The Polarization of the Cosmic Microwave Background Due to Primordial Gravitational Waves

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We review current observational constraints on the polarization of the Cosmic Microwave Background (CMB), with a particular emphasis on detecting the signature of primordial gravitational waves. We present an analytic solution to the Polnarev approximation for CMB polarization produced by primordial gravitational waves. This simplifies the calculation of the curl, or B-mode power spectrum associated with gravitational waves during the epoch of cosmological inflation. We compare our analytic method to existing numerical methods and also make predictions for the sensitivity of upcoming CMB polarization observations to the inflationary gravitational wave background. We show that upcoming experiments should be able either detect the relic gravitational wave background or completely rule out whole classes of inflationary models.

Keywords: cosmic microwave background; polarization; measurements

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1. Introduction

The cosmic microwave background (CMB) is one of the most powerful and precise cosmological probes. Because the CMB photons we observe today probe the physics
of the early universe during the epoch of linear gravity, the CMB is often referred to as a “snapshot” of the primordial universe. The CMB has the promise to address the most fundamental cosmological questions: the geometry and age of the universe, the matter-energy content of the universe, the ionization history and the spectrum of primordial perturbations.

This review addresses the theoretical foundations of CMB polarization generated by cosmological gravitational waves with particular emphasis given to analytic and numerical results which encode its behavior most cogently. Gravitational waves, in contrast to adiabatic perturbations (which are the dominant source of CMB temperature and polarization anisotropy) imprint a unique divergence-free pattern of polarization on the sky. This pattern is called “B-mode” or “curl-mode” polarization. Although there may be a substantial GWB contribution to CMB temperature anisotropy, its effect on the CMB temperature is nearly completely degenerate with other cosmological parameters. This is unsurprising since the CMB temperature is a scalar quantity. However, the tensorial nature of CMB polarization permits separation of scalar fluctuations from tensor, GWB-generated, fluctuations. CMB polarization maps can be decomposed into two terms. One term is the gradient of a scalar potential and is invariant under parity transforms (often called “E-mode” in analogy to the electric field/scalar potential). The second component is the curl of a vector potential (“B-mode”). Scalar perturbations have no handedness so the primary CMB curl-mode component exists only if there is a GWB. Besides the obvious importance of a new method of (indirect) detection of gravitational waves, detection of the B-mode signal provides the cleanest, and perhaps only window into unique predictions of the inflationary cosmological paradigm.

Numerical methods for simulating CMB temperature and polarization power spectra have revolutionized cosmology. Without such codes parameter estimation from CMB data sets would be difficult, if not impossible. However, the complexity of these codes has hidden the underlying physics. For this reason there is considerable interest in analytical approaches for calculating CMB polarization caused by cosmological gravitational waves (see for example Ref. 1, 9, 10, 11, 12).

This review elucidates an analytical approach to the problem using the fundamental physical characteristics of gravitational waves to significantly simplify calculations. Our method simplifies the comparison between theoretical predictions and future observational results. We compare our results to existing numerical methods and summarize current observational results. We conclude with by making predictions for ground, balloon, and space-based observations in the upcoming decade.

2. Inflation and Primordial Gravitational Waves

Inflation is a bold cosmological paradigm which has provided key insights into many observations of modern cosmology. Inflation provides an explanation for the observed spatial-flatness of the universe, motivates the lack of topological defects
such as magnetic monopoles, and explains the exquisite isotropy of the cosmic microwave background (CMB) while simultaneously providing a mechanism to generate the observed fluctuations. However, along with its many successes has come increased scrutiny. Many of inflation’s key predictions are shared by alternative models. While skeptics are increasingly challenged to attack it, inflation’s proponents can only claim circumstantial evidence in its favor. Only new discoveries will provide the data required to break the deadlock. A conclusive detection of the primordial gravitational wave background (GWB) predicted by inflation would be “the smoking gun” confirming the inflationary model beyond a reasonable doubt. No other known cosmological mechanism mimics the imprint of the GWB on the polarization of the CMB.

One of inflation’s successful predictions is its solution to the “horizon problem” - the observation that regions of the universe share the same thermodynamic temperature despite never being in thermal, or even causal, contact. Inflation solves the horizon problem via a superluminal expansion of the universe at very early times, prior to the ordinary Hubble expansion observed today. This rendered the entire observable universe within causal contact initially. This expansion also accounts for the (seemingly) finely-tuned spatial flatness of the universe observed by CMB temperature anisotropy experiments.

This review focuses on inflationary generated gravitational wave, or tensor perturbations. However, inflation also predicts a nearly scale-invariant spectrum of energy density, or scalar, perturbations. That any perturbations remain after the universe expanded by $60 \ e^f$-folds is astonishing! Yet, surprisingly, the observed fluctuation level arises naturally as magnified quantum fluctuations of the scalar field that drove inflation (the inflaton) (see [23] for a review of inflationary perturbation theory). Following the inflationary epoch, the size of the residual fluctuations were imprinted on the surface of last scattering and are observable in the CMB. The observed size of the fluctuations, e.g., [24], and the correlations between the CMB’s temperature and polarization patterns at super-horizon scales motivates, but does not prove, the inflationary paradigm.

Inflation not only produces these perturbations, but also endows them with synchronized initial phases. This is crucial as it allows the perturbations to grow without cancellation, as would occur if the initial phases were uncorrelated. WMAP and other CMB experiments, in combination with large-scale-structure observations, probe the scalar perturbation spectrum and are consistent with inflation.

Regrettably, neither flatness nor smoothness are unique to inflation. Both have long histories in cosmology. The flatness of the universe was anticipated prior to inflation on quasi-anthropic grounds. A scale-invariant primordial matter/energy perturbation spectrum (i.e., deviations from perfect smoothness) was also predicted well before inflation, though no mechanism to produce the perturbations was given in these early works. These perturbations, combined with the universe’s spatial flatness, increase the plausibility of the inflationary paradigm, since the
amount of expansion required for flatness should have also smoothed any initial perturbations to zero.

Two more observations are noteworthy for their consistency with the inflationary paradigm: the low abundance of relicts (e.g., no magnetic monopoles) and the observed Gaussianity of perturbations. Since, as many authors have pointed out, inflation has passed so many observational tests any replacement theory would need to look very much like inflation. However, while there is abundant circumstantial evidence, there is one unique prediction of inflation that cannot be mimicked: a primordial gravitational wave background; thereby cementing its status as “the smoking gun of inflation”.

Inflation posits a new scalar field (the inflaton) and specifies its action potential, and thereby, its dynamics (calculated using the standard methods of scalar field theory). All theories of inflation produce a GWB, though some at unobservable levels. All are scalar field theories, incorporating fluctuations via quantum perturbation theory. While the identity of the inflaton is unknown, specifying the inflaton potential has immediate observational consequences. The most important inflationary measure of the GWB is the tensor-to-scalar ratio, $r$, since it parameterizes the unique prediction of inflation. A detection of $r$ would simultaneously reveal both the epoch of inflation and its energy scale. If, as theorists speculate, inflation is related to the GUT-scale (grand unified theory) then detection of the B-mode signature would probe physics at at the $10^{16}$ GeV scale.

3. CMB Polarization

The CMB has been the most effective tool to appraise inflation because the CMB is the earliest electromagnetic “snapshot” of the universe, and as such, probes the universe in a particularly pristine state - before gravitational and electromagnetic processing. Since gravity is the weakest of the four fundamental forces, gravitational radiation (the inflationary GWB) probes much farther back - to $\sim 10^{-38}$ sec after the Big Bang.

CMB polarization is generated by both scalar and tensor perturbations. The two are related in all models of inflation since both are generated by fluctuations of the same quantum field. Inflation may have occurred at energies too low to detect (or not at all) but the scalar-tensor relation provides a powerful consistency check - insurance against a false-positive claim.

