The Begley-Torvik model difference scheme matrix eigenvalues

L Kirianova
Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia
ludmilakirianova@yadex.ru

Abstract. Two difference schemes for Begley-Torvik models are constructed in the article, depending on the order of the fractional derivative. The eigenvalues are calculated for important special cases encountered in practical problems. Comparison is made with other values obtained by other methods.

1. Introduction
Consider the boundary-value problem for a second-order differential equation with a fractional derivative of the form:

\[ u''(x) + \varepsilon D^\alpha u(x) + \lambda u(x) = 0; \; x \in [0; X]; \]
\[ u(0) = u(X) = 0; \]  
\[ \alpha \in (0; 1) \]

where \( D^\alpha u(x) = \text{Caputo fractional differentiation operator of order } \alpha > 0, \)

\[ D^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \tau)^{1-\alpha} u^{(k)}(\tau) \, d\tau; \]

where \( \Gamma - \text{gamma function, besides } k - 1 < \alpha \leq k, \; k = 1, 2, ... \)

Equation (1) is called the Begley-Torvik equation [1,2] and is used, in particular, to model the polymers damping properties, polymer concrete, glasses, and other viscoelastic materials [3,4].

The eigenvalues of problem (1) - (2), as it was proved in [5], can be found as solutions of the implicit, with respect to \( \lambda \), equations:

\[ X = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-1)^{n+1} \binom{n}{m} e^m \lambda^{n-m} X^{2n+1-m\alpha} / \Gamma(2n - \alpha \cdot m + 2). \]  

In addition (see [6]), there is formula for finding the eigenvalues of problem (1) – (2) for \( \alpha \in (0; 1): \)

\[ \lambda_n = (\pi n/X)^2 - \varepsilon (\pi n/X)^\alpha \sin(\pi (1 - \alpha)/2). \]

2. Methods
We compose a difference scheme for problem (1) – (2) in two cases:

- \( \alpha \in (0; 1), \)
- \( \alpha \in (1; 2). \)

For the first case, a variant of the difference scheme can be found in [7].
Case one: $\alpha \in (0; 1)$.
We divide the segment $[0; X]$ on $N$ equal parts:

$$x_0 = 0; x_k = k \cdot h; \ k = 1; ...; N; \ h = \frac{X}{N}$$

$h$ – uniform grid pitch.

We denote by

$$y_k = y(x_k); \ y_k' = \frac{1}{h} (y_k - y_{k-1}).$$

Consider the fractional differentiation operator (3):

$$D_{\alpha x}^\alpha y = \left[ \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} y'(t) (x_n - t)^{-\alpha} \ dt \right] / \Gamma(1 - \alpha) \approx$$

$$\approx \frac{h^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{n} \left[ y_k - y_{k-1} \right] \cdot \left[ (n - k + 1)^{1-\alpha} - (n - k)^{1-\alpha} \right] = \frac{h^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=1}^{n} a_{nk} y_k,$$

there

$$a_{nk} = [(n - k + 1)^{1-\alpha} - 2(n - k)^{1-\alpha} + (n - k - 1)^{1-\alpha}]; \ 1 \leq k \leq n - 1; \ a_{nn} = 1.$$

Let,

$$y''_n = \frac{1}{h^2} \left( y_{n-1} - 2y_n + y_{n+1} \right); \ 1 \leq k \leq N - 1$$

then we get the system:

$$\begin{cases}
    y_0 = 0; \ y_N = 0 \\
    \frac{1}{h^2} (y_{n-1} - 2y_n + y_{n+1}) + \frac{\epsilon}{\Gamma(2 - \alpha) h^2} \sum_{k=1}^{n} a_{nk} \cdot y_k + \lambda y_n = 0; \\
    \end{cases} \quad (6)$$

System (6) can be solved by setting the values of the parameters included in it.

Next, we write system (6) in matrix form (separating the first and last equations):

$$\begin{pmatrix} \frac{1}{h^2} D + \frac{\epsilon}{\Gamma(2 - \alpha) h^2} A + \lambda E \end{pmatrix} Y = O;$$

there

$$O = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}; \ Y = \begin{pmatrix} y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}; \ E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix};$$

$$D = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 & -2 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ \end{pmatrix};$$

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1,1} & a_{N-1,2} & a_{N-1,3} & \cdots & a_{N-1,N-1} \end{pmatrix}.$$
Note that defining

\[ M = -\frac{1}{h^2} D - \frac{\epsilon}{\Gamma(2-\alpha)h^\alpha} A, \]  

we get

\[ MY = \lambda Y \]

Then the eigenvalues of problem (1) - (2) in the case \( \alpha \in (0; 1) \) should be approximately equal to the eigenvalues of the matrix \( M \).

