ON SCATTERING FOR THE CUBIC DEFOCUSING NONLINEAR SCHRÖDINGER EQUATION ON WAVEGUIDE $\mathbb{R}^2 \times \mathbb{T}$

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ABSTRACT. In this article, we will show the global wellposedness and scattering of the cubic defocusing nonlinear Schrödinger equation on waveguide $\mathbb{R}^2 \times \mathbb{T}$ in $H^1$. We first establish the linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$ motivated by the linear profile decomposition of the mass-critical Schrödinger equation in $L^2(\mathbb{R}^2)$. Then by using the solution of the infinite dimensional vector-valued resonant nonlinear Schrödinger system to approximate the nonlinear profile, we can prove scattering in $H^1$ by using the concentration-compactness/rigidity method.

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1. Introduction

In this article, we will consider the cubic nonlinear Schrödinger equation on \( \mathbb{R}^2 \times \mathbb{T} \):

\[
\begin{aligned}
    i\partial_t u + \Delta_{\mathbb{R}^2 \times \mathbb{T}} u &= |u|^2 u, \\
    u(0) &= u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T}),
\end{aligned}
\]

(1.1)

where \( \Delta_{\mathbb{R}^2 \times \mathbb{T}} \) is the Laplace-Beltrami operator on \( \mathbb{R}^2 \times \mathbb{T} \) and \( u : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \to \mathbb{C} \) is a complex-valued function.

The equation (1.1) has the following conserved quantities:

- mass conservation: \( \mathcal{M}(u(t)) = \int_{\mathbb{R}^2 \times \mathbb{T}} |u(t,x,y)|^2 \, dx \, dy \),
- energy conservation: \( \mathcal{E}(u(t)) = \int_{\mathbb{R}^2 \times \mathbb{T}} \frac{1}{2} |\nabla u(t,x,y)|^2 + \frac{1}{4} |u(t,x,y)|^4 \, dx \, dy \),
- momentum conservation: \( \mathcal{P}(u(t)) = \int_{\mathbb{R}^2 \times \mathbb{T}} \nabla \bar{u}(x,y,t) \cdot \nabla u(x,y,t) \, dx \, dy \).

The equation (1.1) is a special case of the general nonlinear Schrödinger equation on the waveguide \( \mathbb{R}^n \times \mathbb{T}^m \):

\[
\begin{aligned}
    i\partial_t u + \Delta_{\mathbb{R}^n \times \mathbb{T}^m} u &= |u|^{p-1} u, \\
    u(0) &= u_0 \in H^1(\mathbb{R}^n \times \mathbb{T}^m),
\end{aligned}
\]

(1.2)

where \( 1 < p < \infty, m, n \in \mathbb{Z} \), and \( m, n \geq 1 \). This kind of equations arise as models in the study of nonlinear optics (propagation of laser beams through the atmosphere or in a plasma), especially in nonlinear optics of telecommunications [18,19].

We are interested in the range of \( p \) for wellposedness and scattering of (1.2) on \( \mathbb{R}^n \times \mathbb{T}^m \). On one hand, the wellposedness is intuitively determined by the local geometry of the manifold \( \mathbb{R}^n \times \mathbb{T}^m \). Because the manifold is locally just \( \mathbb{R}^n \times \mathbb{R}^m \), we believe the wellposedness is the same as the Euclidean case, that is when \( 1 < p \leq 1 + \frac{4}{m+n-2} \) the wellposedness is expected. Just as the Euclidean case, we call the equation energy-subcritical when \( 1 < p < 1 + \frac{4}{m+n-2}, m, n \geq 1 \) and energy-critical when \( p = 1 + \frac{4}{m+n-2}, m + n \geq 3, m, n \geq 1 \). On the other hand, scattering is expected to be determined by the asymptotic volume growth of ball with radius \( r \) in the manifold \( \mathbb{R}^2 \times \mathbb{T} \) when \( r \to \infty \). From the heuristic that linear solutions with frequency \( \sim N \) initially localized around the origin will disperse at time \( t \) in the ball of radius \( \sim Nt \), scattering is expected to be partly determined by the asymptotic volume growth of balls with respect to their radius. Since \( \inf_{z \in \mathbb{R}^n \times \mathbb{T}^m} \text{Vol}_{\mathbb{R}^n \times \mathbb{T}^m}(B(z,r)) \sim r^n \), as \( r \to \infty \), the linear solution is expected to decay at a rate \( \sim t^{-\frac{n}{2}} \) and based on the scattering theory on \( \mathbb{R}^n \), the solution of (1.2) is expected to scatter for \( p \geq 1 + \frac{4}{n} \). Moreover, modified scattering in the small data case is expected for \( 1 + \frac{2}{n} < p < 1 + \frac{4}{n} \) when \( n \geq 2 \) or \( 2 < p < 5 \) when \( n = 1 \). Therefore, regarding heuristic on the wellposedness and scattering, we expect the solution of (1.2) globally exists and scatters in the range \( 1 + \frac{4}{n} \leq p \leq 1 + \frac{4}{m+n-2} \). For \( 1 + \frac{2}{n} < p < 1 + \frac{4}{n} \) when \( n \geq 2 \) or \( 2 < p < 5 \) when \( n = 1 \), modified scattering is expected as in the Euclidean space case for small data.

The nonlinear Schrödinger equations on the waveguide have been intensively studied in the last decades. When \( n = m = 1 \), H. Takaoka and N. Tzvetkov [20] proved global wellposedness for any data in \( L^2 \) when \( 1 < p < 3 \) and global wellposedness for small data in \( L^2 \) when \( p = 3 \). This work heavily relies on the techniques developed by J. Bourgain [2], where the \( X^{s,b} \) space is used to show the wellposedness when \( 1 < p < 3 \), and the Hardy-Littlewood circle method is used to establish the Strichartz estimate when \( p = 3 \) as in [2]. Later, S. Herr, D. Tataru and N. Tzvetkov [11] considered the cubic nonlinear Schrödinger
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equation on $\mathbb{R}^n \times \mathbb{T}^m$ with $n + m = 4$ and $0 \leq n \leq 3$. In particular, they proved global wellposedness for small data in $H^s$, $s \geq 1$ in the case $\mathbb{R}^2 \times \mathbb{T}^2$, $\mathbb{R}^3 \times \mathbb{T}$ after establishing the corresponding Strichartz estimate, where the trilinear estimates in the context of the $U^p$- and $V^p$-type spaces is used as in [10] to deal with the nonlinear term in the critical space. Recently, Z. Hani, B. Pausader, N. Tzvetkov and N. Visciglia [2] considered the cubic nonlinear Schrödinger equation posed on the spatial domain $\mathbb{R} \times \mathbb{T}^m$, where $1 \leq m \leq 4$. They proved modified scattering and constructed modified wave operators for small initial and final data.

In [14], A. D. Ionescu and B. Pausader proved global wellposedness in $H^1$ for the cubic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^3$. In the article, they first establish a scale invariant Strichartz estimate $\|e^{it\Delta_{\mathbb{R}^3}} P_N f \|_{L^q([-1,1] \times \mathbb{R}^3) \times \mathbb{T}^3} \lesssim N^{2-\frac{q}{4}} \|f\|_{L^2}$, where $q > \frac{18}{5}$, $N \geq 1$. Then a linear profile decomposition consists of $e^{-it_k \Delta_{\mathbb{R}^3}} \left( \lambda_k^{-\frac{4}{3}} \phi_k(\lambda_k^{-\frac{1}{3}} \Psi(x-x_k)) \right)$, where $\lambda_k \to 0$, $x_k \in \mathbb{R} \times \mathbb{T}^3$, $\phi_k \in \dot{H}^1(\mathbb{R}^4)$, $\Psi$ is a local diffeomorphism from $\mathbb{R} \times \mathbb{T}^3$ to $\mathbb{R}^4$ and $\hat{\phi}_k(x-x_k)$, where $\hat{\phi}_k \in \dot{H}^1(\mathbb{R} \times \mathbb{T}^3)$, was established similar to [12][13]. To show the space-time control of the nonlinear profiles associated with the profile $e^{-it_k \Delta_{\mathbb{R}^3}} \left( \lambda_k^{-\frac{4}{3}} \phi_k(\lambda_k^{-\frac{1}{3}} \Psi(x-x_k)) \right)$ in the linear profile decomposition, the scattering of the energy-critical nonlinear Schrödinger equation on $\mathbb{R}^4$ is used. N. Tzvetkov and N. Visciglia [25] studied the Cauchy problem and large data scattering for the energy subcritical nonlinear Schrödinger equation on $\mathbb{R}^n \times \mathbb{T}$ in $H^1$, where $n \geq 1$ and $1 + \frac{4}{n} < p < 1 + \frac{4}{n-1}$. In the article, by the Strichartz estimate, local wellposedness has been established. After proving the Morawetz estimate, global wellposedness and scattering have been established. When $n = 1$ and $m = 2$, Z. Hani and B. Pausader [8] consider the quintic nonlinear Schrödinger equation. They show global wellposedness and large-data scattering in $H^1(\mathbb{R} \times \mathbb{T}^2)$ in the defocusing case. In order to obtain the local existence, a local in time Strichartz estimate $\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}^3} P_N f \|_{L^q([-1,1] \times \mathbb{R}^3 \times \mathbb{T}^3)} \lesssim N^{2-\frac{q}{4}} \|f\|_{L^2}$, where $N \geq 1$ and $q \geq 4$ is sufficient. However, to obtain the asymptotic behavior, they need some global in time Strichartz estimate. By using the Strichartz estimate of the $\mathbb{R}$ direction of the whole manifold $\mathbb{R} \times \mathbb{T}^2$ and the Sobolev embedding, there is a natural global in time Strichartz estimate $\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}^3} f \|_{L^q_x([\gamma,\gamma+1]) (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{T}^3)} \lesssim \|f\|_{H^1_{x,y}^q}$, this loss of $\frac{2}{q}$ derivatives does not allow the local wellposedness theory in $H^1$. Thus, to overcome the difficulty, a global in time integrability and less derivative loss type Strichartz estimate is needed. By $TT^*$ argument, decomposing the relevant inner product into a diagonal part and a non-diagonal part. The diagonal component leads to the loss of derivatives but gives a contribution that has $L^2_q$ time integrability, while the nondiagonal part loses fewer derivatives but brings slower $L^2_q(4 < q < \infty)$ time integrability. They develop a global Strichartz estimate of the form

$$\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}^3} f \|_{L^q_x([\gamma,\gamma+1]) (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{T}^3)} \lesssim \|\nabla\|^{\frac{2}{q}} \|f\|_{L^2_{x,y}^q(\mathbb{R} \times \mathbb{T}^3)},$$

where $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, $4 < q, r < \infty$. To give the existence of the critical element, they establish a linear profile decomposition, where another kind of profile looks like $e^{-it_k \Delta_{\mathbb{R}^3}} (e^{i(x-x_k)} \xi_k \lambda_k^{-\frac{2}{3}} \phi_k(\lambda_k^{-1}(x-x_k, y)))$, where $\lambda_k \to \infty$, $\xi_k \in \mathbb{R}$, $x_k \in \mathbb{R}$, $\phi_k \in H^{0.1}$ was included in the linear profile decomposition. To exclude the nonlinear profile associated with the large-scale profile in the linear profile decomposition, they assume the scattering of the quintic resonant system

$$i\partial_t u_j + \Delta u_j = \sum_{(j_1, j_2, j_3, j_4, j_5) \in \mathcal{Q}(j)} u_{j_1} \bar{u}_{j_2} u_{j_3} \bar{u}_{j_4} u_{j_5}, j \in \mathbb{Z}^2,$$

where $\mathcal{Q}(j) = \{ (j_1, j_2, j_3, j_4, j_5) \in (\mathbb{Z}^2)^5 : j_1 - j_2 + j_3 - j_4 + j_5 = j, |j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 + |j_5|^2 = |j|^2 \}$. We also refer to [23][24] on the wellposedness and scattering of the nonlinear Schrödinger equation on general waveguide $\mathbb{R}^n \times M^m$, where $M^m$ is a compact $m$-dimensional Riemann manifold.
We now give a summary of the known results of NLS on $\mathbb{R}^n \times \mathbb{T}^m$ for $n, m \in \mathbb{Z}$, and $n, m \geq 1$ in the following table:

| $\mathbb{R}^n \times \mathbb{T}^m$ | Results |
|----------------------------------|---------|
| $\mathbb{R} \times \mathbb{T}$  | GWP in $L^2(1 < p < 3)$, GWP in $L^2$, modified scattering ($p = 3$, small data) |
| $\mathbb{R} \times \mathbb{T}^2$ | LWP & GWP in $H^1$, modified scattering ($p = 3$, small data), Scattering in $H^1(p = 5)$ |
| $\mathbb{R}^2 \times \mathbb{T}$  | LWP & GWP in $H^1$, modified scattering ($p = 3$, small data) |
| $\mathbb{R}^2 \times \mathbb{T}^2$ | GWP & Scattering in $H^1(3 < p < 5)$ |
| $\mathbb{R}^4 \times \mathbb{T}$  | GWP & Scattering in $H^1(\frac{4}{3} < p < 3)$ |

**Table 1.** We only state the case for $n + m \leq 4$.

Our main result addresses the scattering for (1.1) in $H^1(\mathbb{R}^2 \times \mathbb{T})$:

**Theorem 1.1** (Scattering in $H^1(\mathbb{R}^2 \times \mathbb{T})$). For any initial data $u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T})$, there exists a solution $u \in C^0_t H^1_{x,y}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ that is global and scatters in the sense that there exist $u^\pm \in H^1(\mathbb{R}^2 \times \mathbb{T})$ such that

$$
\| u(t) - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^\pm \|_{H^1(\mathbb{R}^2 \times \mathbb{T})} \to 0, \text{ as } t \to \pm \infty.
$$

The proof of Theorem 1.1 is based on the concentration compactness/rigidity method developed by C. E. Kenig and F. Merle [15]. For (1.1), the dispersive effect of the $\mathbb{R}^2$-component is strong enough, to give a global Strichartz estimate [24] [25]:

$$(1.3) \quad \| e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} f \|_{L^4_{t,x} H^1_{x,y} \cap L^1_{t,x} W^{1,4}_{x,y} L^2_{y} (\mathbb{R}^2 \times \mathbb{T})} \lesssim \| f \|_{H^1_{x,y}},$$

where $(q,r)$ is $L^2$-admissible on $\mathbb{R}^2$. By the wellposedness and scattering theory, we observe to prove Theorem 1.1, we only need to prove the solution satisfies a weaker space norm $L^4_{t,x} H^1_{y,\epsilon_0}$, where $0 < \epsilon_0 < \frac{1}{2}$ is some fixed number used hereafter. So we only need a linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$ with the remainder in $L^4_{t,x} H^1_{y,\epsilon_0}$, which is essentially equivalent to describe the defect of compactness of

$$(1.4) \quad e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} : H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T}) \hookrightarrow L^4_{t,x} H^1_{y,\epsilon_0}(\mathbb{R}^2 \times \mathbb{T}),$$

we can then establish a linear profile decomposition similar to [8]. However, the argument is mainly based on the argument to establish the linear profile decomposition of the Schrödinger equation in $L^2(\mathbb{R}^2)$.

