Measuring the degree of unitarity for any quantum process

Jing-Xin Cui and Z. D. Wang

Department of Physics and Center of Theoretical and Computational Physics,
The University of Hong Kong, Pokfulam Road, Hong Kong, China

(Dated: April 3, 2014)

Quantum processes can be divided into two categories: unitary and non-unitary ones. For a given quantum process, we can define a degree of the unitarity (DU) of this process to be the fidelity between it and its closest unitary one. The DU, as an intrinsic property of a given quantum process, is able to quantify the distance between the process and the group of unitary ones, and is closely related to the noise of this quantum process. We derive analytical results of DU for qubit unital channels, and obtain the lower and upper bounds in general. The lower bound is tight for most of quantum processes, and is particularly tight when the corresponding DU is sufficiently large. The upper bound is found to be an indicator for the tightness of the lower bound. Moreover, we study the distribution of DU in random quantum processes with different environments. In particular, the relationship between the DU of any quantum process and the non-markovian behavior of it is also addressed.

I. INTRODUCTION

Quantum computation is of great interest in recent years for its great speed up for solving some problems\cite{2,3} and its efficiency in simulating physical systems\cite{1}. Ideally, a quantum computer is a closed quantum system, and the evolution of the system is a unitary operation. However, as any quantum computer is inevitably interacted with the environment, the system is more or less open, and thus the system evolution becomes non-unitary. This is one of major difficulties for making a large scale quantum computer.

When a quantum system becomes open, the information about the initial state may be lost after the evolution of the system, and we cannot undo a quantum operation in general. This is discussed in quantum computation, where quantum channel capacity is introduced to describe the information transfer ability of a quantum channel (see, e.g., Ref. \cite{4}). Quantum channel capacity also illustrates the quantum channel noise. Notably, there are also other ways to study the noise of a quantum channel, such as using the addition of noise to change a quantum map into an entanglement-breaking map \cite{5}.

Since we can divide quantum operations into two categories: the unitary operations which are ideal, and the non-unitary ones which are most of the cases in real quantum systems, it is nature to ask to what extend a general quantum operation deviates from the group of unitary quantum operations, which appears to be fundamentally important. We here treat all of the unitary operations as a group because the information of a general quantum state can be perfectly preserved for all unitary operations.

To study the distance between a given quantum operation and the group of unitary operations, a common way is to find the closest unitary quantum operation for the given one, and use the distance between these two operations as the distance between the given quantum operation and the unitary operation group. The measure of the distance between two quantum processes has already been well studied\cite{6,9}, and we choose quantum process matrix fidelity in this paper. We define the fidelity between a given quantum process and its closest unitary one as the degree of unitarity (DU) of this given quantum process.

We think the study of the measure of the DU is important because of the following reasons. First of all, DU quantifies the difference between a realistic quantum operation and the ideal ones. It is an important intrinsic property for a quantum operation. Also, the DU of a quantum process is closely related to the noise of this process, the quantum capacity, and some other physical quantities such as the non-markovian behavior of a quantum process.

To obtain the DU of a given quantum process, a core problem we need to solve is to find the closest unitary operations of the given quantum process, since an analytical result for the measure of the distance between two quantum processes has already been available\cite{6,9}. However, finding an optimal unitary matrix is in general difficult because the optimization should be taken over all the unitary matrices. Remarkably, we here obtain analytical results for all qubit unital channels. Moreover, we also reveal the upper and lower bounds for the DU of a general quantum process. The lower bound is very tight and deviates from the true value slightly for most quantum processes, and it can be treated as a good approximation of the DU in most cases.

