CHERN-SIMONS THEORY IN THE $SO(5)/U(2)$ HARMONIC SUPER-SPACE

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We consider the superspace of $D=3, N=5$ supersymmetry using $SO(5)/U(2)$ harmonic coordinates. Three analytic $N=5$ gauge superfields depend on three vector and six harmonic bosonic coordinates and also on six Grassmann coordinates. Decomposition of these superfields in Grassmann and harmonic coordinates yields infinite-dimensional supermultiplets including a three-dimensional gauge Chern-Simons field and auxiliary bosonic and fermionic fields carrying $SO(5)$ vector indices. The superfield action of this theory is invariant with respect to $D=3, N=6$ conformal supersymmetry realized on $N=5$ superfields.

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1 Introduction

The three-dimensional gauge Chern-Simons (CS) theory [1, 2] is an interesting example of the topological field theory closely connected with three-dimensional gravity, two-dimensional conformal theories, and topological strings [3, 4]. In the three-dimensional space with the coordinates $x^m$, the Chern-Simons action has the form

$$S_{CS} = \frac{k}{4\pi} \int d^3 x \varepsilon^{mnr} \text{Tr} \{ A_m (\partial_n A_r + \frac{i}{3} [A_n, A_r]) \}, \quad (1.1)$$

where $m, n, r = 0, 1, 2$, $\varepsilon^{mnr}$ is the antisymmetric symbol, and $A_m(x)$ is the non-Abelian gauge field. This action corresponds to the classical solution with zero field-strength of the gauge field. The functional $S_{CS}$ is not defined uniquely. It is invariant only with respect to infinitesimal gauge transformations, but the quantization of the constant $k$ guarantees the uniqueness of the quantity $\exp(iS_{CS})$ in the path integral of the quantum field theory. We represent the Chern-Simons action as an integral of the differential form

$$S_{CS} = \frac{k}{4\pi} \int \text{Tr} \{ dA + \frac{2i}{3} A^3 \}, \quad (1.2)$$

where $A = dx^m A_m$ and $d = dx^m \partial_m$. It follows from this representation that the action is independent on the metric of the three-dimensional space. The term $S_{CS}$ can be regarded as an additional interaction of the gauge field $A_m$ in the generalized three-dimensional Yang-Mills-Chern-Simons theory. In this generalized gauge theory, $S_{CS}$ plays the role of the topological mass term.

Supersymmetric extensions of the $D=3$ Chern-Simons theory were discussed in [5]-[19]. The simplest supersymmetric Chern-Simons theory was constructed in the $D=3, N=1$ superspace with the real coordinates $z = (x^m, \theta^\alpha)$, where the Grassmann coordinate $\theta^\alpha$ has the spinor index $\alpha = 1, 2$ of the group $SL(2, \mathbb{R})$ [5, 6]. The superfield action of this theory is defined on the gauge spinor superfield $A_\alpha(z)$, which contains the vector field $A_m(x)$ and the spinor field $\psi_\alpha(x)$ in the adjoint representation of the gauge group. In addition to (1.1), the component action of the $N=1$ Chern-Simons theory includes the
term $\int d^3x \text{Tr} \psi^\alpha \psi_\alpha$. Hence, the fermion $\psi_\alpha$ is an auxiliary field in the $N=1$ Chern-Simons theory, although this field can describe physical degrees of freedom in the generalized model with the additional supersymmetric Yang-Mills interaction. The superfield action of the $D=3, N=1$ Chern-Simons theory can be interpreted as the superspace integral of the differential Chern-Simons superform $dA + \frac{2}{3}A^3$ in the framework of our theory of superfield integral forms [7]-[10].

The Abelian $D=3, N=2$ CS action was first constructed in the $D=3, N=1$ superspace [5]. The corresponding non-Abelian action was considered in the $D=3, N=2$ superspace with the help of the Hermitian superfield $V(x^m, \theta^\alpha, \bar{\theta}^\dot{\alpha})$, where $\theta^\alpha$ and $\bar{\theta}^\dot{\alpha}$ are the complex conjugated spinor coordinates [7, 11, 12]. In the $N=2$ component-field Lagrangian, bilinear terms with fermionic and scalar fields without derivatives are added to the bosonic action. The unusual dualized form of the $N=2$ CS action contains the second vector field instead of the scalar field[12].

The $D=3, N=3$ CS theory was first analyzed by the harmonic-superspace method [13, 14]. Using the $SU(2)/U(1)$ harmonics [20] in the superspace with three spinor coordinates $\theta^\alpha_k$ allows us to construct the analytic superspace with two spinor coordinates $\theta^{++}_k$ and $\theta^0$, where the fourth supersymmetry can also be realized. The vector $N=3, 4$ supermultiplet is described by the analytic superfield $V^{++}$ of dimension zero, and the corresponding analytic superfield strength $W^{++} = D^{++}A^{++}V^{--}$ has the dimension one and is expressed via spinor derivatives of the harmonic connection $V^{--}$. It is convenient to build the superfield action of the $N=3$ Chern-Simons theory in the full $D=3, N=3$ by analogy with the action of the $D=4, N=2$ Yang-Mills theory in the corresponding harmonic superspace [15, 22]. The variation of the action of the $CS^3_3$-theory can be presented in the full or analytic superspaces

$$\delta S_3^3 = \int d^3x du d^4\theta \text{Tr} \left\{ \delta V^{++}V^{--} \right\} = \int d^3x du d^{(-4)}\theta \text{Tr} \left\{ \delta V^{++}W^{++} \right\} = 0,$$  \hspace{1cm} (1.3)$$

where the measure $d^{(-4)}\theta$ contains derivatives in four analytic Grassmann coordinates. We note that the equations of motion of the $CS^3_3$-theory $W^{++} = 0$ are covariant with respect to the fourth supersymmetry, although the action possesses only three supersymmetries. The supersymmetric action of the $D=3, N=4$ Yang-Mills theory can also be constructed in the $D=3, N=3$ superspace, but the alternative formalism exists in the $N=4$ superspace [23]. The field-component form of the action of the $N=3$ Chern-Simons theory was considered in [16, 17].

The component action of the $D=3, N=8$ Yang-Mills theory is obtained by a dimensional reduction of the $D=4, N=4$ Yang-Mills theory. The action 16 supercharges is defined on the mass shell of these theories. A superfield version of the $D=4, N=3$ Yang-Mills theory $SYM^3_4$ was constructed in the $SU(3)/U(1) \times U(1)$ harmonic superspace [21, 22]; this theory is invariant with respect to 12 supercharges only off the mass shell. It is easy to construct a dimensional reduction of this Yang-Mills theory to the corresponding three-dimensional superspace, but it is unclear whether the Chern-Simons interaction can be included in this superspace. We should say that the integration measure of the three-dimensional variant of the analytic $SU(3)/U(1) \times U(1)$ superspace has the dimension one.

Using the $SO(5)/U(2)$ harmonic superspace for the superfield description of the $D=3, N=5$ Chern-Simons theory was proposed in [24]. An alternative formalism of this theory using the $SO(5)/U(1) \times U(1)$ harmonics and additional harmonic conditions was considered in
where it was shown that the action of this model is invariant under the $D=3, N=6$ superconformal group. It is useful to analyze the superfield formalism of the $N=6$ Chern-Simons theory in terms of the $SO(5)/U(2)$ harmonics in detail.

In Sec. 2, we consider the $SO(5)/U(2)$ harmonics and the corresponding harmonic derivatives. We define six Grassmann coordinates of the analytic superspace using these harmonics and the ten Grassmann coordinates of the full $N=5$ superspace. The coordinates of the analytic superspace are considered real under a special conjugation. Three harmonic derivatives preserve the Grassmann analyticity condition in the harmonic superspace. The integration measure in the analytic $N=5$ superspace has the dimension zero compared with the dimension two of the integration measure in the full $D=3, N=5$ superspace. This property of the measure is very important in constructing the superfield action of the theory.

