Auxiliary fields and hadron dynamics

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Abstract

The relations existing between the auxiliary field (einbein field) formalism and the spinless Salpeter equation are studied in the case of two particles with the same mass, interacting via a confining potential. The problem of non-orthogonality for radial excited states in the auxiliary field formalism is discussed and found to be non-crucial. It is shown that the classical equations of motion of the rotating string model, derived from the QCD lagrangian, reduce exactly to the classical equations of motion of the phenomenological semirelativistic flux tube model, provided all auxiliary fields are eliminated correctly from the rotating string hamiltonian.

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I. INTRODUCTION

A system of relativistic particles can be quantized as a constrained system by using the auxiliary field formalism (also known as einbein field formalism) to get rid of the square root term in the Lagrangian \cite{1,2}. Applied to the QCD Lagrangian, this technique yields Hamiltonians for mesons or baryons $H(\mu_i)$ depending on auxiliary fields $\mu_i$ \cite{3,4,5}. The way of removing these fields for physical applications is not well defined and can lead to ambiguities. In practice, these fields, representing for instance the energy density for the particles, are finally treated as $c$-number and determined from the minimum energy conditions

$$\frac{\partial E(\mu_i)}{\partial \mu_i} = 0,$$  \hspace{1cm} (1)

where $E(\mu_i)$ is an eigenvalue of $H(\mu_i)$.

When the interaction $V$ between particles does not depend on $\mu_i$, the values of $\mu_i$ which minimize the hamiltonian operator $H(\mu_i)$ (and not $E(\mu_i)$) are $\mu_i = \sqrt{\vec{p}_i^2 + m_i^2}$. The eigenvalue equation reduces then to a spinless Salpeter equation with the potential $V$. So, in this case, the auxiliary field formalism can be considered as an approximation of the spinless Salpeter equation \cite{6}.

Although handling a relativistic expression for the kinetic energy term is not a serious problem with nowadays numerical algorithms, a nonrelativistic expression is always simpler, specially for many body calculations. In a sense the auxiliary field formalism simulates a relativistic expression with the simplicity of a nonrelativistic one. The price to pay is a further minimization on eigenenergies. The values of $\mu_i$ are generally different for various radial excited states and this leads to non-orthogonality among them. This is another price to pay. An aim of this paper is to study whether these drawbacks are very inconvenient or not.

In Sec. II we consider the case of two identical particles interacting via a potential independent of auxiliary fields. We solve the eigenvalue equations for hamiltonians written in the auxiliary field formalism and for corresponding hamiltonians in spinless Salpeter equations. Then, we compare the eigenvalues and eigenvectors for these two approaches and discuss the problem of non-orthogonality. The study is first made for a toy model where most of the results are analytical and then in a more realistic one which relies on a precise numerical treatment.
It is now believed that the color field between a quark and an antiquark in a meson can be approximated by a rigid string, carrying both energy and angular momentum. Some years ago, a phenomenological model taking into account this particular string dynamics has been developed: the relativistic flux tube model. The classical equations of motion take the form of a coupled system of nonlinear equations in which a particular quantity is introduced, the transverse velocity of the ends of the string, where the particles are located. More recently, a lagrangian for a rotating string was derived from the QCD lagrangian. Adopting a simple straight line ansatz for the string, another coupled system of nonlinear equations was obtained, containing auxiliary functions for the particle energy density and the string energy density. In this case, the potential does depend on the auxiliary fields. In Sec. III, we show that these two approaches are exactly the same, provided all auxiliary functions are correctly eliminated from the rotating string hamiltonian.

II. AUXILIARY FIELDS AND SPINLESS SALPETER EQUATION

In the formalism of auxiliary fields, the hamiltonian $H^A$ for two particles with the same mass $m$, interacting via a potential $V$ which does not depend on $\mu$, is written \( \hbar = c = 1 \)

$$H^A = \frac{\vec{p}^2 + m^2}{\mu} + \mu + V.$$  \hspace{1cm} (2)

The philosophy of this formalism is to keep the simplicity of the nonrelativistic form of the kinetic energy operator, to get the eigenenergies in terms of auxiliary fields, and to get rid of them through a minimization procedure.

