PROGRESSIVE INTRINSIC ULTRACONTRACTIVITY AND HEAT KERNEL ESTIMATES FOR NON-LOCAL SCHRÖDINGER OPERATORS

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Abstract. We study the long-time asymptotic behaviour of semigroups generated by non-local Schrödinger operators of the form $H = -L + V$; the free operator $L$ is the generator of a symmetric Lévy process in $\mathbb{R}^d$, $d > 1$ (with non-degenerate jump measure) and $V$ is a sufficiently regular confining potential. We establish sharp two-sided estimates of the corresponding heat kernels for large times and identify a new general regularity property, which we call progressive intrinsic ultracontractivity, to describe the large-time evolution of the corresponding Schrödinger semigroup. We discuss various examples and applications of these estimates, for instance we characterize the heat trace and heat content. Our examples cover a wide range of processes and we have to assume only mild restrictions on the growth, resp. decay, of the potential and the jump intensity of the free process. Our approach is based on a combination of probabilistic and analytic methods; our examples include fractional and quasi-relativistic Schrödinger operators.

1. Introduction, assumptions and statement of main results

Over the past decade, there has been strong interest in non-local models involving Schrödinger operators associated with non-local generators of Lévy processes with non-degenerate jump measures. These investigations focus on estimates of the heat kernel and trace [1, 3, 6, 54], gradient estimates of harmonic functions [44], ground states, eigenfunctions and eigenvalues, and spectral bounds [28, 52, 56, 46, 53]; some of these results were, in turn, applied e.g. to quantum field theory [11, 12, 23, 24].

In the presence of so-called confining potentials, currently the only possibility to study heat kernel estimates are the concepts of (asymptotic) intrinsic hyper- and ultracontractivity (see e.g. [15, 37, 45] and the references in these papers). In order to ensure (asymptotic) intrinsic ultracontractivity, one has to make severe restrictions on the nature and the growth properties of the potential. In the present paper we obtain, for the first time, a unified approach to heat kernel estimates which work under rather general assumptions on the growth of the potential. This is possible because of a new type of “progressive” intrinsic ultracontractivity which links space and time.

Typically, the operator is of the form $H = -L + V$; throughout we assume that the potential $V$ is locally bounded and $L$ is the generator of a symmetric Lévy process. The corresponding semigroup $\{e^{tL} : t \geq 0\}$ is a semigroup of convolution operators in $L^2 = L^2(\mathbb{R}^d, dx)$ mapping $L^2$ into $L^\infty$ (if $t$ is large enough). Since $e^{tL}$ is positivity preserving this mapping property is equivalent to either of the following statements: (i) $e^{tL} : L^2 \to L^\infty$ is continuous for large $t$ (see e.g. [39, Corollary 1.3]) or (ii) $X_t$ has a bounded probability density for large $t$. The symmetry of the Lévy process is equivalent to the symmetry of the semigroup operators $e^{-tL}$, $t \geq 0$, and the self-adjointness of the generator $L$, $L$, $\{e^{tL} : t \geq 0\}$ and the corresponding Lévy process are called free generator, semigroup and process. The Schrödinger operator $H$ generates a semigroup of symmetric operators $\{e^{-tH} : t \geq 0\}$ on $L^2$ such that $e^{-tH} : L^2 \to L^\infty$ are bounded for large values of $t$. If the free Lévy process has a sufficiently regular transition density, then $U_t$ is an
integral operator with kernel \( u_t(x,y) \), i.e. \( U_t f(x) = \int u_t(x,y) f(y) \, dy \). [19] Chapter 2.B]. For a
confining potential \( V \), i.e. \( \lim_{|x| \to \infty} V(x) = \infty \), each \( U_t = e^{-tH}, \, t > 0 \), is a compact operator
in \( L^2 \) — see e.g. [33] Lemmas 1 and 9] for a general argument — and the spectra of \( H \) and \( U_t \)
are purely discrete. We denote by \( \lambda_0 = \inf \text{spec}(H) \in \mathbb{R} \) the ground state eigenvalue and by
\( \varphi_0 \in \mathbb{L}^2 \) the corresponding ground state eigenfunction. In particular, it makes sense to study
the spectral regularity — the heat trace or the heat content, and the Hilbert-Schmidt property
— and large-time smoothness properties of the semigroup \( \{ U_t : t \geq 0 \} \) such as intrinsic hyper-
and ultracontractivity [17]. Related to that, it is also a natural question to ask for the behaviour of
\( U_t f \) and \( u_t(x,y) \) as \( t \to \infty \).

In the literature [2, 18, 34] the asymptotic intrinsic ultracontractivity condition (a)IUC is
used to describe the large time behaviour of \( u_t(x,y) \). These are conditions on \( U_t \) which can be
in the equivalently stated form
\[(IUC) \quad \forall t_0 > 0 \exists C = C(t_0) \geq 1 \forall t \geq t_0 \forall x,y \in \mathbb{R}^d : u_t(x,y) \leq C e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y),
\]
\[(aIUC) \quad \exists t_0 > 0 \exists C = C(t_0) \geq 1 \forall t \geq t_0 \forall x,y \in \mathbb{R}^d : u_t(x,y) \leq C e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y).
\]
(\( \sim \)) denotes a two-sided comparison with the constants \( 0 < C^{-1} \leq 1 < C < \infty \).

In the present paper we are mainly interested in the case where \( U_t = e^{-tH} \) fails to be (asymptotically)
IUC and we want to study the asymptotics of \( u_t(x,y) \) as \( t \to \infty \) in the general case. Our main result are sharp two-sided large-time estimates for the kernel \( u_t(x,y) \). Let us first state the result and then discuss the assumptions [A1]–[A3] needed therein.

**Theorem 1.1.** Let \( L \) be the generator of a symmetric Lévy process with Lévy measure \( \nu(dx) = \nu(x) \, dx \) and diffusion matrix \( A = (a_{ij})_{i,j=1,\ldots,n} \), and let \( V \) be a confining potential. Denote by
\( H = -L + V \) the Schrödinger operator and assume [A1]–[A3] with \( t_b > 0, R_0 > 0 \) and the
profile functions \( f(|x|) \) and \( g(|x|) \) which control \( \nu(x) \) and \( V(x) \), respectively. Write \( \lambda_0 \) and \( \varphi_0 \)
for the ground-state eigenvalue and eigenfunction, and \( u_t(x,y) \) for the density of the operator \( U_t = e^{-tH} \). There exist constants \( C \geq 1 \) and \( R > R_0 \) such that for every \( t > 30t_b \) the following assertions hold.

a) If \(|x|,|y| \leq R\), then
\[
\frac{1}{C} e^{-\lambda_0 t} \leq u_t(x,y) \leq Ce^{-\lambda_0 t}.
\]

b) If \(|x| > R \) and \(|y| \leq R\), then
\[
\frac{1}{C} e^{-\lambda_0 t} \frac{\nu(x)}{V(x)} \leq u_t(x,y) \leq Ce^{-\lambda_0 t} \frac{\nu(x)}{V(x)},
\]
by symmetry, if \(|x| \leq R \) and \(|y| > R\), then
\[
\frac{1}{C} e^{-\lambda_0 t} \frac{\nu(y)}{V(y)} \leq u_t(x,y) \leq Ce^{-\lambda_0 t} \frac{\nu(y)}{V(y)}.
\]

c) If \(|x|,|y| > R\), then
\[
\frac{1}{C} F(Kt,x,y) \vee e^{-\lambda_0 t} \nu(x) \nu(y) \leq u_t(x,y) \leq C \frac{F(Kt,x,y) \vee e^{-\lambda_0 t} \nu(x) \nu(y)}{V(x)V(y)},
\]
where \( K = 4C_0 C_2^2 \) — the constants \( C_0, C_2 \) are from [A3] — and
\[
F(\tau, x, y) := \int_{R-1<|z|<|x||y|} (f(|x-z|) \wedge 1) (f(|y-z|) \wedge 1) e^{-\tau g(|z|)} \, dz.
\]

