General Superfield Quantization Method. I. 
Lagrangian Formalism of $\theta$-Superfield Theory of Fields

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Abstract

The rules for constructing Lagrangian formulation for $\theta$-superfield theory of fields ($\theta$-STF) are introduced and considered on the whole in the framework of proposed here new general superfield quantization method for general gauge theories.

Algebraic, group-theoretic and analytic description aspects for supervariables over (Grassmann) algebras containing anticommuting generating element $\theta$ and interpreted further in particular as an ”odd” time are examined.

Superfunction $S_L(\theta) \equiv S_L\left(A(\theta), \frac{dA(\theta)}{d\theta}, \theta\right)$ and its global symmetries are defined on the extended space (supermanifold) $T_{odd}M_cl \times \{\theta\}$ parameterized by local coordinates: superfields $A^i(\theta)$, $\frac{dA^i(\theta)}{d\theta}$, $\theta$. Extremality properties of the superfunctional $Z[A] = \int d\theta S_L(\theta)$ are analyzed together with properties of corresponding Euler-Lagrange equations. System of definitions for Lagrangian formulation of $\theta$-STF including extension of gauge invariance concept is suggested.

The direct and inverse problems of zero locus reduction for extended (anti)symplectic manifolds over $M_{min} = \{(A^i(\theta), C^\alpha(\theta))\}$, with (odd) even brackets, corresponding to initial (special) general type $\theta$-superfield models are employed to construct iteratively the new interconnected models both embedded into manifolds above with reduced brackets and enlarging them with continued ones.

Component (on $\theta$) formulation for $\theta$-STF variables and operations is produced providing the connection with standard gauge field theory. Realization of $\theta$-STF constructions is demonstrated on models of scalar, spinor, vector superfields which are used to formulate the $\theta$-superfield model with abelian two-parametric gauge supergroup generalizing the $\theta$-superfield quantum electrodynamics.

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I Introduction

Investigations in the field of generalization of the Lagrangian and Hamiltonian quantization methods for gauge theories based on using special types of supertransformations the such as
BRST symmetry [1] and BRST-antiBRST (extended BRST) symmetry [2] have been developed in the last 15–20 years sufficiently intensively.

The rules of canonical (BFV [3] and Sp(2) [4]) and Lagrangian (BV [5] and Sp(2) [6]) quantization methods for gauge theories realizing above-mentioned symmetry types have become, in the first place, basic for correct investigations of the quantum properties of concrete modern models of gauge field theory and, in the second, have been found in fact to be fundamental as in exposure of algebraic and geometric-differential aspects inherent in the methods [3–6] as their further advance. As to latter two sentences, then confining ourselves to Lagrangian methods it should be noted a use of algebraic and differential structures (operations) and quantities of the BV method in string theories [7].

Possibilities of deformation theory, deformation quantization and homological perturbation theory were considered in application to generating equations of BV and BFV methods for their study with help of cohomological theory techniques for Lie groups and algebras [8] and for corresponding obtaining of gauge field theory Lagrangians with interaction (see refs.[9]). So, for instance, a quantum deformation (on $\hbar$ degree) of generalized symplectic structures by means of star multiplication [10] was used in order to establish relationship between Fedosov deformation quantization [11] and BFV-BRST quantization of dynamical systems with II class constraints [12].

Ingredients of BV method considered from viewpoint of supermathematics and theory of supermanifolds [13] have found more or less clear classified geometric matter [14] (see paper [15] and references therein as well) being by more complicated analog of symplectic geometry.

The development of quantization scheme [5] in the field of introduction a more general class of gauge conditions than commuting with respect to (w.r.t.) antibracket, i.e. nonabelian hypergauges, was suggested in ref.[16]. In its turn, Sp(2)-covariant Lagrangian quantization method was intensively improved that was expressed both in its generalization in the form of Sp(2)-symmetric method [17], then of triplectic [18], recently modified triplectic [19] (in connection with Fedosov supermanifolds concept [11]) ones and in creation of some variants of the corresponding differential geometry [20]. A presence of $Z_2$-graded differential structures and quantities on superspaces and, more generally, on supermanifolds permitted one to consider the generalization of (ordinary) differential equations concept by means of introduction of the so-called ”Shander’s supertime” [21] $\Gamma = (t, \theta)$. $\Gamma$ includes together with even parameter $t \in \mathbb{R}$, in the sense of fixed $Z_2$-grading, the odd parameter $\theta$ ($\theta^2 = 0$) as well. The fact that variable $\theta$ can be used as ”odd time” for BV method formulation was noted by Ō.F.Dayi [22].

The realization of the fact that BRST (extended BRST) symmetry can be realized, in a some sense, in the form of translations along variable $\theta$ ($\theta, \bar{\theta}$) [23] had led, in first, to extension of $D$-dimensional Minkovski space to superspace parametrized by sets of supernumbers $z^M = (x^\mu, \theta)$ ($z^M = (x^\mu, \theta, \bar{\theta})$), where $x^\mu, \mu = 0, 1, \ldots, D - 1$ are coordinates in $\mathbb{R}^{1,D-1}$ and, in second, to special superfield construction of action functional for Yang-Mills type theories [24]. By distinctive feature of that construction is the point that gauge superfield multiplet has consisted of gauge classical and ghost fields.

One from versions of superfield generalization of the Lagrangian BRST quantization method for arbitrary gauge theories was given in ref.[25] and in [26] for the case of (Sp(2)) BLT one [6]. The extrapolation of the superfield method [25] of BV quantization for nonabelian hypergauges [16] is proposed in ref.[27].

However, given quantization scheme [25] has a number of problematic places (including problems of fundamental nature), which, in general, reduce on component (on $\theta$) formulation level to discrepancy with quantities and relations of BV method. Besides, a superfield realization of some(!) ingredients of BV method both on the whole and for concrete field theory
models was considered in ref.[28].

In Hamiltonian formalism of quantization for dynamical systems with constraints (BFV method) the version of superfield quantization was suggested as well (among them in operator formulation) with its generalization to the case of arbitrary phase space (i.e. with local coordinates not corresponding to Darboux theorem) [29]. This formulation of quantization (with modifications) and its connection with objects and relations from BV method have been considered in ref.[30]. Recently, a so-called superfield algorithm for constructing the master actions in terms of superfields in the framework of BV formulation for a class of special gauge field theories was proposed in refs.[31,32], being based on geometrical description of sigma models given in [33] developed later to the case of models above on manifold with boundary [34].

Quantization variant [25] has important methodological significance consisting in fact that the multiplet contents of superfields and superantifields, defined on the superspace $\mathbb{R}^{1,D-1|1}$ with coordinates $(x^\mu, \theta)$, include into themselves, in a natural way, componentwise w.r.t. expansion in powers of $\theta$ the sets of such (anti)fields which can be identified with all variables of BV method [5] (fields $\phi^A$, auxiliary fields $\lambda^A$, antifields $\phi^*_A$ and sources $J_A$ to fields $\phi^A$). In the second place, an action of all differential-algebraic structures: superantibracket, odd operators $U, V, \Delta$ was realized in the explicit superfield form on superalgebra of functionals with derivation (locally) defined on a supermanifold with coordinates $\Phi^A(\theta), \Phi^*_A(\theta)$. In the third, the generating equation is formulated in terms of above-mentioned objects and generating functional of Green’s functions $Z[\Phi^*]$ (in notations of paper [25]) is constructed, and its properties formally repeating a number of ones for corresponding functional from BV method are established.

The work opens by itself a number of papers devoted to development of new general superfield quantization method for gauge theories in Lagrangian formalism of their description. The complete and noncontradictory formulation of all the statements of the method requires accurate and successive introduction in superfield form of all the quantities being used in quantum field theory, for example, generating functionals of Green’s functions, including effective action, together with correct study of their properties such as gauge invariant renormalization, gauge dependence and so on.

Existence theorems for solutions of generating equations being used in the method and for similar statements in further development of this approach, for instance, for nonabelian hypergauges are the key and more complicated objects for investigation, than in BV scheme.

The correct superfield formulation for classical theory based on variational principle and having by one’s definitely chosen restriction the usual quantum field theory model with standard classical action functional $S_0(A)$ of classical gauge fields $A^i$, composing the zero component w.r.t. expansion on $\theta$ in the superfield multiplet $A^i(\theta)$, is the necessary(!) condition for accurate establishment of general rules for general superfield quantization method (GSQM) in Lagrangian formalism.

Similar realization of the classical objects including additionally to an action the definition of all the gauge algebra structural functions, among them the generators of gauge transformations can be carried out by the way to be different from the construction of the action functionals for standard superfield SUSY field theory models as the superfunctionals over usual superspace coordinatized by $z^a$. By the key point on this way it appears the enlargement of initial $S_0(A)$ to superfunction with values in Grassmann algebra with one generating element $\theta$ and depending upon $A^i(\theta)$, their derivatives w.r.t. $\theta$ and $\theta$. To realize in this direction the noncontradictory description means now of the more general field theory models including the standard ones for $\theta = \partial_\theta A^i(\theta) = 0$ it will be widely used the analogy with Lagrangian classical mechanics and field theory.

Quantities and relations of above-described classical theory will have the adequate corre-
spondence with BV quantization objects and operations. Additionally, given θ-superfield formulation will have ensured the more significant results of the new gauge models construction on the basis of θ-superfield zero locus reduction (ZLR) direct [35] and so-called inverse problems by means of the duality between odd and even Poisson brackets (see [36] and references therein) being embedded each in other in definite sequences on the corresponding manifolds.

The purpose of present work is the construction according to what has been said above of the Lagrangian formulation\(^1\) for θ-STF together with its some extensions and field-theoretic examples. The paper is written in the following way.

In Sec.II elements of algebra on Grassmann algebra \(\Lambda_1(\theta)\) with a single generating element \(\theta\) are considered together with canonical realization of superspace \(\mathcal{M}\) coordinatized by sets \((z^a, \theta)\), where \(z^a\) are the coordinates of usual superspace with space-time supersymmetry. Superfield (in mentioned sense) representations, including (ir)reducible ones, of corresponding supergroup in superspace of superfunctions on \(\mathcal{M}\) are shortly examined. In addition, some technically main questions of algebra and analysis on superalgebras of special superfunctions on \(\mathcal{M}\) are analyzed here.

Section III is devoted to study of algebraic properties of the first order differential operators acting on superalgebra of superfunctions on \(T_{\text{odd}}\mathcal{M}_\text{cl} \times \{\theta\}\). Properly Lagrangian formulation for θ-STF is defined in Sec.IV and is directly connected with possibility of representation of special superfunction \(S_L(A(\theta), \frac{dA(\theta)}{d\theta}, \theta)\) defined on \(T_{\text{odd}}\mathcal{M}_\text{cl} \times \{\theta\}\) together with its maximal global symmetry group.

The detailed systematic research of the Lagrangian formulation for θ-STF is carried out here, being concentrated on the study for Euler-Lagrange equations for superfunctional \(Z[A] = \int d\theta S_L(\theta)\) endowed with introduction a concept on constraints and ideas concerning gauge theories and gauge transformations of general and special types. In Sec.V it is shown as gauge invariance permits to use the BV and BFV generating superfunctional(s) and equations constructions with help of θ-superfield brackets introduction in order to realize, in general, qualitatively the algorithms of the new θ-superfield models obtaining.

The component (on θ) formulation for objects and relations of Lagrangian formalism for θ-STF is suggested in Sec.VI. An application of general statements of Secs.II–VI is demonstrated in Sec.VII on a number of θ-superfield models starting with five basic simple θ-STF ones, among them interacting, describing a massive complex spinless scalar superfield, massive spinor superfield of spin \(\frac{3}{2}\), massless real (of helicity \(1\) for \(D = 4\)) and massive complex vector superfields. Given models can be directly generalized to the case when the corresponding superfields take values in an arbitrary semisimple Lie superalgebra forming some isotopic vectors. In this connection note, that mentioned models appear, in fact, by the base ones for construction of the interacting θ-superfield Yang-Mills type models in realizing of the gauge principle [40]. The such programm is completely realized for the case of superfield theory generalizing the model of Quantum Electrodynamics as well.

Finally, concluding propositions and analogy for θ-STF in Lagrangian formalism with usual classical mechanics complete the paper in Sec.VIII.

Necessary questions from theory of ordinary differential equations with odd differential operator \(\frac{d}{d\theta}\) are considered in appendix A.

\(^{1}\)the term "Lagrangian formulation" of classical theory or θ-STF does not coincide w.r.t. sense with notion "Lagrangian formalism" in the set expression: GSQM in the Lagrangian formalism
atlas is given or its consideration is bounded by a definite neighbourhood in ignoring the topological aspects. As consequence the local supermanifold coordinates are defined globally, and therefore the elements of differential geometry on given supermanifold are not considered in an invariant coordinate free form.

In paper it is used the standard condensed De Witt’s notations [37]. The total left derivative of superfunction \( f(\theta) \) w.r.t. \( \theta \) and superfield partial right derivative of differentiable superfunction \( J(\theta) \equiv J(A(\theta), \tilde{A}(\theta), \theta) \) w.r.t. superfield \( A'(\theta) \) for fixed \( \theta \) are denoted by means of conventions

\[
\frac{\partial f(\theta)}{\partial \theta} \equiv \frac{df(\theta)}{d\theta} \equiv \partial_\theta f(\theta), \quad \frac{\partial_{\theta'} J(\theta)}{\partial A(\theta)} \equiv \frac{\partial J(\theta)}{\partial A'(\theta)} \equiv \mathcal{J}_{\theta} (\theta). \tag{1.1}
\]

II Mathematical grounds

Let us consider a supergroup \( J \) being by the direct product of Lie supergroup \( \tilde{J} \) and one-parameter supergroup \( P \)

\[
J = \tilde{J} \times P, \quad P = \{ h \in P \mid h(\mu) = \exp (\mu p_\theta) \}, \tag{2.1}
\]

with \( \mu \in ^1\Lambda_1(\theta) \), being by subspace of odd elements w.r.t. generating element \( \theta \) from 2-dimensional Grassmann algebra \( \Lambda_1(\theta) \) over number field \( K(\mathbb{R} \text{ or } \mathbb{C}) \), and quantity \( p_\theta \) \( (p_\theta^2 = \frac{1}{2}[p_\theta, p_\theta]^+ = 0) \) as the basis element of Lie superalgebra corresponding to \( P \). The latter can be realized as the translation supergroup acting on Grassmann algebra over \( \Lambda_1(\theta) \) by the formula

\[
h(\mu)g(\theta) = g(\theta + \mu), \quad h(\mu) \in P, \quad g(\theta) \in \tilde{\Lambda}_1(\theta),
\]

\[
\tilde{\Lambda}_1(\theta) = \{ g(\theta) \mid g(\theta) = g_0 + g_1 \theta, g_0, g_1 \in E_K, \}
\]

where \( E_K \) is the algebra of functions over \( K \). From Eq.(2.2) it follows the translation generator \( p_\theta \) may be realized by means of \( \frac{d}{d\theta} : p_\theta = -i \frac{d}{d\theta} \). Regarding that \( \tilde{J} \) is a semidirect product of Lie supergroup \( \tilde{M} \) on a some Lie subsupergroup \( J_\tilde{A} \) from the supergroup \( J_\tilde{A} \) of all automorphisms of \( \tilde{M} \): \( \tilde{J} = \tilde{M} \bowtie \tilde{J}_\tilde{A} \) and taking into account that \( J_\tilde{A} \simeq (e, J_\tilde{A})^2 \) is the Lie subsupergroup in \( \tilde{J} \), we obtain the canonical realization of superspace \( \mathcal{M} \) as the quotient space \( \tilde{J}/J_\tilde{A} \).

In view of \( P \) group commutability it follows the representation for supergroup \( J \) and superspace \( \mathcal{M} \) in the form

\[
J = (\tilde{M} \bowtie \tilde{J}_\tilde{A}) \times P \simeq (\tilde{M} \times P) \bowtie \tilde{J}_\tilde{A}, \quad \mathcal{M} = J/J_\tilde{A} = \tilde{M} \times \tilde{P}, \tag{2.3}
\]

where sign ”×” for \( \mathcal{M} \) denotes a Cartesian product of the superspaces \( \tilde{M} \) and \( \tilde{P} \)

\[
\tilde{M} = \tilde{J}/J_\tilde{A} \simeq (\tilde{J} \times \{ e_P \})/J_\tilde{A}, \quad \tilde{P} \simeq (\{ e \} \times P) \bowtie \tilde{J}_\tilde{A}/J_\tilde{A}. \tag{2.4}
\]

Next, consider as \( \tilde{J} \) the group of space-time supersymmetry, the such that \( \tilde{M} \) is the real superspace, with which one deals in the superfield formulations of supersymmetric field theory models. So choosing \( \tilde{J} \) in the form of Poincare type supergroup acting in

\[
\tilde{\mathcal{M}} = \mathbb{R}^{1,D-1|Nc}, \quad c = 2^{[D/2]}, \tag{2.5}
\]

with \( D, N, [x] \) being by the dimension of Minkowski space, the number of supersymmetries and the integer part of \( x \in \mathbb{R} \) respectively, the global symmetry supergroup can be realized under validity of representation (2.3).

\(^2e_P, e_j, e\) appear by the units in \( P, \tilde{J}, \tilde{M} \) respectively, and ”\( \simeq \)“ is the sign of group isomorphism
More general symmetry supergroups being encountered, for instance, in (super)gravity and (super)string theories can be obtained by localization of $\tilde{J}$ up to supergroup of general coordinate transformations simultaneously with introduction of Riemann metric on $\tilde{M}$.

The elements from $\mathcal{M}$ are parametrized in a basis determined by generators from $\tilde{\mathcal{M}}$ and $\iota p_0$ by the coordinates

$$(z^a, \theta) = (x^\mu, \theta^{A_j}, \theta), \, \mu = 0, 1, \ldots, D - 1, \, A = 1, \ldots, 2^{[D/2]}, \, j = 1, \ldots, N, \quad (2.6)$$

where $\mu, A$ correspond to usual vector $(\mu)$ and spinor $(A)$ Lorentz indices.

The actions of supergroups $\tilde{J}, P$ on the points from $\mathcal{M}$ follows from definitions (2.1), (2.2) and identities $\bar{g} \equiv (\bar{g}, e_P), \, \bar{g} \in \tilde{J}, \, h(\mu) \equiv (e_J, h(\mu))$

$$\forall \bar{g} \in \tilde{J} : \bar{g}(z^a, \theta) = (\bar{g}z^a, \theta) \, , \forall h(\mu) \in P : h(\mu)(z^a, \theta) = (z^a, \theta + \mu). \quad (2.7)$$

Presence of $Z_2$-grading w.r.t. $\theta$ in $\mathcal{M}$ makes the following representation by one-valued

$$\mathcal{M} = 0\mathcal{M} \oplus 1\mathcal{M} \equiv \tilde{\mathcal{M}} \oplus \tilde{\mathcal{P}}, \quad \dim \mathcal{M} = (\dim \tilde{\mathcal{M}}, \dim \tilde{\mathcal{P}}) = (D|Nc, 1). \quad (2.8)$$

The action of boson projectors $P_a(\theta), a = 0, 1$ is defined on $\Lambda_1(\theta)$ with standard properties

$$P_a(\theta)P_b(\theta) = \delta_{ab}P_a(\theta), \, a, b = 0, 1, \sum_a P_a(\theta) = 1, \quad (2.9)$$

dividing $\Lambda_1(\theta)$ into direct sum of their proper subsuperspaces $^a\Lambda_1(\theta) = P_a(\theta)\Lambda_1(\theta)$. Their action is continued in a natural way to the action on $\tilde{\Lambda}_1(\theta)$, so that for any Grassmann function $g(\theta) \in \Lambda_1(\theta)$ (in what follows called the superfunction) the equalities hold

$$P_0(\theta)g(\theta) = g_0 \, , \, P_1(\theta)g(\theta) = g_1\theta, \quad (2.10)$$

remaining valid if instead of $E_\mathbf{K}$ one considers the algebra of functions over $\tilde{\mathcal{M}} ("\mathcal{M} = P_a(\theta)\mathcal{M})$.

From the various realization for $\Lambda_1(\theta)$ elements we will use the representability for any $a(\theta) \in \Lambda_1(\theta)$ as the series in powers of $\theta$ with trivial differentiability w.r.t. this element [13], so that projectors have the form of the 1st order differential operators

$$P_0(\theta) = 1 - \theta\partial_\theta, \, P_1(\theta) = \theta\partial_\theta. \quad (2.11)$$

This realization of $\Lambda_1(\theta)$ is transferred without modifications on $\tilde{\Lambda}_1(\theta)$ and $\Lambda_{D|Nc+1}(z^a, \theta; \mathbf{K})$ being by Grassmann algebra over $\mathbf{K}$ with $D$ even $x^\mu$ and $(Nc+1)$ odd $\theta^{A_j}, \theta$ generating elements [13].

From the main problem of supergroup $J$ finite-dimensional irreducible representations (irreps) study we only note that due to the triviality of group $P$ occurrence into $J$ given question, in fact, is reduced to the study of supergroup $\tilde{J}$ finite-dimensional irreps. So group $J$ superfield irreps are realized (among them) on the superfields of "Lorentz" ($\tilde{J}$) type [38]

$$A^i(\theta), \, i = (\mu_1, \ldots, \mu_k, (A_1 j_1), \ldots, (A_m j_m), z^a), \, \mu_p = 0, 1, \ldots, D - 1, \quad A_r = 1, \ldots, 2^{[D/2]}, \, j_s = 1, \ldots, N, \, p = \bar{1}, k, \, r, s = \bar{1}, m, \quad (12.2)$$

to be regarded as superfunctions on $\Lambda_{D|Nc+1}(z^a, \theta; \mathbf{K})$ with values in the corresponding representation space. Superfields $A^i(\theta)$ are homogeneous w.r.t. Grassmann parity operator $\varepsilon$ acting on $\tilde{\Lambda}_{D|Nc+1}(z^a, \theta; \mathbf{K})$, being by the superalgebra of superfunctions defined on $\Lambda_{D|Nc+1}(z^a, \theta; \mathbf{K})$,

$$\varepsilon : \tilde{\Lambda}_{D|Nc+1}(z^a, \theta; \mathbf{K}) \rightarrow \mathbb{Z}_2, \quad (12.3)$$
being considered as the additive homomorphism of superalgebras. Grassmann parity (grading) \( \varepsilon \) can be represented in the form of direct sum of Grassmann gradings \( \varepsilon_j, \varepsilon_P \)

\[
\varepsilon = \varepsilon_j + \varepsilon_P, \quad \varepsilon_j : \tilde{A}_D|N_c(z^a; K) \rightarrow \mathbb{Z}_2, \quad \varepsilon_P : \tilde{\Lambda}_1(\theta; K) \rightarrow \mathbb{Z}_2,
\]

(2.14)

trivially continued up to mappings on \( \tilde{A}_D|N_{c+1}(z^a, \theta; K) \). Thus, \( \varepsilon_j, \varepsilon_P \) are the Grassmann parities w.r.t. generating elements \( z^a \) and \( \theta \) respectively. Elements from \( \tilde{A}_D|N_c(z^a; K) \) are the superfunctions, which the \( J \) (ir)reducible superfield representation is realized on, being by restriction of the supergroup \( J \) representation \( T \) onto \( \tilde{T} : T_{[J]} \).

In accordance with (2.13), (2.14) \( \tilde{\varepsilon} = (\varepsilon_P, \varepsilon_J, \varepsilon) \) are defined on the generating elements \( z^a, \theta \) in the following way

\[
\tilde{\varepsilon}(x^a) = \tilde{\varepsilon}(\theta^A) + (0, 1, 1) = \tilde{\varepsilon}(\theta) + (1, 0, 1) = (0, 0, 0) = \tilde{0}.
\]

(2.15)

Contents of component fields in \( \mathcal{A}'(\theta) \) are given by the expansion in powers of \( \theta \) [25] together with the values of their \( \tilde{\varepsilon} \) parity

\[
\mathcal{A}'(\theta) = A^i + \lambda^i \theta, \quad \tilde{\varepsilon}(\mathcal{A'}) = \tilde{\varepsilon}(\mathcal{A}'(\theta)) = \tilde{\varepsilon}(\lambda^i) + (1, 0, 1) = ((\varepsilon_P), (\varepsilon_J), \varepsilon_i).
\]

(2.16)

Thus, the homogeneous w.r.t. \( \varepsilon \) superfield \( \mathcal{A}'(\theta) \) has \( \varepsilon_j, \varepsilon_P \) parities as for one’s \( P_0(\theta) \)-component field \( A^i \). In addition to gradings above define the \( \tilde{\varepsilon} \) values for differentials \( (dz^a, d\theta) \) and for differential operators \( (\partial_\theta, \partial_\theta) \) to be the same as in (2.15).

The parities spectrum shows that for \( A^i, \lambda^i \) the connection between spin and statistic is standard w.r.t. \( \varepsilon_j \) for \( \varepsilon_P = 0 \), but w.r.t. \( \varepsilon \) for \( \lambda^i \) is wrong being corresponding as the rule to unphysical degrees of freedom. The latter reflects the nontrivial fact of the generating element \( \theta \) presence and \( \varepsilon_P \neq 0 \).

Whereas classical superfields \( \mathcal{A}'(\theta) \) are transformed w.r.t. a some, in general, group \( J \) reducible superfield finite-dimensional representation \( T \), the group \( P \) irrep is one-dimensional and operators \( T(h(\mu)) \) act on \( \mathcal{A}'(\theta) \) as translations along \( \theta \).

The transformation laws

\[
\mathcal{A}'(\theta) \mapsto \mathcal{A}'(\theta') = (T(e, \tilde{g})A)^i(\theta), \; \tilde{g} \in \tilde{J}_{\tilde{A}},
\]

(2.17)

\[
\mathcal{A}'(\theta) \mapsto \mathcal{A}'(\theta') = (T(e, \tilde{g})A)^i(T(h^{-1}(\mu))\theta) = (T(e, \tilde{g})A)^i(\theta - \mu),
\]

(2.18)

realize the finite-dimensional and infinite-dimensional superfield representations respectively with generator of translations along \( \theta \): \( T(p_\theta) = -i \frac{d}{\theta} \).

Note, firstly, that we do not consider here the other possibilities for supergroup \( \tilde{J} \) nontrivial extension being analogous to the way of \( N = 1 \) supersymmetry group construction and, secondly, the following permutability rule for any \( \varepsilon \)-homogeneous elements holds

\[
a(\theta)b(\theta) = (-1)^{\varepsilon(a(\theta))\varepsilon(b(\theta))}b(\theta)a(\theta), \; a(\theta), b(\theta) \in \tilde{A}_D(N_{c+1}(z^a, \theta; K)).
\]

(2.19)

Starting from supermanifold \( \mathcal{M}_{cl} \) coordinatized by \( \mathcal{A}'(\theta) \) (formally \( i = 1, \ldots, n, n = (n_+, n_-) \); \( n_+(n_-) \) is the number of boson (fermion) w.r.t. \( \varepsilon \) degrees of freedom entering in condensed index\(^3 i \)), being more precisely by a special tensor bundle over \( \mathcal{M} \), let us formally construct the following supermanifolds \( T_{odd}\mathcal{M}_{cl}, T_{odd}\mathcal{M}_{cl} \times \{\theta\} \) parametrized by \( (\mathcal{A}'(\theta), \partial_\theta \mathcal{A}'(\theta)) \), \( (\mathcal{A}'(\theta), \partial_\theta \mathcal{A}'(\theta), \theta) \) respectively.

