FATE OF THE SUPERCONDUCTING GROUND STATE ON THE MOYAL PLANE

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Abstract

It is known that Berry curvature of the band structure of certain crystals can lead to effective noncommutativity between spatial coordinates. Using the techniques of twisted quantum field theory, we investigate the question of the formation of a paired state of twisted fermions in such a system. We find that to leading order in the noncommutativity parameter, the gap between the non-interacting ground state and the paired state is smaller compared to its commutative counterpart. This suggests that BCS type superconductivity, if present in such systems, is more fragile and easier to disrupt.

1 Introduction

Noncommutative quantum theories have the potential to provide us with insights not only at Planck scale physics (see for example [1]) but also in the domain of condensed matter and statistical physics (see [2–4] for a recent reviews). Indeed, apart from the well known noncommutativity of the guiding centre coordinates in the Landau problem, it has been shown by Xiao et. al. [5] that due to Berry curvature of the band structure, the quantum mechanics for the electrons in certain materials is govened by effective noncommutativity between phase space variables. Specifically, this noncommutativity takes the form

\begin{align}
[\hat{x}_i, \hat{x}_j] &= \frac{i \Omega \epsilon_{ij}}{1 + (e/\hbar)B\Omega}, \\
[\hat{k}_i, \hat{k}_j] &= -\frac{i (e/\hbar)B \epsilon_{ij}}{1 + (e/\hbar)B\Omega}, \quad \hat{p}_i \equiv \hbar \hat{k}_i, \\
[\hat{x}_i, \hat{k}_j] &= \frac{i \delta_{ij}}{1 + (e/\hbar)B\Omega} \quad (1.1)
\end{align}

where $i, j = 1, 2$ and $B$ is the external magnetic field (along the third direction) and $\Omega$ the Berry curvature arising due to the breaking of time reversal symmetry (for example in certain ferromagnetic materials) or spatial inversion symmetry (for example in GaAs). Although the

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possibility of superconductivity in such materials has already received some attention in the literature (see for e.g. [6,7]), our treatment of this question will use techniques arising from quantum fields defined on noncommutative spacetimes.

In the absence of external magnetic field ($B = 0$) these commutation relations take the form

\[
\begin{align*}
[\hat{x}_i, \hat{x}_j] &= i\Omega, \\
[\hat{k}_i, \hat{k}_j] &= 0, \\
[\hat{x}_i, \hat{k}_j] &= i\delta_{ij}, \quad i, j = 1, 2.
\end{align*}
\]

Quantum field theories on such a noncommutative space (also known as the Moyal plane) admit a novel quantization called *twisted quantization* [8,9] that implements the underlying spacetime symmetries using a new coproduct, modifying the canonical (anti)commutation relations of the basic oscillator algebra for creation and annihilation operators. Such deformed (anti) commutation relations can have dramatic consequences, like say the violation of the Pauli principle [3,10] at length scales of order $\sqrt{\Omega}$ and the energy shift in the model of degenerate electron gas [11].

In this article, we shall address the question of the formation of paired bound states (even parity spin singlet states of standard BCS theory) of twisted fermions on the Moyal plane. The corresponding question in ordinary space was addressed long ago by Cooper, who showed that formation of such bound states is indeed possible in the presence of arbitrarily weak attractive interaction between fermions of opposite momenta and spins. This makes the Fermi surface unstable, resulting the formation of BCS state which is a coherent state of Cooper pairs. The existence of BCS state explains various key features of superconductivity.

In the next section, we begin by revisiting the usual calculation (due to Cooper) in the second quantized formalism. The discussion is standard textbook material, and may be found for example, in [12]. We then extend it to the Moyal plane in section 3 and derive the new gap equation. This equation can be solved for small values of the parameter $\Omega$, and we find that the presence of $\Omega$ leads to a reduction the the gap, suggesting that it is comparatively easier to disrupt the superconducting state. We conclude in section 4 with a brief discussion of the implications of our results.

