QDYBE: some explicit formulas for exchange matrix and related objects in case of $sl(2), q = 1$

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**Abstract**

This mainly tutorial paper is intended as a somewhat larger example for parts of the theory exposed in the Lectures on the dynamical Yang-Baxter equations by P. Etingof and O. Schiffmann, [math.QA/9908064](http://arxiv.org/abs/math.QA/9908064). We explicitly compute the matrix entries of the intertwining operator, fusion matrix and exchange matrix associated to $sl(2, R)$ for $q = 1$. We also consider the weighted trace function and the dual Macdonald-Ruijsenaars equation for this particular case. As expected, the matrix entries of the exchange matrix turn out to be Racah polynomials. However, the road to their explicit formula via the fusion matrix is quick, and it also yields an interesting way to derive their orthogonality relations.

**1. Introduction**

The quantum dynamical Yang-Baxter equation (QDYBE) was first considered in 1984 by Gervais and Neveu [12], with motivation from physics (for monodromy matrices in Liouville theory). A general form of QDYBE with spectral parameter was presented by Felder [10], [11] at two major congresses in 1994. The corresponding classical dynamical Yang-Baxter equation (CDYBE) was presented there as well. Next Etingof and Varchenko started a program to give geometric interpretations of solutions of CDYBE (see [6]) and of QDYBE (see [7]) in the case without spectral parameter. In the context of this program they pointed out a method to obtain solutions of QDYBE by the so-called exchange construction (see [8]). This uses, for any simple Lie algebra $g$, representation theory of $U(g)$ or of its quantized version $U_q(g)$ in order to define successively the intertwining operator, the fusion matrix and the exchange matrix. The matrix elements of the intertwining operator and of the exchange matrix generalize respectively the Clebsch-Gordan coefficients and the Racah coefficients to the case where the first tensor factor is a Verma module rather than a finite dimensional irreducible module. The exchange matrix is shown to satisfy QDYBE. Etingof and Varchenko also started in [9] a related program to connect the above objects with weighted trace functions and with solutions of the (q-)Knizhnik-Zamolodchikov-Bernard equation (KZB or qKZB). A nice introduction to the topics indicated above was recently given by Etingof and O. Schiffmann [5].

The present paper is intended as a somewhat larger example for parts of the theory in [5]. In particular, while Example 2 of Section 2 of [5] explicitly computes the matrix elements of the intertwining operator, the fusion matrix and the exchange matrix in the case of the 2-dimensional irreducible representation of $sl(2), q = 1$, we will do this for irreps of $sl(2)$ of arbitrary finite dimension. This is the topic of Sections 3–5, after some preliminaries in Section 2. An immediate consequence of the explicit form of the fusion matrix is an explicit expression of the universal fusion matrix for $sl(2), q = 1$, see Section 6,
and of the related universal operator \( Q(\lambda) \), see Section 7. In Section 8 of [5] this particular universal fusion matrix is obtained as a consequence of the ABBR equation in the \( q = 1 \) case. We conclude in Section 8 with the example of the weighted trace function for \( sl(2) \), \( q = 1 \) and the dual Macdonald-Ruijsenaars equation satisfied by it. This illustrates part of Section 9.2 of [5].

The aim of this paper is mainly tutorial, as a complement to [5] for those who like to see larger worked-out examples. The explicit expressions we obtain for the matrix elements of the intertwining operator and the fusion matrix are terminating \( \genfrac{3}{3}{0}{3}{2}(1)'s \) and terminating balanced \( \genfrac{3}{3}{0}{4}{3}(1)'s \), as should be the case since these matrix elements, rational in \( \lambda \in \mathbb{C} \), must be analytic continuations of Clebsch-Gordan coefficients respectively Racah coefficients for finite dimensional irreps of \( sl(2) \). Also, the weighted trace function turns out to be a Gegenbauer function of the second kind, which analytically continues the Gegenbauer polynomial occurring as a weighted trace function associated with finite dimensional irreps of \( sl(2) \).

We would like to point to one observation of particular interest. The fusion matrix \( J_{\delta,\gamma}(\lambda) \), explicitly obtained in Section 4, is triangular and has matrix entries given by elementary expressions, and the same is true for the inverse fusion matrix. The exchange matrix \( R_{\delta,\gamma}(\lambda) \) is defined in terms of the fusion matrix and a transpose of its inverse by equation (5.1). Thus it is a product of an upper triangular and a lower triangular matrix, both with elementary matrix entries. Therefore we arrive very quickly at a single sum expression for the matrix entries of the exchange matrix, which can moreover immediately be recognized as Racah polynomials. This approach seems quicker than other approaches in literature to explicit expressions of Racah coefficients. Moreover, the inverse of the exchange matrix can also be explicitly computed from the above factorization in an easy way, which yields biorthogonality relations for the matrix entries of the exchange matrix. After some transformations this yields the known orthogonality relations for the Racah polynomials involved.

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Notation We will write \( a \lor b \) for \( \max(a, b) \) and \( a \land b \) for \( \min(a, b) \). For notation and results on hypergeometric series we refer to [1] and [4].

2. Preliminaries

Let \( \mathfrak{g} \) be a simple complex Lie algebra and let \( \mathfrak{h} \subset \mathfrak{g} \) be a Cartan subalgebra. Let \( \Delta \subset \mathfrak{h}^* \) be the associated root system and let \( \Delta^+ \) be a choice of positive roots. Define a partial order \( \leq \) on \( \mathfrak{h}^* \) by putting \( \lambda \leq \mu \) if \( \mu - \lambda \) is in the \( \mathbb{Z}_{\geq 0} \) span of \( \Delta^+ \). If \( V \) is any \( \mathfrak{g} \)-module and \( \lambda \in \mathfrak{h}^* \), denote by \( V[\lambda] \) the weight space \( \{ v \in V \mid h \cdot v = \lambda(h) v \text{ for all } h \in \mathfrak{h} \} \). For \( \lambda \in \mathfrak{h}^* \) let \( M_{\lambda} \) be the Verma module with highest weight \( \lambda \) and corresponding highest weight vector \( x_{\lambda} \). Recall that the tensor product \( W \otimes V \) of two \( \mathfrak{g} \)-modules \( W \) and \( V \) becomes a \( \mathfrak{g} \)-module such that \( g \cdot (w \otimes v) = (g \cdot w) \otimes v + w \otimes (g \cdot v) \) \( (g \in \mathfrak{g}, w \in W, v \in V) \). Then
$W \otimes V$ also becomes a $U(\mathfrak{g})$-module and we have

$$g^n \cdot (w \otimes v) = \sum_{j=0}^{n} \binom{n}{j} (g^j \cdot w) \otimes (g^{n-j} \cdot v) \quad (g \in \mathfrak{g}, \ w \in W, \ v \in V, \ n \in \mathbb{Z}_{\geq 0}). \quad (2.1)$$

Take $\mathfrak{g} := sl(2, \mathbb{C})$ with basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

Let $\mathfrak{h}$ be spanned by $h$. Then the map $\lambda \mapsto \lambda(h) : \mathfrak{h}^* \to \mathbb{C}$ identifies $\mathfrak{h}^*$ with $\mathbb{C}$. We have $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Thus $\Delta = \{2, -2\}$. Choose $2 \in \Delta$ as the positive root. In $\mathfrak{h}^*$ we have that $\lambda \leq \mu$ iff $\mu - \lambda$ is an even nonnegative integer. By induction with respect to $n$ we have the following identity in $U(\mathfrak{g})$:

$$ef^n = f^n e + nf^{n-1}(h - n + 1) \quad (n \in \mathbb{Z}_{\geq 1}). \quad (2.2)$$

For $\lambda \in \mathfrak{h}^*$ a basis of the Verma module $M_\lambda$ is given by the elements $f^k \cdot x_\lambda$ ($k \in \mathbb{Z}_{\geq 0}$). Note that $f^k \cdot x_\lambda$ has weight $\lambda - 2k$. Clearly we have $e \cdot x_\lambda = 0$. Let $V$ be a finite dimensional irreducible $sl(2, \mathbb{C})$-module and let $0 \neq v \in V[\beta]$. Then $V = \bigcup_{k \in \mathbb{Z}} V[\beta + 2k]$ and there are $k_0, k_1 \in \mathbb{Z}_{\geq 0}$ such that $\dim V[\beta + 2k] = 1$ for $k = -k_0, \ldots, k_1$ and $\dim V[\beta + 2k] = 0$ otherwise. If $k \in \mathbb{Z}_{\geq 0}$ then $V[\beta + 2k]$ is spanned by $e^k \cdot v$.

