EXISTENCE AND SPATIO-TEMPORAL PATTERNS OF PERIODIC SOLUTIONS TO SECOND ORDER NON-AUTONOMOUS EQUIVARIANT DELAYED SYSTEMS

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Abstract. Existence and spatio-temporal symmetric patterns of periodic solutions to second order reversible equivariant non-autonomous periodic systems with multiple delays are studied under the Hartman-Nagumo growth conditions. The method is based on using the Brouwer $D_1 \times Z_2 \times \Gamma$-equivariant degree theory, where $D_1$ is related to the reversing symmetry, $Z_2$ is related to the oddness of the right-hand-side and $\Gamma$ reflects the symmetric character of the coupling in the corresponding network. Abstract results are supported by a concrete example with $\Gamma = D_n$ — the dihedral group of order $2n$.

2010 AMS Mathematics Subject Classification: 34K13, 37J45, 39A23, 37C80, 47H11.

Key Words: Second order delay-differential equations, multiple delays, Hartman-Nagumo condition, periodic solutions, Brouwer equivariant degree, Burnside ring, reversible systems.

1. Introduction

1.1. Subject and goal. For a long time, the classical forced pendulum equation $\ddot{y} + a \sin(y) = b \sin(t)$ together with its multiple generalizations stand as an important subject, where modern methods and techniques of Nonlinear Analysis are tested (see, for example, [24, 25]). Among them, the variational methods (Lusternik-Schnirelmann Theory, Morse Theory) and topological ones (based on different variants of degree theory) have taken a firm position. For the systematic variational treatment of second order Hamiltonian ODEs, we refer to the monographs [26, 29] and references therein. In [19] (see also [12]), the variational approach was applied for the first time to autonomous second order delay differential equations (in short, DDEs); for non-autonomous second order periodic DDEs (in short, NASODDEs), we refer to [27, 31, 32] and references therein. Also, the case of multiple commensurate periods (i.e., the delays being integer multiples of a given number) was studied in [10], where one can find detailed historical and bibliographical remarks on the subject.

The applications of degree theory based methods require a priori estimates of solutions in appropriate function spaces. These estimates can be guaranteed if the right-hand side of a NASODDE satisfies certain growth restrictions at infinity provided, for example, by the so-called Hartman-Nagumo condition (see, for example, [13, 21, 7, 1] and references therein for ODEs and [20] for (scalar) FDEs independent of the first derivative).

If $k$ identical oscillators described by a NASODDE are coupled $\Gamma$-symmetrically, where $\Gamma$ stands for a finite group, then symmetries of the coupling are reflected in the resulting system in the form of the $\Gamma$-equivariance of the right-hand side, i.e. it commutes with the $k$-dimensional permutation...
Γ-representation canonically corresponding to the coupling. Also, a Γ-coupled system may exhibit additional symmetries: reversibility (cf. \cite{13}), oddness/evenness, to mention a few. Studying the existence of periodic solutions to such systems together with their spatio-temporal symmetric patterns, being a problem of formidable complexity, has not been given enough attention (especially, in non-variational setting). It turns out that there is a topological tool – the so-called Brouwer equivariant degree (see, for example, \cite{15, 9}) – allowing one to effectively study the problem in question in many cases. The goal of the present paper is to open a door to a systematic usage of this tool to study symmetric patterns of periodic solutions to equivariant NASODDEs.

1.2. Methodology. Brouwer degree (in compliance with the guiding function method, see \cite{23}) as well as its different infinite dimensional generalizations (Leray-Schauder degree \cite{22}, coincidence degree \cite{20}, etc.) proved their efficiency in detecting periodic solutions to non-autonomous systems. Theoretically, these degrees can be used in symmetric settings. To be more specific, given an orthogonal G-representation V (here G stands for a compact Lie group) and an admissible G-pair \((f, \Omega)\) in V (i.e., \(\Omega \subset V\) is an open bounded G-invariant set and \(f : V \to V\) a G-equivariant map without zeros on \(\partial \Omega\)), the Brouwer degree \(d_H := \deg(f^H, \Omega^H)\) is well-defined for any \(H \leq G\) (here \(\Omega^H := \{x \in \Omega : hx = x \ \forall h \in H\}\) and \(f^H := f|_{\Omega^H}\)). If for some \(H\), one has \(d_H \neq 0\), then the existence of solutions with symmetry at least \(H\) to equation \(f(x) = 0\) in \(\Omega\) is guaranteed (similar argument can be used for infinite dimensional versions of the Brouwer degree). This approach provides a way to determine the existence of solutions in \(\Omega\), and to distinguish their different orbit types. However, this approach may be inefficient and very exhausting in the case the space \(V\) contains a large number of orbit types.

As it was mentioned above, our method is based on the usage of the Brouwer equivariant degree theory; for the detailed exposition of this theory, we refer to the monographs \cite{3, 15, 14, 17} and survey \cite{2} (see also \cite{4, 5}). In short, the equivariant degree is a topological tool allowing “counting” orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties. To be more explicit, the equivariant degree \(G\text{-deg}(f, \Omega)\) is an element of the free \(\mathbb{Z}\)-module \(A_0(G)\) generated by the conjugacy classes \((H)\) of subgroups \(H\) of \(G\) with a finite Weyl group \(W(H)\):

\[
G\text{-deg}(f, \Omega) = \sum_{(H)} n_H(H), \quad n_H \in \mathbb{Z},
\]

where the coefficients \(n_H\) are given by the following recurrence formula

\[
n_H = \frac{d_H - \sum_{(L) > (H)} n_L n(H, L) |W(L)|}{|W(H)|},
\]

and \(n(H, L)\) denotes the number of subgroups \(L'\) in \(L\) such that \(H \leq L'\) (see \cite{3}). Also, we use the notation

\[
\text{coeff}^H(a) := n_H \text{ for any } a = \sum_{(H)} n_H(H) \in A_0(G).
\]

One can immediately recognize a connection between the two collections: \(\{d_H\}\) and \(\{n_H\}\), where \(H \leq G\) and \(W(H)\) is finite. As a matter of fact, \(G\text{-deg}(f, \Omega)\) satisfies the standard properties expected from any topological degree. However, there is one additional functorial property, which plays a crucial role in computations, namely the multiplicity property. In fact, \(A_0(G)\) has a natural structure of a ring (which is called the Burnside ring of \(G\)), where the multiplication \(\cdot : A_0(G) \times A_0(G) \to A_0(G)\) is defined on generators by \((H) \cdot (K) = \sum_{(L)} m_L(L)\) with

\[
m_L := |(G/H \times G/K)_{(L)}|/G|, \quad \text{where } W(L) \text{ is finite.}
\]
The multiplicity property for two admissible $G$-pairs $(f_1, \Omega_1)$ and $(f_2, \Omega_2)$ means the following equality:

$$G\text{-deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-deg}(f_1, \Omega_1) \cdot G\text{-deg}(f_2, \Omega_2).$$

Given a $G$-equivariant linear isomorphism $A : V \rightarrow V$, formula (5) combined with the equivariant spectral decomposition of $A$, reduces the computations of $G\text{-deg}(A, B(V))$ to the computation of the so-called basic degrees $\deg_{V_i}$, which can be “prefabricated” in advance for any group $G$ (here $\deg_{V_i} := G\text{-deg}(-\text{Id}, B(V_i))$, where $V_i$ is an irreducible $G$-representation and $B(X)$ stands for the unit ball in $X$). All in all, the product property of the Brouwer equivariant degree provides a huge computational advantage in comparison with the usual Brouwer degree approach. In order to facilitate the usage of the Brouwer equivariant degree, a package EquiDeg for GAP programming allowing effective application of the equivariant degree methods, was created by Hao-Pin Wu. This package, being available from https://github.com/psistwu/GAP-equideg (see [30]), is behind our numerical example with dihedral symmetries considered in Subsection 6.2.

1.3. Overview. After the Introduction, the paper is organized as follows. In Section 2 we describe a class of symmetric NASODDEs of our interest (see equation (5) and conditions (7) and (A0)–(A6)), and clarify the role of several symmetry subgroups in studying symmetric patterns of periodic solutions. In Section 3 we establish a priori bound for solutions to problem (8) in the space $C^2(S^1; V)$. In Section 4 we reformulate problem (8) as a $G := D_1 \times Z_2 \times \Gamma$-equivariant fixed point problem in $C^2(S^1; V)$ (see (31)–(32) and conditions (A4), (A6)), and present an abstract equivariant degree based result (see Proposition 4.4; see also Corollary 5.5 dealing with the non-symmetric setting). In Section 5 we explicitly describe a wide class of NASODDEs satisfying hypotheses of Theorem 5.3 and provide an illustrating example with the group $\Gamma = D_n$ – the dihedral group of order $2n$ (see Corollary 6.1). We conclude the paper with an Appendix related to the equivariant topology jargon and equivariant degree background.

2. Setting and symmetries

2.1. Setting. Take the Euclidean space $V = \mathbb{R}^n$ and for given $m \in \mathbb{N}$, put

$$V^m := \underbrace{V \times \cdots \times V}_{m\times}.$$

For $y \in V^m$, denote $y := (y^1, y^2, \ldots, y^m)$, $y^j \in V$, $j = 1, \ldots, m$, and define

$$|y| := \max\{|y^1|, |y^2|, \ldots, |y^m|\}.$$

Assume $\Gamma$ is a subgroup of the symmetric group $S_n$. The group $\Gamma$ acts on vectors $x = (x_1, x_2, \ldots, x_n)$ in $V$ by permuting their coordinates, i.e. for $\gamma \in \Gamma$, one has:

$$\gamma(x_1, x_2, \ldots, x_n) := (x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(n)}).$$

This way $V$ becomes an orthogonal $\Gamma$-representation. Then, the space $V^m$ equipped with the diagonal $\Gamma$-action given by $\gamma(y^1, \ldots, y^m) := (\gamma y^1, \ldots, \gamma y^m)$ is an orthogonal $\Gamma$-representation.

