Trigonometric-type properties and the parity of balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers

Ngo Van Dinh
Thai Nguyen University of Sciences
Tan Thinh Ward, Thai Nguyen City, Vietnam
Email: dinh.ngo [at] tnus.edu.vn

Abstract. Balancing numbers $n$ are originally defined as the solution of the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+\cdots+(n+r)$, where $r$ is called the balancer corresponding to the balancing number $n$. By slightly modifying, $n$ is the cobalancing number with the cobalancer $r$ if $1+2+\cdots+n = (n+1)+\cdots+(n+r)$. Let $B_n$ denote the $n$th balancing number and $b_n$ denote the $n$th cobalancing number. Then $8B_n^2 + 1$ and $8b_n^2 + 8b_n + 1$ are perfect squares. The $n$th Lucas-balancing number $C_n$ and the $n$th Lucas-cobalancing number $c_n$ are the positive roots of $8B_n^2 + 1$ and $8b_n^2 + 8b_n + 1$, respectively. In this paper, we establish some trigonometric-type identities and some new interesting properties of balancing and Lucas-balancing numbers. We also establish some arithmetic properties concerning the parity of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers.

1. Introduction

While studying triangular numbers, Behera and Panda [1] introduced the notion of balancing numbers. An integer $n \in \mathbb{Z}^+$ is a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r),$$

(1.1)

for some $r \in \mathbb{Z}^+$. The number $r$ in (1.1) is called the balancer corresponding to the balancing number $n$. Behera and Panda also found that $n$ is a balancing number if and only if $n^2$ is a triangular number, as well as, $8n^2 + 1$ is a perfect square. Though the definition suggests that no balancing number should be less than 2, we accept 1 as a balancing number being the positive square root of the square triangular number 1 [3]. If $n$ is a balancing number then the positive root of $8n^2 + 1$ is called a Lucas-balancing number [2].

Let $B_n$ and $C_n$ denote the $n$th balancing number and the $n$th Lucas-balancing number, respectively, and set $B_0 = 0, C_0 = 1$. Then we have the recurrence relations

$$B_{n+1} = 6B_n - B_{n-1}, n \geq 1,$$

with $B_0 = 0, B_1 = 1$, and

$$C_{n+1} = 6C_n - C_{n-1}, n \geq 1,$$

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with $C_0 = 1, C_1 = 3$. These recurrence relations give the Binet formulas for balancing and Lucas-balancing numbers as follows:

$$B_1 = 1, B_2 = 6, B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \text{ for all } n \geq 0,$$

and

$$C_1 = 3, C_2 = 17, C_n = \frac{\lambda_1^n + \lambda_2^n}{2}, \text{ for all } n \geq 0,$$

where $\lambda_1 = 3 + \sqrt{8}, \lambda_2 = 3 - \sqrt{8}$.

By slightly modifying (1.1), Panda and Ray [3] defined cobalancing numbers $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r),$$

where $r$ is called the cobalancer corresponding to $n$. An natural number $n$ is a cobalancing number if and only if $8n^2 + 8n + 1$ is a perfect square. So we can accept 0 is the first cobalancing number. Let $b_n$ be the $n^{th}$ cobalancing number. Then the $n^{th}$ Lucas-cobalancing number $c_n$ is the positive root of $8b_n^2 + 8b_n + 1$. Moreover, we have the recurrence relations [4]

$$b_1 = 0, b_2 = 2, b_{n+1} = 6b_n - b_{n-1} + 2, n \geq 2,$$

and

$$c_1 = 1, c_2 = 7, c_{n+1} = 6c_n - c_{n-1}, n \geq 2.$$

The followings are the Binet formulas for cobalancing and Lucas-cobalancing numbers, respectively,

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \text{ and } c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2},$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

In this work, we consider the parity of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers by studying trigonometric-type identities. Some trigonometric-type identities of balancing numbers are established by Panda [2]. We establish some more here and deduce some properties on the parity of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers. We also establish some new interesting properties as Simson formulas and summations of balancing, Lucas-balancing, cobalancing and Lucas-cobalancing numbers.

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2. Main results

Let’s start with two identities established by Panda [2] which look like the trigonometric identities \( \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \). We give here another proof by directly using Binet forms and some elementary calculations.

**Theorem 2.1** ([2, Theorem 2.5, Corollary 2.6]). Let \( n, m \) be non negative integers. Then

i) \( B_{n+m} = B_n C_m + B_m C_n \);

ii) \( B_{n-m} = B_n C_m - B_m C_n \).

