Complete solutions of Altarelli-Parisi evolution equations in leading order and structure functions at low-x

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Abstract

We present complete solutions of singlet and non-singlet Altarelli-Parisi (AP) evolution equations in leading order (LO) at low-x. We obtain t-evolutions of proton and neutron structure functions and x-evolutions of deuteron structure functions at low-x from AP evolution equations. The results of t-evolutions are compared with HERA low-x and low-$Q^2$ data and those of x-evolutions are compared with NMC low-x and low-$Q^2$ data. Also we compare our results with those of general solutions of AP evolution equations.

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1. Introduction:

The Altarelli-Parisi (AP) evolution equations [1-4] are fundamental tools to study the $t(=\ln(Q^2/\Lambda^2))$ and $x$-evolutions of structure functions, where $x$ and $Q^2$ are Bjorken scaling and four momenta transfer in a deep inelastic scattering (DIS) process [5] respectively and $\Lambda$ is the QCD cut-off parameter. On the other hand, the study of structure functions at low-x has become topical in view [6] of high energy collider and supercollider experiments [7]. Solutions of AP evolution equations give quark and gluon structure functions which produce ultimately proton, neutron and deuteron structure functions. Though numerical solutions are available in the literature [8], the explorations of the possibility of obtaining analytical solutions of AP evolution equations are always interesting. In this connection, general solutions of AP evolution equations at low-x in leading order have already been obtained by applying Taylor expansion method [9] and t-evolutions [10] and x- evolutions [11] of structure functions have been presented.

The present paper reports complete solutions of AP evolution equations in leading order at low-x and calculation of $t$ and $x$-evolutions for singlet and non-singlet structure functions and hence $t$- evolutions of proton and neutron structure functions and $x$-evolutions of deuteron structure functions. In calculating structure functions, input data points have been

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taken from experimental data directly unlike the usual practice of using an input distribution function introduced by hand. Results of proton and neutron structure functions are compared with the HERA low-x low-$Q^2$ data and those of deuteron structure functions are compared with the NMC low-x low-$Q^2$ data. Comparisons are also made with the results of general solutions [10-12] of AP evolution equations already obtained. In the present paper, section 1 is the introduction. In section 2 necessary theory has been discussed. Section 3 gives results and discussions.

2. Theory:

Though the basic theory has been discussed elsewhere [10-12], the essential steps of the theory have been presented here for clarity. The AP evolution equations for singlet and non-singlet structure functions in the standard forms are [13]

\[
\frac{\partial F^S_2(x,t)}{\partial t} - \frac{A_f}{t} \left[ (3 + 4 \ln(1-x)) F^S_2(x,t) + I^S_1(x,t) + I^S_2(x,t) \right] = 0
\] (1)

and

\[
\frac{\partial F^{NS}_2(x,t)}{\partial t} - \frac{A_f}{t} \left[ (3 + 4 \ln(1-x)) F^{NS}_2(x,t) + I^{NS}(x,t) \right] = 0,
\] (2)

where,

\[
I^S_1 = 2 \int_x^1 \frac{dw}{1-w} \{(1+w^2)F^S_2(x,t) - 2F^S_2(x,t)\},
\] (3)

\[
I^S_2 = \frac{3}{2} N_f \int_x^1 \{w^2 + (1-w)^2\} G(\frac{x}{w}, t) dw
\] (4)

and

\[
I^{NS} = 2 \int_x^1 \frac{dw}{1-w} \{(1+w^2)F^{NS}_2(\frac{x}{w}, t) - 2F^{NS}_2(x,t)\}.
\] (5)

Here, $t = \ln(Q^2/\Lambda^2)$ and $A_f = 4/(33 - 2N_f)$, $N_f$ being the number of flavours and $\Lambda$ is the QCD cut off parameter.

Using Taylor expansion method [9] and neglecting higher order terms as discussed in our earlier works [10-12], $G(\frac{x}{w}, t)$, $F^S_2(\frac{x}{w}, t)$ and $F^{NS}_2(\frac{x}{w}, t)$ can be approximated for low-x as

\[
G\left(\frac{x}{w}, t\right) \simeq G(x,t) + x \sum_{k=1}^{\infty} u^k \frac{\partial G(x,t)}{\partial x},
\] (6)

\[
F^S_2\left(\frac{x}{w}, t\right) \simeq F^S_2(x,t) + x \sum_{k=1}^{\infty} u^k \frac{\partial F^S_2(x,t)}{\partial x},
\] (7)

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and

\[ F_{2}^{NS} \left( \frac{x}{w}, t \right) \simeq F_{2}^{NS}(x, t) + x \sum_{k=1}^{\infty} u^{k} \frac{\partial F_{2}^{NS}(x, t)}{\partial x} \]  \hspace{1cm} (8)

where \( x = 1 - w \).