Take reals $\tau_j$, $j = 0, 1, 2, \ldots, m$, satisfying

$$0 = \tau_0 < \tau_1 < \cdots < \tau_m < 2\pi, \quad \tau_{m+j+1} = 2\pi - \tau_j \quad \text{for } j = 1, 2, \ldots, m.$$

Notice that for $\zeta_j := e^{i\tau_j}$, $j = 1, 2, \ldots, m$, condition (7) implies $\overline{\zeta_j} = \zeta_{m-j+1}$.
Take a function \( f : \mathbb{R} \times \mathbf{V} \times \mathbf{V}^m \times \mathbf{V} \rightarrow \mathbf{V} \) and consider the following system of NASODDEs:

\[
\dot{x}(t) = f(t, x(t), x_t, \dot{x}(t)), \quad t \in \mathbb{R},
\]

where for a function \( x : \mathbb{R} \rightarrow \mathbf{V} \), we put

\[
x_t := (x(t - \tau_1), x(t - \tau_2), \ldots, x(t - \tau_m)) \in \mathbf{V}^m, \quad t \in \mathbb{R}.
\]

We will assume that \( f \) satisfies the following conditions:

(A0) \( f \) is continuous and \( f(t + 2\pi, x, y, z) = f(t, x, y, z) \) for all \( t \in \mathbb{R}, x, y \in \mathbf{V}, y \in \mathbf{V}^m \);

(A1) There exists \( R > 0 \) such that for all \( x, z \in \mathbf{V}, y \in \mathbf{V}^m \), one has:

\[
|x| \geq R, \quad |y| \leq |x| \quad \text{and} \quad x \cdot z = 0 \quad x \cdot f(t, x, y, z) > 0;
\]

(A2) There exists a continuous function \( \phi : [0, \infty) \rightarrow (0, \infty) \) such that

\[
\int_0^\infty \frac{s \, ds}{\phi(s)} = \infty \quad \text{and} \quad \forall t \in \mathbb{R} \quad \forall \|z\| \leq R \quad |f(t, x, y, z)| \leq \phi(|z|);
\]

(A3) There are constants \( \alpha > 0, K > 0 \) such that

\[
\forall t \in \mathbb{R} \quad \forall |x| \leq R \quad \forall |y| \leq R \quad \forall z \in \mathbf{V} \quad |f(t, x, y, z)| \leq \alpha (x \cdot f(t, x, y, z) + |z|^2) + K;
\]

(A4) For all \( x, z \in \mathbf{V}, y \in \mathbf{V}^m \), one has:

(i) \( f(t, x, y, -z) = f(t, x, y, z) \),

(ii) \( f(t, -x, y, z) = f(t, x, y, z) \),

(iii) \( f(t, x, y_1, y_2, \ldots, y_{m-1}, y_m) = f(t, x, y_m, y_{m-1}, \ldots, y_2, y_1) \),

(iv) \( f(t, -x, -y, -z) = -f(t, x, y, z) \);

(A5) There exist \( n \times n \)-matrices \( A_j, j = 0, 1, \ldots, m \), such that

\[
\forall t \in \mathbb{R} \quad \lim_{(x, y, z) \rightarrow 0} \frac{f(t, x, y, z) - A_0 x - \sum_{j=1}^m A_j y_j}{\|x, y, z\|} = 0;
\]

(A6) \( f(t, \gamma x, \gamma y, \gamma z) = \gamma f(t, x, y, z) \) for all \( \gamma \in \Gamma, x, z \in \mathbf{V}, y \in \mathbf{V}^m \).

**Remark 2.1.** (i) (A0) is the standard condition for looking for (classical) periodic solutions to the non-autonomous system. Conditions (A1)–(A3) are classical Hartman–Nagumo conditions required to establish a priori bounds for periodic solutions to (8) (cf. [13, 20]). The first three conditions in (A4) imply that system (8) is time-reversible. The last condition in (A4) together with condition (A6) imply that system (8) is \( \mathbb{Z}_2 \times \Gamma \)-symmetric. Finally, condition (A5) guarantees the existence of linearization for (8) at zero, which is independent of time \( t \).

(ii) There are many examples of systems (8) of practical meaning (with multiple delays) satisfying (7), (A0)–(A6). For instance, one can consider time delay systems with commensurate delays which play an important role in robust control theory (see, for example, [21] and references therein). Notice that in our case, the delays \( \tau_j \) are not necessarily commensurate (i.e. \( \tau_j \neq \frac{2\pi m}{\omega} \)).

Under the conditions (8), (A1)–(A6), assuming, in addition, that the linearization at zero of system (8) is non-degenerate, our aim is to establish, using equivariant spectral properties of the matrices \( A_j \), the existence of multiple non-constant \( 2\pi \)-periodic solutions with prescribed spatio-temporal symmetries.

### 2.2. Symmetries

We will study periodic solutions to (8) as fixed points of the operator associated with (8) and defined in the space \( \mathcal{E} := C^2_{2\pi}(\mathbb{R}; \mathbf{V}) \) of \( C^2 \)-smooth periodic functions, equipped with the norm

\[
\|u\| := \max\{\|u\|_{\infty}, \|\dot{u}\|_{\infty}, \|\ddot{u}\|_{\infty}\}, \quad \|u\|_{\infty} := \sup\{|u(t)| : t \in \mathbb{R}\}, \quad u \in \mathcal{E}.
\]

Put

\[
\mathcal{G} := O(2) \times \mathbb{Z}_2 \times \Gamma,
\]
where
\[
O(2) = S^1 \cup S^1 \kappa, \quad S^1 = \left\{ e^{i\theta} \simeq \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \ \theta \in [0, 2\pi) \right\}, \quad \kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
\]
and
\[
Z_2 := \{1, -1\}.
\]
Then, \( \mathcal{G} \) admits the natural isometric \( G \)-representation given by
\[
(e^{i\theta}, \pm 1, \gamma)u(t) = \pm \gamma u(t + \theta), \quad (e^{i\theta} \kappa, \pm 1, \gamma)u(t) = \pm \gamma u(-t + \theta), \quad t \in \mathbb{R}.
\]
Each of the following four subgroups of \( G \) will play an important role in what follows:
- \( D_1 := \{1, \kappa\} < O(2) \): this subgroup is related to the reversibility of system \( \mathcal{S} \);
- \( G := D_1 \times Z_2 \times \Gamma \): this subgroup classifies spatio-temporal symmetries of periodic solutions to \( \mathcal{S} \) and as such gives rise to the usage of the Brouwer \( G \)-equivariant degree;
- \( D_1^\gamma = \{(1, 1, \kappa), (\kappa, -1, e)\} < D_1 \times Z_2 \times \{e\} \): this subgroup allows us to distinguish between constant and non-constant periodic solutions to \( \mathcal{S} \) (here \( e \) stands for the neutral element in \( \Gamma \));
- \( \mathcal{G} := O(2) \times \{1\} \times \Gamma \): this subgroup allows us to effectively describe equivariant spectral data contributing to the computation of the \( G \)-equivariant degree associated to the linearization of \( \mathcal{S} \) at zero (by condition \( (A5) \), this linearization is autonomous, and as such admits a \( G \)-equivariant treatment); at the same time, since \( \mathcal{S} \) is a non-autonomous system, the \( S^1 \)-component of \( O(2) \) is irrelevant to classifying spatio-temporal patterns of periodic solutions to \( \mathcal{S} \) (leaving a scene to \( D_1 \)).

3. A Priori Bound

3.1. Auxiliary lemmas. The a priori bound for periodic solutions to a second order system of ODEs was derived in the classical book by P. Hartman [15] (see also [21, 28, 7, 1]). The argument required for our setting (including multiple delays) is close to the one used to treat ODEs. However, since to the best of our knowledge, there is no paper combining (multiple) delays with the Hartman-Nagumo condition, we decided, for the sake of completeness, to include a proof covering this case.

Hereafter, \( p := 2\pi \).

**Theorem 3.1.** Let \( f : \mathbb{R} \times V \times V^m \times V \to V \) satisfy \( (A0) \)---\( (A3) \). Then, there exists a constant \( C > 0 \) such that any \( p \)-periodic solution \( x(t) \) to system \( \mathcal{S} \) satisfies
\[
\forall t \in \mathbb{R} \quad |x(t)|, |\dot{x}(t)|, |\ddot{x}(t)| < C.
\]

To prove Theorem 3.1 we need the following lemmas.

**Lemma 3.2.** Under the assumptions \( (A0) \)---\( (A3) \), any \( p \)-periodic solution \( x(t) \) to \( \mathcal{S} \) satisfies
\[
\forall t \in \mathbb{R} \quad |x(t)| \leq R,
\]
where the constant \( R \) is given in condition \( (A1) \).

**Proof.** Assume that \( x(t) \) is a \( p \)-periodic solution to \( \mathcal{S} \) and consider the function \( r(t) := |x(t)|^2, \ t \in \mathbb{R} \). Obviously,
\[
\dot{r}(t) = 2x(t) \cdot \dot{x}(t), \quad \ddot{r}(t) = 2x(t) \cdot \ddot{x}(t) + 2|\dot{x}(t)|^2.
\]
Assume that \( r(t) \) achieves its maximum at \( t_0 \). Then, \( \dot{r}(t_0) = 0 \) and \( \ddot{r}(t_0) \leq 0 \), which by condition \( (A1) \), implies that \( |x(t_0)| \leq R \). Therefore, \( |x(t)| \leq R \) for all \( t \in \mathbb{R} \). \( \Box \)
Lemma 3.3. Under the assumptions (A0)—(A3), there exists a constant $M := M(\alpha, R, K)$ such that for any solution $x(t)$ to (8), one has

$$\forall t \in \mathbb{R} \quad |\dot{x}(t)| \leq M. \tag{16}$$

Proof. Let $x(t)$ be a $p$-periodic solution to (8). By Lemma 3.2, one has $|x(t)| \leq R$ for all $t \in \mathbb{R}$. Hence (see (8)), $|x(t)| \leq R$ for all $t \in \mathbb{R}$. Combining this with (A2) yields:

$$\forall t \in \mathbb{R} \quad |\dot{x}(t)| = |f(t, x(t), \dot{x}(t))| \leq \phi(|\dot{x}(t)|). \tag{17}$$

In addition, by condition (A3), there exist constants $\alpha > 0$, $K > 0$, such that

$$|\ddot{x}(t)| = |f(t, x(t), \dot{x}(t))| \leq \alpha \left( x(t) \cdot f(t, x(t), \dot{x}(t)) + |\dot{x}(t)|^2 \right) + K = \frac{\alpha}{2} \ddot{r}(t) + K \tag{18}$$

(cf. (18)), hence

$$\forall t \in \mathbb{R} \quad |\ddot{x}(t)| \leq \frac{\alpha}{2} \ddot{r}(t) + K. \tag{19}$$

Next, by using integration by parts and $p$-periodicity of $x(t)$, one obtains

$$\int_{t}^{t+p} (t + p - s) \ddot{x}(s) ds = (t + p - s) \ddot{x}(s) \bigg|_{t}^{t+p} + \int_{t}^{t+p} \ddot{x}(s) ds = x(t + p) - x(t) - p \ddot{x}(t) = -p \ddot{x}(t), \tag{20}$$

i.e.

$$\forall t \in \mathbb{R} \quad p \ddot{x}(t) = - \int_{t}^{t+p} (t + p - s) \ddot{x}(s) ds. \tag{21}$$

