On the Number of Incidences When Avoiding the Klan

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Abstract

Given a set of points $P$ and a set of regions $O$, an incidence is a pair $(p, o) \in P \times O$ such that $p \in o$. We obtain a number of new results on a classical question in combinatorial geometry: What is the number of incidences (under certain restrictive conditions)?

We prove a bound of $O(kn(\log n / \log \log n)^{d-1})$ on the number of incidences between $n$ points and $n$ axis-parallel boxes in $\mathbb{R}^d$, if no $k$ boxes contain $k$ common points, that is, if the incidence graph between the points and the boxes does not contain $K_{k,k}$ as a subgraph. This new bound improves over previous work, by Basit, Chernikov, Starchenko, Tao, and Tran (2021), by more than a factor of $\log d n$ for $d > 2$. Furthermore, it matches a lower bound implied by the work of Chazelle (1990), for $k = 2$, thus settling the question for points and boxes.

We also study several other variants of the problem. For halfspaces, using shallow cuttings, we get a linear bound in two and three dimensions. We also present linear (or near linear) bounds for shapes with low union complexity, such as pseudodisks and fat triangles.

1. Introduction

Problem statement. Let $P$ be a set of $n$ points in $\mathbb{R}^d$, $O$ be a set of $m$ objects in $\mathbb{R}^d$, and let $k$ be a parameter. Assume that there are no $k$ points of $P$ that are all contained in $k$ objects of $O$. Formally, consider the incidence graph

$$G = G(P, O) = (P \cup O, \{p o \mid p \in P, o \in O, \text{ and } p \in o\}),$$

which we assume does not contain the graph $K_{k,k}$. The question is how many edges $G$ has, in the worst case. As $p \in o$ is an incidence, the question can be formulated as bounding the number of incidences between the points of $P$ and the objects of $O$. We denote the number of edges of $G(P, O)$ by $I(P, O)$.

Background. The above problem is interesting because the number of incidences of $n$ points and $n$ lines is bounded by $\Theta(n^{4/3})$, in the worst case, and this graph avoids $K_{2,2}$, as any two lines containing the same two distinct points are identical. The bound on the incidences of lines and points has a long history; see [Szé97] and references therein. Incidence problems are among the most central classes of problems in combinatorial geometry and are closely related to algorithmic problems in computational geometry such as range searching [AE99] (and the so-called “Hopcroft’s problem” [AE99, CZ22]).

Fox et al. [FPS+17] studied the above problem where the objects are defined by semi-algebraic sets. For halfspaces, their bound is $O((mn)^{d/(d+1)+\varepsilon} + n + m)$ for any constant $k$ (however they are studying a more general problem). They also showed that a bound of $O(mn^{1-1/d} + n)$ holds if the set system has VC dimension $d$ (with $n$ elements and $m$ sets). Somewhat surprisingly, Janzer and Pohoata [JP21] improved this bound to

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\( o(n^{2-1/d}) \) for \( m = n \) and \( k \ge d > 2 \). See also follow-up papers by Do [Do19] and Frankl and Kupavskii [FK21], which gave improvements on the hidden dependence on \( k \) in Fox et al.’s bounds.

There was also recent interest in the simpler version of the problem where the ranges are axis-aligned boxes. Basit et al. [BCS+21] showed an upper bound of \( O(n \log^4 n) \) for \( m = n \) in two dimensions, and \( O(n \log^{2d} n) \) for \( d > 2 \), assuming constant \( k \). They proved a lower bound of \( \Omega(n \log n / \log \log n) \) in two dimensions for \( m = n \) and \( k = 2 \). They also studied the case where the objects are polytopes formed by the intersection of \( s \) translated halfspaces, where they show a bound of \( O(n \log^s n) \). Independently, Tomon and Zakharov [TZ21] showed a bound of \( O(kn \log^{2d+3} n) \) for the case \( m = n \) in \( \mathbb{R}^d \). They also showed a better \( O(n \log n) \) bound for the special case \( d = 2 \) if the intersection graph avoids \( K_{2,2} \).

**Our results.** We obtain a plethora of new results as summarized in Table 1. To simplify discussion below, we focus on the main setting when \( m = n \) and \( k \) is constant:

- For the axis-aligned rectangles/boxes case, we prove an \( O(n(\log n / \log \log n)^{d-1}) \) bound. Not only does this improve the previous bounds by Basit et al. [BCS+21] and Tomon and Zakharov [TZ21] by more than a \( \log^d n \) factor, but our bound is also tight for \( d = 2 \) since it matches Basit et al.’s \( \Omega(n \log n / \log \log n) \) lower bound. It is also tight for all larger \( d \), as we observe in Appendix B that an \( \Omega(n(\log n / \log \log n)^{d-1}) \) lower bound for all \( d \ge 2 \) actually follows from a proof by Chazelle [Cha90] on a seemingly different topic (data structure lower bounds), three decades before Basit et al.’s paper.

- For halfspaces in \( d > 3 \) dimensions, we prove an \( O(n^{2(d/2)/([d/2]+1)}) \) bound. Not only is this a significant improvement over the previous \( O(n^{2d/(d+1)+\varepsilon}) \) bound, and it is tight for \( d = 5 \) since there is a matching \( \Omega(n^{4/3}) \) lower bound for 5D halfspaces; see Remark 3.11. (The dependence on \( k \) in our bound is also better than in some previous results.)

- For halfspaces in 2 or 3 dimensions, we prove the first \( O(n) \) bound, which is linear (and thus obviously tight). The 3D halfspace case is particularly important, since 2D disks reduce to 3D halfspaces by a standard lifting transformation.

- For well-behaved shapes in the plane that have linear union complexity (for example, pseudo-disks), we obtain a bound that is very close to linear (and thus very close to tight), namely, \( O(n \log \log n) \). For fat triangles in the plane, our bound is extremely close to linear, namely, \( O(n(\log^4 n)^2) \) (i.e., at most two iterated logarithmic factors away from tight).

**Our techniques.** The combinatorial problem here is closely related to the algorithmic problem of range searching, and techniques developed for the latter will be useful here. The connection with range searching can be formally explained via the notion of **biclique covers** (see [Do19] or the beginning of Section 3), which readily yielded an \( O(n^{2d/(d+1)+\varepsilon}) \) bound for halfspaces and an \( O(n \log^d n) \) bound for axis-aligned boxes. To get better bounds, however, we will solve the problem directly.

For rectangles and boxes, we use simple geometric divide-and-conquer, similar to **range trees** [BCKO08], to obtain an \( O(n \log^{d-1} n) \) bound.

For halfspaces, the key intuition is that there cannot be too many deep points when the incidence graph avoids \( K_{k,k} \); the precise quantitative claim is stated in Lemma 3.9, which we prove using **shallow cuttings** (first introduced by Matoušek [Mat92a] for halfspace range reporting). Now that we know most points are shallow, we can use shallow cuttings again to bound the number of incidences by geometric divide-and-conquer.

For halfspaces in 3D, this approach unfortunately generates an extra logarithmic factor. We present a variant of the approach using a different, interesting recursion in which we reduce the number of points by a fraction. By duality, we can similarly reduce the number of halfspaces by a fraction. In combination, the recurrence then yields a geometric series, summing to a linear bound.

Going back to rectangles and boxes, our shallow-cutting-based approach for 3D halfspaces can also be applied to obtain a linear bound for 2D 3-sided rectangles. Combined with geometric divide-and-conquer with
Lemma 2.1. Let $P$ be a set of $n$ points, and let $O$ be a set of $m$ intervals on the real line. If the graph $G(P,O)$ does not contain $K_{k,k}$, then the maximum number of incidences between $P$ and $O$ is $I_1(n,m) \leq kn + 3km$.

**Proof:** Assume the points of $P$ are $p_1 < p_2 < \cdots < p_n$. We break $P$ into $N = [n/k]$ sets, where the $i$th set is

$$P_i = \{p_{k(i-1)+1}, \ldots, p_{k(i-1)+k}\},$$

a nonconstant branching factor, we can then improve the $O(n \log n)$ bound for 2D general axis-aligned rectangles to the tight bound of $O(n \log n / \log \log n)$, and similarly improve the $O(n \log^{d-1} n)$ bound for boxes in $\mathbb{R}^d$ to $O(n \log n / \log \log n)^{d-1}$.

For shapes with low union complexity in 2D, we can also adapt the shallow cutting approach, but because of the lack of duality, we get a different bound with an extra $\log \log n$ factor. For fat triangles in 2D, one of the $\log^* n$ factors comes from known bounds on the union complexity by Aronov et al. [ABES14]. On the other hand, the second $\log^* n$ factor arises from an entirely different reason, interestingly, from the way we use shallow cuttings. Along the way, the ideas in our proof yield a new data structure for range reporting for 2D fat triangles, which may be of independent interest.

