Long Time Memory of Lagrangian Acceleration Statistics in 2D and 3D Turbulence

Oliver Kamps$^1$ & Michael Wilczek$^2$

$^1$Center for Nonlinear Science, University of Münster, Corrensstraße 2, 48149 Münster, Germany
$^2$Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Str. 9, 48149 Münster, Germany

Abstract. In this paper we report on a comparison of Lagrangian acceleration statistics in the direct energy cascade of three-dimensional turbulence and the corresponding observables for the case of the inverse energy cascade in two dimensions. We focus on the time scales describing the memory of the acceleration statistics of a tracer particle. We show that for both systems the Markov time scale, which is an indicator for the length of the memory of a stochastic process, is in the order of magnitude of the Lagrangian integral time scale. We also show that the decorrelation time for the cross-correlation between the squared components of the acceleration is larger than the integral time scale.

1. Introduction

The Lagrangian description of turbulent flows is the natural frame of reference for describing dispersion and mixing in turbulent flows. Beyond this the Lagrangian picture is also of conceptual interest for studying the basic characteristics of turbulence. With the availability of high quality experimental and numerical data the research in this area made substantial progress over the last years (Yeung, 2002; Toschi & Bodenschatz, 2009). Especially the statistics of the acceleration and velocity increments along tracer trajectories have been in the focus of interest. While these investigations focused strongly on three-dimensional flows, in recent times numerical results for forced (Kamps & Friedrich, 2008; Bos et al., 2010) and decaying (Wilczek et al., 2008; Kadoch et al., 2008) two-dimensional flows analyzing the statistics of these observables have been published.

It is a well-known fact that two- and three-dimensional turbulent flows show some peculiar differences like the opposed direction of energy transfer. Especially the absence of intermittency of the Eulerian increment statistics in the inverse cascade of forced two-dimensional turbulence is a distinguishing feature between both types of flows (see e.g. (Boffetta et al., 2000)). It was shown that in the Lagrangian frame of reference observables like velocity increments or acceleration exhibit similar statistical properties as in three dimensions like intermittency or long time correlations (Kamps & Friedrich, 2008).

The comparison of the relation between Eulerian and Lagrangian statistics in two and three dimensions led to a probabilistic formulation of the translation relation between Eulerian and Lagrangian velocity increments (Kamps et al., 2009) which is a generalization of the types of relations proposed in e.g. (Borgas, 1993; Chevillard et al., 2003; Biferale et al., 2004).
Furthermore this relation is independent of the dimension and to some extent also independent of the type of turbulence (Homann et al., 2009). From this perspective it seems to be fruitful to compare the Lagrangian characteristics of two- and three-dimensional turbulence in a systematic way.

2. Numerical simulations

Our analysis is based on numerical simulations of the two systems in a state where the flow field is statistically stationary, homogeneous and isotropic. In both cases we use a pseudospectral method to integrate the forced Navier-Stokes equation in the vorticity formulation

$$\partial_t \omega(x, t) = \nabla \times [u(x, t) \times \omega(x, t)] + (-1)^{n+1} \nu \Delta^n \omega(x, t) + g^{2D}(x, t) + f^{2D,3D}(x, t)$$

(1)

where $\omega = \nabla \times u$ is the vorticity ($\omega = \omega e_z$ in two dimensions) In three dimensions the forcing $f^{3D}$ is located at large scales leading to a direct energy cascade towards small scales. In two dimensions $f^{2D}$ is designed to inject the energy at small scales to feed the inverse cascade. To remove the energy flowing to large scales an energy sink of the form $g^{2D} = -\gamma \omega e_z$ is introduced to achieve a statistical stationary state in two dimensions. In case of the inverse cascade we have chosen a hyperviscous term of the order $n = 8$ (see e.g. (Goto & Vassilicos, 2004; Boffetta et al., 2000)) and in three dimensions we have $n = 1$. In both cases we use periodic domains with side-length $2\pi$ and a resolution of 1024 grid points in each direction. Time stepping is performed with an memory saving third order Runge-Kutta method (Shu & Osher, 1988) with an integrating factor for the viscous term. To follow the tracer particles, the Eulerian fields are interpolated by a bi- and tri-cubic interpolation scheme, respectively. Further details can be found in (Kamps & Friedrich, 2008) and (Wilczek & Friedrich, 2009). In Fig. 2 the energy spectra $E(k)$ are shown for both cases.

