EXPLICIT SHARP CONSTANTS IN SOBOLEV INEQUALITIES ON Riemannian Manifolds with Nonnegative Ricci Curvature

ALEXANDRU KRISTÁLY

Abstract. Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, i.e. \(0 < \text{AVR}(g) \leq 1\), where \(\text{AVR}(g)\) stands for the asymptotic volume ratio of \((M, g)\). The main purpose of the paper is to prove that the sharp Sobolev constants in various \(L^p\)-Sobolev inequalities on \((M, g)\) (both for \(p < n\) and \(p > n\)) can be explicitly given by means of the geometric invariant \(\text{AVR}(g)\). The equality cases are also characterized by stating that nonzero extremal functions occur if and only if \(\text{AVR}(g) = 1\), i.e. \((M, g)\) is isometric to the standard Euclidean space \((\mathbb{R}^n, g_0)\). The proofs are based on an isoperimetric inequality established by S. Brendle (2020), combined with appropriate symmetrization techniques and optimal volume non-collapsing properties. We also provide examples of Riemannian manifolds with their explicit asymptotic volume ratios, which represent typical geometric settings we are working in.

1. Introduction

Initiated by Aubin [3] in the early seventies, the determination of sharp constants in Sobolev inequalities on curved settings represents one of the major challenges of geometric analysis. Motivated by applications in PDEs on manifolds, a particular interest is devoted to the simplest \(L^p\)-Sobolev inequality
\[
\left( \int_M |u|^{p^*} \, dv_g \right)^{1/p^*} \leq C \left( \int_M |\nabla_g u|^p \, dv_g \right)^{1/p}, \quad \forall u \in C_0^\infty(M), \tag{S}
\]
where \((M, g)\) is an \(n\)-dimensional complete Riemannian manifold endowed with its canonical measure \(dv_g\) and \(C = C(n, p) > 0\) is a universal constant; here \(n \geq 2\), \(1 < p < n\) and \(p^* = \frac{pn}{n-p}\) is the critical Sobolev exponent. When \((M, g)\) is the Euclidean space \(\mathbb{R}^n\) with its usual metric \(g_0\), then Aubin [3] and Talenti [37] identified the sharp constant \(C = \text{AT}(n, p)\) in (S), its value being
\[
\text{AT}(n, p) = \pi^{-\frac{1}{2}} n^{-\frac{1}{p}} \left( \frac{p-1}{n-p} \right)^{1-1/p} \left( \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right)^{1/n};
\]
the form of the whole family of extremals is also known.

In a generic Riemannian manifold \((M, g)\), a local chart analysis shows that the validity of (S) necessarily implies \(C \geq \text{AT}(n, p)\). In order to provide finer properties on the sharp constant in (S), the curvature of \((M, g)\) is going to play an indispensable role.

On one hand, when \((M, g)\) is an \(n\)-dimensional Hadamard manifold (i.e. simply connected, complete Riemannian manifold with nonpositive sectional curvature), then inequality (S)
holds with the sharp constant $C = AT(n,p)$ whenever the Cartan-Hadamard conjecture is valid, see Druet, Hebey and Vaugon [16] and Hebey [20]. The Cartan-Hadamard conjecture is precisely the isoperimetric inequality on $(M,g)$ whose validity is confirmed in the low-dimensional cases, i.e. for $n = 2$ by Beckenbach and Radó [4], for $n = 3$ by Kleiner [21] and for $n = 4$ by Croke [11]; in a recent preprint Ghomi and Spruck [19] claim the validity of the Cartan-Hadamard conjecture in any dimension.

On the other hand, when $(M,g)$ is a complete $n$-dimensional Riemannian manifold with nonnegative Ricci curvature, the existing literature mostly contains rigidity results. In fact, Ledoux [27] proved that if (S) holds with $C = AT(n,p)$, then $(M,g)$ is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$. Moreover, a quantitative form of Ledoux’s result were given by do Carmo and Xia [17] who proved that $(M,g)$ is topologically close to $(\mathbb{R}^n, g_0)$ whenever the constant $C > AT(n,p)$ in (S) is sufficiently close to $AT(n,p)$. In fact, a byproduct of do Carmo and Xia’s approach is that the validity of (S) with a constant $C > 0$ implies the volume non-collapsing property

$$\text{AVR}(g) \geq \left(\frac{AT(n,p)}{C}\right)^n,$$

where the quantity

$$\text{AVR}(g) = \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n}$$

stands for the asymptotic volume ratio of $(M,g)$; here, $B_x(r)$ is the open metric ball on $M$ with center in $x \in M$ and radius $r > 0$, $\text{Vol}_g(S)$ denotes the volume of $S \subset M$ and $\omega_n$ is the volume of the Euclidean unit ball in $\mathbb{R}^n$. By Bishop-Gromov comparison theorem one has that $\text{AVR}(g) \leq 1$ and this number is independent of the choice of $x \in M$, thus it is a global geometric invariant of $(M,g)$; moreover, $\text{AVR}(g) = 1$ if and only if $(M,g)$ is isometric to $(\mathbb{R}^n, g_0)$. In particular, inequality (1.1) implies that for any complete Riemannian manifold $(M,g)$ with nonnegative Ricci curvature supporting the Sobolev inequality (S), one necessarily has that

$\text{AVR}(g) > 0$,

i.e. $(M,g)$ has Euclidean volume growth. The converse is also true, i.e. if $(M,g)$ is any complete Riemannian manifold $(M,g)$ with nonnegative Ricci curvature and $\text{AVR}(g) > 0$ then inequality (S) holds on $(M,g)$ (with some generic constant $C > 0$); see Coulhon and Saloff-Coste [14] for a more general form when the Ricci curvature is bounded from below.

As we know by Anderson [2], Cheeger and Colding [9, 10], Perelman [35], Li [30] and Munn [33], the asymptotic volume ratio of the $n$-dimensional Riemannian manifold $(M,g)$ with nonnegative Ricci curvature: closer value of $\text{AVR}(g)$ to 1 implies homotopically closer space to $(\mathbb{R}^n, g_0)$, which is exploited by do Carmo and Xia [17] in the context of Sobolev inequalities.

Keeping the latter geometric framework, i.e., $(M,g)$ is a noncompact, complete Riemannian manifold with nonnegative Ricci curvature, the purpose of the present paper is to provide various sharp Sobolev inequalities on $(M,g)$. It turns out that the asymptotic volume ratio $\text{AVR}(g) \in (0,1]$ is explicitly encapsulated in the Sobolev inequalities and more spectacularly, they do provide sharp Sobolev constants. In order to get an insight into our achievements, we state two particular results; the first one is an $L^p$-Sobolev inequality in the spirit of (S):
Theorem 1.1. Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, \(n \geq 2\). If \(p \in (1, n)\), then for every \(u \in C^\infty_0(M)\) one has
\[
\left( \int_M |u|^p \, dv_g \right)^{1/p} \leq S_g \left( \int_M |\nabla_g u|^p \, dv_g \right)^{1/p},
\]
where the constant
\[
S_g = AT(n, p) \, AVR(g)^{\frac{1}{n}}
\]
is sharp. Moreover, equality holds in (1.2) for some nonzero function if and only if \(AVR(g) = 1\).

Theorem 1.1 immediately implies the result of do Carmo and Xia [17]. Indeed, if (S) holds for some \(C > 0\), then by the sharpness of \(S_g\) we have \(C \geq S_g\), which is equivalent to (1.1); if \(C = AT(n, p)\) then we obtain \(AVR(g) = 1\), recovering Ledoux’s rigidity result [27] as well.

In fact, inequality (1.2) belongs to a large class of \(L^p\)-Gagliardo-Nirenberg inequalities \((1 < p < n)\) which are going to be treated jointly with their limiting cases (i.e. \(L^p\)-logarithmic Sobolev and \(L^p\)-Faber-Krahn-type inequalities), see Theorems 3.1-3.4. As a counterpart of these inequalities, we also establish sharp \(L^p\)-Morrey-Sobolev inequalities \((p > n)\) in the same Riemannian geometric setting, see Theorems 4.1&4.2.

Another class of problems concerns the Rayleigh-Faber-Krahn inequality. Given a complete \(n\)-dimensional Riemannian manifold \((M, g)\) (with no curvature restriction for the moment), it is well known that the first eigenvalue of the Beltrami-Laplace operator \(-\Delta_g\) for the Dirichlet problem on a smooth bounded open set \(\Omega \subset M\) can be characterized by
\[
\lambda^D_{1, g}(\Omega) = \inf_{u \in C^\infty_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla_g u|^2 \, dv_g}{\int_\Omega u^2 \, dv_g}. \tag{1.3}
\]
According to Carron [7] (see also Hebey [20, Proposition 8.1]), if \(n \geq 3\) and \(Vol_g(M) = +\infty\), the validity of the general Sobolev inequality (S) is equivalent to the validity of a generic Rayleigh-Faber-Krahn inequality on \((M, g)\), i.e. there exists \(\Lambda > 0\) such that for any smooth bounded open set \(\Omega \subset M\) one has
\[
\lambda^D_{1, g}(\Omega) \geq \Lambda Vol_g(\Omega)^{-\frac{2}{n}}. \tag{1.4}
\]
In particular, Theorem 1.1 implies that inequality (1.4) holds whenever \((M, g)\) is a Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth. In fact, in the latter geometric setting we can establish the sharp form of (1.4) (hereafter, \(j_\nu\) stands for the first positive root of the Bessel function \(J_\nu\) of the first kind with degree \(\nu \in \mathbb{R}\)):

Theorem 1.2. Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold as in Theorem 1.1. Then, for every smooth bounded open set \(\Omega \subset M\), we have
\[
\lambda^D_{1, g}(\Omega) \geq \Lambda_g Vol_g(\Omega)^{-\frac{2}{n}}, \tag{1.5}
\]
where the constant
\[
\Lambda_g = \frac{2}{\pi} \left( \frac{\omega_n}{\pi} \right) \frac{\omega_n \, AVR(g)}{2} \tag{1.6}
\]
is sharp. Furthermore, equality holds in (1.5) for some \( \Omega \subset M \) if and only if \((M, g)\) is isometric to \( (\mathbb{R}^n, g_0) \) and \( \Omega \) is isometric to a ball \( B \subset \mathbb{R}^n \) with \( \operatorname{Vol}_g(\Omega) = \operatorname{Vol}_{g_0}(B) \).

