Flexible Sampling of Discrete Scale Invariant Markov Processes: Covariance and Spectrum

N. Modarresi and S. Rezakhah

Abstract

In this paper we consider some flexible discrete sampling of a discrete scale invariant process \( \{X(t), t \in \mathbb{R}^+\} \) with scale \( l > 1 \). By this method we plan to have \( q \) samples at arbitrary points \( s_0, s_1, \ldots, s_{q-1} \) in interval \([1, l)\) and proceed our sampling in the intervals \([l^n, l^{n+1})\) at points \( l^n s_0, l^n s_1, \ldots, l^n s_{q-1}, n \in \mathbb{Z}\). Thus we have a discrete time scale invariant (DT-SI) process and introduce an embedded DT-SI process as \( W(nq+k) = X(l^n s_k), q \in \mathbb{N}, k = 0, \ldots, q-1 \). We also consider \( V(n) = (V_0(n), \ldots, V_{q-1}(n)) \) where \( V_k(n) = W(nq+k) \), as an embedded \( q \)-dimensional discrete time self-similar (DT-SS) process. By introducing quasi Lamperti transformation, we find spectral representation of such process and its spectral density matrix is given. Finally by imposing wide sense Markov property for \( W(\cdot) \) and \( V(\cdot) \), we show that the spectral density matrix of \( V(\cdot) \) and spectral density function of \( W(\cdot) \) can be characterized by \( \{R_j(1), R_j(0), j = 0, \ldots, q-1\} \) where \( R_j(k) = E[W(j+k)W(j)] \).

AMS 2000 Subject Classification: 60G18, 62M15.

Keywords: Discrete scale invariance; Wide sense Markov; Multi-dimensional self-similar processes.

1 Introduction

The concept of stationarity and self-similarity are used as a fundamental property to handle many natural phenomena. Lamperti transformation defines a one to one correspondence between stationary and self-similar processes. Discrete scale invariance (DSI) process can be defined as the Lamperti transform of periodically correlated (PC) process. Many critical systems, like statistical physics, textures in geophysics, network traffic and image processing can be interpreted by these processes [1]. Fourier transform is known as a suited representation for stationarity, but not for self-similarity. A harmonic like representation of self-similar process is introduced by using Mellin transform [4].

*Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran 15914, Iran. E-mail: namomath@aut.ac.ir (N. Modarresi), rezakhah@aut.ac.ir (S. Rezakhah).
A process which is Markov and self-similar, is called self-similar Markov process. These processes are involved in various parts of probability theory, such as branching processes and fragmentation theory [2].

Current authors considered DSI processes in the wide sense with some scale \( l > 1 \). They proposed to have some fixed number of samples, say \( T \), in each scale at points \( \alpha^k, k \in \mathbb{Z} \) where \( l = \alpha^T, T \in \mathbb{N} \). By such sampling they provided a discrete time scale invariant process in the wide sense and found a closed formula for its covariance function [6]. In this paper we consider \( X(\cdot) \) as DSI process with scale \( l > 1 \), and sampling at arbitrary points \( 1 \leq s_0 < s_1 \ldots < s_{q-1} < l \) in the interval \([1, l)\). We also take our samples at points \( l^n s_0, l^n s_1, \ldots, l^n s_{q-1}, n \in \mathbb{Z} \) in the intervals \([l^n, l^{n+1})\). Then we introduce some discrete time embedded scale invariant (DT-ESI) process \( W(nq + k) = X(l^n s_k), q \in \mathbb{N}, k = 0, \ldots, q-1 \) and corresponding multi-dimensional discrete time embedded self-similar (DT-ESS) process as \( V(n) = (V^0(n), \ldots, V^{q-1}(n)) \) where \( V^k(n) = W(nq + k) \). We investigate properties of these processes when they are Markov in the wide sense.

This paper is organized as follows. In section 2, we present a review of multi-dimensional stationary, periodically correlated, self-similar and discrete scale invariant processes. Then we define discrete time self-similar (DT-SS) and scale invariant (DT-SI) processes. We also introduce quasi Lamperti transformation in this section. Section 3 is devoted to the structure of the multi-dimensional DT-SS process resulting from the above method of sampling. We define DT-ESI process and corresponding multi-dimensional DT-ESS process and characterize the spectral density matrix of it in this section. Finally covariance function and spectral density matrix of the discrete time embedded scale invariant Markov (DT-ESIM) processes and corresponding multi-dimensional discrete time embedded self-similar Markov (DT-ESSM) are obtained in section 4.

2 Theoretical framework

This section is organized in tree subsections. First we review the structure of the covariance function and spectral distribution matrix of multi-dimensional stationary processes. We present definitions of DT-SS, DT-SI, wide sense self-similar and scale invariant processes in subsection 2.2. We define quasi Lamperti transformation and present its properties which provide a one to one correspondence between DT-SS and discrete time stationary processes and also between DT-SI and DT-PC processes.

2.1 Stationary and multi-dimensional stationary processes

Definition 2.1 A process \( \{Y(t), t \in \mathbb{R}\} \) is said to be stationary, if for any \( t, \tau \in \mathbb{R} \)

\[
\{Y(t + \tau)\} \overset{d}{=} \{Y(t)\}
\]

where \( \overset{d}{=}) \) is the equality of all finite-dimensional distributions.
If (2.1) holds for some $\tau \in \mathbb{R}$, the process is said to be periodically correlated. The smallest of such $\tau$ is called period of the process.

