POSITIVE SOLUTIONS TO ELLIPTIC EQUATIONS
IN UNBOUNDED CYLINDER

JUN BAO
School of Mathematics and Information Science
Shanghai Lixin University of Commerce
Shanghai, 201620, China

LIHE WANG AND CHUNQIN ZHOU
Department of Mathematics, and MOE-LSC
Shanghai Jiaotong University
Shanghai 200240, China

Abstract. This paper investigates the positive solutions for second order linear elliptic equation in unbounded cylinder with zero boundary condition. We prove there exist two special positive solutions with exponential growth at one end while exponential decay at the other, and all the positive solutions are linear combinations of these two.

1. Introduction. The structure of positive harmonic functions on a domain Ω in \( \mathbb{R}^N \) (\( N \geq 2 \)) has been much studied. Early in 1941 Martin[13] gave a method for uniquely representing any positive harmonic function in an arbitrary domain in \( \mathbb{R}^3 \) by an integral on the minimal Martin boundary. His results have been extended to second order elliptic operators with a zero potential by Shur[16]. In the case where the closure of a domain is compact in the manifold, many mathematicians gave sufficient conditions for the corresponding Martin boundary to be equal to the relative boundary of the domain (see Hunt and Wheeden[9] and Taylor[17]). There are also some investigation fixed on the positive harmonic functions in some special unbounded domain, such as half-space, cone or cylinder. For example, Benedicks[1] has established a harmonic measure criterion that describes when the cone of positive harmonic functions on Ω that vanish on the boundary \( \partial \Omega \) is generated by two linearly independent minimal harmonic functions. Benedicks’ criterion describes when a Denjoy domain behaves like the union of two half-spaces from the point of view of potential theory. Related work, based on sectors, cones or cylinders, may be found in [3, 12, 7, 15]. Landis and Nadirashvili[11] showed that a positive solution to a uniformly elliptic equation in a cone of \( \mathbb{R}^n \) which vanishes at the boundary is unique up to a constant multiple. Murata[14] established a method to construct the Martin boundary and Martin kernel for second order elliptic equations and gave a sufficient condition for an equation in \( \mathbb{R}^n \) or a cone of \( \mathbb{R}^n \) to have a unique (up to a constant multiple) positive solution vanishing at the boundary.

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Assume $C$ is the cylinder $\mathcal{D} \times \mathbb{R}$, where $\mathcal{D}$ is a bounded Lipschitz domain in $\mathbb{R}^{n+1}$, and $x = (x_1, \cdots, x_{n+1}) = (x', y) = (x', y)$ denote a typical point of $\mathbb{R}^n \times \mathbb{R}$. This paper investigates positive solutions of linear elliptic equation defined in $C$. When the cylinder is $U = B' \times \mathbb{R}$ (here $B'$ is the unit ball in $\mathbb{R}^n$), it is known (see [6]) that the cone of positive harmonic functions $h_\pm(x', y) = e^{\pm ay} \phi(x')$, where $a$ is the square root of the first eigenvalue of the operator $-\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on $B'$ and $\phi$ is the corresponding eigenfunction, normalized by $\phi(0) = 1$. We want to show the set of the positive solutions of elliptic equation (1) defined in $C$ has a similar structure.

We consider the following elliptic equation:

$$
\begin{align*}
\begin{cases}
Lu(x) & = 0, & x \in C, \\
u(x) & = 0, & x \in \partial C, \\
u(x) & > 0, & x \in C,
\end{cases}
\end{align*}
$$

(1)

where $L$ stands for second order uniformly elliptic operator of one of the following two types:

$$
Lu(x) = -\sum_{i,j=1}^{n+1} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j},
$$

or

$$
Lu(x) = -\sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u(x)}{\partial x_i}).
$$

We assume that $a_{ij}(x) = a_{ji}(x) \in C^\infty(\bar{C})$, and $L$ is uniformly elliptic,

$$
\Lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n+1} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \text{for some } \Lambda > 0 \text{ and any } \xi \in \mathbb{R}^{n+1}.
$$

The assumption $a_{ij} \in C^\infty$ is qualitative, in the sense that none of our estimates depend on the smoothness of $a_{ij}$. By the standard approximation technique, all of our results are valid to uniformly elliptic equations with measurable coefficients $a_{ij}$.

We are interested in the question of existence and uniqueness (to within a multiplicative constant) of a solution $u$ of problem (1). We also study the precise asymptotic behaviors of the solutions and show finer properties of these solutions in the cylinder. In particular, there are two special solutions with exponential growth at one end while exponential decay at the other and all the positive solutions are linear combinations of these two.

We use $S$ to denote the solution set of problem (1). For $u \in S$, let $\hat{u}(y) := \sup_{x' \in \mathcal{D}} u(x', y), y \in (-\infty, +\infty)$, and $m(u) := \inf_{y \in (-\infty, +\infty)} \hat{u}(y)$. Define: $S^+ := \{u \in S | \lim_{y \to -\infty} u(x', y) = 0\}, S^- := \{u \in S | \lim_{y \to +\infty} u(x', y) = 0\}, S' := \{u \in S | \text{there exists a point } x^* = (x^*, y^*) \in C, \text{ such that, } u(x^*) = m(u) > 0\}$. Without loss of generality, we assume that $0' \in \mathcal{D}$ in the following. Now we can state the results.

