Observing Zitterbewegung with Ultracold Atoms

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We propose an optical lattice scheme which would permit the experimental observation of Zitterbewegung (ZB) with ultracold, neutral atoms. A four-level “tripod” variant of the usual setup for stimulated Raman adiabatic passage (STIRAP) has been proposed for generating non-Abelian gauge fields [1]. Dirac-like Hamiltonians, which exhibit ZB, are simple examples of such non-Abelian gauge fields; we show how a variety of them can arise, and how ZB can be observed, in a tripod system. We predict that the ZB should occur at experimentally accessible frequencies and amplitudes.

A driving force behind the study of ultracold atoms is their potential use as highly tunable quantum simulators for physical systems, ranging from quantum phase transitions in solids [2] to black holes [3]. In particular, the high degree of control over length and time scales in cold atom experiments allows for the possibility of observing phenomena that are experimentally inaccessible in their original counterparts. In this paper, we propose an experiment which simulates the relativistic (and recently, controversial [4]) phenomenon of zitterbewegung (ZB), a jittering motion caused by interference between the positive and negative energy components of the wavefunction of a Dirac fermion.

For a relativistic electron, the ZB frequency is of the order of \(mc^2/\hbar \approx 10^{20}\) s\(^{-1}\), and the amplitude comparable to the Compton wavelength, \(\hbar/me \approx 10^{-12}\) m. ZB has never been observed for free electrons, as these time and length scales render it experimentally inaccessible. The presence of ZB is, however, a general feature of spinor systems with linear dispersion relations. Trapped ions [5] as well as condensed matter systems, including graphene [6, 7, 8] and semiconductor quantum wires [9, 10], have been proposed as candidate systems for observing ZB.

In this paper, we propose a scheme for observing ZB in ultracold neutral atoms. A four-level “tripod” variant of the usual setup for stimulated Raman adiabatic passage (STIRAP) has previously been proposed for generating non-Abelian gauge fields [1]. Dirac-like Hamiltonians, which exhibit ZB, are simple examples of such non-Abelian gauge fields, and we show how a variety of them can arise in a tripod system. The Hamiltonian for atoms in an optical lattice is Dirac-like in the subspace of the tripod’s two degenerate dark states. We predict that an atom’s mean position should thus undergo Dirac-like ZB. However, the characteristic amplitude of tripod ZB is the optical lattice wavelength, vs. the Compton wavelength of Dirac ZB, and the oscillation energy is proportional to the lattice recoil energy vs. the rest mass energy of the electron. This places tripod ZB well within the range of experimental observation, with a characteristic frequency of MHz vs. the THz domain predicted for condensed matter implementations [8]. Although we treat here the case of a noninteracting gas, the ZB persists under the addition of weak interactions: The Hamiltonian separates into center of mass and relative coordinates, and the center of mass Hamiltonian is again Dirac-like. In a dilute atomic cloud, the ZB should thus manifest itself as an oscillation of the cloud’s center of mass.

We consider the tripod STIRAP scheme described in [1] and shown in Fig. 1 The Hamiltonian in the interaction picture is \(H = -\hbar \sum_{j=1}^{3} \Omega_j |0\rangle \langle i | + \text{h.c.} \). Defining \(\Theta_j = \sqrt{\sum_{j=1}^{3} |\Omega_j|^2}\), the dressed states include two dark states degenerate at zero energy: \(|D_1\rangle = \frac{1}{\sqrt{2}} (|\Omega_2|1\rangle - |\Omega_1|2\rangle)\) and \(|D_2\rangle = \frac{1}{\sqrt{2}} (|\Omega_2\rangle|1\rangle - |\Omega_1\rangle|2\rangle - \Theta_2|3\rangle)\) (we have chosen an orthonormal basis). Suppose the atoms are now slowly moving in the field. The degeneracy causes the Born-Oppenheimer approximation to break, yielding an effective U(2) non-Abelian gauge field. The effective Hamiltonian in the 2 \times 2 dark subspace is \(H = \frac{i}{2m} (\vec{p} - \vec{A})^2 + \Phi\) where \(m\) is the atom’s mass, \(\vec{A}_{ij} = \hbar D_j (\vec{V} D_i)\) is an effective vector potential, and \(\Phi\) is a scalar Born-Huang potential resulting from the coupling to the bright subspace. The following choice of Rabi frequencies