Now we turn to \( \alpha \in (1; 2) \), this case was not considered before.

We again split the segment \([0; X]\) on \( N \) equal parts. Then, the fractional differentiation operator (3) can be written as follows:

\[
D_0^\alpha x_n y = \frac{1}{\Gamma(2-\alpha)(2-\alpha)} \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} y''(t) (x_i - t)^{1-\alpha} \, dt =
\]

\[
= \frac{1}{\Gamma(2-\alpha)(2-\alpha)} \sum_{k=1}^{n} [w_k y_k'' + (1 - w_k) y_{k-1}''] \cdot [(x_n - x_{k-1})^{2-\alpha} - (x_n - x_k)^{2-\alpha}]
\]

\[
= \frac{h^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{k=1}^{n} [w_k y_k'' + (1 - w_k) y_{k-1}''] \cdot [(n - k + 1)^{2-\alpha} - (n - k)^{2-\alpha}]
\]

\[
= \frac{b_{n1}}{\Gamma(3-\alpha) h^\alpha} ((3w_1 - 2)y_0 + (1 - 3w_1)y_1 + w_1y_2) +
\]

\[
+ \frac{1}{\Gamma(3-\alpha) h^\alpha} \sum_{k=2}^{n} [(1 - w_k) y_{k-2} + (3w_k - 2)y_{k-1} + (1 - 3w_k)y_k + w_ky_{k-1}] \cdot b_{nk}.
\]

Here the quantities

\[ b_{nk} = (n - k + 1)^{2-\alpha} - (n - k)^{2-\alpha}. \]

Weighting factors \( w_k \in [0; 1], k > 1 \) allow you to make calculations more accurate.

Next, we get a system of \( N + 1 \) equations:

\[
\begin{cases}
 y_0 = 0; \\
 (y_{n-1} - 2y_n + y_{n+1}) / h^2 + \\
 + \frac{\epsilon \cdot h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=2}^{n} b_{nk} [(1 - w_k)y_{k-2} + (3w_k - 2)y_{k-1} + (1 - 3w_k)y_k + w_ky_{k+1}] + \\
 + \frac{\epsilon h^{-\alpha} b_{n1}}{\Gamma(3-\alpha)} ((3w_1 - 2)y_0 + (1 - 3w_1)y_1 + w_1y_2) + \lambda y_n = 0; n = 1; \ldots; N - 1
\end{cases}
\]

\[ y_N = 0. \]

We write system (8) (with the exception of the expressions written in the last line) in matrix form:

\[ \frac{1}{h^2} D \cdot Y + \frac{\epsilon}{\Gamma(3-\alpha) h^\alpha} G \cdot Y + \lambda E \cdot Y = O; \]

there

\[ O = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}; Y = \begin{pmatrix} y_1 \\ \vdots \\ y_{N-1} \end{pmatrix}; E = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1 \end{pmatrix}; \]
\[ D = \begin{pmatrix} -2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix}; \]

\[ B = \begin{pmatrix} b_{11} & 0 & 0 & \ldots & 0 \\ b_{21} & b_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{N-1,1} & b_{N-1,2} & \ldots & b_{N-1,3} & \ldots & b_{N-1,N-1} \end{pmatrix} \]

\[ W = \begin{pmatrix} 1 - 3w_1 & w_1 & 0 & 0 & \ldots & 0 \\ 3w_2 - 2 & 1 - 3w_2 & w_2 & 0 & \ldots & 0 \\ 1 - w_3 & 3w_3 - 2 & 1 - 3w_3 & w_3 & \ldots & 0 \\ 0 & 0 & \ldots & 3w_{N-2} - 2 & 1 - 3w_{N-2} & w_{N-2} \\ 0 & 0 & \ldots & 0 & 1 - w_{N-1} & 3w_{N-1} - 2 & 1 - 3w_{N-1} \end{pmatrix}; \]

\[ G = \sum_{k=1}^{N-1} B_{ik} \cdot W_{k} - \text{fractional differentiation matrix up to a factor } \frac{1}{\Gamma(3-\alpha)\eta^{\alpha}}; \]

\[ B_{ik} - k \text{th column of the matrix } A. \]

\[ W_{k} - k \text{th row of the matrix } W. \]

Note that for \( w_k = 1 \), for any \( k \), the matrices \( D \) and \( W \) coincide, and for \( w_k = 0 \), the matrix \( W \) has the following diagonal form:

\[ W = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -2 & 1 & 0 & \ldots & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & -2 & 1 & 0 \\ 0 & 0 & \ldots & 1 & -2 & 1 \end{pmatrix}. \]

System (8) can be solved by setting the values of its constituent parameters.