One of the key tools to prove our results is a recent isoperimetric inequality provided by Brendle [5], by using a careful limiting argument in the Alexandrov-Bakelman-Pucci method: for every bounded open smooth set \( \Omega \subset M \), one has

\[
\operatorname{Area}_g(\partial \Omega) \geq n \omega_n^{\frac{1}{n}} \operatorname{AVR}(g)^{\frac{1}{n}} \operatorname{Vol}_g(\Omega)^{\frac{n-1}{n}}. \tag{1.6}
\]

We notice that the 3-dimensional version of inequality (1.6) is independently proved recently by Agostiniani, Fogagnolo and Mazzieri [1] by using sharp Willmore-type estimates. The Sobolev inequalities we are dealing with follow by the isoperimetric inequality (1.6) combined with a symmetrization from \((M, g)\) to \((\mathbb{R}^n, g_0)\) in the spirit of Aubin [3].

In fact, the main challenge is to prove the sharpness of the aforementioned inequalities (see e.g. the constants \( S_g \) and \( \Lambda_g \) in (1.2) and (1.5), respectively). In the case of Gagliardo-Nirenberg inequalities, the optimal volume non-collapsing properties of \((M, g)\) established by Kristály [23] provide precisely the required optimalities. However, the sharpness of Sobolev-Morrey and Rayleigh-Faber-Krahn inequalities require fine properties of special functions which will be developed in Sections 4 and 5, respectively.

Although not explicitly stated in the paper of Brendle [5], equality holds in (1.6) if and only if the set \( \Omega \subset M \) is isometric to an Euclidean ball and \( \operatorname{AVR}(g) = 1 \), i.e. the manifold \( M \) itself is isometric to \( \mathbb{R}^n \); we thank S. Brendle for confirming this fact. This characterization is crucial to identify the equality cases in our sharp Sobolev inequalities. We also notice that the equality in (1.6) has been characterized in the 3-dimensional case by Agostiniani, Fogagnolo and Mazzieri [1, Theorem 1.8] with the same statement as before.

The paper is organized as follows. In Section 2 we provide a symmetrization argument, by establishing a sharp Pólya-Szegő inequality that involves the constant \( \operatorname{AVR}(g) \). In Section 3 the sharp \( L^p \)-Gagliardo-Nirenberg inequalities \((p < n)\) are established together with their limit cases (i.e. logarithmic Sobolev and Faber-Krahn inequalities), see Theorems 3.1-3.4. In Section 4 we prove the sharp \( L^p \)-Sobolev-Morrey inequalities \((p > n)\), see Theorems 4.1&4.2. In Section 5, by using fine properties of the Bessel functions, we provide the proof of Theorem 1.2. Section 6 is devoted to two nontrivial Riemannian manifolds with the required geometric properties. The first example is a rotationally invariant metric on \( \mathbb{R}^n \), while the second one is a class of asymptotically locally Euclidean manifolds, i.e. \( n \)-dimensional Riemannian manifolds asymptotic to \((\mathbb{R}^n \setminus \{0\})/G, g_0\) where \( G \) is a finite subgroup of \( SO(n) \) acting freely on \( \mathbb{R}^n \setminus \{0\} \). We conclude the paper with some open questions.

Notations. In the sequel, for every \( p \in (1, n) \) and \( q \in (0, \infty) \) and for enough smooth functions \( u : M \to \mathbb{R} \) and \( v : \mathbb{R}^n \to \mathbb{R} \) we denote

\[
\|u\|_{L^q(M)} = \left( \int_M |u|^q \, dv_g \right)^{\frac{1}{q}}, \quad \|\nabla_g u\|_{L^p(M)} = \left( \int_M |\nabla_g u|^p \, dv_g \right)^{\frac{1}{p}},
\]

\[
\|v\|_{L^q(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |v|^q \, dx \right)^{\frac{1}{q}}, \quad \|\nabla v\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\nabla v|^p \, dx \right)^{\frac{1}{p}}.
\]

The norm \( \|\cdot\|_{L^\infty(M)} \) is the usual sup-norm on \( M \). In addition, \( \operatorname{Vol}_{g_0} \) and \( \operatorname{Area}_{g_0} \), \( \operatorname{Area}_g \), denote the volume in \((\mathbb{R}^n, g_0)\) and the induced \((n-1)\)-dimensional area on \( M \) and \( \mathbb{R}^n \), respectively.
2. Preparatory part

Let \((M, g)\) be a noncompact, complete \(n\)-dimensional Riemannian manifold endowed with its canonical form \(dv_g\), having nonnegative Ricci curvature and Euclidean volume growth, i.e.

\[
\text{AVR}(g) = \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n} > 0.
\]

According to Brendle [5, Corollary 1.2], for every bounded open set \(\Omega \subset M\) with smooth boundary, one has the isoperimetric inequality

\[
\text{Area}_g(\partial \Omega) \geq n \omega_n^{\frac{1}{n}} \text{AVR}(g)^{\frac{1}{n}} \text{Vol}_g(\Omega)^{\frac{n-1}{n}}.
\]

(2.1)

The equality in (2.1) holds if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isometric to an Euclidean ball. The inequality (2.1) can be rewritten into the equivalent form

\[
\text{Area}_g(\partial \Omega) \geq \text{AVR}(g)^{\frac{1}{n}} \text{Area}_{g_0}(\partial B),
\]

(2.2)

where \(B \subset \mathbb{R}^n\) is any ball verifying \(\text{Vol}_{g_0}(B) = \text{Vol}_g(\Omega)\). The 3-dimensional version of (2.1) is stated by Agostiniani, Fogagnolo and Mazzieri [1, Theorem 1.8] together with the characterization of the equality case.

In order to handle Sobolev-type inequalities on \((M, g)\), by Morse theory, Sard’s theorem and density arguments, it is enough to consider continuous test functions \(u : M \to [0, \infty)\) with compact support \(S \subset M\), where \(S\) is smooth enough, \(u\) being of class \(C^2\) in \(S\) and having only nondegenerate critical points in \(S\), see Aubin [3].

Due to Aubin [3] and Druet, Hebey and Vaugon [16], we associate with such a function \(u : M \to [0, \infty)\) its Euclidean rearrangement function \(u^* : \mathbb{R}^n \to [0, \infty)\) which is radially symmetric, non-increasing in \(|x|\), and for every \(t > 0\) is defined by

\[
\text{Vol}_{g_0}(\{x \in \mathbb{R}^n : u^*(x) > t\}) = \text{Vol}_g(\{x \in M : u(x) > t\}).
\]

(2.3)

By the definition (2.3) and the layer cake representation, see Lieb and Loss [28], it turns out that

\[
\text{Vol}_g(\text{supp}(u)) = \text{Vol}_{g_0}(\text{supp}(u^*)),
\]

(2.4)

and

\[
\|u\|_{L^q(M)} = \|u^*\|_{L^q(\mathbb{R}^n)}, \quad \forall q \in (0, \infty].
\]

(2.5)

The key ingredient in our proof is the following Pólya-Szegö-type inequality

\[
\|\nabla_g u\|_{L^p(M)} \geq \text{AVR}(g)^{\frac{1}{n}} \|\nabla u^*\|_{L^p(\mathbb{R}^n)}, \quad \forall p > 1.
\]

(2.6)

In order to prove (2.6) we adapt the argument from Hebey [20] to our setting. Fix arbitrarily \(0 < t < \|u\|_{L^\infty(M)}\). Then we consider the level sets

\[
\Pi_t = u^{-1}(t) \subset S \subset M \quad \text{and} \quad \Pi^*_t = (u^*)^{-1}(t) \subset \mathbb{R}^n,
\]

which are the boundaries of the sets

\[
\Omega_t = \{x \in M : u(x) > t\} \quad \text{and} \quad \Omega^*_t = \{x \in \mathbb{R}^n : u^*(x) > t\},
\]

(2.7)

respectively. Since \(u^*\) is radially symmetric, the set \(\Pi^*_t\) is an \((n - 1)\)-dimensional sphere. Let

\[
\mathcal{V}(t) := \text{Vol}_g(\Omega_t) = \text{Vol}_{g_0}(\Omega^*_t).
\]
The co-area formula (see Chavel [8, Theorem III.5.2]) and the latter relations imply that

\[ V(t) = \int_t^\infty \left( \int_{\Pi_t} \frac{1}{|\nabla g u|^p} d\sigma_g \right) ds = \int_t^\infty \left( \int_{\Pi_t^*} \frac{1}{|\nabla u^*|^p} d\sigma_{g_0} \right) ds, \]

where \( d\sigma_g \) (resp. \( d\sigma_{g_0} \)) denotes the usual \((n-1)\)-dimensional Riemannian (resp. Lebesgue) measure induced by \( dv_g \) (resp. \( dx \)). Accordingly, one has

\[ V'(t) = -\int_{\Pi_t} \frac{1}{|\nabla g u|} d\sigma_g = -\int_{\Pi_t^*} \frac{1}{|\nabla u^*|} d\sigma_{g_0}. \quad (2.8) \]

Since \(|\nabla u^*|\) is constant on the \((n-1)\)-dimensional sphere \( \Pi_t^* \), by (2.8) it follows that

\[ V'(t) = -\frac{\text{Area}_{g_0}(\Pi_t^*)}{|\nabla u^*(x)|}, \quad x \in \Pi_t^*. \quad (2.9) \]

