Improved Support Recovery Guarantees for the Group Lasso
With Applications to Structural Health Monitoring
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Abstract—This paper considers the problem of estimating an unknown high dimensional signal from noisy linear measurements, when the signal is assumed to possess a group-sparse structure in a known, fixed dictionary. We consider signals generated according to a natural probabilistic model, and establish new conditions under which the set of indices of the non-zero groups of the signal (called the group-level support) may be accurately estimated via the group Lasso. Our results strengthen existing coherence-based analyses that exhibit the well-known “square root” bottleneck, allowing for the number of recoverable nonzero groups to be nearly as large as the total number of groups. We also establish a sufficient recovery condition relating the number of nonzero groups and the signal to noise ratio (quantified in terms of the ratio of the squared Euclidean norms of nonzero groups and the variance of the random additive measurement noise), and validate this trend empirically. Finally, we examine the implications of our results in the context of a structural health monitoring application, where the group Lasso approach facilitates demixing of a propagating acoustic waveform, acquired on the material surface by a scanning laser Doppler vibrometer, into antithetical components, one of which indicates the locations of internal material defects.

Keywords—anomaly detection, convex demixing, group Lasso, non-destructive evaluation, primal-dual witness, support recovery

I. INTRODUCTION

In recent years, the recovery of structured signals from noisy linear measurements has been an active area of research in the fields of signal processing, high-dimensional statistics, and machine learning [1]–[4]. Suppose an unknown signal \( \beta^* \in \mathbb{R}^p \) is observed via the noisy linear measurement model

\[
y = X\beta^* + w,
\]

where \( y \in \mathbb{R}^n \) is the vector of observations, \( X \in \mathbb{R}^{n \times p} \) is the dictionary matrix, and \( w \in \mathbb{R}^n \) describes noise and/or model inaccuracies. Many contemporary works assume \( n < p \), in which case it is (in general) impossible to recover general \( \beta^* \) from the measurements. However, exploiting the fact that in many applications the signal of interest exhibits a low-dimensional structure opens the opportunity for using contemporary inference approaches from high dimensional statistics and compressed sensing. The low dimensional structure may be exhibited in different forms; for example, the signal of interest \( \beta^* \) might be entry-wise “sparse,” i.e. it may have only a few non-zero entries, it might be sparse under an appropriate transformation, or it might be “group-wise” sparse, meaning that given a partition of its entries into groups, only a few groups may be non-zero. Remarkable results such as those established in [5], [6] illustrate that, when the signal of interest is sparse and the dictionary \( X \) satisfies certain structural conditions, one can accurately infer \( \beta^* \) by solving the so-called Lasso problem [7] even when the number of non-zero entries of \( \beta^* \) is nearly proportional to the number of measurements.

When the signal of interest is group-sparse, the group Lasso estimator [8],

\[
\hat{\beta} = \arg \min_{\beta \in \mathbb{R}^p} \frac{1}{2} ||y - X\beta||_2^2 + \sum_{g=1}^{G} \lambda_g ||\beta_{I_g}||_2, \quad (2)
\]

can be used to infer the signal. In the formulation of interest here, \( \beta \) is expressed in terms of a given partition of its entries into \( G \) non-overlapping blocks (or groups)

\[
\beta = [(\beta_{I_1})^T(\beta_{I_2})^T\cdots(\beta_{I_G})^T]^T, \quad (3)
\]

where \( \beta_{I_g} \in \mathbb{R}^{d_g} \) represents the \( g \)-th constituent block of \( \beta \) with \( I_g \) denoting the subset of entries of \( \beta \) that belong to the \( g \)-th block, \( d_g \) denotes the cardinality of the \( g \)-th block, the \( \lambda_g > 0 \) are regularization parameters, and \( \| \cdot \|_2 \) denotes the Euclidean norm. This estimator exploits the extra knowledge about the natural grouping of the signal entries, and when this structure is present, its performance can exceed that of the standard Lasso estimator (which amounts to the case where each element of \( \beta \) is a singleton group) [8]–[10].

The existing studies that provide statistical guarantees for the group Lasso problem, when the measurements are generated according to (1), are diverse in terms of their statistical signal generation assumptions and the requirements they prescribe for successful recovery. In terms of the statistical model assumptions, a large body of work is focused on the case where the measurement matrix \( X \) is random [9], e.g., generated according to a Gaussian distribution [11], [12]. Another line of work studies the asymptotic behavior of this recovery procedure when the number of measurements and.
unknown parameters are allowed to tend to infinity [13–15]. In terms of requirements for successful recovery, various conditions have been proposed so far, including the group RIP condition of [16] and the restricted group eigenvalue condition of [3], [17]. Verifying such conditions for structured measurement matrices can be computationally prohibitive [18]; therefore, some existing works do not base their analyses on those requirements and instead use the concept of block coherence [19], which is computable in polynomial time. The recent effort [20] analyzes group-sparse estimation methods using structured dictionary matrices, with a sole focus on providing regression error guarantees.

Here, our investigation is motivated by an application in structural health monitoring, where we model our acquired data via the noisy linear model \( y = X \theta + \epsilon \) for a signal that is assumed group-sparse in a fixed, structured (non-random) dictionary \([21]–[24]\). In this context, and in contrast to the existing works discussed above, we seek finite sample, group-level support recovery guarantees for the group Lasso procedure, in order to pinpoint locations of material defects. We describe our motivating application in detail below.

A. Anomaly Detection for Structural Health Monitoring

In the past few decades, the need to improve the reliability of structural components and reduce their life-management costs has motivated the development of numerous structural diagnostics and structural health monitoring methodologies. Dynamics-based methods include popular techniques based on guided waves that are generated and received by transmitter-receiver pairs distributed over the structure, with detection processes that follow pulse-echo principles [25]. Namely, signatures of wave reflection are captured along each transmitter-receiver path, enabling the triangulation of the position of defects using data from multiple transducer pairs. Within this paradigm, numerous works have examined estimators of damage location likelihood from sparsely positioned sensors (see, e.g., [26]–[29]). However, these methods can suffer when ideality assumptions on the medium are relaxed (common in the context of damage formation and aging materials).

Recently, a powerful new class of diagnostic methodologies has emerged, leveraging the availability of laser-based sensing systems [30], [31]. Through the use of a Scanning Laser Doppler Vibrometer (SLDV) it is possible to perform non-contact measurements at a large number of points on a scanning grid defined on the surface of an object under test, thus providing full spatial reconstruction of the material’s surface dynamic response (e.g., to an induced acoustic excitation). Dedicated image processing techniques have been developed which utilize laser acquired data and meet desired anomaly identification and visualization criteria (see, e.g., [32]–[35]).

Laser-based methods facilitate diagnostic methods in which the inference is performed directly on a data-rich, spatially reconstructed response. Central to this view is the notion that, from a data standpoint, a wavefield is a data cube, slices of which represent snapshots (or frames) of the dynamic response at different temporal instants. The task of locating anomalies in a physical medium, then, can be recast as a problem of identifying atypical patterns in the observed data structures. Such efforts have been among the essential themes in machine learning and computer vision in recent years (see, e.g., [36]).

B. Approach

In this work, we utilize and expand notions from the sparsity-based source separation literature [37]–[40] and group Lasso inference to analyze the damage localization problem. The key observation underlying our approach is that SLDV measurements of a material subjected to narrowband acoustic excitation, acquired in the vicinity of the anomalous regions, exhibit different spatiotemporal behavior than do those acquired in the bulk of the material. We therefore attempt to decompose the acquired wavefield data into two components, one of which is a spatially-localized component arising near the detected areas while the other one is a generally smooth component in the pristine bulk of the structure; Fig. 1 illustrates one nominal measurement frame as well as its constituent components. This facilitates a baseline-free, agnostic inference approach whereby the locations of the defects in a material may be accurately estimated without a priori characterization of (a pristine version of) the medium. This feature distinguishes our method from the recent efforts in [41], [42] which also exploit group sparse inference techniques in the context of Lamb wave-based structural health monitoring but follow pitch-catch principles and require knowledge of the propagation model over the structure.

In order to separate the two structurally-distinct components of each measurement frame, we assume that (upon vectorizing the measurement snapshots) each component can be efficiently expressed as the product of an appropriate dictionary or basis matrix and a coefficient vector. The dictionaries should be chosen such that they capture the structural characteristics of their respective components. In the context of dictionary-based signal representation, this translates to choosing dictionaries that enable the characterization of the respective component in terms of the superposition of a few their columns. Since defects are generally spatially-localized, an appropriate dictionary for the defects is the identity matrix (i.e., the discrete Dirac basis), which comprises columns that are zero at every location except for one. Likewise, the Discrete Cosine Transform (DCT) matrix is one suitable basis for the smooth component of the response from the undamaged regions. In this sense, our model is reminiscent of the basis pairs utilized in the initial works on Basis Pursuit [37], [43].

To further facilitate the task of detecting anomalies, we notice that the effect of anomalies will change the wavefield

![Fig. 1. A snapshot of the wavefield measurement and its structurally-distinct components — the nominally smooth component, which is characteristic of the undamaged bulk of the structure, and the spatially-localized component, which is approximately zero except in the vicinity of the anomaly.](image-url)
characteristics at several pixel locations adjacent to the defect. In other words, one can expect that anomalies manifest themselves as spatially-contiguous pixel blocks of the overall anomaly vector. Therefore, we propose to define a spatial grouping over the domain of the defect component and make use of a spatial block-sparsity-promoting technique over the anomalous component of the measurement decomposition.

Imposing the spatial block-sparsity condition is justified by the fact that the bulk of a medium is undamaged and therefore most of the spatial blocks of the anomalous component should be zero blocks. In addition, since the effect of anomalies is usually persistent across multiple consecutive measurement frames (i.e., across time), we propose to extend the spatial grouping to a spatiotemporal one. This can be accomplished by partitioning the entries of multiple anomaly vectors, corresponding to multiple consecutive frames, into blocks which comprise spatially and temporally adjacent pixels.

In this setting, defect localization may be achieved by identifying the locations of the (nominally few) nonzero spatiotemporal groups describing the anomalous response, called the group-level support of the signal, in their respective dictionaries. Here, we analyze the performance of the group Lasso optimization for this support recovery task.

C. Contributions

We consider two approaches to analyze the performance of (2). In the first (baseline) case we make no specific assumptions on the generative model of the signal except that it be group-sparse as described above. In the second we impose an (arguably natural) generative probabilistic model on the signal. In each case, we identify sufficient conditions under which the group Lasso succeeds in identifying the group-level support of the unknown signal. Motivated by the application outlined above, our specific focus here is on the number of recoverable nonzero groups (relative to the total number of groups), as well as the functional relationship between the group sparsity of the signal and the signal-to-noise ratio (SNR) – quantified in terms of the ratio between the Euclidean norms of the nonzero groups’ coefficients and the additive noise variance – for which group Lasso support recovery provably succeeds.

Our results for the baseline setting are somewhat analogous to existing results analyzing support recovery for the group Lasso (e.g., in [14]), and are provided here largely to facilitate comparison with our results in the second setting, which improve upon the number of recoverable groups (relative to the total number of groups). As in [14], our analyses are based on an application of the primal-dual witness construction approach used in [14], under a predefined coefficient group structure, and for our second (stronger) result, under the specified generative signal model.

The term “baseline” is used frequently in this manuscript to refer to the first presented approach for analyzing the estimation performance of (2). It should not be confused with the “baseline-free” nature of our defect localization approach in the context of our application, as described in subsection 1B.

D. Notation and Organization

Throughout the paper, bold-face lowercase and uppercase letters will be used to denote vectors and matrices, respectively. For a vector $v$, we use $|v|_2$ to denote its Euclidean norm and for a matrix $V$, its spectral and Frobenius norms are denoted by $\|V\|_2 \to 2$ and $\|V\|_F$, respectively. Moreover, the sum of the absolute values of the entries of a matrix $V$ (or a vector $v$) are denoted by $\|V\|_1$ (or $\|v\|_1$) and the maximum absolute value of entries is represented by $\|V\|_\infty$ (or $\|v\|_\infty$).

We use $[m]$ as the shorthand for the set $\{1, 2, \cdots, m\}$, for any integer $m$. If $n$ denotes the length of $\beta$ and the number of columns of $X$, then for the index set $I_g \subset [n]$, $\beta_{I_g}$ will denote the group of entries of $\beta$ whose indices belong to this set and $X_{I_g}$ will denote the submatrix comprised of columns of $X$ indexed by $I_g$. For a column-wise block partitioned matrix $M = \left[ M_{I_g}, M_{I_g}, \cdots, M_{I_g} \right]$ the norm $\|M\|_{B,1}$ is defined as

$$\|M\|_{B,1} := \max_{g \in [G]} \|M_{I_g}\|_2 \to 2.$$ 

Throughout the paper, we will use different notions of support defined as follows:

1. $S(\beta) := \{ \beta_j : \beta_j \neq 0 \}$ will be the support of $\beta \in \mathbb{R}^n$.
2. $G(\beta) := \{ g \in [G] : \beta_{I_g} \neq 0 \}$ will denote the set that contains the indices of the nonzero groups of $\beta$, where $G$ is the total number of groups.
3. $S_g(\beta) := \cup_{g \in [G]} I_g$. In words, $S_g(\beta)$ will denote the set that contains all indices comprising groups that are nonzero (even if there are zero elements at those particular indices). Note that $S(\beta) \subseteq S_g(\beta)$.

We let

$$d_{\min} := \min_{g \in [G]} d_g \quad \text{and} \quad d_{\max} := \max_{g \in [G]} d_g$$

denote the minimum and maximum group sizes, respectively, and

$$d_{\phi}(\beta) := \sum_{g \in [G]} d_g$$

be the total number of entries in the group-level support $G(\beta)$ of $\beta$. Similarly, we define

$$\lambda_{\min} := \min_{g \in [G]} \lambda_g \quad \text{and} \quad \lambda_{\max} := \max_{g \in [G]} \lambda_g$$

to be the minimum and maximum regularization constants, respectively, and let $\lambda_{\phi}(\beta)$ be the $|G(\beta)|$-dimensional vector whose entries are the regularization parameters corresponding to the groups in $G(\beta^*)$. In order to clarify notation, we will use $G^*$, $S_g^*$, and $d_{\phi}^*$ as abbreviations for $G(\beta^*)$, $S_g(\beta^*)$, and $d_{\phi}(\beta^*)$, respectively.

