Gauge Model With Extended Field Transformations
in Euclidean Space

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Abstract

An $SO(4)$ gauge invariant model with extended field transformations is examined in four dimensional Euclidean space. The gauge field is $(A^\mu)^{\alpha\beta} = \frac{1}{2} g^{\mu\nu\lambda} (M^\nu\lambda)^{\alpha\beta}$ where $M^\nu\lambda$ are the $SO(4)$ generators in the fundamental representation. The $SO(4)$ gauge indices also participate in the Euclidean space $SO(4)$ transformations giving the extended field transformations. We provide the decomposition of the reducible field $t^{\mu\nu\lambda}$ in terms of fields irreducible under $SO(4)$. The $SO(4)$ gauge transformations for the irreducible fields mix fields of different spin. Reducible matter fields are introduced in the form of a Dirac field in the fundamental representation of the gauge group and its decomposition in terms of irreducible fields is also provided. The approach is shown to be applicable also to $SO(5)$ gauge models in five dimensional Euclidean space.

1 Introduction

Pure Yang-Mills gauge models with extended field transformations (or extended gauge models) were introduced some time ago [1]. In these models the gauge group indices also participate in the space-time “Lorentz” transformations. Two features of such models of note are the expansion of the gauge field in terms of a number of fields which are “Lorentz” irreducible, and the mixing of these fields in the gauge transformations of the model.

In this paper we examine the special case of an extended gauge model in four dimensional Euclidean space ($4dE$) with gauge group $SO(4)$. The “Lorentz” transformations in $4dE$ are then $SO(4)$ transformations. The $SO(4)$ gauge group indices participate in the $SO(4)$ space-time transformations in a well defined way. This model has recently been examined [2] and shown to have a number of interesting features. We now examine the field content of the model in terms of fields irreducible under the “Lorentz” $SO(4)$ space-time transformations. We also introduce matter fields coupled to the gauge fields: the matter fields being Dirac 4-spinors in the fundamental representation of the $SO(4)$ gauge group. For the matter fields also the gauge group indices participate in the $SO(4)$ space-time transformations. As a
consequence, the matter fields also are $SO(4)$-reducible. Their decomposition in terms of $SO(4)$-irreducible fields is straightforward.

Once the decomposition of the gauge and matter fields has been established, it is a relatively easy exercise to deduce the form of the $SO(4)$ gauge transformations for the irreducible fields. In these gauge transformations we observe the mixing of the different $SO(4)$-irreducible representations. Indeed, in terms of their spin content, we see mixing in the same transformation of

(i) fields of spin 1 and fields of spin 2 and spin 1, and

(ii) fields of spin $\frac{1}{2}$ and fields of spin $\frac{3}{2}$ and $\frac{1}{2}$.

This mixing of fields of different spin is not to be confused with supersymmetry. The set of fields which mix together under the gauge transformations are always either integer spin multiplets, or half-odd-integer spin multiplets. This mixing of fields of different spin under a symmetry transformation provides a second example (in addition to supersymmetry) of how the strictures of the Coleman-Mandula theorem can be circumvented.

In a recent paper [3] we have examined in detail the spinor representations of $SO(4)$ and the associated four- and two-component spinor fields in $4dE$. The spinor representations of $SO(4)$ are quite different from those of $SO(3,1)$, the Lorentz group of space-time transformations in four dimensional Minkowski space. We include in the appendices a review of the salient features of this analysis together with a list of relevant mathematical formulae required in our discussion.

The remainder of this paper is organized as follows. In Section 2 we introduce the extended $SO(4)$ gauge model. In Section 3 we expand the $SO(4)$-reducible gauge field in terms of $SO(4)$-irreducible gauge fields. In Section 4 we carry out the corresponding expansion for the reducible matter spinor fields. In Section 5 we examine the gauge transformations in terms of the $SO(4)$ irreducible fields. We also examine the Lagrangian density in terms of the $SO(4)$ irreducible fields. In Section 6 we discuss some of the general features of the model, and we indicate how the analysis can be extended to higher dimensions.
2 Extended $SO(4)$ Gauge Model

The Lagrangian density for an extended pure Yang-Mills gauge model in $4dE$ has the usual structure in terms of the gauge field $A_\mu$ and the associated field strength $F_{\mu\nu}$, namely

$$L_{YM} = \frac{1}{4} Tr F^{\mu\nu} F_{\mu\nu}$$

(1)

where

$$F^{\mu\nu} = \partial^\lambda A_\lambda - \partial^\rho A_\rho + i[A_\mu, A_\nu].$$

(2)

In ordinary gauge theories the gauge group and the set of fields can be chosen quite independently and arbitrarily. The same is not true in the extended gauge models. In order that the gauge indices participate in the space-time transformations of the fields it is necessary that the dimensionality of the fundamental matrix representation of the gauge group, say $d \times d$, exactly matches the corresponding representations of the space-time symmetry group. This restricts greatly the possible choice of gauge group. We make the simplest choice of gauge group, namely $SO(4)$ itself. The Hermitian generators of $SO(4)$ are, in the fundamental representation,

$$(M^{\mu\nu})^{\alpha\beta} = i \left( \delta^{\mu\alpha} \delta^{\nu\beta} - \delta^{\mu\beta} \delta^{\nu\alpha} \right)$$

(3)

satisfying

$$[M^{\mu\nu}, M^{\lambda\rho}] = i \left( \delta^{\nu\lambda} M^{\mu\rho} + \delta^{\mu\rho} M^{\nu\lambda} - \delta^{\mu\lambda} M^{\nu\rho} - \delta^{\nu\rho} M^{\mu\lambda} \right)$$

(4)

and we will write the gauge field as

$$(A^{\mu})^{\alpha\beta} = \frac{1}{2} t^{\mu\lambda} (M^{\nu\lambda})^{\alpha\beta} \quad (t^{\mu\lambda} = -t^{\mu\lambda}).$$

(5)

$$= i t^{\mu\alpha\beta}.$$

(6)

Despite the occurrence of the factor of $i$ in (6), we note that the $t^{\mu\alpha\beta}$ are real fields, as $(A^{\mu})^{\alpha\beta} = (A^{\mu})^{\beta\alpha*} = (i t^{\mu\alpha\beta})^* = i t^{\mu\alpha\beta}$. In terms of $t$ the field strength is, from (2), given by

$$(F^{\mu\nu})^{\alpha\beta} = \frac{1}{2} \left[ \partial^\rho t^{\mu\lambda\rho} - \partial^\rho t^{\nu\lambda\rho} + \frac{1}{2} \left( t^{\mu\tau\lambda} t^{\nu\tau\rho} - t^{\nu\tau\lambda} t^{\mu\tau\rho} \right) \right] (M^{\lambda\rho})^{\alpha\beta}$$

(7a)

$$= i \left[ \partial^\rho t^{\mu\alpha\beta} - \partial^\rho t^{\nu\alpha\beta} + \frac{1}{2} \left( t^{\mu\tau\alpha} t^{\nu\tau\beta} - t^{\nu\tau\alpha} t^{\mu\tau\beta} \right) \right].$$

(7b)
We now introduce Dirac 4-spinor matter fields in the fundamental representation of \(SO(4)\) \(\Psi^\alpha\). For the matter fields we choose the following \(SO(4)\) gauge invariant, and Hermitian, Lagrangian density in 4\(d_E\)

\[
L_{\text{matter}} = \Psi^\dagger (i\gamma \cdot \partial + \gamma \cdot A) \Psi
= \Psi^{\alpha \dagger} (i\delta^{\alpha\beta} \gamma \cdot \partial + \gamma \cdot A^{\alpha\beta}) \Psi^\beta.
\]  

\[(8a)\]

\[
= i\Psi^{\alpha \dagger} \gamma^\mu (\delta^{\alpha\beta} \partial^\mu + t^{\mu\alpha\beta}) \Psi^\beta.
\]  

\[(8b)\]

We use Hermitian \(\gamma\)-matrices. Our Euclidean space conventions are summarised in Appendix A and are discussed in more detail in [3]. The action

\[
S = \int d^4 x \left( L_{YM} + L_{\text{matter}} \right)
\]  

\[(9)\]

is clearly invariant under the usual local \(SO(4)\) gauge transformations

\[
\Psi(x) \longrightarrow U(\omega(x)) \Psi(x)
\]  

\[(10a)\]

\[
A^\mu(x) \longrightarrow U(\omega(x))(A^\mu + i\partial^\mu)U^{-1}(\omega(x))
\]  

\[(10b)\]

where \(U(\omega(x)) = \exp \left[ \frac{i}{2} \omega^{\lambda\rho}(x) M^{\lambda\rho} \right]\). The model of (9) is an \(SO(4)\) gauge model.

The space-time symmetry group in 4\(d_E\) is \(SO(4)\). In our model we extend the usual field transformations under this \(SO(4)\) group by allowing the gauge group indices \((\alpha\beta\) on \(A^\mu\) and \(\mu\) on \(\Psi\)) to participate in the transformations, as follows,

\[
(A^\mu)^{\alpha\beta}(x) \longrightarrow (A'^\mu)^{\alpha\beta}(x') = \Lambda^{\mu\nu} (U(\lambda) A^\nu(x) U^{-1}(\lambda))^{\alpha\beta}
\]  

\[(11)\]

\[
= \Lambda^{\mu\nu} U(\lambda)^{\alpha\rho} (A^\nu(x))^{\rho\sigma} U^{-1}(\lambda)^{\sigma\beta}
\]

\[
\Psi_i^\mu(x) \longrightarrow \Psi_i'^\mu(x') = S_{ij}(\lambda) (U(\lambda) \Psi_j(x))^\mu
\]  

\[(12)\]

\[
= S_{ij}(\lambda) U(\lambda)^{\mu\rho} \Psi_j^\rho(x)
\]

where

\[
S_{ij}(\lambda) = \exp \left[ \frac{i}{2} \lambda^{\mu\nu} \Sigma_{\mu\nu} \right]_{ij},
\]  

\[(13)\]

\[
U(\lambda)^{\alpha\rho} = \Lambda^{\alpha\rho},
\]  

\[(14)\]
\[ x'^\mu = \Lambda'^{\mu \nu} x^\nu. \]  

(Here, \( \lambda^{\mu \nu} = -\lambda^{\nu \mu} \) is the matrix of four dimensional angles parametrising the general four dimensional rotation matrix \( \Lambda \); for the precise relationship between \( \lambda \) and \( \Lambda \) see \cite{4}.)

