A stochastic modification of the Schrödinger-Newton equation

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ABSTRACT

The Schrödinger-Newton [SN] equation describes the effect of self-gravity on the evolution of a quantum system, and it has been proposed that gravitationally induced decoherence drives the system to one of the stationary solutions of the SN equation. However, the equation by itself lacks a decoherence mechanism, because it does not possess any stochastic feature. In the present work we derive a stochastic modification of the Schrödinger-Newton equation, starting from the Einstein-Langevin equation in the theory of stochastic semiclassical gravity. We specialize this equation to the case of a single massive point particle, and by using Karolyhazy’s phase variance method, we derive the Diosi - Penrose criterion for the decoherence time. We also write down the master equation corresponding to this stochastic SN equation. Lastly, we use physical arguments to obtain expressions for the decoherence length of extended objects.

I. INTRODUCTION

A problem of long-standing interest has been to understand if gravity can help solve the quantum measurement problem; or even if it does not actually cause collapse of the wave-function, can it at least be a source of decoherence. The physical picture being that the gravitational field of space-time cannot be exactly classical but must possess intrinsic quantum uncertainty. Can this uncertainty relate to the quantum evolution described by the Schrödinger equation in such a way that for macroscopic masses the space-time uncertainty induces decoherence and position localisation? The answer seems to be in the affirmative, as has been demonstrated in various studies by Karolyhazy and then followed by Karolyhazy and collaborators in several papers, and also shown by Diosi and collaborators. These authors demonstrate that there is an inherent uncertainty in the space-time geometry, whose origin lies in the quantum nature of...
the sources which produce them. This uncertainty is modelled by introducing a classical stochastic potential in the Schrödinger equation, and it is shown that the stochasticity induces decoherence, whose properties depend on the mass and size of the quantum object under study. [The work of Karolyhazy and of Diósi has recently been compared in [16].]

Having considered the effect on the Schrödinger equation of the space-time uncertainty produced by other objects, it is also important to consider the effect of self-gravity of a quantum object on its Schrödinger evolution. This self-gravity has an intrinsic quantum uncertainty and although we do not quite know at present how to describe the self-gravity of a quantum object, one can attempt to model it. One well-known approach is the Schrödinger-Newton [SN] equation. It is proposed that the self-gravitational potential \( V \) produced by a quantum source in state \( \psi \) satisfies a semiclassical Poisson equation

\[
\nabla^2 V = 4\pi Gm|\psi|^2 (1)
\]

whose solution is incorporated in the potential dependent Schrödinger equation

\[
 i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + mV \psi
\]  

This gives the Schrödinger-Newton equation [17–20]

\[
i\hbar \frac{\partial \psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r,t) - Gm^2 \int \frac{d^3r'}{|r-r'|} |\psi(r',t)|^2 \psi(r,t) \] (3)

This equation has been discussed extensively in the literature [21–31]. The equation is nonlinear and not stochastic. This makes it different in character from the Karolyhazy and the Diósi equation, where by virtue of the stochastic potential the models predict decoherence at the level of the master equation. The SN equation does not predict decoherence but is instead suggested as an equation whose stationary solutions are the ones to which the system will proceed upon decoherence, once a gravity-based decoherence mechanism can be incorporated in this system. What has been shown is that there is a gravitationally induced inhibition of dispersion of an expanding wave-packet, at the critical length \( a_c \sim \hbar^2/Gm^3 \) [22].

The reason why decoherence is not observed in the SN equation is evident: one is only taking into account the mean potential in the semiclassical Poisson equation, whereas the Karolyhazy and Diósi models incorporate stochastic fluctuations by way of the stochastic potential in the Schrödinger equation, which is modelled after the space-time geodesic uncertainty. We need to take into consideration the stochastic fluctuations, apart from the mean, and modify the SN equation accordingly. One possible way to do this is to start from the theory of stochastic gravity [32], which
takes into account corrections to the semiclassical Einstein equations

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu \nu} | \Psi \rangle \]  

by considering the role of the two-point fluctuations of the energy-momentum tensor [going from semiclassical gravity to the Einstein-Langevin equation]. The rest of the present paper develops this idea. The effect of quantum fluctuations of the energy-momentum on the geometry is modelled by defining a classical stochastic field. This field eventually acts as a source for the gravitational potential (which is now stochastic) via a modified semiclassical Poisson equation. This stochastic potential is included in the Schrödinger equation, in the spirit of Karolyhazy and Diósi’s work, and as we shall show, it produces decoherence in the SN equation.

A stochastic modification of the Schrödinger equation has also been proposed in [33]. The application of stochastic gravity to the measurement problem has recently been studied in [34]. The possible role of gravity in the decoherence / collapse of the wave function has been reviewed amongst other places, in [35–37].

II. SEMICLASSICAL GRAVITY AND THE EINSTEIN-LANGEVIN EQUATION

Semiclassical gravity describes the interaction of the gravitational field, which is treated classically, with quantum matter fields. The field equation for the classical metric is the semiclassical Einstein equation, which gives the back reaction of the matter fields on the spacetime; it is a generalization of the Einstein equation where the source is the expectation value in some quantum state of the matter stress-energy tensor operator. The coupling of gravitational field to matter can be modelled by the following equation

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu \nu} | \Psi \rangle . \]  

On applying the weak-field limit

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, \]  

where \( |h_{\mu \nu}| \ll 1 \) and \( \eta_{\mu \nu} \) is the regular Minkowski metric, we obtain the linearised theory of gravity. In a weak field situation one can expand the field equations in powers of \( h_{\mu \nu} \) using a coordinate frame where Eqn. [6] holds and one can keep track of only linear terms.

Using Eqns. [5] and [6] one can arrive at the differential equation for the metric field \( h_{\mu \nu} \), given by [30]

\[ \Box h_{\mu \nu} = - \frac{16\pi G}{c^4} \left( \langle \Psi | \hat{T}_{\mu \nu} | \Psi \rangle - \frac{1}{2} \eta_{\mu \nu} \langle \Psi | \hat{T}_{\alpha \beta} \eta^{\alpha \beta} | \Psi \rangle \right) . \]  

(7)
In Newtonian limit the \( \langle \Psi | \hat{T}_{00} | \Psi \rangle \) component is large compared to other components of the stress-energy tensor. Hence in this limit we get a semiclassical Poisson equation of the form

\[
\nabla^2 V = \frac{4\pi G}{c^2} \langle \Psi | \hat{T}_{00} | \Psi \rangle,
\]

where \( h_{00}c^2/2 = -V \). Using \( \hat{T}_{00} = \frac{c^2}{2} \hat{\rho} \) we get the familiar potential field used in the Schrödinger Newton equation, as explained in [30]. The same follows by noting that for a single particle \( \hat{\rho} = m|\vec{r}\rangle\langle\vec{r}| \).