The nonlinear profiles can be defined to be the solution of the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$, with initial data is each profile in the linear profile decomposition. Just as in [8], the nonlinear profile can be approximated by applying $e^{it\Delta_{\mathbb{R}^2}}$ to the rescaling of the solution of the cubic resonant Schrödinger system

$$(1.5) \quad \begin{cases}
    i\partial_t u_j + \Delta_{\mathbb{R}^2} u_j = \sum_{j_1,j_2,j_3 \in \mathcal{R}(j)} u_{j_1} \bar{u}_{j_2} u_{j_3}, \\
    u_j(0) = u_{0,j}, \quad j \in \mathbb{Z},
  \end{cases}$$

where $\mathcal{R}(j) = \{j_1, j_2, j_3 \in \mathbb{Z} : j_1 - j_2 + j_3 = j, |j_1|^2 - |j_2|^2 + |j_3|^2 = |j|^2 \}$ in the large scale case. To derive the almost-periodic solution, we also need the following scattering theorem of the cubic resonant Schrödinger system, which is proved by using the argument in [7] to deal with the cubic nonlinear Schrödinger equation in $L^2(\mathbb{R}^2)$:
Theorem 1.2 (Scattering of the cubic resonant Schrödinger system, [26]). Let $E > 0$, for any initial data $\tilde{u}_0$ satisfying

$$\|\tilde{u}_0\|_{L^2_t H^1_x} := \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |u_{j,t}|^2 \right)^{\frac{1}{2}} \right\|_{L^2_t L^2_x} \leq E,$$

there exists a global solution to (1.5), where $\tilde{u} = \{u_j\}_{j \in \mathbb{Z}}$, with $\|\tilde{u}(t)\|_{L^2_t H^1_x} = \|\tilde{u}_0\|_{L^2_t H^1_x}$ satisfying

$$\|\tilde{u}\|_{L^4_t L^4_x \times \mathbb{R}^2} := \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |u_j|^2 \right)^{\frac{1}{2}} \right\|_{L^4_t L^4_x} \leq C,$$

for some constant $C$ depends only on $\|\tilde{u}_0\|_{L^2_t H^1_x}$. In addition, the solution scatters in $L^2_t H^1_x$ in the sense that there exists $\{u_j^\pm\}_j \in L^2_t H^1_x$ such that

$$\left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |u_j(t) - e^{it\Delta_{\mathbb{R}^2}} u_j^\pm|^2 \right)^{\frac{1}{2}} \right\|_{L^2_t L^2_x} \to 0, \text{ as } t \to \pm\infty.$$

By using the concentration-contradiction, the existence of an almost-periodic solution is given in $H^1(\mathbb{R}^2 \times \mathbb{T})$. By the interaction Morawetz quantity, the critical element can be killed.

Remark 1.3. The argument in the proof of Theorem 1.2 in fact does not rely on the structure of $\mathbb{T}$ in the manifold $\mathbb{R}^2 \times \mathbb{T}$. Thus, Theorem 1.2 can be generalized to cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{M}$, where $\mathbb{M}$ is a one dimensional compact Riemann manifold as in [24].

The rest of the paper is organized as follows. After introducing some notations and preliminaries, we give the local wellposedness and small data scattering in Section 2. We also give the stability theory in this section. In Section 3, we derive the linear profile decomposition for data in $H^1(\mathbb{R}^2 \times \mathbb{T})$ and analyze the nonlinear profiles. In Section 4, we reduce the non-scattering in $H^1$ to the existence of an almost-periodic solution and show the extinction of such an almost-periodic solution in Section 5.

1.1. Notation and Preliminaries. We will use the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$.

We define the torus to be $\mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z})$.

In the following, we will frequently use some space-time norm, we now give the definition of it.

For any $I \subset \mathbb{R}$, $u(t, x, y) : I \times \mathbb{R}^2 \times \mathbb{T} \to \mathbb{C}$, define the space-time norm

$$\|u\|_{L^4_t L^4_x L^2_y} := \left\| \left( \int_{\mathbb{T}} |u(t, x, y)|^4 \, dy \right)^{\frac{1}{2}} \right\|_{L^4_t L^4_x} \leq \|u\|_{L^4_t L^4_x},$$

$$\|u\|_{H^1_2} = \|\langle \nabla_x \rangle u\|_{L^2_x} + \|\langle \nabla_y \rangle u\|_{L^2_y}.$$

We will frequently use the partial Fourier transform and partial space-time Fourier transform: For $f(x, y) : \mathbb{R}^2 \times \mathbb{T} \to \mathbb{C}$,

$$\mathcal{F}_x f(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi} f(x, y) \, dx.$$

Given $H : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \to \mathbb{C}$, we denote the partial space-time Fourier transform to be

$$\mathcal{F}_{t,x} H(\omega, \xi, \eta) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\omega t - i\xi x} H(t, x, y) \, dx \, dt.$$
We also define the partial Littlewood-Paley projectors $P^x_{\leq N}$ and $P^x_{\geq N}$ as follows: fix a real-valued radially symmetric bump function $\varphi(\xi)$ satisfying

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases}$$

for any dyadic number $N \in 2\mathbb{Z}$, let

$$\mathcal{F}_{x}(P^x_{\leq N}f)(\xi, y) = \varphi\left(\frac{\xi}{N}\right)(\mathcal{F}_{x}f)(\xi, y),$$

$$\mathcal{F}_{x}(P^x_{\geq N}f)(\xi, y) = \left(1 - \varphi\left(\frac{\xi}{N}\right)\right)(\mathcal{F}_{x}f)(\xi, y)$$

We now define the discrete nonisotropic Sobolev space. For $\tilde{\varphi} = \{\phi_k\}_{k \in \mathbb{Z}}$ a sequence of real-variable functions, we define

$$H^s_{x} h^{s_2} = \left\{ \tilde{\varphi} = \{\phi_k\} : \|\tilde{\varphi}\|_{H^s_{x} h^{s_2}} = \left\|\left(\sum_{k \in \mathbb{Z}} |k|^{2s_2}|\phi_k(x)|^2\right)^{\frac{1}{2}}\right\|_{H^s_{x} h^{s_2}} < \infty \right\},$$

where $s_1, s_2 \geq 0$. In particular, when $s_1 = 0$, we denote the space $H^s_{x} h^{s_2}$ to be $L^2_x h^{s_2}$. For $\psi \in L^2_x H^1_y (\mathbb{R}^2 \times \mathbb{T})$, we have the vector $\tilde{\psi} = \{\psi_k\} \in L^2_x h^1$, where $\psi_k$ is the sequence of periodic Fourier coefficients of $\psi$ defined by

$$\psi_k(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}} \psi(x, y) e^{-iky} dy.$$

Throughout the article, $0 < \epsilon_0 < \frac{1}{2}$ is some fixed number.

2. LOCAL WELLPOSEDNESS AND SMALL DATA SCATTERING

In this section, we will review the local wellposedness and small data scattering, that is Theorem 2.3 and Theorem 2.5 These results have been established in [24][25]. We also give the stability theory which will be used in showing the existence of a critical element in Section 4.

We first recall the following Strichartz estimate, which is established in [24].

**Proposition 2.1** (Strichartz estimate).

(2.1) \[
\|e^{it\Delta_{x}^{\alpha \kappa \gamma}} f\|_{L_t^p L_x^q L_y^\gamma} \lesssim \|f\|_{L^2_x (\mathbb{R}^2 \times \mathbb{T}_y)},
\]

(2.2) \[
\left\| \int_0^t e^{i(t-s)\Delta_{x}^{\alpha \kappa \gamma}} F(s, x, y) \, ds \right\|_{L_t^p L_x^q L_y^\gamma} \lesssim \|F\|_{L_t^{p'} L_x^{q'} L_y^\gamma},
\]

where $(p, q)$, $(\bar{p}, \bar{q})$ satisfies $\frac{2}{p} + \frac{2}{q} = 1$, $\frac{2}{\bar{p}} + \frac{2}{\bar{q}} = 1$, and $2 < p, \bar{p} \leq \infty$.

The following nonlinear estimate is useful in showing the local wellposedness.

**Proposition 2.2** (Nonlinear estimate).

(2.3) \[
\|u_1 u_2 u_3\|_{L_t^\frac{4}{1} L_x^\frac{4}{1} H^1_y} \lesssim \|u_1\|_{L_t^2 L_x^2 H^{1-\epsilon_0}_y} \|u_2\|_{L_t^2 L_x^2 H^{1-\epsilon_0}_y} \|u_3\|_{L_t^2 L_x^2 H^{1-\epsilon_0}_y}.
\]
Proof. Since \( H^1_y(\mathbb{T}) \) is an algebra, we have

\[
\| u_1 u_2 u_3 \|_{L^1_y H^1_y} \lesssim \| u_1 \|_{H^1_y} \| u_2 \|_{H^1_y} \| u_3 \|_{H^1_y}.
\]

By the Hölder inequality,

\[
\| u_1 \|_{L^1_y L^1_y H^1_y} \| u_2 \|_{L^1_y L^1_y H^1_y} \| u_3 \|_{L^1_y L^1_y H^1_y} \leq \| u_1 \|_{L^1_y L^1_y H^1_y} \| u_2 \|_{L^1_y L^1_y H^1_y} \| u_3 \|_{L^1_y L^1_y H^1_y},
\]

so we have (2.3). □

By the Strichartz estimate and the nonlinear estimate, we can give the local wellposedness and small data scattering in \( L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}) \) and \( H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T}) \) easily.

**Theorem 2.3** (Local wellposedness). For any \( E > 0 \), suppose that \( \| u_0 \|_{L^2_y H^1_y(\mathbb{R}^2 \times \mathbb{T})} \leq E \), there exists \( \delta_0 = \delta_0(E) > 0 \) such that if

\[
\| e^{it\Delta_{x,y}} u_0 \|_{L^1_x L^4_y H^1_y(\mathbb{R}^2 \times \mathbb{T})} \leq \delta_0,
\]

where \( I \) is a time interval, then there exists a unique solution \( u \in C^0(I \times \mathbb{R}^2 \times \mathbb{T}) \) of (1.1) satisfying

\[
\| u \|_{L^1_x L^4_y H^1_y} \leq 2 \| e^{it\Delta_{x,y}} u_0 \|_{L^1_x L^4_y H^1_y},
\]

\[
\| u \|_{L^\infty_x L^2_y H^1_y} \leq C \| u_0 \|_{L^2_y H^1_y}.
\]

Moreover, if \( u_0 \in H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T}) \), then \( u \in C^0(I \times \mathbb{R}^2 \times \mathbb{T}) \) with

\[
\| u \|_{L^\infty_x H^1_{x,y}} \leq C(E) \| u_0 \|_{H^1_{x,y}}, \quad \| u \|_{L^1_x L^4_y H^1_y \cap L^4_x W^{1,4}_y L^2_x} \leq C(\| u_0 \|_{H^1_y}).
\]

Arguing as in [5,22], we can easily obtain the global wellposedness by the Strichartz estimate together with the conservation of mass and energy:

**Theorem 2.4** (Global wellposedness in \( H^1 \)). For any \( E > 0 \), if \( \| u_0 \|_{H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})} \leq E \), there exists a unique global solution \( u \in C^0(I \times \mathbb{R}^2 \times \mathbb{T}) \) of (1.1) satisfying

\[
\| u \|_{L^1_x L^4_y H^1_y \cap L^4_x W^{1,4}_y L^2_x} \leq C(E) \| u_0 \|_{H^1_y},
\]

\[
\| u \|_{L^\infty_x H^1_{x,y}} \leq C(E) \| u_0 \|_{H^1_{x,y}}.
\]

**Theorem 2.5** (Small data scattering in \( L^2_x H^1_y \)). There exists \( \delta > 0 \) such that if \( u_0 \in H^1_{x,y} \) and \( \| u_0 \|_{L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T})} \leq \delta \), (1.1) has an unique global solution \( u(t, x, y) \in C^0(I \times \mathbb{R}^2 \times \mathbb{T}) \) and there exist \( u_0 \in L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}) \) such that

\[
\| u(t, x, y) - e^{it\Delta_{x,y}} u_0(x, y) \|_{L^2_y H^1_y} \rightarrow 0, \quad \text{as } t \rightarrow \pm \infty.
\]

We now give the stability theory in \( L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}) \).

