A Generalization of Binomial Exponential-2 Distribution: Copula, Properties and Applications

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Abstract: In this paper, we propose a new three-parameter lifetime distribution for modeling symmetric real-life data sets. A simple-type Copula-based construction is presented to derive many bivariate- and multivariate-type distributions. The failure rate function of the new model can be “monotonically asymmetric increasing”, “increasing-constant”, “monotonically asymmetric decreasing” and “upside-down-constant” shaped. We investigate some of mathematical symmetric/asymmetric properties such as the ordinary moments, moment generating function, conditional moment, residual life and reversed residual functions. Bonferroni and Lorenz curves and mean deviations are discussed. The maximum likelihood method is used to estimate the model parameters. Finally, we illustrate the importance of the new model by the study of real data applications to show the flexibility and potentiality of the new model. The kernel density estimation and box plots are used for exploring the symmetry of the used data.

Keywords: type-II half-logistic; binomial exponential-2 distribution; moments; maximum likelihood estimation; Morgenstern family; symmetry; Clayton Copula

1. Introduction

The monotonicity asymmetric failure (hazard) rate function (HRF) of a certain lifetime probabilistic distribution has an important role in modeling real lifetime data. Distributions with the “monotonicity increasing” failure rate (MIFR) function have useful real applications in “pricing” and “supply” chain contracting problems. The MIFR property is a well-known and useful concept in “dynamic programming”, “reliability theory” and other areas of applied probability and statistics (see [1,2]). The paper [3] introduced a new two-parameter lifetime model with MIFR named the binomial-exponential-2 (BE2) model, which is constructed as a model of a random sum (RSm) of independent exponential random variables (RVs) when the sample size has a “zero truncated binomial” distribution. The BE2 distribution can be used as an alternative to the Weibull (W), gamma (Gam), exponentiated exponential (EE), and weighted exponential (WhE) distributions in real life applications.

The survival function (SF) of the binomial exponential-2 (BE2) distribution is given by

\[
\overline{G}_{BE2}(y) = \left( 1 + \frac{\theta \alpha y}{2 - \theta} \right) e^{-\alpha y} \quad \begin{cases} 1 \quad (0 \leq \theta \leq 1), \\ 0 \quad (y > 0), \end{cases}
\] (1)

where \( \alpha > 0 \) is a scale parameter, \( G_{BE2}(y) = 1 - \overline{G}_{BE2}(y) \) is the cumulative distribution function (CDF) of the BE2 model and \( \theta \) is a shape parameter. It is easy to show that the SF in (1) is increasing in
0 ≤ θ ≤ 1 where \( e^{-ay} \leq \overline{G}_{BE2}(y) \leq (ay + 1)e^{-ay} \) (see [2]). The probability density function (PDF) corresponding to (1) is

\[
g_{BE2}(y) = \left[1 + \frac{\theta}{2 - \theta} (ay - 1) \right]\alpha e^{-ay},
\]

which can be expressed as

\[
g_{BE2}(y) = \frac{\alpha}{2 - \theta} [2(1 - \theta) + ay \theta]e^{-ay}.
\]

Since \( \frac{\partial}{\partial \theta} \) is negative, the \( \log[g_{BE2}(y)] \) is “concave” for all \( a \) and \( 0 ≤ \theta ≤ 1 \). As a result, \( g_{BE2}(y) \) is “log-concave” and “unimodal”. Additionally, the PDF (2) can be written as

\[
g_{BE2}(y) = \pi(\theta)\alpha e^{-ay} + \pi(\theta)c^2 y e^{-ay},
\]

where

\[
\pi(\theta) = \frac{2}{2 - \theta} (1 - \theta) \text{ and } \pi(\theta) = 1 - \pi(\theta)\pi(\theta) = \frac{2}{2 - \theta} (1 - \theta) \text{ and } \pi(\theta) = 1 - \pi(\theta).
\]

The BE2 model is a mixture of the standard exponential (with parameter \( \alpha \)) model and standard gamma model (with shape parameter 2 and scale parameter \( \theta \)); when \( \theta = 0 \), we get the standard exponential model, and when \( \theta = 1 \), the BE2 model reduces to the Gam model. In the last few decades, many new G families of continuous distributions have been developed. One of the most famous ones is called the new type II half-logistic (TIIHL-G) family (see [4]). According to [4], the CDF of the TIIHL-G family of distributions is given by

\[
F_{\lambda, \Psi}(y) = \frac{2G_{\Psi}(y)^\lambda}{1 + G_{\Psi}(y)^\lambda},
\]

where \( G_{\Psi}(y) \) is the baseline CDF depending on a parameter vector \( \Psi \) and \( \lambda > 0 \) is an additional shape parameter. For each baseline \( G_{\Psi}(y) \), we can generate a new TIIHL model using (4). The corresponding PDF to (4) is given by

\[
f_{\lambda, \Psi}(y) = 2\lambda G_{\Psi}(y)G_{\Psi}(y)^{\lambda-1}\left[1 + G_{\Psi}(y)^\lambda\right]^{-2},
\]

where \( g_{\Psi}(y) = dG_{\Psi}(y)/dy \) is the baseline PDF. Equation (5) will be most tractable when \( G_{\Psi}(y) \) and \( g_{\Psi}(y) \) have simple expressions. The survival function, the failure (hazard) rate function and the quantile function are \( F_{\lambda, \Psi}(y) = \frac{1-G_{\Psi}(y)^\lambda}{1+G_{\Psi}(y)^\lambda}, h_{\lambda, \Psi}(y) = \frac{2g_{\Psi}(y)G_{\Psi}(y)^{\lambda-1}}{1-G_{\Psi}(y)^\lambda}, \) and \( Q(u) = G^{-1}\sqrt{\frac{u}{2\Psi}}. \)

Equations (4) and (5) are used for generating the new model.

