Due to the discrete nature of electric charge, the current in mesoscopic conductors generally fluctuates. Over the last decade, there has been an increasing interest, theoretical as well as experimental, in the physics of current fluctuations. Most studies have been focused on noise, the second moment of the fluctuations, but recently a considerable interest has been shown for the full distribution of charge fluctuations, the full counting statistics (FCS). A variety of theoretical approaches to the FCS, ranging from quantum mechanical to classical, have been developed. The third moment of current fluctuations was very recently measured, opening up the road, as well, to experimental investigation of the higher moments of the fluctuations.

Noninteracting electrons in purely normal conductors are transferred one by one. In normal-superconducting junctions, the charge transfer mechanism across the normal-superconducting interface, at energies below the superconducting gap, is Andreev reflection. As a consequence, the FCS include terms describing correlated transfer of pairs of electrons. Recently, Belzig and Nazarov studied the FCS in superconducting junctions with a fixed phase difference between the superconducting electrodes. They found that the classical interpretation of the FCS, the probability to transfer a given number of electrons across the junction during the measurement, could imply negative probabilities. Coupling the junction to a detector, they showed that this resulted from an attempt to interpret the phenomena of supercurrent with classical means.

In voltage biased superconducting junctions, the physical situation is quite different. Due to the applied voltage bias, the superconducting phase difference oscillates with the Josephson frequency $2eV/h$, giving rise to both dc and ac-components of the current. For measurement times much longer than the inverse Josephson frequency, only the dc-current, which is dissipative, contributes to the net charge-transport. Microscopically, the charge is transported between the two superconductors via coherent multiple Andreev reflections (MAR). The current has been studied in various junctions, both theoretically and experimentally. Recently, also the noise was studied.

In this paper we present the FCS of charge transfer through a voltage biased superconducting junction, in terms of the amplitudes for quasiparticle scattering. Each quasiparticle scattering process results in an integer number of electron charges being transferred across the junction. As a consequence the FCS can be interpreted in classical terms. At temperatures much lower than the superconducting energy gap $\Delta$, many-quasiparticle scattering processes are exponentially suppressed, resulting in a simple probability distribution, containing only the probabilities for single quasiparticle scattering. This distribution reproduces known results for dc-current and zero-frequency noise. We discuss in detail the third cumulant for single channel junctions and diffusive junctions shorter than the superconducting coherence length.

We consider a superconducting junction consisting of two superconducting reservoirs connected via a normal, mesoscopic conductor (see Fig. 1). For simplicity of notation, we consider a junction with a single transport mode, the multi-mode generalization is discussed below. A voltage $V$ is applied between the two reservoirs.

The single-particle wavefunctions in the junction, solutions to the time dependent Bogoliubov-de Gennes equation, are scattering states labelled by the incoming quasiparticle type, injection energy and reservoir. The scattering states are superpositions of amplitudes for quasiparticles at energies $\pm neV$ from the injection energy, counted from the local chemical potential in each contact. The amplitude for an incoming quasiparticle of type $\alpha$ at energy $E_m$ to exit the junction as a quasiparticle of type $\beta$
at energy $E_{\alpha} = E_m + (n - m)eV$ is denoted $s_{nm}^{\alpha}$ (see Fig. 1). This amplitude is a function of the scattering matrix of the normal conductor and the Andreev reflection amplitudes $\lambda$.

To access the FCS, we make the first important observation that quasiparticle scattering in a voltage biased superconducting junction is formally identical to scattering in a normal voltage biased junction, with an applied harmonic ac-field. The FCS in such a system was investigated in detail by Ivanov and Levitov[4]. Following Ref. [4] we note that for measurement times much longer than the quasiparticle scattering time, the inelastic single mode scattering problem can be mapped onto an elastic scattering problem with many modes. The scattering between all energies and quasiparticle types of a “ladder” (see Fig. 1) is correlated, while different ladders contribute incoherently. We denote the ladder by its energy $E_0$ in the interval $-\Delta - 2eV \leq E_0 \leq -\Delta$ counting its “leg” in the left lead (e.g. $E_0$ in Fig. 1). Thus we may concentrate on the scattering matrix $S$, with elements $s_{nm}^{\alpha \beta}$, of a single ladder, and then integrate over the ladder energy $E_0$. In general $S$ has infinite dimensions, but in our case the vanishing probability of Andreev reflection far outside the gap naturally cuts the number of relevant modes to the order of $\Delta/eV$. Quasiparticle current (but not charge current) is conserved in the scattering processes at the normal-superconductor interfaces. As a consequence $S$ is unitary $\lambda$.

