ABSTRACT. We prove a universal lower bound for the discrepancies of measure-preserving actions of a locally compact group on an atomless probability space. It generalizes the universal lower bound for the discrepancies of measure-preserving actions of a discrete group. Many examples show that the generalization from discrete groups to locally compact groups requires some additional hypothesis on the action (we detail some examples due to Margulis). Well-known examples and results of Kazhdan and Zimmer show that the discrepancies of some actions of Lie groups on homogeneous spaces match exactly the universal lower bounds we prove.

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1. INTRODUCTION

The discrepancies of a group action measure the possible rates of convergence in ergodic von Neumann type theorems. Isometric group actions of free groups with the smallest possible discrepancies (equivalently, with the fastest possible rates of convergence) have been constructed by Lubotzky, Phillips, and Sarnak. Their bounds on the discrepancies
rely on deep results from number theory (Deligne’s solution to the Weil conjecture). See [LPS86]. For discrete groups, there exist universal lower bounds for the discrepancies of measure-preserving actions on atomless probability spaces, which can be expressed with the help of the regular representation. In other words, there is a lower bound for the discrepancies, which is valid for any measure-preserving action of the group on an atomless probability space, and this universal lower bound, which is inherent to the structure of the group, can be extracted from the regular representation of the group. The authors of the present work have established these universal lower bounds in a previous work [PLP21] and have proved that the above mentioned actions considered by Lubotzky, Phillips, and Sarnak, have discrepancies matching exactly the universal lower bounds. Pierre-Emmanuel Caprace asked us if these lower bounds could be generalized to locally compact groups and mentioned to us examples of Margulis showing that a naïve generalization was not possible. What follows is a tentative answer to his question.

1.1. Statement of the results. Let \( G \) be a locally compact group, and let \( \mu_G \) be a left Haar measure on \( G \).

**Definition 1.1 (Discrepancy).** Let \((X, \mathcal{T}, \nu)\) be a probability space, and let us consider a measurable action \( G \rightharpoonup X \) that preserves \( \nu \). Let \( \mu \) be a Borel probability measure on \( G \). We call the number

\[
\delta(\mu) := \sup_{\phi \in L^2(X,\nu)} \left\| x \mapsto \int_G \phi(g^{-1}x) \, d\mu - \int_X \phi \, d\nu \right\|_2.
\]

the discrepancy of \( \mu \).

Similarly, for any continuous, non-negative, compactly-supported function \( f \) on \( G \) of integral 1, we denote by

\[
\delta(f)
\]
the discrepancy of the probability measure with density \( f \) with respect to \( \mu_G \).

Let us denote by

\[
\lambda_G : G \to U(L^2(G,\mu_G)) \quad g \mapsto \phi \mapsto (h \mapsto \phi(g^{-1}h))
\]

the left regular representation of \( G \). Let \( \mathcal{B}(L^2(G,\mu_G)) \) be the involutive algebra of bounded operators on the Hilbert space \( L^2(G,\mu_G) \). The map \( \lambda_G \) is a unitary representation that induces a \( \ast \)-representation of the involutive Banach algebra \( \mathcal{M}(G) \) of complex Borel measures on \( G \) via

\[
\mathcal{M}(G) \to \mathcal{B}(L^2(G,\mu_G)) \quad \mu \mapsto \lambda_G(\mu) := \phi \mapsto (h \mapsto \int_G \phi(g^{-1}h) \, d\mu(g)).
\]

For a symmetric probability measure \( \mu \) on \( G \), the operator \( \lambda(\mu) \) is self-adjoint, and its operator norm is smaller or equal than 1.

Finally, let us recall that an atom of a measure space \((X, \nu)\) is a measurable subset \( A \) such that \( \nu(A) > 0 \) and every measurable subset \( B \) of \( A \) is such that

\[
\nu(B) \in \{0, \nu(A)\}.
\]

A measure space that does not have atoms is called atomless. Atomless probability spaces satisfy the so-called intermediate value property (see Section 2.2).

The following theorem and its corollaries are the main results of the paper.
Theorem 1. Let \((X, \mathcal{B}, \nu)\) be an atomless probability space, \(G \curvearrowright X\) a measurable action that preserves \(\nu\).

We assume that for every compact, symmetric neighborhood \(S\) of the identity element \(e\) of \(G\), there exists a sequence \((B_n)_{n \in \mathbb{N}}\) of measurable subsets of \(X\) and a compact, symmetric neighborhood \(F\) of \(e\) in \(G\) such that
\[
\forall n \in \mathbb{N}, \quad \nu(B_n) > 0;
\]
\[
\inf\{\nu(B) \mid B \in \mathcal{B}, \quad S^n B_n \subset B\} < \frac{1}{2};
\]
\[
\limsup_{n \to \infty} \left( \inf_{g \in F} \frac{\nu(gB_n \cap B_n)}{\nu(B_n)} \right) \geq 1.
\]

Then, for every continuous, compactly-supported, non-negative function \(f\) on \(G\),
\[
\|\lambda_G(f)\|_{2-2} \leq \delta(f).
\]

Remark 1. The reader should note that in condition (2) above, the set \(S^n B_n\) may fail to be measurable (see for example the construction, in \([ES70]\), of subsets \(K\) and \(B\) of \(\mathbb{R}\) such that \(K\) is compact, \(B\) is a \(G_\delta\) - and therefore, Borel - but \(K + B\) is not Borel).

We first notice that the theorem generalizes \([PLP21, \text{Theorem 3.6, p. 77}]\): if \(G\) is discrete we may choose \(F = \{e\}\) to deduce the following corollary.

Corollary 1 (See \([PLP21, \text{Theorem 3.6, p. 77}]\)). Let \((X, \nu)\) be an atomless probability space, \(G\) a discrete group, \(G \curvearrowright X\) a measure-preserving action. Let \(\mu\) be a finitely-supported probability measure on \(G\). Then
\[
\|\lambda_G(\mu)\|_{2-2} \leq \delta(\mu).
\]

From the theorem, we also deduce the following corollary.

Corollary 2. Let \(X\) be a topological space, \(\nu\) be an atomless Borel probability measure on \(X\), \(G \curvearrowright X\) a continuous action that preserves \(\nu\).

We assume that \(G\) is unimodular, and that there exists a point \(x\) in the support of \(\nu\) such that for every compact subset \(K\) of \(G\),
\[
\nu(Kx) = 0
\]
and
\[
\text{Stab}_G(x) \text{ is compact}.
\]

Then, for every non-negative, continuous, compactly-supported function \(f\) on \(G\), we have
\[
\|\lambda_G(f)\|_{2-2} \leq \delta(f).
\]

We deduce the following corollary as a particular case of the previous one.

