorana operators with continuous variables with the equivalent analysis for the bosonic quadrature operators \[1\], we can identify the cause of the problem. In \[1\], the Dirac delta terms in the coefficient functions eventually led to an operator of the form

\[
\hat{a}_R = \int [\hat{a}^\dagger(u)]^2 du,
\]  

where \( \hat{a}_R(u) \) is the bosonic creation operator. The equivalent fermionic operator would obviously vanish due to the anti-commuting property, unless some degree of freedom can be used to introduce an anti-symmetry. Such an anti-symmetry can be obtained if the linear combination of fermionic ladder operators incorporates a relative transformation of their degrees of freedom. However, such a relative transformation implies that the resulting linear combination would not be Hermitian. Since the Hermiticity of the operator is the defining property of a quadrature operator, we are led to abandon the search for fermionic equivalents of the quadrature operators and rather consider fermionic equivalents of the Bogoliubov operators.

The properties of fermionic Bogoliubov operators are expected to be the following:

- **fB1**: Fermionic Bogoliubov operators are not Hermitian; they are pairwise Hermitian adjoints of each other.
• fB2: Pairs of fermionic Bogoliubov operators do not anti-commute; their anti-commutation relations have the same form as those of the fermionic ladder operators from which they are constructed.

• fB3: Each fermionic Bogoliubov operator anti-commutes with itself.

From this point onward, we'll use the fermion ladder operators with all degrees of freedom (spin and three-dimensional wave vectors), as in quantum field theory. As stated before, we'll exclude the anti-fermion ladder operators since they anti-commute with the fermion ladder operators. The Lorentz covariant anti-commutation relations for the fermion ladder operators are

\[
\{\hat{a}_s(k), \hat{a}_s^\dagger(k')\} = \{\hat{a}_s(k), \hat{a}_s^\dagger(k')\} = 0,
\]

\[
\{\hat{a}_s(k), \hat{a}_s^\dagger(k')\} = (2\pi)^3 E_k \delta_{s,r} \delta(k-k'),
\]

with

\[
E_k = \sqrt{m^2 + |k|^2}.
\]

Note that \(k\) is the wave vector (as opposed to the momentum). Therefore, the “energy” \(E_k\) and the “mass” \(m\) both have the units of inverse distance.

There are various ways to define fermionic Bogoliubov operators. Here, we proceed with the line of argument presented above. Therefore, we apply a spin transformation to one of the ladder operators and consider linear combinations of it together with the other ladder operator. So, we define the following two operators

\[
\hat{g}_s(k) = \frac{1}{\sqrt{2}} \left[ \hat{a}_s(k) + \hat{a}_s^\dagger(k) \epsilon_{r,s} \right],
\]

\[
\hat{h}_s(k) = \frac{-i}{\sqrt{2}} \left[ \hat{a}_s(k) - \hat{a}_s^\dagger(k) \epsilon_{r,s} \right].
\]

where \(\epsilon_{r,s}\) is an antisymmetric spin matrix \((\epsilon_{s,r} = -\epsilon_{r,s})\).

We also assume that \(\epsilon_{r,s} = -\epsilon_{s,r}\) and \(\epsilon_{r,s} \epsilon_{s,t} = \delta_{r,t}\). For two-dimensional spin degrees of freedom, \(\epsilon_{r,s}\) is defined as the Pauli \(y\)-matrix

\[
\epsilon_{r,s} = \sigma^y_{r,s} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

We follow the summation convention where repeated indices are summed over.

The Hermitian adjoints of the operators in Eq. (27) are

\[
\hat{g}_s^\dagger(k) = \frac{1}{\sqrt{2}} \left[ \hat{a}_s^\dagger(k) - \hat{a}_r(k) \epsilon_{r,s} \right] \approx -i \hat{h}_s(k) \epsilon_{r,s},
\]

\[
\hat{h}_s^\dagger(k) = \frac{i}{\sqrt{2}} \left[ \hat{a}_s^\dagger(k) + \hat{a}_r(k) \epsilon_{r,s} \right] \approx i \hat{g}_r(k) \epsilon_{r,s}.
\]

We see that the Hermitian adjoints are directly related to the original pair of operators. As a result, we do not expect these Hermitian adjoints to represent additional independent degrees of freedom. However, we'll include them in our analysis.

### A. Fermionic adjoint

Here, we define a fermionic adjoint that includes a spin transformation with the Hermitian adjoint, so that

\[
\hat{a}_s(k) \rightarrow \hat{a}_s^\dagger(k) \epsilon_{r,s} = \hat{a}_s^\dagger(k),
\]

\[
\hat{a}_s^\dagger(k) \rightarrow \epsilon_{r,s} \hat{a}_r(k) = -\hat{a}_r(k) \epsilon_{r,s}.
\]

The symbol \(\dagger\) is used to represent the fermionic adjoint. The fermionic Bogoliubov operators defined in Eq. (27) are fermionic self-adjoint:

\[
\hat{g}_s^\dagger(k) = \frac{1}{\sqrt{2}} \left[ \hat{a}_s^\dagger(k) \epsilon_{r,s} + \hat{a}_r(k) \right] \equiv \hat{g}_s(k),
\]

\[
\hat{h}_s^\dagger(k) = \frac{-i}{\sqrt{2}} \left[ \hat{a}_s^\dagger(k) \epsilon_{r,s} - \hat{a}_r(k) \right] \equiv \hat{h}_s(k).
\]

### B. Anti-commutation relations

The various anti-commutation relations among \(\hat{g}, \hat{h}\) and their Hermitian adjoints are

\[
\{\hat{g}_r(k), \hat{g}_r(k')\} = \{\hat{g}_r^\dagger(k), \hat{g}_r^\dagger(k')\} = 0,
\]

\[
\{\hat{h}_r(k), \hat{h}_r(k')\} = \{\hat{h}_r^\dagger(k), \hat{h}_r^\dagger(k')\} = 0,
\]

\[
\{\hat{g}_r(k), \hat{h}_s(k')\} = \{\hat{g}_r^\dagger(k), \hat{h}_s^\dagger(k')\} = \{\hat{h}_r(k), \hat{h}_s^\dagger(k')\} = \{\hat{h}_r^\dagger(k), \hat{h}_s(k')\} = (2\pi)^3 E_k \epsilon_{r,s} \delta(k-k'),
\]

\[
\{\hat{g}_r(k), \hat{g}_s(k')\} = \{\hat{g}_r^\dagger(k), \hat{g}_s^\dagger(k')\} = \{\hat{h}_r(k), \hat{h}_s(k')\} = \{\hat{h}_r^\dagger(k), \hat{g}_s^\dagger(k')\} = 0.
\]

It shows that these operators satisfy all requirements for fermionic Bogoliubov operators fB1, fB2 and fB3.

### IV. EIGENSTATES

#### A. Bosonized operators

In the attempt to determine the eigenstates for Majorana operators with continuous degrees of freedom, we found that the eigenfunctions need to be Grassmann functions to satisfy the requirement for anti-symmetry in fermionic states consisting of more than one fermion. These Grassmann eigenfunctions would also parameterize their associated eigenstates. Therefore, in analogy to the bosonic case [11], we expect that these eigenstates are represented in terms of bosonized (or Grassmann even) spectral operators, defined as

\[
\hat{A} = \int A_s^*(k) \hat{a}_s(k) \, dk_E = A_s^* \circ \hat{a},
\]

\[
\hat{A}^\dagger = \int \hat{a}_s^\dagger(k) A_s(k) \, dk_E = \hat{a}_s^\dagger \circ A,
\]

where

\[
\left\{ A_s^*(k) \hat{a}_s(k) \right\} = (2\pi)^3 \int E_k \delta(k-k') \left\{ A_s^*(k') \hat{a}_s(k) \right\} = 0.
\]
where \( A_s(k) \) is a Grassmann spectral function,

\[
dk_E = \frac{d^3k}{(2\pi)^3E_k},
\]

and a \( \sigma \)-contraction notation is defined. The combinations of the fermion operators and the Grassmann spectral functions produce operators that follow commutation relations instead of anti-commutation relations and are therefore referred to as being “bosonized.” Indeed, we find that \([A, B] = [\hat{A}^\dagger, \hat{B}^\dagger] = 0\), while

\[
[A, \hat{B}^\dagger] = \int A_s^*(k)B_s(k)\ dk_E = A^* \circ B.
\]

Moreover, we also have commutation relations between spectral operators and fermion ladder operators

\[
\begin{align*}
\begin{bmatrix} \hat{a}_s(k), \hat{A} \end{bmatrix} &= \begin{bmatrix} \hat{A}^\dagger, \hat{a}_s^\dagger(k) \end{bmatrix} = 0, \\
\begin{bmatrix} \hat{a}_s(k), \hat{A}^\dagger \end{bmatrix} &= A_s(k), \\
\begin{bmatrix} \hat{A}, \hat{a}_s(k) \end{bmatrix} &= A_s^*(k).
\end{align*}
\]