In Sec. 3, we define the group of analytic gauge transformations using the harmonic condition of the $U(2)$ invariance for the superfield parameters of these transformations. Basic gauge superfields of the theory (prepotentials) satisfy the conditions of the Grassmann analyticity and the harmonic $U(2)$ covariance. Reality conditions for the gauge group and gauge superfields include the special conjugation of the analytic superspace. We construct the superfield action of the theory in the analytic $D=3, N=5$ superspace [24, 25]. One gauge superfield can be constructed from two other prepotentials; the action of the theory contains two independent superfields in this representation. The classical equations of motion have only pure gauge solutions for the prepotentials in analogy with the $N=1, 2, 3$ superfield Chern-Simons theories. The superfield action of the theory is invariant under transformations of the $D=3, N=5$ superconformal group realized on the prepotentials. Additional transformations of the sixth supersymmetry are defined using the spinor derivative preserving the Grassmann analyticity and $U(2)$ covariance.

In Sec. 4, we study the component structure of the $D=3, N=6$ Chern-Simons theory. We choose the supersymmetric gauge of the $U(2)$ covariant prepotentials analogous to gauges of the Wess-Zumino (WZ) type in other superfield theories. The basic gauge superfield in this gauge includes the vector field $A_m$ and the fermionic field $\psi_\alpha$ in the $SO(5)$ invariant sector and also an infinite number of fermionic and bosonic fields with $SO(5)$ indices in the harmonic and Grassmann decompositions. The component Lagrangian contains the Chern-Simons term for $A_m$ and also simple bilinear and trilinear interactions of other bosonic and fermionic fields. Both the field strength of the gauge field and all other fields vanish on the mass shell defined by the classical equations of motion.

In Sec. 5, we discuss another variant of the superfield action for the Abelian $N=5$ prepotential with the coupling constant of a nontrivial dimension. This action generates the ordinary Maxwell Lagrangian of the vector field and other interactions of physical and auxiliary fields. We consider the $D=3, N=5$ superconformal transformations in the full and analytic superspaces in appendix. We also define the superconformal transformations of the integral measure and harmonic derivatives in the analytic superspace.

The alternative formalism of the $D=3, N=6$ Chern-Simons theory using the $SO(5)/U(1) \times U(1)$ harmonics was investigated in [25]. This formalism assumes introducing additional harmonic constraints for the parameters of the gauge group in accordance with the $U(2)$ covariance. The $U(2)$ covariance is guaranteed in the $SO(5)/U(2)$ harmonic superspace. Another version of the $SO(5)/U(1) \times U(1)$ harmonic superspace without additional harmonic constraint was considered in [26]. This model describes the interaction of the Chern-Simons supermultiplet with unusual three-dimensional matter fields.
2 $SO(5)/U(2)$ harmonic superspace

The homogeneous space $SO(5)/U(2)$ is parametrized by elements of the harmonic $5\times5$ matrix

$$U_a^K = (U_a^{+i}, U_a^{0}, U_{ia}^-) = (U_a^{+1}, U_a^{+2}, U_{a1}, U_{a2}),$$

where $a = 1, \ldots, 5$ is the vector index of the group $SO(5)$, $i = 1,2$ is the spinor index of the group $SU(2)$, and $U(1)$-charges are denoted by symbols $+, -, 0$. The basic relations for these harmonics are

$$U_a^{+i}U_a^{+k} = U_a^{+i}U_a^{0} = 0, \quad U_{ia}^-U_{ia}^- = U_{ia}^-U_{ia}^0 = 0, \quad U_a^{+i}U_{ka}^0 = \delta_i^k, \quad U_a^{0}U_a^{0} = 1,$$

$$U_a^{+i}U_{ib}^- + U_{ia}^-U_{ib}^0 = \delta_{ab}. \quad (2.2)$$

We consider the $SO(5)$ invariant harmonic derivatives with nonzero $U(1)$ charges

$$\partial^{+i} = U_a^{+i} \frac{\partial}{\partial U_a^{0}}, \quad \partial^{+i}U_a^{0} = U_a^{+i}, \quad \partial^{+i}U_{ka}^0 = -\delta_i^k U_a^{0};$$

$$\partial^{++} = U_{ia}^+ \frac{\partial}{\partial U_{ia}^0}, \quad [\partial^{+i}, \partial^{+k}] = \epsilon^{ki}\partial^{++}, \quad \partial^{+i}\partial_i^+ = \partial^{++},$$

$$\partial^- = U_{ia}^+ \frac{\partial}{\partial U_{ia}^0}, \quad \partial^-U_a^{0} = U_{ia}^-, \quad \partial^-U_{ia}^0 = -\delta_i^k U_a^{0},$$

$$\partial^{--} = U_{ia}^+ \frac{\partial}{\partial U_{ia}^0}, \quad [\partial_i^-, \partial_k^-] = \epsilon_{ki}\partial^{--}, \quad \partial^{-k}\partial_k^- = -\partial^{--}, \quad (2.3)$$

where some relations between these harmonic derivatives are defined. The $U(1)$ neutral harmonic derivatives form the Lie algebra $U(2)$

$$\partial_i^k = U_a^{+i} \frac{\partial}{\partial U_a^{+k}} - U_{ka}^- \frac{\partial}{\partial U_{ia}^-}, \quad [\partial^{+i}, \partial_i^k] = -\delta_i^k, \quad (2.4)$$

$$\partial^0 \equiv \partial_i^k = U_a^{+k} \frac{\partial}{\partial U_a^{+i}} - U_{ka}^- \frac{\partial}{\partial U_{ka}^0}, \quad [\partial^{++}, \partial^{--}] = \partial^0,$$

$$\partial_i^kU_a^{+i} = \delta_i^k U_a^{+i}, \quad \partial_i^kU_{ia}^- = -\delta_i^k U_{ia}^- \quad (2.5)$$

The operators $\partial^{+k}, \partial^{++}, \partial_i^-, \partial^{--}$ and $\partial_i^k$ satisfy the commutation relations of the Lie algebra $SO(5)$.

An ordinary complex conjugation is defined on these harmonics

$$\overline{U_a^{+i}} = U_{ia}^-, \quad \overline{U_a^{0}} = U_a^{0}, \quad (2.6)$$

but it is more convenient to use a special conjugation in the harmonic space

$$(U_a^{+i})^\sim = U_a^{+i}, \quad (U_{ia}^-)^\sim = U_{ia}^-, \quad (U_a^{0})^\sim = U_a^{0}. \quad (2.7)$$

All harmonics are real with respect to this conjugation. On harmonic polynomials with complex coefficients $f(U)$, the conjugation $\sim$ acts on these coefficients only, for example, $(zU_a^K)^\sim = \bar{z}U_a^K$, where $\bar{z}$ is the complex conjugation. The special conjugation acts on derivatives of the harmonic functions and allows defining the corresponding Hermitian conjugation $\dagger$ of the harmonic operators

$$(\partial^{+i}f)^\sim = \overline{\partial^{+i}f}, \quad (\partial^{+i})^\dagger = -\overline{\partial^{+i}}, \quad (\partial_i^-f)^\sim = \overline{\partial_i^-f}, \quad (\partial_i^-)^\dagger = -\overline{\partial_i^-},$$

$$(\partial^{\pm\pm}f)^\sim = \overline{\partial^{\pm\pm}f}, \quad (\partial^{\pm\pm})^\dagger = -\overline{\partial^{\pm\pm}}, \quad (2.8)$$

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$$(\partial^{\pm\pm}f)^\sim = \overline{\partial^{\pm\pm}f}, \quad (\partial^{\pm\pm})^\dagger = -\overline{\partial^{\pm\pm}}, \quad (2.8)$$
where \( \partial^\pm_i f = [\partial^\pm_i, f] \), \( \partial^{\pm\pm} f = [\partial^{\pm\pm}, f] \).

