AN ALGEBRAIC INTERPRETATION OF THE INTERTWINING OPERATORS ASSOCIATED WITH THE DISCRETE FOURIER TRANSFORM

MESUMA ATAKISHIYEVA, NATIG ATAKISHIYEV, AND ALEXEI ZHEDANOV

Abstract. We show that intertwining operators for the discrete Fourier transform form a cubic algebra \( C_q \) with \( q \) a root of unity. This algebra is intimately related to the two other well-known realizations of the cubic algebra: the Askey-Wilson algebra and the Askey-Wilson-Heun algebra.

1. Introduction

We begin by recalling first that the discrete (finite) Fourier transform (DFT) based on \( N \) points is represented by an \( N \times N \) unitary symmetric matrix \( \Phi \) with entries (see, for example, \([1]-[6])

\[
\Phi_{kl} = N^{-1/2} q^{kl}, \quad k,l = 0,1,\ldots,N-1,
\]

where \( q = \exp(2\pi i/N) \) is a primitive \( N \)-th root of unity. Note that the matrix \( \Phi \) was introduced by Sylvester \([7]\) back in 1867 and is frequently referred to as Schur’s matrix.

In the present work we discuss some additional findings concerning algebraic properties of two finite-dimensional intertwining operators, associated with the DFT matrix \( \Phi \). These operators are represented by matrices \( A \) and \( B \) of the same size \( N \times N \) such that the intertwining relations

\[
A \Phi = i \Phi A, \quad B \Phi = -i \Phi B,
\]

are valid. The matrices \( A \) and \( B \) have emerged in a paper \([8]\) devoted to the problem of finding an explicit form for the difference operator that governs the eigen vectors of the DFT matrix \( \Phi \).

The purpose of this work is to provide a detailed account of a cubic algebra at roots of unity \( C_q \), which these intertwining operators form.

The lay out of the paper is as follows. Section 2 collects all those known facts about the intertwining operators \( A \) and \( B \), which are needed in section 3 for deriving an explicit form of an algebra \( C_q \), formed by these operators. In section 4 we show how this algebra \( C_q \) is related to the another well-known realization of the cubic algebra – the Askey-Wilson algebra. Section 5 closes the paper with a brief discussion of the interrelation between the algebra \( C_q \) and yet another realization of the cubic algebra – the Askey-Wilson-Heun algebra.

2. Intertwining operators

This section begins by rederiving an explicit form of the intertwining operators \( A \) and \( B \) \([8]\). Let us assume that they are of the most general cyclic 3-diagonal form, that is,

\[
A = \begin{bmatrix}
  b_0 & c_0 & 0 & \cdots & a_{N-1} \\
  a_0 & b_1 & c_1 & \cdots & \vdots \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  c_{N-1} & \cdots & 0 & a_{N-2} & b_{N-1}
\end{bmatrix}
\]

and

\[
B = \begin{bmatrix}
  \tilde{b}_0 & \tilde{c}_0 & 0 & \cdots & \tilde{a}_{N-1} \\
  \tilde{a}_0 & \tilde{b}_1 & \tilde{c}_1 & \cdots & \vdots \\
  0 & \ddots & \ddots & \ddots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  \tilde{c}_{N-1} & \cdots & 0 & \tilde{a}_{N-2} & \tilde{b}_{N-1}
\end{bmatrix},
\]

where \( a_k, b_k, c_k, \tilde{a}_k, \tilde{b}_k, \tilde{c}_k, k = 0,1,\ldots,N-1 \), are complex parameters. Note that no additional relations between \( A \) and \( B \) are assumed at the outset, however we specify \( B \) later on as a Hermitian conjugate matrix with respect to \( A \), i.e., \( B = A^\dagger \).
From the first identity in (1.2) it follows at once that the elements $a_k, b_k, c_k$ of the matrix $A$ are interconnected by the equations
\[ a_{j-1}q^{-k} + b_j + c_j q^k = i \left( q^{-j}c_{k-1} + b_k + a_k q^j \right), \quad (2.1) \]
so one readily concludes that $a_k, b_k, c_k$ are linear combinations of $q^k, q^{-k}$ and a constant. A more detailed analysis of equations (2.1) leads to the general solution
\[ a_k = -i\alpha + \beta \left( iq^k - q^{-k+1} \right), \quad b_k = \alpha \left( q^k - q^{-k} \right), \quad c_k = i\alpha + \beta \left( q^{-k} - iq^{k+1} \right), \quad (2.2) \]
where $\alpha$ and $\beta$ are two arbitrary complex parameters.

