Photon-added Barut-Girardello like coherent states of time-dependent Landau problem

Dedicated to the memory of Lev Davidsovich Landau

Latévi M. Lawson\textsuperscript{a,b}, Komi Sodoga\textsuperscript{b} and Gabriel Y. H. Avossevou\textsuperscript{c}

\textsuperscript{a}African Institute for Mathematical Sciences (AIMS) Ghana, Summerhill Estates, East Legon Hills, P.O. Box LG DTD 20046, Legon, Accra, Ghana

\textsuperscript{b}Laboratoire de Physique des Matériaux et des Composants à Semi-Conducteurs, 02 BP 1515 Lomé, Togo

\textsuperscript{c}Institut de Mathématiques et de Sciences Physiques (IMSP), Laboratoire de Recherche en Physique Théorique (LRPT) 01 BP 613 Porto-Novo, Rep. du Bénin

latevi@aims.edu.gh\textsuperscript{a}; ksodoga@univ-lome.tg\textsuperscript{b} and gabriel.avossevou@imsp-uac.org\textsuperscript{c}

January 22, 2022

Abstract

Recently, we have determined the spectrum and the wave functions of the Hamiltonian of a Landau particle with time-dependent mass and frequency undergoing the influence of a uniform time-dependent electric field [J. Math. Phys. 56, 072104 (2018)]. In the present paper we extend the study of this model that we name the time-dependent Landau problem into the context of coherent states. By means of the traditional factorization method of the eigenfunctions of this system expressed in terms of the generalized Laguerre polynomials, we derive the generators of the $su(1,1)$ Lie algebra and we construct the coherent states à la Barut-Girardello. These states are shown to satisfy the Klauder’s mathematical requirement to build coherent states and some of their statistical properties are calculated and analyzed. We find that these states are sub-Poissonian in nature. We show that, addition of photons from these coherent states, increases the statistical properties and changes the mathematical properties of these states.
1 Introduction

The study of coherent states has remained over the past four decades a constant source of fascination, inspiration and innovation in different branches of physics. In particular, they have found considerable applications in quantum optics \cite{2,3} and quantum information \cite{3,5}. They were first discovered in connection with the quantum harmonic oscillator by Schrödinger in 1926, who referred to them as states of minimum uncertainty product \cite{6}. The importance of coherent states was put forward by Glauber in the framework of quantum optics in early 1960’s \cite{7,8}. According to Glauber, these states can globally be constructed using any of the following three procedures: (i) annihilation operator eigenstates (ii) displacement operator technique (iii) minimum uncertainty states. However, these three approaches are generally nonequivalent and only in the case of standard harmonic oscillator coherent states obtained using any of the three approaches are equivalent. The same states were also reintroduced by Klauder who investigated their mathematical properties \cite{9,10}. He has noted that, these states must satisfy the following minimum conditions: (iv) normalizability, (v) continuity in the label and (vi) existence of a resolution of unity with a positive definite weight function. Nine years later, these coherent states introduced by Glauber have inspired respectively Barut-Girardello \cite{11} and Perelomov \cite{12} in constructing the coherent states for $su(1,1)$ Lie algebraic group basing on the procedures (i) and (ii) respectively. These states also satisfy the Klauder’s minimum conditions (iv,v,vi) and have interesting applications in quantum optics, quantum computation and quantum mechanics \cite{13,14,15,16,17}.

On the other hand, the addition of photons from the coherent states previously introduced by Glauber were objects of extensive studies both in experimental \cite{18,19,20} and theoretical \cite{21,22,23} frameworks. These states originally introduced by Agarwal and Tara \cite{24,25} have received much attention from quantum optics and information \cite{26,27,28} for their statistical properties such as the photon-number distribution, the Mandel parameter, the squeezing parameter, the Wigner function etc. Various generalizations of these states taking into account their statistical properties were also performed \cite{29,30,31,32}. Recently, one of us addressed conjonctly the study of photon added states into the generalized associated hypergeometric coherent states \cite{33} and into a full characterization of shape invariant potentials using algebraic approach based on the supersymmetric quantum mechanics \cite{34}. The original construction of photon added coherent states (PACSs) \cite{24,25} obtained from the conventional coherent states based on the Weyl-Heisenberg group has been extended to a number of Lie groups with square integrable representations. In this sens, Popov constructed and analytically discussed the statistical properties of the photon-added Barut-Giraldeillo coherent states (PABGCs) for the pseudoharmonic oscillator \cite{35}. Daoud extended this construction for exactly solvable Hamiltonian on Gazeau-Klauder and Klauder-Perelomov coherent states \cite{36}. Recently, the mathematical and statistical properties of PACSs are studied for the $SU(2)$ coherent states \cite{22,37}. We extend these results into our recently model studied in \cite{1} namely the time-dependent Landau problem (TDLP). To do so, we organise the paper as follows.

In section 2 we outline the fundamental aspects of the quantization of the TDLP based on the Lewis-Riesenfeld procedure \cite{14,38}. We recall the spectra and the eigenfunctions of the different operators that describe the system. In
section 3, we derive from the solution of the system the hidden generators of the $su(1,1)$ Lie algebra. We proceed by the factorization method as developed in [14, 40, 41] to find the hidden symmetry of the system and derive from the eigenfunctions the related raising and lowering operators which generate the $su(1,1)$ Lie algebra. Section 4 is devoted to the construction of the $SU(1,1)$ coherent states à la Barut-Girardello, the study of their mathematical properties (the non-orthogonality, the continuity, the identity resolution) and to their statistical properties (the photon mean number, the photon distribution, the intensity correlation, the Mandel parameter and the Wigner distribution functions). The section 5 presents in detail the construction of photon added Barut-Girardello like coherent states (PABGLCSs). We then check the effect of adding photons on the above mathematical and statistical properties. Finally, we conclude the paper in section 6.

2 Review of the model

The coming January 22, 2021, the world scientific will celebrate the 113th birth anniversary of Soviet physicist Lev Landau. During the 20th century, Lev Landau made some of the most significant discoveries in physics. In quantum mechanics, he is known for the problem of quantization of the cyclotron orbits of charged particles in constant magnetic field. As a result, the charged particles can only occupy orbits with discrete energy values, called Landau levels. The Hamiltonian of this system is

$$H = \frac{1}{2M} \left[ \vec{p} - q \vec{A}(\vec{x}) \right]^2. \tag{1}$$

where $\vec{p}$ is the canonical momentum operator, $\vec{A}$ is the electromagnetic vector potential, which is related to the magnetic field $\vec{B}$ by $A_i(x_i) = -\frac{1}{2}B\epsilon_{ij}x_j$ (with $i,j = 1,2$) and $M$ is the constant mass of the particle. Recently, we have extended the Hamiltonian of this system (1) to the case of time-dependent mass $M(t)$ and harmonic frequency $\omega(t)$ under the influence of a uniform time-dependent electric field $E(t)$ [1]. This system is described by the Hamiltonian

$$H(t) = \frac{1}{2M(t)} \left[ \vec{p} - q \vec{A}(\vec{x}) \right]^2 + \frac{1}{2}M(t)\omega^2(t)\vec{x}^2 + q\varphi(t). \tag{2}$$

In the symmetric gauge, the vector potential and the time-dependent scalar potential are given by $A_i(x_i) = B\epsilon_{ij}x_j$ and $\varphi(x_i, t) = E_i(t)x_i$, respectively (with $i,j = 1,2$). The Hamiltonian of the system is rewritten as follows

$$H(t) = \frac{1}{2M(t)} \left[ p_1 + \frac{q}{2} Bx_2 \right]^2 + \frac{1}{2M(t)} \left[ p_2 - \frac{q}{2} Bx_1 \right]^2 + \frac{1}{2}M(t)\omega^2(t)(x_1^2 + x_2^2) - q[E_1(t)x_1 + E_2(t)x_2]. \tag{3}$$

Considering the following changes of variables

$$x = x_1 + \frac{qE_1(t)}{M(t)\omega^2(t)}, \quad y = x_2 + \frac{qE_2(t)}{M(t)\omega^2(t)},$$

$$p_x = p_1 - \frac{q^2BE_2(t)}{2M(t)\omega^2(t)}, \quad p_y = p_2 - \frac{q^2BE_1(t)}{2M(t)\omega^2(t)}. \tag{4}$$
and get the transformed Hamiltonian in its simple form rewritten as follows

\[ H(t) = \frac{1}{2M(t)}(p_x^2 + p_y^2) + \frac{\Omega^2(t)M(t)}{2}(x^2 + y^2) - \frac{\omega_r(t)}{2}L_z - \frac{q^2E^2(t)}{2M(t)\omega_r(t)}. \]  

(6)

where \( L_z = xp_y - yp_x \) is the angular momentum, \( \omega_r(t) = \frac{qB}{\mu(t)} \) is the cyclotron frequency of oscillation, \( \Omega(t) = \sqrt{\omega^2(t) + \frac{1}{2}\omega_r^2(t)} \) is the general frequency of oscillation, \( E(t) = \sqrt{E_1^2(t) + E_2^2(t)} \) and \( \hbar = 1 \).