**Theorem 2.6** (Stability theory). Let \( I \) be a compact interval and let \( \tilde{u} \) be an approximate solution to \( i\partial_t u + \Delta_{x,y} u = |u|^2 u \) in the sense that \( i\partial_t \tilde{u} + \Delta_{x,y} \tilde{u} = |\tilde{u}|^2 \tilde{u} + e \) for some function \( e \). Assume that

\[
\tilde{u} \|_{L^\infty_t L^2_y H^1_y} \leq M, \quad \tilde{u} \|_{L^4_x L^4_y H^1_y} \leq L.
\]

for some positive constants \( M \) and \( L \). Let \( t_0 \in I \) and let \( u(t_0) \) obey

\[
\| u(t_0) - \tilde{u}(t_0) \|_{L^2_y H^1_y} \leq M'
\]
for some $M' > 0$. Moreover, assume the smallness conditions

\begin{align}
(2.6) \quad \|e^{i(t-t_0)\Delta_{R^2\times\mathbb{T}}}(u(t_0) - \tilde{u}(t_0))\|_{L^4_t L^2_y H^{1-\epsilon_0}_y} & \leq \epsilon, \\
(2.7) \quad \|e\|_{L^4_t L^2_y H^{1-\epsilon_0}_y} & \leq \epsilon,
\end{align}

for some $0 < \epsilon \leq \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M', L) > 0$ is a small constant. Then, there exists a solution $u$ to

\[ i\partial_t u + \Delta_{R^2\times\mathbb{T}} u = |u|^2 u \text{ on } I \times R^2 \times \mathbb{T} \]

with initial data $u(t_0)$ at time $t = t_0$ satisfying

\[ \|u - \tilde{u}\|_{L^4_t L^2_y H^{1-\epsilon_0}_y} \leq C(M, M', L)\epsilon, \]

\[ \|u - \tilde{u}\|_{L^\infty_t L^2_y H^{1-\epsilon_0}_y} \leq C(M, M', L)M', \]

\[ \|u\|_{L^\infty_t L^2_y H^{1-\epsilon}_y} \leq C(M, M', L), \]

\[ \|u\|_{L^4_t L^2_y H^{1-\epsilon_0}_y} \leq C(M, M', L). \]

We need a short-time perturbation to prove this theorem in $L^2_y H^{1-\epsilon}_y(R^2 \times \mathbb{T})$.

**Lemma 2.7** (Short-time perturbation). Let $I$ be a compact interval and let $\tilde{u}$ be an approximate solution to (1.1) in the sense that $i\partial_t \tilde{u} + \Delta_{R^2\times\mathbb{T}} \tilde{u} = |\tilde{u}|^2 \tilde{u} + e$ for some function $e$. Assume that

\[ \|\tilde{u}\|_{L^\infty_t L^2_y H^{1-\epsilon_0}_y(I \times R^2 \times \mathbb{T})} \leq M \]

for some positive constant $M$. Let $t_0 \in I$ and $u(t_0)$ be such that

\[ \|u(t_0) - \tilde{u}(t_0)\|_{L^2_y H^{1-\epsilon}_y} \leq M' \]

for some $M' > 0$. Assume also the smallness conditions

\begin{align}
(2.8) \quad \|\tilde{u}\|_{L^4_t L^2_y H^{1-\epsilon_0}_y(I \times R^2 \times \mathbb{T})} & \leq \epsilon, \\
(2.9) \quad \|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{L^4_t L^2_y H^{1-\epsilon}_y} & \leq \epsilon, \\
(2.10) \quad \|e\|_{L^4_t L^2_y H^{1-\epsilon_0}_y} & \leq \epsilon,
\end{align}

for some $0 < \epsilon \leq \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M') > 0$ is a small constant. Then, there exists a solution $u$ to (1.1) on $I \times R^2 \times \mathbb{T}$ with initial data $u(t_0)$ at time $t = t_0$ satisfying

\begin{align}
(2.11) \quad \|u - \tilde{u}\|_{L^4_t L^2_y H^{1-\epsilon}_y} & \leq \epsilon, \\
(2.12) \quad \|u - \tilde{u}\|_{L^\infty_t L^2_y H^{1-\epsilon}_y} & \leq M', \\
(2.13) \quad \|u\|_{L^\infty_t L^2_y H^{1-\epsilon}_y} & \leq M + M', \\
(2.14) \quad \|\tilde{u}\|^2 - |\tilde{u}|^2 & \leq \epsilon.
\end{align}

**Proof.** By symmetry, we may assume $t_0 = \inf I$. Let $w = u - \tilde{u}$, then $w$ satisfies the following

\[ \begin{cases} 
    i\partial_t w + \Delta_{R^2\times\mathbb{T}} w = |\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2 \tilde{u} - e, \\
    w(t_0) = u(t_0) - \tilde{u}(t_0).
\end{cases} \]

For $t \in I$, we define

\[ A(t) = \|(|\tilde{u} + w|^2(\tilde{u} + w) - |\tilde{u}|^2 \tilde{u})\|_{L^4_t L^2_y H^{1-\epsilon}_y([t_0, t] \times R^2 \times \mathbb{T})}. \]
By (2.10),
\[
A(t) = \left\| \tilde{u} + w \right\|^2 (\tilde{u} + w) - \left\| \tilde{u} \right\|^2 \tilde{u} + \frac{4}{3} L_1^2 L_2^3 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T)
\]
\[
\lesssim \left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} \left( \left\| \tilde{u} \right\|^2 L_t^2 L_y^4 H_y^{1+\epsilon_0} + \left\| u \right\|^2 L_t^2 L_y^4 H_y^{1+\epsilon_0} \right)
\]
\[
\lesssim \left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T) + \epsilon_1^2 \left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T).
\]
(2.17)

On the other hand, by Strichartz estimate, (2.11) and (2.12), we get
\[
\left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T)
\]
\[
\lesssim e^{(t-t_0)\Delta} w(t_0) \left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T) + A(t) + \epsilon \left\| u \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} ([t_0,t] \times \mathbb{R}^2 \times T)
\]
\[
\lesssim A(t) + \epsilon.
\]
(2.18)

Combining (2.17) and (2.18), we obtain
\[
A(t) \lesssim (A(t) + \epsilon)^3 + \epsilon_1^2 (A(t) + \epsilon).
\]

A standard continuity argument then shows that if \( \epsilon_1 \) is taken sufficiently small,
\[
A(t) \lesssim \epsilon, \quad \forall \ t \in I,
\]
which implies (2.16).

Using (2.16) and (2.18), one easily derives (2.13). Moreover, by Strichartz estimate, (2.9), (2.12) and (2.16),
\[
\left\| u \right\| L_t^\infty L_y^2 H_y^{1+\epsilon_0} (I \times \mathbb{R}^2 \times T)
\]
\[
\lesssim \left\| u(t_0) \right\| L_y^2 H_y^{1+\epsilon_0} + \left\| u + w \right\|^2 u \left\| \tilde{u} \right\|^2 \tilde{u} \left\| L_t^2 L_y^4 H_y^{1+\epsilon_0} + \epsilon \left\| u \right\| L_t^2 L_y^4 H_y^{1+\epsilon_0}
\]
\[
\lesssim \epsilon' + \epsilon,
\]
which establishes (2.14) for \( \epsilon_1 = \epsilon_1 (M') \) sufficiently small.

To prove (2.15), we use Strichartz estimate, (2.8), (2.9), (2.16) and (2.10),
\[
\left\| u \right\| L_t^\infty L_y^2 H_y^{1+\epsilon_0} (I \times \mathbb{R}^2 \times T)
\]
\[
\lesssim \left\| u(t_0) \right\| L_y^2 H_y^{1+\epsilon_0} + \left\| u \right\|^2 \tilde{u} \left\| L_t^2 L_y^4 H_y^{1+\epsilon_0}
\]
\[
\lesssim \epsilon' \left\| \tilde{u} \right\| L_y^2 H_y^{1+\epsilon_0} + \left( u(t_0) - \tilde{u}(t_0) \right) \left\| L_y^2 H_y^{1+\epsilon_0}
\]
\[
+ \left\| u \right\|^2 \tilde{u} - \tilde{u} \left\| \tilde{u} \right\|^2 \tilde{u} \left\| L_t^2 L_y^4 H_y^{1+\epsilon_0} + \left\| u \right\|^2 \tilde{u} \left\| L_t^2 L_y^4 H_y^{1+\epsilon_0}
\]
\[
\lesssim \epsilon' + \epsilon + \left\| u \right\|^3 L_t^4 L_y^4 H_y^{1+\epsilon_0} \lesssim \epsilon + \epsilon' + \epsilon^3.
\]

Choosing \( \epsilon_1 = \epsilon_1 (M, M') \) sufficiently small, this finishes the proof of the lemma.

We now turn to show the stability theory.

Proof. Subdivide \( I \) into \( J \) subintervals \( I_j = [t_j, t_{j+1}] \), \( 0 \leq j \leq J - 1 \) such that
\[
\left\| \tilde{u} \right\| L_t^4 L_y^4 H_y^{1+\epsilon_0} (I_j \times \mathbb{R}^2 \times T) \leq \epsilon_1,
\]
where \( \epsilon_1 = \epsilon_1 (M, 2M') \) is as in Lemma 2.7.

We need to replace \( M' \) by \( 2M' \) as \( \left\| u - \tilde{u} \right\| L_t^\infty L_y^2 H_y^{1+\epsilon_0} \) might grow slightly in time.
By choosing $\epsilon_1$ sufficiently small depending on $J$, $M$ and $M'$, we can apply Lemma 2.7 to obtain for each $j$ and all $0 < \epsilon < \epsilon_1$,

\[
\|u - \tilde{u}\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(j) \epsilon,
\]
\[
\|u - \tilde{u}\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(j) M',
\]
\[
\|u\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(j) (M + M'),
\]
\[
\|u\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(j) \epsilon,
\]

provided we can prove that analogues of (2.5) and (2.6) hold with $t_0$ replaced by $t_j$.

In order to verify this, we use an inductive argument. By Strichartz, (2.5), (2.7) and the inductive hypothesis,

\[
\|u(t_j) - \tilde{u}(t_j)\|_{L^4_y H^{1-\omega}_y} \leq \|u(t_0) - \tilde{u}(t_0)\|_{L^4_y H^{1-\omega}_y} + \|u(t_j) - \tilde{u}(t_j)\|_{L^4_y H^{1-\omega}_y} + \|u\|_{L^4_t L^4_y H^{1-\omega}_y([t_0, t_j] \times \mathbb{R}^2 \times \mathbb{T})} + \|\epsilon\|_{L^4_t L^4_y H^{1-\omega}_y([t_0, t_j] \times \mathbb{R}^2 \times \mathbb{T})}
\]
\[
\leq M' + \sum_{k=0}^{j-1} C(k) \epsilon + \epsilon.
\]

Similarly, by Strichartz, (2.6), (2.7) and the inductive hypothesis,

\[
\|e^{i(t-t_j)\Delta} (u(t_j) - \tilde{u}(t_j))\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq \|e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0))\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} + \|e\|_{L^4_t L^4_y H^{1-\omega}_y([t_0, t_j] \times \mathbb{R}^2 \times \mathbb{T})}
\]
\[
+ \|u^2 u - \tilde{u}^2 \tilde{u}\|_{L^4_t L^4_y H^{1-\omega}_y([t_0, t_j] \times \mathbb{R}^2 \times \mathbb{T})}
\]
\[
\leq \epsilon + \sum_{k=0}^{j-1} C(k) \epsilon.
\]

Choosing $\epsilon_1$ sufficiently small depending on $J$, $M$ and $M'$ to guarantee the hypotheses of Lemma 2.7 continue to hold as $j$ varies. \qed

**Remark 2.8** (Persistence of regularity). *The results in the above theorems can be extended to $H^1(\mathbb{R}^2 \times \mathbb{T})$.***

The following theorem implies that it suffices to show the finiteness of the solution in $L^4_t L^4_y H^{1-\omega}_y$ for the scattering of (1.1) in $H^1$. 

**Theorem 2.9** (Scattering norm). *Suppose that $u \in C^0_t H^1_{x,y}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ is a global solution of (1.1) satisfying $\|u\|_{L^4_t L^4_y H^{1-\omega}_y(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq L$ and $\|u(0)\|_{H^1_{x,y}} \leq M$ for some positive constants $M$, $L$, then $u$ scatters in $H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})$. That is, there exist $u_\pm \in H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})$ such that*

\[
\|u(t, x, y) - e^{it\Delta_{x^2+y^2}} u_\pm(x, y)\|_{H^1_{x,y}} \to 0, \quad \text{as} \ t \to \pm \infty.
\]

**Proof.** By the classical scattering theory as in [5][22], we only need to show

\[
\|u\|_{L^4_t L^4_y H^{1-\omega}_y(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq C(M, L).
\]

By Theorem 2.3, it suffices to prove (2.20) as an a priori bound.