\(^3\)besides of formula (2.12), index \( i \) can certainly contain the discrete indices, characterizing the belonging of \( \mathcal{A}'(\theta) \) to representation space of some other groups, for example, of the Yang-Mills type
Define the superalgebra of superfunctions $K[[T_{\text{odd}}M_{cl} \times \{\theta\}]]$ given on $T_{\text{odd}}M_{cl} \times \{\theta\}$ with elements being by formal power series w.r.t. $A^i(\theta), \partial_\theta A^i(\theta), \theta$. This set contains the superalgebra $K[T_{\text{odd}}M_{cl} \times \{\theta\}]$ of finite polynomials in powers of $A^i(\theta), \partial_\theta A^i(\theta)$. For arbitrary $F(\theta) \in K[[T_{\text{odd}}M_{cl} \times \{\theta\}]]$ the transformation laws hold in acting of the representation $T$ operators, ensuing from Eqs.(2.17), (2.18) respectively

\[
F(A'(\theta), \partial_\theta A'(\theta), \theta') = F((T(e, \tilde{g})A)(\theta), (T(e, \tilde{g})\partial_\theta A)(\theta), \theta), \quad (2.20)
\]

\[
F(A'(\theta), \partial_\theta A'(\theta), \theta) = F((T(e, \tilde{g})A)(\theta - \mu), (T(e, \tilde{g})\partial_\theta A)(\theta), \theta - \mu). \quad (2.21)
\]

To obtain (2.20), (2.21) we have used the formulae for the transformations of $\partial_\theta A^i(\theta)$

\[
\partial_\theta A^i(\theta') = (\partial_\theta e)\partial_\theta A^i(\theta') = (T(e, \tilde{g})\partial_\theta A)(\theta), \quad (\partial_\theta e) = (\partial_\theta (\theta + \mu))^{-1}, \quad (2.22)
\]

\[
\partial_\theta A^i(\theta) = (T(e, \tilde{g})\partial_\theta A)(\theta - \mu) = (T(e, \tilde{g})\partial_\theta A)(\theta), \quad \tilde{g} \in J_{\tilde{A}}. \quad (2.23)
\]

By definition, (local superfunction) $F(\theta)$ is expanded in (finite sum) formal power series

\[
F(A(\theta), \hat{A}(\theta), \theta) = \sum_{l=0}^{1} \frac{1}{l!} F_{(1)}(A(\theta), \theta) \hat{A}^{(l)}(\theta) = \sum_{k,l=0}^{1} \frac{1}{k!l!} F_{(k,l)}(A(\theta), \theta) \hat{A}^{(k,l)}(\theta) \hat{A}^{(l)}(\theta), \quad (2.24)
\]

where we have introduced the notations

\[
F_{(1)}(A(\theta), \theta) \equiv F_{J_{1}\ldots J_{1}}(A(\theta), \theta), \quad F_{(k,l)}(A(\theta), \theta) \equiv F_{J_{1}\ldots J_{k} J_{l}\ldots J_{l}}(\theta), \quad (2.25a)
\]

\[
\hat{A}^{(l)}(\theta) \equiv \prod_{p=1}^{l} A^{p}(\theta), \quad \hat{A}^{(k,l)}(\theta) \equiv \prod_{p=1}^{k} A^{p}(\theta). \quad (2.25b)
\]

Expansion coefficients in (2.24) appear themselves by superfunctions on $K[[M_{cl} \times \{\theta\}]]$ or $\hat{A}_{D \in N_{c+1}}(\theta, A, K)$, and obey to the generalized symmetry properties

\[
F_{(k,l)}(A(\theta), \theta) = (-1)^{(\varepsilon_{s}, s+1)(\varepsilon_{s-1}, s)} F_{(k) J_{1}\ldots J_{r-1} s J_{r} l\ldots l}(\theta) \equiv \prod_{p=1}^{s} A^{p}(\theta), \quad \hat{A}^{(k,l)}(\theta) \equiv \prod_{p=1}^{k} A^{p}(\theta). \quad (2.26)
\]

The superalgebra of the $k$-times differentiated superfunctions $C^k(T_{\text{odd}}M_{cl} \times \{\theta\}) \equiv C^k$, $k \leq \infty$ can be obtained from $K[[T_{\text{odd}}M_{cl} \times \{\theta\}]]$ by means of compatible introduction on the latter the operation of differentiation, the norm structure and the corresponding convergence of series in (2.24). Then for arbitrary $F(\theta) \in C^k$ we will suppose to be valid the expansion in functional Taylor’s series in powers of $\delta A^i(\theta) = (A^i(\theta) - A^i_0(\theta))$ and $\delta \hat{A}^i(\theta) = (\hat{A}^i(\theta) - \hat{A}^i_0(\theta))$ in a some neighbourhood $A^i_0(\theta), \hat{A}^i_0(\theta)$

\[
F(A(\theta), \hat{A}(\theta), \theta) = \sum_{k,l=0}^{1} \frac{1}{k!l!} F_{(k,l)}(A_0(\theta), \hat{A}_0(\theta), \theta) \delta \hat{A}^{(k,l)}(\theta) \delta A^{(k,l)}(\theta), \quad (2.27)
\]

\[
F_{(k,l)}(A(\theta), \hat{A}(\theta), \theta) = \left( \prod_{t=0}^{l-1} \frac{\partial_r}{\partial r A_0^{k+1}}(\theta) \right) \left( \prod_{u=0}^{l-1} \frac{\partial_r}{\partial r A_0^{l+1}}(\theta) \right) F(A(\theta), \hat{A}(\theta), \theta). \quad (2.27)
\]

The nonzero action of the partial right superfield derivatives w.r.t. $A^i(\theta), \partial_\theta A^i(\theta)$ introduced according to (1.1) and acting nontrivially on $F(\theta)$ only for coinciding $\theta$, is defined as follows

\[
\frac{\partial_r A^i(\theta)}{\partial A^i(\theta)} = \delta^i_s, \quad \frac{\partial_r A^i(\theta)}{\partial \partial_\theta A^i(\theta)} = \delta^i_s. \quad (2.28)
\]
At last, it is useful to combine the expansions (2.24), (2.27) for corresponding arguments $\mathcal{A}'(\theta)$, $\partial_\theta \mathcal{A}'(\theta)$. The action of $\{P_a(\theta)\}$ is naturally continued onto $\mathcal{M}_\text{cl}$ ($P_a(\theta)\mathcal{M}_\text{cl} = a\mathcal{M}_\text{cl}$) and $C^k$. Besides, it is convenient to introduce a more detailed system of projectors $\{P_a(\theta), U(\theta)\}, a = 0, 1$ with $C^k$ decomposition\(^4\)

$$
C^k = C^k(P_0(T_{\text{odd}}\mathcal{M}_\text{cl})) \oplus C^k(P_1(T_{\text{odd}}\mathcal{M}_\text{cl})) \oplus C^k(P_0(T_{\text{odd}}\mathcal{M}_\text{cl}) \times \{\theta\})
\equiv 0.0C^k \oplus 1.0C^k \oplus 0.1C^k, \quad \tilde{P}_a(\theta)C^k = 0.0C^k, \quad U(\theta)C^k = 1.0C^k.
$$

By definition $\{\tilde{P}_a(\theta), U(\theta)\}$ are characterized by the relations

$$
\tilde{P}_a(\theta)\tilde{P}_b(\theta) = \delta_{ab}\tilde{P}_b(\theta), \quad \tilde{P}_a(\theta)U(\theta) = 0, \quad U^2(\theta) = U(\theta), \quad \left(\sum_a \tilde{P}_a(\theta)\right) + U(\theta) = 1.
$$

The analytic notation of $\mathcal{F}(\theta)$ by means of relation (2.27) results in representation of the projectors under their action on $C^k$ in the form of the 1st order differential operators

$$
\tilde{P}_0(\theta) = P_0(\theta) = 1 - \theta \frac{d}{d\theta}, \quad U(\theta) = P_1(\theta)\mathcal{A}'(\theta)\frac{\partial_{\mathcal{A}'(\theta)}}{\partial \mathcal{A}'(\theta)}^5, \quad \tilde{P}_1(\theta) = \theta \frac{\partial}{\partial \theta},
$$

permitting to conclude that $\tilde{P}_1(\theta), U(\theta), P_1(\theta)$ appear by the derivations on $C^k$ whereas the $P_0(\theta), (\tilde{P}_0(\theta))$ action on the product of $\mathcal{F}(\theta)$, $\mathcal{J}(\theta)$ from $C^k$ is equal to the product of its action on these elements. In turn, the connection between derivatives $\frac{d}{d\theta}, \frac{\partial}{\partial \theta}$ under their action on $C^k$ is defined by the formulae

$$
\frac{d}{d\theta} = \frac{\partial}{\partial \theta} + \mathcal{A}'(\theta)\frac{\partial_{\mathcal{A}'(\theta)}}{\partial \mathcal{A}'(\theta)} \equiv \frac{\partial}{\partial \theta} + P_0(\theta)\mathcal{U}(\theta), \quad \mathcal{U}(\theta) = \left[\frac{d}{d\theta} U(\theta)\right]_s.
$$

Define a class $C_F$ of analytic over $K$ superfunctionals on $\mathcal{M}_\text{cl}$ by means of the formula

$$
F[\mathcal{A}] = \int d\theta \mathcal{F}(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) \equiv \partial_\theta \mathcal{F}(\theta), \quad F[\mathcal{A}] \in C_F, \mathcal{F}(\theta) \in C^k,
$$

the such that, only $(\mathcal{F}^{1.0}, \mathcal{F}^{0.1}) = (U, \mathcal{P}_1)\mathcal{F}(\theta)$ parts of $\mathcal{F}(\theta)$ give the nontrivial contribution into $F[\mathcal{A}]$. As far as the operator $\partial_\theta$ does not lead out $\mathcal{F}(\theta)$ from $C^k$, then $F[\mathcal{A}]$ belongs to $C^k$ as well. Besides, $F[\mathcal{A}]$ appears now by the scalar under action of representation $T$ operators, if $\mathcal{F}(\theta)$ is transformed according to the rules (2.20) or (2.21).

With help of $\theta$-superfield analog of variational calculus basic lemma one can establish for superfunctionals from $C_F$ the relation between left (right) superfield variational derivative of $F[\mathcal{A}]$ w.r.t. $\mathcal{A}'(\theta)$ and left (right) partial superfield derivatives of its ($F[\mathcal{A}]$) density $\mathcal{F}(\theta)$ w.r.t. $\mathcal{A}'(\theta)$, $\partial_\theta \mathcal{A}'(\theta)$

$$
\frac{\delta \mathcal{F}[\mathcal{A}]}{\delta \mathcal{A}'(\theta)} = \left[\frac{\partial_{(r)}}{\partial \mathcal{A}'(\theta)} - (-1)^{e_1} \frac{\delta_{(r)}}{\delta (\partial_{(r)} \mathcal{A}'(\theta))}\right] \mathcal{F}(\theta) \equiv \mathcal{L}_{(r)}(\theta)\mathcal{F}(\theta).
$$

Note, if to start from priority of the density $\mathcal{F}(\theta)$ in comparison with $F[\mathcal{A}]$ then the preservation of the $J$-covariant total derivative w.r.t. $\theta$ by fixed, if otherwise it is not said, do not gives the nonvanishing contribution in $F[\mathcal{A}]$ and to left-hand side of (2.34) providing the consisteny

\(^4\)the only $0.0C^k$ appears by nontrivial subsuperalgebra, whereas $1.0C^k, 0.1C^k$ are nilpotent ideals in $C^k$

\(^5\) $P_1(\theta)\mathcal{A}'(\theta)$ is regarded as indivisible object in the sense, that $\frac{\partial}{\partial \theta} P_1(\theta)\mathcal{A}'(\theta) = 0$, but $\frac{\partial}{\partial \theta} (\theta \mathcal{A}'(\theta)) = \mathcal{A}'(\theta)$
and additional special properties for $\mathcal{F}(\theta)$. Superfield derivatives have the following values of Grassmann parities according to (2.14)-(2.16)

$$
\hat{\varepsilon} \frac{\partial_l}{\partial A^i(\theta)} = \varepsilon \frac{\partial_l}{\partial(\partial_0 A^i(\theta))} = \varepsilon \frac{\partial_l}{\partial A^i(\theta)} + (1, 0, 1) = ((\varepsilon P), + 1, (\varepsilon j), \varepsilon_1 + 1).
$$

(2.35)

The $k$th superfield variational derivative of superfunctional $F[A]$ w.r.t. $A^i(\theta_1), \ldots, A^i(\theta_k)$ is expressed through partial superfield derivatives w.r.t. $A^i(\theta), \partial_0 A^i(\theta), \ldots, \partial_0 A^i(\theta)$ of its ($F[A]$) density $\mathcal{F}(\theta)$ by the formula

$$
\prod_{l=0}^{k-1} \frac{\delta_r F[A]}{\delta A^{i-l}(\theta_{k-l})} = \left( \prod_{l=0}^{k-2} L_{i-k-l}(\theta_k) \delta(\theta_k - \theta_{k-l-1}) \right) L_{i-1}(\theta_k) \mathcal{F}(\theta_k, \partial_0 A(\theta_k), \theta_k), \quad (2.36)
$$

$$
\delta(\theta' - \theta) = \theta' - \theta, \quad \int d\theta' \delta(\theta' - \theta) y(\theta') = y(\theta), \quad \bar{\theta}_k \equiv \theta_1, \ldots, \theta_k.
$$

(2.37)

The superfield variational derivative w.r.t. $A^i(\theta)$ of $\mathcal{F}(\theta', \partial_0 A(\theta'), \theta')$ for compulsory not coinciding $\theta$, $\theta'$ is determined by (2.36) as well. For its calculation from arbitrary $\mathcal{F}(\theta', \partial_0 A(\theta'), \theta'; \bar{\theta}_k) \in C^k(T_{odd}M_{cl} \times \{\theta'\} \times \{\bar{\theta}_k\})$ it is sufficient to know the variational derivatives of $A^i(\theta'), \partial_0 A^i(\theta')$ whose values follow from (2.34), (2.36) according to (2.28), (2.37)

$$
\frac{\delta_l A^i(\theta')}{\delta A^i(\theta)} = (-1)^{\varepsilon_1} \delta_l \delta(\theta' - \theta), \quad \frac{\delta_l (\partial_0 A^i(\theta'))}{\delta A^i(\theta)} = \delta_l.
$$

(2.38)

### III Operatorial superalgebra $A_{cl}$

A connection between partial superfield derivative of superfunctional $\mathcal{F}(\theta)$ w.r.t. $A^i(\theta)$ and partial derivatives for fixed $\theta$ w.r.t. component fields $P_0(\theta) A^i(\theta)$ and $P_1(\theta) A^i(\theta)$ may be established in many ways. The resultant operatorial formulae being true on $C^k$ read as follows

$$
\frac{\partial_{r(i)}}{\partial A^i(\theta)} = \frac{\partial_{r(i)}}{\partial P_0(\theta) A^i(\theta)} + \frac{\partial_{r(i)}}{\partial P_1(\theta) A^i(\theta)}.
$$

(3.1)

The formulae (3.1) correctness can be readily determined in acting of $\frac{\partial}{\partial A^i(\theta)}$ on $\mathcal{F}(\theta) \in C^k$ taking account of the properties for projectors $P_0(\theta)$ and the following summary of formulae

$$
\frac{\partial_{r(i)}}{\partial P_0(\theta) A^i(\theta)} = P_0(\theta) \delta_{ab} \delta l, \quad \frac{\partial_{r(i)}}{\partial P_0(\theta) A^i(\theta)} = P_0(\theta) \delta_{l}, \quad \frac{\partial_{r(i)}}{\partial P_1(\theta) A^i(\theta)} = P_0(\theta) \delta_{l}.
$$

(3.2)

In fact, it is sufficient to prove the formula (3.1) for superfunctional $\mathcal{J}(\theta) \in C^k$ of the form

$$
\mathcal{J}(A(\theta), \partial_0 A(\theta), \theta) = \mathcal{J}_1(A(\theta), \partial_0 A(\theta), 0) + \mathcal{J}_2(A(\theta), \partial_0 A(\theta), \theta) = \frac{1}{(n-1)!} f_{i(n)}(\partial_0 A(\theta)) \bar{A}^{(i-1)}(\theta) + \frac{1}{(n-1)!} g_{i(n)}(\partial_0 A(\theta), \theta) \bar{A}^{(i-1)}(\theta), \quad P_1(\theta) g_{i(n)}(\theta) = g_{i(n)}(\theta),
$$

(3.3)

with coefficients $f_{i(n)}(\theta), g_{i(n)}(\theta)$ satisfying to properties (2.26). We have successively for right derivatives

$$
\left( \begin{array}{l}
\mathcal{J}_{1+}, \mathcal{J}_{2+}, \mathcal{J}_{1-}, \mathcal{J}_{2-} \end{array} \right)(\theta) = \left( \begin{array}{l}
\left( (n-1)! f_{i(n)}(\partial_0 A(\theta)) \bar{A}^{(i-1)}(\theta), (n-1)! f_{i(n)}(\partial_0 A(\theta)) \bar{A}^{(i-1)}(\theta) \right),
\end{array} \right)
$$

(3.4a)

$$
\frac{\partial_{r(i)}}{\partial P_{0}(\theta) A^i(\theta)} = \frac{1}{(n-1)!} f_{i(n)}(\partial_0 A(\theta)) \bar{A}^{(i-1)}(\theta) P_0(\theta),
$$

(3.4b)

$$
\frac{\partial_{r(i)}}{\partial P_{1}(\theta) A^i(\theta)} = \frac{1}{(n-1)!} f_{i(n)}(\partial_0 A(\theta)) \bar{A}^{(i-1)}(\theta) P_1(\theta),
$$

(3.4c)

$$
\frac{\partial_{r(i)}}{\partial P_{0}(\theta) A^i(\theta)} = \frac{1}{(n-1)!} \left( P_1(\theta) g_{i(n)}(\theta) \bar{A}^{(i-1)}(\theta) P_0(\theta) \right),
$$

(3.4d)
with derivative of \( J_2(\theta) \) w.r.t. \( P_1(\theta)A'\), identical vanishing, that proves the formula (3.1).

With use of the connection for derivatives given by (3.1) it is easily to obtain the component representation for superfield partial derivatives of \( \mathcal{F}(\theta) \in C^k \) w.r.t. \( A'(\theta) \)

\[
\frac{\partial r(t)\mathcal{F}(\theta)}{\partial A'(\theta)} = \frac{\partial r(t)P_0(\theta)A'(\theta)}{\partial P_0(\theta)} + \frac{\partial r(t)P_1(\theta)A'(\theta)}{\partial P_1(\theta)} + \frac{\partial r(t)P_1(\theta)A'(\theta)}{\partial P_1(\theta)}.
\]  

(3.5)

Consider on \( \Lambda_{D|Nc+1}(z^a, \theta; K) \) the special involution acting as isomorphism

\[
(z^a, \theta)^* = (z^a, \theta) = (z^a, \bar{\theta}),
\]

\[
(g(z, \theta) \cdot f(z, \theta))^* = (g(z, \theta))^* (f(z, \theta))^* = g(z, \bar{\theta}) \cdot f(z, \bar{\theta}), ((g(z, \theta))^*)^* = g(z, \theta),
\]

(3.6)

being easily continued onto \( \tilde{\Lambda}_{D|Nc+1}(z^a, \theta; K), C^k \) by the expressions

\[
(A'\theta))^* = A'\theta \equiv (\bar{A}'\theta), \ (\partial_{\theta} A'\theta))^* = \partial_{\theta} A'\theta,
\]

(3.8)

where \( \bar{A}'\theta \) is the superfield being conjugate to \( A'\theta \) w.r.t. \* with components

\[
\bar{A}'\theta = P_0(\theta)A'(\theta) - P_1(\theta)A'(\theta), \ P_1(\theta)\bar{A}'\theta = -P_1(\theta)A'(\theta), \ P_0(\theta) = P_0(\theta).
\]

(3.9)

The restriction of involution onto \( 0.0 C^k \) appears by identity mapping so that this superalgebra is coordinatized by following invariant w.r.t. \* superfields

\[
P_0(\theta)A'(\theta) = \frac{1}{2} \left( A'(\theta) + \bar{A}'\theta \right), \ \partial_{\theta} A'(\theta).
\]

(3.10)

Consider the superspace \( \mathcal{A}_{cl} \) of the 1st order differential operators acting on \( C^k \) of the form

\[
\mathcal{A}_{cl} = \{ \alpha^a U_a(\theta) + \alpha^\pm U_{\pm}(\theta) + \beta^a \bar{U}_a(\theta) + \beta^\pm \bar{U}_{\pm}(\theta), \ \alpha = 0, 1, \ \alpha^a, \alpha^\pm, \beta^a, \beta^\pm \in \mathbb{K} \},
\]

(3.11)

whose elements are given by means of the formulae

\[
U_a(\theta) = P_1(\theta)A'(\theta) \frac{\partial}{\partial P_a(\theta)} A'(\theta), \ U_+(\theta) = \sum_a U_a(\theta) = P_1(\theta)A'(\theta) \frac{\partial}{\partial A'(\theta)},
\]

(3.12)

\[
U_-(\theta) = (U_0 - U_1)(\theta) = P_1(\theta)A'(\theta) \frac{\partial}{\partial A'(\theta)} = -(U_+(\theta))^*,
\]

(3.13)

\[
\hat{U}_j(\theta) = \left[ \frac{d}{d\theta}, U_j(\theta) \right] = \hat{A}'(\theta) \frac{\partial}{\partial K_{j}(\theta)}, \ \hat{U}_-(\theta) = \left( \hat{U}_+(\theta) \right)^*,
\]

(3.14)

\[
\frac{\partial}{\partial A'(\theta)} = \left( \frac{\partial}{\partial A'(\theta)} \right)^*, \ K_{j}(\theta) = \{ P_a(\theta)A'(\theta), A'(\theta), \bar{A}'(\theta) \}, \ j = a, +, -.
\]

The only \( \hat{U}_+ \) and \( \hat{U}_- \) from all these operators are compatible with supergroup \( J \) superfield representation. In particular, \( \hat{U}_+ \) does not lead out the any \( \mathcal{F}(\theta) \in C^k \) from \( C^k \). The only operators \( U_1(\theta), \hat{U}_0(\theta) \) appear to be invariant w.r.t. involution continued by means of relations (3.8), (3.14) onto \( \mathcal{A}_{cl} \).

Elements from \( \mathcal{A}_{cl} \) satisfy to the following algebraic relationships in omitting of fixed \( \theta \) in arguments and intensive use of the formulae (3.2)

\[
1) \ U_0^2 = \delta_{ab} U_a, \ U_0^2 = \pm U_\pm, \ U_+ U_- = U_-, \ U_- U_+ = -U_+;
\]

(3.15a)

\[
2) \ [U_a, U_b]_+ = \varepsilon_{ab} U_0, \ [U_+, U_-] = 2U_0, \ \varepsilon_{ab} = -\varepsilon_{ba}, \ \varepsilon_{10} = 1, \ a, b = 0, 1,
\]

\[
[U_+, U_1]_+ = -2U_1, \ [U_+, U_a]_- = (-1)^a U_0, \ [U_-, U_a]_- = -U_0;
\]

(3.15b)

\[
3) \ [\hat{U}_i, \hat{U}_j]_+ = 0, \ [\hat{U}_j, \hat{U}_i]_+ = j \hat{U}_i, \ i, j \in \{ 0, 1, +, - \}.
\]

(3.15c)
Relationships (3.15a,b) demonstrate the subset $\tilde{A}_{cl}$ in $A_{cl}$
\[
\tilde{A}_{cl} = \{\alpha^{i} U_{i}(\theta), \, i \in \{0, 1, +, -\}, \, \alpha^{i} \in K\} \subset A_{cl}
\] (3.16)
appears, in first, by superalgebra w.r.t. usual multiplication, in second, by module over $C^{k}$, in third, by resolvable Lie subsuperalgebra w.r.t. commutator $[,]$ with radical spanned on $U_{o}(\theta)$. The sets $\{U_{a}(\theta)\}$ and $\{\hat{U}_{a}(\theta)\}$, $a = 0, 1$ are the bases in $\tilde{A}_{cl}$ and $A_{cl}$ respectively.

It should be noted that $\hat{U}_{o}(\theta)$ coincides with the operator $U$ from ref.[25] introduced there from the other grounds. Besides, some analogs of operators from $A_{cl}$, namely $\hat{U}_{1}(\theta)$, are intensively made use in Sp(2)-covariant Lagrangian and (modified) triplectic quantization methods [6,17-20] and their superfield extension [26].

**IV Foundations of $\theta$-STF**

Let us consider a scalar $\Lambda_{1}(\theta; R)$-valued superfunctional $S_{L}(A(\theta), \partial_{b}A(\theta), \theta) \equiv S_{L}(\theta) \in C^{k}$, in what follows called the classical action ($\bar{\varepsilon} S_{L}(\theta) = 0$).

$S_{L}(\theta)$ is invariant under action of the restricted representation $T_{|J}$ operators
\[
S_{L}(A'(\theta), \partial_{b}A'(\theta), \theta) = S_{L}(T(e, \tilde{g})A(\theta), T(e, \tilde{g})\partial_{b}A(\theta), \theta) = S_{L}(A(\theta), \partial_{b}A(\theta), \theta),
\]
whereas w.r.t. action of the $T_{|P}$ operators $S_{L}(\theta)$ is transformed according to the rules (translation along $\theta$) (2.18), (2.21)
\[
S_{L}(A'(\theta), \partial_{b}A'(\theta), \theta) = S_{L}(A(\theta - \mu), \partial_{b}A(\theta), \theta - \mu) \equiv S_{L}(\theta'),
\]
\[
\delta S_{L}(\theta) = S_{L}(\theta') - S_{L}(\theta) = -\mu \partial_{b}S_{L}(\theta) = -\mu \left(\frac{\partial}{\partial \theta} + P_{0}(\theta)\hat{U}_{+}(\theta)\right) S_{L}(\theta).
\]
Formula (4.1) provides the fact that $\bar{J}$ is the maximal in $J$ global symmetry group for $S_{L}(\theta)$. It should be noted that one can write equivalently in Eq.(4.3) instead of $P_{0}(\theta)\hat{U}_{+}(\theta)$ the operator $P_{0}(\theta)\hat{U}_{0}(\theta)$, or $\hat{U}_{1}(\theta)$, but the latter ones are given in nonsuperfield form.

From the assumption on existence of critical point for the fermion superfunctional, being together with $S_{L}(\theta)$ by the central, but not equivalent, $\theta$-STF objects
\[
Z[A] = \int d\theta S_{L}(A(\theta), \partial_{b}A(\theta), \theta), \, \bar{\varepsilon}Z[A] = (1, 0, 1),
\]
it follows the validity of Euler-Lagrange type equations (see Eqs.(2.34))
\[
\frac{\delta_{l}Z[A]}{\delta A'(\theta)} = \left(\frac{\partial}{\partial A'(\theta)} - (-1)^{\varepsilon_{1}} \partial_{b} \frac{\partial}{\partial (\partial_{b}A'(\theta))}\right) S_{L}(\theta) = 0.
\]
Formally, the relations above from the differential equations theory viewpoint are the 2nd order w.r.t. derivatives of superfields $A'(\theta)$ on $\theta(!)$ system of $n$ ordinary differential equations (ODE), in spite of the identity fulfilment
\[
\partial^{2}_{b}A'(\theta) \equiv \overset{{\circ}}{A}'(\theta) \equiv 0.
\]
Abtracting from the fact that system (4.5), in general, is a complicated system of partial differential equations defined by differential operators w.r.t. $(z^{a}, \theta)$ let us single out from them the only operators with $\partial_{b}$ considering others as the zero order on $\theta$ operators.
The analysis of Eqs.(4.5) is based on general statements about the 1st and 2nd orders on θ system of n ODE and on assumptions concerning the $S_L(\theta)$ structure. Therefore the results of the ODE investigation fulfilled in Appendix permit one to lead immediately a classified study of whole let us call the $2\text{NF}$ is controlled by rank value for the supermatrix $n$

System (4.5) is equivalent to one of the 2nd order on $n$ differential constraints in Lagrangian formalism $\text{HCLF}$ respectively for Eqs.(4.5). The system (4.8) on the whole let us call the Lagrangian system (LS) as well.

By virtue of remarks (A.6), (A.14) the solvable DCLF are equivalent to the 1st order on θ 2n ODE system

$$\Theta_i(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) = 0, \quad \text{deg}_{\partial_\theta \mathcal{A}(\theta)} \Theta_i(\theta) = 0$$

we will call the differential constraints in Lagrangian formalism (DCLF) and holonomic constraints in Lagrangian formalism (HCLF) respectively for Eqs.(4.5). The system (4.8) on the whole let us call the Lagrangian system (LS) as well.

Thus, the solvable LS is equivalent to the 2nd order on θ 3n ODE system (4.8a), (4.10).

Thus, the solvable LS is equivalent to the 2nd order on θ 3n ODE system (4.8a), (4.10). Naturally, the 2nd subsystem in (4.10) would not be necessary, if to consider the Eqs.(4.5) as the type (A.1) system.

DCLF restricts an admissible arbitrariness in the choice of 2n initial conditions determining the Cauchy problem for LS for θ = 0

$$\left(\mathcal{A}, \partial_\theta \mathcal{A}\right) \in T_{odd} V, \quad 0 \in \Gamma_{(0,1)} \subset \Lambda(\theta).$$

In its turn, subsystem (4.8a) are not found in NF w.r.t. $\partial_\theta \mathcal{A}(\theta)$. The possibility to pass to NF is controlled by rank value for the supermatrix

$$K(\theta) = \|K_{ij}(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta))\| = \left\| \frac{\partial_i S_L(\theta)}{\partial \theta(\partial_\theta \mathcal{A}(\theta))} \right\|.$$ (4.12)

If $\text{rank}K(\theta) < n$, then there are some constraints, in general, independent from DCLF must be imposed on subsystem (4.8a), complicating the analysis of LS.

The problem of independence for $\Theta_i(\theta)$, appearing by the most important one, requires for its effective resolution in accordance with (A.18), (A.19) to specify the initial assumptions on $Z[A] \left( S_L(\theta) \right)$
1. There exists \((A_0^i(\theta), \partial^i_0(\theta)) \in T_{odd}M_{cl}\) that
\[
\Theta_i(A(\theta), \partial A(\theta), \theta) \bigg|_{(A(\theta), \partial A(\theta)) = (A_0(\theta), \partial A_0(\theta))} = 0;
\]
(4.13)

2. There exists a smooth supersurface \(\Sigma \subset M_{cl}\) at least in a some neighbourhood of
\((A_0^i(\theta), \partial^i_0(\theta))\) the such that,
\[
(A_0^i(\theta), \partial^i_0(\theta)) \in T_{odd} \Sigma, \quad \Theta_i(\theta)_{|T_{odd} \Sigma} = 0,
\]
dim \(\Sigma = m = (m_+, m_-)\), dim \(T_{odd} \Sigma = (m_+ + m_-, m_- + m_+).\)
(4.14b)

Index \(i\) can be divided into 2 groups
\[
\gamma = (A, \alpha), \ A = 1, \ldots, n - m, \ \alpha = n - m + 1, \ldots, n
\]
in such a way, that the following condition almost everywhere on \(\Sigma\) holds
\[
\text{rank} \left\| \frac{\delta l}{\delta A^l(\theta_1)} \frac{\delta Z[A]}{\delta A^l(\theta)} \right\|_\Sigma = \text{rank} \left\| \frac{\delta l}{\delta A^l(\theta_1)} \frac{\delta Z[A]}{\delta A^l(\theta)} \right\|_\Sigma = n - m;
\]
(4.15b)

3. There exists a separation of index \(i\) to be consistient with one from (4.15a)
\[
\gamma = (A, \alpha), \ A = 1, \ldots, n - m, \alpha = n - m + 1, \ldots, n, \ \hat{\alpha} = (\hat{m}_+, \hat{m}_-),
\]
that the relation is valid in superdomain \(V \supset \Sigma\)
\[
\text{rank} \left\| \frac{\partial r}{\partial (\partial^i_0 A^l(\theta_1))} \frac{\partial S_L(\theta)}{\partial (\partial^i_0 A^l(\theta))} \right\|_{T_{odd} V} = \text{rank} \left\| \frac{\partial r}{\partial (\partial^i_0 A^l(\theta_1))} \frac{\partial S_L(\theta)}{\partial (\partial^i_0 A^l(\theta))} \right\|_{T_{odd} V} = n - \hat{m}.
\]
(4.17)

So, the conditions (4.13)–(4.15) mean that for the superfields
\[
\hat{A}^l(\theta) = A^l(\theta) - A_0^l(\theta), \ \hat{A}_0^l(\theta) = 0 \in \Sigma,
\]
(4.18)

the following representation as in (A.22) is valid providing, at least, quadratic dependence upon \(\hat{A}^l(\theta), \partial^i_0 \hat{A}^l(\theta)\) for \(S_L(\theta)\)
\[
\Theta_i(A(\theta), \partial^i_0 A(\theta), \theta) = \Theta_{i \text{lin}}(\hat{A}(\theta), \partial^i_0 \hat{A}(\theta), \theta) + \Theta_{i \text{nlin}}(\hat{A}(\theta), \partial^i_0 \hat{A}(\theta), \theta),
\]
\[
\left( \min \text{deg}_\theta \hat{A}(\theta), \text{deg}_\theta \hat{A}(\theta) \right) \Theta_{i \text{lin}}(\theta) = (1, 1), \ \min \text{deg}_\theta \hat{A}(\theta), \text{deg}_\theta \hat{A}(\theta) \Theta_{i \text{nlin}}(\theta) \geq 2.
\]
(4.19)

Whereas the hypothesis 2 gives the possibility to represent DCLF in the form of 2 subsystems
being especially important for the field (infinite-dimensional) case, when the requirements of locality and covariance w.r.t. index \(i\) appears by obstacles to the condition (4.15) fulfilment.

