On the wedge product of table algebras and applications to association schemes

Javad Bagherian
Department of Mathematics, University of Isfahan,
P.O. Box: 81746-73441, Isfahan, Iran,
bagherian@sci.ui.ac.ir

Abstract
In this paper we will first present a generalization of the wedge product of association schemes to table algebras and give a necessary and sufficient condition for a table algebra to be the wedge product of two table algebras. Then we show that if the duals of two commutative table algebras are table algebras, then the dual of their wedge product is a table algebra, and is also isomorphic to the wedge product of the duals of those table algebras in the reverse order. Some applications to association schemes are also given.

Key words: table algebra, wedge product, dual.
AMS Classification: 20C99; 05E30.

1 Introduction
The wreath product of table algebras provides a way to construct the new table algebras from old ones. This construction is a generalization of the wreath product of association schemes to table algebras. Moreover, if the duals of two commutative table algebras are table algebras, then the dual of their wreath product is a table algebra, and is also isomorphic to the wreath product of the duals of those table algebras in the reverse order [14]. We mention that the dual of a commutative table algebra is a C-algebra but is not a table algebra, in general. (A question in the book by Bannai and Ito [2, p. 104] asks when the dual of a commutative table algebra is also a table algebra.)

Recently, the wedge product of association schemes as a generalization of the wedge product of Schur rings has been given in [9]. This product of association schemes can be considered as a generalization of the wreath product of association schemes.

In this paper, we first give a generalization of the wedge product of association schemes to table algebras. Then we give a necessary and sufficient condition for a table algebra to be the wedge product of two table algebras. We can see that the wreath product of table algebras is a special case of the wedge product of table algebras. Moreover, we prove that if the duals of two table algebras are table algebras then the dual of their wedge product is a table algebra, and is also isomorphic to the wedge
product of the duals of those table algebras in the reverse order. Finally, we show that the complex adjacency algebra of the wedge product of association schemes is isomorphic to the wedge product of the complex adjacency algebras of those association schemes and then we give some applications to association schemes.

2 Preliminaries

In this section, we state some necessary definitions and known results about C-algebras, table algebras, association schemes, homomorphisms of table algebras and the dual of table algebras. Throughout this paper, \( \mathbb{C} \) denotes the complex numbers, \( \mathbb{R} \) the real numbers and \( \mathbb{R}^+ \) the positive real numbers.

2.1 C-algebras, table algebras and association schemes

We follow from [7] for the definition of C-algebras. Hence we deal with C-algebras as the following:

**Definition 2.1.** (See [7, Definition 3.1].) Let \( A \) be a finite dimensional associative algebra over \( \mathbb{C} \) with the identity element \( 1_A \) and a base \( B \) in the linear space sense. Then the pair \( (A, B) \) is called a C-algebra if the following conditions (I)-(IV) hold:

(I) \( 1_A \in B \) and the structure constants of \( B \) are real numbers, i.e., for \( a, b \in B \):

\[
ab = \sum_{c \in B} \lambda_{abc} c, \quad \lambda_{abc} \in \mathbb{R}.
\]

(II) There is a semilinear involutory anti-automorphism (denoted by \( * \)) of \( A \) such that \( B^* = B \).

(III) For \( a, b \in B \) the equality \( \lambda_{ab1_A} = \delta_{ab} |a| \) holds where \( |a| > 0 \) and \( \delta \) is the Kronecker symbol.

(IV) The mapping \( b \rightarrow |b|, b \in B \) is a one dimensional \( * \)-linear representation of the algebra \( A \), which is called the degree map.

**Remark 2.2.** In the definition above if the algebra \( A \) is commutative, then \( (A, B) \) becomes a C-algebra in the sense of [2].

Let \( (A, B) \) be a C-algebra. For any \( x = \sum_{b \in B} x_b b \in A \) we denote by \( \text{Supp}(x) \) the set of all basis elements \( b \in B \) such that \( x_b \neq 0 \). If \( N_1, \ldots, N_m \) are nonempty closed subsets of \( B \), then we set

\[
N_1 N_2 \ldots N_m = \bigcup_{b_1 \in N_1, b_2 \in N_2, \ldots, b_m \in N_m} \text{Supp}(b_1 \ldots b_m).
\]

The set \( N_1 N_2 \ldots N_m \) is called the complex product of closed subsets \( N_i, 1 \leq i \leq m \). If one of the factors in a complex product consists of a single element \( b \), then one
usually writes $b$ for $\{b\}$. A nonempty subset $N \subseteq B$ is called a closed subset, denoted as $N \leq B$, if $N^* N \subseteq N$, where $N^* = \{b^* | b \in N\}$. If $N$ is a closed subset of $B$, then $(<N>,N)$, where $<N>$ is the $\mathbb{C}$-space spanned by $N$, is a $C$-algebra. For every closed subset $N$ of $B$, the order of $N$, $o(N)$, is defined by

$$o(N) = \sum_{b \in N} |b|$$

and $C^+$ is defined by

$$N^+ = \sum_{b \in N} b.$$

If the structure constants of a given $C$-algebra are nonnegative real numbers, then it is called a table algebra in the sense of [1].

Let $(A,B)$ be a table algebra. Let $B' = \{\lambda_b b | b \in B\}$, where $\lambda_1_A = 1$, and $\lambda_b = \lambda_{b^*} \in \mathbb{R}^+$ for all $b \in B$. Then $(A,B')$ is also a table algebra which is called a rescaling of $(A,B)$. Let $(A,B)$ be a table algebra. If $N$ is a closed subset of $B$ such that for any $b \in B$, $bN = Nb$, then $N$ is called a normal closed subset of $B$. It is known that if $N$ is a closed subset of $B$, then $e := o(N)^{-1} N^+$ is an idempotent of $A$, and $N$ is a normal closed subset if and only if $e$ is a central idempotent; see [1, Proposition 2.3(ii)].

Let $(A,B)$ be a table algebra and $N$ be a closed subset of $B$. It follows from [1, Proposition 4.7] that $\{NbN | b \in B\}$ is a partition of $B$. A subset $NbN$ is called a $N$-double coset with respect to the closed subset $N$. Let

$$b//N := o(N)^{-1} (NbN)^+ = o(N)^{-1} \sum_{x \in NbN} x.$$ 

Then the following theorem is an immediate consequence of [1, Theorem 4.9]:

**Theorem 2.3.** Let $(A,B)$ be a table algebra and let $N$ be a closed subset of $B$. Suppose that $\{b_1 = 1_A, \ldots, b_k\}$ is a complete set of representatives of $N$-double cosets. Then the vector space spanned by the elements $b_i//N, 1 \leq i \leq k$, is a table algebra (which is denoted by $A//N$) with a distinguished basis $B//N = \{b_i//N | 1 \leq i \leq k\}$. The structure constants of this algebra are given by the following formula:

$$\gamma_{ijk} = o(N)^{-1} \sum_{r \in Nb_k N, s \in Nb_j N} \lambda_{rst},$$

where $t \in Nb_k N$ is an arbitrary element.

The table algebra $(A//N, B//N)$ is called the quotient table algebra of $(A,B)$ modulo $N$.

Let $(A,B)$ be a table algebra and $N$ be a closed subset of $B$. Put $e = o(N)^{-1} N^+$. Then one can see that $A//N = eAe$ and it follows from [11] that

$$\frac{|b|}{|b//N|} b//N = ebe,$$

for every $b \in B$.

Now we state some necessary definitions and notations for association schemes.
Definition 2.4. Let \( X \) be a finite set and \( G \) be a partition of \( X \times X \). Then the pair \((X,G)\) is called an association scheme on \( X \) if the following properties hold:

(I) \( 1_X \in G \), where \( 1_X := \{(x,x) | x \in X \} \).

(II) For every \( g \in G \), \( g^* \) is also in \( G \), where \( g^* := \{(x,y) | (y,x) \in g \} \).

(III) For every \( g,h,k \in G \), there exists a nonnegative integer \( \lambda_{ghk} \) such that for every \((x,y) \in k\), there exist exactly \( \lambda_{ghk} \) elements \( z \in X \) with \((x,z) \in g \) and \((z,y) \in h\).

Let \((X,G)\) be an association scheme. For each \( g \in G \), we call \( n_g = \lambda_{gg}1_X \) the valency of \( g \). For any nonempty subset \( H \) of \( G \), put \( n_H = \sum_{h \in H} n_h \). Clearly \( n_G = |X| \).

For every \( g \in G \), let \( A(g) \) be the adjacency matrix of \( g \). For every nonempty subset \( H \) of \( G \), put \( A(H) := \{A(h) | h \in H\} \) and let \( \mathbb{C}[H] \) denote the \( \mathbb{C} \)-space spanned by \( A(H) \). It is known that \((\mathbb{C}[G], A(G))\) is a table algebra, called the complex adjacency algebra of \( G \).

Let \((X,G)\) be an association scheme. A nonempty subset \( H \) of \( G \) is called a closed subset of \( G \) if \( A(H) \) is a closed subset of \( \mathbb{C}[G] \). If \( H \) is a closed subset of \( G \), then \((\mathbb{C}[H], A(H))\) is a table algebra.

Let \( H \) be a closed subset of \( G \). For every \( h \in H \) and every \( x \in X \), we define \( xh = \{y \in X | (x,y) \in h\} \). Put \( X/H = \{xH | x \in X\} \), where \( xH = \bigcup_{h \in H} xh \). For \( x \in X \), the subscheme \((X,G)_{xH}\) induced by \( xH \), is an association scheme \((xH, H_{xH})\) where \( H_{xH} = \{h_{xH} | h \in H\} \) and \( h_{xH} = h \cap xH \times xH \). It is known that \( \mathbb{C}[H_{xH}] \cong \mathbb{C}[H] \), as algebras over \( \mathbb{C} \); see [15, Theorem 4.4.5].

