ESTIMATES ON THE NUMBER OF EIGENVALUES OF TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS

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ABSTRACT. Two-particle discrete Schrödinger operators $H(k) = H_0(k) - V$ on the three-dimensional lattice $\mathbb{Z}^3$, $k$ being the two-particle quasi-momentum, are considered. An estimate for the number of the eigenvalues lying outside of the band of $H_0(k)$ via the number of eigenvalues of the potential operator $V$ bigger than the width of the band of $H_0(k)$ is obtained. The existence of non negative eigenvalues below the band of $H_0(k)$ is proven for nontrivial values of the quasi-momentum $k \in \mathbb{T}^3 \equiv (-\pi, \pi)^3$, provided that the operator $H(0)$ has either a zero energy resonance or a zero eigenvalue. It is shown that the operator $H(k)$, $k \in \mathbb{T}^3$, has infinitely many eigenvalues accumulating at the bottom of the band from below if one of the coordinates $k^{(j)}, j = 1, 2, 3$, of $k \in \mathbb{T}^3$ is $\pi$.

Subject Classification: Primary: 81Q10, Secondary: 35P20, 47N50

Keywords and phrases: Spectral properties, two-particle, discrete Schrödinger operators, number of eigenvalues, band spectrum, Birman-Schwinger principle, zero energy resonance, zero eigenvalue, low-lying excitation spectrum.

1. INTRODUCTION

The study of the low-lying excitation spectrum for lattice Hamiltonians of systems with an infinite number of degrees of freedom has recently attracted considerable attention (see, e.g., [7, 10, 20]).

See also [3, 4, 5, 6, 9, 11, 13, 19] for general expositions and the discussion of particular problems of the theory of discrete Schrödinger operators on lattices, including applications to solid state physics.

The main aim of the present paper is to provide a thorough analysis of the dependence of the number of eigenvalues of a two-particle lattice Schrödinger operator with emphasis on its non-trivial dependence on the total quasi-momentum and threshold phenomena.

In the lattice case, similarly as in the continuous case, the two-body problem reduces to a one-body problem through the usual split-off of the center of mass and the introduction of the relative coordinates. As a result, the two-body energy operator is represented as a direct von Neumann integral, where the fiber Hamiltonian $H(k)$ depends parametrically on the internal binding the quasi-momentum $k \in \mathbb{T}^3 \equiv (-\pi, \pi)^3$ (see [3, 4, 5, 6, 8, 13, 19]).

One can observe that the number of eigenvalues of the self-adjoint perturbation $A - V$ of an abstract bounded self-adjoint operator $A$ in the Hilbert space $\mathcal{H}$ depends on the width of the spectrum of the non-perturbed operator $A$ (see Definition \[\ref{def:width}\] below for this concept).

In the present paper we obtain an estimate for the number of eigenvalues of the perturbed operator $A - V$ lying outside of the essential spectrum $\sigma_{\text{ess}}(A - V)$ depending on the number $n_+(w_s(A), |V|)$ of eigenvalues of the perturbation operator $V$, which are larger in absolute value than the width of the essential spectrum $w_s(A)$ of the operator $A$ (Theorem \[\ref{thm:estimate}\]).
As a consequence of these results we obtain an estimate for the number of eigenvalues of the two-particle discrete Schrödinger operator \( H(k) = H_0(k) - V \) via the number of eigenvalues of the interaction operator \( V \) which are bigger than the width \( w_b(H_0(k)) \) of the band spectrum (band) of \( H_0(k) \) (Theorem 4.2).

In the case of continuous Schrödinger operators one observes the emergence of negative eigenvalues from the bottom of the continuous spectrum.

This phenomenon is closely related to the existence of zero-energy resonances and zero eigenvalues at the bottom of continuous spectrum of the two-particle Schrödinger operators (so-called critical Hamiltonians).

The appearance of negative eigenvalues for critical (non-negative) Schrödinger operators under infinitesimally small negative perturbations is remarkable: the presence of a zero-energy resonances in at least two of the two-particle operators (subsystems) leads to the existence of infinitely many eigenvalues accumulating to the bottom of the three-particle continuum for the corresponding three-particle Schrödinger operator (the Efimov effect) (see, e.g., [1, 3, 8, 12, 14, 15, 16, 17]).

The presence of eigenvalues below the bottom of the continuous spectrum is especially remarkable for the three-particle lattice Schrödinger operators (Hamiltonians) \( H_3(K), K \in \mathbb{T}^3 \): the presence in at least one of the two-particle operators (subhamiltonians) implies for \( H_3(K) \) the finiteness of the number of eigenvalues for all sufficiently small nonzero values of the three-particle quasi-momentum \( K \in \mathbb{T}^3 \) even in the case, where all two-particle subhamiltonians have a zero energy resonance (see 3).