Using equations (6), (7) and (8) in equations (3), (4) and (5) and performing \( u \)-integrations we get

\[ I_{1}^{S} = -[(1 - x)(x + 3)]F_{2}^{S}(x, t) + [2xln(\frac{1}{x}) + x(1 - x^{2})]\frac{\partial F_{2}^{S}(x, t)}{\partial x}, \]  \hspace{1cm} (9)

\[ I_{2}^{S} = N_{f}[\frac{1}{2}(1 - x)(2 - x + 2x^{2})G(x, t) + \{-\frac{1}{2}x(1 - x)(5 - 4x + 2x^{2}) + \frac{3}{2}xln(\frac{1}{x})\}\frac{\partial G(x, t)}{\partial x}] \]  \hspace{1cm} (10)

and

\[ I_{NS}^{S} = -[(1 - x)(x + 3)]F_{2}^{NS}(x, t) + [2xln(\frac{1}{x}) + x(1 - x^{2})]\frac{\partial F_{2}^{NS}(x, t)}{\partial x}. \]  \hspace{1cm} (11)

Now using equations (9) and (10) in equation (1) we have,

\[ \frac{\partial F_{2}^{S}(x, t)}{\partial t} - \frac{A_{f}}{t}[A(x)F_{2}^{S}(x, t) + B(x)\frac{\partial F_{2}^{S}(x, t)}{\partial x} + C(x)G(x, t) + D(x)\frac{\partial G(x, t)}{\partial x}] = 0. \]  \hspace{1cm} (12)

Let us assume for simplicity

\[ G(x, t) = K(x)F_{2}^{S}(x, t), \]  \hspace{1cm} (13)

where \( K(x) \) is a function of \( x \). Our earlier assumption as more crude, \( K=\text{constant} \). This assumption gives \( t \)-evolutions of proton structure functions [8, 10, 12] which agree well with data. But it can not produce \( x \)-evolutions of deuteron structure functions [11] which agree well with data, especially in high-\( Q^{2} \) low-\( x \) region. Now equation (12) gives

\[ \frac{\partial F_{2}^{S}(x, t)}{\partial t} - \frac{A_{f}}{t}[L(x)F_{2}^{S} + M(x)\frac{\partial F_{2}^{S}(x, t)}{\partial x}] = 0, \]  \hspace{1cm} (14)

where,

\[ A(x) = 3 + 4ln(1 - x) - (1 - x)(3 + x), \]

\[ B(x) = x(1 - x^{2}) + 2xln(\frac{1}{x}), \]

\[ C(x) = N_{f}(1 - x)(2 - x + 2x^{2}), \]

\[ D(x) = -\frac{1}{2}N_{f}(1 - x)(5 - 4x + 2x^{2}) + \frac{3}{2}ln(\frac{1}{x}), \]

\[ L(x) = A(x) + K(x)C(x) + D(x)\frac{\partial K(x)}{\partial x} \]
and

\[ M(x) = B(x) + K(x)D(x). \]

Secondly using equation (11) in equation (2) we have

\[
\frac{\partial F_2^{NS}(x, t)}{\partial t} - \frac{A_f}{t} [P(x, t)F_2^{NS} + Q(x, t) \frac{\partial F_2^{NS}(x, t)}{\partial x}] = 0,
\]

(15)

where,

\[ P(x) = 3 + 4\ln(1 - x) - (1 - x)(x + 3) \]

and

\[ Q(x) = x(1 - x^2) - 2x\ln x. \]

The general solutions of equation (14) is

\[ F(U, V) = 0 \]

where F is an arbitrary function and

\[ U(x, t, F_2^{S}) = C_1 \]

and

\[ V(x, t, F_2^{S}) = C_2 \]

form a solution of equation

\[
\frac{dx}{A_f M(x)} = \frac{dt}{-t} = \frac{dF_2^{S}(x, t)}{-A_f L(x)F_2^{S}(x, t)}.
\]

(16)

Solving equation (16) we obtain,

\[ U(x, t, F_2^{S}) = t \exp \left[ \frac{1}{A_f} \int \frac{1}{M(x)} dx \right] \]

and

\[ V(x, t, F_2^{S}) = F_2^{S}(x, t) \exp \left[ \int \frac{L(x)}{M(x)} dx \right]. \]

If U and V are two independent solutions of equation (16) and if \( \alpha \) and \( \beta \) are arbitrary constants, then

\[ V = \alpha U + \beta \]

is called a complete solution of equation (16).

