Unified Dark Matter in Scalar Field Cosmologies

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Considering the general Lagrangian of k-essence models, we study and classify them through variables connected to the fluid equation of state parameter \( w_{\kappa} \). This allows to find solutions around which the scalar field describes a mixture of dark matter and cosmological constant-like dark energy, an example being the purely kinetic model proposed by Scherrer. Making the stronger assumption that the scalar field Lagrangian is exactly constant along solutions of the equation of motion, we find a general class of k-essence models whose classical trajectories directly describe a unified dark matter/dark energy (cosmological constant) fluid. While the simplest case of a scalar field with canonical kinetic term unavoidably leads to an effective sound speed \( c_s = 1 \), thereby inhibiting the growth of matter inhomogeneities, more general non-canonical k-essence models allow for the possibility that \( c_s \ll 1 \) whenever matter dominates.

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I. INTRODUCTION

In the current standard cosmological model, two unknown components govern the dynamics of the Universe: the dark matter (DM), responsible for structure formation, and a non-zero cosmological constant \( \Lambda \) (see, e.g. ref. [1]), or a dynamical dark energy (DE) component, that drives cosmic acceleration [2, 3, 4].

If the DE is given by a \( \Lambda \) term, besides having a non-trivial fine-tuning problem to solve (unless one resorts to an anthropic argument), one does not know why \( \Omega_{DM} \) and \( \Omega_{\Lambda} \) are both of order unity today. In these years alternative routes have been followed, for example Quintessence [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and k-essence [17, 18, 19] (a complete list of dark energy models can be found in the recent review [20]). The k-essence is characterized by a Lagrangian with non-canonical kinetic term and it is inspired by earlier studies of k-inflation [21, 22].

Some models of k-essence have solutions which tend toward dynamical attractors in the cosmic evolution so that their late-time behavior becomes insensitive to initial conditions (see, e.g., [17, 24, 24, 23]). Other models, besides having this property allow to avoid fine-tunning and are able to explain the cosmic coincidence problem [15, 19]. Subsequently, it was realized that the latter models [18, 19] have too small a basin of attraction in the radiation era [20] and lead to superluminal propagation of field fluctuations [27].

An important issue is whether the dark matter clustering is influenced by the dark energy and if, when this happens, the dark energy can indirectly smooth the cusp profiles of dark matter at small radii. Another hypothesis is to consider a single fluid that behaves both as dark energy and dark matter. The latter class of models has been dubbed Unified Dark Matter (UDM). Among several models of k-essence considered in the literature there exist two types of UDM models: generalized Chaplygin gas [28, 29, 30, 31, 32, 33, 34] model and the purely kinetic model considered by Scherrer [35]. Alternative approaches to the unification of DM and DE have been proposed in Ref. [36], in the frame of supersymmetry, and in Ref. [37], in connection with the solution of the strong CP problem.

The generalized Chaplygin model can be obtained via a Born-Infeld-type Lagrangian. This “fluid” has the property of behaving like dark matter at high density and like a cosmological constant at low density.

The kinetic model introduced by Scherrer can evolve into a fluid which describes at the same time the dark matter and cosmological constant components. In this case, perturbations do not show instabilities but, at early times, the fluid evolves like radiation, leading to a possible conflict with the constraints coming from primordial nucleosynthesis. Moreover, the parameters of the model have to be fine-tuned in order for the model not to exhibit finite pressure effects in the non-linear stages of structure formation [38].

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In this paper we consider the general Lagrangian of \textit{k-essence} models and classify them through two variables connected to the fluid equation of state parameter $w_\kappa$. This allows to find attractor solutions around which the scalar field is able to describe a mixture of dark matter and cosmological constant-like dark energy, an example being Scherrer’s purely kinetic model. Next, we impose that the Lagrangian of the scalar field is constant, i.e. that $p_\kappa = -\Lambda$, where $\Lambda$ is the cosmological constant, along suitable solutions of the equation of motion, and find a general class of k-essence models whose attractors directly describe a unified dark matter/dark energy fluid. While the simplest of such models, based on a neutral scalar field with canonical kinetic term, unavoidably leads to an effective speed of sound $c_s$ which equals the speed of light, thereby inhibiting the growth of matter perturbations, we find a more general class of non-canonical (k-essence) models which allow for the possibility that $c_s \ll 1$ whenever matter dominates.

