Lipschitz and Fourier type conditions with moduli of continuity in rank 1 symmetric spaces

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Abstract
Sufficient and necessary results have been proven on Lipschitz type integral conditions and bounds of its Fourier transform for an $L^2$ function, in the setting of Riemannian symmetric spaces of rank 1 whose growth depends on a $k$th-order modulus of continuity.

Keywords Generalised Hölder space · Lipschitz type condition · Fourier transform · Moduli of continuity · Translation operator · Symmetric space

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1 Introduction
The inspiration for this research begins with the classical theorem of Titchmarsh [13][Theorem 85] on the characterisation of functions in $L^2(\mathbb{R})$ satisfying an integral Lipschitz-type condition in terms of an asymptotic estimate for the growth of the norm of their Fourier transform. This theorem is stated as follows.
Theorem 1 Titchmarsh [13] Let $f \in L^2(\mathbb{R})$ and $0 < \alpha < 1$. Then the following statements are equivalent.

1. There exists a constant $C > 0$ such that for all sufficiently small $t > 0$,
\[
\int_{-\infty}^{\infty} |f(x + t) - f(x - t)|^2 \, dx \leq Ct^{2\alpha}.
\]

2. There exists a constant $C > 0$ such that for all sufficiently small $t > 0$,
\[
\left( \int_{-\infty}^{-1/t} + \int_{1/t}^{+\infty} \right) |\hat{f}(x)|^2 \, dx \leq Ct^{2\alpha}.
\]

This result has been shown to be a very useful tool to give an alternative description of the Lipschitz or Hölder spaces, see e.g. [3,12] and references therein. Similar results using moduli of continuity for the characterisation of Lipschitz conditions by means of the Fourier transform can be found in [2,6,14] among others.

Following mainly the ideas from [2,12,13], we study and establish growth conditions between functions in $L^2(\mathbb{R})$ satisfying an integral Lipschitz-type condition and its Fourier transforms by means of $k$th-order moduli of continuity over rank 1 Riemannian symmetric spaces. Some examples of the latter spaces are the two-point homogeneous spaces [15, Chapter 8], the $n$-dimensional sphere $\mathbb{S}^n$, and the $n$-dimensional Lobachevskiî space. The results are in the form of sufficient and necessary conditions (under certain assumptions) for Lipschitz-type conditions using asymptotic estimates for the growth of the Fourier transform. In a very particular case, an analogue of Titchmarsh’s original Theorem 1 is recovered.

The paper has the following structure. Section 2 is devoted to setting up all the necessary framework for the problem: firstly all the machinery of Riemannian symmetric spaces, and secondly the essential definitions and properties of $k$th-order moduli of continuity. In Sect. 3, the main results are stated and proved, with all required conditions added for both the necessary and the sufficient condition. Finally, Sect. 4 shows special cases and analogous results on the Euclidean space $\mathbb{R}^n$.

2 Preliminaries

This section is for recalling various definitions and facts which will be used later in proving the main results of the paper. Firstly, a discussion of semi-simple Lie groups and symmetric spaces with particular emphasis on the Fourier transform; for further information about these topics, the reader is recommended to [7,8]. Secondly, a brief introduction to moduli of continuity and $k$th-order moduli of continuity will be given.

2.1 Riemannian symmetric spaces

A noncompact Riemannian symmetric space $X$ can be seen as a quotient space $G/K$ where $G$ is a connected noncompact semi-simple Lie group with finite center and $K$ is a
normalised so that supported functions on $X$ are the Lie subgroups of $G$ as follows: $G$ maximal compact subgroup of $Lipschitz and Fourier type conditions with moduli... 355$. Moreover, $K$ coincides with the stationary subgroup of the point $o = eK \in X$, where $e$ is the identity of $G$.

The Iwasawa decomposition of the group $G$ is written $G = NAK$, where $N, A, K$ are the Lie subgroups of $G$ generated by the Lie algebras $n_0, a_0$ and $t_0$, respectively. Let $M$ be the centraliser of $A$ in $K$ and set $B = K/M$. The $G$-invariant measure on $X$ will be denoted $dx$, and the normalised $K$-invariant measure on $B$ and $K$ will be denoted $db$ and $dk$ respectively.

