MODULI STACK OF ORIENTED FORMAL GROUPS
AND PERIODIC COMPLEX BORDISM

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Abstract. We introduce and study the non-connective spectral stack $M_{FG}$, the moduli stack of oriented formal groups. We realize some results of chromatic homotopy theory in terms of the geometry of this stack. For instance, we show that its descent spectral sequence recovers the Adams-Novikov spectral sequence. For two $E_\infty$-forms of periodic complex bordism $MP$, the Thom spectrum and Snaith construction model, we describe the universal property of the cover $\text{Spec}(MP) \to M_{FG}$. We show that Quillen’s celebrated theorem on complex bordism is equivalent to the assertion that the underlying ordinary stack of $M_{FG}$ is the classical stack of ordinary formal groups $M_{FG}^\odot$. In order to carry out all of the above, we develop foundations of a functor of points approach to non-connective spectral algebraic geometry.

Introduction

The goal of this paper is to show that the connection between the homotopy category of spectra and formal groups, which is at the heart of chromatic homotopy theory, may be manifested in terms of spectral algebraic geometry. By formal group, we mean a smooth 1-dimensional commutative formal group, as is traditional in homotopy theory. These are classified by an algebro-geometric moduli stack, the moduli stack of formal groups, which we shall denote $M_{FG}^\odot$.

Here and in the rest of the paper, we will use the heart symbol $\odot$ in the subscript to emphasize the classical nature of algebro-geometric objects considered, in contrast to spectral algebro-geometric objects which will be our main focus. This notation is motivated by the fact that ordinary abelian groups sit inside spectra as the heart of the usual $t$-structure. For instance, $M_{FG}$ is our notation for the classical moduli stack of formal groups. The superscript $\odot$ is part of the notation here, reminding of us that we are considering it as an object of ordinary algebraic geometry.

The relationship between the moduli stack $M_{FG}$ and the stable homotopy category is well known, and has been one of the main organizing principles in homotopy theory for the last 50 years. Let us quickly review a variant of the usual approach to it, with the only non-standard aspect being our focus on 2-periodic ring spectra (see Remark 3), in the following sense:

Definition 1 ([Ell2, Definition 4.1.5]). A commutative ring spectrum $A$ is said to be weakly 2-periodic if the ring structure induces an isomorphism $\pi_{*+2}(A) \cong \pi_{*}(A) \otimes_{\pi_0(A)} \pi_2(A)$.

The starting point is the observation that any complex orientation on a weakly 2-periodic periodic commutative ring spectrum $A$ gives rise through $A^0(\mathbb{CP}^\infty)$ to a formal group law over $\pi_0(A)$. The universal weakly 2-periodic complex orientable commutative ring is the periodic complex bordism spectrum $MP := \bigoplus_{i \in \mathbb{Z}} \Sigma^i(MU)$. The following celebrated result of Quillen may be seen as the origin of chromatic homotopy theory:

Theorem 2 (Quillen, [Qui69]). There is a canonical isomorphism of graded commutative rings $\pi_0(MP) \cong L$, between the homotopy ring of the periodic complex bordism spectrum.
MP, and the Lazard ring $L$, classifying formal group laws. Under this isomorphism, the formal group law $MP^0(CP^\infty)$ corresponds to the universal formal group law.

Formal group laws give rise to formal groups, and since the Lazard ring $L$ classifies formal groups, this produces a map $\text{Spec}(L) \to M^{\infty}_{FG}$ to the moduli stack of formal groups. The fact that any formal group can be locally presented by a formal group law, implies that $\text{Spec}(L) \to M^{\infty}_{FG}$ is a faithfully flat cover. Quillen’s Theorem may be reformulated in terms of this cover, as asserting that periodic complex bordism exhibits a groupoid presentation for the moduli stack of formal groups as

$$M^{\infty}_{FG} \simeq \lim_{\leftarrow} \left( \text{Spec}(\pi_0(MP \otimes MP) \Rightarrow \text{Spec}(\pi_0(MP)) \right).$$

One crucial consequence of this is that the second page of the Adams-Novikov spectral sequence

$$E_2^{s,t} = \text{Ext}^{s,t}_{\pi_0(MP \otimes MP)}(\pi_\ast(MP), \pi_\ast(MP)) \Rightarrow \pi_{t-s}(S)$$

may be determined purely in terms of algebro-geometric formal group data as

$$E_2^{s,2t} \simeq H^t(M^{\infty}_{FG}; \omega^{\text{et}}_{M^{\infty}_{FG}}).$$

Here $\omega^{\text{et}}_{M^{\infty}_{FG}}$ is the quasi-coherent sheaf on $M^{\infty}_{FG}$, collecting the modules of invariant Kähler differentials on formal groups.

The Adams-Novikov spectral sequence is one of our most powerful computational tools for studying the stable homotopy groups of spheres. Following the insights of Jack Morava, much of the structure of the stable stem uncovered this way has since been explained, using the isomorphism (1), in terms of the structure of the moduli stack $M^{\infty}_{FG}$. More precisely, this was traditionally viewed in terms of formal groups laws instead of the stack of formal groups $M^{\infty}_{FG}$. The stacky perspective on chromatic homotopy theory, which may be seen as a natural outgrowth of Morava’s orbit picture [Rav92, Chapter 4], was first explicitly emphasized and disseminated by Mike Hopkins in [Hop99]. For a more thorough account of chromatic homotopy theory from this perspective, see [Lur10].

Remark 3. Most of the literature on chromatic homotopy theory, e.g. [Rav04] and [Rav92], work with complex oriented ring spectra, without the weakly 2-periodicity assumption we imposed above. In particular, both Quillen’s Theorem and the Adams-Novikov spectral sequence are usually stated in terms of the complex bordism spectrum $MU$, instead of its periodic variant $MP$. However, for instance by [Goe17, Example 7.4], the Adams-Novikov spectral sequence is the same whether formed in terms of $MU$ or in terms of $MP$.

In this paper, we show how the above-discussed relationship between the moduli stack of formal groups $M^{\infty}_{FG}$ and spectra, may be encoded in terms of spectral algebraic geometry. That is a variant of usual algebraic geometry, in which the basic role of ordinary commutative rings as affines is instead replaced by $E_\infty$-ring spectra (called just $E_\infty$-rings for short). These homotopy-coherent analogues of commutative rings, first introduced in [MQR77], have since been shown to accommodate analogues of much of usual commutative algebra, e.g. [EKMM]. For an account from an $\infty$-categorical perspective, see for instance [HA].

Spectral algebraic geometry was first introduced in [HAG2, Chapter 2.4], its foundations laid out extensively in [SAG], and it has since found spectacular applications in [Ell2] by providing elegant approaches to topological modular forms and Lubin-Tate spectra.

Unfortunately, the theory developed in [SAG] works under a connectivity assumption, or alternatively without it, but then restricting itself to Deligne-Mumford stacks. Those are roughly the stacks which admit an étale cover by schemes, and provide a sufficient context for many applications. This is not the case when dealing with the moduli stack of formal groups, or spectral enhancements thereof, since it only admits a flat cover by schemes - see Remark 1.1.4. It similarly does not suffice for us to restrict to the connective setting, since our main spectral stacks of interest $M^{\text{con}}_{FG}$, defined below, will be inherently
non-connective. For this reason, we spend Section 1 of this paper developing an appropriate setting of non-connective spectral algebraic geometry for our desired applications.

We follow a functor of points approach, considering a non-connective spectral stack $X$ to be a certain kind of functor $X : \text{CAlg} \to S$, going from the $\infty$-category of (not-necessarily-connective) $E_{\infty}$-rings $\text{CAlg}$, and into the $\infty$-category of spaces $S$. To every non-connective spectral stack $X$, we associate an underlying ordinary stack $X^\circ : \text{CAlg}^\circ \to S$, a (co)presheaf on the category of ordinary commutative rings $\text{CAlg}^\circ$, as described in the next construction.

**Construction 4.** Write the non-connective spectral stack $X$ as a colimit $X \simeq \lim_i \text{Spec}(A_i)$ in the $\infty$-category of non-connective spectral schemes of some diagram of affine non-connective spectral schemes $\text{Spec}(A_i)$, i.e. functors representable by $E_{\infty}$-rings $A_i$. Then its underlying ordinary stack is defined to be the colimit of the corresponding diagram of ordinary affine schemes

$$X^\circ := \lim_i \text{Spec}(\pi_0(A_i)),$$

with the colimit computed in the $\infty$-category of ordinary stacks.

In the connective case discussed in [SAG], the underlying spectral stack $X^\circ$ of a spectral stack $X$ coincides with the functor restriction $X|_{\text{CAlg}^\circ}$. That no longer holds in the non-connective world, see Remark 2.3.10, making the notion more subtle.

**Remark 5.** Throughout this paper, we adopt the terminology ordinary stack to refer to certain kinds of functors $\text{CAlg}^\circ \to S$ (essentially the higher stacks of Sim96 or To¨e05) from the category of ordinary commutative rings $\text{CAlg}^\circ$. These are objects of ordinary, as opposed to spectral, algebraic geometry, in the sense that they have nothing to do with $E_{\infty}$-rings. On the other hand, they do take values in the $\infty$-category of spaces $S$, while (the functors of points of) algebraic stacks in the algebro-geometric literature are mostly required to being valued in the 2-category of groupoids $\text{Grpd}$. In light of the standard equivalence $\tau_{1}(S) \simeq S_{\text{Grpd}}$ between 1-truncated spaces and groupoids, our notion is a pure generalization - so a groupoid-valued functor like $M_{\text{FG}}^\circ$ is indeed a special case.

Similarly to the above construction of the underlying ordinary stack, any non-connective spectral stack $X$ admits an $\infty$-category of quasi-coherent sheaves $\text{QCoh}(X)$ by the limit formula

$$\text{QCoh}\left(\lim_i \text{Spec}(A_i)\right) \simeq \lim_i \text{Mod}_{A_i},$$

where $\text{Mod}_{A}$ denotes the $\infty$-category of $A$-module spectra. To be precise, this $\text{QCoh}(X)$ is not analogous to the usual abelian category of quasi-coherent sheaves on an ordinary stack, but rather to the derived category of quasi-coherent sheaves. This kind of abuse of language permeates the transition from ordinary to derived or spectral algebraic geometry. Under the assumption that the non-connective spectral stack $X$ is geometric - roughly: it admits a flat cover by affines - a quasi-coherent sheaf $\mathcal{F}$ on $X$ in this sense induces homotopy groups $\pi_t(\mathcal{F})$, which are quasi-coherent sheaves in the usual sense on the underlying ordinary stack $X^\circ$. They are related to the spectrum $\Gamma(X; \mathcal{F})$ of global sections of $\mathcal{F}$ by the descent spectral sequence

$$E_2^{s,t} = H^s(X^\circ; \pi_t(\mathcal{F})) \Rightarrow \pi_{t-s}(\Gamma(X; \mathcal{F})).$$

(2)

This concludes the outline of non-connective spectral algebraic geometry, the details of which we develop Section 1. We now move on to outlining our applications of it in Section 2 to the connection between formal groups and spectra.

Let us start with some motivation. In his work on topological modular forms, sketched in Lur09 and fleshed out in Ell2, one of Lurie’s central ideas was to define a moduli stack $M^{\text{ell}}_{\text{FG}}$ purely inside the realm of spectral algebraic geometry, parameterizing certain spectral analogues of elliptic curves. The Goerss-Hopkins-Miller Theorem on the $E_{\infty}$-structure on
TMF then reduces to a comparison between this spectral stack \( \mathcal{M}_{\text{Ell}}^{\text{or}} \), defined by a universal property, and the ordinary stack of elliptic curves \( \mathcal{M}_{\text{Ell}}^{\text{Ell}} \) in the usual sense. Lurie also proved in \cite{Ell2} that an analogue of this works to define Lubin-Tate spectra, also known as Morava E-theory, or completed Johnson-Wilson theory, together with their \( \mathbb{E}_\infty \)-rings structures.

We show that a similar approach works when elliptic curves are replaced with formal groups. There does indeed exist a well-behaved theory of formal groups in spectral algebraic geometry, but in order to obtain the correct moduli stack, that is not sufficient. Following Lurie, we must impose the following somewhat exotic additional assumption, with no direct classical analogue.

**Definition 6** (\cite{Ell2} Definition 4.3.9, Definition 2.1.7). An orientation on a formal group \( \hat{G} \) over an \( \mathbb{E}_\infty \)-ring \( A \) is an element of \( \pi_2(\hat{G}(\tau_{\geq 0}(A))) \), which induces an equivalence \( \omega_{\hat{G}} \cong \Sigma^{-2}(A) \) of \( A \)-module spectra.

The notion of an orientation of a formal group, which might more accurately be called a 2-shifted orientation, may seem somewhat arbitrary, but see Remark 2.1.8 for some motivation. In terms of this notion, define our central object of interest \( \mathcal{M}_{\text{fg}}^{\text{or}} \) to be the moduli stack of oriented formal groups. That is to say, it is the functor \( \mathcal{M}_{\text{fg}}^{\text{or}} : \text{CAlg} \to \mathcal{S} \), sending any \( \mathbb{E}_\infty \)-ring to the space of oriented formal groups over it. This space turns out to be particularly simple: it is contractible if and only if the \( \mathbb{E}_\infty \)-ring \( A \) is complex periodic (both weakly periodic and complex orientable), and empty otherwise. This is the content of \cite{Ell2} Proposition 4.3.23, summarized here as Proposition 2.1.9 and is some sense the key technical observation underlying this paper.

**Remark 7.** Though the notion of an oriented formal group is due to Lurie, their moduli stack \( \mathcal{M}_{\text{fg}}^{\text{or}} \) does not explicitly appear in his work. Related stacks, such as the moduli of oriented elliptic curves and oriented deformation spaces, are considered and studied in depth in \cite{Ell2}. We summarize their relationship to \( \mathcal{M}_{\text{fg}}^{\text{or}} \) in Example 2.5.9. Therefore, although our analysis of it is based firmly on the Lurie’s foundational work, the consideration of the stack \( \mathcal{M}_{\text{fg}}^{\text{or}} \) is a novelty of the present work.

**Motto 8.** The algebraic geometry of the non-connective spectral stack \( \mathcal{M}_{\text{fg}}^{\text{or}} \) manifests chromatic homotopy theory.

The following is our main result, making precise in what way the non-connective spectral stack \( \mathcal{M}_{\text{fg}}^{\text{or}} \) captures the relationship between formal groups and spectra.

**Theorem 9** (Theorem 2.3.1, Corollary 2.3.9, Proposition 2.4.1, Proposition 2.6.1, and Theorem 2.4.4). The following statements hold for the moduli stack of oriented formal groups \( \mathcal{M}_{\text{fg}}^{\text{or}} \), as defined above:

1. It is a geometric non-connective spectral stack.
2. Its underlying ordinary stack is canonical equivalent to the classical moduli stack of formal groups \( \mathcal{M}_{\text{fg}}^{\text{Ell}} \).
3. There is a canonical equivalence of \( \mathbb{E}_\infty \)-rings \( \mathcal{O}(\mathcal{M}_{\text{fg}}^{\text{or}}) \cong \mathcal{S} \) between the \( \mathbb{E}_\infty \)-ring of global functions on \( \mathcal{M}_{\text{fg}}^{\text{or}} \) and the sphere spectrum.
4. The descent spectral sequence
   \[
   E_2^{s,t} = H^s(\mathcal{M}_{\text{fg}}^{\text{or}}; \pi_t(\mathcal{O}_{\mathcal{M}_{\text{fg}}^{\text{or}}})) \Rightarrow \pi_{t-s}(\mathcal{O}(\mathcal{M}_{\text{fg}}^{\text{or}})).
   \]
   recovers the Adams-Novikov spectral sequence.
5. Under the definition of ind-coherent sheaves as in Definition 2.4.2, there is a canonical equivalence of symmetric monoidal \( \infty \)-categories
   \[
   \text{IndCoh}(\mathcal{M}_{\text{fg}}^{\text{or}}) \cong \text{Sp} \quad \mathcal{F} \mapsto \Lambda(\mathcal{M}_{\text{fg}}^{\text{or}}; \mathcal{F}),
   \]
   given in terms of global sections of ind-coherent sheaves on \( \mathcal{M}_{\text{fg}}^{\text{or}} \).
Sketch of proof. Choosing an $E_\infty$-ring structure on the periodic complex bordism spectrum $\text{MP}$ gives rise to a morphism $\text{Spec}(\text{MP}) \to \mathcal{M}_{\text{FG}}^{\text{or}}$. We first show that this is a flat cover, in particular verifying (i). The Čech nerve of this cover provides a simplicial presentation

$$\mathcal{M}_{\text{FG}}^{\text{or}} \simeq \lim\left( \cdots \longrightarrow \text{Spec}(\text{MP} \otimes \text{MP} \otimes \text{MP}) \longrightarrow \text{Spec}(\text{MP} \otimes \text{MP}) \longrightarrow \text{Spec}(\text{MP}) \right).$$

In light of this presentation, Quillen’s Theorem implies the underlying stack statement (ii). Deducing (iii) boils down to well-known facts about nilpotent completion: namely that the sphere spectrum $S$ equals its own nilpotent completion with respect to the complex bordism spectrum, as first proved by Bousfield [Bou79, Theorem 6.5]. The assertion (v) is a simple consequence of (iii) in light of the definition of ind-coherent sheaves we are using. It remains only to verify (iv), i.e. to identify the descent spectral sequence and the Adams-Novikov spectral sequences. Again through the simplicial presentation, the descent spectral sequence in question is identified with the spectral sequence associated to a cosimplicial spectrum. That cosimplicial spectrum is $\text{MP} \otimes \text{cobar}^{+}$, often called the cobar resolution or the Amitsur complex. This implies (iv), since the standard construction of the Adams-Novikov spectral sequence is as the spectral sequence of that same cosimplicial spectrum, see [MNN17, Proposition 2.14].

Theorem 9 may be viewed as one solution to the realization problem posed in [Goe09]. It asks roughly whether it is possible to enhance the classical stack of formal groups $\mathcal{M}_{\text{FG}}^{\text{or}}$ to a spectral stack, whose descent spectral sequence would recover the Adams-Novikov spectral sequence. The setting for non-connective spectral algebraic geometry in loc. cit. is however slightly different from the ones we use in this paper. In that context, one is looking for an appropriate sheaf of $E_\infty$-rings on the underlying ordinary stack, which then serves as the spectral structure sheaf. Instead of such a vaguely "ringed space" approach to non-connective spectral stacks, we tackle them exclusively from a functor of points perspective. As such, the relationship with their underlying ordinary stacks, while still sufficiently workable, is somewhat less transparent.

On the other hand, the fact that the Adams-Novikov spectral sequence may be interpreted in this way in terms of the simplicially-given non-connective spectral stack given above, is not new. For instance, it is mentioned explicitly in [SAG, Remark 9.3.1.9]. The novelty of our approach is, from this perspective, instead in describing the non-connective spectral stack $\mathcal{M}_{\text{FG}}^{\text{or}}$ directly as a moduli stack, rather than defining it in an ad hoc way in terms of the complex bordism spectrum.

In the last section of the paper, Section 3, we turn to the connection between the moduli stack of oriented formal groups $\mathcal{M}_{\text{FG}}^{\text{or}}$ and periodic complex bordism $\text{MP}$. When neglecting the $E_\infty$-ring structure, the underlying spectrum of $\text{MP}$ admits the following description:

**Corollary 10** ([Remark 2.5.5]). The extraordinary homology theory of periodic complex bordism $\text{MP}_n : S \to \text{Ab}$ may be written for any space $X$ as

$$\text{MP}_n(X) \cong \Gamma\left( \mathcal{M}_{\text{FG}}^{\text{or}}; \pi_n(\mathcal{O}_{\mathcal{M}_{\text{FG}}^{\text{or}}}[X]) \otimes_{\mathcal{O}_{\mathcal{M}_{\text{FG}}^{\text{or}}}} \mathcal{O}_{\mathcal{M}_{\text{FG}}^{\text{or},\text{coord}}}, \right),$$

where $\mathcal{M}_{\text{FG}}^{\text{or},\text{coord}}$ is the moduli stack of coordinatized formal groups, e.g. formal group laws.

The situation becomes much subtler when trying to take an $E_\infty$-structure into account. By [HY19], the spectrum $\text{MP}$ (or even, $E_2$-ring spectrum) admits at least two distinct non-equivalent $E_\infty$-ring structures:

- The *Thom spectrum* $E_\infty$-ring $\text{MUP}$ is the Thom spectrum of the $J$-homomorphism $J : \mathbb{Z} \times BU \to \text{Sp}$, obtained by group completion from the symmetric monoidal functor $\text{Vect}^+ \to \text{Sp}$, which sends a finite dimensional complex vector space $V$ to the suspension spectrum of its one-point compactification $S^V = V \cup \{\infty\}$.
The Snith construction $\text{MP}_{\text{Snith}} := (S[\text{BU}])[\beta^{-1}]$ is obtained from the suspension $\mathbb{E}_\infty$-ring $S[\text{BU}]$ of the $\mathbb{E}_\infty$-space $\text{BU}$, by inverting the canonical Bott element $\beta \in \pi_2(S[\text{BU}])$, arising from the inclusion $S^2 \simeq \mathbb{CP}^1 \subseteq \mathbb{CP}^\infty \simeq \text{BU}(1) \subseteq \text{BU}$.

By the main theorem of [HY19], these two are not equivalent as $\mathbb{E}_\infty$-rings. If we take MP to mean either of these, we obtain as discussed above a flat cover $\text{Spec}(\text{MP}) \to \mathcal{M}_{\text{FG}}^\circ$. In light of it, we can explicitly describe the spectral algebro-geometric moduli problem which periodic complex bordism corresponds to.