The plan of the paper is as follows. In Section 2 we introduce the general class of k-essence models and we propose a new approach to look for attractor solutions. In Section 3 we apply our formalism to obtain the attractors for the purely kinetic case. In Section 4 we generalize our model giving general prescriptions [Eqs. (4.12) and (4.14)] which allow to obtain unified models where the dark matter and a cosmological constant-like dark energy are described by a single scalar field along its attractor solutions. Section 5 contains our main conclusions. The scaling solutions for a particular case of k-essence are discussed in Appendix A.

### II. K-ESSENCE

Let us consider the following action

$$S = S_G + S_\phi + S_m = \int d^4x \sqrt{-g} \left[ \frac{R}{2} + \mathcal{L}(\phi, X) \right] + S_m$$

(2.1)

where

$$X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi.$$  

(2.2)

We use units such that $8\pi G = c^2 = 1$ and signature $(-, +, +, +)$. The energy-momentum tensor of the scalar field $\phi$ is

$$T^\phi_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu \nu}} = \frac{\partial \mathcal{L}(\phi, X)}{\partial X} \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L}(\phi, X) g_{\mu \nu}.$$  

(2.3)

If $X$ is time-like $S_\phi$ describes a perfect fluid with $T^\phi_{\mu \nu} = (\rho_\kappa + p_\kappa) u_\mu u_\nu + p_\kappa g_{\mu \nu}$, where

$$\mathcal{L} = p_\kappa(\phi, X),$$  

(2.4)

is the pressure,

$$\rho_\kappa = \rho_\kappa(\phi, X) \equiv 2X \frac{\partial p_\kappa(\phi, X)}{\partial X} - p_\kappa(\phi, X)$$  

(2.5)

is the energy density and the four-velocity reads

$$u_\mu = \frac{\nabla_\mu \phi}{\sqrt{2X}}.$$  

(2.6)

Now let us assume that our scalar field defines a homogeneous background $X = \frac{1}{2} \dot{\phi}^2$ (where the dot denotes differentiation w.r.t. the cosmic time $t$) and consider a flat Friedman-Robertson-Walker background metric. In such a case, the equation of motion for the homogeneous mode $\phi(t)$ becomes

$$\left( \frac{\partial p_\kappa}{\partial X} + 2X \frac{\partial^2 p_\kappa}{\partial X^2} \right) \ddot{\phi} + \frac{\partial p_\kappa}{\partial X} (3H \dot{\phi}) + \frac{\partial^2 p_\kappa}{\partial \phi \partial X} \dot{\phi}^2 - \frac{\partial p_\kappa}{\partial \phi} = 0.$$  

(2.7)

The k-essence equation of state $w_\kappa \equiv p_\kappa/\rho_\kappa$ is just

$$w_\kappa = \frac{p_\kappa}{2X \frac{\partial \rho_\kappa}{\partial X} - p_\kappa},$$  

(2.8)
while the effective speed of sound, which is the quantity relevant for the growth of perturbations, reads
\[ c_s^2 = \frac{\left(\frac{\partial p_\kappa}{\partial \rho_\kappa}\right)^2}{\left(\frac{\partial \rho_\kappa}{\partial \rho_\kappa}\right)^2} = \frac{\frac{\partial p_\kappa}{\partial X}}{\frac{\partial \rho_\kappa}{\partial \rho_\kappa} + 2X \frac{\partial^2 \rho_\kappa}{\partial X^2}} . \] (2.9)

If we assume that the scalar field Lagrangian depends separately on \( X \) and \( \varphi \), i.e. that it can be written in the form
\[ p_\kappa(\varphi, X) = f(\varphi)g(X) , \] (2.10)
then Eq. (2.5) becomes
\[ \rho_\kappa = f(\varphi)\left[ 2X \frac{dg(X)}{dX} - g(X) \right] = f(\varphi)\beta(X) . \] (2.11)

Notice that the requirement of having a positive energy density imposes a constraint on the function \( g \), namely,
\[ 2X \frac{dg}{dX} > g , \] (2.12)
having assumed \( f > 0 \).

Defining now the variables \( \lambda = \left(\frac{1}{f}\right)\frac{df}{dN} \) and \( \alpha = -d\ln \beta/dN \), where \( N = \ln a \), we can express the energy density as
\[ \rho_\kappa = \bar{K}e^{-3\int^N dN'(w_\kappa(N') + 1)} , \] (2.13)
with \( \bar{K} \) an integration constant. We can also rewrite the energy continuity equation in the form
\[ \frac{d\beta}{dN} + \lambda \beta + 6X \frac{df}{dX} = 0 . \] (2.14)

In terms of \( \alpha \) and \( w_\kappa \), or, equivalently, of \( \alpha \) and \( \lambda \), the effective speed of sound, Eq. (2.9), reads
\[ c_s^2 = -\frac{(w_\kappa + 1) \ln X}{2\alpha} = -\frac{\alpha - \lambda \ln X}{6\alpha} . \] (2.15)

Therefore, in purely kinetic models (\( \lambda = 0 \)) \( X \) can only decrease in time down to its minimum value. The case \( \alpha = 0 \) is analyzed in Appendix A.