Under the additional condition \( \inf_{x \in \mathbb{R}^d} V(x) > 0 \), the cases [a] and [b] can be combined in a single estimate: if \(|x| \leq R \) or \(|y| \leq R\), then
\[
u(x,y) \leq e^{-\lambda_0 t} \left( 1 \wedge \frac{\nu(x)}{V(x)} \right) \left( 1 \wedge \frac{\nu(y)}{V(y)} \right),
\]
Let us now discuss the assumptions and the set-up of Theorem 1.1. Recall that a Lévy process on \( \mathbb{R}^d \) is a stochastic process \( (X_t)_{t \geq 0} \) with values in \( \mathbb{R}^d \), independent and stationary increments, and càdlàg (right-continuous, finite left limits) paths. It is well-known, cf. [27, 29] or [9], that a Lévy process is a Markov process whose transition semigroup is a semigroup of convolution operators

\[
P_t u(x) = E u(X_t + x) = u * \tilde{\mu}_t(x), \quad \tilde{\mu}_t(dy) = \mathbb{P}(-X_t \in dy)
\]

which is a strongly continuous contraction semigroup on \( L^2 = L^2(\mathbb{R}^d, dx) \). Using the Fourier transform we can describe \( P_t \) as a Fourier multiplication operator

\[
P_t u(x) = \mathcal{F}^{-1} \left( e^{-t\psi} \mathcal{F} u \right)(x)
\]

with symbol (multiplier) \( e^{-t\psi(\xi)} \). The semigroup \( \{ P_t : t \geq 0 \} \) is symmetric in \( L^2 \) if, and only if, \( X_t \) is a symmetric Lévy process (i.e. \( \mathbb{P}(X_t \in dy) = \mathbb{P}(-X_t \in dy), t \geq 0 \)) which is equivalent to \( e^{-t\psi} \) or \( \psi \) being real. All real characteristic exponents are given by the Lévy–Khintchine formula

\[
\psi(\xi) = \xi \cdot A \xi + \int_{\mathbb{R}^d \setminus \{0\}} \left( 1 - \cos(\xi \cdot z) \right) \nu(dz), \quad \xi \in \mathbb{R}^d.
\]

where \( A = (a_{ij})_{1 \leq i, j \leq d} \) is a symmetric non-negative definite matrix, and \( \nu \) is a symmetric Lévy measure, i.e. a Radon measure on \( \mathbb{R}^d \setminus \{0\} \) satisfying \( \nu(E) = \nu(-E) \) and \( \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(dz) < \infty \). The matrix \( A \) describes the diffusion part of \( (X_t)_{t \geq 0} \) while \( \nu \) is the jump measure. Throughout this paper we assume that the jump activity is infinite and \( \nu \) is absolutely continuous with respect to Lebesgue measure, i.e.

\[
\nu(\mathbb{R}^d \setminus \{0\}) = \infty \quad \text{and} \quad \nu(dx) = \nu(x) \, dx.
\]

The generator \( L \) is a non-local self-adjoint pseudo-differential operator given by

\[
\mathcal{F}[Lu](\xi) = -\psi(\xi) \mathcal{F} u(\xi), \quad \xi \in \mathbb{R}^d, \quad u \in \mathcal{D}(L) := \left\{ u \in L^2(\mathbb{R}^d) : \psi \mathcal{F} v \in L^2(\mathbb{R}^d) \right\},
\]

Prominent examples of non-local operators (and related jump processes) are fractional Laplacians \( L = -(-\Delta)^{\alpha/2} \), \( \alpha \in (0, 2) \) (isotropic \( \alpha \)-stable processes) and quasi-relativistic operators \( L = -(-\Delta + m^2/\alpha)^{\alpha/2} + m \), \( \alpha \in (0, 2), m > 0 \) (isotropic relativistic \( \alpha \)-stable processes) which play an important role in mathematical physics. These and further examples are discussed in Section 5.

Under (1.2) the process \( (X_t)_{t \geq 0} \) is a strong Feller process, i.e. \( P_t \) maps bounded measurable functions into continuous functions; equivalently, this means that its one-dimensional distributions are absolutely continuous with respect to Lebesgue measure, i.e. there exists a transition density \( p_t(x, y) = p_t(y-x) \) such that \( P^0(\mathbb{R}^d \setminus \{0\}) = \int_E p_t(x) \, dx \) for every Borel set \( E \subset \mathbb{R}^d \), see e.g. [40, Th. 27.7]. Further details on the existence and regularity of transition densities can be found in [43].

We need the following additional regularity assumptions \( (A1)-(A2) \) for the density \( \nu(x) \) of the Lévy measure and the transition density \( p_t(x) \).

**Lévy density.** There exists a profile function \( f : (0, \infty) \to (0, \infty) \) such that

a) there is a constant \( C_1 \geq 1 \) such that \( C_1^{-1} f(|x|) \leq \nu(x) \leq C_1 f(|x|) \) for all \( x \in \mathbb{R}^d \setminus \{0\} \);

b) \( f \) is decreasing and \( \lim_{r \to \infty} f(r) = 0 \);

c) there is a constant \( C_2 \geq 1 \) such that \( f(r) \leq C_2 f(r+1) \) for all \( r \geq 1 \);

d) \( f \) has the direct jump property: there exists a constant \( C_3 > 0 \) such that

\[
\int_{|x-y| > 1} f(|x-y|) f(|y|) \, dy \leq C_3 f(|x|), \quad |x| \geq 1.
\]

Some parts of (A1) are redundant but we prefer to keep it that way for clarity and reference purposes. For instance, under (A1b) the condition (A1d) implies (A1c). Similarly, in (A1b), \( \lim_{r \to \infty} f(r) = 0 \) readily follows from the monotonicity of \( f \) and (A1a).
The convolution property \((A1.d)\) is fundamental for our investigations. It has a very suggestive probabilistic interpretation: the probability to move from 0 to \(x\) in “two large jumps in a row” is smaller than with a “single direct jump”. For this reason we call this condition the direct jump property.

\((A2)\) Transition density of the free process. The function \((t,x) \mapsto p_t(x)\) is continuous on \((0,\infty) \times \mathbb{R}^d\) and there exists some \(t_0 > 0\) such that the following conditions hold.

a) There are constants \(C_4, C_5 > 0\) such that
\[
p_t(x) \leq C_4 \left(e^{C_5 t f(|x|)} \wedge 1\right), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t \geq t_0;
\]
b) For every \(r \in (0,1]\) we have
\[
\sup_{t \in (0,t_0]} \sup_{r \leq |x| \leq 2} p_t(x) < \infty.
\]
An easy-to-check sufficient condition for the time-space continuity of the density \(p_t(x)\) is
\[
e^{-t \psi(t)} \in L^1(\mathbb{R}^d, d\xi)
\]
for all \(t > 0\), see Lemma 2.1 in Section 2 and the discussion following that lemma. Notice that this condition trivially holds as soon as \(\psi\) has a nondegenerate Gaussian part, i.e. \(\det A \neq 0\) in (1.1). The other assumptions in (A2) govern the asymptotic behaviour of the transition density \(p_t(x)\) for the free operator \(L\) and they should be seen as the minimal regularity requirement for the density of the free semigroup. The upper bound on \(p_t(x)\) in (A2.a) is known for a wide range of operators \(L\) whose Lévy measures satisfy (A1), cf. [22, 38, 39, 40, 41]. Similarly, the condition (A2.b) is a small time off-diagonal boundedness property which holds for a large class of semigroups, see e.g. [22, Th. 5.6 and Rem. 5.7]. Under (A2.a) we know that \(\sup_{x \in \mathbb{R}^d} p_t(x) = p_{t_0}(0) < \infty\) — this extends to all \(t \geq t_0\) — and the function \(p_t(\cdot)\) is smooth for all \(t > t_0\); this is a consequence of the fact that \(p_t\) is the convolution of \(p_{t-t_0}\) in \(L^1(\mathbb{R}^d)\) and \(p_{t_0} \in L^\infty(\mathbb{R}^d)\).

We will now introduce the class of potentials which we consider in this paper.

\((A3)\) Confining potential. Let \(V \in L^\infty_{\text{loc}}(\mathbb{R}^d)\) be such that \(\lim_{|x| \to \infty} V(x) = \infty\) and assume that there exist constants \(C_6 \geq 1\) and \(R_0 > 0\), and a profile function \(g : [0,\infty) \to (0,\infty)\) such that

a) \(g|_{[0, R_0]} \equiv 1\) and \(C_6^{-1} g(|x|) \leq V(x) \leq C_6 g(|x|), \ |x| \geq R_0;\)
b) \(g\) is increasing on \([R_0, \infty)\);
c) there exists a constant \(C_7 \geq 1\) such that \(g(r + 1) \leq C_7 g(r), \ r \geq R_0.\)

The uniform growth condition (A3.c) excludes profiles growing like \(\exp(r^2)\) or \(\exp(r^\beta)\), but exponentially and slower growing potentials — for example growth orders \(\log \log r, \log(r^\beta), \ r^\beta\) and \(e^\beta r\), with \(\beta > 0\) — are admissible.

Under (A1)–(A3) \(H = -L + V\) is well defined, bounded below, and self-adjoint in \(L^2 = L^2(\mathbb{R}^d, dx)\). Our standard reference for Schrödinger operators is the monograph [19] by Demuth and van Casteren. The corresponding Schrödinger semigroup \(\{e^{-tH} : t \geq 0\}\) has the following probabilistic Feynman–Kac representation
\[
e^{-tH} f(x) = U_t f(x) := \mathbb{E}^x \left[e^{-\int_0^t V(X_s) \, ds} f(X_t)\right], \quad f \in L^2(\mathbb{R}^d), \ t > 0,
\]
which allows us to use methods from probability theory. Under (A3) the semigroup operators \(U_t, \ t > 0,\) are compact and the spectrum of \(H\) consists of eigenvalues of finite multiplicity without accumulation points. The ground state eigenvalue \(\lambda_0 := \inf \text{spec}(H)\) is simple and the corresponding — unique (when normalized) and positive — \(L^2\)-eigenfunction is denoted by \(\varphi_0\). The operators \(U_t\) have integral kernels, i.e.
\[
U_t f(x) = \int_{\mathbb{R}^d} u_t(x, y) f(y) \, dy, \quad f \in L^2(\mathbb{R}^d), \ t > 0,
\]
and the kernels \(u_t(\cdot, \cdot), \ t > 0,\) are continuous, positive and symmetric functions on \(\mathbb{R}^d \times \mathbb{R}^d\). We call the kernel \(u_t(x, y)\) the Schrödinger heat kernel. Because of (A2.a), \(u_t(x, y)\) is a bounded

\(^1\)Previously [36, 37] this condition has also been called jump paring condition but we prefer the present name as it captures the probabilistic meaning in a more concise way.
function for all $t \geq t_0$. Further properties of $u_t(x, y)$ will be discussed in detail in the next section.

