IDENTIFICATION OF MATRIX DIFFUSION COEFFICIENTS IN A PARABOLIC PDE

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Abstract. We consider an inverse problem of identifying the diffusion coefficient in matrix form in a parabolic PDE. In 2006, Cao and Pereverzev, used a natural linearisation method for identifying a scalar valued diffusion coefficient in a parabolic PDE. In this paper, we make use of that idea for identifying a matrix valued coefficient, namely, using the notion of a weak solution for a parabolic PDE, we transform our non-linear inverse problem into a problem of solving an ill-posed operator equation where the operator depending on the data is linear. For the purpose of obtaining stable approximate solutions, Tikhonov regularization is employed, and error estimates under noisy data are derived. We have also showed the uniqueness of the solution of the inverse problem under some assumptions on the data and obtained explicit representation of adjoint of the linear operator involved. For the obtaining error estimates in the finite dimensional setting, Galerkin method is used, by defining orthogonal projections on the space of matrices with entries from $L^2(\Omega)$, by making use of standard orthogonal projections on $L^2(\Omega)$. For choosing the regularizing parameter, we used the adaptive technique, so that we have an order optimal rate of convergence. Finally, for the relaxed noisy data, we described a procedure for obtaining a smoothed version so as to obtain the error estimates.

Keywords: Weak solution, diffusion matrix, parameter identification, Ill-posed, Tikhonov regularization, parameter choice.

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. For a fixed $\tau > 0$, we denote $\Omega \times (0, \tau)$ by $Q$ and its boundary $\partial\Omega \times (0, \tau)$ by $\partial Q$, also we denote the interval $[0, \tau]$ by $I_\tau$. We consider an inverse problem associated with the PDE

$$u_t - \nabla \cdot (A(x)\nabla u) = f(x, t) \quad \text{in } Q$$

along with the following boundary and initial conditions:

$$u(x, t) = 0 \quad \text{on } \partial Q,$$

$$u(x, 0) = h(x) \quad \text{in } \Omega,$$

where $A$ is a $d \times d$ matrix with entries from $H^1(\Omega)$, $h \in L^2(\Omega)$ and $f \in L^2(0, \tau; L^2(\Omega))$.

In the above, and throughout the paper, if $\mathcal{H}$ is a Hilbert space, then by $L^2(0, \tau; \mathcal{H})$ we mean the space $L^2(I_\tau, \mathcal{H})$, that is, the space of all measurable functions $\psi : [0, \tau] \to \mathcal{H}$, defined a.e. on $[0, \tau]$, such that

$$\int_0^\tau \|\psi(t)\|^2_{\mathcal{H}} dt < \infty.$$

Also, if $\mathcal{H}$ is a Hilbert space of functions or equivalence classes of functions defined a.e. on $\Omega$ and if $\psi : \Omega \times [0, \tau] \to \mathbb{R}$ is an a.e. defined measurable function, then we write $\psi \in L^2(0, \tau; \mathcal{H})$ if and only if $\tilde{\psi} : [0, \tau] \to \mathcal{H}$ defined by

$$\tilde{\psi}(t)(x) = \psi(x, t) \quad \text{for almost all } (x, t) \in \Omega \times [0, \tau],$$

belongs to $L^2(I_\tau, \mathcal{H})$. 
Using the above notation, we may observe that
\[ f \in L^2(0, \tau; L^2(\Omega)) \iff f \in L^2(\Omega \times I_\tau). \]

**Convention:** For \( \psi \in L^2(0, \tau; H^1(\Omega)) \), we use the notation, \( \nabla \psi \) for the gradient of \( \psi \) with respect to the space variable.

**Notation:** For the Sobolev space \( H^1_0(\Omega) \), its dual space will be denoted by standard notation \( H^{-1}(\Omega) \) and for simplicity of the notations, their duality action will be denoted by \( \langle \cdot, \cdot \rangle \), that is for \( \psi \in H^{-1}(\Omega) \) and \( \varphi \in H^1_0(\Omega) \), \( \langle \psi, \varphi \rangle = \psi(\varphi) \).

The usual forward problem associated with (1.1)-(1.3) is to find \( u \) for the parabolic system (1.1)-(1.3)? A natural question is: Under what conditions on \( A \) is \( u \) said to be a weak solution of (1.1)-(1.3), which may be known only approximately with some noise.

**Theorem 1.3.** [6, Theorem 6, Ch. 7] If \( u \in L^2(0, \tau; H^1(\Omega)) \) with \( u_t \in L^2(0, \tau; H^{-1}(\Omega)) \) for all \( \varphi \in H^1_0(\Omega) \), then the R.H.S of (1.4) may be replaced by the duality pairing \( \langle f(\cdot, t), \varphi \rangle \).

A natural question is: Under what conditions on \( A, f, h \) we can ensure the existence of a weak solution for the parabolic system (1.1)-(1.3)? The theorem below (Theorem 1.3) specifies certain conditions under which the above question is answered affirmatively (see Theorem 5, Theorem 6 in Chapter 7 of [6]). Before stating the theorem, let us recall the following standard definitions:

**Definition 1.1.** Let \( A, f, h \) be as in (1.1)-(1.3). Then \( u \) is said to be a **weak solution** of the parabolic system (1.1)-(1.3) if \( u \) satisfies
\[ \langle u_t(\cdot, t), \varphi \rangle + \int_\Omega A(x) \nabla u(\cdot, t) \cdot \nabla \varphi = \int_\Omega f(\cdot, t) \varphi \]
for all \( \varphi \in H^1_0(\Omega) \) and for almost all \( t \in [0, \tau] \), along with \( u(x, 0) = h(x) \) for \( x \in \Omega \). If \( f \) belongs to \( L^2(0, \tau; H^{-1}(\Omega)) \) instead of \( L^2(0, \tau; L^2(\Omega)) \), then the R.H.S of (1.4) may be replaced by the duality pairing \( \langle f(\cdot, t), \varphi \rangle \).

**Definition 1.2.** Let \( B \) be a \( d \times d \) matrix with entries from \( L^2(\Omega) \).

(i) \( B \) is said to be **symmetric** a.e. on \( \Omega \) if the matrix \( B(x) \) is symmetric for almost all \( x \in \Omega \).

(ii) \( B \) is said to satisfy the **uniform ellipticity condition**, if there exists a \( q_0 > 0 \) such that
\[ B(x)\xi \cdot \xi = \sum_{1 \leq i, j \leq d} B_{ij}(x)\xi_i\xi_j \geq q_0\|\xi\|^2 \]
for all \( \xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d \), where \( \|\xi\| = (\xi_1^2 + \xi_2^2 + ... + \xi_d^2)^{1/2} \).

**Theorem 1.3.** [6, Theorem 6, Ch. 7] Let \( A \) be a \( d \times d \) symmetric matrix with entries from \( H^1(\Omega) \) such that it satisfies the uniform ellipticity condition as in Definition 1.2. Then (1.1)-(1.3) has a unique weak solution \( u \in L^2(0, \tau; H^{2m+1}(\Omega)) \) provided \( h \) and \( f \) are such that \( h \in H^{2m+1}(\Omega) \) and
\[ \frac{\partial^k f}{\partial t^k} \in L^2(0, \tau; H^{2m-2k}(\Omega)) \quad \text{for} \quad k = 0, \ldots, m, \quad \text{for some} \quad m \in \mathbb{N} \cup \{0\}, \]
and \( h_0, h_1, \ldots, h_m, \) defined iteratively as follows, belong to \( H^1_0(\Omega) \):
\[ h_0 := h, \quad h_1 := f(\cdot, 0) - \nabla \cdot A \nabla h_0, \quad \ldots, \quad h_m := \left( \frac{\partial^{m-1} f(\cdot, 0)}{\partial t^{m-1}} \right) - \nabla \cdot A \nabla h_{m-1}. \]

In this paper, we are interested in finding regularized approximations for the following inverse problem (IP) associated with (1.1)-(1.3):

**IP:** To identify the matrix diffusion coefficient \( A \) from the knowledge of \( u \), a weak solution of (1.1)-(1.3), which may be known only approximately with some noise.
Clearly, the above inverse problem (IP) is non linear. We shall represent this non linear inverse problem as an equation involving a linear operator and carry out the regularization procedure as in the case of a linear operator equation. We assume that the datum \( u \) is such that this inverse problem (IP) has a solution. More precisely, in view of the Definition 1.1 and Theorem 1.3 we make the following assumption.

**Assumption 1.4.** There exists a \( d \times d \) matrix \( A \) with entries from \( H^1(\Omega) \) such that the parabolic problem (1.1)-(1.3) has a unique weak solution \( u \in L^2(0, \tau; H^1(\Omega)) \) with \( u_t \in L^2(0, \tau; H^{-1}(\Omega)) \) and \( 0 \neq \nabla u \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d)) \).

**Remark 1.5.** It is to be observed that the additional condition assumed for \( u \) in Assumption 1.4, namely, \( \nabla u \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d)) \) is satisfied if \( m > \frac{d}{2} \), and if \( f \) and \( h \) satisfy the hypothesis of Theorem 1.3 for this case, \( \nabla u(\cdot, t) \in H^m(\Omega) \) and \( H^m(\Omega) \) is continuously embedded in \( L^\infty(\Omega) \) (cf. [11, Theorem 2.5.4]).

Parameter identification problem has a vast literature and every problem has its own importance in practical application. One can refer to [8, 4, 10, 18, 19] for more literature on parameter identification problem in parabolic setting. In [4], the authors have considered a method called natural linearisation for identifying a scalar valued diffusion coefficient from final time observation. There, they have suitably reduced the problem involved with parabolic PDE into an elliptic PDE, with some assumptions, and then applied the natural linearisation technique for identifying the parameter.

In this paper, we use similar idea for converting the (IP) into a problem of solving an ill-posed operator equation, where the operator involved is linear, but without converting into an elliptic PDE. We will do the analysis with parabolic PDE setting. Also, we would like to mention that in [4], the authors have worked with Neuman boundary condition, but in this work, we are considering homogeneous Dirichlet boundary condition. Further, we are considering the problem in a more general setting, namely, the identification is that of a \( d \times d \) matrix coefficient with functions from \( H^1(\Omega) \) as entries, instead of scalar valued function. We use the Tikhonov regularization method to obtain stable approximations from the space of all \( d \times d \) matrix with entries from \( L^2(\Omega) \). As a result, while doing analysis in the finite dimensional setting, we found it necessary to define an orthogonal projection on the space of all \( d \times d \) matricies with entries from \( L^2(\Omega) \). Also, we obtain an explicit representation of the adjoint of the linear operator involved (see Theorem 4.4).

For obtaining error estimates for the regularized approximations, first we assume certain regularity conditions on the noisy data. In order to accommodate the case with relaxed noisy data, we considered a smoothing procedure using the Clement operator, which resulted in a reduced accuracy in the error.

We would like to point out that most of the parameter identification problems for parabolic PDE’s are on identifying scalar valued coefficient functions, without paying much attention to the problem of identifying matrix valued coefficient functions. To the best of our knowledge, [19] is the first paper that deals with uniqueness and stability for the inverse problem of identifying a matrix diffusion coefficient in a parabolic PDE from some boundary observations and an intermediate time observation, and not much has been available on identification of matrix coefficients. Thus, this paper intents to make some contribution in that direction. Using some idea employed in [19], we could also establish a uniqueness result for our inverse problem under some additional condition on \( u \) (see Theorem 2.3).

The paper is organized as follows. In Section 2, we introduce the notations that will be used throughout the paper and present some preliminary results that will be required for further analysis, and also establish the uniqueness result for our inverse problem. In Section 3, we reformulate the inverse problem as an operator equation involving a linear operator and prove some results related to boundedness, compactness and rank of the linear operator of our interest. Section 4 deals with regularization and error estimates. We use the standard theory of Tikhonov regularization for regularization purpose. In Section 5 we considered the finite dimensional realizations of the regularized solutions and derived corresponding error estimates. Section 6 deals with the strategy of choosing...
regularizing parameter, based on the adaptive technique. Section 7 is devoted to the procedure of obtaining smoothed version of the observed data, and subsequently, in Section 8 we have given the error estimates for the smoothed version of the noisy data.

