Tables of convolution integrals

A.B. Arbuzov

Bogoliubov Laboratory for Theoretical Physics,
JINR, Dubna, 141980, Russia
e-mail: arbuzov@thsun1.jinr.ru

Abstract

An analytical approach to convolution of functions, which appear in perturbative calculations, is discussed. An extended list of integrals is presented.

1 Introduction

There are many situations, where one can describe a certain probability distribution of a complicated process in a form of a conditional probability involving two or more sub–processes. Typically in quantum physics, a factorization of sub–processes occurs due to the presence of a small (or large) parameter, which allows to suppress the interference of amplitudes, describing different sub–processes. We will consider the dependence on one continuous variable \(0 \leq x \leq 1\), and assume that the conditional probability can be presented as a convolution of the corresponding distributions for the sub–processes. In particular, convolution appear in the so–called evolution equations, arising in the renormalization group approach.

Quite often the convolution is performed by using the Mellin transformation. This approach is very transparent and powerful. On the other hand, it requires more steps (direct and inverse transformations) and involves a considerable number of auxiliary functions in the moment space. Moreover, in a realistic application, one might be interested to change the limits of convolution integrals to separate a certain contribution with a particular physical meaning.

Here I am going to discuss the direct analytical convolution, which is known to work well with a rather wide class of functions, which appear in perturbative calculations. The paper is organized as follows. In the next section I introduce the notation. The tables of convolution integrals of singular and non–singular functions are given in Sect. 3 and Sect. 4, respectively. Possible applications of the Tables are discussed in Conclusions. Properties of polylogarithmic functions are sketched in Appendixes.
2 Preliminaries

Let us consider two functions \( f(x) \) and \( g(y) \), defined on the interval \( 0 \leq x, y \leq 1 \). Their convolution is given by

\[
[f \otimes g](z) = \int_0^1 dx \int_0^1 dy \, \delta(z - xy) f(x) g(y) = \int \frac{dz}{x} f(x) g \left( \frac{z}{x} \right), \quad 0 \leq z \leq 1. \tag{1}
\]

Sometimes we need to perform a convolution of special functions. In particular, functions with the so-called plus prescription can be used. The prescription regularizes a pole singularity at \( x = 1 \). It is defined as follows:

\[
\int \frac{dx}{x_{\text{min}}} [f(x)]_+ g(x) = \int \frac{dx}{0} f(x) [g(x) \Theta(x - x_{\text{min}}) - g(1)], \tag{2}
\]

\[
\Theta(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{for } x < 0 
\end{cases}, \quad 0 \leq x_{\text{min}} < 1.
\]

Integrals of functions, which have poles at \( x = z \) or \( x = 1 \), are divergent. They can be regularized by introducing a small auxiliary parameter \( \Delta \ll 1 \). In the final result of a particular calculation, one has than look for the cancellation of the parameter between different contributions. The parameter can get also a physical meaning as a separator between soft and hard radiation. There is a one-to-one correspondence between the \( \Delta \)-regularization and the plus prescription. In fact, the following definition is equivalent to the prescription (2):

\[
[f(x)]_+ = \lim_{\Delta \to 0} [\delta(1 - x)f_\Delta + \Theta(1 - \Delta - x)f_\Theta(x)], \tag{3}
\]

\[
f_\Delta = - \int_0^{1-\Delta} dx \, f(x), \quad f_\Theta(x) = f(x) \bigg|_{x<1}.
\]

We will call \( f_\Delta \) and \( f_\Theta(x) \) as the \( \delta \)-part and the \( \Theta \)-part of the special function \( f(x) \). The above realization of the plus prescription is convenient in analytical calculations and especially in numerical computations, where the direct implementation of Eq. (2) is problematic. In what follows we will systematically use \( \Delta \)-regularization for divergent integrals, keeping in mind that the cancellation of the parameter will happen after summing with the corresponding contribution of the \( \delta \)-part of the relevant functions as in Eq. (3).

To define the \( \delta \)-part for a given function, one requires a table of definite integrals over the
interval $0 < x < 1 - \Delta$ or $0 < x < 1$ for non–singular functions. They can be found in numerous sources (see, for instance, Ref. [3]).

