Renormalization Analysis for Degenerate Ground States

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Abstract

We consider a Hamilton operator which describes a finite dimensional quantum mechanical system with degenerate eigenvalues coupled to a field of relativistic bosons. We show that the ground state projection and the ground state energy are analytic functions of the coupling constant in a cone with apex at the origin, provided a mild infrared assumption holds. To show the result operator theoretic renormalization is used and extended to degenerate situations.

1 Introduction

Models of quantum field theory which describe low energy phenomena of quantum mechanical matter interacting with a quantized field of massless particles have been mathematically intensively investigated (see for example [27] and references therein, for an early work see [14]). These models are used to study non relativistic matter interacting with the quantized radiation field or electrons in a solid interacting with a field of phonons. Physical properties such as existence of ground states, dispersion relations, and resonances have been treated mathematically rigorous. In particular, the method of operator theoretic renormalization, introduced by Bach Fröhlich and Sigal [7,8], has been used in the literature to study ground states and resonances [3,4,11,13,16,20,22,25]. However, the application of operator theoretic renormalization usually requires that the unperturbed eigenstate is non degenerate or at least protected by a symmetry. In this paper we extend operator theoretic renormalization to situations where the unperturbed eigenvalue is degenerate and the degeneracy is lifted after the interaction is turned on. We note that degenerate situations do occur in physically realistic models, see for example [1]. To
keep notation simple we treat the ground state. Resonances can be treated by the same ideas as used in this paper with additional notational complexity. This is planned to be addressed in a forthcoming paper.

More precisely, we consider a quantum mechanical atomic system described by a Hamilton operator acting on a so called atomic Hilbert space. For simplicity we assume that the atomic Hilbert space is finite dimensional (we expect that this assumption is not essential and can be relaxed in a straightforward way). Furthermore, we assume that the atomic system interacts with a quantized field of massless bosons by means of a linear coupling. The resulting Hamiltonian describing the total system is also referred to as generalized spin boson Hamiltonian. We assume that the interaction satisfies a mild infrared condition. The infrared condition is needed for the renormalization analysis to converge. It can be shown to include realistic models of non relativistic quantum electrodynamics by means of a so called generalized Pauli Fierz transformation [25]. We assume that the Hamiltonian of the atomic subsystem has a degenerate ground state, which is lifted by formal second order perturbation theory in the coupling constant (first order perturbation theory does not affect the ground state energy for models which we consider). We show that the ground state exists for small values of the coupling constant, a result already known in the literature [15, 17, 23, 26]. Furthermore, we show that the ground state projection as well as the ground state energy are analytic as a function of the coupling constant in an open cone with apex at the origin. This result is new and it is in contrast to non degenerate situations, where it has been shown that the ground state projection and the ground state energy are analytic functions of the coupling constant [16]. We do not assume that this is an artefact of our proof. In fact, we conjecture that in the degenerate case there may be situations in which the ground state projection and possibly the ground state energy are not analytic in a neighborhood of zero. In a related model, where a hydrogen atom is minimally coupled to the quantized electromagnetic field, non analyticity in the fine structure constant has been shown [9].

Although we do not obtain analyticity in a neighborhood of zero, analyticity in a cone is of interest in its own right. It is for example a necessary ingredient to show Borel summability. Borel summability methods allow to recover a function from its asymptotic expansion. An asymptotic expansion may for example be obtained using the techniques employed in [2, 5, 6, 10, 18].

In the following section we state the model and the main result. The subsequent sections are devoted to the proof of the main result.
2 Model and Statement of Results

We consider the following model. Let the atomic Hilbert space be modeled by
\[ \mathcal{H}_{\text{at}} = \mathbb{C}^N \]
and equipped with the standard scalar product. Furthermore we equip \( \mathcal{L}(\mathcal{H}_{\text{at}}) \) with the operator norm, which we denote by \( \| \cdot \| \). Let the Fock space
\[ \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathfrak{f}^\otimes n, \]
with \( \mathfrak{f} = L^2(\mathbb{R}^3 \times \mathbb{Z}_2) \) model the quantized radiation field. We denote the Fock vacuum by \( \Omega \) and the Hilbert space of the total system by
\[ \mathcal{H} := \mathcal{H}_{\text{at}} \otimes \mathcal{F}. \]

We assume that \( \mathcal{H}_{\text{at}} \in \mathcal{L}(\mathcal{H}_{\text{at}}) \) is self adjoint. To simplify our notation we define for \( (k, \lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2 \)
\[ k := (k, \lambda), \quad \int dk := \sum_{\lambda=1,2} \int d^3k, \quad \omega(k) := |k| := |k| \tag{2.1} \]
and denote by \( a^*(k) \) and \( a(k) \) the usual creation and annihilation operator satisfying canonical commutation relations. For formal definitions of the annihilation and creation operator and associated field operators we refer the reader to [B]. For \( G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\text{at}})) \) we define
\[ a(G) := \int G^*(k)a(k)dk, \quad a^*(G) := \int G(k)a^*(k)dk \]
which are densely defined closed linear operators in the Hilbert space. We define the free field operator by
\[ H_f := \int \omega(k)a^*(k)a(k)dk, \tag{2.2} \]
which is defined in the sense of forms. For \( g \in \mathbb{C} \) we shall study the following operator
\[ H_g := H_{\text{at}} + H_f + gW, \tag{2.3} \]
where the so called interaction is given by
\[ W = a^*(\omega^{-1/2}G) + a(\omega^{-1/2}G). \tag{2.4} \]
We note that $W$ is infinitesimally bounded with respect to $H_f$ if $\omega^{-1} G, G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at}))$. Let $\epsilon_{at}$ denote the ground state of $H_{at}$, and let $P_{at}$ denote the projection onto the eigenspace of $H_{at}$ with eigenvalue $\epsilon_{at}$, and let $P_{at}^\perp := 1 - P_{at}$. Define
\[
Z_{at} := -\int \frac{dk}{\omega(k)} P_{at} G^*(k) \left[ \frac{P_{at}}{|k|} + \frac{P_{at}^\perp}{H_{at} - \epsilon_{at} + |k|} \right] G(k) P_{at} \uparrow \text{Ran} P_{at},
\]
which is a selfadjoint mapping on the ground state space of $H_{at}$. For $r > 0$ we denote the open disk in the complex plane by
\[
D_r := \{ z \in \mathbb{C} : |z| < r \}.
\]
In order for the renormalization analysis to be applicable we shall need an infrared condition. For this we define for $\mu > 0$
\[
L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at})) := \{ G : \mathbb{R}^3 \times \mathbb{Z}_2 \to \mathcal{L}(\mathcal{H}_{at}) : G \text{ measurable, } \| G \|_\mu < \infty \},
\]
where we defined
\[
\| G \|_\mu := \int \left( \frac{1}{|k|^{3+2\mu} + 1} \right) \| G(k) \|^2 dk.
\]

**Theorem 2.1.** Let $\mu > 0$. Suppose $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at}))$ and let $H_g$ be given by (2.3). Let $\epsilon_{at}^{(2)}$ denote the smallest eigenvalue of $Z_{at}$. Assume that $\epsilon_{at}^{(2)}$ is simple. Let $0 < \delta_0 < \pi/2$, and let $S_{\delta_0} := \{ z \in \mathbb{C} : |\arg(z)| < \delta_0 \text{ or } |\arg(-z)| < \delta_0 \}$. Then there exists a $g_0 > 0$ such that for all $g \in D_{g_0} \cap S_{\delta_0}$ the operator $H_g$ has an eigenvector $\psi_g$ and an eigenvalue $E_g$ such that
\[
E_g = \epsilon_{at} + g^2 \epsilon_{at}^{(2)} + o(|g|^2).
\]
The eigenvalue and eigenprojection are continuous on $S_{\delta_0} \cap D_{g_0}$ and analytic in the interior of $S_{\delta_0} \cap D_{g_0}$. Furthermore for real $g$ the number $E_g$ is the infimum of the spectrum of $H_g$.

**Remark 2.2.** Let $P_{\Omega}$ denote the projection onto the vacuum vector $\Omega$. We note that
\[
Z_{at} \cong -(P_{at} \otimes P_{\Omega}) W(H_0 - \epsilon_{at})^{-1} W(P_{at} \otimes P_{\Omega}) \uparrow \text{Ran} P_{at} \otimes P_{\Omega},
\]
which is exactly the second order energy correction in formal perturbation theory.

**Remark 2.3.** If $N = \sum_\lambda \int a_\lambda(k)^* a_\lambda(k)$ is the number operator we have the symmetry $(-1)^N H_0 (-1)^N = H_{-g}$. This implies that eigenvalues do not depend on the sign of $g$. And if they happen to have an asymptotic expansion it cannot depend on odd powers of $g$. 

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Remark 2.4. Generically one can assume that either a degeneracy of an eigenvalue remains after the interaction is added (which happens if the degeneracy is protected by a symmetry, e.g., spin degeneracy) or it is lifted at some finite order. In this paper we assume that the degeneracy of the ground state is lifted at second order. We believe that the methods used in this paper are also useful to treat degeneracies which are lifted at higher than second order, by possibly inserting several initial Feshbach maps, with energy cutoffs depending on the coupling constant.

Remark 2.5. Borel summability methods allow in certain situations to recover a function from its asymptotic expansion, provided it satisfies a strong asymptotic condition. Theorem 2.1 together with Remark 2.3 can be used to show that the ground state energy as a function of \( g^2 \) satisfies the analyticity requirement of a strong asymptotic condition [24]. Suppose \( \pi/4 < \delta_0 < \pi/2 \) and \( g_0 \) are as in Theorem 2.1. Define the function \( f(w) := E_\sqrt{w} \) with \( |\arg(w)| < 2\delta_0 \) and \( |\sqrt{w}| < g_0 \). Then \( f \) is analytic in the interior of the cone \( S_{2\delta_0} \cap D_{g_0} \) and extends continuously onto the boundary. Moreover \( f \) satisfies the analyticity requirement for a strong asymptotic condition. Now suppose there were \( C \) and \( \sigma \) such that

\[
\left| f(w) - \sum_{n=0}^{N} c_n w^n \right| \leq C\sigma^{N+1}(N+1)!|w|^{N+1}
\] (2.10)

for all \( N \) and all \( w \in S_{2\delta_0} \cap D_{g_0} \). Then \( E_g = f(g^2) \) could be recovered uniquely by the method of Borel summability, see [24] Watson’s theorem.

The remaining part of the paper is devoted to the proof of Theorem 2.1. First we give an overview of the proof, before we comment on the organization of the paper. As already mentioned in the introduction, the proof is based on an operator theoretic renormalization analysis. Such an analysis is based on the so called smooth Feshbach map and its isospectrality properties. Let us introduce the relevant definitions and properties. For additional details we refer the reader to [3,16].

Suppose \( \chi \) and \( \overline{\chi} \) are commuting, nonzero bounded operators, acting on a separable Hilbert space \( \mathcal{K} \) and satisfying \( \chi^2 + \overline{\chi}^2 = 1 \). We shall refer to \( \chi \) and \( \overline{\chi} \) as smoothed projections. By a Feshbach pair \((H,T)\) for \( \chi \) we mean a pair of closed operators in \( \mathcal{K} \) with the same domain such that the following properties hold:

(i) \( \chi \) and \( \overline{\chi} \) commute with \( T \),

(ii) \( T, H_{\overline{\chi}} := T + \overline{\chi}W\overline{\chi} : D(T) \cap \text{Ran}\overline{\chi} \rightarrow \text{Ran}\overline{\chi} \) are bijections with bounded inverse.

(iii) \( \overline{\chi}H_{\overline{\chi}}^{-1}W\chi : D(T) \rightarrow \mathcal{K} \) is a bounded operator.
Given a Feshbach pair \((H,T)\) for \(\chi\), we call the operator
\[
F_\chi(H,T) := H_\chi - \chi W\chi H_\chi^{-1} \chi W\chi : D(T) \to K
\]
the Feshbach operator. The mapping \((H,T) \mapsto F_\chi(H,T)\) is called Feshbach map. Central for our proof is the isospectrality property of the Feshbach map, stated in the following theorem. To formulate it one introduces the so called auxiliary operator
\[
Q_\chi := \chi - \chi H_\chi^{-1} \chi W\chi.
\]

**Theorem 2.6** ([16]). Let \((H,T)\) be a Feshbach pair for \(\chi\) on a separable Hilbert space \(K\). Then
\[
\chi : \ker H \to \ker F_\chi(H,T)
\]
\[
Q_\chi : \ker F_\chi(H,T) \to \ker H
\]
are linear isomorphisms and inverse to each other.

The renormalization analysis is based on a carefully designed iterated application of the smooth Feshbach map. Before one can start this machinery, we need to control the degeneracy. For this, we perform two so called initial Feshbach maps. The first Feshbach map uses a smoothed projection given as a tensor product. The first factor acts on the atomic Hilbert space and projects orthogonally onto \(1_{H_{\text{at}}} = \epsilon_{\text{at}} \mathcal{H}_{\text{at}}\), the degenerate ground state eigenspace of the atomic Hamiltonian. The second factor acts on Fock space and is given by a smooth cutoff function of the free field energy, such that its range is contained in a spectral subspace of field energies between zero and \(\rho_0\). The second Feshbach map uses a smoothed projection given again as a tensor product. The first factor now projects onto the subspace of \(1_{H_{\text{at}}} = \epsilon_{\text{at}} \mathcal{H}_{\text{at}}\) belonging to the lowest energy eigenvalue of \(Z_{\text{at}}\). The second factor is given again by a smooth cutoff function of the free field energy, with range contained in a spectral subspace of even smaller field energies between zero and \(\rho_0 \rho_1\). This procedure will resolve the degeneracy and hence allows us to initiate the usual renormalization analysis in Fock space as introduced in [3].

In the proof we will choose \(\rho_0\) larger than \(|g|\) but \(\rho_0 \rho_1\) smaller than \(|g|^2\). More precisely, we will show the following. For any \(\rho_0 > 0\) there exists a \(\rho_1 > 0\) and positive numbers \(g_-(\rho_0)\), \(g_+(\rho_0)\) with \(g_-(\rho_0) < g_+(\rho_0)\), uniformly in the model parameters, such that for all coupling constants \(g\) in a sectorial region of the complex plane with
\[
g_-(\rho_0) < |g| < g_+(\rho_0)
\]
(2.11)
both Feshbach maps are isospectral and respect necessary Banach space estimates needed for operator theoretic renormalization to be applicable, see Figure 1. Then for $g$ in a sectorial region of an annulus we shall obtain analyticity of the ground state projection and ground state energy by invoking an analyticity result of Griesemer and Hasler [16]. Moreover, we show that we can choose $g_-(\rho_0)$ such that $g_-(\rho_0) \to 0$ as $\rho_0 \to 0$ (at the same time $g_+(\rho_0) \to 0$ but this is no problem as long as $g_-(\rho_0) < g_+(\rho_0)$). Hence the analyticity in a cone with apex at the origin will follow as $\rho_0$ tends to zero.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{For fixed $\rho_0 > 0$, the ground state projection and the ground state energy are analytic functions of the coupling constant in the shaded region.}
\end{figure}

The remaining part of the paper is organized as follows. In Section 3 we define the first Feshbach map with parameter $\rho_0$ and prove the Feshbach pair criterion, which establishes isospectrality. In Section 4 we define Banach spaces of matrix valued integral kernels. These Banach spaces parametrize subspaces of operators in Fock space and allow to control the flow of the operator theoretic renormalization analysis. Thereby we extend the Banach spaces of integral kernels introduced in [4] to include matrix valued integral kernels, in order to adapt degenerate situations. In Section 5 we introduce polydiscs around the free field energy in a subspace parametrized by a Banach space of integral
kernels, \((5.2)\). We then show in Theorem 5.1 that the first Feshbach operator lies in such a polydisc, where the radii of the polydisc can be made arbitrarily small for sufficiently small \(|g|\). In Section 6 we define the second Feshbach map with parameter \(\rho_1\), which will be responsible for lifting the degeneracy. We establish necessary estimates, which will be used to show the Feshbach pair criterion for the second step. We note that for these estimates, in particular Lemma 6.3, we need the coupling constant to lie in a cone. We conclude the section with an abstract Feshbach pair criterion. In Section 7 we prove an abstract Banach space estimate for the second Feshbach step, which will be used to show that the second Feshbach operator lies in a suitable polydisc around the free field energy. Section 8 is devoted to the proof of the main result Theorem 2.1. Having the abstract results of the previous two sections at hand, we will finally choose \(\rho_1\) as a function of \(\rho_0\), such that the second Feshbach map satisfies indeed the Feshbach pair criterion, establishing isospectrality, and such that the second Feshbach operator lies in a suitable polydisc, allowing the application of the analyticity result \([16]\). In particular, we rigorously justify (see \((8.4)\)) the situation outlined in Figure 1 together with \((2.11)\).