If we compare the results in the lattice case with those in the continuous case we see that in the former case there is a mechanism, which causes the emergence of eigenvalues from the threshold of the Hamiltonians, this mechanism has nothing to do with additional (effectively negative) perturbations of the potential term. In fact the mechanism is provided by the nontrivial dependence of the Hamiltonians \( H(k) \) on the quasi-momentum \( k \in \mathbb{T}^3 \). We prove this fact using the lattice analogue of Birman-Schwinger principle (Theorem 6.6).

The number of eigenvalues outside of the band of the two-particle Schrödinger operators essentially depends on the width \( w_{j;k}(\cdot) \) of the band in the direction \( j = 1, 2, 3 \), \((e^j \) being the unit vector along the \( j \)-th direction in \( \mathbb{Z}^3 \)).

We establish that the operator \( H(k) \) has infinitely many positive eigenvalues lying outside of the band spectrum if \( w_{j;k}(\cdot) \) of the band in direction \( j \) vanishes for some \( j = 1, 2, 3 \) (Theorem 6.10). This corresponds to having an effective mass in direction \( j, j = 1, 2, 3 \), which increases to \( +\infty \).

The paper is organized as follows.

In Section 2 we describe the two-particle Hamiltonians in the coordinate representation and two-particle discrete Schrödinger operators \( H(k) \) in the momentum representation. In Section 3 we obtain an estimate for the number of eigenvalues of the perturbed operator \( A - V \) lying outside its essential spectrum by the number of \( V \). In Section 4 we introduce the notions “width of the band” and “width of the band in the direction \( j \)” for the two-particle discrete Schrödinger operators \( H(k) = H_0(k) - V \) and prove Theorem 4.2. In Section 5 we prove the Birman-Schwinger principle for the two-particle discrete Schrödinger operators \( H(k) \). In Section 6 we introduce the notion of a zero-energy resonance for two-particle discrete Schrödinger operators \( H(k) \) and prove Theorems 6.6 and 6.10.
2. Energy Operators of Two Particles on a Lattice in the Coordinate Representation and the Two-Particle Discrete Schrödinger Operators

The free Hamiltonian $\hat{H}_0$ of the system of two quantum particles on the three-dimensional lattice $\mathbb{Z}^3$ is defined by the following (bounded) self-adjoint operator on the Hilbert space $\ell^2((\mathbb{Z}^3)^2)$

$$\hat{H}_0 = \frac{1}{2m_1}\Delta_{x_1} + \frac{1}{2m_2}\Delta_{x_2},$$

with $\Delta_{x_i} = \Delta \otimes I$ and $\Delta_{x_3} = I \otimes \Delta$, where $m_\alpha > 0$ denotes the mass of the particle $\alpha$ and $I$ is the identity operator on $\ell^2(\mathbb{Z}^3)$.

The Laplacian $\Delta$ is a difference operator which describes the transport of a particle from a site to the nearest neighboring site, i.e.,

$$(\Delta\hat{\psi})(x) = \sum_{|s|=1}[\hat{\psi}(x) - \hat{\psi}(x+s)], \quad \hat{\psi} \in \ell^2(\mathbb{Z}^3).$$

The total two-particle Hamiltonian $\hat{H}$ on the Hilbert space $\ell^2((\mathbb{Z}^3)^2)$ describes the interaction between two particles

$$\hat{H} = \hat{H}_0 - \hat{V},$$

where

$$(\hat{V}\hat{\psi})(x_1, x_2) = \hat{v}(x_1 - x_2)\hat{\psi}(x_1, x_2), \quad \hat{\psi} \in \ell^2((\mathbb{Z}^3)^2),$$

and $\hat{v}$ is a real bounded function on $\mathbb{Z}^3$.

**Assumption 2.1.** Assume that $\hat{v}(s)$ is even function on $\mathbb{Z}^3$ verifying

$$\lim_{|s| \to \infty} |s|^{3+\kappa}\hat{v}(s) = 0, \quad \kappa > 1/2.$$

Under above assumptions the two-particle Hamiltonian $\hat{H}$ is a well-defined bounded self-adjoint operator on $\ell^2((\mathbb{Z}^3)^2)$.