In particular,

$$p \ddot{x}(0) = - \int_{0}^{p} (p - s) \ddot{x}(s) ds. \tag{22}$$

Combining (21) with (17) yields:

$$p |\ddot{x}(0)| \leq \int_{0}^{p} (p - s) |\ddot{x}(s)| ds \leq \int_{0}^{p} (p - s) \left( \frac{\alpha}{2} \ddot{r}(s) + K \right) ds = \int_{0}^{p} (p - s) \frac{\alpha}{2} \ddot{r}(s) ds + K \int_{0}^{p} (p - s) ds = -\frac{\alpha}{2} p \ddot{r}(0) + \frac{1}{2} K p^2, \tag{23}$$

i.e.

$$p |\ddot{x}(0)| \leq -\frac{\alpha}{2} p \ddot{r}(0) + \frac{1}{2} K p^2. \tag{24}$$

Similarly,

$$p \ddot{x}(t) = - \int_{t-p}^{t} (t - p - s) \ddot{x}(s) ds. \tag{25}$$

In particular,

$$p \ddot{x}(0) = \int_{-p}^{0} (p + s) \ddot{x}(s) ds. \tag{26}$$

Combining (26) with (17) yields:

$$p |\ddot{x}(0)| \leq \frac{\alpha}{2} p \ddot{r}(0) + \frac{1}{2} K p^2. \tag{27}$$

Adding (24) and (27) leads to

$$2p |\ddot{x}(0)| \leq K p^2 \quad \Leftrightarrow \quad |\ddot{x}(0)| \leq \frac{1}{2} K p. \tag{28}$$
Moreover, by (18), one has:
\[ p|\dot{x}(t)| \leq \int_{t}^{t+p} (t + p - s)|\ddot{x}(s)|ds \leq \int_{t}^{t+p} (t + p - s)\left(\frac{\alpha}{2}r(s) + K\right)ds = -\frac{\alpha}{2}p\dot{r}(t) + \frac{1}{2}Kp^2. \]

The last inequality together with condition (A2) imply
\[ \dot{x}(t) \cdot \dot{x}(t) = \frac{|\dot{x}(t)\cdot\ddot{x}(t)|}{\phi(|\ddot{x}(t)|)} \leq \frac{|\ddot{x}(t)||\dddot{x}(t)|}{\phi(|\dddot{x}(t)|)} \leq |\dddot{x}(t)| \leq \frac{1}{2}Kp - \frac{\alpha}{2}r(t). \]

Next, integrating inequality (24) on \([0, t]\), where \(t \in [0, p]\), one obtains:
\[ \left| \int_{0}^{t} \frac{\dot{x}(s)\cdot\ddot{x}(s)}{\phi(|\dddot{x}(s)|)}ds \right| \leq \int_{0}^{t} \left[ \frac{1}{2}Kp - \frac{\alpha}{2}r(s) \right]ds = \frac{1}{2}Kpt - \frac{\alpha}{2}|r(t) - r(0)| \leq \frac{K}{2}p^2 + \frac{\alpha}{2}2R^2 = \frac{K}{2}p^2 + \alphaR^2. \]

On the other hand, using the substitution \(u := |\dot{x}(s)|\), one obtains:
\[ \left| \int_{0}^{t} \frac{\dot{x}(s)\cdot\ddot{x}(s)}{\phi(|\dddot{x}(s)|)}ds \right| = \left| \int_{0}^{t} \frac{\dot{x}(s)\cdot\ddot{x}(s)}{\phi(|\dddot{x}(s)|)}ds \right| = \frac{1}{2}Kpt - \frac{\alpha}{2}|r(t) - r(0)| \leq \frac{K}{2}p^2 + \alphaR^2. \]

Put \(\Phi(\omega) := \int_{0}^{\omega} \frac{udu}{\phi(u)}\) so we have
\[ \left| \int_{|\dddot{x}(0)|}^{t} \frac{sds}{\phi(s)} \right| = |\Phi(|\dddot{x}(t)|) - \Phi(|\dddot{x}(0)|)|. \]

Combining (26) - (28) yields:
\[ |\Phi(|\dddot{x}(t)|) - \Phi(|\dddot{x}(0)|)| \leq \frac{K}{2}p^2 + \alphaR^2, \]
and consequently (see (24)),
\[ \Phi(|\dddot{x}(t)|) \leq \frac{1}{2}Kp^2 + \alphaR^2 + \Phi(|\dddot{x}(0)|) \leq \frac{1}{2}Kp^2 + \alphaR^2 + \Phi\left(\frac{1}{2}Kp\right). \]

By (A2), \(\lim\limits_{w \to \infty} \Phi(w) = \infty\), hence \(\Phi : [0, \infty) \to [0, \infty)\) is a continuous monotonic bijective function. Combining this with (29) yields:
\[ |\dddot{x}(t)| \leq \Phi^{-1}\left[ \frac{1}{2}Kp^2 + \alphaR^2 + \Phi\left(\frac{1}{2}Kp\right) \right] =: M. \]

Therefore, there exists a constant \(M > 0\) such that
\[ \forall t \in \mathbb{R} \quad |\dddot{x}(t)| \leq M. \]

\[ \Box \]

3.2. **Proof of Theorem 3.1.** Put
\[ N := \max\{\phi(s) : s \in [0, M]\}. \]

Then, by Lemmas 3.2, 3.3 and condition (A2), one has:
\[ |\dddot{x}(t)| = |f(t, x(t), x_{t}, \dddot{x}(t))| \leq \phi(|\dddot{x}(t)|) \leq N \quad \text{for all} \quad t \in \mathbb{R}. \]

Then, clearly, the constant \(C := R + M + N + 1\) satisfies (14).
4. Setting System (8) in Functional Spaces

4.1. Operator reformulation and deformation. Together with the space \( \mathcal{E} = C^2_{2\pi}(\mathbb{R};\mathbf{V}) \) (see (9)), we will use the following functional spaces of \( 2\pi \)-periodic functions:

(i) the space \( \mathcal{E} := C_{2\pi}(\mathbb{R};\mathbf{V} \times \mathbf{V}^m \times \mathbf{V}) \) of continuous \( \mathbf{V} \times \mathbf{V}^m \times \mathbf{V} \)-valued functions with the usual sup-norm;

(ii) the space \( \tilde{\mathcal{E}} := C_{2\pi}(\mathbb{R};\mathbf{V}) \) of continuous \( \mathbf{V} \)-valued functions with the usual sup-norm.

Obviously, formula (13) (see also (10)–(12)) defines on \( \mathcal{G} \) and \( \tilde{\mathcal{E}} \) isometric Banach \( \mathcal{G} \)-representations. To be more specific regarding the \( \kappa \)-action on \( \tilde{\mathcal{E}} \), observe:

(30) \( \kappa u_t = (u(-t-\tau_1), u(-t-\tau_2), ..., u(-t-\tau_m)) = (u(-t+\tau_1), u(-t+\tau_2), ..., u(-t+\tau_m)) \).

Next, define the following operators:

\[
\begin{align*}
    i &: \mathcal{E} \to \mathcal{E}, & iu(t) &= u(t), \\
    L &: \mathcal{E} \to \mathcal{E}, & Lu &= \ddot{u}(t) - iu(t), \\
    J &: \mathcal{E} \to \tilde{\mathcal{E}}, & (Ju)(t) &= (u(t), u_t(t)), \\
    N &: \tilde{\mathcal{E}} \to \mathcal{E}, & N(x,y,z)(t) &= f(t,x(t),y(t),z(t)) - x(t).
\end{align*}
\]

Notice that \( L \) is an isomorphism, \( i \) and \( J \) are compact and \( N \) is continuous. Clearly, system (8) is equivalent to the equation

(31) \( Lx = N(Jx), \quad x \in \mathcal{E}. \)

Put \( \mathcal{F}(x) := x - L^{-1}N(Jx), \quad x \in \mathcal{E}. \) Obviously, \( \mathcal{F} \) is a completely continuous field on \( \mathcal{E} \) and (31) admits an operator reformulation:

(32) \( \mathcal{F}(x) = 0, \quad x \in \mathcal{E}. \)

Consider the deformation \( \mathcal{F}_\lambda : \mathcal{E} \to \mathcal{E} \) given by \( \mathcal{F}_\lambda(x) := x - \lambda L^{-1}N(Jx), \quad x \in \mathcal{E}, \quad \lambda \in [0,1]. \) Then, for each \( \lambda \in [0,1] \), the equation

(33) \( \mathcal{F}_\lambda(x) = 0 \)

is equivalent to the system

\[
\begin{align*}
    \dot{x}(t) &= \lambda f(t,x(t),x_t(t)) + (1-\lambda)x(t), & t \in \mathbb{R} \\
    x(t) &= x(t+2\pi), & \dot{x}(t+2\pi) = \dot{x}(t).
\end{align*}
\]

For a fixed \( \lambda \in (0,1] \), define the function

(34) \( f(t,x,y,z) := \lambda f(t,x,y,z) + (1-\lambda)x, \quad x, z \in \mathbf{V}, \ y \in \mathbf{V}^m, \ t \in \mathbb{R}. \)

**Lemma 4.1.** If the function \( f \) satisfies conditions (A0)–(A3) with the constants \( R, \alpha, \ K \) and the function \( \phi \), then the function \( \check{f} \) given by (34) satisfies conditions (A0)–(A3) with the constants \( R, \alpha, \ K' := K + R \) and the function \( \check{\phi}(s) = \phi(s) + R. \)

**Proof.** Obviously, \( \check{f} \) satisfies (A0). Take \( x, y \) with \( |x| \geq R, \ |y| \leq |x| \) and \( x \cdot z = 0. \) Then,

\[
    x \cdot \check{f}(t,x,y,z) = x \cdot (\lambda f(t,x,y,z) + (1-\lambda)x) = \lambda x \cdot f(t,x,y,z) + (1-\lambda)x \cdot x \geq 0,
\]

thus, \( \check{f} \) satisfies (A1). Also, for any \( x, y \) with \( |x|, |y| \leq R \), and any \( z \), one has:

\[
    |\check{f}(t,x,y,z)| = |\lambda f(t,x,y,z) + (1-\lambda)x| \leq \lambda f(t,x,y,z) + (1-\lambda)|x|
    \leq \lambda \phi(|z|) + (1-\lambda)|x| \leq \phi(|z|) + |x| \leq \phi(|z|) + R =: \check{\phi}(|z|).
\]
Thus, \( f(t, x, y, z) \) satisfies (A2). Finally,

\[
|f(t, x, y, z)| = |\lambda f(t, x, y, z) + (1 - \lambda)x| \leq \lambda|f(t, x, y, z)| + (1 - \lambda)|x|
\leq \lambda[\alpha(x \bullet f(t, x, y, z) + |z|^2) + K] + (1 - \lambda)|x|
= \lambda\alpha x \bullet f(t, x, y, z) + \lambda\alpha|z|^2 + \lambda K + (1 - \lambda)|x|
\leq \alpha\lambda x \bullet f(t, x, y, z) + \alpha(1 - \lambda)|x|^2 + \alpha|z|^2 + K + |x|
\leq \alpha\lambda x \bullet f(t, x, y, z) + \alpha(1 - \lambda)|x|^2 + \alpha|z|^2 + K + R
= \alpha[x \bullet (\lambda f(t, x, y, z) + (1 - \lambda)x) + |z|^2] + K + R
= \alpha[x \bullet f(t, x, y, z) + |z|^2] + K + R.
\]