It is clear that the gauge potential \( t^{\mu \nu \lambda} \) introduced in (5) is reducible under \( SO(4) \), i.e. the gauge potential transforms, under the space-time \( SO(4) \) transformations, according to a non-irreducible representation. (The use of non-irreducible representations of the Lorentz group has previously been suggested by Dirac \cite{5}. ) Indeed, the matter field \( \Psi^\alpha \) is similarly \( SO(4) \)-reducible. We now consider the decomposition of \( t^{\mu \nu \lambda} \) and \( \Psi^\alpha \) into \( SO(4) \)-irreducible components.

The gauge potential \( t^{\mu \nu \lambda} \) decomposes in terms of

- a vector field \( v^\mu \)
- an axial vector field \( a^\mu \)
- two rank 3 tensor fields \( \Delta^{\mu \nu \lambda}_{S,A} \) which are

  (1) anti-symmetric in the final two indices:

  \[ \Delta^{\mu \nu \lambda}_{S,A} = -\Delta^{\mu \lambda \nu}_{S,A} \quad (16) \]

  (2) traceless over the first and either the second or third indices:

  \[ \Delta^{\mu \mu \lambda}_{S,A} = \Delta^{\mu \lambda \mu}_{S,A} = 0 \quad (17) \]

  (3) self-dual \( S \) and anti-self dual \( A \) respectively with respect to the final two indices:

  \[ \ast \Delta^{\mu \nu \lambda}_{S,A} = \frac{1}{2} \epsilon^{\nu \lambda \alpha \beta} \Delta^{\mu \alpha \beta}_{S,A} = \pm \Delta^{\mu \nu \lambda}_{S,A}. \quad (18) \]

The decomposition of the gauge potential is given by

\[ t^{\mu \nu \lambda} = \delta^{\mu \nu} v^\lambda - \delta^{\mu \lambda} v^\nu + \epsilon^{\mu \nu \lambda \rho} a^\rho + \Delta^{\mu \nu \lambda}_{S} + \Delta^{\mu \nu \lambda}_{A}. \quad (19) \]

The matter field 4-spinor \( \Psi^\alpha \) decomposes in terms of

\[ \Psi^\alpha = \cdots \]
• a Dirac 4-spinor $\Psi$, and

• an anti-symmetric tensor-spinor $\Psi^{\alpha\beta}$ where

\[
(1) \quad \Psi^{\alpha\beta}_R = \frac{1}{2} (1 + \gamma_5) \Psi^{\alpha\beta} \text{ is self-dual} \Rightarrow \star \Psi^{\alpha\beta}_R = \frac{1}{2} \epsilon^{\alpha\beta\lambda\rho} \Psi^{\lambda\rho}_R = \Psi^{\lambda\rho}_R \\
(2) \quad \Psi^{\alpha\beta}_L = \frac{1}{2} (1 - \gamma_5) \Psi^{\alpha\beta} \text{ is anti-self dual} \Rightarrow \star \Psi^{\alpha\beta}_L = \frac{1}{2} \epsilon^{\alpha\beta\lambda\rho} \Psi^{\lambda\rho}_L = -\Psi^{\alpha\beta}_L \tag{21}
\]

(together (20) and (21) imply that $\star \Psi^{\alpha\beta} = \gamma_5 \Psi^{\alpha\beta}$) and the decomposition of the matter field is

\[
\Psi^\alpha = \frac{1}{2} \gamma^\alpha \Psi + \gamma^\lambda \Psi^{\lambda\alpha}. \tag{22}
\]

The derivation of these results, (19) and (22), and the spin content of these $SO(4)$ irreducible decompositions are treated in the following sections, and the spin content of these $SO(4)$ irreducible decompositions discussed.

\section{3 Decomposition of Reducible Gauge Potential}

It is well known [6] that $SO(4) = SU(2) \times SU(2)$, and that the representations of $SO(4)$ can be labelled by pairs of $SU(2)$ labels. The occurrence of half-odd-integer labels for $SU(2)$ indicates spinor representations, eg the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations correspond to the two fundamental inequivalent 2-spinor representations of $SO(4)$. The bispinor $(\frac{1}{2}, \frac{1}{2})$ representation corresponds to the fundamental, or 4-vector, representation of $SO(4)$.

The reducible gauge potential $A^{\mu\lambda}$ involves a product of three $(\frac{1}{2}, \frac{1}{2})$ representations. In a product of two $(\frac{1}{2}, \frac{1}{2})$ (representations)

\[
(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) = (1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0) \tag{23}
\]
the anti-symmetric combination of the two 4-vector indices $\nu$ and $\lambda$ corresponds to $(1, 0) \oplus (0, 1)$, while the symmetric combination corresponds to $(1, 1) \oplus (0, 0)$. Consequently, corresponding to $t^{\mu \nu \lambda}$ is the product of representations

$$(\frac{1}{2}, \frac{1}{2}) \otimes [(1, 0) \oplus (0, 1)] = (\frac{3}{2}, \frac{1}{2}) \oplus (\frac{5}{2}, \frac{1}{2}) \oplus (\frac{3}{2}, \frac{3}{2}).$$

Each of the representations on the right-hand side of (24) is irreducible. This is the decomposition that we are seeking.

The spin-content of these irreducible representations can be identified if we focus on the $SO(3)$ subgroup of $SO(4)$ corresponding to spatial rotations: $(\frac{3}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{3}{2})$ each contain spins one and two while $(\frac{1}{2}, \frac{1}{2})$ contains spin zero and one. Thus $t^{\mu \nu \lambda}$ will involve two spin-two states, four spin-one states and two spin-zero states - in all twenty four independent degrees of freedom in agreement with the expected number for $t^{\mu \nu \lambda} = - t^{\mu \lambda \nu}$.

To describe the decomposition of $t^{\mu \nu \lambda}$ corresponding to (24) it is necessary to use the dotted and undotted spinor notation for $SO(4)$. This notation is explained in detail in Appendix [3] and a summary of its salient features is provided in Appendix A. It is crucial to note that the analysis differs considerably from the corresponding analysis for $SO(3, 1)$ in four dimensional Minkowski space.

The first step in the decomposition is to write $t^{\mu \nu \lambda}$ as the sum of self-dual and anti-self-dual parts

$$t^{\mu \nu \lambda} = t^{\mu \nu \lambda}_S + t^{\mu \nu \lambda}_A$$

where

$$t^{\mu \nu \lambda}_S = \frac{1}{2} \left[ t^{\mu \nu \lambda} + * t^{\mu \nu \lambda} \right],$$

$$t^{\mu \nu \lambda}_A = \frac{1}{2} \left[ t^{\mu \nu \lambda} - * t^{\mu \nu \lambda} \right],$$

with

$$* t^{\mu \nu \lambda} = \frac{1}{2} \epsilon^{\nu \lambda \alpha \beta} t_{\mu \alpha \beta}.$$

The second step is to transform from the anti-symmetric $\nu, \lambda$ indices for the self-and anti-self-dual parts of $t^{\mu \nu \lambda}$ to spinor indices (see Appendix B)

$$t^{\mu \nu \lambda}_S = - \frac{1}{2} t_{S \mu}^\nu a (\sigma^\lambda)_b^a,$$
\[ t^\mu_{\nu\lambda} = -\frac{1}{2} t^\mu_A \sigma^b (\delta_{\nu\lambda})^b_\sigma \]  \hspace{1cm} (26b)

where

\[ t^\mu_{S a} = t^\mu_S (\sigma_{\nu\lambda})^a_\nu \]  \hspace{1cm} (26c)
\[ t^\mu_{A \dot{a}} = t^\mu_A (\delta_{\nu\lambda})^a_\sigma \]  \hspace{1cm} (26d)

Next, we transform from the remaining 4-vector index \( \mu \) to bi-spinor indices (see Appendix B)

\[ t^\mu_{S a} = \frac{1}{2} (t_S)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} \]  \hspace{1cm} (27a)
\[ t^\mu_{A \dot{a}} = \frac{1}{2} (t_A)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} \]  \hspace{1cm} (27b)

where

\[ (t_S)_{\dot{m} n a b} = t^\mu_{S a} (\sigma^\mu)^{\dot{m} n m} \]  \hspace{1cm} (27c)
\[ (t_A)_{\dot{m} n a b} = t^\mu_{A \dot{a}} (\sigma^\mu)^{\dot{m} n m} \]  \hspace{1cm} (27d)

Putting these three steps together we find

\[ t^\mu_{\nu\lambda} = -\frac{1}{4} \left[ (t_S)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} (\sigma^\nu_{\lambda})^a_\nu + (t_A)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} (\delta_{\nu\lambda})^{\dot{m} n m} \right] 

= \frac{1}{4} \left[ (t_S)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} (\sigma^\nu_{\lambda})^a_\nu + (t_A)_{\dot{m} n a b} (\sigma^\mu)^{\dot{m} n m} (\sigma^\nu_{\lambda})^{\dot{m} n m} \right]. \]  \hspace{1cm} (28)

\((t_S)_{\dot{m} n a b}\) is explicitly symmetric in \((a, b)\) but has no particular symmetry with respect to \((m, a)\) (or \((m, b)\)). Thus we can expand it in terms of the anti-symmetric matrix \(\epsilon_{ma}\) and the symmetric matrices \((\sigma^{\alpha\beta})_{ma}\)

\[ (t_S)_{\dot{m} n a b} = \frac{1}{2} \left[ p_{\dot{m} n a b} \epsilon_{mb} + p_{\dot{m} n a m} + (\Delta_S)_{m}^{\alpha n a} (\sigma^{\alpha\beta})_{mb} + (\Delta_S)_{m}^{\alpha n a} (\sigma^{\alpha\beta})_{mb} \right] 

= \frac{1}{2} \left[ p_{\dot{m} n a b} \epsilon_{mb} + p_{\dot{m} n a m} + (\Delta_S)_{m}^{\mu a n} \left( (\sigma^\mu)^{\alpha n a} (\sigma^{\alpha\beta})_{mb} + (\sigma^\mu)^{\alpha n a} (\sigma^{\alpha\beta})_{mb} \right) \right]. \]  \hspace{1cm} (29)