The first relation of (2.8) and Hölder’s inequality imply that

\[
\begin{align*}
\text{Area}_g(\Pi_t) &= \int_{\Pi_t} d\sigma_g \\
&= \int_{\Pi_t} \frac{1}{|\nabla g u|^{\frac{p-1}{p}}} |\nabla g u|^{\frac{1}{p}} d\sigma_g \\
&\leq (-V'(t))^{\frac{p-1}{p}} \left( \int_{\Pi_t} |\nabla g u|^{p-1} d\sigma_g \right)^{\frac{1}{p}}.
\end{align*}
\]

Thus, by the isoperimetric inequality (2.2) and relation (2.9) (for every \( x \in \Pi_t^* \)) we have that

\[
\begin{align*}
\int_{\Pi_t} |\nabla g u|^{p-1} d\sigma_g &\geq \text{Area}_g^p(\Pi_t) (-V'(t))^{1-p} \\
&\geq \text{AVR}(g)^\frac{1}{p} \text{Area}_{g_0}^p(\Pi_t^*) \left( \frac{\text{Area}_{g_0}(\Pi_t^*)}{|\nabla u^*(x)|} \right)^{1-p} \\
&= \text{AVR}(g)^\frac{1}{p} \int_{\Pi_t^*} |\nabla u^*|^{p-1} d\sigma_{g_0}. \quad (2.10)
\end{align*}
\]

By combining again the co-area formula with this estimate, it follows that

\[
\begin{align*}
\int_M |\nabla g u|^p dv_g &= \int_0^\infty \int_{\Pi_t} |\nabla g u|^{p-1} d\sigma_g dt \\
&\geq \text{AVR}(g)^\frac{1}{p} \int_0^\infty \int_{\Pi_t^*} |\nabla u^*|^{p-1} d\sigma_{g_0} dt \\
&= \text{AVR}(g)^\frac{1}{p} \int_{\mathbb{R}^n} |\nabla u^*|^p dx,
\end{align*}
\]

which concludes the proof of inequality (2.6).
3. Gagliardo-Nirenberg inequalities and their limits: the case \( p \in (1, n) \)

### 3.1. Gagliardo-Nirenberg inequalities

Sharp Gagliardo-Nirenberg inequalities on \( \mathbb{R}^n \), \( n \geq 2 \), are well known after Del Pino and Dolbeault [15] and Cordero-Erausquin, Nazaret and Villani [13]. More precisely, when \( p \in (1, n) \) and \( 1 < \alpha < \frac{n}{n-p} \), then the sharp Gagliardo-Nirenberg inequality on \( \mathbb{R}^n \) reads as

\[
\|u\|_{L^{p}(\mathbb{R}^n)} \leq G_{\alpha, p, n}\|\nabla u\|_{L^{p}(\mathbb{R}^n)}\|u\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}^{1-\theta} \quad \forall u \in W^{1,p}(\mathbb{R}^n),
\]

where

\[
\theta = \frac{p^*(\alpha - 1)}{\alpha p^* - \alpha p + \alpha - 1}
\]

and the best constant

\[
G_{\alpha, p, n} := (\alpha - 1)^{-\theta} \left( \frac{\alpha}{p^* - \alpha - 1} \right)^{\frac{\theta}{p^*} \left( \frac{\alpha(p-1)+1}{\alpha - 1} - \frac{n}{p^*} \right)^{\frac{\theta}{p^*} \left( \frac{\alpha(p-1)+1}{\alpha - 1} - \frac{n}{p^*} \right)}}
\]

is achieved by the family of functions \( h_{\lambda, p}(x) = (\lambda + |x|^{\frac{n}{p^*}})^{1-\alpha} \), \( x \in \mathbb{R}^n \), \( \lambda > 0 \). Here, \( p^* = \frac{np}{n-p} \), \( p' = \frac{p}{p-1} \) and \( W^{1,p}(\mathbb{R}^n) = \{ u \in L^{p'}(\mathbb{R}^n) : \nabla u \in L^{p}(\mathbb{R}^n) \} \), while \( B(\cdot, \cdot) \) is the Euler beta-function.

Our first main result reads as follows:

**Theorem 3.1.** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold as in Theorem 1.1. Let \( p \in (1, n) \), \( 1 < \alpha < \frac{n}{n-p} \) and the constants \( \theta \) and \( G_{\alpha, p, n} \) given by (3.2) and (3.3), respectively. Then for every \( u \in C^0_c(M) \) one has

\[
\|u\|_{L^{p}(M)} \leq K^{G_{\text{GN}}} \|\nabla g u\|_{L^{p}(M)}\|u\|_{L^{\alpha(p-1)+1}(M)}^{1-\theta},
\]

where the constant

\[
K^{G_{\text{GN}}} = G_{\alpha, p, n} \text{AVR}(g)^{-\frac{\theta}{\alpha}}
\]

is sharp. Moreover, equality holds in (3.4) for some nonzero function if and only if \( \text{AVR}(g) = 1 \).

**Proof.** As we pointed out, it is enough to verify inequality (3.4) for test functions \( u : M \to [0, \infty) \) with the properties from Section 2 (i.e. continuous with a compact support \( S \subset M \), \( S \) being smooth enough and \( u \) of class \( C^2 \) in \( S \) with only non-degenerate critical points in \( S \)).

We recall that its Euclidean rearrangement function \( u^* : \mathbb{R}^n \to [0, \infty) \) satisfies the optimal Gagliardo-Nirenberg inequality (3.1), thus relations (2.5) and (2.6) imply that

\[
\|u\|_{L^{p}(M)} = \|u^*\|_{L^{p}(\mathbb{R}^n)} \leq G_{\alpha, p, n}\|\nabla u^*\|_{L^{p}(\mathbb{R}^n)}\|u^*\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}^{1-\theta} \leq G_{\alpha, p, n} \text{AVR}(g)^{-\frac{\theta}{\alpha}} \|\nabla g u\|_{L^{p}(M)}\|u\|_{L^{\alpha(p-1)+1}(M)}^{1-\theta},
\]

which proves inequality (3.4).
By contradiction, we assume that the constant $K^G_{\alpha,p,n} = \mathcal{G}_{\alpha,p,n}^{\mathcal{AVR}}(g)^{-\frac{\alpha}{p}}$ is not sharp in (3.4), i.e. there exists $0 < C < K^G_{\alpha,p,n}$ such that
\[
\|u\|_{L^p(M)} \leq C \|\nabla u\|_{L^p(M)}^{\theta} \|u\|_1^{1-\theta}, \quad \forall u \in C_0^\infty(M).
\]
Since $(M, g)$ is a complete Riemannian manifold with nonnegative Ricci curvature, the validity of the latter inequality implies the quantitative non-collapsing volume property
\[
\text{Vol}_g(B_x(r)) \geq \left(\frac{\mathcal{G}_{\alpha,p,n}}{C}\right)^{\frac{\alpha}{p}} \omega_n r^n \quad \text{for all } x \in M \text{ and } r \geq 0,
\]
see [39], Kristály [23, Theorem 1.1/(i)], and Kristály and Ohta [25]. By the latter relation, one has that
\[
\text{AVR}(g) = \lim_{r \to \infty} \text{Vol}_g(B_x(r)) \geq \omega_n r^n \quad \text{for a.e. } 0 < t < \|u\|_{L^\infty(M)}, \quad \text{i.e.}
\]
\[
\text{Area}_g(\Omega_t) = \text{AVR}(g)^{\frac{1}{p}} \text{Area}_{g_0}(\Pi_t^*). \quad \text{(3.6)}
\]
Recalling relation (2.3) and the facts that $\Pi_t = \partial \Omega_t$ and $\Pi_t^* = \partial \Omega_t^*$ (see (2.7)), relation (3.6) turns out to be equivalent to
\[
\text{Area}_g(\partial \Omega_t) = \frac{1}{n} \omega_n^{\frac{1}{p}} \text{AVR}(g)^{\frac{1}{p}} \text{Vol}_g(\Omega_t)^{\frac{n-1}{p}} \quad \text{for a.e. } 0 < t < \|u\|_{L^\infty(M)}.
\]
The latter relation means that equality holds in the isoperimetric inequality (2.1) for the set $\Omega_t$, $0 < t < \|u\|_{L^\infty(M)}$, thus $(M, g)$ is isometric to $(\mathbb{R}^n, g_0)$ and $\Omega_t$ are balls for every $0 < t < \|u\|_{L^\infty(M)}$. Moreover, the equality in (3.5) implies that $u^*$ is precisely the $h^{\alpha,p}_\lambda(S_g) = \lambda + |x|^{-\alpha} \text{Vol}(\mathbb{B}_r^\alpha, x \in \mathbb{R}^n, \lambda > 0$ up to a multiplicative factor. Conversely, when $(M, g)$ is isometric to $(\mathbb{R}^n, g_0)$, the statement reduces to the equality in (3.1). \hfill \Box

**Proof of Theorem 1.1.** We choose $p \in (1, n)$ and $\alpha = \frac{n}{n-p}$ in Theorem 3.1. Consequently, it follows that $\theta = 1$ and $\mathcal{G}_{\frac{n}{n-p},p,n} = \mathcal{AT}(n,p)$, thus the Gagliardo-Nirenberg inequality (3.4) reduces to the Sobolev inequality: for every $u \in C_0^\infty(M)$ one has
\[
\|u\|_{L^p(M)} \leq K^G_{G} \|\nabla u\|_{L^p(M)},
\]
where $S_g := K^{GN}_g = \mathcal{AT}(n,p) \mathcal{AVR}(g)^{-\frac{1}{p}}$ is sharp. The equality is similar to Theorem 3.1. \hfill \Box

A complementary Gagliardo-Nirenberg inequality with respect to (3.1) is proven by Cordero-Erausquin, Nazaret and Villani [13]; namely, if $0 < \alpha < 1$ and $p \in (1, n)$, one has
\[
\|u\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)} \leq \mathcal{N}_{\alpha,p,n} \|\nabla u\|_{L^p(\mathbb{R}^n)^\gamma} \|u\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)}, \quad \forall u \in W^{1,p}(\mathbb{R}^n), \quad \text{(3.7)}
\]
where
\[
\gamma = \frac{p^*(1-\alpha)}{(p^* - \alpha p)(\alpha p + 1 - \alpha)}, \quad \text{(3.8)}
\]
Taking a limit, it turns out that

\[ N_{\alpha,p,n} := \left( 1 - \frac{\alpha}{p} \right) \gamma \left( \frac{\alpha}{p} \right)^{\frac{1}{\gamma}} \left( \frac{\alpha(p-1)+1}{1-\alpha} + \frac{n}{p} \right)^{\frac{\gamma}{\gamma-1}} \left( \frac{\alpha(p-1)+1}{1-\alpha} \right)^{\frac{1}{\gamma-1}} \]

is achieved by the family of functions \( h_{\alpha,p}^\lambda(x) = (\lambda - |x|^{\frac{p}{p-1}})^{\frac{1}{1-\alpha}}, \ x \in \mathbb{R}^n, \ \lambda > 0 \). Here, \( s_+ = \max(s,0) \).

