A SEMIAMPLITUDE CRITERION FOR DIVISORS ON $\overline{M}_{0,n}$

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ABSTRACT. We associate to a cyclic integral quadratic form satisfying a certain balancedness condition an infinite sequence of semiample line bundles on moduli spaces of stable pointed curves. We also give a sufficient condition for an $S_n$-symmetric divisor on $\overline{M}_{0,n}$ to be base-point-free.

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1. INTRODUCTION

We give a new sufficient criterion for semiampleness of divisors on $\overline{M}_{0,n}$. To state the criterion, we introduce a series of properties for cyclic quadratic forms, among them balancedness and weak balancedness (Definition 2.4). For a positive definite quadratic form, these conditions can be verified algorithmically. To a balanced quadratic form, we associate an infinite sequence of semiample divisors on various $\overline{M}_{0,n}$ (Theorem 5.1). For example, the balanced quadratic form

$$x_0^2 + x_1^2 + \cdots + x_{m-1}^2$$

is responsible for semiampleness of all $\mathfrak{sl}_m$ level 1 conformal blocks divisors studied in [GG12]. Conversely, to every symmetric divisor $D$ on $\overline{M}_{0,n}$, we associate a cyclic quadratic form $Q_D$ in $n$ variables, whose coefficients depend linearly on the coefficients of $D$. If $Q_D$ is weakly balanced, then $D$ is semiample (Theorem 5.4). Weak balancedness of $Q_D$ is equivalent to finitely many linear

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inequalities on the coefficients of $D$ and so can be verified by a direct computation. This gives a sufficient criterion for semiampleness, which we implement in SAGE [S+14].

Our approach to proving nefness and semiampleness is elementary in that it relies only on Keel’s relations in $\text{Pic}(\overline{M}_{0,n})$ and some quadratic programming. Most of our results are characteristic independent and hold over $\text{Spec}(\mathbb{Z})$.

**Notation and conventions:** We use freely standard divisor theory of $\overline{M}_{0,n}$. The $i^{th}$ cotangent line bundle and its divisor class on $\overline{M}_{0,n}$ is denoted $\psi_i$. We say that a partition $I \sqcup J = [n]$ is proper if $|I|, |J| \geq 2$. The boundary divisor on $\overline{M}_{0,n}$ corresponding to a proper partition $I \sqcup J = [n]$ is denoted $\Delta_{I,J}$. We set $\Delta_i := \sum_{|I| = i} \Delta_{I,J}$ and $\Delta := \sum_{i=2}^{[n/2]} \Delta_i$.

We let $[n] := \{1, \ldots, n\}$ and write $\mathfrak{S}_n$ for the symmetric group on $[n]$. We identify elements of the cyclic group $\mathbb{Z}_m$ with $\{0, 1, \ldots, m-1\}$. For $k \in \mathbb{Z}$, we denote by $\langle k \rangle_m$ the residue of $k$ in $\mathbb{Z}_m$. Given an abelian group $G$, a function $f : G \to \mathbb{Q}$ is called symmetric if $f(a) = f(-a)$ for all $a \in G$. All quadratic forms have rational coefficients, unless specified otherwise. We use the shorthand $(a)_r$ for $a, \ldots, a$.

## 2. Balanced Cyclic Quadratic Forms

A quadratic form $Q(x_0, \ldots, x_{m-1}) = \sum_{0 \leq i, j \leq m-1} Q_{i,j} x_i x_j$ is cyclic if

$$Q(x_0, x_1, \ldots, x_{m-1}) = Q(x_1, \ldots, x_{m-1}, x_0).$$

A form is cyclic if and only if its symmetric matrix is circulant, i.e. $Q_{i,j} = q(i-j)$, where $q : \mathbb{Z}_m \to \mathbb{Q}$ is a symmetric function.

We introduce a series of conditions on cyclic quadratic forms $Q$ in $m$ variables:

**Condition $(n)$:** For $n = mc + r \in \mathbb{Z}$, the minimum value of $Q$ at the integral points of the affine hyperplane

$$\sum_{i=0}^{m-1} x_i = n$$

is achieved at the vector$^1$ $v_n := ( (1+c)r, (c)_{m-r} )$.

**Example 2.1.** $x_0^2 + x_1^2 + \cdots + x_{m-1}^2$ satisfies Condition $(n)$ for all $n \in \mathbb{Z}$.

We make two simple observations:

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$^1$We allow the minimum value to be also achieved at other points; in particular, $Q$ attains the same value at all cyclic shifts of $v_n$. 

Remark 2.2 (Finiteness of conditions). For $Q = \sum_{0 \leq i, j \leq m-1} q(i-j)x_ix_j$, we have
\[
Q(x_0 + c, \ldots, x_{m-1} + c) - Q(x_0, \ldots, x_{m-1}) = 2c \left( \sum_{k=0}^{m-1} q(k) \right) (x_0 + \cdots + x_{m-1}) + Q(c, \ldots, c).
\]
It follows that Condition (r) is satisfied if and only if Condition (n) is satisfied for all $n \equiv \pm r \pmod{m}$.

Remark 2.3 (Reduction to PSD). Condition (0) is equivalent to positive semi-definiteness of the quadratic form
\[
\tilde{Q}_j := Q(x_0, \ldots, x_{j-1}, -\sum_{i \neq j} x_i x_{j+1}, \ldots, x_{m-1})
\]
in variables $x_0, \ldots, \hat{x}_j, \ldots, x_{m-1}$. Furthermore, note that $\tilde{Q} := \sum_{j=0}^{m-1} \tilde{Q}_j$ satisfies
\[
(2.1) \quad \tilde{Q} = mQ + (m - 2)q(0)(x_0 + \cdots + x_{m-1})^2.
\]
Therefore, $Q$ satisfies Condition (0) if and only if $\tilde{Q}$ is positive semi-definite, and $Q$ satisfies Condition (n) if and only if $\tilde{Q}$ does.

Definition 2.4. Let $Q$ be a cyclic quadratic form in $m$ variables.