Our paper is organized as follows. In the next section (Sec. II), we give the definition of the DU of a quantum process and address its properties. In Sec. III, we give analytical results for qubit unital channels and some other special cases. In Sec. IV, we reveal the upper and lower bounds for the DU of a quantum process in general cases, and discuss the tightness of the bounds. In Sec. V, we study the probability distribution of DU of a quantum system interacting with environment of different dimensions. Finally, we present a brief summary and outlook in Sec. VI.
II. DEFINITION AND PROPERTIES

A quantum process $\varepsilon$ can be expressed in Kraus operators\[^{10}\]

$$\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger,$$  \hspace{1cm} (1)

where the operators satisfy $\sum_k E_k^\dagger E_k = I$ to ensure this is a trace preserving map. The Kraus operator representation has clear physical meaning and every part of the summation can be treated as the evolution of the system when the environment is measured and a specific result is obtained. But since we need a group of matrices to represent a quantum process using Kraus operators, sometimes it is more convenient to use quantum process matrix to describe a quantum process which only need one matrix.

Suppose $\{ |m\rangle \}$ is an orthogonal basis set for the system, let $A_j = |m\rangle\langle n |$, then the quantum process $\varepsilon$ can be expressed as

$$\varepsilon(\rho) = \sum_{m,n} (\chi_{\varepsilon})_{mn} A_m \rho A_n^\dagger,$$  \hspace{1cm} (2)

where $\chi$ is called the quantum process matrix which contains all the information of the quantum process.

For a given quantum process $\varepsilon$ and a unitary operation $U$, how to measure the distance between them has been widely discussed\[^{8–9}\]. One way is consider the difference of the output states for $U$ and $\varepsilon$. As we want the distance between two processes to be independent of the initial state, a nature way is to average over all the initial states. If we choose fidelity to measure the difference of the output states, the distance between the processes can be measured by

$$F = \int d|\psi\rangle \langle \psi|U^\dagger \varepsilon(|\psi\rangle \langle \psi|)U|\psi\rangle.$$  \hspace{1cm} (3)

Here $|\psi\rangle$ is the initial state of the system. The integration is carried over the whole Hilbert space of the quantum state of the system, and the volume of the state is calculated according to Harr measure.

The average fidelity measure not only has clear physical meaning, but also has analytical expression.

In Ref.\[^{7}\] they give an analytical result in this formula:

$$F_{\text{ave}}(\varepsilon, U) = \frac{n + \sum_k |\text{tr}(U^\dagger E_k)|^2}{n(n+1)},$$  \hspace{1cm} (4)

where $n$ is the dimension of the system.

Another way to measure the distance is to measure the distance between the quantum process matrices directly. We also choose fidelity to measure the distance between process matrices. Then the fidelity between two processes is

$$F_{\text{pro}}(\varepsilon, U) = F(\chi_{\varepsilon}, \chi_U),$$  \hspace{1cm} (5)

where $F(\chi_{\varepsilon}, \chi_U) = (\text{tr}(\chi_{\varepsilon}^\dagger \chi_U)^2)^2$.

It turns out that the average fidelity measure of two quantum processes and the fidelity between two process matrices is related by the formula below\[^{6}\]:

$$F_{\text{ave}}(\varepsilon, U) = \frac{nF_{\text{pro}}(\varepsilon, U) + 1}{n+1}$$  \hspace{1cm} (6)

Substituting it into Eq. (4), we can get that

$$F_{\text{pro}}(\varepsilon, U) = \sum_k |\text{tr}(U^\dagger E_k)|^2$$  \hspace{1cm} (7)

We choose $F_{\text{pro}}(\varepsilon, U)$ to measure the distance between the quantum process $\varepsilon$ and the unitary evolution $U$ in this paper. The reasons for this will be stated later in this section. In the following, we denote $F_{\text{pro}}(\varepsilon, U)$ as $F(\varepsilon, U)$ for simplicity.

Now we consider the main topic of this paper, the measure of the degree of unitarity (DU) for a given quantum process $\varepsilon$. As it is discussed in Sec. I we choose the nearest unitary operation $U_0$ for the given quantum process $\varepsilon$ and use $F(\varepsilon, U_0)$ to measure the distance between $\varepsilon$ and $U$. We define $F(\varepsilon, U_0)$ as the DU for $\varepsilon$.

This definition can be treated as the geometry measure of the DU, and the geometry measure is also used in other cases such as quantum discord\[^{11}\].

