On Nonlinear Forced Impulsive Differential Equations under Canonical and Non-Canonical Conditions

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Abstract: This study is connected with the nonoscillatory and oscillatory behaviour to the solutions of nonlinear neutral impulsive systems with forcing term which is studied for various ranges of the neutral coefficient. Furthermore, sufficient conditions are obtained for the existence of positive bounded solutions of the impulsive system. The mentioned example shows the feasibility and efficiency of the main results.

Keywords: lebesgue’s dominated converges theorem (LDCT); Banach fixed point theorem; oscillation; neutral; nonoscillation; impulsive systems; nonlinear; delay

1. Introduction

The study of oscillation of solutions by imposing impulse controls can be found in an extensive variety of real phenomena in Applied Sciences and Engineering problems. Impulsive differential systems arise in bifurcation analysis, circuit theory, population dynamics, biotechnology, loss less transmission in computer network, mathematical economic, chemical technology, etc.

Many researchers spend their attentions to dynamical behaviours of a neutral impulsive differential system (IDS) because it has various applications; an interesting study of second-order impulsive differential systems appears in the theory of impact, as there is a good relation between impact and impulse. The term impulse is also used to refer to a fast-acting force or impact. This type of impulse is often idealized so that the change in momentum produced by the force happens with no change in time. Then, models describing viscoelastic bodies colliding systems with delay and impulses are more appropriate (see [1] and references therein for a review). The models appear in the study of several real-world problems (see, for instance, [2–4]). In general, it is well-known that several natural phenomena are driven by impulsive differential equations. Examples of the aforementioned phenomena are related to population dynamics, biological and mechanical systems, pharmacokinetics, biotechnological processes, theoretical physics, chemistry, control theory [5,6] and engineering. Another interesting application is in some vibrational problems [1]. We refer the readers to [7–11] for further details. Many other interesting results concerning nonlinear equations with symmetric kernels with the application of group symmetry have remained beyond the scope of this paper.
Shen et al. [12] considered the IDS of the form:
\[
\begin{cases}
    u'(\zeta) + q(\zeta)u(\zeta - \mu) = 0, \ \theta_i, \ \zeta \geq \zeta_0 \\
    u(\theta_i^+) - u(\theta_i^-) = I_i(u(\theta_i)), \ i \in \mathbb{N}
\end{cases}
\]  
(1)

where \(q, I_i \in C(\mathbb{R}, \mathbb{R})\) for \(i \in \mathbb{N}\), and obtained some conditions to ensure the oscillatory and asymptotic behaviour of the solutions of Equation (1). Graef et al. [13] have studied the IDE of the form:
\[
\begin{cases}
    (u(\zeta) - p(\zeta)u(\zeta - \delta))' + q(\zeta)|u(\zeta - \mu)|^3 \text{sgn} u(\zeta - \mu) = 0, \ \zeta \geq \zeta_0 \\
    u(\theta_i^+) = b_i u(\theta_i), \ i \in \mathbb{N}
\end{cases}
\]
(2)

where \(p(\zeta) \in PC([\zeta_0, \infty), \mathbb{R}_+)^2\) obtained some results for the oscillation to the solutions of the impulsive differential equations in Equation (2).

Shen et al. [14] considered the first-order IDS of the form:
\[
\begin{cases}
    (u(\zeta) - p(\zeta)u(\zeta - \delta))' + q(\zeta)u(\zeta - \mu_1) - v(\zeta)u(\zeta - \mu_2) = 0, \ \mu_1 \geq \mu_2 > 0 \\
    u(\theta_i^+) = I_i(u(\theta_i)), \ i \in \mathbb{N}
\end{cases}
\]
(3)

and established some new sufficient conditions for oscillation of Equation (3) assuming \(p(\zeta) \in PC([\zeta_0, \infty), \mathbb{R}_+)^2\) and \(b_i \leq \frac{I_i(u)}{u} \leq 1\).

In [15], Karpuz et al. have considered the nonhomogeneous counterpart of System (3) with variable delays and extended the results of [14]. Tripathy et al. [16] have studied the oscillation and nonoscillation properties for a class of second-order neutral IDS of the form:
\[
\begin{cases}
    (u(\zeta) - p(u(\zeta - \delta)))'' + qu(\zeta - \mu) = 0, \ \zeta \neq \theta_i, \ i \in \mathbb{N} \\
    \Delta(u(\theta_i^-) - p(u(\theta_i^-)))' + cu(\theta_i^-) - \mu) = 0, \ i \in \mathbb{N}
\end{cases}
\]
(4)

with constant delays and coefficients. Some new characterizations related to the oscillatory and the asymptotic behaviour of solutions of a second-order neutral IDS were established in [17], where tripathy and Santra studied the systems of the form:
\[
\begin{cases}
    (r(\zeta)(u(\zeta) + p(\zeta)u(\zeta - \delta)))' + q(\zeta)g(u(\zeta - \mu)) = 0, \ \zeta \neq \theta_i, \ i \in \mathbb{N} \\
    \Delta(r(\theta_i^-)u(\theta_i^-) + p(\theta_i^-)u(\theta_i^-) - \delta))' + q(\theta_i^-)g(u(\theta_i^-) - \mu) = 0, \ i \in \mathbb{N}
\end{cases}
\]
(5)

Tripathy et al. [18] have considered the first-order neutral IDS of the form
\[
\begin{cases}
    (u(\zeta) - p(u(\zeta - \delta)))' + q(\zeta)g(u(\zeta - \mu)) = 0, \ \zeta \neq \theta_i, \ \zeta \geq \zeta_0 \\
    u(\theta_i^+) = I_i(u(\theta_i)), \ i \in \mathbb{N} \\
    u(\theta_i^+) - \delta = I_i(u(\theta_i^-) - \delta), \ i \in \mathbb{N}
\end{cases}
\]
(6)

and established some new sufficient conditions for the oscillation of Equation (6) for different values of the neutral coefficient \(p\).

Santra et al. [19] obtained some characterizations for the oscillation and the asymptotic properties of the following second-order highly nonlinear IDS:
\[
\begin{cases}
    (r(\zeta)(f'((\zeta)))' + \sum_{j=1}^{m} q_j(\zeta)g_j(u(\mu_j(\zeta)))) = 0, \ \zeta \geq \zeta_0, \ \zeta \neq \theta_i, \ i \in \mathbb{N} \\
    \Delta(r(\theta_i^-)(f'((\theta_i^-)))' + \sum_{j=1}^{m} \tilde{q}_j(\theta_i^-)g_j(u(\mu_j(\theta_i^-)))) = 0,
\end{cases}
\]
(7)

where
\[
f(\zeta) = u(\zeta) + p(\zeta)u(\delta(\zeta)), \ \Delta f(a) = \lim_{\kappa \to a^+} f(\kappa) - \lim_{\kappa \to a^-} f(\kappa), \ -1 \leq p(\zeta) \leq 0.
\]
Tripathy et al. [20] studied the following IDS:

\[
\begin{align*}
&\left( (r(\xi))(f'(\xi))\right)' + \sum_{i=1}^{m} q_i(\xi)u^{\mu_i}(\xi) = 0, \quad \xi \geq \xi_0, \quad \xi \neq \theta_i \\
&\Delta (r(\theta_i))(f'(\theta_i)) + \sum_{i=1}^{m} h_i(\theta_i)u^{\mu_i}(\theta_i) = 0, \quad i \in \mathbb{N}
\end{align*}
\]  

(8)

where \( f(\xi) = u(\xi) + p(\xi)u(\delta(\xi)) \) and \(-1 < p(\xi) \leq 0\) and obtained different conditions for oscillations for different ranges of the neutral coefficient.