As mentioned, the eigenvalues and eigenvectors of this hamiltonian depends on $\mu$

$$H^A |A; nl; \mu\rangle = E_{nl}^A(\mu) |A; nl; \mu\rangle,$$  \hspace{1cm} (3)

where $l$ is the orbital angular momentum and $n$ (0, 1, ...) the vibrational quantum number. The physical solutions are obtained by choosing for each eigenstate the value of $\mu$ which minimizes its energy. We will note $\mu_{nl}$ the value of $\mu$ minimizing the eigenvalue $E_{nl}^A(\mu)$. The following notations will be used

$$E_{nl}^A = E_{nl}^A(\mu_{nl}) \leq E_{nl}^A(\mu) \quad \forall \mu \quad \text{and} \quad |A; nl\rangle = |A; nl; \mu_{nl}\rangle.$$  \hspace{1cm} (4)
Applying the Hellmann-Feynman theorem \[11\] to \(H^A\), we find

\[
\frac{\partial E^A_{nl}(\mu)}{\partial \mu} = -\frac{\langle A; nl; \mu|\vec{p}^2 + m^2|A; nl; \mu \rangle}{\mu^2} + 1.
\]

(5)

It follows immediately that

\[
\mu^2_{nl} = \langle A; nl|\vec{p}^2 + m^2|A; nl \rangle,
\]

(6)

and that

\[
E^A_{nl} = 2\mu_{nl} + \langle A; nl|V|A; nl \rangle.
\]

(7)

Applied to \(H^A\), the virial theorem yields \(2\langle \vec{p}^2 \rangle/\mu = \langle \vec{r} \cdot \vec{\nabla} V \rangle\). If the potential \(V\) is a homogeneous function of \(r\) of degree \(\alpha\), the formula (7) can be simplified. We have then

\[
E^A_{nl} = \frac{2(1 + \alpha)}{\alpha} \mu_{nl} - \frac{2m^2}{\alpha \mu_{nl}} \text{ if } \vec{r} \cdot \vec{\nabla} V = \alpha V.
\]

(8)

The scalar product \(\langle A; n'l'|A; nl \rangle\) is proportional to \(\delta_{ll'}\) because of the orthogonality of the spherical harmonics. But, for a fixed value of \(l\), two eigenstates with different values of \(n\) are not orthogonal since they are characterized by different values of \(\mu\). So, it is interesting to compute the quantity which measures the overlap of two states

\[
P_{n'n'l} = |\langle A; n'l'|A; nl \rangle|^2.
\]

(9)

We have obviously \(P_{nn'l} = P_{n'n'l}^*\) and \(P_{nn'l} = 1\), and we can expect that \(P_{n'n'l} \sim \delta_{n'n}\).

Another way of dealing with auxiliary field is to minimize not the eigenvalues but the hamiltonian operator itself and impose the condition \(\partial H^A/\partial \mu = 0\), \(\mu\) being considered as an operator by its own. In the case of hamiltonian (2), it is easy to determine that the value of the auxiliary field \(\mu\) which realizes this condition is \(\mu = \sqrt{\vec{p}^2 + m^2}\). The minimal hamiltonian (2) then becomes

\[
H^S = 2\sqrt{\vec{p}^2 + m^2} + V,
\]

(10)

which is simply a spinless Salpeter hamiltonian with the same potential \(V\). We note its eigenvalues and eigenstates by

\[
H^S|S; nl \rangle = E^S_{nl}|S; nl \rangle.
\]

(11)

It has been shown in Ref. [6], that we always have \(E^A_{nl} \geq E^S_{nl}\).
If the two ways of treating the auxiliary field were expected to be very similar, one sees that the first method simulates the effect of a relativistic kinetic energy term with a nonrelativistic one; the potential is kept unchanged. The differences $E^A - E^S$ give an indication on the quality of this procedure.