## 2 Commutative Case

Let us consider a many-fermion state in which there are additional two fermions above the filled Fermi sea. These two fermions have equal and opposite momenta and spins, and their energies are (slightly) higher than the Fermi energy. In the second quantized formalism the quantum state representing this system is

\[
|\psi_k\rangle = c_{k,\frac{\pi}{2}}^\dagger c_{-k,\frac{\pi}{2}}^\dagger |F\rangle \quad k > k_F, \quad \text{with}
\]

\[
|F\rangle = \prod_{k \leq k_F, \sigma} c_{k,\sigma}^\dagger |0\rangle, \quad \sigma = \pm \frac{1}{2}
\]
where $c^\dagger_{\vec{k},\sigma}$ is the creation operator of a fermion of momentum $\vec{k}$ and spin $\frac{\hbar}{2}\sigma$. A generic two-particle state $|\psi\rangle$ can be written as

$$|\psi\rangle = \int d^3k g(\vec{k}) |\psi_{\vec{k}}\rangle$$

(2.3)

The effective Hamiltonian describing the dynamics of the fermion pair is given by

$$\hat{H} = \hat{H}^0 + \hat{H}^{\text{int}}, \quad \text{with}$$

$$\hat{H}^0 = \int k F d^3k \epsilon(\vec{k}) c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma}$$

(2.4)

and $\epsilon(\vec{k}) = \frac{\vec{k}^2}{2m}$ is energy of a single free fermion. The interaction part of the Hamiltonian in the position space representation $\hat{H}^{\text{int}}$ is

$$\hat{H}^{\text{int}} = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3x d^3y \psi_{\sigma}(\vec{x}) \psi_{\sigma'}(\vec{y}) \psi_{\sigma'}(\vec{y}) \psi_{\sigma}(\vec{x}) v(\vec{x}, \vec{y})$$

(2.5)

Solving the eigenvalue equation

$$\hat{H} |\psi\rangle = E |\psi\rangle$$

(2.9)

for a generic two-particle state $|\psi\rangle$, we get

$$[2\epsilon(\vec{k}) - E] g(\vec{k}) = - \int d^3k' \tilde{V}(\vec{k} - \vec{k}') g(\vec{k}')$$

(2.10)

Choosing $\tilde{V}(\vec{k} - \vec{k}')$ to be a constant $-V$ for $k_F \leq k, k' \leq k_F + k_D$ and zero otherwise, and setting $\hbar^2 k_D^2/2m = \hbar \omega_D$ (where $\omega_D$ is the Debye frequency) we can rewrite (2.10) as

$$[2\epsilon_k - E] g(\vec{k}) = V \int d^3k' g(\vec{k}')$$

for $k_F \leq k, k' \leq k_F + k_D$ (2.11)

The RHS of (2.11) is independent of $\vec{k}$ and thus can be set to a constant $\Lambda$:

$$[2\epsilon_k - E] g(\vec{k}) = V \int_{k_F}^{k_F+k_D} g(\vec{k}') d^3k' = \Lambda$$

(2.12)
giving
\[ V \int_{\epsilon_F}^{\epsilon_F + \hbar \omega_D} \frac{N(\epsilon) d\epsilon}{2\epsilon - E} = 1 \] (2.13)
where \( N(\epsilon) \) is the density of the states of the (Bloch) fermions at the energy \( \epsilon \), and \( \epsilon_F \) is the Fermi energy. \( N(\epsilon) \approx N(\epsilon_F) \) as \( \hbar \omega_D \ll \epsilon_F \), and the above integral gives us
\[ \ln \frac{\Delta + \hbar \omega_D}{\Delta} = \frac{2}{N(\epsilon_F)V} \] (2.14)
where, \( \Delta = 2\epsilon_F - E \) is the energy gap. For weak coupling, \( N(\epsilon_F)V \ll 1 \) and we get
\[ \Delta \approx 2\hbar \omega_D e^{\frac{\hbar \omega_D}{2\epsilon_F}} \] (2.15)
This energy gap \( \Delta \) is related to the energy difference between normal and superconducting ground states:
\[ \langle E_n \rangle - \langle E_s \rangle = \frac{1}{2} N(\epsilon_F) \Delta^2 \] (2.16)
The full many-electron ground state is a coherent superposition of such paired states.