In addition to the spectral operators, the definition of the eigenstates also needs operators of the form

\[
\begin{align*}
\hat{R} &= \frac{1}{2} \int \hat{a}_s(k)\varepsilon_{s,r}\hat{a}_r(k)\ dk_E = \frac{1}{2}\hat{a} \circ \hat{a}, \\
\hat{R}^\dagger &= \frac{1}{2} \int \hat{a}_s^\dagger(k)\varepsilon_{s,r}\hat{a}_r^\dagger(k)\ dk_E = \frac{1}{2}\hat{a}^\dagger \circ \hat{a}^\dagger,
\end{align*}
\]

where \( \bullet \) represents a contraction that involves the spin transform matrix \( \varepsilon \) defined in Eq. \([25]\). The commutation relations between these operators and fermion ladder operators are

\[
\begin{align*}
\begin{bmatrix} \hat{a}_r(k), \hat{R} \end{bmatrix} &= \varepsilon_{r,s}\hat{a}_s(k), \\
\begin{bmatrix} \hat{R}, \hat{a}_r^\dagger(k) \end{bmatrix} &= \hat{a}_s(k)\varepsilon_{s,r}.
\end{align*}
\]

The commutations among these bosonized operators produce additional operators. To complete the algebra, we also need to define spectral operators that include \( \varepsilon \):

\[
\begin{align*}
\hat{A}_v &= \int A_v(k)\varepsilon_{s,r}\hat{a}_r(k)\ dk_E = \hat{A} \circ \hat{a}, \\
\hat{A}_v^\dagger &= \int \hat{a}^\dagger_s(k)\varepsilon_{s,r}A_r^*(k)\ dk_E = \hat{a}^\dagger \circ \hat{A}^*,
\end{align*}
\]

and a symmetrized number operator

\[
\begin{align*}
\bar{s} &= \frac{1}{2} \int \hat{a}_r^\dagger(k)\hat{a}_s(k) - \hat{a}_s(k)\hat{a}_r^\dagger(k)\ dk_E \\
&= \hat{a}^\dagger \circ \hat{a} - \frac{1}{2}\Omega,
\end{align*}
\]

where the divergent constant \( \Omega \) is given by

\[
\Omega = \int \delta(0)\,d^3k.
\]

The commutation relations among all these bosonized operators are summarized in Appen. \([3]\).

**B. Eigen-equations and their solutions**

The Bogoliubov operators defined in Eq. \([27]\) are not normal. Therefore, their left-eigenstates are not the Hermitian adjoints of their right-eigenstates. As a result, we’ll distinguish between the right- and left-eigenstates with subscript \( R \) and \( L \), respectively. The eigen-equations for the right-eigenstates are

\[
\begin{align*}
\hat{g}_s(k)|g_R\rangle &= |g_R\rangle \hat{g}_s(k), \\
\hat{g}_s^\dagger(k)|g_R\rangle &= |g_R\rangle \hat{g}_s^\dagger(k), \\
\hat{h}_s(k)|h_R\rangle &= |h_R\rangle \hat{h}_s(k), \\
\hat{h}_s^\dagger(k)|h_R\rangle &= |h_R\rangle \hat{h}_s^\dagger(k),
\end{align*}
\]

and for the left-eigenstates, they are

\[
\begin{align*}
\langle g_L|\hat{g}_s(k) &= g_s^\dagger(k)\langle g_L|, \\
\langle g_L|\hat{g}_s^\dagger(k) &= g_s^\dagger(k)\langle g_L|, \\
\langle h_L|\hat{h}_s(k) &= h_s^\dagger(k)\langle h_L|, \\
\langle h_L|\hat{h}_s^\dagger(k) &= h_s^\dagger(k)\langle h_L|.
\end{align*}
\]

The eigenstates are defined in terms of rendering operators that produce the eigenstates from the vacuum. These rendering operators are expressed as exponentiated bosonized operators. For the right-eigenstates, they are generically given by

\[
|\psi_R\rangle = F_0 \exp(\hat{A}^\dagger \pm \hat{R}^\dagger)|\text{vac}\rangle,
\]

where \( F_0 \) is a prefactor to be determined. The form of the rendering operator for each of the Bogoliubov operators is determined by substituting the generic form of the rendering operator into the eigen-equation to determine the correct sign and the relationship between the spectral function and the Grassmann eigenfunction in the equation. These calculations are shown in Appen. \([3]\) leading to the follow expressions for the right-eigenstates in terms of their rendering operators

\[
\begin{align*}
|g_R\rangle &= G_R \exp(\hat{A}^\dagger + \hat{R}^\dagger)|\text{vac}\rangle, \\
|g_R\rangle &= G_R \exp(\hat{A}^\dagger - \hat{R}^\dagger)|\text{vac}\rangle, \\
|h_R\rangle &= H_R \exp(\hat{A}^\dagger - \hat{R}^\dagger)|\text{vac}\rangle, \\
|\bar{h}_R\rangle &= H_R \exp(\hat{A}^\dagger + \hat{R}^\dagger)|\text{vac}\rangle,
\end{align*}
\]

where \( G_R, \bar{G}_R, H_R, \) and \( \bar{H}_R \) are prefactors that we still need to determine. The relationships between the Grassmann parameter functions in the respective spectral operators and the respective eigenfunctions are

\[
\begin{align*}
A_s(k) &= \sqrt{2}g_s(k), \\
A_s(k) &= -\sqrt{2}g_s(k)\varepsilon_{r,s} = \sqrt{2}\varepsilon_{s,r}\bar{g}_s(k), \\
A_s(k) &= i\sqrt{2}h_s(k), \\
A_s(k) &= -i\sqrt{2}\bar{h}_r(k)\varepsilon_{r,s} = i\sqrt{2}\varepsilon_{s,r}h_r(k).
\end{align*}
\]
The same procedure is used for the left-eigenstates of $\tilde{g}_s(k)$, $\tilde{g}_t^\dagger(k)$, $\tilde{h}_s(k)$, and $\tilde{h}_t^\dagger(k)$. The results are given by
\begin{align}
\langle g_L | &= G_L |\text{vac}\rangle \exp\left(\hat{A} - \hat{R}\right), \\
\langle \bar{g}_L | &= \bar{G}_L |\text{vac}\rangle \exp\left(\hat{A} + \hat{R}\right), \\
\langle h_L | &= H_L |\text{vac}\rangle \exp\left(\hat{A} + \hat{R}\right), \\
\langle \bar{h}_L | &= \bar{H}_L |\text{vac}\rangle \exp\left(\hat{A} - \hat{R}\right),
\end{align}
(47)
where $G_L$, $\bar{G}_L$, $H_L$, and $\bar{H}_L$ are prefactors, and the spectral functions related to the eigenfunctions by
\begin{align}
A^+_s(k) &= \sqrt{2}g^*_t(k)\varepsilon_{r,s} - \sqrt{2}\varepsilon_{r,s}g^*_t(k), \\
A^+_t(k) &= -i\sqrt{2}h^*_s(k), \\
A^+_s(k) &= -i\sqrt{2}h^*_t(k),
\end{align}
(48)
where $t$ is an auxiliary variable to be set equal to 1.

Below, the eigenfunctions are used as the parameter functions of the eigenstates.

C. Adjoint

Comparing Eqs. (45) and (47), we find the following relationships among the states and their Hermitian adjoints (assuming their parameter functions are the same):
\begin{align}
\langle g_L | &= (\bar{g}_R)\dagger, \\
\langle \bar{g}_L | &= (g_R)\dagger, \\
\langle h_L | &= (\bar{h}_R)\dagger, \\
\langle \bar{h}_L | &= (h_R)\dagger.
\end{align}
(49)

On the other hand, with the fermionic adjoint, we have
\begin{align}
\langle g_L | &= (\bar{g}_R)\dagger, \\
\langle \bar{g}_L | &= (g_R)\dagger, \\
\langle h_L | &= (\bar{h}_R)\dagger, \\
\langle \bar{h}_L | &= (h_R)\dagger.
\end{align}
(50)

The fermionic adjoint converts the left-eigenstates into the right-eigenstates of the same operator, and vice versa.

V. INNER PRODUCTS

An important aspect of the properties of the eigenstates computed above is the various inner products between the left- and right-eigenstates. It reveals whether there are any sets that are mutually orthogonal or mutually unbiased. It also provides an opportunity to define suitable prefactors.

A. General expression

To compute all the inner products between the left- and right-eigenstates, we consider a generic inner product, given by
\begin{align}
\langle f_L | f_R | &= F_L F_R |\text{vac}\rangle \exp\left(\hat{A} + c_1\hat{R}\right) \\
&\quad \times \exp\left(\hat{B}^\dagger + c_2\hat{R}^\dagger\right) |\text{vac}\rangle.
\end{align}
(51)
Here $F_L$ and $F_R$ are unknown prefactors, $\hat{B}^\dagger$ is a spectral operator with parameter function $B_s(k)$, and $c_{1,2} = \pm 1$, depending on which eigenstates are considered. The general expression for the inner products is calculated in Appendix C. It is given by
\begin{align}
\langle f_L | f_R | &= F_L F_R \left(1 + c_1c_2t^2\right)^{1/2} \exp\left(\frac{t^2A^\dagger \cdot B}{1 + c_1c_2t^2} + \frac{1}{2}c_1c_2t^3A^\dagger \cdot A^*\right),
\end{align}
(52)
where $t$ is an auxiliary variable to be set equal to 1.