In this expansion \(p_{\dot{m} n a}\) corresponds to the \((\frac{1}{2}, \frac{1}{2})\) part of \((t_S)_{\dot{m} n a b}\) while \((\Delta_S)_{m}^{\mu a n}\) \((= \frac{1}{2} (\Delta_S)_{m}^{\alpha n a} (\sigma^\mu)^{\alpha n a})\) corresponds to the \((\frac{3}{2}, \frac{1}{2})\) part. However, the \((\frac{3}{2}, \frac{1}{2})\) part must be totally symmetric in the three spinor indices \((m, a, b)\); as the second term in (29) is not explicitly symmetric in \((m, a)\) we impose this symmetry by contracting the term with \(\epsilon_{am}\) and equating to zero

\[ \Delta_S^{\mu a n} \epsilon_{am} \left[ (\sigma^\mu)^{\alpha n a} (\sigma^{\alpha\beta})_{mb} + (\sigma^\mu)^{\alpha n a} (\sigma^{\alpha\beta})_{mb} \right] = 0 \]  \hspace{1cm} (30)
which immediately gives (see Appendix A) 
\[
0 = \Delta_S^{\mu\alpha\beta} (\sigma^\alpha \sigma^\beta)_{bn} = -\frac{1}{2} \Delta_S^{\mu\alpha\beta} \left[ \delta^{\alpha\mu} \sigma^\beta - \delta^{\beta\mu} \sigma^\alpha + \epsilon^{\alpha\beta\mu\rho} \sigma^\rho \right]_{bn}.
\] (31)
But, as \( \epsilon^{\mu\alpha\beta\rho} \Delta_S^{\mu\alpha\beta} = 2 \ast \Delta_S^{\mu\rho} = 2 \Delta_S^{\mu\rho} \), we find 
\[
\Delta_S^{\mu\rho} \sigma^\rho_{bn} = 0 .
\] (32)
We conclude that the \((\frac{3}{2}, \frac{1}{2})\) part of \((t_S)_{m\bar{n}ab}\) is identified by imposing the tracelessness constraint 
\[
\Delta_S^{\mu\mu\rho} = 0 .
\] (33)
Using (29) in the first term of (28) allows us to identify the self-dual part of \(t^{\mu\nu\lambda}\) as
\[
t_S^{\mu\nu\lambda} = \frac{1}{4} \left[ p^\rho (\sigma^\rho)_{anh} (\sigma^\mu)_{hb} (\sigma^\nu)_{b} a - \Delta_S^{\rho\alpha\beta} (\sigma^\rho)_{anh} (\sigma^\alpha\beta)_{m} b (\sigma^\mu)_{m} (\sigma^\nu)_{b} a \right] \\
= \frac{1}{4} \left( p^\rho \text{Tr} \left[ \sigma^\rho \sigma^\mu \sigma^\nu \right] - \Delta_S^{\rho\alpha\beta} \text{Tr} \left[ \sigma^\rho \sigma^\alpha \sigma^\beta \sigma^\nu \right] \right) .
\] (34)
Evaluating the traces of the products of \(\sigma\)-matrices using the various identities provided in Appendix A we find
\[
t_S^{\mu\nu\lambda} = \frac{1}{4} \left( \delta^{\mu\nu} p^\lambda - \delta^{\mu\lambda} p^\nu + \epsilon^{\mu\nu\lambda\rho} p^\rho \right) + \frac{1}{2} \left( \Delta_S^{\mu\nu\lambda} - \Delta_S^{\nu\lambda\mu} - \Delta_S^{\lambda\mu\nu} \right).
\] (35)
The term in (35) involving \(\Delta_S\) can be simplified using the result (B.4) derived in Appendix B, namely
\[
\Delta_S^{\mu\nu\lambda} + \Delta_S^{\nu\lambda\mu} + \Delta_S^{\lambda\mu\nu} = 0
\]
to give
\[
t_S^{\mu\nu\lambda} = \frac{1}{4} \left( \delta^{\mu\nu} p^\lambda - \delta^{\mu\lambda} p^\nu + \epsilon^{\mu\nu\lambda\rho} p^\rho \right) + \Delta_S^{\mu\nu\lambda} .
\] (37)
We note that \(\Delta_S^{\mu\nu\lambda}\) has \(\frac{1}{2}(4.6) - 4 = 8\) degrees of freedom - the self-duality condition giving rise to the factor \(\frac{1}{2}\) and the tracelessness constraint giving rise to the \(-4\) - appropriate to the \((\frac{3}{2}, \frac{1}{2})\) representation.

The anti-self-dual part of \(t^{\mu\nu\lambda}\) can be expanded in an analogous manner. In that case we find
\[
t_A^{\mu\nu\lambda} = \frac{1}{4} \left( \delta^{\mu\nu} q^\lambda - \delta^{\mu\lambda} q^\nu - \epsilon^{\mu\nu\lambda\rho} q^\rho \right) + \Delta_A^{\mu\nu\lambda}
\] (38)
where \( q^\lambda \) belongs to the \( (\frac{1}{2}, \frac{1}{2}) \) representation and \( \Delta_A^{\mu\nu\lambda} \) is anti-self-dual

\[
*\Delta_A^{\mu\nu\lambda} = \frac{1}{2} \varepsilon^{\nu\lambda\alpha\beta} \Delta_A^{\mu\alpha\beta} = -\Delta_A^{\mu\nu\lambda},
\]

(39a)
satisfies

\[
\Delta_A^{\mu\mu\rho} = 0,
\]

(39b)
and belongs to the \( (\frac{1}{2}, \frac{3}{2}) \) representation.

Combining the results (37) and (38) for \( t_S \) and \( t_A \) we finally obtain for the decomposition of the reducible gauge potential in terms of irreducible \( SO(4) \) components

\[
t^{\mu\nu\lambda} = \delta^{\mu\nu} v^\lambda - \delta^{\mu\lambda} v^\nu + \varepsilon^{\mu\nu\lambda\rho} a^\rho + \Delta_S^{\mu\nu\lambda} + \Delta_A^{\mu\nu\lambda}
\]

(40)
where \( v^\lambda = \frac{1}{4} (p + q)^\lambda \) and \( a^\lambda = \frac{1}{4} (p - q)^\lambda \). We note that the decomposition derived here is somewhat different from that proposed in [1].

To assist in the analysis of the gauge transformations of the fields in Section 5 we indicate now the projection from the reducible gauge potential to each of its irreducible components. The projection for the vector field is straightforward

\[
v^\lambda = \frac{1}{3} t^{\mu\mu\lambda}.
\]

(41)
The projection for the axial vector field can be similarly written, noting the dual of (40) above, namely

\[
*t^{\mu\nu\lambda} = \delta^{\mu\nu} a^\lambda - \delta^{\mu\lambda} a^\nu + \varepsilon^{\mu\nu\lambda\rho} v^\rho + \Delta_S^{\mu\nu\lambda} - \Delta_A^{\mu\nu\lambda}
\]

(42)
so that the roles of \( v \) and \( a \) are interchanged in \( *t^{\mu\nu\lambda} \). The projection is

\[
a^\lambda = \frac{1}{3} *t^{\mu\mu\lambda} = \frac{1}{6} \varepsilon^{\mu\beta\gamma\lambda} t^{\mu\beta\gamma}.
\]

(43)
We introduce two “orthogonal” linear combinations of the fields \( \Delta_S \) and \( \Delta_A \), namely

\[
\Delta_{\pm}^{\mu\nu\lambda} = \Delta_S^{\mu\nu\lambda} \pm \Delta_A^{\mu\nu\lambda}
\]

(44)
which are dual to one another

\[
*\Delta_+^{\mu\nu\lambda} = \frac{1}{2} \varepsilon^{\nu\lambda\alpha\beta} \Delta_+^{\mu\alpha\beta} = \Delta_-^{\mu\nu\lambda}.
\]

(45)
The projection from $t$ to $\Delta_+$ is found by using (41) and (43) above in

$$\Delta_+^{\mu\nu\lambda} = t^{\mu\nu\lambda} - \left(\delta^{\mu\nu} v^\lambda - \delta^{\mu\lambda} v^\nu + \epsilon^{\mu\nu\lambda\rho} a^\rho\right)$$

to find

$$\Delta_+^{\mu\nu\lambda} = \frac{1}{3} \left[2t^{\mu\nu\lambda} - t^{\lambda\mu\nu} - t^{\nu\lambda\mu} - \delta^{\mu\nu} t^{\alpha\alpha\lambda} + \delta^{\mu\lambda} t^{\alpha\alpha\nu}\right]. \quad (46)$$

The relationship between $^*t$ and $\Delta_-$ evident in (42) above leads to a similar projection from $^*t$ to $\Delta_-$, namely

$$\Delta_-^{\mu\nu\lambda} = \frac{1}{3} \left[2^*t^{\mu\nu\lambda} - ^*t^{\lambda\mu\nu} - ^*t^{\nu\lambda\mu} - \delta^{\mu\nu} ^*t^{\alpha\alpha\lambda} + \delta^{\mu\lambda} ^*t^{\alpha\alpha\nu}\right]. \quad (47)$$

Combining (46) and (47) we now find the projections to $\Delta_{S,A}$

$$\Delta_{S,A}^{\mu\nu\lambda} = \frac{1}{6} \left[2(t^{\mu\nu\lambda} \pm ^*t^{\mu\nu\lambda}) - (t^{\lambda\mu\nu} \pm ^*t^{\lambda\mu\nu}) - (t^{\nu\lambda\mu} \pm ^*t^{\nu\lambda\mu})
\quad - \delta^{\mu\nu}(t^{\alpha\alpha\lambda} \pm ^*t^{\alpha\alpha\lambda}) + \delta^{\mu\lambda}(t^{\alpha\alpha\nu} \pm ^*t^{\alpha\alpha\nu})\right]$$

where the $+$ sign is used for $\Delta_S$ and the $-$ sign for $\Delta_A$.

4 Decomposition of Reducible Matter Spinor Fields

The matter 4-spinor field introduced in Section 2 $\Psi_\mu$ carries a 4-vector index in that participates in both the $SO(4)$ gauge and $SO(4)$ Euclidean space transformation, and a Dirac spinor index that participates only in the $SO(4)$ Euclidean space transformation. Thus it is associated with the reducible product of representations

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left[(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\right] = (1, \frac{1}{2}) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 0) \oplus (\frac{1}{2}, 1). \quad (49)$$

Each of the representations on the right hand side of (49) is $SO(4)$-irreducible. To understand this decomposition it is necessary to use the 2-spinor notation in terms of which

$$\Psi_\mu = \begin{bmatrix} \psi_\alpha^\mu \\ \chi^\lambda_{\alpha\mu} \end{bmatrix}. \quad (50)$$
We will focus separately on the undotted and dotted spinor components. The spinors $\psi_{\mu}^\mu$, $\chi^{\dot{a}\mu}$ correspond respectively to the reducible representations

\[
(\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, 0) = (0, \frac{1}{2}) \oplus (1, \frac{1}{2}),
\]

\[
(\frac{1}{2}, \frac{1}{2}) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, 0) \oplus (\frac{1}{2}, 1).
\]

As the $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ representations each contain a spin $\frac{3}{2}$ and spin $\frac{1}{2}$ state we see that each of $\psi_{\mu}^\mu$ and $\chi^{\dot{a}\mu}$ contain one spin $\frac{3}{2}$ state and two spin $\frac{1}{2}$ states. This corresponds to $2.(4 + 2.2) = 16$ states, appropriate to the vector-spinor field $\Psi^\mu$.