Since the Hamiltonian involves time-dependent parameters, we used the so-called Lewis and Riesenfeld method \[14\] to construct a Hermitian operator \( I(t) \) as follows \[1\]

\[ I(t) = \frac{1}{2} \left[ \frac{\kappa^2}{\rho^2}x^2 + \frac{\kappa^2}{\rho^2}y^2 + (\rho x - M \dot{\rho} x)^2 + (\rho y - M \dot{\rho} y)^2 \right]. \]  

(7)

where \( \kappa \) is a constant and the function \( \rho \) is the solution of the so-called nonlinear modified Ermakov-Pinney equation \[42\] which is

\[ \dot{\rho} + \frac{M}{\rho} \dot{\rho} + \Omega^2(t) \rho = \frac{\kappa^2}{M^2 \rho^3}. \]  

(8)

The eigensystems of the invariant are given as follows

\[ \langle \phi_n^{|\ell|}(t)|I(t)|\phi_m^{|\ell|}(t) \rangle = \kappa(2n + |\ell| + 1), \]  

(9)

\[ \phi_n^{|\ell|}(r, \theta, t) = (-)^n (\frac{\kappa}{\rho^{1+|\ell|}})^{\frac{1+|\ell|}{2}} \sqrt{\frac{n!}{\Gamma(n + |\ell| + 1)}} x^{|\ell|} \times \left[ e^{(iM \frac{2}{\rho^2 r^2})} \frac{1}{\pi} L_n^{|\ell|} \left( \frac{\kappa}{\rho^2 r^2} \right) e^{i\theta} \right], \]  

(10)

where \( \ell = n_+ - n_- \), \( n = \min(n_+, n_-) = \frac{1}{2}(n_+ + n_- - |\ell|) \), \( \Gamma(u) \) Euler’s gamma function and \( L_n^{|\ell|}(u) \) are the generalised Laguerre polynomials, while, \( r = \sqrt{x^2 + y^2}, \ \ e^{i\theta} = \frac{x + iy}{\sqrt{x^2 + y^2}}. \)  

(11)

The solution of the Schrödinger equation is obtained by taking the product of the eigenfunction of the invariant operator and the exponential of the complex of the phase function such as

\[ \psi_n^{|\ell|}(r, \theta, t) = (-)^n (\frac{\kappa}{\rho^{1+|\ell|}})^{\frac{1+|\ell|}{2}} \sqrt{\frac{n!}{\Gamma(n + |\ell| + 1)}} x^{|\ell|} \times \left[ e^{(iM \frac{2}{\rho^2 r^2})} \frac{1}{\pi} L_n^{|\ell|} \left( \frac{\kappa}{\rho^2 r^2} \right) e^{i\theta} e^{\gamma_n^{|\ell|}(t)} \right], \]  

(12)

where the phase factor is given by

\[ \gamma_n^{|\ell|}(t) = -\frac{\kappa}{2}(2n + |\ell| + 1) \int_0^t \frac{dt'}{M(t')\rho^2(t')} - \frac{|\ell|}{2} \int_0^t dt' \omega_c(t') + \frac{q^2}{2} \int_0^t \frac{E^2(t')}{M(t')\omega(t')} dt'. \]  

(13)
The Hamiltonian’s expectation values are given by the expression
\[ \langle \phi_n^\ell(t)|H(t)|\phi_n^\ell(t) \rangle = \frac{1}{2\kappa} \left( M\rho^2 + \kappa^2 + M\Omega^2\rho^2 \right) (2n + |\ell| + 1) - \frac{|\ell|}{2} \omega_0 - \frac{\tilde{q}^2 E^2}{2M\omega}. \]

Since the eigenfunction of the system (11) is expressed in terms of the generalized Laguerre functions, it is possible to derive the hidden generators of the \(su(1, 1)\) Lie algebra through factorisation of this eigenfunction. In what follows, we firstly construct the raising and lowering operators from the Hamiltonian’s eigenfunction which generates the \(SU(1, 1)\) hidden Lie group. Secondly, we construct the the Barut-Girardello Coherent states relative to this system. Finally, we introduce the associated PACSs and study their properties.

3 \(su(1,1)\) Lie algebraic treatment

Before constructing the \(SU(1,1)\) coherent states to the system, it is important to review some useful properties to the associated Laguerre polynomials in order to derive from the wavefunctions (9) and (12) the \(su(1,1)\) Lie algebraic treatment of the system.

The generalized Laguerre polynomials \(L_n^\ell(u)\) with \(\ell > 0\) are defined as
\[ L_n^\ell(u) = \frac{1}{n!} e^{u}u^{-\frac{\ell}{2}} \frac{d^n}{du^n} (e^{-u}u^{n+\frac{\ell}{2}}) = \sum_{k=0}^{n} (-)^{k} \binom{n + \ell}{n - k} u^k k!, \quad (14) \]
For \(\ell = 0\), \(L_n^0(u) = L_n(u)\) and for \(n = 0\), \(L_0^0(u) = 1\). The generating function corresponding to the associated Laguerre polynomials is
\[ J_\ell \left( 2\sqrt{uz} \right) e^{u}u^{-\frac{\ell}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + \ell + 1)} L_n^\ell(u), \quad (15) \]
where the \(J_\ell(x)\) is the ordinary Bessel function of \(\kappa\)-order.

The orthogonality relation is
\[ \int_{0}^{\infty} du e^{-u}u^\frac{\ell}{2} L_n^\ell(u) L_m^\ell(u) = \frac{\Gamma(\ell + n + 1)}{n!} \delta_{nm}, \quad (16) \]
The generalized Laguerre polynomials satisfy the following differential equation
\[ \left[ u \frac{d^2}{du^2} + (\ell - u + 1) \frac{d}{du} + n \right] L_n^\ell(u) = 0, \quad (17) \]
and the recurrence relations
\[ (n + 1)L_n^{\ell+1}(u) - (2n + \ell + 1 - u) L_n^\ell(u) + (n + \ell) L_{n-1}^\ell(u) = 0, \quad (18) \]
\[ u \frac{d}{du} L_n^\ell(u) - n L_n^\ell(u) + (n + \ell) L_{n-1}^\ell(u) = 0. \quad (19) \]

With respect to the above equations, we rewrite the eigenfunction of the invariant operator in equation (9) in the form
\[ \phi_n^\ell(u, t) = N_n(\rho, \theta) \sqrt{\frac{n!}{\Gamma(n + \ell + 1)}} u^\frac{\ell}{2} e^{-\frac{\tilde{q}}{2} u} L_n^\ell(u), \quad (20) \]
where \( u = \frac{\rho^2}{r^2} \), \( N_n(\rho, \theta) = (-)^n \sqrt{\frac{\rho}{\pi \rho}} e^{i\ell \theta} \), \( \varpi = 1 - iM(t)\frac{\partial}{\partial \rho} \) and \( \Gamma(n) = (n-1)! \).

