A QUILLEN MODEL STRUCTURE APPROACH TO THE FINITISTIC DIMENSION CONJECTURES

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ABSTRACT. We explore the interlacing between model category structures attained to classes of modules of finite $X$-dimension, for certain classes of modules $X$. As an application we give a model structure approach to the Finitistic Dimension Conjectures and present a new conceptual framework in which these conjectures can be studied.

Let $\Lambda$ be a finite dimensional algebra over a field $k$ (or more generally, let $\Lambda$ be an artin ring). The big finitistic dimension of $\Lambda$, $\text{Findim}(\Lambda)$, is defined as the supremum of the projective dimensions of all modules having finite projective dimension. And the little finitistic dimension of $\Lambda$, $\text{findim}(\Lambda)$, is defined in a similar way by restricting to the subclass of all finitely generated modules of finite projective dimension. In 1960, Bass stated the so-called Finitistic Dimension Conjectures: (I) $\text{Findim}(\Lambda) = \text{findim}(\Lambda)$, and (II) $\text{findim}(\Lambda)$ is finite. The first conjecture was proved to be false by Huisgen-Zimmermann in [19], but the second one still remains open. It has been proved to be true, for instance, for finite-dimensional monomial algebras [15], for Artin algebras with vanishing cube radical [13], or Artin algebras with representation dimension bounded by 3 [21].

In [4] (see also [20]), Auslander proved that the finitistic dimension conjectures hold for Artin algebras in which the class $\mathcal{P}^{<\infty}$ of all finitely generated modules of finite projective dimension is contravariantly finite (equivalently, it is a precovering class in the sense of [10, 14]). In general, $\mathcal{P}^{<\infty}$ does not need to be contravariantly finite, even for Artin algebras satisfying the finitistic dimension conjectures. But, as Angeleri-Hügel and Trlifaj have noticed in [3], it cogenerated a cotorsion pair $(\mathcal{F}, \mathcal{C})$ in which the class $\mathcal{F}$ is precovering in $R\text{-Mod}$. By means of this idea, the authors are able to extend Auslander’s approach to arbitrary artinian rings and obtain a general criterium for an artinian ring to satisfy the finitistic dimension conjectures in terms of Tilting Theory (see [3]). This type of arguments has also been recently extended to more general homologies induced by arbitrary hereditary cotorsion pairs (see [1]).

On the other hand, Hovey has recently shown in [16] that there exists a quite strong relation between the construction of hereditary cotorsion pairs in module categories and the existence of model structures in the sense of Quillen in the

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associated categories of unbounded chain complexes. Moreover, under very general hypotheses, the cohomology functors defined from these model structures coincide with the absolute cohomology functors defined from the injective model structure (in the sense of [16, Example 3.2]). Recall that a model category is a category with three distinguished classes of morphisms (fibrations, cofibrations and weak equivalences) satisfying a certain number of axioms. We refer to [17] for a complete definition and main properties of model categories. One of the main advantages of these model categories is that they allow the construction of the derived category of a ring $R$ as the homotopy category induced by the model structure. This is the approach followed, for instance, in [8], [11], [12] and [13] in order to construct derived categories of Grothendieck categories in absence of enough projective objects (in particular, for the category of quasi-coherent sheaves over a scheme $X$).

The main goal of this paper is to give a new conceptual framework in which the above results concerning the finitistic dimension conjectures can be obtained. This is done by developing Quillen model structures associated to distinguished classes consisting of modules of finite projective dimension. In particular, we show that, given any ring $R$, the cotorsion pair cogenerated by the class of all modules of finite projective dimension induces a Quillen Model Structure in the category $\text{Ch}(R)$ of all unbounded complexes of left $R$-modules, in which the weak equivalences do coincide with the usual homology isomorphisms. In particular, by means of this model structure, the absolute cohomology functors $\text{Ext}^n(M, N)$ can be recovered in terms of certain resolutions of $M$ and $N$ attained to the class of modules of finite projective dimension. This new approach provides a very general framework in which the different approaches given in [1, 3, 4, 24] find their natural setting. We show that, essentially, they correspond to compute the finitistic dimensions of the considered Artin algebra in a different (and more convenient) homologically equivalent model structure. We would like to stress that our approach works for any ring $R$, whereas its main interest appears in the particular case of Artin algebras.

1. Homology relative to a hereditary cotorsion pair

We begin by fixing some notation and terminology. Given a set $X$, we are going to denote its cardinality by $|X|$; and by $\omega$, the first infinite ordinal. The cofinality of an ordinal number $\alpha$ will by denoted by $\text{cf}(\alpha)$. I.e., the least cardinal number which is cofinal in $\alpha$. Recall that an ordinal number is called regular when it coincides with its cofinality (and therefore, it is a cardinal). The symbol $\mid$ will mean restriction.

Along this paper, $R$ will denote a ring with identity and all modules will be left $R$-modules. We will denote by $R$-$\text{Mod}$ the category of all left $R$-modules and by $R$-$\text{mod}$ the subcategory of all modules possessing a projective resolution consisting of finitely generated modules. Morphisms will operate on the right and the composition of $f : A \to B$ and $g : B \to C$ will be denoted by $fg$. Fixed an infinite regular cardinal $\kappa$, a module $M$ is said to be $\kappa$-presented if it has a free presentation with less than $\kappa$ generators and relations. If $\mathcal{X}$ is a nonempty class of modules, $\mathcal{X}^{<\kappa}$ will denote the class of all $\kappa$-presented modules of $\mathcal{X}$. 

Let $\mathcal{X}$ be a nonempty class of modules. We shall consider the Ext-orthogonal classes

$$\mathcal{X}^\perp = \{ Y \in R\text{-Mod} : \text{Ext}^1(X,Y) = 0 \ \forall X \in \mathcal{X} \}$$

and

$$\perp \mathcal{X} = \{ Y \in R\text{-Mod} : \text{Ext}^1(Y,X) = 0 \ \forall X \in \mathcal{X} \}.$$ 

Recall that a module $M$ is called $\mathcal{X}$-filtered if there exists, for some regular cardinal $\kappa$, a continuous chain of submodules of $M$, $\{ M_\alpha : \alpha < \kappa \}$ satisfying that the modules $M_0$ and $\frac{M_{\alpha+1}}{M_\alpha}$ are isomorphic to modules in $\mathcal{X}$, for each $\alpha < \kappa$, and $M = \bigcup_{\alpha<\kappa} M_\alpha$.

An $\mathcal{X}$-precover of a module $M$ is a morphism $\varphi : X \to M$ such that $X \in \mathcal{X}$ and $\text{Hom}(X',\varphi)$ is an epimorphism for every $X' \in \mathcal{X}$. An $\mathcal{X}$-precover $\varphi : X \to M$ of $M$ is called special if it is an epimorphism and $\text{Ker} \varphi \in \mathcal{X}^\perp$ (see e.g., [14]). $\mathcal{X}$-preenvelopes and special $\mathcal{X}$-preenvelopes are defined dually.

