Entanglement entropy of two disjoint blocks in $XY$ chains

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Abstract. We study the Rényi entanglement entropies of two disjoint intervals in $XY$ chains. We exploit the exact solution of the model in terms of free Majorana fermions and we show how to construct the reduced density matrix in the spin variables by taking the Jordan–Wigner string between the two blocks properly into account. From this we can evaluate any Rényi entropy of finite integer order. We study in detail critical $XX$ and Ising chains and we show that the asymptotic results for large blocks agree with recent conformal field theory predictions if corrections to the scaling are included in the analysis correctly. We also report results for the gapped phase and after a quantum quench.

Keywords: conformal field theory (theory), spin chains, ladders and planes (theory), entanglement in extended quantum systems (theory)

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1. Introduction

The bipartite entanglement for a given division of the Hilbert space into a part $A$ and its complement $B$ can be measured in terms of the Rényi entropies \[ S_A^{(\alpha)} = \frac{1}{1 - \alpha} \log \text{Tr} \rho_A^\alpha, \] (1)
where $\rho_A = \text{Tr}_B \rho$ is the reduced density matrix of the subsystem $A$, and $\rho = |\Psi\rangle\langle \Psi|$ is the density matrix of the whole system in a pure state $|\Psi\rangle$. The knowledge of the $S_A^{(\alpha)}$ for different $\alpha$ characterizes the full spectrum of non-zero eigenvalues of $\rho_A$ \[2\], and gives more information about the entanglement than the more commonly studied von Neumann entropy $S_A^{(1)}$. It also gives a fundamental insight into the convergence and scaling of algorithms based on matrix product states \[3\].

In \[4\]–[6] it has been shown that for a one-dimensional critical system whose scaling limit is described by a conformal field theory (CFT), in the case where $A$ is an interval of length $\ell$ in an infinite system, the asymptotic behavior of the quantities determining the Rényi entropies is

\[ \text{Tr} \rho_A^\alpha \simeq c_\alpha \left( \frac{\ell}{a} \right)^{c(\alpha-1/\alpha)/6}, \] (2)
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Figure 1. Typical bipartition that we consider in this paper. The subset $A$ is the union of two disjoint intervals $A_1$ and $A_2$ of lengths $\ell_1$ and $\ell_2$ respectively. The block separating them is denoted as $B_1$, of length $r$. The ‘environment’ is $B = B_1 \cup B_2$. The thermodynamic limit is obtained by sending the total length $L \to \infty$, while $\ell_1, \ell_2, r$ remain finite (i.e. the length of $B_2$ goes to $\infty$).

where $c$ is the central charge of the underlying CFT. Thus the Rényi entropies (and in particular the von Neumann one for $\alpha = 1$) provide one of the best ways of detecting the value of the central charge.

Less attention has been devoted, until now, to the entanglement of two disjoint intervals in a CFT (and also in massive theories). It turned out that entanglement of disjoint intervals is sensitive to universal details of the CFT that are not encoded in the central charge and it is connected with the full spectrum of operators of the CFT underlying the lattice model.

We consider here the case of two disjoint intervals $A = A_1 \cup A_2 = [u_1, v_1] \cup [u_2, v_2]$ depicted in figure 1. By global conformal invariance, in the thermodynamic limit, $\text{Tr} \rho^\alpha_A$ can be always written as

$$\text{Tr} \rho^\alpha_A = c^2_\alpha \left( \frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{(c/6)(\alpha-1/\alpha)} F_\alpha(x),$$

where $x$ is the four-point ratio (for real $u_j$ and $v_j$, $x$ is real)

$$x = \frac{(u_1 - v_1)(u_2 - v_2)}{(u_1 - u_2)(v_1 - v_2)}.$$

Normalizing such that $F_\alpha(0) = 1$, we have that $c_\alpha$ is the same non-universal constant as appears in equation (2). The universal function $F_\alpha(x)$ depends explicitly on the full operator content of the theory and must be calculated case by case. Originally it was proposed that $F_\alpha(x) = 1$ identically [5]. This erroneous prediction has been tested in free
fermion theories \[7\]–\[9\], and only the calculations within more complex theories allowed us to detect this error \[10\]–\[13\].

Furukawa et al \[11\] calculated \(F_2(x)\) for a free boson compactified on a circle of radius \(R\):

\[
F_2(x) = \frac{\theta_3(\eta \tau) \theta_3(\tau/\eta)}{[\theta_3(\tau)]^2},
\]

where \(\tau\) is pure imaginary, and is related to \(x\) via \(x = [\theta_3(\tau)/\theta_3(\tau)]^4\). \(\theta_3\) are Jacobi theta functions. \(\eta\) is a universal critical exponent dependent on the compactification radius \(R\) (in the Luttinger liquid literature \(\eta = 1/(2K)\)). This has been extended to general integer \(\alpha \geq 2\) in \[12\]:

\[
F_\alpha(x) = \frac{\Theta(0|\eta \Gamma) \Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2},
\]

where \(\Gamma\) is an \((\alpha - 1) \times (\alpha - 1)\) matrix obtained in \[12\], \(\eta\) is the same as above, while \(\Theta\) is the Riemann–Siegel theta function

\[
\Theta(0|\Gamma) \equiv \sum_{m \in \mathbb{Z}^{\alpha - 1}} \exp[i \pi m^T \cdot \Gamma \cdot m].
\]

The \(XX\) model considered in the following is described in terms of the compactified boson with \(\eta = 1/2\). We mention that the analytic continuation of this result to real \(\alpha\) for general values of \(\eta\) and \(x\) (in order to obtain the entanglement entropy) is still an open problem, but the results for \(x \ll 1\) and \(\eta \ll 1\) are analytically known \[12\].

Nowadays, the only other example of non-trivial \(F_\alpha(x)\) known exactly is \(F_2(x)\) for the Ising model, which has the rather simple expression \[13\]

\[
F_2(x) = \frac{1}{\sqrt{2}} \left[ \left( \frac{1 + \sqrt{1-x}}{2} \right)^{1/2} + x^{1/4} + ((1-x)x)^{1/4} + (1-x)^{1/4} \right]^{1/2}.
\]

We mention that all of these new results for entanglement entropy were derived by using some old ‘classical’ results for CFT on orbifolds \[14\].

It is worth recalling that in the case of many intervals, the entanglement entropy measures only the entanglement of the disjoint intervals with the rest of the system. It is not a measure of the entanglement of one interval with respect another, which instead requires the introduction of more complicated quantities because \(A_1 \cup A_2\) is in a mixed state (see e.g. \[15\]–\[17\] for a discussion of this and examples).

In this work we report the exact evaluation of the Rényi entropies for integer \(\alpha\) in \(XY\) spin chains with periodic boundary conditions, whose Hamiltonian is given by

\[
H_{XY} = -\sum_{l=1}^{N} \left[ \frac{1 + \gamma}{4} \sigma_i^x \sigma_{i+1}^x + \frac{1 - \gamma}{4} \sigma_i^y \sigma_{i+1}^y + \frac{h}{2} \sigma_i^z \right],
\]

where \(\sigma_i^\alpha\) are the Pauli matrices at the site \(l\), \(h\) is the magnetic field and \(\gamma\) is the so-called anisotropy parameter. For \(\gamma = 1\) the Hamiltonian reduces to the Ising model, while for \(\gamma = 0\) it reduces to the \(XX\) model. Hamiltonian \(9\) can be diagonalized in terms of free fermions (see below) but the fundamental difficulty is that the reduced density matrices of spin and fermion variables are not the same \[13,18\]. While calculating the fermion
matrix is straightforward, we derive here a general method for dealing with spin reduced density matrices and present actual calculations for the Hamiltonian (9).

The paper is organized as follows. In section 2 we give a general description of the density matrix mapping and we introduce our formalism. In section 3 we derive the expressions for the Rényi entropies of integer order. These two sections are rather technical and are intended only for those readers who want to repeat the calculations. The readers interested only in the results can skip directly to section 4 where we report our results for the XX chain and compare them with CFT results by properly detecting corrections to the scaling. In section 5 we present the same analysis for the Ising model and we check the universality while changing $\gamma$. In section 6 we consider non-critical chains. In section 7 we consider the Rényi entropies after a quantum quench. Finally in section 8 we summarize our main results and we discuss topics deserving further investigations. Four appendices contain technical details of the calculations and some background material.

2. The reduced density matrix

Generically a spin $1/2$ chain can be described by means of the Pauli matrices $\sigma^\mu_i$ with $\mu = 0, 1, 2, 3$ ($\sigma^0 = 1$ and $1 \equiv x$, $2 \equiv y$, $3 \equiv z$) and $i$ labels the lattice sites. We are interested in the entanglement of a subsystem $A = \bigcup_{j=1}^{N} A_j$ consisting of $N$ disjoint spin blocks $A_j = [u_j, v_j]$. The reduced density matrix (RDM) can be written as a sum over all operators belonging to the blocks forming $A$:

$$\rho_A = \frac{1}{2^{|A|}} \sum_{\mu_1} \left( \prod_{l \in A} \sigma^\mu_l \right) \prod_{l \in A} \sigma^\mu_l,$$

where $|A| = \sum_j \ell_j$ is the sum of the lengths of the blocks $\ell_j$. The expectation value is taken over the state of which we need to calculate the entanglement. Obtaining knowledge of multipoint correlation functions $\left( \prod_{l \in A} \sigma^\mu_l \right)$ is however a very hard task even for eigenstates of integrable models (see e.g. [19,20] for the XXZ chains) and the above formula is particularly useful only when the spin chain admits a representation in terms of free fermions, where the Wick theorem suffices for calculating all of them.

The mapping between spin variables and fermion ones is obtained via the Jordan–Wigner transformation (that for the XY chains makes the Hamiltonians quadratic)

$$c_l = \left( \prod_{m<l} \sigma^z_m \right) \frac{\sigma^x_l - i \sigma^y_l}{2}, \quad c_l^\dagger = \left( \prod_{m<l} \sigma^z_m \right) \frac{\sigma^x_l + i \sigma^y_l}{2}.\quad (11)$$

It is also useful to define the Majorana fermions [21]

$$a_{2l} = c_l^\dagger + c_l, \quad a_{2l-1} = i(c_l - c_l^\dagger),\quad (12)$$

satisfying anti-commutation relations $\{a_l, a_m^\dagger\} = 2 \delta_{lm}$. For the case of a single interval, a fundamental observation [21] is that the Jordan–Wigner string $\left( \prod_{m<l} \sigma^z_m \right)$ maps the space of the first $\ell$ spins into the space of the first $\ell$ fermions and thus the RDMs of the first $\ell$ spins and fermions are the same, i.e.

$$\rho_A = \frac{1}{2^\ell} \sum_{\mu_1=0,1} \left( \prod_{l=1}^{2\ell} a^\mu_l \right) \left( \prod_{l=1}^{2\ell} a^\mu_l \right)^\dagger,$$

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with \(a_i^0 = 1\) and \(a_i^1 = a_i\). At this point, it is worth recalling that the above equation gives directly all the eigenvalues of \(\rho_A\) and so all Rényi entropies (see e.g. the reviews [22] for more details). Indeed, one recognizes \(\rho_A\) as an exponential form

\[
\rho_A \propto e^{a_i W_{lm} a_m / 4},
\]

and using the Wick theorem (see also appendix A),

\[
tanh \frac{W}{2} = \Gamma, \quad \text{with } \Gamma_{ij} = \langle a_i a_j \rangle - \delta_{ij}.
\]

The \(2\ell\) eigenvalues of the correlation matrix \(\Gamma\) have the form \(\pm \nu_i\) with \(\nu_i \in [-1, 1]\), so one has all \(2\ell\) eigenvalues of \(\rho_A\) as

\[
\lambda_{(s)} = \prod_i \frac{1 + s_i \nu_i}{2},
\]

for the \(2\ell\) possible choices of the variables \(s_i = \pm 1\). The Rényi entropies for any complex \(\alpha\) are then given by

\[
S_A^{(\alpha)}(\sigma) = \frac{1}{1 - \alpha} \sum_{j=1}^{\ell} \ln \left[ \left( \frac{1 + \nu_j}{2} \right)^\alpha + \left( \frac{1 - \nu_j}{2} \right)^\alpha \right].
\]

The calculation of \(\nu_i\) only requires the diagonalization of a \(2\ell \times 2\ell\) matrix, which is an extremely easy task compared to the diagonalization of the full \(2\ell \times 2\ell\) density matrix. This expression can be further analytically massaged to obtain exactly the non-universal constant \(c_\alpha\) [23] and also universal corrections to the scaling [24].

