Finite-dimensional vertex algebra modules over fixed point differential subfields

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Abstract

Let $K$ be a differential field over $\mathbb{C}$ with derivation $D$, $G$ a finite linear automorphism group over $K$ which preserves $D$, and $K^G$ the fixed point subfield of $K$ under the action of $G$. We show that every finite-dimensional vertex algebra $K^G$-module is contained in some twisted vertex algebra $K$-module.

Keywords: vertex algebra; differential field

1 Introduction

In [2], Borcherds defined the notion of vertex algebras and showed that every commutative ring $A$ with an arbitrary derivation $D$ has a structure of vertex algebra. Every ring $A$-module naturally becomes a vertex algebra $A$-module. However, this does not imply that ring $A$-modules and vertex algebra $A$-modules are the same. In fact, a vertex algebra $\mathbb{Z}[z, z^{-1}]$-module which is

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not a ring \( \mathbb{Z}[z, z^{-1}] \)-module was given in [2, Section 8], where \( \mathbb{Z}[z, z^{-1}] \) is the ring of Laurent polynomials over \( \mathbb{Z} \). Moreover, in [9] for the polynomial ring \( \mathbb{C}[s] \) in one variable \( s \) with derivation \( D \), I obtained a necessary and sufficient condition on \( D \) that there exist finite-dimensional vertex algebra \( \mathbb{C}[s] \)-modules which do not come from associative algebra \( \mathbb{C}[s] \)-modules. Thus, in general vertex algebra \( A \)-modules and ring \( A \)-modules are certainly different.

Let \( K \) be a differential field with derivation \( D \), \( G \) a finite linear automorphism group of \( K \) which preserves \( D \), and \( K^G \) the fixed point subfield of \( K \) under the action of \( G \). In this paper, we study vertex algebra \( K^G \)-modules. Here, let us recall the following conjecture on vertex operator algebras: let \( V \) be a vertex operator algebra and \( G \) a finite automorphism group of \( V \). It is conjectured that under some conditions on \( V \), every irreducible module over the fixed point vertex operator subalgebra \( V^G \) is contained in some irreducible \( g \)-twisted \( V \)-module for some \( g \in G \) (cf.[4]). The motivation for studying vertex algebra \( K^G \)-modules is to investigate this conjecture for vertex algebras. In Theorem 1 I shall show that every finite-dimensional indecomposable vertex algebra \( K^G \)-module becomes a \( g \)-twisted vertex algebra \( K \)-module for some \( g \in G \). Namely, the conjecture holds for all finite-dimensional vertex algebra \( K^G \)-modules in a stronger sense.

This paper is organized as follows. In Section 2 we recall some properties of vertex algebras and their modules. In Section 3 we show that every finite-dimensional indecomposable vertex algebra \( K^G \)-module becomes a \( g \)-twisted vertex algebra \( K \)-module for some \( g \in G \). In Section 4 we give the classification of the finite-dimensional vertex algebra \( \mathbb{C}(s) \)-modules where \( \mathbb{C}(s) \) is the field of rational functions in one variable \( s \). In Section 5 for all quadratic extensions \( K \) of \( \mathbb{C}(s) \) and all finite-dimensional indecomposable vertex algebra \( \mathbb{C}(s) \)-modules \( M \) obtained in Section 4, we study twisted vertex algebra \( K \)-module structures over \( M \).

2 Preliminary

We assume that the reader is familiar with the basic knowledge on vertex algebras as presented in [2,3,7].

Throughout this paper, \( \zeta_p \) is a primitive \( p \)-th root of unity for a positive integer \( p \) and \( (V,Y,1) \) is a vertex algebra. Recall that \( V \) is the underlying vector space, \( Y(\cdot,x) \) is the linear map from \( V \) to \( (\text{End} V)[[x,x^{-1}]] \), and \( 1 \) is
the vacuum vector. Let $\mathcal{D}$ be the endomorphism of $V$ defined by $\mathcal{D}v = v_{-1}1$ for $v \in V$.

First, we recall some results in [2] for a vertex algebra constructed from a commutative associative algebra with a derivation.

**Proposition 1.** [2] The following hold:

1. Let $A$ be a commutative associative $\mathbb{C}$-algebra with identity element $1$ and $D$ a derivation of $A$. For $a \in A$, define $Y(a, x) \in (\text{End} A)[[x]]$ by

$$Y(a, x)b = \sum_{i=0}^{\infty} \frac{1}{i!}(D^i a)bx^i$$

for $b \in A$. Then, $(A, Y, 1)$ is a vertex algebra.

2. Let $(V, Y, 1)$ be a vertex algebra such that $Y(u, x) \in (\text{End} V)[[x]]$ for all $u \in V$. Define a multiplication on $V$ by $uv = u_{-1}v$ for $u, v \in V$. Then, $V$ is a commutative associative $\mathbb{C}$-algebra with identity element $1$ and $D$ is a derivation of $V$.

Throughout the rest of this section, $A$ is a commutative associative $\mathbb{C}$-algebra with identity element $1$ over $\mathbb{C}$ and $D$ a derivation of $A$. Let $(A, Y, 1)$ be the vertex algebra constructed from $A$ and $D$ in Proposition 1 and let $(M, Y_M)$ be a module over vertex algebra $A$. We call $M$ a vertex algebra $(A, D)$-module to distinguish between modules over vertex algebra $A$ and modules over associative algebra $A$.

**Proposition 2.** [2] The following hold:

1. Let $M$ be an associative algebra $A$-module. For $a \in A$, define $Y_M(a, x) \in (\text{End}_A M)[[x]]$ by

$$Y(a, x)u = \sum_{i=0}^{\infty} \frac{1}{i!}(D^i a)ux^i$$

for $u \in M$. Then, $(M, Y_M)$ is a vertex algebra $(A, D)$-module.

2. Let $(M, Y_M)$ be a vertex algebra $(A, D)$-module such that $Y(a, x) \in (\text{End}_A M)[[x]]$ for all $a \in A$. Define an action of $A$ on $M$ by $au = a_{-1}u$ for $a \in A$ and $u \in M$. Then, $M$ is an associative algebra $A$-module.
Remark 1. (1) Let us consider the case of $D = 0$. Let $(M, Y_M)$ be an arbitrary vertex algebra $(A, 0)$-module. For all $a \in A$, since $0 = Y_M(0a, x) = dY_M(a, x)/dx$, we see that $Y_M(a, x)$ is constant. Thus, $M$ is an associative algebra $A$-module by Proposition 2.

(2) Let us consider the case that $A$ is finite-dimensional. Let $(M, Y_M)$ be an arbitrary vertex algebra $(A, D)$-module. Suppose that there exists $a \in A$ and $u \in M$ such that $Y_M(D^a, x)u$ is not an element of $M[[x]]$. Since $Y_M(D^a, x) = dY_M(a, x)/dx$ for all $i \geq 0$, we see that $\{Y_M(D^a, x)u \mid i = 0, 1, \ldots\}$ is linearly independent. This contradicts that $A$ is finite-dimensional. Thus, $M$ is an associative algebra $A$-module by Proposition 2.

For a $\mathbb{C}$-linear automorphism $g$ of $V$ of finite order $p$, set $V^r = \{u \in V \mid gu = \zeta_p^r u\}, 0 \leq r \leq p - 1$. We recall the definition of $g$-twisted $V$-modules.

Definition 1. A $g$-twisted $V$-module $M$ is a vector space equipped with a linear map

$$Y_M(\cdot, x): V \ni v \mapsto Y_M(v, x) = \sum_{i \in (1/p)\mathbb{Z}} v_i x^{-i-1} \in (\text{End}_\mathbb{C} M)[[x^{1/p}, x^{-1/p}]]$$

which satisfies the following four conditions:

(1) $Y_M(u, x) = \sum_{i \in r/p + \mathbb{Z}} u_i x^{-i-1}$ for $u \in V^r$.

(2) $Y_M(u, x)w \in M((x^{1/p}))$ for $u \in V$ and $w \in M$.

(3) $Y_M(1, x) = \text{id}_M$.

(4) For $u \in V^r$, $v \in V^s$, $m \in r/T + \mathbb{Z}$, $n \in s/T + \mathbb{Z}$, and $l \in \mathbb{Z}$,

$$\sum_{i=0}^\infty \binom{m}{i} (u_{l+i} v)_{m+n-i}$$

$$= \sum_{i=0}^\infty \binom{l}{i} (-1)^i (u_{l+m-i} v_{n+i} + (-1)^{l+1} u_{l+n-i} v_{m+i}).$$

The following result is well known (cf. [6] Proposition 4.8] and [9] Proposition 3]).
Proposition 3. Let \( g \) be an automorphism of \( V \) of finite order \( p \), \( M \) a vector space, and \( Y_M(\cdot, x) \) a linear map from \( V \) to \( (\text{End}_C M)[[x^{1/p}, x^{-1/p}]] \) such that for all \( 0 \leq r \leq p - 1 \) and all \( u \in V^r \), \( Y_M(u, x) = \sum_{i \in r/p + \mathbb{Z}} u_i x^{-i-1} \). Then, \((M, Y_M)\) is a \( g \)-twisted \( V \)-module if and only if the following five conditions hold:

1. (M1) For \( u \in V \) and \( w \in M \), \( Y_M(u, x)w \in M((x^{1/p})) \).
2. (M2) \( Y_M(1, x) = \text{id}_M \).
3. (M3) For \( u \in V^r \), \( v \in V^s \), \( m \in r/p + \mathbb{Z} \), and \( n \in s/p + \mathbb{Z} \)
   \[ [u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i}. \]
4. (M4) For \( u \in V^r \), \( v \in V^s \), \( m \in r/p + \mathbb{Z} \), and \( n \in s/p + \mathbb{Z} \)
   \[ \sum_{i=0}^{\infty} \binom{m}{i} (u_{-1+i} v)_{m+n-i} = \sum_{i=0}^{\infty} (u_{-1+m-i} v_{n+i} + v_{-1+n-i} u_{m+i}). \]
5. (M5) For \( u \in V \), \( Y_M(Du, x) = dY_M(u, x)/dx \).