2. The New Model and Its Motivation

In this section, we introduce the three-parameter type II half-logistic binomial exponential 2 (TIIHLBE2) distribution. Substituting from (1) into (4), the CDF of the TIIHLBE2 (or expanded BE2 “EBE” for short) model can be expressed as

\[
F_{\lambda, \alpha, \theta}(y) = \frac{2\left[1 - \left(1 + \frac{\theta ay}{2 - \theta}\right)e^{-ay}\right]^\lambda}{1 - \left(1 + \frac{\theta ay}{2 - \theta}\right)e^{-ay}\lambda}.
\]

The corresponding PDF is given by

\[
f_{\lambda, \alpha, \theta}(y) = \frac{2\lambda \alpha e^{-ay} \left(1 + \frac{(ay - 1)\theta}{2 - \theta}\right) \left[1 - \left(1 + \frac{\theta ay}{2 - \theta}\right)e^{-ay}\right]^\lambda}{\left[1 - \left(1 + \frac{\theta ay}{2 - \theta}\right)e^{-ay}\lambda\right]^2}.
\]
Here and henceforth, an RV $Y$ having PDF (7) is denoted by $Y \sim \text{EBE}(\lambda, \alpha, \theta)$. For the EBE distribution, the HRF can be derived as

$$h_{\lambda, \alpha, \theta}(y) = \frac{2\lambda \alpha e^{-\theta y} \left(1 + \frac{\theta y - 1}{1 + \theta y} e^{-\theta y}ight)}{\left[1 - \left(1 + \frac{\theta y}{1 + \theta y} e^{-\theta y}\right)\right]}.$$  

(8)

Figure 1 presents some plots of the PDF of the EBE model for some different values of the parameters $\lambda, \alpha$ and $\theta$. We note that the new PDF can be “right skewed” with different shapes of “skewness” and “kurtosis”.

Figure 2 gives the plots of the HRF of the EBE distribution. We note that the new HRF can be “increasing”, “increasing-constant”, “decreasing” and “upside-down-constant” shaped. Thus, the new model may be useful in modeling different shapes of real data.

Figure 1. Plots of the probability density function (PDF) of EBE distribution.

Figure 2. Plots of the HRF of the EBE distribution.
3. Copula under the EBE Model

3.1. Bivariate EBE (BivEBE) Type via Renyi’s Entropy

Following [5], the joint CDF (JCDF) of the “Renyi’s entropy Copula” can be expressed as
\[ C(u,v) = y_2 u + y_1 v - y_1 y_2; \]
then, the associated BivEBE will be \( \mathcal{H}(t_1, t_2) = \mathcal{H}(F_{\tilde{V}_1}(y_1), F_{\tilde{V}_2}(y_2)) \)
where \( \tilde{V}_1 \) and \( \tilde{V}_2 \) are the parameter vectors for \( F_{\tilde{V}_1}(y_1) \) and \( F_{\tilde{V}_2}(y_2) \), respectively.

3.2. BivEBE Type Using “Farlie-Gumbel-Morgenstern” (FGM) Copula

Consider the JCDF of the bivariate FGM copula, where \( \mathcal{H}_\Delta(u,v) = uv(1 + \Delta w) \), for every \( \Delta \in [-1,1] \). The marginal functions are \( u = F_1(y_1) \in [0,1] \) and \( v = F_2(y_2) \in [0,1] \). The unknown parameter \( \Delta \) is a dependence parameter, and for every \( u, v \in [0,1] \), \( \mathcal{H}_\Delta(u,v) = C_\Delta(0,v) = 0 \), which is the “grounded minimum” property, and \( \mathcal{H}_\Delta(u,1) = u \) and \( \mathcal{H}_\Delta(1,v) = v \), which is “grounded maximum” property. \( \tilde{u} = \tilde{u}_{\tilde{V}_1} = 1 - F_{\tilde{V}_1}(y_1) \) and \( \tilde{v} = \tilde{v}_{\tilde{V}_2} = 1 - F_{\tilde{V}_2}(y_2) \) are then set.

Then, \( F(y_1, y_2) = \mathcal{H}(F_{\tilde{V}_1}(y_1), F_{\tilde{V}_2}(y_2)) \). The joint PDF can be derived from

\[ h_\Delta(u,v) = \Delta u^\ast v^\ast + 1 \] \( (\ast \ast = 1-2\ast + \ast^2) \)

or from

\[ f(y_1, y_2) = h\left(F_{\tilde{V}_1}(y_1), F_{\tilde{V}_2}(y_2)\right) f_{\tilde{V}_1}(y_1) f_{\tilde{V}_2}(y_2). \]

For more details, see [6–12].

3.3. BivEBE Type via “Modified FGM” (MFGM) Copula

The modified JCDF of the bivariate FGM copula can be expressed as

\[ \mathcal{H}_\Delta(u,w) = uw + \Delta \tilde{O}(u) \tilde{\varphi}(w), \]

where \( \tilde{O}(u) = u \tilde{O}(u) \) and \( \tilde{\varphi}(w) = w \tilde{\varphi}(w) \), where \( \tilde{O}(u) \) and \( \tilde{\varphi}(w) \) are two absolutely continuous functions on \((0,1)\) where \( \tilde{O}(0) = \tilde{O}(1) = \tilde{\varphi}(0) = \tilde{\varphi}(1) = 0 \). Let

\[ \alpha = \inf \left\{ \frac{\partial}{\partial u} \tilde{O}(u) : C_1 \right\} < 0, \beta = \sup \left\{ \frac{\partial}{\partial u} \tilde{O}(u) : C_1 \right\} < 0, \]

\[ \xi = \inf \left\{ \frac{\partial}{\partial w} \tilde{\varphi}(w) : C_2 \right\} > 0, \eta = \sup \left\{ \frac{\partial}{\partial w} \tilde{\varphi}(w) : C_2 \right\} > 0. \]

Then,

\[ \min(\beta \alpha, \eta \xi) \geq 1, \]

where

\[ \tilde{O}(u) + u \frac{\partial}{\partial u} \tilde{O}(u) =, \]

\[ C_1 = \left\{ u \in (0,1) : \frac{\partial}{\partial u} \tilde{O}(u) \text{ exists} \right\}, \]

and

\[ C_2 = \left\{ w \in (0,1) : \frac{\partial}{\partial w} \tilde{\varphi}(w) \text{ exists} \right\}. \]