\[
\Omega_1(\vec{r}) = \Omega \sqrt{1 - \varepsilon^2 \cos k_0 z} \\
\Omega_2(\vec{r}) = \Omega \sqrt{1 - \varepsilon^2 \sin (k_0 z + \pi)} \\
\Omega_3(\vec{r}) = \varepsilon \Omega \e^{i k_0 y},
\]

for \(\Omega, \varepsilon, \) and \(k_0\) constant corresponds to the laser beam geometry in Fig. 1(b), and yields a Dirac-like Hamiltonian (a related setup was proposed in [11] in the context of observing spin relaxation effects). Specifically, after some trivial gauge transformations, the two dark states feel an effective vector potential

\[
\hat{A}_y = \frac{\hbar k_0}{2} (1 - \varepsilon^2) \sigma_z \\
\hat{A}_z = -\varepsilon \hbar k_0 \sigma_y
\]
and an effective scalar potential $\hat{\Phi} = V_0(\sigma_z)$, where $V_0 = \frac{\hbar^2 k_0^2}{2m}(1-\varepsilon^2)$. In general, $[\hat{A}_y, \hat{A}_z] \neq 0$, and the field is non-Abelian. The vector potential in Eqs. (4-5) is valid for homogeneous magnetic field along the $\hat{z}$ direction. Ref. [12] proposes an alternate scheme for generating spin-orbit coupling with ultracold atoms.

It is possible to remove the scalar potential by applying a state-dependent external potential to the system. Denoting the potential felt by $|i\rangle$ as $V_i(\vec{r})$, choosing $V_1(\vec{r}) = V_2(\vec{r}) = V(\vec{r})$, and $V_3(\vec{r}) = V(\vec{r}) + V_0(1+\varepsilon^2)/(1-\varepsilon^2)$ subjects the dark states to an additional potential $V = V(\vec{r}) \otimes 1 - \hat{\Phi}$. If the scalar potential is thus removed from Eq. (6), the resulting Hamiltonian is extremely versatile, for two reasons: (1) $\varepsilon$ is tunable and (2) the dark states form a degenerate subspace, for which any basis is equivalent. In fact, depending on the direction we call “spin up,” this Hamiltonian can be viewed as a variety of Dirac-like Hamiltonians (see Table I).

The eigenstates of the system are spinors of the form $e^{ik_yz}e^{ik_z\hat{z}} |i, \vec{k}\rangle$ with $i = a, b$. The dispersion relation of the Hamiltonian in Eq. (6) is

$$E_{\pm}(k_y, k_z) = \frac{\hbar^2 k_y^2}{2m} \pm \frac{\hbar^2 k_0}{m} \sqrt{(1-\varepsilon^2)^2(k_y-k_0)^2 + 2\varepsilon k_z^2}. \tag{7}$$

The energy surfaces in Eq. (7) have a conical intersection at $(k_y, k_z) = (k_0, 0)$ (see Fig. 2). A circuit of the degeneracy in momentum space yields a Berry phase: Defining

$$\tan \xi(\vec{k}) = \frac{2\varepsilon k_z}{(1-\varepsilon^2)(k_y-k_0)}, \tag{8}$$

we designate the eigenfunctions by $|a, \vec{k}\rangle = [i\sin \xi(\vec{k}), \cos \xi(\vec{k})]^T$, $|b, \vec{k}\rangle = [i\cos \xi(\vec{k}), -\sin \xi(\vec{k})]^T$ which are multivalued for a particular $(k_y, k_z)$. The momentum-space Berry phase is much like the one encountered in graphene, which gives rise to phenomena like the half-integer quantum Hall effect [14]. We now examine the role that the Berry phase plays in generating ZB.