The rest of the paper is organized as follows. We provide our main recovery results in Section II and discuss their implications in the context of our motivating application in Section III. We validate our theoretical results experimentally in Section IV, where we evaluate the efficacy of the group Lasso for support recovery on both synthetic data (adhering to our generative signal model) as well as in an FEM (finite element method) simulation of our structural anomaly detection.
problem. Section VII outlines the main steps of the primal-dual witness construction approach, which is used for proving our main recovery result, and how we instantiate this framework under our statistical assumptions. Section VII provides a few brief concluding comments and discussion of some future directions. Intermediate analytical results are relegated to the supplementary material.

II. MAIN THEORETICAL RESULTS

Our main theoretical contribution here comes in the form of a new support recovery guarantee for the group Lasso estimator under a random signal model. As alluded above, we assume measurements acquired according to the linear model (1), and under a random signal model. As alluded above, we assume a new support recovery guarantee for the group Lasso estimator expressed in terms of a given partition of its entries into blocks, as in [5]. We first present a baseline result applicable to deterministic signal models, then proceed to formulating our main result.

In both settings, our recovery guarantees are expressed in terms of the inter-block and intra-block coherence parameters of the dictionary X which are defined with respect to a given column-wise block partition of X.

**Definition II.1.** For any dictionary $X = [X_1, X_2, \ldots, X_G]$ with blocks $X_{g} \in \mathbb{R}^{n \times d_g}$ and whose columns all have unit Euclidean norm, the inter-block coherence constant $\mu_B(X)$ is defined as

$$
\mu_B(X) := \max_{1 \leq g \neq g' \leq G} \|X^T_{g} X_{g'}\|_2 \rightarrow 2,
$$

and the intra-block coherence parameter $\mu_I(X)$ is defined as

$$
\mu_I(X) := \max_{g \in [G]} \|X^T_{g} X_{g} - I_{d_g \times d_g}\|_2 \rightarrow 2.
$$

Notice that $\mu_B(X)$ measures similarity between the blocks of $X$ and reduces to the standard coherence parameter when the groups over the dictionary columns are singletons. Further, $\mu_I(X)$ measures the deviation of the blocks $\{X_{g}\}_{g \in [G]}$ from orthonormal blocks.

A. Baseline Result

Our first theoretical result can be stated as follows; its proof is structurally similar to that of our next main result (though simpler), and is provided in the supplementary material, for completeness.

**Theorem II.1.** Consider the linear measurement model (1) with $w \sim \mathcal{N}(0, \sigma^2 I_{n \times n})$. Assume that

1) $|\mathcal{G}(\beta^*)| \leq \min \left\{ \frac{0.5 - \mu_I(X)}{\mu_B(X)} + 1, \sqrt{\frac{d_{\min}}{d_{\max}}}, \frac{1}{4\mu_B(X)} \right\}$

2) $\|\beta^*_{\mathcal{G}}\|_2 \geq 5 \sigma (1 + \epsilon) \left( \sqrt{d_g} + \sqrt{d_g^*} \right)$, $\forall g \in \mathcal{G}^*$

3) $\lambda_g = 4 \sigma (1 + \epsilon) \sqrt{d_g}$, $\forall g \in [G]$

all hold for some

$$
\epsilon \geq \sqrt{(1 + \mu_I(X)) \log(pG)} / d_{\min}.
$$

Then, the following hold simultaneously, with probability at least $1 - 6 \rho^{-2 \log 2}$:

- the solution $\hat{\beta}$ of problem (2) will have the same group-level support as $\beta^*$; that is, $\mathcal{G}(\hat{\beta}) = \mathcal{G}(\beta^*)$, and
- $\|\hat{\beta}_{\mathcal{G}} - \beta^*_{\mathcal{G}}\|_2 \leq 5 \sigma (1 + \epsilon) \left( \sqrt{d_g} + \sqrt{d_g^*} \right)$, $\forall g \in \mathcal{G}^*$.

**Remark II.1.** As alluded above, this result is reminiscent of a main result of [14], though those results are asymptotic in nature, and the analogous SNR condition there was specified in terms of $\|\beta^*_{\mathcal{G}}\|_\infty$ rather than the group Euclidean norm $\|\beta^*_{\mathcal{G}}\|_2$ as here.

The above theorem provides conditions under which the recovery of the true group-level support is achievable via the group Lasso. Note in particular that the first condition, which limits the group-sparsity level, relates the number of nonzero groups $|\mathcal{G}(\beta^*)|$ to the inverse of the block coherence constant $\mu_B(X)$. This condition leads to sub-optimal scaling between the number of measurements and the number of non-zero groups, as will become clear in the context of our next main result, as well as in the next section, in the context of the material anomaly detection.

B. Strengthened Result

To strengthen the baseline result, we impose some mild statistical assumptions on the generation of the coefficient vector $\beta^*$. Specifically, similar to [20], we assume the group-sparse vector $\beta^* \in \mathbb{R}^p$ in (3) is randomly generated according to the assumptions outlined below:

- $M_1$ The block support $\mathcal{G}^*$ of $\beta^*$ comprises $s := |\mathcal{G}^*|$ non-zero blocks, whose indices are selected uniformly at random from all subsets of $|\mathcal{G}|$ of size $s$.
- $M_2$ The non-zero entries of $\beta^*$ are equally likely to be positive or negative: $\mathbb{E}[|\beta_j^*|] = 0$ for $j \in [p]$.
- $M_3$ The non-zero blocks of $\beta^*$ have statistically independent “directions.” Specifically, it is assumed that

$$
\Pr \left( \bigcap_{g \in \mathcal{G}^*} \frac{\beta_{\mathcal{G}}^*}{\|\beta_{\mathcal{G}}^*\|_F} \in \mathcal{A}_g \right) = \prod_{g \in \mathcal{G}^*} \Pr \left( \frac{\beta_{\mathcal{G}}^*}{\|\beta_{\mathcal{G}}^*\|_F} \in \mathcal{A}_g \right),
$$

where for each $g$, $\mathcal{A}_g \subseteq \mathbb{S}^{d_g - 1}$ with $\mathbb{S}^{d_g - 1}$ representing the unit sphere in $\mathbb{R}^{d_g}$.

Utilizing this model, we obtain the following theorem [45]. Its proof appears in Section V.

**Theorem II.2.** Consider the linear measurement model (1)
with \( w \sim N(0, \sigma^2 I_{n \times n}) \). Assume that

1) \( \mu_I(X) \leq c_0 \) and \( \mu_B(X) \leq \sqrt{d_{\min} \frac{c_1}{d_{\max} \log p}} \),

2) \( |G(\beta^*)| \leq \min \left\{ \frac{c_2}{\| X \|_2 \sqrt{2 \log p}}, \frac{d_{\min} c_2}{d_{\max} c_2} \frac{\mu_B^2(X)}{\log p} \right\} \),

3) \( \forall g \in G(\beta^*) : \left\| \beta_{g, \hat{X}} - \beta_{g, 0} \right\|_2 \geq 10 \sigma (1 + \epsilon) \sqrt{d_g + d_{g'}} \max \left\{ 1, \sqrt{\frac{s}{d_{\max} \log p}} \right\} \),

4) \( \lambda_g = 4 \sigma (1 + \epsilon) \sqrt{d_g}, \forall g \in [G] \),

all hold for some positive constants \( c_0, c_1 \leq 0.001 \),

\[
c_2 = \min \{ c_2, 0.0001 \}, \text{ and some } \epsilon \geq \sqrt{1 + \frac{1}{2} \left( 4 \frac{1}{4} - 3 c_0 - 48 c_1 \right) - 3}^2, \]

Then the following hold simultaneously, with probability at least \( 1 - 12 p^{-2 \log^2 2} \):

- the solution \( \hat{\beta} \) of (3) is unique and has the same group-level support as \( \beta^* \); that is, \( G(\hat{\beta}) = G(\beta) \), and
- \( \left\| \beta_{g, \hat{X}} - \beta_{g, 0} \right\|_2 \leq 5 \sigma (1 + \epsilon) \left( \sqrt{d_g} + \sqrt{d_{g'}} \right), \forall g \in G(\beta^*) \).

**Remark II.2.** As required by the first theorem assumption, the support recovery guarantee relies on the well-conditioning of the dictionary \( X \). We measure the well-conditioning in terms of block coherence constants \( \mu_I(X) \) and \( \mu_B(X) \) of the dictionary. Fortunately, both constants can be computed in polynomial time for a given column-wise partitioned dictionary (unlike other quantities such as restricted isometry constant, which are widely used in proving similar recovery guarantees; but can be NP-hard to compute [18]). Regarding the material anomaly detection framework, this first assumption will impose very mild conditions on the problem parameters, as will be seen in the following sections.

**Remark II.3.** The second condition specifies the requirement on the maximum number of allowable non-zero groups in the group-level support of \( \beta^* \) that can be recovered. Unlike the earlier recovery result, the condition provided here is less stringent since the block coherence parameter appears in the upper-bound in the form of \( \mu_B^{-2}(X) \), which is a significant improvement over similar results, e.g. in [12], [46], that require \( G^* \) be bounded by functions of \( \mu_B^{-1}(X) \).

**Remark II.4.** The third assumption here (like the second assumption in our baseline result) is on the strength of the non-zero groups, which requires their magnitudes to be above a certain threshold depending on the noise variance \( \sigma \). More discussions on this assumption, and its implications in our motivating application are provided in Section IV.

**Remark II.5.** Finally, we note that our choices of the universal constants \( c_0, c_1, c_2, c_2' \) are not optimized here.

### III. Theoretical Results in the Context of Structural Anomaly Detection

As mentioned in the introduction, one goal of this work is to quantify the performance of the group Lasso for laser-enabled anomaly localization in a structural health monitoring application. In this section we apply our main results from the previous section to that problem.

Assume that one vectorized snapshot of wavefield measurements, captured at time instant \( t \in [T] \), is denoted by the vector \( y(t) \in \mathbb{R}^N \), where the integer \( N \) denotes the total number of acquired measurements. In the case where the physical structure is a two-dimensional medium, every snapshot of measurements will be a two-dimensional image with \( N \) denoting the total number of pixels of the image. Moreover, assume that the matrix \( Y = [y(1) y(2) \cdots y(T)] \in \mathbb{R}^{N \times T} \) stores all the measurement vectors for time instants \( 1 \) to \( T \).

As discussed in the introduction, we aim to separate the spatially smooth component of wavefield measurements, which captures the response of the pristine bulk of the medium, from the spatially-localized component, which arises due to the presence of internal material defect(s). To perform the separation, we first assume that both components can be represented in terms of appropriate dictionaries, which capture structural characteristics of their respective components. To make the idea more formal, let \( X_{(1)} \in \mathbb{R}^{N \times p_1} \) and \( X_{(2)} \in \mathbb{R}^{N \times p_2} \) represent the dictionaries that appropriately represent the spatially-smooth and sparse components, respectively. Examples of the appropriate choices for \( X_{(1)} \) and \( X_{(2)} \), as alluded earlier, are the two-dimensional discrete cosine transform (DCT) and identity matrices, respectively (with \( p_1 = p_2 = N \)).

Given the knowledge of appropriate dictionaries, we assume the measurement matrix is generated by the following underlying model

\[
Y = X_{(1)} B_{(1)}^* + X_{(2)} B_{(2)}^* + W, \tag{6}
\]

where \( B_{(1)}^* \in \mathbb{R}^{N \times T} \) and \( B_{(2)}^* \in \mathbb{R}^{N \times T} \) denote the corresponding coefficient matrices and \( W \in \mathbb{R}^{N \times T} \) represents noise and model ambiguities. In this model the first term \( X_{(1)} B_{(1)}^* \) stands for the smooth component of measurements generated by the pristine bulk of the medium and \( X_{(2)} B_{(2)}^* \) models the defect component. Given the above model the problem of anomaly detection reduces to finding the support of the defect component \( X_{(2)} B_{(2)}^* \) (or simply \( B_{(2)}^* \) when \( X_{(2)} = I_N \times N \)).

A practical assumption that improves the performance of the anomaly detection procedure is that defects manifest themselves in the form of spatially-contiguous groups of pixels. Therefore, given a spatial partition of the measurement domain into groups of \( D \geq 1 \) adjacent pixels, one can expect the pixels within a group to be corrupted once a defect is present in that region. In the measurement model expressed by (6), with \( X_{(2)} = I_N \times N \), this implies that each column of \( B_{(2)}^* \)
can be partitioned into \( G_2 := p_2/D = N/D \) groups of size \( D \), where the entries within a group are adjacent pixels in the two-dimensional representation of the measurements. Furthermore, since the effect of anomalies changes the wavefield characteristics across multiple consecutive frames, it makes sense to define a more general spatiotemporal grouping over \( B^*_2 \). Then the imposed grouping will partition the coefficients in \( B^*_2 \) into \( G_2 \) sub-matrices of size \( D \times T \), where the entries of a sub-matrix are spatiotemporally adjacent. On the other hand, a temporal grouping can be applied to the entries of the coefficient matrix \( B^*_1 \) corresponding to the smooth component, with the idea that the same frequencies (i.e. the same columns of the DCT dictionary) should appear in the decomposition of consecutive frames. Doing so, \( B^*_1 \) can be partitioned into \( G_1 := p_1 \) sub-matrices of dimensions \( 1 \times T \). To enable the recovery of \( B^*_1 \) and \( B^*_2 \) from the measurements in \((6)\), we assume both coefficient matrices are block-sparse with respect to the groupings described, i.e. only a few groups in the partition of every coefficient matrix are non-zero.

Given these assumptions we propose to estimate the true coefficient matrices \( B^*_1 \) and \( B^*_2 \) by \( \hat{B}^*_1 \) and \( \hat{B}^*_2 \), which are solutions of the following optimization problem

\[
\min_{B^*_1, B^*_2} \sum_{g \in [G_1]} \| (B^*_1)_{X_{g_1}} \|_F^1 + \lambda_2 \sum_{g \in [G_2]} \| (B^*_2)_{X_{g_2}} \|_F^2
\]

where \( \lambda_1 \) and \( \lambda_2 \) are positive scalars, and \( g_1 \) and \( g_2 \) index the blocks of \( B^*_1 \) and \( B^*_2 \), respectively, which are formed according to the grouping techniques described above. In this formulation, minimizing the first term will ensure that the model fits the measurements; while minimizing the last two terms guarantee that the two components comprise a small number of atoms from the corresponding dictionaries. In particular, minimizing the third term promotes the group sparsity of the recovered anomaly component with respect to the specified spatiotemporal grouping.