All inflationary observables are determined by the inflaton potential. The process is invertible, affording the opportunity to reconstruct, or significantly constrain, the inflaton potential from CMB measurements of the GWB. Assuming the inflationary paradigm is correct, this provides a window into physics at energy scales below $10^{16}$ GeV even if future CMB observations produce a null result. Without observational constraints, the number of potential inflationary models has swelled to daunting levels. Upcoming experiments should be able to rule out many of these models, even with a null result as we discuss in section...
CMB temperature anisotropy measurements can determine the amplitude of the primordial power spectrum, $A_s$, and the scalar power-spectral index, $n_s$, far better than CMB polarimeters but only polarization sensitive experiments can measure the key-prediction of inflation – the tensor-to-scalar ratio, $r$, if it is below $r \lesssim 0.3$, which is near the current 2$\sigma$ upper limits as discussed in §9. A joint detection of $A_s$, $n_s$, and $r$ will serve to reconstruct the inflaton potential (as well as completely rule out the ekpyrotic and variable-speed-of-light models). This observation requires an understanding of the effects of gravitational waves on the CMB, to which we now turn.

4. Primordial Gravitational Wave CMB Perturbations

Gravitational waves perturb the metric tensor describing the geometry of the early universe. The general, perturbed, metric of a flat FLRW universe can be written as

$$ds^2 = a^2(\eta) \left[-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j\right]. \quad (1)$$

The components $h_{ij}(\eta, x)$ represent the gravitational waves and can be expanded into spatial Fourier harmonics $e^{\pm ik \cdot x}$,

$$h_{ij}(\eta, x) = \frac{1}{(2\pi)^{3/2}} \int dk \sum_{s=1,2} s p_{ij}(k) \left[h_k(\eta) e^{ik \cdot x} + h^*_k(\eta) e^{-ik \cdot x}\right], \quad (2)$$

where $k$ is a time-independent wave vector, and $k = (\delta_{ij}k^ik^j)^{1/2}$. The wave number $k$ defines the wavelength measured in units of laboratory standards by $\lambda = 2\pi a/k$. The polarization tensors $p_{ij}(k)$ have different forms depending on whether they represent gravitational waves, rotational perturbations, or density perturbations.[40] In the case of gravitational waves the polarization tensors can be expressed in terms of two mutually orthogonal unit-vectors $(l, m)$ lying in the plane of the wave front (i.e. perpendicular to $k$),

$$\frac{1}{2} p_{ij} = l_i l_j - m_i m_j, \quad \frac{2}{3} p_{ij} = l_i m_j + l_j m_i, \quad (3)$$

and obey the conditions

$$s p_{ij} \delta^{ij} = 0, \quad s p_{ij} k^i = 0, \quad (4)$$

$$s' p_{ij}(k) s^{ij}(k) = 2 \delta_{s's}. \quad$$

For a classical gravitational wave field, the quantities $\hat{c}_k$ and $\hat{c}_k^\dagger$ in Eq. 2 are arbitrary complex (conjugate) numbers.

The Fourier expansion allows us to reduce the problem of evolution of the perturbed gravitational field to the evolution of mode functions $h_k(\eta)$ for each
individual mode $k$. For a single Fourier component the amplitude of the primordial gravitational wave obeys the following wave equation \(^{(11)}\):

$$\frac{d^2 h_k}{d\eta^2} + \frac{2}{a} \frac{da}{d\eta} \frac{dh_k}{d\eta} + k^2 h_k = 0,$$

(5)

This equation ignores the damping effects of anisotropic stress provided by cosmological neutrinos \(^{(12)}\). For $k\eta \ll 1$ (wavelength larger than the cosmological horizon) the last term in the above equation can be dropped out, and the amplitude of the gravitational wave is “frozen” ($h_k \sim \text{const}$). For $k\eta \gg 1$ (wavelength smaller than the horizon) the solutions for $h_k$ are damped plane waves. Since larger $k$ perturbations enter the horizon earlier, their contribution to the tensor spectrum is more heavily damped.

Equation (5) allows us to study the gravitational wave perturbations in two cosmological epochs: 1) in the matter dominated epoch, governed by matter with effective equation of state $p = 0$, where the scale factor behaves as $a(\eta) \propto \eta^2$ and 2) in the radiation dominated epoch, governed by the effective equation of state $p = \epsilon/3$, with $a(\eta) \propto \eta$.

In the general case with cosmic scale factor $a(\eta)$, Eq. (5) might not allow for an analytical solution. For a mixed matter-radiation Universe, the scale factor is given by the expression \(^{(43)}\)

$$a(\eta) = \left[ \frac{4l_H}{1 + \sqrt{1 + 4\eta_*^2}} \right] \eta(\eta + 2\eta_*),$$

(6)

where $\eta_* = (\sqrt{2} - 1)\eta_{eq}$, and $\eta_{eq}$ is the time of matter-radiation equality. The participating constants have been chosen such that the value of the scale factor at the present epoch $\eta_0$ is $a(\eta_0) = 2l_H$, which leads to the value of the time at present epoch $\eta_0 = \left[ (1 - 2\eta_*) + \sqrt{1 + 4\eta_*^2} \right] / 2$. With such a convention, the wave whose wavelength, $\lambda$, today is equal to present Hubble radius, carries the constant wavenumber $k_H = 4\pi$. The scale factor has two asymptotic regions: $\eta \ll \eta_*$ corresponding to the radiation dominated epoch with $a(\eta) \propto \eta$, and $\eta \gg \eta_*$ corresponding to the matter dominated epoch with $a(\eta) \propto \eta^2$.

An elegant method to analyze the evolution of gravitational waves in such a universe is to smoothly approximate the above scale factor with power-law scale factors, for which we can analytically solve the equation (5). The scale factor (Eq. 6) is well approximated by

$$a(\eta) = a_0 \cdot \eta, \quad \eta \leq \eta_*,$$
$$a(\eta) = 2l_H \cdot (\eta - \eta_m)^2, \quad \eta_m \leq \eta.$$  (7)

Imposing continuity of $a(\eta)$ and $a'(\eta)$ at matter-radiation equality, $\eta_*$, fully deter-
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\[ \eta^{**} = -\eta_m = \frac{1}{2\sqrt{1 + z_{eq}}}, \]
\[ = a_0 \frac{4H}{\sqrt{1 + z_{eq}}}. \]  

(8)

The general solution to Eq. 5 for the scale factor (Eq. 7) is given by

\[ h_k(\eta) = \begin{cases} a_0 (C_k e^{-ik\eta} + D_k e^{ik\eta}), & \eta < \eta^{**} \\ \frac{\sqrt{k}}{2H(\eta - \eta_m)^{3/2}} \left[ A_k J_{3/2}(k(\eta - \eta_m)) - iB_k J_{-3/2}(k(\eta - \eta_m)) \right], & \eta^{**} < \eta \end{cases} \]  

(9)

The constants \( C_k \) and \( D_k \) are determined by the evolution of the gravitational waves before radiation domination. Since the radiation dominated era is believed to have been preceded by inflation the values of the coefficients \( C_k \) and \( D_k \), which give the spectral characteristics of the gravitational wave field, are determined by the physics of inflation and initial conditions (see for example 44, 45). It follows that due to the initial stage of rapid expansion \( C_k \approx -D_k \).

The fact that \( C_k \approx -D_k \) shows that the gravitational wave modes, \( k \), are (almost) standing waves at the radiation epoch. To find the coefficients \( A_k \) and \( B_k \) in the solution (9) \( h_k(\eta) \) and \( dh_k(\eta)/d\eta \) must be joined continuously at the transition point \( \eta = \eta^{**} \).

The above analytical approximation (9) sufficiently well approximates the solution to the equation (5) in the case of the scale factor given by (6). Numerical calculations show that the above analytical approximations work very well for wavenumbers \( k \) satisfying \( k\Delta\eta_{eq} < 1 \), where \( \Delta\eta_{eq} \) is the characteristic time scale of change from the radiation-dominated era to matter-dominated era (14).

