Seiberg-Witten Monopole Equations  
on Noncommutative $\mathbb{R}^4$

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Abstract

It is well known that, due to vanishing theorems, there are no nontrivial finite action solutions to the Abelian Seiberg-Witten (SW) monopole equations on Euclidean four-dimensional space $\mathbb{R}^4$. We show that this is no longer true for the noncommutative version of these equations, i.e., on a noncommutative deformation $\mathbb{R}_\theta^4$ of $\mathbb{R}^4$ there exist smooth solutions to the SW equations having nonzero topological charge. We introduce action functionals for the noncommutative SW equations and construct explicit regular solutions. All our solutions have finite energy. We also suggest a possible interpretation of the obtained solutions as codimension four vortex-like solitons representing $D(p-4)$- and $\overline{D(p-4)}$-branes in a $Dp\overline{Dp}$ brane system in type II superstring theory.
1 Introduction

The Seiberg-Witten (SW) monopole equations \cite{1} have been derived in the context of twisted $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory \cite{2,3} in some limit of the coupling constant. Another limit of this theory yields the anti-self-dual Yang-Mills (ASDM) equations. Namely, the ASDYM equations correspond to the weak coupling limit while the SW equations are related to the strong coupling regime obtained by the S-dualization (see, e.g., \cite{1,4,5,6} and references therein). Note that the SW equations are associated with the Abelian group $U(1)$ and have a compact moduli space while the ASDYM equations, considered in Donaldson-Witten (DW) theory \cite{7,2}, possess the non-Abelian gauge group $SU(2)$ and a noncompact moduli space. That is why SW theory is much easier to handle compared to DW theory. A bridge between these theories is provided by the non-Abelian SW equations (see, e.g., \cite{8,9,10,11,6} and references therein) whose moduli space contains both DW and SW moduli spaces as singular submanifolds.

It is well known that, due to a vanishing theorem of the Lichnerowicz-Weitzenböck type, there are no nontrivial finite action solutions to the Abelian SW equations on Riemannian four-manifolds...
with non-negative scalar curvature and, in particular, on $\mathbb{R}^4$ (cf. [1]). This assertion is also true for lower-dimensional reductions of the SW equations, i.e., these reductions also do not exhibit regular solutions on $\mathbb{R}^{n \leq 3}$ with a nonzero topological charge. Nevertheless, one may construct nontrivial non-$L^2$ solutions, as it has been done, e.g., in [12, 13, 14, 15].

Note that for the vanishing spinor field the SW equations specialize to the Abelian ASDYM equations which have no nontrivial regular solutions (instantons) on $\mathbb{R}^4$ either. However, Nekrasov and Schwarz have demonstrated in [16] that smooth Abelian instanton solutions do exist on $\mathbb{R}^4_\theta$ - a noncommutative deformation of $\mathbb{R}^4$ with constant deformation parameters $\theta = (\theta^{\mu\nu})$. Moreover, they have proven that noncommutativity resolves the singularities of the instanton moduli space. In the present paper we observe a similar phenomenon for the SW equations on $\mathbb{R}^4_\theta$ by constructing nontrivial regular solutions to the noncommutative SW equations.

It is well known that the SW equations on Kähler surfaces are similar to the vortex equations in two dimensions. Motivated by this relation, we interpret regular (vortex-like) solutions to the SW equations on $\mathbb{R}^4_\theta$ as $D(p-4)$- and $D(p-4)$-branes in a $Dp$-$D\bar{p}$ brane-antibrane system in type II superstring theory. This interpretation can also be extended to the commutative case of the SW equations on Kähler surfaces.

The paper is organized as follows. In the next section we formulate the SW equations on $\mathbb{R}^4$ and fix our notation. In section 3 we introduce the noncommutatively deformed non-Abelian SW equations. We derive them from properly deformed $U(2)$ self-duality type equations in eight dimensions [17] by a dimensional reduction to four dimensions (cf. [18]). The resulting $U_+(1) \times U_-(1)$, $U_+(1)$ and $U_-(1)$ noncommutative SW equations can also be produced from appropriate action functionals by using a Bogomolny type transformation. We point out that the $U_+(1) \times U_-(1)$, $U_+(1)$ and $U_-(1)$ noncommutative SW equations share the same commutative limit. In section 4 we present a number of regular solutions to the noncommutative SW equations and discuss their $D$-brane interpretation in a string theoretic context. In section 5 we conclude with a brief summary and open problems. Finally, in the Appendix we perform the Bogomolny type transformation for the noncommutative $U_+(1) \times U_-(1)$ SW action functional.

## 2 SW monopole equations on $\mathbb{R}^4$

**SW action functional.** In this paper we consider the SW equations on the Euclidean space $\mathbb{R}^4$, provided with the standard metric $g = (\delta_{\mu\nu})$, where $\mu, \nu, \ldots = 1, \ldots, 4$. The (energy) functional $E = E(A, \Phi)$ for these equations has the form (cf., e.g., [19, 20, 21])

$$E(A, \Phi) = \int_{\mathbb{R}^4} d^4x \left\{ |F_A|^2 + |D_A \Phi|^2 + \frac{1}{4} |\Phi|^4 \right\}.$$  \hspace{1cm} (2.1)

Here $A \in \Omega^1(\mathbb{R}^4, u(1))$ is a connection one-form on $\mathbb{R}^4$ with pure imaginary smooth coefficients and $\Phi \in \mathcal{C}^\infty(\mathbb{R}^4, \mathbb{C}^2)$ is a Weyl spinor given by a smooth complex-valued vector function on $\mathbb{R}^4$. We denote by $F_A^+ \in \Omega^2_+(\mathbb{R}^4, u(1))$ the self-dual part of the curvature $F_A$ of $A$ and by $D_A$ the covariant
derivative associated with $A$. Moreover, we use the abbreviation $|D_A \Phi|^2 = D_\mu \phi_i (D_\mu \phi_i)^\dagger$ and set $|F_A|^2 = \frac{1}{2} F_{\mu \nu} F_{\mu \nu}^\dagger$.

By exploiting a Bogomolny type formula, the energy functional can be rewritten in the form

$$E(A, \Phi) = SW(A, \Phi) - 8\pi^2 Q,$$

where

$$SW(A, \Phi) = \int_{\mathbb{R}^4} d^4x \left\{ |D_A \Phi|^2 + 2 |F_A^+ - \sigma^+(\Phi \otimes \Phi^\dagger)|_0|^2 \right\}$$

is the SW action functional and

$$Q = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} F_A \wedge F_A$$

is the topological charge. In the above formula (2.3) we denote by $D_A$ the Dirac operator associated with $A$. We also use the notation

$$\sigma^+(\Phi \otimes \Phi^\dagger)_0 := \sigma^+(\Phi \otimes \Phi^\dagger) - \frac{1}{2} |\Phi|^2 \text{id},$$

where

$$\sigma^+ : \text{Herm}_0(\mathbb{C}^2) \longrightarrow \Omega^2_+(\mathbb{R}^4, u(1))$$

is a map identifying the space $\text{Herm}_0(\mathbb{C}^2)$ of traceless Hermitian endomorphisms of $\mathbb{C}^2$ with the space $\Omega^2_+(\mathbb{R}^4, u(1))$ of imaginary-valued self-dual two-forms on $\mathbb{R}^4$. The inverse of this map is given by the Clifford multiplication by two-forms (see, e.g., [19, 20, 21]).

It is easy to see that the functionals (2.1) and (2.3) are invariant under gauge transformations of the form

$$A \mapsto A + g^\dagger dg \quad \text{and} \quad \Phi \mapsto g^\dagger \Phi,$$

where $g \in C^\infty(\mathbb{R}^4, U(1))$.

**SW monopole equations.** Since the functional $SW(A, \Phi)$ is positive semi-definite and $Q$ is a topological term, the Bogomolny formula (2.2) implies that the lower bound of the energy $E(A, \Phi)$ is attained on solutions to the equations

$$F_A^+ = \sigma^+(\Phi \otimes \Phi^\dagger)_0,$$

$$D_A \Phi = 0,$$

which are known as the SW monopole equations. They are differential equations of first order and their solutions, which minimize the energy functional $E(A, \Phi)$, automatically satisfy the (second order) Euler-Lagrange equations for the functionals $E(A, \Phi)$ and $SW(A, \Phi)$.

Writing $^t \Phi = (\phi_1, \phi_2)$, one can see that the equations (2.7) are equivalent to (cf. [19])

$$F_{12} + F_{34} = \frac{i}{2} (\phi_1 \bar{\phi}_2 - \phi_2 \bar{\phi}_1),$$

$$F_{13} + F_{42} = -\frac{i}{2} (\phi_2 \bar{\phi}_1 - \phi_1 \bar{\phi}_2),$$

$$F_{14} + F_{23} = \frac{i}{2} (\phi_2 \bar{\phi}_1 + \phi_1 \bar{\phi}_2).$$
and
\[
\begin{pmatrix}
-D_4 + iD_3 & D_2 + iD_1 \\
-D_2 + iD_1 & -D_4 - iD_3
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = 0.
\] (2.8b)

It is easy to prove that these equations have no nontrivial solutions with finite action. Namely, we have the following theorem (see, e.g., [1, 19, 20, 21]):

**Theorem:** Suppose \( A \in \Omega^1(\mathbb{R}^4, u(1)) \) and \( \Phi \in C^\infty(\mathbb{R}^4, \mathbb{C}^2) \) satisfy the equations (2.8). Moreover, we assume that \( \Phi \in L^2(\mathbb{R}^4) \) and \( E(A, \Phi) < \infty \). Then the only solution to (2.8) is the trivial solution \((A, \Phi) = (0, 0)\) modulo the gauge transformations (2.6).

This theorem is also true for lower dimensional reductions of the SW equations (defined on \( \mathbb{R}^{n \leq 3} \)), i.e., these reductions do not exhibit regular nontrivial solutions either. However, as we have already mentioned in the Introduction, one can construct nontrivial non-L^2 solutions [12, 13, 14, 15].

**Perturbed SW action functional and monopole equations.** The gauge group action (2.6) on the space of pairs \((A, \Phi)\) is free, unless \( \Phi \equiv 0 \). In order to avoid solutions of the form \((A, 0)\), which may cause singularities in the moduli space of solutions, we perturb the monopole equations by adding an extra term to the first SW equation,

\[
F_A^+ + \chi^+ = \sigma^+(\Phi \otimes \Phi^\dagger)_0,
\] (2.9a)

\[
D_A \Phi = 0,
\] (2.9b)

where \( \chi^+ \) is the self-dual part of a two-form \( \chi \in \Omega^2(\mathbb{R}^4, u(1)) \) (perturbation). Solutions to these equations minimize the functional

\[
SW_\chi(A, \Phi) = \int_{\mathbb{R}^4} d^4x \left\{ |D_A \Phi|^2 + 2 |F_A^+ + \chi^+ - \sigma^+(\Phi \otimes \Phi^\dagger)_0|^2 \right\}. \tag{2.10}
\]

In components equation (2.9a) reads

\[
\begin{align*}
F_{12} + F_{34} + \chi_{12} + \chi_{34} &= \frac{i}{2}(\phi_1 \tilde{\phi}_1 - \phi_2 \tilde{\phi}_2), \\
F_{13} + F_{42} + \chi_{13} + \chi_{42} &= -\frac{i}{2}(\phi_2 \tilde{\phi}_1 - \phi_1 \tilde{\phi}_2), \\
F_{14} + F_{23} + \chi_{14} + \chi_{23} &= \frac{i}{2}(\phi_2 \tilde{\phi}_1 + \phi_1 \tilde{\phi}_2).
\end{align*} \tag{2.11}
\]

The SW action functional, as in the unperturbed case, is related to an energy functional,

\[
E_\chi(A, \Phi) = \int_{\mathbb{R}^4} d^4x \left\{ |F_A|^2 + |D_A \Phi|^2 + 2 |\chi^+ - \sigma^+(\Phi \otimes \Phi^\dagger)_0|^2 \right\}, \tag{2.12}
\]

via a Bogomolny type formula,

\[
SW_\chi(A, \Phi) = E_\chi(A, \Phi) + 16\pi^2 K_\chi + 8\pi^2 Q. \tag{2.13}
\]

The topological charge \( Q \) is given, as before, by formula (2.4) and the Chern-Simons type term \( K_\chi \) is defined as

\[
K_\chi = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} F_A^+ \wedge \chi^+. \tag{2.14}
\]
3 SW monopole equations on $\mathbb{R}^4_\theta$

3.1 Non-Abelian SW monopole equations

Noncommutative Euclidean space $\mathbb{R}^{2n}_\theta$. Let $\mathcal{A}(\mathbb{R}^{2n})$ be the algebra of polynomial functions on $\mathbb{R}^{2n}$ (which is endowed with the canonical metric $\delta_{\alpha\beta}$) and $\theta = (\theta^{\alpha\beta})$ be a real invertible skew-symmetric $2n \times 2n$ matrix with the inverse matrix $\theta^{-1} = (\theta_{\alpha\beta})$ defined by $\theta_{\alpha\gamma}\theta^{\gamma\beta} = \delta_{\beta\alpha}$ for $\alpha, \beta, \ldots = 1, \ldots, 2n$. Then the deformed algebra $\mathcal{A}_\theta(\mathbb{R}^{2n})$ is defined as

$$\mathcal{A}_\theta(\mathbb{R}^{2n}) := T(\mathbb{R}^{2n})/\langle [x^\alpha, x^\beta] - i\theta^{\alpha\beta} \rangle_{1\leq \alpha, \beta \leq 2n}, \quad (3.1)$$

where $T(\mathbb{R}^{2n})$ is the tensor algebra of $\mathbb{R}^{2n}$ and $\langle [x^\alpha, x^\beta] - i\theta^{\alpha\beta} \rangle_{1\leq \alpha, \beta \leq 2n}$ denotes the two-sided ideal generated by $[x^\alpha, x^\beta] - i\theta^{\alpha\beta} \subset T(\mathbb{R}^{2n})$. For brevity we shall denote $\mathcal{A}_\theta(\mathbb{R}^{2n})$ simply by $\mathbb{R}^{2n}_\theta$ and call it the noncommutative Euclidean 2n-dimensional space.