A cotorsion pair in $R\text{-Mod}$ is a pair of classes of modules, $(\mathcal{F},\mathcal{C})$, such that $\mathcal{F} = \langle \mathcal{C} \rangle$ and $\mathcal{C} = \mathcal{F}^\perp$. The cotorsion pair is said to be cogenerated by a class of modules $\mathcal{X}$ if $\mathcal{X}^\perp = \mathcal{C}$. When this class $\mathcal{X}$ is a set, it is known that every module has a special $\mathcal{F}$-precover and a special $\mathcal{C}$-preenvelope (see e.g. [14 Theorem 3.2.1]). In this case the cotorsion pair is called complete.

Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair cogenerated by a set. Then there exists an infinite regular cardinal $\kappa$ such that $(\mathcal{F},\mathcal{C})$ is cogenerated by $\mathcal{F}^{<\kappa}$. Moreover, by Kaplan-sky’s Theorem for cotorsion pairs (see [23 Theorem 10] or [14 Theorem 4.2.11]), $\mathcal{F}$ consists of all $\mathcal{F}^{<\kappa}$-filtered modules.

A cotorsion pair $(\mathcal{F},\mathcal{C})$ is called hereditary if the class $\mathcal{F}$ is resolving. I. e., it is closed under kernels of surjections and contains all projective modules.

We now recall some well-known facts of the category $\text{Ch}(R)$ of unbounded chain complexes of modules. A complex of $R$-modules,

$$\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots,$$

will be denoted by $(X,d)$, or simply by $X$. And we will denote by $Z_nX = \text{Ker} d_n$, $K_nX = \text{Coker} d_n$, $B_nX = \text{Im} d_{n+1}$ and $H_nX = \frac{Z_nX}{B_nX}$, for every integer $n$. Given other complex $Y$, $\text{Hom}(X,Y)$ will denote the complex defined by

$$\text{Hom}(X,Y)_n = \prod_{k \in \mathbb{Z}} \text{Hom}_R(X_k,Y_{k+n})$$

and $((f)d^n_k)^n_k = f_kd^n_{k+n} - (-1)^n d^n_kf_{k-1}$ for any $n \in \mathbb{Z}$. The class of all exact complexes will be denoted by $\mathcal{E}$.

Let us fix a cotorsion pair $(\mathcal{F},\mathcal{C})$ in $R\text{-Mod}$. We will consider the following subclasses of $\text{Ch}(R)$ (see [13 Definition 3.3]):

1. The class of $\mathcal{F}$-complexes, $\tilde{\mathcal{F}} = \{ X \in \mathcal{E} : Z_nX \in \mathcal{F}, \ \forall n \in \mathbb{Z} \}$.
2. The class of $\mathcal{C}$-complexes, $\tilde{\mathcal{C}} = \{ X \in \mathcal{E} : Z_nX \in \mathcal{C}, \ \forall n \in \mathbb{Z} \}$.
3. The class of $dg$-$\mathcal{F}$ complexes,

$$dg\tilde{\mathcal{F}} = \{ X \in \text{Ch}(R) : X_n \in \mathcal{F} \ \forall n \in \mathbb{Z} \text{ and } \text{Hom}(X,C) \text{ is exact } \forall C \in \tilde{\mathcal{C}} \}.$$
4. The class of $dg$-$\mathcal{C}$ complexes,

$$dg\tilde{\mathcal{C}} = \{ X \in \text{Ch}(R) : X_n \in \mathcal{C} \ \forall n \in \mathbb{Z} \text{ and } \text{Hom}(F,X) \text{ is exact } \forall F \in \tilde{\mathcal{F}} \}.$$
Our next theorem shows that any hereditary cotorsion pair \((\mathcal{F}, \mathcal{C})\) in \(R\text{-Mod}\) cogenerated by a set gives rise to a model structure in \(\text{Ch}(R)\) in which the weak equivalences are the homology isomorphisms. This is essentially due to Hovey \cite{16} Theorem 2.2.

**Theorem 1.1.** Let \((\mathcal{F}, \mathcal{C})\) be a hereditary cotorsion pair in \(R\text{-Mod}\) cogenerated by a set. Then there exists a cofibrantly generated model structure on \(\text{Ch}(R)\) such that:

1. the weak equivalences are the homology isomorphisms;
2. the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are in \(\text{dg}\mathcal{F}\) (resp. \(\mathcal{F}\)); and
3. the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in \(\text{dg}\mathcal{C}\) (resp. \(\mathcal{C}\)).

**Proof.** We are going to apply \cite{16} Theorem 2.2]. So we must check the following conditions:

1. The pairs \((\mathcal{F}, \text{dg}\mathcal{C})\) and \((\text{dg}\mathcal{F}, \mathcal{C})\) are cotorsion pairs.
2. \(\text{dg}\mathcal{F} \cap \mathcal{E} = \mathcal{F}\) and \(\text{dg}\mathcal{C} \cap \mathcal{E} = \mathcal{C}\).
3. The cotorsion pairs \((\mathcal{F}, \text{dg}\mathcal{C})\) and \((\text{dg}\mathcal{F}, \mathcal{C})\) are complete.

By \cite{13} Corollary 3.8] we have induced cotorsion pairs \((\tilde{\mathcal{F}}, \text{dg}\tilde{\mathcal{C}})\) and \((\text{dg}\tilde{\mathcal{F}}, \tilde{\mathcal{C}})\). Now (2) follows from \cite{13} Corollary 3.12]. By \cite{12} Proposition 3.8 and \cite{10} Corollary 6.6] the pair \((\text{dg}\tilde{\mathcal{F}}, \tilde{\mathcal{C}})\) is complete. Finally to see that the pair \((\tilde{\mathcal{F}}, \text{dg}\tilde{\mathcal{C}})\) is complete we appeal to the same proof of \cite{11} Lemma 4.2] (notice that the arguments of the proof of \cite{11} Lemma 4.2] are categorical, so they can be adapted to our setting). □

**Remark 1.2.** Let us note that we cannot use \cite{12} Theorem 4.12] to get the desired model structure since we are not assuming that the class \(\mathcal{F}\) is closed under direct limits. We cannot assume this closure under direct limits since this condition is rarely satisfied in the applications we are interested in.

