WEYL, DEMAZURE AND FUSION MODULES
FOR THE CURRENT ALGEBRA OF $\mathfrak{sl}_{r+1}$

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Abstract. We construct a Poincare-Birkhoff-Witt type basis for the Weyl modules $[CP1]$ of the current algebra of $\mathfrak{sl}_{r+1}$. As a corollary we prove the conjecture made in $[CP1]$, $[CP2]$ on the dimension of the Weyl modules in this case. Further, we relate the Weyl modules to the fusion modules defined in $[FL]$ of the current algebra and the Demazure modules in level one representations of the corresponding affine algebra. In particular, this allows us to establish substantial cases of the conjectures in $[FL]$ on the structure and graded character of the fusion modules.

Introduction

The study of finite–dimensional representations of quantum affine algebras has been of some interest in recent years and there are several different approaches to the subject, $[CP1]$, $[FR]$, $[FM]$, $[N]$, $[Ka1]$ for instance.

An approach that was developed in $[CP1]$ was to study these representations by specializing to the case of classical affine algebras. It was noticed in that paper, that an irreducible finite–dimensional representation of a quantum affine algebra, in general specialized to an indecomposable but reducible representation of the affine Lie algebra. This behavior is seen in the representation theory of simple Lie algebras, when passing from the characteristic zero situation to the case of non–zero characteristic. The modules in non–zero characteristic which are obtained from the irreducible finite–dimensional modules of the simple Lie algebra are called Weyl modules and are given by the same generators and relations as the modules in characteristic zero. This analogy motivated the definition of the Weyl modules in $[CP1]$, in terms of generators and relations for both the classical and quantum affine Lie algebras. It was also conjectured there, that the Weyl modules of the affine algebra are the classical ($q \to 1$) limit of the standard modules of the quantum affine algebra defined in $[N]$. In particular, a proof of the conjecture would give generators and relations for these representations, or equivalently, would prove that the standard modules are isomorphic to the quantum Weyl modules. This latter result was proved recently in $[CdM]$ using some deep geometric results of Nakajima.

However, the results of $[CP1]$ and $[CP2]$ showed that the conjecture on the isomorphism between the quantum Weyl modules and the standard modules would also follow, if a further conjecture on the dimension of the classical Weyl modules could be proved. This latter conjecture was established in $[CP1]$ for $\mathfrak{sl}_2$. Moreover, it was shown that it was enough to study the analogous modules for the current algebra of a simple Lie algebra, namely the natural parabolic subalgebra of the affine Lie algebra.

In this paper we show that the conjecture on the dimension of the classical Weyl modules is true for $\mathfrak{sl}_{r+1}$, by constructing an explicit basis for the Weyl modules over the current algebra of $\mathfrak{sl}_{r+1}$. We use the basis to give a graded fermionic type character formula for the Weyl modules (see also $[HKOTT]$). We then make connections with several other interesting problems in the representation theory of affine and current algebras. Thus, we are able to establish a substantial case of conjectures in $[FL]$ on the fusion modules for a current Lie algebras. The definition of these modules was motivated by studying the space of conformal blocks for affine Lie algebras. The fusion modules are given by a set of representations of the simple Lie algebra and a set of complex points, one for each representation. It was
conjectured that in the case of a simple Lie algebra and where the representations were irreducible and finite–dimensional the fusion modules are independent of these points. This conjecture is established in this paper for the fusion of fundamental representation of $\mathfrak{sl}_{r+1}$ by showing that the fusion module is isomorphic to a Weyl modules.

Another well–studied family of modules are the Demazure modules. These modules for the current algebra are obtained by taking the current algebra module generated by the extremal vectors in modules of positive level of affine Lie algebras. The dimension (and, actually, character) of these modules was computed in [KMOTU], [M]. We use these results to prove that the Weyl modules are isomorphic to the Demazure modules in the level one representations of the affine Lie algebra of $\mathfrak{sl}_{r+1}$. Further, we prove that the graded character of the fusion modules can be written in terms of Kostka polynomials as conjectured in [FL]. Moreover, since the Weyl modules for the current algebra can be regarded as a pull–back of Weyl modules for the affine Lie algebra, it follows that the $\mathfrak{sl}_{r+1}$–structure of the Demazure modules can be extended to a structure of affine Lie algebra modules. This is related to a conjecture in [FoL].

For an arbitrary simple Lie algebra, the conjecture appears to be harder to establish. This is not surprising, since the representation theory of the corresponding quantum affine algebra is much more complicated. In particular the Weyl module associated with a fundamental weight is no longer irreducible as a module for the simple Lie algebra. But we are convinced that the conjecture of [CP1] is true for these modules and can be established in a similar way. Note however, that since the dimension of fusion modules is known from the definition the isomorphism between the Weyl modules, and the corresponding fusion modules would follow as in the case of $\mathfrak{sl}_{r+1}$ once the conjecture on dimension of the Weyl modules is proved.

The paper is organized as follows. In Section 1, we define the Weyl modules and recall some results from [CP1]. We also recall the definition of the fusion modules from [FL] and the Demazure modules and prove that these modules are quotients of Weyl modules. We then state the main theorem on the dimension of the Weyl modules and show that it gives a sufficient condition for an isomorphism to exist between the Weyl modules, the fusion modules and the Demazure modules. In Section 2 we start proving the conjecture by identifying a basis for the Weyl module. The proof that it is actually a basis involves constructing a filtration (which we call a Gelfand–Tsetlin filtration) for the Weyl module, studying the associated graded spaces and using an induction on $r$. Section 3, is devoted to proving that the associated graded space is actually isomorphic to a sum of Weyl modules for $\mathfrak{sl}_r$. This latter result is the motivation for calling it a Gelfand–Tsetlin filtration. We provide an index of notation at the end of the paper.

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**Added in Proof:** There has been progress on the various conjectures since our paper was posted on the web as [math.QA/0502165](http://arxiv.org/abs/math.QA/0502165). Thus, in [math.RT/0509276](http://arxiv.org/abs/math.RT/0509276) the authors prove that the Demazure modules, the Weyl modules and the fusion modules are all isomorphic for simply–laced Lie algebras extending our results for $\mathfrak{sl}_{r+1}$. In particular, this establishes the conjecture on the dimension of the Weyl modules in [CP1] for the simply laced algebras. For the nonsimply–laced case, they give an example to show that the Demazure modules are smaller than the Weyl modules and formulate a natural conjecture for the Demazure modules. They have also have a conjecture for the Weyl modules which would be an immediate consequence of the conjecture in [CP1]. The methods they use are quite different from ours and in particular they do not have an analog of the fermionic character formula that we give in Section 2 of this paper.
1. The main theorem and its applications

1.1. Preliminaries. Throughout the paper we restrict ourselves to the case of the Lie algebra of \((r+1) \times (r+1)\) trace zero matrices. However, we prefer to use the more general notation of simple Lie algebras since we expect the results to hold in that generality.

1.1.1. Let \(Z\) denote the set of integers, \(N\) the set of non–negative integers and \(N_+\) the set of positive integers. Let \(g = sl_{r+1}\) be the Lie algebra of \((r+1) \times (r+1)\)–matrices of trace zero, \(h\) be the Cartan subalgebra of \(g\) consisting of diagonal matrices and \(\alpha_i, 1 \leq i \leq r\), a set of simple roots for \(g\) with respect to \(h\). For \(1 \leq i \leq j \leq r\), let \(\alpha_{i,j} = (\alpha_i + \cdots + \alpha_j)\) and let \(x_{i,j}^+\) (resp. \(x_{i,j}^-\)) be the \((r+1) \times (r+1)\)–matrix with 1 in the \((i,j)^{th}\) (resp. \((j+1,i)^{th}\))–position and 0 elsewhere. Define subalgebras \(n^\pm\) of \(g\) by

\[
n^+ = \bigoplus_{1 \leq i \leq j \leq r} Cx_{i,j}^+.
\]

For \(1 \leq i \leq r + 1\) let \(H_i\) be the diagonal matrix with 1 in the \(i^{th}\) place and zero elsewhere. The elements \(h_i = H_i - H_{i+1}\), \(1 \leq i \leq r\) are a basis of \(h\). Let \(\{\omega_i : 1 \leq i \leq r\}\) be the the set of fundamental weights of \(g\), namely the basis of \(h^*\) which is dual to \(\{\alpha_i : 1 \leq i \leq r\}\). Let \(P = \sum_{i=1}^r Z\omega_i\) (resp. \(Q = \sum_{i=1}^r Z\alpha_i\)) be the weight lattice (resp. root lattice) of \(g\) and set \(P^+ = \sum_{i=1}^r N\omega_i\), (resp. \(Q^+ = \sum_{i=1}^r N\alpha_i\)). Given a Lie algebra \(a\), we let \(U(a)\) denote the universal enveloping algebra \(a\).

1.1.2. Let \(Z[P]\) be the integral group ring of \(P\) with basis \(e(\mu), \mu \in P\). If \(V\) is any finite–dimensional \(g\)–module with

\[
V = \bigoplus_{\mu \in P} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v, \ h \in h\},
\]

let \(ch(V) = \sum_{\mu \in P} \dim(V_\mu)e(\mu) \in Z[P]\) be the character of \(V\).

Given \(\lambda = \sum_{i=1}^r m_i \omega_i \in P^+\), let \(V(\lambda)\) denote the irreducible finite–dimensional \(g\)–module with highest weight \(\lambda\) and highest weight vector \(v_\lambda\). More precisely, \(V(\lambda) = U(g)v_\lambda\) with defining relations:

\[
x_{i,j}^+ v_\lambda = 0, \quad (h - \lambda(h))v_\lambda = 0, \quad (x_{i,i}^-)^{m_i+1}v_\lambda = 0,
\]

for all \(1 \leq i \leq j \leq r\) and \(h \in h\).

1.1.3. Let \(C[t]\) denote the polynomial ring in an indeterminate \(t\) and for any Lie algebra \(a\), set \(a[t] = a \otimes C[t]\). Clearly, \(a[t]\) is a Lie algebra with the Lie bracket being given by \([x \otimes f, y \otimes g] = [x, y] \otimes fg\) for all \(x, y \in a\) and \(f, g \in C[t]\). We regard \(a\) as a subalgebra of \(a[t]\) by mapping \(x \rightarrow x \otimes 1\) for \(x \in a\). We denote the maximal ideal of \(a[t]\) generated by elements of the form \(a \otimes t^n, a \in a, n > 0\) by \(a[t]\). The Lie algebra \(a[t]\) is a \(N\)–graded Lie algebra, the grading being given by powers of \(t\) and hence \(U(a[t])\) is a \(N\)–graded algebra.

Given a \(N\)–graded \(g[t]\)–module \(M = \bigoplus_{s \in N} M_s\). Observe that \(M_s, s \in N\) are \(g\)–submodules of \(M\). In the case when

\[
M = \bigoplus_{(\mu, s) \in P \times N} M_{\mu,s}, \quad M_{\mu,s} = \{v \in M_s : hv = \mu(h)v, \ h \in h\},
\]

recall, that the graded character of \(M\) to be the element of \(Z[P][t]\) is defined as,

\[
ch(M) = \sum_{(\mu, s) \in P \times N} \dim(M_{\mu,s})t^s e(\mu) = \sum_{s \in N} ch_g(M_s)t^s.
\]

1.2. Weyl modules.
1.2.1. We now recall the definition of the Weyl modules.

**Definition.** Given \( \lambda = \sum_{i=1}^{r} m_i \omega_i \in P^+ \), the Weyl module \( W(\lambda) \) is the \( g[t] \)-module generated by an element \( w_\lambda \) with defining relations:

\[
\begin{align*}
\lambda \in \mathbb{Z}, & \quad n^+ [t] w_\lambda = 0, &  \mathfrak{h} t [t] w_\lambda = 0, &  hw_\lambda = \lambda(h) w_\lambda, &  (x_i \otimes 1)^{m_i+1} w_\lambda = 0,
\end{align*}
\]

for all \( h \in \mathfrak{h}, 1 \leq i \leq r \).

**Remark.** The modules \( W(\lambda) \) were defined and initially studied in \([CP1]\) for the affine Lie algebras. All the results of that paper go over with no difficulty to the case of the current algebras, in fact an inspection of the proofs given there show that the results for the affine algebras were obtained by considering the action of the current algebras \( g \otimes \mathbb{C}[t] \). (See Subsection 1.5.3 for further details.) The notation used in this paper is different from that in \([CP1], [CP2]\). In those papers, the modules for the affine Lie algebras were denoted as \( W(\pi) \) where \( \pi \) is an \( r \)-tuple of polynomials in an indeterminate \( u \) with constant term one. For \( \lambda = \sum_{i=1}^{r} m_i \omega_i \in P^+ \) and \( a \in \mathbb{C}^x \), let

\[
\pi_\lambda = ((1-au)^{m_1}, \cdots, (1-au)^{m_r}).
\]

The module \( W(\lambda) \) is obtained from \( W(\pi_\lambda) \) by pulling back \( W(\pi_\lambda) \) by the Lie algebra homomorphism \( g[t] \to g[t] \) which maps \( x \otimes t^a \to x \otimes (t-a)^a \) for all \( x \in g, s \geq 0 \).

1.2.2. The following lemma is an elementary consequence of the fact that the relations which define the Weyl modules are graded.

**Lemma.**

(i) The modules \( W(\lambda) \) admit a \( \mathbb{N} \)-grading induced by the grading on \( g[t] \),

\[
W(\lambda) = \bigoplus_{s \in \mathbb{N}} W(\lambda)_s.
\]

(ii) For all \( s \geq 0 \), the subspaces \( W(\lambda)_s \) are finite–dimensional \( g \)-submodules, and we have,

\[
W(\lambda) = \bigoplus_{(\mu, s) \in P \times \mathbb{N}} W(\lambda)_{\mu, s},
\]

where \( W(\lambda)_{\mu, s} = \{ w \in W(\lambda)_s : hw = \mu(h)w \ \forall \ h \in \mathfrak{h}. \} \).