**Theorem 1.1.** Solution set $S^+$ and $S^-$ of problem (1) are not empty. $S$ is a linear combination of $S^+$ and $S^-$. That is, for any $v \in S^+$, $w \in S^-$, we have

$$
S = S^+ + S^- = \{c_1 v + c_2 w | c_1, c_2 \geq 0, c_1 + c_2 > 0\}.
$$

(2)

The asymptotic behaviors of the solutions:
Theorem 1.2. There exist constants $\alpha$, $\beta$, $C$, $C' > 0$ depending only on $n$, $\Lambda$, $\mathcal{D}$, such that, for any $v \in S^+ \setminus \{0\}$, $w \in S^- \setminus \{0\}$,
\begin{align*}
C + \alpha y &\leq \ln(\frac{\hat{v}(y)}{\hat{v}(0)}) \leq C' + \beta y, \quad y \in (-\infty, +\infty), \quad (3) \\
C' - \beta y &\leq \ln(\frac{\hat{w}(y)}{\hat{w}(0)}) \leq C - \alpha y, \quad y \in (-\infty, +\infty). \quad (4)
\end{align*}

For any $u \in S'$. We assume $\mathbf{x}^* = (x'^*, y^*) \in \mathcal{C}$, such that $u(x^*) = m(u)$. Then
\[ C + \alpha |y - y^*| \leq \ln(\frac{\hat{u}(y)}{\hat{u}(y^*)}) \leq C' + \beta |y - y^*|, \quad y \in (-\infty, +\infty). \quad (5) \]

In order to illustrate the results, we give a simple example which is also a special case of Theorem 1 in Gardiner[6].

Example 1.3. For $n = 1$. Consider area $\mathcal{F} := (0, \pi) \times (-\infty, +\infty) \subset \mathbb{R}^2$ and equation
\[ \begin{cases} \Delta u (x) = 0, & x \in \mathcal{F}, \\ u(x) = 0, & x \in \partial \mathcal{F}, \\ u(x) > 0, & x \in \mathcal{F}. \end{cases} \quad (6) \]

Define $S_{\mathcal{F}}$ is the solution set of problem (6). It is easy to see that $e^y \sin x$, $e^{-y} \sin x \in S_{\mathcal{F}}$. Actually they are the only two nontrivial solutions in $S_{\mathcal{F}}$ in the sense of Theorem 1.1. We also see that the asymptotic behavior of solutions is exponential.

For $A \subset \mathbb{R}$, let $\mathcal{C}_A := \mathcal{D} \times A = \{(x', y) \in \mathbb{R}^{n+1} | x' \in \mathcal{D}, y \in A\}$, $\partial_b \mathcal{C}_A := \partial \mathcal{D} \times A = \{(x', y) \in \mathbb{R}^{n+1} | x' \in \partial \mathcal{D}, y \in A\}$. Particularly, for any $y \in \mathbb{R}$, we use $\mathcal{C}_y := \mathcal{C}_{\{y\}}$, $\mathcal{C}_y^+ := \mathcal{C}_{(0, +\infty)}$, $\mathcal{C}_y^- := \mathcal{C}_{(-\infty, 0)}$, $\mathcal{C}^+ := \mathcal{C}_0^+$, $\mathcal{C}^- := \mathcal{C}_0^-$. We also study the positive bounded solutions defined in half cylinder $\mathcal{C}^+$. They can be approximated by the solution in $S^-$. That is

Theorem 1.4. Suppose $u$ is a bounded solution to problem
\[ \begin{cases} Lu (x) = 0, & x \in \mathcal{C}^+, \\ u(x) = 0, & x \in \partial \mathcal{C}^+ \setminus \mathcal{C}_0, \\ u(x) > 0, & x \in \mathcal{C}^+. \end{cases} \]

Then for $w \in S^-$, there exist constants $\alpha > 0$, which is only dependent of $n$, $\Lambda$, $\mathcal{D}$, and $K$, $C > 0$ which are only dependent of $\hat{u}(1)$, $n$, $\Lambda$, $\mathcal{D}$, such that
\[ |u(x) - K w(x)| \leq Ce^{-\alpha x} |w(x)|, \quad x \in \mathcal{C}_{(1, +\infty)}. \]

The rest of this paper is divided into two parts. In Section 2 we establish some auxiliary results. First we introduce some fundamental notions concerning the positive solutions to equation (1) vanishing at the boundary. The maximum principle for the solutions in cylinder is proved under a bounded condition. We then introduce a so-called boundary Harnack principle which is proved in Fabes et al.[5](Theorem 4.3) in the non-divergence form and in Caffarelli et al.[2](Theorem 1.4) in the divergence form for the positive solutions of equation(1). Through them we are able to compare the solutions.

The main theorem will be proved in Section 3. We study the structure of the set of the positive solutions. We will show the exponential growth and decay for the positive solutions. The existence of the positive solutions is also proved. Any bounded positive solution in half cylinder can be approximated by the solution in the whole cylinder.
2. Preliminaries. In this section, we collect some preliminary results. In Section 2.1, we shall give the maximum principle in cylinder. In Section 2.2, on the basis of the boundary Harnack principle, we give some lemmas to compare the solutions.

2.1. Maximum Principle. According to the well-known Maximum Principle a subharmonic up-bounded continuous function defined in a domain \( \Omega \subset \mathbb{R}^n, \ n \geq 1 \) which is non-positive on the boundary \( \partial \Omega \), is in fact negative everywhere in \( \Omega \) and this result extends to nonnegative solutions of a large class of linear elliptic equation[8]. It is an important tool for us to study the properties of the solutions. Now we want to investigate the validity of the maximum principle for the solutions in cylinder. We begin our investigation with a Harnack-type principle for operator \( L \).