The conditions (A1)–(A3) are needed to prove Theorem 1.1. If we make a further structural assumption on the profile function $g$, we can improve the results of Theorem 1.1 splitting the estimates in two distinct scenarios: the aIUC regime (including the IUC regime) and the non-aIUC regime, see Remark 5.1.

(A4) $V$ is a potential satisfying (A3) with the profile $g$ and $R_0 > 0$ such that $f(R_0) < 1$ and (1.4) $g(r) = h(|\log f(r)|)$, $r \geq R_0$.

for some increasing function $h : \| \log f(R_0) \|, \infty \rightarrow (0, \infty)$ such that $h(s)/s$ is monotone. Examples of such profiles $h$ can be found among functions which are regularly varying at infinity, see [4]. Let us remark that (A4) is, when we compare it with existing results on the asymptotic behaviour of aIUC Schrödinger semigroups [35, 37], a very natural condition.

The large time estimates of the heat kernel $u_t(x, y)$ in the the aIUC vs. the non-aIUC regime are substantially different. This is due to the intricate asymptotic behaviour of the function $F(\tau, x, y)$. In the non-aIUC regime the following result holds true (Corollary 5.4): For every confining potential – no matter how slowly $V$ grows at infinity – there is an increasing function $\rho : (0, \infty) \rightarrow (0, \infty)$ such that $\lim_{t \rightarrow \infty} \rho(t) = \infty$ (cf. Lemma 5.2) and such that the following estimate holds: There is a constant $C \geq 1$ such that for sufficiently large values of $t$ we have

$$u_t(x, y) \leq e^{-\lambda_0 t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \left( 1 \wedge \frac{f(|y|)}{g(|y|)} \right), \quad |x| \wedge |y| < \rho(t).$$

These estimates are equivalent to saying that there is a constant $\tilde{C} \geq 1$ such that for sufficiently large values of $t$ we have

$$u_t(x, y) \leq e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad |x| \wedge |y| < \rho(t).$$

The estimates for $u_t(x, y)$ are essentially different if $|x|, |y| > r(t)$. This means that the regularity of a non-aIUC Schrödinger semigroup improves as soon as the time parameter $t$ increases; note that the constants $C, \tilde{C}$ do not depend on $t$. We believe that this surprising property has not been observed before. In analogy to the asymptotic IUC property, we propose to call this property progressive intrinsic ultracontractivity, pIUC, for short. It seems that pIUC is a regularity property for compact semigroup, in general, and merits a formal definition.

**Definition 1.2.** Let $\{U_t : t \geq 0\}$ be a semigroup of compact operators on $L^2(\mathbb{R}^d)$ with integral kernels $u_t(x, y)$, ground state eigenvalue $\lambda_0$ and ground state eigenfunction $\varphi_0$. The semigroup $\{U_t : t \geq 0\}$ is said to be

- **a) intrinsically ultracontractive** (IUC) if for every $t > 0$ there exists a constant $C > 0$ such that

$$u_t(x, y) \leq e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$

- **b) asymptotically intrinsically ultracontractive** (aIUC) if there exist some $t_0 > 0$ and a constant $C > 0$ such that

$$u_t(x, y) \leq e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad x, y \in \mathbb{R}^d, \quad t \geq t_0.$$

- **c) progressively intrinsically ultracontractive** (pIUC) if there exist some $t_0 > 0$, an increasing function $\rho : [t_0, \infty) \rightarrow (0, \infty]$ such that $\rho(t) \rightarrow \infty$, and a constant $C > 0$ such that

$$u_t(x, y) \leq e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad |x| \wedge |y| < \rho(t), \quad t \geq t_0.$$

Note that IUC always implies aIUC, and aIUC always implies pIUC (with threshold function $\rho \equiv \infty$).

Our paper is organized in the following way. Section 2 contains some basic probabilistic potential theory which is needed in the subsequent sections. In 3.1 we discuss the properties of the profile function $f$ from (A1), in particular we provide some sufficient conditions such that the direct jump property (A1.d) holds. Section 3.2 is about the basic decomposition of
the trajectories of the free process: we use this to obtain estimates of the localized (in space) Feynman–Kac representation. These bounds are essential for the upper and lower estimates in Sections 4.1 and 4.2. They are combined to give sharp two-sided estimates for \( u_t(x,y) \), see Section 4.3. Theorem 4.6 is an extended version of our main Theorem 1.1. Based on these estimates we discuss several applications: we study the decay properties of the functions \( U_t \mathbb{I}_{\mathbb{R}^d}(x) \) (Section 4.4), we show that the present estimates are potent enough to recover known results on aIUC (Section 4.5), and finally we show (Section 4.6) that — for \( t \gg 1 \) — the notions of “operator with finite heat content”, “trace-class operator” and “Hilbert–Schmidt operator” coincide in this setting, and they are equivalent to the condition that \( \int_{|x|>R} e^{-tV(x)} \, dx < \infty \), for some \( R, t > 0 \). Our second main result is presented as Corollary 4.4 in Section 5.2. This is about improved heat-kernel estimates in the pIUC regime. This is only possible if we know about the dependence of the growth at infinity of \( f \) (resp. the jump density \( \nu \) and \( g \) (resp. the potential \( V \)). Here we need the additional assumption (A4), see Section 5.1. The last two sections contain examples: Section 5.4 is about doubling Lévy measures with \( f \) of polynomial type \( f(r) = r^{d-\alpha}(\nu \vee r)^{-\gamma} \) while Section 5.5 considers exponentially decaying Lévy measures with \( f \) of the form \( f(r) = r^{-\gamma}e^{-\varepsilon r} \), \( r \gg 1 \).

**Notation.** Throughout the paper lower case letters \( c, c_1, c_2, \ldots \) denote generic constants; within a proof we indicate changes to constants by increasing their running index. Upper case letters \( C, C_1, \ldots, C_7 \) (they appear in the assumptions (A1–A3)) and \( C_{n,m} \) (\( n,m \) refers to the Theorem, Lemma etc. where \( C_{n,m} \) appears for the first time) denote important constants. This is for cross-referencing and to help keeping track of the dependence of constants in our calculations. The constant \( \tilde{C} \) is from (3.1) on page 10; it serves as an alternative of the constant \( C_2 \) in (A1) and it determines the growth of the functions of class \( \mathcal{C} \) defined in (3.7).

Our basic assumptions (A1), (A2), (A3) and (A4) can be found on pages 3–5. The constant \( t_0 \) is from (A2) and \( R_0 \) is from (A3).

Two-sided estimates between functions are sometimes indicated by

\[
 f(x) \asymp g(x), \quad x \in A \iff \exists C > 0 \forall x \in A : C^{-1}f(x) \leq g(x) \leq Cf(x).
\]

The notation \( f(x) \asymp g(x) \) is used to highlight the comparison constant \( C \).

By \( \lambda_0 \) and \( \varphi_0 \) we denote the ground-state eigenvalue and eigenfunction, see page 5. The Fourier transform and its inverse are defined as

\[
 \mathcal{F}u(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi x} u(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}v(\xi) = \int_{\mathbb{R}^d} e^{i\xi x} v(\xi) \, d\xi.
\]

As usual, we write \( \mathbb{E}(Y; A, B) := \int_{A \cap B} Y \, d\mathbb{P} \); \( |B| \) denotes the Lebesgue measure of a Borel set \( B \subset \mathbb{R}^d \), \( a \land b \) and \( a \lor b \) are the minimum and maximum of \( a \) and \( b \).

2. Preliminaries

We begin with a sufficient condition for the joint continuity of the functions \( p_t(x) \) on the sets \( (t_0, \infty) \times \mathbb{R}^d \), \( t_0 > 0 \).

**Lemma 2.1.** Let \( (X_t)_{t \geq 0} \) be a Lévy process with values in \( \mathbb{R}^d \) and characteristic exponent \( \psi : \mathbb{R}^d \to \mathbb{C} \). If there exists some \( t_0 > 0 \) such that \( e^{-t\psi(\xi)} \in L^1(\mathbb{R}^d, d\xi) \) for all \( t \geq t_0 \), then \( X_t \) admits for \( t \geq t_0 \) a probability density \( p_t(x) \) such that \( (t, x) \mapsto p_t(x) \) is continuous for all \( (t, x) \in (t_0, \infty) \times \mathbb{R}^d \).

