ON MULTIPlicity OF POSITIVE SOLUTIONS
FOR NONLOCAL EQUATIONS WITH CRITICAL NONLINEARITY

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Abstract. This paper deals with existence and multiplicity of positive solutions to the following class of nonlocal equations with critical nonlinearity:

\[
\begin{cases}
(-\Delta)^s u = a(x)|u|^{2^*_s - 2}u + f(x) \quad \text{in } \mathbb{R}^N,
\end{cases}
\]

where \( s \in (0, 1) \), \( N > 2s \), \( 2^*_s := \frac{2N}{N-2s} > 0 \) \( a \in L^\infty(\mathbb{R}^N) \) and \( f \) is a nonnegative nontrivial functional in the dual space of \( H^s \) i.e., \( (H^s)'(f, u)_{H^s} \geq 0 \), whenever \( u \) is a nonnegative function in \( H^s \). We prove existence of a positive solution whose energy is negative. Further, under the additional assumption that \( a \) is a continuous function, \( a(x) \geq 1 \) in \( \mathbb{R}^N \), \( a(x) \to 1 \) as \( |x| \to \infty \) and \( \|f\|_{H^s(\mathbb{R}^N)'} \) is small enough (but \( f \neq 0 \)), we establish existence of at least two positive solutions to \((\mathcal{E})\).

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1. Introduction

In this article we study existence and multiplicity of positive solutions to the following fractional elliptic equation in \( \mathbb{R}^N \)

\[
\begin{cases}
(-\Delta)^s u = a(x)|u|^{2^*_s - 2}u + f(x) \quad \text{in } \mathbb{R}^N,
\end{cases}
\]

where \( s \in (0, 1) \) is fixed parameter, \( N > 2s \), \( 2^*_s := \frac{2N}{N-2s} > 0 \) \( a \in L^\infty(\mathbb{R}^N) \), \( a(x) \to 1 \) as \( |x| \to \infty \) and \( f \neq 0 \) is a nonnegative functional in the dual space of \( H^s(\mathbb{R}^N) \). Here \((-\Delta)^s \) denotes the fractional Laplace operator which can be defined for the Schwartz class functions \( \mathcal{S}(\mathbb{R}^N) \) as follows

\[
(-\Delta)^s u(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \quad c_{N,s} = \frac{4^s \Gamma(N/2 + s)}{\pi^{N/2} \Gamma(-s)}.
\]

Let \( \tilde{H}^s(\mathbb{R}^N) := \{ u \in L^{2^*_s}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy < \infty \} \),

be the homogeneous fractional Sobolev space, endowed with the inner product \( \langle \cdot, \cdot \rangle_{H^s} \) and corresponding Gagliardo norm

\[
\|u\|_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}.
\]

Clearly, \( u \in \tilde{H}^s(\mathbb{R}^N) \) implies \( u \in L^p_{loc}(\mathbb{R}^N) \) for any \( p \in [1, 2^*_s] \).
Definition 1.1. The function $u \in \dot{H}^s(\mathbb{R}^N)$ is said to be a positive weak solution of (2) if $u > 0$ in $\mathbb{R}^N$ and for every $\phi \in \dot{H}^s(\mathbb{R}^N)$ we have,
\[
\int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^{N+2s}} \, dx \, dy = \int_{\mathbb{R}^N} a(x) u^{2^*_s-1} \phi \, dx + \langle \dot{H}^s(\mathbb{R}^N)'(f, \phi) \rangle_{\dot{H}^s},
\]
where $\langle \cdot, \cdot \rangle_{\dot{H}^s}$ denotes the duality bracket between the dual space $\dot{H}^s(\mathbb{R}^N)'$ of $\dot{H}^s(\mathbb{R}^N)$ and $\dot{H}^s(\mathbb{R}^N)$ itself.

Under the stated assumptions equation (2) can be considered as a perturbation problem of the homogeneous equation:
\[
\left\{ \begin{array}{l}
(-\Delta)^s w = w^{2^*_s-1} \text{ in } \mathbb{R}^N, \\
w > 0 \text{ in } \mathbb{R}^N, \\
w \in \dot{H}^s(\mathbb{R}^N).
\end{array} \right. (1.2)
\]

In the celebrated paper [3] Chen, Li and Ou proved that (1.2) has a unique positive solution $W$ (up to translations and dilations). Indeed, any positive solution of (1.2) is radially symmetric, with respect to some point $x_0 \in \mathbb{R}^N$, strictly decreasing in $r = |x-x_0|$, of class $C^\infty(\mathbb{R}^N)$ and so of the explicit parametric form
\[
W(x) = c_{N,s} \left( \frac{\lambda}{\lambda^2 + |x-x_0|^2} \right)^{\frac{N-2s}{2}},
\]
for some $\lambda > 0$.

The main question in this paper is whether positive solutions can still survive for the perturbed equation (2).

When the domain is a bounded subset of $\mathbb{R}^N$, in a pioneering work, Tarantello [10] proved existence of two positive solutions for the following nonhomogeneous problem
\[
-\Delta u = |u|^{N-2s} u + f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]
where $0 \leq f \in H^{-1}(\Omega)$ satisfies suitable condition. In [6, 13] the authors studied existence of sign changing solutions of (1.4). In the nonlocal case, when the domain is a bounded subset of $\mathbb{R}^N$, existence of positive solution of (2) in $\Omega$ with Dirichlet boundary condition has been proved in [10]. Existence of sign changing solutions of
\[
(-\Delta)^s u = |u|^{N-2s} u + \varepsilon f \text{ in } \Omega, \quad u = 0 \in \mathbb{R}^N \setminus \Omega,
\]
where $f \geq 0, f \in L^\infty(\Omega)$ has been studied in [1] and existence of two positive solutions have been established in [20] when $f$ is a continuous function with compact support in $\Omega$.

To the best of our knowledge, so far there has been no papers in the literature, where existence and multiplicity of positive solutions of fractional Laplace equations, with the critical exponents in $\mathbb{R}^N$, have been established in the non homogeneous case $f(x) \neq 0$. The results in this paper are new even in the local case $s = 1$, but we leave the obvious changes, when $s = 1$, to the interested reader.

From now on we assume that $f$ satisfies the following condition
\begin{itemize}
\item [(F)] $f \neq 0$ is a nonnegative functional in the dual space $\dot{H}^s(\mathbb{R}^N)'$ of $\dot{H}^s(\mathbb{R}^N)$.
\end{itemize}

Let us state the main results.

Theorem 1.1. Assume $0 < a \in L^\infty(\mathbb{R}^N)$ and (F) is satisfied. There exists $d > 0$ such that if $\|f\|_{\dot{H}^s} \leq d$, then equation (2) admits a positive solution whose energy is negative.

Next, under an additional hypothesis on $a$, we prove existence of at least two positive solutions.

(A) $a \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a(x) \geq 1$ for all $x \in \mathbb{R}^N$, and $a(x) \to 1$ as $|x| \to \infty$. 

Assume that \( (A) \) and \( (F) \) are satisfied. If
\[
\| f \|_{H^s(\mathbb{R}^N)} < C_0N^\frac{N}{2s}, \quad \text{where} \quad C_0 := \left( \frac{4s}{N + 2s} \right)^{\frac{N-2s}{2s}},
\]
then \( \mathcal{E} \) admits a positive solution.

In addition, if either \( a \equiv 1 \) or \( \| a \|_{L^\infty(\mathbb{R}^N)} \geq \alpha(N,s) \), where \( \alpha(N,s) \) is the second zero of the function
\[
\varphi(t) := \frac{s}{N} t^{\frac{N+2s}{2s}} - \frac{t^2}{2} + \frac{1}{2s},
\]
then \( \mathcal{E} \) admits at least two positive solutions.

As in the local case, the Sobolev embedding \( \dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N) \) is continuous, but not compact. Thus the variational functional associated to \( \mathcal{E} \) fails to satisfy the Palais-Smale condition, briefly called \( (PS) \) condition. The lack of compactness becomes clear, when one looks at the special case \( (1.2) \). Solutions of \( (1.2) \) are invariant under translation and dilation therefore, there is not compactness. Thus the standard variational technique can not be applied directly. Noncompact variational problems have attracted much attention since the late seventies. Among them, the Yamabe [22] and the prescribed scalar curvature problems have played an important role. For those, but also for many related elliptic equations, the loss of compactness is caused by the invariant action of the conformal group, or of one of its subgroups, leading to possible spikes formation. To overcome this difficulty, the a priori knowledge of the energy range where the Palais-Smale condition holds is helpful, and sometimes suffices to construct critical points.

Now let us briefly explain the methodology to obtain our results. In Theorem \( 1.1 \) we establish existence of positive solution as a perturbation of 0 via Mountain Pass theorem. To prove Theorem \( 1.2 \) we first do the Palais-Smale decomposition of the functional associated with \( \mathcal{E} \). Then we decompose \( \dot{H}^s(\mathbb{R}^N) \) into three components which are homeomorphic to the interior, boundary and the exterior of the unit ball in \( \dot{H}^s(\mathbb{R}^N) \) respectively. Thus, using assumption \( (A) \), we prove that the energy functional associated to \( \mathcal{E} \) attains its infimum on one of the components which serves as our first positive solution. The second positive solution is obtained via a careful analysis on the \( (PS) \) sequences associated to the energy functional and we construct a min–max critical level \( \gamma \), where the \( (PS) \) condition holds. That leads to the existence of second positive solution.

This paper has been organised in the following way: In Section 2, we prove the Palais-Smale decomposition theorem associated with the functional corresponding to \( \mathcal{E} \). In Section 3, we show existence of two positive solutions of \( \mathcal{E} \) under the assumption \( (A) \), namely Theorem \( 1.2 \). In Section 4, we prove Theorem \( 1.1 \) Appendix A basic properties of the Morrey spaces.

Theorem 1.2. Assume that \( (A) \) and \( (F) \) are satisfied. If
\[
\| f \|_{H^s(\mathbb{R}^N)} < C_0N^\frac{N}{2s}, \quad \text{where} \quad C_0 := \left( \frac{4s}{N + 2s} \right)^{\frac{N-2s}{2s}},
\]
then \( \mathcal{E} \) admits a positive solution.

In addition, if either \( a \equiv 1 \) or \( \| a \|_{L^\infty(\mathbb{R}^N)} \geq \alpha(N,s) \), where \( \alpha(N,s) \) is the second zero of the function
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Notation: In this paper \( \dot{H}^s(\mathbb{R}^N)' \) (or in short \( (\dot{H}^s)' \)) denotes the dual space of \( \dot{H}^s(\mathbb{R}^N) \), \( C \) denotes the generic constant which may vary from line to line. Moreover, \( u_+ := \max\{u,0\} \) and \( u_- := -\min\{u,0\} \). Therefore, according to our notation \( u = u_+ - u_- \). Finally, \( W \) denotes the unique positive solution of \( (1.2) \) and \( S \) the best Sobolev constant.

2. Palais-Smale Characterization

In this section we study the Palais-Smale sequences (in short, \( (PS) \) sequences) of the functional associated to \( \mathcal{E} \).

\[
I_{a,f}(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \frac{1}{2s} \int_{\mathbb{R}^N} a(x)|u|^{2^*_s} \, dx - \langle \dot{H}^s', (f,u)_{\dot{H}^s} \rangle.
\]

\[
= \frac{1}{2} \| u \|_{\dot{H}^s(\mathbb{R}^N)}^2 - \frac{1}{2s} \int_{\mathbb{R}^N} a(x)|u|^{2^*_s} \, dx - \langle \dot{H}^s', (f,u)_{\dot{H}^s} \rangle. \tag{2.1}
\]
We say that the sequence \((u_k)_k \subset \dot{H}^s(\mathbb{R}^N)\) is a \((PS)\) sequence for \(I_{a,f}\) at level \(\beta\) if \(I_{a,f}(u_k) \to \beta\) and \(I'_{a,f}(u_k) \to 0\) in \((\dot{H}^s(\mathbb{R}^N))^*\). It is easy to see that the weak limit of a \((PS)\) sequence solves \((\mathcal{E})\) except the positivity.