Corollary 3. Let \(G\) be a locally compact group, \(\mu_G\) be a left Haar measure on \(G\), \(H\) a unimodular, closed subgroup of \(G\) such that \(\mu_G(H) = 0\). Let \(\Gamma\) be a lattice in \(G\), such that \(H \cap \Gamma = \{e\}\). Let \(\mu_{G/\Gamma}\) be the unique \(G\)-invariant probability measure on \(G/\Gamma\). We consider the action \(H \curvearrowright G/\Gamma\), that preserves \(\mu_{G/\Gamma}\).

Then, for every non-negative, continuous, compactly-supported function \(f\) on \(H\), we have
\[
\|\lambda_H(f)\|_{2-2} \leq \delta(f).
\]

1.2. Comments.
1.2.1. **On the notion of discrepancy.** The notion of discrepancy can be interpreted as a measure of the ergodicity of the action. For example (see [Sha98, Thm 4.2]), if $G$ is a countable group endowed with the discrete topology, acting on a probability space $(X, T, \nu)$, then the following three assertions are equivalent:

1. there exists a probability measure whose support generates $G$ and whose discrepancy is strictly smaller than 1;
2. every probability measure whose support generates $G$ has discrepancy strictly smaller than 1;
3. the action $G \acts X$ is (strongly) ergodic.

Moreover, assume $G$ is a topological group, acting by measure-preserving transformations on a probability space $(X, \nu)$, and that $\mu$ is a symmetric probability measure on $G$. Let

$$\pi(\mu) : L^2(X, \nu) \to L^2(X, \nu)$$

$$\phi \mapsto \left( x \mapsto \int_G \phi(g^{-1}x) \, d\mu(g) \right)$$

Assume that $\mu$ is such that $\delta(\mu) < 1$. Then there is a uniform exponential rate of convergence in von Neumann’s ergodic theorem; that is, we have

$$\forall \phi \in L^2(X, \nu), \|\phi\|_2 \leq 1, \quad \|\pi(\mu)^n \phi - \int_X \phi \, d\nu\|_2 \leq \delta(\mu)^n.$$  

The proof of this fact easily follows from Proposition 1 below.

1.2.2. **Relation with previous results.** Let us first make the following fundamental remark.

**Remark 2.** Given a measure-preserving action of a locally compact group on a probability space $G \acts (X, \nu)$, let $\pi_0$ denote the subrepresentation on the orthogonal complement of the subspace of constant functions of the Koopman representation (see Definition 2.1 and Proposition 1 below).

Then, for every Borel probability measure $\mu$ on $G$,

$$\delta(\mu) = \|\pi_0(\mu)\|_{2 \to 2};$$

therefore, all the lower bounds on discrepancies presented in this paper are of the form

$$\forall f \in C^0_c(G), f \text{ positive} \implies \|\lambda_G(f)\|_{2 \to 2} \leq \|\pi_0(f)\|_{2 \to 2}.$$  

Note, moreover, that the stronger assertion

$$\forall f \in L^1(G), \quad \|\lambda_G(f)\|_{2 \to 2} \leq \|\pi_0(f)\|_{2 \to 2}$$

is, in fact, equivalent to the weak containment

$$\lambda_G < \pi_0$$

of unitary representations (see [BdLHV08, Thm F.4.4, p.412]).

In [PLP21], a new proof of the following universal minorization is given. This result had been stated without proof in [Sha98] and proved (with additional technical hypothesis) in [DG17] (where the weak containment $\lambda_G < \pi_0$ is proved).

**Theorem** (See [PLP21] Theorem 3.6). Let $G$ be a countable discrete group, let $(X, \nu)$ be an atomless probability space, and let $G \acts X$ a measurable action that preserves $\nu$. Let $\mu$ be a probability measure on $G$.

We have

$$\|\lambda_G(\mu)\|_{2 \to 2} \leq \delta(\mu).$$
When $G$ is amenable, we have, according to Kesten’s work (see [Kes]) that if $\mu$ is a symmetric probability measure, whose support generates $G$
\[ \|\lambda_G(\mu)\|_{2 \to 2} = 1. \]
This allows to state the following corollary.

**Corollary (See [PLP21, Corollary 3.10]).** Let $G$ be a countable, discrete, amenable group, $(X, \nu)$ an atomless probability space and $G \curvearrowright X$ a measurable action that preserves $\nu$. Let $\mu$ be a probability measure on $G$. Then the discrepancy of $\mu$ is 1.

In this same vein, we can make the following remark.

**Remark 3.** Since $\mathbb{R}$ is unimodular and amenable, Corollary 2 implies that for every continuous $\mathbb{R}$-action by measure-preserving transformations with negligible orbits, the discrepancy of any compactly-supported, nonnegative, continuous function is at least 1. For example, this yields the following folklore statement: let $S$ be a compact, hyperbolic surface. Consider $g_t : T_1 S \to T_1 S$ the geodesic flow on the unit tangent bundle, and let $\nu$ be the Liouville measure on $T_1 S$. Then, for every $T \in \mathbb{R}^*$,
\[ \sup_{\phi \in L^2(T_1 S, \nu)} \left\| \left( x \mapsto \frac{1}{T} \int_0^T \phi(g_t x) \, dt - \int_{T_1 S} \phi \, d\nu \right) \right\|_2 = 1. \]
The same proof applies to the geodesic flow of a non-compact locally symmetric space.

In [PLP21], examples are given, and the fact that the inequality is sharp for all countable groups is established (see [PLP21, Corollary 3.12]); that is, for every countable group $G$, there is a measurable action (namely, the Bernoulli shift on $G$) such that for each finitely-supported probability measure $\mu$ on $G$,
\[ \|\lambda_G(\mu)\|_{2 \to 2} = \delta(\mu). \]

In the recent paper of Caprace-Mehrdad-Monod, a weak containment result is proved, for actions of locally compact, totally disconnected groups (see [CMM22, Theorem F]).

### 1.2.3. Interpretation of the technical condition in Theorem 7
We give a heuristic example to illustrate the technical condition in Theorem 7 that we postulate in order to be able to treat actions of locally compact non-discrete groups.

**Example 1.** Let us consider the action of $\mathbb{R}$ on $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ by translations along a line $d$ with irrational slope; that is, let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, and define, for all $t \in \mathbb{R}$, $(a, b) \in \mathbb{T}^2$,
\[ (a + t, b + \alpha t). \]
Let $F := [0, 1]$.

Let us consider long rectangles, centered at the origin, in $\mathbb{R}^2$, such that two opposite sides are parallel to $d$; if $R$ is such a rectangle, we denote by $\overline{R}$ its projection in $\mathbb{T}^2$.