We are going to denote by \(\text{Ch}(R)_{\mathcal{M}_C}\) the category \(\text{Ch}(R)\) endowed with the above model structure induced by the class \(\mathcal{F}\). Let us recall that, if \(\text{Ho}(\text{Ch}(R))\) is the homotopy category associated to \(\text{Ch}(R)\), we can define \(\text{Ext}_R^n(M, N)\) as

\[
\text{Hom}_{\text{Ho(Ch}(R))}(S(M), S^n(N)) = \text{Hom}_{\text{Ch}(R)}(Q_{S(M)}, P_{S^n(N)})/\sim_h,
\]

(see \cite{17} Theorem 1.2.10] where \(Q_{S(M)}\) is a cofibrant replacement of \(S(M)\) (i.e., the complex with \(M\) in the 0’th position and 0 elsewhere), \(P_{S^n(N)}\) is a fibrant replacement of \(S^n(N)\) (the complex with \(N\) in the \(n\)’th position and 0 elsewhere) and \(\sim_h\) denotes the chain homotopy (see \cite{17}).

Given a module \(M\), the standard way of constructing a cofibrant replacement \(Q_M\) of \(S(M)\) (that is, a trivial fibration \(Q_M \to S(M)\), where \(Q_M\) is cofibrant) from a hereditary cotorsion pair \((\mathcal{F}, \mathcal{C})\) as in Theorem \cite{14} is the following: we choose a special \(\mathcal{F}\)-precover \(d_0 : F_0 \to M \to 0\) of \(M\). Then, a special \(\mathcal{F}\)-precover \(d_1 : F_1 \to \text{Ker} d_0 \to 0\) of \(\text{Ker} d_0\) and so on. Proceeding in this way, we get an exact complex

\[
\cdots \xrightarrow{d} F_1 \xrightarrow{d} F_0 \xrightarrow{d} M \to 0,
\]

in which \(F_i \in \mathcal{F}\) and \(\text{Ker} d_i \in \mathcal{F}^\perp = \mathcal{C}\) for every \(i \in \mathbb{N}\). Then, if we denote by \(F^\ast\) (or by \((F^\ast, d)\)) the corresponding deleted complex (which is unique up to chain homotopy equivalence),
we get an epimorphism in $\text{Ch}(R)$, $\tilde{d} : F_\bullet \to S(M)$, with $	ext{Ker}(\tilde{d}) \in \mathcal{C}$ (and $F_\bullet \in \text{id}_{\text{Ch}(R)}$) and therefore, a cofibrant replacement of $S(M)$. Dually, we can get a fibrant replacement of $S(N)$, $C_\bullet$ (unique up to chain homotopy equivalence), from the fact that every module admits a special $C$-preenvelope. Notice that both fibrant and cofibrant replacements are unique in $\text{Ho}(\text{Ch}(R))$ (because they provide unique-up-to-homotopy resolutions and coresolutions, respectively). Then, as

$$\text{Hom}_{\text{Ch}(R)}(Q_{S(M)}, P_{S^n(N)}) \sim h_{-n}\text{Hom}(F_\bullet, C_\bullet),$$

we may identify the Ext groups $\text{Ext}^n_{\mathcal{C}}(M, N)$ with the homology groups of the complex $\text{Hom}(F_\bullet, C_\bullet)$. Using this identification, we can easily deduce the following result.

**Proposition 1.3.** Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair cogenerated by a set, and $M, N$, two modules.

1. If $M \in \mathcal{F}$ and $C_\bullet$ is a fibrant replacement of $S(N)$, then:
   a. $\text{Ext}^n(M, N) = H_{-n}\text{Hom}(M, C_\bullet)$.
   b. $\text{Ext}^n(M, K_{-k}C_\bullet) = \text{Ext}^{n+k+1}(M, N)$ for every $k \leq n$.
2. If $N \in \mathcal{C}$ and $F_\bullet$ is a cofibrant replacement of $S(N)$, then:
   a. $\text{Ext}^n(M, N) = H_{-n}\text{Hom}(F_\bullet, N)$.
   b. $\text{Ext}^n(Z_kF_\bullet, N) = \text{Ext}^{n+k+1}(M, N)$ for every $k \geq n$.
3. If $M \in \mathcal{F}$ and $N \in \mathcal{C}$, then $\text{Ext}^n(M, N) = 0$ for every $n \geq 1$.

The above results suggest the definition of some homological invariants with respect to the classes $\mathcal{F}$ and $\mathcal{C}$. Let $X$ be a complex and $a, b \in \mathbb{Z}$ with $a < b$. We will say that the amplitude of $X$ belongs to $[a, b]$ if $X_n = 0$ for every $n < a$ and every $n > b$.

**Definition 1.4.** Let $(\mathcal{F}, \mathcal{C})$ be a hereditary cotorsion pair cogenerated by a set, and $M, N$, two modules. Then:

1. Given an $n \in \mathbb{N}$, we will say that the $\mathcal{F}$-cofibration dimension of $M$ is bounded by $n$ if there exists a cofibrant replacement $F_\bullet$ of $S(M)$ in $\text{Ch}(R)_{\mathcal{C}}$ with amplitude in $[0, n]$. We will define the $\mathcal{F}$-cofibrant dimension of $M$, denoted by $\text{cofdim}_F(M)$, as the minimal natural number $n$ satisfying this property and $\infty$ otherwise.
2. Given an $n \in \mathbb{N}$, we will say that the projective dimension of $M$ relative to $\mathcal{C}$ is bounded by $n$ if $\text{Ext}^m(M, C) = 0$ for every $m > n$ and $C \in \mathcal{C}$. And we will define the projective dimension of $M$ relative to $\mathcal{C}$, $\text{pd}_\mathcal{C}(M)$, as the minimal $n \in \mathbb{N}$ satisfying this property and $\infty$ otherwise.

Analogously we may define the injective dimension of $M$ relative to $\mathcal{F}$, $\text{id}_\mathcal{F}(M)$, and the $\mathcal{C}$-fibration dimension of $M$, $\text{fibdim}_\mathcal{C}(M)$.

Let $\mathcal{A}$ be any Grothendieck category. If we consider the injective model structure in $\text{Ch}(\mathcal{A})$ introduced by Joyal in [22] and Beke in [6] (see also [17, Theorem 2.3.13]), it is easy to check that the fibrant dimension of an object $M$ of $\mathcal{A}$ is precisely the usual injective dimension of $M$ (note that, in this case, a fibrant replacement of $M$ is an injective coresolution of $M$). Analogously, Hovey has proved in [17, Chapter 4] that there exists a projective monoidal model structure in $\text{Ch}(R)$. And
with respect to this model structure, the cofibrant dimension of a module \( M \) is precisely the usual projective dimension of \( M \). We can obtain these two model structures in \( \text{Ch}(R) \) from Theorem 1.1, by considering the hereditary cotorsion pairs \((\text{Proj}, R\text{-Mod})\) and \((R\text{-Mod}, \text{Inj})\), which are obviously cogenerated by sets. We are going to extend in Proposition 1.6 the usual properties of the classical injective and projective dimension to the dimensions induced from any hereditary and complete cotorsion pair. In order to prove it, we will need the following generalized version of Schanuel’s lemma (see [10, Lemma 8.6.3] for a proof):

**Lemma 1.5.** Let \( \mathcal{F} \) be a class of modules, \( M \) a module, \( F, G \in \mathcal{F} \), \( \varphi : F \to M \) a surjective \( \mathcal{F} \)-precover and \( f : G \to M \), an epimorphism. Then there exists a short exact sequence \( 0 \to \ker f \to \ker \varphi \oplus G \to F \to 0 \).