3 Initial Feshbach Step

Without loss we assume that the distance of \(\epsilon_{at}\) from the rest of the spectrum of \(H_{at}\) is 1, i.e.,

\[
d_{at} := \inf \sigma(H_{at} \setminus \{\epsilon_{at}\}) - \epsilon_{at} = 1. \tag{3.1}
\]

This can always be achieved by a suitable scaling. In this section we define the initial Feshbach operator. For definitions and conventions we refer to \([3, 16]\) and collect for the convenience of the reader basic notions in the appendix. We will show for \(z\) in a neighborhood of \(\epsilon_{at}\) and for \(|g|\) sufficiently small, that \(H_g - z\) and \(H_0 - z\) are a Feshbach pair for a generalized projection, which we will now introduce. We fix two functions \(\chi\) and \(\overline{\chi}\) in \(C^\infty(\mathbb{R}; [0, 1])\) satisfying the following properties:

(i) \(\chi(t) = 1\), if \(t \leq 3/4\), and \(\chi(t) = 0\), if \(t \geq 1\),

(ii) \(\chi^2 + \overline{\chi}^2 = 1\).

For \(\rho > 0\) we define the following operator valued functions

\[
\chi^{(0)}(r) := P_{at} \otimes \chi(r/\rho)
\]

\[
\overline{\chi}^{(0)}(r) := P_{at} \otimes 1 + P_{at} \otimes \overline{\chi}(r/\rho)
\]
and we define by means of the spectral theorem the linear operators in $L(H)$
\[
\chi^{(0)}_\rho := \chi^{(0)}_\rho(H_f),
\]
\[
\bar{\chi}^{(0)}_\rho := \bar{\chi}^{(0)}_\rho(H_f).
\]
It is easily verified that $(\chi^{(0)}_\rho)^2 + (\bar{\chi}^{(0)}_\rho)^2 = 1$.

The following proposition provides us with conditions for which the initial Feshbach operator will be defined.

**Theorem 3.1 (Feshbach Pair Criterion for 1st Iteration).** Let $\omega^{-1/2}G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at}))$, $0 < \rho \leq \frac{1}{4}$ and $z \in D_{\rho/2}(\epsilon_{at})$. The operators $H_g - z$ and $H_0 - z$ are a Feshbach pair for $\chi^{(0)}_\rho$, if
\[
|g| < \frac{\rho^{1/2}}{10\|\omega^{-1/2}G\|}. \tag{3.2}
\]
Furthermore one has the absolutely convergent expansion
\[
F_{\chi^{(0)}_\rho}(H_g - z, H_0 - z) = \epsilon_{at} - z + H_f + \sum_{L=1}^{\infty} (-1)^{L-1} \chi^{(0)}_\rho g W \left( \frac{(\chi^{(0)}_\rho)^2}{H_0 - z} g W \right)^{L-1} \chi^{(0)}_\rho. \tag{3.3}
\]

Let us now provide proof of Theorem 3.1. For this we shall make use of the following two lemmas.

**Lemma 3.2.** Let $\rho \geq 0$. Then for $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at}))$ we have
\[
\|(H_f + \rho)^{-1/2}[a(G) + a^*(G)](H_f + \rho)^{-1/2}\| \leq 2\|\omega^{-1/2}G\|\rho^{-1/2}
\]

**Proof.** This follows from (A.2). For the annihilation operator we find for $r \geq 0$
\[
\|(H_f + r)^{-1/2}a(G)(H_f + r)^{-1/2}\|
\leq \|(H_f + r)^{-1/2}\|\|a(G)H_f^{-1/2}\|\|H_f^{1/2}(H_f + r)^{-1/2}\|
\leq r^{-1/2}\|\omega^{-1/2}G\|,
\]
and likewise we estimate the corresponding expression involving a creation operator. \(\square\)

**Lemma 3.3.** Let $0 < \rho \leq \frac{1}{4}$ and $z \in D_{\rho/2}(\epsilon_{at})$. The operator $H_0 - z$ is invertible on the range of $\bar{\chi}^{(0)}_\rho$ and we have the bound
\[
\|(H_0 - z)^{-1} \upharpoonright \text{Ran}\bar{\chi}^{(0)}_\rho\| \leq \frac{4}{\rho} \tag{3.4}
\]
and for all $\tau \geq 0$ the bound
\[
\|(H_f + \tau)^{1/2}(H_0 - z)^{-1}(H_f + \tau)^{1/2} \upharpoonright \text{Ran}\bar{\chi}^{(0)}_\rho\| \leq 1 + \frac{4\tau}{\rho}. \tag{3.5}
\]
Proof. First we show that $H_0 - z$ is bounded invertible on the range of $\Xi^{(0)}_\rho$. It follows for a normalized $\psi \in \text{Ran}(P_{\text{at}} \otimes \Xi(H_f/\rho))$ that

$$
\| (H_0 - z) \psi \| \geq \inf_{r \geq \frac{3}{4} \rho} |\epsilon_{\text{at}} + r - z| \geq (\frac{3}{4} - 1/2)\rho = \frac{1}{4} \rho,
$$

and it follows from (3.1) that for a normalized $\psi \in \text{Ran}(P_{\text{at}} \otimes 1)$ we have

$$
\| (H_0 - z) \psi \| \geq \inf_{r \geq 0} \| (H_{\text{at}} P_{\text{at}} + r - z) \psi \|
\geq \inf_{r \geq 0} \| (H_{\text{at}} P_{\text{at}} - \epsilon_{\text{at}} + r) \psi \| - |z - \epsilon_{\text{at}}|
\geq 1 - \rho/2.
$$

Thus from the above two inequalities it follows that $H_0 - z$ is bounded invertible on the range of $\Xi^{(0)}_\rho$ and that (3.4) holds. We estimate further,

$$
\| (H_f + \tau)^{1/2} (H_0 - z)^{-1} (H_f + \tau)^{1/2} \upharpoonright \text{Ran}(P_{\text{at}} \otimes \Xi(H_f/\rho)) \|
= \sup_{r \geq \frac{3}{4} \rho} \left| \frac{r + \tau}{\epsilon_{\text{at}} + r - z} \right| \leq 1 + \left| \frac{\tau}{(3/4 - 1/2)\rho} \right| \leq 1 + \frac{4\tau}{\rho}
$$

and if we set $E_1 := \inf \sigma(H_{\text{at}}) \setminus \{\epsilon_{\text{at}}\}$

$$
\| (H_f + \tau)^{1/2} (H_0 - z)^{-1} (H_f + \tau)^{1/2} \upharpoonright \text{Ran}(P_{\text{at}} \otimes 1) \|
= \sup_{r \geq 0} \sup_{\lambda \in \sigma(H_{\text{at}}) \setminus \{\epsilon_{\text{at}}\}} \left| \frac{r + \tau}{\lambda + r - z} \right|
\leq \sup_{r \geq 0} \left| \frac{r + \tau}{E_1 - \epsilon_{\text{at}} - \rho/2 + r} \right|
\leq 1 + \frac{\tau}{1 - \rho/2}.
$$

Now the above two inequalities show (3.5). \hfill \square

Proof of Theorem 3.1. To prove the invertibility, we note that $H_f$ and $H_0$ leave the range of $\Xi^{(0)}_\rho$ invariant and that $(H_f + \tau)^{1/2}$ is bounded invertible on the range of $\Xi^{(0)}_\rho$. We use Lemmas 3.3 and 3.2. We write

$$
A(z, \tau) := (H_f + \tau)^{-1/2} (H_0 - z) (H_f + \tau)^{-1/2}
$$

and

$$
B(z, \tau, \rho) := (H_f + \tau)^{-1/2} \Xi^{(0)}_\rho W \Xi^{(0)}_\rho (H_f + \tau)^{-1/2}
$$

and we use the following identity
\[(H_0 - z + g\overline{\chi}_\rho^{(0)}W\overline{\chi}_\rho^{(0)}) \upharpoonright \text{Ran}\overline{\chi}_\rho^{(0)} = (H_f + \tau)^{1/2}[A(z, \tau) + gB(z, \tau, \rho)](H_f + \tau)^{1/2} \upharpoonright \text{Ran}\overline{\chi}_\rho^{(0)} = (H_f + \tau)^{1/2}A(z, \tau)[1 + gA(z, \tau)^{-1}B(z, \tau, \rho)](H_f + \tau)^{1/2} \upharpoonright \text{Ran}\overline{\chi}_\rho^{(0)}].

Thus the bounded invertibility follows from Neumanns theorem provided
\[\|gA(z, \tau)^{-1}B(z, \tau, \rho)\| < 1.\]

Now using Lemma 3.2 note that \(\overline{\chi}_\rho^{(0)}\) commutes with \(H_f\), and (3.5) of Lemma 3.3 we obtain
\[\|gA(z, \tau)^{-1}B(z, \tau, \rho)\| \leq |g| \left(1 + \frac{4\tau}{\rho}\right)2\|\omega^{-1}G\|\tau^{-1/2}.\]
Thus, if we choose \(\tau = \rho\), then the proposition follows.

4 Banach Space of Integral Kernels

To control the renormalization transformation, in particular proving its convergence, we introduce the following Banach spaces of integral kernels. We note that the choice of these spaces is not unique. We choose the Banach spaces such that they are suitable for the model which we consider. Specifically, the Banach spaces which we introduce below are a straight forward generalization of the spaces defined in [3] or [16] to matrix valued integral kernels. A generalization to matrix valued integral kernels seems to be a canonical choice to accommodate degenerate situations.

For \(d \in \mathbb{N}\) we define the Banach space \(\mathcal{W}_{0,0}^{[d]}\) as the space of continuously differentiable matrix valued functions
\[\mathcal{W}_{0,0}^{[d]} := C^1([0, 1]; \mathcal{L}(\mathbb{C}^d))\]
with norm
\[\|w\|_{C^1} := \|w\|_\infty + \|w\|_\infty',\]
where \((\cdot)'\) stands for the derivative. Let \(B_1 := \{\mathbf{k} \in \mathbb{R}^3 : |\mathbf{k}| \leq 1\}\). For a set \(A \subset \mathbb{R}^3\) we write
\[A := A \times \{1, 2\}, \quad \int_A d\mathbf{k} := \sum_{\lambda=1,2} \int_A d^3 \mathbf{k},\]
where we recall the notation of Eq. (2.1). For \(m, n \in \mathbb{N}\) we write
\[k^{(m)} := (k_1, \ldots, k_m) \in (\mathbb{R}^3 \times \{1, 2\})^m,\]
\[\tilde{k}^{(n)} := (\tilde{k}_1, \ldots, \tilde{k}_n) \in (\mathbb{R}^3 \times \{1, 2\})^n,\]
\[K^{(m,n)} := (k^{(m)}, \tilde{k}^{(n)})\]
\[ dk^{(m)} := dk_1 \cdots dk_m, \]
\[ d\bar{k}^{(n)} := d\bar{k}_1 \cdots d\bar{k}_n \]
\[ dK^{(m,n)} := dk^{(m)}d\bar{k}^{(n)} \]

\[ |k^{(m)}| := |k_1| \cdots |k_m|, \]
\[ |\bar{k}^{(n)}| := |\bar{k}_1| \cdots |\bar{k}_n| \]
\[ |K^{(m,n)}| := |k^{(m)}||\bar{k}^{(n)}|. \]

Moreover, we shall use
\[
\Sigma[k^{(n)}] := \sum_{i=1}^{n} |k_i|, \quad \Sigma[\bar{k}^{(m)}] := \sum_{i=1}^{m} |\bar{k}_i|.
\]

For \( m, n \in \mathbb{N} \) with \( m+n \geq 1 \) and \( \mu > 0 \) let \( \mathcal{W}^{[d]}_{m,n} \) denote the space of measurable functions \( w_{m,n} : B^{m+n}_1 \to \mathcal{W}^{[d]}_{0,0} \) satisfying the following three properties.

(i) \( w_{m,n} \) are symmetric with respect to all permutations of the \( m \) arguments from \( B^m_1 \) and the \( n \) arguments from \( B^n_1 \), respectively;

(ii) for \( m+n \geq 1 \), we have \( w_{m,n}(k^{(m)}, \bar{k}^{(n)})(r) = 0 \), provided \( r \geq 1 - \max(\Sigma[k^{(m)}], \Sigma[\bar{k}^{(n)}]) \);

(iii) The following norm is finite
\[
\|w_{m,n}\|_{\#}^{\#} := \|w_{m,n}\|_{\mu} + \|\partial_r w_{m,n}\|_{\mu},
\]
where
\[
\|w_{m,n}\|_{\mu} := \left( \int_{B^{m+n}_1} \left\| (K^{(m,n)}) \right\|^2_{\infty} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} \right)^{1/2}.
\]

**Remark 4.1.** We note that \( \mathcal{W}^{[d]}_{m,n} \) is given as the subspace of
\[
L^2 \left( B^{m+n}_2, \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} ; \mathcal{W}^{[d]}_{0,0} \right),
\]
consisting of elements satisfying Conditions (i) and (ii) above. Note that the norm \( \| \cdot \|_{\#}^{\#} \) is equivalent to the natural norm of \( (4.1) \) (given by the theory of Banach space valued \( L^p \)-functions), which is the norm chosen in \[16\]. Moreover, we will identify \( (4.1) \) as a subspace of \( L^2 \left( [0,1] \times B^{m+n}_2, \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} ; \mathcal{L}(\mathbb{C}^d) \right) \) by means of
\[
w_{m,n}(r, k^{(m)}, \bar{k}^{(n)}) = w_{m,n}(k^{(m)}, \bar{k}^{(n)})(r). \]

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Henceforth we shall use this identification without comment. Furthermore, we remark that
\[ \|w_{0,0}\|_{C^1} = \|w_{0,0}\|^{\#}_{\mu} \]
with the natural convention that for \(m = n = 0\) the empty Cartesian product consists of a single point and that there is no integration in that case.

For given \(\xi \in (0,1)\) and \(\mu > 0\) we define the Banach space
\[ \mathcal{W}_{\xi}^{[d]} := \bigoplus_{m,n \in \mathbb{N}_0} \mathcal{W}_{m,n}^{[d]} \]
with norm
\[ \|w\|^{\#}_{\mu,\xi} := \|w_{0,0}\|_{C^1} + \sum_{m,n \geq 1} \xi^{-(m+n)} \|w_{m,n}\|^{\#}_{\mu}, \]
for \(w = (w_{m,n})_{m,n \in \mathbb{N}_0} \in \mathcal{W}_{\xi}^{[d]}\).

Next we define a linear mapping \(H : \mathcal{W}_{\xi}^{[d]} \to \mathcal{L}(\mathcal{H}_{\text{red}})\), where \(\mathcal{H}_{\text{red}} := P_{\text{red}}\mathcal{H}\) and \(P_{\text{red}} := 1_{[0,1]}(H_f)\). For this we will use the notation
\[ a^*(k^{(m)}) := \prod_{i=1}^{m} a^*(k_i), \quad a(\tilde{k}^{(n)}) := \prod_{i=1}^{n} a(\tilde{k}_i). \]

If \(w \in \mathcal{W}_{0,0}^{[d]}\) define \(H_{0,0}(w) := w_{0,0}(H_f)\). For \(m + n \geq 1\) and \(w_{m,n} \in \mathcal{W}_{m,n}^{[d]}\) define the operator on \(\mathcal{H}_{\text{red}}\)
\[ H_{m,n}(w_{m,n}) := P_{\text{red}} \left( \int_{B_1^{m+n}} a^*(k^{(m)}) w_{m,n}(H_f, K^{(m,n)}) a(\tilde{k}^{(n)}) \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \right) P_{\text{red}}. \quad (4.3) \]

The following lemma is proven in [3, Theorem 3.1]

**Lemma 4.2.** Let \(\mu > 0\) and \(m, n \in \mathbb{N}_0\) with \(m + n \geq 1\). For \(w_{m,n} \in \mathcal{W}_{m,n}^{[d]}\) we have
\[ \|H_{m,n}(w_{m,n})\| \leq \frac{\|w_{m,n}\|_{\mu}}{\sqrt{n^m m^n}}. \quad (4.4) \]

Using our notation above and the convention that \(p^0 := 1\) for \(p = 0\) it is trivial to extend this lemma to the case \(m + n = 0\). For sequences \(w = (w_{m,n})_{(m,n) \in \mathbb{N}_0^2} \in \mathcal{W}_{\xi}^{[d]}\) we define the operator \(H(w)\) as the sum
\[ H(w) := \sum_{m,n} H_{m,n}(w_{m,n}), \quad (4.5) \]
where the sum converges in operator norm, which can be seen using (4.4).