Thus, (A3) is satisfied with \( K' = K + R \). \( \square \)

Lemma [4.1] together with Theorem [3.3] immediately imply

**Lemma 4.2.** If the function \( f : \mathbb{R} \times \mathbb{V} \times \mathbb{V}^m \times \mathbb{V} \rightarrow \mathbb{V} \) satisfies the assumptions (A0)–(A3), then there exists a constant \( C > 0 \) such that for any \( \lambda \in [0, 1] \) and \( x \in \mathcal{E} \), one has:

\[
\mathcal{F}_\lambda(x) = 0 \quad \Rightarrow \quad ||x|| < C.
\]

### 4.2. Properties of the map \( \mathcal{F} \)

Define the operator \( \mathfrak{A} : \mathcal{E} \rightarrow \mathcal{E} \) by

\[
\mathfrak{A}(x, y, z)(t) = A_0 x(t) + \sum_{j=1}^{m} A_j y^j(t), \quad (x, y, z) \in \mathcal{E}.
\]

and let the operator \( \mathcal{A} : \mathcal{E} \rightarrow \mathcal{E} \) be given by

\[
\mathcal{A}(x) := x - L^{-1}\left(\mathfrak{A}(Jx) - ix\right), \quad x \in \mathcal{E}.
\]

In the statement following below, we summarize properties of the map \( \mathcal{F} \).

**Proposition 4.3.** Assume that the function \( f : \mathbb{R} \times \mathbb{V} \times \mathbb{V}^m \times \mathbb{V} \rightarrow \mathbb{V} \) satisfies the conditions (A0)–(A6). Assume, in addition, that (7) takes place. Then:

(a) The map \( \mathcal{F} \) is a \( G \)-equivariant completely continuous field;
(b) There exists a sufficiently large \( C > 0 \) such that \( \mathcal{F} \) is \( B_C(0) \)-admissibly \( G \)-homotopic to \( \text{Id} \);
(c) The map \( \mathcal{F} \) is differentiable at 0, \( D\mathcal{F}(0) = \mathcal{A} \) and is \( G \)-equivariant;
(d) If \( \mathcal{A} \) is an isomorphism, then there exists \( \varepsilon > 0 \) such that \( \mathcal{F} \) is \( B_\varepsilon(0) \)-admissibly \( G \)-homotopic to \( \mathcal{A} \).

**Proof.** (a) Condition (A0) together with the compactness of \( i \) and \( J \) imply the complete continuity of the field \( \mathcal{F} \). By (A4) (resp. (A6)), \( \mathcal{F} \) is \( \mathbb{Z}_2 \)-equivariant (resp. \( \Gamma \)-equivariant). Let us check that \( \mathcal{F} \) is \( D_1 \)-equivariant. In fact, for all \( t \in \mathbb{R} \) and \( u \in \mathcal{E} \), one has (we skip \( i \) and \( J \) for the sake of
simplicity):
\[ \mathcal{F}(\kappa u)(t) = \kappa u(t) - L^{-1} \left( f(t, \kappa u(t), \kappa u_0, \kappa \dot{u}(t)) - \kappa u(t) \right) \]
\[ = u(-t) - L^{-1} \left( f(t, u(-t), u(-t + \tau_1), \ldots, u(-t + \tau_m), -\dot{u}(-t)) - u(-t) \right) \text{ (by (13) and (30))} \]
\[ = u(-t) - L^{-1} \left( f(t, u(-t), u(-t + 2\pi - \tau_1), \ldots, u(-t + 2\pi - \tau_m), -\dot{u}(-t)) - u(-t) \right) \text{ (by periodicity of } u) \]
\[ = u(-t) - L^{-1} \left( f(t, u(-t), u(-t + \tau_1), \ldots, u(-t - \tau_m), \dot{u}(-t)) - u(-t) \right) \text{ (by (A4)(i))} \]
\[ = u(-t) - L^{-1} \left( f(t, u(-t), u(-t - \tau_1), \ldots, u(-t - \tau_m), \dot{u}(-t)) - u(-t) \right) \text{ (by (A4)(iv))} \]
\[ = u(-t) - L^{-1} \left( f(-t, u(-t), u(-t - \tau_1), \ldots, u(-t - \tau_m), \dot{u}(-t)) - u(-t) \right) \text{ (by (A4)(ii))} \]
\[ = \kappa u(t) - \kappa L^{-1} \left( f(t, u(t), u(t - \tau_1), \ldots, u(t - \tau_m), \dot{u}(t)) - u(t) \right) \text{ (by (30))} \]
\[ = \kappa \left( u(t) - L^{-1} (f(t, u(t), u(t), \dot{u}(t)) - u(t)) \right) \]
\[ = \kappa \mathcal{F}(u)(t). \]

(b) Follows from Lemma 4.2.
(c) Follows from (A5), boundedness of \( L^{-1} \) and \( G \)-equivariance of \( \mathcal{F} \).
(d) Define the linear homotopy \( H : [0, 1] \times \mathcal{E} \to \mathcal{E} \) such that
\[ H(\lambda, u) := (1 - \lambda) \mathcal{A} u + \lambda \mathcal{F}(u). \]

Suppose for contradiction, that there exists a sequence \( \{ \lambda_n, u_n \} \), such that \( \lambda_n \to \lambda_0, u_n \to 0, u_n \neq 0 \), and
\[ 0 = H(\lambda_n, u_n) = (1 - \lambda_n) \mathcal{A} u_n + \lambda_n \mathcal{F}(u_n) = \mathcal{A} u_n + \lambda_n \left( \mathcal{F}(u_n) - \mathcal{A} u_n \right). \]

Dividing (37) by \( \|u_n\| \neq 0 \) yields:
\[ 0 = \mathcal{A} \frac{u_n}{\|u_n\|} + \lambda_n \frac{\mathcal{F}(u_n) - \mathcal{A} u_n}{\|u_n\|}. \]

Put \( v_n := \frac{u_n}{\|u_n\|} \). Then:
\[ 0 = \mathcal{A} v_n + \lambda_n \frac{\mathcal{F}(u_n) - \mathcal{A} u_n}{\|u_n\|}. \]

Since \( \|u_n\| \to 0 \) and \( \{ \lambda_n \} \) is bounded, item (c) implies:
\[ \lim_{n \to \infty} \frac{\mathcal{F}(u_n) - \mathcal{A} u_n}{\|u_n\|} = 0. \]

Therefore (see (38)), one has:
\[ \lim_{n \to \infty} \mathcal{A} v_n = 0. \]

On the other hand, \( \mathcal{A} = \text{Id} - R \) \( (R := L^{-1} \circ (2I \circ J - i \circ \text{Id}) \)

is a compact linear field \((i \text{ and } J \text{ are compact})\). Hence, by passing to a subsequence, one can assume without loss of generality that \( Rv_n \to v_0 \). But this implies \( v_n \to v_0 \) (see (39) and (40)), that is \( v_0 \in \ker \mathcal{A} \). However, \( \|v_0\| = \lim_{n \to \infty} \frac{\|u_n\|}{\|u_n\|} = 1 \), and one arrives at the contradiction to the fact that \( \mathcal{A} \) is an isomorphism. \( \square \)
4.3. Abstract equivariant degree based result. Under the assumptions \(\text{T}1\) and \((A0)-(A6)\), the \(G\)-equivariant degree \(G\text{-deg}(\mathcal{A},B(\mathcal{E})) \in A_0(G)\) is correctly defined provided that \(\mathcal{A}\) is an isomorphism (here \(B(\mathcal{E})\) denotes the unit ball in \(\mathcal{E}\) and \(A_0(G)\) stands for the Burnside ring of \(G\)). Put
\[
(41) \quad \omega := (G) - G\text{-deg}(\mathcal{A},B(\mathcal{E})).
\]
We are now in a position to formulate the abstract result.

**Proposition 4.4.** Assume that \(\text{T}1\) takes place and the function \(f : \mathbb{R} \times V \times V^m \times V \to V\) satisfies the conditions \((A0)-(A6)\). Assume, in addition, that \(\mathcal{A} : \mathcal{E} \to \mathcal{E}\) is an isomorphism (see \(36\)). Assume, finally, that
\[
(42) \quad \omega = n_1(H_1) + n_2(H_2) + \ldots n_k(H_k), \quad n_j \neq 0, \quad (H_j) \in \Phi_0(G), \ j = 1, 2, \ldots, k
\]
(cf. \(11\)). Then:

(a) for every \(j = 1, 2, \ldots, k\) there exists a solution \(u \in \mathcal{E}\) to \((8)\) satisfying \(G_u \geq H_j\);

(b) if, in addition, \(H_j \geq D_{\xi^l} := \{(1, 1, e), (1, -1, e)\}\), then \(u\) is a non-constant periodic solution (here \(e \in \Gamma\) stands for the neutral element of \(\Gamma\)).

**Proof.** (a) Take the constants \(C > 0\) and \(\varepsilon > 0\) provided by Proposition 4.3. Combining the equivariant homotopy invariance of the Brouwer-G-equivariant degree with Proposition 4.3(a),(b), one obtains \(G\text{-deg}(\mathcal{F},B_C(0)) = G\text{-deg}(\text{Id},B_C(0)) = (G)\). By the same reason, Proposition 4.3(a),(c),(d)) implies \(G\text{-deg}(\mathcal{F},B_2(0)) = G\text{-deg}(\mathcal{A},B_2(0)) = G\text{-deg}(\mathcal{A},B(\mathcal{E}))\). Put \(\Omega := B_C(0) \setminus B_2(0)\). Then, by additivity of the Brouwer \(G\)-equivariant degree, one has:
\[
(43) \quad G\text{-deg}(\mathcal{F},\Omega) = G\text{-deg}(\mathcal{F},B_C(0)) - G\text{-deg}(\mathcal{F},B_2(0)) = (G) - G\text{-deg}(\mathcal{A},B_2(0)).
\]
Combining 4.3 with 42 and the existence property of the Brouwer \(G\)-equivariant degree yields statement (a).