According to the definition and Eq. (7), the DU of a quantum process $\varepsilon$ is

$$DU(\varepsilon) = \text{Max}\{ \sum_k |\text{tr}(U^\dagger E_k)|^2 |U| \}.$$  \hspace{1cm} (8)

Generally, it is hard to get analytical result for the above expression as the optimization is over all the unitary matrices. Luckily in this problem for some special cases we can get clean result. We will discuss this in details in the next section.

Here we focus on the properties of DU to illustrate that this is indeed a reasonable definition and measures the similarity between a general quantum process and unitary ones.

First, the $DU$ of $\varepsilon$ should be a property of the process and should not be depended on the choice of Kraus operators $E_k$. This property is assured by that the average fidelity is independent of the choice of the Kraus operators\[^{7}\].

Also, the DU of $\varepsilon$ should not be smaller than the DU $f \circ \varepsilon$, which means that the quantum process cannot become more unitary by subsequent processes. This is also guaranteed by the properties of average fidelity\[^{4}\].
This property also make the DU as an indicator for non-markovian behavior of a quantum process\cite{12}. It can be deduced that if DU increases during a quantum evolution, then this quantum evolution is non-markovian. However, the converse is not true.

Next we consider the extreme values of the DU of $\varepsilon$. When $\varepsilon$ is unitary, the DU should reach its maximum. This can be easily confirmed because in this case the nearest unitary operation is $\varepsilon$ itself and one can get that DU of this unitary process is 1. For the minimum of DU, intuitively, one can choose the quantum process (or quantum channel) to be a maximal depolarizing channel, which maps all the initial states into the maximum mixed state. In this case, one can choose arbitrary unitary operation as the nearest one, and the DU of this process is equal to $\frac{1}{n^2}$.

Finally we discuss the difference between the DU of $\varepsilon$ and $\varepsilon \circ U$, where $U$ is a unitary operation acting on an additional quantum system. One can verify that

$$ DU(\varepsilon) = DU(\varepsilon \circ U), \quad (9) $$

which means that the DU of a given quantum process will not change by adding an unitary operation imposed on an ancillary quantum systems. Also one can verify that if we choose the average fidelity as the measure for two quantum processes, although the nearest unitary operation will not change, the above property will not be satisfied. This is the reason that we choose the process matrix fidelity measure.

One can also refer to Table 1 to get a clearer picture of DU, where the DU of some important qubit channel is listed. Table 1 can be get using the method provided in the next section.

### III. ANALYTICAL RESULT FOR SOME SPECIAL CASES

To get the DU of a quantum process $\varepsilon$, we need to find the nearest $U$ from the group of unitary matrices. This is in general difficult. However, if the quantum process $\varepsilon$ satisfies some properties, we can get analytical result.

We introduce the inner product in the operator space, and denote the inner product between two operators $A$ and $B$ as $\langle A, B \rangle$:

$$ \langle A, B \rangle = \text{tr}(A^\dagger B) \quad (10) $$

Then according to Eq. (8), the DU of the quantum process $\varepsilon$ can be expressed as

$$ DU(\varepsilon) = \text{Max}\left\{ \frac{\sum_k |\langle U, E_k \rangle|^2}{n^2} |U\right\}. \quad (11) $$

So the DU of a quantum process is the sum of the square of the projections of the nearest $U$ on each Kraus operator. If each Kraus operator is a unitary operation multiplied by a constant, and they are orthogonal with each other, then the optimization process can be greatly simplified.

**Theorem 1:**

For a quantum process $\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger$, if $E_k = \alpha U_k$ for all $k$s and $U_k \perp U_j$ for $k \neq j$, then

$$ DU(\varepsilon) = |\alpha_{k_{\text{max}}}|^2, \quad (12) $$

where $\alpha_{k_{\text{max}}}$ is the coefficient with the max norm.