Let \( B \) be a subset of \( A \) which generate \( A \) as a \( C \)-algebra and \( g \) a linear automorphism of \( A \) of finite order \( p \). For a \( g \)-twisted \( A \)-module \((M, Y_M)\), we call \((M, Y_M)\) a \( g \)-twisted vertex algebra \((A, D)\)-module.

Lemma 4. Let \( M \) be a vector space and \( Y_M(\cdot, x) \) a linear map from \( A \) to \( (\text{End}_C M)[[x^{1/p}, x^{-1/p}]] \) such that for all \( 0 \leq r \leq p - 1 \) and all \( a \in A^r \), \( Y_M(a, x) = \sum_{i \in r/p + \mathbb{Z}} a_i x^{-i-1} \). Let \( A_M(A) \) denote the subalgebra of \( \text{End}_C M \) generated by all \( a_i \) where \( a \in A \) and \( i \in (1/p)\mathbb{Z} \). Suppose that \( M \) is a finitely generated \( A_M(A) \)-module.

Then, \((M, Y_M)\) is a \( g \)-twisted vertex algebra \((A, D)\)-module if and only if the following five conditions hold:

1. (1) For \( a \in B \), \( Y_M(a, x) \in (\text{End}_C M)((x^{1/p})) \).
2. (2) \( Y_M(1, x) = \text{id}_M \).
3. (3) For \( a, b \in B \) and \( i, j \in (1/p)\mathbb{Z} \), \( a_i b_j = b_j a_i \).
(4) For $a \in B$ and $b \in A$, $Y_M(ab, x) = Y_M(a, x)Y_M(b, x)$.

(5) For $a \in B$, $Y_M(Da, x) = dY_M(a, x)/dx$.

In this case, $Y_M(\cdot, x)$ is a $\mathbb{C}$-algebra homomorphism from $A$ to $(\text{End}_\mathbb{C} M)((x^{1/p}))$.

Proof. We use Proposition 3. Note that for all $a, b \in A$ and all $i \geq 0$, we have $ab = a_{-1}b$ and $a_i b = 0$. Suppose that $(M, Y_M)$ is a $g$-twisted vertex algebra $(A, D)$-module. We have (3) by (M3). Since $M$ is a finitely generated $\mathcal{A}_M(A)$-module, we have (1) by (M1) and (3). We have (4) by (M4), (1), and (3). The other conditions clearly hold.

Conversely, suppose that $(M, Y_M)$ satisfies the conditions (1)–(5). It follows from (1), (2), and (4) that $Y_M(ab, x) = Y_M(a, x)Y_M(b, x)$ for all $a, b \in A$, namely $Y(\cdot, x)$ is a $\mathbb{C}$-algebra homomorphism from $A$ to $(\text{End}_\mathbb{C} M)((x^{1/p}))$. This shows (M3) by (3) and hence (M4). Let $a, b$ be elements of $A$ which satisfy (M5). Then,

$$Y_M(D(ab), x) = Y_M((Da)b + a(Db), x)$$

$$= Y_M(Da, x)Y_M(b, x) + Y_M(a, x)Y_M(Db, x)$$

$$= (d/dx Y_M(a, x))Y_M(b, x) + Y_M(a, x)(d/dx Y_M(b, x))$$

$$= (d/dx (Y_M(a, x)Y_M(b, x)))$$

$$= (d/dx Y_M(ab, x)).$$

Since $A$ is generated by $B$, (M5) follows from (5). We conclude that $(M, Y_M)$ is a $g$-twisted vertex algebra $(A, D)$-module.

For a $g$-twisted vertex algebra $(A, D)$-module $(M, Y_M)$ and a linear automorphism $h$ of $A$ which preserves $D$, define $(M, Y_M) \circ h = (M \circ h, Y_{M \circ h})$ by $M \circ h = M$ as vector spaces and $Y_{M \circ h}(a, x) = Y_M(ha, x)$ for all $a \in A$. Then, $(M, Y_M) \circ h$ is an $h^{-1}gh$-twisted vertex algebra $(A, D)$-module.

3 Finite-dimensional vertex algebra modules over fixed point differential subfields

Let $K$ be a differential field over $\mathbb{C}$ with derivation $D$ and let $G$ be a finite $\mathbb{C}$-linear automorphism group of $K$ of order $N$ which preserves $D$. 
We fix a primitive element \( \theta \) of \( K \) over \( K^G \) with the minimal polynomial \( P(Z) = \sum_{i=0}^N P_i Z^i \in K^G[Z] \) in \( Z \). For a finite-dimensional vertex algebra \((K^G,D)\)-module \((M,Y_M)\), \( g \in G \) of order \( p \), and a linear map \( \hat{Y}(\cdot, x) \) from \( K \) to \((\text{End}_\mathbb{C} M)((x^{1/p}))\), we call \((M,\hat{Y}_M)\) a \( g \)-twisted vertex algebra \((K,D)\)-module structure over \((M,Y_M)\) if \((M,\hat{Y}_M)\) is a \( g \)-twisted vertex algebra \((K,D)\)-module and if \( \hat{Y}(\cdot, x)|_{K^G} = Y(\cdot, x) \).

In this section, we shall show that every finite-dimensional indecomposable vertex algebra \((K^G,D)\)-module \((M,Y_M)\) has a \( g \)-twisted vertex algebra \((K,D)\)-module structure over \((M,Y_M)\) for some \( g \in G \). Here we give the outline of the proof. It follows from Lemma 4 that for any \( \psi \) with its image under the homomorphism \( (\text{End}_\mathbb{C} M)((x^{1/|\theta|})) \) with the minimal polynomial \( G \) of order \( p \), and a linear map \( \tilde{\psi} \) to construct a \( \mathbb{C} \)-algebra homomorphism from \( K \) to \((\text{End}_\mathbb{C} M)((x^{1/|\theta|}))\) with some conditions. The basic idea to construct such a \( \mathbb{C} \)-algebra homomorphism is to realize the Galois extension \( K/K^G \) in \((\text{End}_\mathbb{C} M)((x^{1/N}))\) where we identify \( K^G \) with its image under the homomorphism \( Y_M(\cdot, x) : K^G \to (\text{End}_\mathbb{C} M)((x^{1/N})) \). To do this, we first define a \( \mathbb{C} \)-algebra homomorphism \( \psi[K^G,(M,Y_M)] : K^G \to \mathbb{C}(x) \) for \((M,Y_M)\) and denote by \( Q \) the image of \( \psi[K^G,(M,Y_M)] \). It is well known that any finite extension of \( \mathbb{C}(x) \) is \( \mathbb{C}(x^{1/j}) \) for some positive integer \( j \) and \( \Omega = \cup_{j=1}^{\infty} \mathbb{C}(x^{1/j}) \) is the algebraic closure of \( \mathbb{C}(x) \) (cf. [5 Corollary 13.15]). These results enable us to construct the extension \( \tilde{K} \) of \( Q \) in \( \Omega \) such that \( \tilde{K} \cong K \) as \( \mathbb{C} \)-algebras. Using \( \theta, P(Z) \), and \( \tilde{K} \), we can construct the desired extension corresponding to \( K \) in \((\text{End}_\mathbb{C} M)((x^{1/N}))\).

We introduce some notation. Let \( R \) be a commutative ring and let \( \text{Mat}_n(R) \) denote the set of all \( n \times n \) matrices with entries in \( R \). Let \( E_n \) denote the \( n \times n \) identity matrix and let \( E_{ij} \) denote the matrix whose \((i, j)\) entry is 1 and all other entries are 0. Define \( \Delta_k(R) = \{(x_{ij}) \in \text{Mat}_n(R) \mid x_{ij} = 0 \text{ if } i + k \neq j \} \) for \( 0 \leq k \leq n \). Then, for \( a \in \Delta_k(R) \) and \( b \in \Delta_l(R) \), we have \( ab \in \Delta_{k+l}(R) \). We have \( X = (x_{ij}) \in \text{Mat}_n(R) \) and \( k = 0, \ldots, n-1 \), define the matrix \( X^{(k)} = \sum_{i=1}^n x_{i,i+k} E_{i,i+k} \in \Delta_k(R) \). For an upper triangular matrix \( X \), we see that \( X = \sum_{k=0}^{n-1} X^{(k)} \) and that the diagonal part of \( X \) is \( X^{(0)} \). For a positive integer \( m \) and \( H = \sum_{i \in (1/m)\mathbb{Z}} H(i)x^i \in (\text{Mat}_n(R))[[x^{1/m}, x^{-1/m}]] \) with \( H(i) \in \text{Mat}_n(R) \), \( H^{(k)} \) denotes \( \sum_{i \in (1/m)\mathbb{Z}} H^{(k)}(i)x^i \in \Delta_k(R)[[x^{1/m}, x^{-1/m}]] \) for \( k = 0, \ldots, n-1 \). For an \( n \)-dimensional vector space \( M \) over \( \mathbb{C} \), we sometimes identify \( \text{End}_\mathbb{C} M \) with \( \text{Mat}_n(\mathbb{C}) \) and \( (\text{End}_\mathbb{C} M)[[x^{1/m}, x^{-1/m}]] \) with \( (\text{Mat}_n(\mathbb{C}))[x^{1/m}, x^{-1/m}]] \) by fixing a basis of \( M \) and use these symbols in the proofs of Theorem 1 and Theorem 2.
Let $A$ be a commutative associative $\mathbb{C}$-algebra, $D$ a derivation of $A$, $g$ a $\mathbb{C}$-linear automorphism of $A$ of finite order $p$. For a vector space $W$ over $\mathbb{C}$ and a linear map $Y_W(\cdot, x)$ from $A$ to $(\text{End} W)[[x^{1/p}, x^{-1/p}]]$, $\mathcal{A}_W(A)$ denotes the subalgebra of $\text{End} W$ generated by all coefficients of $Y_W(a, x)$ where $a$ ranges over all elements of $A$. Let $M$ be a finite-dimensional $g$-twisted vertex algebra $(A, D)$-module. Then, $\mathcal{A}_M(A)$ is a commutative $\mathbb{C}$-algebra by Lemma 4 and $M$ is a finite-dimensional $\mathcal{A}_M(A)$-module. Let $\mathcal{J}_M(A)$ denote the Jacobson radical of $\mathcal{A}_M(A)$. Since $\mathcal{A}_M(A)$ is a finite-dimensional commutative $\mathbb{C}$-algebra, the Wedderburn–Malcev theorem (cf. [8, Section 11.6]) says that $\mathcal{A}_M(A) = \oplus_{i=1}^m \mathbb{C}e_i \oplus \mathcal{J}_M(A)$ and hence

