A Prey – Predator Model for Venture Capital Investment

Letetia Mary Addison¹, Balswaroop Bhatt²*, David Owen³
1,2,3. Department of Mathematics and Statistics, Faculty of Science and Technology, The University of the West Indies, St. Augustine Campus, Trinidad and Tobago
¹letetia.addison@sta.uwi.edu, ²bal.bhatt@sta.uwi.edu, ³drowentt@yahoo.com

Abstract: The Rosenzweig-Macarthur Model is used to partially explain Venture Capital investment cycles. Investment opportunities and the experience of the investors, represent the prey and predator respectively. Stability analysis with respect to the interior equilibrium point is performed and dynamics of the system are investigated using numerical simulations and results are presented. The model shows that parameter variation affects the stability of the system and it experiences bifurcations. The results show that stability analysis is useful to provide a Venture Capitalist with the stability ranges of parameters in the system, to improve the quality of the investment process.

Key words: prey – predator; stability; Hopf bifurcations; Venture Capital

1. Introduction

Venture capital (VC) provides a source for new businesses to obtain financing through accumulated investor funds. According to [1], the VC market plays a significant role in providing capital to a wide variety of enterprises. Many papers have explored different aspects of VC industry with respect to innovation through increased patenting rates [2], methods to estimate risk and return [3] and geographical impact of the location of firms [4].

Another area of interest has been the experience of the Venture Capitalists (VCs) as it relates to the success of investment [5]. Brander and De Bettignies [6] also explored the relationship between investor experience and opportunities using a Lotka – Volterra [7, 8] prey – predator dynamic. Although there is vast literature on VC investment and factors which affect it, little work has been done on the endogenous dynamics of the investment cycles using such a deterministic model. Our work seeks to explore these dynamics further.

The motivation behind the use of such a model can be partly explained by the strategies VCs use to choose an investment opportunity based on experience in that particular industry. The selection process usually consists of the VCs choosing projects which are seemingly low risk to their investments. The complex variability in the system may be due to deterministic and intrinsic factors [9]. Projects which have a larger chance of success in the business world are usually the most appealing. Through ‘learning by doing’, VCs experience not only influences but also increases venture valuation [10].

The notion that the experience of an investor has an effect on the success of the investment is an interesting idea worth further exploration using mathematical models. As such, [6] explored a model which partially captures endogenous cycles of investment where the VCs ‘prey’ on investment opportunities in a particular industry. It is motivated by the ‘boom and bust’ cycles within industry investment, as well as the concentration of venture capital in particular industries.

Although exogenous factors have contributed to these cyclic movements, the idea that endogenous factors occur within the investment model itself may also provide an explanation. Hence, the dynamics of a
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Lotka–Volterra prey–predator model is used to support their claims. At the industry level, VC investment cycles involve the opportunities being ‘consumed’ like prey by the predatory VCs through experience in the industry.

In addition, [6] also explained that technological changes affect industries, and experience with earlier technology would become nearly obsolete for the VCs in these industries. Technological progress would create new investment opportunities as time progresses and the cycle begins again. Hence, the idea that the oscillatory movements of predator-prey cycles are also inherent in the investment model.

Our work seeks to extend this model further by updating the original form of the model to obtain a more precise version based on the real-life behaviour of the system. A Rosenzweig and MacArthur [11] predator-prey model is used, with logistic growth of the prey in the absence of the predator. This is a more realistic update to the exponential growth in the original. It also contains a Holling Type – II [12] functional response of the predator, which is incorporated into the investment function originally proposed. The paper in [13] indicates that the choice of predator functional responses influences the qualitative predictions of mathematical model.

The idea here is that the VC may exercise different behaviours for different investment opportunities. Therefore, the investment function may not always contain a linear functional response as previously described in the original model, for every industry. Hence, this is an important modification of the original model where the investment predatory cycles generated in various industries are dependent of the functional response of the Venture capitalists, based on their experience in that industry.

Our paper first describes the original model and a brief summary of the parameters involved. Then, the modification of the model is provided with an explanation of the additional parameters and their effects on the system. After, the analytical analysis to obtain the positive interior equilibrium point is outlined. Next, the stability of this point and the Hopf bifurcation point is examined. Finally, numerical simulations are provided with graphical interpretations of the results, as well as a discussion of the findings. The stability of the system, with the variation of the Hopf bifurcation parameters, is of particular interest.