2. Preliminaries and a uniqueness result

We shall consider the real vector space
\[ \mathcal{X} = \{ v \in L^2(0, \tau; H_0^1(\Omega)) : v_t \in L^2(0, \tau; H^{-1}(\Omega)) \}. \]
It can be seen that
\[ \|v\|_{\mathcal{X}} = \|v\|_{L^2(0,\tau;H_0^1(\Omega))} + \|v_t\|_{L^2(0,\tau;H^{-1}(\Omega))}, \quad v \in \mathcal{X}, \]
defines a norm on \( \mathcal{X} \) which makes it a Banach space.

Throughout the paper, we shall use the notation \( \mathcal{H}^{d \times d} \) to denote the space of all \( d \times d \) matrices with entries from a Hilbert space \( \mathcal{H} \). Also, we consider the inner product
\[ \langle \mathbf{B}, \mathbf{B} \rangle = \sum_{i,j=1}^{d} \langle b_{ij}, \tilde{b}_{ij} \rangle_{L^2(\Omega)}, \]
and the corresponding norm
\[ \|\mathbf{B}\| = \left( \sum_{i,j=1}^{d} \|b_{ij}\|^2_{L^2(\Omega)} \right)^{1/2}, \]
for \( \mathbf{B} = (b_{ij}) \) and \( \tilde{\mathbf{B}} = (\tilde{b}_{ij}) \) in \( (L^2(\Omega))^{d \times d} \).

Notations: For simplicity of the presentation we shall use the notations \( H^1, W^{1,\infty}, L^2, L^\infty \) for denoting the spaces \( H^1(\Omega), W^{1,\infty}(\Omega), L^2(\Omega) \) and \( L^\infty(\Omega) \), respectively.

Let \( 1 \leq p \leq \infty \). For \( \vec{v} : \Omega \to \mathbb{R}^d \), if \( \vec{v}(x) = (v_1(x), \ldots, v_d(x)) \), \( x \in \Omega \), then we see that
\[ \vec{v} \in L^p(\Omega, \mathbb{R}^d) \iff v_i \in L^p(\Omega) \quad \forall i \in \{1, \ldots, d\} \]
and
\[ \|\vec{v}\|_{L^p} = \begin{cases} \sum_{i=1}^{d} \|v_i\|_{L^p} & \text{for } 1 \leq p < \infty, \\ \max \{\|v_i\|_{L^\infty} : 1 \leq i \leq d\} & \text{for } p = \infty. \end{cases} \]

Let \( \mathbf{A}_0 \in (L^2(\Omega))^{d \times d} \) be symmetric and satisfies the uniform ellipticity condition (see Definition \[1.2\]). For \( f \in L^2(0, \tau; L^2(\Omega)) \) and \( h \in L^2(\Omega) \), consider the PDE
\[ v_t - \nabla \cdot (\mathbf{A}_0(x) \nabla v) = f(x, t) \quad \text{in } Q \]
along with the boundary and initial conditions
\[ v(x, t) = 0 \quad \text{on } \partial Q, \]
\[ v(x, 0) = h(x) \quad \text{in } \Omega. \]
The equation \( [2.2] \) is to be understood in the sense of \textit{trace}. Then we have the following result on existence and uniqueness.

Theorem 2.1. [3] Theorem 2.4, Ch. 7] Let \( \mathbf{A}_0, f \) and \( h \) be as in \( [2.2] \)--\( [2.3] \). Then \( [2.2] \)--\( [2.3] \) has a unique \( v \in L^2(0, \tau; H_0^1(\Omega)) \) with \( v_t \in L^2(0, \tau; H^{-1}(\Omega)) \) satisfying
\[ \langle v_t(\cdot, t), \varphi \rangle + \int_{\Omega} \mathbf{A}_0(x) \nabla v(\cdot, t) \cdot \nabla \varphi = \int_{\Omega} f(\cdot, t) \varphi \]
and \( v(x, 0) = h(x) \) for \( x \in \Omega \), for all \( \varphi \in H_0^1(\Omega) \) and for almost all \( t \in [0, \tau] \). Further, there exist a constant \( C_0 > 0 \) depending on \( \Omega, \tau \) and \( \mathbf{A}_0 \), such that
\[ \|v\|_{L^2(0, \tau; H_0^1(\Omega))} + \|v_t\|_{L^2(0, \tau; H^{-1}(\Omega))} \leq C_0 \left( \|f\|_{L^2(0, \tau; H^{-1}(\Omega))} + \|h\|_{L^2(\Omega)} \right). \]
In this paper we are dealing with spaces of Hilbert space valued functions, for instance $L^2(0, \tau; H^1_0(\Omega))$, $L^2(0, \tau; L^2(\Omega))$. So we would like to have some compact embeddings for these type of function spaces, which will be useful in proving some convergence results. With regard to this, we have the following result, known as Aubin-Lions lemma which gives a compact embedding between certain Banach space valued function spaces. For its proof and more details about Aubin-Lions lemma one may refer to any of [2] [10] [17].

Let $X_0$ and $X_1$ be Banach spaces with $X_0 \subset X_1$ and
\[
W = \{ u \in L^2(0, \tau; X_0) \cap H^1(0, \tau; X_1) : u_t \in L^2(0, \tau; X_1) \}.
\]
Then it can be seen that $W$ is a Banach space with respect to the norm
\[
\| u \|_W = \| u \|_{L^2(0, \tau; X_0)} + \| u_t \|_{L^2(0, \tau; X_1)}, \quad u \in W.
\]

**Lemma 2.2. (Aubin-Lions)** [16, Theorem 1.3] Let $X_0$, $X_1$, $X$ be Banach spaces with $X_0 \subset X \subset X_1$. Then $W$ is compactly embedded in $X$. With regard to this, we have the following lemma which gives a compact embedding between certain Banach spaces with
\[
\| \psi \|_{W} \leq \| \psi \|_{L^2} \quad \text{for each } 1 \leq i \leq d.
\]

**Proof.** Let $\varphi \in C^\infty_c(\Omega)$, the set of all real valued infinitely differentiable functions on $\Omega$ with compact support. Then, for $1 \leq i \leq d$, we have
\[
\left( \frac{\partial \psi}{\partial x_i} \right)(\varphi) = - \int_\Omega \psi \frac{\partial \varphi}{\partial x_i} \quad \text{for all } \varphi \in C^\infty_c(\Omega).
\]
Thus, for all $\varphi \in C^\infty_c(\Omega)$
\[
\left| \frac{\partial \psi}{\partial x_i} \right| \leq \int_\Omega \left| \frac{\partial \varphi}{\partial x_i} \right| \leq \| \psi \|_{L^2} \left| \frac{\partial \varphi}{\partial x_i} \right| \leq \| \psi \|_{L^2} \| \varphi \|_{H^1(\Omega)}.
\]
Since $C^\infty_c(\Omega)$ is dense in $H^1_0(\Omega)$, we have $\| \frac{\partial \psi}{\partial x_i} \|_{H^{-1}} \leq \| \psi \|_{L^2}$ for $1 \leq i \leq d$. \Box

**Remark 2.5.** Let $\psi \in L^2(\Omega)$. From Lemma 2.4 we also have
\[
\| \nabla \psi \|_{H^{-1}}^2 = \sum_{i=1}^d \left| \frac{\partial \psi}{\partial x_i} \right|_{H^{-1}}^2 \leq d \| \psi \|_{L^2}^2.
\]
That is, $\| \nabla \psi \|_{H^{-1}} \leq \sqrt{d} \| \psi \|_{L^2}$.

### 2.1. Uniqueness of the Inverse Problem

We establish the uniqueness result for our inverse problem (IP) under appropriate assumptions. The arguments used here are motivated by those in [19].

Let $A \in (H^1(\Omega))^{d \times d}$ be a solution of the inverse problem (IP) corresponding to the data $u$. Then, we have
\[
u_t + \nabla \cdot A \nabla u = f \quad \text{in } \Omega \times (0, \tau).
\]
Suppose $B \in (H^1(\Omega))^{d \times d}$ is another solution to (IP). Then, from (2.8), we have
\[
- \nabla \cdot (A - B) \nabla u = 0 \quad \text{in } \Omega \times (0, \tau).
\]
Let \( t_i \in (0, \tau) \) for \( i = 1, \ldots, d^2(d + 1) \). We denote \( u(\cdot, t_i) \) and \( \frac{\partial u(t_i)}{\partial x_i} \) by \( u(t_i) \) and \( \partial_i u(t_i) \), respectively, and for \( 1 \leq k \leq d^2 \) and \( 1 \leq i, j \leq d \), we let

\[
D^k_{ij} = \det \begin{pmatrix}
\partial_1 u(t(k-1)(d+1)+1) & \cdots & \partial_d u(t(k-1)(d+1)+1) & \partial_1 \partial_j u(t(k-1)(d+1)+1) \\
\vdots & \ddots & \vdots & \vdots \\
\partial_1 u(t(k-1)(d+1)+d+1) & \cdots & \partial_d u(t(k-1)(d+1)+d+1) & \partial_1 \partial_j u(t(k-1)(d+1)+d+1)
\end{pmatrix}
\]

and

\[
D = \det \begin{pmatrix}
D_{11} & \cdots & D_{1d} & D_{21} & \cdots & D_{2d} & D_{d1} & \cdots & D_{dd} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
D_{d1} & \cdots & D_{dd} & D_{2d} & \cdots & D_{d2} & D_{d2} & \cdots & D_{dd}
\end{pmatrix}
\]

(2.10)

**Theorem 2.6.** Let \( \{t_i : 1 \leq i \leq d^2(d + 1)\} \subset (0, \tau) \) and \( A, B \in (H^1(\Omega))^{d \times d} \) be solutions of the inverse problem (IP) corresponding to the data \( u \in A \). Let \( D \) be as defined in (2.10). If \( D \neq 0 \) a.e. in \( \Omega \), then \( A = B \) a.e. in \( \Omega \).

**Proof.** From the assumptions on \( A \) and \( B \), it follows that they satisfy (2.10). Let \( f_{ij} \) be the \( ij \)-th entry of the matrix \( (A - B) \). Thus, for \( 1 \leq l \leq d^2(d + 1) \), from (2.10), we have

\[
(\sum_{j=1}^{d} \partial_1 f_{ij} \partial_j u(t_l) + \cdots + \sum_{j=1}^{d} \partial_d f_{ij} \partial_j u(t_l)) = - (\sum_{i,j=1}^{d} f_{ij} \partial_i \partial_j u(t_l)).
\]

Therefore, for \( 1 \leq k \leq d^2 \), from (2.11), we have

\[
\begin{pmatrix}
\partial_1 u(t(k-1)(d+1)+1) & \cdots & \partial_d u(t(k-1)(d+1)+1) \\
\vdots & \ddots & \vdots \\
\partial_1 u(t(k-1)(d+1)+d+1) & \cdots & \partial_d u(t(k-1)(d+1)+d+1)
\end{pmatrix} = \begin{pmatrix}
\sum_{j=1}^{d} \partial_j f_{ij} \\
\vdots \\
\sum_{j=1}^{d} \partial_j f_{ij}
\end{pmatrix}
\]

For each \( 1 \leq k \leq d^2 \), the above system of \( d + 1 \) equations in \( d \) variables has a solution, namely

\[
\left( \sum_{i,j=1}^{d} \partial_j f_{ij}, \ldots, \sum_{i,j=1}^{d} \partial_j f_{jd} \right)^T,
\]

therefore, we must have

\[
\det \begin{pmatrix}
\partial_1 u(t(k-1)(d+1)+1) & \cdots & \partial_d u(t(k-1)(d+1)+1) \\
\vdots & \ddots & \vdots \\
\partial_1 u(t(k-1)(d+1)+d+1) & \cdots & \partial_d u(t(k-1)(d+1)+d+1)
\end{pmatrix} = 0.
\]

Now, using the properties of determinant, we obtain the system of \( d^2 \) homogeneous equations with \( d^2 \) variables, namely,

\[
\sum_{i,j=1}^{d} D^k_{ij} f_{ij} = 0, \quad \text{for all } 1 \leq k \leq d^2.
\]

(2.12)

By the hypothesis \( D \neq 0 \) a.e. in \( \Omega \), that is, determinant of the coefficient matrix of the system (2.12) is non-zero. Thus, \( f_{ij} = 0 \) a.e. in \( \Omega \) and hence the proof is complete.

