Linear programming for Decision Processes with Partial Information

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Abstract

Markov Decision Processes (MDPs) are stochastic optimization problems that model situations where a decision maker controls a system based on its state. Partially observed Markov decision processes (POMDPs) are generalizations of MDPs where the decision maker has only partial information on the state of the system. Decomposable POMDPs are specific cases of POMDPs that enable one to model systems with several components. Such problems naturally model a wide range of applications such as predictive maintenance. Finding an optimal policy for a POMDP is PSPACE-hard and practically challenging. We introduce a mixed integer linear programming (MILP) formulation for POMDPs restricted to the policies that only depend on the current observation, as well as valid inequalities that are based on a probabilistic interpretation of the dependence between variables. The linear relaxation provides a good bound for the usual POMDPs where the policies depend on the full history of observations and actions. Solving decomposable POMDPs is especially challenging due to the curse of dimensionality. Leveraging our MILP formulation for POMDPs, we introduce a linear program based on “fluid formulation” for decomposable POMDPs, that provides both a bound on the optimal value and a practically efficient heuristic to find a good policy. Numerical experiments show the efficiency of our approaches to POMDPs and decomposable POMDPs.

1 Introduction

Many real-world problems where a decision maker controls a stochastic system evolving over time can be modeled as Partially Observed Markov Decision Processes (POMDPs). In such problems, at each timestep, the system is in a state \(s\) in some finite state space \(X_S\). The decision maker does not observe \(s\), but has access to an observation \(o\) that belongs to some finite observation space \(X_O\), and is randomly emitted with probability \(p(o|s)\). Based on this observation, the decision-maker chooses an action \(a\) from some finite action space \(X_A\). The system then transits randomly to a new state \(s'\) in \(X_S\) with probability \(p(s'|s,a)\) and the decision maker obtains an immediate reward \(r(s,a,s')\). The goal of the decision maker is to find a policy \(\delta_{o,a}^{s'}\), which represents a conditional probability of taking action \(a\) in \(X_A\) given current observation \(o\) in \(X_O\) at time \(t\), maximizing the expected total reward over a finite horizon \(T\)

\[
\max_{\delta \in \Delta} \mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(S_t, A_t, S_{t+1}) \right],
\]

where \(S_t\) and \(A_t\) are random variables representing the state and the action at time \(t\) and the expectation over \(\delta\) is taken with respect to the distribution \(P_\delta\) induced by the policy \(\delta\) chosen in the set of policies \(\Delta\). In the POMDP literature, the decision maker has perfect-recall, i.e.,
the decision is taken given all history of observation and actions at each time step. Hence the policy lies in a greater set of policies $\Delta_{PR} \supset \Delta$. Hence Problem (1) provides a lower bound on POMDP with perfect-recall. While POMDP problem with perfect-recall is PSPACE-hard Papadimitriou and Tsitsiklis [1987], we restrict to Problem (1) in this work, which is NP-hard Maou et al. [2012]. POMDPs are a generalization of the well-known Markov Decision Processes (MDPs). POMDPs are based on Hidden Markov Models (HMMs), which give a higher power of modeling than the usual Markov Chains (Baum and Petrie [1966]) on which MDPs are based. They appear naturally in the context of predictive maintenance, where a machine evolves over time and at each time step the decision maker decides to replace or not the machine. The decision maker does not have access to the machine’s wear and takes his action while observing output signals. Therefore, the goal is to find an optimal replacement policy (which can be time dependent) minimizing the total expected cost over a finite time horizon. Like MDPs, POMDPs suffer from the curse of dimensionality. Indeed, when the state space $X_S$ is large, the usual exact methods such as dynamic programming become computationally intractable (Puterman [1994]). Leveraging the well-known linear formulation of d’Epenoux [1963] for MDPs, Bertsimas and Misić [2016] propose a tractable heuristic to approximately solve MDPs when the system is composed of several components evolving independently and the state space can be written as the Cartesian product of the individual state space of each component. Such problems are called decomposable MDPs. We introduce the notion of decomposable POMDPs, a generalization where each component evolves independently and individually as a POMDP. For example, in the case of predictive maintenance, a machine is composed of several equipments and each equipment can be modeled individually as a POMDP. Unfortunately, there is no tractable linear program on which we can leverage to generalize the approach of Bertsimas and Misić [2016]. A good reason for that is PSPACE-hardness of POMDP Problem (Papadimitriou and Tsitsiklis [1987]). The first exact algorithms for POMDP use dynamic programming (Sondik [1973]) and consider the POMDP as a MDP where we replace the current state by a belief state, which corresponds to the probability of being in state $s$ at time $t$. However, such methods become computationally expensive even for small state spaces and small observation spaces. More recently, new policy-based solution methods leverage bounds to search efficiently in the policy space (Kaelbling et al. [1998]). Particularly, the finite-state controller algorithms (Poupart and Boutilier [2004]) combine a policy iteration algorithm that enumerates and evaluates policies on the observables history (represented as a new random variable) and gradient ascent method to find local optima, restricting the policy space. Leveraging policy search, Aras et al. [2007] proposed an exact mixed-integer linear program for POMDP problem. However, such a formulation is intractable even for small horizon time. In this paper, we consider exact and approximate linear formulations for Problem (1). Our contributions are as follows:

1. We propose a mixed-integer linear program (MILP) that exactly solves Problem (1). This formulation generalizes the dual linear program for MDP of d’Epenoux [1963].

