The base matrix of hermitian operator order \( n \leq 4 \)

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Abstract. The Eigen problem is a problem that exists when there is an operator that is affected by the function, then that function does not change except being multiplied by the constant in the form of an eigenvalue. In general, the eigenvalue problem uses the Hermitian operators by using observable variable in the form of mathematical operations. Hermitian operator is an operator that happens when the Hermit conjugation is performed, then the operator will return to its original condition. This research aims to determine the base matrix of Hermitian operators with order \( n \leq 4 \) (size \( 2 \times 2 \), \( 3 \times 3 \) and \( 4 \times 4 \)). Entry element of the hermitian matrix operator uses complex numbers of which each order consists of few Hermitian operators. The method used is an analytical method by inserting problems into the eigen equation then normalized to obtain a diagonal matrix, so as to obtain a complete solution of the eigen problem on form of a matrix (including eigenvalues, normalized eigenvectors, and base matrix). Base matrix that has been obtained will be a diagonal matrix with diagonal elements of the eigenvalue. The results of this research indicate that the base matrix obtained in each order forms a square matrix that has entry elements in the form of complex numbers.

1. Introduction

At the end of the 19\(^{th}\) century, classical physics theory was unable to explain some physical phenomena related to the atomic scale in the microscopic domain. The limitations of classical physics in explaining microscopic phenomena that present new theories must be used to describe atomic phenomena [1]. This breakthrough is able to present the quantum theory that contains a component of quantum mechanics, quantum electrodynamics, and quantum thermodynamics [2]. Quantum theory is the most abstract subject. Concepts that are closely related to these properties can be observed in quantum mechanical postulates which consist of probability, wave functions, operators, and so on. They can be observed related to quantum mechanics by wave mechanics by Schrodinger and matrix mechanics by Heisenberg. The cornerstone of quantum mechanics from these two renewals uses mathematical tools such as matrices, special functions, second-order differential differentials, coordinate systems, etc. that can be used to complete all the discussions in quantum theory [3,4].

One of the problems using mathematical devices is the eigen problem. Eigen problem is a problem that happens when operator meets a function, then the function does not change its condition but is multiplied by certain constants [5]. The operator in the eigen problem is a symbol for a mathematical operation that will change a function into a new function. If all of these things are met, then the operator is called an eigen operator, the function of the operator is an eigenfunction and certain constants as eigenvalue. Eigenvalue in quantum theory can be applied in technology such as modeling the introduction of human hand movements with a hybrid model algorithm involving eigenvalues and
eigenvectors by detecting skin and wrist colors that are able to recognize 24 letters [6]. In addition, the eigenvalues and eigenfunctions of linear operators can be used for image test databases in image representations using the PCA method by recognizing hand movements as a form of letter recognition [7].

In general, eigen value problems use hermitian operators with observable variables in the form of mathematical operations in the form of a matrix. Hermitian operator is an operator that will produce the state itself when a transpus is carried out followed by a conjugate operation like this equation: \(X = X^+\) so that \([X] \rightarrow [X]^T \rightarrow [X]^*\). The form of the matrix has a very important role in quantum theory related to the existence of eigen problems [8]. In other cases matrices play an important role in algebraic isomorphism between Hamilton hypercomplex numbers and \(4 \times 4\) complex matrices with linear operators by producing eigenvectors that have matrix shapes that map into hypercomplex numbers. The use of bases as an alternative way to the number four results in a transformation of the similarity of the representation matrix, maintaining the eigenvalue across changes in the basis. This matrix has two pairs of identical and complex conjugate eigen values [9].

The eigen problem has a complete solution term which consists of eigenvalues, normalized eigenvectors and base dates. The base matrix can be defined as a collection of normalized eigenvectors. This base matrix value is adjusted to the shape of the order used. In this research, the matrix form on the order hermitian operator \(n \leq 4\) will be studied. To obtain a complete solution in the eigen value problem the hermitian operator has two conditions of its scalable multiplication operation, including the eigenvalue in the form of real numbers and eigenfunctions \((\Psi_a, \Psi_b)\) when orthogonal applies if there are two different eigen values with different eigen functions. The eigen problem in the hermitian operator can be solved by using the analytical method followed by diagonalization method. Analytic method is a method that is solved by the formulation of algebra that has been standardized from mathematical equations whose end result is in the form of numbers. Then, it can be continued with the diagonalization method which diagonalizes a matrix into the main diagonal [10].

