NON-HAUSDORFF ÉTALE GROUPOIDS AND C*-ALGEBRAS OF LEFT CANCELLATIVE MONOIDS

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Abstract. We study the question whether the representations defined by a dense subset of the unit space of a locally compact étale groupoid are enough to determine the reduced norm on the groupoid C*-algebra. We present sufficient conditions for either conclusion, giving a complete answer when the isotropy groups are torsion-free. As an application we consider the groupoid $G(S)$ associated to a left cancellative monoid $S$ by Spielberg and formulate a sufficient condition, which we call C*-regularity, for the canonical map $C_r^*(G(S)) \to C_r^*(S)$ to be an isomorphism, in which case $S$ has a well-defined full semigroup C*-algebra $C_r^*(S) = C_r^*(G(S))$. We give two related examples of left cancellative monoids $S$ and $T$ such that both are not finitely aligned and have non-Hausdorff associated étale groupoids, but $S$ is C*-regular, while $T$ is not.

Introduction

The C*-algebras of non-Hausdorff locally compact groupoids were introduced by Connes in [Con82], where the main examples were given by the holonomy groupoids of foliations. It is known that some of the basic properties of groupoid C*-algebras of Hausdorff groupoids can fail in the non-Hausdorff case. One of such properties is that to compute the reduced norm it suffices to consider the representations $\rho_x : C_c(G) \to B(L^2(G_x))$ for $x$ running through any dense subset $Y \subset G^{(0)}$. A simple counterexample is provided by the line with a double point. The first systematic study of which extra conditions on $Y$ one needs was carried out by Khoshkam and Skandalis [KS02]. Our starting point is the simple observation, which can be viewed as a reformulation of a result in [KS02], that for étale groupoids it suffices to require that for every point $x \in G^{(0)} \setminus Y$ there is a net in $Y$ converging to $x$ and having no other accumulation points in $G_x^\tau$. As we show, this condition is in general not necessary, but it becomes so if the isotropy groups $G_x^\tau$ do not have too many finite subgroups, in particular, if they are torsion-free.

Our motivation for studying these questions comes from the problem of defining a full semigroup C*-algebra of a left cancellative monoid. Every such monoid $S$ has a regular representation on $l^2(S)$ and hence a well-defined reduced C*-algebra $C_r^*(S)$. It is natural to try to define the full semigroup C*-algebra as a universal C*-algebra generated by isometries $v_s$, $s \in S$, such that $v_s v_t = v_{st}$, but one quickly sees that more relations are needed to get an algebra that is not unreasonably bigger than $C_r^*(S)$. A major progress in this old problem was made by Li [Li12], who realized that in $C_r^*(S)$ there are extra relations coming from the action of $S$ on the constructible ideals of $S$, which are right ideals of the form $s_1^{-1}t_1 \ldots s_n^{-1}t_n S$. Soon afterwards Norling [Nor14] observed that this has an interpretation in terms of the left inverse hull $I_L(S)$ of $S$: the C*-algebra $C_r^*(S)$ is obtained by reducing the reduced C*-algebra of the inverse semigroup $I_L(S)$ to an invariant subspace of its regular representation, and so the new relations in $C_r^*(S)$ arise from those in $C_r^*(I_L(S))$. Since the representations of inverse semigroups are a well-studied subject and the corresponding C*-algebras have groupoid models defined by Paterson [Pat99], this opened the possibility to defining $C^*(S)$ as a groupoid C*-algebra.

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Specifically (see Section 2 for details), the subrepresentation of the regular representation of \( I_t(S) \) defining \( C_r^*(S) \) gives rise to a reduction \( \mathcal{G}_P(S) \) of the Paterson groupoid of \( I_t(S) \) and to a surjective homomorphism \( C_r^*(\mathcal{G}_P(S)) \to C_r^*(S) \). When this map is an isomorphism, it is natural to define \( C^*(S) \) as \( C^*(\mathcal{G}_P(S)) \). This \( C^* \)-algebra can be described in terms of generators and relations, since there is such a description for \( C^*(I_t(S)) \), and simultaneously its definition as a groupoid \( C^* \)-algebra subsumes a number of results on (partial) crossed product decompositions of semigroup \( C^* \)-algebras. The trouble, however, is that this does not work for all \( S \), the map \( C_r^*(\mathcal{G}_P(S)) \to C_r^*(S) \) is not always an isomorphism.

In [Sp20] Spielberg introduced, in a more general context of left cancellative small categories, a quotient \( \mathcal{G}(S) \) of \( \mathcal{G}_P(S) \) that kills some “obvious” elements in the kernel of \( C_r^*(\mathcal{G}_P(S)) \to C_r^*(S) \) (see Proposition 2.13). But as he showed, the canonical homomorphism \( C_r^*(\mathcal{G}(S)) \to C_r^*(S) \) can still have a nontrivial kernel. It should be said that the kernel of \( C_r^*(\mathcal{G}(S)) \to C_r^*(S) \) is small: under rather general assumptions (for example, for all countable \( S \) with trivial group of units) \( C_r^*(S) \) can be identified with the essential groupoid \( C^* \)-algebra \( C^*_{\text{ess}}(\mathcal{G}(S)) \) of \( \mathcal{G}(S) \), as defined by Kwaśniewski and Meyer [KM21]. Still, we do not think that this is enough to call \( C^*(\mathcal{G}(S)) \) the full semigroup \( C^* \)-algebra of \( S \) when \( C_r^*(\mathcal{G}(S)) \neq C_r^*(S) \).

Spielberg showed that there are two sufficient conditions for the equality \( C_r^*(\mathcal{G}(S)) = C_r^*(S) \), one is that \( \mathcal{G}(S) \) is Hausdorff, the other is that \( S \) is finitely aligned, which is equivalent to saying that every constructible ideal of \( S \) is finitely generated. Already the first condition covers, for example, all group embeddable monoids. For such monoids we have \( \mathcal{G}(S) = \mathcal{G}_P(S) \), and the corresponding full semigroup \( C^* \)-algebras \( C^*(S) = C^*(\mathcal{G}(S)) = C^*(\mathcal{G}_P(S)) \) have been recently comprehensively studied by Laca and Schmeling [LS22].

For general \( S \), the question whether \( C_r^*(\mathcal{G}(S)) \to C_r^*(S) \) is an isomorphism is exactly the type of question we started with: can the reduced norm on \( C_r(\mathcal{G}(S)) \) be computed using certain dense subset \( Y = \{ \chi_s \mid s \in S \} \) of \( \mathcal{G}(S)^{(0)} \)? In this formulation it is immediate that the answer is “yes” when \( \mathcal{G}(S) \) is Hausdorff. In the non-Hausdorff case we can try to use our general results to arrive to either conclusion. This leads to a simple (to formulate, but in general not to check) sufficient condition for the equality \( C_r^*(\mathcal{G}(S)) = C_r^*(S) \) that we call \( C^* \)-regularity. We give an example of a \( C^* \)-regular monoid \( S \) that is not finitely aligned and such that the groupoid \( \mathcal{G}(S) \) is non-Hausdorff. A small modification of \( S \) gives a monoid \( T \) with \( C_r^*(\mathcal{G}(T)) \neq C_r^*(T) \). It is interesting that the kernel of \( C_r^*(\mathcal{G}(T)) \to C_r^*(T) \) has nonzero elements already in the \( * \)-algebra generated by the canonical elements \( u_t, t \in T \), so in some sense \( \mathcal{G}(T) \) is a wrong groupoid model for the semigroup \( * \)-algebra of \( T \) already at the purely algebraic level.

Let us finally mention that an interesting related problem is to find groupoid models for boundary quotients of semigroup \( C^* \)-algebras, but we are not going to touch it in the present paper.

1. \( C^* \)-algebras of non-Hausdorff étale groupoids

Assume \( \mathcal{G} \) is a locally compact, not necessarily Hausdorff, étale groupoid. By this we mean that \( \mathcal{G} \) is a groupoid endowed with a locally compact topology such that

- the groupoid operations are continuous;
- the unit space \( \mathcal{G}^{(0)} \) is a locally compact Hausdorff space in the relative topology;
- the range map \( r: \mathcal{G} \to \mathcal{G}^{(0)} \) and the source map \( s: \mathcal{G} \to \mathcal{G}^{(0)} \) are local homeomorphisms.

For an open Hausdorff subset \( V \subset \mathcal{G} \), consider the usual space \( C_c(V) \) of continuous compactly supported functions on \( V \). Every such function can be extended by zero to \( \mathcal{G} \); in general this extension is not a continuous function on \( \mathcal{G} \). This way we can view \( C_c(V) \) as a subspace of the space of functions \( \text{Func}(\mathcal{G}) \) on \( \mathcal{G} \). For arbitrary open subsets \( U \subset \mathcal{G} \) we denote by \( C_c(U) \subset \text{Func}(\mathcal{G}) \) the linear span of the subspaces \( C_c(V) \subset \text{Func}(\mathcal{G}) \) for all open Hausdorff subsets \( V \subset U \). Instead of all possible \( V \) it suffices to take a collection of open bisections covering \( U \).
The space $C_c(G)$ is a $*$-algebra with the convolution product
$$ (f_1 * f_2)(g) := \sum_{h \in G^{r(o)}} f_1(h)f_2(h^{-1}g) \quad \text{for} \quad g \in G, $$
and involution $f^*(g) = \overline{f(\overline{g^{-1}})}$, where $G^x = r^{-1}(x)$. The full groupoid $C^*$-algebra $C^*(G)$ is defined as the $C^*$-enveloping algebra of $C_c(G)$.

For every $x \in G^{(0)}$, define a representation $\rho_x: C_c(G) \to B(\ell^2(G_x))$, where $G_x = s^{-1}(x)$, by
$$ (\rho_x(f)\xi)(g) = \sum_{h \in G^{r(o)}} f(h)\xi(h^{-1}g). $$
Then the reduced $C^*$-algebra $C^r_c(G)$ is defined as the completion of $C_c(G)$ with respect to the norm
$$ \|f\|_r = \sup_{x \in G^{(0)}} \|\rho_x(f)\|. $$

Recall (see, e.g., [Exc08, Section 3]) that for all $f \in C_c(G)$ we have the inequalities $\|f\|_\infty \leq \|f\|_r \leq \|f\|$, where $\|\cdot\|_\infty$ denotes the supremum-norm, and if $f \in C_c(U)$ for an open bisection $U$, then
$$ \|f\| = \|f\|_r = \|f\|_\infty. $$

For a closed (in $G^{(0)}$) invariant subset $X \subset G^{(0)}$, denote by $G_X$ the subgroupoid $r^{-1}(X) = s^{-1}(X) \subset G$. In the second countable case the next result and the subsequent corollary follow easily from Renault’s disintegration theorem, cf. [Ren91] Remark 4.10. The case of étale groupoids allows for the following elementary proof without any extra assumptions on $G$.

**Proposition 1.1.** Assume $G$ is a locally compact étale groupoid and $X \subset G^{(0)}$ is a closed invariant subset. Then the following sets coincide:

1. the kernel of the $*$-homomorphism $C^*(G) \to C^*(G_X)$, $C_c(G) \ni f \mapsto f|_{G_X}$;
2. the closure of $C_c(G \setminus G_X)$ in $C^*(G)$;
3. the closed ideal of $C^*(G)$ generated by $C_0(G^{(0)} \setminus X) \subset C_0(G^{(0)})$.

**Proof.** The sets in (2) and (3) coincide, since $C_c(G \setminus G_X)$ is an ideal in $C_c(G)$ (with respect to the convolution product) and for every $f \in C_c(G \setminus G_X)$ we can find $f' \in C_c(G^{(0)} \setminus X)$ such that $f * f' = f$. It is also clear that $C_c(G \setminus G_X)$ is contained in the kernel of the $*$-homomorphism $C^*(G) \to C^*(G_X)$. It follows that in order to prove the proposition it suffices to show that every representation of $C_c(G)$ on a Hilbert space that vanishes on $G_X$ is an ideal in $C_c(G \setminus G_X)$ factors through $C_c(G_X)$. For this, in turn, it suffices to prove that $C_c(G \setminus G_X)$ is dense, with respect to the norm on $C^*(G)$, in the space of functions $f \in C_c(G)$ such that $f|_{G_X} = 0$.

