On the spin content of the massless Rarita–Schwinger system

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July 8, 2022

Abstract

We analyze the Rarita–Schwinger (RS) massless theory in the Lagrangian and Hamiltonian approaches. At the Lagrangian level, the standard gamma-trace gauge fixing constraint leaves a spin $\frac{3}{2}$ and a spin $\frac{3}{2}$ propagating Poincaré group helicities. At the Hamiltonian level, the result depends on whether the Dirac conjecture, –that all first class constraints generate gauge symmetries–, is assumed or not. In the affirmative case, a secondary first class constraint must be added to the total Hamiltonian and a corresponding gauge fixing condition must be imposed, completely removing the spin–$\frac{3}{2}$ sector. In the opposite case, the spin–$\frac{3}{2}$ field propagates and the Hamilton field equations match the Euler-Lagrange equations.

1 Introduction

In 1939, Markus Fierz and Wolfgang Pauli discussed the obstacles in the attempt to quantize fields of arbitrary spin $\geq 1$ interacting with photons [1]. Two years later, William Rarita and Julian Schwinger simplified the Fierz-Pauli treatment, writing down a set of field equations describing fermions of arbitrary spin $\geq 3/2$ [2]. The Rarita-Schwinger system (RS) described a field of spin $k + 1/2$ as a tensor-spinor of rank $k$, $\psi_{\mu_1 \cdots \mu_k}^\alpha$, symmetric in its tensor indices $\mu_1 \cdots \mu_k$, satisfying a Dirac-like field equation with mass, $(\partial^\mu + M)\psi_\mu = 0$, supplemented by three subsidiary conditions: $\partial^\mu \psi_{\mu \mu_2 \cdots \mu_k} = 0$ (transverse), $\gamma^\mu \psi_{\mu \mu_2 \cdots \mu_k} = 0$ (gamma-traceless), and $\psi^{\mu \mu_2 \cdots \mu_k} = 0$ (traceless). The simplest nontrivial RS system is the spin–$\frac{3}{2}$ field given by a vector-spinor field $\psi_\mu^\alpha$ for which the third subsidiary condition is omitted. These authors also
noted that there is a class of Lagrangians parametrized by the mass \((M)\) and a dimensionless coefficient \((A)\) that gives rise to these equations (see, e.g. \[3–5\]). Rarita and Schwinger then chose some \(A\) “which permits a relatively simple expression of the equations of motion in the presence of electromagnetic fields”. The description of spin-\(\frac{3}{2}\) particles adopted in supergravity, however, traditionally referred to as the Rarita–Schwinger Lagrangian \[6, 7\] (see also \[8\] and references therein) corresponds to a different choice of \(A\) which, in the massless limit gives the Lagrangian

\[
\mathcal{L} := -\frac{i}{2} \psi_{\mu} \gamma^{\mu\nu\lambda} \partial_{\nu} \psi_{\lambda},
\]

(1)

whose corresponding field equations are

\[
\gamma^{\mu\nu\lambda} \partial_{\nu} \psi_{\lambda} = 0.
\]

(2)

The action changes by a boundary term and (2) is invariant under the gauge transformation

\[
\delta \psi_{\mu} = \partial_{\mu} \epsilon.
\]

(3)

Eq. (2) can also be written as \(\partial \psi_{\mu} - \partial_{\mu} \gamma \cdot \psi = 0\), \(\partial \cdot \psi - \partial \gamma \cdot \psi = 0\) and, in the \(\gamma^{\mu} \psi_{\mu} = 0\) gauge, as

\[
\partial \psi_{\mu} = 0, \quad \partial^{\mu} \psi_{\mu} = 0, \quad \gamma^{\mu} \psi_{\mu} \overset{\text{def}}{=} 0.
\]

(4)

The third equation corresponds to the gauge choice that fixes the freedom (3), where the symbol \(\overset{\text{def}}{=}\) reflects this. In the massive RS system, \(\partial^{\mu} \psi_{\mu} = 0\) is a consistency condition of the field equations, hence (4) can be thought of as the massless limit of the original RS equations.\(^2\)

We stress, however, that (4) is not obtained by direct variation of the massless action; they are the consequence of that particular gauge choice. Note that the gauge parameter necessary to reach the gamma-traceless condition \(\gamma^{\mu} \psi_{\mu} = 0\) from a generic configuration, \(\epsilon = -\partial^{-1}(\gamma^{\mu} \psi_{\mu})\), is non-local in time.