Assume the symmetric space $X$ has rank 1, so that the real dual $a_0^*$ of the Lie algebra $a_0$ is 1-dimensional and can be identified isomorphically with $\mathbb{R}$. Let $W$ denote the finite Weyl group acting on $a_0^*$. Let $\Sigma \subset a_0^*$ be the set of all bounded roots, and $\Sigma^+$ be the subset of positive bounded roots. For any $\lambda \in a_0^*$, say $T_\lambda$ is the vector in $a_0$ such that $\lambda(T) = \langle T_\lambda, T \rangle$ for any $T \in a_0$ (here $\langle, \rangle$ is the Killing form on the Lie algebra $g$ of $G$). Then define $a_0^+ = \{ r \in a_0 : \alpha(r) > 0 \text{ for } \alpha \in \Sigma^+ \}$ and $a_{0,+}^* = \{ \lambda \in a_0^* : T_\lambda \in a_0^+ \}$.

Let $W$ denote the set of all bounded roots, and $\Sigma^+$ be the subset of positive bounded roots. For any $\lambda \in a_0^*$, say $T_\lambda$ is the vector in $a_0$ such that $\lambda(T) = \langle T_\lambda, T \rangle$ for any $T \in a_0$ (here $\langle, \rangle$ is the Killing form on the Lie algebra $g$ of $G$). Then define $a_0^+ = \{ r \in a_0 : \alpha(r) > 0 \text{ for } \alpha \in \Sigma^+ \}$ and $a_{0,+}^* = \{ \lambda \in a_0^* : T_\lambda \in a_0^+ \}$.

Under the identification of $a_0^*$ with $\mathbb{R}$, it follows that $a_{0,+}^*$ can be identified with $\mathbb{R}^+$. For more details on this, see e.g. [11,12].

For any $g \in G$, taking into account that $G = NAK$, there is a unique element $A(g) \in a_0$ such that

$$g = n \cdot \exp A(g) \cdot k, \quad n \in N, \ k \in K.$$ 

The map $A : G \to a_0$ thus defined gives rise to a well-defined map $A : X \times B \to a_0$ as follows:

$$A(x, b) := A(k^{-1}g), \quad x = gK \in G/K, \ b = kM \in B = K/M.$$ 

Let $\mathcal{D}(X)$ and $\mathcal{D}(G)$ denote the sets of all infinitely differentiable compactly-supported functions on $X$ and $G$ respectively. Let $dg$ be the Haar measure on $G$, normalised so that

$$\int_X f(x)dx = \int_G f(go)dg, \quad f \in \mathcal{D}(X).$$ 

The Fourier transform on $X$ is defined by

$$\hat{f}(\lambda, b) = \int_X f(x)e^{-i\lambda + \rho}A(x, b) \, dx, \quad \lambda \in a_0^*, \ b \in B = K/M, \ \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha, \quad (1)$$

and the inverse Fourier transform on $X$ is defined by

$$f(x) = \frac{1}{|W|} \int_{a_0^* \times B} \hat{f}(\lambda, b)e^{i\lambda + \rho}A(x, b)|c(\lambda)|^{-2}d\lambda db,$$

where $|W|$ is the order of the Weyl group, $d\lambda$ is the Euclidean measure on $a_0^*$ (identified with $\mathbb{R}$), and $c(\lambda)$ is the Harish-Chandra function. For more details see e.g. [8,9].
The Harish-Chandra formula defines a so-called spherical function \( \varphi_{\lambda} \) on \( G \) for any \( \lambda \in a_0^* \) as follows:

\[
\varphi_{\lambda}(g) = \int_{K} e^{(i\lambda + \rho)A(kg)} dk, \quad g \in G.
\] (2)

One of its properties is that \( \varphi_{\lambda}(k_1 g k_2) = \varphi_{\lambda}(g) \) for any \( k_1, k_2 \in K \) and \( g \in G \). Using this property it can be proved that \( \varphi_{\lambda}(g) \) depends only on the distance \( d(go, o) = t \) which is a non-negative real number. So, where convenient, \( \varphi_{\lambda}(t) \) can be written instead of \( \varphi_{\lambda}(g) \). See the proof of [12, Lemma 3] for more details.