In view of Theorem 2.6, to guarantee the unique solvability of our proposed inverse problem, we also make the one more assumption:
Assumption 2.7. With $u$ as in Assumption 1.4, there exists $t_i \in (0, \tau)$ with $1 \leq i \leq d^2(d + 1)$ such that $D \neq 0$ a.e. in $\Omega$, $D$ is as in (2.10).

3. Operator theoretic formulation

In [4], the authors have used the technique of natural linearisation for identification of a scalar valued diffusion coefficient from final time observation. In this section we show that similar idea, with some modifications in the arguments, can be used for representing our non linear ill-posed problem in an equivalent form that would facilitate us to make use of the theory of regularization for linear operator, as shown in the next section. Essentially, the method of natural linearisation, allows us to obtain solution of the non linear inverse problem in terms of solution of an operator equation, where the operator involved is a linear operator. This fact can be observed as we reach towards the end of this section.

According to our inverse problem, $u$ satisfies (1.1)-(1.3) in the weak sense. Thus, we have

$$
\langle u_t(\cdot, t), \varphi \rangle + \int_{\Omega} A(x) \nabla u(\cdot, t) \cdot \nabla \varphi = \int_{\Omega} f(\cdot, t) \varphi
$$

for all $\varphi \in H^1_0(\Omega)$ and a.a. $t \in [0, \tau]$ along with $u(x, 0) = h(x)$ in $\Omega$. We shall consider an equivalent form of the above equation by making use of another matrix $A_0$, considered as in Theorem 2.1.

So, let $A_0$ be a symmetric $d \times d$-matrix with entries from $L^2(\Omega)$ and satisfies the uniform ellipticity condition. Let $f$, $h$ be as considered in the inverse problem (IP). By Theorem 2.1 there exists a unique weak solution $v_0 \in X$ for (2.2)-(2.4). That is, we have

$$
\langle (v_0)_t(\cdot, t), \varphi \rangle + \int_{\Omega} A_0 \nabla v_0(\cdot, t) \cdot \nabla \varphi = \int_{\Omega} f(\cdot, t) \varphi
$$

for all $\varphi \in H^1_0(\Omega)$ and a.a. $t \in [0, \tau]$ along with $v_0(x, 0) = h(x)$ in $\Omega$. Thus, from (3.1) and (3.2), we have

$$
\langle (v_0 - u)_t(\cdot, t), \varphi \rangle + \int_{\Omega} (A - A_0) \nabla (v_0 - u)(\cdot, t) \cdot \nabla \varphi = \int_{\Omega} (A - A_0) \nabla u(\cdot, t) \cdot \nabla \varphi
$$

for all $\varphi \in H^1_0(\Omega)$ along with $(v_0 - u)(\cdot, 0) = 0$ in $\Omega$ and $(v_0 - u)(\cdot, t) = 0$ on $\partial \Omega$ for a.a. $t \in [0, \tau]$.

The following lemma will be used to show the existence of weak solution of a PDE that arises out in the process of natural linearisation.

**Lemma 3.1.** Let $\Phi \in L^2(0, \tau; L^2(\Omega))$ be such that $\nabla \Phi \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$ and let $C \in (L^2(\Omega))^{d \times d}$. Then $\nabla \cdot C \nabla \Phi \in L^2(0, \tau; H^{-1}(\Omega))$.

**Proof.** Let $C = (c_{ij})_{1 \leq i, j \leq d}$. First we show that $C \nabla \Phi(\cdot, t) \in L^2(\Omega, \mathbb{R}^d)$ for $t \in [0, \tau]$.

Since $\nabla \Phi(\cdot, t) \in L^\infty(\Omega, \mathbb{R}^d)$, for a.a $t \in [0, \tau]$, we have

$$
\left\| c_{ij} \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^2} \leq \left\| c_{ij} \right\|_{L^2} \left\| \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^\infty}, \quad \text{for all } 1 \leq i, j, k \leq d \text{ and a.a } t \in [0, \tau].
$$

Using Cauchy Schwarz inequality we have,

$$
\left\| \sum_{j=1}^{d} c_{ij} \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^2}^2 \leq \left\| \sum_{j=1}^{d} c_{ij} \right\|_{L^2}^2 \left\| \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^\infty}^2 \leq \sum_{j=1}^{d} \left\| c_{ij} \right\|_{L^2}^2 \sum_{j=1}^{d} \left\| \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^\infty}^2
$$

for a.a $t \in [0, \tau]$. Thus, we have

$$
\left\| \nabla C \Phi(\cdot, t) \right\|^2_{L^2(\Omega, \mathbb{R}^d)} \leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| c_{ij} \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^2} \leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| c_{ij} \right\|_{L^2} \left\| \frac{\partial \Phi(\cdot, t)}{\partial x_j} \right\|_{L^\infty},
$$

for a.a $t \in [0, \tau]$, and hence

$$
\left\| \nabla C \Phi(\cdot, t) \right\|_{L^2(\Omega, \mathbb{R}^d)} \leq \left\| C \right\|_{F} \left\| \nabla \Phi(\cdot, t) \right\|_{L^\infty(\Omega, \mathbb{R}^d)}.
$$
Hence, $\nabla \cdot \nabla \Phi (\cdot, t) \in L^2(\Omega, \mathbb{R}^d)$ for a.a $t \in [0, \tau]$. Therefore, $\nabla \cdot \nabla \Phi (\cdot, t) \in H^{-1}(\Omega)$ for a.a $t \in [0, \tau]$.

Now, using Lemma 2.3, for a.a $t \in [0, \tau]$, we have

$$
\left\| \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left( c_{ij} \frac{\partial \Phi (\cdot, t)}{\partial x_j} \right) \right\|_{H^{-1}} \leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| \frac{\partial}{\partial x_i} \left( c_{ij} \frac{\partial \Phi (\cdot, t)}{\partial x_j} \right) \right\|_{H^{-1}}
$$

$$
\leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| c_{ij} \frac{\partial \Phi (\cdot, t)}{\partial x_j} \right\|_{L^2}
\leq \sum_{i=1}^{d} \sum_{j=1}^{d} \left\| c_{ij} \right\|_{L^2} \left\| \frac{\partial \Phi (\cdot, t)}{\partial x_j} \right\|_{L^\infty}
\leq d\|C\|_F \left\| \nabla \Phi (\cdot, t) \right\|_{L^\infty(\Omega; \mathbb{R}^d)}.
$$

Thus,

$$
\|\nabla \cdot \nabla \Phi (\cdot, t)\|_{H^{-1}(\Omega)} \leq d\|C\|_F \|\nabla \Phi (\cdot, t)\|_{L^\infty(\Omega; \mathbb{R}^d)} \quad \text{for a.a } t \in [0, \tau].
$$

Since $\nabla \Phi \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$, we conclude that $\nabla \cdot \nabla \Phi \in L^2(0, \tau; H^{-1}(\Omega))$. \qed

Let $A_0$ be as in 3.2. For $C \in (L^2(\Omega))^{d \times d}$ and $w \in L^2(0, \tau; L^2(\Omega))$, we consider the PDE

$$
v_t - \nabla \cdot A_0 \nabla v = \nabla \cdot C \nabla w \quad \text{in } Q,
$$

along with the conditions

$$
v = 0 \quad \text{on } \partial Q,
$$

$$
v(x, 0) = 0 \quad \text{in } \Omega.
$$

Note that to talk about the existence of a weak solution of (3.5) - (3.7), it is enough to make sure that the R.H.S of (3.5), that is, $\nabla \cdot C \nabla w$ belongs to $L^2(0, \tau; H^{-1}(\Omega))$. In order to satisfy this condition, Lemma 3.1 suggests that, it is enough to assume that $\nabla w \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$. Thus, under this assumption, by Theorem 2.1 for each $C \in (L^2(\Omega))^{d \times d}$, there exist a unique weak solution $v \in X$, of (3.5) - (3.7), where $X$ is defined as in (2.1).

In view of the discussion in the above paragraph, for each $w \in L^2(0, \tau; L^2(\Omega))$ with $\nabla w \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$, we consider the map $T_w : (L^2(\Omega))^{d \times d} \rightarrow X$ defined by

$$
T_w C = v, \quad C \in (L^2(\Omega))^{d \times d},
$$

where $v \in X$ is the unique weak solution of (3.5) - (3.7). Clearly, the map $T_w$ is a linear operator. We now have the following result.

**Theorem 3.2.** Let $w \in L^2(0, \tau; L^2(\Omega))$ be such that $0 \neq \nabla w \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$. Then the linear operator $T_w : (L^2(\Omega))^{d \times d} \rightarrow X$, defined in (3.8), is a bounded linear operator of infinite rank with

$$
\|T_w\| \leq d \sqrt{C_0 \left( \int_0^\tau \|\nabla w(\cdot, t)\|_{L^\infty}^2 dt \right)^{1/2}},
$$

where $C_0$ is the constant as in Theorem 2.1.

**Proof.** We first show that $T_w$ is a bounded operator. Let $C \in (L^2(\Omega))^{d \times d}$ and $T_w C = v$. Then, $v$ is the unique weak solution of (3.5) - (3.7). Hence using the estimate (2.6) of Theorem 2.1 we have

$$
\|T_w C\| = \|v\|_X \leq C_0 \|\nabla \cdot C \nabla u\|_{L^2(0, \tau; H^{-1}(\Omega))}.
$$

Now, using the estimate given in (3.4), we have

$$
\|\nabla \cdot C \nabla w\|_{L^2(0, \tau; H^{-1}(\Omega))} = \int_0^\tau \|\nabla \cdot C \nabla w(\cdot, t)\|_{H^{-1}(\Omega)}^2 dt
\leq d^2 \|C\|_F^2 \int_0^\tau \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}^2 dt.
$$
Thus,
\[ \|T_w C\| \leq d \sqrt{C_0} \left( \int_0^\tau \|\nabla w(.; t)\|^2_{L^\infty} dt \right)^{1/2} \|C\|_F. \]
This shows that \( T_w \) is a bounded linear operator with \( \|T_w\| \leq d \sqrt{C_0} \left( \int_0^\tau \|\nabla w(.; t)\|^2_{L^\infty} dt \right)^{1/2}. \)

We now show that \( T_w \) is of infinite rank. Since \( \nabla w(.; t) \neq 0 \) for almost all \( t \in [0, \tau] \), without loss of generality we assume that \( \frac{\partial w}{\partial x_1} \neq 0 \) for almost all \( t \in [0, \tau] \). Let \( \{\varphi_n\} \) be a sequence of elements in \( C^\infty_0(\Omega) \) such that \( \text{supp } \varphi_m \cap \text{supp } \varphi_n = \emptyset \) for \( m \neq n \). Let \( C_n \) be the \( d \times d \) matrix given by,
\[ C_n = \begin{pmatrix} \varphi_n \\ O_{1 \times (d-1)} \end{pmatrix} \begin{pmatrix} O_{(d-1) \times (d-1)} \\ 0 \end{pmatrix}, \]
where \( O_{k \times l} \) denotes the zero matrix of respective order. Then, clearly \( \{C_n\} \) is a sequence of linearly independent elements in \( (L^2(\Omega))^{d \times d} \). Let \( T_w C_n = v_n \). We claim that \( \{v_n : n \in \mathbb{N}\} \) is linearly independent. Assume, for a moment, that \( \{v_n : n \in \mathbb{N}\} \) is linearly dependent. Then without loss of generality we assume that, for \( m \in \mathbb{N} \), let \( v_1 = \sum_{i=2}^m \beta_i v_i \), where \( \beta_i \)'s are constants not all zero. Using the definition of \( T_w \), \( \{C_n\} \) gives
\[ \nabla \cdot C_1 \nabla w = (v_1)_t - \nabla \cdot A_0 \nabla v_1 = \sum_{i=2}^m \beta_i (v_i)_t - \sum_{i=2}^m \beta_i \nabla \cdot A_0 \nabla v_i = \sum_{i=2}^m \beta_i \nabla \cdot C_i \nabla w. \]
Thus, \( \nabla \cdot (C_1 - \sum_{i=2}^m \beta_i C_i) \nabla w = 0 \). Hence, we have
\[ \int_\Omega (C_1 - \sum_{i=2}^m \beta_i C_i) \nabla w \cdot \nabla \varphi dx = 0 \quad \forall \ \varphi \in H^1_0(\Omega) \quad \text{and for a.a. } t \in (0, \tau). \]

Let \( \varphi \in H^1_0(\Omega) \) be such that \( \frac{\partial \varphi}{\partial x_1} = \varphi_1 = \varphi_1 - \sum_{i=2}^m \beta_i \varphi_i \frac{\partial w}{\partial x_1} \). Then from (3.9), we obtain
\[ \int_\Omega (\varphi_1 - \sum_{i=2}^m \beta_i \varphi_i)^2 \left( \frac{\partial w}{\partial x_1} \right)^2 dx = 0 \quad \text{for a.a. } t \in (0, \tau). \]
Since \( \frac{\partial w}{\partial x_1} \neq 0 \) a.e. in \( (0, \tau) \), we have \( \varphi_1 = \sum_{i=2}^m \beta_i \varphi_i \) a.e. in \( \Omega \). This leads to a contradiction, since \( \varphi_n \)'s are linearly independent. Therefore, \( \{v_n : n \in \mathbb{N}\} \) is a linearly independent set. This shows that \( T_w \) is of infinite rank. \( \square \)

Our next theorem demonstrates one more property of the operator \( T_w \).

