1 Introduction

Let \( p \) be an odd prime and \( k = \mathbb{F}_p(T) \). Let \( \lambda \) be a non-zero root of \( x^p + Tx = 0 \), and \( \gamma \) be a root of \( x^p + Tx = P(T) \), where \( P(T) \in k \) but \( P(T) \neq Q(T)^p + TQ(T) \) for any \( Q(T) \in k \). We have the following lattice of fields:\(^1\)

\[
\begin{array}{c}
\Delta \\
F = k(\gamma) \\
k = \mathbb{F}_p(T)
\end{array}
\quad
\begin{array}{c}
\Omega \\
K = k(\lambda) \\
\cong \Delta
\end{array}
\quad
\begin{array}{c}
\Delta \\
L = k(\lambda, \gamma)
\end{array}
\]

Our main objective is to prove the following theorems about \( \text{Cl}^0_L \), the group of degree-0 divisor classes\(^2\) of \( L \):

**Theorem 1.1.** \( \text{Cl}^0_L \) is isomorphic to a \((p-1)\)-st power of a finite abelian group.

**Theorem 1.2.** The class numbers of \( L \) and \( F \) are related via \( h_L = h_F^{p-1} \) (and in fact this holds when \( p \) is a prime power).

When the \( \ell \)-rank of \( \text{Cl}^0_F \) is 1 for every prime \( \ell \) dividing \( h_F \) (e.g. when \( h_F \) is

---

\(^1\)The edge labels indicate the Galois groups of the corresponding extensions, which will be described in Section 3.

\(^2\)We take the divisor group of a function field to mean the free abelian group indexed by its primes. Its quotient by the principal divisors (those which represent an element of the field) is the divisor class group. The subgroup of elements with total exponent 0 is then the degree-0 divisor class group.
square-free), these combine to say that $Cl_L^0 = (Cl_F^0)^{p-1}$. However, this is not known in general.

In Section 2, we motivate these theorems and provide some background on the fields being considered, and the remaining sections are devoted to the proofs.

2 Preliminaries

2.1 Number field antecedents

Theorem 1.1 is an analogue of a recent result of Schoof in the number field setting. In [8], he proves the following:

\[ \text{Theorem 2.1. Let } p > 2 \text{ be a regular prime and } n \in \mathbb{Z} \text{ not a } p\text{-th power. Suppose that all prime divisors } l \neq p \text{ of } n \text{ are primitive roots mod } p. \text{ Then the ideal class group } Cl_L \text{ of } L = \mathbb{Q}(\zeta_p, \sqrt[p]{n}) \text{ and the kernel of the norm map } N_{L/\mathbb{Q}}(\zeta_p) \text{ fit into the exact sequences} \]

\[ 0 \to V \to \ker(N_{L/\mathbb{Q}}(\zeta_p)) \to A^{p-1} \to 0 \text{ and} \]

\[ 0 \to \ker(N_{L/\mathbb{Q}}(\zeta_p)) \to Cl_L \to Cl_{\mathbb{Q}(\zeta_p)} \to 0, \]

where $A$ is a finite abelian group and $V$ is an $\mathbb{F}_p$-vector space of dimension at most $\left(\frac{p-3}{2}\right)^2$. In particular, if $\#Cl_{\mathbb{Q}(\zeta_p)} = 1$, then $Cl_L/V$ is a $(p-1)$-st power of a finite abelian group.

Theorem 1.2 is inspired by one proved by Honda [5]:

\[ \text{Theorem 2.2. Let } F = \mathbb{Q}(\sqrt[p]{n}), \text{ and } L = \mathbb{Q}(\sqrt[p]{n}, \zeta_3) \text{ be its normal closure. Then } h_L = h_F^2 \text{ or } h_L = \frac{1}{3}h_F^2. \]