2. The Model

It is useful to discuss the original model prior to the modified version in order to provide a clear understanding of the ideas governing it. The model was originally proposed as follows, and a more detailed list of assumptions are outlined in the paper by [6].

\[
\begin{align*}
\frac{dP}{dt} &= \rho P - \alpha X^\beta P \\
\frac{dX}{dt} &= \alpha X^\beta P - \delta X
\end{align*}
\] (1)

where

- $P$ represents the investment opportunities in the industry (prey),
- $X$ represents the VCs experience in the industry (predator),
- $\alpha, \beta, \delta, \rho > 0$, $\beta < 1$, $P, X \geq 0$
- $\alpha$ – investment parameter for each investment,
- $\beta$ – returns to experience,
- $\rho$ – natural opportunity growth rate (due to scientific and technological progress),
- $\delta$ – depreciation rate of experience,

The Cobb-Douglas [14] form of Investment function is defined to be $V = \alpha X^\beta P$.

The greater the experience in an industry, the greater the motivation to invest, such that $\frac{dV}{dx} > 0$.

The larger the investment opportunities pool, the greater the motivation to invest, such that, $\frac{dV}{dp} > 0$. 
2.1 The Rosenweig - Macarthur Modified Model

Our work modifies the original model in [6] in the following way:

$$\frac{dP}{dt} = \rho f(P) - \alpha(P)X^\beta r(P)$$

$$\frac{dX}{dt} = \gamma \alpha(P)X^\beta r(P) - \delta X$$

(2)

where we retain the parameters $\rho, \alpha, \beta, \delta$ representing the natural growth rate of opportunities, the investment ratio, the returns to investment and the depreciation of experience respectively.

Modifications to the model are subsequently made. The parameter $\gamma$ is introduced to represent the efficiency rate of investment, that is, the conversion rate of the investment opportunities into investment.

A modified form of the Cobb- Douglas investment function [14] used in (2) is defined as:

$$V = \alpha(P)X^\beta r(P)$$

(3)

$f(P), r(P)$ are monotonic functions of $P$. In this paper, we define them to be:

$$f(P) = P \left(1 - \frac{P}{K}\right)$$

(4)

with $K > 0$ and

$$r(P) = \frac{P}{1 + ahP}$$

(5)

which is a Holling -Type II functional response [12] with search rate for a particular investment opportunity, $\alpha$, and handling time, $h$, or the average time spent investing in the particular opportunity.

For simplicity, the $\alpha(P)$ is taken to be a constant, that is, $\alpha(P) = \alpha$. All parameters are non-negative with

$$0 < \beta < 1 \text{ and } 0 < \gamma \leq 1.$$  

The Rosenzweig - MacArthur [11] or R-M model modification replaces this with a more realistic assumption of logistic growth of the investment opportunities in the absence of VCs.

The original interaction of the opportunities and VCs via the investment function, $V$, in which there is a linear relationship between the two for a particular value of $\beta$, is modified to reflect a more realistic response.

Consequently, the original functional response between prey and predator is updated from a Holling Type-I to Type-II response, which initially used to show the intake rate of the predator with respect to its prey. This is represented by the function, $r(P)$. The assumption here is that the VCs experience in a particular industry limits earning capacity in those industry-related opportunities. This is similar to the original biological idea that a predator is dependent on its capacity for prey.

The value of $\alpha$, represents the investment ratio or the rate at which the VCs pursue a particular investment opportunity. This rate affects the decision of the investor to invest in the particular opportunity or not. The larger the value of $\alpha$, while keeping other parameters constant, the more profitable the investment is to the VCs. Mathematically, as the product, $ah \gg P$, the function $r(P)$ decreases. This means that the rate at which the investment occurs is affected by the product of the search and investing rates in a particular opportunity.

At high levels of industry opportunities, VCs require little time to pursue investment opportunities. They therefore spend time “handling” or investing in the specific opportunity. A saturation point is reached where they exhaust their available funds to invest in the specific industry. A plateau occurs where investments are no longer pursued. Hence, this functional response, $r(P)$, as well as the value of the investment ratio, $\alpha$, affects the size of investment, $V$.

Finally, the parameter $\gamma$, which represents the conversion factor from opportunities to VC investments, is introduced into the system. This gives
a measure of the rate at which the VCs gain experience investing in the particular industry. The larger this value, the faster the rate at which experience is gained. Hence, the greater the possibility of a profitable investment at a particular point in time.