**Theorem 11** (Theorem 3.3.4 and Proposition 3.5.2). The affine non-connective spectral scheme $\text{Spec}(\text{MP})$ parametrizes oriented formal groups $\mathcal{G}$ over $\mathbb{E}_\infty$-rings $A$, together with

- For $\text{MP} \simeq \text{MUP}$: a system of trivializations $\omega^V \simeq A$ of $A$-module spectra, functorial and symmetric monoidal in terms of finite-dimensional complex vector spaces $V$ and $\mathbb{C}$-linear isomorphisms.
- For $\text{MP} \simeq \text{MP}_{\text{Snith}}$: an $\mathbb{E}_\infty$-ring map $S[\text{BU}] \to A$, which induces an equivalence of $A$-module spectra $\omega^G \simeq A$.

**Remark 12.** Both of the universal properties for $\text{Spec}(\text{MP})$ above have to do with oriented formal groups $\mathcal{G}$ over $\mathbb{E}_\infty$-rings $A$, equipped with an $A$-linear trivialization $\omega^G \simeq A$ of the dualizing line $\omega^G$, together with some built-in unitary group equivariance. The main difference in that in the case of the Thom spectrum $\mathbb{E}_\infty$-ring structure, the unitary group action ultimately stems from the usual action of $U(n)$ on $\mathbb{C}^n$, while in the case of the Snith construction, the trivial $U$-action plays a role instead.

Finally we turn our attention to Quillen’s Theorem itself, isolating its spectral algebro-geometric content as follows:

**Theorem 13** (Theorem 3.6.1). Quillen’s Theorem is equivalent to assertion $[\text{ii}]$ of Theorem 9, which is to say that there exists an equivalence of ordinary stacks

$$(\mathcal{M}_{\text{FG}}^\circ)^\circ \simeq \mathcal{M}_{\text{FG}}^\circ$$

between the underlying ordinary stack $(\mathcal{M}_{\text{FG}}^\circ)^\circ$ of the spectral moduli stack of oriented formal groups $\mathcal{M}_{\text{FG}}^\circ$, and the ordinary stack of formal groups $\mathcal{M}_{\text{FG}}^\circ$.

One striking aspect of this equivalent rephrasing of Quillen’s Theorem, is that complex bordism makes no appearance in it. Indeed, our definition of the moduli stack of oriented formal groups $\mathcal{M}_{\text{FG}}^\circ$ is completely independent of the spectrum MP, whose role is instead only to provide a convenient flat cover of it. The proofs of most fundamental properties of the non-connective spectral stack $\mathcal{M}_{\text{FG}}^\circ$ do still rely on complex bordism, so it and Quillen’s Theorem are still essential, but perhaps their geometric role is clarified somewhat.

It would be interesting to have a direct algebrao-geometric proof of the underlying ordinary stack result of the previous Theorem, which would therefore constitute a new proof of Quillen’s Theorem. We hope to return to this problem in future work. Another topic we will return to in a follow-up [Gre21b] to this paper, is the relationship between chromatic localizations of spectra and the algebraic geometry of the moduli stack of oriented formal groups $\mathcal{M}_{\text{FG}}^\circ$. There we will also clarify the connection between [Gre21a] and the present paper. Let us point out, however, that our work is no way alone in expanding upon the ideas from [Ell2] for chromatic applications; see for instance [Dav20], [Dav21], and [Dev18].

**Relationship to prior work.** The results of this paper constitute validation of an intuitive picture about how the field of chromatic homotopy theory should relate to spectral algebraic geometry. This picture is folklore in the field: it is implicit in [Hop99], indicated in [Lur10, HAG2 Section 2.4], and made center-stage in [Pet18]; but prior to this work, it had never been made rigorous in full generality. Perhaps the first explicit connection to spectral algebraic geometry, though from the perspective of sheaves of $\mathbb{E}_\infty$-rings, is in the work of Goerss-Hopkins-Miller on topological modular forms, as surveyed in [TMF].
Chapter 12 or [Goe10]. The field of spectral algebraic geometry was largely set up, in the foundational monograph [SAG] (and relying further on the groundwork in ∞-category theory and higher algebra [HTT] and [HA] respectively) to clarify this relationship. The precise connection to spectral algebraic geometry in the deformation-theoretic and elliptic-curve settings is therefore due to the work of Lurie, as envisioned in the survey [Lur09], and carried out in the series of papers [Ell1], [Ell2], [Ell3]. Others extended it in several directions: Hill-Lawson to elliptic curves with level structure in [HL16], Behrens-Lawson to certain higher-dimensional Shimura varieties [TAF], Davies to $p$-divisible groups in [Dav20], and more.

The role of this paper is to extend Lurie et al.’s work from the elliptic curve and formal deformation cases, to the entirety of formal groups (though certain remarks [SAG, Example 9.3.1.8, Remark 9.3.1.9] suggest this might have been known to Lurie). As discussed in [Goe09] and [Goe10], the idea of extending the spectral-algebro-geometric story to the entirety of the stack of formal groups had been considered previously, but proved challenging to tackle using the contemporary obstruction-theoretic ringed-space approach. Other than building on the ideas of Lurie from [Ell2], our technical innovation is to instead use the ‘functor of points’ approach to non-connective spectral algebraic geometry (in a setting not developed in [SAG]) to circumvent those difficulties. The present work (as well as its sequel [Gre21b]) therefore, by way of realizing classical chromatic results in spectral-algebro-geometric terms, provides verification of a previously-elusive conjectural relationship between chromatic homotopy and spectral algebraic geometry.

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1. Non-connective spectral algebraic geometry

The first half of this paper is spent setting up sufficient foundations of non-connective spectral algebraic geometry, approached from the “functor of points” perspective, in order to be able to carry out the arguments we wish to in the second half.

1.1. The fpqc topology. Recall from [SAG, Definition B.6.1.1] that a map of (not necessarily connective) $E_\infty$-rings $A \to B$ is faithfully flat if

- the map of ordinary rings $\pi_0(A) \to \pi_0(B)$ is faithfully flat,
- the map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism of graded rings.

This allows us [SAG, Proposition B.6.1.3 and Variant B.6.1.7] to equip the $\infty$-categories $\text{CAlg}^{op}$, $(\text{CAlg}^{cn})^{op}$, and $(\text{CAlg}^c)^{op}$ with the fpqc topology, giving rise to the corresponding $\infty$-topoi $\text{Shv}^{nc}_{\text{fpqc}} := \text{Shv}(\text{CAlg}^{op})$, $\text{Shv}_{\text{fpqc}} := \text{Shv}((\text{CAlg}^{cn})^{op})$, and $\text{Shv}^c_{\text{fpqc}} := \text{Shv}((\text{CAlg}^c)^{op})$ respectively.

Examples 1.1.1. Following the standard convention in functor-of-points algebraic geometry, we call representables affine. An affine non-connective spectral scheme is therefore a functor $\text{Spec}(A) : \text{CAlg} \to \text{S}$, given by $R \mapsto \text{Map}_{\text{CAlg}}(A, R)$, for some $E_\infty$-ring $A$. Similarly, an affine spectral scheme is the functor $\text{Spec}(A) : \text{CAlg}^{cn} \to \text{S}$ given by

\[1\]This stands for fidélément-pla\"et quasi-compacte, French for faithfully flat and quasi-compact.
$R \mapsto \text{Map}_{\text{CAlg}^{\infty}}(A, R)$ for a connective $\mathbb{E}_\infty$-ring $A$, and an affine ordinary scheme if a functor $\text{Spec}(A) : \text{CAlg}^\diamond \to \text{Set} \to S$ given by $R \mapsto \text{Hom}_{\text{CAlg}^{\infty}}(A, R)$.

Remark 1.1.2. The notation $\text{Shv}_{\text{fpqc}}^\diamond$ is potentially misleading. It refers to the full subcategory of $\text{Fun}(\text{CAlg}^\diamond, S)$, spanned by functors satisfying fpqc descent, and not as one might think the underlying ordinary topos $\tau_{\geq 0}(\text{Shv}_{\text{fpqc}}) \subseteq \text{Fun}(\text{CAlg}^\diamond, \text{Set})$ of the $\infty$-topos $\text{Shv}_{\text{fpqc}}$. In the language of Simpson or Toën, see e.g. [Toë05], we might say that $\text{Shv}_{\text{fpqc}}^\diamond$ encodes higher (but not derived or spectral) stacks.

Remark 1.1.3. There are some prickly point-set theoretic difficulties concerning the fpqc topology, making certain aspects of the theory, such as sheafification, require extra care to define correctly. We side-step these difficulties by choosing a Grothendieck universe and implicitly working inside its confines throughout. This kind of restriction would in general be unreasonable, as it is far too restrictive for a satisfactory general theory. Alas, we are in this paper ultimately only concerned with a small number of very specific examples of fpqc sheaves, all of which arise under small colimits from representables, making such stringent restrictions acceptable. We will mostly leave set-theoretic assumptions implicit, but see Remark 1.2.2 for the one instance where some care is in fact required.

Remark 1.1.4. In light of the difficulties alluded to the previous remark, let us explain why we nonetheless use the fpqc topology. In Lemma 2.3.5 below, we will find ourselves implicitly working inside its confines throughout. This kind of restriction would in general be unreasonable, as it is far too restrictive for a satisfactory general theory. Alas, we are in this paper ultimately only concerned with a small number of very specific examples of fpqc sheaves, all of which arise under small colimits from representables, making such stringent restrictions acceptable. We will mostly leave set-theoretic assumptions implicit, but see Remark 1.2.2 for the one instance where some care is in fact required.

1.2. Connective covers and underlying ordinary stacks. The three $\infty$-topoi $\text{Shv}_{\text{fpqc}}^{\infty}$, $\text{Shv}_{\text{fpqc}}$, and $\text{Shv}_{\text{fpqc}}^\diamond$ are related to each other by a series of adjunctions:

Proposition 1.2.1. There are canonical adjunctions

$$\tau_{\geq 0} : \text{Shv}_{\text{fpqc}}^{\infty} \rightleftarrows \text{Shv}_{\text{fpqc}} : \text{Shv}_{\text{fpqc}}^\diamond \rightleftarrows \text{Shv}_{\text{fpqc}} : (-)^{\diamond}$$

with colimit-preserving right adjoints. The unlabeled arrows in both are fully faithful.

Proof. The inclusion of connective $\mathbb{E}_\infty$-rings into all $\mathbb{E}_\infty$-rings and the formation of connective covers form an adjunction $i : \text{CAlg}^{\infty} \rightleftarrows \text{CAlg} : \tau_{\geq 0}$. Restriction and left Kan extension along both of these functors induce adjunctions between presheaf $\infty$-categories

$$i^* : \text{Fun}(\text{CAlg}^{\infty}, S) \rightleftarrows \text{Fun}(\text{CAlg}, S) : i^\dagger, \quad (\tau_{\geq 0})^* : \text{Fun}(\text{CAlg}, S) \rightleftarrows \text{Fun}(\text{CAlg}^{\infty}, S) : (\tau_{\geq 0})^\dagger,$$

and it follows from the adjunction between $i$ and $\tau_{\geq 0}$ by abstract nonsense that $i^* \simeq (\tau_{\geq 0})^*$. Indeed, for any functor $X : \text{CAlg}^{\infty} \to S$ and any $\mathbb{E}_\infty$-ring $A$, the colimit formula for Kan extension tells that $i^*(X)(A) \simeq \lim_{A \in \text{CAlg}^{\infty} \times \text{CAlg} / A} i^!(X)(B)$. Since the connective cover $\tau_{\geq 0}(A) \to A$ is the terminal object of the $\infty$-category $\text{CAlg}^{\infty} \times \text{CAlg} / A$, it follows that $i^*(X)(A) \simeq X(\tau_{\geq 0}(A))$ as desired. From the equivalence $i^* \simeq (\tau_{\geq 0})^*$ it now follows that

$$i^* i^! \simeq i^* \tau_{\geq 0}^* \simeq (\tau_{\geq 0} \circ i)^! \simeq \text{id},$$

showing that $i^!$ is fully faithful.

On the other hand, it is immediate from the definition of faithfully flat $\mathbb{E}_\infty$-ring maps that both of the functors $i$ and $\tau_{\geq 0}$ preserve them. Thus the restriction functors $i^*$ and $(\tau_{\geq 0})^* \simeq i^!$ both preserve the subcategories of sheaves, implying that the above presheaf adjunction restricts to the full subcategories of presheaves, giving rise to

$$(\tau_{\geq 0})^* \simeq i^! : \text{Shv}_{\text{fpqc}} \rightleftarrows \text{Shv}_{\text{fpqc}}^{\infty} : i^*,$$
whose left adjoint is still fully faithful. It also admits a further left adjoint in the form of
\[ \text{Shv}^{\text{nc}}_{\text{fpqc}} \subseteq \text{Fun}(\text{CAlg}, \mathbb{S}) \xrightarrow{(\tau_0)_!} \text{Fun}(\text{CAlg}^{\text{cn}}, \mathbb{S}) \xrightarrow{L} \text{Shv}_{\text{fpqc}}, \]
where \( L \) denotes sheafification. This is the functor \( \tau_0 : \text{Shv}^{\text{nc}}_{\text{fpqc}} \to \text{Shv}_{\text{fpqc}} \) from the statement of the Proposition, and its right adjoint \( (\tau_0)^* \simeq i_! \) indeed preserves all colimits, and is fully faithful.

To obtain the other adjunction in the statement of the Proposition, we repeat the argument, starting with the adjunction \( \pi_0 : \text{CAlg}^{\text{cn}} \xrightarrow{\simeq} \text{CAlg}^\circ : j \). That gives rise to a colimit-preserving functor \( (\pi_0)_! : \text{Shv}_{\text{fpqc}} \to \text{Shv}^\circ_{\text{fpqc}} \) — this is the functor \( (-)^\circ \) from the statement of the Proposition. It admits a left adjoint given by
\[ \text{Shv}^\circ_{\text{fpqc}} \subseteq \text{Fun}(\text{CAlg}^\circ, \mathbb{S}) \xrightarrow{j^!} \text{Fun}(\text{CAlg}^{\text{cn}}, \mathbb{S}) \xrightarrow{L} \text{Shv}_{\text{fpqc}}, \]
where \( L \) is sheafification. It remains to show that this functor is fully faithful. That is, we must show that for any fpqc sheaf \( E \) colimit indexed over all faithfully flat \( \mathbb{S} \)-modules \( L \) an equivalence, and by passage to transfinite composites, we conclude that \( (Y(\mathbb{S}^{\otimes A^{\otimes 1}})) \)
\[ L(Y) \simeq \lim_{\text{BeCAlg}_{/A}} \text{Tot}(Y(B^{\otimes A^{\otimes 1}})), \]
taking totalization of the evaluation of \( Y \) on the Čech nerve of the covers, and with the colimit indexed over all faithfully flat \( \mathbb{E}_{\infty} \)-ring (resp. commutative ring) \( A \) by
\[ Y^+(A) \simeq \lim_{\text{BeCAlg}_{/A}} \text{Tot}(Y(B^{\otimes A^{\otimes 1}})), \]
thus preserving colimits of representables, in which case the fact that all the functors involved in the statement of Proposition \[ \text{Proposition 1.2.1} \]
preserve colimits, ensures that the smallness requirement is in fact obeyed.

**Remark 1.2.2.** Proposition \[ \text{Proposition 1.2.1} \] is the one place in this paper where the issue of set-theoretic difficulties, discussed in Remark \[ \text{Remark 1.1.3} \] actually arise. We gave a naïve treatment, ignoring set-theoretic difficulties. In reality, for fpqc sheafification to exist as we used it in the proof, we must allow ourselves to work in the not-necessarily-small fpqc topos, breaking our implicit smallness assumption established in Remark \[ \text{Remark 1.1.3} \] This is not much of an issue however, as we will only be concerned with fpqc sheaves which are obtained as small colimits of representables, in which case the fact that all the functors involved in the statement of Proposition \[ \text{Proposition 1.2.1} \] preserve colimits, ensures that the smallness requirement is in fact obeyed.

In what follows, we will mostly suppress the fully faithful functors of Proposition \[ \text{Proposition 1.2.1} \] and view the \( \infty \)-topos connective flat stacks (resp. ordinary flat stacks) as a full subcategory \( \text{Shv}_{\text{fpqc}} \subseteq \text{Shv}^{\text{nc}}_{\text{fpqc}} \) (resp. \( \text{Shv}^\circ_{\text{fpqc}} \subseteq \text{Shv}_{\text{fpqc}} \)) of the \( \infty \)-topos of non-connective flat
stacks (resp. connective flat stacks). We also obtain a functor $\text{Shv}^\infty_{\text{fpqc}} \to \text{Shv}^\infty_{\text{fpqc}}$ given by $X \mapsto X^\circ := (\tau_{\geq 0}(X))^\circ$, whose restriction to the full subcategory $\text{Shv}^\infty_{\text{fpqc}} \subseteq \text{Shv}^\infty_{\text{fpqc}}$ recovers the eponymous functor $(-)^\circ$ appearing in Proposition 1.2.1. We refer to $\tau_{\geq 0}(X)$ as the connective cover of $X$, and to $X^\circ$ as its underlying ordinary stack.

Remark 1.2.3. We can read off from the proof of Proposition 1.2.1 that the functors $\tau_{\geq 0} : \text{Shv}^\infty_{\text{fpqc}} \to \text{Shv}^\infty_{\text{fpqc}}$ and $(-)^\circ : \text{Shv}^\infty_{\text{fpqc}} \to \text{Shv}^\infty_{\text{fpqc}}$ are given by left Kan extension (and sheafification), and may as such informally be written as

$$\tau_{\geq 0}(\lim_{i} \text{Spec}(A_i)) \simeq \lim_{i} \text{Spec}(\tau_{\geq 0}(A_i)), \quad (\lim_{i} \text{Spec}(A_i))^\circ \simeq \lim_{i} \text{Spec}(\pi_0(A_i)).$$

In particular, the functors $\tau_{\geq 0}$ and $(-)^\circ$, as well as the inclusions $\text{Shv}^\infty_{\text{fpqc}} \subseteq \text{Shv}^\infty_{\text{fpqc}} \subseteq \text{Shv}^\infty_{\text{fpqc}}$, all preserve the respective notions of affine schemes.

Remark 1.2.4. In contrast to the preceding example, not all ways to pass from non-connective to connective algebraic geometry preserve affines. For any $E_\infty$-ring, the restriction $\text{Spec}(A)|_{\text{CAlg}^\infty}$ is an object of $\text{Shv}^\infty_{\text{fpqc}}$. Under the inclusion $\text{Shv}^\infty_{\text{fpqc}} \subseteq \text{Shv}^\infty_{\text{fpqc}}$, the functor $\text{CAlg} \to S$ given by $R \mapsto \text{Map}_{\text{CAlg}}(A, \tau_{\geq 0}(R))$ is generally not affine unless $A$ itself had been connective to begin with. When $A$ is instead assumed to be coconnective, $\text{Spec}(A)$ recovers the notion of coaffine stacks of [DAGS, Section 4.4] (at least if we were working over a field of characteristic 0), denoted there $c\text{Spec}(A)$. These are closely related to affine stacks of [LMB09], although the latter exist in a cosimplicial, rather than spectral setting. Note in particular that $\text{Spec}(A) \neq c\text{Spec}(A)$ for a coconnective $E_\infty$-ring $A$, i.e. coaffine stacks are not a special case of non-connective affine stacks.

Remark 1.2.5. For a connective fpqc stack $X : \text{CAlg}^\infty \to S$, its underlying ordinary fpqc stack $X^\circ$ admits another description as $X^\circ \simeq X|_{\text{CAlg}^\circ}$. There is also in that case a canonical map $X^\circ \to X$ in $\text{Shv}^\infty_{\text{fpqc}}$, given by the unit of the relevant adjunction. Both of those fail for non-connective stacks; we will see in Remark 2.3.10 an example of a non-connective fpqc stack $X$ for which $X^\circ$ is an interesting non-trivial stack, while $X|_{\text{CAlg}^\circ} = \emptyset$ is the constant empty-set functor. Indeed, $X$ and $X^\circ$ are in general connected only through the cospan $X \rightarrow \tau_{\geq 0}(X) \leftarrow X^\circ$.

Example 14. According to [SAG, Variant 1.1.2.9], non-connective spectral schemes, which embed fully faithfully into spectral stacks, may be described as follows. A non-connective spectral scheme $X$ consists of a $(|X|, O_X)$ of a topological space $|X|$ and a sheaf of $E_\infty$-rings $O_X$ on it, satisfying a spectral analogue of the usual definition of a scheme in the locally ringed space approach. The underlying ordinary stack of $X$ then coincides with the usual scheme $(|X|, \pi_0(O_X))$. An analogous description works for spectral Deligne-Mumford stacks, if topological spaces are replaced with $\infty$-topoi. The underlying ordinary stack construction should be thought of as a functor of points analogue of this.

Remark 1.2.6. The terminology of calling the $\tau_{\geq 0}(X)$ as the connective cover is taken to comply with the standard one for $E_\infty$-rings, recovered for affine stacks. But note that the canonical map goes $X \rightarrow \tau_{\geq 0}(X)$, suggesting a name like “connective quotient” might be more appropriate.

1.3. Geometric stacks. We introduce a convenient class of non-connective spectral stacks, a non-connective version of the geometric stacks of [SAG Definition 9.3.0.1]. Just as Deligne-Mumford stacks are roughly those stacks which admit an étale cover, and Artin stacks are roughly those stacks which admit a smooth cover, so are geometric stacks roughly those stacks which admit a flat cover.

Definition 1.3.1. A non-connective geometric stack is a functor $X : \text{CAlg} \to S$ which satisfies the following conditions:

(a) The functor $X$ satisfies descent for the fpqc topology.
(b) The diagonal map $X \to X \times X$ is affine.
(c) There exists an $E_\infty$-ring $A \in \text{CAlg}$ and a faithfully flat map $\text{Spec}(A) \to X$.

**Remark 1.3.2.** The affine diagonal condition (b) above is as always equivalent to the following statement (see [SAG, Proposition 9.3.1.2] for proof of the analogous result in the connective setting): for any pair of maps $\text{Spec}(A) \to X$ and $\text{Spec}(B) \to X$, the fiber product $\text{Spec}(A) \times_X \text{Spec}(B)$ is affine. This assumption is made largely to simplify various statements, and could be dropped in much of what follows, at the cost of demanding affine representability of various morphisms.