A convolution of two singular functions regularized by the plus prescription can be represented as

$$\left[ [f]_+ \otimes [g]_+ \right](z) = \lim_{\Delta \to 0} \left\{ \int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x} f_\Theta(x) g_\Theta \left( \frac{z}{x} \right) + f_\Delta g_\Theta(z) + f_\Theta(z) g_\Delta \right\}. \quad (4)$$

### 3 Integrals of singular functions

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln^n(1-x)}{1-x} = \frac{1}{n+1} \ln^{n+1}(1-z) - \frac{1}{n+1} \ln^{n+1} \Delta, \quad n = 0, 1, 2, \ldots \quad (5)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{1}{x - z} = -\ln \Delta + \ln(1 - z) - \ln z, \quad (6)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln x}{x - z} = -\ln \Delta \ln z + \text{Li}_2(1 - z) + \ln(1 - z) \ln z \frac{1}{2} \ln^2 z, \quad (7)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln(1-x)}{x - z} = -\ln \Delta \ln(1 - z) - \ln(1 - z) \ln z + \ln^2(1 - z) - \zeta(2), \quad (8)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln^2 x}{x - z} = -\ln \Delta \ln^2 z + 2 \text{S}_{1,2}(1 - z) + 2 \text{Li}_2(1 - z) \ln z$$

$$+ \ln(1 - z) \ln^2 z - \frac{1}{3} \ln^3 z, \quad (9)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln^2(1-x)}{x - z} = -\ln \Delta \ln^2(1 - z) + \ln^3(1 - z) - \ln^2(1 - z) \ln z$$

$$- 2 \zeta(2) \ln(1 - z) + 2 \zeta(3), \quad (10)$$

$$\int_{z/(1-\Delta)}^{1-\Delta} \frac{dx}{x - z} \frac{\ln(1-x) \ln x}{x - z} = -\ln \Delta \ln(1 - z) \ln z + 2 \text{S}_{1,2}(1 - z) - \text{Li}_3(1 - z)$$

$$+ \text{Li}_2(1 - z) \ln(1 - z) + \text{Li}_2(1 - z) \ln z + \ln^2(1 - z) \ln z$$

$$- \frac{1}{2} \ln(1 - z) \ln^2 z - \zeta(2) \ln z, \quad (11)$$
\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\text{Li}_2 (1-x)}{x-z} = \text{Li}_2 (1-z) (\ln(1-z) - \ln z - \ln \Delta)
\]

\(- S_{1,2} (1-z) - \text{Li}_3 (1-z),
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\text{Li}_3 (1-x)}{x-z} = \text{Li}_3 (1-z) (\ln(1-z) - \ln z - \ln \Delta)
\]

\(- S_{2,2} (1-z) - \text{Li}_4 (1-z),
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{S_{1,2} (1-x)}{x-z} = S_{1,2} (1-z) (\ln(1-z) - \ln z - \ln \Delta) + S_{2,2} (1-z)
\]

\(-2 S_{1,3} (1-z) - \frac{1}{2} (\text{Li}_2 (1-z))^2,
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\text{Li}_2 (1-x) \ln(1-x)}{x-z} = \text{Li}_2 (1-z) \ln(1-z) (\ln(1-z) - \ln z - \ln \Delta)
\]

\(+ \text{Li}_4 (1-z) - 2 S_{2,2} (1-z) + \frac{1}{2} (\text{Li}_2 (1-z))^2 - \text{Li}_3 (1-z) \ln(1-z)
\]

\(- S_{1,2} (1-z) \ln(1-z) - \zeta(2) \text{Li}_2 (1-z),
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\text{Li}_2 (1-x) \ln x}{x-z} = \text{Li}_2 (1-z) \ln z (\ln(1-z) - \ln z - \ln \Delta)
\]

\(+ 4 S_{1,3} (1-z) - 4 S_{2,2} (1-z) + (\text{Li}_2 (1-z))^2 - \text{Li}_3 (1-z) \ln z
\]

\(+ S_{1,2} (1-z) \ln z + \frac{1}{2} \text{Li}_2 (1-z) \ln^2 z,
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\ln^2(1-x) \ln x}{x-z} = - \ln \Delta \ln^2(1-z) \ln z + 2 \text{Li}_4 (1-z)
\]

\(- (\text{Li}_2 (1-z))^2 - 2 \text{Li}_3 (1-z) (\ln(1-z) + \ln z) + 4 S_{1,2} (1-z) \ln(1-z)
\]