With help of general relation (2.36) in accordance with (A.19), (A.20) we mean under rank
(4.15b) calculation the rule
\[
\text{rank} \left\| \frac{\delta l}{\delta A^l(\theta_1)} \frac{\delta Z[A]}{\delta A^l(\theta)} \right\|_\Sigma = \text{rank} \left\| \left( \frac{L^l_j[S_L(\theta)](\theta)}{\delta A^l(\theta)} \right) \right\|_{T_{odd} \Sigma} \delta(\theta_1 - \theta)(-1)^{\varepsilon_i} = n - m.
\]
(4.20)

In particular, in view of the same definition (A.20) and its consequence (A.21), the following formula holds for \(F[A]\) of the form (2.33) subject to condition below
\[
\text{rank} \left\| \frac{\delta l}{\delta A^l(\theta)} \frac{\delta F[A]}{\delta A^l(\theta)} \right\|_{T_{odd} \Sigma} \delta(\theta_1 - \theta)(-1)^{\varepsilon_j + \varepsilon(F)} \text{deg}_\theta, \text{deg}_\theta(\frac{L^l_j[F]}{\delta A^l(\theta)})(\theta) = 0.
\]
(4.21)
In the framework of assumptions 1–3 the following fundamental theorem about structure of DCLF $\Theta_i(\theta)$ is valid.

**Theorem** (on reduction of the 1st order on $\theta$ $n$ ODE system to equivalent equations in GNF) A nondegenerate parametrization for superfields $A^i(\theta)$ exists

$$A^i(\theta) = (\delta^i(\theta), \beta^i(\theta), \xi^a(\theta)), \quad i = (i, \xi, \alpha) \equiv (A, \alpha), \quad i = 1, \ldots, n - m,$$

$$\xi = n - m + 1, \ldots, n - m, \quad m = (m_-, m_+), \quad A = 1, \ldots, n - m, \quad \alpha = n - m + 1, \ldots, n,$$  \[(4.22)\]

so that the 1st order on $\theta$ $n$ ODE system w.r.t. unknowns $A(\theta)$ (4.8b) is equivalent to the following independent ODE in generalized normal form (GNF)

$$\partial_\theta \delta^i(\theta) = \phi^i(\delta(\theta), \partial_\theta A^i(\theta), \xi(\theta)), \quad \beta^i(\theta) = \kappa^i(\delta(\theta), \xi(\theta), \theta)$$ \[(4.23)\]

with $\phi^i(\theta), \kappa^i(\theta) \in C^k$ and with arbitrary superfields $\xi^a(\theta)$: $[\xi(\theta)] = m$, $0 < m < n$. Their $(\xi^a(\theta))$ number coincides with one of the differential identities among Eqs. (4.8b)

$$\int d\theta \frac{\delta Z[A]}{\delta A^i(\theta)} \hat{R}_\alpha^i(\theta; \theta') = 0,$$

$$\varepsilon^i \hat{R}_\alpha^i(\theta; \theta') = (1 + (\varepsilon P)_\alpha + (\varepsilon P)_\xi, (\varepsilon J)_\xi, (\varepsilon J)_\alpha, \varepsilon, \xi + \varepsilon + 1)$$ \[(4.24)\]

with a) local and b) functionally independent operators $\hat{R}_\alpha^i(\theta; \theta') \equiv \hat{R}_\alpha^i(\theta; \theta')$:

$$\hat{R}_\alpha^i(\theta; \theta') = \sum_{k=0}^1 \left( \partial_\theta^k \delta_\theta (\theta - \theta') \right) \hat{R}_{k\alpha}^i(\theta; \theta'),$$ \[(4.25a)\]

$$\varepsilon^i \hat{R}_{k\alpha}^i(\theta) = (\delta_{1k} + (\varepsilon P)_\xi + (\varepsilon P)_\alpha, (\varepsilon J)_\xi + (\varepsilon J)_\alpha, \varepsilon, \xi + \varepsilon + \delta_{1k}),$$ \[(4.25b)\]

b) functional equation

$$\int d\theta' \hat{R}_\alpha^i(\theta; \theta') u^a(\theta'), \partial_\theta A^i(\theta'), \theta') = 0$$ \[(4.26)\]

has the unique vanishing solution.

The theorem above appearing by the special case for Theorem from Appendix A and therefore the important consequences follow from it.

**Corollary 1**

In fulfilling of the condition (4.21), written for $Z[A]$, indicating on HCLF dependence

$$\text{rank} \left| \frac{\partial \Theta_i(A(\theta), \theta)}{\partial A^i(\theta)} \right| \equiv n - m < \text{rank} \left[ \Theta_i(\theta) \right],$$ \[(4.27)\]

a nondegenerate parametrization for superfields $A^i(\theta)$ exists

$$A^i(\theta) = (\varphi^A(\theta), \xi^a(\theta)), \quad A = 1, \ldots, n - m, \alpha = n - m + 1, \ldots, n,$$ \[(4.28)\]

so that HCLF are equivalent to the system of algebraically independent constraints, in the sense of differentiation w.r.t. $\theta$

$$\tilde{\Theta}_A(\varphi(\theta), \xi(\theta), \theta) = 0.$$ \[(4.29)\]

The number of superfields $[\xi(\theta)]$ coincides with one of algebraic (on $\theta$) identities among $\Theta_i(\theta)$

$$\Theta_i(A(\theta), \theta) \hat{R}_{\alpha}^i(\theta; \theta') = 0,$$ \[(4.30)\]
where operators $\mathcal{R}_{\alpha}^i(\mathcal{A}(\theta), \theta)$ can be chosen in the form being consistent with (4.25)

$$\hat{\mathcal{R}}_\alpha^i(\mathcal{A}(\theta), \theta; \theta') = -\delta(\theta - \theta')\mathcal{R}_{\alpha}^i(\mathcal{A}(\theta), \theta).$$

(4.31)

Their linear independence means that equation

$$\mathcal{R}_{\alpha}^i(\mathcal{A}(\theta), \theta)u^\alpha(\mathcal{A}(\theta), \theta) = 0$$

(4.32)

has the unique trivial solution.

One from the realization for Corollary 1 is the

**Corollary 2**

If the model of $\theta$-STF is represented by almost natural system defined in the form

$$S_L(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) = T(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta)) - S(\mathcal{A}(\theta), \theta) \equiv T(\theta) - S(\theta),$$

(4.33)

$$T(\theta) = T_1(\partial_\theta \mathcal{A}(\theta)) + \partial_\theta \mathcal{A}^i(\theta)g_{ij}(\theta)\mathcal{A}^j(\theta), \quad g_{ij}(\theta) = P_0(\theta)g_{ij}(\theta) = (-1)^{\epsilon_i\epsilon_j}g_{ij}(\theta),$$

(4.34)

where $(\operatorname{min \ deg}_{\mathcal{A}(\theta)} S(\theta) = \operatorname{min \ deg}_{\partial_\theta \mathcal{A}(\theta)} T_1(\theta) = 2)$, then DCLF

$$\Theta_i(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) \equiv \hat{\Theta}_i(\mathcal{A}(\theta), \theta) = -S_{ij}(\mathcal{A}(\theta), \theta)\epsilon_i = 0$$

(4.35)

appear by the HCLF explicitly depending upon $\theta$. Condition (4.27) in question has the form

$$\operatorname{rank} \|S_{ij}(\mathcal{A}(\theta), \theta)\| \sum = n - m < n,$$

(4.36)

and expressions (4.30)–(4.32) remain valid in this case.

If for DCLF (4.8b) (HCLF $\Theta_i(\mathcal{A}(\theta), \theta)$) the following conditions are fulfilled almost everywhere in any neighbourhood $U : \mathcal{M}_d \supset U \supset \Sigma$ respectively

$$\operatorname{rank} \left\| \frac{\delta L}{\delta \mathcal{A}^i(\theta)} \frac{\delta L[Z[\mathcal{A}]]}{\delta \mathcal{A}^i(\theta)} \right\| U = n, \quad \left( \operatorname{rank} \|\Theta_{ij}(\mathcal{A}(\theta), \theta)\| U = n \right),$$

(4.37)

then $\Theta_i(\theta)$ appear by functionally (linearly) independent and have been already found in GNF.

The performed investigation of LS makes to be justified an introduction of the following terminology:

1) The model of $\theta$-superfield theory of fields (mechanics) given by superfunction $S_L(\theta) \in C^k$ (or, almost equivalently, by superfunctional $Z[\mathcal{A}] \in C_F$) satisfying to the postulates 1–3 ((4.13)–(4.17)) for $m > 0$ is called the gauge theory of general type (GThGT) for superfields $\mathcal{A}^i(\theta)$, and in fulfilling of the 1st condition in (4.37) the nondegenerate theory of general type (ThGT):

2) If, in addition, the Corollary 2 conditions on HCLF (4.27) for $m > 0$ are fulfilled, then the model of $\theta$-superfield theory of fields (mechanics) is called the gauge theory of special type (GThST), and in realizing of the 2nd condition in (4.37) the nondegenerate theory of special type (ThST):

3) Formulation of GThGT and GThST defined by means of $S_L(\theta) \in C^k$ ($Z[\mathcal{A}] \in C_F$) let us call the Lagrangian formalism of description for GThGT and GThST, or equivalently the Lagrangian formalism (formulation) of $\theta$-STF.

Identities (4.24) for GThGT ((4.30) for GThST) with operators $\hat{\mathcal{R}}_\alpha^i(\theta; \theta')$ ($\mathcal{R}_{\alpha}^i(\mathcal{A}(\theta), \theta)$), whose set is complete and functionally (linearly) independent, i.e. is the basis in linear space $Q(Z) = \operatorname{Ker} \left\{ \frac{\delta L[Z[\mathcal{A}]]}{\delta \mathcal{A}^i(\theta)} \right\}$, $\left( Q(S_L) = \operatorname{Ker} \{\Theta_i(\theta)\} \right)$, make to be possible the following interpretation for quantities $\hat{\mathcal{R}}_\alpha^i(\theta; \theta'), (\mathcal{R}_{\alpha}^i(\theta))$:
1) Any quantities \( \hat{\mathcal{R}}^i(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) \equiv \hat{\mathcal{R}}^i(\theta), \mathcal{R}^i_0(\mathcal{A}(\theta), \theta) \equiv \mathcal{R}^i_0(\theta) \) satisfying to identities

\[
\int d\theta \frac{\delta Z[\mathcal{A}]}{\delta A^i(\theta)} \hat{\mathcal{R}}^i(\theta) = 0, \quad \Theta_i(\mathcal{A}(\theta), \theta) \mathcal{R}^i_0(\theta) = 0, \quad \varepsilon \hat{\mathcal{R}}^i(\theta) = \varepsilon \mathcal{R}^i_0(\theta) = ((\varepsilon_p)_i, (\varepsilon_j)_i, \varepsilon_i) \quad (4.38)
\]

are called the generator of general type gauge transformations (GGTGT) and generator of special type gauge transformations (GGST) respectively;

2) The quantities \( \hat{\tau}^i(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) \equiv \hat{\tau}^i(\theta), \tau^i_0(\mathcal{A}(\theta), \theta) \equiv \tau^i_0(\theta) \) being by particular cases for \( \hat{\mathcal{R}}^i(\theta), \mathcal{R}^i_0(\theta) \)

\[
\hat{\tau}^i(\theta) = \int d\theta \frac{\delta Z[\mathcal{A}]}{\delta A^i(\theta)} \hat{E}^{ij}(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta); \theta, \theta'), \quad \tau^i_0(\theta) = \Theta_j(\mathcal{A}(\theta), \theta) E^{ij}_0(\mathcal{A}(\theta), \theta), \quad (4.39)
\]

are called the trivial GGTGT, GGST respectively, where the superfunctions \( E^{ij}_0(\theta) \in C^k(\mathcal{M}_\text{cl} \times \{\theta\}) \) and \( \hat{E}^{ij}(\theta, \theta') \in C^k(T_\text{odd} \mathcal{M}_\text{cl} \times \{\theta, \theta'\}) \) obey to the properties

\[
\hat{E}^{ij}(\theta, \theta') = -(1)^{(\varepsilon_j+1)(\varepsilon_i+1)} \hat{E}^{ij}(\theta', \theta), \quad E^{ij}_0(\theta) = -(1)^{\varepsilon_j} E^{ij}_0(\theta), \quad (4.40a)
\]

\[
\varepsilon(\hat{E}^{ij}_0(\theta)) = \varepsilon(\hat{E}^{ij}(\theta, \theta')) + (1, 0, 1) = ((\varepsilon_p)_i, (\varepsilon_j)_j, (\varepsilon_j)_j, \varepsilon_i + \varepsilon_j). \quad (4.40b)
\]

Trivial GGST \( \tau^i_0(\theta) \) defined by means of \( E^{ij}_0(\theta) \) can be always represented in the form of trivial GGTGT \( \hat{\tau}^i(\mathcal{A}(\theta), \theta) \) with corresponding \( \hat{E}^{ij}(\mathcal{A}(\theta), \theta; \theta') \in C^k(\mathcal{M}_\text{cl} \times \{\theta, \theta'\}) \) \(^6\)

\[
\hat{E}^{ij}(\mathcal{A}(\theta), \theta; \theta') = -\delta(\theta - \theta') E^{ij}_0(\mathcal{A}(\theta), \theta), \quad \hat{\tau}^i(\mathcal{A}(\theta), \theta) = \tau^i_0(\mathcal{A}(\theta), \theta). \quad (4.41)
\]

Taking account of completeness for quantities \( \hat{\mathcal{R}}^i_0(\theta, \theta') \) and \( \mathcal{R}^i_0(\theta) \) and definitions (4.38),(4.39) the any GGTGT, GGST one can represent by the corresponding formulæ

\[
\hat{\mathcal{R}}^i(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) = \int d\theta \hat{\mathcal{R}}^i_0(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta; \theta') \hat{\xi}^\alpha(\mathcal{A}(\theta'), \partial_\theta' \mathcal{A}(\theta'), \theta) + \hat{\tau}^i(\theta), \quad (4.42)
\]

\[
\mathcal{R}^i_0(\mathcal{A}(\theta), \theta) = \mathcal{R}^i_0(\mathcal{A}(\theta), \theta) \xi_0^\alpha(\mathcal{A}(\theta), \theta) + \tau^i_0(\mathcal{A}(\theta), \theta), \quad (4.43)
\]

\[
(\hat{\xi}^\alpha, \xi_0^\alpha) \in (C^k, C^k(\mathcal{M}_\text{cl} \times \{\theta\})) \to \xi_\alpha(\theta) = \xi_\alpha(\theta) = ((\varepsilon_p)_\alpha, (\varepsilon_j)_\alpha, \varepsilon_\alpha),
\]

which turn the linear spaces \( Q(Z), Q(S_L) \) into affine \( C^k(T_\text{odd} \mathcal{M}_\text{cl} \times \{\theta\}) \), \( C^k(\mathcal{M}_\text{cl} \times \{\theta\}) \)-modules respectively.

At last, GGTGT and GGST are defined (as the basis elements of \( Q \)) up to affine transformations of modules \( Q(Z) \) and \( Q(S_L) \) respectively (so-called equivalence transformations)

\[
\hat{\mathcal{R}}^i_\alpha(\theta, \theta') = \int d\theta_1 \hat{\mathcal{R}}^i_\alpha(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta_1; \theta') + \frac{\delta Z[\mathcal{A}]}{\delta A^i(\theta_1)} \hat{E}^{ij}_\alpha(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta, \theta_1; \theta') \quad \hat{\tau}^i(\theta), \quad (4.44a)
\]

\[
\mathcal{R}^i_0(\mathcal{A}(\theta), \theta) = \mathcal{R}^i_0(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta) \xi_0^\alpha(\mathcal{A}(\theta), \theta) + \Theta_j(\mathcal{A}(\theta), \theta) E^{ij}_0(\mathcal{A}(\theta), \theta), \quad \mathcal{R}^i_0(\mathcal{A}(\theta), \theta) \in C^k(T_\text{odd} \mathcal{M}_\text{cl} \times \{\theta, \theta'\}), (4.44b)
\]

\[
(\xi^\alpha, \xi_0^\alpha) \in (C^k, C^k(\mathcal{M}_\text{cl} \times \{\theta\})) \to \xi_\alpha(\theta, \theta') = \xi_\alpha(\theta, \theta') + (1, 0, 1) = ((\varepsilon_p)_\alpha + (\varepsilon_j)_\alpha, (\varepsilon_j)_\alpha, \varepsilon_\alpha + \varepsilon_\beta), \quad (4.45)
\]

where superfunctions \( \xi^\alpha(\theta, \theta') \) belong to \( C^k(T_\text{odd} \mathcal{M}_\text{cl} \times \{\theta, \theta'\}) \), \( \hat{E}^{ij}_\alpha(\theta, \theta') \in C^k(T_\text{odd} \mathcal{M}_\text{cl} \times \{\theta, \theta'\}) \), \( \xi_\alpha(\theta, \theta') \in C^k(\mathcal{M}_\text{cl} \times \{\theta\}) \) and possess by the properties

\[
E^{ij}_0(\mathcal{A}(\theta), \theta) = -(1)^{\varepsilon_j} E^{ij}_0(\mathcal{A}(\theta), \theta), \quad \text{rank} \| \hat{\xi}^\alpha(\theta, \theta') \| = \text{rank} \| \xi_0^\alpha(\theta) \| = [\alpha] = m, \quad (4.46)
\]

\[
E^{ij}_0(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta, \theta_1; \theta') = -(1)^{\varepsilon_j+1} E^{ij}_0(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta_1; \theta').
\]
The general type quantities \( \hat{E}_\alpha^\nu(\mathcal{A}(\theta), \theta'; \theta') \), \( \hat{\xi}_\alpha^\beta(\mathcal{A}(\theta), \theta; \theta') \) (with accuracy up to sign factor \((-1)^K, K \in \mathbb{N}\)

\[
\hat{\xi}_\alpha^\beta(\theta; \theta') = \delta(\theta - \theta')\xi_{\alpha\beta}(\theta), \quad \hat{E}_\alpha^\nu(\mathcal{A}(\theta), \theta'; \theta') = -\delta(\theta_1 - \theta')\delta(\theta_2 - \theta')E_{\alpha\beta}^\nu(\theta)(-1)^{\nu_j},
\]

providing together with relations (4.5), (4.31), (4.41) the conversion of \( Q(S_L) \) into \( C^k(M_{cl} \times \{\theta, \theta'\}) \)-submodule of \( C^k(T_{odd}M_{cl} \times \{\theta, \theta'\}) \)-module \( Q(Z) \). In its turn the locality w.r.t. \( \theta \) for transformed \( \tilde{R}_\alpha^\nu(\theta; \theta') \) (4.44) will be guaranteed by locality of \( \hat{\xi}_\alpha^\beta(\theta; \theta') \), \( \hat{E}_\alpha^\nu(\theta; \theta') \).

Finally, relations (4.24) are easily interpreted for GThGT as \( Z[\mathcal{A}] \) invariance w.r.t. infinitesimal transformations of superfields \( \mathcal{A}(\theta) \)

\[
\mathcal{A}(\theta) \mapsto \mathcal{A}'(\theta) = \mathcal{A}(\theta) + \delta \mathcal{A}(\theta), \quad \delta \mathcal{A}(\theta) = \int d\theta' \hat{R}_\alpha^\nu(\theta; \theta')\xi^\alpha(\theta'),
\]

with arbitrary superfields \( \xi^\alpha(\theta') \in \tilde{\Lambda}_{D[Nc+1]}(\mathbb{Z}_n, \theta; \mathbb{K}) \) whose parities defined as in (4.43). Really, the formula holds

\[
Z[\mathcal{A}] = Z[\mathcal{A}] + \int d\theta' \frac{\delta Z[\mathcal{A}]}{\delta \mathcal{A}'}(\theta) \int d\theta' \hat{R}_\alpha^\nu(\theta; \theta')\xi^\alpha(\theta') + F[\mathcal{A}; \xi] = Z[\mathcal{A}] + F[\mathcal{A}; \xi], \quad \text{min deg}_{\xi(\theta)}F = 2.
\]

The relation (4.30) can not, in general, be interpreted for GThST as the invariance of \( S_L(\mathcal{A}(\theta), \partial_\theta \mathcal{A}(\theta), \theta) \) w.r.t. the transformations with arbitrary \( \xi_0^\alpha(\theta) \)

\[
\mathcal{A}(\theta) \mapsto \mathcal{A}'(\theta) = \mathcal{A}(\theta) + \delta \mathcal{A}(\theta), \quad \delta \mathcal{A}(\theta) = R^\nu_{0\alpha}(\mathcal{A}(\theta), \theta)\xi^\alpha_0(\theta).
\]

However, for superfunction \( S(\mathcal{A}(\theta), \theta) \), defined as in Corollary 2 the real invariance takes place

\[
S(\mathcal{A}'(\theta), \theta) = S(\mathcal{A}(\theta), \theta) + S_\nu(\mathcal{A}(\theta), \theta)R^\nu_{0\alpha}(\mathcal{A}(\theta), \theta)\xi_0^\alpha(\theta) + F(\mathcal{A}(\theta), \xi_0(\theta), \theta) = S(\mathcal{A}(\theta), \theta) + F(\mathcal{A}(\theta), \xi_0(\theta), \theta) \quad \text{min deg}_{\xi_0(\theta)}F = 2.
\]

Therefore in view of definitions above it is natural to call the transformations (4.48), (4.50) as the general type gauge transformations (GThG) for \( Z[\mathcal{A}] \) and the special type gauge transformations (GThST) for \( S(\mathcal{A}(\theta), \theta) \) respectively. The identities (4.24), (4.30) interpretation in terms of GThG, GThST permit to call them by Noether’s identities for GThG, GThST respectively.

Further to be more simplifiedly one can regard that GThGT (GThST) is defined by relations (4.4), (4.15), (4.24) ((4.27), (4.30), (4.17)) and for the existence of only trivial solution for Eqs.(4.26) (Eqs.(4.32)) is called the irreducible GThGT (GThST), otherwise the reducible GThGT (GThST).

V Generating Equations. Zero Locus Reduction Problems

Quantities \( Z[\mathcal{A}], S(\mathcal{A}(\theta), \theta), \) GThGT \( \tilde{R}_\alpha^\nu(\theta; \theta'), \) GThST \( R^\nu_{0\alpha}(\theta) \) together with identities (4.24), (4.30) as the first structural relations of the corresponding general and special types gauge algebras (GA) being by differential-algebraic systems on \( Q(Z), Q(S_L) \) are effectively described by means of 2 special generating (master) equations for not uniquely defined superfunctional \( Z(1)_{\{\Gamma_{min}\}} \equiv Z(1) \) and superfunction \( S(1)_{\{\Gamma_{min}(\theta), \theta\}} \equiv S(1)(\theta) \). The latters appear by corresponding deformations in powers of ghost superfields \( (C^\alpha(\theta), \partial_\theta C^\alpha(\theta)) \) and only \( C^\alpha(\theta) \) respectively into the supermanifolds with following local coordinates specified here for the case of irreducible GThGT, GThST

\[
T_{odd}(T_{odd}^sM_s) \times \{\theta\} = \{(\Gamma^p_S, \theta_B^pS(\theta), \theta)\}, \quad p = 1, 2(n + m), \quad s = min,
\]

\[
T_{odd}^sM_s \times \{\theta\} = \{(\Gamma^p_S(\theta) = (\Phi^B_S, \Phi^B_{Bs}(\theta), \theta))\}, \quad \Phi^B_S(\theta) = (\mathcal{A}', C^\alpha(\theta), B = 1, n + m, (5.1)
\]
The 1st geometric object above may be considered as so-called odd tangent bundle over odd cotangent bundle $T_{odd}^* \mathcal{M}_{min}$, in turn over supermanifold $\mathcal{M}_{min}$ coordinatized by $Φ_{min}^B(θ)$ in the so-called minimal sector [5].

The new superfields $C^α(θ)$, superantifields $(\mathcal{A}^*_α, C^*_α(θ))$ as the elements from $\Lambda_D|N_{c+1}(ε^a, θ; K)$ are transformed w.r.t. $J$ superfield representations $T_ε^J, T^*_J, T^*_ε$ connected with $T$ and transformations for $ξ^α(θ)$ (for instance, $T^*$ is conjugate to $T$ w.r.t. a some bilinear form). Grassmann parities and expansion in powers of $θ$ for $Φ_B^B(θ), Φ_{ Bs}(θ)$ are given as follows

$$(Φ_B^B, Φ_{ Bs})(θ) = (Φ_B^B + λ^B_θ, Φ_{ Bs} - θJ Bs), \quad \varepsilon C^α(θ) = ((ε_p)_α + 1, (ε_J)_α, ε_α + 1),$$

$$\varepsilon(Φ_B^B(θ), J Bs) = \varepsilon(Φ_{ Bs}(θ), λ^B_s) + (1, 0, 1) = ((ε_p)_B, (ε_J)_B, ε_B). \quad (5.2)$$

The superfunctional $Z_1 = \partial_θ S_1(Γ_s(θ), θ, S_1(θ))$ is conjugate to $G_{ThGT}$, its generating equation and $P, J$-even superfunctional $S_1(θ)$ together with one’s $θ$-local master equation for $G_{ThST}$ have the representations for $s = min$ with accuracy up to 1st degree in $C^α(θ)$ (therefore exact for abelian GAs)

$$Z_1[Γ_s] = Z[Α] + \int dθ_2 A^*_α(θ_1) R^*_α(θ_1; θ_2) C^α(θ_2)(-1)ε_1 + O(C^2(θ)), \quad \{Z_1[Γ_s], Z_1[Γ_s]\} = 0; \quad \{Z_1[Γ_s], Z_1[Γ_s]\}_θ = 0. \quad (5.3)$$

The even $\{ , \}$ and odd $θ$-local Poisson brackets are defined on class of superfunctionals $C_{F_{min}} \supset C_F$ on $\mathcal{M}_{min}$ given as in (2.33) via densities on $θ$, i.e. superfunctions from superalgebra $C_k^k(T_{odd}^*|T_{odd}^* \mathcal{M}_{min}) \times \{ \} \equiv D_{min}^k$, with expansion properties to be analogous to (2.24), (2.27) w.r.t. all supervariables, and on the superalgebra $C_k^k(T_{odd}^*|T_{odd}^* \mathcal{M}_{min}) \times \{ \} \equiv C_{min}^k$, respectively by the formulae

$$\{F[Γ_s], G[Γ_s]\} = \int dθ \left[ \frac{δF}{δΦ_B(θ)} \right] \frac{δG}{δΦ_B(θ)} - \left[ \frac{δF}{δΦ_B(θ)} \right] \frac{δG}{δΦ_B(θ)}, \quad (5.5)$$

$$\{F(Γ_s(θ), G(Γ_s(θ), θ))\}_θ = \left[ \frac{δF}{δΦ_B(θ)} \right] \frac{δG}{δΦ_B(θ)} - \left[ \frac{δF}{δΦ_B(θ)} \right] \frac{δG}{δΦ_B(θ)}, \quad (5.6)$$

defining the corresponding QP-structure [14] on the bundles $T_{odd}^*(T_{odd}^* \mathcal{M}_{min}), T_{odd}^* \mathcal{M}_{min}$. The calculation rules for superfield variational in (5.5) and partial (5.6) derivatives w.r.t. $Φ_B^B(θ)$ are based on the expressions (with omitting of index "min")

$$\left( \frac{∂Φ^*_C}{∂Φ^*_B(θ)}, \frac{∂Φ^*_C}{∂Φ^*_B(θ)}, \frac{∂Φ^*_C}{∂Φ^*_B(θ)}, \frac{∂Φ^*_C}{∂Φ^*_B(θ)} \right) = (1, 1, (-1)^{ε_C+1} δ(θ - θ_1), 1) δC^B, \quad (5.7)$$

added by the Euler-Lagrange operator w.r.t. $Φ_B^B(θ)$ written in terms of superfunctional $F[Γ]$ and its density $F \in D_{min}^k$

$$\frac{δ_{(r)}F[Γ]}{δΦ_B(θ)} = \left[ \frac{δ_{(r)}F}{δΦ_B(θ)} \right] - (-1)^{ε_θ+1} δ_{(r)}^θ \left[ \frac{δ_{(r)}Φ^*_B}{δ(δ^r_θ)Φ^*_B(θ)} \right] \quad (5.8)$$

The even bracket (5.5) may be expressed through the new $θ$-local antibracket (5.6) provided that $D[Γ]$ are written via one’s density $D(θ) \in D_{min}^k$ taking the formula (5.8) into account

$$\{F[Γ], G[Γ]\} = \int dθ(F(θ), G(θ))_{\{Γ, θ\}} = \int dθ \left[ \left[ L_B^r F, L_C^r G \right] \right] = \left[ \left[ L_B^r F, L_C^r G \right] \right], \quad (5.9)$$

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Relationship (5.9) appears by natural generalization for the connection of odd and even Poisson brackets from ref. [30] for the case of the densities dependence upon superfields \( \mathcal{A}_p(\theta) \) and contains, in fact, 4 antibrackets.