3.3.1. BivEBE-FGM (Type-I) Model

The BivEBE-FGM (Type-I) model can be derived directly using

\[ \mathcal{H}_\Delta(u,w) = \Delta \tilde{O}(u) \tilde{\varphi}(w) + uw, \]
3.3.2. BivEBE-FGM (Type-II) Model

Consider $O(u)$ and $\varphi(w)$ that satisfy all the conditions stated earlier where

$$O(u)|_{\Delta_1 > 0} = u^{\Delta_1}(1 - u)^{1 - \Delta_1} \quad \text{and} \quad \varphi(w)|_{\Delta_2 > 0} = v^{\Delta_2}(1 - w)^{1 - \Delta_2}.$$

The corresponding BivEBE-FGM (Type-II) copula can be derived from

$$C_{\Delta_1, \Delta_2}(u, w) = u\varphi(w) + \theta_{\Delta_1, \Delta_2}uv,$$

3.3.3. BivEBE-FGM (Type-III) Model

Consider $O(u)$ and $\varphi(w)$ that satisfy all the conditions stated earlier where

$$O(u) = u[\log(1 + u)] \quad \text{and} \quad \varphi(w) = w[\log(1 + w)].$$

In this case, one can also derive a closed form expression for the associated CDF of the BivEBE-FGM (Type-III).

3.3.4. BivEBE-FGM (Type-IV) Model

The JCDF of the BivEBE-FGM (Type-IV) model can be derived from

$$H(u, w) = uF_1(w) - F_1(u)F_1(w).$$

3.4. BivEBE Type via Clayton Copula

The Clayton Copula can be considered as

$$H(u_1, u_2) = \left(\frac{u_1^{-\Delta} + u_2^{-\Delta} - 1}{\Delta\log(\Delta)}\right)^{-\frac{1}{\Delta}} |_{\Delta \in [0, \infty[}.$$

Let $T \sim \text{EBE}(V_1)$ and $W \sim \text{EBE}(V_2)$. Set $u_1 = u(t) = F_{V_1}(t)|_{V_1 > 0}$ and $u_2 = u(w) = F_{V_2}(w)|_{V_2 > 0}$. Then, the BivEBE-type distribution can be derived from $F(t, w) = H(F_{V_1}(t), F_{V_2}(w))$. A straightforward $n$-dimensional extension from the above will be $H(u_i) = \left[1 - n + \sum_{i=1}^{n} u_i^{-\Delta}\right]^{-\frac{1}{\Delta}}$. Many other useful details can be found in [13–22].

4. Properties

4.1. Expansions and Quantile Function (QF)

Consider the series representation

$$\left(\frac{\omega_1}{\omega_2} + 1\right)^{-\omega_3} = \sum_{i=0}^{\infty} \left(\frac{\omega_1}{\omega_2}\right)^i \binom{\omega_3 + i - 1}{i} \left(\frac{\omega_1}{\omega_2}(1 + \omega_3)\right)^i.$$  (9)

expanding $\left[1 + \left(1 + \frac{\omega_y}{2\theta}\right)e^{-ax}\right]^{-2}$, we can write (7) as

$$f(y) = 2\lambda \sum_{i=0}^{\infty} (-1)^i (i + 1)e^{-\theta y} \left[1 - \left(1 + \frac{\theta y}{2 - \theta}\right)e^{-\theta y}\right]^{i(i+1)-1}. \quad (10)$$
Then, consider the power series expansion
\[
\left(1 - \frac{\omega_1}{\omega_2}\right)^{\omega_3 - 1} = \sum_{j=0}^{\infty} \left(-\frac{\omega_1}{\omega_2}\right)^j \binom{\omega_3 - 1}{j} \left(\omega_1 \omega_2 \right)^j.
\] (11)

Using (11) in Equation (10), and after some algebra, the PDF of EBE can be written as
\[
f(y) = 2\lambda 2^{\alpha} \sum_{i,j=0}^{\infty} (-1)^i (i+1) \frac{\lambda(i+1) - 1}{j} \left(1 + \frac{\alpha y - 1}{2 - \theta}\right) \left(1 + \frac{\theta \alpha y}{2 - \theta}\right)^j e^{-\alpha(1+j)y},
\]

Then, we have
\[
\left(1 + \frac{\theta \alpha y}{2 - \theta}\right)^j = \sum_{\kappa=0}^{\infty} \binom{j}{\kappa} \left(\frac{\theta \alpha}{2 - \theta}\right)^\kappa y^\kappa,
\]
therefore, the PDF of the EBE model becomes
\[
f(y) = \sum_{\kappa=0}^{\infty} C_\kappa \Pi_{\theta,\alpha}^{ij,k}(y),
\]
where
\[
C_\kappa = 2\lambda \sum_{i,j=0}^{\infty} \alpha^{1+\kappa} (-1)^i (i+1) \frac{\lambda(i+1) - 1}{j} \binom{j}{\kappa} \frac{\theta^\kappa}{(2 - \theta)^{1+\kappa}}.
\]

And
\[
\Pi_{\theta,\alpha}^{ij,k}(y) = \left[2(1-\theta) y^\kappa + \theta \alpha y^{\kappa+1}\right] e^{-\alpha(1+j)y},
\]

The QF of the EBE model is given by the real solution of the following equation:
\[
\left(1 + \frac{\theta \alpha y_q}{2 - \theta}\right) e^{-\alpha y_q} + \left(\frac{q}{2 - \theta}\right)^{\frac{1}{\lambda}} = 1,
\] (12)

where the above equation has no closed form solution in $y_q$, so we have to use a numerical technique.