The Einstein-Langevin equation [32, 38, 39] describes a back reaction on the space-time metric of the lowest order stress-energy quantum fluctuations. This results in an effective theory which predicts linear stochastic corrections to the semiclassical metric. The equation includes a Gaussian stochastic tensor field \( \xi_{\mu\nu} \) with the following properties

- The stochastic average \( \langle . \rangle_s \) of the field vanishes i.e.
  \[
  \langle \xi_{\mu\nu} \rangle_s = 0.
  \]  

- The two point correlation is given by
  \[
  \langle \xi_{\alpha\beta}(x)\xi_{\mu\nu}(y) \rangle_s = N_{\alpha\beta\mu\nu}(x, y).
  \]

Here \( N_{\alpha\beta\mu\nu}(x, y) \) is called the noise kernel. It is related to stress-energy tensor in the following way

\[
8N_{\alpha\beta\mu\nu}(x, y) = \langle \{ \hat{t}_{\alpha\beta}(x), \hat{t}_{\mu\nu}(y) \} \rangle,
\]

where \( \{.\} \) is the anti-commutator, \( \langle . \rangle \) is the expectation value and \( \hat{t}_{\alpha\beta} = \hat{T}_{\alpha\beta} - \langle \hat{T}_{\alpha\beta} \rangle \).

- Higher moments or cumulants vanish.

The Einstein-Langevin equation takes the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \left( \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle + 2\xi_{\mu\nu} \right).
\]

On applying the weak-field limit Eqn. 6 we obtain the following field equation for the metric \( h_{\mu\nu} \)

\[
\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} \left( \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle - \frac{1}{2} \eta_{\mu\nu} \langle \Psi | \hat{T}_{\alpha\beta} \eta^{\alpha\beta} | \Psi \rangle + 2\xi_{\mu\nu} - \xi_{\alpha\beta} \eta^{\alpha\beta} \eta_{\mu\nu} \right).
\]

In the Newtonian limit, we get the equation

\[
\nabla^2 V = \frac{4\pi G}{c^2} \left( \langle \Psi | \hat{T}_{00} | \Psi \rangle + 2(\xi_{00} + \frac{1}{2} \xi_{\alpha\beta} \eta^{\alpha\beta}) \right)
\]
Now if we consider only the $\xi_{00}$ component (since Newtonian limit has been assumed), we get

$$\nabla^2 V = \frac{4\pi G}{c^2} \left( \langle \hat{T}_{00} | \Psi \rangle + \xi_{00} \right).$$  \hspace{1cm} (15)

As expected, we now have a stochastic source contributing towards the gravitational potential. Using Green’s function, the potential can be written in the form

$$V(\vec{r}) = -Gm \int \frac{\Psi(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}' - \frac{G}{c^2} \int \frac{\xi_{00}(\vec{r}'')}{|\vec{r} - \vec{r}''|} d\vec{r}''. \hspace{1cm} (16)$$

This equation was worked out by going from field theory to quantum mechanics of $N$ particles in non-relativistic limit and then obtaining the SN equation for single particle, following the method outlined in [30]. This modified potential, where the modifying stochastic component originates in the fluctuations of the stress tensor, is responsible for the stochastic SN equation which we write down and discuss shortly.

A couple of striking similarities with Diosi’s model and Karolyhazy’s model can be immediately recognised.

- The total stochastic potential $V$ is linear sum of Newtonian potential and stochastic noise (which is Gaussian).

- The stochastic average of stochastic component of the potential vanishes due to the first property of $\xi_{\mu\nu}$.

Let the stochastic component of the gravitational potential in equation (16) be $V_{st}$ written as

$$V_{st}(\vec{r}) = \frac{G}{c^2} \int \frac{\xi_{00}(\vec{r}'')}{|\vec{r} - \vec{r}''|} d\vec{r}'', \hspace{1cm} (17)$$

Thus the correlation takes the form

$$\langle V_{st}(\vec{r})V_{st}(\vec{r}'') \rangle_s = \frac{G^2}{c^4} \int \frac{\langle \xi_{00}(\vec{x})\xi_{00}(\vec{x}'') \rangle}{|\vec{x} - \vec{r}||\vec{x}' - \vec{r}''|} d\vec{x}'' d\vec{x}. \hspace{1cm} (18)$$

Using equation (10) (for $\xi_{00}$)

$$\langle \xi_{00}(x)\xi_{00}(y) \rangle_s = N_{0000}(x, y), \hspace{1cm} (19)$$

and equation (11)

$$N_{0000}(x, y) = \frac{1}{8} \langle \{ \hat{t}_{00}(x), \hat{t}_{00}(y) \} \rangle, \hspace{1cm} (20)$$

where $\{,\}$ is the anti-commutator, $\langle \rangle$ is the expectation value and $\hat{t}_{00} = \hat{T}_{00} - \langle \hat{T}_{00} \rangle$, one can obtain

$$\langle V_{st}(\vec{r})V_{st}(\vec{r}'') \rangle_s = \frac{G^2}{8} \int \frac{\langle \Psi | (\hat{T}_{00} - \langle \hat{T}_{00} \rangle)(\vec{x}), (\hat{T}_{00} - \langle \hat{T}_{00} \rangle)(\vec{x}'') \rangle |\Psi \rangle}{|\vec{x} - \vec{r}||\vec{x}' - \vec{r}''|} d\vec{x}'' d\vec{x}. \hspace{1cm} (21)$$


III. DENSITY MATRIX EVOLUTION

The corresponding evolution of the state vector can be written as follows

\[
\frac{d\langle\Psi(t)\rangle}{dt} = -\frac{i}{\hbar} \hat{H}_{0} + i\sqrt{\gamma} \int d\vec{k} \langle \hat{A}(\vec{k}) \hat{A}(\vec{k}) \rangle + i\sqrt{\gamma} \int d\vec{k} \hat{\xi}(\vec{k}, t) \hat{A}(\vec{k}) \langle\Psi(t)\rangle,
\]

where \( \hat{A}(\vec{k}) = \frac{m}{\hbar} e^{i\vec{k}\cdot\vec{r}} \), \( \sqrt{\gamma} = \frac{G}{2m^{2}c^{2}} \), \( \langle \hat{A}^{\dagger}(\vec{k}) \rangle = \langle \Psi|\hat{A}^{\dagger}(\vec{k})|\Psi\rangle \) and

\[
\hat{\xi}(\vec{k}, t) = \frac{k}{c^{2}} \int d\vec{r} e^{-i\vec{k}\cdot\vec{r}} \int d\vec{r}^{\prime} \xi_{00}(\vec{r}, t) \langle\vec{r} - \vec{r}^{\prime}\rangle.
\]

is the non-white stochastic noise. From [40], the density-matrix equation corresponding to Eqn. 23 is

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + i\sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^{\dagger}\rangle \hat{\rho} - \hat{\rho} \hat{A}^{\dagger} \langle\hat{A}\rangle \right) - \gamma \int dt' d\vec{k} d\vec{k}' D(\vec{k}, \vec{k}', t', t) \left( \hat{A}(\vec{k}) \hat{A}(\vec{k}', t' - t) \hat{\rho} + \hat{\rho} \hat{A}^{\dagger}(\vec{k}', t' - t) \hat{A}^{\dagger}(\vec{k}) \right),
\]

where \( \hat{\rho} = \langle|\Psi\rangle\langle\Psi|\rangle_{s} \), \( \hat{A}(\vec{k}', t' - t) \) is the interaction picture operator evolved up to time \( (t' - t) \). The proof is given in Appendix I.