For \( \mu \in P \) we set

\[
W(\lambda)_\mu = \bigoplus_{s \in \mathbb{N}} W(\lambda)_{\mu, s} = \{ w \in W(\lambda) : hw = \mu(h)w \ \forall \ h \in \mathfrak{h}. \}.
\]

**Theorem.** \([CP1]\) For all \( \lambda \in P^+ \), the modules \( W(\lambda) \) are finite–dimensional. Moreover, any finite–dimensional \( g[t] \)-module \( V \) generated by an element \( v \in V \) satisfying the relations

\[
\lambda \in \mathbb{Z}, & \quad n^+ [t] v = 0, &  \mathfrak{h} t [t] v = 0, &  hv = \lambda(h)v,
\]

is a quotient of \( W(\lambda) \).

1.3. Fusion modules.

1.3.1. The definition of the fusion product of \( g[t] \)-modules was given in \([FL]\) and we recall the definition in the case of interest to this paper. Given \( a \in \mathbb{C} \), let \( ev_a : g[t] \to g \) be the homomorphism of Lie algebras which maps \( x \otimes t^a \to a^x, x \in g, s \in \mathbb{N} \). Let \( V_a(\lambda) \) be the \( g[t] \)-module obtained by pulling back \( V(\lambda) \) through \( ev_a \). The following proposition is well–known, \([CP], [FL]\) for instance.

**Proposition.** Let \( k \in \mathbb{N}_+, \lambda_s \in P^+, a_s \in \mathbb{C}, 1 \leq s \leq k \) and assume that \( a_s \neq a_s' \) if \( s \neq s' \). Then, \( V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \) is an irreducible \( g[t] \)-module.
1.3.2. Assume that the hypotheses of Proposition 1.3.1 are satisfied and set

\[ V_a(\lambda) = V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k), \quad \mathbf{v} = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}. \]

The module \( V_a(\lambda) \) is quite clearly not a \( \mathbb{N} \)-graded \( \mathfrak{g}[t] \)-module. However, the \( \mathbb{N} \)-grading on \( \mathfrak{g}[t] \) induces a \( \mathfrak{g} \)-equivariant filtration on \( V_a(\lambda) \) as follows: let \( V^n_a(\lambda) \) be the subspace of \( V_a(\lambda) \) spanned by elements of the form \( g^q \mathbf{v} \), where \( g \in \mathbb{U}(\mathfrak{g}[t]) \) has grade at most \( n \).

Set \( V^{-1}_a(\lambda) = 0 \), and set

\[ V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) = \bigoplus_{n \in \mathbb{N}} V^n_a(\lambda)/V^{n-1}_a(\lambda). \]

For \( \mathbf{v}' \in V_a(\lambda) \) let \( \overrightarrow{\mathbf{v}} \) denote its image in \( V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \).

**Lemma.** The following formula defines a \( \mathbb{N} \)-graded \( \mathfrak{g}[t] \)-module structure on \( V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \):

\[ (x \otimes t^s)\overrightarrow{\mathbf{v}} = (x \otimes t^s)\overrightarrow{\mathbf{v}}, \]

for all \( s \in \mathbb{N} \), \( x \in \mathfrak{g} \), \( \overrightarrow{\mathbf{v}} \in V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \).

**Proof.** It suffices to prove that the action is well-defined. But this follows, since \( (x \otimes t^s)V^n_a(\lambda) \subseteq V^{n+s}_a(\lambda) \). □

**Remark.** The resulting \( \mathfrak{g}[t] \)-module is called the fusion product of the modules \( V_{a_1}(\lambda_i), 1 \leq i \leq k \).

1.3.3. The next lemma is an immediate consequence of Proposition 1.3.1.

**Lemma.** We have,

\[ V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) = \mathbb{U}(\mathfrak{g}[t])\nabla. \]

In this paper we prove a significant case of the following conjecture.

**Conjecture.** Let \( k \in \mathbb{N}^+, \lambda_s \in P^+, 1 \leq s \leq k \). Assume that \( a_s, b_s \in \mathbb{C} \), \( 1 \leq s \leq k \) and that \( a_s \neq a_{s'}, b_s \neq b_{s'} \) if \( 1 \leq s \neq s' \leq k \). Then,

\[ V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \cong V_{b_1}(\lambda_1) \otimes \cdots \otimes V_{b_k}(\lambda_k) \]

as \( \mathfrak{g}[t] \)-modules. □

1.3.4. We now prove,

**Proposition.** Suppose that \( k \in \mathbb{N}^+, \lambda_i \in P^+, 1 \leq i \leq k \) and that \( a_1, \cdots, a_k \in \mathbb{C} \) are distinct. The fusion product \( V_{a_1}(\lambda_1) \otimes \cdots \otimes V_{a_k}(\lambda_k) \) is a quotient of \( W(\lambda_1 + \cdots + \lambda_k) \).

**Proof.** It suffices in view of Lemma 1.3.3 to show that \( \nabla \) satisfies the relations in (1.4). Set \( \lambda = \sum_{i=1}^k \lambda_i \).

Since, \( (x_{i,j}^+ \otimes t^s)\mathbf{v} = 0 \) for all \( 1 \leq i \leq j \leq r \) and all \( s \geq 0 \), it follows that \( (x_{i,j}^+ \otimes t^s)\nabla = 0 \). Further, \( h\nabla = \lambda(h)\nabla \).

Finally, we have

\[ (h \otimes t^{s+1})\mathbf{v} = \left( \sum_{j=1}^k \lambda_j(h) a_{s+1}^j \right) \mathbf{v}, \]

which implies that \( (h \otimes t^{s+1})\nabla = 0 \), for all \( s \geq 0 \), thus proving the proposition. □
Corollary. For \( \lambda = \sum_{i=1}^{r} m_i \omega_i \) we have

\[
\dim W(\lambda) \geq \prod_{i=1}^{r} \left( \frac{r + 1}{m_i} \right).
\]

Proof. The proof is immediate from Proposition 1.3.4 by taking \( k = \sum_{i=1}^{r} m_i, \lambda_s = \omega_i, 1 \leq s \leq k \) and we assume that each \( \omega_i \) occurs \( m_i \) times.

The corollary was proved originally in [CP1] by passing to the quantum case and the proof given there was much more complicated.

1.4. Demazure modules.

1.4.1. We now show that the Demazure modules in the highest weight integrable representations of the affine algebra associated to \( \mathfrak{sl}_{r+1} \) are quotients of Weyl modules. We begin by recalling the definition of the affine Kac–Moody algebra. Thus, let \( \mathbb{C}[t, t^{-1}] \) be the ring of Laurent polynomials in an indeterminate \( t \). Let \( \hat{\mathfrak{g}} \) be the affine Lie algebra defined by

\[
\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

where \( c \) is central and

\[
[x \otimes t^s, y \otimes t^t] = [x, y] \otimes t^{s+t} + s\delta_{s+t,0} \langle x, y \rangle c, \quad [d, x \otimes t^s] = sx \otimes t^s,
\]

for all \( x, y \in \mathfrak{g}, s, t \in \mathbb{Z} \) and where \( \langle , \rangle \) is the Killing form of \( \mathfrak{g} \). Set

\[
\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

and regard \( \mathfrak{h}^* \) as a subspace of \( \hat{\mathfrak{h}}^* \) by setting \( \lambda(c) = \lambda(d) = 0 \) for \( \lambda \in \mathfrak{h}^* \).

For \( 0 \leq i \leq r \), define elements \( \Lambda_i \in \hat{\mathfrak{h}}^* \), by

\[
\Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0, \quad \Lambda_i(c) = 1, \quad 1 \leq j \leq r,
\]

and let \( \hat{\mathfrak{P}}^+ \subset \hat{\mathfrak{h}}^* \) be the non-negative integer linear span of \( \Lambda_i, 0 \leq i \leq r \). Let \( \delta \in (\hat{\mathfrak{h}})^* \) be defined by setting

\[
\delta|_\mathfrak{h} = 0, \quad \delta(d) = 1, \quad \delta(c) = 0.
\]

The simple roots for \( \hat{\mathfrak{g}} \) with respect to \( \hat{\mathfrak{h}} \) are \( \alpha_i, 0 \leq i \leq r \), where \( \alpha_0 = \delta - (\alpha_1 + \cdots + \alpha_r) \). Let \( \hat{\mathfrak{Q}}^+ \) be the subset of \( \hat{\mathfrak{P}}^+ \) spanned by the elements \( \alpha_i, 0 \leq i \leq r \). Let \( \widetilde{\mathfrak{W}} \) be the affine Weyl group and \( (\cdot) \) be the \( \mathfrak{W} \)-invariant form on \( \hat{\mathfrak{h}}^* \) obtained by requiring \( (\Lambda_i, \alpha_j) = \delta_{ij} \) for all \( 0 \leq i, j \leq r \) and \( (\delta, \alpha_i) = 0 \) for all \( 1 \leq i \leq r \). The following Lemma is well-known, see [FoL] for instance.

Lemma. Let \( \mu \in \mathbb{P}^+ \). There exists \( 0 \leq i_\mu \leq r \) and \( w(\mu) \in \widetilde{\mathfrak{W}} \) such that \( w(\mu)\Lambda_{i_\mu}|_{\hat{\mathfrak{h}}} = \mu \).

1.4.2. Given \( \Lambda = \sum_{i=0}^{r} m_i \Lambda_i \in \hat{\mathfrak{P}}^+ \), let \( L(\Lambda) \) be the \( \hat{\mathfrak{g}} \)-module generated by an element \( v_\Lambda \) with relations:

\[
\mathfrak{g}t^i v_\Lambda = 0, \quad (n^+ \otimes 1)v_\Lambda = 0, \quad (h \otimes 1)v_\Lambda = \Lambda(h)v_\Lambda, \\
(x_{i}^- \otimes 1)^{m_i+1}v_\Lambda = 0, \quad (x_{1r}^- \otimes t^{-1})^{m_{1r}+1}v_\Lambda = 0,
\]

for \( 1 \leq i \leq r \). This module is known to be irreducible and integrable (see [K]). The next proposition also can be found in [K].

Proposition. Let \( \Lambda \in \hat{\mathfrak{P}}^+ \).

(i) We have

\[
L(\Lambda) = \bigoplus_{\Lambda' \in \hat{\mathfrak{P}}} L(\Lambda)_{\Lambda'}, \quad \text{where} \quad L(\Lambda)_{\Lambda'} = \{ v \in L(\Lambda) : hv = \Lambda'(h)v, \quad h \in \hat{\mathfrak{h}} \}.
\]

Moreover, \( \dim(L(\Lambda)_{\Lambda'}) < \infty \), \( \dim(L(\Lambda))_{\Lambda} = 1 \), and \( L(\Lambda)_{\Lambda'} = 0 \) if \( \Lambda - \Lambda' \notin \hat{\mathfrak{Q}}^+ \).
(ii) The set

$$\text{wt}(L(\Lambda)) = \{ \Lambda' \in \hat{P} : L(\Lambda)_{\Lambda'} \neq 0 \} \subset \Lambda - \hat{Q}^+.$$ 

is preserved by \( \hat{W} \) and

$$\dim(L(\Lambda)_{w\Lambda'}) = \dim(L(\Lambda)_{\Lambda'}) \quad \forall \ w \in \hat{W}, \ \Lambda' \in \hat{P}.$$ 

In particular, \( \dim(L(\Lambda)_{w\Lambda}) = 1 \) for all \( w \in \hat{W} \).

\( \square \)

1.4.3. Given \( w \in \hat{W} \), let \( v_{w\Lambda} \) be a non-zero element in \( L(\Lambda)_{w\Lambda} \) and let

$$D(w\Lambda) = U(g[t])v_{w\Lambda} \subset L(\Lambda).$$

The modules \( D(w\Lambda) \), \( w \in \hat{W} \) are called the Demazure modules. Note that if \( w, w' \in \hat{W} \) are such that \( w^{-1}w' \in W \subset \hat{W} \), then \( D(w\Lambda) = D(w'\Lambda) \). It is clear from Proposition 1.4.2 that \( D(w\Lambda) \) is finite–dimensional for all \( w \in \hat{W} \). We can now prove,

**Proposition.** Let \( \Lambda \in \hat{P}^+ \) and \( w \in \hat{W} \) be such that \( w\Lambda|_h \in P^+ \). Then, \( D(w\Lambda) \) is a quotient of \( W(w\Lambda|_h) \).