1. We say that $Q$ is balanced if Condition (r) holds for every $r \pmod{m}$.
2. We say that $Q$ is weakly balanced if when restricted to $x_i \in \{0,1\}$ Condition (r) holds for all $r = 1, \ldots, m - 1$.
   Equivalently, $Q$ is weakly balanced if and only if for all $r = 1, \ldots, m - 1$, the leading principal $r \times r$ minor of $Q$ has the minimal sum of entries among all principal $r \times r$ minors.
3. We say that $Q$ is $\ell$-balanced if when restricted to $0 \leq x_i \leq \ell$ Condition (r) holds for all $r \pmod{m}$. Thus, 1-balanced forms are weakly balanced and $\infty$-balanced forms are balanced.

Remark 2.5. Suppose $Q$ is a positive definite cyclic quadratic form. Then balancedness of $Q$ can be algorithmically verified by:

1. Finding all vectors $v$ such that $Q(v) \leq \max\{Q(v_r) \mid r = 0, \ldots, m - 1\}$.
2. Testing whether the found vectors violate Conditions (n).

For an arbitrary quadratic form $Q$ in $m$ variables, it is possible to determine whether $Q$ is $\ell$-balanced by a direct evaluation of $Q$ at all integer points of the hypercube $[0, \ell]^m$.

We give several examples of balanced quadratic forms.

Example 2.6. $A(x_0, \ldots, x_{m-1}) := \sum_{i=0}^{m-1} x_i^2$.

Example 2.7. $B(x_0, \ldots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2})^2$, where $m \geq 4$ and $m \not\equiv \pm 1 \pmod{6}$. 
Example 2.8. \( C(x_0, \ldots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i + x_{i+k-1})^2 \), where \( m = 2k \) and \( k \) is odd.

Example 2.9. \( D(x_0, \ldots, x_{m-1}) := \sum_{i=0}^{m-1} (x_i + x_{i+k})^2 \), where \( m = 2k + 1 \).

Note that \( A \) is obviously balanced. We prove balancedness of \( B \) in the following lemma, and leave balancedness of \( C \) and \( D \) as an exercise.

Lemma 2.10. The quadratic form

\[
B(x_0, \ldots, x_{m-1}) = \sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2})^2.
\]

is balanced for all \( m \geq 4 \) such that \( m \not\equiv \pm 1 \pmod{6} \).

Proof. Clearly, \( B \) is positive semi-definite and even positive definite if \( m \not\equiv 0 \pmod{6} \). Thus \( B \) satisfies Condition (0). Next, we verify Condition (r) for \( 2 \leq r \leq m - 2 \). For any \( (a_0, \ldots, a_{m-1}) \in \mathbb{Z}^m \) with \( \sum_{i=0}^{m-1} a_i = r \), we have

\[
B(a_0, \ldots, a_{m-1}) \geq \sum_{i=0}^{m-1} |a_i - a_{i+1} + a_{i+2}|
\]

\[
\geq |\sum_{i=0}^{m-1} (a_i - a_{i+1} + a_{i+2})| = |\sum_{i=0}^{m-1} a_i| = r = B(v_r).
\]

Hence Condition (r) is satisfied for \( 2 \leq r \leq m - 2 \).

When \( r = 1 \), we have \( B(v_1) = 3 \). By the above, for any \( (a_0, \ldots, a_{m-1}) \in \mathbb{Z}^m \) with \( \sum_{i=0}^{m-1} a_i = 1 \), we have the estimate \( B(a_0, \ldots, a_{m-1}) \geq 1 \). In addition, \( B(a_0, \ldots, a_{m-1}) \) is clearly odd. It follows that \( B \) fails to be balanced if and only if there is a vector \( (a_0, \ldots, a_{m-1}) \in \mathbb{Z}^m \) such that \( a_0 - a_1 + a_2 = \pm 1 \) and \( a_i - a_{i+1} + a_{i+2} = 0 \) for all \( i = 1, \ldots, m - 1 \). One checks that such a vector exists only if \( m \equiv 1 \pmod{6} \) or \( m \equiv -1 \pmod{6} \), in which case the vector is either \((-1, -1, 0, 1, 1, 0, -1, -1, 0, 1, 0, \ldots)\) or \((-1, -1, 0, 1, 0, -1, -1, 0, 1, 1, 0, \ldots)\), up to a sign and cyclic shift. \( \square \)

Question 2.11. Describe the convex cone of balanced cyclic quadratic forms. Given an integral linear form \( L(x_0, x_1, \ldots, x_{m-1}) \), determine whether the cyclic quadratic form

\[
Q(x_0, \ldots, x_{m-1}) = \sum_{k=0}^{m-1} L^2(x_k, x_{k+1}, \ldots, x_{k+m-1})
\]

is balanced. For example, for which \( m \) is the following form balanced:

\[
\sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2} - x_{i+3} + x_{i+4})^2
\]
3. F-NEF FUNCTIONS ON ABELIAN GROUPS

**Definition 3.1.** Let $G$ be a finite abelian group (with addition). We say that a symmetric function $f: G \to \mathbb{Q}$ is F-NEF if for any $a, b, c \in G$ we have

\[(3.1) \quad f(a) + f(b) + f(c) + f(a + b + c) \geq f(a + b) + f(a + c) + f(b + c).\]

**3.1. First properties.** For an arbitrary $f: G \to \mathbb{Q}$, we define $d_f: G \times G \to \mathbb{Q}$ by $d_f(a, b) = f(a) + f(b) - f(a + b)$. Then (3.1) is equivalent to subadditivity of $d_f$ in each variable:

\[(3.2) \quad d_f(a, b) + d_f(a, c) \geq d_f(a, b + c).\]

**Lemma 3.2.** Suppose $f: G \to \mathbb{Q}$ is F-NEF. Then $d_f$ is non-negative, and hence $f$ is subadditive. Namely, for any $a, b \in G$, we have

\[f(a) + f(b) \geq f(a + b).\]

**Proof.** Take a positive integer $N$ so that $N b = 0$. Applying (3.2), we obtain

\[(N + 1) d_f(a, b) \geq d_f(a, (N + 1)b) = d_f(a, b).\]

\[\square\]

3.1.1. Cone of F-NEF functions. The F-NEF functions on $G$ form a rational polyhedral cone inside the $\mathbb{Q}$-vector space of all symmetric functions from $G$ to $\mathbb{Q}$. We denote this cone by $C(G)$.