The complete solution \([9]\)

\[ F_2^{S}(x, t) \exp \left[ \int \frac{L(x)}{M(x)} dx \right] = \alpha t \exp \left[ \frac{1}{A_f} \int \frac{1}{M(x)} dx \right] + \beta \]
is a two-parameter family of planes. The one parameter family determined by taking $\beta = \alpha^2$ has equation

$$F_2^S(x, t) \exp\left[\int \frac{L(x)}{M(x)} \, dx\right] = \alpha t \exp\left(\frac{1}{A_f} \int \frac{1}{M(x)} \, dx\right) + \alpha^2. \quad (17)$$

Differentiating equation (17) with respect to $\alpha$, we get

$$\alpha = -\frac{1}{2} t \exp\left[\frac{1}{A_f} \int \frac{1}{M(x)} \, dx\right].$$

Putting the value of $\alpha$ in equation (17), we obtain

$$F_2^S(x, t) = -\frac{1}{4} t^2 \exp\left(\frac{2}{A_f M(x)} - \frac{L(x)}{M(x)}\right) dx. \quad (18)$$

Now, defining

$$F_2^S(x, t_0) = -\frac{1}{4} t_0^2 \exp\left(\frac{2}{A_f M(x)} - \frac{L(x)}{M(x)}\right) dx,$$

at $t = t_0$, where $t_0 = \ln(Q_0^2/\Lambda^2)$ at any lower value $Q = Q_0$, we get from equation (18)

$$F_2^S(x, t) = F_2^S(x, t_0) \left(\frac{t}{t_0}\right)^2 \quad (19)$$

which gives the t-evolution of singlet structure function $F_2^S(x, t)$.

Proceeding exactly in the same way, and defining

$$F_2^{NS}(x, t_0) = -\frac{1}{4} t_0^2 \exp\left(\frac{2}{A_f Q(x)} - \frac{P(x)}{Q(x)}\right) dx,$$

we get for non-singlet structure function

$$F_2^{NS}(x, t) = F_2^{NS}(x, t_0) \left(\frac{t}{t_0}\right)^2, \quad (20)$$

which gives the t-evolution of non-singlet structure function $F_2^{NS}(x, t)$.

Again defining,

$$F_2^S(x_0, t) = -\frac{1}{4} t^2 \exp\left(\frac{2}{A_f M(x)} - \frac{L(x)}{M(x)}\right) dx |_{x = x_0},$$

we obtain from equation (18)

$$F_2^S(x, t) = F_2^S(x_0, t) \exp\left[\int_{x_0}^x \left(\frac{2}{A_f M(x)} - \frac{L(x)}{M(x)}\right) dx\right], \quad (21)$$
which gives the x-evolution of singlet structure function $F_2^S(x, t)$. Similarly defining,

$$F_2^{NS}(x_0, t) = -\frac{1}{4}t^2 \exp\left[\int \left(\frac{2}{A_fQ(x)} - \frac{P(x)}{Q(x)}\right)dx\right]_{x=x_0},$$

we get

$$F_2^{NS}(x, t) = F_2^{NS}(x_0, t) \exp\left[\int_{x_0}^{x} \left(\frac{2}{A_fQ(x)} - \frac{P(x)}{Q(x)}\right)dx\right],$$

(22)

which determines the x-evolution of non-singlet structure function $F_2^{NS}(x, t)$. Deuteron, proton and neutron structure functions measured in deep inelastic electro-production can be written in terms of singlet and non-singlet quark distribution functions in leading order as

$$F_2^d(x, t) = \frac{5}{9} F_2^S(x, t),$$

(23)

$$F_2^p(x, t) = \frac{5}{18} F_2^S(x, t) + \frac{3}{18} F_2^{NS}(x, t)$$

(24)

and

$$F_2^n(x, t) = \frac{5}{18} F_2^S(x, t) - \frac{3}{18} F_2^{NS}(x, t).$$

(25)

