RESEARCH ARTICLE

Design principle for effective mechanical boundary using a resonance band gap under elastic waves

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Abstract

One representative feature of a locally resonant elastic metamaterial (LREM) is that they can prohibit elastic wave propagation at the frequencies inside a band gap, which means that no energy is transmitted. When an incident wave propagates in a host medium at the frequencies inside band gaps, the incident wave is totally reflected at the interface between the host medium and an LREM. However, it remains unexplored what kind of mechanical boundary (e.g. fixed or free) is formed at the interface between the host medium and the LREM. This study thus aims at finding design principles for effective mechanical boundary (EMB) formation and validating the principles by numerical simulation. Conditions for certain EMBs were derived from the magnitude and phase of the reflection coefficient of the LREM. According to the conditions, an LREM is designed and attached to a host medium. It was confirmed from time-harmonic simulation that the velocity at the interface between the host medium and the LREM approached zero when the effective fixed boundary is formed, while the stress at the interface approached zero when the effective free boundary is formed.

Keywords: locally resonant elastic metamaterial; resonance band gap; effective mechanical boundary; mechanical impedance; reflection coefficient

1. Introduction

Over the last few decades, artificially engineered periodic structures have attracted considerable research attention due to their ability to manipulate elastic wave propagation by means of a band gap. A band gap refers to a frequency range within which elastic waves cannot propagate; thus, no wave energy can be transmitted through periodic structures. A locally resonant elastic metamaterial (LREM) is a representative example of artificially engineered periodic structures that exhibit a band gap for elastic waves (Liu et al., 2000, 2015). The LREM is composed of periodically arranged subwavelength unit cells that contain internal elastic resonators (Bae and Oh, 2020; Miranda et al., 2020).

Since the size of the unit cell is much smaller than a wavelength, behaviors of an LREM can be characterized by homogenized effective properties (Zhou and Hu, 2009; Jo et al., 2021; Zhang et al., 2021).

When either the effective mass or the effective stiffness is negative, an LREM exhibits a resonance band gap (Wu et al., 2011; Li et al., 2019b). This implies that a resonance band gap can be mechanically designed by tailoring the mass and stiffness of the elastic resonators internally contained in a unit cell. Accordingly, many research efforts have been devoted to enhancing the frequency tunability of resonance band gaps. For instance, resonance band gap frequencies can be tuned by changing a
temperature (Xia et al., 2016; Li et al., 2018; Zhu et al., 2018),
using a shape memory alloy (Candido de Sousa et al., 2018;
Chuang et al., 2019) and using piezoelectric materials (Casadei
et al., 2012; Chen et al., 2014; Miranda et al., 2021). Oh et al.
(2016a) presented the realization of an elastic metamaterial al-
lowing independent tuning of negative density and stiffness
for elastic waves propagating along a designated direction. On
the other hand, there have been attempts to widen the band-
width of the resonance band gap. Dual- (Tan et al., 2012) and
multi-resonator (Li et al., 2019) elastic metamaterials were pro-
posed to realize the broadband effective mass negativity. To
widen a resonance band gap range, Oh et al. (2016) adjoined
resonance band gaps induced by the negative mass and stiff-
ness. Zhu et al. (2014) showed the broadband vibration sup-
pression capability of a chiral elastic metamaterial that in-
cludes multiple resonators. Wu et al. (2020) presented the flex-
ural wave attenuation in a broad low frequency range by in-
corporating hydro-elastic resonance of internal liquid and coat-
ing layers of an LREM. Jiang et al. (2021) proposed a single-
phase graded elastic metamaterial for broadband vibration
mitigation.

Here, it is worth pointing out that previous studies have
mainly focused on how to improve the frequency tunability of
a resonance band gap or how to achieve a broadband resonance
band gap (Lu et al., 2017). When an incident wave propagates in
a host medium at the frequencies inside a resonance band gap,
the incident wave is totally reflected at the interface between
the host medium and an LREM. In theory, a band gap can be under-
stood as a frequency range in which real wavenumbers do not
correspond to dispersion curves, which means that no wave en-
cy can be transmitted. However, it remains unexplored what
kind of mechanical boundary (e.g. fixed or free) is formed at the
interface between the host medium and the LREM. To answer
this question, we provide a theoretical basis for designing an
“effective mechanical boundary (EMB)” formed by a band gap
exhibited by the LREM. In this study, as representative bound-
aries, effective “fixed” and “free” boundaries are designed under
longitudinal elastic waves. For a fixed boundary, the amplitude
of the reflected wave is identical to that of the incident wave
and opposite in sign, and the displacement at the boundary is zero.
For a free boundary, the amplitude of the reflected wave is iden-
tical to that of the incident wave and the same in sign, and the
stress at the boundary is zero (Graff, 1991). In order to validate
the derived design principle, the velocity and stress fields pro-
duced by the effective fixed and free boundaries using the LREM
are compared with those developed by the corresponding actual
boundary conditions.