**Proof.** To prove the proposition it suffices to show that \( v_{w\Lambda} \) satisfies \([1.4]\). Suppose that

$$(x_{i,j}^+ \otimes t^s)v_{w\Lambda} \neq 0,$$

for some \( 1 \leq i \leq j \leq r \) and \( s \in \mathbb{N} \). It follows that \( w\Lambda + \alpha_{ij} + s\delta \in \text{wt}(L(\Lambda)) \). By Proposition 1.4.2(ii), we get \( \Lambda + w^{-1}(\alpha_{ij}) + s\delta \in \text{wt}(L(\Lambda)) \) and so \( -w^{-1}\alpha_{ij} - s\delta \in \widehat{Q}^+ \). This implies in particular, that \( -w^{-1}\alpha_{ij} \in \widehat{Q}^+ \), and we get

$$(w\Lambda, \alpha_{ij}) = (\Lambda, w^{-1}\alpha_{ij}) \leq (\Lambda, w^{-1}\alpha_{ij} + s\delta) \leq 0$$

which contradicts \( w\Lambda|_h \in P^+ \) unless \( (\Lambda, w^{-1}\alpha_{ij}) = (\Lambda, w^{-1}\alpha_{ij} + s\delta) = 0 \). In this case, it follows that the reflection \( s_{w^{-1}\alpha_{ij} + s\delta} \in \hat{W} \) fixes \( \Lambda \), and so we get

$$\Lambda - (w^{-1}\alpha_{ij} + s\delta) = s_{w^{-1}\alpha_{ij} + s\delta}(\Lambda + w^{-1}\alpha_{ij} + s\delta) \in \text{wt}(L(\Lambda)),$$

which implies \( w^{-1}\alpha_{ij} + s\delta \in \widehat{Q}^+ \), contradicting \( -w^{-1}\alpha_{ij} - s\delta \in \widehat{Q}^+ \). Hence

$$(x_{i,j}^+ \otimes t^s)v_{w\Lambda} = 0, \quad \forall \ 1 \leq i \leq j \leq r, \ s \in \mathbb{N}.$$ 

If \( (h \otimes t^s)v_{w\Lambda} \neq 0 \), then by a similar argument, one sees that \( \Lambda + s\delta \in \text{wt}(L(\Lambda)) \) which is impossible if \( s > 0 \) since \( \delta \in \widehat{Q}^+ \). Since \( D(w\Lambda) \) is a finite–dimensional \( g[t] \)–module the proposition is now immediate from \([1.4]\). \( \square \)

1.4.4. The following is a special case of a result that can be found in [KMOTU],[M],[FoL].

**Theorem.** Assume that \( \Lambda = \Lambda_{i_0} \), \( 0 \leq i_0 \leq r \) and that \( w \in \hat{W} \) is such that \( w\Lambda|_h = \sum_{i=1}^r m_i \omega_i \), where \( m_i \in \mathbb{N} \). Then, we have an isomorphism of \( g \)–modules,

$$D(w\Lambda_{i_0}) \cong V(\omega_1)^{m_1} \otimes \cdots \otimes V(\omega_k)^{m_k}.$$ 

\( \square \)

1.5. Main theorem.
1.5.1. The next result which was conjectured in [CP1] and made precise in [CP2] is the main theorem of this paper.

**Theorem.** Let $\lambda = \sum_{i=1}^{r} m_i \omega_i \in P^+$. We have,

$$\dim(W(\lambda)) = \prod_{i=1}^{r} \left( \frac{r + 1}{i} \right)^{m_i}. \tag{1.5}$$

We prove Theorem 1.5.1 in the following sections. We conclude this section by noting some corollaries of the theorem. Notice that the next corollary establishes the Conjecture 1.3.3 for a substantial family of modules.

**Corollary.** Given $i_1, \cdots, i_k \in \{1, \cdots, r\}$, set $\lambda = \sum_{s=1}^{k} \omega_{i_s}$. Let $w(\lambda) \in \tilde{W}$ and $0 \leq i_\lambda \leq r$ be as in Lemma 1.4.1 so that $w(\lambda)A_{i_\lambda} |_h = \lambda$. Let $a_1, \ldots, a_k \in \mathbb{C}$ be such that $a_i \neq a_j$, $1 \leq i \neq j \leq k$. Then we have an isomorphism of $\mathfrak{g}[t]$–modules

$$V_{a_1}(\omega_{i_1}) \ast \cdots \ast V_{a_k}(\omega_{i_k}) \cong W(\lambda) \cong D(w(\lambda)A_{i_\lambda}).$$

**Proof.** The corollary is immediate from the Theorem 1.5.1 together with Proposition 1.3.3 Proposition 1.4.3 and Theorem 1.4.4.

**Remark.** To prove the conjecture of [PL] for arbitrary irreducible representations by identifying it with a $\mathfrak{g}[t]$–module which is obviously independent of the points, would involve identifying a suitable quotient of the Weyl modules. A possible approach is proposed in [FKL].

1.5.2. For a partition $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{r+1})$ of non–negative integers, set $\lambda_\xi = \sum_{i=1}^{r} (\xi_i - \xi_{i+1}) \omega_i$. Given a partition $\xi$ and regarding $\mu = \sum_{i=1}^{r} n_i \omega_i \in P^+$ as the vector $(n_1, \ldots, n_r)$, let $K_{\mu,\xi}(t)$ be the corresponding Kostka polynomial (see [Mac]). Note that $K_{\mu,\xi}(t) = 0$ unless $\xi_{r+2} = 0$.

**Corollary.** Retain the notation of Corollary 1.5.1. Then,

$$\text{ch}_t(V(\omega_{i_1})_{a_1} \ast \cdots \ast V(\omega_{i_k})_{a_k}) = \text{ch}_t W(\lambda) = \sum_{\xi} K_{\lambda,\xi^{tr}}(t) \cdot \text{ch}_t V(\lambda_{\xi}),$$

where the sum is over all partitions $\xi$ satisfying $\xi_{r+2} = 0$, and $\xi^{tr}$ is the transposed partition.

**Proof.** The second equality follows by first using the identification between the Weyl modules and Demazure modules given in Corollary 1.5.1 and then using Theorem 5.2 in [KMOTU] which gives the formula for $\text{ch}_t(D(w(\lambda))$. The first equality follows from the isomorphism between the Weyl modules and the fusion product.

**Remark.** This corollary establishes a particular case of a conjecture stated in [PL]. In [Ke], another case of the conjecture was proved, namely it was shown that the graded character of module

$$V(n_1 \omega_1) \ast \cdots \ast V(n_k \omega_1), \quad n_j \in \mathbb{N}, \quad 1 \leq j \leq k,$$

is given by a similar formula based on Kostka polynomials.

In the next section we shall give a fermionic character formula for these modules.

1.5.3. We conclude this section by showing that Proposition 1.4.3 and Theorem 1.5.1 are related to Conjecture 1 in [FKL]. Namely, we shall prove that the Demazure module is isomorphic to a cyclic finite–dimensional module for $\hat{\mathfrak{g}}$. For $a \in \mathbb{C}^\times$, $\lambda \in P^+$, define the module $W_a(\lambda)$ for $\hat{\mathfrak{g}}$ as the module with generator $w_{a,\lambda}$ and relations

$$\left(n^+ \otimes \mathbb{C}[t, t^{-1}]\right) w_{a,\lambda} = 0, \quad (h \otimes t^s) w_{a,\lambda} = a^s \lambda(h) w_{a,\lambda}, \quad \left(x_{i,1}^- \otimes 1\right)^{m_i+1} w_{a,\lambda} = 0,$$

for all $h \in \mathfrak{h}$, $1 \leq i \leq r$, $s \in \mathbb{Z}$. It was proved in [CP1] that as $\mathfrak{g}[t]$–modules,

$$W_a(\lambda) \cong U(\mathfrak{g}[t]) w_{a,\lambda},$$
and that
\[ \dim W_a(\lambda) = \dim W_b(\lambda), \quad \forall \, a, b \in C^\times. \]

In fact, it was proved there, that the action of \( x \otimes t^s \) with \( s \in N \) on \( W_a(\lambda) \) was a linear combination of elements from \( U(\mathfrak{g}[t]) \).

Moreover if we pull back \( W_a(\lambda) \) by the automorphism \( \tau_a : \mathfrak{g}[t] \to \mathfrak{g}[t] \) mapping \( x \otimes t^s \) to \( x \otimes (t + a)^s \), we find that \( \tau_a^*(W_a(\lambda)) \) is a quotient of \( W(\lambda) \).

Hence by Theorem 1.5.1 and the lower bound for the dimension of \( W_a(\lambda) \) obtained in \([\text{CPI}]\) we have,
\[ \tau_a^*(W_a(\lambda)) \cong W(\lambda). \]

Let \( N(\lambda) \) be the \( \mathfrak{g}[t] \)-submodule of \( W(\lambda) \) such that the quotient is a Demazure module and let \( N_a(\lambda) = \tau_a^*(N(\lambda)) \) be the corresponding submodule in \( W_a(\lambda) \).

It suffices to prove that \( N_a(\lambda) \) is a \( \mathfrak{g} \otimes C[t, t^{-1}] \)-submodule, but this is now clear, since the action of \( x \otimes t^s \) with \( s \in N \) on \( W_a(\lambda) \) was a linear combination of elements from \( U(\mathfrak{g}[t]) \).

## 2. A BASIS OF \( W(\lambda) \)

We prove Theorem 1.5.1 by giving an explicit basis for \( W(\lambda) \). In this section we define the proposed basis of \( W(\lambda) \) by identifying a suitable subset of the Poincare–Birkhoff–Witt basis of \( U(n^-[t]) \) and prove that it has the desired cardinality. We also deduce the fermionic version of a character formula for \( W(\lambda) \).

### 2.1. The set \( B^r(\lambda) \)

#### 2.1.1. Let \( F \) denote the set of all pairs \((\ell, s)\) which satisfy:
\[ \ell \in N, \quad s = (s(1) \leq \cdots \leq s(\ell)) \in N^\ell, \]
including the pair \((0, \emptyset)\), where \( \emptyset \) is the empty partition. Given \((\ell, s) \in F \) and \( 1 \leq i \leq j \leq r \), let \( x_{i,j}^\pm(\ell, s) \) be the element of \( U(n^-[t]) \) defined by,
\[ x_{i,j}^\pm(\ell, s) = (x_{i,j}^\pm \otimes t^{s(1)}) \cdots (x_{i,j}^\pm \otimes t^{s(\ell)}), \]
if \( \ell > 0 \) and \( x_{i,j}^\pm(0, \emptyset) = 1 \). From now on, given an element \(((\ell_1, s_1), \ldots, (\ell_j, s_j))\) of \( F^j \), \( j > 0 \), we let \((\ell, s)\) be the pair of \( j \)-tuples of integers, \( \ell = (\ell_1, \ldots, \ell_j) \) and \( s = (s_1, \ldots, s_j) \).

**Definition.** Let \( B^r \) be the subset of \( U(n^-[t]) \) consisting of elements of the form
\[ x_{i,1}^\pm(\ell_1, s_1) \cdots x_{i,r}^\pm(\ell_r, s_r), \]
where \((\ell_j, s_j) \in F^j \) for \( 1 \leq j \leq r \).

In other words, writing \((\ell_j, s_j)\) as \(((\ell_{1,j}, s_{1,j}), \ldots, (\ell_{j,j}, s_{j,j}))\), the set \( B^r \) consists of the elements
\[ x_{1,1}^-((\ell_{1,1}, s_{1,1}))x_{1,2}^-((\ell_{1,2}, s_{1,2}))x_{2,2}^-((\ell_{2,2}, s_{2,2}))x_{1,3}^-((\ell_{1,3}, s_{1,3})) \cdots x_{r,r}^-((\ell_{r,r}, s_{r,r})), \]
in \( U(n^-[t]) \) with \((\ell_{i,j}, s_{i,j}) \in F \), \( 1 \leq i \leq j \leq r \).

**Proposition.** The set \( B^r \subset U(n^-[t]) \) is a basis of \( U(n^-[t]) \).

**Proof.** Fix an ordered basis of \( n^-[t] \) as follows: the elements of the basis are \( x_{i,j}^- \otimes t^s \) for \( 1 \leq i \leq j \leq r \), \( s \in N \) and the order is given by, \( x_{i,j}^- \otimes t^s \) precedes \( x_{i',j'}^- \otimes t^{s'} \) when either \( j < j' \) or \( j = j' \), \( i < i' \) or \( j = j', i = i' \), \( s < s' \). The proposition follows from the Poincare–Birkhoff–Witt theorem. \( \square \)
2.1.2. For \( m \in \mathbb{N} \), let
\[
F(m) = \{(\ell, s) : \ell > 0, \ 0 \leq s(i) \leq m - \ell, \ 1 \leq i \leq \ell\} \cup \{(0, \emptyset)\}.
\]
If \( m < 0 \) then we set \( F(m) = \emptyset \). For \( \lambda = \sum_{s=1}^{r} m_i \omega_i \in P^+ \) and \( 1 \leq j \leq r \), let
\[
F^j(\lambda) = \{(\ell, s) \in F^j : (\ell_i, s_i) \in F(m_i), \ 1 \leq i \leq j\}.
\]
Given \( \ell \in \mathbb{N}^j \), set \( \eta_j(\ell) = \sum_{i=1}^{j} \ell_i \alpha_{i,j} \in Q^+ \). For any \( h \in \mathfrak{h} \) we have
\[
[h, \chi^\pm(\ell, s)] = \pm \eta_j(\ell)(h) \cdot \chi^\pm(\ell, s).
\]

**Definition.** For \( \lambda \in P^+ \), let \( B^r(\lambda) \) be the subset of \( B^r \) consisting of those elements in (2.1) which satisfy
\[
(\ell_j, s_j) \in F^j \left( \lambda - \sum_{s=j+1}^{r} \eta_s(\ell_s) \right), \quad 1 \leq j \leq r.
\]
Write \( \ell_j = (\ell_{1,j}, \cdots, \ell_{j,j}) \in \mathbb{N}^j, \ 1 \leq j \leq r \). Observe that if \( \lambda = \sum_{i=1}^{r} m_i \omega_i \in P^+ \), then
\[
\sum_{s=j+1}^{r} \eta_s(\ell_s) - \sum_{i=1}^{j} \left( \sum_{s=j+1}^{r} (\ell_{i,s} - \ell_{i+1,s}) \right) \omega_i \in \sum_{i=j+1}^{r} \mathbb{Z} \omega_i.
\]
It is now easy to check that (2.5) is equivalent to
\[
(\ell_{i,j}, s_{i,j}) \in F \left( m_i + \sum_{s=j+1}^{r} (\ell_{i+1,s} - \ell_{i,s}) \right),
\]
or even more explicitly to: either \( \ell_{i,j} = 0 \), or \( \ell_{i,j} > 0 \) and
\[
s_{i,j}(\ell_{i,j}) \leq m_i + \sum_{s=j+1}^{r} \ell_{i+1,s} - \sum_{s=j}^{r} \ell_{i,s},
\]
for all \( 1 \leq i \leq j \leq r \).