This is no longer true in the case of more disjoint spin blocks. In the fermionic space, the subspace of two disjoint spin blocks is not a local quantity: the non-locality of Jordan–Wigner transformation causes the fermions between the two blocks (i.e. those in \(B_1\)) to contribute to expectation values. For example, considering the product \(\sigma_l^y \sigma_n^y\) with \(l \in A_1\) and \(n \in A_2\) and calling the distance between the blocks \(r\) (see figure 1), we have

\[
\sigma_l^y \sigma_n^y = i a_2 \prod_{j=l+1}^{\ell_1} (ia_{2j-1} a_{2j}) \prod_{j=\ell_1+1}^{\ell_1+r} (ia_{2j-1} a_{2j}) \prod_{j=\ell_1+r+1}^{\ell_2} (ia_{2j-1} a_{2j}) a_{2n-1},
\]

which depends on the ‘unpleasant’ string of Majorana operators

\[
S \equiv \prod_{j=\ell_1+1}^{\ell_1+r} (ia_{2j-1} a_{2j}).
\]

The string contribution cannot be neglected (as also shown in [13,18]) and so the spin representation is not equivalent to the fermionic one.

For practical reasons we limit ourselves to the case of two blocks \(|A_1| = \ell_1\) and \(|A_2| = \ell_2\). We denote as \(|B_1| = r\) the space between the two blocks; see figure 1 for a graphical representation. We focus on spin chains with the ground-state property that only even numbers of fermionic operators have non-vanishing mean values, e.g. those with a Hamiltonian commuting with \(\prod_j \sigma_j^z\). In this case, in equation (10) the product of operators \(\prod_j \sigma_j^{\mu_i}\) does not depend on the string if we have an even number of Majorana operators (i.e. an even number of \(\sigma^{x(y)}\) in each block and it does depend on it if in each
block we have an odd number of Majorana fermions (odd–even configurations give an odd total number of Majorana fermions that have vanishing expectation value by hypothesis). Thus, the density matrix splits into two sums extended only over even and odd numbers of Majorana fermions in each block, i.e.

\[
\rho_{A_1 \cup A_2} = \frac{1}{2^{l_1 + l_2}} \left[ \sum_{\mu_i = 0, 1, \text{even}} \left\langle \prod_{l \in A_1} a_{l}^{\mu_l} \prod_{l \in A_2} a_{l}^{\mu_l} \right\rangle \left( \prod_{l \in A_1} a_{l}^{\mu_l} \prod_{l \in A_2} a_{l}^{\mu_l} \right)^\dagger \right] + \sum_{\mu_i = 0, 1, \text{odd}} \left\langle \prod_{l \in A_1} a_{l}^{\mu_l} S \prod_{l \in A_2} a_{l}^{\mu_l} \right\rangle \left( \prod_{l \in A_1} a_{l}^{\mu_l} S \prod_{l \in A_2} a_{l}^{\mu_l} \right)^\dagger \right]
\]

\[(20)\]

\[
\equiv \frac{1}{2^{l_1 + l_2}} \left[ \sum_{\text{even}} \langle O_1 O_2 \rangle O_2^1 O_1^\dagger + \sum_{\text{odd}} \langle O_1 S O_2 \rangle O_2^1 O_1^\dagger \right],
\]

\[(21)\]

where \(S\) is the complete string of \(\sigma_z\) belonging to the region between the blocks \(19\). We repeat that the two sums are intended over all possible products of Majorana fermions belonging to each interval and even/odd refers to the number of \(\mu_i = 1\), i.e. signaling the presence of a Majorana operator. We introduced also the ‘short’ \(O_{1,2}\) for a general product of Majorana operators belonging to \(A_{1,2}\). Equation (20) is an immediate consequence of the fermionic representation of spin variables and of the considerations above: the string \(S\) appears only in operators with an odd number of \(\sigma^x(y)\) in each block.

Spin variables on different sites commute, and so \([S, O_{1,2}] = 0\). Thus the spin RDM can be written as

\[
\rho_{A_1 \cup A_2} = \frac{1}{2} \left[ \sum_{\text{even}} \langle O_1 O_2 \rangle O_2^1 O_1^\dagger + \sum_{\text{odd}} \langle O_1 S O_2 \rangle O_2^1 O_1^\dagger \right],
\]

\[(22)\]

where \(\rho_{\pm}\) are the fermionic RDMs

\[
\rho_{\pm} = \frac{1}{2} \left[ \sum_{\text{even}} \langle O_1 O_2 \rangle O_2^1 O_1^\dagger \pm \sum_{\text{odd}} \langle O_1 S O_2 \rangle O_2^1 O_1^\dagger \right] \equiv \rho_{\text{even}} \pm \rho_{\text{odd}}.
\]

\[(23)\]

We will be interested in the Rényi entropies and in writing them in terms of the matrices \(\rho_{\pm}\). The orthogonal projectors \((1 \pm S)/2\) commute with \(\rho_{\pm}\) and so the \(n\)th power of the spin RDM is the following combination of the \(n\)th powers of fermionic RDMs:

\[
\rho_{A_1 \cup A_2}^n = \frac{1}{2} \left[ \sum_{\text{even}} \langle O_1 O_2 \rangle O_2^1 O_1^\dagger \pm \sum_{\text{odd}} \langle O_1 S O_2 \rangle O_2^1 O_1^\dagger \right] \equiv \rho_{\text{even}} \pm \rho_{\text{odd}}.
\]

\[(24)\]

We note that \(\rho_{+}\) is unitarily equivalent to \(\rho_{-}\) (the complete string \(S_1\) (or \(S_2\)) of \(\sigma_z\) belonging to \(A_1\) (\(A_2\)) is the unitary operator mapping one into the other, i.e. \(S_1 \rho_{+} S_1 = \rho_{-}\)). As a consequence we have

\[
\text{Tr} \rho_{A_1 \cup A_2}^n = \text{Tr} \rho_{\pm}^n,
\]

\[(25)\]

where we used \(\text{Tr} S = 0\). Still the matrices \(\rho_{\pm}\) are not of the forms (14) that are the only objects that we are able to deal with. Some further manipulations are still needed to bring them into a useful form.

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However, before continuing on the main path, we notice that $\rho_{\pm}$ are the RDMs in fermionic variables of the state

$$|\Psi_{\pm}\rangle = \left(\frac{1 \pm S}{2} + \frac{1 \mp S}{2} S_2\right) |\Psi_0\rangle,$$

where $|\Psi_0\rangle$ is the ground state. These are the ground states of the Hamiltonians

$$H_{\pm} = \left(\frac{1 \pm S}{2} + \frac{1 \mp S}{2} S_2\right) H \left(\frac{1 \pm S}{2} + \frac{1 \mp S}{2} S_2\right).$$

We remark that this unitary transformation is non-trivial only for operators on the contact surface between the blocks and the remaining chain. Summarizing, we can write a fermionic RDM equivalent to the spin RDM at the price of adding a finite number of non-local terms to the Hamiltonian $H$.

Now we write the fermionic RDM $\rho_{\pm}$ in equation (23) as linear combinations of quadratic RDMs of the forms (23) that are the ones that we are able to deal with. The basic intuitive reason for why this is indeed possible is that the $\sigma_z$ string is the exponential of a quadratic form

$$\prod_l \sigma_z^l = \prod_l (2c^\dagger_l c_l - 1) = \exp \left(i \pi \sum_l c_l c^\dagger_l\right) = \exp \left(\frac{\pi}{2} \sum_j (a_{2j-1} a_{2j} + i)\right),$$

and, in the case of a quadratic Hamiltonian, it can be interpreted as a term added to the exponential factor $W_{nm}$ in equation (14). These observations show that the Wick theorem applies in some form for equation (23), but the two-point correlations of $\rho_{\text{odd}}$ are different from the two-point correlations of $\rho_{\text{even}}$.

The RDM’s part $\rho_{\text{odd}}$ includes multipoint correlations in which the string $S$ is inserted. To deal properly with such objects we introduce the operator (that can be thought of as a fake density matrix)

$$\rho^S_A = \frac{\text{Tr}_B[S|\Psi\langle\Psi|]}{\langle S\rangle},$$

where $|\Psi\rangle\langle\Psi|$ is the density matrix of the full system and the denominator is added to ensure the normalization $\text{Tr}_A \rho^S_A = 1$. This fake density matrix could be introduced for any operator, but in the following we need only consider the string $S$. If $O$ is an operator acting on $A$, $\rho^S_A$ satisfies the property $\text{Tr}_A[\rho^S_A O] = \langle OS\rangle / \langle S\rangle$. If the operator $S$ commutes with any operator with support in $A$, then $\rho^S_A$ is Hermitian.

In equation (23) the sums are done only over even and odd numbers of Majorana operators in each block. We need to write these sums in terms of all Majorana fermions. A useful observation is that if we change the sign of all the $a_l$ with $l \in A_1$ (or $l \in A_2$) in the multipoint correlators, then $O_1$ ($O_2$) changes sign in the odd sum, but not in the even one. By taking the appropriate combination of these, we are left with a sum over all possible Majorana fermions and not only over the even/odd ones. The operator that makes this useful change of sign is the Jordan–Wigner string $S_1$ ($S_2$) restricted to the first (second) interval; in fact using also $S_1^{-1} = S_1$, we have

$$S_1 a_l S_1 = \begin{cases} -a_l, & l \in A_1, \\ a_l, & l \notin A_1. \end{cases}$$

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Thus we finally arrive at (1 is the identity matrix)

\[
\rho_{\text{even}} = \frac{\rho_A^1 + S_1 \rho_A^1 S_1}{2}, \quad \rho_{\text{odd}} = \frac{\rho_A^S - S_1 \rho_A^S S_1}{2},
\]

(31)

and so

\[
\rho_{\pm} = \frac{\rho_A^1 + S_1 \rho_A^1 S_1}{2} \pm \frac{\rho_A^S - S_1 \rho_A^S S_1}{2}.
\]

(32)

Notice that \(\rho_A^1\) and \(S_1 \rho_A^1 S_1\) have the same spectrum but different eigenvectors related by the matrix \(S_1\). The same is true for the matrices \(\rho_A^S\) and \(S_1 \rho_A^S S_1\), but they can have eigenvalues smaller than 0 and so in no way can be seen as true density matrices. In appendix B we generalize this form to the case of more disjoint intervals.

