A COUNTEREXAMPLE OF TWO ROMANOV TYPE CONJECTURES

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Abstract. In this note, we disprove two Romanov type conjectures posed by Chen.

1. Introduction

For any subset $A$ of natural numbers set $\mathbb{N}$, let $A(x) = |A \cap [1, x]|$. In 2008, Chen communicated to the authors of [3] and made the following two conjectures therein.

Conjecture 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of positive integers. If there exists a constant $c > 0$ such that $\mathcal{A}(\log x / \log 2) \mathcal{B}(x) > cx$ for all sufficiently large $x$, then the set $\{2^a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$ has a positive lower asymptotic density.

Conjecture 1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of positive integers. If there exists a constant $c > 0$ such that $\mathcal{A}(\log x / \log 2) \mathcal{B}(x) > cx$ for infinitely many $x$, then the set $\{2^a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$ has a positive upper asymptotic density.

The Ramanov theorem [6] offers a positive answer to the conjectures of Chen for $\mathcal{B}$ being the set of primes. However, it will not be the case for general situations. The disavowals of Conjecture 1.1 and 1.2 will be illustrated simultaneously by constructing a concrete counterexample in next section.

2. Counterexample

Let $p_i$ be the $i$-th odd prime and $d_t = p_1 p_2 \cdots p_t$ for $t \in \mathbb{Z}_+$. For any $t \in \mathbb{Z}_+$, define

$$\mathcal{B}_t = \{n : n \in \mathbb{N}, d_t | n\} \cap \left[2^{2^t}, 2^{2^{t+1}}\right], \quad \mathcal{B} = \bigcup_{t=1}^{\infty} \mathcal{B}_t$$

and

$$\mathcal{A} = \mathbb{N}, \quad C = \mathcal{A} + \mathcal{B} = \{2^a + b : a \in \mathcal{A}, b \in \mathcal{B}\}.$$ 

Let $x$ be a large number and $j$ be the number such that

$$2^{2^j} \leq x < 2^{2^{(j+1)}},$$

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which means that
\[
\sqrt{\log \log x} < j \leq 2 \left( \sqrt{\log 2} \right)^{-1} \sqrt{\log \log x}.
\]  
(2.2)

From the Chebyshev estimate, we have
\[
d_j = \exp \left( \sum_{2 \leq p \leq p_j} \log p \right)
\leq \exp (2j \log j
\leq \exp \left( 3 \sqrt{\log \log x \log \log \log x} \right).
\]  
(2.3)

In view of equation (2.1), we know
\[
2^{(j-1)^2} < \sqrt{x}.
\]
So by the construction of \( \mathcal{B} \) and equation (2.3), we have
\[
\mathcal{B}(x) \geq \frac{x - 2^{2^2} - 2^{2^2} - 2^{(j-1)^2}}{d_j} - 2
\geq \frac{x - 2^{(j-1)^2}}{d_j} - 2
\geq \frac{x}{\exp \left( 3 \sqrt{\log \log x \log \log \log x} \right)}.
\]  
(2.4)

It follows that
\[
\mathcal{A} \left( \log x / \log 2 \right) \mathcal{B}(x) \gg \frac{x \log x}{\exp \left( 3 \sqrt{\log \log x \log \log \log x} \right)} > x.
\]  
(2.5)

It remains to prove \( \mathcal{C}(x) = o(x) \) as \( x \to \infty \). It is clear that
\[
\mathcal{C}(x) = \# \{ c \leq x : c = 2^a + b, a \in \mathcal{A}, b \in \mathcal{B} \} \leq S_1(x) + S_2(x),
\]  
(2.6)

where
\[
S_1(x) = \# \{ c \leq x : c = 2^a + b, a \in \mathcal{A}, b \in \mathcal{B} \}
\]  
(2.7)

and
\[
S_2(x) = \# \{ c \leq x : c = 2^a + b, a \in \mathcal{A}, b \notin \mathcal{B} \}.
\]  
(2.8)

Note that if \( c = 2^a + b \) for some \( b \in \mathcal{B}_j \), then \( p_i \nmid c \) for any \( 1 \leq i \leq j \). This fact leads to the following bound
\[
S_1(x) \leq \sum_{c \leq x} \frac{1}{\prod_{p \leq p_j} p} = \sum_{\ell | \prod_{p \leq p_j} p} \mu(\ell) \left\lfloor \frac{x}{\ell} \right\rfloor
\leq x \prod_{p \leq p_j} \left( 1 - \frac{1}{p} \right) + 2^j.
\]  
(2.9)
The same observation yields the following estimate
\[ S_2(x) \leq \# \{ c \leq x : c = 2^a + b, a \in \mathcal{A}, b \in \mathcal{B}_{j-1} \} + 2^{2(j-1)^2} \frac{\log x}{\log 2} \]
\[ \ll x \prod_{p \leq p_{j-1}} \left( 1 - \frac{1}{p} \right) + \sqrt{x} \log x + 2^j. \]  
(2.10)

It can be seen that
\[ 2^j \ll \exp \left( 3\sqrt{\log \log x} \right) \ll \sqrt{x} \]  
(2.11)

from equation (2.2). Therefore, combing equations (2.6), (2.9) and (2.10) we have
\[ \mathcal{C}(x) \ll x \prod_{p \leq p_{j-1}} \left( 1 - \frac{1}{p} \right) + \sqrt{x} \log x \]
\[ \ll x (\log p_{j-1})^{-1} + \sqrt{x} \log x \]
\[ \ll x (\log \log \log x)^{-1}, \]  
(2.12)

where the last but one step follows from the Mertens estimate, which is surely more to expectation than our requirement.

3. Remarks

The Romanov type problems start from the remarkable paper [6] of Romanv, where he proved that there is a positive lower asymptotic density of odd numbers which can be represented by the sum of a prime and a power of 2.

On the opposite direction, van der Corput [1] showed that there is a positive lower asymptotic density of odd numbers none of whose members can be represented by the sum of a prime and a power of 2. Soon, Erdős [2] constructed an arithmetic progression of odd numbers having the same property required in the paper of van der Corput. The results of van der Corput and Erdős give a negative answer to an old conjecture of de Polignac [4, 5].

At present, the author of this note has no answer to the following question. Let \( \mathcal{B} \) be a set of positive integers satisfying \( \mathcal{B}(x) = O(x/ \log x) \), then is it true that
\[ \limsup_{x \to \infty} \frac{\mathcal{C}(x)}{\mathcal{B}(x)} = \infty, \]

where
\[ \mathcal{C} = 2^N + \mathcal{B} = \{ 2^k + b : k \in \mathbb{N}, b \in \mathcal{B} \}. \]

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