Recall that the study of spectral properties of the two-particle Hamiltonian $\hat{H}$ in the momentum representation reduces to the spectral analysis of the two-particle discrete Schrödinger operators $H(k), \ k \in \mathbb{T}^3$, acting in the Hilbert space $L^2(\mathbb{T}^3)$ (see [3, 8, 4])

$$H(k) = H_0(k) - \hat{V}.\quad \quad (2.1)$$

The non-perturbed operator $H_0(k)$ is the multiplication operator by the function $E_k(q)$ on $L^2(\mathbb{T}^3)$

$$(H_0(k)f)(q) = E_k(q)f(q), \quad f \in L^2(\mathbb{T}^3),$$

where

$$E_k(q) = \frac{1}{m_1}\varepsilon(\frac{1}{2}k + q) + \frac{1}{m_2}\varepsilon(\frac{1}{2}k - q),$$

$$\varepsilon(q) = \sum_{j=1}^3(1 - \cos q_j), \quad q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3.$$

The interaction (perturbation) operator $\hat{V}$ acts on the Hilbert space $L^2(\mathbb{T}^3)$ by

$$(\hat{V} f)(q) = (2\pi)^{-3/2}\int_{\mathbb{T}^3} v(q-s)f(s)ds, \quad f \in L^2(\mathbb{T}^3).$$

Here the kernel function $v$ is given by the Fourier series

$$v(q) = (2\pi)^{-3/2}\sum_{s \in \mathbb{Z}^3}\hat{v}(s)e^{i(q,s)},$$
with
\[(q, s) = \sum_{j=1}^{3} q^{(j)} s^{(j)}, \quad q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3, \quad s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3.\]

We see that the operator \(H(k), k \in \mathbb{T}^3\), does not split into the sum of a center-of-mass
term and a relative kinetic energy term contrary to the continuum case, where one as is
center-of-mass.

3. ON THE NUMBER OF EIGENVALUES OF BOUNDED SELF-ADJOINT OPERATORS

We establish some auxiliary results on the number of eigenvalues for an abstract opera-
tor on a Hilbert space \(\mathcal{H}\).

For a bounded self-adjoint operator \(A\) in a Hilbert space \(\mathcal{H}\), we define \(n_+ (\mu, A)\) (resp. \(n_- (\mu, A)\)) as
\[n_+ (\mu, A) = \sup \{ \dim L : L \subset \mathcal{H}; (Af, f) > \mu, f \in L, ||f|| = 1 \}\]
resp.
\[n_- (\mu, A) = \sup \{ \dim L : L \subset \mathcal{H}; (Af, f) < \mu, f \in L, ||f|| = 1 \}\]
The value \(n_+ (\mu, A)\) (resp. \(n_- (\mu, A)\)) is equal to the infinity if \(\mu\) is in the essential spectrum
and \(n_+ (\mu, A)\) (resp. \(n_- (\mu, A)\)) is finite, it is equal to the number of the eigenvalues of \(A\)
bigger (resp. smaller) than \(\mu\).

Now we introduce the notion of ”width of the spectrum of the self-adjoint operator \(A\)”.

**Definition 3.1.** The width of the operator \(A\), denoted by \(w_s (A)\), is defined to be
\[w_s (A) = M(A) - m(A),\]
where \(M(A) = \sup_{||f||=1} (Af, f)\) and \(m(A) = \inf_{||f||=1} (Af, f)\).

**Theorem 3.2.** Let \(A\) be a bounded self-adjoint operator and \(V\) be a compact self-adjoint
operator in a Hilbert space \(\mathcal{H}\). Then
\[n_-(m(A), A-V) \geq n_+ (w_s (A), V) \quad (\text{resp.} \quad n_+ (M(A), A-V) \geq n_- (-w_s (A), V)).\]

**Proof.** Let \(A\) be a scalar operator, i.e., \(A = \mu I, \mu \in \mathbb{R}^1\). Then \(w_s (A) = 0\) and hence we have
\[n_-(\mu, A-V) = n_+ (0, V).\]

Now let \(A\) be a non-scalar operator, i.e., \(A \neq \mu I\). Then \(w_s (A) > 0\). Since \(V\) is a com-
 pact operator the number \(n_+ (w_s (A), V)\) is finite. Let \(\mathcal{H}_{(w_s (A)+\infty)}(V)\) be the subspace
generated by the eigenfunctions of \(V\) associated with the eigenvalues bigger than \(w_s (A)\). Then
\[n_+(w_s (A), V) = \dim \mathcal{H}_{(w_s (A)+\infty)}(V)\]
and for any \(f \in \mathcal{H}_{(w_s (A)+\infty)}(V), ||f|| = 1\) the following relations
\[(Af, f) - (Vf, f) < M(A) - w_s (A) = m(A)\]
hold. This implies that
\[n_-(m(A), A-V) \geq \dim \mathcal{H}_{(w_s (A)+\infty)}(V) = n_+(w_s (A), V).\]
Similarly we get
\[n_+(M(A), A-V) \geq n_- (-w_s (A), V).\]
\[\square\]