However the main difficulty is that the \((PS)\) sequence may not converge strongly and hence the weak limit can be zero even if \(\beta > 0\). The main purpose of this section is to classify \((PS)\) sequences for the functional \(I_{a,f}\). Classification of \((PS)\) sequences has been done for various problems having lack of compactness, to quote a few, we cite [3, 7, 12, 14, 15, 17, 18]. We establish a classification theorem for the \((PS)\) sequences of \((\mathcal{E})\) in the spirit of the above results.

Throughout this section we assume \(0 < a \in L^\infty(\mathbb{R}^N), \ a(x) \to 1\ as \ |x| \to \infty\ and \ f\ is a nontrivial element of \(\dot{H}^s(\mathbb{R}^N)'\).

**Proposition 2.1.** Let \((u_k)_k \subset \dot{H}^s(\mathbb{R}^N)\) be a \((PS)\) sequence for \(I_{a,f}\). Then there exists a subsequence \((u_{k_j})_j\) and \(v \in \dot{H}^s(\mathbb{R}^N)\) such that

\[
(-\Delta)^s v = a(x)|\nabla v|^{2^*_s - 2}v + f \quad \text{in} \quad \mathbb{R}^N
\]

either \(x_{k_j}^1 \to x^1 \in \mathbb{R}^N\) or \(|x_{k_j}^1| \to \infty\), \(r_{k_j}^1 \to 0, \ 1 \leq j \leq m\).

\[
\log \left(\left|\frac{r_{k_j}^1}{r_{k_j}^2}\right| + \left|\frac{x_{k_j}^1 - x_{k_j}^2}{r_{k_j}^1}\right|\right) \to \infty \quad \text{for} \ i \neq j, \ 1 \leq i, j \leq m,
\]

\[
(-\Delta)^s w_j = a(x)|\nabla w_j|^{2^*_s - 2}w_j \quad \text{in} \quad \mathbb{R}^N,
\]

\(w_j \neq 0, \ w_j \in \dot{H}^s(\mathbb{R}^N),\)

\[
\bar{u}_k - \left(\bar{v} + \sum_{j=1}^m a(x^j) \left(\frac{x^j - y}{r_{k_j}^j}\right)\right) \to 0 \ as \ k \to \infty,
\]

where \((w_j)_{1,y} := r^{-\frac{N-2s}{2}}w_j\left(\frac{x-y}{r}\right),\)

\[
I_{a,f}(u_k) \to I_{a,f}(\bar{v}) + \sum_{j=1}^m a(x^j)^{-\frac{N-2s}{2}}I_{1,0}(w_j) \ as \ k \to \infty,
\]

where in the case \(m = 0\) the above expressions hold without \(w_j\), \(x_{k_j}^1\) and \(r_{k_j}^1\). In addition, if \(u_k \geq 0\, then \bar{u} \geq 0\ and \ w_j \geq 0\ for all \ 1 \leq j \leq m\). Therefore, \(w_j = W\ for all \ 1 \leq j \leq m\ due to the uniqueness up to the translation and dilation for the positive solutions of \((\mathcal{E})\).

**Remark 2.1.** From Proposition \((\mathcal{E})\), we see that if \((u_k)_k\) is any nonnegative \((PS)\) sequence for \(I_{a,f}\) at level \(c\), then \((u_k)_k\) satisfies the \((PS)\) condition if \(c\) can not be decomposed as \(c = I_{a,f}(\bar{v}) + \sum_{j=1}^m a(x^j)^{-\frac{N-2s}{2}}I_{1,0}(W)\), where \(m \geq 1\) and \(W\ is the unique positive radial solution of \((\mathcal{E})\).

Before starting the proof of this proposition, we prove an auxiliary lemma

**Lemma 2.1.** Let \((\phi_k)_k\) weakly converge to \(\phi\ in \dot{H}^s(\mathbb{R}^N)\ and a.e. in \mathbb{R}^N, then

\[
a|\phi_k|^{2^*_s - 2}\phi_k - a|\phi|^{2^*_s - 2}\phi \to 0 \ in \ \dot{H}^s(\mathbb{R}^N)'.
\]

**Proof.** Defining \(\psi_k\ as \ \phi_k - \phi,\ we see \(\psi_k \to 0\ in \dot{H}^s(\mathbb{R}^N).\ In particular, \((\psi_k)_k\) is bounded in \(\dot{H}^s(\mathbb{R}^N).\ Thus, up to a subsequence, \(\psi_k \to 0\ in L^q_{\text{loc}}(\mathbb{R}^N)\ for all \ 1 < q < 2^*_s\ and \(\psi_k \to 0\ a.e.\ in \mathbb{R}^N.\ Consequently, \(a|\phi + \psi_k|^{2^*_s - 2}\phi + \psi_k - a|\phi|^{2^*_s - 2}\phi \to 0\ a.e..\ We also observe that for every \(\varepsilon > 0,\ there exists \(C_\varepsilon > 0\ such that

\[
\left|a|\phi + \psi_k|^{2^*_s - 2}(\phi + \psi_k) - a|\phi|^{2^*_s - 2}\phi\right| \leq \varepsilon|\psi_k|^{2^*_s} + C_\varepsilon|\phi|^{2^*_s}.
\]

(2.6)
Moreover, since $\psi_k \to 0$ in $\hat{H}^s(\mathbb{R}^N)$ implies $(\psi_k)_k$ is uniformly bounded in $L^2_s(\mathbb{R}^N)$ and the fact that $|\phi|^{2_s} \in L^1(\mathbb{R}^N)$, using Vitaly’s convergence theorem, it is easy to see from (2.4) that

$$a|\phi + \psi_k|^{2_s-2}(\phi + \psi_k) - a|\phi|^{2_s-2}\phi \to 0 \quad \text{in } L^{2N\over N+2s}_{\text{loc}}(\mathbb{R}^N).$$

Moreover, using (2.6), we also see that given any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R)} \left|a|\phi + \psi_k|^{2_s-2}(\phi + \psi_k) - a|\phi|^{2_s-2}\phi\right|^{{2N\over N+2s}} \, dx < \varepsilon. \quad (2.7)$$

As a result, $a|\phi + \psi_k|^{2_s-2}(\phi + \psi_k) - a|\phi|^{2_s-2}\phi \to 0$ in $L^{2N\over N+2s}_{\text{loc}}(\mathbb{R}^N)$. Since $\hat{H}^s(\mathbb{R}^N)$ is continuously embedded in $L^{2s}_s(\mathbb{R}^N)$, which is the dual space of $L^{2N\over N+2s}(\mathbb{R}^N)$, it follows that $a|\phi + \psi_k|^{2_s-2}(\phi + \psi_k) - a|\phi|^{2_s-2}\phi \to 0$ in $\hat{H}^s(\mathbb{R}^N)'$. \hfill \square

**Proof of Proposition 2.1**

*Proof.* We divide the proof into few steps.

**Step 1:** Using standard arguments it follows that $(PS)$ sequences for $I_{a,f}$ are bounded in $\hat{H}^s(\mathbb{R}^N)$. More precisely, as $k \to \infty$

$$\lim_{k \to \infty} I_{a,f}(u_k) + o(1) + o(1)\|u_k\|_{\hat{H}^s(\mathbb{R}^N)} \geq I_{a,f}(u_k) - \frac{1}{2s} \langle \hat{H}^s(\mathbb{R}^N) \rangle \langle I_{a,f}(u_k), u_k \rangle_{\hat{H}^s(\mathbb{R}^N)}$$

$$= \left(\frac{1}{2s} - \frac{1}{2s}\right)\|u_k\|_{\hat{H}^s(\mathbb{R}^N)}^2 - \left(1 - \frac{1}{2s}\right) \langle f, u_k \rangle_{\hat{H}^s(\mathbb{R}^N)}$$

$$\geq \left(\frac{1}{2s} - \frac{1}{2s}\right)\|u_k\|_{\hat{H}^s(\mathbb{R}^N)}^2 - \left(1 - \frac{1}{2s}\right) \|f\|_{\hat{H}^s(\mathbb{R}^N)} \|u_k\|_{\hat{H}^s(\mathbb{R}^N)}.$$

This immediately implies $(u_k)_k$ is bounded in $\hat{H}^s(\mathbb{R}^N)$. Consequently, up to a subsequence $u_k \to \bar{u}$ in $\hat{H}^s(\mathbb{R}^N)$. Moreover, as $\langle \hat{H}^s(\mathbb{R}^N) \rangle \langle I_{a,f}(u_k), v \rangle_{\hat{H}^s(\mathbb{R}^N)} \to 0$ as $k \to \infty$ for all $v \in \hat{H}^s(\mathbb{R}^N)$, we have

$$(-\Delta)^s u_k - a(x)|u_k|^{2_s-2}u_k - f \to 0 \quad \text{in } \hat{H}^s(\mathbb{R}^N)'. \quad (2.8)$$

**Step 2:** From (2.5) we get by letting $k \to \infty$

$$\int_{\mathbb{R}^N} \frac{(u_k(x) - u_k(y))(\langle v(x) - v(y) \rangle}{|x-y|^{N+2s}} \, dx \, dy - \int_{\mathbb{R}^N} a(x)|u_k|^{2_s-2}u_k v \, dx \to (-\Delta)^s v \to 0, \quad (2.9)$$

for all $v \in \hat{H}^s(\mathbb{R}^N)$. Moreover, $u_k \to \bar{u}$ in $\hat{H}^s(\mathbb{R}^N)$ implies that

$$\int_{\mathbb{R}^N} \frac{(u_k(x) - u_k(y))(\langle v(x) - v(y) \rangle}{|x-y|^{N+2s}} \, dx \, dy \to \int_{\mathbb{R}^N} \frac{(\bar{u}(x) - \bar{u}(y))(\langle v(x) - v(y) \rangle}{|x-y|^{N+2s}} \, dx \, dy.$$

Furthermore, using Lemma 2.1 we conclude

$$\int_{\mathbb{R}^N} a(x)|u_k|^{2_s-2}u_k v \, dx \to \int_{\mathbb{R}^N} a(x)|\bar{u}|^{2_s-2}\bar{u} v \, dx.$$

Therefore, passing the limit in (2.9), we have

$$(-\Delta)^s \bar{u} = a(x)|\bar{u}|^{2_s-2}\bar{u} + f \quad \text{in } \mathbb{R}^N, \quad \bar{u} \in \hat{H}^s(\mathbb{R}^N).$$

**Step 3:** In this step we show that $(u_k - \bar{u})_k$ is a $(PS)$ sequence for $I_{a,0}$ at the level

$$\lim_{k \to \infty} I_{a,f}(u_k) - I_{a,f}(\bar{u}) \quad \text{and} \quad u_k - \bar{u} \to 0 \quad \text{in } \hat{H}^s(\mathbb{R}^N).$$