**Variant 1.3.3.** By substituting $\text{CAlg} \to \text{CAlg}^{\text{cn}}$ in Definition 1.3.1 we recover the notion of a geometric stack from [SAG, Definition 9.3.0.1]. It follows that a non-connective geometric stack that belongs to $\text{Shv}_{\text{fpqc}} \subseteq \text{Shv}_{\text{nc}}$ is precisely a geometric stack. For another variant, we can substitute $\text{CAlg} \to \text{CAlg}^{\circ}$ in Definition 1.3.1 to obtain the notion of an ordinary geometric stack.

**Remark 1.3.4.** As discussed in [SAG] Subsection 9.1.6., the inclusions $\text{Shv}_{\text{fpqc}} \to \text{Shv}_{\text{nc}}$, as well as its right adjoint $X \mapsto X^{\circ}$, both preserve the condition of a stack begin geometric.

Geometric stacks in either of the three variants - non-connective, connective, and ordinary - admit presentations as quotients of flat affine groupoids in their respective setting.

**Proposition 1.3.5.** A functor $X$ in $\text{Shv}_{\text{nc}}$ is a non-connective geometric stack (resp. geometric stack, ordinary geometric stack) if and only if it can be written as a geometric realization $X \simeq |\text{Spec}(A^n)|$ of a groupoid object of the form $\text{Spec}(A^n)$ with $A^n \in \text{CAlg}$ (resp. $\text{CAlg}^{\text{cn}}$, $\text{CAlg}^{\circ}$) for all $n \geq 0$, and such that all of its face maps are faithfully flat.

**Proof.** For a non-connective geometric stack $X$, take a $\text{Spec}(A^n)$ to be the Čech nerve $\check{C}^*(\text{Spec}(A)/X)$ of a faithfully flat map $\text{Spec}(A) \to X$, whose existence is guaranteed by Definition 1.3.1. The induced map $|\text{Spec}(A^n)| \to X$ being an equivalence follows form the abstract nonsense of Lemma 1.3.6.

For the converse direction, the proof of [SAG] Corollary 9.3.1.4 goes through in the non-connective and ordinary setting, since the proof of crucial [SAG] Lemma 9.3.1.1] makes no use of the connectivity hypothesis.

The following simple result could certainly be justified by invoking appropriate passages from [HTT], but we prefer to give a direct and straightforward proof instead.

**Lemma 1.3.6.** Let $X \to Y$ be a a map in a sheaf $\infty$-topos $\mathcal{X} = \text{Shv}(\mathcal{C})$, such that for every map $j(C) \to Y$ with $C \in \mathcal{C}$, the fiber product $j(C) \times_Y Y$ is representable by some $C' \in \mathcal{C}$, and the induced morphism $C \to C'$ is a one-element cover in the topology on $\mathcal{C}$. Then $X \to Y$ is an effective epimorphism, which is to say that the induced map $|\check{C}^*(X/Y)| \to Y$ is an equivalence in $\mathcal{X}$.

**Proof.** Write $Y \simeq \lim_{\to C(C)} j(C)$. Then $X \times_Y j(C) \simeq j(C'/C)$, and the morphisms $C_i \to C'$ are covers in the topology on $\mathcal{C}$ for every $i$. Thus $j(C_i) \to j(C'_C)$ is an effective epimorphism in $\mathcal{X}$ essentially by the definition of $\infty$-categorical sheaves, see [HTT] Proposition 6.2.3.20, hence $|\check{C}^*(j(C'/C))| \to (C_i)$ is an equivalence for every $i$. Furthermore, it follows from the definition of the Čech nerve that it satisfies base-change, so that the canonical map of simplicial objects

$$\check{C}^*(X/Y) \times_Y j(C) \to \check{C}^*(j(C'/C))$$
is an equivalence. Finally we use all of the discussed facts to exhibit the canonical map $|\mathcal{C}^\bullet(X/Y)| \to Y$ as a composition of equivalences in $\mathcal{X}$

$$|\mathcal{C}^\bullet(X/Y)| \cong |\mathcal{C}^\bullet(X/Y)| \times_Y \lim_i j(C_i)$$

$$\cong \lim_i |\mathcal{C}^\bullet(X/Y)| \times_Y j(C_i)$$

$$\cong \lim_i |\mathcal{C}^\bullet(j(C_i)/j(C_i))|$$

$$\cong \lim_i j(C_i) \cong Y,$$

where the one additional fact we used is that pullbacks commute with arbitrary colimits in an $\infty$-topos. \[\square\]

**Corollary 1.3.7.** For any non-connective geometric stack $X$, the connective cover $\tau_{\geq 0}(X)$ is a geometric stack and the underlying ordinary stack $X^0$ is an ordinary geometric stack. Given a groupoid presentation $X \cong |\text{Spec}(A^\bullet)|$ as in Proposition [1.3.3], there are canonical equivalences $\tau_{\geq 0}(X) \cong |\text{Spec}(\tau_{\geq 0}(A^\bullet))|$ and $X^0 \cong |\text{Spec}(\pi_0(A^\bullet))|$.\[\square\]

**Proof.** This follows from Proposition [1.3.3] and Remark [1.2.3] \[\square\]

### 1.4. Quasi-coherent sheaves

Next, we turn our attention to quasi-coherent sheaves in non-connective spectral algebraic geometry.

**Definition 1.4.1.** The functor of quasi-coherent sheaves $\text{Qcoh} : (\text{Shv}_{\text{fqc}}^\text{nc})^{\text{op}} \to \text{CAlg}(\text{P}^1)$ is defined by right Kan extension from the association $A \mapsto \text{Mod}_A$. That is to say, it is given by

$$\text{Qcoh}\left(\lim_i \text{Spec}(A_i)\right) \cong \lim_i \text{Mod}_{A_i}.$$

**Remark 1.4.2.** The above definition works just as well for functors $\text{CAlg} \to S$ that fail to satisfy fpqc descent, but that gains no extra generality since it is invariant under sheafification. Indeed, the proof of the analogous claim in the connective context [SAG Proposition 6.2.3.1] does not use the connectivity hypothesis.

**Remark 1.4.3.** As noted in [SAG Remark 6.2.2.2], the restriction of the functor $\text{Qcoh}$ onto the subcategory $\text{Shv}_{\text{fqc}}^\text{nc} \subseteq \text{Shv}_{\text{fqc}}^\text{nc}$ is equivalent to the definition of quasi-coherent sheaves on a functor in [SAG Definition 6.2.2.1]. This may for instance be seen by noting that it sends $\text{Spec}(A) \mapsto \text{Spec}(A)$ for any connective $E_\infty$-ring $A$, and commutes with colimits by Proposition [1.2.1], hence the Kan extension definition of the functor $\text{Qcoh}$ in both the connective and non-connective context implies that it is preserved under the subcategory inclusion $\text{Shv}_{\text{fqc}}^\text{nc} \subseteq \text{Shv}_{\text{fqc}}^\text{nc}$.

Given a morphism $f : X \to Y$ in $\text{Shv}_{\text{fqc}}^\text{nc}$, we obtain from the definition of quasi-coherent sheaves an adjunction

$$f^* : \text{Qcoh}(Y) \rightleftarrows \text{Qcoh}(X) : f_*,$$

whose left adjoint $f^*$ we call *pullback along* $f$, and whose right adjoint $f_*$ we call *pushforward along* $f$.

**Examples 1.4.4.** Let $p : X \to \text{Spec}(S)$ be the terminal map. Under the equivalence $\text{Qcoh}(\text{Spec}(S)) \cong \text{Sp}$, we use for any $F \in \text{Qcoh}(X)$ and $M \in \text{Sp}$ traditional notation

$$\mathcal{O}_X \otimes M := p^*(M), \quad \Gamma(X; F) := p^*(F),$$

and the terminology *global sections* and *constant quasi-coherent sheaf* respectively. For $M = S$, we obtain the *structure sheaf* $\mathcal{O}_X := p^*(S)$. Because quasi-coherent pullback is symmetric monoidal by construction, $\mathcal{O}_X$ is the monoidal unit for the canonical symmetric
monoidal operation $\otimes_{O_X}$ on $\text{QCoh}(X)$. Its global sections $O(X) := \Gamma(X; O_X)$ are $E_\infty$-ring of functions on $X$, and chasing through the definitions shows it may be computed as

$$O \left( \lim_i \text{Spec}(A_i) \right) \simeq \lim_i A_i.$$  

**Remark 1.4.5.** The global functions functor $O : (\text{Shv}^\text{fpqc}_n)^{\text{op}} \to \text{CAlg}$, introduced in Example 1.4.4 is right adjoint to the fully faithful embedding $\text{Spec} : \text{CAlg} \to (\text{Shv}^\text{fpqc}_n)^{\text{op}}$ discussed in Remark 1.2.3. The limit preservations of right adjoints gives rise to the formula (3). Even for a connective or ordinary fpqc stack $X$, the $E_\infty$-ring $O(X)$ might still be a non-connective $E_\infty$-ring. Indeed, for an ordinary stack, e.g. an ordinary scheme, the homotopy groups $\pi_n(\text{O}(X))$ agree with quasi-coherent sheaf cohomology $H^n(X; O_X)$ for all $i \in \mathbb{Z}$, which is often non-zero for various values of $i > 0$.

**Remark 1.4.6.** The notation $O_X \otimes M$, introduced in Example 1.4.4 is justified in the following way. Since $\text{QCoh}(X)$ is a stable stable $\infty$-category, it is automatically tensored over the $\infty$-category of spectra $\text{Sp}$. Consequently we can form tensoring with a spectrum $M$ for any object $\mathcal{F} \in \text{QCoh}(X)$ to obtain $\mathcal{F} \otimes M \in \text{QCoh}(X)$, determined completely by demanding that the functor $M \mapsto \mathcal{F} \otimes M$ commutes with colimits and that $\mathcal{F} \otimes S \simeq \mathcal{F}$. Because quasi-coherent pullback $p^* : \text{Sp} \to \text{QCoh}(X)$ preserves colimits, and satisfies $f^*(S) \simeq O_X$ on account of being symmetric monoidal, it follows that it is indeed of the form $f^*(M) \simeq O_X \otimes M$ in terms of the tensoring over $\text{Sp}$.

**Definition 1.4.7.** Let $X$ be a non-connective geometric stack, let $\mathcal{F} \in \text{QCoh}(X)$ be a quasi-coherent sheaf on it, and let $n \in \mathbb{Z}$ be an integer. The $n$-th homotopy sheaf of $\mathcal{F}$ is a quasi-coherent sheaf $\pi_n(\mathcal{F}) \in \text{QCoh}(X^\circ)$, defined by the following property:

\[(\ast)\] For some (and consequently any) choice of a faithfully flat cover $f : \text{Spec}(A) \to X$, there is an equivalence

$$f^\circ)^*(\pi_n(\mathcal{F})) \simeq \pi_n(f^*(\mathcal{F}))$$

in $\text{Mod}_{\pi_0(A)}$.

**Remark 1.4.8.** If we choose a groupoid presentation $X \simeq |\text{Spec}(A^\ast)|$ as in Proposition 1.3.7 and the quasi-coherent sheaf $\mathcal{F} \in \text{QCoh}(X)$ corresponds under the equivalence of $\infty$-categories $\text{QCoh}(X) \simeq \text{Tot}(\text{Mod}_{A^\ast})$ to the system of modules $(M^\ast \in \text{Mod}_{A^\ast})$, then the homotopy sheaf $\pi_n(\mathcal{F}) \in \text{QCoh}(X^\circ)$ corresponds to the system $(\pi_n(M^\ast) \in \text{Mod}_{\pi_0(A^\ast)})$ under the equivalence of categories $\text{QCoh}(X^\circ) \simeq \text{Tot}(\text{Mod}_{\pi_0(A^\ast)})$.

**Remark 1.4.9.** Note that, in light of Proposition 1.3.7 and Definition 1.4.7 we have $\pi_0(\text{O}(X)) \simeq \text{O}(X^\circ)$ for any non-connective geometric stack $X$.

Next we show that the pullback property of homotopy sheaves, used to define them above, holds more generally than just for a faithfully flat cover. That in particular implies the claimed independence of the definition from the choice of the cover.

**Lemma 1.4.10.** Let $f : X \to Y$ be an affine flat morphism of non-connective geometric stacks. For any $\mathcal{F} \in \text{QCoh}(Y)$ and any $t \in \mathbb{Z}$, there is a canonical isomorphism $\pi_t(f^*(\mathcal{F})) \simeq (f^\circ)^*(\pi_t(\mathcal{F}))$ in the $\infty$-category $\text{QCoh}(X^\circ)$.

**Proof.** Choose a flat affine cover $i : \text{Spec}(A) \to Y$. Since $f$ is affine, $j : \text{Spec}(A) \times_Y X \simeq \text{Spec}(B) \to X$ is a flat affine cover too, and by the flatness of $f$, the $E_\infty$-ring map $A \to B$ is flat.

By definition, the homotopy sheaf $\pi_t(\mathcal{F}) \in \text{QCoh}(X^\circ)$ is determined by satisfying the condition $(i^\circ)^*(\pi_t(\mathcal{F})) \simeq \pi_t(i^*(\mathcal{F}))$. To show that $(f^\circ)^*(\pi_t(\mathcal{F}))$ satisfies the analogous
defining property of $\pi_t(f^*(\mathcal{F}))$, we consider the chain of equivalences
\[(j^\circ)^*(f^\circ)^*(\pi_t(\mathcal{F})) \cong \pi_0(B) \otimes_{\pi_0(A)} (i^\circ)^*(\pi_t(\mathcal{F})) \]
\[\cong \pi_0(B) \otimes_{\pi_0(A)} \pi_t(i^\circ(\mathcal{F})) \]
\[\cong \pi_t(A \otimes_B i^*(\mathcal{F})) \]
\[\cong \pi_t(f^*(\mathcal{F})),\]
of which the first equivalence uses commutativity of the square
\[
\begin{array}{ccc}
\text{Spec}(\pi_0(B)) & \to & \text{Spec}(\pi_0(A)) \\
\downarrow_{j^\circ} & & \downarrow_{f^\circ} \\
X^\circ & \to & Y^\circ,
\end{array}
\]
(which is a pullback square thanks to the flatness hypothesis on $f$), the second equivalence is the already-discussed defining property of homotopy sheaves, the third equivalence is due to the flatness of the $E_\infty$-ring map $A \to B$, and the final equivalence uses an analogous commuting square to the one we displayed above, but removing $\triangledown$ and $\pi_t$.

\[\square\]

**Proposition 1.4.11.** Let $X$ be a non-connective geometric stack. For any quasi-coherent sheaf $\mathcal{F} \in \text{Q Coh}(X)$, there exists an Adams-graded spectral sequence
\[E_2^{s,t} = \text{H}^s(X^\circ; \pi_t(\mathcal{F})) \Rightarrow \pi_{t-s}(\Gamma(X; \mathcal{F})),\]
called the descent spectral sequence. The second page is Čech cohomology for ordinary quasi-coherent sheaves on $X^\circ$.

**Proof.** Choosing a presentation $X \cong |\text{Spec}(A^\bullet)|$ as in Proposition 1.3.5, this is the Bousfield-Kan spectral sequence of the cosimplicial spectrum $\Gamma(\text{Spec}(A^\bullet); \mathcal{F}|_{\text{Spec}(A^\bullet)})$. To identify the second and infinite page as in the statement, repeat the proof of [Gri21a, Lemma 3.1].

\[\square\]

1.5. **Quasi-coherent sheaves in the connective setting.** The construction of the descent spectral sequence given above depends on the choice of a faithfully flat (hyper)cover. We wish to give an alternative construction of it that is manifestly independent of such a choice, but instead makes use of a $t$-structure (see Construction 1.5.7). Since the $\infty$-category $\text{Mod}_A$ of modules over a non-connective $E_\infty$-ring does not carry a canonical $t$-structure, we can not expect one on $\text{Q Coh}(X)$ for a non-connective geometric stack either. There are no such issues in the connective setting however.

**Proposition 1.5.1.** For any geometric stack $X$, this defines a right and left complete $t$-structure compatible with filtered colimits on the stable $\infty$-category $\text{Q Coh}(X)$.

**Proof.** Though we defined it slightly differently, it follows from [SAG] Remark 9.1.3.4 that this $t$-structure on $\text{Q Coh}(X)$ coincides with the one studied in [SAG] Subsection 9.1.3]. Now the result we are after follows from [SAG] Corollary 9.1.3.2.

**Remark 1.5.2.** Picking a presentation $X \cong |\text{Spec}(A^\bullet)|$ as in Proposition 1.3.5, the $t$-structure in question is explicitly given by $\text{Q Coh}(X)^{\geq n} \cong \text{Tot}(\text{Mod}_A^{\geq n})$. Note that this is sensible because all the degeneracy maps $A^n \to A^m$ are flat, and smash product along flat maps of connective $E_\infty$-rings is left $t$-exact. We thus see that the $t$-structure on quasi-coherent sheaves is induced via affines from the usual $t$-structure on module $\infty$-categories over connective $E_\infty$-rings of [HA] Proposition 7.1.1.13.]

**Remark 1.5.3.** Pullback along the map from the underlying ordinary stack $X^\circ \to X$ induces an equivalence on the heart of the $t$-structure $\text{Q Coh}(X)^{\circ} \cong \text{Q Coh}(X^\circ)^{\circ}$. The latter may be thought as the ordinary abelian category of quasi-coherent sheaves on the underlying ordinary geometric stack $X^\circ$. In fact, even when $X$ is non-connective, it is still clear that $\pi_n(\mathcal{F}) \in \text{Q Coh}(X^\circ)^{\circ}$ for every quasi-coherent sheaf $\mathcal{F} \in \text{Q Coh}(X)$ and $n \in \mathbb{Z}$. 

Remark 1.5.4. Given any geometric stack $X \simeq |\text{Spec}(A^\bullet)|$, presented as in Proposition 1.3.5, we may define via descent

$$\text{QCoh}^\circ(|\text{Spec}(A^\bullet)|) \simeq \text{Tot}(\text{Mod}^\circ_{\tau_0(A^\bullet)}).$$

This gives rise to an abelian category $\text{QCoh}^\circ(X^\circ)$, equivalent to the heart $\text{QCoh}(X^\circ)^\circ \simeq \text{QCoh}(\tau_0(X))^\circ$ of the $t$-structure under discussion. In particular, we view $\text{QCoh}^\circ(X^\circ)$ as the category of ordinary quasi-coherent sheaves on the ordinary geometric stack $X^\circ$.

To make use of the $t$-structure on quasi-coherent sheaves, we must pass from a non-connective geometric stack $X$ to its connective cover $\tau_0(X)$. That is to say, we consider pushforward $\text{QCoh}(X) \to \text{QCoh}(\tau_0(X))$ along the canonical map $X \to \tau_0(X)$. We will abuse notation and not notationally distinguish between a quasi-coherent sheaf on $X$ and its pushforward in $\tau_0(X)$. Indeed, said pushforward may be thought as a forgetful functor, as the following result shows.

Proposition 1.5.5. Let $X$ be a non-connective geometric stack. The map $X \to \tau_0(X)$ induces an equivalence of $\infty$-categories

$$\text{QCoh}(X) \simeq \text{Mod}_{\tau_0}^0(\text{QCoh}(\tau_0(X))).$$

Proof. Denoting the connective cover map by $c : X \to \tau_0(X)$, we must show that the adjunction induced on quasi-coherent sheaves

$$c^* : \text{QCoh}(\tau_0(X)) \rightleftarrows \text{QCoh}(X) : c_*$$

is monadic. Choosing a presentation $X \simeq |\text{Spec}(A^\bullet)|$ as in Proposition 1.3.5, we get by Corollary 1.3.7 that $X \simeq |\text{Spec}(\tau_0(A^\bullet))|$. The adjunction above is induced on totalizations from the adjunction of cosesomimisplicial functors

$$A^\bullet \otimes_{\tau_0(A^\bullet)} - : \text{Mod}_{\tau_0(A^\bullet)} \rightleftarrows \text{Mod}_{A^\bullet} : A^\bullet,$$

induced upon totalizations. In each degree separately, these adjunctions are monadic, as special case of the basic fact that any map of $\mathbb{E}_\infty$-rings $A \to B$ induces an equivalence of $\infty$-categories $\text{Mod}_B(\text{Mod}_A) \simeq \text{Mod}_B$. Because all the face maps $A^i \to A^j$ of the simplicial $\mathbb{E}_\infty$-ring $A^\bullet$ are flat, the commutative diagram

$$\begin{array}{ccc}
\tau_0(A^i) & \longrightarrow & \tau_0(A^j) \\
\downarrow & & \downarrow \\
A^i & \longrightarrow & A^j
\end{array}$$

is in fact a pushout square of $\mathbb{E}_\infty$-rings. This implies that the induced maps on the $\infty$-categories of modules are adjointable in the sense of [HA, Definition 4.7.4.13] (said differently: satisfy the Beck-Chevalley condition), reducing the proof of monadicity to the following general Lemma.

