Coloured Hopf Algebras

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1 Introduction

Since its introduction (for reviews see e.g. [1]), the parametrized Yang-Baxter equation (YBE) has played a crucial role in nonlinear integrable systems in physics, such as exactly solvable statistical mechanics models and low-dimensional integrable field theories. Its constant form has also appeared in knot theory, where it is connected with braid groups. In addition, the YBE has inspired the development of quantum groups and quantum universal enveloping algebras (QUEAs). The latter have appeared in the literature in two different, but related forms: the Faddeev-Reshetikhin-Takhtajan (FRT) and Drinfeld-Jimbo (DJ) formulations.

In recent years, some integrable models with nonadditive-type solutions $R_{\lambda,\mu} \neq R(\lambda - \mu)$ of the YBE have been discovered [2]. The corresponding YBE

$$R_{12}^{\lambda,\mu} R_{13}^{\lambda,\nu} R_{23}^{\mu,\nu} = R_{23}^{\mu,\nu} R_{13}^{\lambda,\nu} R_{12}^{\lambda,\mu} \quad (1)$$

is referred to in the literature as the ‘coloured’ YBE, the nonadditive (in general multicomponent) spectral parameters $\lambda, \mu, \nu$ being considered as ‘colour’ indices. Constructing solutions of Eq. (1) has been achieved by using various approaches (see e.g. [3, 4, 5, 6]). It should be stressed that this coloured YBE is distinct from the so-called ‘colour YBE’ [7] arising in another context, as an extension of the graded YBE to more general gradings than that determined by $\mathbb{Z}_2$.

Extending the definitions of quantum groups and QUEAs by connecting them to coloured $R$-matrices, instead of ordinary ones, has received some attention in the literature. Kundu and Basu-Mallick [5] generalized the FRT formalism for some quantizations of $U(gl(2))$ and $Gl(2)$. In the context of knot theory, Ohtsuki [8] introduced some coloured quasitriangular Hopf algebras, which are characterized by the existence of a coloured universal $R$-matrix, and he applied

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his theory to \( U_q(sl(2)) \). Bonatsos et al.\(^6\) independently considered a rather similar, but nevertheless distinct generalization for some nonlinear deformation of \( U(sl(2)) \). Recently, we extended the DJ formulation of QUEA’s to coloured ones \(^7\) by elaborating on the results of Bonatsos et al.

It is the purpose of the present contribution to review such a generalization. In Sec. 2, we define coloured Hopf algebras in a way that generalizes Ohtsuki’s first attempt. In Sec. 3, we then apply the new concepts to construct two examples of coloured QUEA’s. Finally, Sec. 4 contains some concluding remarks.

2 Coloured Hopf algebras

Let \((\mathcal{H}, \ast, m_q, e_q, S_q, k)\) (or in short \(\mathcal{H}_q\)) be a Hopf algebra over some field \(k\) (= \(\mathbb{C}\) or \(\mathbb{R}\)), depending upon some parameters \(q\). Here \(m_q : \mathcal{H}_q \otimes \mathcal{H}_q \rightarrow \mathcal{H}_q\), \(\epsilon_q : \mathcal{H}_q \rightarrow k\), \(\Delta_q : \mathcal{H}_q \rightarrow \mathcal{H}_q \otimes \mathcal{H}_q\), \(\epsilon_q : \mathcal{H}_q \rightarrow k\), and \(S_q : \mathcal{H}_q \rightarrow \mathcal{H}_q\) denote the multiplication, unit, comultiplication, counit, and antipode maps respectively \(^1\). Whenever \(q\) runs over some set \(Q\), called parameter set \(Q\), we obtain a set of Hopf algebras \(\mathcal{H} = \{ \mathcal{H}_q | q \in Q \}\). We may distinguish between two cases, according to whether \(Q\) contains a single element (fixed-parameter case) or more than one element (varying-parameter case).

Let us assume that there exists a set of one-to-one linear maps \(G = \{ \sigma^\nu : \mathcal{H}_q \rightarrow \mathcal{H}_q^\nu | q, q^\nu \in Q, \nu \in C \}\), defined for any \(\mathcal{H}_q \in \mathcal{H}\). They are labelled by some parameters \(\nu\), called colour parameters \(\nu\), taking values in some set \(C\), called colour set \(C\). The latter may be finite, countably infinite, or uncountably infinite.