Lemma 2.1. Suppose \( u(x) \) is a subsolution of the problem:

\[
\begin{align*}
Lu(x) & \leq 0, \quad x \in C_{(0,2)}, \\
u(x) & = 0, \quad x \in \partial_0 C_{(0,2)}, \\
u(x) & \leq 1, \quad x \in \partial C_{(0,2)}.
\end{align*}
\]

Then there exists a constant \( \delta \in (0,1) \) which is only dependent of \( n, \Lambda, D \), such that

\[
u(x',1) \leq 1 - \delta, \quad x' \in D.
\]

Proof. Consider \( u_+(x) \) which satisfies

\[
\begin{align*}
Lu_+(x) &= 0, \quad x \in C_{(0,2)}, \\
u_+(x) &= \max\{u(x), 0\}, \quad x \in \partial C_{(0,2)}.
\end{align*}
\]

By classical maximum principle, \( u(x) \leq u_+(x) \leq 1, \ x \in C_{(0,2)} \). With Boundary Hölder estimate in Gilbarg and Trudinger[8] (Theorem 6.19 in non-divergence form and Theorem 8.25 in divergence form), there exists a constant \( C_0 \) which only dependent of \( n, \Lambda, D \), such that \( |u_+|_{C^{\alpha}((1, \frac{3}{2}))} \leq C_0, \ x \in C_{1, \frac{3}{2}} \), \( \alpha \in (0,1) \).

So \( |u_+(x)| \leq C_0[\text{dist}(x, \partial C)]^\alpha, \ x \in C_{1, \frac{3}{2}} \). Take \( \varepsilon_0 \) sufficiently small such that \( C_0 \varepsilon_0^\alpha \leq \frac{1}{2} \) and apply Harnack principle to \( 1 - u_+ \) in domain \( C_{(0,2)} \), there exists a constant \( \delta_0 \in (0,1) \), such that \( |u(x)| \leq 1 - \delta_0, \ x \in C_{1, \frac{3}{2}} \). Let \( \delta := \min\{\frac{1}{2}, \delta_0\} \), then \( w(x',1) \leq 1 - \delta, \ x' \in D \).

Remark 1. When \( L \) is in non-divergence form in Lemma 2.1, there is an alternate proof using barrier function. As a matter of fact, assume \( D \subset B_R \subset \mathbb{R}^n \), set

\[
u(x) := 1 - \frac{y(4RA - y)}{4R^2A^2} + \frac{1}{4R^2}(R^2 - |x'|^2), \quad x \in C_{(0,4RA)}.
\]

If we have

\[
\begin{align*}
Lu(x) & \leq 0, \quad x \in C_{(0,4RA)}, \\
u(x) & = 0, \quad x \in \partial C_{(0,4RA)}, \\
u(x) & \leq 1, \quad x \in \partial C_{(0,4RA)}.
\end{align*}
\]

It is easy to check

\[
u(x',2RA) \leq u_+(x',2RA) \leq \frac{1}{4}, \quad x' \in D.
\]

Now we can extend the maximum principle to the solutions in cylinder \( C \).

Lemma 2.2. Suppose \( Lu(x) \leq 0, x \in C \), and \( u \) is bounded from above. Then

\[
\sup_{x \in C} u(x) \leq \sup_{x \in \partial C} u(x).
\]
Lemma 2.4.

Proposition 1.

By definition \( \hat{u} \)

For any \( u(x) := u(x) - \sup_{x \in \partial S} u(x) \) instead. So now we only have to prove

\[ u(x) \leq 0, \quad x \in C. \] (7)

Set \( C_{(N-1,N+1)} = \{ x = (x', y) \in \mathbb{R}^{n+1} | x' \in D, N - 1 < y < N + 1 \}, N \in \mathbb{Z}. \)

We assume \( C \) is the upper bound of \( u \). When \( N = 1 \), by Lemma 2.1, there exists a constant \( \delta \in (0,1) \), such that \( \hat{u}(1) \leq (1 - \delta) \max \{ \hat{u}(0), \hat{u}(2) \} \leq (1 - \delta)C \). For \( N \in \mathbb{Z} \), set \( v_N(x) = v_N(x', y) := u(x', y + N - 1) \). Still with Lemma 2.1, \( \hat{u}(N) = \hat{v}_N(1) \leq (1 - \delta) \max \{ \hat{v}_N(0), \hat{v}_N(2) \} \leq (1 - \delta)C, N \in \mathbb{Z}. \) With the classical maximum principle, \( \hat{u}(y) \leq (1 - \delta)C, y \in \mathbb{R}. \) Do the operation repeatedly, \( \hat{u}(y) \leq (1 - \delta)^k C, x \in \mathbb{R}, k \in \mathbb{N}^+. \) Letting \( k \to \infty \), we get (7).

The solutions defined in half cylinder \( C^+ \) can be proved to satisfy the following maximum principle through the same method.

Lemma 2.3. Suppose \( Lu(x) \leq 0, x \in C^+ \), and \( u \) is bounded from above. Then \( \sup_{x \in C^+} u(x) \leq \sup_{x \in \partial C^+} u(x) \).

Now we are able to show the exponential decay of the bounded solutions in \( C^+ \).

Lemma 2.4. Assume \( u \) is bounded from above and

\[
\begin{cases}
Lu(x) \leq 0, & x \in C^+, \\
u(x) = 0, & x \in \partial D \times (0, +\infty).
\end{cases}
\]

Then there exist constants \( \alpha, C_0 > 0 \) which are only dependent of \( n, \Lambda, D \), such that

\[ u(x) \leq C_0 \hat{u}(0)e^{-\alpha y}, \quad x \in C^+. \]

Proof. For \( u \) is bounded from above, with Lemma 2.3, \( \hat{u}(y) \leq \hat{u}(0), y \in (0, +\infty) \). By Lemma 2.1, there exists \( \delta \in (0,1) \), such that \( \hat{u}(1) \leq (1 - \delta) \hat{u}(0) \). Do the operation repeatedly, then \( \hat{u}(n) \leq (1 - \delta)^n \hat{u}(0), n \in \mathbb{N}^+. \) So \( u(x) \leq \hat{u}(|y|) \leq (1 - \delta)^{|y|} \hat{u}(0) \leq \frac{\hat{u}(0)}{1 - \delta} e^{\ln(1 - \delta) y}, \quad x \in C^+, \) and \( \alpha = -\ln(1 - \delta). \)

Lemma 2.5. For any \( u \in S \), \( \hat{u}(y) \) is continuous in \(( -\infty, +\infty )\).