5. CMB Polarization Observables

Having considered the underlying cosmological behavior of tensor perturbations we now turn to the imprint of tensor gravitational waves on CMB polarization using the equation of radiative transport. To begin we consider a polarized electromagnetic wave with angular frequency, \( \omega \):

\[ \mathbf{E} = E_{y0} \sin(\omega t - \delta_y) \mathbf{\hat{y}} + E_{x0} \sin(\omega t - \delta_x) \mathbf{\hat{x}}. \]

The polarization state of is characterized by the Stokes parameters: \( I, Q, U, \) and \( V \):

\[ I = I_y + I_x, \]

with \( I_y = \langle E_{y0}^2 \rangle \) and \( I_x = \langle E_{x0}^2 \rangle \). \( I \) is the total intensity of the radiation, and is always positive. The other Stokes parameters are defined as

\[ Q = I_y - I_x \]
\[ U = 2E_{y0}E_{x0} \cos(\delta_y - \delta_x) \]
\[ V = 2E_{y0}E_{x0} \sin(\delta_y - \delta_x) \]
where $Q$ and $U$ quantify the linear polarization of the wave, and $V$ quantifies the degree of circular polarization. The polarization fraction is $\Pi = \sqrt{Q^2 + U^2 + V^2}$, and the polarized intensity is $I_{\text{pol}} = \Pi \times I$. The Stokes parameters comprise a symbolic vector $\hat{I}$ introduced by Chandrasekhar and related to the Stokes parameters in the following way:

$$
\hat{I} = \begin{pmatrix}
I_x \\
I_y \\
U \\
V
\end{pmatrix}.
$$

Thomson scattering only produces linear CMB polarization, implying $V = 0$, so we will only consider the symbolic 3-vector: $\hat{I} = \begin{pmatrix} I_x \\ I_y \\ U \end{pmatrix}$.

Polarized radiation in the presence of cosmological metric perturbations is represented as state vector describing the occupation numbers of polarized radiation:

$$
\hat{n} = \frac{c^2}{h\nu^3} \hat{I} = \hat{n}_0 + n_0 \delta \hat{n} = n_0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \delta \hat{n}.
$$

The Boltzmann equation of radiative transfer written in terms of $\hat{n}(\eta, x^\alpha, \theta, \psi)$ is:

$$
\frac{\partial \hat{n}}{\partial \eta} + e^\alpha \cdot \frac{\partial \hat{n}}{\partial x^\alpha} = \frac{\partial \hat{n}}{\partial \nu} \frac{\partial \nu}{\partial \eta} - q(\hat{n} - \hat{J}).
$$

and

$$
\hat{J} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \hat{P}(\theta, \psi, \theta', \psi') \hat{n}(\eta, x^\alpha, \nu, \theta', \psi') \sin \theta' d\theta' d\psi',
$$

where $q = \sigma_T N_e a$, $a$ is the cosmological scale factor, $\hat{J}$ is the “scattering” or “collisional” term which is a function of the angular variables only, with primed angular variables corresponding to the photon direction before scattering and unprimed angular variables corresponding to the photon direction after scattering. $\hat{P}$ is the scattering matrix, $\sigma_T$ is the Thomson cross section, and $N_e$ is the comoving number density of free electrons. The coupling of the gravitational waves to the radiation is manifested in the first term on the right side of Eq. 11:

$$
\frac{1}{\nu} \frac{d\nu}{d\eta} = \frac{1}{2} \frac{\partial \nu}{\partial \eta} e^\alpha e^\beta.
$$

Let us introduce $\hat{n}_0 = n_0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ corresponding to unpolarized isotropic thermal radiation (zero-th order approximation), with $n_0 = [\exp h\nu/k_BT - 1]^{-1}$, which depends only on the photon frequency $\nu$ and corresponds to the Planck spectrum.
The angular symmetry of $\hat{P}$ requires:

$$\hat{J}(\hat{n}_0) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \hat{P}(\theta, \psi, \theta', \psi') \hat{n}_0 \sin \theta' d\theta' d\psi' = \hat{n}_0.$$

(13)

It should be mentioned that the form of the scattering integral in (13) assumes that the chosen reference frame is comoving with the scattering electrons. In the case of scalar perturbations an additional Doppler term may arise due to the movement of electrons with respect to the chosen reference frame. In the case of gravitational waves this Doppler term does not arise, since it is always possible to choose our synchronous reference frame to be comoving with the scattering electrons.

In Eq. (10) $\delta \hat{n}$ is the first order correction to the uniform, isotropic, and unpolarized radiation described by $\hat{n}_0$. This perturbation is comprised of an (unpolarized) term due to angular anisotropy of the photon distribution, $\hat{n}_A$, and a polarized term, $\hat{n}_\Pi$, so $\delta \hat{n} = \hat{n}_A + \hat{n}_\Pi$.

The anisotropic and polarized components are functions of conformal time, $\eta$, co-moving spatial coordinates, $x^\alpha$, photon frequency, $\nu$, and photon propagation direction specified by the unit vector $e^\alpha(\theta, \phi)$ with polar angle, $\theta$, and azimuthal angle, $\phi$. All of the polarization-specific cosmological phenomena discussed in this review can be described by their effect on $\delta \hat{n}$. These effects, if measurable, can be used to evaluate and refine the standard cosmological model. Since CMB polarization depends on anisotropy, it probes all the same underlying physics and, in addition, several cosmological effects are only observable via CMB polarization and not CMB anisotropy. For this reason, CMB polarization is a valuable cosmological tool. The penalty we pay, however, is that the Thomson scattering which produces CMB polarization has a fairly inefficient coupling to cosmological perturbations. We will evaluate the cosmological phenomena to which CMB polarization is sensitive in subsequent sections.

We retain only the zero and first order perturbation terms in $h_{\alpha\beta}$. Since $\frac{d\nu}{d\eta}$ is of the first order, we can replace $\frac{d\hat{n}}{d\nu}$ by $\frac{\partial \hat{n}_0}{\partial \nu_0}$ in Eq. (11) ($\nu_0$ is the unperturbed frequency). This implies that the frequency dependence of both the polarization and anisotropy is given by the same factor.

$$\gamma = \frac{\nu_0 \frac{d\hat{n}_0}{\hat{n}_0}}{\frac{d\nu_0}{\nu_0}}.$$

We now concentrate on the first-order terms. In the following we will identify $\hat{J}(\hat{n}) = \hat{J}_1(\delta \hat{n})$. An arbitrary gravitational wave can be considered as a linear superposition of plane gravitational waves. Due to the linear nature of the problem the anisotropy and polarization generated by an arbitrary gravitational wave is the linear superposition of anisotropy and polarization generated by plane gravitational waves. After spatial Fourier transformation, the first-order Boltzmann equation becomes:

$$\frac{d\delta \hat{n}(\eta, \bm{k})}{d\eta} + ik\mu_k \delta \hat{n}(\eta, \bm{k}) = -\frac{1}{2} \gamma \frac{\partial h_{\alpha\beta}}{\partial \eta} e^\alpha e^\beta - \nu_0 \nu \frac{d\nu_0}{\nu_0} \frac{d\hat{n}_0}{\hat{n}_0} \frac{d\hat{n}_0}{d\nu_0} [\delta \hat{n}(\eta, \bm{k}) - \hat{J}(\eta, \bm{k})].$$

(14)
where, $\mu_k = \frac{\epsilon_\alpha}{k}$, $k = |k|$ and $\phi_k$ is the azimuthal angle of $e^\alpha$ in the plane perpendicular to the vector $k$. In the case of a single gravitational wave we can choose our spherical coordinate system in such a way that $\cos \theta = \mu_k$ and $\phi = \phi_k$ (we shall omit the index $k$ in $\mu$ and $\phi$ when such an omission does not lead to confusion).

6. CMB polarization due to gravitational waves. An analytical approach.

We now focus on the solutions to Eq. 14 and with our attention restricted to primordial gravitational waves (tensor perturbations). The process of generation of polarization occurs after radiation-matter equality. In this case the source in Eq. 14 has the following form:

$$\frac{1}{2} \gamma \partial h_{\alpha\beta} e^\alpha e^\beta = \frac{1}{2} \gamma (1 - \mu^2) \cos 2\phi S(\eta, k),$$  \hspace{1cm} (15)

where $S(\eta, k) = \frac{dh_k(\eta)}{d\eta}$ is the gravitational wave source term.