The rest of the paper is organized as follows: After providing
a brief review of a monoatomic chain model in Section 2.1, the
conditions for realizing EMBs using an LREM are derived in Sec-
ction 2.2. In Section 3, time-harmonic simulation is performed to
numerically validate the proposed concept of EMBs. Conclusions
are outlined in Section 4.

2. Design Principles for Effective Mechanical
Boundary Using a Locally Resonant Elastic
Metamaterial

2.1 A brief review of a monoatomic chain model

Figure 1a shows that elastic waves propagating through a host
medium are totally reflected due to a resonance band gap ex-
hibited by an LREM. In general, since an LREM consists of peri-
dically arranged subwavelength unit cells containing inner res-
onators, it can be modeled as a monoatomic chain model for one-
dimensional longitudinal wave propagation. Figure 1b shows a
monoatomic chain model that consists of identical lumped masses
connected by massless springs that have equal lumped (effective)
stiffness. The equation of motion of the monoatomic chain model

\[
m_{\text{eff}} \frac{\partial^2 u_n}{\partial t^2} = s_{\text{eff}} (u_{n+1} - u_n) - s_{\text{eff}} (u_{n} - u_{n-1}) , \quad \ldots \quad (1)
\]

where \(m_{\text{eff}}\) and \(s_{\text{eff}}\) are the lumped mass and stiffness, re-
spectively, and \(u_n\) is the \(x\)-directional displacement of the \(n\)-th
lumped mass. Assuming time-harmonic wave motions, \(u_n\) can be ex-
pressed as \(u_n = U_n \exp[i(\omega t - kx)]\), where \(U_n\) is the amplitude
of the displacement; \(k\) and \(\omega\) denote the wavenumber and angu-
lar frequency, respectively. Inserting the time-harmonic solution
into equation (1) yields a dispersion relation of the LREM, which
is given by (Brillouin, 2003)

\[
\omega = 2 \sqrt{\frac{s_{\text{eff}}}{m_{\text{eff}}} \sin \left(\frac{\beta}{2}\right)} , \quad \ldots \quad (2)
\]
where \( \beta = kd \) is the normalized wavenumber, \( d \) is the distance between the lumped masses.

The transfer matrix that relates displacements and forces estimated on the left and right ends of one unit cell is given as (Liu et al., 2012)

\[
T = \begin{bmatrix}
\frac{2s_{\text{eff}} - \omega^2 m_{\text{eff}}}{2s_{\text{eff}}} & \frac{4s_{\text{eff}} - \omega^2 m_{\text{eff}}}{2s_{\text{eff}}} \\
-\omega^2 m_{\text{eff}} & -\omega^2 m_{\text{eff}}
\end{bmatrix}.
\]

By substituting equation (2) into equation (3), the transfer matrix \( T \) can be simplified as

\[
T = \begin{bmatrix}
\cos(\beta) & \frac{1}{s_{\text{eff}}} \cos^2 \left( \frac{\beta}{2} \right) \\
-4s_{\text{eff}} \sin^2 \left( \frac{\beta}{2} \right) & \cos(\beta)
\end{bmatrix}.
\]

Then, if the LREM is composed of \( N \) unit cells, the \( N \)-th power of the transfer matrix can be simplified by the Chebyshev identity as (Yeh et al., 1977)

\[
T^N = \begin{bmatrix}
\cos(N\beta) & \frac{1}{s_{\text{eff}}} \cos^2 \left( \frac{\beta}{2} \right) \sin(N\beta) \\
-4s_{\text{eff}} \sin^2 \left( \frac{\beta}{2} \right) & \cos(N\beta)
\end{bmatrix}.
\]