By Rozanov [8], if $Y(t) = \{Y^k(t)\}_{k=1,...,n}$ be an $n$-dimensional stationary process, then

$$Y(t) = \int e^{i\lambda t} \phi(d\lambda)$$

is its spectral representation, where $\phi = \{\varphi_k\}_{k=1,...,n}$ and $\varphi_k$ is the random spectral measure associated with the $k$th component $Y^k$ of the $n$-dimensional process $Y$. Let

$$B_{kr}(\tau) = E[Y^k(\tau + t)Y^r(t)], \quad k, r = 1, \ldots, n$$

and $B(\tau) = [B_{kr}(\tau)]_{k,r=1,...,n}$ be the correlation matrix of $Y$. The components of the correlation matrix of the process $Y$ can be represented as

$$B_{kr}(\tau) = \int e^{i\lambda \tau} F_{kr}(d\lambda), \quad k, r = 1, \ldots, n$$

where for any Borel set $\Delta$, $F_{kr}(\Delta) = E[\varphi_k(\Delta)\overline{\varphi_r(\Delta)}]$ are the complex valued set functions which are $\sigma$-additive and have bounded variation. For any $k, r = 1, \ldots, n$, if the sets $\Delta$ and $\Delta'$ do not intersect, $E[\varphi_k(\Delta)\overline{\varphi_r(\Delta')}] = 0$. For any interval $\Delta = (\lambda_1, \lambda_2)$ when $F_{kr}([\lambda_1]) = F_{kr}([\lambda_2]) = 0$ the following relation holds

$$F_{kr}(\Delta) = \frac{1}{2\pi} \int_{\Delta} \sum_{\tau=-\infty}^{\infty} B_{kr}(\tau) e^{-i\lambda \tau} d\lambda$$

$$= \frac{1}{2\pi} B_{kr}(0)(\lambda_2 - \lambda_1) + \lim_{T \to \infty} \frac{1}{2\pi} \sum_{0<|\tau| \leq T} B_{kr}(\tau) \frac{e^{-i\lambda_2 \tau} - e^{-i\lambda_1 \tau}}{-i\tau}$$

in the discrete parameter case, and

$$F_{kr}(\Delta) = \lim_{a \to \infty} \frac{1}{2\pi} \int_{-a}^{a} \frac{e^{-i\lambda_2 \tau} - e^{-i\lambda_1 \tau}}{-i\tau} B_{kr}(\tau) d\tau$$

in the continuous parameter case.

2.2 Discrete time scale invariant processes

Definition 2.2 A process $\{X(t), t \in \mathbb{R^+}\}$ is said to be self-similar of index $H > 0$, if for any $\lambda > 0$

$$\{\lambda^{-H}X(\lambda t)\} \overset{d}{=} \{X(t)\}. \quad (2.5)$$

The process is said to be DSI of index $H$ and scaling factor $\lambda_0 > 0$ or $(H, \lambda_0)$-DSI, if (2.5) holds for $\lambda = \lambda_0$. 

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As an intuition, self-similarity refers to an invariance with respect to any dilation factor. However, this may be a too strong requirement for capturing in situations that scaling properties are only observed for some preferred dilation factors.

**Definition 2.3** A process \( \{X(k), k \in \mathcal{T}\} \) is called discrete time self-similar (DT-SS) process with parameter space \( \mathcal{T} \), where \( \mathcal{T} \) is any subset of countable distinct points of positive real numbers, if for any \( k_1, k_2 \in \mathcal{T} \)
\[
\{X(k_2)\} \overset{d}{=} \left( \frac{k_2}{k_1} \right)^H \{X(k_1)\}.
\] (2.6)

The process \( X(\cdot) \) is called discrete time scale invariance (DT-SI) with scale \( l > 0 \) and parameter space \( \mathcal{T} \), if for any \( k_1, k_2 = lk_1 \in \mathcal{T} \), (2.6) holds.

**Remark 2.1** If the process \( \{X(t), t \in \mathbb{R}^+\} \) is DSI with scale \( l = \alpha^T \) for fixed \( T \in \mathbb{N} \) and \( \alpha > 1 \), then by sampling of the process at points \( \alpha^k, k \in \mathbb{Z} \), we have \( X(\cdot) \) as a DT-SI process with parameter space \( \mathcal{T} = \{\alpha^k, k \in \mathbb{Z}\} \) and scale \( l = \alpha^T \). If we consider sampling of \( X(\cdot) \) at points \( \alpha^{nT+k}, n \in \mathbb{Z} \), for fixed \( k = 0, 1, \ldots, T-1 \), then \( X(\cdot) \) is a DT-SS process with parameter space \( \hat{T} = \{\alpha^{nT+k}, n \in \mathbb{Z}\} \).

Yazici et.al. [9] introduced wide sense self-similar processes as the following definition, which can be obtained by applying the Lamperti transformation \( \mathcal{L}_H \) to the class of wide-sense stationary processes. This class encompasses all strictly self-similar processes with finite variance, including Gaussian processes such as fractional Brownian motion but no other alpha-stable processes.

**Definition 2.4** A random process \( \{X(t), t \in \mathbb{R}^+\} \) is said to be wide sense self-similar with index \( H \), for some \( H > 0 \) if the following properties are satisfied for each \( c > 0 \), \( t, t_1, t_2 > 0 \)
\[
\begin{align*}
\text{(i)} & \quad E[X^2(t)] < \infty, \\
\text{(ii)} & \quad E[X(ct)] = c^H E[X(t)], \\
\text{(iii)} & \quad E[X(ct_1)X(ct_2)] = c^{2H} E[X(t_1)X(t_2)].
\end{align*}
\]

This process is called wide sense DSI of index \( H \) and scaling factor \( c_0 > 0 \), if the above conditions hold for some \( c = c_0 \).