To see this, first we observe that as $k \to \infty$

$$\|u_k - \bar{u}\|_{\hat{H}^s(\mathbb{R}^N)}^2 \leq \|u_k\|_{\hat{H}^s(\mathbb{R}^N)}^2 - \|\bar{u}\|_{\hat{H}^s(\mathbb{R}^N)}^2 + o(1)$$
and by the Brézis-Lieb lemma
\[ \int_{\mathbb{R}^N} a(x)|u_k - \bar{u}|^2 \, dx = \int_{\mathbb{R}^N} a(x)|u|^2 \, dx - \int_{\mathbb{R}^N} a(x)|\bar{u}|^2 \, dx + o(1). \]
Further as \( u_k \to u \) and \( f \in \dot{H}^s(\mathbb{R}^N)' \), we also have
\[ (\dot{H}^s)'(f, u_k)_{H^s} \to (\dot{H}^s)'(f, u)_{H^s}. \] (2.10)
Using above, it follows that
\[
\tilde{I}_{a,0}(u_k - \bar{u}) = \frac{1}{2} \|u_k\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{2} \|u\|^2_{H^s(\mathbb{R}^N)} - \frac{1}{2} \int_{\mathbb{R}^N} a(x)|u_k - \bar{u}|^2 \, dx - \int_{\mathbb{R}^N} a(x)|u|^2 \, dx + o(1)
= \lim_{k \to \infty} \tilde{I}_{a,f}(u_k) - \tilde{I}_{a,f}(\bar{u}).
\]
As \( (\dot{H}^s)'(\tilde{I}_{a,f}(\bar{u}), v)_{H^s} = 0 \) for any \( v \in \dot{H}^s(\mathbb{R}^N) \), we obtain
\[
(\dot{H}^s)'(\tilde{I}_{a,0}(u_k - \bar{u}), v)_{H^s} = (u_k - \bar{u}, v)_{H^s} - \int_{\mathbb{R}^N} a(x)|u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) v \, dx
= (u_k, v)_{H^s} - \int_{\mathbb{R}^N} a(x)|u_k|^{2^*_s - 2} u_k v - (\dot{H}^s)'(f, v)_{H^s}
+ (\bar{u}, v)_{H^s} + \int_{\mathbb{R}^N} a(x)|\bar{u}|^{2^*_s - 2} \bar{u} v + (\dot{H}^s)'(f, v)_{H^s}
+ \int_{\mathbb{R}^N} a(x) \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) v \, dx
= o(1) + \int_{\mathbb{R}^N} a(x) \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) v \, dx
\]
Claim: \( \int_{\mathbb{R}^N} a(x) \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) v \, dx = o(1), \forall v \in \dot{H}^s(\mathbb{R}^N). \)

To prove the claim, we note that
\[
\left| a \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) \right| \leq C \left( |u_k - \bar{u}|^{2^*_s - 2} |\bar{u}| + |u|^{2^*_s - 2} |u_k - \bar{u}| \right) \]
(2.11)
since \( u_k \to \bar{u} \) in \( \dot{H}^s(\mathbb{R}^N) \) implies \( (u_k - \bar{u})_k \) is uniformly bounded in \( \dot{H}^s(\mathbb{R}^N) \) and thus also bounded in \( L^{2^*_s}(\mathbb{R}^N) \).
Moreover, as \( |\bar{u}|^{2^*_s} \in L^1(\mathbb{R}^N) \), using Hölder inequality on the RHS of (2.11), given \( \varepsilon > 0 \) there exists \( R = R(\varepsilon) > 0 \) such that
\[
\int_{\mathbb{R}^N \setminus B(0,R)} a \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) v \, dx
\leq C \left( \int_{\mathbb{R}^N |u_k - \bar{u}|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s - 2}} \left( \int_{\mathbb{R}^N \setminus B(0,R)} |\bar{u}|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}} \left( \int_{\mathbb{R}^N |v|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}}
+ C \left( \int_{\mathbb{R}^N \setminus B(0,R)} |\bar{u}|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s - 2}} \left( \int_{\mathbb{R}^N |u_k - \bar{u}|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}} \left( \int_{\mathbb{R}^N \setminus B(0,R)} |v|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}}
< \varepsilon. \]
(2.12)
Similarly using (2.11) and Vitaly’s convergence theorem, we also obtain
\[
\int_{B(0,R)} a \left( |u_k|^{2^*_s - 2} u_k - |\bar{u}|^{2^*_s - 2} \bar{u} - |u_k - \bar{u}|^{2^*_s - 2} (u_k - \bar{u}) \right) v \, dx = o(1).
\]
Combining this with (2.12), the claim follows and hence Step 3 follows.
Step 4: Rescaling of \((v_k)_k\) in the nontrivial case.

If \(u_k \to \tilde{u}\) in \(\dot{H}^s(\mathbb{R}^N)\), then the theorem is proved with \(m = 0\). Therefore, we assume \(u_k \not\to \tilde{u}\) in \(\dot{H}^s(\mathbb{R}^N)\). Set

\[ v_k := u_k - \tilde{u}. \]

From Step 3, we have \((-\Delta)^s v_k - a(x)|v_k|^{2^*-2}v_k \to 0\) in \(\dot{H}^s(\mathbb{R}^N)'\). Therefore, as \((v_k)_k\) is uniformly bounded in \(\dot{H}^s(\mathbb{R}^N)\),

\[ \|v_k\|_{\dot{H}^s(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} a(x)|v(x)|^{2^*} dx \leq \|a\|_{L^\infty(\mathbb{R}^N)} \|v_k\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}. \]

Consequently, \(v_k \not\to 0\) in \(L^{2^*}(\mathbb{R}^N)\) and, up to a subsequence,

\[ \inf_k \|v_k\|_{L^{2^*}(\mathbb{R}^N)} \geq C > 0. \] (2.13)

Moreover, since \((v_k)_k\) is bounded in \(\dot{H}^s(\mathbb{R}^N)\) and \(\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \hookrightarrow L^{2^*,2}(\mathbb{R}^N)\) (See Appendix A), we have \(\|v_k\|_{L^{2^*,2}(\mathbb{R}^N)} \leq C\) for some \(C > 0\) (independent of \(k\)). On the other hand, combining (2.13) with Lemma A.1 for \(r = 2\), we readily see that \(\|v_k\|_{L^{2^*,2}(\mathbb{R}^N)} \geq C\), for some \(C > 0\) independent of \(k\). Hence, there exists a positive constant, which we denote by \(C\) again such that, for all \(k\)

\[ C \leq \|v_k\|_{L^{2^*,2}(\mathbb{R}^N)} \leq C^{-1}. \] (2.14)

Combining (2.14) with the definition of \(L^{2^*,2}(\mathbb{R}^N)\), we deduce that for every \(k \in \mathbb{N}\), there exists \(r_k > 0, x_k \in \mathbb{R}^N\) such that

\[ \int_{B(x_k,r_k)} \frac{|v_k|^2}{r_k^{2s}} dx = \int_{B(x_k,r_k)} |v_k|^2 dx \geq \|v_k\|_{L^{2^*,2}(\mathbb{R}^N)}^2 - \frac{C^2}{2k} \geq C, \] (2.15)

for some \(C > 0\) (independent of \(k\)).

Now we define, \(\bar{v}_k := r_k^{-\frac{2-s}{2}} v_k(r_k x + x_k)\). In the view of the scaling invariance of the \(\dot{H}^s(\mathbb{R}^N)\) norm, \((\bar{v}_k)_k\) is a bounded sequence in \(\dot{H}^s(\mathbb{R}^N)\), thus up to a subsequence \(\bar{v}_k \to \bar{v}\) in \(\dot{H}^s(\mathbb{R}^N)\). Consequently, \(\bar{v}_k \to \bar{v}\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\). Therefore, using change of variable, we observe from (2.15)

\[ 0 < \int_{B(x_k,r_k)} \frac{|\bar{v}_k|^2}{r_k^{2s}} dx = \int_{B(0,1)} |\bar{v}_k|^2 dx \to \int_{B(0,1)} |\bar{v}|^2 dx. \]

Hence \(\bar{v} \neq 0\). Clearly, up to a subsequence, either \(x_k \to x_0 \in \mathbb{R}^N\) or \(|x_k| \to \infty\). Also note that \(\bar{v}_k \to \bar{v} \neq 0\) and \(v_k \to 0\) implies \(r_k \to 0\).

Step 5: In this step we prove that \(\bar{v}\) solves

\[ (-\Delta)^s \bar{v} = a(x_0)|\bar{v}|^{2^*-2} \bar{v} \text{ in } \mathbb{R}^N, \quad \bar{v} \in \dot{H}^s(\mathbb{R}^N), \]
or equivalently \(a(x_0)\frac{N-2s}{N} \bar{v}\) solves (2.2), without the sign restriction.

To this aim, it is enough to show that for arbitrarily chosen \(\varphi \in C^\infty_c(\mathbb{R}^N)\) the following holds:

\[ \langle \bar{v}, \varphi \rangle_{\dot{H}^s} = \int_{\mathbb{R}^N} a(x_0)|\bar{v}|^{2^*-2} \bar{v} \varphi. \]

Let \(\varphi \in C^\infty_c(\mathbb{R}^N)\) be arbitrary. By Step 3, we have \(I'_{a,0}(v_k) \to 0\) in \(\dot{H}^s(\mathbb{R}^N)'\). Therefore, as \(\bar{v}_k \to \bar{v}\), using change of variables, we get

\[ \langle \bar{v}, \varphi \rangle_{\dot{H}^s} = \lim_{k \to \infty} \langle \bar{v}_k, \varphi \rangle_{\dot{H}^s} = \lim_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{r_k^{-N-2s}}{|x - y|^{N+2s}} \left( v_k(r_k x + x_k) - v_k(r_k y + x_k) \right) (\varphi(x) - \varphi(y)) dx dy = \lim_{k \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{r_k^{-N-2s}}{|x - y|^{N+2s}} (v_k(x) - v_k(y)) \left( \varphi \left( \frac{x - x_k}{r_k} \right) - \varphi \left( \frac{y - x_k}{r_k} \right) \right) dx dy. \]
In this step we show that the claim follows. Hence, Step 5 is proved. Equivalently
\[ \left| \frac{1}{r_k} \varphi \left( \frac{x - x_k}{r_k} \right) \right| dx \]
\[ = \lim_{k \to \infty} \int_{\mathbb{R}^N} a(x)|v_k(x)|^{2^* - 2} v_k(x) \frac{\varphi}{r_k} \left( \frac{x - x_k}{r_k} \right) dx \]
\[ = \lim_{k \to \infty} \int_{\mathbb{R}^N} a(r_k x + x_k) |\tilde{v}_k(x)|^{2^* - 2} \tilde{v}_k(x) \varphi(x) dx. \]

**Claim:** \( \lim_{k \to \infty} \int_{\mathbb{R}^N} a(r_k x + x_k)|\tilde{v}_k(x)|^{2^* - 2} \tilde{v}_k(x) \varphi(x) dx = a(x_0) \int_{\mathbb{R}^N} |\tilde{v}|^{2^* - 2} \tilde{v} \varphi dx. \)

To see this
\[ \left| \int_{\mathbb{R}^N} a(r_k x + x_k)|\tilde{v}_k(x)|^{2^* - 2} \tilde{v}_k(x) \varphi(x) dx - a(x_0) \int_{\mathbb{R}^N} |\tilde{v}|^{2^* - 2} \tilde{v} \varphi dx \right| \]
\[ \leq \left| \int_{\mathbb{R}^N} a(r_k x + x_k)|\tilde{v}_k(x)|^{2^* - 2} \tilde{v}_k(x) - |\tilde{v}|^{2^* - 2} \tilde{v} \right| \varphi(x) dx + \int_{\mathbb{R}^N} (a(r_k x + x_k) - a(x_0)) |\tilde{v}|^{2^* - 2} \tilde{v} \varphi dx \]
\[ = I_k + J_k. \]

Since \( r_k \to 0, x_k \to x_0, a \in C(\mathbb{R}^N) \) and \( |\tilde{v}|^{2^* - 2} \tilde{v} \varphi \in L^1(\mathbb{R}^N) \) by the Hölder inequality, the dominated convergence theorem gives that \( \lim_{k \to \infty} J_k = 0 \). On the other hand, as \( \varphi \) has compact support and \( \tilde{v}_k \to \tilde{v} \) a.e. by Vitaly's convergence theorem, it is not difficult to see that \( \lim_{k \to \infty} I_k = 0 \). Thus the claim follows. Hence, Step 5 is proved. Equivalently \( a(x_0) \frac{\varphi}{r_k} \tilde{v} \) solves (1.2), without sign requirement.