### III. STUDY OF THE ATTRACTORS FOR PURELY KINETIC SCALAR FIELD LAGRANGIANS

In the \( \lambda = 0 \) case the Lagrangian \( \mathcal{L} \) (i.e. the pressure \( p_\kappa \)) depends only on \( X \), that is we are recovering the equations that describe the purely kinetic model studied in Ref. [35] and the Generalized Chaplygin gas [29, 30]. In this section we want to make a general study of the attractor solutions in this case.

For \( \lambda = 0 \), Eq. (2.14) gives the following nodes,
\[ 1) \quad \left. \frac{dg}{dX} \right|_{\hat{X}} = 0 , \quad 2) \quad X = \hat{X} = 0 , \] (3.1)
with \( \hat{X} \) a constant. Both cases correspond to \( w_\kappa = -1 \), as one can read from Eq. (2.8).

The general solution of the differential equation (2.14) in the \( \lambda \to 0 \) limit is
\[ X \left( \frac{dg}{dX} \right)^2 = k e^{-6N} \] (3.2)
with \( k \) a positive constant. This solution was previously derived although in a different form in Ref. [40]. As \( N \to \infty \), \( X \) or \( \frac{dg}{dX} \) (or both) must tend to zero, which shows that, depending on the specific form of the function \( g(X) \), each particular solution will converge toward one of the nodes above.

In what follows we will provide some examples of stable node solutions of the equation of motion, some of which have been already studied in the literature. The models below are classified on the basis of the stable node to which they asymptotically converge.
A. Case 1): Scherrer solution

For the solution of case 1) we want to study the function $g$ around some $X = \hat{X} \neq 0$. In this case one can approximate $g$ as a parabola with $\frac{dg}{dX} |_{\hat{X}} = 0$

$$g = g_0 + g_2(X - \hat{X})^2.$$  \hfill (3.3)

with $g_0$ and $g_2$ suitable constants. This solution, with $g_0 < 0$ and $g_2 > 0$, coincides with the model studied by Scherrer in Ref. 35.

If we now impose that today $X$ is close to $\hat{X}$ so that $\epsilon \equiv (X - \hat{X})/\hat{X} = (a/a_1)^{-3} \ll 1$, we obtain

$$\rho_\kappa = -g_0 + 4 g_2 \hat{X}^2 \left( \frac{a}{a_1} \right)^{-3}.$$ \hfill (3.4)

In order for the density to be positive at late times, we need to impose $g_0 < 0$. In this case the speed of sound $c_s^2$ turns out to be

$$c_s^2 = \frac{(X - \hat{X})}{3(X - \hat{X})} = \frac{1}{2} \left( \frac{a}{a_1} \right)^{-3},$$ \hfill (3.5)

We notice also that, for $(a/a_1)^{-3} \ll 1$ we have $c_s^2 \ll 1$.

Eq. (3.4) tells us that the background energy density can be written as $\rho_\kappa = \rho_A + \rho_{DM}$, where $\rho_A$ behaves like a "dark energy" component ($\rho_A = const.$) and $\rho_{DM}$ behaves like a "dark matter" component ($\rho_{DM} \propto a^{-3}$). Note that, from Eq. (3.3), $\hat{X}$ must be different from zero in order for the matter term to be there. For this particular case the Hubble parameter $H$ is a function only of the UDM fluid $H^2 = \rho_\kappa/3$.

It is interesting to notice that an alternative model, proposed in Ref. 39 in the frame of extended Born-Infeld dynamics, actually converges to the Scherrer solution in the regime $(X - \hat{X})/\hat{X} \ll 1$.