\(+ 2 \text{Li}_3 (1-z) \ln(1-z) \ln z + \text{Li}_2 (1-z) \ln^2(1-z) + \ln^3(1-z) \ln z
\]

\[- \frac{1}{2} \ln^2(1-z) \ln^2 z - 2 \zeta(2) \ln(1-z) \ln z + 2 \zeta(3) \ln z,
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\ln(1-x) \ln^2 x}{x-z} = - \ln \Delta \ln(1-z) \ln^2 z + 2 S_{2,2} (1-z) - (\text{Li}_2 (1-z))^2
\]

\(- 2 \text{Li}_3 (1-z) \ln z + 2 S_{1,2} (1-z) (\ln(1-z) + \ln z)
\]

\(+ 2 \text{Li}_2 (1-z) \ln(1-z) \ln z + \text{Li}_2 (1-z) \ln^2 z + \ln^2(1-z) \ln^2 z
\]

\[- \frac{1}{3} \ln(1-z) \ln^3 z - \zeta(2) \ln^2 z,
\]

\[
\int_{z/(1-\Delta)}^{1-\Delta} \frac{\ln^3(1-x)}{x-z} = \ln^3(1-z) (\ln(1-z) - \ln z - \ln \Delta) - 3 \zeta(2) \ln^2(1-z)
\]
\[ +6\zeta(3)\ln(1-z) - 6\zeta(4), \quad (19) \]

\[ \int_\frac{1-\Delta}{z/(1-\Delta)} dx \frac{\ln^3 x}{x-z} = -\ln \Delta \ln^3 z + 6 S_{1,3} (1-z) + 6 S_{1,2} (1-z) \ln z \]

\[ +3 \text{Li}_2 (1-z) \ln^2 z + \ln(1-z) \ln^3 z - \frac{1}{4} \ln^4 z. \quad (20) \]

In the above integrals I omitted terms, which are vanishing in the limit \( \Delta \to 0. \)

4 Integrals of non–singular functions

\[ \int_\frac{1}{z} dx x^n = \frac{1}{n+1} (1-z^{n+1}), \quad n \neq -1, \quad (21) \]

\[ \int_\frac{1}{z} dx \frac{\ln^n x}{x} = -\frac{1}{n+1} \ln^{n+1} z, \quad n = 0, 1, 2, \ldots \quad (22) \]

\[ \int_\frac{1}{z} dx \frac{\ln^n(1-x)}{x} = (-1)^n n! \left[ \zeta(n+1) - S_{1,n} (z) \right], \quad n = 1, 2, 3, \ldots \quad (23) \]

\[ \int_\frac{1}{z} dx \frac{\ln(1-x) \ln x}{x} = S_{1,2} (1-z) - \frac{1}{2} \ln(1-z) \ln^2 z, \quad (24) \]

\[ \int_\frac{1}{z} dx x^n \ln x = -\frac{z^{n+1}}{n+1} \ln z - \frac{1}{(n+1)^2} \left(1-z^{n+1}\right), \quad n = 0, 1, 2, \ldots \quad (25) \]

\[ \int_\frac{1}{z} dx \frac{\ln x}{x^n} = \frac{1}{z^{n-1}(n-1)^2} \left( (n-1) \ln z + 1 - z^{n-1} \right), \quad n = 2, 3, 4, \ldots \quad (26) \]

\[ \int_\frac{1}{z} dx \frac{\ln^n x}{1-x} = (-1)^n n! \left[ S_{1,n} (1-z) \right], \quad n = 1, 2, 3, \ldots \quad (27) \]

\[ \int_\frac{1}{z} dx \frac{\ln(1-x) \ln x}{1-x} = \text{Li}_3 (1-z) - \text{Li}_2 (1-z) \ln(1-z), \quad (28) \]

\[ \int_\frac{1}{z} dx x^n \ln(1-x) = \frac{1-z^{n+1}}{n+1} \ln(1-z) + \frac{1}{n+1} \left( \sum_{k=1}^{n+1} \frac{z^k}{k} - S_{1} (n+1) \right), \quad n = 0, 1, 2, \ldots \quad (29) \]