**Proof.** By Binet formulas, we have

\[
B_n C_m + B_m C_n = \frac{\lambda^n - \lambda^m_n}{\lambda_1 - \lambda_2} \cdot \lambda_1^{m} + \lambda_2^{m} + \frac{\lambda^m - \lambda_n^m}{\lambda_1 - \lambda_2} \cdot \lambda_1^{n} + \lambda_2^{n} = \frac{\lambda_{n+m}^n - \lambda_{n+m}^m}{\lambda_1 - \lambda_2} = B_{n+m}.
\]

This implies the first identity. The second is proved by the same method. \( \square \)

The following direct consequence resembles the trigonometric identity

\( \sin(2x) = 2 \sin x \cos x \).

**Corollary 2.2** ([2 Corollary 2.7]). For \( n \) is an non negative integer, we have

\( B_{2n} = 2B_n C_n \).

The following theorem give an identity which has the same type of the trigonometric identity \( \sin x - \sin y = 2 \sin \left( \frac{x - y}{2} \right) \cos \left( \frac{x + y}{2} \right) \).

**Theorem 2.3.** For \( n, m \) are non negative integers such that \( n \geq m \) and having the same parity, we have

\( B_n - B_m = 2 B_{\frac{n-m}{2}} C_{\frac{n+m}{2}} \).

**Proof.** Using Binet formulas, we have

\[
2B_{\frac{n-m}{2}} C_{\frac{n+m}{2}} = 2 \cdot \frac{\lambda^{\frac{n-m}{2}}}{\lambda_1 - \lambda_2} \cdot \lambda_1^{\frac{n-m}{2}} + \lambda_2^{\frac{n-m}{2}} = \frac{\lambda^n - \lambda^m}{\lambda_1 - \lambda_2} \cdot \frac{\lambda^n - \lambda^m}{\lambda_1 - \lambda_2} = B_n - B_m.
\]

This completes the proof. \( \square \)

**Corollary 2.4.** For \( n, m \) are non negative integers such that \( n \geq m \), we have

\( B_{2n} - B_{2m} = 2B_{n-m} C_{n+m} \).
Proof. This is an intermediate consequence of Theorem 2.3.

In Corollary 2.4 by taking \( m = 1 \) we obtain a corollary of which [5, Theorem 2.1] is a particular case.

Corollary 2.5. For \( n \geq 1 \), we have
\[
B_{2n} - 6 = 2B_{n-1}C_{n+1}.
\]

Corollary 2.6. Let \( n, m \) be non negative integers such that \( n \geq m \). Then
\[
B_{2n} = 2(B_{n-m}C_{n+m} + B_mC_m).
\]

Proof. This corollary is an obvious consequence of corollaries 2.2 and 2.4.

With Theorem 2.3 we can see the parity of balancing numbers.

Corollary 2.7. For every integer \( n \geq 0 \), the balancing number \( B_n \) and \( n \) have the same parity.

Proof. If \( n, m \) are integers with the same parity then \( B_n \) and \( B_m \) have the same parity by Theorem 2.3. On the other one, we have \( B_0 = 0, B_1 = 1 \) and \( B_2 = 6 \). It implies that \( B_n \) and \( n \) have the same parity.

We also have an identity of balancing numbers which resembles the trigonometric identity \( \sin x + \sin y = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \).

Theorem 2.8. Let \( n, m \) be non negative integers such that \( n \geq m \) and having the same parity. Then
\[
B_n + B_m = 2B_{n+m}C_{n-m}.
\]

Proof. Using Binet formulas, we have
\[
2B_{n+m}C_{n-m} = 2 \cdot \frac{\lambda_1^{n+m} - \lambda_2^{n+m}}{\lambda_1 - \lambda_2} \cdot \frac{\lambda_1^{n-m} + \lambda_2^{n-m}}{2}
= \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2}
= B_n + B_m.
\]

This is what was to be shown.

Corollary 2.9. For \( n, m \) are non negative integers such that \( n \geq m \), we have
\[
B_{2n} + B_{2m} = 2B_{n+m}C_{n-m}.
\]

Proof. This is a direct consequence of Theorem 2.8.