3 Ground state for the case \( \Omega \neq 0 \)

In Moyal spacetime, functions compose through star-product as
\[ (f * g)(x) \equiv m_0(\mathcal{F}^{-1} f \otimes g)(x) \equiv f(x)e^{\frac{i}{2} \Omega_{ij} \hat{\partial}_i \hat{\partial}_j} g(x), \] (3.1)
where,
\[ \mathcal{F} = e^{\frac{i}{2} \Omega_{ij} \hat{\partial}_i \otimes \hat{\partial}_j} \] (3.2)
is the twist operator and \( m_0 \) is the point-wise multiplication map. Note that here we are considering a simplified form of the noncommutative parameter, so that it is essentially planer with \( \hat{z} \) commuting with both \( \hat{x} \) and \( \hat{y} \). This implies that \( \theta_{ij} = \Omega_{ij} \) (for \( i, j = 1, 2 \)) and \( \theta_{13} = \theta_{23} = 0 \). The implementation of rotational symmetry in the twisted framework now requires that the transformation properties of multiparticle states (in contrast to the single particle state) should also be deformed. This is captured in the deformed coproduct (see [8,9] for the detailed mathematical descriptions) which in turn implies that the exchange operator \( \tau_0 \), defined as \( \tau_0 : \psi \otimes \phi \rightarrow \phi \otimes \psi \), should also be deformed as \( \tau_\theta = \mathcal{F}_\theta^{-1} \tau_0 \mathcal{F}_\theta \). One is thus forced to introduce the concept of twisted fermion/boson by “twist”-(anti) symmetrizing by using the projector \( P_\theta = \frac{1}{2}(1 \pm \tau_\theta) \). It then turns out that in noncommutative space, the fermionic creation and annihilation operators in momentum basis satisfy the twisted anti-commutation relations [8,9]
\[ a_{\mathbf{k},\sigma} a^\dagger_{\mathbf{k}',\sigma'} + e^{i \mathbf{k} \cdot \mathbf{k}'} a_{\mathbf{k}',\sigma'} a_{\mathbf{k},\sigma} = 0, \]
\[ a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k}',\sigma'} + e^{i \mathbf{k} \cdot \mathbf{k}'} a^\dagger_{\mathbf{k}',\sigma'} a^\dagger_{\mathbf{k},\sigma} = 0, \]
\[ a_{\mathbf{k},\sigma} a^\dagger_{\mathbf{k}',\sigma'} + e^{-i \mathbf{k} \cdot \mathbf{k}'} a^\dagger_{\mathbf{k}',\sigma'} a_{\mathbf{k},\sigma} = \delta(\mathbf{k} - \mathbf{k}') \delta_{\sigma \sigma'}, \] (3.3)
where, $\vec{k} \wedge \vec{k}' = \Omega \epsilon_{ij} k_i k'_j$. The twisted oscillators $a_{\vec{k},\sigma}$ are related to their commutative (or untwisted) counterparts $c_{\vec{k},\sigma}$ introduced in [8] as

$$a_{\vec{k},\sigma} = c_{\vec{k},\sigma} e^{-\frac{i}{2} \vec{k} \wedge \vec{P}}, \quad (3.4)$$

$$P_i = \int d^3 k \, k_i a_{\vec{k},\sigma}^\dagger a_{\vec{k},\sigma} = \int d^3 k \, k_i c_{\vec{k},\sigma}^\dagger c_{\vec{k},\sigma} \quad (3.5)$$

where $P_i$ is the Fock space momentum operator. Although the single-particle state created by $a_{\vec{k},\sigma}^\dagger$ is the same as that created by $c_{\vec{k},\sigma}^\dagger$, this is no longer true for multi-particle states: in fact there is no observable that connects a twisted multi-particle state to an untwisted multi-particle state, as these belong to inequivalent superselection sectors [9].