### 3. The intertwining operator

Let $V$ be a finite dimensional $\mathfrak{g}$-module and let $v \in V$ be a weight vector. For $\lambda \in \mathfrak{h}^*$ put $\mu := \lambda - \text{wt}(v)$. As shown in [5, Proposition 2.2], there is for generic $\lambda \in \mathfrak{h}^*$ a unique $\mathfrak{g}$-intertwining operator $\Phi^v_\lambda : M_\lambda \to M_\mu \otimes V$ such that

$$\Phi^v_\lambda(x_\lambda) \in x_\mu \otimes v + \bigoplus_{\nu < \mu} M_\mu[\nu] \otimes V.$$  

Moreover, the operator $\Phi^v_\lambda$ depends rationally on $\lambda$.

#### Lemma 3.1  
Let $V$ be a finite dimensional irreducible $sl(2, \mathbb{C})$-module, $v \in V$, $\beta := \text{wt}(v)$. Then

$$\Phi^v_\lambda(x_\lambda) = \sum_{k=0}^{\infty} \frac{1}{k! (-\lambda + \beta)_k} (f^k \cdot x_{\lambda - \beta}) \otimes (e^k \cdot v) \quad (\lambda \in \mathbb{C}\backslash (\beta + \mathbb{Z}_{\geq 0})). \quad (3.1)$$

#### Proof  
By [5, Proposition 2.2] and by our earlier remarks about $sl(2, \mathbb{C})$-modules, we must have for generic $\lambda \in \mathbb{C}$ that

$$\Phi^v_\lambda(x_\lambda) = \sum_{k=0}^{\infty} a_k (f^k \cdot x_{\lambda - \beta}) \otimes (e^k \cdot v).$$
for certain coefficients $a_k$ depending on $\lambda$ and $\beta$ with $a_0 = 1$. By the intertwining property of $\Phi_\gamma^\lambda$ we have $e \cdot \Phi_\gamma^\lambda(x_\lambda) = 0$. Hence, by (2.2),

$$0 = \sum_{k=1}^{\infty} a_k (e f^k \cdot x_{\lambda - \beta}) \otimes (e^k \cdot v) + \sum_{k=0}^{\infty} a_k (f^k \cdot x_{\lambda - \beta}) \otimes (e^{k+1} \cdot v)$$

$$= \sum_{k=1}^{\infty} a_k k(\lambda - \beta - k + 1) (f^{k-1} \cdot x_{\lambda - \beta}) \otimes (e^k \cdot v) + \sum_{k=0}^{\infty} a_k (f^k \cdot x_{\lambda - \beta}) \otimes (e^{k+1} \cdot v)$$

$$= \sum_{k=1}^{\infty} (a_{k-1} - k(k - \lambda + \beta - 1)a_k) (f^{k-1} \cdot x_{\lambda - \beta}) \otimes (e^k \cdot v).$$

Hence, for $\lambda \in \mathbb{C} \setminus (\beta + \mathbb{Z}_{\geq 0})$ and for $k_1$ such that $e^k \cdot v \neq 0$ if $k \leq k_1$, we obtain the recurrence

$$a_k = \frac{a_{k-1}}{k(k - \lambda + \beta - 1)} \quad (k = 1, \ldots, k_1).$$

By iteration we find that

$$a_k = \frac{1}{k! (-\lambda + \beta)_k}. \quad \Box$$

Let $V_\gamma$ be the finite dimensional irreducible $sl(2, \mathbb{C})$-module with highest weight $\gamma \in \mathbb{Z}_{\geq 0}$. Then we can take a basis of $V_\gamma$ consisting of vectors $v_\gamma^1, v_\gamma^2, \ldots, v_\gamma^n$ such that

$$h \cdot v_\gamma^{-\gamma+2k} = (-\gamma + 2k) v_\gamma^{-\gamma+2k},$$

$$f \cdot v_\gamma^{-\gamma+2k} = k v_\gamma^{-\gamma+2k-2} \quad \text{(or 0 if } k = 0),$$

$$e \cdot v_\gamma^{-\gamma+2k} = (\gamma - k) v_\gamma^{-\gamma+2k+2} \quad \text{(or 0 if } k = \gamma).$$

Then

$$e^i \cdot v_\gamma^{-\gamma+2k} = (-1)^i (-\gamma + k)_i v_\gamma^{-\gamma+2k+i}, \quad f^i \cdot v_\gamma^{-\gamma+2k} = \frac{k!}{(k - i)!} v_\gamma^{-\gamma+2k-2i}. \quad (3.2)$$

Thus formula (3.1) can be rewritten as

$$\Phi_\lambda^{\gamma-\gamma+2k}(x_\lambda) = \sum_{i=0}^{\gamma-k} \frac{(-1)^i (-\gamma + k)_i}{i! (-\lambda - \gamma + 2k)_i} (f^i \cdot x_{\lambda+\gamma-2k}) \otimes v_\gamma^{-\gamma+2k+i}, \quad (3.3)$$

where $\gamma \in \mathbb{Z}_{\geq 0}$, $k \in \{0, 1, \ldots, \gamma\}$, $\lambda \in \mathbb{C} \setminus \{-\gamma + 2k; -\gamma + 2k + 1, \ldots, k\}$.

Let $n \in \mathbb{Z}_{\geq 0}$ and apply $f^n$ to both sides of (3.3). By the intertwining property of $\Phi$, by equation (2.1) and by the second part of equation (3.2) we obtain

$$\Phi_\lambda^{\gamma-\gamma+2k}(f^n \cdot x_\lambda) = \sum_{i=0}^{\gamma-k} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^i (-\gamma + k)_i}{i! (-\lambda - \gamma + 2k)_i} \frac{(k + i)!}{(k + i + j - n)!} \times (f^{i+j} \cdot x_{\lambda+\gamma-2k}) \otimes v_\gamma^{-\gamma+2(k+i+j-n)}$$

$$= \sum_{m=0}^{n+\gamma-k} c_{m,n}^{\lambda,\gamma,k} (f^m \cdot x_{\lambda+\gamma-2k}) \otimes v_\gamma^{-\gamma+2(k+m-n)}, \quad (3.4)$$
where
\[
c_{\lambda,\gamma,k}^{m,n} = \frac{n!k!}{(k+m-n)!} \sum_{i=0}^{(\gamma-k) \wedge m} \frac{(-1)^i (-\gamma+k)_i (k+1)_i}{(\lambda-\gamma+2k)_i (n-m+i)(m-i)!i!} \tag{3.5}
\]
\[
= \frac{n!k!}{(k+m-n)!} \sum_{j=0}^{(n-m+\gamma-k) \wedge n} \frac{(-1)^{j+m-n} (-\gamma+k)_{j+m-n} (k+1)_{j+m-n}}{(-\lambda-\gamma+2k)_{j+m-n} (m-n+j)(n-j)!j!}. \tag{3.6}
\]

We obtained the last part of (3.4) (with (3.5) substituted) by passing from summation variables \(i, j\) to summation variables \(m, i\) with \(m = i + j\). Furthermore, (3.6) is obtained from (3.5) by substituting \(i = j + m - n\) for the summation variable.

In order to write (3.5) or (3.6) in terms of hypergeometric functions, we distinguish two cases.