Let us first prove the following claim. Assume $f = \sum_{i=1}^n f_i \in C_c(G)$ satisfies $\|f|_{G_X}\|_\infty < \varepsilon$ for some $\varepsilon > 0$, $f_i \in C_c(U_i)$ and open bisections $U_i$. Then there exist functions $\tilde{f}_i \in C_c(U_i)$ such that
$$ f - \sum_{i=1}^n \tilde{f}_i \in C_c(G \setminus G_X) \quad \text{and} \quad \|\tilde{f}_i\|_\infty < 2^n \varepsilon \quad \text{for} \quad i = 1, \ldots, n. $$

The proof is by induction on $n$. As the base of induction we take $n = 0$, meaning that $f = 0$. In this case there is nothing to prove. So assume the claim is true for some $n \geq 0$. For the induction step assume $f \in C_c(G)$ satisfies $\|f|_{G_X}\|_\infty < \varepsilon$ and we can write $f = \sum_{i=1}^{n+1} f_i$ for some $f_i \in C_c(U_i)$ and open bisections $U_i$. Let $K_{n+1} \subset U_{n+1}$ be the support of $f_{n+1}|_{U_{n+1}}$. Consider the set $K = K_{n+1} \setminus \bigcup_{i=1}^n U_i$. As $f = f_{n+1}$ on $K$, we have $\|f_{n+1}|_{K \cap G_X}\|_\infty < \varepsilon$. Hence there exists an open neighbourhood $U$ of $K \cap G_X$ in $U_{n+1}$ such that $\|f_{n+1}|_{U}\|\infty < \varepsilon$. Let $V$ be an open neighbourhood of $K \setminus U$ in $U_{n+1}$ such that $V \cap U_{n+1} \cap G_X = \emptyset$. Then the open sets $U_1 \cap U_{n+1}, \ldots, U_n \cap U_{n+1}, U, V$
cover $K_{n+1}$. Hence we can find functions $\rho_1, \ldots, \rho_n, \rho_U, \rho_V \in C_c(U_{n+1})$ taking values in the interval $[0,1]$ such that $supp \rho_i \subset U_i \cap U_{n+1}$, $supp \rho_U \subset U$, $supp \rho_V \subset V$ and
\[ \sum_{i=1}^n \rho_i(g) + \rho_U(g) + \rho_V(g) = 1 \quad \text{for all} \quad g \in K_{n+1}. \]

Define $f'_i = f_i + \rho_i f_{n+1}$ (pointwise product) for $i = 1, \ldots, n$, $f' = \sum_{i=1}^n f'_i$ and $\tilde{f}_{n+1} = \rho_U f_{n+1}$. Then $f'_i \in C_c(U_i)$, $\tilde{f}_{n+1} \in C_c(U_{n+1})$ and we have
\[ f - f' - \tilde{f}_{n+1} = \rho_V f_{n+1} \in C_c(G \setminus G_X). \]
We also have $\|\tilde{f}_{n+1}\|_\infty \leq \|f_{n+1}\|_U < \varepsilon < 2^{n+1}\varepsilon$. It follows that
\[ \|f'|_{G_X} \| = \|(f - \tilde{f}_{n+1})|_{G_X} \| \leq \|f|_{G_X} \| + \|\tilde{f}_{n+1}|_{G_X} \| \leq 2\varepsilon. \]
We can therefore apply the inductive hypothesis to $f'$ and $2\varepsilon$ and find functions $\tilde{f}_i \in C_c(U_i)$, $i = 1, \ldots, n$, such that
\[ f' - \sum_{i=1}^n \tilde{f}_i \in C_c(G \setminus G_X) \quad \text{and} \quad \|\tilde{f}_i\|_\infty < 2^n 2\varepsilon = 2^{n+1}\varepsilon \quad \text{for} \quad i = 1, \ldots, n. \]

Then the functions $\tilde{f}_1, \ldots, \tilde{f}_{n+1}$ have the required properties.

Now, if $f \in C_c(G)$ satisfies $f|_{G_X} = 0$, we write $f = \sum_{i=1}^n f_i$ for some $f_i \in C_c(U_i)$ and open bisections $U_i$ and apply the above claim to an arbitrarily small $\varepsilon > 0$. Recalling that the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ coincide on $C_c(U)$ for any arbitrary small $\varepsilon > 0$. Recalling that the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ coincide on $C_c(U)$ for any open bisection $U$, we conclude that there is a function $\tilde{f} = \sum_{i=1}^n \tilde{f}_i \in C_c(G)$ such that $f - \tilde{f} \in C_c(G \setminus G_X)$ and $\|\tilde{f}\| \leq n2^n \varepsilon$. Hence $f$ lies in the closure of $C_c(G \setminus G_X)$. \hfill \Box

**Corollary 1.2.** We have a short exact sequence
\[ 0 \to C^*(G \setminus G_X) \to C^*(G) \to C^*(G_X) \to 0. \]

**Proof.** Since the restriction map $C_c(G) \to C_c(G_X)$ is surjective, the fact that the sets in (1) and (2) coincide implies that we have an exact sequence $C^*(G \setminus G_X) \to C^*(G) \to C^*(G_X) \to 0$. Therefore we only need to explain why the map $C^*(G \setminus G_X) \to C^*(G)$ is injective. For this it suffices to show that any nondegenerate representation $\pi: C^*(G \setminus G_X) \to B(H)$ extends to $C^*(G)$. Since $C_c(G \setminus G_X)$ is an ideal of $C_c(G)$, we can define a representation $\tilde{\pi}$ of $C_c(G)$ on $\pi(C_c(G \setminus G_X)) H$ by possibly unbounded operators in the standard way: for $f \in C_c(G)$, put $\tilde{\pi}(f)\pi(f')\xi = \pi(f \ast f')\xi$. On $C_c(G(0))$ this agrees with the unique extension of $\pi|_{C_0(G(0) \setminus X)}$ to a representation of $C_0(G(0))$. Hence $\|\tilde{\pi}(f)\| \leq \|f\|_\infty$ for $f \in C_c(G(0))$, and then $\|\tilde{\pi}(f)\| \leq \|f\|$ for any open bisection $U$ and $f \in C_c(U)$, as $f \ast f \in C_c(G(0))$. \hfill \Box

**Remark 1.3 (cf. [CN22] Remark 2.9).** Since the ideal $C_c(G \setminus G_X) \subset C_c(G)$ is dense with respect to the norm on $C^*(G)$ in the space of functions $f \in C_c(G)$ such that $f|_{G_X} = 0$, it is also dense with respect to the reduced norm. It follows that there is a $C^*$-norm on $C_c(G_X)$ dominating the reduced norm such that for the corresponding completion $C^*_e(G_X)$ the sequence
\[ 0 \to C^*_e(G \setminus G_X) \to C^*_e(G) \to C^*_e(G_X) \to 0 \]
is exact. \hfill \Diamond

We now turn to the question when a set of representations $\rho_y$, $y \in Y \subset G(0)$, determines the reduced norm on $C_c(G)$. It is easy to see that if $Y$ is $G$-invariant, which we may always assume since the equivalence class of $\rho_x$ depends only on the orbit of $x$, a necessary condition is that $Y$ is dense in $G(0)$. But this is not enough in the non-Hausdorff case. We start with the following sufficient condition.
Proposition 1.4. Let $\mathcal{G}$ be a locally compact étale groupoid, $Y \subset \mathcal{G}^{(0)}$ and $x \in \mathcal{G}^{(0)} \setminus Y$. Assume there is a net $(y_i)_i$ in $Y$ such that $x$ is the only accumulation point of $(y_i)_i$ in $\mathcal{G}^x = \mathcal{G}_x \cap \mathcal{G}^x$. Then the representation $\rho_x$ of $C^*_r(\mathcal{G})$ is weakly contained in $\bigoplus_{y \in Y} \rho_y$.

Proof. We may assume that $y_i \to x$. For every $g \in \mathcal{G}_x$ we then choose a net $(g_i)_i$ converging to $g$ as follows. Let $U$ be an open bisection containing $g$. Then for all $i$ large enough we have $y_i \in s(U)$, and for every such $i$ we take the unique point $g_i \in U \cap \mathcal{G}_{y_i}$. For all other indices $i$ we put $g_i = y_i$.

Take $g, h \in \mathcal{G}_x$. Observe that by our assumptions if $g \neq h$, then $g_i \neq h_i$ for all $i$ large enough, since otherwise we could first conclude that $r(g) = r(h)$ and then that $g^{-1}h \in \mathcal{G}_x^e$ is an accumulation point of $(y_i)_i$.

Next, take an open bisection $V$ and $f \in C_c(V)$. Then

$$(\rho_x(f)\delta_g, \delta_h) = f(hg^{-1}), \quad (\rho_{y_i}(f)\delta_{g_i}, \delta_{h_i}) = f(h_i^{-1}g_i^{-1}).$$

These equalities and the observation above imply that in order to prove the proposition it suffices to show that $f(h_i^{-1}g_i^{-1}) \to f(hg^{-1})$.

Assume first that $hg^{-1} \notin V$. As $V$ is open and $h_i^{-1}g_i^{-1} \to hg^{-1}$, it follows that eventually $h_i^{-1}g_i^{-1} \notin V$. But then $f(h_i^{-1}g_i^{-1}) \to f(hg^{-1})$ by the continuity of $f$ on $V$.

Assume next that $hg^{-1} \in V$. It is then enough to show that eventually $h_i^{-1}g_i^{-1}$ does not lie in the support $K$ of $f|_V$. Suppose this is not the case. Then by passing to a subnet we may assume that $h_i^{-1}g_i^{-1} \to w$ for some $w \in K$. Since we also have $h_i^{-1}g_i^{-1} \to hg^{-1}$, we must have $r(w) = r(h)$ and $s(w) = r(g)$. Then $h^{-1}wg \in \mathcal{G}_x^e$, $h^{-1}wg \neq x$ and $y_i = h_i^{-1}(h_i^{-1}g_i^{-1})g_i \to h^{-1}wg$, which contradicts our assumptions. \qed

Remark 1.5. The above proposition can also be deduced from results in [KS02, Section 2]. In order to make the connection to [KS02] more transparent, let us reformulate the assumptions of Proposition 1.4 as follows. The functions $f|_{\mathcal{G}^{(0)}}$ for $f \in C_c(\mathcal{G})$ generate a $C^*$-subalgebra $B$ of the algebra of bounded Borel functions on $\mathcal{G}^{(0)}$ equipped with the supremum-norm. Let $Z$ be the spectrum of $B$. As every point of $\mathcal{G}^{(0)}$ defines a character of $B$, we have an injective Borel map $i: \mathcal{G}^{(0)} \to Z$ with dense image. We claim that a net as in Proposition 1.4 exists if and only if $i(x) \in \bar{i(Y)}$.

In order to show this, assume first that $(y_i)_i$ is a net in $Y$ converging to $x$ and having no other accumulation points in $\mathcal{G}_x^e$. We claim that then $i(y_i) \to i(x)$. It suffices to show that $f(y_i) \to f(x)$ for every open bisection $U$ and $f \in C_c(U)$. If $x \in U$, this is true by continuity of $f|_U$. If $x \notin U$, then the net $(y_i)_i$ does not have any accumulation points in $U$ and therefore it eventually lies outside the support of $f|_U$, so in fact $f(y_i) \to f(x)$. Conversely, assume we have a net $(y_i)_i$ in $Y$ such that $i(y_i) \to i(x)$. Then obviously $y_i \to x$. Take $g \in \mathcal{G}_x^e \setminus \{x\}$, an open bisection $U$ containing $g$ and $f \in C_c(U)$ such that $f(g) \neq 0$. As $f(y_i) \to f(x) = 0$ by assumption, we conclude that $g$ cannot be an accumulation point of $(y_i)_i$.

Remark 1.6. For some $n \geq 1$, there are elements $g_1, \ldots, g_n \in \mathcal{G}_x^e \setminus \{x\}$, open bisections $U_1, \ldots, U_n$ such that $g_k \in U_k$ and a neighbourhood $U$ of $x$ in $\mathcal{G}^{(0)}$ satisfying $Y \cap U \subset U_1 \cup \cdots \cup U_n$.

Indeed, if this condition is satisfied, then any net in $Y$ converging to $x$ has one of the elements $g_1, \ldots, g_n$ as its accumulation point. Conversely, assume Condition 1.6 is not satisfied. For every $g \in \mathcal{G}_x^e \setminus \{x\}$ choose an open bisection $U_g$ containing $g$. Then for every finite set $F = \{g_1, \ldots, g_n\} \subset \mathcal{G}_x^e \setminus \{x\}$ and every neighbourhood $U$ of $x$ in $\mathcal{G}^{(0)}$ we can find $y_{F,U} \in (Y \cap U) \setminus (U_{g_1} \cup \cdots \cup U_{g_n})$. Then $(y_{F,U})_{F,U}$, with the obvious partial order defined by inclusion of $F$’s and containment of $U$’s, is the required net.
Remark 1.7. Following the terminology of [KM21], a point \( x \in \mathcal{G}^{(0)} \) is called dangerous if there is a net in \( \mathcal{G}^{(0)} \) converging to \( x \) and to a point in \( G^+ \setminus \{ x \} \). Therefore the set of points \( x \in (\mathcal{G}^{(0)} \cap Y) \setminus Y \) satisfying Condition 1.6 is a subset of dangerous points. As a consequence, if \( Y \) is dense in \( \mathcal{G}^{(0)} \) and \( \mathcal{G} \) can be covered by countably many open bisections, then by [KM21] Lemma 7.15 the set of points \( x \in \mathcal{G}^{(0)} \setminus Y \) satisfying Condition 1.6 is meager in \( \mathcal{G}^{(0)} \).

If \( Y \) is \( \mathcal{G} \)-invariant and Condition 1.6 is satisfied for \( n = 1 \), then \( \rho_x \) is not weakly contained in \( \bigoplus_{y \in Y} \rho_y \). In order to see this, take an open neighbourhood \( V \subset U \) of \( x \) such that \( V \subset \tau(U_1) \cap s(U_1) \) and a function \( f \in C_c(V) \) such that \( f(x) \neq 0 \). Then it is easy to check that \( 0 \neq f * (1_{U_1} - 1_U) * f \in \ker \rho_y \) for all \( y \in Y \). A simple example of such a situation is the real line with a double point at 0, cf. [KS02] Example 2.5.

But in general, as we will see soon, Condition 1.6 is not enough to conclude that \( \rho_x \) is not weakly contained in \( \bigoplus_{y \in Y} \rho_y \). A sufficient extra condition is given by the following proposition.

Proposition 1.8. Assume \( \mathcal{G} \) is a locally compact étale groupoid, \( Y \subset \mathcal{G}^{(0)} \) is a \( \mathcal{G} \)-invariant subset and \( x \in \mathcal{G}^{(0)} \setminus Y \) is a point satisfying Condition 1.6 such that
\[
\sum_{k=1}^{n} \frac{1}{\text{ord}(g_k)} < 1,
\]
where \( \text{ord}(g_k) \) is the order of \( g_k \) in \( G_x^* \). Then \( \rho_x \) is not weakly contained in \( \bigoplus_{y \in Y} \rho_y \).

For the proof we need the following simple lemma.