The transfer matrix connects the physical quantities between successive interfaces, while the scattering matrix describes the relationship between the amplitudes of the wave function bases between the interfaces. The scattering matrix can be thus expressed in terms of the entities of the \( N \)-th power transfer matrix to estimate the wave amplitude in the host medium as

\[
S^N = \begin{bmatrix}
1 & 1 \\
-i\omega Z & i\omega Z
\end{bmatrix}^{-1} \begin{bmatrix}
T^N_{11} & T^N_{12} \\
T^N_{21} & T^N_{22}
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-i\omega Z & i\omega Z
\end{bmatrix},
\]

where \( Z \) denotes the mechanical impedance of the host medium. Note that \( P \) indicates the matrix that relates the vector of displacement and force to the amplitudes of wave function bases.

Assuming that there is no reflection from the right-hand side in Fig. 1a, the reflection coefficient can be derived in terms of the entities of the \( N \)-th power transfer matrix as

\[
R = \frac{S_{21}}{S_{22}} = \frac{-T^N_{11} + T^N_{12} + i \left( \omega Z T^N_{12} + \frac{T^N_{22}}{2} \right)}{T^N_{11} + T^N_{12} + i \left( \omega Z T^N_{12} - \frac{T^N_{22}}{2} \right)} = \frac{i \left( \frac{2s_{\text{eff}}}{m_{\text{eff}}} - \frac{2s_{\text{eff}}}{k} \right) \sin(N\beta)}{2 \cos(N\beta) + i \left( \frac{2s_{\text{eff}}}{m_{\text{eff}}} + \frac{2s_{\text{eff}}}{k} \right) \sin(N\beta)},
\]

where \( Z_{\text{eff}} \) denotes the effective mechanical impedance of the LREM, which is defined as

\[
Z_{\text{eff}} = \frac{\sqrt{m_{\text{eff}} k_{\text{eff}}}}{\cos \left( \frac{\pi}{N} \right)}.
\]

It is worth pointing out that the effective mechanical impedance of the LREM can be tuned by designing the effective mass and stiffness.

### 2.2 Principles for realizing effective fixed and free boundaries

The foundational idea to forming EMBs using a resonance band gap is to make the magnitude and phase of the reflection coefficient have specific values. In this study, the principles for realizing EMBs are derived from the magnitude (the 1st condition) and phase (the 2nd condition) of the reflection coefficient.

It is well known that propagating elastic waves are totally reflected when they reach fixed or free boundaries (Graff, 1991; Kim, 2010). For both effective fixed and free boundaries, the 1st principle states that if the magnitude of the reflection coefficient is unity (i.e. \( |R| = 1 \)), then the incident waves are totally reflected. This 1st principle can be satisfied at the frequencies inside a resonance band gap exhibited by the LREM. Given this, the 2nd principle states that if the phase of the reflection coefficient is zero (i.e. \( R = -1 \)), then the reflected wave is in phase with the incident wave, thereby the effective free boundary is formed. If the phase of the reflection coefficient is \( \pi \) (i.e. \( R = 1 \)), then the reflected wave is 180 degrees out of phase with the incident wave, thereby the effective fixed boundary is formed.

To make the phase of the reflection coefficient zero for the effective free boundary in equation (7), the effective mechanical impedance of the LREM should be zero \( (Z_{\text{eff}} = 0) \). In order to have zero effective mechanical impedance in equation (8), either the effective mass or the effective stiffness should be zero, since the cosine term is a finite value. Note that the normalized wavenumber is a complex value within a band gap. However, since the upper branch of the resonance band gap appears at the frequency where the effective mass is zero (Liu et al., 2011; Su and Sun, 2015; Sang et al., 2019), the LREM can exhibit zero effective mechanical impedance while satisfying the 1st principle, only when it has zero effective stiffness as

\[
s_{\text{eff}} = 0.
\]

To make the phase of the reflection coefficient \( \pi \) for the effective fixed boundary, the effective mechanical impedance of the LREM should be infinite \( (Z_{\text{eff}} = \infty) \), which can be achieved by either infinite mass or infinite stiffness in equation (8) because the cosine term is a finite value. To have a limiting value of the frequency in equation (2), however, the effective mass and the effective stiffness must be infinite simultaneously as

\[
m_{\text{eff}} = s_{\text{eff}} = \infty.
\]