Quite similarly, one can find that the general solution for the operator $B$ is
\[ \tilde{a}_k = i\tilde{\alpha} - \tilde{\beta} \left( iq^k + q^{k+1} \right), \quad \tilde{b}_k = \tilde{\alpha} \left( q^k - q^{-k} \right), \quad \tilde{c}_k = -i\tilde{\alpha} + \tilde{\beta} \left( q^{-k} + iq^{k+1} \right), \quad (2.3) \]
where $\tilde{\alpha}$ and $\tilde{\beta}$ is another pair of arbitrary complex parameters. Assume now that $B$ is the Hermitian conjugate of $A$. Then we have the conditions
\[ \tilde{\alpha} = -\alpha^*, \quad q \tilde{\beta} = -\beta^*. \quad (2.4) \]
If one introduces two linear combinations of the operators $A$ and $A^\dagger$ of the form
\[ X = \frac{1}{2} \left( A + A^\dagger \right) \quad \text{and} \quad Y = \frac{1}{2i} \left( A - A^\dagger \right), \]
then the operators $X$ and $Y$ are Hermitian and play the role of finite-dimensional analogs of the operators of the coordinate and momentum in quantum mechanics.

It is natural to require that the coordinate operator should be diagonal, in complete agreement with the ordinary meaning of the coordinate operator. From the above relations between $a_n, b_n, c_n$ one concludes that the operator $X$ is diagonal if and only if $\beta = 0$ and $\alpha$ is a pure imaginary parameter:
\[ \alpha + \alpha^* = 0, \quad \beta = 0. \quad (2.5) \]
Without loss of generality one can put $\alpha = -i$, then
\[ a_n = -1, \quad b_n = 2\sin(n\theta_N), \quad c_n = 1, \quad \theta_N := \frac{2\pi}{N}. \]
Hence the matrix $X$ is diagonal,
\[ X = 2 \text{diag} \left( 0, s_1, s_2, \ldots, s_{N-2}, s_{N-1} \right), \quad s_n = \frac{q^n - q^{-n}}{2i} = \sin \left( n\theta_N \right), \quad (2.6) \]
whereas the ‘momentum’ matrix $Y$ is tridiagonal (with zero entries on the main diagonal),
\[ Y = i \begin{bmatrix} 0 & -1 & 0 & \cdots & 1 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -1 \\ -1 & \cdots & 0 & 1 & 0 \end{bmatrix}. \quad (2.7) \]

The final form of the intertwining operators
\[ A = X + iY = \begin{bmatrix} 0 & 1 & 0 & \cdots & -1 \\ -1 & 2s_1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2s_{N-2} & 1 \\ 1 & \cdots & 0 & -1 & 2s_{N-1} \end{bmatrix} \quad (2.8) \]
and
\[ A^\dagger \equiv A^\dagger = X - iY = \begin{bmatrix} 0 & -1 & 0 & \cdots & 1 \\ 1 & 2s_1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 2s_{N-2} & -1 \\ -1 & \cdots & 0 & 1 & 2s_{N-1} \end{bmatrix} \quad (2.9) \]
thus coincide with those found in [8]. It remains only to add that the singular (noninvertible) matrices $A$ and $A^\top$ are traceless, because one checks easily that $\sum_{k=1}^{N-1} s_k = 0$. It is a remarkable fact that the rank of the matrix $A$ (which is the same as the rank of the matrix $A^\top$) turns out to be different for the odd and even dimensions $N$:

1'. In the case of odd $N$ the rank of the matrix $A$ is equal to $N - 1$ and the null space of the matrix $A$ is one-dimensional;

2'. In the case of even $N$ the rank of the matrix $A$ is equal to $N - 2$ and the null space of the matrix $A$ is two-dimensional [9].

This unforeseen distinction between the properties of odd and even dimensional DFT’s simply indicates that the discrete reflection symmetry in the vector spaces $\mathbb{R}^N$, spanned by the even-dimensional DFT eigenvectors, is spontaneously broken [10].

Recall that the standard coordinate and momentum operators $x$ and $p$ are known to satisfy the Heisenberg commutation relation $[x, p] = i$. Contrary to this case, the operators $X$ and $Y$ do not satisfy such a simple commutation relation. In the next section we show that these operators satisfy instead a ‘classical’ algebra of the Askey-Wilson type.