We are then, in line with [3, 18], able to directly write down the characteristic function in terms of all different many-particle scattering probabilities

$$\chi_{E_0}^{\beta \alpha}(\Lambda) = \sum_{i, o} e^{i \sum_{m \alpha} \lambda_{m \alpha} + \sum_{m \beta i} \lambda_{m \beta}} P_{i, o}$$

where $\Lambda$ is the set of counting fields $\lambda_{m \alpha}$, one for each mode $m \alpha$. The outer sum runs over all possible sets of incoming modes $i = \{m_1 \alpha_1, m_2 \alpha_2, \ldots\}$ and outgoing modes $o = \{n_1 \beta_1, n_2 \beta_2, \ldots\}$.

The many-particle scattering probabilities are given by

$$P_{i, o} = |s_{i, o}|^2 \prod_{m \in i} f(E_m) \prod_{m' \notin i} (1 - f(E_{m'}))$$

where $f(E) = (1 + \exp[E/kT])$ is the Fermi distribution function and $|s_{i, o}|^2$ is the determinant of the matrix formed by taking the columns $i$ and rows $o$ of $S$.

Eqs. 1 and 2 gives us the FCS for quasiparticle transfer in a voltage biased superconducting junction. The object of main interest is however the FCS of the charge transfer. We then make the second important observation that for measuring times much longer than the inverse Josephson frequency, the net charge transfer is directly related to the quasiparticle transfer. A quasiparticle of type $\alpha$ incident at energy $E_m$, which is scattered into an outgoing quasiparticle of type $\beta$ at energy $E_n$, transports exactly $m - n$ electrical charges across the junction (for $m < n$, the transported charge is thus negative for the bias in Fig. 1). This is independent of the type of quasiparticles $\alpha$ and $\beta$, and also of the charge of the quasiparticles. This can be shown by studying any quasiparticle scattering path (see Fig. 1), keeping in mind that the process of Andreev reflection transfers exactly two electrons across the normal-superconductor interface, while a normal transmission transfers exactly one electron.

This can also be seen from energy conservation: An electron traversing the normal part of the junction from left to right will absorb the energy quantum $eV$ from the electric field, while an electron moving in the opposite direction will emit the quantum $eV$. The effective number of quanta $eV$ absorbed in a quasiparticle scattering process scattered thus equals the number of electrons transferred from left to right. We emphasize that this approach correctly counts all the electrons transferred, including the electrons entering the superconductor as Cooper pairs at energies within the gap.

We can thus count the transferred electrons by counting the transferred quasiparticles weighted by the number of electrons transferred in each quasiparticle scattering event. Thus by choosing the counting fields $\lambda_{m \alpha} = n \lambda$ in Eq. 1 we can directly write down the characteristic function for charge transfer

$$\chi_{E_0}(\Lambda) = \sum_{i, o} e^{i \Lambda (\sum_{m \beta i} n m - \sum_{m \alpha i} m \alpha)} P_{i, o}$$

Following Ref. 3, the characteristic function can be written on a determinant form as

$$\chi_{E_0}(\Lambda) = \det [1 - \bar{f} + fS^{-1} S_{\Lambda}]$$

FIG. 1: Left: A schematic picture of the junction. Right: Multiple Andreev reflection processes in the voltage biased superconducting junction. An electron-like quasiparticle injected at an energy $E_0$ can undergo multiple Andreev reflections before being emitted as an $\alpha$-type (e or h) quasiparticle at an energy $E_\alpha = E_0 + neV$. These scattering processes have the amplitudes $s_{n0}^{\alpha}$, shown in the figure. Notethat all quasiparticles injected from the left at energies $E_0 + 2neV$ and from the right at $E_0 + (2n + 1)eV$ scatter on the same “ladder” in energy space.
where the elements of the matrix $S$ are given by $(S_i)_j^{\alpha\beta} = \sum_{m} s_{mn}^{\alpha\beta} e^{i\lambda(n-m)/2}$ and the diagonal matrix $f$ has elements $f_{m\alpha}^{\alpha\alpha} = f(E_m)$. This results provides a general solution to the temperature and voltage dependence of the full counting statistics of voltage biased superconducting junctions, for measurement times much longer than inverse Josephson frequency, and the quasiparticle scattering time, as defined by the energy dependence of the scattering matrix $\sim \hbar \partial_{E} \ln |s_{mn}^{\alpha\beta}|$, Eqs. 3 and 11 shows that the charge is transferred in correlated quanta of multiple electron charges.

At low temperatures $kT \ll \Delta$ all quasiparticle states below the gap are filled, while all states above the gap are empty. This simplifies Eq. 24 significantly since it fixes the set of incoming modes $i$ to all modes below the gap, but still all different sets of outgoing modes should be considered.