We have a natural $\mathbb{Z}$-action on $C(G)$ given by $(n, f) \mapsto n \ast f$, where

\[(n \ast f)(a) = f(na).\]

3.1.2. Examples of F-NEF functions. In the following examples, we take $G = \mathbb{Z}_m$.

**Example 3.3.** We define the standard function $A_m: \mathbb{Z}_m \to \mathbb{Z}$ by

\[A_m(i) = i(m - i).\]

Given $a, b, c \in \mathbb{Z}_m$, set $d = \langle m - a - b - c \rangle_m$. It is easy to check that

\[A_m(a) + A_m(b) + A_m(c) + A_m(a + b + c) - A_m(a + b) - A_m(a + c) - A_m(b + c)\]

\[= \begin{cases} 0 & \text{if } a + b + c + d = m \text{ or } a + b + c + d = 3m, \\ \min\{a, b, c, d, m - a, m - b, m - c, m - d\}, & \text{otherwise}. \end{cases}\]

(3.3)

Hence $A_m$ is F-NEF.

**Example 3.4.** The second standard function $B_m: \mathbb{Z}_m \to \mathbb{Z}$ is defined by

\[B_m(i) = A_m(i) \text{ if } i \neq 1, m - 1, \text{ and by } B_m(1) = B_m(m - 1) = 3m - 1.\]

It is easy to verify using (3.3) that $f$ is F-NEF for $m \geq 8$ and $m = 4, 6$. 
3.2. F-nef functions and F-nef divisors.

3.2.1. F-nef functions from F-nef divisors. Suppose $D = \sum_{i=1}^{\lfloor n/2 \rfloor} a_i \Delta_i$ is a symmetric F-nef divisor on $\overline{M}_{0,n}$. Define $f : \mathbb{Z}_n \rightarrow \mathbb{Z}$ by

- $f(j) = -c_j$ if $j = 2, \ldots, \lfloor n/2 \rfloor$,
- $f(j) = -c_{n-j}$ if $j = \lceil n/2 \rceil, \ldots, n-2$,
- $f(0) = f(1) = f(n-1) = 0$.

Using (3.3), it is easy to check that $f + cA_n$ is F-nef for $c \gg 0$. Let $c_0 := \min \{ c \in \mathbb{Q} \mid f + cA_n \text{ is F-nef} \}$. We define $f_D := f + c_0A_n$ to be the associated F-nef function of the divisor $D$.

3.2.2. F-nef divisors from F-nef functions. Suppose $f : G \rightarrow \mathbb{Q}$ is an F-nef function. Given $d_1, \ldots, d_n \in G$ satisfying $d_1 + d_2 + \cdots + d_n = 0 \in G$, we define the following divisor on $\overline{M}_{0,n}$:

$$\mathcal{D}(G, f; (d_1, \ldots, d_n)) = \sum_{i=1}^{n} f(d_i) \psi_i - \sum_{I \subseteq [n]} f \left( \sum_{i \in I} d_i \right) \Delta_{I,J},$$

where the summation in the second sum is taken over all proper partitions of $I \sqcup J$. (Note that $f(\sum_{i \in I} d_i) = f(\sum_{j \in J} d_j)$ by our assumption on $d_i$'s and the symmetry of $f$.)

3.2.3. Functoriality with respect to the boundary stratification.

**Proposition 3.6.** Fix an F-nef function $f : G \rightarrow \mathbb{Q}$. Consider $d_i \in G$, $1 \leq i \leq n$, such that $\sum_{i=1}^{n} d_i = 0 \in G$.

1. For any boundary divisor $\Delta_{I,J} \subset \overline{M}_{0,n}$, we have

$$\mathcal{D}(G, f; (d_1, \ldots, d_n))|_{\Delta_{I,J}} = \mathcal{D}(G, f; (\{d_i\}_{i \in I}, \sum_{j \in J} d_j)) \boxtimes \mathcal{D}(G, f; (\sum_{i \in I} d_i, \{d_j\}_{j \in J})),$$

where we use the standard identification $\Delta_{I,J} \simeq \overline{M}_{0, I \cup J} \times \overline{M}_{0, Q \cup J}$.

2. The divisor $\mathcal{D}(G, f; (d_1, \ldots, d_n))$ is F-nef on $\overline{M}_{0,n}$.

**Proof.** (1) follows from standard intersection theory on $\overline{M}_{0,n}$. For (2), consider a partition $I \sqcup J \sqcup K \sqcup L = [n]$ and the corresponding F-curve $F_{I,J,K,L}$. Set $A = \sum_{i \in I} d_i$, $B = \sum_{j \in J} d_j$, $C = \sum_{k \in K} d_k$, $D = \sum_{\ell \in L} d_\ell$. Then by (1), we have

$$\mathcal{D}(G, f; (d_1, \ldots, d_n)) \cdot F_{I,J,K,L} = f(A) + f(B) + f(C) + f(D) - f(A+B) - f(A+C) - f(B+C) \geq 0,$$
Remark 3.7. Every symmetric F-nef divisor on $\overline{M}_{0,n}$ comes from some F-nef function on $\mathbb{Z}_n$. To see this, observe that $\mathcal{D}(\mathbb{Z}_n, \mathbb{A}_n; (1)_n) = 0$. Therefore for every symmetric F-nef divisor $D = \sum_{i=2}^{[n/2]} a_i \Delta_i$ on $\overline{M}_{0,n}$ and its associated F-nef function $f_D$ (cf. 3.2.1), we have
\[ D = \mathcal{D}(\mathbb{Z}_n, f_D; (1)_n). \]

3.3. Quadratic forms of F-nef functions. Suppose $f: \mathbb{Z}_m \to \mathbb{Q}$ is a symmetric function. We define $q_f: \mathbb{Z}_m \to \mathbb{Q}$ by
\begin{equation}
q_f(a) := \frac{1}{2} (f(a+1) + f(a-1) - 2f(a)).
\end{equation}
To $f$ we associate a cyclic quadratic form defined by
\begin{equation}
Q_f(x_0, \ldots, x_{m-1}) := \sum_{0 \leq i,j \leq m-1} q_f(i-j)x_ix_j.
\end{equation}
We call $Q_f$ the associated quadratic form of $f$.