Consider conditions $(1), (2), (3)$, from Theorem 7. Let $g$ be a small translation along the line $d$ (drawn in black). Let $R_{\text{red}}$ be the (projection of) the red rectangle, and $R_{\text{blue}}$ be the (projection of) the blue rectangle. These two rectangles represent possible choices of $B_n$’s in the statement of Theorem 7. One can graphically expect that $gR_{\text{red}} \cap R_{\text{red}}$ can be made quite small, whereas $gR_{\text{blue}} \cap R_{\text{blue}}$ is approximately as big as $R_{\text{blue}}$. Notice that the two opposite sides parallel to $d$ are short in the case of the red rectangle and are long in the case of the blue rectangle.
1.2.4. Comments on the hypotheses of Corollary 2. We do not know to what extent the hypotheses of Corollary 2 are necessary. For example, we do not know if the hypothesis of compactness of a stabilizer is necessary (but it is needed in our proof). However, we show in Section 3.1 by considering an example developed by Margulis, that Corollary 3 fails if we remove the hypothesis \( \mu_G(H) = 0 \). This shows that one cannot remove the hypotheses of negligibility in Corollary 2, and also that [PLP21, Thm 3.6] isn’t valid if we replace “discrete, countable” by “locally compact” without adding hypotheses.

1.2.5. Sharpness of the results. There are examples where the inequality in Corollary 3 is an equality, and we describe one of them in Section 3.2. We do not know if every locally compact group admits an action with a measure with discrepancy equal to our lower bound.

1.3. Outline of the paper. Section 2 is devoted to the proof of the theorem and the corollaries. In Section 3.1, we recall a counter-example of Margulis to a possible generalization of our results. Finally, we describe in Section 3.2 the example of \( \mathbb{R}^2 \times \text{SL}_2(\mathbb{R}) \) acting on a finite-volume locally symmetric space of rank four, where the inequality of Corollary 3 is sharp.

1.4. Acknowledgements. We would like to thank Pierre-Emmanuel Caprace who was at the origin of the questions that we address in this paper; we are grateful to him for helpful discussions and remarks.

2. Proofs

**Notation.** Since we are only concerned with \( L^2 \) spaces, when denoting the norm of an operator \( A \) between such Hilbert spaces, we will, from now on, drop the index, thus writing \( \|A\| \) instead of \( \|A\|_{2 \rightarrow 2} \).

2.1. Measure-preserving actions and unitary representations. Recall that \( G \) denotes a locally compact topological group endowed with a left Haar measure \( \mu_G \).

**Definition 2.1** (Koopman representation). Consider an action of \( G \) on a measure space \((X, \nu)\) by measurable transformations preserving the measure.

The Koopman representation associated to this action is defined by

\[
\pi : G \to U(L^2(X, \nu))
\]

\[
g \mapsto (\phi \mapsto (x \mapsto \phi(g^{-1}x)))
\]
Remark 4. (1) The left regular representation is the Koopman representation for the action of the locally compact group $G$ endowed with a left Haar measure $\mu_G$ by left multiplication on itself.

(2) For every $g \in G$, the operator $\pi(g)$ defined above is unitary and $\pi$ is a morphism. Moreover, if $G$ is $\sigma$-compact, locally compact, if $\nu$ is $\sigma$-finite and $L^2(X, \nu)$ is separable, then the Koopman representation $\pi$ is a unitary representation (that is, it is continuous if we endow $U(L^2(X, \nu))$ with the strong operator topology; see [BdLHV08, Proposition A.6.1]).

(3) If $\nu$ is a finite measure, then $1_X$ is a fixed vector, and its orthogonal complement is a subrepresentation. This subspace is the subset of all (representatives of) functions with zero integral and we denote it by $L^2_0(X, \nu)$. We denote the associated unitary subrepresentation

$$
\pi_0 : G \to U(L^2_0(X, \nu)).
$$

Notation. Let $\pi$ be a unitary representation of $G$ on a Hilbert space $H$, and let $\mu$ be a finite Borel measure on $G$. We define the operator $\pi(\mu)$ in the following way:

$$
\pi(\mu) := \int_G \pi(g) \, d\mu(g),
$$

that is, $\pi(\mu)$ is the only bounded operator on $H$ such that for all $\xi, \eta \in \mathcal{H}$,

$$
\langle \pi(\mu)\xi, \eta \rangle := \int \langle \pi(g)\xi, \eta \rangle \, d\mu(g).
$$

This extension of $\pi$ to the space of finite Borel measures on $G$ is a homomorphism of involutive algebras.

The relevance of the Koopman representation is given by the following proposition.

Proposition 1. Consider an action of $G$ on a probability space $(X, \nu)$ by measurable transformations preserving the measure, and let $\pi_0$ the subrepresentation of the Koopman representation on the orthogonal of the subspace of constant functions (see [3]).

Let $\mu$ be a Borel measure on $G$. Then

$$
\delta(\mu) = \|\pi_0(\mu)\|.
$$

Proof. Let $\phi \in L^2(X, \nu)$, and let us denote

$$
\delta(\mu, \phi) := \|x \mapsto \int_G \phi(g^{-1}x) \, d\mu - \int_X \phi \, d\nu\|_2,
$$

so that

$$
\delta(\mu) = \sup_{\|\phi\|_2 \leq 1} \delta(\mu, \phi).
$$

Let us denote $P_1$ the orthogonal projector on the subspace $\mathbb{C}1_X$ of constant functions, and let

$$
P_0 := I - P_1.
$$

Then it is straightforward to check that $P_0$ is the orthogonal projector on $L^2_0(X)$, for every $f \in L^2(X, \nu),$

$$
P_1(\phi) = \int_X \phi \, d\nu \cdot 1_X,
$$

that

$$
\delta(\mu, \phi) = \delta(\mu, P_0\phi) = \|\pi_0(\mu)\phi\|_2
$$

and the Proposition follows. \qed
Let us recall the following notation from [Dix77, p. 282]: we denote by $\Delta$ the modular function of $G$, and for all continuous, compactly-supported complex-valued $f$ on $G$, we denote by $f^*$ the map defined by

$$\forall g \in G, \quad f^*(g) = \Delta(g^{-1})f(g^{-1}).$$

This is the involution of the involutive algebra $C_c(G)$.

**Lemma 1.** Let $\rho_1, \rho_2$ be two unitary representations of $G$. If

$$\|\rho_1(h \ast h^*)\| \leq \|\rho_2(h \ast h^*)\|$$

holds for all non-negative, continuous, compactly supported function $h$ on $G$, then

$$\|\rho_1(f)\| \leq \|\rho_2(f)\|$$

holds for all non-negative, continuous, compactly-supported function $f$ on $G$.

**Proof.** The proof is straightforward from the $C^*$ equality, that is, for every bounded operator $A$ of a Hilbert space,

$$\|AA^*\| \leq \|A\|^2.$$ 

□

Let us recall the following well-known fact, due to Kesten in the discrete case.

**Theorem 2** (See [Pie, Thm 9.6, p.85]). A locally compact group $G$ is amenable if and only if, for all positive, symmetric probability measure $\mu$ on $G$, the norm of the bounded operator $\lambda_G \mu$ is 1.

