Generalized fractional maximal and integral operators on Orlicz and generalized Orlicz–Morrey spaces of the third kind

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Abstract
In the present paper, we will characterize the boundedness of the generalized fractional integral operators $I_\rho$ and the generalized fractional maximal operators $M_\rho$ on Orlicz spaces, respectively. Moreover, we will give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operators $M_\rho$ and $I_\rho$ on generalized Orlicz–Morrey spaces, respectively. Also we give criteria for the weak versions of the Spanne-type boundedness and the Adams-type boundedness of the operators $M_\rho$ and $I_\rho$ on generalized Orlicz–Morrey spaces.

Keywords Generalized fractional maximal function · Generalized fractional integral · Orlicz spaces · Generalized Orlicz-Morrey spaces

Mathematical Subject Classification 42B20 · 42B25 · 42B35 · 46E30

1 Introduction

The aim of this paper is to obtain the necessary conditions and the sufficient conditions for the generalized fractional maximal operator $M_\rho$ and the generalized fractional integral operator $I_\rho$ to be bounded on Orlicz spaces. Our results can be extended to generalized Orlicz–Morrey spaces of the third kind which will be defined later in this paper.
Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. For a function $\rho : (0, \infty) \to (0, \infty)$, let $I_\rho$ be the generalized fractional integral operator:

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} f(y) dy.$$  

Here $f$ is a suitable measurable function. Note that this type of generalization goes back to [25]. If $\rho(r) = r^\alpha, 0 < \alpha < n$, then $I_\rho$ is the fractional integral operator or the Riesz potential and denoted by $I_\alpha$. Hereafter, we assume that

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty$$  

so that the fractional integrals $I_\rho f$ are well defined, at least for characteristic functions of balls. The operator $I_\rho$ was introduced in [19] and some partial results were announced in [18]. We refer to [16] for the boundedness of $I_\rho$ on Orlicz space $L^\Phi(\Omega)$ with bounded domain $\Omega \subset \mathbb{R}^n$. See also [20–23] for the boundedness of $I_\rho$ on various spaces. In these papers we assumed that $\rho$ satisfies the doubling condition:

$$\frac{1}{C_1} \leq \frac{\rho(r)}{\rho(s)} \leq C_1, \quad \text{if} \quad \frac{1}{2} \leq \frac{r}{s} \leq 2,$$  

and that $r \mapsto \rho(r)/r^n$ is almost decreasing:

$$\frac{\rho(s)}{s^n} \leq C_2 \frac{\rho(r)}{r^n}, \quad \text{if} \quad r < s,$$  

where $C_1$ and $C_2$ are positive constants independent of $r, s \in (0, \infty)$. Under these conditions we proved the boundedness of $I_\rho$ on Orlicz spaces in [18,19].

In this paper, instead of these conditions, we assume that there exist positive constants $C, k_1$ and $k_2$ with $k_1 < k_2$ such that, for all $r > 0$,

$$\sup_{r/2 \leq t \leq r} \rho(t) \leq C \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t} dt =: \tilde{\rho}(r).$$  

The condition (4) was considered in [26] and also used in [31]. If $\rho$ satisfies (2) or (3), then $\rho$ satisfies (4). Let

$$\rho(r) = \begin{cases} r^n (\log(e/r))^{-1/2}, & 0 < r < 1, \\ e^{-(r-1)}, & 1 \leq r < \infty. \end{cases}$$  

Then $\rho$ satisfies (1) and (4), but fails (2) and (3). Therefore, the results in this paper improve ones in [19]. Moreover, we give necessary and sufficient conditions for the boundedness of $I_\rho$ not only on Orlicz spaces but also on Orlicz–Morrey spaces of the third kind.

Next, we define the generalized fractional maximal operator $M_\rho$. For a function $\rho : (0, \infty) \to (0, \infty)$, let

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\[ M_\rho f(x) = \sup_{r > 0} \frac{\rho(r)}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy, \] (6)

where \(|G|\) is the Lebesgue measure of a measurable set \(G \subset \mathbb{R}^n\). We do not assume (1) on the function \(\rho\) in (6). Instead we suppose that \(\rho\) is an increasing function such that \(r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty)\) is decreasing.

If \(\rho \equiv 1\), then \(M_\rho\) is the Hardy-Littlewood maximal operator denoted by \(M\). If \(\rho(r) = r^\alpha\), then \(M_\rho\) is the usual fractional maximal operator denoted by \(M_\alpha\). We give some necessary conditions and some sufficient conditions for the boundedness of \(M_\rho\) on Orlicz and Orlicz–Morrey spaces.

The structure of the remaining part of the present paper is as follows: First we recall Young functions and Orlicz spaces in Sect. 2. In Sect. 3, we investigate the boundedness of generalized fractional integrals on Orlicz spaces. We will give a necessary and sufficient condition for the boundedness of the generalized fractional maximal operators on Orlicz spaces in Sect. 4. In Sect. 5 we discuss some properties of generalized Orlicz–Morrey spaces of the third kind. Moreover, we will give necessary and sufficient conditions for the Spanne and Adams-type boundedness of the generalized fractional integral operators on generalized Orlicz–Morrey spaces of the third kind in Sect. 6. Finally, in Sect. 7 we give criteria for the boundedness of the generalized fractional maximal operators on generalized Orlicz–Morrey spaces of the third kind.

### 2 Young functions and Orlicz spaces

We recall the definition of Young functions.

**Definition 1** A function \(\Phi : [0, \infty] \to [0, \infty]\) is called a Young function if \(\Phi\) is convex, left-continuous, \(\lim_{r \to +0} \Phi(r) = \Phi(0) = 0\) and \(\lim_{r \to \infty} \Phi(r) = \Phi(\infty) = \infty\).

From the non-negativity, convexity and \(\Phi(0) = 0\) it follows that any Young function is increasing. We denote by \(\mathcal{Y}\) the set of all Young functions such that

\[ 0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty. \]

If \(\Phi \in \mathcal{Y}\), then \(\Phi\) is absolutely continuous on every compact interval in \([0, \infty)\) and bijective from \([0, \infty)\) to itself.

Next we recall the generalized inverse of Young function \(\Phi\) in the sense of O’Neil [24, Definition 1.2]. For a Young function \(\Phi\) and \(0 \leq s \leq \infty\), let

\[ \Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}, \quad \text{where} \inf \emptyset = \infty. \]

Note that if \(s < \infty\), then so is \(\Phi^{-1}(s)\). As in [24, p. 301, Remarks], we always have \(\Phi^{-1}(\infty) = \infty\). An important inequality we use is

\[ \Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)). \]
See [24, Property 1.3]. Then $\Phi^{-1}(s)$ is finite for all $s \in [0, \infty)$, continuous on $(0, \infty)$ and right continuous at $s = 0$. Observe that $\Phi^{-1}(\Phi(s)) = r$ if $0 < \Phi(r) < \infty$ and that $\Phi(\Phi^{-1}(s)) = s$ if $s \in [0, \Phi(\inf \{r > 0 : \Phi(r) = \infty\})]$. Furthermore, if $\Phi \in \mathcal{Y}$, then $\Phi^{-1}$ is the usual inverse function of $\Phi$.

**Remark 1** For a Young function $\Phi$, its inverse function $\Phi^{-1}$ is increasing and concave. Hence, we have the following properties:

$$
\begin{align*}
\Phi^{-1}(t) &\geq \Phi^{-1}(\alpha t) \geq \alpha \Phi^{-1}(t), \quad \text{if } 0 < \alpha < 1 \\
\Phi^{-1}(t) &\leq \Phi^{-1}(\alpha t) \leq \alpha \Phi^{-1}(t), \quad \text{if } \alpha > 1.
\end{align*}
$$

Since $\Phi^{-1}$ is increasing, the left inequality is clear. In particular, $\Phi$ satisfies the doubling condition: $\Phi^{-1}(2s) \leq 2\Phi^{-1}(s)$ for all $s \geq 0$.

In fact for $0 < \alpha < 1$,

$$
\Phi^{-1}(\alpha t) = \inf \{s \geq 0 : \Phi(s) > \alpha t\}.
$$

Since $\frac{1}{\alpha} \Phi(s) \leq \Phi\left(\frac{s}{\alpha}\right)$, we have

$$
\Phi^{-1}(\alpha t) \geq \inf \left\{ s \geq 0 : \Phi\left(\frac{s}{\alpha}\right) > \alpha t \right\} = \alpha \inf \{s \geq 0 : \Phi(s) > t\} = \alpha \Phi^{-1}(t).
$$

The right inequality for $\alpha > 1$ is a consequence of the one for $0 < \alpha < 1$.

As in [24, Property 1.6], we have

$$
r \leq \Phi^{-1}(r) \tilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0,
$$

where $\tilde{\Phi}(r)$ is the complementary function of $\Phi$ defined by

$$
\tilde{\Phi}(r) = \begin{cases} 
\sup \{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\
\infty, & r = \infty.
\end{cases}
$$

Then $\tilde{\Phi}$ is also a Young function and $\tilde{\tilde{\Phi}} = \Phi$.