Based on the recurrence relations (18) and (19), we obtain the following equations
\[
\left( -u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\varpi}{2} u \right) \phi_n^\ell(u, t) = \sqrt{n(n + \ell)} \phi_{n-1}^\ell(u, t),
\]
\[
\left( u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\varpi}{2} u + 1 \right) \phi_n^\ell(u, t) = \sqrt{(n + 1)(n + \ell + 1)} \phi_{n+1}^\ell(u, t),
\]
where \( \tilde{\varpi} = 2 - \varpi \). For the sake of simplicity we define the raising operator \( K_- \) and the lowering operator \( K_+ \) acting on the wave function \( \phi_n^\ell(u, t) \) as
\[
K_- = \left( -u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\varpi}{2} u \right),
\]
\[
K_+ = \left( u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\varpi}{2} u + 1 \right),
\]
and hence obtain
\[
K_- \phi_n^\ell(u, t) = \sqrt{n(n + \ell)} \phi_{n-1}^\ell(u, t),
\]
\[
K_+ \phi_n^\ell(u, t) = \sqrt{(n + 1)(n + \ell + 1)} \phi_{n+1}^\ell(u, t).
\]

By multiplying both side of the latter equations by the factor \( e^{i\gamma_n^\ell(t)} \) we obtain
\[
K_- \psi_n^\ell(u, t) = \sqrt{n(n + \ell)} \psi_{n-1}^\ell(u, t),
\]
\[
K_+ \psi_n^\ell(u, t) = \sqrt{(n + 1)(n + \ell + 1)} \psi_{n+1}^\ell(u, t).
\]

By successively applying \( K_+ \) on the ground state \( \psi_0^\ell(u) \), we generate the eigenfunction \( \psi_n^\ell(u, t) \) of the system as follows
\[
\psi_n^\ell(u, t) = \left( \frac{\Gamma(1 + \ell)}{n\Gamma(n + \ell + 1)} \right) (K_+)^n \psi_0^\ell(u, t),
\]
\[
\psi_0^\ell(u, t) = \frac{N(\rho, \theta)}{\sqrt{\Gamma(\ell + 1)}} u^\ell e^{-\varpi u} e^{i\gamma_0^\ell(t)},
\]
\[
K_- \psi_0^\ell(u, t) = 0.
\]

One can also observe that the following relations are satisfied
\[
K_+ K_- \psi_n^\ell(u, t) = n(n + \ell) \psi_n^\ell(u, t),
\]
\[
K_- K_+ \psi_n^\ell(u, t) = (n + 1)(n + \ell + 1) \psi_n^\ell(u, t).
\]

Now, to establish the dynamical Lie algebra associated with the ladder operators \( K_\pm \), we calculate the commutator
\[
[K_-, K_+] \psi_n^\ell(u, t) = (2n + \ell + 1) \psi_n^\ell(u, t).
\]
As a consequence, we can introduce the operator $K_0$ defined to satisfy

$$K_0 \psi^{\ell}_n(u,t) = \frac{1}{2} (2n + \ell + 1) \psi^{\ell}_n(u,t).$$  \hspace{1cm} (36)

The operators $K_\pm$ and $K_0$ satisfy the following commutation relations

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm,$$  \hspace{1cm} (37)

which can be recognized as commutation relation of the generators of a non-compact and non-abelian $SU(1,1)$ Lie group. The corresponding Casimir operator for any irreducible representation is the identity times a number

$$K^2 = K_0^2 - \frac{1}{2} (K_+ K_- + K_- K_+) = \frac{1}{4} (\ell + 1)(\ell - 1).$$  \hspace{1cm} (38)

It satisfies

$$[K^2, K_\pm] = 0 = [K^2, K_0].$$  \hspace{1cm} (39)

If we make the following connection between the physical quantum numbers $(n,\ell)$ and the ordinary $su(1,1)$ group numbers $(n,k)$ such as

$$\ell = 2k - 1,$$  \hspace{1cm} (40)

then we recover the ordinary discrete representations of the $su(1,1)$ Lie algebra

$$K^2 \psi^k_n(u) = k(k-1) \psi^k_n(u),$$  \hspace{1cm} (41)

$$K_- \psi^k_n(u) = \sqrt{n(n+2k-1)} \psi^{k-1}_n(u),$$  \hspace{1cm} (42)

$$K_+ \psi^k_n(u) = \sqrt{(n+1)(n+2k)} \psi^{k+1}_n(u),$$  \hspace{1cm} (43)

$$K_0 \psi^k_n(u) = (n+k) \psi^k_n(u).$$  \hspace{1cm} (44)

Thus, in what follows we use the Bargmann index $\ell$ instead of the ordinary index $k$ in the representation of $su(1,1)$ algebra. Now, with the properties of the generators $K_\pm$ and $K_0$ of this algebra, we are in the position to construct the corresponding coherent states to this system.

### 4 Barut-Girardello like coherent states

#### 4.1 Contraction

Following the Barut and Girardello approach [11], $SU(1,1)$ coherent states are defined to be the eigenstates of the lowering generator $K_-$

$$K_- |\psi^\ell_z(t)\rangle = z |\psi^\ell_z(t)\rangle,$$  \hspace{1cm} (45)

where $z$ is an arbitrary complex number. The normalized Barut-Girardello states can be decomposed over the number-state basis $|\psi^\ell_n(t)\rangle$ as follows

$$|\psi^\ell_z(t)\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n! \Gamma(n+\ell+1)}} |\psi^\ell_n(t)\rangle,$$  \hspace{1cm} (46)

$$\psi^\ell_z(u,t) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\Gamma(n+\ell+1)}} u^\frac{n}{2} e^{-\frac{\omega}{2} u} L_n^\ell(u) e^{i\gamma_n(t)}.$$  \hspace{1cm} (47)
However, in term of the generating function (15), the Barut-Girardello coherent states can be written as follows

$$\psi_{\ell}^{z}(u, t) = \left(\frac{z}{|z|}\right)^{-\frac{1}{2}} N_{\alpha}(\rho, \alpha) e^{z - \frac{\alpha}{2} u} J_{\ell} \left(2\sqrt{|z|}\right) e^{i\gamma_{\ell}(t)}.$$  \hspace{1cm} (48)

As it is seen from Eq.(46), the $$|\psi_{\ell}^{z}(t)\rangle$$ is a linear combination of number states $$|\psi_{n}^{z}(t)\rangle$$. Therefore, the Barut-Girardello like coherent states belong to Hilbert space $$\mathcal{H}_{\ell}$$ indexed by the single real positive number $$\ell$$ which defines the representation of this space

$$\mathcal{H}_{\ell} := \text{span}\{|\psi_{\ell}^{n}(t)\rangle\}_{n \in \mathbb{N}}.$$ \hspace{1cm} (49)

\section{4.2 The mathematical properties}

In the following discussion we will consider various properties of these states including the non-orthogonality, the usual conditions of continuity in the label, normalizability, the resolution of identity by finding the weight function $$\omega^{\ell}$$.

\subsection{4.2.1 The non-orthogonality}

We can see that the scalar product of two coherent states does not vanish

$$\langle \psi_{\ell}^{z_{1}}(t) | \psi_{\ell}^{z_{2}}(t) \rangle = \frac{I_{\ell}(2\sqrt{z_{1}z_{2}})}{I_{\ell}(2|z_{1}|)I_{\ell}(2|z_{2}|)}.$$ \hspace{1cm} (50)

In the case $$z_{1} = z_{2} = z$$, we obtain the normalization

$$\langle \psi_{\ell}^{z}(t) | \psi_{\ell}^{z}(t) \rangle = 1.$$ \hspace{1cm} (51)

Thus, as it is well-known, the states (46) are normalized but are not mutually orthogonal.

\subsection{4.2.2 The Label continuity}

The continuity in label $$z$$ can then be stated as

$$||\psi_{\ell}^{z}(t) - \psi_{\ell}^{z'}(t)||^2 = 2 \left[1 - \Re\left(\langle \psi_{\ell}^{z}(t) | \psi_{\ell}^{z'}(t) \rangle\right)\right] \rightarrow 0,$$

when $$|z - z'|^2 \rightarrow 0.$$ \hspace{1cm} (52)

\subsection{4.2.3 Resolution of unity}

The overcompleteness relation reads as follows

$$\int d\mu(z, \ell) |\psi_{\ell}^{z}(t)\rangle \langle \psi_{\ell}^{z}(t)| = \sum_{n=0}^{\infty} |\psi_{\ell}^{n}(t)\rangle \langle \psi_{\ell}^{n}(t)| = 1_{\ell},$$ \hspace{1cm} (53)

with the measure

$$d\mu(z, \ell) = \frac{2}{\pi} K_{\ell}(2|z|)I_{\ell}(2|z|)d^{2}z.$$ \hspace{1cm} (54)
where $d^2z = d(Rez)d(Imz)$ and $K_v(x)$ is the $v$-order modified Bessel function of the second kind. The weight-function of these BGLCSs are then given by

$$\omega^\ell(|z|) = \frac{2}{\pi} K_\ell(2|z|) I_\ell(2|z|) \quad \ell = 0.5, 1, 1.5, \ldots$$  \hspace{1cm} (55)

The resolution of this identity is easy to demonstrate by using the following integral \cite{43}

$$\int_0^\infty dx x^\mu K_\nu(ax) = 2^{\mu - 1} a^{-\mu - 1} \Gamma \left( \frac{1 + \mu + \nu}{2} \right) \Gamma \left( \frac{1 + \mu - \nu}{2} \right),$$  \hspace{1cm} (56)

where $[\text{Re} (\mu + 1 \pm \nu) > 0, \text{Re} (a) > 0]$ and all the integrals are performed over the whole complex $z$ plane, where $z = re^{i\varphi}, \quad r \in [0, \infty[, \quad \varphi \in [0, 2\pi]$. The asymptotic expression of the weight function $\omega^\ell(|z|)$ for $|z| \gg 1$ is,

$$\omega^\ell(|z|) \simeq \frac{1}{2\pi |z|} \left( 1 + \frac{1}{|z|} \left( \frac{\ell^2}{2} - \frac{1}{8} \right) \right)$$  \hspace{1cm} (57)

as the behaviours of the Bessels functions for $|z| \gg 1$ are

$$K_\mu (x) \simeq e^{-x} \sqrt{\frac{\pi}{2x}} \left( 1 + \frac{1}{x} \left( \frac{\mu^2}{2} - \frac{1}{8} \right) \right)$$

$$I_\mu (x) \simeq \frac{e^x}{\sqrt{2\pi x}}.$$

Since the measure in equation (54) must be necessary positive, the function $\omega^\ell(|z|)$ must be positive. This is confirmed in Fig.1 where we represent the weight functions (55) for different values of the Bargmann index $\ell$, $\{\ell = 0.5, 1, 1.5, 2\}$. We observe that the weight function globally decreases and tends to 0 when $|z|$ increases, as confirmed by the asymptotic expressions (57). We can see also that the weight function decreases while the Bargmann index $\ell$ increases.