One way to realize the noncommutative extension (3.1) of the algebra $\mathcal{A}(\mathbb{R}^{2n})$ is by deformation of the pointwise product between functions via the so-called star (Moyal) product,

$$(f \star g)(x) := f(x) \exp \left\{ i\frac{1}{2} \theta^{\alpha\beta} \partial_\alpha \partial_\beta \right\} g(x), \quad (3.2)$$

where $f, g \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$. In particular, it follows from (3.2) that

$$[x^\alpha, x^\beta] := x^\alpha \star x^\beta - x^\beta \star x^\alpha = i\theta^{\alpha\beta}. \quad (3.3)$$

For later convenience we introduce complex coordinates on $\mathbb{R}^{2n} \cong \mathbb{C}^n$,

$$z^a = x^{2a-1} + ix^{2a} \quad \text{and} \quad \bar{z}^\alpha = x^{2a-1} - ix^{2a}, \quad \text{for} \quad a = 1, \ldots, n, \quad (3.4)$$

and derivatives

$$\partial_{z^a} = \frac{1}{2}(\partial_{2a-1} - i\partial_{2a}) \quad \text{and} \quad \partial_{\bar{z}^\alpha} = \frac{1}{2}(\partial_{2a-1} + i\partial_{2a}). \quad (3.5)$$

Note that by an orthogonal change of coordinates one can always transform $\theta^{\alpha\beta}$ to its canonical (Darboux) form whose only nonzero components are $\theta^{2a-1,2a}$ with $a = 1, \ldots, n$. Then the commutation relations (3.3) translate to

$$[z^a, \bar{z}^\alpha] = \theta^{a\bar{\alpha}}, \quad \text{with} \quad \theta^{a\bar{\alpha}} = 2\theta^{2a-1,2a}, \quad (3.6)$$

and all other commutators are equal to zero.

Self-duality type equations in eight dimensions. A standard way to obtain a noncommutative generalization of a theory is to replace naively the ordinary commutative product between field variables with the noncommutative star product. However, it is well known that this method of translating a commutative theory into a noncommutative one is not uniquely defined when the matter fields are involved. For instance, the scalar fields in noncommutative $U(1)$ gauge theory on $\mathbb{R}^{2n}$ can be regarded in three different ways, namely as elements of a left module (over the algebra
\( \mathcal{A}_0(\mathbb{R}^{2n}) \), or as elements of a right module, or they can transform in the adjoint representation. For this reason we propose deriving the noncommutative SW equations from noncommutative self-duality type Yang-Mills (YM) equations in eight dimensions, which are uniquely defined. Eventually, we will discover the equations corresponding to the above mentioned naive substitution rule by a formal reduction of more general equations. In the commutative case a similar idea has been worked out by the authors of [18].

Let us consider pure \( U(2) \) YM theory on \( \mathbb{R}^8 \). In star-product formulation the components \( F_{\alpha\beta} \) of the YM curvature \( F_A \) read as

\[
F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]_*
\]

and take values in \( u(2) \). Here \( \alpha, \beta, \ldots \), run from 1 to 8. Consider the generalized self-duality equations for YM fields in eight dimensions\(^1\),

\[
F_{\alpha\beta} = \frac{1}{2} T_{\alpha\beta\gamma\delta} F_{\gamma\delta},
\]

where the totally antisymmetric tensor \( T_{\alpha\beta\gamma\delta} \) is determined by the octonionic structure constants \( f_{ijk} \) as\(^2\)

\[
T_{ijkl} = \frac{1}{6} \epsilon_{ijklmnop} f_{mnop} \quad \text{and} \quad T_{8ijk} = f_{8ijk}, \quad \text{for} \quad i, j, k, \ldots = 1, \ldots, 7.
\]

Note that the tensor \( T_{\alpha\beta\gamma\delta} \) and therefore the equations are invariant with respect to the group \( \text{Spin}(7) \) rather than \( \text{SO}(8) \). In fact, it is impossible to construct a totally antisymmetric tensor of rank four in eight dimensions which is invariant under \( \text{SO}(8) \) rotations.

Using the definition, we can write down the generalized self-duality equations in components as follows:

\[
\begin{align*}
F_{12} + F_{34} + F_{56} + F_{78} &= 0, \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0, \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0, \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0, \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0, \\
F_{17} + F_{82} + F_{35} + F_{64} &= 0, \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0.
\end{align*}
\]

With the help of and the definitions

\[
A_{2a} = \frac{1}{2} (A_{2a-1} - iA_{2a}) \quad \text{and} \quad A_{2a} = \frac{1}{2} (A_{2a-1} + iA_{2a}), \quad \text{for} \quad a = 1, \ldots, 4,
\]

\(^1\)In the commutative case these equations were introduced in \([17]\) and discussed, e.g., in \([22, 23]\).

\(^2\)We use \( f_{127} = f_{147} = f_{567} = f_{163} = f_{246} = f_{253} = f_{154} = 1 \).
we rewrite (3.10) as

\[ F_{z \bar{z} 1} + F_{z \bar{z} 2} + F_{z \bar{z} 3} + F_{z \bar{z} 4} = 0, \]  
\[ F_{z \bar{z} 1} + F_{\bar{z} \bar{z} 2} = 0, \]  
\[ F_{z \bar{z} 2} - F_{\bar{z} \bar{z} 3} = 0, \]  
\[ F_{z \bar{z} 3} + F_{\bar{z} \bar{z} 4} = 0. \]  

Note that \( A_{\bar{z} a} = -A^{\dagger}_a \), since the components \( A_{\alpha} \) are skew-Hermitian.

Reduction to four dimensions. Following Baulieu et al. [18], we assume that the gauge potential components \( A_{\bar{z} a} \) for \( a = 1, \ldots, 4 \) do not depend on the coordinates \( z^3, z^4, \bar{z}^3, \bar{z}^4 \) and define \( \Psi := (\Psi_1, \Psi_2) \) with \( \Psi_1 := A_{\bar{z} 3} \) and \( \Psi_2 := A_{\bar{z} 4} \). Then the equations (3.12) dimensionally reduce to

\[ F_{z \bar{z} 1} + F_{\bar{z} \bar{z} 1} = -([\Psi_1, \Psi_1^\dagger], -[\Psi_2, \Psi_2^\dagger]), \]  
\[ F_{z \bar{z} 2} = [\Psi_1, \Psi_2^\dagger], \]  
\[ D_{z \bar{z}} \Psi_1 - D_{\bar{z} \bar{z}} \Psi_2 = 0, \]  
\[ D_{z \bar{z}} \Psi_1 + D_{\bar{z} \bar{z}} \Psi_2 = 0, \]  

where the covariant derivative \( D_{\bar{z} a} \) is defined by \( D_{\bar{z} a} \Psi = \partial_{\bar{z} a} \Psi + [A_{\bar{z} a}, \Psi] \). In the commutative limit these equations coincide with a non-Abelian generalization of the SW equations, considered in [18]. Note that the special case of these equations corresponding to \( \Psi_2 = 0 \) was discussed in [24, 25].

Along with the unperturbed equations (3.13) we shall also consider the perturbed equations (cf. [8, 9, 10]). For that we introduce a \( u(2) \)-valued two-form \( \chi \) and add its self-dual part \( \chi^+ \) to \( F^{+} \),

\[ F_{z \bar{z} 1} + F_{\bar{z} \bar{z} 1} + \chi_{z \bar{z} 1} + \chi_{\bar{z} \bar{z} 1} = -([\Psi_1, \Psi_1^\dagger], -[\Psi_2, \Psi_2^\dagger]), \]  
\[ F_{z \bar{z} 2} + \chi_{z \bar{z} 2} = [\Psi_1, \Psi_2^\dagger], \]  
\[ D_{z \bar{z}} \Psi_1 - D_{\bar{z} \bar{z}} \Psi_2 = 0, \]  
\[ D_{z \bar{z}} \Psi_1 + D_{\bar{z} \bar{z}} \Psi_2 = 0. \]  

3.2 Abelian SW monopole equations

In order to get Abelian SW equations from the non-Abelian ones, we shall consider solutions of a particular type given by a suitable ansatz. This will reduce the gauge group \( U(2) \) to \( U(1) \times U(1) \) and then, further down to \( U(1) \).

Noncommutative \( U_{+}(1) \times U_{-}(1) \) SW monopole equations. Let us consider the \( U(1) \times U(1) \) subgroup of \( U(2) \) with the generators

\[ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]  

(3.15)
and assume that the components of the gauge potential $A_{z^a}$ for $a = 1, 2$ take the form\footnote{If not stated differently, $a, b, \ldots$ run always from 1 to 2 in the sequel.}

$$A_{z^a} := \begin{pmatrix} A_{+z^a} & 0 \\ 0 & A_{-z^a} \end{pmatrix}, \quad \text{with} \quad A_{\pm z^a} = \frac{1}{2}(B_{z^a} \pm A_{z^a}), \quad (3.16)$$

and $A_{z^a}, B_{z^a} \in C^\infty(\mathbb{R}^4, i\mathbb{R} \otimes \mathbb{C})$. Furthermore, we restrict $\Psi_1$ and $\Psi_2$ to the form

$$\Psi_{1,2} := \begin{pmatrix} 0 & \frac{1}{\sqrt{8}}\phi_{1,2} \\ 0 & 0 \end{pmatrix}, \quad (3.17)$$

where $\phi_{1,2} \in C^\infty(\mathbb{R}^4, \mathbb{C})$.

Substituting (3.16) and (3.17) into the equations (3.13), after a straightforward calculation we obtain the equations

$$F_{+z^1 z^1} + F_{+z^2 z^2} = -\frac{1}{8}(\phi_1 \star \phi_1 - \phi_2 \star \phi_2) \quad \text{and} \quad F_{+z^1 z^2} = \frac{1}{8}\phi_1 \star \phi_2, \quad (3.18a)$$

$$F_{-z^1 z^1} + F_{-z^2 z^2} = \frac{1}{8}(\phi_1 \star \phi_1 - \phi_2 \star \phi_2) \quad \text{and} \quad F_{-z^1 z^2} = -\frac{1}{8}\phi_1 \star \phi_2, \quad (3.18b)$$

as well as

$$D_{z^1} \phi_1 - D_{z^2} \phi_2 = 0 \quad \text{and} \quad D_{z^2} \phi_1 + D_{z^1} \phi_2 = 0, \quad (3.19)$$

where we have used the definition

$$D_{z^a} \phi := \partial_{z^a} \phi + A_{+z^a} \star \phi - \phi \star A_{-z^a}. \quad (3.20)$$

Here $F_{z^a z^b}$ are the components of the curvature associated with $A_{z^a}$, i.e.,

$$F_{z^a z^b} = \partial_{z^a} A_{+z^b} - \partial_{z^b} A_{+z^a} + [A_{z^a}, A_{z^b}], \quad (3.21)$$

Analogously, by assuming that $\chi = \text{diag}(\chi_+, \chi_-)$ with $\chi_{\pm} \in \Omega^2(\mathbb{R}^4, i\mathbb{R})$ in (3.14), we obtain the perturbed equations

$$F_{+z^1 z^1} + F_{+z^2 z^2} + \chi_{+z^1 z^1} + \chi_{+z^2 z^2} = -\frac{1}{8}(\phi_1 \star \phi_1 - \phi_2 \star \phi_2), \quad (3.22a)$$

$$F_{+z^1 z^2} + \chi_{+z^1 z^2} = \frac{1}{8}\phi_1 \star \phi_2, \quad (3.22b)$$

$$F_{-z^1 z^2} + F_{-z^2 z^2} + \chi_{-z^1 z^2} + \chi_{-z^2 z^2} = \frac{1}{8}(\phi_1 \star \phi_1 - \phi_2 \star \phi_2), \quad (3.22c)$$

$$F_{-z^1 z^2} + \chi_{-z^1 z^2} = -\frac{1}{8}\phi_2 \star \phi_1. \quad (3.22d)$$

We consider the equations (3.18), (3.19) and (3.22), (3.19) as a noncommutative extension of the unperturbed and perturbed Abelian SW equations, respectively. Since there are two gauge potentials in the equations, we call them the noncommutative unperturbed and perturbed $U_+(1) \times U_-(1)$ SW equations.

It remains to find out what kind of gauge transformations leave the $U_+(1) \times U_-(1)$ SW equations invariant. It is obvious from the explicit form of these equations that $\phi_1$ and $\phi_2$ are in the bifundamental representation of $U_+(1) \times U_-(1)$. Hence, the equations (3.18), (3.19) and (3.22) are invariant under gauge transformations of the form

$$A_{\pm} \mapsto g_{\pm}^\dagger \star A_{\pm} \star g_{\pm} + g_{\pm}^\dagger \star d g_{\pm}, \quad \chi_{\pm} \mapsto g_{\pm}^\dagger \star \chi_{\pm} \star g_{\pm} \quad \text{and} \quad \Phi \mapsto g_{\pm}^\dagger \star \Phi \star g_{\mp}. \quad (3.23)$$
where \( g_\pm \in C^\infty(\mathbb{R}^4, U_\pm(1)) \) and \( \Phi = (\phi_1, \phi_2) \).