**Proof.** Fix \( \epsilon > 0 \). Since \( \xi \mapsto e^{-t_0\psi(\xi)} \) is integrable and \( |e^{-t_0\psi(\xi)}| = e^{-t_0\Re\psi} \), we can use the dominated convergence theorem to ensure that for some \( \delta > 0 \)

\[
2(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t_0\Re\psi(\xi)} |\sin \frac{1}{2} \xi \cdot (x-y)| \, d\xi < \epsilon \quad \text{for all} \quad x, y \in \mathbb{R}^d, \quad |x-y| < \delta,
\]

\[
(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t_0\Re\psi(\xi)} \left| e^{-(t-t_0)\Re\psi(\xi)} - e^{-(u-t_0)\Re\psi(\xi)} \right| \, d\xi < \epsilon \quad \text{for all} \quad t, u > 0, \quad |t-u| < \delta.
\]
Using Fourier inversion we get
\[ p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\psi(\xi)} e^{-ix\cdot\xi} \, d\xi, \quad t \geq t_0, \quad x \in \mathbb{R}^d, \]
which shows the existence of a transition density. For \( x, y \in \mathbb{R}^d \) and \( t > t_0 \) we get, on the one hand
\[ |p_t(x) - p_t(y)| \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Re\psi(\xi)} \left| e^{-ix\cdot\xi} - e^{-iy\cdot\xi} \right| \, d\xi \]
\[ = 2(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Re\psi(\xi)} \left| \sin \frac{1}{2} (x - y) \cdot \xi \right| \, d\xi \]
\[ \leq 2(2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t_0\Re\psi(\xi)} \left| \sin \frac{1}{2} (x - y) \cdot \xi \right| \, d\xi; \]
on the other hand, we have for \( y \in \mathbb{R}^d \) and \( t, u > t_0 \)
\[ |p_t(y) - p_u(y)| \leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t_0\Re\psi(\xi)} \left| e^{-(t-t_0)\Re\psi(\xi)} - e^{-(u-t_0)\Re\psi(\xi)} \right| \, d\xi. \]
If \( x, y \in \mathbb{R}^d \) and \( t, u > t_0 \) satisfy \( |x - y| + |t - u| < \delta \), we finally get
\[ |p_t(x) - p_u(y)| \leq \sup_{t > t_0} \left| p_t(x) - p_u(y) \right| + \sup_{y \in \mathbb{R}^d} \left| p_u(y) - p_u(y) \right| < 2\epsilon, \]
proving joint continuity of \((t, x) \mapsto p_t(x)\) on \((t_0, \infty) \times \mathbb{R}^d\). \( \square \)

**Remark 2.2.** The assumption \( e^{-t\psi(\xi)} \in L^1(d\xi) \) for \( t \geq t_0 \) already appears in [43]. A sufficient condition for this assumption is the Hartman-Wintner condition
\[ (HW_{1/t_0}) \quad \liminf_{|\xi| \to \infty} \frac{\Re\psi(\xi)}{\log(1 + |\xi|)} > \frac{d}{t_0} \]
which stipulates that \( \psi \) grows at infinity at least logarithmically.

Note that the (one-sided) one-dimensional Gamma process has the exponent \( \psi(\xi) = \frac{1}{2} \log(1 + \xi^2) \) and the transition density \( p_t(x) = \Gamma(t)^{-1} x^{t-1} e^{-x^2} \), \( x > 0 \). Clearly, \( p_t(x) \) fails to be continuous on \((0, 1) \times \mathbb{R}\), i.e. logarithmic growth of \( \psi \) seems to be a rather sharp condition for the joint continuity of \((t, x) \mapsto p_t(x)\). Using this, one can also give an example of a symmetric Lévy process on \( \mathbb{R} \) satisfying our basic assumption \([A1]\) but with a density which fails to be time-space continuous for small values of \( t \). A similar picture is true for the one-dimensional symmetric Gamma process, whose transition density is given by \( q_t(x) = \int_0^\infty \frac{1}{\sqrt{4\pi s}} e^{-|x|^2/(4s)} \, p_t(s) \, ds \). For details and further references see [43, Example 2.3].

If we assume \( HW_{\infty} \), i.e. \( t_0 = 0 \), the density \( p_t(x) \) is already smooth in the variable \( x \), cf. [43, Theorem 2.1], indicating that a Hartman–Wintner condition cannot be optimal.

We now collect a few basic properties of the Schrödinger semigroup \( \{U_t : t \geq 0\} \) and facts from potential theory for the free process \( \{X_t\}_{t \geq 0} \). We begin with some fundamental properties of the Schrödinger heat kernel \( u_t(x, y) \) which will be needed in the sequel.

**Lemma 2.3.** Let \( H = -L + V \) be the Schrödinger operator with confining potential \( V \) such that \([A1] - [A3]\) hold. Denote by \( u_t(x, y) \) the density of the operator \( U_t = e^{-tH} \).

a) For every \( x, y \in \mathbb{R}^d \) and \( t > 0 \) we have
\[ u_t(x, y) = \lim_{s \uparrow t} \mathbb{E}^x \left[ e^{-\int_0^t V(X_r) \, dr} \, p_{t-s}(y - X_s) \right]. \]

b) For fixed \( t > 0 \), \( u_t(\cdot, \cdot) \) is a continuous and symmetric function on \( \mathbb{R}^d \times \mathbb{R}^d \).

c) For every \( x, y \in \mathbb{R}^d \) and \( t > 0 \) we have
\[ 0 < u_t(x, y) \leq e^{C_{\infty}} p_t(y - x), \]
where \( C_{\infty} := -\inf_{x \in \mathbb{R}^d} V(x) \wedge 0 \); in particular, all semigroup operators \( U_t, t > 0 \), are positivity improving.
d) For every \( t \geq t_0 \) we have \( \sup_{x,y \in \mathbb{R}^d} u_t(x,y) < \infty \). In particular, \( U_t : L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) is a bounded operator for all \( t \geq t_0 \), that the semigroup \( \{U_t : t \geq 0\} \) is ultracontractive for \( t \geq t_0 \).

Lemma 2.3 is a standard result and we refer for its proof and further details on Feynman–Kac semigroups to the monographs \cite{19, 16}. It shows that the kernel \( u_t(x,y) \) inherits its basic regularity properties from the transition densities \( p_t(y-x) \) of the free Lévy process.

Assumption (A3) guarantees that all semigroup operators \( U_t, t > 0 \), are compact operators on \( L^2(\mathbb{R}^d) \). In particular, there is a ground state, i.e. \( \lambda_0 := \inf \text{spec}(H) \) is an eigenvalue with multiplicity one and there exists a unique eigenfunction \( H\phi_0 = \lambda_0\phi_0 \) — hence \( U_t\phi_0 = e^{-\lambda_0 t}\phi_0 \), \( t > 0 \) — where \( \phi_0 \in L^2(\mathbb{R}^d) \) and ||\( \phi_0 ||_2 = 1; \) Lemma 2.3(d) ensures that \( \phi_0 \in L^\infty(\mathbb{R}^d) \). Moreover, it is known that \( \{U_t : t \geq 0\} \) has the strong Feller property, i.e. \( U_t(L^\infty(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d) \) for \( t > 0 \), which implies that \( \phi_0 \) has a version in \( C_b(\mathbb{R}^d) \) and \( U_t\phi_0(x) = e^{-\lambda_0 t}\phi_0(x) \) has a pointwise meaning. By Lemma 2.3(d), we even have \( \phi_0(x) > 0 \) for all \( x \in \mathbb{R}^d \).

We denote by \( p_D(t,x,y) \) the transition density of the free process \( (X_t)_{t \geq 0} \) killed upon exiting a bounded, open set \( D \subseteq \mathbb{R}^d \); it is given by the Dynkin-Hunt formula

\[
(2.2) \quad p_D(t,x,y) = p_t(y-x) - E^x [p_{t-\tau_D}(y-X_{\tau_D}); \, \tau_D < t], \quad x, y \in D,
\]

where \( \tau_D = \inf \{ t \geq 0 : X_t \notin D \} \) is the first exit time of the process \( X \) from the set \( D \); as usual, we set \( p_D(t,x,y) = 0 \) if \( x \notin D \) or \( y \notin D \). Hence,

\[
(2.3) \quad E^x [f(X_t); \, t < \tau_D] = \int_D f(y)p_D(t,x,y) \, dy, \quad x \in D, \ t > 0,
\]

for every bounded or nonnegative Borel function \( f \) on \( D \). The Green function of the process \( X \) in \( D \) is given by \( G_D(x,y) = \int_0^\infty p_D(t,x,y) \, dt \). If \( D = B_r(0), r > 0 \), then we denote by \( \mu_0(r) \) and \( \psi_{0,r} \in L^2(B_r(0)) \) the ground state eigenvalue and eigenfunction of the process killed upon leaving \( B_r(0) \). It is known that \( \mu_0(r) > 0 \) is the smallest positive number and \( \psi_{0,r} \) is the unique \( L^2 \)-function with \( ||\psi_{0,r}||_2 = 1 \) such that

\[
\int_{B_r(0)} p_{B_r(0)}(t,x,y)\psi_{0,r}(y) \, dy = e^{-\mu_0(r)t}\psi_{0,r}(x), \quad t > 0, \ x \in B_r(0).
\]

This equality entails that \( \psi_{0,r} \) is bounded on \( B_r(0) \), continuous around \( 0 \) (see e.g. \cite{52} proof of Th. 3.4: Claim 1) and \( \psi_{0,r}(0) > 0 \).