![Figure 1: Plots of the weight function (54) of the BGLCS (46) versus $r = |z|$ for different values of the Bargmann index $\ell$.](image-url)
4.3 The statistical properties

After mathematical construction of the BGLCSs, in the present subsection, we investigate some of the quantum statistical properties of these states, such as the photon-number distribution, the mean number of photons, the intensity correlation function, the Mandel parameter and the Wigner function.

4.3.1 The photon-number distribution

The probability of finding the $n^{th}$ photons in the states $|\psi_{\ell}^z(t)\rangle$ is given by

$$P_n(\ell, |z|) = \frac{|\langle \psi_{\ell}^z(t) | \psi_{\ell}^z(t) \rangle|^2}{I_\ell(2|z|n!\Gamma(n + \ell + 1)).}$$

The complexity of the above equation makes it difficult to predict analytically the statistical nature of these states. Therefore, for two limiting cases of the variable $|z|$ ($|z| \ll 1$ and $|z| \gg 1$), the modified Bessel function $I_\mu(x)$ is respectively approximated as follows [38]

$$I_\mu(x) \simeq \frac{1}{\Gamma(\mu + 1)} \left(\frac{x}{2}\right)^\mu$$

and

$$I_\mu(x) \simeq e^x \sqrt{\frac{1}{2\pi x}} \left[1 + O\left(\frac{1}{x}\right)\right].$$

Using these relations, the asymptotical expressions of the photon-number distribution of the BGLCSs (46) are

$$\lim_{|z| \to 0} P_n(\ell, |z|) = \frac{|z|^{2n}}{n! \Gamma(n + \ell + 1)};$$

and

$$\lim_{|z| \to +\infty} P_n(\ell, |z|) = 0.$$

So, for small values of $|z|$ this distribution is smaller than unity. The corresponding BGLCSs have sub-Poissonian statistics, while for large $|z|$, the probabilities $P_n(\ell, |z|)$ is zero. In Figure 2 we plot the PND as a function of the photon number $n$ for different parameters: (a) fixed Bargmann index $\ell = 1.5$ and different values of $|z|^2 = \{6, 9\}$; (b) fixed value of $|z|^2 = 9$ and different values of the Bargmann index $\ell = \{3.5, 5\}$. We see in Figure (a) that as $z$ increases, the peaks decrease and shift to the right. Figure (b) shows that the shift in the PND is less accentuated as the Bargmann index $\ell$ increases.

4.3.2 The intensity correlation function, the Mandel parameter and the Wigner function

The intensity correlation function or equivalently the Mandel Q-parameter yields the information about photon statistics of the quantum states. The intensity correlation function of the BGLCSs (46) is defined by

$$g^{(2)}_{\ell z} = \frac{\langle N^2 \rangle^\ell - \langle N \rangle^\ell}{\langle (N)^2 \rangle^\ell}.$$

where $N$ is the number operator which is defined as the operator which diagonalizes the basis for the number states :

$$N|\psi_{\ell}^z(t)\rangle = n|\psi_{\ell}^z(t)\rangle.$$
Figure 2: Plots of the PND \((58)\) of the BGLCSs \((46)\) versus the photon number \(n\), with parameters: (a) fixed value of \(\ell = 1.5\) with different values of the amplitude \(|z|^2 = 6\) (dot points) and \(|z|^2 = 9\) (scattered points); (b) fixed value of the amplitude \(|z|^2 = 9\) with different values of the Bargmann index \(\ell = 3.5\) (dot points) and \(\ell = 5\) (scattered points).

The Mandel Q-parameter is related to the intensity correlation function by

\[
Q_\ell^z = \frac{\langle N \rangle_\ell^z (g_\ell^z(2) - 1)}{g_\ell^z(2)}.
\]

The intensity correlation function (or the Mandel Q-parameter) determines whether the BGLCSs have a photon number distribution. This latter is sub-Poissonian if \(g^2 < 1\) (or \(-1 \leq Q < 0\)), Poissonian if \(g^2 = 1\) (or \(Q = 0\)), and super-Poissonian if \(g^2 > 1\) (or \(Q > 0\)).

We check that, for BGLCSs \((46)\), the expectation values of \(N\) and \(N^2\) can be computed as \((64, 65)\):

\[
\langle N \rangle_\ell^z = \langle \psi_\ell^z(t)|N|\psi_\ell^z(t) \rangle = |z| \frac{I_{\ell+1}(2|z|)}{I_\ell(2|z|)} \langle \ell \rangle,
\]

\[
\langle N^2 \rangle_\ell^z = \langle \psi_\ell^z(t)|N^2|\psi_\ell^z(t) \rangle = |z|^2 \frac{I_{\ell+1}(2|z|)}{I_\ell(2|z|)} + |z|^2 \frac{I_{\ell+2}(2|z|)}{I_\ell(2|z|)}.
\]

Taking into account the results \((64, 65)\) of the expectation values of the number operator and its square, we obtain

\[
g_\ell^z(2) = \frac{I_{\ell+2}(2|z|)I_\ell(2|z|)}{[I_{\ell+1}(2|z|)]^2}.
\]

Using the above approximation conditions \((59)\) in case of \(|z| \ll 1\) and \(|z| \gg 1\), the intensity correlation function are given by

\[
g_\ell^z(2) \approx \frac{\ell + 1}{\ell + 2} \quad \text{for} \quad |z| \ll 1,
\]

\[
g_\ell^z(2) \approx 1 \quad \text{for} \quad |z| \gg 1.
\]

For all \(\ell\) values, the BGLCSs have sub-Poissonian statistics for small values of \(|z|\), while for large \(|z|\), these states tend to have Poissonian statistics.
The Mandel parameter $Q_z^\ell$ is given by

$$Q_z^\ell = \frac{|z|^{I_{\ell+2}(2|z|)I_\ell(2|z|) - I_{\ell+1}(2|z|)^2}}{I_{\ell+1}(2|z|)^2},$$

$$Q_z^\ell \approx -\frac{|z|^2}{(\ell + 1)(\ell + 2)} \text{ for } |z| \ll 1.$$  \hspace{1cm} (69, 70)

In Figure 3, the intensity correlation function and the Mandel Q-parameter have been plotted in terms of the amplitude $|z|$, in (a) and (b), respectively for different values of the Bargmann index $\ell = \{0.5, 1, 1.5, 2\}$. We can see that the Mandel Q-parameter is always negative and the intensity correlation function $g_2^{(2)} < 1$ that confirms the analytical forms \([69, 70]\) and showing that the BGLCSs \([46]\) have sub-Poissonian statistics.