**Definition 2.5** A random process \( \{X(k), k \in \mathcal{T}\} \) is called DT-SS in the wide sense with index \( H > 0 \) and with parameter space \( \mathcal{T} \), where \( \mathcal{T} \) is any subset of distinct countable points of positive real numbers, if for all \( k, k_1 \in \mathcal{T} \) and all \( c > 0 \), where \( ck, ck_1 \in \mathcal{T} \):
\[
\begin{align*}
\text{(i)} & \quad E[X^2(k)] < \infty, \\
\text{(ii)} & \quad E[X(ck)] = c^H E[X(k)], \\
\text{(iii)} & \quad E[X(ck)X(ck_1)] = c^{2H} E[X(k)X(k_1)].
\end{align*}
\]
If the above conditions hold for some fixed $c = c_0$, then the process is called DT-SI in the wide sense with scale $c_0$.

**Remark 2.2** Let $\{X(t), t \in \mathbb{R}^+\}$ be DSI in the wide sense, with the same scale $l = \alpha^T$. Then $X(\cdot)$ with parameter space $\tilde{T} = \{\alpha^k, k \in \mathbb{Z}\}$ for $\alpha > 1$ is DT-SI in the wide sense and $X(\cdot)$ with parameter space $\tilde{T} = \{\alpha^{nT+k}, n \in \mathbb{Z}\}$ for fixed $T \in \mathbb{N}$, $\alpha > 1$ is DT-SS for $k = 0, \ldots, T - 1$ in the wide sense.

Through this paper we are dealt with wide sense self-similar and wide sense scale invariant process, and for simplicity we omit the term "in the wide sense" hereafter.

### 2.3 Quasi Lamperti transformation

We introduce the quasi Lamperti transformation and its properties by followings.

**Definition 2.6** The quasi Lamperti transform with positive index $H$ and $\alpha > 1$, denoted by $L_{H,\alpha}$ operates on a random process $\{Y(t), t \in \mathbb{R}\}$ as

$$L_{H,\alpha}Y(t) = t^H Y(\log_\alpha t)$$

(2.7)

and the corresponding inverse quasi Lamperti transform $L_{H,\alpha}^{-1}$ on process $\{X(t), t \in \mathbb{R}^+\}$ acts as

$$L_{H,\alpha}^{-1}X(t) = e^{-t^H} X(\alpha^t).$$

(2.8)

**Corollary 2.1** If $\{Y(t), t \in \mathbb{R}\}$ is stationary process, its quasi Lamperti transform $\{L_{H,\alpha}Y(t), t \in \mathbb{R}^+\}$ is self-similar. Conversely if $\{X(t), t \in \mathbb{R}^+\}$ is self-similar process, its inverse quasi Lamperti transform $\{L_{H,\alpha}^{-1}X(t), t \in \mathbb{R}\}$ is stationary.

**Corollary 2.2** If $\{X(t), t \in \mathbb{R}^+\}$ is $(H, \alpha^T)$-DSI then $L_{H,\alpha}^{-1}X(t) = Y(t)$ is PC with period $T > 0$. Conversely if $\{Y(t), t \in \mathbb{R}\}$ is PC with period $T$ then $L_{H,\alpha}Y(t) = X(t)$ is $(H, \alpha^T)$-DSI.

**Remark 2.3** If $X(\cdot)$ is a DT-SS process with parameter space $\tilde{T} = \{l^n, n \in \mathbb{Z}\}$, then its stationary counterpart $Y(\cdot)$ has parameter space $T = \{nT, n \in \mathbb{Z}\}$

$$X(l^n) = L_{H,\alpha}Y(l^n) = l^{nH} Y(\log_\alpha (\alpha^n)) = \alpha^{nT} Y(nT).$$

Also it is clear by the following relation that if $X(\cdot)$ is a DT-SI process with scale $l = \alpha^T$, $T \in \mathbb{N}$ and parameter space $\tilde{T} = \{\alpha^n, n \in \mathbb{Z}\}$, then $Y(\cdot)$ is a discrete time periodically correlated (DT-PC) process with period $T$ and parameter space $\tilde{T} = \{n, n \in \mathbb{Z}\}$

$$Y(n) = L_{H,\alpha}^{-1}X(n) = \alpha^{-nH} X(\alpha^n).$$
3 Structure of the process

In this section we define a multi-dimensional DT-SS process in the wide sense. We also introduce a new method for sampling of a DSI process with scale \( l > 1 \), which provide sampling at arbitrary points in the interval \([1,l] \) and at multiple \( l^n \) of such points in the intervals \([l^n,l^{n+1}]\), \( n \in \mathbb{N} \). We introduce DT-ESI process corresponding to the multi-dimensional DT-ESS process. Finally in Theorem 3.1 we find harmonic like representation and spectral density matrix of the multi-dimensional DT-ESS process.

**Definition 3.1** The process \( U(t) = (U^0(t), U^1(t), \ldots, U^{q-1}(t)) \) with parameter space \( T = \{l^n, n \in \mathbb{Z}\} \), \( l = \alpha^T \), \( \alpha > 1 \) and \( T \in \mathbb{N} \) is a \( q \)-dimensional discrete time self similar process in the wide sense, where

\[
\begin{align*}
&\text{(a)} \quad \{U^j(\cdot)\} \text{ for all } j = 0, 1, \cdots, q - 1 \text{ is DT-SS process with parameter space } \\
&\quad T^j = \{l^n, n \in \mathbb{Z}\}.
\end{align*}
\]

\[
\begin{align*}
&\text{(b)} \quad \text{For every } n, \tau \in \mathbb{Z}, \ j, k = 0, 1, \cdots, q - 1 \\
&\quad \text{Cov}(U^j(l^{n+\tau}), U^k(l^n)) = l^{2nH} \text{Cov}(U^j(l^\tau), U^k(1)).
\end{align*}
\]