**Theorem 3.2** Let the coefficients of the intertwining operator be given by
\[
\Phi^{\gamma+2k}_\lambda (f^n \cdot x_\lambda) = \sum_{m=0}^{n+\gamma-k} c_{m,n}^{\lambda,\gamma,k} (f^m \cdot x_{\lambda+\gamma-2k}) \otimes v^{\gamma}_{-\gamma+2(k+m-n)}. \tag{3.7}
\]
Then
\[
c_{m,n}^{\lambda,\gamma,k} = \frac{n!k!}{m!(n-m)!(k+m-n)!} \, _3F_2 \left[ \begin{array}{c} -m, -\gamma + k, k + 1 \\ -\lambda - \gamma + 2k, n - m + 1 \\ \end{array} ; 1 \right] \tag{3.8}
\]
if \(0 \lor (n-k) \leq m \leq n; \)
\[
c_{m,n}^{\lambda,\gamma,k} = \frac{(1)^{m-n} (-\gamma+k)_{m-n}}{(m-n)!(-\lambda-\gamma+2k)_{m-n}} \, _3F_2 \left[ \begin{array}{c} -n, -\gamma + k + m - n, k + m - n + 1 \\ -\lambda - \gamma + 2k + m - n, m - n + 1 \\ \end{array} ; 1 \right] \tag{3.9}
\]
if \(0 \leq n \leq m \leq n + \gamma - k.

We will later need the special case \(n = 0\) of (3.9) and the special case \(m = 0\) of (3.8) (which are both already an immediate consequence of (3.3)):
\[
c_{m,0}^{\lambda,\gamma,k} = \frac{(-1)^m (-\gamma+k)_m}{m!(-\lambda-\gamma+2k)_m} \quad (m = 0, 1, \ldots, \gamma-k), \tag{3.10}
\]
\[
c_{0,n}^{\lambda,\gamma,k} = \frac{k!}{(k-n)!} \quad (n = 0, 1, \ldots, k). \tag{3.11}
\]
4. The fusion matrix

We define the fusion matrix following [5, §2.1]. For \( g \) a simple complex Lie algebra let \( V \) and \( W \) be finite-dimensional \( g \)-modules, and let \( v \in V \) and \( w \in W \) be weight vectors. Then, for generic \( \lambda \in \mathfrak{h}^* \) there is a unique weight vector \( J_{WV}(\lambda) (w \otimes v) \in W \otimes V \) of weight \( \text{wt}(v) + \text{wt}(w) \) such that

\[
(\Phi^w_{\lambda-\text{ wt}(v)} \otimes 1)(\Phi^v_{\lambda}(x_\lambda)) = \Phi^{J_{WV}(\lambda)(w \otimes v)}_{\lambda}(x_\lambda).
\]  

(4.1)

For generic \( \lambda \) there is a unique extension of \( J_{WV}(\lambda) \) to a \( \mathfrak{h} \)-linear map \( J_{WV}(\lambda) : W \otimes V \to W \otimes V \). The operator \( J_{WV}(\lambda) \) is called the fusion matrix of \( V \) and \( W \). It depends rationally on \( \lambda \).

We will compute the fusion matrix for \( g = sl(2, \mathbb{C}) \). Let \( \gamma, \delta \in \mathbb{Z}_{\geq 0} \), \( k \in \{0, 1, \ldots, \gamma\} \), \( l \in \{0, 1, \ldots, \delta\} \). Let \( \lambda \in \mathbb{C} \) be generic. Then, by (3.7),

\[
(\Phi^\gamma_{\lambda+\gamma-2k} \otimes 1)(\Phi^\gamma_{\lambda+\gamma+2k}(x_\lambda)) = \sum_{r=0}^{\gamma-k} \sum_{r-l=0}^{r-l=0} c_{r,0}^{\lambda,\gamma,k} C_{m,r}^{\lambda,\gamma+\gamma-2k,\delta,l} (f^m \cdot x_{\lambda+\gamma+\gamma-2k-2l}) \otimes v^\gamma_{-\gamma+2k+2r}.
\]

Write \( J_{\delta,\gamma} \) instead of \( J_{V_\delta, \nu_\gamma} \). It follows by combination with (4.1) and by substitution of (3.10), (3.11) that

\[
J_{\delta,\gamma}(\lambda)(v^\delta_{-\gamma+2k} \otimes v^\gamma_{-\gamma+2k}) = \sum_{r=0}^{(\gamma-k) \wedge l} c_{r,0}^{\lambda,\gamma,k} C_{0,r}^{\lambda,\gamma+\gamma-2k,\delta,l} v^\delta_{-\gamma+2k+2r} \otimes v^\gamma_{-\gamma+2k+2r}.
\]

(4.2)

Replace \( l, k, r \) by \( n, s - n, n - m \), respectively. Then we get for \( 0 \leq s \leq \gamma + \delta \) and \((s - \gamma) \vee 0 \leq n \leq \delta \wedge s \) that

\[
J_{\delta,\gamma}(\lambda)(v^\delta_{-\gamma+2n} \otimes v^\gamma_{-\gamma+2s-2n}) = \sum_{m=(s-\gamma) \vee 0}^{n} A_{m,n}^{\lambda-\gamma,s} v^\delta_{-\gamma+2m} \otimes v^\gamma_{-\gamma+2s-2m}
\]

(4.2)

where

\[
A_{m,n}^{\lambda-\gamma,s} = A_{m,n} := \frac{(-1)^{n-m} n! (-\gamma + s - n)_{n-m}}{(n-m)! m! (-\lambda - \gamma + 2s - 2n)_{n-m}} \quad \text{if } m \leq n,
\]

(4.3)

and \( A_{m,n} := 0 \) if \( m > n \). Note that \( A_{m,n} \) is independent of \( \delta \), and that, as a function of \( \lambda, \gamma, s \), it depends only on \( \lambda - \gamma \) and \( \gamma - s \). Note also that \( A_{m,n} \) remains well-defined
for generic complex values of $\gamma$ and $s$ (not necessarily integer) and for $m, n \in \mathbb{Z}_{\geq 0}$. Thus $(A_{m,n})_{m,n \in \mathbb{Z}_{\geq 0}}$ is an infinite upper triangular matrix from which the matrix occurring in (4.2) is obtained by restricting $m$ and $n$ to the finite set $\{(s - \gamma) \lor 0, \ldots, \delta \land s\}$.

We will need later an explicit expression for the inverse $(B_{m,n})$ of the matrix $(A_{m,n})$. By using Maple we found the explicit value of $(B_{m,n})$ for $m, n$ restricted to $\{0, \ldots, s\}$ with $s$ small. From this we conjectured the general expression and next proved it:

**Lemma 4.1** For $\lambda, \gamma, s \in \mathbb{C}$ generic and for $m, n \in \mathbb{Z}_{\geq 0}$ (and moreover $m, n \geq s - \gamma$ if $s - \gamma \in \mathbb{Z}_{> 0}$) let

$$B_{m,n}^{\lambda,\gamma,s} = B_{m,n} := \frac{n!(-\gamma + s - n)_{n-m}}{(n-m)!m!(-\lambda - \gamma + 2s - m - n - 1)_{n-m}} \quad \text{if } m \leq n, \quad (4.4)$$

and $B_{m,n} := 0$ if $m > n$. Then the matrix $(B_{m,n})$ is the inverse of the matrix $(A_{m,n})$.

**Proof** For $m \leq n$ we have

$$\sum_{k=m}^{n} B_{m,k} A_{k,n} = \sum_{l=0}^{n-m} B_{m,m+l} A_{m+l,n} = \sum_{l=0}^{n-m} \frac{(m+l)!(-\gamma + s - m - l)_{l}(-1)^{n-m-l}n!(-\gamma + s - n)_{n-m-l}}{l!m!(-\lambda - \gamma + 2s - 2m - l - 1)_{l}(n-m-l)!m!(\lambda + \gamma - 2s + 2m + 2)_{l}l!}$$

$$= \frac{(-1)^{n-m}n!(-\gamma + s - n)_{n-m}}{m!(n-m)!(-\lambda - \gamma + 2s - 2n)_{n-m}} \sum_{l=0}^{n-m} \frac{(-n+m+l)_{l}(\lambda + \gamma - 2s + n + m + 1)_{l}}{(\lambda + \gamma - 2s + 2m + 2)_{l}l!}$$

$$= \frac{(-1)^{n-m}n!(-\gamma + s - n)_{n-m}}{m!(n-m)!(-\lambda - \gamma + 2s - 2n)_{n-m}} \sum_{l=0}^{n-m} \frac{(-n+m+l)_{l}(\lambda + \gamma - 2s + n + m + 1)_{l}}{(\lambda + \gamma - 2s + 2m + 2)_{l}l!}$$