(b) Condition \(H_j \geq D_{\xi^l} := \{(1, 1, e), (1, -1, e)\}\) implies:
\[
\forall t \in \mathbb{R}, \quad (\kappa, -1)u(t) = -u(-t) = u(t).
\]
Hence, \(u\) is an odd function. Since \(0 \notin \Omega \ni u\), it follows that \(u\) is not a constant function. \(\square\)

5. Computation of \(\omega\) and main results

Proposition 4.4 reduces the study of problem \(8\) to computing the equivariant invariant \(\omega\) (cf. \(11\) and \(12\)). In this section, we, first, analyze the equivariant spectral data required for the computation of \(G\text{-deg}(\mathcal{A},B(\mathcal{E}))\) (see \(11\)). Next, by linking these data to basic \(G\)-equivariant degrees, we present effective results on the existence and symmetric properties of periodic solutions to problem \(8\).

5.1. \(G\)-isotypic decomposition of \(\mathcal{E}\). We will use the notations introduced in Subsection 2.2. With an eye toward determining equivariant spectral data of \(\mathcal{A}\), consider, first, the \(O(2)\)-isotypic decomposition of the space \(\mathcal{E}\) corresponding to its Fourier modes:
\[
(44) \quad \mathcal{E} = \bigoplus_{k=0}^{\infty} \mathcal{E}_k, \quad \mathcal{E}_k := \{\cos(kt)u + \sin(kt)v : u, v \in V\},
\]
where, for \(k \in \mathbb{N}\), the representation \(\mathcal{E}_k\) is equivalent to the complexification \(V^c := V \oplus iV\) of \(V\) (considered as a real \(O(2)\)-representation), where the rotations \(e^{i\theta} \in SO(2)\) act on vectors \(z \in V^c\) by \(e^{i\theta}(z) := e^{-ik\theta} \cdot z\) (here \(\cdot\) stands for complex multiplication) and \(\kappa z := \overline{z}\) (cf. \(11\). \(12\)). Indeed, the linear isomorphism \(\varphi_k : V^c \to \mathcal{E}_k\) given by
\[
(45) \quad \varphi_k(x + iy) := \cos(kt)u + \sin(kt)v, \quad u, v \in V,
\]
is $O(2)$-equivariant. Clearly, $E_0$ can be identified with $V$ with the trivial $O(2)$-action (denote by $W_0$ the trivial one-dimensional $O(2)$-representation). Also, $E_k$, $k = 1, 2, \ldots$, is modeled on the irreducible $O(2)$-representation $W_k \simeq \mathbb{R}^2$, where $SO(2)$ acts by $k$-folded rotations and $\kappa$ acts by complex conjugation. It follows from (44)–(50) and definition of the operator $L$ that

$$L|_{E_0} = -\text{Id} \quad \text{and} \quad L|_{E_k} = -(k^2 + 1)\text{Id} : V^c \to V^c \quad (k > 0).$$

Next, we will refine the $O(2)$-isotypic decomposition (44) to the $O(2) \times \Gamma$-isotypic decomposition of $E$. To this end, consider the $\Gamma$-isotypic decomposition of $E$ (49).

By:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_r,$$

where $V_l$ is modeled on $U_l$ ($l = 0, 1, \ldots, r$; dim $V_l > 0$). Then, the irreducible $\mathcal{G}$-representations $W_k \otimes U_l$ ($k = 0, 1, \ldots, l = 0, \ldots, r$) suggest the $\Gamma$-isotypic components in $E$. More precisely, for the (trivial) $O(2)$-component $E_0 \equiv V$, one has the $\mathcal{G}$-isotypic decomposition

$$E = E_0 \oplus \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r,$$

where the isotypic component $E_0$ is equivalent to $V_0$, and $E_l$ is modeled on the irreducible representation $U_l$. At the same time, for $k > 0$, the $\mathcal{G}$-isotypic decomposition of $E_k$ is

$$E_k = E_{0,k} \oplus \mathcal{E}_{1,k} \oplus \cdots \oplus \mathcal{E}_{r,k},$$

where

$$\mathcal{E}_{l,k} := \{\cos(kt)u_o + \sin(kt)v_o : u_o, v_o \in V_l\}.$$ 

In order to obtain from (44)–(50) the $G := D_1 \times \mathbb{Z}_2 \times \Gamma$-decomposition of $\mathcal{E}$, put $\mathcal{G} := D_1 \times \mathbb{Z}_2 \simeq D_1 \times \mathbb{Z}_2 \times \{e\} \leq G$ and denote by $\mathcal{V}^+ = \mathbb{R}$ and $\mathcal{V}^- = \mathbb{R}$ two irreducible $\mathcal{G}$-representations given by:

$$\forall x \in \mathcal{V}^\pm \quad (\kappa, 1)x = \pm x, \quad (1, -1)x = -x.$$ 

Then, clearly, $\mathcal{G}_x = D_1$ for any non-zero $x \in \mathcal{V}^+$, and $\mathcal{G}_x = D_1^\pm$ for any non-zero $x \in \mathcal{V}^-$. Put

$$\mathcal{U}_l^\pm := \mathcal{V}^\pm \otimes U_l \quad (l = 0, 1, \ldots, r).$$ 

Then (see (43)), the subspace $E_{l,0}$ is isotypic and modeled on $\mathcal{U}_l^+$. At the same time, for $k > 0$ (see (49) and (50)), one has:

$$E_{l,k} = E_{l,k}^+ \oplus E_{l,k}^-,$$

where

$$E_{l,k}^+ = \{\cos(kt)u_o : u_o \in V_l\}, \quad E_{l,k}^- = \{\sin(kt)v_o : v_o \in V_l\}$$

and are modeled on $\mathcal{U}_l^+$ and $\mathcal{U}_l^-$ respectively ($l = 0, 1, \ldots, r$). Thus, the $G$-isotypic decomposition of $\mathcal{E}$ is given by

$$\mathcal{E} = \bigoplus_{l=0}^{r} \left(\mathcal{E}_{l,0}^+ \oplus \mathcal{E}_{l,0}^-\right),$$

where

$$\mathcal{E}_{l,0}^+ = \mathcal{E}_{l,0}^+ \oplus \bigoplus_{k=1}^{\infty} \mathcal{E}_{l,k}^+, \quad \mathcal{E}_{l,0}^- = \bigoplus_{k=1}^{\infty} \mathcal{E}_{l,k}^-,$$

$E_{l,0}^+$ is modeled on $\mathcal{U}_l^+$ and $E_{l,k}^+$ and $E_{l,k}^-$ are described in (52) and are modeled on $\mathcal{U}_l^+$ and $\mathcal{U}_l^-$ respectively ($l = 0, 1, \ldots, r$).
5.2. Spectrum of $\mathcal{A}$. By Proposition 4.3(c), the operator $\mathcal{A}$ is $G$-equivariant. Since the linearization of (8) at the origin is autonomous (cf. (A5) and (35)–(36)), $\mathcal{A}$ is $\mathcal{G}$-equivariant, which can be used to determine spectral properties of $\mathcal{A}$. To be more specific, the $G$-equivariance of the matrices $A_j$ (see assumptions (A5) and (A6)) implies $A_j(V_l) \subset V_l$ ($j = 0, 1, \ldots, m$, $l = 0, 1, \ldots, r$), thus one can put $A_{j,l} := A_j|_{V_l}$. Combining this with (35)–(36) and $\mathcal{G}$-equivariance of $\mathcal{A}$ implies $\mathcal{A}(\delta_{l,k}) \subset \delta_{l,k}$, $(j = 0, 1, \ldots, r, l = 0, 1, 2, \ldots, r)$, thus one can put $\mathcal{A}_{l,k} := \mathcal{A}|_{\delta_{l,k}}$.

To simplify our exposition, we replace assumption (A5) by the following one:

(A7) The $n \times n$-matrices $A_j$, $j = 0, 1, \ldots, m$, satisfy $A_{j,l} = \mu_j^l \text{Id}_{V_l}$ for some number $\mu_j^l \in \mathbb{R}$ and $l = 0, 1, 2, \ldots, r$.

Put $\zeta_j := e^{-i\tau_j}$, $j = 1, 2, \ldots, m$. Then, combining (7), (A0)–(A7), (35)–(36) and (40), one obtains:

$$\mathcal{A}_{l,k} = \left[1 + \frac{1}{k^2 + 1} \left(\mu_0^l + \sum_{j=1}^{m} \mu_j^l \zeta_j^k - 1\right)\right] \text{Id}_{\delta_{l,k}}. \tag{55}$$

Formula (55) implies that the spectrum of $\sigma(\mathcal{A})$ consists of the eigenvalues

$$\xi_{l,k} := 1 + \frac{1}{k^2 + 1} \left(\mu_0^l + \sum_{j=1}^{m} \mu_j^l \zeta_j^k - 1\right) = 1 + \frac{1}{k^2 + 1} \left(\mu_0^l + \sum_{j=1}^{m} 2\mu_j^l \cos(k\tau_j) + \varepsilon_m(-1)^k \mu^l_{m+1} - 1\right), \tag{56}$$

for $k = 0, 1, 2, \ldots$, $l = 0, 1, 2, \ldots, r$, where

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}. \end{cases} \tag{57}$$

**Remark 5.1.** Notice that the (usual) multiplicity $m(\xi_{l,k})$ of $\xi_{l,k} \in \sigma(\mathcal{A})$ is equal to

$$m(\xi_{l,k}) = \begin{cases} \dim V_l & \text{for } k = 0, \\ 2 \dim V_l & \text{for } k > 0. \end{cases} \tag{58}$$

Since $\mathcal{A}$ is a compact linear field, one obtains:

**Corollary 5.2.** Under the assumptions (7), (A0)–(A7), the operator $\mathcal{A}$ is an isomorphism if and only if for every $k = 0, 1, 2, \ldots$ and $l = 0, 1, 2, \ldots, r$, one has

$$-\mu_0 - \sum_{j=1}^{m} 2\mu_j^l \cos(k\tau_j) - \varepsilon_m(-1)^k \mu^l_{m+1} \neq k^2. \tag{58}$$

5.3. Main symmetric result. Given (17), put

$$m_l := \dim V_l/\dim \mathcal{U}_l, \quad l = 0, 1, \ldots, r, \tag{59}$$

and for $k \geq 0$,

$$\nu(l,k) := \begin{cases} m_l & \text{if } \mu_0^l + \sum_{j=1}^{m} 2\mu_j^l \cos(k\tau_j) + \varepsilon_m(-1)^k \mu^l_{m+1} < -k^2, \\ 0 & \text{otherwise}. \end{cases} \tag{60}$$
Define
\begin{equation}
(61) \quad m_l := \sum_{k=1}^{\infty} \nu(l, k), \quad l = 0, 1, \ldots, r.
\end{equation}

Since \( \mathcal{A} \) is a compact vector field, \( \nu(l, k) \) is different from zero only for finitely many pairs of \((l, k)\), hence the integer \( m_l \) is well-defined.