Equation (32) is the main result of this section: the rewriting of the spin RDM as a linear combination of four fermionic RDMs, i.e. exponentials of quadratic form as in equation (14). These four matrices do not commute and so they cannot be diagonalized simultaneously to find all eigenvalues of the spin RDM. However, if we are interested in Rényi entropies with integer \(\alpha\), we can handle this problem in a constructive way: we determine the rules for products between RDMs and then we construct recursively any finite order Rényi entropy. Note that the RDM of the free fermions is just given by \(\rho_A^1\) in equation (32). This matrix can be simply diagonalized in the same way as was explained above for the single interval, as already done in [8].

2.1. The product rule

In this subsection we analyze the algebra of RDMs generated by a quadratic form, i.e.

\[
\rho_W = \frac{1}{Z(W)} \exp \left( \sum_{l,n} a_l W_{ln} a_n / 4 \right),
\]

(33)

where we do not assume \(W\) to be Hermitian (to include the non-Hermitian contribution of the string). Anti-commutation relations of Majorana operators always make \(W\) a complex skew-symmetric matrix, i.e. \(W^T = -W\). The constant

\[
Z(W) = \text{Tr} \exp \left( \sum_{l,n} a_l W_{ln} a_n / 4 \right),
\]

(34)

ensures the normalization \(\text{Tr} \rho_W = 1\). In appendix C we show that this normalization for complex diagonalizable skew-symmetric matrices is

\[
Z(W) = \prod \frac{2 \cosh \left( \frac{w}{2} \right)}{\{w\}_{\pm}},
\]

(35)

where \(\{w\}_{\pm}\) is the set of eigenvalues of \(W\) with halved degeneracy (\(W\) is a skew-symmetric matrix, so any even function of \(W\) has eigenvalues with even degeneracy). We assumed \(Z(W) \neq 0\); pathological cases can be cured as explained in appendix D.

The product of fermionic RDMs of the form (33) is

\[
\rho_W \rho_{W'} = \frac{Z(\log(\exp(W) \exp(W'))) \rho_{\log(\exp(W) \exp(W'))}}{Z(W)Z(W')}.
\]

(36)
Indeed the commutator of operators in the exponent of (33),
\[ \sum_{l,n,j,k} W_{ln} W'_{jk} [a_l a_n, a_j a_k] = \frac{\bar{a}^T [W, W'] \bar{a}}{4}, \] (37)
is the essential ingredient in the Baker–Campbell–Hausdorff formula for the product of exponential of operators.

Fermionic RDMs are specified by correlation matrices
\[ \Gamma_{ij} = \text{Tr}[a_i \rho_W a_j] - \delta_{ij}, \] (38)
which can be written as (see appendix C)
\[ \Gamma = \tanh \left( \frac{W}{2} \right) \rightarrow e^W = \frac{1 + \Gamma}{1 - \Gamma}. \] (39)
Clearly the second equation is true only when \( 1 - \Gamma \) is an invertible matrix. At this point, let us briefly summarize the logic of the following derivation. We can easily calculate/manipulate the correlation matrix \( \Gamma \) that via equation (39) gives the exponential factor \( W \) that defines the quadratic density matrix in equation (33). We need to find what the consequences of the product rule of RDMs for the correlation matrices are, i.e. we need to find the correlation matrix corresponding to the product of two RDMs. While, through the chains of equations above, any \( W \) defines a single \( \rho_A \), the opposite is not true and there are several possible \( W \)s for each \( \rho_A \). Nevertheless, we can give a unique recipe for the composition of correlation matrices.

We indicate this matrix operation as \( \Gamma \times \Gamma' \) (notice that it is not the product of the matrices) and it is formally defined by equation (36) as
\[ \rho[\Gamma] \rho[\Gamma'] = \text{Tr}[\rho[\Gamma] \rho[\Gamma']] \rho[\Gamma \times \Gamma']. \] (40)
To specify this operation we still need two ingredients:
(i) a usable expression for the correlation matrix:
\[ (\Gamma \times \Gamma')_{ij} = \frac{Z(W)Z(W')}{Z(\log(\exp(W) \exp(W'))) \text{Tr} a_i \rho_W \rho_W a_j - \delta_{ij}}, \] (41)
associated with the product \( \rho_W \rho_W' \equiv \rho[\Gamma] \rho[\Gamma'] \);
(ii) an expression for the trace of two fermionic RDMs:
\[ \{\Gamma, \Gamma'\} \equiv \text{Tr} \rho[\Gamma] \rho[\Gamma'] = \frac{Z(W)Z(W')}{Z(\log(\exp(W) \exp(W')))}, \] (42)
in terms of the correlation matrices \( \Gamma \) and \( \Gamma' \).

The first requirement is easily obtained if we assume \( 1 - \Gamma \) and \( 1 - \Gamma' \) invertible. Indeed, if we make explicit the exponential products
\[ \frac{1 + \Gamma \times \Gamma'}{1 - \Gamma \times \Gamma'} \times \frac{1 + \Gamma' \times \Gamma}{1 - \Gamma' \times \Gamma}, \] (43)
after simple algebra we obtain
\[ \Gamma \times \Gamma' = 1 - (1 - \Gamma') \frac{1}{1 + \Gamma'} (1 - \Gamma). \] (44)
\(\Gamma \times \Gamma'\) is a skew-symmetric matrix, even if this is not obvious from the above formula. We checked that this relation remains true also if \(1 - \Gamma\) is not invertible (at least for the kinds of matrices that we are interested in), but a complete rigorous proof of equation (44) is beyond the scope of this paper.

The second requirement is less trivial because the correlation matrix \(\Gamma\) does not determine the matrix \(W\) unequivocally, and the sign of \(Z(W)\) remains ambiguous. However, \(\{\Gamma, \Gamma'\}\) is a functional of \(\Gamma\) and \(\Gamma'\), i.e. it is the product of the eigenvalues of \((1 + \Gamma \Gamma')/2\) with halved degeneracy (the spectrum of \(\Gamma'\) is doubly degenerate [25]):

\[
\{\Gamma, \Gamma'\} = \prod_{\mu \in \text{Spectrum}[\Gamma'\Gamma]/2} \frac{1 + \mu}{2} = \pm \sqrt{\det \left| \frac{1 + \Gamma'\Gamma}{2} \right|}.
\]

The unspecified \(\pm\) sign in front is an ambiguity that (as shown in appendix C) can be solved by rewriting the composition rule as

\[
\{\Gamma, \Gamma'\} = \exp \left( \frac{1}{2} \int_{\gamma(0 \rightarrow 1)} d\lambda \frac{\Gamma'\Gamma - 1}{\lambda \Gamma'\Gamma + 1} \right).
\]

This does not depend on the curve \(\gamma\).

It is evident that the operation \(\times\) is associative and so we are in a position to construct any product of fermionic RDMs:

\[
\prod_{i=1}^{n} \rho[\Gamma_i] = \{\Gamma_1, \ldots, \Gamma_n\} \rho[\Gamma_1 \times \cdots \times \Gamma_n],
\]

where

\[
\{\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_n\} \equiv \text{Tr}[\rho[\Gamma_1][\rho[\Gamma_2] \cdots ] = \{\Gamma_1, \Gamma_2\}\{\Gamma_1 \times \Gamma_2, \Gamma_3, \ldots, \Gamma_n\}.
\]

If, for some \(i\) and \(j\), the matrix \((1 + \Gamma_i \Gamma_j)/2\) is not invertible, equation (47) cannot be applied. It is still possible to isolate the pathological parts and we report the details of this procedure in appendix D.

3. Rényi entropies

From equations (25), (32) and (47), the Rényi entropies for integer \(\alpha\) can be written as

\[
S_\alpha = \frac{1}{1 - \alpha} \log \left[ \frac{1}{2^\alpha} \sum_{\zeta_1, \ldots, \zeta_\alpha} c[\zeta_i] \{\Gamma_{\zeta_1}, \ldots, \Gamma_{\zeta_\alpha}\} \right],
\]

where we defined the variables \(\zeta_i = 1, 2, 3, 4\) (that label which of the terms in equation (49) is taken in the particular product) and we defined the short forms for the two-point correlation matrices:

\[
\Gamma_1 = \Gamma_{\rho^1}, \quad \Gamma_2 = \Gamma_{S_1\rho^1 S_1}, \quad \Gamma_3 = \Gamma_{\rho^S}, \quad \Gamma_4 = \Gamma_{S_1\rho^S S_1},
\]

and

\[
c[\zeta] = \begin{cases} 1 \quad \zeta \in \{1, 2\}, \\ \langle S \rangle \quad \zeta = 3, \\ -\langle S \rangle \quad \zeta = 4. \end{cases}
\]

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In the case of $\alpha = 2$ there are no problems with singular terms (see appendix C) and the above expression can be rewritten as

$$S_2 = -\log \left[ \frac{1}{4} \sum_{\zeta_1, \zeta_2} c[\zeta_1] c[\zeta_2] (\pm) \sqrt{\det \left| \frac{1 - \Gamma_{\phi \phi_1}}{2} - \frac{1 - \Gamma_{\phi \phi_2}}{2} + \frac{1 + \Gamma_{\phi \phi_1}}{2} + \frac{1 + \Gamma_{\phi \phi_2}}{2} \right|} \right],$$

where here and in the following equation we leave the sign ambiguity unspecified. Taking into account the trace’s invariance under cyclic permutations, $S_2$ becomes the logarithm of a sum of 10 terms:

$$e^{-S_2} = \frac{1}{4} \sum_{\zeta} c[\zeta]^2 \sqrt{\det \left| \frac{1 + \Gamma^2}{2} \right|}$$

$$+ \frac{1}{2} \sum_{\zeta_1 > \zeta_2} c[\zeta_1] c[\zeta_2] (\pm) \sqrt{\det \left| \frac{1 - \Gamma_{\phi \phi_1}}{2} - \frac{1 - \Gamma_{\phi \phi_2}}{2} + \frac{1 + \Gamma_{\phi \phi_1}}{2} + \frac{1 + \Gamma_{\phi \phi_2}}{2} \right|}.$$  

These formulae are already usable for a direct computation of Rényi entropies. There are however some simplifications that occur on using the property of the correlation matrices $\Gamma$. The first matrix $\Gamma_1$ is the standard fermionic correlation matrix (i.e. the one corresponding to free fermions in the absence of the Jordan–Wigner string, already considered in [8]); $\Gamma_2$ can be obtained from $\Gamma_1$ as $\Gamma_2 = P_1 \Gamma_1 P_1$, with $P_1$ the Hermitian unitary matrix

$$P_1 \equiv \begin{pmatrix} -1_{A_1} & 0 \\ 0 & 1_{A_2} \end{pmatrix},$$

The same relation applies between the third and the fourth matrices $\Gamma_3 = P_1 \Gamma_3 P_1$. However, $\Gamma_3$ is not trivially related to $\Gamma_1$. In appendix A we prove the following identity:

$$\Gamma_3 = \Gamma_1 - \Gamma_{AB} \Gamma^{-1}_{B_i B_i} \Gamma_{B_i A},$$

where the double subscripts take into account restrictions to rectangular correlation matrices, i.e. the first (second) subscript identifies the region where the row (column) index runs. We show in appendix B that similar properties are valid also for an arbitrary number of intervals.