For a plane gravitational wave perturbation with wavevector $k$, the symbolic vector $\delta \hat{n}(\eta, k)$, describing anisotropy and polarization, can be presented as

$$\delta \hat{n}(\eta, k) = \frac{\gamma}{2} \left[ \alpha(\eta, \mu, k)(1 - \mu^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos 2\phi + \beta(\eta, \mu, k) \begin{pmatrix} (1 + \mu^2) \cos 2\phi \\ -(1 + \mu^2) \cos 2\phi \end{pmatrix} \right], \hspace{1cm} (16)$$

For $k\eta \ll 1$, $\alpha$ and $\beta$ do not depend on $\mu$. Equation 16 will allow for reconstruction of the polarization and temperature power spectra, which can be derived from maps of the CMB's polarization.

Substituting 15 and 16 into the Boltzmann equation, 14, we obtain the following system of coupled integro-differential equations for $\alpha$ and $\beta$

$$\dot{\beta}(\eta, \mu, k) + (q - i k \mu) \beta(\eta, \mu, k) = \frac{3}{16} q(\eta) I(\eta, k), \hspace{1cm} (17)$$

$$\dot{\xi}(\eta, \mu, k) + (q - i k \mu) \xi(\eta, \mu, k) = S(\eta, k), \hspace{1cm} (18)$$

where $\xi(\eta, \mu, k) = \alpha(\eta, \mu, k) + \beta(\eta, \mu, k)$ and

$$I(\eta, k) = \int_{-1}^{1} d\mu' \left[ (1 + \mu^2)^2 \beta(\eta, \mu', k) - \frac{1}{2} (1 - \mu^2)^2 \xi(\eta, \mu', k) \right]. \hspace{1cm} (19)$$

Here

$$q(\eta) = \sigma_T N_c a X_e(\eta) = \frac{\sigma_T \Omega_B \rho_c X_e(\eta)}{m_p a^2},$$

where $\sigma_T$ is the Thomson optical depth, $\Omega_B$ is the baryon fraction, $\rho_c$ is the critical density, $m_p$ is the proton mass, $a$ is the cosmological scale factor, and $X_e(\eta)$ is the
ionization fraction. For $X_e(\eta)$ during decoupling we use the Peebles fitting function \cite{49}.

The formal solution of Eq. (17) is

$$\beta(\eta, \mu, k) = e^{\tau(\eta) + ik\mu\eta} \int_0^\eta \Phi(x, k) e^{-ik\mu x} dx,$$

where $\tau(\eta)$ is optical depth:

$$\tau(\eta) = \int_0^{\eta_0} q(\eta') d\eta'.$$

(21)

The function $\Phi(\eta, k)$ depends primarily on the epoch and duration of decoupling, and is not sensitive to the details of $\tau(\eta)$ \cite{50}.

Taking into account that from Eq. (18)

$$\xi(\eta, \mu, k) = e^{\tau(\eta) + ik\mu\eta} \int_0^\eta S(x, k) e^{-\tau(x) - ik\mu x} dx,$$

we obtain for $\Phi$ the following integral equation:

$$\Phi(\eta, k) = \Phi_0(\eta, k) + \frac{3}{16} q(\eta) \int_0^\eta \Phi(x, k) K_+(\eta - x, k) dx,$$

where

$$\Phi_0(\eta, k) = -\frac{3}{32} q(\eta) \int_0^\eta dx S(x, k) e^{-\tau(x)} K_-(\eta - x, k),$$

(25)

and

$$K_{\pm}(x, k) = \int_{-1}^1 d\mu (1 \pm \mu^2)^2 \cos k\mu x.$$  

(26)

We remind the reader that our goal is to obtain the present value of the symbolic vector $\hat{\delta n}$, from Eq. (16). To do so requires that $\Phi(\eta, k)$ be found for every $k$. The vector $\hat{\delta n}$ is found by by using equations (20), (23) and substituting into Eq. (16), setting $\eta = \eta_0$ and $\tau(\eta_0) = 0$. Once the symbolic vector $\hat{\delta n}$ is found, we proceed to determine the multipole expansion of the GWB-induced polarization, which will eventually be used to construct statistical estimators of observable quantities such as the temperature and polarization power spectra.

7. Multipole expansion of Anisotropy and Polarization due to gravitational waves

This section is largely based on Seljak & Zaldarriaga \cite{3}. The components of the symbolic vector $n(\mu, \phi)$ are related to the fundamental polarization tensor $P_{ab}(\mu, \phi)$
in the following way

\[ P^b_a = \frac{1}{2} \frac{c^2}{h \nu^3} \begin{pmatrix} I + Q & U \\ U & I - Q \end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix} n_1 - \frac{1}{2} n_3 \\ -\frac{1}{2} n_3 & n_2 \end{pmatrix}. \] (27)

From the polarization tensor \( P^{ab}(\mu, \phi) \) we can form three independent scalar fields corresponding to anisotropy and \( E, B \) modes of polarization in the following way:

\[ T(\mu, \phi, k) = g_{ab}(\mu, \phi) P^{ab}(\mu, \phi), \] (28)

\[ E(\mu, \phi, k) = [g_{ab}(\mu, \phi) P^c_c(\mu, \phi) - 2P_{ab}(\mu, \phi)]^{[a;b],} \] (29)

\[ B(\mu, \phi, k) = \epsilon^{a d} [g_{ab}(\mu, \phi) \cdot P^c_c(\mu, \phi) - 2P_{ab}(\mu, \phi)]^{[b;d]}, \] (30)

where repeated indices imply summation, and

\[ g_{ab}(\mu, \phi) = \begin{pmatrix} (1 - \mu^2)^{-1} & 0 \\ 0 & (1 - \mu^2) \end{pmatrix} \]

is the 2-metric on a unit sphere in coordinates \((\mu, \phi)\), and covariant derivatives (;) are effected with respect to this metric.

In terms of our variables \( \alpha(\mu, k) \) and \( \beta(\mu, k) \) (see (16)) the above definitions of \( T, E \) and \( B \) give

\[ T(\mu, \phi, k) = \gamma (1 - \mu^2) \alpha(\mu, k) \cos 2\phi, \]

\[ E(\mu, \phi, k) = -\gamma (1 - \mu^2) \left[ (1 + \mu^2) \frac{d^2 \alpha}{d\mu^2} + 8\mu \frac{d\alpha}{d\mu} + 12 \right] \beta(\mu, k) \cos 2\phi, \]

\[ B(\mu, \phi, k) = \gamma (1 - \mu^2) \left[ 2\mu \frac{d^2 \beta}{d\mu^2} + 8 \frac{d\beta}{d\mu} \right] \beta(\mu, k) \sin 2\phi, \]

The multipole expansion coefficients for anisotropy, and \( E, B \) modes of polarization, in terms of the introduced scalars \( T, E \) and \( B \), are defined as

\[ a^T_{l, m}(k) = \int d\Omega \left[ (Y_{l, m}(\mu, \phi))^* \cdot T(\mu, \phi, k) \right], \]

\[ a^E_{l, m}(k) = \left[ \frac{(l - 2)!}{(l + 2)!} \right]^{\frac{1}{2}} \int d\Omega \left[ (Y_{l, m}(\mu, \phi))^* \cdot E(\mu, \phi, k) \right], \]

\[ a^B_{l, m}(k) = \left[ \frac{(l - 2)!}{(l + 2)!} \right]^{\frac{1}{2}} \int d\Omega \left[ (Y_{l, m}(\mu, \phi))^* \cdot B(\mu, \phi, k) \right], \] (31)
where \( Y_{l,m}(\mu,\phi) \) are the ordinary spherical harmonics. From (31) it follows that the multipole coefficients vanish for \( m \neq \pm 2 \). Taking into account the form expressions for \( \alpha \) and \( \beta \) from (20) and (23), the above expressions can be rewritten as:

\[
a_{T}^{l,m}(k) = -(-i)^{l} \gamma (\delta_{2,m} + \delta_{-2,m}) \sqrt{\pi(2l+1)} \int_{0}^{1} d\eta \; T_k(\eta) \left[ \frac{(l+2)!}{(l-2)!} \frac{j_{l}(\zeta)}{\zeta^{2}} \right],
\]