Let \((X,G)\) and \((Y,S)\) be two association schemes. A scheme epimorphism is a mapping \( \varphi : (X,G) \to (Y,S) \) such that

(i) \( \varphi(X) = Y \) and \( \varphi(G) = S \),

(ii) for every \( x,y \in X \) and \( g \in G \) with \((x,y) \in g\), \((\varphi(x), \varphi(y)) \in \varphi(g)\).

Let \( \varphi : (X,G) \to (Y,S) \) be a scheme epimorphism. The kernel of \( \varphi \) is defined by

\[ \ker \varphi = \{g \in G \mid \varphi(g) = 1_Y \} \]

It is known that \( A(\ker \varphi) \) is a closed subset of \( A(G) \), but \( A(\ker \varphi) \) need not be normal, in general. If \( A(\ker \varphi) \trianglelefteq A(G) \), then the scheme epimorphism \( \varphi \) is called the normal scheme epimorphism. A scheme epimorphism with a trivial kernel is called a scheme isomorphism.

An algebraic isomorphism between two association schemes \((X,G)\) and \((Y,S)\) is a bijection \( \theta : G \to S \) such that it preserves the structure constants, that is \( \lambda_{ghl} = \lambda_{\varphi(g)\varphi(h)\varphi(l)} \), for every \( g,h,l \in G \). It is known that if \( \varphi : (X,G) \to (Y,S) \) is a scheme isomorphism, then \( \varphi \) induces an algebraic isomorphism between \( G \) and \( S \).

2.2 Homomorphisms of table algebras

Here we state some basic definitions and results of homomorphisms of table algebras from [13].
Definition 2.5. (See [13, Definition 3.1].) Let $(A, B)$ and $(C, D)$ be table algebras. A map $\varphi : A \to C$ is called the table algebra homomorphism of $(A, B)$ into $(C, D)$ if

(i) $\varphi : A \to C$ is an algebra homomorphism; and

(ii) $\varphi(B) := \{\varphi(b) | b \in B\}$ consists of positive scalar multiples of elements $D$.

A table algebra homomorphism is called a monomorphism (epimorphism, isomorphism, resp.) if it is injective (surjective, bijective, resp.).

Example 2.6. Let $(A, B)$ and $(C, D)$ be table algebras. Define $\varphi : (A, B) \to (C, D)$ such that $\varphi(b) = |b|d$. Then $\varphi$ is a table algebra homomorphism, and is called the trivial table algebra homomorphism.

Example 2.7. Let $(A, B)$ be a table algebra and $N$ be a normal closed subset of $B$. It follows from [11, Theorem 2.1] that, there is a table algebra epimorphism $\pi : (A, B) \to (A//N, B//N)$ such that $\pi(b) = |b|b/N$, $\forall b \in B$.

The table algebra epimorphism $\pi$ is called the canonical epimorphism from $(A, B)$ to $(A//N, B//N)$.

Two table algebras $(A, B)$ and $(C, D)$ are called isomorphic, denoted by $(A, B) \cong (C, D)$ or simply $B \cong D$, if there exists a table algebra isomorphism $\varphi : (A, B) \to (C, D)$.

Lemma 2.8. [13, Lemma 3.2] Let $(A, B)$ and $(C, D)$ be table algebras and $\varphi : (A, B) \to (C, D)$ be a table algebra homomorphism. Then the following hold:

(i) for every $b \in B$, $|\varphi(b)| = |b|$, 

(ii) for every $b \in B$, if $\text{Supp}(\varphi(b)) = d$, then

$$\varphi(b) = \frac{|b|}{|d|}d.$$ 

The following lemma gives some basis properties of table algebra homomorphisms.

Lemma 2.9. [13, Proposition 3.3] Let $(A, B)$ and $(C, D)$ be table algebras and $\varphi : (A, B) \to (C, D)$ be a table algebra homomorphism. Then the following hold:

(i) $\varphi(1_A) = 1_C$, 

(ii) for every $b \in B$, $\varphi(b^*) = \varphi(b)^*$, 

(iii) for every nonempty closed subset $N$ of $B$, $\varphi(N) = \{\text{Supp}(\varphi(b)) | b \in N\}$ is a closed subset of $C$, 

(iv) for every nonempty closed subset $M$ of $D$, $\varphi^{-1}(M) = \{b \in B | \text{Supp}(\varphi(b)) \in M\}$ is a closed subset of $B$. 

5
Definition 2.10. (See [13, Definition 3.4].) Let \((A, B)\) and \((C, D)\) be table algebras and \(\varphi: (A, B) \to (C, D)\) be a table algebra homomorphism. Then the set \(\varphi^{-1}(1_A)\) is called the kernel of \(\varphi\) in \(B\) and is denoted by \(\ker_B \varphi\).

The next lemma will be needed later.

Lemma 2.11. Let \((A, B)\) and \((C, D)\) be table algebras and \(\varphi: (A, B) \to (C, D)\) be a table algebra homomorphism. Then the following hold:

(i) \(\ker_B \varphi\) is a normal closed subset of \(B\),
(ii) \(\varphi\) is injective if and only if \(\ker_B \varphi = \{1_A\}\).

The next theorem is an isomorphism theorem for table algebras and it can be useful in the theory of table algebras.

Theorem 2.12. [13, Theorem 4.1] Let \((A, B)\) and \((C, D)\) be table algebras and \(\varphi: (A, B) \to (C, D)\) be a table algebra homomorphism. Then \(\varphi\) induces a table algebra homomorphism
\[
\bar{\varphi}: (A/\ker_B \varphi, B/\ker_B \varphi) \to (C, D)
\]
such that \(\bar{\varphi}(eae) = \varphi(a)\), where \(e = o(\ker_B \varphi)^{-1}(\ker_B \varphi)^+\). In particular,
\[
B/\ker_B \varphi \cong \varphi(B).
\]

2.3 Characters of table algebras

Let \((A, B)\) be a table algebra. Then \(A\) is a semisimple algebra; see [1, Theorem 3.11]. Put \(e = o(B)^{-1}B^+\). Then \(Ae\) is a one dimensional \(A\)-module. The character of \(A\) afforded by this module is called the principal character of \(A\) and is denoted by \(\rho\).

The kernel of a character \(\chi\) of \(A\) in \(B\) is defined by
\[
\ker_B(\chi) = \{b \in B | \chi(b) = |b|\chi(1)\}.
\]
It follows from [11, Theorem 4.2] that \(\ker_B(\chi)\) is a closed subset of \(B\). Let \(\text{Irr}(B)\) be the set of irreducible characters of \(A\) and for every closed subset \(N\) of \(B\), let \(\text{Irr}(B//N)\) be the set of irreducible characters of \(A//N\). Then for every normal closed subset \(N\) of \(B\), \(\text{Irr}(B//N) = \{\chi \in \text{Irr}(B) | N \subseteq \ker_B(\chi)\}\); see [11, Theorem 3.6].

Let \((A, B)\) be a table algebra and \(\chi, \psi \in \text{Irr}(B)\). The character product of \(\chi\) and \(\psi\) is defined by
\[
\chi\psi(b) = \frac{1}{|b|} \chi(b)\psi(b),
\]
(see [3]). It is known that this character product need not be a character, in general. But since for every \(a, b \in B\), \(\chi\psi(ab) = \chi\psi(ba)\), it follows that \(\chi\psi\) is a feasible trace and so
\[
\chi\psi = \sum_{\varphi \in \text{Irr}(B)} \lambda_{\chi\psi}^{\varphi} \varphi,
\]
where \(\lambda_{\chi\psi}^{\varphi} \in \mathbb{C}\); see [8].

Let \((A, B)\) be a table algebra. Define a linear function \(\zeta\) on \(A\) by \(\zeta(b) = \delta_{b,1_A}o(B)\), for every \(b \in B\). Then \(\zeta\) is a non-degenerate feasible trace on \(A\) and it follows from [10] that
\[
\zeta = \sum_{\chi \in \text{Irr}(B)} \zeta_{\chi}\chi,
\]
where ζχ ∈ ℂ and all ζχ are nonzero. For every χ, φ ∈ Homℂ(A, ℂ), we define the inner product of χ and φ as follows:

\[ [χ, φ] = \frac{1}{o(B)} \sum_{b ∈ B} \frac{1}{|b|} χ(b)φ(b^*). \]

Then for every χ, φ ∈ Irr(B), it follows from [10, Lemma 3.1(ii)] that

\[ [χ, φ] = δ_{χ,φ} \frac{χ(1)}{ζχ}. \]

So for every χ, ψ, φ ∈ Irr(B) we have

\[ [χψ, φ] = \frac{λ_{χφ}}{ζψ} = \frac{λ_{ψχ}}{ζχ}. \]

Moreover, since

\[ [χψ, φ] = [χφ, ψ] = [ψχ, ψ], \]

it follows that

\[ \frac{λ_{χψ}}{ζψ} = \frac{λ_{χφ}}{ζψ} = \frac{λ_{ψχ}}{ζχ}. \]

### 2.4 Duals of commutative table algebras

In the following, we deal with the dual of a commutative table algebra in the sense of [2].