We first transform the vector index on $\psi_{\mu}^\mu$ to bispinor indices as follows

\[
\psi_{\mu}^\mu = \frac{1}{2} \psi_{\mu}^\mu \sigma^{\mu}_a \psi^b_{\mu} \sigma^b \}
\]

where

\[
\psi_{\mu}^\mu = \psi_{\mu}^\mu (\sigma^{\mu})_{bc}.
\]

The $(0, \frac{1}{2})$ component corresponds to that part of $\psi_{\mu}^\mu$ anti-symmetric in $(a, b)$ while the $(1, \frac{1}{2})$ corresponds to the symmetric part. Thus we have the expansion

\[
\psi_{\mu}^\mu = -\psi_{\mu d}^b \epsilon_{ad} + (\psi_{\mu}^\alpha \sigma^\alpha_{ab})_{ad}
\]

where

\[
\psi_{\mu}^\alpha \sigma^\alpha_{ab} = \psi_{\mu}^\alpha \sigma^\alpha_{ad}
\]

is self-dual. This leads at once to the decomposition of $\psi_{\mu}^\mu$ in terms of its $SO(4)$-irreducible components

\[
\psi_{\mu}^\mu = \frac{1}{2} \sigma_{ab} \psi_{\mu}^b - \sigma_{ab} (\psi_{\mu}^\alpha \sigma^\alpha_{ad}).
\]

The projection from $\psi_{\mu}^\mu$ to the $(0, \frac{1}{2})$ spinor $\psi^b$ is

\[
\psi^b = \frac{1}{2} (\sigma^\mu \sigma^\mu)^{bc} \psi^\mu_{\mu}.
\]

To derive the appropriate projection from $\psi_{\mu}^\mu$ to the $(1, \frac{1}{2})$ spinor $(\psi_{\mu}^\alpha \sigma^\alpha_{ad})$ we use (55) in (53a) to find

\[
(\psi_{\mu}^\alpha \sigma^\alpha_{ad}) = -\psi_{\mu \mu} - \frac{1}{2} (\sigma^\mu \sigma^\mu)^{bc} \psi^\mu_{\mu} \epsilon_{ad}.
\]
Raising the $d$ index in this equation and using (52b) to relate $\psi^a_{c}^\mu$ to $\psi^a_{\mu}$ we obtain

$$
(\psi_S^\alpha^\beta)^b_{a} (\sigma^\alpha^\beta)^c_{a} = - \psi^\mu_{a} (\sigma^\mu)^{bc}_{a} - \frac{1}{2} (\sigma^\mu)^{bc}_{a} \psi^\mu_{a} \delta^c_{a}.
$$

(57)

Multiplying this equation by $((\sigma^\mu)^c_{a})^d_{a}$ and tracing over the $(a,d)$ indices leads to the projection

$$
\begin{align*}
(\psi_S^\mu)^b = & \frac{1}{4} \left[ \delta^\mu^\alpha \delta^\nu^\beta - \delta^\nu^\alpha \delta^\mu^\beta + \epsilon^{\mu\nu\alpha\beta} \right] (\sigma^\alpha)^{ba}_{a} \psi^\beta_{a} \\
= & \frac{1}{4} \left[ (\sigma^\mu)^{ba}_{a} \psi^\nu_{a} - (\sigma^\nu)^{ba}_{a} \psi^\mu_{a} + \epsilon^{\mu\nu\alpha\beta} (\sigma^\alpha)^{ba}_{a} \psi^\beta_{a} \right].
\end{align*}
$$

(58a)

(58b)

The decomposition of $\chi^{\hat{a}\mu}$ into $(\frac{1}{2},0)$ and $(\frac{1}{2},1)$ states proceeds in a very similar manner. We first transform to the spinor indices

$$
\chi^{\hat{a}\mu} = \frac{1}{2} \chi^{\hat{a} \beta \nu} (\sigma^\beta)^{cb}_{a} = \frac{1}{2} \chi^{\hat{a} \beta \nu} (\sigma^\beta)^{cb}_{a} \epsilon^d_{cd} (\sigma^\mu)^{cb}_{a},
$$

(59)

and then expand $\chi^{\hat{a} \beta \mu}$ in terms of a part anti-symmetric in $\hat{a},d$ and an anti-self-dual part symmetric in $\hat{a},d$

$$
\chi^{\hat{a} \beta \mu} = - \chi^{\beta \mu}_{d} \epsilon^{ad}_{cd} + (\chi^{\beta \mu}_{A} )_{b} (\sigma^\mu)^{ad}_{b} \chi^{\hat{a}}_{d} (\sigma^\beta)^{ad}_{b}.
$$

(60)

giving us for the decomposition

$$
\chi^{\hat{a}\mu} = \frac{1}{2} (\sigma^\mu)^{ad}_{b} \chi^{d}_{a} - (\sigma^\mu)^{ad}_{b} (\chi^{\mu a}_{A} )_{b}.
$$

(61)

The projection of $\chi^{\hat{a}\mu}$ to the two $SO(4)$ irreducible components are, then,

$$
\chi^{\mu}_{a} = \frac{1}{2} (\sigma^\mu)^{ad}_{b} \chi^{d}_{a} \tag{62a}
$$

$$
(\chi^{\mu\mu}_{A})_{a} = \frac{1}{4} \left[ (\sigma^\mu)^{ad}_{b} \chi^{d}_{a} - (\sigma^\nu)^{ad}_{b} \chi^{d}_{a} - \epsilon^{\mu\nu\alpha\beta} (\sigma^\alpha)^{ad}_{b} \chi^{d}_{a} \right]. \tag{62b}
$$

The decompositions (54) and (61) can be combined together to give the $SO(4)$ decomposition of the vector-spinor matter field $\Psi^\mu$ in terms of a Dirac 4-spinor

$$
\Psi = \begin{bmatrix} 
\chi_{a} \\
\psi^{\hat{a}} 
\end{bmatrix} \tag{63a}
$$

and a tensor-spinor

$$
\Psi^\alpha^\beta = \begin{bmatrix} 
\chi^{\alpha\beta}_{A} \\
\psi^{\alpha\beta\hat{a}}_{S} 
\end{bmatrix}. \tag{63b}
$$
We find
\[ \Psi^\mu = \frac{1}{2} \gamma^\mu \Psi + \gamma^\alpha \Psi^{\alpha \mu}. \] (63c)

It is clear that the right chirality part of \( \Psi^{\alpha \beta} \) is self-dual
\[ \Psi^{\alpha \beta}_R = \frac{1}{2} (1 + \gamma_5) \Psi^{\alpha \beta} = \begin{bmatrix} 0 \\ \psi^\alpha_{S \beta a} \end{bmatrix} \] (64a)
while the left chirality part of \( \Psi^{\alpha \beta} \) is anti-self-dual
\[ \Psi^{\alpha \beta}_L = \frac{1}{2} (1 - \gamma_5) \Psi^{\alpha \beta} = \begin{bmatrix} \chi^{\alpha \beta}_{A a} \\ 0 \end{bmatrix}, \] (64b)
so that \( *\Psi^{\alpha \beta} = \gamma_5 \Psi^{\alpha \beta} \). The projection from \( \Psi^\mu \) to \( \Psi \) and \( \Psi^{\alpha \beta} \), or to their right- and left-chirality parts, can be written in 4-component form as
\[ \Psi = \frac{1}{2} \gamma^\mu \Psi^\mu \] (65a)
\[ \Psi_R = \frac{1}{2} (1 + \gamma_5) \Psi = \frac{1}{2} \gamma^\mu \Psi_L^\mu \] (65b)
\[ \Psi_L = \frac{1}{2} (1 - \gamma_5) \Psi = \frac{1}{2} \gamma^\mu \Psi_L^\mu \] (65c)

and
\[ \Psi^{\alpha \beta} = \frac{1}{4} \left( \gamma^\alpha \Psi^\beta - \gamma^\beta \Psi^\alpha + \epsilon^{\alpha \beta \lambda \rho} \gamma_5 \gamma^\lambda \Psi^\rho \right) \] (66a)
\[ \Psi^{\alpha \beta}_R = \frac{1}{4} \left( \gamma^\alpha \Psi^\beta_L - \gamma^\beta \Psi^\alpha_L + \epsilon^{\alpha \beta \lambda \rho} \gamma_5 \gamma^\lambda \Psi^\rho_L \right) \] (66b)
\[ \Psi^{\alpha \beta}_L = \frac{1}{4} \left( \gamma^\alpha \Psi^\beta_R - \gamma^\beta \Psi^\alpha_R + \epsilon^{\alpha \beta \lambda \rho} \gamma_5 \gamma^\lambda \Psi^\rho_R \right). \] (66c)

In 4dE the adjoint of the matter vector-spinor field is the Hermitian conjugate \( \Psi^{\mu \dagger} \). The conjugate of the decomposition (63c) is simply
\[ \Psi^{\mu \dagger} = \frac{1}{2} \Psi^{\dagger} \gamma^\mu - \Psi^{\mu \alpha \dagger} \gamma^\alpha \] (67)
as the Dirac \( \gamma \)-matrices have been chosen to be Hermitian. In terms of the \( SU(2) \) 2-spinors these fields are
\[ \Psi^{\mu \dagger} = \begin{bmatrix} \psi^{\mu \alpha}, -\chi^{\mu a} \end{bmatrix}, \] (68a)
The decompositions in terms of $SO(4)$-irreducible components are found by taking Hermitian conjugates of (54) and (61)

$$\overline{\psi}^{\mu a} = -\frac{i}{2} \overline{\psi}_b (\sigma^\mu)^{ba} + \overline{\psi}_S^{\mu a} (\sigma^\alpha)^{ba},$$  \hspace{1cm} (69a)

$$\chi^\mu_a = -\frac{i}{2} \chi^b (\sigma^\mu)_b a + \overline{\chi}_A^{\mu a b} (\sigma^\alpha)_b a.$$ \hspace{1cm} (69b)

In taking Hermitian conjugates of 2-spinors in 4$dE$ we must pay particular attention to the index structure of the fields involved. The rules are derived in [3] and summarised in Appendix A.