2. We introduce an extended formulation with new valid inequalities that improve the resolution of our mixed-integer linear program. Such inequalities come from a probabilistic interpretation of the dependence between random variables. Experiments show their efficiency.

3. We show that the linear relaxation of our MILP provides an upper bound on the usual POMDP with perfect-recall.

4. In the case of decomposable POMDPs, leveraging the MILP previously mentioned, we propose a heuristic that repeatedly solves linear approximations. It extends to POMDPs
to the fluid formulation introduced by Bertsimas and Misic [2016], and strengthens it with valid inequalities.

5. Numerical experiments show the efficiency of our approach.

The paper is organized as follows. Section 2 presents our MILP formulation for Problem (1), and introduces our valid inequalities. We also present the link between our formulation and the usual POMDP with perfect-recall. In Section 3 we define decomposable POMDPs, we present our tractable linear program giving an upper bound on the optimal value, and use it in a heuristic. In Section 4 we present the numerical results. The proofs of the main results are available in Appendix A.

2 Mixed Linear Programming for Partially Observed Markov Decision Processes

In this section, we present an MILP that exactly models Problem (1). Given $N$ in $\mathbb{N}$ we use the notation $[N]$ for $\{1, \ldots, N\}$.

2.1 Problem settings

Let $X_S$, $X_O$, and $X_A$ be three finite sets corresponding respectively to the state space, the observation space and the action space. For a state $s \in X_S$ and an observation $o \in X_O$, let $p(o|s)$ be the conditional probability of observing $o$ given state $s$ $p(o|s) = P(O_t = o|S_t = s) \forall t \in [T].$

Similarly, we define the probability of transition from state $s \in X_S$ to $s' \in X_S$ while taking action $a \in X_A$ $p(s') = P(S_{t+1} = s'|S_t = s, A_t = a) \forall t \in [T].$

We define an immediate reward function $r : X_S \times X_A \times X_S \rightarrow \mathbb{R}$, which associates to a transition $(s, a, s')$ a reward $r(s, a, s')$. The goal is to solve Problem (1), where $\Delta = \left\{ \delta \in \mathbb{R}^{T \times |X_A| \times |X_O|}, \sum_{a \in X_A} \delta_{a|o}^t = 1 \text{ and } \delta_{a|o}^t \geq 0, \forall o \in X_O, a \in X_A, t \in [T] \right\}$.

and $P_\delta$ and $E_\delta$ denote the probability distribution and expectation induced by policy $\delta$ on $(S_t, O_t, A_t)_t$. We define the set of deterministic policies $\Delta^0$ as $\Delta^0 = \left\{ \delta \in \Delta, \delta_{a|o}^t \in \{0, 1\}, \forall o \in X_O : a \in X_A, t \in [T] \right\}$.

Note that $\Delta$ is the convex hull of $\Delta^0$. Any policy in $\Delta \setminus \Delta^0$ is a randomized policy.

2.2 A mixed-integer linear program

It is well-known that there always exists an optimal deterministic policy for MDPs Puterman [1994]. This result can be extended to POMDPs. Lemma 4.3 in Liu [2014] ensures that there always exists an optimal deterministic policy for Problem (1). Equivalently, $\max_{\delta \in \Delta} E_\delta \left[ \sum_{t=1}^{T} r(S_t, A_t, S_{t+1}) \right] = \max_{\delta \in \Delta^0} E_\delta \left[ \sum_{t=1}^{T} r(S_t, A_t, S_{t+1}) \right]$. 

We introduce a collection of variables $\mu = ((\mu_s^t)_{s,a}, (\mu_{soa}^t)_{s,o,a})$, and the following mixed-integer linear program (MILP).

$$\begin{align*}
\max_{\mu, \delta} & \quad \sum_{t=1}^{T} \sum_{s,s', \in X_s} \sum_{a \in X_A} r(s,a,s') p(s'|s,a) \mu_{sa}^t \\
\text{s.t.} & \quad \mu_s^1 = p(s) \quad \forall s \in X_S \\
& \quad \mu_{s,a}^{t+1} = \sum_{s \in X_s, a \in X_A} p(s'|s,a) \mu_{sa}^t \quad \forall s' \in X_S, t \in [T] \\
& \quad \mu_{sa}^t = \sum_{o \in X_O} \mu_{soa}^t \quad \forall s \in X_S, a \in X_A, t \in [T] \\
& \quad \mu_{soa}^t \leq p(o|s) \mu_s^t \quad \forall s \in X_S, o \in X_O, a \in X_A, t \in [T] \\
& \quad \mu_{soa}^t \leq \delta_{a,o}^t \quad \forall s \in X_S, o \in X_O, a \in X_A, t \in [T] \\
& \quad \mu_{soa}^t \geq p(o|s)(\mu_s^t + \delta_{a,o}^t - 1) \quad \forall s \in X_S, o \in X_O, a \in X_A, t \in [T] \\
& \quad \delta \in \Delta^o \\
& \quad \mu \geq 0
\end{align*}$$

(2a)

We show in Appendix that a feasible solution $\mu$ of Problem (2) can be interpreted as a probability distribution over the random variables, i.e., $\mu_s^t, \mu_{sa}^t$ and $\mu_{soa}^t$ respectively represent the probabilities $P_\delta(S_t = s), P_\delta(S_t = s, A_t = a)$ and $P_\delta(S_t = s, O_t = o, A_t = a)$ for all $s$ in $X_S$, $o$ in $X_O$, and $a$ in $X_A$ and $t$ in $[T]$. Let $v^*$ and $z^*$ respectively denote the optimal values of Problem (1) and Problem (2). The following theorem ensures that MILP (2) models Problem (1).