### 2.2. Sierpiński’s theorem on atomless probability spaces.

Let us briefly recall a theorem of Sierpiński on atomless probability spaces.

**Theorem** (See [Sie22]). Let $(X, \mathcal{B}, \nu)$ be an atomless probability space. Then $(X, \nu)$ has the intermediate value property, that is: for all measurable subsets $A, B$ of $X$ such that $A \subset B$, for all $c \in [\nu(A), \nu(B)]$, there exists a measurable subset $C$ of $X$ such that

- $A \subset C \subset B$;
- $\nu(C) = c$.

The proof of this theorem is an amusing exercise in measure theory. The interested reader is invited to prove the following stronger statement, using Zorn’s lemma: for all such $A, B$, there exists a map $f : [\nu(A), \nu(B)] \to \mathcal{B}$ such that $f$ is $(\leq, \subset)$-order-preserving and for all $c \in [\nu(A), \nu(B)]$, $\nu(f(c)) = c$.

### 2.3. Proof of Theorem 1

We first make the following definition.

**Definition 2.2 (Moderate growth).** Let $G$ be a group, $(X, \mathcal{B}, \nu)$ an atomless probability space, and $G \curvearrowright X$ a measure-preserving action. Let $F, S \subset G$ be measurable subsets.

We say that a sequence $(B_n)_{n \in \mathbb{N}}$ of measurable subsets of $X$ is of $(S, F)$-moderate growth if

1. $\forall n \in \mathbb{N}, \nu(B_n) > 0$
2. $\inf\{\nu(B) \mid B \in \mathcal{B}, S^n B \subset B\} < \frac{1}{2}$
3. $\limsup_{n \to \infty} \left( \inf_{g \in F} \frac{\nu(gB_n \cap B_n)}{\nu(B_n)} \right)^{\frac{1}{n}} \geq 1.$

Theorem 1 therefore states that if for any compact, symmetric neighborhood $S$ of $e$ in $G$, there exists a symmetric neighborhood $F$ and an $(S, F)$-moderate growth sequence, then the minorization we are interested in holds.
Proof of Theorem 1. For all measurable \( h : G \to \mathbb{R}^*_+ \), let us denote by \( \mu_h \) (rather than \( h \cdot \mu \)) the measure on \( G \) that has density \( h \) with respect to \( \mu \), that is, \( \forall A \in \mathcal{B}(G), \mu_h(A) = \int_A h(g) \, d\mu_G(g) \).

Let \( f \) be a non-negative, continuous, compactly-supported function on \( G \). We are going to prove that

\[
\|\lambda_G(f)\| \leq \|\pi_0(f)\|.
\]

From Lemma 1 we may assume that \( f = h \ast h^* \) for some non-negative, continuous, compactly-supported function \( h \).

If \( f \) is constant, equal to 0, then the claim is obvious. We therefore assume that \( f \) doesn’t vanish everywhere. Therefore, \( h \) is non-zero, and we therefore have

\[
f(e) = \int_G h(g)h^*(g^{-1}) \, d\mu_G(g) = \int_G h(g)^2 \Delta(g) \, d\mu_G(g) > 0.
\]

Let \( S \) be the support of \( f \). It is a compact, symmetric neighborhood of \( e \) in \( G \). Let \( F \) be a compact, symmetric neighborhood of \( e \) given by the hypothesis. Let \( (B^+_n)_{n \in \mathbb{N}} \) be a sequence of \( F \)-moderate growth. Since \( \nu \) is atomless, according to Sierpiński’s theorem, there exists a subset \( B^-_n \) that has the same \( \nu \)-measure as \( B^+_n \), and such that

\[
\tag{1}
S^n B^+_n \cap B^-_n = \emptyset.
\]

Let

\[
\phi := \frac{1_{B^+_n} - 1_{B^-_n}}{\|1_{B^+_n} - 1_{B^-_n}\|^2_2}.
\]

The function \( \phi \) is obviously in \( L^2_0(X, \nu) \).

We therefore have, denoting by \( f \ast^n \) the \( n \)-fold convolution of \( f \) with itself,

\[
\|\pi_0(f \ast^n)\| \geq \langle \pi_0(f \ast^n) \phi, \phi \rangle 
\]

\[
\geq \frac{1}{\nu(B^+_n) + \nu(B^-_n)} \int_G [\nu(gB^+_n \cap B^+_n) + \nu(gB^-_n \cap B^-_n)] \, d\mu_{f \ast^n}(g) 
\]

\[
- \frac{1}{\nu(B^+_n) + \nu(B^-_n)} \int_G [\nu(gB^+_n \cap B^-_n) + \nu(gB^-_n \cap B^+_n)] \, d\mu_{f \ast^n}(g) 
\]

\[
\geq \frac{1}{\nu(B^+_n) + \nu(B^-_n)} \inf_{g \in F} \nu(gB^+_n \cap B^+_n) \int_F \, d\mu_{f \ast^n}(g) 
\]

\[
\geq \frac{1}{2} \mu_{f \ast^n}(F) \inf_{g \in F} \frac{\nu(gB^+_n \cap B^+_n)}{\nu(B^+_n)}
\]

Now,

\[
\|\pi_0(f)\| = \limsup_{n \to \infty} \|\pi_0(f)^n\|^{1 \over n}
\]

and, according to [BC74],

\[
\limsup_{n \to \infty} (\mu_{f \ast^n}(F))^{1 \over n} = \lim_{n \to \infty} (\mu_{f \ast^n}(F))^{1 \over n} = \|\lambda_G(f)\|.
\]
We can now finish the computation:
\[
\|\pi_0(f)\| = \limsup_{n \to \infty} \|\pi_0(f)^n\|^{\frac{1}{n}} \\
\geq \limsup_{n \to \infty} \|\lambda_G(f)\| \left( \inf_{g \in F} \frac{\nu(gB_n^+ \cap B_n^+)}{\nu(B_n^+)} \right)^{\frac{1}{n}} \\
\geq \|\lambda_G(f)\| \limsup_{n \to \infty} \left( \inf_{g \in F} \frac{\nu(gB_n^+ \cap B_n^+)}{\nu(B_n^+)} \right)^{\frac{1}{n}} \\
\geq \|\lambda_G(f)\|
\]

\[\square\]

2.4. Nets and volume estimates. The goal of this subsection is to establish counting inequalities that allow us to estimate the cardinal of maximal nets, that is, finite sets of evenly-spaced points, in big subsets (that can be thought as balls). The precise technical statement that will be used later in the proofs is Lemma 6.