Lemma 1.9. Let \( A = C^*(a) \) be a \( C^* \)-algebra generated by a contraction \( a \). Assume that for some \( m \in \{ 2, 3, \ldots, +\infty \} \) we have a \(*\)-homomorphism \( \pi: A \to C^*(\mathbb{Z}/m\mathbb{Z}) \) such that \( \pi(a) = u \), where \( u \) is the unitary generator of \( C^*(\mathbb{Z}/m\mathbb{Z}) \). Take numbers \( \alpha > 0 \) and \( \varepsilon > 0 \) and denote by \( \Omega_{\alpha,\varepsilon} \) the convex set of states \( \varphi \) on \( A \) such that
\[
\varphi \geq \alpha \sum_{l=1}^{p} \lambda_l \chi_l
\]
for some \( p \geq 1 \), \( \lambda_1, \ldots, \lambda_p \geq 0 \), \( \sum_{l=1}^{p} \lambda_l = 1 \), and characters \( \chi_1, \ldots, \chi_p: A \to \mathbb{C} \) such that \( |1 - \chi_l(a)| < \varepsilon \) for all \( l \). Then, for every \( \alpha > 1/m \), there is \( \varepsilon > 0 \) depending only on \( m \) and \( \alpha \) such that \( \tau \circ \pi \) does not belong to the weak* closure of \( \Omega_{\alpha,\varepsilon} \), where \( \tau \) is the canonical trace on \( C^*(\mathbb{Z}/m\mathbb{Z}) \).

Here we use the convention \( \mathbb{Z}/m\mathbb{Z} = \mathbb{Z} \) for \( m = +\infty \).

Proof. Assume first that \( m \) is finite. Consider the positive element \( b \in A \) defined by
\[
b = \frac{1}{m^2} \sum_{k,l=1}^{m} (a^k)^* a^l.
\]
Then \( \tau(\pi(b)) = 1/m \). On the other hand, if \( \chi \) is a character on \( A \) such that \( |1 - \chi(a)| < \varepsilon \), then
\[
|1 - \chi(a)^k| \leq k|1 - \chi(a)| < m\varepsilon \quad \text{for all} \quad 1 \leq k \leq m,
\]
hence, assuming \( m\varepsilon < 1 \), we have
\[
\chi(b) = \left| \frac{1}{m} \sum_{k=1}^{m} \chi(a)^k \right|^2 > (1 - m\varepsilon)^2
\]
and therefore
\[
\varphi(b) \geq \alpha(1 - m\varepsilon)^2 \quad \text{for all} \quad \varphi \in \Omega_{\alpha,\varepsilon}.
\]
It follows that \( \tau \circ \pi \notin \Omega_{\alpha,\varepsilon} \) as long as \( \varepsilon \) is small enough so that \( \alpha(1 - m\varepsilon)^2 > 1/m \).

Assume now that \( m = +\infty \). Choose \( m' \geq 1 \) such that \( 1/m' < \alpha \). Then the same arguments as above with \( m \) replaced by \( m' \) show that \( \tau \circ \pi \notin \Omega_{\alpha,\varepsilon} \) as long as \( 1 - m'\varepsilon > 1/\sqrt{m'\alpha} \). \( \square \)
Proof of Proposition 1.8. Let $g_1, \ldots, g_n$ be as in the formulation of the proposition and $U, U_1, \ldots, U_n$ be given by Condition 1.6. Choose functions $f_k \in C_c(U_k)$ such that $0 \leq f_k \leq 1$ and $f_k(g_k) = 1$. Consider the $C^*$-subalgebras $A_k$ of $C^*_r(G)$ generated by $f_k$.

Consider the restriction map $C_c(G) \to C_c(G^*_k)$, $f \mapsto f|G^*_k$. It extends to a completely positive contraction $\vartheta_{x,r} : C^*_r(G) \to C^*_r(G^*_k)$, with the elements $f_k$ contained in its multiplicative domain, see [CN22] Lemmas 1.2 and 1.4. By restricting $\vartheta_{x,r}$ to $A_k$ we therefore get $\ast$-homomorphisms $\pi_k : A_k \to C^*_r(G^*_k)$. The image of $\pi_k$ is $C^*(G_k) \subset C^*_r(G^*_k)$, where $G_k$ is the subgroup of $G^*_k$ generated by $g_k$. Therefore if we let $m_k = \text{ord}(g_k)$, then we can view each $\pi_k$ as a $\ast$-homomorphism $A_k \to C^*(\mathbb{Z}/m_k\mathbb{Z})$.

Choose numbers $\alpha_k > 1/m_k$ such that $\sum_{k=1}^n \alpha_k < 1$. Let $\varepsilon > 0$ be given by Lemma 1.3 for the homomorphism $\pi_k : A_k \to C^*(\mathbb{Z}/m_k\mathbb{Z})$, $\alpha = \alpha_k$ and $a = f_k$. Put $\varepsilon = \min_{1 \leq k \leq n} \varepsilon_k$. Choose an open neighbourhood $V$ of $x$ in $G^{(0)}$ such that $V \subset U$ and

$$f_k(g) > 1 - \varepsilon \quad \text{for all} \quad g \in r^{-1}(V) \cap U_k \quad \text{and} \quad 1 \leq k \leq n.$$  

(1.1)

Let $f \in C_c(V)$ be such that $0 \leq f \leq 1$ and $f(x) = 1$.

Now, denote $\bigoplus_{y \in Y} \rho_y$ by $\rho_Y$ and assume that $\rho_x$ is weakly contained in $\rho_Y$. Denoting the canonical trace on $C^*_r(G^*_k)$ by $\tau$, it follows that $\tau \circ \vartheta_{x,r} = (\rho_x(\cdot)\delta_x, \delta_x)$ lies in the weak* closure of the states $\varphi$ of the form

$$\varphi = \sum_{i=1}^N (\rho_Y(\cdot)\xi_i, \xi_i),$$  

(1.2)

where $\xi_i$ are finitely supported functions on $s^{-1}(Y)$ such that

$$\sum_{i=1}^N ||\xi_i||^2 = \sum_{i=1}^N \sum_{g \in s^{-1}(Y)} |\xi_i(g)|^2 = 1.$$  

As $\tau(\vartheta_{x,r}(f)) = f(x) = 1$, it suffices to consider states such that

$$\varphi(f) > \sum_{k=1}^n \alpha_k.$$  

Since

$$\varphi(f) = \sum_{i=1}^N \sum_{g \in s^{-1}(Y)} f(r(g))|\xi_i(g)|^2 = \sum_{i=1}^N \sum_{g \in r^{-1}(Y)} f(r(g))|\xi_i(g)|^2,$$

where we used that $r^{-1}(Y) = s^{-1}(Y)$ by the invariance of $Y$, and $f$ is zero outside $V$, this implies that

$$\sum_{i=1}^N \sum_{g \in r^{-1}(Y \cap V)} |\xi_i(g)|^2 > \sum_{k=1}^n \alpha_k.$$  

(1.3)

We claim that then for some $k$ we must have $\varphi|_{A_k} \in \Omega^{(k)}_{\alpha_k, \varepsilon}$, where $\Omega^{(k)}_{\alpha_k, \varepsilon}$ is defined as in Lemma 1.9.

Denote by $X_k$ the finite set of pairs $(i, g)$, $1 \leq i \leq N$, $g \in r^{-1}(Y \cap V)$, such that $\xi_i(g) \neq 0$ and $r(g) \in U_k$. As $Y \cap V \subset U_1 \cup \cdots \cup U_n$ by assumption, the inequality (1.3) implies

$$\sum_{k=1}^n \sum_{(i, g) \in X_k} |\xi_i(g)|^2 > \sum_{k=1}^n \alpha_k.$$  

(1.4)

It follows that for some $k$ we have

$$\sum_{(i, g) \in X_k} |\xi_i(g)|^2 > \alpha_k.$$  

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If \((i,g) \in X_k\), then
\[
\rho_Y(f_k)\delta_g = \rho_Y(f_k^g)\delta_g = f(r(g))\delta_g.
\]
Therefore every such point \((i,g)\) defines a one-dimensional subrepresentation of \(\rho_Y|_{A_k}\) and a character \(\chi_{i,g}: A_k \to \mathbb{C}\) satisfying \(\chi_{i,g}(f_k) > 1 - \varepsilon\) by (1.4). Then on \(A_k\) we have
\[
(\rho_Y(\cdot)\xi_i, \xi_i) = (\rho_Y(\cdot)\tilde{\xi}_i, \tilde{\xi}_i) + \sum_{g(i,g) \in X_k} |\xi_i(g)|^2 \chi_{i,g},
\]
where \(\tilde{\xi}_i(g) = \xi_i(g)\) if \((i,g) \notin X_k\) and \(\tilde{\xi}_i(g) = 0\) otherwise. By (1.4) this implies that \(\varphi|_{A_k} \in \Omega_{\alpha_k,\varepsilon}\), proving our claim.

It follows that if there is a net of states of the form (1.2) that converges weakly* to \(\tau \circ \vartheta_{x,r}\), then by passing to a subnet \((\varphi_j)_j\) we can find an index \(k\), \(1 \leq k \leq n\), such that \(\varphi_j|_{A_k} \in \Omega_{\alpha_k,\varepsilon}\) for all \(j\). This contradicts Lemma 1.9.

By combining this with Proposition 1.8 we get the following criterion.

**Corollary 1.10.** Let \(G\) be a locally compact étale groupoid, \(Y \subset G^{(0)}\) a \(G\)-invariant subset and \(x \in (G^{(0)} \cap Y) \setminus Y\). Assume that for every finite set of distinct cyclic nontrivial subgroups \(G_1, \ldots, G_n\) of \(G_x\) we have
\[
\sum_{k=1}^n \frac{1}{|G_k|} < 1. \tag{1.5}
\]
Then \(\rho_x\) is weakly contained in \(\bigoplus_{y \in Y} \rho_y\) if and only if Condition 1.6 is not satisfied (equivalently, if and only if there is a net in \(Y\) such that \(x\) is its unique accumulation point in \(G_x\)).

**Proof.** The “only if” part follows from Proposition 1.8 by observing that if Condition 1.6 is satisfied for \(g_1, \ldots, g_n\) and some elements \(g_k\) and \(g_l\) \((k \neq l)\) generate the same subgroup, then Condition 1.6 is still satisfied if we omit \(g_k\) or \(g_l\). Therefore in Condition 1.6 we may assume in addition that the elements \(g_1, \ldots, g_n\) generate different subgroups.

Condition (1.5) is most probably not optimal, but as the following example shows, some assumptions are needed for the conclusion of the corollary to be true.

**Example 1.11.** Let \(X\) be the disjoint union of countably many copies of \(\{0,1\}\). Consider the involutive map \(S: X \to X\) that acts as a flip on every copy of \(\{0,1\}\). Take three copies \(X_1, X_2, X_3\) of \(X\) and let \(X^+\) be the one-point compactification of the discrete set \(X_1 \sqcup X_2 \sqcup X_3\). Define an action of \(\Gamma = (\mathbb{Z}/2\mathbb{Z})^2\) on \(X^+\) as follows: the element \((1,0)\) acts by \(S\) on \(X_1\) and \(X_2\) and trivially on \(X_3\) and \(\infty\), the element \((0,1)\) acts by \(S\) on \(X_2\) and \(X_3\) and trivially on \(X_1\) and \(\infty\). Consider the corresponding groupoid of germs \(\mathcal{G}\), so \(\mathcal{G}\) is the quotient of the transformation groupoid \(\Gamma \times X^+\) by the equivalence relation defined by \((h,x) \sim (g,x)\) iff \(hy = gx\) for all \(g\) in a neighbourhood of \(x\).
Thus, if we ignore the topology on \(\mathcal{G}\), our groupoid is the disjoint union of \(\Gamma \times \{\infty\} \cong \Gamma\) and three copies of \((\mathbb{Z}/2\mathbb{Z}) \ltimes_s X\).

Consider the set \(Y = X^+ \setminus \{\infty\}\) and the point \(x = \infty\). Condition 1.6 is satisfied for \(n = 3\), since \(Y\) is discrete and for every point \(y \in Y\) there is \(g \in \Gamma \setminus \{0\}\) that acts trivially on \(y\). We claim that nevertheless \(\rho_x\) is weakly contained in \(\bigoplus_{y \in Y} \rho_y\).

We have a short exact sequence
\[
0 \to C^*_r(\mathcal{G} \setminus G_x^+) \to C^*_r(\mathcal{G}) \xrightarrow{\rho_x} C^*(\Gamma) \to 0.
\]
It has a canonical splitting \(\psi: C^*(\Gamma) \to C^*_r(\mathcal{G})\) defined as follows. For every \(g \in \Gamma\) consider the image \(U_g\) of the set \(\{(g,x^+) \mid x^+ \in X^+\} \subset \Gamma \times X^+\) in \(\mathcal{G}\). The sets \(U_g \subset \mathcal{G}\) are bisections and their characteristic functions span a copy of \(C^*(\Gamma)\) in \(C^*_r(\mathcal{G})\). We define \(\psi(\lambda_g) = \mathbb{1}_{U_g}\).
For \( i = 1, 2, 3 \), let \( x_{in} \) be 0 in the \( n \)th copy of \( \{0, 1\} \) in \( X_i \). Then every \( f \in C_c(G \setminus G_x^0) \) is contained in \( \ker \rho_{x_{in}} \) for all \( n \) sufficiently large. On the other hand, \( \rho_{x_{in}} \circ \psi \) is equivalent to the representation \( \lambda_i \) obtained by composing the regular representation of \( C^*(\mathbb{Z}/2\mathbb{Z}) \) with the homomorphism \( C^*(\Gamma) \rightarrow C^*(\mathbb{Z}/2\mathbb{Z}) \) defined by one of the three nontrivial homomorphisms \( \Gamma \rightarrow \mathbb{Z}/2\mathbb{Z} \). Namely, for \( i = 1 \) we get the homomorphism that maps \((1, 0)\) and \((1, 1)\) into \( 1 \in \mathbb{Z}/2\mathbb{Z} \), for \( i = 2 \) it maps \((1, 0)\) and \((0, 1)\) into \( 1 \), and for \( i = 3 \) it maps \((0, 1)\) and \((1, 1)\) into \( 1 \). As \( \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \) is a faithful representation of \( C^*(\Gamma) \) and \( C_c(G \setminus G_x^0) + \psi(C^*(\Gamma)) \) is dense in \( C^*_r(G) \), it follows that \( \rho_x \) is weakly contained in \( \bigoplus_{n=1}^{\infty} \bigoplus_{i=1}^{3} \rho_{x_{in}} \).