A proof of the following theorem can be found in [3] with a modification explained in [19].
Theorem 4.3. Let $\mu > 0$ and $0 < \xi < 1$. Then the map $H : W^{[d]}_\xi \to \mathcal{B}(\mathcal{H}_{red})$ is injective and bounded. For $w \in W^{[d]}_\xi$ and for $\tilde{w} \in W^{[d]}_\xi$ with $\tilde{w}_{0,0} = 0$ we have

$$\|H(w)\| \leq \|w\|_{\mu,\xi}^\#, \quad \|H(\tilde{w})\| \leq \xi \|\tilde{w}\|_{\mu,\xi}^\#.$$  (4.6)

The renormalization transformation will involve a rescaling of the energy. This rescaling is described by means of a dilation operator, which we shall now define.

Definition 4.4. Let $\rho > 0$. We define the operator of dilation on the one particle sector by

$$U_\rho : \mathfrak{h} \to \mathfrak{h}, \quad (U_\rho \varphi)(k) = \rho^{3/2} \varphi(\rho k),$$

where we use the notation $\rho k := (\rho k, \lambda)$. We define the operator of dilation on Fockspace by

$$\Gamma_\rho = \bigoplus_{n=0}^\infty U_\rho \otimes_n \uparrow \mathcal{F}.$$

We define the mapping $S_\rho : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{F})$ called rescaling by dilation by

$$S_\rho(A) := \rho^{-1} \Gamma_\rho (A) \Gamma_\rho^*.$$  (4.7)

Remark 4.5. We note that by definition $\Gamma_\rho \Omega = \Omega$. Moreover, one can show that

$$\Gamma_\rho a^*(k) \Gamma_\rho^* = \rho^{-3/2} a^*(\rho^{-1} k), \quad \Gamma_\rho a(k) \Gamma_\rho^* = \rho^{-3/2} a(\rho^{-1} k).$$

The following lemma relates the scaling transformation to a scaling transformation of the integral kernels. It is straightforward to verify by the substitution formula.

Lemma 4.6. For $w \in W^{[d]}_\xi$ define the scaling transformation of the integral kernel by

$$s_\rho(w_{m,n})[r, K^{(m,n)}] := \rho^{(m+n)-1} w_{m,n}[\rho r, \rho K^{(m,n)}].$$  (4.8)

Then

$$S_\rho(H(w)) = H(s_\rho(w)).$$

Remark 4.7. For $m + n \geq 1$ one finds

$$\|s_\rho(w_{m,n})\|_\mu \leq \rho^{\mu(m+n)} \|w_{m,n}\|_\mu.$$

This illustrates that after a renormalization step, which will be introduced below, the relative size of the perturbative part $\|w_{m,n}\|_\mu$, $m + n \geq 1$, will shrink compared to the unperturbed part $\|w_{0,0}\|_{C^2}$.
5 Banach Space Estimate for the first Step

In Section 3 we showed that the operators $H_g - z$, $H_0 - z$ are a Feshbach pair for $\chi^{(0)}$, provided the coupling constant is in a sufficiently small neighborhood of zero and the spectral parameter is sufficiently close to the unperturbed ground state energy. In particular, if the assumptions of Theorem 3.1 hold, we can define the operator

$$H^{(1,\rho)}_g(z) := S_\rho(F_{\chi^{(0)}}(H_g - z, H_0 - z)),$$

which we call the first Feshbach operator. The goal of this section is to show that the first Feshbach operator is close to the free field energy. The distance will be measured in terms of the norms introduced in the previous section. More precisely, we define the following polydiscs of the free field energy. For given $\alpha, \beta, \gamma \in \mathbb{R}_+$ we define

$$B^{[d]}(\alpha, \beta, \gamma) := \{H(w) : \|w_{0,0}(0)\| \leq \alpha, \|w'_{0,0} - 1\|_\infty \leq \beta, \|w - w_{0,0}\|_{\mu, \xi} \leq \gamma\}.$$  

We shall denote the dimension of the space of ground states of $H_{at}$ by

$$d_0 := \dim(\text{Ran}P_{at}).$$

Moreover, we introduce the following global constant

$$C_F := 10\|\chi'\|_\infty + 20,$$

$$\hat{C}_F := 20,$$

which will be used in various estimates. For the renormalization analysis to be applicable, we need a stronger infrared condition, which is expressed in terms of the norm $\|\cdot\|_{\mu}$. We note that as a consequence of the definition we have

$$\left\|\frac{G}{\omega}\right\| \leq \|G\|_{\mu}.$$  

This inequality shows that the criterion for the Feshbach pair property obtained in Section 3 can be expressed in terms of $\|G\|_{\mu}$. We can now state the main theorem of this section.

**Theorem 5.1 (Banach Space Estimate for 1st Feshbach Operator).** Let $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; L(H_{at}))$, and $\xi \in (0, 1)$. Then there exist constants $C_1, C_2, C_3$, such that, if $0 < \rho < 1/4$, $z \in D_{\rho/2}(\epsilon_{at})$ and

$$|g| < C_0\rho^{1/2},$$

where

$$C_0 := \frac{1}{8\xi^{-1}C_F\|G\|_{\mu}},$$  

(5.6)
the pair of operators \((H_g - z, H_0 - z)\) is a Feshbach pair for \(\chi_\rho^{(1)}\), and

\[
H_g^{(1, \rho)}(z) - \rho^{-1}(\epsilon_{at} - z) \in \mathcal{B}^{[d_0]}(\alpha_0, \beta_0, \gamma_0),
\]

for

\[
\alpha_0 = C_1|g|^2 \rho^{-1}, \quad \beta_0 = C_2|g|^2 \rho^{-1}, \quad \gamma_0 = C_3 \rho^\mu(|g| + \rho^{-1}|g|^2 + \rho^{-2}|g|^3).
\]

Remark 5.2. We note that the explicit form of the constants \(C_1, C_2, \text{ and } C_3\) can be read off from Inequalities \((5.27) - (5.29)\). Only \(C_3\) depends on \(\xi\).

The remaining part of this section is devoted to the proof of Theorem 5.1. First observe that in view of \((5.5)\) and \((5.3)\) we see that assumption \((5.6)\) implies the assumption \((3.2)\) of Theorem 3.1 holds. Thus \((H_g - z, H_0 - z)\) is a Feshbach pair for \(\chi_\rho^{(0)}\) provided \(g\) is in a neighborhood of zero and \(z\) is in a neighborhood of the unperturbed ground state energy. Thus by Theorem 3.1 we can expand the resolvent in the Feshbach operator in an absolutely convergent Neumann series \((3.3)\). We shall bring the resulting Neumann series into normal order by using the pull-through formula, see Lemma A.3, and applying a generalized version of Wicks theorem, see Theorem C.1. First we use an alternative notation for our original Hamiltonian and we define

\[
w_{g,0,0}^{(0)}(z)(r) := H_{at} - z + r,
\]

\[
w_{g,1,0}^{(0)}(z)(r, k) := gG(k),
\]

\[
w_{g,0,1}^{(0)}(z)(r, \tilde{k}) := gG^*(\tilde{k}).
\]

We set \(w_g^{(0)} := (w_{g,m,n})_{0 \leq m+n \leq 1}\) and use the notation

\[
H_{m,n}^{(0)}(w_{m,n}) := \int_{A_{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} a^*(k^{(m)})w_{m,n}(H_f, K^{(m,n)})a(\tilde{k}^{(n)}).
\]

We note that the expression is analogous to \((4.3)\), apart from the fact that the domain of integration is different and there is no projection. To distinguish this difference we use a superscript zeroth order. In the new notation the interaction reads

\[
gW = H_{1,0}^{(0)}(w_{g,1,0}^{(0)}) + H_{0,1}^{(0)}(w_{g,0,1}^{(0)}).
\]

For the bookkeeping of the terms in the Neumann expansion we introduce the following multi-indices for \(L \in \mathbb{N}\)

\[
m := (m_1, \ldots, m_L) \in \mathbb{N}_0^L,
\]

\[
|m| := m_1 + \cdots + m_L,
\]

\[
0 := (0, \ldots, 0) \in \mathbb{N}_0^L.
\]
Now inserting the alternative expression for the interaction, (5.9), into the convergent Neumann Series (3.3), and using the generalized Wick theorem, Theorem C.1, we obtain a sum of terms of the form

$$V_{m,n}^{(0,\rho)}[w](r, K^{(m,n)})$$

(5.10)

$$:= (P_{\rho} \otimes P_{\Omega}) \sum_{l=1}^{L} \left\{ W_{m,n}^{(0,\rho)}[w](\rho K^{(m,n)}) F_{l}^{(0,\rho)}[w](H_{f} + \rho(r + \tilde{r}_{l})) \right\} (P_{\rho} \otimes P_{\Omega}),$$

where the definition of $\tilde{r}_{l}$ is given in (C.6), and where we used the definitions

$$W_{m,n}^{(0,\rho)}[w](K^{(m,n)})$$

(5.11)

$$:= \int_{(\mathbb{R}^{3} \times \{1,2\})^{p+q}} a^{*}(x^{(p)}) w_{m+p,n+q}[K^{(m,n)}, x^{(p)}, \tilde{x}^{(q)}] a(\tilde{x}^{(q)}) \frac{dX(p,q)}{|X(p,q)|^{1/2}},$$

$$F_{l}^{(0,\rho)}[w](r) := F_{0}^{(0,\rho)}[w](r) := \frac{(\chi(0))^{2}(r)}{w_{0,0}(r)},$$

for $l = 1, \ldots, L - 1,$

$$F_{L}^{(0,\rho)}[w](r) := \chi(r/\rho).$$

We use the natural convention that there is no integration if $p = q = 0$ and that the argument $K^{(m,n)}$ is dropped if $m = n = 0$. Note that the appearance of the $\rho$'s in the arguments on the r.h.s. of (5.10) is due to the scaling transformation $S_{\rho}$ in eq. (5.1).

Thus we have shown the algebraic part of the following result.

**Proposition 5.3.** Suppose the assumptions of Theorem 3.1 hold, i.e., let $0 < \rho \leq \frac{1}{4}$, $z \in D_{\rho/2}(\epsilon_{at})$ and (3.2). Define

$$\hat{u}_{g,0,0}^{(1,\rho)}(z)(r)$$

(5.12)

$$:= \rho^{-1} \left( \epsilon_{at} - z + \rho r + \sum_{L=2}^{\infty} (-1)^{L+1} \sum_{p,q \in \mathbb{N}_{0}^{L}} V_{m,n}^{(0,\rho)}[w_{g}^{(0)}(z)](r) \right),$$

and for $M, N \in \mathbb{N}_{0}$ with $M + N \geq 1$ define

$$\hat{u}_{g,M,N}^{(1,\rho)}(z)(r, K^{(M,N)})$$

(5.13)

$$:= \sum_{L=1}^{\infty} (-1)^{L+1} \rho^{M+N-1} \sum_{m,p,n,q \in \mathbb{N}_{0}^{L}: \frac{|m| = M, |n| = N, m_{l} + n_{l} = L_{l} + 1}} V_{m,n}^{(0,\rho)}[w_{g}^{(0)}(z)](r, K^{(M,N)}).$$
Assume that the right hand sides converge with respect to the norm $\| \cdot \|_{\mu, \xi}^\#$ for some $\mu > 0$ and $\xi \in (0, 1)$. Then for the symmetrization $w_g^{(1, \rho)}(z) := [w_g^{(1, \rho)}(z)]^\text{sym}$ we have

$$H_g^{(1, \rho)}(z) = H(w_g^{(1, \rho)}(z)).$$

**Proof.** Let $\hat{w}_g^{(1, \rho, L_0)}$ be defined as the integral kernel obtained by the right hand sides of (5.13) and (5.12), if we sum $L$ only up to $L_0$. Then by the absolute convergence of the Neumann Series (3.3) and the application of the generalized Wick theorem, Theorem C.1, as discussed above, we find

$$H_g^{(1, \rho)}(z) = \lim_{L_0 \to \infty} H(\hat{w}_g^{(1, \rho, L_0)}).$$

A detailed description how to obtain the integral kernels is given in Appendix A of [16], see also [3]. The assumption that the right hand sides of (5.13) and (5.12) converge with respect to the norm $\| \cdot \|_{\mu, \xi}^\#$ imply in view of Theorem 4.3 that

$$\lim_{L_0 \to \infty} H(\hat{w}_g^{(1, \rho, L_0)}) = \lim_{L_0 \to \infty} H([\hat{w}_g^{(1, \rho, L_0)}]^\text{sym}) = H(\lim_{L_0 \to \infty} [\hat{w}_g^{(1, \rho, L_0)}]^\text{sym}) = H(w_g^{(1, \rho)}(z)).$$

Our next goal is to show Inequalities (5.27), (5.28), and (5.29), below. These estimates will on the one hand imply that right hand sides of (5.13) and (5.12) converge with respect to the norm $\| \cdot \|_{\mu, \xi}^\#$, and on the other hand establish a proof of Theorem 5.1. To obtain the desired estimates we will use the bounds, collected in the following lemma.

**Proposition 5.4.** For all $G \in L^2_\mu(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{\text{at}}))$, $\rho \in (0, 1/4)$, $z \in D_{\rho/2}(\epsilon_{\text{at}})$, $L \in \mathbb{N}$, and $m, p, n, q \in \mathbb{N}^L_0$ we have

$$\rho^{-1} |z| \rho^{-1} |z|^{-1} \| V_{\alpha, \beta, \gamma}^{(1, \rho)} [w_g^{(0)}] \|_{\mu}^\# \leq (L + 2) \hat{C}_F^{L-1}(1 + \| \chi' \|_\infty) |g| L \rho^{-L + \frac{1}{2}(|p| + |q|) - q\nu} \chi \times \rho^{(1 + \mu)|m|} \| G \|_{\mu}^\|m| \| |G| \|_{\mu}^{|p| + |q|} \| |G|^{|p| + |q|} \|_{\mu}^{|p| + |q|} \| (5.14)$$

and

$$\rho^{-1} \| V_{\alpha, \beta, \gamma}^{(1, \rho)} [w_g^{(0)}] \|_{\infty} \leq \hat{C}_F^{L-1}|g| L \rho^{-L + \frac{1}{2}(|p| + |q|)} \| G \|_{\omega}^{|p| + |q|}, \quad (5.15)$$

$$\rho^{-1} \| \partial_r V_{\alpha, \beta, \gamma}^{(1, \rho)} [w_g^{(0)}] \|_{\infty} \leq (L + 1) \hat{C}_F^{L-1}(1 + \| \chi' \|_\infty) |g| L \rho^{-L + \frac{1}{2}(|p| + |q|)} \| G \|_{\omega}^{|p| + |q|}, \quad (5.16)$$

where $0 \in \mathbb{N}^L_0$. 

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Remark 5.5. We note that in contrast to Eq. (7.5) in Lemma 7.3 (see also Lemma 3.10) we have an additional factor $\rho^{\frac{1}{2}(|g|^2 - p + q - q_0)}$, which yields an improved estimate. The proof of of Theorem 5.1, which we present, will need this improved estimate.

To show the above proposition we will use the estimates from the following lemma.

Lemma 5.6. For $\rho \geq 0$ let $B_\rho = H_f + \rho$.