The following lemma on some estimates of the function \( \varphi_{\lambda} \) will be very useful later.

**Lemma 1** [11, Lemmas 3.1–3.3] For any \( \lambda \in \mathbb{R} \) (identified with \( a_0^* \)) and any \( t \in \mathbb{R}_+ \):

1. \( |\varphi_{\lambda}(t)| \leq 1 \), with equality just at the point \( t = 0 \), i.e. \( |\varphi_{\lambda}(0)| = 1 \).
2. \( 1 - \varphi_{\lambda}(t) \leq t (\lambda^2 + \rho^2) \).
3. There exists a constant \( C > 0 \), independent of \( \lambda \) and \( t \), such that \( 1 - \varphi_{\lambda}(t) \geq C \) whenever \( \lambda \geq 1/t \).

The abstract Fourier transform on \( X \) defined by (1) satisfies a Plancherel formula on \( X \) just like the classical Fourier transform:

\[
\int_{X} |f(x)|^2 dx = \frac{1}{|W|} \int_{a_0^* \times B} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db = \int_{a_0^* \times B} |\widehat{f}(\lambda, b)|^2 d\mu(\lambda) db,
\]

where \( d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda \) denotes the measure used on \( a_0^* \). Therefore, the continuous mapping \( f \rightarrow \hat{f} \) can be considered as an isomorphism from the space \( L^2(X, dx) \) onto the space \( L^2(a_0^* \times B, d\mu(\lambda) db) \). For more details about this, see [4,5] and also [12, Formula(1.9)].

The results of this paper will be concerned with the Fourier transform defined in (1) above, and also with the translation operator defined in (3) below. The latter is essentially defined by taking averages of a function \( f \) on sphere-like neighbourhoods. So, first it is necessary to define the framework for such neighbourhoods.

For any \( x \in X \) and \( t \in \mathbb{R}_+ \), the sphere of radius \( t \) in \( X \) centred at \( x \) is denoted by

\[
\sigma(x; t) = \{ y \in X : d(x, y) = t \},
\]

where as usual \( d(x, y) \) denotes the distance between the elements \( x, y \in X \). This has an associated area element \( d\sigma_t(y) \) used for integration, and overall surface area \( |\sigma(t)| \) of the whole sphere. The translation operator \( S^t \) is defined as follows, for functions \( f \in C_0(X) \), the set of all continuous compactly-supported functions on \( X \):

\[
(S^t f)(x) = \frac{1}{|\sigma(t)|} \int_{\sigma(x; t)} f(y) d\sigma_t(y), \quad t > 0.
\] (3)

It can be proved that the operator \( S^t \) is bounded from \( C_0(X) \) to \( L^2(X) \), see [12, Lemma 2] and consequences. In [12, Lemma 3], it was shown that the translation oper-
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ator, spherical function, and Fourier transform have the following interrelationship:

\[ \hat{S^t f}(\lambda, b) = \varphi_\lambda(t) \hat{f}(\lambda, b), \quad f \in L^2(X), \quad t \in \mathbb{R}_0^+. \]  \hfill (4)

2.2 Moduli of continuity of kth order

Definition 1 The function \( \omega : I \subset \mathbb{R} \to [0, \infty) \) is called almost increasing if there exists a constant \( C \geq 1 \) such that \( \omega(t) \leq C \omega(s) \) for all \( t, s \in I \) with \( t \leq s \). Moreover, it is called almost decreasing if there exists a constant \( C \geq 1 \) such that \( \omega(t) \leq C \omega(s) \) for all \( t, s \in I \) with \( t \geq s \).

Definition 2 Let \( \delta_0 \in (0, \infty) \) be a fixed real number and \( k \in \mathbb{R}^+ \). The function \( \omega_k : [0, \delta_0] \to [0, \infty) \) is called a \( k \)th-order modulus of continuity if the following conditions hold:

1. \( \omega_k(0) = 0 \) and \( \omega_k(t) \) is continuous on \([0, \delta_0]\).
2. \( \omega_k(t) \) is almost increasing on \( t \in [0, \delta_0] \).
3. \( \frac{\omega_k(t)}{t^k} \) is almost decreasing on \( t \in [0, \delta_0] \).