\[
a_{E}^{l,m}(k) = -(-i)^{l} \gamma (\delta_{2,m} + \delta_{-2,m}) \sqrt{\pi(2l+1)} \int_{0}^{1} d\eta \; \Pi_k(\eta) \left[ \left( 2 - \frac{l(l-1)}{\zeta^{2}} \right) j_{l}(\zeta) - \frac{2}{\zeta} j_{l-1}(\zeta) \right],
\]

\[
a_{B}^{l,m}(k) = -(-i)^{l} \gamma (\delta_{2,m} - \delta_{-2,m}) \sqrt{\pi(2l+1)} \int_{0}^{1} d\eta \; \Pi_k(\eta) \left[ 2 \left\{ -\frac{(l-1)}{\zeta} j_{l}(\zeta) + j_{l-1}(\zeta) \right\} \right],
\]

where

\[
\zeta = k(\eta_0 - \eta),
\]

and

\[
T_k(\eta) = S_k(\eta)e^{-\tau(n)} - \Phi_k(\eta),
\]

\[
\Pi_k(\eta) = \Phi_k(\eta),
\]

and \( j_{l}(\zeta) \) are the spherical bessel functions.

8. Anisotropy and polarization generated by a random field of gravitational waves

The GWB can be considered as an isotropic random superposition of plane waves with the following correlation relationship:

\[
\langle \phi^*_k \phi'_{k'} \rangle = \langle \phi^*_k \phi'_{k'} \rangle = \delta_{ss'} \delta(k - k'), \quad \langle \phi^*_k \phi'_k \rangle = \langle \phi^*_k \phi'_k \rangle = 0
\]

where \( \langle ... \rangle \) means averaging over realizations. Below we assume power law spectrum for Cosmological Gravitational Wave Background, i.e.

\[
4\pi k^3 |h_k(\eta)|^2 \bigg|_{k n_T \ll 1} \sim k^{n_T-1},
\]

with \( n_T = 1 \) corresponding to flat (scale-invariant) Zel’dovich-Harrison spectrum.

Let us consider two scalar fields \( X(\mu,\phi) \) and \( \tilde{X}(\mu,\phi) \) (where \( X \) and \( \tilde{X} \) is any pair of the scalars \( T(\mu,\phi) \), \( E(\mu,\phi) \) and \( B(\mu,\phi) \)). Then their cross correlation is defined as

\[
\Gamma^{X \tilde{X}}(\mu_0) = \frac{1}{8\pi^2} \int d\mu d\mu' d\phi d\phi' \times
\]

\[
\times \delta \left[ \mu_0 - \mu \mu' - \sqrt{(1 - \mu^2)(1 - \mu'^2)} \cos(\phi - \phi') \right] \langle X(\mu,\phi) \tilde{X}(\mu',\phi') \rangle.
\]

(33)
Following Fourier decomposition of $X(\mu, \phi)$

$$X(\mu, \phi) = \frac{1}{(2\pi)^{3/2}} \int dk \sum_{s=1,2} [X_k(\mu_k, \phi_k)e^{ik \cdot \xi_k} + X_k^*(\mu_k, \phi_k)e^{-ik \cdot \xi_k^*}]$$

(34)

and the correlation relations (32), the correlation function is presentable in the form

$$\Gamma^{X \bar{X}}(\mu_0) = 4\pi \int \frac{d^2k}{k^2} \Gamma^{X \bar{X}}(\mu_0, k),$$

(35)

where $\Gamma^{X \bar{X}}(\mu_0, k)$ is the correlation function for a single wave

$$\Gamma^{X \bar{X}}(\mu_0, k) = \frac{1}{8\pi^2} \int d\mu d\mu' d\phi d\phi' \delta \left( \mu_0 - \mu - \sqrt{(1-\mu^2)(1-\mu'^2)} \cos (\phi - \phi') \right) X_k(\mu, \phi) \bar{X}_k^*(\mu', \phi'),$$

(36)

The correlation function being a function of a single angle $\theta_0 = \cos^{-1} \mu_0$, can be expanded over Legendre polynomials to give the following correlation function and power spectrum

$$\Gamma^{X \bar{X}}(\mu_0, k) = \sum_{l=0}^{\infty} C_l^{X \bar{X}}(k) P_l(\mu_0)$$

(37)

$$C_l^{X \bar{X}}(k) = \frac{2}{2l+1} \int_{-1}^{1} d\mu_0 \Gamma^{X \bar{X}}(\mu_0, k) P_l(\mu_0).$$

(38)

Using the addition theorem for Legendre polynomials

$$P_l \left[ \mu' + \sqrt{(1-\mu^2)(1-\mu'^2)} \cos (\phi - \phi') \right] = P_l(\mu) P_l(\mu') + \sum_{m=0}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\mu) P_l^m(\mu') \cos m(\phi - \phi'),$$

we get

$$C_l^{X \bar{X}}(k) = \frac{1}{2l+1} \frac{1}{4\pi^2} \int d\mu_0 P_l(\mu_0) \int d\mu d\mu' d\phi d\phi' \delta \left( \mu_0 - \mu - \sqrt{(1-\mu^2)(1-\mu'^2)} \cos (\phi - \phi') \right) X_k(\mu, \phi) \bar{X}_k^*(\mu', \phi'),$$

$$= \frac{1}{2l+1} \frac{1}{4\pi^2} \int d\mu d\mu' d\phi d\phi' P_l \left( \mu' + \sqrt{(1-\mu^2)(1-\mu'^2)} \cos (\phi - \phi') \right) X(\mu, \phi, k) \bar{X}^*(\mu', \phi', k),$$

$$= \frac{1}{2l+1} \sum_{m=-l}^{l} x_{l,m}(k) \cdot \bar{x}_{l,m}(k),$$

(39)

where

$$x_{l,m}(k) = \int d\Omega \left[ (Y_{l,m}(\mu, \phi))^* \cdot X(\mu, \phi, k) \right],$$

(40)
\[ \tilde{x}_{l,m}(k) = \oint d\Omega \left[ (Y_{l,m}(\mu,\phi))^* \cdot \tilde{X}(\mu,\phi,k) \right], \quad (41) \]

Thus we get for the power spectrum from a single gravitational wave

\[ C_l^{\tilde{X}\tilde{X}}(k) = \frac{1}{2l+1} \sum_{m=-l}^{l} a_{l,m}^X a_{l,m}^{X*} \quad (42) \]

The angular power spectrum from a superposition of gravitational waves is calculated from the above expression by integrating over all the wave numbers \( k \)

\[ C_l^{\tilde{X}\tilde{X}} = 4\pi \int dk \, k^2 \, C_l^{\tilde{X}\tilde{X}}(k), \quad (43) \]

The power spectra is most conveniently determined as coefficients of a Legendre polynomial expansion with index \( l \) rather than an expansion in Fourier modes with wavenumber \( k \). In a flat universe a perturbation of comoving wavelength \( k^{-1} \) at the comoving distance of the last scattering surface (LSS) subtends an angle \( \theta \sim 1/k \). On the other hand \( \ell \sim 1/\theta \) for small \( \theta \). This simple consideration shows that the main contribution to \( C_l^{TT} \) with a given \( \ell \) comes from \( k \sim \ell \).

Figure 3 shows \( C_l^{BB} \) calculated using the CAMB code, which is based on CMBFAST compared to our analytical predictions. The peak multipole of the spectrum \( \ell_{\text{peak}} \approx 90 \) is robust to changes in the inflationary dynamics. The amplitude of the B-mode power spectrum, is determined by \( r \) alone, and its spatial structure is determined solely by the age of the universe at last scattering. This single-parameter dependence makes the B-mode polarization the most robust probe of inflation. While scalar, or mass-energy, perturbations are amplified by gravity, the tensor-GWB is not. Direct detection today (redshift \( z = 0 \)) by, e.g., LIGO/LISA is essentially impossible since, like the CMB, the energy-density of gravitational waves dilutes (redshifts) for waves inside the horizon at, or before, last-scattering as the universe expands. However the GWB imprints curl-mode polarization on the CMB at the surface of last-scattering \( (z \approx 1100) \). Therefore, the GWB energy-density from sub-horizon scale waves at last-scattering was at least one trillion times larger than it is now, which motivates the use of the last-scattering surface as the perhaps the best “detector” of the primordial, inflationary-generated, GWB.