$$= \frac{(-1)^{n-m}n!(-\gamma + s - n)_{n-m}}{m!(n-m)!(-\lambda - \gamma + 2s - 2n)_{n-m}} 2F_1\left[\begin{array}{c} -n+m, \lambda + \gamma - 2s + n + m + 1 \\ \lambda + \gamma - 2s + 2m + 2 \end{array} ; 1 \right]$$

which equals 0 if $m < n$ and equals 1 if $m = n$. In the last identity of the displayed formula we used the Chu-Vandermonde identity

$$2F_1\left[\begin{array}{c} -n, b \\ c \end{array} ; 1 \right] = \frac{(c - b)_n}{(c)_n} \quad (n \in \mathbb{Z}_{\geq 0}), \quad (4.5)$$

see for instance [1, Corollary 2.2.3]. In particular, we have

$$2F_1\left[\begin{array}{c} -n, c + n - 1 \\ c \end{array} ; z \right] = \frac{(-n+1)_n}{(c)_n} = \delta_{n,0} \quad (n \in \mathbb{Z}_{\geq 0}) \quad (4.6) \square$$

**Remark 4.2** Because of Lemma 4.1, $A_{m,n}$ and $B_{m,n}$ as given by (4.3) and (4.4) satisfy

$$\sum_{k=m}^{n} A_{m,k} B_{k,n} = \delta_{m,n} = \sum_{k=m}^{n} B_{m,k} A_{k,n}. \quad (4.7)$$
We showed the second identity in the proof of Lemma 4.1. We may verify the first identity independently as follows.

\[
\sum_{k=m}^{n} A_{m,k} B_{k,n} = \sum_{l=0}^{n-m} A_{m,n-l} B_{n-l,n} = \frac{(-1)^{n-m} n! (-\gamma + s - n)_{n-m}}{m! (n-m)! (-\lambda - \gamma + 2s - 2n)_{n-m}} \\
\times \begin{Bmatrix} -n + m, -\lambda - \gamma + 2s - 2n - 1, \frac{1}{2}(-\lambda - \gamma + 2s - 2n - 1) + 1 \\
-\lambda - \gamma + 2s - n - m, \frac{1}{2}(-\lambda - \gamma + 2s - 2n - 1)
\end{Bmatrix}; 1.
\]

Now observe that, for \( n \in \mathbb{Z}_{\geq 0} \), an application of (4.5) yields:

\[
\begin{aligned}
3F_2 \left[ \begin{array}{c} -n, b, \frac{1}{2}b + 1 \\ b + n + 1, \frac{1}{2}b + 1 \end{array} ; 1 \right] &= 2F_1 \left[ \begin{array}{c} -n, b \\ b + n + 1 \end{array} ; 1 \right] - \frac{2n}{b + n + 1} 2F_1 \left[ \begin{array}{c} -n + 1, b + 1 \\ b + n + 2 \end{array} ; 1 \right] \\
&= \frac{(n + 1)_n}{(b + n + 1)_n} - \frac{2n}{b + n + 1} \frac{(n + 1)_{n-1}}{(b + n + 2)_{n-1}} = \delta_{n,0}.
\end{aligned}
\]

**Remark 4.3** The explicit matrix inversion in Lemma 4.1 is a special case of Gould [13, Theorem 2]. This can be seen if we rewrite (4.3) as

\[
A_{k,n} = \frac{(\lambda + \gamma - 2s + 1)_k}{(\gamma - s + 1)_k} \frac{n! (\gamma - s + 1)_n}{(\lambda + \gamma - 2s + 1)_2n} (-1)^k \binom{n}{k} \left( -\lambda - \gamma + 2s - 1 - k \right),
\]

and (4.4) as

\[
B_{k,n} = \frac{(\lambda + \gamma - 2s + 1)_k}{k! (\gamma - s + 1)_k} \frac{(\gamma - s + 1)_n}{(\lambda + \gamma - 2s + 1)_n} \left( -\lambda - \gamma + 2s - 1 - n \right)^{-1} \\
\times (-1)^k \frac{-\lambda - \gamma + 2s - 1 - 2k}{-\lambda - \gamma + 2s - 1 - n - k} \binom{n}{n - k}.
\]

See further generalizations of formulas for explicit matrix inverses in Riordan [17, Ch. 2,3] and Krattenthaler [16].

We summarize our results in the following theorem.

**Theorem 4.4** Let \((s - \gamma) \vee 0 \leq n \leq \delta \wedge s\). Let \(A_{m,n}^{\lambda-\gamma,-s}\) and \(B_{m,n}^{\lambda-\gamma,-s}\) be given by (4.3) respectively (4.4) if \(0 \leq m \leq n\) and put them equal to 0 for other \(m, n \in \mathbb{Z}_{\geq 0}\). Then the fusion matrix for \(sl(2)\) and its inverse are given by

\[
J_{\delta,\gamma}(\lambda)(v^\delta_{-\delta+2n} \otimes v^\gamma_{-\gamma+2s-2n}) = \sum_{m=(s-\gamma)\vee 0}^{n} A_{m,n}^{\lambda-\gamma,-s} v^\delta_{-\delta+2m} \otimes v^\gamma_{-\gamma+2s-2m},
\]

\[
J_{\delta,\gamma}(\lambda)^{-1}(v^\delta_{-\delta+2n} \otimes v^\gamma_{-\gamma+2s-2n}) = \sum_{m=(s-\gamma)\vee 0}^{n} B_{m,n}^{\lambda-\gamma,-s} v^\delta_{-\delta+2m} \otimes v^\gamma_{-\gamma+2s-2m}.
\]
5. The exchange matrix

Let $\mathfrak{g}$ be a simple complex Lie algebra and let $V$ and $W$ be finite-dimensional $\mathfrak{g}$-modules. Again following [5, §2.1] we define the exchange matrix in terms of the fusion matrix by

$$R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} J_{WV}^{21}(\lambda) \quad (\lambda \in \mathfrak{h}^* \text{ generic}).$$

(5.1)

Here $J_{21} := PJP$ with $P(x \otimes y) := y \otimes x$. Then $R_{VW}(\lambda) : V \otimes W \to V \otimes W$ is an $\mathfrak{h}$-intertwining linear operator, rationally depending on $\lambda$. It follows immediately from (5.1) that

$$R_{VW}(\lambda)^{-1} = (R_{WV}(\lambda))^{21}$$

(5.2)

We will compute the exchange matrix for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $\gamma, \delta \in \mathbb{Z}_{\geq 0}, \ k \in \{0, 1, \ldots, \gamma\}$, $\ l \in \{0, 1, \ldots, \delta\}$. Let $\lambda \in \mathbb{C}$ be generic. Write $R_{\delta, \gamma}$ instead of $R_{V_{\delta}, V_{\gamma}}$. It follows from (4.2) that

$$J_{\gamma, \delta}^{21}(\lambda) (v_{-\delta+2s-2n}^\gamma \otimes v_{-\gamma+2s-2n}^\delta) = P J_{\gamma, \delta}(\lambda) (v_{-\gamma+2n}^\gamma \otimes v_{-\delta+2s-2n}^\delta)$$

$$= \sum_{k=0}^{n} A_{k,n}^{\lambda-\delta, \delta-s} v_{-\delta+2s-2k}^\delta \otimes v_{-\gamma+2k}^\gamma.$$

(5.3)