Corollary 2.10. Let \( n, m \) be non negative integers such that \( n \geq m \). Then
\[
\begin{align*}
i) & \quad B_{n-m}C_n + B_nC_{n-m} = B_{2n-m}; \\
ii) & \quad B_nC_{n-m} - B_{n-m}C_n = B_m.
\end{align*}
\]
Proof. By theorems 2.3 and 2.8, we have
\[ B_{2n-m} - B_m = 2B_{n-m}C_n \quad \text{and} \quad B_{2n-m} + B_m = 2B_nC_{n-m}. \]
Hence we obtain the required identities. □

The following theorem shows that we have an identity of Lucas-balancing numbers which looks like the trigonometric identity
\[ \cos x + \cos y = 2 \cos \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right). \]
However, we have another which resembles, up to a scalar, the trigonometric identity \( \cos x - \cos y = -2 \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right). \)

**Theorem 2.11.** Let \( n, m \) be non negative integers such that \( n \geq m \) and having the same parity. Then
\[ \begin{align*}
\text{i)} & \quad C_n + C_m = 2C_{\frac{n+m}{2}}C_{\frac{n-m}{2}}; \\
\text{ii)} & \quad C_n - C_m = 16B_{\frac{n+m}{2}}B_{\frac{n-m}{2}}.
\end{align*} \]

Proof. Continue using Binet formulas, we have
\[ 2C_{\frac{n+m}{2}}C_{\frac{n-m}{2}} = \frac{\lambda_{\frac{n+m}{2}}^{n+m} + \lambda_{\frac{n-m}{2}}^{n-m}}{\lambda_{\frac{n+m}{2}} - \lambda_{\frac{n-m}{2}}} \cdot \frac{\lambda_{\frac{n+m}{2}}^{n-m} - \lambda_{\frac{n-m}{2}}^{n-m}}{\lambda_{\frac{n+m}{2}} - \lambda_{\frac{n-m}{2}}} \\
= \frac{\lambda_1^n + \lambda_2^n}{2} + \frac{\lambda_1^m + \lambda_2^m}{2} = C_n + C_m. \]
The first identity is proved. To prove the second, we have
\[ B_{\frac{n+m}{2}}B_{\frac{n-m}{2}} = \frac{\lambda_{\frac{n+m}{2}}^{n+m} - \lambda_{\frac{n-m}{2}}^{n-m}}{\lambda_{\frac{n+m}{2}} - \lambda_{\frac{n-m}{2}}} \cdot \frac{\lambda_{\frac{n-m}{2}}^{n-m} - \lambda_{\frac{n-m}{2}}^{n-m}}{\lambda_{\frac{n+m}{2}} - \lambda_{\frac{n-m}{2}}} \]
\[ = \frac{1}{(\lambda_1 - \lambda_2)^2} (\lambda_1^n - \lambda_2^n - \lambda_1^m + \lambda_2^m) \]
\[ = \frac{1}{32} (\lambda_1^n + \lambda_2^n - \lambda_1^m - \lambda_2^m) = \frac{1}{16} (C_n - C_m). \]
This implies the required identity. □

**Corollary 2.12.** For \( n, m \) are non negative integers such that \( n \geq m \), we have
\[ \begin{align*}
\text{i)} & \quad C_{2n} + C_{2m} = 2C_{n+m}C_{n-m}; \\
\text{ii)} & \quad C_{2n} - C_{2m} = 16B_{n+m}B_{n-m}.
\end{align*} \]

Proof. These identities directly follow from Theorem 2.11. □

**Corollary 2.13.** Let \( n, m \) be non negative integers such that \( n \geq m \). Then
\[ \begin{align*}
\text{i)} & \quad C_nC_{n-m} + 8B_nB_{n-m} = C_{2n-m}; \\
\text{ii)} & \quad C_nC_{n-m} - 8B_nB_{n-m} = C_m.
\end{align*} \]
Proof. By Theorem 2.11 we have
\[ C_nC_{n-m} = \frac{1}{2}(C_{2n-m} + C_m) \] and
\[ 8B_nB_{n-m} = \frac{1}{2}(C_{2n-m} - C_m). \]
These imply the required identities. \(\square\)

Now, we can see the parity of Lucas-balancing numbers.

**Corollary 2.14.** For all integer \(n \geq 0\), the Lucas-balancing number \(C_n\) is odd. Moreover, if \(n, m\) are integers with the same parity then the difference between \(C_n\) and \(C_m\) is divided by 16.