### 3.1 Interactions

We will restrict our attention to the case where $v(\vec{x}, \vec{y})$ remains same in noncommutative case. This is a not unreasonable, as the potential $v$ is a function of different spatial points $\vec{x}$ and $\vec{y}$. The $\hat{H}_{\Omega}^{\text{int}}$ in the noncommutative space is thus

$$\hat{H}_{\Omega}^{\text{int}} = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3 x \, \psi_{\sigma}(\vec{x}) \star \left[ \int d^3 y \, \psi_{\sigma'}(\vec{y}) \star v(\vec{x}, \vec{y}) \star \psi_{\sigma'}(\vec{y}) \right] \star \psi_{\sigma}(\vec{x}) \quad (3.6)$$

Writing $\psi_{\sigma}(\vec{x})$ as

$$\psi_{\sigma}(\vec{x}) = \int d^3 k \, e^{-i \vec{k} \cdot \vec{x}} a_{\vec{k},\sigma} \quad (3.7)$$

where, $a_{\vec{k},\sigma}, a_{\vec{k},\sigma}^\dagger$ satisfy the twisted commutation relations (3.3), we can write $\hat{H}_{\Omega}^{\text{int}}$ as

$$\hat{H}_{\Omega}^{\text{int}} = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3 p \, d^3 q \, d^3 r \, d^3 s \langle pq|v|rs \rangle_{nc} a_{\vec{p},\sigma}^\dagger a_{\vec{q},\sigma'}^\dagger a_{\vec{r},\sigma'} a_{\vec{s},\sigma} \quad (3.8)$$

The matrix element $\langle pq|v|rs \rangle_{nc}$ is

$$\langle pq|v|rs \rangle_{nc} = \int d^3 x \, e^{i \vec{p} \cdot \vec{x}} \star \left[ \int d^3 y \, e^{i \vec{q} \cdot \vec{y}} \star v(\vec{x}, \vec{y}) \star e^{-i \vec{r} \cdot \vec{y}} \right] \star e^{-i \vec{s} \cdot \vec{x}} \quad (3.9)$$

$$= e^{-\frac{i}{2} (\vec{p} \wedge \vec{s} + \vec{q} \wedge \vec{r})} \int d^3 \vec{y} \, e^{i \vec{q} \cdot \vec{y}} v(\vec{x}, \vec{y}) e^{-i \vec{r} \cdot \vec{y}} e^{-i \vec{s} \cdot \vec{x}}$$

$$= e^{-\frac{i}{2} (\vec{p} \wedge \vec{q} + \vec{q} \wedge \vec{r} + \vec{r} \wedge \vec{p})} \tilde{V}(\vec{r} - \vec{q}) \delta(\vec{r} - \vec{q}).(\vec{p} - \vec{s}) \quad (3.10)$$

The interaction Hamiltonian $\hat{H}_{\Omega}^{\text{int}}$ can be written in a manifestly twisted symmetrized form as

$$\hat{H}_{\Omega}^{\text{int}} = \frac{1}{2} \sum_{\sigma,\sigma'} \int d^3 p \, d^3 q \, d^3 r \, d^3 s \, \Phi_{\Omega}(p, q, r, s) \, a_{\vec{p},\sigma}^\dagger a_{\vec{q},\sigma'}^\dagger a_{\vec{r},\sigma'} a_{\vec{s},\sigma} \quad (3.11)$$
where
\[
\Phi_\Omega(p, q, r, s) \equiv \langle pq|vr\rangle_{nc} + e^{-i(p\wedge q + r\wedge s)}\langle qp|sr\rangle_{nc} \quad \text{and}
\]
\[
\Phi_\Omega(q, p, s, r) = e^{i(p\wedge q + r\wedge s)}\Phi_\Omega(p, q, r, s).
\]