Using these relations, after some algebraic manipulations one can write down the full sums for the Rényi entropies. It is important to note that $\langle S \rangle = \text{Pf}(\Gamma_{B_1})$ and so $\langle S \rangle^2 = \det(\Gamma_{B_1})$. Furthermore to shorten the notation we define

$$\{\ldots, \Gamma^n_1, \ldots\} = \{\ldots, \underbrace{\Gamma_i, \ldots, \Gamma_i}_n, \ldots\}.$$  

Finally $S_2$ can be written in a rather simple way

$$S_2 = -\log \left[ \frac{\Gamma^2_1}{2} + \{\Gamma_1, \Gamma_2\} \right] + (\det \Gamma_{B_1}) \left\{ \frac{\Gamma^2_3}{2} - \{\Gamma_3, \Gamma_4\} \right\}.$$  

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But, increasing the order $\alpha$, the explicit expressions soon become long. For example, we give the (simplified) formulae for $S_3$:

$$S_3 = -\frac{1}{2} \log \left[ \frac{\{\Gamma_1^3\}}{4} + 3\{\Gamma_1^2, \Gamma_2\} + 3(\det \Gamma_B) \{\Gamma_1, \Gamma_3^2\} + \{\Gamma_2, \Gamma_3^2\} + 2\{\Gamma_1, \Gamma_4, \Gamma_3\} \right],$$

(58)

and for $S_4$:

$$S_4 = -\frac{1}{3} \log \left[ \frac{\{\Gamma_1^4\}}{8} + 4\{\Gamma_1^3, \Gamma_2\} + 2\{\Gamma_2^3, \Gamma_2\} + \{\Gamma_1, \Gamma_2, \Gamma_1, \Gamma_2\} \right]$$

$$+ (\det \Gamma_B) \left( \frac{\{\Gamma_1, \Gamma_3, \Gamma_1, \Gamma_3\}}{4} + \frac{\{\Gamma_1, \Gamma_4, \Gamma_1, \Gamma_4\}}{2} + \frac{\{\Gamma_2, \Gamma_3^2\}}{3} \right)$$

$$+ \left( \frac{\{\Gamma_1, \Gamma_3, \Gamma_2, \Gamma_3\}}{2} - \{\Gamma_3, \Gamma_1, \Gamma_3, \Gamma_1\} - \{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \right)$$

$$+ \left( \frac{\{\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_3\}}{2} + \{\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_3\} - \{\Gamma_3, \Gamma_4, \Gamma_3, \Gamma_4\} \right)$$

$$+ (\det \Gamma_B)^2 \left( \frac{\{\Gamma_1^4\}}{8} - 4\{\Gamma_3^3, \Gamma_4\} + 2\{\Gamma_3^2, \Gamma_2\} + \{\Gamma_3, \Gamma_4, \Gamma_3, \Gamma_4\} \right).$$

(59)

4. The critical $XX$ model

In this section we report the explicit results for the $XX$ chain in zero magnetic field (i.e. $\gamma = 0$ and $h = 0$ in equation (9)). $XX$ chains have been previously analyzed by Furukawa et al [11] by means of exact diagonalization techniques that allow one to explore relatively small chains with at most 30 spins. The asymptotic results from CFT (cf equation (5)) were obscured in this previous analysis by large oscillating corrections to the scaling. The smallness of the systems and the lack of precise knowledge of the form of the correction to the scaling made impossible any finite size scaling analysis to check the CFT predictions (5). In fact these oscillating corrections to the scaling have been widely observed in the last few years [26]–[28], but a precise theory of their origin and their exact form is only recently available [24, 29].

By exploiting the exact solution, we can avoid these problems and explore large enough values of $\ell$ allowing a finite $\ell$ scaling analysis similar to that for a single block [24]. We start from the infinite volume limit. We consider two intervals both of length $\ell = \ell_1 = \ell_2$ at distance $r$. In this case, the four-point ratio $x$ in equation (4) is given by

$$x = \left( \frac{\ell}{\ell + r} \right)^2,$$

(60)

and the scaling of the quantity determining the Rényi entropies is (we use $c = 1$)

$$\text{Tr} \rho_A^\alpha = c_\alpha^2 \left( \frac{\ell + r}{\ell^2 r (2\ell + r)} \right)^{(n-1)/6} F_{\alpha}^{\text{lat}}(x, \ell),$$

(61)

which is an implicit definition of $F_{\alpha}^{\text{lat}}(x, \ell)$ encoding the fact that for finite $\ell$ we expect corrections to the scaling of the form

$$F_{\alpha}^{\text{lat}}(x, \ell) = F_{\alpha}^{\text{CFT}}(x) + \ell^{-\delta_\alpha} f_{\alpha}(x) + \cdots,$$

(62)

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where $F_\alpha^{\text{CFT}}(x)$ is the universal quantity appearing in equation (6). The exponent $\delta_\alpha$ governs the leading correction to the scaling. It has been shown with CFT in [29] that this exponent is equal to $2x/\alpha$ independently of the number of intervals and $x$ is the scaling dimension of an operator introduced by the conical singularity necessary for describing the reduced density matrix. For a single interval, it has been shown by exact analytic calculation that $\delta_\alpha = 2/\alpha$ [24] and thus we expect the same exponent for the double-interval case (the presence of a relevant operator with dimension $x = 1$ has been explained in [29]). In the following analysis we will greatly benefit from the exact knowledge of the quantity $c_\alpha$ derived from the exact solution of the single-interval entanglement [23,30] and so our results will also be a non-trivial check of the equivalence of the two non-universal multiplicative constants in the single- and double-interval cases.

We report in figure 2 results for the function $F_{\text{lat}}^{2,3,4}(x, \ell)$ for various values of $\ell$ and $x$. It is evident that irrespective of the value of $x$, with increasing $\ell$ the results approach the CFT prediction. For odd $\ell$ the asymptotic result is approached from below, while for even ones it is approached from above. These are the already mentioned oscillations that made difficult the analysis based on small chains. These plots are the first direct quantitative tests of the correctness of equation (6). These figures leave no doubt about the correctness of the results of [12]. It is worth mentioning that, as already shown elsewhere [11]–[13], the finite $\ell$ curves do not have the symmetry $x \rightarrow 1-x$, which is restored only in the $\ell \rightarrow \infty$ limit equation (6).

Having established the correctness of the asymptotic form, we can now move to the finite $\ell$ corrections and check whether the prediction $\delta_\alpha = 2/\alpha$ [29] is correct. Achieving precise control of the corrections to the scaling is not an academic task: their analysis is fundamental to providing accurate results when such large system sizes are not available (in previous studies they would have been an important tool) and in cases where the asymptotic form is not known (as in section 5 for the Ising model for $\alpha > 2$). In figure 3 we report the function $|F_\alpha(x) - F_{\text{lat}}^{\alpha}(x)|$ for fixed $x = 1/9$ and $\alpha = 2, 3, 4$. We report results both for the spin Rényi entropies and for the fermionic ones (i.e. without considering the string contribution). We recall that for free fermions we have $F_\alpha(x) = 1$ identically. The results show a power law behavior for large enough $\ell$ with the predicted exponent $2/\alpha$, as for the single interval. Notice that on increasing $\alpha$, the values $\ell$ where the leading asymptotic correction can be identified become larger and larger, in analogy with the single-block case [24] (as is obvious because of the smallness of the exponent $\delta_\alpha$). For the fermionic variables the asymptotic behavior is reached before that for spin degrees of freedom: the string introduces further corrections to the scaling that in the present model are subleading. To show the $x$ independence of this exponent in the last panel of figure 3, we report the same kinds of plots for different values of $x$, showing that, at fixed $\alpha$, the corrections lie on parallel lines.

To conclude this section we present some results for finite systems. It is known that all the formulae above (including the $x$ dependence) at the leading order in finite systems can be described by replacing any distance $u_{ij}$ by the chord length $u_{ij} \rightarrow (L/\pi) \sin((\pi u_{ij})/L)$. We checked that this rescaling is indeed correct and that, for large enough $\ell$ and $L$, the results agree with CFT. These plots give no further information as compared to the ones already presented and we do not report them. We only show the results for a rather small system of length $L = 39$ (that nevertheless is above anything obtainable by exact diagonalization). In figure 4 we report the resulting $F_2^{\text{lat}}$ for all the possible divisions...
in four parts of this chain of length $L = 39$. For such a small chain, the results are obviously very unclear since the corrections to the scaling are obscuring the CFT scaling represented by a continuous line that is surrounded by the points, signaling the oscillatory nature of the corrections (for clarity, compare with the analogous plot for the Ising model in section 5).

Figure 2. Scaling function $F_{\alpha}^\text{lat}(x)$ for $\alpha = 2, 3, 4$ (from top to bottom) for the XX model in the thermodynamic limit and for various $\ell = \ell_1 = \ell_2$ and $r$. Corrections to the scaling show even–odd oscillations with $\ell$ as in the single-interval case. The results converge quickly to the universal CFT prediction $F_{\alpha}(x)$.
Figure 3. Corrections to the scaling for $F^{\text{lat}}(x)$ obtained by subtracting the asymptotic value. The first three panels show that for $\alpha = 2, 3, 4$ at fixed $x = 1/9$, both for fermionic and spin variable corrections to the scaling have exponents $\delta_\alpha = 2/\alpha$, as for one interval. For fermionic degrees of freedom the asymptotic behavior is obtained for smaller values of $\ell$. Last panel: $x$ independence of the corrections to the scaling exponents. All curves with different $x$ at fixed $\alpha$ lie on parallel lines.

5. The critical Ising universality class

The Ising model is given by Hamiltonian (9) with $\gamma = 1$ and it is critical for $h = 1$. Results for this model have been already derived numerically for $\alpha = 2$ in [13] by using a tree tensor network algorithm and Monte Carlo simulations of the two-dimensional classical problem in the same universality class. These calculations allowed a precise determination of $F_2(x)$, but the system sizes explored were not enough for analyzing Rényi entropies with larger values of $\alpha$. In the course of our analysis we always compared our data for small systems with those in [13], in order to check the correctness of both methods.

We report our results obtained in the thermodynamic limit for various values of $\ell = \ell_1 = \ell_2$ at different separations $r$ (resulting in the four-point ratio $x$ given in equation (60)) in figure 5. As already noticed in [13], oppositely to the $XX$ chain case, in the Ising model we have a monotonic finite $\ell$ correction to the scaling. For finite $\ell$, the results do not show the symmetry $x \rightarrow 1 - x$ valid for infinite $\ell$. This is restored
only for the extrapolated data at $\ell \to \infty$. To perform this extrapolation in the most accurate way, we have first to determine the correction to the scaling exponent. For a single interval, it is exactly known that the leading corrections are characterized by the exponent $\delta_\alpha = 2/\alpha$ as for the $XX$ chain (the equality [24] arises due to the equivalence of the two RDMs [31]). However for $\alpha = 2$, it has already been shown [13] that $\delta_2 = 1/2$, different from the single-interval one. This result is quite surprising also in view of the recent CFT analysis [29] predicting the same behavior for any number of intervals. Thus as a first step we check the corrections to the scaling exponent for the Ising model in the absence of the Jordan–Wigner string between the two blocks (i.e. we consider only the correlation matrix $\Gamma_1$, as was done in [8]). In this case, the results reported in figure 6 give compelling evidence that the leading corrections to the scaling are given by $\delta_\alpha = 2/\alpha$ as for the single interval (data not reported here for $\alpha = 3, 4$ show that this remains true).