**Theorem 3.3.** Let \( w \) be as assumed in Theorem 3.2 and \( T_w : (L^2(\Omega))^{d \times d} \rightarrow X \) be as defined in (3.8). Then \( T_w : (L^2(\Omega))^{d \times d} \rightarrow L^2(\Omega) \) is a compact linear operator.

**Proof.** By Theorem 3.2 \( T_w : (L^2(\Omega))^{d \times d} \rightarrow X \) is a bounded linear operator. Also, by Remark 2.3 \( X \) is compactly embedded in \( L^2(0, \tau; L^2(\Omega)) \). Therefore \( T_w : (L^2(\Omega))^{d \times d} \rightarrow L^2(0, \tau; L^2(\Omega)) \) is a composition of a bounded linear operator and a compact linear operator. Hence \( T_w : (L^2(\Omega))^{d \times d} \rightarrow L^2(0, \tau; L^2(\Omega)) \) is a compact linear operator. \( \square \)

We are now in a position to represent our non linear inverse problem in an alternative form as
\[ T_w B = v_0 - u, \]
where \( u \) is as in the Assumption 1.4, the linear operator \( T_u : (L^2(\Omega))^{d \times d} \rightarrow L^2(0, \tau; L^2(\Omega)) \) is as defined in (3.8), \( v_0 \) is as in (3.2), and \( B = A - A_0 \) is a solution of (3.10), where \( A_0 \) is as considered in Theorem 2.1.

Since the operator \( T_u \) depends on the data \( u \) and it is a compact linear operator of infinite rank (cf. Theorem 3.2 and Theorem 3.3), the operator equation (3.10) is ill-posed, that is, small perturbations in the data \( u \) may lead to a large deviation in the solution. But, in practical application, noise in
the data is inevitable. So, a regularization method is necessary for obtaining stable approximate solution, and we employ the method of Tikhonov regularization for the same. For more details about regularization theory, one may look into the books \[7,13\].

4. Regularization and error analysis

As mentioned in the previous section, \(B := A - A_0\) satisfies the equation (\(4.10\)). Thus, we have

\[
\mathcal{T}_u B = v_0 - u.
\]

Now, suppose that \(u\) is available with some noise, say we have \(\tilde{u}\) in place of \(u\).

**Assumption 4.1.** For \(\delta > 0\), let \(\tilde{u} \in L^2(0,\tau; L^2(\Omega))\) be the observed data corresponding to \(u\), satisfying

\[
\nabla \tilde{u} \in L^2(0,\tau; L^\infty(\Omega, \mathbb{R}^d))
\]

along with

\[
\|\tilde{u} - u\|_{L^2(0,\tau; L^2(\Omega))} + \left( \int_0^\tau \|\nabla u(\cdot,t) - \nabla \tilde{u}(\cdot,t)\|_{L^\infty}^2 \, dt \right)^{1/2} \leq \delta.
\]

**Remark 4.2.** Note that in the above assumption we have imposed a regularity condition on the observed data \(\tilde{u}\), namely \(\nabla \tilde{u} \in L^2(0,\tau; L^\infty(\Omega, \mathbb{R}^d))\). But, \(\tilde{u}\) being an observed data, may not satisfy that regularity condition. In order to accommodate the noisy data \(\tilde{u}\) only with the condition that it belongs to \(L^2(0,\tau; L^2(\Omega))\) and

\[
\|\tilde{u} - u\|_{L^2(0,\tau; L^2(\Omega))} \leq \delta
\]

we consider a smoothed version of \(\tilde{u}\), say \(\tilde{\tilde{u}}\), which will satisfy all the conditions in the above assumption and then carry out the analysis with \(\tilde{\tilde{u}}\) in place of \(\tilde{u}\). In Section [7] we have given the procedure for obtaining the smoothed version of noisy data.

**Notation:** In rest of the paper whenever needed, we shall use the notation \(\|u - \tilde{u}\|\) to denote \(\|u - \tilde{u}\|_{L^2(0,\tau; L^2(\Omega))}\).

Corresponding to the noisy data \(\tilde{u}\) satisfying the Assumption 4.1 we would like to obtain a regularized approximation of \(B\). For this purpose we shall consider the Tikhonov regularization (cf. \[7,13\]). That is, for each \(\alpha > 0\), the candidate for the regularized approximation of \(B := A - A_0\) is the unique solution of the equation

\[
(T_{\tilde{u}}^* T_{\tilde{u}} + \alpha I) \tilde{B}_\alpha = T_{\tilde{u}}^* (v_0 - \tilde{u}).
\]

Then we take the regularized approximation of \(A\) as

\[
\hat{A}_\alpha = A_0 + \tilde{B}_\alpha, \quad \alpha > 0.
\]

Let \(B_\alpha\) be the unique solution of

\[
(T_{u}^* T_{u} + \alpha I) B_\alpha = T_{u}^* (v_0 - u).
\]

Also, using (4.1), we have

\[
(T_{u}^* T_{u} + \alpha I) B = T_{u}^* (v_0 - u) + \alpha B.
\]

Then by Theorem 3.2 and using the estimate given in (4.2), we have

\[
\|T_u B - T_u B\| \leq d\sqrt{C_0} \left( \int_0^\tau \|\nabla u(\cdot,t) - \nabla \tilde{u}(\cdot,t)\|_{L^\infty}^2 \, dt \right)^{1/2} \|B\|_F
\]

which implies that

\[
\|T_u - T_u\| \leq d\sqrt{C_0} \left( \int_0^\tau \|\nabla u(\cdot,t) - \nabla \tilde{u}(\cdot,t)\|_{L^\infty}^2 \, dt \right)^{1/2} \leq d\sqrt{C_0}\delta,
\]
where \( C_0 \) be as in (2.6). The equations (4.3) and (4.5) leads to
\[
\begin{align*}
\tilde{B}_\alpha - B_\alpha &= (T_u^* T_u + \alpha I)^{-1} T_u^* (v_0 - \tilde{u}) - (T_u^* T_u + \alpha I)^{-1} T_u^* (v_0 - u) \\
&(= [(T_u^* T_u + \alpha I)^{-1} T_u^* - (T_u^* T_u + \alpha I)^{-1} T_u^*] (v_0 - u) \\
&+ (T_u^* T_u + \alpha I)^{-1} T_u^* (u - \tilde{u}).
\end{align*}
\]
(4.7)

Now, using (4.1), we have
\[
\begin{align*}
(T_u^* T_u + \alpha I)^{-1} T_u^* (v_0 - u) &= (T_u^* T_u + \alpha I)^{-1} T_u^* T_u B \\
(T_u^* T_u + \alpha I)^{-1} T_u^* (v_0 - u) &= (T_u^* T_u + \alpha I)^{-1} T_u^* T_u B \\
\end{align*}
\]

Thus, from (4.7), we have
\[
\tilde{B}_\alpha - B_\alpha = (T_u^* T_u + \alpha I)^{-1} T_u^* (T_u - T_u) T_u^* T_u (T_u^* T_u + \alpha I)^{-1} B \\
+ \alpha (T_u^* T_u + \alpha I)^{-1} (T_u^* T_u + \alpha I)^{-1} (T_u^* - T_u^*) (T_u^* T_u + \alpha I)^{-1} T_u B \\
+ (T_u^* T_u + \alpha I)^{-1} T_u^* (u - \tilde{u}).
\]

Hence, (4.7) takes the form
\[
\begin{align*}
\| (T_u^* T_u + \alpha I)^{-1} T_u^* && \leq \frac{1}{2\sqrt{\alpha}} & \| (T_u^* T_u + \alpha I)^{-1} \| \leq \frac{1}{\alpha} \\
\| T_u^* T_u (T_u^* T_u + \alpha I)^{-1} \| \leq 1 & \| (T_u^* T_u + \alpha I)^{-1} T_u \| \leq \frac{1}{2\sqrt{\alpha}}.
\end{align*}
\]

Using these estimates, we obtain
\[
\| (T_u^* T_u + \alpha I)^{-1} T_u^* - (T_u^* T_u + \alpha I)^{-1} T_u^* \| (v_0 - u) \| \leq \frac{\| T_u - T_u^* \| }{\sqrt{\alpha}} \| B \|
\]

and
\[
\| (T_u^* T_u + \alpha I)^{-1} T_u^* \| (u - \tilde{u}) \| \leq \frac{\| u - \tilde{u} \| }{2\sqrt{\alpha}}.
\]

Thus, from (4.7), we have
\[
\| B_\alpha - \tilde{B}_\alpha \| \leq \frac{\| T_u - T_u^* \| }{\sqrt{\alpha}} \| B \| + \frac{\| u - \tilde{u} \| }{2\sqrt{\alpha}}.
\]

Therefore, by (4.2), we have
\[
\| B_\alpha - \tilde{B}_\alpha \| \leq \frac{d \sqrt{C_0 \delta}}{\sqrt{\alpha}} \| B \| + \frac{\delta}{2 \sqrt{\alpha}} = \left( d \sqrt{C_0} \| B \| + 1/2 \right) \frac{\delta}{\sqrt{\alpha}}.
\]

It is well known (cf. [13]) from the theory of Tikhonov regularization that
\[
\| B - B_\alpha \| \to 0 \quad \text{as} \quad \alpha \to 0.
\]

Thus, since \( A - \tilde{A}_\alpha = B - \tilde{B}_\alpha \) for \( \alpha > 0 \), we have proved the following result.

**Theorem 4.3.** Let \( A_0 \) be as in Theorem 2.4 and \( B \) be as given in (4.1). Let \( B_\alpha \) be the unique solution of (4.3) and \( B_\alpha \) be as given in (4.3). Let \( \delta \) and \( \tilde{u} \) be as given in (4.2). Then we have,
\[
\| A - \tilde{A}_\alpha \| \leq \| B - B_\alpha \| + (d \sqrt{C_0} \| B \| + 1/2) \frac{\delta}{\sqrt{\alpha}},
\]

where \( \| B - B_\alpha \| \to 0 \) as \( \alpha \to 0 \). Further, choose the parameter \( \alpha := \alpha_\delta \) depending on \( \delta \) in such a way that \( \alpha_\delta \to 0 \) and
\[
\frac{d \sqrt{C_0} \delta}{\sqrt{\alpha_\delta}} \| B \| + \frac{\delta}{2 \sqrt{\alpha_\delta}} \to 0 \quad \text{as} \quad \delta \to 0,
\]

then we have \( \| A - \tilde{A}_{\alpha_\delta} \| \to 0 \) as \( \delta \to 0 \).