Our analytic calculations based on the solution of the Boltzmann equation provide clear insight into the underlying physics of polarization and are in good agreement with the results of CMBFAST and CAMB.

9. Current CMB Polarization Results

Figures 1 and 2 show current detections of the E-mode (grad-mode) polarization and the polarization-temperature cross correlation, \( \langle TE \rangle \). Currently, DASI, BOOMERANG, CB, and CAPMAP have detected E-mode polarization, however there are no current detections of the E-mode polarization for \( \ell < 100 \).
Fig. 1. Measurements of the gradient or E-mode polarization power spectrum $C_E^\ell$. The solid line is the polarization power spectrum for the WMAP best fit cosmological model, with $\tau = 0.17$.

where the signature of gravitational waves will be manifest. Only DASI and BOOMERANG have detected the grad-mode polarization at $100 < \ell < 400$.

Both reionization and gravitational waves imprint the polarization of the CMB at large scales. The primary effects of reionization are encoded in the grad-mode and temperature polarization cross-correlation at $\ell \lesssim 50$. In the absence of detections, the most stringent constraint in this range of multipoles is currently $C_E^\ell < 8 \mu K$ at 95% confidence reported by POLAR57 for $2 < \ell < 20$ (assuming no B-modes). WMAP58 reports a large number of highly significant detections, especially at low-$\ell$ due to WMAP’s ability to map the full sky. CBI, DASI, and BOOMERANG69 have also detected the cross-correlation spectrum, mainly at smaller angular scales than WMAP. A complete description of reionization will require detections of CMB E-mode polarization for $\ell < 50$. An ancillary benefit of reionization60 is that it boosts the primary curl-mode power spectrum significantly near $\ell = 10$. Due to reionization, a more stringent limit on the tensor-to-scalar ratio, $r$, in the presence of lensing can be obtained than that calculated in Ref.61 62.
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The tensor to scalar ratio, $r$, is defined here as

$$r = \frac{\Delta h(k_0)}{\Delta R(k_0)}$$

where $\Delta^2_R(k_0)$ and $\Delta^2_h(k_0)$ are the amplitudes of the scalar and primordial power spectra, evaluated at some pivot wavenumber $k_0$. There are other ways of defining the tensor to scalar ratio that appear in the literature. A second definition, $T/S$, is defined as the ratio between the tensor and scalar contributions to the temperature anisotropy’s quadrupole component $T/S = C^T_{2,\text{tens}}/C^T_{2,\text{scal}}$. The relationship between $r$ and $T/S$ depends in a complicated way on the cosmology. For the parameter set used in this paper, $T/S \approx 0.5r$ where the pivot wavenumber used is $0.05 \text{ Mpc}^{-1}$.

CMB temperature anisotropy alone can only detect the tensor to scalar ratio if $r \gtrsim 0.3$ due to cosmic variance. The limit from WMAP data alone is roughly four-times larger than this and approximately two-times larger with the inclusion of...
Thus we are close to the ultimate constraints achievable by CMB temperature anisotropy observations alone.

Further progress requires detection of the B-mode signal. Currently, there are no detections of the B-mode polarization as shown in figure 3. POLAR, CBI, DASI, and BOOMERANG all provide upper limits to the B-mode polarization. The current generation of CMB polarimeters should be able to provide detect or provide much better upper limits to the curl-mode polarization within the next several years.

Fig. 3. Measurements of the curl-mode polarization power spectrum $C_{\ell}^{BB}$. The solid line is the power spectrum for the WMAP best fit cosmological model with $\tau = 0.17^{21}$ and $r = 0.1$. The dotted line are the analytical results of this paper, and the solid curve which peaks at $\ell \sim 1000$, is the B-mode spectrum produced by large scale structure lensing the primary CMB E-mode polarization.

The gravitational lensing of the E-mode polarization into B-mode polarization provides a source of contamination to the primordial B-mode signal. For a noise free, full sky experiment in which the lensing is treated as noise, Ref. 61 found that this contamination sets a detectability limit of $r_{lim} > 10^{-4}$, i.e. if the energy scale of inflation is larger than $3 \times 10^{15} \text{GeV}$. Measurements of 21-cm radiation may be able
to provide a way to de-lens the curl-mode measurements, providing an detectable inflationary energy scale limit lower than one using CMB-only measurements. This energy scale may be as low as $3 \times 10^{14}$ GeV. Work by Seljak & Hirata (2004) indicate that the primary lensing signal can be removed to a level that makes $r = 10^{-6}$ detectable, allowing detection of inflationary GWB at energy scales $< 10^{15}$ GeV using the CMB only.

10. Constraining Inflation with Upcoming CMB Observations

To illustrate the power of upcoming observations to constrain inflationary parameters we consider a hypothetical polarimeter with 1° resolution and a system sensitivity ($NET_{sys}$) = $70 \mu K s^{1/2}$ using the COSMOMC software package. The system sensitivity for polarization is $\sqrt{2}$ times higher, $NEQ = \sqrt{2}NET$. These detector requirements are well within the reach of the current generation of CMB polarimeters. By observing 2.4% of the sky, this experiment could feasibly detect the curl mode signal created by the gravitational wave background (at $\ell < 100$) for $r$ as low as $r = 0.12$ at 95% confidence with no priors. This represents an order of magnitude improvement over the WMAP only results of $r = 1.28$ at 95% confidence, with no priors. We plot the results of our simulations along with predictions for several classes of inflationary models in figure 4. The predictions of the various models of inflation were generated using the method described below.

Detecting the tensor-to-scalar ratio, $r$, will allow the possibility of distinguishing between several classes of inflation, such as negative curvature models, small positive curvature models, etc. As an example we consider the toy experiment described earlier. As mentioned, it can detect $r$ with 95% at 0.12. The $n_s - r$ contours compared to the different classes of inflation models are plotted in Figure 4 assuming the toy experiment detects $r = 0.12$ and $n_s = 0.98$. The contours for the toy experiment were calculated using COSMOMC.

The predictions for the different classes of inflation were generated following the model in Ref. 67, 68 and the different inflation models were separated according to Ref. 26. The observables $r$ and $n_s$ were evaluated at some specific $e$-folding, $N$, of inflation, not a specific wavenumber, $k$. The relation between $N$ and $k$ requires a detailed model of reheating, which has some uncertainty. This uncertainty is marginalized over by calculating the observables at an $e$-fold randomly drawn from 40 to 70. The inflationary flow equations were truncated at sixth order and the observables were calculated to second order in slow roll. Each class of model has a unique color in Fig. 4 as described in the caption, and each realization of that model is indicated by a point in the $r - n_s$ plane.

11. Conclusions

We have developed an analytic method to generate the predictions of the imprint of gravitational waves on the CMB. Using phenomenological models of inflation we
Fig. 4. Shown are 1D marginalized probabilities and 2D joint likelihood contours for $n_s$ and $r$ from the toy experiment described in the text. The purple dots are negative curvature models. The red dots are small positive curvature models. The green dots are intermediate positive curvature models. The black dots are large positive curvature models. WMAP has set an upper limit on $r$ of $r < 1.28$ with no priors. The toy experiment is capable of detecting $r = 0.12$ with 95% confidence with no priors—more than an order of magnitude below the WMAP limits.

predict both the CMB polarization spectra and the derived inflationary parameters: the tensor-to-scalar ratio and spectral index of the scalar perturbations. The combination of the later two observables allows for reconstruction of the dynamics of inflation. With these predictions in hand we have shown that upcoming CMB polarization observations will be able to detect or constrain the cosmological GWB and hence, inflation itself. These new technological advances now position obser-
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At the threshold of an exhilarating era – one in which CMB polarization data will winnow down inflation’s vast model-space and test models of the early universe at energy scales approaching the GUT-scale; nearly one trillion times higher energy than accessible from particle accelerators.