At this point it is natural that the Jordan–Wigner string produces another operator at the conical singularity, which in the Ising model is the leading one. According to [29] all the corrections to the scaling should be of the form $\delta_\alpha = 2x/\alpha$, thus taking the result $\delta_2 = 1/2$ for granted (see also below for further evidence), we conclude that the Jordan–Wigner string introduces an operator with scaling dimension $x = 1/2$. Such an operator in the continuum limit of the Ising model exists and it is the Majorana fermion, which has exactly the same features as the Jordan–Wigner string (i.e. same symmetry and same non-local character). Such an operator is clearly not present in the single-interval case. These considerations allow us to conclude that the leading corrections to the scaling for the double-interval entanglement in the Ising model are described by the exponent

$$\delta_\alpha = \frac{1}{\alpha}.$$  

Unfortunately already for $\alpha = 3$, the value of $\delta_3 = 1/3$ is very low, and subleading corrections to the scaling going with exponents $m\delta_\alpha$ (with $m$ integer) are expected to influence the results considerably. For this reason, in order to have an accurate determination of the asymptotic behavior, at fixed $x$ we consider all corrections to the

Figure 4. The function $F^{\text{lat}}_2(x)$ for a small chain with $L = 39$ spins. Oscillating corrections to the scaling prevent us from seeing the universal CFT prediction shown as a continuous curve.
Figure 5. Scaling function $F_{\alpha}^{\text{lat}}(x)$ for $\alpha = 2, 3, 4$ (from top to bottom) for the Ising model in the thermodynamic limit and for various $\ell$. Corrections to the scaling are monotonic. The top curve in each plot is the extrapolation to $\ell \to \infty$. The convergence to the universal CFT prediction $F_{\alpha}(x)$ is slower than in the $XX$ case, because the leading exponent of corrections to the scaling is $1/\alpha$. 

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Figure 6. $F_2^{\text{Ferm}}(x)$ for different $x$ and $\gamma$. All results present the same leading correction to the scaling exponent $\delta_2 = 1$. The extrapolated data at $\ell \to \infty$ collapse to the single point $F_2(x) = 1$.

scaling up to those with exponent 1. The resulting extrapolated data are the top points in figure 5. In the case of $\alpha = 2$, the CFT prediction $F_2(x)$ (cf equation (8)) agrees perfectly with the extrapolated data, giving strong support both for the procedure for accounting for the subleading corrections terms and for the asymptotic form. For $\alpha \neq 2$ no CFT prediction is yet available. The extrapolated data in figure 5 are the first data for infinite $\ell$.

We now turn to considering the issue of universality. All the critical models ($h = 1$) for any value of $\gamma \neq 0$ are in the Ising universality class. However, the results at finite $\ell$ show a strong dependence on $\gamma$ (as is obvious from the different correlation matrices). In figure 7 we report several data for $F_2^{\text{lat}}(x)$ for different values of $\gamma$ and $\ell$ at fixed $x = 1/4$ and 3/4 (that for the symmetry $x \to 1 - x$ have the same asymptotic value). At finite $\ell$, all results are evidently different. In the figure we report the extrapolation with two corrections to the scaling (i.e. with $\delta_2 = 1/2$ and $2\delta_2$). For $\ell \to \infty$ all data tend to the same value predicted from the CFT $F_2(1/4) = (2 + \sqrt{2}(1 + 3^{1/4}))/4$, confirming in a single plot many results: (i) universality with respect to $\gamma$, (ii) correctness of the correction to the scaling form, (iii) correctness of the CFT prediction equation (8).

To conclude this section we report the data for a finite chain. As usual, in all the scaling variables we substitute distances with the chord distances. In figure 8, we report the values of $F_2^{\text{lat}}(x)$ for all the possible choices of intervals $A_{1,2}$ and $B_{1,2}$ in a chain of $L = 39$ spins. Compared to the analogous plot for the $XX$ chain (figure 4), the figure is much clearer, due to the fact that corrections to the scaling are monotonic. Notice however that the data points lie far below the asymptotic value because the exponent $\delta_2 = 1/2$ is small (also compared to the $XX$ case with $\delta_2 = 1$), resulting in very large corrections to the scaling.

6. Non-critical models

The richness of the phase diagram of the $XY$ model allows us also to explore the gapped phases that are almost everywhere except on the line $|h| = 1$ and the segment $\gamma = 0$.
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Figure 7. Universality with respect to $\gamma$. We plot $F^\text{lat}_2(x)$ for $x = 1/4$ and $3/4$ for different values of $\gamma$ in the Hamiltonian. All results present the same leading correction to the scaling exponent $\delta_2 = 1/2$. The extrapolated data at $\ell \to \infty$ collapse to a single point equal to $F_2(1/4) = F_2(3/4) = 1.31886 \cdots$.

Figure 8. The function $F^\text{lat}_2(x)$ for a small chain with $L = 39$ spins. Corrections to the scaling are monotonic and very large compared to the $XX$ case ones: the data lie far below the asymptotic value predicted from the CFT.

with $h^2 < 1$. These systems have not been considered so much in the literature, until now, because for non-critical models all the correlations and entanglement between the two blocks fall off exponentially (with a decay rate given by the inverse gap or mass). Thus one would always expect

$$S_\alpha(\ell, r, \ell') = S_\alpha(\ell) + S_\alpha(\ell') + O(e^{-r/\Delta}).$$

(64)

However, this is not so obvious because of the importance of the connected part in the correlations. In figure 9 (left) we report $S_2$ for the double-interval case. While for $h > 1$, the spin entropy is the same as double the single interval and the same as the fermionic one, for $h < 1$ there is clearly an offset (that we quantify in $-\log 2$; see below). The best

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way of detecting these unexpected effects is to consider the mutual entropy
\[
\Delta S_\alpha(\ell, r, \ell', r') = S_\alpha(\ell) + S_\alpha(\ell') - S_\alpha(\ell, r, \ell', r'),
\]
that automatically gives zero when factorization occurs.

From numerical data we deduce that, for large blocks, all terms
\[
\prod_{i=1}^\alpha [t_i]\{\Gamma_{\zeta_i}, \ldots, \Gamma_{\zeta_\alpha}\},
\]
in equation (49) have the same absolute value. However, they can have different signs. Furthermore numerical evidence suggests that terms consisting only of the correlation matrices \(\Gamma_1\) and \(\Gamma_2\) always have positive sign. If these observations are generally true, we have
\[
\Delta S_\alpha \sim \frac{1}{\alpha-1} \log \left[ 1 + \frac{1}{2^\alpha} \sum_{\{\zeta\} \text{ s.t. } \#_{3} \neq \#_{4} \neq 0} (-1)^\#_{4} \epsilon[\{\zeta\}] \right],
\]
where \(\epsilon[\{\zeta\}]\) is the sign associated with the element \(\{\Gamma_{\zeta_1}, \ldots, \Gamma_{\zeta_\alpha}\}\) and \(\#_{3(4)}\) is the number of correlation matrices \(\Gamma_{3(4)}\). We expect an eventual discontinuity in \(\Delta S_\alpha\) when crossing a critical line. Thus we study non-critical chains with magnetic field close to \(h = 1\). We found numerically that only terms with an odd number of correlation matrices \(\Gamma_{3(4)}\) display sign changes. Thus (see the right panel of figure 9 for the explicit plot) we conclude from the numerical evidence the behavior
\[
\Delta S_\alpha = \begin{cases} 
\log 2 & \text{if } h^2 < 1, \\
0 & \text{otherwise.}
\end{cases}
\]
This result is not completely unexpected: also $S_A$ for the single interval for $|h| = 0$ tends to $\log 2$, independently of $\ell$ \cite{5,23,30}. Thus in the definition of $\Delta S_\alpha$, since the double interval $S_\alpha$ tends to the same value, we are left with a single $\log 2$.\footnote{This is a consequence of the double degeneracy of the ground state for $|h| < 1$. It is easily understood at $h = 0$, where the ground state is any linear combination of the states all up and all down that we can denote with $|\uparrow\rangle$ and $|\downarrow\rangle$. Since there is no symmetry breaking term in the Hamiltonian, the diagonalization selects a state with zero magnetization, i.e. $(|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$, in which the entanglement of any subsystem, connected or not, is always $\log 2$ as stated in the main text.}

We stress that at $h = 1$ the various $\Delta_\alpha$ cross at a single point if they are characterized by the same four-point ratio $x$ (see the right panel of figure 9) which is the CFT prediction. These kinds of plots could be used to detect the phase transition points in systems where they are not exactly known.

The $\alpha$ independence of the previous expression allows us to analytically continue the result to $\alpha = 1$ and to conjecture the same behavior for the asymptotic von Neumann entanglement entropy in non-critical regions:

$$S_1 = S^\text{ferm}_1 - \theta(1 - h^2) \ln 2 + \cdots,$$

where $S^\text{ferm}_1$ is the fermionic entanglement entropy obtained from the correlation matrix $\Gamma_1$. Notice that this does not exclude the possibility that exponential corrections to this asymptotic form in the fermionic and spin variables could have different amplitudes. It would be interesting to explore this issue with the continuum theory in the form factor approach \cite{32}.

7. Entanglement evolution following a quench

Finally we consider the evolution of Rényi entropies after global quenches. In this problem, the system is prepared in the ground state $|\psi_0\rangle$ of an Hamiltonian $H_0$ of the form (9) and then it is allowed to evolve from time $t = 0$ with a different Hamiltonian $H$, always of the form (9) but with $(h, \gamma) \neq (h_0, \gamma_0)$. The main feature of this non-equilibrium problem is that the initial state differs globally from the ground state and the excess of energy (compared to the ground state of $H$) is extensive. A connected spin block reacts to the quench, increasing the Rényi entropies linearly in time up to each spin in the subsystem becoming entangled with the environment. Then entropies saturate. There is no reason why disjoint blocks should behave in a different way, except for the necessity of subtracting the mutual entropy. Our only goal here is to understand the differences between the spin representation and the fermionic one, postponing any more accurate analysis to further studies. We only report results for the Rényi entropy $S_2$, the value of $\alpha$ being unimportant. We compare the numerical data with the prediction that follows from the interpretation of the entanglement evolution in terms of the motion of quasiparticles \cite{33}. According to this physical scenario, because the excess of energy of the system is extensive, any site acts as a source of quasiparticle excitations. Particles emitted from different points (further apart than the correlation length in the initial state) are incoherent, but pairs of particles moving to the left or right from a given point are entangled. Thus $S_\alpha(t)$ should just be proportional to the number of coherent pairs of particles that are emitted from any point with one reaching a point in $A$ and the other a point in $B$. Since there is a maximum speed for these excitations $v_M$, for a single interval this implies linear growth for $2v_M t < \ell$.