To enable the application of the theoretical results developed in the previous section, we adopt a vectorized representation of the measurement model \((6)\). Specifically, we choose \( y \in \mathbb{R}^n \) to denote the vector of measurements acquired by stacking all the \( T \) columns of \( Y \) in one vector (therefore obtaining a measurement vector of length \( n := NT \)). Upon vectorizing the entire measurement model \((6)\), the new representation becomes

\[
y = \widetilde{X}_Y \beta_1 + \tilde{X}_2 \beta_2 + w,
\]

where \( y = \text{vec}(Y) \in \mathbb{R}^n \), \( \beta_1 = \text{vec}(B^*_1) \in \mathbb{R}^n \), \( \beta_2 = \text{vec}(B^*_2) \in \mathbb{R}^n \), \( w = \text{vec}(W) \in \mathbb{R}^n \) are vectors, with the vectorization operator \( \text{vec}(\cdot) \) stacking the columns of the argument matrix into a single-column vector, and \( \widetilde{X}_Y \) and \( \tilde{X}_2 \) are Kronecker-structured dictionaries given as \( \widetilde{X}_Y = I_{T \times T} \otimes X_i \), for \( i = 1, 2 \). Notice that after the vectorization, the previously-discussed partitions over the entries of \( B^*_1 \) and \( B^*_2 \) result in non-canonical groups, which are either of size \( T \) (for the groups over the smooth component) or of size \( DT \) (for the groups over the second spatially-sparse component). Using vector notation, the problem \((7)\) can be recast as

\[
\min_{\beta_1, \beta_2} \left\{ \frac{1}{2} \| y - \widetilde{X} \beta_1 - \tilde{X}_2 \beta_2 \|_2^2 + \lambda_1 \sum_{g \in [G_1]} \| (\beta_1)_{I_{X_{g_1}}} \|_F^2 + \lambda_2 \sum_{g \in [G_2]} \| (\beta_2)_{I_{X_{g_2}}} \|_F^2 \right\}.
\]

We may write the model \((8)\) in terms of the overall dictionary \( X := \widetilde{X}_Y \mid \tilde{X}_2 \in \mathbb{R}^{n \times p} \), with \( p := 2n \), and the overall coefficient vector \((\beta^*)^T := [(\beta^*_1)^T \mid (\beta^*_2)^T] \in \mathbb{R}^p \) as

\[
y = X \beta^* + w,
\]

which is the linear measurement model discussed in the previous section.

The implications of Theorem \(\text{II.1}^*\) for the anomaly detection scenario discussed above are outlined below.

**Corollary III.1.** Consider the linear measurement model \((6)\) with \( X_1 \) and \( X_2 \) specialized to the 2D-DCT and identity matrices of size \( N \times N \), respectively, and the entries of \( W \) independently drawn from the Gaussian distribution \( \mathcal{N}(0, \sigma^2) \). Moreover, suppose \( B^*_1 \) and \( B^*_2 \) have \( s_1 \) and \( s_2 \) non-zero groups, respectively, which are arbitrarily drawn from the partitions defined over the entries of these matrices. If

\[
1) \ s = s_1 + s_2 \leq \sqrt{N} \ \sqrt{8D}
\]

\[
2) \ \min_{g \in [G_1]} \| (B^*_1)_{I_{X_{g_1}}} \|_F^2 \geq 5 \sigma \sqrt{T}(1 + \epsilon) \left( 1 + \sqrt{s_1 + s_2D} \right)
\]

\[
3) \ \min_{g \in [G_2]} \| (B^*_2)_{I_{X_{g_2}}} \|_F^2 \geq 5 \sigma \sqrt{T}(1 + \epsilon) \left( \sqrt{D} + \sqrt{s_1 + s_2D} \right)
\]

\[
4) \ \lambda_1 = 4 \sigma (1 + \epsilon) \sqrt{T} \ and \ \lambda_2 = 4 \sigma (1 + \epsilon) \sqrt{TD}
\]

all hold for some

\[
\epsilon \geq \left[ \frac{2 \log (2NT)}{T} \right].
\]

then the group-level support of \( \hat{B}^*_1 \) and \( \hat{B}^*_2 \) will exactly match those of \( B^*_1 \) and \( B^*_2 \), respectively, with probability at least \( 1 - 6 (2NT)^{-2} \log^2 \).
Corollary III.2. Consider the linear measurement model (6) with \( X(1) \) and \( X(2) \) specialized to the two-dimensional DCT and identity matrices of size \( N \times N \), respectively, and the entries of \( W \) drawn independently from \( \mathcal{N}(0, \sigma^2) \). Moreover, suppose \( B^* := \{ B^{(1)}_1 | B^{(2)}_2 \} \in \mathbb{R}^{N \times 2T} \) has \( s \) randomly-selected non-zero groups selected according to the statistical assumptions \( M_1, M_2, \) and \( M_3 \). If

1. \( \sqrt{N} \geq \frac{2}{c_1} \left[ \frac{\log(2NT)}{c_2 N^2} \right] \sqrt{D^3 T} \)
2. \( s \leq \frac{TD^3 \log(2NT)}{N} \)
3. \( \forall g \in G_1^* : \left\| (B^{(1)}_1)r_g \right\|_F \geq 10\sigma \sqrt{T} \left( 1 + \sqrt{s_1 + s_2 D} \right) \times \left( 1 + \epsilon \right) \max \left\{ 1, \frac{s}{TD \log(2NT)} \right\} \)
4. \( \forall g \in G_2^* : \left\| (B^{(2)}_2)r_g \right\|_F \geq 10\sigma \sqrt{T} \left( \sqrt{D} + \sqrt{s_1 + s_2 D} \right) \times \left( 1 + \epsilon \right) \max \left\{ 1, \frac{s}{TD \log(2NT)} \right\} \)
5. \( \lambda_1 = 4\sigma(1 + \epsilon) \sqrt{T} \) and \( \lambda_2 = 4\sigma(1 + \epsilon) \sqrt{DT} \)

all hold for \( c_1 \leq 0.001, c_2 \leq 0.0001 \), and

\( \epsilon \geq \sqrt{\frac{2\log(2NT)}{T}} \),

where \( s_1 \) and \( s_2 \) denote the number of nonzero groups selected in \( B^{(1)}_1 \) and \( B^{(2)}_2 \) respectively, then the group-level support of \( B \) will exactly match that of \( B^* \) with probability at least \( 1 - 12(2NT)^{-2\log 2} \).

The above result, whose proof is provided in the supplementary material, is a direct consequence of Theorem II.2 of the previous section. As the theorem asserts, the group-level support of \( B^* \) should be drawn uniformly at random from among the \( \binom{G}{s} \) different subsets of \( G \) comprised of \( s \) elements. The randomness of the (group-level) support of \( B^* \) enables us to bring in tools from the concentration of random variables theory to prove the sufficient conditions of the Theorem. The second condition provides an upper bound on how many anomalous groups can be detected by the convex demixing procedure in (7), and the third gives lower bounds for the strength of the non-zero groups in order for them to be detectable using the group Lasso approach.

IV. NUMERICAL EXPERIMENTS

In this section we test the ability of the group Lasso formulation (8) in recovering the non-zero coefficients \( \beta^s \) for dictionary-based representation of the measurements. The first set of experiments that are presented here are carried out using synthetically generated data adhering to our overall modeling assumptions; the second utilizes simulated data from our motivating structural health monitoring application, obtained via finite element simulation methods.

A. Phase Transition Diagram

We begin by examining the relationship between the group-sparsity level of the unknown coefficient vector and the strength of non-zero groups that guarantees successful recovery. The inspiration for this investigation comes from Conditions 3 and 4 of Corollary III.2 (and similar conditions in Corollary III.1), which outline sufficient lower bounds on \( \| (B^{(1)}_1)r_g \|_2 \) and \( \| (B^{(2)}_2)r_g \|_2 \) for exact support recovery.

Operating under the measurement model assumptions introduced in Section III, we generate measurements according to Equation (6). More specifically, we generate \( T = 8 \) frames of measurements, each of dimension \( 100 \times 100 \), therefore \( N = 10^4 \) in (6). To generate each frame we choose \( X(1) \) to be the \( N \times N \) 2D-DCT matrix, and set \( X(2) \) to be the \( N \times N \) identity matrix (to explain the sizes of the dictionaries we would like to note that \( X(1) \) and \( X(2) \) operate on vectorized images). Once \( X(1) \) and \( X(2) \) are selected, it remains to generate \( B^{(1)}_1 \in \mathbb{R}^{N \times T}, B^{(2)}_2 \in \mathbb{R}^{N \times T} \) and \( W \in \mathbb{R}^{N \times T} \) in order to make the measurement vectors as according to (6).

Inspired by the spatial contiguity assumption of anomalies, we assume each column of \( B^{(2)}_2 \), which corresponds to a vectorized \( 100 \times 100 \) image, is partitioned into groups of size \( D = d^2 \), where each group corresponds to a \( d \times d \) spatially-contiguous block in the original image representation of the column. Here we report the results for \( d = 2 \) (\( D = 4 \)). Also by the assumption of the temporal persistency of anomalies, we extend the grouping across all the frames resulting in the entries of \( B^{(2)}_2 \) be partitioned into groups of size \( d^2 \times T \). Doing so, the total number of blocks over the support of \( B^{(2)}_2 \) becomes \( G_2^* = (N/d)^2 \). For the coefficient matrix \( B^{(1)}_1 \) corresponding to the spatially-smooth component, we assume no spatial grouping structure over its columns; therefore each of its \( G_1 = N \times 10^4 \) rows comprise a group. Next, in order to give values to \( B^* = \{ B^{(1)}_1 | B^{(2)}_2 \} \) we first choose \( s = s_1 + s_2 \) out of the entire \( G = G_1 + G_2 \) blocks uniformly at random (for \( s \) ranging from \( 1 \) to \( 800 \)) and then set the selected entries to i.i.d. standard Gaussian values. Finally, the noise matrix \( W \) is set to have i.i.d. entries generated according to \( \mathcal{N}(0, \alpha^{-2}) \), where \( \alpha \) can be thought as the parameter which defines the signal to noise ratio, and is varied from \( 0 \) to \( 80 \).

For each choice of the \((s, \alpha)\) pair, we generate 100 different realizations and test the performance of the proposed algorithm in recovering the coefficients. The numerical algorithm that we have adopted for solving the corresponding optimization problem (7) is based on alternating minimization with respect to two coefficient matrices \( B^{(1)}_1 \) and \( B^{(2)}_2 \). A pseudocode sketch of the algorithm is detailed in Algorithm [1]. We note that the objective in (7) is jointly-convex in \( B^{(1)}_1 \) and \( B^{(2)}_2 \), and Algorithm [1] can be viewed as an instance of Block Coordinate Descent method. Applying the analysis presented in [7] it can be shown that Algorithm [1] converges linearly to the global minima \( (B^{(1)}_1, B^{(2)}_2) \). We also provide a fully-documented MATLAB software package called Damage Pursuit to supplement this work; it is available for download at http://damagepursuit.umn.edu.

Since by the specific grouping defined over the entries of \( B^{(1)}_1 \) and \( B^{(2)}_2 \) only two distinct group sizes exist, the regu-
Algorithm 1 Alternating minimization routine for solving (7) when $X_{(1)}$ and $X_{(2)}$ are orthonormal bases.

\begin{algorithm}
\begin{algorithmic}
\State Initialize $B_{(1)} \leftarrow 0$ and $B_{(2)} \leftarrow 0$
\Repeat
\State $R_{(1)} \leftarrow X_{(1)}^T (Y - X_{(2)}B_{(2)})$
\State $(B_{(1)})_{x_{g_1}} \leftarrow \left(1 - \frac{\lambda_1}{\|R_{(1)}x_{g_1}\|} \right) \circ (R_{(1)})_{x_{g_1}}$, $\forall g_1 \in G_1$
\State $R_{(2)} \leftarrow X_{(2)}^T (Y - X_{(1)}B_{(1)})$
\State $(B_{(2)})_{x_{g_2}} \leftarrow \left(1 - \frac{\lambda_2}{\|R_{(2)}x_{g_2}\|} \right) \circ (R_{(2)})_{x_{g_2}}$, $\forall g_2 \in G_2$
\Until convergence
\end{algorithmic}
\end{algorithm}

The regularization parameters are set to either $\lambda_1 = \frac{\varepsilon}{s} \sqrt{T}$ for all the groups defined over the support of $B_{(1)}^*$, or $\lambda_2 = \frac{\varepsilon}{s} \sqrt{T} D^2$ for all the groups over the support of $B_{(2)}^*$. The probability of success is then simply defined as the ratio of the number of realizations for which the successful recovery of the group-level support of both $B_{(1)}^*$ and $B_{(2)}^*$ occurs to the total number of trials. To avoid errors due to numerical inaccuracies, we declare the groups of the recovered coefficient matrices as being non-zero if their norms exceed a precision constant $\varepsilon_p = 10^{-6}$ times the norms of their corresponding groups in the ground-truth coefficient matrices.

The left-hand side panel of Fig. 2 shows the phase transition diagram for the described set up. As the number of active non-zero groups increases, one needs to increase the strength of the active groups to enable successful group-level support recovery, as expected. Further, the shape of the curve shows agreement with our theoretical predictions. Indeed, examining conditions 3 and 4 in Corollary III.2, we see that for small $s = s_1 + s_2$, the SNR above which the group Lasso succeeds is on the order of $\sqrt{s_1 + s_2} D$, while for larger values of $s$, the sufficient SNR condition is on the order of $\sqrt{s_1 + s_2} D \cdot \sqrt{s}$ (ignoring leading constants and other factors not depending on $s_1$ and $s_2$). Now, because $D$ is small ($D = 4$ here), we have that the small-$s$ regime should exhibit a sufficient SNR trend that functionally grows like $\sqrt{s}$, and the trend should be nearly linear in $s$. This agrees, empirically, with the observed phase transition.

B. Finite Element Simulations

We also use synthetic wavefield measurements generated by finite element simulations to study the relationship between the number of defects, their severity, and the ability of the proposed group Lasso estimator in successful defect recovery. For this, we model an aluminum plate, with dimensions 100 cm $\times$ 50 cm and thickness 5 mm, which is probed by a flexural wavefield induced by an actuator located in the middle of the left edge of the domain. Localized anomalies are introduced by reducing the Young’s modulus constant of the material of a 1.5 cm $\times$ 1.5 cm region to simulate a soft inclusion. The actuator is set to generate $N_c = 5$ bursts of a narrow-band sine wave at the frequency $f_c = 10^5$. We then record 100 (two milliseconds apart) snapshots of the nodal displacements, over a grid with 160 $\times$ 80 nodes, and store them as columns of a measurements matrix $Y$. Given the grid size and the number of frames, the measurement matrix $Y$ has dimension $N \times 100$, where $N = 160 \times 80 = 12800$.