The harmonic functions on the homogeneous 6-dimensional space \( SO(5)/U(2) \) satisfy the \( U(2) \)-invariance condition

\[
\partial_k f(U) = 0. \tag{2.9}
\]

The full superspace of the \( D=3, N=5 \) supersymmetry has the spinor \( CB \) coordinates \( \theta^a_\alpha \), \( (\alpha = 1, 2) \), \( a = 1, 2, 3, 4, 5 \) in addition to the coordinates \( x^m \) of the three-dimensional Minkowski space. The group \( SL(2,R) \times SO(5) \) acts on the spinor coordinates. We consider the superconformal transformations of these coordinates in appendix.

The \( SO(5)/U(2) \) harmonics allow constructing projections of the spinor coordinates and the partial spinor derivatives

\[
\theta^{i+\alpha} = U_a^i \theta^a_\alpha, \quad \theta^{0\alpha} = U^0_a \theta^a_\alpha, \quad \theta^{-\alpha} = U^{-a}_i \theta^a_\alpha, \quad (\alpha = 1, 2), \quad a = 1, 2, 3, 4, 5 \tag{2.10}
\]

The analytic coordinates \( (AB\text{-representation}) \) in the full harmonic superspace use these projections of ten spinor coordinates \( \theta^{i+\alpha}, \theta^{0\alpha}, \theta^{-\alpha} \) and the following representation of the vector coordinate:

\[
x^m_A = y^m = x^m + i(\theta^{i+k} \gamma^m \theta_k^i) = x^m + i(\theta_a^i \gamma^m \theta_b^i)U^+a_kU^-_{kb}. \tag{2.11}
\]

The analytic coordinates are real under the special conjugation

\[
(\theta^{i+\alpha})^\sim = \theta^{i+\alpha}, \quad (\theta^{0\alpha})^\sim = \theta^{0\alpha}, \quad (\theta^{-\alpha})^\sim = \theta^{-\alpha}, \quad (y^m)^\sim = y^m, \quad (\Phi \Lambda)^\sim = \Lambda^\sim \Phi^\sim, \quad (z \bar{z})^\sim = z^\sim \bar{z}^\sim, \tag{2.12}
\]

where we also define the action of the \( \sim \)-conjugation on arbitrary superfields \( \Phi, \Lambda \) (\( z, \bar{z} \) are conjugate complex numbers).

We define the harmonic derivatives in the \( AB \) representation:

\[
\begin{align*}
D^{+k} &= \partial^{+k} - i(\theta^{+k} \gamma^m \theta_k^0)\partial_m + \theta^{+\alpha} \partial^0 - \theta^0 \partial^{+k}, \\
D^{++} &= \partial^{++} + i(\theta^{+k} \gamma^m \theta_k^0)\partial_m + \theta^{+\alpha} \partial^0, \\
D^{-k} &= \partial^{-k} + i(\theta^{-k} \gamma^m \theta_k^0)\partial_m + \theta^{-\alpha} \partial^0 - \theta^{0\alpha} \partial^{-k}, \\
D^{--} &= \partial^{--} - i(\theta^{-k} \gamma^m \theta_k^0)\partial_m + \theta^{-\alpha} \partial^0 - \theta^{0\alpha} \partial^{-k}, \\
D^k_l &= \partial^k_l + \theta^{+\alpha} \partial^0_l - \theta^{-\alpha} \partial^0_k.
\end{align*} \tag{2.13}
\]

These operators commute with generators of the \( N=5 \) supersymmetry.

The \( U(2) \) operator \( D^k_l \) acts covariantly in the harmonic superspace:

\[
\begin{align*}
D^k_l \theta^{i+\alpha} &= \delta^i_l \theta^{+ka}, & D^k_l \theta^{-\alpha} &= -\delta^i_l \theta^{-ka}, \\
[D^k_l, D^{++}] &= \delta^k_l D^{+k}, & [D^k_l, D^{--}] &= \delta^k_l D^{--}, & [D^k_l, D_i^+] &= -\delta^k_l D_i^-, \\
[D^k_l, D_i^-] &= \delta^k_l D_i^+, & [D^k_l, D^0] &= \delta^k_l D^0.
\end{align*} \tag{2.14}
\]

The commutation relations are easily obtained:

\[
\begin{align*}
[D^{+k}, D^{++}] &= -\xi^{kl} D^{++}, & D^{+k} D^{++} &= D^{+k}, \\
[D^{--}, D_i^-] &= -\xi^{kl} D_i^-, & D^{--} D_i^- &= -D_i^-, \\
[D_i^-, D^{+k}] &= D_i^+ - \frac{1}{2} \xi^{kl} D^0, & [D^{++}, D_i^-] &= D^0.
\end{align*} \tag{2.15}
\]

where \( D^0 = D^k_l \) and \( D^k_l \) are the generators of the subgroups \( U(1) \) and \( SU(2) \).
The relations between harmonic and spinor derivatives are
\[
[D^{+i}, D^0_{\alpha}] = D^{+i}_{\alpha}, \quad [D^{++}, D^0_{\alpha}] = 0, \quad [D^{+i}, D^-_{\alpha}] = -\delta^i_\alpha D^0_{\alpha},
\]
\[
[D^{++}, D^0_{\alpha}] = D^0_{\alpha}, \quad [D^{++}, D^-_{\alpha}] = D^-_{\alpha}, \quad [D^{+i}, D^-_{\alpha}] = -\delta^i_\alpha D^0_{\alpha},
\]
\[
[D^{++}, D^-_{\alpha}] = 0, \quad [D^{++}, D^-_{\alpha}] = D^0_{\alpha}, \quad [D^{+i}, D^-_{\alpha}] = -\delta^i_\alpha D^0_{\alpha},
\]
(2.16)

where
\[
D^{+i}_{\alpha} = \partial^{+i}_{\alpha}, \quad D^-_{\alpha} = \partial^-_{\alpha} + 2i\theta^{-\beta} \partial_{\alpha\beta},
\]
\[
D^0_{\alpha} = \partial^0_{\alpha} + i\theta^{0\beta} \partial_{\alpha\beta}.
\]
(2.17)

The coordinates of the analytic superspace \(\zeta = (y^m, \theta^{+i\alpha}, \theta^{0a}, U^K_a)\) have the Grassmann dimension 6 and dimension of the even space 3+6. The functions \(\Phi(\zeta)\) satisfy the Grassmann analyticity condition in this superspace
\[
D^{+k}_{\alpha} \Phi = 0.
\]
(2.18)

In addition to this condition, the analytic superfields in the \(SO(5)/U(2)\) harmonic superspace also have the \(U(2)\)-covariance. For the \(U(2)\)-scalar superfields, this subsidiary condition is
\[
D^k \Lambda(\zeta) = 0.
\]
(2.19)

For the \(SU(2)\) spinor superfield with the \(U(1)\) charge \(q\), the \(U(2)\) covariance condition has the form
\[
D^0_{\alpha} \Phi^{(q)}_k = q \Phi^{(q)}_k, \quad [D^i_{\alpha}, \Phi^{(q)}_k] = -\delta^i_k \Phi^{(q)}_j + \frac{1}{2} \delta^i_j \Phi^{(q)}_k.
\]
(2.20)

The integration measure in the analytic superspace \(d\mu^{(-4)}\) has the dimension zero
\[
d\mu^{(-4)} = dU d^3 x_A (\partial^0_{\alpha})^2 (\partial^-_{\alpha})^4 = dU d^3 x_A d\theta^{(-4)},
\]
(2.21)

and is imaginary under the special conjugation
\[
(d\mu^{(-4)})^\sim = -d\mu^{(-4)}.
\]
(2.22)

Integrals over the Grassmann and harmonic variables have the simple properties
\[
\int d\theta^{(-4)} (\theta^0)^2 \Theta^{(4)} = 1, \quad \int dU = 1, \quad \int dUU^0_a U^0_b = \frac{1}{5} \delta_{ab},
\]
\[
\int dUU^{+k}_a U^-_{lb} = \frac{1}{5} \delta^i_k \delta_{ab},
\]
(2.23)

where combinations of the spinor coordinates \((\theta^0)^2\) and \(\Theta^{(4)}\) are defined in the appendix.