**Corollary 3.3.** Let \(A\) be a bounded self-adjoint operator and \(V\) be compact self-adjoint
operator in \(\mathcal{H}\). Then
\[n_-(m(A), A-V) + n_+(M(A), A-V) \geq n_+(w_s (A), |V|).\]
4. On the number of eigenvalues of the two-particle discrete Schrödinger operators $H(k), k \in \mathbb{T}^3$

Under Assumption 2.1 the perturbation $V$ of the operator $H_0(k)$ is a trace class operator and therefore in accordance with the well known invariance of the absolutely continuous spectrum under trace class perturbations the absolutely continuous spectrum of the operator $H(k)$ defined by (2.1) fills in the following interval on the real axis:

$$
\sigma_{ac}(H(k)) = [E_{\min}(k), E_{\max}(k)],
$$

where

$$
E_{\min}(k) = \min_{q \in \mathbb{T}^3} E_k(q), \quad E_{\max}(k) = \max_{q \in \mathbb{T}^3} E_k(q).
$$

Now we will apply the results obtained in Section 3 to the two-particle discrete Schrödinger $H(k)$. We recall that the operator $H_0(k)$ has a band.

Let us introduce the notion "width of the band of $H_0(k)$."

**Definition 4.1.** The width of the band of $H_0(k)$, denoted by $w_b(H_0(k))$, is defined to be

$$
w_b(H_0(k)) \equiv \max_{q \in \mathbb{T}^3} E_k(q) - \min_{q \in \mathbb{T}^3} E_k(q).
$$

**Theorem 4.2.** (i) For any $k \in \mathbb{T}^3$ the following inequality holds

$$
n_- (E_{\min}(k), H(k)) \geq n_+(w_b(H_0(k)), V).
$$

(ii) Let $m_1 = m_2$ and $k = \hat{\pi} = (\pi, \pi, \pi)$. Then the equalities hold

$$
n_-(\frac{6}{m}, H(\hat{\pi})) = n_+(0, V) \quad \text{and} \quad n_+(\frac{6}{m}, H(\hat{\pi})) = n_-(0, V).
$$

**Proof.** The part (i) of Theorem 4.2 is a consequence of Theorem 4.2. From the condition (ii) of Theorem 4.2 we conclude that $E_k(p) = 6(m)^{-1}$, hence that $w_b(H_0(\hat{\pi})) = 0$, and finally that $H_0(\hat{\pi}) = 6(m)^{-1} I$ is a scalar operator. Thus the numbers $\{ \frac{6}{m} - \hat{v}(x), x \in \mathbb{Z}^3 \}$ are eigenvalues of the operator $H(\hat{\pi})$.

The Corollary 3.2 yields immediately the following

**Corollary 4.3.** For any $k \in \mathbb{T}^3$ the following inequality

$$
n_- (E_{\min}(k), H(k)) + n_+(E_{\max}(k), H(k)) \geq n_+(w_b(H_0(k)), |V|)
$$

holds.

Now we introduce the notion "width of the band spectrum of $H_0(k)$ in the direction $j = 1, 2, 3$".

**Definition 4.4.** Let $k = (k^{(1)}, k^{(2)}, k^{(3)}) \in \mathbb{T}^3$. By the width $w_{jb}(k^{(j)})$ of the band spectrum in the direction $j = 1, 2, 3$ of $H_0(k)$ we mean

$$
w_{jb}(k^{(j)}) \equiv \max_{q^{(j)} \in (-\pi, \pi]} E_k(q) - \min_{q^{(j)} \in [-\pi, \pi]} E_k(q).
$$
Lemma 4.5. The width of the band spectrum in the direction $w_{jb}(k^{(j)})$, $j = 1, 2, 3$, depends only on $k^{(j)} \in (-\pi, \pi]$ and the equality

$$w_b(H_0(k)) = \sum_{j=1}^{3} w_{jb}(k^{(j)})$$

holds.