The stochastic Eqn. 23 preserves the norm of the statevector \( |\Psi\rangle \). To see that we compute the differential of the stochastic quantity \( \langle\Psi|\Psi\rangle \).

\[
d\langle\Psi|\Psi\rangle = d\langle\Psi|\Psi\rangle + \langle\Psi|d\Psi\rangle
\]

From equations 27 and 28 in Appendix I we have

\[
d\langle\Psi|\Psi\rangle = dt \langle\Psi|\frac{i}{\hbar} \hat{H}_{0}|\Psi\rangle - dt \langle\Psi|i\sqrt{\gamma} \int d\vec{k} \langle \hat{A}^{\dagger}\rangle \hat{A}^{\dagger} |\Psi\rangle - dt \langle\Psi|i\sqrt{\gamma} \int d\vec{k} \hat{\xi}^{\dagger} \hat{A}^{\dagger} |\Psi\rangle
\]

\[
- \langle\Psi|\frac{i}{\hbar} \hat{H}_{0}|\Psi\rangle dt + i\sqrt{\gamma} \langle\Psi| \int d\vec{k} \langle \hat{A}^{\dagger}\rangle \hat{A}^{\dagger} |\Psi\rangle dt + i\sqrt{\gamma} \langle\Psi| \int d\vec{k} \hat{\xi}^{\dagger} \hat{A}^{\dagger} |\Psi\rangle dt
\]

\[
= i\sqrt{\gamma} \int d\vec{k} \left( \hat{\xi}(\langle\Psi|\hat{A}^{\dagger} \Psi\rangle - \hat{\xi}^{\ast}(\langle\Psi|\hat{A}^{\dagger} |\Psi\rangle) \right) dt.
\]

\[
= 0
\]
It is challenging to demonstrate decoherence from this master equation, but an alternative method applied by Karolyhazy, which we call the phase variance method, comes to our rescue. The equivalence of the phase variance method and the Markovian master equation of Diósí’s model has been demonstrated by us in [16] and we believe this equivalence holds in general for non-Markovian master equations as well.

IV. PHASE VARIANCE METHOD AND PROOF OF DECOHERENCE

In this section we adopt the scheme used by Karolyhazy in his work to find the decoherence effect due to the stochastic potential in the modified SN equation. In our calculation, we have found,

\[ i\hbar \frac{\partial \Psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + V_{Tot}(r, t) \Psi(r, t) \]  

(28)

where

\[ V_{Tot}(r, t) = -Gm^2 \int \frac{|\psi(r', t)|^2}{|r - r'|} d^3r' - \frac{Gm}{c^2} \int \frac{\xi_{00}(r', t)}{|r - r'|} d^3r' \]  

(29)

Following the argument of Karolyhazy, we assume that the solution of this stochastic equation can be given in the form

\[ \Psi_{st}(r, t) = \psi_{free}(r, t) \exp(i\phi_{st}(r, t)) \]  

(30)

where \( \psi_{free}(r, t) \) is the free solution without any gravitational back-reaction and \( \phi_{st}(r, t) \) is the phase of the actual solution with respect to the free wavefunction, and is given by,

\[ \phi_{st}(r, t) = -\frac{1}{\hbar} \int_0^t V_{Tot}(r, t') dt' \]  

(31)

We emphasise an important approximation that has been introduced here, in order to be able to make progress. In the potential \( V_{Tot} \), the exact state \( \Psi \) has been replaced by the free wave function \( \psi \); thus the gravitational effect is being calculated iteratively. Thus in our calculations we have used

\[ V_{Tot}(r, t) = -Gm^2 \int \frac{|\psi(r', t)|^2}{|r - r'|} d^3r' - \frac{Gm}{c^2} \int \frac{\xi_{00}(r', t)}{|r - r'|} d^3r' \]  

(32)

[For completeness, we outline the justification of this result of Karolyhazy. The Schrödinger equation in a fluctuating spacetime is given by

\[ i\hbar \frac{\partial \Psi_\beta(x, t)}{\partial t} = H_\beta(x, t) \Psi_\beta(x, t) \]  

(33)\]
where the time dependent Hamiltonian is given by,

\[ H_\beta(x,t) = H_0(x) + Mc^2\gamma_\beta(x,t)/2 \]  

(34)

where the second term denotes a small perturbation around the time-independent Hamiltonian. To solve this equation, Karolyhazy considers the perturbation to be switched on adiabatically (see, for instance, Karolyhazy’s detailed paper of 1990, pg. 223 [4]). Hence we can assume the “adiabatic approximation” to be valid i.e., if a system starts with a certain state, it will remain in that same state after the perturbation is introduced adiabatically.

Now, let the initial state be \( \Psi_\beta(x,0) = \Psi_0(x,0) \) where \( \Psi_0(x,t) \) is the solution without the perturbation so that \( \Psi_0(x,t) = \Psi_0(x,0) \exp(-iEt/\hbar) \). If now, the perturbation is turned on adiabatically, the system essentially remains in the same state, with only some phase factors introduced, similar to the unperturbed case. Using the adiabatic approximation, we can write (see Griffiths [41] chap. 10, Eqns. 10.12, 10.13 and 10.23)

\[ \Psi_\beta(x,t) = \Psi_0(x,0) \exp[i\theta_\beta(x,t)] \exp[i\Omega_\beta(x,t)] \]  

(35)

where \( \theta_\beta(x,t) = -\frac{i}{\hbar} \left[ E_0 t + \int_0^t E_P(x,t') \, dt' \right] \), \( E_0 \) being the unperturbed eigenenergy and \( E_P \) is due to perturbation. The phase \( \Omega_\beta(x,t) \) is given by Eqn 10.22 of Griffiths (denoted by \( \gamma \) there). Plugging this into the above equation, we get,

\[ \Psi_\beta(x,t) = \Psi_0(x,t) \exp \left[ -\frac{i}{\hbar} \int_0^t E_P(x,t') \, dt' \right] \exp[i\Omega_\beta(x,t)] \]  

(36)