Lemma 3.8. $Q_f$ is weakly balanced if and only if for every $S \subset \mathbb{Z}_m$, we have
\begin{equation}
f(|S| - 1) - f(0) \geq \frac{1}{2} \sum_{i,j \in S} \left( f(i-j-1) + f(i-j+1) - 2f(i-j) \right).
\end{equation}

Proof. This is immediate from Definition 2.4 (2).

We note the following implications:
\begin{equation}
Q_f \text{ is balanced } \implies Q_f \text{ is weakly balanced} \\
Q_f \text{ is weakly balanced and } f(0) = 0 \\
\implies f(a) + f(b) + f(c) + f(a+b+c) \geq f(a+b) + f(a+c) + f(b+c) \\
\text{for all } a, b, c \in \mathbb{Z}_m \text{ such that } a + b + c \leq m.
\end{equation}
The last implication follows by taking $S = \{0, \ldots, a-1\} \cup \{a+b, \ldots, a+b+c-1\}$ in Lemma 3.8.

4. Weighted graphs and effective F-nef functions

4.1. Weighted graphs. Let $P_n$ be the set of vertices of a regular $n$-gon, with a fixed clockwise numbering by $[n]$. We denote by $\Gamma(P_n)$ the complete graph on $P_n$ and by $E(P_n)$ the set of all edges of $\Gamma(P_n)$. We write $(i \sim j)$ to denote the edge joining vertices $i$ and $j$. We say that a partition $I \sqcup J = P_n$ is contiguous if $I$ is a set of contiguous vertices in $P_n$.

A degree function on $P_n$ is a function $d: P_n \to G$, where $G$ is a finite abelian group and $\sum_{i \in P_n} d(i) = 0 \in G$. We set $d(I) := \sum_{i \in I} d(i)$ for every $I \subset P_n$. 

by F-nefness of $f$. 

□
A weighting or a weight function on \( P_n \) is a function \( w : E(P_n) \rightarrow \mathbb{Q} \). We write \( w(i \sim j) \) to denote the weight of \((i \sim j)\). Given a weight function \( w \) on \( P_n \), we make the following definitions:

1. The \( w \)-flow through a vertex \( k \in P_n \) is defined to be
   \[
   w(k) := \sum_{i \neq k} w(k \sim i).
   \]

2. The \( w \)-flow across a partition \( I \sqcup J = P_n \) is defined to be
   \[
   w(I \mid J) = \sum_{i \in I, j \in J} w(i \sim j).
   \]

### 4.2. Effective boundary.

Every line bundle on \( \overline{M}_{0,n} \) can be written as
\[
\sum_{i=1}^{n} a_i \psi_i - \sum_{I,J} b_{I,J} \Delta_{I,J}.
\]
This representation is far from unique because we have the following relation in \( \text{Pic}(\overline{M}_{0,n}) \) for every \( i \neq j \):

\[
(4.1) \quad \psi_i + \psi_j = \sum_{I \in I, J \in J} \Delta_{I,J}.
\]

Relations (4.1) generate the module of all relations among \( \{\psi_i\}_{i=1}^{n} \) and \( \{\Delta_{I,J}\} \); this follows, for example, from \cite[Theorem 2.2(d)]{AC98}, which in turn follows from Keel’s relations \cite{Kee92}. (We note that the above representation is unique if we impose an additional condition \(|I|, |J| \geq 3\); see \cite[Lemma 2]{FG03}.)

We now state a simple observation that we will use repeatedly in the sequel.

**Lemma 4.1** (Effective boundary criterion). Let \( R = \mathbb{Z} \) or \( R = \mathbb{Q} \). A divisor \( D = \sum_{i=1}^{n} a_i \psi_i - \sum_{I,J} b_{I,J} \Delta_{I,J} \) is \( R \)-linearly equivalent to \( \sum_{I,J} c_{I,J} \Delta_{I,J} \) if and only if there is an \( R \)-valued weighting of \( P_n \) such that the flow through the vertex \( i \) is \( a_i \) and the flow across each proper partition \( I \sqcup J = P_n \) is \( b_{I,J} + c_{I,J} \).

In particular, \( D \) is \( R \)-linearly equivalent to an effective linear combination of the boundary divisors on \( \overline{M}_{0,n} \) if and only if there exists an \( R \)-valued weighting of \( P_n \) such that the flow through the vertex \( i \) is \( a_i \) and the flow across each proper partition \( I \sqcup J = P_n \) is at least \( b_{I,J} \).

**Proof.** Identify the \( n \) marked points with \( P_n \). Suppose that for each \( i \neq j \) we use the relation (4.1) \( w(i \sim j) \) times to rewrite \( D \) as \( \sum_{I,J} c_{I,J} \Delta_{I,J} \). Then in the free \( R \)-module generated by \( \{\psi_i\}_{i=1}^{n} \) and \( \{\Delta_{I,J}\} \) we have

\[
\sum_{I,J} c_{I,J} \Delta_{I,J} = D - \sum_{i \neq j} w(i \sim j) \left( \psi_i + \psi_j - \sum_{I \in I, J \in J} \Delta_{I,J} \right) = \sum_{i=1}^{n} (a_i - w(i)) \psi_i - \sum_{I,J} (b_{I,J} - w(I \mid J)) \Delta_{I,J}.
\]
The claim follows. □

**Definition 4.2.** We say that an F-nef function $f: G \to \mathbb{Q}$ is **effective with respect to the degree function** $d: P_n \to G$ if there exists a $\mathbb{Q}$-weighting on $P_n$ such that:

1. The flow through the vertex $i \in P_n$ is $f(d(i))$.
2. The flow across each proper partition $I \sqcup J = P_n$ is at least $f(d(I))$.