Proof. The function $E_k(p)$ can be rewritten as

$$(4.1) \quad E_k(p) = 3\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - 3 \sum_{j=1}^{3} (a(k^{(j)}) \cos p^{(j)} + b(k^{(j)}) \sin p^{(j)}),$$

where the coefficients $a(k^{(j)})$ and $b(k^{(j)})$ are given by

$$a(k^{(j)}) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \cos \frac{k^{(j)}}{2}, \quad b(k^{(j)}) = \left(\frac{1}{m_1} - \frac{1}{m_2}\right) \sin \frac{k^{(j)}}{2}.$$  

The equality (4.1) implies the following representation for $E_k(p)$

$$(4.2) \quad E_k(p) = 3\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \sum_{j=1}^{3} r(k^{(j)}) \cos(p^{(j)} - p^{(k^{(j)})}),$$

where

$$r(k^{(j)}) = \sqrt{a^2(k^{(j)}) + b^2(k^{(j)})}, \quad p^{(k^{(j)})} = \arcsin \frac{b(k^{(j)})}{r(k^{(j)})}, \quad k^{(j)} \in (-\pi, \pi].$$

From Definition 3.3 of the width $w_{jb}(k^{(j)})$ of the band in the direction $j = 1, 2, 3$ and the representation (4.2) for $E_k(p)$ we conclude that

$$w_{jb}(k^{(j)}) = 2r(k^{(j)}) \quad \text{and} \quad w_b(H_0(k)) = \sum_{j=1}^{3} 2r(k^{(j)}).$$

The proof of Lemma 4.5 implies the following

Corollary 4.6. (i) The functions $w_{jb}(k^{(j)}) \equiv 2r(k^{(j)})$, $j = 1, 2, 3$, are real analytic, even and positive on the interval $(-\pi, \pi)$.

(ii) The equality

$$w_{jb}(k^{(j)}) = 0$$

holds if and only if $m_1 = m_2 = m$ and $k^{(j)} = \pi$ for some $j = 1, 2, 3$.

5. The Birman-Schwinger Principle for the Two-Particle Discrete Schrödinger Operators $H(k)$, $k \in \mathbb{T}^3$

Assumption 5.1. Assume that the interaction operator $V$ is positive.

Since the operator $\hat{V}$ is the multiplication operator by the positive function $\hat{v}(s)$ its positive root $\hat{V}^{\frac{1}{2}}$ is the multiplication operator by $\hat{v}^{\frac{1}{2}}(s)$. Hence the positive root $V^{\frac{1}{2}}$ of the positive operator $V$ has form

$$(V^{\frac{1}{2}}f)(p) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{T}^3} \hat{v}^{\frac{1}{2}}(p - p')f(p')dp',$$
where the kernel function $\hat{v}^\pm(p)$ is the inverse Fourier transform of the function $\hat{v}^\pm(s)$, i.e.,

$$\hat{v}^\pm(p) = \frac{1}{(2\pi)^{3/2}} \sum_{n \in \mathbb{Z}^3} \hat{v}^\pm(s)e^{i(p,s)}.$$ 

We define for any $k \in (-\pi, \pi)^3$ and $z \leq E_{\min}(k)$ (and also for any $k \in \mathbb{T}^3 \setminus (-\pi, \pi)^3$ and $z < E_{\min}(k)$) the integral operator $G(k, z)$ resp. $G_k^+(k, z)$ with the kernel $G(k, z; p, q)$ resp. $G_k^+(k, z; p, q)$

$$G(k, z; p, q) = \frac{1}{(2\pi)^{3}} \int_{\mathbb{T}^3} \hat{v}^\pm(p-t)(E_k(t)-z)^{-1/2}v^\pm(t-q)dt$$

resp.

$$G_k^+(k, z; p, q) = \frac{1}{(2\pi)^{3/2}} \hat{v}^\pm(p-q)(E_k(q)-z)^{-1/2}.$$ 

The proof of the following variant of the Birman-Schwinger principle for two-particle discrete Schrödinger operator $H(k)$ follows similar lines as in the case of quantum particles moving on $\mathbb{R}^3$.

**Lemma 5.2.** For any $k \in \mathbb{T}^3$ and $z < E_{\min}(k)$ the operator $G(k, z)$ acts on $L^2(\mathbb{T}^3)$, is positive and the equality

$$n_-(z, H(k)) = n_+(1, G(k, z))$$

holds.

Now we are going to obtain a generalization of the Birman-Schwinger principle for the two-particle Schrödinger operators on the lattice $\mathbb{Z}^3$.

**Theorem 5.3.** For any $k \in (-\pi, \pi)^3$ the operator $G(k, E_{\min}(k))$ acts on $L^2(\mathbb{T}^3)$, is positive, trace class and the equality

$$n_-(E_{\min}(k), H(k)) = n_+(1, G(k, E_{\min}(k)))$$

holds.