**Theorem 3.2.** Let \((M,g)\) be an \(n\)-dimensional Riemannian manifold as in Theorem 1.1, \( p \in (1,n), \ 0 < \alpha < 1 \) and the constants \( \gamma \) and \( N_{\alpha,p,n} \) given by (3.8) and (3.9), respectively. Then for every \( u \in C^0_\infty(M) \) one has

\[ \|u\|_{L^{\alpha(p-1)+1}(M)} \leq C_{\alpha,p,n}^{GN} \|\nabla u\|_{L^p(M)}^{\gamma} \|u\|_{L^{\alpha(p-1)+1}(M)}^{1-\gamma}, \]

where the constant

\[ C_{\alpha,p,n}^{GN} = N_{\alpha,p,n} \text{AVR}(g)^{-\frac{\gamma}{\gamma-1}} \]

is sharp. Furthermore, equality holds in (3.10) for some nonzero function if and only if \( \text{AVR}(g) = 1 \).

**Proof.** The proof of inequality (3.10) follows as in Theorem 3.1, by using the optimal Gagliardo-Nirenberg inequality (3.7) and relations (2.5) and (2.6), respectively.

If the constant \( C_{\alpha,p,n}^{GN} = N_{\alpha,p,n} \text{AVR}(g)^{-\frac{\gamma}{\gamma-1}} \) is not sharp in (3.10), there exists \( 0 < C < C_{\alpha,p,n}^{GN} \) such that

\[ \|u\|_{L^{\alpha(p-1)+1}(M)} \leq C \|\nabla u\|_{L^p(M)}^{\gamma} \|u\|_{L^{\alpha(p-1)+1}(M)}^{1-\gamma}, \ \forall u \in C^0_\infty(M). \]

The latter inequality and the result of Kristály [23, Theorem 1.1/(ii)] imply the non-collapsing volume estimate

\[ \text{Vol}_g(B_x(r)) \geq \left( \frac{N_{\alpha,p,n}}{C} \right)^{\frac{n}{\gamma}} \omega_n r^n \text{ for all } x \in M \text{ and } r \geq 0. \]

Taking a limit, it turns out that

\[ \text{AVR}(g) = \lim_{r \to \infty} \frac{\text{Vol}_g(B_x(r))}{\omega_n r^n} \geq \left( \frac{N_{\alpha,p,n}}{C} \right)^{\frac{n}{\gamma}}. \]

This estimate is equivalent to \( C \geq C_{\alpha,p,n}^{GN} \), contradicting the assumption \( 0 < C < K_{\alpha,p,n}^{GN} \). Therefore, \( C_{\alpha,p,n}^{GN} \) is sharp in inequality (3.10). The discussion of the equality case is similar to that of Theorem 3.1. \( \square \)

### 3.2. Limiting cases in Gagliardo-Nirenberg inequalities.

The inequalities (3.1) and (3.7) degenerate to the optimal \( L^p \)-logarithmic Sobolev inequality whenever \( \alpha \to 1 \) (called also as the entropy-energy inequality involving the Shannon entropy), see Del Pino and Dolbeault [15] and Gentil [18]. More precisely, for every \( p \in (1,n) \) one has

\[ \text{Ent}_{dx}(|u|^p) = \int_{\mathbb{R}^n} |u|^p \log |u|^p dx \leq \frac{n}{p} \log \left( \mathcal{L}_{p,n} \|\nabla u\|_{L^p(\mathbb{R}^n)} \right), \ \forall u \in W^{1,p}(\mathbb{R}^n), \ \|u\|_{L^p(\mathbb{R}^n)} = 1, \]

(3.11)
where the best constant
\[ L_{p,n} := \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \left( \omega_n \Gamma \left( \frac{n}{p'} + 1 \right) \right)^{-\frac{n}{p}} \] (3.12)
is achieved by the family of Gaussian functions
\[ l_{p}^{\lambda}(x) := \lambda^{\frac{1}{p'}} \omega_n^{-\frac{1}{p}} \Gamma \left( \frac{n}{p'} + 1 \right)^{-\frac{1}{p}} e^{-\frac{\lambda}{p} |x|^{p'}} , \lambda > 0. \]

**Theorem 3.3.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold as in Theorem 1.1, \(p \in (1, n)\) and the constant \(L_{p,n}\) given by (3.12). Then for every \(u \in C_{0}^{\infty}(M)\) with \(\|u\|_{L^{p}(M)} = 1\) one has
\[ \text{Ent}_{dv_{g}}(|u|^{p}) = \int_{M} |u|^{p} \log |u|^{p} dv_{g} \leq \frac{n}{p} \log \left( L_{g} \|\nabla_{g} u\|_{L^{p}(M)}^{p} \right), \] (3.13)
where the constant
\[ L_{g} = L_{p,n} \text{AVR}(g)^{-\frac{n}{p}} \]
is sharp. In addition, equality holds in (3.13) for some nonzero function if and only if \(\text{AVR}(g) = 1\).

**Proof.** Let \(u : (M \rightarrow [0, \infty)\) be a nonzero test function with the properties as in Section 2 and \(\|u\|_{L^{p}(M)} = 1\), and \(u^* : \mathbb{R}^{n} \rightarrow [0, \infty)\) its Euclidean rearrangement function; in particular, \(\|u^*\|_{L^{p}(\mathbb{R}^{n})} = 1\) due to (2.5). In spite of the fact that the entropy-conversation by means of the symmetrization (2.3) is expected, we have not found any reference in this subject; therefore, we describe the main steps to prove
\[ \text{Ent}_{dv_{g}}(u^{p}) = \text{Ent}_{dx}((u^*)^{p}). \] (3.14)

For convenience, we recall the notations \(\Omega_{t}\) and \(\Omega_{t}^{*}\) from (2.7), \(0 < t < \|u\|_{L^{\infty}(M)}\). Since the function \(s \mapsto s \log s\) is positive and increasing on \((1, \infty)\), by the layer cake representation and relation (2.3) it turns out that
\[ \int_{\Omega_{t}} u^{p} \log u^{p} dv_{g} = \int_{\Omega_{t}^{*}} (u^*)^{p} \log (u^*)^{p} dx. \]
Clearly, we implicitly assumed that in the above step we have \(\|u\|_{L^{\infty}(M)} > 1\); otherwise, both integrals vanish as \(\Omega_{t}^{*} = \Omega_{t} = \emptyset\). Furthermore, a standard argument based again on (2.3) implies that
\[ \int_{M \setminus \Omega_{1}} u^{p} dv_{g} = \int_{\mathbb{R}^{n} \setminus \Omega_{1}^{*}} (u^*)^{p} dx; \]
by the properties of \(u\) and \(u^*\) the latter terms turn out to be finite. Since \(s \mapsto s(1 - \log s)\) is positive and increasing on \((0, 1)\), one has by the layer cake representation and (2.3) that
\[ \int_{M \setminus \Omega_{1}} u^{p}(1 - \log u^{p}) dv_{g} = \int_{\mathbb{R}^{n} \setminus \Omega_{1}^{*}} (u^*)^{p}(1 - \log(u^*)^{p}) dx. \]
Combining the latter three relations, (3.14) follows at once.
Thus, by (3.14), (2.6) and inequality (3.11) one has that
\[
\text{Ent}_{dvg}(u^p) = \text{Ent}_{dx}((u^*)^p) \\
\leq \frac{n}{p} \log \left( \mathcal{L}_{p,n} \| \nabla u^* \|_{L^p(\mathbb{R}^n)}^p \right) \\
\leq \frac{n}{p} \log \left( \mathcal{L}_{p,n} \text{AVR}(g)^{-\frac{p}{n}} \| \nabla g u \|_{L^p(M)}^p \right),
\]
which proves inequality (3.13).

We now assume that \( L_g > 0 \) is not sharp in (3.13), i.e. there exists \( 0 < C < L_g \) such that
\[
\text{Ent}_{dvg}(|u|^p) \leq \frac{n}{p} \log \left( C \| \nabla g u \|_{L^p(M)}^p \right), \quad \forall u \in C_0^\infty(M), \|u\|_{L^p(M)} = 1.
\]
By Kristály [23, Theorem 1.2], the latter inequality implies that
\[
\text{Vol}_g(B_x(r)) \geq \left( \frac{\mathcal{L}_{p,n}}{C} \right)^{\frac{1}{p'}} \omega_n r^n \quad \text{for all } x \in M \text{ and } r \geq 0.
\]
This inequality leads us to \( \text{AVR}(g) \geq \left( \frac{\mathcal{L}_{p,n}}{C} \right)^{\frac{1}{p'}} \), which contradicts \( 0 < C < L_g \). Therefore, the constant \( L_g = \mathcal{L}_{p,n} \text{AVR}(g)^{-\frac{p}{n}} \) is optimal in (3.13). For the equality case we argue in the same way as in the proof of Theorem 3.1. \( \square \)

The last question within this section is the limiting case in (3.7) whenever \( \alpha \to 0 \); this is referred as a Faber-Krahn-type inequality, see [13, p. 320]. More precisely, for every \( p \in (1, n) \), one has
\[
\|u\|_{L^1(\mathbb{R}^n)} \leq \mathcal{F}_{p,n} \| \nabla u \|_{L^p(\mathbb{R}^n)} \text{Vol}_g(\text{supp}(u))^{1-\frac{1}{p'}}, \quad \forall u \in W^{1,p}(\mathbb{R}^n),
\]
and the sharp constant
\[
\mathcal{F}_{p,n} = \lim_{\alpha \to 0} \mathcal{N}_{\alpha,p,n} = n^{-\frac{1}{p'}} \omega_n \left( p' + n \right)^{-\frac{1}{p'}}
\]
is achieved by the family of functions
\[
f^\lambda_p(x) = \lim_{\alpha \to 0} h^\lambda_{\alpha,p}(x) = (\lambda - |x|^{p-1})_+, \quad \lambda > 0, \ x \in \mathbb{R}^n,
\]
where \( \text{supp}(u) \) stands for the support of \( u \).

