IR Duality in $d = 3$ $N = 2$ Supersymmetric $USp(2N_c)$ and $U(N_c)$ Gauge Theories

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We suggest IR-dual descriptions for $d = 3$ $N = 2$ supersymmetric gauge theories with gauge groups $USp(2N_c)$ and $U(N_c)$ and matter in the fundamental representation. We relate this duality to the IR duality of $d = 4$ $N = 1$ SQCD theories, and in one case also to mirror symmetry.
1. Introduction

In the past few years, many exact (non-perturbative) results have been found regarding the moduli spaces and superpotentials of $d = 4$ $N = 1$ supersymmetric gauge theories (see [1] for a review). One of the most interesting results is the existence of pairs of gauge theories which flow to the same theory in the IR limit, first discovered in [2]. If one of the pair of theories is IR-free, it provides a good description of the theory in the IR, while otherwise they both flow to the same non-trivial fixed point.

Similar dualities, which were called “mirror symmetries”, were recently discovered also in $d = 3$ supersymmetric theories, both with $N = 4$ supersymmetry and with $N = 2$ supersymmetry [3,4,5]. These dualities also relate the IR behaviors of two different gauge theories, and interchange their Higgs and Coulomb branches (which are distinguishable in $N = 4$ theories but not in $N = 2$ theories). In three dimensions all gauge theories are strongly coupled in the IR, so dualities between gauge theories never give effective IR descriptions, but only theories flowing to the same strongly coupled fixed points.

The moduli spaces of $d = 3$ $N = 2$ SQCD theories (analyzed for $SU(N_c)$ and $U(N_c)$ gauge theories in [3], and for $USp(2N_c)$ gauge theories in [5]), show many similarities to their $d = 4$ $N = 1$ counterparts (except for the existence of an additional Coulomb branch). As in the $d = 4$ $N = 1$ theories (up to a shift of $N_f$ by one), for small number of flavors there is no supersymmetric vacuum, for $N_f = N_c - 1$ (in $SU(N_c)$ and $U(N_c)$ theories, or $N_f = N_c$ in $USp(2N_c)$ theories) there is a smooth moduli space, for $N_f = N_c$ ($N_f = N_c + 1$) there is a dual description of the origin of moduli space using only chiral multiplets, while for larger values of $N_f$ there is a moduli space with a singularity at the origin, for which no dual description is known. The $d = 4$ $N = 1$ results suggest that a dual description should exist for these singularities at the origin of moduli space, and we will show that (at least for $U(N_c)$ and $USp(2N_c)$ gauge theories) this is indeed the case. A similar (but not identical) duality for $USp(2N_c)$ gauge theories was recently suggested in [3].

We should emphasize that the dual descriptions that we describe here are not expected to be unique (thus, the word “duality” is perhaps a misnomer). For instance, all the theories we discuss here have also a “mirror” description (which one can flow to from the $N = 4$ “mirror” description) which is not identical with their “dual” description (except in special cases). The full set of relationships between $d = 3$ $N = 2$ theories has yet to be explored. The “mirror” symmetry for Abelian theories was interpreted as exchanging
bound states of the original “electric” variables with Nielsen-Olesen vortices in [3]. Perhaps similar interpretations exist also for other dualities, which might shed light also on the duality in $d = 4 N = 1$ theories.

The $d = 4 N = 1$ dualities were given an interpretation in terms of brane constructions (generalizing the brane constructions of [7]) in [8,9]. These constructions may easily be generalized to the $d = 3 N = 2$ case, and they suggest dualities which are similar to the ones described here (they involve the same gauge groups and charged matter content, but it is difficult to see the superpotentials involving the quantum variables on the Coulomb branch in the brane construction). The brane construction also suggests a duality between $SO(N_c)$ gauge groups and $SO(N_f - N_c + 2)$ gauge groups, which we will not discuss here. As discussed in [9], the brane construction cannot be viewed as a proof of these dualities, since it only shows that there is a smooth interpolation between the two theories (which is not always an IR-equivalence, even if both theories are asymptotically free). It would be interesting to understand the conditions for a brane construction of this type to give a consistent field theory duality.