Suppose that (A, B) is a commutative table algebra of dimension d with the set of primitive idempotents \{εχ | χ ∈ Irr(B)\}. Then from [2, Section 2.5] there are two matrices \( P = (p_b(χ)) \) and \( Q = (q_b(χ)) \) in Matd(ℂ), where \( b ∈ B \) and \( χ ∈ Irr(B) \), such that \( PQ = QP = o(B)I \), where I is the identity matrix in Matd(ℂ), and

\[ b = \sum_{χ ∈ Irr(B)} p_b(χ)εχ \quad \text{and} \quad εχ = \frac{1}{o(B)} \sum_{b ∈ B} q_b(χ)b. \]

The dual of (A, B) in the sense of [2] is as follows: with each linear representation \( Δ_χ : b ↦ p_b(χ) \), we associate the linear mapping \( Δ_χ^* : b ↦ q_b(χ) = \frac{εχ(b^*)}{|b|} \). Since the matrix Q = (q_b(χ)) is non-singular, the set \( \hat{B} = \{Δ_χ^* : χ ∈ Irr(B)\} \) is linearly independent and so forms a base of the set of all linear mappings \( \hat{A} \) of A into ℂ. From [2, Thorem 5.9] the pair \( (\hat{A}, \hat{B}) \) is a C-algebra with the identity 1\( \hat{A} = Δ^*_ρ \), where ρ is the principal character of A, and involutory automorphism * which maps \( Δ_χ^* \) to \( Δ_{χ^*}^* \), where \( χ^* \) is the complex conjugate to χ. The C-algebra \( (\hat{A}, \hat{B}) \) is called the dual of (A, B). Moreover, for every χ, ψ ∈ Irr(B) we have

\[ Δ_χ^*Δ_ψ^* = \sum_{Δφ ∈ \hat{B}} q_{χφ}^*Δ_φ^*, \]
where the structure constants $q^\varphi_\chi_{\psi} \in \text{Irr}(B)$, are real numbers. Furthermore, it follows from the Duality Theorem [2, Theorem 5.10] that $(\hat{A}, \hat{B}) \cong (A, B)$.

Let $(\hat{A}, \hat{B})$ be the dual of $(A, B)$ and $N$ be a closed subset of $B$. Put

$$\text{ker}(N) = \{ \Delta_\chi^* \in \hat{B} \mid \chi(b) = |b|, \text{ for every } b \in N \}.$$  

Then ker$(N)$ is a closed subset of $\hat{B}$; see [6]. Moreover, since $N \subseteq \text{ker}_B(\chi)$ for every $\Delta_\chi^* \in \text{ker}(N)$, we conclude that

$$\text{ker}(N) = \{ \Delta_\chi^* \in \hat{B} \mid N \subseteq \text{ker}_B(\chi) \} = \{ \Delta_\chi^* \in \hat{B} \mid \chi \in \text{Irr}(B/\!\!\!/N) \}.$$  

Suppose that $(A, B)$ is a table algebra such that $(\hat{A}, \hat{B})$ is also a table algebra. Then for closed subsets $N \leq M$ of $B$, it follows from [12, Theorem 4.8] that

$$\hat{M/\!\!\!/N} \cong \text{ker}(N)\!\!\!/\text{ker}(M).$$

In particular, we have

$$\text{ker}(N) \cong \hat{B/\!\!\!/N},$$

and

$$\hat{N} \cong \hat{B/\!\!\!/\text{ker}(N)}.$$  

### 3 A wedge product of table algebras

In this section we will first define the wedge product of table algebras and then give a necessary and sufficient condition for a table algebra to be the wedge product of two table algebras.

Let $(A, B)$ be a table algebra and $N$ be a closed subset of $B$. Suppose that $(C, D)$ is a table algebra and

$$\varphi : (C, D) \rightarrow (\langle N \rangle, N)$$

is a table algebra epimorphism. Put $K = \text{ker}_D \varphi = \{ d \in D \mid \text{Supp}(\varphi(d)) = 1_A \}$. Then it follows from Lemma 2.8 and Theorem 2.11 that

(i) for every $d \in D$ with Supp$(\varphi(d)) = h$, we have

$$\varphi(d) = \frac{|d|}{|h|}h.$$  

In particular, for every $d \in K$, $\varphi(d) = |d|1_A$.

(ii) $K \trianglelefteq D$ and

$$(C/\!\!\!/K, D/\!\!\!/K) \cong (\langle N \rangle, N).$$
For every $1_A \neq b \in B$, put $\overline{b} = o(K)b$. Suppose that $(A, \overline{B})$ is a rescaling of $(A, B)$ where

$$\overline{B} = \{1_A\} \cup \{\overline{b} \mid b \in B \setminus \{1_A\}\}.$$ 

Put $X = D \cup \overline{B}$ and let $\widetilde{A}$ be the $C$-space spanned by $X$. Suppose that $D = \{d_1, d_2, \ldots, d_n\}$, $B = \{b_1, b_2, \ldots, b_m\}$ and the sets $\{\lambda_{xyz} \mid x, y, z \in B\}$ and $\{\mu_{xyz} \mid x, y, z \in D\}$ are the structure constants of $(A, B)$ and $(C, D)$, respectively. We define a multiplication “$\cdot$” on the elements of $X$ as follows:

(i) for every $d_i, d_j \in D$,

$$d_i \cdot d_j = \sum_{z=1}^{n} \mu_{d_idjdz}d_z,$$

(ii) for every $\overline{b}_i, \overline{b}_j \in \overline{B}$,

$$\overline{b}_i \cdot \overline{b}_j = o(K) \sum_{t=1}^{m} \lambda_{b_ibjt}\overline{b}_t,$$

(iii) for every $d_i \in D$ with $\text{Supp}(\varphi(d_i)) = h_i$, and $\overline{b}_j \in \overline{B}$,

$$d_i \cdot \overline{b}_j = o(K)\varphi(d_i)b_j = \frac{|d_i|}{|h_i|} \sum_{t=1}^{m} \lambda_{h_ibjb_t}\overline{b}_t,$$

and similarly,

$$\overline{b}_j \cdot d_i = o(K)b_j\varphi(d_i) = \frac{|d_i|}{|h_i|} \sum_{t=1}^{m} \lambda_{b_jh_id_t}\overline{b}_t.$$

If we extend “$\cdot$” linearly to all $\widetilde{A}$, then it defines the structure of a $C$-algebra on $\widetilde{A}$.

**Lemma 3.1.** $\widetilde{A}$ is an associative $C$-algebra.

**Proof.** Since $A$ and $C$ are associative $C$-algebras, it suffices to show that:

(i) for every $d_i, d_j \in D$ and $\overline{b}_t \in \overline{B}$, $(d_i \cdot d_j) \cdot \overline{b}_t = d_i \cdot (d_j \cdot \overline{b}_t)$,

(ii) for every $d_i \in D$ and $\overline{b}_j, \overline{b}_t \in \overline{B}$, $d_i \cdot (\overline{b}_j \cdot \overline{b}_t) = (d_i \cdot \overline{b}_j) \cdot \overline{b}_t$.

To prove (i), assume that $d_i, d_j \in D$ with $\text{Supp}(d_j) = h_j$ and $\overline{b}_t \in \overline{B}$. Since $\varphi(d_idj) = \varphi(d_i)\varphi(d_j)$ and $A$ is an associative algebra, we have
\[(d_i \cdot d_j) \cdot \overline{b_t} = (\sum_{s=1}^{n} \mu_{d_i,d_j,d_s} \cdot \overline{b_t}) = \sum_{s=1}^{n} \mu_{d_i,d_j,d_s} \cdot \overline{b_t}\]

\[= o(K) \sum_{s=1}^{n} \mu_{d_i,d_j,d_s} \varphi(d_s)b_t = o(K) \varphi(\sum_{s=1}^{n} \mu_{d_i,d_j,d_s} b_t)\]

\[= o(K) \varphi(d_i)(\varphi(d_j)b_t) = o(K) \varphi(d_i)(\sum_{t=1}^{|d_j|} \varphi(b_t))\]

\[= o(K) \varphi(d_i)(\varphi(b_t)) = o(K) \varphi(d_i)(\sum_{t=1}^{m} \lambda_{h_i b_t} b_r)\]

\[= o(K) \varphi(d_i)(\sum_{t=1}^{m} \lambda_{h_i b_t} b_r) = o(K) \varphi(d_i)(\sum_{t=1}^{m} \lambda_{h_i b_t} b_r)\]

\[= d_i \cdot o(K) \varphi(d_j)b_t = d_i \cdot (d_j \cdot \overline{b_t}).\]

Similarly, to prove \((ii)\), assume that \(d_i \in D\) and \(\overline{b}_j, \overline{b}_t \in \overline{B}\). Then we have

\[d_i \cdot (\overline{b}_j \cdot \overline{b}_t) = d_i \cdot (\sum_{r=1}^{m} \lambda_{h_i b_t} b_r) = o(K) \sum_{r=1}^{m} \lambda_{h_i b_t} b_r\]

\[= o(K) \sum_{r=1}^{m} \lambda_{h_i b_t} \varphi(d_r)b_t = o(K) \sum_{r=1}^{m} \lambda_{h_i b_t} \varphi(d_r)b_t\]

\[= o(K) \sum_{r=1}^{m} \lambda_{h_i b_t} \varphi(d_r)b_t = o(K) \sum_{r=1}^{m} \lambda_{h_i b_t} \varphi(d_r)b_t\]

\[= (d_i \cdot \overline{b}_j \cdot \overline{b}_t).\]

\[\square\]

In the rest of this paper, for convenience, we write \(xy\) in place of \(x \cdot y\), for every \(x, y \in \overline{B}\).

Let \(d \in D\) such that \(\text{Supp}(\varphi(d)) = h\). Since \(\varphi(K^+) = o(K)\), we have

\[\varphi(dK^+) = o(K) \varphi(d) = \frac{|d|}{|h|} \overline{h}.\]

Then for every \(\overline{b} \in \overline{B}\),

\[(dK^+)\overline{b} = o(K) \varphi(dK^+)b = \frac{|d|}{|h|} \overline{h} \overline{b}.\]

In particular,

\[(dK^+)1_A = \frac{|d|}{|h|} \overline{h}.\]

If we identify \((dK^+)1_A\) with \(dK^+\), then we can assume that \(dK^+ = \frac{|d|}{|h|} \overline{h}\). By this identification we can see that \(\overline{B} = D \cup (\overline{B} \setminus \overline{N})\) is a base for the algebra \(\overline{A}\).
Moreover, for every $\vec{b} \in \overline{B} \setminus \overline{N}$, we have $1_D \cdot \vec{b} = o(K)\varphi(1_D)b$. But by Lemma 2.9, $\varphi(1_D) = 1_A$. So $1_D \cdot \vec{b} = o(K)1_Ab = \vec{b}$. Similarly, $\vec{b} \cdot 1_D = \vec{b}$. Hence $1_D \in \tilde{B}$ is the identity element of $\tilde{A}$. So in the rest of this paper, we denote $1_D$ by $1_{\tilde{A}}$.