In the commutative limit the covariant derivative (3.20) turns into

\[
D_{za} \phi = \partial_{za} \phi + (A_{za} + A_{-za}) \phi = \partial_{za} \phi + A_{za} \phi,
\]

i.e., the gauge potential \( B \) disappears from the equations (3.19). In other words, one copy of \( U(1) \) decouples from \( U(2) \) and the matter field \( \Phi \) interacts only with the \( SU(2) \) part. Hence, in the commutative case \( \Phi \) is charged with respect to the diagonal \( U(1) \) subgroup of \( U_+(1) \times U_-(1) \) corresponding to the gauge potential \( A = A_+ - A_- \). Furthermore, the commutator in the expression for the (Abelian) curvature vanishes and hence as a corollary we have \( F_{\pm z^a z^b} = \frac{1}{2} (F_{B z^a z^b} \pm F_{A z^a z^b}) \).

In the commutative limit one may choose the perturbations \( \chi_{\pm \pm} \) so that

\[
\chi_{\pm z^a z^b} = -\frac{1}{2} F_{B z^a z^b} \pm \frac{1}{2} \chi_{z^a z^b} \quad \text{and} \quad \chi_{\pm z^1 z^2} = -\frac{1}{2} F_{B z^1 z^2} \pm \frac{1}{2} \chi_{z^1 z^2},
\]

where \( \chi \in \Omega^2(\mathbb{R}^4, u(1)) \) is some other perturbation. Then from equations (3.22) we obtain

\[
F_{A z^1 z^1} + F_{A z^2 z^2} + \chi_{z^1 z^1} + \chi_{z^2 z^2} = -\frac{1}{4} (\phi_1 \bar{\phi}_1 - \phi_2 \bar{\phi}_2) \quad \text{and} \quad F_{A z^1 z^2} + \chi_{z^1 z^2} = \frac{1}{4} \phi_1 \bar{\phi}_2.
\]

Thus, we recover the perturbed \( \text{SW} \) equations (2.8b) and (2.11) (written in complex coordinates).

Of course, the choice \( \chi \equiv 0 \) corresponds to the unperturbed equations.

**Remark.** Consider the unperturbed equations (3.18) and (3.19). In the commutative limit we arrive at the standard unperturbed \( \text{SW} \) equations for configurations \( (A, \Phi) \) plus the Abelian ASDYM equations for \( B \), i.e.,

\[
F_{B z^1 z^1} + F_{B z^2 z^2} = 0 \quad \text{and} \quad F_{B z^1 z^2} = 0.
\]

Taking the trivial solution \( B = 0 \) (recall that there are no Abelian instantons on \( \mathbb{R}^4 \)) we remain with the standard unperturbed \( \text{SW} \) equations. More generally, any pure gauge configuration for \( B \) will do the same. The noncommutative version of the latter statement is, however, nontrivial. If we choose \( B_{za} \) in (3.10) of the form

\[
B_{za} = \frac{1}{2} (b^\dagger \star \partial_{za} b - b \star \partial_{za} b^\dagger),
\]

with \( b \in C^\infty(\mathbb{R}^4, U(1)) \), then it will correspond in the commutative limit to a pure gauge configuration but, of course, it is not pure gauge in the noncommutative case. Only in the commutative limit the curvature \( F_B \) disappears and we arrive at the unperturbed \( \text{SW} \) equations (2.8).

**Noncommutative \( U_\pm(1) \) SW monopole equations.** In equations (3.18) - (3.23) the field \( \Phi \) is regarded as an element of a \( \mathbb{R}^4_g \)-bimodule transforming in the bi-fundamental representation of the gauge group \( U_+(1) \times U_-(1) \). However, in the noncommutative setup the matter field \( \Phi \) can also be thought of either as an element of a right \( \mathbb{R}^4_g \)-module (the \( U_+(1) \) case) or as an element of a left-module (the \( U_-(1) \) case).

\[\]
\( \mathbb{R}^4 \) -module (the \( U_+ (1) \) case). These two cases can easily be read off the equations and respectively. Namely, consider the equations

\[
F_{z_1 z_1} + F_{z_2 z_2} + \chi_{z_2 z_1} + \chi_{z_1 z_2} = -\frac{1}{4} (\phi_1 \star \phi_1 - \phi_2 \star \phi_2) \quad \text{and} \quad F_{z_1 z_2} + \chi_{z_1 z_2} = \frac{1}{4} \phi_1 \star \phi_2, \tag{3.29a}
\]
as well as

\[
D_{z_1} \phi_1 - D_{z_2} \phi_2 = 0 \quad \text{and} \quad D_{z_2} \phi_1 + D_{z_1} \phi_2 = 0, \tag{3.29b}
\]
where \( \chi \in \Omega^2 (\mathbb{R}^4, i \mathbb{R}) \) and the (right) covariant derivative reads

\[
D_{z_1} \phi = \partial_{z_1} \phi + A_{z_1} \star \phi. \tag{3.29c}
\]

Note that the curvature \( F_A \) is now computed from \( A, \) i.e., \( F_{\mu \nu} = \partial \mu A_\nu - \partial \nu A_\mu + [A_\mu, A_\nu]. \)

Similarly, we may introduce the equations\(^6\)

\[
F_{z_1 z_1} + F_{z_2 z_2} + \chi_{z_2 z_1} + \chi_{z_1 z_2} = \frac{1}{4} (\phi_1 \star \phi_1 - \phi_2 \star \phi_2) \quad \text{and} \quad F_{z_1 z_2} + \chi_{z_1 z_2} = -\frac{1}{4} \phi_1 \star \phi_2, \tag{3.30a}
\]
and

\[
D_{z_1} \phi_1 - D_{z_2} \phi_2 = 0 \quad \text{and} \quad D_{z_2} \phi_1 + D_{z_1} \phi_2 = 0, \tag{3.30b}
\]
where \( \chi \in \Omega^2 (\mathbb{R}^4, i \mathbb{R}) \) and

\[
D_{z_1} \phi = \partial_{z_1} \phi - \phi \star B_{z_1}. \tag{3.30c}
\]
is the (left) derivative. Now the curvature \( F_B \) is associated with the gauge potential \( B, \) i.e., \( F_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \)

We shall call \((3.29)\) and \((3.30)\) the perturbed noncommutative \( U_+ (1) \) and \( U_-(1) \) SW equations. Obviously, the unperturbed equations appear for \( \chi \equiv 0. \) Note that the systems \((3.29)\) and \((3.30)\) are totally equivalent and the only difference between them is an artifact of noncommutativity. The commutative limits of both cases are, of course, identical and produce \((2.1)\). Moreover, in the commutative case the gauge transformations \((3.28)\) reduce to the standard ones, i.e., one may choose the identity either for \( g_- \) or for \( g_+ \).

### 3.3 Operator form of the Abelian SW monopole equations

**Weyl transform.** Due to the nonlocal nature of the star product, explicit calculations might be quite tedious. It is therefore convenient to pass over to the operator formalism via the Weyl ordering \( W \) given by

\[
W : \hat{f}(k) \mapsto \hat{f}(\hat{x}) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \hat{f}(k) e^{ik\cdot\hat{x}} \, dk, \quad (3.31a)
\]

\[
W^{-1} : \hat{f}(\hat{x}) \mapsto \hat{f}(k) = |\text{Pf}(2\pi i\theta)| \text{Tr} \{ e^{-ik\cdot\hat{x}} \hat{f}(\hat{x}) \}, \quad (3.31b)
\]

\(^5\)Formally, these equations can be obtained from \((3.19) - (3.22)\) by choosing \( B_{\pm} = A_{\pm} \) (i.e., \( A_{\pm} = 0 \)), taking \( \chi \) such that \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \) and \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), where \( \chi \in \Omega^2 (\mathbb{R}^4, i \mathbb{R}) \), and rescaling \( \Phi \mapsto \sqrt{2} \Phi. \)

\(^6\)These equations can be formally obtained from \((3.19) - (3.22)\) by choosing \( B_{\pm} = - A_{\pm} \) (i.e., \( A_{\pm} = 0 \)) and putting \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \) and \( \chi_{\pm, \pm} = \chi_{\pm, \pm} \), where \( \chi \in \Omega^2 (\mathbb{R}^4, i \mathbb{R}) \). Again we should rescale \( \Phi \mapsto \sqrt{2} \Phi. \)
where $\tilde{f}(k)$ stands for the Fourier transform of $f(x) \in \mathcal{S}(\mathbb{R}^{2n})$,

$$f(x) \mapsto \tilde{f}(k) = \int_{\mathbb{R}^{2n}} e^{2\pi i k \cdot x} f(x) e^{-ikx}.$$  

(3.32)

Here $\mathcal{S}(\mathbb{R}^{2n})$ is the Schwartz space of fast decreasing functions\(^7\) on $\mathbb{R}^{2n}$. ‘Tr’ denotes the trace in the operator representation of the noncommutative algebra and ‘Pf’ is the Pfaffian of $(2\pi \theta^{\alpha\beta})$. Also, in these equations $kx$ is a shorthand notation for $k_\alpha x^\alpha$. One can verify (see, e.g., [20]) that the following relations are true:

$$\mathcal{W} : f \star g \mapsto \hat{f}\hat{g} \quad \text{and} \quad \int_{\mathbb{R}^{2n}} e^{2\pi i k \cdot x} f = \text{Tr} \hat{f}.$$  

(3.33)

We may regard the coordinates $\hat{x}^\alpha$ as operators which act on some Fock space $\mathcal{H}$, specified in section [4] and satisfy the commutation relations $[\hat{x}^\alpha, \hat{x}^\beta] = i\theta^{\alpha\beta}$. With a proper choice of coordinates the parameters $\theta^{\alpha\beta}$ will have the canonical form (3.6). For the complex coordinates $\hat{z}^\alpha$, also considered as operators in $\mathcal{H}$, we then get

$$[\hat{z}^a, \hat{z}^b] = 0 \quad \text{and} \quad [\hat{z}^a, \hat{z}^\alpha] = \theta^{\alpha\bar{a}}, \quad \text{for} \quad a, b = 1, \ldots, n.$$  

(3.34)

A straightforward calculation shows that coordinate derivatives are now inner derivations of this algebra, i.e.,

$$\hat{\partial}_{z^a} \hat{f} = \theta_{\bar{a}a} \hat{[\hat{z}^\alpha, \hat{f}]} \quad \text{and} \quad \hat{\partial}_{\hat{z}^a} \hat{f} = \theta_{\bar{a}a} \hat{[\hat{z}^\alpha, \hat{f}]}.$$  

(3.35)

In the operator formulation, the perturbed noncommutative $U_+(1) \times U_-(1)$ SW equations (3.19) and (3.22) retain their form,

\[\begin{align*}
\hat{F}_{+z_1z_2} &+ \hat{F}_{+z_2z_1} + \hat{\chi}_{+z_1z_2} + \hat{\chi}_{+z_2z_1} = -\frac{1}{8} (\hat{\phi}_1 \hat{\phi}_1 - \hat{\phi}_2 \hat{\phi}_2), \\
\hat{F}_{-z_1z_2} &+ \hat{\chi}_{+z_1z_2} + \hat{\chi}_{+z_2z_1} = \frac{1}{8} \hat{\phi}_1 \hat{\phi}_2, \\
\hat{F}_{-z_1z_2} &+ \hat{\chi}_{+z_1z_2} + \hat{\chi}_{+z_2z_1} = \frac{1}{8} (\hat{\phi}_1 \hat{\phi}_1 - \hat{\phi}_2 \hat{\phi}_2), \\
\hat{F}_{-z_1z_2} &+ \hat{\chi}_{-z_1z_2} = -\frac{1}{8} \hat{\phi}_2 \hat{\phi}_1,
\end{align*}\]

(3.36)

and

\[\begin{align*}
\hat{D}_{z_1} \hat{\phi}_1 - \hat{D}_{z_2} \hat{\phi}_2 = 0 \quad \text{and} \quad \hat{D}_{z_2} \hat{\phi}_1 + \hat{D}_{z_1} \hat{\phi}_2 = 0,
\end{align*}\]

(3.36e)

where

$$\hat{D}_{z^a} \hat{\phi} = \hat{\partial}_{z^a} \hat{\phi} + \hat{A}_{+z^a} \hat{\phi} - \hat{\phi} \hat{A}_{-z^a}.$$  

(3.37)

In order to simplify our notation, from now on we omit the hats over the operators.

**$U_+(1) \times U_-(1)$ SW action functional.** Having introduced the $U_+(1) \times U_-(1)$ noncommutative extension of the SW equations (3.36), we shall define an appropriate action functional. For this purpose we switch back to real coordinates.

---

\(^7\)However, in later considerations we shall make suitable choices for $f$ which are weaker.
Let \( \Phi = (\phi_1, \phi_2) \) and \( \Phi^* := (\phi_1^\dagger, -\phi_2^\dagger) \). Then the noncommutative deformation of the action functional (3.10) will have the form

\[
SW_\chi(A_+, A_-, \Phi; \theta) = \frac{1}{2} |\text{Pf}(2\pi \theta)| \text{Tr} \left\{ |D_{A_+, A_-} \Phi|^2 + |(D_{A_+, A_-} \Phi)^\dagger|^2 + 8 |F_{A_+}^+ + \chi_+^+ - \sigma^+(\Phi \otimes \Phi^\dagger)|0|^2 + 8 |F_{A_-}^+ + \chi_-^+ - \sigma^+(\Phi^* \otimes (\Phi^*)^\dagger)|0|^2 \right\},
\]

(3.38)

where

\[
|\psi|^2 := |\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2| \quad \text{and} \quad |\psi|^2 := |\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2|,
\]

(3.39)

for any \( \psi = (\psi_1, \psi_2) \). Here, \( D_{A_+, A_-} \) denotes the Dirac operator depending on the two gauge potentials \( A_+ \) and \( A_- \), i.e.,

\[
D_{A_+, A_-} \Phi = \begin{pmatrix} -D_4 + iD_3 & D_2 + iD_1 \\ -D_2 + iD_1 & -D_4 - iD_3 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},
\]

(3.40)

where the covariant derivatives \( D_\mu \) are given by (3.37). Note that the prefactors in (3.38) are adjusted in such a way that we recover (2.10) in the commutative limit. It is not difficult to see that the (perturbed) \( U_+(1) \times U_-(1) \) SW equations following from (3.38) are given by the equations

\[
D_{A_+, A_-} \Phi = 0, \quad F_{A_+}^+ + \chi_+^+ = \sigma^+(\Phi \otimes \Phi^\dagger)|0| \quad \text{and} \quad F_{A_-}^+ + \chi_-^+ = \sigma^+(\Phi^* \otimes (\Phi^*)^\dagger)|0|,
\]

(3.41)

whose solutions minimize the action functional (3.38). In components these equations coincide with (3.36).