Since the minimal energy eigenvalues of $H^A$ are upper bound of the eigenvalues of $H^S$ and since these two hamiltonians have in some sense the same physical content, their corresponding eigenstates must be very similar. By computing the overlap

$$T_{nl} = |\langle A; nl | S; nl \rangle|^2,$$

we can expect that $T_{nl} \sim 1$.

If we define

$$M_{nl} = \langle S; nl | \sqrt{\vec{p}^2 + m^2} | S; nl \rangle,$$

we can also expect that $M_{nl} \approx \mu_{nl}$, because of the relation (6). In both models, $m$ is sometimes considered as the current mass, and $M_{nl}$ or $\mu_{nl}$ are then interpreted as the constituent mass, which, in this formalism, is state dependent.

For two massless particles, the relativistic virial theorem \[12\] yields $2\langle \sqrt{\vec{p}^2} \rangle = \langle \vec{r} \cdot \vec{\nabla} V \rangle$. If the potential $V$ is a homogeneous function of $r$ of degree $\alpha$, new relations can be written. We have then

$$E^S_{nl} = (1 + \alpha) \langle S; nl | V | S; nl \rangle = \frac{2(1 + \alpha)}{\alpha} M_{nl} \quad \text{if} \quad \vec{r} \cdot \vec{\nabla} V = \alpha V \quad \text{and} \quad m = 0. \quad (14)$$

When the mass $m$ increases, the differences existing between the two approaches $H^A$ and $H^S$ tend to vanish and $\mu_{nl} \approx m$ for all values of quantum numbers. In order to quantify these differences, we will study two models containing a confining potential, which is the relevant kind of interaction for the hadronic physics. In particular, we will focus our attention on the case $m = 0$, for which we expect the largest discrepancies between the two approaches.

**A. Toy model**

We first consider a “toy model” for which most of the calculations are analytical. Let us choose $V = kr^2$ and $m = 0$. In this case, $H^A$ is a harmonic oscillator hamiltonian. Its solutions are well-known and we have

$$E^A_{nl}(\mu) = 2\sqrt{\frac{k}{\mu}}(2n + l + 3/2) + \mu. \quad (15)$$
The regularized radial part of the wave function is noted

\[ u^A_{nl}(r) = (\mu k)^{3/8} r O_{nl}((\mu k)^{1/4} r), \]  

(16)

where \( O_{nl}(x) \) is the usual radial harmonic oscillator wave function and where dimensioned factors ensure that the eigenstates \( |A; nl; \mu\rangle \) are normalized.

The values of the parameter \( \mu \) minimizing the energies are given by

\[ \mu_{nl} = k^{1/3}(2n + l + 3/2)^{2/3}, \]  

(17)

and the calculation of the eigenenergies yields

\[ E^A_{nl} = 3 k^{1/3}(2n + l + 3/2)^{2/3} = 3 \mu_{nl}. \]  

(18)

The relation \( E^A_{nl} = 3 \mu_{nl} \) can also be obtained directly from formula (8).

The overlap of two eigenstates can be computed easily. Knowing the values of \( \mu_{nl} \), we have

\[ P_{n'n'l} = F^2_{n'nl} \left( \left( \frac{2n + l + 3/2}{2n' + l + 3/2} \right)^{1/6} \right), \]  

(19)

where the function \( F_{n'nl}(x) \) is given, with its properties, in Ref. [13]. The quantity \( P_{n'n'l} \) is independent of \( k \) and decreases with increasing values of \( l \) and \( |n - n'| \). For instance, we have \( P_{100} \approx 0.0287 \), which is the worse case. Considering the states as orthogonal is thus always a good approximation.

In order to solve the corresponding hamiltonian \( H^S \) for our toy model, let us consider the following nonrelativistic hamiltonian

\[ H^N = \frac{\vec{p}^2}{2 \nu} + a r. \]  

(20)

Solutions for \( l = 0 \) are known [14]. The eigenenergies are given by

\[ E^N_{n0} = - \left( \frac{a^2}{2 \nu} \right)^{1/3} x_n, \]  

(21)

where \( x_n \) is the \((n + 1)\)th zero of the Airy function \( \text{Ai}(x) \) \( (x_0 \approx -2.338, x_1 \approx -4.088, \ldots) \).