We use Eqs. (46) and (48) to represent $B_s(k)$ and $A^*_s(k)$ in terms of the parameter functions for the respective eigenstates. To distinguish them, $g_s(k)$, $\bar{g}_s(k)$, $h_s(k)$, and $\bar{h}_s(k)$ denote those of the respective right-eigenstates and $\gamma^*_s(k), \bar{\gamma}_s^*(k), \theta_s^*(k)$, and $\bar{\theta}_s^*(k)$ are those for the respective left-eigenstates. The self-contractions $\hat{B} \cdot \hat{B}$ for the respective eigenstates are
\begin{align}
\hat{B} \cdot \hat{B} &= 2g \cdot g, \\
\hat{B} \cdot \hat{B} &= -2\bar{g} \cdot \bar{g}, \\
\hat{B} \cdot \hat{B} &= -2\bar{h} \cdot \bar{h},
\end{align}
(53)
and those for $A^* \cdot A^*$ are
\begin{align}
A^* \cdot A^* &= -2\gamma^* \cdot \gamma^*, \\
A^* \cdot A^* &= 2\bar{\gamma}^* \cdot \bar{\gamma}^*, \\
A^* \cdot A^* &= -2\bar{\theta}^* \cdot \bar{\theta}^*.
\end{align}
(54)

B. Inner products with common signs

Considering those inner products for which $c_1 = c_2$, we have two sets. Those with $c_1 = c_2 = 1$ are $\langle \bar{g}_L | g_R \rangle$, $\langle h_L | g_R \rangle$, $\langle \bar{g}_L | h_R \rangle$, and $\langle h_L | h_R \rangle$, and for $c_1 = c_2 = -1$ they are $\langle g_L | \bar{g}_R \rangle$, $\langle \bar{h}_L | g_R \rangle$, $\langle g_L | h_R \rangle$, and $\langle \bar{h}_L | h_R \rangle$. For $t = 1$, they produce expressions of the form
\begin{align}
\langle f_L | f_R | &= F_L F_R \left(1 + c_1c_2t^2\right)^{1/2} \exp\left(\frac{t^2A^\dagger \cdot B}{1 + c_1c_2t^2} + \frac{1}{2}c_1c_2t^3A^\dagger \cdot A^*\right),
\end{align}
(55)
where the signs are given by $c_1 = c_2$.

When we combine the self-contractions in Eqs. (49) and (51) with the appropriate signs given by $c_1$ and $c_2$ as stipulated in Eq. (52), we find that they are all positive. We can remove them from the exponent by defining the prefactors as functionals with the negative exponentiated self-contractions times a constant to remove the
remaining constant factor:

\[
G_{L}[\gamma] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\gamma}^* \cdot \gamma^* \right), \\
H_{L}[\theta] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\theta}^* \cdot \theta^* \right), \\
\bar{G}_{L}[\bar{\gamma}] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\gamma}^* \cdot \bar{\gamma}^* \right), \\
\bar{H}_{L}[\bar{\theta}] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\theta}^* \cdot \bar{\theta}^* \right), \\
G_{R}[\gamma] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\gamma} \cdot \gamma \right), \\
H_{R}[\theta] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\theta} \cdot \theta \right), \\
\bar{G}_{R}[\bar{\gamma}] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\gamma} \cdot \bar{\gamma} \right), \\
\bar{H}_{R}[\bar{\theta}] = 2^{-\Omega/4} \exp \left(-\frac{1}{2} \hat{\theta} \cdot \bar{\theta} \right).
\]

(56)

With these definitions, together with the contractions \( A^\ast \circ B \) in terms of the parameter functions in Eqs. (46) and (48), the inner products with \( c_1 = c_2 = 1 \) become

\[
\langle \bar{g}_{L}[g_{R}] \rangle = \exp \left( \bar{\gamma}^* \circ g \right), \\
\langle h_{L}[g_{R}] \rangle = \exp \left( -i \hat{\theta}^* \circ g \right), \\
\langle \bar{g}_{L}[\bar{h}_{R}] \rangle = \exp \left( i \hat{\gamma}^* \circ \bar{h} \right), \\
\langle h_{L}[\bar{h}_{R}] \rangle = \exp \left( \theta^* \circ \bar{h} \right),
\]

(57)

and those with \( c_1 = c_2 = -1 \) become

\[
\langle g_{L}[\bar{g}_{R}] \rangle = \exp \left( \gamma^* \circ \bar{g} \right), \\
\langle h_{L}[\bar{g}_{R}] \rangle = \exp \left( -i \hat{\theta} \circ \bar{g} \right), \\
\langle g_{L}[h_{R}] \rangle = \exp \left( i \gamma^* \circ h \right), \\
\langle h_{L}[h_{R}] \rangle = \exp \left( \theta^* \circ h \right).
\]

(58)

If all the parameter functions are real-valued Grassmann functions, then all these inner products produce unitary functionals of these parameter functions, because, for example

\[
(\gamma \circ \bar{g})^\dagger = \bar{g} \circ \gamma = - \gamma \circ \bar{g}, \\
(i\gamma \circ \theta)^\dagger = -i \theta \circ \gamma = -i \gamma \circ \theta.
\]

(59)

The modulus squares of these inner products are thus equal to 1. Hence, with real-valued Grassmann parameter functions, the respective eigenstates in these inner products represent mutually unbiased bases.

C. Inner products with opposite signs

The inner products where the signs in the eigenstates are opposite \((c_1 = -c_2)\) include all the fermionic self-overlaps \( \langle g_{L}[g_{R}] \rangle, \langle \bar{g}_{L}[\bar{g}_{R}] \rangle, \langle h_{L}[h_{R}] \rangle, \) and \( \langle \bar{h}_{L}[\bar{h}_{R}] \rangle, \) and the four remaining inner products \( \langle h_{L}[\bar{g}_{R}] \rangle, \langle \bar{h}_{L}[\bar{g}_{R}] \rangle, \langle \bar{g}_{L}[h_{R}] \rangle \) and \( \langle g_{L}[h_{R}] \rangle. \) With opposite signs for \( c_1 \) and \( c_2, \) the expression in Eq. (58) becomes singular at \( t = 1. \) Therefore, we can’t set \( t = 1 \) and thus need to consider the limit

\[
t \to 1. \quad \text{The expression for the inner product becomes}
\]

\[
\langle f_{L}[f_{R}] \rangle = \lim_{t \to 1} F_{L} F_{R} \left( 1 - t^2 \right)^{\Omega/2} \exp \left( \frac{t^2 A^* \circ B}{1 - t^2} + \frac{c_1}{2} \frac{t^2 B \circ B}{1 - t^2} + \frac{c_2}{2} t^2 A^* \circ A^* \right).
\]

(60)

The expressions for the contractions \( A^* \circ B \) are obtained with Eqs. (46) and (48). The combination of the self-contractions in Eqs. (63) and (64) with the appropriate signs given by \( c_1 \) and \( c_2 \) for the current inner products make them all negative. However, when the prefactors are incorporated the exponents can be factorized. The resulting expressions for these eight inner products all have the form

\[
\langle f_{L}[f_{R}] \rangle = p[f] = \lim_{t \to 1} \left( 1 - t^2 \right)^{\Omega/2} \exp \left( -\frac{t^2 f \circ f}{1 - t^2} \right).
\]

(61)

For the fermionic self-overlaps, we get

\[
\langle g_{L}[g_{R}] \rangle = \exp [\gamma^* - g], \\
\langle g_{L}[\bar{g}_{R}] \rangle = \exp [\bar{\gamma}^* - \bar{g}], \\
\langle h_{L}[h_{R}] \rangle = \exp [\theta^* - h], \\
\langle \bar{h}_{L}[\bar{h}_{R}] \rangle = \exp [\bar{\theta}^* - \bar{h}].
\]

(62)

The expressions for the remaining four inner products are

\[
\langle \bar{g}_{L}[g_{R}] \rangle = \exp [\bar{\gamma}^* + i \varepsilon \cdot g], \\
\langle \bar{g}_{L}[\bar{g}_{R}] \rangle = \exp [\gamma^* - i \varepsilon \cdot \bar{g}], \\
\langle g_{L}[h_{R}] \rangle = \exp [\gamma^* - i \varepsilon \cdot h], \\
\langle h_{L}[\bar{g}_{R}] \rangle = \exp [\theta^* + i \varepsilon \cdot \bar{g}].
\]

(63)

Note that, for the contraction \( f \circ f, \) any spin transformation in the first \( f \) needs to be converted to \( \varepsilon \cdot g \rightarrow -g \cdot \varepsilon. \)