A Young function $\Phi$ is said to satisfy the $\Delta_2$-condition, denoted also by $\Phi \in \Delta_2$, if

$$
\Phi(2r) \leq C \Phi(r), \quad r > 0
$$

for some $C \geq 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function $\Phi$ is said to satisfy the $\nabla_2$-condition, denoted also by $\Phi \in \nabla_2$, if

$$
\Phi(r) \leq \frac{1}{2C} \Phi(Cr), \quad r \geq 0
$$

for some $C > 1$.

We denote by $\chi_G$ the characteristic function of the set $G \subset \mathbb{R}^n$. 

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Definition 2 (Orlicz space) For a Young function \( \Phi \), the Orlicz space \( L^\Phi(\mathbb{R}^n) \) is defined by:

\[
L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}.
\]

The space \( L^\Phi_{\text{loc}}(\mathbb{R}^n) \) is defined as the set of all measurable functions \( f \) such that \( f \chi_B \in L^\Phi(\mathbb{R}^n) \) for all balls \( B \subset \mathbb{R}^n \).

If \( \Phi \) is a Young function, then \( L^\Phi(\mathbb{R}^n) \) is a Banach space under the Luxemburg–Nakano norm

\[
\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right)dx \leq 1 \right\}.
\]

For example, if \( \Phi(r) = r^p, 1 \leq p < \infty \), then \( L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n) \). If \( \Phi(r) = 0, (0 \leq r \leq 1) \) and \( \Phi(r) = \infty, (r > 1) \), then \( L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \).

For a measurable set \( \Omega \subset \mathbb{R}^n \), a measurable function \( f \) and \( t > 0 \), let

\[
m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.
\]

In the case \( \Omega = \mathbb{R}^n \), we abbreviate it to \( m(f, t) \).

\( L^0(\mathbb{R}^n) \) is the set of all measurable functions.

Definition 3 For a Young function \( \Phi \), the weak Orlicz space

\[
WL^\Phi(\mathbb{R}^n) = \{ f \in L^0(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty \}
\]

is defined by the quasi-norm

\[
\|f\|_{WL^\Phi} = \sup_{\lambda > 0} \|\lambda \chi_{(\lambda, \infty)}(|f|)\|_{L^\Phi}.
\]

For \( \Omega \subset \mathbb{R}^n \), let

\[
\|f\|_{L^\Phi(\Omega)} := \|f \chi_{\Omega}\|_{L^\Phi} \quad \text{and} \quad \|f\|_{WL^\Phi(\Omega)} := \|f \chi_{\Omega}\|_{WL^\Phi}.
\]

A tacit understanding is that \( f \) is defined to be zero outside \( \Omega \).

We note that \( \|f\|_{WL^\Phi(\Omega)} \leq \|f\|_{L^\Phi(\Omega)} \), that

\[
\|f\|_{WL^\Phi} = \sup_{t > 0} \Phi(t)m(\Omega, f, t) = \sup_{t > 0} t m(\Omega, f, \Phi^{-1}(t)) = \sup_{t > 0} t m(\Omega, \Phi(|f|), t)
\]
and that
\[
\int_{\Omega} \Phi \left( \frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}} \right) dx \leq 1, \quad \sup_{t>0} \Phi(t)m \left( \Omega, \frac{f}{\|f\|_{WL^\Phi(\Omega)}}, t \right) \leq 1 \tag{8}
\]
according to [13, Proposition 4.2].

The following analogue of the Hölder inequality is well known; see [32] as well as the paper [24, § II] and the textbooks [14,29].

**Theorem 1** Let \( \Omega \subset \mathbb{R}^n \) be a measurable set and, let \( f \) and \( g \) be measurable functions on \( \Omega \). For a Young function \( \Phi \) and its complementary function \( \tilde{\Phi} \), the following inequality is valid:

\[
\int_{\Omega} |f(x)g(x)| dx \leq 2 \|f\|_{L^\Phi(\Omega)} \|g\|_{L^{\tilde{\Phi}}(\Omega)}.
\]

By elementary calculations we have the following property:

**Lemma 1** Let \( \Phi \) be a Young function and let \( B \) be a set in \( \mathbb{R}^n \) with finite Lebesgue measure. Then

\[
\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{WL^\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}.
\]

By Theorem 1, Lemma 1 and (7) we get the following estimate:

**Lemma 2** For a Young function \( \Phi \) and \( B = B(x, r) \), the following inequality is valid:

\[
\int_{B} |f(y)| dy \leq 2|B|\Phi^{-1} \left( |B|^{-1} \right) \|f\|_{L^\Phi(B)}.
\]

We recall the boundedness property of \( M \) on Orlicz spaces since we use it later.

**Theorem 2** Let \( \Phi \) be a Young function.

1. [2, Theorem 1] The maximal operator \( M \) is bounded from \( L^\Phi(\mathbb{R}^n) \) to \( WL^\Phi(\mathbb{R}^n) \), and the inequality

\[
\|Mf\|_{WL^\Phi} \leq C_0 \|f\|_{L^\Phi}
\]

holds with constant \( C_0 \) independent of \( f \).

2. [2, Theorem 1], [11, Corollary 3.3] The maximal operator \( M \) is bounded on \( L^\Phi(\mathbb{R}^n) \), and the inequality

\[
\|Mf\|_{L^\Phi} \leq C_0 \|f\|_{L^\Phi}
\]

holds with constant \( C_0 \) independent of \( f \) if and only if \( \Phi \in \nabla_2 \).

See the textbooks [14,15,27,29] for more about Orlicz spaces.
3 Generalized fractional integrals on Orlicz spaces

The following theorem is one of our main results and gives necessary and sufficient conditions for the boundedness of the operator $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

**Theorem 3** Let $\Phi, \Psi$ be Young functions.

1. Let $\rho$ satisfy the conditions (1) and (4). Then the condition
   \[ \Phi^{-1}(r^{-n}) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \rho(t) \Phi^{-1}(t^{-n}) \frac{dt}{t} \leq C \Psi^{-1}(r^{-n}) \]  
   for all $r > 0$, where $C > 0$ does not depend on $r$, is sufficient for the boundedness of $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the condition (11) is also sufficient for the boundedness of $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. The condition
   \[ \Phi^{-1}(r^{-n}) \int_0^r \frac{\rho(t)}{t} dt \leq C \Psi^{-1}(r^{-n}) \]  
   for all $r > 0$, where $C > 0$ does not depend on $r$, is necessary for the boundedness of $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. Let $\rho$ satisfy the conditions (1) and (4). Assume the condition
   \[ \int_r^\infty \rho(t) \Phi^{-1}(t^{-n}) \frac{dt}{t} \leq C \Psi^{-1}(r^{-n}) \]  
   holds for all $r > 0$, where $C > 0$ does not depend on $r$. Then condition (12) is necessary and sufficient for the boundedness of $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the condition (12) is necessary and sufficient for the boundedness of $I_{\rho}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

**Remark 2** We cannot replace $\int_0^r \frac{\rho(t)}{t} dt$ by $\rho(r)$ in (11), see [23, Section 5].

We need a couple of auxiliary estimates. The following lemma was proved in [6, Lemma 2.1]:

**Lemma 3** There exist a constant $C > 0$ such that for all $x \in B(0, r/2)$ and $r > 0$,

\[ \int_0^{r/2} \frac{\rho(t)}{t} dt \leq C I_{\rho} \chi_{B(0,r)}(x) \]

holds.

**Proposition 1** Let $\rho$ satisfy (4). Define

\[ \tilde{\rho}(r) = \int_{k_1 r}^{k_2 r} \rho(s) \frac{ds}{s} \quad (r > 0). \]
Let $\tau : (0, \infty) \to (0, \infty)$ be a doubling function in the sense that $\tau(r) \sim \tau(s)$ if $0 < s \leq r \leq 2s$. Then, for each $r > 0$,

$$
\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) \lesssim \int_0^{k_2 r} \frac{\rho(s)}{s} ds, \quad (15)
$$

$$
\sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \tau ((2^j r)^{-n}) \lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau(s^{-n}) ds. \quad (16)
$$

**Proof** We invoke the overlapping property in [31] and by Remark 1 we have

$$
\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) = \sum_{j=-\infty}^{-1} \int_{2^j k_1 r}^{2^j k_2 r} \rho(s) ds
$$

$$
\leq \int_0^{k_2 r} \left( \sum_{j=-\infty}^{-1} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} ds
$$

and

$$
\sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \tau ((2^j r)^{-n}) = \int_{k_1 r}^{\infty} \left( \sum_{j=0}^{\infty} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} \tau((2^j r)^{-n}) ds
$$

$$
\lesssim \int_{k_1 r}^{\infty} \left( \sum_{j=0}^{\infty} \chi_{[2^j k_1 r, 2^j k_2 r]}(s) \right) \frac{\rho(s)}{s} \tau(s^{-n}) ds
$$

$$
\lesssim \int_{k_1 r}^{\infty} \frac{\rho(s)}{s} \tau(s^{-n}) ds.
$$

To prove Theorem 3, we need the following estimate of Hedberg-type [12]:

**Proposition 2** Under the assumption of Theorem 3, for any positive constant $C_0$, there exists a positive constant $C_1$ such that, for all nonnegative functions $f \in L^\Phi(\mathbb{R}^n)$ with $f \neq 0$,

$$
I_\rho f(x) \leq C_1 \|f\|_{L^\Phi} \Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) (x \in \mathbb{R}^n). \quad (17)
$$

**Proof** The idea of the proof comes from [6]. First note that

$$
0 < \Phi^{-1}(0) \int_0^{\infty} \frac{\rho(t)}{t} dt \lesssim \Psi^{-1}(0)
$$

as long as $\Phi^{-1}(0) > 0$.
Let $x \in \mathbb{R}^n$. Keeping in mind that $Mf(x) > 0$, we may assume

$$0 < \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} < \infty, \quad 0 \leq \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) < \infty;$$

otherwise there is nothing to prove. If

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) = 0,$$

then

$$\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \leq \sup\{u \geq 0 : \Phi(u) = 0\} = \Phi^{-1}(0)$$

and hence

$$I_\rho f(x) \leq C \sum_{j=-\infty}^{\infty} \frac{\hat{\rho}(2^j)}{2^{jn}} \int_{|x-y|<2^j} |f(y)| dy \leq C \left( \int_0^\infty \frac{\rho(s)}{s} ds \right) Mf(x) \leq C \frac{\Psi^{-1}(0)}{\Phi^{-1}(0)} Mf(x) \leq C \frac{1}{\Phi^{-1}(0)} \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \right) Mf(x) \leq C \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) \right) \|f\|_{L^\Phi}.$$

So, this case the result is valid.