The MAR ladder forms a simple mode for transport in energy space 17. The scattering amplitudes $s_{nm}^{\beta\alpha}$ can be decomposed into amplitudes for entering the normal region, $t_{mn}^{++}$, propagation along the MAR ladder, $t_{mn}^{1/2}$, and leaving the normal region, $t_{mn}^{\alpha\alpha}$, as

$$s_{nm}^{\beta\alpha} = t_{mn}^{++} t_{mn}^{1/2} t_{mn}^{\alpha\alpha} \quad \text{and analogously for } E_n < E_m.$$ (5)

and analogously for $E_n < E_m$. Using this decomposition we find that choosing two or more outgoing modes above the gap in the set $i$ gives $P_{i\alpha} = 0$, since the matrix $s_i^{\alpha\alpha}$ then contains two or more parallel rows. In other words, the many-particle scattering process of two or more quasiparticles passing the gap through the single mode in energy space is prohibited by the Pauli exclusion principle. For the outgoing sets with all outgoing modes below the gap $|s_i^{\alpha\alpha}|$ is evaluated through making $s_i^{\alpha\alpha}$ unitary by inserting a row and a column with the amplitudes $t_{mn}^{++} t_{mn}^{1/2}$, where $E_0$ is the first energy above the gap in the ladder. Finally, the sets of one outgoing mode above the gap, using similar manipulations, give the single-particle scattering probabilities. The characteristic function can then be expressed in single-particle scattering probabilities

$$\chi_{E_0}(\lambda) = \sum_{m\leq0,n>0} e^{i(n-m)\lambda} \sum_{\alpha,\beta} |s_{nm}^{\alpha\beta}|^2.$$ (6)

This simple expression is the main result of this paper.

The cumulants of the charge distribution function are obtained by taking derivatives of $\ln \chi_{E_0}(\lambda)$, and summing over all ladders (integrating over the energy $E_0$),

$$\langle (n(\tau)^m) \rangle = \frac{\tau}{\hbar} \int_{-\Delta}^{-\Delta-2eV} dE_0 \langle -i\partial_{E} \rangle^m \ln \chi_{E_0}(\lambda)|_{\lambda=0},$$ (7)

where $\tau$ is the measurement time.

The integrands for the various moments are conveniently expressed in the physically relevant $n$-electron

![FIG. 3: The zero-frequency current noise (upper panel) and the third cumulant (lower panel) as a function of voltage for $D = 0.1$, 0.5, 0.9, 0.99 and 1. All curves are for a single mode junction at temperatures $kT \ll \Delta$.]

The integrands for the first three cumulants in Eq. 7 are the spectral current density $I(E) = \sum_{n=0}^{\infty} nP_n(E)$, the noise spectral density $S_I(E) = \sum_{n=0}^{\infty} n^2 P_n(E) - I(E)^2$, and the spectral density of the third cumulant $C_3(E) = \sum_{n=0}^{\infty} n^3 P_n(E) - I(E)(3S_I(E) + I(E)^2)$. The expression for the current is identical to the one presented in Ref. 17, and the expression for the zero-frequency
current noise reproduces the known results [14] (see the upper panel in Fig. 3).

The noise and the third cumulant, calculated numerically, are plotted in Fig. 4, for different transparencies of the normal contact. We see that the subgap structure is generally more pronounced in the third cumulant, compared to the current and noise. Furthermore, $C_3$ changes sign both as a function of voltage and transparency.

In the tunnel limit $D \ll 1$ where all scattering probabilities in Eq. (6) are small, the corresponding probability distribution is Poissonian, i.e., describes an uncorrelated transfer of quanta of multiple charge. Within the $n$-th subgap region, i.e., $n - 1 < 2\Delta/eV < n$, $n$-electron transfer dominates, giving $S = 2enI$ and $C_3 = (2en)^2I$, see the upper panel in Fig. 4. As seen in the right panel of Fig. 4 the picture is more complicated on the borders of the regions. Due to resonances up to three adjacent $n$-particle processes have similar strength [17], and the FCS is no longer Poissonian.

So far we considered only single mode junctions. In the limit of a short junction, when the scattering matrix of the normal conductor is independent on energy on the scale of $\Delta$, it is possible, just as for the current and noise [15], to write the generating function in terms of the transmission eigenvalues $D_n$ of the normal conductor only. For all mesoscopic conductors where the transmission eigenvalue distribution is known, the FCS can thus be obtained via averaging the single mode result in Eq. (6). As an example, we present in the lower panel of Fig. 4 the first three moments for a short, diffusive superconducting junction.

We note that the third moment is positive for the full counting statistics of a voltage biased superconducting junction, describing a correlated transfer of quanta of multiple electron charge. It is found that the counting statistics can be expressed in terms of probabilities for quasiparticles to scatter between the two superconductors.

In conclusion, we have derived an expression for the full counting statistics of a voltage biased superconducting junction, describing a correlated transfer of quanta of multiple electron charge. It is found that the counting statistics can be expressed in terms of probabilities for quasiparticles to scatter between the two superconductors.

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