In the next subsection, we determine an explicit representation of \( f T_u^* \), the adjoint of \( T_u \).
4.1. Explicit representation of the adjoint. Let $\mathcal{T}_w : (L^2(\Omega))^{d \times d} \to L^2(0, \tau; L^2(\Omega))$ be as in Theorem 3.3. Let $C \in (L^2(\Omega))^{d \times d}$ and $v = \mathcal{T}_w C$. Then, in view of (3.6) - (3.8), $v \in L^2(0, \tau; H^1_0(\Omega))$ and we have

\begin{align*}
\langle v_t, \varphi \rangle + \int_{\Omega} A_0 \nabla v \cdot \nabla \varphi \, dx &= - \int_{\Omega} C \nabla w \cdot \nabla \varphi \, dx \quad \forall \varphi \in L^2(0, \tau; H^1_0(\Omega)) \text{ and for a.a. } t \in [0, \tau],
\end{align*}

where $v_t \in L^2(0, \tau; H^{-1}(\Omega))$, and $\langle \cdot , \cdot \rangle$ in the above is the duality pairing corresponds to $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Now, let $\phi \in L^2(0, \tau; L^2(\Omega))$ and consider the PDE

\begin{align*}
\begin{cases}
  z_t + \nabla \cdot A_0 \nabla z = \phi & \text{in } Q \\
  z = 0 & \text{on } \partial Q \\
  z(\cdot, \tau) = 0 & \text{in } \Omega.
\end{cases}
\end{align*}

By reversing the time direction for the above PDE, and by Theorem 2.1, we know that there exists a unique $z \in L^2(0, \tau; H^1_0(\Omega))$ with $z_t \in L^2(0, \tau; H^{-1}(\Omega))$ such that $z(\cdot, \tau) = 0$ and

\begin{align*}
\langle z_t, \varphi \rangle - \int_{\Omega} A_0 \nabla z \cdot \nabla \varphi \, dx &= \int_{\Omega} \phi \varphi \, dx \quad \text{for a.a. } t \in [0, \tau] \text{ and } \varphi \in H^1_0(\Omega).
\end{align*}

Since $v(\cdot, t) \in H^1_0(\Omega)$ for a.a. $t \in [0, \tau]$, from (4.10), we obtain

\begin{align*}
\langle z_t, v \rangle - \int_{\Omega} A_0 \nabla z \cdot \nabla v \, dx &= \int_{\Omega} \phi v \, dx \quad \text{for a.a. } t \in [0, \tau].
\end{align*}

Now, integrating with respect to $t$, and using the fact $v(\cdot, 0) = 0 = z(\cdot, \tau)$, we obtain

\begin{align*}
- \int_{0}^{\tau} \langle z_t, v_t \rangle \, dt - \int_{0}^{\tau} \int_{\Omega} A_0 \nabla z \cdot \nabla v \, dx \, dt &= \int_{0}^{\tau} \int_{\Omega} \phi v \, dx \, dt = \langle \mathcal{T}_w C, \phi \rangle_{L^2(0, \tau; L^2(\Omega))}.
\end{align*}

Since $z \in L^2(0, \tau; H^1_0(\Omega))$, from (4.8), we obtain

\begin{align*}
\langle z_t, v \rangle + \int_{\Omega} A_0 \nabla v \cdot \nabla z \, dx &= - \int_{\Omega} C \nabla w \cdot \nabla z \, dx \quad \text{for a.a. } t \in [0, \tau],
\end{align*}

and hence

\begin{align*}
\int_{0}^{\tau} \langle z_t, v \rangle \, dt + \int_{0}^{\tau} \int_{\Omega} A_0 \nabla v \cdot \nabla z \, dx \, dt &= \int_{0}^{\tau} \int_{\Omega} C \nabla w \cdot \nabla z \, dx \, dt.
\end{align*}

Therefore, from (4.11) and (4.12), using the fact that $A_0$ is symmetric, we obtain

\begin{align*}
\int_{0}^{\tau} \int_{\Omega} \phi v \, dx \, dt &= \int_{0}^{\tau} \int_{\Omega} C \nabla w \cdot \nabla z \, dx \, dt.
\end{align*}

Now,

\begin{align*}
\int_{0}^{\tau} \int_{\Omega} C \nabla w \cdot \nabla z \, dx \, dt &= \int_{0}^{\tau} \int_{\Omega} (C \nabla w, \nabla z)_{\mathbb{R}^d} \, dx \, dt = \int_{0}^{\tau} \int_{\Omega} (C \nabla w)^T \nabla z \, dx \, dt \\
&= \int_{0}^{\tau} \int_{\Omega} (\nabla w)^T C \nabla z \, dx \, dt = \int_{0}^{\tau} \int_{\Omega} (\nabla w)^T (\nabla z)^T C \, dx \, dt \\
&= \int_{0}^{\tau} \int_{\Omega} (C \nabla w)^T (\nabla z) \, dx \, dt = \int_{0}^{\tau} \int_{\Omega} C \nabla (\nabla w)^T \, dx \, dt \\
&= \langle C, \int_{0}^{\tau} \nabla (\nabla w)^T \, dt \rangle_{(L^2(\Omega))^{d \times d}}.
\end{align*}

Thus, from (4.13), we have

\begin{align*}
\langle \mathcal{T}_w C, \phi \rangle_{L^2(0, \tau; L^2(\Omega))} &= \int_{0}^{\tau} \int_{\Omega} \phi v \, dx \, dt = \langle C, \int_{0}^{\tau} \nabla (\nabla w)^T \, dt \rangle.
\end{align*}

Therefore, we have proved the following result.

**Theorem 4.4.** Let $w$ be as in Theorem 3.3 and $\mathcal{T}_w : (L^2(\Omega))^{d \times d} \to L^2(0, \tau; L^2(\Omega))$ be as in Theorem 3.3. Then the operator $\mathcal{T}_w^*$, the adjoint of $\mathcal{T}_w$, is given by

\begin{align*}
\mathcal{T}_w^* \phi &= \int_{0}^{\tau} \nabla (\nabla w)^T \, dt, \quad \phi \in L^2(0, \tau; L^2(\Omega)),
\end{align*}
where $z \in L^2(0, \tau; H^1_0(\Omega))$ is the unique weak solution of the PDE:

$$z_t + \nabla \cdot \mathbf{A}_0 \nabla z = \phi \quad \text{in } \Omega \times (0, \tau)$$

along with the conditions

$$z = 0 \text{ on } \partial \Omega \times (0, \tau), \quad z(\cdot, \tau) = 0 \text{ in } \Omega.$$

5. Finite dimensional setting

So far we have theoretically obtained stable approximation for $\mathbf{A}$, namely $\tilde{\mathbf{A}}_\alpha := \mathbf{A}_0 + \mathbf{B}_\alpha$, and we have seen our underlying spaces were all infinite dimensional. But, in the context of applications one has to work with finite dimensional spaces. From that point of view it becomes necessary to realize our analysis in finite dimensional setting. For this purpose, we employ Galerkin method to obtain finite dimensional approximations of $\tilde{\mathbf{B}}_\alpha$, which in turn will give approximations for $\mathbf{A}$ with suitable choice of the regularization parameter $\alpha$.

Let $(X_n)$ be a sequence of finite dimensional subspaces of $L^2(\Omega)$ such that

(i) $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$ and
(ii) $\bigcup_{n=1}^{\infty} X_n$ is dense in $L^2(\Omega)$.

For each $n \in \mathbb{N}$, let $P_n : L^2(\Omega) \to L^2(\Omega)$ be an orthogonal projection onto $X_n$. Then, with the assumptions (i) and (ii) on $X_n$, we have

$$P_n \varphi \to \varphi \quad \text{as } n \to \infty$$

for all $\varphi \in L^2(\Omega)$. For each $n \in \mathbb{N}$, we define $P_n : (L^2(\Omega))^{d \times d} \to (L^2(\Omega))^{d \times d}$ by

$$P_n \mathbf{C} = (P_n c_{ij})_{1 \leq i,j \leq d}$$

(5.1)

for all $\mathbf{C} = (c_{ij})_{1 \leq i,j \leq d} \in (L^2(\Omega))^{d \times d}$.

We now have the following result.

**Theorem 5.1.** For each $n \in \mathbb{N}$, let $P_n : (L^2(\Omega))^{d \times d} \to (L^2(\Omega))^{d \times d}$ be as defined in (5.1). Then $P_n$ is an orthogonal projection and for every $\mathbf{C} \in (L^2(\Omega))^{d \times d}$,

$$\lim_{n \to \infty} \|P_n \mathbf{C} - \mathbf{C}\| = 0.$$

**Proof.** From the definition of $P_n$, it follows that $P_n$ is a projection. Also, $X_n \subseteq X_{n+1}$ for all $n \in \mathbb{N}$ ensures that $R(P_n) \subseteq R(P_{n+1})$ for all $n \in \mathbb{N}$, where $R(P_n)$ denotes the range space of $P_n$. Since $P_n : L^2(\Omega) \to L^2(\Omega)$ is an orthogonal projection, for any $\mathbf{C}, \mathbf{D} \in (L^2(\Omega))^{d \times d}$, we have

$$\langle \mathbf{C}, P_n \mathbf{D} \rangle = \sum_{i,j=1}^{d} \langle c_{ij}, P_n d_{ij} \rangle_{L^2} = \sum_{i,j=1}^{d} \langle P_n c_{ij}, d_{ij} \rangle_{L^2} = \langle \mathbf{P}_n \mathbf{C}, \mathbf{D} \rangle.$$

Hence, $P_n : (L^2(\Omega))^{d \times d} \to (L^2(\Omega))^{d \times d}$ is an orthogonal projection. Also, we have

$$\|P_n \mathbf{C} - \mathbf{C}\|^2 = \sum_{i,j=1}^{d} \|P_n c_{ij} - c_{ij}\|_{L^2}.$$

Since $P_n$ converges pointwise to the identity in $L^2(\Omega)$, we have

$$\lim_{n \to \infty} \|P_n c_{ij} - c_{ij}\|_{L^2} = 0 \quad \text{for all } 1 \leq i,j \leq d.$$

Therefore, we have $\lim_{n \to \infty} \|P_n \mathbf{C} - \mathbf{C}\| = 0$ for every $\mathbf{C} \in (L^2(\Omega))^{d \times d}$. \qed

**Corollary 5.2.** For each $n \in \mathbb{N}$, let $P_n : (L^2(\Omega))^{d \times d} \to (L^2(\Omega))^{d \times d}$ be as defined in (5.1). Then for every $w \in L^2(0, \tau; L^2(\Omega))$ with $0 \neq \nabla w \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$,

$$\|T_w - T_{\alpha} P_n\| \to 0 \quad \text{as } n \to \infty.$$
\textbf{Proof.} We know, by Theorem 3.3, that $\mathcal{T}_n$ is a compact operator and by Theorem 5.1 that $\mathbf{P}_n$ is an orthogonal projection for each $n \in \mathbb{N}$ satisfying $\lim_{n \to \infty} \|\mathbf{P}_n \mathbf{C} - \mathbf{C}\| \to 0$. Since $\mathcal{T}_n^*$ is also a compact operator, by a standard result in Functional analysis, as a consequence of uniform boundedness principle (see, e.g., \cite{12}, Corollary 6.6), we have
\[
\|(I - \mathbf{P}_n)\mathcal{T}_n^*\| \to 0 \quad \text{as} \quad n \to \infty.
\]
Since $\|\mathcal{T}_n - \mathcal{T}_n \mathbf{P}_n\| = \|(I - \mathbf{P}_n)\mathcal{T}_n^*\|$, the conclusion in the corollary follows. \hfill \square

Next we observe that (4.3) holds iff\footnote{Equivalently, we look for $\tilde{\mathbf{B}}_{\alpha,n} \in \mathcal{R}(\mathbf{P}_n)$ such that
\[
(\mathbf{P}_n \mathcal{T}_n^* \mathcal{T}_n \mathbf{P}_n + \alpha \mathbf{I})\tilde{\mathbf{B}}_{\alpha,n} = \mathbf{P}_n \mathcal{T}_n^*(v_0 - \tilde{u}).
\]
For each $n \in \mathbb{N}$, let $\dim(X_n) = n$. Then, it can be seen that $\dim(\mathcal{R}(\mathbf{P}_n)) = nd^2$. Indeed, if $\{\varphi_1, ..., \varphi_n\}$ is a basis of $X_n$, then it can be seen that $\{E_{ij}^{(\nu)} : 1 \leq i, j \leq d, 1 \leq \nu \leq n\}$ forms a basis for $\mathcal{R}(\mathbf{P}_n)$, where
\[
(E_{ij}^{(\nu)})_{pq} = \begin{cases} \varphi_{ij} & \text{if } p = i, q = j \\ 0 & \text{if } p \neq i \text{ or } q \neq j. \end{cases}
\]
Also, it can be seen that, if $\{\varphi_1, ..., \varphi_n\}$ is an orthonormal basis of $X_n$, then $\{E_{ij}^{(\nu)} : 1 \leq i, j \leq d, 1 \leq \nu \leq n\}$ too becomes an orthonormal basis for $\mathcal{R}(\mathbf{P}_n)$.

