Antichains of positive roots and Heaviside functions

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Abstract

The ring of locally-constant integer-valued functions on the dominant chamber of the Shi arrangement is endowed with a filtration and a new basis, compatible with this filtration, is found. This basis is compared to the trivial basis. The ring is given a presentation by generators and relations.

0 Introduction

The aim of this short paper is to study some rings associated to finite root systems. These rings are defined starting from a well-known hyperplane arrangement associated to a root system, which is called the Shi arrangement [8, 9]. The number of connected components of the Shi arrangement which are contained in the dominant Weyl chamber is known to be the number of antichains of the poset of positive roots for the standard order and is equal to the generalized Catalan number which also appears in the theory of cluster algebras of Fomin and Zelevinsky [5, 6], see [1, 2, 4] and references therein. The main object in the present article is the ring of locally-constant integer-valued functions on the intersection of the dominant Weyl chamber with the Shi arrangement. This commutative ring, which is a free abelian group of rank given by the generalized Catalan number above, is endowed with a filtration by a general construction on hyperplane arrangements due to Gelfand and Varchenko [10]. A presentation by generators and relations is given, which leads to a new basis indexed by antichains and compatible with the filtration. The relation between the basis of idempotents and the new basis is explained using a natural partial order on antichains.

1 The root order

Let $X_n$ be a finite Dynkin diagram, i.e. $X_n$ is either in the classical series $A_n, B_n, C_n, D_n$ or one of the exceptionals $E_6, E_7, E_8, F_4, G_2$.

Let $\Phi_+$ be the set of positive roots for $X_n$. This set is endowed with the standard partial order $\leq$ defined by $\alpha \leq \beta$ if the difference $\beta - \alpha$ has non-negative coefficients in the basis of simple roots.

Let $A_{X_n}$ be the set of antichains in the poset $(\Phi_+, \leq)$. The elements of $A_{X_n}$ are called non-nesting partitions in the literature [11, 12, 7].
In any poset, there is a simple bijection between antichains and upper ideals. An antichain is mapped to the upper ideal it generates and an upper ideal is mapped to the set of its minimal elements, which is an antichain. The upper ideal associated to an antichain $p$ is denoted by $I_p$.

2 The Shi hyperplane arrangements

The roots in $\Phi_+$ are considered as linear forms on a real vector space of dimension $n$ in the usual way.

The Shi hyperplane arrangement associated to $X_n$ is the collection formed by all the hyperplanes $\alpha(x) = 0$ and $\alpha(x) = 1$ where $\alpha$ describes $\Phi_+$.

Let us call region a connected component of the complement of the union of these hyperplanes and dominant region a region contained in the dominant Weyl chamber.

By works of Athanasiadis and Cellini-Papi (see [2] and references therein), the number of dominant regions is known to be the generalized Catalan number associated to $X_n$, which is

$$C_{X_n} = \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1},$$

where $h$ is the Coxeter number and $e_1, \ldots, e_n$ the exponents of $X_n$.

Let us recall here the bijection between dominant regions and antichains. An antichain $p$ is mapped to the dominant region defined by $\alpha(x) > 1$ for all $\alpha \in I_p$ and $0 < \alpha(x) < 1$ elsewhere. One recovers the ideal $I_p$ from a dominant region as the set of roots which take values greater than 1 on this region.

Define a partial order $\preceq$ on antichains by inclusion of associated upper ideals, i.e. set $p \preceq q$ if $I_p \subseteq I_q$ as a set. This is just the lattice of upper ideals of $(\Phi_+, \leq)$ for the inclusion order.

Fig. 1 displays the Shi arrangement and the dominant Weyl chamber for $B_2$. 

![Figure 1: Shi arrangement for $B_2$](image)
3 Heaviside functions and filtration

Let $V_{X_n}$ be the ring of locally-constant functions with integer values on the complement of the Shi arrangement of $X_n$.