As the consequence the generating equation (5.3) written for the superfunctional \( Z_{(0)}[\Gamma] = \int d\theta S_{(1)}(\theta) \) is embedded into the same equation but for \( Z_{(1)}[\Gamma] \) providing, at least for abelian GAs of GTGT and GTST, the embedding of the latter GA into former one

\[
\{Z_{(0)}[\Gamma_{\text{min}}], Z_{(0)}[\Gamma_{\text{min}}]\} = \int d\theta (S_{(1)}(\theta), S_{(1)}(\theta))_\theta = 0, \quad Z_{(1)}[\Gamma_{\text{min}}] = Z_{(0)}[\Gamma_{\text{min}}] + \ldots \quad (5.10)
\]

Numbering even and odd Poisson brackets as \( k = 0, k = 1 \) respectively it is easy to check the standard properties validity of generalized antisymmetry, Leibnitz rule, Jacobi identity

\[
\{D, E\}_k = -(1)^{(\varepsilon(D)+k)(\varepsilon(E)+k)}\{E, D\}_k,
\]

\[
\{DE, K\}_k = D\{E, K\}_k + \{D, K\}_k E(-1)^{(\varepsilon(E)+k)(\varepsilon(K)+k)},
\]

\[
\{\{D, E\}_k, K\}_k (-1)^{(\varepsilon(D)+k)(\varepsilon(K)+k)} + \text{cycl.perm.}(D, E, K) = 0,
\]

with superfunctionals for \( k = 0 \) and \( \theta \)-local superfunctions for \( k = 1 \) instead of \( D, E, K \).

The master equations (5.3), (5.4) allow to solve the \( \theta \)-superfield problem of ZLR [35] (through duality between superbrackets [36]) simultaneously with new procedure to obtain the \( \theta \)-superfield (and therefore almost standard for \( \theta = 0 \)) models on the reduced (anti)symplectic spaces starting from initial GTGT, GTST. Indeed, from validity of (5.3), (5.4) it follows the definition for new superfunctional antibracket and even \( \theta \)-local bracket on the corresponding (at least locally) supermanifolds given by zero locus \( Z_{Q^0} \) for \( (\varepsilon_p, \varepsilon) \) odd interrelated nilpotent vector fields \( Q^b, b = 0, 1 \)

\[
Q^0(\theta) = (S_{(1)}(\theta), \ )_\theta, \quad Q^1 = \{Z_{(1)}[\Gamma], \}
\]

\[
\{f(\theta), g(\theta)\}_\theta = (\mathcal{F}(\theta), (S_{(1)}(\theta), G(\theta))_\theta|_{Z_{Q^0}}, \mathcal{F}, G \in C^k_{\text{min}}, (f; g) = (\mathcal{F}; G)|_{Z_{Q^0}}, \quad (5.13)
\]

\[
(f[\Gamma], g[\Gamma]) = \{F[\Gamma], (Z_{(1)}[\Gamma], G[\Gamma])\}|_{Z_{Q^1}}, \quad (\tilde{f}; \tilde{g}) = (F; G)|_{Z_{Q^1}}.
\]

Given this, the explicit form for nonlinear \( \theta \)-local even bracket together with specification for \( Z_{Q^0} \) structure in \( T_{\text{odd}}^{\ast}\mathcal{M}_{\text{min}} \) are defined by the equations with accuracy up to \( O(\mathcal{A}^4 \mathcal{C}) \) and terms, at least, linear in \( \mathcal{A}^4 \), \( \mathcal{C} \)

\[
Q^0(\theta)\Gamma_{\text{min}}^p(\theta) = 0 \Rightarrow \left\{ \mathcal{R}_{\alpha\beta}(\theta)C^o(\theta) = 0,
\right.

\[
S_{\alpha}(\theta) + \mathcal{A}_{\beta}(\theta)\mathcal{R}_{\alpha\beta}(\theta)C^o(\theta)(-1)^{\varepsilon_\beta(\varepsilon_\alpha+1)} = 0,
\left. \mathcal{A}_{\beta}(\theta)\mathcal{R}_{\alpha\beta}(\theta) = 0. \quad (5.15) \right\}
\]

The irreducibility of GGTST (4.32) permits to solve (5.15) in the form

\[
C^o(\theta) = 0, \quad S_{\alpha}(\theta) = 0, \quad \mathcal{A}_{\beta}(\theta)\mathcal{R}_{\alpha\beta}(\theta)|_{S_{\alpha}(\theta)=0} = 0.
\]

and therefore to get the representation for \( \{ , \}_\theta \)

\[
\{f(\theta), g(\theta)\}_\theta = \left[ \frac{\partial_r \mathcal{F}(\theta)}{\partial \mathcal{A}_{\beta}(\theta)}S_{r\beta}(\theta)\frac{\partial \mathcal{G}(\theta)}{\partial \mathcal{A}_{\gamma}(\theta)}(-1)^{\varepsilon_{\gamma}} + \left( \frac{\partial \mathcal{F}(\theta)}{\partial \mathcal{A}_{\beta}(\theta)}\mathcal{R}_{\alpha\beta}(\theta)\frac{\partial \mathcal{G}(\theta)}{\partial \mathcal{C}_{\gamma}(\theta)} \right) \right]_{|Z_{Q^0}}.
\]
In turn, the structure of $Z_{Q^1}$ embedded in $T_{odd}(T_{odd}^*\mathcal{M}_{min})$ are given by the equations with the same accuracy as for $Z_{Q^0}$ in (5.15) extended for $\partial_b(\mathcal{A}^*, C, C^*)(\theta)$ as well

$$Q^1\Gamma_{min}^p(\theta) = 0 \Rightarrow \begin{cases} \int d\theta' \hat{R}_\alpha^1(\theta; \theta') C^{\alpha}(\theta') = 0, \\ (L_\epsilon^* S_L)(\theta) - \int d\theta_2 A_2^\epsilon(\theta_1) \frac{\partial \hat{R}^\alpha_\epsilon(\theta_2)}{\partial A^\epsilon(\theta_1)} C^{\alpha}(\theta_2)(-1)^{\epsilon_1(\epsilon_1+1)} = 0, \\ (-1)^{\epsilon_1} \int d\theta A_1^\epsilon(\theta) \hat{R}^\alpha_\epsilon(\theta; \theta') = 0. \end{cases}$$  

(5.18)

With regard for Eq.(4.26) trivial solution for irreducible GThGT the system above is reduced to the form

1) $C^{\alpha}(\theta) = 0, \ L_\epsilon^*(\theta) S_L(\theta) = 0, \ \partial_b A_1^\epsilon(\theta) \hat{R}^\alpha_\epsilon(\theta; \theta') = 0,$

2) $\partial_b C^{\alpha}(\theta) = 0, \ (\frac{\partial}{\partial \theta} + \hat{U}_+ (\theta)) S_L(\theta) = 0, \ \partial_b \partial_{\theta'} A_1^\epsilon(\theta) \hat{R}^\alpha_\epsilon(\theta; \theta') = 0,$

(5.19)

providing the explicit structure for antibracket (5.14)

$$\begin{align*}
(f[\Gamma], \tilde{g}[\Gamma]) &= \int d\theta \left[ \frac{\delta F[\Gamma]}{\delta A_i^*(\theta)} \int d\theta_1 \frac{\delta Z[A_1^\epsilon(\theta)]}{\delta A^\epsilon(\theta_1)} \frac{\delta G[\Gamma]}{\delta A^\epsilon(\theta_1)} \right] (-1)^{\epsilon_1} - \left( \frac{\delta F[\Gamma]}{\delta A^\epsilon(\theta)} \right) \int d\theta_1 \hat{R}^\alpha_\epsilon(\theta; \theta_1) \frac{\delta G[\Gamma]}{\delta C^\alpha_\epsilon(\theta_1)} \\
&\quad - \frac{\delta F[\Gamma]}{\delta A_i^*(\theta)} \int d\theta_2 A_2^\epsilon(\theta_1) \frac{\delta \hat{R}^\alpha_\epsilon(\theta_2)}{\delta A^\epsilon(\theta_1)} \frac{\delta G[\Gamma]}{\delta C^\alpha_\epsilon(\theta_2)} (-1)^{\epsilon_1, \epsilon_2} - (-1)^{(\epsilon(F) + 1)(\epsilon(G) + 1)} (F \leftrightarrow G) \right] \bigg|_{Z_{Q^1}}.
\end{align*}$$

(5.20)

The validity of the properties (5.11) for brackets on $Z_{Q^0}$ directly ensues from master equations (5.3), (5.4) and the same properties above but for initial brackets (5.5), (5.6). The 1st 2 summands (1st in square and 1st in round parentheses) in (5.17), (5.20) permit to define on $Z_{Q^0}, Z_{Q^1}$ correspondingly the nondegenerate 2-forms $\omega^\alpha_1(\theta), \omega^\alpha_2(\theta)$ (and the 1st term in (5.17) corresponds for $\epsilon = 0$ in appropriate basis for superfields $A^\epsilon(\theta)$ of the form (4.15a), specially compatible with other superfields from $\Gamma^p(\theta)$, to the symmetric part $\wedge^2 (\alpha) \triangleleft \wedge^2 (\alpha)$ of even 2-form $\omega^\alpha_2(\theta), A = 1, n-m [21]$). The rest terms guarantee the Jacobi identities validity, i.e. to be closed for $\omega^\alpha_2(\theta)$ and hence to be for $(Z_{Q^0}, Z_{Q^1})$ not only Poisson but by the (anti)symplectic, at least locally, supermanifolds.

There exists the inverse inclusion of the new even and odd brackets (5.13), (5.14) given on the corresponding superalgebras of superfunction(al)s modulo the superfunction(al)s vanishing on $Z_{Q^0}$. Really, having restricted GThGT with $Z_{(1)}[\Gamma]$ (5.3) to GThST with $Z_{(0)}[\Gamma]$ satisfying to (5.10) we immediately find the relationship, taking correspondence between superfunction(al)s in (5.13), (5.14) into account and under condition for $F[\Gamma], G[\Gamma]$ in (5.14) to be defined through densities only on $T_{odd}^*\mathcal{M}_{min} \times \{\theta\}$, $\mathcal{Q}^1 = \{Z_{(0)}[\Gamma], G[\Gamma]\}$

$$\begin{align*}
(f[\Gamma], \tilde{g}[\Gamma])|^{Z_{(0)}[\Gamma]} &= - \int d\theta \{f(\Gamma(\theta), \theta), g(\Gamma(\theta), \theta)\}_{\theta}, \\
(f[\Gamma], \tilde{g}[\Gamma])|^{Z_{(0)}[\Gamma]} &= \{F[\Gamma], \{Z_{(0)}[\Gamma], G[\Gamma]\}\}|^{Z_{Q^1}}, \quad \mathcal{Q}^1 = \{Z_{(0)}[\Gamma], \}
\end{align*}$$

(5.21) (5.22)

in turn based on being easily derived from (5.16) and (5.5), (5.6) expressions

$$\begin{align*}
\{Z_{(0)}[\Gamma], G[\Gamma]\} &= - \int d\theta \{S_{(1)}(\Gamma(\theta), \theta), G(\Gamma(\theta), \theta)\}_{\theta}, \quad G = \partial_b g, (G|_{Z_{Q^1}}) = (\tilde{g}, g), \\
Z_{Q^1} \cap T_{odd}^*\mathcal{M}_{min} = Z_{Q^1} \cap T_{odd}^*\mathcal{M}_{min} = Z_{Q^0}.
\end{align*}$$

(5.23)

In general, even bracket (5.13) is embedded into antibracket (5.14).

The hierarchy of superbrackets permits to suggest a some different ways to construct the new $\theta$-superfield models starting from initial GThST and GThGT, firstly, as embedded into
depending upon $\Gamma^1$ as the BFV-BRST generators in minimal sector further enlarged in more wide than space coordinatized by $\Phi^\text{tot}$ analogs of Faddeev-Popov supermatrices in the 0-level even bracket (5.5) given in enlarged $Z_{\text{gauge fermion superfunctions}} \Psi_1$ of multiplier superfields and its superantifields. Next it is necessary to define the such $Z_\text{s for generalized Poisson sigma models [31]}$ and results in the 1st level supertime $T^{1}$ defined on $H_\text{Z}$ the result we arrive, in fact, at the action superfunction being similar to superfunction in (5.4) $S_\text{Z}$ The following dynamical equations arise from variational principle for both actions (5.26), (5.27) $\delta_{\theta^1} S_{\text{H}}^{1}(\theta, \Gamma^1) = \partial_{\theta^1} \Gamma^p_{\text{tot}}(\theta, \Gamma^1) = \{ \Gamma^p_{\text{tot}}(\theta, \Gamma^1), \mathcal{H}_1(\Gamma^1) \}_{\Gamma^1}$, (5.28) $\partial^p_{\Gamma^1} \mathcal{H}_1(\Gamma^1) = \{ \Gamma^p_{\text{tot}}(\theta, \Gamma^1), Z_\text{s}(\Gamma^1) \}_{\Gamma^1}, s = \text{min, tot}$, (5.29) with superfield variational derivatives w.r.t. $\Gamma^p_{\text{tot}}(\theta, \Gamma^1)$ for fixed $\theta^1$ in (5.28) and the next level superfunctional one for $s = \text{min, tot}$ in (5.29).
Firstly, note the commutation of $H_{(a)}(\Gamma^1)$ with $Z_{(a)}(\Gamma^1)$ w.r.t. bracket (5.26) leads to compatibility of the Eqs.(5.28),(5.29) for $s = tot$, i.e. the commutator $[\partial_{\theta_{1}}, \partial_{\theta_{2}}]_{\Gamma_{s}^{l}}(\theta, \Gamma^1)$ vanishes on their solutions in view of (5.3), (5.26) equations fulfilment, antisymmetry and Jacobi identity properties (5.11a,c) for $k = 0$. Besides, the generating equations (5.3), (5.10) provide the $\theta$-superfield solvability of the Eqs.(5.28), (5.29) in the sense of (A.6) relation.

Moreover, the actions $S_{(a)}^{l}[\Gamma_{s}]$ obey to the next level master equation with new superfunctional antibracket in the supermanifold $T_{odd}^{(1)}(T_{odd}(T_{odd}^{*}\mathcal{M}_{s}))$

$$\left(\mathcal{L}_{(a)}^{l}[\Gamma_{s}], S_{(a)}^{l}[\Gamma_{s}]\right)^{l} = \int d\Gamma^{l} \left( d\theta_{1}[\partial_{\theta_{1}}(\Phi_{B_{s}}^{l}[\Gamma_{s}])\theta(\Gamma^1) + \{Z_{(a)}(\Gamma^1), Z_{(a)}(\Gamma^1)\}_{\Gamma_{s}} \right)$$

$$+ \partial_{\theta_{1}}Z_{(a)}(\Gamma^1) = 0, s = min, tot, \quad (5.30)$$

$$\left(F^{l}[\Gamma_{s}], G^{l}[\Gamma_{s}]\right)^{l} = \int d\Gamma^{l} d\theta \left[ \frac{\delta_{r}F^{l}[\Gamma_{s}]}{\delta_{r}\Phi_{B_{s}}^{l}(\theta, \Gamma^1)} - \frac{\delta_{l}G^{l}[\Gamma_{s}]}{\delta_{l}\Phi_{B_{s}}^{l}(\theta, \Gamma^1)} \right]$$

$$= \int d\Gamma^{l} d\theta \left[ \mathcal{L}_{B}^{l}(\theta, \Gamma^1)F^{l}(\Gamma^1) - \mathcal{L}_{G}^{l}(\theta, \Gamma^1)G^{l}(\Gamma^1) \right]$$

$$= \int d\Gamma^{l} \left( F^{l}(\Gamma^1), G^{l}(\Gamma^1) \right)_{\Gamma_{s}}^{l} + \delta_{1}(\Gamma_{s}) = \int d\Gamma^{l} D(\Gamma_{s}(\Gamma^1), \partial_{\Gamma_{s}}\Gamma_{s}(\Gamma^1), \Gamma^1), \quad (5.31)$$

where we could not include $t^{1}$ variable in definition of antibracket (5.31) to get the type (5.21) connection and therefore instead of $S_{(a)}^{l}[\Gamma_{s}]$ we should write its density w.r.t. $t^{1}$ in direct analog of the Eq.(5.30). At last, under Euler-Lagrange operator, for instance, for $\Phi_{B_{s}}^{l}(\theta, \Gamma^1)$ in (5.31) we mean the analog of expression (5.8) written for superfunctional w.r.t. $\theta$: $\mathcal{F}(\Gamma^1)$, defined as in (5.31) (with $\delta$-symbol $\delta_{a_{1}}$)

$$\mathcal{L}_{B}^{(r)}(\theta, \Gamma^1)\mathcal{F}(\Gamma^1) = \left[ \frac{\delta_{(r)}^{l}(\theta, \Gamma^1)}{\delta_{(r)}\Phi_{B}^{l}(\theta, \Gamma^1)} - (-1)^{\varepsilon_{B}\delta_{a_{1}}} \frac{\delta_{(r)}^{l}(\theta, \Gamma^1)}{\delta_{(r)}\Phi_{B}^{l}(\theta, \Gamma^1)} \right] \mathcal{F}(\Gamma^1). \quad (5.32)$$

**Resume:**

1) The action $S_{(a)}^{l}(5.27)$ construction defined on the odd tangent bundle with new antibracket (5.31) starting from $Z_{(a)}(\Gamma_{s}(\Gamma^1))$ given on symplectic supermanifold $T_{odd}(T_{odd}^{*}\mathcal{M}_{min})$ with even bracket (5.5) appears by none other than the essence of so-called inverse ZLR problem.

2) Together the superfunction(al)s $S_{(a)}^{l}(H_{(a)}(\Gamma_{tot}(\theta)))$ (5.26), $S_{(a)}^{l}[\Gamma_{s}]$ (5.27) and their extremals (5.28), (5.29) describe in the new superfield manner the BFV-BRST classical construction of the corresponding dynamical system initially defined on $T_{odd}(T_{odd}^{*}\mathcal{M}_{cl})$ and enlarged into space parametrized by $\Gamma_{tot}(\theta, \Gamma^1)$ subject to constraints encoded by $Z_{(a)}(\Gamma_{tot}(\Gamma^1))$ and $Z_{(a)}(\Gamma_{min}(\Gamma^1))$ (5.3), (5.10), in turn, constructed from $S_{L}(\mathcal{A}(\theta), \partial_{\theta}\mathcal{A}(\theta), \theta)$. The dynamics of this external model is provided by the 1st level supertime $\Gamma_{s} = (t^{1}, \theta^{1})$ presence in comparison with initial $t \in \iota$, $\theta$ describing the original so-called internal model.

As to the direct ZLR problem then we can use the constructed brackets (5.13), (5.14) to define on the corresponding $Z_{Q}$ the BFV-BRST charge $Z^{(-1)}(\theta)$ and on $Z_{Q}$ the new superfunction $S^{(-1)}$ which must satisfy to the corresponding generating equations with appropriate new 



$n(-1)$-level" supertime $\Gamma^{-1}$ introduction. Shortly, these new objects can be obtained by correlated with each other as well as their initial analogs $Z_{(a)}[\Gamma], S_{(1)}(\Gamma(\theta), \theta), S_{(1)L}(\Gamma(\theta), \partial_{\theta}\Gamma(\theta), \theta)$ or in correspondence with anzatz (5.27).

If there exist the special coordinates on $Z_{Q,1}$ with values of $\varepsilon_{P}^{(-1)} = \varepsilon_{P_{1}} + \varepsilon_{P_{0}} = 0$ then we able to restrict the action superfunction $S^{(-1)}$ to depend only upon them and therefore to be the classical new action.
Certainly, under appropriate extension of $Z_{Q^0}$ in analogy with superfunctions (5.26) introduction we can to construct $\theta$-superfield BFV similar triple $(Z^{(-1)}, H^{(-1)}, \Psi^{(-1)})(\theta, \Gamma^{-1})$ and $S_H^{(-1)}(\Gamma_{tot}(\theta^{-1}))$ as for inverse ZLR problem if the structure of $Z_{Q^0}$ allows to define $Z^{(-1)}(\theta)$, i.e. $Z_{Q^0}$ appears by supermanifold.

The more profound and sufficiently perspective investigation of the produced schemes for construction of the embedded and enlarged new $\theta$-superfield models and their detailed properties (including the relationships between the corresponding observables making use the ghost number prescription) requires the special efforts in view of both the models nontrivial connection and, for instance, the correspondence with superfield BFV method [29]. In addition, note on the particular similar features between the models construction above with the results from ref.[30].

**VI $\theta$-STF Component Formulation**

Let us continue a particular started in Sec.III programm of establishment the correspondence between superfield and component field quantities and relations. From representation (2.33) for superfunctional on $T_{odd}M_{cl} \times \{\theta\}$ find the expression for their densities in terms of the formers themselves

$$F(A(\theta), \partial_{\theta}A(\theta), \theta) = P_{\theta}(\theta)F(\theta) + \theta F[A] \equiv F(A, \lambda, 0) + \theta F[A, \lambda], \quad (6.1)$$

where the bar means the standard change of the form of dependence from the component arguments ($F[A] = \bar{F}[P_{\theta}A, \partial_{\theta}A] = \bar{F}[A, \lambda]$).

Taking account of the connection (3.1) for partial superfield and component derivatives the component expression for supermatrix of the 2nd partial superfield derivatives of $F(\theta) \equiv F(A(\theta), \partial_{\theta}A(\theta), \theta)$ w.r.t. $A^i(\theta), A^j(\theta)$ has the form

$$\left| \frac{\partial_r}{\partial_{A^i}(\theta)} \frac{\partial_l}{\partial_{A^j}(\theta)} F(\theta) \right| = \left| \frac{\partial_r}{\partial_{P_{\theta}A^i}(\theta)} \frac{\partial_l}{\partial_{P_{\theta}A^j}(\theta)} F(\theta) \right|, \quad \frac{\partial_r}{\partial_{P_{\theta}A^i}(\theta)} \frac{\partial_l}{\partial_{P_{\theta}A^j}(\theta)} = \left( \frac{\delta_r}{\delta_{A^i}}, \frac{\delta_r}{\delta_{A^j}} \right), \quad (6.2)$$

where in the right-hand side the usual variational derivatives w.r.t. $A^i$ and composite objects ($\lambda^\theta$) are written. The rank of supermatrix above is determined by one of its subsupermatrix for $a = b = 0, \theta = 0$ with trivial subsuperscript for $a = b = 1$.

As far as the relations hold

$$\frac{\partial_r}{\partial (\partial_{\theta}A^i(\theta))} = \frac{\delta_r}{\delta \lambda^i}, \quad \frac{\partial_l}{\partial (\partial_{\theta}A^i(\theta))} = \frac{\delta_l}{\delta \lambda^i}(-1)^{\varepsilon_i+1}, \quad (6.3)$$

it is convenient to introduce the variational component derivatives for connection with component quantities

$$\frac{\delta_r}{\delta (\lambda^\theta)} = \left( \frac{\partial_{\theta}}{\partial \lambda^i} \frac{\delta_r}{\delta \lambda^i} \right) (-1)^{\varepsilon_i+1}, \quad \frac{\delta_l}{\delta (\lambda^\theta)} = \left( \frac{\partial_{\theta}}{\partial \lambda^i} \frac{\delta_l}{\delta \lambda^i} \right) \quad (6.4)$$

in view of density $F(\theta)$ dependence on fields $\lambda^i$ through two arguments $A^i(\theta)$ and $\partial_{\theta}A^i(\theta)$. The differential operators $\left( \frac{\partial_{\theta}}{\partial \lambda^i}, \frac{\partial_{\theta}}{\partial \lambda^i} \right)$ it should be considered as the whole objects.

The connection of superfield variational derivative on $A^i(\theta)$ with variational component derivatives w.r.t. $A^i, \lambda^i$ coincides with proposed in ref.[25] and follows from (2.34), (6.1), (6.3)
and identities (4.6) being valid on $C^k$

$$
\left. \frac{\delta_\theta F[A]}{\delta A^i(\theta)} = \theta \frac{\delta_\epsilon F[A, \lambda]}{\delta A^i} + \frac{\delta_\lambda F[A, \lambda]}{\delta \lambda^i} (-1)^{\epsilon(F)} \right|_{\theta = 0},
$$

(6.5)

$$
\left[ \partial_\theta, \frac{\partial_l}{\partial (\partial_\theta A^i(\theta))} \right] = \left[ \frac{\delta}{\delta U_+}, \frac{\partial_l}{\partial (\partial_\theta A^i(\theta))} \right] = (-1)^{\epsilon_i} \frac{\partial_l}{\partial (\partial_\theta A^i(\theta))},
$$

(6.6)

with obvious restriction of the last formula to apply in $^{0,0}C^k$. The differential consequence of the formula (6.5) w.r.t. $\theta$ permits to express the superfield variational derivatives of the form (2.33) superfunctionals and of superfunctions both with partial superfield derivatives w.r.t. $A^i$,

$$
\partial_\theta \frac{\delta_\epsilon F[A]}{\delta A^i(\theta)} = (-1)^{\epsilon_i + \epsilon(F)} \frac{\delta F[A]}{\delta A^i(\theta)} = (-1)^{\epsilon_i + \epsilon(F)} \frac{\delta F[A, \lambda]}{\delta A^i},
$$

(6.7)

$$
\partial_\theta \frac{\delta_\epsilon F(\theta)}{\delta A^i(\theta)} = \partial_\theta \left( \delta(\theta_1 - \theta) \frac{\delta F(\theta_1)}{\delta A^i(\theta_1)} \right) = (-1)^{\epsilon(F) + \epsilon_i} \frac{\delta F(\theta)}{\delta P_0 A^i(\theta_1)},
$$

(6.8)

with use of the relationships (2.11) and trivial identities: $(P_0(\theta_1), \partial_\theta F(\theta_1)) \equiv (P_0(\theta), \partial_\theta F(\theta))$ to derive (6.8).

For the operators from $\mathcal{A}_cl$ investigated in Sec.III let us indicate the component expressions for only basis operators $\{U_0(\theta), \tilde{U}_0(\theta), \theta, a = 0, 1\}$ in acting on $C^k$

$$
\left( U_0, \tilde{U}_0, U_1, \tilde{U}_1 \right)(\theta) = \left( \lambda' \theta \frac{\delta}{\delta A^i}, \frac{\delta}{\delta \lambda^i}, (-1)^{\epsilon_i + 1} \lambda' \frac{\delta}{\delta \lambda^i}, \lambda' \theta \left( \frac{\delta}{\delta \theta \lambda^i} \right), (-1)^{\epsilon_i + 1} \lambda' \left( \frac{\delta}{\delta \theta \lambda^i} \right) \right).
$$

(6.9)

With regard for the formulae above, consider the physically main from the standard conventional gauge fields theory restriction $\theta = 0$, imposed on the structure of $\mathcal{M}_cl, \mathcal{T}_{odd} \mathcal{M}_cl$, classical actions $S_L(\theta)$ (4.1), $S(\theta)$ (4.3.3), Euler-Lagrange equations

1) $S_L(A(\theta), \theta, \theta)(\theta) = S_L(A, \lambda, 0), \quad (T_{odd} \mathcal{M}_cl)_{\theta = 0} = \{(A^i, \lambda^i)\},

(6.10a)

$S(A(\theta), \theta)(\theta) = S(A, 0), \quad \mathcal{M}_cl_{\theta = 0} = \{(A^i)\},

(6.10b)

2) $\frac{\delta_\theta Z[A]}{\delta A^i(\theta)} = - \frac{\delta_\lambda Z[A, \lambda]}{\delta \lambda^i} (-1)^{\epsilon_i} = - \frac{\delta_\lambda S_{\theta L}(A, \lambda)}{\delta \lambda^i} (-1)^{\epsilon_i},

(6.11a)

$S_{\alpha, \lambda}(A(\theta), \theta)(\theta) = S_{\alpha, \lambda}(A, 0) = 0,

(6.11b)

gauge transformations (4.48), (4.50), their generators $\mathcal{R}_{\alpha}^{\prime}(\theta; \theta'), \mathcal{R}_{0, \alpha}^{\prime}(\theta)$ (for $\xi^\alpha(\theta) = \xi^\alpha_0 + \xi^\alpha_1 \theta$)

3) $\delta_\theta A^i = - \mathcal{R}_{0, \alpha}^{\prime}(\theta; \theta') \mathcal{R}_{\alpha}^{\prime}(A, \lambda, 0) + (-1)^{\epsilon_i} \mathcal{R}_{0, \alpha}^{\prime}(A, 0, \lambda) \xi^\alpha_0$, $\delta_\lambda \lambda^i = - \left( \mathcal{R}_{\alpha}^{\prime}(A, \lambda, 0) + \mathcal{R}_{1, \alpha}^{\prime}(A, \lambda) \right) \xi^\alpha_1 + (-1)^{\epsilon_i} \mathcal{R}_{0, \alpha}^{\prime}(A, 0) \xi^\alpha_1$, $\delta_\lambda \lambda^i = - \left( \mathcal{R}_{\alpha}^{\prime}(A, \lambda, 0) + \mathcal{R}_{1, \alpha}^{\prime}(A, \lambda) \right) \xi^\alpha_1 + (-1)^{\epsilon_i} \mathcal{R}_{0, \alpha}^{\prime}(A, 0) \xi^\alpha_1$,

(6.12a)

$\frac{\delta_\theta A^i}{\delta \lambda^i} = \left( \mathcal{R}_{\alpha}^{\prime}(A, 0, \lambda) \xi^\alpha_0, \quad \mathcal{R}_{\alpha}^{\prime}(A, 0, \lambda) \xi^\alpha_1 - (-1)^{\epsilon_i} \mathcal{R}_{0, \alpha}^{\prime}(A, 0) \xi^\alpha_0 \right),

(6.12b)

4) $\mathcal{R}_{\alpha}^{\prime}(A(\theta), \theta, \theta')(\theta) = - \theta \mathcal{R}_{0, \alpha}^{\prime}(A, \lambda, 0) + \mathcal{R}_{1, \alpha}^{\prime}(A, \lambda, 0)$,

(6.13a)

$\mathcal{R}_{\alpha}^{\prime}(A(\theta), \theta)(\theta) = \mathcal{R}_{\alpha}^{\prime}(A, 0)$,

(6.13b)

Noether’s identities (4.24), (4.30)

5) $\int d\theta \left. \frac{\delta_\theta Z[A]}{\delta A^i(\theta)} \right|_{\theta = 0} = - \frac{\delta_\lambda S_{\theta L}(A, \lambda)}{\delta \lambda^i} \left[ \mathcal{R}_{0, \alpha}^{\prime}(A, \lambda, 0) + \mathcal{R}_{1, \alpha}^{\prime}(A, \lambda) \right]$

$$
+ \frac{\delta_\lambda S_{\theta L}(A, \lambda)}{\delta \lambda^i} \mathcal{R}_{1, \alpha}^{\prime}(A, \lambda, 0) = 0,
$$

(6.14a)

$S_{\alpha, \lambda}(A(\theta), \theta) \mathcal{R}_{\alpha}^{\prime}(A(\theta), \theta)(\theta) = S_{\alpha, \lambda}(A, 0) \mathcal{R}_{\alpha}^{\prime}(A, 0) = 0,$

(6.14b)
where the notation was used for arbitrary \( f(A(\theta), \partial_\theta A(\theta), \theta) \)

\[
    f'_\theta(A, \lambda) \equiv \partial_\theta f(\theta) = -\lambda^i f_{,i}(A, \lambda)(-1)^{e_\varepsilon(f)} + \frac{\partial}{\partial \theta} f(\theta). \tag{6.15}
\]

Note, the Eqs.(6.11a) in terms of fields \( A^i, \lambda^i \) have not the form of the usual component Euler-Lagrange equations for functional \( S_L(A, \lambda, 0) \) in view of the different \( \theta \)-superfield origin of the arguments \( A^i, \lambda^i \) in \( S_L \).