The interplay between combinatorial geometry and computational geometry is nice to see, and is one of our reasons for studying this class of problems in the first place. In some sense, the incidence problems here are “cleaner” abstractions of the algorithmic problem of range searching, as one does not have to be concerned with computational cost of certain operations such as point location. But there are subtle differences, giving rise to different bounds. The connection to range searching is also what led us to realize that a lower bound argument in Chazelle’s paper [Cha90] had addressed the same combinatorial problem.

### 2. Intervals, Rectangles and Boxes

#### 2.1. Intervals

**Lemma 2.1.** Let $P$ be a set of $n$ points, and let $O$ be a set of $m$ intervals on the real line. If the graph $G(P,O)$ does not contain $K_{k,k}$, then the maximum number of incidences between $P$ and $O$ is $I_1(n,m) \leq kn + 3km$.

**Proof:** Assume the points of $P$ are $p_1 < p_2 < \cdots < p_n$. We break $P$ into $N = [n/k]$ sets, where the $i$th set is

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| Dimension | Objects | Bound | ref |
|-----------|---------|-------|-----|
| $d = 1$   | intervals | $\leq kn + 3km$ | Lemma 2.1 |
| $d > 1$   | axis-aligned boxes | $O(kn(\log n / \log \log n)^{d-1} + km \log^{d-2+\epsilon} n)$ | Theorem 4.5 |
| $d > 1$   | $\delta$-polyhedra | $O(kn(\log n / \log \log n)^{\delta-1} + km \log^{\delta-2+\epsilon} n)$ | Lemma 2.4/T4.5 |
| $d = 2, 3$ | halfplanes | $O(k(n + m))$ | Lemma 3.12 |
| $d > 3$   | halfspaces | $O(k^2/(\lfloor d/2 \rfloor + 1)(\lfloor d/2 \rfloor) + k(n + m))$ | Lemma 3.10 |
| $d = 2$   | disks    | $O(k(n + m))$ | Corollary 3.14 (I) |
| $d > 3$   | balls    | $O(k^2/(\lfloor d/2 \rfloor + 1)(\lfloor d/2 \rfloor) + k(n + m))$ | Corollary 3.14 (II) |
| $d = 2$   | shapes with union complexity $U(m)$ | $O(kn + kU(m)(\log \log m + \log k))$ | Theorem 5.7 |
| $d = 2$   | pseudo-disks | $O(kn + km(\log \log m + \log k))$ | Corollary 5.8 |
| $d = 2$   | fat triangles | $O(kn + km(\log^* m)(\log^* m + \log \log k))$ | Corollary 5.13 |
for \( i = 1, \ldots, N \), where all the sets contain \( k \) points, except the last set which might contain fewer points. Now, let \( \mathcal{O}_i = \{ \ell \in \mathcal{O} \mid \ell \cap P_i \neq \emptyset \} \), and \( m_i = |\mathcal{O}_i| \). For \( i < N \), at most \( k - 1 \) intervals of \( \mathcal{O}_i \) contains all the points of \( P_i \), and all other intervals of \( \mathcal{O}_i \) must have an endpoint in \( \text{ch}(P_i) = [\min P_i, \max P_i] \). In particular, let \( e_i \) be the number of intervals of \( \mathcal{O}_i \) with an endpoint in \( \text{ch}(P_i) \) that do not contain this interval fully. We have

\[
I(P, \mathcal{O}) = \sum_{i=1}^{N-1} I(\mathcal{O}_i, P_i) + I(\mathcal{O}_N, P_N) \leq \sum_{i=1}^{N-1} (e_i(k - 1) + (k - 1)\ell) + |P_N|m
\]

\[
\leq 2m(k - 1) + k(k - 1)(N - 1) + km < k(n + 3m).
\]

\[\blacksquare\]

### 2.2. Axis-aligned boxes

For integers \( \alpha \leq \beta \), let \([\alpha : \beta] = \{\alpha, \alpha + 1, \ldots, \beta\} \) denote the range between \( \alpha \) and \( \beta \). A dyadic range is a range \( J = \left[s2^i : (s + 1)2^i - 1\right] \), for some non-negative integers \( s \) and \( i \), where \( i \) is the rank of \( J \). Such a range has two children – specifically, the dyadic ranges \( \left[s2^i : s2^i + 2^{i-1} - 1\right] \) and \( \left[s2^i + 2^{i-1} : (s + 1)2^i - 1\right] \). Let \( \mathcal{D}(n) \) be the set of all dyadic ranges contained in \([n] = [1 : n] \).

The following is well known, and we provide a proof in Appendix A for completeness.

**Lemma 2.2.** Let \( n > 1 \) be an integer, and consider an arbitrary range \( I = [\alpha : \beta] \subseteq [n] \). Let \( \text{dy}(I) \) denote a minimal (cardinality wise) union of disjoint dyadic ranges that covers \( I \). The set \( \text{dy}(I) \) is unique and has \( \leq 2 \log n \) ranges, where log is in base 2.

A \( d \)-dimensional box is the Cartesian product of \( d \) intervals (we consider here only axis-aligned boxes). We can now apply an inductive argument on the dimension, to get a bound on the number of incidences between points and boxes.

**Lemma 2.3.** Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \), and let \( \mathcal{O} \) be a set of \( m \) boxes in \( \mathbb{R}^d \). If the graph \( G(P, \mathcal{O}) \) does not contain \( K_{k,k} \), then the maximum number of incidences between \( P \) and \( \mathcal{O} \) is \( I_d(n, m) = O(k(n + m) \log^{d-1} n) \).

**Proof:** Denote the points of \( P \) in increased order of their \( x \) value by \( p_1, p_2, \ldots, p_n \). For any range \( \nu = [\alpha : \beta] \in \mathcal{D}(n) \), let \( P_\nu = \{p_\alpha, \ldots, p_\beta\} \). For a box \( \ell \in \mathcal{O} \), let \( P_\ell \) be the set of all points of \( P \) with \( x \)-coordinate in the \( x \)-interval of \( \ell \). Clearly, there is a range \([\alpha : \beta] \subseteq [n] \) (potentially empty), such that \( P_\ell = P_{[\alpha : \beta]} \). The corresponding decomposition into dyadic sets of \( P_\ell \) is

\[
\mathcal{P}(\ell) = \{P_\nu \mid \nu \in \text{dy}([\alpha : \beta])\}.
\]

This gives rise to a natural decomposition of the original incidence graph into dyadic incidences graphs. Specifically, for every \( \nu \in \mathcal{D}(n) \), let \( \mathcal{O}_\nu \) be the set of all boxes \( \ell \) such that \( P_\nu \in \mathcal{P}(\ell) \). The incidence graph \( G_\nu = G(P_\nu, \mathcal{O}_\nu) \), has the property that all the boxes of \( \mathcal{O}_\nu \) have intervals on the \( x \)-axis that contains the \( x \)-coordinates of all the points of \( P_\nu \). In particular, one can interpret this as a \( d - 1 \) dimensional instance of boxes and points, by projecting the points and the boxes into a \((d - 1)\)-dimensional halfplane orthogonal to the \( x \)-axis. Thus, for \( n_\nu = |P_\nu| \) and \( m_\nu = |\mathcal{O}_\nu| \), we have that \( I_d(P_\nu, \mathcal{O}_\nu) \leq I_{d-1}(n_\nu, m_\nu) \).

A key property we need is that any number \( i \in [1 : n] \) participates in at most \( \lceil \log n \rceil \) dyadic intervals of \( \mathcal{D}(n) \). This implies that \( \sum_{\nu \in \mathcal{D}(n)} n_\nu = n \lceil \log n \rceil \). Similarly, by Lemma 2.2, we have \( \sum_{\nu \in \mathcal{D}(n)} m_\nu \leq 2m \log n \).

For \( d = 2 \), by Lemma 2.1, we have

\[
I_2(P, \mathcal{O}) \leq \sum_{\nu \in \mathcal{D}(n)} I_2(P_\nu, \mathcal{O}_\nu) \leq \sum_{\nu \in \mathcal{D}(n)} I_1(n_\nu, m_\nu) \leq k \sum_{\nu \in \mathcal{D}(n)} (n_\nu + 3m_\nu) \leq k(n + 6m) \lceil \log n \rceil,
\]

For \( d > 2 \), we use induction on the dimension, which implies

\[
I_d(P, \mathcal{O}) \leq \sum_{\nu \in \mathcal{D}(n)} I_{d-1}(n_\nu, m_\nu) \leq k \sum_{\nu \in \mathcal{D}(n)} O\left((n_\nu + m_\nu) \log^{d-2} n\right) = O\left(k(n + m) \log^{d-1} n\right).
\]

\[\blacksquare\]
2.3. Polytopes formed by the intersection of $s$ translated halfspaces

Let $H = \{h_1, \ldots, h_s\}$ be a set of $s$ halfspaces in $\mathbb{R}^d$. Let $Z(H) = \{\cap_i (h_i + v_i) \mid v_i \in \mathbb{R}^d\}$ be the family of all polytopes formed by the intersection of translates of the halfspaces of $H$. Let the dimension $\delta$ of $H$ be the size of smallest set of vectors orthogonal to the boundary hyperplanes of $H$ (thus, if two halfspaces of $H$ have parallel boundaries, then they contribute only one to the dimension). Thus, the family of $d$-dimensional boxes in $\mathbb{R}^d$ is generated by a set $H$ of $2d$ halfspaces of dimension $\delta = d$.