Finally, we mention the recent work [21] by Marianna et al., where they studied the nonlinear IDS with canonical and non-canonical operators of the form

\[
\begin{align*}
&\left( (r(\xi))(u(\xi)) + p(\xi)u(\xi - \delta)\right)' + q(\xi)g(u(\xi - \mu)) = 0, \quad \xi \neq \theta_i, \quad i \in \mathbb{N} \\
&\Delta (r(\theta_i))(u(\theta_i)) + p(\theta_i)u(\theta_i - \delta)' + q(\theta_i)g(u(\theta_i - \mu)) = 0, \quad i \in \mathbb{N}
\end{align*}
\]  

(9)

and established new sufficient conditions for the oscillation of solutions of Equation (9) for various ranges of the neutral coefficient \( p \).

For further details on neutral IDS, we refer the reader to the papers [22–35] and to the references therein.

In the above studies, we have noticed that most of the works have considered only the homogeneous counterpart of the IDS (S), and only a few have considered the forcing term. Hence, in this work, we considered the forced impulsive systems (S) and established some new sufficient conditions for the oscillation and asymptotic properties of solutions to a second-order forced nonlinear IDS in the form

\[
(S) \quad \begin{cases}
(\delta(\xi))(u(\xi)) + p(\xi)u(\xi - \delta)' + q(\xi)g(u(\xi - \mu)) = f(\xi), \quad \xi \neq \theta_i, \quad i \in \mathbb{N}, \\
\Delta (r(\theta_i))(u(\theta_i)) + p(\theta_i)u(\theta_i - \delta)' + h(\theta_i)g(u(\theta_i - \mu)) = g(\theta_i), \quad i \in \mathbb{N},
\end{cases}
\]

where \( \delta > 0, \mu \geq 0 \) are real constants, \( G \in C(\mathbb{R}, \mathbb{R}) \) is nondecreasing with \( vG(v) > 0 \) for \( v \neq 0 \), \( q, r, h \in C(\mathbb{R}_+, \mathbb{R}_+) \), \( p \in PC(\mathbb{R}_+, \mathbb{R}) \) are the neutral coefficients, \( p(\theta_i), r(\theta_i), f, g \in C(\mathbb{R}, \mathbb{R}), q(\theta_i) \) and \( h(\theta_i) \) are constants \( i \in \mathbb{N} \), \( \theta_i \) with \( \theta_1 < \theta_2 < \cdots < \theta_i < \ldots \), and \( \lim_{i \to \infty} \theta_i = \infty \) are impulses. For (S), \( \Delta \) is defined by

\[
\Delta (a(\theta_i)(b'(\theta_i))) = a(\theta_i + 0)b'(\theta_i + 0) - a(\theta_i - 0)b'(\theta_i - 0); \\
u(\theta_i - 0) = u(\theta_i) \quad \text{and} \quad u(\theta_i - \delta - 0) = u(\theta_i - \delta), \quad i \in \mathbb{N}.
\]

Throughout the work, we need the following hypotheses:

**Hypothesis 1.** Let \( F \in C(\mathbb{R}, \mathbb{R}) \) so that \( (r(\xi))F'(\xi) \in C(\mathbb{R}, \mathbb{R}), \ (r(\xi))F'(\xi) \to f(\xi) \) and \( \Delta (r(\theta_i))F'(\theta_i)) = g(\theta_i) \). In addition, we assume that \( F(\xi) \) changes sign with \( -\infty < \lim_{\xi \to \infty} F(\xi) < 0 < \lim_{\xi \to \infty} F(\xi) < \infty \);

**Hypothesis 2.** There exists \( \mu_1 > 0 \) such that \( G(p) + G(q) \geq \mu_1 G(p + q) \) for \( p, q > 0 \);

**Hypothesis 3.** \( G(pq) \leq G(p)G(q) \) for \( p, q \in \mathbb{R}_+ \);

**Hypothesis 4.** \( G(-p) = -G(p) \) for \( p \in \mathbb{R}_+ \);

**Hypothesis 5.** \( F^+(\xi) = \max\{F(\xi), 0\} \) and \( F^-(\xi) = \max\{-F(\xi), 0\} \);

**Hypothesis 6.** \( \int_0^\infty \frac{dp}{r(p)} + \sum_{i=1}^{\infty} \frac{1}{r(\theta_i)} = \infty \);
Hypothesis 7. \( \int_0^\infty Q(\eta)G(\eta - \mu)d\eta + \sum_{i=1}^{\infty} H_k G(F^+(\eta_i - \mu)) = \infty, T > 0, \)
where \( Q(\xi) = \min\{ q(\xi), q(\xi - \delta) \}, \eta \geq \delta \) and \( H_k = \min\{ h(\theta_i), h(\theta_i - \delta) \}, i \in \mathbb{N}; \)

Hypothesis 8. \( \int_0^\infty Q(\eta)G(\eta - \mu)d\eta + \sum_{i=1}^{\infty} H_k G(F^+(\theta_i - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 9. \( \int_0^\infty q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(F^+(\theta_i - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 10. \( \int_0^\infty q(\eta)G(F^+(\eta - \delta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(F^+(\theta_i - \delta - \mu)) = \infty \) where \( T > 0; \)

Hypothesis 11. \( \int_0^\infty q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(F^-(\theta_i - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 12. \( \int_0^\infty q(\eta)G(F^+(\eta + \delta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(F^+(\theta_i + \delta - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 13. \( \int_0^\infty q(\eta)G(\frac{1}{2}F^-(\eta + \delta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(\frac{1}{2}F^-(\theta_i + \delta - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 14. \( \int_0^\infty q(\eta)G(\frac{1}{2}F^+(\eta + \delta - \mu))d\eta + \sum_{i=1}^{\infty} h(\theta_i) G(\frac{1}{2}F^+(\theta_i + \delta - \mu)) = \infty, \) where \( T > 0; \)

Hypothesis 15. \( \int_0^\infty \frac{dy}{n(y)} + \sum_{i=1}^{\infty} \frac{1}{n(y_i)} < \infty; \)

Let \( R(\xi) = \int_0^\infty \frac{dy}{n(y)}. Then \int_0^\infty \frac{dy}{n(y)} < \infty implies that \( R(\xi) \to 0 \) as \( \xi \to \infty \) since \( R(\xi) \) is nonincreasing.