The kernel \( \nu(z-X_{s-}(\omega)) \, dz \, ds \) is the Lévy system for \( (X_t)_{t \geq 0} \); it is uniquely characterized by the identity

\[
(2.4) \quad E^x \sum_{s \in (0,t] \atop |\Delta X_s| \neq 0} f(s,X_{s-},X_s) = E^x \int_0^t \int_{\mathbb{R}^d} f(s,X_{s-},z)\nu(z-X_{s-}) \, dz \, ds,
\]

where \( f : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty) \). If we use \( f(s,y,z) = 1_I(s)1_E(y)1_F(z) \), where \( I \) is a bounded interval, and \( E \subseteq D \), \( F \subseteq D^c \) are Borel subsets of \( \mathbb{R}^d \) with \( \text{dist}(E,F) > 0 \), the functional

\[
M_t := \sum_{s \in (0,t]} f(s,X_{s-},X_s) - \int_0^t \int_{\mathbb{R}^d} f(s,X_{s-},z)\nu(z-X_{s-}) \, dz \, ds
\]

is a uniformly integrable martingale, see e.g. \cite{31} Chapter II.1d, II.2a, II.4c. By a stopping argument we get

\[
(2.5) \quad P^x(\tau_D \in dt, X_{\tau_D-} \in dy, X_{\tau_D} \in dz) = p_D(t,x,y) \, dt \, 1_{\{|z-y| > 0\}}(y,z) \, \nu(z-y) \, dy \, dz,
\]
on $(0, \infty) \times D \times (D)^c$. This is usually called the Ikeda-Watanabe formula, see [25, Th. 1] for the original version. We will use this formula mainly in the following setting: for every $\theta > 0$ and every bounded or non-negative Borel function $h$ on $\mathbb{R}^d$ such that $\text{dist(\text{supp} \, h, D)} > 0$, one has

\begin{equation}
\mathbb{E}^x \left[ e^{-\theta \tau_D} h(X_{\tau_D}) \right] = \int_D \int_0^\infty e^{-\theta t} p_D(t, x, y) \, dt \int_{D^c} h(z) \nu(z - y) \, dz \, dy, \quad x \in D.
\end{equation}

We will also need the concept of $(X, \theta)$-harmonic functions, $\theta > 0$. A Borel function $f$ on $\mathbb{R}^d$ is called $(X, \theta)$-harmonic in an open set $D \subset \mathbb{R}^d$ if

\begin{equation}
f(x) = \mathbb{E}^x \left[ e^{-\theta \tau_D} f(X_{\tau_D}); \tau_U < \infty \right], \quad x \in U,
\end{equation}

for every open (possibly unbounded) set $U$ such that $\overline{U} \subset D$; $f$ is called regular $(X, \theta)$-harmonic in $D$, if (2.7) holds for $U = D$. We will always assume that the expected value in (2.7) is absolutely convergent. By the strong Markov property every regular $(X, \theta)$-harmonic function in $D$ is $(X, \theta)$-harmonic in $D$.

The next lemma provides a uniform estimate for $(X, \theta)$-harmonic functions which is often called the boundary Harnack inequality, see [8, Lem. 3.2(b) and Th. 3.5].

**Lemma 2.4.** Assume [A1(a,b,c)] and [A2]. There exists a universal constant $C > 0$ such that

\[ h(y) \leq \frac{C}{\theta} \int_{B^c_{1/4}(x)} h(z) \nu(x - z) \, dz, \quad y \in D \cap B_{1/4}(x) \]

holds for all $x \in \mathbb{R}^d$, all open sets $D \subset B_1(x)$ and any nonnegative, regular $(X, \theta)$-harmonic function $h$ in $D$ such that $h$ vanishes in $B_1(x) \setminus D$.

**Proof.** This result follows from a combination of Lemma 3.2, Theorem 3.5 and the discussion in Example 3.9 in [8]. We only need to check the assumptions (A)–(D) in that paper. Since $X$ is a symmetric Lévy process, (A)–(C) always hold. In order to justify (D), let $B = B_R(0)$, $0 < r < R \leq 1$ and $x, y \in B$ be such that $|x - y| > r$. We have

\[ G_B(x, y) = \int_0^\infty p_B(t, x, y) \, dt = \int_0^{t_0} p_B(t, x, y) \, dt + \int_{t_0}^\infty p_B(t + t_0, x, y) \, dt \]

\[ \leq \int_0^{t_0} p_t(y - x) \, dt + \int_{B_R(0)} p_B(t_b, x, z) p_B(t, z, y) \, dz \, dt \]

\[ \leq t_b \sup_{t \in (0, t_b]} \sup_{r < |x| \leq 2} p_t(x) + \sup_{x \in \mathbb{R}^d} p_{t_b}(x) \int_0^\infty \mathbb{P}^y(\tau_B > t) \, dt. \]

Observe that

\[ \int_0^\infty \mathbb{P}^y(\tau_B > t) \, dt = \mathbb{E}^y \tau_B \leq \mathbb{E}^y \tau_{B_{2R}(y)} = \mathbb{E}^0 \tau_{B_{2R}(0)}, \quad y \in B. \]

By [51] Rem. 4.8 the last mean exit time is finite. Thus, under (A2), we have

\[ \sup_{x, y \in B, |x - y| > r} G_B(x, y) < \infty, \]

for every $0 < r < R$. This proves (D) from [8]. \qed

Finally, we will need the following technical estimate.

**Lemma 2.5.** Assume [A1(a,b,c)]. For every $t > 0$ there exists a constant $C_{t, \infty} > 0$ such that $p_t(x) \geq C_{t, \infty} \mathcal{L}(x)$ for every $|x| \geq 1$.

**Proof.** Fix $t > 0$ and denote by $\mu_{1,t}(dx)$ and $\mu_{2,t}(dx)$ the measures determined by their characteristic functions (inverse Fourier transforms)

\[ \mathcal{F}^{-1}(t) \mu_{1,t}(\xi) = \exp \left( -t \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot y)) \nu^{0}_{1/2}(y) \, dy \right), \quad \xi \in \mathbb{R}^d, \]
and 
\[ \mathcal{F}^{-1} \mu_{2,t}(\xi) = \exp \left( -t \int (1 - \cos(\xi \cdot y)) \nu_{1/2}(y) dy \right), \quad \xi \in \mathbb{R}^d, \]

with 
\[ \nu_{1/2}^0(x) := \nu(x) 1_{B_{1/2}(0)}(x) \quad \text{and} \quad \nu_{1/2}(x) := \nu(x) 1_{\mathbb{R}^d \setminus B_{1/2}(0)}(x). \]

Because of (A1) both measures are non-degenerate. Recall that 
\[ \mu_{2,t}(dx) = \exp \left[ \frac{1}{t} \nu_{1/2} - \frac{1}{t} \nu_{1/2}[\delta_0] \right] (dx) = e^{-\frac{1}{t} \nu_{1/2}[\delta_0]}(dx) + p_{2,t}(x) dx \]

with 
\[ p_{2,t}(x) := e^{-\frac{1}{t} \nu_{1/2}} \sum_{n=1}^{\infty} \frac{t^n \nu_{1/2}^n(x)}{n!}; \]
\[ \nu_{1/2}^n(x) \] denotes the density of the n-fold convolution \( \nu_{1/2}^n(dx) \). Moreover, due to [50, Th. 27.7] the measure \( \mu_{1,t}(dx) \) is absolutely continuous with respect to Lebesgue measure. We denote the corresponding density by \( p_{1,t}(x) \).

Let \( A \in \mathbb{R}^{n \times n} \), be a symmetric, positive semi-definite matrix and denote by \( \gamma_t(dx) \) the Gaussian measure with characteristic function 
\[ \mathcal{F}^{-1} \gamma_t(\xi) = \exp \left( -t (\xi \cdot A \xi) \right), \quad \xi \in \mathbb{R}^d. \]

Due to (1.1) and (2.8), the density \( p_t \) is of the form 
\[ p_t(x) = e^{-\frac{1}{t} \nu_{1/2}}(p_{1,t} * \gamma_t)(x) + (p_{1,t} * p_{2,t} * \gamma_t)(x). \]

Assume that \( |x| \geq 1 \). By the above representation and (A1,b,c) we get 
\[ p_t(x) \geq \int_{\mathbb{R}^d} p_{2,t}(x - y) \int_{\mathbb{R}^d} p_{1,t}(y - z) \gamma_t(dz) dy \]
\[ \geq t e^{-\frac{1}{t} \nu_{1/2}} \int_{\mathbb{R}^d} \nu_{1/2}(x - y) \int_{\mathbb{R}^d} p_{1,t}(y - z) \gamma_t(dz) dy \]
\[ \geq c_1 \int_{|y| < 1/2} \nu(x - y) \int_{\mathbb{R}^d} p_{1,t}(y - z) \gamma_t(dz) dy \]
\[ \geq c_2 \nu(x) \int_{|y| < 1/2} \int_{\mathbb{R}^d} p_{1,t}(y - z) \gamma_t(dz) dy. \]

Since \( \int_{|y| < 1/2} \int_{\mathbb{R}^d} p_{1,t}(y - z) \gamma_t(dz) dy > 0 \), the claimed bound follows. \( \square \)

3. Structure and estimates of large jumps of the process

3.1. Properties of the profile function \( f \). Sometimes it is convenient to replace the profile function \( f(r) \), for large values of \( r \), by its truncation 
\[ f_1 = f \wedge 1. \]

In this section we are going to show that \( f_1 \) still enjoys the basic assumptions (A1,b,c,d).

If \( f \) satisfies (A1b,c), then so does \( f_1 \): it is again a decreasing function and there exists a constant \( \tilde{C}_2 \geq C_2 \) such that the following uniform growth condition holds 
\[ f_1(r) \leq \tilde{C}_2 f_1(r + 1), \quad r > 0. \]

Note that, under (A1a,b), the conditions (A1c) and (3.1) are equivalent.