**Theorem 1.** Let $(\mu, \delta)$ be a feasible solution of Problem (2). Then $\mu$ is equal to the distribution $P_\delta$ induced by $\delta$, and $(\mu, \delta)$ is an optimal solution of Problem (2) if, and only if $\delta$ is an optimal deterministic policy of Problem (1). Furthermore, $v^* = z^*$.

### 2.3 Valid inequalities

The difficulty of POMDP comes from the fact that action variable $A_t$ is independent from state $S_t$ given observation $O_t$, which induces non-linearities. A corollary of these independences is that

$$A_t \text{ is independent from } S_t \text{ given } O_t, A_{t-1} \text{ and } S_{t-1}. \quad (4)$$

Theorem (1) ensures that these independences are satisfied by the distribution $\mu$ corresponding to an integer solution $(\mu, \delta)$ of MILP (2). If the component $\mu$ of a solution $(\mu, \delta)$ of the linear relaxation of MILP (2) can still be interpreted as a distribution, independences (3) and (4) are unfortunately no longer satisfied according to this distribution. We now introduce an extended formulation and valid inequalities that enable to restore independences (4).

We introduce new variables $\mu_{sa}^{t,a,o,sa}$ that can be interpreted as the probabilities

$$P(S_{t-1} = s', A_{t-1} = a', S_t = s, O_t = o, A_t = a).$$
Consider the following equalities

\( \sum_{s' \in X, a' \in A} \mu_{s'a'\text{soa}}^{t} = \mu_{\text{soa}}^{t}, \quad \forall s \in X_{S}, o \in X_{O}, a \in X_{A}, \quad (5a) \)

\( \sum_{a \in X_{A}} \mu_{s'a'\text{soa}}^{t} = p(o|s)p(s|s', a')\mu_{s'a'}^{t-1}, \quad \forall s', s \in X_{S}, o \in X_{O}, a' \in X_{A}, \quad (5b) \)

\( \mu_{s'a'\text{soa}}^{t} = p(s|s', a', o) \sum_{\pi \in \mathcal{A}} \mu_{s'a''\text{soa}}^{t}, \quad \forall s', s \in X_{S}, o \in X_{O}, a', a' \in X_{A}, \quad (5c) \)

where

\( p(s'|s', a', o) = \mathbb{P}(S_{t} = s|S_{t-1} = s', A_{t-1} = a', O_{t} = o). \)

Note that \( p(s'|s', a', o') \) does not depend on the policy \( \delta \) and can be easily computed using Bayes rules. Therefore, constraints in (5) are linear.

**Proposition 2.** Equalities (5) are valid for Problem (2), and there exists solution \( \mu \) of the linear relaxation of (2) that does not satisfy constraints (5).

Hence, constraints (5) strengthen the linear relaxation. Numerical experiments in Section 4 show the efficiency of such valid inequalities.

**Remark 1.** Our extended formulation with valid inequalities has many more constraints than the initial one. Its linear relaxation therefore takes longer to solve.

### 2.4 Link with perfect-recall

In the previous section, we assumed the decision maker forgets all history of information. It would be interested to measure the policy obtained by solving Problem (2) against the POMDP with perfect-recall. Let \( v_{PR}^{*} \) be the optimal value of perfect-recall POMDP, i.e., by optimizing the expectation of (1) over \( \Delta_{PR} \). Let \( z_{LR}^{*} \) be the value of the linear relaxation of MILP (2).

**Theorem 3.** The linear relaxation of MILP (2) is an upper bound on the POMDP value with perfect-recall:

\( z^{*} \leq v_{PR}^{*} \leq z_{LR}^{*} \quad (6) \)

The proof is based on the influence diagram representation of POMDP problem. The authors in [Parmentier et al., 2019, e.g. Theorem 14] prove that the linear relaxation of our MILP is equivalent to the MDP relaxation of POMDP. Therefore, the integrity gap obtained by solving MILP (2) bounds the gap with the value of perfect-recall POMDP. By adding equalities (5), we obtain a better gap.

### 3 Decomposable Partially Observed Markov Decision Processes

Due to the curse of dimensionality, some applications have exponentially large \( X_{S} \) and cannot be solved using MILP (2). In this section, we propose a tractable heuristic for POMDPs with large but decomposable state space \( X_{S} \).
3.1 Problem settings