Hypothesis 16. \( \int_0^\infty \frac{1}{n(y)} \left[ \int_{\xi_1}^{\infty} Q(\xi)G(F^+(\xi - \mu))d\xi + \sum_{i=1}^{\infty} H_k G(F^+(\xi_i - \mu)) \right] d\eta = \infty \) where \( T, \xi_1 > 0; \)

Hypothesis 17. \( \int_0^\infty \frac{1}{n(y)} \left[ \int_{\xi_1}^{\infty} Q(\xi)G(F^-(\xi - \mu))d\xi + \sum_{i=1}^{\infty} H_k G(F^-(\xi_i - \mu)) \right] d\eta = \infty \) where \( T, \xi_1 > 0; \)

Hypothesis 18. \( \int_0^\infty \frac{1}{n(y)} \int_{\xi_1}^{\infty} q(\xi)G(F^+(\xi + \delta - \mu))d\xi d\eta + R(\xi) \sum_{i=1}^{\infty} h(\theta_i) G(F^+(\theta_i + \delta - \mu)) = \infty, \) where \( T, \xi_1 > 0; \)

Hypothesis 19. \( \int_0^\infty \frac{1}{n(y)} \int_{\xi_1}^{\infty} q(\xi)G(F^-(\xi + \delta - \mu))d\xi d\eta + R(\xi) \sum_{i=1}^{\infty} h(\theta_i) G(F^-(\theta_i + \delta - \mu)) = \infty, \) where \( T, \xi_1 > 0; \)

Hypothesis 20. \( \int_0^\infty \frac{1}{n(y)} \int_{\xi_1}^{\infty} q(\xi)G(F^+(\xi - \mu))d\xi d\eta + R(\xi) \sum_{i=1}^{\infty} h(\theta_i) G(F^+(\theta_i - \mu)) = \infty, \) where \( T, \xi_1 > 0; \)

Hypothesis 21. \( \int_0^\infty \frac{1}{n(y)} \int_{\xi_1}^{\infty} q(\xi)G(F^-(\xi - \mu))d\xi d\eta + R(\xi) \sum_{i=1}^{\infty} h(\theta_i) G(F^-(\theta_i - \mu)) = \infty, \) where \( T, \xi_1 > 0; \)
**Hypothesis 22.** \( \int_T^\infty \frac{1}{r(\eta)} \int_{\zeta_1}^\eta q(\zeta)G\left(\frac{1}{r} F^+(\zeta + \delta - \mu)\right) d\zeta d\eta + R(\zeta) \sum_{i=1}^\infty h(\theta_i)G\left(\frac{1}{r} F^+(\theta_i + \delta - \mu)\right) = \infty \), where \( T, \zeta_1 > 0 \);

**Hypothesis 23.** \( \int_T^\infty \frac{1}{r(\eta)} \int_{\zeta_1}^\eta q(\zeta)G\left(\frac{1}{r} F^-(\zeta + \delta - \mu)\right) d\zeta d\eta + R(\zeta) \sum_{i=1}^\infty h(\theta_i)G\left(\frac{1}{r} F^-(\theta_i + \delta - \mu)\right) = \infty \), where \( T, \zeta_1 > 0 \);

**Hypothesis 24.** \( \int_T^\infty \frac{1}{r(\eta)} \left[ \int_\eta^\infty q(\kappa) d\kappa + \sum_{i=1}^\infty h(\theta_i) \right] d\eta < \infty \).

### 2. Qualitative Behaviour under the Canonical Operator

This section deals with the sufficient conditions for the oscillatory and asymptotic properties of solutions of a nonlinear second-order forced neutral IDS of the form (\( S \)) under the canonical operator (H5).

**Theorem 1.** Consider \( 0 \leq p(\zeta) \leq a < \infty \), \( \zeta \in \mathbb{R}_+ \) and (H1)–(H8) hold. Then each solution of the system (\( S \)) is oscillatory.

**Proof.** For the sake of contradiction, let the solution be nonoscillatory. Therefore, for \( \zeta_0 > \rho \), we have \( u(\zeta) > 0, u(\zeta - \delta) > 0 \) and \( u(\zeta - \mu) > 0 \), where \( \zeta \geq \zeta_0 \). Setting

\[
\begin{align*}
z(\zeta) &= u(\zeta) + p(\zeta)u(\zeta - \delta), \; \zeta \neq \theta_i, \; i \in \mathbb{N} \\
z(\theta_i) &= u(\theta_i) + p(\theta_i)u(\theta_i - \delta), \; i \in \mathbb{N},
\end{align*}
\]

and

\[
\nabla(\zeta) = z(\zeta) - F(\zeta), \quad \nabla(\theta_i) = z(\theta_i) - F(\theta_i)
\]

due to (H1), it follows from (\( S \)) that

\[
(\rho(\zeta) \nabla'(\zeta))' = -q(\zeta)G(u(\zeta - \mu)) \leq 0, \quad \zeta \neq \theta_i, \; k \in \mathbb{N}
\]

(12)

\[
\Delta(\rho(\theta_i) \nabla'(\theta_i)) = -h(\theta_i)G(u(\theta_i - \mu)) \leq 0, \; i \in \mathbb{N}
\]

(13)

for \( \zeta \geq \zeta_1 > \zeta_0 + \mu \). Consequently, \( (\rho(\zeta) \nabla'(\zeta)) \) is nonincreasing, and \( \nabla'(\zeta), \; \nabla(\zeta) \) are of either eventually positive or eventually negative on \( [\zeta_2, \infty) \), where \( \zeta_2 > \zeta_1 \). Since \( z(\zeta) > 0 \), then \( \nabla(\zeta) < 0 \) for \( \zeta \geq \zeta_2 \), that is, \( F(\zeta) > 0 \) for \( \zeta \geq \zeta_2 \), which is not possible. Hence, \( \nabla(\zeta) > 0 \) for \( \zeta \geq \zeta_2 \). For the next, we assume the cases \( (\rho(\zeta) \nabla'(\zeta)) < 0 \) or \( > 0 \) for \( \zeta \geq \zeta_2 \). Let the former hold for \( \zeta \geq \zeta_2 \). Therefore, there exist \( C > 0 \) and \( \zeta_3 > \zeta_2 \) such that \( (\rho(\zeta) \nabla'(\zeta)) \leq -C \) for \( \zeta \geq \zeta_3 \). Finally, \( (\rho(\theta_i) \nabla'(\theta_i)) \leq -C \). Integrating the relation \( \nabla'(\zeta) \leq -\frac{C}{r(\zeta)} \), \( \zeta \geq \zeta_3 \) from \( \zeta_3 \) to \( \zeta > \zeta_3 \), we obtain

\[
\nabla(\zeta) - \nabla(\zeta_3) - \sum_{\zeta_3 \leq \theta_i < \zeta} \nabla'(\theta_i) \leq -C \int_{\zeta_3}^{\zeta} \frac{d\eta}{r(\eta)},
\]

that is,

\[
\nabla(\zeta) \leq \nabla(\zeta_3) - C \left[ \int_{\zeta_3}^{\zeta} \frac{d\eta}{r(\eta)} + \sum_{\zeta_3 \leq \theta_i < \zeta} \frac{1}{r(\theta_i)} \right] \to -\infty \quad \text{as} \quad \zeta \to \infty,
\]
a contradiction to $\nabla (\zeta) > 0$ for $\zeta \geq \zeta_2$. Hence, $(r(\zeta)\nabla' (\zeta)) > 0$ for $\zeta \geq \zeta_2$. Ultimately, $z(\zeta) > F(\zeta)$, and hence, $z(\zeta) > \max\{0, F(\zeta)\} = F^+(\zeta)$ for $\zeta \geq \zeta_2$. Due to Equations (10) and (11), Equation (12) becomes