**Proof.** For any $z \leq E_{\min}(k)$ the operator $G_k^+(k, z)$ is Hilbert-Schmidt. Since the equality

$$G(k, z) = G_k^+(k, z)(G_k^+(k, z))^*$$

holds the operator $G(k, z)$ is positive and belongs to the trace class. It is easy to show that there exists $C > 0$ so that the inequality

$$||G(k, E_{\min}(k)) - G(k, z)|| \leq C\sqrt{E_{\min}(k) - z}$$

holds.

Let us show that $n_-(E_{\min}(k), H(k)) = n_+(1, G(k, E_{\min}(k)))$. Since $G(k, E_{\min}(k))$ is a compact operator the number $n_+(1, G(k, E_{\min}(k)))$ is finite. For any $\psi \in L^2(\mathbb{T}^3)$ and $z < E_{\min}(k)$ the following relations

$$(G(k, z)\psi, \psi) = \int_{\mathbb{T}^3} \frac{|V^{1/2}\psi(p)|^2 dp}{E_k(p) - z} \leq \int_{\mathbb{T}^3} \frac{|V^{1/2}\psi(p)|^2 dp}{E_k(p) - E_{\min}(k)} = (G(k, E_{\min}(k))\psi, \psi)$$

hold.

Consequently we have

$$n_+(1, G(k, z)) \leq n_+(1, G(k, E_{\min}(k))).$$
Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 1 \) be the eigenvalues and \( \psi_1, \psi_2, \ldots, \psi_n \) the corresponding eigenfunctions of the operator \( G(k, E_{\text{min}}(k)) \). Denote by \( \mathcal{H}_n \) the subspace generated by the eigenfunctions \( \psi_1, \psi_2, \ldots, \psi_n \). Since for all \( z \in U_\delta(E_{\text{min}}(k)) = (E_{\text{min}}(k) - \delta, E_{\text{min}}(k)) \), \( \delta > 0 \) sufficiently small, the inequality

\[
\|G(k, z) - G(k, E_{\text{min}}(k))\| < \lambda_n - 1
\]

holds for any \( \psi \in \mathcal{H}_n \), we have

\[
(G(k, z)\psi, \psi) = (G(k, E_{\text{min}}(k))\psi, \psi) - ([G(k, E_{\text{min}}(k)) - G(k, z)]\psi, \psi) > (\psi, \psi).
\]

From (5.4) we obtain that

\[
n_+(1, G(k, z)) \geq \dim \mathcal{H}_n = n_+(1, G(k, E_{\text{min}}(k)))
\]

for any \( z \in U_\delta(E_{\text{min}}(k)), \delta > 0 \) sufficiently small.

Combining (5.3) and (5.5) we obtain

\[
n_+(1, G(k, z)) = n_+(1, G(k, E_{\text{min}}(k)))
\]

for all \( z \in U_\delta(E_{\text{min}}(k)) \).

By Birman-Schwinger’s principle (Lemma 5.2) and the equality (5.6) we have

\[
\lim_{z \to E_{\text{min}}(k) - 0} n_-(z, H(k)) = n_-(E_{\text{min}}(k), H(k)).
\]

Taking into account (5.7) and (5.6) we obtain the equality (5.2).  

6. Threshold Analysis of \( H(k), k \in \mathbb{T}^3 \)

**Proposition 6.1.** Assume Assumption 2.1. Then the operator \( H(0) \) has a nontrivial kernel if and only if the integral operator \( G(0, 0) \) has the eigenvalue \( \lambda = 1 \) and the corresponding eigenfunction \( \psi \in L^2(\mathbb{T}^3) \) satisfies the condition

\[
\int_{\mathbb{T}^3} v^{1/2}(t)\psi(t)dt = 0.
\]

**Proof.** See [4].  

**Definition 6.2.** Let Assumption 2.1 be fulfilled. The operator \( H(0) \) is said to have a zero energy resonance if the integral operator \( G(0, 0) \) has the eigenvalue \( \lambda = 1 \) and the associated eigenfunction \( \psi \) satisfies the condition

\[
(v^{1/2}, \psi) = \int_{\mathbb{T}^3} v^{1/2}(p)\psi(p)dp \neq 0.
\]

**Remark 6.3.** Our definition 6.2 of a zero energy resonance is a direct analogue of that in the continuous case (see, e.g., [2], [14], [16] and [18] and references therein).

**Remark 6.4.** If the Hamiltonian \( H(0) \) has a zero energy resonance then the function

\[
f(p) = \frac{(V^{1/2}\psi)(p)}{E_0(p)}
\]

obeys the equation

\[
H(0)f = 0
\]

and \( f \) belongs to \( L^r(\mathbb{T}^3) \), \( 1 \leq r < 3/2 \).