$$
\mathcal{A}_M(A)((x^{1/p})) = \oplus_{i=1}^m \mathbb{C}((x^{1/p}))e_i \oplus \mathcal{J}_M(A)((x^{1/p}))
$$

where $e_1, \ldots, e_m$ are primitive orthogonal idempotents of $\mathcal{A}_M(A)$. In the case of $m = 1$, which is equivalent to $\mathcal{A}_M(A)$ being a local $\mathbb{C}$-algebra, we shall often identify the subalgebra $\mathbb{C}((x^{1/p}))$ id of $\mathcal{A}_M(K)((x^{1/p}))$ with $\mathbb{C}((x^{1/p}))$. For $X \in \mathcal{A}_M(A)$, $X^{[0]}$ denotes the image of $X$ under the projection

$$
\mathcal{A}_M(A) = \oplus_{i=1}^m \mathbb{C}e_i \oplus \mathcal{J}_M(A)
$$

$$
\rightarrow \oplus_{i=1}^m \mathbb{C}e_i \cong \mathbb{C}^m
$$

and is called the semisimple part of $X$. For $H = \sum_{i \in (1/p)^\mathbb{Z}} H_{(i)} x^i \in \mathcal{A}_M(A)((x^{1/p}))$ with $H_{(i)} \in \mathcal{A}_M(A)$, $H^{[0]}$ denotes $\sum_{i \in (1/p)^\mathbb{Z}} H_{(i)}^{[0]} x^i \in \mathbb{C}((x^{1/p}))^{\oplus m}$ and is called the semisimple part of $H$. We denote by $\psi[A, (M, Y_M)]$ the $\mathbb{C}$-algebra homomorphism $Y_M(\cdot, x)^{[0]}$ from $A$ to $\mathbb{C}((x^{1/p}))^{\oplus m}$. Note that $\mathcal{J}_M(A)^n((x^{1/p})) = 0$ where $n = \dim_{\mathbb{C}} M$. If $A$ is a field, then $\psi[A, (M, Y_M)]$ is an injective homomorphism from $A$ to $\mathbb{C}((x^{1/p}))^{\oplus m}$. Since $\mathcal{A}_M(A)$ is commutative, we sometimes identify $\text{End}_{\mathbb{C}} M$ with $\text{Mat}_n(\mathbb{C})$ by fixing a basis of $M$ so that the representation matrix of each element of $\mathcal{A}_M(A)$ with respect to the basis is an upper triangular matrix. Under this identification, for $H \in \mathcal{A}_M(A)((x^{1/p})) \subset (\text{End}_{\mathbb{C}} M)((x^{1/p}))$ we see that $H^{[0]} = H^{[0]}$, which is the diagonal part of $H$ defined above.

Let $M$ be a finite-dimensional indecomposable vertex algebra $(K^G, D)$-module. Since $\mathcal{A}_M(K^G)$ is a commutative algebra, $\mathcal{A}_M(K^G)$ is a local algebra and hence $\mathcal{A}_M(K^G) = \mathbb{C} \text{id} \oplus \mathcal{J}_M(K^G)$. Let $(M, Y_M)$ be a $g$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$. Since $\mathcal{A}_M(K^G)$ is a subalgebra of $\mathcal{A}_M(K)$, $M$ is an indecomposable $\mathcal{A}_M(K)$-module. Therefore, $\mathcal{A}_M(K)$ is local since $\mathcal{A}_M(K)$ is commutative. Thus, $\mathcal{A}_M(K) = \mathbb{C} \text{id} \oplus \mathcal{J}_M(K)$. and
hence

$$\psi[K, (M, \tilde{Y}_M)]|_{K^G} = \psi[K^G, (M, Y_M)].$$  \hspace{1cm} (3.2)

Symbol $Q$ denotes the image of the homomorphism $\psi[K^G, (M, Y_M)] : K^G \to \mathbb{C}((x))$. For a polynomial $F(Z) \in K^G[Z]$, $\hat{F}(Z)$ denotes the image of $F(Z)$ under the map $Y_M(\cdot, x) : K^G[Z] \to (\mathcal{A}_M(K^G)((x)))[Z]$ and $\hat{F}^{[0]}(Z)$ denotes the image of $F(Z)$ under the map $\psi[K^G, (M, Y_M)] : K^G[Z] \to Q[Z] \subset \mathbb{C}((x))[Z]$. We write

$$\hat{F}(Z) = \sum_{i \geq 0} \hat{F}_i(x)Z^i, \hat{F}_i(x) \in \mathcal{A}_M(K^G)((x)) \quad \text{and}$$

$$\hat{F}^{[0]}(Z) = \sum_{i \geq 0} \hat{F}_i(x)^{[0]}Z^i, \hat{F}_i(x)^{[0]} \in Q.$$

Since $P(Z)$ is an irreducible polynomial over $K^G$, so is $\hat{P}^{[0]}(Z)$ over $Q$.

Now we state our main theorem.

**Theorem 1.** Let $K$ be a differential field over $\mathbb{C}$ with derivation $D$, $G$ a finite linear automorphism group of $K$ which preserves $D$, and $(M, Y_M)$ a non-zero finite-dimensional indecomposable vertex algebra $(K^G, D)$-module. Then, we have the following results:

1. $M$ has a $g$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$ for some $g \in G$.

2. Let $g$ be an element of $G$ and let $(M, \tilde{Y}_M^1), (M, \tilde{Y}_M^2)$ be two $g$-twisted vertex algebra $(K, D)$-module structures over $(M, Y_M)$ such that $\psi[K, (M, \tilde{Y}_M^1)] = \psi[K, (M, \tilde{Y}_M^2)]$. Then, $(M, \tilde{Y}_M^1) \cong (M, \tilde{Y}_M^2)$ as $g$-twisted vertex algebra $(K, D)$-modules.

3. Let $g$ be an element of $G$ and let $(M, \tilde{Y}_M)$ be a $g$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$. Then, $\tilde{Y}_M \circ h, h \in G$, are all distinct homomorphisms from $K$ to $(\text{End}_\mathbb{C} M)((x^{1/|g|}))$.

4. For $k = 1, 2$, let $g_k \in G$ of order $p_k$ and let $(M, \tilde{Y}_M^k)$ be a $g_k$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$. Then, $(M, \tilde{Y}_M^1) \circ h \cong (M, \tilde{Y}_M^2)$ for some $h \in G$. 
Proof. Set \( n = \dim_\mathbb{C} M \) and \( |G| = N \). Let the notation be as above. Since \( K^G \) is a field, \( \psi[K^G,(M,Y_M)] \) is an injective homomorphism from \( K^G \) to the field \( \mathbb{C}((x)) \) and hence \( K^G \cong Q \) as \( \mathbb{C} \)-algebras. It is well known that any finite extension of \( \mathbb{C}((x)) \) is \( \mathbb{C}((x^{1/j})) \) for some positive integer \( j \) and \( \Omega = \cup_{j=1}^{\infty} \mathbb{C}((x^{1/j})) \) is the algebraic closure of \( \mathbb{C}((x)) \) (cf. [2, Corollary 13.15]). The field \( \mathbb{C}((x^{1/j})) \) becomes a Galois extension of \( \mathbb{C}((x)) \) whose Galois group is the cyclic group generated by the automorphism sending \( x^{1/j} \) to \( \zeta_j x^{1/j} \).

Let \( \hat{K} \) denote the splitting field for \( \hat{P}^{[0]}(Z) \in Q[Z] \) in \( \Omega \) and let \( \mathcal{S} \) denote the set of all \( \mathbb{C} \)-algebra isomorphisms from \( K \) to \( \hat{K} \) whose restrictions to \( K^G \) are equal to \( \psi[K^G,(M,Y_M)] \). Since \( \theta \) is a primitive element of \( K \) over \( K^G \), any map of \( \mathcal{S} \) is uniquely determined by the image of \( \theta \) under that map. Since \( \hat{K} \) is the splitting field for \( P(Z) \in K^G[Z] \), Galois theory implies that \( |\mathcal{S}| = N \) and for any two \( \phi_1, \phi_2 \in \mathcal{S} \) there exists \( h \in G \) such that \( \phi_1(ha) = \phi_2(a) \) for all \( a \in K \).