For any hyperplane arrangement, an increasing filtration on the ring of locally-constant integer-valued functions has been introduced by Gelfand and Varchenko in [10]. In the case of the Shi arrangement, this filtration is defined as follows. First for each root $\alpha \in \Phi_+$, define two locally-constant functions $h^0_\alpha$ and $h^1_\alpha$ as follows.

\[ h^0_\alpha(x) = \begin{cases} 
0 & \text{if } \alpha(x) < 0, \\
1 & \text{if } \alpha(x) > 0, 
\end{cases} \quad \text{and} \quad h^1_\alpha(x) = \begin{cases} 
0 & \text{if } \alpha(x) < 1, \\
1 & \text{if } \alpha(x) > 1. 
\end{cases} \tag{2} \]

These are called Heaviside functions by Gelfand and Varchenko [10] by similarity with the step-function of Heaviside.

Then the filtration $F$ is

\[ Z^1 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \ldots, \tag{3} \]

where $F_1$ is the linear span of the functions $1$, $h^0_\alpha$, $h^1_\alpha$ for all positive roots $\alpha$ and the space $F_k$ is defined to be $(F_1)^k$ for all $k > 1$.

A key result of Gelfand and Varchenko in [10] says that the filtration reaches the full ring at step $n$, i.e. $F_n = V_{X_n}$. This implies that $V_{X_n}$ is generated by the Heaviside functions $h^0_\alpha$, $h^1_\alpha$.

4 Restriction to the dominant chamber

Let $H_{X_n}$ be the ring of locally-constant functions with integer values on the intersection of the complement of the Shi arrangement and the dominant Weyl chamber. The rank of $H_{X_n}$ is therefore the number $C_{X_n}$ of dominant regions of the Shi arrangement.

Let $p$ be an antichain. By the correspondence between antichains and dominant regions, one can define a function $\delta_p$ in $H_{X_n}$ which has value 1 on the region corresponding to $p$ and vanishes elsewhere.

The set of functions $(\delta_p)_p$ where $p$ describes $A_{X_n}$ is of course a basis of $H_{X_n}$ made of orthogonal idempotents, called the trivial basis.

As the set of dominant regions is a subset of the set of regions, there is a surjective restriction morphism from $V_{X_n}$ to $H_{X_n}$. The filtration $F$ of $V_{X_n}$ induces a filtration on $H_{X_n}$ still denoted by $F$. Remark that the image of $h^0_\alpha$ is 1 for all $\alpha$. Denote simply by $h_\alpha$ the image of $h^1_\alpha$. So the filtration $F$ on $H_{X_n}$ is given by

\[ Z^1 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \ldots, \tag{4} \]

where $F_1$ is the linear span of the functions $1, h_\alpha$ for all positive roots $\alpha$ and the space $F_k$ is defined to be $(F_1)^k$ for all $k > 1$.

From surjectivity and the similar result for $V_{X_n}$, one has $F_n = H_{X_n}$. In particular $H_{X_n}$ is generated by the functions $h_\alpha$.

Lemma 4.1 In the basis $(\delta_p)_p$ of $H_{X_n}$, the function $h_\alpha$ is given by

\[ h_\alpha = \sum_{\{\alpha\} \subseteq p} \delta_p. \tag{5} \]
Proof. From its definition by restriction of a Heaviside function, \( h_\alpha \) is the characteristic function of the regions where \( \alpha \) takes values greater than 1. Therefore it is the sum of \( \delta_p \) over each dominant region \( p \) where \( \alpha \) takes values greater than 1. This condition means exactly that \( \alpha \in I_p \) or \( I\{\alpha\} \subseteq I_p \), i.e. \( \{\alpha\} \preceq p \).

Lemma 4.2 The map \( \alpha \mapsto \{\alpha\} \) gives an order reversing injection of the poset of roots \((\Phi_+, \preceq)\) in the poset of antichains \((A_{X_n}, \preceq)\).

