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Exploring one loop amplitude at four-points vertices by the OPP method

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Abstract. We construct the general formula of one-loop amplitude at four-point vertices using Ossola, Papadopoulos, and Pittau (OPP) method. The incoming and outgoing particles are defined as arbitrary massless particles, and the intermediate state contains arbitrary particles inside. In this work, the amplitude is reconstructed via finding four-type rational coefficients. First, box coefficient is extracted using the four-cut technique with linear algebra. We found that triangle and bubble coefficients can be extracted using three-cut and two-cut technique with Cauchy residue theorem instead of with discrete Fourier sum like the original version of the OPP. Tadpole coefficient can be dropped out, because its scalar integral, which contains only UV divergence, is completely absorbed by renormalization.

1. Introduction
Nowadays, searching for a sign of new physics in particle collision experiment is done through high experimental statistics. The highly efficient next to leading order (NLO) calculation is required in order to check theoretical and experimental matching. However, direct NLO calculation is not easy, when the number of vertices increases. The number of loop integral is increased extremely. In present, the Ossola, Papadopoulos, and Pittau (OPP) \cite{1} method is found to be an effective tool to construct one-loop amplitude in form of a linear combination of 4-types of Master Integral (MI) \cite{2} multiplied with rational coefficients.

In this research, we will explore details calculation of one-loop four-point vertices amplitude using the OPP method. All external particles are assigned to be massless particles while the internal intermediate particles are arbitrary. The amplitude will be written in form of a linear combination of four master integral multiplied with rational coefficients, i.e., box, triangle, bubble, and tadpole. In the next section, we will introduce the basic construction of the OPP method based on the integrand reduction technique by the multiple-cut method. In section 3, the box coefficient is calculated by solving linear algebra. The triangle and bubble coefficients are calculable by the Cauchy residue theorem while in the original version of OPP calculation used discrete Fourier summation. The reason is that within triangle and bubble numerators, both can be rewritten as Laurent series \cite{3} when the amplitude is cut by the Cutkorsky’s rule \cite{4}. Each coefficient will be on the different complex pole order. Therefore, they can be extracted using the Cauchy residue theorem. This is the different point to the original OPP calculation.
2. The OPP method

In general, the 4-point one-loop amplitude is generated by Feynman’s rule. It contains many types of tensor integral, which depend on the numerator $N(p)$. It can be generally written as

$$iA = \int \frac{d^4p}{(2\pi)^4} \frac{N(p)}{D_0 D_1 D_2 D_3},$$

where $p$ is a loop momentum. Traditionally, the propagator is defined by momentum flow in the loop as $D_0 = p^2 - m_0^2$, $D_1 = (p + k_1)^2 - m_1^2$, $D_2 = (p + k_1 + k_2)^2 - m_2^2$, $D_3 = (p - k_3)^2 - m_3^2$, when we take all external legs are incoming $k_1 + k_2 + k_3 + k_4 = 0$. We would like to express the equation (1) in the form of the linear combination of MI multiplied by the rational coefficient as

$$iA = c_0^{(0123)} I_{0123} + \sum_{i<j<k} c_{\epsilon_{00}}^{(ijk)} I_{ijk} + \sum_{i<j} c_{\epsilon_{000}}^{(ij)} I_{ij} + \sum_{i} c_{\epsilon_{0000}}^{(i)} I_i + \mathcal{R},$$

where $c_0, c_{\epsilon_{00}}, c_{\epsilon_{000}}, c_{\epsilon_{0000}}$ and $\mathcal{R}$ are box-, triangle-, bubble-, tadpole-coefficients and rational part respectively. The numerator of equation (1) is rewritable in term of rational coefficient plus irreducible scalar product (ISP) via Aguila and Pittau works [5] as

$$N(p) = \Delta_0^{(0123)}(p) + \sum_{i<j<k} \Delta_{\epsilon_{00}}^{(ijk)}(p) \prod_{l \neq ij} D_l + \sum_{i<j} \Delta_{\epsilon_{000}}^{(ij)}(p) \prod_{k \neq ij} D_k + \sum_{i} \Delta_{\epsilon_{0000}}^{(i)}(p) \prod_{j \neq i} D_j,$$

when $\Delta_{\epsilon_{00}}^{(i_0i_1..i_n)}(p)$ contains a rational coefficient plus ISP, $\tilde{c}(p)$. It will not be reduced to $n - 1$ point MI, however, it will vanish after integrating out of loop momentum.

$$\int \frac{d^4p}{(2\pi)^4} \Delta_{\epsilon_{00}}^{(i_0i_1..i_n)}(p) = \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{c}^{(i_0i_1..i_n)}(p)}{D_{i_0} D_{i_1} \ldots D_{i_n}} = c^{(i_0i_1..i_n)} I_{i_0i_1..i_n}$$