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1. A. G. Polnarev. Polarization and Anisotropy Induced in the Microwave Background by Cosmological Gravitational Waves. Soviet Astronomy, 29:607–+, December 1985.
2. U. Seljak. Measuring Polarization in the Cosmic Microwave Background. Astrophys. J., 482:6–+, June 1997.
3. M. Zaldarriaga and U. Seljak. All-sky analysis of polarization in the microwave background. Phys. Rev. D, 55:1830–1840, February 1997.
4. M. Kamionkowski, A. Kosowsky, and A. Stebbins. Statistics of cosmic microwave background polarization. Phys. Rev. D, 55:7368–7388, June 1997.
5. M. Kamionkowski, A. Kosowsky, and A. Stebbins. A Probe of Primordial Gravity Waves and Vorticity. Physical Review Letters, 78:2058–2061, March 1997.
6. U. Seljak and M. Zaldarriaga. Signature of Gravity Waves in the Polarization of the Microwave Background. Physical Review Letters, 78:2054–2057, March 1997.
7. Uros Seljak and Matias Zaldarriaga. A line of sight approach to cosmic microwave background anisotropies. Astrophys. J., 469:437–444, 1996.
8. Antony Lewis and Sarah Bridle. Cosmological parameters from CMB and other data: a Monte-Carlo approach. Phys. Rev., D66:103511, 2002.
9. M. M. Basko and A. G. Polnarev. Polarization and anisotropy of the RELICT radiation in an anisotropic universe. MNRAS, 191:207–215, April 1980.
10. P. Coles, R. A. Frewin, and A. G. Polnarev. CMBR Polarization from Gravitational Waves. LNP Vol. 455: Birth of the Universe and Fundamental Physics, 455:273–+, 1995.
11. B. Keating, P. Timbie, A. Polnarev, and J. Steinberger. Large angular scale polarization of the cosmic microwave background radiation and the feasibility of its detection. Astrophys. J., 495:580–+, March 1998.
12. J. R. Pritchard and M. Kamionkowski. Cosmic microwave background fluctuations from gravitational waves: An analytic approach. Annals of Physics, 318:2–36, July 2005.
13. W. Zhao and Y. Zhang. An analytic approach to the polarization of the cosmic microwave background generated by relic gravitational waves. astro-ph/0508345, 2005.
14. A. H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. Phys. Rev. D, 23:347–356, January 1981.
15. J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok. Ekpyrotic universe: Colliding branes and the origin of the hot big bang. Phys. Rev. D, 64(12):123522–+, December 2001.
16. J. Magueijo. Faster than the speed of light: the story of a scientific speculation. Faster
than the speed of light: the story of a scientific speculation, by Joao Magueijo. Cambridge, MA: Perseus Books, 2003, 2003.

17. P. de Bernardis, P. A. R. Ade, J. J. Bock, J. R. Bond, J. Borrill, A. Boscaleri, K. Coble, B. P. Crill, G. De Gasperis, P. C. Freese, P. G. Ferreira, K. Ganga, M. Giacometti, E. Hivon, V. V. Hristov, A. Jacongeli, A. H. Jaffe, A. E. Lange, L. Martinis, S. Masi, P. V. Mason, P. D. Mauskopf, A. Melchiorri, L. Miglio, T. Montroy, C. B. Netterfield, E. Pascale, F. Piacentini, D. Pogosyan, S. Prunet, S. Rao, G. Romeo, J. E. Ruhl, F. Scaramuzza, D. Sforna, and N. Vittorio. A flat Universe from high-resolution maps of the cosmic microwave background radiation. Nature, 404: 955–959, April 2000.

18. A. Balbi, P. Ade, J. Bock, J. Borrill, A. Boscaleri, P. De Bernardis, P. G. Ferreira, S. Hanany, V. Hristov, A. H. Jaffe, A. T. Lee, S. Oh, E. Pascale, B. Rabii, P. L. Richards, G. F. Smoot, R. Stompor, C. D. Winant, and J. H. P. Wu. Constraints on Cosmological Parameters from MAXIMA-1. Astrophys. J. Lett., 545:L1–L4, December 2000.

19. C. Pryke, N. W. Halverson, E. M. Leitch, J. Kovac, J. E. Carlstrom, W. L. Holzapfel, and M. Dragovan. Cosmological Parameter Extraction from the First Season of Observations with the Degree Angular Scale Interferometer. Astrophys. J., 568:46–51, March 2002.

20. J. L. Sievers, J. R. Bond, J. K. Cartwright, C. R. Contaldi, B. S. Mason, S. T. Myers, S. Padin, T. J. Pearson, U.-L. Pen, D. Pogosyan, S. Prunet, A. C. S. Readhead, M. C. Shepherd, P. S. Udomprasert, L. Bronfman, W. L. Holzapfel, and J. May. Cosmological Parameters from Cosmic Background Imager Observations and Comparisons with BOOMERANG, DASI, and MAXIMA. Astrophys. J., 591:599–622, July 2003.

21. D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright. First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters. Astrophysical Journal Supplement Series, 148:175–194, September 2003.

22. J. H. Goldstein, P. A. R. Ade, J. J. Bock, J. R. Bond, C. Cantalupo, C. R. Contaldi, M. D. Daub, W. L. Holzapfel, C. Kuo, A. E. Lange, M. Lueker, M. Newcomb, J. B. Peterson, D. Pogosyan, J. E. Ruhl, M. C. Runyan, and E. Torbet. Estimates of Cosmological Parameters Using the Cosmic Microwave Background Angular Power Spectrum of ACBAR. Astrophys. J., 599:773–785, December 2003.

23. A. R. Liddle and D. H. Lyth. Cosmological Inflation and Large-Scale Structure. Cosmological Inflation and Large-Scale Structure, by Andrew R. Liddle and David H. Lyth, pp. 414. ISBN 052166022X. Cambridge, UK: Cambridge University Press, April 2000., April 2000.

24. G. Hinshaw, A. J. Banday, C. L. Bennett, K. M. Gorski, A. Kogut, G. F. Smoot, and E. L. Wright. Band Power Spectra in the COBE DMR Four-Year Anisotropy Maps. Astrophys. J. Lett., 464:L17+, June 1996.

25. D. N. Spergel and M. Zaldarriaga. Cosmic Microwave Background Polarization as a Direct Test of Inflation. Physical Review Letters, 79:2180–2183, September 1997.

26. H. V. Peiris, E. Komatsu, L. Verde, D. N. Spergel, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, E. Wollack, and E. L. Wright. First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Implications For Inflation. Astrophys. J. Supplement., 148:213–231, September 2003.

27. D J Eisenstein, I Zehavi, D W Hogg, R Scoccimarro, M R Blanton, R C Nichol, R Scranton, H Seo, M Tegmark, Z Zheng, S Anderson, J Annis, N Bahcall, J Brinkmann, S Burles, F J Castander, A Connolly, I Csabai, M Doi, M Fukugita,
J A Frieman, K Glazebrook, J E Gunn, J S Hendry, G Hennessy, Z Ivezic, S Kent, G R Knapp, H Lin, Y Loh, R H Lupton, B Margon, T McKay, A Meiksin, J A Munn, A Pope, M Richmond, D Schlegel, D Schneider, K Shimatsuk, C Stoughton, M Strauss, M Subbarao, A S Szalay, I Szapudi, D Tucker, B Yanny, and D York. Detection of the baryon acoustic peak in the large-scale correlation function of sdss luminous red galaxies. submitted to ApJ, astro-ph/0501171, 2005.

28. S. Cole, W. J. Percival, J. A. Peacock, P. Norberg, C. M. Baugh, C. S. Frenk, I. Baldry, J. Bland-Hawthorn, T. Bridges, R. Cannon, M. Colless, C. Collins, W. Couch, N. J. G. Cross, G. Dalton, V. R. Eke, R. De Propris, S. P. Driver, G. Efstathiou, R. S. Ellis, K. Glazebrook, C. Jackson, A. Jenkins, O. Lahav, I. Lewis, S. Lumsden, S. Maddox, D. Madgwick, B. A. Peterson, W. Sutherland, and K. Taylor. The 2dF Galaxy Redshift Survey: power-spectrum analysis of the final data set and cosmological implications. MNRAS, 362:505–534, September 2005.