Proof. If \(n, m\) are integers with the same parity then the difference between \(C_n\) and \(C_m\) is divided by 16 by the second identity of Theorem 2.11. This also means that \(C_n\) and \(C_m\) have the same parity. On the other hand, we have \(C_0 = 1, C_1 = 3, C_2 = 17\). It implies that \(C_n\) is odd for all \(n\). \(\square\)

We can not find identities for Lucas-balancing numbers which resemble the trigonometric identities \(\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y\). But we establish the following interesting theorem.

**Theorem 2.15.** Let \(n, m\) be non negative integers such that \(n \geq m\). Then
\[ \text{i) } 16(C_nC_m - B_nB_m) = 7C_{n+m} + 9C_{n-m}; \]
\[ \text{ii) } 16(C_nC_m + B_nB_m) = 9C_{n+m} + 7C_{n-m}. \]

Proof. Applying Binet forms, we have
\[ C_nC_m - B_nB_m = \frac{\lambda_1^n + \lambda_2^n}{2} \frac{\lambda_1^m + \lambda_2^m}{2} - \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \frac{\lambda_1^m - \lambda_2^m}{\lambda_1 - \lambda_2} \]
\[ = \frac{\lambda_1^{n+m} + \lambda_2^{n+m} + \lambda_1^{-m} + \lambda_2^{-m}}{4} - \frac{\lambda_1^{n+m} + \lambda_2^{n+m} - \lambda_1^{-m} - \lambda_2^{-m}}{32} \]
\[ = \frac{7}{16} \frac{\lambda_1^{n+m} + \lambda_2^{n+m}}{2} + \frac{9}{16} \frac{\lambda_1^{-m} + \lambda_2^{-m}}{2} = \frac{7C_{n+m} + 9C_{n-m}}{16}. \]
It follows the first identity. The second is proved by similar calculations. \(\square\)

Motived by the above results, we establish some identities of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers. We also obtain some properties on the parity of these numbers. The following theorem give us relations between sums of Lucas-balancing numbers and Lucas-cobalancing numbers.

**Theorem 2.16.** For \(n, m\) are integers such that \(n \geq m \geq 1\), we have
\[ \text{i) } C_{n+m-1} - C_{n-m} = 2c_{n}c_{m}; \]
\[ \text{ii) } C_{n+m-1} + C_{n-m} = 16b_{n}b_{m} + 8(b_{n} + b_{m}) + 4. \]
Proof. Using Binet forms with remark that $\alpha_1\alpha_2 = -1$, we have

\[ c_n c_m = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \cdot \frac{\alpha_1^{2m-1} + \alpha_2^{2m-1}}{2} \]
\[ = \frac{\alpha_1^{2(n+m-1)} + \alpha_2^{2(n+m-1)}}{2} - \frac{\alpha_1^{2(n-m)} + \alpha_2^{2(n-m)}}{2} \]
\[ = \frac{1}{2} (C_{n+m-1} - C_{n-m}). \]

Imply the first identity. Similarly, we can prove the second identity. □

By ii) of Theorem 2.16, we have the following consequence about sum of two consecutive Lucas-balancing numbers.

Corollary 2.17. For all integer $n \geq 1$, the sum of $(n-1)^{th}$ and $n^{th}$ Lucas-balancing numbers is divided by 4.

The following theorem is an interesting property of sums of two cobalancing numbers from which we can see the parity of cobalancing numbers.

Theorem 2.18. Let $n, m$ be positive integers.

i) If $n > m$ then $b_{n+m} - b_{n-m} = 2c_n B_m$;
ii) If $n \leq m$ then $b_{n+m} - b_{m-n+1} = 2c_n B_m$.

Proof. By Binet formulas, we have

\[ c_n B_m = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \cdot \frac{\alpha_1^{2m} - \alpha_2^{2m}}{4\sqrt{2}} \]
\[ = \frac{\alpha_1^{2(n+m)-1} - \alpha_2^{2(n+m)-1}}{8\sqrt{2}} - \frac{\alpha_1^{2(n-m)-1} - \alpha_2^{2(n-m)-1}}{8\sqrt{2}} \]
\[ = \begin{cases} \frac{1}{2}(b_{n+m} - b_{n-m}), & \text{if } n > m, \\ \frac{1}{2}(b_{n+m} - b_{m-n+1}), & \text{otherwise}. \end{cases} \]

Imply what was to be demonstrated. □

Corollary 2.19. The cobalancing numbers are even. Moreover, for all $m \geq 1$, the difference between the $(2m+1)^{th}$ and $(2m)^{th}$ cobalancing numbers is divided by 4.