Define,
\[ z_k(x) := v_k(x) - r_k \frac{\varphi}{r_k} \left( \frac{x - x_k}{r_k} \right). \]

**Step 6:** In this step we show that \( (z_k) \) is a \( (PS) \) sequence for \( \tilde{I}_{a,0} \) at the level \( \tilde{I}_{a,0} \) of \( \tilde{I}_{a,f}(\tilde{u}) - \tilde{I}_{a,f}(\tilde{u}) - a(x_0) \frac{\varphi}{r_k} \tilde{v} \), where \( w \) is a solution of (1.2), without the sign condition.

To see this, first we observe that if we define, \( \tilde{z}_k := r_k \frac{\varphi}{r_k} z_k(r_k x + x_k) \), then it is easy to check that \( \tilde{z}_k = \tilde{v}_k - \tilde{v} \). Therefore, the scaling invariance in the norm of \( \tilde{H}^s(\mathbb{R}^N) \) gives
\[ \| z_k \|_{\tilde{H}^s(\mathbb{R}^N)} = \| \tilde{z}_k \|_{\tilde{H}^s(\mathbb{R}^N)} = \| \tilde{v}_k - \tilde{v} \|_{\tilde{H}^s(\mathbb{R}^N)}. \]  

\[ \text{(2.16)} \]

**Claim 1:**
\[ \int_{\mathbb{R}^N} a(r_k x + x_k)|\tilde{v}_k(x) - \tilde{v}(x)|^{2^*} dx = \int_{\mathbb{R}^N} a(r_k x + x_k)|\tilde{v}_k(x)|^{2^*} dx - \int_{\mathbb{R}^N} a(x_0)|\tilde{v}(x)|^{2^*} dx + o(1). \]  

(2.17)

To prove the claim, we set \( a_k := a(r_k x + x_k) \). An elementary analysis yields for any \( p > 1 \),
\[ \left| \tilde{v}_k^{p-1} \tilde{v}_k - |\tilde{v}|^{p-1} \tilde{v} - |\tilde{v}_k - \tilde{v}|^{p-1} (\tilde{v}_k - \tilde{v}) \right| \leq C \left( |\tilde{v}_k - \tilde{v}|^{p-1} |\tilde{v}| + |\tilde{v}_k - \tilde{v}| |\tilde{v}|^{p-1} \right) \]  

(2.18)

Thus,
\[ \left| a_k^{\frac{1}{p}} \tilde{v}_k^{2^*-1} - |a_k^{\frac{1}{p}} \tilde{v}|^{2^*-1} - a_k^{\frac{1}{p}} (\tilde{v}_k^{2^*-1} - |\tilde{v}_k - \tilde{v}|^{2^*-1}) \right| \leq C \left( a_k^{\frac{1}{p}} |\tilde{v}_k - \tilde{v}|^{2^*-1} |a_k^{\frac{1}{p}} \tilde{v}| + a_k^{\frac{1}{p}} (\tilde{v}_k - \tilde{v}) |a_k^{\frac{1}{p}} \tilde{v}|^{2^*-1} \right) \]
\[ \leq C ||a||_{L^\infty(\mathbb{R}^N)} \left( |\tilde{v}_k - \tilde{v}|^{2^*-1} |\tilde{v}| + |\tilde{v}_k - \tilde{v}| |\tilde{v}|^{2^*-1} \right) \]  

(2.19)

Using the dominated convergence theorem, we immediately have \( \lim_{k \to \infty} \int_{\mathbb{R}^N} a_k^{\frac{1}{p}} |\tilde{v}_k|^{2^*} dx = \int_{\mathbb{R}^N} a(x_0) |\tilde{v}|^{2^*} dx \). Therefore, to prove the claim, it is enough to show that
\[ \int_{\mathbb{R}^N} |\tilde{v}_k - \tilde{v}|^{2^*-1} |\tilde{v}| dx = o(1) \quad \text{and} \quad \int_{\mathbb{R}^N} |\tilde{v}_k - \tilde{v}| |\tilde{v}|^{2^*-1} dx = o(1). \]
For this, given any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0, R)} |\tilde{v}_k - \tilde{v}|^{2^* - 1} |\tilde{v}| \, dx \leq \left( \int_{\mathbb{R}^N} |\tilde{v}_k - \tilde{v}|^{2^*} \, dx \right)^{\frac{2^* - 1}{2^*}} \left( \int_{\mathbb{R}^N \setminus B(0, R)} |\tilde{v}|^{2^*} \, dx \right)^{\frac{1}{2^*}} < \varepsilon,$$

since $\tilde{v}_k \to \tilde{v}$ in $\dot{H}^{s}(\mathbb{R}^N)$ implies that $(\tilde{v}_k - \tilde{v})_k$ is uniformly bounded in $L^{2^*_s}(\mathbb{R}^N)$. Similarly using Vitaly’s convergence theorem via the Hölder inequality, it can be also shown that $\int_{B(0, R)} |\tilde{v}_k - \tilde{v}|^{2^* - 1} |\tilde{v}| \, dx = o(1)$. Thus (i) holds. Similarly (ii) can also be proved. Hence the claim follows.

Applying (2.16) and (2.17), we have

$$\begin{align*}
I_{a,0}(z_k) &= \frac{1}{2} \|z_k\|_{H^{s}(\mathbb{R}^N)}^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} a(x) |z_k(x)|^{2^*} \, dx \\
&= \frac{1}{2} \|\tilde{v}_k - \tilde{v}\|_{H^{s}(\mathbb{R}^N)}^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} a(r_k x + x_k) |z_k(r_n x + y_n)|^{2^*_s} \, dx \\
&= \frac{1}{2} \|\tilde{v}_k - \tilde{v}\|_{H^{s}(\mathbb{R}^N)}^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} a(r_n x + y_n) \|\tilde{v}_k - \tilde{v}\|^{2^*_s} \, dx \\
&= \frac{1}{2} \left\{ \|\tilde{v}_k\|_{H^{s}(\mathbb{R}^N)}^2 - \|\tilde{v}\|_{H^{s}(\mathbb{R}^N)}^2 \right\} - \frac{1}{2^*} \int_{\mathbb{R}^N} a(r_k x + x_k) |\tilde{v}_k(x)|^{2^*} \, dx \\
&\quad + \frac{1}{2^*} \int_{\mathbb{R}^N} a(x_0) |\tilde{v}(x)|^{2^*} \, dx + o(1) \\
&= \frac{1}{2} \|v_k\|_{H^{s}(\mathbb{R}^N)}^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} a(x) |v_k(x)|^{2^*} \, dx \\
&\quad - a(x_0) \frac{N - 2s}{2^*} \int_{\mathbb{R}^N} a(\tilde{v})^{\frac{N - 2s}{2^*}} \|\tilde{v}\|_{H^{s}(\mathbb{R}^N)}^{2^*} - a(x_0) \frac{N - 2s}{2^*} \int_{\mathbb{R}^N} |\tilde{v}|^{2^*} \, dx \right) + o(1) \\
&= I_{a,0}(v_k) - a(x_0) \frac{N - 2s}{2^*} I_{1,0}(a(x_0)^{\frac{N - 2s}{2^*}} \tilde{v}) + o(1) \\
&= \lim_{k \to \infty} I_{a,f}(u_k) - I_{a,f}(\tilde{v}) - a(x_0) \frac{N - 2s}{2^*} I_{1,0}(w),
\end{align*}$$

where $w$ is a solution of (1.2) without the sign condition. From the above energy estimate of $z_k$, we also observe that

$$I_{a,0}(z_k) = I_{a,0}(v_k) - a(x_0) \frac{N - 2s}{2^*} I_{1,0}(w) \leq I_{a,0}(v_k) - a(x_0) \frac{N - 2s}{2^*} I_{1,0}(W),$$

where $W$ is the unique positive solution of (1.2), which also has the minimum energy among all the solutions of (1.2) with or without the sign condition. Further as $I_{1,0}(W) = \frac{1}{2} \|S\|_{H^s}$ (see (2.24) and the comments below to it) and $a > 0$, we obtain $I_{a,0}(z_k) < I_{a,0}(v_k)$.

Next, we estimate $(H^s)'(I'_{1,0}(z_k), \varphi)_{H^s}$ for any arbitrarily chosen $\varphi \in C_c^{\infty}(\mathbb{R}^N)$. Towards this, first we observe that an easy computation yields $(z_k, \varphi)_{H^s}(\mathbb{R}^N) = (\tilde{v}_k, \varphi)_H(\mathbb{R}^N)$, where $\varphi(x) := r_k \varphi(r_k x + x_k)$. Clearly, $\|\varphi\|_{H^s(\mathbb{R}^N)} = \|\varphi\|_{H^s(\mathbb{R}^N)}$ and $\varphi_k \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ as $r_k \to 0$. Using these and the fact that $\tilde{z}_k = \tilde{v}_k - \tilde{v}$, we obtain

$$(H^s)'(I'_{1,0}(z_k), \varphi)_{H^s} = (z_k, \varphi)_{H^s}(\mathbb{R}^N) - \int_{\mathbb{R}^N} a(x) |z_k|^{2^* - 2} z_k \varphi \, dx$$

$$= (\tilde{v}_k, \varphi_k)_{H^s}(\mathbb{R}^N) - \int_{\mathbb{R}^N} a(r_k x + x_k) |z_k(r_k x + x_k)|^{2^* - 2} z_k(r_k x + x_k) \varphi(r_k x + x_k) r_k^N \, dx$$

$$= (\tilde{v}_k - \tilde{v}, \varphi_k)_{H^s}(\mathbb{R}^N) - \int_{\mathbb{R}^N} a(r_k x + x_k) |\tilde{z}_k|^{2^* - 2} \tilde{z}_k \varphi_k \, dx$$

$$= (\tilde{v}_k - \tilde{v}, \varphi_k)_{H^s}(\mathbb{R}^N) - \int_{\mathbb{R}^N} a(r_k x + x_k) |\tilde{z}_k|^{2^* - 2} \tilde{z}_k \varphi_k \, dx$$
where in the last line we have used the fact that $\varphi_k \to 0$ in $\dot{H}^s(\mathbb{R}^N)$ and $a_k(x) = o(r_k x + x_k)$. Now, using (2.15) with $p = 2^*_s - 1$ and the following an argument similar to the proof of Claim 1, it can be shown that

$$\int_{\mathbb{R}^N} a_k |\tilde{v}_k - \tilde{v}|^{2^*_s - 2}(\tilde{v}_k - \tilde{v}) \varphi_k \, dx = \int_{\mathbb{R}^N} a_k |\tilde{v}_k|^{2^*_s - 2} \tilde{v}_k \varphi_k \, dx - \int_{\mathbb{R}^N} a_k |\tilde{v}|^{2^*_s - 2} \tilde{v} \varphi_k \, dx + o(1), \quad (2.21)$$

Since, at a level which is strictly lower than the previous one, with a fixed minimum amount of decrease.