2.2 Function field analogue of $\mathbb{Q}(\zeta_p)$

We return now to considering extensions of $k = \mathbb{F}_p(T)$. Let $\Lambda$ denote the roots (in $k$, an algebraic closure) of $x^p + Tx = 0$. Fixing a nonzero root $\lambda$, we see that $\Lambda = \{m\lambda \mid m \in \mathbb{F}_p\} \cong \mathbb{F}_p^+$, and that $k(\Lambda) = k(\lambda)$ is a degree $p - 1$ cyclic extension of $k$. These definitions should be reminiscent of those for the $p$-th roots of unity $\mu_p = \{\zeta_p^i \mid i \in \mathbb{F}_p\}$, and in fact they are part of a rich theory of ‘cyclotomic’ function fields first developed by Hayes [4] based on work by Carlitz [2]. For a more thorough discussion of the connection, we refer to [3].

We note two properties which will be important to the proofs of Theorems 1.1 and 1.2 and which contribute to their comparative simplicity vis-à-vis Theorems 2.1 and 2.2. The first is that $k(\Lambda) = k(\lambda)$ has genus 0 (since $T = -\lambda^{p-1}$), and therefore its degree-0 divisor class group is trivial. The second is that there are
exactly two primes which ramify in \( k(\lambda)/k \), both of which are totally ramified: the prime \((T) = (\lambda)^{p-1}\), and the distinguished infinite prime \((1/T) = (1/\lambda)^{p-1}\). This is fairly easy to see in our case from the discriminant, but also applies to a more general class of such extensions [1].

2.3 Function field analogue of \( \mathbb{Q}(\sqrt[n]{n}) \)

Now let \( \gamma \) denote a root of \( x^p + Tx = P(T) \), where \( P(T) \in k \) but \( P(T) \neq Q(T)^p + TQ(T) \) for any \( Q(T) \in k \). Then \( \gamma + \lambda, \gamma + 2\lambda, \ldots, \gamma + (p-1)\lambda \) are the other roots, and the distinguished infinite prime \((1/T) = (1/\lambda)^{p-1}\).

By discriminant considerations, the only primes that possibly ramify in \( k(\lambda, \gamma)/k(\lambda) \) are those lying above \((T)\) and \((1/T)\). Therefore every prime of \( k \) which ramifies in \( k(\lambda, \gamma)/k \) is totally ramified in \( k(\lambda)/k \).

3 Proof of Theorem 1.1

As in the introduction, we set \( k = \mathbb{F}_p(T), K = k(\lambda), F = k(\gamma), \) and \( L = k(\lambda, \gamma). \) Define \( \Omega = \text{Gal}(L/k), G = \text{Gal}(L/K), \) and \( \Delta = \text{Gal}(L/F) \cong \text{Gal}(K/k). \) These groups have the presentations

- \( G = \langle \tau | \tau^p = 1 \rangle \cong \mathbb{F}_p^\times, \)
- \( \Delta = \langle \sigma | \sigma^{p-1} = 1 \rangle \cong \mathbb{F}_p^\times, \)
- \( \Omega = \langle \sigma, \tau | \sigma^{p-1} = 1, \tau^p = 1, \sigma \tau \sigma^{-1} = \tau^{\omega(\sigma)} \rangle, \)

where \( \omega : \Delta \to \mathbb{F}_p^\times \) is the cyclotomic character defined by \( \sigma(\lambda) = \omega(\sigma)\lambda. \) Refer to the field diagram in the introduction for a depiction of these relationships.

Naturally, there is a Galois action of \( \Omega \) on \( \text{Cl}^0_L \). The norm element \( N_G = \sum G \tau \) of \( G \) gives a map \( \text{Cl}^0_L \to \text{Cl}^0_K \) that factors through \( \text{Cl}^0_K \), which is trivial. Thus \( \text{Cl}^0_L \) is a module over the group ring \( \mathbb{Z}[\Omega]/(N_G) \), or alternately a module over \( \mathbb{Z}[\Delta]/(N_G) \) with a twisted action of \( \Delta \) (by which we mean that \( \Delta \) acts on \( \text{Cl}^0_L \) in a way that is consistent with the action of \( \Delta \) on \( \mathbb{Z}[\Delta]/(N_G) \)). Now, \( \mathbb{Z}[\Delta]/(N_G) \cong \mathbb{Z}[\zeta_p] \) as a \( \Delta \)-module (because \( N_G \) is the \( p \)-th cyclotomic polynomial evaluated at \( \tau \)), so we may freely apply standard facts about \( \zeta_p \) to \( \mathbb{Z}[\zeta_p] \).