We finish the section with a short discussion of essential groupoid \( C^* \)-algebras. Following [KM21], define
\[
J_{\text{sing}} = \{ a \in C^*_r(G) \mid \text{the set of } x \in G^{(0)} \text{ such that } \rho_x(a) \delta_x \neq 0 \text{ is meager} \}.
\]
This is a closed ideal in \( C^*_r(G) \); in order to see that it is a right ideal, note that if \( U \subset G \) is an open bisection and \( f \in C_c(U) \), then for all \( x \in s(U) \) we have
\[
\| \rho_x(a \ast f) \delta_x \| = |f(g_x)| \| \rho_{T(x)}(a) \delta_{T(x)} \|,
\]
where \( g_x \) is the unique element in \( U \cap G_x \) and \( T : s(U) \rightarrow r(U) \) is the homeomorphism defined by \( T(x) = r(g_x) \). The essential groupoid \( C^* \)-algebra of \( G \) is defined by
\[
C^*_{\text{ess}}(G) = C^*_r(G)/J_{\text{sing}}.
\]

**Proposition 1.12** (cf. [KM21 Proposition 7.18]). Assume \( G \) is a locally compact étale groupoid that can be covered by countably many open bisections. Let \( D_0 \subset G^{(0)} \) be the set of points \( x \in G^{(0)} \) satisfying the following property: there exist elements \( g_1, \ldots, g_n \in G_x^0 \setminus \{x\} \), open bisections \( U_1, \ldots, U_n \) and an open neighbourhood \( U \) of \( x \) in \( G^{(0)} \) such that \( g_k \in U_k \) for all \( k \) and \( U \setminus (U_1 \cup \cdots \cup U_n) \) has empty interior. Let \( Y \) be a dense subset of \( G^{(0)} \setminus D_0 \). Then
\[
J_{\text{sing}} = \{ a \in C^*_r(G) \mid \text{the set of } x \in G^{(0)} \text{ such that } \rho_x(a) \neq 0 \text{ is meager} \} = \bigcap_{y \in Y} \ker \rho_y.
\]
In particular, if \( D_0 = \emptyset \), then \( J_{\text{sing}} = 0 \).

Before we turn to the proof, let us make the connection to [KM21] more explicit. Let \( D \subset G^{(0)} \) be the set of dangerous points [KM21], that is, points \( x \in G^{(0)} \) such that there is a net in \( G^{(0)} \) converging to \( x \) and to an element of \( G_x^0 \setminus \{x\} \). It is easy to see then that \( D_0 \subset D \). (Should the points of \( D_0 \) be called extremely dangerous?)

**Proof of Proposition 1.12.** Let \( (U_n) \) be a sequence of open bisections covering \( G \). For every \( n \), let \( T_n : s(U_n) \rightarrow r(U_n) \) be the homeomorphism defined by \( U_n \) and \( g_n : s(U_n) \rightarrow U_n \) be the inverse of \( s : U_n \rightarrow s(U_n) \), so \( T_n(x) = r(g_n(x)) \). Similarly to (1.6), for all \( a \in C^*_r(G) \) and \( x \in s(U_n) \), we have
\[
\| \rho_x(a) \delta_{g_n(x)} \| = \| \rho_{T_n(x)}(a) \delta_{T_n(x)} \|.
\]
It follows that if \( a \in J_{\text{sing}} \), then the set \( A_n \) of points \( x \in s(U_n) \) such that \( \rho_x(a) \delta_{g_n(x)} \neq 0 \) is meager in \( G^{(0)} \). Then the set \( \cup_n A_n \) is meager as well. Since it coincides with the set of points \( x \in G^{(0)} \) such that \( \rho_x(a) \neq 0 \), this proves the first equality of the proposition.

For the second equality, observe first that if \( x \in G^{(0)} \setminus D \) and \( \rho_x(a) \neq 0 \) for some \( a \in C^*_r(G) \), then \( \rho_z(a) \neq 0 \) for all \( z \) close to \( x \). This follows from [KM21] Lemma 7.15 or our Proposition 1.14 since otherwise we could find a net \( (x_i) \) converging to \( x \) such that \( \rho_x(a) = 0 \) for all \( i \) and then conclude that \( \rho_x(a) = 0 \), as \( \rho_x \) is weakly contained in \( \bigoplus_{i=1}^{\infty} \rho_{x_i} \).

The observation implies that if \( a \in \cap_{y \in Y} \ker \rho_y \), then \( \rho_x(a) = 0 \) for all \( x \in \bar{Y} \setminus D \). The set \( D \) is meager by [KM21] Lemma 7.15. As \( D_0 \subset D \), it follows that \( Y \) is dense in \( G^{(0)} \). Therefore if \( a \in \cap_{y \in Y} \ker \rho_y \), then \( \rho_x(a) \) can be nonzero only for elements \( x \) of the meager set \( D \), hence \( a \in J_{\text{sing}} \).
Conversely, assume \( a \in J_{\text{sing}} \). Then the observation above implies that \( \rho_x(a) = 0 \) for all \( x \in G^{(0)} \setminus D \). Therefore to finish the proof it suffices to show that \( \rho_x(a) = 0 \) for all \( x \in D \setminus D_0 \). By Proposition \[4.3\] for this, in turn, it suffices to show that for every \( x \in D \setminus D_0 \) Condition \[4.6\] is not satisfied for \( Y = G^{(0)} \setminus D \). Assume this condition is satisfied for some \( x \in D \), that is, there exist elements \( g_1, \ldots, g_n \in G_{x}^{(0)} \setminus \{x\} \), open bisections \( U_1, \ldots, U_n \) such that \( g_k \in U_k \) and a neighbourhood \( U \) of \( x \) in \( G^{(0)} \) satisfying \( U \setminus D \subset U_1 \cup \cdots \cup U_n \). As the set \( D \) is meager, this implies that \( x \in D_0 \). □

2. \( \text{C}^* \)-algebras associated with left cancellative monoids

Let \( S \) be a left cancellative monoid with identity element \( e \). Consider its left regular representation
\[
\lambda : S \to B(\ell^2(S)), \quad \lambda_s \delta_t = \delta_{st}.
\]
The reduced \( \text{C}^* \)-algebra \( C_r^*(S) \) of \( S \) is defined as the \( \text{C}^* \)-algebra generated by the operators \( \lambda_s, s \in S \).

Consider the left inverse hull \( I_e(S) \) of \( S \), that is, the inverse semigroup of partial bijections on \( S \) generated by the left translations \( S \to S \). Whenever convenient we view \( S \) as a subset of \( I_e(S) \) by identifying \( s \) with the left translation by \( s \). For \( s \in S \), we denote by \( s^{-1} \in I_e(S) \) the bijection \( ss^{-1} \to S \) inverse to the bijection \( S \to ss^{-1}, t \mapsto st \). If the map with the empty domain is present in \( I_e(S) \), we denote it by 0.

Let \( E(S) \) be the abelian semigroup of idempotents in \( I_e(S) \). Every element of \( E(S) \) is the identity map on its domain of definition \( X \subseteq S \), which is a right ideal in \( S \) of the form
\[
X = s_1^{-1}t_1 \ldots s_n^{-1}t_nS
\]
for some \( s_1, \ldots, s_n, t_1, \ldots, t_n \in S \). Such right ideals are called constructible \[Li12\]. We denote by \( J(S) \) the collection of all right constructible ideals. It is a semigroup under the operation of intersection, and we have an isomorphism \( E(S) \cong J(S) \). Denote by \( p_X \in E(S) \) the idempotent corresponding to \( X \in J(S) \).

Denote by \( \widehat{E(S)} \) the collection of semi-characters of \( E(S) \), that is, semigroup homomorphisms \( E(S) \to \{0,1\} \) that are not identically zero, where \( \{0,1\} \) is considered as a semigroup under multiplication. Note that every semi-character \( \chi \in \widehat{E(S)} \) must satisfy \( \chi(p_S) = 1 \). If \( 0 \in I_e(S) \), then denote by \( \chi_0 \) the semi-character that is identically one. This is the unique semi-character satisfying \( \chi_0(0) = 1 \). The set \( \widehat{E(S)} \) is compact Hausdorff in the topology of pointwise convergence.

Consider the Paterson groupoid \( G(I_e(S)) \) associated with \( I_e(S) \) \[Pat99\]:
\[
G(I_e(S)) = \Sigma / \sim_P, \quad \text{where} \quad \Sigma = \{(g, \chi) \in I_e(S) \times \widehat{E(S)} | \chi(g^{-1}g) = 1\}
\]
and the equivalence relation \( \sim_P \) is defined by declaring \( (g_1, \chi_1) \) and \( (g_2, \chi_2) \) to be equivalent if and only if
\[
\chi_1 = \chi_2 \quad \text{and there exists} \quad p \in E(S) \quad \text{such that} \quad g_1p = g_2p \quad \text{and} \quad \chi_1(p) = 1.
\]
We denote by \([g, \chi]\) the class of \((g, \chi) \in \Sigma \) in \( G(I_e(S)) \). The product is defined by
\[
[g, \chi][h, \psi] = [gh, \psi] \quad \text{if} \quad \chi = \psi(h^{-1} \cdot h).
\]
In particular, the unit space \( G(I_e(S))^{(0)} \) can be identified with \( \widehat{E(S)} \) via the map \( \widehat{E(S)} \to G(I_e(S)) \), \( \chi \mapsto [p_S, \chi] \), the source and range maps are given by
\[
s([g, \chi]) = \chi, \quad r([g, \chi]) = \chi(g^{-1} \cdot g),
\]
while the inverse is given by \([g, \chi]^{-1} = [g^{-1}, \chi(g^{-1} \cdot g)]\).

For a subset \( U \) of \( \widehat{E(S)} \), define
\[
D(g, U) = \{[g, \chi] \in G(I_e(S)) | \chi \in U\}.
\]
Then the topology on $G(I_{\ell}(S))$ is defined by taking as a basis the sets $D(g,U)$, where $g \in I_{\ell}(S)$ and $U$ is an open subset of the clopen set $\{\chi \in \hat{E}(S) \mid \chi(g^{-1}g) = 1\}$. This turns $G(I_{\ell}(S))$ into a locally compact, but not necessarily Hausdorff, étale groupoid.

For every $s \in S$ define a semi-character $\chi_s \in \hat{E}(S)$ by

$$\chi_s(p_X) = 1_X(s).$$

The following lemma is a groupoid version of the observation of Norling [Nor14, Section 3] on a connection between the regular representations of $S$ and $I_{\ell}(S)$. A closely related result was also proved by Spielberg [Spi20, Proposition 11.4].

**Lemma 2.1.** Put $G = G(I_{\ell}(S))$ and $Z = G^{(0)} = \widehat{E(S)}$. Then the map $S \to G_{\chi_e}$, $s \mapsto [s, \chi_e]$, is a bijection. If we identify $S$ with $G_{\chi_e}$ using this map, so that the representation $\rho_{\chi_e}$ of $C^*_r(G)$ is viewed as a representation on $\ell^2(S)$, then

$$\rho_{\chi_e}(C^*_r(G)) = C^*_r(S) \text{ and } \rho_{\chi_e}(1_D(s,Z)) = \lambda_s \text{ for all } s \in S.$$

**Proof.** Since $\chi_e(p_J) = 0$ for every constructible ideal $J$ different from $S$, we have $[s, \chi_e] = [t, \chi_e]$ only if $s = t$. This shows that the map $S \to G_{\chi_e}$, $s \mapsto [s, \chi_e]$, is injective. In order to prove that it is surjective, assume that $(g, \chi_e) \in S$, that is, $\chi_e(g^{-1}g) = 1$, for some $g \in I_{\ell}(S)$. This means that the domain of definition of $g$ contains $e$, and since this domain is a right ideal, it must coincide with $S$. But then if $s \in S$ is the image of $e$ under the action of $g$, we must have $g(t) = g(e)t = st$ for all $t \in S$, so $g = s$, which proves the surjectivity.

Next, as $\chi_e(t^{-1}, t) = \chi_t$ and $[s, \chi_e][t, \chi_e] = [st, \chi_e]$, we immediately get that $\rho_{\chi_e}(1_D(s,Z)) = \lambda_s$ for all $s \in S$. It is not difficult to see that the $C^*$-algebra $C^*_r(G)$ is generated by the elements $1_D(s,Z)$. (One can also refer to [Pat99, Theorem 4.4.2] that shows that the $C^*$-algebra $C^*_r(G)$ is the reduced $C^*$-algebra of the inverse semigroup $I_{\ell}(S)$, which is generated by $S$.) Hence $\rho_{\chi_e}(C^*_r(G))$ is exactly $C^*_r(S)$. \qed

This lemma leads naturally to a candidate for a groupoid model for $C^*_r(S)$: define

$$G_P(S) := G(I_{\ell}(S))\Omega(S),$$

where $\Omega(S) \subset G(I_{\ell}(S))^{(0)}$ is the closure of the $G(I_{\ell}(S))$-orbit of $\chi_e$. As $\chi_e(s^{-1} \cdot s) = \chi_s$, by Lemma 2.1 this orbit is exactly the set of semi-characters $\chi_s$, $s \in S$. The closure of this set is known and easy to find, cf. [CELY17, Corollary 5.6.26]: the set $\Omega(S) = \{\chi_s \mid s \in S\} \subset \hat{E}(S)$ consists of the semi-characters $\chi$ satisfying the properties

(i) if $0 \in I_{\ell}(S)$, then $\chi \neq \chi_0$ (equivalently, $\chi(0) = 0$);
(ii) if $\chi(p_X) = 1$ and $X = X_1 \cup \cdots \cup X_n$ for some $X, X_1, \ldots, X_n \in J(S)$, then $\chi(p_{X_i}) = 1$ for at least one index $i$.

