MHD flow in a circular pipe with arbitrarily conducting slipping walls

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Abstract. In this study, the magnetohydrodynamic (MHD) flow is simulated in a circular pipe with slipping and arbitrarily conducting boundary. The 2D governing coupled equations in terms of the velocity and the induced magnetic field are solved by the Dual Reciprocity Boundary Element Method (DRBEM). The discretized system of equations is solved in one stroke without introducing an iteration which reduces the computational cost. It is shown that, the flow decelerates, Hartmann layers enlarge through the top and the bottom of the pipe and induced current lines align as the wall conductivity or Hartmann number increases. An increase in the slip length accelerates the flow, shrinks the stagnant region, diminishes the boundary layers and retards the effect of the wall conductivity increase. The DRBEM is an advantageous method in solving MHD flow especially with slipping and arbitrarily conducting boundary conditions, since it enables to insert both the unknowns and their normal derivatives in slip and conductivity wall conditions.

1. Introduction

Magnetohydrodynamics is the scientific branch investigating the flow of electrically conducting fluids under magnetic field (MHD flow). It has many engineering and industrial applications as electromagnetic pumps, MHD generators, nuclear reactors and microfluidics. The derivation of the governing equations for the laminar MHD flow in channels are presented in [1]. Analytical solutions for the MHD flow are available only for simple geometries and simple boundary conditions. The need of approximate and experimental solutions have been arisen in the case of high Hartmann number limit. Asymptotic solutions for the MHD pipe flow when Hartmann number is high are presented in [2]. The improvement in the numerical methods brings new approximate solutions for more complicated flow configurations. Among these with finite difference method [3], finite volume method [4], finite element method [5] can be counted. The DRBEM is an advantageous method in the solutions of MHD channel and pipe flow problems due to its boundary only nature which results in discretized matrix-vector equations small in size. The DRBEM transforms the partial differential equations into the boundary integral equations using the fundamental solution of Laplace equation and approximating the inhomogeneity using radial basis functions. There are quite a lot numerical studies in MHD using the DRBEM such as [6, 7]. In all these numerical and analytical studies no-slip boundary condition is considered for the velocity. However, in microfluidics and surface roughness slip may occur on the boundary exhibiting also varying electrical conductivity.

In this study, we present a DRBEM solution for the MHD pipe flow with arbitrarily conducting, slipping walls. The coupled MHD equations are discretized by taking constant boundary elements and arbitrary number of interior nodes. It is found that, Hartmann and Robert layers
are developed as Hartmann number increases in the no-slip case but the increase in the slip weakens the Hartmann layers. The flow behaves as if the walls are perfectly conducting when the wall conductance ratio increases.

2. The physical problem and the mathematical model
We consider laminar, fully developed flow of an electrically conducting, incompressible fluid in a circular pipe with thin arbitrarily conducting and slipping walls. The fluid is pumped within the pipe due to a constant pressure gradient $\partial P/\partial z$ in the pipe axis direction and the flow is under the influence of a horizontally applied uniform magnetic field $B_0$. Being a fully developed flow the problem is modeled in the 2D cross-section of the pipe which is a unit disk. The governing coupled MHD equations for the third components of the velocity and the magnetic induction fields in nondimensional form [1, 5], and the boundary conditions are

$$\nabla^2 V + Ha \frac{\partial B}{\partial x} = -1 \quad V + \alpha \frac{\partial V}{\partial n} = 0 \quad \text{in } \Omega$$

$$\nabla^2 B + Ha \frac{\partial V}{\partial x} = 0 \quad B + \beta \frac{\partial B}{\partial n} = 0 \quad \text{on } \partial \Omega = \Gamma$$

where $V(x, y)$ is the velocity and $B(x, y)$ is the induced magnetic field. The non-dimensional parameter $Ha = B_0 L \frac{\sqrt{\sigma/\rho \nu}}{\lambda}$ is called the Hartmann number with $L$, $\sigma$, $\rho$, $\nu$ being the radius of the circular pipe, electrical conductivity, density and kinematic viscosity of the fluid, respectively. The slip at the wall is measured by the nondimensional slip length $\alpha$ which is the distance from the fluid to the surface within the solid phase where the extrapolated flow velocity vanishes $[8]$. $\beta$ in the electromagnetic boundary condition is called the wall conductance ratio defined as $\beta = (\sigma_w t_w)/(\sigma L)$ where $t_w$ and $\sigma_w$ are the thickness and the electrical conductivity of the wall, respectively. Notice that for insulated walls ($\sigma_w = 0$, $\beta = 0$) the condition is reduced to $B = 0$ whereas for perfectly conducting walls ($\sigma_w \to \infty$, $\beta \to \infty$) the condition becomes $\partial B/\partial n = 0$.