Similar to the previous sub-section, we aim to generate a phase transition diagram for the successful recovery rate of our procedure, with the horizontal and vertical axes indicating the number of defects and their severity level, respectively. In this experiment, we vary the number of anomalies between 1 and 30, and place them at randomly selected locations over the surface of the simulated structure. To change defects’ severity at those locations, we modulate the Young’s modulus constant of the bulk structure by a scalar parameter $\eta \in (0, 1)$ to obtain the Young’s modulus constant of the defected regions. On the vertical axis of the phase transition diagram the defect severity is changed by raising $\eta$ to different integer powers $i$, where $i$ takes values between 1 and 30. Intuitively speaking, as the integer power $i$ increases the defect severity increases as well, since the Young’s modulus constant of defected regions become a smaller fraction of that corresponding to the healthy regions of the structure, which in turn makes recovery easier.

In the current experiment we set $\eta = 0.9$. We solve the group Lasso problem (7) for five consecutive frames, i.e. $T = 5$, and adopt a partitioning of the defect component coefficient vectors into spatial groups of size four pixels. The regularization parameters were experimentally tuned to $\lambda_1 = 0.005$ and $\lambda_2 = 0.12$ for the groups over the smooth and sparse components, respectively. We repeat the experiment 50 times for every specialization of the number of defects and their severity level. Fig. 2 (b) shows the phase transition diagram for this experiment. Interestingly, the overall trend of the phase transition diagram resembles the diagram of the former sub-section. In fact, by increasing the mismatch between the Young’s modulus constant of defects and the rest of the medium, local displacements at the place of anomalies increase. The displacements are effectively captured by the sparse coefficient matrix of our decomposition model and therefore contribute to stronger coefficient values in this matrix. For illustration, Fig. 2 (c) illustrates the schematic of the simulated plate, a wavefield snapshot, and the recovered defect component for that snapshot.

Finally, we would like to note modifying the Young’s modulus is at one principled approach to adjust the strength of an anomaly in a physical setting. Properly speaking, by adjusting this parameter, we are varying the contrast in elastic properties (acoustic mismatch). By extension, we can also model partial holes via this approach (see [24]), but omit those evaluations here due to space constraints.

V. PROOF OF THEOREM II.2

In this section we present the proof of Theorem II.2. The proof of Theorem II.1 is similar (and simpler), so is omitted here, though we do comment but included in the supplementary material for completeness.
A. Overview of Approach

Our analysis utilizes a basic result for characterizing the optimal solutions of the group Lasso problem [3]. We state the result here as a lemma; its proof follows what are now fairly standard methods in convex analysis so we omit it here.

Lemma VI.1. A vector $\beta$ solves problem (2) if and only if

$$X^T X (\hat{\beta} - \beta^*) - X^T z = 0, \quad \forall \ g \in [G]$$

holds for some vector $\hat{z}$, whose elements satisfy

$$\hat{z}_g = \frac{\beta_g}{\|\beta_g\|_2}, \quad \text{if } \|\beta_g\|_2 > 0$$

$$\|\hat{z}_g\|_2 \leq 1, \quad \text{otherwise}$$

If $\|\hat{z}_g\|_2 < 1$ for all $g \notin G(\hat{\beta})$ then any optimal solution $\hat{\beta}$ to (2) satisfies $\hat{z}_g = 0$ for all $g \notin G(\hat{\beta})$; if, in addition, the matrix $X^T S(\hat{\beta}) X S(\hat{\beta})$ is invertible, then $\hat{\beta}$ is the unique solution to (2).

Note that the optimality condition (11) can be written in matrix form, as

$$X^T X (\hat{\beta} - \beta^*) - X^T z = 0,$$

where $\Lambda$ is the $p \times p$ diagonal matrix whose $j$-th diagonal entry $\Lambda_{j,j} = \lambda_g(j)$, where $g(j) = \{g \in [G] : j \in I_g\}$. In other words, the diagonal elements of $\Lambda$ are, for each index $j$, the regularization parameters associated with the group to which the corresponding element $\beta_j$ of $\beta$ belongs. We will find this formulation convenient in the analysis that follows.

The ultimate goal of this section is to find conditions under which the group-level support of $\beta$ and $\beta^*$ are identical, i.e., $G(\hat{\beta}) = G(\beta^*)$. Our proof follows the so-called Primal-Dual Witness (PDW) technique utilized in [44] for the analysis of the Lasso problem and also in [11] for the analysis of the group Lasso problem arising in the context of multivariate regression. In our setting, a primal-dual certificate pair $(\hat{\beta}, \hat{z})$ is constructed according to the following steps:

1) We identify the solution of a restricted group Lasso problem over the true “group-level” support $S^*_R(\beta^*)$. Specifically, we consider $\beta_{S^*_R} \in \mathbb{R}^{G_R}$ obtained via

$$\beta_{S^*_R} = \arg \min_{\beta_{S^*_R} \in \mathbb{R}^{G_R}} \frac{1}{2} \|y - X_{S^*_R} \beta_{S^*_R}\|^2 + \sum_{g \in G^*} \lambda_g \|\beta_{S^*_R}\|_2.$$ (14)

Note that if $X_{S^*_R}$ has full column-rank, there will be a unique vector $\beta_{S^*_R}$ that solves (14).

2) We choose $\hat{z}_{S^*_R} \in \mathbb{R}^{d_R}$ to be the optimal dual solution of the restricted group Lasso problem (14) such that the primal-dual pair $(\beta_{S^*_R}, \hat{z}_{S^*_R})$ satisfies the optimality conditions of the restricted problem.

3) We set the “off group-level support” primal variable $\beta_{(S^*_R)^c}$ to be zero.

4) Finally, we solve for an “off group-level support” dual variable $\hat{z}_{(S^*_R)^c} \in \mathbb{R}^{n-d_R}$ which satisfies the optimality conditions for the full (unrestricted) group Lasso problem, as specified in (11) and (12), and identify conditions under which this vector satisfies $\|\hat{z}_{S^*_R}\|_2 < 1$ for all $g \notin G^*$.

Overall, the PDW approach can be viewed, essentially, as a method for evaluating the feasibility of one particular candidate solution $\beta$ to the original group Lasso problem (2), constructed in a piece-wise manner. The first two steps identify conditions that the elements of the candidate solution must adhere to on the true “group-level” support. The strict dual feasibility condition ($\|\hat{z}_{S^*_R}\|_2 < 1$ for all $g \notin G^*$) in step 4 together with step 3 ensure that no “spurious” nonzero groups are present in $\beta$. In other words, the success of the PDW approach outlined above ensures that the primal-dual pair $(\beta, \hat{z})$ satisfies the optimality conditions of the general group Lasso problem.
for any fixed group index \( g \notin \mathcal{G}^* \) we have

\[
\tilde{z}_{I_g} = \frac{1}{\lambda_g} X_{I_g}^T \left[ X_{S_g^c} (\beta_{S_g^c}^* - \tilde{\beta}_{S_g^c}) + w \right]
\]

\[
= \frac{1}{\lambda_g} X_{I_g}^T \left[ X_{S_g^c} (X_{S_g^c}^T X_{S_g^c})^{-1} (\Lambda_{S_g^c} z_{S_g^c} - X_{S_g^c}^T w) + w \right]
\]

\[
= \frac{1}{\lambda_g} X_{I_g}^T \left[ X_{S_g^c} (X_{S_g^c}^T X_{S_g^c})^{-1} \Lambda_{S_g^c} z_{S_g^c} + \Pi_{(S_g^c)^c} (w) \right],
\]

where the second equality follows from the incorporation of (19), and the third one makes use of the definition \( \Pi_{(S_g^c)^c} (w) := (I - X_{S_g^c} (X_{S_g^c}^T X_{S_g^c})^{-1} X_{S_g^c}^T) w \).

Now, we exploit our statistical assumptions, i.e. that the “direction” vectors \( \beta_{I_g}^* / \| \beta_{I_g}^* \|_2 \) associated with every nonzero block of \( \beta^* \) indexed by \( g \in \mathcal{G}^* \) are random, and statistically independent. To this aim, we need to express the elements of the vector \( z_{S_g^c}^* \) (or more specifically, its individual blocks) in terms of the “direction” vectors associated with the corresponding nonzero blocks of the true vector \( \beta_{S_g^c} \). In particular, the following Lemma, proved in the supplementary material, states that every block of \( z_{S_g^c}^* \) can be expressed as the sum of the corresponding true direction vector and a bounded perturbation.

**Lemma V.2:** Suppose that the group-level support \( \mathcal{G}^* \) is fixed and that the event \( E_1 \) occurs. Defining \( h_{g'} := \beta_{I_g^c}^* - \beta_{I_g}^* \), for every \( g' \in \mathcal{G}^* \), it follows that

\[
\| h_{g'} \|_2 \leq \| X_{I_g^c}^T w \|_2 + \lambda_{g'} + \| X_{S_g^c}^T w \|_2 + \| \lambda_{g'} \|_2,
\]

(21)

where \( \lambda_{g'} \in \mathbb{R}^{d_g^c} \) is a vector whose entries are the elements \( \{ \lambda_{g'} \}_{g' \in \mathcal{G}^*} \). Moreover, the blocks of the dual vector over the true support set \( \mathcal{G}^* \) can be expressed as

\[
\tilde{z}_{I_g'} = \frac{\beta_{I_g'}^*}{\| \beta_{I_g'}^* \|_2} + u_{g'},
\]

and if \( \| h_{g'} \|_2 \leq \frac{1}{2} \| \beta_{I_g'}^* \|_2 \) for \( g' \in \mathcal{G}^* \), then \( \| u_{g'} \|_2 \leq 4 \| h_{g'} \|_2 / \| \beta_{I_g'}^* \|_2 \).

As the Lemma asserts, for each \( g' \in \mathcal{G}^* \) for which it holds that \( \| \beta_{I_g'}^* - \beta_{I_g}^* \|_2 \leq \frac{1}{2} \| \beta_{I_g'}^* \|_2 \), we can write

\[
\tilde{z}_{I_g'} = \left( \beta_{I_g'}^* / \| \beta_{I_g'}^* \|_2 \right) + u_{g'},
\]

where the norm of \( u_{g'} \) can be controlled in terms of the norm of the difference \( \beta_{I_g'} - \beta_{I_g} \).

We can also express the condition (22) in the following compact form over the entire support \( \mathcal{G}^* \)

\[
\tilde{z}_{S_g^c} = \beta_{S_g^c} + u_{S_g^c},
\]

where \( \beta_{S_g^c} \) is obtained by concatenating the direction vectors \( \beta_{I_{g'}^c} / \| \beta_{I_{g'}^c} \|_2 \) for all \( g' \in \mathcal{G}^* \) and similarly \( u_{S_g^c} \) is the result of stacking all \( \{ u_{g'} \}_{g' \in \mathcal{G}^*} \). With this, we have overall that for
each $g \notin G^*$, we can write
\[ \| z_{x_g} \|_2 \leq \frac{1}{\lambda_g} \left\| X_{x_g}^T X_{S_g^c} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} \beta_{S_g^c} \right\|_2 + \frac{1}{\lambda_g} \left\| X_{x_g}^T X_{S_g} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} \beta_{S_g^c} \right\|_2 + \frac{1}{\lambda_g} \left\| X_{x_g}^T \Pi_{S_g^c}(w) \right\|_2. \quad (23) \]

Now, by establishing that the right-hand side is strictly less than 1 for each $g \notin G^*$, we ensure no “spurious” groups will be identified by the group Lasso procedure. This strategy is central to the proof of Theorem [11.2] which employs concentration theory arguments to control the terms in the above upper bound. The rest of the current section describes the proof of this theorem; we relegate the (simpler) proof of Theorem [11.1] to the supplementary material.

C. Bounding the Terms

In addition to the well-conditioning event $E_1$, we condition also on the event that
\[ E_2 := \left\{ \left\| X_{S_g}^T X_{S_g^c} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} \beta_{S_g^c} \right\|_2 \leq \frac{\lambda_n}{4}, \forall g \notin G^* \right\}, \]
for the specific choice of
\[ \gamma = \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}} \cdot \frac{e_4}{\sqrt{d_{\max} \cdot \log p}}, \]
where $e_4$ is a positive constant (independent of problem parameters) that satisfies
\[ e_4 \leq \frac{1}{8 \sqrt{2(1 + 4 \log 2)}}, \]
as required later in the proof. When this event holds, (shown to be the case, with high probability, via Lemma [V.3] in the supplementary material), we are assured that blocks over the true group-level support $G^*$ are distinct enough from the remaining blocks.

Now, conditioned on the events $E_1$ and $E_2$, to prove the strict dual feasibility condition we will show that for any $g \notin G^*$, each of the terms appearing in the upper bound in (23) can be further bounded (e.g. by the constant 1/4) under the assumptions $M_2$ and $M_3$ of our statistical model. To better organize the proof, we also define the three following probabilistic events, which correspond to the terms of the upper bound in (23):
\[ E_3 := \left\{ \left\| X_{x_g}^T X_{S_g^c} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} \beta_{S_g^c} \right\|_2 \leq \frac{\lambda_g}{4}, \forall g \notin G^* \right\}, \]
\[ E_4 := \left\{ \left\| X_{x_g}^T X_{S_g} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} u_{S_g^c} \right\|_2 \leq \frac{\lambda_n}{4}, \forall g \notin G^* \right\}, \]
\[ E_5 := \left\{ \left\| X_{x_g}^T \Pi_{S_g^c}(w) \right\|_2 \leq \frac{\lambda_n}{4}, \forall g \notin G^* \right\}. \]

Lemmas [V.3], [V.5], and [V.6] below describe conditions under which these events each hold with high probability. With these, the probabilistic guarantee of the strict dual feasibility condition will naturally follow using a simple union bound argument.

1) Event $E_3$: The following lemma provides a condition under which the event $E_3$, which corresponds to the first term of the upper bound (23), holds with high probability. The proof of the Lemma is in the supplementary material.