Assuming that \( \phi_{1,2} \) and \( F_{\pm \mu \nu} \) are of proper trace-class, e.g., \( |\text{Tr} \phi_{1,2}| < \infty \) and \( |\text{Tr} F_{\pm \mu \nu}| < \infty \), we can show that

\[
SW_\chi(A_+, A_-, \Phi; \theta) = E_\chi(A_+, A_-, \Phi; \theta) + 16\pi^2 K_\chi - |\text{Pf}(2\pi \theta)| \text{Tr} \mathcal{T},
\]

(3.42)

where the functionals \( E_\chi \) and \( K_\chi \) are given by

\[
E_\chi(A_+, A_-, \Phi; \theta) = |\text{Pf}(2\pi \theta)| \text{Tr} \left\{ 2 |F_{A_+}|^2 + 2 |F_{A_-}|^2 + \frac{1}{2} |D_{A_+, A_-} \Phi|^2 + \frac{1}{2} |(D_{A_+, A_-} \Phi)^\dagger|^2 + 4 |\chi_+^+ - \sigma^+(\Phi \otimes \Phi^\dagger)|0|^2 + 4 |\chi_-^+ - \sigma^+(\Phi^* \otimes (\Phi^*)^\dagger)|0|^2 \right\},
\]

(3.43)

and

\[
K_\chi = -\frac{1}{8\pi^2} |\text{Pf}(2\pi \theta)| \text{Tr} \sum_{i=\pm} \{ F_{i \mu \nu}^+, \chi_{i \mu \nu}^+ \}.
\]

(3.44)

The topological term \( \mathcal{T} \) reads

\[
\mathcal{T} = \sum_{i=\pm} (F_{i \mu \nu}^* F_{i \mu \nu} + \nabla_{i \mu} \mathcal{F}_{i \mu}).
\]

(3.45a)

Here

\[
\nabla_{\pm \mu} := \partial_{\mu} \cdot + [A_{\pm \mu}, \cdot]
\]

(3.45b)
‘*’ denotes the Hodge operator and \( \{A, B\} := AB + BA \). The currents \( J_{\pm \mu} \) depend in a particular fashion on the fields \( \phi_1 \) and \( \phi_2 \), their derivatives and on the gauge potentials \( A_{\pm \mu} \). The explicit derivation of (3.42) and the expressions for the currents \( J_{\pm \mu} \) are given in the Appendix. Note that similarly to the commutative case the functionals \( E_\chi + 16\pi^2 K_\chi \) and \( SW_\chi \) yield the same equations of motion. The integration of (3.45a) yields the topological charge

\[
Q = -\frac{1}{8\pi^2} |\text{Pf}(2\pi\theta)| \text{Tr } T,
\]

for the considered field configuration \((A_+, A_-, \Phi) \) on \( \mathbb{R}^4_\theta \).

**Generalized coupled vortex equations.** Let us put one component of \( \Phi \) to zero, e.g. consider the case \( ^4\Phi = (\phi_1, \phi_2) =: (\phi, 0) \). Moreover, we choose

\[
\chi_{\pm z^1 z^2} = 0 \quad \text{and} \quad \chi_{\pm z^1 \bar{z}^1} + \chi_{\pm z^2 \bar{z}^2} = \mp \frac{1}{8} v_\pm,
\]

where \( v_\pm \) are some Hermitian operators acting on \( \mathcal{H} \). Then the energy functional (3.43) turns into

\[
E_\chi(A_+, A_-, \Phi; \theta) = |\text{Pf}(2\pi\theta)| \text{Tr } \left\{ F_{+\mu\nu} F_{+\mu\nu}^\dagger + F_{-\mu\nu} F_{-\mu\nu}^\dagger + \frac{1}{2} D_\mu \phi (D_\mu \phi)^\dagger + \frac{1}{2} (D_\mu \phi)^\dagger D_\mu \phi + \frac{1}{8} (v_+ - \phi \phi^\dagger)^2 + \frac{1}{8} (v_- - \phi^\dagger \phi)^2 \right\},
\]

and \( K_\chi \) is given by

\[
K_\chi = -\frac{i}{32\pi^2} |\text{Pf}(2\pi\theta)| \text{Tr } \left\{ \{F_{+12} + F_{+34}, v_+\} - \{F_{-12} + F_{-34}, v_-\} \right\}.
\]

Also the currents \( J_{\pm \mu} \) have a fairly simple form\(^{10}\),

\[
J_{+\mu} = \frac{i}{4} (\epsilon_{\mu\nu12} + \epsilon_{\mu\nu34}) (\phi (D_\nu \phi)^\dagger - (D_\nu \phi) \phi^\dagger),
\]

\[
J_{-\mu} = -\frac{i}{4} (\epsilon_{\mu\nu12} + \epsilon_{\mu\nu34}) (\phi^\dagger (D_\nu \phi) - (D_\nu \phi)^\dagger \phi),
\]

where \( \epsilon_{\mu\nu\lambda\sigma} \) is the Levi-Civita symbol with \( \epsilon_{1234} = 1 \).

For our choices of \( \Phi \) and \( \chi_\pm \) the perturbed \( U_+(1) \times U_-(1) \) SW equations in complex coordinates read

\[
F_{+z^1 z^1} + F_{+z^2 z^2} = \frac{1}{8} (v_+ - \phi \phi^\dagger) \quad \text{and} \quad F_{+z^1 \bar{z}^2} = 0,
\]

\[
F_{-z^1 z^1} + F_{-z^2 z^2} = -\frac{1}{8} (v_- - \phi^\dagger \phi) \quad \text{and} \quad F_{-z^1 \bar{z}^2} = 0,
\]

\[
D_{\bar{z}^1} \phi = 0 \quad \text{and} \quad D_{\bar{z}^2} \phi = 0.
\]

In the commutative case these equations were considered, e.g., in \([9]\). \( ^9 \)Note that in the case of Kähler manifolds (\( \mathbb{R}^4 \) is trivially Kähler) the field \( \phi \) can be regarded as a scalar \([20, 21]\). \( ^{10} \)Cf. the Appendix.
$U_{\pm}(1)$ SW action functionals. Having introduced the $U_{\pm}(1) \times U_{\pm}(1)$ SW functionals, we are now interested in proper functionals for the $U_{\pm}(1)$ cases \(3.29\) and \(3.30\). Let us first discuss the perturbed $U_{\pm}(1)$ SW equations. In this case the SW action functional takes the following form:

$$SW_{\chi}(A, \Phi; \theta) = |\text{Pf}(2\pi \theta)| \text{Tr}\{|D_{\mu}A|^2 + 2|F_{A\mu}^+ + \chi^+ - \sigma^+(\Phi \otimes \Phi^\dagger)\phi^0|^2\}.$$  

(3.51)

Note that now the Dirac operator depends only on $A$ and the covariant derivatives are given by \(3.29\). Also here the prefactors have been chosen such that the correct commutative limit will be obtained. As before, the functional $SW_{\chi}$ may be rewritten as

$$SW_{\chi}(A, \Phi; \theta) = E_{\chi}(A, \Phi; \theta) + 16\pi^2 K_{\chi} - |\text{Pf}(2\pi \theta)| \text{Tr}\, T,$$  

(3.52)

where $E_{\chi}$ turns out to be

$$E_{\chi}(A, \Phi; \theta) = |\text{Pf}(2\pi \theta)| \text{Tr}\left\{|F_A|^2 + |D_{\mu}A|^2 + 2|\chi^+ - \sigma^+(\Phi \otimes \Phi^\dagger)\phi^0|^2\right\}.$$  

(3.53)

The Chern-Simons term $K_{\chi}$ reads as

$$K_{\chi} = -\frac{1}{16\pi^2} |\text{Pf}(2\pi \theta)| \text{Tr}\left\{F^+_{\mu\nu}, \chi^\mu_{\nu}\right\},$$  

(3.54)

and the topological term $T$ is given by

$$T = \frac{i}{4}\{F_{\mu\nu}, *F_{\mu\nu}\} + \nabla_{\mu}J_{\mu},$$  

with

$$\nabla_{\mu}J_{\mu} = \partial_{\mu}J_{\mu} + [A_{\mu}, J_{\mu}],$$  

(3.55)

implying the charge

$$Q = -\frac{1}{8\pi^2} |\text{Pf}(2\pi \theta)| \text{Tr}\, T.$$  

(3.56)

Again all equations simplify essentially if one chooses $^t\Phi = (\phi_1, \phi_2) = (\phi, 0)$, $\chi_{z_1z_2} = 0$ and $\chi_{z_1\bar{z}_1} + \chi_{z_2\bar{z}_2} = -\frac{1}{2}v$, where $v$ is some Hermitian operator. Then $E_{\chi}$ and $K_{\chi}$ are given by

$$E_{\chi}(A, \Phi; \theta) = |\text{Pf}(2\pi \theta)| \text{Tr}\left\{\frac{1}{2}F_{\mu\nu}F_{\mu\nu}^{\dagger} + D_{\mu}\phi(D_{\mu}\phi)^{\dagger} + \frac{1}{4}(v - \phi\phi^{\dagger})^2\right\}$$  

(3.57a)

and

$$K_{\chi} = -\frac{i}{32\pi^2} |\text{Pf}(2\pi \theta)| \text{Tr}\{F_{12} + F_{34}, v\}.$$  

(3.57b)

The current $J_{\mu}$ reduces in this case to

$$J_{\mu} = \frac{i}{2}(\epsilon_{\mu12} + \epsilon_{\mu34})(\phi(D_{\nu}\phi)^{\dagger} - (D_{\nu}\phi)^{\dagger}).$$  

(3.58)

Finally, the perturbed $U_{\pm}(1)$ SW equations read

$$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = \frac{i}{4}(v - \phi\phi^{\dagger})$$  

and

$$F_{\bar{z}_1z_1} = 0,$$  

(3.59a)

$$D_{\bar{z}_1}\phi = 0$$  

and

$$D_{\bar{z}_2}\phi = 0.$$  

(3.59b)

In the commutative case these $v$-vortex equations in four dimensions were considered, e.g., in [9]. Note that in a similar manner one can write down the functionals corresponding to the $U_{-}(1)$ case \(3.30\). Since they look essentially the same as the above-introduced $U_{\pm}(1)$ functionals we refrain from writing down their explicit form.
4 Particular solutions

4.1 Operator realization

The form of the Heisenberg algebra type commutation relations \( [a, c] = 1 \) suggests that the algebra \( \mathbb{R}^d \) may be represented by a pair of harmonic oscillators in the two-oscillator Fock space \( \mathcal{H} \). We introduce, as usual, annihilation and creation operators \( \{ c_a, c_a^\dagger \} \), satisfying \( [c_a, c_b^\dagger] = 1 \). They act on \( \mathcal{H} \) and are defined by the relations

\[
\begin{align*}
    c_1 |n_1, n_2\rangle &= \sqrt{n_1} |n_1-1, n_2\rangle \quad \text{and} \quad c_1^\dagger |n_1, n_2\rangle = \sqrt{n_1+1} |n_1+1, n_2\rangle, \\
    c_2 |n_1, n_2\rangle &= \sqrt{n_2} |n_1, n_2-1\rangle \quad \text{and} \quad c_2^\dagger |n_1, n_2\rangle = \sqrt{n_2+1} |n_1, n_2+1\rangle,
\end{align*}
\]

where \( \{|n_1, n_2\rangle | n_1, n_2 \in \mathbb{N}_0 \} \) form an orthonormal basis in \( \mathcal{H} \). The commutation relations imply that the operators \( \{ c_a, c_a^\dagger \} \) have the form

\[
\begin{align*}
    c_1 &= \frac{\hat{z}^1}{\sqrt{\theta^{11}}} \frac{1 + \text{sgn}(\theta^{11})}{2} + \frac{\hat{z}^1}{\sqrt{\theta^{11}}} \frac{1 - \text{sgn}(\theta^{11})}{2} \quad \text{and} \quad c_1^\dagger = \frac{\hat{z}^1}{\sqrt{\theta^{11}}} \frac{1 + \text{sgn}(\theta^{11})}{2} + \frac{\hat{z}^1}{\sqrt{\theta^{11}}} \frac{1 - \text{sgn}(\theta^{11})}{2}, \\
    c_2 &= \frac{\hat{z}^2}{\sqrt{\theta^{22}}} \frac{1 + \text{sgn}(\theta^{22})}{2} + \frac{\hat{z}^2}{\sqrt{\theta^{22}}} \frac{1 - \text{sgn}(\theta^{22})}{2} \quad \text{and} \quad c_2^\dagger = \frac{\hat{z}^2}{\sqrt{\theta^{22}}} \frac{1 + \text{sgn}(\theta^{22})}{2} + \frac{\hat{z}^2}{\sqrt{\theta^{22}}} \frac{1 - \text{sgn}(\theta^{22})}{2}.
\end{align*}
\]

We introduce so-called shift operators \( S^{(a)} \) acting on the Fock spaces \( \mathcal{H}_a \). They are partially isometric operators sending the Fock space \( \mathcal{H}_a \) to its subspace \( (1^{(a)} - P_0^{(a)})\mathcal{H}_a \), where we denote by \( P_0^{(a)} = |0\rangle_a \langle 0|_a \) the orthogonal projector onto the ground state of \( \mathcal{H}_a \) and \( 1^{(a)} - P_0^{(a)} \) is the complement projector. Then

\[
S^{(a)} : \mathcal{H}_a \rightarrow (1^{(a)} - P_0^{(a)})\mathcal{H}_a, \quad \text{with} \quad S^{(a)\dagger} S^{(a)} = 1^{(a)} \quad \text{and} \quad S^{(a)} S^{(a)\dagger} = 1^{(a)} - P_0^{(a)}.
\]

The operator \( S^{(a)} \) may be given by the explicit formula

\[
S^{(a)} = \sum_{n \geq 0} |n+1\rangle_a \langle n|_a.
\]

We will sometimes drop the index ‘\( a \)’ on the state \( |n\rangle_a \) in the following if the meaning is clear from the context.