Note that for \( \lambda \in P^+ \), and \( \ell \in \mathbb{N}^{\ell} \), we have
\[
B^{r-1}(\lambda - \eta_r(\ell)) \subset B^{r}(\lambda).
\]

**Proposition.** (i) We have
\[
B^r(\lambda) = \bigcup_{(\ell, s) \in F^r(\lambda)} B^{r-1}(\lambda - \eta_r(\ell)) \chi^-(\ell, s),
\]
(ii) We have
\[
|B^r(\lambda)| = \prod_{i=1}^{r} \left( r + 1 \right)^{m_i}.
\]

**Proof.** Part (i) is clear. For (ii), recall that for a fixed pair \( \ell, n \) of integers, the number of elements \( s \) of the form, \( (s = (s(1) \leq \cdots \leq s(\ell) \leq n) \) is equal to \( \binom{n+\ell}{\ell} \), and so we have
\[
|\{(s : (\ell, s) \in F(m))\}| = \binom{m}{\ell}, \quad |F(m)| = \sum_{\ell=0}^{m} \binom{m}{\ell} = 2^m.
\]
We now proceed by induction on $r$ with (2.7) showing that induction begins at $r = 1$. Using (i) and (2.7), we get

$$|\mathcal{B}^r(\lambda)| = \sum_{\xi} |\mathcal{B}^{r-1}(\lambda - \eta_r(\xi))| \prod_{i=1}^r \binom{m_i}{\ell_i},$$

where the sum is over all $\xi = (\ell_1, \ldots, \ell_r)$ such that $0 \leq \ell_i \leq m_i$ for $1 \leq i \leq r$. This gives,

$$|\mathcal{B}^r(\lambda)| = \sum_{\xi} \prod_{i=1}^{r-1} \binom{r}{i}^{m_i-\ell_i+\ell_{i+1}} \prod_{i=1}^r \binom{m_i}{\ell_i} = \prod_{i=1}^r \sum_{l_i=0}^{m_i} \binom{m_i}{l_i} \binom{r}{l_i} \binom{r-\ell_i}{l_i-1} = \prod_{i=1}^r \binom{r+1}{l_i}^{m_i},$$

where the last equality is a consequence of the fact that $\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1}$ which implies

$$\binom{r+1}{i} = \sum_{l=0}^i \binom{n}{l} \binom{r}{l} \binom{r-1}{l-1},$$

and hence the proposition is proved.

The following corollary is immediate from Corollary 2.1.4.

**Corollary.** For $\lambda \in P^+$, we have

$$\dim(W(\lambda)) \geq |\mathcal{B}^r(\lambda)|.$$

**2.1.3.** The next theorem now obviously establishes Theorem 2.1.4.

**Theorem.** Let $\lambda = \sum_{i=1}^r m_i \omega_i \in P^+$. The set $\{bw_\lambda: b \in \mathcal{B}^r(\lambda)\}$ is a basis of $W(\lambda)$.

**Remark.** The theorem was proved in [CP1] when $r = 1$. In the rest of the paper we shall use that case to prove Theorem 2.1.3 when $r \geq 2$.

Again, we postpone the proof of the theorem and deduce some corollaries.

**Corollary.** The elements $(x_{1,1})^{\ell_{1,1}}(x_{1,2})^{\ell_{1,2}}(x_{2,2})^{\ell_{2,2}}(x_{1,3})^{\ell_{1,3}}(x_{2,3})^{\ell_{2,3}} \cdots (x_{r,r})^{\ell_{r,r}} v_\lambda$ where $\ell_{i,j} \in \mathbb{N}^+$ satisfy

$$m_i + \sum_{s=j+1}^r \ell_{i+1,s} - \sum_{s=j}^r \ell_{i,s} \geq 0$$

are a basis $V(\lambda)$.

**Proof.** It suffices to observe that the subspace of $W(\lambda)$ with $\mathbb{N}$–grading zero is isomorphic to $V(\lambda)$. 

**2.1.4.** Using this basis we get a fermionic formula for the character of $W(\lambda)$.

Given an integer $n \in \mathbb{N}$, set

$$[n]_t = \frac{1-t^n}{1-t} \in \mathbb{N}[t], \quad [n]_t ! = [1]_t \cdots [n]_t, \quad \left[ n \atop s \right]_t = \frac{[n]_t !}{[s]_t ![n-s]_t !}.$$

Clearly all these elements are in $\mathbb{N}[t]$.

**Proposition.** We have,

$$\text{ch}_t W(\lambda) = \sum_{(1,1) \in \mathbb{N}^{(r+1)/2}} \left( \prod_{1 \leq i \leq r} \left[ m_i + \sum_{s=j+1}^r \ell_{i+1,s} - \sum_{s=j}^r \ell_{i,s} \right]_{t, \ell_{i,j}} \right) e \left( \lambda - \sum_{1 \leq i \leq j \leq r} \ell_{i,j} a_{i,j} \right).$$
Proof. By Theorem 2.1.3 it follows that
\[
\text{ch}_t(W(\lambda)) = \sum_{b \in B'(\lambda)} e(\mu(b)) t^{s(b)},
\]
where \(\mu(b) \in P\), \(s(b) \in \mathbb{N}\) are defined as follows. If
\[
b = x_{1,1}^{-}(\ell_{1,1}, s_{1,1}) x_{1,2}^{-}(\ell_{1,2}, s_{1,2}) x_{2,2}^{-}(\ell_{2,2}, s_{2,2}) x_{1,3}^{-}(\ell_{1,3}, s_{1,3}) \cdots x_{r,r}^{-}(\ell_{r,r}, s_{r,r}),
\]
for some \((\ell_{i,j}, s_{i,j}) \in \mathcal{F}\), \(1 \leq i \leq j \leq r\), then
\[
s(b) = \sum_{1 \leq i \leq j \leq r} \ell_{i,j}, \quad \mu(b) = \lambda - \sum_{1 \leq i \leq j \leq r} \ell_{i,j} \alpha_{i,j}.
\]
Hence we get
\[
\text{ch}_t W(\lambda) = \sum_{(\ell_{i,j}) \in \mathbb{N}^{(r+1)/2}} \sum_{1 \leq i \leq j \leq r} e\left(\lambda - \sum_{1 \leq i \leq j \leq r} \ell_{i,j} \alpha_{i,j}\right) \prod_{1 \leq i \leq j \leq r} \sum_{s: (\ell, s) \in \mathcal{F}(m_{i,j})} t^{s(1)+\cdots+s(t)},
\]
where \(m_{i,j} = m + \sum_{s=j+1}^{r} \ell_{i+1,s} - \sum_{s=j}^{r} \ell_{i,s}\). But now, the result follows by observing that for a fixed pair of integers \(\ell, m \in \mathbb{N}\) we have
\[
\sum_{s: (\ell, s) \in \mathcal{F}(m)} t^{s(1)+\cdots+s(t)} = \sum_{0 \leq s(1) \leq \cdots \leq s(t) \leq m-\ell} [m]_{t}.
\]
\[\square\]

Remark. It follows from [HHKOTY, Proposition 5.3] that the right hand side of (2.9) is equal to \(\sum K_{\lambda, \ell}(t) \cdot \text{ch}_0 W(\lambda_\ell)\). Thus, the preceding proposition gives an alternate proof of [KMOTU, Theorem 5.2].

2.2. A Gelfand–Tsetlin type filtration for \(W(\lambda)\). In this section we construct a filtration indexed by \(\mathbb{N}^{2r}\) of \(W(\lambda)\) and show that Theorem 2.1.3 follows from Proposition 2.2.3 which studies the associated graded space of the filtration. As a consequence, we see that the associated graded module is isomorphic to a direct sum of Weyl modules for \(\mathfrak{sl}_r[t] \subset \mathfrak{g}[t]\).

2.2.1. Let \(\mathfrak{g}_{r-1}\) be the subalgebra of \(\mathfrak{g}\) isomorphic to \(\mathfrak{sl}_r\) which is generated by the elements \(x_{i,j}^{\pm}\) \(1 \leq i \leq j \leq r-1\). Set
\[
n_{r-1}^{\pm} = \bigoplus_{1 \leq i \leq j \leq r-1} \mathbb{C} x_{i,j}^{\pm}, \quad u_r^{\pm} = \bigoplus_{i=1}^{r} \mathbb{C} x_{i,r}^{\pm}.
\]
Clearly the elements \(\{x_{\ell, s}^{\pm}(\lambda, \mathbf{g}) : (\ell, s) \in \mathcal{F}^r\}\) are a basis for \(U(u_r^{\pm}[t])\), and we have by using the PBW–theorem that
\[
W(\lambda) = \sum_{(\ell, s) \in \mathcal{F}^r} U(\mathfrak{g}_{r-1}[t]) x_{\ell, s}^{\pm}(\lambda, \mathbf{g}).
\]
2.2.2. Let \( \Gamma = \mathbb{N}^r \times \mathbb{N}^r \). Introduce an order on \( \Gamma = \{(\ell, d)\} \) in the following way. We say that

\[
\ell \triangleright \ell' \quad \text{if} \quad \ell_1 = \ell'_1, \ldots, \ell_{s-1} = \ell'_{s-1}, \quad \ell_s < \ell'_s,
\]

\[
d \triangleright d' \quad \text{if} \quad d_r = d'_r, \ldots, d_{s+1} = d'_{s+1}, \quad d_s > d'_s
\]

for some \( 1 \leq s \leq r \). Finally we say that

\[
(\ell, d) > (\ell', d') \quad \text{if} \quad \ell > \ell' \quad \text{or} \quad \ell = \ell' \quad d > d'.
\]

For, \( (\ell, s) \in F \) set \( |s| = \sum_{p=1}^{\ell} s(p) \), if \( \ell > 0 \) and \( |\emptyset| = 0 \). Given \( (\ell, s) \in F^r \) let \( |s| = (|s_1|, \ldots, |s_r|) \). Note that \( (\ell, |s|) \in \Gamma' \). Given \( i \in \Gamma' \), define the \( g_{r-1}[t] \)-modules,

\[
W(\lambda)^{\geq i} = \sum_{\{(s, w) \in F^r : (s, w) \geq i\}} U(g_{r-1}[t]) x^{-}(\ell, s) w_{\lambda},
\]

\[
W(\lambda)^{> i} = \sum_{\{(s, w) \in F^r : (s, w) > i\}} U(g_{r-1}[t]) x^{-}(\ell, s) w_{\lambda},
\]

\[
\text{gr}(W(\lambda)) = \bigoplus_{i \in \Gamma'} W(\lambda)^{\geq i}/W(\lambda)^{> i}.
\]

Let

\[
\text{gr}^i : W(\lambda)^{\geq i} \rightarrow W(\lambda)^{\geq i}/W(\lambda)^{> i}
\]

the canonical projection which is clearly a map of \( g_{r-1}[t] \)-modules. The next result indicates the usefulness of this construction.

**Proposition.** For all \( \lambda \in P^+ \), we have an isomorphism of vector spaces,

\[
W(\lambda) \cong \text{gr}(W(\lambda)).
\]

More precisely, let \( \{v_1, \ldots, v_N\} \in W(\lambda) \) be a set of vectors such that \( v_j \in W(\lambda)^{i_j} \) for some \( i_j \in \Gamma' \), \( 1 \leq j \leq N \). Suppose that \( \{\text{gr}^{i_1}(v_1), \ldots, \text{gr}^{i_N}(v_N)\} \) span \( \text{gr}(W(\lambda)) \). Then \( \{v_1, \ldots, v_N\} \) span \( W(\lambda) \).

**Proof.** Let \( \Gamma'(\lambda) \) be the subset of \( \Gamma' \) defined by,

\[
\Gamma'(\lambda) = \left\{ i \in \Gamma' : \sum_{\{(s, w) \in F^r : (s, w) = i\}} U(g_{r-1}[t]) x^{-}(\ell, s) w_{\lambda} \neq 0 \right\}.
\]

We first show that \( \Gamma'(\lambda) \) is finite. For this, it suffices to prove that \( x^{-}(\ell, s) w_{\lambda} = 0 \) for all but finitely many pairs \( (\ell, s) \in F^r \). Note that

\[
x^{-}(\ell, s) w_{\lambda} \in W(\lambda)_{\mu_k, k_2}, \quad \mu_2 = \lambda - \eta_\ell(\ell), \quad k_2 = \sum_{i=1}^r |s_i|.
\]

Since \( W(\lambda) \) is a finite–dimensional \( g[t] \)-module, we can choose a finite set of pairs \( (\mu_1, k_1), \ldots, (\mu_M, k_M) \) such that \( W(\lambda)_{\mu, k} = 0 \) unless \( \mu = \mu_j \) and \( k = k_j \) for some \( 1 \leq j \leq M \). Then it is enough to show that the sets

\[
S_j = \{ (\ell, s) \in F^r : \mu_2 = \mu_j, \quad k_2 = k_j \}, \quad 1 \leq j \leq M,
\]

are finite. Since the elements \( \alpha_{i,r} \in Q^+ \), \( 1 \leq i \leq r \) are linearly independent, it follows immediately that for \( 1 \leq j \leq M \), either \( S_j \) is empty (in which case there is nothing to prove), or there exists unique \( \ell(j) \in \mathbb{N}^r \), such that \( \mu_{\ell(j)} = \mu_j \), so

\[
S_j = \{ (\ell(j), s) \in F^r : |s_1| + \cdots + |s_r| = k_j \}.
\]

Since each \( s_i \) is a tuple of non-negative integers, it follows now that \( S_j \) is finite for all \( 1 \leq j \leq M \).
Let $i_1 < i_2 < \cdots < i_R$ be an ordered enumeration of the elements of $\Gamma(\lambda)$. Let $i_{R+1}$ be any element of $\Gamma$ greater than $i_R$. Then we have

\[ W(\lambda)^{\geq i_1} = W(\lambda), \quad W(\lambda)^{\geq i_{R+1}} = 0, \quad W(\lambda)^{\geq i_k} = W(\lambda)^{\geq i_{k+1}}, \quad 1 \leq k \leq R. \]

Thus, we get

\[ \text{gr}(W(\lambda)) = \bigoplus_{i \in \Gamma(\lambda)} W(\lambda)^{\geq i}/W(\lambda)^{> i} = \bigoplus_{k=1}^{r} W(\lambda)^{\geq i_k}/W(\lambda)^{\geq i_{k+1}}, \]

which is the associated graded space of a usual increasing filtration

\[ 0 \subset W(\lambda)^{\geq i_1} \subset \cdots \subset W(\lambda)^{\geq i_R} = W(\lambda), \]

and for such a filtration the statement of the proposition is standard. \hfill \Box

2.2.3. We shall deduce Theorem 2.1.3 from the next proposition.

**Proposition.** Let $r \geq 2$.

(i) For $(\ell, s) \in F^r$, there exists a map of $\mathfrak{g}_{r-1}[t]$-modules $\psi_{\ell, s} : W(\lambda - \eta_r(\ell)) \to \text{gr}(W(\lambda))$ given by extending

\[ \psi_{\ell, s}(w_{\lambda - \eta_r(\ell)}) = \text{gr}(\ell, s)(x_-(\ell, s)w_{\lambda}). \]

Moreover,

\[ \text{gr}(W(\lambda)) = \bigoplus_{(\ell, s) \in F^r} \text{Im}(\psi_{\ell, s}). \]

(ii) The images of $\psi_{\ell, s}$ for $(\ell, s) \in F^r(\lambda)$ span $\text{gr}(W(\lambda))$.