**Theorem 3.4.** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold as in Theorem 1.1, \( p \in (1, n) \) and the constant \( \mathcal{F}_{p,n} \) given by (3.16). Then for every \( u \in C_0^\infty(M) \) one has
\[
\|u\|_{L^1(M)} \leq \mathcal{F}_g \| \nabla g u \|_{L^p(M)} \text{Vol}_g(\text{supp}(u))^{1-\frac{1}{p'}},
\]
where the constant
\[
\mathcal{F}_g = \mathcal{F}_{p,n} \text{AVR}(g)^{-\frac{1}{p'}}
\]
is sharp. Furthermore, equality holds in (3.17) for some nonzero function if and only if \( \text{AVR}(g) = 1 \).
Proof. The proof of (3.17) follows by relations (2.4), (2.6) and (3.15), respectively. If the constant \(F_g = F_{p,n}\) would not be sharp in (3.17), then there exists \(0 < C < F_g\) such that
\[
\|u\|_{L^1(M)} \leq C \|\nabla_g u\|_{L^p(M)} \text{Vol}_g(\text{supp}(u))^{1 - \frac{1}{p'}}, \quad \forall u \in C_0^\infty(M).
\]
By Kristály [23, Theorem 1.3], one has that
\[
\text{Vol}_g(B_x(r)) \geq \left(\frac{F_{p,n}}{C}\right)^n \omega_n r^n \quad \text{for all } x \in M \text{ and } r \geq 0.
\]
Taking \(r \to \infty\), it follows that \(\text{AVR}(g) \geq \left(\frac{F_{p,n}}{C}\right)^n\). The latter inequality contradicts \(0 < C < F_g\), which proves that the constant \(F_g = F_{p,n}\) is sharp in (3.17). The equality case is discussed as in the previous results. \(\square\)

4. Morrey-Sobolev inequalities: the case \(p > n\)

Before presenting the Morrey-Sobolev inequalities, the following auxiliary result will be used whose proof follows in a straightforward way by the layer cake representation (for further use, \(d_g : M \times M \to \mathbb{R}\) denotes the usual metric function on the Riemannian manifold \((M, g)\):

**Lemma 4.1.** Let \((M, g)\) be a complete Riemannian manifold, \(R > 0\) and \(x_0 \in M\) are arbitrarily fixed, and \(f : (0, R] \to \mathbb{R}\) be a \(C^1\)-function on \((0, R)\). Then
\[
\int_{B_{x_0}(R)} f(d_g(x_0, x)) dv_g = f(R) \text{Vol}_g(B_{x_0}(R)) - \int_0^R f'(r) \text{Vol}_g(B_{x_0}(r)) dr.
\]

In the sequel we prove two different types of \(L^p\)-Sobolev-Morrey inequalities on noncompact, complete \(n\)-dimensional Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growths, \(p > n \geq 2\); the first is a kind of Gagliardo-Nirenberg interpolation inequality while the second one is a Faber-Krahn-type inequality.

4.1. Morrey-Sobolev interpolation: sharp \(L^1\)-bound. Let \(p > n \geq 2\). Talenti [38, Theorem 2.C] proved the following Morrey-Sobolev inequality: for every \(u \in C_0^\infty(\mathbb{R}^n)\) one has
\[
\|u\|_{L^\infty(\mathbb{R}^n)} \leq M_{p,n} \|u\|_{L^1(\mathbb{R}^n)}^{1 - \eta} \|\nabla u\|_{L^p(\mathbb{R}^n)}^\eta, \quad (4.1)
\]
where by scaling
\[
\eta = \frac{np}{np + p - n} \quad (4.2)
\]
and the sharp constant
\[
M_{p,n} = (n\omega_n)^{\frac{1}{n}} \cdot \frac{np' - n}{n + p'} \left(\frac{1}{n} + \frac{1}{p'}\right) \left(\frac{1}{n} - \frac{1}{p'}\right)^{\frac{(n-1)p' - n}{n + p'}} \left(\mathcal{B}\left(1 - \frac{n}{n}p' + 1, p' + 1\right)\right)^{\frac{n}{n + p'}} \quad (4.3)
\]
is achieved by the function
\[
u(x) = \begin{cases} \int_0^{1} r^{\frac{1-n}{p-n}}(1 - r^n)^{\frac{1}{p-n}} dr, & \text{if } |x| \leq 1; \\ 0, & \text{otherwise.} \end{cases}
\]
We prove the following sharp result on Riemannian manifolds with nonnegative Ricci curvature:

**Theorem 4.1.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold as in Theorem 1.1, \(p > n \geq 2\) and the constants \(\eta\) and \(M_{p,n}\) given by (4.2) and (4.3), respectively. Then for every \(u \in C_0^\infty(M)\) one has

\[
\|u\|_{L^\infty(M)} \leq K_{g}^{MS} \|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)},
\]

where the constant

\[
K_{g}^{MS} = M_{p,n} \text{AVR}(g)^{-\frac{n}{n}}
\]

is sharp. Moreover, equality holds in (4.4) for some nonzero function if and only if \(\text{AVR}(g) = 1\).

**Proof.** The proof of (4.4) directly follows by relations (2.4)-(2.6) and (4.1). We assume now that the constant \(K_{g}^{MS} = M_{p,n} \text{AVR}(g)^{-\frac{n}{n}}\) is not sharp in (4.4), i.e. there exist \(C < K_{g}^{MS}\) such that

\[
\|u\|_{L^\infty(M)} \leq C \|u\|_{L^1(M)}^{1-\eta} \|\nabla_g u\|_{L^p(M)}, \quad \forall u \in C_0^\infty(M).
\]

Although there is a volume-non-collapsing estimate on Riemannian manifolds with nonnegative Ricci curvature that support (4.5), see Kristály [24], it is not in a sharp form, being in this way not appropriate for our purposes; accordingly, we provide in the sequel its optimal form. To do this, let \(h, H : (0, 1] \to \mathbb{R}\) be the functions

\[
h(r) = r^{\frac{1-n}{p-1}}(1 - r^n)^{\frac{1}{p-1}} \quad \text{and} \quad H(s) = \int_0^s h(r)dr.
\]

Let \(x_0 \in M\) and \(R > 0\) be fixed, and consider the function \(u_R : M \to \mathbb{R}\) defined by

\[
u_R(x) = \begin{cases} H(1) - H \left( \frac{d_g(x_0, x)}{R} \right), & \text{if } x \in B_{x_0}(R); \\ 0, & \text{otherwise.} \end{cases}
\]

We have first that

\[
\|u_R\|_{L^\infty(M)} = H(1) = \int_0^1 h(r)dr = \frac{1}{n} B \left( \frac{1-n}{n}, p' + 1, p' \right).
\]

By using Lemma 4.1 and a change of variables, it follows that

\[
\|u_R\|_{L^1(M)} = \int_{B_{x_0}(R)} \left( H(1) - H \left( \frac{d_g(x_0, x)}{R} \right) \right) dv_g = \frac{1}{R} \int_0^R \text{Vol}_g(B_{x_0}(R)) H' \left( \frac{r}{R} \right) dr
\]

and

\[
\|u_R\|_{L^p(M)}^p = \frac{1}{R^p} \int_{B_{x_0}(R)} h^p \left( \frac{d_g(x_0, x)}{R} \right) dv_g = -\frac{1}{R^p} \int_0^1 \text{Vol}_g(B_{x_0}(R))(h^p)'(t) dt.
\]
By density reasons, the function \( u_R \) can be used as a test function in (4.5), i.e.

\[
\|u_R\|_{L^\infty(M)} \leq C \|u_R\|_{L^1(M)}^{1-\eta/\gamma} \|\nabla u_R\|_{L^\gamma(M)}^\gamma.
\]

Furthermore, the Lebesgue’s dominated convergence theorem and relations (4.7) and (4.8) imply that

\[
\lim_{R \to \infty} \frac{\|u_R\|_{L^1(M)}}{R^n} = \omega_n \text{AVR}(g) \int_0^1 t^n h(t) \, dt = \omega_n \text{AVR}(g) \frac{1}{n} B \left( \frac{1-n}{n} p' + 2, p' \right),
\]

and

\[
\lim_{R \to \infty} \frac{\|\nabla u_R\|_{L^p(M)}}{R^{n-p}} = -\omega_n \text{AVR}(g) \int_0^1 t^n (h^p)'(t) \, dt = \omega_n \text{AVR}(g) B \left( \frac{1-n}{n} p' + 1, p' + 1 \right).
\]

Therefore, by using the latter limits and relations (4.6) and (4.3), a straightforward manipulation of the above terms implies \( \text{M}_{p,n} \leq C \text{AVR}(g)^{1-\eta+\frac{p}{2}} \). Since \( 1-\eta + \frac{p}{n} = \frac{n}{n} \) (see (4.2)), the latter inequality contradicts our initial assumption \( C < \text{K}^{\text{MS}}_g \), which concludes the sharpness of \( \text{K}^{\text{MS}}_g \) in (4.4). The equality case can be similarly discussed as in the previous results. \( \Box \)

4.2. **Morrey-Sobolev interpolation: sharp support-bound.** Let \( p > n \geq 2 \). The following *Morrey-Sobolev inequality* is proved also by Talenti [38, Theorem 2.1], stating that

\[
\|u\|_{L^\infty(\mathbb{R}^n)} \leq T_{p,n} \|\nabla u\|_{L^p(\mathbb{R}^n)} \text{Vol}_g(\text{supp } u)^{\frac{1}{n} - \frac{1}{p}}
\]

for every \( u \in C^\infty_0(\mathbb{R}^n) \), where \( \text{supp } u \subset \mathbb{R}^n \) stands for the support of \( u \), while the sharp constant

\[
T_{p,n} = n^{-\frac{1}{p}} \omega_n^{-\frac{1}{n}} \left( \frac{p-1}{p-n} \right)^{-\frac{1}{p}}
\]

is achieved by the function \( u(x) = \left(1 - |x|^\frac{p-n}{p} \right)^+ \); compare (4.10) with (3.15).