Let \( g \) be an element of \( G \) and let \( (M,\hat{Y}_M) \) be a \( g \)-twisted vertex algebra \( (K,D) \)-module structure over \( (M,Y_M) \). By [3,2], for all root \( a \in K \) of the polynomial \( P(Z) \) we have

\[
0 = \psi[K,(M,\hat{Y}_M)](P(a)) = \hat{P}^{[0]}(\psi[K,(M,\hat{Y}_M)](a)).
\]

Since \( \hat{K} \) is the splitting field for \( \hat{P}^{[0]}(Z) \in Q[Z] \) in \( \Omega \), the image of \( K \) under the map \( \psi[K,(M,\hat{Y}_M)] \) is equal to \( \hat{K} \) and hence \( \psi[K,(M,\hat{Y}_M)] \) is an element of \( \mathcal{S} \).

We identify \( G \) with \( \text{Gal}(\hat{K}/Q) \) throughout the argument below.

(1) Let \( \phi \) be an arbitrary element of \( \mathcal{S} \). We use Lemma [3] by taking \( \mathcal{B} = K^G \cup \{ \theta \} \). We shall construct a \( \mathbb{C} \)-algebra homomorphism \( \hat{Y}_M(\cdot,x) \) from \( K \cong K^G[Z]/(P(Z)) \) to \( (\text{End}_\mathbb{C} M)((x^{1/N})) \). We first find \( g \in G \) such that for all \( b \in K \) with \( gb = \zeta_i b, i \in \mathbb{Z}, \phi(b) \) is an element of \( x^{-i/p}\mathbb{C}(\mathcal{N}) \). Since \( K \) is a finite extension of \( K^G \) and \( Q \) is a subfield of \( \hat{K} \cap \mathbb{C}((x)) \) and \( \hat{K}\mathbb{C}((x)) = \mathbb{C}((x^{1/p})) \) for some positive integer \( p \). The isomorphism

\[
\text{Gal}(\mathbb{C}((x^{1/p}))/\mathbb{C}((x))) \ni \sigma \mapsto \sigma|_{\hat{K}} \in \text{Gal}(\hat{K}/\hat{K} \cap \mathbb{C}((x)))
\]

implies that \( \text{Gal}(\hat{K}/\hat{K} \cap \mathbb{C}((x))) \) is the cyclic group of order \( p \). Let \( g \in G \) be a generator of \( \text{Gal}(\hat{K}/\hat{K} \cap \mathbb{C}((x))) \) which is the homomorphic image of the element in \( \text{Gal}(\mathbb{C}((x^{1/p}))/\mathbb{C}((x))) \) sending \( x^{1/p} \) to \( \zeta_p x^{1/p} \).
under the isomorphism \( \hat{\psi} \). We have the eigenspace decomposition \( K = \bigoplus_{j=0}^{p-1} K^{(g^j)} \) for \( g \) where \( K^{(g^j)} = \{ a \in K \mid ga = \zeta_p^j a \} \). For \( a \in K^{(g^i)} \) and \( b \in K^{(g^j)}, 1 \leq i, j \leq p - 1 \), taking two integers \( i_0, j_0 \) such that \( i_0 i + j_0 j = (i, j) \), we have \( g(a^{i_0} b^{j_0}) = \zeta_p^{(i,j)} a^{i_0} b^{j_0} \) where \((i, j)\) is the greatest common divisor of \( i \) and \( j \). This implies that we can take a nonzero element \( a \) of \( K^{(g^1)} \) since the order of \( g \) is equal to \( p \). We fix a nonzero element \( a \) of \( K^{(g^1)} \). It follows from \( \phi(a^p) \in \hat{K}^{(g)} = \hat{K} \cap \mathbb{C}(x) \) that \( \phi(a) \) is a root of the polynomial \( Z^p - \phi(a^p) \in \mathbb{C}(x)[Z] \). Thus, \( \phi(a) \) is an element of \( x^{-r/p} \mathbb{C}(x) \) for some integer \( r \). For all \( i = 1, \ldots, p - 1 \), it follows by \( a^i \notin K^{(g)} \) that \( \phi(a^i) \notin \hat{K}^{(g)} = \hat{K} \cap \mathbb{C}(x) \). Therefore, we have \((r, p) = 1\). Taking two integers \( \gamma, \delta \) such that \( \gamma r + \delta p = 1 \) and replacing \( a \) by \( a^r \), we have \( ga = \zeta_p^\gamma a \) and \( \phi(a) \in x^{-1/p} \mathbb{C}(x) \). For all \( b \in \hat{K} \) with \( gb = \zeta_p^\delta b \), since \( a^{-i} b \) is an element of \( K^{(g)} = \hat{K} \cap \mathbb{C}(x) \), we have \( \phi(b) \) is an element of \( x^{-1/p} \mathbb{C}(x) \).

Set \( T(x)^{[0]} = \phi(\theta) \in \hat{K} \), which is a root of \( \hat{P}^{[0]}(Z) \in Q[Z] \). Since \( \hat{P}^{[0]}(Z) \) is the image of \( P(Z) \) under the map \( \psi[K^G, (M, Y_M)] \), it is a separable polynomial and hence \( (d\hat{P}^{[0]}/dZ)(T(x)^{[0]}) \) is not zero. We set

\[
\begin{align*}
\hat{P}_1(x)^{[1]} &= \hat{P}_1(x) - \hat{P}_1(x)^{[0]} \in \mathcal{J}_M(K^G)(x) & \text{and} \\
\hat{P}_1(x)^{[k]} &= 0 \in \mathcal{J}_M(K^G)^k((x))
\end{align*}
\]

for all \( i = 0, \ldots, N \) and \( k = 2, 3, \ldots \) for convenience. For \( k = 1, 2, \ldots, n - 1 \) we inductively define \( T(x)^{[k]} \in \mathcal{J}_M(K^G)^k((x^{1/p})) \) by

\[
T(x)^{[k]} = -\left( \frac{d\hat{P}^{[0]}}{dZ}(T(x)^{[0]}) \right)^{-1} \times \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{j_0 + j_1 + \cdots + j_i = k} \hat{P}_1(x)^{[j_0]} T(x)^{[j_1]} \cdots T(x)^{[j_i]}.
\] (3.4)

Set \( T(x) = \sum_{k=0}^{n-1} T(x)^{[k]} \in \mathcal{A}_M(K^G)((x^{1/p})) \). It follows from the definition of \( T(x) \) that \( T(x)^{[0]} \) is exactly the semisimple part of \( T(x) \) defined
arter (3.1). Since $\mathcal{J}_M(K^G)^n((x)) = 0$, we have

$$\hat{P}(T(x)) = \sum_{i=0}^{N} \hat{P}_i(x)(T(x))^i$$

$$= \sum_{i=0}^{N} \sum_{j=0}^{n-1} \hat{P}_i(x)^{[j]} \left( \sum_{k=0}^{n-1} T(x)^{[k]} \right)^i$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{N} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq n} \hat{P}_i(x)^{[j_0]} T(x)^{[j_1]} \ldots T(x)^{[j_i]}$$

$$= \hat{P}^0(T(x)^0)$$

$$+ \sum_{k=1}^{n-1} \sum_{i=0}^{N} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k} \hat{P}_i(x)^{[j_0]} T(x)^{[j_1]} \ldots T(x)^{[j_i]}$$

$$= 0 + \sum_{k=1}^{n-1} \left( T(x)^{[k]} \frac{d\hat{P}^0}{dZ} (T(x)^0) \right)$$

$$+ \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{0 \leq j_1, \ldots, j_{k-i} \leq k} \hat{P}_i(x)^{[j_0]} T(x)^{[j_1]} \ldots T(x)^{[j_i]}$$

$$= 0. \quad (3.5)$$

Thus, we have an injective homomorphism $\bar{Y}_M(\cdot, x)$ from $K \cong K^G[Z]/(P(Z))$ to $\mathcal{A}_M(K^G)((x^{1/p}))$ sending $\theta$ to $T(x)$. Since $T(x)$ is an element of $\mathcal{A}_M(K^G)((x^{1/p}))$, the $C$-algebra $\mathcal{A}_M(K)$ for $\bar{Y}_M(\cdot, x)$ is equal to $\mathcal{A}_M(K^G)$. In particular, $\mathcal{A}_M(K)$ is a commutative $C$-algebra. Moreover, it follows from $\phi(\theta) = T(x)^0$ that $\psi[K, (\bar{Y}_M, M)] = \phi$.