If

$$\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) > 0,$$

choose $r \in (0, \infty)$ so that

$$r^{-n} = \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right).$$

We have

$$I_\rho f(x) \leq C \left[ \sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\hat{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^j r} f(y) dy \right] = C(I + II)$$
for given $x \in \mathbb{R}^n$ and $r > 0$.

Then from Proposition 1

$$I \leq C \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) M f(x) \leq C \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) M f(x)$$

$$II \leq C \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \Phi^{-1}((2^j r)^{-n}) \|f\|_{L^\Phi(B(x,2^j r))} \leq C \|f\|_{L^\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}(s^{-n}) \frac{\rho(s)}{s} ds.$$

Consequently, we have

$$I_\rho f(x) \lesssim \left( \int_0^{k_2 r} \frac{\rho(s)}{s} ds \right) M f(x) + \|f\|_{L^\Phi} \int_{k_1 r}^{\infty} \Phi^{-1}(s^{-n}) \frac{\rho(s)}{s} ds.$$

Thus, by the doubling property of $\Phi^{-1}$ and $\Psi^{-1}$, (11) and Remark 1 we obtain

$$I_\rho f(x) \lesssim M f(x) \frac{\Psi^{-1}((k_2 r)^{-n})}{\Phi^{-1}((k_2 r)^{-n})} + \|f\|_{L^\Phi} \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} \lesssim M f(x) + \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}).$$

Recall that $\Phi^{-1}(\Phi(r)) = r$ if $0 < \Phi(r) < \infty$. Thus $\Phi^{-1}(r^{-n}) = \frac{M f(x)}{C_0 \|f\|_{L^\Phi}}$ and

$$I_\rho f(x) \lesssim \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}) = \|f\|_{L^\Phi} \Psi^{-1} \left( \Phi \left( \frac{M f(x)}{C_0 \|f\|_{L^\Phi}} \right) \right).$$

Therefore, we get (17).

Now we move on to the proof of Theorem 3. The third statement is a consequence of the remaining statements. So we concentrate on the first and the second ones.

– Let $C_0$ be as in (9). Let $f$ be a non-negative measurable function. Then by (9) and (17),

$$\sup_{r>0} \frac{\Psi(r) m \left( \frac{I_\rho f(x)}{C_1 \|f\|_{L^\Phi}}, r \right)}{r} = \sup_{r>0} \frac{m \left( \frac{I_\rho f(x)}{C_1 \|f\|_{L^\Phi}}, r \right)}{r} \leq \sup_{r>0} \left( \frac{M f(x)}{C_0 \|f\|_{L^\Phi}} \right) \leq \sup_{r>0} \frac{\Phi(r) m \left( \frac{M f(x)}{\|f\|_{W^L \Phi}}, r \right)}{r} \leq 1.$$
\[ \|I_\rho f\|_{W_L^\Phi} \lesssim \|f\|_{L_\Phi}. \]

- Assume in addition that \( \Phi \in \nabla_2 \), so that we have (10). By (10), we have
\[
\int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L_\Phi}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left( \frac{Mf(x)}{\|Mf\|_{L_\Phi}} \right) dx \leq 1,
\]
i.e.
\[ \|I_\rho f\|_{L_\Psi} \lesssim \|f\|_{L_\Phi}. \]

- We can and do concentrate on the boundedness of \( I_\rho \) from \( L_\Phi(\mathbb{R}^n) \) to \( W_L^\Psi(\mathbb{R}^n) \), since the boundedness of \( I_\rho \) from \( L_\Phi(\mathbb{R}^n) \) to \( L_\Psi(\mathbb{R}^n) \) is stronger than the boundedness of \( I_\rho \) from \( L_\Phi(\mathbb{R}^n) \) to \( W_L^\Psi(\mathbb{R}^n) \). With this in mind, assume that \( I_\rho \) is bounded from \( L_\Phi(\mathbb{R}^n) \) to \( L_\Psi(\mathbb{R}^n) \). Then we have by Lemma 3
\[
\int_0^{r/2} \frac{\rho(s)}{s} ds \|\chi_{B(0,r/2)}\|_{W_L^\Psi(B(0,r/2))} \lesssim \|I_\rho \chi_{B(0,r)}\|_{W_L^\Psi(B(0,r/2))}.
\]
Therefore, by the doubling property of \( \Phi^{-1} \) and Lemma 1, we have
\[
\int_0^{r/2} \frac{\rho(s)}{s} ds \lesssim \psi^{-1}(r^{-n}) \|I_\rho \chi_{B(0,r)}\|_{W_L^\Psi(B(0,r/2))} \lesssim \psi^{-1}(r^{-n}) \|I_\rho \chi_{B(0,r)}\|_{W_L^\Psi(B(0,r/2))} \lesssim \psi^{-1}(r^{-n}) \|\chi_{B(0,r)}\|_{L_\Phi} \lesssim \psi^{-1}(r^{-n}) \Phi^{-1}(r^{-n}).
\]

**Remark 3** In [19, Corollary 3.2] the third author found the sufficient conditions which ensures the boundedness of the operator \( I_\rho \) from \( L_\Phi(\mathbb{R}^n) \) to \( L_\Psi(\mathbb{R}^n) \), including its weak version. Theorem 3 improves the third author’s result in that Theorem 3 also covers the necessity by imposing a weaker condition on \( \rho \).

**Remark 4** In the case \( \Phi(t) = t^p \), Theorem 3 was proved in [6, Corollary 1.5].

**Example 1** Let \( \rho \) be as in (5) and
\[
\Phi(t) = \begin{cases} 
  t^{3/2}, & 0 \leq t \leq 1, \\
  t (\log(et))^{1/2}, & t > 1, 
\end{cases} \quad \Psi(t) = \begin{cases} 
  \frac{2e}{3} t^{3/2}, & 0 \leq t \leq 1, \\
  \frac{2e}{3} \exp \exp(t), & t > 1. 
\end{cases}
\]
Then the pair \((\rho, \Phi, \Psi)\) satisfies (11). In fact, we have

\[
\Phi^{-1}(u) \sim \begin{cases} 
  u^{2/3}, & 0 \leq u \leq 1, \\
  u(\log(eu))^{-1/2}, & u > 1,
\end{cases}
\]

\[
\Psi^{-1}(u) \sim \begin{cases} 
  u^{2/3}, & 0 \leq u \leq 1, \\
  \log(\log(eu)), & u > 1,
\end{cases}
\]

and for all \(r > 0\)

\[
\int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}(1/r^n) \lesssim \min(1, r^{-2n/3}),
\]

\[
\int_r^{\infty} \frac{\rho(t) \Phi^{-1}(1/t^n)}{t} \, dt \lesssim \min(\log \log(e^r/r), r^{-2n/3}),
\]

\[
\Psi^{-1}(1/r^n) \sim \min(\log \log(e^r/r), r^{-2n/3}).
\]

See [20] for other examples.

4 Generalized fractional maximal operators on Orlicz spaces

We recall that, for a function \(\rho : (0, \infty) \to (0, \infty)\), \(M_\rho\) is defined by (6). Here we suppose that \(\rho\) is an increasing function such that \(r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty)\) is decreasing.

Under this assumption, we have the following localized estimate:

**Lemma 4** There exists a positive constant \(C\) such that, for all balls \(B = B(x, r)\) and all measurable functions \(f\) supported on \(B\),

\[
M_\rho f(x) \leq C \rho(r) Mf(x). \tag{18}
\]

**Proof** Let \(B(R) = B(x, R)\) with \(x = 0\) for \(R > 0\). By the definition of \(M_\rho\), we have

\[
M_\rho f(x)
= \max \left\{ \sup_{0 < R < 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| \, dy, \sup_{R \geq 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| \, dy \right\}.
\]

For the first term, we use the fact that \(\rho\) is increasing and doubling to have

\[
\sup_{0 < R < 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)| \, dy \lesssim \sup_{0 < R < 3r} \frac{\rho(r)}{|B(x, R)|} \int_{B(x, R)} |f(y)| \, dy
\leq \rho(r) Mf(x).
\]
For the second term, since \( r \in (0, \infty) \mapsto r^{-n} \rho(r) \in (0, \infty) \) is decreasing and \( f \) is supported on \( B(x, r) \)

\[
\sup_{R \geq 3r} \frac{\rho(R)}{|B(R)|} \int_{B(x, R)} |f(y)|dy \leq \sup_{R \geq 3r} \frac{\rho(3r)}{|B(x, 3r)|} \int_{B(x, R)} |f(y)|dy \\
\leq \rho(r)Mf(x).
\]

Thus, combining these estimates, we obtain the desired result.