In the following we will focus on the Cartesian components of the acceleration $a_i$ and velocity $v_i$. Assuming isotropy we will skip the index and write $a$ and $v$. By $T = \int_0^\infty \text{d}\tau C_v(\tau)$ we define the Lagrangian integral time scale via the autocorrelation

$$C_v(\tau) = \frac{\langle v(t)v(t + \tau) \rangle - \langle v(t) \rangle \langle v(t + \tau) \rangle}{\langle v^2(t) \rangle - \langle v(t) \rangle^2}$$

(2)

of the velocity components, which is a very useful quantity to compare time scales between the two systems under investigation.
3. Long time correlations

The autocorrelation function is an often used quantity to characterize the time-dependent evolution of the acceleration of a tracer particle. It has been shown that in three-dimensional as well as in two-dimensional turbulence the acceleration components and their magnitudes are long time correlated quantities (Mordant et al., 2002; Kamps & Friedrich, 2008; Bos et al., 2010). In (Mordant et al., 2002) it has been argued that the connection of long time accelerations and a non-Gaussian acceleration probability density function (PDF) leads to intermittency in the Lagrangian frame, showing the importance of these quantities for the velocity increment statistics. Here we compare the correlation of order $n$ defined as

$$C_a^n(\tau) = \frac{\langle a^n(t)a^n(t+\tau)\rangle - \langle a^n(t)\rangle\langle a^n(t+\tau)\rangle}{\langle a^{2n}(t)\rangle - \langle a^n(t)\rangle^2}$$

for the two-dimensional and the three-dimensional flow. The results can be seen in Fig. 3. In both cases the correlation $C_a^1(\tau)$ shows the characteristic undershoot. This undershoot is a consequence of the fact that the integral over the correlation function has to be zero when the velocity statistics is stationary. The different positions of the minimum are not relevant for our comparison because for the example of three-dimensional turbulence we know that their position depends on the Reynolds number. The important point is that time scale of the decay is very similar for both functions.

A closer look at the correlation of the squared components reveals a significant difference between both types of flow. While in the three-dimensional case both correlation functions decay on a time scale of the order of 1.5 to 2T, in two dimensions $C_a^2(\tau)$ decays much slower than $C_a^1(\tau)$. The time scale for the decay of the correlation of the squared acceleration component lies in the order of 18 or 20 times the integral scale. At this point we can only speculate about the reason for the difference, but we think the fact that in two dimensions the spiraling motion is confined to the plane leads to a more persistent behavior in two dimensions than in three dimensions where the particle has more degrees of freedom to escape the circular motion.

4. Long time memory

The correlation function describes how much information on past states is carried along with the tracer, but it does not describe which information from the past, here denoted as memory,
The high-dimensional PDF can be decomposed into a product of two-time conditional PDFs even when \( N \) by the CFL criterion. Nevertheless, a description in terms of a stochastic process is meaningful. In numerical simulations such a sampling time could be the time step given on which the signal is smooth. This assumption seems to contradict the description in terms of a complete. However, it is sufficient to choose the sampling interval smaller than the time scale \( t \approx \tau \) and the direct estimated PDF differs significant in its structure from the PDF computed from the Chapman-Kolmogorov equation while for larger \( \tau \) both PDFs differ only in their width.

![Figure 3](image-url) | ![Figure 3](image-url)
---|---

**Figure 3.** Contour plots of transition probabilities \( p(a_i|a_{i-2}) \) (red) and \( p_{CK}(a_i|a_{i-2}) \) (blue) for \( \tau \approx 0.13T \) (left) and \( \tau \approx 0.52T \) (right) in the three-dimensional case. For the smaller time lag the direct estimated PDF differs significant in its structure from the PDF computed from the Chapman-Kolmogorov equation while for larger \( \tau \) both PDFs differ only in their width.

is needed to predict the next state. For example, a simple stochastic process like the Ornstein-Uhlenbeck process can have a very long correlation, but to compute the next state we only have to know the present state. If we want to address this aspect, we have to consider the full statistical description of the acceleration of a tracer particle along its trajectory. This is given by the \( N \)-time PDF \( f(a_N, \ldots, a_1) \), where \( a_i \) denotes \( a(t_i) \) and the sampling interval is \( t_{i+1} - t_i = \Delta t \). In the limit \( \Delta t \to 0 \) (and correspondingly \( N \to \infty \)) the information would be complete. However, it is sufficient to choose the sampling interval smaller than the time scale on which the signal is smooth. This assumption seems to contradict the description in terms of a stochastic process, but as we consider the whole ensemble of tracer particles, a probabilistic theory is meaningful. In numerical simulations such a sampling time could be the time step given by the CFL criterion. Nevertheless, a description in terms of a \( N \)-time PDF is very complex even when \( N \) is countable. In the case that \( a(t_i) \) can be described by a Markov process this high-dimensional PDF can be decomposed into a product of two-time conditional PDFs

\[
f(a_N, \ldots, a_1) = p(a_N|a_{N-1}) \times \cdots \times p(a_2|a_1)f(a_1). \tag{4}\]