4.2. Moments

Theorem 1. If $Y \sim$ EBE ($\lambda, \alpha, \theta$), then the $r^{th}$ moment of $Y$ is given by
\[
\mu_r'(y) = \sum_{\kappa=0}^{\infty} C_\kappa \Gamma(r+\kappa+1),
\] (13)

where
\[
C_\kappa' = C_\kappa \frac{2\alpha(1+j)(1-\theta) + \theta \alpha (r+\kappa+1)}{[\alpha (1+j)]^{r+\kappa+2}}.
\]

Proof. Let $Y$ be an RV following the EBE distribution. The $r^{th}$ ordinary moment can be obtained using the well-known formula
\[
\mu_r'(y) = \int_0^\infty f(y) y^r dy = \sum_{\kappa=0}^{\infty} C_\kappa \int_0^\infty y^r \Pi_{\theta,\alpha}^{ij,k}(y) dy,
\]
then

\[ \mu'_r(y) = \sum_{\kappa=0}^{\infty} C_{\kappa} \int_{0}^{\infty} \left[ 2(1 - \theta)y^{r+\kappa} + \theta \alpha y^{r+\kappa+1} \right] e^{-\alpha(1+j)y} dy. \]

Setting \( x = \alpha(1 + j)y \), after some algebra, we obtain

\[ \mu'_r(y) = \sum_{\kappa=0}^{j} C_{\kappa} \Gamma(r + \kappa + 1). \quad (14) \]

If we set \( r = 1 \), we obtain the mean of the EBE distribution. Variance, skewness and kurtosis measures can be easily derived from the well-known relationships. Three-dimensional plots of the skewness and kurtosis of the EBE model are presented in Figures 3 and 4.

![Figure 3. Three-dimensional plot for the skewness of the EBE model.](image)

![Figure 4. Three-dimensional plot for the kurtosis of the EBE model.](image)

These plots indicate that both measures depend very much on the shape parameter \( \theta \). The first four moments and the skewness and kurtosis of the EBE distribution for different values of parameters are represented in Table 1.
Table 1. Moments, skewness and kurtosis of the EBE model.

| α  | θ  | λ  | µ′_1 | µ′_2 | µ′_3 | µ′_4 | Skewness | Kurtosis |
|----|----|----|------|------|------|------|----------|----------|
| 0.5| 0.5| 0.5| 1.066| 3.009| 16.07| 149.755| 2.752   | 12.349 |
| 0.5| 0.7| 0.7| 1.660| 4.313| 23.29| 277.509| 2.189   | 9.1470 |
| 0.6| 0.7| 1.5| 2.414| 4.008| 14.47| 125.278| 1.669   | 9.2000 |
| 0.7| 0.3| 2.7| 2.273| 2.749| 8.148| 60.481| 1.627   | 6.8450 |
| 1.5| 0.5| 0.5| 0.243| 0.316| 0.545| 1.589| 2.753   | 12.231 |
| 1.5| 0.2| 0.7| 0.381| 0.294| 0.458| 1.269| 2.539   | 11.139 |
| 2.6| 0.2| 1.5| 0.404| 0.149| 0.121| 0.233| 1.929   | 8.0020 |
| 2.7| 0.4| 2.7| 0.626| 0.195| 0.153| 0.325| 1.571   | 6.3250 |

Theorem 2. The moment generating function $M_Y(\tau)$ of the EBE is given by

$$M_Y(\tau) = \sum_{\kappa=0}^{\infty} C^{(r,\tau)}_{\kappa} \Gamma(\kappa + 1),$$

where

$$C^{(r,\tau)}_{\kappa} = C_\kappa \frac{2(1-\theta)\alpha(1+j)-\tau + \theta \alpha(\kappa+1)}{[\alpha(1+j)]^{\kappa+2}}.$$

Proof. Starting with

$$M_Y(\tau) = E(e^{\tau Y}) = \int_0^{\infty} e^{\tau y} f(y) dy,$$

then

$$M_Y(\tau) = \sum_{\kappa=0}^{\infty} C_\kappa \int_0^{\tau} \left[ 2(1-\theta)y^{\kappa+1} + \theta \alpha y^{\kappa+2} \right] e^{-\alpha(1+j)y} dy,$$

finally, we get

$$M_Y(\tau) = \sum_{\kappa=0}^{\infty} C^{(r,\tau)}_{\kappa} \Gamma(\kappa + 1).$$

In the same way, the characteristic function of the EBE distribution becomes

$$\phi_Y(\tau) = M_Y(i\tau)$$

where $i = \sqrt{-1}$ is the unit imaginary number. □

4.3. Incomplete Moments

The $s^{th}$ lower and upper incomplete moments of $Y$ are defined by

$$v_{s,Y}(\tau) = E(Y^s | Y < \tau) = \int_0^\tau y^s f(y) dy,$$

and

$$v_{s,Y}(\tau) = E(Y^s | Y > \tau) = \int_\tau^{\infty} y^s f(y) dy,$$

respectively, for any real $s > 0$. The $s^{th}$ lower incomplete moment of the EBE distribution is

$$v_{s,Y}(\tau) = \int_0^\tau y^s f(y) dy = \sum_{\kappa=0}^{\infty} C_\kappa \int_0^\tau \left[ 2(1-\theta)y^{\kappa+1} + \theta \alpha y^{\kappa+2} \right] e^{-\alpha(1+j)y} dy,$$
then
\[
v_{s,Y}(\tau) = \sum_{k=0}^{\infty} C_k \left[ c_{(s,j)}^{(1)} \gamma(s + \kappa + 1, \alpha (1 + j) \tau) + c_{(s,j)}^{(2)} \gamma(s + \kappa + 2, \alpha (1 + j) \tau) \right],
\]
where
\[
c_{(\Delta,j)}^{(1)} = 2(1 - \theta) \frac{1}{[\alpha (1 + j)]^{\alpha + \kappa + 1}} c_{(\Delta,j)}^{(2)} = \theta \alpha \frac{1}{[\alpha (1 + j)]^{\alpha + \kappa + 2}}
\]
and \(\gamma(s, \tau) = \int_{\tau}^{\infty} y^{1-e^{-y} dy} \) is the lower incomplete gamma function. Similarly, the \(s_{jh}\) upper incomplete moment of the EBE distribution is
\[
\eta_{s}(\tau) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} C_k \left[ c_{(s,j)}^{(1)} \zeta(s + \kappa + 1, \alpha (1 + j) \tau) + c_{(s,j)}^{(2)} \zeta(s + \kappa + 2, \alpha (1 + j) \tau) \right],
\]
where
\[
\zeta(s, \tau) = \int_{\tau}^{\infty} e^{-y} y^{-1} dy,
\]
is the upper incomplete gamma function.