Two conditions are imposed on the \(\sigma^\nu\)’s:

(i) Every \(\sigma^\nu\) is an algebra isomorphism, i.e.,

\[
\sigma^\nu \circ m_q = m_{q^\nu} \circ (\sigma^\nu \otimes \sigma^\nu), \quad \sigma^\nu \circ \epsilon_q = \epsilon_{q^\nu};
\]

(ii) \(G\) is a group (called colour group) with respect to the composition of maps, i.e.,

\[
\forall \nu, \nu' \in C, \exists \nu'' \in C : \sigma^{\nu''} = \sigma^{\nu'} \circ \sigma^\nu : \mathcal{H}_q \rightarrow \mathcal{H}_{q^{\nu''}} = \mathcal{H}_{q^\nu \cdot \nu'}, \quad (3)
\]

\[
\exists \nu^0 \in C : \sigma^{\nu^0} = \text{id} : \mathcal{H}_q \rightarrow \mathcal{H}_{q^{\nu^0}} = \mathcal{H}_q, \quad (4)
\]

\[
\forall \nu \in C, \exists \nu' \in C : \sigma^{\nu'} = \sigma_\nu \equiv (\sigma^\nu)^{-1} : \mathcal{H}_{q^\nu} \rightarrow \mathcal{H}_q. \quad (5)
\]

In Eqs. (3) and (4), \(\nu''\) and \(\nu^0\) will be denoted by \(\nu \circ \nu'\) and \(\nu\), respectively.

\(\mathcal{H}, C, \) and \(G\) can be combined into

**Definition 2.1** The maps \(\Delta^\lambda_{q,\nu} : \mathcal{H}_{q^\nu} \rightarrow \mathcal{H}_q^\nu \otimes \mathcal{H}_q^\nu\), \(\epsilon_{q,\nu} : \mathcal{H}_q^\nu \rightarrow k\), and \(S^\mu_{q,\nu} : \mathcal{H}_q^\nu \rightarrow \mathcal{H}_q^\nu\), defined by

\[
\Delta^\lambda_{q,\nu} \equiv (\sigma^\lambda \otimes \sigma^\nu) \circ \Delta_q \circ \sigma_\nu, \quad \epsilon_{q,\nu} \equiv \epsilon_q \circ \sigma_\nu, \quad S^\mu_{q,\nu} \equiv \sigma^\mu \circ S_q \circ \sigma_\nu, \quad (6)
\]
for any \( q \in \mathbb{Q} \), and any \( \lambda, \mu, \nu \in \mathbb{C} \), are called coloured comultiplication, counit, and antipode respectively.

It is easy to prove the following proposition:

**Proposition 2.2** The coloured comultiplication, counit, and antipode maps, defined in Eq. (6), transform under the colour group \( \mathcal{G} \) as

\[
\left( \sigma^\lambda_\alpha \otimes \sigma^\mu_\beta \right) \circ \Delta^\alpha,\beta_{q,\nu} = \Delta^\lambda,\mu_{q,\gamma} \circ \sigma^\gamma_\nu,
\]

\[
\epsilon_{q,\alpha} \circ \sigma^\alpha_\nu = \epsilon_{q,\nu},
\]

\[
\sigma^\alpha_\nu \circ S^\alpha_{q,\nu} = S^\mu_{q,\nu} \circ \sigma^\beta_\nu,
\]