Next, we write the matrix equation (9) in the form

\[ \tilde{M}Y = -\lambda Y. \]

Where

\[ \tilde{M} = -\frac{1}{\eta^2} D - \frac{\varepsilon}{\Gamma(3-\alpha)\eta^{\alpha}} G \] (10)

Then the eigenvalues of problem (1) – (2) in the case \( \alpha \in (1; 2) \) should be close enough to the eigenvalues of the matrix \( \tilde{M} \).

3. Results

Let us verify the statement on the approximate equality of the eigenvalues of the matrix of the difference scheme and the eigenvalues of the boundary value problem of the differential equation containing the fractional derivative numerically using the high-level technical calculation language MATLAB, choosing different values of the parameters. As the segment \( [0; X] \) we take the segment \( [0; 1] \).

We first consider the case \( \alpha \in (0; 1) \).

The following Table 1 shows the first seven ascending eigenvalues for the values \( \alpha = 0.5 \) and \( \varepsilon = 1.8 \), calculated in three ways:

- approximately by the formula (4);
- according to the exact formula (5);
as the eigenvalues of the matrix (7) of the difference scheme.

Table 1. The first seven ascending eigenvalues for the values \( \alpha = 0.5 \) and \( \varepsilon = 1.8 \).

| Eigenvalue according (4) | Difference scheme. \((h = 0.1)\) | Difference scheme. \((h = 0.01)\) | Eigenvalues by formula (5) |
|--------------------------|----------------------------------|----------------------------------|-----------------------------|
| \( \lambda_1 \)         | 8.1                              | 8.1                              | 7.6                         |
| \( \lambda_2 \)         | 36.5                             | 36.5                             | 36.3                        |
| \( \lambda_3 \)         | 85.1                             | 85.0                             | 84.9                        |
| \( \lambda_4 \)         | 153.5                            | 153.3                            | 153.4                       |
| \( \lambda_5 \)         | 241.9                            | 241.3                            | 241.7                       |
| \( \lambda_6 \)         | 349.8                            | 348.7                            | 349.8                       |
| \( \lambda_7 \)         | 477.8                            | 475.7                            | 477.6                       |

The figure 1 shows the absolute value of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the difference scheme matrix (7) for \( h = 0.01 \). The abscissa axis is the number of eigenvalues.

Figure 1. The modulus of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the matrix (7) for \( h = 0.01 \).

The figure 2 shows the graphs of the absolute value of the difference between the eigenvalues calculated according to the exact formula (5) and approximately according to the formula (4). The abscissa axis is the number of eigenvalues.

Figure 2. The modulus of the difference between the eigenvalues calculated according to the exact formula (5) and approximately according to the formula (4).

For the first five eigenvalues, the absolute value of the difference between the eigenvalues calculated according to the exact formula (5) and approximately according to the formula (4) and the absolute value of the difference between the eigenvalues calculated according to the exact formula (5)
and the eigenvalues of the difference matrix (7) for $h = 0.01$ do not exceed the value 0.5, which shows the possibility of using the difference scheme in practical calculations at step 0.01.

Now consider the case $\alpha \in (1; 2)$.
In this case, only the implicit equation (4) and the corresponding difference scheme described above can be used.

First, we present the calculations and their visualization for the case when the coefficient before the fractional derivative is less than 1. The following table shows the eigenvalues of the boundary value problem (1) - (2) calculated using the implicit equation (4) and the matrix of the difference scheme for $\varepsilon = 1/7$, $\alpha = 3/2$.