**Theorem 4.2.** Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold as in Theorem 1.1, \( p > n \geq 2 \) and the constant \( T_{p,n} \) given by (4.11). Then for every \( u \in C^\infty_0(M) \) one has

\[
\|u\|_{L^\infty(M)} \leq C_g^{\text{MS}} \|\nabla u\|_{L^p(M)} \text{Vol}_g(\text{supp } u)^{\frac{1}{n} - \frac{1}{p}},
\]

where the constant

\[
C_g^{\text{MS}} = T_{p,n} \text{AVR}(g)^{-\frac{1}{n}}
\]

is sharp. Furthermore, equality holds in (4.12) for some nonzero function if and only if \( \text{AVR}(g) = 1 \).

**Proof.** Inequality (4.12) follows by (4.10) and relations (2.4)-(2.6). We assume by contradiction that there exists a constant \( \hat{C} < C_g^{\text{MS}} \) such that

\[
\|u\|_{L^\infty(M)} \leq \hat{C} \|\nabla u\|_{L^p(M)} \text{Vol}_g(\text{supp } u)^{\frac{1}{n} - \frac{1}{p}}, \quad \forall u \in C^\infty_0(M).
\]

We fix \( x_0 \in M \) and \( R > 0 \), and define the function \( u_R : M \to \mathbb{R} \) by

\[
u_R(x) = \left(1 - \left(\frac{d_g(x_0, x)}{R} \right)^{\frac{p-n}{p}}\right)^+,
\]

\( x \in M \).
By the eikonal equation and Lemma 4.1 it follows that
\[
\|\nabla_g u_R\|_{L^p(M)}^p = \left(\frac{p-n}{p-1}\right)^p \int_{B_{x_0}(R)} \left(\frac{d_g(x_0, x)}{R}\right)^{(1-n)p'} \, dv_g
\]
\[
= \frac{1}{R^p} \left(\frac{p-n}{p-1}\right)^p \left(\text{Vol}_g(B_{x_0}(R)) - (1-n)p' \int_0^1 t^{(1-n)p'-1}\text{Vol}_g(B_{x_0}(Rt)) \, dt\right).
\]
The latter relation and the Lebesgue’s dominated convergence theorem imply that
\[
\lim_{R \to \infty} \frac{\|\nabla_g u_R\|_{L^p(M)}^p}{R^{n-p}} = n\omega_n \left(\frac{p-n}{p-1}\right)^{p-1} \text{AVR}(g).
\]
Note that \(\|u_R\|_{L^\infty(M)} = 1\) and \(\text{supp} \, u_R = B_{x_0}(R)\). If we use \(u_R\) as a test function in (4.13), a limiting argument via the latter limit gives that
\[
1 \leq C \omega_n^{\frac{1}{p}} n^{\frac{1}{p}} \left(\frac{p-n}{p-1}\right)^{\frac{1}{p}} \text{AVR}(g)^{\frac{1}{p}},
\]
which is equivalent to \(C \geq T_{p,n} \text{AVR}(g)^{-\frac{1}{p}} = C_g^{\text{MS}}\), contradicting our initial assumption. The equality case is treated in a similar manner as above. \(\square\)

5. Rayleigh-Faber-Krahn inequality: first eigenvalues in sharp form

In this section we are going to prove Theorem 1.2. In fact, we shall prove first:

**Theorem 5.1.** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold as in Theorem 1.1. Then for every smooth bounded open set \(\Omega \subset M\) and \(u \in C^\infty_0(\Omega)\) we have
\[
\text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 \, dv_g \leq R_g \int_{\Omega} |\nabla_g u|^2 \, dv_g,
\]
where the constant
\[
R_g = j^{-2}_{\frac{n}{p}-1}(\omega_n \text{AVR}(g))^{-\frac{2}{n}}
\]
is sharp. In addition, equality holds in (5.1) for some set \(\Omega \subset M\) and for some nonzero function if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, g_0)\) and \(\Omega\) is isometric to a ball \(B \subset \mathbb{R}^n\) with \(\text{Vol}_g(\Omega) = \text{Vol}_{g_0}(B)\).

**Proof.** Let \(\Omega \subset M\) be any smooth bounded open set and \(B \subset \mathbb{R}^n\) be a ball with \(\text{Vol}_g(\Omega) = \text{Vol}_{g_0}(B)\). If \(u : \Omega \to \mathbb{R}\) is any nonzero function with the properties from Section 2, its Euclidean rearrangement function \(u^* : B \to [0, \infty)\) satisfies properties (2.4)-(2.6); in particular, the latter relations combined with the Euclidean Rayleigh-Faber-Krahn inequality immediately gives
\[
\frac{\int_{\Omega} |\nabla_g u|^2 \, dv_g}{\text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 \, dv_g} \geq \frac{\text{AVR}(g)^{\frac{2}{n}}}{\text{Vol}_{g_0}(B)^{-\frac{2}{n}} \int_B (u^*)^2 \, dx} \geq j^{-2}_{\frac{n}{p}-1}(\omega_n \text{AVR}(g))^{\frac{2}{n}} = R_g^{-1},
\]
which is exactly inequality (5.1).
The sharpness of the constant $R_g$ in (5.1) is more delicate, which requires fine properties of Bessel functions of the first kind $J_\nu, \nu \in \mathbb{R}$. For completeness, we first recall those properties which are crucial in our arguments; for details, see e.g. Olver et al. [34].

The Bessel function of the first kind with order $\nu \in \mathbb{R}$, denoted by $J_\nu$, is the nonsingular solution of the differential equation

$$t^2 y''(t) + ty'(t) + (t^2 - \nu^2) = 0.$$  

The function $J_\nu$ has the following useful properties

$$J'_\nu(t) = -J_{\nu+1}(t) + \frac{\nu}{t}J_\nu(t), \quad t > 0,$$  

$$\int_0^1 tj_{\nu+1}^2(j_\nu t)dt = \int_0^1 tj_{\nu}^2(j_\nu t)dt = \frac{J_{\nu+1}^2(j_\nu)}{2},$$  

where $J'_\nu$ is the derivative of $J_\nu$ and $j_\nu$ stands for the first positive root of $J_\nu$.

We are going to prove the sharpness of $R_g = \frac{\nu^2 - 1}{\nu^2}$ in (5.1). By contradiction, we assume there exists $C < R_g$ such that for every smooth bounded open set $\Omega \subset M$, \n
$$\text{Vol}_g(\Omega)^{\frac{2}{\nu}} \int_\Omega u^2 dv_g \leq C \int_\Omega |\nabla_g u|^2 dv_g, \quad \forall u \in C_0^\infty(\Omega).$$  

(5.5)

For convenience of notation, we choose $\nu = \frac{\tau}{2} - 1 \geq 0$. For every $R > 0$ and $x_0 \in M$, we consider the function $u_R : B_{x_0}(R) \rightarrow \mathbb{R}$ defined by

$$u_R(x) = d_g(x_0, x)^{-\nu}J_\nu\left(j_\nu \frac{d_g(x_0, x)}{R}\right), \quad x \in B_{x_0}(R).$$

By density arguments, it turns out that (5.5) can be applied for the function $u_R$ and set $\Omega = B_{x_0}(R)$, i.e. one has

$$\text{Vol}_g(B_{x_0}(R))^{-\frac{2}{\nu}} \int_{B_{x_0}(R)} u_R^2 dv_g \leq C \int_{B_{x_0}(R)} |\nabla_g u_R|^2 dv_g.$$  

(5.6)

Relation (5.3) implies that for a.e. $x \in B_{x_0}(R)$,

$$\nabla_g u_R(x) = d_g(x_0, x)^{-\nu}\left(-\frac{\nu}{d_g(x_0, x)}J_\nu\left(j_\nu \frac{d_g(x_0, x)}{R}\right) + \frac{j_\nu}{R}J'_\nu\left(j_\nu \frac{d_g(x_0, x)}{R}\right)\right)\nabla_g d_g(x_0, x).$$

By using the eikonal equation it follows that

$$\int_{B_{x_0}(R)} |\nabla_g u_R|^2 dv_g = \frac{\nu^2}{R^2} \int_{B_{x_0}(R)} d_g(x_0, x)^{-2\nu}j_{\nu+1}^2\left(j_\nu \frac{d_g(x_0, x)}{R}\right) dv_g.$$  

(5.7)

Applying Lemma 4.1, by a change of variables and the Lebesgue’s dominated convergence theorem one has

$$\lim_{R \to \infty} \int_{B_{x_0}(R)} |\nabla_g u_R|^2 dv_g = j_\nu^2 \omega_n \text{AVR}(g) \left(J_{\nu+1}^2(j_\nu) - \int_0^1 t^n \frac{d}{dt}(t^{-2\nu}J_{\nu+1}^2(j_\nu t))dt\right)$$

$$= j_\nu^2 n \omega_n \text{AVR}(g) \int_0^1 tJ_{\nu+1}^2(j_\nu t)dt.$$
Since $J_{
u}(j_{
u}) = 0$, a similar reasoning as above shows that

$$\lim_{R \to \infty} \frac{\int_{B_{x_0}(R)} u^2 dv_g}{R^2} = n \omega_n \text{AVR}(g) \int_0^1 t J_{
u}(j_{
u}, t) dt.$$  

Letting $R \to \infty$ in (5.6) and taking into account the latter relations and the integral identity (5.4), we obtain $(\omega_n \text{AVR}(g))^{- \frac{2}{n}} \leq C j_{
u}^2$. This inequality contradicts $C < R_g = j_{
u}^{-2}(\omega_n \text{AVR}(g))^{- \frac{2}{n}}$, concluding the sharpness of $R_g$ in (5.1).