(a) Let $\omega^{-1/2}G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; L(H_{\text{at}}))$. Then for all $m, n, p, q \in \mathbb{N}_0$, with $m+n+p+q = 1$, all $K^{(m,n)} \in B_1^{m+n}$, and $\rho \geq 0$

$$\|B_\rho^{-1/2}W^{(m,n)}_{p,q}[w_g^{(0)}](K^{(m,n)})B_\rho^{-1/2}\| \leq \left\| \frac{G}{\omega} \right\|^{p+q} |g| \left\{ \|G(k_1)\| \right\}^m \left\{ \|G(\hat{k}_1)\| \right\}^n \rho^{\frac{1}{2}(p+q)-1}, \quad (5.17)$$

$$\|1_{[0,1]}(H_f)W^{(m,n)}_{p,q}[w_g^{(0)}](K^{(m,n)})B_\rho^{-1/2}\| \leq \left\| \frac{G}{\omega} \right\|^{p+q} |g| \left\{ \|G(k_1)\| \right\}^m \left\{ \|G(\hat{k}_1)\| \right\}^n \rho^{\frac{1}{2}(q-1)}, \quad (5.18)$$

$$\|B_\rho^{-1/2}W^{(m,n)}_{p,q}[w_g^{(0)}](K^{(m,n)})1_{[0,1]}(H_f)\| \leq \left\| \frac{G}{\omega} \right\|^{p+q} |g| \left\{ \|G(k_1)\| \right\}^m \left\{ \|G(\hat{k}_1)\| \right\}^n \rho^{\frac{1}{2}(q-1)}. \quad (5.19)$$

(b) For all $\rho \in (0, 1/4)$, $z \in D_{\rho/2}(\epsilon_{\text{at}})$ and $r \in [0, \infty)$ we have

$$\|B_\rho^{1/2}F^{(0,\rho)}[w_g^{(0)}](z)(H_f + \rho r)B_\rho^{1/2}\| \leq \hat{C}_F, \quad (5.20)$$

$$\|B_\rho^{1/2}\partial_r F^{(0,\rho)}[w_g^{(0)}](z)(H_f + \rho r)B_\rho^{1/2}\| \leq 20 + 10\|\chi\|_{\infty} \leq \hat{C}_F(1 + \|\chi\|_{\infty}). \quad (5.21)$$

Proof. Eq. (5.17) follows from the proof of Lemma 3.2. Eqns. (5.18) and (5.19) follow also from the proof of Lemma 3.2 and (A.1). Estimate (5.20) follows from Lemma 3.3.

To show Estimate (5.21) we calculate the derivative

$$\partial_r F^{(0,\rho)}[w_g^{(0)}](H_f + \rho r) = \frac{2\mathcal{X}_\rho^{(0)}(H_f + \rho r)(\mathcal{X}^{-1}_1)(\rho^{-1}H_f + r)}{H_{\text{at}} + H_f + \rho r - z} + \frac{(\mathcal{X}_\rho^{(0)})(H_f + \rho r)(H_{\text{at}} + H_f + \rho r - z)^2}{(H_{\text{at}} + H_f + \rho r - z)^2}. \quad (5.22)$$

To estimate the terms on the right hand side we use again Lemma 3.3 together with

$$\frac{\|B_\rho^{1/2}\|}{\|B_\rho^{1/2}r\|} \leq 1. \quad \Box$$
Proof of Proposition 5.4. Let $B_\rho = H_f + \rho$. We estimate $\|V_{m,p,n,q}^{(0,\rho)}[w_g^{(0)}]\|_\mu$ using

$$
\|\langle \varphi_{at} \otimes \Omega, A_1 A_2 \cdots A_n \varphi_{at} \otimes \Omega \rangle \| \leq \|A_1\| \|A_2\| \cdots \|A_n\|,
$$

(5.23)

where $\| \cdot \|$ denotes the operator norm, and Inequalities (5.17)-(5.21). First we have for $r \geq 0$

\[
\begin{align*}
\| V_{m,p,n,q}^{(0,\rho)}[w_g^{(0)}](r, K^{[m_1,n_1]}_0) \| & \leq \left\| \left( P_{at} \otimes P_{\Omega} \right) F_0^{(0,\rho)}[w_g^{(0)}](H_f + \rho(r + \tilde{r}_0)) W_{p_1,q_1}^{(0,m_1,n_1)}[w_g^{(0)}](\rho K^{(m_1,n_1)}_0) B_\rho^{1/2} \right. \\
& \quad \times B_\rho^{1/2} F_1^{(0,\rho)}[w_g^{(0)}](H_f + \rho(r + \tilde{r}_1)) B_\rho^{1/2} \\
& \quad \times \prod_{l=2}^{L-1} \left\{ B_\rho^{1/2} W_{p_l,q_l}^{(0,m_l,n_l)}[w_g^{(0)}](\rho K^{(m_l,n_l)}_0) B_\rho^{1/2} B_\rho^{1/2} F_{l+1}^{(0,\rho)}[w_g^{(0)}](H_f + \rho(r + \tilde{r}_l)) B_\rho^{1/2} \right\} \\
& \quad \times B_\rho^{1/2} W_{p_L,q_L}^{(0,m_L,n_L)}[w_g^{(0)}](\rho K^{(m_L,n_L)}_0) F_L^{(0,\rho)}[w_g^{(0)}](H_f + \rho(r + \tilde{r}_L))(P_{at} \otimes P_{\Omega}) \\
& \leq \hat{C}_F^{L-1} \left\| \frac{G}{\omega} \right\| \left\| g \right\|^{\frac{1}{2}+\frac{|g|}{2}} \prod_{l=1}^{L} \left[ \|G(\rho k_{m_l})\|^{m_l} \|G(\rho \tilde{k}_{m_l})\|^{n_l} \right] \\
& \quad \times \left[ \prod_{l=2}^{L-1} \left( \rho^{\frac{1}{2}(p_l+q_l)-1} \right) \right] \left( \rho^{\frac{1}{2}(q_1+p_L)-1} \right) \\
& \leq \hat{C}_F^{L-1} \left\| \frac{G}{\omega} \right\| \left\| g \right\|^{\frac{1}{2}+\frac{|g|}{2}} \prod_{l=1}^{L} \left[ \|G(\rho k_{m_l})\|^{m_l} \|G(\rho \tilde{k}_{m_l})\|^{n_l} \right].
\end{align*}
\]

(5.24)

To calculate the derivative we use Leibniz rule and we obtain similarly using again equation (5.23) and Inequalities (5.17)-(5.21). We find for $r \geq 0$

\[
\begin{align*}
\| \partial_r V_{m,p,n,q}^{(0,\rho)}[w_g^{(0)}](r, K^{[m_1,n_1]}_0) \| & \leq \hat{C}_F^{L-1} (1 + \| \chi \|_\infty)(L + 1) \left\| \frac{G}{\omega} \right\| \left\| g \right\|^{\frac{1}{2}+\frac{|g|}{2}} \prod_{l=1}^{L} \left[ \|G(\rho k_{m_l})\|^{m_l} \|G(\rho \tilde{k}_{m_l})\|^{n_l} \right] \\
& \quad \times \prod_{l=1}^{L} \left[ \|G(\rho k_{m_l})\|^{m_l} \|G(\rho \tilde{k}_{m_l})\|^{n_l} \right].
\end{align*}
\]

(5.25)
Now we can estimate inserting \((5.24)\) and \((5.25)\), respectively,
\[
\| V_{m-p,n,q}^{(0,\rho)} [w_g^{(0)}] \|_\mu \\
= \left( \int_{B_{m-|\!\!\!|n|\!\!\!|+|\!\!\!|n|\!\!\!|}} \| \partial_{n} V_{m-p,n,q}^{(0,\rho)} [w_g^{(0)}] (K^{(m,|\!\!\!|n|\!\!\!|,|\!\!\!|n|\!\!\!|)}) \|_\infty^2 \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} \right)^{1/2} \\
\leq \hat{C}_F^{L-1} |g|^L \left\| \frac{G}{\omega} \right\|^{p+|q|} \rho^{-L+1+\frac{1}{2}(|p|-p_1+|q|-q_L)} \\
\times \left( \int_{B_{m-|\!\!\!|n|\!\!\!|+|\!\!\!|n|\!\!\!|}} \prod_{l=1}^L \left\{ \| G(\rho k_{m_l}) \|^{2m_l} \| G(\rho \tilde{k}_{n_l}) \| \right\}^{2n_l} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} \right)^{1/2} \\
\leq \hat{C}^{L-1} F^{-1} (1 + \| \chi' \|_\infty) |g|^L \left\| \frac{G}{\omega} \right\|^{p+|q|} \rho^{-L+1+\frac{1}{2}(|p|-p_1+|q|-q_L)} \\
\times \left( \int_{B_{m-|\!\!\!|n|\!\!\!|+|\!\!\!|n|\!\!\!|}} \prod_{l=1}^L \left\{ \| G(\rho k_{m_l}) \|^{2m_l} \| G(\rho \tilde{k}_{n_l}) \| \right\}^{2n_l} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{3+2\mu}} \right)^{1/2} \\
\leq \hat{C}^{L-1} F^{-1} (1 + \| \chi' \|_\infty) |g|^L \left\| \frac{G}{\omega} \right\|^{p+|q|} \rho^{-L+1+\frac{1}{2}(|p|-p_1+|q|-q_L)} \mu^{(m+|\!\!\!|n|\!\!\!|)} G \|^{m+|\!\!\!|n|\!\!\!|)} .
\]

Adding above estimates yields \((5.14)\). Eqs. \((5.15)\) and \((5.16)\) follow similarly noting that \(|m|=0\) and \(|n|=0\) can only occur if \(L\) is even and on the very left we have a annihilation operator and on the very right a creation operator. \(\square\)

**Proof of Theorem** \([5.1]\) It suffices to establish inequalities \((5.27)\)–\((5.29)\), below. Let \(S_{M,N}^L\) denote the set of tuples \((m,p,n,q) \in \mathbb{N}_0^4\) with \(|m|=M\), \(|n|=N\), and
\[
m_t + p_t + q_t + n_t = 1. \hspace{1cm} (5.26)
\]

Such tuples obviously satisfy \(|m| + |n| + |n| + |q| = L\). Using this identity, we now estimate
the norm of \((5.13)\) using \((5.14)\) and \((5.5)\). This yields
\[
\| (u^{(1,\rho)}_{g,M,N})_{M+N \geq 1}(z) \|_{\#}^\mu \xi \\
= \sum_{M+N \geq 1} \xi^{-(M+N)} \| w^{(1,\rho)}_{g,M,N}(z) \|_{\#}^\mu \\
\leq \sum_{M+N \geq 1} \sum_{L=1}^{\infty} \sum_{(m,p,n,q) \in S^L_{M,N}} \xi^{-(M+N)} \rho^{M+N} \| V^{(0,\rho)}_{m,p,n,q,\xi}[w^{(0)}_g(z)] \|_{\#}^\mu \\
\leq \sum_{L=1}^{\infty} \sum_{M+N \geq 1} \sum_{(m,p,n,q) \in S^L_{M,N}} \xi^{-(M+N)} \rho^{M+N} \| V^{(0,\rho)}_{m,p,n,q,\xi}[w^{(0)}_g(z)] \|_{\#}^\mu \\
\preceq \rho^\mu (1 + \| \chi' \|_\infty) \left[ \rho^{1/2} \xi^{-1} |g| \rho^{-1/2} G \|_{\mu} + 64 \left( \xi^{-1} |g| \rho^{-1/2} \hat{C}_F G \|_{\mu} \right)^2 \\
+ \rho^{-1/2} \sum_{L=3}^{\infty} (L+2) \left( 4 \xi^{-1} |g| \rho^{-1/2} \hat{C}_F G \|_{\mu} \right)^L \right],
\] (5.27)

where in the last inequality we estimated the summands \(L = 1, L = 2\) separately and summmed over the terms with \(L \geq 3\), as we now outline. First we note that \((5.26)\) implies that \(S^L_{M,N}\) is empty unless \(M+N \leq L\), and that the number of elements \((m,p,n,q) \in \mathbb{N}_0^{4L}\) with \((5.26)\) is bounded above by \(4^L\). Specifically for \(L = 1\), we have only two terms: \((m_1, p_1, n_1, q_1)\) equal \((1,0,0,0)\) or equal \((0,0,1,0)\). For \(L = 2\) we use that \(L \leq M + N\) implies \(p_1 + q_1 \leq 1\). For \(L \geq 3\) we use \(|m_1| + |n_1| - (p_1 + q_1) \geq -1\). These considerations establish \((5.27)\). We now estimate the norm of \((5.12)\) using \((5.16)\), by means of a similar but simpler estimate
\[
\sup_{r \in [0,1]} | \partial_r w^{(1,\rho)}_{g,0,0}(z)(r) - 1 | \leq \rho^{-1} \sum_{L=2}^{\infty} \sum_{p,q \in \mathbb{N}_0^L; \ p+q=1} \| \partial_r V_{0,p,q,\xi}^{(0,\rho)}[w^{(0)}_g(z)] \|_\infty \\
\leq (1 + \| \chi' \|_\infty) \sum_{L=2}^{\infty} (L+1) \left( 2 \left\| \frac{G}{\omega} \right\| |g| \rho^{-1/2} \hat{C}_F \right)^L.
\] (5.28)

Analogously we have using \((5.15)\)
\[
| w^{(1,\rho)}_{g,0,0}(z)(0) + \rho^{-1}(z - c_{at}) | \leq \rho^{-1} \sum_{L=2}^{\infty} \sum_{p,q \in \mathbb{N}_0^L; \ p+q=1} \| V_{0,p,q,\xi}^{(0,\rho)}[w^{(0)}_g(z)] \|_\infty \\
\leq \sum_{L=2}^{\infty} \left( 2 \left\| \frac{G}{\omega} \right\| |g| \rho^{-1/2} \hat{C}_F \right)^L.
\] (5.29)
The series on the right hand sides in (5.27)–(5.29) converge if (5.6) holds. Thus Theorem 5.1 now follows in view of equations (5.27)–(5.29).

6 Second Feshbach Step

In this section we perform our second Feshbach step. We want to note that the field energy cutoff of the first Feshbach step will be henceforth denoted by \( \rho_0 \) while the field energy cutoff of the second Feshbach step will be denoted by \( \rho_1 \). First we approximate \( w_{g,0,0}(z) \), which is the content of the following lemma. We recall the mapping \( Z_{at} \), which was defined in (2.5), and that its ground state energy, \( \epsilon_{at}(2) \), is by assumption a simple eigenvalue.

**Lemma 6.1** (Free Approx. to 1st Feshbach). There exists a constant \( C \) such that the following holds. Let \( G \in L^2_{\mu}(\mathbb{R}^3 \times \mathbb{Z}_2; L(H_{at})) \), \( 0 < \rho_0 < 1/4 \), and suppose

\[
|g| < \frac{\rho_0^{1/2}}{4C_F \|\omega^{-1}G\|}.
\]

If \( z \in D_{\rho_0/2}(\epsilon_{at}) \), then the defining series of \( w_{g,0,0}^{(1,\rho_0)}(z) \), i.e., the r.h.s. of (5.12), converges absolutely and

\[
\sup_{0 \leq r \leq 1} \| \rho_0^{-1}(\epsilon_{at} - z + \rho_0 r + \chi^2(r)g^2Z_{at}) - w_{g,0,0}^{(1,\rho_0)}(z)(r) \| \leq C(\|G\|_\mu |g|^2 + \|G\|_\mu^4 C_1^4 |g|^4 \rho_0^{-2}).
\]

**Remark 6.2.** We note that if we replaced the \( \chi^2(r) \) in front of \( Z_{at} \) by one, then the proof given below would yield a bound only of order \( |g|^2 \rho_0^{-1} \).