Our method of sampling is to provide enough flexibility to choose arbitrary sample points of a discrete time scale invariant process. So, one could decide to have \( T \) partitions in each scale interval \( I_n = [l^n, l^{n+1}] \), \( n \in \mathbb{Z} \) of a continuous time DSI process \( X(\cdot) \) with scale \( l > 1 \) and find \( \alpha \) by \( l = \alpha^T \). Then our partitions in scale interval \( I_n \) are

\[
[\alpha^{nT}, \alpha^{nT+1}), [\alpha^{nT+1}, \alpha^{nT+2}), \ldots, [\alpha^{nT+T-1}, \alpha^{(n+1)T}).
\]

So we consider to have \( n_k \) samples in partition \([\alpha^{nT+k}, \alpha^{nT+k+1})\) at points

\[
\alpha^{nT+k}s_{k_1}, \alpha^{nT+k}s_{k_2}, \ldots, \alpha^{nT+k}s_{k_{n_k}}
\]

where \( 1 \leq s_{k_1} < s_{k_2} < \ldots < s_{k_{n_k}} < \alpha, k = 0, \ldots, T - 1 \) and \( q = \sum_{i=0}^{T-1} n_i \). Now we can state the following remark.

**Remark 3.1** Let \( U^k(l^n) = X(l^n s_u) \) in Definition 3.1, where \( s_u = \alpha^k s_{k_i} \) in which \( \sum_{i=0}^{k-1} n_i \leq u < \sum_{i=1}^{k} n_i, n_{-1} = 0 \) and \( u = x + \sum_{i=0}^{k-1} n_i, x = 1, \ldots, n_k \). Thus \( X(l^n s_u) \) for \( u = 0, \ldots, q - 1 \) is a DT-SS process and \( U(l^n) = (X(l^n s_0), \ldots, X(l^n s_{q-1})) \) is a \( q \)-dimensional DT-SS process.

By such method of sampling at discrete points we provide a \( q \)-dimensional DT-ESS process \( V(n) \) as

\[
V(n) = (V^0(n), V^1(n), \ldots, V^{q-1}(n)), \quad n \in \mathbb{Z}
\]
where \( q = \sum_{i=0}^{T-1} n_i \) and
\[
V^u(n) := X(\alpha^T s_u)
\] (3.1)
\[
\sum_{i=-1}^{k-1} n_i \leq u < \sum_{i=-1}^{k} n_i, \ n_{-1} = 0, \ s_u = \alpha^k s_{k_x} \text{ and } x = u - \sum_{i=0}^{k-1} n_i, \ u = 0, \ldots, q - 1.
\]

**Remark 3.2** Corresponding to the \( q \)-dimensional DT-ESS process \( V(n) \) there exist a DT-ESI process \( W(\kappa) \) with scale \( l = \alpha^T \) as
\[
W(\kappa) := V^u(n) = X(\alpha^T s_u) \quad \kappa \in \mathbb{Z}
\] (3.2)
where \( u = \kappa - q[\frac{\kappa}{q}], \ n = [\frac{\kappa}{q}] \) and \( \kappa = nq + u, \) since by (3.1) and (3.2)
\[
W(\kappa + q) = X(\alpha^{(n+1)T} s_u) \overset{d}{=} \alpha^{TH} X(\alpha^{nT} s_u) = l^H W(\kappa).
\]

By the following theorem, the spectral density matrix of the \( q \)-dimensional DT-ESS process and harmonic like representation of each column is obtained.

**Theorem 3.1** Let \( X(\cdot) \) be a DSI process with scale \( l = \alpha^T \) and \( 1 \leq s_0 < s_1 < \ldots < s_q < \alpha^T, \) then \( V(n) = (V^0(n), \ldots, V^{q-1}(n)) \), where \( V^u(n) = X(\alpha^T s_u), \ n \in \mathbb{Z} \) and \( u = 0, \ldots, q - 1 \) is a \( q \)-dimensional DT-ESS process and

(i) The harmonic like representation of \( V^u(n) \) is
\[
V^u(n) = (\alpha^T s_u)^H \int_0^{2\pi} e^{i\omega u} d\phi_u(\omega)
\] (3.3)
where \( \phi_u(\omega) \) is an orthogonal spectral measure, that is \( E[d\phi_u(\omega) d\phi_\nu(\omega') ] = 0, \ u, \nu = 0, \ldots, q - 1 \) when \( \omega \neq \omega' \).

(ii) The corresponding spectral density matrix of \( V(n) \) is \( g^H(\omega) = [g^H_{u, \nu}(\omega)]_{u, \nu = 0, \ldots, q - 1}, \)

where
\[
g^H_{u, \nu}(\omega) = \frac{(s_u s_u)^{-H}}{2\pi} \sum_{\tau = -\infty}^{\infty} \alpha^{-TH} e^{-i\tau} Q^H_{u, \nu}(\tau)
\] (3.4)
\( \tau \in \mathbb{N} \) and \( Q^H_{u, \nu}(\tau) \) is the covariance function of \( V^u(\tau) \) and \( V^\nu(0) \).