3 Gauge Chern-Simons theory in the \(SO(5)/U(2)\) harmonic superspace

3.1 Superfield action

The harmonic derivatives \(D^{+k}\) and \(D^{++}\) together with the spinor derivatives \(D^{+k}_{\alpha}\) determine the \(CR\)-structure of the harmonic \(SO(5)/U(2)\) superspace. This \(U(2)\)-covariant
The CR-structure is invariant under the $N=5$ supersymmetry. The CR-structure should be preserved in the superfield gauge theory.

The gauge superfields (prepotentials) $V^{+k}(\zeta)$ and $V^{++}(\zeta)$ in the harmonic $SO(5)/U(2)$ superspace satisfy the Grassmann analyticity and $U(2)$-covariance conditions

$$D_\alpha^{+k}V^{+k} = D_\alpha^{+k}V^{++} = 0, \quad D_j^{i}V^{+k} = \delta_j^{i}V^{+i}, \quad D_j^{i}V^{++} = \delta_j^{i}V^{++}. \quad (3.1)$$

In the gauge group $SU(n)$, these traceless matrix superfields are anti-Hermitian

$$(V^{+k})^\dagger = -V^{+k}, \quad (V^{++})^\dagger = -V^{++}, \quad (3.2)$$

where operation $\dagger$ includes transposition and the special conjugation.

The analytic superfield parameters of the gauge group $SU(n)$ satisfy the generalized CR analyticity conditions

$$D_\alpha^{+k}\Lambda = D_j^{i}\Lambda = 0, \quad (3.3)$$

and are traceless and anti-Hermitian $\Lambda^\dagger = -\Lambda$. We can use the decomposition of these superfield parameters and prepotentials in terms of generators of the adjoint representation of the gauge group $T_A$, for instance,

$$V^{+k} = V_A^{+k}T_A, \quad [T_A,T_B] = if_{ABC}T_C, \quad \text{Tr} (T_A T_B) = \delta_{AB}, \quad (3.4)$$

where $f_{ABC}$ are the group structure constants.

We treat these prepotentials as connections in the covariant gauge derivatives

$$\nabla^{+i} = D^{+i} + V^{+i}, \quad \nabla^{++} = D^{++} + V^{++},$$

$$\delta_\Lambda V^{+i} = D^{+i}\Lambda + [\Lambda, V^{+i}], \quad \delta_\Lambda V^{++} = D^{++}\Lambda + [\Lambda, V^{++}], \quad (3.5)$$

where the infinitesimal gauge transformations of the gauge superfields are defined. These covariant derivatives commute with the spinor derivatives $D^{+k}_\alpha$ and preserve the CR-structure in the harmonic superspace.

We can construct three analytic superfield strengths off the mass shell

$$F^{++} = \nabla^{++} + \frac{1}{2}e^{ki}[\nabla^{+i}, \nabla^{+k}] = V^{++} - D^{+k}V^{++} - V^{+k}V^{++},$$

$$F^{(n)k} = [\nabla^{++}, \nabla^{+k}] = D^{++}V^{+k} - D^{+k}V^{++} + [V^{++}, V^{+k}], \quad (3.6)$$

$$\delta_\Lambda F^{++} = [\Lambda, F^{++}], \quad \delta_\Lambda F^{(n)k} = [\Lambda, F^{(n)k}].$$

The superfield action in the analytic $SO(5)/U(2)$ superspace is defined on three prepotentials $V^{+k}$ and $V^{++}$ by analogy with the off-shell action of the SYM $M^4$ theory [21]

$$S_1 = \frac{ik}{12\pi} \int d\mu(-4) \text{Tr} \left\{ V^{+j}D^{++}V^{j+} + 2V^{++}D^{+j}V^{+j} + (V^{++})^2 + V^{++}[V^{j+}, V^{+j}] \right\}, \quad (3.7)$$

where $k$ is the coupling constant and a numerical factor is chosen to guarantee the correct normalization of the vector-field action. This action is invariant under the infinitesimal gauge transformations of the prepotentials (3.5). The idea of construction of the superfield action in the harmonic $SO(5)/U(2)$ was proposed in [24], although the detailed construction of the superfield Chern-Simons theory was not discussed there. An equivalent superfield action was considered in the alternative harmonic formalism [25].
The action $S_1$ yields superfield equations of motion meaning that the superfield strengths of the theory are trivial
\[
F_{k}^{(+3)} = D^{;++} V_{k}^{++} - D_{k}^{++} V^{++} + [V^{++}, V_{k}^{+}] = 0, \\
F^{++} = V^{++} - D^{++} V_{k}^{+} - V^{+} V_{k}^{+} = 0.
\] (3.8)
These classical superfield equations have only pure gauge solutions for the prepotentials
\[
V^{+} = e^{-\Lambda} D^{++} e^{\Lambda}, \\
V^{++} = e^{-\Lambda} D^{++} e^{\Lambda},
\] (3.9)
where $\Lambda$ is an arbitrary analytic superfield. To prove this, we study the corresponding classical field-component equations of motion in Sec. 4. We note that formally similar superfield equations of the $SYM_4^3$ theory have nontrivial solutions for the gauge superfields [21]. The main reason for the differences in the character of the solutions of these two theories is connected with the different Grassmann dimensions of the corresponding analytic superspaces.

We rewrite action (3.7) in the equivalent form in terms of the prepotentials and analytic strengths (3.6)
\[
S'_{1} = \frac{ik}{12\pi} \int d\mu^{(-4)} \text{Tr} \left\{ V^{+} F_{k}^{(+3)} + V^{++} F^{++} + V^{++} V^{+} V_{k}^{+} \right\} = \int d\mu^{(-4)} L^{+4},
\] (3.10)
where $L^{(+4)}$ is the superfield density of the action.

We consider the composite superfield
\[
\hat{V}^{++} = D^{++} V_{k}^{+} + V^{+} V_{k}^{+}, \\
\delta \Lambda \hat{V}^{++} = D^{++} \Lambda + [\Lambda, \hat{V}^{++}],
\] (3.11)
constructed from two other superfields. Substituting $V^{++} \rightarrow \hat{V}^{++}$ in the action $S_1$, we can construct the alternative action on two independent prepotentials
\[
S_2 = \frac{ik}{12\pi} \int d\mu^{(-4)} \text{Tr} \left\{ V^{+} D^{++} V_{j}^{+} - (\hat{V}^{++})^2 \right\}
\]
\[
= \frac{ik}{12\pi} \int d\mu^{(-4)} \text{Tr} \left\{ V^{+} D^{++} V_{j}^{+} - (D^{++} V_{j}^{+} + V^{+} V_{j}^{+})^2 \right\}.
\] (3.12)
This action corresponds to the equation of motion with the second-order harmonic derivatives
\[
\hat{F}_{k}^{(+3)} = D^{++} V_{k}^{+} - D_{k}^{++} \hat{V}^{++} + [\hat{V}^{++}, V_{k}^{+}] = 0,
\] (3.13)
which is equivalent to equations (3.8).

### 3.2 Superconformal invariance of the action

The superconformal $SC^5_3$ transformations of the analytic coordinates are considered in the appendix. Using these transformations, in particular, allows verifying the superconformal invariance of the integration measure $d\mu^{(-4)}$ in the analytic $SO(5)/U(2)$ superspace (see (A.16)).