**Proof.** From (5.12) we see that

\[
w_{g,0,0}^{(1,\rho_0)}(z)(r) = \rho_0^{-1} \left( \epsilon_{at} - z + \rho_0 r + \sum_{L=2}^{\infty} (-1)^{L+1} \sum_{p,q \in \mathbb{N}_0^3 : p+q=1} V_{p,0,0}^{(0,\rho_0)}[w_g^{(0)}(z)](r) \right).
\]

From the definition (5.10) we see, that the summand with \( L = 2 \) is only non-vanishing if \( p = (0,1) \) and \( q = (1,0) \) and that summands with odd \( L \) vanish. Using this we can write

\[
\rho_0^{-1}(\epsilon_{at} - z + \rho_0 r + \chi^2(r)g^2Z_{at}) - w_{g,0,0}^{(1,\rho_0)}(z)(r)
\]

\[
= \rho_0^{-1} \left( \chi^2(r)g^2Z_{at} - X_g^{\rho_0}(z)(r) \right) - Y_g^{\rho_0}(z)(r)
\]

\[
= \rho_0^{-1} \left( \chi^2(r)g^2Z_{at} - X_g^{\rho_0}(\epsilon_{at})(r) + X_g^{\rho_0}(\epsilon_{at})(r) - X_g^{\rho_0}(z)(r) \right) - Y_g^{\rho_0}(z)(r)
\]

(6.2)
where we introduced the notation

\[ X_{g}^{\rho_0}(z)(r) := -V_{(0,\rho_0)}(\rho_0,1,0)\{w_g^{(0)}(z)\}(r) \]

\[ Y_{g}^{\rho_0}(z)(r) := \rho_0^{-1} \sum_{L=4}^{\infty} (-1)^{L+1} \sum_{p,q \in \mathbb{N}_0: p+q+1} V_{(p,q,0,0)}(\rho_0,0,0)\{w_g^{(0)}(z)\}(r). \]

The second term can be estimated similarly to (5.29), i.e., using (5.15) we find

\[ \sup_{r \in [0,1]} \|Y_{g}^{\rho_0}(z)(r)\| \leq \sum_{L=4}^{\infty} \left( \frac{2|g|}{\omega} \right) \left( \hat{C}_F \rho_0^{-1} \right)^{L/2} \]  

(6.3)

In order to obtain a suitable estimate for \( X_{g}^{\rho_0} \) we use that \((\overline{\chi}_{\rho_0})^2\) has a natural decomposition into a sum of two terms and we calculate the vacuum expectation using the pull-through formula

\[ X_{g}^{\rho_0}(z)(r) = g^2 \chi(r) P_{at} \int \frac{dk}{|k|} G^*(k) P_{at} \frac{\overline{\chi}^2(\rho_0^{-1}|k| + r)}{\epsilon_{at} - z + |k| + \rho_0 r} P_{at} G(k) P_{at} \chi(r) \]

\[ + g^2 \chi(r) P_{at} \int \frac{dk}{|k|} G^*(k) P_{at} \frac{1}{H_{at} - z + |k| + \rho_0 r} P_{at} G(k) P_{at} \chi(r). \]

(6.4)

First we estimate the relative error if we replace \( z \) by \( \epsilon_{at} \), that is we show for \( z \in D_{\rho_0/2}(\epsilon_{at}) \)

\[ \sup_{r \in [0,1]} \|R_{g}^{\rho_0}(z)(r) - R_{g}^{\rho_0}(\epsilon_{at})(r)\| \leq |g|^2 \rho_0 (3 + 1) \|G\|_{\mu} \]  

(6.5)

To estimate the first term in (6.4) we use common denominators

\[ \frac{1}{\epsilon_{at} - z + |k| + \rho_0 r} - \frac{1}{|k| + \rho_0 r} \]

\[ = \frac{1}{\epsilon_{at} - z + |k| + \rho_0 r} \frac{1}{|k| + \rho_0 r} \]

\[ = \frac{1}{\epsilon_{at} - z + |k| + \rho_0 r} \frac{1}{z - \epsilon_{at}} \]

\[ = \left( \frac{1}{\epsilon_{at} - z + |k| + \rho_0 r} \frac{1}{z - \epsilon_{at}} \right) \frac{1}{(|k| + \rho_0 r)^2}. \]

(6.6)

This yields

\[ \overline{\chi}^2(\rho_0^{-1}|k| + r) \mid \text{l.h.s. of (6.6)} \mid \leq \rho_0 \left( 1 + \frac{\rho_0}{\frac{3}{2} \rho_0 - 1} \right) |k|^{-2}. \]

This explains the first contribution on the right hand side of (6.5). We estimate the second term in (6.4) similarly. For \( E \in \sigma(H_{at}) \setminus \{\epsilon_{at}\} \) we write

\[ \frac{1}{E - z + |k| + \rho_0 r} - \frac{1}{E - \epsilon_{at} + |k| + \rho_0 r} \]

\[ = \frac{1}{E - z + |k| + \rho_0 r} \frac{1}{E - \epsilon_{at} + |k| + \rho_0 r} \]

(6.7)
This yields

\[ | \text{l.h.s. of (6.7)} | \leq \rho_0 |k|^{-2}. \]

This explains the second contribution on the right hand side of (6.5). Next we show that

\[ \sup_{r \in [0,1]} \| \chi^2(r)g^2 Z_{at} - R^\rho_0 (\epsilon_{at})(r) \| \leq 3 \|G\|_\mu |g|^2 \rho_0. \] (6.8)

To estimate the first term in (6.4) with \( z = \epsilon_{at} \) we use

\[ \frac{1}{|k| + \rho_0 r} - \frac{1}{|k|} = \frac{1}{|k| + \rho_0 r} \left( \frac{1}{|k|} \right) \]

and make use of

\[ |\chi^2(\rho_0^{-1}|k| + r) - 1| \leq \begin{cases} 0, & |k| \geq \rho_0 \\ 1, & |k| \leq \rho_0. \end{cases} \]

We estimate the second term in (6.4) with \( z = \epsilon_{at} \) using for \( E \in \sigma(H_{at}) \setminus \{\epsilon_{at}\} \) that

\[ \frac{1}{E - \epsilon_{at} + |k| + \rho_0 r} - \frac{1}{E - \epsilon_{at} + |k|} = \frac{1}{E - \epsilon_{at} + |k| + \rho_0 r} \left( \frac{1}{E - \epsilon_{at} + |k|} \right). \]

This gives (6.8). Finally, inserting estimates of (6.3), (6.5), and (6.8) into (6.2) shows the lemma.

Let \( P_{at}^{(2)} \) denote the projection onto the one dimensional eigenspace of \( Z_{at} \) with eigenvalue \( \epsilon_{at}^{(2)} \) and let \( \overline{P}_{at}^{(2)} = 1 - P_{at}^{(2)} \). We mention that the superscript (2) originates from the fact that these expressions are obtained by formal second order perturbation theory. For \( \rho_1 > 0 \) define

\[ \chi^{(1)}_{\rho_1}(r) = P_{at}^{(2)} \otimes \chi(r/\rho_1) \]

\[ \overline{\chi}^{(1)}_{\rho_1}(r) = \overline{P}_{at}^{(2)} \otimes 1 + P_{at}^{(2)} \otimes \chi(r/\rho_1) \]

and

\[ \chi^{(1)}_{\rho_1} = \chi^{(1)}_{\rho_1}(H_f) \]

\[ \overline{\chi}^{(1)}_{\rho_1} = \overline{\chi}^{(1)}_{\rho_1}(H_f). \] (6.9)

(6.10)

Recall that we assumed that the distance, \( d_{at} \), from the lowest to the second lowest eigenvalue of \( H_{at} \) is one. By the assumption \( 0 \leq \delta_0 < \pi/2 \) the following expression is positive

\[ c_{\delta_0} := \inf_{g \in S_{\delta_0}} |d_{at} + g^{-2}| > 0, \] (6.11)
which follows from an easy minimization problem, yielding

\[
c_{\delta_0} = \begin{cases} 
  d_{at}, & \text{if } 0 \leq \delta_0 \leq \pi/4 \\
  d_{at} \sin(\pi - 2\delta_0), & \text{if } \pi/4 < \delta_0 < \pi/2.
\end{cases}
\]

Let \( \rho_0 > 0 \). We shall assume the following inequalities

\[
\rho_0^{-1} |g|^2 < \frac{1}{4} \frac{\|Z_{at}\| + c_{\delta_0}}{} \tag{6.12}
\]

and

\[
\rho_1 \rho_0 \leq |g|^2 c_{\delta_0} \tag{6.13}
\]

hold. Next we show the following lemma, which will establish the required invertibility.

**Lemma 6.3** (Invertibility of Free Approx to 1st Feshbach). Suppose \( \rho_0, \rho_1 \in (0, 1/2] \). Let \( g \in S_{\delta_0} \) satisfy \( (6.12) \) and \( (6.13) \). Then for \( z \in D_{\rho_0 \rho_1/2}(\epsilon_{at} + g^2 \epsilon_{at}^{(2)}) \) we have

\[
\|\left( \rho_0^{-1}(\epsilon_{at} - z + \rho_0 H_f + \chi^2(H_f)g^2 Z_{at}) \mid \text{Ran}X_{\rho_1}^{(1)} \right)^{-1}\| \leq \frac{4}{\rho_1}. \tag{6.14}
\]

**Proof.** For notational simplicity we shall write

\[
X(\rho_0, g, z) := \rho_0^{-1}(\epsilon_{at} - z + \rho_0 H_f + \chi^2(H_f)g^2 Z_{at}).
\]

For normalized \( \psi \) in the range of \( Q_1 := \overline{\mathcal{D}_{at}^{(2)}} \otimes 1_{[0, 1]}(H_f) \) we have

\[
\|X(\rho_0, g, z)\psi\| \geq \inf_{\eta \in \text{Ran}\overline{\mathcal{D}_{at}^{(2)}}} \inf_{|\eta| = 1} \|(-\rho_0^{-1}g^2 \epsilon_{at}^{(2)} + r + \rho_0^{-1}\chi^2(r)g^2 Z_{at})\eta\|
\]

\[
- |\rho_0^{-1} (\epsilon_{at} + g^2 \epsilon_{at}^{(2)} - z)|
\]

\[
\geq \rho_0^{-1} |g|^2 c_{\delta_0} - \rho_1/2, \tag{6.15}
\]

where in the last inequality we used on the one hand that for \( r \in [0, 3/4] \) we have by \( (6.11) \)

\[
\|(-\rho_0^{-1}g^2 \epsilon_{at}^{(2)} + r + \rho_0^{-1}\chi^2(r)g^2 Z_{at})\eta\| = \rho_0^{-1} |g|^2 \|(-\epsilon_{at}^{(2)} + g^{-2}\rho_0 r + Z_{at})\eta\|
\]

\[
\geq \rho_0^{-1} |g|^2 c_{\delta_0},
\]

and that on the other hand we used that for \( r \in [3/4, 1] \) we have

\[
\|(-\rho_0^{-1}g^2 \epsilon_{at}^{(2)} + r + \rho_0^{-1}\chi^2(r)g^2 Z_{at})\eta\| \geq 3/4 - \rho_0^{-1} |g|^2 Z_{at} - \epsilon_{at}^{(2)} \| Z_{at} - \epsilon_{at}^{(2)} \| \geq \rho_0^{-1} |g|^2 c_{\delta_0}
\]

by \( (6.12) \). Using \( (6.13) \) in \( (6.15) \) it now follows that

\[
\|(X(\rho_0, g, z) \mid \text{Ran}Q_1)^{-1}\| \leq \frac{2}{\rho_1}.
\]

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Next we consider for normalized $\psi$ in the range of $Q_2 := P_{at}^{(2)} \otimes \chi(H_f/\rho_1)$

$$
\|X(\rho_0, g, z)\psi\| \geq \inf_{\eta \in \text{Ran} P_{at}^{(2)}, \|\eta\|=1, 3\rho_1/4 \leq r \leq 1} \|(-\rho_0^{-1}g^2\epsilon_{at}^{(2)} + r + \chi^2(r)\rho_0^{-1}g^2\epsilon_{at}^{(2)})\eta\|
- |\rho_0^{-1}(\epsilon_{at} + g^2\epsilon_{at} - z)|
\geq \frac{3}{4}\rho_1 - \frac{1}{2}\rho_1 = \frac{1}{4}\rho_1,
$$

where we used that on the one hand side we have for $r \in [3\rho_1/4, 3/4]$

$$
\|(-\rho_0^{-1}g^2\epsilon_{at}^{(2)} + r + \chi^2(r)\rho_0^{-1}g^2\epsilon_{at}^{(2)})\eta\| = r \geq \frac{3}{4}\rho_1,
$$

and on the other hand we have for $r \in [3/4, 1]$

$$
\|(-\rho_0^{-1}g^2\epsilon_{at}^{(2)} + r + \chi^2(r)\rho_0^{-1}g^2\epsilon_{at}^{(2)})\eta\| \geq \frac{3}{4} - \rho_0^{-1}|g|^2|\epsilon_{at}^{(2)}| \geq \frac{1}{2} \geq \frac{3}{4}\rho_1,
$$

by (6.12). Thus we can invert the operator $X(\rho_0, g, z)$ on the range of $\chi_{\rho_1}^{(1)}$. \qed

In order to perform a second Feshbach iteration, we choose the following decomposition

$$
H^{(1,\rho_0)}_g = T_g^{(1,\rho_0)} + W_g^{(1,\rho_0)}
$$

(6.16)

where

$$
T_g^{(1,\rho_0)} := H_{0,0}(t_g^{(1,\rho_0)}),
W_g^{(1,\rho_0)} := H(w_g^{(1,\rho_0)})
$$

(6.17)

(6.18)

where

$$
t_g^{(1,\rho_0)} := P_{at}^{(2)} w_g^{(1,\rho_0)} P_{at}^{(2)} + \overline{P}_{at}^{(2)} w_g^{(1,\rho_0)} \overline{P}_{at}^{(2)}
$$

$$
w_g^{(1,\rho_0)} := (P_{at}^{(2)} w_{g,0,0}^{(1,\rho_0)} P_{at}^{(2)} + \overline{P}_{at}^{(2)} w_{g,0,0}^{(1,\rho_0)} \overline{P}_{at}^{(2)}, w_{g,m+n \geq 1}^{(1,\rho_0)})
$$

Remark 6.4. We note that decomposition (6.16) into free part and interacting part is not unique. The isospectrality property of the smooth Feshbach merely requires that the free part commutes with the smoothed projections. This issue is pointed out in Remark 2.4 in [3]. Alternatively, we could use the decomposition according to

$$
t_g^{(1,\rho_0)}(z)(r) := \rho_0^{-1}(\epsilon_{at} - z + \rho_0 r + \chi^2(r)g^2Z_{at})
$$

\begin{equation}
(6.19)
\end{equation}

$$
w_g^{(1,\rho_0)} := (w_{g,m+n \geq 1}^{(1,\rho_0)}, t_g^{(1,\rho_0)} Z_{at})
$$

The proof given in this paper would carry through also with this decomposition, with only notational modifications.
Thus we can now prove the main result of this section.

**Theorem 6.5** (Abstract Feshbach Pair Criterion for 2nd Iteration). Assume that the smallest eigenvalue of $Z_{at}$ is simple. Let $\rho_1 \in (0, 1]$. Suppose

\[
t \in C([0, 1]; \mathcal{L}(P_{at}^{(2)} \mathcal{H}_{at}) \oplus \mathcal{L}(P_{at}^{(2)} P_{at} \mathcal{H}_{at}))
\]

and $w \in \mathcal{W}^{[d]}$. Then the operators $H_{0,0}(t)$ and $H(w)$ are a Feshbach pair for $\chi^{(1)}_{\rho_1}$, provided

(i) $H_{0,0}(t)$ is invertible on the closure of $\text{Ran} \chi^{(1)}_{\rho_1}$ and

\[
\| (H_{0,0}(t) \upharpoonright \text{Ran} \chi^{(1)}_{\rho_1})^{-1} \| \leq \frac{8}{\rho_1},
\]

(ii) the following inequality holds

\[
\| H(w) \| < \frac{\rho_1}{8}. \tag{6.20}
\]

In this case we have the absolutely convergent expansion

\[
F_{\chi^{(1)}_{\rho_1}}(H_{0,0}(t), H(w)) = H_{0,0}(t) + \sum_{L=1}^{\infty} (-1)^{L-1} \chi^{(1)}_{\rho_1} H(w) \left( \frac{(\chi^{(1)}_{\rho_1})^2}{H_{0,0}(t)} H(w) \right)^{L-1} \chi^{(1)}_{\rho_1}.
\]

**Proof.** First observe that $H_{0,0}(t)$ commutes with $\chi^{(1)}_{\rho_1}$. The Feshbach pair property follows from (i) and (ii) and Neumann’s theorem. The second claim follows again by Neumann’s theorem. $\square$

**Remark 6.6.** In the proof of the main theorem, in Section 8, we will determine an explicit relation among $\rho_0$ and $\rho_1$ and $g$. Using this relation we will verify assumption (i) and (ii) of the above theorem by the help of Lemmas 6.1 and 6.3.