The next step is to construct a shift operator \( S \) on \( \mathcal{H} \) such that

\[
S : \mathcal{H} \rightarrow (1 - P_0)\mathcal{H}, \quad \text{with} \quad P_0 = |0, 0\rangle \langle 0, 0|,
\]

and \( S^\dagger S = 1 \) and \( SS^\dagger = 1 - P_0 \). A naive idea to take simply the tensor product \( S^{(1)} \otimes S^{(2)} \) does not work. One possible realization of the required operator \( S \) is given by the formula

\[
S = 1 + \sum_{n \geq 0} (|n+1\rangle \langle n| - |n\rangle \langle n|) \otimes P_0^{(2)} = 1 + (1^{(1)} - P_0^{(1)}) \otimes P_0^{(2)}.
\]

but there are also other realizations (see, e.g., \[23\]).

For later convenience we introduce the operators

\[
X_{\pm a} := A_{\pm a} + \theta_{aa} z^a \quad \text{and} \quad X_{\pm a} := A_{\pm a} + \theta_{aa} z^a, \quad \text{for} \quad a = 1, 2.
\]
Then a short calculation of the YM curvature yields

\[ F_{\pm\pm22} = [X_{\pm\pm2}, X_{\pm\pm2}], \quad F_{\pm\pm22} = [X_{\pm\pm2}, X_{\pm\pm2}], \quad F_{\pm\pm22} = [X_{\pm\pm2}, X_{\pm\pm2}], \]  \hspace{1cm} (4.8a)

\[ F_{\pm\pm22} = [X_{\pm\pm2}, X_{\pm\pm2}] + \theta_{11}, \quad F_{\pm\pm22} = [X_{\pm\pm2}, X_{\pm\pm2}] + \theta_{22}, \]  \hspace{1cm} (4.8b)

and the covariant derivatives become

\[ D_{a} \phi = X_{+a} \phi - \phi X_{-a} \quad \text{and} \quad D_{\bar{a}} \phi = X_{+\bar{a}} \phi - \phi X_{-\bar{a}}. \]  \hspace{1cm} (4.9)

4.2 Solutions to the perturbed SW equations

\[ U_+(1) \times U_-(1) \text{ SW monopole equations.} \]  

Let us consider the equations (3.50) rewritten in operator form. For the operators (4.7) we take (cf., e.g., [29, 30])

\[ X_{\pm\pm a} = \theta_{a\bar{a}} S^N \bar{z}^\alpha (S^\dagger)^N + \sum_{n=0}^{N-1} \lambda_{a,n} |n\rangle \langle n| \otimes P^{(2)}_0, \]  \hspace{1cm} (4.10)

where the shift operator \( S \) is given by (4.6), \( \lambda_{a,n} \in \mathbb{C} \) and \( N \in \mathbb{N} \). Then the commutator [\( X_{\pm\pm a}, X_{\pm\pm \bar{a}} \)] is readily computed to be

\[ [X_{\pm\pm a}, X_{\pm\pm \bar{a}}] = -\theta_{a\bar{a}} (1 - \mathcal{P}_N), \]  \hspace{1cm} (4.11)

where \( \mathcal{P}_N \) is given by

\[ \mathcal{P}_N := \sum_{n=0}^{N-1} |n\rangle \langle n| \otimes P^{(2)}_0. \]  \hspace{1cm} (4.12)

It easy to see that the second equations of (3.50a) and (3.50b) are trivially satisfied. Choosing \( \nu_- \equiv 0 \) and

\[ \phi = \sqrt{8(\theta_{11} + \theta_{22})} \mathcal{P}_N, \]  \hspace{1cm} (4.13)

we can solve (3.50a) and the first equation of (3.50b) consistently, while the first equation of (3.50a) implies that

\[ \nu_+ = 16(\theta_{11} + \theta_{22}) \mathcal{P}_N. \]  \hspace{1cm} (4.14)

Note that the moduli \( \lambda_{a,n} \) in (4.10) can be interpreted as position parameters (see, e.g., [29, 31, 32]).

Obviously, the components (1.8) of the curvature and the field \( \phi \) are of proper trace-class. Moreover, it can be easily checked that \( \phi \) is covariantly constant, i.e., along with \( D_{\bar{a}} \phi = 0 \), required by the SW equations, we also have \( D_{a} \phi = 0 \). This means that the currents \( J_{\pm \mu} \) (4.9) vanish identically. Thus, it is straightforward to evaluate the topological charge (3.46) for this configuration. What we find for the topological term (3.45a) is

\[ \mathcal{T} = -8 \frac{1}{\theta_{12} \theta_{34}} \mathcal{P}_N. \]  \hspace{1cm} (4.15)
Using $|\text{Pf}(2\pi \theta)| = 4\pi^2 |\theta^{12} \theta^{34}|$, we get immediately a charge\textsuperscript{11},
\[
Q = 4\epsilon_1 \epsilon_2 N, \quad \text{with} \quad \epsilon_1 := \frac{|\theta^{12}|}{\theta^{12}} \quad \text{and} \quad \epsilon_2 := \frac{|\theta^{34}|}{\theta^{34}}.
\] (4.16)

Note that $K_\chi$ given by (3.48b) is also finite. Therefore, the considered field configuration has finite energy
\[
E_\chi = 32\pi^2 f(\theta)N \quad \text{with} \quad f(\theta) := |\theta^{12} \theta^{34}| \left[ \left( \frac{1}{\theta^{12}} \right)^2 + \frac{1}{\theta^{12} \theta^{34}} + \left( \frac{1}{\theta^{34}} \right)^2 \right].
\] (4.17)

Let us now consider a slight generalization of the ansatz (4.10). We take first
\[
X_{+z^a} = \theta_{a\bar{a}} S_N^z \bar{z}^a (S^\dagger)^N_+ + \sum_{n=0}^{N-1} \lambda_{a,n} |n\rangle \langle n| \otimes P_0^{(2)},
\] (4.18a)

and find that $[X_{+z^a}, X_{+z^b}] = -\theta_{a\bar{a}} (1 - P_{N_+})$. Second, choosing
\[
\phi = \sqrt{8(\theta_{11} + \theta_{22})} S^N_+ (S^\dagger)^N_- \quad \text{(4.18b)}
\]

and the perturbations $v_{\pm}$ such that
\[
v_+ = 8(\theta_{11} + \theta_{22}) \quad \text{and} \quad v_- = 8(\theta_{11} + \theta_{22})(1 - 2P_{N_-}),
\] (4.19)

one can easily show that our equations are solved consistently. Again, all our operators are of proper trace-class for $N_+ \neq N_-$. In the case of $N_+ = N_-$ one encounters a slight subtlety since the field $\phi$ is not of trace-class contrary to our assumption. However, potentially dangerous terms, like $\phi F_{-12} \phi^\dagger$ for instance\textsuperscript{12}, which occurred in (3.43), are obviously zero when $N_+ = N_-$. Moreover, the field $\phi$ is covariantly constant, as one can readily check. Therefore, the currents $J_{\pm \mu}$ (3.49) are identically zero. The topological term (3.45a) for this configuration thus reads as
\[
T = -4 \frac{1}{\theta^{12} \theta^{34}} (P_{N_+} + P_{N_-}),
\] (4.20)

which produces a topological charge\textsuperscript{11} $Q = 2\epsilon_1 \epsilon_2 (N_+ + N_-)$. The functional $E_\chi$ for this solution computes to
\[
E_\chi = 16\pi^2 f(\theta)(N_+ + N_-),
\] (4.21)

where $f(\theta)$ is given by (4.17).

\textbf{$U_+ (1)$ SW monopole equations.} Consider now the equations (3.59) and choose the ansatz
\[
X_{z^a} = \theta_{a\bar{a}} S_N^a \bar{z}^a (S^\dagger)^N + \sum_{n=0}^{N-1} \lambda_{a,n} |n\rangle \langle n| \otimes P_0^{(2)} \quad \text{and} \quad \phi = \gamma S^N,
\] (4.22)

with $\lambda_{a,n} \in \mathbb{C}$ and $\gamma \in \mathbb{R}$. Then the first equation of (3.59a) can be solved by $\theta_{11} + \theta_{22} = \frac{1}{\bar{\gamma}^2} = \frac{1}{v}$, while the other two equations are trivially satisfied. The only nonvanishing components of the

\textsuperscript{11}Note that the definition of the charge (3.46) differs by a factor of 2 in comparison with the standard one.

\textsuperscript{12}Cf. the Appendix.
curvature are \( F_{z\bar{a}} = \theta_{\bar{a}a} P_N \). Note that the expression for \( \phi \) is covariantly constant which implies the vanishing of the current (3.58). The charge (3.56) is equal to \( Q = \epsilon_1 \epsilon_2 N \) and the energy \( E_\chi \) is \( 8\pi^2 f(\theta) N \), where \( f(\theta) \) is given by (4.17).

Consider again the equations (3.59). Besides the shift type solution for \( \phi \) we can also find a projector type solution. Namely, we choose \( \phi = \gamma P_0 \) with \( \gamma \in \mathbb{C} \),

\[
v = (4(\theta_{1\bar{1}} + \theta_{2\bar{2}}) + \gamma^2) P_0, \tag{4.23}
\]

and the \( X_{z^a} \)'s as previously. With this choice the first two equations of (3.59) are trivially satisfied, while (3.59b) yields the condition \( P_0 z^a = 0 \). Hence, we have to set \( \theta_{a\bar{a}} < 0 \). Again, the only nonvanishing components of the curvature are \( F_{z\bar{a}} = \theta_{a\bar{a}} P_N \). Moreover, we have

\[
D_1 \phi = -iD_2 \phi = -\sqrt{\theta_{1\bar{1}}} \gamma |0, 0 \rangle \langle 1, 0| \quad \text{and} \quad D_3 \phi = -iD_4 \phi = -\sqrt{\theta_{2\bar{2}}} \gamma |0, 0 \rangle \langle 0, 1|, \tag{4.24}
\]

which imply the vanishing of current the \( J_\mu \) (3.58). The topological charge for this configuration is \( Q = \epsilon_1 \epsilon_2 \).

### 4.3 Solutions to the unperturbed SW equations

In this section we shall consider solutions to the unperturbed SW equations. We concentrate on the case of the unperturbed \( U_+(1) \) equations, i.e., (3.59) with \( v \equiv 0 \), as an illustrative example.

**\( U_+(1) \) SW monopole equations with \( \Phi \equiv 0 \).** In this case we obtain the noncommutative Abelian ASDYM equations. Solutions to these equations, i.e., noncommutative Abelian instantons, have been known for quite some time [16]. However, for the sake of completeness we briefly review their construction. Note that in the case \( \Phi \equiv 0 \) the current \( J_\mu \) is identically zero.

In terms of the operators (4.7) the Abelian ASDYM equations read

\[
[X_{z^1}, X_{\bar{z}^1}] + [X_{z^2}, X_{\bar{z}^2}] + \theta_{1\bar{1}} + \theta_{2\bar{2}} = 0 \quad \text{and} \quad [X_{z^1}, X_{z^2}] = 0. \tag{4.25}
\]

Again we consider an ansatz for \( X_{z^a} \) of the form (4.22). This ansatz yields solutions to the equations (4.25) if the deformation tensor \( \theta^{\mu\nu} \) is anti-self-dual which trivially follows from (4.22). The only nonvanishing components of the curvature are \( F_{z^a\bar{z}^\bar{a}} = \theta_{a\bar{a}} P_N \) implying that the charge (3.56) is equal to \( Q = -N \). The moduli \( \lambda_{a,n} \) entering the solution are position parameters showing the location of the noncommutative instantons [33] (see also [32] for a recent review).

It is also possible to consider the case of ASDYM equations on a self-dual background for which \( \theta_{1\bar{1}} = \theta_{2\bar{2}} =: \theta \). Let us assume that \( \theta > 0 \), which leads to the definitions \( c_1 = z^1 / \sqrt{\theta} \) and \( c_2 = z^2 / \sqrt{\theta} \). We choose the ansatz [34, 35]

\[
X_{z^1} = -\frac{1}{\theta} S^d \bar{z} f(N) S = -\frac{1}{\sqrt{\theta}} S^d c_1^\dagger f(N) S \quad \text{and} \quad X_{z^2} = -\frac{1}{\theta} S^d \bar{z}^2 f(N) S = -\frac{1}{\sqrt{\theta}} S^d c_2^\dagger f(N) S, \tag{4.26}
\]
and assume that $f(N)|0\rangle = f(0)|0\rangle = 0$, where $N := N_1 + N_2 := c_1^\dagger c_1 + c_2^\dagger c_2$. Then the equation $[X_{z\bar{z}}, X_{z\bar{z}}] = 0$ is trivially satisfied. A short calculation yields for $f(N)$ the result \[ f^2(N) = \frac{N(N+3)}{(N+1)(N+2)}. \] (4.27)

The nonvanishing components of the curvature in this case are

\[
\begin{align}
F_{z\bar{z}z} &= \frac{1}{\theta^2} S_t c_a^\dagger (f^2(N+1) - f^2(N)) c_b S, \\
F_{z\bar{z}z} &= \frac{1}{\theta^2} S_t [(N_a + 1)f^2(N) - N_a f^2(N-1) - 1] S.
\end{align}
\] (4.28a)

Using these expressions, we compute the topological charge \[ [34, 35] \] to be $-1$. A straightforward extension of the above ansatz allows one to construct multi-instanton configurations, as well \[ [34, 35] \].