To analyze the behavior on phase space $(u, p) \in \mathbb{R}^2$ of the Barut-Girardello like coherent states associated with this system, we use the Wigner quasiprobability distribution function \([44]\)

$$W_\ell^u(u, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-i\pi \frac{p}{\hbar}} \psi_\ell^*(u - \frac{v}{2}, t) \psi_\ell(u + \frac{v}{2}, t) dv,$$

$$= \frac{|z|^{I_{\ell+1}(2|z|)^2} \sum_{n, n' = 0}^{\infty}}{2\pi \hbar I_\ell(2|z|)^2} \sqrt{n!n'!} \Gamma(n + \ell + 1) \Gamma(n' + \ell + 1)$$

$$\times \int_{-\infty}^{+\infty} e^{-i\pi \frac{p}{\hbar}} \psi_{n'}^*(u - \frac{v}{2}, t) \psi_n(u + \frac{v}{2}, t) dv.$$  \hspace{1cm} (71)

In order to compute the integral of the wave functions $\psi_\ell^u(u, t)$ in (71) it is convenient to use together the following change of variables $r = e^{-\pi \frac{u}{\hbar}}$ and $\xi = e^{-\pi \frac{p}{\hbar}}$, so

$$W_\ell^u(u, p, t) = \frac{|z|^{I_{\ell+1}(2|z|)^2}}{\pi \hbar I_\ell(2|z|)^2} \sum_{n, n' = 0}^{\infty} N_n(\rho, \theta) N_{n'}(\rho, \theta)$$

$$\times \frac{e^{\frac{n-n'}{2\pi}} e^{(\gamma_n(0) - \gamma_{n'}(0))}}{\Gamma(n + \ell + 1) \Gamma(n' + \ell + 1)}$$

$$\times \int_0^\infty e^{-\frac{\xi^2}{2}(r + r^{-1})} L_n^\ell(\xi r^{-1}) L_{n'}^\ell(\xi) r^{\frac{n-n'}{2\pi}} dr.$$  \hspace{1cm} (72)
Then, expanding the associated Laguerre polynomials in their finite series with binomial coefficients \(^{14}\) and using the following relation \(^{32}\)

\[
\int_0^\infty dx x^{\nu-1}e^{-x-\tau x} = 2 \left( \frac{\lambda}{\tau} \right)^{\frac{\nu}{2}} K_\lambda(2\sqrt{\lambda\tau}),
\]

(73)

we get the final result

\[
\mathcal{W}_z^\ell(u,p,t) = 2 |z|^\ell \pi \hbar I_\ell |z| K_\lambda(2\sqrt{\lambda\tau}),
\]

(74)

where \(K_\lambda(\xi)\) are the modified Bessel functions of the third kind. This result is related to the similar discussions of the Wigner function of Morse potential \(^{44}\).

5 The photon added coherent states

5.1 Construction

Photon-added coherent states \(|\psi_\ell^{z_m}(t)\rangle\) are defined by the repeated application of the raising operator \(K_+\) to the BGLCSs \(^{16}\) of the TDLP

\[
|\psi_\ell^{z_m}(t)\rangle = \frac{(K_+)^m|\psi_\ell^z(t)\rangle}{\sqrt{\langle\psi_\ell^z(t)|K_-(K_+)^m|\psi_\ell^z(t)\rangle}} = \mathcal{M}_m(|z|) \sum_{n=0}^\infty \frac{z^n}{\sqrt{n!\Gamma(n+\ell+1)}} (K_+)^m|\psi_n^z(t)\rangle
\]

(75)

where \(m\) is a positive integer being the number of added quanta (or added photons). \(\mathcal{M}_m(|z|)\) is the normalization constant such as

\[
\mathcal{M}_m(|z|) = \sqrt{\frac{\langle\psi_\ell^z(t)|K_-(K_+)^m|\psi_\ell^z(t)\rangle}{\langle\psi_\ell^z(t)|K_-(K_+)^m|\psi_\ell^z(t)\rangle}}.
\]

(76)

For \(m = 0\), we recover the normalization constante of the states \(^{47}\). Making use of the expressions

\[
(K_+)^m|\psi_n^z(t)\rangle = \sqrt{\frac{\Gamma(n+m+1)\Gamma(n+\ell+m+1)}{\Gamma(n+1)\Gamma(n+\ell+1)}} |\psi_{n+m}^z(t)\rangle,
\]

(77)

we obtain

\[
|\psi_\ell^{z_m}(t)\rangle_m = \mathcal{M}_m(|z|) \sum_{n=0}^\infty \frac{z^n}{\sqrt{F_m(\ell,n)}} |\psi_{n+m}^z(t)\rangle,
\]

(78)

and

\[
F_m(\ell,n) = \frac{[\Gamma(n+\ell+1)]^2[\Gamma(n+1)]^2}{\Gamma(n+m+\ell+1)\Gamma(n+m+1)}
\]

(79)
The corresponding eigenfunctions of the states are termed
\[ \psi_{zm}^\ell(u, t) = \mathcal{M}_m^\ell(|z|) u^\ell e^{-\frac{z^2}{2}} \sum_{n=0}^{\infty} N_{n+m}(\theta, \rho) \frac{\Gamma(n + m + 1)}{\Gamma(n + 1)\Gamma(n + \ell + 1)} \times z^n L_{n+m}^\ell(u) e^{i\ell n+m(t)}. \] (80)

As it is seen from Eq. (78), the states \(|\psi_{n+m}^\ell(t)\rangle\) are linear combination of number states \(|\psi_{n+m}^\ell(t)\rangle\). Therefore, the PABGLCSs belong to the Hilbert space \(\mathcal{H}_m^\ell\) indexed by the single real positive number \(\ell\) which defines the representation of these spaces
\[ \mathcal{H}_m^\ell := \text{span}\{\psi_{n+m}^\ell(t)\}_{n \in \mathbb{N}}. \] (81)

In other words, the application of \((K_+)^m\) transfers the coherent states of Barut-Girardello from \(\mathcal{H}_m^\ell\) to \(\mathcal{H}_m^\ell\).

5.2 The mathematical properties
In this subsection we are interested to examine the change of the mathematical properties of the states \(|\psi_{zm}^\ell(t)\rangle\) compared to the previous properties of BGLCSs.

5.2.1 The non-orthogonality
The states \(|\psi_{zm}^\ell(t)\rangle\) of this system must be normalized but not orthogonal. The non-orthogonality is expressed as
\[ \langle \psi_{z'm}^\ell(t) | \psi_{zm}^\ell(t) \rangle = \mathcal{M}_m^\ell(|z'|) \mathcal{M}_m^\ell(|z|) \sum_{n'=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^n(z')^n}{\sqrt{(\ell, n')F_m(\ell, n)}} \times \langle \psi_{n'+m'}^\ell(t) | \psi_{n+m}^\ell(t) \rangle. \] (82)

In the case of the state \(|\psi_{zm}^\ell(t)\rangle\), due to the orthogonality relation of the number vectors \(|\psi_n^\ell(t)\rangle\), it follows that
\[ \langle \psi_{z'm}^\ell(t) | \psi_{zm}^\ell(t) \rangle = \mathcal{M}_m^\ell(|z'|) \mathcal{M}_m^\ell(|z|) (z')^{m-m'} \frac{\Gamma(m + 1)\Gamma(m + \ell + 1)}{\Gamma(m - m' + 1)\Gamma(m - m' + \ell + 1)\Gamma(\ell + 1)} \times \mathcal{F}_3(m + 1, m + \ell + 1; m - m' + 1, m + \ell + 1, \ell + 1; z'z), \] (83)

where \(\mathcal{F}_3\) is the generalized hypergeometric function and with \(m > m'\). This relation can be obtained in a more explicit way in terms of Meijer’s G-function by
\[ \langle \psi_{z'm}^\ell(t) | \psi_{zm}^\ell(t) \rangle = \mathcal{M}_m^\ell(|z'|) \mathcal{M}_m^\ell(|z|) (z')^{m-m'} \mathcal{G}^{1,2}_{1,4} \left( -z'z; 0, -m, -m' - \ell, -m' - \ell, -\ell \right), \] (84)

where we used the following relation between the generalized hypergeometric function and the Meijer’s G-function
\[ {}_p\mathcal{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) = \prod_{j=1}^{q} \Gamma(b_j) \prod_{j=1}^{p} \Gamma(a_j) \mathcal{G}^{1,2}_{p, q+1} \left( 0, x; 1-a_p, 1-b_q \right). \] (85)
This relation prove that the PAGBLCSs are not mutually orthogonal. Performing the normalization condition of these states, such that

$$
\langle \psi^\ell_{zm}(t) | \psi^\ell_{zm}(t) \rangle = 1, \quad (86)
$$

we determine the constants $M^\ell_m(|z|) \equiv [G_{2,4}^{1,2} \left( -|z|^2, 0, -\ell \right)]^{-\frac{1}{2}}$. (87)

From these results, for $m = 0$ the normalization constant (88) becomes

$$
M^\ell_0(|z|) \equiv \left[ G_{2,4}^{1,2} \left( -|z|^2, 0, -\ell \right) \right]^{-\frac{1}{2}} = \sqrt{|z|^\ell I^\ell_0(2|z|)}, \quad (88)
$$

which is the normalization constant of the states $|psi^\ell_{z}(t)\rangle$. Therefore, with this condition the non-orthogonality (84) is reduced to the one obtained (50) for the states $|\psi^\ell_{z}(t)\rangle$. All this argument shows that the addition of photons changes the non-orthogonality of the BGLCSs.