$$0 = (r(\zeta)\nabla (\zeta))' + q(\zeta)G(u(\zeta - \mu)) + G(\zeta)[(r(\zeta - \delta)\nabla (\zeta - \delta))' + q(\zeta - \delta)G(u(\zeta - \delta - \mu))]$$

for $\zeta \geq \zeta_2$ and because of (H2) and (H3), we find that

$$0 \geq (r(\zeta)\nabla (\zeta))' + G(a)(r(\zeta - \delta)\nabla (\zeta - \delta))' + Q(\zeta)[G(u(\zeta - \mu)) + G(a(\zeta - \delta - \mu))]$$

$$\geq (r(\zeta)\nabla (\zeta))' + G(a)(r(\zeta - \delta)\nabla (\zeta - \delta))' + \mu_1 Q(\zeta)G(z(\zeta - \mu))$$

(14)

for $\zeta \geq \zeta_3 > \zeta_2 + \mu$. Similarly from Equation (13), we obtain

$$0 \geq \Delta (r(\theta_i)\nabla' (\theta_i)) + G(a)\Delta (r(\theta_i - \delta)\nabla' (\theta_i - \delta)) + \mu_1 H_k G(z(\theta_i - \mu))$$

(15)

for $i \in \mathbb{N}$. Integrating Equation (14) from $\zeta_3$ to $+\infty$, we obtain

$$\mu_1 \int_{\zeta_3}^{\infty} Q(\eta)G(z(\eta - \mu))d\eta \leq \left[\frac{[r(\eta)\nabla (\eta) + G(a)(r(\eta - \delta)\nabla (\eta - \delta))]}{\zeta_3^{\infty}}\right.$$}

$$+ \sum_{\zeta_3 \leq \theta_i < +\infty} \Delta [r(\theta_i)\nabla (\theta_i)] + G(a)(r(\theta_i - \delta)\nabla (\theta_i - \delta))$$

$$\leq \left[\frac{- [r(\eta)\nabla (\eta) + G(a)(r(\eta - \delta)\nabla (\eta - \delta))]}{\zeta_3^{\infty}}\right.$$}

$$- \mu \sum_{\zeta_3 \leq \theta_i < +\infty} H_k G(z(\theta_i - \mu))$$

due to Equation (15). Since $\lim_{\xi \to +\infty} (r(\zeta)\nabla' (\zeta))$ exists, then the above inequality becomes

$$\mu_1 \left[\int_{\zeta_3}^{\infty} Q(\eta)G(z(\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < +\infty} H_k G(z(\theta_i - \mu))\right] < \infty,$$

that is,

$$\mu_1 \left[\int_{\zeta_3}^{\infty} Q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < +\infty} H_k G(F^+(\theta_i - \mu))\right] < \infty$$

which contradicts (H7).

If $u(\zeta) < 0$ for $\zeta \geq \zeta_0$, then we set $x(\zeta) = -u(\zeta)$ for $\zeta \geq \zeta_0$ in (S), and we obtain that

$$\left(\bar{E}\right) \begin{cases} (r(\zeta)(x(\zeta) + p(\zeta)x(\zeta - \delta))' + q(\zeta)G(x(\zeta - \mu)) = f(\zeta), \xi \neq \delta, i \in \mathbb{N} \\
\bar{\Delta}(r(\theta_i)x(\theta_i) + p(\theta_i)x(\theta_i - \delta))' + h(\theta_i)G(x(\theta_i - \mu)) = \bar{g}(\theta_i), i \in \mathbb{N}, \end{cases}$$

where $\bar{f}(\zeta) = -f(\zeta)$, $\bar{g}(\theta_i) = -g(\theta_i)$ due to (H4). Let $\bar{F}(\zeta) = -F(\zeta)$, then

$$-\infty < \liminf_{\zeta \to +\infty} \bar{F}(\zeta) < 0 < \limsup_{\zeta \to +\infty} \bar{F}(\zeta) < \infty$$

and $(r(\zeta)\bar{F}'(\zeta))' = \bar{f}(\zeta)$, $\bar{\Delta}(r(\theta_i)\bar{F}'(\theta_i)) = \bar{g}(\theta_i)$ hold. Similar to ($\bar{E}$), we can find a contradiction to (H8). This completes the proof. \(\square\)

**Theorem 2.** Assume that (H1), (H4)–(H6) and (H9)–(H12) hold, and $-1 \leq p(\zeta) \leq 0, \zeta \in \mathbb{R}_+$. Then each solution of (S) is oscillatory.

**Proof.** For the contradiction, we follow the proof of the Theorem 1 to get $\nabla (\zeta)$ and $(r(\zeta)\nabla' (\zeta))$ are of either eventually negative or positive on $[\zeta_2, \infty)$. Let $\nabla' (\zeta) < 0$ for
\( \zeta \geq \zeta_2 \). Then as in Theorem 1, we have \( \nabla(\zeta) < 0 \) and \( \lim_{\zeta \to \infty} \nabla(\zeta) = -\infty \). Hence, for \( \zeta_3 > \zeta_2 \) we have \( z(\zeta) < F(\zeta) \) where \( \zeta \geq \zeta_3 \). Considering \( z(\zeta) > 0 \) we have \( F(\zeta) > 0 \), which is not possible. Thus, \( z(\zeta) < 0 \) and \( z(\zeta) < F(\zeta) \) for \( \zeta \geq \zeta_3 \). Again, \( z(\zeta) < 0 \) for \( \zeta \geq \zeta_3 \) implies that

\[
 u(\zeta) \leq -p(\zeta)u(\zeta - \delta) \leq u(\zeta - \delta) \leq u(\zeta - 2\delta) \leq \cdots \leq u(\zeta_3), \quad \zeta \neq \theta_i
\]

and also

\[
 u(\theta_i) \leq u(\theta_i - \delta) \leq \cdots \leq u(\zeta_3), \quad \zeta \neq \theta_i, \quad i \in \mathbb{N},
\]

that is, \( u(\zeta) \) is bounded on \([\zeta_3, \infty)\). Consequently, \( \lim_{\zeta \to \infty} \nabla(\zeta) \) hold and that is a contradiction.

Finally, \( \nabla(\zeta) > 0 \) for \( \zeta \geq \zeta_2 \). So, we have following two cases \( \nabla(\zeta) < 0 \), \( (r(\zeta)\nabla(\zeta)) > 0 \) and \( \nabla(\zeta) > 0 \), \( (r(\zeta)\nabla(\zeta)) > 0 \) on \([\zeta_3, \infty)\), \( \zeta_3 > \zeta_2 \). For the first case \( \nabla(\zeta) < 0 \), we have \( z(\zeta) < F(\zeta) \) and \( \lim_{\zeta \to \infty} (r(\zeta)\nabla(\zeta)) \) exists. Let \( z(\zeta) > 0 \) we have \( F(\zeta) > 0 \), a contradiction. 