Now, let $\{\varphi_1, ..., \varphi_n\}$ be an orthonormal basis of $X_n$, and let $\{E_{ij}^{(\nu)} : 1 \leq i, j \leq d, 1 \leq \nu \leq n\}$, which is an orthonormal basis of $\mathcal{R}(\mathbf{P}_n)$. Let us write the solution of the well-posed equation (5.4) as
\[
\tilde{\mathbf{B}}_{\alpha,n} = \sum_{1 \leq i, j \leq d} \sum_{l=1}^{n} c_{ij}^{(l)} E_{ij}^{(l)}
\]
for some constants $c_{ij}^{(l)}$, $1 \leq i, j \leq d, 1 \leq l \leq n$. Then, from (5.4), we have
\[
\left(\mathbf{P}_n \mathcal{T}_n^* \mathcal{T}_n \mathbf{P}_n + \alpha \mathbf{I}\right)\tilde{\mathbf{B}}_{\alpha,n} = \mathbf{P}_n \mathcal{T}_n^*(v_0 - \tilde{u}), E_{ij}^{(l)}
\]
for all $1 \leq i, j \leq d, 1 \leq l \leq n$. The equation (5.6) can be written in the matrix form as
\[
U\tilde{c} + \alpha D\tilde{c} = \tilde{b},
\]
where $U, D$ are matrices and $\tilde{c}, \tilde{b}$ are column vectors given by
\[
U = \left[\langle \mathcal{T}_n E_{pq}^{(k)}, E_{ij}^{(l)} \rangle\right], \quad D = \left[\langle E_{pq}^{(k)}, E_{ij}^{(l)} \rangle\right], \quad \tilde{c} = \left[c_{pq}^{(k)}\right], \quad \tilde{b} = \left[\langle v_0 - \tilde{u}, \mathcal{T}_n E_{ij}^{(l)} \rangle\right]
\]
for $1 \leq p, q \leq d, 1 \leq k \leq n$ and $1 \leq i, j \leq d, 1 \leq l \leq n$.

Thus, in order to obtain $\tilde{\mathbf{B}}_{\alpha,n}$, we need to solve the matrix equation (5.7). Since $\mathbf{P}_n \mathcal{T}_n^* \mathcal{T}_n \mathbf{P}_n$ is a bounded positive self adjoint operator, for each $\alpha > 0$, there exists a unique solution for the equation (5.4), in other words $\tilde{\mathbf{B}}_{\alpha,n}$ is determined uniquely and hence the matrix equation (5.7) also has a unique solution. Thus, $\tilde{c}$ is determined uniquely and hence $\tilde{\mathbf{B}}_{\alpha,n}$ is obtained explicitly using (5.5).
Theorem 5.3. Let $B_\alpha$ and $\tilde{B}_{\alpha,n}$ be the unique solutions of (4.4) and (5.4), respectively. Let $C_0$ be as in Theorem 2.7, and $B$ be as in (4.1). Let $\tilde{u}$ be as in Assumption 4.1, and $\epsilon_n > 0$ be such that $\|T_{\tilde{u}} - T_{\tilde{u}}P_n\| \leq \epsilon_n$ and $\delta > 0$ be as given in (4.2). Then

$$\|B_\alpha - \tilde{B}_{\alpha,n}\| \leq \frac{d\sqrt{C_0}}{\sqrt{\alpha}} \|B\| + \frac{\epsilon_n}{\sqrt{\alpha}} \|B\| + \frac{\delta}{2\sqrt{\alpha}}.$$ 

Proof. We have

$$B_\alpha - \tilde{B}_{\alpha,n} = (T_{\tilde{u}} + \alpha I)^{-1}T_{\tilde{u}}^*(v_0 - u) - (P_n T_{\tilde{u}}^* T_{\tilde{u}} + \alpha I)^{-1}P_n T_{\tilde{u}}^*(v_0 - \tilde{u}),$$

(5.8)

Now, using (4.4), we have

$$(P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}P_n T_{\tilde{u}}^*(v_0 - u) = (P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}P_n T_{\tilde{u}}^* T_{\tilde{u}} B.$$ 

Also,

$$(P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}P_n T_{\tilde{u}}^* T_{\tilde{u}} B = (P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}P_n T_{\tilde{u}}^* (T_{\tilde{u}} - T_{\tilde{u}} P_n) T_{\tilde{u}} + \alpha(P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}(P_n T_{\tilde{u}}^* - T_{\tilde{u}}) T_{\tilde{u}} B.$$ 

We now recall the following estimates from [13],

$$\|(P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}P_n T_{\tilde{u}}^*\| \leq \frac{1}{2\sqrt{\alpha}}, \quad \|(P_n T_{\tilde{u}}^* T_{\tilde{u}} P_n + \alpha I)^{-1}\| \leq \frac{1}{\alpha},$$

$$\|T_{\tilde{u}}^* T_{\tilde{u}} (T_{\tilde{u}}^* T_{\tilde{u}} + \alpha I)^{-1}\| \leq 1, \quad \|(T_{\tilde{u}}^* T_{\tilde{u}} + \alpha I)^{-1}T_{\tilde{u}}\| \leq \frac{1}{2\sqrt{\alpha}}.$$ 

Using these estimates and using (5.9), we obtain from (5.8),

$$\|B_\alpha - \tilde{B}_{\alpha,n}\| \leq \frac{\|T_{\tilde{u}} - T_{\tilde{u}} P_n\|}{\sqrt{\alpha}} \|B\| + \frac{\|P_n T_{\tilde{u}}^* - T_{\tilde{u}}^*\|}{\sqrt{\alpha}} \|B\| + \frac{\|\tilde{u} - u\|}{\sqrt{2\alpha}}.$$ 

Now, using the fact $\|P_n T_{\tilde{u}}^* - T_{\tilde{u}}^*\| = \|T_{\tilde{u}} - T_{\tilde{u}} P_n\|$, we have

$$\|B_\alpha - \tilde{B}_{\alpha,n}\| = \frac{\|T_{\tilde{u}} - T_{\tilde{u}} P_n\|}{\sqrt{\alpha}} \|B\| + \frac{\|\tilde{u} - u\|}{\sqrt{2\alpha}} \|B\| + \frac{\|\tilde{u} - u\|}{\sqrt{2\alpha}}.$$ 

Therefore, using (4.2) and (4.6), we have

$$\|B_\alpha - \tilde{B}_{\alpha,n}\| \leq \frac{d\sqrt{C_0}}{\sqrt{\alpha}} \|B\| + \frac{\epsilon_n}{\sqrt{\alpha}} \|B\| + \frac{\delta}{2\sqrt{\alpha}}.$$
Let 
\[ \tilde{A}_{\alpha,n} := \tilde{B}_{\alpha,n} + A_0. \]
Then, our next theorem will give an estimate of \( \|A - \tilde{A}_{n,\alpha}\| \), the proof follows from Theorem 5.3 and using the fact
\[ \|A - \tilde{A}_{\alpha,n}\| = \|B - \tilde{B}_{\alpha,n}\| \leq \|B - B_\alpha\| + \|B_\alpha - \tilde{B}_{\alpha,n}\|. \]

**Theorem 5.4.** Let \( B_\alpha \) and \( \tilde{B}_{\alpha,n} \) be the unique solutions of (4.4) and (5.4), respectively. Let \( C_0 \) be as in Theorem 2.7 and \( B \) be as in (4.1). Let \( \varepsilon_n > 0 \) be such that \( \|T_u - T_{\tilde{u}} P_n\| \leq \varepsilon_n \) and \( \delta > 0 \) be as given in (4.2). Then
\[ \|A - \tilde{A}_{\alpha,n}\| \leq \|B - B_\alpha\| + \frac{d\sqrt{C_0\delta}}{\sqrt{\alpha}}\|B\| + \frac{\varepsilon_n}{\sqrt{\alpha}}\|B\| + \frac{\delta}{2\sqrt{\alpha}}. \]

**Remark 5.5.** By Corollary 5.2 we know that \( \|T_u - T_{\tilde{u}} P_n\| \to 0 \) as \( n \to \infty \). Hence \( \varepsilon_n > 0 \) can be chosen in such a way that \( \|T_u - T_{\tilde{u}} P_n\| \leq \varepsilon_n \) and \( \varepsilon_n \to 0 \) as \( n \to \infty \). Thus, Theorem 5.4 shows that, by an appropriate choice of \( \alpha \) and \( n \), depending on \( \delta \), \( A_{\alpha,n} \) is an approximation of \( A \).

### 6. Adaptive choice of Parameters

Recall that, our aim is to show \( \tilde{A}_{\alpha,n} \) is a stable approximation to \( A \) for some chosen parameters \( n \) and \( \alpha \) compatible with the noise level \( \delta \). In other words, we need to choose the parameters \( n \) and \( \alpha \), depending on \( \delta \), in such a way that \( \tilde{A}_{\alpha,n} \) converges to \( A \) as the noise level \( \delta \to 0 \). In this regard, we observe that Theorem 5.3 will show us a direction for choosing the parameters suitably. Parameter choice strategy in regularization theory has a vast literature, for more details one may look into [7], [13], [15] and the references therein. In this paper we will use the procedure adopted in [15].

Now, we would like to recall that \( B = A - A_0 \) and \( B_\alpha \) is the solution of (4.4), that is, a Tikhonov regularized solution. Therefore, from the theory of Tikhonov regularization, it is known that (cf. [13])
\[ \|B - B_\alpha\| \to 0 \quad \text{as} \quad \alpha \to 0. \]
But, for obtaining an estimate for \( \|B - B_\alpha\| \), it is necessary to assume some source condition on \( B \).
So, we make the following general assumption for source condition.

**Source condition:** Let
\[ B = \varphi(T_u^* T_u) C, \quad \|C\| \leq \rho \]
for some \( \rho > 0 \) and for some monotonically increasing function \( \varphi \) defined on \((0, \gamma]\), where \( \gamma \geq \|T_u\|^2 \), such that \( \lim_{\lambda \to 0} \varphi(\lambda) = 0 \) and
\[ \sup_{0 \leq \lambda \leq \gamma} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha) \quad \text{for all} \quad \alpha > 0. \]
A typical case of such a situation is when \( B \) is in the range of \( \varphi(T_u^* T_u) \), where \( \varphi(\lambda) := \lambda^\nu \) for some \( \nu \in (0, 1] \) or \( \varphi(\lambda) := [\log(1/\lambda)]^{-p} \) for some \( p > 0 \) (see, for example, [7] [13]).

Using (4.3), (4.4) and the above assumption, we have
\[ B - B_\alpha = \alpha(T_u^* T_u + \alpha I)^{-1} B - \alpha(T_u^* T_u + \alpha I)^{-1} \varphi(T_u^* T_u) C \]
so that
\[ \|B - B_\alpha\| = \|\alpha(T_u^* T_u + \alpha I)^{-1} \varphi(T_u^* T_u) C\| \leq \sup_{0 \leq \lambda \leq \gamma} \frac{\alpha \varphi(\lambda)}{\lambda + \alpha} \|C\| \leq \rho \varphi(\alpha). \]
Thus, with the above source condition, and by Theorem 5.4 we obtain
\[ \|B - \tilde{B}_{\alpha,n}\| \leq \rho \varphi(\alpha) + C_B \frac{\delta(d+1) + \varepsilon_n}{\sqrt{\alpha}}, \]
where \( C_B = \max \left\{ 1/2, \sqrt{C_0} \|B\| \right\} \).