2.2 Mathematical Preliminaries

Boundedness of the system – This shows that the system is well-behaved with respect the restriction limited resources in the environment places on natural growth.

Theorem 1: The system defined in (6) is bounded.

Proof:
Let \( W(t) = P(t) + X(t) \). Then, assuming \( 0 < \eta < \delta \),

\[
\frac{dW}{dt} + \eta W = \frac{dP}{dt} + \frac{dX}{dt} + \eta W
\]

(7)

\[
\frac{dW}{dt} + \eta W = (\rho + \eta)P - \frac{\rho P^2}{K} - \frac{\alpha X^\theta P}{1 + ahP} + \frac{\gamma}{1 + ahP} + (-\delta + \eta)P
\]

\[
\frac{dW}{dt} + \eta W = -\frac{\rho}{K} \left[ P - \frac{K(\rho + \eta)}{2\rho} \right]^2 + \frac{K^2(\rho + \eta)^2}{4\rho^2}
- \left( \frac{(1 - \gamma)\alpha X^\theta P}{1 + ahP} \right) + (-\delta + \eta)P
\]

Since it is assumed that the conversion rate of the opportunities into investment, \( 0 < \gamma \leq 1 \), then clearly \( \frac{dW}{dt} + \eta W \leq \mu \), where \( \mu = \frac{K^2(\rho + \eta)^2}{4\rho^2} \), ( a positive constant).

Solving this differential inequality and observing the behaviour as \( t \to \infty \), we see that:

\[ 0 < W \leq \frac{\mu}{\eta} \]  

(8)

Hence, \( \Omega = \{ (P,X) \in \mathbb{R}_+^2 : 0 < W = P(t) + X(t) \leq \frac{\mu}{\eta} \} \). Therefore, all trajectories with positive initial conditions are bounded.

2.3 Existence of Interior Equilibrium Point

Let the interior equilibrium point be \( (P^*,X^*) \). This is the non-zero, positive solution to the system (6) when \( \frac{dP}{dt} = \frac{dX}{dt} = 0 \). The superscripts have been dropped in calculations, for simplicity.

\[
\rho P \left( 1 - \frac{P}{K} \right) - \frac{\alpha X^\theta P}{1 + ahP} = 0
\]

(9)

\[
\gamma \frac{\alpha X^\theta P}{1 + ahP} - \delta X = 0
\]

(10)

Using equation (9):

\[
\frac{\alpha X^\theta P}{1 + ahP} = \rho P \left( 1 - \frac{P}{K} \right)
\]

(11)

Also,

\[
X = \left( \frac{\rho \left( 1 - \frac{P}{K} \right) (1 + ahP)}{\alpha} \right)^{\frac{1}{\theta}}
\]

(12)

Substituting (11) and (12) into equation (10) gives:

\[
\gamma \rho P \left( 1 - \frac{P}{K} \right) - \delta \left( \frac{\rho \left( 1 - \frac{P}{K} \right) (1 + ahP)}{\alpha} \right)^{\frac{1}{\theta}} = 0
\]

(13)

From equation (9), since \( X > 0 \), the following condition must hold:

\[ K > \frac{P}{\rho} \]  

(14)

Then, equation (13) can be solved to obtain \( P > 0 \). This value can then be substituted into equation (12) to find \( X \), once the condition (14) holds. Thus, we write the following Lemma, resuming the use of the superscript, \( ^* \), to define the positive interior equilibrium point, \( E \).

Lemma 1:

The positive equilibrium point \( E = (P^*,X^*) \) of the system in (6) exists and represents real populations if \( P^* > 0, X^* > 0 \) are solutions to the equations (12) and (13) and satisfy the inequality in (14).

2.4 Stability Analysis of Interior Equilibrium Point

The stability of the positive interior equilibrium point, \( E = (P^*,X^*) \), is examined by first linearizing the equations in (6). The following substitutions are used:

\[ P = P^* + u, \]

(15)
where $u, v$ are small perturbations about the equilibrium point. Assuming Taylor’s Theorem, all terms are expanded about the equilibrium point, while neglecting higher order terms of $u$ and $v$.