The last phase term \( \Omega_\beta(x,t) \) (called the geometric phase) can be neglected as it depends weakly on \( x \) (see Karolyhazy 1990 paper). With the identification \( E_P = V_\beta \) we get,

\[ \Psi_\beta(x,t) = \Psi_0(x,t) \exp \left[ -\frac{i}{\hbar} \int_0^t V_\beta(x,t') \, dt' \right] \]  

(37)

A few remarks are in order, with regard to the application of this phase variance method to the present case of the stochastic SN equation (28). Both the terms in the potential (32), namely the mean as well as the fluctuation, are treated on the same footing (32), as in principle there can be circumstances when they are comparable. Thus, from the viewpoint of application of the adiabatic approximation assumed while writing down the ansatz (30) it is assumed that the total potential can be treated as a perturbation on the free particle. While this is definitely evident for microscopic objects, the assumption maybe considered to be a reasonable one in the macroscopic case as well, so long as the gravitational field in question is sufficiently weak. The plausibility of the results that we derive below serves to justify the validity of the ansatz (30).
We now calculate the phase variance between two points \( \vec{r}_1 \) and \( \vec{r}_2 \) with the formula

\[
\Delta \phi^2 = \langle (\phi_{st}(\vec{r}_1, t) - \phi_{st}(\vec{r}_2, t))^2 \rangle_s
\]  

(38)

We find the time \( t \) for which the above quantity would be \( \sim \pi^2 \) and that will give us the decoherence time \( \tau \). Calculating the variance, we get,

\[
\Delta \phi^2 = \frac{3G^2m^4}{4\hbar^2} \left[ \int \int \frac{\left| \psi(\vec{r}', t') \right|^2 \left| \psi(\vec{r}'', t'') \right|^2}{|\vec{r}_1 - \vec{r}'||\vec{r}_2 - \vec{r}''|} d^3r'_d d^3r''_d t' dt'' + \int \int \frac{\left| \psi(\vec{r}', t') \right|^2 \left| \psi(\vec{r}'', t'') \right|^2}{|\vec{r}_2 - \vec{r}'||\vec{r}_2 - \vec{r}''|} d^3r'_d d^3r''_d t' dt'' - 2 \int \int \frac{\left| \psi(\vec{r}', t') \right|^2 \left| \psi(\vec{r}'', t'') \right|^2}{|\vec{r}_1 - \vec{r}'||\vec{r}_2 - \vec{r}''|} d^3r'_d d^3r''_d t' dt'' \right]
\]

(39)

where the time integration is done up to a time \( T \) and the volume integrations are over the whole space.

To calculate the phase variance, we use the free gaussian solution of the wavefunction \[22\] which is given by [we recall from our remark above \[31\] that we are finding the gravitational effect perturbatively]

\[
\psi(r, t) = (\pi a^2)^{-3/4} \left( 1 + \frac{i\hbar t}{ma^2} \right)^{-3/2} \exp \left( -\frac{r^2}{2a^2(1 + \frac{\hbar t}{ma^2})} \right).
\]  

(40)

Let us consider the first term in \[39\]. The integration can be separated into two integrals of \( r' \) and \( r'' \). The integration can be performed in spherical polar co-ordinates. Similarly, all the other terms can also be calculated. Some steps of the calculations are as follows. The first term can be separated into two integrals of \( r' \) and \( r'' \) of the form \( \int \int \frac{\left| \psi(\vec{r}', t') \right|^2}{|\vec{r}_1 - \vec{r}'|} d^3r'_d t' \). We have,

\[
\left| \psi(\vec{r}', t') \right|^2 = \pi^{-3/2} \frac{1}{a^3(1 + \frac{\hbar^2}{m^2a^4})^{3/2}} \exp \left( -\frac{r'^2}{a^2(1 + \frac{\hbar^2}{m^2a^4})} \right)
\]  

(41)

Now let \( \vec{r}_1 - \vec{r}' = \vec{r} \) so that \( r'^2 = r_1^2 + r^2 - 2r_1r \cos \theta \). This gives

\[
\left| \psi(\vec{r}', t') \right|^2 = \pi^{-3/2} \frac{1}{a^3} \alpha^{3/2} \exp \left( -\frac{\alpha}{a^2}(r_1^2 + r^2 - 2r_1r \cos \theta) \right)
\]  

(42)

where \( \alpha = 1/(1 + \frac{\hbar^2}{m^2a^4}) \). The integration is now straightforward. Other terms can also be calculated in the same fashion.

Small time approximation: In our calculation, in order to obtain an analytical result, the time integration has been calculated using small time approximation i.e. the interval \( T \) is such that

\[
e^{-r_1^2/a^2} \frac{\hbar^2 r_1}{m^2a^4} \frac{T^2}{3a\sqrt{\pi}} \ll Erf(r_1/a)
\]  

(43)
FIG. 1. The plot of functions $f_1$ and $f_2$.

where the error function $Erf(x)$ is given, along with the imaginary error function $Erfi(x)$, used below, as

$$
Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt; \quad Erfi(x) = -iErf(ix) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} \, dt
$$

This condition given in (43) is well satisfied for $r_1/a \to \infty$.

After putting small time approximation we get

$$
\Delta \phi^2 = \frac{3G^2m^4}{4\hbar^2 T^2} \left( \frac{1}{r_1^2} \left( Erf \left( \frac{r_1}{a} \right) \right)^2 + \frac{1}{r_2^2} \left( Erf \left( \frac{r_2}{a} \right) \right)^2 - \frac{2}{r_1 r_2} Erf \left( \frac{r_1}{a} \right) Erf \left( \frac{r_2}{a} \right) \right)
$$

$$
+ \frac{G^2m^4}{4\hbar^2 T^2} \left( \frac{\sqrt{\pi}}{r_1 a} e^{-\frac{r_1^2}{a^2}} Erfi \left( \frac{r_1}{a} \right) + \frac{\sqrt{\pi}}{r_2 a} e^{-\frac{r_2^2}{a^2}} Erfi \left( \frac{r_2}{a} \right) \right)
$$

$$
- \frac{2\sqrt{\pi}}{a \sqrt{r_1 r_2}} e^{-1/2(r_1^2/a^2+r_2^2/a^2)} \sqrt{Erfi(r_1/a)Erfi(r_2/a)}
$$

The last term in the above equation has not been calculated explicitly as it was very complicated. So we used the argument that since it was initially obtained by breaking a squared term, the final result should be a perfect square. Putting the last term in this form gives the desired result.

Let $f_1 = \frac{r_1^2}{a^2} \left( Erf \left( \frac{r_2}{a} \right) \right)^2$ and $f_2 = \frac{r_1^2}{a^2} Erfi \left( \frac{r_1}{a} \right)$. The plot of $f_1$ and $f_2$ is shown in Figure 1 where $x$ denotes $r_1/a$. We see that we cannot neglect $f_2$ for large values of $r_1/a$ because both functions become comparable. We now calculate term by term in Eqn. (45) in the limit $r_1/a \to \infty$ and $r_2/a \to \infty$.