### 3. Numerical Demonstration

This section demonstrates the effective free and fixed boundaries using an LREM through numerical simulation. Many studies have been reported on the tunability of effective properties (i.e. effective mass and effective stiffness) of LREMs (Pope and Laaie, 2014; Oh et al., 2016a, b, 2017; Zhu et al., 2016; Chen et al., 2017; Lee et al., 2018). As an illustrative example, an LREM allowing independent tuning of effective properties, proposed in the literature (Oh et al., 2016b), was adopted in this study, with some modifications. Details of the design of the LREM and its lumped model can be found in the appendix. Of course, any kind of LREMs can be used as long as it can realize effective properties for the effective free boundary in equation (9) or for the effective fixed boundary in equation (10).

Figure 2a presents the dispersion relation of the unit cell used to construct the LREM. The gray shaded area indicates the band gap ranging from 22.0 to 34.2 kHz. The upper and lower parts of
The properties of the unit cell used for the LREM: (a) the dispersion curve, (b) the effective mass, (c) the effective stiffness, and (d) the effective mechanical impedance.

Figure 2: The properties of the unit cell used for the LREM: (a) the dispersion curve, (b) the effective mass, (c) the effective stiffness, and (d) the effective mechanical impedance.

Figure 3: The magnitude (a) and the phase (b) of the reflection coefficient of the LREM. The shaded area represents the range of the band gap.

The band gap are attributed to the negative mass (blue shaded area) and the negative stiffness (red shaded area), respectively, as shown in Fig. 2b and c. Note that the lowest frequency (22.0 kHz) of the band gap in Fig. 2a and the zero-stiffness frequency (24 kHz) in Fig. 2c are different from each other. This is because the band gap can be formed when an LREM exhibits very small stiffness as well as negative stiffness.

Figure 3 shows the magnitude and phase of the reflection coefficient of the LREM, which are calculated by using equation (7). The formation of the band gap can be confirmed from Fig. 3a that the magnitude of the reflection coefficient is unity in the frequency range from 24.0 to 34.2 kHz.

At the frequency of 30.2 kHz, since the LREM is designed to possess the effective mass and stiffness of infinite values (Fig. 2b and c), its effective mechanical impedance becomes very large (Fig. 2d). At the frequency of 24.0 kHz, since the LREM is designed to possess the effective stiffness of zero value (Fig. 2c), it exhibits zero effective mechanical impedance (Fig. 2d). Therefore, one can expect that the effective fixed boundary can be formed at the interface between the host medium and the elastic metamaterial at the frequency of 30.2 kHz, while the effective free boundary can be formed at the frequency of 24.0 kHz. Accordingly, the phases of the reflection coefficients at each corresponding frequency become π and zero, respectively, as shown in Fig. 3b.

To demonstrate the formation of the EMBs, time-harmonic analysis was performed using a commercially available finite element analysis (FEA) software, a solid mechanics module in COMSOL Multiphysics 5.4, as shown in Fig. 4. It has been confirmed in many works that elastic wave simulation using COMSOL Multiphysics shows accurate predictive capability compared to actual experiments (Tol et al., 2017; Lee et al., 2018; Jo et al., 2021). Ten unit cells were used in total, since the magnitude of the reflection coefficient in equation (7) became unity over the band gap frequencies when more than eight unit cells were used. The effects of the number of unit cells on the reflection can be found in Appendix B. The longitudinal wave was excited at the left end with a constant velocity of $3\pi$ mm/s, which corresponds to the displacement of 50 nm near 30 kHz. In general, several tens of nanometers can be observed in an elastic wave propagation experiment (Park et al., 2019; Lee et
Periodic boundary conditions were applied to the top and bottom surfaces. Perfectly matched layers, artificial absorbing layers for elastic waves, were imposed at the left and right ends to mimic nonreflecting boundary (Givoli, 1991, 2013). The LREM-embedded system is discretized by hexahedron meshes; the maximum mesh size is less than one-thirtieth of the wavelength at 40 kHz to ensure accuracy of the FEA (at the highest frequency).