(7)

and satisfy generalized coassociativity, counit, and antipode axioms

\[
\left( \Delta^\alpha,\beta_{q,\lambda} \otimes \sigma^\gamma_\mu \right) \circ \Delta^\lambda,\mu_{q,\nu} = \left( \sigma^\alpha_\lambda \otimes \Delta^\beta,\gamma_{q,\nu} \right) \circ \Delta^{\lambda,\mu'}_{q,\nu},
\]

\[
\left( \epsilon_{q,\lambda} \otimes \sigma^\alpha_\mu \right) \circ \Delta^\lambda,\mu_{q,\nu} = \left( \sigma^\alpha_\nu \otimes \epsilon_{q,\mu'} \right) \circ \Delta^{\lambda,\mu'}_{q,\nu} = \sigma^\alpha_\nu,
\]

\[
m^\alpha_{q,\nu} \circ \left( S^\alpha_{q,\lambda} \otimes \sigma^\alpha_\mu \right) \circ \Delta^\lambda,\mu_{q,\nu} = \epsilon_{q,\nu} \circ \left( \sigma^\alpha_\nu \otimes \Delta^\alpha_{q,\nu} \right) \circ \Delta^{\lambda,\mu'}_{q,\nu} = \epsilon_{q,\nu} \circ \epsilon_{q,\nu},
\]

(8)

as well as generalized bialgebra axioms

\[
\Delta^\lambda,\mu_{q,\nu} \circ m^\nu_{q,\nu} = \left( m^\lambda_{q,\nu} \otimes m^\nu_{q,\nu} \right) \circ \left( \text{id} \otimes \tau \otimes \text{id} \right) \circ \left( \Delta^\lambda,\mu_{q,\nu} \otimes \Delta^\lambda,\mu_{q,\nu} \right),
\]

\[
\Delta^\lambda,\mu_{q,\nu} \circ \iota^\nu_{q,\nu} = \iota^\lambda_{q,\nu} \otimes \iota^\mu_{q,\nu},
\]

\[
\epsilon_{q,\nu} \circ m^\nu_{q,\nu} = \epsilon_{q,\nu} \otimes \epsilon_{q,\nu},
\]

\[
\epsilon_{q,\nu} \circ \iota^\nu_{q,\nu} = 1_k.
\]

(9)

Here \( \sigma^\lambda_\mu \) is the element of \( \mathcal{G} \) defined by

\[
\sigma^\lambda_\mu \equiv \sigma^\lambda \circ \sigma^\mu,
\]

(10)

\( \tau \) is the twist map, i.e., \( \tau(a \otimes b) = b \otimes a \), \( 1_k \) denotes the unit of \( k \), and no summation is implied over repeated indices.

From Proposition 2.2, it is straightforward to obtain

**Corollary 2.3** If Eqs. (4)–(6) are satisfied, then for any \( q \in \mathbb{Q} \), any \( \nu \in \mathbb{C} \), and \( q_\nu \equiv q^{\nu} \), \( (\mathcal{H}_q, +, m_q, \tau_q, \Delta^\nu,\nu_{q,\nu}, \epsilon_{q,\nu}, S^\nu_{q,\nu}; k) \) is a Hopf algebra over \( k \) with co-multiplication \( \Delta^\nu,\nu_{q,\nu} \), counit \( \epsilon_{q,\nu} \), and antipode \( S^\nu_{q,\nu} \), defined by particularizing Eq. (7).

Remark. In particular, for \( \nu = \nu^0 \), we get back the original Hopf structure of \( \mathcal{H}_q \).

Generalizing the result contained in Corollary 2.3, we are led to introduce
Definition 2.4 A set of Hopf algebras \( \mathcal{H} \), endowed with coloured comultiplication, counit, and antipode maps \( \Delta_{q,\lambda,\mu}, \epsilon_{q,\lambda,\mu}, S_{q,\lambda,\mu} \), as defined in (1), is called coloured Hopf algebra, and denoted by any one of the symbols \( (\mathcal{H}_q, +, \rho, \epsilon_{q,\lambda,\mu}, S_{q,\lambda,\mu}, k, \mathbb{Q}, \mathbb{C}, \mathcal{G}) \), (\( \mathcal{H}, \mathbb{C}, \mathcal{G} \)), or \( \mathcal{H}^c \).

Let us now assume that the members of the Hopf algebra set \( \mathcal{H} \) are almost cocommutative Hopf algebras \([1]\), i.e., for any \( q \in \mathbb{Q} \) there exists an invertible element \( R_q \in H_q \otimes H_q \) (completed tensor product), such that

\[
\tau \circ \Delta_q(a) = R_q \Delta_q(a) R_q^{-1}
\]
for any \( a \in H_q \).