We are now ready to develop the proof of Theorem 1.1. We proceed by separately considering the \( p \) part and the non-\( p \) part of \( \text{Cl}^0_L \).

\[^3\text{We opt to keep the notation in terms of } \tau \text{ rather than } \zeta_p, \text{ to maintain coherence with the function field setting.}\]
3.1 The non-$p$ part of $Cl^0_L$

In this section, let $M$ denote the non-$p$ part of the degree-0 divisor class group of $L$. The following proposition describes the $\Delta$-module structure of $M$.

**Proposition 3.1.** The map

$$\varphi : M^{\Delta} \otimes_{\mathbb{Z}[G]/(N_G)} \mathbb{Z} \rightarrow M$$

given by $\sum_i m_i \otimes [\tau^i] \mapsto \sum_i \tau^i m_i$ is an isomorphism of $\Delta$-modules.

**Proof.** Suppose first that $\sum \tau^i m_i = 0$ (and note that since $N_G$ acts trivially, this sum can be assumed to be over $1 \leq i \leq p - 1$). Then $\sum \tau^{i\omega(\sigma)} m_i = 0$ for all $\sigma$, and thus for $1 \leq j \leq p - 1$,

$$0 = \sum_{\sigma \in \Delta} \tau^{-j\omega(\sigma)} (1 - \tau^{j\omega(\sigma)}) \sum_i \tau^{i\omega(\sigma)} m_i$$

$$= \sum_i \sum_{\sigma \in \Delta} (1 - \tau^{j\omega(\sigma)}) \tau^{(i-j)\omega(\sigma)} m_i.$$

The sum $\sum_{\sigma} (1 - \tau^{j\omega(\sigma)}) \tau^{(i-j)\omega(\sigma)}$ acts as $p - 1 + (1 - N_G) = p$ for $i = j$, and as 0 for $i \neq j$, so this says $pm_j = 0$, and thus $m_j = 0$, for all $j$ (since $M$ has order prime to $p$). Therefore $\varphi$ is injective.

Now suppose $m \in M$. Then for any $i$, $N_{\Delta}(\tau^i m) = \sum_\sigma \tau^{i\omega(\sigma)} \sigma(m) \in M^{\Delta}$, and accordingly,

$$\sum_{i=1}^{p-1} \tau^{-i\omega(\sigma')} (1 - \tau^{i\omega(\sigma')}) \sum_\sigma \tau^{i\omega(\sigma)} \sigma(m)$$

$$= \sum_\sigma \sum_{i=1}^{p-1} (1 - \tau^{i\omega(\sigma')}) \tau^{(\omega(\sigma) - \omega(\sigma'))} \sigma(m) \in im(\varphi) \text{ for all } \sigma' \in \Delta.$$

The sum $\sum_{\sigma'} (1 - \tau^{i\omega(\sigma')}) \tau^{(\omega(\sigma) - \omega(\sigma'))}$ acts as $p$ for $\sigma' = \sigma$ and as 0 for $\sigma' \neq \sigma$, so this says that each $p\sigma'(m)$, and in particular $pm$, is in $im(\varphi)$. Therefore $\varphi$ is surjective (again using that $M$ has order prime to $p$).

Ignoring the module structure gives $M \cong (M^{\Delta})^{p-1}$ as abelian groups, which settles the non-$p$ part of Theorem 1.1.