The Hedberg inequality for \( M_\rho \) and \( L^\Phi \) can be stated as follows:

**Lemma 5** Let \( \Phi, \Psi \) be Young functions. Assume that there exists a positive constant \( C \) such that, for all \( r > 0 \),

\[
\rho(r) \leq C \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})}.
\]  

(19)

Then, for any positive constant \( C_0 \), there exists a positive constant \( C_1 \) such that, for all \( f \in L^\Phi(\mathbb{R}^n) \) with \( f \neq 0 \),

\[
M_\rho f(x) \leq C_1 \|f\|_{L^\Phi}(\Psi^{-1} \circ \Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right)) \quad (x \in \mathbb{R}^n).
\]

(20)

**Proof** First note that

\[
\lim_{r \to \infty} \rho(r) \lesssim \frac{\Psi^{-1}(0)}{\Phi^{-1}(0)}
\]

(21)

if \( \Phi^{-1}(0) > 0 \). Let \( x \in \mathbb{R}^n \) be an arbitrary point. We may assume that \( 0 < Mf(x) < \infty \) keeping in mind that \( f \) does not vanish on a set of positive measure. Furthermore, we can assume that

\[
\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) < \infty;
\]

otherwise there is nothing to do since \( \Psi^{-1}(\infty) = \infty \). If

\[
\Phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) = 0,
\]

then

\[
\Phi^{-1}(0) \geq \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} > 0.
\]
according to the definition of $\Phi^{-1}$. Thus, thanks to (21)

$$
\lim_{r \to \infty} \rho(r) \lesssim \frac{\psi^{-1}(0)}{\phi^{-1}(0)} \lesssim \frac{1}{\phi^{-1}(0)} \psi^{-1} \circ \phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) \lesssim \frac{C_0 \|f\|_{L^\phi}}{Mf(x)} \psi^{-1} \circ \phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right).
$$

Thus by (18) we have

$$
M_\rho f(x) \lesssim \lim_{r \to \infty} \rho(r) Mf(x) \lesssim \|f\|_{L^\phi} \psi^{-1} \circ \phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right).
$$

It thus remains to handle the case where

$$
0 < \phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right) < \infty.
$$

In the case we can choose $r > 0$ such that

$$
r^{-n} = \phi \left( \frac{Mf(x)}{C_0 \|f\|_{L^\phi}} \right).
$$

Let $B = B(x, r)$ and represent $f$ as

$$
f = f_1 + f_2, \quad f_1 = f \chi_B, \quad f_2 = f \chi_{\mathbb{R}^n \setminus B}
$$

so that $M_\rho f(x) \leq M_\rho f_1(x) + M_\rho f_2(x)$.

We have (18) for $f_1$. Meanwhile by Lemma 2,

$$
M_\rho f_2(x) = \sup_{t > 0} \frac{\rho(t)}{|B(x, t)|} \int_{B(x, t) \cap \mathbb{R}^n \setminus B(x, r)} |f(z)|dz
\lesssim \sup_{r < t < \infty} \rho(t) \phi^{-1}(|B(x, t)|^{-1}) \|f\|_{L^\phi(B(x, t))}
\lesssim \|f\|_{L^\phi} \sup_{r < t < \infty} \rho(t) \phi^{-1}(t^{-n}).
$$

Consequently we have by Lemma 4

$$
M_\rho f(x) \lesssim \rho(r) Mf(x) + \|f\|_{L^\phi} \sup_{r < t < \infty} \rho(t) \phi^{-1}(t^{-n}).
$$
Thus, by (19) and the monotonicity of $\Psi^{-1}$ we obtain
\[
M_\rho f(x) \lesssim Mf(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}).
\]
Since $\Phi^{-1}(r^{-n}) = \frac{Mf(x)}{C_0\|f\|_{L^\Phi}}$, we have
\[
M_\rho f(x) \lesssim \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}) = \|f\|_{L^\Phi} \Psi^{-1}\left(\frac{Mf(x)}{C_0\|f\|_{L^\Phi}}\right).
\]
Therefore, we get (20).

In [10] we obtain a counterpart to generalized Orlicz–Morrey spaces of the second kind defined in [7]. However, as is written in [7] generalized Orlicz–Morrey spaces of the second kind do not cover $L^2(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$. So, the following theorem can be viewed as a different theorem from [7]:

**Theorem 4** Let $\Phi, \Psi$ be Young functions. Assume that $\rho$ is increasing and that $r \mapsto r^{-n} \rho(r)$ is decreasing. Then the condition (19) is necessary and sufficient for the boundedness of $M_\rho$ from $L^\Phi(\mathbb{R}^n)$ to $W^L_\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the condition (19) is necessary and sufficient for the boundedness of $M_\rho$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

**Proof** We start with the necessity. For the necessity, we can concentrate on the boundedness of $M_\rho$ from $L^\Phi(\mathbb{R}^n)$ to $W^L_\Psi(\mathbb{R}^n)$, since the boundedness of $M_\rho$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ is stronger than the boundedness of $M_\rho$ from $L^\Phi(\mathbb{R}^n)$ to $W^L_\Psi(\mathbb{R}^n)$. With this in mind, assume that $M_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $W^L_\Psi(\mathbb{R}^n)$. We utilize a trivial pointwise estimate
\[
\rho(r) \chi_{B(0,r)} \leq M_\rho \chi_{B(0,2r)}.
\]

Therefore, by the doubling property of $\Phi^{-1}$ and Lemma 1, we have
\[
\rho(r) \lesssim \Psi^{-1}(r^{-n}) \|M_\rho \chi_{B(0,2r)}\|_{W^L_\Psi(B(0,r))}
\lesssim \Psi^{-1}(r^{-n}) \|M_\rho \chi_{B(0,2r)}\|_{W^L_\Psi}
\lesssim \Psi^{-1}(r^{-n}) \|\chi_{B(0,2r)}\|_{L^\Phi}
\lesssim \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})}.
\]

We move on to the sufficiency. Here and below we let $f$ be a nonzero measurable function.
– Let $C_0$ be as in (9). Then by (9) and (20), we have
\[
\sup_{r > 0} \Psi(r) m\left( \frac{M_\rho f(y)}{C_1 \| f \|_{L^\Phi}}, r \right) = \sup_{r > 0} r m\left( \frac{M_\rho f(y)}{C_1 \| f \|_{L^\Phi}}, r \right) \leq \sup_{r > 0} r m\left( \frac{Mf(y)}{C_0 \| f \|_{L^\Phi}}, r \right) \leq \sup_{r > 0} \Phi(r) m\left( \frac{Mf(y)}{\| Mf \|_{W^\Phi}}, r \right) \leq 1,
\]
i.e.
\[
\| M_\rho f \|_{W^\Psi} \lesssim \| f \|_{L^\Phi}. \tag{23}
\]
– Assume in addition that $\Phi \in \nabla_2$. Let $C_0$ be as in (10). By (10) and (20), we have
\[
\int_{\mathbb{R}^n} \Psi\left( \frac{M_\rho f(y)}{C_1 \| f \|_{L^\Phi}} \right) dy \leq \int_{\mathbb{R}^n} \Phi\left( \frac{Mf(y)}{C_0 \| f \|_{L^\Phi}} \right) dy \leq \int_{\mathbb{R}^n} \Phi\left( \frac{Mf(y)}{\| Mf \|_{L^\Phi}} \right) dy \leq 1,
\]
i.e.
\[
\| M_\rho f \|_{L^\Psi} \lesssim \| f \|_{L^\Phi}. \tag{24}
\]

5 Generalized Orlicz–Morrey spaces of the third kind

In [3], the generalized Orlicz–Morrey space $M^{\Phi, \Psi}(\mathbb{R}^n)$ was introduced to unify Orlicz spaces and generalized Morrey spaces. Other definitions of generalized Orlicz–Morrey spaces can be found in [22,31]. In words of [8], our generalized Orlicz–Morrey space is the third kind and the ones in [22,31] are the first kind and the second kind, respectively. Notice that the definition of the space of the third kind relies only on the fact that $L^\Phi(\mathbb{R}^n)$ is a normed linear space, which is independent of the condition that it is generated by modulars.