**Proposition 1.6.** Let \((\mathcal{F}, \mathcal{C})\) be an hereditary cotorsion pair cogenerated by a set, \( M \), a module and \( n \), a natural number. The following assertions are equivalent:

1. \( \text{cofdim}_\mathcal{F}(M) \leq n \).
2. \( \text{pd}_\mathcal{C}(M) \leq n \).
3. Each cofibrant replacement \( Q \) of \( S(M) \) satisfies that \( Z_{n-1}Q \in \mathcal{F} \).
4. There exists a short exact sequence \( 0 \to G_n \xrightarrow{d_n} \cdots \xrightarrow{d_0} G_0 \xrightarrow{\delta} M \xrightarrow{0} \)

with \( G_i \in \mathcal{F} \) for every \( i \).

**Proof.** (1) \( \iff \) (2) follows from Proposition 1.3. (1) \( \iff \) (3) is clear since we can compute Ext functors by using any cofibrant replacement of \( S(M) \). (3) \( \Rightarrow \) (4) is trivial. In order to finish the proof we need to show that (4) \( \Rightarrow \) (3). Let \( Q \) be a cofibrant replacement of \( S(M) \). We can assume that \( Q \) is bounded below, and thus it is of the form \((F_* , \varphi)\). We will induct on \( n \in \mathbb{N} \). If \( n = 0 \), then \( M \in \mathcal{F} \) and the result is clear. Suppose now that the result is true for any module \( M \) with \( \text{cofdim}_\mathcal{F}(M) \leq n \) and let us prove it for \( \text{cofdim}_\mathcal{F}(M) \leq n + 1 \). Then there exists an exact sequence with \( n + 2 \) terms, \( 0 \to G_{n+1} \xrightarrow{\delta_{n+1}} G_n \to \cdots \to G_1 \xrightarrow{\delta_1} G_0 \xrightarrow{\delta_0} M \to 0 \) with \( G_i \in \mathcal{F} \) for every \( i \). Let us denote by \( \psi_1 : F_1 \oplus G_0 \to \ker \varphi_0 \oplus G_0 \) the direct sum \( \psi_1 \oplus \mathcal{I}_{G_0} \), and let us note that it is a special \( \mathcal{F} \)-precover. Then, if we construct the pullback of the short exact sequence obtained in Lemma 1.5 for the epimorphism \( \delta_0 \) and the \( \mathcal{F} \)-precover \( \varphi_0 \), we get the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & \to & P & \xrightarrow{\pi} & F_1 \oplus G_0 & \xrightarrow{\psi_1 \beta} & F_0 & \to 0 \\
| & & \downarrow \psi_1 & & \downarrow \psi_1 & & \\
0 & \to & \ker \delta_0 & \xrightarrow{\alpha} & \ker \varphi_0 \oplus G_0 & \xrightarrow{\beta} & F_0 & \to 0 \\
\end{array}
\]

Now note that, since \( \mathcal{F} \) is resolving, \( P \in \mathcal{F} \). And, as the bottom square is a pullback, \( \psi_1 \) is a special \( \mathcal{F} \)-precover. Therefore, \( \ker \delta_0 \) has a cofibrant replacement
Let \( F, C \) be a hereditary cotorsion pair cogenerated by a set and \( M \), a module. Then \( \mathrm{cofdim}_F(M) = \mathrm{pd}_C(M) \) and \( \mathrm{fibdim}_C(M) = \mathrm{id}_F(M) \).

2. Model structures from finite relative dimensional modules

Throughout this section, we will fix a hereditary cotorsion pair \((F, C)\) cogenerated by a set. Let \( D \) be the class of all \( k \)-th cosyzygies of objects of \( C \) \((k \geq n)\). Then a module \( L \in D\) if and only if \( \text{Ext}^m(L, C) = 0 \), for each \( C \in C \) and each \( m > n \). And this happens if and only if \( L \in P_n \). By [7, Theorem 1.2] it follows that \( P_n \) is closed under \( P_n\)-filtrations.

The main goal of this section will be to show that the above cotorsion pairs induce model structures in \( \text{Ch}(R) \). By Theorem 1.1 we only need to prove that they are hereditary and cogenerated by a set. Let us note that the cotorsion pairs \((A^{<\infty}, B^{<\infty})\) and \((A_n^{<\infty}, B_n^{<\infty})\) (for each \( n \in \mathbb{N} \)) are cogenerated by a set by definition. The next result shows that the same holds for \((A, B)\) and \((A_n, B_n)\) \((n \in \mathbb{N})\).

**Theorem 2.2.** Let \((F, C)\) be a hereditary cotorsion pair cogenerated by a set. Then \((A, B)\) and \((A_n, B_n)\) are also cogenerated by a set, for each \( n \in \mathbb{N} \).

**Proof.** It suffices to show that \((A_n, B_n)\) is cogenerated by a set for each \( n \in \mathbb{N} \), since \( A = \bigcup_{n \in \mathbb{N}} A_n \). Let us fix an \( n \in \mathbb{N} \). Since \((F, C)\) is cogenerated by a set, there exists by [13, Lema 4.2.10] an infinite regular cardinal \( \kappa \) such that each module in \( F \) is \( F^{<\kappa}\)-filtered. We are going to prove that each module in \( P_n \) is \( P_n^{<\kappa}\)-filtered. Then, if \( S \) is a set of representatives of the isomorphism classes of modules in \( P_n^{<\kappa} \), we get from [7, Theorem 1.2] that the cotorsion pair \((A_n, B_n)\) is cogenerated by \( S \).

Let us fix an \( A \in P_n \) and an exact sequence,

\[
0 \to F_n \xrightarrow{d_n} F_{n-1} \to \ldots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} A \to 0
\]

with \( F_i \in F \), \( 1 \leq i \leq n \). We know that each \( F_i \) is \( F^{<\kappa}\)-filtered. Let us denote by \( F_i \) a family of submodules of \( F_i \) given by the Hill Lemma (see [14, Theorem 4.2.6]).
We will follow the notation used in [14, Theorem 4.2.6]. In particular, we will refer to the properties satisfied by this family as properties (H1), (H2), (H3) and (H4).