3.2 The $p$ part of $Cl^0_L$

From this section forward, $M$ will denote the $p$ part of the degree-0 divisor class group of $L$. Having only $p$-power torsion allows us to strengthen the $\mathbb{Z}[G]/(N_G)$-module structure previously described to a $\mathbb{Z}_p[G]/(N_G)$-module structure, still
Proof. Suppose first that the isomorphism holds. Then such that \( \Delta \)-action:  

\[
\text{Lemma 3.2.} \quad \text{Let } M' \text{ be a finite } A\text{-module with twisted } \Delta\text{-action. Then } \Delta \text{ acts trivially on } M'/((\tau-1)M') \text{ if and only if there exist } n_1, n_2, \ldots, n_t \geq 1 \text{ such that } \\
M' \cong \bigoplus_{i=1}^{t} A/((\tau-1)^{n_i}A).
\]

Proof. Suppose first that the isomorphism holds. Then \( M'/((\tau-1)M') \) is a direct sum of \( A/((\tau-1)A) \cong \mathbb{F}_p \) terms with trivial \( \Delta \)-action.

Conversely, suppose that \( \Delta \) acts trivially on \( M'/((\tau-1)M') \). Then the map \( M'\Delta \rightarrow (M'/((\tau-1)M'))\Delta = M'/((\tau-1)M') \) is surjective (its cokernel is the first \( \Delta \)-cohomology group of \( (\tau-1)M' \), which is trivial because \( \Delta \) and \( M' \) have coprime orders). This says that there are \( \Delta \)-invariant elements \( v_1, \ldots, v_t \) which generate \( M' \) over \( A \), i.e. that there is a surjective map from \( A^t \rightarrow M' \) taking 1 in the \( i \)-th coordinate to \( v_i \). By the finiteness of \( M' \), this descends to a surjective map  

\[
\varphi : \bigoplus_{i=1}^{t} A/((\tau-1)^{n_i}A) \rightarrow M'.
\]

If \( \varphi \) is not injective, there is a nonzero element \( x \) in the kernel, and we may assume \( x = (x_1, \ldots, x_t) \in \bigoplus_{i=1}^{t} (\tau-1)^{m_i+1}A/((\tau-1)^{n_i}A) \). Now, \( \Delta \) acts on \( x \) by \( \omega^m \) for some \( m \), but by \( \omega^{n_i-1} \) on each of these summands, so \( x_i \) is zero unless \( n_i - 1 = m + k_i(p-1) \) for some \( k_i \). Reordering if necessary, let the first \( s \) coordinates of \( x \) be exactly those which are nonzero, and choose \( n_1 \) to be minimal among \( n_1, \ldots, n_s \). For \( i = 1, \ldots, s \), define \( \mu_i \) such that \( x_i = (\tau-1)^{m_i}p^{k_i}\mu_i \), and \( m_i \in \mathbb{Z} \) such that \( m_i \equiv \mu_i \mod (\tau-1)^N \), where \( N \) is the maximum of the \( n_i \). Notice that \( \mu_i \), and thus \( m_i \), is a unit in \( A \).

We are now able to construct a new map  

\[
\varphi' : A/((\tau-1)^{n_1+1}A \oplus \bigoplus_{i=2}^{t} A/((\tau-1)^{n_i}A) \rightarrow M',
\]
which takes the basis vector \( e_1 = (1, 0, \ldots, 0) \) to \( \sum_{i=1}^{s} m_i p^{k_i} v_i \), and \( e_i \) to \( v_i \) for \( i \neq 1 \). This is well-defined because

\[
\varphi'((\tau - 1)^m p^{k_1} e_1) = \sum_{i=1}^{s} (\tau - 1)^m p^{k_1} m_i v_i = \sum_{i=1}^{s} (\tau - 1)^m p^{k_1} \mu_i v_i = \sum_{i=1}^{s} x_i v_i = \varphi(x) = 0.
\]

Furthermore, \( \varphi' \) is surjective because, by the surjectivity of \( \varphi \), \( m_1 v_1 \in \text{im}(\varphi') \), and \( m_1 \) is invertible. If \( \varphi' \) is not injective, we repeat this procedure until we reach a map that is, at which point we will have found a direct sum of finite quotients of \( A \) which is isomorphic to \( M' \).