**Lemma 3.1.** Let \( f \) be as in (A1). Condition (A1d) can be replaced by the following equivalent condition: there exists a uniform constant \( C_{f1} > 0 \) such that 
\[ \int_{\mathbb{R}^d} f_1(|x - z|) f_1(|z|) dz \leq C_{f1} f_1(|x|), \quad x \in \mathbb{R}^d. \]
In particular, (3.2) implies
\[
(3.3) \quad f_1(|x-w|)f_1(|w|) \leq \frac{(C_2)^2 \cdot C_{\text{d}}}{|B_1(0)|} f_1(|x|), \quad x, w \in \mathbb{R}^d,
\]
and there exists a constant $\tilde{C}_{\text{d}} > 0$ such that
\[
(3.4) \quad f(|x|)f(|y|) \leq \tilde{C}_{\text{d}} f_1(|x-y|), \quad |x|, |y| > 1.
\]

Proof. Let us first establish the equivalence of (3.2) and (A1.d). Assume that (3.2) holds. Since $f$ is decreasing, we have
\[
(3.5) \quad cf(|z|) \leq f_1(|z|) \leq f(|z|), \quad |z| > 1,
\]
where $c := 1/(1 \vee f(1))$. Therefore, (3.2) implies
\[
c^2 \int_{|x-y| > 1} f(|x-y|)f(|y|) \, dy \leq \int_{\mathbb{R}^d} f_1(|x-z|)f_1(|z|) \, dz \leq C_{\text{d}} f_1(|x|) \leq f(|x|), \quad |x| > 1,
\]
and (A1.d) follows.

In order to see the opposite implication, we note that (A1.a) implies $\int_{|y| > 1} f(|y|) \, dy < \infty$, hence $\int_{\mathbb{R}^d} f_1(|y|) \, dy < \infty$. If $|x| < 1$, then
\[
\int_{\mathbb{R}^d} f_1(|x-z|)f_1(|z|) \, dz \leq \frac{\int_{\mathbb{R}^d} f_1(|y|) \, dy}{f_1(1)} f_1(1) \leq c_1 f_1(|x|).
\]
Moreover, (3.1) shows for $|x| > 1$
\[
\int_{|z-x| \leq 1} f_1(|x-z|)f_1(|z|) \, dz = \int_{|z| \leq 1} f_1(|z|)f_1(|x-z|) \, dz \leq c_2 f_1(|x|), \quad x \in \mathbb{R}^d.
\]
Finally, combining (3.5) and (A1.d) we see that there exists a constant $c_3 > 0$ such that
\[
\int_{|z-x| > 1} f_1(|x-z|)f_1(|z|) \, dz \leq c_3 f_1(|x|), \quad |x| > 1,
\]
and (3.2) follows.

The inequality (3.3) is a direct consequence of (3.2) and (3.1):
\[
\frac{|B_1(w)|}{(C_2)^2} f_1(|x-w|)f_1(|w|) \leq \int_{B_1(w)} f_1(|x-z|)f_1(|z|) \, dz
\]
\[
\leq \int_{\mathbb{R}^d} f_1(|x-z|)f_1(|z|) \, dz \leq C_{\text{d}} f_1(|x|), \quad x, w \in \mathbb{R}^d.
\]
Finally, (3.4) follows from (3.3) and (3.5). \qed

The next lemma gives simple sufficient conditions under which the convolution condition (A1.d) holds. Recall that a decreasing function $f$ satisfies the doubling property, if there exists a constant $C \geq 1$ such that $f(r) \leq Cf(2r)$ for all $r > 0$.

Lemma 3.2. Let $f : (0, \infty) \to (0, \infty)$ be a decreasing function. Under each of the following conditions, $f$ satisfies (A1.d).

a) **Doubling profiles:** $f$ has the doubling property and $f|_{(1, \infty)} \in L^1((1, \infty), r^{d-1} \, dr)$.

b) **Tempered profiles:** $f(r) = e^{-m r} h(r)$ for some $m > 0$ and $h : (0, \infty) \to (0, \infty)$ is a decreasing function with doubling property and $h|_{(1, \infty)} \in L^1((1, \infty), r^{d-1} \, dr)$.

c) **Log-convex profiles:** $f$ is log-convex\(^3\) on $(1, \infty)$ and satisfies
\[
(3.6) \quad \sup_{|x| > 1} \left\{ \int_{|y| > 1} e^{-\frac{f(|y|)}{f(|x|) + y}} f(|y|) \, dy \right\} = \int_{|y| > 1} e^{-\frac{f(|y|)}{f(|y|) + 1}} f(|y|) \, dy < \infty.
\]

\(^2\)Since the doubling property is only used in connection with the direct jump property (A1.d) it is, in fact, enough to require the doubling property for large $r$, e.g. $r > \frac{1}{2}$.

\(^3\)By log-convexity, $f'$ exists Lebesgue almost everywhere.
Proof. [a] Since \( y \) and \( x - y \) play symmetric roles in the integral in (A1.d), we have
\[
\int_{\|x-y\| > |y| > 1} f(|x-y|) f(|y|) \, dy = 2 \int_{\|x-y\| > 1, |y| > |x-y|} f(|x-y|) f(|y|) \, dy.
\]
From \(|x-y| > |y| = |y-x+x| \geq |x| - |x-y|\), we get \(|x-y| > \frac{1}{2} |x|\). The monotonicity and doubling property of \( f \) show for all \(|x| \geq 1\)
\[
\int_{\|x-y\| > |y| > 1} f(|x-y|) f(|y|) \, dy \leq f \left( \frac{1}{2} |x| \right) \int_{|y| > 1} f(|y|) \, dy \leq C f(|x|) \int_{|y| > 1} f(|y|) \, dy.
\]
This gives (A1.d).

[b] Using \(|x| \leq |x-y| + |y|\), \( x, y \in \mathbb{R}^d \), we get
\[
\int_{|y| > 1} \frac{1}{1+|y|} \left( f(|x|) + f(|y|) \right) \, dy \leq \int_{|y| > 1} e^{-m(|x-y|+|y|)} h(|x-y|) h(|y|) \, dy
\leq e^{-m|x|} \int_{|y| > 1} h(|x-y|) h(|y|) \, dy.
\]
Now we can use part [a] for the last integral involving \( h \), and (A1.d) follows.

[c] The condition (3.6) guarantees that \( C := \int_{|y| > 1} f(|y|) \, dy < \infty \). Indeed, since \( f' / f \leq 0, f(|\cdot|) \) is integrable on the set \( \{ y : |y| > 1, y_1 \geq 0 \} \), hence on \( \{ |y| > 1 \} \) because of rotational symmetry. Write
\[
\int_{|y| > 1} f(|x-y|) f(|y|) \, dy \leq \int_{<|x-y|<|x|} f(|x-y|) f(|y|) \, dy + 2 \int_{|y| > |x|} f(|x-y|) f(|y|) \, dy
\]
\[=: I + II.\]
By the monotonicity of \( f \), we have \( II \leq 2 f(|x|) \int_{|y| > 1} f(|y|) \, dy \leq 2 C f(|y|) \). In order to estimate \( I \), we consider two cases: \( 1 \leq |x| \leq 2 \) and \(|x| > 2\).

Case 1: \( 1 \leq |x| \leq 2 \). We have
\[
I \leq f(1) \int_{|y| > 1} f(|y|) \, dy \leq \frac{f(1)}{f(2)} f(|x|) \int_{|y| > 1} f(|y|) \, dy \leq \frac{f(1)}{f(2)} C f(|x|).
\]

Case 2: \(|x| > 2\). We can use again the symmetry of \( y \) and \( x - y \) in the integrand to see
\[
I = 2 f(|x|) \int_{1 < |x-y| < |x|, 1 < |y| < |x|, \frac{f(|x-y|)}{f(|x|)}} f(|y|) \, dy
\]
\[= 2 f(|x|) \int_{1 < |x-y| < |x|, 1 < |y| < |x|} e^{-(\log f(|x|)-\log f(|x-y|))} f(|y|) \, dy.
\]
Using the log-convexity of \( f \) on \((1, \infty)\), we get for \(|x| > 2\)
\[
\log f(|x|) - \log f(|x-y|) \geq \frac{f'(|x-y|)}{f(|x-y|)} (|x| - |x-y|),
\]
almost everywhere on the domain of the above integration. Since \( f' / f \) is increasing and negative, and since \( 0 < |x| - |x-y| \leq (x \cdot y) / |x| \) holds on the domain of integration, we obtain
\[
\log f(|x|) - \log f(|x-y|) \geq \frac{f'(|y|) x \cdot y}{f(|y|)} \frac{|x|}{|x|}
\]
almost everywhere. This gives the estimate
\[
I \leq 2 f(|x|) \int_{1 < |x-y| < |x|, 1 < |y| < |x|} e^{\frac{f'(|y|) x \cdot y}{f(|y|)} \frac{|x|}{|x|}} f(|y|) \, dy \leq 2 f(|x|) \int_{|y| > 1} \frac{f'(|y|) x \cdot y}{f(|y|)} \frac{|x|}{|x|} f(|y|) \, dy
\]
for all \(|x| > 2\). Thus, (3.6) implies (A1.d).
3.2. Decomposition of the paths of the process and related estimates. In view of the Feynman–Kac representation of the semigroup \( \{U_t : t \geq 0\} \) and Lemma 2.3.b, we can estimate \( u_t(x, y) \) if we can control the behaviour of the sample path of the process. For this, we decompose the paths using exit times from and entrance times into certain annuli. Such decompositions appeared for the first time in [10] and they were also used in [45, 35]. Let \( k, n_0 \in \mathbb{N}, k \geq n_0 \geq 2 \), and define

\[
A_{k-1}^k := \begin{cases} 
\{ x \in \mathbb{R}^d : k-1 < |x| \leq k \} & \text{if } k \geq n_0 + 2, \\
\{ x \in \mathbb{R}^d : |x| \leq n_0 + 1 \} & \text{if } k = n_0 + 1, \\
\{ x \in \mathbb{R}^d : |x| \leq n_0 \} & \text{if } k = n_0,
\end{cases}
\]

and

\[
S_{k-2}^\infty := \begin{cases} 
\{ x \in \mathbb{R}^d : |x| > k - 2 \} & \text{if } k \geq n_0 + 2, \\
\mathbb{R}^d & \text{if } k \in \{n_0, n_0 + 1\}.
\end{cases}
\]