Lemma 2.4. Let $H$ be a set of $s$ halfspaces of $\mathbb{R}^d$, with dimension $\delta$. Let $P$ be a set of $n$ points in $\mathbb{R}^d$, and let $O \subseteq Z(H)$ be a set of $m$ polytopes. If the graph $G(P, O)$ does not contain $K_{k,k}$, then $I(P, O) = O(k(n + m) \log^{k-1} n)$.

Proof: Let $v_1, \ldots, v_8$ be the vectors orthogonal to the boundary hyperplanes of the halfspaces of $H$. Let $f$ be the mapping of a point $p \in \mathbb{R}^d$ to $(v_1 \cdot p, \ldots, v_8 \cdot p)$. Clearly, this maps a polytope of $O$ to an axis-parallel box in $\mathbb{R}^d$. A point $p$ is inside such a polytope $\iff f(p)$ is inside the mapped box. The result now readily follows from Lemma 2.3.

3. Halfspaces

In this section, we consider the version of the incidence problem where $P$ is a set of $n$ points in $\mathbb{R}^d$, and $O$ is a set of $m$ halfspaces in $\mathbb{R}^d$. We first mention two simple approaches giving weaker bounds, before describing our best approach using shallow cuttings.

Approach I: Reduction to intervals. One can reduce the problem to one-dimensional problem involving intervals, using the result of Welzl [Wel92], which provides a spanning path of the points, such that any hyperplane crosses it only $O(n^{1-1/d})$ times. Then each halfplane becomes a set of $O(n^{1-1/d})$ intervals on this spanning path, and one apply the above bound for intervals. This yields a bound of $O(kn + kmn^{1-1/d})$ on the incidences – since this bound is inferior, we provide no further details.

Approach II: Reduction to biclique covers. We next mention a different approach, also observed by Do [Do19], based on the following concept about compact representations of graphs:

Definition 3.1 (Biclique covers). Given a graph $G = (V, E)$, a biclique cover is a collection of pairs of vertex subsets $\{(A_1, B_1), \ldots, (A_{\ell}, B_{\ell})\}$ such that $E = \bigcup_{i=1}^{\ell} (A_i \times B_i)$. The size of the cover refers to $\sum_{i=1}^{\ell}(|A_i| + |B_i|)$.

Remarks. Biclique covers [AAAS94] are closely related to range searching, where the goal is to build data structures for a set $P$ of $n$ points, so that given a query object $\phi$, we can quickly report or count the points in $P \cap \phi$. Many known data structures for range searching produce a collection of so-called “canonical subsets” of $P$, so that for any query object $\phi$, the points in $P \cap \phi$ can be expressed as a union of a small number of canonical subsets. From such a data structure, we can form a biclique cover of $G(P, \phi)$ by letting the $A_i$’s be the canonical subsets, and letting $B_i$ be the subset of all query objects $\phi \in O$ that “use” the canonical subset $A_i$ in their answers. Here, $\sum_i |A_i|$ roughly corresponds to the space or preprocessing time of the data structure, and $\sum_i |B_i|$ roughly corresponds to the total query time.

For halfspace range searching, there are known data structures [Mat92b, Mat93] that can answer $m$ queries on $n$ points in $O((n + m + (nm)^{d/(d+1)}) \log^{O(1)} n)$ total time, and these data structures yield biclique covers of size $O((n + m + (nm)^{d/(d+1)}) \log^{O(1)} n)$. For orthogonal range searching (where the ranges are axis-aligned boxes), known data structures (namely, range trees) [BCKO08] yield biclique covers of size $O((n + m) \log^d n)$.

Lemma 3.3. Let $P$ be a set of points, and let $O$ be a set of objects. If the graph $G(P, O)$ has a biclique cover of size $X$ and does not contain $K_{k,k}$, then $I(P, O) = O(kX)$. 


Proof: Let \((A_1, B_1), \ldots, (A_\ell, B_\ell)\) be a biclique cover. For each \(i\), we must have \(\min\{|A_i|, |B_i|\} < k\), because otherwise \(G(P, \mathcal{O})\) would contain \(K_{k,k}\). Thus, \(I(P, \mathcal{O}) \leq \sum_{i=1}^\ell |A_i||B_i| \leq \sum_{i=1}^\ell k(|A_i| + |B_i|).\)

By the above lemma and known results on biclique covers, we immediately obtain an \(O(k(n + m) \log^d n)\) incidence bound for \(n\) points and \(m\) boxes in \(\mathbb{R}^d\); this is a logarithmic-factor worse than the bound from the previous section (but is already better than results from previous papers). We also immediately obtain an \(O(k(n + m + (nm)^{d/(d+1)}) \log^{O(1)} n)\) incidence bound (similar to Fox et al.’s bound [FPS+17]) for \(n\) points and \(m\) halfspaces in \(\mathbb{R}^d\); this improves the bound from the previous subsection, but we will do even better (both as a function of \(n\) and \(m\), as well as in terms of the dependence on \(k\)) in the next subsection for halfspaces, by using specific known techniques developed for halfspace range reporting – namely, shallow cuttings.

3.1. Better bounds using shallow cuttings

3.1.1. Preliminaries

Definition 3.4 (Duality). The dual hyperplane of a point \(p = (p_1, \ldots, p_d) \in \mathbb{R}^d\) is the hyperplane \(p^*\) defined by the equation \(x_d = -p_d + \sum_{i=1}^{d-1} x_i p_i\). The dual point of a hyperplane \(h\) defined by \(x_d = a_d + \sum_{i=1}^{d-1} a_i x_i\) is the point \(h^* = (a_1, a_2, \ldots, a_{d-1}, -a_d)\).

Fact 3.5. Let \(p\) be a point and let \(h\) be a hyperplane. Then \(p\) lies above \(h\) \iff the hyperplane \(p^*\) lies below the point \(h^*\).

Given a set of objects \(T\) (e.g., points in \(\mathbb{R}^d\)), let \(T^* = \{x^* \mid x \in T\}\) denote the dual set of objects.

Definition 3.6 (Levels). For a collection of hyperplanes \(H\) in \(\mathbb{R}^d\), the (bottom) level of a point \(p \in \mathbb{R}^d\), denoted by level\((p)\), is the number of hyperplanes of \(H\) lying on or below \(p\). The (bottom) \(k\)-level of \(H\), denoted by \(L_k(H)\), is the closure of the set formed by the union of all the points in \(\cup H\) with level \(k\).

The \((\leq k)\)-levels, denoted by \(L_{\leq k}(H) = \cup_{i=0}^k L_i(H)\), is the union of all levels up to \(k\).

By Fact 3.5, if \(h\) is a hyperplane which contains \(k\) points of \(P\) lying on or above it (and at least one point lies on it), then the dual point \(h^*\) is a member of the \(k\)-level of \(P^*\).

Definition 3.7 (Cuttings). Let \(H\) be a set of \(m\) hyperplanes in \(\mathbb{R}^d\). For a parameter \(r \in [1, m]\), a \(1/r\)-cutting is a decomposition of \(\mathbb{R}^d\), such that each simplex intersects \(\leq m/r\) hyperplanes of \(H\).

A somewhat more refined concept with better bounds is the following.

Definition 3.8 (Shallow cuttings). Let \(H\) be a set of \(m\) hyperplanes in \(\mathbb{R}^d\). A \(k\)-shallow \(1/r\)-cutting is a collection of simplices such that:

(i) the union of the simplices covers the \((\leq k)\)-levels of \(H\) (see Definition 3.6), and

(ii) each simplex intersects at most \(m/r\) hyperplanes of \(H\).

Chazelle and Friedman [CF90] showed how to compute \(1/r\)-cuttings with \(O(r^d)\) simplices. Matoušek [Mat92a] was the first to show how to compute \(k\)-shallow \(1/r\)-cuttings of size \(O((rk/m + 1)^d(m/k)^{d/2})\). In particular, there exist \((m/r)\)-shallow \(1/r\)-cuttings of size \(O(r^{(d/2)})\).

3.1.2. Halfspaces for \(d > 3\)

To obtain better incidence bounds for halfspaces, the following lemma is the key and states that under the \(K_{k,k}\)-free assumption, most points of \(P\) are shallow, i.e., there are fewer points of \(P\) at higher levels.