Let $b$ be an element of $K$ with the minimal polynomial $R(Z) \in K^G[Z]$ over $K^G$ and let $B(x)$ denotes $\bar{Y}_M(b, x)$. We write $R(Z) = \sum_{i=0}^{m} R_i Z^i \in K^G[Z], R_i \in K^G$. We shall show that if $B(x)^0$ is an element of $\mathbb{C}((x))$ id, then $B(x)$ is an element of $(\text{End}_C M)((x))$. By definition, $\bar{R}(B(x)) = 0$ in $(\text{End}_C M)((x^{1/p}))$ and $\bar{R}^0(B(x)^0) = 0$ in $\Omega$ id. We identify $\text{End}_C M$ with $\text{Mat}_n(\mathbb{C})$ by fixing a basis of $M$ so that the representation matrix of each element of $\mathcal{A}_M(K^G)$ with respect to the basis is an upper triangular matrix. We use the expansion $B(x) = \sum_{k=0}^{n-1} B(x)^{(k)}, B(x)^{(k)} \in \Delta_k(\mathbb{C}((x^{1/p})))$. We recall $B(x)^0 = B(x)^{(0)}$. 
and \( \hat{R}_i(x)^{(0)} = \hat{R}_i(x)^{(0)}, i = 0, \ldots, m \). It may be possible that \( B(x)^{(k)} \) and \( \hat{R}_i(x)^{(k)} \) are not elements of \( \mathcal{A}_M(K^G)((x^{1/p})) \) for \( k = 1, \ldots, n - 1 \). Since \( B(x)^{(0)} \) and \( \hat{R}_i(x)^{(0)} \) are elements of \( \mathbb{C}((x^{1/p}))E_n \), they commute any element of \( (\text{End}_C M)((x^{1/p})) \). We denote the diagonal part of \( \hat{R}(Z) \) by \( \hat{R}^{(0)}(Z) \), namely, \( \hat{R}^{(0)}(Z) = \sum_{i=0}^{m} \hat{R}_i(x)^{(0)} Z^i \). Under the identification of \( \text{End}_C M \) with \( \text{Mat}_n(\mathbb{C}) \) above, \( \hat{R}^{(0)}(Z) \) is equal to \( \hat{R}^{(0)}(Z) \). The same computation as (3.5) shows

\[
0 = \hat{R}(B(x)) = \sum_{i=0}^{m} \hat{R}_i(x)(B(x))^i \\
= \sum_{i=0}^{m} \left( \sum_{j=0}^{n-1} \hat{R}_i(x)^{(j)} \right) \left( \sum_{k=0}^{i-1} B(x)^{(k)} \right)^i \\
= \sum_{k=0}^{n-1} \sum_{i=0}^{m} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq n} \hat{R}_i(x)^{(j_0)} B(x)^{(j_1)} \cdots B(x)^{(j_i)} \\
= \hat{R}^{(0)}(B(x)^{(0)}) \\
+ \sum_{k=1}^{n-1} \sum_{i=0}^{m} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k} \hat{R}_i(x)^{(j_0)} B(x)^{(j_1)} \cdots B(x)^{(j_i)} \\
= \sum_{k=1}^{n-1} \left( B(x)^{(k)} \frac{d\hat{R}^{(0)}}{dZ}(B(x)^{(0)}) \right) \\
+ \sum_{i=0}^{m} \sum_{j_0=0}^{k} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k} \hat{R}_i(x)^{(j_0)} B(x)^{(j_1)} \cdots B(x)^{(j_i)} \right) . \quad (3.6)
\]

Since \( \hat{R}^{(0)}(Z) \in Q[Z] \) is the image of \( R(Z) \) under the map \( \psi[K^G, (M, Y_M)] \), \( \hat{R}^{(0)}(Z) \) is a separable polynomial and hence \( (d\hat{R}^{(0)}/dZ)(B(x)^{(0)}) \) is not zero. Thus, we have

\[
B(x)^{(k)} = -\left( \frac{d\hat{R}^{(0)}}{dZ}(B(x)^{(0)}) \right)^{-1} \\
\times \sum_{i=0}^{m} \sum_{j_0=0}^{k} \sum_{0 \leq j_1, \ldots, j_i \leq k} \hat{R}_i(x)^{(j_0)} B(x)^{(j_1)} \cdots B(x)^{(j_i)}. 
\]
Since $B(x)^{(0)} = \phi(b) \in \mathbb{C}((x))E_n$ and $\hat{R}_i(x) \in (\text{End}_\mathbb{C} M)((x))$, it follows by induction on $k$ that $B(x)^{(k)}$ is an element of $(\text{End}_\mathbb{C} M)((x))$. We conclude that $B(x)$ is an element of $(\text{End}_\mathbb{C} M)((x))$.

For all $b \in K$ with $gb = \zeta_p b$, we shall show that $\tilde{Y}_M(b, x)$ is an element of $x^{-i/p}(\text{End}_\mathbb{C} M)((x))$. It follows from $b^p \in K^{(g)}$ that $C(x)^{(0)} = (B(x)^{(0)})^p$ is an element of $\hat{K}^{(g)} = \hat{K} \cap \mathbb{C}((x))$. The argument above shows $C(x)$ is an element of $(\text{End}_\mathbb{C} M)((x))$. We identify $\text{End}_\mathbb{C} M$ with $\text{Mat}_n(\mathbb{C})$ by fixing a basis of $M$ so that the representation matrix of each element of $A_M(K^G)$ with respect to the basis is an upper triangular matrix. We use the expansion $B(x) = \sum_{k=0}^{n-1} B(x)^{(k)}B(x)^{(k)} \in \Delta_k(\mathbb{C}((x)^{1/p})))$. We have already seen in the first part of the proof of (1) that $B(x)^{(0)} = \phi(b)$ is an element of $x^{-i/p}(\text{End}_\mathbb{C} M)((x))$. By $C(x) = B(x)^p$, the same computation as (3.6) shows

$$B(x)^{(k)} = -p^{-1}(B(x)^{(0)})^{-p+1} \times (C(x)^{(k)} + \sum_{0 \leq j_1, \ldots, j_p < k} B(x)^{(j_1)} \cdots B(x)^{(j_p)}),$$

for all $k = 1, \ldots, n - 1$. It follows by induction on $k$ that $B(x)^{(k)}$ is an element of $x^{-i/p}(\text{End}_\mathbb{C} M)((x))$. We conclude that $B(x)$ is an element of $x^{-i/p}(\text{End}_\mathbb{C} M)((x))$.

We shall show $\tilde{Y}_M(D\theta, x) = d\tilde{Y}_M(\theta, x)/dx$. It follows from $P(\theta) = 0$ that

$$0 = D(P(\theta)) = \sum_{i=0}^{N} (DP_i)\theta^i + \left(\frac{dP}{dZ}(\theta)\right)(D\theta).$$

and hence

$$0 = \sum_{i=0}^{N} \tilde{Y}_M(DP_i, x)\tilde{Y}_M(\theta, x)^i + \tilde{Y}_M\left(\frac{dP}{dZ}(\theta), x\right)\tilde{Y}_M(D\theta, x). \quad (3.7)$$
We also have

\[
0 = \frac{d}{dx} \tilde{Y}_M(P(\theta), x)
= \frac{d}{dx} \sum_{i=0}^N \tilde{Y}_M(P_i, x)\tilde{Y}_M(\theta, x)^i
= \sum_{i=0}^N \frac{d\tilde{Y}_M(P_i, x)}{dx} \tilde{Y}_M(\theta, x)^i + \sum_{i=0}^N \tilde{Y}_M(P_i, x)i\tilde{Y}_M(\theta, x)^i-1 \frac{d\tilde{Y}_M(\theta, x)}{dx}
= \sum_{i=0}^N \frac{dY_M(P_i, x)}{dx} \tilde{Y}_M(\theta, x)^i + Y_M\left(\frac{dP}{dZ}(\theta), x\right)\tilde{Y}_M(\theta, x)
\]

(3.8)

Since \(P(Z) \in K^G[Z]\) is a separable polynomial in \(Z\), \((dP/dZ)(\theta)\) is not zero and hence \(\tilde{Y}_M((dP/dZ)(\theta), x)\) is an invertible element of \((\text{End}_C M)((x^{1/p}))\). Since \(Y_M(DP_i, x) = dY_M(P_i, x)/dx\) for all \(i\), we have \(\tilde{Y}_M(D\theta, x) = d\tilde{Y}_M(\theta, x)/dx\) by (3.7) and (3.8).

By Lemma 4, we conclude that \((M, \tilde{Y}_M)\) is a \(g\)-twisted vertex algebra \((K, D)\)-module structure over \((M, Y_M)\).

(2) We denote the order of \(g\) by \(p\) and \(\psi[K, (M, \tilde{Y}_M)] = \psi[K, (M, \tilde{Y}_M^2)]\) by \(\Psi\) for simplicity. We recall \(\Psi\) is an element of \(S\) as mentioned before starting the proof of (1).

Let \((M, \tilde{Y}_M)\) be a \(g\)-twisted vertex algebra \((K, D)\)-module structure over \((M, Y_M)\) constructed in (1) by taking \(\Psi\) as \(\phi\). It is enough to show that \(\tilde{Y}_M = Y_M^1\). We denote \(\mathcal{A}_M(K)\) for \((M, \tilde{Y}_M)\) by \(\mathcal{A}^0\) and \(\mathcal{A}_M(K)\) for \((M, Y_M^1)\) by \(\mathcal{A}^1\). We have seen in (1) that \(\mathcal{A}^0 = \mathcal{A}_M(K^G)\). Thus, \(\mathcal{A}^0\) is a subalgebra of \(\mathcal{A}^1\). We identify \(\text{End}_C M\) with \(\text{Mat}_n(C)\) by fixing a basis of \(M\) so that the representation matrix of each element of \(\mathcal{A}^1\) with respect to the basis is an upper triangular matrix. We denote \(\tilde{Y}_M^1(\theta, x)\) by \(U(x)\). We use the expansions

\[
U(x) = \sum_{k=0}^{n-1} U(x)^{(k)}, U(x)^{(k)} \in \Delta_k(C((x^{1/p}))) \text{ and }
\]

\[
\hat{P}_i(x) = \sum_{k=0}^{n-1} \hat{P}_i(x)^{(k)}, \hat{P}_i(x)^{(k)} \in \Delta_k(C((x^{1/p}))).
\]
We denote the diagonal part of $\hat{P}(Z)$ by $\hat{P}(0)(Z)$, namely, $\hat{P}(0)(Z) = \sum_{i=0}^{N} \hat{P}_i(x)(0)Z^i$. Under the identification of $\text{End}_\mathbb{C} M$ with $\text{Mat}_n(\mathbb{C})$ above, $\hat{P}(0)(Z)$ is equal to $\tilde{P}(0)(Z)$. Note that we do not assume $U(x) \in A_M(K^G)((x^{1/p}))$. We recall $\Psi(\theta) = U(x)(0)$ by definition. We have $\hat{P}(0)(U(x)(0)) = \Psi(P(\theta)) = 0$ and $(d\hat{P}(0)/dZ)(U(x)(0))$ is not zero since $\tilde{P}(0)(Z)$ is a separable polynomial. The same computation as (3.6) shows