\[ \int_\frac{1}{z} dx \frac{\ln(1-x)}{x^2} = \ln z + \frac{1-z}{z} \ln(1-z), \quad (30) \]
\[
\int_{x} \ln(1-x) x^n = \frac{1}{n-1} \left( \ln z + \frac{1 - z^{n-1}}{z^{n-1}} \ln(1-z) - \sum_{k=1}^{n-2} \frac{1}{k z^k} + S_1(n-2) \right),
\]
\[n = 3, 4, 5, \ldots\]
\[
\int_{x} x^n \ln^2 x = \frac{2}{(n+1)^2} (1 - z^{n+1}) + \frac{2z^{n+1}}{(n+1)^2} \ln z - \frac{z^{n+1}}{n+1} \ln^2 z, \quad n \neq -1,
\]
\[
\int_{x} x^n \ln^2(1-x) = \frac{1 - z^{n+1}}{n+1} \ln^2(1-z) + \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k \frac{(1-z)^{k+1}}{(k+1)^2} \left( \frac{2}{k+1} - 2 \ln(1-z) \right), \quad n = 0, 1, 2, \ldots
\]
\[
\int_{x} x^n \ln(1-x) \ln x = -\frac{1}{n+1} \text{Li}_2(1-z) - \frac{z^{n+1}}{n+1} \ln(1-z) \ln z
\]
\[
- \frac{1 - z^{n+1}}{(n+1)^2} \ln(1-z) + \frac{1}{n+1} \ln z \sum_{k=1}^{n+1} \frac{z^k}{k}
\]
\[
+ \frac{1}{(n+1)^2} \sum_{k=1}^{n+1} (1 - z^k) \frac{n+k+1}{k^2}, \quad n = 0, 1, 2, \ldots
\]
\[
\int_{x} \ln x \ln(1-x) = \frac{1}{n-1} \text{Li}_2(1-z) + \frac{1}{z^{n-1}(n-1)} \ln z \ln(1-z)
\]
\[
+ \ln z \left( \frac{1}{n-1} - \frac{1}{n-1} \sum_{k=1}^{n-2} \frac{z^k}{k} \right) + \frac{1 - z^{n-1}}{z^{n-1}(n-1)^2} \ln(1-z)
\]
\[
+ \frac{1}{2(n-1)} \ln^2 z - \frac{1}{(n-1)^2} \sum_{k=1}^{n-2} \frac{(1-z^k)(n+k-1)}{z^k k^2}, \quad n = 2, 3, 4, \ldots
\]
\[
\int_{x} \frac{\ln^2(1-x)}{x^2} = \frac{1 - z}{z} \ln^2(1-z) + 2 \text{Li}_2(1-z) + 2 \ln(1-z) \ln z,
\]
\[
\int_{x} \frac{\ln^2(1-x)}{x^n} = \frac{1 - z^{n-1}}{(n-1)z^{n-1}} \ln^2(1-z) + \frac{2}{n-1} \left( \text{Li}_2(1-z) + \ln(1-z) \ln z \right)
\]
\[- \frac{2}{n-1} \sum_{k=2}^{n-1} \int_{x} \frac{\ln(1-x)}{x^k} \quad n = 3, 4, 5, \ldots
\]
\[
\int_{x} \ln^2(1-x) \ln x = 2 \text{Li}_3(1-z) - 2 \text{Li}_2(1-z) \ln(1-z)
\]
\[- z \ln^2(1-z) \ln z + 2 \text{Li}_2(1-z) + 2z \ln(1-z) \ln z - (1-z) \ln^2(1-z)
\]
\[+ 4(1-z) \ln(1-z) - 2z \ln z + 6z - 6,
\]
\[n = 3, 4, 5, \ldots
\]
\[
\int \frac{\ln^2(1 - x) \ln x}{x} \, dx = 2 S_{1,2} (1 - z) \ln(1 - z) - 2 S_{2,2} (1 - z) - \frac{1}{2} \ln^2(1 - z) \ln^2 z
\] (39)
\[
\int \frac{\ln^2(1 - x) \ln x}{x^2} \, dx = -2 S_{1,2} (1 - z) - 2 \text{Li}_3 (1 - z) + 2 \text{Li}_2 (1 - z) \ln(1 - z)
+ \frac{1}{z} \ln^2(1 - z) \ln z + \ln(1 - z) \ln^2 z + 2 \text{Li}_2 (1 - z) + 2 \ln(1 - z) \ln z
+ \frac{1 - z}{z} \ln^2(1 - z),
\] (40)
\[
\int \frac{\ln^2(1 - x) \ln x}{1 - x} \, dx = -2 \text{Li}_4 (1 - z) + 2 \text{Li}_3 (1 - z) \ln(1 - z)
- \text{Li}_2 (1 - z) \ln^2(1 - z),
\] (41)
\[
\int \frac{\ln(1 - x) \ln^2 x}{x} \, dx = 2 S_{1,2} (1 - z) - z \ln(1 - z) \ln^2 z + 2 \text{Li}_2 (1 - z)
+ 2 z \ln(1 - z) \ln^2 z + 2(1 - z) \ln(1 - z) - 4 z \ln z - 6 + 6 z,
\] (42)
\[
\int \frac{\ln(1 - x) \ln^2 x}{x^2} \, dx = -2 S_{1,3} (1 - z) - \frac{1}{3} \ln(1 - z) \ln^3 z,
\] (43)
\[
\int \frac{\ln(1 - x) \ln^2 x}{x^2} \, dx = -2 S_{1,2} (1 - z) + \frac{1}{z} \ln(1 - z) \ln^2 z + \frac{1}{3} \ln^3 z + 2 \text{Li}_2 (1 - z)
+ \frac{2}{z} \ln(1 - z) \ln z + \ln^2 z + 2 \frac{1 - z}{z} \ln(1 - z) + 2 \ln z,
\] (44)
\[
\int \frac{\ln(1 - x) \ln^2 x}{1 - x} \, dx = -2 S_{2,2} (1 - z) + 2 S_{1,2} (1 - z) \ln(1 - z),
\] (45)
\[
\int \frac{\ln^3(1 - x)}{x} \, dx = (1 - z)(\ln^3(1 - z) - 3 \ln^2(1 - z) + 6 \ln(1 - z) - 6),
\] (46)
\[
\int \frac{\ln^3(1 - x)}{x^2} \, dx = -6 \text{Li}_3 (1 - z) + 6 \text{Li}_2 (1 - z) \ln(1 - z) + 3 \ln^2(1 - z) \ln z
+ \frac{1 - z}{z} \ln^3(1 - z),
\] (47)
\[
\int \frac{\ln^3 x}{x} \, dx = -z \ln^3 z + 3z \ln^2 z - 6z \ln z + 6z - 6,
\] (48)
\[
\int \frac{\ln^3 x}{x^2} \, dx = \frac{1}{z}(\ln^3 z + 3 \ln^2 z + 6 \ln z - 6z + 6).
\] (49)