2. Duality in $USp(2N_c)$ gauge theories

In this section we propose an IR-dual description for $d = 3 N = 2 USp(2N_c)$ gauge theories with $2N_f$ fundamental flavors. The duality is similar in spirit to the Seiberg duality in four dimensions [2,10], and is connected to it when we discuss the $d = 4$ theory on a circle of finite radius, as described below. Another duality for these theories was proposed in [6], which is similar to ours but involves an extra $SU(2)$ symmetry whose role is not clear (and which seems to destroy the duality when included in the dynamics). Generally, we could have many different theories flowing to the same IRFP, so many other dual descriptions might also exist.

The “original” theory will be a $USp(2N_c)$ $N = 2$ gauge theory, with chiral multiplets $Q_i (i = 1, \ldots, 2N_f)$ in the fundamental ($2N_c$) representation. The gauge invariant operators parametrizing the Higgs branch are $M_{ij} = Q_i Q_j$, and the Coulomb branch (after part of it is lifted by instantons) is parametrized by a chiral superfield $Y$ (as in [5,6]). The quantum-corrected global charges of the various fields are (with a convenient choice for the $U(1)_R$ symmetry):

$$
\begin{array}{cccc}
Q & U(1)_R & U(1)_A & SU(2N_f) \\
M & 0 & 1 & 2N_f \\
Y & 2(N_f - N_c) & -2N_f & N_f(2N_f - 1) \\
& & & 1
\end{array}
$$

(2.1)
The “dual” theory we propose is a $USp(2(N_f - N_c - 1))$ gauge theory, with $2N_f$ chiral multiplets $q_i$ in the fundamental representation, and with additional singlet chiral multiplets $M$ and $Y$ (corresponding to the fields $M$ and $Y$ defined above). The Coulomb branch parameter of the “dual” theory will be denoted by $\tilde{Y}$, and we will choose the quantum numbers of the dual fields such that a $Mqq$ superpotential is possible. This leads to the following charge assignments for the new fields appearing in the “dual” theory:

$$
\begin{array}{ccc}
U(1)_R & U(1)_A & SU(2N_f) \\
1 & -1 & 2N_f \\
2(N_f - N_c - 1) & 2N_f & 1
\end{array}
$$

(2.2)

We suggest that the “dual” theory has a superpotential of the form\footnote{All scales are set to one in this paper for simplicity.}

$$W = M_{ij} q_i q_j + Y \tilde{Y},$$

(2.3)

which is consistent with the global symmetries (note that in non-Abelian theories there is no exact $U(1)_J$ symmetry acting on the “dual photons” $Y$). As in $d = 4$ $N = 1$ duality, performing the duality transformation twice returns us to the original theory. It is not clear how to define the superpotential (2.3) microscopically, since $\tilde{Y}$ is not a gauge-invariant variable (but only an effective variable on the Coulomb branch). Perhaps the chiral superfield $S = W^2_\alpha$ of the “dual” theory, which is related to $\tilde{Y}$ (they depend on the same “fundamental” fields), can be used for such a microscopic definition of the “dual” theory. In any case, since we will only be discussing the IR behavior of the theory here, we will be content with using the superpotential (2.3).