We say that $f: G \to \mathbb{Z}$ is **strongly effective with respect to the degree function** $d: P_n \to G$ if there exists a $\mathbb{Z}$-weighting on $P_n$ satisfying (1–2) and

3. The flow across each contiguous partition $I \sqcup J = P_n$ is exactly $f(d(I))$.

An F-nef function $f: G \to \mathbb{Q}$ (resp., $f: G \to \mathbb{Z}$) that is effective (resp., strongly effective) with respect to all degree functions $d: P_n \to G$, for all possible $n$, is called **effective** (resp., strongly effective).

**Remark 4.3.** Suppose $f$ is an F-nef function with $f(0) = 0$. Then by subadditivity of $f$ (Lemma 3.2), to check effectivity of $f$, it suffices to consider only degree functions $d: P_n \to G$ for which there is no $S \subsetneq P_n$ satisfying $\sum_{i \in S} d(i) = 0 \in G$. In this case, for a given group $G$, there are only finitely many $n$ and degree functions to check. However, because edge weights are allowed to be arbitrary rational numbers, we do not know an algorithm for checking effectivity of $f$ even with respect to a fixed degree function $d: P_n \to G$.

As we will see in Propositions 4.4 and 4.5, effective (resp., strongly effective) F-nef functions give rise to nef (resp., base-point-free) divisors on $\mathcal{M}_{0,n}$. Although the statements of the two theorems are parallel, the proofs we give differ drastically.

**Proposition 4.4.** Suppose $f: G \to \mathbb{Q}$ is an effective F-nef function. Then for all $n$ and all $\{d_i\}_{i=1}^n$, where $d_i \in G$ satisfy $\sum_{i=1}^n d_i = 0 \in G$, the divisor

$$\mathcal{D}(G,f;(d_1,\ldots,d_n)) = \sum_{i=1}^n f(d_i) \psi_i - \sum_{I \sqcup J = [n]} f\left(\sum_{i \in I} d_i\right) \Delta_{I,J}$$

is nef on $\overline{\mathcal{M}}_{0,n}$.

**Proof.** By the functoriality of $\mathcal{D}(G,f;(d_1,\ldots,d_n))$ with respect to the boundary stratification as given in Proposition 3.6 (1), we only need to prove that $\mathcal{D}(G,f;(d_1,\ldots,d_n))$ intersects non-negatively every irreducible curve in $\overline{\mathcal{M}}_{0,n}$ with a generic point in $M_{0,n}$. To do this, identify the $n$ marked points with $P_n$. Observe that the effectivity of $f$ implies that the divisor $\mathcal{D}(G,f;(d_1,\ldots,d_n))$ is linearly equivalent to an effective combination of the boundary divisors by Lemma 4.1. We are done. □

A small elaboration of Lemma 4.1 gives us a bit more:
Proposition 4.5 (Base-point-freeness criterion). Suppose \( d_i \in G, 1 \leq i \leq n, \) satisfy \( \sum_{i=1}^{n} d_i = 0 \in G. \) Suppose \( f : G \to \mathbb{Z} \) is an F-nef function such that for every \( \sigma \in \mathcal{S}_n \) and the degree function \( d^\sigma : P_n \to G \) defined by \( d^\sigma(\sigma(i)) = d_i, \) we have that \( f \) is strongly effective with respect to \( d^\sigma. \) Then the divisor
\[
\mathcal{D}(G, f; (d_1, \ldots, d_n)) = \sum_{i=1}^{n} f(d_i) \psi_i - \sum_{I \sqcup J = [n]} f\left(\sum_{i \in I} d_i\right) \Delta_{I,J}
\]
is base-point-free on \( \overline{\mathcal{M}}_{0,n}. \)

Proof. We prove base-point-freeness by exhibiting for each \([C] \in \overline{\mathcal{M}}_{0,n}\) an effective divisor \( D_C \) such that \( \mathcal{D}(G, f; (d_1, \ldots, d_n)) \sim D_C \) and \( [C] \notin \text{Supp}(D_C). \)

The key observation is that for every \([C] \in \overline{\mathcal{M}}_{0,n},\) there exists a bijection \( \sigma : [n] \to P_n, \) between the set of \( n \) marked points of \( C \) and \( P_n, \) such that for every \( \Delta_{I,J} \) satisfying \([C] \in \Delta_{I,J},\) the partition \( \sigma(I) \sqcup \sigma(J) = P_n \) is contiguous. To see this, consider a planar realization of the dual graph \( G(C) \) of \( C, \) with vertices of \( G(C) \) corresponding to the irreducible components of \( C, \) edges of \( G(C) \) to the nodes of \( C, \) and half-edges of \( G(C) \) to the marked points. Choose a simple loop in the plane around \( G(C). \) The order in which the half-edges are encountered as one goes around the loop in the clockwise direction gives a requisite \( \sigma : [n] \to P_n. \) Indeed, stretching the edge of \( G(C) \) corresponding to \( \Delta_{I,J} \) shows that points of \( \sigma(I) \) are contiguous in \( P_n. \)

Let \( \sigma : [n] \to P_n \) be as in the previous paragraph. Since \( f \) is strongly effective with respect to \( d^\sigma \) by assumption, there exists a \( \mathbb{Z}\)-weighting of \( P_n \) satisfying (1–3) of Definition 4.2. Lemma 4.1 now shows that
\[
\mathcal{D}(G, f; (d_1, \ldots, d_n)) \sim \sum_{I \sqcup J = [n]} c_{I,J} \Delta_{I,J} =: D_C,
\]
where \( c_{I,J} \geq 0 \) and \( c_{I,J} = 0 \) for any \( I \sqcup J = [n] \) such that \([C] \in \Delta_{I,J}.\) We are done.