Figure 5 shows the time-harmonic analysis results—the normalized velocity and stress—at the interface between the host medium and the LREM; both are normalized by each maximum value. Here, the reason that the normalized velocity was calculated rather than the normalized displacement is that the constant velocity was imposed as the input loading. It is observed that the normalized velocity approaches almost zero at the frequency of 29.9 kHz where the effective mechanical impedance becomes infinite, while the normalized stress approaches nearly zero at the frequency of 24.3 kHz where the effective mechanical impedance becomes zero. It can be thus confirmed that the interface between the LREM and the host medium becomes the effective fixed and free boundaries at the frequencies of 29.9 and 24.3 kHz, respectively. Table 1 summarizes the percentage difference between the frequencies at which the EMBs are formed, estimated by the monoatomic chain model and the FEA.

The reason that the normalized velocity and stress calculated at the interface between the host medium and the LREM are not exactly zero at the frequencies of 29.9 and 24.3 kHz, respectively, could be explained by a local effect that might appear near the interface. The effective medium method is applicable in the long-wavelength limit, i.e. when the wavelength is much longer than the lattice constant of the unit cell (Ni and Cheng, 2005; Fokin et al., 2007; Popa and Cummer, 2009). At the frequency of 29.9 kHz for the effective fixed boundary, the length of the unit cell is 0.11λ. This implies that the wave field near the interface could be affected by the geometry of the unit cell.

Figure 6a shows the velocity field of the host medium when an actual free boundary condition is imposed at the right end, while Fig. 6b shows the velocity field of the LREM-embedded host medium when the effective free boundary is formed at the interface between the host medium and the LREM. Overall, the velocity field of the LREM-embedded host medium is in a very good agreement with that of the host medium with the actual free boundary condition at the frequency of 24.3 kHz.

On the other hand, Fig. 6d shows the velocity field of the host medium when an actual fixed boundary condition is imposed at
Figure 6: Velocity fields at various frequencies (only two unit cells are shown for simplicity): (a) 24.3 kHz (actual free boundary); (b) 24.3 kHz (effective free boundary); (c) 28.1 kHz (center of the band gap); (d) 29.9 kHz (actual fixed boundary); (e) 29.9 kHz (effective fixed boundary); and (f) 35.5 kHz (outside the band gap).

the right end, while Fig. 6e shows the velocity field of the LREM-embedded host medium when the effective fixed boundary is formed at the interface between the host medium and the LREM. Overall, the velocity field of the LREM-embedded host medium is in a good agreement with that of the host medium with the actual fixed boundary condition at the frequency of 29.9 kHz.

Figure 6c and f show the velocity fields of the LREM-embedded host medium at the frequencies of 28.1 kHz (the center of the band gap) and 35.5 kHz (outside the band gap), respectively. For both cases, it can be seen that the interfaces between the host medium and the LREM are not fixed, nor free boundaries. At the frequency of 28.1 kHz (the center of the band gap), even if the magnitude of the reflection coefficient is unity due to the band gap, the phase of the reflection coefficient is neither 0 nor π.

At the frequency of 35.5 kHz (outside the band gap), since the incident waves are reflected due to not the band gap but the mechanical impedance mismatching, the incident waves are not totally reflected but partially transmitted; this implies that the magnitude of the reflection coefficient is not unity at the frequency of 35.5 kHz. Subsequently, the amplitude of the velocity field at the frequency of 35.5 kHz is smaller than those at frequencies inside the band gap (Shin et al., 2021).

Figures 6a and 7a show the velocity and stress fields of the host medium when an actual free boundary condition is imposed at the right end, respectively, while Figs 6b and 7b show the velocity and stress fields of the LREM-embedded host medium, respectively, when the effective free boundary is formed at the interface between the host medium and the LREM. On the other hand, Figs 6d and 7d show the velocity and stress fields of the host medium when an actual fixed boundary condition is imposed at the right end, respectively, while Figs 6e and 7e show the velocity and stress fields of the LREM-embedded host medium, respectively, when the effective fixed boundary is formed at the interface between the host medium and the LREM. Overall, the velocity fields of the LREM-embedded host medium are in very good agreement with those of the host medium with the actual free and fixed boundaries at the frequencies of 24.3 and 29.9 kHz, respectively. Even though the overall stress fields of the LREM-embedded host medium are in good agreement with those of the host medium with the actual free and fixed boundaries at the frequencies of 24.3 and 29.9 kHz, however, there is a relatively large difference in the stress fields near the interface between the host medium and the LREM. The discrepancy in the stress fields at the interface seems to be due to the local effect that could yield localized stress due to the effect of the geometry of the unit cell.