# 5 Gauge Transformations and Gauge Invariant Lagrangian in terms of Irreducible Fields.

The $SO(4)$ gauge model with extended field transformations was described in Section 2 in terms of the reducible gauge potential $A^\mu$ and vector-spinor matter field $\Psi^\mu$. Using the results of Sections 3 and 4 we now consider the $SO(4)$ gauge transformations (10a,b) and the $SO(4)$ gauge invariant action (9) in terms of the irreducible fields.

Equations (10a,b) are the finite gauge transformations. In this section we restrict our attention to the transformations for which the gauge parameters $\omega^{\alpha \beta}(x)(= -\omega^{\beta \alpha}(x))$ are infinitesimal. The corresponding gauge transformations for the reducible gauge potential $t^{\mu \alpha \beta}$ can be written in three equivalent ways

$$\delta t^{\mu \alpha \beta} = \left\{ \begin{array}{l}
\partial^\mu \omega^{\alpha \beta} + t^{\mu \rho \lambda} \omega^{\rho \beta} - t^{\mu \beta \rho} \omega^{\rho \alpha} \\
\partial^\mu \omega^{\alpha \beta} + \epsilon^{\alpha \beta \gamma \delta} t^{\mu \gamma \lambda} \omega^{\rho \rho} \\
\partial^\mu \omega^{\alpha \beta} + * t^{\mu \alpha \rho} \omega^{\rho \beta} - * t^{\mu \beta \rho} \omega^{\rho \alpha}.
\end{array} \right.$$ \hspace{1cm} (70a)

(70b)

The equivalence of these three transformations, and the equivalence in a number of other cases below, can easily be established using the result (B.5) in Appendix B. A similar gauge transformation can be established for $* t^{\mu \alpha \beta}$.
In a very obvious way we can deduce the following equivalent gauge transformations for the vector field $v^\mu$ and the axial vector field $a^\mu$

$$
\delta v^\mu = \left\{ \begin{array}{ll}
\frac{1}{3} \left[ \partial^\alpha \omega^{\alpha \mu} + 2 v^\alpha \omega^{\alpha \mu} + 2 a^\alpha \omega^{\alpha \mu} + \Delta^{+ \mu \alpha \beta} \omega^{\alpha \beta} \right] \\
\frac{1}{3} \left[ \partial^\alpha \omega^{\alpha \mu} + 2 v^\alpha \omega^{\alpha \mu} + 2 a^\alpha \omega^{\alpha \mu} + \Delta^{- \mu \alpha \beta} \omega^{\alpha \beta} \right]
\end{array} \right. \quad (72a)
$$

$$
\delta a^\mu = \left\{ \begin{array}{ll}
\frac{1}{3} \left[ \partial^\alpha \omega^{\alpha \mu} + 2 v^\alpha \omega^{\alpha \mu} + 2 a^\alpha \omega^{\alpha \mu} + \Delta^{+ \mu \alpha \beta} \right]
\end{array} \right. \quad (73a)
$$

The gauge transformations for $\Delta^{\mu \nu \lambda}$ are more tedious to derive; we give here just one of the equivalent forms in each case. The transformations are

$$
\delta \Delta^{+ \mu \nu \lambda} = \frac{1}{3} \left\{ (\partial^\alpha - v^\alpha)(\omega^{\alpha \lambda} - a^\alpha \omega^{\lambda \mu}) - (\partial^\nu - v^\nu)(\omega^{\lambda \mu} - a^\nu \omega^{\lambda \mu}) - \delta^{\mu \nu} \Delta^{+ \alpha \lambda \beta} \omega^{\alpha \beta} + (\Delta^{+ \mu \alpha \lambda} + \Delta^{+ \nu \alpha \lambda}) \omega^{\alpha \lambda} + (\Delta^{+ \nu \lambda \alpha} + \Delta^{+ \nu \lambda \alpha}) \omega^{\alpha \lambda} \right\} (\nu \leftrightarrow \lambda) \quad (74)
$$

$$
\delta \Delta^{- \mu \nu \lambda} = \frac{1}{3} \left\{ (\partial^\alpha - v^\alpha)(\omega^{\alpha \lambda} - a^\alpha \omega^{\lambda \mu}) - (\partial^\nu - v^\nu)(\omega^{\lambda \mu} - a^\nu \omega^{\lambda \mu}) - \delta^{\mu \nu} \Delta^{- \alpha \lambda \beta} \omega^{\alpha \beta} + (\Delta^{- \mu \alpha \lambda} + \Delta^{- \nu \alpha \lambda}) \omega^{\alpha \lambda} + (\Delta^{- \nu \lambda \alpha} + \Delta^{- \lambda \alpha \lambda}) \omega^{\alpha \lambda} \right\} (\nu \leftrightarrow \lambda) \quad (75)
$$

These transformations can be used to find the gauge transformations of the $SO(4)$ irreducible fields $\Delta_{S,A}$

$$
\delta \Delta^{\mu \nu \lambda}_{S,A} = \frac{1}{6} \left\{ (\partial^\mu - v^\mu - a^\mu) (\omega^{\nu \lambda} \pm * \omega^{\nu \lambda}) - (\partial^\nu - v^\nu - a^\nu) (\omega^{\lambda \mu} \pm * \omega^{\lambda \mu}) - \delta^{\mu \nu} (\Delta_{S} + \Delta_{A})^{\alpha \beta} (\omega^{\alpha \beta} \pm * \omega^{\alpha \beta}) \right\}
$$
+ [(\Delta_S + \Delta_A)^{\mu\alpha} + (\Delta_S + \Delta_A)^{\nu\alpha}] (\omega^{\alpha\lambda} \pm \ast \omega^{\alpha\lambda})
+ (\Delta_S + \Delta_A)^{\lambda\alpha} (\omega^{\alpha\mu} \pm \ast \omega^{\alpha\mu}) \} - (\nu \leftrightarrow \lambda) \quad (76)

where the + sign is used for \( \delta \Delta_S \) and the − sign for \( \delta \Delta_A \). By inspection of the \( SO(4) \) gauge transformations (72) - (76) we see that the gauge transformations mix together the different \( SO(4) \) irreducible fields. For instance in eq. (72a) the vector \( v^\mu \) is mixed with the axial vector \( a^\mu \) and the \( \Delta_+ \) field, which contains both spin 2 and spin 1 components.

In a similar vein, using the projections derived in the previous section from the reducible matter field \( \Psi^\mu \) to the “irreducible” 4-spinors \( \Psi \) and \( \Psi^{\alpha\beta} \) we can deduce from eq. (10a) the following \( SO(4) \) gauge transformations

\[
\delta \Psi = \frac{i}{2} \omega^{\mu\nu} \Sigma^{\mu\nu} \Psi - i \Sigma^{\mu\alpha} \omega^{\alpha\nu} \Psi^\mu
\]

\[
\delta \Psi^{\alpha\beta} = (\omega^{\alpha\beta} + \ast \omega^{\alpha\beta} \gamma_5) \Psi - \frac{1}{2} \epsilon^{\alpha\beta\lambda\mu} \left( \ast \omega^{\lambda\mu} + \gamma_5 \omega^{\lambda\mu} \right) \Psi^\mu
+ \frac{i}{2} \left[ \Sigma^{\alpha\mu} \omega^{\beta\nu} - \Sigma^{\beta\mu} \omega^{\alpha\nu} \right] + \epsilon^{\alpha\beta\lambda\rho} \gamma_5 \Sigma^{\lambda\mu} \omega^{\rho\nu} \right] \Psi^\mu. \quad (78)
\]

Again, we see the inevitable mixing of \( \Psi \) and \( \Psi^{\mu\nu} \) in the gauge transformations.

The Lagrangian density is very simple in terms of the reducible fields as seen in \( L_{YM} \) and \( L_{matter} \)-eqs. (1) and (8). In terms of the irreducible fields, however, we find a large number of induced couplings. It must be emphasized that the resulting Lagrangian density, though complicated, is gauge invariant under the gauge transformations derived above. We now examine \( L_{YM} \) and \( L_{matter} \) separately.

Using the decomposition of the reducible gauge potential \( h^{\mu\nu\lambda} \) derived in Section 3 in the field strength \( (F^{\mu\nu})^{\alpha\beta} \) of eq. (7b), we can see that the pure Yang-Mills Lagrangian density is

\[
L_{YM} = \frac{1}{4} \left[ X_{(1)}^{\mu\nu\alpha\beta} + X_{(2)}^{\mu\nu\alpha\beta} - (\mu \leftrightarrow \nu) \right]^2 \quad (79)
\]

where \( X_{(1)}^{\mu\nu\alpha\beta} \) is linear in the irreducible fields

\[
X_{(1)}^{\mu\nu\alpha\beta} = \delta^{\nu\alpha} \partial^\mu v^\beta - \delta^{\nu\beta} \partial^\mu v^\alpha + \epsilon^{\nu\alpha\beta\lambda} \partial^\mu a^\lambda + \partial^\mu \Delta_+^{\nu\alpha\beta}\quad (80)
\]
and $X^{\mu\nu\rho\gamma}_{(2)}$ is quadratic in the irreducible fields

$$X^{\mu\nu\rho\gamma}_{(2)} = \frac{1}{2} \left[ \delta^{\mu\beta} v^\nu v^\alpha - \delta^{\mu\alpha} v^\nu v^\beta + \delta^{\mu\alpha} \delta^{\nu\beta} (v \cdot v) + \delta^{\mu\beta} a^\nu a^\alpha - \delta^{\mu\alpha} a^\nu a^\beta + \delta^{\mu\alpha} \delta^{\nu\beta} (a \cdot a) + \Delta^\mu_+ \Delta^\nu_+ \right]$$

$$+ \left( \epsilon^{\mu\nu\rho\gamma} v^\beta - \epsilon^{\mu\nu\beta\gamma} v^\rho \right) a^\rho + \left( \delta^{\mu\beta} \epsilon^{\alpha\nu\lambda\rho} - \delta^{\mu\alpha} \epsilon^{\beta\nu\lambda\rho} \right) v^\lambda a^\rho$$

$$+ \left( \Delta^\mu_+ v^\beta - \Delta^\nu_+ v^\alpha \right) + \left( \delta^{\mu\beta} \Delta^\nu_+ - \delta^{\mu\nu} \Delta^\rho_+ \right) v^\rho$$

$$+ \left( \Delta^\mu_+ \epsilon^{\nu\lambda\beta\rho} - \Delta^\nu_+ \epsilon^{\nu\lambda\alpha\rho} \right) a^\rho \right]. \quad (81)$$

