Debye screening mass of hot Yang-Mills theory to three loop order

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Frontiers in pQFT, Bielefeld 2012
Debye screening mass

- It parametrizes the dynamically generated screening of chromo-electric fields in hot QCD.
- Here defined as a matching coefficient of electrostatic QCD.
- The result contributes to the coefficient of $O(g^7)$ of the pressure in hot QCD.
- During computation we developed new methods for computing (scalar as well as tensor) 3-loop sum-integrals.
Electrostatic QCD

- Thermodynamical equilibrium $\Rightarrow$ static fields $\tau = 0 \Rightarrow d = 3$.
- Write down the most general Lagrangian that resembles all the symmetries of the original $\mathcal{L}_{\text{QCD}}$ [Ginsparg 1980, Appelquist, Pisarski, 1981].

$$\mathcal{L}_{\text{EQCD}}^{3d} = \frac{1}{4} F_{ij}^a F_{ij}^a + \text{Tr}[D_i, A_0]^2 + m_E^2 \text{Tr}[A_0^2] + \lambda_E^{(1)} (\text{Tr}[A_0^2])^2 + ...$$

$$D_i = \partial_i - ig_E A_i, \ i, j = 1, 2, 3$$

- It is a super-renormalizable, universal theory.
- Describes dynamics at intermediate scale $p \leq 2\pi T$.
- Contains 2 scales: $\approx gT, \approx g^2 T$. 
$m_E$ to $\mathcal{O}(g^6)$

- Define the Debye mass as the pole of the static gluon propagator.
- on the QCD side:
  
  $p^2 + \Pi_{00}(0, p)_{p^2=-m_{el}^2} = 0$

- on the EQCD side:
  
  $p^2 + m_{E,\text{ren}}^2 + \delta m_{E}^2 + \Pi_{\text{EQCD}}(p^2)_{p^2=-m_{el}^2} = 0$

  Leading to:

  $m_{E,\text{ren}}^2 = m_{el}^2 - \delta m_{E}^2 - \Pi_{\text{EQCD}}(-m_{el}^2)$

- $m_E$ requires computation of the gluon self-energy.
Mass renormalization

- Mass parameter renormalization due to known UV divergences from $SU(N_c) +$ Adjoint Higgs Theory ($N_c \equiv C_A$):

$$\delta m_E^2 = \frac{2(N^2 + 1)}{(4\pi)^2} \frac{\mu^{-4\epsilon}}{4\epsilon} (-g_E^2 \lambda^{(1)} C_A + \lambda^{(1)}^2)$$

- Using $g_E$ and $\lambda^{(1)}$ from matching EQCD to QCD $g_E = gT$ and $\lambda^{(1)} = \frac{20}{3} \frac{N^2}{N^2 + 1} \frac{g^4 T}{(4\pi)^2}$:

$$\delta m_E^2 = -\frac{10}{3} \frac{g^6 T^2}{(4\pi)^4} \frac{C_A^3}{\epsilon}$$
Motivation

Matching computations

Master sum-integrals

Results

Conclusion

Gluon self-energy

- **4d QCD**: Perform an expansion both in $g$ and around $p = 0$ of gluon self-energy:

\[
\Pi^{QCD}_{\mu\nu}(p^2) = \sum_{n=1}^{\infty} \Pi_{\mu\nu,n}(0) (g^2)^n + p^2 \sum_{n=1}^{\infty} \Pi'_{\mu\nu,n}(0) (g^2)^n + \ldots
\]
Gluon self-energy

- **4d QCD**: Perform an expansion both in $g$ and around $p = 0$ of gluon self-energy:

$$\Pi^{QCD}_{\mu\nu}(p^2) = \sum_{n=1}^{\infty} \Pi_{\mu\nu,n}(0) (g^2)^n + p^2 \sum_{n=1}^{\infty} \Pi'_{\mu\nu,n}(0) (g^2)^n + \ldots$$