First we look at the terms involving $r_1$. So, the first and fourth terms of the above expression together give,

$$
\frac{G^2m^4}{4\hbar^2} \frac{T^2}{r_1^2} \left[ \sqrt{\pi} \frac{r_1}{a} e^{-r_1^2/a^2} Erfi \left( \frac{r_1}{a} \right) + 3 Erf \left( \frac{r_1}{a} \right) \right]
$$
As $r_1/a \to \infty$, the term inside $[..]$ goes to 4. So, we have, this term as

$$
\frac{G^2 m^4 T^2}{4 \hbar^2} \cdot 4 \quad (47)
$$

Similarly, by adding second and fifth terms, we get,

$$
\frac{G^2 m^4 T^2}{4 \hbar^2} \cdot 4 \quad (48)
$$

Now let us consider the third and sixth terms together:

$$
- \frac{G^2 m^4 2 T^2}{\hbar^2} \frac{2}{r_1 r_2} \left[ 3 \text{Erf} \left( \frac{r_1}{a} \right) \text{Erf} \left( \frac{r_2}{a} \right) + \sqrt{\pi} \frac{r_1 r_2}{a^2} e^{-\frac{1}{2}(r_1^2/a^2 + r_2^2/a^2)} \sqrt{\text{Erfi} \left( \frac{r_1}{a} \right) \text{Erfi} \left( \frac{r_2}{a} \right)} \right] \quad (49)
$$

Again the term in square brackets goes to 4 for $\frac{r_1}{a} \to \infty$ and $\frac{r_2}{a} \to \infty$. We get,

$$
- \frac{G^2 m^4 2 T^2}{\hbar^2} \frac{2}{r_1 r_2} \cdot 4 \quad (50)
$$

Finally, adding all the terms gives,

$$
\frac{G^2 m^4}{\hbar^2} T^2 \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \quad (51)
$$

We demand the phase variance to be of the order $\pi^2 \sim 1$. This gives,

$$
T \sim \frac{\hbar}{G m^2} \cdot \frac{1}{|r_1 - r_2|} \quad (52)
$$

So, $T = \frac{\hbar}{|\Delta E|}$ where $\Delta E = \frac{G m^2}{r_1} - \frac{G m^2}{r_2}$. Thus, from the stochastic Schrödinger-Newton equation we have deduced the Diósi-Penrose criterion, namely that the decoherence time is of the order of the inverse of the gravitational potential energy difference at the two locations. This strongly suggests that gravitational decoherence drives the system to one of the solutions of the SN equation, as originally proposed by Diósi and Penrose. It is interesting that this result has been derived starting from the Einstein-Langevin equation, and it highlights the significance of considering quantum fluctuations in the stress tensor, while also encouraging us to believe that stochastic gravity is intimately connected with the stochastic SN equation.

Here, it is evident that we certainly cannot talk of the coherence length as simply the distance between the two points $|r_1 - r_2|$ because the gravitational potential energy difference is involved. So, unlike the other gravity induced collapse models, where the decoherence time and coherence length depend upon the distance between two points, here they actually depend upon the difference $|\frac{1}{r_1} - \frac{1}{r_2}|$. So for this case, we introduce a length scale $a_c$, which is the coherence length, defined as:

$$
\frac{1}{a_c} = |\frac{1}{r_1} - \frac{1}{r_2}|
$$
Then from the above calculation we can write,

$$T = \frac{\hbar a_c}{Gm^2} \quad (53)$$

Again we know that $T \sim ma_c^2/\hbar$. These two together give,

$$a_c = \frac{\hbar^2}{Gm^3} \quad (54)$$

a well-known result. The calculations can be repeated for the case $\frac{r_1}{a} \approx 1$ and $\frac{r_2}{a} \approx 1$ and the results come out to be the same. The only problem in this limit seems to be that the small time approximation is not valid for microscopic masses. For large masses, it is still valid.

In our calculation, we have found that the decoherence time between two points $\vec{r}_1$ and $\vec{r}_2$ depends on the difference between the gravitational potential energy difference at these two points. This is somewhat similar, although not exactly same as, the so called Diósi - Penrose criterion. The latter says that if we have two stationary states represented by two mass configurations then their superposition will decay to one of the solutions of the SN equation in a time scale $\tau \approx \hbar/\Delta E_g$ where $\Delta E_g$ is the gravitational self-energy of the difference between the mass distributions of the two states. Penrose suggests in [17] that if we have two mass distributions representing the two states characterised by the mass densities $\rho$ and $\rho'$ respectively, then the measure of incompatibility between them would be

$$\Delta = -4\pi G \int \int (\rho(x) - \rho'(x))(\rho(y) - \rho'(y))/|x - y| d^3x d^3y \quad (55)$$

He suggests that the decay time would be of the form $\tau \sim \hbar/\Delta E_g$ where $\Delta E_g$ is $\Delta$ or some multiple of that quantity. Diósi also obtains the same form as has been given in [14, 42] where he finds that the decay time should be $\tau \sim \hbar/\Delta E_g$, with $\Delta E_g$ having a form

$$\Delta E_g = G \int \int [f(x|X) - f(x'|X')][f(x'|X) - f(x'|X')]/|x - x'| d^3x d^3x' \quad (56)$$

Here $f(x|X)$ denotes the mass density at $x$ for a configuration denoted by $X$.

V. HEURISTIC INTERPRETATION OF THE STOCHASTIC SN EQUATION

We consider the relative significance of terms in the stochastic SN equation:

\[
\begin{array}{l}
\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t) - Gm^2 \int \frac{|\psi(\vec{r'}, t)|^2}{|\vec{r} - \vec{r'}|} d\vec{r'} \Psi(\vec{r}, t) - \frac{Gm}{c^2} \int \frac{\xi_{00}(\vec{r'})}{|\vec{r} - \vec{r'}|} d\vec{r'} \Psi(\vec{r}, t).
\end{array}
\]

Quantum evolution
Non-linear evolution (self gravity)
Stochastic evolution
Next, we consider the state function

$$\Psi(\vec{r}, t) = (\pi a^2)^{-3/4} \left(1 + \frac{i \hbar}{ma^2} \right)^{-3/2} \exp \left( - \frac{r^2}{2a^2 \left(1 + \frac{i \hbar}{ma^2} \right)} \right). \quad (57)$$

The probability density is given by

$$|\Psi(\vec{r}, t)|^2 = \pi^3 \beta^3(t) e^{-\beta r^2}, \quad (58)$$

where $$\beta(t) = \frac{1}{a^2 \left(1 + \frac{\hbar^2 m^2}{4a^4} \right)}$$. On inserting the state function in stochastic SN equation we get

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = -\frac{\hbar^2}{m} \beta (2\beta r^2 - 3) \Psi(\vec{r}, t) - \frac{Gm^2}{2\pi r} \text{erf} \left( r \sqrt{\beta} \right) \Psi(\vec{r}, t) - \frac{Gm}{c^2} \int \frac{\xi_{00}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\vec{r}' \Psi(\vec{r}, t) \quad (59)$$

The evolution of the state function has three pieces as shown. We see that these three pieces are dependent on mass in different ways (inversely proportional for the first term and quadratically proportional for the second and third terms; it is also clear from Eqn. [10] that $$\xi_{00} \propto m$$). It suggests a separation of scales. It means that for a range of the mass only the quantum evolution dominates, and for other range, the nonlinear self gravity evolution with stochastic evolution dominates.