Divide the time interval $\mathbb{R}$ into $N \sim (1 + \frac{L}{\delta})^{10}$ subintervals $I_j = [t_j, t_{j+1}]$ such that

\[
\|u\|_{L^4_t H^{1-\omega}_y(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq \delta,
\]

For the subintervals $I_j$, we have

\[
\|\dot{u}\|_{L^4_t L^4_y H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(M, L),
\]

and

\[
\|u\|_{L^4_t H^{1-\omega}_y(I_j \times \mathbb{R}^2 \times \mathbb{T})} \leq C(M, L).
\]

This completes the proof of Theorem 2.9.
where $\delta > 0$ will be chosen later.

On each $I_j$, by Strichartz estimate, Sobolev embedding and (2.21), we have

$$\|u\|_{L^4_t L^4_x L^2_y(\mathbb{R}^2 \times \mathbb{T})} \leq C\left(\|u(t_j)\|_{H^1} + \|u(t_j)^2\|_{L^4_t H^1_x} + \|u(t_j)^3\|_{L^4_t H^1_x} \right) \leq C\left(\|u(t_j)\|_{H^1} + \|u(t_j)^2\|_{L^4_t H^1_x} \right) \leq C\left(\|u(t_j)\|_{H^1} + \|u(t_j)^2\|_{L^4_t L^4_x H^1_y} \right),$$

choosing $\delta \leq (\frac{1}{2C})^2$, we have

$$\|u\|_{L^4_t L^4_x L^2_y(\mathbb{R}^2 \times \mathbb{T})} \leq 2C\|u(t_j)\|_{H^1_x},$$

The bound now follows by adding up the bounds on each subintervals $I_j$, which gives (2.20). \hfill \Box

### 3. Profile decomposition

In this section, we will establish the linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$ in Subsection 3.1 which will heavily rely on the linear profile decomposition in $L^2(\mathbb{R})$. The linear profile decomposition in $L^2(\mathbb{R})$ for the mass-critical nonlinear Schrödinger equation is established by F. Merle and L. Vega [17]. After the work of R. Carles and S. Keraani [4] on the one-dimensional case, P. Bégout and A. Vargas [11] establish the linear profile decomposition of the mass-critical nonlinear Schrödinger equation for arbitrary dimensions by the refined Strichartz inequality [3] and bilinear restriction estimate [21]. We also refer to [16] for a version of the proof of the linear profile decomposition. We then analyze the nonlinear profiles in Subsection 3.2 which is important to show the existence of the almost-periodic solution in Section 4.

#### 3.1. Linear profile decomposition

In this subsection, we will establish the linear profile decomposition. The linear profile decomposition is mainly inspired by the mass-critical profile decomposition in $\mathbb{R}^2$, which we refer to [1][3][4][16][17].

**Proposition 3.1** (Linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$). Let $\{u_n\}_{n \geq 1}$ be a bounded sequence in $H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})$. Then after passing to a subsequence if necessary, there exist $J^* \in \{0, 1, \ldots \} \cup \{\infty\}$, functions $\phi_j$ in $L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T})$ and mutually orthogonal frames $(\lambda_{n,j}, t_{n,j}, x_{n,j}, \xi_{n,j})_{n \geq 1} \in (0, \infty) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, which means

$$\lambda_{n,j}^k = \lambda_{n,j}^k + \lambda_{n,k}^j \frac{\xi_{n,j}}{\xi_{n,k}} - \xi_{n,k}^j \frac{\xi_{n,k}}{\xi_{n,j}} + \frac{|x_{n,j} - x_{n,k}|^2}{\lambda_{n,k}^2} + \frac{|(\lambda_{n,j}^k)^2 t_{n,j}^k - (\lambda_{n,k}^j)^2 t_{n,k}^j|}{\lambda_{n,j}^k \lambda_{n,k}^j} \to \infty, \text{ as } n \to \infty, \text{ for } j \neq k,$$

with $\lambda_{n,j} \to 1$ or $\infty$, as $n \to \infty$, $|\xi_{n,j}^j| \leq C_j$, for every $1 \leq j \leq J$, and for every $J \leq J^*$ a sequence $r_{n,j} \in L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T})$, such that

$$u_n(x, y) = \sum_{j=1}^{J} \frac{1}{\lambda_{n,j}} e^{ix\xi_{n,j}} \left( e^{i\mu_{n,j}^2} \Delta_{n,j} \phi_j \right) \left( \frac{x - x_{n,j}}{\lambda_{n,j}}, y \right) + r_{n,j}(x, y).$$
Moreover,

\[ \lim_{n \to \infty} \left( \| u_n \|_{L^2_t H^s_x}^2 - \sum_{j=1}^J \| \phi^j \|_{L^2_t H^s_x}^2 - \| r_n^j \|_{L^2_t H^s_x}^2 \right) = 0, \tag{3.2} \]

\[ \lambda_n^j e^{-it_n^j \Delta_{R^2}} \left( e^{-i(\lambda_n^j x + x_n^j)} e^{-it_n^j} (\lambda_n^j x + x_n^j, y) \right) \to 0 \text{ in } L^2_t H^1_y, \text{ as } n \to \infty, \text{ for each } j \leq J, \tag{3.3} \]

\[ \limsup_{n \to \infty} \| e^{it \Delta_{R^2}} r_n^j \|_{L^1_t L^2_x L^2_y} \to 0, \text{ as } J \to J^*. \tag{3.4} \]

**Proof.** By Proposition 3.9 and Remark 3.10, we have the decomposition

\[ u_n(x,y) = \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{ix \xi_n^j} (e^{it \Delta_{R^2}} \phi^j) \left( \frac{x-x_n^j}{\lambda_n^j}, y-y_n^j \right) + w_n^j(x,y), \]

with \( \lambda_n^j \geq 1 \), and \( |\xi_n^j| \leq C_j \), for every \( 1 \leq j \leq J \), and for \( j \neq j' \),

\[ \lambda_n^j + \frac{\lambda_n^{j'}}{\lambda_n^j} + \lambda_n^j |\xi_n^j - \xi_n^{j'}|/\lambda_n^j + |x_n^j - x_n^{j'}|^2/\lambda_n^j + |y_n^j - y_n^{j'}|^2/\lambda_n^j \to \infty, \text{ as } n \to \infty. \]

Since \( \mathbb{T} \) is compact, we may assume \( y_n^j \to y^*_j \), as \( n \to \infty \). Then we may replace \( \phi^j(\cdot,y) \) by \( \phi^j(\cdot,y-y_n^j) \), and set \( y_n^j \equiv 0 \). If \( \lambda_n^j \) does not converge to \( \infty \), suppose \( \lambda_n^j \to \lambda_\infty^j \in [1, \infty) \). Thus, we may replace \( \phi^j(x,\cdot) \) by \( \frac{1}{\lambda_n^j} \phi^j \left( \frac{x}{\lambda_n^j}, \cdot \right) \) and set \( \lambda_n^j \equiv 1 \), whilst retaining the conclusion of Proposition 3.1. \( \square \)

Before presenting Proposition 3.9 that is the linear profile decomposition in \( L^2_t H^1_y (\mathbb{R}^2 \times \mathbb{T}) \). We will first establish the refined Strichartz estimate in Proposition 3.5. We will collect some basic facts appeared in 16.

**Definition 3.2.** Given \( j \in \mathbb{Z} \), we write \( D_j \) for the set of all dyadic cubes of side-length \( 2^j \) in \( \mathbb{R}^2 \),

\[ D_j = \left\{ \left[ 2^j k_1, 2^j (k_1 + 1) \right) \times \left[ 2^j k_2, 2^j (k_2 + 1) \right) : (k_1, k_2) \in \mathbb{Z}^2 \right\}. \]

We also write \( D = \bigcup_j D_j \). Given \( Q \in D \), we define \( f_Q(x,y) \) by \( \mathcal{F}_x(f_Q) = \chi_Q \mathcal{F}_x f \).

By the bilinear Strichartz estimate on \( \mathbb{R}^2 \) and interpolation, we have

**Corollary 3.3.** Suppose \( Q, Q' \in D \) with \( \text{dist}(Q,Q') \geq \text{diam}(Q) = \text{diam}(Q') \), then for some \( 1 < p < 2 \),

\[ \| e^{it \Delta_{R^2}} f_Q \cdot e^{it \Delta_{R^2}} f_{Q'} \|_{L^1_{t,x} L^p_y} \lesssim |Q|^{1/p - 2} \| \mathcal{F}_x f \|_{L^p_y(Q)} \| \mathcal{F}_x f \|_{L^p_y(Q')} \]

By Plancherel’s theorem, we have

**Lemma 3.4.** Let \( \{ R_k \} \) be a family of parallelepipeds in \( \mathbb{R}^2 \) obeying \( \sup_{\xi} \sum_{k} \chi_{\alpha R_k}(\xi) \lesssim 1 \) for some \( \alpha > 1 \). Then

\[ \left\| \sum_k P_{R_k} f_k \right\|_{L^2} \lesssim \sum_k \| f_k \|_{L^2}, \text{ for any } \{ f_k \} \subseteq L^2(\mathbb{R}^2). \]

**Proposition 3.5** (Refined Strichartz estimate).

\[ \| e^{it \Delta_{R^2}} f \|_{L^4_{t,x,y}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \| f \|_{L^2_{1/2} L^4_{1/2} H^s_y} \left( \sup_{Q \in D} |Q|^{1/4} \| e^{it \Delta_{R^2}} f_Q \|_{L^4_{t,x,y}} \right)^{1/2}. \]
Proof. Given distinct $\xi, \xi' \in \mathbb{R}^2$, there is a unique maximal pair of dyadic cubes $Q \uplus \xi$ and $Q' \uplus \xi'$ obeying $|Q| = |Q'|$ and $4 \text{diam}(Q) \leq \text{dist}(Q, Q')$.

Let $\mathcal{W}$ denote the family of all such pairs as $\xi \neq \xi'$ vary over $\mathbb{R}^2$. According to this definition,

$$
\sum_{(Q, Q') \in \mathcal{W}} \chi_Q(\xi) \chi_{Q'}(\xi') = 1, \text{ for a.e. } (\xi, \xi') \in \mathbb{R}^2 \times \mathbb{R}^2.
$$

Note that since $Q$ and $Q'$ are maximal, $\text{dist}(Q, Q') \leq 10 \text{diam}(Q)$. In addition, this shows that given $Q$ there are a bounded number of $Q'$ so that $(Q, Q') \in \mathcal{W}$, that is,

$$
\forall Q \in \mathcal{W}, \# \{Q' : (Q, Q') \in \mathcal{W}\} \leq 1.
$$

In view of (3.5), we can write

$$
(e^{it\Delta_{\mathbb{R}^2}} f)^2 = \sum_{(Q, Q') \in \mathcal{W}} e^{it\Delta_{\mathbb{R}^2}} f_Q \cdot e^{it\Delta_{\mathbb{R}^2}} f_{Q'}.
$$

We have the support of the partial space-time transform of $e^{it\Delta_{\mathbb{R}^2}} f_Q \cdot e^{it\Delta_{\mathbb{R}^2}} f_{Q'}$ satisfies

$$
\text{supp}_{\omega, \xi} \left( F_{t,x} \left( e^{it\Delta_{\mathbb{R}^2}} f_Q \cdot e^{it\Delta_{\mathbb{R}^2}} f_{Q'} \right) \right) (\omega, \xi) \subseteq R(Q + Q'),
$$

where $Q + Q'$ denotes the Minkowski sum and

$$
R(Q + Q') = \left\{ (\omega, \eta) : \eta \in Q + Q', 2 \leq \frac{\omega - \frac{1}{2} |c(Q + Q')|^2 - c(Q + Q')(\eta - c(Q + Q'))}{\text{diam}(Q + Q')^2} \leq 19 \right\},
$$

where $c(Q + Q')$ denotes the center of the cube $Q + Q'$. We also remind that $\text{diam}(Q + Q') = \text{diam}(Q) + \text{diam}(Q') = 2 \text{diam}(Q)$.

We need to control the overlap of the Fourier supports, or rather, of the enclosing parallelepipeds. By [16], for any $\alpha \leq 1.01$,

$$
\sup_{\omega, \eta} \sum_{(Q, Q') \in \mathcal{W}} \chi_{\alpha R(Q + Q')}(\omega, \eta) \leq 1,
$$

where $\alpha R$ denotes the $\alpha$–dilate of $R$ with the same center.