As in $d = 4$ $N = 1$ theories, we can test this duality in various ways. The global symmetries and gauge invariant chiral superfields are obviously the same in the two theories. Next, let us compare the moduli spaces of the two theories. In the “original” theory, for $Y = 0$ the meson $M$ can obtain a VEV with $\text{rank}(M) \leq 2N_c$, while for $Y \neq 0$ it satisfies $\text{rank}(M) \leq 2(N_c - 1)$. In the “dual” theory, naively $M$ can obtain any VEV (up to rank $2N_f$). The equation of motion of $\tilde{Y}$ appears to set $Y = 0$, but we should remember that for $N_f > N_c + 1$ we have no good description of the region of moduli space near the origin ($\tilde{Y} = 0$) in which $\tilde{Y}$ is a fundamental variable, so we cannot simply use this equation of motion. Using the “effective” superpotential on the moduli space, of the form $W \sim (\tilde{Y} \text{Pf}(q_i q_j))^{1/(N_f - N_c - 1)}$ (which gives a good description of the region of moduli space away from the origin), suggests that $Y$ can generically obtain any VEV in the “dual”
theory (as long as rank($M$) < 2$N_c$, as discussed below). However, since $Y$ is a singlet chiral superfield in the “dual” theory, we can use its equation of motion and find $\tilde{Y} = 0$, so the dual Coulomb branch is lifted. Now, if we give $M$ a VEV of rank $2N_c$, we are left (using the superpotential) with $2(N_f - N_c)$ massless quarks, and the low-energy theory then has a dual description in terms of a superpotential $\tilde{W}$ of the form $W = -\tilde{Y}\text{Pf}(q_iq_j)$. Together with the original superpotential (2.3) this now sets $Y = \text{Pf}(q_iq_j) = 0$, in agreement with the “original” theory. For rank($M$) = 2($N_c + 1$), instantons in the low energy theory generate a constraint of the form $\tilde{Y}\text{Pf}(q_iq_j) = 1$ [6], which is inconsistent with the superpotential, so there are no supersymmetric vacua, as in the “original” theory. Thus, the moduli spaces of the two theories are the same.

It is easy to check that, just as in four dimensions, the duality behaves in an appropriate way under complex mass perturbations (reducing $N_f$) and flows along its Higgs branch (reducing $N_f$ and $N_c$). The duality obeys the parity anomaly matching conditions discussed in [3]. The effect of adding real masses, which can also be identified in the “dual” theory since they can be thought of as background global vector fields [5], is more complicated and will not be discussed here.

The “dual” theory exists for $N_f > N_c + 1$. For $N_f = N_c + 1$, a dual description without gauge fields exists, as discussed in [5]. If we start with the $N_f = N_c + 2$ theory, whose dual is a $USp(2)$ theory, and add a mass term $mM$ for one of the quarks, the dual gauge theory breaks completely, and after integrating out the (now massive) dual quarks and $\tilde{Y}$, the global symmetries guarantee that we will go over to the known dual description using just $Y$ and $M$ [6], which is $W = -Y\text{Pf}(M)$.

Our $d = 3$ results can be connected with the $d = 4$ duality for the same gauge groups [10]. As discussed in [4,5], the description of the $d = 4 N = 1$ theory compactified on a circle of radius $R$ is the same as that of the $d = 3$ theory, except for an additional “twisted instanton” [11] contribution to the superpotential of the form $W = \eta Y$, where $\eta \sim e^{-1/Rg^2} \sim e^{-1/g_s^2}$. Due to the superpotential of the “dual” theory, we do not know how to analyze this theory directly at finite radius. However, if we add the correction to the superpotential of the “original” theory also to the “dual” theory, we find that for finite radius $\tilde{Y} = -\eta$, so the “dual” theory is always on its Coulomb branch, where classically the gauge symmetry is broken to $USp(2(N_f - N_c - 2)) \times U(1)$. In the $d = 4$ limit, we can integrate out the fields $Y$ and $\tilde{Y}$, and we are left with the duality of $d = 4$ [10]: a dual $USp(2(N_f - N_c - 2))$ gauge group with a superpotential $W = M_{ij}q_iq_j$ (the remaining $U(1)$ gauge field is free and decouples in this limit).
3. Duality in \( U(N_c) \) gauge theories

The duality for \( U(N_c) \) gauge theories is similar to the one described in the previous section. The “original” theory has \( N_f \) chiral multiplets \( Q_i \) in the \( N_c \) representation, and \( N_f \) chiral multiplets \( \tilde{Q}_\tilde{i} \) in the \( \overline{N_c} \) representation. The Higgs branch may be parametrized by the gauge-invariant superfields \( M_{\tilde{i}i} = Q_i \tilde{Q}_\tilde{i} \). The Coulomb branch remaining after the instanton corrections is now parametrized by two chiral superfields, \( V_+ \) and \( V_- \), and the quantum-corrected global charges are:

\[
\begin{array}{cccccc}
U(1)_R & U(1)_A & SU(N_f) & SU(N_f) & U(1)_J \\
Q & 0 & 1 & N_f & 1 & 0 \\
\tilde{Q} & 0 & 1 & \overline{N_f} & 0 \\
M & 0 & 2 & N_f & \overline{N_f} & 0 \\
V_\pm & N_f - N_c + 1 & -N_f & 1 & 1 & \pm 1.
\end{array}
\] (3.1)