In terms of Poincaré group the vector-spinor \(\psi_{\mu}^{\alpha}\) carries a reducible representation of spins \((1 \oplus 0) \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}\). The latter splitting can be achieved explicitly in terms of non-local Ogievetsky-Sokatchev projectors \[8, 9\], which also involve nonlocal operators in time. Non-locality along time directions is a problematic aspect of the Lorentz-covariant approaches: they might be incompatible with the integration of the field equations under initial conditions.

\(^1\)Here \(\gamma_{\mu}, \{\gamma_{\mu}, \gamma_{\nu}\} = 2\eta_{\mu\nu}\), are Dirac matrices, with \(\eta_{\mu\nu} = \text{diag}(-1,1,\ldots)\) and \(\gamma_{\mu\ldots\nu} = \gamma_{\mu} \cdots \gamma_{\nu}\) are completely anti-symmetric products. We assume the Majorana reality condition \(\psi^{\dagger} = \psi, \overline{\psi} = \psi^{\dagger} C\), were \(C^{t} = -C\).

\(^2\)This is analogous to the transverse condition \(\partial^{\mu} A_{\mu} = 0\) that is required by consistency of the Proca equations but is only a gauge option in Maxwell’s theory.
on a Cauchy surface. Spatial nonlocality, on the other hand, is compatible with the Cauchy data because it leaves the initial surface intact and therefore gauge transformations or field redefinitions involving nonlocal spatial operators such as $\nabla^{-1} := (\gamma^i \partial_i)^{-1}$, do not lead to inconsistencies.

### 2 Space and time splitting

In order to avoid nonlocal time operators, one can separate the spinor vector $\psi_\mu$ into $\psi_0$ and $\psi_i$. The spatial components $\psi_i$, in turn, can be split into three more pieces, the spatial divergence $\partial^i \psi_i$, the $\gamma$–trace $\gamma^i \psi_i$, and a spatial $\gamma$–traceless and divergenceless vector-spinor $\xi_i$ ($\gamma^i \xi_i = 0 = \partial^i \xi_i$). Thus we have one spin–$\frac{1}{2}$ field $\xi_i$, and three spin–$\frac{1}{2}$ representations of the spatial rotation group, $\psi_0$, $\partial^i \psi_i$ and $\gamma^i \psi_i$. We shall see that the $\gamma$–traceless and divergenceless conditions (4) remove one spin–$\frac{1}{2}$ representations of the rotation group each, whilst the third is a propagating spin–$\frac{1}{2}$ irreducible representation of the Poincaré group.

Let us consider a decomposition of the identity acting on the space of spin-vectors into three orthogonal projectors,

$$
\mathbb{1} = \mathcal{P}^\perp + \mathcal{P}^N + \mathcal{P}^L,
$$

where

$$
(P^N)_{ij} := \frac{1}{D-2} N_i N_j, \quad (P^L)_{ij} := L_i L_j, \quad P^\perp = \mathbb{1} - P^N - P^L,
$$

with $N_i := \gamma_i - L_i$ and $L_i := \nabla^{-1} \partial_i$. These are space-like Ogievetsky–Sokatchev projectors that split the spatial vector-spinor as

$$
\psi_i = \xi_i + N_i \zeta + L_i \lambda, \quad \text{where} \quad \xi_i = P^\perp \psi_j, \quad \zeta = \frac{1}{D-2} N^i \psi_i, \quad \lambda = L^i \psi_i,
$$

which can be verified with the help of the identities $N_i N^i = D - 2, L_i L^i = 1, N_i L^i = 0$.

The gamma traceless relation is equivalent to $\psi_0 = -\gamma_0 \gamma^i \psi_i = -\gamma_0 ((D - 2) \zeta + \lambda)$ which, together with the decomposition (6), reduce (4) to

$$
\phi \dot{\xi}_i = 0, \quad \phi \dot{\lambda} = 0, \quad \dot{\zeta} = 0, \quad \nabla \zeta = 0,
$$

where $\dot{\lambda} = \gamma^0 \lambda$. It follows that fields $\xi_i$ and $\lambda$ propagate, whilst $\zeta$ is a constant spinor –and must therefore vanish on-shell–, and $\psi_0 = \dot{\lambda}$ is not an independent field. Hence, the massless RS equations in the gauge $\gamma^\mu \psi_\mu = 0$ eliminates two spin–$\frac{1}{2}$ modes, $\psi_0$ and $\zeta$, whilst one spin–$\frac{1}{2}$ and one spin–$\frac{3}{2}$ field propagate. Thus, there are no gauge orbits left in the system to be fixed.