- **3d EQCD**: $\Pi^{EQCD}_{\mu\nu}(p^2) = 0$ due to the absence of any scale in the vacuum integrals.
Gluon self-energy

- **4d QCD**: Perform an expansion both in \( g \) and around \( p = 0 \) of gluon self-energy:

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\Pi^{QCD}_{\mu\nu}(p^2) = \sum_{n=1}^{\infty} \Pi_{\mu\nu,n}(0) (g^2)^n + p^2 \sum_{n=1}^{\infty} \Pi'_{\mu\nu,n}(0) (g^2)^n + \ldots
\]

- **3d EQCD**: \( \Pi^{EQCD}(p^2) = 0 \) due to the absence of any scale in the vacuum integrals.

- This gives \( (\Pi_{00} \equiv \Pi) \):

\[
m_{el}^2 = g^2 \Pi_1(0) + g^4 \left[ \Pi_2(0) - \Pi'_1(0) \Pi_1(0) \right] + g^6 \left[ \Pi_3(0) - \Pi'_1(0) \Pi_2(0) - \Pi'_2(0) \Pi_1(0) \right.
\]
\[
+ \Pi''_1(0) \Pi_1(0)^2 + \Pi_1(0) \Pi'_1(0)^2 \right] + \mathcal{O}(g^8).
\]
\( \Pi_{\mu\nu} \) to 3-loop order

- \( \Pi_{00} \) enters \( m_E \), \( \Pi_{ij} \) enters \( g_E \).

Automatized procedure:

- \( \sim 500 \) Feynman diagrams are generated.
- Taylor expansion, Lorentz contraction, color algebra yield \( \approx 10^7 \) sum-integrals.
- Systematic method for reducing the no. of sum-integrals: Integration By Parts [Laporta 2000].
  - Result: \( \mathcal{O}(10) \) master sum-integrals with rational functions in \( d = 3 - 2\epsilon \) as coefficients.
  - Problem: All coefficients divergent in the limit \( \epsilon \to 0 \).
  - Non-trivial task: Trade divergent coefficients for more complicated sum-integrals via a suitable basis transformation

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_diagram}
\end{array}
\end{align*}
\]

\[
\frac{94 - 48d + 6d^2}{3(d-3)^2(d-4)} + \frac{16}{3(d-3)^2}
\]
Solving master sum-integrals

- Remaining master sum-integrals

- Laurent expansion in dim. reg. \((d = 3 - 2\epsilon)\) parameter \(\epsilon\):
  \[
  \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \# + \ldots
  \]

- No automatized procedure, everything beyond \(\epsilon^0\) very difficult to compute due to discrete Matsubara modes, \(\sum_{p_0}\).

- Crucial: exploit 1-loop sub-structure [Arnold, Zhai 1994]:

\[
\begin{align*}
\bigcirc &= \sum_{p_0 = -\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{[\Pi_{bce}(P)]^2}{(P^2)^a} \\
\Pi_{bce}(P^2) &= \sum_{q_0 = -\infty}^{\infty} \int \frac{d^d q}{(2\pi)^d} \frac{q_0^e}{(Q^2)^b[(P + Q)^2]^c}, \ P = (p_0, p).
\end{align*}
\]
UV and IR divergences

- Subtract 0-temperature and leading UV parts from $\Pi_{bce}$ to make sum-integrals finite.
- $\Pi^{0-temp}, \Pi_{UV} \propto \frac{1}{(p^2)^\alpha}$. Simple propagator-like structure: all UV divergences expressed in terms of $\Gamma$ and $\zeta$ functions.
- IR divergences from $p_0 = 0$ mode due to

\[
\int_p \frac{1}{(p^2)^a} \Pi_{bce}(p_0 = 0)^2
\]

- "distribute" them on the other propagators through IBP's.