So, \( z(\zeta) < 0 \). Clearly, \( -z(\zeta) > -F(\zeta) \) implies that \( -z(\zeta) > \max\{0, -F(\zeta)\} = F(\zeta) \). Therefore, for \( \zeta \geq \zeta_3 \)

\[
 -u(\zeta - \delta) \leq p(\zeta)u(\zeta - \delta) \leq z(\zeta) < -F(\zeta),
\]

that is, \( u(\zeta - \mu) > F(\zeta - (\zeta + \delta - \mu)), \quad \zeta \geq \zeta_4 > \zeta_3 \) and Equations (12) and (13) reduce to

\[
 (r(\zeta)\nabla(\zeta))' + q(\zeta)G(F^- (\zeta + \delta - \mu)) \leq 0, \quad \zeta \neq \theta_i, \quad i \in \mathbb{N}
\]

\[
 \Delta(r(\theta_i)\nabla(\theta_j)) + h(\theta_i)G(F^- (\theta_i + \delta - \mu)) \leq 0, \quad i \in \mathbb{N}
\]

for \( \zeta \geq \zeta_4 \). Integrating the inequality from \( \zeta_4 \) to \( +\infty \), we have

\[
 \int_{\zeta_4}^{\infty} q(\eta)G(F^- (\eta + \delta - \mu))d\eta + \sum_{\zeta_4 \leq \theta_i < \infty} h(\theta_i)G(F^- (\theta_i + \delta - \mu)) < \infty
\]

which contradicts \((H10)\). With the latter case, it follows that \( z(\zeta) > F(\zeta) \). Let \( z(\zeta) < 0 \) we have \( F(\zeta) < 0 \), a contraction. Hence, \( z(\zeta) > 0 \) and \( z(\zeta) \leq u(\zeta) \) for \( \zeta \geq \zeta_3 > \zeta_2 \). In this case, \( \lim_{\zeta \to \infty} (r(\zeta)\nabla(\zeta)) \) exists. Since \( F^+(\zeta) = \max\{F(\zeta), 0\} < z(\zeta) \leq u(\zeta) \) for \( \zeta \geq \zeta_3 \), then Equations (12) and (13) can be viewed as

\[
 (r(\zeta)\nabla(\zeta))' + q(\zeta)G(F^+ (\zeta - \mu)) \leq 0, \quad \zeta \neq \theta_i, \quad i \in \mathbb{N}
\]

\[
 \Delta(r(\theta_i)\nabla(\theta_j)) + h(\theta_i)G(F^+ (\theta_i - \mu)) \leq 0, \quad i \in \mathbb{N}
\]

Integrating the above impulsive system from \( \zeta_3 \) to \( +\infty \), we obtain

\[
 \int_{\zeta_3}^{\infty} q(\eta)G(F^+ (\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < \infty} h(\theta_i)G(F^+ (\theta_i - \mu)) < \infty
\]

which is a contradiction to \((H9)\). The case \( u(\zeta) < 0 \) for \( \zeta \geq \zeta_0 \) is similar. Thus, the theorem is proved. \( \square \)

**Theorem 3.** Consider \(-\infty < -b \leq p(\zeta) \leq -1, \quad \zeta \in \mathbb{R}_+, \quad b > 0\). Assume that \((H1), (H4)-(H6), (H9), (H11), (H13) \) and \((H14)\) hold. Then each bounded solution of \((S)\) is oscillatory.

### 3. Qualitative Behaviour under the Noncanonical Operator

In the following, we establish sufficient conditions that guarantee the oscillation and some asymptotic properties of solutions of the IDS \((S)\) under the noncanonical condition \((H15)\).
Theorem 4. Let \( 0 \leq p(\zeta) \leq a < \infty, \zeta \in \mathbb{R}_+ \). Assume that (H1)–(H5), (H7), (H8), (H15), (H16) and (H17) hold. Then each solution of (S) is oscillatory.

Proof. Let \( u(\zeta) \) be a nonoscillatory solution of the impulsive system (S). Proceeding as in Theorem 1, we obtain Equations (12) and (13) for \( \zeta \geq \zeta_1 \). In what follows, \((r(\zeta)\nabla'(\zeta))\) and \(\nabla'(\zeta)\) are monotone functions on \([\zeta_2, \infty)\), where \(\zeta_2 < \zeta_1\). Consider the case when \((r(\zeta)\nabla'(\zeta)) < 0, \nabla'(\zeta) > 0 \) for \( \zeta \geq \zeta_2 \). Therefore, for \( s \geq t > \zeta_2 \), \((r(s)\nabla'(s)) \leq (r(\zeta)\nabla'(\zeta)) \) implies that \(\nabla'(s) \leq \frac{\nabla'(\zeta)}{r(s)}\), that is,

\[
\nabla'(s) \leq \frac{\nabla'(\zeta)}{r(s)} 
\]

Since \((r(\zeta)\nabla'(\zeta))\) is nonincreasing, there exists a constant \(C > 0\) such that \((r(\zeta)\nabla'(\zeta)) \leq -C\) for \( \zeta \geq \zeta_2 \). As a result, \(\nabla'(s) \leq \frac{\nabla'(\zeta)}{C} \int_T^s \frac{d\theta}{r(\theta)}\). For \( s \to \infty\), it follows that \(0 \leq \nabla'(\zeta) - CR(\zeta)\) for \( \zeta \geq \zeta_2 \). Clearly, \(\nabla'(\theta_i) \geq CR(\theta_i), i \in \mathbb{N}\). So, \(z(\zeta) \geq F(\zeta) + CR(\zeta)\) and hence \(z(\zeta) - CR(\zeta) \geq F(\zeta)\). Considering \(z(\zeta) - CR(\zeta) < 0\) we have \(F(\zeta) < 0\), a contradiction. So, \(z(\zeta) - CR(\zeta) > 0\) implies that \(z(\zeta) \geq CR(\zeta) + F^+(\zeta) \geq F^+(\zeta)\). Furthermore, \(z(\theta_i) \geq F^+(\theta_i), i \in \mathbb{N}\). Consequently, Equations (14) and (15) reduce to

\[
(r(\zeta)\nabla'(\zeta))' + G(a)(r(\zeta - \delta)\nabla'(\zeta - \delta))' + \mu Q(\zeta)G(F^+(\zeta - \mu)) \leq 0
\]

\[
\Delta(r(\theta_i)\nabla'(\theta_i)) + G(a)\Delta(r(\theta_i - \delta)\nabla'(\theta_i - \delta)) + \mu H_k(G(F^+(\theta_i - \mu)) \leq 0
\]

for \( \zeta_3 \geq \zeta_2, \zeta \neq \theta_i, i \in \mathbb{N}\). Integrating the last inequality from \( \zeta_3 \) to \( \zeta > \zeta_3 \), we find

\[
G(a) \sum_{\zeta_3 \leq \theta_i < \zeta} \Delta(r(\theta_i - \delta)\nabla'(\theta_i - \delta)) + \mu \int_{\zeta_3}^\zeta Q(\eta)G(F^+(\eta - \mu))d\eta \leq 0,
\]

that is,

\[
\mu \left[ \int_{\zeta_3}^\zeta Q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < \zeta} H_k(G(F^+(\theta_i - \mu))) \right]
\]

\[
\leq - \left[ (r(\zeta)\nabla'(\zeta)) + G(a)(r(\zeta - \delta)\nabla'(\zeta - \delta)) \right]_{\zeta_3}^\zeta
\]

\[
\leq - \left[ (r(\zeta)\nabla'(\zeta)) + G(a)(r(\zeta - \delta)\nabla'(\zeta - \delta)) \right]_{\zeta_3}^\zeta
\]

\[
\leq -(1 + G(a))(r(\zeta)\nabla'(\zeta))
\]

implies that

\[
\frac{\mu}{1 + G(a)} \frac{1}{r(\zeta)} \left[ \int_{\zeta_3}^\zeta Q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < \zeta} H_k(G(F^+(\theta_i - \mu))) \right] \leq -\nabla'(\zeta).
\]