Consider the time evolution of a Gaussian wavepacket prepared in a superposition of dark states. In the non-Abelian case ($0 < \varepsilon < 1$), the eigenvectors have an associated Berry phase, and are both $\vec{k}$ dependent and multi-valued. However, the initial spin state must be single valued, forcing its expansion coefficients to be $\vec{k}$ dependent (and also multivalued). The presence of the Berry phase thus translates into a $\vec{k}$ dependence of $\xi(\vec{k})$ in Eq. (8); we later show that it is this nonvanishing $\vec{\nabla}_k \xi(\vec{k})$ that gives rise, directly, to ZB.

Considering an initial wavepacket

$$\tilde{\psi}(\vec{r}; 0) = \frac{1}{\sqrt{2}} \int d\vec{k} g(\vec{k}; 0)e^{i\vec{k} \cdot \vec{r}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{9}$$

it is straightforward to show that its time evolution is

$$\tilde{\psi}(\vec{r}; t) = \frac{1}{\sqrt{2}} \int d\vec{k} g(\vec{k}; 0)e^{i\vec{k} \cdot \vec{r} - i\omega(\vec{k})t/\hbar} \times \begin{pmatrix} \cos [\omega(\vec{k})t] \begin{pmatrix} 1 \\ 1 \end{pmatrix} - ie^{-i\xi(\vec{k})} \sin [\omega(\vec{k})t] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}. \tag{10}$$

where $\omega(\vec{k}) = \frac{m}{\hbar^2} (E_+(\vec{k}) - E_-(\vec{k}))$.

ZB, in the Dirac equation, is an oscillation of the average position $\langle \vec{r}(t) \rangle$. The usual method of understanding the Dirac equation, and related equations is to derive equations of motion for the Heisenberg operators, and show that they oscillate in time [7,8,9,10]. We instead work in the Schrödinger picture, which makes explicit the connection to Berry phase. As $[\hat{H}, \hat{p}] = 0$, it is convenient to work in the momentum basis, and calculate

$$\langle \vec{r}(t) \rangle = i \int d\vec{k} \tilde{\phi}^*(\vec{k}; t) \cdot \vec{\nabla}_k \tilde{\phi}(\vec{k}; t)$$

where $\tilde{\psi}(\vec{r}; t) = \int d\vec{k} \tilde{\phi}(\vec{k}; t)e^{i\vec{k} \cdot \vec{r}/(2\pi)}$ is the spinor wavefunction. After some algebra, we find

$$\langle \vec{r}(t) \rangle = \langle \vec{r}(0) \rangle + \frac{\hbar \langle \vec{k}(0) \rangle \cdot \vec{r}}{m} + \frac{1}{2} \int d\vec{k} \left| g(\vec{k}; 0) \right|^2 \left( \vec{\nabla}_k \xi(\vec{k}) \right) [1 - \cos 2\omega(\vec{k})t].$$

Figure 2: (Color online) Energy surfaces in momentum space for $^{85}$Rb with $k_0 = 2\pi/820 \text{nm}^{-1}$ and $\varepsilon = \varepsilon_R$. A conical intersection occurs at $(k_y, k_z) = (k_0, 0)$; a circuit of this conical intersection in $\vec{k}$ space gives a Berry phase.
where the final term, which oscillates in time, is ZB. The amplitude of the oscillation is proportional to \(\tilde{V}_y\xi(k)\). We had previously shown that the \(\tilde{k}\) dependence of \(\xi(k)\) occurs as a direct consequence of the eigenfunctions being multivalued. The Schrodinger picture thus illuminates what is not evident in the Heisenberg representation—that the ZB here can be viewed as a measurable consequence of the momentum-space Berry phase.