We may then introduce

Definition 2.5 Let \( R^c \) denote the set of elements \( R_{q,\lambda,\mu}^\lambda \in H_q^\lambda \otimes H_q^\mu \), defined by

\[
R_{q,\lambda,\mu}^\lambda \equiv \left( \sigma^\lambda \otimes \sigma^\mu \right) (R_q),
\]
where \( q \) runs over \( \mathbb{Q} \), and \( \lambda, \mu \) over \( \mathcal{C} \).

The following result can be easily obtained:

Proposition 2.6 If the Hopf algebras \( H_q \) of \( \mathcal{H} \) are almost cocommutative, then \( R_{q,\lambda,\mu}^\lambda \), as defined in (12), is invertible with \( (R_q^\lambda)^{-1} \) given by

\[
(R_q^\lambda)^{-1} = \left( \sigma^\lambda \otimes \sigma^\mu \right) (R_q)^{-1},
\]
and

\[
\tau \circ \Delta_{q^{\lambda,\mu}}(a) = R_q^{\lambda,\mu} \Delta_{q^{\lambda,\mu}}(a) (R_q^{\lambda,\mu})^{-1}
\]
for any \( a \in H_q^\nu \).

Hence we have

Definition 2.7 A coloured, almost cocommutative Hopf algebra is a pair \( \left( \mathcal{H}^c, R^c \right) \), where \( \mathcal{H}^c \) is a coloured Hopf algebra, \( R^c = \{ R_{q,\lambda,\mu}^\lambda \mid q \in \mathbb{Q}, \lambda, \mu \in \mathcal{C} \} \), and \( R_{q,\lambda,\mu}^\lambda \), defined in (13), satisfies Eqs. (14) and (15).

In the same way, we may define coloured, quasitriangular (or coboundary, or triangular) Hopf algebras by starting with an appropriate \( R_q \):

Definition 2.8 A coloured, almost cocommutative Hopf algebra \( \left( \mathcal{H}^c, R^c \right) \) is said to be quasitriangular if

\[
\begin{align*}
\left( \Delta_{q,\lambda,\mu}^{a,\beta,\gamma} \otimes \sigma_{\mu}^\beta \right) (R_q^{\lambda,\mu}) &= R_{q,\gamma,13}^a R_{q,\gamma,21}, \\
\left( \sigma_{\lambda}^\gamma \otimes \Delta_{q,\lambda,\mu}^{\beta,\gamma} \right) (R_q^{\lambda,\mu}) &= R_{q,\gamma,13}^a R_{q,\gamma,12}.
\end{align*}
\]

The set \( R^c \) is called the coloured universal \( R \)-matrix of \( \left( \mathcal{H}^c, R^c \right) \).
The terminology used for $\mathcal{R}^c$ in Definition 2.8 is justified by the following proposition, which shows among others that the elements of $\mathcal{R}^c$ satisfy the coloured YBE, as given in (1):

**Proposition 2.9** Let $(\mathcal{H}^c, \mathcal{R}^c)$ be a coloured quasitriangular Hopf algebra. Then

$$
\mathcal{R}_{q,12}^{\lambda,\mu} \mathcal{R}_{q,13}^{\lambda,\nu} \mathcal{R}_{q,23}^{\mu,\nu} = \mathcal{R}_{q,23}^{\mu,\nu} \mathcal{R}_{q,13}^{\lambda,\nu} \mathcal{R}_{q,12}^{\lambda,\mu},
$$

$$(\epsilon_{q,\lambda} \otimes \sigma^\alpha_\mu) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \left( \sigma^\alpha_\lambda \otimes \epsilon_{q,\mu} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = 1_{q^\alpha},$$

$$(S_{q,\lambda}^\alpha \otimes \sigma^\beta_\mu) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \left( \sigma^\alpha_\lambda \otimes \left( S_{q,\beta}^\mu \right)^{-1} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \left( \mathcal{R}_{q}^{\alpha,\beta} \right)^{-1},$$

where $1_{q^\alpha}$ denotes the unit element of $\mathcal{H}_{q^\alpha}$, and $(S_{q,\nu}^\mu)^{-1} : \mathcal{H}_{q^\nu} \to \mathcal{H}_{q^\nu}$ is given by $(S_{q,\mu}^\nu)^{-1} = \sigma^\nu \circ S^{-1} \circ \sigma_\mu$.