Thus substituting (2.22) into (2.20) yields

$$\lim_{k \to \infty} \int_{\mathbb{R}^N} a_k |\tilde{v}_k|^{2^*_s - 2} \tilde{v}_k \varphi_k \, dx = \int_{\mathbb{R}^N} a(x)|v_k(x)|^{2^*_s - 2}v_k(x)\varphi(x)dx + o(1). \quad (2.22)$$

Substituting these into (2.21), we obtain

$$\int_{\mathbb{R}^N} a_k |\tilde{v}_k - \tilde{v}|^{2^*_s - 2}(\tilde{v}_k - \tilde{v}) \varphi_k \, dx = \int_{\mathbb{R}^N} a(x)|v_k(x)|^{2^*_s - 2}v_k(x)\varphi(x)dx + o(1). \quad (2.22)$$

Thus substituting (2.22) into (2.21) yields

$$\langle \dot{H}^s \rangle [I_{a,0}(z_k), \varphi]_{\dot{H}^s} = \langle v_k, \varphi \rangle_{\dot{H}^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} a(x)|v_k(x)|^{2^*_s - 2}v_k(x)\varphi(x)dx + o(1) = I_{a,0}(v_k)(\varphi) = 0,$$

for the last equality we have used Step 3. This completes the proof of Step 6.

Now, starting from a $(PS)$ sequence $(v_k)_k$ for $I_{a,0}$ we have extracted another $(PS)$ sequence $(z_k)_k$ at a level which is strictly lower than the previous one, with a fixed minimum amount of decrease. Since, $\sup_k \|v_k\|_{\dot{H}^s(\mathbb{R}^N)} \leq C$ (finite), hence the process should terminate after finitely many steps and the last $(PS)$ sequence strongly converges to 0. Further, $\lim_{k \to \infty} \int_{\mathbb{R}^N} a_k |\tilde{v}_k|^{2^*_s - 2} \tilde{v}_k \varphi_k \, dx = 0, \quad (2.24)$

where for the last equality we have used Step 3. This completes the proof of Step 6.

We end this section with the definition of some functions which will be used throughout the rest of the paper. We define,

$$J(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2}{\left(\int_{\mathbb{R}^N} a(x)|u(x)|^{2^*_s} \, dx\right)^{\frac{s}{2^*_s}}} \quad J_\infty(u) := \frac{\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2}{\left(\int_{\mathbb{R}^N} |u(x)|^{2^*_s} \, dx\right)^{\frac{s}{2^*_s}}} \quad (2.23)$$

$$J_\infty(u) := \inf_{u \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}} J_\infty(u), \quad (2.24)$$

i.e., $S$ is the best Sobolev constant. From [54], it is known that $S$ is achieved by the unique positive solution (up to translation and dilation) $W$ of (1.12). Further, as already noted in the above proof, $W$ is radially symmetric positive decreasing smooth function satisfying (1.3) and

$$I_{1,0}(W) = \frac{s}{N} S^{\frac{N}{2}} > 0. \quad (2.25)$$

3. PROOF OF THEOREM 1.2

In this section we prove Theorem 1.2. To this aim we first establish existence of two positive critical points in the spirit of [3] for the following functional:

$$I_{a,f}(u) = \frac{1}{2}\|u\|_{\dot{H}^s(\mathbb{R}^N)}^2 - \frac{1}{2^*_s} \int_{\mathbb{R}^N} a(x)u_+^{2^*_s} \, dx - \langle \dot{H}^s \rangle \langle f, u \rangle_{\dot{H}^s}, \quad (3.1)$$

where $u_+ := \max\{u, 0\}$ and $u_- := -\min\{u, 0\}$ and $f \in \dot{H}^s(\mathbb{R}^N)'$ is a nonnegative nontrivial functional.
Clearly, if $u$ is a critical point of $I_{a,f}$, then $u$ solves
\[
\begin{cases}
(-\Delta)^s u = a(x)u^{2_s^* - 1} + f(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]
(3.2)

**Remark 3.1.** If $u$ is a weak solution of (3.2) and $f$ is a nonnegative functional, then taking $v = u_-$ as a test function in (3.2) we obtain
\[
-\|u_-\|^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \frac{u_+(y)u_-(x) + u_+(x)u_-(y)}{|x - y|^{N + 2s}} \, dx \, dy = \langle \mathcal{H}(f, u_-) \rangle_{H^s} \geq 0.
\]

This in turn implies $u_- = 0$, i.e., $u \geq 0$. Therefore, using maximum principle [3 Theorem 1.2], it follows that, $u$ is a positive solution to (3.2). Hence $u$ is a solution to (3).

To establish the existence of two critical points for $I_{a,f}$, we first need to prove some auxiliary results. Towards that, we partition $\mathcal{H}(\mathbb{R}^N)$ into three disjoint sets. Let $g : H^s(\mathbb{R}^N) \to \mathbb{R}$ be defined by
\[
g(u) := \|u\|^2_{H^s(\mathbb{R}^N)} - (2_s^* - 1)\|a\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*}.
\]

Now, put
\[
U_1 := \{u \in \mathcal{H}(\mathbb{R}^N) : u = 0 \text{ or } g(u) > 0\}, \quad U_2 := \{u \in \mathcal{H}(\mathbb{R}^N) : g(u) < 0\},
\]
\[
U := \{u \in \mathcal{H}(\mathbb{R}^N) \setminus \{0\} : g(u) = 0\}.
\]

**Remark 3.2.** Using the Sobolev inequality, it is easy to see that $\|u\|_{H^s(\mathbb{R}^N)}$ and $\|u\|_{L^{2_s^*}(\mathbb{R}^N)}$ are bounded away from 0 for all $u \in U$.

Set
\[
c_0 := \inf_U I_{a,f}(u) \quad \text{and} \quad c_1 := \inf_{U_1} I_{a,f}(u).
\]
(3.3)

**Remark 3.3.** Clearly, $g(tu) = t^2\|u\|^2_{H^s(\mathbb{R}^N)} - t^{2_s^*}(2_s^* - 1)\|a\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^{2_s^*}$ for any $t > 0$ and $u \in \mathcal{H}(\mathbb{R}^N)$. Moreover $g(0) = 0$ and $t \mapsto g(tu)$ is a strictly concave function. Thus, for any $u \in \mathcal{H}(\mathbb{R}^N)$, with $\|u\|_{H^s(\mathbb{R}^N)} = 1$, there exists unique $t = t(u)$ such that $tu \in U$. On the other hand, $g(tu) = (t^2 - t^{2_s^*})\|u\|^2_{H^s(\mathbb{R}^N)}$ for any $u \in U$. This implies that
\[
tu \in U_1 \quad \text{for all } t \in (0,1) \quad \text{and} \quad tu \in U_2 \quad \text{for all } t > 1.
\]

**Lemma 3.1.** Assume that $C_0$ is defined as in Theorem 1.2. Then
\[
\frac{4s}{N + 2s}\|u\|_{H^s(\mathbb{R}^N)} \geq C_0S_{\mathbb{R}^N}^{\frac{N}{2s}} \quad \text{for all } u \in U,
\]
where $S$ is defined in (2.24). \[\]

**Proof.** Note that
\[
\|u\|_{L^{2_s^*}(\mathbb{R}^N)} = \left(\left(\frac{\|u\|_{H^s(\mathbb{R}^N)}}{(2_s^* - 1)\|a\|_{L^\infty(\mathbb{R}^N)}}\right)^{\frac{s}{2_s^*}}\right)^{\frac{2_s^*}{s}}
\]
whenever $u \in U$.

Therefore, combining this with the definition of $S$, we have
\[
\|u\|_{H^s(\mathbb{R}^N)} \geq S_{\mathbb{R}^N}^{\frac{s}{2}}\|u\|_{L^{2_s^*}(\mathbb{R}^N)} = S_{\mathbb{R}^N}^{\frac{s}{2}}\left(\left(\frac{\|u\|_{H^s(\mathbb{R}^N)}}{(2_s^* - 1)\|a\|_{L^\infty(\mathbb{R}^N)}}\right)^{\frac{s}{2_s^*}}\right)^{\frac{2_s^*}{s}}
\]
for all $u \in U$. From here, using the definition of $C_0$, the lemma follows. \[\]
Lemma 3.2. Assume $C_0$ is defined as in Theorem 1.2 and $c_0$ and $c_1$ are defined as in (3.3). Further, if

$$\inf_{u \in H^s(\mathbb{R}^N), \|u\|_{L^{2^*_s}(\mathbb{R}^N)} = 1} \left\{ C_0 \|u\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{2s}} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s} \right\} > 0,$$  

(3.4)

then $c_0 < c_1$.

Proof. Define

$$\tilde{J}(u) := \frac{1}{2} \|u\|_{H^s(\mathbb{R}^N)}^2 - \|a\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^{2^*_s}(\mathbb{R}^N)}^{2^*_s} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s}, \quad u \in \dot{H}^s(\mathbb{R}^N).$$  

(3.5)

**Step 1:** In this step we prove that there exists $\alpha > 0$ such that

$$\frac{d}{dt} \tilde{J}(tu) \big|_{t=1} \geq \alpha \quad \text{for all } u \in U.$$

From the definition of $\tilde{J}$, we have $\frac{d}{dt} \tilde{J}(tu) \big|_{t=1} = \|u\|_{H^s(\mathbb{R}^N)}^2 - \|a\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^{2^*_s}(\mathbb{R}^N)}^{2^*_s} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s}$. Therefore, using the definition of $U$ and the value of $C_0$, we have for $u \in U$

$$\frac{d}{dt} \tilde{J}(tu) \big|_{t=1} = \frac{4s}{N+2s} \|u\|_{H^s(\mathbb{R}^N)}^2 - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s}$$

$$= \left(\frac{2^*_s - 1}{2^*_s}\right)\|u\|_{L^\infty(\mathbb{R}^N)} \frac{N+2s}{2s} C_0 \|u\|_{H^s(\mathbb{R}^N)}^2 - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s}$$

(3.6)

$$= C_0 \|u\|_{H^s(\mathbb{R}^N)} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s}.$$

Further, (3.3) implies there exists $d > 0$ such that

$$\inf_{u \in H^s(\mathbb{R}^N), \|u\|_{L^{2^*_s}(\mathbb{R}^N)} = 1} \left\{ C_0 \|u\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{2s}} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s} \right\} \geq d.$$  

(3.7)

Now,

$$C_0 \|u\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{2s}} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s} \geq d, \quad \|u\|_{L^{2^*_s}(\mathbb{R}^N)} = 1$$

$$\iff C_0 \|u\|_{H^s(\mathbb{R}^N)}^{\frac{N+2s}{2s}} - \langle \mathbb{H}^s \rangle \langle f, u \rangle_{H^s} \geq d \quad \|u\|_{L^{2^*_s}(\mathbb{R}^N)} \in \dot{H}^s(\mathbb{R}^N) \setminus \{0\}.$$

Hence, plugging back the above estimate into (3.3) and using Remark 3.2 we complete the proof of Step 1.

**Step 2:** Let $(u_n)_n$ be a minimizing sequence for $I_{a,f}$ on $U$, i.e., $I_{a,f}(u_n) \to c_1$ and $\|u_n\|_{H^s(\mathbb{R}^N)} = (2^*_s - 1)\|a\|_{L^\infty(\mathbb{R}^N)} \|u_n\|_{L^{2^*_s}(\mathbb{R}^N)}$. Therefore, for large $n$

$$c_1 + o(1) \geq I_{a,f}(u_n) \geq \tilde{J}(u_n) \geq \left(\frac{1}{2} - \frac{1}{2^*_s(2^*_s - 1)}\right) \|u_n\|_{H^s(\mathbb{R}^N)}^2 - \|f\|_{L^{\infty}(\mathbb{R}^N)} \|u_n\|_{H^s(\mathbb{R}^N)}.$$

This implies that $(\tilde{J}(u_n))_n$ is a bounded sequence and $(\|u_n\|_{H^s(\mathbb{R}^N)})_n$ and $(\|u_n\|_{L^{2^*_s}(\mathbb{R}^N)})_n$ are bounded.

**Claim:** $c_0 < 0$. \\

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Indeed, to prove the claim, it is enough to show that there exists \( v \in U_1 \) such that \( I_{a,f}(v) < 0 \). Note that, thanks to Remark 3.3 we can choose \( t \) enough. Thus, by Step 1, there exists \( t \). Thus, by Step 1, there exists \( t \in \mathbb{R} \) such that \( t \). Hence the claim follows.