Proof. Obvious. \( \blacksquare \)

Proposition 4.3 The functions \( h_\alpha \) satisfy
\[
h_\alpha h_\beta = h_{\min(\alpha, \beta)},
\]
whenever \( \{\alpha, \beta\} \) is not an antichain.

Proof. Assume for example that \( \alpha \leq \beta \), so that \( \min(\alpha, \beta) = \alpha \). Then \( I\{\beta\} \subseteq I\{\alpha\} \) and \( \beta \preceq \alpha \). Therefore using the description of \( h_\alpha \) as a sum of idempotents given in Lemma 4.1 one has \( h_\alpha h_\beta = h_\alpha \).

Let \( \text{gr} \ H_{X_n} \) be the graded ring associated to the filtration \( F \) of the ring \( H_{X_n} \). Then \( \text{gr} \ H_{X_n} \) is generated by elements \( \tilde{h}_\alpha \) which satisfy
\[
\tilde{h}_\alpha \tilde{h}_\beta = 0,
\]
whenever \( \{\alpha, \beta\} \) is not an antichain.

5 Rings presented by generators and relations

Consider the commutative ring \( U_{X_n} \) generated by the variables \( u_\alpha \) for \( \alpha \in \Phi_+ \) modulo the relations
\[
u_\alpha u_\beta = u_{\min(\alpha, \beta)},
\]
whenever \( \{\alpha, \beta\} \) is not an antichain.

One can define an increasing filtration \( F \) on \( U \) as follows
\[
\mathbb{Z}\mathbb{1} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots,
\]
where \( F_1 \) is the linear span of the elements \( 1, u_\alpha \) for all positive roots \( \alpha \) and the space \( F_k \) is defined to be \( (F_1)^k \) for all \( k > 1 \).

Let \( \text{gr} \ U_{X_n} \) be the associated graded ring. Then \( \text{gr} \ U_{X_n} \) is presented by the generators \( \tilde{u}_\alpha \) modulo the relations
\[
\tilde{u}_\alpha \tilde{u}_\beta = 0,
\]
whenever \( \{\alpha, \beta\} \) is not an antichain.

Let us define an element \( u_p \) of \( U_{X_n} \) for each antichain \( p \) as follows. If \( p = \{\alpha_1, \ldots, \alpha_k\} \) then set \( u_p = u_{\alpha_1} \cdots u_{\alpha_k} \). Note that there is no conflict in notation. In the same way, one defines elements \( \tilde{u}_p \) in \( \text{gr} \ U_{X_n} \).

Proposition 5.1 The set \( (\tilde{u}_p)_p \) is a basis of the ring \( \text{gr} \ U_{X_n} \).
Proof. This set of monomials spans $\text{gr} \ U_X^n$ because any monomial containing two comparable roots vanish by relations (10). Now it follows immediately from the shape of relations (10) that the ring $\text{gr} \ U_X^n$ is in fact graded by the free abelian group generated by the set $\Phi_+$. So each of the monomials $\tilde{u}_p$ has a different weight. Hence there can be no linear relation between these monomials except maybe that some of them may vanish. But one can see that monomials which vanish because of relations (10) are precisely the monomials containing two comparable roots.

Proposition 5.2 The set $(u_p)_p$ is a basis of the ring $U_X^n$.

Proof. This is a direct corollary of Theorem 5.1.

Therefore the rank of $U_X^n$ is the generalized Catalan number $C_{X^n}$. Furthermore the rank of the graded component of degree $k$ of $\text{gr} \ U_X^n$ is the number of antichains of cardinal $k$.

6 Isomorphism

Proposition 6.1 There exists a unique morphism of filtered rings $\rho$ from $U_X^n$ to $H_X^n$ which sends $u_\alpha$ to $h_\alpha$.

Proof. It is enough to check that the relations (8) are satisfied in $H_X^n$. But this is exactly (9). The filtrations are clearly mapped one to another.

Proposition 6.2 The morphism $\rho$ is surjective.

Proof. The image of $\rho$ contains the elements $h_\alpha$ which generate $H_X^n$.