### 3 Examples of coloured quantum universal enveloping algebras

In the present section, we construct two examples of coloured quasitriangular Hopf algebras, for which the underlying Hopf algebras $\mathcal{H}_q$ are QUEAs of Lie algebras $U_q(g)$.

#### 3.1 The two-parameter quantum algebra $U_{q,s}(gl(2))$

The first example deals with the two-parameter deformation of $U(gl(2))$ [10], whose universal $\mathcal{R}$-matrix was given in Ref. [11]. Such an example is quite significant since $U_{q,s}(gl(2))$ and its corresponding quantum group have played an important role both in generating some matrix solutions of the coloured YBE [3, 5], and in constructing a coloured extension of the FRT formalism [5].

The quantum algebra $U_{q,s}(gl(2))$, for which $k = \mathbb{C}$ and $q, s \in \mathbb{C} \setminus \{0\}$, is generated by four operators $J_3, J_\pm, Z$, with commutation relations

$$
\left[ J_3, J_\pm \right] = \pm J_\pm, \quad \left[ J_+, J_- \right] = [2J_3]_q, \quad [Z, J_3] = [Z, J_\pm] = 0, \quad (17)
$$

and coalgebra and antipode depending upon both parameters $q$ and $s$.

Eq. (17) is left invariant under the transformations

$$
\sigma^\nu (J_3) = J_3, \quad \sigma^\nu (J_\pm) = J_\pm, \quad \sigma^\nu (Z) = \nu Z, \quad (18)
$$

where $\nu \in \mathbb{C} = \mathbb{C} \setminus \{0\}$, and $(q^\nu, s^\nu) = (q, s)$ (fixed-parameter case). Since $\nu^0 = 1, \nu^i = \nu^{i-1}$, the colour group $\mathcal{G}$ is isomorphic to the abelian
The coloured maps and universal $\mathcal{R}$-matrix are easily obtained as

\[
\Delta_{q,s,\nu}^{\lambda,\mu}(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \quad \Delta_{q,s,\nu}^{\lambda,\mu}(Z) = \frac{\lambda}{\nu} Z \otimes 1 + \frac{\mu}{\nu} 1 \otimes Z,
\]

\[
\Delta_{q,s,\nu}^{\lambda,\mu}(J_{\pm}) = J_{\pm} \otimes q^{J_3} \left( \frac{s}{q} \right)^{\pm \mu Z} + q^{-J_3(qs)^{\pm \lambda Z}} \otimes J_{\pm},
\]

\[
\epsilon_{q,s,\nu}(X) = 0, \quad X \in \{J_3, J_{\pm}, Z\},
\]

\[
S_{q,s,\nu}^{\mu}(J_3) = -J_3, \quad S_{q,s,\nu}^{\mu}(Z) = -\frac{\mu}{\nu} Z, \quad S_{q,s,\nu}^{\mu}(J_{\pm}) = -q^{\pm 1} s^{\pm 2 \mu Z} J_{\pm},
\]

\[
\mathcal{R}_{q,s}^{\lambda,\mu} = q^{2(J_3 \otimes J_3 - \lambda Z \otimes J_3 + J_3 \otimes Z)} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^n (n-1)/2
\]

\[
	imes \left( q^{J_3(qs)^{-\lambda Z} J_+} \right)^n \otimes \left( q^{-J_3(qs)^{-\lambda Z} J_-} \right)^n.
\]

The matrix representation of the coloured universal $\mathcal{R}$-matrix in any finite-dimensional representation of $U_{q,s}(gl(2))$ provides us with a matrix solution $\mathcal{R}_{q,s}^{\lambda,\mu}$ of the coloured YBE (1). For instance, in the two-dimensional representation of $U_{q,s}(gl(2))$,

\[
D(J_3) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
D(Z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we get a (renormalized) 4 × 4 coloured $R$-matrix $R_{q,s}^{\lambda,\mu} \equiv q^{1/2}(D \otimes D) (\mathcal{R}_{q,s}^{\lambda,\mu})$, given by

\[
R_{q,s}^{\lambda,\mu} = \begin{pmatrix} q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{\lambda+\mu} & (q - q^{-1}) s^{-\lambda+\mu} & 0 \\ 0 & 0 & q^{-\lambda-\mu} & 0 \\ 0 & 0 & 0 & q^{1+\lambda-\mu} \end{pmatrix}.
\]