The regularized and normalized radial part of the eigenvectors is written as [15]

\[ u^N_{n0}(r) = (2 \nu a)^{1/6} A_n \left( (2 \nu a)^{1/3} r \right) \quad \text{where} \quad A_n(x) = \frac{\text{Ai}(x + x_n)}{\sqrt{\int_{x_n}^{\infty} \text{Ai}^2(q) \, dq}}. \]  

(22)

Let us note that we have \( \int_{x_n}^{\infty} \text{Ai}^2(q) \, dq = \text{Ai}^2(x_n) \).
The hamiltonian $H^N$ is converted into the hamiltonian $H^S$ with $V = k r^2$ and $m = 0$ by the duality transformation $|\vec{p}| \leftrightarrow \frac{1}{2} a r$, provided the parameters $k$, $a$, and $\nu$ are related by $k = \frac{a^2}{8 \nu}$. Due to this relation, we have immediately the property

$$E_{n0}^S = -\left(4 k\right)^{1/3} x_n.$$  \hfill (23)

In order to compare energies $E_{n0}^A$ and $E_{n0}^S$ given respectively by relations (18) and (23), we can use the following approximation for the numbers $x_n$ \cite{16}

$$x_n \approx -\left[\frac{3\pi}{2} (n + 3/4)\right]^{2/3}. \hfill (24)$$

With this formula, whose error decreases when $n$ increases, relation (23) can be written

$$E_{n0}^S \approx \left(\frac{3\pi}{2}\right)^{2/3} k^{1/3} (2n + 3/2)^{2/3}. \hfill (25)$$

So we have immediately the ratio (independent of $k$),

$$\frac{E_{n0}^S}{E_{n0}^A} \approx \frac{1}{3} \left(\frac{3\pi}{2}\right)^{2/3} \approx 0.937. \hfill (26)$$

Numerically, we found that the ratio $E_{n0}^S/E_{n0}^A$ is around 0.944 for $n = 0$ and tends rapidly towards the asymptotical value (26) as $n$ increases.

From relation (22) and the duality transformation, the regularized function of S-wave eigenvectors of $H^S$ can be obtained in the momentum space

$$u_{n0}^S(p) = \left(\frac{2}{k}\right)^{1/6} A_n \left(\frac{2}{k}\right)^{1/3} p. \hfill (27)$$

Consequently, the quantity $M_{n0} = \langle S; nl | \sqrt{\vec{p}^2} | S; nl \rangle$ can be calculated by the following integral ($\sqrt{\vec{p}^2} = |\vec{p}|$)

$$M_{n0} = k^{1/3} \frac{1}{2^{1/3}} \int_0^\infty dq q A_n^2(q) = -\left(4 k\right)^{1/3} x_n^3 = \frac{E_{n0}^S}{3}, \hfill (28)$$

and we find that $M_{n0}/\mu_{n0} = E_{n0}^S/E_{n0}^A \approx 0.937$. The relation $E_{n0}^S = 3 M_{n0}$ can also be obtained directly from formula (14).

Since the Fourier transform of a harmonic oscillator is also a harmonic oscillator (with a phase factor), the overlap $T_{n0}$ can be computed in the momentum space. We obtain

$$T_{n0} = \frac{2^{1/3}}{\sqrt{2n + 3/2}} \left[ \int_0^\infty dq q O_{n0} \left(\frac{q}{(2n + 3/2)^{1/6}}\right) A_n (2^{1/3} q) \right]^2, \hfill (29)$$
which is independent of $k$. We have checked numerically that $T_{00} \approx 0.997$ and that $T_{n0}$ decreases monotonically when $n$ increases. For instance, $T_{90} \approx 0.523$. One sees that, when $n$ increases, the eigenvalues for the two formalisms are almost the same (to within 6%), whereas the corresponding wave functions can differ appreciably.