The definition of generalized Orlicz–Morrey spaces of the third kind is as follows:

**Definition 4** Let $\varphi$ be a positive measurable function on $(0, \infty)$ and $\Phi$ any Young function. We denote by $M^{\Phi, \varphi}(\mathbb{R}^n)$ the generalized Orlicz–Morrey space of the third kind, the space of all functions $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$ with finite norm
\[
\| f \|_{M^{\Phi, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \| f \|_{L^\Phi(B(x,r))}.
\]
Also by $W^{\Phi, \varphi}(\mathbb{R}^n)$ we denote the weak generalized Orlicz–Morrey space of the third kind of all measurable functions $f \in WL^\Phi_{\text{loc}}(\mathbb{R}^n)$ for which
\[
\| f \|_{W^{\Phi, \varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \| f \|_{WL^\Phi(B(x,r))} < \infty.
\]
A function \( \varphi : (0, \infty) \to (0, \infty) \) is said to be almost increasing (resp. almost decreasing) if there exists a constant \( C > 0 \) such that

\[
\varphi(r) \leq C \varphi(s) \quad (\text{resp. } \varphi(r) \geq C \varphi(s)) \quad \text{for } r \leq s.
\]

For a Young function \( \Phi \), we denote by \( \mathcal{G}_\Phi \) the set of all \( \varphi : (0, \infty) \to (0, \infty) \) functions such that \( t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-n})t^n} \) is almost increasing and \( t \in (0, \infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-n})t^n} \) is almost decreasing. Note that \( \varphi \in \mathcal{G}_\Phi \) implies doubling condition of \( \varphi \).

We investigate the structure of \( \mathcal{M}_{\Phi, \varphi}(\mathbb{R}^n) \). We denote by \( \Theta \) the set of all measurable functions equivalent to 0 on \( \mathbb{R}^n \). To exclude some trivial cases, we can use the following lemma was proved in [4]:

**Lemma 6** Let \( \Phi \) be a Young function and \( \varphi \) be a positive measurable function on \( (0, \infty) \).

(i) If

\[
\sup_{t < r < \infty} \frac{\Phi^{-1}(r^{-n})}{\varphi(r)} = \infty \quad \text{for some } t > 0
\]

then \( \mathcal{M}_{\Phi, \varphi}(\mathbb{R}^n) = \Theta \).

(ii) If

\[
\sup_{0 < r < \tau} \varphi(r)^{-1} = \infty \quad \text{for some } \tau > 0
\]

then \( \mathcal{M}_{\Phi, \varphi}(\mathbb{R}^n) = \Theta \).

**Remark 5** If

\[
\sup_{0 < r \leq t} \frac{\Phi^{-1}(r^{-n})r^n}{\varphi(r)} = \infty \quad \text{for some } t > 0,
\]

then \( \mathcal{M}_{\Phi, \varphi} = \Theta \). Actually, by Remark 1 one has

\[
\Phi^{-1}(r^{-n})r^n \leq \Phi^{-1}(t^{-n})t^n < \infty
\]

and then

\[
\sup_{0 < r < t} \varphi(r)^{-1} = \infty.
\]

**Remark 6** Based on Lemma 6 and Remark 5 and an observation similar to the one made by Nakai [17, p. 446] it can be assumed that \( \varphi \in \mathcal{G}_\Phi \) in the definition of \( \mathcal{M}_{\Phi, \varphi}(\mathbb{R}^n) \). More explicitly, we have the following observation:
(i) By Lemma 6 we may assume that \( \inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(r^{-n})} > 0 \) for every \( r > 0 \). Let

\[
\psi(r) = \Phi^{-1}(r^{-n}) \inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(t^{-n})}, \quad r > 0.
\]

Then \( r \in (0, \infty) \mapsto \frac{\psi(r)}{\Phi^{-1}(r^{-n})} \) is increasing and \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) = \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \) with equivalent norms. Indeed, it is clear that \( \psi(r) \leq \varphi(r) \) by the definition of \( \psi \). Hence \( \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) and \( \|f\|_{\mathcal{M}^{\Phi, \psi}} \leq \|f\|_{\mathcal{M}^{\Phi, \varphi}} \). On the other hand,

\[
\sup_{r > 0} \psi(r)^{-1} \Phi^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x, r))} = \sup_{r > 0} \frac{1}{\inf_{r \leq t < \infty} \frac{\varphi(t)}{\Phi^{-1}(r^{-n})}} \|f\|_{L^\Phi(B(x, r))}
\leq \sup_{t > 0} \psi(t)^{-1} \Phi^{-1}(t^{-n}) \|f\|_{L^\Phi(B(x, t))}.
\]

Hence \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \) and \( \|f\|_{\mathcal{M}^{\Phi, \varphi}} \leq \|f\|_{\mathcal{M}^{\Phi, \psi}} \).

(ii) By Remark 5 we may assume that \( \inf_{0 < t \leq r} \frac{\varphi(t)}{\Phi^{-1}(r^{-n})} > 0 \) for every \( r > 0 \). Define \( \psi(r) \) by the formula:

\[
\sup_{t \in (0, r]} \frac{\Phi^{-1}(t^{-n})r^n}{\varphi(t)} = \frac{\Phi^{-1}(r^{-n})r^n}{\psi(r)}.
\]

It is easy to see that \( \psi(r) \leq \varphi(r) \) for any \( r > 0 \). Thus, \( \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \subset \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) and \( \|f\|_{\mathcal{M}^{\Phi, \psi}} \leq \|f\|_{\mathcal{M}^{\Phi, \varphi}} \). Conversely, let \( f \in \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \). For any \( r \in (0, \infty) \), choose \( t \in (0, r] \) so that

\[
\frac{\Phi^{-1}(r^{-n})r^n}{\psi(r)} \leq 2 \frac{\Phi^{-1}(t^{-n})r^n}{\varphi(t)},
\]

and cover \( B(x, r) \) with a family of \( N \) balls \( \{B(x_j, t)\}_{j=1}^N \), where \( N \lesssim r^{-n}t^n \). Let \( j_0 \) be such that

\[
\|f\|_{L^\Phi(B(x, r))} \leq N \|f\|_{L^\Phi(B(x_{j_0}, t))} \lesssim t^{-n}r^n \|f\|_{L^\Phi(B(x_{j_0}, r))}.
\]
Thus,
\[
\frac{\Phi^{-1}(r^{-n})}{\psi(r)} \| f \|_{L^\Phi(B(x, r))} \lesssim \frac{r^n \Phi^{-1}(r^{-n})}{\psi(r)t^n} \| f \|_{L^\Phi(B(x_0, t))} \\
\leq 2 \frac{\Phi^{-1}(t^{-n})}{\varphi(t)} \| f \|_{L^\Phi(B(x_0, t))} \\
\leq 2 \| f \|_{\mathcal{M}^{\Phi, \psi}},
\]
implying \( f \in \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \) and \( \| f \|_{\mathcal{M}^{\Phi, \psi}} \leq \| f \|_{\mathcal{M}^{\Phi, \psi}}. \) Thus, \( \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^{\Phi, \psi}(\mathbb{R}^n). \)

As the following lemma shows, \( \mathcal{G}_\Phi \) is useful:

**Lemma 7** ([5]) Let \( B_0 := B(x_0, r_0). \) If \( \varphi \in \mathcal{G}_\Phi \) is almost decreasing, then there exist \( C > 0 \) such that
\[
\frac{1}{\varphi(r_0)} \leq \| \chi_{B_0} \|_{\mathcal{M}^{\Phi, \psi}} \leq \frac{C}{\varphi(r_0)}.
\]

### 6 Generalized fractional integrals on generalized Orlicz–Morrey spaces

We remark that there are two types of the boundedness of the fractional integral operators. One is the Spanne-type boundedness obtained in [28]. Another boundedness is of Adams-type obtained by Adams [1]. In the classical case due to the fact that Morrey spaces are nested we can say that the Adams-type boundedness is stronger than the Spanne-type boundedness. However, we need to depend on the pointwise estimate of Hedberg-type [12], so the Adams-type boundedness is unavailable for local Morrey spaces. In this section we give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operator \( I_\rho \) on generalized Orlicz–Morrey spaces, respectively.

#### 6.1 Spanne-type result

We need the following lemma is valid:

**Lemma 8** Let \( \Phi, \Psi \) be Young functions, and let \( \rho \) satisfy (1) and (4). Assume that the condition (11) is fulfilled. Then there exists a positive constant \( C \) such that, for all \( f \in L^{\Phi}_{\text{loc}}(\mathbb{R}^n) \) and \( B = B(x, r) \),
\[
\| I_\rho f \|_{W^\Psi(B)} \\
\leq C \| f \|_{L^\Phi(2B)} + \frac{C}{\psi^{-1}(r^{-n})} \int_{2kr}^\infty \| f \|_{L^\Phi(B(x, t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}.
\]
Moreover if we assume \( \Phi \in \nabla_2 \), the following inequality is also valid:

\[
\| I_\rho f \|_{L^\Psi(B)} \leq C \| f \|_{L^\Phi(2B)} + \frac{C}{\Psi^{-1}(r^{-n})} \int_{2k_1r}^\infty \| f \|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}.
\]  

(28)

**Proof** We represent \( f \) as

\[
f = f_1 + f_2, \quad f_1 = f \chi_{2B}, \quad f_2 = f - f_1, \quad r > 0.
\]

Then we have

\[
\| I_\rho f \|_{W^L\Psi(B)} \leq 2(\| I_\rho f_1 \|_{W^L\Psi(B)} + \| I_\rho f_2 \|_{W^L\Psi(B)}).
\]

From the boundedness of \( I_\rho \) from \( L^\Phi(\mathbb{R}^n) \) to \( W^L\Psi(\mathbb{R}^n) \) (see Theorem 3) it follows that:

\[
\| I_\rho f_1 \|_{W^L\Psi(B)} \lesssim \| I_\rho f_1 \|_{W^L\Psi(\mathbb{R}^n)} \leq C \| f_1 \|_{L^\Phi(\mathbb{R}^n)} = C \| f \|_{L^\Phi(2B)},
\]

where constant \( C > 0 \) is independent of \( f \).