In addition the component form of the identities (4.24) in (6.14a) has the specific character, on the one hand showing that for particular case of the natural system (4.33) and \( \mathcal{R}_{k_\alpha} = 0 \) this relation passes into the usual GThST formula (6.14b), and on the other hand they represent the extension of the functional \( S(A, 0) \) invariance w.r.t. only transformations with \( \delta A^i \) in (6.12b) up to invariance of \( \bar{Z}[A, \lambda] \) w.r.t. transformations (6.12a) with doubled numbers of the generators \( \mathcal{R}_{k_\alpha}(A, \lambda, 0) \) and functions \( \xi^0, \xi^1 \) of opposite parities.

At the same time for the special type quantities and relations depending only upon superfields \( \mathcal{A}^i(\theta) \) the component relationships (6.10b)-(6.14b) have the usual form in view of the purely gauge character of the fields \( \lambda^i \) in this case.

The component form for the transformations (2.18) of \( A^i, \lambda^i \) w.r.t. \( P \) group translations and induced by them the transformation of superfunction \( \mathcal{F}(\theta) \) (2.21) with taking account of the rules (4.2), (4.3), (6.15) are defined for \( \theta = 0 \) by the formulae

\[
    \delta(A^i, \lambda^i) = (-\lambda^i \mu, 0), \quad \delta \mathcal{F}(A(\theta), \partial_\theta A(\theta), \theta)|_{\theta = 0} = -\mu \mathcal{F}'(A, \lambda). \tag{6.16}
\]

Next, list only some component expressions for Sec.V objects and relations devoted to the GAs of GThGT, GThST and ZLR problem. So, the superfunction (5.4) being by \( \theta \)-superfield extension of BV action and superfunctional (5.3) have the form

\[
    S_{(1)}[\Gamma_{\min}(\theta), \theta]|_{\theta = 0} = S(A, 0) + A^*_k \mathcal{R}^*_\alpha(A, 0)C^\alpha + O(C^2); \tag{6.17a}
\]

\[
    \bar{Z}_{(1)}[\Gamma_{\min}] = S'_{\mathcal{L}}(A, \lambda) - A^*_k \left[ \mathcal{R}^*_\alpha(A, \lambda)C^\alpha(1)^{\varepsilon_1} \right. + \left. (\mathcal{R}^*_k(A, \lambda) + \mathcal{R}^*_\alpha(A, \lambda, 0))\lambda^\alpha \right] - J_k \left[ \mathcal{R}^*_\alpha(A, \lambda, \lambda^\alpha) + (1)^{\varepsilon_1} \mathcal{R}^*_k(A, \lambda, 0)\lambda^\alpha \right] + O(C^{2-t}(\lambda^\alpha)^{1}). \tag{6.17b}
\]

Note, for the simplest case of nondegenerate ThST \( (m = 0) \) for which the only 1st terms in (6.17ab) survive, so that the extremals \( S_{n}(A, 0) = 0 \) for (6.17a) appear by the 1st class constraints w.r.t. the 2nd Poisson bracket in (6.18b) for BFV generator \( \bar{Z}_{(0)}(A, \lambda) = \lambda^i S_{n}(A, 0) - \frac{\partial}{\partial \theta} S(A(\theta), \theta) \), revealing the physical significance of \( \lambda^i \) to be the ghost fields which are different from \( C^\alpha, \lambda^\alpha \) corresponding to the nontrivial GThST invariance.

The corresponding \( \theta \)-local antibracket (5.6) given on the functional superalgebra of the restricted supermanifold \( T_{odd}^* \mathcal{M}_{min|\theta = 0} = \{(\phi^B, \phi^*_B)\} \) and superfunctional even bracket (5.5) calculated on superfunctionals defined on \( T_{odd}^*(\mathcal{M}_{min}) \) with coordinates \( \{(\phi^B, \phi^*_B), (\lambda^B, J_B)\} \) forming the flat phase space structure read as follows with regard for obvious generalization of (6.1), (6.3), (6.5) to the case of superfields \( \Phi^B(\theta), \Phi^*_B(\theta) \)

\[
    (\mathcal{F}(\phi, \phi^*, 0), \mathcal{G}(\phi, \phi^*, 0)) = \frac{\delta \mathcal{F}(\phi, \phi^*, 0)}{\delta \phi^B} \frac{\delta \mathcal{G}(\phi, \phi^*, 0)}{\delta \phi^*_B} - \frac{\delta_r \mathcal{F}(\phi, \phi^*, 0)}{\delta \phi^*_B} \frac{\delta \mathcal{G}(\phi, \phi^*, 0)}{\delta \phi^*_B} - \frac{\delta_r \mathcal{F}(\phi, \phi^*, 0)}{\delta \phi^*_B} \frac{\delta \mathcal{G}(\phi, \phi^*, 0)}{\delta \phi^*_B} \tag{6.18a}
\]

\[
    \{\bar{F}, \bar{G}\} = \{\bar{F}, \bar{G}\}^{(\lambda, \phi^*)} - \{\bar{F}, \bar{G}\}^{(\lambda, \phi)} = \left( \frac{\delta_r \mathcal{F}}{\delta \lambda^B} \frac{\delta \bar{G}}{\delta \phi^*_B} - (1)^{\varepsilon_1} \frac{\delta_r \bar{F}}{\delta \phi^*_B} \frac{\delta \bar{G}}{\delta \lambda^B} \right) \tag{6.18b}
\]
The last bracket is presented in terms of 2 ones so that the role of coordinates and momenta play \((\phi^B, \Lambda^B)\) and \((J_B, \phi^B)\).

It is easy to produce the component form on \(Z_Q\) for \(\theta\)-local even bracket (5.13) for \(\theta = 0\) and therefore let us only fulfill it for the antibracket (5.14) on \(Z_Q\), taking the explicit expressions (5.19), (5.20) for the most important 1st term in square and 1st one in round parentheses in (5.20) (being exact for GGTGT not depending upon \(\mathcal{A}^i, \partial_\theta \mathcal{A}^i\))

\[
(f [\Gamma], \tilde{g} [\Gamma]) = (-1)^{\varepsilon_i} \left( \frac{\delta_i F}{\delta A^i} \left( \frac{\delta_i \delta Z}{\delta A^j} \right) \frac{\delta G}{\delta J_j} - \frac{\delta_i F}{\delta J_i} \left( \frac{\delta_i \delta Z}{\delta A^j} \right) \frac{\delta G}{\delta J_j} \right) + \frac{\delta_i F}{\delta A^i} \left( \frac{\delta_i \delta Z}{\delta A^j} \delta G \right) \frac{\delta G}{\delta J_j} \right) \]

\[
\left. (-1)^{\varepsilon_i+1} \right|_{Z_Q} \] (6.19)

Zero locus \(Z_Q\) is defined by the equations

\[
\left( \frac{\delta_i \tilde{Z}}{\delta A^i}, \frac{\delta_i \tilde{Z}}{\delta A^i}, C^\alpha, \lambda^\alpha \right) = \tilde{\partial}_4, \quad \left( (-1)^{\varepsilon_i} A^i \left( \tilde{\mathcal{R}}_{0\theta} + \tilde{\mathcal{R}}_{1\theta} \right) + J_i \tilde{\mathcal{R}}_{1\theta} + A^i \tilde{\mathcal{R}}_{0\theta} \right) = \tilde{\partial}_2. \] (6.20)

whose solutions are ambiguous by virtue of the identities (6.14a). In particular, having considered as \(\tilde{Z}_{(0)}\) the functional \(\tilde{Z}_{(0)}\) from (5.10) we get the first 2 summands in (6.19) vanish together with \(\tilde{\mathcal{R}}_{1\theta}, \tilde{\mathcal{R}}_{1\theta}\), so that we obtain the representation for antibracket (5.21) \(\tilde{Z}_{(0)}\) with definition therefore the even bracket (5.13) as well but in terms of component superfunctionals (5.23).

\section{Lagrangian \(\theta\)-STF Models}

\subsection{Massive Complex Scalar Superfield Models}

Let us choose as the supergroups \(\tilde{J}, \tilde{M}, \tilde{J}_\Lambda\) (2.3) the following Lie groups

\[
\tilde{J} = \Pi(1, 3)^\dagger, \quad \tilde{M} = T(1, 3), \quad \tilde{J}_\Lambda = SO(1, 3)^\dagger,
\] (7.1)

to be respectively by proper Poincare group, group of space-time translations and proper Lorentz group. As the group \(\tilde{J}_\Lambda\) one can take \(SL(2, \mathbb{C})\) being by the universal covering group for \(SO(1, 3)^\dagger\). The corresponding quotient superspace has the form

\[
\mathcal{M} = \mathbb{R}^{1.3} \times \tilde{P} = \{(x^\mu, \theta)\}, \quad \text{diag } \eta_{\mu\nu} = (1, -1, -1, -1).
\] (7.2)

An action of \(\Pi(1, 3)^\dagger \times P\) has the standard character of Poincare transformations on \(\mathbb{R}^{1.3}\) (with identical action of \(P\)) and on \(\tilde{P}\) is given as in (2.7).

Choose as the Lorentz type superfields \(\mathcal{A}^i(\theta)\) the complex scalar superfield \(\varphi(x, \theta) \in \tilde{A}_{40+1}(x^\mu, \theta; \mathbb{C})\)

\[
\varphi(x, \theta) = \varphi_1(x, \theta) + \nu \varphi_2(x, \theta) = \varphi(x) + \lambda(x) \theta, \quad \varphi_j(x, \theta) = \varphi_j(x) + \lambda_j(x) \theta,
\] \(j = 1, 2, \quad i^2 = -1\). (7.3)

The index \(i\) condensed contents (2.12) and vector of Grassmann gradings \(\varepsilon = (\varepsilon_P, \varepsilon_H, \varepsilon)\) for \(\varphi(x, \theta)\) and its complex components on \(\theta\) are written in the form

\[
i = ([\varphi], [\overline{\varphi}], x), \quad n = (2, 0), \quad \varepsilon \varphi(x) = \varepsilon \varphi(x, \theta) = \varepsilon \lambda(x) + (1, 0, 1) = \bar{0}.
\] (7.4)
Superfield \( \varphi(x, \theta) \) and its \( \theta \)-component fields are transformed in a standard way w.r.t. restriction onto \( \Pi(1,3) \) of the supergroup \( J \) \( \theta \)-superfield representation \( T \) as the spin 0 and mass \( m \) elements of Poincare group representation [38]. As to restricted representation \( T|_{P} \), then only \((\varphi, \bar{\varphi})(x, \theta)\) are transformed nontrivially according to the general rule (2.18)

\[
\delta(\varphi, \bar{\varphi})(x, \theta) = (\varphi, \bar{\varphi})'(x, \theta) - (\varphi, \bar{\varphi})(x, \theta) = -\mu(\partial_\theta \varphi, \partial_\theta \bar{\varphi})(x, \theta) = \mu(\lambda, \Lambda)(x). \tag{7.5}
\]

As the classical action \( S_L\left((\varphi, \bar{\varphi}, \partial_\theta \varphi, \partial_\theta \bar{\varphi})(\theta)\right) \in \Lambda_1(\theta; \mathbb{R}) \) for free superfields \((\varphi, \bar{\varphi})(x, \theta)\), describing 2 opposite charged spinless massive particles, let us construct the superfunction having the type (4.33) almost natural system with \( g(\theta) = \text{antidiag}(\nu, \nu) \) \((\bar{\epsilon} \nu = (1, 0, 1))\) in (4.34) and real dimensional in the units of length (for \( \hbar = c = 1 \)) constants \( a_\varphi, b_\varphi \) \([a_\varphi]_l = [\varphi]_l + 1 = [b_\varphi]_l = -2\) in order to \( T(\theta) \) and therefore \( S_L(\theta) \) would be by dimensionless

\[
S_L(\theta) = S_L((\varphi, \bar{\varphi}, \partial_\theta \varphi, \partial_\theta \bar{\varphi})(\theta)) = T((\varphi, \bar{\varphi}, \partial_\theta \varphi, \partial_\theta \bar{\varphi})(\theta)) - S_0((\varphi, \bar{\varphi})(\theta))
= \int d^4x \left[ (-\partial_\mu \varphi \partial_\mu \varphi + b_\varphi (\partial_\mu \varphi \varphi + \partial_\mu \varphi \bar{\varphi})) \right] (x, \theta). \tag{7.6}
\]

Note, the requirement on \( a_\varphi, b_\varphi \), \( T(\theta) \) above are naturally derived from realization of \( J \) as direct product of \( J \), \( P \) and therefore from condition \([\partial_\theta]_l = [\theta]_l = 0\) and, secondly, the reality of \( S_L(\theta) \) is provided by the natural continuation of the complex conjugation from the \( P_0(\theta) \) real component fields \( \varphi_j(x) \) (in general \( A^i \)) up to the same property fulfilment for \( P_1(\theta) \) component \( \lambda_j(x) \) (in general \( \lambda^i \)) written in the form

\[
(\lambda_j(x))^* = \lambda_j(x), \quad \theta^* = \theta. \tag{7.7}
\]

Superfunction \( S_L(\theta) \) being considered as the more fundamental object than \( Z[\varphi, \bar{\varphi}] \) (see remark after (2.34)) is invariant w.r.t. Poincare transformations, but w.r.t. \( P \) group ones (7.5) is transformed according to (4.3) as follows simultaneously with operator \( \hat{U}_+(\theta) \) realization

\[
\delta S_L(\theta) = \mu \int d^4x \left[ \varphi(x, \theta) P_0(\theta) \frac{\partial S_L(\theta)}{\partial \varphi(x, \theta)} + \bar{\varphi}(x, \theta) P_0(\theta) \frac{\partial S_L(\theta)}{\partial \bar{\varphi}(x, \theta)} \right]
= \mu \int d^4x \left[ \partial_\theta \varphi(x, \theta)(\Box + m^2)\varphi(x, \theta) + c.c. \right], \quad \Delta = \eta_{\mu \nu} \partial_\mu \partial_\nu. \tag{7.8}
\]

The invariance of \( S_L(\theta) \) w.r.t. above transformations is restored on the solutions for following independent Euler-Lagrange equation of the form (4.5) appearing by virtue of (4.8), (4.9) by one from HCLF \( \Theta_\varphi((\varphi, \bar{\varphi})(x, \theta)) = 0, \Theta_{-\varphi}((\varphi, \bar{\varphi})(x, \theta)) = 0 \)

\[
\frac{\delta Z[\varphi, \bar{\varphi}]}{\partial \bar{\varphi}(x, \theta)} = -\frac{\partial_\theta S_0(\theta)}{\partial \bar{\varphi}(x, \theta)} - \partial_\theta \frac{\partial T(\theta)}{\partial \bar{\varphi}(x, \theta)} = (a_\varphi \partial_\theta^2 + (\Box + m^2))\varphi(x, \theta) = 0, \tag{7.9}
\]

where we preserve the form for the 1st summand in \( T(\theta) \in \text{Ker}\{\partial_\theta\} \) by fixed that does not influence on \( Z[\varphi, \bar{\varphi}] \) value and therefore on LS (7.9) structure as in (4.7) but demonstrate the nondegeneracy of (4.17) in question.

Really, the supermatrix (4.17) has the form in this case

\[
K(\theta, x, y) = \begin{bmatrix}
\delta r \\
\partial_r (\partial_\theta E_a(x, \theta)) \\
\partial_\theta (\partial_\theta E_b(y, \theta))
\end{bmatrix}
= \begin{bmatrix}
0 & a_\varphi \\
-a_\varphi & 0
\end{bmatrix}
\delta(x - y), \quad E_b = (\varphi, \bar{\varphi}), \tag{7.10}
\]

being by the usual matrix w.r.t. \( \varepsilon_{II} \) grading and by the supermatrix w.r.t. \( \varepsilon \) one \( K(\theta, x, y) \) with only odd-odd nontrivial block by virtue of (7.4).
Solutions for Eq. (7.9) being by the superfield (on $\theta$) generalization of the Klein-Gordon equation exist providing the assumption (4.13) fulfilment in question. The rank of supermatrix (4.15) is calculated by the rule (4.21) according to Corollary 2 and formulae (4.27), (4.36)

$$\text{rank } \left| \frac{\partial_r}{\partial E_a(x,\theta)} \frac{\partial_l S_0(\theta)}{\partial E_b(y,\theta)} \right| = \text{rank } \begin{pmatrix} \square + m^2 & 0 \\ 0 & -(\square + m^2) \end{pmatrix} \delta(x-y) = 2 .$$  \hspace{1cm} (7.11)

From the last formula being valid almost everywhere in $\mathcal{M}_{cl}$ it follows that there are not differential identities among two HCLF and the number of physical degrees of freedom is equal to 2. As the Cauchy problem for the 2nd order w.r.t. $x^\mu, \theta$ independent part of the LS (7.9) one can choose the initial conditions

$$\left( \varphi, \partial_0 \varphi, \delta \varphi, \partial_0 \delta \varphi \right)(x, \theta)_{|x^0=\theta=0} = \left( \varphi_0, \varphi_1, \delta \varphi_1, \lambda_1 \right)(x^i), \hspace{1cm} x^\mu = (x^0, x^i).$$ \hspace{1cm} (7.12)

Therefore according to terminology introduced in Sec.IV, given $\theta$-STF model belongs to the class of nondegenerate ThSTs.

A generalization of the model (7.6) onto interacting theory case may be realized in terms of the local superfunction, for instance, by means of addition to $S_0(\theta)$ at least the cubic w.r.t. $(\varphi, \overline{\varphi})(x, \theta)$ polynomial $V(\theta)$ without derivatives on $(x^\mu, \theta)$ with real constants $\zeta, \eta$

$$S_{0M}(\theta) = S_0(\theta) - V((\varphi, \overline{\varphi})(\theta)), \hspace{1cm} V(\theta) = \int d^4x \left( \frac{\zeta}{3} \overline{\varphi} \varphi + \frac{\eta}{2} (\overline{\varphi} \varphi)^2 + \ldots \right)(x, \theta).$$ \hspace{1cm} (7.13)

The corresponding independent dynamical equation for $Z_M[\varphi, \overline{\varphi}] = Z[\varphi, \overline{\varphi}] + \int d\theta \overline{\varphi} V(\theta)$ appears by nonlinear

$$\frac{\delta_i Z_M[\varphi, \overline{\varphi}]}{\delta \varphi(x, \theta)} = \left( -ia_x \partial_\theta^2 + \square + m^2 + \frac{\zeta}{3} (\overline{\varphi} + 2 \varphi) + \eta (\overline{\varphi} \varphi) + \ldots \right) \overline{\varphi}(x, \theta) = 0 ,$$ \hspace{1cm} (7.14)

so that, if the $(P_0, P_1)(\theta)$ components $\varphi(x), \lambda(x)$ for free superfield had satisfied to the same Klein-Gordon equation (7.9) respectively, in fact, (formally for $\lambda(x)$) describing 2 identical of the same name charged spinless massive particles then component (on $\theta$) equations following from (7.14) lead to nontrivial interaction for $\overline{\lambda}(x)$ with fields $\varphi(x), \overline{\varphi}(x)$

$$\left( \square + m^2 \right) \overline{\varphi}(x) = -\frac{\delta_i P_0(\theta) V(\theta)}{\delta \varphi(x)} ,$$ \hspace{1cm} (7.15a)

$$\left( \square + m^2 \right) \overline{\lambda}(x) = -\int d^4y \left[ \overline{\lambda}(y) \frac{\delta_i}{\delta \varphi(y)} + \lambda(y) \frac{\delta_i}{\delta \varphi(y)} \right] \frac{\delta_i P_0(\theta) V(\theta)}{\delta \varphi(x)} ,$$ \hspace{1cm} (7.15b)

resulting in different dynamics for the particles corresponding to the fields $\overline{\varphi}(x), \overline{\lambda}(x)$. In deriving of Eqs.(7.15) the formulae (6.5) have been taken into account and the component form for $U_0(\theta)$ (6.9) in question is defined by the last equation.

The requirement of invariance for $S_{0M}(\theta)$ w.r.t. transformations from $U(1)$ group results in restriction $\zeta = 0$ in (7.13). The only relationships (7.8), (7.11) above are changed in an obvious way with allowance made for Eq.(7.14) so that the rank condition (7.11) remains invariable together with classification for given interacting model as ThST.

The nondegeneracy, for instance, of free model (7.6) with BRST similar charge (5.10)

$$Z_{(0)}[\varphi, \overline{\varphi}, \varphi^*, \overline{\varphi}^*] = \bar{Z}[\varphi, \overline{\varphi}, \lambda, \overline{\lambda}] = -\int d^4x \left( \overline{\lambda}(x) \left( \square + m^2 \right) \varphi(x) + c.c. \right)$$ \hspace{1cm} (7.16)
and $S_{(1)}((\varphi, \overline{\varphi}, \varphi^*, \overline{\varphi}^*) (\theta)) \equiv S_0((\varphi, \overline{\varphi})(\theta))$ (5.4) leads to trivial of the form (5.4), (5.10) generating equations given by means of standard superbrackets (5.6), (5.5) defined in question on the antisymplectic $T^\oddc_\cdot M_d = \{((\varphi, \varphi^*), (\overline{\varphi}, \overline{\varphi}^*)) (x, \theta)\} \equiv \{\Gamma^p_d(x, \theta)\}$ and symplectic supermanifolds $T^\oddc_\cdot (T^\oddc_\cdot M_d) = \{\Gamma^p_d, \partial_{\theta}\Gamma^p_d\}(x, \theta)\}, p = 1, 4$ based on the complex superantifields introduction

$$
(\varphi^*, \overline{\varphi}^*)(x, \theta) = (\varphi^*, \overline{\varphi}^*)(x) - \theta (J_\varphi, J_{\overline{\varphi}})(x) = (\varphi_1^*, \overline{\varphi}_1^*)(x, \theta) + \iota (\varphi_2^*, -\overline{\varphi}_2^*)(x, \theta),
$$

$$
\varphi^*_\theta(x, \theta) \in \tilde{A}_{40+1}(x^\mu, \theta; \mathbb{R}), \quad \varepsilon(\varphi^*, \overline{\varphi}^*) = \varepsilon(J_\varphi, J_{\overline{\varphi}}) + (1, 0, 1) = (1, 0, 1).
$$

(7.17)

Corresponding ZLR odd and even brackets are given by the particular $\theta$-superfield relationships (5.22), (5.13) with fulfilment of the property (5.21) (for $z = (x, \theta)$)

$$
(f[\varphi^*, \overline{\varphi}^*], g[\varphi^*, \overline{\varphi}^*])_{Z^{(0)}} = \int dz \left[ \frac{\delta f}{\delta \varphi^*(z)}(\square + m^2) \frac{\delta g}{\delta \overline{\varphi}^*(z)} + \frac{\delta f}{\delta \overline{\varphi}^*(z)}(\square + m^2) \frac{\delta g}{\delta \varphi^*(z)} \right],
$$

$$
Z_{\tilde{Q}^1} = \{(\varphi^*, \overline{\varphi}^*, \partial_0(\varphi^*, \overline{\varphi}^*)) (z)\} \supset Z_{\tilde{Q}^0} = \{((\varphi^*, \overline{\varphi}^*) (z))\},
$$

$$
\{f((\varphi^*, \overline{\varphi}^*)(\theta)), g((\varphi^*, \overline{\varphi}^*)(\theta))\}_{\tilde{Q}^0} = \int d^4x \left( \frac{\partial f(\theta)}{\partial \varphi^*(z)}(\square + m^2) \frac{\partial g(\theta)}{\partial \overline{\varphi}^*(z)} + (\varphi^* \leftrightarrow \overline{\varphi}^*) \right),
$$

(7.18)

(7.19)

where $Z_{\tilde{Q}^1}, Z_{\tilde{Q}^0}$ are determined on the solution $(\varphi_0, \overline{\varphi}_0)(z)$ of Eqs.(7.9).

Since $Z_{\tilde{Q}^0}$ is parametrized by only $\varepsilon$-odd variables being by coordinates in the fiber over $(\varphi_0, \overline{\varphi}_0)(z)$ in $T^\oddc_\cdot M_d$ then the corresponding $Z^{(-1)}((\varphi^*, \overline{\varphi}^*)(\theta))$ is trivial. As the classical physical action given on $Z_{\tilde{Q}^1}$ for direct ZLR problem by virtue of $\varepsilon$-vector distribution in (7.17), satisfying to the master equation with antibracket (7.18), one can choose the superfunction by introducing of only $\theta^{-1}$ odd time

$$
S^{(-1)}((J_\varphi, J_{\overline{\varphi}})(\theta^{-1})) = \int d^4x \left( \partial_\mu J_\varphi \partial^\mu J_{\overline{\varphi}} - m^2 J_\varphi J_{\overline{\varphi}} \right)(x, \theta^{-1}).
$$

(7.20)

Having added the kinetic term $T(\theta^{-1})$ with derivatives on $\theta^{-1}$ to the quantity above one can obtain the coincidence of $S^{(-1)}_L(\theta^{-1}) = T(\theta^{-1}) - S^{(-1)}_L(\theta^{-1})$ with $S_L(\theta)$ (7.6) under identification $\theta \leftrightarrow \theta^{-1}, (J_\varphi, J_{\overline{\varphi}}) \leftrightarrow (\varphi, \overline{\varphi})$, reflecting the fact of the special duality for these superfunctions in solving the ZLR problem.

The similar construction may be carried out for interacting model with obvious change of the operator $(\square + m^2)$, for instance, for $\overline{\varphi}(z)$ in (7.16)–(7.20) onto $\left[(\square + m^2) + \frac{\partial}{\partial \overline{\varphi}(z)}, \frac{\partial V(\theta)}{\partial \overline{\varphi}(z)} \right]$.

At last, setting $\theta = 0$ in $S_L(\theta)$ (7.6) or (7.13) we get for $\partial_\theta(\varphi, \overline{\varphi})(x, \theta) = 0$ the standard action $S_0(\varphi, \overline{\varphi})$ for complex scalar fields whereas without the latter restriction we shall have the additional (of the topological nature) presence of the background nonpropagating fields $\lambda_j(x)$ in accordance with (6.10a).

VII.2 Massive Spinor Superfield of Spin $\frac{1}{2}$ Models

In the framework of the Subsec.VI.1 representations for $J, M$ choose as $A^i(\theta)$ the Dirac spinor superfields $\Psi(x, \theta) \in \tilde{A}_{40+1}(x^\mu, \theta; \mathbb{C})$ and its Dirac conjugate $\overline{\Psi}(x, \theta)$ in the 4- and 2-component spinor formalisms

$$
\Psi(x, \theta) = (\psi_\gamma(x, \theta), \chi^\gamma(x, \theta))^T, \quad \Psi(x, \theta) = \psi(x) + \psi_1(x)\theta, \quad \gamma = 1, 2, \quad \gamma = 1, 2, \quad \gamma = 1, 2, \quad \gamma = 1, 2, \quad \gamma = 1, 2,
$$

$$
\overline{\Psi}(x, \theta) = (\overline{\chi}^\beta(x, \theta), \overline{\varphi}_\beta(x, \theta)), \quad \overline{\Psi}(x, \theta) = \overline{\psi}(x) + \overline{\varphi}_1(x)\theta, \quad \beta = 1, 2, \quad \beta = 1, 2, \quad \beta = 1, 2, \quad \beta = 1, 2,
$$

(7.21)

being by elements of $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ Lorentz group reducible representation. The index $\iota$ condensed contents and Grassmann gradings for superfields $(\Psi, \overline{\Psi})(x, \theta)$ and its $\theta$-component fields $((\psi, \overline{\psi}), (\psi_1, \overline{\psi}_1))(x)$ read in the form

$$
\iota = (\gamma, \overline{\gamma}, \beta, \overline{\beta}, \chi, \overline{\chi}), \quad n = (0, 8), \quad \varepsilon(\psi, \overline{\psi}) = \varepsilon(\Psi, \overline{\Psi}) = \varepsilon(\psi_1, \overline{\psi}_1) + (1, 0, 1) = (0, 1, 1).
$$

(7.22)
If \((\Psi, \overline{\Psi})(x, \theta)\) and their \(\theta\)-components are transformed in the standard way w.r.t. \(\theta\)-superfield representation \(T_{|\Pi(1,3)^+}\) as spin \(\frac{1}{2}\) and mass \(m\) elements of Poincare group representation [38], then w.r.t. \(T_P\) operators the only \((\Psi, \overline{\Psi})(x, \theta)\) have the nontrivial transformation law following from (2.18)
\[
\delta(\Psi, \overline{\Psi})(x, \theta) = (\Psi', \overline{\Psi}')(x, \theta) - (\Psi, \overline{\Psi})(x, \theta) = -\mu \partial_\theta(\Psi, \overline{\Psi})(x, \theta) = -\mu(\psi_1, \overline{\psi}_1)(x) .
\]
(7.23)

As the classical actions \(S_L^l((\Psi, \overline{\Psi}, \partial_\theta(\Psi, \overline{\Psi}))(\theta))\), \(l = 1, 2\) for free spinor superfields describing massive particle and its antiparticle with spin \(\frac{1}{2}\), let us construct the superfunctions consisting only of quadratic w.r.t. \((\Psi, \overline{\Psi}, \partial_\theta(\Psi, \overline{\Psi}))(x, \theta)\) parts without writing of the special dimensional constants, one from which appears by the 1st order w.r.t. derivatives on \((x^\mu, \theta)\) superfunction with \(g_{\mu}(\theta) = -(-1)^{\epsilon_\mu}g_{\mu}(\theta)\) in (4.34) and another with \(g_{\mu}(\theta) = 0\)
\[
S_L^{(1)}(\theta) \equiv S_L^1((\Psi, \overline{\Psi}, \partial_\theta(\Psi, \overline{\Psi}))(\theta)) = T(\partial_\theta(\Psi, \overline{\Psi})(\theta)) - S_0((\Psi, \overline{\Psi})(\theta))
\]
(7.24a)
\[
S_L^{(2)}(\theta) \equiv S_L^2((\Psi, \overline{\Psi}, \partial_\theta(\Psi, \overline{\Psi}))(\theta)) = -\int d^4x(\overline{\Psi}i\Gamma^A\partial_\mu - m)\Psi)(x, \theta) ,
\]
(7.24b)
\[
\Gamma^A = (\Gamma^\mu, \hat{\mu}_4) , \quad A = (\nu, \theta) , \quad \partial_\Lambda = (\partial_\nu, \partial_\theta) , \quad \bar{\varepsilon}\hat{\mu} = (1, 0, 1) , \quad (\Gamma^\theta)^+ = \Gamma^\theta , \\
\Gamma^A\Gamma^B + (-1)^{\epsilon_4\epsilon_B}\Gamma^B\Gamma^A = 2\eta^{AB}\mathbf{1}_4 , \quad \eta^{\mu\nu}\mathbf{1}_4 = \hat{\mu}\Gamma^\nu , \quad \eta^{\theta\theta} = 0 ,
\]
(7.24c)
where \(\Gamma^\mu\) are the Dirac matrices in the ref.[39] notations. The superfunctions \(S_L^{(1)}(\theta)\) are invariant w.r.t. Poincare transformations whereas w.r.t. \(P\) group ones \(S_L^{(1)}(\theta)\) are transformed according to (4.3) respectively
\[
\delta S_L^{(1)}(\theta) = \mu \int d^4x \left[ P(0, \theta) \frac{\partial S_0^{(1)}(\theta)}{\partial \Psi(x, \theta)} \overline{\Psi}(x, \theta) + \overline{\Psi}(x, \theta) P(0, \theta) \frac{\partial S_0^{(1)}(\theta)}{\partial \overline{\Psi}(x, \theta)} \right] ,
\]
(7.25a)
\[
\delta S_L^{(2)}(\theta) = \delta S_L^{(1)}(\theta) - \mu \int d^4x (\partial_\theta \overline{\Psi}(x, \theta) \hat{\mu} \partial_\theta \Psi)(x, \theta) ,
\]
(7.25b)
with writing in (7.25a) the expression for operator \(\hat{U}_+(\theta)\).