As mentioned in Remark 5.5, \( \varepsilon_n \to 0 \) as \( n \to \infty \). Let \( n_\delta \in \mathbb{N} \) be such that \( \varepsilon_n < \delta \) for all \( n \geq n_\delta \). Then by (6.2), we have

\[
\|B - \hat{B}_{\alpha, n_\delta}\| \leq \rho \varphi(\alpha) + C_B \delta \sqrt{\alpha}. \tag{6.3}
\]

Let \( C > 1/2 \) be a constant such that \( \rho \varphi(\gamma) \leq C \). Then, using (6.1), we obtain \( \|B\| \leq C \), and hence

\[
C_B = \max \left\{ 1/2, \sqrt{C_0} \|B\| \right\} \leq C. \tag{6.3}
\]

Thus, by (6.3) we have

\[
\|B - \hat{B}_{\alpha, n_\delta}\| \leq \rho \varphi(\alpha) + C \frac{\delta(d + 2)}{\sqrt{\alpha}}. \tag{6.4}
\]

We now apply the adaptive technique for choosing the parameter \( \alpha \) a posteriori, so that the rate of convergence is order optimal. Here we would like to point out that, the technique is elaborately explained in general setting in [15]. Also, whatever result we will obtain, can be derived easily from the results already obtained in [9], [15]. But, in order to keep this paper self contained we are giving the details, presented in our own way.

Let \( \delta > 0 \) be such that \( \delta^2(d + 2)^2 \leq \gamma \). Let \( \alpha_0 = \delta^2(d + 2)^2 \). Since \( \varphi \) is an increasing function on \( (0, \gamma] \) and \( \alpha_0 \leq \gamma \), we have

\[
\rho \varphi(\alpha_0) \leq \rho \varphi(\gamma) \leq C.
\]

Let \( \mu > 1 \) be any fixed real number and \( N \in \mathbb{N} \) be fixed. Let

\[
\alpha_i = \mu^{2i} \alpha_0, \quad i = 1, 2, ..., N. \tag{6.5}
\]

Then, clearly we have

\[
0 < \alpha_0 < \alpha_1 < ... < \alpha_N
\]

and

\[
\sqrt{\alpha_i} \leq \mu \sqrt{\alpha_{i-1}} \quad \text{for all } i = 1, 2, ..., N.
\]

Let

\[
l = \max \left\{ i \in \{0, 1, ..., N - 1\} : \rho \mu^i \varphi(\alpha_i) \leq C \right\}. \tag{6.6}
\]

We now establish the following lemma.

**Lemma 6.1.** Let \( n_\delta \) and \( l \) be as in (6.3) and (6.6), respectively. Then, for any \( j \in \{0, 1, ..., l\} \), we have

\[
\|B_{\alpha_j, n_\delta} - \hat{B}_{\alpha_j, n_\delta}\| \leq \frac{4C}{\mu^j}. \tag{6.7}
\]

**Proof.** First we note that for any \( j \leq l \), we have

\[
\rho \mu^j \varphi(\alpha_j) \leq \rho \mu^j \varphi(\alpha_l).
\]

Then, using (6.4), (6.3) and (6.6), we have

\[
\|\hat{B}_{\alpha_j, n_\delta} - \hat{B}_{\alpha_j, n_\delta}\| \leq \|B_{n_\delta, \alpha_l} - B\| + \|B_{n_\delta, \alpha_j} - B\|
\leq \rho \varphi(\alpha_l) + C \frac{\delta(d + 2)}{\sqrt{\alpha}} + \rho \varphi(\alpha_j) + C \frac{\delta(d + 2)}{\sqrt{\alpha}}
\leq \frac{4C}{\mu^j}.
\]

\( \square \)
Let
\begin{equation}
(6.7) \quad k = \max \left\{ i \in \{0, 1, \ldots, N\} : \| \tilde{B}_{n_i, \alpha_i} - \tilde{B}_{n_j, \alpha_j} \| \leq \frac{4C}{\mu^j}, j = 0, 1, \ldots, i \right\}.
\end{equation}

Then, Lemma 6.1 ensures that \( l \leq k \). Now, using (6.4), (6.6), (6.7) we have
\[ \| B - \tilde{B}_{\alpha_k, n} \| \leq \| B - \tilde{B}_{\alpha_l, n} \| + \| \tilde{B}_{\alpha_l, n} - \tilde{B}_{\alpha_k, n} \| \leq \rho \phi(\alpha_l) + \frac{4C}{\mu^l} \leq \frac{2C}{\mu^l} + \frac{4C}{\mu^l} = \frac{6C}{\mu^l}. \]

Let \( \alpha_\delta \) be such that \( \rho \phi(\alpha_\delta) = C \frac{\delta(d+2)}{\sqrt{\alpha_\delta}} \). Then, it is clear that \( \rho \phi(\alpha) + C \frac{\delta(d+2)}{\sqrt{\alpha}} \) attains its minimum at \( \alpha_\delta \). Now, using the definition of \( l \), we have
\[ \phi(\alpha_\delta) \sqrt{\alpha_\delta} = C \frac{\delta(d+2)}{\rho} \leq \delta(d+2) \mu^{l+1} \phi(\alpha_{l+1}) = \sqrt{\delta^2(d+2)^2 \mu^{2(l+1)}} \phi(\alpha_{l+1}) = \sqrt{\alpha_{l+1}} \phi(\alpha_{l+1}). \]

Now, using the fact that \( \phi \) is an increasing function, we obtain \( \alpha_\delta < \alpha_{l+1} \).

Thus, we have
\[ \sqrt{\alpha_\delta} < \sqrt{\alpha_{l+1}} = \sqrt{\alpha_0 \mu^{l+1}} = \delta(d+2) \mu^{l+1}. \]

Therefore,
\begin{equation}
(6.8) \quad \| B - \tilde{B}_{\alpha_k, n} \| \leq \frac{6C}{\mu^l} \leq \frac{6C \delta(d+2)}{\sqrt{\alpha_\delta}} = 6\mu \rho \phi(\alpha_\delta).
\end{equation}

Let
\begin{equation}
(6.9) \quad \Psi(\lambda) = \frac{\rho \lambda \sqrt{\phi^{-1}(\lambda)}}{C(d+2)}, \quad 0 < \lambda \leq \gamma.
\end{equation}

Then, we have
\[ \delta = \frac{\rho \phi(\alpha_\delta) \sqrt{\alpha_\delta}}{C(d+2)} = \Psi(\phi(\alpha_\delta)), \]
so that \( \phi(\alpha_\delta) = \Psi^{-1}(\delta) \). Thus, using (6.8), we have
\begin{equation}
(6.10) \quad \| B - \tilde{B}_{\alpha_k, n} \| \leq 6\mu \rho \Psi^{-1}(\delta).
\end{equation}

Since \( B = A - A_0 \), therefore, using (6.10), we have obtained the following theorem.

**Theorem 6.2.** Let \( \rho, \mu, k \) be as defined in (6.1), (6.5) and (6.7), respectively and \( A_0 \) be as in (6.2). Let \( n_\delta \) and \( \alpha_k \) be as defined in (6.3) and (6.5), respectively, and let \( \Psi \) be as defined in (6.9). Let \( \hat{A}_{\alpha_k, n} := A_0 + \tilde{B}_{\alpha_k, n} \). Then, we have
\[ \| A - \hat{A}_{\alpha_k, n} \| \leq 6\mu \rho \Psi^{-1}(\delta). \]

The above theorem ensures that the a posteriori chosen parameter \( \alpha_k \) satisfies the optimal rate of convergence with respect to \( \delta \), without the knowledge of any a priori source function. So, the procedure of adaptive choosing of the regularizing parameter is effective in our analysis of obtaining the a posteriori regularizing parameter.
7. Smoothing of noisy data

Recall that in Remark 1.2, we have mentioned the need for a smoothed version of the noisy data $\tilde{u} \in L^2(0, \tau; L^2(\Omega))$. This section is devoted to that purpose. In [14], the authors have used the Clement operator (see [5]) for obtaining a smoothed version of a noisy data in the context of a parameter identification problem for an elliptic PDE. Here we will do similar kind of smoothing, but we will be doing for the parabolic case.

Let $u$ be as considered in the Assumption 1.4. Then $u \in L^2(0, \tau; L^2(\Omega))$ along with $\nabla u \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$. In order to do our smoothing analysis, we assume that $u \in L^2(0, \tau; H^4(\Omega))$. Let $\tilde{u} \in L^2(0, \tau; L^2(\Omega))$ be the observed data corresponding to $u$. We want to obtain an element, say $z \in L^2(0, \tau; L^2(\Omega))$ such that $\nabla z \in L^2(0, \tau; L^\infty(\Omega, \mathbb{R}^d))$, and we will call this $z$ as a smoothed version of $\tilde{u}$. For this we assume that $\Omega$ is a polygonal domain in $\mathbb{R}^2$.

Let $\mathcal{L}$ be an element of a family of quasi uniform triangulation of $\Omega$. Then there exist a constant $\gamma_0 > 0$ such that
\begin{equation}
\min_{S \in \mathcal{L}} \frac{\text{diam}(S)}{h} \geq \gamma_0 > 0,
\end{equation}
where $\text{diam}(S)$ is the diameter of triangle $S \in \mathcal{L}$ and $h = \max_{S \in \mathcal{L}} \text{diam}(S)$, is the mesh size. Let $\Pi$ be the Clement operator (see [5]) that maps $L^2(\Omega)$ to the space of all polynomials of degree less than or equal to 3. We now state an important result required for our analysis, for its proof refer to [5].

**Theorem 7.1.** ([5], Theorem 1) Let $v \in L^2(\Omega)$. Then $\Pi v \in W^{1,\infty}(\Omega)$ and there exist constants $C_1, C_2 > 0$ such that
\[ \|v - \Pi v\|_{L^2} \leq C_1 \|v\|_{L^2} \quad \text{for all } v \in L^2(\Omega) \]
and
\[ \|v - \Pi v\|_{H^3} \leq C_2 h \|v\|_{H^4} \quad \text{for all } v \in H^4(\Omega). \]

We know that if $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary, then for every $k$ with $d < 2k$, we have the continuous embedding
\[ H^k(\Omega) \hookrightarrow L^\infty(\Omega). \]
In our case, since $u \in L^2(0, \tau; H^4(\Omega))$ and $\Omega \subset \mathbb{R}^2$, therefore we have the continuous embedding
\[ H^4(\Omega) \hookrightarrow H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega), \]
and hence there exist a constant $C_3 > 0$ such that
\begin{equation}
\|u(\cdot, t) - \Pi u(\cdot, t)\|_{W^{1,\infty}} \leq C_3 \|u(\cdot, t) - \Pi u(\cdot, t)\|_{H^3} \quad \text{for a.a } t \in [0, \tau].
\end{equation}
Let $C_4 = C_2 C_3$. Then, using Theorem 7.1 and (7.2), we have
\begin{equation}
\|u(\cdot, t) - \Pi u(\cdot, t)\|_{W^{1,\infty}} \leq C_4 h \|u(\cdot, t)\|_{H^4} \quad \text{for a.a } t \in [0, \tau].
\end{equation}
We now state a result, which is a reformulation of Theorem 4.5.11 in [3].

**Lemma 7.2.** For any $S \in \mathcal{L}$ and for all $v \in L^2(S)$, we have
\[ \|\Pi v\|_{W^{1,\infty}(S)} \leq \frac{1}{(\text{diam}(S))^2} \|\Pi v\|_{L^2(S)}. \]

Let $v \in L^2(\Omega)$. Then by the above Lemma, we have
\[ \|\Pi v\|_{W^{1,\infty}(S)} \leq \frac{1}{(\text{diam}(S))^2} \|\Pi v\|_{L^2(S)} \leq \frac{1}{h^2 \gamma_0} \|\Pi v\|_{L^2(S)} \]
\[ \leq \frac{1}{h^2 \gamma_0} \|\Pi v\|_{L^2(S)} \leq \frac{1}{h^2 \gamma_0} \left(\|1 - \Pi v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) \]
\[ \leq \frac{C_1 + 1}{h^2 \gamma_0} \|v\|_{L^2(\Omega)}, \]
where $C_1$ is as in Theorem 7.1 and $\gamma_0$ is as in (7.1). Thus, using the fact that
$$||\Pi v||_{W^{1,\infty}(\Omega)} \leq \max_{S \in L} ||\Pi v||_{W^{1,\infty}(S)}$$
we have proved the following theorem.