We now suggest a possible experimental demonstration of ZB using ultracold atoms. Suppose an ensemble of atoms is prepared in the vibrational ground state of a harmonic trap. A Raman pulse with space-dependent Rabi couplings is applied, as suggested in [11], to put the atom in a superposition of dark states, after which the trap is switched off to allow ballistic expansion. The initial wavepacket can be approximated by a Gaussian function \(g(\tilde{k};0) = \frac{d}{\sqrt{2\pi}} e^{-\frac{1}{2} \tilde{k}^2} d^2\), where \(d\) is the oscillator length of the trap, and \(\tilde{k}(0)\) is a momentum boost (which is zero for the case of a stationary trap). For this wavepacket, the expectation values of \(y\) and \(z\) oscillate as

\[
\begin{align*}
\langle y(t) \rangle &= \frac{\hbar \tilde{k}(0)}{m} t + \frac{d}{2\pi} \int d\tilde{k} e^{-(\tilde{k}^2 + \tilde{k}(0)^2) t} \frac{1}{\tilde{k}^2} \left[1 - \cos 2\omega(\tilde{k}) t\right] \\
\langle z(t) \rangle &= \frac{\hbar \tilde{k}(0)}{m} t + \frac{d}{2\pi} \int d\tilde{k} e^{-(\tilde{k}^2 + \tilde{k}(0)^2) t} \frac{1}{\tilde{k}^2} \left[1 - \cos 2\omega(\tilde{k}) t\right] \frac{1}{\tilde{k}^2} \left[1 - \cos 2\omega(\tilde{k})(0) t\right]
\end{align*}
\]

where we have defined \(\tilde{k}_y = (1 - \varepsilon^2)k_y, \tilde{k}_z = 2\varepsilon k_z,\) and \(\tilde{k} = \sqrt{\tilde{k}_y^2 + \tilde{k}_z^2}\). Eq. (11) shows that the ZB vanishes in the Abelian cases, \(\varepsilon = 0\) or \(\varepsilon = 1\).

It is useful to consider the limit \(d \to \infty\), where \(g(\tilde{k};0) \to \delta(\tilde{k} - \tilde{k}(0))\), i.e., the initial wavepacket approaches a plane wave. The integrals in Eqs. (11) become trivial, and we find that

\[
\begin{align*}
\langle y(t) \rangle &= \frac{\hbar \tilde{k}(0)}{m} t + \frac{1}{2 (\tilde{k}(0))^2} \left[1 - \cos 2\omega(\tilde{k}(0)) t\right] \\
\langle z(t) \rangle &= \frac{\hbar \tilde{k}(0)}{m} t + \frac{1}{2 (\tilde{k}(0))^2} \left[1 - \cos 2\omega(\tilde{k}(0)) t\right] \frac{1}{\tilde{k}(0)} \left[1 - \cos 2\omega(\tilde{k}(0)) t\right]
\end{align*}
\]

In the opposite limit, \(d \to 0\), the ZB vanishes, and for intermediate values the energy spread causes damping, as can be shown analytically for bilayer graphene [8].

Due to the induced Born-Huang field, ZB will occur in this system (unlike its condensed matter counterparts [7, 8, 9]) even if the wavepacket has an initial zero group velocity. Supposing \(^{85}\)Rb atoms, we take the lattice wavenumber to be \(k_0 = (2\pi/820)\) nm\(^{-1}\), and a Gaussian with \(\tilde{k}(0) = 0\) and width \(k_0d = 16.2\), corresponding to the ground state of a trap with trap frequency 112 Hz [15]. Fig. 3 shows that a pronounced oscillation would occur in the \(z\) direction before damping out. A typical time scale of the ZB here would be \(\mu s\) rather than the fs predicted in e.g. graphene and related systems [8].