The reality of \(S^{(1)}(\theta)\) follows from the complex conjugation relations for the case of bispinors having the form (without writing of arguments)
\[
(\overline{\Psi}_1\Psi_2, \overline{\Psi}_1\hat{\Psi}_2, \overline{\Psi}_1\overline{\hat{\Psi}}_2, \overline{\Psi}_1\overline{\hat{\Psi}}_2) = (\overline{\Psi}_2\Psi_1, \overline{\Psi}_2\hat{\Psi}_1, \overline{\Psi}_2\overline{\hat{\Psi}}_1, \theta \hat{\Psi}, \hat{\mu}, -\overline{\hat{\Psi}}\Gamma^\mu \hat{\Psi}) ,
\]
(7.26)
with accuracy up to total derivative w.r.t. \(\theta\) in \(S_L^{(1)}(\theta)\) and w.r.t. \(x^\mu\) for \(S_0^{(1)}(\theta)\).

Euler-Lagrange equations (4.5) have the form, with the same comments on \(T^{(1)}(\theta)\) role as after Eq.(7.9), in the corresponding superfunctional \(Z^\dagger[\Psi, \overline{\Psi}] = \int d\theta S_L^{(1)}(\theta)\)
\[
\delta_{\theta} Z^1[\Psi, \overline{\Psi}] = \frac{\delta_{\theta} S_0^{(1)}(\theta)}{\delta \overline{\Psi}(x, \theta)} + \frac{\delta_{\theta} T^{(1)}(\theta)}{\delta \partial_\theta \overline{\Psi}(x, \theta)} = -\left( i \overline{\Psi} \Gamma^\mu + m \overline{\Psi} - \overline{\partial_\theta \Psi} \right)(x, \theta) = 0 , \quad (7.27a)
\]
\[
\delta_{\theta} Z^2[\Psi, \overline{\Psi}] = \frac{\delta_{\theta} S_0^{(1)}(\theta)}{\delta \Psi(x, \theta)} + \frac{\delta_{\theta} T^{(1)}(\theta)}{\delta \partial_\theta \Psi(x, \theta)} = \left( \partial_\theta^2 - (i \Gamma^\mu \partial_\mu - m) \right) \Psi(x, \theta) = 0 , \quad (7.27b)
\]
\[
\frac{\delta_{\theta} Z^1[\Psi, \overline{\Psi}]}{\delta \Psi(x, \theta)} = \frac{\delta_{\theta} S_0^{(1)}(\theta)}{\delta \Psi(x, \theta)} + \frac{\delta_{\theta} T^{(1)}(\theta)}{\delta \partial_\theta \Psi(x, \theta)} = -\left( i \partial_\nu \overline{\Psi} \Gamma^A + m \overline{\Psi} \right)(x, \theta) = 0 , \quad (7.28a)
\]
\[
\frac{\delta_{\theta} Z^2[\Psi, \overline{\Psi}]}{\delta \overline{\Psi}(x, \theta)} = -\left( i \Gamma^A \partial_\Lambda - m \right) \Psi(x, \theta) = 0 , \quad (7.28b)
\]
and represent, by virtue of (4.8), (4.9) the HCLF \( \Theta^1((\Psi, \bar{\Psi}, \partial_\mu \Psi, \partial_\mu \bar{\Psi})(x, \theta)) = 0 \) and DCLF(!) \( \Theta^2((\Psi, \bar{\Psi}, \partial_A \Psi, \partial_A \bar{\Psi})(x, \theta)) = 0 \), containing 2 equations in terms of bispinors, respectively.

The 2nd derivatives of \( S_L^{(1)}(\theta) \) w.r.t. \( \partial_\mu \Psi(x, \theta), \partial_\mu \bar{\Psi}(x, \theta) \) supermatrix (4.17) has the following block form in 4-component spinor formalism

\[
K^{(1)}(\theta, x, y) = \begin{bmatrix}
\frac{\partial_r}{\partial_\mu \bar{E}_a(x, \theta)} & \frac{\partial_t T^{(1)}(\theta)}{\partial_\mu \bar{E}_b(y, \theta)} \\
0 & 1_4 & -i\Gamma^\mu \partial_\mu - m 1_4 & 0_4 \\
1_4 & 0_4 & \delta(x - y) \\
0_4 & \delta(x - y), (E_1, E_2) = (\Psi, \bar{\Psi}), (7.29)
\end{bmatrix}
\]

appearing by the usual nondegenerate matrix w.r.t. \( \varepsilon \) grading and by the supermatrix with only odd-odd nonvanishing block for the case of \( \varepsilon_\Pi \) parity. The corresponding supermatrix for \( S_L^{(12)}(\theta) \) will be trivial \( (n = \tilde{m}) \) complicating the analysis of DCLF (7.28) in comparison with HCLF (7.27).

Solutions for Eqs.(7.27), (7.28) (for instance, trivial) exist, providing the fulfilment of assumption (4.13) in question. Since the Eqs.(7.27) are the HCLF, then in view of Corollary 2 and formula (4.36) validity the rank (4.15b) for \( S_L^{(1)}(\theta) \) is calculated by the rule (4.21)

\[
\begin{bmatrix}
\frac{\partial_r}{\partial_\mu \bar{E}_a(x, \theta)} & \frac{\partial_t S_L^{(1)}(\theta)}{\partial_\mu \bar{E}_b(y, \theta)} \\
0_4 & -i\Gamma^\mu \partial_\mu - m 1_4 & 0_4 \\
1_4 & 0_4 & \delta(x - y)
\end{bmatrix}
\]

It is well known fact that on mass-shell \( \Sigma_\psi^{(1)} \) the ranks of the 4 \times 4 matrices \( (i\Gamma^\mu \partial_\mu - m 1_4) \) and \( (i\Gamma^\mu \partial_\mu + m 1_4) \) equal to 2 so that supermatrix (7.30) contains only odd-odd block with rank being equal to 4 in terms of Weyl spinor’s components. From the equality \( \dim \Sigma_\psi^{(1)} = 0 \) it follows that \( m = 0 \) in (4.14b) and therefore there are not differential identities among Eqs.(7.27).

Although the same conclusion can be derived from the investigation results for the model with \( S_L^{(12)}(\theta) \), the structure of Eqs.(7.28) as DCLF leads to the definition for given theory status from the general grounds developed in Sec.IV. So, the rank of the supermatrix (4.17) for \( S_L^{(12)}(\theta) \) will be 4 matrices \((i\Gamma^\mu \partial_\mu - m 1_4)\) equal to 2 in terms of \( \varepsilon_\Pi \) grading and by the supermatrix with only odd-odd nonvanishing block for the case of \( \varepsilon_\Pi \) parity. The corresponding supermatrix for \( S_L^{(12)}(\theta) \) will be trivial \( (n = \tilde{m}) \) complicating the analysis of DCLF (7.28) in comparison with HCLF (7.27).

Solutions for Eqs.(7.27), (7.28) (for instance, trivial) exist, providing the fulfilment of assumption (4.13) in question. Since the Eqs.(7.27) are the HCLF, then in view of Corollary 2 and formula (4.36) validity the rank (4.15b) for \( S_L^{(1)}(\theta) \) is calculated by the rule (4.21)

\[
\begin{bmatrix}
\frac{\partial_r}{\partial_\mu \bar{E}_a(x, \theta)} & \frac{\partial_t S_L^{(1)}(\theta)}{\partial_\mu \bar{E}_b(y, \theta)} \\
0_4 & -i\Gamma^\mu \partial_\mu - m 1_4 & 0_4 \\
1_4 & 0_4 & \delta(x - y)
\end{bmatrix}
\]

It is well known fact that on mass-shell \( \Sigma_\psi^{(1)} \) the ranks of the 4 \times 4 matrices \((i\Gamma^\mu \partial_\mu - m 1_4)\) and \((i\Gamma^\mu \partial_\mu + m 1_4)\) equal to 2 so that supermatrix (7.30) contains only odd-odd block with rank being equal to 4 in terms of Weyl spinor’s components. From the equality \( \dim \Sigma_\psi^{(1)} = 0 \) it follows that \( m = 0 \) in (4.14b) and therefore there are not differential identities among Eqs.(7.27).

Although the same conclusion can be derived from the investigation results for the model with \( S_L^{(12)}(\theta) \), the structure of Eqs.(7.28) as DCLF leads to the definition for given theory status from the general grounds developed in Sec.IV. So, the rank of the supermatrix (4.17) for \( Z^2[\Psi, \bar{\Psi}] \) may be calculated by the rule (4.20) with regard for notations in (7.29) in the form \((z_a = (x_a, \theta_a), a = 1, 2)\)

\[
\begin{bmatrix}
\frac{\partial_r}{\partial_\mu \bar{E}_a(z_1)} & \frac{\partial_t Z^2[\Psi, \bar{\Psi}]}{\partial_\mu \bar{E}_b(z_2)} \\
0_4 & -i\Gamma^A \partial_A + m 1_4 & 0_4 \\
1_4 & 0_4 & \delta(z_1 - z_2)
\end{bmatrix}
\]

The rank value for supermatrix (7.31) on mass-shell \( \Sigma_\psi^{(2)} \) for Eqs.(7.28) coincides with rank for supermatrix (7.30) on \( \Sigma_\psi^{(1)} \) (i.e. equal to 4) in view of nilpotent character for term with \( \tilde{m} 1_4 \), therefore not affecting on rank value. As in the previous case the supermatrix (7.31) appear by nondegenerate in any neighbourhood of \( \Sigma_\psi^{(2)} \) reflecting the fact of the type (4.24) differential identities absence for this model.

By independent initial conditions for the 1st order w.r.t. derivatives on \( x^\mu \) and the 2nd order w.r.t. derivative on \( \theta \) \((\partial_\mu \Psi(x, \theta) = 0, \partial_\mu \bar{\Psi}(x, \theta) = 0)\) partial differential equations (HCLF) (7.27) and for the DCLF (7.28) being by the 1st order w.r.t. derivatives on \((x^\mu, \theta)\) superfield differential equations one can choose the expressions respectively

\[
(\Psi, \partial_\mu \Psi)(x, \theta)|_{x^\alpha = \theta = 0} = (\Psi, \bar{\Psi})(x^i), \quad \Psi(x, \theta)|_{x^\alpha = \theta = 0} = \bar{\Psi}(x^i).
\]

In the framework of Sec.IV terminology the \( \theta \)-superfield models described by the actions \( S_L^{(1)}(\theta) \) (7.24a) and \( S_L^{(12)}(\theta) \) (7.24b) belong to the classes of nondegenerate ThSTs and nondegenerate ThGTs with vanishing supermatrix (4.17) respectively. Really, given theories for \( \partial_\mu \Psi(x, \theta) = \)
$\partial_\theta \Psi(x, \theta) = \theta = 0$ have 2 second-class constraints, in terms of Dirac spinors leading to survival of only 4 physical degrees of freedom in terms of Weyl spinor's components for every model.

The interacting $\theta$-superfield spinor ThST and ThGT may be constructed in the framework of local theory by means of addition to $S^{(1)i}_L(\theta)$ (7.24a,b) at least quadratic combinations w.r.t. product $\langle \Psi \Psi \rangle(x, \theta)$ without derivatives w.r.t. $(x^\mu, \theta)$

$$S^{(1)i}_{LM}(\theta) = S^{(1)i}_L(\theta) + V(\langle \Psi, \overline{\Psi} \rangle(\theta)), \quad V^{(1)}(\theta) \equiv V(\langle \Psi, \overline{\Psi} \rangle(\theta))$$

$$= \int d^4x \left[ \frac{\lambda_1}{2} (\overline{\Psi} \Psi)^2 + \frac{\lambda_2}{2} (\overline{\Psi} \Gamma^\mu \Psi)(\overline{\Psi} \Gamma_\mu \Psi) \right](x, \theta), \quad \lambda_1, \lambda_2 \in \mathbb{R}, \quad (7.33)$$

including the $\theta$-superfield generalization of the Fermi interaction term (polynomial at $\lambda_1$).

Corresponding nonlinear independent Euler-Lagrange equations have the form for $Z^l_M[\Psi, \overline{\Psi}]$

$$Z^l_M[\Psi, \overline{\Psi}] + \int d\theta V^{(1)}(\theta)$$

$$\frac{\delta \{Z^l_M[\Psi, \overline{\Psi}] \}}{\delta \psi(y(x, \theta))} = -\left\{ \left[ i \partial_\mu \Gamma^\mu - \delta_{\alpha 1} \hat{\mu} \partial_\theta - m - \delta_{\alpha 1} \partial_\theta^2 - \lambda_1 (\overline{\Psi} \Psi) - \lambda_2 (\overline{\Psi} \Gamma^\mu \Psi) \Gamma_\mu \right] \Psi \right\}(x, \theta) = 0. \quad (7.34)$$

While in view of (4.8), (4.9) the linear HCLF (7.27) describe 2 pairs of the opposite charged particles (electrons $e^-$ and positrons $e^+$) corresponding to $\psi(x)$ and formally to $\psi_1(x)$, the nonlinear HCLF for $l = 1$ in (7.34) contains the following $P_a(\theta)$ components

$$-[(i \partial_\mu \Gamma^\mu - m) \psi(x) = \frac{\delta P_0 V^{(1)}(\theta)}{\delta \psi(x)}], \quad (7.35a)$$

$$-[(i \partial_\mu \Gamma^\mu - m) \psi_1(x) = \int d^4y \frac{\delta P_0 V^{(1)}(\theta)}{\delta \psi(y)} \frac{\delta P_0 V^{(1)}(\theta)}{\delta \psi(y)} \psi_1(y) + \delta \psi(y) \frac{\delta P_0 V^{(1)}(\theta)}{\delta \psi(x)} \psi_1(y)], \quad (7.35b)$$

with simultaneous definition of the component form for $\hat{U}_0(\theta)$ (6.9). One can consider that Eq. (7.35b) for $\psi_1(x)$ is given in an external field being determined by a solution of the Eq. (7.35a) for ordinary spinor $\psi(x)$.

The structure of linear DCLF (7.28) is more difficult in view of the superfields $\partial_\theta (\overline{\Psi} \Psi)(x, \theta)$ nontrivial occurrence which complicates the $P_0(\theta)$ component of Eqs. (7.34) for $l = 2$ in comparison with HCLF. On the other hand the $P_1(\theta)$ component of Eqs. (7.34) has the form (7.35b). The relationships (7.25), (7.30), (7.31) are changed taking (7.33), (7.34) into account in an evident way for interacting models.

It is not difficult to repeat here the all computations for ZLR and its direct problem made for preceding model in (7.16)–(7.20). Let us demonstrate a some moments, for only free ThGT with $S^{(1)2}_L(\theta)$ (7.24b) with taking notations (7.25) into account. So, the superfunctional

$$Z^{(1)}[\Psi, \overline{\Psi}, \Psi^*, \overline{\Psi}^*] = Z^2[\Psi, \overline{\Psi}] = \frac{\partial}{\partial \theta} S^{(1)2}_L(\theta) \quad (7.36)$$

and superfunction $S^{(1)}(\langle \Psi, \overline{\Psi}, \Psi^*, \overline{\Psi}^* \rangle(\theta)) \equiv S_0(\langle \Psi, \overline{\Psi} \rangle(\theta))$ lead to the trivial generating equations (5.3) (for $\Gamma^\theta = 0$ to the Eqs. (5.10)), (5.4) defined on the symplectic $T_{odd}(T_{odd}^* M_{cl}) = \{(\Gamma_{cl}, \partial_\theta \Gamma_{cl}(z))\}$ and antisymplectic $T_{odd}^* M_{cl} = \{((\Psi, \Psi^*), (\overline{\Psi}, \overline{\Psi}^*)) \} \equiv \Gamma_{cl}(z)$ supermanifolds, where we have introduced the spinor superantifields (for $z = (x, \theta)$)

$$(\Psi^*, \overline{\Psi}^*)(z) = (\psi^*, \overline{\psi}^*)(x) - \theta (J_{\Psi}, J_{\overline{\Psi}})(x), \quad \varepsilon(\Psi^*, \overline{\Psi}^*) = \varepsilon(J_{\Psi}, J_{\overline{\Psi}}) + (1, 0, 1) = (1, 1, 0). \quad (7.37)$$

Let us only find here the more general than in (7.18) form for antibracket (5.14) on $Z_{Q^l} = \{((\Psi^*, \overline{\Psi}^*), \partial_\theta (\Psi^*, \overline{\Psi}^*))(z)\}$ defined by $Q^l = Z^2[\Psi, \overline{\Psi}], \quad \{\}$ over a some solution $(\Psi_0, \overline{\Psi}_0)(z)$ for
Eqs. (7.28) (for $Z_{(0)} = Z^1[\Psi, \overline{\Psi}]$)

$$(\tilde{f}[\Psi^*, \overline{\Psi}^*], \tilde{g}[\Psi^*, \overline{\Psi}^*]) = \int dz \left[ \frac{\delta_{\Psi} \tilde{f}}{\delta_{\Psi^*}(z)} (i \Gamma^A \partial_A - m) \frac{\delta_{\Psi} \tilde{g}}{\delta_{\Psi^*}(z)} - (-1)^{(e(f)+1)(e(g)+1)} (\tilde{f} \leftrightarrow \tilde{g}) \right]$$

$$= \int dz \left[ \frac{\delta_{\Psi} \tilde{f}}{\delta_{\Psi^*}(z)} i \mu \partial_\mu \frac{\delta_{\Psi} \tilde{g}}{\delta_{\Psi^*}(z)} - (-1)^{(e(f)+1)(e(g)+1)} (\tilde{f} \leftrightarrow \tilde{g}) \right] + (\tilde{f}[\Psi^*, \overline{\Psi}^*], \tilde{g}[\Psi^*, \overline{\Psi}^*]) Z_{(0)}.$$ (7.38)

Again the corresponding BRST charge $Z^{(-1)}(\theta)$ on $Z_{Q^0} = \{ (\Psi^*, \overline{\Psi}^*) \}$ is trivial whereas the superfunction $S^{(-1)2}((J_\Psi, J_{\overline{\Psi}})(\theta^{-1}))$ being analogous to (7.20) continued up to $S_L^{(-1)2}(\theta^{-1})$ exists

$$S_L^{(-1)2}((J_\Psi, J_{\overline{\Psi}}, \partial_{\theta^{-1}}(J_\Psi, J_{\overline{\Psi}}))(\theta^{-1})) = - \int d^4 x (J_\Psi (i \Gamma^A \partial_A - m) J_{\overline{\Psi}})(x, \theta^{-1}),$$ (7.39)

satisfying to the master equation with antibracket (7.38) and being dual to $S_L^{(1)2}(\theta)$ (7.24b).

### VII.3 Free Vector Superfield Models

Setting for supergroups and quotient space (2.3)

$$\tilde{J} = \Pi(1, D - 1)^\dagger, \tilde{M} = T(1, D - 1), \tilde{J}_A = SO(1, D - 1)^\dagger, D \geq 2, D \in N,$$

$$\mathcal{M} = R^{1,D-1} \times \tilde{P} = \{ (x^\mu, \theta) \}, \text{ diag } \eta_{\mu\nu} = (1, -1, \ldots, -1), \mu, \nu = 0, 1, \ldots, D - 1,$$ (7.41)

we consider as $A^\mu(x, \theta)$ the real vector superfield $A^\mu(x, \theta)$

$$A^\mu(x, \theta) = A^\mu(x) + A^\mu_1(x) \theta, A^\mu(x, \theta) \in \tilde{A}_{D;0+1}(x^\mu, \theta; \mathbb{R}),$$ (7.42)

being by element of $\Pi(1, D - 1)^\dagger$ group massless irrep space and encoding $n = n_+ = D$ real degrees of freedom. The index $i$ contents, Grassmann vector values for quantities above and the type (7.7) obvious properties of conjugation read as follows

$$i = (\mu, x), \tilde{\varepsilon}(A^\mu(x), A^\mu(x, \theta)) = \tilde{\varepsilon}(A^\mu_1(x)) + (1, 0, 1) = 0, (\partial_\theta A^\mu)(x, \theta) = \partial_\theta A^\mu(x, \theta).$$ (7.43)

The superfields $(A^\mu, \partial_\theta A^\mu)(x, \theta)$ are transformed, in a standard way, as Lorentz vectors w.r.t. $T_{\Pi(1, D - 1)^\dagger}$-superfield representation whereas the only $A^\mu(x, \theta)$ have nontrivial transformation law w.r.t. $T_{|\tilde{P}}$ action of the form (2.18)

$$\delta A^\mu(x, \theta) = A^{\mu'}(x, \theta) - A^\mu(x, \theta) = - \mu \partial_\theta A^\mu(x, \theta) = \mu A^\mu_1(x).$$ (7.44)

As the classical $\Lambda_1(\theta, \mathbb{R})$-valued action $S_L((A^\mu, \partial_\theta A^\mu)(\theta)) \equiv S_L^{(2)}(\theta)$ for free vector superfield, describing massless particle (helicity $\lambda = \pm 1$ for $D = 4$), choose the local superfunction in the natural system form with $g_{ij}(\theta) = 0$ in (4.34) not explicitly depending upon $\theta$, without dimensional constants as in (7.6) and with antisymmetric $\varepsilon_{\mu\nu}$

$$S_L^{(2)}(\theta) = T(\partial_\theta A^\mu(\theta)) - S_0(A^\mu(\theta)) = \int d^D x \left( \frac{1}{2} \varepsilon_{\mu\nu} \partial_\theta A^\nu \partial_\theta A^\mu - \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right)(x, \theta).$$ (7.45)

The transformation law for $\Pi(1, D - 1)^\dagger$-scalar $S_L^{(2)}(\theta)$ w.r.t. $T_{|\tilde{P}}$ action has the form in agreement with (4.3) realizing among them the superfield structure for operator $\tilde{U}(\theta)$

$$\delta S_L^{(2)}(\theta) = \mu \int d^D x A^\nu(x, \theta) P_0(\theta) \frac{\partial \tilde{S}_0(A^\mu(\theta))}{\partial A^\nu(x, \theta)} = \mu \int d^D x A^\nu(x, \theta) \partial^\mu F_{\mu\nu}(x, 0).$$ (7.46)
The invariance of $S_L^{(2)}(\theta)$ w.r.t. $T_{IP}$ are restored on the Euler-Lagrange equations

$$\frac{\delta Z[A^\mu]}{\delta A^\mu(x, \theta)} = -\partial_\mu S_0(A^\mu(\theta)) - \partial_\mu \frac{\delta T}{\delta A^\mu(x, \theta)} = (\varepsilon_{\mu\nu} \partial_\mu A^\nu - \partial^\mu F_{\mu\nu})(x, \theta) = 0, \quad (7.47)$$

containing in view of identical fulfilment $\partial_\mu A^\mu = 0$ the HCLF $\Theta_{\nu} (A^\mu (x, \theta)) = 0$ being by the 2nd order on $(x^\mu, \theta)$ $D$ linear homogeneous partial differential equations.

The supermatrix (4.12) for $S_L^{(2)}(\theta)$ has the form in question

$$K^{(2)}(\theta, x, y) = \frac{\partial_t}{\partial(\partial_t A^\mu(x, \theta))} \frac{\partial S_L(A^\mu(\theta), \partial_\theta A^\mu(\theta))}{\partial(\partial_\theta A^\mu(y, \theta))} = \|\varepsilon_{\mu\nu}\| \delta(x - y), \quad (7.48)$$

being by the usual matrix w.r.t. $\varepsilon_H$ grading and by the supermatrix with only nontrivial odd-odd block w.r.t. $\varepsilon$ parity. Rank of $K^{(2)}(\theta, x, y)$ depends on values of $D = \dim \mathbb{R}^{1,D-1}$. So, for odd $D$, this supermatrix is always degenerate ($\tilde{m} > 0$) in view of skew-symmetry for $\varepsilon_{\mu\nu}$ whereas the choice for $\varepsilon_{\mu\nu}$ for even $D$, for instance, in the form

$$\|\varepsilon_{\mu\nu}\| = \text{antidiag}(-1_k, 1_k), \quad D = 2k, \quad k \in \mathbb{N}, \quad (7.49)$$

yields the nondegenerate $K^{(2)}(\theta, x, y)$.

The solutions for Eqs.(7.47) exist, providing the fulfilment of assumption (4.13) in question. In its turn the rank of supermatrix (4.15) for given model is equal to

$$\text{rank} \left\| \frac{\partial_t}{\partial A^\mu(x, \theta)} \frac{\partial S_L(A^\mu(\theta), \partial_\theta A^\mu(\theta))}{\partial A^\mu(y, \theta)} \right\|_{\Sigma_A} = \text{rank} \|\left(\Box \eta_{\mu
u} - \partial_\mu \partial_\nu\right)\|_{\Sigma_A} \delta(x - y) = D - 1, \quad (7.50)$$

being always strictly less than $n$ in the whole $M_{cl} = \{A^\mu(x, \theta)\}$.