The characteristic equation has the form

$$ P(\lambda) = \lambda^2 + a_1 \lambda + a_2 = 0 \quad (17) $$

where

$$ a_1 = -(j_{11} + j_{22}), \quad a_2 = j_{11}j_{22} - j_{12}j_{21}, \quad (18) $$

And

$$ j_{11} = \rho \left(1 - \frac{P}{K}\right) - \frac{\rho P}{K} - \frac{\alpha X^\beta}{1 + ahP} + \frac{\alpha^2 X^\beta \rho h}{(1 + ahP)^2} \quad (19) $$

$$ j_{12} = -\frac{\alpha X^\beta \rho P}{X(1 + ahP)} \quad (20) $$

$$ j_{21} = \frac{\gamma \alpha X^\beta}{1 + ahP} - \frac{\gamma \alpha^2 X^\beta \rho h}{(1 + ahP)^2} $$

$$ j_{22} = \frac{\gamma \alpha X^\beta \rho P}{X(1 + ahP)} - \delta $$

For stability, it is necessary for the eigenvalues, $\lambda$ in equation (17) to have negative real parts. These conditions are satisfied by the Routh-Hurwitz [15] criteria, which states that a stable equilibrium occurs if and only if

$$ a_1 > 0, \quad a_2 > 0 \quad (21) $$

We therefore have the following theorem:

**Theorem 2:** Given an equilibrium point $(P', X')$ satisfying the equations of system (6), then once Lemma holds and $j_{11}, j_{12}, j_{21}, j_{22}$ are defined by equation (20), then the equilibrium point exists and is stable if and only if $a_1 > 0, a_2 > 0$, where $a_1, a_2$ have been defined in (18) and (19).

**Theorem 3: Hopf Bifurcation Theorem [16]**

Consider the general system $y' = \varphi(y, \mu)$, and suppose that for each parameter $\mu$ in the relevant interval, it has an isolated equilibrium point $y_e = y_e(\mu)$. Assume that the Jacobian matrix of $\varphi$ with respect to $y$, evaluated at $(y_e(\mu), \mu)$ has the following properties:

a. it possesses a pair of simple complex conjugate eigenvalues $\theta(\mu) \pm iw(\mu)$ that become imaginary at the critical value $\mu_0$ of the parameter – that is, $\theta(\mu_0) = 0$, while $w(\mu_0) \neq 0$ and no other eigenvalues with zero real part exist at $(y_e(\mu_0), \mu_0)$;

b. $\frac{d\theta(\mu)}{d\mu} \neq 0$, at $\mu = \mu_0$

Then the system $y' = \varphi(y, \mu)$ has a family of periodic solutions. The critical value $\mu_0$ is called the Hopf bifurcation of the system.

**Hopf Bifurcations near interior equilibrium point, $E = (P', X')$**

Consider the values for $a_1$ and $a_2$ in our system. Note the superscripts in the interior equilibrium point, $E$, have been dropped for simplicity:

$$ a_1 = \rho - \frac{2\rho P}{K} - \frac{\alpha X^\beta}{1 + ahP} \left(1 - \frac{\alpha Ph}{1 + ahP} - \frac{\gamma \beta P}{X}\right), \quad (22) $$

$$ a_2 = \rho - \frac{2\rho P}{K} - \frac{\alpha X^\beta}{1 + ahP} \left(1 - \frac{\alpha Ph}{1 + ahP}\right) \left(\frac{\gamma \alpha X^\beta \rho P}{X(1 + ahP)} - \delta\right) $$

$$ + \frac{\gamma \alpha^2 X^\beta \rho P^2}{X(1 + ahP)^2} \left(1 - \frac{\alpha Ph}{1 + ahP} - \frac{\gamma \beta P}{X}\right) $$

Considering the roots of equation (17),

$$ \lambda_{1,2} = -\frac{1}{2} a_1 \pm \frac{1}{2} \sqrt{a_1^2 - 4a_2} \quad (23) $$

In order to obtain a pair of imaginary roots for the eigenvalues at this equilibrium point,

$$ \rho - \frac{2\rho P}{K} - \frac{\alpha X^\beta}{1 + ahP} \left(1 - \frac{\alpha Ph}{1 + ahP} - \frac{\gamma \beta P}{X}\right) $$

$$ - \delta $$

$$ = 0 \quad (24) $$

and
\[
\begin{align*}
\rho - \frac{2\rho P}{K} - \frac{\alpha X^\beta}{1 + ahP} \left(1 - \frac{\alpha Ph}{1 + ahP}\right) \\
\left(\frac{\gamma \alpha X^\beta \beta P}{X(1 + ahP)} - \delta\right) + \frac{\gamma a^2 X^{2\beta} \beta P}{X(1 + ahP)} \left(1 - \frac{\alpha Ph}{1 + ahP}\right) > 0
\end{align*}
\]

(25)

This satisfies Theorem 3 a. and it can be easily verified that Theorem 3 b. holds when the model is parameterised in terms of one parameter.