\[
\int \frac{\partial}{\partial p_i} p_i \frac{\Pi_{bc0}(p)^2}{(p^2)^a} = 0
\]

\[
\Rightarrow \int_p \frac{\Pi_{bc0}(p)^2}{(p^2)^a} = \#_1 \int_p \frac{\Pi_{bc0}(p)^2}{(p^2)^{a-1}} + \#_2 \int_p \frac{\Pi_{b-1c0}(p)^2}{(p^2)^a}
\]
Result

\[ \sum_{p} \frac{\Pi_{bce}^{0,\text{UV}} \Pi_{fgh}^{0;\text{UV}} + \text{other comb. thereof}}{(P^2)^a} + \int_{p} \frac{\Pi_{bce}(p_0 = 0) \Pi_{fgh}(p_0 = 0)}{(p^2)^a} \bigg|_{\text{IBP}} \]

\[ + \sum_{p_0 \neq 0} \int_{p} \frac{(\Pi_{bce} - \Pi_{bce}^0 - \Pi_{bce}^{\text{UV}})^2}{(P^2)^a} \bigg] \mathcal{O}(\epsilon^0) \]

- Divergent part done in momentum space.
- Finite part done in configuration space via Fourier transform.
Handling tensor structures

- Of the form:

\[
\int_{PQR} \frac{\{Q_{\mu} R_{\mu}, (Q_{\mu} R_{\mu})^2\}}{(P^2)^a Q^2 R^2 (P + Q)^2 (P + R)^2} \rightarrow \int_{Q} \frac{\{Q_{\mu}, Q_{\mu} Q_{\nu}\}}{Q^2 (P + Q)^2}
\]

- Usual projection techniques introduce \(1/p^2\)-like structures in sum-integrals:

\[
\int_{Q} \frac{q_0 q}{Q^2 (P + Q)^2} = \frac{p_0 p}{p^2} \int_{Q} \left( \frac{1}{Q^2} + \frac{P^2/2 - p_0^2}{Q^2 (P + Q)^2} \right)
\]

- Idea from 0-temperature technique [Tarasov, 1996].

- Remove tensor-structure of master sum-integral:
  tensor sum-integral in \(d\) dim. = scalar sum-integral in \(d + 2(4)\)-dim.
From tensor to scalar

Define:

\[ I_{\nu_1 \ldots \nu_5}^d = \int_{\mathbb{R}^3} \frac{e^{-2\alpha \bar{q} \cdot \bar{r}}}{(p^2 + m_1^2)^{\nu_1}(q^2 + m_2^2)^{\nu_2}(r^2 + m_3^2)^{\nu_3}((p + q)^2 + m_4^2)^{\nu_4}((p + r)^2 + m_5^2)^{\nu_5}} \]

Via \( \Gamma(\nu)A^{-1} = \int_0^\infty d\alpha \alpha^{\nu-1} e^{-\alpha A} \) we get:

\[ I_{\nu_1 \ldots \nu_5}^d \propto \int_0^\infty \frac{\prod_{i=1}^5 d\alpha_i \alpha_i^{\nu_i-1} e^{-\alpha_i m_i^2}}{[D(\alpha) + 2\alpha \alpha_4 \alpha_5 + \alpha^2 \ldots]^{d/2}} \]
From tensor to scalar

Define:

$$I_{\nu_1...\nu_5}^d = \int_{pqr} \frac{e^{-2\alpha \vec{q}\vec{r}}}{(p^2 + m_1^2)^{\nu_1}(q^2 + m_2^2)^{\nu_2}(r^2 + m_3^2)^{\nu_3}((p + q)^2 + m_4^2)^{\nu_4}((p + r)^2 + m_5^2)^{\nu_5}}$$

Via $\Gamma(\nu)A^{-1} = \int_0^\infty d\alpha \alpha^{\nu - 1} e^{-\alpha A}$ we get:

$$I_{\nu_1...\nu_5}^d \propto \int_0^\infty \prod_{i=1}^5 d\alpha_i \alpha_i^{\nu_i - 1} e^{-\alpha_i m_i^2} [D(\alpha) + 2\alpha\alpha_4\alpha_5 + \alpha^2...]^{d/2}$$