Assume now that equality holds in (5.1) for some $\Omega \subset M$ and some nonzero function $u : \Omega \to \mathbb{R}$; in particular, equality in (2.6) also holds for its Euclidean rearrangement function $u^* : B \to \mathbb{R}$, where $B \subset \mathbb{R}^n$ is a ball with $\text{Vol}_g(\Omega) = \text{Vol}_{y_0}(B)$, cf. Section 2. Consequently, one has $\text{AVR}(g) = 1$ and $\Omega$ is isometric to $B \subset \mathbb{R}^n$. The converse is trivial. □

**Proof of Theorem 1.2.** The proof rests upon Theorem 5.1. Indeed, the sharp estimate of the first eigenvalue $\lambda_{1,\nu}^P(\Omega)$ in (1.5) immediately follows by (5.1), noticing that $R_g = \Lambda_{\nu}^{-1}$. Standard compactness and variational argument show that the infimum in (1.3) is achieved. Thus, if equality holds in (1.5) for some $\Omega \subset M$, then there exists some nonzero function $u : \Omega \to \mathbb{R}$ which produces equality in (5.1) as well, concluding the proof, cf. Theorem 5.1. □

6. Examples and final comments

We notice that Riemannian manifolds with nonnegative Ricci curvature have been widely studied in the literature, stating various classifications and topological rigidities, see e.g. Anderson [2], Cheeger and Colding [9], Colding [12], Li [30], Liu [29], Menguy [31], Perelman [35], Reiris [36], Zhu [40]. In the sequel we first present two Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growths, providing as well their explicit asymptotically volume growths. We then conclude the paper by some comments.

6.1. **Rotationally invariant metric on** $\mathbb{R}^n$. Let $n \geq 3$ and $f : [0, \infty) \to [0, 1]$ be a smooth nonincreasing function such that $f(0) = 1$ and $\lim_{s \to \infty} f(s) = a \in [0, 1]$. We consider the rotationally invariant metric on $\mathbb{R}^n$ defined by the warped product metric

$$g = dr^2 + F(r)^2 d\theta^2,$$

where $F(r) = \int_0^r f(s) ds$ and $d\theta^2$ is the standard metric on the sphere $S^{n-1}$. If $x = (x_1, \theta_1)$ and $\tilde{x} = (x_2, \theta_2)$ are two point in $\mathbb{R}^n$, it turns out that $d_g(x, \tilde{x}) \geq |x_1 - x_2|$, which implies that $(M, g)$ is complete. Furthermore, it is well known that the curvature operator of $g$ has two eigenvalues, i.e.,

$$-\frac{F''(r)}{F(r)} = -\frac{f'(r)}{F(r)} \geq 0$$

and

$$\frac{1 - (F')^2(r)}{F^2(r)} = \frac{1 - f(r)^2}{F(r)^2} \geq 0$$

for every $r > 0$, see Carron [6]. Accordingly, the sectional (thus, the Ricci) curvature of $(\mathbb{R}^n, g)$ is nonnegative.
For $R \gg 1$, one has that
\[
\text{Vol}_g(B_0(R)) = \int_{B_0(R)} dv_g \sim n\omega_n \int_0^R F(r)^{n-1} dr.
\]
The latter estimate and L'Hôpital's rule give that
\[
\text{AVR}(g) = \lim_{R \to \infty} \frac{\text{Vol}_g(B_0(R))}{\omega_n R^n} = \lim_{R \to \infty} \frac{n \int_0^R F(r)^{n-1} dr}{R^n} = \lim_{R \to \infty} \frac{F(R)^{n-1}}{R^{n-1}} = a^{n-1}.
\]
In particular, when $a = 1$, i.e. $\text{AVR}(g) = 1$, by our assumption it turns out that $f \equiv 1$ on $[0, \infty)$; thus $F(r) = r$ for every $r > 0$, and the metric becomes the Euclidean one, i.e. $g = g_0 = dr^2 + r^2 d\theta^2$.

When $a \in (0, 1)$, i.e. $(\mathbb{R}^n, g)$ has Euclidean volume growth, then the sharp Sobolev-type inequalities stated in Sections 3-5 for $(\mathbb{R}^n, g)$ are all valid, with no extremal functions.

When $a = 0$, thus $\text{AVR}(g) = 0$, as we already stated in the Introduction, no Sobolev inequalities are expected on $(\mathbb{R}^n, g)$. In fact, one can also prove that for every $\theta \in S^{n-1}$,
\[
\lim_{x \to \infty} \text{Vol}_g(B_{(x, \theta)}(1)) = 0,
\]
which confirms the failure of Sobolev inequalities on $(\mathbb{R}^n, g)$, see Hebey [20, Theorem 3.3/(ii)].

6.2. Asymptotically locally Euclidean manifolds. Following Agostiniani, Fogagnolo and Mazzieri [1, Definition 4.13], a complete, noncompact Riemannian manifold $(M, g)$ is asymptotically locally Euclidean manifold if there exist a compact set $K \subset M$, a ball $B \subset \mathbb{R}^n$, a diffeomorphism $\Psi : M \subset K \to \mathbb{R}^n \setminus B$, a number $\tau > 0$ and a finite subgroup $G$ of $SO(n)$ acting freely on $\mathbb{R}^n \setminus B$ such that
\[
(\Psi^{-1} \circ \pi)^* g(z) = g_0 + O(|z|)^{-\tau};
\]
\[
|\partial_i ((\Psi^{-1} \circ \pi)^* g)(z) = O(|z|)^{-\tau-1};
\]
\[
|\partial_i \partial_j ((\Psi^{-1} \circ \pi)^* g)(z) = O(|z|)^{-\tau-2},
\]
where $\pi : \mathbb{R}^n \to \mathbb{R}^n/G$ stands for the natural projection, $z \in \mathbb{R}^n \setminus B$ and $i, j \in \{1, \ldots, n\}$.

Due to assumptions (6.1)-(6.3), it turns out that $(M, g)$ has Euclidean volume growth; furthermore, one has that
\[
\text{AVR}(g) = \frac{1}{\text{Card}(G)},
\]
see [1, rel. (4.31)]. In particular, $(M, g)$ is isometric to $(\mathbb{R}^n, g_0)$ if and only if $G = \{\text{Id}\} \subset SO(n)$; otherwise, $0 < \text{AVR}(g) < 1$ and no extremal functions exist in the sharp Sobolev inequalities on $(M, g)$ stated in Sections 3-5.

When $n = 3$, the finite subgroups of $SO(3)$ are isomorphic to either a cyclic group $\mathbb{Z}/m = \mathbb{Z}_m$ ($m \in \mathbb{N} \setminus \{0, 1\}$), a dihedral group $D_m$, or the rotational symmetry group of a regular solid, i.e. (a) the symmetry group of the tetrahedron $A_4$, (b) the symmetry group of the cube $S_4$ (or octahedron), (c) the symmetry group of the dodecahedron $A_5$ (or icosahedron). According to (6.4), one can establish various sharp Sobolev inequalities on 3-dimensional asymptotically locally Euclidean manifolds depending on the choice of the finite subgroup of $SO(3)$ provided by the above list.
6.3. Final remarks and further questions. We conclude the paper with two questions; in both cases we assume that \((M, g)\) is an \(n\)-dimensional complete Riemannian manifold with nonnegative Ricci curvature and Euclidean volume growth, \(n \geq 2\).

6.3.1. Moser-Trudinger inequality. In Sections 3 and 4 we discussed sharp \(L^p\)-Sobolev inequalities on \(n\)-dimensional Riemannian manifolds, both for \(p < n\) and \(p > n\), respectively. It is natural to ask about the case \(p = n\), when a Moser-Trudinger inequality is obtained. In fact, due to Kristály [22], one can prove that for every smooth bounded open set \(\Omega \subset M\) and for every \(\alpha \in [0, \alpha_n(g)]\) and \(\in C_0^\infty(\Omega)\) with \(\|\nabla_g u\|_{L^n(\Omega)} \leq 1\) one has

\[
\int_{\Omega} \Phi_n \left( \alpha |u|^{\frac{n}{n-1}} \right) \, dv_g \leq C_0 \|\nabla_g u\|_{L^n(\Omega)} \, \text{Vol}_g(\Omega),
\]

where \(\alpha_n(g) = n(\omega_{n-1}\text{AVR}(g))^{\frac{1}{n-1}}\), the number \(C_0 = C_0(n) > 0\) depends only on \(n \geq 2\) and

\[
\Phi_n(t) = e^t - \sum_{k=0}^{n-2} \frac{t^k}{k!}, \quad t \geq 0.
\]

When \((M, g)\) is the Euclidean space, the constant \(\alpha_n(g) = \alpha_n(g_0) = n\omega_{n-1}^{\frac{1}{n-1}}\) turns out to be sharp, see Moser [32]. However, no information is available on the sharpness of \(\alpha_n(g)\) whenever \((M, g)\) is not isometric to \((\mathbb{R}^n, g_0)\).