**Proposition 3.3.** Let \( M' \) be a finite \( A \)-module with twisted \( \Delta \)-action, such that \( \Delta \) acts trivially on \( M'/((\tau - 1) A) \) and by \( \omega^{-1} \) on \( M'[(\tau - 1)] \). Then there exists a finite abelian \( p \)-group \( H \) such that

\[
M \cong H \otimes_{\mathbb{Z}} A.
\]

**Proof.** Suppose \( M' \cong A/((\tau - 1)^n A) \) for some positive integer \( n \). Then \( M'[(\tau - 1)] = (\tau - 1)^{n-1} A/((\tau - 1)^n A) \cong \mathbb{F}_p(\omega^{n-1}) \). By assumption, this requires \( n = (p - 1)m \) for some positive integer \( m \), whereby \( A/((\tau - 1)^n A) \cong A/p^m A \cong \mathbb{Z}/p^m \mathbb{Z} \otimes_{\mathbb{Z}} A \).

By the previous lemma, a general \( M' \) satisfying the condition on \( M'/((\tau - 1) A) \) is a direct sum of such \( \mathbb{Z}/p^m \mathbb{Z} \otimes_{\mathbb{Z}} A \), proving the proposition.

It remains to show that this proposition can be applied to (a twist of) \( M \). We will achieve this by exploring the Galois cohomology of \( \text{Cl}_L^0 \) and related objects.

### 3.3 Galois cohomology of \( \text{Cl}_L^0 \)

In this section, \( \hat{H}^i(X) \) is the \( i \)-th Tate \( G \)-cohomology group of \( X \). We fix the notations

- \( P_L \), the principal \( L \)-divisors
- \( D_L^0 \), the \( L \)-divisors of degree 0
- \( \mathbb{N}_L \), the ideles of total valuation 0
- \( C_L^0 = \mathbb{N}_L/L^\times \), the idele classes of total valuation 0
- \( U_L \), the product of the local unit groups \( U_L \) of \( L \).
We have the following commutative diagram of $\Omega$-modules in which the rows and columns are exact:

\[
\begin{array}{ccc}
1 & 1 & 0 \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & F^\times_p \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & U_L \\
\downarrow & \downarrow & \downarrow \\
1 & \rightarrow & U_L/F^\times_p \\
\end{array}
\]

We will make use of various long exact sequences induced in cohomology by the above diagram in order to study $\hat{H}^i(C\Omega^1_L)$ for $i = -1, 0$. We remark that a $G$-cohomology group of an $\Omega$-module is an $F_p[\Delta]$-module (because it is killed by $p$ and is $G$-invariant), and any map induced in cohomology by any of the maps in the diagram is $\Delta$-equivariant. Furthermore, since $G$ is a cyclic group, we have $\Delta$-isomorphisms $\hat{H}^i(X) \rightarrow \hat{H}^{i-2}(X)(\omega^{-1})$ for every $i \in \mathbb{Z}$ and $\Omega$-module $X$, given by cupping with a generator of $\hat{H}^{-2}(\mathbb{Z}) \cong H_1(\mathbb{Z}) \cong G \cong \mathbb{Z}/p\mathbb{Z}(\omega)$.

**Lemma 3.4.** $C\Omega^1_L$ and $F^\times_p$ have trivial Tate $G$-cohomology.

**Proof.** The degree map on the idele class group gives rise to the sequence

\[0 \rightarrow C\Omega^1_L \rightarrow C_L \rightarrow \mathbb{Z} \rightarrow 0.\]

This gives rise to a long exact sequence which includes

\[H^0(C_L) \rightarrow H^0(\mathbb{Z}) \rightarrow \hat{H}^1(C\Omega^1_L) \rightarrow \hat{H}^1(C_L),\]

where the first two terms are standard (i.e. non-Tate) cohomology. We have that $H^0(C_L) = C\Omega^1_L = C_K$ [11 p. 2]. The leftmost map, then, is the degree map on $C_K$ as a subgroup of $C_L$. An idele class of $C_K$ which has valuation 1 at an inert prime and valuation 0 elsewhere maintains this property when extended to $C_L$. Since $G$ is cyclic, there are (infinitely many) primes inert in $L/K$ by the Chebotarev density theorem, and thus the map $H^0(C_L) \rightarrow H^0(\mathbb{Z})$ is surjective. We also have $\hat{H}^1(C_L) = 0$ [11 p. 19], and so $\hat{H}^1(C\Omega^1_L) = 0$.