The groupoid $G_P(S)$ is denoted by $I_{\ell} \ltimes \Omega$ in [Li21].

We are now in the setting of Section 1 with $G = G_P(S)$ and $Y = \{\chi_s \mid s \in S\}$ a dense invariant subset of $G^{(0)}$. Negation of Condition $1.6$ leads to the following definition.

**Definition 2.2.** We say that $S$ is strongly $C^*$-regular if, given elements $h_1, \ldots, h_n \in I_{\ell}(S)$ and constructible ideals $X, X_1, \ldots, X_m \in J(S)$ satisfying

$$\emptyset \neq X \setminus \bigcup_{i=1}^m X_i \subset \bigcup_{k=1}^n \{s \in S : h_k s = s\},$$

then
there are constructible ideals $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ and indices $1 \leq k_j \leq n \ (j = 1, \ldots, l)$ such that
\[ X \setminus \bigcup_{i=1}^m X_i \subseteq \bigcup_{j=1}^l Y_j \quad \text{and} \quad h_{kj}p_{Y_j} = p_{Y_j} \quad \text{for all} \quad 1 \leq j \leq l. \quad (2.2) \]

**Lemma 2.3.** Condition 1.6 is not satisfied for $\mathcal{G} = \mathcal{G}_F(S), Y = \{\chi_s \mid s \in S\}$ and every $x \in \mathcal{G}^{(0)} \setminus Y$ if and only if $S$ is strongly $C^*$-regular.

**Proof.** Assume first that $S$ is strongly $C^*$-regular. Suppose there is $\chi \in \mathcal{G}^{(0)} \setminus Y$ such that Condition 1.6 is satisfied for $x = \chi$, and let $g_k = [h_k, \chi], U_k (1 \leq k \leq n)$ and $U$ be as in that condition. We may assume that $U_k = D(h_k, \Omega(S))$ and
\[ U = \{\eta \in \Omega(S) \mid \eta(p_X) = 1, \eta(p_{X_s}) = 0 \text{ for } i = 1, \ldots, m\} \quad (2.3) \]
for some $X, X_1, \ldots, X_m \in \mathcal{J}(S)$. Then Condition 1.6 says that for every $s \in X \setminus \bigcup_{i=1}^m X_i$ there is $k$ such that $\chi_s \in U_k$, that is, $h_k s = s$. By the strong $C^*$-regularity we can find $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ satisfying (2.2). As $\chi \in \Omega(S)$, there must exist $j$ such that $\chi(p_{Y_j}) = 1$. But then $g_{kj} = [h_{kj}, \chi] = \chi$, which contradicts the assumption that $g_1, \ldots, g_n$ are nontrivial elements of the isotropy group $G^*_X$.

Assume now that $S$ is not strongly $C^*$-regular, so there are elements $h_1, \ldots, h_n \in I_\ell(S)$ and constructible ideals $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ such that (2.1) holds but (2.2) doesn’t for any choice of $Y_j$ and $k_j$. In other words, if we consider the set $\mathcal{F}$ of all constructible ideals $J$ such that there is $k$ (depending on $J$) satisfying $h_{kj}p_J = p_J$, then for any finite set $F \subset \mathcal{F}$ we have
\[ X \setminus \left( \bigcup_{i=1}^m X_i \cup \bigcup_{J \in F} J \right) \neq \emptyset. \]

Pick a point $s_F$ in the above set and consider a cluster point $\chi$ of the net $(\chi_{s_F})_F$, where $F$’s are partially ordered by inclusion. Then $\chi$ lies in the set $U$ defined by (2.2), and $\chi(p_{s_F}) = 0$ for all $J \in \mathcal{F}$. The semi-character $\chi$ cannot be of the form $\chi_s$, since otherwise we must have $s \in X \setminus \bigcup_{i=1}^m X_i$, and then $s \in \mathcal{F}$ and $\chi(p_{s \in \mathcal{F}}) = \chi(p_{s \in \mathcal{F}}) = 1$, which is a contradiction.

We claim that it is possible to replace $U$ by a smaller neighbourhood of $\chi$ and discard some of the elements $h_k$ in such a way that Condition 1.6 gets satisfied for $x = \chi, g_k = [h_k, \chi]$ and $U_k = D(h_k, \Omega(S))$. Namely, if $(h_k, \chi) \not\subseteq \Sigma$ for some $k$, then we add dom $h_k$ to the collection $\{X_1, \ldots, X_m\}$ and discard such $h_k$. If $(h_k, \chi) \subseteq \Sigma$ but $\chi(h_k^{-1} \cdot h_k) \neq \chi$, then $\chi(h_k^{-1} \cdot h_k) \neq \chi$ for all $\chi$ close $\chi$, so by replacing $X$ by a smaller ideal and adding more constructible ideals to $\{X_1, \ldots, X_m\}$ we may assume that $\chi(h_k^{-1} \cdot h_k) = \chi_s$ for all $\chi_s \in U$ and again discard such $h_k$. For the remaining elements $h_k$ and the new $U$ we have that for every $s \in S$ such that $\chi_s \in U$ there is an index $k$ satisfying $h_k s = s$. Then, in order to show that Condition 1.6 is satisfied, it remains to check that the elements $g_k = [h_k, \chi]$ of $G^*_X$ are nontrivial. But this is clearly true, since $\chi(p_{s_F}) = 0$ for every $J \in \mathcal{J}(S)$ such that $h_{kj}p_J = p_J$. \hfill \qed

**Remark 2.4.** From the last part of the proof we see that in Definition 2.2 we may assume in addition that $X \subseteq \text{dom } h_k$ for all $k$. More directly this can be seen as follows. Assume (2.1) is satisfied. Consider the nonempty subsets $F \subset \{1, \ldots, n\}$ such that
\[ X_F := X \cap \left( \bigcap_{k \in F} \text{dom } h_k \right) \not\subseteq \bigcup_{i=1}^m X_i \cup \bigcup_{k \not\in F} \text{dom } h_k. \]
Then (2.1) is satisfied for $X_F, \{X_1, \ldots, X_m, \text{dom } h_k \ (k \not\in F)\}$ and $\{h_k \ (k \in F)\}$ in place of $X, \{X_1, \ldots, X_m\}$ and $\{h_k \ (1 \leq k \leq n)\}$. Since
\[ X \setminus \bigcup_{i=1}^m X_i \subseteq \bigcup_{F} \left( X_F \setminus \left( \bigcup_{i=1}^m X_i \cup \bigcup_{k \not\in F} \text{dom } h_k \right) \right), \]
we conclude that if for every $F$ condition $[22]$ can be satisfied for $X_F$, \{X_1, \ldots, X_m, \text{dom} \ h_k \ (k \notin F)\}$ and \{h_k \ (k \in F)\}, then it can be satisfied for $X$, \{X_1, \ldots, X_m\} and \{h_k \ (1 \leq k \leq n)\} as well. \hfill \Box

Since the points $\chi_s$, $s \in S$, lie on the same $G_P(S)$-orbit, the corresponding representations $\rho_{\chi_s}$ of $C^*_\text{e}(G_P(S))$ are mutually equivalent. Thus, by Lemma 2.1 and Proposition 1.4 we get the following result.

Proposition 2.5. If $S$ is a strongly $C^*$-regular left cancellative monoid, then the representation $\rho_{\chi_e}$ of $C^*_\text{e}(G_P(S))$ defines an isomorphism $C^*_\text{e}(G_P(S)) \cong C^*(S)$.

Therefore if $S$ is strongly $C^*$-regular, it is natural to define the full semigroup $C^*$-algebra of $S$ by

$$C^*(S) := C^*(G_P(S)).$$

As $C^*(G(I_\ell(S)))$ has a known description in terms of generators and relations [Pat99], we can quickly obtain such a description for $C^*(G_P(S))$ as well.

Proposition 2.6 (cf. [Spi20] Theorem 9.4,[LS22] Definition 3.6). Assume $S$ is a countable left cancellative monoid. Consider the elements $v_s = 1_{p_s(\Omega(S)))} \in C^*(G_P(S))$, $s \in S$. Then $C^*(G_P(S))$ is a universal unital $C^*$-algebra generated by the elements $v_s$, $s \in S$, satisfying the following relations:

(R1) $v_e = 1$;
(R2) for every $g = s_1^{-1}t_1 \ldots s_n^{-1}t_n \in I_\ell(S)$, the element $v_g := v_{s_1}^*v_{t_1} \ldots v_{s_n}^*v_{t_n}$ is independent of the presentation of $g$;
(R3) if $0 \in I_\ell(S)$, then $v_0 = 0$;
(R4) if $X = X_1 \cup \cdots \cup X_n$ for some $X, X_1, \ldots, X_n \in \mathcal{J}(S)$, then

$$\prod_{i=1}^n (v_{p_{X_i}} - v_{p_{X}}) = 0.$$  

Note that relation (R4) is unambiguous, since relation (R2) implies that the elements $v_{p_{X_i}}$, $X \in \mathcal{J}(S)$, are mutually commuting projections.

Proof. Consider a universal unital $C^*$-algebra with generators $v_s$, $s \in S$, satisfying relations (R1) and (R2). This is nothing else than the full $C^*$-algebra $C^*(I_\ell(S))$ of the inverse semigroup $I_\ell(S)$, which is by definition generated by elements $v_g$, $g \in I_\ell(S)$, satisfying the relations

$$v_g v_h = v_{gh}, \quad v_g^* = v_{g^{-1}}.$$  

By [Pat99] Theorem 4.4.1, we can identify $C^*(I_\ell(S))$ with $C^*(G(I_\ell(S)))$. As $G_P(S) = G(I_\ell(S))_{\Omega(S)}$, it follows that in order to prove the proposition it remains to show that relations (R3) and (R4) describe the quotient $C^*(G(I_\ell(S))_{\Omega(S)})$ of $C^*(G(I_\ell(S)))$. By Proposition 1.4, the kernel of the map $C^*(G(I_\ell(S))) \to C^*(G(I_\ell(S))_{\Omega(S)})$ is generated as a closed ideal by the functions

$$f \in C(G(I_\ell(S))_{\Omega(S)}) \to C(\bar{E}(S))$$  

vanishing on $\Omega(S)$. The $C^*$-algebra $C(\bar{E}(S))$ is a universal $C^*$-algebra generated by the projections $e_X := 1_{U_X}$, $X \in \mathcal{J}(S)$, where $U_X = \{\eta \in E(S) : \eta(p_X) = 1\}$, satisfying the relations $e_X e_Y = e_{X \cap Y}$. By the definition of $\Omega(S)$, relations (R3) and (R4), with $e_X$ instead of $v_{p_{X}}$, describe the quotient $C(\Omega(S))$ of $C(\bar{E}(S))$. This gives the result. \hfill \Box

Remark 2.7. The assumption of countability of $S$ is certainly not needed in the above proposition, we added it to be able to formally apply results of [Pat99].

Remark 2.8. Relation (R2) can be slightly relaxed, cf. [LS22] Definition 3.6: it suffices to require that the elements $v_g$ are well-defined only for $g = p_X$, $X \in \mathcal{J}(S)$. Indeed, then, given $g = a_1^{-1}b_1 \ldots a_n^{-1}b_n = c_1^{-1}d_1 \ldots c_m^{-1}d_m \in I_\ell(S)$, for the elements $v = v_{a_1}^*v_{b_1} \ldots v_{a_n}^*v_{b_n}$ and $w = v_{c_1}^*v_{d_1} \ldots v_{c_m}^*v_{d_m}$ we have $v^*v = v^*w = w^*v = w^*w$, hence $(v - w)^*(v - w) = 0$ and $v = w$. \hfill \Box
Let us next give a few sufficient conditions for strong $C^*$-regularity. Recall that a left cancellative monoid $S$ is called finitely aligned [Spi20], or right (ideal) Howson [ES18], if the right ideal $sS \cap tS$ is finitely generated for all $s, t \in S$. If $S$ is finitely aligned, then by induction on $n$ one can see that the constructible ideals $s_1^{-1}t_1 \ldots s_n^{-1}t_nS$ are finitely generated.

**Proposition 2.9.** A left cancellative monoid $S$ is strongly $C^*$-regular if either of the following conditions is satisfied:

1. the groupoid $G_P(S)$ is Hausdorff;
2. the monoid $S$ is group embeddable;
3. the monoid $S$ is finitely aligned.

We remark that apart from the easy implication (2) $\Rightarrow$ (1), which will be explained shortly, there are no relations between conditions (1)–(3). For example, certain Baumslag–Solitar monoids are group embeddable but are not finitely aligned [Spi12, Lemma 2.12]. Examples of finitely aligned monoids with non-Hausdorff $G_P(S)$ can be found among the Zappa–Szép products $G \bowtie X^*$ defined by self-similar actions of groups on free monoids with infinite number of generators [Law08]; see Remark 3.9 for a related in spirit example with trivial group of units. Nevertheless, if $S$ is finitely aligned and right cancellative, then $G_P(S)$ is Hausdorff, see [Spi20, Lemma 7.1] or [Li21, Remark 4.3].