In the following we will show that the pair $(\tilde{A}, \tilde{B})$ is a table algebra.

Suppose that $\ast_1$ and $\ast_2$ are semilinear involuntary anti-automorphisms of table algebras $(A, B)$ and $(C, D)$, respectively. Then we can define a semilinear involuntary anti-automorphism $\ast$ on $\overline{B}$ as follows:

(i) for every $d \in D$, $d^* := d^{*2}$,

(ii) for every $\vec{b} \in \overline{B}$, $(\vec{b})^* := \overline{b}^{*1} = o(K)b^{*1}$.

Note that $(\vec{b}d)^* = o(K)(b\varphi(d))^* = o(K)\varphi(d)^*b^{*1} = o(K)\varphi(d^{*2})b^{*1} = d^*(\vec{b})^*$; see Lemma 2.9. Similarly, $(d\vec{b})^* = (\vec{b})^*d^*$.

Moreover, if $|.|_A$ and $|.|_C$ are the degree maps of $(A, B)$ and $(C, D)$, respectively, then we can define a linear map $|.| : \tilde{A} \to C$ as follows:

(i) for every $d \in D$, $|d| := |d|_C$,

(ii) for every $\vec{b} \in \overline{B}$, $|\vec{b}| := o(K)|b|_A$.

We show that $|.|$ is an algebra homomorphism. To do so, suppose that $d_i, d_j \in D$ and $\vec{b}_i, \vec{b}_j \in \overline{B} \setminus \overline{N}$. Then we have

$$|d_id_j| = \sum_{r=1}^{n} \mu_{d_id_jd_r}d_r$$

$$= \sum_{r=1}^{n} \mu_{d_id_jd_r}|d_r| = \sum_{r=1}^{n} \mu_{d_id_jd_r}|d_r|_C$$

$$= |d_i|_C|d_j|_C = |d_i||d_j|$$

and

$$|\vec{b}_i\vec{b}_j| = o(K)|\sum_{r=1}^{m} \lambda_{\vec{b}_i\vec{b}_j\vec{b}_r}|$$

$$= o(K)|\sum_{r=1}^{m} \lambda_{\vec{b}_i\vec{b}_j\vec{b}_r}| = o(K)^2(|\sum_{r=1}^{m} \lambda_{\vec{b}_i\vec{b}_j\vec{b}_r}|_A)$$

$$= o(K)^2(|\vec{b}_i|_A|\vec{b}_j|_A) = |\vec{b}_i||\vec{b}_j|.$$
Moreover, if $\text{Supp}(d_j) = h_j$, then

$$|\tilde{b}_i d_j| = o(K)|b_i \varphi(d_j)| = o(K)\frac{|d_j|c}{|h_j| A} |b_i h_j|$$

$$= o(K)\frac{|d_j|c}{|h_j| A} \left(\sum_{r=1}^{m} \lambda_{b_i h_j t_r} |t_r|\right) = o(K)\frac{|d_j|c}{|h_j| A} \sum_{r=1}^{m} \lambda_{b_i h_j t_r} |t_r|$$

$$= o(K)\frac{|d_j|c}{|h_j| A} \sum_{r=1}^{m} \lambda_{b_i h_j t_r} |t_r| A = o(K)\frac{|d_j|c}{|h_j| A} |b_i h_j| A$$

$$= o(K)\frac{|d_j|c}{|h_j| A} |b_i| A |h_j| A = (o(K)|b_i| A)|d_j|c$$

$$= |\tilde{b}_i||d_j|.$$  

Similarly, $|d_j \tilde{b}_i| = |d_j||\tilde{b}_i|$. So we conclude that $| | : \tilde{A} \to \mathbb{C}$ is an algebra homomorphism. Furthermore, since $| |_A$ and $| |_C$ are $*$-linear representations, it follows that $| | : \tilde{A} \to \mathbb{C}$ is also a $*$-linear representation of $\tilde{A}$.

In the theorem below we show that the pair $(\tilde{A}, \tilde{B})$ is a table algebra.

**Theorem 3.2.** The algebra $\tilde{A}$ with the basis $\tilde{B}$ is a table algebra.

**Proof.** It follows from Lemma 3.1 that $\tilde{A}$ is an associative algebra. Moreover, the identity element of $\tilde{B}$, $1_{\tilde{A}}$, is in $\tilde{B}$ and for every $x, y \in \tilde{B}$, we have

$$xy = \sum_{z \in \tilde{B}} \alpha_{xyz} z,$$

where $\alpha_{xyz}, z \in \tilde{B}$, are nonnegative real numbers. By the preceding remarks, there is a semilinear involuntary anti-automorphism $*$ on $\tilde{B}$ such that $(\tilde{B})^* = \tilde{B}$. Furthermore, there exists a degree map $| | : \tilde{A} \to \mathbb{C}$ such that for every $d, c \in D$,

$$\mu_{dc}^{*1_{\tilde{A}}} = \mu_{dc c 1_{\tilde{A}}} = \delta_{dc} |d|,$$

and for every $\overline{a}, \overline{b} \in \overline{B} \setminus \overline{N}$,

$$\overline{a \overline{b}} = \overline{a^* b^*} = o(K)^2(ab^*) = o(K)(K^+ ab^*) = \delta_{ab^*} o(K)|b| 1_{\tilde{A}} + o(K) \sum_{1_{\tilde{A}} \neq 1_{\tilde{A}}} \lambda_{ab^* c}\overline{c}.$$

Note that $K^+ a = \varphi(K^+) a = o(K) a$. So $|\overline{b}| = o(K)|b| = \lambda_{bb^* 1_{\tilde{A}}}$. Thus we conclude that the pair $(\tilde{A}, \tilde{B})$ is a table algebra.

**Definition 3.3.** With the notation above, the table algebra $(\tilde{A}, \tilde{B})$ is called the wedge product of table algebras $(C, D)$ and $(A, B)$ relative to $\varphi$.

In the following we give some properties of the wedge product of table algebras.
Lemma 3.4. Let \((\tilde{A}, \tilde{B})\) be the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to \(\varphi\). Then \(\tilde{B}\) contains a closed subset \(K \leq D\) such that \(K \subseteq \tilde{B}\), and for every \(x \in \tilde{B} \setminus D\) and every \(k \in K\), \(kx = |k|x = xk\). In particular, for every \(x \in \tilde{B} \setminus D\), \(xK^+ = o(K)x = K^+x\).

**Proof.** Put \(K = \ker_D \varphi\). It follows from Lemma 2.11 that \(K \subseteq D\). Since for every \(b \in B \setminus N\) and \(k \in K\), we have \(k\overline{b} = o(K)\varphi(k)b = o(K)|k|b\), we conclude that \(k\overline{b} = |k|\overline{b} = \overline{bk}\) and \(K \subseteq \tilde{B}\). In particular, for every \(b \in \overline{B} \setminus N\),

\[K^+\overline{b} = o(K)\overline{b} = \overline{b}K^+.\]

Lemma 3.5. Let \((\tilde{A}, \tilde{B})\) be the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to \(\varphi\). Then \((\tilde{A}/K, \tilde{B}/K) \cong (A, B)\), where \(K = \ker_D \varphi\).

**Proof.** Put \(e = o(K)^{-1}K^+\). Then it follows from Lemma 3.4 that \(e\) is a central idempotent of \(\tilde{A}\). Define

\[\theta : e\tilde{A}e \to A\]

such that \(\theta(ede) = \varphi(d)\) for every \(d \in D\), and \(\theta(ebe) = o(K)b\) for every \(b \in \overline{B}\). We first show that \(\theta\) is an algebra homomorphism. To do so, first assume that \(d_1, d_2 \in D\). Then

\[
\theta((ed_1e)(ed_2e)) = \theta(ed_1d_2e) = \theta(\sum_{c \in D} \mu_{d_1d_2c} ece) = \sum_{c \in D} \mu_{d_1d_2c} \theta(ce e) = \sum_{c \in D} \mu_{d_1d_2c} \varphi(e).
\]