3. Cubic algebra, generated by the operators $A$ and $A^\top$

The operators $A$ and $A^\top$, defined as in (2.8) and (2.9), respectively, do satisfy a simple cubic algebra under the commutation relation. Indeed, let us choose the third operator $C$ as the commutator

$$C = [A, A^\top] = AA^\top - A^\top A. \quad (3.1)$$

The operator $C$ is represented by a 3-diagonal cyclic matrix,

$$C = 4 \begin{bmatrix}
0 & s_1 & 0 & \cdots & s_1 \\
s_1 & 0 & s_2 - s_1 & \cdots & \cdots \\
0 & s_2 - s_1 & 0 & s_3 - s_2 & \cdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & s_{N-2} - s_{N-3} & 0 & s_{N-1} - s_{N-2} \\
s_1 & \cdots & 0 & s_{N-1} - s_{N-2} & 0
\end{bmatrix},$$

and it is not hard to verify directly that

$$[C, A] = \beta_1 AA^\top A + \beta_2 A - \beta_1 (A^\top)^3, \quad [A^\top, C] = \beta_1 A^\top AA^\top + \beta_2 A^\top - \beta_1 A^3, \quad (3.2)$$

where

$$\beta_1 = \frac{(1 - q)^2}{1 + q^2}, \quad \beta_2 = -4 \frac{(q - q^{-1})^2}{q + q^{-1}}. \quad (3.3)$$

To check whether the three operators $A, A^\top$ and $C$ form a closed algebra, one should verify their compatibility with the Jacobi identity

$$[A, [A^\top, C]] + [C, [A, A^\top]] + [A^\top, [C, A]] = 0. \quad (3.4)$$

Note that in our particular case the second double commutator in the Jacobi identity (3.4) vanishes because of definition (3.1). Hence the Jacobi identity (3.4) reduces to the sum of only two double commutators, the first and the third one. So one evaluates next that

$$[A, [A^\top, C]] = \beta_1 \left( (AA^\top)^2 - (A^\top A)^2 \right) + \beta_2 C \quad (3.5)$$

and

$$[A^\top, [C, A]] = \beta_1 \left( (A^\top A)^2 - (AA^\top)^2 \right) - \beta_2 C. \quad (3.6)$$

Consequently

$$[A, [A^\top, C]] + [A^\top, [C, A]] = 0, \quad (3.7)$$

and the Jacobi identity (3.4) holds.

This algebra possesses the Casimir operator

$$Q_1 = C^2 + r_1 \{A^2, (A^\top)^2\} + r_2 \{A, A^\top\} - r_1 (A^4 + (A^\top)^4) \quad (3.8)$$

with

$$r_1 = \frac{(1 - q)^2}{(1 + q)^2}, \quad r_2 = -4 \frac{(1 + q^2)(q - q^{-1})^2}{(1 + q)^2}. \quad (3.9)$$
When $N \to \infty$, the parameter $q$ goes to 1 and the coefficients $\beta_1$ and $\beta_2$ tend to zero. This means that the commutation relations (3.2) become trivial,

$$[C, A] = [A^\dagger, C] = 0,$$

(3.10)
as happens in the case of the linear quantum harmonic oscillator with the Heisenberg commutation relation $[a, a^\dagger] = 1$ for the lowering and raising operators $a$ and $a^\dagger$. For arbitrary natural $N$, however, we see that the algebra (3.2) is not classical and does not even belong to the HAW type. Nevertheless, this algebra is still very simple and can be exploited to derive further useful relations concerning discrete Fourier transform.

4. Askey-Wilson algebra for the operators $X, Y$

It is a remarkable fact that the operators $X$ and $Y$ are ‘classical’ operators with nice spectral properties. For $X$ this is obvious because the spectrum of $X$ is

$$x_n = -i(q^n - q^{-n}) = 2 s_n, \quad n = 0, 1, \ldots, N - 1,$$

(4.1)

which indicates that $x_n$ belongs to the class of the Askey-Wilson spectra of the type

$$\lambda_n = C_1 q^n + C_2 q^{-n} + C_0.$$

(4.2)

Observe also that the eigenvectors of the operator $X$ are represented by the euclidean $N$-column orthonormal vectors $e_k$ with the components $(e_k)_l = \delta_{kl}$, $k, l = 0, 1, \ldots, N - 1$, that is,

$$X e_k = x_k e_k.$$

(4.3)

The spectrum $y_n$ of the matrix $Y$ belongs to the same Askey-Wilson family too. Moreover, it turns out that the spectra of the matrices $Y$ and $X$ are equal. The reason for this similarity is that the operators $X$ and $Y$ are unitary equivalent. Indeed, from the intertwining relations (1.2) it follows that [8]

$$Y \Phi = \Phi X, \quad X \Phi = -\Phi Y.$$  

(4.4)

This means that

$$Y = \Phi X \Phi^\dagger,$$

(4.5)
i.e., that the operators $X$ and $Y$ are unitary equivalent and hence isospectral.

Note that the spectrum of $X$ is simple (i.e., nondegenerate) only if $N$ is odd, whereas for $N$ even the spectrum of $X$ is doubly degenerate. Evidently, the same statement is true for the operator $Y$.