Their definitions are defined as following below

$$\Delta_0^{(0123)}(p) = c_0^{(0123)} + \tilde{c}_{0}(0123) p \omega,$$

$$\Delta_{\epsilon_{00}}^{(ijk)}(p) = c_{\epsilon_{00}}^{(ijk)} + \sum_{l=1}^{3} \left( c_{00} ((p + r_0).e_{ab})^l + c_{00} ((p + r_0).e_{ab})^l \right),$$

$$\Delta_{\epsilon_{000}}^{(i+2)}(p) = c_{\epsilon_{000}}^{(i+2)} + \tilde{c}_{\epsilon_{000}}^{(i+2)} (2((p + r_0).e_{ab}(k_a - k_b))^2 - (p + r_0).e_{ab}(p + r_0).e_{ab}) + \ldots,$$

where $\omega^\mu = [2\gamma^\mu 1][13][32] - (2\gamma^\mu 1)[13][32]$ and $e_{ab} = [k_a \gamma^\mu k_b]$. The choices of $a, b$ have to correspond with the equation (4). Then, we apply the multiple cuts to the equation (1). The propagator $D_i^{-1}$ which is cut will be substituted by the Dirac delta function $\delta(D_i)$, and the amplitude which is cut will be turned to the imaginary part. We cut 4-propagators (0123), 3-propagators $(ijk)$, and 2-propagators $(i, i + 2)$ respectively. We obtain

$$\Delta_0^{(0123)}(p) = N(p)|_{D_0 = D_1 = D_2 = D_3 = 0},$$

$$\Delta_{\epsilon_{00}}^{(ijk)}(p) = D_i^{-1} \left(N(p) - \Delta_0^{(0123)}(p)\right)|_{D_i = D_1 = D_2 = 0},$$

$$\Delta_{\epsilon_{000}}^{(i+2)}(p) = D_i^{-1} D_i^{-1} \left(N(p) - \Delta_0^{(0123)}(p) - \sum_{i_0 < i_1 < i_2} \Delta_{\epsilon_{00}}^{(i_0i_1i_2)}(p) D_{i_2}\right)|_{D_i = D_1 = D_2 = 0},$$

where $i_0 \neq i, j, k$, $i_4 \neq i_1, i_2, i_3$ and $i_5, i_6 \neq i, i + 2$. We drop out the 1-cut propagator and the adjacent 2-cut propagators because their scalar integrals contain only an UV-divergence. They can be canceled out by cyclic permutation when the amplitude is sum overall possible channel.
3. Rational coefficients
4-cut propagators appear four-constrained equations, $D_0 = D_1 = D_2 = D_3 = 0$. It gives us the two solutions of loop momentum $p_\pm$, which is shown in the equation (A.1). The overall $x_i$ are completely solved, thus the coefficient can be solved using a linear equation.

\[
\begin{align*}
C^{(0123)}_0 &= \frac{N(p_+)+N(p_-)}{2} \\
\tilde{C}^{(0123)}_0 &= \frac{N(p_+)-N(p_-)}{2p_+\omega}
\end{align*}
\]

3-cut propagators, the constrained equation is not enough to solve all variables of $x_i$ on equation (A.2). However, the equation (6) can be expanded as Laurent series of $x_3$ or $x_4$. We found that each coefficient is attached on different complex pole order, thus it is solvable by Cauchy residue.

\[
\begin{align*}
C^{(ijk)}_{00} &= \text{Res} \left( x_3^{-1} \Delta^{(ijk)}_{00} (p_1) \right)_{x_3=0} , \\
C^{(ijk)}_{00} &= (-1)^l (C_{ijk})^l \text{Res} \left( x_3^{l-1} \Delta^{(ijk)}_{00} (p_1) \right)_{x_3=0}, \\
C^{(ijk)}_{00} &= (-1)^l (C_{ijk})^l \text{Res} \left( x_4^{l-1} \Delta^{(ijk)}_{00} (p_2) \right)_{x_4=0},
\end{align*}
\]

The definitions of $p_{1,2}$ and $C_{ijk}$ are shown in (A.2) and Table (A1). Finally, 2-cut propagators use Cauchy residue theorem similar to 3-cut propagators.

\[
C^{(i,i+2)}_{000} = \text{Res} \left( \text{Res} \left( \tilde{C}_{i,i+2}(x_2,x_4)\Delta^{(i,i+2)}_{000}(p_3) \right)_{x_2=0} \right)_{x_4=0},
\]

where $\tilde{C}_{i,i+2}(x_2,x_4) = x_2^{-1} x_4^{-1} + (4/3)x_2^{-1} x_4^{-1} x_3^{(i,i+2)}(0,1)$ and $p_3, x_3^{(i,i+2)}(x_2,x_4)$ are shown on equation (A.3) and Table (A2). However, rational part $R$ cannot be detected by unitarity cut. Their origin comes from the small dimensional part of the amplitude. We can follow the reconstruction $R$ in [6].