## 7 Banach Space Estimate for the second Step

Using estimates of the previous section we will show in Section 8 that $(T_{\rho_1}^{(1,\rho_0)}(z), W_{g}^{(1,\rho_0)}(z))$ is indeed a Feshbach pair for $\chi^{(1)}_{\rho_1}$. For the moment we will assume that the Feshbach property is satisfied. In that case we can define

\[
H_{g}^{(2,\rho_1)}(z) := S_{\rho_1} \left( F_{\chi^{(1)}_{\rho_1}}(T_{g}^{(1,\rho_0)}(z), W_{g}^{(1,\rho_0)}(z)) \right)
\]

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provided the right sides exist. Similar to Section 5 we now want to show that there exists a sequence of integrals kernels \( w^{(2, \rho_1)}_g(z) \) such that

\[
H(w^{(2, \rho_1)}_g(z)) = H_{g}^{(2, \rho_1)}(z) \upharpoonright \text{Ran}\chi^{(1)}_{\rho_1}.
\]

This will follow as a conclusion of the following theorem.

For notational compactness we introduce in this section the following constant

\[
C_{\chi} := 20\sqrt{2}.
\]

**Theorem 7.1** (Abstract Banach Space Estimate for 2nd Feshbach Operator). Let \( 0 < \xi \leq 1/4 \) and assume that the smallest eigenvalue of \( Z_{at} \) is simple. Let \( \rho_1 \in (0, 1) \). Suppose

\[
t \in C^1([0, 1]; \mathcal{L}(P^{(2)}_{at} \mathcal{H}_{at}) \oplus \mathcal{L}(\overline{P^{(2)}_{at} P_{at} \mathcal{H}_{at}}))
\]

and \( w \in \mathcal{W}^{[d]}_{\xi} \).

(i) \( H_{0,0}(t) \) is invertible on the closure of \( \text{Ran}\chi^{(1)}_{\rho_1} \) and

\[
\|(H_{0,0}(t) \upharpoonright \text{Ran}\chi^{(1)}_{\rho_1})^{-1}\| \leq \frac{8}{\rho_1},
\]

(ii) \( \|H(w)\| < \frac{\rho_1}{8} \).

Then \( H(t) \) and \( H(w) \) are a Feshbach pair for \( \chi^{(1)}_{\rho_1} \). Moreover, suppose

\[
\gamma < \frac{\rho_1}{8C_{\chi}}
\]

and

\[
\|w\|_{\rho, \xi}^{#} \leq \gamma, \\
\|t'\|_{\infty} \leq \tau_0, \\
\|P^{(2)}_{at} t' P^{(2)}_{at} - 1\| \leq \tau_1.
\]

Then

\[
S_{\rho_1}(F_{\chi^{(1)}_{\rho_1}}(H(t), H(w))) - \rho_1^{-1}P^{(2)}_{at} t(0)P^{(2)}_{at} \in \mathcal{B}^{[1]}(\alpha_1, \beta_1, \gamma_1),
\]

where

\[
\alpha_1 = 12(1 + 2\|\chi'\|_{\infty} + 8\tau_0)C_{\chi}\gamma\rho_1^{-1}, \\
\beta_1 = \tau_1 + 12(1 + 2\|\chi'\|_{\infty} + 8\tau_0)C_{\chi}\gamma\rho_1^{-1}, \\
\gamma_1 = 96(1 + 2\|\chi'\|_{\infty} + 8\tau_0)\rho_1^{\mu}C_{\chi}\gamma.
\]
In order to prove Theorem 7.1 we first use the Neumann expansion given in (6.21) and we normal order the resulting expression using the generalized Wick Theorem, stated in the appendix. The result is given in Proposition 7.2 below, and can be obtained as in [3, Theorem 3.7]. To state it we recall the following notation

\[ m := (m_1, ..., m_L) \in \mathbb{N}_0^L, \]
\[ |m| := m_1 + \cdots + m_L, \]
\[ 0 := (0, ..., 0) \in \mathbb{N}_0^L, \]

for \( L \in \mathbb{N}. \) We define

\[ V_{m,p,n,q}(t, w)(r, K^{(m,|n|)}) := \left( P_{at}^{(2)} \otimes P_{\Omega} \right) F_{0}^{(1,\rho_1)}[t](H_f + \rho_1(r + \tilde{r}_0)) \]
\[ \prod_{l=1}^{L} \left\{ W_{p_l,q_l}^{m_l,n_l}[w](\rho_1(r + \tilde{r}_l), \rho_1 K^{(m_l,n_l)}) F_l^{(1,\rho_1)}[t](H_f + \rho_1(r + \tilde{r}_l)) \right\} \]
\[ \left( P_{at}^{(2)} \otimes P_{\Omega} \right). \]

For \( w \in W_{m+p,n+q}^{d} \) we defined

\[ W_{p,q}^{m,n}[w](r, K^{(m,n)}) := 1_{[0,1]}(H_f) \int_{B_{1}^{p+q}} \frac{dX^{(p,q)}}{|X^{(p,q)}|^{1/2}} a^*(x^{(p)}) \]
\[ w_{m+p,n+q}[H_f + r, k^{(m)}, x^{(p)}, \tilde{k}^{(n)}, \tilde{x}^{(q)}] a(\tilde{x}^{(q)})) 1_{[0,1]}(H_f), \]

where we use the natural convention that there is no integration if \( p = q = 0 \) and that the argument \( K^{(m,n)} \) is dropped if \( m = n = 0. \) Moreover we defined for \( l = 0, L \) the expressions \( F_l^{(1,\rho_1)}[t](r) := \chi(r/\rho_1) \) and for \( l = 1, ..., L - 1 \) we have set

\[ F_l^{(1,\rho_1)}[t](r) := F^{(1,\rho_1)}[t](r) = \frac{(\tilde{X}_{\rho_1}^{(1)}(r))^{2}}{t(r)}, \]

and \( \tilde{r}_l \) is defined as in (C.6). We note that we used similar notation as in the previous section.

**Proposition 7.2.** Define

\[ \hat{w}_{0,0}^{(2,\rho_1)}(r) := \rho_1^{-1} \left( t(\rho_1 r) + \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{p,q \in \mathbb{N}_0^L} V_{0,p,0,q}^{(1,\rho_1)}[t, w](r) \right), \]
and for \( M + N \geq 1 \) define

\[
\hat{w}^{(2, \rho_1)}(r, K^{(M,N)}) := \sum_{L=1}^{\infty} (-1)^{L+1} \rho_1^{M+N-1} \sum_{\substack{m, p, n, q \in \mathbb{N}_0^L \mid |m| = M, |n| = N}} \prod_{l=1}^{L} \left\{ \left( m_l + p_l \right) \left( n_l + q_l \right) \right\} \sqrt{w_{m_l+p_l,n_l+q_l}}[t, w](r, K^{(M,N)}). \tag{7.4}
\]

Assume that the right hand side converges with respect to the norm \( \| \cdot \|^{\#}_{\mu_\xi} \). Let \( w^{(2, \rho_1)} \) be the symmetrization w.r.t. \( k^{(M)} \) and \( \tilde{k}^{(N)} \) of \( \hat{w}^{(2, \rho_1)} \). Then

\[
S_{\rho_1}(F_{x^{(1)}_{\rho_1}}(H(t), H(w))) = H(w^{(2, \rho_1)}).
\]

The proof of this proposition is analogous to the proof of Proposition 5.3. To prove Theorem 7.1, we shall need the estimate given in the following lemma.

**Lemma 7.3.** Suppose the assumptions of Theorem 7.1 hold. For fixed \( L \in \mathbb{N} \) and \( m, p, n, q \in \mathbb{N}_0^L \) and \( w \in W^{[d]}_{\mu_\xi} \) we have

\[
\rho_1^{(1+\mu)\left( |m|+|n| \right) - L} \left\| \sqrt{w_{m_l+p_l,n_l+q_l}} \right\|^{\#}_{\mu} \leq (L+2)2^{L/2}C_{\chi}^{L-1} \left( 1 + \| \chi' \|_{\infty} + 4\| t' \|_{\infty} \right) + \left( \left\| \chi^{(1)}_{\rho_1}(u+\rho_1 r) \right\| \right)^2 \leq \rho_1^{1+\rho_1} \left( |m|+|n| \right) - L \prod_{l=1}^{L} \left\| \sqrt{w_{m_l+p_l,n_l+q_l}} \right\|^{\#}_{\mu}.
\]

with \( C_{\chi} = 20 \) and the convention that \( p^0 := 1 \) for \( p = 0 \).

**Remark 7.4.** We note that in contrast to [3] we do not have in (7.4) and (7.3) the conditions \( m_l + p_l + q_l + n_l \geq 1 \) and \( p_l + q_l \geq 1 \), respectively. The proof of Lemma 7.3 is still similar to the proof of Lemma 3.10 in [3] however we have to take into account more terms. These terms are hidden in our notation, see Chapter 4.

**Proof of Lemma 7.3.** We start by estimating the resolvents. Let \( 0 \leq u + \rho_1 r \leq 1 \) for \( u, r \geq 0 \). Then for \( l = 0 \) and \( l = L \) we have

\[
|F_0^{(1, \rho_1)}[t](u + \rho_1 r)| \leq 1, \quad |\partial_r F_0^{(1, \rho_1)}[t](u + \rho_1 r)| \leq \| \chi' \|_{\infty}
\]

and for \( l = 1, \ldots, L - 1 \),

\[
\left\| F_l^{(1, \rho_1)}[t](u + \rho_1 r) \right\| \leq \left\| \frac{\left( x_{\rho_1}^{(1)}(u + \rho_1 r) \right)^2}{t(u + \rho_1 r)} \right\| \leq \frac{8}{\rho_1} \leq \frac{C_{\chi}}{\rho_1},
\]

with \( \hat{C}_{\chi} = 20 \).
\[
\| \partial_r F^{(1, \rho_1)}_I[t](u + \rho_1 r) \|
\leq \left\| \frac{2 \chi^{(1)} \rho_1 (u + \rho_1 r) \partial_r \chi^{(1)}(u + \rho_1 r)}{t(u + \rho_1 r)} \right\| + \left\| \frac{\rho_1 t'(u + \rho_1 r)}{(t(u + \rho_1 r))^2} \right\|
\leq \frac{16}{\rho_1} \| \chi' \|_\infty + \frac{64}{\rho_1} \| t' \|_\infty \leq \frac{\widehat{C}_\chi}{\rho_1} (\| \chi' \|_\infty + 4 \| t' \|_\infty),
\]

where we used a similar equation as (5.22).

Now we estimate \( \| V^{(1, \rho_1)}_{m,p,n,q}[t, w] \| \) and \( \| \partial_r V^{(1, \rho_1)}_{m,p,n,q}[t, w] \| \) using a variant of (5.23).

\[
\| V^{(1, \rho_1)}_{m,p,n,q}[t, w](r, K^{(m, n)}) \|
\leq \prod_{l=0}^{L} \left\| F^{(1, \rho_1)}_l[t](H_f + \rho_1 (r + \tilde{r}_l)) \right\| \prod_{l=1}^{L} \left\| W^{m_i,n_i}_{p_i,q_i}[w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_i,n_i)}) \right\|
\leq \widehat{C}_\chi \rho_1^{-L+1} \prod_{l=1}^{L} \left\| W^{m_i,n_i}_{p_i,q_i}[w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_i,n_i)}) \right\|. \tag{7.6}
\]
Similarly we get with Leibniz’ rule

\[
\| \partial_t V^{(1, \rho_1)}_{m, p, q} [t, w](r, K^{(|m|, |q|)}) \|
\leq \left\{ \sum_{j=0}^L \| \partial_t F_j^{(1, \rho_1)}[t](H_f + \rho_1 (r + \tilde{r}_l)) \| \prod_{l=0}^L \| F_l^{(1, \rho_1)}[t](H_f + \rho_1 (r + \tilde{r}_l)) \| \\
\times \prod_{l=1}^L \| W^{m_l, n_l}_{p_l, q_l} [w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \|
\right\}
\]

\[
+ \prod_{l=0}^L \| F_l^{(1, \rho_1)}[t](H_f + \rho_1 (r + \tilde{r}_l)) \|
\times \left\{ \sum_{j=1}^L \| \partial_t W^{m_l, n_l}_{p_l, q_l} [w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \|
\left\} \right.
\]

\[
\leq (L + 1) \hat{C}_\chi^{-L-1} (\| \chi' \|_\infty + 4 \| t' \|_\infty) \rho_1^{-L+1}
\]

\[
\prod_{l=1}^L \| W^{m_l, n_l}_{p_l, q_l} [w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \|
\]

\[
+ \hat{C}_\chi^{-L-1} \rho_1^{-L+1} \left\{ \rho_1 \sum_{j=1}^L \| \partial_t W^{m_l, n_l}_{p_l, q_l} [\partial_t w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \|
\left\} \right.
\]

\[
\leq (L + 1) \hat{C}_\chi^{-L-1} (1 + \| \chi' \|_\infty + 4 \| t' \|_\infty) \rho_1^{-L+1}
\]

\[
\times \prod_{l=1}^L \{ \| W^{m_l, n_l}_{p_l, q_l} [w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \| + \rho_1 \| \partial_t W^{m_l, n_l}_{p_l, q_l} [\partial_t w](\rho_1 (r + \tilde{r}_l), \rho_1 K^{(m_l, n_l)}) \| \}.
\]

(7.7)
To estimate the $\| \cdot \|_\mu$ norm we shall use the following estimate

\[
\int_{B^{m_1+n_1}} \sup_{r \in [0,1]} \left\| W^{m_1,n_1}_{\rho_1,q_1}[w](r, \rho_1 K^{(m_1,n_1)}) \right\|^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}}
\]

\[
= \rho_1^{2\mu(m_1+n_1)} \int_{B^{m_1+n_1}} \sup_{r \in [0,1]} \left\| W^{m_1,n_1}_{\rho_1,q_1}[w](r, K^{(m_1,n_1)}) \right\|^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}}
\]

\[
\leq \rho_1^{2\mu(m_1+n_1)} \frac{1}{p_1^{m_1} q_1^{n_1}} \int_{B^{m_1+n_1}} \left\| w_{m_1+p_1,n_1+q_1}(\cdot, K^{(m_1,n_1)}) \right\|^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}}
\]

\[
\leq \rho_1^{2\mu(m_1+n_1)} \frac{1}{p_1^{m_1} q_1^{n_1}} \left\| w_{m_1+p_1,n_1+q_1} \right\|^2,
\]

(7.8)

where the first equality follows by the substitution formula for integrals, the second line follows from the estimate in Lemma 4.2, and the last line follows from Fubini’s theorem. Using (7.6) together with (7.8) we find

\[
\left\| V^{(1,\rho_1)}_{m_1,p_1,n_1,q_1}[t, w] \right\|_\mu = \left( \int_{B^{[m_1+n_1]}_2} \left\| V^{(1,\rho_1)}_{m_1,p_1,n_1,q_1}[t, w](K^{(m_1,n_1)}) \right\|^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}} \right)^{1/2}
\]

\[
\leq \tilde{C}^{L-1} \rho_1^{-L+1} \prod_{l=1}^L \left\{ \int_{B^{[m_1+n_1]}_1} \sup_{r \in [0,1]} \left\| W^{m_1,n_1}_{\rho_1,q_1}[w](r, \rho_1 K^{(m_1,n_1)}) \right\|^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}} \right\}^{1/2}
\]

\[
\leq \tilde{C}^{L-1} \rho_1^{-L+1} \rho_1^\mu \prod_{l=1}^L \left\{ \frac{1}{\sqrt{p_1^{m_1} q_1^{n_1}}} \left\| w_{m_1+p_1,n_1+q_1} \right\| \right\}.
\]

(7.9)
Similarly using (7.7) together with (7.8) we find
\[
\| \partial_t V^{(1,\varrho_1)}_{m,p,q} [t, w] (K^{(|m|,|q|)}) \|_\mu \\
= \left( \int_{B^{[m]+[q]}} \| \partial_t V^{(1,\varrho_1)}_{m,p,q} [t, w] (K^{(|m|,|q|)}) \|^2 \frac{dK^{(|m|,|q|)}}{|K^{(|m|,|q|)}|^{3+2\mu}} \right)^{1/2} \\
\leq (L + 1) \tilde{C}_\chi^{L-1} (1 + \| \chi' \|_\infty + 4 \| t' \|_\infty) \varrho_1^{-L+1} \\
\times \prod_{l=1}^L \left\{ \int_{B^{[m]+[q]}} \sup_{r \in [0,1]} \left\{ \| W^{m_1,n_1}_{p_1,q_1} [r, \varrho_1 K^{(m_1,n_1)}] \| \\
+ \rho_1 \| W^{m_1,n_1}_{p_1,q_1} [\partial_r w] (r, \varrho_1 K^{(m_1,n_1)}) \| \right\}^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}} \right\}^{1/2} \\
\leq (L + 1) \tilde{C}_\chi^{L-1} (1 + \| \chi' \|_\infty + 4 \| t' \|_\infty) \varrho_1^{-L+1} \\
\times \prod_{l=1}^L \left\{ 2 \int_{B^{[m]+[q]}} \left( \sup_{r \in [0,1]} \| W^{m_1,n_1}_{p_1,q_1} [r, \varrho_1 K^{(m_1,n_1)}] \| \right)^2 \frac{dK^{(m_1,n_1)}}{|K^{(m_1,n_1)}|^{3+2\mu}} \right\}^{1/2} \\
\leq (L + 1) \tilde{C}_\chi^{L-1} (1 + \| \chi' \|_\infty + 4 \| t' \|_\infty) \varrho_1^{-L+1} \rho_1^{\mu (|m| + |q|)} \prod_{l=1}^L \left\{ \frac{\sqrt{2}}{\sqrt{p_l^{\mu} q_l}} \| w^{m_1+p_1,n_1+q_1} \|_\mu \right\} .
\]
(7.10)

Adding Inequalities (7.9) and (7.10) establishes the estimate in (7.5).