Passing the first exact sequence to Tate cohomology and recognizing that $\hat{H}^0(C_L) \cong \mathbb{Z}/p\mathbb{Z}$ [11 p. 19], $\hat{H}^0(\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$, and $\hat{H}^{-1}(\mathbb{Z}) \cong \hat{H}^1(\mathbb{Z}) = 0$, we have that $\hat{H}^0(C\Omega^1_L) = 0$ as well.
As for $\mathbb{F}_p$, it is a finite module of order prime to the order of $G$, and so has trivial Tate $G$-cohomology.

Applying this lemma to the long exact sequences induced by the leftmost column and bottom row of the diagram immediately gives:

**Corollary 3.4.1.** $\hat{H}^{-1}(Cl_L^0) \cong \hat{H}^0(U_L)$ and $\hat{H}^0(Cl_L^0) \cong \hat{H}^1(U_L)$.

**Lemma 3.5.** $\Delta$ acts trivially on $\hat{H}^1(U_L)$ and $\hat{H}^2(U_L)$.

**Proof.** We write $\ell$ for a prime of $k$, $l$ for a prime of $K$ above $\ell$, and $L$ for a prime of $L$ above $l$. We have decomposition groups $\Omega_L = D(L/\ell)$, $G_L = D(L/l)$, and $\Delta_l = D(l/\ell)$. For each $i$, $\hat{H}^i(U_L)$ can be expressed as a product of local cohomology groups:

$$\hat{H}^i(U_L) = \hat{H}^i(\prod_{\mathfrak{p}} U_{\mathfrak{p}})$$

$$= \hat{H}^i(\prod_{l \text{ ram in } L} \bigoplus_{e \mid l} U_{\mathfrak{p}})$$

$$= \bigoplus_{\ell \text{ ram in } L} \bigoplus_{1 \ell} \hat{H}^i(G_{\mathfrak{p}}, U_{\mathfrak{p}}),$$

with the last equality by applying Shapiro’s Lemma to $\bigoplus_{\mathfrak{p} \mid l} U_{\mathfrak{p}} = \text{Ind}_{G_{\mathfrak{p}}}^{G_L} U_{\mathfrak{p}}$. Furthermore, we saw in Section 2 that any prime that ramifies in $L$ ramifies totally in $K$, which means that $\Delta_l = \Delta$ and any action it has on $\hat{H}^1(U_L)$ is on a single summand $\hat{H}^i(G_{\mathfrak{p}}, U_{\mathfrak{p}})$. Thus for $i = 1, 2$, it is sufficient to show that $\Delta$ acts trivially on $\hat{H}^1(G_{\mathfrak{p}}, U_{\mathfrak{p}})$.

We look first at $i = 1$. Since $G$ and $\Delta$ have coprime orders, the inflation-restriction sequence

$$0 \to \hat{H}^1(\Delta_l, U_l) \to \hat{H}^1(\Omega_L, U_{\mathfrak{p}}) \to \hat{H}^1(G_{\mathfrak{p}}, U_{\mathfrak{p}})^{\Delta_l} \to 0$$

is exact. Now, from local class field theory, $\hat{H}^1(G_{\mathfrak{p}}, U_{\mathfrak{p}})$ is cyclic of order equal to the ramification index $e_{\mathfrak{p}/l}$ [11, p. 9], and likewise $\hat{H}^1(\Omega_{\mathfrak{p}}, U_{\mathfrak{p}}) \cong \mathbb{Z}/e_{\mathfrak{p}/l}\mathbb{Z}$ and $\hat{H}^1(\Delta_l, U_l) \cong \mathbb{Z}/e_{\mathfrak{p}/l}\mathbb{Z}$. This forces $\hat{H}^1(G_{\mathfrak{p}}, U_{\mathfrak{p}})^{\Delta_l} = \hat{H}^1(G_{\mathfrak{p}}, U_{\mathfrak{p}})$, and so the action of $\Delta = \Delta_l$ on $\hat{H}^1(U_L)$ is trivial.