Lemma V.3. Suppose the group-level support $G^*$ is given such that the events $E_1$ and $E_2$ hold for the sub-dictionary $X_{S_g^c}$ of the dictionary $X \in \mathbb{R}^{n \times p}$. Then assuming $\beta_{S_g^c}$ is a random vector generated according to the statistical model assumptions $M_2$ and $M_3$ described earlier we have that
\[ \Pr \left( \bigcup_{g \notin G^*} \left\{ \left\| X_{x_g}^T X_{S_g^c} (X_{S_g}^T X_{S_g})^{-1} A_{S_g^c} \beta_{S_g^c} \right\|_2 > \frac{\lambda_n}{4} \right\} \right) \leq 2p^{-4 \log 2}. \quad (24) \]

2) Event $E_4$: Next, we derive conditions under which the event $E_4$, which is associated with the second term of the upper bound in (23), holds with high probability. In order to show this, we leverage Lemma [V.2] to control the size of the $\{u_g\} \forall g \in G^*$, vectors and in turn the size of the $\{h_g, g' \in G^*\}$ vectors. Since the upper bound in (24) for $h_g, g' \in G^*$, is in terms of the noise-related terms $\left\| X_{x_g}^T w \right\|_2$ and $\left\| X_{x_g}^T w \right\|_2$, we will start by providing probabilistic bounds on these quantities.

Lemma V.4. Suppose the group-level support $G^*$ is fixed, and $w \sim \mathcal{N}(0, \sigma^2 I_{n \times n})$. Then there exists a universal constant $c_7 \in (3, 7)$ for which the following holds: for any $t \geq 1$ and
\[ \epsilon \geq \frac{1}{c_7 d_{\min}} \left( 1 + \mu_1(X) \right) \cdot \log (p^t |G^*|), \]
the following events hold simultaneously with probability at least $1 - 2 p^{-t} - 2 \exp \left(-c_7^2 d_{\min}^2 / 2\right)$:
\[ \|X_{S_g^c}^T w\|_2 \leq \sigma(1 + \epsilon) \sqrt{d_{S_g^c}} \]
\[ \bigcap_{g' \in G^*} \left\{ \|X_{S_g^c}^T w\|_2 \leq \sigma(1 + \epsilon) \sqrt{d_{g'}} \right\}. \]

The proof of this Lemma is in the supplementary material. Now, by using this lemma together with Lemma [V.2] we obtain the following result on the norm of the difference vectors $h_g = \beta_{x_g} - \beta_{x_{g'}}$ for $g' \in G^*$.

Corollary V.1. Suppose the group-level support $G^*$ is given such that the event $E_1$ holds. Furthermore, assume that $w \sim \mathcal{N}(0, \sigma^2 I_{n \times n})$. There exists a universal finite constant $c_7 > 0$ for which the following holds: for any $t \geq 1$ and
\[ \epsilon \geq \frac{1}{c_7 d_{\min}} \left( 1 + \mu_1(X) \right) \cdot \log (p^t |G^*|), \]
we have that
\[ \|h_{g'}\|_2 \leq \sigma(1 + \epsilon) \left( \sqrt{d_{S_g^c}} + \sqrt{d_{g'}} \right) + \lambda_{g'} + \|\lambda_{g'}\|_2 \quad (25) \]
holds simultaneously for every $g' \in G^*$ with probability at least $1 - 2 p^{-t} - 2 \exp \left(-c_7^2 d_{\min}^2 / 2\right)$.

Leveraging the above Corollary, we are able to bound the norm of the second term of the upper bound in (23).
Lemma V.5. Suppose the group-level support \( G^* \) is given such that both events \( E_1 \) and \( E_2 \) hold for the sub-dictionary \( X_{S^0} \) of \( X \). Furthermore, assume \( w \sim \mathcal{N}(0, \sigma^2 I_{n\times n}) \) and that \( \|\beta_{2g}'\|_2 \geq t_2\|h_g'\|_2 \) holds for all \( g' \in G^* \), for some value of \( t_2 \) satisfying
\[
t_2 \geq \max\left\{ 2, c_8 \frac{\|G^*\|}{d_{\max} \log p} \right\},
\]
where \( c_8 \) is a universal constant which satisfies \( c_8 \geq 4/\sqrt{2(1 + 4 \log 2)} \). Then, we have that for all \( g \notin G^* \)
\[
\frac{1}{\lambda_g} \left\| X_{I_g}^T X_{S^0} (X_{S^0}^T X_{S^0})^{-1} A_{S^0} u_{S^0} \right\|_2 \leq \frac{1}{4}.
\]
Putting the result of Corollary V.1 together with the above Lemma and also setting \( c_8 = 2 > 4/\sqrt{2(1 + 4 \log 2)} \), we immediately obtain the following.

Corollary V.2. Suppose the group-level support \( G^* \) is given such that both events \( E_1 \) and \( E_2 \) hold for the sub-dictionary \( X_{S^0} \) of \( X \). Furthermore, assume \( w \sim \mathcal{N}(0, \sigma^2 I_{n\times n}) \), \( \epsilon \) is set as in Theorem II.1 and for all \( g' \in G^* \)
\[
\|\beta_{2g'}\|_2 \geq \max \left\{ 2, 2 \frac{\|G^*\|}{d_{\max} \cdot \log p} \right\} \times \left\{ \sigma (1 + \epsilon) \left( \sqrt{d_g' + 1} + \lambda_g' + \|\lambda_{g'}\|_2 \right) \right\},
\]
Then
\[
\left\| X_{I_g}^T X_{S^0} (X_{S^0}^T X_{S^0})^{-1} A_{S^0} u_{S^0} \right\|_2 \leq \frac{\lambda_g}{4},
\]
holds with probability at least \( 1 - 2p^{-t} - 2 \exp \left( -c_7 \epsilon^2 d_{\max}^2 / 2 \right) \).

4) Completing the Proof of Theorem II.2. Now we can put all the proof ingredients together to complete the overall argument. Let \( E \) denote the event that the group-level support \( G^* \) is exactly recovered via solving the group Lasso problem \( [2] \). As explained in Section V.C to ensure \( E \) happens our approach is to first find conditions that guarantee \( E_1 \) and \( E_2 \) hold true; then conditioned on those two events, we impose extra assumptions to ensure \( E_3, E_4 \) and \( E_5 \) occur as well. Using a union bound then implies the following upper bound:
\[
\Pr(E^c) \leq \Pr(E_1^c) + \Pr(E_2^c) + \Pr(E_3^c | E_1 \cap E_2) + \Pr(E_4^c | E_1 \cap E_2) + \Pr(E_5^c | E_1 \cap E_2).
\]

The rest of the proof briefly reviews conditions under which the probability terms on the right-hand side of the above inequality are appropriately bounded. First, by Lemma VI.2, we know that if there exist positive constants \( c_0 \) and \( c_1 \) such that \( \mu_1(X) \leq c_0 \), \( \mu_B(X) \leq c_1 / \log p \), and
\[
s \leq \min \left\{ c_2 \frac{\mu_B^2(X)}{\|X\|_{2 \rightarrow 2} \log p}, c_3 G \right\},
\]
where \( c_2 \) and \( c_3 \) are such that
\[
(48 c_1 + 6 \sqrt{2(c_2 + c_3) + 2 c_3 + 3c_0}) \leq \frac{1}{4},
\]
then \( \Pr(E_1^c) \leq 2p^{-4 \log 2} \). Notice that the relationship in (32) requires \( c_0 \) and \( c_1 \) to be such that \( 48 c_1 + 3 c_0 \leq 1 / 4 \). Given this condition, a valid choice for \( c_2 \) and \( c_3 \) that satisfies (32) is
\[
c_2 = c_3 \leq \left[ \frac{3}{2} \left( \frac{1}{2} - 4 c_0 - 48 c_1 \right) - 3 \right].
\]
Second, utilizing Lemma VI.3 with \( \lambda_g = 4 \sigma (1 + \epsilon) \sqrt{d_g} \),
\[
g := \frac{\lambda_{\min}}{\lambda_{\max}} = \frac{c_4}{\sqrt{d_{\max} \cdot \log p}},
\]
and
\[
c_4 = \frac{1}{8 \sqrt{2(1 + 4 \log 2)}},
\]
it follows that as long as
\[
\mu_B(X) \leq \frac{d_{\min}}{d_{\max}} c_5 \frac{c_6}{\log p} \text{ and } s \leq \frac{d_{\min}}{d_{\max}} G \frac{c_6^2}{\mu_B^2(X) \log p},
\]
with constants \( c_5 \) and \( c_6 \) chosen such that \( 4 \sqrt{2} c_5 + c_6 \leq c_4 / 2 \), then \( \Pr(E_2^c) \leq 2p^{-4 \log 2} \). In particular,
\[
c_5 = 0.001, \ c_6 = 0.01
\]
3To show the inequality, notice that for two probabilistic events \( A \) and \( B \), we can write \( A^c \cup B^c = A^c \cup (B^c \cap A) \). Setting \( A = E_3 \cap E_4 \) and \( B = E_3 \cap E_4 \cap E_5 \) and using the fact that \( \Pr(A^c \cup B^c) \leq \Pr(A^c) + \Pr(B^c \cap A) \leq \Pr(A^c) + \Pr(B^c | A) \) concludes the proof.
are valid choices here. To express the upper bounds in (31) and (33) on the maximum possible group-sparsity level $s$ more compactly, notice that since $d_{\min}/d_{\max}^2 \leq 1$, we have that
\[ s \leq \frac{d_{\min}}{d_{\max}^2} \cdot \min\{c_2, c_2\} \]
with $s \leq c_2 G/((\|X\|_{2 \rightarrow 2}^2 \cdot \log p))$ guarantees the requirements on $s$ are met. Similarly, $\sqrt{d_{\min}/d_{\max}^2} \leq 1$ implies that imposing
\[ \mu_B(X) \leq \sqrt{\frac{d_{\min}}{d_{\max}^2}} \cdot \frac{c_2}{\log p} \]
will ensure the block coherence parameter meets $\mu_B(X) \leq c_1/\log p$ for $c_1 \leq 0.001$.

Third, Lemma V.3 implies that $\Pr(E^c_3|E_1 \cap E_2) \leq 2p^{-4\log^2 2}$. Fourth, Corollary V.2 with $\lambda_g = 4\sigma(1 + \epsilon)\sqrt{d_g}$ and $t = 4\log 2$, implies that as long as for
\[ \epsilon \geq \sqrt{\frac{1 + \mu_1(X)}{\mu_2(X)}} \log (G \cdot p^{1+2}) \]
we have
\[ \|\beta_{\tilde{g}}\|_2 \geq 10\sigma(1 + \epsilon) \left( \sqrt{\lambda_g} + \sqrt{\lambda_g} \right) \times \max\left\{1, \sqrt{\frac{s}{d_{\max} \cdot \log p}} \right\} \]
for every $\tilde{g} \in G^*$, then $\Pr(E^c_3|E_1 \cap E_2) \leq 2p^{-4\log^2 2} + 2\exp(-c_7\epsilon^2d_g^2/2)$. Finally, by Lemma V.6 we have that $\Pr(E^c_3|E_1 \cap E_2) \leq 2p^{-4\log^2 2}$ whenever $\lambda_g = 4\sigma(1 + \epsilon)\sqrt{d_g}$ for all $g \notin G^*$. Therefore, under the stated conditions of the theorem we have
\[ \Pr(E^c) \leq 10p^{-4\log^2 2} + 2\exp(-c_7\epsilon^2d_g^2/2) \leq 12p^{-2\log^2 2} \]
where the last inequality follows from the lower bound on $\epsilon$, namely that $\exp(-c_7\epsilon^2d_g^2/2) \leq p^{-2\log^2 2}$. Finally, since $c_7 > 3$, the choice of $\epsilon$ in the theorem statement is always above the threshold in (34).

VI. DISCUSSION AND CONCLUSIONS

In this paper we examined recovery of group-sparse signals from low-dimensional noisy linear measurements using the group Lasso estimation procedure, motivated by a defect localization application in non-destructive evaluation. We established practically relevant group-level support recovery guarantees for non-asymptotic regimes in terms of the block coherence parameter, and validated our analytical results via simulation on both synthetic data, as well as simulated data generated according to a realistic model for our motivating defect localization application.

Our main theoretical contribution improved upon existing results for support recovery via group Lasso in settings where the dictionary matrix is fixed. As with other results in the sparse inference literature, the use of a random signal model here allows us to break the “square root” bottleneck from which deterministic coherence-based analyses are known to suffer.

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A. Well-Conditioning of the Selected Sub-Dictionary

In this section we establish a general guarantee for the well-conditioning of the sub-dictionaries selected from an arbitrary dictionary $X$ (assumed, throughout, to have unit normed columns) in terms of its intra- and inter-block coherence parameters, $\mu_I(X)$ and $\mu_B(X)$, given in Definition II.1.

We begin by restating (and slightly expanding) the notation needed to develop our analysis. A predefined column-wise partition of the dictionary $X \in \mathbb{R}^{n \times p}$ into $G$ column blocks is given by $X = [X_{g1} \, X_{g2} \, \cdots \, X_{gG}]$, where $I_g$ denotes the indices for columns belonging to block $g$, $X_{gi} \in \mathbb{R}^{n \times d_g}$ denotes the corresponding dictionary block, and $\sum_{g \in G} d_g = p$.

Letting $G \subset [G]$ be a set of group indices and $S_G = \cup_{g \in G} I_g$ be the corresponding column indices, $X_{S_G}$ will denote the sub-dictionary whose columns are in $S_G$. With this notation, we establish the following Lemma.

**Lemma VI.1.** Suppose $X$ denotes an arbitrary dictionary which is column-wise partitioned as above. For any sub-dictionary $X_{S_G}$ of $X$ comprising $|G|$ blocks, we have

$$\|X_{S_G}^T X_{S_G} - I_{d_G \times d_G}\|_{2 \rightarrow 2} \leq \mu_I(X) + (|G| - 1)\mu_B(X),$$

where $d_G = \sum_{g \in G} d_g$ is the total number of columns of $X_{S_G}$.

**Proof:** Let $\lambda$ denote an arbitrary eigenvalue of the hollow Gram matrix associated with $X_{S_G}$; namely assume that for $H := X_{S_G}^T X_{S_G} - I_{d_G \times d_G}$ there exists some (non-zero) eigenvector $x \in \mathbb{R}^{d_G}$ such that we have $H x = \lambda x$. Furthermore, let $x = [x_1 \, x_2^T \, \cdots \, x_{|G|}^T]^T$ denote the partitioning of $x$ with respect to the blocks incorporated in $H$, with $x_i \in \mathbb{R}^{d_i}$, for $i = 1, 2, \cdots, |G|$. Since $x$ is a non-zero vector, there must exist a block $g_i$ with maximum Euclidean norm, i.e. we can find $1 \leq i \leq |G|$ such that $\|x_i\|_2 = \max_{1 \leq j \leq |G|}\|x_j\|_2$.