The proof of Theorem 7.1 is similar to the proof of Theorem 3.8 given in [3].

Proof of Theorem 7.1. The operators \( H(t) \) and \( H(w) \) are a Feshbach pair for \( \mathcal{X}_{\varrho_1}^{(1)} \) by Theorem 6.5. First we use Lemma 7.3 to estimate (7.4). We shall write \( C_t := 1 + 2 \| \chi' \|_\infty + 8 \tau_0 \). Since \( C_\chi := \sqrt{2} \tilde{C}_\chi \) and \( \tilde{C}_\chi \geq 1 \), we have \( 2^{L/2} \tilde{C}_\chi^{L-1} \leq C_\chi^L \). Moreover, we use that \( \frac{\mu (|m| + |q|)}{p} \leq 2^{m+p} \). With this we find for \( M+N \geq 1 \)

\[
\| w^{(2,\varrho_1)}_{M,N} \|_\mu \\
\leq \sum_{L=1}^\infty \sum_{m,p,q \in \mathbb{N}_0^L, |m|=M, |q|=N} \left\{ \binom{m+1}{p} \binom{n+q}{q} \right\} (L + 2) C_t^L C_\chi (1+\mu)(M+N) \| w^{m+p,n+q} \|_\mu \\
\leq \sum_{L=1}^\infty \left\{ (L + 2) \left( \frac{C_\chi}{\varrho_1} \right)^L C_t (2\rho_1^{(1+\mu)})^{M+N} \sum_{m,p,q \in \mathbb{N}_0^L, |m|=M, |q|=N} \prod_{l=1}^L \left\{ \left( \frac{2}{\sqrt{p_l}} \right)^m \left( \frac{2}{\sqrt{q_l}} \right)^q \right\} \| w^{m+p,n+q} \|_\mu \right\} .
\]
(7.11)
Inserting this in $\| \cdot \|_{\mu,\xi}^#$ we obtain the following bound

$$
\| (w_{M,N}^{(2,\rho_1)})_{M+N \geq 1} \|_{\mu,\xi}^#
\leq C_t \sum_{M+N \geq 1} \xi^{-(M+N)} \| w_{M,N}^{(2,\rho_1)} \|_{\mu}^#
\leq 2C_t \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L + 2) \left( \frac{C_\chi}{\rho_1} \right)^L \sum_{M+N \geq 1} \prod_{l=1}^{L} \left\{ \left( \frac{2\xi}{\sqrt{p_l}} \right)^p \left( \frac{2\xi}{\sqrt{q_l}} \right)^q \xi^{-(m_l+p_l+n_l+q_l)} \| w_{m_l+p_l,n_l+q_l} \|_{\mu}^# \right\}
\leq 2C_t \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L + 2) \left( \frac{C_\chi}{\rho_1} \right)^L \left\{ \sum_{m,n,p,q \in N_0} \left( \sum_{p=0}^{m} \left( \frac{2\xi}{\sqrt{p}} \right)^p \right) \left( \sum_{q=0}^{m} \left( \frac{2\xi}{\sqrt{q}} \right)^q \right) \xi^{-(m+n)} \| w_{m,n} \|_{\mu}^# \right\}
\leq 2C_t \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L + 2) \left( \frac{C_\chi}{\rho_1} \right)^L 4^L \left( \| w \|_{\mu,\xi}^# \right)^L,
$$

(7.12)

where in the second last inequality we used a substitution of summation variables and in the last inequality we used that $\sum_{p=0}^{\infty} \left( \frac{2\xi}{\sqrt{p}} \right)^p \leq \sum_{p=0}^{\infty} (2\xi)^p = \frac{1}{1-2\xi} \leq 2$ since $0 < \xi \leq 1/4$.
By the assumptions of the theorem we have

$$
\| w \|_{\mu,\xi}^# \leq \gamma.
$$

Inserting this into (7.12) we find

$$
\| (w_{M,N}^{(2,\rho_1)})_{M+N \geq 1} \|_{\mu,\xi}^# \leq 2C_t \rho_1^{(1+\mu)} \sum_{L=1}^{\infty} (L + 2) \left( \frac{4C_\chi \gamma}{\rho_1} \right)^L
\leq 24C_t \rho_1^{(1+\mu)} \left( \frac{4C_\chi \gamma}{\rho_1} \right)^2,
$$

(7.13)

where we used that by assumption

$$
0 \leq \frac{4C_\chi \gamma}{\rho_1} < 1,
$$
and moreover we used that $\sum_{L=1}^{\infty} (L+2)a^L = \sum_{L=2}^{\infty} La^{L-2} = a^{-1} \frac{d}{da} \sum_{L=2}^{\infty} a^L = a^{-1} \frac{d}{da} \frac{a^3}{(1-a)^2}$ for $a \in (0, 1)$.

It remains to estimate $\eqref{eq:7.3}$. To this end we recall that for $m = n = 0$ we see from Lemma 7.3 that

$$\rho_1^{-1} \|V_{0,p,0,q}^{(1,\rho_1)}[t, w]\|_\mu^# \leq (L + 2) C^L_x C_t \rho_1^{-L} \prod_{l=1}^{L} \frac{\|w_{p_l,q_l}\|_\mu^#}{\sqrt{P_{l,t}^q q_l^q}}. \quad \text{(7.14)}$$

Using this we find for the derivative

$$\sup_{r \in [0,1]} |\partial_t \tilde{w}_{0,0}^{(2,\rho_1)}(r) - 1| = \sup_{r \in [0,1]} |\rho_1^{-1} \partial_t t(r) + \rho_1^{-1} \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{p,q \in N_0^L} \partial_t V_{0,p,q}^{(1,\rho_1)}[t, w](r)|$$

$$\leq \|t' - 1\|_\infty + \rho_1^{-1} \sum_{L=1}^{\infty} \sum_{p,q \in N_0^L} \|V_{0,p,q}^{(1,\rho_1)}[t, w]\|_\mu^#$$

$$\leq \tau_1 + \sum_{L=1}^{\infty} (L + 2) C^L_x C_t \rho_1^{-L} \sum_{p,q \in N_0^L} \prod_{l=1}^{L} \frac{\|w_{p_l,q_l}\|_\mu^#}{\sqrt{P_{l,t}^q q_l^q}}$$

$$\leq \tau_1 + \sum_{L=1}^{\infty} (L + 2) C^L_x C_t \rho_1^{-L} \left( \sum_{p,q \in N_0^L} \|w_{p,q}\|_\mu^# \right)^L$$

$$\leq \tau_1 + \sum_{L=1}^{\infty} (L + 2) C^L_x C_t \rho_1^{-L} \left( \sum_{p,q \in N_0} \|w_{p,q}\|_{\mu,\xi} \right)^L$$

$$\leq \tau_1 + \sum_{L=1}^{\infty} (L + 2) C_t \left( \frac{C_x \gamma}{\rho_1} \right)^L$$

$$\leq \rho_1^{-1} \tau_1 + 3 C_t \frac{C_x \gamma}{\rho_1} (1 - \frac{C_x \gamma}{\rho_1})^{-2}, \quad \text{(7.15)}$$

where we used that by assumption

$$0 \leq \frac{C_x \gamma}{\rho_1} < 1. \quad \text{(7.16)}$$

Analogously we estimate

$$|\tilde{w}_{0,0}^{(2,\rho_1)}(0) - \rho_1^{-1} t(0)| \leq |\rho_1^{-1} \sum_{L=1}^{\infty} (-1)^{L-1} \sum_{p,q \in N_0^L} V_{0,p,q}^{(1,\rho_1)][t, w](0)|$$

$$\leq \rho_1^{-1} \sum_{L=1}^{\infty} \sum_{p,q \in N_0^L} \|V_{0,p,q}^{(1,\rho_1)}[t, w]\|_\infty$$

$$\leq 3 C_t \frac{C_x \gamma}{\rho_1} (1 - \frac{C_x \gamma}{\rho_1})^{-2}, \quad \text{(7.17)}$$
provided (7.16) holds. The claim now follows from Equations (7.13), (7.15) and (7.17).

8 Proof of the Main Theorem

Let $B_R$ denote the closed ball with radius $R$.

Theorem 8.1 (Analyticity Theorem of Griesemer-Hasler [16]). Let $\mu > 0$, and let $\rho_{GH} \in (0, 1)$ be sufficiently small. Then for $\xi \in (0, 1)$ sufficiently small there exist positive constants $\alpha_0, \beta_0$ and $\gamma_0$ such that the following holds.

Let $V$ be an open subset of $\mathbb{C}$ and let $E_{at} : V \to \mathbb{C}$ be an analytic function. Let $U$ be an open subset of $\mathbb{C} \times \mathbb{C}$ such that for all $s \in V$ we have

$$\{s\} \times B_{\rho_{GH}}(E_{at}(s)) \subset U.$$  \hfill (8.1)

Suppose $H(\cdot, \cdot)$ is an $L(\mathcal{F})$ valued analytic function on $U$, such that for all $(s, \zeta) \in U$

$$H(s, \zeta) - (E_{at}(s) - \zeta) \in \mathcal{B}^{[1]}(\alpha_0, \beta_0, \gamma_0).$$

Then there exist analytic functions $\zeta_\infty : V \to B_{\rho_{GH}}(s)$ and $\psi_\infty : V \to \mathcal{H}_{\text{red}}$, nowhere vanishing, such that for all $s \in V$

$$H(s, \zeta_\infty(s))\psi_\infty(s) = 0.$$

If furthermore $H(s, \zeta)^* = H(\bar{s}, \bar{\zeta})$ on $U$, then for real $s \in V$ the operator $H(s, \lambda)$ has a bounded inverse for all $\lambda \in (E_{at}(s) - \rho_{GH}, \zeta_\infty(s))$.

The proof of this theorem follows directly from the proof of Theorem 1 in [16, pp. 610-611]. The only difference being that the theorem given above does not involve the initial Feshbach step and has therefore a shorter proof. In the application of Theorem 8.1 which we have in mind, the parameter $s$ in the theorem will be played by the coupling constant $g$. Furthermore the energy cutoff $\rho_{GH}$ will be an order one quantity, i.e., independent of the coupling constant. Now we are ready to prove the main result of this paper.

Proof of Theorem 2.1. Fix $\mu > 0$. Let $\rho_{GH} \in (0, 1/2]$ and $\xi \in (0, 1/4]$ by chosen sufficiently small such that the assertion of Theorem 8.1 holds. The idea will be to choose energy cutoffs, $\rho_0$ and $\rho_1$, at the first and second Feshbach step so that for $g$ in a sectorial region of an annulus, we can apply Theorem 8.1. As we let the outer and inner radius of the annulus tend to zero we will obtain the desired result.
Choose $\epsilon \in (0, \mu)$ and $\alpha \in (0, \min(\mu - \epsilon, 1))$. For $\rho_0 > 0$ we define

$$\rho_1 := \rho_0^{1+2\epsilon+\alpha}. \quad (8.2)$$

We shall assume that

$$\rho_0 \in (0, 1/4). \quad (8.3)$$

We consider the following sectorial region of an annulus, determined by the conditions $g \in S_{\delta_0}$ and

$$c_{\delta_0}^{-1/2} \rho_0^{1+\epsilon+\frac{\alpha}{2}} < |g| < \min(\rho_0^{1+\epsilon}, (8\|Z_{at}\| + 4c_{\delta_0})^{-1} \rho_0^{1/2}, (8\xi^{-1} C_F\|G\|_\mu)^{-1} \rho_0^{1/2}). \quad (8.4)$$

In view of the upper bound in (8.4), we conclude from Theorem 5.1 (Banach Space Estimate for 1st Feshbach Operator) that there exists a finite constant $C^{(1)}$ such that for $\rho_0$ satisfying (8.3) and all $g \in S_{\delta_0}$ with (8.4) and $z \in D_{\rho_0/2}(\epsilon_{at})$ we have

$$H_g^{(1, \rho_0)}(z) - \rho_0^{-1}(\epsilon_{at} - z) \in B^{(d_0)}(C^{(1)} \rho_0^{\frac{1}{2}+\epsilon}, C^{(1)} \rho_0^{\frac{1}{2}+\epsilon}, C^{(1)} \rho_0^{\mu+1+\epsilon}). \quad (8.5)$$

Next we want to use Theorem 6.5 (Feshbach Pair Criterion for 2nd Iteration). Observe that by (8.3) the assumptions on the $\rho$’s and by (8.3) the assumptions (6.12) and (6.13) of Theorem 6.5 are satisfied for $g \in S_{\delta_0}$ with (8.4). To apply the theorem we consider the decomposition defined in (6.16) where

$$H_g^{(1, \rho_0)}(z) = T_g^{(1, \rho_0)}(z) + W_g^{(1, \rho_0)}(z).$$

By Lemma 6.1 (Free Approx to 1st Feshbach) we see that for the difference to the free approximation (6.19) there is some constant $C$ such that

$$\|t_g^{(1, \rho_0)}(z) - t_g^{(1, \rho_0)}(z)\| \leq C\rho_0^{2+2\epsilon}, \quad (8.6)$$

$$\|P_{at}^{(2)} w_{g, 0, 0}^{(1, \rho_0)} P_{at}^{(2)} + P_{at}^{(2)} w_{g, 0, 0}^{(1, \rho_0)} P_{at}^{(2)}\| \leq C\rho_0^{2+2\epsilon}, \quad (8.7)$$

where we used the upper bound in (8.4). By (8.2) we see that the right hand side of (8.6) is less than $\rho_1/4$ provided $\rho_0$ is sufficiently small. Thus by Neumann’s theorem and Lemma 6.3 we see that $T_g^{(1, \rho_0)}(z)$ is invertible on $\text{Ran} \mathbf{X}_{\rho_1}$ and

$$\|(H_{0, 0}(t_g^{(1, \rho_0)}) \upharpoonright \text{Ran} \mathbf{X}_{\rho_1}^{(1)})^{-1}\| \leq \frac{8}{\rho_1}.$$

Furthermore, we see from (8.7) and (8.5) that

$$\|w_{g, \text{int}}^{(1, \rho_0)}(z)\|_{\mu, \xi} \leq C\rho_0^{2+2\epsilon} + C^{(1)} \rho_0^{\mu+1+\epsilon} =: \gamma.$$
We note that
\[ \frac{\gamma}{\rho_1} = C \rho_0^{1-\alpha} + C^{(1)} \rho_0^{-\epsilon-\alpha}, \]
tends to zero for \( \rho_0 \to 0 \). In view of (4.6) we see that condition (6.20) of Theorem 6.5 is satisfied for \( \rho_0 \) small. Thus \( T_g^{(1,\rho_0)}(z) \) and \( W_g^{(1,\rho_0)}(z) \) are a Feshbach pair for \( \chi^{(1)}_{\rho_1} \), provided \( g \in S_{\delta_0}, \) (8.4) holds, and \( z \in D_{\rho_0^{1/2}}(\epsilon_{\text{at}} + g^2 \epsilon_{\text{at}}^{(2)}) \). It now follows from Theorem 7.1 (Analyticity Theorem of Griesemer-Hasler) that for some constant \( C^{(2)} \) we have

\[
S_{\rho_1}(F_{\chi^{(1)}_{\rho_1}}(T_g^{(1,\rho_0)}(z), W_g^{(1,\rho_0)}(z))) = (1) \rho_0^{-1}(\epsilon_{\text{at}} - z + g^2 \epsilon_{\text{at}}^{(2)})
\]

\( \in B^{[1]}(C^{(2)}, \rho_0^{1/2} + C^{(2)} \rho_0^0, C^{(2)} \rho_0^{1/\alpha}) \)

for all \( z \in D_{\rho_0^{1/2}}(\epsilon_{\text{at}} + g^2 \epsilon_{\text{at}}^{(2)}) \) and \( g \in S_{\delta_0} \) with (8.4).