Proof. Suppose \( u \in S \). We only have to show \( \hat{u}(y) \) is continuous at \( y = 0 \).

For \( \hat{u}(0) > 0 \), we assume that \( u(x_0,0) = \sup_{x \in D} u(0), x_0 \in D \). For \( u(x) \) is continuous, any \( \varepsilon > 0 \), there exist \( \delta_1 > 0, |x| < \delta_1, u(x) > u(x_0,0) - \varepsilon \). By definition, \( \hat{u}(y) := \sup_{x \in D} u(x',y), y \in ( -\infty, +\infty) \), so \( \hat{u}(y) > \hat{u}(0) - \varepsilon, \) \( |y| \leq \delta_1 \).

By definition of \( x_0, u(x_0',0) \leq u(x_0,0) \), \( x' \in D \). By continuity of \( u(x) \), there exists \( \delta_2 > 0, u(x) < u(x',0) + \frac{\varepsilon}{2} \), \( |x - x'| \leq \delta_2 \). For \( \{ (x',0) | x' \in D \} \) is a compact set, we can choose a \( \delta_2 > 0 \), such that if \( |y| < \delta_2 \), then \( u(x) < u(x',0) + \frac{\varepsilon}{2} \).

By definition \( \hat{u}(y) \leq u(x',0) + \frac{\varepsilon}{2} < \hat{u}(0) + \varepsilon \).

Now we can divide \( S \) into three subsets.

Proposition 1. Assume \( u \in S \). Then \( m(u) = \inf_{y \in ( -\infty, +\infty)} \hat{u}(y) \geq 0 \).

1. If \( m(u) = 0 \), then either of the following alternatives holds.

(a) There exists a sequence \( \{ (x_j', y_j) \} \subset C \) such that \( \lim_{j \to \infty} y_j = +\infty \), and

\[ \lim_{j \to \infty} u(x_j', y_j) = +\infty, \quad \hat{u}(y) \text{ is a strictly increasing function}. \]

(b) There exists a sequence \( \{ (x_j', y_j) \} \subset C \), such that \( \lim_{j \to \infty} y_j = -\infty \),

\[ \lim_{j \to \infty} u(x_j', y_j) = +\infty, \quad \hat{u}(y) \text{ is a strictly decreasing function}. \]
2. If \( m(u) > 0 \), then there exists \( \mathbf{x}^* = (\mathbf{x}^*, y^*) \in \mathcal{C} \), such that \( u(\mathbf{x}^*) = m(u) \).

\( \hat{u}(y) \) is strictly decreasing in \((-\infty, y^*]\), and strictly increasing in \((y^*, +\infty)\).

**Proof.** By the definition of \( m(u) \) and \( \hat{u}(y) \), we assume the minimize sequence \( \{x_j = (x_j', y_j)\}_{j=1}^{\infty} \subset \mathcal{C} \), such that \( u(x_j') = \sup_{x \in \mathcal{D}} u(x, y) \), \( \lim_{j \to \infty} u(x_j) = m(u) \).

1° If \( m(u) = 0 \), then \( \lim_{j \to +\infty} u(x_j) = 0 \).

We claim that there exists a subsequence of \( \{(x_j', y_j)\}_j \) (we still denote it \( \{(x_j', y_j)\}_j \)), such that \( \lim_{j \to \infty} y_j = -\infty \) or \( \lim_{j \to \infty} y_j = +\infty \).

By contradiction, suppose \( \{(x_j', y_j)\}_j \) is bounded in \( \mathbb{R}^{n+1} \), then we get a point \( \mathbf{x}^{**} = (\mathbf{x}^{**}, y^{**}) \in \mathcal{C} \) and a subsequence of \( \{(x_j', y_j)\}_j \) (still denote it \( \{(x_j', y_j)\}_j \)), such that \( \lim_{j \to \infty} x_j = \mathbf{x}^{**} \), so \( u(\mathbf{x}^{**}) = 0 \). So \( \mathbf{x}^{**} \in \partial \mathcal{C} \). We assume that \( \mathbf{x}^{**} = (x_0', y_0), x_0' \in \partial \mathcal{D} \).

For \( \hat{u}(y_0) > 0 \), by Lemma 2.5 there exist \( \varepsilon_0, \delta_0 > 0 \), such that \( \hat{u}(y) > \varepsilon_0 \), \( |y - y_0| < \delta_0 \). So there exists a sufficient large \( K_0 > 0, u(x_j') > \frac{\varepsilon_0}{2} \), \( j \geq K_0 \). So \( u(\mathbf{x}^{**}) = \lim_{j \to \infty} u(x_j) \geq \frac{\varepsilon_0}{2} > 0 \), a contradiction.

Now if we assume \( \lim_{j \to \infty} y_j = -\infty \) (in subsequence sense). Claim that \( \hat{u}(y) \) is a strictly monotone increasing function.

By contradiction, suppose there exist \(-\infty < \tilde{y} < \bar{y} < +\infty, \hat{u}(\tilde{y}) \geq \hat{u}(\bar{y}) > 0 \). With \( \lim_{j \to \infty} \hat{u}(y_j) = 0 \) and \( \lim_{j \to \infty} y_j = -\infty \), take \( K > 0 \) sufficiently large, such that \( y_K < \tilde{y} < \bar{y}, u(x', y_K) < \hat{u}(\bar{y}), x' \in \mathcal{D} \). Consider the area \( \Omega := \mathcal{C}(y_K, \bar{y}) \), by the definition of \( \hat{u}(y) \), we can get a point \( \hat{x} \in \mathcal{C} \), such that \( u(\hat{x}) = \hat{u}(\bar{y}) \geq \sup_{\partial \Omega} u(x) \), this contradicts with maximum principle. Therefore we get the strictly increasing property of \( \hat{u}(x) \), and (1b) will not happen in this case. And \( \lim_{y \to -\infty} \hat{u}(y) = 0 \).