### B. Realistic model

The previous toy model was interesting because most of the quantities can be calculated analytically. However, it lacks some physics. First the real confining potential is more alike a linear one, and second there exist one gluon exchange contributions which add a coulombic term, important at short distance. In order to stick more to the true physical situation, we switch, in this section, to a more realistic model. Of course, this study needs a complete numerical treatment.

In the framework of a semirelativistic potential model, a more realistic interaction to simulate the dynamics between a quark and an antiquark inside a meson is certainly what is called the funnel potential

$$V(r) = -\frac{\kappa}{r} + ar \quad \text{with} \quad \kappa = 0.5 \quad \text{and} \quad a = 0.2 \text{ GeV}^2. \quad (30)$$

The values chosen here for the parameters $\kappa$ and $a$ can be considered as typical. The eigenvalue equations for hamiltonians $H^A$ and $H^S$ with this potential have been numerically solved by the Lagrange-mesh method. This technique is very accurate and can be implemented easily for both nonrelativistic or semirelativistic kinematics.

In Fig. the quantities $\Delta E = (E^A - E^S)/E^S$ are presented, as a function of the mass $m$, for the three lowest states ($1S$, $2S$, $1P$) obtained with the potential (30). As expected, $\Delta E$ decreases when the quark mass increases. The corresponding curves for the quantities $E^A - E^S$ present the same profiles. Thus we only present more results for the case of a vanishing quark mass which maximizes the error.

In Table some energies $E^S_{nl}$ and $E^A_{nl}$ are given for $n \leq 3$ and $l \leq 2$, for the potential (30) and $m = 0$. The difference $E^A_{nl} - E^S_{nl}$ can be quite large, around 100 MeV for the ground state. It increases for increasing values of $n$ and decreases for increasing values of $l$. The quantity $M_{nl}$ is also given in this Table; it increases with $n$ and $l$. The parameter $\mu_{nl}$ can differ from $M_{nl}$ by several tens of MeV, but the curves $E^A_{nl}(\mu)$ always present a flat minimum.
around $\mu_{nl}$. So it is not necessary to obtain a precise value of $\mu_{nl}$ to obtain a good value of $E_{nl}^A$.

The overlap $P_{nn'l}$ is given in Table II for $n, n' \leq 3$ and for $l \leq 1$, for the potential (30) and $m = 0$. Except in the case $|n - n'| = 1$, the overlap is very small. So the eigenstates $|A; nl\rangle$ can be considered as quasi-orthogonal. This conclusion is the same than in the toy model, and we can hope that it is a universal conclusion, valid whatever the potential $V$.

In table III the overlap $T_{nl}$ is presented for $n \leq 3$ and $l \leq 2$, for the potential (30) and $m = 0$. It increases for increasing values of $l$ and decreases for increasing values of $n$, and present similar features to the toy model. For small values of the vibrational quantum number, the eigenvectors $|A; nl\rangle$ and $|S; nl\rangle$ are rather similar.

III. ROTATING QCD STRING

Up to now, the confining term was put by hand. There exist more serious explanations of this term, based on QCD arguments. In particular, one can consider a string with an energy density between the quark and the antiquark. This string is itself a dynamical object. Here we examine two such approaches.

We first quickly present the relativistic flux tube model, which is essentially phenomenological, and then the rotating string model within the formalism of auxiliary fields, which has a more firm foundation on QCD theory. Lastly, we show that these two models are completely equivalent.