Given a \( G \)-representation \( W \), denote by \( \mathcal{M}_G(W) \) the set of all maximal orbit types in \( W \setminus \{0\} \). Under the assumptions (7), (A0)–(A7) and (58) (i.e. the operator \( \mathcal{A} \) is an isomorphism), put
\begin{equation}
(62) \quad U^\pm := U_0^\pm \oplus U_1^\pm \oplus \cdots \oplus U_r^\pm
\end{equation}
(cf. (61)). Then, \( \mathcal{M}_G(U^-) \subset \mathcal{M}_G(U^- \oplus U^+) = \mathcal{M}_G(\delta) \). Finally, for \( (H) \in \mathcal{M}_G(U^-) \), define (see (61)):
\begin{equation}
(63) \quad m(H) := \sum_{l=0}^{r} m_l(H), \quad \text{where } m_l(H) := \begin{cases} m_l & \text{if } \mathrm{coeff}^H(\deg U^-_l) \neq 0, \\ 0 & \text{otherwise}. \end{cases}
\end{equation}

We are now in a position to formulate our main symmetric result.

**Theorem 5.3.** Let \( f : \mathbb{R} \times V \times V^m \times V \to V \) satisfy conditions (7), (A0)–(A7) and (58). Suppose that for some \( (H) \in \mathcal{M}_G(U^-) \), the number \( m(H) \) is odd (cf. (59)–(63)). Then, system (8) admits a non-constant periodic solution \( u \in \delta \) such that \( G_u = H \).

**Proof.** For a subspace \( K \subset \delta \), denote by \( B(K) \) the (open) unit ball in \( K \). Denote by \( \sigma_{-}(\mathcal{A}) \) the negative spectrum of \( \mathcal{A} \) and for \( \lambda \in \sigma_{-}(\mathcal{A}) \), denote by \( E(\lambda) \) the eigenspace of \( \mathcal{A} \) corresponding to \( \lambda \). Then, by (59), \( \lambda := \xi_{l,k} \) for some \( k = 0, 1, 2, \ldots \) and \( l = 0, 1, \ldots, r \), where
\[ \xi_{l,k} = 1 + \frac{1}{k^2 + 1} \left( \mu_0^l + \sum_{j=1}^{\infty} 2\mu_j^l \cos(k\tau_j) + \varepsilon_m(-1)^{l}\mu_{l+1}^k - 1 \right) < 0. \]

By Proposition 4.1, it is sufficient to show that \( \mathrm{coeff}^H(\omega) \neq 0 \). Combining the Multiplicativity property of the \( G \)-equivariant Brouwer degree (see Theorem A.1 in the Appendix) with formula (54), yields:
\[ \omega = (G) - G-\deg(\mathcal{A}, B(\delta)) = (G) - \prod_{\lambda \in \sigma_{-}(\mathcal{A})} G-\deg(-\text{Id}, B(E(\lambda))) \]
\[ = (G) - \prod_{k=0}^{\infty} \prod_{l=0}^{r} (G-\deg(-\text{Id}, B(\xi_{l,k})))^{\nu(l,k)} \]
\[ = (G) - \prod_{k=0}^{\infty} \prod_{l=0}^{r} \left( \deg U^-_l \right)^{\nu(l,k)} \prod_{k=1}^{r} \prod_{l=0}^{r} \left( \deg U^-_l \right)^{\nu(l,k)} \]
\[ = (G) - \prod_{k=0}^{\infty} \prod_{l=0}^{r} \left( \deg U^-_l \right)^{\nu(l,k)} \prod_{l \in \mathcal{J}} \left( \deg U^-_l \right)^{\nu(l,k)}, \]
where “\( \ast \)” stands for the product in the Burnside ring \( A_0(G) \) and the (finite) set \( \mathcal{J} \) is given by
\[ \mathcal{J} := \left\{ (l, k) : k = 1, 2, \ldots; l = 0, 1, \ldots, r; \nu(l, k) > 0 \right\}. \]

Put
\[ \Lambda_H := \left\{ (l, k) \in \mathcal{J} : \mathrm{coeff}^H(\deg U^-_l) \neq 0 \right\}, \quad \Lambda^c_H := \mathcal{J} \setminus \Lambda_H. \]
To be more precise, there exist subspaces $V_l$ under the assumption $(A7)'$ by the following more general assumption: we are interested in the non-equivariant setting of problem (8). Also, we replace condition $(A7)$ and the conclusion follows from the Existence property of the Brouwer Non-symmetric result.

5.4. Theorem A.1 in the Appendix). □

Using (66), one can easily establish the following

Lemma 5.4. Let $(H) \in \mathfrak{M}_G(U^-)$ and $l, l' \in \{0, 1, \ldots, r\}$.

(a) If $\text{coeff}^H(\text{deg}_{U^-}) 
eq 0$ and $\text{coeff}^H(\text{deg}_{U'^-}) 
eq 0$, then

$$\text{coeff}^H(\text{deg}_{U'^-} \cdot \text{deg}_{U^-}) = 0.$$

(b) If $\text{coeff}^H(\text{deg}_{U'^-}) \neq 0$ and $\text{coeff}^H(\text{deg}_{U^-}) = 0$, then

$$\text{coeff}^H(\text{deg}_{U'^-} \cdot \text{deg}_{U^-}) = -x_o$$

(in particular, different from zero).

Combining Lemma 5.4 with formula (65) and the assumption that $m(H)$ is odd implies

$$\text{coeff}^H \left[ -b \cdot \prod_{p=1}^{m(H)} ((G) - x_o(H) + c_p) \right] = -x_o.$$

Therefore,

$$\text{coeff}^H (\omega)) = x_o \neq 0,$$

and the conclusion follows from the Existence property of the Brouwer $G$-equivariant degree (see Theorem A.1 in the Appendix).

5.4. Non-symmetric result. In the remaining part of this section, we assume that $\Gamma = \{e\}$, i.e. we are interested in the non-equivariant setting of problem (9). Also, we replace condition $(A7)$ by the following more general assumption:

$(A7)'$ For all $j, j' \in \{0, 1, 2, \ldots, m\}$ we have $A_j A_{j'} = A_{j'} A_j$.

Under the assumption $(A7)'$, the matrices $A_j$, $j = 0, 1, \ldots, m$, share their generalized eigenspaces. To be more precise, there exist subspaces $V_l \subset V$, $l = 1, 2, \ldots, s$, such that:

(i) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$;

(ii) $\sigma(A_j) = \{\mu_j^1, \mu_j^2, \ldots, \mu_j^s\}$ for any $j = 0, 1, \ldots, m$;

(iii) the generalized eigenspace $E(\mu_j^l)$ of $\mu_j^l$ is exactly $V_l$ for any $j = 0, 1, \ldots, m$ and $l = 1, 2, \ldots, s$. 

and

$$b := \prod_{k=0}^{\infty} \prod_{l=0}^{r} \left( \text{deg}_{U^+}^{(l,k)} \right)^{\nu(l,k)} \cdot \prod_{(l,k) \in \Lambda_H^+} \left( \text{deg}_{U^-}^{(l,k)} \right)^{\nu(l,k)}.$$
Next, put \( m_l := \dim V_l \) and define for any \( k = 0, 1, \ldots \)

\[
\nu(l, k) := \begin{cases} 
  m_l & \text{if } \mu_0^l + \sum_{j=1}^{[\frac{k}{2}]} 2\mu_j^l \cos(k\tau_j) + \varepsilon_m(-1)^k \mu^l_{m+1} < -k^2, \\
  0 & \text{otherwise.}
\end{cases}
\]

Finally, put

\[
m := \sum_{k=1}^{\infty} \sum_{l=1}^{s} \nu(k, l).
\]

**Corollary 5.5.** Let \( f : \mathbb{R} \times V \times V^m \times V \to V \) satisfy conditions (67), (A0)–(A5), (A7)' (cf. conditions (i)–(iii) above). Assume that for each \( l = 1, 2, \ldots, s \) and \( k = 0, 1, \ldots \)

\[
-\mu_0^l - \sum_{j=1}^{[\frac{k}{2}]} 2\mu_j^l \cos(k\tau_j) - \varepsilon_m(-1)^k \mu^l_{m+1} \neq k^2.
\]

Finally, suppose that the number \( m \) is odd (see (67)–(68)). Then, system (8) admits a non-constant periodic solution \( u \in \mathcal{E} \).

**Proof.** Apply the same arguments as in the proof of Theorem 5.3 with \( H := D^+_1 \). \( \square \)

### 6. Examples

#### 6.1. Examples of \( f \) satisfying conditions (A0)–(A7)

Consider the space \( V := \mathbb{R}^n \) (with the norm \( \max \)). For the purpose of presenting an example of a map \( f \) satisfying all the required assumptions, consider a map \( F : \mathbb{R} \times V \times V^m \times V \to V \) given by

\[
F(t, x, y, z) = (p_1(t, x, y, z), p_2(t, x, y, z), \ldots, p_n(t, x, y, z))^T \in V,
\]

where

\[
p_i(t, x, y, z) = x_i^T Q_i(t, x, y, z) + q_i(t, x, y) + z_i^2 q_k(t, x, y) \quad (i = 1, \ldots, n)
\]

satisfies the following conditions (which can be easily provided!):

(i) \( r \geq 3 \) is an odd integer;

(ii) \( Q_i(t, x, y, z) \) is continuous, 2\( \pi \)-periodic in \( t \)-variable and

\[
\overline{Q} := \inf \{Q_i(t, x, y, z) : i = 1, \ldots, n; t \in \mathbb{R}; x, z \in V; y \in V^m \} > 0;
\]

(iii) for any fixed \( x, y \) and \( t \), the function \( |Q_i(t, x, y, z)| \) is bounded with respect to \( z \);

(iv) \( q_i(t, x, y) \) is 2\( \pi \)-periodic in \( t \)-variable and for any fixed \( t \in \mathbb{R} \), the map \( q_i(t, x, y) \) is a homogeneous polynomial of degree \( d \) such that \( r > d > 1 \);

(v) \( x_i \cdot q_i(t, x, y) \geq 0 \) for all \( t \in \mathbb{R}, x \in V, y \in V^m \).