Lemma 3.9. Let \(P\) be a set of \(n\) points, and let \(\mathcal{O}\) be a set of \(m\) upper halfspaces, both in \(\mathbb{R}^d\). Let \(H\) be the hyperplanes bounding \(\mathcal{O}\). If the graph \(G(P, \mathcal{O})\) does not contain \(K_{k,k}\), then for \(r \leq m/(2k)\), the number of points of \(P\) between the \(m/r\)-level and the \(2m/r\)-level of \(H\) is at most \(O(k \cdot r^{(d/2)})\).
Proof: Compute a $2m/r$-shallow $1/(2r)$-cutting $\Xi$ of $H$, consisting of $O(r^{[d/2]})$ simplices. The simplices in $\Xi$ cover all points below the $2m/r$-level. Consider a simplex $\nabla \in \Xi$ that contains at least one point between the $m/r$-level and the $2m/r$-level. Since $\nabla$ contains a point of level at least $m/r$ and intersects at most $m/(2r)$ hyperplanes, the number of hyperplanes completely below $\nabla$ is at least $m/r - m/(2r) = m/(2r) \geq k$. By the $K_{k,k}$-free assumption, we must have $|\nabla \cap P| < k$. 

We now prove our main result for halfspaces in dimensions $d > 3$:

**Lemma 3.10.** Let $P$ be a set of $n$ points, and let $\Omega$ be a set of $m$ halfspaces, both in $\mathbb{R}^d$ for a constant $d > 3$. If the graph $G(P, \Omega)$ does not contain $K_{k,k}$, then the maximum number of incidences between $P$ and $\Omega$ is $O(k^{2/[([d/2]+1)\lfloor d/[2]\rfloor]}(mn)^{[d/2]/([d/2]+1)} + k(n + m))$.

Proof: Without loss of generality, assume that all halfspaces in $\Omega$ are upper halfspaces (since we can treat the lower halfspaces separately in a similar way and add the two incidence bounds). Let $I(n, m)$ denote the maximum number of incidences between $n$ points and $m$ upper halfspaces, under the $K_{k,k}$-free assumption. Let $r \leq m/(2k)$ be a parameter.

Let $H$ be the set of hyperplanes bounding $\Omega$. We break $P$ into $O(\log r)$ classes. Specifically, a point $p \in P$ is in $P_0$ if $p$ is below the $m/r$-level of $H$. For $i > 0$, $p \in P_i$ if it is between the $2^{i-1}m/r$-level and the $2^im/r$-level of $H$.

For $i > 0$, we have $|P_i| = O(k(r/2^i)^{[d/2]})$ by Lemma 3.9. It follows that

$$I(P_i, \Omega) \leq O(|P_i| \cdot 2^i m/r) = O(km(r/2^i)^{[d/2]}).$$

Summing over all $i > 0$ gives $\sum_{i > 0} I(P_i, \Omega) = O(kmr^{[d/2]})$ (assuming $d > 3$).

For $i = 0$, compute an $m/r$-shallow $1/r$-cutting $\Xi_0$ of $H$, consisting of $O(r^{[d/2]})$ simplices. The simplices in $\Xi_0$ cover all points of $P_0$. Consider a simplex $\nabla \in \Xi_0$ that intersects $P_0$. Since $\nabla$ contains a point of level at most $m/r$ and intersects at most $m/r$ hyperplanes, the number of hyperplanes intersecting or below $\nabla$ is $O(m/r)$. By subdividing the simplices into subcells, we can ensure that each subcell contains $O\left(\left\lceil \frac{n}{r^{[d/2]}} \right\rceil\right)$ points of $P_0$, while keeping the number of subcells $O(r^{[d/2]})$. It follows that

$$I(P_0, \Omega) \leq O(r^{[d/2]}) \cdot I\left(\left\lceil \frac{n}{r^{[d/2]}} \right\rceil, \frac{m}{r}\right).$$

Thus, for any $r \leq m/(2k)$,

$$I(n, m) \leq O(r^{[d/2]}) \cdot I\left(\left\lceil \frac{n}{r^{[d/2]}} \right\rceil, \frac{m}{r}\right) + O(kmr^{[d/2]}).$$

We first set $r = m/(2k)$ in Eq. (1) and use the naive bound $I(n', 2k) \leq n' \cdot 2k$ to obtain

$$I(n, m) \leq O(kn + m^{[d/2]}/k^{[d/2]}).$$

In particular, $I(n, m) = O(kn)$ if $n = \Omega(m^{[d/2]}/k^{[d/2]})$. By duality, $I(m, n) = I(n, m) = O(km)$ if $m = \Omega(n^{[d/2]}/k^{[d/2]})$.

We next set $r$ in Eq. (1) so that $m/r = (n/r)^{[d/2]}/k^{[d/2]}$, i.e., $r = (n^{[d/2]}/(k^{[d/2] - 1}m))^{1/(d/2)^2 - 1}$. Assuming $n \ll m^{[d/2]}/k^{[d/2]}$ and $m \gg n^{[d/2]}/k^{[d/2]}$, we indeed have $1 \ll r \ll m/k$. We then obtain

$$I(n, m) = O(k^{2/[([d/2]+1)\lfloor d/[2]\rfloor]}(mn)^{[d/2]/([d/2]+1)}).$$

**Remark 3.11.** For example, for halfspaces in dimension $d = 5$, the above bound is $O(k^2m^2n^2k^2 + k(n + m))$. This bound is tight, at least for $d = 5$ and for constant $k \geq 2$, since $\Omega(m^{2/3}n^{2/3} + m + n)$ is a lower bound. This follows from a known reduction of 2D point-line incidences to 5D point-halfspace incidences [Eri95, Eri96]: Take a set $P$ of $n$ points and a set $L$ of $m$ lines in the plane with $\Omega(m^{2/3}n^{2/3} + n + m)$ incidences [Mat02]. The incidence graph avoids $K_{2,2}$, and thus $K_{k,k}$ for any $k \geq 2$. Observe that for a sufficiently small constant $\epsilon > 0$, a point $(px, py) \in P$ lies on a line $y = ax + b$ in $L$ iff $(py - apx - b)^2 \leq \epsilon$, i.e., $a^2p_x^2 + p_y^2 - 2apxp_y + 2abp_x - 2bpy + b^2 \leq \epsilon$, iff the point $(p_x^2, p_y^2, p_xp_y, p_x, p_y) \in \mathbb{R}^5$ lies in the halfspace $\{(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) | a^2\xi_1 + \xi_2 - 2a\xi_3 + 2ab\xi_4 - 2b\xi_5 + b^2 \leq \epsilon\}$. Thus, we obtain $n$ points and $m$ halfspaces in $\mathbb{R}^5$ with the same incidence graph.
3.1.3. Halfspaces for $d \leq 3$

For $d \leq 3$, the approach in Section 3.1.2 generates an extra logarithmic factor (since $I(P_i, \mathcal{O}) = O(km)$ for each $i$ and so $\sum_{i>0} I(P_i, \mathcal{O}) = O(km \log m)$). We propose a different recursive strategy which eliminates the extra factor and achieves a linear bound.

Lemma 3.12. Let $P$ be a set of $n$ points, and let $\mathcal{O}$ be a set of $m$ halfspaces, both in $\mathbb{R}^d$ for $d \leq 3$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then the maximum number of incidences between $P$ and $\mathcal{O}$ is $O(k(n+m))$.

Proof: We modify the proof of Lemma 3.10, for the case $d = 3$ (as the bound for $d = 2$ is readily implied by the 3d bound). Recall that, for $i > 0$, $p \in P_i$ if it is between the $2^{i-1}m/r$-level and the $2^i m/r$-level of the planes of $\mathcal{O}$, and $|P_i| = O(kr/2)$. Thus, $|\bigcup_{i>0} P_i| = O(kr)$, which can be made less than $m/2$ by setting $r = m/(ck)$ for a sufficiently large constant $c$. So, $I(\bigcup_{i>0} P_i, \mathcal{O}) \leq I(m/2, m)$. For $i = 0$, we use the trivial upper bound $I(P_0, \mathcal{O}) \leq O(|P_0| \cdot m/r) = O(kn)$. It follows that

$$I(n, m) \leq I(m/2, m) + O(kn).$$

By duality, we also obtain $I(n, m) \leq I(n, n/2) + O(km)$. By alternating between these two recurrences, we have $I(m, m) \leq I(m/2, m/2) + O(km)$, implying $I(m, m) = O(km)$ and $I(n, m) = O(k(n+m))$. \hfill \blacksquare

Remark 3.13. The above recursion, which uses duality to obtain a linear bound, is interesting. A loosely similar recursion appeared in a recent paper [Cha21] on an algorithmic problem related to 2D pseudo-halfplane range reporting.

3.2. Disks and balls

The mapping of a point $p \in \mathbb{R}^d$ to the point $(p, \|p\|^2) \in \mathbb{R}^{d+1}$ can be interpreted as mapping balls into halfspaces [BCKO08]. Applying Lemma 3.12 and Lemma 3.10 in $d + 1$ dimensions gives the following.