A. Phenomenological model

In the simplest version of the relativistic flux tube model [7], a quark and an antiquark, with the same mass $m$, move being attached with a rigid flux tube, assumed to be linear with a uniform constant energy density $a$. The system rotates in a plane with a constant angular velocity around the center of mass, which is assumed to be stationary. Denoting $r$ the distance between the two particles, $p_r$ the radial momentum $\left(p_r^2 = -\frac{1}{r} \frac{\partial^2}{\partial r^2} r\right)$, $L$ the total angular momentum, $v_\perp$ the transverse velocity of the quark or the antiquark (relative to the string direction), $\gamma_\perp = (1 - v_\perp^2)^{-1/2}$, and $U(r)$ a potential used to describe dynamical effects coming from mechanisms other than the flux tube, the classical equations of motion
of the system are given by

$$\frac{L}{r} = v_\perp \gamma \sqrt{p_r^2 + m^2} + ar f(v_\perp), \quad (31)$$

$$H = 2\gamma_\perp \sqrt{p_r^2 + m^2} + ar \frac{\arcsin v_\perp}{v_\perp} + U(r), \quad (32)$$

where

$$f(x) = \frac{1}{4x^2} \left( \arcsin x - x\sqrt{1 - x^2} \right), \quad (33)$$

is a very important function in the formalism. These equations have been generalized in
the case of asymmetrical systems [8]. Numerical solutions of the quantized versions of these
equations have been obtained [7, 8, 19, 20] (for a practical calculation $L$ must be replaced
by $\sqrt{l(l+1)}$). Equations (31)-(32) are coupled nonlinear equations. In practice, the value
$v_\perp(m, a, L; p_r, r)$ is extracted from Eq. (31) and injected into Eq. (32), giving a $L$ dependent
hamiltonian $H(m, a, L; p_r, r)$, which is diagonalized afterwards.

When the relativistic flux tube hamiltonian is supplemented by appropriate potentials
(Coulomb-like, instanton induced effects) and when it is assumed that each extremity of
the flux tube can give a constant energy contribution, rather good meson spectra can be
obtained [19, 20].

B. Auxiliary fields for a rotating string

Starting from the QCD lagrangian, a lagrangian for a meson can be derived taking into
account the dynamical degrees of freedom of the string [10]. For a system containing a quark
and an antiquark with the same mass, with the hypothesis of a straight line configuration for
the minimal string, and introducing an auxiliary field $\mu$ for the energy density of the quarks
and another auxiliary field $\nu$ for the energy density of the string, the following classical
hamiltonian can be obtained [10]

$$H = \frac{p_r^2 + m^2}{\mu(\tau)} + \mu(\tau) + \frac{L^2/r^2}{\mu(\tau) + 2 \int_0^1 (\beta - \frac{1}{2})^2 \nu(\beta, \tau) \, d\beta}
+ \frac{\alpha^2 r^2}{2} \int_0^1 \frac{d\beta}{\nu(\beta, \tau)} + \int_0^1 \frac{\nu(\beta, \tau)}{2} \, d\beta + U(r), \quad (34)$$

where $U(r)$ is a potential describing dynamical effects coming from mechanisms other than
the string. In this formula, $\tau$ is the common proper time of the two particles and $\beta$ is a
coordinate along the string. Within this formalism, the two auxiliary functions $\mu$ and $\nu$ are to be varied and to be found from the minimum of $H$.

Let us first remark that, if we neglect the integral in the denominator of the term depending on $L^2$ or if we put $L = 0$, then $H$ is minimum for $\nu = ar$, independent of $\beta$. In this case, because of the relation $p_r^2 + \frac{L^2}{r^2} = \tilde{p}^2$, $H$ reduces simply to $H^A$ with $V(r) = ar + U(r)$. The linear confining term appears naturally in the formalism. A further minimization on $\mu$, as demonstrated in second section, gives rise to $H^S$ with the same potential $V(r)$.

This result is interesting and simple, but indeed one can make an exact minimization on the field $\nu$. A straightforward calculation shows that the minimization condition $\partial H / \partial \nu = 0$ is fulfilled if the field $\nu$ is set to $\nu_0$ with

$$
\nu_0(\beta) = \frac{ar}{\sqrt{1 - 4y^2(\beta - \frac{1}{2})^2}},
$$

(35)

where $y$ is to be found from the transcendental equation

$$
\frac{L}{ar^2} = f(y) + \frac{\mu y}{ar}.
$$

(36)

The function $f$ defined previously by relation (33) appears again curiously (for a practical calculation $L$ must also be replaced by $\sqrt{l(l+1)}$). Using expression (35), one obtains from relation (34)

$$
H = \frac{p_r^2 + m^2}{\mu} + \mu(1 + y^2) + ar \frac{\arcsin y}{y} + U(r).
$$

(37)

WKB solutions for the system (36) and (37) are obtained in Ref. [10] for $m = 0$ and $U(r) = 0$. Regge trajectories are computed in agreement with experimental data.