Scholium 4.6 (Effective base-point-freeness). Suppose \( D \) is a divisor on \( \overline{\mathcal{M}}_{0,n} \) from Proposition 4.5. Let \( \{B_i\}_{i=1}^{N} \) be the zero-dimensional boundary strata of \( \overline{\mathcal{M}}_{0,n}. \) Then there exists a linear series \( \{D_i\}_{i=1}^{N} \subset |D|, \) such that each \( D_i \) is an effective linear combination of boundary and \( B_i \notin \text{Supp}(D_i). \) In particular, there is a morphism \( \overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{N-1} \) such that \( D \) is the pullback of \( \mathcal{O}_{\mathbb{P}^{N}}(1) \) and the pullback of every standard hyperplane in \( \mathbb{P}^{N} \) is supported on the boundary of \( \overline{\mathcal{M}}_{0,n}. \)

4.3. Cyclic weighting. We proceed to give a sufficient criterion for an F-nef function to be strongly effective.

Proposition 4.7 (Cyclic weighting). Let \( f : \mathbb{Z}_m \to \mathbb{Z} \) be a symmetric function such that \( f(0) = 0. \) If the associated quadratic form \( Q_f \) is integral and balanced, then \( f \) is a strongly effective F-nef function.
We now compute the weight on $(	ext{weight of text})$

Figure 1. A planar realization of a dual graph of a 13-pointed rational curve $C$. The half-edges correspond to the marked points. To obtain $\sigma : [13] \to P_{13}$ such that $\sigma(I)$ is contiguous for all partitions $I \cup J$ satisfying $[C] \in \Delta_{I,J}$, choose a loop around the graph. The order in which the half-edges are encountered as one goes around the loop gives a requisite $\sigma$.

If $Q_f$ is integral and $\ell$-balanced, then $f$ is strongly effective with respect to all degree functions $d : P_n \to \mathbb{Z}_m$ satisfying $\sum_{i=1}^{n} d(i) \leq m\ell$.

**Remark 4.8.** The integrality condition on $Q_f$ is essentially harmless and can always be achieved by doubling $f$ (cf. (3.5)).

**Proof.** Consider a degree function $d : P_n \to \mathbb{Z}_m$. Suppose $\sum_{i=1}^{n} d(i) = m\ell$. The key observation is that a weighting on $P_{m\ell}$ satisfying conditions (1–3) of Definition 4.2 with respect to the constant degree function $P_{m\ell} \mapsto 1 \in \mathbb{Z}_m$ induces a weighting on $P_n$ that satisfies conditions (1–3) with respect to $d$: Let the vertex $i \in P_n$ correspond to $d(i)$ adjacent vertices $S_i \subset P_{m\ell}$ in such a way that $\bigcup_{i=1}^{n} S_i = P_{m\ell}$ and $S_i$'s occur in the clockwise order in $P_{m\ell}$. Then define the weight of $(i \sim j)$ in $E(P_n)$ to be the sum of weights of the edges in $E(P_{m\ell})$ joining $S_i$ and $S_j$.

We have reduced to proving the assertion of the proposition in the case when $d_1 = \cdots = d_n = 1$ and $m \mid n$. Let $q_f$ be as in (3.5). We define the weight function $w$ on $P_n$ by

$$w(i \sim j) := -q_f(i-j) = \frac{1}{2} (2f(i-j) - f(i-j-1) - f(i-j+1)).$$

Since $f$ is a function on $\mathbb{Z}_m$, $w$ is invariant under rotations of $P_n$ by the angle $2\pi/m$. In particular,

- The weight $w(i \sim j)$ depends only on $i-j \pmod{m}$.
- All vertices have the same $w$-flow.
- The $w$-flow across a contiguous partition $I \cup J$ depends only on $|I|$.

We now compute the $w$-flow through the vertex $1 \in [n]$:

$$w(1) = \sum_{j=2}^{n} w(1 \sim j) = -\sum_{j=2}^{n} q_f(j-1) = \frac{1}{2} \sum_{j=2}^{n} (2f(j-1) - f(j) - f(j-2)) = f(1).$$
It follows that \( w \) satisfies Condition (1) of Definition 4.2.

Suppose now that \( I \cup J \) is an arbitrary partition of \( P_{n+1} \). For \( k = 0, \ldots, m - 1 \), let \( x_k \) be the number of vertices in \( I \) whose index is congruent to \( k \) (mod \( m \)). Then \( x_0 + \cdots + x_{m-1} = d(I) = mc + r \) for some \( c \) and \( 0 \leq r \leq m - 1 \). Tracing through the construction we see that

\[
(4.2) \quad w(I \mid J) = - \sum_{0 \leq i, j \leq m-1} q_f(i - j)x_i(m - x_j)
\]

\[
= -m \left( \sum_{i=0}^{m-1} x_i \right) \left( \sum_{k=0}^{m-1} q_f(k) \right) + \sum_{0 \leq i, j \leq m-1} q_f(i - j)x_i x_j
\]

\[
= Q_f(x_0, \ldots, x_{m-1}).
\]

If \( Q_f \) is balanced, then \( w(I \mid J) \) is minimized at \( x_0 = x_1 = \cdots = x_{r-1} = c + 1 \) and \( x_r = \cdots = x_{m-1} = c \). In particular, this is achieved when \( I \cup J \) is contiguous. We conclude that the minimum value of \( w(I \mid J) \) under the constraint \( d(I) = mc + r \) is achieved for contiguous partitions and equals to

\[
Q_f((1 + c)_{r, (c)_{m-r}}) = Q_f((1)_{r, (0)_{m-r}}) = f(r) = f(d(I)).
\]

It follows that \( w \) satisfies Conditions (2–3) of Definition 4.2.