4.4. Mean Deviation and Bonferroni and Lorenz Curve

The mean deviations about the mean \(\mu = E(Y)\) and the mean deviations about the median \(M\) can be written as
\[
\delta_1(y) = 2[\mu F(\mu) - \omega(\mu)] = \int_{0}^{\infty} f(y) |y - \mu| dy,
\]
and
\[
\delta_2(y) = \mu - 2 \omega(M) = \int_{0}^{\infty} f(y) |y - M| dy,
\]
respectively, where
\[
\omega(d) = \int_{0}^{d} y f(y) dy = \sum_{k=0}^{\infty} C_k \left[ c_{(1,j)}^{(1)} \gamma(\kappa + 2, \alpha (1 + j) d) + c_{(1,j)}^{(2)} \gamma(\kappa + 3, \alpha (1 + j) d) \right].
\]
The Lorenz curve for a positive RV \(Y\) is defined as
\[
L(p) = \frac{1}{\mu} \int_{0}^{h} y f(y) dy = \frac{\omega(h)}{\mu} = \frac{1}{\mu} \sum_{k=0}^{\infty} C_k \left[ c_{(1,j)}^{(1)} \gamma(\kappa + 2, \alpha (1 + j) h) + c_{(1,j)}^{(2)} \gamma(\kappa + 3, \alpha (1 + j) h) \right],
\]
where \(h = G^{-1}(p).\) Additionally, the Bonferroni curve is defined by
\[
B(p) = \frac{1}{\mu p} \int_{0}^{h} y f(y) dy = \frac{\omega(h)}{\mu p} = \frac{1}{\mu p} \sum_{k=0}^{\infty} C_k \left[ c_{(1,j)}^{(1)} \gamma(\kappa + 2, \alpha (1 + j) h) + c_{(1,j)}^{(2)} \gamma(\kappa + 3, \alpha (1 + j) h) \right],
\]
4.5. Residual Life and Reversed Residual Life Functions

The \(r_{th}\) moment of the residual life via the general formula is given by
\[
\mu_{r,Y}(\tau) = E\left( \left( Y - \tau \right)^r \right)_{(Y > \tau)} = \frac{1}{F(\tau)} \int_{\tau}^{\infty} f(y) (y - \tau)^r dy \bigg|_{(r \geq 1)}
\]
then

\[ \mu_{r,Y}(\tau) = \frac{1}{F(\tau)} \sum_{h=0}^{r} \sum_{k=0}^{\infty} C_k \left( -1 \right)^{r-h} \binom{r}{h} \int_{\tau}^{\infty} \left[ c_{(r,j)}^{(1)} \zeta(r + \kappa + 1, \alpha(1 + j)\tau) + c_{(r,j)}^{(2)} \zeta(r + \kappa + 2, \alpha(1 + j)\tau) \right] \]

The mean residual life (MRL) of the EBE distribution is given by

\[ \mu_{1,Y}(\tau) = \frac{1}{F(\tau)} \sum_{k=0}^{\infty} C_k \left( -1 \right)^{r-h} \binom{r}{h} \int_{\tau}^{\infty} \left[ c_{(1,j)}^{(1)} \zeta(\kappa + 2, \alpha(1 + j)\tau) + c_{(1,j)}^{(2)} \zeta(\kappa + 3, \alpha(1 + j)\tau) \right] - \tau \]

The \( r \)th order moment of the reversed residual life can be obtained by the well-known formula

\[ m_{r,Y}(\tau) = E((\tau - Y)^r) = \frac{1}{F(\tau)} \int_0^\tau f(y)(\tau - y)^r dy \bigg|_{y \leq \tau}. \]

Applying the binomial expansion of \((\tau - y)^r\) and substituting \( f_{\lambda,\alpha,\theta}(y) \) given by (7) into the above formula gives

\[ m_{r,Y}(\tau) = \frac{1}{F(\tau)} \sum_{r=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \int_{\tau}^{\infty} \left[ 2(1 - \theta) y^{r+k} + \theta \alpha y^{r+k+1} \right] e^{-\alpha(1+j)y} dy, \]

then

\[ m_{r,Y}(\tau) = \frac{1}{F(\tau)} \sum_{k=0}^{\infty} C_k \left( -1 \right)^{r-h} \binom{r}{h} \int_{\tau}^{\infty} \left[ \sum_{j=1}^{r} c_{(r,j)}^{(1)} \gamma(r + \kappa + 1, \alpha(1 + j)\tau) + c_{(r,j)}^{(2)} \gamma(r + \kappa + 2, \alpha(1 + j)\tau) \right], \]

where

\[ \gamma(s,p) = \int_0^p w^{s-1} e^{-w} dw, \]

is the lower incomplete gamma function. The mean waiting time of the EBE distribution is given by

\[ m_1(\tau) = \tau - \frac{1}{F(\tau)} \sum_{k=0}^{\infty} C_k \left[ c_{(1,j)}^{(1)} \gamma(\kappa + 2, \alpha(1 + j)\tau) + c_{(1,j)}^{(2)} \gamma(\kappa + 3, \alpha(1 + j)\tau) \right]. \]

Using \( m_{1,Y}(\tau) \) and \( m_{2,Y}(\tau) \), one can obtain the “variance” and the “coefficient of variation” of the reversed residual life of the EBE distribution.