The dynamical equations (7) completely determine the evolution of the fields, provided initial data is given on a Cauchy surface. The number of degrees of freedom in the system is defined by one half the number of functions in the set $\{ \xi_i^\alpha(t_0, \vec{x}), \lambda^\alpha(t_0, \vec{x}) \}$ necessary to specify
the evolution. These functions are: \( k \times (D - 1) - 2k \) components of \( \xi_\alpha \), \( \alpha = 1, \ldots, k \), and \( k = 2^{[D/2]} \) components in \( \lambda^\alpha \); in both cases the Dirac equation restricts half of them. In total, we are left with \( k(D - 2)/2 \) degrees of freedom which are equivalent to two massless states of spin-\( \frac{1}{2} \) and spin-\( \frac{3}{2} \), respectively.

3 Hamiltonian analysis

Splitting \( \psi_\mu \) as in (6) one obtains, up to boundary terms,

\[
\mathcal{L} = -i \bar{\psi}_0 \gamma^{0ij} \partial_\mu \psi_j + \frac{i}{2} \bar{\psi}_0 \gamma^{0ij} \dot{\psi}_j - \frac{i}{2} \bar{\psi}_i \gamma^{ijk} \partial_j \psi_k .
\]

The definition of momenta, \( \pi^\mu := \partial \mathcal{L}/\partial \dot{\psi}_\mu \), yields the primary constraints

\[
\pi^0 \approx 0 , \tag{9}
\]

\[
\chi^i := \pi^i - \frac{i}{2} C^{ij}_\alpha \psi_j \approx 0 , \tag{10}
\]

where \( C^{ij}_\alpha := -(C \gamma^{0ij})_{\alpha \beta} = C^{ji}_\alpha \) is invertible, \( C^{ij}_\alpha (C^{-1})^{\beta \kappa} := \delta^\beta_\alpha \delta^\kappa_\alpha \),

\[
(C^{-1})^{i \beta} = \left( - \frac{1}{(D - 2)} \gamma^i \gamma_0 C^{-1} + \delta^i_\beta \gamma_0 C^{-1} \right)^{\alpha \beta} . \tag{11}
\]

The constraint (9) states that \( \psi_0 \) is a Lagrange multiplier and (10) is a consequence of the first order character of the system. The Hamiltonian, including a linear combination of the primary constraints is

\[
H = \int d^{D-1}x \left( i \bar{\psi}_0 \gamma^{0ij} \partial_\mu \psi_j + \frac{i}{2} \bar{\psi}_0 \gamma^{0ij} \dot{\psi}_j + \chi^i_\alpha \mu_\alpha + \pi^0_\alpha \mu_0 \right) , \tag{12}
\]

where \( \mu_\alpha \) and \( \mu_0 \) are arbitrary spinorial Lagrange multipliers. Preservation in time of the primary constraint \( \pi^0 \approx 0 \) yields a secondary constraint \( \bar{\psi}_0 \partial_\mu \psi_0 \approx 0 \),

\[
\dot{\pi}^0 = -\{ H, \pi^0 \} = -\frac{\delta H}{\delta \psi_0} = -i C \gamma^{0ij} \partial_\mu \psi_j \approx 0 \quad \Rightarrow \quad \varphi := -i C^{ij}_\alpha \partial_\mu \psi_j \approx 0 , \tag{13}
\]

which is equivalent to the equation of motion obtained varying (8) with respect to \( \psi_0 \). Preservation in time of the other primary constraints, \( \chi_\alpha^i \approx 0 \), and of the secondary one, \( \varphi \approx 0 \), yield conditions that determine the Lagrange multipliers \( \mu_\alpha \) in terms of the phase space fields, and there are no further constraints. It is easily checked that \( \pi^0 \) and \( \varphi + i \partial_\mu \chi^i \) are first class and \( \chi^i \) second class. The latter will be eventually dropped, leaving \( \pi^0 \approx 0 \) and \( \varphi \approx 0 \) as the only remaining first class constraints.