Thanks to the above claim, \( I_{a,f}(u_n) < 0 \) for large \( n \). Consequently, \( 0 > I_{a,f}(u_n) \geq \left( \frac{1}{2} - \frac{1}{2(2^*_s - 1)} \right) \| u_n \|_{H^s(\mathbb{R}^N)}^2 - (H^*)^\prime \langle f, u_n \rangle_{H^s} \). Moreover, it is easy to check that for all \( u \in U \), the function \( \frac{dt}{dt} \tilde{\mathcal{J}}(tu) \) is strictly increasing in \( t \in [0,1] \) and therefore we can conclude that \( t_n \) is unique.

**Step 3:** In this step we show that

\[
\lim_{n \to \infty} \left\{ \tilde{\mathcal{J}}(u_n) - \tilde{\mathcal{J}}(t_nu_n) \right\} > 0. \tag{3.8}
\]

Observe that \( \tilde{\mathcal{J}}(u_n) - \tilde{\mathcal{J}}(t_nu_n) = \int_{t_n}^{1} \frac{d}{dt} \tilde{\mathcal{J}}(tu_n) \, dt \) and that for all \( n \in \mathbb{N} \) there is \( \xi_n > 0 \) such that \( t_n \in (0, 1 - 2\xi_n) \) and \( \frac{dt}{dt} \tilde{\mathcal{J}}(tu_n) \geq \alpha/2 \) for \( t \in [1 - \xi_n, 1] \).

To establish (3.8) it is enough to show that \( \xi_n > 0 \) can be chosen independent of \( n \in \mathbb{N} \). But this is true since, \( \frac{dt}{dt} \tilde{\mathcal{J}}(tu_n) \bigg|_{t=1} \geq \alpha \) and by the boundedness of \( \{u_n\} \),

\[
\left| \frac{d^2}{dt^2} \tilde{\mathcal{J}}(tu_n) \right| = \left| \| u_n \|_{H^s(\mathbb{R}^N)}^2 - (2^*_s - 1) \| u \|_{L^\infty(\mathbb{R}^N)}^2 \cdot 2 \int_{\mathbb{R}^N} |u_n|^2 \, dx \right| \leq \left| 1 - i^2 \right| \| u_n \|_{H^s(\mathbb{R}^N)}^2 \leq C,
\]

for all \( n \geq 1 \) and \( t \in [0,1] \).

**Step 4:** From the definition of \( I_{a,f} \) and \( \tilde{\mathcal{J}} \), it immediately follows that \( \frac{dt}{dt} I_{a,f}(tu) \geq \frac{dt}{dt} \tilde{\mathcal{J}}(tu) \) for all \( u \in \dot{H}^s(\mathbb{R}^N) \) and for all \( t > 0 \). Hence,

\[
I_{a,f}(u_n) - I_{a,f}(t_nu_n) = \int_{t_n}^{1} \frac{d}{dt} I_{a,f}(tu_n) \, dt \geq \int_{t_n}^{1} \frac{d}{dt} \tilde{\mathcal{J}}(tu_n) \, dt = \tilde{\mathcal{J}}(u_n) - \tilde{\mathcal{J}}(t_nu_n).
\]

Since \( (u_n)_n \subset U \) is a minimizing sequence for \( I_{a,f} \) on \( U \), and \( t_nu_n \in U_1 \), we conclude using (3.8) that

\[
c_0 = \inf_{u \in U_1} I_{a,f}(u) < \inf_{u \in U} I_{a,f}(u) = c_1
\]

\[\square\]

Next, we introduce the equation at infinity associated to (3.2):

\[
(-\Delta)^s u = u^{2^*_s - 1} \quad \text{in} \quad \mathbb{R}^N, \quad u \in \dot{H}^s(\mathbb{R}^N), \tag{3.9}
\]

and the corresponding functional \( I_{1,0} : \dot{H}^s(\mathbb{R}^N) \to \mathbb{R} \) defined by

\[
I_{1,0}(u) = \frac{1}{2} \| u \|^2_{H^s(\mathbb{R}^N)} - \frac{1}{2^*_s} \int_{\mathbb{R}^N} u^{2^*_s} \, dx.
\]

Arguing as in Remark 3.4, it immediately follows that solutions of (3.9) are the positive solutions of (1.2).

**Proposition 3.1.** Assume that (3.4) holds. Then \( I_{a,f} \) has a critical point \( u_0 \in U_1 \), with \( I_{a,f}(u_0) = c_0 \). In particular, \( u_0 \) is a positive weak solution to (4).
Proof. We decompose the proof into few steps.

Step 1: $c_0 > -\infty$.

Note that $I_{a,f}(u) \geq \bar{J}(u)$, where $\bar{J}$ is defined as in (3.6). Therefore, in order to prove Step 1, it is enough to show that $\bar{J}$ is bounded from below. From definition of $U_1$, it immediately follows that

$$\bar{J}(u) \geq \left[ \frac{1}{2} - \frac{1}{2s(2s - 1)} \right] \|u\|^2_{H^s(\mathbb{R}^N)} - \|f\|_{H^s(\mathbb{R}^N)} \|u\|_{H^s(\mathbb{R}^N)} \text{ for all } u \in U_1. \quad (3.10)$$

As RHS is quadratic function in $\|u\|_{H^s(\mathbb{R}^N)}$, $\bar{J}$ is bounded from below. Hence Step 1 follows.

Step 2: In this step we show that there exists a bounded nonnegative $(PS)$ sequence $(u_n)_n \subset U_1$ for $I_{a,f}$ at level $c_0$.

Let $(u_n)_n \subset U_1$ such that $I_{a,f}(u_n) \to c_0$. Since Lemma 3.2 implies that $c_0 < c_1$, without restriction we can assume $(u_n)_n \subset U_1$. Further, using Ekeland’s variational principle from $(u_n)_n$, we can extract a $(PS)$ sequence in $U_1$ for $I_{a,f}$ at level $c_0$. We again call it by $(u_n)_n$. Moreover, as $I_{a,f}(u) \geq \bar{J}(u)$, from (3.10) it follows that $(u_n)_n$ is a bounded sequence. Therefore, up to a subsequence $u_n \rightharpoonup u_0$ in $H^s(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$. In particular, $(u_n)_+ \to (u_0)_+$ and $(u_n)_- \to (u_0)_-$ a.e. in $\mathbb{R}^N$. Moreover, as $f$ is a nonnegative functional, we have

$$0(1) = \langle H^{s}, (u_n)_+ - (u_n)_- \rangle_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} a(x)(u_n)_+ - (u_n)_- \rangle_{H^s} \leq -\|u_n\|^2_{H^s(\mathbb{R}^N)} - \int_{\mathbb{R}^N} \|u_n\|_{H^s(\mathbb{R}^N)}^2 \|u_n\|_{H^s(\mathbb{R}^N)} dy \leq -\|u_n\|^2_{H^s(\mathbb{R}^N)}.$$

Therefore, $(u_n)_+$ strongly converges to $0$ in $H^s(\mathbb{R}^N)$ and so $(u_n)_- \to 0$ a.e. in $\mathbb{R}^N$ and also $(u_n)_- = 0$ a.e. in $\mathbb{R}^N$. In other words, $u_0 \geq 0$ a.e. in $\mathbb{R}^N$. Consequently, without loss of generality, we can assume that $(u_n)_n$ is a nonnegative sequence. This completes the proof of Step 2.

Step 3: In this step we show that $u_n \to u_0$ in $H^s(\mathbb{R}^N)$ and $u_0 \in U_1$.

Applying Proposition 2.1, we get

$$u_n = \left( u_0 + \sum_{j=1}^m a(x_j)^{-\frac{N+2s}{2}} W^{r_j,x_j} \right) \to 0 \quad \text{in } H^s(\mathbb{R}^N). \quad (3.11)$$

with $I_{a,f}(u_0) = 0$, $W$ is the unique positive solution of (1.2) and some appropriate sequences $(x_j)_n$, $(r_j)_n$, with either $x_j \to x$ or $|x_j| \to \infty$ and $r_j \to 0$. To prove Step 3, we need to show that $m = 0$. Arguing by contradiction, suppose that $j \neq 0$ in (3.11). Then,

$$g \left( a(x) \right) \frac{a(x)}{\|W\|_{L^\infty(\mathbb{R}^N)}} \left( a(x) \right)^{-\frac{N+2s}{2}} \|W\|_{L^\infty(\mathbb{R}^N)} = a(x)^{-\frac{N+2s}{2}} \|W\|_{L^\infty(\mathbb{R}^N)} - (2s - 1) \|a\|_{L^\infty(\mathbb{R}^N)} a(x)^{-\frac{N+2s}{2}} \|W\|_{L^\infty(\mathbb{R}^N)}^2 < 0. \quad (3.12)$$

From Proposition 2.1 we also have

$$c_0 = I_{a,f}(u_0) \to I_{a,f}(u_0) + \sum_{j=1}^m a(x_j)^{-\frac{N+2s}{2}} I_{1,0}(W).$$

As $a > 0$ and in force of (2.25), from the above expression we obtain $I_{a,f}(u_0) < c_0$. This in turn yields $u_0 \not\in U_1$ and

$$g(u_0) \leq 0. \quad (3.13)$$
Next, we evaluate\( g\left(u_0 + \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right) \). Since\( u_n \in U_1 \), we have\( g(u_n) \geq 0 \). Therefore, the uniform continuity of\( g \) and (3.11) give

\[
0 \leq \liminf_{n \to \infty} g(u_n) = \liminf_{n \to \infty} g\left(u_0 + \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right).
\]

We also note that if\( u_0 \neq 0 \) then using Remark 3.1, we can say that\( u_0 \) is nonnegative. Therefore,

\[
g\left(u_0 + \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right) \leq \|u_0\|_{H^s(\mathbb{R}^N)}^2 + \| \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \|_{H^s(\mathbb{R}^N)}^2
\]

\[
+ 2 \left\langle u_0, \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)}
\]

\[
- (2^*_s - 1) \|a\|_{L^\infty(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |u_0|^2 dx \right)
\]

\[
+ \int_{\mathbb{R}^N} \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) |x|^2 dx
\]

\[
= g(u_0) + g\left( \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right)
\]

\[
+ 2 \left\langle u_0, \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)}
\]

Similarly, it can be also shown that

\[
g\left( \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right) \leq \sum_{j=1}^{m} g\left(a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right)
\]

\[
+ 2 \sum_{i,j=1}^{m} \left\langle a(x^i) - \frac{N-2s}{4s} \left( W_{r_n^i,x_n^i}^s \right), a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)}
\]

Substituting this into (3.15) yields

\[
g\left(u_0 + \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right) \leq g(u_0) + \sum_{j=1}^{m} g\left(a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right)
\]

\[
+ 2 \sum_{i,j=1}^{m} \left\langle a(x^i) - \frac{N-2s}{4s} \left( W_{r_n^i,x_n^i}^s \right), a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)}
\]

\[
+ 2 \left\langle u_0, \sum_{j=1}^{m} a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)}
\]

Claim:

(i) \( \left\langle u_0, a(x^j) - \frac{N-2s}{4s} \left( W_{r_n^j,x_n^j}^s \right) \right\rangle_{H^s(\mathbb{R}^N)} = o(1) \).

(ii) \( \left\langle a(x^i) - \frac{N-2s}{4s} W_{r_n^i,x_n^i}^s, a(x^j) - \frac{N-2s}{4s} W_{r_n^j,x_n^j}^s \right\rangle_{H^s(\mathbb{R}^N)} = o(1) \).