Let $e_k$, $k = 0, 1, \ldots, N - 1$, be a set of the eigenvectors of the operator $Y$,

$$Y e_n = x_n e_n, \quad n = 0, 1, \ldots, N - 1.$$  

(4.6)

From the relations (4.3) and (4.5) it follows at once that the expansion relation

$$e_n = \Phi e_n = \sum_{k=0}^{N-1} \Phi_{kn} e_k = N^{-1/2} \left(1, q^n, q^{2n}, \ldots, q^{(N-1)n}\right)^\dagger$$

(4.7)

between the sets of the eigenvectors $e_k$ and $e_n$ is valid. We thus see that the discrete Fourier matrix $\Phi_{kn}$ can be also defined as the matrix of the overlap coefficients between eigenbases of the operators $X$ and $Y$. Observe also that it is not hard to show that the operator $Y$ is two-diagonal in the eigenbasis of the operator $X$:

$$Y e_n = i (e_{n+1} - e_{n-1}),$$

(4.8)

where $e_{-1} = e_{N-1}$ and $e_N = e_0$. Moreover, from (4.8) and (4.4) it follows that the operator $X$ is similarly two-diagonal in the eigenbasis of the operator $Y$:

$$X e_n = i (e_{n-1} - e_{n+1}).$$

(4.9)

Note that this symmetry between operators $X$ and $Y$ can be explained from the algebraic point of view in the following way.

**Proposition 1.** The operators $X$ and $Y$ provide a representation of the Askey-Wilson algebra with commutation relations

$$X^2 Y + Y X^2 - (q + q^{-1}) X Y X = -\left(q - q^{-1}\right)^2 Y,$$

$$Y^2 X + X Y^2 - (q + q^{-1}) Y X Y = -\left(q - q^{-1}\right)^2 X.$$  

(4.10)
\textit{Proof.} Taking into account that the componentwise structures of the operators \( X \) and \( Y \) in the euclidean basis \( e_k \) are of the form (cf (4.3) and (4.8), respectively)

\[
X_{kl} = x_k \delta_{k,l}, \quad Y_{kl} = i \left( \delta_{k,l+1} - \delta_{k,l-1} \right),
\]

one evaluates first that

\[
\left( X^2 Y + Y X^2 \right)_{kl} = \left( x_k^2 + x_l^2 \right) Y_{kl}, \quad \left( XYX \right)_{kl} = x_k x_l Y_{kl}.
\]

Then

\[
\left( X^2 Y + Y X^2 - (q - q^{-1}) X Y X \right)_{kl} = \left( x_k^2 + x_l^2 - 2 c_1 x_k x_l \right) Y_{kl} = i \left( a_k \delta_{k,l+1} - a_k \delta_{k,l-1} \right), \quad (4.11)
\]

where \( a_k = x_k^2 + x_{k-1}^2 - 2 c_1 x_k x_{k-1} \). The last step is to show, by using the trigonometric identity

\[
2 c_k c_{2k-1} = c_{2k} + c_{2k-2}, \quad c_k := \cos k \theta_N \text{, that } a_k \text{ does not actually depend on the index } k \text{ since } a_k = 4 s_k^2 = - (q - q^{-1})^2. \]

This means that the right-hand side in (4.11) is equal to \(- (q - q^{-1})^2 Y_{kl} \) and the first identity in (4.10) is proved.

Similarly,

\[
\left( Y^2 X + X Y^2 \right)_{kl} = (x_k + x_l) Y_{kl}^2 = (x_k + x_l) \left( 2 \delta_{k,l} - \delta_{k,l+2} - \delta_{k,l-2} \right),
\]

whereas

\[
\left( Y Y X \right)_{kl} = x_j Y_{kj} Y_{jl} = (x_{k+1} + x_{k-1}) \delta_{k,l} - x_{k-1} \delta_{k,l+2} - x_{k+1} \delta_{k,l-2}.
\]

Hence one concludes, upon using another trigonometric identity \( x_{k+1} + x_{k-1} = 4 c_1 s_k \), that

\[
\left( Y^2 X + X Y^2 - 2 c_1 Y X Y \right)_{kl} = 8 s_k^2 \delta_{k,l} = 4 s_k^2 X_{kl} = - (q^{-1})^2 Y_{kl}.
\]

Thus the second identity in (4.10) is proved as well. \( \square \)

\textit{Remark.} The generic Askey-Wilson algebra was introduced in [11]. The above form of the Askey-Wilson algebra is due to Terwilliger [12]. Let us draw attention to the remarkable symmetry property of relations (4.10) each can be obtained from another by the transposition \( X \leftrightarrow Y \).