\[\text{The Poisson bracket is } \{ f(t, \bar{x}), g(t, \bar{y}) \} := (-1)^{|f|} \int d^{D-1}z \left( \frac{\delta f(t, \bar{x})}{\delta \psi_0(t, \bar{y})} \frac{\delta g(t, \bar{y})}{\delta \psi_0(t, \bar{x})} + \frac{\delta f(t, \bar{x})}{\delta \psi_0(t, \bar{x})} \frac{\delta g(t, \bar{y})}{\delta \psi_0(t, \bar{y})} \right).\]
The system can be reduced to the surface of the second class constraints \([10]\) by strongly setting \(\chi^i = 0\) and replacing Poisson by Dirac brackets, \(\{ f, g \}_D := \{ f, g \} - \{ f, \chi^i \} C^{-1}_{ij} \{ \chi^j, g \}\), which in the variables \(\psi_0, \pi^0, \xi, \zeta\) and \(\lambda\), reads

\[
\{ f, g \}_D = (-1)^f \int d^{D-1}z \left[ -\frac{\delta f}{\delta \xi_i} P_{ij} \gamma_0 C^{-1} \frac{\delta g}{\delta \xi_j} - i \frac{D - 3}{D - 2} \frac{\delta f}{\delta \lambda} \gamma_0 C^{-1} \frac{\delta g}{\delta \lambda} + \frac{i}{2} \frac{\delta f}{\delta \psi_0} \frac{\delta \pi^0}{\delta \alpha^0} + \frac{\delta f}{\delta \pi^0} \frac{\delta \gamma_0}{\delta \psi_0} \right],
\]

and the first class Hamiltonian \([12]\) reduces to

\[
H_1 = \int d^{D-1}x \left( i(D - 2)\overline{\psi}_0 \gamma^0 \nabla \zeta - \frac{i(D - 2)(D - 3)}{2} \overline{\zeta} \nabla \psi_0 + \frac{i}{2} \overline{\zeta} \nabla \xi_i + \pi^0 \mu^0 \right),
\]

where the first class secondary constraint \(\varphi \approx 0\) is equivalent to \(\nabla \zeta \approx 0\).

The question now is whether one should add this secondary first class constraint to the Hamiltonian as an independent gauge generator. This is equivalent to asking whether the Dirac conjecture \((DC)\) holds in this case, namely, whether all secondary first class constraints generate gauge transformations. If the conjecture is valid, the gauge transformations generated by \(\varphi\) would require gauge fixing; if that is not the case, \(\varphi\) does not generate gauge transformations, it should not be included in the Hamiltonian and no gauge fixing would be required.

One can examine the effect of adding \(\varphi\) to the Hamiltonian \((15)\) with a Lagrange multiplier. The time evolution defined by \(\dot{f} = \{ f, H' \}\), with \(H' := H_1 + \tau^\alpha \varphi_\alpha\), is

\[
\dot{\xi}_i = -\gamma_0 \nabla \psi_i, \quad \dot{\lambda} = -(D - 3)\gamma_0 \nabla \zeta + \nabla \psi_0 + \nabla \tau, \\
\dot{\psi}_0 = -\mu_0, \quad \pi^0 = 0, \quad \dot{\zeta} = 0, \quad \nabla \zeta = 0.
\]

The gauge symmetry generated by \(\pi^0\) is fixed by specifying \(\psi_0\), which can be chosen to implement the standard \(\gamma\)-traceless condition in \((11)\) as \(\psi_0 + \gamma_0 \gamma^i \psi_i \approx 0\). This, together with \(\pi^0 \approx 0\), form a pair of second class constraints that can be readily eliminated from the phase space. This gauge choice is accessible since \(\pi^0\) generates arbitrary shifts in \(\psi_0\) and, in particular, the shift \(\delta \psi_0 = -(\psi_0 + \gamma_0 \gamma^i \psi_i)\), renders \(\psi'_\mu = \psi_\mu + \delta \psi_\mu\) \(\gamma\)-traceless. Thus \((10)\), in the phase space spanned by variables \(\xi_i, \zeta, \lambda\), reduces to

\[
\dot{\xi}_i = -\gamma_0 \nabla \psi_i, \quad \dot{\lambda} = \gamma_0 \nabla \lambda + \nabla \tau, \quad \dot{\zeta} = 0 = \nabla \zeta.
\]

Assuming the DC as valid, implies that evolution of \(\lambda\) is indeterminate and therefore an external gauge condition, in convolution with the constraint \(\varphi \approx 0\), is necessary to fix the gauge freedom represented by \(\tau\). This means that \(\lambda\) is gauge field which can be removed by choosing
\( \lambda = 0 \). Then, the only propagating field left is \( \xi_i \), and in this case the Hamilton equations do not match the Euler-Lagrange equations (7).