Further integration of the above inequality, we obtain that

\[
\frac{\mu}{(1 + G(a))} \int_{\zeta_3}^\zeta \frac{1}{r(\kappa)} \left[ \int_{\zeta_3}^\kappa Q(\eta)G(F^+(\eta - \mu))d\eta + \sum_{\zeta_3 \leq \theta_i < \kappa} H_k(G(F^+(\theta_i - \mu))) \right] d\kappa
\]

\[
\leq - \left[ \nabla'(\eta) \right]_{\zeta_3}^\zeta + \sum_{\zeta_3 \leq \theta_i < \kappa} \Delta \nabla(\theta_i)
\]

\[
= - \left[ \nabla'(\eta) \right]_{\zeta_3}^\zeta + \sum_{\zeta_3 \leq \theta_i < \kappa} [\nabla(\theta_i + 0) - \nabla(\theta_i - 0)]
\]

\[
\leq \nabla(\zeta_3) + \sum_{\zeta_3 \leq \theta_i < \kappa} \nabla(\theta_i + 0).
\]
Since \( \nabla(\xi) \) is monotonic and bounded, hence,
\[
\int_{\xi_3}^{\infty} \frac{1}{r(\kappa)} \left[ \int_{\xi_3}^{\kappa} q(\eta) G(F^+(\eta - \mu)) d\eta + \sum_{i=1}^{\infty} H_i G(F^+(\theta_i - \mu)) \right] d\kappa < \infty,
\]
which contradicts to (H16). The rest of the proof follows from the proof Theorem 1. This completes the proof of the theorem. \( \Box \)

**Theorem 5.** Assume that (H1), (H4), (H5), (H9)–(H12), (H15) and (H18)–(H21) hold and \(-1 \leq p(\xi) \leq 0, \xi \in \mathbb{R}_+ \). Then each solution of (S) is oscillatory.

**Proof.** For contrary, let \( u(\xi) \) be a nonoscillatory solution of (S). Then preceding as in the proof of Theorem 2, we obtain \( \nabla(\xi) \) and \( (r(\xi) \nabla'(\xi)) \) are monotonic on \([\xi_2, \infty)\).

If \( \nabla(\xi) < 0 \) and \( (r(\xi) \nabla'(\xi)) < 0 \) for \( \xi \geq \xi_3 > \xi_2 \), then we use the same type of argument as in Theorem 2 to obtain that \( u(\xi) \) is bounded, that is, \( \lim_{\xi \to \infty} \nabla(\xi) \) exists. Clearly, \( z(\xi) < 0 \).

So, \(-z(\xi) > -F(\xi)\), and hence, \(-z(\xi) > F^-(\xi)\). So, for \( \xi \geq \xi_3 \)
\[
-u(\xi - \delta) \leq p(\xi) u(\xi - \delta) \leq z(\xi) < -F^-(\xi),
\]
Consequently, \( u(\xi - \mu) > F^-(\xi + \delta - \mu) \), \( \xi \geq \xi_4 > \xi_3 \) and Equations (12) and (13) yield
\[
\begin{align*}
(r(\xi) \nabla'(\xi))' + q(\xi) G(F^-(\xi + \delta - \mu)) & \leq 0, \quad \xi \neq \theta_i, \ i \in \mathbb{N} \\
\Delta(r(\theta_i) \nabla'(\theta_i)) + h(\theta_i) G(F^-(\theta_i + \delta - \mu)) & \leq 0, \ i \in \mathbb{N}
\end{align*}
\]
for \( \xi \geq \xi_4 \). Integrating the preceding impulsive system from \( \xi_4 \) to \( +\infty \), we obtain
\[
\int_{\xi_4}^{\infty} q(\eta) G(F^-(\eta + \delta - \mu)) d\eta + \sum_{\xi_4 \leq \theta_i < \infty} h(\theta_i) G(F^-(\theta_i + \delta - \mu)) < -r(\xi) \nabla'(\xi),
\]
that is,
\[
\frac{1}{r(\xi)} \left[ \int_{\xi_4}^{\infty} q(\eta) G(F^-(\eta + \delta - \mu)) d\eta + \sum_{\xi_4 \leq \theta_i < \infty} h(\theta_i) G(F^-(\theta_i + \delta - \mu)) \right] < -\nabla'(\xi).
\]
From further integration of the last inequality, we find
\[
\int_{\xi_4}^{\infty} \frac{1}{r(\eta)} \left[ \int_{\xi_4}^{\infty} q(\eta) G(F^-(\kappa + \delta - \mu)) d\kappa + \sum_{\xi_4 \leq \theta_i < \infty} h(\theta_i) G(F^-(\theta_i + \delta - \mu)) \right] d\eta < \infty
\]
which contradicts (H19). If \( \nabla(\xi) > 0 \) and \( (r(\xi) \nabla'(\xi)) < 0 \) for \( \xi \geq \xi_3 \), then following Theorem 4, we find \( z(\xi) \geq F^+(\xi) + CR(\xi) F^+(\xi) \) and \( z(\xi) > 0 \), that is, \( u(\xi) \geq F^+(\xi) \).

The rest of the proof follows from the proof of Theorem 2. Thus, the theorem is proved. \( \Box \)

**Theorem 6.** Consider \(-\infty < -b \leq p(\xi) \leq -1, \xi \in \mathbb{R}_+ \). Assume that (H1), (H4), (H5), (H9)–(H12), (H15), (H20) and (H21)–(H23) hold. Then each bounded solution of (S) is oscillatory.

**Proof.** The proof of the theorem follows the proof of Theorem 5. \( \Box \)

4. **Sufficient Conditions for Nonoscillation**

This section deals with the existence of positive solutions to show that the IDS (S) has positive solution. nonincreasing.