5.2.2 The label continuity

The label continuity condition of the $|\psi^\ell_{zm}(t)\rangle$ can then be stated as:

$$
|||\psi^\ell_{zm}(t) - |\psi^\ell_{z'm}(t)\rangle||^2 = 2 \left[ 1 - \text{Re} \left( \langle \psi^\ell_{z'm}(t) | \psi^\ell_{zm}(t) \rangle \right) \right] \to 0 \quad (89)
$$

when $|z - z'| \to 0$ and $|m - m'| \to 0$.

5.2.3 Overcompleteness

We have to search for non-negative weight functions $W^\ell_m(|z|)$ such that the overcompleteness or the resolution of the identity

$$
\int_C \frac{d^2z}{\pi} |\psi^\ell_{zm}(t)\rangle W^\ell_m(|z|) \langle \psi^\ell_{zm}(t) | \rangle = \eta^\ell_m \quad (90)
$$

holds, where

$$
\eta^\ell_m = \sum_{n=0}^\infty |\psi^\ell_n(t)\rangle \langle \psi^\ell_{n+m}(t) |. \quad (91)
$$

For PABGLCSs case, by substituting equation (78) into equation (90) we obtain

$$
\int_C \frac{d^2z}{\pi} W^\ell_m(|z|) \left[ M^\ell_m(|z|) \right]^2 \frac{|z|^n (z^*)^{n'}}{\sqrt{F^m_0(t, n') F^m_0(t, n)}} \times |\psi^\ell_{n+m}(t)\rangle \langle \psi^\ell_{n+m}(t) | = \eta^\ell_m. \quad (92)
$$

By means of a change of the complex variables in terms of polar coordinates $z = re^{i\theta}$ where $r \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$, and $d^2z = r dr d\theta$, this equation becomes

$$
\sum_{n, n'=0}^\infty \frac{1}{\sqrt{F^m_0(t, n') F^m_0(t, n)}} \int_0^{2\pi} d\theta \int_0^\infty dr \ e^{i(n-n')\theta} \ |z|^n (z^*)^{n'} W^\ell_m(|z|) \left[ M^\ell_m(|z|) \right]^2 \int_0^\infty d\theta \ e^{i(n-n')\theta}.
$$

15
By performing the angular integration, i.e.

\[ \int_0^{2\pi} \frac{d\theta}{\pi} e^{i(n-n')\theta} = 2\delta_{nn'} , \]

the resolution of the identity operator is

\[ 2 \sum_{n=0}^{\infty} \left[ \frac{1}{F_m(\ell, n)} \int_0^\infty dr r^{1+2n} W_m^\ell (r^2) |M_m^\ell (r^2)|^2 \right] \times |\psi_{n+m}^\ell (t) \rangle \langle \psi_{n+m}^\ell (t) | = I_m^\ell . \]

Setting the weight function such as

\[ W_m^\ell (r^2) = \frac{1}{|M_m^\ell (r^2)|^2} r^{2m} g_m^\ell (r^2) , \]

and using the completeness of the states \(|\psi_{n+m}^\ell (t) \rangle\) and performing the variable change \(x = r^2\) and \(n + m = s - 1\), the integral from the above equation is called the Mellin transform

\[ \int_0^\infty dx x^{s-1} g_m^\ell (x) = F_m(\ell, s - m - 1) = \frac{[\Gamma(s + \ell - m)]^2 [\Gamma(s - m)]^2}{\Gamma(s + \ell) \Gamma(s)} . \]

Using the definition of Meijer G-function, it follows that

\[ \int_0^\infty dx x^{s-1} G_{p+q}^{m+n} \left[ \frac{\alpha x}{b_1, \ldots, b_m, b_{m+1}, \ldots, b_q} \right] = \frac{1}{\alpha^s} \prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s) \prod_{j=m+1}^{n+p} \Gamma(a_j + s) . \]

Comparing equations (97) and (98), we obtain that

\[ g_m^\ell (x) = G_{2,4}^{4,0} (x |_{-m}, \ell, -m, \ell - m) . \]

Using the multiplication formula of Meijer’s G-function

\[ x^\sigma G_{p,q}^{m,n} \left[ \frac{\alpha x}{b_1, \ldots, b_m, b_{m+1}, \ldots, b_q} \right] = G_{p,q}^{m,n} \left[ \frac{\alpha x}{b_1, \ldots, b_m, b_{m+1}, \ldots, b_q} \right] , \]

then, the weight function (96) becomes

\[ W_m^\ell (|z|) = \frac{1}{|M_m^\ell (|z|)|^2} G_{2,4}^{4,0} (|z|^{2m}, m+\ell, 0, \ell, \ell) , \]

and the overcompleteness (99) can be explicitly expressed as follows

\[ \int_{\mathbb{C}} \frac{d^2 z}{|M_m^\ell (|z|)|^2} G_{2,4}^{4,0} (|z|^{2m}, m+\ell, 0, \ell, \ell) |\psi_{z,m}^\ell (t) \rangle \langle \psi_{z,m}^\ell (t) | = I_m^\ell . \]

Thus, at the limit \(m = 0\), this weight function is reduced to the one obtained for the ordinary Barut-Girardello like coherent states (55).

\[ \omega' = \frac{1}{\pi} W_m^\ell (|z|) = \frac{1}{\pi} \left( M_0^\ell (|z|) \right)^{-2} G_{2,4}^{4,0} (|z|^{2}, 0, \ell, \ell, \ell) , \]

\[ = \frac{2}{\pi} K\varepsilon(2|z|) I_\ell (2|z|) . \]
In Fig. 4, we plot the weight function \( W^{\ell m}(95) \) versus \( x = |z| \) for fixed value of photon-added number \( m = 3 \) and various values of the Bargmann index \( \ell \) on figure (a) and fixed value of the parameter \( \ell = 2.5 \) and for various values of \( m \) on figure (b). All the curves are positive, this confirms the positivity of the weight function for half integer values of the parameter \( \ell = 0.5, 1.5, \ldots \). All the curves on figure (a) have the same behaviour, this shows that the Bargmann index \( \ell \) do not affect the general form of the curves. But we can see that the weight function decreases while \( \ell \) increases. Figure (b) shows that increasing the photon-added number \( m \) increases the weight function. We see also that all the curves tends asymptoptically to the weight function of the conventional BGLCSs (55).

**Figure 4:** Plots of the weight function \( W^{\ell m}(95) \) of the PABGLCSs versus \( |z| \): (a) for fixed value of the photon-added number \( m = 3 \) and different values of the the Bargmann index \( \ell \); (b) for fixed value of the the Bargmann index \( \ell = 2.5 \) of the photon-added number \( m = 3 \) and different values of the photon-added number \( m \).

### 5.3 The statistical properties

We have seen that the process of photon addition has changed the mathematical properties of the BGLCSs. Here, we analyze the effect of this process on the statistics of the original coherent states.

#### 5.3.1 The photon-number distribution

The probability of finding \( n \) photons in the PAGBLCSs in the Hilbert spaces \( \mathcal{H}_m \) are given by:

\[
P_{nm}(\ell, z) = |\langle \psi_n^\ell(t) | \psi_m^\ell(t) \rangle|^2 = |\mathcal{M}_m^\ell(|z|)|^2 \frac{|z|^{2n}}{F_m(\ell, n)}. \tag{105}
\]

The addition of photons drastically increases the photon distribution (58) such as

\[
P_{n+m}(\ell, z) = \frac{|z|^{2n}}{G_{2,4}^1 \left( \left[ -|z|^2 \right]_{0, 0}^{m, m - \ell, -\ell, -\ell} \right) \times \frac{\Gamma(n + m + \ell + 1)\Gamma(n + m + 1)}{\left[ \Gamma(n + m + 1) \right]^2 \left[ \Gamma(n + 1) \right]^2}}. \tag{106}
\]
At the limit \( m = 0 \), we recover the result of the photon distribution (58) of the ordinary coherent states. This situation reflects the fact that \( m \) photons have been added. In Figure 5, the photon number distribution of the PABGLCSs (Figure 5), as function of \( n \) is depicted in (a) with fixed parameters \( \ell = 3.5, m = 2 \) different amplitude values \( |z| = \{4, 9\} \); in (b) with fixed values of \( \ell = 4.5, |z| = 9 \) and different values of the photon-added number \( m = \{1, 7\} \). We see in Figure (a) that as \( |z| \) and \( m \) increase, the peaks decrease and shift to the right. Figure (b) shows that the shift of the peaks increase as the photon-added number increase.