Corollary 3.14. (I) Let $P$ be a set of $n$ points, and $\mathcal{O}$ be a set of $m$ disks, both in the plane. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k(n+m))$.

(II) Let $P$ be a set of $n$ points in $\mathbb{R}^d$, and $\mathcal{O}$ be a set of $m$ balls in $\mathbb{R}^d$, for $d > 2$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k^{2/\lceil[d/2]+1\rceil}(mn)^{\lceil[d/2]+1\rceil} + k(n+m))$.

4. Back to rectangles and boxes

In this section, we return to the case of rectangles and boxes, and apply our method for 3D halfspaces from Section 3.1.3 to slightly improve the bounds in Section 2.2.

First, we can obtain a linear bound for the case of 3D orthants, by modifying the proof using known versions of shallow cuttings for orthants [AT18], or by directly mapping from 3D orthants to 3D halfspaces [CLP11, PT11]: Without loss of generality, assume that the orthants are of the form $(-\infty, q_x] \times (-\infty, q_y] \times (-\infty, q_z]$ (since we can treat the other 7 types of orthants separately in a similar way, and add the bounds). Assume that all coordinates of the points and orthants are all integers (for example, by replacing coordinate values with ranks). Then the point $(p_x, p_y, p_z)$ is in the orthant $(-\infty, q_x] \times (-\infty, q_y] \times (-\infty, q_z]$ iff the point $(4^{p_x}, 4^{p_y}, 4^{p_z})$ is in the halfspace $x/4^{p_x} + y/4^{p_y} + z/4^{p_z} \leq 3$.

Corollary 4.1. Let $P$ be a set of $n$ points, and $\mathcal{O}$ be a set of $m$ orthants, both in $\mathbb{R}^3$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k(n+m))$.

We can also obtain a linear bound for the case of 2D 3-sided (axis-aligned) rectangles, by modifying the proof using known (simpler) versions of shallow cuttings for 2D 3-sided rectangles [JL11], or by directly mapping from 2D 3-sided rectangles to 3D orthants: Without loss of generality, assume that the 3-sided rectangles are of the
form \([a, b] \times (-\infty, h]\) (since we can treat the other 3 types of 3-sided rectangles separately in a similar way, and add the bounds). Then the point \((p_x, p_y)\) is in the 3-sided rectangle \([a, b] \times (-\infty, h]\) iff the point \((-p_x, p_x, p_y)\) in the orthant \((-\infty, -a) \times (-\infty, b) \times (-\infty, h]\).

**Corollary 4.2.** Let \(P\) be a set of \(n\) points, and \(O\) be a set of \(m\) 3-sided rectangles, both in \(\mathbb{R}^2\). If the graph \(G(P, O)\) does not contain \(K_{k,k}\), then \(I(P, O) = O(k(n + m))\).

**Remark 4.3.** Alternatively, Corollary 4.2 can be derived from the following known result by Fox and Pach [FP08], and reproved by Mustafa and Pach [MP16]: if the intersection graph of \(N\) line segments in the plane does not contain \(K_{k,k}\) for a constant \(k\), then the graph has \(O(N)\) edges. To see the connection, assume that the 3-sided rectangles in \(O\) are unbounded from below. For each 3-sided rectangle in \(O\), we take the horizontal line segment bounding its top side, and for each point in \(P\), we take the vertical upward ray from the point. The resulting intersection graph with \(N = n + m\) corresponds to the incidence graph (since there are no crossings between two horizontal segments, nor between two vertical rays). The dependence on \(k\) was not explicitly analyzed in the previous papers, but also appears to be \(O(k)\).

Using this result for 2D 3-sided (axis-aligned) rectangles, we obtain an improved result for 2D general 4-sided (axis-aligned) rectangles, by using a divide-and-conquer with a nonconstant branching factor \(b\). Although the improvement is a small \(\log \log n\) factor, the bound is tight for \(m = n\) and constant \(k\), as it matches Basit et al. [BCS+21] lower bound \(\Omega(n \log n / \log \log n)\).

**Lemma 4.4.** Let \(P\) be a set of \(n\) points in \(\mathbb{R}^2\), and let \(O\) be a set of \(m\) rectangles in \(\mathbb{R}^2\). If the graph \(G(P, O)\) does not contain \(K_{k,k}\), then the maximum number of incidences between \(P\) and \(O\) is \(I(n, m) = O(kn \log n / \log \log n + km \log^\varepsilon n)\) for any constant \(\varepsilon > 0\).

**Proof:** Divide the plane into \(b\) vertical slabs \(\sigma_1, \ldots, \sigma_b\), each containing \(n/b\) points of \(P\). For each \(i = 1, \ldots, b\), let \(P_i = P \cap \sigma_i\), let \(O_i\) be the set of all rectangles of \(O\) that are completely inside \(\sigma_i\), and let \(O_i'\) be the set of all rectangles of \(O\) that intersect \(\sigma_i\) but are not completely inside \(\sigma_i\). Inside \(\sigma_i\), the rectangles in \(O_i'\) may be viewed as 3-sided, and so \(I(P_i, O_i') = O(k(|P_i| + |O_i'|))\) by Corollary 4.2. Thus, \(\sum_{i=1}^{b} I(P_i, O_i') = O(k(n + bm_0))\) where \(m_0 = \left| \bigcup_{i=1}^{b} O_i' \right|\). We also have \(\sum_{i=1}^{b} I(P_i, O_i) \leq \sum_{i=1}^{b} I(n/b, m_i)\) where \(m_i = |O_i|\). We obtain the recurrence

\[
I(n, m) \leq m_0, \ldots, m_b \max_{m_0 + \cdots + m_b \leq m} \left( \sum_{i=1}^{b} I(n/b, m_i) + O(kn + bkm_0) \right),
\]

which solves to \(I(n, m) \leq O(kn \log b n + bkm)\). Setting \(b = \log^\varepsilon n\) gives the result.

The improvement in 2D also implies an improvement for boxes in higher dimensions. Again, we can use a divide-and-conquer with a larger branching factor \(b\).

**Theorem 4.5.** Let \(P\) be a set of \(n\) points in \(\mathbb{R}^d\), and let \(O\) be a set of \(m\) boxes in \(\mathbb{R}^d\). If the graph \(G(P, O)\) does not contain \(K_{k,k}\), then the maximum number of incidences between \(P\) and \(O\) is \(O(kn \log n / \log \log n)^{d-1} + km \log^{d-2+\varepsilon} n)\).

**Proof:** Assume that \(P\) is a set of \(n\) points inside a vertical slab \(\sigma\), \(O\) is a set of \(m\) boxes each having at least one vertex in \(\sigma\). Let \(v\) be the number of vertices of the boxes that are inside \(\sigma\). Then \(v \leq 2^d m\). Let \(I_d(n, v)\) denote the maximum number of point-box incidences in this setting.

Divide \(\sigma\) into \(b\) vertical slabs \(\sigma_1, \ldots, \sigma_b\), each containing \(n/b\) points of \(P\). For each \(i = 1, \ldots, b\), let \(P_i = P \cap \sigma_i\), let \(O_i\) be the set of all boxes of \(O\) that have at least one vertex in \(\sigma_i\), and let \(O_i'\) be the set of all “long” boxes of \(O\) that intersect \(\sigma_i\) but are not in \(O_i\) (i.e., that completely cut across \(\sigma_i\)). Inside \(\sigma_i\), the boxes in \(O_i'\) are liftings of \((d-1)\)-dimensional boxes, and so \(I(P_i, O_i') \leq I_{d-1}(n/b, v)\). We can recursively bound \(\sum_{i=1}^{b} I(P_i, O_i)\) by \(\sum_{i=1}^{b} I_d(n/b, v_i)\) where \(v_i\) is the number of vertices in \(\sigma_i\). We obtain the recurrence

\[
I_d(n, v) \leq \max_{v_1, \ldots, v_b: v_1 + \cdots + v_b \leq v} \left( \sum_{i=1}^{b} I(n/b, v_i) + b I_{d-1}(n/b, v_i) \right),
\]
with the base case \( I_2(n, v) = O(kn \log_b n + bkv) \) from Lemma 4.4. This solves to
\[
I_d(n, m) \leq O\left( kn \left( \log_b n \right)^{d-1} + b^{d-1} kv \left( \log_b n \right)^{d-2} \right).
\]

Setting \( b = \log^{\varepsilon/(d-1)} n \) gives the result.

\[ \square \]

We immediately obtain a similar improvement of Lemma 2.4 for \( \delta \)-polyhedra.