The elements used in hermitian operators are complex number entry elements consisting of imaginary and real in a matrix. Hermitian operator matrix with order \(n \leq 4\) starts with order \(2 \times 2, 3 \times 3\) and \(4 \times 4\). The use of orders in this research does not begin with \(1 \times 1\) because the determinant of the matrix will produce the value itself. In the hermitian operator the resulting eigen value must be a real number while the entry element used is in the form of a complex number. The highest order limitation in this research is up to \(4 \times 4\) because with the higher order the use of the method used will switch to using a computing device. The increase of order is very influential with the eigenvectors formed [11]. This research will discuss the matrix form of a hermitian operator with entry elements in the form of complex numbers. Then it will examine the basis matrix form obtained from the eigenvalues and eigenvectors.

2. Methods

The eigen problem is an operator imposed on a function so that it produces certain constants that are multiplied by the function of the initial state. Eigen problems can apply the equation below:

\[ A\Psi = \alpha\Psi \]  

(1)

If \(A\) is an Hermitian matrix operator, it will apply the equation as follows:

\[ A = A^+ \]  

(2)

Where \(A\) is a hermitian operator and \(A^+\) as a “deger” operator (hermitian conjugate) from \(A\). In a expectation value of \(A\) can be written as follows:

\[ \langle A \rangle_\Psi = \langle A \rangle = \int \Psi^* X \Psi \, dv = \langle A \rangle^* \]  

(3)
\langle A \rangle^* = \left( \int \Psi^* X \Psi \, dv \right)^* \\
= \int (\Psi^*)^* (A \Psi)^* \, dv \\
= \int \Psi (A \Psi)^* \, dv \\
\int \Psi (A \Psi)^* \, dv = \int \Psi A^+ \Psi^* \, dv \quad (4)

The completion of the basic matrix in the discussion of the hermitian operator eigen at low order can be done by analytic methods and continued with the diagonalisation method which starts by determining for eigen values, normalizing eigen functions and matrix bases: Determining eigen values by substituting operators into conversation (1) “a” can be written as "\( \lambda \)" which forms:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \ldots & a_{14} \\
    a_{21} & a_{22} & \ldots & a_{24} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{l1} & a_{l2} & \ldots & a_{lm}
\end{bmatrix}
\begin{bmatrix}
    \Psi_1 \\
    \Psi_2 \\
    \vdots \\
    \Psi_l
\end{bmatrix}
= \lambda
\begin{bmatrix}
    \Psi_1 \\
    \Psi_2 \\
    \vdots \\
    \Psi_l
\end{bmatrix} \quad (6)
\]

The eigenvalue or constant "\( \lambda \)" is moved to the left-hand side so that the right-hand side will equal to zero:

\[
\begin{bmatrix}
    (a_{11} - \lambda) & a_{12} & \ldots & a_{14} \\
    a_{21} & (a_{22} - \lambda) & \ldots & a_{24} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{l1} & a_{l2} & \ldots & (a_{lm} - \lambda)
\end{bmatrix}
\begin{bmatrix}
    \Psi_1 \\
    \Psi_2 \\
    \vdots \\
    \Psi_l
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
0 \end{bmatrix} \quad (7)
\]

The eigenvalue can be obtained if the determinant of the square matrix equation (7) must be equal to zero as in equation (8):

\[
|\det A| = \sum_{\sigma \in \Sigma_l} \left( \text{sgn}(\sigma) \prod_{i=1}^l a_{i,\sigma_i} \right) = 0 \quad (8)
\]

Equation of (8) is used for orders \( n \times n \) that are too high. The determinant results are the results of eigen values. The eigen values obtained greatly influence the order that forms the hermitian operator matrix.

If the eigen value has been obtained, then it can be continued with determining the eigen function. Eigen functions can be determined by substituting eigen values into equation (7). The eigen function will become as follows:

\[
\Psi = \begin{bmatrix}
    \Psi_1 \\
    \Psi_2 \\
    \vdots \\
    \Psi_l
\end{bmatrix} = \begin{bmatrix}
p \Psi_1 \\
q \Psi_2 \\
\vdots \\
z \Psi_l
\end{bmatrix} \quad (9)
\]

\( p, q, \) and \( z \) are the results of an un normalized eigen function. To get a normalized eigen function, the function can be performed using equations below:
\[ \psi^+ \psi = 1 \] (10)

Equation (10) is a step to normalize eigen functions by mathematical operations in the form of transpose and continued with conjugate mathematical operations.