Now we want to apply Theorem 8.1 (Analyticity Theorem of Griesemer-Hasler). To this end we express the Hamiltonian in terms of the variables \( \zeta = \rho_0^{-1} \rho_1^{-1} z \) and \( s = g \). We define the function \( E_{\text{at}}(s) = \rho_0^{-1} \rho_1^{-1}(\epsilon_{\text{at}} + s^2 \epsilon_{\text{at}}^{(2)}) \) on \( \mathbb{C} \) and the function

\[
H(s, \zeta) = S_{\rho_1}(F_{\chi^{(1)}_{\rho_1}}(T_s^{(1,\rho_0)}(\rho_0 \rho_1 \zeta), W_s^{(1,\rho_0)}(\rho_0 \rho_1 \zeta)))
\]

for \( (s, \zeta) \in U := \bigcup_{g \in S_{\delta_0}} D_{1/2}(\epsilon_{\text{at}}(g)) \). Since we expressed \( H(s, \zeta) \) in terms of uniformly convergent Neumann series (Theorems 3.1 and 6.5), one concludes that \( H(s, \zeta) \) is jointly analytic on \( U \) and that the conjugation property \( H(s, \zeta)^* = H(\bar{s}, \bar{\zeta}) \) holds, since each term in the convergent expansion has that property. We conclude now by Theorem 8.1 (Analyticity Theorem of Griesemer-Hasler) (if \( \rho_{GH} \) is less than 1/2, then (8.1) holds) that there exist analytic functions

\[ g \mapsto \zeta_\infty(g), \quad g \mapsto \psi_\infty(g) \]

for \( g \in S_{\delta_0} \) with (8.4) such that

\[ H(g, \zeta_\infty(g)) \psi_\infty(g) = 0. \]

and moreover \( \zeta_\infty(g) \in D_{1/2}(\epsilon_{\text{at}}(g)) \). Expressed in terms of the original variables we obtain from the isospectrality of the Feshbach map, that

\[ E_g := \rho_0 \rho_1 \zeta_\infty(g) \quad \text{and} \quad \psi_g := Q_g^{(0,\rho_0)}(E_g) Q_g^{(1,\rho_1)}(E_g) \psi_\infty(g), \]

where

\[
Q_g^{(0,\rho_0)}(z) := \chi^{(0)}_{\rho_0} - \overline{\chi^{(0)}_{\rho_0}} \left( H_0 - z + \overline{\chi^{(0)}_{\rho_0}} WT^{(1,\rho_0)} \chi^{(0)}_{\rho_0} \right)^{-1} g \overline{\chi^{(0)}_{\rho_0}}
\]

\[
Q_g^{(1,\rho_1)}(z) := \chi^{(1)}_{\rho_1} - \overline{\chi^{(1)}_{\rho_1}} \left( T_g^{(1,\rho_0)}(z) + \overline{\chi^{(1)}_{\rho_1}} W_g^{(1,\rho_0)}(z) \chi^{(1)}_{\rho_1} \right)^{-1} W_g^{(1,\rho_0)} \chi^{(1)}_{\rho_1},
\]

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are eigenvalue and eigenvector of $H_g$. It now follows that $E_g$ and $\psi_g$ are analytic functions of $g \in S_{\delta_0}$ with (8.4) since $Q_0^{(0,\rho_0)}(z)$ and $Q_0^{(1,\rho_1)}(z)$ are analytic functions of $g$ and $z$, as they are given by convergent expansions of jointly analytic functions. Furthermore in terms of the original spectral parameter we have $E_g \in D_{\rho_0 \rho_1/2}(\epsilon_{at} + g^2 \epsilon_{at}^{(2)})$, which implies (2.8). The conjugation property of $H(s, \zeta)$, the last statement in Theorem 8.1, and the isospectrality property of the Feshbach map imply that $E_g$ is the ground state energy of $H_g$. As we take $\rho_0$ to zero, the theorem follows. \hfill \Box

Remark 8.2. We note that the choice for $\rho_0$ and $\rho_1$ in the proof corresponds to $\rho_0$ being larger than $|g|$ and $\rho_1$ being smaller than $|g|$ such that their product is smaller than $|g|^2$.

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A Elementary Estimates

Lemma A.1. We have the following estimate

$$
\|a(G)1_{H_f \leq r}\| \leq \left( \int_{|k| \leq r} \frac{\|G(k)\|^2}{|k|} dk \right)^{1/2} r^{1/2}
$$

(A.1)

Proof. This follows from the following estimate

$$
\|a(G)1_{H_f \leq r}\| = \|a(G1_{|k| \leq r})1_{H_f \leq r}\| = \|a(G1_{|k| \leq r})H_f^{-1/2}H_f^{1/2}1_{H_f \leq r}\|
\leq \|a(G1_{|k| \leq r})H_f^{-1/2}\|\|H_f^{1/2}1_{H_f \leq r}\|
\leq \left( \int_{|k| \leq r} \frac{\|G(k)\|^2}{|k|} dk \right)^{1/2} r^{1/2}.
$$

In [16] the following Lemma is shown.

Lemma A.2. For $G \in L^2(\mathbb{R}^3 \times \mathbb{Z}_2; \mathcal{L}(\mathcal{H}_{at}))$ we have

$$
\|a(G)H_f^{-1/2}\| \leq \|\omega^{-1/2}G\|,
\|a^*(G)(H_f + 1)^{-1/2}\| \leq \|(\omega^{-1} + 1)^{1/2}G\|.
$$

(A.2)
Proof. We will use the notation of eq. \((2.1)\). We set
\[ N := \int a^*(k)a(k)dk \]
and let \( \psi \in \Phi \) for some \( n \in \mathbb{N} \). To prove the first inequality we estimate
\[
\|a(G)\psi\| \leq \int \|G(k)a(k)\psi\|dk = \int \|G(k)|^{-1/2}|k|^{1/2}a(k)\psi\|dk \\
\leq \left( \int |k|\|a(k)\psi\|^2dk \right)^{1/2} \left( \int |k|^{-1}\|G(k)\|^2dk \right)^{1/2} \\
= \left( \int |k|^{-1}\|G(k)\|^2dk \right)^{1/2} \|H^{1/2}\psi\|.
\]
To prove the second we use the commutation relations
\[
\|a^*(G)\psi\|^2 = \langle a^*(G)\psi, a^*(G)\psi \rangle = \langle \psi, a(G)a^*(G)\psi \rangle \\
= \langle \psi, (a^*(G)a(G) + \int \|G(k)\|^2dk)\psi \rangle \\
\leq \left( \int |k|^{-1}\|G(k)\|^2dk \right) \|H^{1/2}\psi\|^2 + \int \|G(k)\|^2dk\|\psi\|^2.
\]

The following lemma states the well known pull-through formula. For a proof see for example [8,22].

Lemma A.3. Let \( f : \mathbb{R}_+ \to \mathbb{C} \) be a bounded measurable function. Then for all \( K \in \mathbb{R}^3 \times \mathbb{Z}_2 \)
\[
f(H_f)a^*(K) = a^*(K)f(H_f + \omega(K)), \quad a(K)f(H_f) = f(H_f + \omega(K))a(K).
\]

### B Field Operators Associated to Integral Kernels

In what follows we shall give a precise meaning to field operators defined by integral kernels. Let \( X := \mathbb{R}^3 \times \mathbb{Z}_2 \). For \( \psi \in \Phi \) having finitely many particles we have
\[
[a(K_1) \cdots a(K_m)\psi]_n(K_{m+1}, \ldots, K_{m+n}) = \sqrt{\frac{(m+n)!}{n!}} \psi_{m+n}(K_1, \ldots, K_{m+n}), \quad (B.1)
\]
for all \( K_1, \ldots, K_{m+n} \in X \), and using Fubini’s theorem it is elementary to see that the vector valued map \( (K_1, \ldots, K_m) \mapsto a(K_1) \cdots a(K_m)\psi \) is an element of \( L^2(X^m; \Phi) \). For measurable functions \( w_{m,n} \) on \( (\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n} \) with values in the linear operators of \( \mathcal{H}_{\text{sat}} \) we define the sesquilinear form
\[
\int_{(\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n}} \frac{dK^{(m,n)}}{K^{(m,n)}|^{1/2}} \langle a(K^{(m)})\varphi, w_{m,n}(K^{(m,n)})a(K^{(n)})\psi \rangle,
\]
defined for all \( \varphi \) and \( \psi \) in \( \mathcal{H} \), for which the integrand on the right hand side is integrable. This yields by the representation theorem a densely defined linear operator, which can easily be shown to be closable. We denote by \( H^{(0)}(w_{m,n}) \) the closure of the operator. By adjusting the notation this also provides a precise definition of the field operators \( a(G) \) and \( a^*(G) \). To define field operators which depend on the free field energy we consider measurable functions \( w_{m,n} \) on \( \mathbb{R}_+ \times (\mathbb{R}^3 \times \mathbb{Z}_2)^{n+m} \) with values in the linear operators of \( \mathcal{H}_{at} \). To such a function we associate the sesquilinear form

\[
q_{w_{m,n}}(\varphi, \psi) := \int_{(\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^{1/2}} \left\langle a(K^{(m)})\varphi, w_{m,n}(H_f, K^{(m,n)})a(\tilde{K}^{(n)})\psi \right\rangle,
\]
defined for all \( \varphi \) and \( \psi \) in \( \mathcal{H} \), for which the integrand on the right hand side is integrable. If the integral kernel decays sufficiently fast as a function of the free field energy, the sesquilinear form defines a bounded operator. For this we can use the following lemma and the identification in (4.2).

**Lemma B.1.** For measurable \( w : X^{m+n} \to C_\infty[0, \infty) \), define

\[
\|w_{m,n}\|_2 := \int_{X^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|^2} \times \sup_{r \geq 0} \left[ \|w_{m,n}(r, K^{(m,n)})\|^2 \prod_{l=1}^m \{r + \Sigma[K^{(l)}]\} \prod_{l=1}^n \{r + \Sigma[\tilde{K}^{(l)}]\} \right] .
\]

Then for all \( \varphi, \psi \in \mathcal{H} \) with finitely many particles

\[
|q_{w_{m,n}}(\varphi, \psi)| \leq \|w_{m,n}\|_2 \|\varphi\| \|\psi\| . \tag{B.2}
\]

In particular if \( \|w_{m,n}\|_2 < \infty \) the form \( q_{w_{m,n}} \) determines uniquely a bounded linear operator \( H_{m,n}(w_{m,n}) \) such that

\[
q_{w_{m,n}}(\varphi, \psi) = \langle \varphi, H_{m,n}(w_{m,n})\psi \rangle ,
\]

for all \( \varphi, \psi \in \mathcal{H} \). Moreover, \( \|H_{m,n}(w_{m,n})\| \leq \|w_{m,n}\|_2 \).

**Proof.** We set \( P[K^{(n)}] := \prod_{l=1}^n (H_f + \Sigma[K^{(l)}])^{1/2} \) and insert 1’s to obtain the trivial identity

\[
|q_{w_{m,n}}(\varphi, \psi)| = \left| \int_{X^{m+n}} \frac{dK^{(m,n)}}{|K^{(m,n)}|} \left\langle P[K^{(m)}]P[K^{(m)}]^{-1}|K^{(m)}|^{1/2} \right.
\]

\[
\times a(K^{(m)})\varphi, w_{m,n}(H_f, K^{(m,n)})P[\tilde{K}^{(n)}]P[\tilde{K}^{(n)}]^{-1}|\tilde{K}^{(n)}|^{1/2}a(\tilde{K}^{(n)})\psi \right\rangle .
\]
The lemma now follows using the Cauchy-Schwarz inequality and the following well known
identity for \( n \geq 1 \) and \( \phi \in \mathcal{F} \),
\[
\int_X dK^{(n)}|K^{(n)}| \left| \prod_{l=1}^n \left[ H_f + \Sigma[K^{(l)}] \right]^{-1/2} a(K^{(n)})\phi \right|^2 = \| P_{\Omega}^+ \phi \|^2, \quad (B.3)
\]
where \( P_{\Omega}^+ := |\Omega \rangle \langle \Omega| \). A proof of (B.3) can for example be found in [22] Appendix A. The
last statement of the lemma follows from the first and the representation theorem. \( \square \)

\section*{C Generalized Wick Theorem}

For \( m, n \in \mathbb{N}_0 \) let \( \mathcal{M}_{m,n} \) denote the space of measurable functions on \( \mathbb{R}_+ \times (\mathbb{R}^3 \times \mathbb{Z}_2)^{m+n} \)
with values in the linear operators of \( \mathcal{H}_\text{at} \). Let
\[
\mathcal{M} = \bigoplus_{m+n=1} \mathcal{M}_{m,n}.
\]
Then using the notation introduced in Section 5 we define for \( w \in \mathcal{M} \)
\[
W[w] := \sum_{m+n=1} H_{m,n}^{(0)}(w). \quad (C.1)
\]
Moreover for \( w \in \mathcal{W}_{\xi}^{[d]} \) we define, similar to (4.5),
\[
W[w] := \sum_{m,n \in \mathbb{N}_0} H_{m,n}(w). \quad (C.2)
\]
The following Theorem is from [8]. It is a generalization of Wick’s Theorem.

**Theorem C.1.** Let \( w \in \mathcal{M} \) or \( w \in \mathcal{W}_{\xi}^{[d]} \) and let \( F_0, F_1, ..., F_L \in \mathcal{M}_{0,0} \) resp. \( F_0, F_1, ..., F_L \in \mathcal{W}_{0,0}^{[d]} \). Then as a formal identity
\[
F_0(H_f)W[w]F_1(H_f)W[w] \cdots W[w]F_{L-1}(H_f)W[w]F_L(H_f) = H(\tilde{w}^{(\text{sym})}),
\]
where \( \tilde{w}^{(\text{sym})} \) is the symmetrization w.r.t. \( k^{(M)} \) and \( \tilde{k}^{(N)} \) of
\[
\tilde{w}_{M,N}(r;K^{(M,N)}) = \sum_{m_1+...+m_L=M} \sum_{n_1+...+n_L=N} \prod_{l=1}^L \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \right\} \times F_0(r+\tilde{r}_0)\langle \Omega | \prod_{l=1}^{L-1} \left\{ W_{m_l,q_l}[r+q_l] F_l(H_f+r+\tilde{r}_l) \right\} \times W_{m_L,n_L}[r+r_L; K_L^{(m_L,n_L)}] \Omega \rangle F_L(r+\tilde{r}_L), \quad (C.3)
\]
with
\[ K^{(M,N)} := (K_1^{(m_1,n_1)}, \ldots, K_L^{(m_L,n_L)}), \quad K_t^{(m_l,n_l)} := (k_t^{(m_l)}, \tilde{k}_t^{(n_l)}), \tag{C.4} \]
\[ r_t := \sum k_t^{(n_1)} + \cdots + \sum k_{t+1}^{(n_{l-1})} + k_{t+1}^{(m_{l-1})} + \cdots + k_L^{(m_L)}, \tag{C.5} \]
\[ \tilde{r}_t := \sum \tilde{k}_t^{(n_1)} + \cdots + \sum \tilde{k}_t^{(n_l)} + \sum \tilde{k}_L^{(m_{l-1})} + \cdots + \tilde{k}_L^{(m_L)}. \tag{C.6} \]

In the case (C.1) we set
\[ W^{m_l,n_l}[w](\cdot; K_t^{(m_l,n_l)}) = W^{(0)}_{m_l,n_l}[w](K_t^{(m_l,n_l)}), \]
which was defined in (5.11) and in case (C.2) we use (7.2), i.e.
\[ W^{m_l,n_l}[w](r; K_t^{(m_l,n_l)}) = W^{m_l,n_l}[w](r, K_t^{(m_l,n_l)}). \]

A proof can be found in [8]. We note that the proof is essentially the same as the proof of Theorem 3.6 in [3] or Theorem 7.2 in [22].

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