D. Orthogonality

The general result of the inner product in Eq. (61) is proportional to a Grassmann Dirac delta functional, for which it suffices to have

\[
\delta[f - f'] = \prod_{\Omega} (f - f') \circ (f - f').
\]

(64)

In App. E, we show that when the inner product in Eq. (61) is multiplied by an arbitrary Grassmann functional and integrated over a common Grassmann field variable, the part of the exponential function that contributes has the form of such a product and thus acts as a Grassmann Dirac delta functional. Therefore, Eq. (61) represents an orthogonality condition. Formally, we’ll represent it as

\[
\langle f[f'] \rangle = \lim_{t \to 1} \left( 1 - t^2 \right)^{\Omega/2} \exp \left[ -\frac{t^2 f \circ f}{1 - t^2} - (f - f') \right] = \delta[f - f'].
\]

(65)
The inner products associated with the fermionic self-overlaps are then represented as

\[
\begin{align*}
\langle g_L|g_R \rangle &= \delta(\gamma^* - g), \\
\langle \bar{g}_L|g_R \rangle &= \delta(\bar{\gamma}^* - \bar{g}), \\
\langle h_L|h_R \rangle &= \delta(\theta^* - h), \\
\langle h_L|\bar{h}_R \rangle &= \delta(\bar{\theta}^* - \bar{h}),
\end{align*}
\]

(66)

and those for the remaining four inner products are

\[
\begin{align*}
\langle \bar{h}_L|g_R \rangle &= \delta(\bar{\theta}^* + i\varepsilon \cdot g), \\
\langle \bar{g}_L|h_R \rangle &= \delta(\bar{\gamma}^* - i\varepsilon \cdot h), \\
\langle g_L|\bar{h}_R \rangle &= \delta(\gamma^* - i\varepsilon \cdot \bar{h}), \\
\langle h_L|\bar{g}_R \rangle &= \delta(\theta^* + i\varepsilon \cdot \bar{g}).
\end{align*}
\]

(67)

Here, the argument of the Dirac delta functional contains the differences between one parameter function and the complex conjugate of another. It suggests that these parameter functions should be real valued.

VI. COMPLETENESS

A. Notation

To consider the completeness of eigenstates, we need to distinguish between the type of eigenstate and its parameter function. Therefore, the real-valued Grassmann parameter functions, represented by q’s or p’s, are shown as functional dependences in the states. It then follows that

\[
\begin{align*}
|g_R[i\varepsilon \cdot q]\rangle &= |\bar{h}_R[q]\rangle, \\
|\bar{g}_R[i\varepsilon \cdot \bar{p}]\rangle &= |h_R[p]\rangle, \\
|g_L[-i\varepsilon \cdot \bar{q}]\rangle &= |\bar{h}_L[q]\rangle, \\
|\bar{g}_L[-i\varepsilon \cdot q]\rangle &= |h_L[p]\rangle.
\end{align*}
\]

(68)

Since \(i\varepsilon\) is a real valued matrix, the spin-transformed real valued parameter functions remain real valued. Therefore, these relationships show that we only need four of these eigenstates. The other four are related to them via spin transformations of the parameter functions. The four eigenstates that we select are the unbarred eigenstates \(|g_R[q]\rangle, |g_L[q]\rangle, |h_R[p]\rangle,\) and \(|h_L[p]\rangle\).

B. Resolving the identity

Based on their orthogonality conditions in Eq. (66), the associated completeness conditions composed from these eigenstates are

\[
\begin{align*}
\int |g_R[q]\rangle \langle g_L[q]| D[q] &= 1, \\
\int |h_R[p]\rangle \langle h_L[p]| D[p] &= 1.
\end{align*}
\]

(69)

The Grassmann parameter functions become Grassmann integration field variables. The identity operator for the Hilbert space is denoted by \(1\).

To test whether Eq. (69) represents valid resolutions of the identity, we overlap them by two arbitrary states and see whether the result gives the expected inner product.

Considering the first expression in Eq. (69), we have the four inner products among the eigenstates:

\[
\begin{align*}
\langle g_L[q_2]|1|g_R[q_1]\rangle &= \int \langle g_L[q_2]|g_R[q]\rangle \langle g_L[q]|g_R[q]\rangle D[q] \\
&= \int \delta(q - q_2)\delta(q_1 - q)D[q] = \delta(q_1 - q_2), \\
\langle g_L[q_2]|1|h_R[p_1]\rangle &= \int \langle g_L[q_2]|g_R[q]\rangle \langle g_L[q]|h_R[p]\rangle D[q] \\
&= \int \delta(q - q_2)\exp(iq \cdot p_1)D[q] = \exp(iq_2 \cdot p_1), \\
\langle h_L[p_2]|1|g_R[q_1]\rangle &= \int \langle h_L[p_2]|g_R[q]\rangle \langle g_L[q]|g_R[q]\rangle D[q] \\
&= \int \exp(-ip_2 \cdot q)\delta(q_1 - q)D[q] = \exp(-ip_2 \cdot q_1), \\
\langle h_L[p_2]|1|h_R[p_1]\rangle &= \int \langle h_L[p_2]|g_R[q]\rangle \langle g_L[q]|h_R[p]\rangle D[q] \\
&= \int \exp(-ip_2 \cdot q)\exp(ip \cdot p_1)D[q] = \delta(p_1 - p_2).
\end{align*}
\]

The Grassmann Dirac delta functionals alleviate the evaluation of the integrations in the first three cases. They produce the expected results, thanks to the orthogonality conditions. The fourth inner product produces a result that is consistent with the result that would be obtained from direct Grassmann integration. The equivalent inner products obtained from the second expression in Eq. (69) follow the same pattern.
VII. QUADRATURE BASES

Henceforth, we use the four sets of eigenstates to represent two mutually unbiased bases and their fermionic duals and we call them the fermionic quadrature bases. They are directly denoted by their parameter functions:

$$\{ |q\rangle \} = \{ |g_{R}[q]\rangle \}, \quad \{ |q\rangle \} = \{ |g_{L}[q]\rangle \},
\{ |p\rangle \} = \{ |h_{R}[p]\rangle \}, \quad \{ |p\rangle \} = \{ |h_{L}[p]\rangle \}.$$  \hspace{1cm} (70)

The dual states are given by the fermionic adjoints

$$\langle q | = (|q\rangle)\dagger, \quad \langle p | = (|p\rangle)\dagger.$$ \hspace{1cm} (71)

The inner products among these bases elements are

$$\langle q | q' \rangle = \delta[q - q'], \quad \langle p | p' \rangle = \delta[p - p'], \quad \langle q | p \rangle = \exp(iq \bullet p), \quad \langle p | q \rangle = \exp(-ip \bullet q).$$ \hspace{1cm} (72)

The first two represent orthogonality conditions and the last two indicate that the bases are mutually unbiased. It then also follows that

$$\langle p | 1 | p' \rangle = \int \langle p | q \rangle \langle q | p' \rangle \mathcal{D}[q]$$
$$= \int \exp(-ip \bullet q) \exp(iq \bullet p') \mathcal{D}[q]$$
$$= \delta[p - p'].$$ \hspace{1cm} (73)

The fermionic quadrature operators are identified as the following two fermionic Bogoliubov operators

$$\hat{q}_{s}(k) = \frac{1}{\sqrt{2}} [\hat{a}_{s}(k) + \hat{a}_{s}^\dagger(k) \varepsilon_{r,s}] = \hat{g}_{s}(k),$$
$$\hat{p}_{s}(k) = \frac{-i}{\sqrt{2}} [\hat{a}_{s}(k) - \hat{a}_{s}^\dagger(k) \varepsilon_{r,s}] = \hat{h}_{s}(k).$$ \hspace{1cm} (74)

The ladder operators are linear combinations of the quadrature operators, given by

$$\hat{a}_{s}(k) = \frac{1}{\sqrt{2}} [\hat{g}_{s}(k) + i\hat{p}_{s}(k)],$$
$$\hat{a}_{s}^\dagger(k) = \frac{1}{\sqrt{2}} [\hat{g}_{s}(k) - i\hat{p}_{s}(k)] \varepsilon_{r,s}. \hspace{1cm} (75)$$

The quadrature operators can be represented in terms of their eigenstates as

$$\hat{q}_{s}(k) = \int \langle q | q_{s}(k) \rangle \langle q | \mathcal{D}[q],$$
$$\hat{p}_{s}(k) = \int \langle p | p_{s}(k) \rangle \langle p | \mathcal{D}[p].$$ \hspace{1cm} (76)

VIII. GRASSMANN FOURIER ANALYSIS

A. Grassmann Dirac delta functionals

The expression in Eq. (73) is a definition of the Grassmann Dirac delta functional. An equivalent expression follows by interchanging $|p\rangle$ and $|q\rangle$. The two definitions can be represented as

$$\int \exp(iq \bullet p) \mathcal{D}[q] = \delta[p],$$
$$\int \exp(-iq \bullet p) \mathcal{D}[q] = \delta[p].$$ \hspace{1cm} (77)