In the case of a stationary process \( p(a_i|a_{i-1}) \) would be identical for all \( i \), which implies that the whole process can be described by a single two-time conditional PDF. The question is for which time lag \( \tau = t_{i+1} - t_i \) this factorization can be performed. For a pure Markov process we would have \( \tau = 0 \) by definition, which would imply that even on the time scale of the smallest step \( \Delta t \) the Markov property is fulfilled. For most systems, however, \( \tau \) is larger than \( \Delta t \). For instance, in the case of the direct cascade of three-dimensional turbulence it has been shown in (Friedrich & Peinke, 1997; Renner et al., 2001) that the evolution of the longitudinal velocity increments in scale can be described by a non-stationary Markov process when the step between two increments on different scales is larger than the Taylor-length. A necessary condition to check for the Markov property of the acceleration is the Chapman-Kolmogorov equation

\[
p_{CK}(a_i|a_{i-2}) = \int da_{i-1} p(a_i|a_{i-1})p(a_{i-1}|a_{i-2}). \tag{5}\]

When the Markov property is fulfilled, \( p_{CK}(a_i|a_{i-2}) \) has to be equal to the transition probability \( p(a_i|a_{i-2}) \) directly estimated from the data. In Fig. 4 we see two examples of \( p_{CK}(a_i|a_{i-2}) \)
Figure 4. Correlation measure $M(\tau)$ for for three dimensions (red) and two dimensions (blue).

together with the corresponding $p(a_i|a_{i-2})$. From these plots one can conclude that for small time lags the Markov property is not fulfilled. In the past several methods have been developed to test for the equality of the two transition probabilities (Nawroth & Peinke, 2006). Here we use the correlation measure (Kamps et al., 2010)

$$\tilde{M}(a_{i-2}, \tau) = \frac{\int da_i p(a_i|a_{i-2})pCK(a_i|a_{i-2})}{\sqrt{\int da_i p^2(a_i|a_{i-2})}\sqrt{\int da_i p^2_{CK}(a_i|a_{i-2})}}.$$  

(6)

for fixed $a_{i-2}$ and $\tau$. By computing

$$M(\tau) = N^{-1} \int da_{i-2} \tilde{M}(a_{i-2}, \tau) \quad \text{with} \quad N = a_{i-2}^{\max} - a_{i-2}^{\min}$$

(7)

we average over the condition $a_{i-2}$ to get a single-valued measure only depending on $\tau$. If the Markov assumption is correct, $M(\tau)$ converges to 1. In Fig. 6 the result of this test is depicted. The time for the acceleration to become indistinguishable from a Markov process is $2T$ to $2.5T$ and $1T$ to $1.5T$ for three and two dimensions, respectively. Although for two dimensions this time is a little bit shorter, it lies in the order of magnitude of the integral time scale in both cases. Not only the correlation but also the memory of the acceleration process has a decay time that is very long. That means that the acceleration retains its memory over the whole inertial range.

5. Cross-correlations

Up to know we have considered only one component of the acceleration. To investigate the connection between the different components we now analyze the cross-correlation between them by studying the cross-correlation defined as

$$C_{a{i,a_j}}^n(\tau) = \frac{\langle a^n_i(t)a^n_j(t+\tau) \rangle - \langle a^n_i(t) \rangle \langle a^n_j(t+\tau) \rangle}{\sqrt{\langle (a^{2n}_i(t)) - \langle a^n_i(t) \rangle^2 \rangle} \sqrt{\langle (a^{2n}_j(t)) - \langle a^n_j(t) \rangle^2 \rangle}}.$$  

(8)

We only consider the case $n = 2$ because the correlation function $C_{a{i,a_j}}^1$ is zero in the case of homogeneous isotropic turbulence for $i \neq j$. A pronounced feature of $C_{a{i,a_j}}^2$ both in two and three dimensions is the peak occurring after a short time (for three dimensions this time is in the
order of 0.25-0.3τη) which was already indicated for three-dimensional turbulence in (Mordant et al., 2004). For our analysis the time scale of the decay after the peak is the central quantity. One can see that this time scale is for both systems similar to the time scale connected to Cα2(τ), e.g. in the two-dimensional case the decay is much slower than in three dimensions. We think that the same reasons leading to the differences in the behavior of Cα2(τ) are responsible for this observation.

6. Summary and outlook

To conclude, in this paper we have investigated the time scales of the acceleration of a Lagrangian particle in two- and three-dimensional homogeneous isotropic turbulence. We have shown that in both flows the acceleration exhibits long time correlations. The main difference is that the magnitude of the acceleration is more persistent in two dimensions. The central point of our investigation is that in both cases the acceleration can only be described by a Markov process for time lags in the order of magnitude of the integral time scale. This is accompanied by long time cross-correlations coupling the different components of the acceleration.

The next step would be to get a better understanding of the physical phenomena behind our findings. Beyond this it would be necessary to investigate the Reynolds number dependence of the Markov time scale and the time scales of the cross-correlations. It would be interesting to see whether the relation between these scales and the integral time scale remains constant or is a function of the Reynolds number. We think that these findings should be accounted for in Lagrangian modeling, for example, by introducing more spatial dimensions in the model equations.