Finally, if \( \sum_{i=1}^{n} d_i = m\ell \), then \( 0 \leq x_k \leq \ell \) for every \( k \), and the same conclusions hold under the assumption that \( Q_f \) is \( \ell \)-balanced. \( \square \)

5. Semiample Divisors on \( \overline{M}_{0,n} \)

In this section, we state and prove two semiampleness criteria for divisors on \( \overline{M}_{0,n} \). Our first result says that a single balanced cyclic quadratic form gives rise to an infinite sequence of semiample divisors:

**Theorem 5.1.** Suppose \( f : \mathbb{Z}_m \to \mathbb{Z} \) is an F-nef function such that the associated quadratic form \( Q_f \) is balanced. Then

\[
2\mathcal{O}(\mathbb{Z}_m \setminus f; (d_1, \ldots, d_n))
\]

is base-point-free on \( \overline{M}_{0,n} \) for all \( d_1, \ldots, d_n \in \mathbb{Z}_m \) such that \( m \mid \sum_{i=1}^{n} d_i \).

**Proof.** Suppose \( f : \mathbb{Z}_m \to \mathbb{Z} \) is an F-nef function such that the associated quadratic form \( Q_f \) is balanced. Then \( 2f \) is strongly effective by Proposition 4.7. It follows by Proposition 4.5 that \( 2\mathcal{O}(\mathbb{Z}_m \setminus f; (d_1, \ldots, d_n)) \) is base-point-free on \( \overline{M}_{0,n} \). \( \square \)

**Example 5.2.** The quadratic form associated to the standard function (Example 3.3) \( f_m : \mathbb{Z}_m \to \mathbb{Z} \) is

\[
Q_{f_m}(x_0, \ldots, x_{m-1}) = m(x_0^2 + x_1^2 + \cdots + x_{m-1}^2) - \left( \sum_{i=0}^{m-1} x_i \right)^2.
\]
Note that $Q_{A_m}$ is balanced because $x_0^2 + x_1^2 + \cdots + x_{m-1}^2$ is balanced. We note that every $s_m$ level 1 conformal blocks divisor (see [GG12]) can be written as a multiple of $\mathcal{D}(Z_m, A_m; (d_1, \ldots, d_n))$ for some $d_1, \ldots, d_n \in \mathbb{Z}_m$ such that $m \mid \sum_{i=1}^{n} d_i$. Thus Theorem 5.1 gives a new proof of semiampleness for these divisors.

Example 5.3. The quadratic form associated to the 2nd standard function (Example 3.4) $B_m$ is

$$Q_{B_m}(x_0, \ldots, x_{m-1}) = m \sum_{i=0}^{m-1} (x_i - x_{i+1} + x_{i+2})^2 - \left( \sum_{i=0}^{m-1} x_i \right)^2.$$  

Recall that $Q_{B_m}$ is balanced for all $m \geq 4$ such that $m \not\equiv \pm 1 \pmod{6}$ (Lemma 2.10). We list some examples of semiample divisors associated to $B_m$ by Theorem 5.1:

$$\mathcal{D}(Z_{15}, B_{15}; (1)_{15}) = 13\Delta_2 + 18\Delta_3 + 22\Delta_4 + 25\Delta_5 + 27\Delta_6 + 28\Delta_7$$

is a semiample extremal ray of $\text{Nef}(\overline{M}_{0,15})^{\oplus_{15}}$.

$$\mathcal{D}(Z_{16}, B_{16}; (1)_{16}) = 14\Delta_2 + 17\Delta_3 + 24\Delta_4 + 30\Delta_5 + 30\Delta_6 + 34\Delta_7 + 32\Delta_8$$

is a semiample extremal ray of $\text{Nef}(\overline{M}_{0,16})^{\oplus_{16}}$.

$$\mathcal{D}(Z_{18}, B_{18}; (1)_{18}) = 32\Delta_2 + 45\Delta_3 + 56\Delta_4 + 65\Delta_5 + 72\Delta_6 + 77\Delta_7 + 80\Delta_8 + 81\Delta_9$$

is a semiample extremal ray of $\text{Nef}(\overline{M}_{0,18})^{\oplus_{18}}$.

We also note that although

$$\mathcal{D}(Z_{17}, B_{17}; (1)_{17}) = 15\Delta_2 + 21\Delta_3 + 26\Delta_4 + 30\Delta_5 + 33\Delta_6 + 35\Delta_7 + 36\Delta_8$$

is an extremal ray of the symmetric F-nef cone of $\overline{M}_{0,17}$, our Theorem 5.1 does not prove its semiampleness because the quadratic form $Q_{B_{17}}$ is not balanced by Lemma 2.10.

Since checking whether a given quadratic form is balanced can be difficult, we proceed to give a simple sufficient criterion for a symmetric divisor on $\overline{M}_{0,n}$ to be semiample.

Suppose $D = \sum_{i=2}^{\lfloor n/2 \rfloor} a_i \Delta_i$ is a symmetric divisor on $\overline{M}_{0,n}$. Set $a_0 = a_1 = 0$ and $a_i = a_{n-i}$ for $\lfloor n/2 \rfloor + 1 \leq i \leq n - 1$. Define a function $q_D: \mathbb{Z}_n \to \mathbb{Q}$ by

$$q_D(i) := \frac{1}{2}(2a_i - a_{i-1} - a_{i+1}).$$

and a quadratic form

$$(5.1) \quad Q_D(x_0, \ldots, x_{n-1}) := \sum_{0 \leq i, j \leq n-1} q_D(i - j)x_i x_j.$$  

We call $Q_D$ the associated quadratic form of $D$.