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and saturation for very large times. For a general bipartition the result is [33]
\[
S_\alpha(t) \approx \int_{x' \in A} dx' \int_{x'' \in B} dx'' \int_{-\infty}^{\infty} dx \int H_\alpha(p) dp \delta(x' - x - v(p)t)\delta(x'' - x - v(p)t),
\]
(70)
where we assumed momentum conservation. For a double interval, this formula predicts a series of linear behaviors with different slopes, that finally saturates at late time [33]. The function $H_\alpha$ does not depend on the subsystem length and topology. It has been exactly derived for a single interval [34]:
\[
H_\alpha(\cos \Delta) = \frac{1}{1 - \alpha} \log \left[ \left( \frac{1 + \cos \Delta}{2} \right)^\alpha + \left( \frac{1 - \cos \Delta}{2} \right)^\alpha \right],
\]
(71)
where $\Delta(p)$ is the difference between the Bogoliubov angles before and after the quench, whose explicit expression in terms of the dispersion relation $\varepsilon(\varphi)$ is reported in [34]. Thus, plugging everything together, the leading order of Rényi entropies is
\[
S_{\ell_1, r, \ell_2}(t) \sim S_{\ell_1}(t) + S_{\ell_2}(t) - \Delta S[\ell_1, r, \ell_2](t),
\]
(72)
with $(\chi_{[a,b]}(x) \equiv \theta(x - a)\theta(b - x))$
\[
\Delta S(t) = \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} H_\alpha(\cos \Delta) \int_r^{2|\varepsilon'|t} dv (\chi_{[r,r+\min(\ell_1, \ell_2)]}(v) - \chi_{[r+\max(\ell_1, \ell_2), r+\ell_1+\ell_2]}(v))
\]
and [34]
\[
S_{\ell}(t) = t \int_{2|\varepsilon'|t < \ell} \frac{d\varphi}{2\pi} 2|\varepsilon'| H_\alpha(\cos \Delta) + \ell \int_{2|\varepsilon'|t > \ell} \frac{d\varphi}{2\pi} H_\alpha(\cos \Delta).
\]
(73)

Let us now go back to the direct computation of the entanglement in two blocks after a quench; we observe that not all terms contribute to the leading order in block lengths and distances: the string expectation value decays exponentially and only terms constructed with $\Gamma_1$ and $\Gamma_2$ survive. For instance, the Rényi entropy $S_2$ is simply the logarithm of two terms:
\[
S_2 \sim -\log \left[ \frac{1}{2} \exp \left( \frac{1}{2} \text{Tr} \log \frac{1 + \Gamma_1^2}{2} \right) + \frac{1}{2} \exp \left( \frac{1}{2} \text{Tr} \log \frac{1 + \Gamma_2^2}{2} \right) \right]
\]
(74)
\[
\sim \begin{cases} 
-\frac{1}{2} \text{Tr} \log \frac{1 + \Gamma_1^2}{2} + \log 2 + c & \text{if } \text{Tr} \log \frac{1 + \Gamma_1\Gamma_2}{2} < \text{Tr} \log \frac{1 + \Gamma_2^2}{2} \\
-\frac{1}{2} \text{Tr} \log \frac{1 + \Gamma_2^2}{2} + c & \text{if } \text{Tr} \log \frac{1 + \Gamma_1\Gamma_2}{2} \sim \text{Tr} \log \frac{1 + \Gamma_2^2}{2} \\
-\frac{1}{2} \text{Tr} \log \frac{1 + \Gamma_1\Gamma_2}{2} + c + \log 2 & \text{otherwise}.
\end{cases}
\]
(75)

Analyzing several data (we report only a single example in figure 10), we deduce that in the scaling limit the term constructed with the standard fermionic correlations is never negligible: finite order Rényi entropies are controlled by fermionic correlations. When other terms are comparable with $\{\Gamma_1, \ldots, \Gamma_1\}$, they give additive $O(1)$ contributions of the form $\sim \log k$, where $k$ is the coefficient in front of the factor $\{\cdot, \cdot\}$, and $k = 2$ for $S_2$.

The quasiparticle interpretation [33] perfectly agrees (up to a constant) with the fermionic representation $\rho_\Gamma$ (see figure 10). In other words the non-locality of the Jordan–Wigner transformation does not influence (as one expects) the leading order of the time

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![Figure 10](image)

**Figure 10.** Evolution of the Rényi entropy $S_2$ after a global quench from a non-critical system to a critical Ising one. The asymptotic prediction is given by equation (72) while the fermionic entanglement is obtained neglecting the string contribution. When $\{\Gamma_1, \Gamma_2\}$ becomes negligible with respect to $\{\Gamma_1^2\}$, a jump of about $\sim \log 2$ is seen. In the graph this happens in the neighborhood of the time $t \sim 11$. The asymptotic prediction presents a small offset due to the finite entanglement in the initial state.

The evolution of the entropies after a global quench. However, as in the non-critical equilibrium case, we observe an additive $\log 2$ contribution (in figure 10 a stick of width $\log 2$ is included to help us appreciate this difference) from the string and we believe that this is a general feature that will persist also away from criticality. As a final comment, we mention that the $O(1)$ offset of the numerical data at finite $\ell$ compared with the asymptotic form is a consequence of the initial entanglement. This subleading term is present also for the single-interval case [34], and can be included in the asymptotic result [35].

8. Discussion

We presented a general method for calculating Rényi entropies for integer $\alpha$ in the ground state of the $XY$ Hamiltonian (9) in the case where $A$ consists of two (but the method works for even more) disjoint intervals. We carefully analyzed the results in the critical cases of $XX$ and Ising universality classes. We found that the asymptotic results for large intervals and separations are described correctly by conformal field theory when results are known. We also provided results for the asymptotic scaling of the $\alpha = 3, 4$ Rényi entropies of the Ising model, for which no analytic prediction is yet available. In order to check CFT predictions we had to carefully include corrections to the scaling in the analysis. We found that for the $XX$ model, the leading corrections to the scaling are governed by the exponent $\delta_\alpha = 2/\alpha$, as for the single-interval case [24], in agreement with general predictions [29]. Oppositely, for the critical Ising model we found $\delta_\alpha = 1/\alpha$, which is half of the single-interval value. This can be understood in terms of the general theory of the corrections to the scaling [29] by interpreting the Jordan–Wigner string in the lattice model as a non-local fermion in the Ising CFT. We have also briefly considered the
entanglement in the gapped phase and the time evolution following a quantum quench. The exact representation of the reduced density matrix of spin variables provided could also be useful for the calculation of the entanglement between $A_1$ and $A_2$ measured via the negativity [15].

A major problem remains open. We are unable to find all the eigenvalues of the reduced density matrix for two intervals and so also the Rényi entropies for non-integer $\alpha$ and in particular for $\alpha = 1$ that would give the widely studied von Neumann entropy. The same problem is also present in conformal calculations. It is possible to obtain $\mathrm{Tr} \rho_A^\alpha$ only for $\alpha$ integer [12] and the analytic continuation of the result to general complex values remains a major open problem.

The contribution of the Jordan–Wigner string also affects the entanglement entropy of a single interval in a system with one or more boundaries if the block $A$ does not include the boundary. In this case, the correlation matrix is slightly more complicated and we are currently analyzing the problem. Among the other things, this would be relevant also for some quench problems where indeed only the fermionic entanglement has been calculated [36].

Another interesting open question concerns the behavior of the Rényi entropies in systems with disorder for which in the single-block case it is known that the entanglement simply follows from the counting of the singlets [37] between $A$ and $B$. Whether this remains true in the case of more intervals is not yet known.

Appendix A. Correlation matrices

We report in this appendix some simple properties of the correlation matrices:

\begin{equation}
\Gamma^S_{ij} = \mathrm{Tr}[a_i \rho^S a_j] - \delta_{ij}, \quad i, j \in A,
\end{equation}

with $\rho_S$ defined in equation (29). The elements of $\Gamma^S$ are ratios of two expectation values ($n \neq 0$):

\begin{equation}
\Gamma^S_{ll+n} = \frac{\langle a_{l+n} \prod_{j \in B_1} (a_{2j-1} a_{2j}) a_l \rangle}{\langle \prod_{j \in B_1} (a_{2j-1} a_{2j}) \rangle},
\end{equation}

which can be evaluated by means of the Wick theorem. We show that $\Gamma^S$ can be expressed in terms of the system correlation matrix $\Gamma \equiv \Gamma^1$, where $\mathbf{1}$ is the identity matrix. By isolating the two-point correlations between $a_l$ and the other Majorana operators in the numerator of (A.2) we obtain

\begin{equation}
\langle a_{l+n} \prod_{j \in B_1} (a_{2j-1} a_{2j}) a_l \rangle = \langle S \rangle \Gamma_{ll+n} + [\Lambda \Gamma]_{ll+n},
\end{equation}

with

\begin{equation}
\Lambda_{ij} = (-1)^{i-j} \operatorname{sgn}(j-i) \left( \prod_{k \in B_{ij}^*} a_k \right),
\end{equation}

where $B_{ij}^*$ denotes the set of all indices $\{2l\} \cup \{2l-1\}$ with $l \in B_1$ and with $i$ and $j$ removed. The expectation value in (A.4) is the Pfaffian of the system correlation matrix

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restricted to the region $B^\circ_{ij}$. Checking the Pfaffian sign we have

$$\Lambda_{ij} = -\langle S \rangle [\Gamma_{B_i}^{-1}]_{ij},$$

(A.5)

with $\Gamma_{B_i}$ the correlation matrix restricted to the region $B_i$. Substituting this expression into (A.2) we get

$$\Gamma^S = \Gamma_A - \Gamma_{AB_i} \Gamma_{B_i A} = \Gamma_{A \cup B_i} / \Gamma_{B_i},$$ (A.6)

where the double subscript takes into account restrictions to rectangular correlation matrices, i.e. the first (second) subscript identifies the region where the row (column) index runs. Notice that $\Gamma^S$ is the Schur complement of $\Gamma_{B_i}$ in $\Gamma_{A \cup B_i}$. This proves equation (55).

Appendix B. Generalization to many disjoint spin blocks

We consider in this appendix how to generalize our approach to the case where the subsystem consists of $n$ disjoint blocks of lengths $\ell_i$, with $i = 1, \ldots, n$. We indicate with $O^+_{\ell_i}$ a product of an even number of Majorana operators lying in the $i$th block $A_i$ and with $O^-_{\ell_i}$ an odd product. $S^B_i$ is the complete $\sigma_z$ string associated with the $i$th region between consecutive blocks $B_i$ and $S^A_i$ is the string corresponding to the block $A_i$ (see figure 1 for the double-block case). Equation (20) can be easily generalized by observing that the reduced density matrix has the following expansion:

$$\rho_{\cup_i A_i} = \frac{1}{2^{\sum_i \ell_i}} \sum_{\{s\}} \langle S[\{s\}] | O^+_{\ell_1} \cdots O^+_{\ell_n} (O^-_{\ell_1} \cdots O^-_{\ell_n})^\dagger S[\{s\}] \rangle,$$ (B.1)

where $s_i = \pm 1$ and $S[\{s\}]$ are $\sigma_z$ strings determined by the conditions

$$S[\{s_1, \ldots, s_k, \ldots, s_n\}] = S[\{s_1, \ldots, -s_k, \ldots, s_n\}] \prod_{i=1}^{k-1} S^B_i, \quad S[\{0, \ldots, 0\}] = 1.$$ (B.2)

Indeed, any spin operator that can be written as a product of Majorana operators that are odd in number if restricted to the $k$th block is constructed with an odd number of $\sigma^x(y)$ of the $k$th block. Spin variables $\sigma^x$ and $\sigma^y$ are non-local in the fermionic space:

$$\sigma^x_{l \in A_k} = \Lambda(A_1, \ldots, A_{k-1}) \prod_{i=1}^{k-1} S_i a_{2l(2l-1)},$$ (B.3)

where $\Lambda(A_1, \ldots, A_{k-1})$ is an operator that depends on the Majorana fermions of the blocks $A_1, \ldots, A_{k-1}$. When multiplying an odd number of $\sigma^x(y)$, the string $\prod_{i=1}^{k-1} S_i$ does not cancel (we recall that $S_2^2 = 1$) so we obtain the conditions (B.2). The construction of the fermionic equivalent reduced density matrix is analogous to the double-block case. In fact the unitary transformation

$$U = \sum_{\{s\}_n} \prod_{i=1}^{n} \Pi_{s_i} \quad s_i \in \{+,-\},$$

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with
\[ \Pi^+_i = \frac{1 + S^A_i}{2}, \quad \Pi^-_i = \prod_{j=1}^{i-1} S^B_j \frac{1 - S^A_i}{2}, \] (B.4)

brings the spin reduced density matrix to a fermionic form. In particular we obtain the simple result
\[ \rho_{\cup_{s} A_i} \sim \frac{1}{2N} \sum_{\{s\}} \sum_{O_1^{s_1} \cdots O_n^{s_n}} \langle S[\{s\}] O_1^{s_1} \cdots O_n^{s_n} \rangle (O_1^{s_1} \cdots O_n^{s_n})^\dagger. \] (B.5)