The Askey-Wilson algebra can be presented in an equivalent to (4.10) cyclic form if one introduces the third Hermitian operator \( Z \) defined as

\[
Z = q^{1/2} \begin{bmatrix}
0 & q^{-1} & 0 & \cdots & 0 & 0 & 0 & q^{-1} \\
1 & 0 & q^{-2} & \cdots & 0 & 0 & 0 & 0 \\
0 & q & 0 & \ddots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & q^{-4} & 0 & q & \ddots & 0 \\
0 & 0 & 0 & \cdots & q^{-3} & 0 & q & 0 \\
1 & 0 & 0 & \cdots & 0 & q^{-2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & q^{-1} & 0 & 0
\end{bmatrix}.
\]

(4.12)

Then the three Hermitian operators \( X, Y, Z \) satisfy the Askey-Wilson algebra in the cyclic \( \mathbb{Z}_3 \) form

\[
pXY - p^{-1} YX = (q - q^{-1})Z, \quad pZX - p^{-1} XZ = (q - q^{-1})Y,
\]

\[
pYZ - p^{-1} ZY = (q - q^{-1})X,
\]

(4.13)

where \( p = q^{1/2} = \exp(\pi i / N) \). This algebra can be considered as a q-analog of the rotation algebra \( o(3) \). For \( q \) a root of unity this algebra and its representations were considered in [13].

Similar to the operator \( Y \), the operator \( Z \) is two-diagonal in the eigenbasis of the operator \( X \):

\[
Z e_n = q^{1/2} \left( q^n e_{n+1} + q^{-n} e_{n-1} \right).
\]

(4.14)

Despite the completely symmetric form of the algebra (4.13), the spectral properties of the operators \( X, Y, Z \) are slightly different.
**Proposition 2.** By a similarity transformation $S^{-1} Z S$ the matrix $Z$ can be transformed into the simple circulant matrix $(-1)^N \tilde{Z}$.

$$
\tilde{Z} = 
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
$$

(4.15)

provided that the diagonal matrices $S$ and $S^{-1}$ are chosen componentwise as $S_{k,l} = x_k \delta_{k,l}$ and $S^{-1}_{k,l} = x_k^{-1} \delta_{k,l}$, with $x_k = (-1)^k N q^{k^2/2}$, $0 \leq k, l \leq N - 1$.

**Proof.** Taking into account that the componentwise structure of the matrix $Z$ in the Euclidean basis $e_k$ is of the form $Z_{k,l} = q^{-1/2} \left( q^k \delta_{k+1,l} + q^{-k} \delta_{k-1,l} \right)$ (cf (4.14)), one evaluates that

$$
\left( S^{-1} Z S \right)_{k,l} = S^{-1}_{k,m} S_{m,l} = x_k^{-1} x_l Z_{k,l}
$$

(4.16)

where $b_k = q^{k^2/2} x_k^{-1} x_{k-1}$. Let us assume now that, in complete analogy with the Proposition 1 (cf (4.11)), the coefficient $b_k$ does not depend on the index $k$ and it is equal to $(-1)^N$:

$$
b_k = q^{k^2/2} x_k^{-1} x_{k-1} = (-1)^N.
$$

(4.17)

Then from (4.17) it follows that at once that $x_k$ in this case have to satisfy a recurrence relation $x_{k+1} = (-1)^N q^{k^2/2} x_k$. Hence $x_k = (-1)^k N q^{k^2/2}$, $k = 0, 1, 2, \ldots, N - 1$, and formula (4.16) reduces to

$$
\left( S^{-1} Z S \right)_{k,l} = (-1)^N \left( \delta_{k+1,l} + \delta_{k-1,l} \right) = (-1)^N \tilde{Z}_{k,l}.
$$

(4.18)

This completes the proof of the similarity between matrices $Z$ and $(-1)^N \tilde{Z}$. \hfill \Box

The spectrum $\zeta_n$ of the matrix $\tilde{Z}$ is well known [14],

$$
\zeta_n = 2 \cos n \theta_N = 2 c_n, \quad n = 0, 1, 2, \ldots, N - 1.
$$

So it is evident that apart from the eigenvalues ±2, all other eigenvalues $\zeta_k$ of the matrix $Z$ are doubly degenerate because $\zeta_k = \zeta_{N-k}$, $k = 1, 2, \ldots, N - 1$, by definition.

The Casimir operator of this algebra (4.13), which commutes with $X, Y, Z$, has the expression

$$
Q = p XY Z - q \left( X \cdot Z + Z \cdot X \right) - q^{-1} Y^2.
$$

(4.19)

It is directly verified that $Q$ is Hermitian matrix, i.e., $Q^\dagger = Q$, and that for the given representation this operator is proportional to the identity matrix,

$$
Q = -2 \left( q + q^{-1} \right) I = -2 \cos \theta_N I.
$$

(4.20)

We thus have associated the ‘classical’ Askey–Wilson algebra to the discrete Fourier transform.