By means of identical relations (see Appendix B) we reduce the arguments of the polylog-
arithm functions to \((1 - x)\). On the right hand side of the integrals we perform the same reduction of arguments. It’s worth to note, that there are certain physical arguments in favor of the \((1 - x)\) argument with respect to the simple \(x\). Namely, the point \(x = 1\) corresponds usually to a singularity of a fragmentation or structure function, remind e.g. the common lowest order splitting function

\[
P^{(0)}(x) = \left[\frac{1 + x^2}{1 - x}\right]_+.
\] (50)

So in analytical results of a certain convolution, the limit \(x \to 1\) has usually a principal importance, and the choice of the argument of polylogarithms helps to analyze this limit. Anyway, conversion of functions depending on \((1 - x)\) into the ones depending on \(x\) can be performed using simple formulae from Appendix B. The relevant integrals of polylogarithmic functions are

\[
\int z \, dx \, x^n \, \text{Li}_2 (1 - x) = \frac{1 - z^{n+1}}{n+1} - \frac{1}{n+1} \sum_{k=1}^{n+1} \left( \frac{x^k}{k} \ln z + \frac{1 - x^k}{k^2} \right),
\] (51)

\[
\int z \, dx \, \frac{\text{Li}_2 (1 - x)}{1 - x} = \text{Li}_3 (1 - z),
\] (52)

\[
\int z \, dx \, \frac{\text{Li}_2 (1 - x)}{x} = -2 \, S_{1,2} \, (1 - z) - \text{Li}_2 \, (1 - z) \ln z,
\] (53)

\[
\int z \, dx \, \frac{\text{Li}_2 (1 - x)}{x^n} = \frac{1 - z^{n-1}}{z^{n-1}(n-1)} \, \text{Li}_2 (1 - z) - \frac{1}{n-1} \, \left[ \sum_{k=1}^{n-2} \left( \frac{1 - x^k}{z^k k^2} + \frac{\ln z}{z^k k} \right) - \frac{1}{2} \, \ln^2 z \right], \quad n = 2, 3, 4, \ldots
\] (54)