Remark 4.6. Similar ideas of using shallow cuttings for 3D orthants and 2D 3-sided rectangles, reducing the 4-sided to the 3-sided case, and doing divide-and-conquer with \( \log\log n \) factors in Lemma 4.4 may look surprising at first, but it is not unprecedented in the range searching literature. For example, Chazelle [Cha90] proved an \( \Omega(\log n / \log \log n) \) space lower bound for orthogonal range reporting in pointer machines, and in Appendix B, we note that his argument actually implies a lower bound for our incidence problem.

Remark 4.7. It is possible to obtain a further improved bound of \( O(n \log n / \log \log n)^{d-1} + m \log^{d-3+\varepsilon} n \) for boxes in dimension \( d \geq 3 \) and for constant \( k \). The idea is to prove an \( O(n \log_b n + bm \log \log n) \) bound for the 3D 5-sided case by employing a known recursive grid approach introduced by Alstrup et al. [ABR00] and adapted by Chan et al. [CNRT18], using the bound for 3D orthants as the base case. However, this does not imply any improvement for the main case \( m = n \), and so we will omit the details.

5. When the union complexity is low

In this section, we consider the version of the incidence problem for set \( P \) of \( n \) points and a set \( F \) of \( m \) well-behaved shapes with near-linear union complexity. We will adapt the shallow cutting approach from Section 3.1.

5.1. Preliminaries

Definition 5.1. Let \( F \) be a set of \( m \) shapes in \( \mathbb{R}^d \). The family \( F \) is well-behaved if
(I) The union complexity for any subset of \( F \) of size \( r \) is (at most) \( U(r) \), for some function \( U(r) \) such that \( U(r)/r \) is nondecreasing.
(II) The total complexity of an arrangement of any \( r \) shapes of \( F \) is \( O(r^d) \).
(III) For any set \( X \subseteq \mathbb{R}^d \), and any subset \( G \subseteq F \), one can perform a decomposition of the cells of the arrangement \( A(G) \) that intersects \( X \), into cells of constant descriptive complexity (e.g., vertical trapezoids), and the complexity of this decomposition is proportional to the number of vertices of the cells of \( A(G) \) that intersects \( X \).
(IV) The arrangement of any \( d \) shapes of \( F \) has constant complexity.

The following minor variant of Matoušek’s shallow cuttings [Mat92a] is from Chekuri et al. [CCH12].

Theorem 5.2. Given a set \( F \) of \( m \) well-behaved shapes in \( \mathbb{R}^d \), and parameters \( r \) and \( k \), one can compute a decomposition \( \Xi \) of space into \( O(r^d) \) cells of constant descriptive complexity, such that total weight of boundaries of shapes of \( F \) intersecting a single cell is at most \( m/r \). The decomposition \( \Xi \) is a 1/r-cutting of \( F \). Furthermore, the total number of cells of \( \Xi \) containing points of depth smaller than \( k \) is \( O((rk/m+1)^dU(m/k)) \). Here, the depth of a point \( p \) is the number of shapes in \( F \) that contains \( p \).

Definition 5.3. A set \( F \) of \( m \) shapes has primal shatter dimension \( b \) if for every point set \( P \), we have \( | \{ P \cap o \mid o \in F \} | = O(|P|^b) \).

Lemma 5.4. Let \( P \) be a set of \( n \) points, and let \( F \) be a set of \( m \) shapes with primal shatter dimension \( b \). If the graph \( G(P,F) \) does not contain \( K_{k,k} \), then the maximum number of incidences between \( P \) and \( F \) is
\[ I(n, m) = O(k(n^{b+1} + m)). \]
Proof: Let \( F_0 \) be the set of all shapes of \( F \) that contain more than \( k \) points. Call two shapes \( \phi \) and \( \phi' \) equivalent if \( P \cap \phi = P \cap \phi' \). There are \( O(n^b) \) equivalence classes of shapes. Each equivalence class contains at most \( k \) shapes in \( F_0 \), by the \( K_{k,k} \)-free assumption. Thus, \(|F_0| \leq O(kn^b)\), and \( I(P,F_0) \leq O(kn^{b+1})\). On the other hand, \( I(P,F \setminus F_0) \leq O(km)\).

\[ \text{Remark 5.5. If } G(P,F) \text{ does not contain } K_{k,k}, \text{ it is not difficult to see that the primal shatter dimension must be bounded by } O(k). \text{ However, most families of shapes encountered in low-dimensional geometry have } O(1) \text{ VC dimension, and thus by Sauer’s lemma [Mat02], actually have } O(1) \text{ primal shatter dimension independent of } k. \]

### 5.2. Shapes with low union complexity

We now modify our approach for 3D halfspaces to prove incidence bounds for 2D well-behaved shapes with low (near-linear) union complexity. We do not have duality now, and as a result, a straightforward adaptation of the proof of Lemma 3.10 would result in an extra logarithmic factor. We describe a nontrivial modification to lower the extra factor to \( \log \log m + \log k \).

**Lemma 5.6.** Let \( P \) be a set of \( n \) points, and let \( F \) be a set \( m \) well-behaved shapes in the plane, with union complexity \( O(U(m)) \). If the graph \( G(P,F) \) does not contain \( K_{k,k} \), then for any \( r \leq m/(2k) \), the number of points of \( P \) having depth between \( m/r \) and \( 2m/r \) is at most \( O(k \cdot U(r)) \).

**Proof:** Compute a \( 1/(2r) \)-cutting \( \Xi \) by Theorem 5.2. Let \( \Xi \) be the cells of the cutting that contain at least one point with depth between \( m/r \) and \( 2m/r \); there are \( O(U(r)) \) such cells. Consider a cell \( \nabla \in \Xi \). Since \( \nabla \) contains a point of depth at least \( m/r \) and intersects the boundaries of at most \( m/(2r) \) shapes, the number of shapes completely containing \( \nabla \) is at least \( m/r - m/(2r) = m/(2r) \geq k \). By the \( K_{k,k} \)-free assumption, we must have \(|\nabla \cap P| < k\).

**Theorem 5.7.** Let \( P \) be a set of \( n \) points in the plane, and let \( F \) be a set \( m \) well-behaved shapes in the plane, with union complexity \( O(U(m)) \) and constant primal shatter dimension. If the graph \( G(P,F) \) does not contain \( K_{k,k} \), then the maximum number of incidences between \( P \) and \( F \) is \( I(n,m) = O(kn + kU(m)(\log \log m + \log k)) \).

**Proof:** Let \( \phi(m) = U(m)/m \). Let \( t_0, t_1, \ldots, t_\ell \) be an increasing sequence of parameters, with \( t_0 = 2k \) and \( t_\ell \geq m \).

We break \( P \) into \( O(\ell) \) classes. Specifically, a point \( p \in P \) is in \( P_0 \) if its depth is below \( t_0 \). For \( i > 0 \), \( p \in P_i \) if its depth is between \( t_{i-1} \) and \( t_i \). For \( i > 0 \), we have \(|P_i| = O(k \cdot (U(m)/(2t_{i-1})) + U(m)/(4t_{i-1}) + \cdots)) \) by Lemma 3.9; this gives \(|P_i| = O(k(m/t_{i-1})\phi(m))\). Compute a \( t_i/m \)-cutting \( \Xi \) by Theorem 5.2. Let \( \Xi \) be the cells of the cutting that intersect \( P_i \); there are \( O(U(m/t_i)) \leq O((m/t_i)\phi(m)) \) such cells. Consider a cell \( \nabla \in \Xi \). Since \( \nabla \) contains a point of depth at most \( t_i \) and intersects the boundaries of at most \( t_i \) shapes, the number of shapes intersecting or containing \( \nabla \) is \( O(t_i) \). By subdividing the simplices into subcells, we can ensure that each subcell contains at most \( O\left(\frac{k(m/t_{i-1})\phi(m)}{(m/t_i)\phi(m)}\right) \) points of \( P_i \), while keeping the number of subcells \( O((m/t_i)\phi(m)) \).

It follows that for \( i > 0 \),

\[ I(P_i, \mathcal{O}) \leq O((m/t_i)\phi(m)) \cdot I(t_i/t_{i-1}, t_i). \]

For \( i = 0 \), we use the trivial upper bound \( I(P_0, \mathcal{O}) \leq O(|P_0| \cdot t_0) = O(kn). \)

It follows that

\[ I(n,m) \leq \sum_{i=1}^{\ell} O((m/t_i)\phi(m)) \cdot I(t_i/t_{i-1}, t_i) + O(kn). \quad (2) \]

We choose the sequence \( t_0 = 2k \), \( t_i = 2t_{i-1} \) for \( 0 < i \leq c \log k \), and \( t_i = \frac{c}{i-1} \) for \( i > c \log k \) for some constant \( c \). The number of terms is \( \ell = O(\log k + \log \log m) \). For \( i \leq c \log k \), we have \( I(t_i/t_{i-1}, t_i) = I(2k, t_i) = O(kt_i) \). For \( i > c \log k \), we have \( t_i \geq k^c \) and \( I(t_i/t_{i-1}, t_i) = I(t_i^{1/c}, t_i) \leq I(t_i^{2/c}, t_i) = O(kt_i) \) by Lemma 5.4 if \( c \) is sufficiently large. Then Eq. (2) implies \( I(n,m) = O(\ell km\phi(m) + kn) = O(km\phi(m)(\log k + \log \log m) + kn) \).
A set $\mathcal{F}$ of simply-connected shapes in the plane is a set of pseudo-disks, if for every pair of shapes, their boundaries intersect at most twice. It known that the union complexity of pseudo-disks is linear [SA95] and the primal shatter dimension is $O(1)$ [Mat02]. For technical reasons it is convenient to assume that the pseudo-disks are $y$-monotone – that is, any vertical line either avoids a pseudo-disk, or intersects it in an interval. We thus get the following.