Letting $H_i$ denote the $d_{g_i} \times d_{g_i}$ row sub-matrix of $H$ corresponding to the $i$-th block of $X$, we have $\lambda = H_i x_i = \sum_{j \neq i} H_{ij} x_j$. Taking the Euclidean norm of both sides and using the triangle inequality leads to

$$\lambda \|x_i\|_2 \leq \sum_{j = 1}^{\max(1, |G|)} \|H_{ij}\|_2 \|x_j\|_2 \leq \sum_{j = 1}^{\max(1, |G|)} \|H_{ij}\|_2 \|x_j\|_2,$$

where the last inequality uses $\|x_i\|_2 \geq \|x_j\|_2$ for any $j$. Since

$$\sum_{j = 1}^{\max(1, |G|)} \|H_{ij}\|_2 \|x_j\|_2 = \|H_{i1}\|_2 \|x_1\|_2 + \sum_{j \neq i} \|H_{ij}\|_2 \|x_j\|_2,$$

where $\|H_{i1}\|_2 \|x_1\|_2 = \|X_{S_G}^T X_{S_G} - I_{d_G \times d_G}\|_{2 \rightarrow 2} \leq \mu_I(X)$ by the definition of the intra-block coherence, and $\|H_{ij}\|_2 \|x_i\|_2 \leq \mu_B(X)$ for $i \neq j$ by the definition of the inter-block coherence $\mu_B(X)$, it follows that $\lambda \leq \mu_I(X) + (|G| - 1)\mu_B(X)$.

Notice that since

$$\|X_{S_G}^T X_{S_G} - I_{d_G \times d_G}\|_{2 \rightarrow 2} = \max \left\{ \mu^2 \max(X_{S_G}) - 1, 1 - \sigma^2_{\min}(X_{S_G}) \right\},$$

the implication of having $\|X_{S_G}^T X_{S_G} - I_{d_G \times d_G}\|_{2 \rightarrow 2} \leq \delta$ for some $\delta \in [0, 1]$ is that the singular values of $X_{S_G}$ lie within the interval $[\sqrt{T - \delta}, \sqrt{T + \delta}]$. Now according to the above lemma, if

$$|G| \leq \frac{\delta - \mu_I(X)}{\mu_B(X)} + 1$$

with the upper bound being strictly positive, then the well-conditioning of the sub-dictionary $X_{S_G}$ will be implied.

Using the statistical modeling assumption $M_3$ of [20], and adapting Theorem 1 of [20], this result, stated in the above lemma, is similar to the following.

**Lemma VI.2 (Theorem 1 of [20]).** Suppose that the $n \times p$ dictionary $X = [X_{g1} \, X_{g2} \, \cdots \, X_{gG}]$ satisfies $\mu_I(X) \leq c_0$ and $\mu_B(X) \leq c_1 / \log p$, with positive constants $c_0$ and $c_1$. Suppose that $G^*$ is a subset of size $s := |G^*|$ of the set $|G| = \{1, 2, \cdots, G\}$, drawn uniformly at random. Then, provided

$$s \leq \min \left\{ \frac{c_2}{\mu_B(X) \log p}, \frac{c_3 G}{\|X\|_{2 \rightarrow 2}^2 \log p} \right\}$$

for some positive constants $c_2$ and $c_3$ that only depend on $c_0$ and $c_1$, we have that the singular values of the sub-dictionary $X_{S_G}$ lie within the interval $[\sqrt{1/2}, \sqrt{3/2}]$ (equivalently, $\|X_{S_G}^T X_{S_G} - I_{d_G \times d_G}\|_{2 \rightarrow 2} \leq 1/2$) with probability at least $1 - 2p^{-4 \log 2}$.

The above lemma is essentially identical to Theorem 1 of [20], with the difference in (39) we have replaced the so-called quadratic-mean block coherence $\mu_B(X)$ in [20] by $\mu_B(X)$. This change yields a slightly more restrictive condition, since $\mu_B(X) \geq \mu_B(X)$, but it does not cause a significant difference in the context of our demixing problem. As a consequence of this lemma, it directly follows that under the conditions above, $\|X_{S_G}^T X_{S_G}^{-1}\|_{2 \rightarrow 2} \leq 2$ with high probability. Finally, we note that in the above Lemma, $c_2$ and $c_3$ are selected such that $(48c_1 + 6\sqrt{2}(c_2 + c_3) + 2c_3 + 3c_0) \leq 1/4$ holds true. This limits the allowable ranges of $c_0$ and $c_1$, as well.

The following lemma, which provides a probabilistic upper bound on the quantity $\|X_{S_G}^T X_{S_G}^{-1}\|_{2 \rightarrow 2} = \max_{g \in G^*} \|X_{S_G}^T X_{S_G}^{-1}\|_{2 \rightarrow 2}$, is also useful in the proof of the strict dual feasibility condition.

**Lemma VI.3.** Suppose the $n \times p$ dictionary $X$ is column-wise partitioned into $G$ blocks as $X = [X_{g1} \, X_{g2} \, \cdots \, X_{gG}]$. Assume further that $G^*$ is a subset of size $s := |G^*|$ of the...
set $\{G\} = \{1, 2, \ldots, G\}$, which is drawn uniformly at random. Then for $\gamma > 0$ it follows that
\[
\Pr \left( \left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1} > \gamma \right) 
\leq 2 \left\{ \frac{\mu_B(X)}{\gamma} \cdot \left( 4\sqrt{2\log p} + \sqrt{s} \right) \right\}^{4\log p}.
\]
In particular, for the choice
\[
\gamma := \frac{\lambda_{\min}}{\lambda_{\max}} \cdot \frac{c_4 \cdot \sqrt{d_{\max} \cdot \log p}}{d_{\max} \cdot \log p},
\]
where $c_4$ is an arbitrary positive constant, it holds that $\left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1} \leq \gamma$, with probability at least $1 - 2 p^{-4\log^2}$, as long as
\[
\mu_B(X) \leq \frac{\lambda_{\min} \cdot 1}{\lambda_{\max} \cdot \sqrt{d_{\max} \cdot \log p}} \cdot \min \left\{ \frac{c_5 \cdot \sqrt{\log p}}{c_6 \cdot \sqrt{s}} \right\},
\]
where $c_5$ and $c_6$ are small enough universal constants that satisfy $4\sqrt{2} c_5 + c_6 \leq c_4/2$.

**Proof:** The proof first utilizes Lemma A.5 in [20] to show that
\[
\Pr \left( \left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1} > \gamma \right) \leq 2\gamma^q \mathbb{E} \left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1}^q 
\leq 2\gamma^q \mathbb{E} \left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1}^q 
\leq 2\gamma^q \left( 2^{1.5} \sqrt{q} \mu_B(X) + \sqrt{s} \mu_B(X)^q \right),
\]
where $q := 4\log p$, the first inequality is due to the Markov inequality and a Poissonization argument (a similar argument is used in the proof of Theorems 1 and 2 in [20]), the second inequality is due to the fact that $X_{(S_0)^c}$ is a sub-dictionary of $X$, and the third inequality is by Lemma A.5 in [20] along with the fact that $\mu_B(X) \geq \mathbb{P}_B(X)$. Rearranging the terms completes the proof of the first part.

Now, setting $\gamma = c_4/(\lambda_{\min} \cdot \sqrt{d_{\max} \cdot \log p})$ will convert the upper bound of (40) into
\[
\Pr \left( \left\| X_{S_0}^T X_{(S_0)^c} \right\|_{B,1} > \gamma \right) 
\leq 2 \left\{ \frac{\mu_B(X)}{\lambda_{\min} \cdot \sqrt{d_{\max} \cdot \log p}} \cdot \left( 4\sqrt{2\log p} + \sqrt{s} \right) \right\}^{4\log p} 
\leq 2 \left( 4\sqrt{2} \frac{c_5 \cdot \sqrt{\log p}}{c_6 \cdot \sqrt{s}} \right)^{4\log p} \leq 2 p^{-4\log^2}
\]
where the second inequality is by the condition (41) on $\mu_B(X)$ and the third one holds since $4\sqrt{2} c_5/c_6 + c_6/c_4 < 1/2$.

**B. Proof of Lemma [12]**

Using the relationship established in [19], and defining $S_g \in \mathbb{R}^{d_g \times d_{G^*}}$ as the selector matrix which selects indices corresponding to the block $g \in G^*$, we have that for each $g \in G^*$,
\[
\beta_{I_g} = \beta_{I_g}^* + S_g (X_{S_0}^T X_{S_0}^{-1}) (X_{S_0}^T w - \Lambda_{S_0} \hat{z}_{S_0}).
\]
We use an implication of [16] (that essentially follows from Weyl’s inequality), writing $(X_{S_0}^T X_{S_0}^{-1})^{-1} = I_{d_g \times d_{G^*}} + \Delta$, where $\|\Delta\|_{2 \to 2} \leq 1$, and note that
\[
\|h_g\|_2 = \|S_g (X_{S_0}^T w - \Lambda_{S_0} \hat{z}_{S_0}) + S_g \Delta (X_{S_0}^T w - \Lambda_{S_0} \hat{z}_{S_0})\|_2 
\leq \|X_{S_0}^T w\|_2 + \|\lambda_g \hat{z}_{S_0}\|_2 
+ \|S_g\|_{2 \to 2} \|\Delta\|_{2 \to 2} \|X_{S_0}^T w\|_2 + \|\Lambda_{S_0} \hat{z}_{S_0}\|_2.
\]
The first result follows from the facts that $\|\Delta\|_{2 \to 2} \leq 1$, and $\|S_g\|_{2 \to 2} \leq 1$, and that
\[
\|\Lambda_{S_0} \hat{z}_{S_0}\|_2 = \left( \sum_{e \in G^*} \hat{\lambda}_e^2 \|\hat{z}_{S_0}\|_2^2 \right)^{1/2} \leq \|\hat{\lambda}_{G^*}\|_2,
\]
where we have used the definition of $\hat{\lambda}_{G^*}$, and the subgradient condition on each group of $\hat{z}$.

The second result follows from a similar argument as that given for Lemma 3 in [11]. We include a proof here for completeness. First notice that $\|h_g\|_2 \leq \frac{1}{2} \|\beta_{I_g}^*\|_2$ implies $\beta_{I_g'} \neq 0$ and so $\hat{z}_{I_g'} = \frac{\beta_{I_g'}}{\|\beta_{I_g'}\|_2}$. Given this, we have that
\[
u_{g'}' = \hat{z}_{I_g'} - \frac{\beta_{I_g'}^*}{\|\beta_{I_g'}^*\|_2} = \frac{\beta_{I_g'} - \beta_{I_g'}}{\|\beta_{I_g'}\|_2}.
\]
Now, since the function $f(\beta, h) := 1/\|\beta + h\|_2$, for $\beta \neq 0$, is differentiable with respect to the vector $h$, with gradient $\nabla_h f(\beta, h) = -\frac{\beta + h}{2\|\beta + h\|_2^2}$. By the mean value theorem, there must exist a scalar $\alpha \in [0, 1]$ such that
\[
\frac{1}{\|\beta + h\|_2} - \frac{1}{\|\beta\|_2} = f(\beta, h) - f(\beta, 0) = -\frac{\alpha \beta}{2\|\alpha \beta\|_2}.
\]
This, together with the last expression in the above, implies
\[
u_{g'} = \beta_{I_g'} \left( -\frac{\beta_{I_g'} + \alpha h_{g'}}{2\|\beta_{I_g'} + \alpha h_{g'}\|_2^2} \right) + \frac{h_{g'}}{\|\beta_{I_g'} + h_{g'}\|_2^2}.
\]
Using the Cauchy-Schwartz inequality then yields
\[
\|\nu_{g'}\|_2 \leq \frac{\|\beta_{I_g'}\|_2 \cdot \|h_{g'}\|_2 + \|h_{g'}\|_2}{2\|\beta_{I_g'} + \alpha h_{g'}\|_2^2} \leq \frac{\|h_{g'}\|_2}{\|\beta_{I_g'}\|_2^2} + \frac{\|h_{g'}\|_2}{\|\beta_{I_g'} + h_{g'}\|_2} \leq \frac{4\|h_{g'}\|_2}{\|\beta_{I_g'}\|_2^2}.
\]
where the last two inequalities follows from the fact that since 
\[ \|h_x\|_2 \leq \frac{1}{2} \|\beta_{z_x}'\|_2, \]
we have 
\[ \|\beta_{z_x}' + \alpha h_x\|_2 \geq \|\beta_{z_x}'\|_2 - \alpha \|h_x\|_2 \]
\[ \geq \|\beta_{z_x}'\|_2 - \|h_x\|_2 \geq \frac{1}{2} \|\beta_{z_x}'\|_2, \]
for any \( \alpha \in [0, 1] \).

C. Proof of Lemma V.3

The proof essentially follows the last step in the proof of 
Theorem 2 in [20]. First notice that the event in (24) is 
equivalent to the event that 
\[ \left\| \Lambda^{-1}(S_g) X^T(S_g') X S_g (X S_g X^T S_g) - 1 \Lambda S_g^{\top} \right\|_2 \leq \frac{1}{4}, \]
where for a block-wise partitioned arbitrary vector 
\( a = \left[ a_1^T, a_2^T, \cdots, a_T^T \right]^T \), \( \|a\|_{2,\infty} \) denotes the maximum 
Euclidean norm of its constituent blocks, i.e. \( \|a\|_{2,\infty} := \max_{g \in [G]} \|a_g\|_2 \). Furthermore, since \( \|a\|_{2,\infty} \leq \sqrt{d_{\max}} \|a\|_{\infty} \)
with \( d_{\max} \) denoting the maximum block size, it is sufficient to show that 
\[ v := \left\| \Lambda^{-1}(S_g) X^T(S_g') X S_g (X S_g X^T S_g) - 1 \Lambda S_g^{\top} \right\|_{2,\infty} \]
\[ \leq \frac{1}{4 \sqrt{d_{\max}}}, \]
holds with probability at least \( 1 - 2p^{-4 \log 2} \).