**Fock spaces with indefinite norm.** Let us now discuss the case when $\Phi$ does not vanish identically. For that we relax the condition of positivity of the norm of the Fock spaces $H_{1,2}$ and assume instead that at least one of them has an indefinite norm (cf. \[ 36 \]). For instance, we can introduce a collection of the creation and annihilation operators $\{c_a, c_a^\dagger\}_{a=1,2}$, satisfying $[c_1, c_1^\dagger] = 1, [c_2, c_2^\dagger] = -1$, and defined by the relations

\[
\begin{align}
|n_1, n_2\rangle &= \sqrt{n_1} |n_1 - 1, n_2\rangle \quad \text{and} \quad c_1^\dagger |n_1, n_2\rangle = \sqrt{n_1 + 1} |n_1 + 1, n_2\rangle, \\
|n_1, n_2\rangle &= -\sqrt{n_2} |n_1, n_2 - 1\rangle \quad \text{and} \quad c_2^\dagger |n_1, n_2\rangle = \sqrt{n_2 + 1} |n_1, n_2 + 1\rangle,
\end{align}
\] (4.29a)

substituting (4.11). The normalization condition is modified to $\langle n_1, n_2 | m_1, m_2 \rangle = (-1)^{n_2} \delta_{n_1 m_1} \delta_{n_2 m_2}$. The identity operator is given by

\[
1 = \sum_{n_1, n_2} (-1)^{n_2} |n_1, n_2\rangle \langle n_1, n_2|.
\] (4.30)

Moreover, we have to redefine the relations between $\{c_a, c_a^\dagger\}_{a=1,2}$ and $\{z^a, z^\dagger a\}_{a=1,2}$ so that

\[
\begin{align}
c_1 := \frac{z_1^1}{\sqrt{\theta^{11}}} \frac{1 + \text{sgn}(\theta^{11})}{2} + \frac{z_1^\dagger}{\sqrt{\theta^{11}}} \frac{1 - \text{sgn}(\theta^{11})}{2} \quad \text{and} \quad c_1^\dagger := \frac{z_1^1}{\sqrt{\theta^{11}}} \frac{1 + \text{sgn}(\theta^{11})}{2} + \frac{z_1^\dagger}{\sqrt{\theta^{11}}} \frac{1 - \text{sgn}(\theta^{11})}{2}, \\
c_2 := \frac{z_2^2}{\sqrt{\theta^{22}}} \frac{1 + \text{sgn}(\theta^{22})}{2} + \frac{z_2^\dagger}{\sqrt{\theta^{22}}} \frac{1 - \text{sgn}(\theta^{22})}{2} \quad \text{and} \quad c_2^\dagger := \frac{z_2^2}{\sqrt{\theta^{22}}} \frac{1 + \text{sgn}(\theta^{22})}{2} + \frac{z_2^\dagger}{\sqrt{\theta^{22}}} \frac{1 - \text{sgn}(\theta^{22})}{2}.
\end{align}
\] (4.31a)

The definition of the shift operator $S^{(1)}$ remains the same, while for $S^{(2)}$ we obtain

\[
S^{(2)} : H_2 \rightarrow (1 - P_0^{(2)}) H_2 \quad \text{with} \quad S^{(2)} S^{(2)} = -1^{(2)} \quad \text{and} \quad S^{(2)} S^{(2)} = -(1^{(2)} - P_0^{(2)}),
\] (4.32)

with the representation

\[
S^{(2)} = \sum_{n \geq 0} (-1)^n |n + 1\rangle_2 \langle n_2|.
\] (4.33)

For an operator $S : H \rightarrow (1 - P_0) H$ on $H$ with the properties $SS^\dagger = 1 - P_0$ and $S^\dagger S = 1$, we can use the old expression (4.10), since the indefiniteness of the norm does not affect the verification of these properties. Indeed, $S$ contains only the identity $1$, given by (4.30), and the projector $P_0^{(2)}$ on $H_2$. 

19
$U_+(1)$ SW monopole equations with $^t\Phi = (\phi,0)$. We start from (3.34) with $\nu \equiv 0$. In this case the SW equations read as

$$F_{z_1z_2} + F_{z_2z_1} = -\frac{1}{4}\phi\phi^\dagger, \quad F_{z_1z_2} = 0 \quad \text{and} \quad Dz^a\phi = 0,$$  \hspace{1cm} (4.34)

where the covariant derivative $Dz^a$ acts as $Dz^a\phi = -\theta_{ab}\phi\bar{z}^a + X_z^a\phi$. Suppose that $\mathcal{H}_1$ is positive normed while $\mathcal{H}_2$ is endowed with an indefinite norm. Using the definition (4.10) and the ansatz $X_z^a = \theta_{ab}S^aS^b\dagger$, we find again (3.11). Assuming that $\phi = \gamma P_0$ with $\gamma \in \mathbb{R}$, we can solve the last equation of (3.34) if $P_0z^a = 0$, i.e., $z^a \sim c^\dagger_a$. For $a = 2$ this is satisfied because of our choice $\theta^{22} > 0$ and for $a = 1$ it implies $\theta^{11} < 0$. The second equation of (3.34) is again identically satisfied, while the first one yields the condition $\theta_{11} + \theta_{22} = -\frac{1}{4}\gamma^2$, which makes sense due to the different signs of $\theta_{11}$ and $\theta_{22}$. The nonvanishing components of the curvature are $F_{z^a\bar{z}^b} = \theta_{ab}P_0$ and one can readily verify that the current $J_\mu$ from (3.58) has no contributions to the topological charge (3.56). Moreover, the computation of the topological charge gives $-1$. Note that the introduction of an indefinite norm on $\mathcal{H}_2$ was needed for having solutions to the equation $\theta_{11} + \theta_{22} = -\frac{1}{4}\gamma^2$ on the components of $\theta_{\mu\nu}$.

$U_+(1)$ SW monopole equations and vortices. Finally we discuss a case which is slightly different from the cases described above, in the sense that we fix the explicit form of the solutions on a two-dimensional subspace of $\mathbb{R}_\theta^4$ from the very beginning. This eventually results in the $U_+(1)$ vortex equations on the complementary two-dimensional subspace.

Again, let $\mathcal{H}_1$ be positive normed and $\mathcal{H}_2$ endowed with an indefinite norm, such that $[z^2, \bar{z}^2] = \theta^{22} > 0$ and $[c_2, \bar{c}_2^\dagger] = -1$. Furthermore, we choose an ansatz for $X_{z^2}$ of the form $X_{z^2} = \theta_{22}\mathbf{1}^{(1)} \otimes S^{(2)}z^2S^{(2)}\dagger$. A short calculation yields the result $F_{z^2\bar{z}^2} = \theta_{22}\mathbf{1}^{(1)} \otimes P_0^{(2)}$. Assuming that $A_{z^1} = A_{z^1} \otimes P_0^{(2)}$, we obtain $F_{z^1z^1} = F_{\bar{z}^1\bar{z}^1} \otimes P_0^{(2)}$. Similarly, we take $\phi = \psi \otimes P_0^{(2)}$. Then the equation $Dz^a\phi = 0$ implies the condition $P_0^{(2)}z^2 = 0$, which is identically satisfied due to the relation $z^2 \sim c^\dagger_2$. Moreover, the equation $F_{z^1z^2} = 0$ is trivially satisfied. Using $\theta_{22} = -1/\theta^{22}$, we finally arrive at the equations\footnote{Note that these equations can also be obtained in the context of the perturbed SW equations if one chooses the perturbation proportional to $P_0^{(2)}$. Then there is no necessity to introduce an indefinite normed space.}

$$\left\{ F_{z^1z^1} - \left( \frac{1}{\theta^{22}} - \frac{1}{4}\psi\psi^\dagger \right) \right\} \otimes P_0^{(2)} = 0, \hspace{2cm} (4.35a)$$

$$(\partial_{z^1} + A_{z^1})\psi \otimes P_0^{(2)} = 0. \hspace{2cm} (4.35b)$$

Rescaling $\psi \mapsto 2\tilde{\psi}/\sqrt{\theta^{22}}$ and introducing $\beta := 4/\theta^{22}$, we obtain the equations

$$F_{z^1z^1} = \frac{\beta}{4}(1 - \tilde{\psi}\tilde{\psi}^\dagger) \quad \text{and} \quad (\partial_{z^1} + A_{z^1})\tilde{\psi} = 0, \hspace{2cm} (4.36)$$

on $\mathbb{R}_\theta^4 \subset \mathbb{R}_\theta^4$. For $\beta = 1$ they coincide with the standard $U_+(1)$ vortex equations. The equations (4.36) with $\theta^{11} = 2012 \neq 0$ and their explicit solutions were extensively discussed in the literature (see, e.g., [37, 38, 30]). Note that the choice of the projector $P_0^{(2)}$ on $\mathcal{H}_2$ ensures a finite charge.
for this solution. To exemplify this case let us rewrite $A_{z_1}$ via $\chi_{z_1} = A_{z_1} + \theta_{11}\tilde{z}^a$ with $\chi_{z_1} = \theta_{11}(S^{(1)})^N\tilde{z}^a(S^{(1)*})^N$. Moreover, suppose that $\tilde{\psi} = (S^{(1)})^N$. Then one readily verifies that $\theta_{11} = \beta/4 = 1/\theta^{22}$. Therefore, we end up with $F_{z_1\bar{z}_1} = \theta_{11}\mathcal{P}_N$ and $\phi = 2\sqrt{\theta^{22}}(S^{(1)})^N \otimes P_0^{(2)}$. Note that $\theta_{11} < 0$. The topological charge $3.46$ for this configuration is $Q = -N$.

To summarize, we have described solutions on $\mathbb{R}^2_{0,12} \times \mathbb{R}^2_{0,23}$. It is also allowed to put $\theta_{12}$ to zero in order to have solutions on $\mathbb{R}^2_{0,12} \times \mathbb{R}^2_{0,23}$. Then the second equation of (4.36) reduces to $A_{z_1} = -\partial_{\bar{z}_1}\log \psi$. Assuming that $\tilde{\psi} = e^{(u+i\nu)/2}$ has zeros at points $z_1^n$ in the complex plane, we obtain from the first equation of (4.36) with $\beta = 1$ (see, e.g., [39, 21])

$$\Delta u = e^u - 1 + 4\pi \sum_{n=1}^N \delta^{(2)}(z_1-\bar{z}_n, \bar{z}_1-n),$$

(4.37)
i.e., the standard Liouville type equation on $\mathbb{R}^2 \cong \mathbb{C}$. The moduli $z_1^n$ are position parameters of the $N$-vortex solution on the $z_1$-plane. It is well known that this equation exhibits regular $N$-vortex solutions [39].

### 4.4 Noncommutative solitons and $D$-branes

Without refering to string theory the $U_+(1) \times U_+(1)$ and $U_-(1)$ SW monopole equations are simply understood as noncommutative generalizations of the Abelian SW equations. In this section we shall discuss how one can interpret solutions to the noncommutative SW equations as $D$-brane configurations in a stringy context.

**Brane-antibrane effective action.** In the simplest case of type II superstrings living in the target space $\mathbb{R}^{9,1}$ a $Dp$-brane with a world volume $\mathbb{R}^{p,1} \to \mathbb{R}^{9,1}$ may be defined via a relative map,$$
\varphi : (\Sigma_2, \partial \Sigma_2) \to (\mathbb{R}^{9,1}, \mathbb{R}^{p,1}),
$$
where $\Sigma_2$ is a string world sheet with boundary $\partial \Sigma_2$. One may also consider $\overline{Dp}$-branes (= anti-$Dp$-branes) which are $Dp$-branes with opposite orientation and Ramond-Ramond (RR) charge.

It is well known that there are stable BPS $Dp$-branes in type IIA (even $p$) and type IIB (odd $p$) superstring theory. Besides that, it is also well known that a system consisting of coincident $Dp$-brane and $\overline{Dp}$-brane is unstable since open strings ending on these branes have a tachyonic mode ($M^2 < 0$) in the spectrum (see, e.g., [40] [41] [42] [43] [44] and references therein). This instability can be seen in the low energy effective action for the brane-antibrane pair. Namely, the effective field theory describing light excitations of this system contains two Abelian gauge potentials $A_{\pm}$ and a complex scalar $\phi$ (tachyon). The latter one is associated with modes of the open string stretched between brane and antibrane, and is believed to be subject to a “Mexican hat” potential. The tachyon carries charge one under the diagonal $U(1)$ subgroup of the group $U_+(1) \times U_-(1)$ with the generator $\text{diag}(i, -i)$ corresponding to the gauge potential $A_+ - A_-$. After turning on a constant
$B$-field (generating a noncommutativity tensor $\theta$ on $\mathbb{R}^{p,1}$) the theory becomes noncommutative and the tachyon field transforms in the bi-fundamental representation of $U_+(1) \times U_-(1)$.

In perturbative string theory the first computations of the brane-antibrane effective action were performed in [47]. The resulting effective Lagrangian reads as

$$\mathcal{L}^{(2)} = F_{+\dot{\alpha}\dot{\beta}} F^{+\dot{\alpha}\dot{\beta}} + F_{-\dot{\alpha}\dot{\beta}} F^{-\dot{\alpha}\dot{\beta}} + \frac{1}{2} (D_{\dot{\alpha}} \phi)(D_{\dot{\alpha}} \phi)^\dagger + \frac{1}{8} (\tau^2 - \phi^2)^2,$$

(4.39)

where the covariant derivative is given by $D_{\dot{\alpha}} \phi = \partial_{\dot{\alpha}} \phi + (A_{+\dot{\alpha}} - A_{-\dot{\alpha}}) \phi$, the overbar denotes complex conjugation, $\tau^2$ is a constant and $\dot{\alpha}, \dot{\beta}, \ldots = 0, 1, \ldots, p$. This is not the full effective Lagrangian since the coupling between closed string RR fields and open strings gives an additional term $\mathcal{L}_{CS}$ called the Chern-Simons term [48]. For constant components of the RR fields, which we now consider, this term does not contribute to the equations of motion, and therefore we will not discuss it here. Note that the terms given by (2.14) and (3.44) are of such a kind.