The exact value of \( n_0 \) will be chosen later on. We need two families of stopping times

\[
s_{k-1}^k := \inf \left\{ t \geq 0 : X_t \in A_{k-1}^k \right\} \quad \text{and} \quad \tau_{k-2}^\infty := \inf \left\{ t \geq 0 : X_t \notin S_{k-2}^\infty \right\};
\]

as \( A_{n_0-2}^\infty = A_{n_0-1}^\infty = \mathbb{R}^d \), we have \( \tau_{n_0-2}^\infty = \tau_{n_0-1}^\infty = \infty \).

With these stopping times we can count the number of annuli which are visited by the process \( X \) on its way from \( A_{n_0-2}^\infty \) to \( A_{k-1}^k \) for \( n-2 \geq k \) moving ‘inward’, i.e. when the modulus \( |X_s| \) reaches a new minimum — see Fig. 2. More precisely, for \( n-2 \geq k \geq n_0 \) and \( t > 0 \), we set

\[
S(A_{n_0-2}^\infty, A_{k-1}^k; 1; t) := \left\{ X_{n_0-2}^\infty \in A_{k-1}^k, s_{k-1}^k < t \right\},
\]

\[
S(A_{n_0-2}^\infty, A_{k-1}^k; l; t) := \bigcup_{p=k+2}^{n-2} S(A_{n_0-2}^\infty, A_{p-1}^p; l-1; t) \cap S(A_{p-2}^\infty, A_{k-1}^k; 1; t), \quad l > 1.
\]

The first set is the event that the process moves to the annulus \( A_{k-1}^k \) before time \( t \) upon exiting \( A_{n_0-2}^\infty \). The second set is defined recursively: \( S(A_{n_0-2}^\infty, A_{p-1}^p; l-1; t) \cap S(A_{p-2}^\infty, A_{k-1}^k; 1; t) \) describes those paths which move, before time \( t \), from \( A_{n_0-2}^\infty \) to the annulus \( A_{p-1}^p \), visiting on their way exactly \( l-1 \) annuli in-between \( A_{n_0-2}^\infty \) and \( A_{p-1}^p \) (including the final destination). In the end, but still before time \( t \), the process moves directly from \( A_{n_0-2}^\infty \supset A_{p-1}^p \) to \( A_{k-1}^k \).
Proof of Lemma 3.3.
We will consider two cases: we have

There is a constant $C$.

Assume $\lambda > 0$ and let $W \in C$ and $n, k \in \mathbb{N}$ be such that $n - 2 \geq k \geq n_0$. There is a constant $C > 0$ and $\vartheta_0 \geq 1$ such that for every $t > 0$, for all $x \in A_{n-1}^\infty$ and $\vartheta > \vartheta_0$ we have

$$
\mathbb{E}^x \left[ e^{-\vartheta \tau_{n-2}^\infty} W(X_{\tau_{n-2}^\infty}) ; \tau_{n-2}^\infty < t, X_{\tau_{n-2}^\infty} \in A_{k-1}^\infty \right] \leq \frac{C}{\vartheta} \int_{A_{k-1}^\infty} f(|x-z|) W(z) \, dz;
$$

$C$ depends on $W$ only through the growth constant $\tilde{C}_2$ appearing in (3.1) and (3.7).

**Remark 3.4.** On the set $\{ \tau_{n-2}^\infty < t, X_{\tau_{n-2}^\infty} \in A_{k-1}^\infty \}$ we have $\tau_{n-2}^\infty = \sigma_{k-1}^\infty$.

**Proof of Lemma 3.3.** Define for $\vartheta > 0$

$$
u(y) := \mathbb{E}^y \left[ e^{-\vartheta \tau_{n-2}^\infty} W(X_{\tau_{n-2}^\infty}) ; \tau_{n-2}^\infty < \infty, X_{\tau_{n-2}^\infty} \in A_{k-1}^\infty \right], \quad y \in A_{n-2}^\infty.
$$

We will consider two cases: $k = n - 2$ and $k < n - 2$. 

---

**Figure 2.** A typical path contained in the set $S(A_{n-2}^\infty, A_{k-1}^k; 3; t)$ comprising sample paths moving from $A_{n-2}^\infty$ to $A_{k-1}^k$ via $A_p^p$ and $A_{p-1}^p$ (for simplicity, we show only jump entries into new annuli). Notice that the number $l = 3$ only counts first visits to annuli in a ‘downward’ or ‘inward’ movement where the modulus $|X_s|$ reaches new minima, i.e. only the visits $n - 2 \geq p > p - 2 \geq p' > p'' - 2 \geq k$ are counted. Those annuli, which are visited (even for the first time) when going ‘outward’, are not counted.
Estimation of Markov property, we have and we will estimate $\varrho$.
The above

$$v_x(y) := \mathbb{E}^y \left[ e^{-\vartheta \tau_{n-2}^\infty} W(X_{n-2}^\infty); \tau_{n-2} < \infty, X_{n-2}^\infty \in A_{k-1}^k \cap B_1(x) \right], \quad y \in A_{n-2}^\infty,$$

and

$$h_x(y) := \begin{cases} 
\mathbb{E}^y \left[ e^{-\vartheta \tau_{n-2}^\infty} W(X_{n-2}^\infty); \tau_{n-2} < \infty, X_{n-2}^\infty \in A_{k-1}^k \setminus B_1(x) \right], & y \in A_{n-2}^\infty, \\
W(y) \mathbb{1}_{A_{k-1}^k \setminus B_1(x)}(y), & y \notin A_{n-2}^\infty.
\end{cases}$$

Clearly, we have the following decomposition

$$u(y) = v_x(y) + h_x(y), \quad y \in A_{n-2}^\infty,$$

and we will estimate $v_x$ and $h_x$ separately.

**Estimation of $v_x$:** If $x \in A_{n-1}^\infty$, then $v_x \equiv 0$ since $B_1(x) \cap A_{k-1}^k = B_1(x) \cap A_{n-3}^n = \emptyset$. On the other hand, for $x \in A_{n-2}^\infty \setminus A_{n-1}^\infty$, we get from (3.7), and the definition of $v_x$ that $v_x(y) \leq \tilde{C}_2 W(x)$. Pick $\varrho \in (0,1)$ such that $B_{1/2}(\varrho x) \subset A_{k-1}^k$ and $\sup_{z \in B_{1/2}(\varrho x)} |x - z| \leq 2$ and let $c_1 := |B_{1/2}(0)|$, see Fig. 3. The monotonicity of $f_1$ and (3.7) give

$$W(x) = \frac{f_1(2)W(x)|B_{1/2}(\varrho x)|}{c_1 f_1(2)} = \frac{1}{c_1 f_1(2)} \int_{B_{1/2}(\varrho x)} f_1(2) W(x) \, dz \leq \frac{(\tilde{C}_2)^2}{c_1 f_1(2)} \int_{B_{1/2}(\varrho x)} f_1(|x - z|) W(z) \, dz.$$ 

Therefore,

$$v_x(y) \leq \frac{(\tilde{C}_2)^3}{c_1 f_1(2)} \int_{A_{k-1}^k} f_1(|x - z|) W(z) \, dz, \quad x \in A_{n-2}^\infty \setminus A_{n-1}^\infty.$$ 

**Estimation of $h_x$:** Set $D_{n,x} = A_{n-2}^\infty \setminus B_1(x)$ and observe that $\tau_{n-2}^\infty \geq \tau := \tau_{D_{n,x}}$. By the strong Markov property, $h_x(y) = \mathbb{E}^y \left[ e^{-\vartheta \tau_{n-2}^\infty} h_x(X_\tau) \right]$ for every $y \in D_{n,x}$, i.e. $h_x$ is regular $(X, \vartheta)$-harmonic in $D_{n,x}$. On the other hand, $h_x(y) = 0$ for every $y \in B_1(x) \setminus D_{n,x}$. By (3.5), Lemma 2.4 and (A1), we have

$$h_x(x) \leq \frac{c_2}{\vartheta} \int_{B_{1/2}(x)} h_x(z) f(|x - z|) \, dz,$$

$$\leq \frac{c_3}{\vartheta} \int_{B_{1/2}(x)} h_x(z) f_1(|x - z|) \, dz,$$

$$\leq \frac{c_3}{\vartheta} \int_{A_{k-1}^k} f_1(|x - z|) W(z) \, dz + \frac{c_3}{\vartheta} \int_{A_{n-2}^\infty} h_x(z) f_1(|x - z|) \, dz,$$
(recall that \( k = n - 2 \)) with absolute constants \( c_2 \) and \( c_3 \).

