QCD Sum Rule and the Validity of Phenomenological Models

Tsuneki Matsuki * and Koichi Yazaki †

Department of Physics, University of Tokyo

Bunkyo-ku, Tokyo 113, Japan

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Abstract

The consistency of effective models with QCD is investigated through the use of the QCD sum rule. Taking the potential model for the heavy quark system, we apply the method to two phenomenologically successful parameter sets, and obtain the dependences of the model parameters on the QCD scale Λ. Comparison with the expected scaling laws allows us to reject one of the two sets. The method is applicable to any model which reproduces the low lying spectra of hadronic systems.

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* email: matsuki@nt.phys.s.u-tokyo.ac.jp

† email: yazaki@phys.s.u-tokyo.ac.jp

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Heavy quark systems have been the most successful place in the study of strong interaction physics. There exist many methods that give quantitative agreement with experiment, both in the QCD based methods and in the QCD inspired models. As for the former, the heavy quark effective theory has related various observables in the framework of perturbative QCD, and the QCD sum rule has been applied to various channels to calculate the masses of the heavy mesons. As for the latter, the non-relativistic quark model (QM) has proved itself to be accurate enough to reproduce a vast number of physical observables, with the bonus that it stays very close to intuition. It is the mass of the heavy quark itself that sets the scale in the meson masses, the role of the strong interaction being only to give the splitting among them.

A problem about the QM is that there are usually several sets that seem to work equally well in a phenomenological sense. Being a model, it is natural to expect that it reflects the effective degrees of freedom when all the complicated degrees of freedom in QCD are integrated out, and ultimately its parameters be calculated starting from the fundamental theory. At this stage, no such calculation has been done, which leaves us to rely either on their self-consistencies \cite{3}, or on their phenomenological successes. In this paper, we propose the use of QCD sum rules to investigate the consistency of various models with QCD. Evaluation of the vector current polarization function in the $c\bar{c}$ channel enables us to exclude one of the two specific sets in the potential model for the charmonium system. The method is in principle applicable for any phenomenological model, the only requirement being that it reproduces the observed mass spectrum and leptonic decay widths.

The QCD sum rule basically deals with the polarization function, evaluating it on the one hand by the operator product expansion (OPE) in perturbative QCD and by a phenomenological model representing the observed spectrum on the other. Here we consider the vector polarization function

$$\Pi_{\mu\nu}(q^2) \equiv i \int_{-\infty}^{+\infty} d^4 x \, e^{i q \cdot x} \langle 0 \mid T (j_\mu (x) \bar{j}_\nu (0)) \mid 0 \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \quad (1)$$

where $j_\mu (x) = \bar{\psi}_c (x) \gamma_\mu \psi_c (x)$ and $q_\mu$ is the 4-momentum carried by the system. Instead
of extracting the physical masses by using the simplified spectral function, we substitute a more realistic spectrum calculated by the model on the phenomenological side to see how the parameters in the model vary with the quantities appearing in the OPE. More specifically, we calculate the variation of $\Pi(Q^2)$ with the QCD scale parameter $\Lambda^*$ and fit it for a certain range of $Q^2$ with the variations in the QM side with respect to the parameters of the model. It is essential that we fit the change in the polarization function and not the function itself, since the latter is usually saturated by the first few resonances, and is insensitive to the parameters as long as they accurately reproduce the first few spectra. The validity of the model parameters is investigated through their $\Lambda$ dependences.

For our analysis, we take the simple 'Cornell' type potential

$$V(r) = -\frac{a}{r} + \kappa r + V_0,$$

with two sets of parameters as our candidates. Set A has four adjustable parameters and reproduces $m_{\psi(1S)}, m_{\psi(2S)}, \Gamma_{\psi(1S)\to e^+e^-}$ and $\Gamma_{\psi(2S)\to e^+e^-}$, with the radiative correction factor $(1 - \frac{4\pi}{a})$ for the widths. The parameters take the values

$$\{m = 2.04[GeV], a = 0.579[1], \kappa = 0.172[GeV^2], V_0 = -1.12[GeV]\},$$

which are essentially the ones employed by [5]. Set B, on the other hand, has one restriction $a = \frac{4}{3}\alpha_s = 0.27$, and is thus adjusted to reproduce only $m_{1S}, m_{2S}$ and $\Gamma_{\psi(1S)\to e^+e^-}$. The parameters are given by

$$\{m, a, \kappa, V_0\} = \{1.78, 0.270, 0.222, -1.00\}$$

[3]. Their characteristics are summarised in Table 1.