\[
0 = \hat{P}(U(x)) \\
= \hat{P}(0)(U(x)(0)) \\
+ \sum_{k=1}^{n-1} \sum_{i=0}^{N} \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k} \hat{P}_i(x)(j_0)U(x)(j_1) \cdots U(x)(j_i)
\]

\[
= \sum_{k=1}^{n-1} \left( U(x)(k) \frac{d\hat{P}(0)}{dZ}(U(x)(0)) \right) \\
+ \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{0 \leq j_1, \ldots, j_i < k} \hat{P}_i(x)(j_0)U(x)(j_1) \cdots U(x)(j_i)
\]

and hence

\[
U(x)(k) = -\left( \frac{d\hat{P}(0)}{dZ}(\Psi(\theta))^{-1} \right) \\
\times \sum_{i=0}^{N} \sum_{j_0=0}^{k} \sum_{0 \leq j_1, \ldots, j_i < k} \hat{P}_i(x)(j_0)U(x)(j_1) \cdots U(x)(j_i).
\]

for all $k = 1, \ldots, n-1$. It follows by induction on $k$ that $U(x) = \sum_{k=0}^{n-1} U(x)(k)$ is uniquely determined by $\Psi(\theta)$ and $\hat{P}(Z)$. By definition of $\Psi$, we have $\tilde{Y}_M(\theta, x) = U(x) = \tilde{Y}_M^1(\theta, x)$ and hence $\tilde{Y}_M = \tilde{Y}_M^1$.

(3) Let $h \in G$ with $h \neq 1$. Since $h^{-1}(\theta) \neq \theta$ and $\tilde{Y}_{M,\theta h}(h^{-1}\theta, x) = \tilde{Y}_M(\theta, x)$, $\tilde{Y}_{M,\theta h}(\cdot, x)$ is distinct from $\tilde{Y}_M(\cdot, x)$. This implies that $\tilde{Y}_M \circ h, h \in G$, are all distinct homomorphisms from $K$ to $(\text{End}_\mathbb{C} M)((x^{1/|g|}))$.

(4) For $k = 1, 2$, let $g_k \in G$ of order $p_k$ and let $(M, \tilde{Y}_M^k)$ be a $g_k$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$. We denote $\psi[K, (M, \tilde{Y}_M^k)]$ by $\psi_k$ and $\psi[K^G, (M, Y_M)]$ by $\psi$ briefly.
Since \( \psi_1, \psi_2 \in S \), there exists \( h \in G \) such that \( \psi_1(ha) = \psi_2(a) \) for all \( a \in K \). By definition of \( (M, \bar{Y}_M^1) \circ h \), we have \( \psi[K, (M, \bar{Y}_M^1) \circ h](a) = \psi_1(ha) = \psi_2(a) \) for all \( a \in K \). We conclude that \( (M, \bar{Y}_M^1) \circ h \cong (M, \bar{Y}_M^2) \) by (2).

\[ \square \]

4 Finite-dimensional vertex algebra \( \mathbb{C}(s) \)-modules

Throughout the rest of this paper, \( \mathbb{C}(s) \) is the field of rational functions in one variable \( s \). In this section we classify the finite-dimensional vertex algebra \( \mathbb{C}(s) \)-modules. We use the notation introduced in Section 3. It is easy to see that every non-zero derivation \( D \) of \( \mathbb{C}(s) \) can be expressed as \( D = (p(s)/q(s))d/ds \), where \( p(s) \) and \( q(s) \) are non-zero coprime elements of \( \mathbb{C}[s] \). We write

\[
p(s) = \sum_{i=L_p}^{N_p} p_is^i \quad \text{and} \quad q(s) = \sum_{i=L_q}^{N_q} q_is^i
\]

where \( p_{L_p}, p_{N_p}, q_{L_q}, q_{N_q} \) are all non-zero complex numbers.

The following lemma is a corollary of Lemma 4.

**Lemma 5.** Let the notation be as above. Let \( M \) be a finite-dimensional vector space and let \( S(x) = \sum_{i \in \mathbb{Z}} S(i)x^i \) be an element of \( \text{End}_\mathbb{C} M(\langle x \rangle) \). Then, there exists a vertex algebra \( (\mathbb{C}(s), D) \)-module \( (M, Y_M) \) with \( Y_M(s, x) = S(x) \) if and only if the following three conditions hold:

(i) For all \( \alpha \in \mathbb{C} \), \( S(x) - \alpha \) is an invertible element of \( \text{End}_\mathbb{C} M(\langle x \rangle) \).

(ii) For all \( i, j \in \mathbb{Z} \), \( S(i)S(j) = S(j)S(i) \).

(iii) \( dS(x)/dx = p(S(x))/q(S(x)) \).

In this case, for \( u(s) \in \mathbb{C}(s) \) we have \( Y_M(u(s), x) = u(S(x)) \) and hence \( (M, Y_M) \) is uniquely determined by \( S(x) \).

**Proof.** We use Lemma 4 by taking \( B = \{s\} \cup \{(s-\alpha)^{-1} \mid \alpha \in \mathbb{C}\} \). Suppose that \( (M, Y_M) \) is a vertex algebra \( (\mathbb{C}(s), D) \)-module. Then, \( Y_M(\cdot, x) \) is a \( \mathbb{C} \)-algebra homomorphism from \( \mathbb{C}(s) \) to \( \text{End}_\mathbb{C} M(\langle x \rangle) \). For all \( \alpha \in \mathbb{C} \), since...
s − α is an invertible element in \( \mathbb{C}(s) \), so is \( Y_M(s - \alpha, x) = S(x) - \alpha \) in \((\text{End}_C M)(x)\). The other conditions in (ii) and (iii) are clearly hold.

Conversely, suppose that \((M, Y_M)\) satisfies the conditions (i)–(iii). Since \( S(x) - \alpha \) is an invertible element of \((\text{End}_C M)(x)\) for all \( \alpha \in \mathbb{C} \), we obtain \( q(S(x)) \neq 0 \) and we can define the \( \mathbb{C} \)-algebra homomorphism \( Y_M : \mathbb{C}(s) \ni u(s) \mapsto u(S(x)) \in (\text{End}_C M)(x) \). Since for all \( \alpha \in \mathbb{C} \) each coefficient of \( x^j, j \in \mathbb{Z} \) in the expansion of \((S(x) - \alpha)^{-1}\) is a polynomial in \( \{S(k) \mid k \in \mathbb{Z}\} \), \( \mathcal{A}_M(\mathbb{C}(s)) \) is commutative. For all \( \alpha \in \mathbb{C} \) we have

\[
Y_M\left(D((s-\alpha)^{-1}), x\right) = Y_M\left(-Ds, (s-\alpha)^{-2}, x\right) = -Y_M(Ds, x)(Y_M(s, x) - \alpha)^{-2} = \frac{d}{dx}(Y_M(s, x) - \alpha)^{-1}.
\]

We conclude that \((M, Y_M)\) is a vertex algebra \((\mathbb{C}(s), D)\)-module. \(\square\)

Note that there is no nontrivial finite-dimensional associative algebra \( \mathbb{C}(s) \)-module since \( \mathbb{C}(s) \) is an infinite-dimensional \( \mathbb{C} \)-vector space and \( \mathbb{C}(s) \) is a field.

Let \( M \) be a finite-dimensional vector space over \( \mathbb{C} \). For \( X \in \text{End}_C M, X^{[0]} \) denotes the semisimple part of \( X \) and \( X^{[1]} \) denotes the nilpotent part of \( X \). For \( H(x) = \sum_{i \in \mathbb{Z}} H(i)x^i \in (\text{End}_C M)[[x, x^{-1}]], H(x)^{[0]} = \sum_{i \in \mathbb{Z}} H(i)^{[0]}x^i \) and \( H(x)^{[1]} = \sum_{i \in \mathbb{Z}} H(i)^{[1]}x^i \). We call \( H(x)^{[0]} \) the semisimple part of \( H(x) \). For \( H(x) = \sum_{i \in \mathbb{Z}}^\infty H(i)x^i \in (\text{End}_C M)((x)) \) with \( H_{(L)} \neq 0, \text{ld}(H(x)) \) denotes \( L \) and \( \text{lc}(H(x)) \) denotes \( H_{(L)} \).

For a finite-dimensional indecomposable vertex algebra \( \mathbb{C}(s) \)-module \((M, Y_M)\), we denote \( Y_M(s, x) \) by \( S(x) \). Let \( J_n \) denote the following \( n \times n \) matrix:

\[
J_n = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 1 & \ddots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
\]

**Theorem 2.** Let the notation be as above. Let \( \alpha \) be a non-zero complex number. We have the following results:

1. There exists a non-zero finite-dimensional indecomposable vertex algebra \((\mathbb{C}(s), D)\)-module \( M \) with \( \text{ld}(S(x)^{[0]}) > 0 \) and with \( \text{lc}(S(x)^{[0]}) = \alpha \)
if and only if \( p(0)q(0) \neq 0 \) and \( \alpha = p(0)/q(0) \). Moreover, in this case \( \text{ld}(S(x)^{[0]}) = 1 \) and \( S(x) \in (\text{End}_C M)[[x]] \).

(2) There exists a non-zero finite-dimensional indecomposable vertex algebra \((C(s), D)\)-module \( M \) with \( \text{ld}(S(x)^{[0]}) = 0 \) and with \( \text{lc}(S(x)^{[0]}) = \alpha \) if and only if \( p(\alpha)q(\alpha) \neq 0 \). Moreover, in this case \( S(x) \in (\text{End}_C M)[[x]] \) and \( S^{[0]} = p(\alpha)/q(\alpha) \).