Similar to the argument in [16], we may apply Lemma 3.4 Hölder’s inequality, Corollary 3.3 and (3.6) as follows:

$$
\|e^{it\Delta_{\mathbb{R}^2}} f\|_{L^4_tL^2_{x,y}}^4 = \left\| \sum_{(Q, Q') \in \mathcal{W}} e^{it\Delta_{\mathbb{R}^2}} f_Q \cdot e^{it\Delta_{\mathbb{R}^2}} f_{Q'} \right\|_{L^2_tL^4_{x,y}}^2
\leq \sum_{(Q, Q') \in \mathcal{W}} \left\| e^{it\Delta_{\mathbb{R}^2}} f_Q \right\|_{L^4_{t,x,y}}^2 \left\| e^{it\Delta_{\mathbb{R}^2}} f_{Q'} \right\|_{L^4_{t,x,y}}^2
\leq \left\{ \sup_{Q \in \mathcal{D}} |Q|^{-\frac{2}{p}} \left\| e^{it\Delta_{\mathbb{R}^2}} f_Q \right\|_{L^4_{t,x,y}} \right\} \sum_{(Q, Q') \in \mathcal{W}} \left( |Q|^{-\frac{2}{p}} \right)^2 \left\| F_x f(\xi, \eta) \right\|_{L^p_\xi(Q)}^2 \left\| F_y f(\xi, \eta) \right\|_{L^p_\eta(Q')}^2
\leq \left\{ \sup_{Q \in \mathcal{D}} |Q|^{-\frac{2}{p}} \left\| e^{it\Delta_{\mathbb{R}^2}} f_Q \right\|_{L^4_{t,x,y}} \right\} \sum_{Q \in \mathcal{D}} \left( |Q|^{-\frac{2}{p}} \right)^2 \left\| F_x f(\xi, \eta) \right\|_{L^p_\xi(Q \times \mathbb{T})}^2 \left\| F_y f(\xi, \eta) \right\|_{L^p_\eta(Q \times \mathbb{T})}^2, \text{ where } 1 < p < 2.
$$
As in [16], we have
\[
\sum_{Q \in \mathcal{D}} \left( |Q|^{-\frac{2-p}{p}} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})} \right)^{\frac{3}{2}} \lesssim \left\| f \right\|^{3}_{L^2_x H^{1-\frac{3}{p}}_y}.
\]

In fact, we now assume \( \|f\|^{\frac{1}{p}}_{L^p_x H^{1-\frac{3}{p}}_y} = 1 \) and break
\[
\|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) = \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \geq 2^{-j} \right\} (\xi) + \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \leq 2^{-j} \right\} (\xi),
\]
where the side-length of \( Q \) is \( 2^j \). We see by Hölder, Sobolev, Plancherel,
\[
\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} \left( |Q|^{-\frac{2-p}{p}} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})} \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \geq 2^{-j} \right\} (\xi) \right)^{\frac{3}{2}} \lesssim \left( \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} |Q|^{-\frac{2-p}{p}} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})} \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \geq 2^{-j} \right\} (\xi) \right)^{\frac{3}{p}} \lesssim \left( \int_{\mathbb{R}^2} 2^{-j(2-p)} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})}^{1 \frac{3}{p}} d\xi \right)^{\frac{3}{p}} \lesssim \left( \int_{\mathbb{R}^2} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})}^{1 \frac{3}{p}} d\xi \right)^{\frac{3}{p}} \lesssim \left\| f \right\|_{L^2_x H^{1-\frac{3}{p}}_y}^{\frac{3}{2}} \lesssim 1.
\]

Similarly, by Hölder, Sobolev, Plancherel, we have
\[
\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} \left( |Q|^{-\frac{2-p}{p}} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})} \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \leq 2^{-j} \right\} (\xi) \right)^{\frac{3}{2}} \lesssim \left( \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j} |Q|^{\frac{3}{2}} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})} \chi \left\{ \|\mathcal{F}_x f(\xi, y)\|_{L^p y}^{1 \frac{3}{p}} (T) \leq 2^{-j} \right\} (\xi) \right)^{\frac{3}{p}} \lesssim \left( \int_{\mathbb{R}^2} 2^{-j(2-p)} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})}^{1 \frac{3}{p}} d\xi \right)^{\frac{3}{p}} \lesssim \left( \int_{\mathbb{R}^2} \left\| \mathcal{F}_x f(\xi, y) \right\|_{L_p^p (Q \times \mathbb{T})}^{1 \frac{3}{p}} d\xi \right)^{\frac{3}{p}} \lesssim \left\| f \right\|_{L^2_x H^{1-\frac{3}{p}}_y}^{\frac{3}{2}} \lesssim 1.
\]

Therefore, the proof is completed. \( \square \)

To prove the inverse Strichartz estimate, we also need the following facts:

**Lemma 3.6** (Refined Fatou). Suppose \( \{f_n\} \subseteq L^1(\mathbb{R}^3 \times \mathbb{T}) \) with \( \limsup_{n \to \infty} \|f_n\|_{L^1} < \infty \). If \( f_n \to f \) almost everywhere, then
\[
\|f_n\|_{L^1}^4 - \|f_n - f\|_{L^1}^4 - \|f\|_{L^1}^4 \to 0, \text{ as } n \to \infty.
\]

**Proposition 3.7** (Local smoothing estimate). Fix \( \epsilon > 0 \), we have
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^2 \times \mathbb{T}} \left| \left| \nabla_x \right|^{\frac{1}{2}} e^{it\Delta_x} f \left( x, y \right) \right|^2 \langle x \rangle^{-1-\epsilon} \, dx \, dy \, dt \leq \epsilon \left\| f \right\|_{L^2_x L^q_y(\mathbb{R}^2 \times \mathbb{T})}^2.
\]
Furthermore, if \( \epsilon \geq 1 \), we have
\[
\int_{\mathbb{R}} \int_{\mathbb{R}^2 \times \mathbb{T}} \left| \left( (\nabla_x) \frac{i}{2} e^{it\Delta_{R^2}} f \right)(x,y) \right|^2 dx dy dt \lesssim \epsilon \| f \|_{L^2_{x,y}(\mathbb{R}^2 \times \mathbb{T})}^2.
\]

We can now prove the inverse Strichartz estimate.

**Proposition 3.8 (Inverse Strichartz estimate).** For \( \{f_n\} \subseteq L^2_y H^\frac{1-n}{2}_x \) satisfying
\[
\lim_{n \to \infty} \| f_n \|_{L^2_y H^\frac{1-n}{2}_x} = A \quad \text{and} \quad \lim_{n \to \infty} \| e^{it\Delta_{R^2}} f_n \|_{L^4_{x,y}} = \epsilon,
\]
there exist a subsequence in \( n, \phi \in L^2_y H^\frac{1-n}{2}_x \), \( \{\lambda_n\} \subseteq (0, \infty) \), \( \xi_n \in \mathbb{R}^2 \), and \( (t_n, x_n, y_n) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \) so that along the subsequence, we have the following:

\[
\lambda_n e^{-i\xi_n (\lambda_n x + x_n)} \left( e^{it_n \Delta_{R^2}} f_n \right)(\lambda_n x + x_n, y + y_n) \to \phi(x, y) \quad \text{in} \quad L^2_y H^\frac{1-n}{2}_x,
\]

\[
\lim_{n \to \infty} \left( \| f_n(x,y) \|_{L^2_y H^{\frac{1-n}{2}}_x}^2 - \| f_n - \phi_n \|_{L^2_y H^{\frac{1-n}{2}}_x}^2 \right) = \| \phi \|_{L^2_y H^{\frac{1-n}{2}}_x}^2 \gtrsim A^2 \left( \frac{\epsilon}{A} \right)^{24},
\]

\[
\limsup_{n \to \infty} \| e^{it\Delta_{R^2}} (f_n - \phi_n) \|_{L^4_{x,y}}^4 \leq c^4 \left( 1 - c \left( \frac{\epsilon}{A} \right)^\beta \right),
\]

where \( c \) and \( \beta \) are constants,

\[
\phi_n(x, y) = \frac{1}{\lambda_n} e^{i\xi_n \left( e^{\frac{-ix}{\lambda_n^2}} \phi \right)} \left( \frac{x - x_n}{\lambda_n}, y - y_n \right).
\]

Moreover, if \( \{f_n\} \) is bounded in \( H^1_{x,y} (\mathbb{R}^2 \times \mathbb{T}) \), we can take \( \lambda_n \gtrsim 1 \), \( |\xi_n| \lesssim 1 \), with \( \phi \in L^2_y H^\frac{1}{2}_x (\mathbb{R}^2 \times \mathbb{T}) \), and

\[
\lambda_n e^{-i\xi_n (\lambda_n x + x_n)} \left( e^{it_n \Delta_{R^2}} f_n \right)(\lambda_n x + x_n, y + y_n) \to \phi(x, y) \quad \text{in} \quad L^2_y H^1_x,
\]

\[
\lim_{n \to \infty} \left( \| f_n(x,y) \|_{L^2_y H^1_x}^2 - \| f_n - \phi_n \|_{L^2_y H^1_x}^2 \right) = \| \phi \|_{L^2_y H^1_x}^2 \gtrsim A^2 \left( \frac{\epsilon}{A} \right)^{24}.
\]

**Proof.** By Proposition 3.3 there exists \( \{Q_n\} \subseteq \mathcal{D} \) so that

\[
\epsilon^4 A^{-3} \lesssim \liminf_{n \to \infty} |Q_n|^\frac{1}{24} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}}.
\]

Let \( \lambda_n \) to be the inverse of the side-length of \( Q_n \), which implies \( |Q_n| = \lambda_n^2 \). We also set \( \xi_n = c(Q_n) \), which is the center of the cube. By Hölder’s inequality, we have

\[
\liminf_{n \to \infty} |Q_n|^\frac{1}{24} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}} \lesssim \liminf_{n \to \infty} |Q_n|^\frac{1}{24} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}}.
\]

\[
\lesssim \liminf_{n \to \infty} \lambda_n^\frac{11}{24} \| e^{it\Delta_{R^2}} (f_n) Q_n \|_{L^4_{x,y}}^\frac{1}{24}.
\]

Thus by (3.15), there exists \( (t_n, x_n, y_n) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \) so that

\[
\liminf_{n \to \infty} \lambda_n \langle e^{it_n \Delta_{R^2}} (f_n) Q_n \rangle_{(x_n, y_n)} \gtrsim \epsilon^{12} A^{-11}.
\]

By the weak compactness of \( L^2_y H^\frac{1-n}{2}_x \), we have

\[
\lambda_n e^{-i\xi_n (\lambda_n x + x_n)} \left( e^{it_n \Delta_{R^2}} f_n \right)(\lambda_n x + x_n, y + y_n) \to \phi(x, y) \quad \text{in} \quad L^2_y H^\frac{1-n}{2}_x,
\]
as \( n \to \infty \).
Define \( h \) so that \( \mathcal{F} \gamma h \) is the characteristic function of the cube \( [-\frac{1}{2}, \frac{1}{2}]^2 \). Then \( h(x)\delta_0(y) \in L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T}) \). From (3.17), we obtain
\[
|\langle h(x)\delta_0(y), \phi(x, y) \rangle_{x,y}| = \lim_{n \to \infty} \left| \int_{\mathbb{R}^2 \times \mathbb{T}} \delta_0(y) h(x) \lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{x,y} f_n}(\lambda_n x + x_n, y + y_n)) \, dx dy \right|
\]
\[
= \lim_{n \to \infty} \lambda_n \left| e^{it_n \Delta_{x,y}} (f_n)_{Q_n}(x_n, y_n) \right| \geq \varepsilon^{12} A^{-1},
\]
which implies (3.11), and \( \phi \) carries non-trivial norm in \( L^2_y H_y^{1-\frac{\mu}{2}} \). We are left to verify (3.12). By Proposition 3.7 and the Rellich-Kondrashov theorem, we have
\[
e^{it \Delta_{x,y}} (\lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{x,y} f_n}(\lambda_n x + x_n, y + y_n))) \to e^{it \Delta_{x,y}} \phi(x, y), \quad \text{a.e.} \ (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}, \quad \text{as} \ n \to \infty.
\]
Then by applying Lemma 3.6, we obtain
\[
\left\| e^{it \Delta_{x,y}} f_n \right\|_{L^4_{t,x,y}} = \left\| e^{it \Delta_{x,y}} (f_n - \phi_n) \right\|_{L^4_{t,x,y}} + \left\| e^{it \Delta_{x,y}} \phi_n \right\|_{L^4_{t,x,y}} \to 0, \quad \text{as} \ n \to \infty.
\]
Together with (3.18), we obtain (3.12).