The latter coincides with the coloured $R$-matrix previously derived by Burdík and Hellinger [3] by considering $2 \times 2$ representations of $U_{q,s}(gl(2))$ characterized by different eigenvalues $\lambda, \mu$ of $Z$.

### 3.2 The standard quantum oscillator algebra $U_z^{(s)}(h(4))$

The second example is the standard deformation $U_z^{(s)}(h(4))$ of the oscillator algebra $U(h(4))$, which was first derived by contracting $U_q(gl(2))$ [12], then recently obtained in a more convenient basis [13]. This quantum algebra has
been used to construct a solution of the coloured YBE connected with some link
invariants [4].

\[ U_s (h(4)) \] is generated by four operators \( N, M, A_\pm \), satisfying the commu-
tation relations

\[ [N, A_\pm] = \pm A_\pm, \quad [A_-, A_+] = \frac{\sinh(zM)}{z}, \quad [M, N] = [M, A_\pm] = 0. \quad (22) \]

Here we assume \( k = C \) and \( z \in \mathbb{C} \setminus \{0\} \). The algebra defining relations (22) are
left invariant under the transformations

\[ \sigma^\nu(N) = N, \quad \sigma^\nu(M) = \nu_+ \nu_- M, \quad \sigma^\nu(A_+) = \nu_+ A_+, \quad \sigma^\nu(A_-) = \nu_- A_-, \quad (23) \]

where \( \nu \equiv (\nu_+, \nu_-) \), provided \( z \) is changed into \( z^\nu = \nu_+ \nu_- z \). Hence the param-
eter set is \( \mathcal{Q} = \mathbb{C} \setminus \{0\} \) (varying-parameter case), the colour set is the cartesian
product \( \mathbb{C} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \), and the colour group is the direct product
group \( \mathcal{G} = \text{Gl}(1, \mathbb{C}) \otimes \text{Gl}(1, \mathbb{C}) \).

The corresponding coloured maps and universal \( \mathcal{R} \)-matrix are given by

\[ \Delta_{z,\nu}^{\lambda,\mu}(N) = N \otimes 1 + 1 \otimes N, \quad \Delta_{z,\nu}^{\lambda,\mu}(M) = \frac{\lambda_+ \lambda_-}{\nu_+ \nu_-} M \otimes 1 + \frac{\mu_+ \mu_-}{\nu_+ \nu_-} 1 \otimes M, \]

\[ \Delta_{z,\nu}^{\lambda,\mu}(A_+) = \frac{\lambda_+}{\nu_+} A_+ \otimes 1 + \frac{\mu_+}{\nu_+} e^{-\lambda_+ \lambda_- zM} \otimes A_+, \]

\[ \Delta_{z,\nu}^{\lambda,\mu}(A_-) = \frac{\lambda_-}{\nu_-} A_- \otimes e^{\mu_+ \mu_- zM} + \frac{\mu_-}{\nu_-} 1 \otimes A_-, \]

\[ \epsilon_{z,\nu}(X) = 0, \quad X \in \{ N, M, A_\pm \}, \]

\[ S_{z,\nu}^{\mu}(N) = -N, \quad S_{z,\nu}^{\mu}(M) = -\frac{\mu_+ \mu_-}{\nu_+ \nu_-} M, \]

\[ S_{z,\nu}^{\mu}(A_\pm) = -\frac{\mu_+ \mu_-}{\nu_+ \nu_-} A_\pm e^{\pm \mu_+ \mu_- zM}, \]

\[ \mathcal{R}_{z}^{\mu} = \exp\{-\lambda_+ \lambda_- zM \otimes N\} \exp\{-\mu_+ \mu_- zN \otimes M\} \times \exp\{2 \lambda_- \mu_+ zA_- \otimes A_+\}. \quad (24) \]