We close this section with the following extended remark about the algebra, defined by (4.13). Recall first that the Askey–Wilson polynomials reveal themselves within the overlap functions between the two dual bases in the Askey–Wilson algebra [11]. On the other hand, the algebra (4.13) can be considered as a $q$-analogue of the rotation algebra (2.3). Indeed, if one introduces three operators $K_0 = \frac{1}{2s_1} X$, $K_1 = \frac{1}{2s_1} Y$ and $K_2 = \frac{1}{2s_1} Z$, then the commutation relations in (4.13) can be rewritten as

$$
[K_0, K_1]_q = K_2, \quad [K_0, K_2]_q = -K_1, \quad [K_1, K_2]_q = -K_0,
$$

(4.21)

where $[A, B]_q := q^{1/2} AB - q^{-1/2} BA$ by definition. The commutation relations (4.21) are known to define the $so(3, q)$ associative algebra for $q$ a root of unity. This algebra and its representations were considered in [13], where it was found that the above-mentioned overlap functions between the two dual bases of the operators $K_0$ and $K_1$ are expressed in this case in terms of the $q$-ultraspherical polynomials.

So the question can be then posed what is explicit form of a polynomial family, associated with the overlap functions between the two dual eigenbases of the operators $K_0 = \frac{1}{2s_1} X$ and $K_1 = \frac{1}{2s_1} Y$.
in our case. Now observe that from the relations (4.3) and (4.7) it is evident that those overlap functions are equal to

$$\langle \epsilon_k, \epsilon_l \rangle = N^{-1/2} \sum_{j=0}^{N-1} q^{kj} \delta_{ij} = N^{-1/2} q^{-kl} = \langle \epsilon_k, \epsilon_0 \rangle P_l(\mu_k). \quad (4.22)$$

Thus in the case under study the overlap functions between the two dual eigenbases of the operators $K_0$ and $K_1$ turn out to be the monomials $P_l(\mu_k) = \mu_k^l$ in the discrete argument $\mu_k = q^{-k}$. These monomials $P_l(\mu_k)$ form a complete and orthogonal system since

$$N^{-1/2} \sum_{j=0}^{N-1} P_l(\mu_j)^* P_l(\mu_j) = \delta_{kl}. \quad (4.23)$$

5. Askey-Wilson-Heun operators and algebras

Starting from the pair of operators $X$, which satisfy AW-algebra, one can construct the algebraic Heun operator $W$ as an arbitrary bilinear combination [15]

$$W = \tau_1XY + \tau_2XY + \tau_3X + \tau_4Y + \tau_5I \quad (5.1)$$

In [16] it was shown that the pairs of operators $X, W$ or $Y, W$ constitute a cubic algebra closed under commutation relations. This algebra is the Askey-Wilson-Heun (briefly AWH) algebra.

We can apply this observation to our situation. Indeed, it is convenient to choose $X$ as one of the AWH generators because $X$ is the diagonal matrix with ‘classical’ AW-spectrum. We can also choose $W$ as an operator commuting with $\Phi$. It is easily seen that the only possibility to get the operator $W$ commuting with $\Phi$ is to put $\tau_3 = \tau_4 = 0$ and $\tau_2 = -\tau_1$. This is equivalent to the choice of taking the commutator

$$W = -2i [X, Y] = [A, A^\dagger].$$

We show that the pair $X, W$ forms a simple Heun-type algebra.

Indeed, it is directly verified that the following relations hold

$$X^2W + WX^2 - (q + q^{-1})WXW = g_1W, \quad W^2X + XW^2 - (q + q^{-1})WXW = g_2X^3 + g_3X, \quad (5.2)$$

where

$$g_1 = 16s_1^2, \quad g_2 = -16c_1(1 + c_1)(1 - c_1)^2, \quad g_3 = 64(1 + c_1)(1 - c_1)^2(3c_1 + 1). \quad (5.3)$$

These relations correspond to a special case of the Askey-Wilson-Heun (AWH) algebra introduced in [16].