Next we take $i = 2$. Again by the coprime orders of $G$ and $\Delta$, and also using that $\hat{H}^1(G_{\mathfrak{p}}, L_{\mathfrak{p}}) = 0$ by Hilbert Theorem 90, the sequence

$$0 \to \hat{H}^2(\Delta_l, K_l) \to \hat{H}^2(\Omega_L, L_{\mathfrak{p}}) \to \hat{H}^2(G_{\mathfrak{p}}, L_{\mathfrak{p}})^{\Delta_l} \to 0$$
is exact. But local class field theory gives us that these cohomology groups are dual to the decomposition groups that define them [1, p. 9], and so by order considerations, we must have $\hat{H}^2(G_L, L_L) \cong \hat{H}^2(G_L, L_L)$. Since the inclusion-induced map $\hat{H}^2(G_L, U_L) \to \hat{H}^2(G_L, L_L)$ is injective (its kernel is $\hat{H}^1(G_L, \mathbb{Z})$, which is trivial), we conclude that $\hat{H}^2(U_L)$ is $\Delta$-invariant as well.

We are now ready to connect the cohomological theory back to $M$, the $p$-part of $\text{Cl}_0^0$.

**Corollary 3.5.1.** $\Delta$ acts trivially on $M[\tau - 1]$ and via $\omega$ on $M/(\tau - 1)M$.

**Proof.** By Corollary 3.4.1 and the fact that $N_G$ kills $M$, we have

$$M[\tau - 1] \cong \hat{H}^0(\text{Cl}_0^0) \cong \hat{H}^1(U_L)$$

and

$$M/(\tau - 1)M \cong \hat{H}^{-1}(\text{Cl}_0^0) \cong \hat{H}^0(U_L) \cong \hat{H}^2(U_L)(\omega),$$

which have the claimed actions by Lemma 3.5.

Finally, we can prove the $p$ part of our result:

**Proposition 3.6.** $M$ is a $(p - 1)$-st power of some finite abelian $p$-group.

**Proof.** We consider the twist $M' = M(\omega^{-1})$. The previous result says that $\Delta$ acts trivially on $M'/((\tau - 1))M'$ and by $\omega^{-1}$ on $M'[\tau - 1]$. Thus Proposition 3.3 may be applied to $M'$. As abelian groups, $M \cong M'$, so we are done.

Combining this with Proposition 3.1, we have that the $p$ part and non-$p$ part of $\text{Cl}_0^0$ are each the $(p - 1)$-st power of an abelian group, and so the proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

4.1 Character and zeta function relations

Henceforth let $q$ be a power of an odd prime. The Galois groups $G$, $\Delta$, and $\Omega$ may still be defined as in the beginning of Section 1.1, with the caveat that $G$ is no longer a cyclic group when $q$ is not prime. Instead, we have an isomorphism $\nu : G \to \mathbb{F}_q$, where $\nu(\tau)$ is the element of $\mathbb{F}_q$ such that $\tau(\gamma) = \gamma + \nu(\tau)\lambda$.

Now, $\Omega$ can be conveniently realized as the matrix group

$$\left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \bigg| \ a, x \in \mathbb{F}_q, a \neq 0 \right\},$$
with \( \sigma \leftrightarrow \begin{pmatrix} \omega(\sigma) & 0 \\ 0 & 1 \end{pmatrix} \) for \( \sigma \in \Delta \) and \( \tau \leftrightarrow \begin{pmatrix} 1 & \nu(\tau) \\ 0 & 1 \end{pmatrix} \) for \( \tau \in G \). Thus the elements with \( a = 1 \) are identified with the elements of \( G \), and those with \( x = 0 \) with the elements of \( \Delta \).