\[
\int z \, dx \, \text{Li}_3 (1 - x) = (1 - z) \, \text{Li}_3 (1 - z) - (1 - z) \, \text{Li}_2 (1 - z) + z \ln z + 1 - z,
\] (55)

\[
\int z \, dx \, \frac{\text{Li}_3 (1 - x)}{x} = - \, \text{Li}_3 (1 - z) \ln z - \frac{1}{2} \, (\text{Li}_2 (1 - z))^2,
\] (56)

\[
\int z \, dx \, \frac{\text{Li}_3 (1 - x)}{x^2} = \frac{1 - z}{z} \, \text{Li}_3 (1 - z) + 2 \, S_{1,2} (1 - z) + \text{Li}_2 (1 - z) \ln z,
\] (57)

\[
\int z \, dx \, \frac{\text{Li}_3 (1 - x)}{1 - x} = \text{Li}_4 (1 - z),
\] (58)
\[ \int_{1}^{z} \, S_{1,2} (1 - x) = (1 - z) S_{1,2} (1 - z) + \frac{z}{2} \ln^2 z - z \ln z + z - 1, \]  
\[ \int_{1}^{z} \, \frac{S_{1,2} (1 - x)}{x} = -3 S_{1,3} (1 - z) - S_{1,2} (1 - z) \ln z, \]  
\[ \int_{1}^{z} \, \frac{S_{1,2} (1 - x)}{x^2} = \frac{1 - z}{z} S_{1,2} (1 - z) + \frac{1}{6} \ln^3 z, \]  
\[ \int_{1}^{z} \, \frac{S_{1,2} (1 - x)}{1 - x} = S_{2,2} (1 - z), \]

\[ \int_{1}^{z} \, \text{Li}_2 (1 - x) \ln(1 - x) = (1 - z) \text{Li}_2 (1 - z) \ln(1 - x) + (z - 2) \text{Li}_2 (1 - z) 
- z \ln(1 - z) \ln z - (1 - z) \ln(1 - z) + 2z \ln z + 3 - 3z, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln(1 - x)}{x} = 2 S_{2,2} (1 - z) - \frac{1}{2} (\text{Li}_2 (1 - z))^2 
- 2 S_{1,2} (1 - z) \ln(1 - z) - \text{Li}_2 (1 - z) \ln(1 - z) \ln z, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln(1 - x)}{x^2} = 3 S_{1,2} (1 - z) + \text{Li}_2 (1 - z) \ln z 
+ \frac{1 - z}{z} \text{Li}_2 (1 - z) \ln(1 - z) - \frac{1}{2} \ln(1 - z) \ln^2 z, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln(1 - x)}{1 - x} = - \text{Li}_4 (1 - z) + \text{Li}_3 (1 - z) \ln(1 - z), \]  
\[ \int_{1}^{z} \, \text{Li}_2 (1 - x) \ln x = -2 S_{1,2} (1 - z) - (z \ln z - z + 1) \text{Li}_2 (1 - z) - z \ln^2 z 
+ 3z \ln z - 3z + 3, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln x}{x} = 3 S_{1,3} (1 - z) - \frac{1}{2} \text{Li}_2 (1 - z) \ln^2 z, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln x}{x^2} = 2 S_{1,2} (1 - z) + \frac{1}{z} (\ln z - z + 1) \text{Li}_2 (1 - z) 
- \frac{1}{3} \ln^3 z - \frac{1}{2} \ln^2 z, \]  
\[ \int_{1}^{z} \, \frac{\text{Li}_2 (1 - x) \ln x}{1 - x} = - \frac{1}{2} (\text{Li}_2 (1 - z))^2. \]  