It is important to note the following fact. Any \( k \)th-order modulus of continuity is also an \( m \)th-order modulus of continuity for all \( m \geq k \): conditions 1 and 2 are the same, and dividing an almost decreasing function by a positive power of \( t \) gives another almost decreasing function. This fact can be illustrated by considering power functions: for any given \( k \), the function \( \omega(t) = t^\gamma \) is a \( k \)th-order modulus of continuity whenever \( 0 < \gamma < k \).

The following Zygmund type conditions of \( k \)th order will also frequently be used:

1. There exists a constant \( C \) such that for all \( t \in [0, \delta_0] \),

\[ \int_0^t \frac{\omega_k(x)}{x} \mathrm{d}x \leq C \omega_k(t). \]  \hfill (5)

2. There exists a constant \( C \) such that for all \( t \in [0, \delta_0] \),

\[ \int_t^{\delta_0} \frac{\omega_k(x)}{x^{1+k}} \mathrm{d}x \leq C \frac{\omega_k(t)}{t^k}. \]  \hfill (6)

In the book [10, Definition 2.9], the notation \( Z_k \) is used for functions satisfying (6), and the condition (5) says that \( \omega \in Z^0 \) (in general \( Z^\beta \) would be defined by replacing the integrand and right-hand side in (5) with those from (6)).

The Matuszewska-Orlicz type (MO) lower and upper indices of the function \( \omega_k(t) \) is also related to these function spaces. The definitions are [10]:

\[ m(\omega_k) = \sup_{0 < t < 1} \log \left( \lim_{\varepsilon \to 0} \frac{\omega_k(\varepsilon t)}{\omega_k(\varepsilon)} \right) = \lim_{t \to 0} \log \left( \frac{\lim_{\varepsilon \to 0} \omega_k(\varepsilon t)}{\omega_k(\varepsilon)} \right), \]
\[ M(\omega_k) = \sup_{t>1} \frac{\log \left( \limsup_{\varepsilon \to 0} \frac{\omega_k(\varepsilon t)}{\omega_k(\varepsilon)} \right)}{\log t} = \lim_{t \to \infty} \frac{\log \left( \limsup_{\varepsilon \to 0} \frac{\omega_k(\varepsilon t)}{\omega_k(\varepsilon)} \right)}{\log t}, \]

and these are related to the Zygmund conditions and spaces by the following result [10, Theorem 2.10]:

**Theorem 2** Let \( \omega_k \) be a \( k \)th-order modulus of continuity as above, and \( \delta \in \mathbb{R} \). Then \( \omega_k \in Z^\delta \) if and only if \( m(\omega_k) > \delta \), while \( \omega_k \in Z^\delta \) if and only if \( M(\omega_k) < \delta \). Moreover, we have

\[ m(\omega_k) = \sup \left\{ \delta > 0 : \frac{\omega_k(t)}{t^\delta} \text{ is almost increasing} \right\}, \]

\[ M(\omega_k) = \inf \left\{ \delta > 0 : \frac{\omega_k(t)}{t^\delta} \text{ is almost decreasing} \right\}. \]

If \( \omega \) is a modulus of continuity and satisfies the extra Zygmund type conditions (5) and (6) then \( \omega \) belongs to the so-called Bary–Stechkin class [1]. Some classical examples of functions in the Bary–Stechkin class are \( t^\gamma, t^\gamma (\log 1/t)^\lambda, \) and \( t^\gamma (\log \log 1/t)^\lambda, \) where \( \gamma \in (0, 1) \) and \( \lambda \in \mathbb{R} \). These examples can be adapted easily to the \( k \)th-order case, simply replacing the condition \( \gamma \in (0, 1) \) by the more general condition \( \gamma \in (0, k) \).

The most important behaviour of the functions above is near zero. But, when it is necessary to do some estimations on these functions over \([\delta_0, \infty)\), it will be assumed that \( \omega_k(t) \) is bounded below on \([\delta_0, \infty)\) by a positive number, and that \( \omega_k(t)^2/t^5 \in L^1([\delta_0, \infty))\), without loss of generality of the results.