The former is naturally better in reproducing the experimental data, and is the most frequently used set in modern quark model analyses. The latter on the other hand has the strong point that the Coulomb force is directly correlated with the one-gluon exchange of

*\(\Lambda\) stands for $\Lambda_{\text{mom}}$ in this particular case.
QCD. Set B with the coupling with $DD$ channel will be discussed later. With these models at hand, we calculate

$$
\Pi(Q^2) = \frac{1}{\pi} \int_0^\infty ds \frac{\text{Im} \Pi(s)}{(s + Q^2)},
$$

with $\text{Im} \Pi(s)$ given by

$$
\text{Im} \Pi(s) = \sum_n \frac{s}{3\pi} |\langle 0 | j^\mu(0) | 0, n \rangle|^2 \delta(s - M_n^2).
$$

$k, n'$ denotes the $n$th bound state with momentum $k$, with the normalization given by

$$
\langle p, n | p', n' \rangle = (2\pi)^3 2p_0 \delta^3(p - p') \delta_{nn'}.
$$

$n$ includes the possible polarizations of the state. The matrix elements are evaluated with the use of the van Royen-Weisskopf formula $[\Pi]$. On the other hand, the OPE side of the QCD sum rule is calculated in the usual manner $[2],$

$$
\Pi(Q^2) = C_0(Q^2) + C_G(Q^2) \langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu} | 0 \rangle + ....
$$

where the dots indicate higher dimensional contributions $[\dagger]$. Neglecting light quarks, we have two parameters on this side, i.e. the QCD scale parameter $\Lambda$ and the heavy quark current mass, $m_c$. We are here interested in the variants of $\Pi$ with respect to $\Lambda$, with $m_c$ fixed. The essential point is that we control the strength of the interaction solely through the parameter $\Lambda$. This parameter comes in through the quantities $\alpha_s$ and the gluon condensates ($\langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu} | 0 \rangle$ etc.). $\langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu} | 0 \rangle$ behaves as $\Lambda^4$ since it has no anomalous dimension (we are neglecting light quarks) and $\alpha_s$ will behave as $1/\ln(m_c/\Lambda)$. It is easy to see that taking $\Lambda \rightarrow 0$ will give 0 for both quantities, which means that the interaction is turned off and we are left with only the free particle contribution to the polarization function.

Now we can compare both sides concerning their variants with respect to $\Lambda$,

$$
\frac{\partial \Pi(Q^2)}{\partial \Lambda} |_{m_c} = \frac{\partial \Pi(Q^2)}{\partial m} |_{a, \kappa} \frac{\partial m}{\partial \Lambda} |_{m_c} + \frac{\partial \Pi(Q^2)}{\partial a} |_{m, \kappa} \frac{\partial a}{\partial \Lambda} |_{m_c} + \frac{\partial \Pi(Q^2)}{\partial \kappa} |_{a, m} \frac{\partial \kappa}{\partial \Lambda} |_{m_c}.
$$

The derivatives of $\Pi(Q^2)$ with respect to the parameters are calculated by numerical differentiation, at some fixed values of $\Lambda, m_c, a, \kappa, \kappa$, etc. The coefficients ($\frac{\partial m}{\partial \Lambda} |_{m_c}, ...$) are obtained $[\dagger]$

$\dagger$ We take into account up to operators of dimension 8. Heavy quark condensates are reduced to gluon condensates with the use of the heavy quark mass expansion $[2,3].$

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as a result of the fit in the region $Q^2 = 0.5 \sim 3.0\text{GeV}^2$ [2]. In the actual calculation, we evaluate the moments of $\Pi(Q^2)$, i.e. $\Pi_n(Q^2) \equiv \frac{1}{n!}(-\frac{d}{dQ^2})^n\Pi(Q^2)$ since this has an effect of emphasizing the low energy part of the physical spectrum [2]. We also evaluate for the parameter $O$,

$$d_O = \frac{\partial(\ln O)}{\partial(\ln \Lambda)}$$

instead of $\frac{\partial O}{\partial \Lambda}$ for reasons of convenience. This implies that the parameter behaves as $O \sim \Lambda^{d_O}$ near the actual value of $\Lambda$.