2.2.4. **Proof of Theorem 2.1.3** The theorem is proved by induction on $r$. Note that induction starts at $r = 1$ by \cite[Theorem 6.1.3]{CP}. Assume the result for $r - 1$. Using Proposition 2.2.2 and the induction hypothesis we see that $\text{gr}(W(\lambda))$ is spanned by the sets

\[ \text{gr}(\ell, s)(\mathfrak{B}^{-1}(\lambda - \eta_r(\ell))x_-(\ell, s)w_{\lambda}), \quad \text{where } (\ell, s) \in F(\lambda). \]

Proposition 2.2.2 together with Proposition 2.1.2 (i) now shows that $\mathfrak{B}(\lambda)$ spans $W(\lambda)$ and hence $\dim W(\lambda) \leq |\mathfrak{B}(\lambda)|$. Since Corollary 2.1.2 established the reverse inequality, it follows that $\dim W(\lambda) = |\mathfrak{B}(\lambda)|$. Hence $\mathfrak{B}(\lambda)$ is a basis of $W(\lambda)$ and the theorem is proved. \hfill \Box

2.2.5. We conclude this section with a final corollary of Theorem 2.1.3 and Proposition 2.2.3

**Corollary.** For $r \geq 2$ and $\lambda = \sum m_i \omega_i$, we have an isomorphism of $\mathfrak{g}_{r-1}[t]$-modules

\[ \text{gr}(W(\lambda)) \cong \bigoplus_{(\ell, s) \in F^r} m_{\ell, s} W(\lambda - \eta_r(\ell)), \quad \text{where} \quad m_{\ell, s} = \left( m_1 \ell_1 \right) \cdots \left( m_r \ell_r \right). \]

In particular, the module $W(\lambda)$ admits a filtration of $\mathfrak{g}_{r-1}[t]$-modules such that the composition factors are Weyl modules for $\mathfrak{g}_{r-1}[t]$. \hfill \Box
3. Proof of Proposition 2.2.3

The proof of the proposition is quite complicated and requires a fairly detailed analysis of the structure of \( U(g[t]) \) and of the canonical projection \( pr^- : U(g[t]) \to U(n^-[t]) \) defined below. One has to study in some detail the commutation relations in the algebras and their behavior with respect to the ordered set \( \Gamma \). Section 3.1 is a collection of elementary properties of \( pr^- \), Section 3.2 analyses the \( g_{r-1}[t] \)-module structure of \( W(\lambda) \) and concludes by proving Proposition 2.2.3 (i). Section 3.3 studies the finite-dimensional irreducible \( g \)-module \( V(\lambda) \). The point of this section is to obtain a suitable spanning set for this module and to use it later in the study of \( W(\lambda) \) which we recall can be written as \( W(\lambda) = U(gt[t])V(\lambda) \). Section 3.4 is devoted to the study of \( U(sl_2[t]) \), more precisely it describes the subspace of \( U(n^-[t]) \) annihilating \( w_\lambda \). This is then used in the general case, to divide \( U(n^-[t]) \) into two different subspaces whose elements have different behavior when applied to \( w_\lambda \). After some further technical results, including the crucial lemma in Section 3.5, which calculates \( pr^- \) modulo higher terms in the Gelfand-Tsetlin filtration, the proof of Proposition 2.2.3 (ii) is completed in Section 3.6.

3.1. The projection \( pr^- \).

3.1.1. Let \( b^+ \) be the subalgebra \( h \oplus n^+ \) of \( g \) and note that \( g = b^+ \oplus n^- \). Let \( pr^- : U(g[t]) \to U(n^-[t]) \) be the projection corresponding to the vector space decomposition

\[
U(g[t]) = U(n^-[t]) \oplus U(g[t])(b^+[t]),
\]

given by the Poincare–Birkhoff–Witt theorem. Clearly \( pr^- \) is a \( N \)-graded linear map. The next result collects some properties of \( pr^- \) which are immediate from the definition.

**Proposition.** (i) For all \( g^- \in U(n^-[t]) \) and \( x \in U(g[t]) \), we have \( pr^- (g^- x) = g^- pr^- (x) \).

(ii) For all \( g^+ \in U(b^+[t])(b^+[t]) \), \( x \in U(g[t]) \), we have \( pr^- (xg^+) = 0 \) and hence \( pr^- (g^+ x) = pr^- (g^+ x) \).

(iii) For all \( g_1, g_2 \in U(g[t]) \) we have \( pr^- (g_1 g_2) = pr^- (g_1 pr^- (g_2)) \).

\( \square \)

Note that \( n^-[t] \oplus b^+ [t] \) is a subalgebra of \( g[t] \).

**Lemma.** Let \( \lambda \in P^+ \). Then for all \( x \in U(n^-[t] \oplus b^+ [t]) \) we have \( x w_\lambda = pr^- (x) w_\lambda \) in \( W(\lambda) \).

**Proof.** The proof is immediate from the observation that \( (n^+[t] \oplus h[t]) w_\lambda = 0 \).

\( \square \)

3.1.2. Let

\[
U(n^-[t])^i = \sum_{(\ell, \alpha) \in \Gamma : (\ell, \alpha) \leq i} U(n_{r-1}^-[t] \chi^- _{(\ell, \alpha)}),
\]

\[
U(n^-[t])^{\geq i} = \sum_{(\ell, \alpha) \in \Gamma : (\ell, \alpha) \geq i} U(n_{r-1}^-[t] \chi^- _{(\ell, \alpha)}),
\]

\[
U(n^-[t])^{> i} = \sum_{(\ell, \alpha) \in \Gamma : (\ell, \alpha) > i} U(n_{r-1}^-[t] \chi^- _{(\ell, \alpha)}).
\]

Clearly we have

\[
W(\lambda)^{\geq i} \supset U(n^-[t])^{\geq i} w_\lambda, \quad W(\lambda)^{> i} \supset U(n^-[t])^{> i} w_\lambda.
\]

We will see in Proposition 3.2.2 that the reverse inclusions hold as well.
3.1.3. The algebra $\mathbf{U}(\mathfrak{g}[t])$ is a sum of eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}[t]$ and the decomposition is compatible with the $\mathbb{N}$-grading on $\mathbf{U}(\mathfrak{g}[t])$. More precisely, we can write

$$\mathbf{U}(\mathfrak{g}[t]) = \bigoplus_{\mu \in \mathfrak{b}^*} \mathbf{U}(\mathfrak{g}[t])_{\mu} = \bigoplus_{\mu \in \mathfrak{b}^*, s \in \mathbb{N}} \mathbf{U}(\mathfrak{g}[t])_{\mu, s}.$$  

The subspaces $\mathbf{U}(\mathfrak{n}^\pm[t])_{\mu, s}$ are defined analogously. Note that $\mathbf{U}(\mathfrak{g}[t])_{\mu, s} = 0$ if $\mu \notin Q$ and that $\mathbf{U}(\mathfrak{n}^\pm[t])_{\pm \mu, s} = 0$ if $\mu \notin Q^+$. Again, the next lemma collects together some elementary properties of these decompositions.

Lemma. (i) For $\lambda \in P^+, \mu \in P$ and $s \in \mathbb{N}$, we have $W(\lambda)_{\mu, s} = \mathbf{U}(\mathfrak{n}^-[t])_{\mu - \lambda, s}w_\lambda$.

(ii) Let $\eta, \eta' \in Q$, $s, s' \in \mathbb{N}$. We have $\mathbf{U}(\mathfrak{g}[t])_{\eta, s} \mathbf{U}(\mathfrak{g}[t])_{\eta', s'} \subset \mathbf{U}(\mathfrak{g}[t])_{\eta + \eta', s + s'}$. Moreover, if $g_1, g_2 \in \mathbf{U}(\mathfrak{n}^-[t]),$ then

$$g_1 \in \mathbf{U}(\mathfrak{n}^-[t])_{\eta, s}, \quad g_1 g_2 \in \mathbf{U}(\mathfrak{n}^-[t])_{\eta', s'} \quad \text{implies} \quad g_2 \in \mathbf{U}(\mathfrak{n}^-[t])_{\eta' - \eta, s' - s}.$$  

(iii) For all $\eta \in Q$, $s \in \mathbb{N}$ we have $\mathbf{U}(\mathfrak{n}^-[t])_{\eta, s} \subset \mathbf{U}(\mathfrak{g}[t])_{\eta, s}$ and $\mathbf{pr}^- (\mathbf{U}(\mathfrak{g}[t])_{\eta, s}) \subset \mathbf{U}(\mathfrak{n}^-[t])_{\eta, s}$.

(iv) For all $1 \leq i \leq j \leq r$, and $(\ell, s) \in \mathbf{F}$, we have, $x^\pm_{i,j} (\ell, s) \in \mathbf{U}(\mathfrak{n}^-[t])_{\pm \ell, \pm |s|}$.

3.1.4. Let $\mathbf{F}_+$ denotes the subset of $\mathbf{F}$ consisting of pairs $(\ell, s) \in \mathbf{F}$ such that $s(i) > 0$ for $1 \leq i \leq \ell$, together with the pair $(0, \emptyset)$. The next proposition is used repeatedly in this section.

Proposition. Let $1 \leq i \leq r$, $(\ell^+, s^+) \in \mathbf{F}_+$, $(\ell, s) \in \mathbf{F}$. The element $\mathbf{pr}^- \left( x^+_{i,r} (\ell^+, s^+) x^-_{i,r} (\ell, s) \right)$ is a linear combination of elements of the form $x^\pm_{i,r} (\ell - \ell', s')$, where $s'$ satisfies the condition $|s'| = |s| + |s^+|$.

Proof. Note that the subspace $\mathfrak{s}_i[t]$ of $\mathfrak{g}[t]$ spanned by the elements $x^\pm_{i,r} \otimes t^k$, $h_i \otimes t^k$, $k \in \mathbb{N}$ is a subalgebra of $\mathfrak{g}[t]$ which is isomorphic to $\mathfrak{s}_2[t]$. Further, $\mathbf{pr}^-$ maps $\mathbf{U}(\mathfrak{s}_i[t])$ onto $\mathbf{U}(\mathfrak{n}^-_{i,r}[t])$, where $\mathfrak{n}^-_{i,r}[t]$ is the subalgebra of $\mathfrak{n}^-[t]$ spanned by elements of the form $x^-_{i,r} \otimes t^k$, $k \in \mathbb{N}$. This proves that $\mathbf{pr}^- (x^+_{i,r} (\ell^+, s^+) x^-_{i,r} (\ell, s))$ is a linear combination of elements of the form $x^\pm_{i,r} (\ell', s')$ for some $(\ell', s') \in \mathbf{F}$. But now, we see that since $x^+_{i,r} (\ell^+, s^+) x^-_{i,r} (\ell, s) \in \mathbf{U}(\mathfrak{g}[t])_{(\ell - \ell') \alpha_{i,r}, |s| + |s^+|}$ the same holds for $\mathbf{pr}^- (x^+_{i,r} (\ell^+, s^+) x^-_{i,r} (\ell, s))$. This means that we can assume that $\ell' = \ell^+ - \ell$ and $|s'| = |s| + |s^+|$.

3.2. Action of $\mathfrak{g}_{-1}[t]$ on $W(\lambda)$.

3.2.1. Let $e_i \in \mathbb{N}^1$, $1 \leq i \leq r$, denote the standard basis vectors.

Lemma. Let $(\ell, s) \in \mathbf{F}^r, g \in b^+_{-1}[t]$.

(i) The commutator $[g, x^-_{\ell, s}]$ is a linear combination of terms of the form $x^-_{\ell', s'}$ with $(\ell', s') \geq (\ell, s)$. Moreover, if $g \in n^+_{-1}[t] \otimes h_{-1} t[t]$ then we have $(\ell', s') > (\ell, s)$.

(ii) Let $\lambda \in P^+$. The element $g x^-_{\ell, s} w_\lambda \in W(\lambda)$ is a linear combination of terms of the form $x^-_{\ell', s'} w_\lambda$ with $(\ell', s') \geq (\ell, s)$. Moreover, if $g \in n^+_{-1}[t] \otimes h_{-1} t[t]$ then we have $(\ell', s') > (\ell, s)$.