**Theorem 7.** Consider \( p \in C(\mathbb{R}_+, [-1, 0]) \) and assume that (H1) holds. If (H24) holds, then the IDS (S) has a positive solution.
Proof. (i) Consider \(-1 < -b \leq p(\zeta) \leq 0, \zeta \in \mathbb{R}_+\) where \(b > 0\). For (H24), we can find a \(\zeta > \rho = \max\{\delta, \mu\}\) such that

\[
\int_T^\zeta \frac{1}{r(\eta)} \left[ \int_\eta^\infty q(\xi) d\xi + \sum_{i=1}^\infty h(\theta_i) \right] d\eta < \frac{1-b}{10G(1)}.
\]

We consider the set

\[
M = \left\{ u : u \in C([\zeta - \rho, +\infty), \mathbb{R}), u(\zeta) = 0 \text{ for } \zeta \in [\zeta - \rho, \zeta] \text{ and } \frac{1-b}{20} \leq u(\zeta) \leq 1 \right\}
\]

and define \(\Phi : M \to C([\zeta - \rho, +\infty), \mathbb{R})\) by

\[
(\Phi u)(\zeta) = \begin{cases} 
0, & \zeta \in [\zeta - \rho, \zeta) \\
-p(\zeta)u(\zeta - \delta) + \int_T^\zeta \frac{1}{r(\eta)} \left[ \int_\eta^\infty q(\xi) G(u(\xi - \mu)) d\xi + \sum_{i=1}^\infty h(\theta_i) G(u(\theta_i - \mu)) \right] d\eta + F(\zeta) + \frac{1-b}{10}, & \zeta \geq T,
\end{cases}
\]

where \(F(\zeta)\) is such that \(|F(\zeta)| \leq \frac{1-b}{20}\). For every \(u \in M\),

\[
(\Phi u)(\zeta) \leq -p(\zeta)u(\zeta - \delta) + G(1) \int_T^\zeta \frac{1}{r(\xi)} \left[ \int_\xi^\infty q(\eta) d\eta + \sum_{i=1}^\infty h(\theta_i) \right] d\eta + \frac{1-b}{20} + \frac{1-b}{10}
\]

\[
\leq b + \frac{1-b}{10} + \frac{1-b}{20} + \frac{1-b}{10} \leq 1 + \frac{3b}{4} < 1,
\]

and

\[
(\Phi u)(\zeta) \geq F(\zeta) + \frac{1-b}{10} \leq -\frac{1-b}{20} + \frac{1-b}{10} = \frac{1-b}{20}
\]

implies that \((\Phi u)(\zeta) \in M\). Define \(v_n : [\zeta - \rho, +\infty) \to \mathbb{R}\) by

\[
v_n(\zeta) = (\Phi v_{n-1})(\zeta), \quad n \geq 1
\]

with

\[
v_0(\zeta) = \begin{cases} 
0, & \zeta \in [\zeta - \rho, \zeta) \\
\frac{1-b}{20}, & \zeta \geq T.
\end{cases}
\]

Inductively,

\[
\frac{1-b}{20} \leq v_{n-1}(\zeta) \leq u_n(\zeta) \leq 1.
\]

for \(\zeta \geq T\). Therefore, for \(\zeta \geq \zeta - \rho\), \(\lim_{n \to \infty} v_n(\zeta)\) exists. Let \(\lim_{n \to \infty} v_n(\zeta) = v(\zeta)\) for \(\zeta \geq \zeta - \rho\).

By the LDCT, we have \(u \in M\) and \((\Phi u)(\zeta) = u(\zeta)\), where \(u(\zeta)\) is a solution of the impulsive system \((S)\) on \([\zeta - \rho, \infty)\) such that \(u(\zeta) > 0\).

(ii) If \(p(\zeta) \equiv -1, \zeta \in \mathbb{R}_+\), we choose \(-1 < p_0 < 0\) such that \(p_0 \neq -\frac{1}{2}\). For this case, we can use the same method. Here, we need the following settings

\[
\int_T^\zeta \frac{1}{r(\eta)} \left[ \int_\eta^\infty q(\xi) d\xi + \sum_{i=1}^\infty h(\theta_i) \right] d\eta < \frac{1+2p_0}{10G(-p_0)} \quad \text{and} \quad -\frac{1+2p_0}{40} \leq F(\zeta) \leq \frac{1+2p_0}{20}.
\]

We set

\[
M = \left\{ u : u \in C([\zeta - \rho, +\infty), \mathbb{R}), u(\zeta) = 0 \text{ for } t \in [\zeta - \rho, \zeta] \text{ and } \frac{7+2p_0}{40} \leq u(\zeta) \leq -p_0 \right\}
\]
and \( \Phi : M \to C([\zeta - \rho, +\infty), \mathbb{R}) \) defined by

\[
(\Phi u)(\zeta) = \begin{cases} 
0, & \zeta \in [\zeta - \rho, \rho) \\
\int_{\zeta}^{\infty} \frac{1}{r(\eta)} \left[ \int_{\eta}^{\infty} q(\xi) G(u(\xi - \mu)) d\xi \\
+ \sum_{i=1}^{\infty} h(\theta_i) G(u(\theta_i - \mu)) \right] d\eta + F(\xi) + \frac{2 + p_0}{10}, & \zeta \geq T.
\end{cases}
\]

Thus, the proof is completed. \( \blacksquare \)

**Theorem 8.** Consider \( p \in C[\mathbb{R}_+, [0, 1]] \) and \( G \) are Lipschitzian on the interval \([a, b]\), where \( 0 < a < b < \infty \). If (H1) and (H24) hold, then the IDS \((S)\) has a positive solution.

**Proof.** Consider \( 0 \leq p(\zeta) \leq a < 1 \). Then we can find \( \zeta_1 > 0 \) so that

\[
\int_{\zeta_1}^{\infty} \frac{1}{r(\xi)} \left[ \int_{\xi}^{\infty} q(\eta) d\eta + \sum_{i=1}^{\infty} h(\theta_i) \right] d\xi < 1 - a \frac{1}{5K},
\]

where \( K = \max\{K_1, G(1)\} \), \( K_1 \) is the Lipschitz constant on \([\frac{3}{4}(1 - a), 1]\). Let \( |F(\xi)| < \frac{1-a}{10} \) for \( \zeta \geq \zeta_2 \). For \( \zeta_3 > \max\{\zeta_1, \zeta_2\} \), we set \( X = BC([\zeta, \infty), \mathbb{R}) \), the space of real valued continuous functions on \([\zeta, \infty)\). Clearly, \( X \) is a Banach space with respect to the sup norm defined by

\[
\|u\| = \sup\{|u(\zeta)| : \zeta \geq \zeta_3\}.
\]

We consider the set

\[
S = \{u \in X : 3 \frac{1}{5}(1 - a) \leq u(\zeta) \leq 1, \zeta \geq \zeta_3\}.
\]

It is clear that \( S \) is the closed and convex subspace of \( X \). Let us define \( \Phi : S \to S \) by

\[
(\Phi u)(\zeta) = \begin{cases} 
(\Phi u)(\zeta_3 + \rho), & \zeta \in [\zeta_3, \zeta_3 + \rho] \\
-\frac{p(\zeta) u(\zeta - \delta)}{1 - a} + K_1 + F(\zeta) & \zeta \geq \zeta_3 + \rho.
\end{cases}
\]