If we relabel the last expression $p \leftrightarrow q$, and change the order in the exponent $q \bullet p = p \bullet q$, we get

$$\int \exp(-iq \bullet p) \mathcal{D}[q] = \delta[p],$ \quad (78)
$$\int \exp(ixq \bullet p) \mathcal{D}[q] = \delta[p].$$ \hspace{1cm} (79)

It shows that the Grassmann Dirac delta functional is symmetric, just like its bosonic counterpart. The Grassmann Dirac delta functionals are therefore given by

$$\int \exp(ixq \bullet p) \mathcal{D}[q] = \delta[p],$$
$$\int \exp(iq \bullet p) \mathcal{D}[p] = \delta[q].$$ \hspace{1cm} (80)

B. Grassmann functional Fourier transform

We can also interpret Eq. (73) as an orthogonality condition for the exponential functionals. It allows us to define Grassmann functional Fourier integrals. The Grassmann functional Fourier transform of an arbitrary Grassmann functional $W[q]$ and its associated inverse are thus represented by

$$\hat{W}[p] = \int W[q] \exp(-iq \bullet p) \mathcal{D}[q],$$
$$W[q] = \int \hat{W}[p] \exp(iq \bullet p) \mathcal{D}[p]. \hspace{1cm} (80)$$

When the inverse Fourier transform is applied to the result of the Fourier transform, it gives

$$\int \hat{W}[p] \exp(iq' \bullet p) \mathcal{D}[p]$$
$$= \int W[q] \exp[-i(q - q') \bullet p] \mathcal{D}[p,q]$$
$$= \int W[q] \delta[q - q'] \mathcal{D}[q] = W[q'].$$ \hspace{1cm} (81)

By operating with the identity operators resolved in the opposite bases, the elements of the different bases can be represented in terms of each other

$$|q\rangle = 1 |q\rangle = \int |p\rangle \exp(-ip \bullet q) \mathcal{D}[p],$$
$$|p\rangle = 1 |p\rangle = \int |q\rangle \exp(iq \bullet p) \mathcal{D}[q].$$ \hspace{1cm} (82)

It shows that there exists a Fourier relationship between the two mutually unbiased bases.

The Fourier relationship between the two quadrature bases allows us to interpret their parameter functions as the mutually orthogonal coordinates of a functional phase space. The stage is set to introduce distributions that are defined as functionals on this phase space.


IX. FERMIONIC WIGNER FUNCTIONAL

A. Definition of the Wigner functional

In analogy with the bosonic definition, together with the knowledge of the fermionic quadrature bases, we can now define the fermionic Wigner functional as

\[
W[q,p] = \int \langle q + \frac{1}{2}x | \hat{\rho} | q - \frac{1}{2}x \rangle \exp(-ix \cdot p) \mathcal{D}[x].
\]

(83)

Here \( \hat{\rho} \) (nominally a density operator) can represent any operator defined on the Hilbert space of all fermionic states incorporating all the spin, spatiotemporal and particle-number degrees of freedom. The density operator \( \hat{\rho} \) can be represented as a density ‘matrix’, which we refer to as a density functional

\[
\rho[q_1, q_2] = \langle q_1 | \hat{\rho} | q_2 \rangle.
\]

(84)

In the case of pure states, the density functional becomes a product of wave functionals

\[
\rho[q_1, q_2] = \psi[q_1] \psi^*[q_2],
\]

(85)

where \( \psi[q] = \langle q | \psi \rangle \).

B. Quadrature and ladder operators

We compute the fermionic Wigner functionals for the quadrature operators as examples. Using the eigen-equation for the quadrature operator \( \hat{q}_s(k) \), we get

\[
W_{\hat{q}}[q,p] = \int \langle q + \frac{1}{2}x | \hat{q}_s(k) | q - \frac{1}{2}x \rangle \exp(-ix \cdot p) \mathcal{D}[x]
\]

\[
= \int \langle q + \frac{1}{2}x | q - \frac{1}{2}x \rangle [q_s(k) - \frac{1}{2}x_s(k)] \exp(-ix \cdot p) \mathcal{D}[x]
\]

\[
= \int \delta[x] [q_s(k) - \frac{1}{2}x_s(k)] \exp(-ix \cdot p) \mathcal{D}[x]
\]

\[
= q_s(k).
\]

(86)

For the Wigner functional of \( \hat{p}_s(k) \), we represent it as in Eq. (74). Then

\[
W_{\hat{p}}[q,p] = \int \langle q + \frac{1}{2}x | \hat{p}_s(k) | q - \frac{1}{2}x \rangle \exp(-ix \cdot p) \mathcal{D}[x]
\]

\[
= \int \langle q + \frac{1}{2}x | p' \rangle [p'_s(k) | p' - \frac{1}{2}x \rangle \exp(-ix \cdot p) \mathcal{D}[p',x]
\]

\[
= \int p'_s(k) \exp[ix \cdot (p' - p)] \mathcal{D}[p',x]
\]

\[
= \int p'_s(k) \delta[p' - p] \mathcal{D}[p'] = p_s(k).
\]

(87)

Here, we used Eqs. (72) and (79). We see that the sign of the exponent in the definition of the Wigner functional needs to be negative, otherwise we get the negative of what we have here.

The Wigner functionals of linear combinations of operators are the linear combinations of the Wigner functionals of those operators. Hence, in terms of Eq. (75), the Wigner functionals of the ladder operators are

\[
W_a[q,p] = \frac{1}{\sqrt{2}} [q_s(k) + ip_s(k)],
\]

\[
W_\alpha[q,p] = \frac{1}{\sqrt{2}} [q_s(k) - ip_s(k)] \varepsilon_{r,s}.
\]

C. Weyl transformations

The Weyl transformation represents the inverse process whereby a fermion operator is reproduced from its Grassmann Wigner functional. The inverse Fourier transform to the Wigner functional gives

\[
\rho[q_1, q_2] = \int W \left[ \frac{1}{2} q_1 + \frac{1}{2} q_2, p \right] \times \exp((q_1 - q_2) \cdot p) \mathcal{D}[p].
\]

(89)

The Fourier transform of Eq. (89) with respect to \( x = q_1 - q_2 \) reproduces the expression for the Grassmann Wigner functional in Eq. (83). The Weyl transformation then represents an operator as

\[
\hat{\rho} = \int |q_1| W \left[ \frac{1}{2} q_1 + \frac{1}{2} q_2, p \right] \times \exp(i(q_1 - q_2) \cdot p) \mathcal{D}[p,q_1,q_2]
\]

\[
\times \exp(ix \cdot p) \mathcal{D}[p,q,x],
\]

(90)

in terms of its Wigner functional.

It follows that the trace of an operator is given by

\[
\text{tr}\{\hat{A}\} = \int W_{A}(q,p) \mathcal{D}[q,p].
\]

(91)

The normalization of a density operator thus implies

\[
\text{tr}(\hat{\rho}) = \int W_{\hat{\rho}}[q,p] \mathcal{D}[q,p] = 1.
\]

(92)

D. Characteristic functional

The characteristic functional is the symplectic Grassmann functional Fourier transform of the Wigner functional with respect to both \( q \) and \( p \). It reads

\[
\chi[\xi,\zeta] = \int W[q,p] \exp(i\xi \cdot p - iq \cdot \xi) \mathcal{D}[q,p].
\]

(93)

The Wigner functional is recovered from the characteristic functional via the inverse symplectic Grassmann functional Fourier transform

\[
W[q,p] = \int \chi[\xi,\zeta] \exp(iq \cdot \xi - i\zeta \cdot p) \mathcal{D}[\xi,\zeta].
\]

(94)
E. Grassmann star-products

The way Grassmann Wigner functionals are combined to represent the products of operators leads to the definition of the Grassmann star-product or Grassmann Moyal product. In Appendix E we derive integral expressions for the star-products of two and three Wigner functionals, respectively. For two Wigner functionals, it reads

\[
W_{\hat{A}\hat{B}} = \frac{1}{N_0^2} \int W_{\hat{A}}[q_1, p_1] W_{\hat{B}}[q_2, p_2] \times \exp\{i2(q - q_2) \cdot p_1 + i2(q_1 - q) \cdot p_2 + i2(q_2 - q_1) \cdot p\} D[q_1, p_1, q_2, p_2],
\]  

(95)

where \( N_0 = 2^{\Omega} \). The star-product of three Wigner functionals is given by

\[
W_{\hat{A}\hat{B}\hat{C}} = \int W_{\hat{A}}\left[\frac{q + q_a + q_b}{2}, \frac{p + p_a + p_b}{2}\right] W_{\hat{B}}[q_a, p_a] \times W_{\hat{C}}\left[\frac{q - q_a + q_b}{2}, \frac{p - p_a - p_b}{2}\right] \times \exp\{i((q - q_a) \cdot p_b - iq_b \cdot (p - p_a))\} \times D[q_a, p_a, q_b, p_b].
\]  

(96)

X. CONCLUSION

A Grassmann Wigner functional formalism is developed for fermions. Much of the development involves the identification of suitable quadrature bases. We show that the eigenstates of the Majorana operators are not suitable for this purpose. Instead, we find that a set of fermionic Bogoliubov operators is analogues to the bosonic quadrature bases, provided that the Hermitian adjoint is replaced by a so-called fermionic adjoint, which incorporates a spin transformation.