6.3.2. Sobolev inequalities involving singular terms. Another class of problems represents the Sobolev inequalities involving Hardy-type singular terms of the form \(x \mapsto d_g(x_0, x)^{\gamma}\) with \(\gamma < 0\) and fixed \(x_0 \in M\). In particular, if \(\Omega \subset M\) is any smooth bounded open set and \(x_0 \in \Omega\), for sufficiently small \(\mu \geq 0\) we consider the Brezis-Vázquez-Poincaré inequality

\[
\int_{\Omega} |\nabla_g u|^2 \, dv_g - \mu \int_{\Omega} \frac{u^2}{d_g(x_0, x)^2} \, dv_g \geq S_{\mu}(g) \, \text{Vol}_g(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 \, dv_g, \quad \forall u \in C_0^\infty(\Omega),
\]

where \(S_{\mu}(g) > 0\). In fact, the optimal range for \(\mu\) seems to be \([0, \frac{(n-2)^2}{4} \text{AVR}(g)]\). Indeed, since by the Bishop-Gromov comparison principle we have \(\text{Vol}_g(B_{x_0}, R) \leq \text{Vol}_{g_0}(B_0(R))\) for every \(R > 0\), a standard Hardy-Littlewood argument (see Lieb and Loss [28, Theorem 3.4]) combined with the symmetrization (2.3) shows that for every function \(u : \Omega \to R\) with the properties from Section 2, one has

\[
\int_{\Omega} \frac{u^2}{d_g(x_0, x)^2} \, dv_g \leq \int_B (u^*)^2 \, \frac{dx}{|x|^2},
\]

where \(u^* : B \to \mathbb{R}\) stands for the Euclidean rearrangement of \(u\) and \(B \subset \mathbb{R}^n\) is the ball with center in \(0 \in \mathbb{R}^n\) and \(\text{Vol}_g(\Omega) = \text{Vol}_{g_0}(B)\). Thus, by (2.6) and the Euclidean Hardy inequality, the left hand side of (6.6) turns out to be nonnegative whenever \(\mu \leq \frac{(n-2)^2}{4} \text{AVR}(g)^{\frac{2}{n}}\). The latter arguments together with the results of Kristály and Szakál [26] imply that if \(\mu \in\)
\[0, \frac{(n-2)^2}{4} \text{AVR}(g) \frac{2}{\pi}\]

one has
\[
\int_{\Omega} |\nabla g u|^2 dv_g - \mu \int_{\Omega} \frac{u^2}{d_g(x_0, x)^2} dv_g \geq \text{AVR}(g) \frac{2}{\pi} \int_{B} |\nabla u^*|^2 dx - \mu \int_{B} \frac{(u^*)^2}{|x|^2} dx
\]
\[
\geq \text{AVR}(g)^\frac{2}{n} P_\mu(g) \text{Vol}_{g_0}(B)^{-\frac{2}{n}} \int_{B} (u^*)^2 dx
\]
\[
= \text{AVR}(g)^\frac{2}{n} P_\mu(g) \text{Vol}_{g}(\Omega)^{-\frac{2}{n}} \int_{\Omega} u^2 dv_g,
\]
where
\[
P_\mu(g) = j_2^2 \omega_n^\frac{2}{n} \quad \text{and} \quad \overline{\mu} = \sqrt{\frac{(n - 2)^2}{4} - \mu \text{AVR}(g)^\frac{2}{n}}.
\]

Therefore, (6.6) holds with the constant
\[
S_\mu(g) = \text{AVR}(g)^\frac{2}{n} P_\mu(g).
\]
We notice that \(S_\mu(g)\) is sharp both when \(\mu = 0\) (cf. Theorem 1.2) and when \(\text{AVR}(g) = 1\) (cf. [26]). However, in general, the sharpness of \(S_\mu(g)\) is an open question as the argument similar to the one from the proof of Theorem 1.2 does not provide the expected optimality.

**Acknowledgments.** The author thanks Z. Balogh, S. Brendle, G. Carron and L. Mazzieri for stimulating conversations.

**Final statement.** The author states that there is no conflict of interest.

**References**

[1] V. Agostiniani, M. Fogagnolo, L. Mazzieri, Sharp geometric inequalities for closed hypersurfaces in manifolds with nonnegative Ricci curvature. Invent. Math. 222 (2020), no. 3, 1033–1101.
[2] M. Anderson, On the topology of complete manifold of nonnegative Ricci curvature. Topology 3 (1990), 41–55.
[3] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geometry 11 (1976), no. 4, 573–598.
[4] E. F. Beckenbach, T. Radó, Subharmonic functions and surfaces of negative curvature. Trans. Amer. Math. Soc. 35 (1933), no. 3, 662–674.
[5] S. Brendle, Sobolev inequalities in manifolds with nonnegative curvature. Preprint, October 2020. Available: https://arxiv.org/pdf/2009.13717.pdf
[6] G. Carron, Euclidean volume growth for complete Riemannian manifolds. Milan J. Math. 88 (2020), 455–478.
[7] G. Carron, Inegalités isopérimétriques de Faber-Krahn et consequences. Publications de l’Institut Fourier, 220, 1992.
[8] I. Chavel, Riemannian Geometry. A Modern Introduction. Second Edition. 2006.
[9] J. Cheeger, T. H. Colding, Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math. (2) 144 (1996), no. 1, 189–237.
[10] J. Cheeger, T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I. J. Differential Geom. 46 (1997), no. 3, 406–480.
[11] C. Croke, A sharp four-dimensional isoperimetric inequality. Comment. Math. Helv. 59 (1984), no. 2, 187–192.
[12] T. H. Colding, Ricci curvature and volume convergence. Ann. of Math. (2) 145 (1997), no. 3, 477–501.
[13] D. Cordero-Erausquin, B. Nazaret, C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. Adv. Math. 182 (2004), no. 2, 307–332.
[14] T. Coulhon, L. Saloff-Coste, Isopérimétrie pour les groupes et les variétés. Rev. Mat. Iberoamericana 9 (1993), no. 2, 293–314.
[15] M. Del Pino, J. Dolbeault, The optimal Euclidean $L^p$-Sobolev logarithmic inequality. J. Funct. Anal. 197 (2003) 151–161.
[16] O. Druet, E. Hebey, M. Vaugon, Optimal Nash’s inequalities on Riemannian manifolds: the influence of geometry. Int. Math. Res. Not. IMRN, 14, 1999, 735–779.
[17] M. P. do Carmo, C. Xia, Complete manifolds with non-negative Ricci curvature and the Caffarelli-Kohn-Nirenberg inequalities. Compos. Math. 140 (2004), 818–826.
[18] I. Gentil, The general optimal $L^p$-Euclidean logarithmic Sobolev inequality by Hamilton-Jacobi equations. J. Funct. Anal. 202 (2003), no. 2, 591–599.
[19] M. Ghomi, J. Spruck, Total curvature and the isoperimetric inequality in Cartan-Hadamard manifolds. Preprint, August 2019. Available: https://arxiv.org/abs/1908.09814.
[20] E. Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities. Courant Lecture Notes in Mathematics, 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.
[21] B. Kleiner, An isoperimetric comparison theorem. Invent. Math. 108 (1992), no. 1, 37–47.
[22] A. Kristály, New geometric aspects of Moser-Trudinger inequalities on Riemannian manifolds: the non-compact case. J. Funct. Anal. 276 (2019), no. 8, 2359–2396.
[23] A. Kristály, Metric measure spaces supporting Gagliardo-Nirenberg inequalities: volume non-collapsing and rigidities. Calc. Var. Partial Differential Equations 55 (2016), no. 5, Art. 112, 27 pp.
[24] A. Kristály, Sharp Morrey-Sobolev inequalities on complete Riemannian manifolds. Potential Anal. 42 (2015), no. 1, 141–154.
[25] A. Kristály, S. Ohta, Caffarelli-Kohn-Nirenberg inequality on metric measure spaces with applications. Math. Ann. 357 (2013), no. 2, 711–726.
[26] A. Kristály, A. Szakáll, Interpolation between Brezis-Vázquez and Poincaré inequalities on nonnegatively curved spaces: sharpness and rigidities. J. Differential Equations 266 (2019), no. 10, 6621–6646.
[27] M. Ledoux, On manifolds with non-negative Ricci curvature and Sobolev inequalities. Comm. Anal. Geom. 7 (1999), no. 2, 347–353.
[28] E. H. Lieb, M. Loss, Analysis, Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
[29] G. Liu, 3-manifolds with nonnegative Ricci curvature, Invent. Math. 193 (2013), no. 2, 367–375.
[30] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature. Ann. of Math. (2) 124 (1986), no. 1, 1–21.
[31] X. Menguy, Noncollapsing examples with positive Ricci curvature and infinite topological type. Geom. Funct. Anal. 10 (2000), no. 3, 600–627.
[32] J. Moser, A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. 20 (1971) 1077–1091.
[33] M. Munn, Volume growth and the topology of manifolds with nonnegative Ricci curvature. J. Geom. Anal. 20 (2010), no. 3, 723–750.
[34] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[35] G. Perelman, Manifolds of positive Ricci curvature with almost maximal volume. J. Am. Math. Soc. 7 (1994), 299–305.
[36] M. Reiris, On Ricci curvature and volume growth in dimension three. J. Differential Geom. 99 (2015), no. 2, 313–357.
[37] G. Talenti, Best constant in Sobolev inequality. Ann. Mat. Pura Appl. 110 (1976), 353–372.
[38] G. Talenti, Inequalities in rearrangement invariant function spaces. Nonlinear Analysis, Function Spaces and Applications, vol. 5 (Prague, 1994), 177–230. Prometheus, Prague (1994).
[39] C. Xia, The Gagliardo-Nirenberg inequalities and manifolds of non-negative Ricci curvature. J. Funct. Anal. 224 (2005), no. 1, 230–241.
[40] S.-H. Zhu, A finiteness theorem for Ricci curvature in dimension three. J. Differential Geom. 37 (1993), no. 3, 711–727.
Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary & Department of Economics, Babeș-Bolyai University, 400591 Cluj-Napoca, Romania

Email address: kristaly.alexandru@nik.uni-obuda.hu; alex.kristaly@econ.ubbcluj.ro.