The tachyon potential given by (4.39) has the quartic form

$$V(\phi, \bar{\phi}) = \frac{1}{4} (\tau^2 - \phi\bar{\phi})^2.$$  

(4.40)

Computations in level truncated superstring field theory yield the same result [49]. Any minimum $\phi\bar{\phi} = \tau^2$ of the tachyon potential describes the closed string vacuum (tachyon condensate). The perturbative spectrum around this vacuum is conjectured not to contain any open string excitations. As it was discussed in [46, 30, 50], in the presence of a $B$-field background a noncommutative generalization of the effective theory (4.39) might be of the form

$$\mathcal{L}_{nc}^{(2)} = F_{+\dot{\alpha}\dot{\beta}} F^{+\dot{\alpha}\dot{\beta}} + F_{-\dot{\alpha}\dot{\beta}} F^{-\dot{\alpha}\dot{\beta}} + \frac{1}{2} (D_{\dot{\alpha}} \phi)(D_{\dot{\alpha}} \phi)^\dagger + \frac{1}{8} (\tau^2 - \phi\phi^\dagger)^2 + \frac{1}{8} (\tau^2 - \phi^\dagger\phi)^2,$$

(4.41)

where $D_{\dot{\alpha}} \phi = \partial_{\dot{\alpha}} \phi + (A_{+\dot{\alpha}} - A_{-\dot{\alpha}}) \phi$. Higher order corrections to the quartic tachyon potential are known (see, e.g., [51, 43]). The result is qualitatively similar (“Mexican hat” form with minima at $\phi\bar{\phi} < \infty$), and (4.40) is the leading order term. Note that a quartic form of the potential was also obtained in [52, 45, 53, 54] for bound states in $D(p - 2)$-$Dp$ and $D(p - 4)$-$Dp$ systems by using scattering calculations in string theory, level truncated superstring field theory and by analyzing the fluctuation spectrum around (noncommutative) vortex and instanton solutions.

Quite different results have been obtained in boundary string field theory (BSFT) (see, e.g., [55, 59] and references therein). There (as in [57]) the Lagrangian density is proportional to the tachyon potential itself, and the potential has the form $V \sim \exp(-\phi\bar{\phi})$ with a ring of minima at $\phi\bar{\phi} \to \infty$. It is believed that the level truncation scheme and the BSFT approach can be related by a field redefinition involving all components of the string field [55]. Various ‘improved versions’ of the effective action of the brane-antibrane system have been introduced (see, e.g., [58, 52]), and in the literature there is no final agreement on its form. Thus, the action based on the Lagrangian (4.39) (or (4.41) in the presence of a $B$-field) might be considered as an approximation of the low energy effective action of the brane-antibrane system. In any case the discussed Yang-Mills-Higgs theory provides a simple field theoretic model of the more complicated string field theory description of brane-antibrane systems.
Noncommutative vortices and ABS solitons. Recall that branes and antibranes have opposite RR charge and therefore they can annihilate into the closed string vacuum state with \( A_\pm = 0 \) and \( \phi \phi^\dagger = \phi^\dagger \phi = \tau^2 \). However, instead of taking the vacuum solution, one can choose as the ground state a (tachyon) soliton solution to the equations of motion for the Lagrangian (4.41). Such kinds of solutions are interpreted as bound states of \( D_p \)-branes and \( \overline{D_p} \)-branes, equivalent to \( D \)-branes of lower dimensionality. Here we discuss the main example of such solutions obtained via the Atiyah-Bott-Shapiro (ABS) construction [60, 40, 61, 62]. These ABS (noncommutative) solitons live in \( 2n \leq p \) dimensions, and for \( n = 1 \) they coincide with noncommutative vortices on \( \mathbb{R}^2_\theta \). For \( n = 2 \) the ABS solitons are non-Abelian and therefore differ from those (Abelian) SW solutions which also solve the field equations following from (4.41).

To describe the ABS solitons on \( \mathbb{R}^2_\theta \) we consider a non-Abelian generalization of the Lagrangian (4.41) with a \( U_+(q) \times U_-(q) \) gauge group for \( q = 2^{n-1} \) and the tachyon field \( \phi \) in the bi-fundamental representation \((q, \bar{q})\) of this group. This model describes two Hermitian rank \( q \) vector bundles \( E_\pm \rightarrow \mathbb{R}^n \) with connection one-forms \( A_\pm \) and \( \phi \in \text{Hom}(E_-, E_+) \) and is associated with a system of \( q \) \( D_p \)-branes and \( q \) \( \overline{D_p} \)-branes with common world volume \( \mathbb{R}^{p-1} \). We assume that \( 2n \leq p \) and introduce an ansatz for ABS solitons related to Clifford algebras.

Consider the Clifford algebra of the Euclidean space \((\mathbb{R}^{2n}, \delta_{\alpha\beta})\), generated by unity and elements \( \Gamma_\alpha \) such that
\[
\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = -2\delta_{\alpha\beta},
\]
with \( \alpha, \beta, \ldots = 1, \ldots, 2n \). In the \( 2^n \times 2^n \) matrix representation of this algebra the generators \( \Gamma_\alpha \) can be chosen of the form
\[
\Gamma_\alpha = \begin{pmatrix}
0 & \gamma_\alpha^\dagger \\
-\gamma_\alpha & 0
\end{pmatrix},
\]
where the \( \gamma_\alpha \)'s are \( q \times q \) gamma matrices with \( q = 2^{n-1} \). In this representation the spinor space \( W \cong \mathbb{C}^{2q} \) can be decomposed into a direct sum \( W \cong W^+ \oplus W^- \) of semi-spinor spaces \( W^\pm \) (the spaces of Weyl spinors). Note that for \( n = 1 \) we have \( \gamma_1 = 1 \) and \( \gamma_2 = i \).

Considering the noncommutative deformation \( \mathbb{R}^n_\theta \) of \( \mathbb{R}^{2n} \) discussed in subsections 3.1 and 3.3, we introduce the operators
\[
T = \gamma_\alpha^\dagger x^\alpha \sqrt{\frac{1}{(\gamma x)(\gamma x)^\dagger}}, \quad \text{and} \quad T^\dagger = \sqrt{\frac{1}{(\gamma x)(\gamma x)^\dagger}} \gamma_\alpha x^\alpha,
\]
where \( \gamma x \) is a shorthand notation for \( \gamma_\alpha x^\alpha \). The operator \( T \) defines a map,
\[
\hat{T} : \mathcal{H} \otimes W^- \rightarrow \mathcal{H} \otimes W^+.
\]
Here \( \mathcal{H} \) is the Hilbert space realized as a representation space of \( n \) oscillators defined by formulas similar to (4.14) and (4.22), and \( W^\pm \) are the semi-spinor spaces introduced above. In matrix realization \( \hat{T} \) looks as
\[
\hat{T} = \begin{pmatrix}
0 & T \\
0 & 0
\end{pmatrix},
\]
and hence, following the authors of [40, 63, 62, 50], we will not distinguish $T$ and $\hat{T}$ in the sequel.

It is not difficult to see that

$$T^\dagger T = 1_q \quad \text{and} \quad TT^\dagger = 1_q - P_0,$$

where $P_0$ is the projector onto the kernel of $T^\dagger$. This state is the tensor product of the oscillator ground state with itself and the lowest weight spinor of $SO(2n)$ (the fermion ground state). Also, by introducing $T_N := T^N$ with $T_1 \equiv T$, we have

$$T_N^\dagger T_N = 1_q \quad \text{and} \quad T_N T_N^\dagger = 1_q - P_{N-1},$$

(4.48)

where $P_{N-1}$ is the projector onto the kernel of $T_N^\dagger$, an $N$-dimensional subspace in $\mathcal{H} \otimes W$. One can show that

$$\dim \ker T_N = 0 \quad \text{and} \quad \dim \coker T_N := \dim \ker T_N^\dagger = N,$$

(4.49)

and therefore the index of $T_N$ is given by

$$\text{ind} T_N := \dim \ker T_N - \dim \coker T_N = -N.$$  

(4.50)

The operators $T_N$ and $T_N^\dagger$ are Toeplitz operators.

Now we reduce the equations of motion for the Lagrangian (4.41) to the space $\mathbb{R}_q^{2n} \rightarrow \mathbb{R}_q^{p+1}$ by assuming that all fields depend only on $x^\alpha$ and by taking $A_{\pm \alpha}$ as the only nonvanishing components of the gauge potentials $A_{\pm}$. To solve the reduced field equations we consider $T_N$ as $q \times q$ matrices with operator entries and introduce the ansatz,

$$A_{+z^a} = - (A_{+\bar{z}^a})^\dagger = \tau \theta_{\bar{a}a}(T_{N_+} z^0 T_N^\dagger - \bar{z}^0), \quad A_{-z^a} = 0 = A_{-\bar{z}^a} \quad \text{and} \quad \phi = \tau T_{N_+},$$

(4.51)

which solves the equation of motion for (4.41). More general solutions can also be constructed. For $n = 1$ the solution describes $N_+$ vortices on $\mathbb{R}_q^2$. In the case $n = 2$ it gives a solution for the $U_+(2) \times U_-(2)$ Yang-Mills-Higgs model on $\mathbb{R}_q^4$. Some noncommutative SW solutions (e.g., (4.18a), (4.18b) with $N_- = 0$) solve the above equations as well but for the gauge group $U_+(1) \times U_-(1)$. These solutions can therefore be regarded as a new kind of tachyon solitons.

**Noncommutative SW solitons.** Note that the SW equations (3.30) with $v_+ = v_- = \tau^2 = \text{const}$ coincide with the first order BPS equations for the Lagrangian (4.41), and therefore their solutions also satisfy the equations of motion for (4.41). In particular, the solution given by (4.18) with $N_+ \geq 1$ and $N_- = 0$ is such a solution. Following the general logic of Sen’s proposal, it is natural to interpret this solution as a configuration of $N_+$ stable $D(p - 4)$-branes, corresponding to the topologically stable codimension four SW soliton on a $Dp - \overline{Dp}$ brane pair.

The more general configuration described by (4.18) with $N_+ \geq 1$ can be interpreted in two different ways. First, we may again choose $v_+ = v_- = \tau^2 = \text{const}$. Then one can show that this configuration satisfies the second order equations of motion for the Lagrangian (4.41) but does
not satisfy the first order equations with constant $v_\pm$. This is natural since this solution corresponds to a system of $N_+ D(p - 4)$-branes and $N_- \overline{D(p - 4)}$-branes which is not a BPS configuration from the point of view of the Lagrangian (3.41). Second, one may consider the situation where $v_+$ depends on $\phi \phi^\dagger$ and $v_-$ on $\phi^\dagger \phi$ and take as the low energy effective action the sum of the functional (3.48a) and the Chern-Simons term (3.48b), where the latter contributes to the equations of motion for nonconstant $v_\pm$. Then for proper choices of $v_\pm$ one can obtain configurations which satisfy the noncommutative SW equations. For instance, the choice

$$v_+ = \tau^2 \quad \text{and} \quad v_- = -\tau^2 + 2\phi^\dagger \phi$$

(4.52)

corresponds to the same tachyon potential $V \sim (\tau^2 - \phi \phi^\dagger)^2 + (\tau^2 - \phi^\dagger \phi)^2$ as in (4.41). However, for the above choice of $v_\pm$ the Chern-Simons term (3.48b) becomes nontrivial and contributes to the equations (3.50) which are the BPS equations for the action $E_\chi + 16\pi^2 K_\chi$ with $E_\chi$ and $K_\chi$ determined by (3.48). Therefore, for $v_\pm$ given by (4.52) the configuration (4.18) with $N_\pm \geq 1$ is a solution to the SW equations (3.50). So, in both cases the configuration (4.18) may be interpreted as $N_+ D(p - 4)$-branes and $N_- \overline{D(p - 4)}$-branes. Other solutions to the noncommutative SW equations can be analyzed similarly.

5 Concluding remarks

In this paper we have discussed different noncommutative deformations of the (perturbed) SW monopole equations on Euclidean four-dimensional space. Namely, starting from properly deformed $U(2)$ self-duality type YM equations in eight dimensions, we performed a reduction to $U(2)$ noncommutative SW equations on $\mathbb{R}^4_\theta$ with the matter field in the adjoint representation of the gauge group. We then concentrated on the $U_+(1) \times U_-(1) \subset U(2)$ noncommutative SW equations with the matter in the bi-fundamental representation of $U_+(1) \times U_-(1)$. Perturbed versions of these equations have also been discussed. Then, by considering the matter field $\Phi$ as an element of a right or left $\mathbb{R}^4_\theta$-module, we have introduced the (perturbed) $U_+(1)$ and $U_-(1)$ SW equations. The commutative limits of all these three Abelian cases are identical to the standard (perturbed) Abelian SW equations on $\mathbb{R}^4$. In summary we may write down the following diagram:
This picture shows the connection between the theories discussed in this paper. Note that the noncommutative vortex equations in two dimensions can easily be obtained via dimensional reduction from the noncommutative SW equations with properly chosen perturbations.

It has been shown that $\mathbb{R}^4_\theta$ supports regular finite-action solutions to the SW equations even if there are no such solutions in the commutative case. This is a well known phenomenon related to the fact that, due to the noncommutativity tensor $\theta$, an additional length scale enters the theory. We have constructed explicit solutions to the $U_+\times U_-$ and $U_\pm$ noncommutative SW equations and interpreted them as D-brane configurations in type II superstring theory. It would be illuminating to generalize the present results to non-Abelian noncommutative SW theory and discuss the latter’s relation to superstring theory.