For \( f_2 \) we have

\[
\| I_\rho f_2 \|_{W^L\Psi(B)} \lesssim \| f_2 \|_{W^L\Psi(\mathbb{R}^n)} \leq C \| f_2 \|_{L^\Phi(\mathbb{R}^n)} = C \| f \|_{L^\Phi(2B)}.
\]

A geometric observation shows that \( y \in B, \ z \in \mathbb{R}^n \setminus (2B) \) implies

\[
\frac{1}{2} |x - z| \leq |y - z| \leq \frac{3}{2} |x - z|.
\]

Using (4) and Lemma 2, we have

\[
\int_{2^{j+1}B \setminus 2^jB} \frac{\rho(|y - z|)}{|y - z|^n} |f(z)| \, dz \leq \left( \sup_{2^{j-1}r \leq t \leq 3 \cdot 2^j r} \rho(t) \right) \frac{1}{2^j B} \int_{2^{j+1}B \setminus 2^jB} |f(z)| \, dz.
\]

Then

\[
\| I_\rho f_2 \|_{W^L\Psi(B)} \lesssim \int_{2^{j+1}B \setminus 2^jB} \| f \|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}.
\]

(30)
Thus by Lemma 1 we have
\[ \|I\rho f\|_{W^\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r-n)} \int_0^\infty \|f\|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}. \] (31)

Therefore we obtain (27) by (29) and (31).

If \( \Phi \in \nabla_2 \), then we can use strong type inequality instead of (29) and obtain (28) by using the same argument.

**Remark 7** In the case \( \Phi(t) = t^p \) \((1 \leq p < \infty)\) Lemma 8 was proved in [9].

The following theorem gives necessary and sufficient conditions for Spanne-type boundedness of the operator \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \).

**Theorem 5** (Spanne-type result) Let \( \Phi, \Psi \) be Young functions, and let \( \varphi_1 \in \mathcal{G}_\Phi \) and \( \varphi_2 \in \mathcal{G}_\Psi \).

1. Let \( \rho \) satisfy (1) and (4). Assume that (11) is fulfilled. Then the conditions
\[ \frac{\varphi_1(r)}{\Phi^{-1}(r-n)} \leq C \frac{\varphi_2(r)}{\Psi^{-1}(r-n)}, \] (32)
\[ \int_r^\infty \varphi_1(t) \rho(t) \frac{dt}{t} \leq C \varphi_2(r), \] (33)

for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), are sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{W} \mathcal{M}^{1,\varphi_2}(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then the conditions (32) and (33) are sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \).

2. Let \( \varphi_1 \) be almost decreasing. Then the condition
\[ \varphi_1(r) \int_0^r \rho(t) \frac{dt}{t} \leq C \varphi_2(r), \] (34)

for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), is necessary for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{W} \mathcal{M}^{1,\varphi_2}(\mathbb{R}^n) \) and hence \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \).

3. Let \( \rho \) satisfy (1) and (4). Assume that (11) is fulfilled, that \( \varphi_1 \) is almost decreasing and that \( \varphi_1 \) and \( \varphi_2 \) satisfy (32). Assume also that \( \varphi_1 \) and \( \rho \) satisfy the condition
\[ \int_r^\infty \varphi_1(t) \rho(t) \frac{dt}{t} \leq C \varphi_1(r) \rho(r), \] (35)

for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \). Then the condition (34) is necessary and sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{W} \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then the condition (34) is necessary and sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \).
Proof 1. By (27), (32) and (33) we have
\[
\| I_\rho f \|_{WM_\Psi, \eta} \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \beta^{-1}(r^{-n}) \| f \|_{L^\Phi(B(x,2r))} \\
+ \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \int_0^\infty \| f \|_{L^\Phi(B(x,t))} \Phi^{-1}(t^{-n}) \rho(t) \frac{dt}{t}
\]
\[
\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(r)^{-1} \beta^{-1}(r^{-n}) \| f \|_{L^\Phi(B(x,r))} \\
+ \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \int_0^\infty \varphi_1(t)^{-1} \rho(t) \frac{dt}{t} \| f \|_{M_\Phi, \varphi_1}
\]
\[
\lesssim \| f \|_{M_\Phi, \varphi_1}.
\]

Simply replace \( W^\Psi(B) \) with \( L^\Psi(B) \) and \( WM_\Psi, \eta(\mathbb{R}^n) \) with \( M_\Psi, \eta(\mathbb{R}^n) \) and use (28), (32) and (33) for the strong estimate.

2. We will now prove the second part. Let \( B_R = B(0, R) \) and \( x \in B_{R/2} \). By Lemma 3 we have
\[
\rho^*(R/2) := \int_0^{R/2} \frac{\rho(t)}{t} dt \leq CI_\rho \chi_{B_R}(x).
\]

Therefore, by Lemma 7 and the doubling property of \( \varphi_1 \),
\[
\rho^*(R/2) \lesssim |B_{R/2}|^{-1} \| I_\rho \chi_{B_R} \|_{W^1_1(B_{R/2})} \lesssim \varphi_2(R/2) \| I_\rho \chi_{B_R} \|_{WM^{1,2}}
\]
\[
\lesssim \varphi_2(R/2) \| \chi_{B_0} \|_{M_{\Phi, \varphi_1}} \lesssim \frac{\varphi_2(R/2)}{\varphi_1(R)} \lesssim \frac{\varphi_2(R/2)}{\varphi_1(R/2)}.
\]

Since this is true for every \( R > 0 \), we are done.

3. The third statement of the theorem follows from the first and second parts of the theorem.

6.2 Adams-type result

The following theorem was proved in [3, Theorem 4.6]:

Theorem 6 1. Let \( \varphi \in G_\Phi \) be almost decreasing. Then the maximal operator \( M \) is bounded from \( M_\Phi, \varphi(\mathbb{R}^n) \) to \( WM_\Psi, \eta(\mathbb{R}^n) \).

2. Let \( \Phi \in \nabla^2 \) and \( \varphi \in G_\Phi \) be almost decreasing. Then the maximal operator \( M \) is bounded on \( M_\Phi, \varphi(\mathbb{R}^n) \).

The following theorem gives necessary and sufficient conditions for Adams-type boundedness of the operator \( I_\rho \) from \( M_\Phi, \varphi(\mathbb{R}^n) \) to \( M_\Psi, \eta(\mathbb{R}^n) \):

Theorem 7 (Adams-type result) Let \( \Phi \) be a Youn function, and let \( \varphi \in G_\Phi \) be almost decreasing. Let \( \beta \in (0, 1) \) and define \( \eta(t) \equiv \varphi(t)^\beta \) for \( t > 0 \) and \( \Psi(t) \equiv \Phi(t^{1/\beta}) \) for \( t > 0 \).
1. Let \( \rho \) satisfy (1) and (4). Then the condition
\[
\varphi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \rho(t) \varphi(t) \frac{dt}{t} \leq C \eta(r),
\]
for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), is sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{W}^{\Psi, \eta}(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then the condition (36) is sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi, \eta}(\mathbb{R}^n) \).

2. The condition
\[
\varphi(r) \int_0^r \frac{\rho(t)}{t} dt \leq C \eta(r),
\]
for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), is necessary for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{W}^{\Psi, \eta}(\mathbb{R}^n) \) and hence for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi, \eta}(\mathbb{R}^n) \).

3. Let \( \rho \) satisfy (1) and (4). Assume that \( \varphi \) satisfies the condition
\[
\int_r^\infty \rho(t) \varphi(t) \frac{dt}{t} \leq C \rho(r) \varphi(r),
\]
for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \). Then the condition (37) is necessary and sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi, \eta}(\mathbb{R}^n) \).

Moreover, if \( \Phi \in \nabla_2 \), then the condition (37) is necessary and sufficient for the boundedness of \( I_\rho \) from \( \mathcal{M}^{\Phi, \varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi, \eta}(\mathbb{R}^n) \).