![Figure 3: ZB for an atom with zero momentum spread (dashed), and for a momentum spread corresponding to the velocity spread of a cloud initially in a trap of frequency 112 Hz (solid). ZB oscillations for finite momentum spread damp out over time, but persist over several periods.](image-url)

We have shown that the mean position of the atom oscillates sinusoidally. However, ZB can also be viewed in terms of state-resolved spatial dynamics. For the Gaussian initial wavepacket, it is not difficult to show that as the center of mass of the cloud is oscillating in the \(z\) direction, in the \(y\) direction, the wavepacket separates by spin, such that \(\langle y_{1,2}(t) \rangle = \pm \hbar k_0 t/m\) (see Fig. 4). This spin separation, which coexists with the ZB, is a manifestation of the atomic spin Hall effect proposed in a different setup [10]; a related effect occurs in velocity-selective coherent population trapping [11].

Fig. 5 shows the dynamics of the “spin-up” component of the wavepacket in a representative non-Abelian case. In essence, the effective magnetic field deflects spin-up and spin-down in opposite directions. As the two wavepackets separ-
rate, the coupling between the components results in oscillating “tails” on the wavepackets, giving rise to ZB. The ZB decays as the wavepackets separate and cease to interfere.

Figure 4: (Color online) Spin separation and ZB for $\varepsilon = 0(0.2)1$, as indicated on the right colorbar. For each value of $\varepsilon$ the dashed trajectory corresponds to the atom’s mean position in “spin-up,” while the solid trajectory corresponds to the mean position of the atom in “spin-down;” open and closed circles indicate the respective ends of these trajectories. In the Abelian cases, $\varepsilon = 0$ and 1, the trajectories are straight lines; the trajectory for $\varepsilon = 0$ is the vertical line $y = 0$.

Figure 5: As seen in Fig. 4, the wavepacket separates by internal state as it jitters. This figure shows the time-evolved probability distribution of the “spin-up” component of the wavefunction for an initial Gaussian wavepacket ($\varepsilon = \varepsilon_R, k_{0d} = 5, \hbar k_0/m = 10$). The black dot indicates the mean position, which has drifted to the right. The mixing between internal states gives rise to “tails” on the wavepacket, resulting in ZB. The ZB damps as the internal states separate.

We note that the Hamiltonian for $N$ particles in the non-Abelian gauge field, interacting via two-body interactions, separates in center-of-mass and relative coordinates, $\vec{R}$ and $\vec{\sigma}$ respectively. The Hamiltonian is then $\hat{H} = \hat{H}_{CM} + \hat{H}_{\vec{\rho}}$, where $\hat{H}_{\vec{\rho}}$ is a function only of the relative coordinates $\vec{\sigma}$, and

$$\hat{H}_{CM} = \frac{1}{2mN} \vec{P}_R^2 - \frac{\hbar k_0}{2m} \sum_i (1 - \varepsilon^2)(P_i + \vec{h}k_0)\Sigma_z - 2\varepsilon P_i \Sigma_y,$$

where $\vec{P}$ is the center-of-mass momentum and $\Sigma = \sum_i \sigma^{(i)}$ is the total spin. The Hamiltonian for the center of mass is thus again of a single-particle Dirac form (with a higher spin), and in a dilute cloud of $N$ particles with two-body interactions, the center of mass of the cloud undergoes ZB.

In this paper, we have examined the dynamics of an atom in a tripod level scheme on an optical lattice; this common experimental setup gives rise to a non-Abelian gauge field which is isomorphic to the spin-orbit interaction in 2D electron gases. The idea of “atomtronics,” or engineering atomic versions of semiconductor devices, has generated recent interest [18, 19]. The prospect of engineering artificial spin-orbit couplings suggests the possibility of atomic “spintronics,” for example engineering atomic counterparts of devices such as the Datta-Das transistor [20], which have yet to be successfully realized with electrons. The tripod system here exhibits atomic ZB, with an amplitude many orders of magnitude larger than that offered by Dirac electrons or recently discussed condensed matter systems. We believe it is a promising candidate for the experimental observation of ZB.

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