Integrals of some functions, which depend on \((1 + x)\), are required in certain cases (we
consider only the real part of the corresponding functions):

\[
\int \frac{1}{1 + x} \, dx = \ln 2 - \ln(1 + z), \quad (71)
\]

\[
\int \frac{\ln x}{1 + x} \, dx = \text{Li}_2(1 + z) - \frac{3}{2} \zeta(2), \quad (72)
\]

\[
\int \frac{\ln^2 x}{1 + x} \, dx = \frac{7}{2} \zeta(3) - 2 S_{1,2} (1 + z), \quad (73)
\]

\[
\int \frac{\text{Li}_2 (1 + x)}{1 + x} \, dx = -(1 + z) \text{Li}_2 (1 + z) - z \ln z - 1 + z + 3 \zeta(2), \quad (74)
\]

\[
\int \frac{\text{Li}_2 (1 + x)}{x} \, dx = -\text{Li}_3 (1 + z) + \frac{7}{8} \zeta(3) + \frac{3}{2} \zeta(2) \ln 2, \quad (75)
\]

\[
\int \frac{\text{Li}_2 (1 + x)}{x} \, dx = \frac{7}{2} \zeta(3) - 2 S_{1,2} (1 + z) - \text{Li}_2 (1 + z) \ln z, \quad (76)
\]

\[
\int \frac{\text{Li}_2 (1 + x)}{x^n} \, dx = -\frac{3}{2(n - 1)} \zeta(2) + \frac{1}{z^{n-1}(n-1)} \text{Li}_2 (1 + z) \]

\[-\frac{(-1)^{n-1}}{n-1} \left( \text{Li}_2 (1 + z) + \frac{1}{2} \ln^2 z - \frac{3}{2} \zeta(2) \right) \]

\[-\frac{1}{n-1} \sum_{k=1}^{n-2} (-1)^{n+k} \left( 1 - \frac{z^k}{z^{k^2}} + \frac{\ln z}{z^k} \right), \quad n = 2, 3, 4, \ldots \quad (77)
\]

5 Conclusions

The tables of integrals were implemented in a FORM [4] subroutine and used to perform analytical calculations of various convolutions in Refs. [5,6,7,8]. The subroutine can be used to construct an automated program for convolution of a rather wide class of functions, which appear in perturbative QED and QCD calculations. In particular, a possibility to include the leading and next-to-leading logarithmic corrections into the SANC project [1] is considered. It can be done by means of an automated convolution of perturbative coefficient functions with relevant structure and fragmentation functions.

Most of the presented integrals can be found in other sources, including automatic integrators in MATHEMATICA [9] and other packages. But I hope that the tables can come in handy in further analytical calculations.
Acknowledgements

This work is supported by the RFBR grant 03-02-17077.

Appendix A
Notation for polylogarithm and other functions

The short notation for partial sums is

$$S_k(n) = \sum_{j=1}^{n} \frac{1}{j^k}, \quad S_k(0) = 0. \quad (A.1)$$

Binomial coefficients are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (A.2)$$

The Riemann $\zeta$-functions are defined as usual:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) \approx 1.20205690315959,$$

$$\zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) \approx 1.03692775514337. \quad (A.3)$$

Following the notation of Refs. [2,3], the general Nielsen's polylogarithm is

$$S_{n,m}(z) = \frac{(-1)^{n+m-1}}{(n-1)! m!} \int_{0}^{1} dx \frac{\ln^{n-1}(x) \ln^{m}(1-xz)}{x},$$

$$n = 1, 2, 3 \ldots, \quad m = 1, 2, 3 \ldots \quad (A.4)$$

In particular,

$$\text{Li}_2(z) \equiv S_{1,1}(z) = -\int_{0}^{1} dx \frac{\ln(1-xz)}{x}, \quad S_{1,2}(z) = \frac{1}{2} \int_{0}^{1} dx \frac{\ln^2(1-xz)}{x},$$
Some explicit expressions for polylogarithms of constant argument $s$ can be useful:

\[
\text{Li}_3 (z) \equiv S_{2,1} (z) = \int_0^1 \frac{dx}{x} \ln(x) \ln (1-xz) = \int_0^z \frac{dx}{x} \text{Li}_2 (x),
\]

\[
\text{Li}_4 (z) \equiv S_{3,1} (z) = -\frac{1}{2} \int_0^1 \frac{dx}{x} \ln^2 (x) \ln (1-xz), \quad S_{1,3} (z) = -\frac{1}{6} \int_0^1 \frac{dx}{x} \ln^3 (1-xz),
\]

\[
S_{2,2} (z) = -\frac{1}{2} \int_0^1 \frac{dx}{x} \ln(x) \ln^2 (1-xz).
\]  \hspace{1cm} (A.5)