Similarly, for every \(\overline{b}_1, \overline{b}_2 \in \overline{B} \setminus N\) we have

\[
\theta((e\overline{b}_1e)(e\overline{b}_2e)) = \theta(e\overline{b}_1\overline{b}_2e) = o(K)\theta(\sum_{t \in B} \lambda_{b_1b_2t} e\overline{t}e) = o(K) \sum_{t \in B} \lambda_{b_1b_2t} \theta(e\overline{t}e) = o(K) \sum_{t \in B} \lambda_{b_1b_2t} o(K)t = o(K)^2 b_1b_2 \theta(e\overline{b}_1e)\theta(e\overline{b}_2e).
\]
Moreover, for every $d \in D$ and $b \in \overline{B} \setminus N$ with $\text{Supp}(\varphi(d)) = h$, we have
\[
\theta((ede)(e\overline{b}e)) = \theta(ed\overline{b}e) = o(K)\theta(e\varphi(d)be) = o(K)\frac{|d|}{|h|}\theta(ehbe) = o(K)\frac{|d|}{|h|}\left(\sum_{t \in B \setminus N} \lambda_{hbt}ete\right) = \frac{|d|}{|h|}\sum_{t \in B \setminus N} \lambda_{hbt}\theta(e\overline{t}e) = \frac{|d|}{|h|}\sum_{t \in B \setminus N} \lambda_{hbt}t = o(K)\frac{|d|}{|h|}hb = o(K)\varphi(d)b = \theta(ed)\theta(e\overline{b}e).
\]
Similarly, $\theta((e\overline{b}e)(ede)) = \theta(e\overline{b}e)\theta(ede)$. So $\theta$ is an algebra homomorphism. Since $e\overline{A}e = \overline{A//K}$ and for every $x \in \overline{B}$,
\[
exe = \frac{|x|}{|x//K|}x//K,
\]
we can define $\overline{\varphi} : (\overline{A//K}, \overline{B//K}) \to (A, B)$ such that
\[
\overline{\varphi}(x//K) = \frac{|x//K|}{|x|}\varphi(exe).
\]
Since $\varphi$ is an algebra homomorphism, it follows that $\overline{\varphi}$ is a table algebra homomorphism. Now suppose that $x \in \overline{B}$ such that $\text{Supp}(\overline{\varphi}(x//K)) = \{1_A\}$. If $x \in D$, then one can see that $\text{Supp}(\varphi(d)) = \{1_A\}$ and so $d \in \ker_D \varphi = K$. Thus $x//K = 1\overline{A//K}$.
Moreover, if $x \in \overline{B} \setminus N$ such that $\text{Supp}(\overline{\varphi}(x//K)) = \{1_A\}$, then
\[
\text{Supp}(\frac{|x//K|}{|x|}x) = \{1_A\}
\]
and hence $x = o(K)1_A$ which is a contradiction, since $x \in \overline{B} \setminus N$. So we conclude that
\[
\ker_{\overline{B//K}} \overline{\varphi} = \{1_{\overline{A//K}}\}
\]
and it follows from Lemma 2.11 that $\overline{\varphi}$ is a table algebra monomorphism.
Moreover, since $\varphi : (C, D) \to (< N >, N)$ is a table algebra epimorphism, and for every $b \in B \setminus N$ we have
\[
\overline{b//K} = o(K)^{-1}(K\overline{b}K)^+ = o(K)^{-1}o(K)b = b,
\]
we conclude that $\theta$ is a table algebra epimorphism. Thus $\theta$ is a table algebra isomorphism and so $(\overline{A//K}, \overline{B//K}) \cong (A, B)$. 14
Theorem 3.6. Let \((A, B)\) be a table algebra and \(K \leq D\) closed subsets of \(B\) such that \(K \subseteq B\) and for every \(b \in B \setminus D\), \(bK^+ = o(K)b = K^+b\). Then \((A, B)\) is the wedge product of table algebras \((< D >, D)\) and \((A//K, B//K)\) relative to the canonical epimorphism \(\pi : (< D >, D) \to (< D >//K, D//K)\).

Proof. Consider the table algebra \((A//K, B//K)\). Then \(D//K\) is a closed subset of \(B//K\) and we can define the canonical epimorphism \(\pi : (< D >, D) \to (< D >//K, D//K)\) such that

\[
\pi(d) = \frac{|d|}{|d//K|}d//K.
\]

Put \(e = o(K)^{-1}K^+\). Since \(e\) is a central idempotent of \(A\) and \(\frac{|d|}{|d//K|}d//K = ede\) we have

\[
\ker_D \pi = \{d \in D \mid \pi(d) = |d|e\} = \{d \in D \mid ede = |d|e\} = \{d \in D \mid dK^+ = |d|K^+\} = K.
\]

Now let \((\tilde{A}, \tilde{B})\) be the wedge product of \((< D >, D)\) and \((A//K, B//K)\) relative to \(\pi\). Then

\[
\tilde{B} = D \cup \{o(K)(b//K) \mid b//K \in B//K \setminus D//K\}.
\]

Since for every \(b//K \in B//K \setminus D//K\), we have \(o(K)(b//K) = o(K)o(K)^{-1}(KbK)^+ = b\) and so \(\tilde{B} = D \cup (B \setminus D) = B\). Moreover, for every \(b \in B \setminus D\) and \(d \in D\),

\[
o(K)(b//K)d = o(K)(b//K)\pi(d) = o(K)(b//K)\frac{|d|}{|d//K|}d//K = b(ede) = (be)d = o(K)^{-1}K^+bd = (o(K)^{-1}o(K)b)d = bd.
\]

Thus we conclude that \((\tilde{A}, \tilde{B}) = (A, B)\) and so \((A, B)\) is the wedge product of \((< D >, D)\) and \((A//K, B//K)\) relative to \(\pi\).

As a direct consequence of Lemma 3.4 and Theorem 3.6, we can give a necessary and sufficient condition for a table algebra to be the wedge product of two table algebras.

Corollary 3.7. Let \((A, B)\) be a table algebra and \(K \leq D\) closed subsets of \(B\). Then the following are equivalent:

(i) \(K \subseteq B\), and for every \(b \in B \setminus D\), \(bK^+ = o(K)b = K^+b\),

(ii) \((A, B)\) is the wedge product of \((< D >, D)\) and \((A//K, B//K)\) relative to the canonical epimorphism \(\pi : (< D >, D) \to (< D >//K, D//K)\).

Proof. (i) \(\Rightarrow\) (ii) follows directly from Theorem 3.6.

(ii) \(\Rightarrow\) (i) Since \((A, B)\) is the wedge product of \((< D >, D)\) and \((A//K, B//K)\) we have \(B = D \cup \{o(K)b//K \mid b//K \in B//K \setminus D//K\}\), and it follows from Lemma 3.4 that \(K \leq B\) and for every \(b//K \in B//K \setminus D//K\)

\[
K^+(o(K)b//K) = o(K)(o(K)b//K) = (o(K)b//K)K^+.
\]
Then for every \( b/K \in B/K \setminus D/K \),
\[
K^+(b/K) = o(K)(b/K) = (b/K)K^+
\]
and so
\[
K^+(KbK)^+ = o(K)(KbK)^+ = (KbK)^+K^+.
\]
Thus for every \( b \in B \setminus D \), \( bK^+ = o(K)b = K^+b \), and (i) holds.

\[\blacksquare\]

**Remark 3.8.** Let \((\tilde{A}, \tilde{B})\) be the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to \(\varphi\). If \(\ker_D \varphi = D\), then for every \(d \in D\) and every \(x \in \tilde{B} \setminus D\) we have \(xd = |d|x = dx\). So it follows from [4, Definition 1.2] that \((\tilde{A}, \tilde{B})\) is a wreath product \((\tilde{B}, D)\). Thus the wreath product of table algebras is a partial case of the wedge product of table algebras whenever \(\varphi\) is the trivial table algebra homomorphism; see Example 2.6.

### 4 The dual of wedge product

In this section we first give a sufficient condition for which the dual of a commutative table algebra is also a table algebra. Then we will show that if the duals of two commutative table algebras are table algebras, then the dual of their wedge product is a table algebra.

The following easy lemma is useful.

**Lemma 4.1.** Let \((A, B)\) be a commutative table algebra. Then \((\hat{A}, \hat{B})\) is a table algebra if and only if for every \(\chi, \psi \in \text{Irr}(B)\), \(\chi\psi\) is a linear combination of \(\text{Irr}(B)\) with the nonnegative real number coefficients.

**Proof.** For every \(\chi, \psi \in \text{Irr}(B)\), we have
\[
\Delta^*_\chi \Delta^*_\psi = \sum_{\varphi \in \text{Irr}(B)} q_{\chi\psi}^\varphi \Delta^*_\varphi,
\]
where \(q_{\chi\psi}^\varphi, \varphi \in \text{Irr}(B)\), are real numbers. One the other hand, for every \(b \in B\),
\[
\Delta^*_\chi \Delta^*_\psi (b^*) = \Delta^*_{\chi(b^*)} \Delta^*_\psi (b^*) = \frac{\zeta_{\chi(b^*)} \zeta_{\psi(b^*)}}{|b|} = \frac{\zeta_{\chi\psi}}{|b|} \chi\psi(b^*). \tag{1}
\]

Since
\[
\chi\psi = \sum_{\varphi \in \text{Irr}(B)} \lambda_{\chi\psi}^\varphi \varphi, \tag{2}
\]
it follows from equalities (1) and (2) that
\[
\Delta^* \chi \Delta^* \psi(b^*) = \frac{\xi \xi \psi}{b} \sum_{\varphi \in \text{Irr}(B)} \lambda_{\chi \psi}^\varphi \varphi(b^*)
\]
and hence
\[
\Delta^* \chi \Delta^* \psi(b^*) = \sum_{\varphi \in \text{Irr}(B)} \frac{\xi \xi \psi}{\xi \varphi} \lambda_{\chi \psi}^\varphi \Delta^* \varphi(b^*).
\]
This implies that
\[
\Delta^* \chi \Delta^* \psi = \sum_{\varphi \in \text{Irr}(B)} \frac{\xi \xi \psi}{\xi \varphi} \lambda_{\chi \psi}^\varphi \Delta^* \varphi.
\]
Thus we conclude that
\[
\frac{\xi \xi \psi}{\xi \varphi} \lambda_{\chi \psi}^\varphi = q_{\chi \psi}^\varphi.
\] (3)
Since \(\frac{\xi \xi \psi}{\xi \varphi} > 0\), equality (3) shows that \(q_{\chi \psi}^\varphi, \varphi \in \text{Irr}(B)\), are nonnegative real numbers if and only if so are \(\lambda_{\chi \psi}^\varphi, \varphi \in \text{Irr}(B)\).

Let \((A, B)\) be a table algebra and \(H \leq B\). For every \(b \in B\), define
\[
\text{St}_H(b) = \{x \in H \mid xb = |x|b = bx\},
\]
and for every subset \(U \subseteq B\), put \(\text{St}_H(U) = \bigcap_{b \in U} \text{St}_H(b)\).