**Proof:**

Expand $\{U_k\}$ into a set of orthogonal basis in the operator space with operations $\{V_j\}$. Then any unitary operation $U$ can be expressed as a linear combination of $\{U_k, V_j\}$,

$$ U = \sum_k a_k U_k + \sum_j b_j V_j, \quad (13) $$

where $\sum_k |a_k|^2 + \sum_j |b_j|^2 = 1$. Then

$$ \sum_k |\langle U, E_k \rangle|^2 = \sum_k |(\sum_i a_i U_i + \sum_j b_j V_j) E_k|^2 $$

$$ = \sum_k |(\sum_i a_i U_i) E_k|^2 $$

$$ = \sum_k \sum_i |a_i|^2 \delta_{ki} |\alpha_k|^2 n^2 $$

$$ = n^2 \sum_k |a_k|^2 |\alpha_k|^2 \quad (14) $$

Suppose $\alpha_1$ is the coefficient with the biggest norm. As $\sum_k |a_k|^2 \leq 1$, one can see that the above expression reaches its maximum when $|\alpha_1| = 1$.

In this case, according to the expression of the DU of a quantum process in Eq. (8),

$$ DU(\varepsilon) = \text{Max}\left\{ \frac{\sum_k |\langle U, E_k \rangle|^2}{n^2} |U\right\} $$

$$ = |\alpha_1|^2. \quad (15) $$

This completes the proof.

Of course, the situation in Theorem 1 is a very special case, it requires all the Kraus operators to be proportional to unitary operations and are orthogonal to each other at the same time. But one can notice that the representation of a quantum process using Kraus operators has its freedom and it turns out that every quantum process can be represented in orthogonal Kraus operators\cite{13}.

Suppose a quantum process $\varepsilon(\rho) = \sum_k E_k \rho E_k^\dagger$, let $W$ be the correlation matrix with $W_{jk} = \langle E_j, E_k \rangle$. One can note that $W$ is a Hermitian matrix. We can diagonalize $W$ with unitary matrix $u$

$$ D = uWu^\dagger \quad (16) $$
It can be proved that the rank of $D$ is $n^2$ at most, where $n$ is the dimension of the system.

Let

$$F_i = \sum_j u_{ij}^* E_j \quad (17)$$

One can show that $\{F_i\}$ is equivalent with $\{E_k\}$ and

$$\langle F_i, F_k \rangle = \delta_{ik} D_{kk} \quad (18)$$

Further more, for a qubit channel, if the original Kraus operators $E_k$s are proportional to unitary operations, then after diagonalization, the operators $F_k$ are still unitary operations. This theorem can be found in Ref. [13].

Here we give a different proof using the notations in this paper.

Proof:

Suppose the Kraus operators are $E_k = \sqrt{p_k} U_k$, where $\sum_k p_k = 1$, $U_k^\dagger U_k = I$.

Follow the diagonalization process stated above, we can get the Kraus operators after diagonalization $\{F_k\}$. Then

$$F_k^\dagger F_k = \sum_{ij} u_{kj} u_{ki}^* E_j^\dagger E_i \quad (19)$$

One can note that if $E_k$ is replaced by $\alpha E_k$ with $|\alpha| = 1$, the quantum process will be the same as before. So we can treat all the $U_k$s as SU(2) matrices. For a SU(2) matrix, one can represent it as

$$U = \begin{pmatrix} \cos \theta e^{-i\phi} & \sin \theta e^{i\gamma} \\ -\sin \theta e^{-i\gamma} & \cos \theta e^{-i\phi} \end{pmatrix} \quad (20)$$

One can verify that $\text{tr}(U)$ is a real, and

$$U + U^\dagger = \begin{pmatrix} 2 \cos \theta \cos \phi & 0 \\ 0 & 2 \cos \theta \cos \phi \end{pmatrix} \quad (21)$$

Also one can notice that $W$ is an orthogonal matrix in this case, so $u$ is real too. On the r.h.s. of Eq. (19), if $i = j$, $u_{kj} u_{ki}^* E_j^\dagger E_i$ is proportional to $I$. If $i \neq j$, as $u$ is real and $E_j^\dagger E_i + E_i^\dagger E_j$ is proportional to $I$, $u_{kj} u_{ki}^* E_j^\dagger E_i + u_{ki} u_{kj}^* E_i^\dagger E_j$ is also proportional to $I$. So we can get

$$F_k^\dagger F_k = \alpha I, \quad (22)$$

where $\alpha$ is a complex number. This condition implies that $F_k$ is proportional to a unitary operation.