Proof. For (i) it is enough to consider the case $g = h \otimes t^s$, where $h \in \mathfrak{h}_{-1}$ and $s \in \mathbb{N}$, and the case $g = x^+_{i,j} \otimes t^s$, where $1 \leq i \leq j < r$ and $s \in \mathbb{N}$. Recall the adjoint action of $\mathfrak{g}[t]$ on $\mathbf{U}(\mathfrak{g}[t])$ is a derivation, i.e

$$[y, x_1 \cdots x_s] = \sum_{j=1}^s x_1 \cdots \hat{x}_j \cdots x_s [y, x_j] x_{j+1} \cdots x_s,$$

for all $y, x_1, \cdots, x_s \in \mathfrak{g}$. Now, since

$$[h \otimes t^s, x^-_{k,r} \otimes t^m] = -\alpha_{k,r} (h) x^-_{k,r} \otimes t^{m+s},$$

(3.1)
for all \( h \in \mathfrak{h}, m, s \in \mathbb{N}, 1 \leq k \leq r \), we see that \( [h \otimes t^s, \mathbf{x}_-^{\leq s}(\ell, \mathbf{s})] \) is a linear combination of terms \( \mathbf{x}_-^{\leq s}(\ell', \mathbf{s}') \) with \( \ell' = \ell, |\mathbf{s}'| = |\mathbf{s}| + se_j, 1 \leq j \leq r \) and hence \( (\ell', |\mathbf{s}'|) \geq (\ell, |\mathbf{s}|) \) with the inequality being strict if \( s > 0 \).

Next for \( 1 \leq i \leq j < r \) we have \( [x_{i,j}^+ \otimes t^s, x_{i,r}^- \otimes t^m] = -\delta_{k,i} x_{i+1,j}^- \otimes t^{m+s} \), and it follows again that 

\[
[x_{i,j}^+ \otimes t^s, \mathbf{x}_-^{\leq s}(\ell, \mathbf{s})] \]

is a linear combination of terms \( \mathbf{x}_-^{\leq s}(\ell', \mathbf{s}') \) with

\[
\ell' = \ell + e_{j+1} - e_i, \quad |\mathbf{s}'| = |\mathbf{s}| + (m + s)e_{j+1} - me_i.
\]

Further, \( (\ell', \mathbf{s}') > (\ell, \mathbf{s}) \).

For (ii) note that

\[
\delta\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = \mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda + [\delta, \mathbf{x}_-^{\leq s}(\ell, \mathbf{s})]w_\lambda.
\]

The first term is proportional to \( w_\lambda \) and is zero if \( g \in n_{r-1}^+ [t] \oplus \mathfrak{h}_{r-1} [t] \) and hence is of the desired form. Using part (i), we see that the second term is also in the correct form and the lemma is proved. \( \Box \)

3.2.2. Proposition. We have

\[
W(\lambda)^{\geq i} = U(n^- [t])^{\geq i}w_\lambda, \quad W(\lambda)^{> i} = U(n^- [t])^{> i}w_\lambda.
\]

Proof. It is enough to show that \( U(\mathfrak{g}_{r-1} [t])\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda \) is contained in \( U(n^- [t])^{\geq (\ell, \mathbf{s})}w_\lambda \). Since, \[U(\mathfrak{g}_{r-1} [t]) = U(n_{r-1}^- [t])U(b_{r-1}^+[t]),\]

it suffices to prove that \[U(b_{r-1}^+[t])\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda \subset U(n^- [t])^{\geq (\ell, \mathbf{s})}w_\lambda.\]

But this follows from Lemma 3.2.1 (ii). \( \Box \)

3.2.3. Proof of Proposition 3.2.2. (i). Let \( (\ell, \mathbf{s}) \in \mathbb{F}^r \). Using Lemma 3.2.1 (ii), we see that for all \( h \in \mathfrak{h}, k \in \mathbb{N}, s \in \mathbb{N}_+, 1 \leq i \leq j \leq r \), we have

\[
(x_{i,j}^+ \otimes t^k) \cdot \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = 0, \quad (h \otimes t^s) \cdot \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = 0,
\]

\[
h \cdot \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = \left( \lambda(h) - \sum_{j=1}^r \epsilon_j \alpha_{j,r}(h) \right) \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = (\lambda - \eta_r(\ell))(h)\text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda.
\]

This means that the element \( \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda \) is in \( \text{gr}(W(\lambda)) \) satisfies the relations in (1.4) with the weight \( \lambda - \eta_r(\ell) \). Theorem 1.2.2 now implies the existence of the map \( \psi_\ell^\mathbf{s} \). Since,

\[
\text{Im} \psi_\ell^\mathbf{s} = U(\mathfrak{g}_{r-1} [t]) \cdot \text{gr}(\ell, \mathbf{s})_s\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda = \text{gr}(\ell, \mathbf{s})_s(U(\mathfrak{g}_{r-1} [t])\mathbf{x}_-^{\leq s}(\ell, \mathbf{s})w_\lambda),
\]

the second statement follows.

3.3. The case of \( V(\lambda) \). It is necessary in this section alone, to work with the Lie algebra \( \mathfrak{g}_l_{r+1} = \mathfrak{g} \oplus \mathbb{C}c \).

3.3.1. It is well–known that the finite–dimensional irreducible representations of \( \mathfrak{g}_l_{r+1} \) are parameterized by partitions \( \xi = (\xi_1 \geq \cdots \geq \xi_r \geq \xi_{r+1} \geq 0) \) of non–negative integers. Let \( V(\xi) \) denote the corresponding representation, which is generated as a \( \mathfrak{g}_l_{r+1} \)–module by an element \( v_\xi \) with the relations:

\[
x_{i,j}^+ v_\xi = 0, \quad H_i v_\xi = \xi_i v_\xi, \quad (x_{i,j}^-)^{\xi_i - \xi_{i+1} + 1} v_\xi = 0,
\]

for all \( 1 \leq i \leq j \leq r \). Note that \( V(\xi) = U(n^-)v_\xi \). Moreover, we have \( V(\xi) \cong V(\lambda_\xi) \) as \( \mathfrak{g} \)–modules, where

\[
\lambda_\xi = \sum_{i=1}^r (\xi_i - \xi_{i+1}) \omega_i \in P^+.
\]
Conversely, given \( \lambda = \sum_{i} m_i \omega_i \in P^+ \), we have \( V(\lambda) \cong V(\xi^\lambda) \) where \( \xi^\lambda = (\xi^\lambda_1 \geq \cdots \geq \xi^\lambda_r \geq \xi^\lambda_{r+1} \geq 0) \) is defined by \( \xi^\lambda_i = \sum_{j=1}^{r} m_j \) and \( \xi^\lambda_{r+1} = 0 \).

**Theorem.** As a module for \( \mathfrak{gl}_r \subset \mathfrak{gl}_{r+1} \) we have
\[
V(\xi) = \bigoplus_{\eta} V(\eta),
\]
where the sum is over all partitions \( \eta = (\eta_1 \geq \cdots \geq \eta_r) \) of non-negative integers satisfying, \( \xi_i \geq \eta_i \geq \xi_{i+1} \) for all \( 1 \leq i \leq r \).

The inductive construction of the Gelfand–Tsetlin basis for the \( \mathfrak{gl}_{r+1} \)-module \( V(\xi^\lambda) \) (see \cite{GT}) is based on this theorem and it motivated our definition of the Gelfand–Tsetlin filtration of the \( \mathfrak{gl}[t] \)-modules \( W(\lambda) \).

**3.3.2.** The next two results are probably well-known, but we include a proof here, since it is basic for the results of this paper.

**Lemma.** Let \( \lambda = \sum m_i \omega_i \). Then we have,
\[
V(\lambda) = \sum_{k_1, \ldots, k_r \in \mathbb{N}} U(\mathfrak{n}^-_{r-1}) (x^-_{1,\tau})^{k_1} \cdots (x^-_{r,\tau})^{k_r} v_\lambda.
\]

**Proof.** To prove the lemma, it suffices to establish it for the \( \mathfrak{gl}_{r+1} \)-module \( V(\xi^\lambda) \), i.e to prove that that
\[
V(\xi^\lambda) = \sum_{k_1, \ldots, k_r \in \mathbb{N}} U(\mathfrak{n}^-_{r-1}) (x^-_{1,\tau})^{k_1} \cdots (x^-_{r,\tau})^{k_r} v_\xi.
\]

Using Theorem 3.3.1 we see that \( V(\xi^\lambda) \) is spanned by the sets \( U(\mathfrak{n}^-_{r-1}) v_\eta \), where \( \eta = (\eta_1 \geq \cdots \geq \eta_r) \) are non-negative integers \( \xi^\lambda_1 \geq \eta_1 \geq \xi^\lambda_{i+1} \), \( 1 \leq i \leq r \), and \( v_\eta \in V(\xi^\lambda) \) satisfies
\[
x^-_{i,\tau} v_\eta = 0, \quad H_m v_\eta = \eta_m v_\eta.
\]

Writing \( v_\eta \) as a \( U(\mathfrak{n}^-_{r-1}) \)-linear combination of elements \( (x^-_{1,\tau})^{k_1} \cdots (x^-_{r,\tau})^{k_r} v_\xi \), \( 1 \leq k \leq r \), we find easily that \( \eta_1 = \xi^\lambda_1 - k_1 - k \) for some \( k \geq 0 \). Hence \( k_1 \leq \xi^\lambda_1 - \eta_1 \leq \xi^\lambda_1 - \xi^\lambda_2 = m_1 \) and the lemma is proved.

**3.3.3.** We can now prove the following stronger statement.

**Proposition.** Let \( \lambda = \sum m_i \omega_i \in P^+ \). Then we have,
\[
V(\lambda) = \sum_{\{k_r \leq k_i \leq m_i, 1 \leq i \leq r\}} U(\mathfrak{n}^-_{r-1}) (x^-_{1,\tau})^{k_1} \cdots (x^-_{r,\tau})^{k_r} v_\lambda.
\]

**Proof.** Let \( \mathfrak{n}^+_{m,n} \) be the subalgebra of \( \mathfrak{g} \) spanned by \( x^+_{i,j} \), \( m \leq i < j \leq n \), let \( \mathfrak{h}_{m,n} \) be the subalgebra spanned by \( h_i \), \( m \leq i \leq n \) and, finally, let \( \mathfrak{g}_{m,n} = \mathfrak{n}^+_{m,n} \oplus \mathfrak{h}_{m,n} \oplus \mathfrak{n}^-_{m,n} \). Clearly, \( \mathfrak{g}_{m,n} \) is isomorphic to \( \mathfrak{sl}_{n-m+2} \). We proceed by induction on \( r \). For \( r = 1 \), the result follows from the defining relations of \( V(\lambda) \). Note that \( U(\mathfrak{g}_{2,r}) v_\lambda \cong V(\lambda|_{\mathfrak{g}_{2,r}}) \) as \( \mathfrak{g}_{2,r} \)-modules, and hence we have by the induction hypothesis
\[
U(\mathfrak{n}^-_{r-1}) v_\lambda = \sum_{\{k_1 \leq k_i \leq m, 2 \leq i \leq r\}} U(\mathfrak{n}^-_{r-1}) (x^-_{2,\tau})^{k_1} \cdots (x^-_{r,\tau})^{k_r} v_\lambda.
\]
Combining this with Lemma 3.3.2 we find that
\[
V(\lambda) = \sum_{\{k_i \leq k_i \leq m_i, 1 \leq i \leq r\}} U(\mathfrak{n}^-_{r-1}) (x^-_{1,\tau})^{k_1} U(\mathfrak{n}^-_{r-1}) (x^-_{2,\tau})^{k_2} \cdots (x^-_{r,\tau})^{k_r} v_\lambda.
\]
Since \( \mathfrak{n}^-_{2,r-1} \subset \mathfrak{n}^-_{r-1} \) and \( [x^-_{1,\tau}, \mathfrak{n}^-_{2,r-1}] = 0 \), the proposition follows.
3.4. Some results on $U(sl_2[t])$ and their consequences.

3.4.1. Assume in this section that $g = sl_2$ and set $\omega_1, x^\pm = x_{1,1}^\pm$. For $n \in \mathbb{N}$, let $I_n \subset U(n^{-}[t])$ be the ideal generated by the elements $pr^-(x^+ \otimes t)^k(x^- \otimes 1)^l$, where $k \geq 0$ and $l > n$.

**Theorem.** [CP1, Section 6].
(i) The elements $\{x^-(\ell, s)w_n, (\ell, s) \in F(n)\}$ are a basis of $W(n\omega)$.
(ii) The map $U(n^{-}[t]) \rightarrow W(n\omega)$ sending $g$ to $gw_n$ induces an isomorphism of vector spaces
$$U(n^{-}[t])/I_n \cong W(n\omega).$$

For $n \in \mathbb{N}$, let $J_n$ be the subspace of $U(n^{-}[t])$ spanned by the set
$$\{pr^-(x^+(\ell, s)(x^- \otimes 1)^m : m > n, (\ell, s) \in F_+\}.$$ Note that if $m > n$, we have $(x^- \otimes t^m)w_n = 0$ and hence
$$x^+(\ell, s)(x^- \otimes 1)^m w_n = 0,$$
which is equivalent by Lemma 3.1.1 to
$$pr^-(x^+(\ell, s)(x^- \otimes 1)^m)w_n = 0,$$ i.e. $pr^-(x^+(\ell, s)(x^- \otimes 1)^m) \in I_n$ and hence $J_n \subset I_n$. In what follows, we shall prove the reverse inclusion.

3.4.2. We begin with,

**Lemma.** Let $m, s \in \mathbb{N}$ and assume that $m > n$.
(i) We have $(x^- \otimes t^s)(x^- \otimes 1)^m \in J_n$.
(ii) We have $pr^-((h \otimes t^s)(x^- \otimes 1)^m) \in J_n$.
(iii) For $g \in J_n, (\ell, s) \in F_+$ we have $pr^- (x^+(\ell, s)g) \in J_n$.