Using (3.10), \(\hat{H}^\text{int}_\Omega\) takes the form
\[
\hat{H}^\text{int}_\Omega = \frac{1}{2} \sum_{\sigma, \sigma'} \int d^3 p d^3 q d^3 r \tilde{V}_\Omega(\vec{p}, \vec{q}, \vec{r}) a^\dagger_{p, \sigma} a^\dagger_{q, \sigma'} a_{r, \sigma'} a_{(p+q-r), \sigma}
\]

where \(V_\Omega\) is given by,
\[
\tilde{V}_\Omega = e^{-\frac{1}{2}(p\wedge q + q\wedge r + r\wedge p)} \tilde{V}(\vec{r} - \vec{q})
\]

The free part of the Hamiltonian
\[
\hat{H}^0_\Omega = \int d^3 k \epsilon(\vec{k}) a^\dagger_{\vec{k}, \sigma} a_{\vec{k}, \sigma} = \int d^3 k \epsilon(\vec{k}) c^\dagger_{\vec{k}, \sigma} c_{\vec{k}, \sigma} = \hat{H}^0
\]
is the same as in the commutative case because of (3.4). Therefore the full Hamiltonian is
\[
\hat{H}_\Omega = \hat{H}^0_\Omega + \hat{H}^\text{int}_\Omega
\]

3.2 Gap equation

We wish to solve the eigenvalue equation
\[
(\hat{H}^0_\Omega + \hat{H}^\text{int}_\Omega)|\psi\rangle = E|\psi\rangle
\]

for \(|\psi\rangle = \int d^3 k g_\Omega(\vec{k})|\psi_{\vec{k}}\rangle = \int d^3 k g_\Omega(\vec{k})a^\dagger_{\vec{k}, \frac{1}{2}} a_{\vec{k}, -\frac{1}{2}}|F\rangle\) a paired state of twisted fermions. Note that the ansatz (3.18) has the similar form as that of its commutative counterpart (2.3) with deformed coefficient \(g_\Omega(\vec{k})\). This is because here we are interested in understanding the robustness of standard BCS type of superconductivity (in the regime of small Berry curvature), where one considers the even parity spin singlet states (particularly \(l = 0\) state i.e. s-wave superconductivity). Indeed, it has been shown explicitly in [7] that higher pairing channels e.g. p-wave channel do not get activated in presence of small Berry curvature \(\Omega\); the activation of higher pairing channels require the presence of large \(\Omega\). Furthermore, the twist operator (3.2) shows that it acts only on the configuration space when twist symmetrizing, leaving spin part untouched so that the spin singlet (or triplet) states can be constructed in the usual manner see [8, 9].

Because of the map (3.4), the number operator in noncommutative case is the same as its commutative counterpart \((a^\dagger_{\vec{k}} a_{\vec{k}} = c^\dagger_{\vec{k}, \frac{1}{2}} c_{\vec{k}, -\frac{1}{2}})\), giving us \(\hat{H}^0_\Omega|\psi_{\vec{k}}\rangle = 2\epsilon(\vec{k})|\psi_{\vec{k}}\rangle\). But, since the operators \(a^\dagger_{\vec{k}}, a_{\vec{k}}\) satisfy the twisted anti-commutation relations (3.3), we get
\[
\hat{H}^\text{int}_\Omega|\psi\rangle = \int d^3 k g_\Omega(\vec{k})\tilde{H}^\text{int}_\Omega|\psi_{\vec{k}}\rangle
\]
\[
= \int d^3 p d^3 k V_\Omega(\vec{p}, \vec{k}) g_\Omega(\vec{k}) |\psi_{\vec{p}}\rangle
\]

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where, \( V_\Omega(\vec{p}, \vec{k}) \) is given, in terms of \( \tilde{V}_\Omega \) (3.15), by

\[
V_\Omega(\vec{p}, \vec{k}) = \frac{1}{2} \left[ \tilde{V}_\Omega(\vec{p}, -\vec{p}, -\vec{k}) + \tilde{V}_\Omega(-\vec{p}, \vec{p}, \vec{k}) \right] \tag{3.20}
\]