To make the statement more precise, we introduce a length scale $$a_c$$ in the equation in the following way

Non-linear evolution = $$Gm^2 a_c \int \frac{|\psi(\vec{r}', t)|^2}{a_c |\vec{r} - \vec{r}'|} d\vec{r}' \Psi(\vec{r}, t) \quad (60)$$

and

Stochastic evolution = $$\frac{Gm a_c}{c^2} \int \frac{\xi_{00}(\vec{r}')}{a_c |\vec{r} - \vec{r}'|} d\vec{r}' \Psi(\vec{r}, t). \quad (61)$$

Now keeping these points in mind let us compare the evolution. The Quantum evolution dominates chiefly when its coefficient ($$\hbar^2/m$$) is much greater than coefficient of non-linear evolution which is $$Gm^2 a_c$$.

$$\frac{\hbar^2}{m} \gg Gm^2 a_c \rightarrow \frac{\hbar^2}{Gm^3} \gg a_c \quad (62)$$

If we identify $$a_c$$ with the coherence length this reproduces the anticipated condition for decoherence.

VI. SOME QUALITATIVE ESTIMATES FROM THE SN EQUATION

In an effort to obtain some estimates as to how the coherence length $$a_c$$ depends on the mass $$m$$ and size $$R$$ in case of an extended object, and in order to compare with the results of Karolyhazy, and of Diósi, we make some qualitative estimates.
Let us recall first the case of the point particle treated in \[22\] according to the SN equation:

\[
\frac{i\hbar}{\partial t} \psi(r, t) = \hat{H}_o \psi(r, t) - Gm^2 \left( \int \frac{\psi(r', t)^2}{|r - r'|} dr' \right) \psi(r, t).
\] (63)

An estimate of the critical length and decay time can be obtained from this equation considering the self gravitation of an expanding wave packet. If we start with a spherically symmetric Gaussian wave packet of width \(a\),

\[
\psi(r, 0) = (\pi a^2)^{-3/4} \exp\left(-\frac{r^2}{2a^2}\right)
\] (64)

then after time \(t\), the wave function evolves, via Schrödinger equation (in the absence of any gravitational potential), as,

\[
\psi(r, t) = (\pi a^2)^{-3/4} \left(1 + \frac{i\hbar t}{ma^2}\right)^{-3/2} \exp\left(-\frac{r^2}{2a^2(1 + \frac{i\hbar t}{ma^2})}\right)
\] (65)

The radial probability density is maximum at \(r_p = a\sqrt{1 + \frac{\hbar^2 t^2}{ma^4}}\). So, the peak shifts with time with an acceleration given by the following equation:

\[
\ddot{r}_p = \frac{\hbar^2}{ma^4 r_p^3}
\] (66)

This gives the acceleration of the wave packet when it expands freely. Now, the acceleration due to gravity for a point mass \(m\) at a distance \(r_p\) is

\[
\ddot{r}_p = \frac{Gm}{r_p^2}
\] (67)

Let, after a certain time, these two accelerations become equal. We call the width of the wave packet in such an equilibrium as \(a_{equil}\). Then equating the two accelerations we get, for a mass \(m\),

\[
m = \left(\frac{\hbar^2}{G a_{equil}}\right)^{1/3}
\] (68)

So, for a given mass, one can calculate this equilibrium width. If \(r_p < a_{equil}\) then usual quantum evolution dominates while for \(r_p > a_{equil}\) gravity becomes more significant and collapse of the wave function takes place. Hence, \(a_{equil}\) can be thought of as the critical length \(a_c\).

We now focus our attention to the behaviour of the wave function at \(t = 0\). At \(t = 0\) we have \(r_p = a\). Now if we start with a mass \(m\) such that

\[
\frac{\hbar^2}{ma^4} > \frac{Gm}{a^2}
\]

then the Scrödinger-like expansion keeps on accelerating unless the two effects become equal. In that case we have \(a_{equil} > a\). But if \(\frac{\hbar^2}{ma^4} < \frac{Gm}{a^2}\) then the wave packet starts contracting right from the beginning and we get \(a_{equil} < a\). This can be summarized as below:
If \( m < \left( \frac{\hbar^2}{Gm} \right)^{1/3} \) then \( a_{\text{equil}} > a \)

If \( m > \left( \frac{\hbar^2}{Gm} \right)^{1/3} \) then \( a_{\text{equil}} < a \)

So, for an already contracted wave function, we must have \( m > \left( \frac{\hbar^2}{Gm} \right)^{1/3} \). \( m = \left( \frac{\hbar^2}{Gm} \right)^{1/3} \) is the threshold mass for collapse.

**Critical Length for Extended Objects:**

We have already seen that the critical length for a single point-like mass \( m \) is given by

\[
a_c = \frac{\hbar^2}{Gm^3} \tag{69}
\]

Now we calculate the same for an extended object of mass \( m \) and size \( R \) [we use the notations \( a_c \) and \( a_{\text{equil}} \) interchangeably - they both represent the same quantity]. We consider two cases:

- \( a_{\text{equil}} \gg R \)

In this case, the gravitational acceleration on the wave packet which extends outside the mass \( m \) is,

\[
g = \frac{Gm}{r_p^2} \tag{70}
\]

For the critical length \( a_c \) we have,

\[
\frac{\hbar^2}{m^2 a_c^5} = \frac{Gm}{a_c^2} \tag{71}
\]

This essentially gives the same critical length as in case of a single point mass

\[
a_c = \frac{\hbar^2}{Gm^3} \tag{72}
\]

- \( a_{\text{equil}} \ll R \)

When \( a_{\text{equil}} \ll R \), the wave packet lies inside the extended mass \( m \). Considering the mass \( m \) having uniform density within the radius \( R \) we get the gravitational acceleration on the wave packet as,

\[
g = \frac{Gmr_p}{R^3} \tag{73}
\]

Again at the critical length,

\[
\frac{\hbar^2}{m^2 a_c^5} = \frac{Gma_c}{R^3} \tag{74}
\]
This gives the critical length as:

\[ a_c = \left( \frac{\hbar}{Gm^3} \right)^{1/4} R^{3/4} \]  

(75)

Interestingly, this same expression for \( a_c \) is obtained by Diósi in his treatment of gravitational decoherence, while it differs from the expression obtained by Karolyhazy. What this is telling about possible similarity between the stochastic SN equation and the Diósi approach is not clear to us at present.