Lemma 1.5.6. Consider a small diagram $\mathcal{J} \to \text{Fun}(\Delta^1, \mathcal{P}_1^\mathbb{L})$, i.e. a collection of adjunction of presentable $\infty$-categories $F_i : \mathcal{C}_i \rightleftarrows \mathcal{D}_i : G_i$ for all $i \in \mathcal{J}$, and morphisms $f_{ij} : \mathcal{C}_i \to \mathcal{C}_j$ and $g_{ij} : \mathcal{D}_i \to \mathcal{D}_j$ for any morphism $i \to j$ in $\mathcal{J}$, such that all the diagrams of $\infty$-categories

$$\begin{array}{ccc}
\mathcal{C}_i & \xrightarrow{f_{ij}} & \mathcal{D}_i \\
\downarrow f_{ij} & & \downarrow g_{ij} \\
\mathcal{C}_j & \xrightarrow{g_{ij}} & \mathcal{D}_j
\end{array}$$

$$\begin{array}{ccc}
\mathcal{C}_i & \xleftarrow{G_i} & \mathcal{D}_i \\
\downarrow f_{ij} & & \downarrow g_{ij} \\
\mathcal{C}_j & \xleftarrow{G_j} & \mathcal{D}_j
\end{array}$$

commute (i.e. the commutative squares are right and left adjointable respectively). Let us denote the limit of the functor $\mathcal{J} \to \text{Fun}(\Delta^1, \mathcal{P}_1^\mathbb{L})$ by $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$. Suppose that each adjunction $F_i : \mathcal{C}_i \rightleftarrows \mathcal{D}_i : G_i$ is monadic for every $i \in \mathcal{J}$. Then the adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is also monadic.
Proof. Let us first observe that, since the forgetful functor $\mathcal{P}_t^L \in \text{Cat}_{\infty}$, of presentable $\infty$-categories (with colimit-preserving functors) into all $\infty$-categories, preserves limits, we have canonical equivalences of $\infty$-categories $\mathcal{C} \cong \varprojlim_{i \in I} \mathcal{C}_i$ and $\mathcal{D} \cong \varprojlim_{i \in I} \mathcal{D}_i$. Thanks to the adjointability of the adjunctions between $F_i$ and $G_i$, the functor $G : \mathcal{D} \to \mathcal{C}$ is induced from the functors $G_i : \mathcal{D}_i \to \mathcal{C}_i$ by [HA Proposition 4.7.4.19].

To prove that the adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is monadic, we must by the Barr-Beck Theorem [HA Theorem 4.7.3.5] show that the functor $G : \mathcal{D} \to \mathcal{C}$ is conservative, and preserves $G$-split totalizations. Recalling the definition of split simplicial objects from [HA Definition 4.7.2.2], the fact that $\text{Fun}((\Delta_{\infty})^{op}, \mathcal{C}) \cong \varprojlim_{i \in I} \text{Fun}((\Delta_{\infty})^{op}, \mathcal{C}_i)$ implies that a $G$-split simplicial object $X^\bullet$ in $\mathcal{C}$ corresponds to a functorial collection of $G_i$-split simplicial objects $X_i^\bullet$ in $\mathcal{C}_i$. Since each $G_i$ preserves the totalization of $X_i^\bullet$, it follows that $G$ preserves the totalization of $X^\bullet$. To show that $G$ is conservative, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{G} & \mathcal{C} \\
\downarrow & & \downarrow \\
\varprojlim_{i \in I} \mathcal{D}_i & \xrightarrow{\varprojlim_{i \in I} G_i} & \varprojlim_{i \in I} \mathcal{C}_i \\
\Pi_{i \in I} \mathcal{D}_i & \xrightarrow{\Pi_{i \in I} G_i} & \Pi_{i \in I} \mathcal{C}_i,
\end{array}
$$

Here the unlabeled arrows are the usual inclusions of limits into products, which is to say, the functor induced on limits by the inclusion $\text{ob}(\mathcal{I}) \subseteq \mathcal{I}$ of the set of objects into the indexing $\infty$-category $\mathcal{I}$. These functors are always conservative, and since each $G_i$ are conservative for all $i \in \mathcal{I}$, so is $\prod_{i \in I} G_i$. It therefore follows from the commutativity of the above diagram that $G$ must also be conservative. \qed

It follows in particular from (the proof of) Proposition [1.5.5] that the homotopy sheaves $\pi_i(\mathcal{F})$ agree regardless of whether we start with $\mathcal{F} \in \text{QCoh}(X)$, or if we take its push-forward to $\text{QCoh}(\tau_{\geq 0}(X))$. With that, we may give a construction of the descent spectral sequence that makes no reference to a choice of a flat (hyper)cover:

**Construction 1.5.7.** Let $X$ be a non-connective geometric stack, and let $\mathcal{F} \in \text{QCoh}(X)$. Viewing $\mathcal{F}$ as a quasi-coherent sheaf on the geometric stack $\tau_{\geq 0}(X)$, the Postnikov tower for the $t$-structure on $\text{QCoh}(\tau_{\geq 0}(X))$ gives rise to the filtered object

$$Z \ni n \mapsto \Gamma(\tau_{\geq 0}(X); \tau_{-n}(\mathcal{F})) \in \text{QCoh}(\tau_{\geq 0}(X)).$$

Its associated spectral sequence is of the form (in the homological grading)

$$E_1^{p,q} = \pi_{p+q}(\Gamma(\tau_{\geq 0}(X); \Sigma^p(\pi_{-p}(\mathcal{F})))) \Rightarrow \pi_{p+q}(\Gamma(X; \mathcal{F})).$$

By re-grade via $s = -(2p + q)$, $t = -p$, and $r \mapsto r + 1$, the spectral sequence is brought into (an Adams-graded) form

$$E_2^{s,t} = \pi_{-s}(\Gamma(\tau_{\geq 0}(X); \pi_t(\mathcal{F}))) \Rightarrow \pi_{-s}(\Gamma(X; \mathcal{F})).$$

Note that the homotopy sheaves $\pi_t(\mathcal{F})$ actually belong to $\text{QCoh}(X^\circ)$, and are being secretly pushed forward along $X^\circ \to \tau_{\geq 0}(X)$. Thus $\Gamma(\tau_{\geq 0}(X); \pi_t(\mathcal{F})) \cong \Gamma(X^\circ; \pi_t(\mathcal{F}))$, and since $\pi_t(\mathcal{F}) \in \text{QCoh}(X^\circ)^\circ$, this is just the complex computing sheaf cohomology of $\pi_t(\mathcal{F})$ on the ordinary stack $X^\circ$. With that, the spectral sequence becomes

$$E_2^{s,t} = H^s(X^\circ; \pi_t(\mathcal{F})) \Rightarrow \pi_{-s}(\Gamma(X; \mathcal{F})).$$

Under some light assumptions on $X^\circ$, which ensure that Čech cohomology agrees with derived-functor cohomology for quasi-coherent sheaves, the spectral sequence thus obtained is equivalent to the descent spectral sequence of Proposition [1.4.11]. For a proof of such a claim in a related setting, see [Ant1].
2. Moduli of formal groups and chromatic homotopy theory

Having taken the time to set up the necessary basics of non-connective spectral algebraic geometry, we now apply it to study a non-connective stack of particular interest to chromatic homotopy theory.

2.1. Formal groups in spectral algebraic geometry. Before getting to that though, we review the theory of formal groups over $\mathbb{E}_{\infty}$-rings, as developed in [Ell2, Chapter 1], and needed in the rest of this paper. See loc. cit. for details and a more complete discussion.

Definition 2.1.1 ([Ell2 Definition 1.2.4]). A smooth coalgebra of dimension $r$ over an $\mathbb{E}_{\infty}$-ring $A$ is a cocommutative coalgebra object in the $\infty$-category of flat $A$-modules

$$C \in \mathrm{cCAlg}_A^\flat \cong \mathrm{CAlg}((\mathrm{Mod}_A^\flat)^{\mathrm{op}})^{\mathrm{op}},$$

such that there exists a projective $\pi_0(A)$-module $E$ of finite rank $r$, and an isomorphism $\pi_0(C) \cong \Gamma^{\ast}_{\pi_0(A)}(E)$ of coalgebras over $\pi_0(A)$. Here $\Gamma^{\ast}_{\pi_0(A)}(E)$ denotes the free divided power coalgebra, see for instance [Ell2 Construction 1.1.11]. Smooth coalgebras over $A$ form a full subcategory $\mathrm{cCAlg}_A^{\mathrm{sm}} \subseteq \mathrm{cCAlg}_A^\flat$.

The $\infty$-category of smooth coalgebras $\mathrm{cCAlg}_A^{\mathrm{sm}}$ may be viewed as formal hyperplanes over $A$, i.e. an incarnation of smooth formal varieties. Formal groups should therefore be defined to be some sort of commutative algebra objects in this $\infty$-category. But we must be careful about picking the correct sort.

Definition 2.1.2 ([Ell1 Definition 1.2.4]). Let $\mathrm{Lat}$ denote the category of lattices, i.e. the full subcategory of abelian groups spanned by $\{\mathbb{Z}^I\}_{I \geq 0}$. Let $\mathcal{C}$ be any $\infty$-category with finite products. An abelian group object in $\mathcal{C}$ is any functor $\mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}$ that preserves finite products. Abelian group objects form the full subcategory $\mathrm{Ab}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}}, \mathcal{C})$.

Remark 2.1.3. Contrast the notion of abelian group objects with the more familiar one of commutative (i.e. $\mathbb{E}_{\infty}$-)monoid objects in an $\infty$-category with finite products $\mathcal{C}$. The latter are given by product-preserving functors $\mathrm{Fin} \to \mathcal{C}$ from the category of finite sets. By pre-composing with the map $\mathrm{Fin} \to \mathrm{Lat}^{\mathrm{op}}$, given by $I \mapsto \mathbb{Z}^I$, we obtain a “forgetful functor” $\mathrm{Ab}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{C})$. For $\mathcal{C}$ the $\infty$-category of spaces $\mathcal{S}$, this recovers the inclusion of topological abelian groups into $\mathbb{E}_{\infty}$-spaces.

Though the definition of formal groups in [Ell2 Definition 1.6.1] exhibits them manifestly as objects of functor-of-points-style algebraic geometry, it will be more convenient for us to use an equivalent “Hopf algebra” definition instead.

Definition 2.1.4 ([Ell2 Remark 1.6.6]). The $\infty$-category of formal groups over an $\mathbb{E}_{\infty}$-ring $A$ is defined to be the $\infty$-category $\mathrm{FGrp}(A) := \mathrm{Ab}(\mathrm{cCAlg}_A^{\mathrm{sm}})$ of abelian group objects in smooth coalgebras over $A$. The full subcategory of all formal groups whose underlying smooth coalgebras have dimension $r$ is denoted $\mathrm{FGrp}_{\mathrm{dim}(r)}(A)$.

Remark 2.1.5. In [Ell2 Variant 1.6.2], formal groups over an $\mathbb{E}_{\infty}$-ring $A$ are defined as formal groups over its connective cover $\tau_{\pi_0}(A)$. But since the extension of scalars $\mathrm{cCAlg}_A^{\mathrm{sm}} \to \mathrm{cCAlg}_{\tau_{\pi_0}(A)}^{\mathrm{sm}}$ is an equivalence of $\infty$-categories by [Ell2 Proposition 1.2.8], it follows that our definition is no less general. Instead, we find that the canonical functor $\mathrm{FGrp}(\tau_{\pi_0}(A)) \to \mathrm{FGrp}(A)$ is an equivalence of $\infty$-categories for any $\mathbb{E}_{\infty}$-ring $A$ as a consequence, rather than as definition.

Examples 2.1.6. Our interest in this paper is restricted to two classes of formal groups:

- Let $A$ be an ordinary commutative ring. If we view it as a discrete $\mathbb{E}_{\infty}$-ring, then formal groups over it, in the sense of Definition 2.1.4, coincide with formal groups over $A$ in the usual sense, e.g. see [Smi11 Subsection 2.6], [Goe08 Section 2] or [Lur10 Lecture 11].
Let $A$ be a complex periodic $\mathbb{E}_\infty$-ring, i.e. complex orientable and such that $\pi_2(A)$ is a locally free $\pi_0(A)$ module of rank 1. The Quillen formal group of $A$, denoted $\mathcal{G}_A^\infty$, is in terms of Definition 2.1.4 defined to be the smooth coalgebra $C_*(\mathbb{CP}^\infty; A)$ over $A$. The abelian group object structure is inherited from the topological abelian group structure on $\mathbb{CP}^\infty$; see [Ell2 Subsection 4.1.3] for a proof that this actually defines a formal group over $A$.

Definition 2.1.7 ([Ell2 Definition 4.3.9]). Let $\mathcal{G}$ be a 1-dimensional formal group over an $\mathbb{E}_\infty$-ring $A$. Let $\omega_0$ be the dualizing line of $\mathcal{G}$, in the sense of [Ell2 Definition 4.2.14]. An orientation on $\mathcal{G}$ is an equivalence $\omega_0 \simeq \Sigma^{-2}(A)$ in the $\infty$-category $\text{Mod}_A$, coming via the linearization construction of [Ell2 Construction 4.2.9] from the choice of an element in $\pi_2(\mathcal{G}(\tau_{\infty}(A))$ (which is also part of the orientation data). Let $\text{FGrp}^\text{or}(A)$ denote the $\infty$-category of oriented formal groups over $A$.

Remark 2.1.8. The notion of an oriented formal group may be motivated as follows. Suppose that $X$ is a non-connector spectral stack, classifying formal groups with perhaps some additional structure over $\mathbb{E}_\infty$-rings. Assume that $X$ satisfies conditions [1] - [iv] of Theorem [9] from the Introduction. In light of the isomorphism [1], the condition [iv] amounts to demanding the equality of quasi-coherent sheaf cohomology groups

$$H^s(X^\varnothing; \pi_2(\mathcal{O}_X)) \simeq H^s(M_{\text{FG}}^\varnothing; \omega_0^\otimes t)$$

for all $s \geq 0, t \in \mathbb{Z}$. Since we have by [ii] an equivalence of ordinary stacks $X^\varnothing \simeq M_{\text{FG}}^\varnothing$, this will be satisfied if there are isomorphisms of (usual) quasi-coherent sheaves

$$\pi_2(\mathcal{O}_X) \simeq \omega_0^\otimes t$$

for all $t \in \mathbb{Z}$. Let $A$ be an $\mathbb{E}_\infty$-ring and $\mathcal{G}$ be a formal group over $A$, which is classified by $X$. The quasi-coherent sheaf isomorphisms discussed above then give rise $\pi_0(A)$-module isomorphisms

$$\pi_0(\Sigma^{-2t}(A)) \simeq \pi_0(\mathcal{G}) \simeq \pi_0(\omega_0^\otimes t) \simeq \pi_0(\omega_0^\otimes)^t$$

for all $t \in \mathbb{Z}$. Fixing $t$, and looking at the left- and right-most terms, we might hope that this $\pi_0(A)$-module isomorphism was a reflection of an equivalence $\Sigma^{-2t}(A) \simeq \omega_0^\otimes t$ on the level of $A$-module spectra. If we had such an equivalence for $t = 1$, we could obtain it for all $t \in \mathbb{Z}$ by smash powers, consequently it suffices to assume that $\Sigma^{-2}(A) \simeq \omega_0^\varnothing$. But of course, that is almost the notion of an orientation on the formal group $\mathcal{G}$, in the sense of Definition 2.1.7 (or equivalently, Definition [1] from the Introduction), ignoring only the technical issue of the isomorphism needing to arise from a homotopy class on $\mathcal{G}$.

Proposition 2.1.9 ([Ell2 Proposition 4.3.23]). Let $\mathcal{G}$ be a 1-dimensional formal group over an $\mathbb{E}_\infty$-ring $A$. Then $\mathcal{G}$ is oriented if and only if $A$ is complex-periodic and $\mathcal{G} \simeq \mathcal{G}_A^\infty$.

Proof sketch. Consider a preoriented formal group $\mathcal{G} \in \text{FGrp}(A)$, i.e. a formal group over $A$ equipped with a map $\beta : \omega_0 \rightarrow \Sigma^{-2}(A)$ in $\text{Mod}_A$, arising via linearization as in [Ell2 Construction 4.2.9] from an element in $\pi_2(\mathcal{G}(\tau_{\infty}(A))).$ Using the facts that that $\omega_0^\varnothing \simeq C^*_\text{red} (S^2; A) \simeq \Sigma^{-2}(A)$, and that $\mathbb{CP}^\infty$ is the free topological abelian group generated by the pointed space $\mathbb{CP}^1 \simeq S^2$, the data of such a map $\beta$ may be seen to be equivalent to a map $\mathcal{G}_A^0 \rightarrow \mathcal{G}$. Since a map of formal groups $\mathcal{G} \rightarrow \mathcal{G}'$ is an equivalence if and only if it induces an equivalence on dualizing lines $\omega_0^\varnothing \rightarrow \omega_0^\varnothing$, it follows that an orientation on $\mathcal{G}$ is indeed equivalent to an equivalence of formal groups $\mathcal{G}_A^\varnothing \simeq \mathcal{G}$. On the other hand, note that the Quillen formal group $\mathcal{G}_A^\infty$ is a 1-dimensional formal group over $A$ if and only if the $\mathbb{E}_\infty$-ring $A$ is complex-periodic.

Remark 2.1.10. For any formal group $\mathcal{G}$ over a complex-periodic $\mathbb{E}_\infty$-ring $A$, the space of formal group maps $\mathcal{G}_A^\infty \rightarrow \mathcal{G}$ is by [Ell2 Proposition 4.3.21] equivalent to the space of
preorientations on $\hat{G}$, i.e. maps $S^2 \to \hat{G}(\tau_{s0}(A))$ whose linearization $\beta: \omega_G \to \Sigma^{-2}(A)$ is not required to be an $A$-linear equivalence. Consequently, the Quillen formal group $\hat{G}_A^Q$ of a complex-periodic $E_{\infty}$-ring $A$ has no automorphisms as on oriented formal group over $A$.

2.2. Descent for formal groups. In this section we will show that various functors $\text{CAlg}_{\infty} \to \text{Cat}_{\infty}$ that we considered in the previous subsection satisfy faithfully flat descent.

**Proposition 2.2.1.** The functor $\text{A} \mapsto \text{FGrp}(A)$ satisfies descent for the fpqc topology.

*Proof.* The functor $\text{A} \mapsto \text{Mod}_{\text{A}}$ satisfies flat descent by [SAG Theorem D.6.3.5], and since flatness if local for the fpqc topology, it follows that $\text{A} \mapsto \text{Mod}_{\text{A}}$ does as well. The same thus holds for $\text{A} \mapsto \text{cCAlg}_{\text{A}}^f \simeq \text{CAlg}(\text{Mod}_{\text{A}})$, and, because the construction of abelian group objects $\mathcal{E} \mapsto \text{Ab}(\mathcal{E})$ from Definition 2.1.2 preserves limits, also for $\text{A} \mapsto \text{Ab}(\text{cCAlg}_{\text{A}}^f)$. Now recall from Definition 2.1.1 that the $\infty$-category of smooth $E_{\infty}$-coalgebras factors as

$$\text{cCAlg}_{\text{A}}^{\text{sm}} \simeq \text{cCAlg}_{\text{A}}^f \times_{\text{cCAlg}_{\text{sg}}^f} \text{cCAlg}_{\text{sg}}^{\text{sm}}(A).$$

Thanks to the already-mentioned fact that the $\mathcal{E} \mapsto \text{Ab}(\mathcal{E})$ commutes with limits, and we may write $\text{FGrp}(A) \simeq \text{Ab}(\text{cCAlg}_{\text{A}}^{\text{sm}})$ by [Ell2 Remark 1.6.6], we find that

$$\text{FGrp}(A) \simeq \text{Ab}(\text{cCAlg}_{\text{A}}^{\text{sm}}) \times_{\text{Ab}(\text{cCAlg}_{\text{sg}}^{\text{sm}}(A))} \text{Ab}(\text{cCAlg}_{\text{sg}}^{\text{sm}}(A)).$$

It now suffices to show that $\text{A} \mapsto \text{Ab}(\text{cCAlg}_{\text{sg}}^{\text{sm}}(A))$ satisfies fpqc descent. By design, the construction $\text{Ab}(\text{cCAlg}_{\text{sg}}^{\text{sm}}(A)) \simeq \text{FGrp}(R)$ recovers the usual category of formal groups for an ordinary commutative ring $R$. This can be outsourced e.g. to [Smi11 Corollary 2.6.6], or [Goe08 Theorem 2.30] (though the assumption there is that we are only considering 1-dimensional formal groups, the argument works just as well for arbitrary finite-dimensional ones), or easily verified directly by using [Ell2 Lemma 1.1.20] - applicable since any Hopf algebra has at least one grouplike element: its multiplicative unit.

**Remark 2.2.2.** The functor $\text{A} \mapsto \text{cCAlg}_{\text{A}}^{\text{sm}}$ only satisfies étale descent, but not fpqc descent. The issue is addressed in [Ell2 Warning 1.12], and boils down to the possibility of a coalgebra having no grouplike elements. As we observed in the proof above however, formal groups correspond to Hopf algebras, for which this issue does not arise.

**Corollary 2.2.3.** The functor $\text{A} \mapsto \text{FGrp}_{\text{dim}=r}(A)$ satisfies descent for the fpqc topology for any $r \geq 0$.

*Proof.* Follows from from Proposition 2.2.1 and fpqc descent for finitely generated projective modules over ordinary commutative rings [EGA4 Proposition 2.5.2] or [Stacks Tag 10.83].

**Proposition 2.2.4.** The functor $\text{A} \mapsto \text{FGrp}_{\text{or}}(A)$ satisfies descent for the fpqc topology.

*Proof.* Consider the forgetful functor $\text{FGrp}_{\text{or}} \to \text{FGrp}_{\text{dim}=1}$. A map Spec$(A) \to \text{FGrp}_{\text{dim}=1}$ is equivalent to specifying a 1-dimensional formal group $\hat{G} \in \text{FGrp}(A)$, and the pullback Spec$(A) \times_{\text{FGrp}_{\text{dim}=1}} \text{FGrp}_{\text{or}}$, viewed as a functor $\text{CAlg}_{\infty} \to \mathcal{S}$, is by definition given by $B \mapsto \text{OrDat}(\hat{G}_B)$. By [Ell2 Proposition 4.3.13] we have $\text{OrDat}(\hat{G}_B) \simeq \text{Map}_{\text{CAlg}_{\infty}}(\text{D}_{\hat{G}}, B)$ for an $E_{\infty}$-algebra $\text{D}_{\hat{G}}$ over $A$. Thus Spec$(A) \times_{\text{FGrp}_{\text{dim}=1}} \text{FGrp}_{\text{or}} \simeq \text{Spec}(\text{D}_{\hat{G}})$ is (representable by) an affine spectral $A$-scheme, and so the map $\text{FGrp}_{\text{or}} \to \text{FGrp}_{\text{dim}=1}$ is affine. Thus $\text{FGrp}_{\text{or}}$ satisfies fpqc descent by virtue of that holding for $\text{FGrp}$ by Corollary 2.2.3.