**Proof of Proposition 2.9.** Condition (1) is obviously sufficient for strong $C^*$-regularity, since Condition 1.6 can be satisfied only for non-Hausdorff groupoids.

Condition (2) is known, and is easily seen, to be stronger than (1): if $S$ is a submonoid of a group $G$, then every nonzero element $g$ of $I_t(S)$ acts by the left translation by an element $h_g \in G$, and we have either $h_g = e$ and $D(g, \Omega(S)) \subset G_P(S)(0)$ or $h_g \neq e$ and $D(g, \Omega(S)) \cap G_P(S)(0) = \emptyset$.

Assume now that $S$ is finitely aligned and (2.1) is satisfied. Choose a finite set of generators of the right ideal $X$. Let $s_1, \ldots, s_l$ be those generators that do not lie in $X_1 \cup \ldots \cup X_m$. By assumption, for every $1 \leq j \leq l$ we can find $1 \leq k_j \leq n$ such that $h_{k_j}s_j = s_j$. Then (2.2) is satisfied for $Y_j = s_jS$.

**Remark 2.10.** By [Li21, Lemma 4.1], the groupoid $G_P(S)$ is Hausdorff if and only if whenever $g \in I_t(S)$ and $\{s \in S \mid gs = s\} \neq \emptyset$, there are $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ such that $\{s \in S \mid gs = s\} = Y_1 \cup \ldots \cup Y_l$. Using this characterization one can easily see that (1) implies strong $C^*$-regularity without relying on Lemma 2.3.

In addition to $G_P(S)$ there is another closely related groupoid associated to $S$, which was introduced by Spielberg [Spi20] and which we will now turn to.

Consider the collection $\mathcal{J}(S)$ of subsets of $S$ obtained by adding to $\mathcal{J}(S)$ all sets of the form $X \setminus \bigcup_{i=1}^m X_i$ for $X, X_1, \ldots, X_m \in \mathcal{J}(S)$. It is a semigroup under intersection. When we want to view $E(S)$ as its subsemigroup, we will write $p_X$ instead of $X$ for the elements of $\mathcal{J}(S)$. Every $\chi \in \Omega(S)$ extends to a semi-character on $\mathcal{J}(S)$ by letting $\chi(p_X \setminus \bigcup_{i=1}^m X_i) = 1$ for $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ if $\chi(p_X) = 1$ and $\chi(p_{X_i}) = 0$ for all $i$, and $\chi(p_X \setminus \bigcup_{i=1}^m X_i) = 0$ otherwise.

Now, define an equivalence relation $\sim$ on $G_P(S)$ by declaring $[g, \chi] \sim [h, \chi]$ iff there exists $X \in \mathcal{J}(S)$ such that $\chi(p_X) = 1$ and $g|_X = h|_X$. Consider the quotient groupoid

$$G(S) := G_P(S)/\sim.$$  

This groupoid is denoted by $G_2(S)$ in [Spi20] and by $I_t \bowtie \Omega$ in [Li21].

Similarly to Lemma 2.1, the representation $\rho_{\chi_s}$ of $C^*_\sigma(G(S))$ defines a surjective $*$-homomorphism $C^*_\sigma(G(S)) \to C^*_\sigma(S)$. Negation of Condition 1.6 for $G = G(S)$, $Y = \{\chi_s \mid s \in S\}$ and all $x \in G^{(0)} \setminus Y$ leads to the following definition.

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**Definition 2.11.** We say that $S$ is $C^*$-regular if, given $h_1, \ldots, h_n \in I_\ell(S)$ and $X \in \mathcal{J}(S)$ satisfying
\[
\emptyset \neq X \subset \bigcup_{k=1}^{n} \{s \in \{h_k s = s\},
\]
there are sets $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ and indices $1 \leq k_j \leq n$ ($j = 1, \ldots, l$) such that
\[
X \subset \bigcup_{j=1}^{l} Y_j \quad \text{and} \quad h_{k_j}|_{Y_j} = \text{id} \quad \text{for all} \quad 1 \leq j \leq l.
\]

Note that by the same argument as in Remark 2.4 in order to check $C^*$-regularity it suffices to consider $X = X_0 \setminus (X_1 \cup \cdots \cup X_m)$ ($X_i \in \mathcal{J}(S)$) and $h_k$ such that $X_0 \subset \text{dom} h_k$ for all $k$. Note also that the only difference between $C^*$-regularity and strong $C^*$-regularity is that the sets $Y_j$ are required to be in $\mathcal{J}(S)$ in the first case and in $\mathcal{J}(S)$ in the second. In particular, strong $C^*$-regularity implies $C^*$-regularity.

Similarly to Proposition 2.5 we get the following result.

**Proposition 2.12.** If $S$ is a $C^*$-regular left cancellative monoid, then the representation $\rho_{\chi_e}$ of $C^*_r(\mathcal{G}(S))$ defines an isomorphism $C^*_r(\mathcal{G}(S)) \cong C^*_r(S)$.

Thus, if $S$ is $C^*$-regular, we can define, following [Spi20], the full semigroup $C^*$-algebra of $S$ by
\[
C^*(S) := C^*(\mathcal{G}(S)).
\]

A presentation of $C^*(\mathcal{G}(S))$ in terms of generators and relations is given in [Spi20] Theorem 9.4.

We therefore have two candidates for a groupoid model of $C^*_r(S)$, and hence two potentially different definitions of full semigroup $C^*$-algebras associated with $S$. As the following result shows, it is $\mathcal{G}(S)$ which is the preferred model and we have only one candidate for $C^*(S)$.

**Proposition 2.13.** Assume $S$ is a left cancellative monoid such that $\rho_{\chi_e}$ defines an isomorphism $C^*_r(\mathcal{G}_P(S)) \cong C^*_r(S)$. Then $\mathcal{G}_P(S) = \mathcal{G}(S)$.

**Proof.** Assume $\mathcal{G}_P(S) \neq \mathcal{G}(S)$. Then there exists $[g, \chi] \in \mathcal{G}_P(S)$ such that $[g, \chi] \neq \chi$ but $[g, \chi] \sim \chi$. Let $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ be such that $\chi(p_X) = 1, \chi(p_{X_i}) = 0$ for all $i$ and $gs = s$ for all $s \in X \setminus \bigcup_{i=1}^{m} X_i$. Consider the clopen set $U \subset \mathcal{G}_P(S)^{(0)}$ defined by (2.3) and the function $f = 1_{\mathcal{G}_P(S)} - 1_U$ on $\mathcal{G}_P(S)$. Then $f \neq 0$, but if $g = s_1^{-1} t_1 \cdots s_m^{-1} t_n$ and we identify $\ell^2(\mathcal{G}_P(S)_{\chi_e})$ with $\ell^2(S)$, then
\[
\rho_{\chi_e}(f) = (\lambda_{s_1} \lambda_{t_1} \cdots \lambda_{s_n} \lambda_{t_n} - 1) 1_{X \setminus \bigcup_{i=1}^{m} X_i} = 0.
\]

Therefore $\rho_{\chi_e} : C^*_r(\mathcal{G}_P(S)) \to C^*_r(S)$ has a nontrivial kernel. \hfill $\Box$

**Corollary 2.14.** If $S$ is strongly $C^*$-regular, then $S$ is $C^*$-regular and $\mathcal{G}_P(S) = \mathcal{G}(S)$.

**Proof.** The first statement follows from the definitions, as was already observed after Definition 2.11. The equality $\mathcal{G}_P(S) = \mathcal{G}(S)$ follows from Propositions 2.5 and 2.13. \hfill $\Box$

In particular, by Proposition 2.9 if $\mathcal{G}_P(S)$ is Hausdorff or $S$ is finitely aligned, then $\mathcal{G}_P(S) = \mathcal{G}(S)$. This has been already known, see [Li21] Lemma 3.2.

The equality $\mathcal{G}_P(S) = \mathcal{G}(S)$ in the strong $C^*$-regular case is also an immediate consequence of the following criterion.

**Lemma 2.15.** For every left cancellative monoid $S$, we have $\mathcal{G}_P(S) = \mathcal{G}(S)$ if and only if $S$ has the following property: given $g \in I_\ell(S)$ and $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ such that $gs = s$ for all $s \in X \setminus \bigcup_{i=1}^{m} X_i \neq \emptyset$, there are $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ such that
\[
X \setminus \bigcup_{i=1}^{m} X_i \subset \bigcup_{j=1}^{l} Y_j \quad \text{and} \quad g \mathsf{p} Y_j = p Y_j \quad \text{for all} \quad 1 \leq j \leq l.
\]
Proof. Assume first that the condition in the formulation of the lemma is satisfied. In order to prove that $\mathcal{G}_P(S) = \mathcal{G}(S)$, we have to show that if $[g, \chi] \sim \chi$ for some $g \in I_t(S)$ and $\chi \in \Omega(S)$, then $[g, \chi] = \chi$. Let $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ be such that $\chi(p_X) = 1$, $\chi(p_{X_i}) = 0$ for all $i$ and $gs = s$ for all $s \in X \setminus \bigcup_{i=1}^m X_i$. By assumption, there are $Y_1, \ldots, Y_l \in \mathcal{J}(S)$ such that $X \setminus \bigcup_{i=1}^m X_i \subset \bigcup_{j=1}^l Y_j$ and $gp_{Y_j} = p_{Y_j}$ for all $1 \leq j \leq l$. But then $\chi(p_{Y_j}) = 1$ for some $j$, hence $[g, \chi] = \chi$.

Assume now that the condition in the formulation is not satisfied. Then, similarly to the proof of Lemma 2.3, we can find $g \in I_t(S)$, $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ and $\chi \in \Omega(S)$ such that $\chi(p_X) = 1$, $\chi(p_{X_i}) = 0$ for all $i$, $\chi(p_J) = 0$ for all $J \in \mathcal{J}(S)$ satisfying $gp_J = p_J$, and $gs = s$ for all $s \in X \setminus \bigcup_{i=1}^m X_i$. Then $[g, \chi] \sim \chi$ and $[g, \chi] \neq \chi$, so $\mathcal{G}_P(S) \neq \mathcal{G}(S)$. □

We finish this section by showing that under rather general assumptions $C^*_r(S)$ coincides with the essential groupoid $C^*$-algebras of $\mathcal{G}_P(S)$ and $\mathcal{G}(S)$.

Proposition 2.16. Let $S$ be a countable left cancellative monoid. Assume that for any nontrivial units $u_1, \ldots, u_k \in S^* \setminus \{e\}$ and every $X \in \mathcal{J}(S)$ containing $e$, there is $Z \in \mathcal{J}(S)$ such that $\emptyset \neq Z \subset X$ and $u_is \neq s$ for all $s \in Z$ and $i = 1, \ldots, k$. Then the maps $\rho_{\chi_e}: C^*_r(\mathcal{G}_P(S)) \to C^*_r(S)$ and $\rho_{\chi_e}: C^*_r(\mathcal{G}(S)) \to C^*_r(S)$ define isomorphisms

$$C^*_r(\mathcal{G}_P(S)) \cong C^*_r(S) \cong C^*_r(\mathcal{G}(S)).$$

Proof. Consider the groupoid $\mathcal{G}_P(S)$. Let $Y = \{\chi_s | s \in S\}$ and $D_0$ be the set defined in Proposition 1.12. As $Y$ is dense in $\Omega(S)$ and $\ker \rho_{\chi_e}$ is independent of $s$, by Proposition 1.12 in order to prove the first isomorphism it suffices to show that $D_0 \cap Y = \emptyset$.

Since both sets $D_0$ and $Y$ are invariant, it is enough to show that $\chi_e \notin D_0$. By Lemma 2.1, the isotropy group $\mathcal{G}_P(S)_{\chi_e}$ consists of the elements $[u, \chi_e], u \in S^*$. Therefore we need to show that if $u_1, \ldots, u_k \in S^* \setminus \{e\}$ and $U$ is a neighbourhood of $\chi_e$ in $\Omega(S)$, then the set $U \setminus \bigcup_{i=1}^k D(u_i, \Omega(S))$ has nonempty interior. By the definition of the topology on $\Omega(S)$, we can find $X \in \mathcal{J}(S)$ such $e \in X$ and $\{\chi | \chi(p_X) = 1\} \subset U$. Let $Z \in \mathcal{J}(S)$ be as in the formulation of the proposition. Then the clopen set $V = \{\chi | \chi(p_Z) = 1\}$ is contained in $U$. We claim that it does not intersect $D(u_i, \Omega(S))$ for all $i$.

Assume $\chi \in V \cap D(u_i, \Omega(S))$ for some $i$. This means that $[u_i, \chi] = [e, \chi]$, that is, there is $W \in \mathcal{J}(S)$ such that $\chi(p_W) = 1$ and $u_is = s$ for all $s \in W$. But then $\chi(p_{Z \cap W}) = 1$, so $Z \cap W$ is nonempty, contradicting the property $u_is \neq s$ for all $s \in Z$.

This proves the proposition for $\mathcal{G}_P(S)$, the proof for $\mathcal{G}(S)$ is essentially the same. □

3. Example of a regular monoid

Consider the monoid $S$ given by the monoid presentation

$$S = \langle a, b, x_n, y_n \, | \, n \in \mathbb{Z} \rangle : abx_n = bx_n, aby_n = by_{n+1} \, (n \in \mathbb{Z}) \rangle. \quad (3.1)$$

Our goal is to prove the following.

Proposition 3.1. The monoid $S$ defined by (3.1) is left cancellative and strongly $C^*$-regular. It is not finitely aligned and the groupoid $\mathcal{G}_P(S) = \mathcal{G}(S)$ is not Hausdorff.