**Proof of (i):** Remark 2.3 implies that
\[
V^u(n) = X(\alpha^T s_u) = \mathcal{L}_{H, \alpha} Y(\alpha^T s_u) = (\alpha^T s_u)^H \eta^u(n)
\]
where \( \eta^u(n) = Y(nT + \log_n s_u) \). Thus \( V^u(n) \) for every \( u = 0, 1, \ldots, q - 1 \) is a DT-ESS process in \( n \), where its discrete time stationary counterpart \( \eta^u(n) \) for fixed \( u = 0, 1, \ldots, q - 1 \) has spectral representation \( \eta^u(n) = \int_0^{2\pi} e^{i\omega n} d\phi_u(\omega) \).
**Proof of (ii):** The covariance matrix of \( V(n) \) is denoted by \( Q^H(n, \tau) = [Q^H_{u,\nu}(n, \tau)]_{u,\nu=0,\ldots,q-1} \) where

\[
Q^H_{u,\nu}(n, \tau) = E[V^u(n + \tau)V^\nu(n)] = E[X(\alpha^{n+\tau}T s_u)X(\alpha^nT s_\nu)]
\]

By the scale invariant property of the process \( X(\cdot) \) we have that

\[
Q^H_{u,\nu}(n, \tau) = \alpha^{2nT H} E[X(\alpha^{T} s_u)X(s_\nu)] = \alpha^{2nT H} Q^H_{u,\nu}(\tau) \tag{3.5}
\]

where \( Q^H_{u,\nu}(\tau) = Q^H_{u,\nu}(0, \tau) = E[V^u(\tau)V^\nu(0)] \), then by (3.3)

\[
Q^H_{u,\nu}(\tau) = E[(\alpha^{T} s_u)^H(s_\nu)^H \int_0^{2\pi} e^{i\omega\tau} d\phi_u(\omega) \int_0^{2\pi} d\phi_\nu(\omega)]
\]

\[
= \alpha^{T H} (s_u s_\nu)^H \int_0^{2\pi} e^{i\omega\tau} dG^H_{u,\nu}(\omega) \tag{3.6}
\]

where \( E[d\phi_u(\omega)d\phi_\nu(\omega)] = dG^H_{u,\nu}(\omega) \) when \( \omega = \omega' \) and is 0 when \( \omega \neq \omega' \).

On the other hand, by the definition of \( \eta^u(n) \) in the proof of part (i)

\[
Q^H_{u,\nu}(\tau) = E[X(\alpha^{T} s_u)X(s_\nu)] = E[\mathcal{L}_{H,\alpha} Y(\alpha^{T} s_u)\mathcal{L}_{H,\alpha} Y(s_\nu)]
\]

\[
= (\alpha^{T} s_u s_\nu)^H E[Y(\tau T + \log_\alpha s_u)Y(\log_\alpha s_\nu)]
\]

\[
= (\alpha^{T} s_u s_\nu)^H E[\eta^u(\tau)\eta^\nu(0)] = (\alpha^{T} s_u s_\nu)^H B_{u,\nu}(\tau).
\]

Then by (3.6)

\[
B_{u,\nu}(\tau) = \int_0^{2\pi} e^{i\omega\tau} dG^H_{u,\nu}(\omega), \quad u, \nu = 0, \ldots, q-1
\]

Now by (2.3) and (2.4) for \( u, \nu = 0, \ldots, q-1 \) we have that

\[
G^H_{u,\nu}(A) = \frac{1}{2\pi} \int_A \sum_{\tau=-\infty}^\infty B_{u,\nu}(\tau)e^{-i\lambda\tau} d\lambda.
\]

By substituting \( B_{u,\nu}(\tau) = (\alpha^{T} s_u s_\nu)^{-H} Q^H_{u,\nu}(\tau) \), the elements of the spectral distribution function, \( G^H_{u,\nu}(\cdot) \) has the following representation

\[
G^H_{u,\nu}(A) = \frac{(s_u s_\nu)^{-H}}{2\pi} \int_A \sum_{\tau=-\infty}^\infty \alpha^{-T H \tau} e^{-i\lambda\tau} Q^H_{u,\nu}(\tau) d\lambda. \tag{3.7}
\]
Let $A = (\omega, \omega + d\omega]$, then the elements of the spectral density matrix, $g^H_{u,v}(\omega)$ are
\[
g^H_{u,v}(\omega) := \frac{G^H_{u,v}(d\omega)}{2\pi} = \frac{(s_us_v)^{-H}}{2\pi} \sum_{\tau = -\infty}^{\infty} \alpha^{-TH\tau} \left( \frac{1}{d\omega} \int_{\omega}^{\omega+d\omega} e^{-i\lambda\tau} d\lambda \right) Q^H_{u,v}(\tau)
\]
\[
= \frac{(s_us_v)^{-H}}{2\pi} \sum_{\tau = -\infty}^{\infty} \alpha^{-TH\tau} \left( \lim_{d\omega \to 0} \frac{e^{-i(\omega+d\omega)\tau} - e^{-i\omega\tau}}{d\omega} \right) Q^H_{u,v}(\tau)
\]
\[
= \frac{(s_us_v)^{-H}}{2\pi} \sum_{\tau = -\infty}^{\infty} \alpha^{-TH\tau} \left( \left( \frac{1}{-i\tau} \right)(-i\tau) e^{-i\omega\tau} \right) Q^H_{u,v}(\tau).
\]

Thus we get to the assertion of part (ii) of the theorem. □

4 Multi-dimensional DT-ESSM process

Using our method of sampling in section 3, we find the covariance function of the DT-ESI process $W(\cdot)$, which is defined in (3.2) and its corresponding multi-dimensional DT-ESS process $V(\cdot)$, defined in (3.1) for the case that they are Markov in the wide sense as well, which we call them DT-ESIM and DT-ESSM respectively in subsection 4.1. We find the spectral density matrix of these processes in subsection 4.2.

4.1 Covariance function of DT-ESIM

Here we characterize the covariance function of the DT-ESIM process $\{W(\kappa), \kappa \in \mathbb{Z}\}$ in Theorem 4.1 and the covariance function of the associated $q$-dimensional DT-ESSM process in Theorem 4.2.