Lemma 2. Let \( G \) be a topological group, \( A \) and \( B \) two compact subsets, such that \( B \) is a neighborhood of \( e \). Then, for all \( k \in \mathbb{N} \), there exists a finite subset \( S \subset G \) such that we have
\[
B^k A \subset BS.
\]

Proof. Let \( k \in \mathbb{N} \). Let \( U \) be an open set in \( G \) containing \( e \) and contained in \( B \). Then the compact subset \( B^k A \), is covered by the sets \( Uh \) for \( h \in B^k A \). Therefore, there is a finite subset \( S \) of \( B^k A \) such that \( B^k A \subset US \). Since \( US \subset BS \), we therefore have \( B^k A \subset BS \), hence, for all \( n \in \mathbb{N}^* \), \( B^{k+n-1} A \subset B^n S \).

Note that in the following Lemma, we consider a right-Haar measure.

Lemma 3. Let \( G \) be a locally compact group, let \( \mu_G \) be a right-Haar measure on \( G \), let \( A, B \) be compact subsets of \( G \), such that \( B \) is a neighborhood of \( e \). Then, for all \( k \in \mathbb{N} \), there is a constant \( c \) such that
\[
\forall n \in \mathbb{N}^*, \ \mu_G(B^{n+k-1} A) \leq c \mu_G(B^n).
\]

Proof. The claim follows immediately from Lemma 2: it is enough to take \( c := |S| \) where \( S \) is a finite subset given by Lemma 2.\[\square\]

Definition 2.3 (Net). Let \( A, B, N \subset G \). We say that \( N \) is an \( A \)-separated net of \( B \) if

1. \( N \) is a finite subset of \( B \) ;
2. \( \forall n_1, n_2 \in N, \ n_1 A \cap n_2 A \neq \emptyset \Rightarrow n_1 = n_2. \)

Let us denote by \( N(B,A) \) the set of \( A \)-separated nets in \( B \). We endow \( N(B,A) \) with the order given by inclusion and denote by \( N_{max}(B,A) \) the set of maximal nets (that is, nets that are not strictly contained in a bigger net).

Lemma 4. Let \( G \) be a group, and let \( A, B \subset G \). We have
\[
\forall N \in N_{max}(B,A), \ B \subset NAA^{-1}.
\]

Proof. Let \( N \in N_{max}(A,B) \). Let us show that we have \( B \subset NAA^{-1} \). We proceed by contradiction. Assume that \( b \in B \setminus NAA^{-1} \). Let us show that \( N \cup \{b\} \) is an \( A \)-separated net of \( B \). It is enough to check that for every \( n \in N \), \( nA \cap bA = \emptyset \). Let \( n \in N \). If \( nA \cap bA \neq \emptyset \), there exists \( a_1, a_2 \in A \) such that \( na_1 = ba_2 \). Let \( a_1, a_2 \) as such. Then \( b = na_1a_2^{-1} \in NAA^{-1} \), which is a contradiction. So \( nA \cap bA = \emptyset \). Therefore, \( N \cup \{b\} \) is an \( A \)-separated net of \( B \), and this contradicts the maximality of \( N \). So we have \( B \subset NAA^{-1} \).

\[\square\]
Lemma 6. Let $G$ be a locally compact group, and $\mu_G$ be a left-Haar measure on $G$. Let $A, B \subset G$ such that $A B$ and $B A$ are measurable. We then have
\[
\forall N \in N(B, A), \quad |N| \mu_G(A) \leq \mu_G(BA),
\]
and
\[
\forall N \in N_{\text{max}}(B, A), \quad \mu_G(B) \leq |N| \mu_G(AA^{-1}).
\]
Proof. Let us prove the first item. From the definition of a net, $N A$ is the disjoint union of the $n A$ for $n \in N$, so $\mu_G(N A) = |N| \mu_G(A)$. But since $N \subset B$, $N A \subset B A$, so we get the desired inequality.

Let us now prove the second item. From Lemma 4 $B \subset N A A^{-1}$. So, we have $\mu_G(B) \leq \mu_G(N A A^{-1}) \leq \mu_G(AA^{-1})|N|$. □

Lemma 6. Let $G$ be a locally compact, unimodular group, $A_1, A_2, B$ compact neighborhoods of $e$ in $G$. Then there are constants $c_1, c_2 > 0$ such that
\[
\forall n \geq 3, \quad \forall N_1 \in N_{\text{max}}(B^n, A_1), \forall N_2 \in N_{\text{max}}(B^{n-2}, A_2), \quad c_1 \leq \frac{|N_2|}{|N_1|} \leq c_2.
\]
Proof. From Lemma 5 there are strictly positive $k_1, k_2, k_3$ such that
\[
(2) \quad \forall n \in \mathbb{N}, \quad \mu_G(B^n A_1) \leq k_1 \mu_G(B^n),
\]
\[
(3) \quad \forall n \geq 3, \quad \mu_G(B^{n-2} A_2) \leq k_2 \mu_G(B^n),
\]
and
\[
(4) \quad \forall n \in \mathbb{N}, \quad \mu_G(B^{n+2}) \leq k_3 \mu_G(B^n).
\]
Let $n \geq 3, N_1 \in N_{\text{max}}(B^n, A_1), N_2 \in N_{\text{max}}(B^{n-2}, A_2)$. We have
\[
(n) \quad |N_2| \leq \frac{\mu_G(B^{n-2} A_2)}{\mu_G(A_2)} \leq \frac{k_2}{\mu_G(A_2)} \mu_G(B^n)
\]
Moreover,
\[
(6) \quad \frac{1}{\mu_G(A_2 A_2^{-1})} \leq \frac{1}{\mu_G(B^n)} \leq \frac{1}{\mu_G(A_2 A_2^{-1})} \leq \frac{1}{\mu_G(A_1 A_1^{-1})} \leq \frac{1}{\mu_G(A_1)}.
\]
So, we have
\[
(7) \quad \frac{1}{k_3 \mu_G(A_2 A_2^{-1})} \mu_G(B^n) \leq \frac{1}{\mu_G(A_2 A_2^{-1})} \leq \frac{1}{\mu_G(A_1 A_1^{-1})} \leq \frac{1}{\mu_G(A_1)} \mu_G(B^n).
\]
On the other hand, $|N_1| \leq \frac{\mu_G(B^n A_1)}{\mu_G(A_1)}$, so $|N_1| \leq \frac{k_1}{\mu_G(A_1)} \mu_G(B^n)$. Moreover,
\[
(8) \quad \mu_G(B^n) \leq \frac{1}{\mu_G(A_1 A_1^{-1})} \leq \frac{1}{\mu_G(A_1)} \mu_G(B^n).
\]
So we have
\[
\mu_G(B^n) \leq \frac{1}{\mu_G(A_1 A_1^{-1})} \leq |N_1| \leq \frac{1}{\mu_G(A_1)} \mu_G(B^n).
\]
We therefore obtain
\[
\frac{\mu_G(A_1)}{k_3 k_1 \mu_G(A_2 A_2^{-1})} \leq \frac{|N_2|}{|N_1|} \leq \frac{k_2 \mu_G(A_1 A_1^{-1})}{\mu_G(A_2)}.
\] □
2.5. **Proof of Corollaries**\(^2\) and \(^3\) In order to prove Corollary\(^2\) it is obviously enough to prove the following proposition:

**Proposition 2.** Let \(G\) be a locally compact, unimodular group. Let \(X\) be a Hausdorff topological space, \(\nu\) an atomless regular Borel measure on \(X\). Consider a continuous measure-preserving action \(G \curvearrowright X\). We assume that there exists \(x_0\) in the support of \(\nu\) such that

- \(\text{Stab}_G(x_0)\) is compact;
- for all compact subset \(K\) of \(G\),
  \[\nu(Kx_0) = 0.\]

Then for every compact, symmetric neighborhood of \(e\) in \(G\), there exists a compact, symmetric neighborhood \(F\) of \(e\), and a sequence of neighborhoods of \(x_0\) that is of \((S,F)\)-moderate growth.