Due to the complicated nature of the expressions, the analysis with respect to these eigenvalues is performed numerically.

3. Results and Discussion

Each parameter is varied individually while keeping the others constant for a given simulated dataset. The results are recorded where there is a change in stability for specific parameters. The Routh-Hurwitz Criteria [15] in Theorem 2 is used to determine if the dataset is stable or unstable. The analytical results are confirmed using numerical integration in the commercial software package MATLAB [17], where the initial condition is the equilibrium point, \( E \), with slight perturbation.

Table 1 shows the results related to changes in stability when parameters are varied individually in the dataset. In each case, a time series plot is generated to show the behaviour of the system near the equilibrium point. The graphical approach assists in the bifurcation analysis of the model. The following dataset is applied to the model:

**Dataset**

\( \alpha = 1, \quad \beta = 0.9, \quad \delta = 0.05, \quad \rho = 0.05, \quad K = 10, \quad h = 1, \quad \gamma = 1 \)

Equilibrium point: \( E = (0.0373, 0.0371) \)

Figure 1 shows the time series plot for the opportunities and experience for the parameters in the dataset. The graph has large oscillations initially which gradually reduce over time. The opportunities and experience of the VCs co-exist for the equilibrium value \( E = (0.0373, 0.0371) \) and the Routh-Hurwitz Criteria confirms analytically that the system eventually stabilizes.

In the dataset, we first vary the value of the parameter \( \beta \). The Hopf bifurcation point here is the value of \( \beta \) where the qualitative nature of the system shifts from stable to unstable. The system remains stable for values of \( \beta \) in the interval \( 0 < \beta < 0.98 \), while it is unstable for \( \beta \geq 0.98 \). The Routh–Hurwitz Criteria confirm stability conditions are met in the stable interval and violated in the unstable interval for this dataset.

This is also supported by the time series graph in Figure 1 for the value of \( \beta = 0.9 \) in the stable region. The amplitudes of the fluctuations are decreasing over time. Figure 2 shows the times series for \( \beta = 0.98 \) which is an unstable trajectory with drastically increasing amplitudes for the populations of opportunities and experience over time.

Similarly, other bifurcation parameters are varied in a similar way numerically, to analyse their effects on the stability of the system. These results are shown in Table 1. The system remains stable for \( 0 < h < 2.89 \) and \( \gamma \geq 0.32 \) respectively and is unstable otherwise. Figures 3, 4, 5 and 6 show the time series graphs for a unique value of \( h \) and \( \gamma \) within each region.

| Parameter | Stable (S) / Unstable(U) Region | Hopf Bifurcation Point |
|-----------|---------------------------------|------------------------|
| \( \beta \) | \( 0 < \beta < 0.98 \) (S), \( 0.98 \leq \beta < 1 \) (U) | 0.98 |
| \( \gamma \) | \( 0 < \gamma < 0.32 \) (U), \( \gamma \geq 0.32 \) (S) | 0.32 |
| \( h \) | \( 0 < h < 2.89 \) (S), \( h \geq 2.89 \) (U) | 2.89 |
Fig. 1  Time series for opportunities and experience of Venture Capitalists in dataset where $\beta = 0.9$.

Fig. 2  Time series for opportunities and Experience of Venture Capitalists in dataset where $\beta = 0.98$. 
Fig. 3  Time series for opportunities and Experience of Venture Capitalists for dataset where $\gamma = 0.3$.

Fig. 4  Time series for opportunities and Experience of Venture Capitalists for dataset where $\gamma = 0.95$. 
Fig. 5  Time series for opportunities and Experience of Venture Capitalists for dataset where $h = 2.0$

Fig. 6  Time series for opportunities and Experience of Venture Capitalists for dataset where $h = 3.5$
The knowledge of these values and regions provides the investor with insight into a particular investment and the trend it will follow over time, assuming little interference from large exogenous effects such as recessions. The investor has an idea of the ranges of values to keep the parameters within if he wants his Venture Capital investments to remain stable. The parameters in the dataset are an example of a real-world application used in the original model and these values can be altered depending on the system under consideration.