Combined with Theorem 1 together, we can get that if a qubit channel is a convex combination of unitary channels, we can get analytical result for the DU of this channel.

For qubit channels, it can be shown that every unital channel is a convex combination of unitary channels, where a unital channel is a channel that maps $\rho = \frac{1}{n} I$ into $\rho' = \frac{1}{n} I$. This is related to Birkhoff conjecture, which is only true for $n = 2$ [13]. Since we can get analytical result for convex combination of unitary channels, we can get analytical result of the DU for all qubit unital channels.

Convex combination of unitary channels are of great interest as they are the only channels that can be perfectly inverted by monitoring the environment [13]. The calculation of DU for these channels offers a new perspective for the information and the noise of these channels.

Now we list the analytical result of DU in Table 1 for some important qubit channels, including depolarizing channel, bit flip channel, phase flip channel, and amplitude damping channel. We can see that the DU of these quantum channel is between $\frac{1}{2}$ and 1, as it is shown in Sec. IV. For depolarizing channel, bit flip channel and phase flip channel, the nearest unitary operation changes when the parameter of the channel changes. For amplitude damping channel, the nearest unitary operation is always the identity operation.

IV. LOWER BOUND AND UPPER BOUND FOR DU IN GENERAL CASES

In general cases, it is hard to get analytical result for the DU of a quantum process. We try to get the lower bound and upper bound for DU in this section.

To get the lower bound, we try to find a unitary operation that is very close to the quantum process, although it may not be the closest. As it is hard to find the closest $U$ for all these Kraus operators, we try to find the closest $U$ for one of them.

One option is to find the nearest $U$ for the Kraus operator with the largest norm after diagonalization. Suppose the Kraus operators of $\varepsilon$ after diagonalization are $\{F_i\}$, and the norm of $F_i$, $\sqrt{\text{tr}(F_i^\dagger F_i)}$, is the largest among all the $F_i$s. Then the closest unitary matrix for $F_i$, which is the unitary matrix that maximize $|\text{tr}(U^\dagger F_i)|$, can be found though polar decomposition.

Suppose the polar decomposition of $F_i$ is

$$F_i = U_i \sqrt{F_i^\dagger F_i}, \quad (23)$$

Then

$$|\text{tr}(U^\dagger F_i)| \leq |\text{tr}(U^\dagger U_1 \sqrt{F_i^\dagger F_i})|$$

$$= |\text{tr}(U^\dagger U_1 \sqrt{F_i^\dagger F_i})|$$

$$\leq \sqrt{\text{tr}(F_i^\dagger F_i) \text{tr}(U^\dagger U_1 F_i^\dagger U_1 U)}$$

$$= \text{tr}(F_i^\dagger F_i) \quad (24)$$
where the equality is obtained when $U = U_1$. One can note that $U_1$ is the nearest unitary matrix for $F_1$.

We choose $U_1$ as the approximation of the nearest unitary operation for $\varepsilon$, and calculate the DU of this quantum process. Then we get the lower bound for DU of $\varepsilon$,

$$DU(\varepsilon) \geq \frac{\sum_{i} |\text{tr}(U_i^† F_i)|^2}{n^2}$$  \hspace{1cm} (25)

Also as $F_i$s are orthogonal with each other, the contribution of $|\text{tr}(U_i^† F_i)|^2$ is relatively small if $i$ is not 1, i.e., if $F_i$ is not the Kraus operator with the largest norm. So we can also choose $\sum_{i} |\text{tr}(U_i^† F_i)|^2$ as a simplified lower bound, which equals $\frac{(\sum_{i} \sigma_i)^2}{n^2}$, where $\sigma_i$s are the singular value of $F_i$. This lower bound equals the largest visibility the interference between $\varepsilon$ and unitary channels, and it can be got experimentally [10].