Hence, by (5.1), (5.3) and (4.9) we obtain for $(s - \delta) \vee 0 \leq n \leq \gamma \wedge s$ that

$$R_{\delta, \gamma}(\lambda) (v_{-\delta+2s-2n}^\gamma \otimes v_{-\gamma+2s-2n}^\delta) = \sum_{k=0}^{n} A_{k,n}^{\lambda-\delta, \delta-s} J_{\delta, \gamma}(\lambda)^{-1} (v_{-\delta+2s-2k}^\delta \otimes v_{-\gamma+2k}^\gamma)$$

$$= \sum_{k=0}^{n} \sum_{m=k}^{\gamma \wedge s} B_{s-m, s-k}^{\lambda-\gamma, \gamma-s} A_{k,n}^{\lambda-\delta, \delta-s} v_{-\delta+2s-2m}^\delta \otimes v_{-\gamma+2m}^\gamma$$

$$= \sum_{m=0}^{\gamma \wedge s} C_{m,n}^{\lambda, \gamma, \delta, s} v_{-\delta+2s-2m}^\delta \otimes v_{-\gamma+2m}^\gamma,$$

where

$$C_{m,n}^{\lambda, \gamma, \delta, s} := \sum_{k=0}^{m \wedge n} B_{s-m, s-k}^{\lambda-\gamma, \gamma-s} A_{k,n}^{\lambda-\delta, \delta-s}$$

(5.4)

$$= \sum_{k=0}^{m \wedge n} \frac{(s-k)! (-\gamma+k)_{m-k} (-1)^{n-k} n! (-\delta + s - n)_{n-k}}{(m-k)! (s-m)! (-\lambda - \gamma + m + k - 1)_{m-k} (n-k)! (-\lambda - \delta + 2s - 2n)_{n-k}}$$

$$= \frac{(-1)^{m} (-s)_{m} (-\gamma)_{m}}{m! (-\lambda - \gamma + m - 1)_{m} (-\lambda - \delta + 2s - 2n)_{n}} \sum_{k=0}^{m \wedge n} (\delta - s + k + 1)_{n-k}$$

$$\times \frac{(-m)_{k} (-\lambda - \gamma + m - 1)_{k} (-n)_{k} (\lambda + \delta - 2s + n + 1)_{k}}{(-s)_{k} (-\gamma)_{k} k!}.$$

(5.5)

Now we can write (5.5) in terms of hypergeometric functions, where we will distinguish two cases (for the second case replace the summation variable $k$ in (5.5) by $k + s - \delta$).
Theorem 5.1. Let the coefficients $C_{m,n}^{\lambda,\gamma,\delta,s}$ of the exchange matrix for $sl(2)$ be defined by

$$R_{\delta,\gamma}(\lambda)(v_{-\delta+2s-2n} \otimes v_{-\gamma+2n}) = \sum_{m=0}^{\gamma \land s} C_{m,n}^{\lambda,\gamma,\delta,s} v_{-\delta+2s-2m} \otimes v_{-\gamma+2m}. \quad (5.6)$$

Then the following holds.

If $s \leq \delta$ and $m, n \in \{0, \ldots, \gamma \land s\}$ then

$$C_{m,n}^{\lambda,\gamma,\delta,s} = \frac{(-1)^m (-s)_m (-\gamma)_m (\delta - s + 1)_n}{m!(-\lambda - \gamma + m - 1)_m (-\lambda - \delta + 2s - 2n)_n} \times \genfrac{[}{]}{0pt}{}{-m,-\lambda - \gamma + m - 1,-n,\lambda + \delta - 2s + n + 1}{-s,-\gamma,\delta - s + 1}; 1, \quad (5.7)$$

if $s \geq \delta$ and $m, n \in \{s - \delta, \ldots, \gamma \land s\}$ then

$$C_{m,n}^{\lambda,\gamma,\delta,s} = \frac{(-1)^{n+\delta-s} (-\delta)_{m+\delta-s} (s - \gamma - \delta)_{m+\delta-s} n!}{(\lambda + \gamma - 2m + 2)_{m+\delta-s} (\lambda - s + n + 1)_{n+\delta-s} (m + \delta - s)! (s - \delta)!} \times \genfrac{[}{]}{0pt}{}{-m + s - \delta,-\lambda - \gamma - \delta + s + m - 1,-n + s - \delta,\lambda - s + n + 1}{-\delta,-\gamma - \delta + s,\delta - s + 1}; 1. \quad (5.8)$$

The $\genfrac{[}{]}{0pt}{}{4}{3}$ in (5.7) can be recognized as a Racah polynomial, see [18] and [14, §1.2]. In the notation of [14, (1.2.1)] we have

$$\genfrac{[}{]}{0pt}{}{4}{3} \left[-m,-\lambda - \gamma + m - 1,-x,\lambda + \delta - 2s + x + 1
\begin{array}{c}
-m, -\lambda - \gamma + m - 1, -n, \lambda + \delta - 2s + n + 1 \\
-s, -\gamma, \delta - s + 1
\end{array}
; 1
\right]
=R_m(x(x + \lambda + \delta - 2s + 1); -\gamma - 1, -\lambda - 1, -s - 1, \lambda + \delta - s + 1). \quad (5.9)$$

From (5.2) we obtain that $R_{\delta,\gamma}^{-1}(\lambda) = (R_{\gamma,\delta}(\lambda))^{21}$. Hence it follows from (5.6) that

$$R_{\delta,\gamma}^{-1}(\lambda)(v_{-\delta+2s-2n} \otimes v_{-\gamma+2n}) = \sum_{m=0}^{\gamma \land s} C_{s-m,s-n}^{\lambda,\gamma,\delta,s} v_{-\delta+2s-2m} \otimes v_{-\gamma+2m}. \quad (5.10)$$

Let us restrict ourselves from now on, for convenience, to the case that $s \leq \gamma \land \delta$. Then we are in the situation of (5.7) and we have that $m, n \in \{0, 1, \ldots, s\}$. By (5.7) and (5.10) the matrix elements of $R_{\delta,\gamma}^{-1}(\lambda)$ can be expressed in terms of

$$C_{s-m,s-n}^{\lambda,\gamma,\delta,s} = \frac{(-1)^{s-m} (-s)_{s-m} (-\delta)_{s-m} (\gamma - s + 1)_{s-n}}{(s - m)!(-\lambda - \delta + s - m - 1)_{s-m} (-\lambda - \gamma + 2n)_{s-n}} \times \genfrac{[}{]}{0pt}{}{s-m,-\lambda - \delta + s - m - 1,n - s,\lambda + \gamma - s - n + 1}{-s,-\delta,\gamma - s + 1}; 1. \quad (5.11)$$

It will turn out that the balanced $\genfrac{[}{]}{0pt}{}{4}{3}$ in (5.11) can be rewritten in terms of the balanced $\genfrac{[}{]}{0pt}{}{4}{3}$ in (5.7) with $m$ and $n$ interchanged. This will follow by two successive applications of Whipple’s $\genfrac{[}{]}{0pt}{}{4}{3}$ transform

$$\genfrac{[}{]}{0pt}{}{4}{3} \left[-n,a,b,c
\begin{array}{c}
d, e, f
\end{array}
; 1
\right] = \frac{(e - a)_n (f - a)_n}{(e)_{n} (f)_{n}} \genfrac{[}{]}{0pt}{}{4}{3} \left[\begin{array}{c}
-n,a,d - b,d - c
\end{array}
\begin{array}{c}
d, 1 + a - e - n, 1 + a - f - n
\end{array}
; 1
\right], \quad (5.12)$$
where \( n \in \mathbb{Z}_{\geq 0} \) and \( a + b + c - n + 1 = d + e + f \) (see for instance \([1, \text{Theorem 3.3.3}]\)). In fact, we have

\[
\begin{align*}
4F3 & \left[ \begin{array}{c} n-s, \lambda + \gamma - s - n + 1, m-s, -\lambda - \delta + s - m - 1 \\ -s, -\delta, \gamma - s + 1 
\end{array} ; 1 \right] = \frac{(-\lambda + n)_{s-n}}{(-\delta)_{s-n}} \\
\times \frac{(-\lambda - \gamma - \delta + s + n - 1)_{s-n}}{(\gamma - s + 1)_{s-n}} 4F3 & \left[ \begin{array}{c} n-s, \lambda + \gamma - s - n + 1, -m, \lambda + \delta - 2s + m + 1 \\ -s, \lambda + \gamma + \delta - 2s + 2, \lambda - s + 1 
\end{array} ; 1 \right]
\end{align*}
\]

and

\[
\begin{align*}
4F3 & \left[ \begin{array}{c} -m, \lambda + \delta - 2s + m + 1, n-s, \lambda + \gamma - s - n + 1 \\ -s, \lambda + \gamma + \delta - 2s + 2, \lambda - s + 1 
\end{array} ; 1 \right] = \frac{(\gamma - m + 1)_m}{(\lambda + \gamma + \delta - 2s + 2)_m} \\
\times \frac{(-\delta + s - m)_m}{(\lambda - s + 1)_m} 4F3 & \left[ \begin{array}{c} -m, \lambda + \delta - 2s + m + 1, -n, -\lambda - \gamma + n - 1 \\ -s, -\gamma, \delta - s + 1 
\end{array} ; 1 \right].
\end{align*}
\]