**Corollary 5.8.** Let $P$ be a set of $n$ points in the plane, and let $\mathcal{F}$ be a set of $m$ $y$-monotone pseudo-disks in the plane. If the graph $G(P, \mathcal{F})$ does not contain $K_{k,k}$, then the maximum number of incidences between $P$ and $\mathcal{F}$ is bounded by $I(n, m) = O(kn + km(\log \log m + \log k))$. 

A natural open question is to improve the bound to linear for pseudodisks.

**Remark 5.9.** The idea of taking a sequence of $O(\log \log m)$ shallow cuttings was also used in some known data structures for 3D halfspace range reporting [Cha00, Ram99] – the above proof is partly inspired by these algorithmic works.

### 5.3. Fat triangles

A set of triangles $\mathcal{F}$ is fat if the smallest angle in any triangle of $\mathcal{F}$ is bounded away from 0 by some absolute positive constant. Aronov et al. [ABES14] showed that the union complexity here is $U(m) = O(m \log^* m)$. So, Theorem 5.7 immediately implies $I(n, m) = O(kn + km \log^* m(\log \log m + \log k))$. In this subsection, we show how to further lower the log log $m$ factor to $\log^* m$ for fat triangles.

We first prove a weaker bound $I(n, m) = O(kn \log^2 n + km)$ for fat triangles. To this end, we start with the following lemma:

**Lemma 5.10.** (I) Let $P$ be a set $n$ points in $\mathbb{R}^3$, and $\mathcal{O}$ be a set of $m$ wedges of the form $\{(x, y, z) \mid y \leq ax + b, z \leq c\}$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k(n + m))$.

(II) Let $P$ be a set $n$ points in $\mathbb{R}^2$, and $\mathcal{O}$ be a set of $m$ wedges of the form $\{(x, y) \mid y \leq ax + b, x \leq c\}$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k(n + m))$.

(III) Let $P$ be a set $n$ points in $\mathbb{R}^2$, and $\mathcal{O}$ be a set of $m$ “curtains” of the form $\{(x, y) \mid y \leq ax + b, c \leq x \leq c'\}$. If the graph $G(P, \mathcal{O})$ does not contain $K_{k,k}$, then $I(P, \mathcal{O}) = O(k(n \log n + m))$.

**Proof:** For (I), the union complexity for $m$ wedges of this type is $O(m)$. To see this, imagine moving a horizontal sweep plane from top to bottom. On the sweep plane, the complement of the union is an intersection of halfplanes. As the sweep plane moves, we encounter $m$ insertions of halfplanes; each insertion creates at most two vertices to the wedge. Having established linear union complexity, we could now apply Theorem 5.7, but we can do better by observing that we have duality here: a point $(p_x, p_y, p_z)$ is in the wedge $\{(x, y, z) \mid y \leq ax + b, z \leq c\}$ iff the point $(a, -b, -c)$ is in the wedge $\{(\alpha, \beta, \gamma) \mid \beta \leq p_x \alpha - p_y, \gamma \leq -p_z\}$. So, we can adapt the proof in Lemma 3.12 to obtain the $O(k(n + m))$ incidence bound.

(II) is a special case of (I): a point $(p_x, p_y)$ is in the wedge $\{(x, y) \mid y \leq ax + b, x \leq c\}$ iff the point $(p_x, p_y, p_z)$ is in the wedge $\{(x, y, z) \mid y \leq ax + b, z \leq c\}$.

For (III), we use divide-and-conquer to reduce to (II). Divide the plane into two vertical slabs $\sigma_1$ and $\sigma_2$, each containing $n/2$ points of $P$. For each $i \in \{1, 2\}$, let $P_i = P \cap \sigma_i$, and let $\mathcal{O}_i$ be the set of all curtains of $\mathcal{O}$ that are completely inside $\sigma_i$. Let $\mathcal{O}_0$ be the set of all curtains of $\mathcal{O}$ that intersect both $\sigma_1$ and $\sigma_2$. Inside $\sigma_i$, the curtains in $\mathcal{O}_0$ may be viewed as wedges, and so $I(P_i, \mathcal{O}_0) = O(k(|P_i| + |\mathcal{O}_0|))$ by (II). Thus, $I(P, \mathcal{O}) \leq I(P_1, \mathcal{O}_1) + I(P_2, \mathcal{O}_2) + O(k(n + |\mathcal{O}_0|))$. We obtain the recurrence

$$I(n, m) \leq \max_{m_0, m_1, m_2: m_0 + m_1 + m_2 = m} \left( I(n/2, m_1) + I(n/2, m_2) + O(kn + km_0) \right),$$

which solves to $I(n, m) \leq O(kn \log n + km)$. 

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The special case of triangles containing the origin can be reduced to the case of curtains:

**Corollary 5.11.** Let \( P \) be a set of \( n \) points in the plane, and let \( F \) be a set of \( m \) triangles in the plane, such that all triangles contain the origin. If the graph \( G(P, F) \) does not contain \( K_{k,k} \), then the maximum number of incidences between \( P \) and \( F \) is bounded by \( O(k(n \log n + m)) \).

**Proof:** Without loss of generality, assume that the triangles all have the origin as one of its vertices, since a triangle containing the origin can be decomposed into three such triangles. Also assume that the points in \( P \) are all above the \( x\)-axis (since we can treat the points below the \( x\)-axis in a similar way, and add the two bounds). The affine transformation \((x, y) \mapsto (-1/x, y/x)\) then maps triangles with the origin as a vertex to curtains. □

**Lemma 5.12.** Let \( P \) be a set of \( n \) points in the plane, and let \( F \) be a set of \( m \) fat triangles in the plane. If the graph \( G(P, F) \) does not contain \( K_{k,k} \), then the maximum number of incidences between \( P \) and \( F \) is bounded by \( I(n, m) = O(k(n \log^2 n + m)) \).

**Proof:** By rescaling, assume that all points and triangles are contained in \([0,1]^2\).

A **quadtree square** is a square of the form \([i/2^\ell, (i + 1)/2^\ell] \times [j/2^\ell, (j + 1)/2^\ell]\) for integers \( i, j, \ell \). A shape \( \phi \) with diameter \( r \) is **aligned** if it is inside a quadtree square of side length \( 4r \).

For any shape \( \phi \) contained in \([0,1]^2\), it is known [Cha03] that \( \phi \) is aligned, or \( \phi + (1/3, 1/3) \) is aligned, or \( \phi + (2/3, 2/3) \) is aligned. Without loss of generality, assume that all triangles in \( F \) are aligned, since we can separately treat the triangles \( \phi \) where \( \phi + (1/3, 1/3) \) is aligned and the triangles \( \phi \) where \( \phi + (2/3, 2/3) \) (by shifting all the triangles and points) and add the three bounds.

We use a balanced divide-and-conquer. First it is known [AMN+98] that there exists a quadtree square \( s \) that has at most \( 4n/5 \) points inside and at most \( 4n/5 \) points outside. (Such a square \( s \) can be obtained by taking a “centroid” in the quadtree; or, to be more self-contained, we can let \( s \) be the smallest quadtree square that has at least \( n/5 \) points inside.) Let \( P_1 \) be the set of all points of \( P \) inside \( s \), and \( P_2 \) be the set of all points of \( P \) outside \( s \). Let \( F_1 \) be the set of all triangles of \( F \) completely inside \( s \), \( F_2 \) be the set of all triangles of \( F \) completely outside \( s \), and \( F_0 \) be the set of all triangles of \( F \) that intersect the boundary of \( s \).

Let \( r \) be the side length of \( s \). Because all triangles are aligned, the triangles of \( F_0 \) must have diameter at least \( r/4 \). Because the triangles are fat and intersect a square \( s \) of side length \( r \), they can be stabbed by \( O(1) \) points. We can bound incidences for the triangles containing each of the \( O(1) \) stabbing points by invoking Corollary 5.11 (after translating the origin). It follows that \( I(P, F_0) = O(k(n \log n + |F_0|)) \). Thus, \( I(P, F) \leq I(P_1, F_1) + I(P_2, F_2) + O(k(n + |F_0|)) \). We obtain the recurrence

\[
I(n, m) \leq \max_{n_1, n_2, m_0, m_1, m_2 : n_1, n_2 \leq 4n/5, \ n_1 + n_2 = n, \ m_0 + m_1 + m_2 = m} \ (I(n_1, m_1) + I(n_2, m_2) + O(kn \log n + km_0)),
\]

which solves to \( I(n, m) \leq O(kn \log^2 n + km) \).