4. Calculational example
The example is QED photon-photon scattering, which leading order appears at four pointed one-loop level. All $m_i$ in $D_i$ are set equally. The numerator is writable from Feynman’s rule

\[
N_{++++}(p) = -e^4 \text{Tr}[\hat{p} + m] f^+_2 \hat{p} + \hat{k}_2 + m) f^+_4 (\hat{p} - \hat{k}_4 + m).\]

The other two channels of amplitude are obtained by interchanging the momentum index $3 \leftrightarrow 4$ and $2 \leftrightarrow 3$. Putting equation (17) into (8)-(16) and using the method in [6]

\[
iA_{++++} = -8ie^4 m^4 (I_{box}(s_{12}, s_{14}, m) + I_{box}(s_{12}, s_{13}, m) + I_{box}(s_{13}, s_{14}, m)) + \frac{e^4}{4\pi^2},
\]

where $s_{ij} = (k_i + k_j)^2$ and $I_{box}(s_{12}, s_{14}, m) = I_{0123}$. The scalar MIIs existing are symbolically calculated in the condition $s_{ij} \ll m$, where $\omega$ is incoming photon’s energy.

\[
iA_{++++} \approx -\frac{e^4(s_{12}^2 + s_{14}^2 + s_{12}^2)}{240\pi^2 m^4} = -\frac{\alpha^2(3 + \cos^2 \theta)\omega^4}{15m^4},
\]

where $\theta$ is the scattering angle. This result is in agreement with the result of Karplus [7].
5. Conclusions
We found that box-coefficient is solvable using linear algebra while triangle and bubble coefficients are solvable using the Cauchy residue theorem. However, the adjacent bubble and tadpole coefficients are ignored in our processes, the amplitude is obtained correctly.

Appendix A.
This section shows the parametrized loop momentum in various cut channel and its solution.

\[ p_{\pm}^\mu = k_{\pm}^\mu x_1 + k_{\pm}^\nu x_2 + \epsilon_{123}^\mu x_3 \pm \epsilon_{214}^\mu x_4, \]  
\[ (A.1) \]
where \( x_1 = s_{12}^{-1}M_{21} - 1, \ x_2 = s_{12}^{-1}M_{10}, \ x_3 = (4\epsilon_{12}k_d)^{-1} F_\pm, \ x_4 = (4\epsilon_{21}k_d)^{-1} F_\pm, \) which \( M_{ij} = m_i^2 - m_j^2, \ c = M_03 - s_{14}x_1 - s_{13}x_2, \ F_\pm = 1 \pm \sqrt{1 + 4s_{13}s_{14}c^2 (s_{12}^{-1}m_0^2 - x_1x_2)}. \)

\[ p_1^\mu = -r_0^\mu + k_a^\mu d_1 + k_b^\nu d_2 + \epsilon_{ab}^\nu x_3 + (2s_{12}C_{ijk}x_3)^{-1} e_{ab}^\mu, \]
\[ p_2^\mu = -r_0^\mu + k_a^\mu d_1 + k_b^\nu d_2 + (2s_{ab}C_{ijk}x_4)^{-1} e_{ab}^\mu + e_{ba}^\mu x_4, \]  
\[ (A.2) \]

Table A1. \( a \) and \( b \) are the choices of parametrized momentum, and \( d_1, d_2, C_{ijk} \) are solutions of (A.2) in several cut channels of 2-cut propagators.

| Cut (ijk) | \( r_0 \) | \( a \) | \( b \) | \( d_1 \) | \( d_2 \) | \( C_{ijk} \) |
|----------|----------|----------|----------|----------|----------|----------|
| 012      | 0        | 1        | 2        | \( s_{12}^{-1}M_{21} - 1 \) | \( s_{12}^{-1}M_{10} \) | \( 2s_{12}(M_{01}M_{12} - m_0^2s_{12})^{-1} \) |
| 013      | 0        | 1        | 4        | \( s_{14}^{-1}M_{03} \) | \( s_{14}^{-1}M_{10} \) | \( 2s_{14}(M_{01}M_{03} - m_0^2s_{14})^{-1} \) |
| 023      | 0        | 3        | 4        | \( s_{14}^{-1}M_{03} \) | \( s_{14}^{-1}M_{32} + 1 \) | \( 2s_{12}(M_{03}M_{03} - m_0^2s_{12})^{-1} \) |
| 123      | \( k_1 \) | 2        | 3        | \( s_{14}^{-1}M_{32} - 1 \) | \( s_{14}^{-1}M_{21} \) | \( 2s_{14}(M_{12}M_{23} - m_0^2s_{14})^{-1} \) |

\[ p_3^{(i,i+2)}(x_2, x_4) = \frac{1}{16s_{ab}^2x_4} \left( s_{ab}^2(1 - 4x_2^2) + M_{i,i+2}^2 \right) \left( M_{i,i+2}^2 + 2s_{ab}(m_i^2 + m_{i+2}^2) \right), \]  
\[ (A.3) \]

Table A2. \( a \) and \( b \) are the choices of parametrized momentum, and \( n \) are solutions of (A.3) in several cut channels of the nonadjacent 2-cut propagators.

| Cut (ij) | \( r_0 \) | \( a \) | \( b \) | \( n \) |
|----------|----------|----------|----------|----------|
| 02       | 0        | 1        | 2        | \( (2s_{12})^{-1}M_{20} - 2^{-1} \) |
| 13       | \( k_1 \) | 1        | 4        | \( (2s_{14})^{-1}M_{13} + 2^{-1} \) |

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