**Theorem 4.1** Let $\{W(\kappa), \kappa \in \mathbb{Z}\}$, defined in (3.2), be DT-ESI and Markov in the wide sense DT-ESIM, with scale $\alpha^T$. Then for $\tau \in \mathbb{W} = \{0, 1, \ldots\}$, $\kappa = nq + \nu$, $\kappa + \tau = mq + u$, $u, \nu = 0, \ldots, q - 1$ and $n, m \in \mathbb{Z}$, the covariance function
\[
R_\kappa(\tau) := E[W(\kappa + \tau)W(\kappa)] = E[X(\alpha^{nT}s_u)X(\alpha^{mT}s_\nu)]
\]
can be characterized as
\[
R_\kappa(tq + s) = [\tilde{f}(q - 1)]^t \tilde{f}(\kappa + s - 1)[\tilde{f}(\kappa - 1)]^{-1} R_\kappa(0)
\]
\[
R_\kappa(-tq + s) = \alpha^{-2tqH} R_{\kappa+s}((t - 1)q + q - s)
\]
where $1 \leq s_0 < s_1 < \ldots < s_{q-1} < \alpha^T$, $t \in \mathbb{Z}$, $s = 0, \ldots, q - 1$

\[
\tilde{f}(r) = \prod_{j=0}^{r} f(j) = \prod_{j=0}^{r} R_j(1)/R_j(0), \quad r \in \mathbb{Z}
\]
and $\tilde{f}(-1) = 1$. 

9
Before proceeding to the proof of the theorem we present the main property of covariance function of the wide sense Markov process.

Let \( \{X(n), n \in \mathbb{Z}\} \) be a second order process of centered random variables, \( E[X(n)] = 0 \) and \( E[|X(n)|^2] < \infty, n \in \mathbb{Z} \). Following Doob \[3\], the real valued second order process \( X(\cdot) \) is Markov in the wide sense if

\[
R(n_1, n_2) = G(\min(n_1, n_2))H(\max(n_1, n_2))
\] (4.4)

where \( R(n_1, n_2) := E[X(n_1)X(n_2)] \) is the covariance function of \( X(\cdot) \) and \( G \) and \( H \) are defined uniquely up to a constant multiple and the ratio \( G/H \) is a positive nondecreasing function.

**Proof of the theorem:** As \( \{W(\kappa), \kappa \in \mathbb{Z}\} \) is DT-ESI with scale \( \alpha^T \), this theorem fully characterize the covariance function of the DT-ESIM process. From the Markov property (4.4), \( R_\kappa(\tau) \) defined in (4.1), satisfies

\[
R_\kappa(\tau) = G(\alpha^{n_T} s_\nu)H(\alpha^{m_T} s_u), \quad \tau \in \mathbb{Z}, \alpha > 1
\] (4.5)

By substituting \( \tau = 0 \) in the above relation we have \( m = n \), then

\[
G(\alpha^{n_T} s_\nu) = \frac{R_\kappa(0)}{H(\alpha^{n_T} s_u)}. \tag{4.6}
\]

Therefore

\[
R_\kappa(\tau) = \frac{H(\alpha^{m_T} s_u)}{H(\alpha^{n_T} s_u)}R_\kappa(0), \quad \tau \in \mathbb{Z}
\]

Thus

\[
H(\alpha^{m_T} s_u) = \frac{R_\kappa(\tau)}{R_\kappa(0)}H(\alpha^{n_T} s_u).
\]

So

\[
H(\kappa + \tau) = \frac{R_\kappa(\tau)}{R_\kappa(0)}H(\kappa)
\]

where \( H(\kappa) = H(\alpha^{n_T} s_\nu) \) and \( H(\kappa + \tau) = H(\alpha^{m_T} s_u) \). Therefore

\[
R_\kappa(\tau) = \frac{H(\kappa + \tau)}{H(\kappa)}R_\kappa(0).
\] (4.7)

For \( \tau = 1 \), we have

\[
H(\kappa + 1) = \frac{R_\kappa(1)}{R_\kappa(0)}H(\kappa).
\]

By the recursive relation, it follows that

\[
H(\kappa + 1) = \frac{R_\kappa(1) R_{\kappa-1}(1)}{R_\kappa(0) R_{\kappa-1}(0)} \cdots \frac{R_0(1)}{R_0(0)}H(0) = H(0) \prod_{j=0}^{\kappa} f(j)
\]

10
and

\[ H(\kappa) = H(0) \prod_{j=0}^{\kappa-1} f(j) \]

where \( f(j) = R_j(1)/R_j(0) \). By the assumptions \( n = [\kappa/q] \), \( \nu = \kappa - q[\kappa/q] \) we have \( \kappa = nq + \nu \), then

\[ H(nq + \nu) = H(0) \prod_{j=0}^{nq+\nu-1} f(j). \quad (4.8) \]

As mentioned in Remark 3.2, \( \{W(\kappa), \kappa \in \mathbb{Z}\} \) is DT-ESI with scale \( l \), then

\[ f(\kappa + q) = \frac{R_{\kappa+q}(1)}{R_{\kappa+q}(0)} = \frac{E[W(\kappa + q + 1)W(\kappa + q)]}{E[W(\kappa + q)W(\kappa + q + 1)]} = \frac{\alpha^{2TH}E[W(\kappa + 1)W(\kappa)]}{\alpha^{2TH}E[W(\kappa)W(\kappa + 1)]} = \frac{R_{\kappa+1}}{R_{\kappa}} = f(\kappa). \]

Hence by (4.8)

\[ H(nq + \nu) = H(0) \prod_{j=0}^{q-1} f(j) \prod_{j=0}^{\nu-1} f(j), \quad \nu \geq 1 \]