It is immediate to verify that in the early universe case $(X \gg \hat{X}$ i.e. $\rho_\kappa \gg (g_0))$ the k-essence behaves like radiation. Scherrer 35 imposed the constraint $\epsilon_0 = \epsilon(a_0) = -g_0/(8g_2\hat{X}^2) \ll 10^{-10}$, requiring that the k-essence behaves as a matter component at least from the epoch of matter-radiation equality. The stronger bound $\epsilon_0 \leq 10^{-18}$ is obtained by Giannakis and Hu 38, who considered the small-scale constraint that enough low-mass dark matter halos are produced to reionize the universe. One should also consider the usual constraint imposed by primordial nucleosynthesis on extra radiation degrees of freedom, which however leads to a weaker constraint.

B. Case 1): Generalized Scherrer solution

Starting from the condition that we are near the attractor $X = \hat{X} \neq 0$, we can generalize the definition of $g$, extending the Scherrer model in the following way

$$\rho_\kappa = g = g_0 + g_n(X - \hat{X})^n,$$ \hfill (3.6)

with $n \geq 2$ and $g_0$ and $g_n$ suitable constants. The density reads

$$\rho_\kappa = (2n - 1)g_n(X - \hat{X})^n + 2\hat{X}ng_n(X - \hat{X})^{n-1} - g_0.$$ \hfill (3.7)

If $\epsilon^n = [(X - \hat{X})/\hat{X}]^n \ll 1$, Eq. 36 reduces to

$$X = \hat{X} \left[ 1 + \left( \frac{a}{a_{n-1}} \right)^{-3/(n-1)} \right]$$ \hfill (3.8)

(where $a_{n-1} \ll a$) and so $\rho_\kappa$ becomes

$$\rho_\kappa \simeq 2n\hat{X}^n g_n \left( \frac{a}{a_{n-1}} \right)^{-3} - g_0.$$ \hfill (3.9)
with \((1/a_{n-1})^{-3} = (1/(ng_n))(k/X^{2n-1})^{1/2}\) for \(\epsilon \ll 1\).

We have therefore obtained the important result that this attractor leads exactly to the same terms found in the purely kinetic model of Ref. [35], i.e. a cosmological constant and a matter term. We can therefore extend the constraint of Ref. [35] to this case obtaining \((\epsilon_0)^{n-1} = -g_0/(4nX^n g_n) \leq 10^{-10}\). A stronger constraint would clearly also apply to our model by considering the small-scale constraint imposed by the universe reionization, as in Ref. [38].

If we write the general expressions for \(w_\kappa\) and \(c_s^2\) we have

\[
w_\kappa = -\left[1 + \left(\frac{g_n}{g_0}\right) (X - \hat{X})^n\right]\left[1 - 2n\hat{X} \left(\frac{g_n}{g_0}\right) (X - \hat{X})^{n-1} - (2n - 1) \left(\frac{g_n}{g_0}\right) (X - \hat{X})^n\right]^{-1}
\]

\[
c_s^2 = \frac{(X - \hat{X})}{2(n - 1)\hat{X} + (2n - 1)(X - \hat{X})}.
\]

For \(\epsilon \ll 1\) we obtain a result similar to that of Ref. [35], namely

\[
w_\kappa \simeq -1 + 2n \left(\frac{g_n}{g_0}\right) \left(\frac{a}{a_{n-1}}\right)^{-3},
\]

\[
c_s^2 \simeq \frac{1}{2(n - 1)}\epsilon.
\]

On the contrary, when \(X \gg \hat{X}\) we obtain

\[
w_\kappa \simeq c_s^2 \simeq \frac{1}{2n - 1}
\]

In this case we can impose a bound on \(n\) so that at early times and/or at high density the k-essence evolves like dark matter. In other words, when \(n \gg 1\), unlike the purely kinetic case of Ref. [35], the model is well behaved also at high densities.

### C. Case 2): Generalized Chaplygin gas

An example of case 2) is provided by the Generalized Chaplygin (GC) model (see e.g. Refs. [28, 29, 30, 31, 32, 33, 34]), whose equation of state has the form

\[
p_{GC} = -\rho_\ast \left(\frac{\rho_{GC}}{\rho_\ast}\right)^{\frac{1}{\gamma}},
\]

where now \(p_{GC} = p_\kappa\) and \(\rho_{GC} = \rho_\kappa\) and \(\rho_\ast\) and \(p_\ast\) are suitable constants.