Hence

\[
\begin{align*}
4F3 & \left[ \begin{array}{c} n-s, \lambda + \gamma - s - n + 1, m-s, -\lambda - \delta + s - m - 1 \\ -s, -\delta, \gamma - s + 1 
\end{array} ; 1 \right] = \frac{(-\lambda - \gamma - \delta + s + n - 1)_{s-n} (-\lambda + n)_{s-n} (\gamma - m + 1)_m (-\delta + s - m)_m}{(-\delta)_{s-n} (\gamma - s + 1)_{s-n} (\lambda + \gamma + \delta - 2s + 2)_m (\lambda - s + 1)_m} \\
\times 4F3 & \left[ \begin{array}{c} -m, \lambda + \delta - 2s + m + 1, -n, -\lambda - \gamma + n - 1 \\ -s, -\gamma, \delta - s + 1 
\end{array} ; 1 \right]. \tag{5.13}
\end{align*}
\]

From (5.6) and (5.10) we obtain (for \( s \leq \gamma \wedge \delta \)):

\[
\sum_{x=0}^{\gamma \wedge s} C_{m,x}^{\lambda,\gamma,\delta,s} C_{s-x,s-n}^{\lambda,\delta,\gamma,s} = \delta_{m,n}. \tag{5.14}
\]

We can consider (5.14) as biorthogonality relations between the two systems of functions on \( \{0, \ldots, s\} \) given by \( x \mapsto C_{n,x}^{\lambda,\gamma,\delta,s} \) and \( x \mapsto C_{s-x,s-n}^{\lambda,\delta,\gamma,s} \) (\( n \in \{0, \ldots, s\} \)). If we substitute (5.7) and (5.11) in (5.14), next substitute (5.13), and finally substitute (5.9) then, after a computation, the resulting orthogonality relations for Racah polynomials precisely coincide with those given in \([14, (1.2.2)]\) (replace \( \alpha, \beta, \gamma, \delta, N \) in \([14, (1.2.2)]\) by \( -\gamma - 1, -\lambda - 1, -s - 1, \lambda + \delta - s + 1, s \)).
6. The universal fusion matrix

From (4.2) we can compute the universal fusion matrix for \( sl(2) \). For a simple complex Lie algebra the universal fusion matrix is a suitable generalized element \( J(\lambda) \) of \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) such that for all finite dimensional \( \mathfrak{g} \)-modules \( V,W \) we have \( J(\lambda)|_{W \otimes V} = J_{WV}(\lambda) \), see [5, §8.1].

In fact, it follows from (4.2), (3.2) and (4.3) that

\[
J_{\delta,\gamma}(\lambda) (v_{\delta-\delta+2n}^\gamma \otimes v_{\gamma-\gamma+2s-2n}^\gamma) = \sum_{m=(s-\gamma)\lor 0}^{n} \frac{(-1)^{n-m} m!}{n!((-\gamma+s-n)_{n-m})} f^{n-m} \cdot v_{\delta-\delta+2n}^\gamma \otimes e^{n-m} \cdot v_{\gamma-\gamma+2s-2n}^\gamma
\]

Hence

\[
J_{WV}(\lambda) (w \otimes v) = \left( \sum_{n=0}^{\infty} \frac{1}{n!((-\lambda+\text{wt}(v))_{n}} f^n \otimes e^n \right) \cdot (w \otimes v)
\]

\[
= \sum_{n=0}^{\infty} \left( f^n \otimes \frac{1}{n!((-\lambda+h-2n)_{n}} e^n \right) \cdot (w \otimes v),
\]

if \( w \) and \( v \) are weight vectors in finite dimensional \( sl(2) \)-modules \( W \) resp. \( V \).

Since \( (-\lambda + h - 2n)_n = (-1)^n (\lambda - h + n + 1)_n \) we obtain:

**Theorem 6.1** The universal fusion matrix \( J(\lambda) \) for \( sl(2) \) equals

\[
J(\lambda) = \sum_{n=0}^{\infty} f^n \otimes \frac{(-1)^n}{n!(\lambda-h+n+1)_n} e^n. \tag{6.1}
\]

Formula (6.1) is in agreement with the formula at the end of §8.1 in [5]. Earlier, a quantum analogue of (6.1) was given by Babelon [2], see also Babelon, Bernard & Billey [3, §2].

In a quite similar way we can derive from (4.9) the universal inverse fusion matrix \( J(\lambda)^{-1} \). We successively obtain:

\[
J_{\delta,\gamma}(\lambda)^{-1} (v_{\delta-\delta+2n}^\gamma \otimes v_{\gamma-\gamma+2s-2n}^\gamma) = \sum_{m=(s-\gamma)\lor 0}^{n} \frac{(-1)^{n-m}}{(n-m)!((-\lambda-\gamma+2s-m-n-1)_{n-m})} \times (f^{n-m} \otimes e^{n-m}) \cdot (v_{\delta-\delta+2n}^\gamma \otimes v_{\gamma-\gamma+2s-2n}^\gamma);
\]

\[
J_{WV}(\lambda)^{-1} (w \otimes v) = \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!((-\lambda+\text{wt}(v)+n-1)_n} f^n \otimes e^n \right) \cdot (w \otimes v)
\]

\[
= \sum_{n=0}^{\infty} \left( f^n \otimes \frac{(-1)^n}{n!((-\lambda+h-n-1)_n} e^n \right) \cdot (w \otimes v);
\]
so the universal inverse fusion matrix equals

\[ J(\lambda)^{-1} = \sum_{n=0}^{\infty} f^n \otimes \frac{1}{n! (\lambda - h + 2)^n} e^n. \] (6.2)

A quantum analogue of (6.2) was given in [3, §2].

At least formally, it should hold now that

\[ J(\lambda) J(\lambda)^{-1} = 1 \otimes 1 = J(\lambda)^{-1} J(\lambda). \] (6.3)

We can also verify these identities independently, quite parallel to the verification of the two identities in (4.7). For the proof of the second identity in (6.3) note that

\[
J(\lambda)^{-1} J(\lambda) = \sum_{k,l=0}^{\infty} f^{k+l} \otimes \frac{1}{k! (\lambda - h + 2)^k} \frac{(-1)^l}{l! (\lambda - h + l + 1)^l} e^l
\]

\[
= \sum_{n=0}^{\infty} \left( 1 \otimes \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k! (\lambda - h + 2)_k (n-k)! (\lambda - h + n + k + 1)_{n-k}} \right) (f^n \otimes e^n) = 1 \otimes 1,
\]

since, by (4.6),

\[
\sum_{k=0}^{n} \frac{(-1)^{n-k}}{k! (\lambda - h + 2)_k (n-k)! (\lambda - h + n + k + 1)_{n-k}}
\]

\[
= \frac{(-1)^n}{n! (\lambda - h + n + 1)_n} \binom{-n, \lambda - h + n + 1}{\lambda - h + 2} = \delta_{n,0}.
\]

For the proof of the first identity in (6.3) note that

\[
J(\lambda) J(\lambda)^{-1} = \sum_{k,l=0}^{\infty} f^{k+l} \otimes \frac{(-1)^l}{l! (\lambda - h + l + 1)^l} e^l \frac{1}{k! (\lambda - h + 2)^k} e^k
\]

\[
= \sum_{n=0}^{\infty} \left( 1 \otimes \sum_{l=0}^{n} \frac{(-1)^l}{l! (\lambda - h + l + 1)_l (n-l)! (\lambda - h + 2l + 2)_{n-l}} \right) (f^n \otimes e^n) = 1 \otimes 1,
\]

since by (4.8),

\[
\sum_{l=0}^{n} \frac{(-1)^l}{l! (\lambda - h + l + 1)_l (n-l)! (\lambda - h + 2l + 2)_{n-l}}
\]

\[
= \frac{1}{n! (\lambda - h + 2)_n} \binom{-n, \lambda - h + 1, \frac{1}{2}(\lambda - h + 1) + 1}{\lambda - h + n + 2, \frac{1}{2}(\lambda - h + 1)} = \delta_{n,0}.
\]
7. The operator \( Q(\lambda) \)