We define the superconformal $SC^5_3$ transformations of the covariant harmonic derivatives by analogy with formulas (A.18)
\[
\delta_{sc} \nabla^{+} = -\lambda_{+}^{+} D_{+}^{+}, \\
\delta_{sc} \nabla^{++} = \lambda^{++} D^{++}_{k} - \lambda^{+}_{k} \nabla^{+},
\] (3.14)
where $\lambda_k^+$, $\lambda^{++}$ are superfield parameters (A.13). The corresponding transformations of the analytic prepotentials are

$$\delta_{sc}V^{+k} = 0, \quad \delta_{sc}V^{++} = -\lambda_k^+V^{+k}. \quad (3.15)$$

Analytic superfield strengths (3.6) transform covariantly in the superconformal group

$$\delta_{sc}F^{(+3)k} = \lambda^{+k}F^{++}, \quad \delta_{sc}F^{++} = 0. \quad (3.16)$$

Action (3.10) is invariant under the $SC^5$ transformations

$$\delta_{sc}S'_1 = \frac{ik}{12\pi} \int d\mu^{(-4)} \text{Tr} \left\{ V^{+k} \lambda_k^+ F^{++} - \lambda_k^+ V^{k+} + \lambda^{+k} V^{++} - \lambda^{+k} V^{+k} V^{k+} \right\} = 0, \quad (3.17)$$

where we use the identity

$$\text{Tr} \left( V^{+i} V^{+k} V^{k+} \right) = i f_{ABC} V^{+i} V^{+k} V^{k+}_C = 0.$$

The transformation of the sixth supersymmetry can be defined on the analytic $N=5$ superfields

$$\delta_6 V^{++} = \epsilon^6_\alpha D^{0}_\alpha V^{++}, \quad \delta_6 V^{+k} = \epsilon^6_\alpha D^{0}_\alpha V^{+k}, \quad (3.18)$$

$$\delta_6 D^{+k} V^{+l} = \epsilon^6_\alpha D^{0}_\alpha D^{+k} V^{+l}, \quad \delta_6 D^{++} V^{+l} = \epsilon^6_\alpha D^{0}_\alpha D^{++} V^{+l}, \quad (3.19)$$

where $\epsilon^6_\alpha$ are the corresponding odd parameters. This transformation preserves the Grassmann analyticity and $U(2)$-covariance

$$\{D^{0}_\alpha, D^{+k}_\beta\} = 0, \quad [D^{+k}_\alpha, D^{0}_\alpha] = 0, \quad [D^{+k}_\alpha, D^{+k}_\alpha] = D^{+k}_\alpha, \quad [D^{++}, D^{0}_\alpha] = 0. \quad (3.20)$$

The action (3.10) is invariant with respect to this sixth supersymmetry

$$\delta_6 S'_1 = \int d\mu^{(-4)} \epsilon^6_\alpha D^{0}_\alpha L^{(+4)} = 0. \quad (3.21)$$

We can therefore speak on the $N=6$ Chern-Simons theory in the $N=5$ superspace.

### 4 Investigation of the component structure of the action

The harmonic decomposition of the superfield gauge $U(1)$ parameter has the following form:

$$\Lambda = \Lambda_0 + \Lambda_1 + O(U^2), \quad D^{0}_k \Lambda = 0,$$

$$\Lambda_0 = i a + \theta^{\alpha d} \beta^{kd}_\alpha + (\theta^0)^2 d,$$

$$\Lambda_1 = i a^b U^0_b + \theta^{\alpha d} U^0_b \beta^{kd}_\alpha + \theta^{+k\alpha} U^{-b}_{kzb} + (\theta^0)^2 U^0_b d^a + (\theta^{+k\alpha} U^{-b}_{kza} + (\theta^0)^2 \theta^{+k\alpha} U^{-b}_{kza}). \quad (4.1)$$

where all local parameters $a(y), a^b(y), \beta_\alpha(y), \beta^{kd}_\alpha(y)$ ... are chosen real to ensure the superfield condition $\Lambda^o = -\Lambda$. In the group $SU(n)$, the corresponding local parameters are Hermitian matrices with a zero trace. We also use conditions of the $U(2)$-invariance
\[ \mathcal{D}_k^p \Lambda_p = 0 \text{ in any degree } p \text{ of the harmonic decomposition. The local parameters in } \Lambda_0 \text{ are } \text{SO}(5) \text{ invariant and } \Lambda_p \text{ contain parameters with } \text{SO}(5) \text{ tensor indices.} \]

We study the component decomposition of the action \( S_2 \) depending on two prepotentials \( V^{+k} \). In any gauge, these superfields can be decomposed in degrees of harmonics

\[ V^{+k} = V_0^{+k} + V_1^{+k} + O(U^2), \tag{4.2} \]

where each term contains Grassmann coordinates \( \theta^{0\alpha}, \theta^{+k\alpha} \) and the corresponding degrees of harmonics with the \( U(2) \)-covariance condition \( \mathcal{D}_k^i V^{+k} = \delta_k^i V^{+i} \) taken into account. The omitted terms contain an infinite number of fields with tensor indices of the group \( \text{SO}(5) \).

We note that the \( N=5 \) supersymmetry transformations connect harmonic terms \( V_p^{+k} \) with different degrees of harmonics. The supersymmetric \( WZ \)-gauge of each harmonic term can be obtained using the transformations \( \delta V^{+k}_p = \mathcal{D}^{+k} \Lambda_p \); for instance, the transformation

\[ \delta V_0^{+k} = \mathcal{D}^{+k} \Lambda_0 = \theta^{+k\alpha} \beta_\alpha + (\theta^{+k\gamma} \gamma^m \theta^0) \partial_m a + \frac{i}{2} \theta^{+k\gamma} (\theta^0)^2 \partial_\gamma \beta_\alpha + 2(\theta^{+k} \theta^0) d \tag{4.3} \]

yields the gauge choice

\[ V_0^{+k} = (\theta^{+k\gamma} \gamma^m \theta^0) A_m + i(\theta^0)^2 \theta^{+k\alpha} \psi_\alpha. \tag{4.4} \]

The \( WZ \) gauge of the term linear in harmonics contains fields with the \( \text{SO}(5) \) vector index

\[
\begin{align*}
V_1^{+k} &= (\theta^0)^2 U_a^+ B_a + (\theta^{+l\gamma} \gamma^m \theta^0)^2 U_a^+ C_m \\
&+ i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0}) U_{(\alpha \beta \gamma \delta \epsilon)} + i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0}) U_{(\alpha \beta \gamma \delta \epsilon)} - i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0}) U_{(\alpha \beta \gamma \delta \epsilon)} \\
&+ i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0}) U_{(\alpha \beta \gamma \delta \epsilon)} - i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0}) U_{(\alpha \beta \gamma \delta \epsilon)} \\
&+ i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0})(\theta^0)^2 U_{(\alpha \beta \gamma \delta \epsilon)} - i(\theta^{+l(\alpha \beta \gamma \delta \epsilon) \theta^0})(\theta^0)^2 U_{(\alpha \beta \gamma \delta \epsilon)} \tag{4.5}
\end{align*}
\]

We note that this decomposition contains the fermionic field \( \Psi_{(\alpha \beta \gamma \delta \epsilon)} \) with three spinor indices.

The harmonic-independent term of the superfield action (3.12) is

\[ S_0 = \frac{i}{12 \pi} \int d\mu(-4) \text{Tr} \{V_0^{+ij} \mathcal{D}^{+m} V_0^{+m} - (\mathcal{D}^{+m} V_0^{+m} + V_0^{+ij} V_0^{+ij})^2 \}, \tag{4.6} \]

where the gauge (4.4) is chosen. The corresponding component Lagrangian can be obtained by integrating over harmonics and spinor coordinates

\[ L_0 = \frac{k}{4 \pi} \int d^3 x e^{mr} \text{Tr} \{ A_m (\partial_n A_r + \frac{i}{3} [A_n, A_r]) + i \psi^\alpha \psi_\alpha \}. \tag{4.7} \]

In addition to the standard Chern-Simons term, this Lagrangian contains the bilinear interaction of the auxiliary field.