**Remark 5.** If \(G\) is \(\sigma\)-compact, then \(\nu(Kx) = 0\) for every compact \(K\) if and only if the orbit of \(x_0\) is a null-set.

We will need the following lemma.

**Lemma 7.** Let \(G\) be a topological group, \(X\) a Hausdorff topological space. Let \(G \curvearrowright X\) be a continuous action (that is, the map \(G \times X \to X\) that defines it is continuous).

Let \(x_0 \in X\), let \(K\) be a compact subset of \(G\), and \(U\) a neighborhood of \(e\) in \(G\). Then there exists a neighborhood \(V\) of \(x_0\) in \(X\) such that

\[\forall g \in K \setminus \text{Stab}_G(x_0)U, \quad gV \cap V = \emptyset.\]

**Proof.** First of all, let us notice that \(\text{Stab}_G(x_0)U\) is open, and therefore \(K \setminus \text{Stab}_G(x_0)U\) is compact.

Let \(E\) be the set of triples \((g,W,V)\) such that

- \(g \in K \setminus \text{Stab}_G(x_0)U\);
- \(W\) is a neighborhood of \(g\) in \(G\);
- \(V\) is a neighborhood of \(x_0\) in \(X\);
- \(\forall w \in W, \quad wV \cap V = \emptyset\).

For all \(g \in K \setminus \text{Stab}_G(x_0)U\), there exists \(W, V\) such that \((g,W,V) \in E\). Indeed, \(g \notin \text{Stab}_G(x_0)\), so \(gx_0 \neq x_0\). Let \(Y\) be a closed neighborhood of \(x_0\) that does not contain \(gx_0\). By continuity, the set of \((h,x)\) such that \(hx \notin Y\) is an open subset of \(G \times X\); moreover, it contains \((g,x_0)\), so there are neighborhoods \(W, Z\) of \(g\) and \(x_0\) such that \(WZ \subset Y^c\). Let \(V := Z \cap Y\). Then \(WV \subset Y^c \subset V^c\), so \(WV \cap V = \emptyset\).

Therefore, the set of second coordinates of \(E\) covers \(K \setminus \text{Stab}_G(x_0)U\). By compactness, there exists \(n \in \mathbb{N}^*, \ g_1, \ldots, g_n, \ W_1, \ldots, W_n, \ V_1, \ldots, V_n\) such that \(K \setminus \text{Stab}_G(x_0)U \subset \bigcup_i W_i\). Let us define \(V := \bigcap_i V_i\). This subset \(V\) satisfies the needed requirements: it is a neighborhood of \(x_0\), and if \(g \in K \setminus \text{Stab}_G(x_0)U\), then there exists \(i\) such that \(g \in W_i\). Therefore \(gV_i \cap V_i = \emptyset\), so \(gV \cap V = \emptyset\).

**Proof of Proposition**\(^3\) Let \(S\) be a compact, symmetric neighborhood of \(G\). We would like to find a compact symmetric neighborhood \(F\) of \(e\) and a sequence \((B_n)_{n \geq 3}\) that has \((S,F)\)-moderate growth. Let \(n \geq 3\).

First of all, since \(\nu(S^{2n}x_0) = 0\), and since \(\nu\) is regular, there exists an open subset \(U\) of \(X\) containing \(S^{2n}x_0\), that has measure strictly smaller than \(\frac{1}{2}\).

We claim that there exists a neighborhood \(W_n\) of \(x_0\) such that \(S^{2n}W_n \subset U\). Indeed, by continuity of the action, for all \(g \in S^{2n}\), there exists a neighborhood \(D_g\) of \(g\) and a
neighborhood $E_g$ of $x_0$ such that $D_g E_g \subset U$. By compactness of $S^{2n}$, there exists a finite subset $I$ of $S^{2n}$ such that $S^{2n} \subset \bigcup_{i \in I} D_i$. Let us now define $W_n := \bigcap_{i \in I} E_i$. We have
\[ S^{2n} W_n \subset \bigcup_{i \in I} D_i W_n \subset \bigcup_{i \in I} D_i E_i \subset U, \]
which gives the claim.

Now, taking $F = S$, according to Lemma 7, there exists a neighborhood $V_n$ of $x_0$ such that
\[ \forall g \in S^{2n-2} \setminus \text{Stab}_G(x_0)S, \quad gV_n \cap V_n = \emptyset; \]
moreover, up to taking a smaller neighborhood instead of $V_n$, we can assume that $V_n$ is contained in $W_n$.

Let us then define
\[ B_n := S^n V_n. \]
We now check that the sequence $(B_n)_{n \geq 0}$ fulfills the three requirements of $(S, F)$-moderate growth.

First of all, since $V_n$ is a neighborhood of $x_0$ and since $x_0$ is in the support of $\nu$, $\nu(B_n) > 0$, so the first condition is verified. Moreover, by construction, $S^n B_n = S^{2n} V_n \subset U$; and $\nu(U) < \frac{1}{3}$ so the second condition is satisfied.

We will now estimate, for all $g \in F$,
\[ \frac{\nu(g B_n \cap B_n)}{\nu(B_n)} \]
by considering nets.

Let $N_1 \in N_{\text{max}}(S^{n-2}, S \text{Stab}_G(x_0)S^2)$. We have $N_1 F \subset F^{n-1}$, so
\[ \nu(N_1 F V_n) \leq \nu(F^{n-1} V_n). \]
Moreover, we have
\[ N_1 F V_n = \bigcup_{g \in N_1} g F V_n; \]
indeed, on the one hand, let $g_1, g_2 \in N_1$, $f_1, f_2 \in F$, $v_1, v_2 \in V_n$ such that $g_1 f_1 v_1 = g_2 f_2 v_2$. We then have $f_2^{-1} g_2^{-1} g_1 f_1 v_1 = v_2$. On the other hand, $f_2^{-1} g_2^{-1} g_1 f_1 \in F^{2n-2}$, so, by condition (8) on $V_n$, we have $f_2^{-1} g_2^{-1} g_1 f_1 \in \text{Stab}_G(x_0)F$. So, $g_1 \in g_2 F \text{Stab}_G(x_0)F^2$, and this is only possible if $g_1 = g_2$.