The next step is to prove the results of the eigen function by doing an orthogonal check and proving the condition equation. Then, it can be arranged with the base matrix like the equation below:

\[
\psi = \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_l
\end{bmatrix} = \psi_l \begin{bmatrix}
\psi_{l1} & \psi_{l2} & \psi_{l3} & \psi_{l4} \\
\vdots & \vdots & \vdots & \vdots \\
\psi_{l1} & \psi_{l2} & \psi_{l3} & \psi_{l4}
\end{bmatrix}
\] (11)

If a basic matrix such as Equation (11) has been obtained, then the basic matrix can be proceed by diagonalizing the matrix into a diagonal matrix through substituting into

\[
D = A^{-1}XA
\] (12)

Into shapes:

\[
D = \begin{bmatrix}
\lambda_{l1} & 0 & 0 & 0 \\
0 & \lambda_{l2} & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_{kl}
\end{bmatrix}
\] (13)

3. Results and discussion

Following are the results of the base matrix on hermitian operators with the order 2 x 2, 3 x 3 and with the order 4 x 4 based on the analysis of equations (1) to (13). Hermitian operators of the order of 2 x 2 can be expressed as follows:

\[
Q = \begin{bmatrix}
p_{11} & p_{12} \\
p_{21} & p_{22}
\end{bmatrix}
\] (14)

The complex number entry element can be shown as follows:

\[
\begin{bmatrix}
p_{11} & ip_{12} \\
-ip_{21} & p_{22}
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
p_{11} & -ip_{12} \\
-ip_{21} & p_{22}
\end{bmatrix}
\] (15)

Thehermitian operators of the order 3 x 3 can be shown like the equation below:

\[
Q = \begin{bmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{bmatrix}
\] (16)

If the complex number entry element uses the order 3 x 3, then the formula can be shown as follows:

\[
\begin{bmatrix}
p_{11} & ip_{12} & p_{13} \\
-ip_{21} & p_{22} & ip_{23} \\
p_{31} & ip_{32} & p_{33}
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
p_{11} & -ip_{12} & p_{13} \\
-ip_{21} & p_{22} & p_{23} \\
p_{31} & -ip_{32} & p_{33}
\end{bmatrix}
\] (17)

And the hermitian operators of the order 4 x 4 becomes like this:

\[
Q = \begin{bmatrix}
p_{11} & p_{12} & p_{13} & p_{14} \\
p_{21} & p_{22} & p_{23} & p_{24} \\
p_{31} & p_{32} & p_{33} & p_{34} \\
p_{41} & p_{42} & p_{43} & p_{44}
\end{bmatrix}
\] (18)

If the complex number entry element uses the order 4 x 4, then the formula can be shown as follows:
All three operators have met the requirements, so that they can be used in eigen problems. Eigen problems consist of eigenvalues, normalized eigen functions and base matrices. The solution to the order of eigenvalues $2 \times 2, 3 \times 3$ and $4 \times 4$ begins with obtaining the eigenvalue. Eigenvalues indicate a characteristic value of a hermitian operator. If the order of $2 \times 2$ is substituted into equation (1) the order can be shown as follows:

$$\begin{bmatrix}
    p_{11} - \lambda & p_{12} \\
    p_{21} & p_{22} - \lambda
\end{bmatrix}
\begin{bmatrix}
    \varphi_1 \\
    \varphi_2
\end{bmatrix} = 0$$

If the order $3 \times 3$ and $4 \times 4$ are used on the hermitian operator, then the form of the characteristic equation becomes as follows:

$$\begin{bmatrix}
    (p_{11} - \lambda) & p_{12} & p_{13} \\
    p_{21} & (p_{22} - \lambda) & p_{23} \\
    p_{31} & p_{32} & (p_{33} - \lambda)
\end{bmatrix}
\begin{bmatrix}
    \varphi_1 \\
    \varphi_2 \\
    \varphi_3
\end{bmatrix} = 0 ;$$

$$\begin{bmatrix}
    (p_{11} - \lambda) & p_{12} & p_{13} & p_{14} \\
    p_{21} & (p_{22} - \lambda) & p_{23} & p_{24} \\
    p_{31} & p_{32} & (p_{33} - \lambda) & p_{34} \\
    p_{41} & p_{42} & p_{43} & (p_{44} - \lambda)
\end{bmatrix}
\begin{bmatrix}
    \varphi_1 \\
    \varphi_2 \\
    \varphi_3 \\
    \varphi_4
\end{bmatrix} = 0$$

Based on the matrix above, we obtain two eigenvalues ($\lambda$) in $\lambda_1$ and $\lambda_2$. If orders $3 \times 3$ and $4 \times 4$ each get three eigenvalues ($\lambda$) in $\lambda_1, \lambda_2$ and $\lambda_3$ and 4 eigenvalue ($\lambda$) in $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$. The acquisition of eigenvalue is due to the determinant of the matrix which has a square form. The results of this eigenvalue have numbers in the form of real numbers even though the entry elements are complex numbers. This is because the operator that used is a Hermitian operator. If the eigenvalues generated are not real numbers, we can be sure that the operator used is not a Hermitian operator.