The following lemma expresses an important feat of the two-particle Schrödinger operators.
Lemma 6.5. Let $m_1 = m_2 = m$ and let $H(0)$ be positive. Then for all $k \in \mathbb{T}^3, k \neq 0$ the operator $H(k)$ is positive.

Proof. Since

$$E_k(p) = \frac{1}{m} \left( \varepsilon \left( \frac{k}{2} - p \right) + \varepsilon \left( \frac{k}{2} + p \right) \right)$$

and $v(p)$ are even functions, the subspace $L^2_e(\mathbb{T}^3)$ resp. $L^2_o(\mathbb{T}^3)$ of the even resp. odd functions of $L^2(\mathbb{T}^3)$ is an invariant subspace for the operator $H(k)$.

Then for any $f \in L^2(\mathbb{T}^3)$ we have

$$\langle H(k)f, f \rangle = \langle H(k)f_e, f_e \rangle + \langle H(k)f_o, f_o \rangle,$$

where

$$f_e(p) = \frac{f(p) + f(-p)}{2}, \quad f_o(p) = \frac{f(p) - f(-p)}{2}.$$

We remark that $f_e \in L^2_e(\mathbb{T}^3)$ and $f_o \in L^2_o(\mathbb{T}^3)$. Therefore making a change of variables on the r.h.s of (6.1) we have

$$\langle H(k)f, f \rangle = \langle H(0)f_{ek}, f_{ek} \rangle + \langle H(0)f_{ok}, f_{ok} \rangle \geq 0$$

for $f \neq 0$,

where

$$f_{ek}(p) = f_e(k/2 - p), \quad f_{ok}(p) = f_o(k/2 - p).$$

We note that $f_{ek}, f_{ok} \in L^2(\mathbb{T}^3).$ \hfill $\square$

Our main nonperturbative result is the following

Theorem 6.6. Let $m_1 = m_2 = m$ and assume that the operator $H(0)$ is positive. Assume moreover that $H(0)$ has a zero eigenvalue of multiplicity $n$ and a zero energy resonance. Then for all nonzero $k \in (-\pi, \pi)^3$ the operator $H(k)$ has at least $n + 1$ nonnegative eigenvalues lying below the bottom $E_{\min}(k)$ of the band spectrum of $H(k)$.

Proof. Under the assumptions of Theorem 6.6 the equation $G(0,0)\psi = \psi$ has $n + 1$ solutions $\psi_0, \psi_1, ..., \psi_n$ in the Hilbert space $L^2(\mathbb{T}^3)$. Let $\psi \in L^2(\mathbb{T}^3)$ be one of the solutions. Then

$$\int_{\mathbb{T}^3} (E_0(p))^{-1} \left| (V^{1/2}\psi)(p) \right|^2 dp = \langle G(0,0)\psi, \psi \rangle = \langle \psi, \psi \rangle.$$ 

For any nonzero $k \in (-\pi, \pi)^3$ and $p \neq 0$ the inequalities

$$0 < E_k(p) - E_{\min}(k) < E_0(p)$$

hold. Therefore, by the definition (5.1) of the operator $G(k, E_{\min}(k))$ we have

$$\langle \psi, \psi \rangle = \int_{\mathbb{T}^3} \frac{|(V^{1/2}\psi)(p)|^2 dp}{E_0(p)} \lesssim \int_{\mathbb{T}^3} \frac{|(V^{1/2}\psi)(p)|^2 dp}{E_k(p) - E_{\min}(k)} = \langle G(k, E_{\min}(k))\psi, \psi \rangle$$

for all nonzero $k \in (-\pi, \pi)^3$. This means

$$n_+(1, G(k, E_{\min}(k))) \geq n + 1.$$ 

Applying the Birman-Schwinger principle we conclude that for all nonzero $k \in (-\pi, \pi)^3$ the operator $H(k)$ has at least $n + 1$ eigenvalues lying below the bottom of the essential spectrum. Since by Lemma 6.5 the operator $H(k)$ is positive we have that all eigenvalues of $H(k)$ are nonnegative.

$\square$
Corollary 6.7. Let $m_1 = m_2 = m$ and assume that the operator $H(0)$ is positive. Assume that $H(0)$ has a zero eigenvalue of multiplicity $n$. Then for all nonzero $k \in (-\pi, \pi)^3$ the operator $H(k)$ has at least $n$ nonnegative eigenvalues lying below the bottom $E_{\min}(k)$ of the band spectrum of $H(k)$.

Remark 6.8. We note that the interaction operator $V$ is positive and hence $H(k)$ has no eigenvalue bigger than $E_{\max}(k)$.