We follow [5, §9.2] and [9, §1.2] by defining, for generic \( \lambda \in \mathfrak{h}^* \), a generalized element \( Q(\lambda) \) in \( U(\mathfrak{g}) \) in terms of the universal fusion matrix as follows:

\[
Q(\lambda) := (m \circ P \circ (1 \otimes S^{-1})) J(\lambda).
\]

(7.1)

Here \( m \) denotes the multiplication operator \( x \otimes y \mapsto xy \) in \( U(\mathfrak{g}) \) and \( S \) denotes the antipode in \( U(\mathfrak{g}) \). It follows from (6.1) that \( Q(\lambda) \) for \( \mathfrak{sl}(2) \) takes the following form:

\[
Q(\lambda) = \sum_{n=0}^{\infty} e_n \frac{1}{n! (\lambda + h + n + 1)} f^n. 
\]

(7.2)

For \( \gamma \in \mathbb{Z}_{\geq 0} \) let \( Q_\gamma(\lambda) \) denote \( Q(\lambda) \) acting on \( V_\gamma \). It follows from (7.2) and (3.2) that, for \( k \in \{0, 1, \ldots, \gamma\} \),

\[
Q_\gamma(\lambda) v_{-\gamma+2k} = \left( \sum_{n=0}^{k} \frac{(-1)^n(-\gamma+k-n)k!}{n! (\lambda - \gamma + 2k - n + 1)(k-n)!} \right) v_{-\gamma+2k}
\]

The coefficient of \( v_{-\gamma+2k} \) on the right can be rewritten as \( _2F_1(-k, \gamma-k+1; -\lambda + \gamma - 2k; 1) = \frac{(-\lambda - k - 1)_k}{(-\lambda + \gamma - 2k)_k} \) by (4.5). So we have obtained:

**Theorem 7.1**  The operator \( Q(\lambda) \) acting on \( V_\gamma \) is explicitly given by

\[
Q_\gamma(\lambda) v_{-\gamma+2k} = \frac{(-\lambda-k-1)_k}{(-\lambda + \gamma - 2k)_k} v_{-\gamma+2k} \quad (k \in \{0, 1, \ldots, \gamma\}).
\]

(7.3)

8. Weighted trace functions

Weighted trace functions for \( q = 1 \) are defined in [9, §10.1]. This is by analogy to or as a limit case of the \( q \)-case. Weighted trace functions in the \( q \)-case are defined in [5, §1.2] and in [9, §1.2]. Let us state here once more the definition of weighted trace function for \( q = 1 \).

For \( \lambda \in \mathfrak{h}^* \) and \( U \) a \( \mathfrak{g} \)-module let \( \text{exp}_\lambda \) be the endomorphism of \( U \) sending a weight vector \( u \) in \( U \) to \( e^{\langle \lambda, \text{wt}(u) \rangle} u \). Let \( V \) be a finite dimensional \( \mathfrak{g} \)-module, let \( B[0] \) be a basis of \( V[0] \) and let \( v^* \in V^*[0] \) be the dual basis vector corresponding to a basis vector \( v \) in \( B[0] \). For \( \mu \in \mathfrak{h}^* \) generic let

\[
\Phi_{\mu}^V[0] := \sum_{v \in B[0]} \Phi_{\mu}^v \otimes v^*
\]

(8.1)

and define

\[
\Psi_V(\lambda, \mu) := \text{Tr}|_{M_\mu}(\Phi_{\mu}^V[0] \circ \text{exp}_\lambda) \in V[0] \otimes V^*[0].
\]

(8.2)

Because we have taken the trace in (8.2), we will generally obtain an infinite sum which is a priori a formal power series in the variables \( e^{-\langle \lambda, \alpha_i \rangle} \), where \( \alpha_1, \ldots, \alpha_{\dim \mathfrak{h}} \) are the simple roots.
Let \( \rho \in h^* \) be half the sum of the positive roots of \( g \). Let \( \langle , \rangle \) be the nondegenerate symmetric bilinear form on \( h^* \) induced, up to a constant factor, by the Killing form on \( g \) such that \( \langle \alpha, \alpha \rangle = 2 \) if \( \alpha \) is a long root. Let the Weyl denominator be given by

\[
\delta(\lambda) := e^{\langle \lambda, \rho \rangle} \prod_{\alpha > 0} (1 - e^{-\langle \lambda, \alpha \rangle}).
\]  

(8.3)

For \( sl(2, \mathbb{C}) \), where \( h^* \) is identified with \( \mathbb{C} \) and 2 is the only positive root, we get \( \rho = 1 \), \( \langle \lambda, \mu \rangle = 1/2 \lambda \mu \) and

\[
\delta(\lambda) = e^{1/2 \lambda} - e^{-1/2 \lambda}.
\]  

(8.4)

Now define the weighted trace function by

\[
F_V(\lambda, \mu) := (id_V \otimes Q^{-1}_{V^*}(-\mu - \rho)) \Psi_V(\lambda, -\mu - \rho) \delta(\lambda).
\]  

(8.5)

This is again in \( V[0] \otimes V^*[0] \).

Let us compute \( \Psi(\lambda, \mu) \), i.e. (8.2) for \( g = sl(2, \mathbb{C}) \) and \( V = V_\gamma \) with \( \gamma \) an even nonnegative integer. From (3.7) we obtain

\[
\Phi_\mu^{V_0}(\exp \lambda \cdot f^n \cdot x_\mu) = e^{1/2 \lambda (\mu - 2n)} \sum_{m=0}^{n+1/2} c_{m,n}^{\mu,\gamma} (f^m \cdot x_\mu) \otimes v_{m-n}^\gamma.
\]

Hence, by (8.1),

\[
\Phi_\mu^{V_0}[0](\exp \lambda \cdot f^n \cdot x_\mu) = \Phi_\mu^{V_0}(\exp \lambda \cdot f^n \cdot x_\mu) \otimes (v_0^\gamma)^*.
\]

Hence, by (8.2),

\[
\Psi(\lambda, \mu) = e^{1/2 \lambda \mu} \sum_{n=0}^{\infty} c_{n}^{\mu,\gamma} e^{-n\lambda},
\]  

(8.6)

where we omitted \( v_0^\gamma \otimes (v_0^\gamma)^* \) on the right-hand side, since the one-dimensional vector space \( V_\gamma[0] \otimes V_\gamma^*[0] \) can be identified with \( \mathbb{C} \). By (3.8) we have

\[
\sum_{n=0}^{\infty} c_{n}^{\mu,\gamma} e^{-n\lambda} = \sum_{n=0}^{\infty} e^{-n\lambda} 3F_2 \left[ -n, -\frac{1}{2} \gamma, \frac{1}{2} \gamma + 1 \right],
\]

\[
= \sum_{n=0}^{\infty} e^{-n\lambda} \sum_{k=0}^{n} \frac{(-n)_k (-\frac{1}{2} \gamma)_k (\frac{1}{2} \gamma + 1)_k}{(-\mu)_k k! k!}.
\]

Now interchange the two summations and next substitute \( n = m + k \) for the summation variable \( n \). The above double sum becomes

\[
\sum_{k=0}^{1/2 \gamma} \sum_{m=0}^{\infty} \frac{(k + 1)_m}{m!} (e^{-\lambda})^m \frac{(-\frac{1}{2} \gamma)_k (\frac{1}{2} \gamma + 1)_k}{(-\mu)_k k!} (-e^{-\lambda})^k.
\]
The inner sum converges absolutely for $\Re\lambda > 0$. Therefore, by dominated convergence, we see that for $\Re\lambda > 0$ the interchange of summation was justified and the sum in (8.6) converges absolutely. The inner sum equals $(1 - e^{-\lambda})^{-k-1}$. So the double sum equals

$$(1 - e^{-\lambda})^{-1} \sum_{k=0}^{\frac{1}{2}\gamma} \frac{(-\frac{1}{2}\gamma)_{k} (\frac{1}{2}\gamma + 1)_{k}}{(-\mu)_{k} k!} (1 - e^{\lambda})^{-k}.$$ 