We notice that the function \( \varphi \) and \( \eta \) come into play, unlike Spanne-type. Similar to Lemma 5, we have the following pointwise estimate:

**Lemma 9** Let \( \Phi \) be a Young function, \( \varphi \in \mathcal{G}_{\Phi} \), \( \beta \in (0, 1) \), \( \eta(t) \equiv \varphi(t)^{\beta} \) and \( \Psi(t) \equiv \Phi(t^{1/\beta}) \). If (36) holds, then there exists a positive constant \( C \) such that, for all non-negative measurable functions \( f \) and for every \( x \in \mathbb{R}^n \),
\[
I_\rho f(x) \leq C (Mf(x))^{\beta - 1} \| f \|_{\mathcal{M}^{\Phi, \varphi}}.
\]

**Proof** Let \( \tilde{\rho} \) be defined by (14). We have
\[
I_\rho f(x) \leq C \left[ \sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\tilde{\rho}(2^j r)}{(2^j r)^\alpha} \int_{|x-y| < 2^j r} f(y) dy \right] = C (I + II)
\]
for given \( x \in \mathbb{R}^n \) and \( r > 0 \). Thus from (15) and (16) with \( \tau = \varphi \) we deduce
\[
I \leq C \sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) Mf(x) \leq C \left( \int_0^{k2^j r} \frac{\rho(s)}{s} ds \right) Mf(x)
\]
and
\[
II \leq C \sum_{j=0}^{\infty} \tilde{\rho}(2^j r) \varphi(2^j r) \| f \|_{\mathcal{M}^{\Phi, \varphi}} \leq C \| f \|_{\mathcal{M}^{\Phi, \varphi}} \int_{k2^j r}^\infty \varphi(s) \frac{\rho(s)}{s} ds.
\]
Consequently we have
\[ I_\rho f(x) \lesssim \left( \int_0^{k_2r} \frac{\rho(s)}{s} ds \right) Mf(x) + \| f \|_{M^\Phi, \psi} \int_{k_1r}^\infty \varphi(s) \frac{\rho(s)}{s} ds. \]

Thus, the technique in [30, p. 6492] by (36) and the doubling property of \( \varphi \) we obtain
\[
I_\rho f(x) \lesssim \min \left\{ \varphi(r)^{\beta-1} Mf(x), \varphi(r)^\beta \| f \|_{M^\Phi, \psi} \right\}
\]
\[
\lesssim \sup_{s>0} \min \left\{ s^{\beta-1} Mf(x), s^\beta \| f \|_{M^\Phi, \psi} \right\}
\]
\[
= (Mf(x))^\beta \| f \|_{M^\Phi, \psi}^{-1-\beta},
\]
where we have used that the supremum is achieved when the minimum parts are balanced. Hence we have
\[
I_\rho f(x) \lesssim (Mf(x))^\beta \| f \|_{M^\Phi, \psi}^{-1-\beta}.
\]

We have the following scaling law:

**Lemma 10** Let \( \beta > 0 \). Let \( \Psi \) and \( \Phi \) be Young functions, and let \( B \) be a ball. Then
\[
\| |f|^\beta \|_{L^\Phi(B)} = \| f \|_{L^\Phi(B)}^\beta \quad \text{and} \quad \| |f|^\beta \|_{W^\Psi(B)} = \| f \|_{W^\Psi(B)}^\beta
\]
for all measurable functions \( f \).

**Proof** Simply note that
\[
\int_B \Psi \left( \frac{|f(x)|^\beta}{\| f \|_{L^\Phi(B)}^\beta} \right) dx = \int_B \Phi \left( \frac{|f(x)|}{\| f \|_{L^\Phi(B)}} \right) dx
\]
for \( L^\Phi(B) \). The equality for weak spaces can be proved similarly.

**Proof of Theorem 7** 1.

– We deal with the weak-type estimate. By using inequality (39) we have for an arbitrary ball \( B \)
\[
\| I_\rho f \|_{W^\Psi(B)} \lesssim (Mf)^\beta \| f \|_{W^\Psi(B)}^{-1-\beta}.
\]

Consequently by using this inequality and Lemma 10 we have
\[
\| I_\rho f \|_{W^\Psi(B)} \lesssim \| Mf \|_{W^\Phi(B)}^\beta \| f \|_{M^\Phi, \psi}^{-1-\beta}. \tag{40}
\]

From Theorem 6 and (40), we get
\[
\| I_\rho f \|_{W^\Psi, \eta(B)} = \sup_B \eta(r)^{-1} \Psi^{-1}(r^{-n}) \| I_\rho f \|_{W^\Psi(B)}
\]
\[
\lesssim \| f \|_{M^\Phi, \psi} \sup_B \eta(r)^{-1} \Psi^{-1}(r^{-n}) \| Mf \|_{W^\Phi(B)}^\beta
\]
\[
= \| f \|_{M^\Phi, \psi} \left( \sup_B \varphi(r)^{-1} \Phi^{-1}(r^{-n}) \| Mf \|_{W^\Phi(B)} \right)^\beta
\]
\[
\lesssim \| f \|_{M^\Phi, \psi}.
\]
Simply replace $WL^\Psi(B)$ with $L^\Psi(B)$ and $W\mathcal{M}^\Psi,\eta(\mathbb{R}^n)$ with $\mathcal{M}^\Psi,\eta(\mathbb{R}^n)$ for the strong estimate.

2. We will now prove the second part. Let $B_R = B(0, R)$ and $x \in B_{R/2}$. By Lemmas 1, 3 and 7 and the doubling property of $\varphi$, we have

$$\rho^*(R/2) \leq C|B_{R/2}|^{-1}\|I_\rho \chi_{B_R}\|_{W^1_{L\Psi}(B_{R/2})} \leq C\eta(R/2)\|I_\rho \chi_{B_R}\|_{W\mathcal{M}^1,\eta} \leq C\eta(R/2)\|\chi_{B_R}\|_{\mathcal{M}^\varphi,\varphi} \leq C\rho(R/2) = C\varphi(R/2)^{\beta-1}.$$  

Since this is true for every $R > 0$, the proof is complete.

3. This part follows from the first and second parts.

7 Generalized fractional maximal operators on generalized Orlicz–Morrey spaces

In this section we give a characterization for the Spanne-type boundedness and the Adams-type boundedness of the operator $M_\rho$ on generalized Orlicz–Morrey spaces, respectively.

7.1 Spanne-type result

We use the following lemma:

**Lemma 11** Let $\Phi, \Psi$ be Young functions. Assume that $\rho$ is increasing and that $r \mapsto r^{-n}\rho(r)$ is decreasing. Assume also that the condition (19) is fulfilled. Then there exists a positive constant $C$ such that, for all $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$ and $B = B(x, r)$,

$$\|M_\rho f\|_{W^\Psi L(\mathbb{R}^n)} \leq C\|f\|_{L^\Phi(2B)} + \frac{C}{\psi^{-1}(r^{-n})} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x, 2t))} \Phi^{-1}(t^{-n}) \rho(t).$$  

Moreover if we assume $\Phi \in \nabla_2$, the following inequality is also valid:

$$\|M_\rho f\|_{L^\Psi(\mathbb{R}^n)} \leq C\|f\|_{L^\Phi(2B)} + \frac{C}{\psi^{-1}(r^{-n})} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x, 2t))} \Phi^{-1}(t^{-n}) \rho(t).$$  

**Proof** We represent $f$ as

$$f = f_1 + f_2, \quad f_1 = f \chi_{2B}, \quad f_2 = f - f_1, \quad r > 0.$$  

Then we have

$$\|M_\rho f\|_{W^\Psi L(\mathbb{R}^n)} \lesssim \|M_\rho f_1\|_{W^\Psi L(\mathbb{R}^n)} + \|M_\rho f_2\|_{W^\Psi L(\mathbb{R}^n)}.$$
From the boundedness of $M_\rho$ from $L^\Phi (\mathbb{R}^n)$ to $WL^\Psi (\mathbb{R}^n)$ (see Theorem 4) it follows that:

$$
\| M_\rho f_1 \|_{WL^\Psi (B)} \leq \| M_\rho f_1 \|_{WL^\Psi (\mathbb{R}^n)} \leq C \| f_1 \|_{L^\Phi (\mathbb{R}^n)} = C \| f \|_{L^\Phi (2B)},
$$

(43)

where constant $C > 0$ is independent of $f$.

If $y \in B$ and $r < t$, then $B(y, t) \subset B(x, 2t)$. Then, using Lemma 2, we have

$$
M_\rho f_2 (y) = \sup_{t > 0} \frac{\rho (t)}{|B(y, t)|} \int_{B(y, t) \setminus (B(x, 2t))} |f(z)| dz \\
\leq \sup_{t > r} \frac{\rho (t)}{|B(x, t)|} \int_{B(x, 2t)} |f(z)| dz \\
\lesssim \sup_{r < t < \infty} \rho (t) \Phi^{-1} (t^{-n}) \| f \|_{L^\Phi (B(x, 2t))} \text{ for } y \in B.
$$

(44)

Thus by Lemma 1 we have

$$
\| M_\rho f_2 \|_{WL^\Psi (B)} \lesssim \frac{1}{\Psi^{-1} (r^{-n})} \sup_{r < t < \infty} \rho (t) \Phi^{-1} (t^{-n}) \| f \|_{L^\Phi (B(x, 2t))}.
$$

(45)

Therefore we obtain (41) by (43) and (45).

If $\Phi \in \nabla_2$, then we can use strong type inequality instead of (43) and obtain (42) by using the same argument.

The following theorem gives a necessary and sufficient condition for Spanne-type boundedness of the operator $M_\rho$ from $\mathcal{M}^{\Phi, \varphi_1} (\mathbb{R}^n)$ to $\mathcal{M}^{\Psi, \varphi_2} (\mathbb{R}^n)$: We notice that the requirement is the same as the Orlicz spaces.

**Theorem 8** (Spanne-type result) Let $\Phi, \Psi$ be Young functions, and let $\varphi_1 \in \mathcal{G}_\Phi$ and $\varphi_2 \in \mathcal{G}_\Psi$.