### 3 Main results

This section begins by introducing the space to be considered in this paper.

**Definition 3** Let \( \omega_k \) be a \( k \)th-order modulus of continuity. Let \( X \) be a Riemannian symmetric space of rank 1 with \( n = \dim X \). A function \( f \in L^2(X) \) is said to be in the generalised Lipschitz class \( \text{Lip}(\omega_k) \) if there exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),

\[ \|S^t f - f\|_{L^2(X)} \leq C \omega_k(t). \]

Below is one of the main results of this paper, an estimate of the Fourier transform of any function in \( \text{Lip}(\omega_k) \), or in other words a necessary condition in terms of Fourier transforms for a function to be in this generalised Lipschitz class.

**Theorem 3** Let \( \omega_k \) be a \( k \)th-order modulus of continuity. Let \( X \) be a Riemannian symmetric space of rank 1 with \( n = \dim X \). If \( f \in \text{Lip}(\omega_k) \), then there exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),

\[ \|\hat{S^t f} - \hat{f}\|_{L^2(\mathbb{R}^n)} \leq C \omega_k(t). \]
\[
\int_{1/t}^{+\infty} \int_{B} |\hat{f}(\lambda, b)|^2 d\lambda db \leq C\omega_k(t)^2 t^{n-1}. \tag{7}
\]

**Proof** Notice that [12, Formula (3.2)], which follows from the Plancherel formula and the identity (4), gives the following:

\[
\|S^t f - f\|^2_{L^2(X)} = \int_{X} |S^t f(x) - f(x)|^2 dx = \int_{0}^{+\infty} |1 - \varphi_\lambda(t)|^2 H(\lambda) d\mu(\lambda), \tag{8}
\]

where

\[
H(\lambda) = \int_{B} |\hat{f}(\lambda, b)|^2 db \quad \text{and} \quad d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda.
\]

By the representation (8), part 3 of Lemma 1, and the fact that \(f \in \text{Lip}(\omega_k)\), we have

\[
\int_{1/t}^{+\infty} H(\lambda) d\mu(\lambda) \leq \frac{1}{C_1^2} \int_{1/t}^{+\infty} |1 - \varphi_\lambda(t)|^2 H(\lambda) d\mu(\lambda)
\]

\[
\leq \frac{1}{C_2} \int_{0}^{+\infty} |1 - \varphi_\lambda(t)|^2 H(\lambda) d\mu(\lambda) \leq C_2 \omega_k(t)^2. \tag{9}
\]

Besides, by [12, formula (3.6)] it is known that

\[
|c(\lambda)|^{-2} \approx \lambda^{n-1}. \tag{10}
\]

Therefore, by (9), (10) and \(\lambda \geq 1/t\) it follows that

\[
\int_{1/t}^{+\infty} \int_{B} |\hat{f}(\lambda, b)|^2 d\lambda db = \int_{1/t}^{+\infty} H(\lambda) d\lambda = t^{n-1} \int_{1/t}^{+\infty} H(\lambda) \left(\frac{1}{t}\right)^{n-1} d\lambda
\]

\[
\leq t^{n-1} \int_{1/t}^{+\infty} H(\lambda) \lambda^{n-1} d\lambda \approx t^{n-1} \int_{1/t}^{+\infty} H(\lambda) d\mu(\lambda) \leq C_3 t^{n-1} \omega_k(t)^2.
\]

For the converse statement of Theorem 3, it is necessary to add an extra assumption on \(k\). The result is given by the following theorem.

**Theorem 4** Let \(k \leq 2\), and let \(\omega_k\) be a modulus of continuity of order \(k\), satisfying the Zygmund conditions (5) and (6). It is also assumed that \(\omega_k(t)\) is bounded below on \([\delta_0, \infty)\) by a positive number, and that \(\omega_k(t)^2/t^5 \in L^1([\delta_0, \infty))\). Let \(X\) be a Riemannian symmetric space of rank 1 with \(n = \dim X\). If there exists a constant \(C > 0\) such that for all sufficiently small \(t > 0\) the condition (7), then \(f \in \text{Lip}(\omega_k)\).