The proof is divided into several lemmas. But first we need to introduce some notation. Consider the set $S$ of finite words (including the empty word) in the alphabet $\{a, b, x_n, y_n \, | \, n \in \mathbb{Z}\}$. We say that two words are equivalent if they represent the same element of $S$. Let $\tau \subset S \times S$ be the symmetric set of relations defining $S$, so

$$\tau = \{(abx_n, bx_n), (bx_n, abx_n), (aby_n, by_{n+1}), (by_{n+1}, aby_n) \, | \, n \in \mathbb{Z} \}.$$
By a $\tau$-sequence we mean a finite sequence $s_0, \ldots, s_n$ of words such that for every $i = 1, \ldots, n$ we can write $z_{i-1} = c_ip_i d_i$ and $z_i = c_i q_i d_i$ with $(p_i, q_i) \in \tau$. Then by definition two words $s$ and $t$ are equivalent if and only if there is a $\tau$-sequence $s_0, \ldots, s_n$ with $s_0 = s$ and $s_n = t$.

**Lemma 3.2.** The monoid $S$ is left cancellative.

*Proof.* It will be convenient to use the following notation. For words $s$ and $t$, let us write $s \perp_0 t$ if every word $abx_n, bx_n, aby_n, by_n$ ($n \in \mathbb{Z}$) in $st$ that begins in $s$ ends in $s$. We write $s \perp t$ if for all words $s'$ and $t'$ such that $s \sim s'$ and $t \sim t'$, we have $s' \perp_0 t'$. We will repeatedly use that if $s \perp t$ and $st \sim w$ for a word $w$, then $w = s't'$ for some $s' \sim s$ and $t' \sim t$.

In order to prove the lemma, it suffices to show that for all letters $x$ and words $w$, $w'$, the equivalence $xw \sim xw'$ implies that $w \sim w'$.\vspace{1ex}

Case $x = x_n y_n$:
The only word equivalent to $x$ is $x$ itself, and we have $x \perp w$, so the equivalence $xw \sim xw'$ furnishes the equivalence $w \sim w'$.

Case $x = a$:
Write $xw = a^kv$ ($k \geq 1$), with $v$ not starting with $a$. Consider several subcases.

Assume $v$ is empty or starts with $x_n$ or $y_n$. Then $a^k \perp v$. The only word equivalent to $a^k$ is $a^k$ itself. It follows that $xw' = a^kv'$, with $v' \sim v$, and therefore $w = a^{k-1}v \sim a^{k-1}v' = w'$.

Assume now that $v$ starts with $b$ and write $xw = a^kbv$. Assume $u$ is empty or starts with $a$ or $b$. Then $a^kb \perp u$. The only word equivalent to $a^kb$ is $a^kb$ itself. It follows that $xw' = a^kbv'$, with $v' \sim u$, hence $w \sim w'$.

Assume next that $u$ starts with $x_n$ and write $xw = a^k bv_n z$. Then $a^k bv_n \perp z$. The words equivalent to $a^k bv_n$ are $a^m bv_n$ ($m \geq 0$). It follows that $xw' = a^m bv_n z'$ for some $m \geq 1$ and $z' \sim z$, hence $w \sim w'$.

Similarly, if $u$ starts with $y_n$, write $xw = a^kby_n z$. Then $a^kby_n \perp z$. The words equivalent to $a^kby_n$ are $a^mby_n$ ($m \geq 0$). It follows that $xw' = a^mby_n z'$ for some $m \geq 1$ and $z' \sim z$, hence $w \sim w'$.

Case $x = b$:
If $w$ is empty or starts with $a$ or $b$, then $x \perp w$ and we are done similarly to the first case above.

If $w$ starts with $x_n$, write $xw = bx_nv$. Then $bx_n \perp v$. The words equivalent to $bx_n$ are $a^m bx_n$ ($m \geq 0$), and the only one among them beginning with $b$ is $bx_n$ itself. It follows that $xw' = bx_nv'$, with $v' \sim v$, hence $w \sim w'$. The case when $w$ starts with $y_n$ is similar, since the only word beginning with $b$ that is equivalent to $by_n$ is $by_n$ itself. □

Next we want to describe the constructible ideals of $S$. We start with the following lemma.

**Lemma 3.3.** We have:

1. if $x \in \{x_n, y_n (n \in \mathbb{Z})\}$, $y \in \{a, b, x_n, y_n (n \in \mathbb{Z})\}$ and $x \neq y$, then $xS \cap yS = \emptyset$;

2. for all $k \geq 1$,
   $$bS \cap a^kS = bS \cap a^kS = \bigcup_{n \in \mathbb{Z}} bx_nS \cup \bigcup_{n \in \mathbb{Z}} by_nS.$$  

*Proof.* (1) None of the words occurring in the defining relations of $S$ begins with $x_n$ or $y_n$. From this it follows that a word beginning with $x_n$, resp., $y_n$, can only be equivalent to a word beginning with $x_n$, resp., $y_n$. Thus, $xS \cap yS = \emptyset$.

(2) Since we have $bx_n = a^k bx_n$ and $by_n = a^k by_n - k$ for all $n$ and $k$, it is clear that

$$\bigcup_{n} bx_nS \cup \bigcup_{n} by_nS \subset bS \cap a^k bS \subset bS \cap a^k S.$$
To prove the opposite inclusions, assume \( s \in bS \cap a^kS \). Take words \( w \) and \( w' \) in \( S \) such that \( s \) is represented by \( bw \) and \( a^kw' \). There is a \( \tau \)-sequence \( z_0, \ldots, z_m \) such that \( bw = z_0 \) and \( a^kw' = z_m \). We have \( z_{i-1} = c_ip_id_i \) and \( z_i = c_iq_id_i \), with \( (p_i, q_i) \in \tau \). There must be an index \( i \) such that \( c_i = 0 \) and \( p_i \) starts with \( b \). But then \( p_i = bx_n \) or \( p_i = by_n \) for some \( n \), since these are the only words in the defining relations of \( S \) that begin with \( b \). Therefore \( s \) lies in \( bx_nS \) or in \( by_nS \).

This lemma already implies that \( S \) is not finitely aligned, since \( b^{-1}aS = \bigcup_n x_nS \cup \bigcup_n y_nS \) by (2) and the sets \( x_nS, y_mS \) are disjoint for all \( n \) and \( m \) by (1), so the right ideal \( b^{-1}aS \) is not finitely generated.

**Lemma 3.4.** The constructible ideals of \( S \) are

\[
\emptyset, \quad sS, \quad \bigcup_{n \in \mathbb{Z}} sx_nS \cup \bigcup_{n \in \mathbb{Z}} sy_nS \quad (s \in S).
\]  

**(3.2)**

**Proof.** By Lemma 3.3, the ideals in (3.2) are constructible. In order to prove the lemma it is then enough to show that for every \( x \in \{a, b, x_n, y_n \ (n \in \mathbb{Z})\} \) and \( s \in S \), the right ideals \( x^{-1}sx_nS \) and \( \bigcup_n x^{-1}sx_nS \cup \bigcup_n x^{-1}sy_nS \) are again of the form (3.2). This is obviously true when \( s \in xS \). By Lemma 3.3(1) this is also true if \( s = e \). So from now on we assume that \( s \notin xS \) and \( s \neq e \).

**Case** \( x = x_n, y_n \):

In this case, from Lemma 3.3(1) we see that the sets \( x^{-1}sx_nS \) and \( \bigcup_n x^{-1}sx_nS \cup \bigcup_n x^{-1}sy_nS \) are empty.

**Case** \( x = a \):

Again, Lemma 3.3(1) tells us that the sets \( a^{-1}sa \) and \( \bigcup_n a^{-1}sx_nS \cup \bigcup_n a^{-1}sy_nS \) are empty if \( s \in x_mS \) or \( s \in y_mS \) for some \( m \). As \( s \notin aS \) and \( s \neq e \), we may therefore assume that \( s \in bS \).

Consider several subcases.

Assume \( s = b \). Then, by Lemma 3.3(2), \( aS \cap bS = \bigcup_n bx_nS \cup \bigcup_n by_nS \). As \( a \) maps this set onto itself, we conclude that both \( a^{-1}bS \) and \( \bigcup_n x^{-1}bx_nS \cup \bigcup_n a^{-1}by_nS \) are equal to \( \bigcup_n bx_nS \cup \bigcup_n by_nS \).

Next, assume \( s \in baS \) or \( s \in b^2S \). From the defining relations we see that every word in \( S \) that starts with \( ba \) or \( b^2 \) can only be equivalent to a word that again starts with \( ba \) or \( b^2 \). Hence the sets \( a^{-1}baS \) and \( a^{-1}b^2S \) are empty, and therefore \( a^{-1}S \) and \( \bigcup_n a^{-1}sx_nS \cup \bigcup_n a^{-1}sy_nS \) are empty as well.

It remains to consider the subcase when \( s \in bx_mS \) or \( s \in by_mS \). But then \( s \in aS \), which contradicts our assumption on \( s \).

**Case** \( x = b \):

Similarly to the previous case, we may assume that \( s \in aS \). Write \( s = a^kt \) for some \( k \geq 1 \) and \( t \in S \) such that \( t \) can be represented by a word not starting with \( a \). Consider several subcases.

Assume \( t = e \). Then, using Lemma 3.3(2), we get

\[
b^{-1}a^kS = b^{-1}(bS \cap a^kS) = b^{-1}\left( \bigcup_n bx_nS \cup \bigcup_n by_nS \right) = \bigcup_n x_nS \cup \bigcup_n y_nS.
\]

Every word in \( S \) that starts with \( a^kx_n \) can only be equivalent to a word that again starts with \( a^kx_n \). The same is true for \( y_n \) in place of \( x_n \). Hence

\[
\bigcup_n b^{-1}a^kx_nS \cup \bigcup_n b^{-1}a^ky_nS = \emptyset. \tag{3.3}
\]

Next, assume \( t \in x_mS \) or \( t \in y_mS \). Then (3.3) implies that both \( b^{-1}a^ktS \) and \( \bigcup_n b^{-1}a^ktx_nS \cup \bigcup_n b^{-1}a^ky_nS \) are empty.

It remains to consider the subcase \( t \in bS \). This splits into several subsubcases.
If \( t = b \), then using Lemma 3.3(2) again,
\[
b^{-1} a^k b S = b^{-1} \left( b S \cap a^k b S \right) = b^{-1} \left( \bigcup_n b x_n S \cup \bigcup_n b y_n S \right) = \bigcup_n x_n S \cup \bigcup_n y_n S.
\]
As \( a \) maps \( \bigcup_n b x_n S \cup \bigcup_n b y_n S \) onto itself, we also have
\[
\bigcup_n b^{-1} a^k b x_n S \cup \bigcup_n b^{-1} a^k b y_n S = \bigcup_n x_n S \cup \bigcup_n y_n S.
\]
Assume next that \( t \in baS \) or \( t \in b^2 S \). Every word in \( S \) that starts with \( a^k b a \) or \( b^2 \) can only be equivalent to a word that again starts with \( a^k b a \) or \( b^2 \). Hence the sets \( b^{-1} a^k b a S \) and \( b^{-1} a^k b^2 S \) are empty, and therefore \( b^{-1} a^k b t S \) and \( \bigcup_n b^{-1} a^k b x_n S \cup \bigcup_n b^{-1} a^k b y_n S \) are empty as well.

Finally, assume \( t \in bx_m S \) or \( t \in by_m S \). Then \( s = a^k t \in b S \), which contradicts our assumption on \( s \).

We now look at the topology on \( G_P(S) \). We will need the following lemma.

**Lemma 3.5.** The constructible ideals that contain at least two elements \( bx_n \) are
\[
S, \quad a^k b S \ (k \geq 0), \quad \bigcup_{n \in \mathbb{Z}} bx_n S \cup \bigcup_{n \in \mathbb{Z}} by_n S. \tag{3.4}
\]

**Proof.** It is clear that the ideals in (3.4) contain \( bx_n \) for all \( n \in \mathbb{Z} \). Assume \( X \in J(S) \) contains \( bx_l \) and \( bx_m \) for some \( l \neq m \). Observe that the words in \( S \) equivalent to \( bx_l \) for a fixed \( l \) are \( a^k bx_l \), \( k \geq 0 \). It follows that if \( bx_l \in s S \) for some \( s \), then any word in \( S \) representing \( s \) has the form \( a^k \), \( a^k b \) or \( a^k bx_l \) for some \( k \geq 0 \). Consider two cases.

Assume \( X = s S \). By the above observation, the only possibilities for \( s \) to have \( bx_l, bx_m \in s S \) are \( s = a^k \) or \( s = a^k b \) for some \( k \geq 0 \), so \( X \) has the required form.

Assume \( X = \bigcup_n s x_n S \cup \bigcup_n s y_n S \). If \( bx_l \in s x_n S \) for some \( n \), then by the above observation \( n = l \) and \( s = a^k b \) for some \( k \geq 0 \), so \( X \) has the required form. Otherwise we must have \( bx_l \in s y_n S \) for some \( n \), but this is not possible, again by the above observation.

This lemma implies that the semi-characters \( \chi_{bx_n} \) on \( E(S) \) converge as \( n \to \pm \infty \) to a semi-character \( \chi \) such that \( \chi(p_X) = 1 \) for all \( X \) in (3.4) and \( \chi(p_X) = 0 \) for all other constructible ideals.

**Lemma 3.6.** The semi-characters \( \chi_{bx_n} \) converge in \( G_P(S) \) to the different elements \( \chi \) and \([a, \chi]\), so \( G_P(S) \) is not Hausdorff.

**Proof.** The convergence \( \chi_{bx_n} \to [a, \chi] \) follows from the fact that \([a, \chi_{bx_n}] = [e, \chi_{bx_n}]\) for all \( n \), because \( a \) fixes \( bx_n \). The element \([a, \chi] \in G_P(S)^{\chi}\) is nontrivial, since by Lemma 3.5 if \( X \in J(S) \) and \( \chi(p_X) = 1 \), then \( X \) contains the elements \( by_n \) that \( a \) does not fix.