Let \( \{ x_\alpha : \alpha < \mu \} \) be a generating set of \( A \) with \( x_0 = 0 \). For each \( \alpha < \mu \), we are going to construct an exact sequence \( S_\alpha \),

\[
0 \to F_\alpha^0 \xrightarrow{d_\alpha^0} F_\alpha^1 \xrightarrow{d_\alpha^1} \cdots \xrightarrow{d_\alpha^n} F_\alpha^0 \xrightarrow{d_\alpha} A_\alpha \to 0
\]

such that \( x_\alpha \in A_\alpha \), \( d_\alpha^i \) is the restriction of \( d_i \), \( \{ F_\alpha^i : \alpha < \mu \} \) is an \( \mathcal{F}^< \)-filtration of \( \bigcup_{\alpha<\mu} F_\alpha^i \) for every \( i \in \{0, \ldots, n\} \), and \( \{ A_\alpha : \alpha < \mu \} \) is an \( A_\alpha^< \)-filtration of \( A \).

We will make this construction by transfinite induction on \( \alpha < \mu \). For \( \alpha = 0 \), let us fix \( A_0 = 0 \). Assume now that \( 0 \leq \alpha < \mu \) and that \( S_\alpha \) has been already constructed. We are going to find, for each \( i \in \{0, \ldots, n\} \), two chains of submodules of \( F_i \), \( \{ X^\alpha_{i,m} : m \in \mathbb{N} \} \) and \( \{ Y^\alpha_{i+1} : m \in \mathbb{N} \} \), satisfying the following properties:

a) \( A_\alpha \cup \{ x_{\alpha+1} \} \leq (X^\alpha_{0,0})_d_0 \) and \( F_\alpha^0 \leq X^\alpha_{i,0} \) for each \( i \in \{0, \ldots, n\} \).

b) \( X^\alpha_{i,m} \leq F_\alpha^i \leq X^\alpha_{i+1} \) for each \( m \in \mathbb{N} \) and each \( i \in \{0, \ldots, n\} \).

c) \( \text{Ker} \ (d_i \mid X^\alpha_{i,m+1}) \leq (X^\alpha_{i,m+1})_d_{i+1} \) for each \( i \in \{0, \ldots, n-1\} \) and each \( m \in \mathbb{N} \).

d) \( (Y^\alpha_{i+1})_d_i \leq Y^\alpha_{i+1} \) for each \( i \in \{0, \ldots, n-1\} \) and each \( m \in \mathbb{N} \).

e) The quotients \( \frac{X^\alpha_{i,m}}{F^i_{i,m}} \) and \( \frac{Y^\alpha_{i+1}}{F^i_{i+1}} \) belong to \( \mathcal{F}^< \) for each \( i \in \{0, \ldots, n\} \) and each \( m \in \mathbb{N} \).

We will proceed by induction on \( m \in \mathbb{N} \). If \( m = 0 \), take \( y_{\alpha+1} \in F_0^\alpha \) with \( (y_{\alpha+1})_d_0 = x_{\alpha+1} \). Since \( F_0^\alpha \in \mathcal{F}_0 \) there exists, by (H4), an \( X^\alpha_{0,0} \in \mathcal{F}_0 \) with \( F_0^\alpha \cup \{ y_{\alpha+1} \} \leq X^\alpha_{0,0} \) and \( \left| \frac{X^\alpha_{0,0}}{F_0^\alpha} \right| < \kappa \). Note that, by (H3), this module is \( \mathcal{F} \)-filtered and therefore, it belongs to \( \mathcal{F} \) by hypothesis.

In order to construct \( X^\alpha_{i+1} \) note that, as \( \left| \frac{X^\alpha_{0,0}}{F_0^\alpha} \right| < \kappa \), we deduce from the Ker-Coker Lemma that \( \left| \frac{\text{Ker} \ d_0 \mid X^\alpha_{0,0}}{\text{Ker} \ d_0 \mid F_0^\alpha} \right| < \kappa \). Thus, there exists a submodule \( K \leq F_1 \) with cardinality strictly smaller than \( \kappa \) such that \( \text{Ker} \ d_0 \mid X^\alpha_{0,0} \leq (F_1 \cup K) d_1 \). Using again (H4), we may find an \( X^\alpha_{1,0} \in \mathcal{F}_1 \) with \( F_1 \cup K \leq X^\alpha_{1,0} \) and \( \left| \frac{X^\alpha_{1,0}}{F_1} \right| < \kappa \). Moreover, this module is \( \mathcal{F} \)-filtered by (H3) and therefore, it belongs to \( \mathcal{F} \). Repeating a similar argument we may find \( X^\alpha_{2,0}, \ldots, X^\alpha_{n,0} \) satisfying the desired properties.

Now we proceed to construct the modules \( Y \)'s. Let us take \( Y^\alpha_{n,0} = X^\alpha_{n,0} \). Using that \( \frac{Y^\alpha_{n,0}}{X^\alpha_{n,0}} \) has cardinality strictly smaller than \( \kappa \), it is easy to check that \( \frac{(y^\alpha_{n,0})_d_{n-1} + X^\alpha_{n-1,0}}{X^\alpha_{n-1,0}} \) also has cardinality strictly smaller than \( \kappa \). So, by (H4), there exists an \( Y^\alpha_{n-1,0} \in \mathcal{F}_{n-1} \) with \( (y^\alpha_{n,0})_d_{n-1} + X^\alpha_{n-1,0} \leq Y^\alpha_{n-1,0} \) and \( \left| \frac{Y^\alpha_{n-1,0}}{X^\alpha_{n-1,0}} \right| < \kappa \). Note that, in addition, this quotient belongs to \( \mathcal{F} \) since it is \( \mathcal{F} \)-filtered by (H3). Repeating this procedure, we get \( Y^\alpha_{n-2,0}, \ldots, Y^\alpha_{0,0} \). This finishes the case \( m = 0 \).
Suppose now that we have constructed the chains for some \( m \in \mathbb{N} \) and take \( X^\alpha_{0,m+1} = Y^\alpha_{0,m} \). Since \( \frac{X^\alpha_{0,m+1}}{F_0} < \kappa \), then \( \frac{\text{Ker } d_0}{X^\alpha_{0,m+1} + (Y^\alpha_{1,m+1} + 1)} < \kappa \) and, by (H4), there exists \( X^\alpha_{1,m+1} \in \mathcal{F}_1 \) with \( \text{Ker } d_0 \mid X^\alpha_{0,m+1} + Y^\alpha_{1,m+1} \leq X^\alpha_{1,m+1} \) and \( \frac{X^\alpha_{1,m+1}}{Y^\alpha_{1,m+1}} < \kappa \). Note that, in addition, this module belongs to \( \mathcal{F} \) since it is \( \mathcal{F} \)-filtered by (H3). Repeating this process we get \( X^\alpha_{2,m+1}, \ldots, X^\alpha_{n,m+1} \).