4. Conclusion

This study proposed the design principles for EMB formation using an LREM. The principles for designing EMBs were derived from the magnitude and phase of the reflection coefficient. The first principle states that if the magnitude of the reflection coefficient is unity, then the incident waves are totally reflected due to the band gap. Given this, the second principle states that if the phase of the reflection coefficient is zero, then the effective free boundary is formed; however, if the phase of the reflection coefficient is π, then the reflected wave is 180 degrees out of phase with the incident wave, thereby the effective fixed boundary is formed. To make the phase of the reflection coefficient zero, the effective mechanical impedance of the elastic metamaterial should be zero, which can be achieved by having zero effective stiffness. To make the phase of the reflection coefficient π, the effective mechanical impedance should be infinite, which can be achieved by simultaneously having...
Effective mechanical boundary by a resonance band gap

Figure 7: Stress fields at various frequencies (only two unit cells are shown for simplicity): The actual free and fixed boundary conditions are applied to the right ends of the host medium in (a) and (d), respectively. The other stress fields are for the elastic-metamaterial-embedded system. (a) 24.3 kHz; (b) 24.3 kHz (effective free boundary frequency); (c) 28.1 kHz (center of the band gap); (d) 29.9 kHz; (e) 29.9 kHz (effective fixed boundary frequency); and (f) 35.5 kHz (outside the band gap).

infinite effective mass and stiffness. The LREM was attached to the host medium to configure the system to validate the concept of the EMB. Time-harmonic analysis was performed to confirm that the velocity neared zero at the frequency where the effective mechanical impedance was infinite, and that the stress approached zero at the frequency where the effective mechanical impedance was zero. As a result, it was shown that the overall wave fields produced when the LREM was attached to the host medium were in good agreement with those developed when actual boundary conditions were imposed on the right end of the host medium. This paper presented a fundamental study that derived design principles of effective boundary formation and numerically validated them. Future works may include experiments to explore the effects of damping on effective boundary formation.

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Conflict of interest statement

None declared.

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**Appendix 1. Modeling of a Locally Resonant Elastic Metamaterial**

In this study, an LREM is introduced, since the derived conditions for the EMB demand extreme values, such as infinite mass and zero stiffness. There are several candidate metamaterial designs in the literature (Pope et al., 2014; Zhu et al., 2016; Oh et al., 2016a, b, 2017; Chen et al., 2017; Lee et al., 2018) that satisfy the
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required conditions for effective fixed and free boundaries. In this study, the LREM design in Oh et al. (2016b) is adopted, with some modifications. Figure A1 shows the modified unit cell design of the LREM. Vertical bars are added to the right and left ends to fully connect the host medium with the LREM in order to not force a free boundary condition at the interface between the host medium and the LREM. Figure A2 represents a lumped-parameter model for the LREM. The equations of motion for the masses can be written as

\[ m_1 \frac{\partial^2 u_1|_{n,j}}{\partial t^2} = (u_2|_{n,j} - u_1|_{n,j}) \alpha - (u_2|_{n-1,j} - u_1|_{n,j}) \alpha + (u_2|_{n,j} - u_1|_{n,j}) \alpha \]
\[ - (u_2|_{n-1,j} - u_1|_{n,j}) \alpha + \gamma v_2|_{n,j} - \gamma v_2|_{n-1,j} + \gamma v_2|_{n-1,j} \]
\[ + (u_3|_{n,j} - u_1|_{n,j}) 2 \beta + (u_3|_{n,j} - u_1|_{n,j}) 2 \beta. \]  
\[ (A.1.1) \]

\[ m_2 \frac{\partial^2 u_2|_{n,j}}{\partial t^2} = \alpha (u_1|_{n,j} + u_1|_{n+1,j} - 2 u_2|_{n,j}). \]  
\[ (A.1.2) \]

\[ m_2 \frac{\partial^2 u_2|_{n,j}}{\partial t^2} = \alpha (u_1|_{n,j} + u_1|_{n+1,j} - 2 u_2|_{n,j}). \]  
\[ (A.1.3) \]