The fit is obtained in the simplest way through a linear $\chi^2$ fit formula (matrix inversion). We have found that the matrix is practically singular, which means that the effect of the parameters are redundant when reflected in the behavior of the polarization function. This is not surprising when we recall that the function itself is saturated by the first few resonances, which leaves us with only $2n$ degrees of freedom where $n$ is the number of resonances required to saturate the function. Actually, the lowest one is dominant and we found that only two out of four parameters were independent. This forces us to fix two of the $\Lambda$ dependences of the parameters, which we choose to be $\kappa \sim \Lambda^2$ and $V_0 \sim \Lambda$. This corresponds to taking the quenched approximation for the interaction, which is in agreement with the spirit of the quark model [1].

For set A, fitting the curves shown in figure [1] gives the scaling behaviors of the model parameters expressed as $O \sim \Lambda^{d_O}$, with $d_O$

$$\{d_m, d_a, d_\kappa, d_v\} = \{0.28 \pm 0.02, 0.6 \pm 0.3, 2., 1.\}$$

where $d_\kappa$ and $d_v$ are inputs. The error is due to the following ambiguities [2];

- Region of $Q^2$ where the fit is performed.

\^Of course we do not explicitly handle the dynamical (radiative) effects of the gluons [13]. We take the view that their effects are integrated out and arise as a change in the potential and the quark mass when we restrict ourselves to the two fermion sector.
- Which moment we take: \( \Pi_6 \sim \Pi_{10} \)

- Value of \( \alpha_s(4m_c^2) \), or equivalently, \( \Lambda_{\text{mom}} \): \( 130 \sim 220 [\text{MeV}] \)

- Value of \( \langle 0 | \frac{\alpha_s}{\pi} G_{\mu\nu} G_{\mu\nu} | 0 \rangle \): \( (360 \pm 50 [\text{MeV}])^4 \)

Although we know of no correct formula for the analytic behavior of \( m \) against \( \Lambda \), we can give an estimate for the \( \Lambda \) dependence from the following arguments. First, it is natural to expect that \( m \) reduces to \( m_c \) (the ’on shell’ mass) in the limit \( \Lambda \to 0 \), which is the free limit. This determines the sign of \( d_m \). Second, from dimensional analysis we expect that \( \Delta m = m - m_c \sim c_m \Lambda \) where \( c_m \) is some constant of proportionality. This requires that

\[
\frac{d_m}{\partial (\ln m)} = \frac{\Delta m}{m}
\]

which, after substituition of \( m = 2.0 [\text{GeV}] \) and \( m_c = 1.5 [\text{GeV}] \), gives \( d_m = 0.25 \) which is clearly close to the value of set A. (allowing a dependence of \( m \sim \ln(\Lambda/m_c) \) does not alter the essence of this estimate.) Also the dependence of \( a \) on \( \Lambda \) is consistent with that of the QCD \( \alpha_s \) parameter (\( da \sim d_{\alpha_s} \)).

On the other hand, for set B we obtain

\[
\{d_m, d_a, d_k, d_v\} = \{-0.2 \pm 0.1, -4.9 \pm 1.9, 2., 1.\}
\]

We notice that they fulfil neither of the requirements discussed above. They are not appropriate firstly since their signs are the opposite, and secondly because we would not expect such a strong dependence of \( a \) on \( \Lambda \).

These results clearly show that we should take set A as our phenomenological model in order to meet the requirements of QCD. This is rather contrary to our intuition since it seems that set B is more easily justified from the viewpoint of perturbative QCD. We have considered the following facts that might be the reasons for the failure of set B:

1. Set B does not reproduce \( \Gamma_{\psi(2S) \to e^+ e^-} \) (Table [I]). The effect of the second bound state accounts for a non-negligible portion of the polarization function.
2. The ‘radiative’ correction \((1 - \frac{4}{\pi}a)\) differs largely in the two cases. Taking \(a\) to be the value of set A, this gives a correction of more than 50% of the magnitude. While this is often considered as a weak point in a quark model analysis, we on the other hand have obtained a result that supports this factor.