**Lemma 4.2.** Let \((A, B)\) be a commutative table algebra. Suppose that \(K \leq D\) are closed subsets of \(B\) such that \(K \subseteq \text{St}_B(B \setminus D)\). Then for every \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B//K)\) and every \(b \in B \setminus D\), \(\chi(b) = 0\).

**Proof.** Suppose \(\chi \in \text{Irr}(B) \setminus \text{Irr}(B//K)\). Then there exists \(k \in K\) such that \(\chi(k) \neq |k|\). So for every \(b \in B \setminus D\), the equality \(kb = |k|b\) implies that \(\chi(k)\chi(b) = |k|\chi(b)\). Thus we conclude that \(\chi(b) = 0\).

**Theorem 4.3.** Let \((A, B)\) be a commutative table algebra. Suppose that \(K\) and \(D\) are closed subsets of \(B\) such that \(K \leq D\) and \(K \subseteq \text{St}_B(B \setminus D)\). If the duals of table algebras \((<D>, D)\) and \((A//K, B//K)\) are table algebras, then \((\hat{A}, \hat{B})\) is also a table algebra.

**Proof.** From Lemma 4.1, it is enough to prove that for every \(\chi, \psi \in \text{Irr}(B)\), the coefficients \(\lambda_{\chi \psi}^\varphi, \varphi \in \text{Irr}(B)\) in the following product
\[
\chi \psi = \sum_{\varphi \in \text{Irr}(B)} \lambda_{\chi \psi}^\varphi \varphi.
\]
are nonnegative real numbers. To do this, first assume that $\chi,\psi \in \text{Irr}(B/K)$. Since $\widehat{B/K} \cong \ker(K)$ is a closed subset of $\widehat{B}$, it follows that

$$\Delta^*_\chi \Delta^*_\psi = \sum_{\Delta^*_\varphi \in \ker(K)} q^\varphi_{\chi\psi} \Delta^*_\varphi,$$

and so

$$\chi \psi = \sum_{\varphi \in \text{Irr}(B/K)} \lambda^\varphi_{\chi\psi} \varphi.$$

But the dual of table algebra $(A/K,B/K)$ is a table algebra. Then it follows from Lemma 4.1 that $\lambda^\varphi_{\chi\psi}, \varphi \in \text{Irr}(B)$, are nonnegative real numbers, as desired. So we consider the case that $\chi \in \text{Irr}(B/K)$ and $\psi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$. Again, since $\widehat{B/K} \cong \ker(K)$ is a closed subset of $\widehat{B}$, it follows that

$$\Delta^*_\chi \Delta^*_\psi = \sum_{\Delta^*_\varphi \in B - \ker(K)} q^\varphi_{\chi\psi} \Delta^*_\varphi.$$

Then

$$\chi \psi = \sum_{\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)} \lambda^\varphi_{\chi\psi} \varphi.$$

But it follows from Lemma 4.2 that for every $b \in B \setminus D$ and every $\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$ where $\lambda^\varphi_{\chi\psi} \neq 0$, we have $\varphi(b) = 0$. This implies that for every distinct irreducible characters $\varphi, \varphi' \in \text{Irr}(B) \setminus \text{Irr}(B/K)$ where $\lambda^\varphi_{\chi\psi}$ and $\lambda^\varphi'_{\chi\psi}$ are nonzero, $\varphi_D \neq \varphi'_D$, otherwise, if $\varphi_D = \varphi'_D$, then since $\varphi(b) = \varphi'(b) = 0$, for every $b \in B \setminus D$, we have $\varphi = \varphi'$, a contradiction. Thus we conclude that

$$\chi_D \psi_D = \sum_{\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)} \lambda^\varphi_{\chi\psi} \varphi_D.$$

But the dual of table algebra $(<D>,D)$ is a table algebra. It follows that $\lambda^\varphi_{\chi\psi} \geq 0$ for every $\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$ and we are done.

To complete the proof it remains to consider $\chi, \psi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$. Then

$$\chi \psi = \sum_{\varphi \in \text{Irr}(B/K)} \lambda^\varphi_{\chi\psi} \varphi + \sum_{\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)} \mu^\varphi_{\chi\psi} \varphi.$$

Suppose that $\varphi \in \text{Irr}(B/K)$ such that $\lambda^\varphi_{\chi\psi} \neq 0$. Since

$$\frac{\lambda^\varphi_{\chi\psi}}{\zeta_{\varphi}} = \frac{\lambda^\varphi_{\psi}}{\zeta_{\psi}},$$

we have $\lambda^\varphi_{\chi\psi} \neq 0$. But since $\chi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$ and $\varphi \in \text{Irr}(B/K)$, it follows from the preceding case that $\lambda^\varphi_{\chi\psi} \geq 0$. This shows that $\lambda^\varphi_{\chi\psi} \geq 0$. Finally, we prove that $\mu^\varphi_{\chi\psi} \geq 0$ for every $\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/K)$. To do this, we observe that for distinct
irreducible characters \( \varphi, \varphi' \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K) \) such that \( \mu^\varphi_{\chi \psi} \) and \( \mu^\varphi'_{\chi \psi} \) are nonzero, if \( \varphi_D = \varphi'_D \), then since for every \( b \in B \setminus D \), \( \varphi(b) = \varphi'(b) = 0 \), we must have \( \varphi = \varphi' \), a contradiction. Thus we conclude that

\[
\chi_D \psi_D = \sum_{\varphi \in \text{Irr}(B/\!/K)} a^\varphi_{\chi \psi} \varphi_D + \sum_{\varphi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K)} \mu^\varphi_{\chi \psi} \varphi_D.
\]

But the dual of table algebra \((< D >, D)\) is a table algebra. So \( \mu^\varphi_{\chi \psi} \geq 0 \) for every \( \varphi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K) \) and we are done.

**Remark 4.4.** Let \((A, B)\) be a commutative table algebra and \( K \leq D \) be closed subsets of \( B \) such that the duals of \((< D >, D)\) and \((A/\!/K, B/\!/K)\) are table algebras. Since

\[
(< \ker(K), \ker(K)) \cong (A/\!/K, B/\!/K)
\]

and

\[
(\widehat{A/\!/K} \ker(D), \widehat{B/\!/K} \ker(D)) \cong (\widehat{< D >}, \widehat{D}),
\]

it follows that \((\widehat{A}, \widehat{B})\) is a \( C \)-algebra such that \((< \ker(K), \ker(K))\) and \((\widehat{A/\!/K} \ker(D), \widehat{B/\!/K} \ker(D))\) are table algebras. But \((\widehat{A}, \widehat{B})\) need not be a table algebra, in general; see [12, Example 3.5]. So the condition \( K \subseteq \text{St}_B(B \setminus D) \) in the Theorem 4.3 is a necessary condition.

**Corollary 4.5.** Let the commutative table algebra \((U, V)\) be the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to \( \varphi \). If the duals of table algebras \((C, D)\) and \((A, B)\) are table algebras, then the dual of \((U, V)\) is also a table algebra.

**Proof.** It follows from Lemma 3.4 that \( V \) contains the closed subset \( K \) such that \( K \subseteq \text{St}_V(V \setminus D) \) and \( V/\!/K \cong B \). Then since the dual of \((A, B)\) is a table algebra, the dual of \((U/\!/K, V/\!/K)\) is also a table algebra. So Theorem 4.3 yields that the dual of \((U, V)\) is a table algebra.

**Lemma 4.6.** Let \((A, B)\) be a commutative table algebra and \( K \leq D \) be closed subsets of \( B \) such that \( K \subseteq \text{St}_B(B \setminus D) \). Then \( \ker(D) \leq \ker(K) \) and \( \ker(D) \subseteq \text{St}_B(\widehat{B} \setminus \ker(K)) \).

**Proof.** Since \( K \leq D \), we have

\[
\ker(D) = \{ \Delta^*_\chi | \chi \in \text{Irr}(V/\!/D) \} \subseteq \{ \Delta^*_\chi | \chi \in \text{Irr}(V/\!/K) \} = \ker(K).
\]

Moreover, since \( K \subseteq \text{St}_B(B \setminus D) \), it follows from Lemma 4.2 that for every \( b \in B \setminus D \) and every \( \chi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K) \), \( \chi(b) = 0 \). Let \( \chi \in \text{Irr}(B) \setminus \text{Irr}(B/\!/K) \) and \( \psi \in \text{Irr}(B/\!/D) \). Since for every \( b \in D \), \( \psi(b) = |b| \) we have

\[
\chi \psi(b) = \frac{\chi(b) \psi(b)}{|b|} = \chi(b).
\]

On the other hand, for every \( b \in B \setminus D \),

\[
\chi \psi(b) = \frac{\chi(b) \psi(b)}{|b|} = 0 = \chi(b).
\]
Thus we conclude that $\chi_\psi = \chi$. This implies that for every $b \in B$,

$$
\Delta^*_\chi \Delta^*_\psi (b) = \Delta^*_\chi (b) \Delta^*_\psi (b) = \frac{\xi \chi (b^*) \xi \psi (b^*)}{|b|} = \frac{\xi \chi \xi \psi (b^*)}{|b|} = \xi \Delta^*_\chi (b).
$$

So $\Delta^*_\chi \Delta^*_\psi = \xi \Delta^*_\chi$. Since $\hat{B} \setminus \ker(K) = \{ \Delta^*_\chi | \chi \in \text{Irr}(B) \setminus \text{Irr}(B/\langle K \rangle) \}$ we conclude that

$$
\Delta^*_\psi \in \text{St}_B(\hat{B} \setminus \ker(K)).
$$

Hence

$$
\ker(D) = \{ \Delta^*_\psi | \psi \in \text{Irr}(B/\langle D \rangle) \} \subseteq \text{St}_B(\hat{B} \setminus \ker(K)).
$$

\begin{proof}

It follows from Corollary 4.5 that $(\hat{U}, \hat{V})$ is a table algebra. Moreover, from Lemma 3.4 we see that there exists a closed subset $K \leq D$ such that $K \subseteq \text{St}_V(V \setminus D)$. Then Lemma 4.6 shows that $\ker(D) \leq \ker(K)$ and $\ker(D) \subseteq \text{St}_\hat{V}(\hat{V} \setminus \ker(K))$. So the result follows from Theorem 3.6.