Theorem 5.4 (Semiampleness Criterion). Suppose $D$ is a symmetric divisor on $\overline{M}_{0,n}$ such that the associated quadratic form $Q_D$ is weakly balanced. Then $D$ is semiample.
Proof. Suppose $D$ is a symmetric divisor on $\overline{M}_{0,n}$ such that the associated quadratic form $Q_D$ is weakly balanced. Then $D$ is F-nef by (3.8). Let $f_D$ be the associated F-nef function of $D$ (see 3.2.1). Then 

$$Q_{f_D} = Q_D + c_0(x_0 + \cdots + x_{m-1})^2,$$

is weakly balanced. Proposition 4.7 implies that some positive multiple of $f_D$ is strongly effective with respect to the constant degree function $P_n \mapsto 1 \in \mathbb{Z}_n$. Finally, Proposition 4.5 implies that $D = D(\mathbb{Z}_n, f_D; (1)_n)$ is semiample. \hfill \Box

As observed in Lemma 3.8, weak balancedness of $Q_D$ translates into a system of linear inequalities on the coefficients of $D$ defining a convex polyhedral cone in $\text{NS}(\overline{M}_{0,n})^{\mathbb{S}_n}$. The number of such inequalities growth as the number of distinct principal minors of $Q_D$, which is exponential in $n$.

6. Extensions and Computations

To obtain more semiample symmetric divisors on $\overline{M}_{0,n}$ than the ones covered by Theorem 5.4, we will use Kawamata base-point-freeness theorem and a by now classical result of Keel and McKernan.

Recall that $K_{\overline{M}_{0,n}} = \psi - 2\Delta$. We say that a symmetric divisor $D$ on $\overline{M}_{0,n}$ is log terminal (resp., log canonical), if for some positive rational $a$ we can write

$$aD = K_{\overline{M}_{0,n}} + \sum_{i=2}^{[n/2]} c_i \Delta_i,$$

where $0 \leq c_i < 1$ (resp., $0 \leq c_i \leq 1$).

Proposition 6.1 (char = 0). Suppose $D$ is a non-zero F-nef divisor on $\overline{M}_{0,n}$ which can be written as $D = L + N$, where $L$ is log terminal (resp., log canonical) and $N$ is a nef divisor on $\overline{M}_{0,n}$. Then $D$ is semiample (resp., nef).

Proof. Suppose $D = L + N$ is F-nef, where $L$ is log terminal and $N$ is nef. Then $D$ is nef by [KM13, Theorem 1.2.2]. Recalling that a non-zero symmetric nef divisor on $\overline{M}_{0,n}$ is necessarily big (see [KM13, Theorem 1.3.2]), we apply the Kawamata base-point-freeness theorem [KM98, Theorem 3.3] to conclude that $D$ is semiample. This establishes the first part of the proposition.

Let $C$ be the open convex cone in $\text{NS}(\overline{M}_{0,n})^{\mathbb{S}_n}$ generated by divisors of the form $K_{\overline{M}_{0,n}} + \sum c_i \Delta_i + A$, where $0 < c_i < 1$ and $A$ is an ample divisor. Let $\mathcal{F}$ be the (closed and convex) symmetric F-nef cone of $\overline{M}_{0,n}$. We have already established that $C \cap \mathcal{F} \subset \text{Nef}(\overline{M}_{0,n})^{\mathbb{S}_n}$. Since $C \cap \mathcal{F}$ is a full-dimensional convex cone inside $\text{NS}(\overline{M}_{0,n})^{\mathbb{S}_n}$, we conclude that $C \cap \mathcal{F} = \overline{C \cap \mathcal{F}} \subset \text{Nef}(\overline{M}_{0,n})^{\mathbb{S}_n}$. The remaining part of the proposition follows. \hfill \Box

Corollary 6.2 (char = 0). Let $S$ be the cone spanned by all log canonical divisors and by the extremal rays of the symmetric F-nef cone of $\overline{M}_{0,n}$ that are semiample by Theorem
5.4. Then every F-nef divisor in the interior of $S$ is semiample and every F-nef divisor in $S$ is nef.

| $n$ | Extremal rays of F-nef cone | Semiample rays from Theorem 5.4 (arbitrary char.) | Semiample rays from Corollary 6.2 | Additional nef rays Corollary 6.2 | Undetermined |
|-----|----------------------------|-----------------------------------------------|-------------------------------|-------------------------------|-------------|
| 14  | 27                         | 13                                            | 27                            | 0                             | 0           |
| 15  | 26                         | 11                                            | 26                            | 0                             | 0           |
| 16  | 74                         | 19                                            | 72                            | 1                             | 1           |
| 17  | 113                        | 22                                            | 107                           | 5                             | 1           |
| 18  | 159                        | 26                                            | 157                           | 1                             | 1           |
| 19  | 371                        | 48                                            | 354                           | 12                            | 5           |
| 20  | 739                        | 60                                            | 723                           | 9                             | 7           |
| 21  | 905                        | 40                                            | 864                           | 32                            | 9           |
| 22  | 3082                       | 125                                           | 3039                          | 16                            | 27          |

**Table 1.** Experimental results

6.1. **Experimental results.** We wrote a simple worksheet in SAGE \[S^{+14}\] (which in turn uses PARI/GP \[The14\] for certain computations with quadratic forms), available at [https://www2.bc.edu/maksym-fedorchuk/Semiampleness-criterion.sws](https://www2.bc.edu/maksym-fedorchuk/Semiampleness-criterion.sws), that can compute, among other things, whether:

- A positive definite cyclic quadratic form is good.\(^2\)
- A given symmetric divisor on $\overline{M}_{0,n}$ satisfies semiampleness criterion of Theorem 5.4.
- A given symmetric divisor on $\overline{M}_{0,n}$ satisfies semiampleness criterion of Corollary 6.2.

We compute the number of extremal rays of the symmetric F-nef cone of $\overline{M}_{0,n}$ that are semiample (or nef) by Theorem 5.4 or Corollary 6.2 in Table 1. The output for a given $n$ is obtained by running the command `SemiampleAndNef(n)` of the worksheet.

**Remark 6.3** (Running time). The first row of the table takes about 11 seconds and the last row takes about 2 days to run.

\(^2\)Unfortunately, SAGE or PARI/GP quickly runs out of memory for more interesting quadratic forms.
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