We refrain from expanding out $L_{YM}$ in full detail satisfying ourselves instead with the kinetic part which is quadratic in the fields

$$L^{(2)}_{YM} = \frac{1}{4} \left( X^{\mu\nu\rho\gamma}_{(1)} \right)^2$$

$$= -v^\mu (2 \Box \delta^\mu - \partial^\mu \partial^\nu - \partial^\mu \partial^\nu) v^\nu - a^\mu (2 \Box \partial^\mu - \partial^\mu \partial^\nu) a^\nu$$

$$- \Delta^\mu_+ \left( \Box \delta^\mu - \partial^\mu \partial^\nu \right) \Delta^\nu_+ - v^\alpha \partial^\mu \partial^\nu \Delta^\mu_+$$

$$- \Delta^\mu_+ \partial^\mu \partial^\nu v^\alpha - \Delta^{\mu\nu\rho} \partial^\mu \partial^\nu a^\rho - a^\rho \partial^\mu \partial^\nu \Delta^{\mu\nu\rho}. \quad (82)$$

The Lagrangian density for the matter fields is simpler to derive. We consider separately the kinetic and interaction parts, which upon using $\gamma^\alpha \gamma^\beta \gamma^\gamma = \delta^{\alpha\beta} \gamma^\gamma - \delta^{\alpha\gamma} \gamma^\beta + \delta^{\beta\gamma} \gamma^\alpha + \epsilon^{\alpha\beta\gamma} \gamma^\delta$ becomes

$$L^{(2)}_{matter} = - \frac{i}{2} \Psi^\dagger \partial \Psi + i \Psi^\dagger \alpha \mu (2 \gamma^\mu \partial^\nu - \delta^\mu^\nu \partial^\mu) \Psi^\alpha \nu$$

$$- i \Psi^\dagger \mu \nu (\gamma^\mu \partial^\nu) \Psi - i \Psi^\dagger (\gamma^\nu \partial^\mu) \Psi^\mu \nu$$

$$L^{(3)}_{matter} = - \frac{1}{2} \Psi^\dagger \gamma \cdot a_5 \gamma \Psi + \Psi^\dagger \mu \nu \lambda \gamma \Sigma_{\alpha \lambda \mu} \gamma^\nu \psi^\beta - v^\alpha \Psi^\alpha \mu \nu \gamma \Delta^\nu_+ \psi^\beta$$

$$+ i \epsilon^{\mu \nu \rho} \Psi^\alpha \lambda \gamma \lambda \gamma^\mu \psi^\nu \psi^\beta - i \Delta^\mu_+ \psi^\alpha \lambda \gamma \lambda \gamma^\mu \psi^\nu \psi^\beta$$

$$+ 2 i \Psi^\dagger \gamma^\mu \psi^\mu \psi^\nu \psi^\rho$$

$$+ \frac{1}{2} \Psi^\dagger \gamma \cdot v \Sigma_{\mu \nu \gamma \delta} \psi^\mu \psi^\nu \partial^\alpha - \psi^\alpha \mu \nu \gamma \Sigma_{\nu \gamma} \psi^\alpha \mu \nu \partial^\gamma$$

$$+ 2 i \psi^\alpha \Psi^\mu \nu \gamma \psi^\mu \psi^\nu$$

$$+ \frac{1}{2} \psi^\alpha \mu \nu \gamma \psi^\mu \psi^\nu \partial^\alpha$$

$$+ \frac{1}{2} \left( \Psi^\dagger \gamma^\alpha \gamma^\mu \gamma^\nu \psi^\beta \right) \Delta^\mu_+ \psi^\alpha \nu \gamma \psi^\mu \gamma \psi^\beta.$$  \quad (84)

It is curious to note that the only term in (84) which is bilinear in $\Psi$ and $\Psi^\dagger$ involves just the axial field $a_\mu$ and does not involve the vector field $v_\mu$ or the tensor field $\Delta^\mu_+ \psi^\alpha \nu \gamma \psi^\mu \gamma$. 19
6 Discussion

In this paper we have examined a particular gauge invariant model in four dimensional Euclidean space which has extended field transformations: we allow the gauge group indices on the fields to participate in the Euclidean space \( SO(4) \) field transformations. This requires, of necessity, that the gauge group in question has a four-dimensional representation. In this paper we have chosen \( SO(4) \) itself as the gauge group, availing of the fundamental, or vector, representation. Gabrielli [1] availed of the gauge group \( U(4) \).

The \( SO(4) \) gauge model is defined by its action and the gauge transformations under which the action is invariant. The structure of the \( SO(4) \) gauge model is very standard – we choose as action density the sum of a pure \( SO(4) \) Yang-Mills \( F^2 \) term and a standard (for Euclidean space) gauge-invariant matter-gauge field term \( \Psi^\dagger(\not p + \not A)\Psi \) where the matter field is in the fundamental representation of the gauge group. However, the gauge potential \( (A^\mu)^{\alpha\beta} \) and the matter field \( \Psi^\mu \) are now reducible under \( SO(4) \).

In field theory models we always work with fields which transform irreducibly under the symmetry transformations of the space the model is built on. Consequently we investigated the decomposition of the reducible gauge potential \( t^{\mu\nu\lambda} = -i(A^\mu)^{\nu\lambda} \) and the reducible matter field \( \Psi^\mu \) in terms of \( SO(4) \) irreducible component fields. The results of this analysis can be presented in two ways which may be related by parity transformations.

We consider first the decomposition of the gauge potential. We found the \( t^{\mu\nu\lambda} \) can be decomposed in terms of a vector field \( v^\mu \), an axial vector field \( a^\mu \), a rank 3 tensor \( \Delta_S^{\mu\nu\lambda} (= -\Delta_S^{\lambda\mu\nu}) \), self-dual with respect to the \((\nu,\lambda)\) indices and traceless over the \((\mu,\nu)\) indices, and another rank 3 tensor \( \Delta_A^{\mu\nu\lambda} (= -\Delta_A^{\lambda\mu\nu}) \), anti-self-dual with respect to the \((\nu,\lambda)\) indices and traceless over the \((\mu,\nu)\) indices.

This decomposition corresponds to

\[
\left( \frac{1}{2}, \frac{1}{2} \right) \otimes \left[ (1, 0) \oplus (0, 1) \right] = \left( \frac{1}{2}, \frac{1}{2} \right)_v \oplus \left( \frac{1}{2}, \frac{1}{2} \right)_a \oplus \left( \frac{3}{2}, \frac{3}{2} \right)_S \oplus \left( \frac{1}{2}, \frac{3}{2} \right)_A
\]

Alternatively we can consider two linear combinations of \( \Delta_S \) and \( \Delta_A \) which are dual to one another, \( \Delta_\pm = \Delta_S \pm \Delta_A, \ast \Delta_+ = \Delta_- \).
In treating the matter fields $\Psi^\mu$ we can choose to use 2-component or 4-component spinor notation. In the former case we found that $\Psi^\mu$ can be decomposed in terms of 2-component spinors in the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ representations, $\chi$ and $\psi$ respectively, and 2-component spinors in the $\left(\frac{1}{2},1\right)$ and $\left(1,\frac{1}{2}\right)$ representations, namely $\chi_A^\alpha(= -\chi_A^\beta)$ and $\psi_A^\alpha(= -\psi_A^\beta)$ which are, respectively, anti-self-dual and self-dual. In the 4-component notation we have $\Psi$ in the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ representation and $\Psi^{\alpha\beta}(= -\Psi^{\beta\alpha})$ in the $\left(\frac{1}{2},1\right) \oplus \left(1,\frac{1}{2}\right)$ representation as the two components in the decomposition of $\Psi^\mu$.

Having the $SO(4)$-irreducible field structure of the model in hand, we proceeded to an examination of the induced gauge transformations for each of these fields. The resulting gauge transformations, while complicated, have the distinctive feature of mixing the fields. In a sense, $(v^\mu, a^\mu, \Delta^{\mu\lambda}_S, \Delta^{\mu\lambda}_A)$ can be thought of as a multiplet of fields under the $SO(4)$ gauge transformations. The same comment applies to $(\chi, \chi^\alpha_A)$ and $(\psi, \psi^\alpha_S)$.

Finally, we considered the standard action density in terms of the irreducible field components.

Although Gabrielli [1] considered a different gauge group, and consequently the multiplet of gauge fields in his case is different to the one we have found, our $t^{\mu\alpha\beta}$ corresponds to his $C_{\mu\alpha\beta}$. In his paper the action density involves not only the $L_{YM}$ which we have in eq. (79) but also the induced interactions with the other fields in his multiplet ($\phi, \bar{\phi}, A, \bar{A}, T^S, \bar{T}^S, T^A, \bar{T}^A$) and their self-interaction and kinetic terms. The explicit form of the action density in terms of the fields contained within $C_{\mu\alpha\beta}$ was not considered.

It is interesting to note that we could have made a different choice of gauge group representation for both the gauge potential and the matter field $\Psi$. If instead of the fundamental representation of $SO(4)$ we had chosen the 4-component spinorial representation, the resulting decomposition in terms of $SO(4)$ irreducible fields would be very different. In that case we would use $\frac{1}{2}\Sigma^{\mu\nu}$ in place of $M^{\mu\nu}$ throughout. The gauge potential would correspond exactly to that considered by Gabrielli. Furthermore the extended $SO(4)$ field transformations would be

$$(A^\mu)^{ij}(x) \rightarrow (A'^\mu)^{ij}(x') = \Lambda^{\mu\nu} \left[ S(\lambda) A^\nu(x) S^{-1}(\lambda) \right]^{ij}$$
\[ \Psi_{i}^{k}(x) \rightarrow \Psi_{i}^{k}(x') = S_{ij}(\lambda) [S(\lambda)\Psi_{j}(x)]^{k} \]
\[ = S_{ij}(\lambda)S^{km}(\lambda)\Psi_{m}(x). \] (87)

The decomposition of \( \Psi_{i}^{k} \) into irreducible component fields inevitably includes fermionic scalar and vector fields. This type of result has previously been found in applying these ideas in three Euclidean dimensions [7].

In an examination of the pure Yang-Mills sector of the model presented here, the presence of instantons has previously been noted [2].