If \( \{f_n\} \) is bounded in \( H_{1,y}^1(\mathbb{R}^2 \times \mathbb{T}) \), we have
\[
\limsup_{n \to \infty} \left\| P_{\geq R} f_n \right\|_{L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T})} \leq \limsup_{n \to \infty} \left\| R \right\|_{\mathcal{F} \gamma} \left\| f_n \right\|_{H_{1,y}^1(\mathbb{R}^2 \times \mathbb{T})} \to 0, \quad \text{as} \ R \to \infty.
\]

For \( R \in 2\mathbb{Z} \) large enough depending on \( A, \varepsilon \), by (3.9), \( P_{\geq R} f_n \) satisfies
\[
\lim_{n \to \infty} \left\| P_{\geq R} f_n \right\|_{L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T})} \geq \lim_{n \to \infty} \left\| f_n \right\|_{L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T})} - \lim_{n \to \infty} \left\| P_{\geq R} f_n \right\|_{L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T})} \geq \frac{1}{2} A,
\]
and
\[
\lim_{n \to \infty} \left\| e^{it \Delta_{x,y}} P_{\geq R} f_n \right\|_{L^4_{t,x,y}} \geq \lim_{n \to \infty} \left\| e^{it \Delta_{x,y}} f_n \right\|_{L^4_{t,x,y}} - \lim_{n \to \infty} \left\| e^{it \Delta_{x,y}} P_{\geq R} f_n \right\|_{L^4_{t,x,y}} \geq \lim_{n \to \infty} \left\| e^{it \Delta_{x,y}} f_n \right\|_{L^4_{t,x,y}} - \lim_{n \to \infty} \left\| C \right\|_{\mathcal{F} \gamma} \left\| P_{\geq R} f_n \right\|_{L^2_y H^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T})} \geq \varepsilon \frac{2}{2}.
\]
So we can replace \( f_n \) by \( P_{\geq R} f_n \) in the assumption of the proposition, for \( R = R(A, \varepsilon) > 0 \) large enough, then we can take \( \{Q_n\} \subseteq \mathcal{D} \) and \( |Q_n| \lesssim R^2 \), then \( \lambda_n \gtrsim R^{-1} \), and \( |\xi_n| = |c(Q_n)| \lesssim R \). In the proof above, we see since \( \{f_n\} \) is bounded in \( H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T}) \), we can obtain
\[
\lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it \Delta_{x,y} f_n}(\lambda_n x + x_n, y + y_n)) \to \phi(x, y)
\]
holds for some \( \phi \in L^2_y H_y^1(\mathbb{R}^2 \times \mathbb{T}) \). We also see \( h(x)\delta_0(y) \in L^2_y H_y^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T}) \), then as (3.18), we have (3.14).

By using Proposition 3.8 repeatedly until the remainder has asymptotically trivial linear evolution in \( L^1_{t,x,y} \), we can obtain the following result:

**Proposition 3.9 (Linear profile decomposition in \( L^2_y H_y^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T}) \)).** Let \( \{u_n\} \) be a bounded sequence in \( L^2_y H_y^{1-\frac{\mu}{2}}(\mathbb{R}^2 \times \mathbb{T}) \). Then (after passing to a subsequence if necessary) there exists \( J^* \in \{0, 1, \ldots\} \cup \{\infty\} \), functions \( \phi^j \in L^2_y H_y^{1-\frac{\mu}{2}} \), \( (\lambda^j_n, t^j_n, x^j_n, y^j_n)_{n \geq 1} \subseteq (0, \infty) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \), so that for any \( J \leq J^* \), defining \( w_n^J \) by
\[
u_n(x, y) = \sum_{j=1}^{J} \frac{1}{\lambda^j_n} e^{it^j_n \Delta_{x,y} \phi^j} \left( \frac{x - x^j_n}{\lambda^j_n}, y - y^j_n \right) + w_n^J(x, y),
\]
we have the following properties:

\[
\limsup_{n \to \infty} \|e^{it\Delta_{x^2}} w_n\|_{L^4_{x,y} (\mathbb{R}^2)} = 0, \quad \text{as } J \to \infty,
\]

\[
\lambda^2 \lambda' \lambda' \left( e^{-i(\lambda_n x + x_n') y} w_n \right) \to 0 \text{ in } L^2_{x} H^1_y(\mathbb{R}^2), \quad n \to \infty, \quad \text{for each } j \leq J,
\]

\[
\sup \lim_{n \to \infty} \left( \|u_n\|^2 - \sum_{j=1}^{J} \|\phi_j\|^2 - \|w_n\|^2 \right) = 0,
\]

and lastly, for \( j \neq j' \), and \( n \to \infty, \)

\[
\frac{\lambda^2_n}{\lambda_n} + \frac{\lambda^2_n}{\lambda_n} |\xi_n - \xi'_n|^2 + \frac{|x_n - x'_n|^2}{\lambda_n^2} + |y_n - y'_n|^2 \to \infty.
\]

Moreover, if \( u_n \) is bounded in \( H^1(\mathbb{R}^2) \), we can take \( \lambda_n \geq 1 \), and \( |\xi_n| \leq C_j \), for every \( 1 \leq j \leq J \). We also have \( \{\phi_j\}_{j=1}^{J} \subseteq L^2 H^1_y \),

\[
\frac{\lambda^2_n}{\lambda_n} \left( e^{-i(\lambda_n x + x_n') y} w_n \right) \to 0 \text{ in } L^2_{x} H^1_y(\mathbb{R}^2), \quad n \to \infty, \quad \text{for each } j \leq J,
\]

\[
\sup \lim_{n \to \infty} \left( \|u_n\|^2 - \sum_{j=1}^{J} \|\phi_j\|^2 - \|w_n\|^2 \right) = 0.
\]

**Remark 3.10.** By using Plancherel, interpolation, the Hölder inequality and Proposition 3.9, we have

\[
\limsup_{n \to \infty} \|e^{it\Delta_{x^2}} w_n\|_{L^4_{x,y} H^1_y(\mathbb{R}^2)} = \limsup_{n \to \infty} \|e^{it\Delta_{x^2}} w_n\|_{L^4_{x,y} H^1_y(\mathbb{R}^2)} \leq \limsup_{n \to \infty} \|w_n\|_{L^4_{x,y} H^1_y(\mathbb{R}^2)} \to 0, \quad \text{as } J \to \infty.
\]

### 3.2. Approximation of profiles.

In this subsection, by using the solution of the resonant Schrödinger system in [26] to approximate the nonlinear Schrödinger equation with initial data be the bubble in the linear profile decomposition, we can show the nonlinear profile have a bounded space-time norm.

**Lemma 3.11** (Large-scale profiles). Let \( \phi \in L^2 H^1_x(\mathbb{R}^2) \) be given, then

(i) There is \( \lambda_0 = \lambda_0(\phi) \) sufficiently large such that for \( \lambda \geq \lambda_0 \), there is a unique global solution \( U_\lambda \in C^0_t L^2 \cap L^\infty \) of

\[
\begin{cases}
i \partial_t U_\lambda + \Delta_{x^2} U_\lambda = |U_\lambda|^2 U_\lambda, \\
U_\lambda(0, x, y) = \frac{\lambda}{\lambda_0} \phi(\frac{x}{\lambda_0}, \frac{y}{\lambda_0}).
\end{cases}
\]

Moreover, for \( \lambda \geq \lambda_0 \),

\[
\|U_\lambda\|_{L^\infty_t L^2_x H^1_y(\mathbb{R}^2)} \leq \|\phi\|_{L^2_x H^1_y}.
\]

(ii) Assume \( \epsilon_1 \) is a sufficiently small positive constant depending only on \( \|\phi\|_{L^2_x H^1_y} \), \( \bar{v}_0 \in H^2_x h^1 \), and

\[
\|\phi - \bar{v}_0\|_{L^2_x H^1} \leq \epsilon_1.
\]

Let \( \bar{v} \in C^0_t H^1_x(\mathbb{R} \times \mathbb{R}^2) \) denote the solution of

\[
\begin{cases}
i \partial_t v_j + \Delta_{x^2} v_j = \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3}, \\
v_j(0) = v_{0,j}, \quad j \in \mathbb{Z}.
\end{cases}
\]
For $\lambda \geq 1$, we define
\[
v_{j,\lambda}(t, x) = \frac{1}{\lambda} v_j \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2,
\]
\[
V_\lambda(t, x, y) = \sum_{j \in \mathbb{Z}} e^{-itj^2} e^{i(y \cdot j)} v_{j,\lambda}(t, x), \quad (t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}.
\]

Then
\[
\lim_{\lambda \to \infty} \sup \|U_\lambda - V_\lambda\|_{L^4_t L^4_x H^1_y (\mathbb{R}^2 \times \mathbb{T})} \leq \|\phi\|_{L^2_x H^1_y} \epsilon_1.
\]

Proof. We will show $V_\lambda$ is an approximate solution to the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$ in the sense of Theorem 2.6. By noting $v_j$ satisfies (3.22) and an easy computation, we have
\[
e_\lambda = (i \partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}}) V_\lambda - |V_\lambda|^2 V_\lambda
\]
(3.23)
\[
i \epsilon_\lambda = - \sum_{j \in \mathbb{Z}} e^{-itj^2} e^{i(y \cdot j)} \sum_{\lambda \in \mathcal{N}(j)} e^{-i(tj_1^2 - \lambda j_2^2 + j_3^2 - |j|^2)} (v_{j_1,\lambda} \overline{v}_{j_2,\lambda} v_{j_3,\lambda})(t, x),
\]
where
\[
\mathcal{N}(j) = \{(j_1, j_2, j_3) \in \mathbb{Z}^3 : j_1 = j_2 + j_3 - j = 0, |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \neq 0\}.
\]
We first decompose $e_\lambda = P_{\geq 2-10} e_\lambda + P_{< 2-10} e_\lambda$. By Bernstein’s inequality, Plancherel theorem, (3.23), Leibnitz’s rule, and Hölder’s inequality, we have
\[
\|P_{\geq 2-10} e_\lambda\|_{L^4_t L^4_x H^1_y (\mathbb{R} \times \mathbb{T})} \leq \|P_{\geq 2-10} \nabla_x e_\lambda\|_{L^4_t L^4_x H^1_y (\mathbb{R} \times \mathbb{T})}
\]
(3.24)
\[
\leq \frac{1}{\lambda} \left\| \sum_j \left( \sum_{\lambda \in \mathcal{N}(j)} e^{-i\lambda^2 t(j_1^2 - j_2^2 + j_3^2 - |j|^2)} (v_{j_1,\lambda} \nabla_x v_{j_2,\lambda} \cdot v_{j_3,\lambda})(t, x) \right)^2 \right\|_{L^4_t L^4_x} + \frac{1}{\lambda} \left\| \sum_j \left( \sum_{\lambda \in \mathcal{N}(j)} e^{-i\lambda^2 t(j_1^2 - j_2^2 + j_3^2 - |j|^2)} (v_{j_1,\lambda} \nabla_x v_{j_2,\lambda} \cdot v_{j_3,\lambda})(t, x) \right)^2 \right\|_{L^4_t L^4_x} + \frac{1}{\lambda} \left\| \sum_j \left( \sum_{\lambda \in \mathcal{N}(j)} e^{-i\lambda^2 t(j_1^2 - j_2^2 + j_3^2 - |j|^2)} (\nabla_x v_{j_1,\lambda} \cdot \nabla_{\nabla_x} v_{j_2,\lambda} \cdot v_{j_3,\lambda})(t, x) \right)^2 \right\|_{L^4_t L^4_x}.
\]
Thus $P_{\geq 2^{-10}}^x e_\lambda$ is acceptable when $\lambda \geq \lambda_0$. We turn to the estimate of $P_{\leq 2^{-10}}^x e_\lambda$. By integrating by parts, we have

$$\int_0^t e^{i(t-\tau)\Delta_{\R^2 \times \T}} P_{\leq 2^{-10}}^x e_\lambda(\tau) \, d\tau$$

$$= - \sum_{j \in \mathcal{E}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} \int_0^t e^{i(t-\tau)(\Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} e^{-i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \rho P_{\leq 2^{-10}}^x (v_{j_1, j_2} \bar{v}_{j_3, \lambda})(\tau, x) \, d\tau$$

$$- \sum_{j \in \mathcal{E}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \rho \int_0^t e^{i(t-\tau)(\Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \rho P_{\leq 2^{-10}}^x (v_{j_1, j_2} \bar{v}_{j_3, \lambda})(\tau, x) \, d\tau$$

$$+ \sum_{j \in \mathcal{E}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \rho \int_0^t e^{i(t-\tau)(\Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \rho P_{\leq 2^{-10}}^x (v_{j_1, j_2} \bar{v}_{j_3, \lambda})(\tau, x) \, d\tau$$

$$= A_1 + A_2 + A_3.$$

For $A_1$, by Plancherel’s theorem, the boundedness of the operator $\frac{P_{\leq 2^{-10}}^x}{\Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2}$ on $L^2_2(\R^2)$ when $(j_1, j_2, j_3) \in \mathcal{N}(j)$, Hölder’s inequality and the Sobolev embedding theorem, we have

$$\left\| \sum_{j \in \mathcal{E}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i\tau(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \rho \int_0^t e^{i(t-\tau)(\Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \Delta_{\R^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \rho P_{\leq 2^{-10}}^x (v_{j_1, j_2} \bar{v}_{j_3, \lambda})(0, x) \right\|_{L^2_2} \leq \frac{1}{\lambda^2} \| \bar{v}_0 \|_{H^{3/4}_{1, \lambda}}^3.$$
We now consider $A_3$. By Hölder’s inequality, Leibnitz’s rule and (3.22), we have

\[
\left\| \sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i(y, j)} \int_0^t e^{i(t-\tau)(\Delta_{_{\mathbb{R}^2}} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \frac{\partial_x P_{\leq 2^{-10}}^{\epsilon}(v_{j_1, \lambda} \bar{v}_{j_2, \lambda} v_{j_3, \lambda})}{\partial \tau} \, d\tau \right\|_{L_t^\infty L_x^2 H_y^1} \\
+ \left\| \sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i(y, j)} \int_0^t e^{i(t-\tau)(\Delta_{_{\mathbb{R}^2}} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \frac{\partial_x P_{\leq 2^{-10}}^{\epsilon}(v_{j_1, \lambda} \bar{v}_{j_2, \lambda} v_{j_3, \lambda})}{\partial \tau} \, d\tau \right\|_{L_t^4 L_x^2 H_y^1}
\]

\[
\lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |\partial_t v_{j, \lambda}(t, x)| \right)^2 \right\|_{L_t^2 L_x^6} \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |v_{j, \lambda}(t, x)| \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}} \lesssim \lambda^{-2} \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 \left( -\Delta_{_{\mathbb{R}^2}} v_{j} + \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3} \right)(t, x) \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}} \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |v_{j}(t, x)| \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}}
\]