To prove (i), first we define \( u_0^n(x) := \left( r_n^j \right)^{\frac{N-2s}{4s}} u_0(x_n^j + r_n^j x) \). As \( r_n^j \to 0 \) and \( u_0 \in \dot{H}^s(\mathbb{R}^N) \), it is easy to see that \( u_0^n \to 0 \) in \( \dot{H}^s(\mathbb{R}^N) \). Thus,

\[
\left\langle u_0, a(x^j) - \frac{N-2s}{4s} W_{r_n^j,x_n^j}^s \right\rangle_{H^s(\mathbb{R}^N)}
\]
\[ a(x^j)^{-\frac{N+s}{4}} (r_n^j)^{-\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} \frac{(u_0(x) - u_0(y))(W(x) - W(y))}{|x - y|^{N+2s}} \, dx \, dy \]

\[ = a(x^j)^{-\frac{N+s}{4}} (r_n^j)^{-\frac{N+2s}{2}} \int_{\mathbb{R}^{2N}} (u_0(x_n^j + r_n^j x) - u_0(x_n^j + r_n^j y))(W(x) - W(y)) \, dx \, dy \]

\[ = a(x^j)^{-\frac{N+s}{4}} \langle u_0^N, W \rangle_{H^s(\mathbb{R}^N)} \]

\[ = o(1). \]

Similarly,

\[ \left\langle a(x^j)^{-\frac{N+s}{4}} (W^{j}_{x_n^j, x_n^j}), a(x^j)^{-\frac{N+s}{4}} (W^{j}_{x_n^j, x_n^j}) \right\rangle_{H^s(\mathbb{R}^N)} \]

\[ = a(x^j)^{-\frac{N+s}{4}} a(x^j)^{-\frac{N+s}{4}} (r_n^j)^{-\frac{N+2s}{2}} (r_n^j)^{-\frac{N+2s}{2}} \times \int_{\mathbb{R}^{2N}} (W(x) - W(y)) (W(x) - W(y)) (W(x) - W(y)) \, dx \, dy \]

\[ = a(x^j)^{-\frac{N+s}{4}} a(x^j)^{-\frac{N+s}{4}} \langle W, W \rangle_{H^s(\mathbb{R}^N)}, \]

where \( W_n := (\frac{r_n^j}{r_k^j})^{\frac{N+s}{4}} W(\frac{x_n^j}{r_n^j} + \frac{x_n^j}{r_k^j}) \). Further, we observe that using the following

\[ \log(\frac{r_n^j}{r_k^j}) \rightarrow \infty \]

from Proposition 3.1 it is easy to see that \( W_n \rightarrow 0 \) in \( H^s(\mathbb{R}^N) \). Hence Claim (ii) follows.

A combination of the above claim along with 3.12 and 3.13 contradicts 3.14. Therefore, \( j = 0 \) in 3.11. Hence, \( u_n \rightarrow u_0 \) in \( H^s(\mathbb{R}^N) \). Consequently, \( g(u_n) \rightarrow g(u_0) \), which in turn implies \( u_0 \in U_1 \). But, since \( c_0 < c_1 \), we can conclude \( u_0 \in U_1 \). Thus Step 3 follows.

**Step 4:** From the previous steps we conclude that \( I_{a,f}(u_0) = c_0 \) and \( I^*_{a,f}(u_0) = 0 \). Therefore, \( u_0 \) is a weak solution to 3.2. Combining this with Remark 3.1 we conclude the proof of the proposition. \( \square \)

**Proposition 3.2.** Assume 3.3 holds and either \( a \equiv 1 \) or \( \|a\|_{L^{\infty}(\mathbb{R}^N)} \geq \alpha(N,s) \), where \( \alpha(N,s) \) is the second zero of the function 1.5. Then \( I_{a,f} \) has a second critical point \( v_0 \neq u_0 \). In particular, \( v_0 \) is a positive solution to 2.

**Proof.** Let \( u_0 \) be the critical point obtained in Proposition 3.1 and \( W \) be the unique positive solution of 1.2. Set, \( w_t(x) := W(\frac{x}{t}) \) and let \( \tilde{x}_0 \in \mathbb{R}^N \) such that \( a(\tilde{x}_0) = \|a\|_{L^{\infty}(\mathbb{R}^N)} \).

**Claim 1:** \( u_0 + a(\tilde{x}_0)^{-\frac{N+s}{4}} w_t \in U_2 \) for \( t > 0 \) large enough.

Indeed, as \( \|a\|_{L^{\infty}(\mathbb{R}^N)} \geq 1 \) and \( u_0, w_t > 0 \), using Young inequality with \( \varepsilon > 0 \), we obtain

\[ g(u_0 + a(\tilde{x}_0)^{-\frac{N+s}{4}} w_t) \leq \|u_0\|_{H^s(\mathbb{R}^N)}^2 + a(\tilde{x}_0)^{-\frac{N+s}{4}} \|w_t\|_{H^s(\mathbb{R}^N)}^2 + 2a(\tilde{x}_0)^{-\frac{N+s}{4}} \langle u_0, w_t \rangle_{H^s(\mathbb{R}^N)} \]

\[ - (2^*_s - 1)(\|u_0\|_{H^s(\mathbb{R}^N)}^2 + a(\tilde{x}_0)^{-\frac{N+s}{4}} \|w_t\|_{H^s(\mathbb{R}^N)}^2 \]

\[ - (2^*_s - 1)(\|u_0\|_{H^s(\mathbb{R}^N)}^2 + a(\tilde{x}_0)^{-\frac{N+s}{4}} \|w_t\|_{H^s(\mathbb{R}^N)}^2 \]

\[ = (1 + C(\varepsilon)) \|u_0\|_{H^s(\mathbb{R}^N)}^2 - (2^*_s - 1)(\|u_0\|_{H^s(\mathbb{R}^N)}^2 \]

\[ + \|W\|_{H^s(\mathbb{R}^N)}^2 (1 + C(\varepsilon))(1 + C(\varepsilon)) \|u_0\|_{H^s(\mathbb{R}^N)}^2 - (2^*_s - 1) a(\tilde{x}_0)^{-\frac{N+s}{4}} l^N \]

\[ \leq 0 \] for \( t > 0 \) large enough.

Therefore, \( g(u_0 + a(\tilde{x}_0)^{-\frac{N+s}{4}} w_t) < 0 \) for \( t \) large enough. Hence the claim follows.

**Remark 3.1.**
Claim 2: $I_{a,f}(u_0 + a(\bar{x}_0) - \frac{N-2}{4} w_t) < I_{a,f}(u_0) + I_{1,0} \left( a(\bar{x}_0) - \frac{N-2}{4} w_t \right), \forall t > 0$.

Indeed, since $u_0$, $w_t > 0$, taking $a(\bar{x}_0) - \frac{N-2}{4} w_t$ as the test function for (3.17) yields

$$\langle u_0, a(\bar{x}_0) - \frac{N-2}{4} w_t \rangle_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} a(x) a(\bar{x}_0) - \frac{N-2}{4} w_t^2 dx + a(\bar{x}_0) - \frac{N-2}{4} (f, w_t)_{H^s}$$

Consequently, using the above expression and the fact that $a \geq 1$, we obtain

$$I_{a,f}(u_0 + a(\bar{x}_0) - \frac{N-2}{4} w_t) = \frac{1}{2} \|u_0\|_{H^s(\mathbb{R}^N)}^2 + \frac{a(\bar{x}_0)}{2} \|w_t\|_{H^s(\mathbb{R}^N)}^2 + a(\bar{x}_0) - \frac{N-2}{4} \langle u_0, w_t \rangle_{H^s(\mathbb{R}^N)}$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} a(x) a(\bar{x}_0) - \frac{N-2}{4} w_t^2 dx - (\langle f, w_t \rangle_{H^s})$$

$$= I_{a,f}(u_0) + I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t) + a(\bar{x}_0) - \frac{N-2}{4} \langle u_0, w_t \rangle_{H^s(\mathbb{R}^N)}$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} a(x) u_0^2 dx + \frac{a(\bar{x}_0)}{2} \int_{\mathbb{R}^N} w_t^2 dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} a(x) u_0^2 dx - a(\bar{x}_0) - \frac{N-2}{4} (f, w_t)_{H^s}$$

$$\leq I_{a,f}(u_0) + I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t) + \frac{1}{2} \int_{\mathbb{R}^N} a(x) \left[ 2s \right]_a a(\bar{x}_0) - \frac{N-2}{4} w_t^2 dx - (u_0 + a(\bar{x}_0) - \frac{N-2}{4} w_t) tw_t$$

$$< I_{a,f}(u_0) + I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t).$$

Hence the Claim follows.

A direct computation shows that

$$\lim_{t \to \infty} I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t) = -\infty \quad (3.17)$$

From (3.17) and using the relation

$$\|w_t\|_{H^s(\mathbb{R}^N)}^2 = t^{N-2s}\|W\|_{H^s(\mathbb{R}^N)}^2, \quad \|w_t\|_{L_2^2(\mathbb{R}^N)}^2 = t^N\|W\|_{H^s(\mathbb{R}^N)}^2,$$

a straightforward computation yields that

$$\sup_{t > 0} I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t) = I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_{t_{\max}}), \text{ where } t_{\max} = a(\bar{x}_0)^{-\frac{1}{s}}.$$

Therefore, substituting the value of $t_{\max}$ in the definition of $I_{a,f}$, it is not difficult to check that

$$\sup_{t > 0} I_{1,0}(a(\bar{x}_0) - \frac{N-2}{4} w_t) = a(\bar{x}_0) - \frac{N-2}{4} \left( a(\bar{x}_0)^2 - \frac{1}{2s} \right) \|W\|_{H^s(\mathbb{R}^N)}^2.$$
Hence, using the hypothesis of Proposition 3.2, we have that
\[ \max_{t \in [0,1]} I_{a,f}(i(t)) \geq I_{a,f}(i(t_i)) \geq \inf_{U} I_{a,f}(u) = c_1. \]
Thus, \( \gamma \geq c_1 \geq c_0 = I_{a,f}(u_0) \). Here in the last inequality we have used Lemma 3.2.

**Claim 3:** \( \gamma < I_{a,f}(u_0) + a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)}. \)

It is easy to see that \( \lim_{t \to 0} \| w_t \|_{H^s(\mathbb{R}^N)} = 0 \). Thus, if we define \( \bar{t}(t) = u_0 + a(\bar{x}_0) - \frac{s}{N} \| w_{\bar{t}_0} \|^2, \) then \( \lim_{t \to 0} \| \bar{t}(t) - u_0 \|_{H^s(\mathbb{R}^N)} = 0 \). Consequently, \( \bar{t} \in \Gamma \). Therefore, using (3.18), we obtain
\[
\gamma \leq \max_{t \in [0,1]} I_{a,f}(\bar{t}(t)) = \max_{t \in [0,1]} I_{a,f}(u_0 + a(\bar{x}_0) - \frac{s}{N} \| w_{\bar{t}_0} \|^2) < I_{a,f}(u_0) + a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)}. \]
Thus the claim follows.

Hence
\[ I_{a,f}(u_0) < \gamma < I_{a,f}(u_0) + a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)}. \] (3.19)

**Claim 4:** \( a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)} \leq a(\bar{x}_0) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W). \)

Since \( I_{1,0}(W) = \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)}, \) we observe that
\[ a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)} \leq a(\bar{x}_0) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W) \]
if and only if
\[ \frac{s}{N} a(\bar{x}_0) - \frac{a(\bar{x}_0)^2}{2} \leq 1 + \frac{1}{2s} \geq 0. \] (3.20)

Define
\[ \varphi(t) := \frac{s}{N} \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} - t^2. \]

Then an easy analysis shows that \( \varphi(1) = 0 \) and there exists \( \alpha(N,s) > 1 \) such that \( \varphi(t) > 0 \) for all \( t > \alpha(N,s) \), \( \varphi(t) < 0 \) for \( t \in (1, \alpha(N,s)) \).