**Remark 2.2.5.** Since $\text{FGrp}_{\text{or}}(A)$ is either contractible if the $E_{\infty}$-ring $A$ is complex-periodic (see Remark 2.1.10), or empty otherwise, Proposition 2.2.4 is equivalent to the following assertion: if an $E_{\infty}$-ring $A$ admits a faithfully flat $E_{\infty}$-ring map $A \to B$ into a complex-periodic $E_{\infty}$-ring $B$, then $A$ is complex-periodic. The analogous statement for weak 2-periodicity is tautological from the definition of flatness for $E_{\infty}$-rings, but complex orientability is less immediate.
In light of the preceding results, we will refer to the functors $A \mapsto \text{FGp}_0(M_{\text{dim-1}}(A))$ and $A \mapsto \text{FGp}_0(M_{\text{dim-1}}(A)) \simeq \text{FGp}_0(M_{\text{dim-1}}(A))$ as the moduli stack of (resp. oriented) formal groups, and denote them by $M_{\text{FG}}, M_{\text{FG}}^0 \in \text{Sp}_{\text{fgc}}$.

**Remark 2.2.6.** With this terminology, we embrace the tradition in homotopy theory to drop the adjective “1-dimensional” from the term “1-dimensional formal group”, since all the formal groups we will care about will be 1-dimensional.

2.3. **The stack of oriented formal groups.** This section is dedicated to proving the following result:

**Theorem 2.3.1.** The moduli stack of oriented formal groups $M_{\text{FG}}^0$ is a non-connective geometric stack.

We already showed that $M_{\text{FG}}^0$ satisfies fqc descent in Proposition 2.2.4. According to Definition 1.3.1, it remains to prove that its has affine diagonal, and that it admits a faithfully flat cover by an affine. The first of those is easy:

**Lemma 2.3.2.** The diagonal morphism $M_{\text{FG}}^0 \to M_{\text{FG}}^0 \times M_{\text{FG}}^0$ is affine.

**Proof.** For any $E_\infty$-ring $A$, there exists an essentially unique map $\text{Spec}(A) \to M_{\text{FG}}^0$ if and only if $A$ is complex-periodic. It follows from this that, for any pair of complex-periodic $E_\infty$-rings $A$ and $B$, the fibered product $\text{Spec}(A) \times_{M_{\text{FG}}^0} \text{Spec}(B)$ is equivalent to $\text{Spec}(A \otimes B)$. By Remark 1.3.2, this is what we needed to show. \□

For the purpose of finding an affine atlas for $M_{\text{FG}}^0$, we introduce a distinguished class of $E_\infty$-rings. Recall e.g. from [HY19, Remark 2.2] that the 2-periodic spectrum $S[\beta^+1] := \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n}(S)$ carries a canonical $\mathbb{E}_2$-ring structure by [HY19, Remark 2.2].

**Definition 2.3.3.** An $E_\infty$-ring is a form of periodic complex bordism if it is equivalent to $\text{MU} \otimes S[\beta^+1]$ as an $\mathbb{E}_2$-ring.

**Examples 2.3.4.** The following constructions all give rise to forms of periodic complex bordism, and by [HY19, Theorem 1.3], at least the first two are not equivalent to each other as $E_\infty$-rings.

- The Thom spectrum $M_{\text{UP}} \simeq \lim_{\to} (BU \times \mathbb{Z} \cong (\text{Vect}_V)_{\text{sp}} \xrightarrow{J} \text{Pic}(\mathbb{P}) \subseteq \text{Sp})$, given by the colimit of the map induced upon group completion from the symmetric monoidal functor $\text{Vect}_V \to \text{Pic}(\mathbb{P})$ given by $V \mapsto S^V$. This is equivalent to the $E_\infty$-ring appearing under the name MP in [E12].
- Snaith’s construction $(S[BU])[\beta^{-1}]$, obtained by inverting the Bott element $\beta \in \pi_2(S[BU])$, induced from $S^2 \cong \mathbb{CP}^1 \subseteq \mathbb{CP}^\infty \simeq BU(1) \to BU$, in the suspension spectrum $S[BU]$, which inherits the structure of an $E_\infty$-ring from the infinite loop space structure on $BU \simeq \Omega^\infty(\Sigma^2(\text{ku}))$.
- The Tate spectrum $MU\langle \Sigma^1 \rangle$, with its $E_\infty$-ring structure induced via the lax symmetric monoidality of the Tate construction from the usual Thom spectrum $E_\infty$-ring structure on the complex bordism spectrum $MU \simeq \lim_{\to} (BU \xrightarrow{J} \text{Sp})$. According to [HY19, Remark 1.13], this form of periodic complex bordism was suggested by Tyler Lawson.

From now on, and throughout the rest of this note, we fix MP to be an arbitrary form of periodic complex bordism.

**Lemma 2.3.5.** There exists an essentially unique map $\text{Spec}(MP) \to M_{\text{FG}}^0$, and this map is faithfully flat.
Proof. The existence of a map \( \text{Spec}(\mathcal{M}) \to \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) is equivalent to \( \mathcal{M} \) being complex-periodic, a condition which only depends on the underlying commutative ring spectrum. The complex bordism spectrum \( \mathcal{M} \) is the universal complex orientable spectrum, and the spectrum \( S[\beta^{+1}] \) is 2-periodic, implying that both of those hold for the smash product \( \mathcal{M} \otimes S[\beta^{+1}] \approx \mathcal{M} \) as well.

To show that the map in question is faithfully flat, it suffices to prove that after pullback along an arbitrary map \( \text{Spec}(A) \to \mathcal{M}_{\mathcal{F}G}^{\text{or}} \). As we saw in the proof of Lemma 2.3.2, this means proving that assertion the map \( A \to \mathcal{M} \otimes \mathcal{M} \) is faithfully flat for every complex-periodic \( E_{\infty} \)-ring \( A \). That is the content of [Eli2 Theorem 5.3.13], but it is really just the result of a standard computation in chromatic homotopy theory; that of the \( \mathcal{M} \) homology of a complex oriented ring spectrum [Rav04 Lemma 4.1.7, Lemma 4.1.8, and Corollary 4.1.9], [Lur10 Lecture 7], or [Hop99 Proposition 6.2], □

Remark 2.3.6. In [Eli2 Theorem 5.3.13], the claim we used in the proof above is only asserted for a specific choice of a periodic form of complex bordism \( \mathcal{M} \approx \mathcal{M}^{\text{P}} \). But since being faithfully flat is a condition fully determined on the level of the underlying homotopy-commutative ring spectra, that restriction makes no difference.

With that, Theorem 2.3.1 is proven, and we may begin to reap the rewards.

Corollary 2.3.7. There is a canonical equivalence

\[ \mathcal{M}_{\mathcal{F}G}^{\text{or}} \simeq |\tilde{C}(\text{Spec}(\mathcal{M}))/\text{Spec}(S)|, \]

with the geometric realization formed in the \( \infty \)-topos \( \text{Shv}^{\text{nc}}_{fqc} \).

Proof. This is an instance of Proposition 1.3.5 or more specifically, an application of Lemma 1.3.6. □

Remark 2.3.8. The Čech groupoid appearing in the statement of Corollary 2.3.7 is the so-called Amitsur complex. It encodes the descent data along the terminal map of stacks \( \text{Spec}(\mathcal{M}) \to \text{Spec}(S) \). But since this map is not flat itself (i.e. \( \mathcal{M} \) is not a flat spectrum), descent along it does not return \( \text{Spec}(S) \) itself, but rather \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) as we saw.

Corollary 2.3.9. The canonical map \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \to \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) induces an equivalence on underlying ordinary stacks. In particular, \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) is the ordinary moduli stack of formal groups.

Proof. It follows from Corollary 2.3.7 and Corollary 1.3.7 that the underlying ordinary stack of \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) is given by

\[ (\mathcal{M}_{\mathcal{F}G}^{\text{or}})^{\circ} \simeq \text{Spec} \left( \pi_0(\mathcal{M}^{\otimes(\bullet+1)}) \right). \]

By Quillen’s Theorem, we have canonical isomorphisms \( \pi_0(\mathcal{M}) \simeq L \) with the Lazard ring, classifying formal group laws, and \( \pi_0(\mathcal{M} \otimes \mathcal{M}) \simeq W \) is the groupoid of (non-strict) isomorphisms of formal group laws. In particular, the standard groupoid presentation of the ordinary moduli stack of formal groups

\[ \mathcal{M}_{\mathcal{F}G}^{\circ} \simeq \lim (\text{Spec}(W) \Rightarrow \text{Spec}(L)), \]

e.g. from [Smi11 Theorem 2.6.4] or [Goe08 Theorem 2.3.4], gives rise to the desired identification \( (\mathcal{M}_{\mathcal{F}G}^{\text{or}})^{\circ} \simeq \mathcal{M}_{\mathcal{F}G}^{\circ} \) - see also [SAG Example 9.3.1.8]. □

Remark 2.3.10. The underlying ordinary stack \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \) is the classical stack of formal group. As a functor \( \mathcal{M}_{\mathcal{F}G}^{\circ} : \text{CAlg}^{\circ} \to \text{Grpd} \simeq \pi_1(S) \subset S \), it is highly non-trivial. It is thus very far from the restriction we have \( \mathcal{M}_{\mathcal{F}G}^{\text{or}} \subset \text{CAlg}^{\circ} \) onto the subcategory \( \text{CAlg}^{\circ} \subset \text{CAlg} \) of discrete \( E_{\infty} \)-rings, since \( \mathcal{M}_{\mathcal{F}G}^{\text{or}}(A) = \emptyset \) for any \( E_{\infty} \)-ring that is not complex-periodic, which includes every ordinary commutative ring.
2.4. Quasi-coherent sheaves on the stack of oriented formal groups. Let us begin this section with a straightforward computation, going back ostensibly to Bousfield [Bon79, Theorem 6.5], implied by the results of the previous one.

**Proposition 2.4.1.** The initial map of $E_{\infty}$-rings $S \to \mathcal{O}(M_{\text{FG}})$ is an equivalence.

**Proof.** By applying the limit-preserving functor $\mathcal{O}: (\text{Shv}_{\text{fpqc}}^{\text{ac}})^{\text{op}} \to \text{CAlg}$ to the equivalence of Corollary 2.3.7, we obtain an equivalence of $E_{\infty}$-rings

\[
\mathcal{O}(M_{\text{FG}}) \cong \mathcal{O}(\mathcal{O}(\text{Spec}(\text{MP})/\text{Spec}(S)))
\]

\[
\cong \text{Tot}(\mathcal{O}(\text{Spec}(\text{MP})^\times(\times 1)))
\]

\[
\cong \text{Tot}(\text{MP}^\times(\times 1)) =: S_{\text{MP}}^\wedge
\]

with the nilpotent completion of the sphere spectrum along MP. We thus need to show that the canonical map of $E_{\infty}$-rings $S \to S_{\text{MP}}^\wedge$ is an equivalence. Nilpotent completion depends only on the $E_2$-structure on MP, hence it suffices to choose any particular form of periodic complex bordism. For the Thom spectrum $\text{MUP}$, introduced in Example 2.3.4, the unit map $S \to \text{MUP}$ is a Hopf-Galois extension by [Rog05, Remark 12.2.3], from which the desired completion claim follows from [Rog05, Proposition 12.1.8].

This suggests a close connection between quasi-coherent sheaves on $M_{\text{FG}}$ and the $\infty$-category of spectra. To make that precise with Theorem 2.4.4, we need to first single out a certain class of quasi-coherent sheaves.

**Definition 2.4.2.** For any fpqc stack $X$, let $\text{IndCoh}(X)$ denote the ind-completion of the thick subcategory of $\text{QCoh}(X)$, spanned by the structure sheaf $\mathcal{O}_X$, i.e. the smallest stable full subcategory of $\text{QCoh}(X)$ that is contains $\mathcal{O}_X$ and is closed under retracts.

**Remark 2.4.3.** The preceding definition, partially inspired by [BHV18, Definition 5.39] is quite fanciful. In particular, it will almost certainly fail to specify many of the good properties of its namesake from derived algebraic geometry, as discussed for instance in [GR17]. But in our context, with our very specific scope of interest, the above terminology is highly convenient and at least somewhat appropriate to indicate the difference of the $\infty$-category in question from $\text{QCoh}(X)$.

**Theorem 2.4.4.** The functor $\text{Sp} \to \text{QCoh}(M_{\text{FG}})$ given by $X \mapsto \mathcal{O}_{M_{\text{FG}}} \otimes X$ induces an equivalence of $\infty$-categories $\text{IndCoh}(M_{\text{FG}}) \simeq \text{Sp}$.

**Proof.** Recall from Example 1.4.4 that quasi-coherent sheaves on $M_{\text{FG}}$ and spectra are related by the adjunction

\[
\mathcal{O}_{M_{\text{FG}}} \otimes - : \text{Sp} \rightleftarrows \text{QCoh}(M_{\text{FG}}) : \Gamma(M_{\text{FG}}; -),
\]

obtained by quasi-coherent pullback and pushforward along the terminal map of stacks $M_{\text{FG}} \to \text{Spec}(S)$. We can identify the unit of this adjunction on the subcategory $\text{Sp}^{\text{fin}} \subseteq \text{Sp}$ as the identity functor by Lemma 2.4.5. Therefore the restriction of the left adjoint to this subcategory is fully faithful. On the other hand, its essential image is precisely the thick subcategory in $\text{QCoh}(X)$ spanned by $\mathcal{O}_X$. In light of Definition 2.4.2, the result follows by passing to the ind-completion on both sides.

**Lemma 2.4.5.** For any finite spectrum $M$, the canonical map $M \to \Gamma(M_{\text{FG}}; \mathcal{O}_{M_{\text{FG}}} \otimes M)$ is an equivalence of spectra.

**Proof.** The functors $\mathcal{F} \mapsto \Gamma(M_{\text{FG}}; \mathcal{F})$ and $M \mapsto \mathcal{O}_{M_{\text{FG}}} \otimes M$ form an adjunction, and therefore preserve colimits and limits respectively. Since both Sp and $\text{QCoh}(M_{\text{FG}})$ are stable $\infty$-categories, this implies that both functors preserve finite (co)limits. Recalling that the subcategory of finite spectra $\text{Sp}^{\text{fin}} \subseteq \text{Sp}$ is spanned by the sphere spectrum $S$ under finite colimits, we have reduced to proving the claim for $M = S$, in which case it follows from Proposition 2.4.1. □
Remark 2.4.6. The conclusion of Lemma 2.4.5 may, in light of the proof of Proposition 2.4.1, be restated as the claim that the canonical map \( M \to M'_{\text{MP}} \) is an equivalence of spectra for any \( M \in \text{Sp}^{\text{fin}} \). That is to say, every finite spectrum is MP-nilpotent complete.

Remark 2.4.7. Another way to view Theorem 2.4.4 is as an instance of the Schwede-Shipley Recognition Theorem [HA, Theorem 7.1.2.1] for compactly generated stable \( \infty \)-categories. Indeed, the \( \infty \)-category \( \text{IndCoh}(M'_{\text{FG}}) \) is stable and compactly generated by the structure sheaf \( O_{M'_{\text{FG}}} \) by construction, hence the aforementioned result identifies it with \( \text{Mod}_{\mathcal{O}(M'_{\text{FG}}))} \), which Lemma 2.4.1 shows to be Sp.

We should also mention the comodule-theoretic description of the \( \infty \)-category of quasi-coherent sheaves on \( M'_{\text{FG}} \), analogous to the classical comodule-theoretic description of the abelian category of ordinary quasi-coherent sheaves on \( M'_{\text{FG}} \) that underlies most approaches to the ANSS.

Construction 2.4.8. Let \( A \) be an \( \mathbb{E}_\infty \)-ring. The forgetful functor \( \text{Mod}_A \to \text{Sp} \) admits a left adjoint \( M \mapsto M \otimes A \). This adjunction induces the comonad \( T : \text{Mod}_A \to \text{Mod}_A \) given by

\[
T(M) \cong M \otimes A \cong M \otimes_A (A \otimes A).
\]

The comonad structure on \( T \) is therefore equivalent to an \( \mathbb{E}_1 \)-coalgebra structure on \( A \otimes A \otimes A \) over \( A \) (see [Tor16] for a precise treatment thereof). The \( \infty \)-category of comodules over it, defined as

\[
\text{cMod}_{A \otimes A}(\text{Mod}_A) := \text{Mod}_T(\text{Mod}^{\text{op}}_A)^{\text{op}},
\]

may be identified by the Beck-Chevalley theory of comonadic descent, in the form of [HA, Theorem 4.7.5.2], with the totalization \( \text{Tot}(A^{\otimes (n+1)}) \) of the Amitsur complex of \( A \). Indeed, the latter may be identified with the cobar construction of the coalgebra \( A \otimes A \otimes A \) over \( A \).

Proposition 2.4.9. Pullback along the cover \( \text{Spec}(\text{MP}) \to M'_{\text{FG}} \) induces an equivalence of \( \infty \)-categories \( \text{QCoh}(M'_{\text{FG}}) \cong \text{cMod}_{\mathcal{O}(\text{MP})}(\text{Mod}_\text{MP}) \). By passing to \( \pi_0 \), this recovers the traditional equivalence of abelian categories \( \text{QCoh}(M'_{\text{FG}}) \cong \text{cMod}_W(\text{Mod}^\vee_\ell) \).

Proof. Follows from the Beck-Chevalley description of the comodule \( \infty \)-category, since we already know by Corollary 2.3.7 that \( \text{QCoh}(M'_{\text{FG}}) \cong \text{Tot}\left(\text{Mod}_{\mathcal{O}(\text{MP})^{\otimes (n+1)}}\right) \). \( \square \)

Remark 2.4.10. In terms of the comodule description of quasi-coherent sheaves on \( M'_{\text{FG}} \), the \( \infty \)-category \( \text{IndCoh}(M'_{\text{FG}}) \) is a version of Hovey’s stable category of comodules; see [BHV18] for a discussion. A modern treatment can be found in [Kra18, Definition 2.4], where this is called the compactly generated category of comodules. Our Theorem 2.4.4 may be viewed as a special case of Krause’s [Kra18, Theorem 2.44]; see in particular [Kra18, Example 2.45] for a version with \( \text{MU} \) instead of \( \text{MP} \).

2.5. Landweber exactness. Let \( X \) be any fixed spectrum. The non-connective geometric stack \( M'_{\text{FG}} \) allows us to define a quasi-coherent sheaf

\[
\mathcal{F}_n(X) := \pi_n(\mathcal{O}_{M'_{\text{FG}}} \otimes X) \in \text{QCoh}^\vee(M'_{\text{FG}}).
\]

It satisfies \( \mathcal{F}(\Sigma^{-n}(X)) \cong \pi_n(\mathcal{O}_{M'_{\text{FG}}} \otimes X) \), hence it is encoding the \( \text{QCoh}^\vee(M'_{\text{FG}}) \)-valued homology theory corresponding to the structure sheaf \( \mathcal{O}_{M'_{\text{FG}}} \).

Remark 2.5.1. The collection of sheaves \( \mathcal{F}_n(X) \) for all spectra \( X \) contain already for \( n = 0 \) all the information encoded in those for other \( n \in \mathbb{Z} \), since

\[
\mathcal{F}_n(X) \cong \pi_0(\Sigma^{-n}(\mathcal{O}_{M'_{\text{FG}}} \otimes X)) \cong \pi_0(\mathcal{O}_{M'_{\text{FG}}} \otimes \Sigma^{-n}(X)) \cong \mathcal{F}_0(\Sigma^{-n}(X)).
\]

Remark 2.5.2. Let \( \omega_{M'_{\text{FG}}} \in \text{QCoh}(M'_{\text{FG}}) \) be the dualizing line, i.e. the module of invariant differentials on the universal oriented formal group, pulled back from the analogous \( \omega_{M'_{\text{FG}}} \) along the canonical map \( M'_{\text{FG}} \to M_{\text{FG}} \). These quasi-coherent sheaves are flat by definition, and in follows from Corollary 2.3.3 that \( \pi_0(\omega_{M'_{\text{FG}}}) = \omega_{M'_{\text{FG}}} \in \text{QCoh}^\vee(M'_{\text{FG}}) \) is the sheaf
of invariant differentials on the universal ordinary formal group. On the other hand, we have by the definition of orientability for formal groups that \( \omega^\otimes_{M_{FG}} \simeq \Sigma^{-2}(O_{M_{FG}}) \), and consequently \( \omega^\otimes_{M_{FG}} \simeq \Sigma^{-2n}(O_{M_{FG}}) \) for every \( n \in \mathbb{Z} \). For any spectrum \( X \) and any \( n \in \mathbb{Z} \), it follows that

\[
\pi_{2n}(X) \simeq \pi_0(\Sigma^{-2n}(O_{M_{FG}}) \otimes X) \\
\simeq \pi_0(\omega^\otimes_{M_{FG}} \otimes X) \\
\simeq \omega^\otimes_{M_{FG}} \otimes O_{M_{FG}} \pi_0(O_{M_{FG}} \otimes X) \\
\simeq \omega^\otimes_{M_{FG}} \otimes O_{M_{FG}} \mathcal{F}_0(X),
\]

and, either by an analogous consideration or by use of the last remark, similarly for odd-degree homotopy groups

\[
\pi_{2n+1}(X) \simeq \omega^\otimes_{M_{FG}} \otimes O_{M_{FG}} \mathcal{F}_1(X).
\]

Thus already for any fixed spectrum \( X \), the sheaves \( \mathcal{F}_n(X) \) and \( \mathcal{F}_1(X) \) contain the information of all the other \( \mathcal{F}_n(X) \).

**Proposition 2.5.3.** Let \( f^\circ : \text{Spec}(L) \to M_{FG}^\circ \) be the cover of the ordinary stack of formal groups by formal group laws. For any spectrum \( X \), there is a canonical isomorphism of L-modules \((f^\circ)^*(\mathcal{F}_n(X)) \simeq \text{MP}_n(X)\).