Proof. By ii) of Theorem 2.18, we can see that $b_n$ and $b_{n+1}$ have the same parity for all $n \geq 1$. It follows that $b_n$ is even for all $n \geq 1$ since $b_1 = 0$. Moreover, from ii) of Theorem 2.18 we also obtain the second affirmation since $B_{2m}$ is even by Corollary 2.7. □

The following theorem is another property of sums of two cobalancing numbers.
Theorem 2.20. Let \( n, m \) be positive integers.

i) If \( n > m \) then \( b_{n+m} + b_{n-m} = 2b_nC_m + C_m - 1 \);

ii) If \( n \leq m \) then \( b_{n+m} - b_{m+n+1} = 2b_mC_m + C_m - 1 \).

Proof. By Binet forms, we have

\[
b_nC_m = \left( \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \right) \frac{\alpha_1^{2m} + \alpha_2^{2m}}{2} \\
= \alpha_1^{2(n+m)-1} + \frac{\alpha_1^{2(n-m)-1} - \alpha_2^{2(n-m)-1}}{8\sqrt{2}} - \frac{1}{2}C_m \\
= \left\{ \begin{array}{ll}
\frac{1}{2}(b_{n+m} + b_{n-m} - C_m + 1), & \text{if } n > m, \\
\frac{1}{2}(b_{n+m} + b_{m+n+1} - C_m + 1), & \text{otherwise}.
\end{array} \right.
\]

Imply the required identities. \( \square \)

In the following theorem, we again get an identity which looks like the trigonometric identity \( \sin x + \sin y = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right) \).

Theorem 2.21. Let \( n, m \) be positive integers.

i) If \( n > m \) then \( c_{n+m} + c_{n-m} = 2c_nC_m \);

ii) If \( n \leq m \) then \( c_{n+m} - c_{m+n+1} = 2c_mC_m \).

Proof. By Binet forms, we have

\[
c_nC_m = \left( \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2} \right) \frac{\alpha_1^{2m} + \alpha_2^{2m}}{2} \\
= \alpha_1^{2n} + \frac{\alpha_1^{2n-m} - \alpha_2^{2n-m}}{4} + \frac{\alpha_1^{2(m-n)} + \alpha_2^{2(m-n)}}{4} \\
= \left\{ \begin{array}{ll}
\frac{1}{2}(c_{n+m} + c_{n-m}), & \text{if } n > m, \\
\frac{1}{2}(c_{n+m} - c_{m+n+1}), & \text{otherwise}.
\end{array} \right.
\]

This completes the proof. \( \square \)

Corollary 2.22. The Lucas-cobalancing numbers are odd.

Proof. By ii) of Theorem 2.21 we can see that \( c_n \) and \( c_{n+1} \) have the same parity for all \( n \geq 1 \). It follows that \( c_n \) is odd for all \( n \geq 1 \) since \( c_1 = 1 \). \( \square \)

The following theorem give us a better property on the parity of Lucas-cobalancing numbers of even index. It shows that the \( (2n)^{th} \) Lucas-cobalancing number is congruent to \( -1 \) modulo 8 and the \( (4n)^{th} \) Lucas-cobalancing number is congruent to \( -1 \) modulo 16, for all \( n \geq 1 \).

Theorem 2.23. For integer \( n \geq 1 \), we have

\[ c_{2n} + 1 = 8(2b_n + 1)B_n. \]
Proof. By Binet forms, we have

\[ b_n B_n = \left( \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \right) \cdot \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \]

\[ = \frac{\alpha_1^{4n-1} + \alpha_2^{4n-1}}{32} - \frac{\alpha_1^{-1} + \alpha_2^{-1}}{32} - \frac{1}{2} B_n \]

\[ = \frac{1}{16}(c_{2n} + 1) - \frac{1}{2} B_n. \]

Hence we obtain the required identity. \(\square\)

The next theorem is another identity on sums of balancing numbers.

**Theorem 2.24.** For \( n, m \) are integers such that \( n \geq m \geq 1 \), we have

\[ B_{n+m-1} - B_{n-m} = (2b_n + 1)c_m. \]

**Proof.** By Binet forms, we have

\[ b_n c_m = \left( \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2} \right) \cdot \frac{\alpha_1^{2m-1} + \alpha_2^{2m-1}}{2} \]

\[ = \frac{\alpha_1^{2(n+m-1)} - \alpha_2^{2(n+m-1)}}{8\sqrt{2}} - \frac{\alpha_1^{2(n-m)} + \alpha_2^{2(n-m)}}{8\sqrt{2}} - \frac{1}{2} c_m \]

\[ = \frac{1}{2}(B_{n+m-1} - B_{n-m} - c_m). \]

Imply the required identity. \(\square\)

The following theorem is an interesting property of products of two balancing numbers.