**Corollary 5.13.** Let \( P \) be a set of \( n \) points in the plane, and let \( F \) be a set of \( m \) fat triangles in the plane, such that \( G(P, F) \) does not contain \( K_{k,k} \). Then, the maximum number of incidences between \( P \) and \( F \) is bounded by \( I(n, m) = O(kn + km \log^* m)(\log^* m + \log k) \).

**Proof:** We follow the proof of Theorem 5.7, but use a sparser sequence: \( t_0 = 2k \), \( t_i = 2t_{i-1} \) for \( 0 < i \leq t_0 \), and \( t_i = 2\sqrt{t_{i-1}/k} \) for \( i > t_0 \). This \( t_0 \) is \( [3 \log \log k] \). The number of terms is \( \ell = O(\log \log k + \log^* m) \). (To see this, observe that \( t_{i_0} \geq k \log^3 k \), and \( t_{i_0+1} \geq k^{6/5} \), and \( t_{i_0+1} \geq 2^{t_{i+1}/k} \) for \( i > t_0 \).) For \( i \leq t_0 \), we have \( I(kt_i/t_{i-1}, t_i) = I(2k, t_i) = O(kt_i) \). For \( i > t_0 \), by Lemma 5.12 we have \( I(kt_i/t_{i-1}, t_i) = O(k \cdot ((kt_i/t_{i-1}) \log^2 t_i + t_i)) = O(kt_i) \). Then Eq. (2) implies \( I(n, m) = O(\ell k m(\log^* m + kn) = O(km \log^* m(\log \log k + \log^* m + kn) \), where \( \phi(m) = O(\log^* m) \) by Aronov et al. [ABES14]. □
Remark 5.14. As noted, some of our proofs for the incidence problem are related to known data structure techniques for range searching. Interestingly, the ideas in the above proof actually imply a new data structure for the problem of range reporting for points in 2D in the case where the query ranges are fat triangles:

First, Chazelle and Guibas [CG86] have already given a data structure for range reporting for the case of curtains in 2D (they called this case “slanted range search”) with $O(n)$ space and $O( \log n + K)$ query time, where $K$ denotes the output size. The same bound thus holds for 2D triangle range reporting where the query ranges are triangles containing the origin (incidentally, this improves a result from [ISB08]). The quadtree-based approach in the proof of Lemma 5.12 then gives a data structure for 2D fat-triangle range reporting with $O(n \log n)$ space and $O( \log n + K)$ query time. This result may be of independent interest (see [SS11] for previous results which have a larger query time, though with linear space).

Lemma 5.15. Let $P$ be a set of $n$ points in the plane. We can construct a data structure with $O(n \log n)$ space, so that we can report all $K$ points inside any query fat triangle in $O(\log n + K)$ time.

6. Final Remarks

It is not clear how to get a near linear bound on the number of incidences between points and unit balls in $\mathbb{R}^3$ when avoiding $K_{k,k}$ (Corollary 3.14 implies a bound of $O(k^{2/3}n^{4/3})$). In particular, consider the following variant of the unit distance problem, which we leave as an open problem for further research.

Problem 6.1 (At most unit distance when avoiding $K_{k,k}$). Let $P$ and $P'$ be two sets of $n$ red and blue points, respectively, in $\mathbb{R}^d$ for $d > 2$. What is the maximum number of bichromatic pairs of points with distance $\leq 1$, under the assumption that the bipartite graph $G(P, P')$, connecting all pairs of points in distance $\leq 1$, does not contain $K_{k,k}$?

(The non-bichromatic version of this problem has a linear bound, as the point set cannot have balls of radius $1/2$ that contain $k$ points.) The $\Omega(n^{4/3})$ lower bound for 5D halfspaces from Remark 3.11 also applies to 5D unit balls (since we can replace halfspaces with balls of a sufficiently large radius, and then rescale the balls and the points), but our upper bound for 5D unit balls from Corollary 3.14 is only $O(k^{1/2}n^{3/2})$.

Another open question is whether lower bounds could be proved to show tightness of our result for halfspaces in dimensions other than 2, 3, and 5.

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A. Proof of Lemma 2.2

Restatement of Lemma 2.2. Let n > 1 be an integer, and consider an arbitrary range I = [α : β] ⊆ [n]. Let dy(I) denote a minimal (cardinality wise) union of disjoint dyadic ranges that covers I. The set dy(I) is unique and has \( \leq 2 \log_2 n \) ranges, where \( \log \) is in base 2.

Proof: This is well known, and we include a proof for the sake of completeness. Assume that \( n = 2^h \). A dyadic range of \( D(n) \) is a boundary range if it is not contained in I, but one of its children is contained in I. Observe that there can be at most two boundary ranges of the same rank, as they each cover an endpoint of I. Replacing each boundary range by its child included in I, results in the desired decomposition dy(I) of I into dyadic ranges (this fails only for the case where I = [n] – but then dy([n]) is a single range).
We are left with bounding the number of boundary ranges when $I \subseteq [n]$. Observe that a range of rank 0 can not be a boundary range. Furthermore, there is only a single range of rank $h$, specially $[n]$, and if it is not a boundary range, then the number of boundary ranges is at most $2(h-1) = 2h - 2$. If $[n]$ is a boundary range, then there is at most one range of each rank that is a boundary range (since the other endpoint of $I$ is “unreachable” to be covered by another boundary range), which implies a bound of $h$ on the number of boundary ranges in this case. Thus the number of boundary ranges is bounded by $\max(2h - 2, h) \leq 2h - 2$.

As for the uniqueness of $dy(I)$, observe that any dyadic range properly containing a range of $dy(I)$, must also contain one of the endpoints of $I$ in its “interior”, which readily implies the minimality of $dy(I)$, as any other dyadic decomposition of $I$ must refine the ranges in $dy(I)$.

Finally, for $n$ that is not a power of two, observe that $2 \lceil \log_2 n \rceil - 2 \leq 2 \log_2 n$.

B. On Chazelle’s Lower Bound

Chazelle [Cha90] proved that in a pointer machine model of computation, any data structure for the $d$-dimensional orthogonal range reporting problem with $O(\log^{O(1)} n + K)$ query time (where $K$ denotes the output size of the query) requires at least $\Omega(n(\log n/\log \log n)^{d-1})$ space. His framework for proving such lower bounds has been adapted in various subsequent works (e.g., see [CR95, AAL10, AC21]).

For a given set $P$ of $n$ points, Chazelle called a set of boxes $O = \{q_1, \ldots, q_m\}$ $b$-favorable if for each $i, j$ with $i \neq j$,

1. $|P \cap q_i| \geq \log^b n$ and
2. $|P \cap q_i \cap q_j| \leq 1$.

On his way to proving his data structure lower bound, he constructed a set $P$ of $n$ points and a $b$-favorable set $O$ of $m$ boxes with

$$m = \Omega(n(\log n)^{d-b-1}/(\log \log n)^{d-1})$$

for given any $b, d = O(1)$ and sufficiently large $n$ [Cha90, Lemma 4.6].

It is easy to see that condition (ii) of favorability is equivalent to the statement that the incidence graph $G(P, O)$ does not contain $K_{2,2}$. On the other hand, condition (i) implies that the total number of incidences is $\Omega(m \log^b n)$.

By setting $b$ with $\log^b n = \Theta((\log n/\log \log n)^{d-1})$, we immediately obtain from Chazelle’s construction a set $P$ of $n$ points and a set $O$ of $n$ boxes in $\mathbb{R}^d$ such that $G(P, O)$ does not contain $K_{2,2}$ and the number of incidences is $\Omega(n(\log n/\log \log n)^{d-1})$. (Thus, Chazelle’s argument from years earlier not only implied the two-dimensional lower bound from Basit et al.’s recent paper, but also answered their question about extension to higher dimensions [BCS+21].)

Similarly, it should be possible to obtain lower bounds for certain other classes of objects by using subsequent results that follow Chazelle’s framework (e.g., simplices [CR95]).

Remark B.1. It might also be possible to combine Chazelle’s framework with existing upper bounds on orthogonal range reporting in pointer machines [Cha90, AAL10] to obtain upper bounds on our incidence problem for boxes. But query time bounds on pointer machines are $\Omega(\log n)$ for $d = 2$, so one cannot rederive our $O(kn \log n/\log \log n)$ upper bound in two dimensions this way. Also, the dependence on $k$ appears to be significantly worse. It is better to directly modify the ideas used in known data structures for orthogonal range reporting (which is what we did in Section 4).