We now introduce the notion of decomposable POMDP. We consider a system that can be decomposed into $M$ components, so that the system state space $\mathcal{X}_S$ and observation space $\mathcal{X}_O$ can be written as the Cartesian product of individual state spaces $\mathcal{X}_S^m$ and observation spaces $\mathcal{X}_O^m$ for $m \in [M]$, i.e., $\mathcal{X}_S = \mathcal{X}_S^1 \times \cdots \times \mathcal{X}_S^M$ and $\mathcal{X}_O = \mathcal{X}_O^1 \times \cdots \times \mathcal{X}_O^M$. Let $S_t^m$ and $O_t^m$ be the random variables that represent the state and the observation of component $m \in [M]$ at time $t \in [T]$, and $S_t = (S_1^t, \ldots, S_M^t)$ and $O_t = (O_1^t, \ldots, O_M^t)$ the state and observation of the complete system. At each time step $t$, a component $m$ is in state $S_t^m = s$, and emits an observation $O_t^m = o$ with probability $\mathbb{P}(O_t^m = o | S_t^m = s) = p_m(o | s)$. Then, based on the observations of the full system $O_t = (O_1^t, \ldots, O_M^t)$, the decision maker takes an action $A_t = a$. Each component $m$ then evolves independently from state $S_t^m = s$ to state $S_{t+1}^m = s'$ with probability $p_m(s' | s, a)$, and the decision maker perceives reward $r_m(s, a, s')$.

To sum things up, a decomposable POMDP is a POMDP with state space $\mathcal{X}_S = \mathcal{X}_S^1 \times \cdots \times \mathcal{X}_S^M$, observation space $\mathcal{X}_O = \mathcal{X}_O^1 \times \cdots \times \mathcal{X}_O^M$, action space $\mathcal{X}_A$, and such that the probabilities of emission factorize as

$$\mathbb{P}(O_t = o | S_t = s) = \prod_{m=1}^M p_m(o^m | s^m), \quad \forall t \in [T],$$

the probabilities of transition factorize as

$$\mathbb{P}(S_1 = s) = \prod_{m=1}^M p_m(s^m), \quad \text{and} \quad \mathbb{P}(S_{t+1} = s' | S_t = s, A_t = a) = \prod_{m=1}^M p_m(s'^m | s^m, a),$$

and the reward decomposes additively

$$r(s, a, s') = \sum_{m=1}^M r_m(s^m, a, s'^m).$$

The goal is still to find a policy maximizing the total expected reward

$$\max_{\delta \in \Delta} \mathbb{E}_\delta \left[ \sum_{t=1}^T \sum_{m=1}^M r_m(S_t^m, A_t, S_{t+1}^m) \right]. \quad (7)$$

3.2 Linear formulation

As a decomposable POMDP is a POMDP, we can solve Problem (7) using MILP (2). However, the number of constraints and variables grows exponentially with $M$, and even the linear relaxation of MILP (2) becomes quickly intractable. We propose a heuristic that repeatedly solves a tractable linear program. We introduce new variables $\tau = \left( \tau_{s}^{t,m}, \tau_{sa}^{t,m}, \tau_{soa}^{t,m} \right)_{m \in [M], t}$ and the following linear program.
\[
\max_{\tau} \quad \sum_{t=1}^{T} \sum_{m=1}^{M} \sum_{s',a,m} r_{m}(s'^m,a,s^m)p(s'^m|s^m,a)\tau_{s'a}^{t,m} \tag{8a}
\]

s.t. \[ \tau_{s'a}^{t,m} = \sum_{s \in X^m_S, a \in X_A} \tau_{sa}^{t,m} \quad \forall s \in X^m_S, a \in X_A, m \in [M], t \in [T] \tag{8b} \]
\[ \tau_{s'a}^{t,m} = \sum_{o \in X^m_O} \tau_{soa}^{t,m} \quad \forall s \in X^m_S, a \in X_A, m \in [M], t \in [T] \tag{8c} \]
\[ \sum_{s \in X^m_S} \tau_{sa}^{t,m} = \tau_{sa}^{t} \quad \forall a \in X_A, m \in [M], t \in [T] \tag{8d} \]
\[ \sum_{a \in X_A} \tau_{sa}^{t,m} = \tau_{s}^{t,m} \quad \forall s \in X^m_S, o \in X^m_O, m \in [M], t \in [T] \tag{8e} \]
\[ \tau_{s}^{t,m} = p_m(s) \quad \forall s \in X^m_S, m \in [M] \tag{8f} \]
\[ \tau \geq 0 \tag{8g} \]

We show in Appendix that \((\tau_{s'}^{t,m})_{s',a}, (\tau_{s}^{t,m})_{s,a}, (\tau_{soa}^{t,m})_{soa}, \text{ and } (\tau_{s}^{t})_{a}\) can still be interpreted as probability distributions on \(X_S, X_{S,m} \times X_A, X_{S,m} \times X_{O,m} \times X_A, \) and \(X_A,\) which coincide when marginalized on the intersection of their domains. However, there is no guarantee that there exists a joint distribution on \((X_S \times X_O \times X_A)^T\) from which they can be derived as marginal distributions. Let \(v^*_M\) and \(z^*_M\) be respectively the optimal values of Problem (7) and Problem (8).

**Theorem 4.** Problem (8) is a relaxation of Problem (7), and \(v^*_M \leq z^*_M.\)

This inequality is not an equality in general, as we will see in the numerical experiments. Linear formulation (8) is a generalization to decomposable POMDPs of the “fluid” formulation proposed by Bertsimas and Misić [2016] for decomposable MDPs.