Master formula:

$$\partial_{-2\alpha} I_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^d \bigg|_{\alpha=0} = \int_{pqr} \frac{\vec{q}\vec{r}}{(p^2 + m_1^2)^{\nu_1}((p + r)^2 + m_5^2)^{\nu_5}}$$

$$= \int \left( \prod_{i=1}^5 d\alpha_i \alpha_i^{\nu_i - 1} e^{-\alpha_i m_i^2} \right) \frac{\alpha_4\alpha_5}{[D(\alpha)]^{(d+2)/2}}$$

$$= \partial_{m_4^2} \partial_{m_5^2} I_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5}^{d+2} \bigg|_{\alpha=0} = I_{\nu_1,\nu_2,\nu_3,\nu_4+1,\nu_5+1}^{d+2} \bigg|_{\alpha=0}$$
Applying to sum-integrals

Via:

\[
\sum \int_P \frac{\vec{q} \cdot \vec{r}}{(P^2)^a} \prod_{bce} \prod_{fgh} = T^3 \sum_{p_0, q_0, r_0} (q_0)^e (r_0)^h \partial_{-2 \alpha} l^{3-2 \epsilon}_{a, b, c, f, g} \bigg|_{m_1 = p_0, ..., \alpha = 0}
\]

Example of sum-integral containing a tensor-structure:

\[
\begin{array}{c}
\text{Example: } \sum \int_{PQR} \frac{(Q \cdot R)^2}{P^6 Q^2 R^2 (P + Q)^2 (P + R)^2} - 2 \left( \begin{array}{c}
\end{array} \right)
\end{array}
\]

\[
= 4 \sum \int_{PQR} \frac{(Q \cdot R)^2}{P^6 Q^2 R^2 (P + Q)^2 (P + R)^2} - 2 \left( \begin{array}{c}
\end{array} \right)
\]

\[
+ 2 \left( \begin{array}{c}
\end{array} \right)
\]

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\end{array} \right)
\]
...Result

\[ t = -\frac{5}{36} \frac{T^2}{(4\pi)^4} \left( \frac{\mu^2}{4\pi T^2} \right)^3 \epsilon \frac{1}{\epsilon^2} \times \left[ 1 + \epsilon \left( \frac{71}{30} + \gamma_E + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) + 44.6299(1)\epsilon^2 + \mathcal{O}(\epsilon^3) \right] \]
Renormalized mass parameter

\[
m^2_{\text{E, ren}} = T^2 g^2(\bar{\mu}) \frac{C_A}{3} \left\{ 1 + \frac{g^2(\bar{\mu})}{(4\pi)^2} \frac{C_A}{3} \left( 22 \ln \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} + 5 \right) 
+ \frac{g^4(\bar{\mu})}{(4\pi)^4} \left( \frac{C_A}{3} \right)^2 \left( 484 \ln^2 \frac{\bar{\mu} e^{\gamma_E}}{T} - 116 \ln \frac{\bar{\mu} e^{\gamma_E}}{T} + \frac{1091}{2} 
- 144 \gamma_E + 324 \ln 4\pi - 180 \frac{\zeta'(-1)}{\zeta(-1)} - \frac{56}{5} \zeta(3) \right) + \mathcal{O}(g^6) \right\}
\]

- Gauge invariant.
- Band due to variation of arbitrary scale \(\bar{\mu}\).
- Shows better convergence properties in comparison to 1 and 2 loop result.
Conclusion and outlook

- Debye mass calculated via a matching computation between QCD and EQCD. It enters also the $\mathcal{O}(g^7)$ coefficient of the QCD pressure.
- Side benefit: We generalized the computation of a class of three loop sum-integrals.
- In addition: New method for computing tensor sum-integrals.

JHEP 09 (2012) 016
hep-ph/1208.0284

Next steps:
- Find a suitable basis transformation for $\Pi_{ij}$ which enters the computation of $g_E$.
- Compute $g_E$ with the methods developed here and use it in determining the spatial string tension of QCD.