We can get the lower bound in another way. We choose the Kraus operator that may contribute most to $DU(\varepsilon)$ as the major part, of which the value $\varepsilon$ is large. We denote the major Kraus operator by this new standard as $F_0$, and the nearest unitary operation of $F_0$ as $U_0$, then the new low bound for $DU(\varepsilon)$ is

$$DU(\varepsilon) \geq \frac{\sum_{i} |\text{tr}(U_i^† F_i)|^2}{n^2}$$  \hspace{1cm} (26)

Now we consider the upper bound of the DU of a quantum process. We assume the nearest $U$ of $\varepsilon$ is the nearest unitary operation of all the Kraus operator $F_i$s, which means that there is a unitary operation that makes $|\text{tr}(U_i^† F_i)|^2$ reaches its largest value for all $F_i$s. Apparently, this is not possible for most cases. But it give a upper bound for the DU of this quantum process.

According to Eq. (21), the largest value of $|\text{tr}(U_i^† F_i)|^2$ is $(\sum_{j} \sigma_j)^2$. So we can get a upper bound for $DU(\varepsilon)$,

$$DU(\varepsilon) \leq \frac{\sum_{i} (\sum_{j} \sigma_{ij})^2}{n^2} \hspace{1cm} (27)$$

We can show that the upper bound of $DU(\varepsilon)$ is less or equal than 1. As

$$\frac{\sum_{i} (\sum_{j} \sigma_{ij})^2}{n^2} \leq \frac{n (\sum_{i} \sigma_{ij}^2)}{n^2} = \frac{\text{tr}(\sum_i F_i^† F_i)}{n} = 1 \hspace{1cm} (28)$$

Now we consider the tightness of the lower bound and upper bound. We denote the lower bound in Eq. (25) as lower bound 1, and the one in Eq. (26) as lower bound 2.

We randomly choose a quantum process and calculate the DU of the quantum process, two lower bounds of the DU, and the upper bound of DU. Also, in order to study the bound behavior as a function of the DU, we choose approximately equal amount of quantum processes for each interval of the DU of quantum processes. As shown in Fig. 1(a), the lower bound of DU(red stars) is a good approximation of DU(black cross) in almost all the cases. Particularly, when DU is larger than 0.4, the lower bound of DU(red stars) is almost the same as DU. When the DU of a quantum process is very small, the lower bound may differs from the true value for about 10%.

On the other hand, the upper bound of DU is quite loose. But it can be shown that the upper bound may be a indicator of the behavior of the lower bound. As shown in Fig. 1(b), the difference of two lower bounds of DU from the true value are almost zero when the upper bound of DU is bigger than 0.8. When the upper bound of is small, the error of the lower bound may be large.

| Channel                  | Kraus Operators                                                                 | DU of this channel |
|-------------------------|---------------------------------------------------------------------------------|--------------------|
| depolarizing channel    | $\sqrt{1 - \frac{3p}{4}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{\frac{p}{4}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sqrt{\frac{p}{4}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\text{Max}\{\frac{p}{4}, 1 - \frac{3p}{4}\}$ |
| bit flip channel        | $\sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{1 - p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  | $\text{Max}\{p, 1 - p\}$                 |
| phase flip channel      | $\sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sqrt{1 - p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  | $\text{Max}\{p, 1 - p\}$                 |
| amplitude damping channel| $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \gamma} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$ | $(1 + \sqrt{1 - \gamma})^2$            |

Table I: The DU of some important qubit channels.
Quantum Process Number
DU and Bound of DU

DU of quantum process
Lower bound 1 of DU
Upper bound of DU

(a)

Quantum Process Number

DU
Count Number

(b)

DU
Count Number

Figure 1: (a) The DU of random quantum processes and the lower bound and upper bound of the DU of the quantum process. The random quantum processes are generated by a random unitary operation for two qubits with one qubit traced off. The data are rearranged with the DU of them in ascending order. (b) The error of two lower bounds of the DU of quantum processes. The data are rearranged with the upper bound of DU in ascending order.