If we assume \( \lim_{j \to \infty} y_j = +\infty \), (1b) can be proved with the same method.

2° If \( m(u) > 0 \), we claim that there exists a subsequence of \( \{(x_j', y_j)\}_j \) (still denote it \( \{(x_j', y_j)\}_j \)) and \( \mathbf{x}^* \in \mathcal{C} \), such that \( \lim_{j \to \infty} x_j = \mathbf{x}^* \), \( \lim_{j \to \infty} u(x_j) = u(\mathbf{x}^*) = m(u) > 0 \).

By contradiction, if there exists a subsequence of \( \{(x_j', y_j)\}_j \) (still denote it \( \{(x_j', y_j)\}_j \)), such that \( \lim_{j \to \infty} y_j = -\infty \),

\[
\lim_{j \to \infty} \hat{u}(y_j) = m(u) > 0.
\] (8)

With a similar argue to 1° we can prove \( \hat{u}(y) \) is strictly increasing in \( \mathbb{R} \). Therefore \( u(x) \) is bounded in area \( \mathcal{C}(-\infty, 0) \). With Lemma 2.4, \( \lim_{x \to -\infty} \hat{u}(x) = 0 \), this contradicts with (8). If there exists a subsequence of \( \{(x_j', y_j)\}_j \) (still denote it \( \{(x_j', y_j)\}_j \)), such that \( \lim_{j \to \infty} y_j = -\infty \), and \( \lim_{j \to \infty} \hat{u}(y_j) = m(u) > 0 \). With a similar argue we get a contradiction.

So there exists \( \mathbf{x}^* = (\mathbf{x}^*, y^*) \in \mathcal{C} \) and a subsequence of \( \{(x_j', y_j)\}_j \) (still denote it \( \{(x_j', y_j)\}_j \)), such that \( \lim_{j \to \infty} x_j = \mathbf{x}^*, u(\mathbf{x}^*) = m(u) > 0 \).

We claim that \( \hat{u}(y) \) is strictly monotone decreasing in \((-\infty, y^*]\). By contradiction, suppose there exist \( \tilde{y}, \bar{y} \in (-\infty, y^*) \), such that \( \hat{u}(\tilde{y}) \leq \hat{u}(\bar{y}) \), we can get a local maximum point in the area \( \mathcal{C}(y^*, y^*) \), which contradicts with the maximum principle. The increasing property of \( \hat{u} \) in \((y^*, +\infty)\) can be proved in the same way. □

**Remark 2.** From Proposition 1, we actually get \( S = S^+ \cup S^- \cup S^\vee \), \( S^+ \cap S^- = S^- \cap S^\vee = S^\vee \cap S^\vee = \emptyset \).
2.2. Boundary Harnack Principle. Boundary Harnack principle of harmonic functions for Lipschitz domains was proved in [18, 4]. Jerison and Kenig [10] extend the results to nontangentially accessible domains. Moreover, Boundary Harnack principle also holds for the positive solutions of equation (1). From Theorem 4.3 in Fabes et al. [5] in the non-divergence case, and Theorem 1.4 in Caffarelli et al. [2] in the divergence case, we have:

**Lemma 2.6.** There is a universal constant $C > 0$ which is only dependent of $n, \Lambda, D$ such that $u_1, u_2 \in S$, $u_1(0', 0) = u_2(0', 0)$ implies that

$$1 \leq \frac{u_2(x)}{C} \leq u_1(x), \quad x \in C_{[-1, 1]}.$$

**Remark 3.** If the condition $u_1(0', 0) = u_2(0', 0)$ in Lemma 2.6 is replaced by inequality $u_1(0', 0) \leq (\geq) u_2(0', 0)$, then the conclusion should be

$$u_1(x) \leq C u_2(x) (u_1(x) \geq \frac{1}{C} u_2(x)), \quad x \in C_{[-1, 1]}.$$

From Theorem 3.3 in Fabes et al. [5] in the non-divergence case, and Theorem 1.1 in Caffarelli et al. [2] in the divergence case, we have the following Harnack principle in the boundary.

**Lemma 2.7.** Assume $u \in S$. There is a universal constant $C$, for any $y_0 \in (-\infty, +\infty)$,

$$u(x) \leq C u(0', y_0), \quad x \in C_{(y_0 - 2, y_0 + 2)}.$$

From Lemma 2.6, we get a lemma to compare the solutions. That is

**Lemma 2.8.** For any $u_1, u_2 \in S$, if there is a point $x_0 = (x_0', y_0) \in C$, $u_1(x_0) = u_2(x_0)$, then there exists a universal constant $\epsilon$ such that

$$u_1(x', y_0) \geq \epsilon u_2(x', y_0), \quad x' \in D.$$

**Proof.** Set $\bar{u}_1(x', y) = u_1(x', y - y_0)$, $\bar{u}_2(x', y) = u_2(x', y - y_0)$, $\alpha = u(0', 0)$ and $\beta = v(0', 0)$, then $\bar{u}_1/\alpha$, $\bar{u}_2/\beta$ satisfy all the conditions in Lemma 2.6, so there exists a constant $C$, such that $\frac{1}{C} \frac{\bar{u}_1(x', y_0)}{\alpha} \leq \frac{\bar{u}_1(x', y_0)}{\alpha} \leq C \frac{\bar{u}_2(x', y_0)}{\beta}$, and $\bar{u}_1(x', 0) \geq \frac{1}{C} \bar{u}_1(x', 0) \geq \frac{1}{C} \frac{\bar{u}_2(x', y_0)}{\beta}$. With the condition $u_1(x_0) = u_2(x_0)$, $\bar{u}_1(x', 0) = \bar{u}_2(0', 0)$, so $\frac{\alpha}{\beta} \geq \frac{\bar{u}_1(x', 0)}{\bar{u}_2(x', 0)} = \frac{1}{C} \bar{u}_1(x', 0) \geq \frac{1}{C} \bar{u}_2(x', 0)$. Take $\epsilon = \frac{1}{C}$, we get the conclusion. \(\square\)

**Remark 4.** If the condition $u_1(x_0) = u_2(x_0)$ in Lemma 2.8 is replaced by inequality $u_1(x_0) \geq u_2(x_0)$, the conclusion still holds.