In the noncommutative case the eigenvalue equation \( \hat{H}_\Omega |\psi\rangle = E |\psi\rangle \) gives

\[
[2\epsilon(\vec{k}) - E]g_\Omega(\vec{k}) = -\int d^3k' V_\Omega(\vec{k}, \vec{k}') g_\Omega(\vec{k}') \tag{3.22}
\]

Assuming that \( V(\vec{p} - \vec{k}) \) is constant \((-V)\) when \( k_F \leq k, k' \leq k_F + \hbar \omega_D \) and otherwise 0, (3.22) takes the form

\[
[2\epsilon(\vec{k}) - E]g_\Omega(\vec{k}) = V \int d^3k' \ g_\Omega(\vec{k}') e^{-i\epsilon_{ij}k_i k_j} \tag{3.23}
\]

### 3.3 Solution of the gap equation

It is easy to see that for modes of equal and opposite momenta, the deformed anti-commutation relations are

\[
a^\dagger_{-\vec{k}, \sigma} a^\dagger_{\vec{k}, \sigma} = -e^{i\epsilon_{ij}(k_i)(-k_j)} a^\dagger_{\vec{k}, \sigma} a^\dagger_{-\vec{k}, \sigma'} = -a^\dagger_{\vec{k}, \sigma} a^\dagger_{-\vec{k}, \sigma'} \tag{3.24}
\]

implying that the composite wave function of two twisted fermions with equal and opposite momenta is anti-symmetric like its commutative counterpart. As the fermions are in a spin-singlet state, the spin part of the wave-function is anti-symmetric, thus forcing the momentum (or position) space wave function to be symmetric. This requires that \( g(\vec{k}) = g(-\vec{k}) \). We can thus write the gap equation (3.23) in the form

\[
[2\epsilon(\vec{k}) - E]g_\Omega(\vec{k}) = V \int d^3k' \cos(\epsilon_{ij}k_i k_j) g_\Omega(\vec{k}') \tag{3.25}
\]

We now solve the gap equation (3.25) perturbatively in \( \Omega \). To that end, let us write \( g(\vec{k}) \) and \( E \) in a series expansion as

\[
g_\Omega(\vec{k}) = g_0(\vec{k}) + \sum_{n=1}^{\infty} \Omega^n g_n(\vec{k}), \tag{3.26}
\]

\[
E = E_0 + \sum_{n=1}^{\infty} \Omega^n E_n. \tag{3.27}
\]

Expanding \( \cos(\Omega(k_1 k_2' - k_2 k_1')) \) in a Taylor series in \( \Omega \) and equating coefficients of various powers...
of Ω on the both sides of (3.25) we get

\[ f_0(k)g_0(\vec{k}) = V \int d^3k' g_0(\vec{k}') , \]  
\[ f_0(\vec{k})g_1(\vec{k}) - E_1g_0(\vec{k}) = V \int d^3k' g_1(\vec{k}') , \]  
\[ f_0(\vec{k})g_2(\vec{k}) - E_1g_1(\vec{k}) - E_2g_0(\vec{k}) = \frac{V}{2} \left[ 2 \int g_2(\vec{k}')d^3k' - k_1^2 \int k_2^2 g_0(\vec{k}')d^3k' 
- k_2^2 \int k_1^2 g_0(\vec{k}') + 2k_1k_2 \int k_1'k_2'g_0(\vec{k}')(d^3k') \right] \]  

(3.30)

where \( f_0(k) = \epsilon(\vec{k}) - E_0 = \frac{\vec{k}^2}{2m} - E_0 \). From (3.28) we see that \( g_0(\vec{k}) \) is the same as \( g(\vec{k}) \) in the commutative case and is spherically symmetric. Using the spherical symmetry of \( g_0 \) we are able write (3.30) in a more simplified form as

\[ f_0(\vec{k})g_2(\vec{k}) - E_1g_1(\vec{k}) - E_2g_0(\vec{k}) = V \int g_2(\vec{k}')d^3k' - Vk^2 \int k_1'g_0(\vec{k}')d^3k' \]