**Proof.** Note that
$$[x^+ \otimes t^s, (x^- \otimes 1)^{m+2}] = -(m + 2)(m + 1)(x^- \otimes t^s)(x^- \otimes 1)^m.$$ Proposition 3.1.1 (ii) gives,
$$pr^-((x^+ \otimes t^s)(x^- \otimes 1)^{m+2}) = pr^-([x^+ \otimes t^s, (x^- \otimes 1)^{m+2}]) = -(m + 2)(m + 1)(x^- \otimes t^s)(x^- \otimes 1)^m,$$
and hence (i) follows. The proof of (ii) is similar, using
$$[h \otimes t^s, (x^- \otimes 1)^m] = -2m(x^- \otimes t^s)(x^- \otimes 1)^{m-1} = \frac{2}{m+1}pr^-((x^+ \otimes t^s)(x^- \otimes 1)^{m+1}).$$
Next, note that Proposition 3.1.1 (iii) gives,
$$pr^- (x^+(\ell, s)pr^- (x^+(\ell', s')(x^- \otimes 1)^m)) = pr^- (x^+(\ell, s)x^+(\ell', s')(x^- \otimes 1)^m).$$
So Part (iii) follows since $x^+(\ell, s)x^+(\ell', s') = x^+(\ell'', s'')$ where $\ell'' = \ell + \ell'$ and $s''$ is obtained by concatenating $s$ and $s'$ into a partition. \qed
3.4.3. Note that \( J_n \) contains the generators of \( I_n \) and so if we show that \( J_n \) is an ideal we establish that \( J_n = I_n \).

**Proposition.** For all \( n \in \mathbb{N} \), the subspace \( J_n \) is an ideal in \( U(n^{-}[t]) \) and hence \( J_n = I_n \).

**Proof.** Let \( k, m \in \mathbb{N} \) and \( m > n \). Then,

\[
(x^{-} \otimes t^k) \cdot \mathbf{pr}^{-}(x^{+}(\ell, s)(x^{-} \otimes 1)^m) = \mathbf{pr}^{-}((x^{-} \otimes t^k)x^{+}(\ell, s)(x^{-} \otimes 1)^m),
\]

\[
= \mathbf{pr}^{-}(x^{+}(\ell, s)(x^{-} \otimes t^k)(x^{-} \otimes 1)^m)
\]

\[
+ \mathbf{pr}^{-}([x^{-} \otimes t^k, x^{+}(\ell, s)])(x^{-} \otimes 1)^m).
\]

Lemma 3.4.2 (i), (iii) implies that \( \mathbf{pr}^{-}(x^{+}(\ell, s)(x^{-} \otimes t^k)(x^{-} \otimes 1)^m) \in J_n \). For the second term, on the right hand side of the preceding equation, note that \([x^{-} \otimes t^k, x^{+}(\ell', s')] \) is a linear combination of terms of the form \( x^{+}(\ell', s') \) and \( x^{+}(\ell', s')(h \otimes t^p) \) where \((\ell', s') \in F_+, \ p > 0 \). Now, if \( m > n \), the element \( \mathbf{pr}^{-}(x^{+}(\ell', s')(x^{-} \otimes 1)^m) \in J_n \) by definition. By Lemma 3.4.2 (ii), (iii) we have

\[
\mathbf{pr}^{-}(x^{+}(\ell', s')(h \otimes t^p)(x^{-} \otimes 1)^m) = \mathbf{pr}^{-}(x^{+}(\ell', s') \mathbf{pr}^{-}((h \otimes t^p)(x^{-} \otimes 1)^m)) \in J_n.
\]

This proves that \( (x^{-} \otimes t^k) \cdot \mathbf{pr}^{-}(x^{+}(\ell, s)(x^{-} \otimes 1)^m) \in J_n \), establishing the proposition. \( \square \)

3.4.4. The following is now an immediate corollary of Theorem 3.4.1 and Proposition 3.4.3

**Proposition.** Assume that \( g \) is of type \( \mathfrak{sl}_2 \) and that \( n \geq 0 \). Then, \( U(n^{-}[t]) \) is spanned by elements of the set

\[
\{ x^{-}(\ell, s) : (\ell, s) \in F(n) \} \cup \{ \mathbf{pr}^{-}(x^{+}(\ell, s)(x^{-} \otimes 1)^m) : m > n, \ (\ell, s) \in F_+ \}.
\]

\( \square \)

We now return to the case of \( \mathfrak{sl}_{r+1} \). Fix \( 1 \leq i \leq r \). Applying Proposition 3.4.4 to the subalgebra of \( \mathfrak{sl}_{r+1} \) generated by the elements \( x_{r,i}^{\pm} \), and using Proposition 3.1.4, we obtain the following result.

**Corollary.** Let \( 1 \leq i \leq r \) and \( n \geq 0 \). For all \( (\ell, s) \in F \), the element \( x_{r,i}^{-}(\ell, s) \) is in the span of the union of

\[
\{ x_{r,i}^{-}(\ell, s') : (\ell, s') \in F(n), \ |s'| = |s| \},
\]

and

\[
\{ \mathbf{pr}^{-}(x_{r,i}^{-}(m - \ell, s')(x_{r,i}^{-} \otimes 1)^m) : m > n, \ (m - \ell, s') \in F_+, \ |s'| = |s| \}.
\]

In particular, the element \( x_{r,i}^{-}(\ell, s) \), is in the span of

\[
\{ \mathbf{pr}^{-}(x_{r,i}^{+}(m - \ell, s')(x_{r,i}^{-} \otimes 1)^m) : m \geq 0, \ (m - \ell, s') \in F_+, \ |s'| = |s| \}.
\]

\( \square \)

3.5. A crucial Lemma.

3.5.1. The goal of this section is to prove the following statement

**Lemma.** Let \( (\ell, s)^+ \in F_r^+ \). For all \( (\ell, s) \in F_r^+ \) with \( \ell_i \geq \ell_i^+ \), \( 1 \leq i \leq r \), we have

\[
(3.6) \mathbf{pr}^{-}(x_{r,i}^{+}(\ell, s^{+})x_{r,i}^{-}(\ell, s)) - \prod_{i=1}^{r} \mathbf{pr}^{-}(x_{i,r}^{+}(\ell_i^+, s_i^+)x_{i,r}^{-}(\ell_i, s_i)) \in U(n^{-}[t])^{\ell_1^+ + \cdots + \ell_r^+, |s^{+}| + |s^{-}|)}
\]

In particular,

\[
\mathbf{pr}^{-}(x_{r,i}^{+}(\ell, s^{+})x_{r,i}^{-}(\ell, s)) \in U(n^{-}[t])^{\ell_1^+ + \cdots + \ell_r^+, |s^{+}| + |s^{-}|)}.
\]

We shall prove the first statement by induction on \( |\ell^{+}| = \ell_1^+ + \cdots + \ell_r^+ \). Proposition 3.1.4 gives

\[
\prod_{i=1}^{r} \mathbf{pr}^{-}(x_{i,r}^{+}(\ell_i^+, s_i^+)x_{i,r}^{-}(\ell_i, s_i)) \in U(n^{-}[t])^{\ell_1^+ + \cdots + \ell_r^+, |s^{+}| + |s^{-}|)},
\]

which proves the second statement of the proposition.
3.5.2. We shall need the following. For a partition $s = (s(1) \leq \cdots \leq s(\ell))$ let

$$s^{(i)} = (s(1) \leq \cdots \leq s(i-1) \leq s(i+1) \leq \cdots \leq s(\ell)).$$

**Lemma.** Let $(\ell, s) \in F$, $s \in N$ and $1 \leq i \leq j \leq r$. Then

$$[x^+_{i,r} \otimes t^s, x^-_{j,r}(\ell, s)] = \begin{cases} \sum_{k=1}^{\ell} x^+_{j,r}(\ell-1, s^{(k)})(x^+_{i,j-1} \otimes t^{s(k)}) & i < j, \\
\sum_{k=1}^{\ell} x^-_{j,r}(\ell-1, s^{(k)})(x^-_{i,j-1} \otimes t^{s(k)}) & i > j, \\
\sum_{k=1}^{\ell} x^+_{j,r}(\ell-1, s^{(k)})(h_{i,r} \otimes t^{s(k)}) + \text{pr}^-(j_{i,r})x^-_{j,r}(\ell, s) & i = j, \end{cases}$$

where $h_{i,r} = h_i + \cdots + h_r \in h$.

**Proof.** The commutators are calculated as usual by using the fact that $g[t]$ acts on $U(g[t])$ as derivations (see (3.1)). The case $i < j$ follows from

$$[x^+_{i,r} \otimes t^s, x^-_{j,r} \otimes t^s] = x^+_{i,j-1} \otimes t^{s'+s}, \quad i < j,$$

and the observation that $x^+_{i,j-1} \otimes t^{s'+s}$ commutes with the factors of $x^-_{j,r}(\ell, s)$.

The case $i > j$ follows from

$$[x^+_{i,r} \otimes t^s, x^-_{j,r} \otimes t^s] = x^-_{j,i-1} \otimes t^{s'+s}, \quad i > j,$$

and the observation that $x^-_{j,i-1} \otimes t^{s'+s}$ commutes with the factors of $x^-_{j,r}(\ell, s)$.

Finally consider the case $i = j$. We have

$$[x^+_{i,r} \otimes t^s, x^-_{i,r} \otimes t^s] = h_{i,r} \otimes t^{s'+s},$$

which gives

$$[x^+_{i,r} \otimes t^s, x^-_{i,r}(\ell, s)] = \sum_{k=1}^{\ell} \prod_{n=1}^{k-1} x^+_{i,r}(\ell-1, s^{(n)})(h_{i,r} \otimes t^{s(k)}) \left( \prod_{n=k+1}^{\ell} x^-_{i,r}(\ell, s^{(n)}) \right)$$

$$= \sum_{k=1}^{\ell} x^-_{i,r}(\ell-1, s^{(k)})(h_{i,r} \otimes t^{s(k)})$$

$$+ \sum_{k=1}^{\ell} \prod_{n=1}^{k-1} x^+_{i,r}(\ell, s^{(n)}) \left( h_{i,r} \otimes t^{s(k)} \right) \left( \prod_{n=k+1}^{\ell} x^+_{i,r}(\ell, s^{(n)}) \right).$$

The first term on the right hand side of the preceding equality is in $U(n^{-}[-t])h[t]$. Since $[h[t], n^{-}[-t]] \subset n^{-}[-t]$ the second term is in $U(n^{-}[-t])$ and hence is equal to $\text{pr}^-(x^+_{i,r} \otimes t^s, x^-_{i,r}(\ell, s))$. Since the latter term is also equal to $\text{pr}^-(x^+_{i,r} \otimes t^s, x^-_{i,r}(\ell, s))$, the proof of the Lemma is complete.

**3.5.3. The case $|\ell^+| = 1$ in the proof of Lemma 3.5.1.** In this case, we have $x^+_{i,r}(\ell^+, s^+) = x^+_{i,r} \otimes t^s$ for some $1 \leq i \leq r$ and $s > 0$. For $1 \leq j \leq r$, set

$$g_j = x^-_{1,r}(1, s^1) \cdots x^-_{j-1,r}(j-1, s^{j-1}), \quad g'_j = x^-_{j+1,r}(j+1, s^j) \cdots x^-_{r,r}(r, s^r),$$

where we understand that $g_1 = 1$ and $g'_r = 1$. Using (3.1) we get,

$$\text{pr}^- \left( x^+_{i,r} \otimes t^s, x^-_{j,r}(\ell, s) \right) = \text{pr}^- \left( x^+_{i,r} \otimes t^s, x^-_{j,r}(\ell, s) \right)$$

$$= \sum_{j=1}^{r} \text{pr}^- \left( g_j[x^+_{i,r} \otimes t^s, x^-_{j,r}(\ell, s)]g'_j \right).$$

If $i < j$, then by using Lemma 3.5.2 we see that

$$\text{pr}^- \left( g_j[x^+_{i,r} \otimes t^s, x^-_{j,r}(\ell, s)]g'_j \right) = g_j \sum_{k=1}^{\ell} \text{pr}^- \left( x^-_{j,r}(\ell-1, s^{(k)})(x^+_{i,j-1} \otimes t^{s(k)})g'_j \right).$$
where the last equality follows since \(x_{i,j-1}^+ \otimes t^{s+s_j(k)}\) belongs to \(b^+ t\) and commutes with \(g'_j\).

If \(i > j\), then by Lemma \([3.5.2]\) we get

\[
pr^-(g_j [x_{i,r}^+ \otimes t^s, x_{i,r}^-(l_j, s_j)]g'_j) = g_j \sum_{k=1}^{\ell_j} x_{i,r}^-(l_j, s_j)(x_{j,i-1}^- \otimes t^{s+s_j(k)})g'_j,
\]

where the second equality follows since \(x_{j,i-1}^- \otimes t^{s+s_j(k)}\) commutes with both \(x_{j,r}^-(l_j, s_j)\) and \(g_j\). Moreover, the right hand side of the second equation clearly lies in \(\sum_{k=1}^{\ell_j} U(n^- t)^{l_k}\) where

\[
i_k = (\ell - e_j, |s| - s_j(k)e_j) > (\ell - e_j, |s| + se_i).
\]

Finally, if \(i = j\) we have

\[
pr^-(g_i [x_{i,r}^+ \otimes t^s, x_{i,r}^-(l_i, s_i)]g'_j) = g_i \sum_{k=1}^{\ell_i} x_{i,r}^-(l_i, s_i)(h_{i,r} \otimes t^{s+s_i(k)})g'_j + g_i pr^-((x_{i,r}^+ \otimes t^s)x_{i,r}^-(l_i, s_i))g'_j
\]

It is now not hard to see that the first term is in

\[
\sum_{i < m \leq r} U(n^- t)^{\langle \ell - e_i, |s| - s_i(k)e_i + (s+s_i(k))e_m \rangle} \subset U(n^- t)\).
\]

And the second term is equal to \(g_i pr^-((x_{i,r}^+ \otimes t^s)x_{i,r}^-(l_i, s_i))g'_j\) since both \(pr^-((x_{i,r}^+ \otimes t^s)x_{i,r}^-(l_i, s_i))\) and \(g'_j\) belongs to \(U(n^- t)\).