**Theorem 2.25.** Let \( n, m, r \) be non negative integers such that \( m \geq r \). Then

\[ B_r B_{m+n} = B_mB_{n+r} - B_{m-r}B_n. \]

**Proof.** Since \( \lambda_1 \lambda_2 = 1 \), we have

\[ \lambda_1^m \lambda_2^{n+r} = \lambda_1^{m-r} \lambda_2^n \text{ and } \lambda_2^m \lambda_1^{n+r} = \lambda_2^{m-r} \lambda_1^n. \]

It follows that

\[ (\lambda_1^m - \lambda_2^m)(\lambda_1^{n+r} - \lambda_2^{n+r}) = (\lambda_1^{m-r} - \lambda_2^{m-r})(\lambda_1 - \lambda_2) \]

\[ = \lambda_1^{m+n+r} + \lambda_2^{m+n+r} - \lambda_1^n \lambda_2^{n+r} - \lambda_2^m \lambda_1^{n+r} \]

\[ - \lambda_1^{m+n-r} - \lambda_2^{m+n-r} - \lambda_1^m \lambda_2^{n-r} - \lambda_2^m \lambda_1^{n-r} \]

\[ = \lambda_1^{m+n}(\lambda_1^r - \lambda_2^r) - \lambda_2^{m+n}(\lambda_2^r - \lambda_1^r) \]

\[ = \lambda_1^{m+n}(\lambda_1^r - \lambda_2^r) - \lambda_2^{m+n}(\lambda_2^r - \lambda_1^r) \]

\[ = (\lambda_1^{m+n} - \lambda_2^{m+n})(\lambda_1^r - \lambda_2^r). \]
Now, using the Binet formula, we have

\[ B_mB_{n+r} - B_{m-r}B_m = B_{m+n}B_r. \]

The theorem is proved.

**Corollary 2.26.** Let \( n, m \) be non negative integers such that \( m \geq 1 \). Then

\[ B_{m+n} = B_mB_{n+1} - B_{m-1}B_n. \]

**Proof.** This is a direct consequence of Theorem 2.25 by taking \( r = 1 \).

**Theorem 2.27** (Simson formulas). Let \( n \) be a positive integer. Then

i) \( B_{n+1}B_{n-1} - B_n^2 = -1; \)

ii) \( b_{n+1}b_{n-1} - b_n^2 = -2b_n; \)

iii) \( C_{n+1}C_{n-1} - C_n^2 = 8; \)

iv) \( c_{n+1}c_{n-1} - c_n^2 = -8. \)

**Proof.** By ii) of Theorem 2.11 we have

\[ B_{n+1}B_{n-1} = \frac{1}{16}(C_{2n} - C_2) \] \( \text{and} \)
\[ B_n^2 = \frac{1}{16}(C_{2n} - C_0). \]

This implies the first identity. Similarly, we can easily obtain the others by using i) of Theorem 2.11 and Theorem 2.16.

Using Binet forms once again, we obtain the following theorem about the summations of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers.

**Theorem 2.28.** For all positive integer \( n \), we have

i) \( \sum_{i=1}^{n} B_i = \frac{b_{n+1}}{2}; \)

ii) \( \sum_{i=1}^{n} b_i = \frac{B_n}{2}; \)

iii) \( \sum_{i=0}^{n} C_i = \frac{c_{n+1}}{2} + \frac{1}{2}; \)

iv) \( \sum_{i=1}^{n} c_i = \frac{C_n}{2} - \frac{1}{2}. \)
Proof. We prove the first identity. The others are proved by the same way. By Binet forms, we have

\[
\sum_{i=1}^{n} B_i = \frac{1}{\lambda_1 - \lambda_2} \sum_{i=1}^{n} (\lambda_1^i - \lambda_2^i) \\
= \frac{1}{\lambda_1 - \lambda_2} \left( \frac{\lambda_1^n}{\lambda_1 - 1} - \frac{\lambda_2^n}{\lambda_2 - 1} \right) \\
= \frac{b_{n+1}}{2}.
\]

The proof is completed. \qed

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