By the definition of \( \tilde{f} \) in (4.3)

\[ H(nq + \nu) = H(0)[\tilde{f}(q-1)]^n \tilde{f}(\nu - 1). \quad (4.9) \]

By a similar method one can verify that

\[ H(-nq + \nu) = H(0)[\tilde{f}(q-1)]^{-n} \tilde{f}(\nu - 1). \]

Let \( \tau = tq + s \) in (4.7), \( t \in \mathbb{W} \) and \( s = 0, 1, \ldots, q - 1 \), then it follows from (4.9) that

\[ R_\kappa(tq + s) = \frac{H(\kappa + tq + s)}{H(\kappa)} R_\kappa(0) = \frac{H(0)[\tilde{f}(q-1)]^t \tilde{f}(\kappa + s - 1)}{H(0)f(\kappa - 1)} R_\kappa(0). \]

For \( \tau = -tq + s \) we have that

\[ R_\kappa(-tq + s) = E[X(\alpha^{-tq+s})X(\alpha^\kappa)] = \alpha^{-2tqH} E[X(\alpha^{\kappa+s})X(\alpha^{tq+s})] = \alpha^{-2tqH} R_{\kappa+s}(tq - s) = \alpha^{-2tqH} R_{\kappa+s}((t - 1)q + q - s). \quad \square \]

Now we can use this theorem to prove the next result as follows, for \( q \)-dimensional DT-ESSM process.
**Theorem 4.2** Let \( \{W(\kappa), \kappa \in \mathbb{Z}\} \) be a DT-ESIM process, and \( \{V(n), n \in \mathbb{Z}\} \) be its associated \( q \)-dimensional DT-ESSM process with covariance matrix \( Q^H(n, \tau) \) which is defined by (3.5). Then

\[
Q^H(n, \tau) = \alpha^{2nTH} [\tilde{f}(q - 1)]^\tau CR, \quad \tau \in \mathbb{Z} \tag{4.10}
\]

where \( \tilde{f}(\cdot) \) is defined in (4.3) and the matrices \( C \) and \( R \) are given by \( C = [C_{u,\nu}]_{u,\nu=0,\ldots,q-1} \), where \( C_{u,\nu} = \tilde{f}(u - 1)[\tilde{f}(\nu - 1)]^{-1} \), and \( R \) is a diagonal matrix with diagonal elements \( R_{\nu}(0), \nu = 0, 1, \ldots, q - 1 \), which is defined in (4.1).

**Proof:** As \( W(\cdot) \) is DT-ESI with scale \( l \), (3.2) and (3.5) indicate that \( Q^H_{u,\nu}(n, \tau) = \alpha^{2nTH} Q^H_{u,\nu}(\tau) \). Now by the assumption \( \kappa = nq + \nu \) and \( \kappa + \tau = mq + u \) where \( m, n \in \mathbb{Z}, \tau \in \mathbb{W} \), we have \( \tau = (m - n)q + u - \nu \) and therefore

\[
R_{\kappa}(\tau) = R_{nq+\nu}((m-n)q+u-\nu) = E[W(mq+u)W(nq+\nu)]
= E[X(\alpha^{mT} s_u)X(\alpha^{nT} s_\nu)].
\]

Hence

\[
Q^H_{u,\nu}(\tau) = E[X(\alpha^{mT} s_u)X(\alpha^{nT} s_\nu)] = R_{\nu}(\tau q + u - \nu) \tag{4.11}
\]

and by the Markov property of \( W(\cdot) \) from (4.2) we have

\[
R_{\nu}(\tau q + u - \nu) = [\tilde{f}(q - 1)]^\tau \tilde{f}(u - 1)[\tilde{f}(\nu - 1)]^{-1} R_{\nu}(0)
\]

for \( u, \nu = 0, \ldots, q - 1 \). Let \( C_{u,\nu} = \tilde{f}(u - 1)[\tilde{f}(\nu - 1)]^{-1} \), so

\[
Q^H_{u,\nu}(\tau) = [\tilde{f}(q - 1)]^\tau C_{u,\nu} R_{\nu}(0). \tag{4.12}
\]

Thus we can represent the elements of the covariance matrix of \( q \)-dimensional DT-ESSM process as

\[
Q^H_{u,\nu}(n, \tau) = \alpha^{2nTH} [\tilde{f}(q - 1)]^\tau C_{u,\nu} R_{\nu}(0). \Box
\]

### 4.2 Spectral representation of the process

The spectral density matrix of the \( q \)-dimensional DT-ESSM process is characterized by the following lemma which is proved in [7].

**Lemma 4.1** The spectral density matrix \( g^H(\omega) = [g^H_{u,\nu}(\omega)]_{u,\nu=0,\ldots,q-1} \) of the \( q \)-dimensional DT-ESSM process \( V(n) \) is specified by

\[
g^H_{u,\nu}(\omega) = \frac{(s_u s_\nu)^{-H}}{2\pi} \left[ \frac{\tilde{f}(u - 1) R_{\nu}(0)}{\tilde{f}(\nu - 1)(1 - e^{-i\omega \alpha^{HT} \tilde{f}(q - 1)})} - \frac{\tilde{f}(\nu - 1) R_{\nu}(0)}{\tilde{f}(u - 1)(1 - e^{-i\omega \alpha^{HT} \tilde{f}^{-1}(q - 1)})} \right]
\]

where \( R_{\kappa}(0) \) is the variance of \( W(k) \) and \( \tilde{f}(\cdot) \) is defined by (4.3).
Proof: By applying (3.4) and (4.12), the spectral density matrix of the process \( \{V(n), n \in \mathbb{Z} \} \) which is denoted by \( g^H(\omega) = [g^H_{u,v}(\omega)]_{u,v=0,\ldots,q-1} \) can be written as