5. Estimation and Inference

Let \( Y_1, Y_2, \ldots, Y_n \) be a random sample of size \( n \) from EBE \( \psi \). The log likelihood function for the vector of parameters \( \lambda, \alpha \) and \( \theta \) can be written as

\[ \log L = n \log(2\lambda) + n \log(2 - 2\theta) - a \sum_{i=1}^{n} y_i + (\lambda - 1) \sum_{j=1}^{n} \log(1 - m_i) - 2 \sum_{j=1}^{n} \log \left[ 1 + (1 - m_i)^{\lambda} \right] \]

where \( m_i = 1 + \frac{\partial \log y_i}{\partial \theta} \) and \( s_i = e^{-\lambda y_i} \). The associated score function is given by

\[ U_n(\psi) = \left[ \frac{\partial \log L}{\partial \lambda}, \frac{\partial \log L}{\partial \alpha}, \frac{\partial \log L}{\partial \theta} \right]^T. \]

The \( \log L \) in (17) can be maximized by solving the nonlinear likelihood equations obtained by differentiating (17). The components of the score vector are given by
\[ U_{\lambda} = \frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log(1 - m_i s_i) - 2 \sum_{i=1}^{n} \frac{(1 - m_i s_i)^\lambda \log(1 - m_i s_i)}{1 - (1 - m_i s_i)\lambda}, \]

\[ U_{\alpha} = \frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} y_i + (\lambda - 1) \sum_{i=1}^{n} \frac{s_i \left[ a m_i \frac{y_i}{1 - m_i s_i} \right]}{1 - (1 - m_i s_i)^{\lambda - 1}} \]

\[ -2 \lambda \sum_{i=1}^{n} \frac{s_i [1 - m_i s_i]^{1-\lambda}}{1 + (1 - m_i s_i)^{\lambda}} \]

\[ U_{\theta} = \frac{\partial \log L}{\partial \theta} = \frac{n}{\alpha - \beta} - 2(\lambda - 1) \sum_{i=1}^{n} \frac{y_i}{\left( \beta - \alpha \right)^2} + (\lambda - 1) \sum_{i=1}^{n} \frac{\alpha \log(y_i) y_i e^{-\alpha y_i} - y_i^\beta}{1 - e^{-\alpha y_i} y_i^\beta} \]

\[ -4 \lambda \sum_{i=1}^{n} \frac{\alpha y_i}{\left( \beta - \alpha \right)^2} \left[ 1 - m_i s_i \right]^{\lambda - 1} \]

\[ \left[ 1 - e^{-\alpha y_i} y_i^\beta \right] \]

\[ \lambda \left[ 1 - m_i s_i \right] \]

\[ + \left( \lambda - 1 \right) \sum_{i=1}^{n} \frac{\alpha \log(y_i) y_i e^{-\alpha y_i} - y_i^\beta}{1 - e^{-\alpha y_i} y_i^\beta} \]

\[ -4 \lambda \sum_{i=1}^{n} \frac{\alpha y_i}{\left( \beta - \alpha \right)^2} \left[ 1 - m_i s_i \right]^{\lambda - 1} \]

6. Simulation

The “inverse transform algorithm” is used to generate random data from the EBE distribution. We generated samples of sizes \( n = 50, 100, 200, 500 \) and 1000, and the simulations were repeated \( N = 1000 \) times from the EBE model for some parameter values. Tables 2 and 3 give the mean square errors (MSEs) and the biases, respectively. The average values of estimates (AVs), estimated average length (EAL) and the coverage probability (CP) are listed in Tables 4–6, respectively. From Table 2, we note that the AVs of estimates approach the initial values as \( n \to \infty \), the MSEs for each parameter decrease to zero as \( n \to \infty \), and the coverage lengths for each parameter decrease to zero as \( n \to \infty \). From Table 3, we note that the biases for each parameter are generally positive and decrease to zero as \( n \to \infty \), and the coverage probabilities for each parameter approach the nominal level as \( n \to \infty \).

### Table 2. Mean square errors (MSEs) for \( n = 50, 100, 200, 500 \) and 1000.

| \( n \) | \( \lambda \) | \( \alpha \) | \( \theta \) | MSEs |
|---|---|---|---|---|
| 50 | 0.5 | 0.5 | 0.5 | 0.11892, 0.23647, 0.06169 |
| 100 | 0.08304, 0.17158, 0.04192 |
| 200 | 0.05794, 0.12764, 0.02949 |
| 500 | 0.03636, 0.0876, 0.01866 |
| 1000 | 0.02595, 0.06105, 0.01325 |
| 50 | 0.7 | 0.7 | 0.7 | 0.08813, 0.17030, 0.08578 |
| 100 | 0.06136, 0.11879, 0.05894 |
| 200 | 0.04320, 0.08131, 0.04139 |
| 500 | 0.02764, 0.04874, 0.02616 |
| 1000 | 0.01964, 0.03346, 0.01856 |
| 50 | 0.7 | 0.3 | 2.7 | 0.06866, 0.12976, 0.18044 |
| 100 | 0.04885, 0.08743, 0.12657 |
| 200 | 0.03427, 0.08270, 0.08909 |
| 500 | 0.02180, 0.03709, 0.05538 |
| 1000 | 0.01550, 0.02515, 0.03903 |
| 50 | 0.7 | 0.3 | 2.7 | 0.07211, 0.13178, 0.29484 |
| 100 | 0.05105, 0.09550, 0.20625 |
| 200 | 0.03537, 0.07094, 0.14854 |
| 500 | 0.02227, 0.04645, 0.09524 |
| 1000 | 0.01545, 0.03457, 0.06935 |
| 50 | 0.5 | 0.5 | 0.5 | 0.11892, 0.23647, 0.06169 |
| 100 | 0.08304, 0.17158, 0.04192 |
| 200 | 0.05794, 0.12764, 0.02949 |
| 500 | 0.03636, 0.0876, 0.01866 |
| 1000 | 0.02595, 0.06105, 0.01325 |
Table 3. Biases for $n = 50, 100, 200, 500$ and $1000$.