The quantum moduli space of these theories was discussed in [5]. For \( N_f = N_c \), there is a dual description in terms of a superpotential \( W = -V_+V_- \det(M) \), from which we can flow to theories with smaller values of \( N_f \). For higher \( N_f \), the only superpotential consistent with the global symmetries is \( W \sim (V_+V_- \det(M))^{1/(N_f-N_c+1)} \), which describes the moduli space correctly but is singular at the origin. In particular, for \( V_+ = V_- = 0 \), \( M \) obeys \( \text{rank}(M) \leq N_c \), while if one of them is non-zero we find \( \text{rank}(M) \leq (N_c - 1) \), and if both \( V_+ \) and \( V_- \) are non-zero, \( \text{rank}(M) \leq (N_c - 2) \).

The “dual” theory we propose is a \( U(N_f - N_c) \) theory with \( N_f \) flavors \( q^i \) (in the \( N_c^{-} \) representation) and \( \tilde{q}_\tilde{i} \) (in the \( \overline{N_c}^{-} \) representation), much as in \( d = 4 \) [2]. As in the previous section, we will take \( M, V_+ \) and \( V_- \) to be singlet fields in the “dual” theory, and choose the global charges of \( q \) and \( \tilde{q} \) to be consistent with a \( Mq\tilde{q} \) superpotential. This leads to the quantum-corrected global charges:

\[
\begin{array}{cccccc}
U(1)_R & U(1)_A & SU(N_f) & SU(N_f) & U(1)_J \\
q & 1 & -1 & \overline{N_f} & 1 & 0 \\
\tilde{q} & 1 & -1 & 1 & N_f & 0 \\
V_\pm & N_c - N_f + 1 & N_f & 1 & 1 & \pm 1.
\end{array}
\] (3.2)

We suggest that the superpotential of the “dual” theory is

\[ W = M^\tilde{i}q^i\tilde{q}_\tilde{i} + V_+\tilde{V}_- + V_-\tilde{V}_+ , \] (3.3)

which is consistent with the global symmetries described above. As in \( d = 4 \) \( N = 1 \) duality, performing the duality transformation twice returns us to the original theory.
The tests we can perform of this duality are the same as the ones described in the previous section. The only small difference is in comparing the Higgs branches of the two theories. Now, when we give $M$ a VEV of rank $N_c$, we flow at low energies to the $U(N_f - N_c)$ theory with $N_f - N_c$ flavors, which has a dual description in terms of a superpotential $W = \tilde{V}_+ \tilde{V}_- \text{det}(q\tilde{q})$. Adding this to (3.3), we find that the equations of motion set $V_+ = V_- = 0$, as in the “original” theory. If rank$(M) = (N_c - 1)$ things are a bit more complicated, since we have no consistent dual description of the low energy theory in this case. However, using the “effective” low energy superpotential $W \sim (\tilde{V}_+ \tilde{V}_- \text{det}(q\tilde{q}))^{1/2}$ for this case, we find (using (3.3)) that $V_+$ and $V_-$ can be non-zero but $V_+V_- = 0$, again as in the “original” theory. For rank($M$) $> N_c$ we find, as before, no stable vacua because of instanton effects. The standard Coulomb and Higgs branches of the “dual” theory are obviously lifted by (3.3). Thus, the moduli spaces of the two theories are the same.