Take matrices \( A_j : V \to V, j = 0, 1, \ldots, m \) (to be specified later on), and consider the map

\[
f(t, x, y, z) := A_0 x + \sum_{j=1}^{m} A_j y^j + F(t, x, y, z), \quad x, z \in V, \ y = (y^1, \ldots, y^m)^T \in V^m, \ t \in \mathbb{R}.
\]

Clearly, if \( F \) is given by (70) and satisfies (ii) and (iv), then \( f \) satisfies (A0). Let us show that if \( F \) is given by (70) and satisfies (i)–(v), then \( f \) satisfies the Hartman-Nagumo conditions (A1)–(A3).
In fact, if $|x| \geq |y|$, then (ii) and (v) imply:

$$x \cdot f(t, x, y, z) = \sum_{i=1}^{n} \left( x_i^{r+1} Q_i(t, x, y, z) + x_i q_i(t, x, y) + \sum_{j=1}^{m} A_j y^j \right)$$

$$\geq \sum_{i=1}^{n} Q x_i^{r+1} - \sum_{i=1}^{n} x_i q_i(t, x, y) - |A_0| |x|^2 - \sum_{j=1}^{m} |A_j| |y|^j |x|$$

(73)

$$\geq (\sum_{i=1}^{n} Q x_i^{r+1} - \sum_{i=1}^{n} x_i q_i(t, x, y) - \left( |A_0| + \sum_{j=1}^{m} |A_j| \right) |x|^2$$

By (iv), for any $t \in \mathbb{R}$, the powers in monomials of $q_i(t, x, y)$ satisfy

$$d := \sum_{j=0}^{m} \sum_{i=1}^{n} \alpha_i^j < r.$$

Hence (recall, we assume $|x| \geq |y|$),

(74)

$$\left| x_k \prod_{i=1}^{n} x_i^{-\alpha_i} \cdot \prod_{j=1}^{m} (q_j)^{-\alpha_j} \right| \leq |x|^{d+1} \leq (1 + |x|)^r.$$

Combining (73) and (74) implies that there exists a constant $a > 0$ such that

$$x \cdot f(t, x, y, z) \geq (\sum_{i=1}^{n} Q x_i^{r+1} - \sum_{i=1}^{n} x_i q_i(t, x, y) - \left( |A_0| + \sum_{j=1}^{m} |A_j| \right) |x|^2$$

for any $x \in \mathbb{V}$, which implies (see (i)) that for a sufficiently large $R > 0$, if $|x| > R$, then $x \cdot f(t, x, y, z) > 0$, so condition (A1) is satisfied.

Put

$$C := \max \left\{ |x_i^{r+1} Q_i(t, x, y, z)| + |q_i(t, x, y)| + \sum_{j=0}^{m} |A_j| : i = 1, \ldots, n; |x| \leq R; |y| \leq R; t \in \mathbb{R} \right\},$$

$$D := \max \{ |q_i(t, x, y)| : i = 1, \ldots, n; |x| \leq R; |y| \leq R; t \in \mathbb{R} \},$$

and observe that by (iii), $C$ is correctly defined. Clearly, condition (A2) is satisfied with $\phi(s) = C + D|s|^2$, $s \in \mathbb{R}$.

Next, put

$$K := \min \left\{ \sum_{i=1}^{n} \left( x_i^{r+1} Q_i(t, x, y, z) + x_i q_i(t, x, y) + x \cdot A_0 x + x \cdot \sum_{j=1}^{m} A_j y^j \right) : |x| \leq R, |y| \leq R, t \in \mathbb{R} \right\}$$

(again, $K$ is correctly defined by (iii)). Then, for all $|x| \leq R, |y| \leq R, t \in \mathbb{R}$, one has:

$$|f(t, x, y, z)| \leq x \cdot f(t, x, y, z) + K - D|z|^2$$

$$\leq (D + 1) (x \cdot f(t, x, y, z) + |z|^2) + K$$

so that condition (A3) is satisfied.

It follows immediately from (70)–(72) and (i) that condition (A5) is satisfied. Notice that it is possible to choose the related polynomials in such a way that condition (A4) is also satisfied.
Assume that \( V = \mathbb{R}^n \) is an orthogonal \( \Gamma \)-representation where \( \Gamma \leq S_n \) acts on the vectors in \( \mathbb{R}^n \) by permuting their coordinates. One can easily assure, by identifying the \( \Gamma \)-symmetric interactions between the coordinates in \( V \), that condition (A6) is satisfied. To fulfill condition (A7), it is enough to suppose that the \( \Gamma \)-isotypic decomposition of \( V \) is of the form

\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_r,
\]

where \( V_l \cong U_l \) is an irreducible \( \Gamma \)-representation of real type. Then, for each of the \( \Gamma \)-equivariant matrices \( A_j \), one necessarily has \( A_j|_{V_l} = \mu_j^l \Id_{V_l} \), so that condition (A7) is satisfied.

### 6.2. Example of system \((8)\) with dihedral symmetries

For the sake of simplicity, in what follows, we will assume that the dimension \( n \) of the space \( V \) is odd.

Let us consider, as a particular case of the group \( \Gamma \), the dihedral group \( D_n \leq S_n \), where the rotation \( \gamma := e^{\frac{2\pi}{n}} \) is identified with the permutation \( (1, 2, 3, \ldots, n) \) and the reflection \( \kappa, \kappa z = \overline{z} \), with \( (2, n)(3, n-1) \ldots \). We will also assume that the \( D_n \)-equivariant matrices \( A_j \) are given by

\[
A_j := \begin{pmatrix}
  a_j & b_j & 0 & \cdots & 0 & b_j \\
  b_j & a_j & b_j & \cdots & 0 & 0 \\
  & b_j & a_j & \cdots & 0 & 0 \\
  & & \ddots & \ddots & \ddots & \ddots \\
  & & & b_j & a_j & b_j \\
  & & & & b_j & a_j \\
  & & & & & b_j & a_j \\
\end{pmatrix}, \quad j = 0, 1, 2, \ldots, m.
\]

Then, the \( D_n \)-isotypic decomposition of \( V \) is given by

\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_r, \quad r = \left\lfloor \frac{n}{2} \right\rfloor,
\]

where \( V_0 \cong U_0 \) is the trivial one-dimensional \( D_n \)-representation and \( V_l \cong U_l, l = 1, \ldots, r, \) is the irreducible two-dimensional \( D_n \)-representation, where \( \gamma \) acts on \( \mathbb{R}^2 \cong \mathbb{C} \) by usual complex multiplication by \( \gamma^l \). Since \( V_l, l = 0, 1, \ldots, r, \) are irreducible, it follows that

\[
A_{j, l} = \mu_j^l \Id_{V_l}, \quad \text{where } \mu_j^l = a_j + 2b_j \cos \frac{2\pi l}{n}, \quad l = 0, 1, \ldots, r, \quad j = 0, 1, \ldots, m.
\]

Clearly, the orbit type \( (D_2 \times D_3) \) is maximal in \( \mathcal{S} \setminus \{0\} \). Suppose \( n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \), where \( \varepsilon_s > 0 \), and \( p_s, s = 1, 2, \ldots, k \), are the prime numbers such that \( 2 < p_1 < p_2 < \cdots < p_k \). Then, for \( s \in \{1, 2, \ldots, k\} \), put

\[
n_s := \frac{n}{p_s} \quad \text{and} \quad H_s := (D_1 \times \mathbb{Z}_2)^{D_2^{s_2}} \times \mathbb{Z}_2^{n_s} D_{n_s}.
\]

(see Appendix, formulas \((79)\)–\((80)\), for the amalgamated notation used here). One can easily see that for each \( s = 1, 2, \ldots, k \), the orbit type \( (H_s) \) is maximal in \( \mathcal{S} \setminus \{0\} \) and \( (H_s) \in \mathfrak{M}_C(\mathcal{U}^-) \). Notice that for \( l = 0, 1, 2, \ldots, r \), one has \( m_l = 1 \) (see \((39)\)). Therefore, we have the following immediate consequence of Theorem 5.3.

**Corollary 6.1.** Assume that \( \Gamma = D_n \leq S_n \) (here \( n \) is an odd number) acts on \( V := \mathbb{R}^n \) by permuting coordinates of vectors and suppose that \( n = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \), where \( \varepsilon_s > 0 \), and \( p_s, s = 1, 2, \ldots, k \), are prime numbers such that \( 2 < p_1 < p_2 < \cdots < p_k \). Let \( f : \mathbb{R} \times V \times V^m \times V \to V \) satisfy conditions \((7)\), \((A0)\)–\((A7)\) and \((58)\). Assume, finally, that for some \( s \in \{1, 2, \ldots, k\} \), the number \( m(H_s) \) is odd (cf. \((20)\) and \((63)\)). Then, system \((8)\) admits a non-constant periodic solution \( u \in \mathcal{S} \) such that \( G_u = H_s \).
Example: case $m = 4$ and $n = 3$: We choose $a_0 = -1$, $b_0 = -2$, $a_1 = a_4 = -2$, $b_1 = b_4 = -4$, $a_2 = a_3 = -3$, $b_2 = b_3 = -5$. We will also assume that $\tau_j := \frac{2\pi j}{g}$, $j = 0, 1, 2, 3, 4$ (see condition (77)). In this case,

$$\begin{align*}
\mu_0 &= -5, \quad \mu_1 = -10, \quad \mu_2 = -13, \quad \mu_3 = -13, \quad \mu_4 = -10 \\
\mu_0 &= 1, \quad \mu_1 = 1, \quad \mu_2 = 2, \quad \mu_3 = 2, \quad \mu_4 = 2.
\end{align*}$$

Notice that

$$\cos(\tau_1 k) = \begin{cases} \frac{\sqrt{5} - 1}{4} & \text{if } k = 5p \pm 1; \\ -\frac{\sqrt{5}}{4} & \text{if } k = 5p \pm 3; \\ 1 & \text{if } k = 5p \\ \end{cases}$$

$(p = 0, 1, \ldots)$, and the eigenvalues of the operator $\mathcal{A}$ are given by

$$\xi_{1,k} = 1 + \frac{\mu_0}{k^2 + 1} + \frac{2\mu_1}{k^2 + 1} \cos(\tau_1 k) + \frac{2\mu_2}{k^2 + 1} \cos(\tau_1 2k) + \frac{2\mu_3}{k^2 + 1} \cos(\tau_1 3k) + \frac{2\mu_4}{k^2 + 1} \cos(\tau_1 4k) - 1$$

$$\xi_{0,k} = \begin{cases} -\frac{98}{k^2 + 1} & \text{if } k = 5l, \\
\frac{16}{k^2 + 1} & \text{if } k = 5l + 1, 5l + 2, \\
\frac{16\sqrt{5}}{k^2 + 1} & \text{if } k = 5l + 3, 5l + 4. \\
\end{cases}$$

Hence,

$$\xi_{0,0} = 1, \xi_{1,1} = 7, \xi_{1,2} = -11, \xi_{1,3} = -3.$$

Consequently, the negative spectrum of the operator $\mathcal{A}$ is:

$$\sigma_-(\mathcal{A}) = \left\{ \xi_{0,0} = -97, \xi_{1,1} = 7, \xi_{1,2} = -\frac{11}{5}, \xi_{1,3} = -\frac{3}{5} \right\}.$$