Figure 2: The probability distribution of the DU of the evolution of a qubit system interacting with an auxiliary system with the dimension of (a) 2, (b) 4. The evolution of the total system is a random unitary operation according to Harr measure. The number of the random quantum process is one million for each case.

For simplicity, we only consider the cases when \( d = 2, 4 \). We generate a random unitary operation for the total system according to the Harr measure, and calculate the DU for the corresponding quantum operations imposed on the principle qubit system. As it is already tested in Sec. [IV] that the lower bound of DU is very tight, we treated the lower bound of DU as DU in this calculation. As shown in Fig. 2, when the dimension of the system become bigger, the expected value for the DU is decreased. This means that the system is expected to be more open and the evolution of the system is expect to be more non-unitary.

VI. SUMMARY AND OUTLOOK

In conclusion, we have investigated a key problem of the DU for any quantum process. We have introduced a definition of the DU of a quantum process and addressed its properties. The DU of a quantum process quantifies the distance between a given quantum process and the group of all unitary ones. It is closely related to the noise of the quantum process and is an indicator for non-markovian behavior of a quantum system. For qubit unital channels, we have obtained an analytical result of the DU. For general cases, we have derived the lower and upper bounds for the DU. We have presented two different lower bounds for the DU, both being quite tight in most cases. The upper bound of DU can be treated as an indicator for the tightness of DU. When the upper bound is low, the lower bound of DU may be less tight. We have also discussed the probability distribution of the DU of a qubit system interacting with different environments, and found that the DU tends to become smaller when the dimension of the environment become bigger.

The study of DU of a quantum process raises many related issues to be investigated in near future, such as the relationship between DU of a quantum process and the noise of a quantum process according to other measures, the DU’s evolution behavior for a real physical system, the DU for the quantum process in an open quantum system based quantum algorithm[17, 18] and so on.
Acknowledgments

We thank Y. Hu and Z. Y. Xue for helpful discussions. This work was supported by the RGC of Hong Kong un-

der Grant No. HKU7058/11P.

[1] R. Feynman, Int. J. Theor. Phys. 21, 467 (1982).
[2] P. W. Shor, in Proc. 35th Symposium on the Foundations of Computer Science, (IEEE Computer Society Press), 124 (1994).
[3] L.K. Grover, Phys. Rev. Lett. 79, 325 (1997).
[4] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[5] A. De Pasquale and V. Giovannetti, Phys. Rev. A 86, 052302 (2012).
[6] A. Gilchrist, N. K. Langford and M.A. Nielsen, Phys. Rev. A 71, 062310(2005).
[7] M. A. Nielsen, Phys. Lett. A 303, 249(2002).
[8] M.D. Grace, J. Dominy, R. L Kosut et. al, New Journal of Physics 12, 015001(2010)
[9] Z. Puchała, J. A. Miszczak, P. Gawron et al, Quantum Information Processing 10(1), 1-12(2011).
[10] Karl Kraus, A. Bo’hm, J. D. Dollard, and W. H. Woot-
ters, States, Effects, and Operations Fundamental No-
tions of Quantum Theory: Lectures in Mathematical Physics at the University of Texas at Austin, Lecture Notes in Physics, Vol. 190 (Springer, Berlin, 1983).
[11] S. Luo and S. Fu, Phys. Rev. A 82, 034302(2010).
[12] H. P. Breuer, E. M. Laine, and J. Piilo, Phys. Rev. Lett. 103, 210401(2009).
[13] I. Bengtsson and K. Zyczkowski, Geometry of Quantum States: an Intro- duction to Quantum Entanglement (Cambridge University Press, 2006)
[14] L.J. Landau and R.F. Streater, Lin. Alg. Applic. 193, 107-127 (1993).
[15] M. Gregoratti and R.F. Werner, J. Mod. Opt. 50, 915 (2003).
[16] D. K. L. Oi Phys. Rev. Lett. 91, 067902(2003).
[17] A. Mizel, Phys. Rev. Lett. 102,150501(2009).
[18] G. L. Long, Int. J. Theor. Phys. 50(2011).