Therefore, if \( a(\bar{x}_0) = 1 \) (which is equivalent to \( a \equiv 1 \)) or \( \| a \|_{L^\infty(\mathbb{R}^N)} \geq \alpha(N,s) \) then 3.20 holds.

Hence, using the hypothesis of Proposition 3.2, we have
\[ a(\bar{x}_0) - \frac{s}{N} \left( \frac{a(\bar{x}_0)^2}{2} - \frac{1}{2s} \right) \| W \|^2_{H^s(\mathbb{R}^N)} \leq a(\bar{x}_0) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W) \leq a(x) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W), \]
for all \( x \in \mathbb{R}^N \). Substituting this into (3.19), yields
\[ I_{a,f}(u_0) < \gamma < I_{a,f}(u_0) + a(x) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W) \] for all \( x \in \mathbb{R}^N \).

Using Ekeland’s variational principle, there exists a \((PS)\) sequence \((u_n)_n\) for \( I_{a,f} \) at level \( \gamma \). Doing a standard computation yields \((u_n)_n\) is bounded sequence. Further as, \( \gamma < I_{a,f}(u_0) + a(x) - \frac{s}{N} \| W \|^2_{H^s(\mathbb{R}^N)} I_{1,0}(W) \) (for any \( x \in \mathbb{R}^N \)), from Proposition 2.1 we can conclude that \( u_n \to v_0 \), for some \( v_0 \in H^s(\mathbb{R}^N) \) such that \( I'_{a,f}(v_0) = 0 \) and \( I_{a,f}(v_0) = \gamma \). Further, as \( I_{a,f}(u_0) < \gamma \), we conclude \( v_0 \neq u_0 \).

\[ I'_{a,f}(v_0) = 0 \implies v_0 \text{ is a weak solution to (3.2).} \]

Combining this with Remark 3.1 we conclude the proof of the proposition.

**Lemma 3.3.** If \( \| f \|_{(H^s)'} < C_0 S^\frac{4}{N} \), then (3.3) holds.

**Proof.** Using the given hypothesis, we can obtain \( \varepsilon > 0 \) such that \( \| f \|_{(H^s)'} < C_0 S^\frac{4}{N} - \varepsilon \). Therefore, using Lemma 3.1 we have
\[ \langle f, u \rangle_{H^s} \leq \| f \|_{(H^s)'} \| u \|_{H^s(\mathbb{R}^N)} < \left[ C_0 S^\frac{4}{N} - \varepsilon \right] \| u \|_{H^s(\mathbb{R}^N)} \leq \frac{4s}{N + 2s} \| u \|^2_{H^s(\mathbb{R}^N)} - \varepsilon \| u \|_{H^s(\mathbb{R}^N)}, \]
for all \( u \in U \). Therefore,
\[
\frac{4s}{N + 2s} \|u\|_{H^s(\mathbb{R}^N)}^2 - (H^s)'(f, u)_{H^s} > \varepsilon \|u\|_{H^s(\mathbb{R}^N)}
\]
for all \( u \in U \), i.e.,
\[
\inf_U \left[ \frac{4s}{N + 2s} \|u\|_{H^s(\mathbb{R}^N)}^2 - (H^s)'(f, u)_{H^s} \right] \geq \varepsilon \inf_U \|u\|_{H^s(\mathbb{R}^N)}.
\]
Since, by Remark 3.2 we have \( \|u\|_{H^s(\mathbb{R}^N)} \) is bounded away from 0 on \( U \), the above expression implies
\[
\inf_U \left[ \frac{4s}{N + 2s} \|u\|_{H^s(\mathbb{R}^N)}^2 - (H^s)'(f, u)_{H^s} \right] > 0. \tag{3.21}
\]
On the other hand,
\[
\text{(3.1)} \quad \iff \quad C_0 \frac{\|u\|_{H^s(\mathbb{R}^N)}^{N+2s}}{\|u\|_{L^2(\mathbb{R}^N)}^{N+2s}} - (H^s)'(f, u)_{H^s} > 0 \quad \text{for} \quad \|u\|_{L^2(\mathbb{R}^N)} = 1
\]
\[
\iff \quad C_0 \frac{\|u\|_{H^s(\mathbb{R}^N)}^{N+2s}}{\|u\|_{L^2(\mathbb{R}^N)}^{N+2s}} - (H^s)'(f, u)_{H^s} > 0 \quad \text{for} \quad u \in U
\]
\[
\iff \quad \frac{4s}{N + 2s} \|u\|_{H^s(\mathbb{R}^N)}^2 - (H^s)'(f, u)_{H^s} > 0 \quad \text{for} \quad u \in U. \tag{3.22}
\]
Clearly, (3.21) insures RHS of (3.22) holds. Hence the lemma follows. \( \square \)

**Proof of Theorem 1.2 completed:**

Proof. Combining Proposition 3.1 and Proposition 3.2 with Lemma 4.3 we conclude the proof of Theorem 1.2. \( \square \)

4. PROOF OF THEOREM 1.1

Proof. We observe that
\[
I''_{a, f}(u)(h, h) = \|h\|_{H^s(\mathbb{R}^N)}^2 - (2s - 1) \int_{\mathbb{R}^N} a(x)u_+^{2s - 2}h^2 \, dx \tag{4.1}
\]
Since \( a \in L^{\infty}(\mathbb{R}^N) \), using Hölder inequality and Sobolev inequality, we estimate the second term on the RHS as follows
\[
\int_{\mathbb{R}^N} a(x)u_+^{2s - 2}h^2 \, dx \leq \|a\|_{L^{\infty}(\mathbb{R}^N)} \left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right)^{\frac{2}{2s}} \left( \int_{\mathbb{R}^N} |h|^{2s} \, dx \right)^{\frac{2}{2s}}
\]
\[
\leq \|a\|_{L^{\infty}(\mathbb{R}^N)} S^{-\frac{2}{2s}} \|u\|_{L^2(\mathbb{R}^N)}^{2s - 2} \|h\|_{H^s(\mathbb{R}^N)}^2.
\]
Thus substituting the above in (4.1) we obtain
\[
I''_{a, f}(u)(h, h) \geq \left( 1 - (2s - 1)\|a\|_{L^{\infty}(\mathbb{R}^N)} S^{-\frac{2}{2s}} \|u\|_{L^2(\mathbb{R}^N)}^{2s - 2} \|h\|_{H^s(\mathbb{R}^N)}^2 \right) \|h\|_{H^s(\mathbb{R}^N)}^2.
\]
Therefore, \( I''_{a, f}(u) \) is positive definite for \( u \in B(r_1) \), with \( r_1 = \left( (2s - 1)\|a\|_{L^{\infty}(\mathbb{R}^N)} S^{-\frac{2}{2s}} \right)^{-\frac{1}{2s - 2}} S^{\frac{2}{2s}} \) and hence \( I_{a, f} \) is strictly convex in \( B(r_1) \).

For \( \|u\|_{H^s(\mathbb{R}^N)} = r_1 \),
\[
I_{a, f}(u) = \frac{1}{2} \|u\|_{H^s(\mathbb{R}^N)}^2 - \frac{1}{2s} \int_{\mathbb{R}^N} a(x)u_+^{2s} \, dx - (H^s)'(f, u)_{H^s}.
\]
Since, \( r_1^{2r^* - 2} = (\frac{1}{(2r^*-1)c_0})\frac{1}{\|a\|_{L^\infty(\mathbb{R}^N)}}S_2 \), we obtain
\[
I_{a,f}(u) \geq \left( \frac{1}{2} - \frac{1}{2c_0(2r^*-1)} \right) r_1^2 - r_1\|f\|_{(H^r)^*}.
\]

Thus there exists \( d > 0 \) such that
\[
\inf_{\|u\|_{H^r(\mathbb{R}^N)} = r_1} I_{a,f}(u) > 0, \quad \text{provided that} \quad 0 < \|f\|_{(H^r)^*} \leq d.
\]

Since \( I_{a,f} \) is strictly convex in \( B(r_1) \) and \( \inf_{\|u\|_{H^r(\mathbb{R}^N)} = r_1} I_{a,f}(u) > 0 = I_{a,f}(0) \), there exists a unique critical point \( u_0 \) of \( I_{a,f} \) in \( B(r_1) \) and it satisfies
\[
I_{a,f}(u_0) = \inf_{\|u\|_{H^r(\mathbb{R}^N)} < r_1} I_{a,f}(u) < I_{a,f}(0), \tag{4.2}
\]
where the last inequality is due to the strict convexity of \( I_{a,f} \) in \( B(r_1) \). Combining this with Remark [5.1] we conclude the proof of the theorem. \( \square \)

**Appendix A. Morrey space**

We recall the definition of the homogeneous Morrey spaces \( L^{r,\gamma}(\mathbb{R}^N) \), introduced by Morrey as a refinement homogeneous of the usual Lebesgue spaces. A measurable function \( u : \mathbb{R}^N \to \mathbb{R} \) belongs to the Morrey space \( L^{r,\gamma}(\mathbb{R}^N) \), with \( r \in [1, \infty) \) and \( \gamma \in [0, N] \) if and only if
\[
\|u\|_{L^{r,\gamma}(\mathbb{R}^N)} := \sup_{R > 0, x \in \mathbb{R}^N} R^{\gamma} \int_{B(x, R)} |u|^r dy.
\]  

(A.1)

From the above definition, it is clear that if \( \gamma = N \) then \( L^{r,N}(\mathbb{R}^N) \) coincides with usual Lebesgue space \( L^r(\mathbb{R}^N) \) for any \( r \geq 1 \) and similarly \( L^{r,0}(\mathbb{R}^N) \) coincides with \( L^\infty(\mathbb{R}^N) \). It is interesting to note that \( L^{r,\gamma} \) experiences same translation and dilation invariance as in \( L^2 \) and therefore of \( \dot{H}^s(\mathbb{R}^N) \) if \( \frac{1}{s} = \frac{N-2r^*}{2r^*} \). Let \( (u)^{r_0,r} \) be the function defined by (2.2). By change of variable formula, one can see that the following equality holds
\[
\|u\|_{L^{r,\gamma}(\mathbb{R}^N)} \equiv \|u\|_{L^{r,\gamma}(\mathbb{R}^N)} = \|u\|_{L^{r,\gamma}(\mathbb{R}^N)},
\]
for any \( r \in [1, 2r^*] \). Next, we recall a result from [14].

**Lemma A.1.** [14] Theorem 1 For any \( 0 < s < N/2 \) there exists a constant \( C \) depending only on \( N \) and \( s \) such that, for any \( 2/2^* \leq \theta < 1 \) and for any \( 1 \leq r < 2^* \),
\[
\|u\|_{L^{2r^*,s}(\mathbb{R}^N)} \leq C\|u\|_{\dot{H}^s(\mathbb{R}^N)}\|u\|^{1-\theta}_{L^{r,s-2/s}(\mathbb{R}^N)}\|u\|^{\theta}_{L^{2r^*,r-2/r}(\mathbb{R}^N)} \quad \text{for all } u \in \dot{H}^s(\mathbb{R}^N).
\]

Note that using the Hölder inequality, we also have \( L^{2r^*,s}(\mathbb{R}^N) \hookrightarrow L^{r,s-2/r}(\mathbb{R}^N) \) is continuous, i.e., there exists a constant \( C = C(N,s) \) such that
\[
\|u\|_{L^{r,s-2/r}(\mathbb{R}^N)} \leq C\|u\|_{L^{2r^*,s}(\mathbb{R}^N)} \quad \text{for all } u \in L^{2r^*,s}(\mathbb{R}^N).
\]  

(A.2)

For more details about the Morrey spaces, we refer to [14].

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