**Proof.** The cover in question is induced upon the underlying ordinary stacks by the cover of non-connective geometric stacks \( f : \text{Spec}(MP) \to M_{FG}^\otimes \). Using Lemma \[1.4.10\] and the fact that quasi-coherent pullback commutes with colimits, we obtain the series of natural equivalences

\[
(f^\circ)^*(\mathcal{F}_n(X)) \simeq (f^\circ)^* \pi_n(O_{M_{FG}} \otimes X) \\
\simeq \pi_n(f^*(O_{M_{FG}} \otimes X)) \\
\simeq \pi_n(f^*(O_{M_{FG}}^\otimes \otimes X)) \\
\simeq \pi_n(MP \otimes X),
\]

where the final term is the \( n \)-th MP-homology of the spectrum \( X \) by definition. \( \square \)

**Remark 2.5.4.** It follows from Proposition \[2.5.3\] that the sheaves \( \mathcal{F}_n(X) \) on \( M_{FG}^\circ \) may equivalently be defined in terms of the usual coordinatized presentation of formal groups in terms of formal group laws, which is to say the groupoid presentation of the moduli stack \( M_{FG}^\otimes \simeq \text{colim} (\text{Spec}(W) \simeq \text{Spec}(L)) \). In combination with the isomorphism of Hopf algebroids \((\pi_0(MP), \pi_0(MP \otimes MP)) \simeq (L, W) \) of Quillen’s Theorem, the sheaf \( \mathcal{F}_n(X) \) therefore corresponds to the \( L = \pi_0(MP) \)-module \( \text{MP}_n(X) = \pi_0(MP \otimes X) \), equipped with its usual of \( W = MP_0(MP) \simeq \pi_0(MP \otimes MP) \) as a generalized Steenrod algebra. It is in this guise that the sheaves \( \mathcal{F}_n(X) \) appear as foundations for chromatic homotopy theory in \[Lur10\]. The advantage of our approach is that it does not require explicitly distinguishing the complex bordism spectrum \( MP \), but instead proceeded from the non-connective spectral stack \( M_{FG}^\otimes \).

**Remark 2.5.5.** From another perspective, Proposition \[2.5.3\] allows us to construct the homology theory \( \text{MP}_n \) corresponding to the periodic complex bordism spectrum solely in terms of the stack \( M_{FG}^\otimes \) and the classical stack \( M_{FG}^\otimes_{\text{coord}} \) of coordinatized classical formal groups (equivalently: formal group laws - see Definition \[3.1.1\])

\[
\text{MP}_n(X) \simeq (f^\circ)^*(\pi_n(O_{M_{FG}}^\otimes \otimes X)) \simeq \Gamma(M_{FG}^\otimes; \pi_n(O_{M_{FG}}^\otimes \otimes X) \otimes O_{M_{FG}^\otimes_{\text{coord}}}).
\]

This might not recover the \( \mathbb{E}_\infty \)-ring structure, but then we should not expect it would; after all, we are working with an arbitrary form of periodic complex bordism \( MP \).
Remark 2.5.6. The construction of the preceding Remark hints at an attractive possibility of a path towards an alternative, perhaps more insightful, proof of Quillen’s Theorem. Nevertheless, it relies crucially on the observation that the underlying ordinary stack of \( \mathcal{M}^{\text{or}}_{FG} \) is the ordinary stack of formal groups \( \mathcal{M}^{\circ}_F \), our proof of which in Corollary 2.3.9 ultimately reduced to an application of Quillen’s Theorem. If we were able to leverage the homology theories come directly from (2.3.9) which does not involve complex bordisms, we have not succeeded to provide a more insightful proof of Quillen’s Theorem, we have succeeded to reduce it to a purely spectral-algebro-geometric statement (that of Corollary 2.3.9) which does not involve complex bordisms.

The example of Remark 2.5.5 may be extended to show that all Landweber exact homology theories come directly from \( \mathcal{M}^{\text{or}}_{FG} \).

Corollary 2.5.7. Let \( R \) be a commutative ring and \( \mathcal{G} \in \text{FGrp}_{\dim=1}(R) \) a Landweber-exact 1-dimensional formal group law. That is to say, suppose that its classifying map \( \eta_{\mathcal{G}} : \text{Spec}(R) \to \mathcal{M}^{\circ}_F \) is flat. If the associated even-periodic Landweber exact spectrum is denoted \( E_R \), then its value as a homology theory \( (E_R)_n : \text{Sp} \to \text{Ab} \) is given by

\[
(E_R)_n(X) \simeq \eta_{\mathcal{G}}^*(\mathcal{F}_n(X))
\]

Proof. In light of the characterization of the sheaves \( \mathcal{F}_n(X) \) given by Proposition 2.5.3 this is just how even-periodic Landweber exact homology theories are defined, see for instance [Lur10, Lecture 18, Proposition 6].

With more hypothesis, we can exhibit \( \text{E}_\infty \)-structures on Landweber spectra, coming from upgrading the underlying ordinary groups to sufficiently nice spectral formal groups. From another perspective, we relate Lurie’s orientation classifier construction \( \Omega_{\mathcal{G}} \) for a formal group \( \mathcal{G} \) from [Ell2] Definition 4.3.14], with a Landweber exact spectrum under certain extra assumption.

Recall here that a 1-dimensional formal group law \( \mathcal{G} \) over an \( \text{E}_\infty \)-ring \( A \) is balanced in the sense of [Ell2] Definition 6.4.1] if both of the following hold:

- The unit map \( A \to \Omega_{\mathcal{G}} \) induces an isomorphism of commutative rings \( \pi_0(A) \simeq \Omega_{\mathcal{G}} \).
- The homotopy groups of \( \Omega_{\mathcal{G}} \) are concentrated in even degrees.

Proposition 2.5.8. Let \( \mathcal{G} \in \text{FGrp}_{\dim=1}(A) \) be a balanced 1-dimensional formal group over an \( \text{E}_\infty \)-ring \( A \), such that the underlying ordinary formal group \( G^0 \in \text{FGrp}_{\dim=1}(\pi_0(A)) \) is Landweber exact. Then the orientation classifier \( \Omega_{\mathcal{G}} \) exhibits an \( \text{E}_\infty \)-ring structure on the Landweber exact even-periodic spectrum \( E_{\pi_0(A)} \) associated to \( \mathcal{G}^0 \).

Proof. By definition, the \( \text{E}_\infty \)-ring \( \Omega_{\mathcal{G}} \) is complex periodic. Under the assumption that \( \Omega_{\mathcal{G}} \) is balanced, it follows from passing to underlying stacks from the commutative diagram of non-connective fpqc stacks

\[
\begin{array}{ccc}
\text{Spec}(\Omega_{\mathcal{G}}) & \xrightarrow{f'} & \mathcal{M}^{\text{or}}_{FG} \\
\downarrow^{\nu'} & & \downarrow^{u} \\
\text{Spec}(A) & \xrightarrow{f} & \mathcal{M}^\circ_{FG},
\end{array}
\]

that \( (f')^\circ : \text{Spec}(\pi_0(\Omega_{\mathcal{G}})) \to (\mathcal{M}^{\text{or}}_{FG})^\circ \simeq \mathcal{M}^\circ_{FG} \) recovers the map \( f^\circ : \text{Spec}(\pi_0(A)) \to \mathcal{M}^\circ_{FG} \) classifying \( \mathcal{G}^0 \). Landweber exactness of the latter means that \( f^\circ \) is flat. On the other hand, since balancedness also implies that \( \pi_{2i+1}(\Omega_{\mathcal{G}}) = 0 \) for all \( i \), and quasi-coherent sheaves on \( \mathcal{M}^{\text{or}}_{FG} \) are all weakly 2-periodic by design, it follows that the morphism \( f' \) is flat itself.
Just as in the proof of Proposition 2.5.3, we now find that
\[(E_{\pi_0(A)})_n(X) \simeq (f^\circ)^*(\pi_n(O_{M_{FG}} \otimes X))\]
\[\simeq \pi_n((f^\circ)^*(O_{M_{FG}} \otimes X))\]
\[\simeq \pi_n(f^*(O_{M_{FG}} \otimes X))\]
\[\simeq \pi_n(O_{\hat{G}} \otimes X),\]
proving that the spectrum $O_{\hat{G}}$ indeed represents the homology theory $E_{\pi_0(A)}$. \[\square\]

**Examples 2.5.9.** The proof of Proposition 2.5.8 shows that, when complex periodic $E_\infty$-rings arise from the orientation classifier construction, they may be obtain by pullback along the forgetful map $u : M_{FG}^{\text{gr}} \to M_{FG}$. This happens in a number of cases:

1. By [Ell2, Corollary 4.3.27] and Snaith’s Theorem [Ell2, Theorem 6.5.1], there is a canonical equivalence $KU \simeq O_{\hat{G}_m}$ between the complex $K$-theory spectrum, and the orientation classifier of the multiplicative formal group $\hat{G}_m$ over the sphere spectrum $S$. The latter is a balanced formal group by [Ell2, Proposition 6.5.2] (from which Lurie is able to deduce Snaith’s Theorem), and so there is a canonical pullback square of non-connective spectral stacks

\[
\begin{array}{ccc}
\text{Spec}(KU) & \longrightarrow & M_{FG}^{\text{gr}} \\
\downarrow & & \downarrow u \\
\text{Spec}(S) & \longrightarrow & M_{FG}
\end{array}
\]

2. For any perfect field $\kappa$ of characteristic $p > 0$, and any formal group $\hat{G}_0$ of finite height over $\kappa$, there exists by [Ell2, Theorem 3.0.11] the spectral deformation $E_\infty$-ring $R_{\hat{G}_0}^{\text{un}}$, supporting a universal deformation $\hat{G}$ of $G_0$. This is a spectral enhancement of the better-known Lubin-Tate deformation ring, which is recovered on $\pi_0$. By [Ell2, Theorem 6.4.7], the formal group $\hat{G}$ is balanced. It hence follows that there is a pullback square

\[
\begin{array}{ccc}
\text{Spf}(E(\kappa, \hat{G}_0)) & \longrightarrow & M_{FG}^{\text{gr}} \\
\downarrow & & \downarrow u \\
\text{Spf}(R_{\hat{G}_0}^{\text{un}}) & \longrightarrow & M_{FG},
\end{array}
\]

exhibiting the relationship between Lubin-Tate spectrum $E(\kappa, \hat{G}_0) \simeq O_{\hat{G}}$ and the non-connective spectral stack $M_{FG}^{\text{gr}}$.

3. Though not quite fitting into the paradigm of Proposition 2.5.8, the construction of the $E_\infty$-ring of topological modular forms from [Ell2, Chapter 7] is analogous to the previous two examples. It start with the moduli stack $M_{\text{Ell}}^{\text{gr}}$ of strict elliptic curves, which are in [Ell1, Definition 2.0.2] defined as abelian group objects in varieties over $E_\infty$-ring; see [Ell1, Definition 1.1.1] for the latter. As discussed in [Ell2, Section 7.1], we may extract from any strict elliptic curve $E$ over an $E_\infty$-ring $A$ a formal group $\hat{E}$ over $A$. That gives rise to a map of stacks $M_{\text{Ell}}^{\text{gr}} \to M_{FG}$, classifying the formal group of the universal elliptic curve. By [Ell2, Theorem 7.3.1], this formal group is balanced. This implies, just like Proposition 2.5.8, that the stack of oriented elliptic curves, defined in [Ell2, Definition 7.2.9], fits into a pullback square

\[
\begin{array}{ccc}
M_{\text{Ell}}^{\text{gr}} & \longrightarrow & M_{FG}^{\text{gr}} \\
\downarrow & & \downarrow u \\
M_{\text{Ell}}^{\text{gr}} & \longrightarrow & M_{FG}.
\end{array}
\]
in the ∞-category of non-connective spectral stacks. Though the main idea is the same, this is not quite an instance of Proposition 2.5.8. Indeed, the non-connective (by [Ell2] Proposition 7.2.10 Deligne-Mumford) spectral stack \( M_{\text{or}}^\text{or} \) has as its underlying ordinary stack being the usual stack of elliptic curves \( M_{\text{Ell}}^\text{or} \), and is thus not affine. Of course, the main interest in it is that, thanks to [Ell2] Theorem 7.3.1, its ring of functions \( \mathcal{O}(M_{\text{Ell}}^\text{or}) \simeq \text{TMF} \) provides an approach to the \( \mathbb{E}_\infty \)-ring of topological modular forms.

2.6. The Adams-Novikov spectral sequence. We finally consider the descent spectral sequence on the non-connective spectral stack \( M_{\text{or}}^\text{or} \), and recover the ANSS, returning to the womb of chromatic homotopy theory.

**Proposition 2.6.1.** The descent spectral sequence

\[
E_2^{s,t} = H^s(M_{\text{or}}^\text{or}; \mathcal{O}(\mathcal{O}_{M_{\text{or}}^\text{or}})) \Rightarrow \pi_{t-s}(\mathcal{O}(M_{\text{or}}^\text{or}))
\]

is isomorphic to the ANSS

\[
E_2^{s,t} = \text{Ext}^{s,t}_{\pi_*(\pi_1(MU))]\text{(MU, } \pi_*(MU))} \Rightarrow \pi_{t-s}(S).
\]

**Proof.** It follows from the proof of Lemma 2.3.5 that fibered products of non-connective affines over \( M_{\text{or}}^\text{or} \) is canonically equivalent to their categorical product, i.e. fibered products over the terminal object \( \text{Spec}(S) \). In particular, the canonical map

\[
\hat{\mathbb{C}}^*(\text{Spec(MP)}/M_{\text{or}}^\text{or}) \to \hat{\mathbb{C}}^*(\text{Spec(MP)}/\text{Spec(S)}) \tag{5}
\]

is an equivalence of simplicial objects.

By passing to global functions on the left-hand side of the equivalence (5), we obtain the cosimplicial spectrum whose associated Bousfield-Kan spectral sequence is, according to the proof of Proposition 1.4.11, the descent spectral sequence. Its second page is sheaf cohomology on the underlying ordinary stack of \( M_{\text{or}}^\text{or} \), which is \( M_{\text{or}}^\text{or} \) by Corollary 2.3.9.

By passing to global functions on the right-hand side of (5) though, we recover as

\[
\mathcal{O}(\hat{\mathbb{C}}^*(\text{Spec(MP)}/\text{Spec(S)})) \simeq \text{MP}^{\otimes (s+1)},
\]

the Amitsur complex (also known as the cobar complex) of MP. That is precisely the cosimplicial spectrum whose Bousfield-Kan spectral sequence is the MP-based Adams spectral sequence, see e.g. [Rav04] Section 2.2, [Hop99] Section 5, or [Lur10] Lecture 8.

It remains to show that the MP-based and the MB-based Adams spectral sequences agree. The \( \mathbb{E}_2 \)-ring unit map \( S \to S[\beta^{-1}] \) induces a map of \( \mathbb{E}_2 \)-rings \( \text{MU} \simeq \text{MU} \otimes S \to \text{MU} \otimes S[\beta^{-1}] \simeq \text{MP} \) induces a map on Amitsur complexes \( \text{MU}^{\otimes (s+1)} \to \text{MB}^{\otimes (s+1)} \), which then gives rise to a map between the Adams spectral sequences. To prove that a map of spectral sequences is an equivalence, it suffices to show that this happens on the second page. That is a classical change-of-rings observation, see for instance [Goe17] Example 7.4, but it also follows from any other way of identifying the \( E_2 \)-page of the Adams-Novikov (i.e. MB-based Adams) spectral sequence with sheaf cohomology on the stack of formal groups \( M_{\text{or}}^\text{or} \), e.g. [Lur10] Lectures 10 & 11 or [Goe08] Remark 3.14, since we already know the latter agree with the \( E_2 \)-page of the MP-based Adams spectral sequence thanks to the preceding discussion.

**Remark 2.6.2.** In the proof of Proposition 2.6.1 we used the fpqc cover \( \text{Spec}(\text{MP}) \to M_{\text{or}}^\text{or} \) to obtain the descent spectral sequence. Thus it amounts to little more than a redressing of the usual connection between \( \text{MU} \) and formal group, stemming ultimately from Quillen’s Theorem. But if we instead invoke Construction 1.5.7, we obtain the descent spectral sequence starting purely from the non-connective geometric stack \( M_{\text{or}}^\text{or} \). In particular, the complex bordism spectrum plays no distinguished role in setting up the ANSS from the perspective.
Remark 2.6.3. A simple modification of the proof of Proposition 2.6.1 shows that for any spectrum $X$, the descent spectral sequence for the quasi-coherent sheaves $\mathcal{O}_{M_{FG}^c} \otimes X$ on $\mathcal{M}_{FG}^c$ gives rise to the ANSS for $X$. Using the conventions and results from Subsection 2.5. its second page may be written as

$$E_2^{s,t} = \text{H}^s(\mathcal{M}_{FG}^c; \mathcal{F}_t(X)) = \text{H}^s(\mathcal{M}_{FG}^c; \omega_{\mathcal{M}_{FG}^c} \otimes \mathcal{O}_{\mathcal{M}_{FG}^c} \mathcal{F}_{t(\text{mod}2)}(X)) .$$

Specifying this back to the case of $X = S$, we find that the ANSS may be rewritten in a way to highlight how the second page is determined purely by ordinary formal group data as

$$E_2^{s,t} = \text{H}^s(\mathcal{M}_{FG}^c; \omega_{\mathcal{M}_{FG}^c}^{t}) \Rightarrow \pi_{2s-t}(S).$$

3. Universal properties of periodic complex bordism

So far, we have been using any form of periodic complex bordism $\mathcal{M}_{FG}$. In this subsection, we instead fix two specific $\mathbb{E}_\infty$-ring forms of $\mathcal{M}_{FG}$ that we discussed in Example 2.3.4: the Thom spectrum $\mathbb{M}_{UP}$ and the Snaith construction $\mathbb{M}_{Snaith} := (S[\mathbb{B}U])[\beta^{-1}]$. We give interpretations of the corresponding non-connective affine spectral schemes in terms of oriented formal groups. Finally we discuss a possible path towards using the ideas of this paper to prove Quillen’s Theorem.

3.1. Classical picture of coordinatized formal groups. To motivate the discussion of universal properties of $\mathbb{E}_\infty$-forms of $\mathcal{M}_{FG}$, let us first recall how such an identification in terms of formal groups works on the level of $\pi_0$.

Definition 3.1.1 ([Ell2, Definition 5.3.5]). Let $R$ be a commutative ring. A coordinatized formal group over $R$ consists of a pair $(\tilde{G}, t)$ of a formal group $\tilde{G}$ over $R$ and coordinate $t$ on it. Here a coordinate on $\tilde{G}$ means any element $t \in \mathcal{O}_{\tilde{G}}(-e)$, whose image under the quotient map $\mathcal{O}_{\tilde{G}}(-e) \to \mathcal{O}_{\tilde{G}}/\mathcal{O}_{\tilde{G}}^2 = \omega_{\tilde{G}}$ induces a trivialization of the dualizing line, i.e. an $R$-module isomorphism $R \cong \omega_{\tilde{G}}$. Coordinates on $\tilde{G}$ form a subset $\text{Coord}_{\tilde{G}} \subseteq \mathcal{O}_{\tilde{G}}(-e)$.

For any ring homomorphism $f : R \to R'$, the extension of scalars along $f$ preserves coordinates, and hence gives rise to a map $\text{Coord}_{\tilde{G}} \to \text{Coord}_{f^!(\tilde{G})}$. This defines a functor $\text{Coord}(\tilde{G}) : \text{CAlg}_R^\mathbb{Z} \to \text{Set}$, or equivalently, a map of ordinary stacks $\text{Coord}(\tilde{G}) \to \text{Spec}(R)$.

From this perspective, Quillen’s Theorem ([Ell2, Theorem 5.3.10]) is the following assertion:

Theorem 3.1.2 (Quillen). The commutative ring $L = \pi_0(\mathcal{M}_{FG})$ corepresents the moduli of coordinatized formal groups. More precisely, for any formal group $\tilde{G}$ over a commutative ring $R$, classified by a map of stacks $\eta_{\tilde{G}} : \text{Spec}(R) \to \mathcal{M}_{FG}$, the fpqc cover $\text{Spec}(L) \to \mathcal{M}_{FG}$ participates in the pullback square of ordinary stacks

$$\begin{array}{ccc}
\text{Coord}(\tilde{G}) & \longrightarrow & \text{Spec}(L) \\
\downarrow & & \downarrow \\
\text{Spec}(R) & \longrightarrow & \mathcal{M}_{FG}^\circ.
\end{array}$$

Remark 3.1.3. The reader may be better accustomed to the claim that $\text{Spec}(L)$ classifies formal group laws. As discussed in [Ell2, Remark 5.3.6], formal group laws are equivalent to coordinatized group laws up to equivalence. Indeed, the data of a coordinatization is equivalent to an isomorphism of pointed formal $R$-schemes $\tilde{A}_L^1 \cong \tilde{G}$. We instead prefer the perspective of coordinatized formal groups given in Definition 3.1.1 because it matches better with the characterizations of $\text{Spec}(\mathcal{M}_{FG})$ in Theorem 3.3.4 and Proposition 3.5.2.

Remark 3.1.4. The notion a coordinatized formal group that we are using is the same as the one discussed in [Goe08, Section 2.3]. The prestack of coordinatized formal groups $\mathcal{M}_{\text{coord}}$ from [Goe08, Definition 2.16], however, disagrees with our meaning of the moduli
of coordinatized formal groups. Indeed, while the objects of the groupoid \( \mathcal{M}_{\text{coord}}(R) \) are coordinatized formal groups, the morphisms are all isomorphisms of formal groups, not merely those which respect the chosen coordinatization.

### 3.2. Invertible and complex-exponentiable quasi-coherent sheaves

To consider an analogue of the above story in spectral algebraic geometry, we must find an analogue of trivializing a line bundle. That necessitates a brief digression on invertible sheaves in spectral algebraic geometry. In what follows, let \( X \) be a non-connective spectral stack.