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A Noncommutative Bogomolny transformation

In order to derive (3.42) we need the trivial identities

\[
(D_\mu \phi) \psi^\dagger = -\phi (D_\mu \psi)^\dagger + \nabla_+ \mu (\phi \psi^\dagger) \quad \text{and} \quad (D_\mu \phi) \psi = -\phi^\dagger (D_\mu \psi) + \nabla_- \mu (\phi^\dagger \psi),
\]

where the covariant derivatives $\nabla_{\pm \mu}$ are given by (3.45b). Furthermore, we have

\[
[D_\mu, D_\nu] \phi = F_{\mu \nu} \phi - \phi F_{-\mu -\nu}.
\]

Let us consider

\[
4|F_{A^+} + \chi^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_0|^2 = -2(F_{\pm \mu \nu} + \chi_{\pm \mu \nu} - \sigma_{\pm \mu \nu} (\Phi \otimes \Phi^\dagger)_0)^2
\]

\[
= -2(F_{\pm \mu \nu})^2 - 2\{F_{\pm \mu \nu}, \chi_{\pm \mu \nu}\} + 2\{F_{\pm \mu \nu}, \sigma_{\pm \mu \nu} (\Phi \otimes \Phi^\dagger)_0\}
\]

\[
- 2(\chi_{\pm \mu \nu} - \sigma_{\pm \mu \nu} (\Phi \otimes \Phi^\dagger)_0)^2.
\]

This expression contains

\[
(F_{\pm \mu \nu})^2 = \frac{1}{4}(F_{\pm \mu \nu} + *F_{\pm \mu \nu})^2 = \frac{1}{2}(F_{\pm \mu \nu})^2 + \frac{1}{4}\{F_{\pm \mu \nu}, *F_{\pm \mu \nu}\},
\]

\[
\{F_{\pm \mu \nu}, \sigma_{\pm \mu \nu} (\Phi \otimes \Phi^\dagger)_0\} = \frac{1}{2}\{F_{+12}, \phi_1 \phi_1^\dagger - \phi_2 \phi_2^\dagger\} - \frac{1}{2}\{F_{+13}, \phi_2 \phi_1^\dagger - \phi_1 \phi_2^\dagger\}
\]

\[
+ \frac{1}{2}\{F_{+14}, \phi_1 \phi_2^\dagger + \phi_2 \phi_1^\dagger\},
\]

\[
-(\chi_{\pm \mu \nu} - \sigma_{\pm \mu \nu} (\Phi \otimes \Phi^\dagger)_0)^2 = 2|\chi^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_0|^2.
\]
Similarly,

\[4|F^+_{A-} + \chi^+ - \sigma^+(\Phi^* \otimes (\Phi^*)^\dagger)_0|^2 = -2(F^+_{-\mu\nu} + \chi^+_{-\mu\nu} - \sigma^+_{-\mu\nu}(\Phi^* \otimes (\Phi^*)^\dagger)_0)^2\]
\[= -2(F^+_{-\mu\nu})^2 - 2\{F^+_{-\mu\nu}, \chi^+_{-\mu\nu}\} + 2\{F^+_{-\mu\nu}, \sigma^+_{-\mu\nu}(\Phi^* \otimes (\Phi^*)^\dagger)_0\}
\[+ 2(\chi^+_{-\mu\nu} - \sigma^+_{-\mu\nu}(\Phi^* \otimes (\Phi^*)^\dagger)_0)^2 \] (A.5)

with

\[(F^+_{-\mu\nu})^2 = \frac{1}{2}(F_{-\mu\nu})^2 + \frac{1}{4}F_{-\mu\nu}, F_{-\mu\nu},\]
\[\{F^+_{-\mu\nu}, \sigma^+_{-\mu\nu}(\Phi^* \otimes (\Phi^*)^\dagger)_0\} = -\frac{1}{2}\{F^+_{12}, \phi^1_1 \phi_1 - \phi^1_2 \phi_2\} + \frac{1}{2}\{F^+_{13}, \phi^1_1 \phi_2 - \phi^1_2 \phi_1\}
\[- \frac{1}{2}\{F^+_{14}, \phi^1_1 \phi_2 + \phi^1_2 \phi_1\}\] (A.6b)
\[-(\chi^+_{-\mu\nu} - \sigma^+_{-\mu\nu}(\Phi^* \otimes (\Phi^*)^\dagger)_0)^2 = 2|\chi^+_{-\mu\nu} - \sigma^+(\Phi^* \otimes (\Phi^*)^\dagger)_0|^2.\] (A.6c)

A lengthy but straightforward calculation exploiting (A.1) and (A.2) yields for \(\frac{1}{2}|D_{A+,A-}|^2\) the expression

\[\frac{1}{2}|D_{A+,A-}|^2 = \frac{i}{2}|D \phi_1 - D_\phi_1 + i D \phi_2 + D_\phi_2|^2 + \frac{i}{2}|D \phi_1 - D_\phi_2 + i D \phi_2 - D_\phi_2|^2\]
\[= -\frac{1}{2}\{F^+_{12}, \phi^1_1 \phi_1 - \phi^1_2 \phi_2\} + \frac{1}{2}\{F^+_{13}, \phi^1_1 \phi_2 - \phi^1_2 \phi_1\}
\[- \frac{1}{2}\{F^+_{14}, \phi^1_1 \phi_2 + \phi^1_2 \phi_1\}\]
\[+ \frac{1}{2}|D_{A+,A-}|^2 - \nabla \mu J_{+\mu} + C_+,\] (A.7)

where

\[\mathcal{J}_{+\mu} = \frac{i}{4}(\epsilon_{\mu\nu 12} + \epsilon_{\mu\nu 34})\mathcal{J}^{(1)}_{+\nu} + \frac{1}{4}(\epsilon_{\mu\nu 31} + \epsilon_{\mu\nu 24})\mathcal{J}^{(2)}_{+\nu} + \frac{i}{4}(\epsilon_{\mu\nu 23} + \epsilon_{\mu\nu 14})\mathcal{J}^{(3)}_{+\nu}\] (A.8a)

and

\[\mathcal{J}^{(1)}_{+\nu} = \phi_1 (D_\nu \phi_1)^\dagger - (D_\nu \phi_1)^\dagger \phi_1 - \phi_2 (D_\nu \phi_2)^\dagger + (D_\nu \phi_2)^\dagger \phi_2,\] (A.8b)
\[\mathcal{J}^{(2)}_{+\nu} = -\phi_1 (D_\nu \phi_2)^\dagger + (D_\nu \phi_1)^\dagger \phi_2 + \phi_2 (D_\nu \phi_1)^\dagger - (D_\nu \phi_2)^\dagger \phi_1,\] (A.8c)
\[\mathcal{J}^{(3)}_{+\nu} = \phi_2 (D_\nu \phi_1)^\dagger + (D_\nu \phi_2)^\dagger \phi_1 + \phi_1 (D_\nu \phi_2)^\dagger - (D_\nu \phi_1)^\dagger \phi_2.\] (A.8d)

The term \(C_+\) is given by

\[C_+ = -i \phi_2 F^+_{12} \phi^1_2 + i \phi_1 F^+_{12} \phi^1_1 - \phi_2 F^+_{13} \phi^1_1 + i \phi_1 F^+_{13} \phi^1_2 + i \phi_1 F^+_{14} \phi^1_2 + \phi_1 F^+_{14} \phi^1_1.\] (A.9)

In a similar manner, we also have

\[\frac{1}{2}|(D_{A+,A-})^\dagger|^2 = \frac{i}{2}\{F^+_{12}, \phi^1_1 \phi_1 - \phi^1_2 \phi_2\} - \frac{1}{2}\{F^+_{13}, \phi^1_1 \phi_2 - \phi^1_2 \phi_1\}
\[+ \frac{1}{2}\{F^+_{14}, \phi^1_1 \phi_2 + \phi^1_2 \phi_1\}\]
\[+ \frac{1}{2}|(D_{A+,A-})^\dagger|^2 - \nabla \mu J_{-\mu} + C_-\] (A.10)

where

\[J_{-\mu} = \frac{i}{4}(\epsilon_{\mu\nu 12} + \epsilon_{\mu\nu 34})\mathcal{J}^{(1)}_{-\nu} - \frac{1}{4}(\epsilon_{\mu\nu 31} + \epsilon_{\mu\nu 24})\mathcal{J}^{(2)}_{-\nu} - \frac{i}{4}(\epsilon_{\mu\nu 23} + \epsilon_{\mu\nu 14})\mathcal{J}^{(3)}_{-\nu}\] (A.11a)
and
\begin{align*}
J_{-2}^{(1)} &= \phi_1^\dagger (D_\mu \phi_1) - (D_\mu \phi_1)^\dagger \phi_1 - \phi_2^\dagger (D_\mu \phi_2) + (D_\mu \phi_2)^\dagger \phi_2, \\
J_{-2}^{(2)} &= -\phi_2^\dagger (D_\mu \phi_1) + (D_\mu \phi_2)^\dagger \phi_1 + \phi_1^\dagger (D_\mu \phi_2) - (D_\mu \phi_1)^\dagger \phi_2, \\
J_{-2}^{(3)} &= \phi_1^\dagger (D_\mu \phi_2) - (D_\mu \phi_1)^\dagger \phi_2 + \phi_2^\dagger (D_\mu \phi_1) - (D_\mu \phi_2)^\dagger \phi_1.
\end{align*}
(A.11b)

The term $C_-$ is
\[ C_- = i\phi_2^\dagger F_{+12}^+ \phi_2 - i\phi_1^\dagger F_{+13}^+ \phi_1 - \phi_2^\dagger F_{+14}^+ \phi_1 - i\phi_1^\dagger F_{+14}^+ \phi_2 + \phi_1^\dagger F_{+13}^+ \phi_2. \] (A.12)

Therefore, we discover that
\[
\begin{align*}
\frac{1}{2} |D_{A_+ A_-} \Phi|^2 + \frac{1}{2} |(D_{A_+ A_-} \Phi)^\dagger|^2 + 4 |F_{A_+}^+ + \chi_+^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_{0}|^2 + 4 |F_{A_-}^+ + \chi_-^+ - \sigma^+ (\Phi^* \otimes (\Phi^*)^\dagger)_{0}|^2 \\
= \frac{1}{2} |D_{A_+ A_-} \Phi|^2 + \frac{1}{2} |(D_{A_+ A_-} \Phi)^\dagger|^2 + 2 |F_{A_+}^+|^2 + 2 |F_{A_-}^+|^2 + 4 |\chi_-^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_{0}|^2 \\
+ 4 |\chi_-^+ - \sigma^+ (\Phi^* \otimes (\Phi^*)^\dagger)_{0}|^2 - \frac{1}{2} \{F_{-\mu \nu}, \Phi_{-\mu \nu}^\dagger\} - 2 \{F_{+\mu \nu}, \chi_{-\mu \nu}^+\} - \nabla_{\mu} J_{\mu} \\
- \frac{1}{2} \{F_{-\mu \nu}, \Phi_{-\mu \nu}^\dagger\} - 2 \{F_{+\mu \nu}, \chi_-^+ \} - \nabla_{-\mu} J_{-\mu} \\
+ \frac{1}{8} \{F_{+12}^+, \phi_1^\dagger \phi_1^\dagger - \phi_2^\dagger \phi_2^\dagger\} - \frac{1}{2} \{F_{+13}^+, \phi_2^\dagger \phi_1^\dagger - \phi_1^\dagger \phi_2^\dagger\} + \frac{1}{8} \{F_{+14}^+, \phi_2^\dagger \phi_1^\dagger + \phi_1^\dagger \phi_2^\dagger\} + C_+ \\
- \frac{1}{8} \{F_{-12}^+, \phi_1^\dagger \phi_1^\dagger + \phi_2^\dagger \phi_2^\dagger\} + \frac{1}{2} \{F_{-13}^+, \phi_2^\dagger \phi_2^\dagger - \phi_1^\dagger \phi_1^\dagger\} - \frac{1}{8} \{F_{-14}^+, \phi_1^\dagger \phi_2^\dagger + \phi_2^\dagger \phi_1^\dagger\} + C_- \}
\end{align*}
\]

Now suppose that all operators entering this formula are of proper trace-class, e.g., $|\text{Tr} \phi_{1,2}| < \infty$ and $|\text{Tr} F_{+\mu \nu}| < \infty$. Then
\[
\text{Tr} \left\{ \frac{1}{8} |D_{A_+ A_-} \Phi|^2 + \frac{1}{2} |(D_{A_+ A_-} \Phi)^\dagger|^2 + 4 |F_{A_+}^+ + \chi_+^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_{0}|^2 \\
+ 4 |F_{A_-}^+ + \chi_-^+ - \sigma^+ (\Phi^* \otimes (\Phi^*)^\dagger)_{0}|^2 \right\} \\
= \text{Tr} \left\{ \frac{1}{2} |D_{A_+ A_-} \Phi|^2 + \frac{1}{2} |(D_{A_+ A_-} \Phi)^\dagger|^2 + 2 |F_{A_+}^+|^2 + 2 |F_{A_-}^+|^2 + 4 |\chi_-^+ - \sigma^+ (\Phi \otimes \Phi^\dagger)_{0}|^2 \\
+ 4 |\chi_-^+ - \sigma^+ (\Phi^* \otimes (\Phi^*)^\dagger)_{0}|^2 \right\} - \text{Tr} \left( \frac{16\pi^2}{|\text{Pf}(2\pi\theta)|} \right) K_{\chi},
\] (A.13)

which is the desired result. Note that the choice $\phi_1 = \phi$ and $\phi_2 = 0$ yields the expressions \ref{3.49a} and \ref{3.49b} for the currents $J_{\pm \mu}$. 

28
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