We are interested in four characters of \( \Omega \) which we will show fit an arithmetic relation. These are:

- \( \chi_L \), for the regular representation (permutation representation on \( \Omega \))
- \( \chi_K \), for the permutation representation on \( \Omega/G \)
- \( \chi_F \), for the permutation representation on \( \Omega/\Delta \)
- \( \chi_k \), for the trivial representation (permutation representation on \( \Omega/\Omega \)).

**Proposition 4.1.**

\[
\chi_L - \chi_k = \chi_K - \chi_k + (q - 1)(\chi_F - \chi_k).
\]

**Proof.** We know of course that \( \chi_k \) takes the value 1 on every element of \( \Omega \), and that \( \chi_L \) takes \( |\Omega| = q(q - 1) \) on the identity and 0 elsewhere.

Now, \( \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}x \\ 0 & 1 \end{pmatrix} \). This says that each coset of \( \Omega/\Delta \) can be represented by a unique element of \( G \), and each coset of \( \Omega/G \) by a unique element of \( \Delta \).

We have \( \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (ab)^{-1}x \\ 0 & 1 \end{pmatrix} \), so an element of \( \Omega \) fixes a coset of \( \Omega/G \) if and only if \( a = 1 \). This says that \( \chi_K \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} = |\Omega/G| = q - 1 \) for \( a = 1 \), and 0 for \( a \neq 1 \).

On the other hand, \( \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax + y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \), so an element of \( \Omega \) fixes a coset of \( \Omega/\Delta \) if and only if \( x = y(1-a) \). This means that \( \chi_F \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} = |\Omega/\Delta| = q \) for \( a = 1, x = 0 \), 0 for \( a = 1, x \neq 0 \), and 1 for \( a \neq 1 \).

Using the values ascertained above, the relation holds for each element \( \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \) of \( \Omega \) as follows:

- For \( a = 1, x = 0 \): \( q(q - 1) - 1 = (q - 1) - 1 + (q - 1)(q - 1) \).
- For \( a = 1, x \neq 0 \): \( 0 - 1 = (q - 1) - 1 + (q - 1)(0 - 1) \).
- For \( a \neq 1 \): \( 0 - 1 = 0 - 1 + (q - 1)(1 - 1) \).
This arithmetic relation between characters gives rise to a corresponding multiplicative relation between L-functions, and thus zeta functions \[9\]:

**Corollary 4.1.1.** Let \( \zeta_* \) denote the zeta function for the field \(*\). Then

\[
\frac{\zeta_L}{\zeta_k} = \frac{\zeta_K}{\zeta_k} \cdot \left( \frac{\zeta_F}{\zeta_k} \right)^{q-1}.
\]

### 4.2 Residues of zeta functions

Schmidt \[7\] gives the residue formula

\[
\lim_{s \to 1} (s-1)\zeta(s) = \frac{q^{1-g}h}{(q-1)\log q}
\]

and the functional equation

\[
\zeta(1-s) = q^{(g-1)(2s-1)} \zeta(s)
\]

for the zeta function of a function field in positive characteristic, where \( g \) and \( h \) denote the genus and class number, respectively.\[4\]. These combine to give a formula for the residue of \( \zeta(1-s) \) at 1:

\[
\lim_{s \to 1} (s-1)\zeta(1-s) = \lim_{s \to 1} (s-1)q^{(g-1)(2s-1)}\zeta(s) = \frac{h}{(q-1)\log q}.
\]

As \( K \) and \( k \) have trivial class group, applying this equation to the relation in Corollary 4.1.1 gives \( h_L = h_F^{q-1} \), completing the proof of Theorem 1.2.

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\[4\]See Roquette \[6\] for an English reference summarizing these results, but note a typo in the numerator of the residue formula (reversing the sign of the exponent).
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