In the \( 3 \times 3 \) matrix representation of \( U_s(h(4)) \) defined by

\[ D(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(M) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ D(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (25) \]
the coloured universal $R$-matrix is represented by the $9 \times 9$ matrix

$$R^\lambda_\mu \equiv (D \otimes D) (R^\lambda_\mu) = \begin{pmatrix}
1_3 & 2\lambda - \mu_+ zD(A_+) & -\lambda + \lambda_+ zD(N) \\
0_3 & 1_3 - \mu_+ \mu_- zD(M) & 0_3 \\
0_3 & 0_3 & 1_3
\end{pmatrix}, \quad (26)$$

where $1_3$ and $0_3$ denote the $3 \times 3$ unit and null matrices respectively.

4 Conclusion

In this contribution, we did present some new algebraic structures, termed coloured Hopf algebras $[9]$, by combining the coalgebra structures and antipodes of a standard Hopf algebra set with the transformations of an algebra isomorphism group, called colour group. We did also show that quasitriangular Hopf algebras can be extended into coloured ones, characterized by the existence of a coloured universal $R$-matrix, satisfying the coloured YBE. Finally, we did apply the new concepts to QUEAs of Lie algebras by constructing two examples of coloured QUEAs.

As shown elsewhere $[9]$, the coloured Hopf algebras defined here generalize those previously introduced by Ohtsuki $[8]$, which are restricted to abelian colour groups, in which case they reduce to substructures of the present ones. Many more examples of coloured QUEAs, corresponding to finite or infinite, abelian or nonabelian colour groups have been constructed $[9]$, thereby providing some new solutions of the coloured YBE, which might be of interest in the context of integrable models.

Extending the present formalism to QUEAs of Lie superalgebras, as well as defining and constructing duals to coloured Hopf algebras are under current investigation $[14]$.

References

[1] S. Majid, Int. J. Mod. Phys. A 5 (1990) 1; V. Chari and A. Pressley, A Guide to Quantum Groups, (Cambridge U.P., Cambridge, 1994).
[2] V. V. Bazhanov and Yu. G. Stroganov, Theor. Math. Phys. 62 (1985) 253.
[3] Č. Burdík and P. Hellinger, J. Phys. A 25 (1992) L1023.
[4] C. Gómez and G. Sierra, J. Math. Phys. 34 (1993) 2119.
[5] A. Kundu and B. Basu-Mallick, J. Phys. A 27 (1994) 3091; B. Basu-Mallick, Mod. Phys. Lett. A 9 (1994) 2733; Int. J. Mod. Phys. A 10 (1995) 2851.
[6] D. Bonatsos, C. Daskaloyannis, P. Kolokotronis, A. Ludu, and C. Quesne, “A nonlinear deformed su(2) algebra with a two-colour quasitriangular
Hopf structure”, *J. Math. Phys.* (in press); D. Bonatsos, P. Kolokotronis, C. Daskaloyannis, A. Ludu, and C. Quesne, “Nonlinear deformed su(2) algebras involving two deforming functions”, *Czech. J. Phys.* (in press).

[7] D. S. McAnally, in: *Proc. Yamada Conf. XL, XX Int. Coll. on Group Theoretical Methods in Physics, Toyonaka, Japan, July 4–9, 1994*, eds. A. Arima, T. Eguchi, and N. Nakanishi, p. 339, (World Scientific, Singapore, 1995).

[8] T. Ohtsuki, *J. Knot Theor. Its Rami.* 2 (1993) 211.

[9] C. Quesne, “Coloured quantum universal enveloping algebras”, Université Libre de Bruxelles preprint (1996).

[10] A. Schirrmacher, J. Wess, and B. Zumino, *Z. Phys.* C 49 (1991) 317.

[11] Č. Burdík and P. Hellinger, *J. Phys.* A 25 (1992) L629.

[12] E. Celeghini, R. Giachetti, E. Sorace, and M. Tarlini, *J. Math. Phys.* 31 (1990) 2548; 32 (1991) 1155.

[13] A. Ballesteros and F. J. Herranz, *J. Phys.* A 29 (1996) 4307.

[14] C. Quesne, unpublished.