Therefore there is only one $(m = 1)$ differential identity among Eqs.(7.47) with standard choice for linear independent generator compatible with the conclusions from Corollary 1

$$R^\mu(x, y) = \partial^\mu \delta(x - y), \quad \alpha = y. \quad (7.51)$$

Given $\theta$-STF model is the GThST with GGTST above in the Sec.IV terminology. GTST being invariance transformation for only $S_0(A^\mu(\theta))$ has the form of standard gradient transformation $\delta A^\mu(x, \theta) = \partial^\mu \xi(x, \theta)$ with $\varepsilon$-boson arbitrary superfield $\xi(x, \theta)$. As consequence, not all from the following initial conditions for LS (7.47) are independent

$$\left(A^\mu, \partial_\theta A^\mu, \partial_\theta A^\mu, \partial_\theta \partial_\theta A^\mu\right)(x, \theta)|_{x^\theta = \theta = 0} = \left(A^\mu, A^\mu_i, \tilde{A}^\mu, \lambda^\mu_i\right)(x^\lambda), \quad i = 1, D - 1. \quad (7.52)$$

As the another example of a vector model consider the theory of free complex massive vector superfield for arbitrary $D \geq 2$ (for $D = 4$ describing two mass $m$ charged particles of spin 1). In this case the configuration space $M_{cl}$ coordinatized by $(A^\mu, \tilde{A}^\mu)(x, \theta) \in \Lambda_{D+1}(x^\mu, \theta; \mathbb{C})$ describing $2D$ real degree of freedom

$$A^\mu(x, \theta) = A^\mu_i(x, \theta) + i \Lambda^\mu_j(x, \theta) = A^\mu(x) + \lambda^\mu(x) \theta, \quad A^\mu_j(x, \theta) = A^\mu_j(x) + \lambda_j^\mu(x) \theta, \quad \Lambda^\mu_j(x, \theta) \in \Lambda_{D+2}(x^\mu, \theta; \mathbb{R}), \quad j = 1, 2. \quad (7.53)$$

The condensed index $i$ contents is extended up to $i = (\mu, \nu, x) \mapsto ([A^\mu], \overline{[A^\nu]}], x)$ in comparison with (7.43) whereas the Grassmann gradings appear by the same but for complex (super)fields. The properties (7.7) for scalar $P_1(\theta)$-component fields remain valid for $\lambda^\mu_j(x)$. 
The $\Lambda_1(\theta, R)$-valued superfunction

$$S^{(2)}_{Lm}(\theta) = S^{(2)}_{Lm}(\{A^\mu, \bar{A}^\mu, \partial_\theta(A^\mu, \bar{A}^\mu)\}(\theta)) = T(\partial_\theta(A^\mu, \bar{A}^\mu)(\theta)) - S_{0m}(\{A^\mu, \bar{A}^\mu\}(\theta))$$

$$= \int d^D x \left((\varepsilon_{\mu\nu}\partial_\theta \bar{A}^\nu \partial_\theta A^\mu - i\partial_\theta \bar{A}^\nu \partial_\theta A_\mu) - \left( -\frac{1}{2} \bar{T}_{\mu\nu} F^{\mu\nu} + m^2 \bar{A}_\mu A_\mu \right) \right)(x, \theta) \quad (7.54)$$

may be chosen as the classical action leading to the 2nd order w.r.t. $(x^\mu, \theta)$ complex linear partial differential equations (LS) written by means of superfunctional $Z_m[A^\mu, \bar{A}^\mu] = \int d\theta S^{(2)}_{Lm}(\theta)$

$$\frac{\delta Z_m[A^\mu, \bar{A}^\mu]}{\delta \bar{A}^\mu(x, \theta)} = (\varepsilon_{\nu\mu} + \eta_{\nu\mu})\partial_\theta^2 A^\mu(x, \theta) - (\partial^\mu F_{\mu\nu} + m^2 A_\nu)(x, \theta) = 0 \quad (7.55)$$

providing the property for $S^{(2)}_{Lm}(\theta)$ to be integral w.r.t. $\hat{T}_1$ global transformations with simultaneous realization of $\hat{U}_+(\theta)$ for this model

$$\delta S^{(2)}_{Lm}(\theta) = \mu \int d^D x \left[ \hat{A}^\nu(x, \theta) P_0(\theta) \frac{\partial}{\partial \bar{A}^\nu(x, \theta)} + \hat{\bar{A}}^\nu(x, \theta) P_0(\theta) \frac{\partial}{\partial A^\nu(x, \theta)} \right] S_{0m}(\theta)$$

$$= \mu \left[ \partial_\theta A^\mu(x, \theta) (\partial^\mu \bar{T}_{\mu\nu} + m^2 \bar{A}_\nu)(x, 0) + (c.c.) \right] \quad (7.56)$$

As the independent initial conditions for complex LS (7.55) one can take the complexified Cauchy problem (7.52). Really, in first, the solution for HCLF in (7.55) exists, in second, the supermatrix (4.12) in question for $E_b \in \{A, \bar{A}\}$

$$K^{(2)}_m(\theta, x, y) = \begin{vmatrix} \frac{\partial}{\partial(\partial_\theta E^\mu_b(x, \theta))} \frac{\partial S^{(2)}_{Lm}(\theta)}{\partial(\partial_\theta E^\nu_b(y, \theta))} \end{vmatrix} = \begin{vmatrix} 0 & \varepsilon_{\mu\nu} + \eta_{\mu\nu} \\ \varepsilon_{\mu\nu} - \eta_{\mu\nu} & 0 \end{vmatrix} \delta(x - y) \quad (7.57)$$

may be chosen by nondegenerate for any $D$ and, in third, the rank of supermatrix (4.15) is calculated by the rule (4.21) in the form, being differed from the double value for rank of the previous vector model supermatrix (7.50),

$$\begin{vmatrix} \frac{\partial}{\partial E^\mu_b(x, \theta)} \frac{\partial S_{0m}(\theta)}{\partial E^\nu_b(y, \theta)} \end{vmatrix} = \begin{vmatrix} 0 & (\Box + m^2) \eta_{\mu\nu} - \partial_\mu \partial_\nu \\ (\Box + m^2) \eta_{\mu\nu} - \partial_\nu \partial_\mu & 0 \end{vmatrix} \delta(x - y) \quad (7.58)$$

and equal to $2D$ almost everywhere in $\mathcal{M}_{cl}$ in view of zero-dimensionality of the mass-shell $\Sigma_{A_m}$ ($m = 0$) in question.

In contrast to massless case the model has 6 physical degrees of freedom for $D = 4$ and appears by singular [39] nondegenerate (i.e. nongauge) ThST because of the 2 second class constraints presence in applying of Dirac-Bergmann algorithm.

The dynamical equations (7.47), (7.55) appear by the same for both their $P_0(\theta)$ components $a = 0, 1$, thus describing formally the similar dynamics for corresponding to the fields $A^\mu(x)$, $A_1^\mu(x)$ particles.

The generalization of the massive complex vector model to interacting theory appears by evident as it was made for the examples with scalar and spinor superfields.

Since the ZLR brackets and objects construction for superfunction (7.54), in fact, repeats the scalar superfield properties then we consider only the analogous problem for GThST with $S^{(2)}(\theta)$ (7.45). Corresponding exact $\theta$-superfield BRST charge as in (5.10) and, in fact, BV action (5.4) have the form in $T_{odd}(T_{odd}^{M_{min}}) = \{ (\Gamma^{(0)}_{min}, \partial_\theta \Gamma^{(0)}_{min})(z) \}, p = 1, 2(4 + 1)$ and $T_{odd}^{*} \mathcal{M}_{min} = \{ (A^\mu, C), (A^\mu_1, C_1)(z) \} \equiv \{ \Gamma^{(0)}_{min}(z) \}$ respectively

$$Z(0)[\Gamma_{min}] = -\partial_\theta S(1)(\Gamma_{min}(\theta)) = \int d^D x \left(A^1_\mu \partial^\mu F_{\mu\nu} + J_\mu \partial^\mu C + A_\mu^* \partial^\mu \lambda \right)(x), \quad (7.59)$$

$$S(1)(\Gamma_{min}(\theta)) = \int d^D x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu^* \partial^\mu C \right)(x, \theta). \quad (7.60)$$
The above quantities satisfy to the generating equations (5.10), (5.4) with simple superbrackets (5.5), (5.6). Whereas the corresponding new even bracket (5.13) on \( Z_{Q^0} \) (dim \( Z_{Q^0} = (2, 3) \)) and odd (5.22) on \( Z_{Q^1} \) (dim \( Z_{Q^1} = (2 + 3 + 2) \)) are written as follows (for \( z = (x^\mu, \theta) \))

\[
\{ f, g \}_\theta = \int d^D x \left[ \frac{\partial F(\theta)}{\partial A^a_k(x)} (\Box \eta_{\mu
u} - \partial_\mu \partial_\nu) + \left( \frac{\partial F(\theta)}{\partial A_\mu(x)} \partial_\mu \partial C^\ast(x) + (C^\ast \leftrightarrow A^\mu) \right) \right]_{Z_{Q^0}},
\]

\[
\{ f, g \} = \int d^D x \left[ \frac{\delta F[\Gamma]}{\delta A^a_k(x)} (\Box \eta_{\mu
u} - \partial_\mu \partial_\nu) + \left( \frac{\delta F[\Gamma]}{\delta A_\mu(x)} \partial_\mu \partial C^\ast(x) + (C^\ast \leftrightarrow A^\mu) \right) \right]_{Z_{Q^1}},
\]

where \( Z_{Q^0} \) may be parametrized by \( C^\ast(z) \), 3 antifields \( \tilde{A}^a_k(z) \) from \( A^a_k(z) \) (determined by equation \( \partial_\mu A^a_k(z) = 0 \)) and 1 from superfield \( A_\mu(z) \) not being conjugate w.r.t. initial antibracket to \( \tilde{A}^a_k(z) \). In turn, the \( Z_{Q^1} \) therefore may be coordinatized by the variables above and their derivatives on \( \theta \).

In view of \( Z_{Q^0} \) structure the new BRST charge \( Z_{(-1)}(\theta) \) vanishes again. The new dual classical action representing the GThST on \( Z_{Q^1} \) may be determined by the formula with introduction of new \( \theta^{-1} \)

\[
S_{(-1)}^L \left( \partial_\theta A^a_\mu, \partial_{\theta^{-1}} A^a_\mu \right)(\theta^{-1}) = \int d^D x \left( \frac{1}{2} \varepsilon^{\mu
u} \partial_\theta \partial_{\theta^{-1}} A^a_\mu \partial_\theta \partial_{\theta^{-1}} A^a_\mu + \frac{1}{4} F_{\mu
u} F^\ast_{\mu\nu} \right)(x, \theta^{-1}),
\]

where the superantifield strength \( F_{\mu
u}^\ast(x, \theta^{-1}) = [\partial_\theta, \partial_{\theta^{-1}} A^a_\mu(z, \theta^{-1})] \) is invariant w.r.t. standard gradient transformation for \( \partial_{\theta^{-1}} A^a_\mu(z, \theta^{-1}) \) with arbitrary \( \xi(z, \theta^{-1}) \).

### VII.4 \( \theta \)-superfield \( U(1|0) \times U(0|1) \) Abelian GThST Model

As the initial model we choose the any from the models above with complex superfields for \( D = 4 \) admitting the realization of the global transformations generated by two-parametric supergroup (without sum on \( a \) in (7.64))

\[
U^{11} = U(1|0) \times U(0|1), \ U(\delta_{d0}|\delta_{d1}) = \{ \exp \{ i \xi^a \epsilon^a \} | e^0, \xi^0 \in \mathbb{R}, e^1, \xi^1 \in \Lambda_1(\theta; \mathbb{R}) \}, \ (e^a, \xi^a)^* = (e^a, \xi^a), \ (\epsilon^a, \xi^a) = (a, 0, a), a = 0, 1,
\]

for composite superfields \( (\Phi, \overline{\Phi}) \in \{ (\varphi, \Psi, A^\mu), (\overline{\varphi}, \overline{\Psi}, \overline{A}^\mu) \} \) in the form

\[
(\Phi, \overline{\Phi})(x, \theta) \mapsto (\Phi', \overline{\Phi}') (x, \theta) = (\exp \{ e^a \xi^a \} \Phi, \overline{\Phi} \exp \{ -i \xi^a \epsilon^a \} ) (x, \theta) = (g \Phi, \overline{g}^\ast) (x, \theta).
\]

The set of transformations (7.65) leaves the actions (7.6), (7.24a,b), (7.54) by invariant.

The realization of the Yang-Mills type gauge principle [40], for instance, for the case of spinor superfields is based on the change of superparameters \( \xi^a \) onto arbitrary superfields \( \xi^a(x, \theta) \) in such a way that the resultant action \( S_{LC}(\theta) \), which we shall seek now, must be invariant w.r.t. GTG for all the superfields parametrizing the new extended configuration space \( M_{cl} \) (n = (n_+, n_-) = (4 + 1, 8 + 4 + 1) = ([\Lambda^{\mu\nu}], [\Lambda^{0\ast}], [\Lambda^\mu], [\Lambda^\ast])\):

\[
(\Psi, \overline{\Psi}, A^{a\mu})(x, \theta) \mapsto (\Psi', \overline{\Psi}', A'^{a\mu})(x, \theta) = (g \Psi, \overline{g}^\ast, A^a + \partial A^a \xi^a)(x, \theta),
\]

\[
A^{a\mu}(x, \theta) = A^{a\mu}(x) + A_1^{a\mu}(x, \theta) = (A^{\mu\ast}, C^{a\ast}_0)(x, \theta) \in \Lambda_{40 + 1}(x^\mu, \theta; \mathbb{R}),
\]

\[
\varepsilon(A^{a\mu}(x, \theta), A^{a\mu}(x)) = \varepsilon(A_1^{a\mu}(x)) + (1, 0, 1) = ((\varepsilon_P) + a, 0, a + \varepsilon_A).
\]

In contrast to the models above the \( \varepsilon_P \) Grassmann parity spectrum is nontrivial even for the case of standard \( U(1|0) \equiv U(1) \) transformations in view of ghost superfield \( C^{a\ast}_0(x, \theta) \equiv C^{0\ast}_0(x, \theta) \).
inclusion (being differed from the role of $C(x, \theta)$ in Sec.V) on the initial level of the model formulation.

Adapting the general form of infinitesimal GTGT (4.48) and GGTGT (4.25) one can write their realization together with specification of the index $i, \alpha$ contents as follows

$$\delta_{\theta}A'(\theta) = \int d\theta' R^i_{ib}(A(\theta), \theta; \theta') \delta \xi^b(\theta'),$$

$$A'(\theta) = (A^{A_{}\alpha}, \bar{\Psi}, \Psi)(x, \theta), \quad (\bar{\xi}, \xi) = (i, x), \quad \epsilon = (b, y),$$

$$R^i_{ib}(A(x, \theta), x; y, \theta') = \sum_{k \geq 0} \left( (\partial_{\theta})^k \delta (\theta - \theta') \right) R^i_{kb}(A(\theta, x))(\delta (x - y),$$

$$\tilde{R}^i_{\mu b}(A(x, \theta)) = \begin{cases} -\partial^\mu b_{\alpha}, & i = (\mu, a) \\ ic^\mu \bar{\Psi}(x, \theta), & i = (\beta, \tilde{\beta}) \\ -ic^\mu \Psi(x, \theta), & i = (\gamma, \tilde{\gamma}, x) \end{cases},$$

$$\tilde{R}^{\hat{\mu} b}_{\alpha}(A(x, \theta)) = \delta^{\hat{\mu} b}_{\alpha}, \quad (\epsilon_\mu) = \epsilon_a = b, \quad m = (m_+, m_-) = (1, 1).$$

The easily obtained only trivial solution for superfunctions $u^b(A(\theta), \partial_{\theta}A(\theta), \theta)$ in Eq.(4.26) implies the set of GGTGT above is the functionally independent and forms the gauge algebra of GTGT with abelian gauge supergroup $U^{11}$.

To construct the classical action, realizing the minimal inclusion of interaction for spinor superfields by means of connectedness coefficients $A^{A_{}\alpha}(x, \theta)$, let us consider the not Lorentz type covariant derivatives

$$D^A \equiv \partial^A - \iota A^{A_{}\alpha}(x, \theta)e^a = (D^\mu, D^\theta),$$

whose supercommutator permit to obtain the GTGT-invariant $\theta$-superfield strength in the almost standard manner

$$\mathcal{F}_{AB}^a(x, \theta) = \iota \frac{d}{d\epsilon} [D_A, D_B]_s = (\partial_\alpha A^a_{\beta} - (-1)^{\epsilon A_{\beta} B_{\alpha}} \partial_B A^a_{\alpha})(x, \theta),$$

$$A^{A_{}\alpha}(x, \theta) = \delta^{A_{}\alpha} B_{\beta}(x, \theta), \quad \hat{\eta}^{AB} = \text{diag}(1, -1, -1, -1|1) = (\hat{\eta}^{\mu\nu}, \hat{\eta}^{\theta\theta}),$$

$$\mathcal{F}_{AB}^a(x, \theta) = \begin{vmatrix} F_{\mu a} & F_{\nu a} \\ F_{\theta a} & F_{\eta a} \end{vmatrix}(x, \theta) = \begin{vmatrix} \partial_{[\mu} A_{\nu]} & \partial_{\nu} C_{(0)}^\alpha - \partial_{\mu} A_{\alpha}^a \\ \partial_{\mu} A_{\nu} - \partial_{\nu} C_{(0)}^\alpha & 2\partial_{\nu} C_{(0)}^\alpha \end{vmatrix}(x, \theta),$$

$$= -(-1)^{\epsilon A_{\beta} B_{\alpha}} \mathcal{F}_{BA}^a(x, \theta), \quad (A, B) = ((\mu, \theta), (\nu, \theta)).$$

In view of special structure for gauge algebra the following $\Lambda_1(x^\mu, \theta; \mathbf{R})$-valued quadratic w.r.t. $\mathcal{F}_{AB}^a(x, \theta)$ superfunctions possess by the properties of Poincare and GTGT invariances

1) \begin{equation} (\mathcal{F}_{AB}^a \mathcal{F}_{AB}^b)(x, \theta) = \left( F_{\mu a}^0 F_{\mu a}^{00} + 2F_{\mu a}^1 F_{\mu a}^{01} + 4(\partial_{\theta} C_{(0)}^0)^2 \right)(x, \theta), \end{equation}

\begin{equation} F_{\mu a}^1 F_{\mu a}^{01}(x, \theta) = \left( \partial_{\mu} C_{(0)}^1 \partial_{\theta} C_{(0)}^1 + \partial_{\mu} A_{\alpha}^a \partial_{\theta} A_{\alpha}^a - 2(\partial_{\theta} A_{\mu a}^a \partial_{\mu} C_{(0)}^1 \right)(x, \theta), \end{equation}

2) \begin{equation} \varepsilon_{ABC} \mathcal{F}_{AB}^a \mathcal{F}_{CD}^a \varepsilon_{\delta a}(x, \theta) = 4 \left( \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{00} + \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{0 \theta} + \varepsilon_{\mu a \rho} F_{\rho a}^{0 \theta} \right)(x, \theta), \end{equation}

$$\varepsilon_{\hat{\mu} \hat{\lambda} \hat{\rho} \hat{\theta}} \mathcal{F}_{AB}^a \mathcal{F}_{CD}^a \varepsilon_{\delta a}(x, \theta) = \left( \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{00} + \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{0 \theta} + \varepsilon_{\mu a \rho} F_{\rho a}^{0 \theta} \right)(x, \theta),$$

$$\varepsilon_{\hat{\mu} \hat{\lambda} \hat{\rho} \hat{\theta}} \mathcal{F}_{AB}^a \mathcal{F}_{CD}^a \varepsilon_{\delta a}(x, \theta) = \left( \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{00} + \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{0 \theta} + \varepsilon_{\mu a \rho} F_{\rho a}^{0 \theta} \right)(x, \theta),$$

$$\varepsilon_{\hat{\mu} \hat{\lambda} \hat{\rho} \hat{\theta}} \mathcal{F}_{AB}^a \mathcal{F}_{CD}^a \varepsilon_{\delta a}(x, \theta) = \left( \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{00} + \varepsilon_{\mu a \rho} F_{\mu a}^{0 \rho} F_{\rho a}^{0 \theta} + \varepsilon_{\mu a \rho} F_{\rho a}^{0 \theta} \right)(x, \theta),$$

with $\varepsilon_{ab}$ defined in (3.15b). The such choice for quantities above is stipulated by the general solutions for 3 independent equations, following from the trivial requirement for $\mathcal{F}_{AB}^a \mathcal{F}_{AB}^a$. 


$\varepsilon_{ABCD} F^{ABa} F^{CDe} , \bar{\varepsilon}_{ABCD} F^{ABa} F^{CDe} \varepsilon_{ab}$ to be invariant w.r.t. permutation of their comultipliers $F^{ABa}$ taking the gradings distribution (7.68) into account

$$
1) \quad \varepsilon_A + \varepsilon_B + a = 0 \quad \Leftrightarrow \quad \begin{cases} 
\varepsilon_A = \varepsilon_B , \quad a = 0 \\
\varepsilon_A = \varepsilon_B + 1 , \quad a = 1 
\end{cases} ,
$$

(7.78a)

$$
2) \quad a(\varepsilon_A + \varepsilon_B + \varepsilon_C + \varepsilon_D + 1) = 0 \quad \Leftrightarrow \quad \begin{cases} 
\forall A, B, C, D , \quad a = 0 \\
\varepsilon_A + \varepsilon_B = \varepsilon_C + \varepsilon_D + 1 , \quad a = 1 
\end{cases} ,
$$

(7.78b)

$$
3) \quad a(\varepsilon_A + \varepsilon_B + \varepsilon_C + \varepsilon_D) + \varepsilon_A + \varepsilon_B = 1 \quad \Leftrightarrow \quad \begin{cases} 
\varepsilon_A = \varepsilon_B + 1 , \forall A, B , \quad a = 0 \\
\varepsilon_C = \varepsilon_D + 1 , \forall A, B , \quad a = 1 
\end{cases} .
$$

(7.78c)

Let us restrict for simplicity now the choice for components of totally superantisymmetric constant tensors $(\varepsilon, \bar{\varepsilon})_{ABCD}$ to be only with even w.r.t. $(\varepsilon, \bar{\varepsilon})_{ABCD}$ parities for the 1st tensor and with vanishing for the 2nd tensor values respectively

$$
\varepsilon_{0123} = \varepsilon_{0000} = 1 , \quad (\varepsilon, \bar{\varepsilon})_{\mu\nu\rho\sigma} = (\varepsilon, \bar{\varepsilon})_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma} = 0 , \quad \varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu}^{(1)} = \varepsilon_{\mu\nu}^{(1)} .
$$

(7.79)

That representation permit to simplify the only nonzero terms in (7.75) as follows

$$
(\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu0} F^{\rho\sigma0} + 4\varepsilon_{\mu\nu}^{(1)} (F^{\nu\sigma\delta} \partial_\delta C_{(0)}^0 + F^{\mu\sigma0} F^{\nu\delta0}) + 4(\partial_\delta C_{(0)}^0)^2)(x, \theta) .
$$

(7.80)

Note, firstly, the density $(\partial_\delta C_{(0)}^0)^2(x, \theta)$ appears by the self-dual, secondly, the summands with $\varepsilon_{\mu\nu}^{(1)}$ factor are reduced with accuracy up to total derivatives w.r.t. $(x^\mu, \theta)$ to the form

$$
\varepsilon_{\mu\nu}^{(1)} (F^{\nu\sigma\delta} \partial_\delta C_{(0)}^0 + F^{\mu\sigma0} F^{\nu\delta0}) (x, \theta) = \varepsilon_{\mu\nu}^{(1)} (\partial_\delta A^{\mu0} \partial_\delta A^{\nu0})(x, \theta) ,
$$

(7.81)

and, thirdly, we transform the 3rd term in $F^{\mu01} F^{\mu01}$ (7.74) up to the same accuracy to $2A^{\mu01} \partial_\mu \partial_\nu C_{(0)}^{01}$.

Now we have all means in order to construct the GTGT invariant superfunction $S_{LG}(\theta)$ defining the GTHGT with incorporation both the ghost superfield with its real even scalar superpartner $C_{(1)}^0(x, \theta)$ and electromagnetic superfield $A^{\mu0}(x, \theta)$ with its real odd vector superpartner $A_{(1)}^{a}(x, \theta)$ into $\theta$-superfield multiplet of the gauge classical superfields $A^{(\theta)}$. Besides, making use of the inclusion into $S_{LG}(\theta)$ the quantities (7.80) by means of the “$\theta$-term” (vacuum angle) addition that leads to application in the electromagnetic duality theory, see for instance ref.[41], we choose in the action in the form (for $D_\chi^a \overline{\Psi} = (\partial_\alpha + i A^{a}_A \bar{\psi}^a)(\overline{\Psi})$

$$
S_{LG}(\theta) = S_{LG}((A^a_A, \partial_\theta A^a_A, \overline{\Psi}, \overline{\partial_\theta \Psi}, \partial_\theta \overline{\Psi}, \partial_\theta \overline{\Psi}))(\theta) = T_{inv}((D_\theta \Psi, D^*_\theta \overline{\Psi}))(\theta) \\
+ S_{inv}^{(1)}((\overline{\Psi}, \overline{\partial_\theta \Psi}, \overline{A}^a_A))(\theta) + S_{inv}^{(1)}((A^a_A, \partial_\theta A^a_A))(\theta) \\
= \int d^4 x \left[ (D_\theta^\theta \overline{\Psi}) (D_\theta \Psi) - \left( \overline{\Psi}(i \Gamma A \partial_\theta - m) \Psi \right) + \left( \frac{1}{4} \partial_\theta A^a_A \partial_\theta A^{ab} + L_{\theta} \right) \right] (x, \theta) ,
$$

(7.82)

$$
L_{\theta} = \frac{\bar{\theta}(\varepsilon)^2}{32\pi^2} \varepsilon_{ABCD} F^{AB0} F^{CD0} = \frac{\bar{\theta}(\varepsilon)^2}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu0} F^{\rho\sigma0} + 4 \varepsilon_{\mu\nu}^{(1)} \partial_\delta A^{\mu0} \partial_\delta A^{\nu0} + 4(\partial_\delta C_{(0)}^0)^2 .
$$

(7.83)

Every summand in (7.82) is invariant w.r.t. GTGT (7.66). Omitting the expression for $\delta S_{LG}(\theta)$ under $T_{IP}$ transformations of the type (4.3), having the more complicated form than in formulae (7.8), (7.25), (7.46) for the models above and now nonvanishing on one’s mass-shell, we write down the Euler-Lagrange equations for $Z_{LG}[A^a_A, \overline{\Psi}, \overline{\Psi}] = \int d\theta S_{LG}(\theta)$ taking account of the notation (2.34) (by omitting $(x, \theta)$ on the right)

$$
\frac{\delta_i Z_{LG}}{\delta \overline{\Psi}(x, \theta)} = L_\theta(x, \theta)(T_{inv} + S_{inv}^{(1)})(\theta) = -\left[ i D^a_A \overline{\Psi} \Gamma^a - (\partial_\theta^a - m + i \epsilon^a \partial_\theta C_{(0)}^a) \overline{\Psi} \right] = 0 ,
$$

(7.84)
\[
\frac{\delta_{\mu}Z_{LG}}{\delta \Psi(x, \theta)}(T_{\text{inv}} + S_{\text{inv}}^{(1)}(\theta)) = \left[ (i \Gamma^{A} D_{A} + m + \xi^{a} \partial_{\theta} C_{(0)}^{a} \partial_{\theta}^{2} \Psi) \right] = 0 , \quad (7.85)
\]
\[
\frac{\delta_{\mu}Z_{LG}}{\delta \mu_0(x, \theta)}S_{LG}(\theta) = \left[ j^{\mu 0} + \partial_{\mu} F^{\nu \mu 0} + \frac{\hat{\epsilon}(\epsilon)^{2}}{4\pi^{2}} \epsilon^{(1) \mu \nu} \partial_{\theta}^{2} \mathcal{A}_{\nu}(0) \right] = 0 , \quad (7.86)
\]
\[
\frac{\delta_{\mu}Z_{LG}}{\delta A_{\mu}(x, \theta)}S_{LG}(\theta) = \left[ j^{\mu 1} + \partial_{\theta}^{2} C_{(0)}^{1} \right] = 0 , \quad (7.87)
\]
\[
\frac{\delta_{\mu}Z_{LG}}{\delta \mu_0(0)(x, \theta)}S_{LG}(\theta) = \left[ j^{\mu 0} - 2 \epsilon \epsilon^{2} \epsilon^{0} \partial_{\theta}^{2} C_{(0)}^{0} \right] = 0 , \quad (7.88)
\]
\[
\frac{\delta_{\mu}Z_{LG}}{\delta C_{(1)}(x, \theta)}S_{LG}(\theta) = \left[ j^{\mu 1} + \Box C_{(0)}^{1} \right] = 0 . \quad (7.89)
\]

In (7.86)-(7.89) we have introduced the \( \theta \)-superfield generalization of the standard electromagnetic current \( j^\mu(x) \equiv j^{\mu 0}(x) \), extended to \((5+5)\) even and odd superfield components w.r.t. \( \varepsilon \) parity and following from the 1st Noether’s theorem analog applied to the \( S_{LG}(\theta) \) invariance w.r.t. global \( U^{11} \) transformations
\[
\mathcal{J}^{A_\alpha}(x, \theta) = - \frac{\partial (S_{\text{inv}}^{(1)} + T_{\text{inv}}^{(1)}(\theta))}{\partial A_{\alpha}(x, \theta)} \left( \epsilon \epsilon^{\gamma} \gamma^A \Psi + i \gamma^A \epsilon^{\gamma} \partial_{\theta} (\bar{\Psi} \Psi) \right)(x, \theta) , \quad (7.90)
\]
which are conserved on the solutions for corresponding dynamical equations
\[
(\partial_{\mu} j^{\mu 0}(x, \theta))_{\mathcal{L}_{\mathcal{C}}^{\mu 0} S_{LG} = 0} , \quad j^{\mu 0}(x, \theta)_{\mathcal{L}_{\mathcal{C}}^{\mu 0} S_{LG} = 0} , \quad (\partial_{\mu} j^{\mu 1}(x, \theta))_{\mathcal{L}_{\mathcal{C}}^{\mu 1} S_{LG} = 0} = 0 . \quad (7.91)
\]
The solution for the 2nd algebraic equation w.r.t. \( x^\mu \) exists with use of (7.24c) in the form
\[
\psi_1(x) = i\psi_1(x) \mu \Rightarrow i\psi_1(x) \mu , \quad (7.92)
\]
permitting to express \((\psi_1, \bar{\psi}_1)(x)\) in all the equations in LS in terms of only \((\psi, \bar{\psi})(x)\).

Given LS contains after representation (4.8) application the 18 \((8+5)\) odd and \((4+1)\) even w.r.t. \( \varepsilon \) of the 2nd order w.r.t. \( \varepsilon \) of all the superfields, nonlinear equations including the 2nd \((1)\) order w.r.t. \( x^\mu(\theta) \) DCLF. So, subject to assumption \( T_{\text{inv}}(\theta) = 0 \) the spinor subsystem (7.84), (7.85) will pass to the 1st order w.r.t. \( (x^\mu, \theta) \)-superfield generalization of Dirac equations in presence of dynamical composite superfields \( \mathcal{A}_{\alpha}(x, \theta) \).

The both supermatrices (4.17) and (4.15b) are degenerate (the former in sector of \( C_{(0)}^{1}(x, \theta) \)) complicating the analysis of DCLF in (7.84)-(7.89), in particular, restricting the Cauchy problem setting, therefore providing the classification for the theory as the irreducible GThGT with degenerate supermatrix (4.17). Really there are 2 Noether’s identities in correspondence with general formula (4.24) among DCLF
\[
\partial_{\mu} \left( \mathcal{L}_{\mathcal{A}_{\alpha}}^{\mu \beta}(y, \theta) S_{LG}(\theta) \right) - \partial_{\beta} \left( \mathcal{L}_{\mathcal{C}}^{\mu \beta}(y, \theta) S_{LG}(\theta) \right) (1)^{b} \]
\[
- i(1)^{b} e^{\beta} \left( \bar{\Psi}(y, \theta) \mathcal{L}_{\mathcal{C}}^{\beta}(y, \theta) S_{LG}(\theta) + (\mathcal{L}_{\Psi}(y, \theta) S_{LG}(\theta)) \Psi(y, \theta) \right) \equiv 0 , \quad b = 0,1 . \quad (7.93)
\]
The number of real physical degrees of freedom is equal to
\[
[\Psi] + \frac{1}{2} [A^{\mu 0}] + [C^{1}] = 4 + 2 + 1 = 7 ,
\]
that is greater on one degree than for standard quantum electrodynamics in view of the structure for the action \( S_{\text{inv}}(\theta) \) in (7.82) and Eq.(7.89).
In turn, under reduction of the given $\mathcal{M}_{cl}$ onto hypersurface $\mathcal{A}^{11}(x, \theta) = 0$, the theory become by the irreducible GThGT with $U(1)$ gauge group ($\partial A^1 = 0$) and nondegenerate supermatrix (4.17) for $T_{\text{inv}}(\theta) \neq 0$, therefore representing now the $\theta$-superfield generalization of quantum electrodynamics at least on the classical level. The GTGT, GGTGT, classical action, Euler-Lagrange equations, $\theta$-superfield current $j^{A0}(x, \theta)$ and its conservation, Noether’s identity can be easily obtained from the formulae (7.66)–(7.93) subject to conditions $\mathcal{A}^{11}(x, \theta) = 0$.