Theorem 6.3 The morphism $\rho$ is an isomorphism of filtered rings between $U_X^n$ and $H_X^n$.

Proof. The morphism $\rho$ preserves filtrations, is surjective and both rings have the same rank given by the generalized Catalan number $C_{X^n}$.

Proposition 6.4 The induced morphism $\rho$ from $\text{gr} \ U_X^n$ to $\text{gr} \ H_X^n$ is an isomorphism of graded rings.

Proof. This is a corollary of Theorem 6.3.

7 Change of basis

The aim of this section is to study the relation between the bases $(h_p)_p$ and $(\delta_p)_p$ of $H_X^n$, both indexed by the set $\mathcal{A}_X^n$ of antichains.

Proposition 7.1 Let $p$ be an antichain. One has

$$h_p = \sum_{p \preceq q} \delta_q.$$  

(11)
Proof. By induction on the cardinal of the antichain. This works for the empty antichain. This is also true if the antichain is a singleton by Lemma 4.1. Assume that it is proven for antichains with less elements. Let \( p = p' \sqcup \{ \alpha \} \), so that \( h_{p'} = h_p h_{\alpha} \). Then

\[
h_p = \sum_{p' \sqcup \{ \alpha \} \leq q} \delta_q \delta_r.
\]

(12)

From the idempotency and orthogonality of the basis \((\delta_p)_p\), one has

\[
h_p = \sum_{(p' \sqcup \{ \alpha \}) \leq q} \delta_q,
\]

(13)

because the union (which is also the supremum) of the ideals \( I_{p'} \) and \( I_{\{\alpha\}} \) is the ideal \( I_{p' \sqcup \{ \alpha \}} = I_p \).

So the coefficient matrix of the basis \((h_p)_p\) in the basis \((\delta_p)_p\) is given by the poset matrix for \( \preceq \).

Hence, conversely, the coefficients of the idempotents \( \delta_p \) in the basis \((h_p)_p\) are described by the Möbius matrix of the poset \((A_{X_n}, \preceq)\).

Fig. 2 displays the poset of antichains for the Dynkin diagram \( A_3 \).

Let us remark that there is a simple bijection between antichains of type \( A_n \) and Dyck paths which maps the cardinal of the upper ideal associated to an antichain to the area above the Dyck path. Therefore the antichains of \( X_n \) counted according to the cardinal of the upper ideal give a possible generalization to root systems of the classical \( q \)-Catalan numbers corresponding to Dyck paths and area [3].

References

[1] Christos A. Athanasiadis. On noncrossing and nonnesting partitions for classical reflection groups. *Electron. J. Combin.*, 5(1):Research Paper 42, 16 pp. (electronic), 1998.

[2] Christos A. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. preprint, 2002.

[3] L. Carlitz and J. Riordan. Two element lattice permutation numbers and their \( q \)-generalization. *Duke Math. J.*, 31:371–388, 1964.
[4] P. Cellini and P. Papi. Ad-nilpotent ideals of a Borel subalgebra II. *J. Algebra*, (258):112–121, 2002.

[5] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[6] Sergey Fomin and Andrei Zelevinsky. Cluster algebras II: Finite type classification. *Inventiones Mathematicae*, 2003. arXiv:math.RA/0208229.

[7] Victor Reiner. Non-crossing partitions for classical reflection groups. *Discrete Math.*, 177(1-3):195–222, 1997.

[8] Jian Yi Shi. Sign types corresponding to an affine Weyl group. *J. London Math. Soc. (2)*, 35(1):56–74, 1987.

[9] Jian-Yi Shi. The number of $\oplus$-sign types. *Quart. J. Math. Oxford Ser. (2)*, 48(189):93–105, 1997.

[10] A. N. Varchenko and I. M. Gelfand. Heaviside functions of a configuration of hyperplanes. *Funktsional. Anal. i Prilozhen.*, 21(4):1–18, 96, 1987.