Letting \( v_{g,j} := \frac{1}{\sqrt{\gamma_g}} \left( X S_g X^T S_g - 1 \right) \Lambda S_g^{\top} \beta_{z_g}', \)
where \( x_{g,j} \) denotes the \( j \)-th column in the block sub-dictionary 
\( X_{S_g} \in \mathbb{R}^{n \times d_{S_g}}, \) with \( j \in [d_{S_g}] \), we may write \( v = \max_{g \in G^*, \ j \in [d_g]} |v_{g,j}|. \)
Moreover, defining the vector \( u_{g,j} := \frac{1}{\lambda_g} \left( X S_g X^T S_g - 1 \right) x_{g,j} \) for \( g \notin G^* \) and \( j \in [d_g] \), we can express each \( v_{g,j} \) as an inner product of the form 
\[ v_{g,j} = u_{g,j}^{\top} \Lambda S_g^{\top} \beta_{z_g}'. \]

Notice that in the current lemma we are proceeding under the
condition that the selected block support \( G^* \) is fixed, and therefore the only random vector that appears on the right-hand
side of the last expression is \( \beta_{z_g}' \). Now, by utilizing the
definition of \( \beta_{z_g}' \) and that \( u_{g,j} \) is the concatenation of block
vectors \( u_{g,j,g'} \in \mathbb{R}^{d_{g'}} \) (with \( g' \in G^* \)) corresponding to rowwise
blocks in the partition of \( \left( X S_g X^T S_g - 1 \right) \), we can express \( v_{g,j} \) as 
\[ v_{g,j} = \sum_{g' \in G^*} \lambda_{g'} u_{g,j,g'}^{\top} \left( \beta_{z_g}' / \|\beta_{z_g}'\|_2 \right). \]
Since \( v_{g,j} \) is now expressed in the form of the summation of random variables, its absolute value can be bounded by
utilizing probabilistic concentration tools. To do so, first we apply the Cauchy-Schwartz inequality to every term in the
summation to yield 
\[ \lambda_{g'} u_{g,j,g'}^{\top} \left( \beta_{z_g}' / \|\beta_{z_g}'\|_2 \right) \leq \lambda_{g'} \|u_{g,j,g'}\|_2, \]
where we also employed the fact that \( \beta_{z_g}' / \|\beta_{z_g}'\|_2 \) is a
unit-norm vector. Then since \( \mathbb{E} \left[ \|\beta_{z_g}' / \|\beta_{z_g}'\|_2 \|v_{g,j}\|_2 \right] = 0 \) for every \( g' \in G^* \),
Hoefding’s inequality implies 
\[ \Pr \left( |v_{g,j}| \geq t \right) \leq 2 \exp \left( \frac{-t^2}{2 \sum_{g' \in G^*} \lambda_{g'}^2 \|u_{g,j,g'}\|_2^2} \right) \]
\[ = 2 \exp \left( \frac{-t^2}{2 \|\Lambda S_g^{\top} u_{g,j}\|_2^2} \right). \]
Now, choosing \( \kappa \geq \max_{g \notin G^*, \ j \in [d_g]} \|\Lambda S_g^{\top} u_{g,j}\|_2 \) and applying
a union bound we obtain \( \Pr \left( v \geq t \right) \leq 2 \exp \left( -t^2 / 2 \kappa^2 \right) \). 
To find an appropriate choice for \( \kappa \) that is explicitly in terms of
our defining parameters, we explore upper bounds on \( u_{g,j} \) as follows:
\[ \|u_{g,j}\|_2 \leq \frac{1}{\lambda_g} \left( \left( X S_g X^T S_g - 1 \right) \right) \left( X S_g X^T S_g - 1 \right) \left( X S_g X^T S_g - 1 \right) \]
\[ \leq \frac{1}{\lambda_g} \left( X S_g X^T S_g - 1 \right) \left( X S_g X^T S_g - 1 \right) \left( X S_g X^T S_g - 1 \right) \]
\[ \leq \frac{1}{\lambda_g} \left( X S_g X^T S_g - 1 \right) \left( X S_g X^T S_g - 1 \right) \left( X S_g X^T S_g - 1 \right) \]
Now, given that the selected sub-dictionary is well-conditioned, i.e. \( \left\| \left( X S_g X^T S_g - 1 \right) \right\|_2 \leq 2 \), as guaranteed by
\( E_1 \), and moreover that \( \left\| \left( X S_g X^T S_g - 1 \right) \right\|_2 \leq \gamma \), as guaranteed by
\( E_2 \), we obtain that \( \|u_{g,j}\|_2 \leq 2 / \lambda_g \leq \gamma / \lambda_{\min} \). Therefore
an appropriate choice for \( \kappa \) is \( \kappa = 2 \gamma (\lambda_{\min} / \lambda_{\max}) \) (also
utilizing the fact that \( \|\Lambda S_g^{\top} u_{g,j}\|_2 \leq \lambda_{\max} \|u_{g,j}\|_2 \))
Therefore, setting \( t = 1 / (4 \sqrt{d_{\max}}) \) and
\[ \gamma = \frac{\lambda_{\min}}{\lambda_{\max}} \cdot \frac{c_4}{\sqrt{d_{\max} \cdot \log p}} \]
implies 
\[ \Pr \left( v \geq \frac{1}{4 \sqrt{d_{\max}}} \right) \leq 2p \cdot \exp \left( \frac{-1}{32 \kappa^2 d_{\max}} \right) \]
\[ = 2p \cdot \exp \left( \frac{-1}{128 d_{\max} (\lambda_{\max} / \lambda_{\min})^2} \right) \]
\[ = 2p \left( 1 - \frac{1}{128 c_4} \right) \]
Thus, assuming \( c_4 \) satisfies \( 1 - \frac{1}{128 c_4} \leq -4 \log 2 \), we have that the
last expression on the right hand-side is less than \( 2p^{-4 \log 2} \),
which completes the proof.

D. Proof of Lemma V.4

We establish that the events \( \left\{ \|X_{S_g} w\|_2 \leq \sigma (1 + \epsilon) \sqrt{d_{g'}} \right\} \)
and \( \left\{ \|X_{S_g} w\|_2 \leq \sigma (1 + \epsilon) \sqrt{d_{g'}}, \forall g' \in G^* \right\} \)
hold with the specified probability using the Hanson-Wright Inequality. We state a useful (for our purposes) version of this inequality here
as a lemma.
Lemma VI.4 (Hanson Wright Inequality; From Thm. 2.1 of [48]). Let $A$ be a fixed matrix, and let $x$ be a vector whose elements are iid $\mathcal{N}(0,1)$ random variables (which are thus subgaussian). Then, there exists a finite constant $c_7 > 0$ such that for any $\tau > 0$,

$$\Pr \left( \|Ax\|_2 - \|A\|_F > \tau \right) \leq 2 \exp \left( - \frac{c^2 \tau^2}{\|A\|_2^2} \right). \quad (44)$$

First fix any $g' \in G^*$ and note that

$$\Pr \left( \|X_{T_g}^T \cdot w\|_2 > \sigma(1 + \epsilon) \sqrt{d_{g'}} \right) \leq \Pr \left( \|X_{T_g}^T \cdot w\|_2 - \sigma \sqrt{d_{g'}} \geq \epsilon \sigma \sqrt{d_{g'}} \right) \leq 2 \exp \left( - \frac{c_7 \epsilon^2 d_{g'}}{1 + \mu_I(X)} \right),$$

where the second inequality follows directly from the Hanson-Wright inequality (specifically, setting $x = w/\sigma$ and $A = \sigma X_{T_g}^T$, and noting that $\|A\|_F = \sigma \sqrt{d_{g'}}$ and $\|A\|_2 \leq \sigma \sqrt{1 + \mu_I(X)}$). Next, note that

$$\Pr \left( \|X_{T_g}^T \cdot w\|_2 > \sigma(1 + \epsilon) \sqrt{d_{g'}} \right) \leq \Pr \left( \|X_{T_g}^T \cdot w\|_2 - \sigma \sqrt{d_{g'}} \geq \epsilon \sigma \sqrt{d_{g'}} \right) \leq 2 \exp \left( - \frac{2c_7 \epsilon^2 d_{g'}}{3} \right).$$

Here, the second inequality follows again from the Hanson-Wright inequality, setting $x = w/\sigma$, and $A = \sigma X_{T_g}^T$, and noting that $\|A\|_F = \sigma \sqrt{d_{g'}}$ (since each row of $A$ is unit-norm) and $\|A\|_2 \leq \sigma \sqrt{3}/2$, which follows from the event $E_1$ in [18].

Thus, by a union bound, both of the stated claims hold, except in an event of probability no larger than $2 \exp \left( -\frac{c_7 \epsilon^2 d_{g'}}{3} \right) + 2 \sum_{g' \in G^*} \exp \left( -c_7 \epsilon^2 d_{g'}/(1 + \mu_I(X)) \right)$, which itself is upper-bounded by

$$2 \exp \left( - \frac{c_7 \epsilon^2 d_{g'}}{2} \right) + 2 |G^*| \exp \left( - \frac{c_7 \epsilon^2 d_{\min}}{1 + \mu_I(X)} \right),$$

where $d_{\min} := \min_{g' \in G} d_{g'}$. Finally, note that whenever

$$\epsilon \geq \sqrt{(1 + \mu_I(X)) \cdot \log (p' |G^*|)} \quad \frac{c_7 d_{\min}}{\epsilon},$$

for any $t \geq 1$, we have

$$2 |G^*| \exp \left( - \frac{c_7 \epsilon^2 d_{\min}}{1 + \mu_I(X)} \right) \leq 2 p'^{-t},$$

and the result follows.

E. Proof of Lemma [V3]

We begin by using the sub-multiplicativity property of the spectral norm to obtain

$$\frac{1}{\lambda_g} \|X_{I_g}^T \cdot X_{S_g} (X_{S_g}^T \cdot X_{S_g})^{-1} \cdot A_{S_g} u_{S_g}\|_2 \leq \frac{1}{\lambda_g} \|X_{I_g}^T \cdot X_{S_g}\|_{2 \to 2} \cdot \left\|X_{S_g}^T \cdot X_{S_g}^{-1}\right\|_{2 \to 2} \cdot \left\|A_{S_g} u_{S_g}\right\|_2 \leq 2 \frac{\gamma_1}{\lambda_g} \left\|A_{S_g} u_{S_g}\right\|_2 \leq 2 \frac{\lambda_{\max}}{\lambda_{\min}} \cdot \left\|u_{S_g}\right\|_2$$

where the second inequality follows since we assume $E_1$ and $E_2$ hold true (therefore $\|X_{S_g}^T \cdot X_{S_g}\|_{2 \to 2} \leq 2$ and $\|X_{I_g}^T \cdot X_{S_g}\|_{2 \to 2} \leq \|X_{I_g}^T \cdot X_{S_g}\|_{2 \to 2} \leq \|X_{S_g}^T \cdot X_{S_g}^{-1}\|_{2 \to 2}$ and the third inequality follows by the fact that $\|A_{S_g}\|_{2 \to 2} = \lambda_{\max}$ (and therefore $\|A_{S_g} u_{S_g}\|_2 \leq \lambda_{\max} \|u_{S_g}\|_2$)). In addition, note that by assuming $\|\beta_{I_g}^\ast\|_2 \geq t_2 \|h_g\|_2 \geq 2 \|h_g\|_2$ for all $g' \in G^*$, Lemma [V2] implies

$$\left\|u_{g'}\right\|_2 \leq 4 \frac{\|h_{g'}\|_2}{\beta_{I_g}^\ast} \leq \frac{4}{t_2} \left(\frac{\lambda_{\max}}{\lambda_{\min}} \cdot \frac{\gamma_1}{\sqrt{\gamma_{\star}}} \right). \quad (46)$$

for all $g' \in G^*$ and therefore $\|u_{S_g}\|_2 \leq 4 \sqrt{|G^*|}/t_2$. Combining all of these results we obtain that

$$\frac{1}{\lambda_g} \|X_{I_g}^T \cdot X_{S_g} (X_{S_g}^T \cdot X_{S_g})^{-1} \cdot A_{S_g} u_{S_g}\|_2 \leq \frac{\lambda_{\max}}{\lambda_{\min}} \cdot \frac{8 \gamma_1 \sqrt{|G^*|}}{t_2}$$

Therefore, assuming the event $E_2$ holds for the choice of

$$\gamma = \frac{c_4}{\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) \sqrt{d_{\max} \cdot \log p}},$$

where

$$c_4 \leq \frac{1}{8} \sqrt{2(1 + 4 \log 2)}$$

is a finite positive constant as appeared in the proof of Lemma [V3] will ensure that

$$\frac{1}{\lambda_g} \|X_{I_g}^T \cdot X_{S_g} (X_{S_g}^T \cdot X_{S_g})^{-1} \cdot A_{S_g} u_{S_g}\|_2 \leq \frac{8 c_4}{t_2} \sqrt{|G^*| \cdot \frac{\lambda_{\max}}{\lambda_{\min}} d_{\max} \cdot \log p} \cdot \frac{\lambda_{\max}}{\lambda_{\min}} \cdot \frac{8 \gamma_1 \sqrt{|G^*|}}{t_2}$$

Then choosing $t_2 \geq \tilde{c}_8 \sqrt{|G^*|/d_{\max} \log p}$ as specified by the statement of the lemma (with $\tilde{c}_8 := \frac{32}{c_4}$) completes the proof.

F. Proof of Lemma [V6]

Fix any $g \notin G^*$. Note that for any $\tau > 0$,

$$\Pr \left( \|X_{I_g}^T \cdot \Pi_{(S_g)^+} \cdot w\|_2 > \sigma \sqrt{d_g} + \tau \right) \leq \Pr \left( \|X_{I_g}^T \cdot \Pi_{(S_g)^+} \cdot w\|_2 > \sigma \|X_{I_g}^T \cdot \Pi_{(S_g)^+}\|_F + \tau \right) \leq \Pr \left( \|X_{I_g}^T \cdot \Pi_{(S_g)^+} \cdot w\|_2 > \sigma \|X_{I_g}^T \cdot \Pi_{(S_g)^+}\|_F > \tau \right),$$

where the first inequality follows from the fact that $\|X_{I_g}^T \cdot \Pi_{(S_g)^+}\|_F \leq \sqrt{d_g}$ (which is easy to verify by considering $\|\Pi_{(S_g)^+}^T \cdot X_{I_g}\|_F^2$, arranging the sum that arises in the definition of the squared Frobenius norm into a sum of
sums over columns of $X_{g^*}$, and applying standard matrix inequalities along with the fact that $\|\Pi(S_g^*)\|_{2\rightarrow 2} = 1$.