\[
g^H_{u,v}(\omega) = \frac{(s_us_v)^{-H}}{2\pi} \left[ \sum_{\tau=0}^{\infty} \alpha^{-TH\tau} e^{-i\omega\tau} Q^H_{u,v}(\tau) + \sum_{\tau=0}^{-1} \alpha^{-TH\tau} e^{-i\omega\tau} Q^H_{u,v}(\tau) \right] = g^H_{u,v,1}(\omega) + g^H_{u,v,2}(\omega)
\]

where

\[
g^H_{u,v,1}(\omega) = \frac{(s_us_v)^{H}}{2\pi} \sum_{\tau=0}^{\infty} \alpha^{-TH\tau} e^{-i\omega\tau} [\tilde{f}(q-1)]^\tau \tilde{f}(u-1)[\tilde{f}(\nu-1)]^{-1} R_v(0)
\]

\[
= \frac{(s_us_v)^{H}}{2\pi} \frac{\tilde{f}(u-1)R_v(0)}{\tilde{f}(\nu-1)} \sum_{\tau=0}^{\infty} \alpha^{-TH\tau} e^{-i\omega\tau} (\tilde{f}(q-1))^\tau. \tag{4.13}
\]

By Remark 3.2, the scale invariant property of \( W(k) \) and the assumption, that at least one of the Corr \( [W(j)W(j+1)] \) be smaller than one, we have that \( |\tilde{f}(q-1)| < \alpha^{TH} \) for \( j = 0, \ldots, q-1 \). Thus

\[
|e^{-i\omega\alpha^{-TH}\tilde{f}(q-1)}| = |\alpha^{-TH}\tilde{f}(q-1)| < 1,
\]

and (4.13) for \( \tau \in \mathbb{W} \) is convergent. By the equality

\[
Q_{u,v}(-\tau) = E[X(\alpha^{-T} s_u)X(s_v)] = \alpha^{-2rTH} E[X(\alpha^{T} s_v)X(s_u)] = \alpha^{-2rTH} Q_{v,u}(\tau),
\]

convergence of \( g^H_{u,v,2}(\omega) \) follows by a similar method. Therefore

\[
g^H_{u,v}(\omega) = \frac{(s_us_v)^{H}}{2\pi} \left[ \frac{R_v(0)\tilde{f}(u-1)}{\tilde{f}(\nu-1)} \sum_{\tau=0}^{\infty} (\alpha^{-TH} e^{-i\omega} \tilde{f}(q-1))^\tau \right.
\]

\[
+ \frac{R_v(0)\tilde{f}(\nu-1)}{\tilde{f}(u-1)} \sum_{\tau=1}^{\infty} (\alpha^{-TH} e^{i\omega} \tilde{f}(q-1))^\tau \left. \right]
\]

\[
= \frac{(s_us_v)^{H}}{2\pi} \left[ \frac{R_v(0)\tilde{f}(u-1)}{\tilde{f}(\nu-1)(1 - \alpha^{-TH} e^{-i\omega} \tilde{f}(q-1))} + \frac{R_v(0)\tilde{f}(\nu-1)}{\tilde{f}(u-1)(1 - \alpha^{-TH} e^{i\omega} \tilde{f}(q-1))} \right],
\]

so we arrive at the assertion of the lemma. □

Example 4.1 Let

\[
X(t) = \sum_{n=1}^{\infty} \lambda_n^{(H-\frac{1}{2})} I_{[\lambda^{n-1},\lambda^{n})}(t)B(t)
\]
where $B(\cdot)$ is the standard Brownian motion, $I(\cdot)$ indicator function, $H > 0$ and $\lambda > 1$. We call this process Simple Brownian Motion. We showed in [7] that $\{X(t), t \in R^+\}$ is DSI and Markov with Hurst index $H$ and scale $\lambda$.

We showed in [7] that $\{X(t), t \in R^+\}$ is DSI and Markov with Hurst index $H$ and scale $\lambda$.

By sampling of this process at points $\alpha^n T_s, n \in W$, where $1 \leq s_0 \leq s_1, \cdots, s_q-1 < \alpha T$, and by assuming $\lambda = \alpha T$,

$$W(\kappa) := X(\alpha^n T_s),$$

is a DT-ESIM process, and $V(n) = (V^0(n), \ldots, V_{q-1}(n))$ where $V^u(n) = W(\kappa)$ is the associated $q$-dimensional DT-ESSM process where $u = k - q[k \frac{\kappa}{q}]$, $n = [k \frac{\kappa}{q}]$. By (4.1) we have that $R^H_j(0) = R^H_j(1) = \alpha^{2TH}\beta_j$ for $j = 0, \cdots, q-2$ and $R^H_q-1(1) = \alpha^{TH}R^H_q-1(0) = \alpha^{3TH}\beta_{q-1}$, where $H' = H - \frac{1}{2}$. So $R_u(0) = \alpha^{2TH}\beta_u$, $R_\nu(0) = \alpha^{2TH}\beta_\nu$. Also (4.3) implies that $\hat{f}(u-1) = \hat{f}(\nu-1) = 1$, $\hat{f}(q-1) = \alpha^{TH}$. Thus By Lemma 4.1, the spectral density matrix of $V(n)$ is

$$g^{H}_{u,\nu}(\omega) = \frac{(s_u s_\nu)^{-H/2} \alpha^{2TH}}{2\pi} \left[ \frac{s_\nu}{1 - e^{-i\omega \alpha^{-T}/2}} - \frac{s_u}{1 - e^{-i\omega \alpha^{-T}/2}} \right].$$

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