\end{proof}

\begin{corollary}

Let commutative table algebra $(U, V)$ be the wedge product of table algebras $(C, D)$ and $(A, B)$ relative to $\varphi$. If the duals of table algebras $(C, D)$ and $(A, B)$ are table algebras, then $(\hat{U}, \hat{V})$ is a table algebra, and is also the wedge product of $(< \ker(K) >, \ker(K))$ and $(\hat{U} \setminus \ker(D), \hat{V} \setminus \ker(D))$ relative to the canonical epimorphism

$$
\pi : (< \ker(K) >, \ker(K)) \to (< \ker(K) > \text{ker}(D), \ker(K) \text{ker}(D)).
$$

\begin{proof}

Suppose $(i)$ holds. Then it follows from Corollary 4.7 that $(\hat{U}, \hat{V})$ is a table algebra, and is also the wedge product of $(< \ker(K) >, \ker(K))$ and $(\hat{U} \setminus \ker(D), \hat{V} \setminus \ker(D))$ relative to the canonical epimorphism

$$
\pi : (< \ker(K) >, \ker(K)) \to (< \ker(K) > \text{ker}(D), \ker(K) \text{ker}(D)).
$$

But from Lemma 3.5 we have $V/\langle K \rangle \cong B$ and so

$$
\ker(K) = V/\langle K \rangle \cong \hat{B}.
$$

Then $(< \ker(K) >, \ker(K)) \cong (\hat{A}, \hat{B})$. Moreover, it follows from Lemma 3.5 that

$$(\hat{U} \setminus \ker(D), \hat{V} \setminus \ker(D)) \cong (\hat{C}, \hat{D}).$$

\end{proof}

\end{corollary}
So \((\hat{U}, \hat{V})\) is the wedge product of \((\hat{A}, \hat{B})\) and \((\hat{C}, \hat{D})\) relative to

\[
\pi : (\hat{A}, \hat{B}) \to (\ker(K)/\ker(D), \ker(K)/\ker(D)).
\]

On the other hand, \(\ker(K)/\ker(D) \cong \hat{D}/\hat{K}\) and \(D/K \cong N\). So we deduce that \(\ker(K)/\ker(D) \cong \hat{N}\). Hence \((\hat{U}, \hat{V})\) is the wedge product of \((\hat{A}, \hat{B})\) and \((\hat{C}, \hat{D})\) relative to

\[
\hat{\varphi} : (\hat{A}, \hat{B}) \to (\langle N \rangle, \hat{N}).
\]

Now assume that \((ii)\) holds. Since \((\hat{A}, \hat{B}) \cong (A, B)\) and \((\hat{C}, \hat{D}) \cong (C, D)\) are table algebras, it follows from the first part of the proof that \((\hat{U}, \hat{V}) \cong (U, V)\) is the wedge product of table algebras \((C, D)\) and \((A, B)\) relative to

\[
\hat{\varphi} = \varphi : (C, D) \to (\langle N \rangle, N)
\]

and thus \((i)\) holds. \(\blacksquare\)

5 Applications to association schemes

The wedge product of association schemes which provides a way to construct new association schemes from old ones, has been given by Muzychuk in [9]. In the following we have a look at the wedge product of association schemes. The reader is referred to [9] for more details.

Let \((X, G)\) be an association scheme and \(D \leq G\). Suppose that \(X/D = \{x_1D, \ldots, x_mD\}\). Put \(X_i = x_iD\) and \(D_i = D_{X_i} = \{d_{X_i} \mid d \in D\}\), where \(d_{X_i} = d \cap X_i \times X_i\). Consider the bijection \(\varepsilon_i : D \to D_i\) such that \(\varepsilon_i(d) = d_{X_i}\). Then \(\varepsilon_j\varepsilon_i^{-1} : D_i \to D_j\) is an algebraic isomorphism between association schemes \((X_i, D_i)\) and \((X_j, D_j)\). Assume that for every \(i\), there exists an association scheme \((Y_i, B_i)\) and a scheme normal epimorphism \(\psi_i : Y_i \cup B_i \to X_i \cup D_i\). Moreover, assume that there exist algebraic isomorphisms \(\varphi_i : B_1 \to B_i\) such that the diagram

\[
\begin{array}{ccc}
B_1 & \xrightarrow{\psi_i} & B_i \\
\downarrow \psi_1 & & \downarrow \psi_i \\
D_1 & \xrightarrow{\varepsilon_i\varepsilon_i^{-1}} & D_i
\end{array}
\]

is commutative for every \(i\). Assume that \(Y_i, 1 \leq i \leq m, \) are pairwise disjoint. Put \(Y = Y_1 \cup \cdots \cup Y_m, \psi = \psi_1 \cup \cdots \cup \psi_m, \bar{G} = \{g \mid g \in G\}, \) where \(\bar{g} = \psi^{-1}(g)\), and for every \(b \in B_1, \bar{b} = \biguplus_{i=1}^m \varphi_i(b)\). Let \(\mathbb{C}[\bar{B}]\) be the \(\mathbb{C}\)-space spanned by \(\bar{B} = \{\bar{b} \mid b \in B_1\}\). Then it follows from [9, Theorem 2.2] that \(U = \mathbb{C}[\bar{G}] + \mathbb{C}[\bar{B}]\) is the adjacency algebra.
of an association scheme \((Y, \widetilde{B}_1 \cup (C \setminus D))\), which is called the *wedge* product of \((Y_i, B_i), 1 \leq i \leq m, \) and \((X, \widetilde{G})\).

In the rest of this section, we show that the complex adjacency algebra of the wedge product of \((Y_i, B_i), 1 \leq i \leq m\), and \((X, \widetilde{G})\) is the wedge product of table algebras \((\mathbb{C}[\widetilde{B}_1], A(\widetilde{B}_1))\) and \((\mathbb{C}[G], A(G))\). We also study applications to association schemes.

Put \(K = \ker \psi_1\) and \(V = \{A(x) \mid x \in \widetilde{B}_1 \cup (C \setminus D)\}\). Then \((U, V)\) is a table algebra and \(A(\tilde{K}) \leq A(\widetilde{B}_1)\). It follows from [9, Theorem 2.2] that \(A(\tilde{K}) \leq V\) and \(A(\tilde{K}) \leq \text{St}_V(V \setminus A(\widetilde{B}_1))\). Since \((\mathbb{C}[\widetilde{B}_1], A(\widetilde{B}_1))\) is a table algebra, it follows from Corollary 3.7 that \((U, V)\) is the wedge product of table algebras \((\mathbb{C}[\widetilde{B}_1], A(\widetilde{B}_1))\) and \((U//A(\tilde{K}), V//A(\tilde{K}))\) relative to the canonical epimorphism

\[
\pi : (\mathbb{C}[\widetilde{B}_1], A(\widetilde{B}_1)) \to (\mathbb{C}[\widetilde{B}_1]//A(\tilde{K}), A(\widetilde{B}_1)//A(\tilde{K})).
\]

Since \(\psi_1 : (Y_1, B_1) \to (X_1, D_1)\) is a scheme epimorphism and

\[
\epsilon_1^{-1} : (\mathbb{C}[D_1], A(D_1)) \to (\mathbb{C}[D], A(D))
\]

is an algebra isomorphism, it follows that there is a table algebra epimorphism

\[
\overline{\psi}_1 : (\mathbb{C}[B_1], A(B_1)) \to (\mathbb{C}[D], A(D))
\]

such that

\[
\overline{\psi}_1(A(b)) = \frac{n_b}{n_{\psi_1(b)}} A(d)
\]

where \(\text{Supp}(\overline{\psi}_1(A(b))) = A(d)\); see Lemma 2.8. Then we can define a linear map

\[
\varphi : (\mathbb{C}[\widetilde{B}_1], A(\widetilde{B}_1)) \to (\mathbb{C}[D], A(D))
\]

by \(\varphi(A(\tilde{b})) = \overline{\psi}_1(A(b))\).

**Lemma 5.1.** With the notation above, \(\varphi\) is a table algebra epimorphism such that \(\ker_{A(\widetilde{B}_1)} \varphi = A(\tilde{K})\).

**Proof.** For every \(\tilde{a}, \tilde{b} \in \widetilde{B}_1\), it follows from [9] that \(A(\tilde{a})A(\tilde{b}) = \sum_{c \in B_1} \lambda_{abc} A(c)\), where \(\lambda_{abc}, c \in B_1\), are the structure constants of \((\mathbb{C}[B_1], A(B_1))\). Then we have

\[
\varphi(A(\tilde{a})A(\tilde{b})) = \varphi(\sum_{c \in B_1} \lambda_{abc} A(c))
\]

\[
= \sum_{c \in B_1} \lambda_{abc} \varphi(A(c)) = \sum_{c \in B_1} \lambda_{abc} \overline{\psi}_1(A(c))
\]

\[
= \overline{\psi}_1(\sum_{c \in B_1} \lambda_{abc} A(c)) = \overline{\psi}_1(A(\tilde{a})A(\tilde{b}))
\]

\[
= \overline{\psi}_1(A(\tilde{a}))(\overline{\psi}_1(A(\tilde{b}))) = \varphi(A(\tilde{a})) \varphi(A(\tilde{b})).
\]
So $\varphi$ is a table algebra epimorphism. Note that $\psi_1$ is a scheme epimorphism. Moreover, we have

$$\ker_{A(B_1)} \varphi = \{ A(\tilde{b}) \in A(B_1) \mid \text{Supp}(\varphi(A(\tilde{b}))) = A(1_X) \}$$

$$= \{ A(\tilde{b}) \in A(B_1) \mid \psi_1(A(b)) = n_b A(1_X) \}$$

$$= \{ A(\tilde{b}) \in A(B_1) \mid b \in \ker_{A(B_1)}(\psi_1) \}$$

$$= \{ A(\tilde{b}) \in A(B_1) \mid b \in \ker(\psi_1) \}$$

$$= A(K).$$

Lemma 5.2. With the notation above, we have $n_K = \frac{|Y|}{|X|}$.