Putting \( a_c \approx R \) gives the transition between micro region and macro region:

- micro regime\( (m^3R \ll \frac{\hbar^2}{G}) \): critical length \( a_c = \frac{\hbar^2}{Gm^3} \)
- macro regime\( (m^3R \gg \frac{\hbar^2}{G}) \): critical length \( a_c = \left( \frac{\hbar^2}{Gm^3} \right)^{1/4} R^{3/4} \)
- Transition occurs at \( R \approx \frac{\hbar^2}{Gm} \)

At \( r_p = a_c \) the two opposite accelerations become equal and so, the wave function evolves at a constant rate. Let us consider a mass for which \( m \) is below the threshold value i.e, \( m < \left( \frac{\hbar^2}{Ga} \right)^{1/3} \). Initially when \( r_p < a_c \), the wave packet expands at an accelerating rate until it reaches \( r_p = a_c \). At this point, as the two forces become equal, it stops accelerating and starts expanding at a constant rate. As soon as \( r_p \) becomes greater than \( a_c \), again, gravity dominates and the wave packet shrinks to size \( r_p = a_c \). Again at this point onwards, it keeps on contracting at a constant rate until \( r_p < a_c \) where again Schrödinger like expansion takes over. Thus, there should be an oscillation around the equilibrium width \( a_c \) which has been found through numerical simulations in [22].

VII. CONCLUDING REMARKS

We believe that the stochastic SN equation we have found makes a useful contribution to the subject of gravity induced decoherence. The equation is possibly robust, because its origin lies in the well-defined theory of stochastic semiclassical gravity. It correctly predicts the Diósi-Penrose decoherence criterion which relates decoherence time to the gravitational potential energy difference between the two points under consideration.

The correlation function in this instance is state dependent, unlike in the case of the stochastic potentials introduced by Karolyhazy and by Diósi. Furthermore, the noise is not white, unlike in the case of Diósi’s model, where the noise is white in time. And yet, the final physical results match in the two cases - the reasons for this intriguing feature are not clear to us at present.
Further investigations should include studying the stochastic SN equation for states other than Gaussian wave-packets, and for extended objects. It will be useful also to see if there is some way to convert the stochastic equation into a collapse equation, and to compare it with CSL. One should also attempt to work out how the stochastic SN equation can be tested in experiments such as molecular interferometry and via gravitationally induced random diffusion [43, 44].

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Author Contribution: TPS suggested the idea for this work. RM worked out the non-relativistic limit of the Einstein-Langevin equation, and the master equation. SB developed the phase-variance method, and SB and RM found the decoherence result. SB found the qualitative estimates for the SN equation. All authors took part in discussions and contributed to the writing of the paper.

Appendix I : Master equation for the density matrix

We have

$$\frac{d|\Psi(t)\rangle}{dt} = \left[ -\frac{i}{\hbar} \hat{H}_0 + i\sqrt{\gamma} \int d\vec{k} \langle \hat{A}^\dagger(\vec{k}) \rangle \hat{A}(\vec{k}) + i\sqrt{\gamma} \int d\vec{k} \xi(\vec{k}, t) \hat{A}(\vec{k}) \right] |\Psi(t)\rangle.$$  

(76)

It implies that

$$d|\Psi\rangle = -\frac{i}{\hbar} \hat{H}_0 |\Psi\rangle dt + i\sqrt{\gamma} \int d\vec{k} \langle \hat{A}^\dagger(\vec{k}) \rangle \hat{A}(\vec{k}) |\Psi\rangle dt + i\sqrt{\gamma} \int d\vec{k} \xi(\vec{k}, t) \hat{A}(\vec{k}) |\Psi(t)\rangle dt,$$  

(77)

and, on taking the conjugate of the equation

$$d\langle \Psi \rangle = dt \langle \Psi | \frac{i}{\hbar} \hat{H}_0 - d\langle \Psi | i\sqrt{\gamma} \int d\vec{k} \langle \hat{A}(\vec{k}) \rangle \hat{A}^\dagger(\vec{k}) - dt \langle \Psi | i\sqrt{\gamma} \int d\vec{k} \xi^*(\vec{k}, t) \hat{A}^\dagger(\vec{k}) \rangle.$$  

(78)

Let $\hat{\rho}_{st} = |\Psi\rangle\langle\Psi|$. Then

$$d\hat{\rho}_{st} = d|\Psi\rangle\langle\Psi| + |\Psi\rangle d\langle\Psi|$$

$$= -\frac{i}{\hbar} [\hat{H}_0, \hat{\rho}_{st}] dt + i\sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger \rangle \hat{A} \hat{\rho}_{st} - \hat{\rho}_{st} \langle \hat{A} \rangle \hat{A}^\dagger \right) dt$$

$$+ i\sqrt{\gamma} \int d\vec{k} \left( \hat{A} \xi \hat{\rho}_{st} - \hat{\rho}_{st} \hat{A}^\dagger \xi^* \right) dt.$$  

(79)

The density matrix of the system is the stochastic average of the $\hat{\rho}_{st}$

$$\hat{\rho} = \langle \hat{\rho}_{st} \rangle_s.$$  

(80)
We now take the stochastic average of Eqn. [79] and retain terms up to the order of $\sqrt{\gamma}$. It simply means that $\langle \hat{A} \rangle \approx \langle \Psi_0 | \hat{A} | \Psi_0 \rangle$ where we have the expansion $| \Psi \rangle = | \Psi_0 \rangle + \sqrt{\gamma} | \psi_1 \rangle + \gamma | \Psi_2 \rangle + \ldots$

\[
\langle d\hat{\rho}_{st} \rangle_s = \left\langle -\frac{i}{\hbar} [\hat{H}_0, \hat{\rho}_{st}] dt + i\sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger \hat{\rho}_{st} - \hat{\rho}_{st} \hat{A}^\dagger \rangle \right) dt \\
+ i\sqrt{\gamma} \int d\vec{k} \left( \hat{\xi} \hat{\rho}_{st} - \hat{\rho}_{st} \hat{\xi}^\dagger \hat{A}^\dagger \right) dt \right\rangle_s
\]

\[
\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + i\sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger \hat{\rho} - \hat{\rho} \hat{A}^\dagger \rangle \right) + i\sqrt{\gamma} \int d\vec{k} \left( \hat{\xi} \langle \hat{\rho}_{st} \rangle - \langle \hat{\rho}_{st} \hat{\xi}^\dagger \rangle \hat{A}^\dagger \right),
\]

(81)