C. Equivalence between the two string models

In principle the way of solving the system (36)-(37) is the following. One extracts $y(a, L; r; \mu)$ from Eq. (33) and inject it into Eq. (37) leading to $H(m, a, L; p_r, r; \mu)$. This hamiltonian is then diagonalized and, for each state, the optimal value of $\mu$ must be determined to get the physical eigenenergies. In view of the form for various expressions, this procedure seems hopeless. Fortunately this is not so and one can go one step further and perform the minimization on $\mu$. 

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The partial derivative of Eq. (36) with respect to \( \mu \) gives the following relation

\[
\frac{y^2}{a r} + \frac{\partial y}{\partial \mu} \left[ \frac{\mu y}{a r} + \frac{1}{2 y \sqrt{1 - y^2}} - \frac{\arcsin y}{2 y^2} \right] = 0,
\]

(38)
in which the expression \( \partial y/\partial \mu \) appears. Now imposing \( \partial H/\partial \mu = 0 \), we obtain from Eq. (37)

\[
-y_r^2 + m^2 + 1 + y^2 + \frac{\partial y}{\partial \mu} \left[ 2 \mu y + \frac{a r}{y \sqrt{1 - y^2}} - a r \frac{\arcsin y}{y^2} \right] = 0.
\]

(39)

Combining the two last formulas, we obtain

\[
\mu = \sqrt{\frac{p_r^2 + m^2}{1 - y^2}}.
\]

(40)

Using this relation, it can be verified that Eqs. (38) and (39) are well defined for \( y \in [0, 1] \).

Equation (40) giving \( \mu(y) \) and the presence of the function \( f \) in Eq. (36) are the clues for the identification of both methods. Replacing \( y \) by \( v_\perp \) and \( \mu \) by \( \gamma_\perp \sqrt{p_r^2 + m^2} \) transforms Eq. (36) into Eq. (31) and Eq. (37) into Eq. (32), proving thus the complete equivalence between both approaches. In passing we have now a clear idea of the physical content for the auxiliary field \( \mu \) and of the mysterious variable \( y \) appearing in the theory.

IV. CONCLUSION

Effective hamiltonians for hadron dynamics can be derived from the original QCD lagrangian. These hamiltonians can depend on auxiliary fields, representing for instance the quark energy density [4, 5]. When the interaction does not depend on these fields (exactly or by approximation), it is possible to reduce the effective hamiltonian to a semirelativistic one (spinless Salpeter).

In the first part of this paper, we have compared the eigenvalues and eigenvectors for hamiltonians written in the auxiliary field formalism \( H^A \) and for the corresponding spinless Salpeter hamiltonians \( H^S \). We have only considered the case of two particles with the same mass and two different confining interaction: A “toy” quadratic potential and a more realistic funnel potential. We have shown that the eigenvalues of \( H^A \) are close to the eigenvalues of \( H^S \), but the relative differences can exceed 10%. The overlap of corresponding eigenvectors for the two approaches is generally close to unity for ground states, but it deteriorates when the vibrational quantum number increases. It appears that, if precise calculations
are desired, it is better to solve directly a spinless Salpeter equations: Accurate numerical techniques exist for two-body \[18\], and many-body \[21, 22\] problems. Hamiltonians with auxiliary fields can be useful if one searches to obtain analytical results, as it is the case in Ref. \[5\]. The non-orthogonality resulting in this formalism is not a serious problem, the radial excited states being quasi-orthogonal.