**Theorem 7.3.** Let $C_5 := \frac{C_1 + 1}{\gamma_0^2}$, where $C_1$ is as in Theorem 7.1 and $\gamma_0$ is as in (7.1). Then for all $v \in L^2(\Omega)$,
$$||\Pi v||_{W^{1,\infty}(\Omega)} \leq \frac{C_5}{h^2} ||v||_{L^2(\Omega)}.$$

8. Modified error estimates

Let $\Pi$ be as considered in Section 7 and $u$ be as considered in Assumption 1.4. Let the noisy data $\tilde{u} \in L^2(0, \tau; L^2(\Omega))$ be such that
$$(8.1) \quad ||u - \tilde{u}||_{L^2(0, \tau; L^2(\Omega))} \leq \delta.$$  

Using the results of Section 7 we now obtain error estimates with the noisy data $\tilde{u}$.

**Theorem 8.1.** Let $\delta > 0$ be as considered in (8.1). Let $C_4$ and $C_5$ be as in (7.3) and Theorem 7.3 respectively. Let $C = \max \{C_4, C_5\}$. Then
$$\left(\int_0^\tau ||\nabla u(\cdot, t) - \nabla \Pi \tilde{u}(\cdot, t)||_{L^\infty}^2 dt\right)^{1/2} \leq C\left(h ||u||_{L^2(0, \tau; H^4(\Omega))} + \frac{\delta}{h^2}\right).$$

**Proof.** Using (7.3) and Theorem 7.3 we have
$$||u(\cdot, t) - \Pi \tilde{u}(\cdot, t)||_{W^{1,\infty}} \leq ||u(\cdot, t) - \Pi u(\cdot, t)||_{W^{1,\infty}} + ||\Pi u(\cdot, t) - \Pi \tilde{u}(\cdot, t)||_{W^{1,\infty}} \leq C_4 h ||u(\cdot, t)||_{H^4} + \frac{C_5}{h^2} ||u(\cdot, t) - \tilde{u}(\cdot, t)||_{L^2}$$
for a.a $t \in [0, \tau]$. Thus, we have
$$\int_0^\tau ||\nabla u(\cdot, t) - \nabla \Pi \tilde{u}(\cdot, t)||_{L^\infty}^2 dt \leq \int_0^\tau ||u(\cdot, t) - \Pi \tilde{u}(\cdot, t)||_{W^{1,\infty}}^2 dt$$
$$= C_4^2 h^2 ||u||_{L^2(0, \tau; H^4(\Omega))}^2 + \frac{C_5^2}{h^2} \int_0^\tau ||u(\cdot, t) - \tilde{u}(\cdot, t)||_{L^2}^2 dt$$
$$+ 2C_4 C_5 \int_0^\tau ||u(\cdot, t)||_{H^4} ||u(\cdot, t) - \tilde{u}(\cdot, t)||_{L^2} dt$$
$$\leq C_4^2 h^2 ||u||_{L^2(0, \tau; H^4(\Omega))}^2 + \frac{C_5^2}{h^2} ||u - \tilde{u}||_{L^2(0, \tau; L^2(\Omega))}^2$$
$$+ 2C_4 C_5 \int_0^\tau ||u||_{L^2(0, \tau; H^4(\Omega))} ||u - \tilde{u}||_{L^2(0, \tau; L^2(\Omega))}$$
$$= \left(C_4 h ||u||_{L^2(0, \tau; H^4(\Omega))} + \frac{C_5}{h^2} ||u - \tilde{u}||_{L^2(0, \tau; L^2(\Omega))}\right)^2$$
and hence
$$\left(\int_0^\tau ||\nabla u(\cdot, t) - \nabla \Pi \tilde{u}(\cdot, t)||_{L^\infty}^2 dt\right)^{1/2} \leq C\left(h ||u||_{L^2(0, \tau; H^4(\Omega))} + \frac{\delta}{h^2}\right). \quad \square$$

We define $\hat{\Pi} : L^2(0, \tau; L^2(\Omega)) \to L^2(0, \tau; L^2(\Omega))$ by
$$(\hat{\Pi} \psi)(t) := \Pi(\psi(t), \cdot) \quad \text{for} \ t \in [0, \tau].$$

With this definition of $\hat{\Pi}$ we have the following theorem.

**Theorem 8.2.** Let $\delta > 0$ be as considered in (8.1). Let $C_4$ and $C_5$ be as in (7.3) and Theorem 7.3 respectively. Let $C = \max \{C_4, C_5\}$ and $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$ and $C' = C\sqrt{|\Omega|}$. Then
$$||u - \hat{\Pi} \tilde{u}||_{L^2(0, \tau; L^2(\Omega))} \leq C'\left(h ||u||_{L^2(0, \tau; H^4(\Omega))} + \frac{\delta}{h^2}\right).$$
Proof. First we observe that
\[ \|u - \tilde{u}\|_{L^2(0,\tau;L^2(\Omega))} = \int_0^\tau \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^2(\Omega)}^2 dt \leq |\Omega| \int_0^\tau \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{W^{1,\infty}(\Omega)}^2 dt. \]

Now, following the proof of Theorem 8.1 we obtain
\[ \int_0^\tau \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{W^{1,\infty}(\Omega)}^2 dt \leq C^2 \left( h\|u\|_{L^2(0,\tau;H^4(\Omega))} + \frac{\delta}{h^2} \right)^2. \]

Thus, we have
\[ \|u - \tilde{u}\|_{L^2(0,\tau;L^2(\Omega))} \leq C' \left( h\|u\|_{L^2(0,\tau;H^4(\Omega))} + \frac{\delta}{h^2} \right). \]

By Theorem 8.1 and Theorem 8.2 we have
\[ \|u - \tilde{u}\|_{L^2(0,\tau;L^2(\Omega))} + \left( \int_0^\tau \|\nabla u(\cdot,t) - \nabla \tilde{u}(\cdot,t)\|_{L^\infty}^2 dt \right)^{1/2} \leq 2\tilde{C}\delta h, \]

where
\[ \tilde{C} = \max \left\{ C\|u\|_{L^2(0,\tau;L^2(\Omega))}, \ C'\|u\|_{L^2(0,\tau;L^2(\Omega))} \right\} \]

and
\[ \delta_h := \max \left\{ h, \frac{\delta}{h^2} \right\}. \]

Now, corresponding to the Assumption 4.1 on \( \tilde{u} \), we have the inequality
\[ \|u - \tilde{u}\|_{L^2(0,\tau;L^2(\Omega))} + \left( \int_0^\tau \|\nabla u(\cdot,t) - \nabla \tilde{u}(\cdot,t)\|_{L^\infty}^2 dt \right)^{1/2} \leq 2\tilde{C}\delta h, \]

for \( \tilde{u} \), where \( \tilde{C} \) is as in (8.2). Then carrying out the analysis as done in Section 4 and Section 5 with \( \tilde{u} \) and \( \delta \) replaced by \( \tilde{u} \) and \( \delta_h \), respectively, by Theorem 8.1 we obtain the following theorem.

**Theorem 8.3.** Let \( B_\alpha \) be the unique solution of (4.1) and \( \tilde{B}_{\alpha,n} \) be the unique solution of (5.4), with \( \tilde{u} \) replaced by \( \tilde{u} \). Let \( C_0 \) be as in Theorem 2.7. Let \( B \) be as in (4.1) and \( \tilde{A}_{\alpha,n} := \tilde{B}_{\alpha,n} + A_0 \). Let \( \tilde{\varepsilon}_n > 0 \) be such that \( \|\tilde{T}_{\tilde{u}} - T_{\tilde{u}} P_n\| \leq \tilde{\varepsilon}_n \) and \( \tilde{\delta}_h > 0 \) be as given in (8.3). Then
\[ \|A - \tilde{A}_{\alpha,n}\| \leq \|B - B_\alpha\| + C'' \frac{\delta_h + \tilde{\varepsilon}_n}{\sqrt{\alpha}} \]

where \( \|B - B_\alpha\| \to 0 \) as \( \alpha \to 0 \) and \( C'' \) is a positive constant depending only on the constants \( \|B\|, \|u\|_{L^2(0,\tau;L^2(\Omega))}, d, \tilde{C} \) and \( \sqrt{C_0} \).

**Remark 8.4.** Note that, if we fix the mesh size \( h \) first, and then choose the error level \( \delta \) in such a way that \( \delta \leq h^3 \), then \( \delta_h = h \) and hence the estimate in (8.4) becomes
\[ \|A - \tilde{A}_{\alpha,n}\| \leq \|B - B_\alpha\| + O\left( \frac{h + \tilde{\varepsilon}_n}{\sqrt{\alpha}} \right). \]

9. **Conclusion**

We have considered an inverse problem of identifying a coefficient \( A \in (H^1(\Omega))^{d \times d} \) of a parabolic PDE with Dirichlet boundary condition. Under specific assumption, a uniqueness result for the solution of inverse problem is obtained. By making use of a weak formulation, we have reduced our inverse problem into solving an ill-posed operator equation, where the operator involved is linear, and we have explicitly obtained a representation for the adjoint of the corresponding linear operator. In order to obtain stable approximations for the sought coefficient \( A \), we have used the theory of Tikhonov regularization. Also, we have given a finite dimensional realization of the method for practical implementation. For the parameter choice, we have used the adaptive technique to obtain the regularizing parameter effectively. Finally, we have demonstrated a procedure to obtain a smoothed version of a noisy data by making use of Clement operator. But, we would like to mention that for smoothing, we have assumed a higher regularity of the data, namely \( u \in L^2(0,\tau;H^4(\Omega)) \).
References

[1] R. Adams, Sobolev spaces, Pur Appl Math. 65, Academic press, New York, 1975.
[2] J.P. Aubin, Un théorême de compacté, C. R. Acad. Sci. 256 (1963), 5042-5044.
[3] S.C. Brenner and L.R. Scott, The mathematical theory of finite element methods, Third edition, Springer, New York, 2008.
[4] H. Cao and S.V. Pereverzev, Natural linearization for the identification of a diffusion coefficient in a quasi-linear parabolic system from short-time observations, Inverse Problems 22 (2006), 2311-2330.
[5] P. Clement, Approximation by finite element functions using local regularization, Rev. Franc. Automat. Inform. Rech. Operat. 9(1975), no. R2, 77-84.
[6] L.C. Evans, Partial differential equations, Second edition, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp. ISBN: 978-0-8218-4974-3 35-01.
[7] H.W. Engl, M. Hanke, and A. Neubauer, Regularization of inverse problems, Kluwer, Dordrecht, 1996.
[8] H.W. Engl and J. Zou, A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction, Inverse Problems 16 (2000), 1907-1923.
[9] S. George and M.T. Nair, A modified Newton-Lavrentiev regularization for nonlinear ill-posed Hammerstein-type operator equations, J. Complexity 24 (2008), no. 2, 228-240.
[10] M. Hanke and O. Scherzer, Error analysis of an equation error method for the identification of the diffusion coefficient in a quasi-linear parabolic differential equation, SIAM J. Appl. Math. 59(1999).
[11] S. Kesavan, Topics in functional analysis and applications, John Wiley Sons, Inc., New York, 1989. xii+267 pp. ISBN: 0-470-21050-8 46-01 (Revised edition).
[12] M.T. Nair, Functional Analysis: A First Course, PHI-Learning, New delhi, 2002 (Fourth print, 2014).
[13] M.T. Nair, Linear Operator Equations: Approximation and Regularization, World Scientific, Hackensack, 2009.
[14] M.T. Nair and S.D. Roy, A linear regularization method for a nonlinear parameter identification problem, J. Inverse Ill-Posed Probl. 25 (2017), no. 6, 687-701.
[15] S.V. Pereverzev and E. Schock, On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM J. Numer. Anal. 43 (2005), no. 5, 2060-2076.
[16] R. Serrano, An alternative proof of the Aubin-Lions lemma, Arch. Math. 101 (2013), 253-257.
[17] J. Simon, Compact sets in the space $L^p(0; T; B)$, Annali di Matematica pura ed applicata (1986) 146: 65.
[18] X.C. Tai and T. Kärkkäinen, Identification of a nonlinear parameter in a parabolic equation from a linear equation, Mat. Apl. Comput. 14 (1995), no. 2, 157-184.
[19] G. Yuan and M. Yamamoto, Lipschitz stability in the determination of the principal part of a parabolic equation, ESAIM Control Optim. Calc. Var. 15 (2009), no. 3, 525554.

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