We deduce from this
\[ \nu(N_1 F V_n) = |N_1| \nu(F V_n), \]
so
\[ |N_1| \nu(F V_n) \leq \nu(F^{n-1} V_n). \]
Now, for any $g \in F$, we have $F^{n-1} V_n \subset g F^n V_n \cap F^n V_n = g B_n \cap B_n$, so
\[ \nu(g B_n \cap B_n) \geq |N_1| \nu(F V_n). \]

Finally, let $C$ be a compact neighborhood of $e$ in $G$ such that $CC^{-1} \subset F$. Let $N_2 \in N_{\text{max}}(F^n, C)$. Then
\[ B_n = F^n V_n \overset{\text{Lemma 8}}{\subset} N_2 CC^{-1} V_n \]
so
\[ \nu(B_n) \leq |N_2| \nu(F V_n). \]
We therefore have, for all $n \geq 3$,
\[ \frac{\nu(g B_n \cap B_n)}{\nu(B_n)} \geq \left| \frac{N_1}{N_2} \right|. \]
so
\[ \left( \frac{\nu(gB_n \cap B_n)}{\nu(B_n)} \right)^{\frac{1}{n}} \geq \left( \frac{|N_1|}{|N_2|} \right)^{\frac{1}{n}}. \]

The third requirement follows now from Lemma \[6\].

We now add a few words to show that Corollary 3 follows in a straightforward way from Corollary \[2\].

Proof of Corollary \[3\]. It is immediate to check the hypotheses of Corollary \[2\] are satisfied: let \( G \) be a locally compact group, \( H \) a closed subgroup of \( G \) such that
\[ \mu_G(H) = 0, \]
and \( \Gamma \) a lattice in \( G \) such that
\[ H \cap \Gamma = \{ e \}. \]

The action of \( H \) on \( G/\Gamma \) by left-translations is continuous, its orbits are null-sets, and the stabilizer of the coset \( e\Gamma \), \( H \cap \Gamma \), is finite, hence compact. \( \square \)

3. Examples and counterexamples

3.1. Margulis’ counterexample. In this section, we prove that Theorem \[PLP21, Thm 3.6\] quoted in Section 1.2.2 cannot extend to the case where \( G \) is locally compact without making further hypotheses.

Example 2. Consider the action by left-translation of \( \mathbb{R} \) on \( \mathbb{R}/\mathbb{Z} \). Let, for \( t \in \mathbb{R} \), \( \pi_0(t) \) be the operator that sends every zero-integral \( \phi \in L^2(\mathbb{R}/\mathbb{Z}, \text{Leb}) \) to the map \( s \mapsto f(s-t) \). Let \( \mu \) be the uniform probability measure on \([0, 1]\). Then, obviously, \( \pi_0(\mu) \) is the zero operator, whereas \( \lambda_\mathbb{R}(\mu) \) is not.

In fact, we have the following generalization, that follows from the discussion that one can find in pages 107–112 of the book \[Mar91\]. We give a sketch of the proof for the convenience of the reader.

Proposition 3 (Margulis). Let \( G \) be a locally compact, \( \sigma \)-compact group, and let \( H \) be a closed cocompact subgroup of \( G \). We consider the action by left-multiplication of \( G \) on \( G/\Gamma \) and we suppose that there is a \( G \)-invariant probability measure \( \nu \) on \( G/\Gamma \). Let \( \pi \) denote the associated Koopman representation, and \( \pi_0 \) the subrepresentation on the subspace of zero integral. Then there is a continuous, compactly-supported, nonnegative function \( f \) on \( G \) that has integral 1, such that
\[ \| \pi_0(f) \| < 1. \]

Proof sketch. Assume, by contradiction, that for every continuous, compactly-supported, nonnegative function \( f \) on \( G \) that has integral 1, we have
\[ \| \pi_0(f) \| = 1. \]

From \[Mar91\] (1.3) Proposition, p. 109, there exists a sequence \( (p_i)_{i \in \mathbb{N}} \) that is asymptotically \( \pi_0 \)-invariant, that is, each \( p_i \) is an element of \( L^2_0(G/\Gamma) \), has norm 1, and we have, for every compact \( K \) in \( G \),
\[ \lim_{i \to \infty} \sup_{g \in K} \| \pi_0(g)p_i - p_i \|_2 = 0. \]

Let \( (p_i)_{i \in \mathbb{N}} \) be such a sequence. First of all, we have
\[ \lim_{i \to \infty} \sup_{g \in \text{supp}(f)} \| \pi_0(g)p_i - p_i \|_2 = \lim_{i \to \infty} \sup_{g \in \text{supp}(f)} \| \pi_0(g)p_i - p_i \|_2 = 0. \]
Let $K$ be a compact subset of $G$. We then have
\[
\limsup_{i \to \infty} \sup_{g \in K} \| \pi_0(g) \pi_0(f) p_i - \pi_0(f) p_i \|_2 \leq \limsup_{i \to \infty} \sup_{g \in K} (\| \pi_0(g) \pi_0(f) p_i - \pi_0(g) p_i \|_2 \\
+ \| \pi_0(g) p_i - p_i \|_2 + \| p_i - \pi_0(f) p_i \|_2) \\
= \limsup_{i \to \infty} \sup_{g \in K} (2 \| \pi_0(f) p_i - p_i \|_2 \\
+ \| \pi_0(g) p_i - p_i \|_2) \\
= 0
\]
so the sequence
\[
\left( \frac{\pi_0(f) p_i}{\| \pi_0(f) p_i \|_2} \right)_{i \in \mathbb{N}}
\]
is also asymptotically $\pi_0$-invariant. Let us denote, for all $i \in \mathbb{N}$, $q_i$ the composition of
\[
\frac{\pi_0(f) p_i}{\| \pi_0(f) p_i \|_2}
\]
with the canonical surjection $G \to G/H$. So, from [Mar91] (1.7) Lemma, p. 110, the sequence $(q_i)_{i \in \mathbb{N}}$ is equicontinuous and uniformly bounded on $G$. From Ascoli’s theorem, up to extraction, one can assume that it converges uniformly on compact subsets to a continuous, $H$-invariant on the right function (since each term of the sequence is $H$-invariant on the right). This limit gives, on the quotient $G/H$, a function that we denote by $p$. Since $G/H$ is compact, we have the uniform convergence
\[
\frac{\pi_0(f) p_i}{\| \pi_0(f) p_i \|_2} \to p.
\]
We deduce from all this
- that $p$ has zero integral (since it is the limit of a sequence of zero-integral functions);
- that $p$ has $L^2$-norm 1 (as a uniform limit of uniformly bounded functions of $L^2$-norm 1);
- that $p$ is $\pi_0$-invariant (as a uniform limit of $\pi_0$-invariant functions) and therefore is constant.