An additional deformation we can perform in the $U(N_c)$ case is to add a Fayet-Iliopoulos term to the “original” theory. As described in [5], this term can be thought of as a background $U(1)_J$ vector field, so in the “dual” theory it becomes both a Fayet-Iliopoulos term and a real mass term for $V_+$ and $V_-$. The Fayet-Iliopoulos term forces the gauge symmetries of both theories to be completely broken. The simplest flat direction now involves taking only $Q$’s to be non-zero in the “original” theory (assuming the Fayet-Iliopoulos term is positive) – this is the “baryonic” flat direction, discussed in [12,13] (and used in $d = 4$ $N = 1$ dual theories). In the “original” theory, the gauge symmetry is completely broken, some of the $Q$’s are swallowed by the Higgs mechanism, and $N_c(N_f - N_c)$ $Q$’s and $N_cN_f$ $\tilde{Q}$’s remain massless (and free in the IR). In the “dual” theory, again only $q$’s obtain VEVs, breaking the gauge group completely. Some of the $q$’s are swallowed by the Higgs mechanism, and the $\tilde{q}$’s and some of the $M$’s are given a mass by the superpotential. We remain with $N_c(N_f - N_c)$ massless $q$’s and $N_fN_c$ massless $M$’s, which are again free in the IR, and which may be identified with the remaining fields of the “original” theory [12]. The Coulomb branches of both theories are lifted (including the branch with non-zero $V_\pm$ in the “dual” theory, due to their real masses), so the duality becomes a trivial IR identification of free fields in this case (like in $d = 4$ [12]).

The “dual” gauge theory description exists for $N_f > N_c$, while for $N_f = N_c$ a dual description is known involving just $V_+, V_-$ and $M$ with no gauge fields [5]. If we start with the $N_f = N_c + 1$ theory, whose dual is a $U(1)$ gauge theory, and add a mass term $mM$ for one of the quarks, the “dual” gauge theory breaks completely. Integrating out the massive
quarks and $\tilde{V}_\pm$ fields, the global symmetries guarantee that we go over to the known dual description $W = -V_+ V_- \det(M)$ for the $N_f = N_c$ theory.

Flowing to $d = 4$ in this case involves adding a “twisted instanton” superpotential of the form $W = \eta V_+ V_-$ (for $N_c > 1$). Again, it is not clear how this term arises in the “dual” theory, but we will assume that it arises there as well as in the “original” theory. Now, in the $d = 4$ limit, we integrate out the Coulomb branch fields, and remain with the standard duality of $d = 4$ $U(N_c)$ gauge theories [2] (which follows from the duality of $SU(N_c)$ theories by gauging the global $U(1)_B$ symmetry). Note that this works even for $N_f = N_c + 1$. In this case we end up in $d = 4$ with the “effective” descriptions of the $N_f = N_c + 1$ SQCD theories [14], again with the $U(1)_B$ symmetry gauged (and free in the IR).

For the $U(1)$ gauge theory with $N_f = 2$, the “dual” theory is also a $U(1)$ theory with two flavors, and the same is true also for the “mirror” description of this theory [3,4,5]. The “mirror” description involves a superpotential of the form $W = S_1 q^1 \tilde{q}_1 + S_2 q^2 \tilde{q}_2$ with two singlets $S_1$ and $S_2$, which are identified with two of the meson fields of the “original” theory (say, $M_1$ and $M_2$). The “dual” description (defined above) differs from this by having four additional singlet fields in the “dual” theory, $M_2^1, M_1^2, V_+$ and $V_-$, and by having a manifest $SU(2) \times SU(2)$ global symmetry (which is not manifest in the “mirror” theory). The equations of motion, however, relate these additional fields to the other “dual” fields (for instance, they set $V_+ V_- \sim \det(q^i \tilde{q}_i)$), so we can think of them as auxiliary fields, and then the “dual” description seems to be very similar to the “mirror” description. This relationship suggests that perhaps the duality we describe here can also be understood in terms of classical vortex solutions, like the “mirror” symmetry [3].

Presumably, it should be possible to generalize this duality also to $SU(N_c)$ gauge groups, but it is not obvious how to do this. A duality for $SU(N_c)$ gauge groups may easily be turned into a $U(N_c)$ duality by gauging a global symmetry, but “ungauging” symmetries is more difficult, and the $U(1)$ gauge field has important dynamics (such as confinement) in $d = 3$. Generalizations to other cases (with or without known $d = 4$ dualities) should presumably also be possible.

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