Appendix B

Relations between polylogarithms

To convert polylogarithms of $z$ into the ones of $(1-z)$, one can use the following relations:

\[
\text{Li}_2 (z) = -\text{Li}_2 (1-z) - \ln z \ln (1-z) + \zeta (2),
\]

\[
S_{1,2} (z) = -\text{Li}_3 (1-z) + \text{Li}_2 (1-z) \ln (1-z) + \frac{1}{2} \ln^2 (1-z) \ln z + \zeta (3),
\]

\[
\text{Li}_3 (z) = -S_{1,2} (1-z) - \text{Li}_2 (1-z) \ln z - \frac{1}{2} \ln (1-z) \ln^2 z + \zeta (2) \ln z + \zeta (3),
\]

\[
S_{1,3} (z) = -\text{Li}_4 (1-z) + \text{Li}_3 (1-z) \ln (1-z) - \frac{1}{2} \text{Li}_2 (1-z) \ln^2 (1-z)
- \frac{1}{6} \ln^3 (1-z) \ln z + \zeta (4),
\]

\[
S_{2,2} (z) = -S_{2,2} (1-z) - \text{Li}_3 (1-z) \ln z + S_{1,2} (1-z) \ln (1-z)
+ \text{Li}_2 (1-z) \ln (1-z) \ln z + \frac{1}{4} \ln (1-z) \ln^2 z + \zeta (3) \ln z + \frac{1}{4} \zeta (4),
\]

\[
\text{Li}_4 (z) = -S_{1,3} (1-z) - S_{1,2} (1-z) \ln z - \frac{1}{2} \text{Li}_2 (1-z) \ln^2 z
- \frac{1}{6} \ln (1-z) \ln^3 z + \frac{1}{2} \zeta (2) \ln^2 z + \zeta (3) \ln z + \zeta (4).
\]  \hspace{1cm} (B.1)

Some explicit expressions for polylogarithms of constant arguments can be useful:

\[
S_{n,m} (0) = 0, \quad n = 1, 2, 3 \ldots, \quad m = 1, 2, 3 \ldots
\]

\[
\text{Li}_n (1) = \zeta (n), \quad n = 2, 3, 4, \ldots
\]

\[
S_{2,2} (1) = \frac{1}{4} \zeta (4), \quad S_{1,n} (1) = \zeta (n+1), \quad n = 1, 2, 3 \ldots
\]

\[
\text{Re} \text{Li}_2 (2) = \frac{3}{2} \zeta (2), \quad \text{Li}_2 (-1) = -\frac{1}{2} \zeta (2), \quad \text{Li}_2 \left( \frac{1}{2} \right) = \frac{1}{2} \zeta (2) - \frac{1}{2} \ln^2 2,
\]

\[
\text{Re} \text{Li}_3 (2) = \frac{7}{8} \zeta (3) + \frac{3}{2} \ln 2 \zeta (2), \quad \text{Re} \text{Li}_3 (-1) = -\frac{3}{4} \zeta (3),
\]
\[
\text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{1}{2}\zeta(2) \ln 2 + \frac{1}{6} \ln^3 2,
\]
\[
\text{Re} S_{1,2}(2) = \frac{7}{4}\zeta(3), \quad S_{1,2}(-1) = \frac{1}{8}\zeta(3), \quad S_{1,2}\left(\frac{1}{2}\right) = \frac{1}{8}\zeta(3) - \frac{1}{6} \ln^3 2. \quad (B.2)
\]

References

[1] A. Andonov, D. Bardin, S. Bondarenko, P. Christova, L. Kalinovskaya, G. Nanava, and G. Passarino, hep-ph/0209297.

[2] K.S. Kolbig, J.A. Mignaco, and E. Remiddi, B. I. T. 10 (1970) 38.

[3] A. Devoto and D.W. Duke, Riv. Nuovo Cim. 7 (1984) 1.

[4] J. A. Vermaseren, math-ph/0010025.

[5] A. B. Arbuzov, Phys. Lett. B 470 (1999) 252.

[6] A. Arbuzov, A. Czarnecki, and A. Gaponenko, Phys. Rev. D 65 (2002) 113006.

[7] A. Arbuzov and K. Melnikov, Phys. Rev. D 66 (2002) 093003.

[8] A. Arbuzov, JHEP 03 (2003) 063.

[9] S. Wolfram, Mathematica - A system for doing mathematics by computer, Addison - Wesley Publishing Company Inc., 1998.