Now using equations (19) and (21) in equation (23) we will get t and x-evolution of deuteron structure function $F_2^d(x, t)$ at low-x as

$$F_2^d(x, t) = F_2^d(x, t_0) \left(\frac{t}{t_0}\right)^2$$

(26)

and

$$F_2^d(x, t) = F_2^d(x_0, t) \exp\left[\int_{x_0}^{x} \left(\frac{2}{A_fM(x)} - \frac{L(x)}{M(x)}\right)dx\right],$$

(27)

where, the input functions are

$$F_2^d(x, t_0) = \frac{5}{9} F_2^S(x, t_0)$$

and

$$F_2^d(x_0, t) = \frac{5}{9} F_2^S(x_0, t).$$

The corresponding results for general solutions of AP evolution equations obtained earlier [10-12] are

$$F_2^d(x, t) = F_2^d(x, t_0) \left(\frac{t}{t_0}\right)^2$$

(28)
and
\[ F_2^d(x, t) = F_2^d(x_0, t) \exp[ \int_{x_0}^x \left( \frac{1}{A_f M(x)} - \frac{L(x)}{M(x)} \right) dx]. \]  

(29)

These have been obtained by taking arbitrary linear combinations of \( U \) and \( V \) of arbitrary function \( F(U, V) = 0 \) as solutions of equations (14) and (15) for singlet and non-singlet structure functions respectively.

Similarly using equations (19) and (20) in equations (24) and (25) we get the \( t \)-evolutions of proton and neutron structure functions at low-x as
\[ F_2^p(x, t) = F_2^p(x, t_0) \left( \frac{t}{t_0} \right)^2 \]  

(30)

and
\[ F_2^n(x, t) = F_2^n(x, t_0) \left( \frac{t}{t_0} \right)^2, \]  

(31)

where, the input functions are
\[ F_2^p(x, t_0) = \frac{5}{18} F_2^S(x, t_0) + \frac{3}{18} F_2^{NS}(x, t_0) \]  

and
\[ F_2^n(x, t_0) = \frac{5}{18} F_2^S(x, t_0) - \frac{3}{18} F_2^{NS}(x, t_0). \]

The corresponding results for general solutions of AP evolution equations are
\[ F_2^p(x, t) = F_2^p(x, t_0) \left( \frac{t}{t_0} \right) \]  

(32)

and
\[ F_2^n(x, t) = F_2^n(x, t_0) \left( \frac{t}{t_0} \right). \]  

(33)

But the \( x \)-evolutions of proton and neutron structure functions like those of deuteron structure function is not possible by this methodology, because to extract the \( x \)-evolution of proton and neutron structure functions we are to put equations (21) and (22) in equations (24) and (25). But as the functions inside the integral sign of equations (21) and (22) are different, we need to separate the input functions \( F_2^S(x, t_0) \) and \( F_2^{NS}(x_0, t) \) from the data points to extract the \( x \)-evolutions of the proton and neutron structure functions, which is not possible.

3. Results and Discussion:

In the present paper, we compare our results of \( t \)-evolutions of proton and neutron structure functions from equations (30) and (31) respectively with the HERA low-x, low-\( Q^2 \) data
Here proton structure functions $F_p^2(x, Q^2, z)$ measured in the range $2 \leq Q^2 \leq 50 \text{ GeV}^2$, $0.73 \leq z \leq 0.88$ and neutron structure functions $F_n^2(x, Q^2, z)$ measured in the range $2 \leq Q^2 \leq 50 \text{ GeV}^2$, $0.3 \leq z \leq 0.9$ have been used. Moreover here $P_T \leq 200 \text{ Mev}$, where $P_T$ is the transverse momentum of the final state baryon and $z = 1 - q. (p - p')/(q.p)$, where $p, q$ are the four momenta of the incident proton and the exchanged vector boson coupling to the positron and $p'$ is the four-momentum of the final state baryon.

In figures 1(a-d) we present our results of t-evolutions of proton structure functions $F_p^2$ (solid lines) for the representative values of $x$ given in the figures. Data points at lowest-$Q^2$ values in the figures are taken as input to test the evolution equation (30). Agreement is found to be excellent. In the same figures we also plot the results of t-evolutions of proton structure functions $F_p^2$ (dashed lines) for the general solutions from equation (32) of AP evolution equations. We observe that results of complete solutions are better than those of general solutions.