So we obtain

$$\Psi_\gamma(\lambda, \mu) = e^{\frac{1}{2}\lambda\mu} (1 - e^{-\lambda})^{-1} {}_2F_1 \left[ \frac{-1}{2}\gamma, \frac{1}{2}\gamma + 1; (1 - e^{\lambda})^{-1} \right]$$

$$= e^{\frac{1}{2}\lambda\mu} (1 - e^{-\lambda})^{\frac{1}{2}\gamma} {}_2F_1 \left[ \frac{1}{2}\gamma - \mu, \frac{1}{2}\gamma + 1; e^{-\lambda} \right],$$

where we used Pfaff’s transformation ${}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1(c - a, b; c; z/(z - 1))$, see for instance [1, (2.2.6)]. Combination with (8.5), (8.4) and (7.3) yields:

**Theorem 8.1** The weighted trace function for $sl(2, \mathbb{C})$ and $V_\gamma(\gamma$ nonnegative even integer) is given by

$$F_\gamma(\lambda, \mu) = \frac{(-1)^{\frac{1}{2}\gamma} (\mu + 1)^{\frac{1}{2}\gamma}}{(-\mu + 1)^{\frac{1}{2}\gamma}} e^{-\frac{1}{2}\lambda\mu} {}_2F_1 \left[ \frac{-1}{2}\gamma, \frac{1}{2}\gamma + 1; (1 - e^{\lambda})^{-1} \right]$$

$$= \frac{(-1)^{\frac{1}{2}\gamma} (\mu + 1)^{\frac{1}{2}\gamma}}{(-\mu + 1)^{\frac{1}{2}\gamma}} e^{-\frac{1}{2}\lambda\mu} (1 - e^{-\lambda})^{\frac{1}{2}\gamma + 1} {}_2F_1 \left[ \frac{1}{2}\gamma + \mu + 1, \frac{1}{2}\gamma + 1; e^{-\lambda} \right].$$

Note that the restriction $\Re\lambda > 0$ is no longer needed for convergence in (8.7) or (8.9). The quantum analogue of (8.9) was obtained in [9, Proposition 7.3].

By application to (8.10) of the quadratic transformation formula [4, 2.11(36)] (see also [1, (3.1.11)]), and by next comparing with [15, (2.15)], we can express $F_\gamma(\lambda, \mu)$ in terms of a Jacobi function of the second kind:

$$\frac{(-1)^{\frac{1}{2}\gamma} (\mu + 1)^{\frac{1}{2}\gamma}}{(-\mu + 1)^{\frac{1}{2}\gamma}} (e^{\frac{1}{2}\lambda} - e^{-\frac{1}{2}\lambda})^{-\frac{1}{2}\gamma - 1} F_\gamma(\lambda, \mu)$$

$$= (e^{\frac{1}{2}\lambda} + e^{-\frac{1}{2}\lambda})^{-\gamma - 2\mu - 2} {}_2F_1 \left[ \frac{1}{2}\gamma + \mu + 1, \frac{1}{2}; \frac{1}{\cosh^2(\frac{1}{4}\lambda)} \right]$$

$$= \Phi^{(\frac{1}{2}\gamma + \frac{1}{2}, \frac{1}{2}\gamma + \frac{1}{2})}(\frac{1}{4}\lambda).$$

Finally we want to check [5, Theorem 9.2] (the dual Macdonald-Ruijsenaars equations) for $sl(2, \mathbb{C})$ (so $q = 1$). The theorem states that for $V$ and $W$ finite-dimensional $g$-modules we have

$$\mathcal{D}^\mu_{W, V} F_V(\lambda, \mu) = \chi_W(e^{-\lambda}) F_V(\lambda, \mu),$$

(8.12)
where
\[ \chi_W(e^\lambda) := \text{Tr}|_W \exp = \sum_{\nu} \text{dim}(W[\nu]) e^{(\lambda,\nu)}. \]  
(8.13)

and \( D^\mu,V^* \) is a difference operator
\[
D^\mu,V^* := \sum_{\nu \in b^*} \text{Tr}|_W[\nu] (R_{W^*}(-\mu - \rho)) T^\mu_{\nu} = \sum_{\nu \in b^*} \text{Tr}|_W[\nu] (R_{W[\nu]},V^*[0]:W[\nu],V^*[0](-\mu - \rho)) T^\mu_{\nu}.
\]  
(8.14)

In (8.14) \( T^\mu_{\nu} \) denotes the shift operator defined by \( (T^\mu_{\nu}) (\mu) := f(\mu + \nu) \). Furthermore, in (8.14) \( R_{W[\nu],V^*[\sigma];W[\nu'],V^*[\sigma']} \) denotes the block of the matrix \( \mathbb{R}_{W,V^*} \) corresponding to the weight spaces \( W[\nu],V^*[\sigma];W[\nu'],V^*[\sigma'] \) (which block will be zero unless \( \nu + \sigma = \nu' + \sigma' \)).

For the case of \( sl(2,\mathbb{C}) \) we obtain from (8.14) and (5.6) that
\[
D^\mu,V^*_{\delta,\gamma} = \sum_{s=\frac{1}{\gamma}}^{\frac{1}{\gamma}+\delta} C^{-\mu-1,\gamma,\delta,s}_{\frac{1}{\gamma},\frac{1}{\gamma}} T^{-\delta-\gamma+2s}_{\mu}.
\]  
(8.15)

where \( \gamma \) is even and the coefficients \( C_{m,n,\gamma}^{(\lambda,\gamma)} \) of the exchange matrix are explicitly given by (5.5), or alternatively by (5.7) if \( s \leq \delta \) or (5.8) if \( s \geq \delta \). Also, for \( W = W_{\delta} \), formula (8.13) becomes
\[
\chi_{\delta}(e^\lambda) = \sum_{k=0}^{\delta} e^{\frac{1}{\gamma} \lambda(-\delta+2k)} = \frac{e^{\frac{1}{\gamma}(\delta+1)\lambda} - e^{-\frac{1}{\gamma}(\delta+1)\lambda}}{e^{\frac{1}{\gamma}\lambda} - e^{-\frac{1}{\gamma}\lambda}}.
\]  
(8.16)

Thus the general theory yields that the difference equations (8.12) hold with (8.9), (8.15) and (8.16) substituted. For general \( \delta \) we don’t know if this formula was presented earlier. However, for \( \delta = 1 \) we can reduce the formula to a well-known contiguous relation for Gaussian hypergeometric series.

Indeed, for \( \delta = 1 \) we get
\[
C^{-\mu-1,\gamma,1}_{\frac{1}{2},\frac{1}{2}\gamma} = 1; \quad C^{-\mu-1,\gamma,1}_{\frac{1}{2},\frac{1}{2}\gamma} = \frac{(\mu - \frac{1}{2}\gamma - 1)(\mu + \frac{1}{2}\gamma)}{(\mu - 1)\mu}.
\]  
(8.17)

Thus
\[
F_{\gamma}(\lambda,\mu + 1) + \frac{(\mu - \frac{1}{2}\gamma - 1)(\mu + \frac{1}{2}\gamma)}{(\mu - 1)\mu} F_{\gamma}(\lambda,\mu - 1) = (e^{\frac{1}{2}\lambda} + e^{-\frac{1}{2}\lambda}) F_{\gamma}(\lambda,\mu)
\]  
(8.18)

with \( F_{\gamma}(\lambda,\mu) \) given by (8.9). This coincides (after appropriate substitution of parameters and argument) with the contiguous relation
\[
c(c - 1)(z - 1) 2F_1(a, b; c - 1; z) + c(c - 1 - (2c - a - b - 1)z) 2F_1(a, b; c; z) + (c - a)(c - b) 2F_1(a, b; c + 1; z) = 0,
\]  
(8.19)

see [4, 2.8(30)].
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