![Figure 5: Plots of the PND of the PABGLCSs versus the photon number, with parameters : (a) fixed value of \( \ell = 3.5 \) and \( m = 2 \) and different values of the amplitude \( |z| = \{2, 3\} \) (dot points) and \( |z| = 3 \) (scattered points); (b) fixed value of \( \ell = 4.5 \) and \( |z| = 9 \) with different values of photon-added number \( m = 1 \) (dot points) and \( m = 7 \) (scattered points).]

5.3.2 The intensity correlation function, the Mandel parameter and the Wigner function

The intensity correlation function \( g^{(2)}_{\ell,zm} \) and the Mandel \( Q_{2m}^\ell \)-parameter are given by:

\[
g^{(2)}_{\ell,zm} = \frac{\langle \psi_{\ell,zm}^{\ell}(t)|N^2|\psi_{\ell,zm}^{\ell}(t)\rangle - \langle \psi_{\ell,zm}^{\ell}|N|\psi_{\ell,zm}^{\ell}\rangle^2}{\langle \psi_{\ell,zm}^{\ell}|N|\psi_{\ell,zm}^{\ell}\rangle^2}, \tag{107}
\]

\[
Q_{2m}^\ell = \langle \psi_{\ell,zm}^{\ell}(t)|N|\psi_{\ell,zm}^{\ell}(t)\rangle \left[ g^{(2)}_{\ell} - 1 \right]. \tag{108}
\]

The expectation values of the operator number \( N \) and its square in the states \( \psi_{n+m}^\ell(t) \) are given by:

\[
\langle \psi_{n+m}^\ell(t)|N|\psi_{n+m}^\ell(t)\rangle = n + m, \tag{109}
\]

\[
\langle \psi_{n+m}^\ell(t)|N^2|\psi_{n+m}^\ell(t)\rangle = (n + m)^2. \tag{110}
\]

Based on the references [33], these expectation values in the \( |\psi_{2m}^\ell(t)\rangle \) states give

\[
\langle \psi_{2m}^\ell(t)|N|\psi_{2m}^\ell(t)\rangle = m \frac{G_m'(1)}{G_m'(0)}, \tag{111}
\]

\[
\langle \psi_{2m}^\ell(t)|N^2|\psi_{2m}^\ell(t)\rangle = m^2 - (2m + 1) \frac{G_m'(1)}{G_m'(0)} + \frac{G_m''(2)}{G_m''(0)} \tag{112}
\]
where
\[ G_{\ell m}(i) = G_{2,4}^{1,2}(\frac{|z|^2}{l_0}, \frac{-m-\ell}{i}, -\ell, -\ell), \quad i = 0, 1, 2. \] (113)

For the special case \( m = 0 \), we have
\[ G_{\ell 0}(0) = G_{2,4}^{1,2}(\frac{|z|^2}{l_0}, -\ell, -\ell, -\ell) = |z|^{-\ell} I_\ell(2 |z|), \] (114)
\[ G_{\ell 0}(1) = G_{2,4}^{1,2}(\frac{|z|^2}{l_0}, -\ell, -\ell, -\ell) = |z|^{-1-\ell} I_{\ell+1}(2 |z|), \] (115)
\[ G_{\ell 0}(2) = G_{2,4}^{1,2}(\frac{|z|^2}{l_0}, -\ell, -\ell, -\ell) = |z|^{2-\ell} I_{\ell+2}(2 |z|). \] (116)

Using the results of the expectation values (111) and (112), we obtain the intensity correlation function and the Mandel \( Q \)-parameter of the PABGCSs as
\[ g_{\ell,zm}^{(2)} = \frac{m(m-1) - 2m G_{m}^{(1)} + G_{m}^{(2)}}{G_{m}^{(0)}} \] (117)
\[ Q_{zm} = \frac{m^2 - 2m + (1 - 2m) G_{m}^{(1)} + G_{m}^{(2)}}{m - G_{m}^{(1)}}. \] (118)

For \( m = 0 \), we obtain \( g_{\ell,z0}^{(2)} = g_{\ell,z}^{(2)} \) and \( Q_{z0} = Q_{z} \). In fact the statistical behaviors of the states \( |\psi_{n+m}\rangle \) are difficult to guess since the analytical properties of the intensity correlation function (117) and the Mandel parameter (118) depend on the ratio of Meijer’s \( G \)-functions. Therefore, we examine the statistical nature of these states through the numerical computation. Let remark that, the expectation value of the number operator \( N \) (124) vanishes for a certain value \( |z_0| \) depending on the parameters \( \ell \) and \( m \) as shown in Figure 6.

Figure 6: Plots of the expectation values \( \langle N \rangle \) (124) versus \( |z| \) for different values of the photon-number \( m \) and with fixed parameter \( \ell = 1.5 \)

So the intensity correlation and the Q-Mandel functions are not defined for this value \( |z_0| \), since their denominators depend on this expectation value. We limit then the analysis of the characteristics of \( g_{\ell,zm}^{(2)} \) and \( Q_{zm} \) on the domain of sufficiently high values of \( |z_0| \) to avoid the undetermined value.
In Figure 7, we plot (a) the intensity correlation function (117) and (b) the Q-Mandel parameter (118) versus $x = |z|$, with fixed parameter $\ell = 2.5$ and various values of the photon-added number $m$. All the curves show that for sufficiently high values of $|z|$ $g_{\ell,zm}^{(2)} < 1$ and $Q_{\ell,zm}^{\ell} < 0$. We conclude then that PABGLCSs have also the sub-Poissonian statistics as the conventional BGLCSs. Figures (a) and (b) show that increasing the photon-added number $m$ decreases both the intensity correlation and the Q-Mandel functions. So adding more photons increases the depth of the non-classicality of these states.

In a similar manner as (71), it is possible to determine the Wigner distribution function corresponding to the states (78). Referring to the Ref. [45], we obtain:

$$W_{\ell,zm}^{\ell}(u,p,t) = \frac{2\left|\mathcal{M}_{\ell}^{m}(|z|)\right|^2}{\pi\varpi\hbar} \xi^{\ell-2m} \sum_{n=0}^{\infty} \sum_{n'=0}^{n} N_{n+m}(\rho,\theta)N_{n'+m}(\rho,\theta) \times e^{i(\gamma_{n+m}^{\ell}(t)-\gamma_{n'+m}^{\ell}(t))} z_{n+n'}^{\ell} \varphi^{\ell} \times \sum_{k'=0}^{n'+m} \sum_{k=0}^{n+m} \left(\frac{n'+m+\ell}{n'+m-k'}\right) \left(\frac{n+m+\ell}{n+m-k}\right) \times \frac{(-\xi)^{k+k'}}{k!k!} K_{k-k'}^{\ell} (\varpi\xi),$$

(119)

where

$$\varphi^{\ell} = \frac{\Gamma(n'+m+1)\Gamma(n+m+1)}{\Gamma(n'+\ell+1)\Gamma(n+\ell+1)\Gamma(n'+1)\Gamma(n+1)}.\quad (120)$$

This result for the Wigner distribution fits similarly with the obtained result of the reference [45].

6 Conclusion

In the present paper, we have constructed a new kind of $SU(1,1)$ coherent states à la Barut-Girardello deriving from the factorisation method of the TDLP
The minimal set of Klauder’s conditions required to build coherent states i.e., the label continuity, the normalizability and the overcompleteness [9, 10] has been studied and discussed. Statistical properties like the photon number distribution, the Mandel parameter and the second-order correlation function are examined and analyzed. These statistical predictions were confirmed by the numerical investigation and we have found that these states are sub-Poissonian in nature. In addition to these statistical properties, we have also examined the behavior of these states in the phase space from the viewpoint of the Wigner distribution function and this result is found to be similar to the discussions of the Wigner function of Morse potential [14, 15].

On the other hand, we have also constructed the $m$-PACSs associated with the BGLCSs of the TDLP. We remarked that both states satisfy also the Klauder’s minimum conditions. A comprehensive study of the statistics of the $m$-PACSs are presented and are expressed in terms of the modified Bessel function of the third kind, in Gamma function, and in Meijer G-function. We have found that the addition of photons from the BGLCSs increases the statistical properties and changes the mathematical properties.

The sub-Poissonian statistics of our introduced states are established through the nonclassicality sign of the Mandel parameter or the the second-order correlation function. However, one can also investigate some others nonclassicality criteria which are usually used in the relevant literatures [46, 47]. To achieve this aim, we refer to the normal squeezing and amplitude-squared squeezing.