To investigate the first effect, we have performed a coupled channel analysis similar to the one in [5]. For simplicity, we have taken only the \(DD\) channel into account, with the coupling potential

\[
\langle \bar{c}c | V | DD \rangle = g \, e^{-\mu r} \frac{r_i}{r} \varepsilon_i,
\]

where \(\varepsilon_i\) denotes the polarization vector of the \(\bar{c}c\) system. This allows us to reproduce the decay width \(\Gamma_{\psi(2S) \rightarrow e^+e^-}\). Going through the same analysis but now with a larger parameter space \(\{m, a, \kappa, V_0, \mu, g, m_D\}\), we obtain figure 2, which shows the contribution from the changes in the model parameters \(\mu, g, m_D\) and \(\kappa\) respectively. The difference in their magnitudes is clear. Requiring \(\mu, g\) and \(m_D\) to take appropriate values \((d_\mu = d_g = 1, d_{m_D} = .2)\) \(^1\), they clearly have only a small effect on the fit. We gain the values

\[
\{d_m, d_a, d_\kappa, d_v\} = \{-0.15 \pm 0.06, -3.8 \pm 0.7, 2., 1.\},
\]

which is essentially unchanged from the previous fit. Thus, the modified set B(including the coupled channel effect) works equally well phenomenologically as the set A but the scaling behavior of the parameters is still inconsistent with that expected from QCD.

One can easily diminish the effect of the factor \((1 - \frac{4}{\pi}a)\) by simply replacing \(\Pi_8\) with the ratio \(\Pi_{n+1}/\Pi_n\). The previous result obviously satisfies the fit, but due to the cancellation there might still be a set of parameters that is consistent with OPE. Notice that we have even fewer degrees of freedom (namely one), since taking a ratio means losing the information on the magnitude. The analysis gives the result

\[
\{d_m, d_a, d_\kappa, d_v\} = \{0.10 \pm 0.04, 0.2, 2., 1.\},
\]

\(^1d_{m_D}\) is evaluated as \(\{m_D - (m_c + m_u)\}/m_D = 0.2\).
where we have substituted $d_a = 0.2$. Varying $d_a$ gives a change in the result which is included in the error. We have thus obtained the scaling behavior of set B close to that of set A, but this is only due to the fact that we are looking at a fit with fewer requirements. Requiring the individual moments to behave correctly will force the parameters to go to the inappropriate value. This indicates that it is indeed the 'radiative correction' factor $(1 - \frac{4}{\pi} a)$ that prohibits the use of set B. In other words, the factor is essential in achieving the consistency with OPE.

In this paper, we have shown that it is set A of the model parameters that is consistent with the OPE, and is therefore appropriate for use in calculating physical quantities. This is in agreement with the choice of [5], who justifies this set through comparison with experimental data. We have also shown that it is the 'radiative correction' factor itself that was responsible for this result, although its 'physical' meaning remains unclear. Our analysis shows that in one way or the other, the total magnitude of Im$\Pi(s)$ must be appreciably modified from the simple potential picture.

The method is applicable in exactly the same fashion for a model that reproduces the spectrum and the leptonic decay widths. Applying finite temperature QCD sum rules, the method can also be used to extract the temperature dependences of the model parameters. In this case, one has to substitute two of the temperature dependences.

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TABLES

TABLE I. Calculated Spectrum

|                  | set A   | set B   | B with $D\bar{D}$ | Exp. |
|------------------|---------|---------|--------------------|------|
| $m_{1S}$ [GeV]   | 3.097   | 3.097   | 3.097              | 3.097|
| $m_{2S}$ [GeV]   | 3.685   | 3.685   | 3.685              | 3.685|
| $m_{1D}$ [GeV]   | 3.82    | 3.775   | 3.75               | 3.770|
| $m_{4040}$ [GeV] | 4.09    | 4.14    | 4.2                | 4.040|
| $\Gamma_{\psi(1S)\rightarrow e^+e^-}$ [keV] | 5.26    | 5.26    | 5.26               | 5.26 |
| $\Gamma_{\psi(2S)\rightarrow e^+e^-}$ [keV] | 2.14    | 3.00    | 2.14               | 2.14 |
FIG. 1. Variation with respect to $\Lambda, m, a, \kappa$ and $V_0$. Plotted are $\partial \ln \Pi_{OPE}/\partial \ln \Lambda, \partial \ln \Pi_{QM}/\partial \ln m$ and its counterparts.

FIG. 2. Variation with respect to $\kappa, \mu, g$ and $m_D$. Plotted are $\partial \ln \Pi_{QM}/\partial \ln \kappa, \partial \ln \Pi_{QM}/\partial \ln \mu$ and its counterparts.