The quality of the bound \(z^*_M\) can be improved by generalizing the valid inequalities introduced in Section 2.3. We introduce new variables \(\tau_{s'a' soa}^{t,m}\), that can be interpreted as the probability

\[ P(\ell_{t-1} = s', A_{t-1} = a', S_t = s, O_t^m = o, A_t = a). \]

Consider the following linear inequalities.

\[ \sum_{s' \in X_S, a' \in X_A} \tau_{s'a' soa}^{t,m} = \tau_{soa}^{t,m}, \quad \forall s \in X^m_S, o \in X^m_O, a \in X_A, \tag{9a} \]
\[ \sum_{a \in X_A} \tau_{soa}^{t,m} = p_m(o|s)p_m(s'|s,a')\tau_{s'a}^{t-1,m}, \quad \forall s, s' \in X^m_S, o \in X^m_O, a \in X_A, \tag{9b} \]
\[ \tau_{s'a' soa}^{t,m} = p_m(s'|s,a') \sum_{s' \in X_S, a' \in X_A} \tau_{s'a}^{t,m}, \quad \forall s, s' \in X^m_S, o \in X^m_O, a, a' \in X_A. \tag{9c} \]

**Proposition 5.** Theorem 4 remains true if we add equalities (9) to (8) for all \(m \in [M]\) and \(t \in [T].\) There are solutions of (8) that do not satisfy (9).

Since decomposable POMDPs are POMDPs, we would like to compare our policies with optimal value of POMDP with perfect-recall. Let \(v^*_{PR}\) be the optimal value of POMDP with perfect-recall, i.e., the value of the expectation in (7) by optimizing over \(\Delta_{PR}\).

**Theorem 6.** The linear formulation (8) gives an upper bound on the POMDP value with perfect-recall, i.e., \(v^*_{PR} \leq z^*_M.\)
3.3 Heuristic

Consider an optimal solution $\tau$ of Problem (8). Solution $\tau$ is achievable if there exists a policy $\delta \in \Delta$ such that

$$\delta^t_{o|\tilde{o}} = \frac{\tau^t_m}{\tau_{soa}^t}, \quad \forall s \in X_S^n, o \in X_O, a \in X_A, m \in [M], t \in [T].$$

(10)

The following proposition generalizes Proposition 3 of Bertsimas and Misić [2016].

**Proposition 7.** Let $\tau$ be an optimal solution of Problem (8). If $\tau$ is achievable, then $v^*_M = z^*_M$.

In the remainder of this section, we assume that the decision maker observes the initial observation $\tilde{o}$ in $X_O$. This assumption imposes conditioning all probabilities on the event $\{O_1 = \tilde{o}\}$ in Problem (7) and Problem (8). Therefore, it suffices to replace $p_m(o|s)$ by $1_{y(o)}$ at time $t = 1$ where $1_y(x)$ is the indicator function of $y$ evaluated in $x$.

**Theorem 8.** Let $\tau$ be an optimal solution of Problem (8) with an initial observation $\tilde{o} \in X_O$. Suppose that $\tau$ is achievable and let $\delta$ be the policy achieving $\tau$. We define the initial policy $\delta^1$:

$$\delta^1_{o|\tilde{o}} = 1 \text{ if } a^* = \arg\max_{a \in X_A} \tau^1_{a}$$

(11)

Then the policy $(\delta^1, \delta^2, \ldots, \delta^T)$ is optimal for Problem (7).

Under condition (10), Theorem 8 says that we can take an optimal deterministic action at time $t = 1$. However, an optimal solution $\tau$ of Problem (8) is not always achievable. Nevertheless, a feasible solution of Problem (8) satisfying the valid inequalities (9) respects the transition of all independent components and almost all independences between random variables. Therefore, $\tau$ may be close to being achievable. Consequently, we propose a heuristic similar to the algorithm presented in Bertsimas and Misić [2016], that repeatedly solves Problem (8) and selects action $\arg\max_{a \in X_A} \tau^1_{a}$ (that depends on initial observation $\tilde{o}$). Algorithm 1 states our heuristic policy, which we expect to provide good performances.

**Algorithm 1** Heuristic policy for Problem (7)

1: Input $T, p_m, r_m$ for all $m \in [M]$, current observation $\tilde{o} \in X_O$
2: Solve Problem (8) with initial observation $\tilde{o}_t$ to obtain an optimal solution $\tau$.
3: Take action $a^* = \arg\max_{a \in X_A} \tau^1_{a}$.

Note that each iteration of Algorithm 1 solves a linear program with a polynomial number of constraints and variables. Numerical experiments in Section 4 show the efficiency of this heuristic.