Through the equation \(\rho_\kappa = 2X \frac{dg(X)}{dX} - g(X)\) and the continuity equation \(\frac{\rho_{GC}}{dX} + (\rho_{GC} + p_{GC}) = 0\) we can write \(p_{GC}\) and \(\rho_{GC}\) as functions of either \(X\) or \(a\). When the pressure and the energy density are considered as functions of \(X\) one gets [30]

\[
p_{GC} = -\left(\frac{-p_\ast}{\rho_\ast}\right)^{1/(1 - \gamma)} \left[1 - \mu X^{1 + \gamma} \right]^{\frac{1}{1 - \gamma}}
\]

\[
\rho_{GC} = \left(\frac{-p_\ast}{\rho_\ast}\right)^{1/(1 - \gamma)} \left[1 - \mu X^{1 + \gamma} \right]^{\frac{\gamma}{1 - \gamma}}
\]

with \(\mu = \text{const}\).

It is necessary for our scopes to consider the case \(\gamma < 0\), so that \(c_s^2 > 0\). Note that \(\gamma = -1\) corresponds to the standard “Chaplygin gas” model.

Another model that falls into this class of solution is the one proposed in Ref. [24], in which \(g = b\sqrt{2X - \Lambda}\) (with \(b\) a suitable constant) which satisfies the constraint that \(p = -\Lambda\) along the attractor solution \(X = \hat{X} = 0\). This model, however is well-known to imply a diverging speed of sound.
IV. UNIFIED DARK MATTER FROM A SCALAR FIELD WITH NON-CANONICAL KINETIC TERM

Starting from the barotropic equation of state \( p = p(\rho) \) we can describe the system either through a purely kinetic k-essence Lagrangian (if the inverse function \( \rho = \rho(p) \) exists) or through a Lagrangian with canonical kinetic term, as in quintessence-like models. The same problem has been solved in Ref. [20], although with a different procedure and for a different class of models. In the first case we have to solve the equation
\[
\rho(p(X)) = 2X \frac{dp(X)}{dX} - p(X)
\]
when \( X \) is time-like. In the second case we have to solve the two differential equations
\[
\begin{align*}
\chi - V(\phi) &= p(\phi, \chi) \\
\chi + V(\phi) &= \rho(\phi, \chi)
\end{align*}
\]
where \( \chi = \dot{\phi}^2/2 \) is time-like. In particular if we assume that our model describes a unified dark matter/dark energy fluid we can proceed as follows: starting from \( \dot{\rho} = -3H(p + \rho) = -\sqrt{3\rho}(p + \rho) \) and \( 2\chi = (p + \rho) = (d\phi/d\rho)^2 \dot{\rho}^2 \) we get
\[
\phi = \pm \frac{1}{\sqrt{3}} \int_{\rho_0}^\rho \frac{dp'/\sqrt{p'}}{(p'(\rho') + \rho')^{1/2}}.
\]
up to an additive constant which can be dropped without any loss of generality. Inverting the Eq. (4.4), i.e. writing \( \rho = \rho(\phi) \) we are able to get \( V(\phi) = [\rho(\phi) - \rho(\rho(\phi))]/2 \). If one requires the exact condition that our unified DM fluid has a constant pressure term \( p = -\Lambda \) and looks for a scalar field model with canonical kinetic term, one arrives at an exact solution with potential \( V(\phi) = (\Lambda/2)[\cosh^2(\sqrt{3}/2 \phi) + 1] \) (see also Ref. [43, 44]). Using standard criteria (e.g. Ref. [20]) it is immediate to verify that the above trajectory corresponds to a stable node even in the presence of an extra-fluid (e.g. radiation) with equation of state \( w_{\text{fluid}} \equiv p_{\text{fluid}}/\rho_{\text{fluid}} > 0 \), where \( p_{\text{fluid}} \) and \( \rho_{\text{fluid}} \) are the fluid pressure and energy density, respectively. Along the above attractor trajectory our scalar field behaves precisely like a mixture of pressure-less matter and cosmological constant. Such a “quartessence” model therefore behaves exactly like a mixture of dark matter and dark energy along the attractor solution, whose matter sector, however is unable to cluster on sub-horizon scales (at least as long as linear perturbations are considered).  

A closely related solution was found by Salopek & Stewart [45], using the Hamiltonian formalism. Like any scalar field with canonical kinetic term [46], such a UDM model however predicts \( c_s^2 = 1 \), as it is clear from Eq. (2.9), which inhibits the growth of matter inhomogeneities. Such a “quartessence” model therefore behaves exactly like a mixture of dark matter and dark energy along the attractor solution, whose matter sector, however is unable to cluster on sub-horizon scales (at least as long as linear perturbations are considered).