(3.31)

Solving (3.28, 3.29, 3.30) we get

\[ E_1 = 0 , \]
\[ E_2 = V \frac{\beta^2}{\gamma} \]  

(3.32)

where

\[ \beta = \int \frac{k_1^2}{\epsilon(\vec{k}) - E_0} d^3k , \]  
\[ \gamma = \int_{k_F}^{k_F + \hbar \omega_D} \frac{1}{[2\epsilon(\vec{k}) - E_0]^2} d^3k 
= \int_{\epsilon_F}^{\epsilon_F + \hbar \omega_D} \frac{N(\epsilon)d\epsilon}{[2\epsilon - E_0]^2} \]  

(3.33)

(3.34)

We can finally write the energy gap in for the \( \Omega \neq 0 \) case as

\[ \Delta_{\Omega} = \Delta_0 - \Omega^2E_2 \]  

(3.35)

where, \( \Delta_0 \) is the gap in the commutative case and is given by,

\[ \Delta_0 = \frac{k_F^2}{2m} - E_0 \approx \frac{2\hbar \omega_D e^{\frac{\epsilon_F^2}{2\hbar}}} {V} \]  

(3.36)

This shows that to second order in \( \Omega \), the energy gap reduces in the presence of noncommutativity. This is reminiscent of the presence of an external magnetic field [6]: in either case time-reversal symmetry is broken (see [13] for time reversal symmetry breaking in noncommutative system and its impact on lifting of degeneracy).
4 Conclusion

In this paper we have investigated the effect of spatial noncommutativity (of Moyal type) in the Cooper-like problem of BCS superconductivity, and find that at least in the second order of the noncommutative parameter there is an effective reduction in the gap energy. This is reminiscent of the behaviour of a superconductor in the presence of an external magnetic field, and it is tempting to suggest that noncommutativity provides a very simple model of this property. As has been pointed out in the introduction, that such an investigation is not of mere academic interest; genuine (but effective) noncommutativity can indeed be induced by the Berry curvature in a class of condensed matter systems. This analysis therefore gives a prominent and explicit role to the topological/geometrical properties of band structure and the consequent implications of this properties to the algebraic structures in quantum field theory. We would like to emphasize here that the effect of such noncommutativity stems from both the Moyal product and the twisted anti-commutation relations. It would be interesting to see if our results can be realized in experiments on ferromagnetic or GaAs crystals.

Finally we would like to mention that the noncommutative structure given in eq.(1.1) or its simplified version eq.(1.2) in the absence of the external magnetic field ($B = 0$) was obtained, in turn, from the semiclassical Lagrangian derived in Sundaram & and Niu [14] by using the Dirac bracket formalism appropriate for the second class constraints obtained from this Lagrangian. But this semiclassical Lagrangian itself was obtained within the approximations where terms involving higher moments of the wave packets i.e. those containing higher order gradient terms in the perturbations were neglected. It is quite plausible that retaining those higher order terms and running through the entire constraint analysis again may indeed produce terms involving momentum space metric, which in turn involve second order derivatives like the Berry curvature. Our analysis is limited to the regime in which this derivation of ref.[5] holds. In this context, we would like to mention that the momentum-space metric $g_{\alpha\beta}$ and Berry curvature corresponds to the real and imaginary parts of a certain gauge invariant quantity as has been shown in [15]. If a small Berry curvature $\Omega$ is due to a small variation of the periodic part of the Bloch wave function w.r.t Bloch momentum, then correspondingly the metric $g_{\alpha\beta}$ will be nearly flat. As mentioned earlier, our analysis in this paper was essentially restricted to this domain. It will definitely be interesting to study the case of strong Berry curvature, where terms involving the momentum-space metric will not be ignorable any more. In addition, one would have to consider the effects of activating higher channel pairings.

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