(3) There exists a non-zero finite-dimensional indecomposable vertex algebra \((C(s), D)\)-module \( M \) with \( \text{ld}(S(x)^{[0]}) < 0 \) and with \( \text{lc}(S(x)^{[0]}) = \alpha \) if and only if \( \deg p = \deg q + 2 \) and \( \alpha = -q_N/p_N \). Moreover, in this case \( \text{ld}(S(x)^{[0]}) = -1 \).

In each case, for all positive integers \( n \), there exists a unique \( n \)-dimensional indecomposable vertex algebra \((C(s), D)\)-module which satisfies the conditions up to isomorphism.

**Proof.** (1) Let \( n \) be a positive integer and let \((M, Y_M)\) be an \( n \)-dimensional indecomposable vertex algebra \((C(s), D)\)-module and let \( S(x) \) denotes \( Y_M(s, x) \). Since \( M \) is indecomposable, \( S(x)^{[0]} \) is an element of \( C((x)) \text{id} \). If \( S(x)^{[0]} = 0 \), then \( S(x) \) is nilpotent. This is impossible since \( C(s) \) is a field. Thus, we write \( S(x)^{[0]} = \sum_{i=L}^{\infty} S^{[0]}_{(i)} x^i \) where \( L = \text{ld}(S(x)^{[0]}) \). Since \( D = (p(s)/q(s))d/ds \), we have

\[
\frac{dS(x)}{dx} = q(S(x))^{-1} p(S(x)) \quad (4.1)
\]

or equivalently

\[
q(S(x)) \frac{dS(x)}{dx} = p(S(x)). \quad (4.2)
\]

Since \( q(S(x)) \) is an invertible element of \((\text{End}_C M)((x))\), \( q(S(x)^{[0]}) \) is not zero. Taking the semisimple part of (4.2), we have

\[
q(S(x)^{[0]}) \frac{dS(x)^{[0]}}{dx} = p(S(x)^{[0]}). \quad (4.3)
\]

Suppose that \( L \) is a positive integer, namely \( S(x)^{[0]} \) is an element of \( xC[[x]] \). Then, \( S^{[0]}_{(0)} = 0 \) and hence \( S_{(0)} \) is a nilpotent element of \( \text{End}_C M \).
We shall show $p(0)q(0) \neq 0$, $\text{lc}(S(x)^0) = 1$, and $\text{lc}(S(x)^0) = p(0)/q(0)$. In (4.3), the term with the lowest degree of the left-hand side is $q_{L_q}L(S(0)^0)_{L_q+1}xL(L_q+1)^{-1}$ and the term with the lowest degree of the right-hand side is $p_{L_p}(S(0)^0)_{L_q}xL\Sigma L_p$. Comparing these terms, we have $L(L_p-L_q-1) = -1$. Therefore, $L = 1$ and $L_p = L_q$. We have $L_p = L_q = 0$ since $p(x)$ and $q(x)$ are coprime. Thus, both $p_0, q_0$ are not zero. Comparing the coefficients of these terms, we also have $S^{0}_{(1)} = p_0/q_0 = p(0)/q(0)$.

We shall show that $S(x)$ is an element of $(\text{End}_C M)[[x]]$. In order to do that, we identify $\text{End}_C M$ with $\text{Mat}_n(\mathbb{C})$ by fixing a basis of $M$ so that the representation matrix of each element of $A_M(\mathbb{C}(s))$ with respect to the basis is upper triangular matrix for a while. We write the expansion $S(x) = \sum_{k=0}^{n-1} S(x)^{(k)}, S(x)^{(k)} \in \Delta_k(\mathbb{C}(x))$ as in Section 3. We recall that $S(x)^0$ is equal to the diagonal part $S(x)^{(0)}$. We need to expand $(dS(x)/dx)q(S(x))$ and $p(S(x))$. By the same computation as (3.5), we have

$$\frac{dS(x)}{dx}q(S(x))$$

$$= \sum_{j_0=0}^{n-1} \frac{dS(x)^{(j_0)}}{dx} \left( \sum_{i=0}^{N_q} q_i \sum_{0 \leq j_1, \ldots, j_i \leq n-1} S(x)^{(j_1)} \cdots S(x)^{(j_i)} \right)$$

$$= \sum_{i=0}^{N_q} q_i \sum_{0 \leq j_0, j_1, \ldots, j_i \leq n-1} \frac{dS(x)^{(j_0)}}{dx} S(x)^{(j_1)} \cdots S(x)^{(j_i)}$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{N_q} q_i \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k, j_0 + j_1 + \cdots + j_i = k} \frac{dS(x)^{(j_0)}}{dx} S(x)^{(j_1)} \cdots S(x)^{(j_i)}$$

$$= \frac{dS(x)^{(0)}}{dx}q(S(x)^{(0)})$$

$$+ \sum_{k=1}^{n-1} \left( \frac{dS(x)^{(k)}}{dx}q(S(x)^{(0)}) + \frac{dS(x)^{(0)}}{dx} \frac{dq}{ds}(S(x)^{(0)})S(x)^{(k)} \right)$$

$$+ \sum_{i=0}^{N_q} q_i \sum_{0 \leq j_0, j_1, \ldots, j_i \leq k, j_0 + j_1 + \cdots + j_i = k} \frac{dS(x)^{(j_0)}}{dx} S(x)^{(j_1)} \cdots S(x)^{(j_i)}$$
and

\[ p(S(x)) = p(S(x)^{(0)}) + \sum_{k=1}^{n-1} \left( \frac{dp}{ds}(S(x)^{(0)})S(x)^{(k)} \right. \]

\[ \left. + \sum_{i=0}^{N_p} p_i \sum_{0 \leq j_1, \ldots, j_i < k} S(x)^{(j_1)} \cdots S(x)^{(j_i)} \right). \]

Thus, it follows from (4.2) that for all \( k = 1, 2, \ldots, n - 1 \)

\[
\frac{dS(x)^{(k)}}{dx} = q(S(x)^{(0)})^{-1} \left( - \frac{dS(x)^{(0)}}{dx} \frac{dq}{ds}(S(x)^{(0)}) + \frac{dp}{ds}(S(x)^{(0)}) \right) S(x)^{(k)} \]

\[ - \sum_{i=0}^{N_q} q_i \sum_{0 \leq j_0, j_1, \ldots, j_i < k} \frac{dS(x)^{(j_0)}}{dx} S(x)^{(j_1)} \cdots S(x)^{(j_i)} \]

\[ + \sum_{i=0}^{N_p} p_i \sum_{0 \leq j_1, \ldots, j_i < k} S(x)^{(j_1)} \cdots S(x)^{(j_i)} \]. \quad (4.4)

Now we show that \( S(x)^{(k)} \) is an element of \((\text{End} C M)[[x]]\) by induction on \( k \). The case \( k = 0 \) follows from \( \text{ld}(S(x)^{(0)}) = 1 \). For \( k > 0 \), suppose that \( \text{ld}(S(x)^{(k)}) < 0 \). This implies \( \text{ld}(dS(x)^{(k)}/dx) = \text{ld}(S(x)^{(k)}) - 1 \). Since \( q_0 \neq 0 \) and \( S(x)^{(0)} \) is an element of \( xC[[x]] \), \( q(S(x)^{(0)})^{-1} \) is an element of \( C[[x]] \). Thus, the lowest degree of the right-hand side of (4.4) is greater than or equal to \( \text{ld}(S(x)^{(k)}) \) by the induction assumption. This contradicts \( \text{ld}(dS(x)^{(k)}/dx) = \text{ld}(S(x)^{(k)}) - 1 \). We conclude that \( S(x) \) is an element of \((\text{End} C M)[[x]]\).

We shall show that \( S(x) \) is uniquely determined by \( D \) and \( S(0) \). For all positive integer \( m \), we expand \( (dS(x)/dx)q(S(x)) \) and \( p(S(x)) \) modulo
$x^m \mathbb{C}[[x]]$:

\[
\frac{dS(x)}{dx} q(S(x)) \equiv \left( \sum_{j=1}^{m} j S_j x^{j-1} \right) \sum_{i=0}^{N_q} q_i (\sum_{j=0}^{m-1} S_j x^j)^i
\]

\[
\equiv \sum_{k=0}^{m-1} \sum_{i=0}^{N_q} q_i \sum_{j_0=1}^{m} \sum_{0 \leq j_1, \ldots, j_i \leq m-1} \sum_{j_0 + j_1 + \cdots + j_i = k+1} j_0 S_{(j_0)} S_{(j_1)} \cdots S_{(j_i)} x^k
\]

\[
= mS_{(m)} q(S(0)) x^{m-1}
\]

\[
+ \sum_{k=0}^{m-1} \sum_{i=0}^{N_q} q_i \sum_{j_1=0}^{m-1} \sum_{0 \leq j_1, \ldots, j_i \leq m-1} \sum_{j_0 + j_1 + \cdots + j_i = k+1} j_0 S_{(j_0)} S_{(j_1)} \cdots S_{(j_i)} x^k \pmod{x^m \mathbb{C}[[x]]}
\]

(4.5)

and

\[
p(S(x)) \equiv \sum_{k=0}^{m-1} \sum_{i=0}^{N_p} p_i \sum_{0 \leq j_1, \ldots, j_i \leq m-1} \sum_{j_1 + \cdots + j_i = k} S_{(j_1)} \cdots S_{(j_i)} x^k \pmod{x^m \mathbb{C}[[x]]}.
\]

(4.6)

Comparing the coefficients of $x^{m-1}$ in (4.5) and (4.6), it follows from (4.2) that

\[
mS_{(m)} q(S(0))
\]

\[
= - \sum_{i=0}^{N_q} q_i \sum_{j_0=1}^{m-1} \sum_{0 \leq j_1, \ldots, j_i \leq m-1} \sum_{j_0 + j_1 + \cdots + j_i = m} j_0 S_{(j_0)} S_{(j_1)} \cdots S_{(j_i)}
\]

\[
+ \sum_{i=0}^{N_p} p_i \sum_{0 \leq j_1, \ldots, j_i \leq m-1} \sum_{j_1 + \cdots + j_i = m-1} S_{(j_1)} \cdots S_{(j_i)}.
\]

(4.7)

Since $q_0 \neq 0$ and $S(0)$ is nilpotent, $q(S(0))$ is an invertible element of $\text{End}_\mathbb{C} M$ and $q(S(0))^{-1}$ is a polynomial in $S(0)$. Thus, it follows by induction on $m$ that every $S_{(m)}$ is a polynomial in $S(0)$ and is uniquely determined by $D$ and $S(0)$. We conclude that $S(x)$ is uniquely determined by $D$ and $S(0)$. 