\[ m_2 \frac{\partial^2 v_2|_{n,j}}{\partial t^2} = -\gamma u_1|_{n,j} + \gamma u_1|_{n+1,j} - 2 \beta v_2|_{n,j}. \]  
\[ (A.1.4) \]

\[ m_2 \frac{\partial^2 v_2|_{n,j}}{\partial t^2} = \gamma u_1|_{n,j} + \gamma u_1|_{n+1,j} - 2 \beta v_2|_{n,j}. \]  
\[ (A.1.5) \]

\[ m_3 \frac{\partial^2 u_3|_{n,j}}{\partial t^2} = \beta (u_1|_{n,j} - u_3|_{n,j}) + \beta (u_1|_{n,j-1} - u_3|_{n,j}). \]  
\[ (A.1.6) \]

\[ m_3 \frac{\partial^2 u_3|_{n,j}}{\partial t^2} = \beta (u_1|_{n,j} - u_3|_{n,j}) + \beta (u_1|_{n,j-1} - u_3|_{n,j}). \]  
\[ (A.1.7) \]

Figure A1: Schematics of the unit cell of the designed LREM with the geometric dimensions.

Figure A2: Schematics of the lumped model for the designed LREM.
where $u_i$ and $v_i$ denote $x$- and $y$-directional displacements, respectively; $\alpha$, $\beta$, and $\gamma$ are the stiffness matrix elements of the skew beam; $n$ and $j$ represent the location of the unit cells. By applying the time-harmonic solutions $u_{i|n,j} = U_{i|n,j} \exp[i(\omega t - kx)]$ and periodic condition in the $x$-direction $u_{i|n+1,j} = \exp[i(\omega t - kx)]u_{i|n,j}$, the dispersion equation can be derived as
\begin{equation}
-\omega^2 \left( m_1 + \frac{4\alpha m_2}{2\alpha - \omega^2 m_2} + \frac{4\beta m_3}{2\beta - \omega^2 m_3} \right) \\
= \left( \frac{2\alpha^2}{2\alpha - \omega^2 m_2} - \frac{2\gamma^2}{2\beta - \omega^2 m_3} \right) (e^{ikd} + e^{-ikd} - 2). \tag{A.2}
\end{equation}

Since the frequency of interest is much lower than $\sqrt{2\alpha/m_2}$, it can be assumed that $2\alpha - \omega^2 m_2 \approx 2\alpha$. Then, the dispersion relation can be simplified as
\begin{equation}
-\omega^2 \left( m_1 + 2m_2 + \frac{2\omega_0^2 m_3}{\omega_0^2 - \omega^2} \right) \\
= \left( \alpha - \frac{2\gamma^2}{\omega_0^2 - \omega^2} \right) (e^{ikd} + e^{-ikd} - 2). \tag{A.3}
\end{equation}

By comparing the dispersion relation with that of the monoatomic chain model (Brillouin, 2003), the effective mass and the effective stiffness of the unit cell can be derived as
\begin{align*}
m_{\text{eff}} &= m_1 + 2m_2 + \frac{2\omega_0^2 m_3}{\omega_0^2 - \omega^2}, \tag{A.4} \\
s_{\text{eff}} &= \alpha - \frac{2\gamma^2}{\omega_0^2 - \omega^2}. \tag{A.5}
\end{align*}

Appendix 2. The Effect of the Number of Unit Cells on Reflection

A band gap of an LREM is formed due to periodic deployment of its unit cells. In this study, the periodic boundary conditions are applied in the process of deriving equation (A.2). Since it is practically not possible to use the infinite number of unit cells, it is required to study on the number of unit cells that can effectively realize total reflection due to the band gap.

Figure B1 shows the magnitude of the reflection coefficient for various numbers of unit cells. The gray shaded area indicates the band gap region estimated by equation (A.3). As the number of unit cells increases, the magnitude of the reflection coefficient converges to unity over the entire band gap region. In this study, the magnitude of the reflection coefficient becomes unity when more than eight unit cells are used to construct the LREM.
Figure B1: The magnitude of the reflection coefficient for various numbers of unit cells. (a) 1 unit cell; (b) 2 unit cells; (c) 3 unit cells; (d) 5 unit cells; (e) 8 unit cells; and (f) 10 unit cells.