4 Numerical experiments

We now provide experiments showing the practical efficiency of our approaches to POMDPs and decomposable POMDPs. All linear programs have been implemented in Julia with JuMP interface and solved using Gurobi 7.5.2. Experiments have been run on a server with 192Gb of RAM and 32 cores at 3.30GHz.
| $|\mathcal{X}_S|$ | $|\mathcal{X}_O|$ | $|\mathcal{X}_A|$ | $T$ | Nb. Policies | Prog. | Int. Gap (%) | Final Gap (%) | Time (s) |
|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 10 | $10^4$ | (2) and (5) | 3.57 | Opt | 0.85 |
| | | | | | (2) and (5) | 0.43 | Opt | 0.16 |
| 20 | $10^{28}$ | (2) | 3.53 | Opt | 11.79 |
| | | | | | (2) and (5) | 0.22 | Opt | 0.46 |
| 3 | 4 | 4 | 10 | $10^{24}$ | (2) | 2.74 | Opt | 1.82 |
| | | | | | (2) and (5) | 0.61 | Opt | 1.35 |
| 20 | $10^{28}$ | (2) | 2.67 | Opt | 326.71 |
| | | | | | (2) and (5) | 0.53 | Opt | 5.85 |
| 3 | 5 | 5 | 10 | $10^{34}$ | (2) | 8.88 | Opt | 12 |
| | | | | | (2) and (5) | 2.61 | Opt | 3.58 |
| 20 | $10^{50}$ | (2) | 9.06 | Opt | 2.56 |
| | | | | | (2) and (5) | 2.45 | Opt | 116.54 |
| 4 | 8 | 8 | 10 | $10^{72}$ | (2) and (5) | 15.50 | 8.64 |
| | | | | | (2) and (5) | 1.77 | Opt | 383.32 |
| 20 | $10^{134}$ | (2) | 15.64 | 12.57 |
| | | | | | (2) and (5) | 1.29 | 0.62 |

Table 1: POMDP results using MILP (2) with and without (5), with a time limit TL=3600s

### 4.1 Instances

Each instance is generated by first choosing $|\mathcal{X}_S|$, $|\mathcal{X}_O|$, $|\mathcal{X}_A|$, and finally $M$ when we consider decomposable MDPs. We then randomly generate the initial probability $p_m(s)$, the transition probability $p_m(s'|s,a)$, the emission probability $p_m(o|s)$ and the immediate reward function $r_m(s,a,s')$ for all $m \in [M]$.

**Remark 2.** In our numerical experiments, we use instances with short horizons $T$ in $\{5, 10, 20\}$. This is not a limitation in practice, as the length $T$ of the horizon is not the main challenge when designing heuristics for decision processes. Bertsimas and Misic [2016] solve large or infinite horizon instances of decomposable MDPs using rolling horizon approaches. They obtain good performances by solving at each time steps problems on a horizon $T = 10$. Similar rolling techniques can be used to adapt Algorithm 1 to large or infinite horizon decomposable POMDPs.

### 4.2 Simulated experiments on single POMDP

We solve Problem (2) with and without valid equalities (5). Algorithms were stopped after a Time Limit (TL) of 600s. Table 1 shows the efficiency of MILP (2) to solve (1). The first four columns indicate the size of state space $|\mathcal{X}_S|$, observation space $|\mathcal{X}_O|$, action space $|\mathcal{X}_A|$ and time horizon $T$. The fifth column indicates the mathematical program used to solve Problem (2) with or without constraints (5). In the last three columns, we report the integrity gap, the final gap and the computation time for each instance.

### 4.3 Simulated experiments on decomposable POMDPs

We now compare two heuristics to solve decomposable POMDPs: Algorithm 1 solving Problem (8) without valid inequalities (9), and a greedy algorithm, which infers the most probable state and takes the action maximizing the expected immediate reward of the next state. On each instance, we compute an upper bound $z_{M,k}$ on the optimal value by solving Problem (8) with constraints (9). We test our two heuristics under $K = 100$ different scenarios.
Table 2 summarizes the results obtained. The first five columns describe the instance. They indicate the sizes of state space $|\mathcal{X}_S|$, observation space $|\mathcal{X}_O|$, and action space $|\mathcal{X}_A|$, the horizon time $T$, and the number of components $M$. The sixth column indicates the heuristic used. The next column provides the average computation time needed at each time step to take a decision. When using Algorithm 1, this is the time needed to solve MILP (8). Finally, column “Av. gap (%)” provides the gap

$$100 \times \frac{z^*_{M,k} - R_{k,h}}{z^*_{M,k}},$$

between the upper bound $z^*_{M,k}$ and total reward $R_{k,h}$ obtained using the heuristic on average on the $K$ scenarios.

| $M$ | $|\mathcal{X}_S|$ | $|\mathcal{X}_O|$ | $|\mathcal{X}_A|$ | $T$ | Heuristic (h) | CPU time (s) | Av. gap (%) |
|-----|-----------------|-----------------|-----------------|-----|----------------|--------------|--------------|
| 3   | 5               | 5               | 5               | 5   | Greedy         | 0.00         | 22.43        |
|     |                 |                 |                 |     | Alg. (1)       | 2.52e-2      | 4.95         |
| 10  |                 |                 |                 |     | Greedy         | 0.00         | 21.17        |
|     |                 |                 |                 |     | Alg. (1)       | 8.07e-2      | 4.00         |
| 4   | 3               | 5               | 5               | 5   | Greedy         | 0.00         | 18.53        |
|     |                 |                 |                 |     | Alg. (1)       | 2.17e-2      | 8.73         |
| 10  |                 |                 |                 |     | Greedy         | 0.00         | 18.40        |
|     |                 |                 |                 |     | Alg. (1)       | 7.66e-2      | 7.97         |
| 5   | 5               | 5               | 5               | 5   | Greedy         | 0.00         | 17.3         |
|     |                 |                 |                 |     | Alg. (1)       | $5.35 \times 10^{-2}$ | 7.24 |
| 10  |                 |                 |                 |     | Greedy         | 0.00         | 17.9         |
|     |                 |                 |                 |     | Alg. (1)       | $1.73 \times 10^{-1}$ | 6.11 |
| 10  | 5               | 5               | 5               | 5   | Greedy         | 0.00         | 14.7         |
|     |                 |                 |                 |     | Alg. (1)       | $1.69 \times 10^{-1}$ | 5.01 |

Table 2: Heuristic performances on decomposable POMDPs ($M > 1$)

Note that when the number of components grows, Algorithm 1 outperforms the standard greedy algorithm.