From results of [16] it is possible to derive the expression of the Casimir operator $Q$ which commutes with both $X$ and $Y$:

$$Q = [X, W]^2 + \rho_1 \left( (XW)^2 + (WX)^2 \right) + \rho_2W^2 + \rho_3X^4 + \rho_4X^2, \quad (5.4)$$

where

$$\rho_1 = -\frac{(1 - q)^2}{1 + q^2} = \frac{1}{c_1} - 1, \quad \rho_2 = 4(q - q^{-1})^2 = -16s_1^2, \quad \rho_3 = \left( q + q^{-1} \right)^4 \left( \frac{1 + q^2}{1 + q} \right) = 16s_1^4 \left( 1 + c_1^{-1} \right),$$

$$\rho_4 = -4 \frac{(1 + q)^2(5q^4 + 2q^3 + 2q^2 + 2q + 5)(q - 1)^4}{(q^2 + 1)^4} = 64s_1^2 \left( 1 - c_1 \right) \left( 5c_1^2 + c_1 - 2 \right). \quad (5.5)$$

It is readily verified that for the given representations (2.6) and (2.7) of the matrices $X$ and $Y$, the operator $Q$ becomes the identity matrix to within a constant:

$$Q = -64(q - q^{-1})^4 = -1024s_1^4. \quad (5.6)$$

As is shown in [16], one can deduce useful information about the shape of the operators $X, W$, starting only with the commutation relations (5.2). Namely, it is possible to show that there exists a basis $e_n$ where the operator $X$ is diagonal while the operator $W$ is tridiagonal. On the other hand, in the ‘dual’ basis $\tilde{e}_n$ (for which the operator $W$ is diagonal: $W\tilde{e}_n = \mu_n\tilde{e}_n$) the matrix of the operator $X$ will have, in general, a nonlocal shape with all entries being nonzero.
6. Concluding remarks

To summarize, we have demonstrated that the ‘position’ and ‘momentum’ operators $X$ and $Y$ of the discrete Fourier transform form a special case of the Askey-Wilson algebra $\mathfrak{AW}(3)$. On the other hand, the creation and annihilation operators $A$ and $A^\dagger$ generate more complicated and ‘non-classical’ algebra with cubic members in commutation relations. Moreover, we have shown that the position operator $X$ together with the operator $W = [A, A^\dagger]$, which commutes with the discrete Fourier transform, constitute an algebra of Heun type.

These results clarify what is an algebraic distinction of the discrete Fourier transform from the continuous one. In the classical case the commutator of creation and annihilation operators is equal to a constant and this leads to the well known Heisenberg-Weyl algebra, which generates exact solutions of the quantum harmonic oscillator in terms of the Hermite polynomials $H_n(x)$, times the Gaussian factor $\exp(-x^2/2)$.

Contrary to the continuous case, the creation and annihilation operators for the discrete Fourier transform $A^\dagger$ and $A$ do not form neither Lie algebra nor any of ‘classical’ nonlinear algebras of Askey-Wilson type. This makes the problem of discrete Fourier harmonic oscillator, governed by the standard Hamiltonian $H = A^\dagger A + AA^\dagger$, hardly exactly solvable. This phenomenon of non-solvability of $H$ was observed earlier by a number of authors. Here we propose a simple algebraic explanation of this phenomenon. It remains but to mention in this connection that there is still a possibility to construct a version of the discrete Fourier harmonic oscillator but in terms of the eigenvectors of the difference operator $A^\dagger A$, upon interpreting it as a discrete analog of the number operator $N = a^\dagger a$ for the harmonic oscillator. Then it is possible to use the same procedure of constructing the eigenvectors of $A^\dagger A$ in the form of the ladder-type hierarchy, as in the harmonic oscillator case in quantum mechanics: $a \psi_0(x) = 0$ and $\psi_n(x) = 1/\sqrt{n!}a^\dagger \psi_{n-1}(x), n = 1, 2, 3,...$. An important aspect to observe is that the operator $A^\dagger A$ commutes, $[A^\dagger A, P_d] = 0$, with the discrete reflection operator $P_d$ for an arbitrary dimension $N$. Nevertheless, thus constructed family of the eigenvectors of $A^\dagger A$ turn out to be $P_d$-symmetric only for odd dimensions $N = 2L + 1$. In the all even cases with $N = 2L$ the $P_d$-symmetry in the space of the eigenvectors of $A^\dagger A$ is spontaneously broken. This essential distinction between odd and even dimensions has been shown to be consistent with the old formula [1]-[3] for the multiplicities of the eigenvalues, associated with the $N$-dimensional DFT (consult [8]-[10] for a detailed discussion of this approach).

Note that the operators $X$ and $Y$ under discussion are similar to those considered earlier by Grünbaum [17]. These operators were exploited in [17] in order to solve the minimization problem of the uncertainty product $\Delta X \Delta Y$. Using a general approach for noncommuting compact operators $X, Y$ (see, e.g., [18]), one can show that such minimization is achieved for generalized ‘coherent’ states, that is, for the eigenvectors of the operator $V = X + i\gamma Y$ with an arbitrary real nonzero parameter $\gamma$. In the classical case, when $X$ is the coordinate operator and $Y$ the momentum, the solution of this problem for an arbitrary value of $\gamma$ is explicit and simple; it actually corresponds to the so-called squeezed states [19]. However, in our case the operator $V$ belongs to the class of Heun operators defined by (5.1). This leads to the conclusion that an explicit expression for such eigenvectors is hardly possible.