**Definition 3.2.1** ([SAG, Definition 2.9.5.1]). A quasi-coherent sheaf \( \mathscr{L} \) on \( X \) is said to be **invertible** if it is invertible in the symmetric monoidal \( \infty \)-category \( \text{QCoh}(X) \). That is to say, if there exists a quasi-coherent sheaf \( \mathscr{L}^{-1} \) such that \( \mathscr{L} \mathcal{O}_X \mathscr{L}^{-1} \cong \mathcal{O}_X \). Invertible sheaves form a full subcategory \( \text{Pic}^1(X) \subset \text{QCoh}(X)^{\text{op}} \).

**Remark 3.2.2.** When \( X \) is connective, there exists another closely related notion: a **line bundle** on \( X \) is such an invertible sheaf on \( X \) for which both \( \mathscr{L} \) and \( \mathscr{L}^{-1} \) belong to the full subcategory \( \text{QCoh}(X)_{\text{cn}} \subset \text{QCoh}(X) \) of connective quasi-coherent sheaves. At least when \( X \) is a spectral Deligne-Mumford stack, this is shown in [SAG, Proposition 2.9.4.2] to be equivalent to the quasi-coherent sheaf \( \mathscr{L} \) being locally free of rank 1. In particular, if we denote by \( \text{Pic}(X) \subset \text{Pic}^1(X) \) the subspace of line bundles, then \( \text{Pic}(X) \) recovers the usual Picard groupoid for an ordinary scheme \( X \). The space \( \text{Pic}^1(X) \) is always larger however, as it contains invertible sheaves such as \( \Sigma \mathcal{O}_X \) for any \( n \in \mathbb{Z} \).

**Remark 3.2.3.** Recall that the adjunction \( \mathcal{S} \rightleftarrows \text{Cat}_\infty : (-)^\ast \), between spaces viewed as \( \infty \)-groupoids and \( \infty \)-categories, is Cartesian symmetric monoidal. It therefore induces an adjunction \( \text{CMon}(\mathcal{S}) \rightleftarrows \text{CMon}(\text{Cat}_\infty) : (-)^\ast \) between \( \mathcal{E}_\infty \)-spaces and symmetric monoidal \( \infty \)-categories. In particular, the relative smash product symmetric monoidal structure on \( \text{QCoh}(X) \) equips the maximal underlying subspace \( \text{QCoh}(X)^{\text{op}} \) with the structure of an \( \mathcal{E}_\infty \)-space. The subspace \( \text{Pic}^1(X) \) inherits a group-like \( \mathcal{E}_\infty \)-structure.

Let \( \text{Vect}^C_\mathbb{C} \) denote the the topological category of finite-dimensional complex vector spaces and linear isomorphisms, viewed as an \( \infty \)-category through an implicit application of the nerve construction of [HTT, Definition 1.1.5.5]. This is an \( \infty \)-groupoid, explicitly \( \text{Vect}^C_\mathbb{C} \cong \coprod_{n \geq 0} \text{BU}(n) \), and direct sum of vector spaces makes it into an \( \mathcal{E}_\infty \)-space.

**Definition 3.2.4.** Let \( X \) be a non-connective spectral stack. A **complex-exponentiable quasi-coherent sheaf on** \( X \) is a symmetric monoidal functor \( \text{Vect}^C_\mathbb{C} \to \text{QCoh}(X) \). They form an \( \infty \)-category \( \text{QCoh}^C(X) \).

**Remark 3.2.5.** Let \( \mathcal{F}^\circ : \text{Vect}^C_\mathbb{C} \to \text{QCoh}(X) \) be a complex-exponentiable quasi-coherent sheaf. We will view the functor value \( \mathcal{F} = \mathcal{F}^\circ(C) \) as the underlying quasi-coherent sheaf of \( \mathcal{F} \). By symmetric monoidality, this determines the functor object-wise as \( \mathcal{F}^\circ : (\text{Vect}^C_\mathbb{C})^n \to \mathcal{F}^\circ^n \). The complex-exponentiable structure amounts to specifying appropriately coherently compatible system of a \( U(n) \)-action on \( \mathcal{F}^\circ^n \) for every \( n \geq 0 \). Said differently, a complex-exponentiable structure on an underlying quasi-coherent sheaf \( \mathcal{F} \) consists of a functorial system of powers \( \mathcal{F}^\circ V \cong \mathcal{F}^\circ_{\text{dim}C}(V) \) for finite dimensional complex vector spaces \( V \).

**Remark 3.2.6.** If the underlying quasi-coherent sheaf \( \mathcal{F} \) of a complex-exponentiable quasi-coherent sheaf is invertible, then there is a canonical symmetric monoidal factorization \( \mathcal{F}^\circ : \text{Vect}^C_\mathbb{C} \to \text{Pic}^1(X) \subset \text{QCoh}(X) \). Because the \( \mathcal{E}_\infty \)-space \( \text{Pic}^1(X) \) is group-like, this further factors through the group completion (\( \text{Vect}^C_\mathbb{C})^\circ \cong \Omega^\infty(\text{ku}) \), see [Ell2, Section 6.5]. That is to say, a complex-exponentiable invertible sheaf is equivalent to an \( \mathcal{E}_\infty \)-space map \( \Omega^\infty(\text{ku}) \to \text{Pic}^1(X) \), or yet equivalently a map of connective spectra \( \text{ku} \to \text{pic}^1(X) \).

### 3.3. Universal property of the Thom spectrum

**MUP.** The notion of a complex-exponentiable quasi-coherent sheaf, introduced in the last section, will enable us to formulate the analogue of Theorem 3.1.2 for the Thom spectrum \( \mathcal{E}_\infty \)-ring MUP.
Proposition 3.3.1. Let \( \hat{\mathcal{G}} \) be an oriented formal group over an \( \mathbb{E}_\infty \)-ring \( A \). The dualizing line \( \omega_{\hat{\mathcal{G}}} \) admits a canonical enhancement to a complex-exponentiable quasi-coherent sheaf \( \omega_{\hat{\mathcal{G}}}^\otimes \) on \( \text{Spec}(A) \).

Proof. Let us write \( X \simeq \text{Spec}(A) \). The functor \( \mathcal{O}_X \otimes - : \mathcal{S} \to \text{QCoh}(X) \) from Example 1.4.4 is symmetric monoidal, and as such induces an (also symmetric monoidal) functor \( \mathcal{O}_X \otimes - : \text{Pic}^1(S) \to \text{Pic}^1(X) \) between invertible objects. Consider the \( J \)-homomorphism, viewed as map of \( \mathbb{E}_\infty \)-spaces \( J : \text{Vect}_C^\otimes \to \text{Pic}^1(S) \), given by \( V \mapsto S^V = \Sigma^V(S) \). Combining these two, and the multiplicative inverse self-equivalence \( (\cdot)^{-1} : \text{Pic}^1(S) \to \text{Pic}^1(S) \), whose existence we owe to the fact that \( \text{Pic}^1(S) \) is a group-like \( \mathbb{E}_\infty \)-space, we obtain a symmetric monoidal composite functor
\[
\text{Vect}_C^\otimes \xrightarrow{J} \text{Pic}^1(S) \xrightarrow{(\cdot)^{-1}} \text{Pic}^1(S) \xrightarrow{\mathcal{O}_X \otimes -} \text{Pic}^1(X) \subseteq \text{QCoh}(X).
\]
This exhibits an complex-exponentiable structure on the quasi-coherent sheaf \( \Sigma^{-2}(\mathcal{O}_X) \) on \( X \). Because \( \hat{\mathcal{G}} \) is an oriented formal group, this is equivalent to the dualizing line \( \omega_{\hat{\mathcal{G}}} \). □

Remark 3.3.2. The proof of Proposition 3.3.1 shows that the quasi-coherent sheaf \( \Sigma^2(\mathcal{O}_X) \) always admits a complex-exponentiable structure for any non-compact spectral stack \( X \). Indeed, in the universal case \( X = \text{Spec}(S) \), the \( J \)-homomorphism may be viewed as exhibiting \( \Sigma^2(S) \) to be complex-exponentiable.

Construction 3.3.3. For any oriented formal group \( \hat{\mathcal{G}} \) over an \( \mathbb{E}_\infty \)-ring \( A \), there are thanks to Proposition 3.3.1 two complex-exponentiable quasi-coherent sheaves on \( \text{Spec}(A) \): the dualizing line \( \omega_{\hat{\mathcal{G}}} \) and the trivial functor \( \mathcal{O}^\otimes_{\text{Spec}(A)} \). Given any \( \mathbb{E}_\infty \)-ring map \( f : A \to B \), the base-change \( f^* \hat{\mathcal{G}} \) is a formal group on \( B \), and all the other structures in sight respect base-change. Let therefore \( \text{Triv}^\otimes_{\hat{\mathcal{G}}}(\omega_{\hat{\mathcal{G}}}) \in \text{Fun}(\text{CAlg}_A, S) \) be the presheaf given by
\[
B \mapsto \text{Map}^\otimes_{\text{QCoh}^c(\text{Spec}(B))}(\mathcal{O}^\otimes_{\text{Spec}(B)}, \omega^\otimes_{f^* \hat{\mathcal{G}}}).
\]
That is to say, \( \text{Triv}^\otimes_{\hat{\mathcal{G}}}(\omega_{\hat{\mathcal{G}}}) \) classifies trivializations of the dualizing line \( \omega_{\hat{\mathcal{G}}} \) as a complex-exponentiable quasi-coherent sheaf.

Theorem 3.3.4. Let \( \hat{\mathcal{G}} \) be an oriented formal group over an \( \mathbb{E}_\infty \)-ring \( A \), classified by a map of non-compact spectral stacks \( \eta_{\hat{\mathcal{G}}} : \text{Spec}(A) \to \mathcal{M}^\text{er}_{\text{FG}} \). The canonical morphism \( \text{Spec}(\text{MUP}) \to \mathcal{M}^\text{er}_{\text{FG}} \), stemming from complex-orientability of the Thom spectrum \( \text{MUP} \), induces a pullback square of non-compact stacks
\[
\begin{array}{ccc}
\text{Triv}^\otimes_{\hat{\mathcal{G}}}(\omega_{\hat{\mathcal{G}}}) & \longrightarrow & \text{Spec}(\text{MUP}) \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \xrightarrow{\eta_{\hat{\mathcal{G}}}^\otimes} & \mathcal{M}^\text{er}_{\text{FG}}.
\end{array}
\]

Proof. There is an essentially unique map \( \text{Spec}(A) \to \mathcal{M}^\text{er}_{\text{FG}} \) if and only if the \( \mathbb{E}_\infty \)-ring \( A \) is complex-orientable, therefore the pullback statement will follow immediately from a description of the functor of points \( \text{Spec}(\text{MUP}) : \text{CAlg} \to \mathcal{S} \). That is to say, suffices to verify that the canonical maps induce the homotopy pullback equivalence
\[
\text{Spec}(\text{MUP})(A) \times_{\mathcal{M}^\text{er}_{\text{FG}}(A)} \{\hat{\mathcal{G}}\} \simeq \text{Map}^\otimes_{\text{QCoh}^c(\text{Spec}(A))}(\mathcal{O}^\otimes_{\text{Spec}(A)}, \omega^\otimes_{\hat{\mathcal{G}}}).
\]
in the \( \infty \)-category of spaces.

By definition, the \( \mathbb{E}_\infty \)-ring \( \text{MUP} \) is defined as the Thom spectrum of the symmetric monoidal functor \( J : (\text{Vect}_C)^{\text{sp}} \to \mathcal{S} \). By the \( \infty \)-categorical perspective on Thom spectra, as developed in \text{ABG}^+14 and \text{ABG}18, an \( \mathbb{E}_\infty \)-ring map \( \text{MUP} \to A \) is therefore equivalent to an equivalence on the \( \infty \)-category of symmetric monoidal functors \( \text{Fun}^\otimes((\text{Vect}_C)^{\text{sp}}, \text{Mod}_A) \) between the composite
\[
(\text{Vect}_C)^{\text{sp}} \xrightarrow{J} \mathcal{S} \xrightarrow{\mathbb{A}^\otimes} \text{Mod}_A
\]
and the corresponding constant functor with value $A$.

Note that, for instance on the account of the the idempotence of the $E_\infty$-space map $(-)^{-1} : (\text{Vect}_C)_{CSp} \rightarrow (\text{Vect}_C)_{CSp}$, pre-composition with $(-)^{-1}$ induces a self-equivalence on the $\infty$-category $\text{Fun}^\otimes((\text{Vect}_C)_{CSp}, \text{Mod}_A)$. The constant functor with value $A$ is invariant under this self-equivalence, allowing us to conclude that an $E_\infty$-ring map $\text{MUP} \rightarrow A$ is also equivalent to the data of an equivalence in $\text{Fun}^\otimes((\text{Vect}_C)_{CSp}, \text{Mod}_A)$ between

$$(\text{Vect}_C)_{CSp} \xrightarrow{(-)^{-1}} (\text{Vect}_C)_{CSp} \xrightarrow{J} \text{Sp} \xrightarrow{A \otimes -} \text{Mod}_A$$

and the constant functor with value $A$. Since $J$ is a map of group-like $E_\infty$-spaces, there is a canonical commutative square of $E_\infty$-spaces

$$(\text{Vect}_C)_{CSp} \xrightarrow{(-)^{-1}} (\text{Vect}_C)_{CSp} \xrightarrow{J} \text{Sp} \xrightarrow{(-)^{-1}} \text{Sp},$$

which we may use to re-write the first of the two functors in question as

$$(\text{Vect}_C)_{CSp} \xrightarrow{J} \text{Sp} \xrightarrow{(-)^{-1}} \text{Sp} \xrightarrow{A \otimes -} \text{Mod}_A.$$

Both of the symmetric monoidal functors in question take values in the full symmetric monoidal subcategory $\text{Pic}^1(A) \subseteq \text{Mod}_A$. It is therefore equivalent to look for an equivalence between the two functors inside the full subcategory of symmetric monoidal functors $(\text{Vect}_C)_{CSp} \rightarrow \text{Pic}^1(A)$. The universal property of group completion garners a homotopy equivalence

$$\text{Map}_{\text{CMon}^\otimes(\mathcal{S})}((\text{Vect}_C)_{CSp}, \text{Pic}^1(A)) \simeq \text{Map}_{\text{CMon}^\otimes(\mathcal{S})}(\text{Vect}_C, \text{Pic}^1(A)),$$

and the right-hand side once again embeds fully faithfully into $\text{Fun}^\otimes(\text{Vect}_C, \text{Mod}_A)$. In conclusion, a map of $E_\infty$-rings $\text{MUP} \rightarrow A$ equivalently corresponds to the data of an equivalence in the $\infty$-category $\text{Fun}^\otimes(\text{Vect}_C, \text{Mod}_A) \simeq \text{QCoh}^\mathcal{C}(\text{Spec}(A))$ between the composite functor

$$\text{Vect}_C \xrightarrow{J} \text{Sp} \xrightarrow{(-)^{-1}} \text{Sp} \xrightarrow{A \otimes -} \text{Mod}_A$$

and the constant functor with value $A$. We may recognize the first of these functors as exhibiting the complex-exponentiable structure on the dualizing line $\omega_\mathcal{G}$ by the proof of Proposition 3.3.1. The constant symmetric monoidal functor with value $A$ similarly encodes the complex-exponentiable structure on the trivial bundle $\mathcal{O}_\text{Spec}(A)^\otimes$, leading to the conclusion of the Theorem.

**Remark 3.3.5.** The conclusion of Theorem 3.3.4 may be expressed as the assertion that the non-connector spectral scheme $\text{Spec}(\text{MUP})$ parametrizes the data of an oriented formal group $\mathcal{G}$, together with a trivialization of the dualizing line $\omega_\mathcal{G}$ as a complex-exponentiable quasi-coherent sheaf. Informally, an $E_\infty$-ring map $\text{MUP} \rightarrow A$ amounts to specifying an oriented formal group $\mathcal{G}$ over $A$, and a system of equivalences $\omega_\mathcal{G}^V \simeq A$ in the $\infty$-category $\text{Mod}_A$, equivariant and symmetric monoidal in $V \in \text{Vect}_C$. Yet more explicitly, the trivialization data consists of an $U(n)$-equivariant $A$-module equivalence $\theta_n : \omega_\mathcal{G}^n \simeq A$ for every $n \geq 0$, satisfying $\theta_n \otimes \theta_m = \theta_{n+m}$ and $\theta_0 \simeq \text{id}_A$.

**Remark 3.3.6.** In light of the proof of Theorem 3.3.4, the final informal description from Remark 3.3.5 may be upgraded, using the homotopy equivalence

$$(\text{Vect}_C)_{CSp} \simeq \mathbb{Z} \times \text{BU}.$$
the splitting \( \emptyset \) does not hold on the level of \( E_\infty \)-spaces, making the compatibility and coherence precise is less straightforward in this formulation than that of Remark 3.3.5.

### 3.4. Connection with coordinatized formal groups

We would like to relate the structure \( \text{triv}_C^G(\bar{G}) \), highlighted in Theorem 3.3.4, with the notion of coordinatized formal groups from Definition 3.1.1. First we must extend it to the spectral setting.

**Definition 3.4.1.** The space of coordinates on a formal group \( \bar{G} \) over an \( E_\infty \)-ring \( A \) is

\[
\text{Coord}_{\bar{G}} \simeq \Omega^\infty(\varnothing_{\bar{G}}(-\epsilon)) \times \Omega^\infty(\omega_{\bar{G}}) \text{Map}_{\text{Mod}_A}^\ast(A, \omega_{\bar{G}}).
\]

**Remark 3.4.2.** Equivalently, \( \text{Coord}_{\bar{G}} \) is the union of those path-connected components of the space \( \Omega^\infty(\varnothing_{\bar{G}}(-\epsilon)) \), which correspond to coordinates on the underlying formal group \( \bar{G}^0 \) over \( \pi_0(A) \). That is because \( \omega_{\bar{G}} \) is a flat \( A \)-module, hence an \( A \)-module map \( A \to \omega_{\bar{G}} \) is an equivalence if and only if it induces an equivalence on \( \pi_0 \), where it gives rise to a \( \pi_0(A) \)-linear map \( \pi_0(A) \to \omega_{\bar{G}}^0 \). As a consequence, the space \( \text{Coord}_{\bar{G}} \) is discrete whenever the base ring \( A \) is, recovering its meaning from Definition 3.1.1.

Consider the subcategory \( \text{Vect}_{C}^{\text{dim}=1} \subseteq \text{Vect}_C^\ast \) spanned by 1-dimensional complex vector spaces, and restriction of previously-discussed functors to this subcategory.

**Proposition 3.4.3.** Let \( \bar{G} \) be an oriented formal group over an \( E_\infty \)-ring \( A \). There is a canonical homotopy equivalence

\[
\text{Coord}_{\bar{G}} \simeq \text{Map}_{\text{Fun}(\text{Vect}_{C}^{\text{dim}=1}, \text{Mod}_A)}(A^\otimes, \omega_{\bar{G}}^\otimes).
\]

**Proof.** In light of the homotopy equivalence \( \text{Vect}_{C}^{\text{dim}=1} \simeq \text{BU}(1) \), an equivalence \( A^\otimes \simeq \omega_{\bar{G}}^\otimes \) of functors \( \text{Vect}_{C}^{\text{dim}=1} \to \text{Mod}_A \) amounts to two things:

(a) An \( A \)-module equivalence \( \theta : A \simeq \omega_{\bar{G}} \);

(b) A map of spectra \( \tau : \Sigma^\infty(\text{BU}(1)) \to A \).

These two are not unrelated, however. Restriction of \( \tau \) along the inclusion

\[
S^2 \simeq \text{CP}^1 \subseteq \text{CP}^\infty \simeq \text{BU}(1)
\]

gives rise to a map of \( A \)-modules \( \Sigma^2(A) \to A \), which must, through the orientation equivalence \( \Sigma^{-2}(A) \simeq \omega_{\bar{G}} \), induce the \( A \)-module isomorphism \( \theta \). Conversely, since orientation of \( \bar{G} \) implies that it is the Quillen formal group of the complex oriented \( E_\infty \)-ring \( A \), the identification \( \omega_{\bar{G}} \simeq \Sigma^{-2}(A) \) extends to an equivalence \( \varnothing_{\bar{G}}(-\epsilon) \simeq C^\ast_{\text{red}}(\text{CP}^\infty; A) \). Thus the map of spectra \( \tau \) is equivalent to an \( A \)-linear map \( t : A \to \varnothing_{\bar{G}}(-\epsilon) \). In conclusion, an equivalence \( A^\otimes \simeq \omega_{\bar{G}}^\otimes \) in the \( \infty \)-category \( \text{Fun}(\text{Vect}_{C}^{\text{dim}=1}, \text{Mod}_A) \) is equivalent to:

(*) An \( A \)-module map \( t : A \to \varnothing_{\bar{G}}(-\epsilon) \), whose composite with the canonical map

\[
\varnothing_{\bar{G}}(-\epsilon) \to \omega_{\bar{G}}
\]

induces a trivialization of the dualizing line of the formal group \( \bar{G} \), which is to say, an \( A \)-module equivalence \( A \simeq \omega_{\bar{G}} \).

That is precisely the data of a coordinate on \( \bar{G} \).

As in the setting of ordinary algebraic geometry, the space of coordinates in the sense of Definition 3.4.1 is compatible with base-change along \( E_\infty \)-ring maps \( A \to B \), giving rise to a functor \( \text{Coord}(\bar{G}) : \text{CAlg}_A \to S \). Under the canonical equivalence of \( \infty \)-categories \( \text{Fun}(\text{CAlg}_A, S) \simeq \text{Fun}(\text{CAlg}_S, S)/\text{Spec}(A) \), this may equivalently be seen as a map \( \text{Coord}(\bar{G}) \to \text{Spec}(A) \) of functors \( \text{CAlg} \to S \). This is analogous to the prestack \( \text{Triv}^\ast_C(\omega_{\bar{G}}) \) from Construction 3.3.3 and Proposition 3.4.3 allows us to relate the two.