The $\theta$-superfield free GThGT with $U_1^{11}$ gauge group described by only $\mathcal{A}^{Aa}(x, \theta)$ and, in particular, the $\theta$-superfield free electrodynamics with $U(1)$ gauge group for $\mathcal{A}^{11}(x, \theta) = 0$ are yielded from the formulae (7.82)–(7.89) in the form of superfunctionals respectively

$$Z_{FLG}[A^0_a] = Z_{LG}[A^0_a, \Psi, \bar{\Psi}|_{\Psi = 0} = \int d\theta S_{\text{Lin}}((A^0_a, \partial_\theta A^0_a)(\theta)),$$

$$Z_{FED}[A^0_a] = Z_{FLG}[A^0_a]|_{A^0_a=0} = \int d\theta S_{\text{Lin}}((A^0_a, \partial_\theta A^0_a)(\theta)).$$

Subject to constraints $C_{(0)}^0(x, \theta) = 0$ the every from free GThGTs become by the irreducible GThSTs representing now the examples of the type (4.33) natural systems and revealing here the validity of the general Theorem and its corollary 2 application in these cases. Evidently, that with accuracy up to total derivative w.r.t. $\theta$, that is the term $(\partial_\theta A^A\partial_\theta A^1)$, the classical actions above coincide for $C_{(0)}^1(x, \theta) = 0$.

The presence of boson scalar superfield $C^1_{(0)}(x, \theta)$ increases the number of physical degrees of freedom on 1 for the 1st model (7.94a) in contrast to (7.94b) reflecting the nontrivial character of $U(0|1)$ group realization as gauge one in spite of $e^1$ charge nilpotency.

Next, setting $C_{(0)}^0(x, \theta) = 0$ together with absence of the topological summand $\varepsilon_{\mu\nu\rho\sigma} \times (F^{\mu\nu}F^{\rho\sigma})(x, \theta)$, we can derive from (7.94b) the GThST described for $D = 4$ by massless vector superfield with action (7.45) under identification $\varepsilon_{\mu\nu} = -\frac{\delta(e^0)^2}{4\pi^2}\varepsilon^{(1)}$.

As the consequence, consider in the model (7.82) the only potential terms having singled out them by means of the constraints $\partial_\theta(\Psi, \bar{\Psi}, A^0_a)(x, \theta) = 0$

$$S((\Psi, \bar{\Psi}, A^0_a, C_{(0)}^0)(\theta)) = -\int d^4x [\bar{\Psi}(i\Gamma^\mu \partial_\mu - m)\Psi + A^a_\mu j^{Aa}$$

$$-\left(\frac{1}{4}F^{\mu\nu}F^{\mu\nu} + \frac{\delta(e^0)^2}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{1}{2}\partial_\mu C_{(0)}^0\partial_\mu C_{(1)}^0\right)(x, \theta).$$

As it follows from Corollary 2 application and formula (4.51) the superfunction (7.95) is invariant w.r.t. (7.66) GTST now with nongauge ghost superfields $C_{(0)}^0(x, \theta)$, whose infinitesimal form written by means of GGTST have the form as independent $R_{\theta\theta}^2(\mathcal{A}(x, \theta))$ with opposite sign in (7.70) and $\hat{R}_{\theta\theta}^1(\mathcal{A}(x, \theta)) = 0$, providing the fulfilment of the relation (4.31) in question

$$\hat{R}_{\theta\theta}^1(\mathcal{A}(x, \theta), x, \theta; y, \theta') = -\delta(\theta - \theta')R_{\theta\theta}^2(\mathcal{A}(x, \theta))\delta(x - y).$$

One exist the another invariant possibility to extract the potential term in action (7.82) based on the superfield BRST symmetry realization for Yang–Mills type theories [23,24]. To this end one serve the independent constraints onto matter superfields and strength components (pointed out in [23,24] for $a = 0$

$$\left(F^{a\theta}_\theta, F^{a\theta}_\mu, \partial_\theta\bar{\Psi}\right)(x, \theta) = 0 \iff \partial_\theta\left(C_{(0)}^0, A^a_\mu, \bar{\Psi}\right)(x, \theta) = 0, \partial_\mu C_{(0)}^0, tC_{(0)}^0 e^a)(x, \theta)$$

satisfying to the type (A.6) solvability conditions and permitting to write the BRST transformations for all superfields on the solutions $\mathcal{A}^i(\theta)$ for Eqs.(7.97) in the form of translations along $\theta$ on odd constant $\mu$ as in (2.18)

$$\delta_\mu A^i(\theta) = \mu \partial_\theta \mathcal{A}^i(\theta).$$
These superfield transformations become by the component BRST ones for $\theta = 0$.

In limiting onto ”BRST surface” (7.97), the action (7.82), with allowance made for strength’s transformations (7.80), (7.81) and remark after (7.81), passes into superfunction (7.95) depending upon $A^a(\theta)$ with vanishing $j^a$, with regards of the fact that $(\varepsilon_{\mu}^{(1)})^{a}_{\mu} \partial^\mu C^a_{(0)}(\theta) \partial^\nu C^a_{(0)}(\theta))$ is equal to the total derivative w.r.t. $x^\mu$ (therefore omitted) and without $\frac{1}{2}$ factor before $(\partial{}_{\mu} C^a_{(0)}(\theta) \partial{}^\mu C^a_{(0)}(\theta))$. Moreover the purely electromagnetic terms with $A^a_{\mu}$ do not depend upon $\theta$ by virtue of (7.97) $(\ddot{E}_{\mu}^0(x, \theta) = \partial{}_{[\mu} \dot{A}^0_{\nu]}(x, \theta) = E_{\mu}^0(x))$.

At last, for $\theta = 0$ we obtain from (7.95) the spinor electrodynamics extended by the topological $\theta$-term with not interacting scalar nongauge field $C^a_{(0)}(x)$ and additional current’s summand $A^1_{\mu}e^1_\nu \Gamma^{}_{\mu\nu}$. Note that fermion nature for the vector field $A^1_{\mu}(x)$ is not so unusual from physical viewpoint if to recollect that to describe the (super)particle models it is widely used the twistor variables being by boson spinors.

In turn, for massive nongauge models from Secs.VII.1,2,3 the restriction $\partial_\theta(\varphi, \overline{\varphi}, \Psi, \overline{\Psi}, \mathcal{A}^\mu, \overline{\mathcal{A}}^\mu) = 0$ to get the standard action functionals is associated with superfield form of the trivial BRST transformations considered as in (7.97) for the matter superfields.

The embedding of GTST given by classical action (7.95) $S(\theta)$ into GTGT with classical $Z_{LT}[\mathcal{A}^a, \Psi, \overline{\Psi}]$ is realized by means of the type (5.4), (5.3) real quantities construction

$$Z_{(1)}[\Gamma_{min}] = \int d\theta \left(S_{LT}(\theta) - \int d^4 x [\mathcal{A}^a_{\mu} \partial{}^\mu C^a + i e^a (\Psi C^a - \overline{\Psi} \overline{C}^a)] \right)$$

$$S_{(1)}(\Gamma_{min}(\theta)) = S(\theta) + \int d^4 x [\mathcal{A}^a_{\mu} \partial{}^\mu C^a + i e^a (\Psi C^a - \overline{\Psi} \overline{C}^a)](x, \theta),$$

defined on $T^{odd}_{(1)}{\mathcal{M}}_{min} = \{(\mathcal{A}^a_{\mu}, \overline{\mathcal{A}}^a_{(0)}, \Psi, \overline{\Psi}, C^a)\}$ and $T^{odd}_{(1)}{\mathcal{M}}_{min} = \{(\Gamma_{min}, \partial{}_{\mu} \Gamma_{min}(\theta))\}$ respectively. The configuration space $\mathcal{M}_{min}$ in question is enlarged by ghost superfields $C^a(x, \theta)$ with the same Grassmann gradings as for the classical ghosts $C^a_{(0)}(x, \theta)$.

By construction, the superfunction(a)l above appear by the exact solutions of corresponding master equations (5.4), (5.3) with easily defined for this case superfield brackets (5.6), (5.5).

The problems of ZLR brackets and new models construction appear by the more complicated than for the GThST and ThSTs from Sec.VII.1,2,3 and allows, in particular, the existence of the nontrivial BRST change $Z^{(-1)}(\theta^{-1})$ defined on the corresponding zero locus $Z_{Q}(\varphi^0(\theta) = (S_{(1)}(\theta), \gamma_\theta)$, in turn, being the local supermanifold with nontrivial Bose–Fermi distribution for one’s coordinates (for $C^{a*}$ and, for instance, for two from $\mathcal{A}^a_{(0)}$). It is interesting to more carefully investigate this problem for $U^{11}$ model independently.

**VIII Conclusion**

The programm of Lagrangian formulation of $\theta$-STF realized on the whole in the paper constitutes the 1st step in order to construct the general superfield quantization method for gauge theories in the usual Lagrangian formalism. By the next large effort to resolve the last problem one will appear the construction a so-called Hamiltonian formulation of $\theta$-STF (whose elements, in part, have been used in Secs.V–VII for ZLR problems and to describe the GAs structure) based on the powerful use on the classical level the superantifields $\mathcal{A}^a_{(0)}(x)$ permitting to reformulate, not always equivalently, the $\theta$-superfield models given in the Lagrangian $\theta$-STF.

The noncontradictory possible description of an arbitrary superfield model, being by natural extension of one from the usual field theory, is guaranteed by a number of mathematical tools,
leading to the classical action superfunction $S_L(\theta)$ construction defined on the odd tangent bundle $T_{\text{odd}}\mathcal{M}_{\text{cl}} \times \{\theta\}$ over configuration space $\mathcal{M}_{\text{cl}}$ of classical superfields $\mathcal{A}(\theta)$.

Concepts for the Lagrangian description of the GThGT, GThST and nondegenerate theories also together with the statements and their corollaries on structure of dynamical equations (the so-called constraints) allow to extend the notions from standard gauge fields theory onto $\theta$-STF. Simultaneously the concept of gauge invariance is prolonged onto superfield case, based on the embedding of the local on $\theta$ special type objects and relations (being equivalent for $\theta = 0$ to the ones from usual field theory) into their general type analogs.

As it have been waited the $\theta$-STF provides the iterative algorithms to construct the new $\theta$-superfield models by means of the introduction the earlier hidden supertimes $\Gamma^l = (t^l, \theta^l)$, $l = \ldots, -1, 0, 1, \ldots$ in the framework of so-called direct and inverse ZLR problems with revealing the nontrivial role for such variables as $\lambda^l, J_l$. Additionally, it is demonstrated the possibility of the another superfield form for BFV method construction.

The component formulation for Lagrangian $\theta$-STF and ZLR problems, in fact, completely establishes the connection of treatment of the new $\theta$-superfield models with their description in terms of component objects.

The statements and consequences of $\theta$-STF have found one’s confirmation of their validity and interconnection with each other on the example of the $\theta$-superfield models construction in Sec.VII. A number from them have sufficiently simple illustrative character, for instance, for the models of scalar and vector superfields realizing the nondegenerate and gauge ThSTs. At the same time, the one from the spinor models with action $S_L^{(1/2)}(\theta)$ appears now by nondegenerate general type theory.

The gauge principle have allowed to construct from these models the abelian GThGT with $U^{(1)}$ gauge group leading, in first, to ghost superfield $C^0(x, \theta)$ inclusion into model on the classical level, in second, the scalar boson superpartner $C^1(x, \theta)$ for $C^0(x, \theta)$, connected with scalar parameter $\xi^1$, have provided the appearance of the additional physical degree of freedom within interacting theory. This model contains both the $\theta$-superfield generalization of quantum electrodynamics and the $\theta$-superfield QED itself in complete accordance with Sec.IV general statements. The generalization of the last model onto case with nonabelian gauge group together with possible interpretations appear by very perspective.

As to the ways to construct the $\theta$-superfield arbitrary model starting from usual relativistic field theory in the form of natural system then the following algorithm may be applied. It is sufficient to extend the fields $A^i$ up to superfields $\mathcal{A}(\theta)$, having built the superfunction $S(\theta)$. Next, we must add the "kinetic" $T_{[j}T_{l]}$-invariant term in superfield form $T(\partial_b \mathcal{A}(\theta))$. Resultant action $S_L(\mathcal{A}(\theta), \partial_b \mathcal{A}(\theta))$ appears by their difference and defines GThST or nondegenerate ThST, which leads to the HCLF, in turn, being by the nontrivial (as it have been shown for interacting models(!)) $\theta$-superfield extension of the initial $P_0(\theta)$-component dynamical equations. Original field theory model is obtained from superfield one, in first, under restriction from $C^k(T_{\text{odd}}\mathcal{M}_{\text{cl}} \times \{\theta\})$ down to $C^k(\mathcal{M}_{\text{cl}} \times \{\theta\})$, and then the ordinary functionals are singled out from the superfunctions, in the invariant way, by means of involution $*$ (3.6), (3.8) or, equivalently, by setting $\theta = 0$. The another way to do this is the imposing of the type (7.97) constraints realizing simultaneously the $\theta$-superfield BRST transformations. At last, the models from the class of GThGT may be obtained, for example, by means of gauge principle application [40] to the corresponding ThST or GThST.

These models appear by appropriate proving ground for demonstration of the general constructions validity and efficiency both of Hamiltonian $\theta$-STF formulation and next the quantization procedure itself.

As it was mentioned in introduction we have intensively used the following analogy be-
between quantities and relations of the Lagrangian $\theta$-STF and classical mechanics in the usual Lagrangian formulation with $\varepsilon_p$-even objects organized in the table form

| usual classical mechanics | $\theta$ – STF |
|--------------------------|-----------------|
| 1. $t \in \mathbb{R}$ – time $(\varepsilon_j, \varepsilon) t = (0, 0)$ | $\theta \in \Lambda_1(\theta)$ – odd time $(\varepsilon_p, \varepsilon_j, \varepsilon) \theta = (1, 0, 1)$ |
| 2. $q^a(t)$ – generalized coordinates $(\varepsilon_j, \varepsilon) q^a(t) = (\varepsilon_a, \varepsilon_a)$ | $\mathcal{A}^i(\theta)$ – superfields ($i$ contains $t$) |
| 3. $\dot{q}^a(t)$ – generalized velocities $\varepsilon(\dot{q}^a(t)) = \varepsilon_a$ | $\mathcal{A}^i(\theta)$ – odd generalized velocities |
| 4. $\mathcal{M} = \{q^a\}, T\mathcal{M} = \{(q^a, \dot{q}^a)\}$ – configuration space and tangent bundle | $\mathcal{M}_{cl} = \{\mathcal{A}^i(\theta)\}, T_{odd}\mathcal{M}_{cl} = \{(\mathcal{A}^i(\theta), \mathcal{A}^i(\theta))\}$ – configuration space and odd tangent bundle |
| 5. $L(q, \dot{q}, t) \equiv L(t) : T\mathcal{M} \times \{t\} \to \mathbb{R}$ – Lagrange function, $(\varepsilon_j, \varepsilon) L(t) = (0, 0)$ | $S_L(\mathcal{A}(\theta), \mathcal{A}(\theta), \theta) \equiv S_L(\theta) : T_{odd}\mathcal{M}_{cl} \times \{\theta\} \to \Lambda_1(\theta, \mathbb{R})$ – superfunction of Lagrangian classical action, $(\varepsilon_p, \varepsilon_j, \varepsilon) S_L(\theta) = (0, 0, 1)$ |
| 6. $S[q] = \int dt L(q, \dot{q}, t)$ – classical action functional, $\varepsilon(S) = 0$ | $Z[\mathcal{A}] = \int d\theta S_L(\theta), (\varepsilon_p, \varepsilon_j, \varepsilon) Z = (1, 0, 1)$ |
| 7. $\frac{\delta S}{\delta q^a(t)} = \left( \frac{\partial}{\partial q^a(t)} - \frac{d}{dt} \frac{\partial}{\partial \dot{q}^a(t)} \right) L(q, \dot{q}, t) = 0$ – usual Euler-Lagrange equations | $\frac{\delta L}{\delta \mathcal{A}^i(\theta)} = \mathcal{L}^i_{\theta} S_L(\theta) = 0$ – Euler-Lagrange equations for $Z[\mathcal{A}]$ |

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**A ODE with odd operator** $\frac{d}{d\theta}$

**Statement 1** (Existence and Uniqueness theorem for solutions of the 1st order on $\theta$ systems of $N$ ODE)

1. The general solutions of $N$ ODE systems in normal form (NF) of the 1st order w.r.t. unknowns $g^j(\theta), j = 1, \ldots, N = (N_+, N_-)$ in the domain $U$ of the supermanifold $\mathcal{N}$ locally coordinatized by $g^j(\theta)$, for $\theta \in \Gamma_{(0, 1)} \subset 1\Lambda_1(\theta)$

   a) $\partial_\theta g^j(\theta) = P_0(\theta) f^j(g(\theta), \theta), \ f^j(g(\theta), \theta), \ g^j(\theta) \in C^2(\mathcal{N} \times \{\theta\})$, \hspace{1cm} (A.1)

   b) $\partial_\theta g^j(\theta) = h^j(g(\theta), \theta), \ h^j(g(\theta), \theta) \in C^2(\mathcal{N} \times \{\theta\})$ \hspace{1cm} (A.2)

   exist and have the form respectively

   a) $g^j(\theta) = P_0(\theta) k^j(\theta) + \theta f^j(g(\theta), \theta), \ \varepsilon(g^j(\theta)) = \varepsilon(k^j(\theta)) = \varepsilon(f^j(\theta)) + 1$, \hspace{1cm} (A.3)

   b) $g^j(\theta) = P_0(\theta) l^j(\theta) + \theta h^j(g(\theta), \theta), \ \varepsilon(g^j(\theta)) = \varepsilon(l^j(\theta)) = \varepsilon(h^j(\theta)) + 1$, \hspace{1cm} (A.4)

   with arbitrary superfunctions $k^j(\theta), l^j(\theta) \in C^2(\mathcal{N})$ restricted on $U$.

2. The integral curve $\dot{g}^j(\theta)$ for system (A.1), (A.2) satisfying to Cauchy problem for $\theta = 0^+$ in the domain $U \subset \mathcal{N}$

   \begin{equation}
   P_0(\theta) \dot{g}^j(\theta) = \dot{g}^j(\theta)|_{\theta = 0} = \overline{g}^j, \ 0 \in \Gamma_{(0, 1)}
   \end{equation} \hspace{1cm} (A.5)

\(^7\)having represented solution for (A.2) in the form $g^j(\theta - \theta_0) = (\theta - \theta_0) h^j(g(\theta - \theta_0), \theta - \theta_0) + P_0(\theta - \theta_0) l^j(\theta - \theta_0)$ one can set up the Cauchy problem for arbitrary $\theta = \theta_0 \in \Gamma_{(0, 1)}$
is unique.

To be solvable in explicit superfield form it is necessary, in first, to impose a so-called necessary solvability conditions for system (A.2) in view of the identity (4.6) validity for $g^i(\theta)$. The condition above has the form of the 1st order on $\theta$ 2N ODE system

$$\partial_\theta g^i(\theta) = h^i(g(\theta), \theta), \quad \partial_\theta h^i(g(\theta), \theta) = 0,$$  

(A.6)

so that the properly solvability conditions written in the 2nd subsystem means that superfunctions $h^i(g(\theta), \theta)$ on the possible solutions for Eqs.(A.2) appear by the integrals for this system.

**Statement 2** (Existence and Uniqueness theorem for solution of the 2nd order on $\theta$ N ODE system)

1. The general solution for the 2nd order on $\theta$ N ODE system in NF

$$\partial^2_\theta g^i(\theta) = 0, \quad j = 1, \ldots, N,$$  

(A.7)

in the domain $U$ of supermanifold $N$ with unknowns $g^i(\theta) \in C^2(N \times \{\theta\})$ exists in the form

$$g^i(\theta) = k^i(\theta), \quad \varepsilon(g^i(\theta)) = \varepsilon(k^i(\theta)),$$  

(A.8)

with arbitrary superfunctions $k^i(\theta) \in C^2(N \times \{\theta\})$.

2. Particular solution $\hat{g}^i(\theta)$ for system (A.7) satisfying to the initial conditions for $\theta = 0$

$$(\hat{g}^i(\theta), \partial_\theta \hat{g}^i(\theta))|_{\theta=0} = (\hat{g}^i, \partial_\theta \hat{g}^i)$$

(A.9)

is unique.

By more complicated of the 2nd order on $\theta$ N ODE system in NF one appear the equations

$$\partial^2_\theta g^i(\theta) = f^i_1(g(\theta), \partial_\theta g(\theta), \theta), \quad f^i_1(\theta) \in C^1(T_{odd}N \times \{\theta\}),$$  

(A.10)

being equivalent to the following 2N ODE system

$$\partial^2_\theta g^i(\theta) = 0, \quad (A.11)$$

$$f^i_1(g(\theta), \partial_\theta g(\theta), \theta) = 0.$$  

(A.12)

Eqs.(A.12) appear by the 1st order on $\theta$ differential constraints both for the initial conditions values (A.9) and onto possible values of $g^i(\theta), \partial_\theta g^i(\theta)$ obtained by resolution of Eqs.(A.11) for all $\theta \in \Gamma_{(0,1)}$.

It should be noted the $N$ ODE system with given superfunctions

$$m^j(g(\theta), \partial_\theta g(\theta), \theta) = 0, \quad m^j(g(\theta), \partial_\theta g(\theta), \theta) \in C^1(T_{odd}N \times \{\theta\})$$  

(A.13)

may be written in the equivalent form of 2N ODE system by means of projectors $P_a(\theta)$ action

$$P_0(\theta) m^j(g(\theta), \partial_\theta g(\theta), \theta) = m^j(P_0 g(\theta), \partial_\theta g(\theta), 0) = 0, \quad \partial_\theta m^j(g(\theta), \partial_\theta g(\theta), \theta) = 0.$$  

(A.14)

Therefore a solution for the 2nd subsystem (A.14) must belong to a set of solutions for the 1st one, if the solvability conditions is true for Eqs.(A.13).

It is convenient to introduce the following single-valued functions of degree and least degree w.r.t. any from elements $C(\theta) \in \{g(\theta), \partial_\theta g(\theta), g\partial_\theta g(\theta), \ldots\}$ respectively

$$\text{(deg}_{C(\theta)}, \text{min}\text{deg}_{C(\theta)}) : C^k(T_{odd}N \times \{\theta\}) \to N_0,$$  

(A.15)
acting on arbitrary $\mathcal{F}(\theta)$, with representation of the form (2.27) by the rule
\[
(\deg_b(\theta), \min \deg_b(\theta), \deg_g(\theta)\partial_b g(\theta), \min \deg_g(\theta)\partial_b g(\theta)) \mathcal{F}(\theta) = (A.16)
\]
\[
(\max p, \min p, \max (l + k), \min (l + k)), \ b(\theta) \in \{g(\theta), \partial_b g(\theta)\},
\]
where the symbols $\max \{p, (l + k), \min \{p, (l + k)\}$ appear by the most and the least values of
degree order for $\tilde{g}^{(j)p}(\theta), (\tilde{g}^{(j)}\partial_b g^{(j)k})(\theta)$, $i, j = 1, \ldots, N$ respectively. By means of the functions
above the so-called holonomic $f^i_1(g(\theta), \theta)$ and linearized $f^i_{1\text{lin}}(\theta)$ constraints are singled out from
$f^i_1(\theta)$ (A.12) by the relations respectively
\[
\deg_{\partial_b g(\theta)} f^i_1(\theta) = 0, \ \deg_{g(\theta)\partial_b g(\theta)} f^i_1(\theta) = 1. \quad (A.17)
\]
Not all from differential constraints (A.12) appear by (functionally) independent. To investiga-
te this problem let us assume the following postulates fulfilment
\[
1) \ (\tilde{g}^j(\theta), \partial_b \tilde{g}^j(\theta)) = (0, 0) \in T_{\text{odd}}\Phi, \ \Phi \subset \mathcal{N}, \quad (A.18)
\]
with $\Phi$ being the set of solutions for Eqs. (A.12);
2) $f^i_1(\theta) = 0$ define the 1st order supersurface, which the conditions hold on
\[
f^i_1(g(\theta), \partial_b g(\theta), \theta)|_{T_{\text{odd}}\Phi} = 0, \ \text{rank} \left| \frac{\delta f^i_1(g(\theta), \partial_b g(\theta), \theta)}{\delta g^j(\theta_1)} \right|_{T_{\text{odd}}\Phi} \leq N. \quad (A.19)
\]
By definition under rank of supermatrix in (A.19) simultaneously with allowance made for the
connection of superfield variational derivative of $f^i_1(\theta)$ w.r.t. $g^j(\theta_1)$ with partial superfield ones
w.r.t. $g^j(\theta_1), \partial_{\theta_1} g^j(\theta_1)$ in correspondence with (2.34) we mean
\[
\text{rank} \left| \left( \frac{\partial}{\partial g^j(\theta_1)} - (-1)^{\varepsilon(\theta)} \partial_{\theta_1} \frac{\partial}{\partial (\partial_{\theta_1} g^j(\theta_1))} \right) f^i_1(\theta_1) \right|_{T_{\text{odd}}\Phi} \delta(\theta_1 - \theta)(-1)^{\varepsilon(f^i_1)}. \quad (A.20)
\]
For holonomic constraints $f^i_1(\theta)$ the rank (A.20) pass into almost standard form [39]
\[
\text{rank} \left| \frac{\partial f^i_1(g(\theta_1), \theta_1)}{\partial g^j(\theta_1)} \right|_{\Phi} \delta(\theta_1 - \theta)(-1)^{\varepsilon(f^i_1)}. \quad (A.21)
\]
Hypothesis 2 permit to present $f^i_1(\theta)$ in the form
\[
f^i_1(\theta) = f^i_{1\text{lin}}(\theta) + f^i_{1\text{lin}}(\theta), \ \text{min} \deg_{g(\theta)\partial_b g(\theta)} f^i_{1\text{lin}}(\theta) \geq 2, \quad (A.22)
\]
so that $f^i_1(\theta)$ appear by perturbation of functionally dependent linearized constraints by means
of nonlinear components.

An effective analysis of the constraints (A.22), considered as the 1st order on $\theta$ N ODE
system not being reduced to NF of the form (A.2), is based on the fundamental

**Theorem:**
The 1st order on $\theta$ N ODE system w.r.t. unknowns $g^j(\theta)$ (A.12) subject to conditions (A.18),
(A.19) is reduced to equivalent system of independent equations in so-called generalized normal
form (GNF) under following parametrization for $g^j(\theta) = (\alpha^j(\theta), \beta^j(\theta), \gamma^j(\theta), \sigma^j) = (\bar{j}, \underline{j}, \sigma)
\[
\partial_{\theta} \alpha^j(\theta) = \varphi^j(\alpha(\theta), \gamma(\theta), \partial_{\theta} \gamma(\theta), \theta), \ \beta^j(\theta) = \kappa^j(\alpha(\theta), \gamma(\theta), \theta), \quad (A.23)
\]
with arbitrary superfunctions $\gamma^\alpha(\theta)$. The number of $\gamma^\alpha$: $[\gamma]$ coincides with one of differential identities among Eqs.(A.12)

$$\int d\theta f^j_1(g(\theta), \partial_\theta g(\theta), \theta) \mathcal{R}_{j\sigma}(g(\theta), \partial_\theta g(\theta), \theta; \theta') = 0,$$

(A.24)

where quantities $\mathcal{R}_{j\sigma}(g(\theta), \partial_\theta g(\theta), \theta; \theta')$ are a) local on $\theta$ and b) functionally independent operators

a) $\mathcal{R}_{j\sigma}(g(\theta), \partial_\theta g(\theta), \theta; \theta') \equiv \mathcal{R}_{j\sigma}(\theta; \theta') = \sum_{k=0}^{1} ((\partial_\theta)^k \delta(\theta - \theta')) \mathcal{R}^k_{j\sigma}(g(\theta), \partial_\theta g(\theta), \theta),$

(A.25)

b) functional equation

$$\int d\theta' \mathcal{R}_{j\sigma}(\theta; \theta') u^\sigma(g(\theta'), \partial_\theta' g(\theta'), \theta') = 0$$

(A.26)

has unique solution $u^\sigma(\theta') = 0$.

In order to be solvable in superfield form the system (A.23) must satisfy to the solvability conditions written additionally to these equations

$$\partial_\theta \phi^j(\alpha(\theta), \gamma(\theta), \partial_\theta \gamma(\theta), \theta) = 0.$$

(A.27)

The detailed proof of the Theorem and its related consequences will be considered in another paper.

Corollary:

For dependent holonomic constraints $f^j_1(\theta)$ from the condition (A.19) it follows the existence of equivalent system of constraints under following parametrization for $g^j(\theta)$

$$g^j(\theta) = (\alpha^A(\theta), \gamma^\sigma(\theta))/ j = (A, \sigma), \sigma = 1, \ldots, [\gamma], A = 1, \ldots, [\alpha],$$

(A.28)

by means of relationships

$$\Phi^A(\alpha(\theta), \gamma(\theta), \theta) = 0.$$

(A.29)

The number $[\gamma]$ coincides with one of algebraic (in the sense of differentiation w.r.t. $\theta$) identities among $f^j_1(\theta)$

$$f^j_1(g(\theta), \theta) \mathcal{R}^0_{j\sigma}(g(\theta), \theta) = 0,$$

(A.30)

being obtained by integration on $\theta$ of Eqs.(A.24) with allowance made for the type (A.25) connection of $\mathcal{R}_{j\sigma}(\theta; \theta')$ with $\mathcal{R}^0_{j\sigma}(\theta)$

$$\mathcal{R}_{j\sigma}(g(\theta), \theta; \theta') = \delta(\theta - \theta') \mathcal{R}^0_{j\sigma}(g(\theta), \theta)(-1)^{\varepsilon(f^j_1(\theta))}.$$

(A.31)

A dependence on $\partial_\theta g^j(\theta)$ in (A.30), (A.31) may be only parametric one.

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