Finally, it may be interesting to note that some general properties of representations of the Askey-Wilson algebra for roots of unity were considered in [20] and [21]. Also, after completing this work, V. Spiridonov has drawn our attention to the paper [22], where the intertwining relations for generators of Sklyanin algebra were established with respect to the so-called elliptic Fourier transform. It would be interesting to interrelate the results of the present work with those obtained in [22].

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References

[1] J.H.McClellan and T.W.Parks, *Eigenvalue and eigenvector decomposition of the discrete Fourier transform*, IEEE Trans. Audio Electroac., AU-20, 66–74, 1972.

[2] L.Auslander and R.Tolimieri, *Is computing with the finite Fourier transform pure or applied mathematics?* Bull. Amer. Math. Soc., 1, 847–897, 1979.

[3] B.W.Dickinson and K.Steiglitz, *Eigenvectors and functions of the discrete Fourier transform* IEEE Trans. Acoust. Speech, 30, 25–31, 1982.

[4] M.L.Mehta, *Eigenvalues and eigenvectors of the finite Fourier transform*, J. Math. Phys., 28, 781–785, 1987.

[5] V.B.Matveev, *Interwining relations between the Fourier transform and discrete Fourier transform, the related functional identities and beyond*, Inverse Probl., 17, 633–657, 2001.

[6] N.M.Atakishiyev, *On q-extensions of Mehta’s eigenvectors of the finite Fourier transform*, Int. J. Mod. Phys. A, 21, 4993–5006, 2006.

[7] J.J.Sylvester, *Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton’s rule, ornamental tile-work, and the theory of numbers*, Philos. Mag., 34, 461–475, 1867.

[8] M.K.Atakishiyeva and N.M.Atakishiyev, *On the raising and lowering difference operators for eigenvectors of the finite Fourier transform*, J. Phys: Conf. Ser., 597, 012012, 2015.

[9] M.K.Atakishiyeva, N.M.Atakishiyev and J.Loreto-Hernández, *More on algebraic properties of the discrete Fourier transform raising and lowering operators*, 4 Open, 2, 1–11, 2019.

[10] M.K.Atakishiyeva, N.M.Atakishiyev and J.Loreto-Hernández, *On the discrete Fourier transform eigenvectors and spontaneous symmetry breaking*, Springer Proceedings in Mathematics & Statistics, 333, 549–569, 2020.

[11] A.S.Zhedanov, *“Hidden symmetry” of Askey-Wilson polynomials, Theoretical and Mathematical Physics 89, 1146–1157, 1991.*

[12] P.Terwilliger, *The Universal Askey-Wilson Algebra*, SIGMA 7, 069, 2011, arXiv:1104.2813.

[13] V.Spiridonov and A.Zhedanov, *q-Ultraspherical polynomials for q a root of unity*, Lett.Math.Phys. 37, 173–180, 1996.

[14] P.J.Davis, *Circulant Matrices*, Wiley, New York, 1970.

[15] F. A. Grünbaum, L. Vinet, and A. Zhedanov, *Algebraic Heun Operator and Band-Time Limiting*, Communications in Mathematical Physics 364, 1041–1068, 2018, arXiv: 1711.07862.

[16] P. Baseilhac, S. Tsujimoto, L. Vinet, and A. Zhedanov, *The Heun-Askey-Wilson Algebra and the Heun Operator of Askey-Wilson Type*, Annales Henri Poincare 20, 3091–3112, 2019, arXiv: 1811.11407.

[17] A.F. Grünbaum, *The Heisenberg inequality for the discrete Fourier transform*, Appl. Comput. Harmon. Anal. 15, 163–167, 2003.

[18] E.K.Ifantis, *Minimal Uncertainty States for Bounded Observables*, J.Math.Phys. 12, 2512–2516, 1971.

[19] M.M.Nieto and D.R.Truax, *Squeezed states for general systems*, Phys. Rev. Lett. 71, 28–43, 1993.

[20] H.Huang, *Finite-dimensional irreducible modules of the universal Askey-Wilson algebra at roots of unity*, Journal of Algebra 569, 12–29, 2021.

[21] H.Huang, *Center of the universal Askey-Wilson algebra at roots of unity*, Nucl. Phys. B 909, 260–296, 2016.

[22] S.E.Derkachov and V.P.Spiridonov, *Yang-Baxter equation, parameter permutations, and the elliptic beta integral*, Russ. Math. Surv. 68, 10–27, 2013, arXiv: 1205.3520v2.