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A Proofs

**Proof of Theorem 1.** Let \((\mu, \delta)\) be a feasible solution of Problem (2). We prove by induction on \(t\) that \(\mu_t^s = \mathbb{P}_\delta(S_t = s)\), \(\mu_t^{sa} = \mathbb{P}_\delta(S_t = s, A_t = a)\) and \(\mu_t^{soa} = \mathbb{P}_\delta(S_t = s, O_t = o, A_t = a)\).

At time \(t = 0\), the statement is immediate. Suppose that it holds up to \(t - 1\). Constraints (2a) and induction hypothesis ensure that

\[
\mu_t^s = \sum_{s',a'} p(s'|s,a') \mathbb{P}_\delta(S_{t-1} = s', A_{t-1} = a)
= \mathbb{P}_\delta(S_t = s)
\]

where the last equality comes from the law of total probability. Since \(\delta_{a|o}^t\) are binary, constraints (2a), (2b) and (2c) ensure that:

\[
\mu_{t}^{soa} = \delta_{a|o}^t p(o|s) \mu_t^s
= \mathbb{P}_\delta(S_t = s, O_t = o, A_t = a).
\]

Finally, constraints (2d) ensure that \(\mu_t^{sa} = \mathbb{P}_\delta(S_t = s, A_t = a)\). Therefore, any feasible solution \(\mu\) of (2) is equal to the distribution \(\mathbb{P}_\delta\). Consequently,

\[
\sum_{t=1}^{T} \sum_{s,s' \in X_S} \sum_{a \in A_A} r(s,a,s') p(s'|s,a) \mu_t^{sa} = \mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(S_t, A_t, S_{t+1}) \right],
\]

which implies that \(\delta\) is optimal if and only if \((\mu, \delta)\) is optimal for Problem (2) and \(v^* = z^*\).

\[\square\]
Proof of the validity of equalities\(\text{[5]}\). Let \((\bm{\mu}, \bm{\delta})\) be a feasible solution of problem \((2)\). We define
\[
\mu_{s'a'sao}^{t} = \delta_{a'|a}^{t} p(o|s)p(s'|s, a') \mu_{s'a}^{t-1}
\]
for all \((s', a', s, o, a) \in \mathcal{X}_S \times \mathcal{X}_A \times \mathcal{X}_S \times \mathcal{X}_O \times \mathcal{X}_A\), \(t \in [T]\). These new variables satisfy constraints in \((5)\):
\[
\sum_{a \in \mathcal{X}_A} \mu_{s'a'sao}^{t} = \left( \sum_{a \in \mathcal{X}_A} \delta_{a'|a}^{t} p(o|s)p(s'|s, a') \mu_{s'a}^{t-1} \right) = p(o|s)p(s'|s, a') \mu_{s'a}^{t-1}
\]
\[
\sum_{a' \in \mathcal{X}_A, s' \in \mathcal{X}_S} \mu_{s'a'sao}^{t} = \left( \sum_{a' \in \mathcal{X}_A, s' \in \mathcal{X}_S} p(s'|s, a') \mu_{s'a}^{t-1} \right) \delta_{a'|a}^{t} p(o|s) = \mu_{s'o}^{t} \delta_{a'|a}^{t} p(o|s) = \mu_{s'o}^{t}
\]
The remaining constraint \((5c)\) is obtained using the following observation:
\[
\sum_{s \in \mathcal{X}_S} \mu_{s'a'sao}^{t} = \frac{p(o|s)p(s'|s, a')}{\sum_{s \in \mathcal{X}_S} p(o|s)p(s'|s, a')}
\]
By setting \(p(s'|s, a', o) = \sum_{s \in \mathcal{X}_S} p(o|s)p(s'|s, a')\), equality \((5c)\) holds. \(\Box\)

### A.1 Decomposable Partially Observed Markov Decision Processes

Proof of Theorem \([4]\). We prove that an optimal solution \((\bm{\mu}, \bm{\delta})\) of Problem \((2)\) is a feasible solution of Problem \((3)\). For each \(m \in [M]\), we define \(\tau_{s,m}^{t} = \sum_{s^{-m} \in \prod_{j \neq m, i} \mathcal{X}_S} \mu_{s}^{t}\) where \(s^{-m}\) is the vector \(s\) without the \(m\)-th coordinate corresponding to component \(m\). Similarly, we define \(\tau_{sa}^{t} = \sum_{s^{-m} \in \prod_{j \neq m, i} \mathcal{X}_S} \mu_{s}^{t}\) and \(\mu_{s}^{t} = \sum_{s \in \mathcal{X}_S} \mu_{s}^{t}\). This solution is indeed a feasible solution of Problem \((3)\). \(\Box\)