The term associated with the configuration \( \{ s \} \):
\[ \rho^{s[\{s\}]}_{\{s\}} = \frac{1}{2N} \sum_{s_1} \sum_{O_1^{s_1} \cdots O_n^{s_n}} \langle S[\{s\}] O_1^{s_1} \cdots O_n^{s_n} \rangle (O_1^{s_1} \cdots O_n^{s_n})^\dagger \] (B.6)
is a sum of \( 2^{n-1} \) fermionic RDMs. This is because, for fixed \( S[\{s\}] \), the further dependence on \( s_1 \) can be handled in the following way (recall that \( S_k A_k S_k^A = s O_k^A \)):
\[ \rho^{s[\{s\}]}_{\{s\}} \left( S_{\{s_1 \cdots s_n \}} \right) = \frac{[\rho^{s[\{s\}]}_{\{s_1 \cdots s_n \}} + \rho^{s[\{s\}]}_{\{s_{\cdots s_n} \}}]}{2} + \frac{S^A_k \rho^{s[\{s\}]}_{\{s_1 \cdots s_n \}} + \rho^{s[\{s\}]}_{\{s_{\cdots s_n} \}} \rho^{s[\{s\}]}_{\{s_{\cdots s_n} \}} S^A_k}{2}, \] (B.7)
and, for any string \( S^B \) and \( S^A \), the operator
\[ S^A \sum_{\{s\}} \rho^{s[\{s\}]}_{\{s\}} S^A, \] (B.8)
is a fermionic RDM. We obtain \( 2^{n-1} \) and not \( 2^n \) fermionic RDMs only because \( S[\{s\}] \) does not depend on \( s_1 \) (there is no string contribution for operators lying in the first block). Thus we can write
\[ \rho_{\cup_{s} A_i} \sim \frac{1}{2^{n-1}} \sum_{\{i\} \text{ w/o repetition}} \text{Pf}[\Gamma_{\cup_{i} (\{i\} B_i)}] \sum_{\{s\}} \epsilon_{\{i\}}^{\{s\}} \rho^{\Gamma[\{s\}]}_{\{i\}} \quad s_i \in \{-1, 1\}. \] (B.9)

The explicit form of the correlation matrices \( \Gamma^{[\sigma]}_{\{i\}} \) is a simple generalization of the double-block case:
\[ \Gamma^{[\sigma]}_{\{i\}} = P[s_1, \ldots, s_{n-1}] [A_{\cup_{i} (\{i\} B_i)}] \Gamma_{\cup_{i} (\{i\} B_i)} P[s_1, \ldots, s_{n-1}], \] (B.10)
where we indicate with \( A/B \) the Schur complement of \( B \) in \( A \) (see appendix A). \( \text{Pf} \) denotes the Pfaffian, and \( P \) is the diagonal matrix
\[ P[s_1, \ldots, s_{n-1}] = \begin{pmatrix} s_1 A_1 & s_2 A_2 & \cdots & s_{n-1} A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 A_n \end{pmatrix}. \] (B.11)

The index \( i \in \{0, n - 1\} \) runs over the environment blocks lying between the first and the last subsystem block, and \( B_0 \equiv \emptyset \) so \( \Gamma^{[\{0\}] = \text{standard fermionic correlation matrix} \)
\( \Gamma_A \equiv \Gamma_{\bigcup A_i} \) . The sign of \( \epsilon \) is determined by the conditions
\[
\epsilon^{\{s\}}_{\{i\}} = \begin{cases} 
1 & \{i\} = \{0\} \vee \{s\} = \{1, \ldots, 1\}, \\
(-1)^{j \in \{i\}} \epsilon^{\{s\}}_{\{i\}} & \text{otherwise,}
\end{cases}
\]
where \((-1)^{j \in \{i\}}\) is equal to \(-1\) if \( j \in \{i\} \) and 1 otherwise, and
\[
\{s\}^{s_j \rightarrow -s_j} = \{s_1, \ldots, s_{j-1}, -s_j, s_{j+1}, \ldots, s_n\}.
\]
Rényi entropies can be computed in the same way as when the subsystem consists of just two disjoint blocks. But many more terms contribute. And when the number of blocks is comparable with the chain size we expect an extensive behavior of Rényi entropies, as observed in [18], that is reminiscent of the huge number of fermionic RDMs needed to represent the spin RDM. Finally, as in the double-block case, Rényi entropies do not depend on the sign of the Pfaffians because any sign change is the result of a unitary transformation (that is just a \( \sigma_z \) string).

Appendix C. Some technical details for the product rules

The reduced density matrix of a given number of disjoint spin blocks in a free chain is unitarily equivalent to a finite sum of operators that behave well under multiplication. In fact one of our results is that, for some \( \sigma_z \) strings \( S_j \) and constants \( C_j \), the relation
\[
\rho \sim \sum_j C_j \frac{\text{Tr}_{\bigcup A} S_j \rho_0}{\langle S_j \rangle} \equiv \sum_j C_j \rho^{S_j},
\]
holds. Like in the single-block case, each operator \( \rho^{S_j} \) turns out to be a normalized exponential of a quadratic form in the Majorana fermions. Up to a constant factor, these matrices are closed under multiplication and we identify any of them with the skew-symmetric matrix \( W \) in the exponent in equation (33). We recall in this appendix some basic properties of skew-symmetric diagonalizable matrices.

If \( \lambda \) is an eigenvalue of \( W \), then also \(-\lambda\) is, because
\[
0 = \det |W - \lambda I| = \det |W^T - \lambda I| = \det | -W - \lambda I | .
\]
If \( \tilde{w}_\lambda \) and \( \tilde{w}_\mu \) are two eigenvectors with eigenvalues \( \lambda \) and \( \mu \), then
\[
\mu \tilde{w}_\lambda \cdot \tilde{w}_\mu = \tilde{w}_\lambda^T W \tilde{w}_\mu = -\tilde{w}_\mu^T W \tilde{w}_\lambda = -\lambda \tilde{w}_\mu \cdot \tilde{w}_\lambda , \Rightarrow (\mu + \lambda) \tilde{w}_\mu \cdot \tilde{w}_\lambda = 0.
\]
The vectors \( \tilde{w}_\mu \) and \( \tilde{w}_\lambda \) are orthogonal (with respect to the scalar product \( \tilde{w}_\mu \cdot \tilde{w}_\lambda = \sum_j (w_\mu)_j (w_\lambda)_j \), which is not the standard one because of the absence of complex conjugation) unless \( \mu = -\lambda \). If the eigenvalue \( \lambda \) (and so \(-\lambda\)) is \( n \)-times degenerate, in general one can redefine the eigenvectors in the same \( n \)-dimensional eigenspace so that each vector \( \tilde{w}_\lambda \) becomes orthogonal to the other \( n-1 \) ones. Thus the following decomposition holds:
\[
W = \sum_{\{\lambda\}/\pm} \lambda \tilde{w}_\lambda \otimes \tilde{w}_{-\lambda} - \tilde{w}_{-\lambda} \otimes \tilde{w}_\lambda = \sum_{\{\lambda\}/\pm} \lambda \Pi_{\lambda},
\]
where \( \{ \lambda \}/_{\pm} \) means that each couple \( \pm \lambda \) contributes only once in the sum and the symbol \( \vec{v} \otimes \vec{w} \) identifies the matrix with elements \( v_i w_j \). On introducing the auxiliary matrices

\[
\Pi^+_{\lambda} = \frac{\vec{w}_{\lambda} \otimes \vec{w}_{-\lambda} + \vec{w}_{-\lambda} \otimes \vec{w}_{\lambda}}{\vec{w}_{\lambda} \cdot \vec{w}_{-\lambda}},
\]

which satisfy together with \( \Pi^-_{\lambda} \) the orthogonality relations

\[
\begin{vmatrix}
\Pi^+_{\lambda} & \Pi^-_{\lambda} \\
\Pi^+_{\mu} & \Pi^-_{\mu}
\end{vmatrix} = \frac{\delta_{\mu+\lambda \Pi^+_{\mu}}}{\delta_{\mu+\lambda \Pi^-_{\mu}}},
\]

any function of \( W \) can be expressed in terms of \( \Pi^\pm_{\lambda} \):

\[
f(W) = \sum_{\{\lambda\}/_{\pm}} (f_e(\lambda)\Pi^+_{\lambda} + f_o(\lambda)\Pi^-_{\lambda}) = \sum_{\{\lambda\}/_{\pm}} \frac{f(\lambda)\vec{w}_{\lambda} \otimes \vec{w}_{-\lambda} + f(-\lambda)\vec{w}_{-\lambda} \otimes \vec{w}_{\lambda}}{\vec{w}_{\lambda} \cdot \vec{w}_{-\lambda}},
\]

with \( f_e \) and \( f_o \) the even and odd parts of \( f \).

The decomposition \((C.4)\) allows us to express \( \rho_W \) in a factorized form:

\[
Z(W)\rho_W = \exp \left( \sum_{\{\lambda\}/_{\pm}} \frac{\lambda}{2} \left[ \vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a} \right] \right) = \prod_{\{\lambda\}/_{\pm}} \exp \left( \frac{\lambda}{2} \left[ \vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a} \right] \right)
\]

\[
= \prod_{\{\lambda\}/_{\pm}} \left[ \cosh \left( \frac{\lambda}{2} \right) + \sinh \left( \frac{\lambda}{2} \right) \frac{\vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a}}{2\vec{w}_{\lambda} \cdot \vec{w}_{-\lambda}} \right],
\]

where we made use of the orthogonality of \( \vec{w}_{\lambda} \) \((C.3)\):

\[
[[\vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a}], [\vec{w}_{\mu} \cdot \vec{a}, \vec{w}_{-\mu} \cdot \vec{a}]] = 0, \quad [\vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a}]^2 = 4(\vec{w}_{\lambda} \cdot \vec{w}_{-\lambda})^2.
\]

We obtain \( Z(W) \) by imposing the normalization of \( \rho_W \):

\[
Z(W) = \text{Tr} \exp \left( \frac{a W a}{4} \right) = \prod_{\{\lambda\}/_{\pm}} 2 \cosh \left( \frac{\lambda}{2} \right) = \pm \sqrt{\det |e^{W/2} + e^{-W/2}|}.
\]