Bilinear and trilinear interaction terms of auxiliary fields with an arbitrary number of \( \text{SO}(5) \) indices can also be obtained from the superfield action (3.12) in the \( WZ \)-gauge, although these calculations are rather tedious. On the mass shell, all auxiliary fields vanish and the gauge field \( A_m \) has the zero field-strength.
5 Alternative superfield action of the $N=5$ theory

In the three-dimensional $N = 1, 2$ and $3$ supersymmetries, the off-mass-shell vector multiplets can be simultaneously used in the actions of the supersymmetric Chern-Simons and Yang-Mills theories. A dimensional reduction of the $D=4, N=3$ superfield gauge theory to the three-dimensional space yields the corresponding supersymmetric $D=3, N=6$ Yang-Mills theory $SYM^6_3$ which preserves the automorphism group $U(3)$ and breaks the superconformal invariance. The equations of motion of this theory have an additional supersymmetry [27]. After an infinite number of auxiliary fields are excluded, there remain the three-dimensional vector field, seven scalar fields and eight two-component spinor fields, which form the $D=3, N=8$ multiplet of the physical fields. We do not know how to construct the Chern-Simons action on the harmonic superfields of the $SYM^6_3$ theory.

We consider the Abelian gauge superfield in the $D=3, N=5$ supersymmetry

$$\delta_\Lambda V^+ = D^+ \Lambda. \quad (5.1)$$

In addition to the Abelian version of the Chern-Simons action (3.12), we can also construct the alternative superfield action of the $U(1)$ gauge superfield

$$S_E = \frac{1}{12g^2} \int d\mu(-4)(D^{0\alpha}D^{0\alpha}V^{++k})(D^{++}V^+_k - D^+_k D^{++}V^+_i), \quad (5.2)$$

where $g$ is a coupling constant of the dimension $1/2$. This action is invariant with respect to the gauge transformation (5.1)

$$\delta_\Lambda S_E \sim \int d\mu(-4)(D^{0\alpha}D^{0\alpha}D^{++k} \Lambda)(D^{++}V^+_k - D^+_k D^{++}V^+_i) = 0, \quad (5.3)$$

where we use the relations $D^{0\alpha}D^{0\alpha}D^{++k} \Lambda = D^{++k}D^{0\alpha}D^{0\alpha} \Lambda$, $D^{++} = D^{++k}D^+_k$ and integrate by parts. The interaction $S_E$ breaks the sixth supersymmetry (3.19) and also the conformal invariance.

In components, the action (5.2) contains the Lagrangian of the three-dimensional electrodynamics and also the kinetic term of the fermion field $\psi^\alpha$. In the sector of the $SO(5)$ vector fields (4.5), the action $S_E$ yields bilinear terms of the bosonic and fermionic fields. Compared with the supersymmetric $N = 1, 2, 4$ and $6$ generalizations of electrodynamics, the action (5.2) contains an unusual interaction of the chirality $3/2$ field $\Psi^{\alpha\beta\gamma}_{\lambda}$. In contrast to supergravity models, this field arises without additional local supersymmetry here. The equation of motion for the action $S_E$ has nontrivial solutions.

6 Conclusions

We used the harmonic coordinates describing the $SO(5)/U(2)$ space in the framework of the $D=3, N=5$ supersymmetry and investigated the geometry of the gauge theory in the analytic $SO(5)/U(2)$ superspace based on the Grassmann analyticity and $U(2)$-covariance of the gauge superfields in detail. The gauge transformations of the superfield potentials of the theory preserve the $CR$-structure of the spinor and harmonic covariant derivatives in analogy with other examples of gauge theories in harmonic superspaces. We constructed equivalent representations of the superfield action of the gauge theory with three and two independent superfields. We showed that these superfield actions
are invariant under the $N=5$ conformal supersymmetry and the additional sixth supersymmetry. We studied the supersymmetric $WZ$-gauge of the superfield prepotential; it defines an infinite number of the field components, including the vector gauge field and auxiliary bosonic and fermionic fields. The superfield equations of motion of this theory have only the pure gauge solutions. The component equations of motion yield trivial (zero) solutions for the field-strength of the vector field and all other auxiliary fields in the Chern-Simons supermultiplet. The superfield formulation can be useful in the study of the quantum properties of the $N=6$ Chern-Simons theory.

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Appendix. Superconformal transformations

A.1. Superconformal $CB$ transformations

We use the real symmetric representation for the $D=3$ gamma-matrices

\[
(\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\rho}(\gamma_m)^{\rho\beta} = (\gamma_m)_{\alpha}^{\beta}, \quad (\gamma^m)_{\alpha\beta}(\gamma^m)_{\rho\gamma} = \delta^\rho_\alpha \delta^\gamma_\beta + \delta^\rho_\beta \delta^\gamma_\alpha,
\]

\[
(\gamma_m\gamma_n)_{\alpha}^{\beta} = (\gamma_m)_{\alpha}^{\rho}(\gamma_n)^{\rho\beta} = -\eta_{mn}\delta^\beta_\alpha + \varepsilon_{mnp}(\gamma^p)_{\alpha}^{\beta}, \quad (\gamma_m\gamma_r)_{\alpha}^{\beta} = -\varepsilon_{mnr}\delta^\beta_\alpha - \eta_{mn}(\gamma_r)_{\alpha}^{\beta} - \eta_{nr}(\gamma_m)_{\alpha}^{\beta} + \eta_{mr}(\gamma_n)_{\alpha}^{\beta},
\]

where $\eta_{mn} = \text{diag}(1,-1,-1)$ is the Minkowski metric, and $\varepsilon_{mnp}$ is the antisymmetric symbol in the $3D$ space. The spinor $SL(2,R)$ indices are raised and lowered using the antisymmetric symbol

\[
\psi^\alpha = \varepsilon^{\alpha\beta}\psi^\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta, \quad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma.
\]

We use the notation

\[
(\psi_\xi) = \psi^\alpha\xi_\alpha, \quad (\psi^{\gamma_m}\xi) = \psi^\alpha(\gamma_m)_{\alpha\beta}\xi^\beta
\]

for the scalar and vector combinations of two spinors.

The bispinor representations of the three-dimensional coordinates and derivatives are

\[
x^{\alpha\beta} = (\gamma_m)^{\alpha\beta}x^m, \quad \partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m
\]

and the corresponding expression for the $N=5$ spinor derivatives are

\[
D_{\alpha\beta} = \partial_{\alpha\beta} + i\theta^a_{\beta}\partial_{\alpha a}, \quad \partial_{\alpha a}\theta^a_{\beta} = \delta_{ab}\delta^c_{\alpha}.
\]

In the full superspace with the $CB$-coordinates $z = (x^m, \theta^a_m)$, we define the $N=5$ superconformal transformations ($SC^5_{3}$-transformations)

\[
\delta_{sc}x^{\alpha\beta} = c^{\alpha\beta} + \frac{1}{2}\theta^a_{\gamma}\gamma^{\gamma\beta} - \frac{i}{2}\eta_{\alpha}^{\beta}\theta^c_{a} + \frac{1}{2}x^{\alpha\beta} - i\theta^a_{\beta}\partial_{\alpha a} - i\theta^a_{\alpha}\partial_{\beta a} + \frac{1}{2}x^{\alpha\gamma}k_{\gamma\rho}x^{\rho\beta} - \frac{1}{2}\theta^a_{\alpha}\theta^b_{\beta}\theta^c_{\gamma}k_{\gamma\rho},
\]

\[
\delta_{sc}\theta^a_{\alpha} = \frac{1}{2}\theta^a_{\beta}\theta^c_{\gamma}\eta_{\alpha\gamma} - \frac{1}{2}\theta^a_{\beta}\theta^c_{\gamma}\eta_{\gamma\alpha} + \frac{1}{2}x^{\alpha\beta}\theta^a_{\gamma}k_{\gamma\rho} - \frac{1}{2}\theta^a_{\alpha}\theta^b_{\beta}\theta^c_{\gamma}k_{\gamma\rho} + \frac{1}{2}\theta^a_{\alpha}\theta^b_{\beta}\theta^c_{\gamma}k_{\gamma\rho} + \epsilon_{\alpha}
\]

\[
+ \frac{1}{2}x^{\alpha\beta}\eta_{\alpha\beta} - i\theta^a_{\beta}\theta^c_{\alpha}\eta_{\gamma} + \frac{1}{2}\theta^a_{\alpha}\theta^b_{\beta}\eta_{\alpha\beta},
\]