Proof of Proposition \([7]\). We prove that we can build a feasible solution of Problem \((2)\) for \(M\) components. We build such a solution by induction on \(t\). For \(t = 1\), we define \(\mu_{s}^{1} = \prod_{m=1}^{M} \tau_{s,m}^{1,}\), \(\mu_{so}^{1} = p(o|s)\mu_{s}^{1}\), \(\mu_{so}^{1} = \delta_{a'|a}^{1} \mu_{so}^{1}\) and \(\mu_{sa}^{1} = \sum_{o \in \mathcal{X}_O} \mu_{so}^{1}\). We define the following induction equation:
\[
\mu_{s'}^{t+1} = \sum_{s \in \mathcal{X}_S} \sum_{a \in \mathcal{X}_A} p(s'|s, a) \mu_{sa}^{t}
\]
\[
\mu_{so}^{t+1} = p(o|s) \mu_{s}^{t+1}
\]
\[
\mu_{so}^{t+1} = \delta_{a'|a}^{t} \mu_{so}^{t+1}
\]
\[
\mu_{sa}^{t+1} = \sum_{o \in \mathcal{X}_O} \mu_{so}^{t+1}
\]
It is easy to observe that all constraints of problem \((2)\) are satisfied. Therefore, we build a feasible solution of Problem \((2)\) with the same expected reward. Since Problem \((7)\) can be
exactly solved by Problem (2) by considering the complete system and the probabilities over \( X_S \) and \( X_O \). Using Proposition (1), there is equality \( z_M = v_M \). \(\square\)

**Proof of Theorem (8).** We define \( \delta^* = (\delta, \delta^2, \ldots, \delta^T) \). We prove that:

\[
\mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta \right] = \mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta \right]
\]

We have:

\[
\mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta \right] = \mathbb{E}_\delta \left[ r(s_1, a_1, s_2) | o_1 = \delta \right] + \mathbb{E}_\delta \left[ \sum_{t=2}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta \right]
\]

\[
= \sum_{a \in A} \delta_{a|\delta} \sum_{s,s' \in X_S} p(s'|s,a) p(\delta'|s)p(s) \left( r(s,a,s') + \mathbb{E}_\delta \left[ \sum_{t=2}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta, s_2 = s' \right] \right)
\]

where

\[
\alpha(a, \delta) = \sum_{s,s' \in X_S} p(s'|s,a) p(\delta'|s)p(s) \left( r(s,a,s') + \mathbb{E}_\delta \left[ \sum_{t=2}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta, s_2 = s' \right] \right)
\]

Since all variables \((S_t, O_t, A_t)_{2 \leq t \leq T+1}\) are conditionally independent from \((S_1, O_1, A_1)\) given \( S_2 \) we have:

\[
\alpha(a, \delta) = \sum_{s,s' \in X_S} p(s'|s,a) p(\delta'|s)p(s) \left( r(s,a,s') + \mathbb{E}_{\delta^{-1}} \left[ \sum_{t=2}^{T} r(s_t, a_t, s_{t+1}) | s_2 = s' \right] \right)
\]

where \( \delta^{-1} = (\delta^2, \ldots, \delta^T) \). Therefore, \( \alpha \) does not depend on \( \delta^1 \). Note that:

\[
\tau_{a}^{1} = \sum_{s \in X_{S}^{m}, \sigma \in X_{\sigma}^{m}} \tau_{s,\sigma,0}^{1, m} \tau_{a}^{1, m}
\]

\[
= \sum_{s \in X_{S}^{m}} \tau_{s,\delta,0}^{1, m} \tau_{a}^{1, m}
\]

\[
= \delta_{a|\delta} \sum_{s \in X_{S}^{m}, \sigma \in X_{\sigma}^{m}} \tau_{a}^{1, m}
\]

\[
= \delta_{a|\delta} \tau_{a}^{1}
\]

where the last equality holds because the decision maker observes the initial observation. Therefore, we obtain:

\[
\mathbb{E}_\delta \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \delta \right] = \sum_{a \in A} \alpha(a, \delta) \tau_{a}^{1}
\]

\[
= \max_{a \in A} \alpha(a, \delta)
\]
where the last equality holds because the right-hand side is always greater than the left-hand side since \( \sum_{a \in \mathcal{X}_A} \tau^1_a = 1 \) and if \( \tau^1_a > 0 \) then \( a \in \arg\max_{a \in \mathcal{X}_A} \alpha(a, \tilde{o}) \). Indeed, suppose that \( \tau^1_a > 0 \) and \( a \notin \arg\max_{a \in \mathcal{X}_A} \alpha(a, \tilde{o}) \). Let \( a^* \in \arg\max_{a \in \mathcal{X}_A} \alpha(a, \tilde{o}) \), then the solution \( \tilde{\tau}_a = 0 \) and \( \tilde{\tau}_{a^*} = \tau_{a^*} + \tau_a \). Therefore,
\[
\sum_{a \in \mathcal{X}_A} \tau^1_a \alpha(a, \tilde{o}) < \sum_{a \in \mathcal{X}_A} \tilde{\tau}^1_a \alpha(a, \tilde{o})
\]
which contradicts the optimality assumption of \( \tau \). We deduce that:
\[
\mathbb{E}_d \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \tilde{o} \right] = \mathbb{E}_{d^*} \left[ \sum_{t=1}^{T} r(s_t, a_t, s_{t+1}) | o_1 = \tilde{o} \right]
\]
because \( \max_{a \in \mathcal{X}_A} \tau^1_a > 0 \) since \( \sum_{a \in \mathcal{X}_A} \tau^1_a = 1 \). \( \square \)