There are a number of directions in which the approach of this paper can be developed. First however, we should note a direction that is unlikely to be useful, namely, applying the analogous approach in flat four dimensional Minkowski space: \( SO(3, 1) \) is not a compact group and, as a consequence, there will be unwelcome ghost states [8]. A more promising direction of further study would be to extend the gauge group from \( SO(4) \) to \( SO(4) \otimes G \) where \( G \) could be any compact Lie group.

The approach of gauge models with extended field transformations can also be applied in higher dimensional Euclidean spaces. For example, in five dimensions the gauge potential \( (A^{\mu})^{\alpha\beta} = -(A^{\mu})^{\beta\alpha} \) has 50 independent components. The decomposition of the gauge potential analogous to our discussion in Section 3 is
\[ (A^{\mu})^{\alpha\beta} = iT^{\mu\alpha\beta} = i \left[ \delta^{\mu\alpha}V^{\beta} - \delta^{\mu\beta}V^{\alpha} + \epsilon^{\mu\alpha\beta\lambda\sigma}E_{\lambda\sigma} + D^{\mu\alpha\beta} \right] \] (88)
where
\[ D^{\mu\alpha\beta} = -D^{\mu\beta\alpha}, \] (89a)
\[ D^{\mu\mu\beta} = 0, \] (89b)
\[ E^{\lambda\sigma} = \epsilon^{\lambda\sigma\alpha\beta\gamma}D^{\alpha\beta\gamma}. \] (89c)

The vector field \( V^{\mu} \) has 5 components, the anti-symmetric tensor \( E^{\lambda\sigma} \) has 10 components while the rank 3 tensor field \( D^{\mu\alpha\beta} \) has 45 components. The constraint (89c) leads to 50
independent components in total, effectively saying that the $E^{\lambda \sigma}$ are not independent components.

It is of interest to consider the further decomposition of $V^\mu$ and $D^{\mu \alpha \beta}$ into $SO(4)$ irreducible fields, thereby making contact with the analysis of Section 3. It is understood that the $SO(4)$ rotations are in the space spanned by $x^1, x^2, x^3, x^4$. We also identify the $SO(3)$ spin content where the $SO(3)$ rotations are in the space spanned by $x^1, x^2, x^3$. In what follows Greek indices are understood to take values 1 to 5, while Latin indices take the values 1 to 4. For the vector field we find

$$V^\mu = (V^i, V^5)$$

where

$$V^i = (\text{spin 1, spin 0})$$

$$V^5 = (\text{spin 0}).$$

Next consider the fields $D^{\mu \alpha \beta}$ when none of the indices has the value 5, namely $D^{ijk} = -D^{ikj}$. This is just the gauge potential $t^{ijk}$ considered in detail in Section 3. We identify

(i) the trace over the first pair of indices $v^k = \frac{1}{3} D^{ijk}$

$$v^k = (\text{spin 1, spin 0})$$

(ii) the trace over the first pair of indices of the dual $a^k = \frac{1}{3} * D^{ijk}$

$$a^k = (\text{spin 1, spin 0})$$

(iii) the self-dual and anti-self-dual parts of the traceless part of $D^{ijk}$, $\Delta^{ijk} = D^{ijk} - \frac{1}{3} \delta^{ij} D^{ik}$

$$\Delta_S^{ijk} = (\text{spin 2, spin 1})$$

$$\Delta_A^{ijk} = (\text{spin 2, spin 1}).$$

We need only consider one of the indices of $D^{\mu \alpha \beta}$ taking the value 5, as $D^{\mu 55} = 0$ and $D^{55k} = -D^{iik}$ due to the constraint (89b). We first consider $D^{ij5}$ and note that this is traceless as (89b) implies $D^{\alpha \alpha 5} = D^{ii5} = 0$. We can then identify
(iv) the symmetric traceless part $T^{ij} = \frac{1}{2}(D^{ij5} + D^{ji5})$

$T^{ij} = \text{(spin 2, spin 1, spin 0)}$

(v) the self-dual and anti-self-dual parts of the anti-symmetric part of $D^{ij5}$, namely $B_{S}^{ij}$, $B_{A}^{ij}$ where

$B_{S}^{ij} = \frac{1}{2}(D^{ij5} - D^{ji5}).$

$B_{S}^{ij} = \text{(spin 1)}$

$B_{A}^{ij} = \text{(spin 1)}.$

Finally, we consider $D^{5ij} = -D^{5ji} \equiv C^{ij}$

(vi) the self-dual and anti-self-dual parts $C_{S}^{ij}$, $C_{A}^{ij}$

$C_{S}^{ij} = \text{(spin 1)}$

$C_{A}^{ij} = \text{(spin 1)}.$

In all we count 3 spin 2 fields, 10 spin 1 fields and 5 spin 0 fields arriving at 50 independent components as expected. The mixing of the above fields under an $SO(5)$ Yang-Mills gauge transformation is inevitable. It is interesting to note that the maximum spin we need to consider is spin two, in both four and five dimensions.

The quantization of extended gauge models of the type proposed in this paper has not yet been considered. One of the first questions to be addressed in that procedure will be the most appropriate form of gauge fixing. It is important to identify which of the field components are dynamical and which are gauge artifacts. In this way it will be possible to identify clearly the dynamical degrees of freedom. Although it seems more efficient to do explicit calculations of quantum effects in terms of the original reducible Yang-Mills and matter fields rather than in terms of the $SO(4)$ irreducible fields, this will depend on the identification of dynamical degrees of freedom. We note in the action density many occurrences of $\epsilon^{\mu\nu\alpha\beta}$ and $\gamma_{5}$; both quantities are intrinsically connected with four dimensions. This indicates that the regularization scheme to be used should be one which preserves the gauge invariance and does not alter the number of spatial dimensions. Operator regularization [9] is one such regularization scheme.
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Appendix A. Spinor Notation in 4dE.

In this appendix we summarize the spinor notation in 4dE developed in [3].

A.1 $\gamma$-matrix conventions

$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\mu, \nu = 1, \ldots, 4$

$\gamma^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma^\nu \gamma^\rho \gamma^\sigma$

$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix}$

$\sigma^\mu = (i\overline{\tau}, 1), \quad \overline{\sigma}^\mu = (-i\overline{\tau}, 1) = (\sigma^\mu)^\dagger$

$\gamma_5 = \frac{1}{4!}\epsilon^{\mu\nu\lambda\rho}\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\epsilon^{1234} = 1$

$\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \Sigma^{\mu\nu\dagger} = -\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\Sigma^{\alpha\beta} \gamma_5 = \begin{bmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & -i\overline{\sigma}^{\mu\nu} \end{bmatrix}$

$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^{\mu\nu} - \sigma^{\nu\mu})$

$\overline{\sigma}^{\mu\nu} = \frac{1}{4}(\overline{\sigma}^{\mu\nu} - \overline{\sigma}^{\nu\mu})$

$C = \begin{bmatrix} -i\tau_2 & 0 \\ 0 & i\tau_2 \end{bmatrix} = -C^T = -C^\dagger = -C^{-1} = C^*$

A.2 2-spinors and 4-spinors

$\Psi = \begin{bmatrix} \psi_a \\ \chi^\dot{a} \end{bmatrix}$, $\Psi^\dagger = [\overline{\psi}, -\overline{\chi^\dot{a}}]$

$\Psi_C = \begin{bmatrix} -\overline{\psi_a} \\ -\overline{\chi^\dot{a}} \end{bmatrix}$, $\Psi_C^\dagger = [\psi^a, \chi_{\dot{a}}]$

Raising and lowering 2-spinor indices:

$\psi_a = \epsilon_{ab}\psi^b$, $\psi^b = \epsilon^{ba}\psi_a$, $\chi_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\chi^\dot{b}$, $\chi^\dot{b} = \epsilon^{\dot{a}\dot{b}}\chi_{\dot{a}}$

$\epsilon_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{ab}$, $\epsilon_{\dot{a}\dot{b}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_{\dot{a}\dot{b}}$
\[ \epsilon_{ab} \epsilon^{bc} = \delta_a^c \]
\[ \epsilon_{\dot{a}\dot{b}} \epsilon^{\dot{b}\dot{c}} = \delta_{\dot{a}}^{\dot{c}} \]

Index summation:
undotted indices are summed top to bottom: \( \psi^a \lambda_a = -\bar{\psi}_a \lambda^a = \lambda^a \psi_a \)
dotted indices are summed bottom to top: \( \chi_{\dot{a}} \phi_{\dot{a}} = -\bar{\chi}_{\dot{a}} \phi_{\dot{a}} = \phi_{\dot{a}} \chi_{\dot{a}} \)

Complex conjugation of 2-spinors:
* conjugation raises/lowers 2-spinor indices accompanied by a (±) (+ if the index is in its “natural position”, – if not) and adds a conjugation bar above the spinor (or removes the bar already there). For an unconjugated spinor the natural position for either an undotted or dotted index is as a lower index, so

\[
(\psi_a)^* = \overline{\psi}^a \implies (\overline{\psi})^* = \psi_a
\]
\[
(\chi_{\dot{a}})^* = \overline{\chi}^\dot{a} \implies (\overline{\chi})^* = \chi_{\dot{a}}
\]

but

\[
(\psi^a)^* = -\overline{\psi}_a \implies (\overline{\psi}_a)^* = -\psi^a
\]
\[
(\chi^\dot{a})^* = -\overline{\chi}_{\dot{a}} \implies (\overline{\chi}_{\dot{a}})^* = -\chi^\dot{a}
\]

and we deduce that for a conjugated spinor the natural position for an index is as an upper index.