Now, the final upper bound above is of the form controllable by the Hanson-Wright inequality (c.f., Lemma [VI.4]). Specifically, setting $x = w/\sigma$, and $A = \sigma X_{g^*}^T \Pi(S_g^*)$, and using the fact that $\|X_{g^*}^T \Pi(S_g^*)\|_{2\rightarrow 2} \leq \sigma \sqrt{1 + \mu_1(X)}$ (which is easy to verify using the sub-multiplicativity of the spectral norm), we obtain overall that for the universal finite constant $c_7 > 0$, and the specific choice $\tau = c_7 \sqrt{d_g}$,

$$\Pr \left( \|X_{g^*}^T \Pi(S_g^*)^T w\|_2 > \sigma (1 + \epsilon) \sqrt{d_g} \right) \leq 2 \exp \left( -c_7^2 d_g \right).$$

Thus, it follows that

$$\Pr \left( \bigcup_{g \notin G^*} \left\{ \|X_{g^*}^T \Pi(S_g^*)^T w\|_2 > \sigma (1 + \epsilon) \sqrt{d_g} \right\} \right) \leq 2 \sum_{g \notin G^*} \exp \left( -c_7^2 d_g \right) \leq 2(G - |G^*|) \exp \left( -c_7^2 d_{\min} \right).$$

Next, note that whenever

$$\epsilon \geq \sqrt{\frac{\log \left( p^t |G - |G^*|\right)}{c_7 d_{\min}}},$$

the last term is no larger than $2p^{-t}$. Finally, note that the stated result holds if $\lambda_g \geq 4\sigma(1 + \epsilon)\sqrt{d_g}$ for all $g \notin G^*$.

**G. Proof of Theorem [II.1]**

Since the number of non-zero groups of the true parameter vector $\beta^*$ satisfies the condition $|G^*| \leq \frac{0.5 - \mu_1(X)}{\mu_1(X)} + 1$, Lemma [VI.1] implies that the sub-dictionary $X_{g^*}$, which incorporates the blocks of $X$ corresponding to $\beta_{g^*}$ will be well-conditioned and therefore the event $E_1$ holds true deterministically. We can then control the norm of the dual variable blocks $\tilde{z}_{g^*}$ for $g \notin G^*$ by leveraging the inequality

$$\|\tilde{z}_{g^*}\|_2 \leq \frac{1}{\lambda_g} X_{g^*}^T X_{g^*} (X_{g^*}^T X_{g^*})^{-1} \Lambda_{S_{g^*}^*} \tilde{z}_{g^*} \|_2 + \frac{1}{\lambda_g} X_{g^*}^T \Pi(S_g^*) (w) \|_2,$$

(47)

that follows from (20), and bounding each of the terms.

We bound the first term on the right-hand side utilizing standard matrix norm inequalities, and exploiting only magnitude information about the vectors $\tilde{z}_{g^*}$ (e.g., that $\|\tilde{z}_{g^*}\|_2 \leq 1$ for all $g' \in G^*$). This strategy relies on the following lemma.

**Lemma VI.5. Suppose the group-level support $G^*$, with $|G^*| = s$, is fixed and that the event $E_1$ in (16) occurs. If

$$s \leq \frac{\lambda_{\min}}{\lambda_{\max}} \cdot \frac{1}{4\mu_B(X)},$$

then $\|\frac{1}{\lambda_g} X_{g^*}^T X_{g^*} (X_{g^*}^T X_{g^*})^{-1} \Lambda_{S_{g^*}^*} \tilde{z}_{g^*} \|_2 \leq 1/2$ for every $g \notin G^*$.

**Proof:** Using sub-multiplicativity of the spectral norm and standard arguments, we have that

$$\|\frac{1}{\lambda_g} X_{g^*}^T X_{g^*} (X_{g^*}^T X_{g^*})^{-1} \Lambda_{S_{g^*}^*} \tilde{z}_{g^*} \|_2 \leq \frac{1}{\lambda_{\min}} \|X_{g^*}^T X_{g^*}\|_{2\rightarrow 2} \|X_{g^*}^T X_{g^*}\|_{2\rightarrow 2} \|\Lambda_{S_{g^*}^*} \tilde{z}_{g^*} \|_2.$$

By Lemma [VI.6] (provided in the sequel) we have that $\|X_{g^*}^T X_{g^*}\|_{2\rightarrow 2} \leq \sqrt{\mu_B(X)}$, while well-conditioning implies $\|X_{g^*}^T X_{g^*}\|_{2\rightarrow 2} \leq 2$. Further, since every diagonal element of $\Lambda_{S_{g^*}^*}$ is no larger than $\lambda_{\max}$ and $\tilde{z}_{g^*}$ is composed of $s$ sub-vectors whose norms are less than or equal to one, it follows that

$$\|\Lambda_{S_{g^*}^*} \tilde{z}_{g^*} \|_2 \leq \lambda_{\max} \sqrt{s}.$$

The result follows.

Coupled with the result of Lemma [VI.6] this implies that when

$$s = |G^*| \leq \frac{\lambda_{\min}}{\lambda_{\max}} \cdot \frac{1}{4\mu_B(X)}$$

and

$$\lambda_g \geq 4\sigma(1 + \epsilon)\sqrt{d_g}$$

for all $g \in [G]$ and some

$$\epsilon \geq \sqrt{\frac{\log (p^t (|G - |G^*|))}{c_7 d_{\min}}},$$

with $t \geq 1$, that

$$\|\tilde{z}_{g^*}\|_2 < 1$$

holds for every $g \notin G^*$ simultaneously with probability at least $1 - 2p^{-t}$.

Now, by Corollary [VI.1] we know that if

$$\|\beta_{g^*}' \|_2 \geq \sigma (1 + \epsilon) \left( \sqrt{d_g^*} + \sqrt{d_g^*} \right) + \lambda_g' \|\beta_{g^*}' \|_2,$$

for every $g' \in G^*$ for some

$$\epsilon \geq \sqrt{\frac{\log (p^t |G^*|)}{c_7 d_{\min}}},$$

then $\|\beta_{g^*} - \beta_{g^*}' \|_2 \leq \|\beta_{g^*}' \|_2$ simultaneously for every $g' \in G^*$, with probability at least

$$1 - 4p^{-t/2} \leq 1 - 2p^{-t} - 2 \exp \left( -c_7^2 d_{\min}^2 / 2 \right).$$

Now by using a union bounding argument, we have that if the assumptions of Theorem [II.1] are met, then with probability at least $1 - 6p^{-t/2}$, the sub-dictionary $X_{g^*}$ will be well-conditioned, $\|\tilde{z}_{g^*}\|_2 < 1$ for every $g \notin G^*$, and $\|\beta_{g'} - \beta_{g'}' \|_2 \leq \|\beta_{g'}' \|_2$ for every $g' \in G^*$, which further imply that $\beta$ will be the unique solution to the problem (2) and its group-level support will be exactly that of $\beta'$, i.e., $G(\beta) = G(\beta')$.

To justify the choice for $\epsilon$ in the theorem statement, notice that
specializing $t = 4 \log 2$ along with the fact that $c_7 > 3$ implies that
\[ \epsilon = \sqrt{\frac{(1 + \mu_B(X)) \cdot \log(pG)}{d_{\min}}} \]
meets both the above requirements on $\epsilon$.

**H. Proof of Corollary III.1**

This is a direct consequence of Theorem II.1 for the anomaly detection framework in Section III. There we assumed $X = [\tilde{X}(1) | \tilde{X}(2)]$, where $\tilde{X}(1) = I_{T \times T} \otimes X(1)$ and $\tilde{X}(2) = I_{T \times T} \otimes X(2)$, with $X(1)$ and $X(2)$ specialized to two-dimensional DCT and identity matrices of size $N \times N$, respectively. Since in this setup $d_g$ is either $T$ (for the temporal groups defined over the support of the smooth component) or $DT$ (for the spatiotemporal groups defined over the support of the anomaly component), we set $\lambda_1 = 4\sigma(1 + \epsilon)\sqrt{T}$ and $\lambda_2 = 4\sigma(1 + \epsilon)\sqrt{DT}$ as in the statement of Theorem II.1.

Moreover, under the assumptions on the dictionary, we have $\|p = 2NT\|_{\frac{1}{2} - 2} = 2$, the intra-block coherence parameter $\mu_B(X)$ will be zero and upper bounding $\mu_B(X)$ will amount to finding upper bounds on
\[ \left\| \left( \tilde{X}(1) \right)^T_{i_1} \left( \tilde{X}(2) \right)_{i_2} \right\|_{2 - 2}, \]
where $\left( \tilde{X}(1) \right)_{i_1}$ and $\left( \tilde{X}(2) \right)_{i_2}$ represent two column sub-matrices of $\tilde{X}(1)$ and $\tilde{X}(2)$ whose numbers of columns are given by the defined partition. More specifically, since the groups over the smooth component are temporal, we may write
\[ \left( \tilde{X}(1) \right)_{i_1} = I_{T \times T} \otimes \left( X(1) \right)_{i_1} \in \mathbb{R}^{NT \times T} \]
for the $T \times T$ identity matrix $I_{T \times T}$ and some column of $X(1)$ denoted by $\left( X(1) \right)_{i_1}$. Also, since spatiotemporal groups are defined over the anomalous component, we may write
\[ \left( \tilde{X}(2) \right)_{i_2} = I_{T \times T} \otimes \left( X(2) \right)_{i_2} \].

Given these expressions for the sub-matrices of the two dictionaries, the associated inner products may be simplified as
\[ \left( \tilde{X}(1) \right)^T_{i_1} \left( \tilde{X}(2) \right)_{i_2} = I_{T \times T} \otimes \left( \left( X(1) \right)^T_{i_1} \left( X(2) \right)_{i_2} \right), \]
and it follows that
\[ \left\| I_{T \times T} \otimes \left( \left( X(1) \right)^T_{i_1} \left( X(2) \right)_{i_2} \right) \right\|_{2 - 2} = \left\| \left( X(1) \right)^T_{i_1} \left( X(2) \right)_{i_2} \right\|_2. \]

Next, as $X(1) \in \mathbb{R}^{N \times N}$ is a two-dimensional DCT matrix, the absolute value of its largest entry is no larger than $\sqrt{4N}$; see also [6]. Then since $\left( X(2) \right)_{i_2}$ comprises $D$ columns of the identity matrix, the Euclidean norm on the right-hand side of the above expression will not exceed $\sqrt{4D}N$. Therefore, the block coherence parameter satisfies $\mu_B(X) \leq \sqrt{4D}N$.

The sufficient conditions stated in Corollary III.1 are then simplifications of the conditions in Theorem II.1 by specializing $d_g$ and $\lambda_g$ to their values mentioned in the above, and replacing $\mu_B(X)$ by its upper bound $\sqrt{4D}N$. In particular, one can show that
\[ \min \left\{ \frac{0.5 - \mu_B(X)}{\mu_B(X)} + 1, \frac{\lambda_{\min}}{\lambda_{\max}} , \frac{1}{4\mu_B(X)} \right\} = \frac{\lambda_{\min}}{\lambda_{\max}} , \frac{1}{4\mu_B(X)} \geq \frac{\sqrt{N}}{8D} \]
and that the condition on the norms of the blocks of the coefficient matrices takes the form
\[ \min_{g \in [G]|} \left\| \left( B(1) \right)_{i_{g_1}} \right\|_F \geq 5\sigma(1 + \epsilon) \left( \sqrt{4T} + \sqrt{s_1T + s_2DT} \right), \]
for the anomalous component and
\[ \min_{g \in [G]} \left\| \left( B(2) \right)_{i_{g_2}} \right\|_F \geq 5\sigma(1 + \epsilon) \left( \sqrt{T} + \sqrt{s_1T + s_2DT} \right), \]
for the nominally smooth component.

**I. Proof of Corollary III.2**

Similar to the proof of Corollary III.1 in the context of the discussed anomaly detection problem we have $\|X\|_{\frac{1}{2} - 2} = 2$, $\mu_1(X) = 0$, and $\mu_B(X) \leq 4\sqrt{D}N$. The sufficient conditions stated in Corollary III.2 are then simplifications of the conditions in Theorem II.2. In particular, one can show that by imposing
\[ \sqrt{N} \geq \frac{2 \log(2NT)}{c_1} \sqrt{D^3T}, \]
we are ensured
\[ \mu_B(X) \leq \sqrt{\frac{d_{\min}}{d_{\max}}} \cdot \frac{c_1}{\log(2NT)}. \]

Furthermore, the fact that $\mu_B(X) \leq \sqrt{4D}N$, along with that $d_{\max}/d_{\min} = D^3T$, can be used to demonstrate
\[ \frac{d_{\min}}{d_{\max}} \cdot \frac{c_2 \mu_B^2(X)}{\log(2NT)} \geq \frac{c_2}{4 \log(2NT)} \cdot \frac{N}{TD^3}. \]

Then the condition on the group-level sparsity in Theorem II.2 will be ensured by imposing
\[ s = |G^*| \leq \frac{c_2^2 N}{4TD^3 \log(2NT)} \]
\[ = \min \left\{ \frac{c_2^2 N}{4TD^3 \log(2NT)} , \frac{c_2 G}{2 \log(2NT)} \right\}, \]

since $G = N(1 + 1/D) \geq N$, $c_0 = 0$, and
\[ c_2 \leq 0.00028 \]
\[ \leq \left\lfloor \sqrt{9 + \frac{1}{2} \left( \frac{1}{4} - 3c_0 - 48c_1 \right) - 3} \right\rfloor^2 \]
so that $c_2 = 0.0001 = \min\{c_2, 0.0001\}$.
Lemma VI.6. Let \( X = [X_{I_1}, \ldots, X_{I_G}] \) be a column-wise partitioned matrix with \( X_{I_g} \in \mathbb{R}^{n \times d_g} \) for \( g \in [G] \). Then it is always true that
\[
\|X\|_2^2 \leq \sum_{g \in [G]} \|X_{I_g}\|_2^2 \tag{49}
\]

Proof: Let \( \beta = [\beta^T_{I_1}, \beta^T_{I_2}, \ldots, \beta^T_{I_G}]^T \) be an arbitrary unit norm vector partitioned according to the prescribed partition of \( X \). It follows by the triangle inequality that
\[
\|X\beta\|_2^2 = \| \sum_{g \in [G]} X_{I_g} \beta_{I_g} \|_2^2 \leq \left( \sum_{g \in [G]} \|X_{I_g} \beta_{I_g}\|_2 \right)^2
\]
\[
\leq \left( \sum_{g \in [G]} \|X_{I_g}\|_2 \| \beta_{I_g}\|_2 \right)^2
\]

Then by using the Cauchy-Schwartz inequality we have that
\[
\|X\beta\|_2^2 \leq \left( \sum_{g \in [G]} \|X_{I_g}\|_2 \right) \left( \sum_{g \in [G]} \| \beta_{I_g}\|_2 \right), \tag{50}
\]
and since \( \beta \) is an arbitrary unit norm vector, the result follows by the definition of the spectral norm.
\[\blacksquare\]