4. Conclusions

This modified Venture Capital Prey - Predator Model is a means of representing the endogenous dynamics of a more complex system. Many assumptions are made with respect to the exclusion of exogenous factors affecting the Venture Capital Industry as outlined by [6]. This paper presented a more generalized form of the original model with a Rosenweig-MacArthur [11] predator-prey model where the investment function contains a functional response, which can change depending on the industry dynamics.

Future work would involve experimenting with different functions involving the rate of investment, $\alpha$, as well as different functional responses in the investment function, $V$. This analysis is meant to be a guide for Venture Capital investors with respect to the particular dataset which fits their investment scenarios.

Although the dynamics within the investment system are complex, our model seeks to showcase how parameter variation affects the stability of the system and hence the investment. This is an example of the importance of dynamical analysis in the Venture Capital industry and it can be further extended to fit different scenarios.

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References

[1] V. H. Fried and R. D. Hisrich, "Toward a Model of Venture Capital Investment Decision Making," Financial Management, vol. 23, no. 3, pp. 28-37, 1994.
[2] S. Kortum and L. Lerner, "Assessing the contribution of venture capital to innovation," Rand Journal of Economics, vol. 31, no. 4, pp. 674-692, 2000.
[3] J. H. Cochrane, "The risk and return of venture capital," Journal of Financial Economics, vol. 75, no. 1, pp. 3-52, 2005.
[4] H. Chen, G. P. K. A and J. Lerner, "Buy local? The geography of venture capital," Journal of Urban Economics, vol. 67, no. 1, pp. 90 - 102, 2010.
[5] M. Sorenson, "How Smart Is Smart Money? A Two-Sided Matching Model of Venture Capital," Journal of Finance, vol. 62, no. 6, pp. 2725 - 2762, 2007.
[6] J. A. Brander and J.-E. De Bettignies, "Venture capital investment: The role of predator - prey dynamics with learning by doing," Economics of Innovation and New Technology, vol. 18, no. 1, pp. 1 - 19, 2009.
[7] A. J. Lotka, Elements of Physical Biology, Baltimore: Williams and Wilkins, 1925.
[8] V. Volterra, "Variazioni e fluttuazioni del numero d’individui in specie animali conviventi," Mem. Acad. Lincei Roma, vol. 2, pp. 31-113, 1926.
[9] S. Hallegatte, M. Ghil, P. Dumas and J.-C. Hourcade, "Business cycles, bifurcations and chaos in a neo-classical model with investment dynamics," Journal of Economic Behavior & Organization, vol. 67, no. 1, pp. 57-77, 2008.
[10] D. H. Hsu, "Experienced entrepreneurial founders, organizational capital, and venture capital funding," Research Policy, vol. 36, no. 5, pp. 722 - 741, 2007.
[11] M. L. Rosenzweig and R. H. MacArthur, "Graphical representation and stability conditions of predator-prey interactions," The American Naturalist, vol. 97, no. 895, pp. 209-223, 1963.
[12] C. Holling, "The Components of Predation as Revealed by a Study of Small-Mammal Predation of the European Pine Sawfly," The Canadian Entomologist, vol. 91, no. 5, p. 293–320, 1959.
[13] C. V. Pavan Kumar, K. Shiva Reddy and M. A. S. Srinivas, "Dynamics of prey predator with Holling interactions and stochastic influences," Alexandria Engineering Journal, 2017.
[14] C. W. Cobb and P. H. Douglas, "A Theory of Production," The American Economic Review, vol. 18, no. 1, pp. 139-165, 1928.
[15] R. M. May, Stability and Complexity in Ecosystems, P. a. Oxford, Ed., Princeton University Press, 1973.
[16] G. Gandolfo, Economic Dynamics, Springer-Verlag Berlin Heidelberg, 2009.
[17] E. J. Routh, A Treatise on the Stability of a Given State of Motion: Particularly Steady Motion, Macmillan, 1877.
[18] A. Hurwitz, "Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt," Mathematische Annalen, pp. 273-284, 1895.
[19] N. Khanna and R. D. Mathews, "Can herding improve investment decisions?," RAND Journal of Economics, vol. 42, no. 1, pp. 150-174, 2011.
[20] MATLAB, Numerical software package, Version 7.12.0 (R2011a) ed., Natick, Massachusetts: The MathWorks Inc., 2011.