The phenomenological semirelativistic flux tube hamiltonian has been developed in order to take into account the string dynamics in a meson \[7, 8\]. Another model derived from the QCD lagrangian, the rotating string, has been more recently developed \[10\]. But the rotating string hamiltonian depends on auxiliary fields.

In the second part of this paper, we have shown that the classical equations of motion of the rotating string reduce exactly to the classical equations of motion of the semirelativistic flux tube, provided all auxiliary fields are correctly eliminated. The complete equivalence of both models has two virtues: to provide a natural interpretation of the auxiliary field in term of the transverse velocity and to reinforce the relevance of the relativistic flux tube model (which was considered up to now as a phenomenological one), since it is equivalent to a model based on firm QCD grounds.

The set of coupled equations for these two models are very complicated to solve numerically \[7, 8, 19, 20\]. The difficulties are essentially the same for both approaches, with the supplementary necessity to perform a minimization within the rotating string formalism in order to determine the value of the auxiliary field for the quark energy density. It is then more interesting to work directly with the semirelativistic flux tube model.

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TABLE I: Values in GeV for the quantities $E^{S}_{nl}$, $E^{A}_{nl}$ and $M_{nl}$ (see Sec. II), as a function of the quantum numbers $n$ and $l$, for the potential (30) and $m = 0$.

| $n$ | $l = 0$ | 0 | 1 | 2 | 3 |
|-----|--------|---|---|---|---|
|     | $E^{S}_{nl}$ | 1.197 | 1.912 | 2.461 | 2.914 |
|     | $E^{A}_{nl}$ | 1.294 | 2.109 | 2.700 | 3.185 |
|     | $M_{nl}$ | 0.422 | 0.588 | 0.712 | 0.817 |
|     | $l = 1$ |  |  |  |  |
|     | $E^{S}_{nl}$ | 1.759 | 2.316 | 2.782 | 3.187 |
|     | $E^{A}_{nl}$ | 1.826 | 2.472 | 2.990 | 3.434 |
|     | $M_{nl}$ | 0.508 | 0.644 | 0.757 | 0.855 |
|     | $l = 2$ |  |  |  |  |
|     | $E^{S}_{nl}$ | 2.170 | 2.643 | 3.056 | 3.426 |
|     | $E^{A}_{nl}$ | 2.225 | 2.780 | 3.248 | 3.659 |
|     | $M_{nl}$ | 0.594 | 0.711 | 0.813 | 0.904 |

TABLE II: Values of $P_{nn'l}$ (see Sec. III), as a function of the quantum numbers $n$, $n'$, and $l$, for the potential (30) and $m = 0$. Values for $l = 0$ ($l = 1$) are given in the upper-right (lower-left) triangle of the Table ($P_{nn'l} = P_{n'n'l}$ and $P_{n nl} = 1$).

| $P_{nn'l}$ | $n = 0$ | 1 | 2 | 3 |
|------------|--------|---|---|---|
| $n' = 0$   | 1      | 0.022 | $1.6 \times 10^{-4}$ | $2.4 \times 10^{-4}$ |
|            | 1      | 0.017 | 1 | 0.022 | $5.7 \times 10^{-5}$ |
|            | 2      | $2.0 \times 10^{-6}$ | 0.023 | 1 | 0.028 |
|            | 3      | $5.9 \times 10^{-5}$ | $2.5 \times 10^{-7}$ | 0.019 | 1 |

TABLE III: Values of $T_{nl}$ (see Sec. III), as a function of the quantum numbers $n$ and $l$, for the potential (30) and $m = 0$.

| $T_{nl}$ | $n = 0$ | 1 | 2 | 3 |
|----------|--------|---|---|---|
| $l = 0$  | 0.9837 | 0.9639 | 0.9389 | 0.8982 |
|          | 1      | 0.9908 | 0.9722 | 0.9406 | 0.9005 |
|          | 2      | 0.9925 | 0.9746 | 0.9443 | 0.9008 |
FIG. 1: $\Delta E = (E^A - E^S)/E^S$ (see Sec. II) as a function of the mass $m$, for 1S, 2S, and 1P states with potential (30).