For every \( u \in X \), \( (\Phi u)(\zeta) \leq F(\zeta) + \frac{9a}{10} \leq 1 \) and

\[
(\Phi u)(\zeta) \geq -p(\zeta) u(\zeta - \delta) - G(1) \int_{\zeta}^{\infty} \frac{1}{r(\eta)} \left[ \int_{\eta}^{\infty} q(\xi) d\xi + \sum_{i=1}^{\infty} h(\theta_i) \right] d\eta + F(\zeta) + \frac{9 + a}{10} \]

\[
\geq -a - \frac{1-a}{5} - \frac{1-a}{10} + \frac{9 + a}{10} = 3 \frac{1}{5}(1 - a)
\]

implies that \( (\Phi u) \in S \). Now for \( u_1 \) and \( u_2 \) in \( S \), we have

\[
|(\Phi u_1)(\zeta) - (\Phi u_2)(\zeta)| \leq a|u_1(\zeta - \delta) - u_2(\zeta - \delta)|
\]

\[
+ \int_{\zeta}^{\infty} \frac{1}{r(\xi)} \left[ \int_{\xi}^{\infty} q(\eta) |G(u_1(\eta - \mu)) - G(u_2(\eta - \mu))| d\eta + \sum_{i=1}^{\infty} h(\theta_i) |G(u_1(\theta_i - \mu)) - G(u_2(\theta_i - \mu))| \right] d\xi,
\]

where \( 0 < a < b < \infty \). If (H1) and (H24) hold, then the IDS \((S)\) has a positive solution.
that is,
\[ |(\Phi u_1)(\zeta) - (\Phi u_2)(\zeta)| \leq a\|u_1 - u_2\| + \|u_1 - u_2\| K \int_{-T}^{T} \frac{1}{r(\kappa)} \left[ \int_{k}^{\infty} q(\eta) d\eta + \sum_{i=1}^{\infty} h(\theta_i) \right] d\kappa \]
\[ \leq \left( a + \frac{1 - a}{5} \right) \|u_1 - u_2\| \]
\[ = \frac{4a + 1}{5} \|u_1 - u_2\|. \]

Therefore, \( \|\Phi u_1 - (\Phi u_2)\| \leq \frac{4a + 1}{5} \|u_1 - u_2\| \) implies that \( \Phi \) is a contraction and \( \Phi \) has a unique fixed point \( u(\zeta) \) in \([\frac{3}{4}(1 - a), 1]\) by Banach’s fixed point theorem. Hence, \( (\Phi u) = u \). Thus, the theorem is proved. \( \square \)

**Remark 1.** It is not possible to use the Lebesgue’s dominated convergence theorem for another intervals of the neutral coefficient except \(-1 \leq p(\zeta) \leq 0\) as there are different solutions in different ranges. But, one can use Banach’s fixed point theorem for another intervals of the neutral coefficient similar to Theorem 8.

5. Discussion and Example

In this paper, we have seen that (H7)–(H14) and (H16)–(H23) are the new sufficient conditions for oscillatory behaviour of solutions of (S), in which we are depending explicitly on the forcing function. The results of this paper are not only true for (S) but also for its homogeneous counterpart.

Next, we mentioning examples to show feasibility and efficiency of main results.

**Example 1.** Consider the IDS

\[
(S_1) \quad \begin{cases} 
(u(\zeta) + u(\zeta - \pi))'' + u(\zeta - \frac{\pi}{4}) = \cos(\zeta - \frac{3\pi}{4}), & t > \frac{\pi}{4}, \\
\Delta(u(\theta_i) + u(\theta_i - \pi))' + h(\theta_i)u(\theta_i - \frac{\pi}{4}) = 2\sin(h) \cos(k - \frac{\pi}{4}),
\end{cases}
\]

where \( h(\theta_i) = \frac{2}{1 + \cot(h)}, \theta_i = i, i \in \mathbb{N}, G(u) = u \) and \( f(\zeta) = \cos(\zeta - \frac{\pi}{4}). \) Indeed, if we choose \( F(\zeta) = -\cos(\zeta - \frac{\pi}{4}), \) then \( (r(\zeta)F'(\zeta))' = F''(\zeta) = f(\zeta) \) and

\[
\Delta(r(\theta_i)F'(\theta_i)) = F'(\theta_i + \epsilon) - F'(\theta_i - \epsilon) \\
= F'(i + \epsilon) - F'(i - \epsilon) \\
= \sqrt{2} \sin(\epsilon) (\sin(i) + \cos(i)) = g(\theta_i), \quad i \in \mathbb{N}.
\]

Now, it is clear that

\[
F^+(\zeta) = \begin{cases} 
-\cos(\zeta - \frac{\pi}{4}), & 2n\pi + \frac{3\pi}{4} \leq \zeta \leq 2n\pi + \frac{7\pi}{4} \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
F^-\left(\zeta - \frac{\pi}{4}\right) = \begin{cases} 
\cos(\zeta - \frac{\pi}{4}), & 2n\pi + \frac{7\pi}{4} \leq \zeta \leq 2n\pi + \frac{11\pi}{4} \\
0, & \text{otherwise}
\end{cases}
\]

implies that

\[
F^+(\zeta - \frac{\pi}{4}) = \begin{cases} 
-\sin(\zeta), & 2n\pi + \pi \leq \zeta \leq 2n\pi + 2\pi \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
F^-\left(\zeta - \frac{\pi}{2}\right) = \begin{cases} 
\sin(\zeta), & 2n\pi + 2\pi \leq \zeta \leq 2n\pi + 3\pi \\
0, & \text{otherwise}
\end{cases}
\]
Since
\[ \int_{\pi/4}^{\infty} F^+ \left( \eta - \frac{\pi}{4} \right) d\eta = \sum_{n=0}^{\infty} \int_{2n\pi + \pi}^{2n\pi + 2\pi} \left[ - \sin(\eta) \right] d\eta = \infty, \]
then for \( n = 0, 1, 2 \ldots \), we obtain
\[ \int_{\pi/4}^{\infty} F^+ \left( \eta - \frac{\pi}{4} \right) d\eta + \sum_{i=1}^{\infty} \left( \frac{2}{1 + \cot(\theta_i)} \right) F^+ \left( k - \frac{\pi}{4} \right) = \infty. \]
Thus, every condition of Theorem 1 is satisfied, and hence, each solution of \((S_1)\) is oscillatory by Theorem 1.

**Example 2.** Consider the impulsive system
\[(S_2) \begin{cases} (r(\zeta)(u(\zeta) + p(\zeta)u(t-1)))' + q(\zeta)u(\zeta - 1) = 0, \quad \zeta \neq \theta_i; \\ \Delta r(\theta_i)(u(\theta_i)) + p(\theta_i)u(\theta_i - 1))' + h(\theta_i)u(\theta_i - 1) = 0, \quad i \in \mathbb{N}, \end{cases} \]
where \( 1 \leq p(\zeta) = e^{\zeta} + 1 \leq \infty, \quad q(\zeta) = e^{-\zeta}, \quad r(\zeta) = e^{\zeta}, \quad G(u) = u, \quad \rho = 1 \) and \( \theta_i = 2^i, \quad i \in \mathbb{N}. \)
Clearly, all conditions of Theorem 4 are satisfied. Thus, by Theorem 4, every solution of the system \((S_2)\) oscillates.

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