So, $p$ is identically 0, but has $L^2$-norm 1. This is a contradiction. \hfill \qed

**Remark 6.** In particular, if $G$ is a locally compact, $\sigma$-compact amenable group, if $H$ is a cocompact closed subgroup of $G$ such that $G \to G/H$ has an invariant probability measure, and if $\pi_0$ denotes the subrepresentation of the Koopman representation on the subspace of zero-integral functions, then there is a continuous, positive, symmetric function $h$ of integral 1 such that the inequality
\[
\| \lambda_G(h) \| \leq \| \pi_0(h) \|
\]
doesn’t hold: take $f$ as in Margulis’ theorem, and let
\[
g := \frac{1}{\int_G f^* \ast f \, d\mu_G} f^* \ast f.
\]
Then $g$ satisfies the aforementioned properties. Moreover, from Kesten’s theorem, since $G$ is amenable, the operator $\lambda(f)$ has norm 1; and, according to Margulis’ theorem, $\pi_0(f)$ has norm strictly less than 1.
3.2. An action on a finite-volume locally symmetric space of rank four with optimal discrepancy. In this section, we give an example of an action of a non-amenable locally compact group that acts continuously on a measured topological space by preserving the measure, where the discrepancy inequality is an equality.

Let $G := \text{SL}_3(\mathbb{R}) \times \text{SL}_3(\mathbb{R})$, and consider the subgroup
$$\left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in \text{SL}_2(\mathbb{R}), \ v \in \mathbb{R}^2 \right\} \cong \mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{R})$$
of $\text{SL}_3(\mathbb{R})$ and consider its embedding $H$ in the first factor of $G$.

Consider now the unique (unital) ring morphism $\sigma : \mathbb{Z} \to \mathbb{C}$ sending $\sqrt{2}$ to $-\sqrt{2}$. For a matrix $A := (a_{ij})_{i,j=1,2,3} \in \text{SL}_3(\mathbb{Z}[\sqrt{2}])$, let us denote also by $\sigma(A)$ the matrix $(\sigma(a_{ij}))_{i,j=1,2,3}$. Since $\sigma$ is a ring morphism,
$$\sigma(A) \in \text{SL}_3(\mathbb{Z}[\sqrt{2}]).$$

Consider now the map
$$j : \text{SL}_3(\mathbb{Z}[\sqrt{2}]) \to G, \quad A \mapsto (A, \sigma(A)).$$

Then $j$ is an injective group morphism, and $\Gamma := j(\text{SL}_3(\mathbb{Z}[\sqrt{2}]))$ is an irreducible lattice in $G$ (see [Zim84, Example 2.2.5, p.18]). Obviously, as $H$ is contained in the first factor of $G$, $\Gamma \cap H = \{e\}$.

**Proposition 4.** Consider the action $H \curvearrowright (G/\Gamma, \mu_{G/\Gamma})$ and the subrepresentation $\pi_0$ of the Koopman representation of $H$ on $L^2_0(G/\Gamma, \mu_{G/\Gamma})$.

Then, for every continuous, non-negative, compactly-supported function $f$ on $H$, the inequality in Corollary 3 is optimal, that is, we have
$$\|\lambda_H(f)\| = \delta(f).$$

**Proof.** By construction, we can apply Corollary 3 so we have, for every continuous, non-negative, compactly-supported function $f$ on $H$,
$$\|\lambda_H(f)\| \leq \delta(f).$$

For the other inequality, notice that since $\Gamma$ is irreducible, the action $G \curvearrowright G/\Gamma$ is mixing (see for example [PL19, Corollary 3]), and therefore the restriction of this action to any noncompact subgroup is ergodic.

In particular, the restriction of $\pi_0$ to the embedded copy of $\mathbb{R}^2$ in $H$ has no nontrivial invariant vectors. So, according to [Zim84, Theorem 7.3.9, p.146] we have the weak containment
$$\pi_0 \prec \lambda_H.$$As already mentioned (in Remark 2) this implies the inequality we are interested in. □

**REFERENCES**

[B.C74] C. Berg and J. P. R. Christensen. Sur la norme des opérateurs de convolution. *Inventiones math.*, 23:173–178, 1974.

[BdLHV08] B. Bekka, P. de La Harpe, and A. Valette. *Kazhdan’s Property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, 2008.

[CMM22] P.E. Caprace, K. Mehrdad, and N. Monod. A type I conjecture and boundary representations of hyperbolic groups. *https://arxiv.org/abs/2110.00190*, 2022.

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1 The reader willing to read in Zimmer’s book the result we quote should be warned that in the litterature, the symbol $\prec$ can denote two slightly different notions of weak containment (see [BdLHV08, Remark F.1.2 (ix), p. 397] for a detailed account of the differences between the two).
A UNIVERSAL LOWER BOUND FOR THE DISCREPANCIES OF ACTIONS OF A L.C. GROUP

[A. Dudko and R. Grigorchuk. On spectra of Koopman, groupoid and quasi-regular representations. *J. Mod. Dyn.*, 11:99–123, 2017.]

[J. Dixmier. *C*-algebras. Modern Birkhäuser Classics. North-Holland, 1977.]

[P. Erdős and A. H. Stone. On the sum of two Borel sets. *Proceedings of the American Mathematical Society*, 25(2):304–306, 1970.]

[H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.]

[A. Lubotzky, R. Phillips, and P. Sarnak. Hecke operators and distributing points on the sphere. I. *Comm. Pure Applied Math.*, 39:S149–S186, 1986.]

[G. A. Margulis. *Discrete Subgroups of Semisimple Lie Groups*. Springer-Verlag, 1991.]

[J. P. Pier. *Amenable Locally Compact Groups*. John Wiley & Sons, Inc., New York, 1984.]

[A. Pinochet Lobos. On a generalization of the Howe-Moore property. *arXiv:1904.00953*, 2019.]

[A. Pinochet Lobos and Ch. Pittet. The exact rate in the ergodic theorem of Lubotzky-Phillips-Sarnak and a universal lower bound for the discrepancy. *Ens. Mat.*, 67:63–94, 2021.]

[Y. Shalom. Random ergodic theorems, invariant means and unitary representation. Lie groups and ergodic theory (Mumbai, 1996). Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Bombay, 14:273–314, 1998.]

[W. Sierpiński. Sur les fonctions d’ensemble additives et continues. *Fundamenta Mathematicae*, 3:240–246, 1922.]

[R. Zimmer. *Ergodic theory and semisimple groups*. Birkhäuser, 1984.]