**Proof** With the same notation \(H(\lambda)\) as used in the proof of Theorem 3, the condition (7) implies

\[
\int_{1/t}^{2/t} H(\lambda) \lambda^{n-1} d\lambda \leq \left(\frac{2}{t}\right)^{n-1} \int_{1/t}^{2/t} H(\lambda) d\lambda \leq \left(\frac{2}{t}\right)^{n-1} \int_{1/t}^{+\infty} H(\lambda) d\lambda \leq C_1 \omega_k(t)^2.
\]
Hence, substituting \( t/2^j \) for \( t \) and then combining the results for different \( j \),

\[
\int_{1/t}^{+\infty} H(\lambda)\lambda^{n-1}d\lambda = \sum_{j=0}^{+\infty} \int_{2^j/t}^{2^{j+1}/t} H(\lambda)\lambda^{n-1}d\lambda \leq C_1 \sum_{j=0}^{+\infty} \omega_k \left( \frac{t}{2^j} \right)^2
\]
\[
\leq C_2 \omega_k(t)^2 \sum_{j=0}^{+\infty} \left( \frac{t}{2^j} \right)^{2\delta} \leq C\omega_k(t)^2,
\]

where \( \delta > 0 \) is less than the MO index \( m(\omega) \), so that \( \frac{\omega(t)}{t^\delta} \) is almost increasing by Theorem 2. Consequently, using (10), the following is obtained:

\[
\int_{1/t}^{+\infty} H(\lambda)d\mu(\lambda) \leq C_3\omega_k(t)^2. \tag{11}
\]

The integral on the right-hand side of (8) can be split as follows:

\[
\|S^t f - f\|^2_{L^2(\mathcal{H})} = J_1 + J_2,
\]

where

\[
J_1 = \int_0^{1/t} |1 - \varphi_\lambda(t)|^2 H(\lambda)d\mu(\lambda), \quad J_2 = \int_{1/t}^{+\infty} |1 - \varphi_\lambda(t)|^2 H(\lambda)d\mu(\lambda).
\]

The second term \( J_2 \) can be estimated using (11) and the first part of Lemma 1:

\[
J_2 \leq 4 \int_{1/t}^{+\infty} H(\lambda)d\mu(\lambda) \leq C_3\omega_k(t)^2.
\]

For the first term \( J_1 \), use the second part of Lemma 1:

\[
J_1 \leq C_4 t^4 \int_0^{1/t} (\lambda^4 + \rho^4) H(\lambda)d\mu(\lambda) = K_1 + K_2,
\]

where

\[
K_1 = C_4 t^4 \int_0^{1/t} \lambda^4 H(\lambda)d\mu(\lambda), \quad K_2 = C_4 t^4 \rho^4 \int_0^{1/t} H(\lambda)d\mu(\lambda).
\]

Here the second term \( K_2 \) can be estimated using Plancherel’s theorem:

\[
K_2 \leq C_4 t^4 \rho^4 \int_0^{+\infty} H(\lambda)d\mu(\lambda) \leq C_4 t^4 \rho^4 \|f\|_2^2 \leq C_5\omega_k(t)^2,
\]

since \( t^2 \leq C\omega_k(t) \). Here the assumption \( k \leq 2 \) is used. It remains only to estimate \( K_1 \).
The formula (11) is assumed to hold for all sufficiently small \( t > 0 \). But, by the Plancherel formula and the fact that \( \omega_k \) is bounded below on any closed interval in the positive reals, it can equally well be assumed that (11) holds for all \( t \in \mathbb{R}^+ \). Using the latter fact and setting \( \phi(t) = \int_t^{+\infty} H(\lambda) \, d\mu(\lambda) \) for any \( t > 0 \), the estimating of \( K_1 \) can be done as follows:

\[
K_1 = C_4 t^4 \int_0^{1/t} (-s^4 \phi'(s)) \, ds = C_4 t^4 \left( -\frac{1}{t^4} \phi \left( \frac{1}{t} \right) + 4 \int_0^{1/t} s^3 \phi(s) \, ds \right) 
\leq 4C_4 t^4 \int_0^{1/t} s^3 \phi(s) \, ds \leq C_5 t^4 \int_0^{1/t} s^3 \omega_k(1/s)^2 \, ds = C_5 t^4 \int_t^{+\infty} \frac{\omega_k(u)^2}{u^5} \, du 
= C_5 t^4 \left( \int_t^{\delta_0} \frac{\omega_k(u)^2}{u^5} \, du + \int_{\delta_0}^{+\infty} \frac{\omega_k(u)^2}{u^5} \, du \right) 
\leq C_6 \left( t^4 \cdot \frac{\omega_k(t)^2}{t^{4-k}} \int_t^{\delta_0} \frac{\omega_k(u)}{u^{k+1}} \, du + t^4 \right) \leq C \omega_k(t)^2,
\]

where in the last two lines the following were used: the fact that \( \omega_k(t)/t^{4-k} \) is almost decreasing (since \( 4-k \geq k \) because \( k \leq 2 \), the assumption \( \omega_k(t)^2/t^5 \in L^1[\delta_0, +\infty) \), the Zygmund condition (6), and the fact that \( t^2 \leq C \omega_k(t) \) since \( k \leq 2 \).

So finally it is established that \( \|S^t f - f\|_{L^2(X)} \leq C \omega_k(t) \).

Combining Theorem 3 and Theorem 4 gives the following equivalence relation between bounding conditions on the translation operator and on the Fourier transform.

**Theorem 5** Let \( X \) be a Riemannian symmetric space of rank 1 with \( n = \dim X \). Let \( \omega_k \) be a \( k \)-th-order modulus of continuity, with \( k \leq 2 \), satisfying the Zygmund conditions (5) and (6). It is also assumed that \( \omega_k(t) \) is bounded below on \([\delta_0, \infty)\) by a positive number, and that \( \omega_k(t)^2/t^5 \in L^1([\delta_0, \infty)) \). If \( f \) is a function in \( L^2(X) \), then the following statements are equivalent.

1. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),

\[
\|S^t f - f\|_{L^2(X)} \leq C \omega_k(t).
\]

2. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),

\[
\int_{1/t}^{+\infty} \int_B |\hat{f}(\lambda, b)|^2 \, d\lambda \, db \leq C \omega_k(t)^2 t^{n-1}.
\]

**Proof** Both directions of the equivalence are proved in Theorems 3, 4 respectively.

As an example of the above results, consider the function \( \omega(t) = t^\alpha \) for any \( 0 < \alpha < 2 \) and \( t \in [0, \delta_0] \). This is a 2nd-order modulus of continuity \( (k = 2) \). To ensure the appropriate conditions on \([\delta_0, +\infty)\), assume that \( \omega(t) \equiv W \) for any \( t \geq \delta_0 \) and some constant \( W > 0 \). Then the following result is obtained as a corollary of Theorem 5.
Corollary 1  Let $X$ be a Riemannian symmetric space of rank $1$ with $n = \dim X$, and let $0 < \alpha < 2$. If $f$ is a function in $L^2(X)$, then the following assertions are equivalent:

1. There exists a constant $C > 0$ such that for all sufficiently small $t > 0$,

$$\|S^t f - f\|_{L^2(X)} \leq C t^\alpha.$$ 

2. There exists a constant $C > 0$ such that for all sufficiently small $t > 0$,

$$\int_{1/t}^{+\infty} \int_B |\hat{f}(\lambda, b)|^2 d\lambda \, db \leq C t^{2\alpha+n-1}.$$ 

4 Special cases on $\mathbb{R}^n$

The symmetric spaces considered in this paper and the Euclidean spaces $\mathbb{R}^n$ ($n \geq 1$) belong to the noncompact two-point homogeneous spaces [15]. Thus, similar results as those given in Section 3 can be established for the space $\mathbb{R}^n$.

The following notation will be used. The usual inner product and norm on $\mathbb{R}^n$ are given by $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$, $|x| = \sqrt{\langle x, x \rangle}$, for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$. The Fourier transform on $\mathbb{R}^n$ is defined as

$$\hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle y, x \rangle} \, dx, \quad y \in \mathbb{R}^n,$$

for any $f \in C_0(\mathbb{R}^n)$. Moreover, by continuity the Fourier transform can be extended to the Hilbert space $L^2(\mathbb{R}^n)$.