**3.5.4. The inductive step.** For the inductive step, fix \(1 \leq i_0 \leq r\) such that \(\ell_{i_0}^+ \neq 0\). Then we can write, \(x_{i_0,r}^+(\vec{l}^+, \vec{s}^+) = (x_{i_0,r}^+ \otimes t^s) x_{i_0,r}^+(\ell', \vec{s}')\) for \(\ell' = \ell - e_{i_0}\) and some \(s \in \mathbb{N}, \vec{s}' \in \mathbb{F}^r\). Note that by Proposition \([3.1.1]\) (iii) we have,

\[
pr^-((x_{i_0,r}^+ \otimes t^s)x_{i_0,r}^+(\ell', \vec{s}')x_{i_0,r}^-(\ell, \vec{s})) = pr^-((x_{i_0,r}^+ \otimes t^s)x_{i_0,r}^-(\ell, \vec{s}))
\]

The result follows by first using the induction hypothesis for \(pr^-((x_{i_0,r}^+ \otimes t^s)x_{i_0,r}^-(\ell, \vec{s}))\) and then the result when \(|\ell^+| = 1\).

**3.6. Proof of Proposition [2.2.2.3] (ii).**

**3.6.1.** To complete the proof of the proposition, we need some more results. We begin with two lemmas related to the order.

**Lemma.** Let \(\ell, \ell' \in \mathbb{N}^r\).

(i) We have \(\eta_i(\ell) = \eta_i(\ell')\) if and only if \(\ell = \ell'\).

(ii) If \(\eta_i(\ell) - \eta_i(\ell') \in Q_+\), but \(\eta_i(\ell) \neq \eta_i(\ell')\) then \(\ell' > \ell\).

**Proof.** Part (i) follows since \(\alpha_{i,r}, 1 \leq i \leq r\) are linearly independent. For part (ii), write \(\eta_i(\ell) - \eta_i(\ell') = \sum_{j=1}^{r} m_i \alpha_{ij}\), where \(m_i = \sum_{j<i} (\ell_j - \ell'_j)\). Since \(m_i \geq 0\) for all \(1 \leq i \leq r\), we can take \(1 \leq s \leq r\) minimal such that \(m_s > 0\). This means that \(\ell' = \ell_i\) for \(i < s\) and \(\ell'_s < \ell_s\), so the lemma is proved. \(\square\)
3.6.2.

Lemma. Assume that \( g \in \mathbf{U}(\mathfrak{n}_{-1}[t])_\eta \), \( \eta \neq 0 \) and let \( (\ell, s) \in \mathbf{F}_+^r \). Then, \([x^+(\ell, s), g] \) is a linear combination of elements of the form \( g'x^+(\ell', s') \) where \( g' \in \mathbf{U}(\mathfrak{n}_{-1}[t]) \), \( (\ell', s') \in \mathbf{F}_+^r \) and \( s' \triangleright s \).

Proof. It suffices to prove the result when \( g \) is a product of terms of the form \( x_{j,k}^- \otimes t^s \), where \( 1 \leq j \leq k \leq r - 1 \) and \( s \geq 0 \). We prove this by induction on the number \( m \) of terms in the product. If \( g = x_{j,k}^- \otimes t^s \), \( 1 \leq j \leq k < r \), then \([x_{j,k}^- \otimes t^s, g] = -\delta_{ij} x_{k+1,j}^+ \otimes t^{s+s'} \). It follows from (5.1) that \([x^+(\ell, s), g] \) is a linear combination of terms of the form \( x^+(\ell', s') \) where \( (\ell', s') \in \mathbf{F}_+^r \) and

\[
|s'| = |s| - s_j(m)e_j + (s_j(m) + s)e_{k+1}, \quad 1 \leq m \leq \ell_j.
\]

Since \( s_j(m) > 0 \) for \( 1 \leq m \leq \ell_j \), we have \( s' \triangleright s \), hence induction begins.

For the inductive step, write \( g = (x_{j,k}^- \otimes t^s)g_0 \), where \( g_0 \) is a product of \( (m - 1) \)-elements of the form \( x_{j,k}^- \otimes t^s \), where \( 1 \leq j \leq k \leq r - 1 \) and \( s' \geq 0 \). We have,

\[
[x^+(\ell, s), (x_{j,k}^- \otimes t^s)g_0] = [x^+(\ell, s), (x_{j,k}^- \otimes t^s)]g_0 + (x_{j,k}^- \otimes t^s)[x^+(\ell, s), g_0].
\]

The second term has the required form by the induction hypothesis. For the first one we have (by using the induction hypothesis for \( g = x_{j,k}^- \otimes t^s \)) that

\[
[x^+(\ell, s), (x_{j,k}^- \otimes t^s)]g_0 = \sum_{\ell', s'} a_{\ell', s'} x^+(\ell', s')g_0 = \sum_{\ell', s'} a_{\ell', s'} g_0 x^+(\ell', s') + \sum_{\ell', s'} a_{\ell', s'} x^+(\ell', s'),
\]

where \( a_{\ell', s'} \in \mathbb{C} \). The result follows by using the induction hypothesis again. \( \square \)

3.6.3.

Proposition. Let \( \lambda = \sum_{i=1}^{r} m_i \omega_i \in \mathbb{P}^+ \). Fix \( 1 \leq k \leq r \) and \( n \in \mathbb{N} \) with \( n > m_k \). Let \( (\ell_i, s_i) \in \mathbf{F} \), \( 1 \leq i \neq k \leq r \), and \( (n - \ell_k, s_k) \in \mathbf{F}_+^r \). Then,

\[
\prod_{i=1}^{k-1} x_{i,r}^-(\ell_i, s_i) \prod_{i=k+1}^{r} x_{i,r}^- (\ell_i, s_i) w_{\lambda} \in W(\lambda, \mathcal{D}, \mathcal{U}^+)\]

Proof. Applying Corollary 3.4.3 for all \( 1 \leq i \leq r \) except \( i = k \), we see that it suffices to prove that elements of the form

\[
\prod_{i=1}^{n} x_{i,r}^-(n_i - \ell_i, s_i^+) (x_{i,r}^- \otimes 1)^{n_i} w_{\lambda}, \quad (n_i - \ell_i, s_i^+) \in \mathbf{F}_+^r, \quad n_i \geq \ell_i, \quad 1 \leq i \leq r,
\]

are in \( W(\lambda, \mathcal{D}, \mathcal{U}^+) \) if \( n_k > m_k \). By Lemma 3.3.1 and Lemma 3.1.1 this is equivalent to proving that

\[
w_1 = \prod_{i=1}^{n} x_{i,r}^+(n_i - \ell_i, s_i^+) \prod_{i=1}^{n} (x_{i,r}^- \otimes 1)^{n_i} w_{\lambda} \in W(\lambda, \mathcal{D}, \mathcal{U}^+)\]

Note that \( w_1 \in W_{\eta_\mathcal{D}}(\mathcal{D}) \). Now, Proposition 3.3.3 implies that \( w_1 \) is a linear combination of elements from \( W_{\eta_\mathcal{D}}(\mathcal{D}) \) of the form

\[
w_2 = \prod_{i=1}^{n} x_{i,r}^+(n_i - \ell_i, s_i^+) \prod_{i=1}^{n} (x_{i,r}^- \otimes 1)^{n_i} w_{\lambda}
\]

with \( n'_k \leq m_k < n_k \) and \( g \in \mathbf{U}(\mathfrak{n}_{-1}^- \otimes 1) \). This gives \( g \in \mathbf{U}(\mathfrak{n}_{-\nu})_\nu \), where

\[
\nu = \sum_{j=1}^{r} (n_j - n'_j) \alpha_{j,r}.
\]
Since \( n_i' < n_k \) and \( \alpha_{i,r} \), \( i = 1 \ldots r \), are linearly independent, we see that \( \nu \neq 0 \), in particular, \( g \) is not a constant. Now, Lemma 3.6.4 implies that \( w_2 \) is a linear combination of elements from \( W_{\eta_r(\ell)} \) of the form

\[
w_3 = g' \prod_{i=1}^{r} x_{i,r}^+(n_i - \ell_i, s_i') \cdot \prod_{i=1}^{r} (x_{i,r}^+ \otimes 1)^{n_i'} w_{\lambda}, \quad g' \in U(n_{-1}[\ell]),
\]

where either \( g' = g, \ell' = \ell, s' = s^+ \) or \(|s'| \parallel |s^+|\).

By Lemma 3.5.1 we have

\[
\prod_{i=1}^{r} x_{i,r}^+(n_i - \ell_i, s_i') \cdot \prod_{i=1}^{r} (x_{i,r}^+ \otimes 1)^{n_i'} \in U(n^{-}[\ell]) \geq (\ell', |s'|), \quad \text{where} \quad \ell'' = \ell' - n_i + n_i',
\]

therefore \( w_3 \in W(\lambda) \geq (\ell', |s'|) \), and hence the proposition follows if we show that \((\ell'', |s'|) > (\ell, |s^+|)\).

For this, observe that since \( w_3 \in W(\lambda) \geq (\ell', |s'|) \), we have \( g' \in U(n^{-}[\ell]) \geq (\ell', |s'|) \), where \( \nu' = \eta_r(\ell') - \eta_r(\ell'') \). Since \( \nu' \in Q^+ \), by Lemma 3.6.1 we have either \( \ell'' \triangleright \ell \) or \( \nu' = 0, \ell'' = \ell \). In the first case which includes the case \( g' = g \), we have \((\ell'', |s'|) > (\ell, |s^+|)\) by the definition of the order, and in the second case we have \( \ell'' = \ell, \) but \(|s'| \parallel |s^+|\), so \((\ell'', |s'|) > (\ell, |s^+|)\) and the proof is complete.

\subsection*{Proof of Proposition 3.2.2 (ii)}

Let \( \lambda = \sum_{i=1}^{r} m_i \omega_i \in P^+ \). Using Corollary 3.4.1 simultaneously for \( 1 \leq i \leq r \) and for \( n = m_1, \ldots, m_r \), we see that \( U(n^{-}[\ell]) \) is spanned by elements from

\[
(3.7) \quad g \mathfrak{L}_{\ell, s}, \quad g \in U(n_{-1}[\ell]), \quad |s| = d, (\ell, s) \in \mathfrak{F}^r(\lambda)
\]

together with the elements from

\[
(3.8) \quad g \prod_{i=1}^{k-1} x_{i,r}^+(\ell_i, s_i) \cdot pr^{-}((x_{i,r}^+(m - \ell_k, s)(x_{i,r} \otimes 1)^m) \prod_{i=k+1}^{r} x_{i,r}^+(\ell_i, s_i), \quad m > m_k, \quad |s| = d.
\]

where \( g \in U(n_{-1}[\ell]), (\ell_i, s_i) \in \mathfrak{F}, 1 \leq i \neq k \leq r, \) and \((m - \ell_k, s_k) \in F^+ \). Proposition 3.6.3 implies that the elements of \( (3.8) \) applied to \( w_{\lambda} \) lie in \( W(\lambda) \geq (\ell, s) \) and hence, \( W(\lambda) \geq (\ell, s) \) is spanned by the elements in \( (3.7) \) applied to \( w_{\lambda} \). In turn, this means that \( \text{Gr} W(\lambda) \) is spanned by \( g x_{\ell, s}^+ w_{\lambda}, \) with \((\ell, s) \in \mathfrak{F}^r(\lambda), g \in U(n_{-1}[\ell]), \) that is by the images of \( \psi_{\ell, s} \) for such \((\ell, s)\) and the proof is complete.

\section*{Index of Notation}

We provide for the readers convenience a brief index of the notation which is used repeatedly in this paper.

Section 1.1.1: \( \mathbb{Z}, \mathbb{N}, \mathbb{N}_+, g, h, n^+, x_{i,j}^+, H_i, h_i, \omega_i, \alpha_i, P, P^+, Q, Q^+, U(a) \).

Section 1.1.2: \( \mathbb{Z}[P], e(\mu), ch_2(V), V(\lambda), v_{\lambda} \).

Section 1.1.3: \( a[t], at[t], ch_4(M) \).

Section 1.2.1: \( W(\lambda), w_{\lambda} \).

Section 1.2.2: \( W(\lambda)_s, W(\lambda)_\mu, W(\lambda)_{\mu,s} \).

Section 2.1.1: \( F, (\ell, s), (0, \emptyset), x_{i,j}^+(\ell, s), (\ell, s), x_{i,j}^+(\ell, s), \mathcal{B}^r \).

Section 2.1.2: \( F(m), F^r(\lambda), \eta_j(\ell), \mathcal{B}^r(\lambda) \).
Section 2.2.1: $\mathfrak{g}_{r-1}, \mathfrak{n}^\pm_{r-1}, \mathfrak{u}^\pm_r$.

Section 2.2.2: $\Gamma, \blacksquare, \triangleright, \triangleright, \triangleright, |s|, |\bar{s}|, W(\lambda)^{\geq 1}, W(\lambda)^{> 1}, \text{gr}(W(\lambda)), \text{gr}^1$. 

Section 2.2.3: $\psi^L\mathfrak{a}$.

Section 3.1.1: $\mathfrak{b}^+_{r}, \mathfrak{pr}^{-}$. 

Section 3.1.2: $U(\mathfrak{n}^{-}[t])^\mu, U(\mathfrak{n}^{-}[t])^{\geq \mu}, U(\mathfrak{n}^{-}[t])^{> \mu}$.

Section 3.1.3: $U(\mathfrak{g}[t])^{\mu}, U(\mathfrak{g}[t])_{\mu,s}, U(\mathfrak{n}^\pm[t])^{\mu}, U(\mathfrak{n}^\pm[t])_{\mu,s}$.

Section 3.1.4: $F^+$.

Section 3.2.1: $e_i$.

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