The eigenstates of the fermionic Bogoliubov operators are obtained with the aid of a closed algebra of bosonized operators, constructed from the fermion ladder operators and Grassmann spectral functions. Each of the resulting eigenstates is associated with a Grassmann eigenvalue function that carries spin and spatiotemporal degrees of freedom. These eigenvalue functions eventually serve as field variables of the Grassmann functional phase space.

The eigenstates are similar to their bosonic counterparts, but with some differences. The most important difference is the fact that the dual space is not given by the Hermitian adjoint, but by the fermionic adjoint. The eigenstates do not satisfy orthogonality relationships with their Hermitian adjoints. However, the left and right eigenstates of the operators, which are related via the fermionic adjoint, do satisfy orthogonality relationships for real-valued parameter functions.

From the calculation of all the inner products among these eigenstates, we find that these eigenstates form orthogonal bases. We select suitable quadrature bases among the sets of eigenstates of the fermionic Bogoliubov operators. These fermionic quadrature bases obey the required orthogonality and completeness conditions. They are suitable to act as fermionic quadrature bases for the formulation of a fermionic Wigner functional theory.

The completeness relations for the fermionic quadrature bases, together with their orthogonality conditions, lead to a Fourier relationship between the two mutually unbiased bases. It allows for the definition of a functional Fourier transformation that transforms functionals of field variables associated with one quadrature basis into those associated with the other quadrature basis.

The Grassmann Wigner functional is then defined with the aid of the quadrature basis and the functional Fourier transform. It can represent any operator defined on the Hilbert space of the fermions. A Grassmann Weyl transformation is defined, converting a Grassmann Wigner functional back into an operator. The Grassmann characteristic functional is given by the symplectic functional Fourier transformation of a Grassmann Wigner functional with respect to both field variables. We also derive functional integral expressions for the star products that combine Wigner functionals of operators to obtain the Wigner functional for the products of these operators.

Appendix A: Commutation relations

The complete algebra of bosonized operators obeys the following commutation relations. Those among spectral operators are

\[
[\hat{A}, \hat{B}^\dagger] = A^* \circ B, \quad [\hat{A}_\varepsilon, \hat{B}^\dagger] = A \circ B,
\]

\[
[\hat{A}, \hat{B}^\dagger] = A^* \circ B^*, \quad [\hat{A}_\varepsilon, \hat{B}_\varepsilon^\dagger] = -B^* \circ A.
\]

(A1)

Those including \( \hat{R} \) and \( \hat{R}^\dagger \), are

\[
[\hat{A}, \hat{R}^\dagger] = \hat{A}_\varepsilon^\dagger, \quad [\hat{R}, \hat{A}^\dagger] = -\hat{A},
\]

\[
[\hat{R}, \hat{A}^\dagger] = \hat{A}_\varepsilon^\dagger, \quad [\hat{R}, \hat{R}^\dagger] = -\hat{A}^\dagger,
\]

\[
[\hat{R}, \hat{R}^\dagger] = -\hat{s},
\]

(A2)

and those including \( \hat{s} \), are

\[
[\hat{A}, \hat{s}] = \hat{A}, \quad [\hat{A}_\varepsilon, \hat{s}] = \hat{A}_\varepsilon, \quad [\hat{R}, \hat{s}] = 2\hat{R},
\]

\[
[\hat{s}, \hat{A}^\dagger] = \hat{A}_\varepsilon^\dagger, \quad [\hat{s}, \hat{A}_\varepsilon^\dagger] = \hat{A}_\varepsilon^\dagger, \quad [\hat{s}, \hat{R}^\dagger] = 2\hat{R}^\dagger.
\]

(A3)

Note that \( \hat{R}, \hat{R}^\dagger \), and \( \hat{s} \), form an algebra for SU(1,1).

Appendix B: Eigenstates

Combining the commutation relations in Eqs. (A1) and Eq. (A3), we have

\[
[\hat{a}_s(k), \hat{A}^\dagger] = A_s(k) \pm \hat{a}_s(k) \varepsilon_{r,s},
\]

\[
[\hat{A} \pm \hat{R}, \hat{a}_s(k)] = A_s^*(k) \pm \hat{a}_s(k) \varepsilon_{r,s}.
\]

(B1)
These commutation relations and the identity
\[ \exp(\hat{X})\hat{Y}\exp(-\hat{X}) = \hat{Y} + [\hat{X}, \hat{Y}] + \frac{1}{2!}[\hat{X}, [\hat{X}, \hat{Y}]] + \frac{1}{3!}[\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]] + \ldots, \] (B2)
allow us to obtain
\[ \hat{a}_s(k) \exp \left( \hat{A}^\dagger \pm \hat{R}^\dagger \right) = \exp \left( \hat{A}^\dagger \pm \hat{R}^\dagger \right) \left[ \hat{a}_s(k) + A_s(k) \mp \hat{a}_s(k) \varepsilon_{r,s} \right], \]
\[ \exp \left( \hat{A} \pm \hat{R} \right) \hat{a}_s(k) = \left[ \hat{a}_s(k) + A^*_s(k) \mp \hat{a}_r(k) \varepsilon_{r,s} \right] \exp \left( \hat{A} \pm \hat{R} \right). \] (B3)

Considering the generic operators \( \hat{a}_s = a_1 \hat{a}_s + a_2 \hat{a}_s \pm \varepsilon_{r,s} \) and \( \hat{v}_s = b_1 \hat{a}_r \pm \varepsilon_{r,s} + b_2 \hat{a}_s \), where \( a_1, a_2, b_1, \) and \( b_2 \) are arbitrary constants, we obtain solutions for the eigen-equations, starting with those for the right-eigenstates
\[ \hat{u}_s(k)|u\rangle = \left[ a_1 \hat{a}_s(k) + a_2 \hat{a}_s(k) \varepsilon_{r,s} \right] F_0 \exp \left( \hat{A}^\dagger + c_0 \hat{R}^\dagger \right) |\text{vac}\rangle \]
\[ = |u\rangle a_1 A_s(k) = |u\rangle u_s(k) \quad \text{for} \quad c_0 a_0 = a_2, \]
\[ \hat{v}_s(k)|v\rangle = \left[ b_1 \hat{a}_r(k) \varepsilon_{r,s} + b_2 \hat{a}_s(k) \right] F_0 \exp \left( \hat{A}^\dagger + c_0 \hat{R}^\dagger \right) |\text{vac}\rangle \]
\[ = |v\rangle b_1 A_r(k) \varepsilon_{r,s} = |v\rangle v_s(k) \quad \text{for} \quad c_0 b_1 = b_2, \] (B4)
where \( c_0 = \pm 1 \). For the left-eigenstates, we have
\[ \langle u| \hat{u}_s(k) = \langle \text{vac}| F_0 \exp \left( \hat{A} + c_0 \hat{R} \right) \left[ a_1 \hat{a}_s(k) + a_2 \hat{a}_s(k) \varepsilon_{r,s} \right] \]
\[ = F_0 \langle \text{vac}| a_1 A_s(k) + a_2 A^*_s(k) \varepsilon_{r,s} + a_2 c_0 \hat{a}_t(k) \varepsilon_{t,r,s} + a_2 c_0 \hat{a}_t(k) \varepsilon_{t,r,s} \rangle \exp \left( \hat{A} + c_0 \hat{R} \right) \]
\[ = a_2 A^*_s(k) \varepsilon_{r,s} |u\rangle = u_s^*(k) |u\rangle \quad \text{for} \quad c_0 a_2 = -a_1, \]
\[ \langle v| \hat{v}_s(k) = \langle \text{vac}| F_0 \exp \left( \hat{A} + c_0 \hat{R} \right) \left[ b_1 \hat{a}_r(k) \varepsilon_{r,s} + b_2 \hat{a}_s(k) \right] \]
\[ = F_0 \langle \text{vac}| b_1 A_r(k) \varepsilon_{r,s} + b_2 A^*_s(k) \varepsilon_{t,r,s} + b_2 c_0 \hat{a}_t(k) \varepsilon_{t,r,s} \rangle \exp \left( \hat{A} + c_0 \hat{R} \right) \]
\[ = b_2 A^*_s(k) \langle v\rangle = v_s^*(k) \langle v\rangle \quad \text{for} \quad c_0 b_2 = -b_1. \] (B5)