In order to construct the \( Y' \)'s, set \( Y^\alpha_{n,m+1} = X^\alpha_{n,m+1} \). Using that \( \frac{Y^\alpha_{n,m+1}}{F_0} < \kappa \) we conclude easily that \( \frac{(Y^\alpha_{n,m+1})}{d_0 + X^\alpha_{n-1,m+1}} < \kappa \) and, applying (H4) again, there exists an \( Y^\alpha_{n-1,m+1} \in \mathcal{F}_{n-1} \) with \( (Y^\alpha_{n,m+1}) d_0 + X^\alpha_{n-1,m+1} \leq Y^\alpha_{n-1,m+1} \) and \( \frac{Y^\alpha_{n-1,m+1}}{X^\alpha_{n-1,m+1}} < \kappa \). Note that, in addition, this module belongs to \( \mathcal{F} \) as it is \( \mathcal{F} \)-filtered.

Proceeding in this way we construct \( Y^\alpha_{n-2,m+1}, \ldots, Y^\alpha_{0,m+1} \).

Set \( F^\alpha_{+1} = \bigcup_{m \in \mathbb{N}} X^\alpha_{0,m+1} = \bigcup_{m \in \mathbb{N}} Y^\alpha_{0,m+1} \) for each \( m \in \mathbb{N} \) and note that the sequence \( S^\alpha_{+1} \) is exact by c) and d). Since \( S^\alpha \) is a subcomplex of \( S^\alpha_{+1} \), the corresponding quotient complex is an exact sequence with each term, except the first one, in \( \mathcal{F}^{< \kappa} \) by d). This means that \( \frac{A^\alpha_{+1}}{A^\alpha} \in \mathcal{P}^{< \kappa} \). This finishes the case \( \alpha + 1 \).

Finally, suppose that \( \alpha \) is a limit ordinal and define \( S^\alpha \) taking \( F^\alpha_i = \bigcup_{\gamma < \alpha} F^\alpha_i \) and \( A^\alpha = \bigcup_{\gamma < \alpha} A^\gamma \). It is easy to see that \( S^\alpha \) satisfy the desired properties. \( \square \)

Corollary 2.3. Let \( (\mathcal{F}, \mathcal{C}) \) be a hereditary cotorsion pair cogenerated by a set. Then \( \mathcal{A}_n = \mathcal{P}_n \) for each \( n \in \mathbb{N} \).

Proof. As the cotorsion pair \( (\mathcal{A}_n, \mathcal{B}_n) \) is cogenerated by a set \( \mathcal{S} \subseteq \mathcal{P}_n \), the result follows from Lemma 2.1 and [14] Corollary 3.2.4] (notice that \( \mathcal{P}_n \) is closed under direct summands). \( \square \)

Let us now check that the above cotorsion pairs are hereditary. In order to prove it, we will need to use the following two propositions. The proof of the first one is straightforward.

Proposition 2.4. Let \( (\mathcal{F}, \mathcal{C}) \) be a cotorsion pair. The following assertions are equivalent:

1. The cotorsion pair is hereditary.
2. Every projective presentation \( \varphi : P \rightarrow F \) of any element \( F \in \mathcal{F} \) verifies that \( \text{Ker } \varphi \in \mathcal{F} \).
3. For every \( F \in \mathcal{F} \), there exists a projective presentation \( \varphi : P \rightarrow F \) with \( \text{Ker } \varphi \in \mathcal{F} \).

Proposition 2.5. Let \( \mathcal{X} \) be a class of modules such that for each \( X \in \mathcal{X} \), there exists a free presentation \( \varphi : R^{(\ell)} \rightarrow X \) with \( \text{Ker } \varphi \in \mathcal{X} \). If \( A \) is a direct summand of an \( \mathcal{X} \)-filtered module, then there exists a free presentation \( \psi : R^{(\ell)} \rightarrow A \) in which \( \text{Ker } \psi \) is also a direct summand of an \( \mathcal{X} \)-filtered module.
Proof. Let us first assume that $A$ is an $\mathcal{X}$-filtered module and let $\{A_\alpha : \alpha \leq \sigma\}$ be an $\mathcal{X}$-filtration of $A$. We are going to construct, by transfinite induction on $\alpha$, commutative diagrams

$$
\begin{array}{ccccccccc}
0 & \rightarrow & K_\gamma & \rightarrow & \bigoplus_{\delta \leq \gamma} R^{(I_\delta)} & \rightarrow & A_\gamma & \rightarrow & 0 \\
\downarrow k_{\gamma \alpha} & & \downarrow j_{\gamma \alpha} & & \downarrow i_{\gamma \alpha} & & & & \rightarrow \rightarrow \rightarrow \\
0 & \rightarrow & K_\alpha & \rightarrow & \bigoplus_{\delta \leq \alpha} R^{(I_\delta)} & \rightarrow & A_\alpha & \rightarrow & 0 \\
\end{array}
$$

in which $I_\gamma$ is a set, $j_{\gamma \alpha}$ and $i_{\gamma \alpha}$ are the inclusions and $k_{\gamma \alpha}$ is a monomorphism for each $\gamma < \alpha < \sigma$, satisfying the following two conditions:

1. If $\alpha < \sigma$ is a limit ordinal, then the corresponding row is the directed colimit of the previous ones.
2. If $\alpha = \gamma + 1$ is a successor ordinal, then $\text{Coker } k_{\gamma, \gamma + 1} \in \mathcal{X}$.

Note that the sequence constructed in step $\sigma$ will produce the desired presentation, since $K_\sigma$ is an $\mathcal{X}$-filtered module by construction.