(A.6)
where \( c^{\alpha \beta} = c(m(\gamma_m)^{\alpha \beta}, k^{\alpha \beta} = k(m(\gamma_m)^{\alpha \beta}, a^{\alpha} = a(m(\gamma_m)^{\alpha} \text{ and } l \text{ are the parameters of the D}=3 \text{ conformal group, } \omega_{ab} \text{ are the } SO(5) \text{ parameters, and the Grassmann parameters } e^{\alpha}_a \text{ and } \eta^{\alpha}_a \text{ correspond to the transformations of the } Q\text{-supersymmetries and } S\text{-supersymmetries.}

The generators of the \( N=5 \) superconformal group \( SC^5_3 \) in the CB-representation are

\[
P_m = \partial_m, \quad Q^\alpha_a = \theta^\alpha_a - i \theta^{[\alpha}(\gamma_m)_{\alpha \beta} \partial_m, \quad T_{ab} = \theta^{\alpha}_b \partial_{a\alpha} - \theta^{\alpha}_a \partial_{b\alpha},
\]
\[
L_m = \varepsilon_{mnp} x^p \partial^n + \frac{1}{2} \theta^{[\alpha}_a (\gamma_m)_{\alpha \beta} \partial^p = \frac{1}{2} \varepsilon_{mnp} L^{np}, \quad D = x^m \partial_m + \frac{1}{2} \theta^{\alpha}_a Q^\alpha_a,
\]
\[
K_m = x_m x^n \partial_n - \frac{1}{2} x^2 \partial_m - \frac{1}{4} (\theta_a \gamma_m \theta_b)(\theta_a \gamma_n \theta_b) \partial_n
\]
\[
+ \frac{1}{2} x_m \theta^\alpha_a \partial_n - \frac{1}{2} x^n \varepsilon_{mnp}(\gamma_p)^{\alpha \beta} \theta^\beta_a + \frac{1}{2} (\theta_a \gamma_m \theta_a) \theta^\alpha_a \partial_a,
\]
\[
S_a = \frac{i}{2} \theta_{a\alpha} x_m \partial_m + \frac{i}{2} \theta^{\alpha}_a \varepsilon_{mnp}(\gamma_p)^{\alpha \beta} x_n \partial_m + \frac{1}{2} \theta_{a\beta} \theta^\beta_a (\gamma^m)^{\alpha \beta} \partial_m
\]
\[
- \frac{1}{2} x^m (\gamma^m)_\alpha \partial_{a\alpha} - i \theta_{a\alpha} \theta^\alpha_a \partial_{a\beta} + \frac{i}{2} \theta_{a\beta} \theta^\beta_a \partial_{a\beta}.
\]

The local (active) transformations of superfields are determined by these generators

\[
\tilde{\delta}_{sc} \Phi = \delta_{sc} \Phi - (e^m P_m + a^m L_m + \frac{1}{2} \omega_{ab} T_{ab} + k^m K_m + l D + e^a Q_a + \eta^{a}_a S_{a\alpha}) \Phi,
\]

where \( \delta_{sc} \Phi \) are the coordinate (passive) superconformal transformations.

It is easy to obtain the (anti)commutator relations of the Lie superalgebra \( SC^5_3 \)

\[
[L_m, L_n] = \varepsilon_{mnp} L^p, \quad [L_m, P_n] = \varepsilon_{mnp} P^p, \quad [L_m, K_n] = \varepsilon_{mnp} K_p, \quad [L_m, D] = 0,
\]
\[
[L_m, Q^\alpha_a] = -\frac{1}{2} (\gamma_m)^{\alpha \beta} Q^\beta_a, \quad [L_m, S^a_a] = -\frac{1}{2} (\gamma_m)^{\alpha \beta} S^a_a, \quad [P_m, Q^\alpha_a] = 0, \quad [P_m, D] = P_m,
\]
\[
[K_m, L_n] = \varepsilon_{mnp} L^p + \eta_{mn} D, \quad [K_m, S^a_a] = -\frac{1}{2} (\gamma_m)^{\alpha \beta} Q^\beta_a, \quad [Q^a_a, P_m] = \frac{1}{2} Q^a_a,
\]
\[
[K_m, S^a_a] = \frac{1}{2} (\gamma_m)^{\alpha \beta} S^\beta_a, \quad \{Q^a_a, S^b_b\} = -\frac{1}{2} \delta_{ab} (\gamma_m)^{\alpha \beta} L_m - i \delta_{a\beta} \varepsilon_{a\beta} D - i \varepsilon_{a\beta} T_{ab}
\]
\[
\{Q^a_a, Q^b_b\} = -2 i \delta^{ab} (\gamma^m)^{\alpha \beta} P_m, \quad \{S^a_a, S^b_b\} = -i \delta^{ab} (\gamma^m)^{\alpha \beta} K_m, \quad \{S^a_a, K_m\} = 0.
\]

### A.2. \( SC^5_3 \) -transformations in analytic coordinates

The superconformal \( N=5 \) transformations of the analytic coordinates have the form

\[
\tilde{\delta}_{sc} e^{\alpha \beta}_A = \delta_{sc} e^{\alpha \beta}_A = \epsilon^{\alpha \beta}_A + L^2 y^{\alpha \beta} + L^2 y^{\alpha \gamma} + \frac{1}{2} y^{\alpha \gamma} k_{\gamma \rho} y^{\beta \rho} + 2 l y^{\alpha \beta} + i \omega_{ab} U^0_a U_{kb}^0 \theta^{+k} \theta^{\alpha \beta}
\]
\[
+ i \omega_{ab} U^0_a U^0_{kb} \theta^{+k} \theta^{\alpha \beta} - i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - i \epsilon^{\alpha \beta} \theta^{\alpha \beta} + 2 i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - 2 i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta}
\]
\[
+ \frac{1}{2} y^{\alpha \beta} \theta^{\alpha \beta} \eta^\gamma - \frac{1}{2} y^{\alpha \beta} \theta^{\alpha \beta} \eta^\gamma + i y^{\alpha \gamma} \theta^{+k} \eta^\gamma - i y^{\beta \gamma} \theta^{+k} \eta^\gamma.
\]

\[
\tilde{\delta}_{sc} \theta^0 \alpha = \epsilon^0 \alpha + \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - \omega_{ab} U^0_a U_{kb}^0 - \theta^{+k} \theta^{\alpha \beta}
\]
\[
+ \theta^0 \alpha + \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - i \epsilon^{\alpha \beta} \theta^{\alpha \beta} + 2 i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - 2 i \epsilon^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta}
\]
\[
+ \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} \eta^\gamma - \frac{1}{2} y^{\alpha \beta} \theta^{\alpha \beta} \eta^\gamma + i y^{\alpha \gamma} \theta^{+k} \eta^\gamma - i y^{\beta \gamma} \theta^{+k} \eta^\gamma.
\]

\[
\tilde{\delta}_{sc} \theta^{+k} \alpha = \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} + \theta^{+k} \alpha + \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} + \frac{1}{4} (\theta^0)^2 \theta^{+k} \theta^{\alpha \beta}
\]
\[
- \omega_{ab} U^0_a U_{kb}^0 \theta^{+k} \theta^{\alpha \beta} + \epsilon^{+k} \alpha + \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} - i \epsilon^{+k} \alpha - i \epsilon^{+k} \beta - i \epsilon^{+k} \beta - i \epsilon^{+k} \beta - i \epsilon^{+k} \beta
\]
\[
- \theta^{+k} \alpha + \frac{1}{2} y^{\alpha \beta} \theta^{+k} \theta^{\alpha \beta} + \frac{1}{4} (\theta^0)^2 \theta^{+k} \theta^{\alpha \beta},
\]