| $n$  | $\lambda$ | $\alpha$ | $\theta$ | MSEs          |
|------|-----------|-----------|-----------|---------------|
| 50   | 0.5       | 0.5       | 0.5       | 0.18356, 0.31326, 0.06034 |
| 100  | 0.14037   | 0.29166   | 0.02491   |               |
| 200  | 0.10718   | 0.27047   | 0.00952   |               |
| 500  | 0.07427   | 0.22547   | 0.00392   |               |
| 1000 | 0.05854   | 0.18297   | 0.00431   |               |
| 50   | 0.5       | 0.7       | 0.7       | 0.13103, 0.32039, 0.16804 |
| 100  | 0.09571   | 0.29243   | 0.12804   |               |
| 200  | 0.07617   | 0.25160   | 0.11495   |               |
| 500  | 0.05278   | 0.17909   | 0.10177   |               |
| 1000 | 0.03559   | 0.11151   | 0.09647   |               |
| 50   | 0.6       | 0.7       | 1.5       | 0.11728, 0.32816, 1.12796 |
| 100  | 0.09252   | 0.29394   | 1.03834   |               |
| 200  | 0.07715   | 0.27919   | 0.98811   |               |
| 500  | 0.05480   | 0.21775   | 0.91763   |               |
| 1000 | 0.03922   | 0.17326   | 0.88793   |               |
| 50   | 0.7       | 0.3       | 2.7       | 0.16086, 0.36788, 2.90744 |
| 100  | 0.13715   | 0.35482   | 2.70484   |               |
| 200  | 0.10661   | 0.32348   | 2.66995   |               |
| 500  | 0.08491   | 0.27404   | 2.64048   |               |
| 1000 | 0.07009   | 0.22686   | 2.72810   |               |
| 50   | 0.5       | 0.5       | 0.5       | 0.18356, 0.31326, 0.06034 |
| 100  | 0.14037   | 0.29166   | 0.02491   |               |
| 200  | 0.10718   | 0.27047   | 0.00952   |               |
| 500  | 0.07427   | 0.22547   | 0.00392   |               |
| 1000 | 0.05854   | 0.18297   | 0.00431   |               |

Table 4. Average values (AVs) for $n = 50, 100, 200, 500$ and $1000$.

| $n$  | $\lambda$ | $\alpha$ | $\theta$ | MSEs          |
|------|-----------|-----------|-----------|---------------|
| 50   | 0.5       | 0.5       | 0.5       | 0.55465, 0.50616, 0.52281 |
| 100  | 0.53818   | 0.51109   | 0.50207   |               |
| 200  | 0.52237   | 0.48520   | 0.49930   |               |
| 500  | 0.50325   | 0.47041   | 0.49911   |               |
| 1000 | 0.49813   | 0.46471   | 0.50127   |               |
| 50   | 0.5       | 0.7       | 0.7       | 0.50550, 0.58874, 0.72531 |
| 100  | 0.49190   | 0.60765   | 0.70470   |               |
| 200  | 0.48878   | 0.63157   | 0.69991   |               |
| 500  | 0.49378   | 0.66331   | 0.69945   |               |
| 1000 | 0.49624   | 0.67996   | 0.70161   |               |
| 50   | 0.6       | 0.7       | 1.5       | 0.57701, 0.58491, 1.52331 |
| 100  | 0.58530   | 0.61671   | 1.51090   |               |
| 200  | 0.57826   | 0.61435   | 1.50499   |               |
| 500  | 0.58310   | 0.65758   | 1.47994   |               |
| 1000 | 0.58988   | 0.68285   | 1.47548   |               |
| 50   | 0.7       | 0.3       | 2.7       | 0.74776, 0.45747, 2.49293 |
| 100  | 0.74369   | 0.44427   | 2.46686   |               |
| 200  | 0.72548   | 0.39578   | 2.51294   |               |
| 500  | 0.71662   | 0.36901   | 2.54710   |               |
| 1000 | 0.70309   | 0.31662   | 2.62308   |               |
| 50   | 0.5       | 0.5       | 0.5       | 0.55465, 0.50616, 0.52281 |
| 100  | 0.53818   | 0.51109   | 0.50207   |               |
| 200  | 0.52237   | 0.48520   | 0.49930   |               |
| 500  | 0.50325   | 0.47041   | 0.49911   |               |
| 1000 | 0.49813   | 0.46471   | 0.50127   |               |
Table 5. Estimated average lengths (EALs) for $n = 50, 100, 200, 500$ and $1000$.

| $n$ | $\lambda$ | $\alpha$ | $\theta$ | MSEs |
|-----|----------|----------|----------|------|
| 50  | 0.5      | 0.5      | 0.5      | 0.46616, 0.92696, 0.24180 |
| 100 | 0.32550  | 0.67258  | 0.16431  |
| 200 | 0.22713  | 0.50034  | 0.11560  |
| 500 | 0.14251  | 0.33226  | 0.07514  |
| 1000| 0.10172  | 0.23930  | 0.05195  |
| 50  | 0.5      | 0.7      | 0.7      | 0.34545, 0.66757, 0.33625 |
| 100 | 0.24054  | 0.46565  | 0.23104  |
| 200 | 0.16935  | 0.31873  | 0.16226  |
| 500 | 0.10834  | 0.19105  | 0.10255  |
| 1000| 0.07700  | 0.13116  | 0.07274  |
| 50  | 0.6      | 0.7      | 1.5      | 0.26914, 0.50864, 0.70733 |
| 100 | 0.19150  | 0.34271  | 0.49614  |
| 200 | 0.13434  | 0.24579  | 0.34924  |
| 500 | 0.08545  | 0.14539  | 0.21710  |
| 1000| 0.06077  | 0.09860  | 0.15300  |
| 50  | 0.7      | 0.3      | 2.7      | 0.28268, 0.51657, 1.15574 |
| 100 | 0.20011  | 0.37434  | 0.80847  |
| 200 | 0.13865  | 0.27809  | 0.58226  |
| 500 | 0.08730  | 0.18209  | 0.37332  |
| 1000| 0.06055  | 0.13552  | 0.27184  |