**Proposition 2.** For any $u \in S'$, $v \in S^+$, $w \in S^-$, there exist constants $a, b > 0$, such that

$$av(x) \leq u(x), bw(x) \leq u(x), \quad x \in C.$$

**Proof.** From Proposition 1, for $u \in S'$, there exists $x^* = (x^*, y^*) \in C$, $\hat{u}(y)$ is decreasing in $(-\infty, y^*)$, and increasing in $(y^*, +\infty)$.

For $v \in S^+$, $\hat{v}(y)$ is strictly increasing. So $v(x)$ is bounded in $C_{(-\infty, 0)}$. By Lemma 2.8, we can choose a constant $k_0 > 0$, such that $k_0 v(x) \leq u(x), x \in C_{(0)}$. With the maximum principle,

$$k_0 v(x) \leq u(x), \quad x \in C_{(-\infty, 0)}.$$

(9)

We claim there exists a constant $a > 0$, such that $av(x) \leq u(x), x \in C$. By contradiction, suppose there exist $\{x_j = (x_j', y_j')\}_{j=1}^{\infty} \subset C$, such that $\frac{1}{a} v(x_j) >
For any $u, v \in S^+$, there exists a constant $c > 0$, such that
\[
|u(x) - v(x)| \leq c |u(x)|, \quad x \in C.
\] (11)

Proof. With Lemma 2.8, for any $y \in \mathbb{R}$, there exists a constant $c_y$, such that
\[
u(x) \leq c_y v(x), \quad x \in C(y).
\]
With the maximum principle, $u(x) \leq c_y v(x), \quad x \in C(-\infty, y]$. Take $x = 0$,
\[
u(x) \leq c_0 v(x), \quad x \in C(-\infty, 0].
\] (12)

We prove (11) by contradiction. Suppose there exist $(x_j, y_j)_{j=1}^{+\infty} \subseteq C$, such that $u(x_j) > v(x_j)$. With Lemma 2.8, there exist a constant $\varepsilon$, such that $u(x', y_j) \geq \varepsilon v(x', y_j), x' \in D$. If we choose $j = N$ big enough such that $\varepsilon N > c_0 + 1$, then
\nu(x) \geq (c_0 + 1) v(x), \quad x \in C(-\infty, y_N],
\]
this contradicts with (12). \qed

3. Proof of the main theorem. With the preparations in above section, we will finish the proof of the main theorems in the section.

Proof of Theorem 1.1. We divide the proof into three parts.

1° First we study the structure of $S^+$ and $S^-$. For any $u, v \in S^+$, set
\[
E := \{k > 0|u(x) \leq k v(x), x \in C\}, \quad K = \inf E.
\]
With Proposition 3, \( c \in E, E \neq \emptyset \). Consider \( K v(x) - u(x) \), by definition \( K v(x) - u(x) \geq 0, x \in C \). We claim that

\[
K v(x) - u(x) = 0, \quad x \in C.
\]

By contradiction. If \( K v - u \neq 0 \), then \( K v - u \in S^+ \). With Proposition 3, there exists a constant \( K_1 > 0 \), such that \( v(x) \leq K_1(K v(x) - v(x)) \), and

\[
(K - \frac{1}{K_1}) v(x) - u(x) \geq 0.
\]

Then \( K - \frac{1}{K_1} \in E \), and this contradicts with the definition of \( K \). Therefore we get \( u = K v \), and \( S^+ = \{a v | a > 0\} \). With the similar method we can also get \( S^- = \{b w | b > 0\} \).

2° Next we study the structure of the solution set \( S \).

From Proposition 1, \( S = S^+ \cup S^- \cup S^\lor \lor, S^+ \cap S^\lor S^\lor = S^- \cap S^\lor = S^+ \cap S^\lor = \emptyset \).

For any \( x \in S \), if \( u \in S^+ \) or \( u \in S^- \), then there is \( a > 0 \) or \( b > 0 \), such that \( u = a v \) or \( u = b w \). Next we suppose \( u \in S^\lor \). Assume

\[
E := \{k > 0 | kv \leq w\}, \quad a^* = \sup E.
\]

With Proposition 2, \( E \neq \emptyset \), so \( a^* > 0 \). We also have \( a^* < +\infty \). Actually if not we can follow the method in Proposition 3 to get a contradiction.

Consider the function \( u - a^* v \). By the continuity,

\[
u(x) - a^* v(x) \geq 0, \quad x \in C.
\]

From \( u \notin S^+ \) and Harnack principle,

\[
u(x) - a^* v(x) > 0, \quad x \in C.
\]

For \( u - a^* v \in S \), we claim that \( u - a^* v \in S^- \). Then there exists \( b > 0 \), such that \( u - a^* v = b w \). It is easy to see that

\[
\{av + bw | a, b \in \mathbb{R}, a \geq 0, b \geq 0, a + b > 0\} \subset S.
\]

So we get the conclusion.