In figures 2(a-d) we present our results of t-evolutions of neutron structure functions $F_n^2$ (solid lines) for the representative values of $x$ given in the figures. Data points at lowest-$Q^2$ values in the figures are taken as input to test the evolution equation (31). Agreement is found to be excellent. In the same figures we also plot the results of t-evolutions of neutron structure functions $F_n^2$ (dashed lines) for the general solutions from equation (33) of AP evolution equations. We observe that in this case also results of complete solutions are better than those of general solutions.

For a quantitative analysis of $x$-distributions of structure functions, we evaluate the integrals that occured in equation (27) and present the results in figures 3(a-d) for $N_f = 4$ and representative values of $Q^2$ given in each figure, and compare them with NMC deuteron low-$x$ low-$Q^2$ data [15]. In each figure the data point for $x$-value just below 0.1 has been taken as input $F_2^d(x_0, t)$. If we take $K(x)=$constant, agreement of the results with experimental data becomes poor as before [11] especially in high-$Q^2$ lower-$x$ region. We, therefore, consider here $K(x)=a.\exp(-bx)$ where $a=1$ and $b=1$ for simplicity which gives better results. But explicit form of $K(x)$ can actually be obtained only by solving coupled AP evolution equations for singlet and gluon structure functions, and works are going on in this regard.

Traditionally the AP equations provide a means of calculating the manner in which the parton distributions change at fixed $x$ as $Q^2$ varies. This change comes about because of the various types of parton branching emission processes and the $x$-distributions are modified as the initial momentum is shared among the various daughter partons. However the exact rate of modifications of $x$-distributions at fixed $Q^2$ cannot be obtained from the AP equations since it depends not only on the initial $x$ but also on the rate of change of parton distributions with respect to $x$, $d^n F(x)/dx^n$ ($n=1$ to $\infty$), up to infinite order. Physically this implies that at high-$x$, the parton has a large momentum fraction at its disposal and as a result it radiates partons including gluons in innumerable ways, some of them involving complicated
QCD mechanisms. However for low-x, many of the radiation processes will cease to occur due to momentum constraints and the x-evolutions get simplified. It is then possible to visualise a situation in which the modification of the x-distribution simply depends on its initial value and its first derivative. In this simplified situation, the AP equations give information on the shapes of the x-distribution as demonstrated in this paper. The clearer testing of our results of x-evolution is actually the equation (22) which is free from the additional assumption equation (13). But non-singlet data is not sufficiently available in low-x to test our result. It is observed as a whole that the results of complete solutions of AP evolution equations are improved than those of general solutions, especially in t-evolution calculations. Of course, this is a leading order calculation. Its natural improvement will be the calculation considering next-to-leading order terms and we plan to do so in our subsequent works.

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Fig. 1(a) $x=0.00033$, $z=0.732$

Fig. 1(b) $x=0.00033$, $z=0.878$
Figure 1: In figures 1(a-d) t-evolutions of proton structure functions $F_p^p$ (solid lines) for the representative values of x given in the figures. Data points at lowest-*$Q^2$ values in the figures are taken as input to test the evolution equation (30). To test the evolution equation (30). In the same figures we also plot the results of t-evolutions of proton structure functions $F_p^p$ (dashed lines) for the general solutions from equation (32) of AP evolution equations.
Fig. 2(a) $x=0.00033$, $z=0.3$

Fig. 2(b) $x=0.00104$, $z=0.3$
Figure 2: In figures 2(a-d) t-evolutions of neutron structure functions $F_2^n$ (solid lines) for the representative values of $x$ given in the figures. Data points at lowest-$Q^2$ values in the figures are taken as input to test the evolution equation (31). In the same figures we also plot the results of t-evolutions of neutron structure functions $F_2^n$ (dashed lines) for the general solutions from equation (33) of AP evolution equations.
Fig-3(a), $Q^2=0.75 \text{ GeV}^2$

Fig 3(b), $Q^2=5.5 \text{ GeV}^2$
Figure 3: In figures 3(a-d) x-distributions of deuteron structure functions (solid lines) for $N_f = 4$ and the representative values of $Q^2$ given in each figure and compared them with NMC deuteron low-x low $Q^2$ data. In each figure the data point for x-value just below 0.1 has been taken as input $F_2^d(x_0, t)$. 