Let $\sigma$ be the unit sphere on $\mathbb{R}^n$, with $dx$ denoting the $(n-1)$-dimensional area element of the sphere and $|\sigma|$ the hypersurface area of the whole sphere. For $f \in C_0(\mathbb{R}^n)$ the translation operator $S^t$ is defined as

$$(S^t f)(x) = \frac{1}{|\sigma|} \int_{\sigma} f(x + ty) \, dy, \quad t \geq 0. \quad (12)$$

This operator is also called the spherical mean operator. By continuity the operator $S^t$ can be extended to the Hilbert space $L^2(\mathbb{R}^n)$.

Let $\omega_k$ be a $k$th-order modulus of continuity. It is also assumed that $\omega_k(t)$ is bounded below on $[\delta_0, \infty)$ by a positive number, and that $\omega_k(t)^2/t^5 \leq L^1([\delta_0, \infty))$. 

Theorem 6  Let $\omega_k$ be a $k$th-order modulus of continuity. It is also assumed that $\omega_k(t)$ is bounded below on $[\delta_0, \infty)$ by a positive number, and that $\omega_k(t)^2/t^5 \leq L^1([\delta_0, \infty))$. 

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If \( f \in \text{Lips}_{\mathbb{R}^n}^k (\omega_k) \), then there exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),
\[
\int_{1/t}^{+\infty} \int_{\sigma} |\hat{f}(\lambda x)|^2 \, d\lambda \, dx \leq C \omega_k(t)^2 t^{n-1}.
\]

(13)

**Theorem 7** Let \( \omega_k \) be a modulus of continuity of order \( k \leq 2 \) satisfying the conditions (5) and (6). It is also assumed that \( \omega_k(t) \) is bounded below on \([\delta_0, \infty)\) by a positive number, and that \( \omega_k(t)^2/t^5 \in L^1([\delta_0, \infty)) \). If \( f \in L^2(\mathbb{R}^n) \) and there exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \) the condition (13) holds, then \( f \in \text{Lips}_{\mathbb{R}^n}^k (\omega_k) \).

**Theorem 8** If \( \omega_k \) is a \( k \)th-order modulus of continuity, for \( k \leq 2 \), satisfying the Zygmund conditions (5) and (6), and \( \omega_k(t) \) is bounded below on \([\delta_0, \infty)\) by a positive number, and that \( \omega_k(t)^2/t^5 \in L^1([\delta_0, \infty)) \). If \( f \) is a function in \( L^2(\mathbb{R}^n) \), then the following statements are equivalent.

1. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),
\[
\|S^t f - f\|_{L^2(\mathbb{R}^n)} \leq C \omega_2(t).
\]

2. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),
\[
\int_{1/t}^{+\infty} \int_{\sigma} |\hat{f}(tx)|^2 \, dt \, dx \leq C \omega_k(t)^2 t^{n-1}.
\]

The above is an analogue of the classical theorem of Titchmarsh [13, Theorem 85], extended to \( \mathbb{R}^n \)-dimensional space and with bounds given by generalised \( k \)th-order moduli of continuity rather than just power functions. The following corollary, given like Corollary 1 above by the special case \( \omega(t) = t^\alpha \) with \( 0 < \alpha < 2 \), looks even more similar to the theorem of Titchmarsh, but still taking place in \( \mathbb{R}^n \)-dimensional space.

**Corollary 2** For \( \alpha \in (0, 2) \) and \( f \in L^2(\mathbb{R}^n) \), the following assertions are equivalent.

1. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),
\[
\|S^t f - f\|_{L^2(\mathbb{R}^n)} \leq C t^\alpha.
\]

2. There exists a constant \( C > 0 \) such that for all sufficiently small \( t > 0 \),
\[
\int_{1/t}^{+\infty} \int_{\sigma} |\hat{f}(tx)|^2 \, dt \, dx \leq C t^{2\alpha+n-1}.
\]

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