Remark 6.9. A result related to Theorem 6.6 has been obtained in [4]. In the latter reference only existence of eigenvalues below $E_{\min}(k)$ has been in fact proven, using a different method, however it was not proven that these eigenvalues are nonnegative. See also [4] for examples, which in particular show the existence of a zero-energy resonance and zero-eigenvalue of multiplicity 2 or 3.

Theorem 6.10. Assume $m_1 = m_2$ and $\kappa^{(j)} = \pi$ for some $j = 1, 2, 3$. Then $\omega_{j\kappa}(\pi) = 0$ and the following inequality
\[
\inf_{s \in \mathbb{Z}^1} (s \in \mathbb{Z}^1 : \hat{\nu}(se^j) > 0)
\]
holds.

Proof. Under the assumptions of Theorem 6.10 the function $E_k(p)$ does not depend on $p^{(j)}$ and $\omega_{j\kappa}(\pi) = 0$. Note that for any $x \in \mathbb{Z}^3$ the value $\hat{\nu}(x)$ is an eigenvalue of the operator $V$. Let $\mathcal{H}_j = \{s \in \mathbb{Z}^1 : \hat{\nu}(se^j) > 0\}$. Then the associated eigenfunctions
\[
\psi_s(p^{(j)}) = (2\pi)^{-3/2} e^{i p^{(j)}}, \quad s \in \mathcal{H}_j
\]
of the operator $H(k)$ depend only on $p^{(j)}$. Let $\mathcal{H}_n$ be the $n(n \leq \text{card } \mathcal{H}_j)$-dimensional subspace of the Hilbert space $L^2(\mathbb{T}^3)$ generated by the eigenfunctions $\{\psi_s(p^{(j)})\}_{i=1}^n$, $s_i \in \mathcal{H}_j$. Then the subspace $\mathcal{H}_n$ is invariant with respect to the operator $V^{1/2}$ and the equality
\[
\inf_{\psi \in \mathcal{H}_n} \|V^{1/2}_z\psi\|^2 = \inf_{\psi \in \mathcal{H}_n} \|V^{1/2}_z\psi, V^{1/2}_z\psi\|^2 = \inf_{\psi \in \mathcal{H}_n} (V\psi, \psi) = \min_{1 \leq s \leq n} \hat{\nu}(s_i e^j)\|\psi\|^2
\]
holds. Now we will prove that for any natural $n \leq \text{card } \mathcal{H}_j$ there is $z_n < E_{\min}(k)$ such that
\[
\inf_{\psi \in \mathcal{H}_n, \|\psi\| = 1} (G(k, z_n)\psi, \psi) > 1.
\]
By Birman-Schwinger principle this means
\[
n_+(E_{\min}(k), H(k)) \geq n_+(1, G(k, z_n)) \geq \text{card } \mathcal{H}_j.
\]
Indeed
\[
\inf_{\psi \in \mathcal{H}_n, \|\psi\| = 1} (G(k, z)\psi, \psi) = \frac{1}{(2\pi)^3} \min_{1 \leq s \leq n} \hat{\nu}(s_i e^j) \int_{\mathbb{T}^3} (E_k(p) - z)^{-1} \frac{dp^{(1)} dp^{(2)} dp^{(3)}}{dp^{(j)}}.
\]
One can check
\[
\lim_{z \to E_{\min}(k) - 0} \int_{\mathbb{T}^3} \frac{dp^{(i)} dp^{(l)}}{E_k(p) - z} = \lim_{z \to E_{\min}(k) - 0} \left( C_0(k) - \frac{2\pi \log(E_{\min}(k) - z)}{\sqrt{\cos \frac{k^{(i)}}{2} \cos \frac{k^{(l)}}{2}}} \right) = +\infty,
\]
where $i, l = 1, 2, 3$, $i \neq l, i \neq l$, $i \neq j$ and $C_0(k)$ is a positive number. Therefore there exists a number $z_n < E_{\min}(k)$ satisfying inequality (6.2). \qed
Theorem 6.10 immediately yields the following

**Corollary 6.11.** Let the perturbation operator $V$ have infinitely many positive eigenvalues. Then under assumptions of Theorem 6.10 the operator $H(k)$ has infinitely many eigenvalues below the bottom $E_{min}(k)$ of the band.

**Acknowledgments** The authors are grateful to Prof. Dr. K. A. Makarov and Dr. Z. I. Muminov for useful discussions. This work was partially supported by the DFG 436 USB 113/4 Project and the Fundamental Science Foundation of Uzbekistan. S. N. Lakaev and J. I. Abdullaev gratefully acknowledge the hospitality of the Institute of Applied Mathematics and of the IZKS of the University of Bonn.

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