In order to study $SU(1, 1)$ normal squeezing, we consider the following Hermitian quadrature operators

$$X_1 = \frac{K_+ + K_-}{2}, \quad P_1 = \frac{K_- - K_+}{2i}. \quad (121)$$

The squeezing parameters can be defined as

$$S_\gamma = \frac{(\Delta \gamma)^2}{\sqrt{\frac{1}{4}([X_1, P_1])^2}} - 1, \quad \gamma = X_1, P_1, \quad (\Delta \gamma)^2 = \langle \gamma^2 \rangle - \langle \gamma \rangle^2. \quad (122)$$

In order to evaluate $SU(1, 1)$ amplitude-squared squeezing, we define the Hermitian operators

$$X_2 = \frac{K_+^2 + K_-^2}{2}, \quad P_2 = \frac{K_+^2 - K_-^2}{2i}. \quad (123)$$

with the squeezing parameters

$$S_l = \frac{(\Delta l)^2}{\sqrt{\frac{1}{4}([X_2, P_2])^2}} - 1, \quad l = X_2, P_2, \quad (\Delta l)^2 = \langle l^2 \rangle - \langle l \rangle^2. \quad (124)$$

The continuation of these computations are under investigation, we hope to report these aspects elsewhere.

Acknowledgments

L.M Lawson acknowledges support from AIMS-Ghana under the Postdoctoral fellow/teaching assistance (Tutor) grant
References

[1] L. Lawson and G. Avossevou, Landau problem with time dependent mass in time dependent electric and harmonic background fields, J. Math. Phys. 59, 042109 (2018)

[2] F. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Atomic Coherent States in Quantum Optics, Phys. Rev. A 6, 2211 (1972)

[3] Xue-xiang XuaHong-chunYuanb, Generating single-photon catalyzed coherent states with quantum-optical catalysis, Phys. Lett A 380, 2342-2348 (2016)

[4] D. Popov, I. Zaharie, V. Sajfert, I. Luminosu and D. Popov, Quantum Information in the Frame of Coherent States Representation, Int J Theor Phys 47, 1441 (2008)

[5] J-P Gazeau, Coherent states in Quantum Information: An example of experimental manipulations, J. Phys.: Conf. Ser. 213, 012013 (2010)

[6] E. Schrödinger, Naturwissenschaften 14, 664 (1926)

[7] R. Glauber, The Quantum Theory of Optical Coherence, Phys. Rev. 130, 130, 2529 (1963).

[8] R. Glauber, Coherent and Incoherent States of the Radiation Field”, Phys. Rev. 131, 2766 (1963)

[9] J. Klauder, Continuous-Representation Theory. I. Postulates of Continuous-Representation Theory, J. Math. Phys, 4, 1055 (1963)

[10] J. Klauder, Continuous-Representation Theory. II. Postulates of Continuous-Representation Theory, J. Math. Phys, 4, 1058 (1963)

[11] O. Barut and L. Girardello, New “coherent” states associated with non-compact groups, Commun. Math. Phys. 21, 41 (1971)

[12] A. Perelomov, Coherent states for arbitrary Lie group, Commun. Math. Phys. 26, 222-236 (1972)

[13] M. Daoud and L. Gouba: Generalized Grassmann variables for quantum $k$-level systems and Barut-Girardello coherent states for $su(r+1)$ algebras, J. Math. Phys 58, 053509 (2017)

[14] L. Lawson, G. Avossevou, L. Gouba, Lewis-Riesenfeld quantization and SU(1,1) coherent states for 2D damped harmonic oscillator, J.Math.Phys. 59,112101 (2018)

[15] Z. Sazonova and R. Singh, The role of SU(1,1) in quantum optics: I, Laser Physics 10, 765-769 (2000)

[16] G. Chiribella, G. D’Ariano, P. Perinotti, Applications of the group SU(1,1) for quantum computation and tomography, Quantum Inf. Quantum Comput 16, 1572 -1581 (2006)
[17] B. Yurke, S. McCall, J. Klauder, SU(2) and SU(1,1) interferometers Phys. Rev. A 33, 4033-4054 (1986)
[18] A. Zavatta V. Parigi, M. Kim and M. Bellini: Subtracting photons from arbitrary light fields: experimental test of coherent state invariance by single-photon annihilation, New Journal of Physics 10, 123006 (2008)
[19] P. Pinheiro and R. Ramos, Quantum communication with photon-added coherent states, Quantum Inf Process DOI 10.1007/s11128-012-0400-0
[20] M. Dakna, L. Knöll and D.G. Welsch, Photon-added state preparation via conditional measurement on a beam splitter; Optics Communications 145, 309-321 (1998)
[21] S. Sivakumar, Photon-added coherent states as nonlinear coherent states, J. Phys. A 32, 3441 (1999)
[22] A. Mahdifar, E. Amooghorban, and M. Jafari, Photon-added and photon-subtracted coherent states on a sphere”, J. Math. Phys. 59, 072109 (2018)
[23] B. Mojaveri and A. Delghani, Generation of photon-added coherent states via photon-subtracted generalised coherent states, Eur. Phys. J. D 68, 315 (2014)
[24] G.S. Agarwal, K. Tara, Nonclassical properties of states generated by excitation on a coherent state, Phys. Rev. A 43, 492 (1991)
[25] G. S. Agarwal, Negative binomial states of the field-operator representation and production by state reduction in optical processes, Phys. Rev. A 45, 1787 (1992).
[26] S. Loepp and W. K. Wootters, Protecting Information (Cambridge University, Cambridge, England, 2006).
[27] G. Van Assche, Quantum Cryptography and Secret-Key Distillation (Cambridge University, Cambridge, England, 2006).
[28] E. Lantz, Quantum-enhanced interferometry with weak thermal light: comment, Optica 4 4, 1314 (2017)
[29] M.N Hounkonnou and E. B. Ngompe-Nkouankam, Generalized hypergeometric photon-added and photon-depleted coherent states, J. Phys. A: Math. Theor 42, 025206 (2009)
[30] R. Román-Ancheyta, C. González Gutiérrez, and J. Récamier, Photon-added nonlinear coherent states for a one-mode field in a Kerr medium, Journal of the Optical Society of America B 31, Issue 1, pp. 38-44 (2014)
[31] L. Hong, Statistical properties of photon-added and photon-subtracted two-mode squeezed vacuum state, Phys. Lett. A 264, 265-269 (1999)
[32] S. Barnett, G. Ferenczi, C. Gilson and F. Speirits, Statistics of photon-subtracted and photon-added states, Phy Rev A 98, 013809 (2018)
[33] K. Sodoga, M. Hounkonnou and I. Aremua, Photon-added coherent states for shape invariant systems, Eur. Phys. J. D 72, 105 (2018)
[34] K. Sodoga, I. Aremua and M. Hounkonnou, Generalized photon-added associated hypergeometric coherent states: characterization and relevant properties, Eur. Phys. J. D 72, 172 (2018)

[35] D. Popov, Photon-added Barut–Girardello coherent states of the pseudo-harmonic oscillator, J. Phys. A: Math. Gen. 35, 7205–7223 (2002)

[36] M. Daoud, Photon-added coherent states for exactly solvable Hamiltonian”, Phys Lett A 305, 135-143 (2002)

[37] K. Berrada, Construction of photon-added spin coherent states and their statistical properties, J. Math. Phys. 56, 072104 (2015)

[38] H. Lewis and W. Riesenfeld, An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field, J. Math. Phys. 10, 1458-1473 (1969)

[39] L. Lawson, Coherent states of position-dependent mass in strong quantum gravitational background fields, arXiv:2021.10551v1 [help-th] 18 Dec 2020

[40] Hossein Motavalli and Amin Rezaei Akbarieh, Factorization Method and Special Orthogonal Functions”, Int. J. Theor Phys 49, 2074-2079 (2010) doi: 10.1007/s10773-010-0393-3

[41] D. Miguel Lorente, Raising and lowering operators, factorization and differential/difference operators of hypergeometric type, J. Phys. A Math. Gen. 34, 569 (2001)

[42] E. Pinney, The nonlinear differential equation: $y'' + p(x)y + cy^{-3}$, Proc. Amer. Math. Soc. 1, 681, (1950).

[43] I. Gradshteyn and M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, San Diego, California, USA, 8th edition 2015 .

[44] A. Frank, A. Rivera and K. Wolf, Wigner function of Morse potential eigenstates, Phy Rev A 61, 054102 ( 2000)

[45] O de los Santos-Sánchez and J. Récamier, Phase space picture of Morse-like coherent states based upon the Wigner function, J. Phys. A: Math. Theor. 45, 415310 (2012)

[46] D. F. Walls, Squeezed states of light, Nature 306, 141 (1983)

[47] L. Mandel, Sub-Poissonian photon statistics in resonance fluorescence, Opt. Lett. 4, 205 (1979)