**Table 2.** The first seven ascending eigenvalues for the values $\alpha = 0.5$ and $\varepsilon = 1.8$.

| Eigenvalue according to (4) | Difference scheme. $w_k=0$; $h=0.1$ | Difference scheme. $w_k=0$; $h=0.01$ | Difference scheme. $w_k=0$; $h=0.001$ | Difference scheme. $w_k=1$; $h=0.01$ | Difference scheme. $w_k=1$; $h=0.001$ |
|-----------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\lambda_1$                | 10.3                            | 10.8                            | 10.9                            | 10.9                            | 10.3                            | 10.4                            |
| $\lambda_2$                | 40.9                            | 40.3                            | 41.3                            | 41.3                            | 41.0                            | 41.0                            |
| $\lambda_3$                | 91.7                            | 86.7                            | 92.5                            | 92.4                            | 91.6                            | 91.6                            |
| $\lambda_4$                | 162.3                           | 144.3                           | 162.9                           | 162.8                           | 162.3                           | 162.3                           |
| $\lambda_5$                | 252.9                           | 208.6                           | 253.8                           | 253.8                           | 252.7                           | 252.9                           |
| $\lambda_6$                | 363.5                           | 272.5                           | 363.8                           | 364.0                           | 362.9                           | 363.5                           |
| $\lambda_7$                | 493.9                           | 330.3                           | 490.1                           | 494.9                           | 492.7                           | 494.0                           |

On the figure 3 - the absolute value of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the difference scheme matrix (7) at $h = 0.01$; $w_k = 1$. The abscissa axis is the number of eigenvalues.

**Figure 3.** The modulus of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the matrix (7) at $h = 0.01$; $w_k = 1$.

The figure 4 shows the graphs of the absolute value of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the difference scheme matrix (7) for $h = 0.001$; $w_k = 0$. The abscissa axis is the number of eigenvalues.

**Figure 4.** The modulus of the difference between the eigenvalues calculated according to the exact formula (5) and the eigenvalues of the matrix (7) for $h = 0.001$; $w_k = 0$.
Next, we consider the case $\alpha = 1.47$ and $\varepsilon = 1.8$, which is important for practical applications, which corresponds to problem (1) - (2), which describes the change in the deformation-strength properties of polymer concrete.

The following table 3 shows the first (ascending) six eigenvalues at $\alpha = 1.47$ and $\varepsilon = 1.8$. The last row $\Delta$ shows the sum error over six values, calculated as the sum of the absolute values of the eigenvalues, calculated in two ways: as a solution to the implicit equation and as the eigenvalues of the difference scheme matrix for $h = 0.01$.

| Eigenvalue according (4) | Difference scheme. ($w_k=0$) | Difference scheme. ($w_k=0.5$) | Difference scheme. ($w_k=1$) | Difference scheme. ($w_k=(k-1)/(N-1)$) | Difference scheme. ($w_k=(N-k)/(N-1)$) |
|-------------------------|-------------------------------|-------------------------------|-------------------------------|-----------------------------------------|-----------------------------------------|
| $\lambda_1$            | 16.6                          | 16.4                          | 25.0                          | 24.9                                    | 16.7                                    | 13.7                                    |
| $\lambda_2$            | 59.5                          | 59.1                          | 60.2                          | 60.6                                    | 61.0                                    | 60.7                                    |
| $\lambda_3$            | 125.1                         | 124.0                         | 141.5                         | 141.4                                   | 128.2                                   | 124.0                                   |
| $\lambda_4$            | 213.4                         | 210.8                         | 210.9                         | 214.1                                   | 218.9                                   | 220.0                                   |
| $\lambda_5$            | 323.4                         | 318.6                         | 347.9                         | 348.5                                   | 331.1                                   | 324.9                                   |
| $\lambda_6$            | 442.2                         | 447.1                         | 445.9                         | 456.0                                   | 465.9                                   | 469.5                                   |
| $\Delta$               | -                             | 14                            | 56.2                          | 65.3                                    | 41.6                                    | 40.6                                    |

The table shows that, the sum error $\Delta$ for the first six eigenvalues is minimal for the values of the weight coefficients $w_k = 0$.

4. Conclusions
The calculations lead to the following conclusions.

- A decrease in the partition step from $h = 0.01$ to $h = 0.001$ practically does not change the values of the first in increasing values of the eigenvalues of the matrix of the difference scheme.
- The value of the weight coefficient has a significant effect with the same step of the partition.
- Methods for selecting weights require further research.
- The error grows with increasing order of the fractional derivative and the modulus coefficient with it.
- The proposed difference scheme can be used for practical calculations of the eigenvalues of the problem (1) – (2).

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