We can then summarize our findings so far by stating that purely kinetic k-essence cannot produce a model which exactly describes a unified fluid of dark matter and cosmological constant, while scalar field models with canonical kinetic term, while containing such an exact description, unavoidably lead to \( c_s^2 = 1 \), in conflict with cosmological structure formation. In order to find an exact UDM model with acceptable speed of sound we consider more general scalar field Lagrangians.

A. Lagrangians of the type \( \mathcal{L}(\varphi, X) = g(X) - V(\varphi) \)

Let us consider Lagrangians with non-canonical kinetic term and a potential term, in the form
\[
\mathcal{L}(\varphi, X) = g(X) - V(\varphi).
\]
The energy density then reads
\[
\rho = 2X \frac{dg(X)}{dX} - g(X) + V(\varphi),
\]
while the speed of sound keeps the form of Eq. (2.9). The equation of motion for the homogeneous mode reads
\[
\left( \frac{dg}{dX} + 2X \frac{d^2g}{dX^2} \right) \frac{dX}{dN} + 3 \left( 2X \frac{dg}{dX} \right) = -\frac{dV}{dN}.
\]
One immediately finds
\[ p + \rho = 2X \frac{dg(X)}{dX} \equiv 2\mathcal{F}(X) \, . \] (4.9)

We can rewrite the equation of motion, Eq. (4.8), in the form
\[ \left[ 2X \frac{d\mathcal{F}}{dX} - \mathcal{F} \right] \frac{dX}{dN} + X \left( 6\mathcal{F} + \frac{dV}{dN} \right) = 0 . \] (4.10)

It is easy to see that this equation admits 2 nodes, namely: 1) \( \frac{dg}{dX}|_{\tilde{X}} = 0 \) and 2) \( \tilde{X} = 0 \). In all cases, for \( N \to \infty \), the potential \( V \) should tend to a constant, while the kinetic term can be written around the attractor in the form
\[ g(X) = M^4 \left( \frac{X - \tilde{X}}{M^4} \right)^n \quad n \geq 2 , \] (4.11)

with \( M \) a suitable mass-scale and the constant \( \tilde{X} \) can be either zero or non-zero. The trivial case \( g(X) = X \) obviously reduces to the one of Section 4.

Following the same procedure adopted in the previous section we impose the constraint \( p = -\Lambda \), which yields the general solution \( \rho_m = 2\mathcal{F}(X) \).

This allows to define \( \varphi = \varphi(\rho_m) \) as a solution of the differential equation
\[ \rho_m = 2\mathcal{F} \left[ \frac{3}{2} (\rho_m + \Lambda) \rho_m^2 \left( \frac{d\varphi}{d\rho_m} \right)^2 \right] . \] (4.12)

As found in the case of k-essence, the most interesting behavior corresponds to the limit of large \( n \) and \( \tilde{X} = 0 \) in Eq. (4.11), for which we obtain
\[ \rho_m \approx \Lambda \sinh^{-2} \left[ \left( \frac{3\Lambda}{8M^4} \right)^{1/2} \varphi \right] , \] (4.13)

leading to \( V(\varphi) \approx \rho_m/2n - \Lambda \), and \( c_s^2 = 1/(2n - 1) \approx 0 \). The Lagrangian of this model is similar to that analyzed in Ref. [41].

B. Lagrangians of the type \( \mathcal{L}(\varphi, X) = f(\varphi)g(X) \)

Let us now consider Lagrangians with a non-canonical kinetic term of the form of Eq. (2.10), namely \( \mathcal{L}(\varphi, X) = f(\varphi)g(X) \).

Imposing the constraint \( p = -\Lambda \), we obtain \( f(\varphi) = -\Lambda/g(X) \), which inserted in the equation of motion yields the general solution
\[ X \frac{d\ln g}{dX} = -\frac{\rho_m}{2\Lambda} . \] (4.14)

The latter equation, together with Eq. (4.12), define our general prescription to get UDM models describing both DM and cosmological constant-like DE.

As an example of the general law in Eq. (4.14) let us consider an explicit solution. Assuming that the kinetic term is of Born-Infeld type, as in Refs. [43, 44, 47, 48, 49],
\[ g(X) = -\sqrt{1 - 2X/M^4} , \] (4.15)

with \( M \) a suitable mass-scale, which implies \( \rho = f(\varphi)/\sqrt{1 - 2X/M^4} \), we get
\[ X(a) = \frac{M^4}{2} \frac{\bar{k}a^{-3}}{1 + \bar{k}a^{-3}} , \] (4.16)
where $k = \rho_m(a_*) a_*^3/\Lambda$ and $a_*$ is the scale-factor at a generic time $t_*$. In order to obtain an expression for $\varphi(a)$, we impose that the Universe is dominated by our UDM fluid, i.e. $H^2 = \rho/3$. This gives

$$\varphi(a) = \frac{2M^2}{\sqrt{3\Lambda}} \left\{ \arctan \left[ (ka)^{-3/2} \right] - \frac{\pi}{2} \right\}, \quad (4.17)$$

which, replaced in our initial ansatz $p = -\Lambda$ allows to obtain the expression (see also Ref. 43, 44)

$$f(\varphi) = \frac{\Lambda}{\cos \left( \frac{3\Lambda}{2M^2} \varphi \right)}. \quad (4.18)$$