Proof. It follows from [9, Proposition 2.1] that $A(i_X^A)A(g) = \frac{|Y|}{|X|} A(g)$. But $i_X = \psi^{-1}(i_X) = K$. So $A(i_X) = A(K)$ and thus $n_K = \frac{|Y|}{|X|}$.

Now consider table algebras $(\mathbb{C}[B_1], A(B_1))$ and $(\mathbb{C}[G], A(G))$. Since $A(D) \leq A(G)$ and there exists the table algebra epimorphism $\varphi : (\mathbb{C}[B_1], A(B_1)) \to (\mathbb{C}[D], A(D))$, we can construct the wedge product of $(\mathbb{C}[B_1], A(B_1))$ and $(\mathbb{C}[G], A(G))$ relative to $\varphi$, say $(E, F)$. Then

$$F = A(B_1) \cup \{ n_K A(g) \mid g \in G \setminus D \}.$$  

Note that for every $A(\tilde{b}) \in A(B_1)$ with $\text{Supp}(\psi_1(A(b))) = A(d)$, and $A(g) \in A(G) \setminus A(D)$ we have

$$A(\tilde{b})(n_K A(g)) = n_K \varphi(A(\tilde{b})) A(g) = n_K \psi_1(A(b)) A(g) = n_K \frac{n_b}{n_{\psi_1}(b)} A(d) A(g), \quad (4)$$

and similarly,

$$(n_K A(g))A(\tilde{b}) = n_K A(g) \psi_1(A(b)) = n_K \frac{n_b}{n_{\psi_1}(b)} A(g) A(d).$$

Theorem 5.3. The table algebras $(U, V)$ and $(E, F)$ are isomorphic.

Proof. Define $\theta : (U, V) \to (E, F)$ by $\theta(A(\tilde{b})) = A(\tilde{b})$ and $\theta(A(\overline{g})) = n_K A(g)$, for every $\tilde{b} \in \widetilde{B}_1$ and every $\overline{g} \in \overline{C} \setminus \overline{D}$. We first show that $\theta$ is a table algebra homomorphism. To do so, first assume that $\overline{g}, \overline{h} \in \overline{G} \setminus \overline{D}$. Then it follows from [9, Proposition 2.1] that

$$A(\overline{g})A(\overline{h}) = \frac{|Y|}{|X|} \sum_{l \in G} \lambda_{ghl} A(l).$$

So we have
\[
\theta(A(\overline{g})A(\overline{h})) = \theta\left(\frac{|Y|}{|X|} \sum_{l \in G} \lambda_{ghl}A(l)\right)
\]
\[
= \frac{|Y|}{|X|} \sum_{l \in G} \lambda_{ghl} \theta(A(l)) = \frac{|Y|}{|X|} n_{\overline{K}} \sum_{l \in G} \lambda_{ghl} A(l)
\]
\[
= n_{\overline{K}}^2 (A(\overline{g})A(\overline{h})) = \theta(A(\overline{g})) \theta(A(\overline{h})).
\]

Note that from Lemma 5.2 we have \(n_{\overline{K}} = \frac{|Y|}{|X|}\). Now suppose that \(\overline{b} \in \overline{B}_1\) and \(\overline{g} \in \overline{G}\setminus \overline{D}\). Then it follows from [9, Theorem 2.2] that
\[
A(\overline{b})A(\overline{g}) = \frac{n_b |X|}{n_{\psi_1(b)} |Y|} A(\overline{d})A(\overline{g}).
\]

Moreover, from [9, Proposition 2.1] we have
\[
A(\overline{d})A(\overline{g}) = \frac{|Y|}{|X|} \sum_{l \in G} \lambda_{dgl} A(l).
\]

Thus by applying equality (4) we see that
\[
\theta(A(\overline{b})A(\overline{g})) = \frac{n_b |X|}{n_{\psi_1(b)} |Y|} \theta(A(\overline{d})A(\overline{g}))
\]
\[
= \frac{n_b |X|}{n_{\psi_1(b)} |Y|} \theta\left(\frac{|Y|}{|X|} \sum_{l \in G} \lambda_{dgl} A(l)\right) = \frac{n_b}{n_{\psi_1(b)}} \sum_{l \in G} \lambda_{dgl} \theta(A(l))
\]
\[
= \frac{n_b}{n_{\psi_1(b)}} n_{\overline{K}} \sum_{l \in G} \lambda_{dgl} A(l) = \frac{n_b}{n_{\psi_1(b)}} n_{\overline{K}} A(d)A(g)
\]
\[
= \left(\frac{n_b}{n_{\psi_1(b)}} A(d)\right)(n_{\overline{K}} A(g)) = A(\overline{b})(n_{\overline{K}} A(g))
\]
\[
= \theta(A(\overline{b})) \theta(A(\overline{g})).
\]

Similarly, \(\theta(A(\overline{g})A(\overline{b})) = \theta(A(\overline{g})) \theta(A(\overline{b}))\). So \(\theta\) is a table algebra homomorphism. But \(\theta\) is also a bijection from \(V\) onto \(F\). Thus \(\theta\) is a table algebra isomorphism, as desired.

\begin{corollary}
Let \(U = \mathbb{C}[\overline{G}] + \mathbb{C}[\overline{B}_1]\) be the complex adjacency algebra of the wedge product of \((Y_i, B_i), 1 \leq i \leq m,\) and \((X, G)\). Then table algebra \((U, V)\), where \(V = \{A(x) \mid x \in B_1 \cup (\overline{C} \setminus \overline{D})\}\), is the wedge product of table algebras \((\mathbb{C}[B_1], A(\overline{B}_1))\) and \((\mathbb{C}[G], A(G))\) relative to \(\varphi\), where
\[
\varphi : (\mathbb{C}[\overline{B}_1], A(\overline{B}_1)) \to (\mathbb{C}[D], A(D))
\]
such that \(\varphi(A(\overline{b})) = \overline{\psi_1(A(b))}\).

By applying Theorem 3.5 and Corollary 5.4, we can give the following result.
Theorem 5.5. Let \( U = \mathbb{C}[\mathcal{G}] + \mathbb{C}[\widetilde{B}_1] \) be the complex adjacency algebra of the wedge product of \((Y_i, B_i), 1 \leq i \leq m, \) and \((X, G)\). Then
\[
U/\mathbb{K} \cong A(G).
\]

As a direct consequence of Theorem 4.8 and Corollary 5.4, we have the following:

Theorem 5.6. Suppose that \( U = \mathbb{C}[\mathcal{G}] + \mathbb{C}[\widetilde{B}_1] \), the complex adjacency algebra of the wedge product of \((Y_i, B_i), 1 \leq i \leq m, \) and \((X, G)\), is commutative. Then the table algebra \((\widehat{U}, \widehat{V})\), where \( V = \{ A(x) \mid x \in \widetilde{B}_1 \cup (\mathcal{C} \setminus \mathcal{D}) \} \), is the wedge product of table algebras \((\widehat{\mathbb{C}[\mathcal{G}]}, \widehat{A(G)})\) and \((\widehat{\mathbb{C}[\widetilde{B}_1]}, \widehat{A(\widetilde{B}_1)})\) relative to \( \widehat{\varphi} \), where
\[
\widehat{\varphi} : (\widehat{\mathbb{C}[\mathcal{G}]}, \widehat{A(G)}) \to (\widehat{\mathbb{C}[\mathcal{D}]}, \widehat{A(D)}).
\]

References

[1] Z. Arad, E. Fisman, M. Muzychuk, Generalized table algebras, Israel J. Math. 114 (1999) 29-60.

[2] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, Menlo Park, 1984.

[3] J. Bagherian, A. R. Barghi, Burnside-Brauer theorem for table algebras, Electron. J. Combin. 18(2011), P204.

[4] H. I. Blau, B. Xu, Irreducible characters of wreath products in reality-based algebras and applications to association schemes, J. Algebra 412 (2014) 155-172.

[5] H. I. Blau, Table algebras, European J. Combin. 30 (2009) 1426-1455.

[6] H. I. Blau, Quotient structures in C-algebras, J. Algebra 177 (1995) 297-337.

[7] S. Evdokimov, I. Ponomarenko and A. Vershik, Algebras in plancherel duality and algebraic combinatorics, Functional Analysis and its Applications, 31 no. 4 (1997) 34-46.

[8] D. G. Higman, Coherent algebras, J. Linear Algebra Appl. 93 (1987) 209-239.

[9] M. Muzychuk, A wedge product of association schemes, European J. Combin. 30 (2009) 705-715.

[10] A. Rahnamai Barghi, J. Bagherian, Standard character condition for table algebras, Electron. J. Combin. 17. (2010) 1-13.

[11] B. Xu, Characters of table algebras and applications to association schemes. J. Combin. Theory Ser. A 115 (2008), 1258-1373.

[12] B. Xu, Morphisms and functor ker f in C-algebras, Algebra Colloq. 15 (2008) 145-166.
[13] B. Xu, Some structure theory of table algebras and applications to association schemes, J. Algebra 325 (2011) 97-131.

[14] B. Xu, On wreath products of C-algebras, table algebras, and association schemes, J. Algebra 344 (2011) 268-283.

[15] P.-H. Zieschang, An Algebraic Approach to Association Schemes, in: Lect. Notes in Math., vol. 1628, Springer-Verlag, Berlin, 1996.