According to Furutsu Novikov formula

\[
\left\langle F[\hat{\xi}] \hat{\xi}(\vec{k}, t) \right\rangle_s = \int d\vec{k}' \int dt' D(\vec{k}, \vec{k}', t, t') \left\langle \frac{\delta F[\hat{\xi}]}{\delta \hat{\xi}(\vec{k}', t')} \right\rangle_s,
\]

(82)

where $F[\hat{\xi}]$ is an arbitrary functional of the gaussian stochastic noise and $D(k, k', t, s)$ is the correlation function of $\xi(\vec{k}, t)$

\[
D(k, k', t, t') = \langle \xi(\vec{k}, t), \xi(\vec{k}', t') \rangle_c
\]

\[
= \left\langle kk' \int d\vec{r}_1' d\vec{r}_2' e^{-i\vec{k}.\vec{r}_1'} e^{-i\vec{k}'.\vec{r}_2'} \int d\vec{r}_3' d\vec{r}_4' \xi_{00}(\vec{r}_3', t) \xi_{00}(\vec{r}_4', t') \right\rangle_c
\]

\[
= kk' \int d\vec{r}_1' d\vec{r}_2' e^{-i\vec{k}.\vec{r}_1'} e^{-i\vec{k}'.\vec{r}_2'} \int d\vec{r}_3' d\vec{r}_4' \frac{N_{0000}(\vec{r}_3', \vec{r}_4', t, t')}{|\vec{r}_1' - \vec{r}_3'| |\vec{r}_2' - \vec{r}_4'|}
\]

(83)

where $N_{0000}(\vec{r}, \vec{r}', t)$ is given by equation [10]

Let $F[\hat{\xi}] = \hat{\rho}_{st}$ then, we are presuming a system with spherical symmetry.

\[
\left\langle \hat{\rho}_{st} \hat{\xi}(k, t) \right\rangle_s = 4\pi \int dk' k'^2 \int dt' D(k, k', t, t') \left\langle \frac{\delta \hat{\rho}_{st}}{\delta \hat{\xi}(k', t')} \right\rangle_s
\]

(84)

We now transit to the interaction picture where the operators and state vectors are defined as

\[
\hat{A}(\vec{k}, t) = e^{i\frac{\hat{H}_I t}{\hbar}} \hat{A}(\vec{k}) e^{-i\frac{\hat{H}_I t}{\hbar}}, |\Psi(t)\rangle_I = e^{i\frac{\hat{H}_I t}{\hbar}} |\Psi(t)\rangle.
\]

(85)

We, then, expand $| \Psi \rangle_I$ with the parameter $\sqrt{\gamma}$ as follows

\[
| \Psi \rangle_I = | \Psi_0 \rangle_I + \sqrt{\gamma} | \psi_1 \rangle_I + \gamma | \Psi_2 \rangle_I + \ldots
\]

(86)

In the interaction picture equation [76] takes the form

\[
\frac{d|\Psi(t)\rangle_I}{dt} = i\sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger(\vec{k}) \rangle + \hat{\xi}(\vec{k}, t) \right) \hat{A}(\vec{k}, t) |\Psi(t)\rangle_I.
\]

(87)

On plugging equation [86] in equation [87] and collecting the terms of order $0$, $\sqrt{\gamma}$ and $\gamma$ we get
\[ \mathcal{O}(0) : \quad \frac{\partial}{\partial t} |\Psi_0\rangle_I = 0 \quad (88) \]

\[ \mathcal{O}(\sqrt{\gamma}) : \quad \frac{\partial}{\partial t} |\Psi_1\rangle_I = i \int d\vec{k} \left( \langle \hat{A}^\dagger(\vec{k}) + \bar{\xi}(\vec{k},t) \rangle \hat{A}(\vec{k},t) |\Psi_0\rangle_I \right) \quad (89) \]

\[ |\Psi_1(t)\rangle = i \int_0^t dt' \int d\vec{k} \left( \langle \hat{A}^\dagger(\vec{k}) \rangle + \bar{\tilde{\xi}}(\vec{k},s) \right) \hat{A}(\vec{k},t' - t) |\Psi_0(t)\rangle \]

\[ \mathcal{O}(\gamma) : \text{and so on...} \]

The density matrix in Schrödinger picture \( \hat{\rho}_{st} = |\Psi\rangle\langle\Psi| \), when expanded perturbatively upto order \( \sqrt{\gamma} \) is

\[ \hat{\rho}_{st} = |\Psi_0\rangle\langle\Psi_0| + \sqrt{\gamma}(|\Psi_1\rangle\langle\Psi_0| + |\Psi_0\rangle\langle\Psi_1|). \quad (90) \]

Using equation (89), we can evaluate

\[ \frac{\delta \hat{\rho}_{st}}{\delta \xi(\vec{k}',t')} = i \sqrt{\gamma} \langle \hat{A}(\vec{k}',t' - t) \rho_{st}^0 \rangle, \quad (91) \]

and

\[ \frac{\delta \hat{\rho}_{st}}{\delta \xi^*(\vec{k}',t')} = -i \sqrt{\gamma} \langle \rho_{st}^0 \hat{A}^\dagger(\vec{k}',t' - t) \rangle, \quad (92) \]

where \( \rho_{st}^0 = |\Psi_0\rangle\langle\Psi_0| \).

Hence from equation (81), we have

\[ \langle \hat{\rho}_{st} \xi(k,t) \rangle_s = 4\pi i \sqrt{\gamma} \int dk' k'^2 \int dt' D(k,k',t,t') \langle \hat{A}(\vec{k}',t' - t) \rho_{st}^0 \rangle_s, \quad (93) \]

Finally from Eqn. (81), we have

\[ \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + i \sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger \rangle \hat{A} \hat{\rho} - \hat{\rho} \hat{A}^\dagger \langle \hat{A} \rangle \right) + i \sqrt{\gamma} \int d\vec{k} \left( \hat{\xi} \rho_{st}^0 \right)_{s} \hat{\xi}^\dagger - \langle \hat{\rho}_{st} \xi \rangle_s \hat{A}^\dagger \]

\[ = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] + i \sqrt{\gamma} \int d\vec{k} \left( \langle \hat{A}^\dagger \rangle \hat{A} \hat{\rho} - \hat{\rho} \hat{A}^\dagger \langle \hat{A} \rangle \right) \]

\[ - \gamma \int dt' dk' k'^2 \hat{D}(k,k',t,t') \left( \hat{A}(k) \hat{A}(k',t' - t) \hat{\rho} + \hat{\rho} \hat{A}^\dagger(k',t' - t) \hat{A}(k) \right), \quad (94) \]

where \( \hat{D}(k,k',t,t') = 16\pi^2 k^2 k'^2 D(k,k',t,t') \). We note that the calculation has been carried out perturbatively, by expanding the state in powers of \( \sqrt{\gamma} \), and is exact up to order \( \sqrt{\gamma} \).
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