Also, $G := D_1 \times \mathbb{Z}_2 \times D_3$ and the related basic degrees are:

$$\deg_G = (G) - (D_1^2 \times D_3),$$

$$\deg_G = (G) - (D_1 \times D_3),$$

$$\deg_G = (G) - (D_1^2 \times D_1) - ((D_1 \times \mathbb{Z}_2)D_1 \times (\mathbb{Z}_2^3 \times D_1)), (D_1 \times \mathbb{Z}_2 \times D_1),$$

$$\deg_G = (G) - (D_1 \times D_1) - ((D_1 \times \mathbb{Z}_2)D_1 \times (\mathbb{Z}_2^3 \times D_1)), (D_1 \times \mathbb{Z}_2 \times D_1),$$

where $\{e\}$ stands for the unit subgroup in $D_3$. The maximal orbit types in $\mathcal{E} \setminus \{0\}$ are:

$$\mathcal{M}_G = \left\{ (D_1^2 \times D_3), (D_1^2 \times D_3), ((D_1 \times \mathbb{Z}_2)D_1 \times (\mathbb{Z}_2^3 \times D_1)), ((D_1 \times \mathbb{Z}_2)D_1 \times (\mathbb{Z}_2^3 \times D_1)) \right\}$$

and

$$\mathcal{M}_G = \left\{ (D_1^2 \times D_3), (D_1 \times \mathbb{Z}_2)D_1 \times (\mathbb{Z}_2^3 \times D_1) \right\}.$$

On the other hand,

$$G - \deg(\mathcal{A}, B_1(0)) = \deg_G - \deg_G \cdot \deg_G \cdot \deg_G = \deg_G - \deg_G \cdot \deg_G \cdot \deg_G,$$

which implies that for $H := (D_1 \times \mathbb{Z}_2)^{\mathcal{E}} \times (\mathbb{Z}_2^3 \times D_1)$, one has $\mathcal{M}(H) = 1$. Hence,

$$\text{coeff}_H(\omega) = \text{coeff}_H \left( (G) - \deg_G \cdot \deg_G \cdot \deg_G \right) = 1.$$

Therefore, there exists an orbit of non-constant periodic solutions to system (3) with the orbit type exactly $(H)$.

To double check the obtained result, one can also use the GAP package EquiDeg, as it is presented below:
GAP Code: $G := D_1 \times Z_2 \times D_3$. 

```gcode
LoadPackage( "EquiDeg" );
gr1 := SymmetricGroup( 2 );
# create the product of $D_1$ and $Z_2$
gr2 := DirectProduct( gr1, gr1 );
# create group $G$
gr3 := pDihedralGroup( 3 );
G := DirectProduct( gr2, gr3 );
# create and name CCSs of $gr2$ and $gr3$
ccs_gr2 := ConjugacyClassesSubgroups( gr2 );
ccs_gr2_names := [ "Z1", "Z1p", "D1", "D1z", "D1p" ];
ccs_gr3 := ConjugacyClassesSubgroups( gr3 );
ccs_gr3_names := [ "Z1", "D1", "Z3", "D3" ];
SetCCSsAbbrv( gr2, ccs_gr2_names );
SetCCSsAbbrv( gr3, ccs_gr3_names );
ccs := ConjugacyClassesSubgroups( G );
# create characters of irreducible $G$-representations
irr := Irr( G );
# compute the corresponding to $irr[k]$ basic degree
deg0m := BasicDegree( irr[6] );
deg0p := BasicDegree( irr[7] );
deg1m := BasicDegree( irr[9] );
deg1p := BasicDegree( irr[10] );
# obtaining amalgamation symbols
Print( AmalgamationSymbol( ccs[16]));
# maximal orbit types in $E$
max := MaximalOrbitTypes( irr[6] + irr[7] + irr[9] + irr[10] );
# unit element in Burnside ring $AG$
u := -BasicDegree( irr[1] );
# compute $gdeg$ of $F$ on $Omega$
deg := u - deg0p*deg1p*deg1m;
```

### Appendix A. Equivariant Brouwer Degree Background

(a) **Amalgamated Notation.** Given two groups $G_1$ and $G_2$, the well-known result of É. Goursat (see [11]) provides the following description of a subgroup $\mathcal{U} \leq G_1 \times G_2$: there exist subgroups $H \leq G_1$ and $K \leq G_2$, a group $L$, and two epimorphisms $\varphi : H \to L$ and $\psi : K \to L$ such that

$$\mathcal{U} = \{ (h,k) \in H \times K : \varphi(h) = \psi(k) \}.$$  

The widely used notation for $\mathcal{U}$ is

(79) $$\mathcal{U} := H \varphi \times L K,$$

in which case $H \varphi \times L K$ is called an **amalgamated** subgroup of $G_1 \times G_2$.

In this paper, we are interested in describing conjugacy classes of $\mathcal{U}$. Therefore, to make notation (79) simpler and self-contained, it is enough to indicate $L$, $Z = \text{Ker}(\varphi)$ and $R = \text{Ker}(\psi)$. Hence, instead of (79), we use the following notation:

(80) $$\mathcal{U} := HZ \times LR K.$$  

(b) **Equivariant Notation.** Below $G$ stands for a compact Lie group. For a subgroup $H$ of $G$, denote by $N(H)$ the normalizer of $H$ in $G$ and by $W(H) = N(H)/H$ the Weyl group of $H$. The symbol $(H)$ stands for the conjugacy class of $H$ in $G$. Put $\Phi(G) := \{(H) : H \leq G\}$. 

The set \( \Phi(G) \) has a natural partial order defined by \((H) \leq (K) \iff \exists g \in G \; gHg^{-1} \leq K \). Put \( \Phi_0(G) := \{(H) \in \Phi(G) : W(H) \) is finite\}. 

For a \( G \)-space \( X \) and \( x \in X \), denote by \( G_x := \{g \in G : gx = x\} \) the isotropy group of \( x \) and call \((G_x)\) the orbit type of \( x \) in \( X \). Put \( \Phi(G,X) := \{(H) \in \Phi_0(G) : (H) = (G_x) \) for some \( x \in X\} \) and \( \Phi_0(G,X) := \Phi(G,X) \cap \Phi_0(G) \). For a subgroup \( H \leq G \), the subspace \( X^H := \{x \in X : G_x \geq H\} \) is called the \( H \)-fixed-point subspace of \( X \). If \( Y \) is another \( G \)-space, then a continuous map \( f : X \to Y \) is called equivariant if \( f(gx) = gf(x) \) for each \( x \in X \) and \( g \in G \). Let \( V \) be a finite-dimensional \( G \)-representation (without loss of generality, orthogonal). Then, \( V \) decomposes into a direct sum

\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_r,
\]

where each component \( V_i \) is modeled on the irreducible \( G \)-representation \( \mathcal{V}_i \), \( i = 0, 1, 2, \ldots, r \), that is, \( V_i \) contains all the irreducible subrepresentations of \( V \) equivalent to \( \mathcal{V}_i \). Decomposition (81) is called \( G \)-isotypic decomposition of \( V \).

(b) Axioms of Equivariant Brouwer Degree. Denote by \( \mathcal{M}^G \) the set of all admissible \( G \)-pairs and let \( A_0(G) \) stand for the Burnside ring of \( G \) (see Introduction, items (a) and (b) respectively). The following result (cf. [3]) can be considered as an axiomatic definition of the \( G \)-equivariant Brouwer degree.

**Theorem A.1.** There exists a unique map \( G \)-deg : \( \mathcal{M}^G \to A_0(G) \), which assigns to every admissible \( G \)-pair \((f, \Omega)\) an element \( G \)-deg\((f, \Omega) \in A_0(G)\)

\[
G \text{-deg}(f, \Omega) = \sum_{(H)} n_H(H) = n_{H_1}(H_1) + \cdots + n_{H_m}(H_m),
\]

satisfying the following properties:

- **(Existence)** If \( G \)-deg\((f, \Omega) \neq 0 \), i.e., \( n_{H_i} \neq 0 \) for some \( i \) in (82), then there exists \( x \in \Omega \) such that \( f(x) = 0 \) and \((G_x) \supset (H_i)\).

- **(Additivity)** Let \( \Omega_1 \) and \( \Omega_2 \) be two disjoint open \( G \)-invariant subsets of \( \Omega \) such that \( f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2 \). Then,

\[
G \text{-deg}(f, \Omega) = G \text{-deg}(f, \Omega_1) + G \text{-deg}(f, \Omega_2).
\]

- **(Homotopy)** If \( h : [0,1] \times V \to V \) is an \( \Omega \)-admissible \( G \)-homotopy, then

\[
G \text{-deg}(h, \Omega) = \text{constant}.
\]

- **(Normalization)** Let \( \Omega \) be a \( G \)-invariant open bounded neighborhood of \( 0 \) in \( V \). Then,

\[
G \text{-deg}(\text{Id}, \Omega) = (G).
\]

- **(Multiplicativity)** For any \((f_1, \Omega_1), (f_2, \Omega_2) \in \mathcal{M}^G\),

\[
G \text{-deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = G \text{-deg}(f_1, \Omega_1) \cdot G \text{-deg}(f_2, \Omega_2),
\]

where the multiplication \( \cdot \) is taken in the Burnside ring \( A_0(G) \).

- **(Recurrence Formula)** For an admissible \( G \)-pair \((f, \Omega)\), the \( G \)-degree (82) can be computed using the following Recurrence Formula:

\[
n_H = \frac{\deg(f^H, \Omega^H) - \sum_{(K) \succ (H)} n_K n(H,K) |W(K)|}{|W(H)|},
\]

where \(|X| \) stands for the number of elements in the set \( X \) and \( \deg(f^H, \Omega^H) \) is the Brouwer degree of the map \( f^H := f|_{\Omega^H} \) on the set \( \Omega^H \subset V^H \).
The $G$-deg$(f,\Omega)$ is called the $G$-equivariant Brouwer degree of $f$ in $\Omega$.

**(c) Computation of Brouwer Equivariant Degree.** Consider a $G$-equivariant linear isomorphism $T : V \to V$ and assume that $V$ has a $G$-isotypic decomposition $\sum_i V_i$. Then, by the Multiplicativity property,

$$G\text{-deg}(T, B(V)) = \prod_{i=0}^{r} \text{deg}(T_i, B(V_i)) = \prod_{i=0}^{r} \prod_{\mu \in \sigma_- (T)} (\text{deg}_{V_i})^{m_i(\mu)}$$

where $T_i = T|_{V_i}$, $\sigma_- (T)$ denotes the real negative spectrum of $T$ (i.e., $\sigma_- (T) = \{ \mu \in \sigma (T) : \mu < 0 \}$) and $m_i(\mu) = \dim(E(\mu) \cap V_i)$ (here $E(\mu)$ stands for the generalized eigenspace of $T$ corresponding to $\mu$). Notice that the basic degrees can be effectively computed from (83):

$$\text{deg}_{V_i} = \sum_{(H)} n_H (H),$$

where

$$(85) \quad n_H = \frac{(-1)^{\dim V_i} - \sum_{H < K} n_K} {||W(H)||} n(H, K) ||W(K)||.$$

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