3. Application of the DRBEM
The coupled MHD equations in (1) are discretized by the DRBEM using at most 160 constant boundary elements. All the terms other than the Laplacian are taken as inhomogeneity and the differential equations are transformed to boundary integral equations by weighting them with the fundamental solution of the Laplace equation ($u = \frac{1}{2 \pi} \ln(1/r)$) $[9]$ and applying Green’s first identity two times. The inhomogeneities are approximated by the radial basis functions $f_j(r) = 1 + r_j$ which are connected to the particular solutions with $\nabla^2 \hat{u}_j = f_j$, $r_j$ being the distance between source and the field points $[9]$. The discretization of the boundary with $N$ constant elements and taking $L$ arbitrary interior points, one achieves the matrix-vector equations

$$H V - G \frac{\partial V}{\partial n} = -(\hat{H} \hat{U} - G \hat{Q}) F^{-1} \{ Ha \frac{\partial B}{\partial x} + 1 \}$$

$$H B - G \frac{\partial B}{\partial n} = -(\hat{H} \hat{U} - G \hat{Q}) F^{-1} \{ Ha \frac{\partial V}{\partial x} \}$$

in terms of the DRBEM matrices

$$H_{i j} = \int_{\Gamma_j} q^* d\Gamma_j, \quad H_{i i} = c_i, \quad G_{ij} = \int_{\Gamma_j} u^* d\Gamma_j, \quad G_{ii} = \frac{l}{2 \pi} (\ln(\frac{1}{l}) + 1)$$

with $l$ being the length of the element and $q^* = \partial u^*/\partial n$. The matrices $\hat{U}$, $\hat{Q}$ are constructed by taking each vector $\hat{u}_j$ and $\hat{q}_j$ as columns, respectively. The space derivatives of the unknowns $V$ and $B$ are approximated by the coordinate matrix $F$ as $\partial V/\partial x = (\partial F/\partial x) F^{-1} V$, $\partial B/\partial x = (\partial F/\partial x) F^{-1} B$ where $F$ is constructed with $f_j$’s column
wise. The two matrix-vector equations are combined and after the insertion of boundary conditions, a linear system $Ax = b$ is obtained and solved as a whole. The obtained linear system is small in size due to the boundary only nature of the DRBEM. This procedure provides the nodal solutions for the velocity and the induced magnetic field at all the boundary and interior nodes in one stroke and reduces the computational cost.

4. Numerical results

The numerical solutions are presented in terms of equivelocity and current isolines for various $Ha$, $\alpha$ and $\beta$ values. In the MHD pipe flow three regions occur: core region where the flow is almost stagnant, Robert layers and the Hartmann layers [10]. Robert layers are located adjacent to the wall where its normal is perpendicular to the external magnetic field ($B_0$) and Hartmann layers are in the vicinity of the pipe wall in which the normal vector is not perpendicular to $B_0$. The presence of these boundary layers are well observed in Figure 1 when the walls are electrically insulated ($\beta = 0$). As Hartmann number increases, flow is flattened, core region enlarges through the Hartmann walls and the extend of the Robert layers decreases leaving their places to Hartmann layers. Two counter current loops are observed with centers close to the walls and these centers shift through the walls developing boundary layers with an increase in the external magnetic field strength. These behaviors are in very well agreement with the ones in [3].

![Figure 1: Velocity and induced magnetic field profiles $\alpha = 0$, $\beta = 0$.](image)

An increase in the wall conductance ratio (Figure 2), decreases the flow speed and directs the currents between the walls diminishing the boundary layers and enlarges the core region vertically. As $\beta$ reaches to 10, the flow and induced magnetic field behaviors become similar to the case that the walls are electrically conducting comparing the results with the one in [4].

![Figure 2: Velocity and induced magnetic field profiles $Ha = 10$, $\alpha = 0$.](image)
When the pipe wall admits slip and the walls are insulated \((\beta = 0\), Figure 3) Hartmann layers weaken and flow accelerates similar to the case of rectangular duct flow with insulated walls [8]. In strong slip condition \(\alpha = 0.4\), the slip phenomenon becomes significant especially in the vicinity of the Hartmann walls which squeezes the core region vertically.

In Figure 4 the influence of both the slip and the wall conductance ratio are shown. It is observed that, the vertical enlargement of the core region with an increase in the wall conductance ratio is retarded even for small slip length. When \(\alpha = 0.4\) the slip at the wall dominates the flow behavior.

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