**Corollary 3.4.4.** Let \( \bar{G} \) be an oriented formal group on an \( E_\infty \)-ring \( A \). There is a canonical map of non-connective spectral prestacks \( \text{Triv}^\ast_C(\omega_{\bar{G}}) \to \text{Coord}(\bar{G}) \).
**Proof.** In light of Theorem 3.3.4 and Proposition 3.4.3, this is induced by the composition 
\[
\text{Qcoh}_C(\text{Spec}(A)) \cong \text{Fun}^\otimes(\text{Vect}_C^\infty, \text{Mod}_A) \to \text{Fun}(\text{Vect}_C^\infty, \text{Mod}_A) \to \text{Fun}(\text{Vect}_C^{\text{dim}=1}, \text{Mod}_A)
\]
of the forgetful functor from symmetric monoidal to non-symmetric-monoidal functors, with the functor restriction along the subcategory inclusion \(\text{Vect}_C^{\text{dim}=1} \subseteq \text{Vect}_C^\infty\).

**Remark 3.4.5.** A coordinate on an oriented formal group \(\hat{G}\) over an \(E_\infty\)-ring \(A\) is by Proposition 3.4.3 equivalent to an \(U(1)\)-equivariant equivalence \(\omega_G \cong A\). This gives rise to a symmetric monoidal system of \(U(n)\)-equivariant equivalences \(\omega^{\otimes n}_G \cong A\) for every \(n \geq 0\). The additional structure encoded in a symmetric monoidal \(U(n)\)-equivariant equivalences \(\omega^{\otimes n} \cong A\) for every \(n \geq 0\), equivalent according to the discussion of Remark 3.3.5 to an \(E_\infty\)-map \(\omega\) by Theorem 3.3.4 is therefore in extending the \(U(1)^n\)-equivariance to an \(U(n)\)-equivariance, along the diagonal matrix inclusion \(U(1)^n \subseteq U(n)\).

3.5. **Universal property of the Snaith construction for MP.** Another form of periodic complex bordism is the Snaith \(E_\infty\)-ring \(\text{MP}_{\text{Snaith}} := (S[BU])[\beta^{-1}]\). Here we use the series of inclusions
\[
S^2 \cong \mathbb{CP}^1 \subseteq \mathbb{CP}^\infty \cong BU(1) \subseteq BU
\]
inducing a map of spectra \(\beta : \Sigma^2(S) \cong \Sigma^\infty(S^2) \to \Sigma^\infty(BU) \to S[BU]\), which is equivalent to an element \(\beta \in \pi_2(S[BU])\). The Snaith \(E_\infty\)-ring is not equivalent to the Thom spectrum \(MUP\) by [HY19]. We can describe the map of non-connective spectral stacks \(\text{Spec}(\text{MP}_{\text{Snaith}}) \to \mathcal{M}_{\mathbb{FG}}^\omega\) in a way somewhat analogous to the description of the map \(\text{Spec}(L) \to \mathcal{M}_{\mathbb{FG}}^\omega\) through Definition 3.1.1. But compared to Theorem 3.3.4 the \(U\)-action is incorporated in a somewhat *ad hoc* manner.

**Construction 3.5.1.** Let \(\hat{G}\) be an oriented formal group over an \(E_\infty\)-ring \(A\). Any fixed map of \(E_\infty\)-rings \(S[BU] \to A\) induces an \(A\)-module map \(\tau : A \to \omega_G\) by
\[
A \xrightarrow{\Sigma^{-2}\beta} \Sigma^{-2}(A) \cong \omega_G,
\]
where the unlabeled equivalence exhibits the orientation of \(\hat{G}\). This gives rise to a canonical map of spaces \(\text{Map}_{\text{CAlg}}(S[BU], A) \to \text{Map}_{\text{Mod}_A}(A, \omega_G)\).

**Proposition 3.5.2.** Let \(\hat{G}\) be an oriented formal group over an \(E_\infty\)-ring \(A\). The canonical map of non-connective spectral stacks \(\text{Spec}(\text{MP}_{\text{Snaith}}) \to \mathcal{M}_{\mathbb{FG}}^\omega\), stemming from complex-orientability of the Thom spectrum \(MUP\), induces a homotopy pullback square
\[
\text{Map}_{\text{CAlg}}(S[BU], A) \times_{\text{Map}_{\text{Mod}_A}(A, \omega_G)} \text{Map}_{\text{Mod}_A}^\omega(A, \omega_G) \longrightarrow \text{Spec}(\text{MP}_{\text{Snaith}})(A)
\]
in the \(\infty\)-category of spaces.

**Proof.** Once again, as there is an essentially unique map \(\text{Spec}(A) \to \mathcal{M}_{\mathbb{FG}}^\omega\) if and only if the \(E_\infty\)-ring \(A\) is complex-orientable, the pullback statement will follow immediately from a description of the functor of points \(\text{Spec}(\text{MP}_{\text{Snaith}}) : \text{CAlg} \to \delta\).

By the universal property of localization of \(E_\infty\)-rings, e.g. [Ell2, Proposition 4.3.17], an \(E_\infty\)-ring map \(\text{MP}_{\text{Snaith}} = (S[BU])[\beta^{-1}] \to A\) is equivalent to an \(E_\infty\)-ring map \(S[BU] \to A\) for which the induced \(A\)-module map \(\beta : \Sigma^{-2}(A) \to A\) is an equivalence. In light of the definition of the map \(\tau : A \to \omega_G\) as
\[
A \xrightarrow{\beta} \Sigma^{-2}(A) \xrightarrow{\tau} \omega_G,
\]
it is immediate that $\beta$ is invertible if and only if $\tau$ is. That proves the claim. \qed

**Remark 3.5.3.** Using the perspective of Subsection 3.4, the universal property of the Snaith construction from Proposition 3.4.3 may be expressed in terms of coordinates. Indeed, a comparison between the proof of Proposition 3.4.3 and Construction 3.5.1 shows that the map $\text{Map}_{\text{CAlg}}(S[BU], A) \to \text{Map}_{\text{Mod}_A}(A, \omega_{\tilde{G}}) \simeq \Omega^\infty(\omega_{\tilde{G}})$ for $A$ a complex-oriented $E_\infty$-ring with Quillen formal group $\tilde{G}$, naturally factors through a map

$$\text{Map}_{\text{CAlg}}(S[BU], A) \to \text{Map}_{\text{Sp}}(\Sigma^\infty(\mathbb{CP}^\infty), A) \simeq \Omega^\infty(\omega_{\tilde{G}}(-\epsilon)).$$

In light of this, the conclusion of Proposition 3.5.2 may now be expressed as

$$\text{Map}_{\text{CAlg}}(\text{MP}_{\text{Snaith}}, A) \simeq \text{Map}_{\text{CAlg}}(S[BU], A) \times_{\Omega^\infty(\omega_{\tilde{G}}(-\epsilon))} \text{Coord}_{\tilde{G}}.$$

Similarly to Remark 3.4.5 for the Thom cover $\text{Spec}(\text{MUP}) \to \text{M}_{\text{FG}}\text{or}$, the Snaith fpqc cover $\text{Spec}(\text{MP}_{\text{Snaith}}) \to \text{M}_{\text{FG}}\text{or}$ thus extends the coordinate-discarding map $\text{Coord}(\tilde{G}) \to \text{M}_{\text{FG}}\text{or}$, for $\tilde{G}$ the universal oriented formal group over $\text{M}_{\text{FG}}\text{or}$. Then Construction 3.5.1 may be seen as giving rise to a map of non-connective spectral stacks

$$\text{Spec}(S[BU]) \times \text{M}_{\text{FG}}\text{or} \to \text{Spec}(\text{Sym}^*(\Sigma^\infty(\mathbb{CP}^\infty))) \times \text{M}_{\text{FG}}\text{or} \simeq \text{Map}_{\text{sp}}(\tilde{G}, \tilde{A}^1),$$

and Proposition 3.5.2 expresses the universal property of the Snaith $E_\infty$-ring as

$$\text{Spec}(\text{MP}_{\text{Snaith}}) \simeq (\text{Spec}(S[BU]) \times \text{M}_{\text{FG}}\text{or}) \times_{\text{Map}_{\text{sp}}}(\tilde{G}, \tilde{A}^1) \text{Coord}(\tilde{G}).$$

3.6. Toward an algebro-geometric proof of Quillen’s Theorem. The key result we have used to prove the basic properties of the stack $\text{M}_{\text{FG}}\text{or}$ is Quillen’s Theorem, identifying $\pi_0(\text{MP}) \simeq L$ with the Lazard ring, classifying coordinatized formal groups. It is through this theorem that complex bordisms really enter the discussion of either oriented formal groups in particular, or chromatic homotopy theory in general.

Recall that the standard proof of Quillen’s Theorem, e.g. [Ada74 Part II], [Rav04 Section 4.1], or [Lur10 Lectures 7 - 10], relies on combining two computations: Lazard’s Theorem on $L$ and Milnor’s Theorem on $\pi_0(\text{MP}) = \pi_*(\text{MU})$, which identify both with polynomial rings in countably many variables. It is this “matching of the two sides” and the involved computations that go into proving these results that make Quillen’s Theorem rather mysterious, even while it is an undeniable foundation of a large chunk of modern homotopy theory.

While our spectral algebro-geometric methods are sadly not capable of producing an alternative more insightful proof of Quillen’s Theorem, we are able to isolate its geometric content. We show that it is equivalent to the following statement purely about the non-connective spectral stack $\text{M}_{\text{FG}}\text{or}$, with no trace of complex bordisms.

**Theorem 3.6.1.** Quillen’s Theorem (i.e. Theorem 3.1.2) is equivalent to the assertion (of Corollary 2.3.9) that the canonical map $\text{M}_{\text{FG}}\text{or} \to \text{M}_{\text{FG}}$ induces an equivalence

$$(\text{M}_{\text{FG}}\text{or})^\circ \simeq \text{M}_{\text{FG}}$$

upon underlying ordinary stacks, exhibiting the ordinary stack of formal groups as the underlying ordinary stack of $\text{M}_{\text{FG}}\text{or}$. 
Proof. We have already seen above in the proof of Corollary 2.3.9 how the desired statement about underlying ordinary stacks follows from Quillen’s Theorem.

To go the other way, let us assume that the map \( \text{Spec}_\text{FG} \to \text{Spec}_\text{FG} \) induces an equivalence between underlying mapping stacks. Let the \( \mathbb{E}_{\infty}\)-ring \( M \) be any form of periodic complex bordism. It is complex periodic, hence there is a canonical map of non-connective spectral stacks \( \text{Spec}(M) \to \text{Spec}_\text{FG} \). Observe that we can write

\[
\text{Spec}_\text{FG} \cong \lim_{A \in C} \text{Spec}(A),
\]

indexed over the full subcategory \( C \subseteq \text{CAlg} \) spanned by complex periodic \( \mathbb{E}_{\infty}\)-rings. Using this, we obtain a series of equivalences of non-connective spectral stacks

\[
\text{Spec}(M) \cong \text{Spec}(M) \times_{\text{Spec}_\text{FG}} \text{Spec}_\text{FG} \cong \lim_{A \in C} \text{Spec}(A) \\
\cong \lim_{A \in C} \text{Spec}(A) \times_{\text{Spec}_\text{FG}} \text{Spec}(A) \\
\cong \lim_{A \in C} \text{Spec}(M \otimes A),
\]

in which the third equivalence is due to the fact that \( \text{Shv}_{\text{Spec}} \) is an \( \infty\)-topos and hence pullbacks in it are universal [HTT, Theorem 6.1.0.6], while the fourth and final equivalence follows from the observation that fiber products of affines over \( \text{Spec}_\text{FG} \) is given by the smash product, that we had originally made in the proof of Lemma 2.3.2. We may use this colimit formula for \( \text{Spec}(M) \), together with the fact that passage to the underlying ordinary stack by definition commutes with colimits, see Remark 1.2.3, to conclude that

\[
\text{Spec}(\pi_0(M)) \cong \lim_{A \in C} \text{Spec}(\pi_0(M \otimes A)).
\]

The colimit formula [7], combined with the assumption on underlying ordinary stacks, similarly implies that

\[
\text{Spec}_\text{FG} \cong (\text{Spec}_\text{FG})^\varnothing \cong \lim_{A \in C} \text{Spec}(\pi_0(A)).
\]

On the other hand, consider the ordinary moduli stack of coordinatized formal groups \( \text{Spec}_\text{FG}^{\text{coord}} \). It admits a canonical map \( \text{Spec}_\text{FG}^{\text{coord}} \to \text{Spec}_\text{FG} \) by discarding the choice of coordinate. The fiber of this map along any map of ordinary stacks \( \text{Spec}(R) \to \text{Spec}_\text{FG} \), classifying a formal group \( \mathcal{G} \) over the commutative ring \( R \), is by definition

\[
\text{Spec}_\text{FG}^{\text{coord}} \times_{\text{Spec}_\text{FG}} \text{Spec}(R) \cong \text{Coord}(\mathcal{G}).
\]

Using the same reasoning as we did for \( \text{Spec}(M) \) above, we obtain

\[
\text{Spec}_\text{FG}^{\text{coord}} \cong \lim_{A \in C} \text{Spec}(\pi_0(A)).
\]

The one new ingredient, justifying the final equivalence, is the following observation. The map \( \text{Spec}(A) \to \text{Spec}_\text{FG} \) induces upon underlying ordinary stacks the map \( \text{Spec}(\pi_0(A)) \to (\text{Spec}_\text{FG})^\varnothing \cong \text{Spec}_\text{FG} \), classifying the ordinary Quillen formal group \( \mathcal{G}_A^{\text{Q}_0} = \text{Spf}(A[\text{CP}_\infty]) \) of the complex periodic \( \mathbb{E}_{\infty}\)-ring \( A \), see [Zil] Notation 4.1.14.

The desired Theorem of Quillen, which is to say that the map \( \text{Spec}(\pi_0(M)) \to \text{Spec}_\text{FG}^{\text{coord}} \), induced by the canonical coordinate on \( \mathcal{G}_A^{\text{Q}_0} \), is an equivalence of ordinary stacks, now
follows from the colimit formulas derived above, together with one classical homotopical computation. That is the fact [Ell2 Theorem 5.3.13] that the canonical coordinate on \( \hat{G}^\varnothing_{MP} \) induces the equivalence of ordinary stacks

\[
\text{Spec}(\pi_0(M \otimes A)) \simeq \text{Coord}(\hat{G}^\varnothing_A).
\]

for any complex periodic E\(_\infty\)-ring (or even just commutative ring spectrum) \( A \). For a proof, which does not require Quillen’s Theorem, see [Rez07 Proposition 7.11], [Lur10 Lecture 7], or [Hop99 Proposition 6.5].

**Remark 3.6.2.** The idea behind Theorem 3.6.1 is in some shape well-known to the experts. For instance, [Pet11] is (modulo the non-periodic setting) essentially a concise summary of the proof we have just given.

Having a proof of Quillen’s Theorem directly through Theorem 3.6.1 would of course be wonderful, but it seems to be beyond our current reach. We sketch one possible approach below, reducing it to a concrete computation, collected in Claim 3.6.3 yet one which we are completely incapable of tackling.

For this, we first recall that the canonical map \( \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG} \) admits a factorization \( \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG} \) through the stack of preoriented formal groups \( \mathcal{M}_{FG} \). Here a a pre-orientation on a formal group \( \hat{G} \) over an \( E_\infty \)-ring \( A \), in the sense of [Ell2 Definition 4.3.1], is a map of pointed spaces \( S^2 \to \hat{G}(\tau_{\leq 0}(A)) \). By a linearization procedure [Ell2 Construction 4.3.7], a preorientation gives rise to an \( A \)-module map \( \omega_{\hat{G}} \to \Sigma^{-2}(A) \). Demanding that this map is an equivalence produces an orientation on \( \hat{G} \), hence the map \( \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG}^{\text{pre}} \). Quillen’s Theorem in light of Theorem 3.6.1 (i.e. the conclusion of Corollary 2.3.9) now boils down to two facts:

(i) The map \( \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG} \) induces an equivalence upon underlying ordinary stacks.

(ii) The map \( \mathcal{M}_{FG}^{\text{pre}} \to \mathcal{M}_{FG}^{\text{pre}} \) induces an equivalence upon underlying ordinary stacks.

We offer two different proofs of the first one:

**Proof of (i) (via abstract nonsense).** Claim (i) unlike (ii) concerns only connective spectral stacks, hence we may restrict to the connective setting for the length of discussing it. By Remark 1.2.5 the underlying ordinary stacks may thus be computed by restriction along the subcategory inclusion \( \text{CAlg}_{\varnothing} \subseteq \text{CAlg} \). For any commutative ring \( R \), the space \( \mathcal{M}_{FG}^{\text{pre}}(R) \) consists of formal groups \( \hat{G} \in \mathcal{M}_{FG}(R) \), together with preorientation maps \( S^2 \to \hat{G}(R) \). But the space \( \hat{G}(R) \) is discrete, hence \( \pi_2(\hat{G}(R)) = 0 \) and there is no non-trivial preorientation data.

**Proof of (i) (via explicit construction).** Once again we restrict ourselves to the connective setting. Recall from [Ell2 Lemma 4.3.16] that the stack of preoriented formal groups may be expressed in terms of the universal formal group \( \hat{G} \) over \( \mathcal{M}_{FG} \) as the double based loop space of the zero-section \( \mathcal{M}_{FG} \to \hat{G} \) (viewed as a map of spectral stacks). That is to say, we have an equivalence of spectral stacks

\[
\mathcal{M}_{FG}^{\text{pre}} \simeq \mathcal{M}_{FG} \times_{\mathcal{M}_{FG} \times_{\hat{G}} \mathcal{M}_{FG}} \mathcal{M}_{FG}.
\]

On the other hand, since any formal group is formally affine, we have \( \hat{G} \simeq \text{Spf}_{\mathcal{M}_{FG}}(\mathcal{O}_{\hat{G}}) \), hence the stack of preoriented formal groups may be written as

\[
\mathcal{M}_{FG}^{\text{pre}} \simeq \text{Spf}_{\mathcal{M}_{FG}}(\text{HH}(\mathcal{O}_{\mathcal{M}_{FG}}/\mathcal{O}_{\hat{G}}))
\]

in terms of Hochschild homology \( \text{HH}(A/R) \simeq A \otimes_{A \otimes R} A \). Its underlying ordinary stack is therefore

\[
(\mathcal{M}_{FG}^{\text{pre}})^{\varnothing} \simeq \text{Spf}_{\mathcal{M}_{FG}^{\varnothing}}(\pi_0(\text{HH}(\mathcal{O}_{\mathcal{M}_{FG}}/\mathcal{O}_{\hat{G}}))).
\]

For any pair of connective \( E_\infty \)-ring map \( R \to A \) we have

\[
\pi_0(\text{HH}(A/R)) \simeq \text{HH}_0(\pi_0(A)/\pi_0(R)) \simeq \pi_0(A),
\]

as the double based loop space a map of pointed spaces.
hence the connectivity assumption implies that $\pi_0(\text{HH}(\mathcal{O}_{\mathcal{M}^\text{pre}}/\mathcal{O}_G)) \cong \mathcal{O}_{\mathcal{M}^\text{pre}}^\circ$, proving assertion [ii].\hfill $\square$

In order to discuss assertion [ii], we must recall how to construct $\mathcal{M}_{\text{FG}}^\text{or}$ from $\mathcal{M}_{\text{FG}}^\text{pre}$. Let $\widehat{G}^\text{pre}$ denote the universal preoriented formal group over $\mathcal{M}_{\text{FG}}^\text{pre}$. Its preorientation gives rise to the universal Bott map $\beta: \omega_{G^\text{pre}} \to \Sigma^2(\mathcal{O}_{\mathcal{M}_{\text{FG}}^\text{pre}})$, which must be inverted in order to make $\widehat{G}^\text{pre}$ oriented and hence pass to $\mathcal{M}_{\text{FG}}^\text{or}$. That is to say, by [Ell2, Proposition 4.3.17], the non-connective stack of oriented formal groups is obtained from the stack of preoriented ones by

$$\mathcal{M}_{\text{FG}}^\text{or} \cong \text{Spec}_{\mathcal{M}_{\text{FG}}^\text{pre}} \left( \lim_{n \to \infty} \text{HH}_{2n}(\mathcal{O}_{\mathcal{M}^\text{pre}}/\mathcal{O}_G) \otimes_{\mathcal{O}_{\mathcal{M}_{\text{FG}}^\text{pre}}} (\omega_{\mathcal{M}_{\text{FG}}^\text{pre}}^{-1})^n \right).$$

To find the underlying ordinary stack, we therefore need to determine $\pi_0$ of the quasi-coherent sheaf $\mathcal{O}_{\mathcal{M}_{\text{FG}}^\text{or}}$ on $\mathcal{M}_{\text{FG}}^\text{pre}$. The use of continuity of the functor $\pi_0$, flatness of the dualizing sheaf, and the explicit Hochschild formula for $\mathcal{O}_{\mathcal{M}_{\text{FG}}^\text{pre}}$ from the second proof above, combine to show that [ii] (and therefore Quillen’s Theorem) is equivalent to

**Claim 3.6.3.** Let $\widehat{G}$ denote the universal formal group on the spectral stack $\mathcal{M}_{\text{FG}}$. The canonical map of ordinary quasi-coherent sheaves on $\mathcal{M}_{\text{FG}}^\circ$,

$$\mathcal{O}_{\mathcal{M}_{\text{FG}}} \to \lim_{n \to \infty} \text{HH}_{2n}(\mathcal{O}_{\mathcal{M}_{\text{FG}}}^\circ/\mathcal{O}_G) \otimes_{\mathcal{O}_{\mathcal{M}_{\text{FG}}}^\circ} (\omega_{\mathcal{M}_{\text{FG}}}^{-1})^n$$

is an isomorphism.

Attempting to prove this claim directly would however require a greater degree of understanding of the spectral moduli stack of formal groups $\mathcal{M}_{\text{FG}}$, and the universal formal group $\widehat{G}$ over it, than we currently possess.

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