If we expand $f(\varphi)$ around $\varphi = 0$, and $X/M^4 \ll 1$ we get the approximate Lagrangian

$$\mathcal{L} \approx \frac{\Lambda}{2M^4} \dot{\varphi}^2 - \Lambda \left[ 1 + \frac{3\Lambda}{8M^4} \varphi^2 \right]. \quad (4.19)$$

Note that our Lagrangian depends only on the combination $\varphi/M^2$, so that one is free to reabsorb a change of the mass-scale in the definition of the filed variable. Without any loss of generality we can then set $M = \Lambda^{1/4}$, so that the kinetic term takes the canonical form in the limit $X \ll 1$. We can then rewrite our Lagrangian as

$$\mathcal{L} = -\Lambda \sqrt{1 - 2X/\Lambda} \left| \frac{\dot{\varphi}}{\cos \left( \frac{\sqrt{3} \varphi}{2} \right)} \right|. \quad (4.20)$$

This model implies that for values of $\sqrt{3} \varphi \approx -\pi$ and $2X/\Lambda \approx 1$,

$$\cos \left( \frac{\sqrt{3}}{2} \varphi \right) \propto a^{3/2}, \quad \sqrt{1 - 2X/\Lambda} \propto a^{-3/2}, \quad (4.21)$$

the scalar field mimics a dark matter fluid. In this regime the effective speed of sound is $c_s^2 = 1 - 2X/\Lambda \approx 0$, as desired.

To understand whether our scalar field model gives rise to a cosmologically viable UDM solution, we need to check if in a Universe filled with a scalar field with Lagrangian $\mathcal{L} = \frac{\Lambda}{2M^4} \dot{\varphi}^2 - \Lambda \left[ 1 + \frac{3\Lambda}{8M^4} \varphi^2 \right]$, plus a background fluid of e.g. radiation, the system displays the desired solution where the scalar field mimics both the DM and DE components. Notice that the model does not contain any free parameter to specify the present content of the Universe. This implies that the relative amounts of DM and DE that characterize the present universe are fully determined by the value of $\varphi_0 \equiv \varphi(t_0)$. In other words, to reproduce the present universe, one has to tune the value of $f(\varphi)$ in the early Universe. However, a numerical analysis shows that, once the initial value of $\varphi$ is fixed, there is still a large basin of attraction in terms of the initial value of $d\varphi/dt$, which can take any value such that $2X/\Lambda \ll 1$.

The results of a numerical integration of our system including scalar field and radiation are shown in Figures 1 - 3. Figure 1 shows the density parameter, $\Omega_{UDM}$ as a function of redshift, having chosen the initial value of $\varphi$ so that today the scalar field reproduces the observed values $\Omega_{DM} \approx 0.268$, $\Omega_{DE} \approx 0.732$ [33]. Notice that the time evolution of the scalar field energy density is practically indistinguishable from that of a standard DM plus Lambda (LCDM) model with the same relative abundances today. Figure 2 shows the evolution equation of state parameter $w_{UDM}$; once again the behavior of our model is almost identical to that of a standard LCDM model for $1 + z < 10^8$. Notice that, since $c_s^2 = -w_{UDM}$, the effective speed of sound of our model is close to zero, as long as matter dominates, as required. In Figure 3 we finally show the redshift evolution of the scalar field variables $X = \dot{\varphi}^2/2$ and $\varphi$: one can easily check that the evolution of both quantities is accurately described by the analytical solutions above, Eqs. (4.16) and (4.17), respectively (the latter being obviously valid only after the epoch of matter-radiation equality).