We prove the claim by contradiction, if \( u - a^* v \in S^\lor \), then by Proposition 2, there is a constant \( \tilde{a} > 0 \), such that \( u - a^* v \geq \tilde{a} v, u - (a^* + \tilde{a}) v \geq 0 \). So \( a^* + \tilde{a} \in E \), this contradicts with \( a^* = \sup E \).

3° The existence of the solutions:

For \( H \in \mathbb{N}^+ \), consider the equation

\[
\begin{cases}
Lu(x) = 0, & x \in C_{(-H,H)}, \\
u(x) = 0, & x \in \partial H C_{(-H,H)} \cup C_{-H}, \\
u(x) = C \text{dist}(x', \partial D), & x \in C_{H}.
\end{cases}
\]

With the classical elliptic theory, there exists a unique positive solution \( u_H(x) \) to equation (13) and \( u_H(x) > 0, x \in C_{(-H,+H)} \). For (13) is linear, we can adjust the constant \( C \), such that \( u_H(0', 0) = 1 \).

For \( M \in \mathbb{N}^+ \), we use the Harnack estimate in \( C_{(-M-1,M+1)} \),

\[
u_H(x) \leq C_M, \quad x \in C_{(-M,M)}, \quad H > M + 1.
\]

With Boundary Hölder estimate in Gilbarg and Trudinger [8] (Theorem 6.19 in non-divergence form and Theorem 8.25 in divergence form), there exists a constant \( C_0 > 0 \) which is only dependent of \( n, \Lambda, \mathcal{D} \), such that

\[
[u_H(x)]_{C^0(\mathcal{C}_{(-M,M)})} \leq C_0, \quad H > M + 1, \quad \alpha \in (0, 1).
\]
When $M = 1$, we use Arzela-Ascoli theorem, and there exists a subsequence of $\{u_H(x)\}$ (we denote it $\{u_L^{(1)}(x)\}_{L=1}^{\infty}$) such that $\{u_L^{(1)}(x)\}$ uniformly converges in $C_{[-1,1]}$. When $M = 2$, we can also get a subsequence of $\{u_L^{(2)}(x)\}_{L=1}^{\infty}$ (denote it $\{u_L^{(2)}(x)\}_{L=1}^{\infty}$) such that $\{u_L^{(2)}(x)\}$ uniformly converges in $C_{[-2,2]}$. We do the similar operations when $M \in \mathbb{N}^+$, and get a sequence $\{u_L^{(M)}(x)\}_{L=1}^{\infty}$ uniformly converges in $C_{[-M,M]}$. Then we subtract the dialogged sequence $\{u_L^{(L)}(x)\}_{L=1}^{\infty}$.

$\{u_L^{(L)}(x)\}_{L=1}^{\infty}$ is uniformly converging in $C_{[-M,M]}$, $M \in \mathbb{N}^+$. We denote the limit by $u(x)$ which is defined in $C$. So $u(x)$ satisfies (1). Therefore $u(x) \in S$. Moreover we can prove that $u(x) \in S^+$. For $u_H(0',0) = 1$, by Lemma 2.6, there exists a constant $C > 0$ which is independent of $H$, such that $u_H(x) \leq C$, $x \in C_{(0)}$. By the maximum principle $u_H(x) \leq C$, $x \in C_{(-H,0)}$. Taking limit, we get $u(x) \leq C$, $x \in C_{(-H,0)}$.

For $H$ is any number in $\mathbb{N}^+$, therefore $u(x) \leq C$, $x \in C_{(-\infty,0)}$. With Lemma 2.4, we get $\lim_{y \to -\infty} \hat{u}(y) = 0$, therefore $u(x) \in S^+$. We can prove (14) with $w(x) \in S^-$. It is easy to check $(u + w)(x) \in S^- \neq \emptyset$.

**Proof of Theorem 1.2.** We divide the proof into two steps. In Step 1, we prove that the growing and decaying rate of positive solutions to (1) is at least exponential in the infinity. In Step 2, we prove the rate of asymptotic behaviors of these positive solutions is at most exponential in the infinity.

**Step 1.** For any $v \in S^+$. With Proposition 1, we know that $\hat{v}(y)$ is strictly monotone increasing in $(-\infty, +\infty)$. We claim there exists a constant $\eta > 0$ which is only dependent of $n$, $\Lambda$, $D$, such that

$$
(1 + \eta)\hat{v}(y) \leq \hat{v}(y + 1), \quad y \in (-\infty, +\infty).
$$

Then

$$
\hat{v}(y) \geq (1 + \eta)^{|y|} \hat{v}(0) \geq C\hat{v}(0)e^{\beta_-|y|}, \quad y \in (0, +\infty), \quad \beta_- = \ln(1 + \eta).
$$

Therefore we get a part of the inequality in (3). With the similar method we get a part of the inequality in (4).

For any $u \in S^+$, with Proposition 1 we assume that $x^* = (x^*, y^*) \in C$, such that $u(x^* + m) \leq u(x^*)$. $\hat{u}(y)$ is strictly monotone decreasing in $(-\infty, y^*]$, and strictly monotone increasing in $(y^*, +\infty)$. We also claim that

$$
(1 + \eta)\hat{u}(y^* + m) \leq \hat{u}(y^* + (m + 1)), \quad m \in \mathbb{N}^+.
$$

Then with Lemma 2.6, we can get

$$
\hat{u}(y) \geq (1 + \eta)^{|y - y^*|} \hat{u}(y^* + 1) \geq C_0\hat{u}(y^* + (m + 1)), \quad y \in (y^*, +\infty).
$$

With a similar argue we also get this estimate in $(-\infty, y^*)$. Therefore we get a part of (5).