(3.25)

\[= \lambda^{-2} \cdot I \cdot III.\]

By Hölder’s inequality, Sobolev’s inequality, together with the scattering of the cubic resonant Schrödinger system and persistence of regularity in [26], we have

\[
I \lesssim \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 |\Delta_{_{\mathbb{R}^2}} v_{j}(t, x)| \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}} + \left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 \left( \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3}(t, x) \right) \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}}
\]

\[
\lesssim \| \bar{v}_0 \|_{H_{2}^{2,1}} + \left\| \left( \sum_{j} (j)^2 |v_{j}(t, x)| \right)^2 \right\|_{L_t^2 L_x^6}^{\frac{1}{2}} \left\| \left( \sum_{j} (j)^2 |v_{j}(t, x)| \right)^2 \right\|_{L_t^\infty L_x^{18}}^{\frac{1}{2}}
\]

(3.26)

\[
\lesssim \| \bar{v}_0 \|^3_{H_{2}^{2,1}}.
\]

By the scattering result in [26], we also have

\[
(3.27) \quad II \lesssim \| \phi \|^2_{L_{t}^2 L_x^{1} H_y^{1}}.
\]

So by (3.25), (3.26) and (3.27), we have

\[
\left\| \sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i(y, j)} \int_0^t e^{i(t-\tau)(\Delta_{_{\mathbb{R}^2}} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \frac{\partial_x P_{\leq 2^{-10}}^{\epsilon}(v_{j_1, \lambda} \bar{v}_{j_2, \lambda} v_{j_3, \lambda})}{\partial \tau} \, d\tau \right\|_{L_t^\infty L_x^2 H_y^1}
\]

\[
+ \left\| \sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{N}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{i(y, j)} \int_0^t e^{i(t-\tau)(\Delta_{_{\mathbb{R}^2}} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} \frac{\partial_x P_{\leq 2^{-10}}^{\epsilon}(v_{j_1, \lambda} \bar{v}_{j_2, \lambda} v_{j_3, \lambda})}{\partial \tau} \, d\tau \right\|_{L_t^4 L_x^2 H_y^1}
\]

\[
\lesssim \| \bar{v}_0 \|^2_{H_{2}^{2,1}} \lambda^{-2} \left( \| \bar{v}_0 \|^3_{H_{2}^{2,1}} + \| \bar{v}_0 \|^3_{H_{2}^{2,1}} \right);
\]

So $P_{\leq 2^{-10}}^{\epsilon}(v_{\lambda})$ is acceptable when $\lambda \geq \lambda_0$. Therefore, for $\lambda \geq \lambda_0$, $\int_0^t e^{i(t-\tau)d\lambda} e_{\lambda}(\tau) \, d\tau$ is small enough in $L_t^\infty L_x^{2} H_y^1 \cap L_t^\infty L_x^{2} H_y^1 (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$.

We still need to verify the easier assumptions of Theorem 2.6.
By Plancherel, (3.22), we have
\[
\| V_\lambda \|_{L^\infty_tL^2_xH^1_y(R \times R^2 \times T)} \lesssim \left( \sum_{j \in \mathbb{Z}} (j-2)^2 \| v_j(0,x) \|_{L^2_x}^2 + \sum_{j_1,j_2 \neq j} \| v_{j_1} v_{j_2} v_{j_3} \|_{L^4_{t,x}}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim \| \tilde{v}_0 \|_{L^2_x} + \left( \sum_{j} (j-2)^2 \| v_j \|_{L^2_x}^2 \right)^{\frac{1}{2}} \lesssim \| \tilde{v}_0 \|_{L^2_x} \| \tilde{v}_0 \|_{L^2_x}^{3/2},
\]
and
\[
\| V_\lambda \|_{L^1_tL^4_xH^1_y} \lesssim \left( \sum_{j \in \mathbb{Z}} e^{i(y \cdot j)} v_j(0,x) \right)_{L^2_x}^2 + \left( \sum_{j \in \mathbb{Z}} e^{-i(y \cdot j)} e^{i(y \cdot j)} (i \partial_t + \Delta) v_j(t,x) \right)_{L^4_tL^2_xH^1_y}^2
\]
\[
\sim \left( \sum_{j \in \mathbb{Z}} (j-2)^2 \| v_j(0,x) \|_{L^2_x}^2 \right)^{\frac{1}{2}} + \left( \sum_{j \in \mathbb{Z}} (j-2)^2 \| (i \partial_t + \Delta) v_j(t,x) \|_{L^2_x}^2 \right)^{\frac{1}{2}} \lesssim \| \tilde{v}_0 \|_{L^2_x} + \left( \sum_{j} (j-2)^2 \| v_j(t,x) \|_{L^2_x} \right)^{\frac{1}{2}} \lesssim \| \tilde{v}_0 \|_{L^2_x} + \| \tilde{v}_0 \|_{L^2_x}^{3/2}.
\]

Moreover, by Plancherel, (3.21), we have
\[
\| U_\lambda(0) - V_\lambda(0) \|_{L^2_xH^1_y} = \| \tilde{v} - \tilde{v}_0 \|_{L^2_x} \lesssim \epsilon_1.
\]
Applying Theorem 2.6 we conclude that for \( \lambda \) (depending on \( \tilde{v}_0 \)) large enough, the solution \( U_\lambda \) of (3.20) exists globally and
\[
\| U_\lambda - V_\lambda \|_{L^\infty_tL^4_xH^1_y \cap L^4_tL^4_xH^1_y(R \times R^2 \times T)} \lesssim \epsilon_1,
\]
which ends the proof.

After spatial translation, time translation, and Galilean transformation, we have

**Proposition 3.12.** For any \( \phi \in L^2_xH^1_y(R \times \mathbb{T}) \), \( (\lambda_n, t_n, x_n, \xi_n)_{n \geq 1} \subseteq (0, \infty) \times \mathbb{R} \times \mathbb{R}^2 \), \( \lambda_n \to \infty \), as \( n \to \infty \), \( |\xi_n| \leq 1 \), and let
\[
u_n(0,x,y) = \frac{1}{\lambda_n} e^{ix\xi_n} \left( e^{it \Delta} \phi \left( \frac{x-x_n}{\lambda_n}, y \right) \right).
\]

(i) For \( n \) large enough, there is a solution \( u_n \in C^0_tL^2_xH^1_y \cap L^4_tL^4_xH^1_y(R \times \mathbb{T}) \) of (1.1), satisfying
\[
\| U_n \|_{L^\infty_tL^2_xH^1_y \cap L^4_tL^4_xH^1_y(R \times \mathbb{T})} \lesssim \| \tilde{v} \|_{L^2_xH^1_y}.
\]

(ii) There exists a solution \( \tilde{v} \in C^0_tL^2_xh^1(R \times \mathbb{Z}) \) of the resonant Schrödinger system such that: for any \( \epsilon > 0 \), it holds that
\[
\| u_n(t) - W_n(t) \|_{L^\infty_tL^2_xH^1_y \cap L^4_tL^4_xH^1_y(R \times \mathbb{T})} \leq \epsilon,
\]
\[
\| u_n \|_{L^\infty_tL^2_xH^1_y \cap L^4_tL^4_xH^1_y(R \times \mathbb{T})} \leq 1.
\]
for \( n \) large enough, where
\[
W_n(t,x,y) = e^{-it \Delta} V_{\lambda_n}(t - t_n, x - x_n, y).
\]
Proposition 4.1. Show a Palais-Smale type theorem.

Let \( \{ u_n \} \) be a sequence of solutions to (1.1) obeying \( \sup_{t \in \mathbb{R}} \| u(t) \|_{L^2_x H^1_y}^2 \leq L \).

By the local wellposedness theory, \( \Lambda(L) < \infty \) for \( L \) sufficiently small. In addition, define \( L_{\max} = \sup \{ L : \Lambda(L) < \infty \} \). Our goal is to prove \( L_{\max} = \infty \). Suppose to the contradiction \( L_{\max} < \infty \), we will show a Palais-Smale type theorem.

Proposition 4.1 (Palais-Smale condition modulo symmetries in \( L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}) \)). Assume that \( L_{\max} < \infty \). Let \( \{ t_n \} \) be an arbitrary sequence of real numbers and \( \{ u_n \} \) be a sequence of solutions to (1.1) satisfying

\[
(4.1) \quad \sup_{t \in (-\infty, \infty)} \| u_n(t) \|_{L^2_x H^1_y} \to L_{\max},
\]

\[
(4.2) \quad \|u_n\|_{L^4_t L^4_x H^1_y(\mathbb{R}^2 \times \mathbb{T})} \to \infty; \quad \| u_n \|_{L^4_t L^4_y H^1_y((\mathbb{R}^2 \times \mathbb{T}) \times \mathbb{R}^2 \times \mathbb{T})} \to \infty, \quad \text{as } n \to \infty,
\]

and such that \( u_n \in C^0_t H^1_{x,y}((-\infty, \infty) \times \mathbb{R}^2 \times \mathbb{T}) \). Then, after passing to a subsequence, there exists a sequence \( x_n \in \mathbb{R}^2 \) and \( w \in H^1(\mathbb{R}^2 \times \mathbb{T}) \) such that

\[
\lim_{n \to \infty} u_n(x + x_n, y, t_n) = w(x, y) \quad \text{in } L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}), \quad \text{as } n \to \infty.
\]

Proof. By replacing \( u_n(t) \) with \( u_n(t + t_n) \), we may assume \( t_n = 0 \). Applying Proposition 3.1 to \( \{ u_n(0) \} \), after passing to a subsequence, we have

\[
u_n(0, x, y) = \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{i x \xi_n} \left( e^{it_n \Delta_x} \phi^j \right) \left( \frac{x - x_n}{\lambda_n^j}, y \right) + w_n^j(x, y).
\]

The remainder has asymptotically trivial linear evolution

\[
(4.3) \quad \lim_{n \to \infty} \sup_{t \in \mathbb{R}} \| e^{it \Delta_x} w_n^j \|_{L^2_x L^4_y H^1_y(\mathbb{R}^2 \times \mathbb{T})} \to 0, \quad \text{as } J \to \infty,
\]

and we also have asymptotic decoupling of the \( L^2_x H^1_y \) norm:

\[
(4.4) \quad \lim_{n \to \infty} \left( \| u_n(0) \|_{L^2_x H^1_y}^2 - \sum_{j=1}^J \| \phi^j \|_{L^4_x L^4_y H^1_y}^2 - \| w_n^j \|_{L^2_x H^1_y}^2 \right) = 0, \quad \forall J.
\]

There are two possibilities:

**Case 1.** \( \sup_{j \to \infty} \limsup_{n \to \infty} \| \phi^j \|_{L^2_x H^1_y}^2 = L_{\max} \). Combining (4.4) with the fact that \( \phi^j \) are nontrivial in \( L^2_x H^1_y \), we deduce that \( u_n(0, x, y) = \frac{1}{\lambda_n} e^{i x \xi_n} \left( e^{it_n \Delta_x} \phi \right) \left( \frac{x - x_n}{\lambda_n}, y \right) + w_n(x, y) \), \( \lim_{n \to \infty} \| w_n \|_{L^2_x H^1_y} = 0 \). We will show that \( \lambda_n \to 1 \), otherwise \( \lambda_n \to \infty \).
Proposition 3.12 implies that for all large \( n \), there exists a unique solution \( u_n \) on \( \mathbb{R} \) with \( u_n(0, x, y) = \frac{1}{\lambda_n} e^{ix \xi_n / \lambda_n} (e^{it_n \Delta_{R^2}} \phi)(x, y) \) and
\[
\limsup_{n \to \infty} \left\| u_n \right\|_{L^4_t L^2_y H^1_{-\theta}}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}) \leq C(L_{\max}),
\]
which is a contradiction with (4.2).

Therefore, \( \lambda_n \equiv 1 \), and \( u_n(0, x, y) = e^{ix \xi_n / \lambda_n} (e^{i t_n \Delta_{R^2}} \phi)(x, y) + w_n(x, y) \). If \( t_n \equiv 0 \), by the fact \( \xi_n \) is bounded, this is precisely the conclusion. If \( t_n \to -\infty \), by the Galilean transform
\[
e^{it_t \Delta_{R^2}} e^{i x \xi_0} \phi(x) = e^{-i t_0 \xi_0} e^{i x \xi_0} (e^{it_0 \Delta_{R^2}} \phi)(x - 2t_0 \xi_0),
\]
we observe
\[
\left\| e^{it \Delta_{R^2}} \left(e^{i x \xi_0} \phi(x - x_n, y)\right) \right\|_{L^4_t L^2_y H^1_{-\theta}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})}
= \left\| e^{-i t \xi_n \phi} e^{i x \xi_0} (e^{it \Delta_{R^2}} \phi)(x - 2t \xi_n) \right\|_{L^4_t L^2_y H^1_{-\theta}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})}
= \left\| e^{i(\xi_0 + t \xi_n \phi} \phi \right\|_{L^4_t L^2_y H^1_{-\theta}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \to 0, \text{ as } n \to \infty.
\]
Using Theorem 2.3 we see that, for \( n \) large enough,
\[
\left\| u_n \right\|_{L^4_t L^2_y H^1_{-\theta}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq 2\delta_0 < \infty,
\]
which contradicts (4.2). The case \( t_n \to \infty \) is similar.