1. Assume that $\rho$ is increasing and that $r \mapsto r^{-n} \rho (r)$ is decreasing. Assume also that the conditions (19) and (32) are satisfied. Then the condition

$$
\sup_{r < t < \infty} \varphi_1 (t) \rho (t) \leq C \varphi_2 (r),
$$

(46)

for all $r > 0$, where $C > 0$ does not depend on $r$, are sufficient for the boundedness of $M_\rho$ from $\mathcal{M}^{\Phi, \varphi_1} (\mathbb{R}^n)$ to $\mathcal{W} \mathcal{M}^{\Psi, \varphi_2} (\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the condition (46) is sufficient for the boundedness of $M_\rho$ from $\mathcal{M}^{\Phi, \varphi_1} (\mathbb{R}^n)$ to $\mathcal{M}^{\Psi, \varphi_2} (\mathbb{R}^n)$.

2. Let $\varphi_1$ be almost decreasing. Then the condition

$$
\varphi_1 (r) \rho (r) \leq C \varphi_2 (r),
$$

(47)

for all $r > 0$, where $C > 0$ does not depend on $r$, is necessary for the boundedness of $M_\rho$ from $\mathcal{M}^{\Phi, \varphi_1} (\mathbb{R}^n)$ to $\mathcal{W} \mathcal{M}^{\Psi, \varphi_2} (\mathbb{R}^n)$ and hence $\mathcal{M}^{\Phi, \varphi_1} (\mathbb{R}^n)$ to $\mathcal{M}^{\Psi, \varphi_2} (\mathbb{R}^n)$. 

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3. Assume that \( \rho \) is increasing and that \( r \mapsto r^{-n}\rho(r) \) is decreasing. Assume also that the conditions (19) and (32) are satisfied. Let \( \varphi_1 \) and \( \varphi_2 \) be almost decreasing. Then the condition (47) is necessary and sufficient for the boundedness of \( M_\rho \) from \( \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) \) to \( \mathcal{W}\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then the condition (47) is necessary and sufficient for the boundedness of \( M_\rho \) from \( \mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n) \).

**Proof** 1. By (32), (41), (46) and the doubling properties of \( \varphi_1 \) and \( \Phi^{-1} \) we have

\[
\|M_\rho f\|_{\mathcal{W}\mathcal{M}^{\Psi,\varphi_2}} \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(r)^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x,2r))} + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \sup_{r < t < \infty} \|f\|_{L^\Phi(B(x,2t))} \Phi^{-1}(t^{-n}) \rho(t) \\
\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi_1(r)^{-1}(r^{-n}) \|f\|_{L^\Phi(B(x,r))} + \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(r)^{-1} \varphi_1(t) \rho(t) \|f\|_{\mathcal{M}^{\Psi,\varphi_1}} \\
\lesssim \|f\|_{\mathcal{M}^{\Phi,\varphi_1}}.
\]

Simply replace \( WL^\Psi(B) \) with \( L^\Psi(B) \) and \( \mathcal{W}\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n) \) with \( \mathcal{M}^{\Psi,\eta}(\mathbb{R}^n) \) for the strong estimate.

2. We will now prove the second part. We utilize (22). By Lemma 7, we have

\[
\rho(r) \lesssim |B(0, r)|^{-1} \|M_\rho \chi_B(0,2r)\|_{WL^1(B(0,r))} \lesssim \varphi_2(r) \|M_\rho \chi_B(0,2r)\|_{\mathcal{W}\mathcal{M}^{\varphi_2}} \\
\lesssim \varphi_2(r) \|\chi_B(0,2r)\|_{\mathcal{M}^{\Phi,\varphi_1}} \lesssim \frac{\varphi_2(r)}{\varphi_1(r)}.
\]

3. Since \( \varphi_2 \) is almost decreasing, (46) and (47) are equivalent. Then the third statement of the theorem follows from the first and second parts of the theorem.

### 7.2 Adams-type result

The following theorem gives necessary and sufficient conditions for Adams-type boundedness of the operator \( M_\rho \) from \( \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\eta}(\mathbb{R}^n) \).

Here we suppose that \( \rho \) is an increasing function such that \( r \in (0, \infty) \mapsto r^{-n}\rho(r) \in (0, \infty) \) is decreasing.

**Theorem 9** Let \( \Phi \) be a Young function, and let \( \varphi \in \mathcal{G}_\Phi \) be almost decreasing. Assume that \( \rho \) is increasing and that \( r \mapsto r^{-n}\rho(r) \) is decreasing. Let \( \beta \in (0, 1) \), \( \eta(t) \equiv \varphi(t)^\beta \) and \( \Psi(t) \equiv \Phi(t^{1/\beta}) \). Then the condition

\[
\rho(t) \lesssim \varphi(t)^{\beta^{-1}}, \tag{48}
\]

is necessary and sufficient for the boundedness of \( M_\rho \) from \( \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) \) to \( \mathcal{W}\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n) \). Moreover, if \( \Phi \in \nabla_2 \), then the condition (48) is necessary and sufficient for the boundedness of \( M_\rho \) from \( \mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n) \) to \( \mathcal{M}^{\Psi,\eta}(\mathbb{R}^n) \).
As before, we start with an auxiliary pointwise estimate.

**Lemma 12** Under the assumption of Theorem 9 including (48), there exists a positive constant C such that, for all \( f \in \mathcal{M}^{\Phi,\psi}(\mathbb{R}^n) \) and all \( x \in \mathbb{R}^n \),

\[
M_\rho f(x) \leq C (Mf(x))^{\beta} \|f\|_{\mathcal{M}^{\Phi,\psi}}^{1-\beta}.
\]  

**Proof** For arbitrary ball \( B = B(x, r) \) we represent \( f \) as

\[
f = f_1 + f_2, \quad f_1 = f \chi_{B(x, r)}^{\circ}, \quad f_2 = f - f_1, \quad r > 0,
\]

so that

\[
M_\rho f(x) \leq M_\rho f_1(x) + M_\rho f_2(x).
\]

Hence by Lemma 2,

\[
M_\rho f_2(x) = \sup_{t>0} \frac{\rho(t)}{|B(x,t)|} \int_{B(x,t) \setminus (B(x,r))} |f(z)|dz
\]

\[
\leq \sup_{t>r} \frac{\rho(t)}{|B(x,t)|} \int_{B(x,t)} |f(z)|dz
\]

\[
\lesssim \sup_{t>r} \rho(t) \Phi^{-1}(|B(x,t)|^{-1}) \|f\|_{L^\Phi(B(x,t))}
\]

\[
\lesssim \|f\|_{\mathcal{M}^{\Phi,\psi}} \sup_{t>r} \rho(t) \varphi(t).
\]

Consequently by Lemma 4 we have

\[
M_\rho f(x) \lesssim \rho(r) Mf(x) + \|f\|_{\mathcal{M}^{\Phi,\psi}} \sup_{t>r} \rho(t) \varphi(t).
\]

Thus, using the technique in [30, p. 6492] as before and (48) we obtain

\[
M_\rho f(x) \lesssim \min \{[\varphi(r)]^{\beta-1} Mf(x), \varphi(r) \beta \|f\|_{\mathcal{M}^{\Phi,\psi}}\}
\]

\[
\lesssim \sup_{s>0} \min \{s^{\beta-1} Mf(x), s^{\beta} \|f\|_{\mathcal{M}^{\Phi,\psi}}\}
\]

\[
= (Mf(x))^{\beta} \|f\|_{\mathcal{M}^{\Phi,\psi}}^{1-\beta},
\]

where we have used that the supremum is achieved when the minimum parts are balanced. This shows (49).

We prove Theorem 9.

**Proof of Theorem 9** By using inequality (49) and Lemma 10 we have, for all balls \( B \),

\[
\|M_\rho f\|_{WL^\Phi(B)} \lesssim \|(Mf)^{\beta}\|_{WL^\Phi(B)} \|f\|_{\mathcal{M}^{\Phi,\psi}}^{1-\beta} = \|Mf\|_{WL^\Phi(B)}^{\beta} \|f\|_{\mathcal{M}^{\Phi,\psi}}^{1-\beta}.
\]
Consequently, by using the boundedness of the maximal operator \( M \), we get

\[
\eta(r)^{-1} \psi^{-1}(|B|^{-1}) \| M_\rho f \|_{WL^\psi(B)} \lesssim \eta(r)^{-1} \psi^{-1}(|B|^{-1}) \| Mf \|_{WL^\psi(B)}^\beta \| f \|_{M^\psi}\Psi \lesssim \eta(r)^{-1} \psi^{-1}(|B|^{-1}) \| Mf \|_{WL^\psi(B)}^\beta \| f \|_{M^\psi}. \\
\]

By taking the supremum over all balls \( B \), we get the desired result. Moreover, if \( \Phi \in \nabla_2 \), then we have the strong type estimate.

We will now prove the necessity. We utilize (22). By Lemmas 1 and 7, we have

\[
\rho(r) \lesssim \psi^{-1}(r^{-\eta}) \| M_\rho \chi_B(0,2r) \|_{WL^\psi(B(0,r))} \lesssim \eta(r) \| M_\rho \chi_B(0,2r) \|_{W^\psi} \lesssim \eta(r)^{-1} \psi^{-1}(r^{-\eta}) \| f \|_{M^\psi}. \\
\]

Then the proof is complete.

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