Since every $S_{(m)}$, $m \in \mathbb{Z}$ is a polynomial in $S_{(0)}$ and $M$ is an indecomposable $A_M(\mathcal{C}(s))$-module, the nilpotent element $S_{(0)}$ conjugates to $J_n$ under the identification of $\text{End}_C M$ with $\text{Mat}_n(\mathbb{C})$. Therefore, Lemma 5 implies that $S(x)$ and hence $(M, Y_M)$ with $\dim_C M = n$ is uniquely determined up to isomorphism under the conditions in (1).

Conversely, suppose that $p(0)q(0) \neq 0$ and $\alpha = p(0)/q(0)$. We shall construct $S(x) \in (\text{End}_C M)[[x]]$ which satisfies the conditions in (1). We identify $\text{End}_C M$ with $\text{Mat}_n(\mathbb{C})$ by fixing a basis of $M$ and we use Lemma 5. Set $S_{(0)} = J_n$. By (4.7) we can inductively define $S_{(m)}$ for $m = 1, 2, \ldots$. Reversing the argument used to get (4.7) above, it is easy to see that the obtained upper triangular matrix $S(x) = \sum_{m=0}^{\infty} S_{(m)} x^m \in (\text{Mat}_n(\mathbb{C}))[x]$ satisfies (4.2). Taking the semisimple part of (4.7), we obtain the formula by replacing $S_{(j)}$ with $S_{(j)}[0]$ for all $j \in \mathbb{Z}$ in (4.7). By this formula for $m = 1$, we have $S_{(1)}[0] = p_0/q_0 = \alpha$. By this formula again, an inductive argument on $m$ shows $S(x)[0] = \sum_{m=0}^{\infty} S_{(m)}[0] x^m$ is a non-constant element of $\mathbb{C}[[x]]E_n$. Thus, $q(S(x))$ is an invertible element of $(\text{Mat}_n(\mathbb{C}))(x)$ and hence (4.1) holds. Since all coefficients of $S(x)$ are polynomials in $S_{(0)} = J_n$, $S_{(i)}S_{(j)} = S_{(j)}S_{(i)}$ for all $i, j \in \mathbb{Z}$. By Lemma 5, we have obtained an $n$-dimensional vertex algebra $(\mathcal{C}(s), D)$-module $M$ with $\text{ld}(S(x)[0]) = 1$ and with $\text{lc}(S(x)[0]) = \alpha$. This completes the proof of (1).

(2) Next, suppose that $L = 0$. Setting $\bar{s} = s - \alpha$, we have

$$D = \frac{p(\bar{s} + \alpha)}{q(\bar{s} + \alpha)} \frac{d}{d\bar{s}}.$$

Thus, this case reduces to the case of $L > 0$. It follows by (1) that $S_{(1)}[0] = p(\alpha)/q(\alpha)$.

(3) Finally, suppose that $L$ is a negative integer. Setting $\bar{s} = 1/s$, we have $Y_M(\bar{s}, x)[0] = 1/S(x)[0] \in x\mathbb{C}((x))$ and

$$D = \frac{p(1/\bar{s})}{q(1/\bar{s})} (-\bar{s}^2) \frac{d}{d\bar{s}}.$$
It follows by $S(x)^{[0]} \neq 0$ that $S(x)$ is invertible in $(\text{End}_\mathbb{C} M)((x)))$ and that $(S(x)^{-1})^{[0]} = (S(x)^{[0]})^{-1}$. Since $S(x)^{-1}$ is a polynomial in $S(x)$, all coefficients in $S(x)^{-1}$ are commutative. Thus, this case also reduces to the case of $L > 0$.

\[ \square \]

5 Examples

Throughout this section, $D$ is a non-zero derivation of $\mathbb{C}(s)$ and $f(s) = \sum_{j=0}^m f_j s^j \in \mathbb{C}[s]$ is a square-free polynomial of degree $m$ at least 3. Set $K = \mathbb{C}(s)[t]/(t^2 - f(s))$, which is a quadratic extension of $\mathbb{C}(s)$. Let $\sigma$ be the generator of the Galois group of $K$ over \( \mathbb{C}(s) \) mapping $t$ to $-t$. Since $K$ is a finite Galois extension of $\mathbb{C}(s)$, $D$ can be uniquely extends to a derivation of $K$, which is also denoted by $D$.

Let $(M, Y_M)$ be a finite-dimensional indecomposable $(\mathbb{C}(s), D)$-module. In this section, we shall investigate untwisted or $\sigma$-twisted vertex algebra $(K, D)$-module structures over $(M, Y_M)$. We denote $Y_M(s, x)$ by $S(x)$ and its semisimple part by $S(x)^{[0]} = \sum_{i=\ell}^{\infty} S_{(i)}^{[0]} x^i$ with $S_{(0)}^{[0]} \neq 0$ as in Section 4. Theorem 1 says that $L = \text{ld}(S(x)^{[0]}) = 1, 0, \text{ or } -1$.

**Proposition 6.** Let $M$ be a finite-dimensional indecomposable vertex algebra $(\mathbb{C}(s), D)$-module and let $(M, Y_M)$ be a $g$-twisted vertex algebra $(K, D)$-module structure over $(M, Y_M)$ in Theorem 2, where $g = 1$ or $\sigma$. Then

1. In the case of $\text{ld}(S(x)^{[0]}) = 1$, $g = \sigma$ if and only if $f_0 = 0$.

2. In the case of $\text{ld}(S(x)^{[0]}) = 0$, $g = \sigma$ if and only if $f(S_{(0)}^{[0]}) = 0$.

3. In the case of $\text{ld}(S(x)^{[0]}) = -1$, $g = \sigma$ if and only if $\deg f$ is odd.

**Proof.** We denote by $\sqrt{f(S(x)^{[0]})}$ a square root of $f(S(x)^{[0]})$ in $\Omega = \bigcup_{i=1}^{\infty} \mathbb{C}((x^{1/i}))$. We denote by $\psi$ the $\mathbb{C}$-algebra homomorphism $\psi[\mathbb{C}(s), (M, Y_M)] : \mathbb{C}(s) \to \mathbb{C}(x)$ defined just after (3.3) and by $Q$ the image of $\psi$. Let $\hat{K}$ denote the splitting field for $Z^2 - \psi(f(s)) = Z^2 - f(S(x)^{[0]}) \in Q[Z]$ in $\Omega$ as in the proof of Theorem 2. We have $\hat{K} \cap \mathbb{C}((x)) = \hat{K}$ if and only if $\sqrt{f(S(x)^{[0]})} \in \mathbb{C}((x))$. By the argument just after (3.3), $g$ is a generator of $\text{Gal}(\hat{K}/\hat{K} \cap \mathbb{C}((x)))$. Thus, $g = 1$ if and only if $\sqrt{f(S(x)^{[0]})} \in \mathbb{C}((x))$. We use the following
expansion of \( f(S(x)^{[0]}) \):

\[
f(S(x)^{[0]}) = \sum_{j=0}^{m} f_j \left( \sum_{i=L}^{\infty} S_{(i)}^{[0]} x^i \right)^j
\]

\[
= \begin{cases} 
  f_0 + f_1 S_{(1)}^{[0]} x + \cdots, & \text{if } L = 1, \\
  f(S_{(0)}^{[0]}) + S_{(1)}^{[0]} f'(S_{(0)}^{[0]}) x + \cdots, & \text{if } L = 0, \\
  f_m (S_{(-1)}^{[0]})^m x^{-m} + \cdots, & \text{if } L = -1.
\end{cases}
\] (5.1)

(1) In this case if \( f_0 \neq 0 \), then \( \sqrt{f(S(x)^{[0]})} \in \mathbb{C}(x) \). If \( f_0 = 0 \), then \( f_1 \neq 0 \) since \( f \) is square-free. Thus, \( \sqrt{f(S(x)^{[0]})} \not\in \mathbb{C}(x) \).

(2) In this case \( S_{(1)}^{[0]} \neq 0 \) by Theorem \( \text{(2)} \). If \( f(S_{(0)}^{[0]}) \neq 0 \), then \( \sqrt{f(S(x)^{[0]})} \in \mathbb{C}(x) \). If \( f(S_{(0)}^{[0]}) = 0 \), then \( \sqrt{f(S(x)^{[0]})} \not\in \mathbb{C}(x) \) since \( f \) is square-free and \( S_{(1)}^{[0]} \neq 0 \).

(3) In this case, the assertion follows easily from (5.1).

\[\square\]

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