29. R. H. Dicke and P. J. Peebles. Gravitation and Space Science. Space Science Reviews, 4:419–+, 1965.

30. E. R Harrison. Fluctuations at the Threshold of Classical Cosmology. Phys. Rev. D, 1:2726, 1970.

31. P. J. E. Peebles and J. T. Yu. Primeval Adiabatic Perturbation in an Expanding Universe. Astrophys. J., 162:815–+, December 1970.

32. Y. B. Zeldovich. A hypothesis, unifying the structure and the entropy of the Universe. MNRAS, 160:1P–+, 1972.

33. G. Boerner. The early universe: facts and fiction. Berlin: Springer, 2003.

34. E. Komatsu, A. Kogut, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, M. Limon, S. S. Meyer, L. Page, D. N. Spergel, G. S. Tucker, L. Verde, E. Wollack, and E. L. Wright. First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Tests of Gaussianity. Astrophysical Journal Supplement Series, 148:119–134, September 2003.

35. Michael S. Turner. The new cosmology: Mid-term report card for inflation. Annales Henri Poincare, 4:S333–S346, 2003.

36. R. Penrose. The road to reality: a complete guide to the laws of the universe. London: Jonathan Cape, 2004, 2004.

37. M. Kamionkowski and A. Kosowsky. The Cosmic Microwave Background and Particle Physics. Annual Reviews of Nuclear and Particle Science, 47:77, 1999.

38. W. H. Kinney. Hamilton-Jacobi approach to non-slow-roll inflation. Phys. Rev. D, 56:2002–2009, August 1997.

39. W. H. Kinney. The energy scale of inflation: is the hunt for the primordial B-mode a waste of time? New Astronomy Review, 47:967–975, December 2003.

40. S. Bose and L. P. Grishchuk. Observational determination of squeezing in relic gravitational waves and primordial density perturbations. Phys. Rev. D, 66(4):043529–+, August 2002.

41. S. Dodelson. Modern cosmology. Modern cosmology / Scott Dodelson. Amsterdam (Netherlands): Academic Press. ISBN 0-12-219141-2, 2003, XIII + 440 p., 2003.

42. S. Weinberg. Damping of tensor modes in cosmology. Phys. Rev. D, 69(2):023503–+, January 2004.

43. A. D. Chernin. A Model of a Universe Filled by Radiation and Dustlike Matter. Soviet Astronomy, 9:871–+, April 1966.

44. L. P. Grishchuk. Relic gravitational waves and limits on inflation. Phys. Rev. D,
48:3513–3516, October 1993.

45. B. Allen and S. Koranda. CBR anisotropy from primordial gravitational waves in inflationary cosmologies. Phys. Rev. D, 50:3713–3737, September 1994.

46. K.-W. Ng and A. D. Spokiotopulos. Cosmological evolution of scale-invariant gravity waves. Phys. Rev. D, 52:2112–2117, August 1995.

47. S. Chandrasekhar. *Radiative transfer*. New York: Dover, 1960, 1960.

48. J. R. Bond and G. Efstathiou. Cosmic background radiation anisotropies in universes dominated by nonbaryonic dark matter. Astrophys. J. Lett., 285:L45–L48, October 1984.

49. P. J. E. Peebles. *Principles of physical cosmology*. Princeton Series in Physics, Princeton, NJ: Princeton University Press, —c1993, 1993.

50. P. D. Nasel’skii and A. G. Polnarev. Anisotropy and Polarization of the Microwave Background Radiation as a Test of Nonequilibrium Ionization of the Pregalactic Plasma. Astrophysics, 26:327–+, November 1987.

51. Antony Lewis, Anthony Challinor, and Anthony Lasenby. Efficient computation of CMB anisotropies in closed FRW models. Astrophys. J., 538:473–476, 2000.

52. Peiris H. Smith, T. and A. Cooray. Deciphering inflation with gravitational waves: Cosmic microwave background polarization vs. direct detection with laser interferometers. astro-ph/0602137, 2006.

53. E. M. Leitch, J. M. Kovac, N. W. Halverson, J. E. Carlstrom, C. Pryke, and M. W. E. Smith. Degree Angular Scale Interferometer 3 Year Cosmic Microwave Background Polarization Results. Astrophys. J., 624:10–20, May 2005.

54. T.E. Montroy et al. A measurement of the cmb ee spectrum from the 2003 flight of boomerang. ApJ, submitted, 2005.

55. A. C. S. Readhead, S. T. Myers, T. J. Pearson, J. L. Sievers, B. S. Mason, C. R. Contaldi, J. R. Bond, R. Bustos, P. Altamirano, C. Aghermann, L. Bronfman, J. E. Carlstrom, J. K. Cartwright, S. Casassus, C. Dickinson, W. L. Holzapfel, J. M. Kovac, E. M. Leitch, J. May, S. Padin, D. Fosgusón, M. Pospelov, C. Pryke, R. Reeves, M. C. Shepherd, and S. Torres. Polarization Observations with the Cosmic Background Imager. Science, 306:836–844, October 2004.

56. D. Barkats, C. Bischoff, P. Farese, L. Fitzpatrick, T. Gaier, J. O. Gundersen, M. M. Hedman, L. Hyatt, J. J. McMahon, D. Stagges, K. Vanderlinde, and B. Weinstein. First measurements of the polarization of the cosmic microwave background radiation at small angular scales from capmap. Astrophys. J. Lett., 619:L127–L130, February 2005.

57. B. G. Keating, C. W. O’Dell, A. de Oliveira-Costa, S. Klaskowski, N. Stebor, L. Piccirillo, M. Tegmark, and P. T. Timbie. A limit on the large angular scale polarization of the cosmic microwave background. Astrophys. J. Lett., 560:L1–L4, 2001.

58. A. Koget, D. N. Spergel, C. Barnes, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, E. Wollack, and E. L. Wright. First-year wilkinson microwave anisotropy probe (wmap) observations: Temperature-polarization correlation. Astrophysical Journal Supplement Series, 148:161–173, September 2003.

59. F. Piacentini et al. A measurement of the polarization-temperature angular cross power spectrum of the cosmic microwave background from the 2003 flight of boomerang. ApJ, submitted, 2005.

60. M. Kaplinghat, L. Knox, and Y.-S. Song. Determining Neutrino Mass from the Cosmic Microwave Background Alone. Physical Review Letters, 91(24):241301–+, December 2003.

61. L. Knox and Y.-S. Song. Limit on the detectability of the energy scale of inflation.
Phys. Rev. Lett., 89(1):011303–+, July 2002.

62. M. Kesden, A. Cooray, and M. Kamionkowski. Separation of Gravitational-Wave and Cosmic-Shear Contributions to Cosmic Microwave Background Polarization. Phys. Rev. Lett., 89(1):011304–+, July 2002.

63. L. Knox and M. S. Turner. Detectability of tensor perturbations through anisotropy of the cosmic background radiation. Physical Review Letters, 73:3347–3350, December 1994.

64. Kris Sigurdson and Asantha Cooray. Cosmic 21-cm delensing of microwave background polarization and the minimum detectable energy scale of inflation. Submitted to Phys. Rev. Lett, 2005.

65. U. Seljak and C. M. Hirata. Gravitational lensing as a contaminant of the gravity wave signal in the CMB. Phys. Rev. D, 69(4):043005–+, February 2004.

66. Antony Lewis and Sarah Bridle. Cosmological parameters from CMB and other data: a Monte Carlo approach. Phys. Rev. D, 66:103511, 2002.

67. William H. Kinney. Inflation: Flow, fixed points and observables to arbitrary order in slow roll. Phys. Rev. D, 66:083508, 2002.

68. Richard Easther and William H. Kinney. Monte carlo reconstruction of the inflationary potential. Phys. Rev. D, 67:043511, 2003.