where we introduce the notation

\[
e^{\alpha}_a = U^0_a \epsilon^{\alpha}_a, \quad e^{\alpha}_k = U^0_k \epsilon^{\alpha}_a, \quad \eta^{\alpha}_a = U^0_a \eta^{\alpha}_a, \quad \eta^{\alpha}_k = U^0_k \eta^{\alpha}_a.
\]
We also write the superconformal transformations of the harmonics
\[
\delta_{sc} U^{+k} = \lambda^{+k} U^{+}_a - \lambda^{++} U^{-}_a, \quad \delta_{sc} U^{0} = -\lambda^{+k} U^{-}_a, \quad \delta_{sc} U^{-k} = 0,
\]
\[
\lambda^{+k} = ik_{\alpha\beta} \theta^{+k\alpha} \theta^{0\beta} + U^{+}_a U^{0}_b \omega_{ab} + i(\theta^{+k\alpha} U^{0}_a - \theta^{0\alpha} U^{+}_a) \eta_{ab},
\]
\[
\lambda^{++} = \frac{1}{2} k_{\alpha\beta} (\theta^{+k\alpha} \theta^{+}\beta) + \frac{1}{2} U^{+}_a U^{+}_b \omega_{ab} + i \theta^{+k\alpha} U^{+}_a \eta_{ab} = \frac{1}{2} D^+ \lambda^{+},
\]
\[
D^{+i} \lambda^{+k} = \varepsilon^{ik} \lambda^{++}, \quad D^{+i} \lambda^{++} = 0, \quad D^{i}_j \lambda^{+k} = \delta^k_i \lambda^{++}, \quad D^{i}_j \lambda^{++} = \delta^i_j \lambda^{++}. \tag{A.14}
\]

It is easy to obtain expressions for the $SC^5_3$ generators in the analytic superspace; for instance, the generator of the special conformal transformations has the form
\[
K^A_m = y^m \partial^\alpha - \frac{1}{2} y^2 \partial_m + \frac{1}{2} y^\alpha \eta^\alpha_0 \partial^0 + \frac{1}{2} \eta^{+k\alpha} \eta^{0\alpha} \partial^{+k} - i \theta^0 \theta^+ \eta^{+} + \frac{i}{4} \theta^0 \theta^+ \eta^{+} \partial^{-}.
\tag{A.15}
\]

### A.3. $SC^5_3$-transformations of analytic measure

Analytic integration measure (2.21) is invariant with respect to the $SC^5_3$ transformations
\[
\delta_{sc} d\mu^{-4} = d\mu^{-4} \left( \partial_m \delta_{sc} y^m + \frac{\partial}{\partial U^a} \delta_{sc} U_0^a + \frac{\partial}{\partial U^{-k}_a} \delta_{sc} U_-^k \right) - \partial_0^\alpha \delta_{sc} \theta^{0\alpha} - \partial_{-\alpha} \delta_{sc} \theta^{+\alpha} = 0. \tag{A.16}
\]

Here we use the formulas
\[
\partial_m \delta_{sc} y^m = 3 k_m y^m + 6 l + \frac{3i}{2} \theta^{0\alpha} \eta^0_{\alpha} + 3i \theta^{+k\alpha} \eta^{-}_{\alpha},
\]
\[
\frac{\partial}{\partial U^a} \delta_{sc} U_0^a + \frac{\partial}{\partial U^{-k}_a} \delta_{sc} U_-^k = -2 \omega_{ab} U^{+}_a U^{-}_b - 2i \theta^{0\alpha} \eta_a^0 - 2i \theta^{+k\alpha} \eta^{-}_{\alpha}, \tag{A.17}
\]
\[
-\partial_0^\alpha \delta_{sc} \theta^{0\alpha} - \partial_{-\alpha} \delta_{sc} \theta^{+\alpha} = -6l - 3 y^m k_m + 2 \omega_{ab} U^{+}_a U^{-}_b + \frac{i}{2} \theta^{0\alpha} \eta^0_{\alpha} - i \theta^{+k\alpha} \eta^{-}_{\alpha},
\]
derived by differentiating the infinitesimal transformations of the analytic coordinates.

### A.4. $SC^5_3$-transformations of harmonic derivatives

The $SC^5_3$-transformations of the analytic coordinate generate the superconformal transformations of the harmonic derivatives (2.13)
\[
\delta_{sc} D^{+k} = -\lambda^{+l} D^{+l}_k, \quad \delta_{sc} D^{++} = \lambda^{++} D^{+}_k - \lambda^{++} D^{l}_k, \tag{A.18}
\]
\[
\delta_{sc} D^{-} = (D^{-}_k \lambda^{+l}) D^{+}_l - (D^{+}_k \lambda^{++} - \lambda^{++} D^{+}_k) D^{--},
\]
\[
\delta_{sc} D^{--} = (D^{--} \lambda^{+l}) D^{+}_l + (D^{--} \lambda^{++} - \lambda^{++} D^{+}_k) D^{--}, \quad \delta_{sc} D^{l}_l = 0, \tag{A.19}
\]
where the superfield parameters $\lambda^{+l}, \lambda^{++}$ are defined in the transformations of harmonics (A.13).

### A.5. Algebra of spinor coordinates
We consider the independent combinations of the Grassmann coordinates \( \theta^{+k\alpha} \)

\[
(\theta^{+k\alpha}) = \theta^{+k\alpha} \theta^{+l} = (\theta^{+l} \theta^{+k}), \quad (\theta^{+\alpha} \theta^{+\beta}) = \theta^{+k\alpha} \theta^{+\beta} = (\theta^{+\beta} \theta^{+\alpha}),
\]

\[
\Theta^{3\alpha} = \frac{1}{3}(\theta^{+k\alpha}) \theta^{+\alpha} = -\frac{1}{3} \theta^{+l}(\theta^{+\alpha} \theta^{+\beta}),
\]

\[
\Theta^{(+4)} = \frac{1}{3}(\theta^{+k\alpha}) (\theta^{+l} \theta^{+\alpha}) = (\theta^{+1})^2 (\theta^{+2})^2 = -\frac{1}{3}(\theta^{+\alpha} \theta^{+\beta}) (\theta^{+\alpha} \theta^{+\beta}).
\]

Including the neutral spinor coordinate, we can also obtain the combinations

\[
(\theta^{0})^2 = \theta^{0\alpha} \theta^{0\alpha}, \quad (\theta^{+i\alpha}) = \theta^{+i\alpha} \theta^{0\alpha}, \quad (\theta^{+i\gamma m} \theta^{0\alpha}) = -(\theta^{0\gamma m} \theta^{+i}),
\]

\[
\Theta^{+i\alpha} = (\theta^{0})^2 \theta^{+i\alpha}, \quad \Theta^{+i\alpha} = (\theta^{+i\gamma m} \theta^{0\alpha}),
\]

\[
\Theta^{+\gamma \alpha} = \theta^{+k\alpha} \theta^{+\beta} \theta^{0\beta}, \quad \Theta^{+\gamma \alpha} = (\theta^{+k\alpha} \theta^{+\beta} \theta^{0\beta}),
\]

where the last quantity is totally symmetric in spinor indices. The special conjugation of the Grassmann polynomials is defined by (2.12)

\[
(\theta^{+k\alpha})^\sim = -(\theta^{+k\alpha}), \quad (\theta^{+\alpha} \theta^{+\beta})^\sim = -(\theta^{+\alpha} \theta^{+\beta}), \quad ([\theta^{0}]^2)^\sim = -(\theta^{0})^2,
\]

\[
(\theta^{+i\alpha})^\sim = -(\theta^{+i\alpha}), \quad (\theta^{+i\gamma m} \theta^{0\alpha})^\sim = -(\theta^{+i\gamma m} \theta^{0\alpha}), \quad (\Theta^{(4)})^\sim = \Theta^{(4)},
\]

\[
(\Theta^{+i\alpha})^\sim = -\Theta^{+i\alpha}, \quad (\Theta^{++\gamma \alpha})^\sim = -\Theta^{++\gamma \alpha}, \quad (\Theta^{++\gamma \alpha})^\sim = \Theta^{++\gamma \alpha}.
\]

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