The correlation matrix \( \Gamma_{ij} = \delta_{ij} - \langle a_i a_j \rangle \) comes directly from the quadratic part of expression \((C.8)\):

\[
Z(W)\Gamma_{ij} = \text{Tr} \left[ \frac{a_i a_j}{2} \exp \left( \frac{\vec{a}^T W \vec{a}}{4} \right) \right]
\]

\[
= \prod_{\{\lambda\}/_{\pm}} \cosh \left( \frac{\lambda}{2} \right) \sum_{\{\lambda\}/_{\pm}} \tanh \left( \frac{\lambda}{2} \right) \text{Tr} \left[ \frac{a_i a_j}{2} \frac{\vec{w}_{\lambda} \cdot \vec{a}, \vec{w}_{-\lambda} \cdot \vec{a}}{2\vec{w}_{\lambda} \cdot \vec{w}_{-\lambda}} \right]
\]

\[
= \left[ \prod_{\{\lambda\}/_{\pm}} 2 \cosh \left( \frac{\lambda}{2} \right) \right] \left[ \tanh \left( \frac{W}{2} \right) \right]_{ij},
\]

that is

\[
\Gamma = \tanh \left( \frac{W}{2} \right).
\]
The correlation matrix does not determine unequivocally the function \(Z(W)\). Indeed we obtain the expression

\[
Z(W) \sim \frac{2^N}{\sqrt{\det |1 - \Gamma|^2}}
\]  

(C.13)

with an evident phase ambiguity (that is actually a sign ambiguity). In spite of the ambiguity in \(Z(W) \sim Z[\Gamma]\), we believe that the trace of two fermionic RDMs,

\[
\text{Tr}[\rho_W \rho_{W'}] = \frac{Z(\log(e^W e^{W'}))}{Z(W)Z(W')},
\]

(C.14)

is a functional of \(\Gamma = \tanh(W/2)\) and \(\Gamma' = \tanh(W'/2)\). In fact, we argue that

\[
\{\Gamma, \Gamma'\} = \text{Tr} \rho[\Gamma]\rho[\Gamma'] = \frac{1}{2N} \prod_{\{\lambda\}/2} \lambda
\]

(C.15)

where the product is extended over all eigenvalues \(\lambda\) of \(1 + \Gamma\Gamma'\) with degeneracy (always even) reduced by half. We do not have a rigorous proof of equation (C.15), but we give an argument to justify it. After infinitesimal variations of \(\Gamma\) and \(\Gamma'\), function \(Z(W)\) changes in the following way:

\[
d \log Z(W) = \frac{1}{2} \text{Tr} \frac{\Gamma d\Gamma}{1 - \Gamma^2},
\]

(C.16)

and thus from equation (C.13) it follows that

\[
d \log \text{Tr}[\rho_W \rho_{W'}] = \frac{1}{2} \text{Tr} \frac{\Gamma' \times \Gamma d(\Gamma \times \Gamma')} {1 - (\Gamma \times \Gamma')^2} - \frac{\Gamma d\Gamma}{1 - \Gamma^2} - \frac{\Gamma' d\Gamma'}{1 - \Gamma'^2}.
\]

(C.17)

Using equation (44) for \(\Gamma \times \Gamma'\), many simplifications occur, leading to

\[
d \log \text{Tr}[\rho_W \rho_{W'}] = \frac{1}{2} \text{Tr} \frac{\Gamma' \times \Gamma d\Gamma} {1 + \Gamma' \Gamma} + \frac{1}{2} \text{Tr} \frac{\Gamma d\Gamma'} {1 + \Gamma' \Gamma} = \frac{1}{2} \text{Tr} \frac{d(\Gamma \Gamma')} {1 + \Gamma \Gamma'}.
\]

(C.18)

The variations \(d\Gamma\) and \(d\Gamma'\) are skew-symmetric by construction. Our task is to integrate the differential equation above. We consider all possible sets of \(2N\) smooth simple curves joining the \(\Gamma \Gamma'\) eigenvalues \(\lambda_i\) from \(\Gamma' = 0\) (when \(\text{Tr} \rho_W \rho_{W'} = 2^{-N}\)) to our desired values. Despite the initial ambiguity in the sign of \(Z(W)\), the expression for the logarithm,

\[
\log \text{Tr}[\rho_W \rho_{W'}] = \frac{1}{2} \sum_{i=1}^{2N} \int_{\gamma_i} \frac{d\lambda_i}{1 + \lambda_i} - N \log 2,
\]

(C.19)

is well defined, in view of the fact that \(\Gamma \Gamma'\) has a doubly degenerate spectrum and the ambiguity related to the choice of the curves \(\gamma_i\) (avoiding any singular point) gives simply an additive constant proportional to \(2\pi i\). Considering the path characterized by \(d(\Gamma \Gamma') = d\lambda(\Gamma \Gamma' - 1)/2\), the product rule can be expressed in the following integral form:

\[
\{\Gamma, \Gamma'\} = \exp \left[ \frac{1}{2} \int_{\gamma[0\rightarrow 1]} \text{Tr} \frac{\Gamma \Gamma' - 1}{2 - \lambda + \lambda \Gamma \Gamma'} d\lambda \right] = \exp \left[ \frac{1}{2} \int_{\gamma[0\rightarrow 1]} \frac{1}{1 + \lambda} \text{Tr} \frac{\Gamma \Gamma' - 1}{1 + \lambda \Gamma \Gamma'} d\lambda \right],
\]

where \(\gamma\) is a smooth simple path in \(\mathbb{C}\) which joins 0 to 1 and avoids any of the points at which \(1 + \lambda \Gamma \Gamma'\) fails to be invertible. Because of the factor 1/2 in front of the integral,
we get equation (C.15). For this proof we used several properties of the Pfaffian reviewed in [38]. Summarizing,

\[
\rho[\Gamma]\rho[\Gamma'] = \exp \left( \frac{1}{2} \int_{\gamma=0}^{\gamma=1} \frac{1}{1 + \lambda} \frac{\text{Tr} \Gamma \Gamma' - 1}{\lambda \Gamma' + 1} \text{d}\lambda \right) \rho[\Gamma \times \Gamma'].
\]  

(C.20)

A final remark about the XY correlation matrices in the double-block case: \( \Gamma_1 = \Gamma_A \) and \( \Gamma_3 = \Gamma_A \cup B_1 / \Gamma_B \) (appendix A). The quantity

\[
s \equiv \frac{\{\Gamma_1, \Gamma_3\}}{\sqrt{\det |(1 + \Gamma_1 \Gamma_3)/2|}}
\]  

(C.21)

can be 1 or \(-1\) and numerical data suggest that it is independent of the subsystem and of the chain length. In particular we find

\[
s = \begin{cases} 
1 & |h| \geq 1 \text{ or } \gamma = 0, \\
-1 & \text{otherwise},
\end{cases}
\]

(C.22)

and so

\[
\{\Gamma_1, \Gamma_3\} = [\text{sign}(|h| - 1) + 2\delta_{\gamma0}(1 - |h|)] \sqrt{\det \left| \frac{1 + \Gamma_1 \Gamma_3}{2} \right|}.
\]

(C.23)

We close this appendix with some speculations and some simplified (wrong) expressions at which one could arrive by treating correlation matrices too naively. If we did not worry about the signs and non-invertibility, we could directly write the result for a generic product like equation (48) as

\[
\{\Gamma_1, \ldots, \Gamma_n\} \sim \sqrt{\det \left| \prod_{i=1}^{n} e^{W_i} + 1 \right|} \prod_{i=1}^{n} \det \left| \frac{1 - \Gamma_i}{2} \right| \det \left| \prod_{i=1}^{n} \frac{1 + \Gamma_i}{1 - \Gamma_i} + 1 \right|.
\]

(C.24)
This expression is generally wrong. On substituting the chain correlation matrices, 
\( (1 - \Gamma_{1,3}) \) is usually non-invertible (figure C.1). Thus, except for \( S_2 \), equation (C.24) is wrong.

By using this wrong expression one can try to see whether any simplification occurs in the general Rényi entropy. In general it is not possible to push this approach, unless one makes further wrong assumptions. Assuming for example the (wrong) hypothesis of commuting \( \Gamma \) matrices, after trivial algebra we have the (wrong) expression

\[
S_{\alpha} \approx \frac{1}{1 - \alpha} \log \left[ \frac{1}{2^n} \sum_{\zeta_1 \cdots \zeta_n} \prod_{i=1}^{\alpha} \zeta_i [1 \pm \Gamma_{\zeta_i}] \left( \frac{\det {\prod_{i=1}^{\alpha} \frac{1 - \Gamma_{\zeta_i}}{2} + \prod_{i=1}^{\alpha} \frac{1 + \Gamma_{\zeta_i}}{2}}} \right) \right].
\]

We wrote this expression out here, because one could have wondered whether it (despite being wrong) could have been defined for any real \( \alpha \) (i.e. depending only on the eigenvalues of the matrices \( \Gamma \)) to have an expression analytically continuable to \( \alpha \to 1 \) (and maybe correct in this limit). However, even this simplified wrong expression is only calculable for integer \( \alpha \) and thus we did not persist with this direction and we preferred to exploit the exact solution without assumptions.

### Appendix D. Accidental singularities

When the correlation matrix of the region interposed between the blocks \( \Gamma_{B_1B_2} \) is non-invertible, that is the \( \sigma_z \) string mean value vanishes: \( \langle S \rangle = 0 \), some problems arise. The density matrix (32) is not well defined and all successive discussions are meaningless. Here we give some tricks for solving these problems. We consider a generic operator

\[
Q_W = \frac{1}{2^L} \sum_{O \in A} \langle e^{a W a/4} O \rangle O.
\]

If \( \langle e^{a W a/4} \rangle \neq 0 \), as we observed earlier, \( Q_W \) is proportional to an effective free density matrix:

\[
Q_W = \langle e^{a W a/4} \rangle \rho_{\Gamma[W]}.
\]

But when \( \langle e^{a W a/4} \rangle \) vanishes (this can happen only when \( W \) is not Hermitian) the previous expression is meaningless. The trick is to control the singularities, introducing the traceless string operator \( S_E \equiv \prod_{j \in E} \sum_{a_{2j-1} a_{2j}} \langle S_E^2 \rangle = 1 \) that belongs to the subsystem \( E \subset A \):

\[
Q_W = S_E \frac{1}{2^L} \sum_{O \in A} \langle e^{a W a/4} O \rangle S_E O = S_E \frac{1}{2^L} \sum_{O \in A} \langle e^{a W a/4} S_E O \rangle O = \langle e^{a W a/4} S_E \rangle S_E \rho_{\Gamma},
\]

and such that the mean value \( \langle e^{a W a/4} S_E \rangle \) is different from 0. The correlation matrix

\[
\Gamma = 1 - \frac{\langle e^{a W a/4} S_E a \otimes a \rangle}{\langle e^{a W a/4} S_E \rangle}
\]

can be used to perform any product at the price of having the string \( S_E \). We observe that

\[
S_E \rho_{\Gamma} = \rho [P_E \Gamma P_E] S_E,
\]

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where $P_E$ is the diagonal operator with elements $i = -1$ if $i \in E$ and 1 otherwise. In fact, exploiting the commutation relation, products of operators like $Q_{W_i}$ can be carried to the form

$$\prod_{i=1}^{m} Q_{W_i} = \prod_{i=1}^{m} \langle e^{a W_i a/4} S_{E_i} \rangle \prod_{i=1}^{m} \rho[\Gamma_i] S_{E_i},$$

(D.6)

where $S_{E_i}$ is the final $\sigma_z$ string. The trace

$$\text{Tr} \prod_{i=1}^{m} Q_{W_i} = \prod_{i=1}^{m} \langle e^{a W_i a/4} S_{E_i} \rangle \text{Tr} \prod_{i=1}^{m} \rho[\Gamma_i] S_{E_i}$$

$$= \prod_{i=1}^{m} \langle e^{a W_i a/4} S_{E_i} \rangle \{\Gamma_1, \ldots, \Gamma_m\} \text{Tr} \rho[\Gamma_1 \times \cdots \times \Gamma_m] S_{E_i}$$

(D.7)

is free from pathologies.

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