Appendix C: Commuting the operators for inner products

To compute the expression for the generic inner product given in Eq. \ref{eq:51}, we commute the exponentiated operators so that they appear in normal order, producing a more complex expression with new operators generated by the current ones. The expected operators include \( \hat{s}, \hat{A}^\dagger \), and \( \hat{B}_z \). The general expression has the form
\[ \exp(\hat{A}) \exp(c_1 \hat{R}) \exp(\hat{B}^\dagger) \exp(c_2 \hat{R}^\dagger) = \exp(h_0) \exp(h_1 \hat{B}^\dagger) \exp(h_2 \hat{A}^\dagger) \exp(h_3 \hat{R}^\dagger) \]
\[ \times \exp(h_4 \hat{s}) \exp(h_5 \hat{B}_z) \exp(h_6 \hat{A}) \exp(h_7 \hat{R}), \] (C1)
where the \( h_n \)'s are unknown constants. We introduce an auxiliary variable \( t \) into the exponents on the left-hand side and convert the unknown constants into unknown functions of \( t \) on the right-hand side. The equation then reads
\[ \exp(t \hat{A}) \exp(tc_1 \hat{R}) \exp(t \hat{B}^\dagger) \exp(tc_2 \hat{R}^\dagger) = \exp[h_0(t)] \exp[h_1(t) \hat{B}^\dagger] \exp[h_2(t) \hat{A}^\dagger] \exp[h_3(t) \hat{R}^\dagger] \]
\[ \times \exp[h_4(t) \hat{s}] \exp[h_5(t) \hat{B}_z] \exp[h_6(t) \hat{A}] \exp[h_7(t) \hat{R}]. \] (C2)
The unknown functions become zero for \( t = 0 \), and for \( t = 1 \), we recover the original expression.
Substituting Eq. \ref{eq:C2} into Eq. \ref{eq:51} and evaluating the contractions with the vacuum state, we get
\[ \langle f_L| f_R \rangle = F_{f_L} F_{f_R} \exp[h_0(t)] \exp\left[-\frac{1}{2} h_4(t) \Omega \right], \] (C3)
where we used the fact that any exponentiated normal ordered operator \( \hat{O} \) produces \( \exp(K \hat{O}) |\text{vac}\rangle = |\text{vac}\rangle \) and \( |\text{vac}\rangle \exp(K \hat{O}) = |\text{vac}\rangle \), regardless of the value of \( K \). We used Eq. \ref{eq:40} to express \( \hat{s} \) in normal order. Therefore
\[ \langle v\rangle \exp(K \hat{s}) |\text{vac}\rangle = \exp\left(-\frac{1}{2} K \Omega \right). \] (C4)
A derivative with respect to $t$ is applied on both sides of Eq. (2) and as many of the exponentiated operators as possible are removed by operating with the respective inverse operators on the right-hand sides of both sides of the equations. The result reads

\[
\hat{A} + c_1 \hat{R} + \exp(t\hat{A}) \exp(t_1 \hat{R}) \exp(-t \hat{A}) + \exp(t \hat{A}) \exp(t_1 \hat{R}) c_2 \hat{R}^d \exp(-t_1 \hat{R}) \exp(-t \hat{A}) \\
= \partial_t h_0 + \partial_t h_1 \hat{B}^d + \partial_t h_2 \hat{A}_1^d + \partial_t h_3 \hat{R}^d + \exp(h_1 \hat{B}^d) \exp(h_2 \hat{A}_1^d) \exp(h_3 \hat{R}^d) \partial_t h_4 \hat{s} \exp(-h_4 \hat{R}^d) \exp(-h_2 \hat{A}_1^d) \exp(-h_1 \hat{B}^d) \\
+ \exp(h_1 \hat{B}^d) \exp(h_2 \hat{A}_1^d) \exp(h_3 \hat{R}^d) \exp(h_4 \hat{s}) \partial_t h_5 \hat{s} \exp(-h_4 \hat{R}^d) \exp(-h_2 \hat{A}_1^d) \exp(-h_1 \hat{B}^d) \\
+ \exp(h_1 \hat{B}^d) \exp(h_2 \hat{A}_1^d) \exp(h_3 \hat{R}^d) \exp(h_4 \hat{s}) \partial_t h_6 \hat{A} \exp(-h_4 \hat{R}^d) \exp(-h_3 \hat{R}^d) \exp(-h_2 \hat{A}_1^d) \exp(-h_1 \hat{B}^d) \\
+ \exp(h_1 \hat{B}^d) \exp(h_2 \hat{A}_1^d) \exp(h_3 \hat{R}^d) \exp(h_4 \hat{s}) \partial_t h_7 \hat{R} \exp(-h_4 \hat{R}^d) \exp(-h_3 \hat{R}^d) \exp(-h_2 \hat{A}_1^d) \exp(-h_1 \hat{B}^d). \\
\text{(C5)}
\]

Using the identities derived in Appen. [1] we convert the products of operators into sums of operators. It leads to

\[
\hat{A} + c_1 \hat{R} + \hat{B}^d + tA^\ast \circ B + tc_1 \hat{B}_e + c_2 \hat{R}^d + tc_2 \hat{A}_1^d + \frac{1}{2} t^2 c_2 A^\ast \bullet A^\ast - tc_1 c_2 \hat{s} - t^2 c_1 c_2 \hat{A} - t^2 c_1 c_2 \hat{R} \\
\mathcal{F}R_{14}
\]

This equation can be divided into separate equations for the different operators. With some simplifications, we obtain a set of eight differential equations, one for each of the $h$-functions. They are given by

\[
\partial_t h_0 = A^\ast \circ (B + Th_1)(1 - c_1 h_2) + B \bullet B(tc_1 h_1 + \frac{1}{2} c_1 Th_2^2) + A^\ast \bullet A^\ast(\frac{1}{2} t^2 c_2 + Th_2 - \frac{1}{2} c_1 Th_2^2), \\
\partial_t h_1 = 1 - tc_1 c_2 h_1 - c_1 h_3 - c_1 T h_1 h_3, \\
\partial_t h_2 = tc_2 - tc_1 c_2 h_2 + Th_3 - c_1 Th_3 h_3, \\
\partial_t h_3 = -2tc_1 c_2 h_3 - c_1 Th_3^2, \\
\partial_t h_4 = tc_2 - c_1 Th_3, \\
\partial_t h_5 = c_1 \exp(h_4) + c_1 T h_1 \exp(h_4), \\
\partial_t h_6 = T(1 - c_1 h_2) \exp(h_4), \\
\partial_t h_7 = c_1 T \exp(2h_4),
\text{(C7)}
\]

where $T = 1 - c_1 c_2 t^2$. These differential equations can be solved in sequence to obtain the solutions for the $h$-functions. These solutions are

\[
h_0(t) = \frac{(A^\ast \circ B + \frac{1}{2} c_1 t B \bullet B + \frac{1}{2} c_2 t A^\ast \bullet A^\ast) t^2}{1 + c_1 c_2 t^2}, \quad h_1(t) = \frac{t}{1 + c_1 c_2 t^2}, \quad h_2(t) = \frac{c_2 t^2}{1 + c_1 c_2 t^2},
\text{ (C8)}
\]

\[
h_3(t) = \frac{c_2 t}{1 + c_1 c_2 t^2}, \quad h_4(t) = -\ln(1 + c_1 c_2 t^2), \quad h_5(t) = \frac{c_1 t^2}{1 + c_1 c_2 t^2}, \quad h_6(t) = \frac{t}{1 + c_1 c_2 t^2}, \quad h_7(t) = \frac{c_1 t}{1 + c_1 c_2 t^2}.
\]

Substituted into Eq. (33), these solutions produce the general expression for the inner products among the eigenstates:

\[
\langle f_1 | f_R \rangle = F_L F_R(1 + c_1 c_2 t^2)^{n/2} \exp\left(\frac{t^2 A^\ast \circ B + \frac{1}{2} c_1 t^3 B \bullet B + \frac{1}{2} c_2 t^3 A^\ast \bullet A^\ast}{1 + c_1 c_2 t^2}\right).
\text{(C9)}
\]