The results obtained after gaining the eigenvalue are continued with the eigenfunction which shows the characteristic function of the hermitian operator. The eigenfunctions of the normalized $2 \times 2, 3 \times 3$ and $4 \times 4$ orders, which have been normalized through the help of equation (10):

$$\varphi_n = c \begin{bmatrix}
    \varphi_1 \\
    \varphi_2
\end{bmatrix}, \varphi_n = c \begin{bmatrix}
    \varphi_1 \\
    \varphi_2 \\
    \varphi_3
\end{bmatrix}, \varphi_n = c \begin{bmatrix}
    \varphi_1 \\
    \varphi_2 \\
    \varphi_3 \\
    \varphi_4
\end{bmatrix}$$

The eigenfunction produces real values and contains imaginary in one matrix which means complex numbers. The eigenfunction has a normalized constant ($c$). Each eigen function has different results according to the state of the system and its entry elements. From all the forms of a hermitian operator in each order there is a non degeneration system which results in the scalability forming orthogonal products. In the $4 \times 4$ order form equation (19) experiences a degeneration system because there is one eigenvalue that has more different eigenfunctions. All eigenfunctions produced will be orthonormal.
The existence of eigen functions in each order is able to form a base matrix that acts as an operator. The form of the base matrix in each order \((2 \times 2, 3 \times 3\) and \(4 \times 4\)) is shown as follows:

\[
A = \begin{bmatrix} \varphi_{21} & \varphi_{12} \\ \varphi_{22} & \varphi_{12} \end{bmatrix} \quad (25)
\]

\[
A = \begin{bmatrix} \varphi_{31} & \varphi_{21} & \varphi_{11} \\ \varphi_{32} & \varphi_{22} & \varphi_{12} \\ \varphi_{33} & \varphi_{23} & \varphi_{13} \end{bmatrix} \quad (26)
\]

\[
A = \begin{bmatrix} \varphi_{41} & \varphi_{31} & \varphi_{21} & \varphi_{11} \\ \varphi_{42} & \varphi_{32} & \varphi_{22} & \varphi_{12} \\ \varphi_{43} & \varphi_{33} & \varphi_{23} & \varphi_{13} \\ \varphi_{44} & \varphi_{34} & \varphi_{24} & \varphi_{14} \end{bmatrix} \quad (27)
\]

The obtained base matrix can be diagonalized by mathematical operations as in equation (12). As a result of the diagonalization, it forms a diagonal matrix with the main diagonal of the eigenvalue itself.

\[
A^{-1}QA = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix} \quad (28)
\]

\[
A^{-1}QA = \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \quad (29)
\]

\[
A^{-1}QA = \begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \quad (30)
\]

Based on the above results the number of hermitian operators in quantum theory is not limited in number but not all matrix operators can belong to the hermitian operators. Operators with real number entry elements with complex numbers are clearly different. The terms and conditions that the Hermitian operator must have vary in each entry element. For complex number entry elements, the hermitian operator has the most important characteristic, which is the main diagonal of the real number element. The top and bottom parts of each order are complex conjugate numbers. For real numbers hermitian operators, the requirements exist if the transpus is carried out and then surged it will return to the operator itself. The result of eigenvalues correspond to the conditions of the hermitian operator. Real elements or complex elements still have real eigenvalue. The eigenfunctions produced by real elements will produce real numbers while complex elements will produce complex numbers. The result of the base matrix is an entry element used in the hermitian operator. Each element of entry real number and complex number in the hermitian operator produces a base matrix that can diagonalise the matrix into its main diagonal element in the form of real numbers, so that the base matrix shows the only solution in the hermitian operator.

4. Conclusions

Based on the discussion, it can be concluded that the hermitian operator base matrix is determined in accordance with the order used to form a square matrix. The higher the order that is used, the higher the base matrix that can be obtained. The result of the base matrix has entry elements in the form of complex numbers which consists of real and imaginary numbers, according to the elements used in the hermitian operator. In this research, the order used starts from \(2 \times 2\) to \(4 \times 4\) because size \(1 \times 1\) cannot form a base matrix with complex numbers hermitian operators. The \(4 \times 4\) size becomes a
barrier in the order used in this research because the higher the order is, the higher need for a computer device to solve the problem. This is because in the process of finding eigenvalues and base matrices with diagonalization method in high order, it requires great accuracy and great precision so that the help of computing devices is needed. Suggestions that can be offered are to implement the order $n \geq 5$ by using computing devices with entry elements which can be real, imaginary, or complex numbers and can use other matrix operators such as non-Hermitian operators.

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