A.3 Various properties of the \( \sigma \)-matrices:

Products of matrices \( \sigma^{\mu^\nu}_{\dot{a}\dot{b}} \), \( \overline{\sigma}^{\mu^\nu}_{\dot{a}\dot{b}} \), \( \sigma_{\mu^\nu}^a b \), \( \overline{\sigma}_{\mu^\nu}^a b \)

\[
\sigma^{\mu^\nu} = \delta^{\mu^\nu} 1 + 2\sigma^{\mu^\nu}
\]
\[
\overline{\sigma}^{\mu^\nu} = \delta^{\mu^\nu} 1 + 2\overline{\sigma}^{\mu^\nu}
\]
\[
\sigma^{\mu^\nu} \sigma^{\kappa^\lambda} = -\frac{1}{4} (\delta^{\mu^\kappa} \delta^{\nu^\lambda} - \delta^{\mu^\lambda} \delta^{\nu^\kappa}) 1 - \frac{1}{4} \epsilon^{\mu^\nu^\kappa^\lambda} 1
\]
\[
-\frac{1}{2} (\delta^{\mu^\kappa} \sigma^{\nu^\lambda} + \delta^{\nu^\lambda} \sigma^{\mu^\kappa} - \delta^{\mu^\lambda} \sigma^{\nu^\kappa} - \delta^{\nu^\kappa} \sigma^{\mu^\lambda})
\]
\[
\overline{\sigma}^{\mu^\nu} \overline{\sigma}^{\kappa^\lambda} = -\frac{1}{4} (\delta^{\mu^\kappa} \delta^{\nu^\lambda} - \delta^{\mu^\lambda} \delta^{\nu^\kappa}) 1 + \frac{1}{4} \epsilon^{\mu^\nu^\kappa^\lambda} 1
\]
\[
-\frac{1}{2} (\delta^{\mu^\kappa} \overline{\sigma}^{\nu^\lambda} + \delta^{\nu^\lambda} \overline{\sigma}^{\mu^\kappa} - \delta^{\mu^\lambda} \overline{\sigma}^{\nu^\kappa} - \delta^{\nu^\kappa} \overline{\sigma}^{\mu^\lambda})
\]
\[ \sigma^{\alpha\beta}\gamma = -\frac{1}{2}(\delta^{\alpha\gamma}\sigma^{\beta} - \delta^{\beta\gamma}\sigma^{\alpha}) - \frac{1}{2}\epsilon^{\alpha\beta\gamma}\rho\sigma^{\rho} \]

\[ \dot{\sigma}^{\alpha\beta}\gamma = -\frac{1}{2}(\delta^{\alpha\gamma}\sigma^{\beta} - \delta^{\beta\gamma}\sigma^{\alpha}) + \frac{1}{2}\epsilon^{\alpha\beta\gamma}\rho\sigma^{\rho} \]

\[ \sigma^{\alpha}\dot{\sigma}^{\beta}\gamma = \frac{1}{2}(\delta^{\alpha\beta}\sigma^{\gamma} - \delta^{\alpha\gamma}\sigma^{\beta}) - \frac{1}{2}\epsilon^{\alpha\beta\gamma}\rho\sigma^{\rho} \]

\[ \sigma^{\alpha}\sigma^{\beta}\gamma = \frac{1}{2}(\delta^{\alpha\beta}\sigma^{\gamma} - \delta^{\alpha\gamma}\sigma^{\beta}) + \frac{1}{2}\epsilon^{\alpha\beta\gamma}\rho\sigma^{\rho} \]

Fierz identities:

\[ \delta_{a}^{\ b}\delta_{f}^{\ e} = \frac{1}{2}(\delta_{a}^{\ e}\delta_{f}^{\ b} - \sigma^{\mu\nu}\epsilon_{a}^{\ e}\sigma^{\mu\nu}\epsilon_{f}^{\ b}) \]

\[ \delta_{a}^{\ b}\delta_{d}^{\ c} = \frac{1}{2}\sigma^{\mu}_{\ ad}\epsilon^{\mu\nu\delta\epsilon} \]

Duality Relations

\[ *\sigma^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}\sigma^{\lambda\sigma} = \sigma^{\lambda\sigma} \]

\[ *\dot{\sigma}^{\mu\nu} = -\dot{\sigma}^{\mu\nu} \]

Other identities

\[ (\dot{\sigma}^{\mu\nu})_{\ a}^{\ \dot{a}} = 0 \]

\[ (\sigma^{\mu\nu})_{\ a}^{\ a} = 0 \]

\[ (\sigma^{\mu})_{\ a}^{\ ba} = (\sigma^{\mu})_{\ ba} \]

\[ (\sigma^{\mu})_{\ c}^{\ b} = (\sigma^{\mu})_{\ c}^{\ b} \]

\[ (\sigma^{\lambda})_{\ ba} = -\sigma^{\alpha}\sigma^{\lambda}_{\ ba} \]

\[ \sigma_{\ ba}^{\lambda}\sigma_{\ b}^{\ \dot{c}} = (\sigma^{\lambda})_{\ ba}^{\ \dot{c}} = -\sigma_{\ ba}^{\lambda} \]

A.4 Complex conjugation of multi-spinors

By a multi-spinor we mean any quantity which has more than one spinorial index (dotted or undotted, upper or lower). The action of * on such an object is to

(i) raise a lower index,

(ii) lower an upper index

in each case taking account of the natural position of the index,
(iii) invert the order of the indices

(iv) place a conjugation bar on the multispinor (or remove it from an already conjugated multispinor).

We can include the $\sigma$-matrices in this discussion by noting that if the multispinor is Hermitian there is no need to introduce a bar. Thus,

\[(V_a^b)_* = \nabla^{ba}, (\nabla^{ba})_* = V_{ab}, (V_a^b)_* = -\nabla_b^a\]

\[(\sigma^\mu)_{ab}^* = (\bar{\sigma}^\mu)^{ba}\]

\[(\sigma^{\mu\nu})_{ab}^* = -(\sigma^{\mu\nu})_{ba}\]

\[(\hat{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}^* = -(\hat{\sigma}^{\mu\nu})_{\dot{b}\dot{a}}\]

\[(\hat{\sigma}^{\mu\nu})_{\dot{a}\dot{b}}^* = -(\hat{\sigma}^{\mu\nu})_{\dot{b}\dot{a}}\]

\[\epsilon_{ab}^* = \epsilon^{ba}, (\psi_{ab}^c)^* = -\bar{\psi}_{cb}^a\]

\[(\psi_{ab}^c)^* = \bar{\psi}_{cb}^{a}, (\bar{\psi}_{ab}^{c})^* = -\bar{\psi}_{cb}^{a}, \text{ etc.}\]

Appendix B.

B.1 2-spinor indices for a 4-vector

\[V_{ab} = V^\mu (\sigma^\mu)_{ab} \implies V^\mu = \frac{1}{2} V_{ab} (\bar{\sigma}^\mu)^{ba}\]

\[W^{\dot{a}\dot{b}} = W^\mu (\bar{\sigma}^\mu)^{\dot{a}\dot{b}} \implies W^\mu = \frac{1}{2} W^{\dot{a}\dot{b}} (\sigma^\mu)_{\dot{b}\dot{a}}\]

B.2 2-spinor indices for self-dual part of rank-2 anti-symmetric tensor

\[S^{\mu\nu} = -S^{\nu\mu}, S^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} S^{\alpha\beta} = S^{\mu\nu}\]

\[S^{\mu\nu} = \frac{1}{2} (\sigma^{\mu\nu})^\mu_a\]

\[\implies S^{\mu\nu} Tr[\sigma^{\mu\nu} \sigma^{\alpha\beta}]\]
\[ S^{\alpha\beta} = -\frac{1}{2} S_{a}^{\ b}(\sigma^{\alpha\beta})_{\ a} \]

We note \( S_{ab} = S_{ba} \).

**B.3 2-spinor indices for anti-self dual part of rank-2 anti-symmetric tensor**

\[ A^{\mu\nu} = -A^{\nu\mu}, \quad *A^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} A^{\alpha\beta} = -A^{\mu\nu} \]

\[ A_{\hat{a}}^{\hat{b}} = A^{\mu\nu}(\hat{\sigma}^{\mu\nu})_{\hat{a}}^{\hat{b}} \]

\[ \implies A_{\hat{a}}^{\hat{b}}(\hat{\sigma}^{\alpha\beta})_{\hat{b}}^{\hat{a}} = A^{\mu\nu} Tr[\hat{\sigma}^{\mu\nu}\hat{\sigma}^{\alpha\beta}] \]

\[ \implies A^{\alpha\beta} = -\frac{1}{2} A_{\hat{a}}^{\hat{b}}(\hat{\sigma}^{\alpha\beta})_{\hat{b}}^{\hat{a}} \]

We note \( A_{\hat{a}b} = A_{b\hat{a}} \).

**B.4 Special cyclic sum property of \( \Delta_{S}, \Delta_{A}, \Delta_{+}, \) and \( \Delta_{-} \)**

\[ \Delta_{i}^{\mu\nu\lambda} + \Delta_{i}^{\nu\lambda\mu} + \Delta_{i}^{\lambda\mu\nu} = 0 \quad i = S, A, +, - \]

where \( \Delta_{i}^{\mu\nu\lambda} = -\Delta_{i}^{\nu\lambda\mu} \) and \( \Delta_{i}^{\mu\lambda\nu} = 0 \). To see this we define

\[ H_{i}^{\mu\nu\lambda} = \Delta_{i}^{\mu\nu\lambda} + \Delta_{i}^{\nu\lambda\mu} + \Delta_{i}^{\lambda\mu\nu} \]

and consider the dual of \( H_{i}^{\mu\nu\lambda} \) with respect to \((\nu\lambda)\) for \( i = S, A \). We find

\[ *H_{i}^{\mu\nu\lambda} = \frac{1}{2} \epsilon^{\nu\lambda\alpha\beta} \left[ \Delta_{i}^{\mu\alpha\beta} + \Delta_{i}^{\alpha\beta\mu} + \Delta_{i}^{\beta\mu\alpha} \right] \]

\[ = \pm \left[ \Delta_{i}^{\mu\alpha\beta} \mp \epsilon^{\nu\lambda\alpha\beta} \Delta_{i}^{\alpha\beta\mu} \right] \]

\[ = \pm \left[ \Delta_{i}^{\mu\alpha\beta} + \frac{1}{2} \epsilon^{\nu\lambda\alpha\beta} \epsilon^{\beta\rho\tau} \Delta_{i}^{\rho\tau} \right] \]

\[ = \pm \left[ \Delta_{i}^{\mu\alpha\beta} - \Delta_{i}^{\mu\alpha\beta} \right] \]

\[ = 0 \]

from which the result follows.

**B.5 Special property of anti-symmetric tensors**

If \( A^{\mu\nu} = -A^{\nu\mu} \) and \( B^{\mu\nu} = -B^{\nu\mu} \) and the anti-symmetric tensors \( T \) and \( X \) are defined by

\[ T^{\alpha\beta} = A^{\mu\alpha} B^{\mu\beta} - A^{\mu\beta} B^{\mu\alpha} \]
\[ X^{\alpha\beta} = *A^{\mu\alpha}B^{\mu\beta} - *A^{\mu\beta}B^{\mu\alpha} \]

then \( X \) is the dual of \( T \),

\[ *X^{\alpha\beta} = T^{\alpha\beta}. \]