**Case 2.** \( \limsup_{n \to \infty} \left\| \phi_n \right\|_{L^2_y H^1_y} \leq L_{\max} - 2\delta \) for some \( \delta > 0 \).

By the definition of \( L_{\max} \), there exist global solution \( v_n^j \) to
\[
\begin{cases}
\Delta_{R^2} v_n^j + \Delta_{R^2} x T v_n^j = |v_n^j|^2 v_n^j, \\
v_n^j(0, x, y) = \frac{1}{\lambda_n} e^{ix \xi_n / \lambda_n} (e^{it \Delta_{R^2} \phi})(x, y),
\end{cases}
\]
satisfying
\[
(4.5) \quad \left\| v_n^j \right\|_{L^4_t L^2_y H^1_{-\theta}} \leq L_{\max} \delta \left\| \phi_n \right\|_{L^2_y H^1_y}.
\]
Put
\[
u_n^j = \sum_{j=1}^{J} v_n^j + e^{i t \Delta_{R^2} x T} u_n^j.
\]
Then we have \( u_n^j(0) = u_n(0) \). We claim that for sufficiently large \( J \) and \( n \), \( u_n^j \) is an approximate solution to \( u_n \) in the sense of the Theorem 2.3. Then we have the finiteness of \( \left\| u_n \right\|_{L^4_t L^2_y H^1_{-\theta}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \), which contradicts with (4.2).

To verify the claim, we only need to check that \( u_n^j \) satisfies the following properties:
(1) \( \limsup_{n \to \infty} \left\| u_n^j \right\|_{L^4_t L^2_y H^1_{-\theta}} \leq L_{max}, \delta \right\| 1, \text{ uniformly in } J; \]
(2) \( \limsup_{n \to \infty} \left\| e^{i t_n \Delta_{R^2}} u_n^j \right\|_{L^4_t L^2_y H^1_{-\theta}} \to 0, \text{ as } J \to J^*, \text{ where } \left( i \phi_n + \Delta_{R^2} x T \right) u_n^j = |u_n^j|^2 u_n^j. \)

The verification of (1) relies on the asymptotic decoupling of the nonlinear profiles \( v_n^j \), which we record in the following two lemmas.
Lemma 4.2 (Orthogonality). Suppose that two frames $\Gamma^j = (t^j, x^j, \xi^j, \lambda^j)$, $\Gamma^k = (t^k, x^k, \xi^k, \lambda^k)$ are orthogonal, then for $\psi^j, \psi^k \in C_0^\infty (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$,

$$\left\| \frac{1}{\lambda^j_n} e^{ix\xi^j_n} (\langle \nabla_y \rangle^{1-\epsilon_0} \psi^j) \left( \frac{t - t^j_n}{\lambda^j_n}, \frac{x - x^j_n}{\lambda^j_n}, y \right) \right\|_{L^2_{t,x,y}} = 0, \text{as } n \to \infty.$$ 

Similarly to the proof in [8] to deal with the quintic nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$, we can obtain the following lemma from Proposition 3.12. We also refer to [6] for similar argument.

Lemma 4.3 (Decoupling of nonlinear profiles). Let $\nu^j_n$ be the nonlinear solutions defined above, then for $j \neq k$,

$$\left\| \langle \nabla_y \rangle^{1-\epsilon_0} \nu^j_n \cdot \langle \nabla_y \rangle^{1-\epsilon_0} \nu^k_n \right\|_{L^2_{t,x,y}} \to 0, \text{ and } \left\| \nu^k_n \cdot \langle \nabla_y \rangle^{1-\epsilon_0} \nu^j_n \right\|_{L^2_{t,x,y}} \to 0, \text{ as } n \to \infty.$$ 

Let us verify claim (i) above. We see

$$\left\| u^j_n \right\|_{L^4_{t,x} H^1_{y=0}} \leq \sum_{j=1}^J \left\| v^j_n \right\|_{L^4_{t,x} H^1_{y=0}} + \left\| e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^j_n \right\|_{L^4_{t,x} H^1_{y=0}}.$$ 

By (4.5) and Lemma 4.3 we have

$$\left\| \sum_{j=1}^J \frac{t^j_n}{\phi^j_{L^2_{t,x} H^1_y}} \right\|_{L^4_{t,x} H^1_{y=0}} \leq \left( \sum_{j=1}^J \left\| v^j_n \right\|_{L^4_{t,x} H^1_{y=0}} \right)^2 + \left( \sum_{j \neq k} \left\| \langle \nabla_y \rangle^{1-\epsilon_0} \nu^j_n \cdot \langle \nabla_y \rangle^{1-\epsilon_0} \nu^k_n \right\|_{L^2_{t,x,y}} \right)^2 \leq \left( \sum_{j=1}^J \phi^j_{L^2_{t,x} H^1_y} + o_j(1) \right)^2.$$ 

Since the $L^2_{t,x} H^1_y$ norm decoupling implies

$$\limsup_{n \to \infty} \sum_{j=1}^J \phi^j_{L^2_{t,x} H^1_y} \leq L_{\text{max}},$$

together with (3.4), we obtain

$$\lim_{j \to J^*} \limsup_{n \to \infty} \left\| u^j_n \right\|_{L^4_{t,x} H^1_{y=0}} \leq L_{\text{max}},$$

It remains to check property (ii) above, by the definition of $u^j_n$, we decompose

$$e^J_n = (i \partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}}) u^J_n - \left| u^J_n \right|^2 u^J_n = \sum_{j=1}^J \left| v^j_n \right|^2 v^j_n - \sum_{j=1}^J \sum_{j=1}^J \left| v^j_n \right| \left| v^j_n \right|^2 \left( u^J_n - e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^J_n \right)^2 \left( u^J_n - e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^J_n \right) - \left| u^J_n \right|^2 u^J_n.$$ 

First consider

$$\sum_{j=1}^J \left| v^j_n \right|^2 v^j_n = \sum_{j=1}^J \sum_{n=1}^J \sum_{n=1}^J v^j_n.$$
Thus by the fractional chain rule, Minkowski, Hölder, Sobolev, (4.6) and (4.5),

\[(4.8)\]

\[
\sum_{j+k} \|v_n^k(\nabla_y)^{1-\epsilon_0} v_n^j\|_{L_{t,x}^2}^2 + \sum_{j+k} \|v_n^k(\nabla_y)^{1-\epsilon_0} v_n^j\|_{L_{t,x}^4}^4 H_y^{1-\epsilon_0} \lesssim o_f(1), \text{ as } n \to \infty.
\]

We now estimate \(|u_n^f - e^{it\Delta}\nabla_x^2 w_n^f|^2 (u_n^f - e^{it\Delta}\nabla_x^2 w_n^f) - |u_n^f|^2 u_n^f|\). By the fractional chain rule, Hölder, Sobolev, we have

\[
\left\|\left| u_n^f - e^{it\Delta}\nabla_x^2 w_n^f\right|_{H_y^{1-\epsilon_0}}^2 \left( u_n^f - e^{it\Delta}\nabla_x^2 w_n^f\right) - |u_n^f|^2 u_n^f\right\|_{L_{t,x}^4}^4 \lesssim \left\| u_n^f\right|_{H_y^{1-\epsilon_0}}^2 \left\| e^{it\Delta}\nabla_x^2 w_n^f\right|_{H_y^{1-\epsilon_0}}^2 \|e^{it\Delta}\nabla_x^2 w_n^f\|_{H_y^{1-\epsilon_0}}^2 \|e^{it\Delta}\nabla_x^2 w_n^f\|_{H_y^{1-\epsilon_0}}^2.
\]

Using (4.7), and the decay property (4.3), we get

\[
\lim_{n \to \infty} \left\| u_n^f - e^{it\Delta}\nabla_x^2 w_n^f\right|_{H_y^{1-\epsilon_0}}^2 (u_n^f - e^{it\Delta}\nabla_x^2 w_n^f) - |u_n^f|^2 u_n^f\right\|_{L_{t,x}^4}^4 \to 0, \text{ as } J \to J^*.
\]

Arguing as in [6], the proof of Proposition 4.1 implies the following result:

**Theorem 4.4** (Existence of the almost-periodic solution). Assume that \(L_{\max} < \infty\), then there exists \(u_c \in C_0^0 H^{1.5}_x(R \times R^2 \times T)\) solving (1.1) satisfying

\[(4.9)\]

\[
\sup_{t \in \mathbb{R}} \left\| u_c(t)\right\|_{L_{t,x}^2}^2 = L_{\max}, \quad \left\| u_c\right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}(R \times R^2 \times T)} = \infty.
\]

Furthermore, \(u_c\) is almost periodic in the sense that \(\forall \eta > 0\), there is \(C(\eta) > 0\) such that

\[(4.10)\]

\[
\int_{|x-x(t)| \geq C(\eta)} \left\| u_c(t, x, y)\right\|_{H_y^1(T)}^2 dx < \eta
\]

for all \(t \in \mathbb{R}\), where \(x : \mathbb{R} \to \mathbb{R}^2\) is a Lipschitz function with \(\sup_{t \in \mathbb{R}} |x'(t)| \leq 1\).

5. Rigidity Theorem

In this section, we will exclude the almost-periodic solution in Theorem 4.4 by using the interaction Morawetz action with the weight function taken appropriately.

**Proposition 5.1** (Non-existence of the almost-periodic solution). The almost-periodic solution \(u_c\) in Theorem 4.4 does not exist.

**Proof.** Define the interaction Morawetz action

\[
M(t) = \int_{\mathbb{R}^2 \times T} \int_{\mathbb{R}^2 \times T} J(\hat{u}_c(t, x, y) \nabla_x u_c(t, x, y)) \nabla_x a(x - \tilde{x}) |u_c(t, \tilde{x}, \tilde{y})|^2 dx dy d\tilde{x} d\tilde{y},
\]

where \(a\) is a radial function defined on \(\mathbb{R}^2\) with

\[
a(r) = \begin{cases} \frac{r^2}{2r_0^2}(1 + \frac{1}{2} \log \frac{r_0}{r}), & r < r_0, \\ r - \frac{r_0}{2}, & r \geq r_0, \end{cases}
\]

Thus by the fractional chain rule, Minkowski, Hölder, Sobolev, (4.6) and (4.5),
Therefore, let $r_0 \to 0$, we have $\forall T_0 > 0$,
\begin{equation}
\int_{-T_0}^{T_0} \int_{\mathbb{R}^2 \times \mathbb{T}} \left| \nabla_x \right|^2 \left( |u_c(t, x, y)|^2 \right) \, dx \, dy \, dt \lesssim \int_{-T_0}^{T_0} M'(t) \, dt \lesssim \sup_{t \in [-T_0, T_0]} |M(t)|.
\end{equation}

By Hölder, Minkowski, Sobolev, and the fact $\mathbb{T}$ is compact, we have
\begin{equation}
\int_{|x| \leq C\left(\frac{m_0}{100}\right)} \|u_c(t, x, y)\|_{L_y^2}^2 \, dx \lesssim C\left(\frac{m_0}{100}\right) \|u_c(t, x, y)\|_{L_y^2}^2 \lesssim C\left(\frac{m_0}{100}\right) \|\nabla_x \left( |u_c(t, x, y)|^2 \right)\|_{L_y^2}^2 \lesssim C\left(\frac{m_0}{100}\right) \|\nabla_x \left( |u_c(t, x, y)|^2 \right)\|_{L_y^2},
\end{equation}
where $m_0 = M(u_c)$.

By (4.10) and conservation of mass, we have
\begin{equation}
\frac{m_0}{2} \leq \int_{|x| \leq C\left(\frac{m_0}{100}\right)} \|u_c(t, x, y)\|_{L_y^2}^2 \, dx.
\end{equation}

By Hölder, the fact $|\nabla a|$ is bounded, and $u_c \in C^0_t H^1_{x,y} (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$,
\begin{equation}
|M(t)| \lesssim \|u_c\|_{L_y^\infty L_{x,y}^2}^2 \|\nabla_x u_c\|_{L_y^\infty L_{x,y}^2} \lesssim 1.
\end{equation}

Therefore, $\forall T_0 > 0$, by (5.3), (5.2), (5.1) and (5.4),
\begin{align*}
m_0^2 T_0 &= \int_0^{T_0} m_0^2 \, dt \lesssim \int_0^{T_0} \left( \int_{|x| \leq C\left(\frac{m_0}{100}\right)} \|u_c(t, x, y)\|_{L_y^2}^2 \, dx \right)^2 \, dt \\
&\lesssim C\left(\frac{m_0}{100}\right)^3 \int_0^{T_0} \|\nabla_x \left( |u_c(t, x, y)|^2 \right)\|_{L_y^2}^2 \, dt \\
&\lesssim C\left(\frac{m_0}{100}\right)^3 \|\nabla_x \left( |u_c(t, x, y)|^2 \right)\|_{L_y^2}^2 \lesssim C\left(\frac{m_0}{100}\right) \sup_{t \in [0,T_0]} |M(t)| \lesssim C\left(\frac{m_0}{100}\right).
\end{align*}

Let $T_0 \to \infty$, we obtain a contradiction unless $u_c \equiv 0$, which is impossible due to $\|u_c\|_{L_y^\infty H^1_{x,y}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} = \infty$.

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