The case $\alpha = 0$ is trivial. Suppose now that $\alpha = \gamma + 1 \leq \sigma$ is a successor ordinal and that we have constructed the corresponding commutative diagrams for $\gamma$. Choose a presentation $\overline{f}_\alpha : R^{(I_\alpha)} \rightarrow \bigoplus_{\delta \leq \alpha} R^{(I_\delta)}$ with $\overline{f}_\alpha \in \mathcal{X}$ and, using the projectivity of $R^{(I_\alpha)}$, construct $h_\alpha : R^{(I_\alpha)} \rightarrow A_\alpha$ with $h_\alpha \pi_\alpha = \overline{f}_\alpha$, where $\pi_\alpha$ is the canonical projection. Let $f_\alpha : \bigoplus_{\delta \leq \alpha} R^{(I_\delta)} \rightarrow A_\alpha$ be the map induced by $f_\gamma$ and $h_\alpha$, $K_\alpha = \ker f_\alpha$. Call $\overline{K}_\alpha = \ker \overline{f}_\alpha$ and let $g_\alpha : K_\alpha \rightarrow \bigoplus_{\delta \leq \alpha} R^{(I_\delta)}$ and $\overline{g}_\alpha : \overline{K}_\alpha \rightarrow R^{(I_\alpha)}$ be the inclusions. Then using the Ker-Coker Lemma, we get the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \rightarrow \\
0 & \rightarrow & K_\gamma & \rightarrow & \bigoplus_{\delta \leq \gamma} R^{(I_\delta)} & \rightarrow & A_\gamma & \rightarrow & 0 \\
\downarrow k_{\gamma \alpha} & & \downarrow j_{\gamma \alpha} & & \downarrow i_{\gamma \alpha} & & & & \rightarrow \rightarrow \rightarrow \\
0 & \rightarrow & K_\alpha & \rightarrow & \bigoplus_{\delta \leq \alpha} R^{(I_\delta)} & \rightarrow & A_\alpha & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \rightarrow \rightarrow \rightarrow \\
0 & \rightarrow & \overline{K}_\alpha & \rightarrow & R^{(I_\alpha)} & \rightarrow & \overline{A}_\alpha & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \rightarrow \rightarrow \rightarrow \\
0 & 0 & 0 & 0 & & & & & \\
\end{array}
$$

Set $k_{\delta \alpha} = k_{\delta \gamma} k_{\gamma \alpha}$, for each $\delta < \alpha$, and note that $\text{Coker } k_{\gamma \alpha} \in \mathcal{X}$, since $\ker \overline{f}_\alpha \in \mathcal{X}$.

Finally, if $\alpha \leq \sigma$ is a limit ordinal, the directed colimit of the exact sequences for $\gamma < \alpha$ gives the desired sequence.

Therefore, the result is true for $\mathcal{X}$-filtered modules. Let us now assume that $B$ is a direct summand of an $\mathcal{X}$-filtered module $A$. Then we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & R^{(I)} & \rightarrow & A & \rightarrow & 0 \\
\downarrow k & & \downarrow 1 & & \downarrow p & & & & \rightarrow \rightarrow \rightarrow \\
0 & \rightarrow & K' & \rightarrow & R^{(I)} & \rightarrow & B & \rightarrow & 0 \\
\end{array}
$$
in which \( K \) is \( X \)-filtered and \( p \) is a splitting epimorphism. Therefore, \( k \) is also a splitting epimorphism and thus \( K' \) is a direct summand of an \( X \)-filtered module. In particular, it is also \( X' \)-filtered. □

We can now prove the following result.

**Theorem 2.6.** Let \( S \) be a set of modules containing the regular module \( R \) and such that every module \( S \in S \) has a free presentation \( \varphi : R^{(I)} \to S \) with \( \text{Ker} \varphi \in S \). Then the cotorsion pair cogenerated by \( S \) is hereditary.

**Proof.** Follows from [25, Theorem 2.2] and propositions 2.5 and 2.4. □

**Corollary 2.7.** The cotorsion pairs \((A, B), (A^{<\infty}, B^{<\infty}), (A_n, B_n)\) and \((A^{<\infty}_n, B^{<\infty}_n)\), for each \( n \in \mathbb{N} \), are hereditary.

**Proof.** Let \( n \in \mathbb{N} \). By Theorem 2.2, the cotorsion pair \((A_n, B_n)\) is cogenerated by the set \( \{ S \in P_n : |S| < \kappa \} \), for some infinite regular cardinal \( \kappa \). This set satisfies the hypothesis of the above theorem. Analogously, for each \( S \in P_n^{<\infty} \), we have that \( |S| \leq |R| \). So the cotorsion pair \((A_n^{<\infty}, B_n^{<\infty})\) is cogenerated by the set \( \{ S \in P_n^{<\infty} : |S| < |R|^+ \} \) and this set also satisfies the hypothesis of the previous theorem. □

We get now the desired model structures on \( \text{Ch}(R) \).

**Corollary 2.8.**

1. The cotorsion pair \((A, B)\) (resp. \((A_n, B_n)\), for each \( n \in \mathbb{N} \)) induces a model category structure on \( \text{Ch}(R) \) in which the weak equivalences are the homology isomorphisms, the cofibrations are the monomorphisms whose cokernels are in \( \text{dg}A \) (resp. \( \text{dg}A_n \)), and the fibrations are the epimorphisms whose kernels are in \( \text{dg}B \) (resp. \( \text{dg}B_n \)).

2. The cotorsion pair \((A^{<\infty}, B^{<\infty})\) (resp. \((A_n^{<\infty}, B_n^{<\infty})\), for each \( n \in \mathbb{N} \)) induces a model category structure on \( \text{Ch}(R) \) in which the weak equivalences are the homology isomorphisms, the cofibrations are the monomorphisms whose cokernels are in \( \text{dg}A^{<\infty} \) (resp. \( \text{dg}A_n^{<\infty} \)), and the fibrations are the epimorphisms whose kernels are in \( \text{dg}B^{<\infty} \) (resp. \( \text{dg}B_n^{<\infty} \)).

**Proof.** Apply Theorem 1.1. □

### 3. Applications to the Finitistic Dimension

We finish this paper by applying our results to the calculus of the finitistic dimensions of rings and algebras. From now on, we will assume that \( \mathcal{F} \) is the class of all projective modules and consequently, \( \mathcal{C} = R\text{-Mod} \). Therefore, \( \mathcal{P} \) (resp. \( \mathcal{P}^{<\infty} \)) is the class of all modules (resp. all modules in \( R\text{-mod} \)) with finite projective dimension. Recall that the left big finitistic dimension of \( R \) is defined as

\[
\text{Findim}(R) = \sup \{pd(P) : P \in \mathcal{P} \}
\]

and the left little finitistic dimension of \( R \) is

\[
\text{findim}(R) = \sup \{pd(P) : P \in \mathcal{P}^{<\infty} \}.
\]
Our goal is to characterize in Theorem 3.2 when the big finitistic dimension is finite. In order to do it, we will use the following result which is an immediate consequence of the arguments given in [9, Lemma 2.2 and Proposition 2.3].

**Proposition 3.1.** Let $\kappa$ be an infinite regular cardinal and $M$, a module such that both $M$ and each left ideal of $R$ are $<\kappa$-presented. Then $\operatorname{Ext}^n (M, \lim_{\alpha<\kappa} M_\alpha) \cong \lim_{\alpha<\kappa} \operatorname{Ext}^n (M, M_\alpha)$, for every $n \in \mathbb{N}$ and every directed system of morphisms $(M_\alpha, f_{\alpha\beta})_{\alpha<\beta<\kappa}$.

**Theorem 3.2.** Let $n$ be a natural number. The following assertions are equivalent:

1. $\operatorname{Findim} (R) \leq n$.
2. $\operatorname{pd}(A) \leq n$ for every $A \in \mathcal{A}$.
3. $\operatorname{fibdim}_R (M) \leq n$ for every $M \in R\text{-Mod}$.
4. $\operatorname{fibdim}_R (R^{(\mathcal{I})}) \leq n$.