Table 6. Coverage probabilities (CPs) for $n = 50, 100, 200, 500$ and $1000$.

| $n$ | $\lambda$ | $\alpha$ | $\theta$ | MSEs |
|-----|----------|----------|----------|------|
| 50  | 0.5      | 0.5      | 0.5      | 0.82686, 0.71564, 0.79176 |
| 100 | 0.76642  | 0.65414  | 0.75986  |
| 200 | 0.69834  | 0.57143  | 0.79418  |
| 500 | 0.66517  | 0.51762  | 0.82334  |
| 1000| 0.57302  | 0.51648  | 0.81022  |
| 50  | 0.5      | 0.7      | 0.7      | 0.81610, 0.73529, 0.76823 |
| 100 | 0.78664  | 0.58513  | 0.72796  |
| 200 | 0.73695  | 0.52505  | 0.62213  |
| 500 | 0.72222  | 0.47172  | 0.60909  |
| 1000| 0.74675  | 0.49449  | 0.61361  |
| 50  | 0.5      | 0.7      | 1.5      | 0.63454, 0.50188, 0.00000 |
| 100 | 0.51110  | 0.28606  | 0.00000  |
| 200 | 0.42228  | 0.23378  | 0.00000  |
| 500 | 0.31797  | 0.22104  | 0.00000  |
| 1000| 0.29115  | 0.20381  | 0.00000  |
| 50  | 0.7      | 0.3      | 2.7      | 0.082686, 0.71564, 0.79176 |
| 100 | 0.76642  | 0.65414  | 0.75986  |
| 200 | 0.69834  | 0.57143  | 0.79418  |
| 500 | 0.66517  | 0.51762  | 0.82334  |
| 1000| 0.57302  | 0.51648  | 0.81022  |
7. Modeling Stress-Rupture Life of Kevlar 49/Epoxy Strands Data

In this section, we illustrate the performance of the EBE distribution as compared to some alternative distributions using a real data application. The goodness-of-fit (GOF) statistics for this distribution are compared with other competitive distributions, and the maximum likelihood estimations (MLEs) of the distribution parameters are determined numerically. We compare the fits of the EBE distribution with the Burr type X (Burr X) distribution, Burr type XII (Burr XII) distribution, beta log logistic Weibull distribution (BLLW), beta Weibull log logistic (BWLL) and beta log logistic, beta linear failure rate geometric (ELFRG), exponentiated linear failure rate geometric (ELFRG), beta Rayleigh (BR), and beta Weibull geometric distributions (BWG) (see [23]). In order to compare the distributions, we consider the measures of GOF including the Akaike Information Criterion ($C_1$), Bayesian Information Criterion ($C_2$), Consistent Akaike Information Criterion ($C_4$) and Hannan–Quinn Information Criterion ($C_3$) statistics.

The following real data set represents the stress-rupture life of Kevlar 49/epoxy strands that are subjected to constant sustained pressure at the 90% stress level until all have failed that were provided by [24], given as 0.01, 0.08, 0.09, 0.09, 0.10, 0.02, 0.02, 0.03, 0.03, 0.05, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 1.00, 0.06, 1.34, 0.10, 1.45, 1.50, 1.51, 0.63, 0.72, 0.99, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 4.20, 4.69, 7.89, 0.07, 0.07, 0.36, 0.38, 0.40, 0.65, 0.67, 0.68, 0.79, 0.80, 0.80, 0.83, 0.72, 0.42, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 1.01, 1.02, 1.03, 0.72, 0.73, 0.79, 0.85, 0.90, 0.92, 0.95, 1.05, 0.11, 0.24, 0.29, 0.34, 0.35, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 0.11, 0.01, 0.02, 1.40, 1.43 and 1.33. Table 7 gives the MLE for all the models corresponds to the failure times data set. Table 8 shows the statistics for the failure times of the Kevlar data set. Figure 5 gives the kernel density estimation and box plot for exploring the symmetry of the stress-rupture life data. Figure 6 provides the fitted PDF in the left panel and fitted CDF in the right panel.

Table 7. The MLE for all the models corresponds to the failure times data set.

| Model   | Estimates       |
|---------|-----------------|
| Burr X  | a 0.462891      |
| Burr XII| a 1.14571       |
| ELLG    | a 1.211659      |
| BLLG    | a 0.581650      |
| EBE     | α θ λ           |
| BLLGW   | α β             |
| BWLLG   | a 0.70773       |

Table 8. Statistics for failure times of Kevlar data set.

| Model | −2 logL | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|-------|---------|-------|-------|-------|-------|
| EBE   | 143.3996| 149.4100| 152.3784| 152.3784| 149.6087|
| Burr XII | 145.4801| 149.4801| 154.4119| 152.4660| 149.6229|
| BLLGW | 204.0771| 214.0771| 227.1527| 219.3705| 214.7087|
| BWLLG | 204.8205| 214.8205| 227.8961| 220.1139| 215.4521|
| Burr X | 285.8730| 287.8730| 290.3389| 288.8659| 287.9200|
| ELLG | 587.6830| 591.6830| 596.9133| 593.8004| 591.8055|
| BLLG | 462.1078| 468.1078| 175.9531| 471.2838| 468.3552|
8. Conclusions

A new three-parameter lifetime distribution is proposed and studied. A simple-type Copula-based construction is presented to derive many bivariate- and multivariate-type distributions. We investigated some of mathematical properties such as the ordinary moments, moment generating function and conditional moment. Bonferroni and Lorenz curves and mean deviations are discussed. Residual life and reversed residual functions are also obtained. Some bivariate- and multivariate-type extensions are proposed. The maximum likelihood method is used to estimate the model parameters. Finally, we illustrate the importance of the new model by studying real data applications to show the flexibility and potentiality of the new model.
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**Sample Availability:** The data used to support the findings in this study are included within the paper.

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