**Appendix D: Products with exponentiated operators**

For products of exponentiated operators, we use the identity in Eq. (122). Those with exponentiated operators involving only spectral operators are

\[
\exp(c\hat{A})\hat{B}^d \exp(-c\hat{A}) = \hat{B}^d + cA \bullet B, \quad \exp(c\hat{A}_e)\hat{B}^d \exp(-c\hat{A}_e) = \hat{B}^d + cA \bullet B, \\
\exp(c\hat{A})\hat{B}_e^d \exp(-c\hat{A}) = \hat{B}_e^d + cA \bullet B^\ast, \quad \exp(c\hat{A}_e)\hat{B}_e^d \exp(-c\hat{A}_e) = \hat{B}_e^d + cB^\ast \bullet A, \\
\text{(D1)}
\]

where $c$ is an arbitrary complex constant. Including $\hat{R}$ and $\hat{R}^d$, we have

\[
\exp(c\hat{R})\hat{A}^d \exp(-c\hat{R}) = \hat{A}^d + cA, \quad \exp(c\hat{A})\hat{R}^d \exp(-c\hat{A}) = \hat{R}^d + c\hat{A}^d + \frac{1}{2} c^2 A^\ast \bullet A^\ast, \\
\exp(c\hat{R})\hat{A}_e^d \exp(-c\hat{R}) = \hat{A}_e^d - cA, \quad \exp(c\hat{A}_e)\hat{R}^d \exp(-c\hat{A}_e) = \hat{R}^d - c\hat{A}_e^d - \frac{1}{2} c^2 A \bullet A, \\
\exp(c\hat{R})\hat{R}^d \exp(-c\hat{R}) = \hat{R}^d - cs - c^2 R, \\
\text{(D2)}
\]
and then with $s$, we have

$$\exp(c\hat{A})s\exp(-c\hat{A}) = s + c\hat{A}, \quad \exp(c\hat{A}_s)\hat{s}\exp(-c\hat{A}_s) = s + c\hat{A}_s, \quad \exp(c\hat{R})\hat{s}\exp(-c\hat{R}) = s + 2c\hat{R},$$

$$\exp(c\hat{s})\hat{A}^\dagger\exp(-c\hat{s}) = \exp(c)\hat{A}^\dagger, \quad \exp(c\hat{s})\hat{A}^\dagger\exp(-c\hat{s}) = \exp(c)\hat{A}^\dagger, \quad \exp(c\hat{s})\hat{R}^\dagger\exp(-c\hat{s}) = \exp(2c)\hat{R}^\dagger.$$  \hfill (D3)

### Appendix E: Grassmann Dirac delta functionals

The generic form for the inner product among eigenstates with opposite signs is given by

$$\langle f|f' \rangle = \lim_{\epsilon \to 0} \epsilon^{\Omega/2} \exp \left[-\frac{1}{2\epsilon}(f - f') \cdot (f - f')\right],$$  \hfill (E1)

where we substitute $t = 1 - \epsilon$. To be proportional to a Dirac delta functional, the inner product must satisfy

$$\int W[f] \langle f|f' \rangle \mathcal{D}[f] = \Lambda W[f'],$$  \hfill (E2)

where $W[f]$ is an arbitrary functional of $f$, and $\Lambda$ is a constant. Using Eq. (E1), we obtain

$$\int W[f] \langle f|f' \rangle \mathcal{D}[f] = \lim_{\epsilon \to 0} \epsilon^{\Omega/2} \int W[f] \exp \left\{ \frac{1}{2\epsilon}(f - f') \cdot (f - f') \right\} \mathcal{D}[f]$$

$$= \lim_{\epsilon \to 0} \epsilon^{\Omega/2} \int W[f_0 + f'] \exp \left( \frac{1}{2\epsilon} f_0 \cdot f_0 \right) \mathcal{D}[f_0]$$

$$= \lim_{\epsilon \to 0} \int W[\sqrt{\epsilon}f_1 + f'] \exp \left( \frac{1}{2\epsilon} f_1 \cdot f_1 \right) \mathcal{D}[f_1]$$

$$= W[f'] \int \exp \left( \frac{1}{2\epsilon} f_1 \cdot f_1 \right) \mathcal{D}[f_1] = \Lambda W[f'],$$  \hfill (E3)

where

$$\Lambda = \int \exp \left( \frac{1}{2\epsilon} f \cdot f \right) \mathcal{D}[f].$$  \hfill (E4)

Here, we first shifted $f'$ into $f$, making it emerge in the argument of $W[f]$. Then we absorbed the $\epsilon$-factor into a new field variable $f_0$, using the change of Grassmann integration variable given by

$$\int W[\theta] \mathcal{D}[\theta] = \int W[c\theta'] \frac{1}{c} \mathcal{D}[\theta'].$$

Hence, the constant emerges from the Grassmann measure, but as the inverse of what it would have been for non-Grassmann measures. As a result, it cancels the $\epsilon$-factor in front. The limit can be evaluated without effect and the argument of the functional becomes independent of the integration field. Therefore, we can remove it from the functional integral. The rest of the expression is the Grassmann functional integral without $W[f']$, which we define as the constant $\Lambda$.

To evaluate the Grassmann functional integral for the constant in Eq. (E4), we assume a two-dimensional spin matrix. Then we can write

$$f \cdot f = if_2 \circ f_1 - if_1 \circ f_2 = -i2f_1 \circ f_2,$$  \hfill (E6)

where the subscripts denote spin components. Assuming that the cardinality of the Grassmann numbers is $\Omega$ (counting all the degrees of freedom including spin), we have

$$\int \exp (-if_1 \circ f_2) \mathcal{D}[f] = (-i)^{\Omega/2} = \exp(-i\frac{\pi}{4}\Omega).$$  \hfill (E7)

We discard the global phase factor. So, the constant becomes $\Lambda = 1$. 

Appendix F: Grassmann star-products

The Wigner functional for the product of two operators, each of which is expressed as a Weyl transformation of a Wigner function, gives

\[
W_{\hat{A}\hat{B}}[q,p] = \int \langle q + \frac{1}{2}x | q_a \rangle W_{\hat{A}} \left[ \frac{q_a + q_b}{2}, p_1 \right] \exp[i(q_a - q_b) \cdot p_1] \langle q_b | q_c \rangle W_{\hat{B}} \left[ \frac{q_c + q_d}{2}, p_2 \right] \\
\times \exp[i(q_c - q_d) \cdot p_2] \langle q_d | q - \frac{1}{2}x \rangle \exp(-ix \cdot p) \mathcal{D}[p_1, p_2, q_a, q_b, q_c, q_d, x]
\]

\[
= \int \delta \left[ q + \frac{1}{2}x - q_a \right] W_{\hat{A}} \left[ \frac{q_a + q_b}{2}, p_1 \right] \exp[i(q_a - q_b) \cdot p_1] \delta \left[ q_b - q_c \right] W_{\hat{B}} \left[ \frac{q_c + q_d}{2}, p_2 \right] \\
\times \exp[i(q_c - q_d) \cdot p_2] \delta \left[ q_d - q + \frac{1}{2}x \right] \exp(-ix \cdot p) \mathcal{D}[p_1, p_2, q_a, q_b, q_c, q_d, x]
\]

\[
= \int W_{\hat{A}} \left[ \frac{1}{2}q + \frac{1}{2}q_3 + \frac{1}{2}x, p_1 \right] \exp[i(q + \frac{1}{2}x - q_b) \cdot p_1] W_{\hat{B}} \left[ \frac{1}{2}q + \frac{1}{2}q_b - \frac{1}{4}x, p_2 \right] \\
\times \exp[i(q_b - q + \frac{1}{2}x) \cdot p_2] \mathcal{D}[p_1, p_2, q_a, q_b]
\]

\[
= \exp[i2(q - q_3) \cdot p_1 + i2(q_1 - q) \cdot p_2 + i2(q_2 - q_1) \cdot p] \\
\times W_{\hat{A}}[q_1, p_1] W_{\hat{B}}[q_3, p_3] W_{\hat{C}}[q_4, p_4] \mathcal{D}[q_1, q_2, p_2, q_3, p_3, q_4, p_4]
\]

where we used the transformation of Grassmann integration variables given in Eq. (E5). For the product of three operators, it then follows that

\[
W_{\hat{A}\hat{B}\hat{C}}[q,p] = 2^{-4\Omega} \int \exp[i2(q - q_3) \cdot p_1 + i2(q_1 - q) \cdot p_2 + i2(q_2 - q_1) \cdot p] \\
\times \exp[i2(q - q_4) \cdot p_3 + i2(q_3 - q_2) \cdot p_4 + i2(q_4 - q_3) \cdot p_2] \\
\times W_{\hat{A}}[q_1, p_1] W_{\hat{B}}[q_3, p_3] W_{\hat{C}}[q_4, p_4] \mathcal{D}[q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4]
\]

\[
= 2^{-2\Omega} \int \exp[i2(q - q_3) \cdot p_1 + i2q_1 \cdot (p_3 - p) - i2q \cdot p_3 + i2q_3 \cdot p] \\
\times W_{\hat{A}}[q_1, p_1] W_{\hat{B}}[q_3, p_3] W_{\hat{C}}[q - q_1 + q_3, p - p_3 + p] \mathcal{D}[q_1, q_2, q_3, p_1, p_2, p_3, p_4]
\]

\[
= \int \exp[i(q - q_3) \cdot p_2 + i(p - p_3)] W_{\hat{A}} \left[ \frac{1}{2}q + \frac{1}{2}q_3 + \frac{1}{2}p + \frac{1}{2}p_3 + \frac{1}{2}p_2 \right] W_{\hat{B}}[q, p_3]
\]

\[
\times W_{\hat{C}} \left[ \frac{1}{2}q + \frac{1}{2}q_3 - \frac{1}{2}q_2 + \frac{1}{2}p_3 + \frac{1}{2}p_2 - \frac{1}{2}p_1 \right] \mathcal{D}[q, p_3, q_2, p_2, p_1].
\]

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