References

Biferale, L., Boffetta, G., Celani, A., Devenish, B. J., Lanotte, A. & Toschi, F. 2004 Multifractal Statistics of Lagrangian Velocity and Acceleration in Turbulence. Phys. Rev. Lett. 93 (6), 064502.

Boffetta, G., Celani, A. & Vergassola, M. 2000 Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior. Phys. Rev. E 61 (1), R29–R32.
BORGAS, M. S. 1993 The Multifractal Lagrangian Nature of Turbulence. *Philosophical Transactions: Physical Sciences and Engineering* **342** (1665), 379–411.

BOS, W.J.T., KADOCH, B., NEFFAA, S. & SCHNEIDER, K. 2010 Lagrangian dynamics of drift-wave turbulence. *Physica D: Nonlinear Phenomena* **239** (14), 1269 – 1277.

CHEVILLARD, L., ROUX, S. G., LÉVEQUE, E., MORDANT, N., PINTON, J.-F. & ARNEODO, A. 2003 Lagrangian velocity statistics in turbulent flows: Effects of dissipation. *Phys. Rev. Lett.* **91** (21), 214502.

FRIEDRICH, R. & PEINKE, J. 1997 Description of a Turbulent Cascade by a Fokker-Planck Equation. *Phys. Rev. Lett.* **78** (5), 863–866.

GOTO, S. & VASSILICOS, J C 2004 Particle pair diffusion and persistent streamline topology in two-dimensional turbulence. *New Journal of Physics* **6**, 65.

HOMANN, H., KAMPS, O., FRIEDRICH, R. & GRAUER, R 2009 Bridging from Eulerian to Lagrangian statistics in 3d hydro- and magnetohydrodynamic turbulent flows. *New Journal of Physics* **11** (7), 073020 (15pp).

KADOCH, BENJAMIN, BOS, WOUTER J. T. & SCHNEIDER, KAI 2008 Extreme Lagrangian Acceleration in Confined Turbulent Flow. *Phys. Rev. Lett.* **100** (18), 184503.

KAMPS, O. & FRIEDRICH, R. 2008 Lagrangian statistics in forced two-dimensional turbulence. *Phys. Rev. E* **78** (3), 036321.

KAMPS, O., FRIEDRICH, R. & GRAUER, R. 2009 Exact relation between Eulerian and Lagrangian velocity increment statistics. *Phys. Rev. E* **79** (6), 066301.

KAMPS, O., WILCZEK, M. & FRIEDRICH, R. 2010 Lagrangian acceleration statistics in 2d and 3d turbulence. In *Proceedings of the iTi Conference in Turbulence 2010* (ed. Joachim Peinke, Martin Oberlack & Alessandro Talamelli). Accepted.

MORDANT, N., DELOUR, J., LÉVEQUE, E., ARNÉODO, A. & PINTON, J.-F. 2002 Long Time Correlations in Lagrangian Dynamics: A Key to Intermittency in Turbulence. *Phys. Rev. Lett.* **89** (25), 254502.

MORDANT, N., LEVEQUE, E. & PINTON, J.-F. 2004 Experimental and numerical study of the Lagrangian dynamics of high Reynolds turbulence. *New Journal of Physics* **6**, 116.

NAWROTH, A.P. & PEINKE, J. 2006 Small scale behavior of financial data. *The European Physical Journal B* **50** (1-2), 147–151.

RENNER, C., PEINKE, J. & FRIEDRICH, R. 2001 Experimental indications for Markov properties of small-scale turbulence. *Journal of Fluid Mechanics* **433**, 383–409.

SHU, CHI-WANG & Osher, STANLEY 1988 Efficient implementation of essentially non-oscillatory shock-capturing schemes. *Journal of Computational Physics* **77** (2), 439 – 471.

TOSCHI, F. & BODENSCHATZ, E. 2009 Lagrangian Properties of Particles in Turbulence. *Annual Review of Fluid Mechanics* **41** (1), 375–404.

WILCZEK, MICHAEL & FRIEDRICH, RUDOLF 2009 Dynamical origins for non-Gaussian vorticity distributions in turbulent flows. *Phys. Rev. E* **80** (1), 016316.

WILCZEK, MICHAEL, KAMPS, OLIVER & FRIEDRICH, RUDOLF 2008 Lagrangian investigation of two-dimensional decaying turbulence. *Physica D: Nonlinear Phenomena* **237** (14-17), 2090 – 2094, Euler Equations: 250 Years On - Proceedings of an international conference.

YEUNG, P. K. 2002 Lagrangian Investigations of Turbulence. *Annual Review of Fluid Mechanics* **34** (1), 115–142.