If the DC is not assumed, \( \varphi \) is not regarded as a gauge generator and should not be added to \( H_1 \). This is equivalent to setting \( \tau = 0 \) in (18), in agreement with the Euler-Lagrange equations (7), in which case \( \lambda \) does not have an undetermined evolution.

The two scenarios presented above are consistent. Although in the first case the resulting Hamiltonian evolution is not equivalent to the Lagrangian dynamics, this is not necessarily unphysical. There are Lagrangian systems whose Hamiltonian formulation leads to secondary first class constraints that do not generate gauge transformations [10–12]. For those counterexamples to the DC it is still possible to postulate the validity of the conjecture without running into inconsistencies. Moreover it has been argued that not adopting the Dirac conjecture might lead to problems in the quantization, which supports the idea that it would be safer to assume the validity of the DC in general [12].

On the other hand, it does not seem necessary to postulate the DC in our case; the resulting system is still consistent and in agreement with the Lagrangian description. And the Dirac bracket (14) does not lead to quantization problems of the sort found in the counterexample to the DC in [12]. If the DC is not valid because some secondary first class constraints do not generate gauge transformations, there is no need to provide a gauge condition for those constraints, and the standard formula for the counting of degrees of freedom [12–14] generalizes as

\[
2 \times \left[ \text{Number of d.o.f.} \right] = \left[ \text{Dimension of phase space} \right] - \left[ \text{2nd class constraints} \right] - 2 \times \left[ \text{1st class gauge generators} \right] - \left[ \text{1st class non-gauge generators} \right].
\]

Note that the last term on the right hand side could be odd, leading to a paradoxical quantum scenario. However in systems of spinors, first class constraints have an even number of components. Hence, there are no a priori inconsistencies in our case. For the RS system in 4 dimensions, this counting gives \((16 \times 2 - 12 - 2 \times 4 - 4)/2 = 4\) degrees of freedom, which correspond to two spin-\(\frac{3}{2}\) helicities plus two spin-\(\frac{1}{2}\) helicities. In references [8,15–18], on the other hand, the DC is assumed to be valid, concluding that there are only 2 degrees of freedom, those of a massless spin-\(\frac{3}{2}\) field.

The apparent presence of a propagating spin-\(\frac{1}{2}\) mode in the RS system contradicts the expectation that the spin-\(\frac{1}{2}\) field is a pure gauge mode. A dynamical spin-\(\frac{1}{2}\) mode in RS sounds similar to the claim that there is a propagating spin-0 field in the Maxwell theory.
However, in contrast to what happens in gauge theories like Maxwell, Yang-Mills or Chern-Simons when evaluated on a pure gauge configuration like $A_{\mu} = \Lambda^{-1} \partial_{\mu} \Lambda$, the RS action neither vanishes nor reduces to a boundary term when evaluated on $\psi_{\mu} = \gamma_{\mu} \zeta$ for a generic $\zeta$. This means that configurations $\psi_{\mu} = \gamma_{\mu} \zeta$ are not zero-modes of the action, unlike what happens in gauge theories for pure gauge configurations. The reduction $\psi_{\mu} = \gamma_{\mu} \zeta$ is precisely what is done in unconventional supersymmetry \cite{19,23}, while in supergravity the complementary option is selected by imposing $\gamma_{\mu} \psi_{\mu} = 0$ \cite{8}.

So far we have assumed a flat spacetime, although the generalization to a curved background is straightforward. In the light of these results, it would be interesting to consider supergravity theories without enforcing the validity of the Dirac conjecture, which must contain a spin-$\frac{1}{2}$ excitation along with the gravitino. The spin-$\frac{1}{2}$ sector will inherit the gravity and gauge interactions of the vector spinor, which would generate new supergravity phenomenology.

Acknowledgements We thanks L. Andrianopoli, N. Boulanger, C. Bunster, R. Matrecano, R. Noris, M. Rausch de Traubenberg, P. Sundell, M. Trigiante for their challenging questions, comments and suggestions. This work was partially funded by grant FONDECYT 1220862.

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