V. CONCLUSIONS

In this paper we have investigated the possibility that the dynamics of a single scalar field can account for a unified description of the dark matter and dark energy sectors. In particular, we have studied the case of purely kinetic k-essence, showing that these models have only one late-time attractor with equation of state $w_k = -1$ (cosmological constant). Studying all possible solutions near the attractor we have found a generalization of the Scherrer model [35], which describes a unified dark matter fluid.
FIG. 1: Evolution of the scalar field density parameter vs. redshift. The continuous line shows the UDM density parameter; the dashed line is the density parameter of the DM + DE components in a standard ΛCDM model; the dotted line is the radiation density parameter.

FIG. 2: The redshift evolution of the scalar field equation of state parameter $w_{\text{UDM}}$ (continuous line) is compared with that of the sum of the DM + DE components in a standard ΛCDM model (dashed line).

Generalizing our analysis to the case where the Lagrangian is not purely kinetic, we have given general prescriptions [Eqs. (4.12) and (4.14)] to obtain unified models where the dark matter and a cosmological constant-like dark energy are described by a single scalar field along its attractor solutions. Moreover, we have given explicit examples for which the effective speed of sound is small enough whenever matter dominates, thus allowing for the onset of gravitational instability. Studying the detailed consequences of such unified dark matter model for cosmic microwave background anisotropies and for the formation of the large-scale structure of the Universe will be the subject of a subsequent analysis.

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FIG. 3: Redshift evolution of the scalar field of the scalar field variables $X = \dot{\varphi}^2/2$ (top) and $\varphi$ (bottom).

**APPENDIX A: STUDY OF THE SOLUTIONS FOR $\alpha = 0$**

If $X$ is constant $(dX/dN = 0)$ then, from Eq. (2.14), $\alpha = 0$. In this situation we have that

$$\frac{1}{f} \frac{df}{dN} = \frac{1}{N} \frac{df}{dt} = \lambda = -3(w_\kappa + 1)$$

(A.1)

is constant because $w_\kappa$ is function only of $X$.

Now if we consider the case in which the universe is dominated by a fluid with constant equation of state $w_B$ then $H = \dot{N} \sim 2/[3(w_B + 1)t]$ and Eq. (A.1) becomes $d\ln f/d\ln t \sim -2(w_\kappa + 1)/(w_B + 1)$.

Therefore we get $f \sim t^{-\frac{w_B+1}{3}} \sim \varphi^{-\frac{w_B+1}{3}}$ (because if $\dot{\varphi}$ is constant then $\varphi \sim \sqrt{2Xt}$). In other words, we have recovered, although in a more general way, the result of Ref. [17]. These models have been dubbed “scaling k-essence” (see also Ref. [51]). In such a case, the equation of state $w_\kappa$ can be written as $w_\kappa = \beta(w_B + 1)/2 - 1$, where $f = \varphi^{-\beta}$.

If $w_B = w_\kappa$ we have only the k-essence as background and we get $\beta = 2$.

Using the latter approach it is simpler to see that when $\alpha \rightarrow 0$ all the viable solutions 1 converge to the scaling solution. In fact, if a priori $\alpha \neq 0$ and $f = \varphi^{-\beta}$ we have that

$$\alpha = 3(w_\kappa + 1) - \frac{3}{2}\beta(w_B + 1)\frac{\sqrt{2Xt}}{\varphi}$$

(A.2)

Starting from Eq. (2.14), we note that when $\alpha \rightarrow 0$ then $2X \frac{dg}{dx} - g \rightarrow \text{const.} \neq 0$. It is then easy to see that $X$ must be a constant 2. Therefore, provided that $g \neq b\sqrt{2X} + \text{const.}$, the solution of the equation of motion converges to the scaling solution.

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1. If $\alpha \rightarrow c > 0$ for $N \rightarrow +\infty$ we have $2X \frac{dg}{dx} - g \rightarrow 0$ and we get three possible solutions: i) if $g \rightarrow \text{const.}$ we have $w_\kappa \rightarrow -\infty$, which is not acceptable; ii) if $g \rightarrow 0$ and $w_\kappa \rightarrow \text{const.} \neq 0$ we get $w_\kappa \rightarrow c_s^2 > 0$ [23]; this solution cannot be used to describe an accelerated universe; iii) if $g \rightarrow 0$ and $w_\kappa \rightarrow 0$ we get $0 \leq c_s^2 \leq 1$; obviously, also in this case the universe cannot accelerate.

2. When $2X \frac{dg}{dx} - g = \text{const.}$, we have two possibilities: either $X$ is constant or we need resolve directly this differential equation. In this last case we obtain $g = b\sqrt{2X} + \text{const.}$, which implies constant $X$ but diverging speed of sound $c_s^2$ [24].
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