It remains to show that \( S \) is strongly C*-regular. Recall that by Definition 2.2 and Remark 2.3, this means that, given elements \( h_1, \ldots, h_N \in I_\ell(S) \) and constructible ideals \( X, X_1, \ldots, X_M \in J(S) \) satisfying
\[
X \subset \bigcap_{k=1}^N \text{dom} \ h_k, \quad \emptyset \neq X \setminus \bigcup_{i=1}^M X_i \subset \bigcup_{k=1}^M \{ s \in S : h_k s = s \},
\]
we need to show that there are constructible ideals \( Y_1, \ldots, Y_L \in J(S) \) and indices \( 1 \leq k_j \leq N \) \((j = 1, \ldots, L)\) such that
\[
X \setminus \bigcup_{i=1}^M X_i \subset \bigcup_{j=1}^L Y_j \quad \text{and} \quad h_{k_j} p_{Y_j} = p_{Y_j} \quad \text{for all} \quad 1 \leq j \leq L.
\]
We will actually show more: there is an index \( k \) such that \( h_k p_X = p_X \).
This is obviously true when \( X \) is a principal right ideal, cf. Proposition 2.9(3). Therefore we need only to consider \( X = \bigcup_{n \in \mathbb{Z}} x_n S \cup \bigcup_{n \in \mathbb{Z}} y_n S \). By replacing \( X, X_i \) and \( h_k \) by \( s^{-1}X, s^{-1}X_i \) and \( s^{-1}h_k s \) we may assume that

\[
X = \bigcup_{n \in \mathbb{Z}} x_n S \cup \bigcup_{n \in \mathbb{Z}} y_n S.
\]

**Lemma 3.7.** The only constructible ideals that contain \( y_n \) for a fixed \( n \) are \( S, y_n S \) and \( X \).

*Proof.* A principal ideal \( sS \) contains \( y_n \) only if \( s = e \) or \( s = y_n \), since the only word in \( S \) that is equivalent to \( y_n \) is \( y_n \) itself. For the same reason if an ideal \( \bigcup_k s x_k S \cup \bigcup_k s y_k S \) contains \( y_n \), then we must have \( s = e \), so the ideal is \( X \).

Since by assumption \( X \backslash \bigcup_{i=1}^M X_i \neq \emptyset \), it follows that every ideal \( X_i \) contains at most one element \( y_n \). Therefore \( X \backslash \bigcup_{i=1}^M X_i \) contains \( y_n \) for all but finitely many \( n \)'s. In particular, there are indices \( m \) and \( k \) such that \( h_k y_m = y_m \). To finish the proof of strong \( C^* \)-regularity it is now enough to establish the following.

**Lemma 3.8.** If \( h \in I_\ell(S) \) fixes \( y_m \) for some \( m \) and satisfies \( \text{dom } h \supset X = \bigcup_n x_n S \cup \bigcup_n y_n S \), then \( h = p_S = e \) or \( h = p_X \).

*Proof.* For \( k \geq 1 \), consider the element \( g_k = a^{-k}b \in I_\ell(S) \). Note that by Lemma 3.3(2) we have \( \text{dom } g_k = X \) and \( g_k X = b X \). For \( k \in \mathbb{Z} \setminus \{0\} \), consider \( g_k' = b^{-1}a^k b \in I_\ell(S) \). The domain and range of \( g_k' \) is \( X \).

We will show that if \( h \in I_\ell(S) \) satisfies \( \text{dom } h \supset X \), then \( h \) must be of the form

\[
sg_k p_J, \quad sg_k' p_J \quad \text{or} \quad sp_J \quad (s \in S, J \in \mathcal{J}(S)).
\]

This will yield the proposition. Indeed, first of all, we then have \( J = S \) or \( J = X \), since by Lemma 3.7 these are the only constructible ideals containing \( X \). Then \( h y_m = s b y_{m+k}, h y_m = s y_{m+k} \) or \( h y_m = s y_m \), resp., and this equals \( y_m \) only in the third case with \( s = e \).

Take \( h \in I_\ell(X) \) that is not of the form (3.5). We will show that \( \text{dom } h \not\supset X \). Begin by writing \( h \) as a word in the generators of \( S \) and their inverses. We may assume that there are no occurrences of \( x^{-1}x \) or \( xx^{-1} \) in this word for each generator \( x \). Indeed, \( x^{-1}x = e \) can be omitted, while \( xx^{-1} = p_s S \) and if \( h = h_1 h_2 p_s h_2 \), then \( h = h_1 h_2 p_s h_2 \). Let's consider \( h_1 h_2 \). We may also assume that there are no occurrences of \( a^k b x_n \) and \( a^k b y_n \) for \( k \in \mathbb{Z} \setminus \{0\} \) and \( n \in \mathbb{Z} \), as these can be simplified to \( bx_n \) and \( by_{n+k} \), resp. We then may say that \( h \) (or, more precisely, our word for \( h \)) is reduced.

The assumption that \( h \) is not of the form (3.5) implies that our word for \( h \) has the form \( h_1 x^{-1} w g \), where \( h_1 \) is a word in the generators and their inverses, \( x \) is one of the generators, \( w \) is a word in the generators and we have one of the following options for \( g \): (i) \( g = g_k' \); (ii) \( g = g_k \) and either \( w \) is not trivial or \( x \neq a, b \); (iii) \( g = e \) and neither \( x = a \) and \( w = b \) nor \( x = b \) and \( w = a^m b \) for some \( m \geq 1 \). Without loss of generality we may assume that \( h_1 \) is trivial. We are not going to distinguish between the word \( w \in S \) and the element of \( S \) it represents.

**Case** \( x = x_n, y_m \):

By the proof of Lemma 3.4 for every word \( v \) in the generators, we have \( x^{-1} v S = \emptyset \) unless \( v \) is empty or starts with \( x \). Since \( w \) does not start with \( x \) and we have \( g_k X = b X \), it follows that \( x^{-1} w g X = \emptyset \) unless \( w \) is empty and either \( g = g_k' \) or \( g = e \). In order to deal with the remaining cases we have to show that \( g_k' X = X \not\subseteq x S \), which is clearly true by Lemma 3.3(1).

**Case** \( x = a \):

By the proof of Lemma 3.4 for every word \( v \) in the generators, we have \( a^{-1} v S = \emptyset \) unless \( v \) is empty, \( v = b \) or \( v \) starts with \( a, bx_n \) or \( by_n \). Since \( w \) cannot start with \( a, bx_n \) or \( by_n \), it follows that \( a^{-1} w g X = \emptyset \) unless we have one of the following: (i) \( w \) is empty and \( g = g_k' \); (ii) \( w = b \) and \( g = g_k' \);
(iii) $w = b$ and $g = e$. Cases (i) and (iii) are not possible by our assumptions on $x$, $w$ and $g$. Case (ii) is not possible either, as the word $bg_k' = bb^{-1}a^kb$ is not reduced.

Case $x = b$.

By the proof of Lemma 4.1, for every word $v$ in the generators, we have $b^{-1}vS = \emptyset$ unless $v$ is empty, $v = a^m$, or $v$ starts with $b$, $a^{m}bx_n$ or $a^{m}by_n$ ($m \geq 1$, $n \in \mathbb{Z}$). Since $w$ cannot start with $b$, $a^{m}bx_n$ or $a^{m}by_n$, it follows that $b^{-1}wgX = \emptyset$ unless we have one of the following: (i) $w$ is empty and $g = g_k$; (ii) $w = a^m$ and $g = g_k$; (iii) $w = a^mb$ and $g = g_k'$; (iv) $w = a^mb$ and $g = e$. Cases (i) and (iv) are not possible by our assumptions on $x$, $w$ and $g$. Cases (ii) and (iii) are not possible either, as the words $a^{m}g_k = a^{m}a^{-k}b$ and $a^{m}bg_k' = a^{m}bb^{-1}a^{k}b$ are not reduced. □

This finishes the proof of Proposition 3.1

Remark 3.9. If we drop the generator $b$ and consider

$$
\tilde{S} = \langle a, x_n, y_n \mid n \in \mathbb{Z} \rangle : ax_n = x_n, ay_n = y_{n+1} \quad (n \in \mathbb{Z}),
$$

then we get a left cancellative right LCM monoid, meaning that every constructible ideal is either empty or principal. Similarly to Lemma 3.6, the semi-characters $\chi_{x_n}$ converge to different elements $\chi$ and $[a, \chi]$, so the groupoid $\mathcal{G}_p(\tilde{S}) = \mathcal{G}(\tilde{S})$ is not Hausdorff.

4. EXAMPLE OF A NONREGULAR MONOID

Consider a small modification $T$ of the monoid $S$ from the previous section:

$$
T = \langle a, b, c, x_n, y_n \mid n \in \mathbb{Z} \rangle : abx_n = bx_n, aby_n = by_{n+1},
$$

$$
cbx_n = bx_{n+1}, cby_n = by_n \quad (n \in \mathbb{Z}).
$$

(4.1)

For this monoid we have the following result.

Proposition 4.1. The monoid $T$ defined by (4.1) is left cancellative. It is not $C^*$-regular, furthermore, the homomorphism $\rho_\chi : C^*_r(\mathcal{G}(T)) \to C^*_r(T)$ has nontrivial kernel.

A large part of the analysis of $T$ is similar to that of $S$, so we will omit most of it.

The claim that $T$ is left cancellative is proved similarly to Lemma 3.2, the main difference being that powers of $a$ get replaced by products of $a$ and $c$. The description of the constructible ideals is exactly the same as for $S$ (Lemma 3.4):

Lemma 4.2. The constructible ideals of $T$ are

$$
\emptyset, \quad tT, \quad \bigcup_{n \in \mathbb{Z}} tx_nT \cup \bigcup_{n \in \mathbb{Z}} ty_nT \quad (t \in T).
$$

The next result is similar to Lemma 3.5.

Lemma 4.3. The constructible ideals of $T$ that contain at least two elements $bx_n$ or at least two elements $by_n$ are

$$
T, \quad tby_n \quad (t \text{ is a product of } a, c), \quad \bigcup_{n \in \mathbb{Z}} bx_nT \cup \bigcup_{n \in \mathbb{Z}} by_nT.
$$

(4.2)

This lemma implies that the semi-characters $\chi_{bx_n}$ converge as $n \to \pm \infty$ to a semi-character $\chi$ such that $\chi(x) = 1$ for all $X$ in (4.2) and $\chi(x) = 0$ for all other $X \in \mathcal{J}(T)$. The semi-characters $\chi_{by_n}$ converge to $\chi$ as well. The following lemma finishes the proof of Proposition 4.1.

Lemma 4.4. The representation $\rho_\chi$ of $C^*_r(\mathcal{G}(T))$ is not weakly contained in $\rho_{\chi_0}$, hence

$$
\rho_\chi : C^*_r(\mathcal{G}(T)) \to C^*_r(T)
$$

has nontrivial kernel.
Proof. Consider the constructible ideal \( X = \bigcup_n bx_n T \cup \bigcup_n by_n T \), the neighbourhood \( U = \{ \eta \in \Omega(T) : \eta(p_X) = 1 \} \) of \( \chi \) and the elements \( g_1 = [a, \chi] \) and \( g_2 = [c, \chi] \) of \( G(T)_X^\chi \). That \( g_1 \) and \( g_2 \) are indeed nontrivial elements of the isotropy group is proved as in Lemma \( \ref{lemma:nontrivial_elements} \). Furthermore, we can see that these are elements of infinite order generating a copy of \( \mathbb{Z}^2 \). Now, if \( \chi_t \in U \) for some \( t \in T \), then \( t \in X \) and hence \( t \) is fixed by \( a \) or \( b \). It follows that either \( \chi_t \in D(a, U) \) or \( \chi_t \in D(c, U) \). We thus see that \( x = \chi \) satisfies Condition \( \ref{condition:condition1} \) for \( Y = \{ \chi_t : t \in T \} \). By Proposition \( \ref{proposition:condition} \) we conclude that \( \rho \chi \) is not weakly contained in \( \rho \chi_e \).

At this point it is actually not difficult to exhibit an explicit nonzero element of \( \ker \rho \chi_e \): consider, for example, the function

\[
(\mathbb{1}_{D(a, \Omega(T))} - \mathbb{1}_{\Omega(T)}) * (\mathbb{1}_{D(c, \Omega(T))} - \mathbb{1}_{\Omega(T)}) * \mathbb{1}_U \in C_c(G(T)).
\]

Its restriction to \( G(T)_X^\chi \) is nonzero, since \( g_1 \) and \( g_2 \) are nontrivial elements of \( G(T)_X^\chi \). It lies in the kernel of \( \rho \chi_e \), since

\[ (\lambda_a - 1)(\lambda_c - 1)\mathbb{1}_X = 0 \quad \text{ on } \ell^2(T). \]

Therefore \( G(T) \) does not provide a groupoid model for \( C^*_r(T) \). Since the unit group of \( T \) is trivial, by Proposition \( \ref{proposition:trivial_unit} \) we nevertheless have

\[ C^*_r(T) \cong C^*_{es}(G(T)). \]

Remark 4.5. An argument similar to the proof of Lemma \( \ref{lemma:nontrivial_elements} \) shows that if \( h \in I_e(T) \) fixes \( x_k \) and \( y_m \) for some \( k \) and \( m \) and satisfies \( \text{dom } h \supseteq X = \bigcup_n x_n T \cup \bigcup_n y_n T \), then \( h = p_T \) or \( h = p_X \). Using this it is not difficult to show that \( G_P(T) = G(T) \).

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