**Proof.** 1) $\Rightarrow$ 2) This follows from the facts that modules in $\mathcal{A}$ are direct summands of $\mathcal{P}$-filtered modules (see [25, Theorem 2.2]), the class $\mathcal{P} = \mathcal{P}_n$ by hypothesis, and $\mathcal{P}_n$ is closed under direct summands and $\mathcal{P}_n$-filtrations (see [4, Proposition 3]).

2) $\Leftrightarrow$ 3) Let $M$ be any module and take a fibrant replacement of $S(M)$ in $\operatorname{Ch}(R)_{\mathcal{M}^\omega}$, $0 \to M \to B_0 \to B_1 \to \cdots$. By Proposition 1.3 we get that $\operatorname{Ext}^1 (A, B_{-n+1}) = \operatorname{Ext}^{n+1} (A, M) = 0$, for every $A \in \mathcal{A}$ (since $\operatorname{pd}(A) \leq n$). This means that $B_{-n+1} \in \mathcal{B}$ and the result follows from the dual version of Proposition 1.6.

3) $\Rightarrow$ 4) This is clear.

4) $\Rightarrow$ 1) By Proposition 1.3 $\operatorname{Ext}^m (P, R^{(\mathcal{I})}) = 0$ for every $P \in \mathcal{P}$ and every $m > n$. We deduce that $\operatorname{Ext}^m (P, R^{(J)}) = 0$ for each $P \in \mathcal{P}$, each set $I$ and each $m > n$. Let us fix an $m > n$ and a $P \in \mathcal{P}$ and let us denote by $\kappa = |R^{(\mathcal{I})}|$. Note that, by the proof of Theorem 2.2 we can assume that $P$ is $<\kappa$-presented.

We are going to induct on the cardinality of $I$. If $|I| \leq |R|$, the result is obvious. So assume that $|I| = \lambda > |R|$ and that $\operatorname{Ext}^m (P, R^{(J)}) = 0$ for every set $J$ of cardinality strictly smaller than $\lambda$. Set $\mu_0 = \max \{\kappa, \operatorname{cf}(\lambda)\}$ and note that $\mu_0$ and $P$ satisfy the hypotheses of the above proposition. Moreover, since $\mu_0 \geq \operatorname{cf}(\lambda)$, we can find an increasing sequence of ordinals, $\{\alpha_\nu : \nu < \mu_0\} \subseteq \lambda$, which is cofinal in $\lambda$. Therefore,

$$\operatorname{Ext}^n (P, R^{(\lambda)}) \cong \lim_{\alpha<\mu_0} \operatorname{Ext}^n (P, R^{(\alpha)}) = 0$$

by the induction hypothesis.

Finally, let us choose a module $P \in \mathcal{P}$ and let us suppose that $\operatorname{pd}(P) = m > n$. Let $M$ be any module and take a free presentation $0 \to K \to R^{(I)} \to M \to 0$ of $M$. Then, applying the functor $\operatorname{Ext}^m (P, \_)$ to this sequence, we get the exact sequence $\operatorname{Ext}^m (P, R^{(I)}) \to \operatorname{Ext}^m (P, M) \to \operatorname{Ext}^{m+1} (P, K)$ in which $\operatorname{Ext}^m (P, R^{(I)}) = 0$ by the previous considerations, and $\operatorname{Ext}^{m+1} (P, K) = 0$ since $\operatorname{pd}(P) = m$. But, as $M$ is arbitrary, this means that $\operatorname{pd}(P) < m$. A contradiction that shows that $\operatorname{pd}(P) \leq n$.

$\square$
The above theorem can be easily improved when the ring $R$ is left perfect and right coherent. Recall that, in this case, any direct product of projective modules is projective ([5, Theorem 3.3]).

**Corollary 3.3.** Let $R$ be a left perfect and right coherent ring, and $n$, a natural number. Then \( \text{Findim}(R) \leq n \) if and only if \( \text{cofdim}_B(R) \leq n \).

**Proof.** In this case, \( R^I \) is a direct summand of \( R^I \) for every index set \( I \). So the hypotheses of the corollary imply that \( \text{Ext}^m(P, R^I) = 0 \) for every \( P \in \mathcal{P} \), every set \( I \) and every \( m > n \). Therefore, we can mimic the arguments used in the proof of 4) \( \Rightarrow \) 1) in the above theorem in order to prove the result. \( \square \)

Our next result gives analogous results for the little finitistic dimension.

**Theorem 3.4.** Let \( n \) be a natural number. The following assertions are equivalent:

1. \( \text{findim}(R) \leq n \).
2. \( \text{pd}(A) \leq n \) for every \( A \in \mathcal{A}^<\infty \).
3. \( \text{fibdim}_{B^<\infty}(M) \leq n \) for every \( M \in R\text{-Mod} \).
4. \( \text{fibdim}_{B^<\infty}(R^I) \leq n \).

**Proof.** The proof is analogous to that of Theorem 3.2. The only difference is that, in order to prove 4) \( \Rightarrow \) 1), we need to use that \( \mathcal{A}^<\infty, \mathcal{B}^<\infty \) is cogenerated by \( \{ A \in \mathcal{P}^<\infty : |A| < |R|^+ \} \). \( \square \)

This theorem can be improved for left coherent rings.

**Corollary 3.5.** Let $R$ be a left coherent ring and $n$, a natural number. Then \( \text{findim}(R) \leq n \) if and only if \( \text{fibdim}_{B^<\infty}(R) \leq n \).

**Proof.** Since each module \( P \in \mathcal{P}^<\infty \) is finitely presented and \( R \) is left coherent, the functor \( \text{Ext}^n(P, -) \) commutes with direct limits for each \( n \in \mathbb{N} \). Now, the fact that \( \text{Ext}^m(P, R) = 0 \) for each \( m > n \) implies that \( \text{Ext}^m(P, R^I) = 0 \) for each index set \( I \). Finally, the proof of 4) \( \Rightarrow \) 1) in Theorem 3.2 gives the result. \( \square \)

**Corollary 3.6.** Let $R$ be a two-sided artinian ring.

1. The small finitistic dimension of $R$ is finite if and only if \( \text{fibdim}_{B^<\infty}(R) \) is finite. Moreover, in this case, both dimensions do coincide.
2. The big finitistic dimension of $R$ is finite if and only if \( \text{fibdim}_B(R) \) is finite. Moreover, in this case, both dimensions do coincide.
3. \( \text{findim}(R) = \text{Findim}(R) \) if and only if \( \text{fibdim}_{B^<\infty}(R) = \text{fibdim}_B(R) \).

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