The proof of inequality (15): We prove by contradiction. Suppose $\hat{u}(y^* + (m_k + 1)) \leq (1 + \frac{1}{k})\hat{u}(y^* + m_k), k \in \mathbb{N}^+$. With Lemma 2.1, there exists a constant $\delta \in (0, 1)$ which is only dependent of $n$, $\Lambda$, $D$, such that

$$
\hat{u}(y^* + (m_k)) \leq (1 - \delta)\hat{u}(y^* + (m_k + 1)) \leq (1 - \delta)(1 + \frac{1}{k})\hat{u}(y^* + (m_k + 1)).
$$

We take $k$ large enough such that $(1 - \delta)(1 + \frac{1}{k}) < 1$. Then we get a contradiction. So we have $\hat{u}(y^* + (m + 1)) \geq \frac{1}{2\delta(1 - \delta)}\hat{u}(y^* + m), m \in \mathbb{N}^+$. We can prove (14) with the same method. With Lemma 2.4, we get another part of (3) and (4).
Step 2. For \( u \in S \), we claim there exists constant \( \theta > 0 \), which is only dependent of \( n \), \( \Lambda \), \( D \), such that
\[
\hat{u}(y_0 + 1) \leq (1 + \theta)\hat{u}(y_0), \quad \hat{u}(y_0 - 1) \leq (1 + \theta)\hat{u}(y_0), \quad y_0 \in \mathbb{R}.
\] (16)
Therefore we get
\[
\hat{u}(y_0 + 1) \geq \frac{1}{1 + \theta}\hat{u}(y_0), \quad \hat{u}(y_0 - 1) \geq \frac{1}{1 + \theta}\hat{u}(y_0), \quad y_0 \in \mathbb{R}.
\] (17)
Then for any \( u \in S^+ \),
\[
\hat{u}(y) \leq (1 + \theta)^{|y| + 1}\hat{u}(0) \leq C\hat{u}(0)e^{\beta_y|y|}, \quad y \in (0, +\infty), \quad \beta_y = \ln(1 + \theta).
\]
Therefore we get the rest of (3) through (17). With a similar argue we get the estimate in \((-\infty, y^*)\). Therefore we get the rest of (5).

The proof of Claim (16):
With Lemma 2.7, there exist a constant \( C > 0 \), such that for \( y_0 \in \mathbb{R} \),
\[
u(x) \leq Cu(0', y_0) \leq C\hat{u}(y_0), \quad x \in C_{(y_0-2, y_0+2)}.
\]
Thus there exists a constant \( \theta > 0 \), such that
\[
\hat{u}(y_0 + 1) \leq (1 + \theta)\hat{u}(y_0), \quad \hat{u}(y_0 - 1) \leq (1 + \theta)\hat{u}(y_0), \quad y_0 \in \mathbb{R}.
\]

**Proof of Theorem 1.4.** For \( j \in \mathbb{N}^+ \), define:
\[
E_j := \{ k > 0 | w(x) \leq kw(x), x \in C_{(j, +\infty)} \}, \quad K_j := \inf E_j.
\]
\[
F_j := \{ l > 0 | w(x) \geq lw(x), x \in C_{(j, +\infty)} \}, \quad L_j := \sup F_j.
\]
With the same method in Proposition 3, there exist constants \( k_{j, 0}, l_{j, 0} > 0 \), such that \( k_{j, 0} \in E_j, \ l_{j, 0} \in F_j \). So we have \( 0 < L_j \leq K_j < +\infty \).

We claim that there exists a constant \( 0 < \delta < 1 \) which is only dependent of \( n \), \( \Lambda \), \( D \), such that
\[
K_j+1 - L_{j+1} \leq \delta(K_j - L_j). \tag{18}
\]
Notice that \( \{ K_j \}_{j=1}^{\infty} \) is a decreasing sequence and \( \{ L_j \}_{j=1}^{\infty} \) is an increasing sequence, so there is a constant \( K > 0 \), such that \( \lim_{n \to \infty} K_j = K = \lim_{n \to \infty} L_j \).
\[
K_j - L_j \leq \delta^{j-1}(K_1 - L_1) \leq Ce^{\alpha \ln \delta}.
\]
Therefore,
\[
|u(x) - Kw(x)| \leq (K_{|y|} - L_{|y|})w(x) \leq Ce^{\alpha |y|\ln \delta} \leq Ce^{-\alpha |y|}, \quad \alpha = -\ln \delta.
\]

The proof of Claim (18): we consider any point \( x_0 \in \mathcal{D} \), by definition,
\[
0 < L_j w(x_0', j + 1) \leq u(x_0', j + 1) \leq K_j w(x_0', j + 1), \quad j \in \mathbb{N}^+.
\]
therefore we have
\[
u(x_0', j + 1) \geq L_j w(x_0', j + 1) + \frac{1}{2}(K_j - L_j)w(x_0', j + 1). \quad \tag{19}
\]
or
\[
u(x_0', j + 1) \leq K_j w(x_0', j + 1) - \frac{1}{2}(K_j - L_j)w(x_0', j + 1) \quad \tag{20}
\]
If (19) is satisfied, by Lemma 2.6, there exists constant \( C > 0 \), such that
\[
u(x', j + 1) - L_j \nu(x', j + 1) \geq \frac{1}{2C}(K_j - L_j)\nu(x', j + 1), \quad x' \in \mathcal{D}.
\]
We use Lemma 2.3 and get
\[
L_j \nu(x) + \frac{1}{2C}(K_j - L_j)\nu(x) \leq \nu(x) \leq K_j \nu(x), \quad x \in C_{j+1, +\infty}.
\]
So
\[
K_{j+1} - L_{j+1} \leq (1 - \frac{1}{2C})(K_j - L_j).
\] (21)
If (20) is satisfied, with a similar argue we also get (21).

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Received April 2014; revised August 2015.
E-mail address: jbao.ms@gmail.com
E-mail address: wanglihe@sjtu.edu.cn
E-mail address: cqzhou@sjtu.edu.cn