Combining the estimates from above, we get

\[
(3.8) \quad u(x) = h_x(x) \leq \frac{c_3}{\vartheta} \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz + \frac{c_3}{\vartheta} \int_{A_{n-2}^\infty} u(z)f_1(|x-z|) \, dz, \quad x \in A_{n-1}^\infty,
\]

and

\[
(3.9) \quad u(x) = v_x(x) + h_x(x)
\]

\[
\leq c_4 \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz + \frac{c_3}{\vartheta} \int_{A_{n-2}^\infty} u(z)f_1(|x-z|) \, dz, \quad x \in A_{n-2}^\infty,
\]

where \( c_4 = c_3 \vartheta^{-1} + (C_2)^3(c_1 f_1(2))^{-1} \).

Using induction in \( p \in \mathbb{N}_0 \), we will prove for \( p \in \mathbb{N}_0 \) and \( x \in A_{n-2}^\infty \) the estimate

\[
(3.10) \quad u(x) \leq c_4 \sum_{i=0}^p \left( \frac{c_3 C_{(3.11)}}{\vartheta} \right)^i \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz + \|W\|_\infty \left( \frac{c_3 \|f_1\|_1}{\vartheta} \right)^{p+1}.
\]

For \( p = 0 \) the inequality (3.10) follows from (3.9) and the definition of the function \( u \). Suppose now that (3.10) holds true for some \( p \geq 0 \). Inserting it into (3.9), we get

\[
u(x) \leq c_4 \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz
\]

\[
+ c_3 c_4 \sum_{i=0}^p \left( \frac{c_3 C_{(3.11)}}{\vartheta} \right)^i \int_{A_{n-2}^\infty} f_1(|x-w|)W(w) \, dw \int_{A_{k-1}^k} f_1(|x-z|) \, dz
\]

\[
+ \|W\|_\infty \left( \frac{c_3 \|f_1\|_1}{\vartheta} \right)^{p+2},
\]

for every \( x \in A_{n-2}^\infty \). By Tonelli and (3.2) we have

\[
J(x) = \int_{A_{k-1}^k} \left( \int_{A_{n-2}^\infty} f_1(|x-z|)f_1(|x-w|) \, dz \right) W(w) \, dw \leq C_{(3.11)} \int_{A_{k-1}^k} f_1(|x-w|)W(w) \, dw.
\]

Returning to the previous estimate, we get

\[
u(x) \leq c_4 \left[ 1 + c_3 c_4 \sum_{i=0}^p \left( \frac{c_3 C_{(3.11)}}{\vartheta} \right)^i \right] \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz + \|W\|_\infty \left( \frac{c_3 \|f_1\|_1}{\vartheta} \right)^{p+2}
\]

\[
= c_4 \sum_{i=0}^{p+1} \left( \frac{c_3 C_{(3.11)}}{\vartheta} \right)^i \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz + \|W\|_\infty \left( \frac{c_3 \|f_1\|_1}{\vartheta} \right)^{p+2}.
\]

This is exactly (3.10) with \( p \sim p + 1 \).

Taking \( \vartheta > \max \{2c_3 C_{(3.11)} c_3 \|f_1\|_1 \} \) and letting \( p \to \infty \), we obtain

\[
(3.11) \quad u(x) \leq \frac{c_4 \vartheta}{\vartheta - c_3 C_{(3.11)}} \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz \leq 2c_4 \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz, \quad x \in A_{n-2}^\infty.
\]

Inserting (3.11) into (3.8) and a further application of Tonelli’s theorem and (3.2) (as above) yields

\[
u(x) \leq \frac{C_{(3.8)}}{\vartheta} \int_{A_{k-1}^k} f_1(|x-z|)W(z) \, dz \leq \frac{C_{(3.8)}}{\vartheta} \int_{A_{k-1}^k} f(|x-z|)W(z) \, dz, \quad x \in A_{n-1}^\infty,
\]

for every \( \vartheta > 2c_3 C_{(3.11)} + c_3 \|f_1\|_1 \) with the constant \( C_{(3.8)} = c_3(1 + 2c_4 C_{(3.11)}) \). This completes the proof in the case \( k = n - 2 \).
Case 2: \( k < n - 2 \). From the Ikeda–Watanabe formula (2.6) we get

\[
u(x) = \int_{A_{n-2}^k} \int_0^\infty e^{-\theta t} p_{A_{n-2}^k}(t, x, y) dt \int_{A_{k-1}^k} W(z)\nu(z-y) dz \, dy, \quad x \in A_{n-2}^k.
\]

It follows from Lemma 2.5 that

(3.12) \( C_{\text{Ik}}(w) \leq p_{0b}(w), \quad |w| \geq 1. \)

Observe that \( p_{A_{n-2}^k}(t, x, y) \leq p_t(x-y) \). Since \( \text{dist}(A_{k-1}^k, A_{n-2}^k) \geq 1 \), the estimate (3.12), Tonelli’s theorem and the Chapman–Kolmogorov equation give

\[
u(x) \leq C_{\text{Ik}}^{-1} \int_{A_{k-1}^k} \int_0^\infty e^{-\theta t} \left( \int_{A_{n-2}^k} p_t(x-y) p_{0b}(y-z) dy \right) dt \int_{A_{k-1}^k} W(z) \, dz
\]

\[
\leq C_{\text{Ik}}^{-1} \int_{A_{k-1}^k} \int_0^\infty e^{-\theta t} p_t + t_{0b}(x-z) dt \int_{A_{k-1}^k} W(z) \, dz.
\]

Finally, using (A2) and (A1) we get

\[
u(x) \leq c_5 \int_0^\infty e^{-(\theta - C_\theta) t} dt \int_{A_{k-1}^k} f((x-z)) W(z) \, dz, \quad x \in A_{n-1}^k.
\]

Setting \( \vartheta_0 := 2c_3 C_{\text{Ik}}^{-1} + c_3 \|f\|_1 + 2C_5 \) and \( C_{\text{Ik}} \) as from (3.7). For \( n, k, l \in \mathbb{N}, n-1 < |x| \leq n, n_0 < k \leq n-2 \), and \( t > 0 \), the following estimate holds with \( C_{\text{Ik}} := 2C_6 C_{\text{Ik}}^{-1} \)

\[
E^x \left[ e^{-\int_0^{\tau_{k-1}^n} V(x_s) ds} W(X_{\sigma_{k-1}^n}^k); S(A_{n-2}^k, A_{k-1}^k, l, t) \right] \leq \frac{C_{\text{Ik}}}{2^l g(n-2)} \int_{A_{k-1}^k} f((x-z)) W(z) \, dz.
\]

Proof. We use induction in \( l \in \mathbb{N} \).

Let \( l = 1 \). By definition, \( S(A_{n-2}^k, A_{k-1}^k, 1, t) = \{ X_{\tau_{n-2}^k} \in A_{k-1}^k, \sigma_{k-1}^k < t \} \) and \( \sigma_{k-1}^k = \tau_{n-2}^k \) on this set. From (A3) and Lemma 3.3 with \( \theta = g(n-2)/C_6 \) we get

\[
E^x \left[ e^{-\int_0^{\tau_{k-1}^n} V(x_s) ds} W(X_{\sigma_{k-1}^n}^k); S(A_{n-2}^k, A_{k-1}^k, 1, t) \right]
\]

\[
\leq E^x \left[ e^{-((n-2)/C_6)\tau_{n-2}^k} W(X_{\tau_{n-2}^k}^k); \quad X_{\tau_{n-2}^k}^k \in A_{k-1}^k, \sigma_{k-1}^k < t \right]
\]

\[
\leq \frac{2C_6 C_{\text{Ik}}}{2^l g(n-2)} \int_{A_{k-1}^k} f((x-z)) W(z) \, dz.
\]

This means that the claimed bound holds for \( l = 1 \), all \( n, k, x, t \) as detailed in the statement of the lemma and all functions \( W \) from the class \( \mathcal{C} \) (recall that \( \mathcal{C} \) includes all functions of the form \( W_g(x) = f_t(|x-y|), \quad y \in \mathbb{R}^d \)).

Now assume that the induction assumption holds for \( 1, 2, ..., l-1 \) with \( l \geq 2 \). Using the decomposition of paths introduced at the beginning of Section 3.2 we may write

\[
E^x \left[ e^{-\int_0^{\tau_{k-1}^n} V(x_s) ds} W(X_{\sigma_{k-1}^n}^k); S(A_{n-2}^k, A_{k-1}^k, l, t) \right]
\]

\[
\leq \sum_{p=k+2}^{n-2} E^x \left[ e^{-\int_0^{\tau_{k-1}^n} V(x_s) ds} W(X_{\sigma_{k-1}^n}^k); S(A_{n-2}^k, A_{p-1}^k, l-1, t) \cap S(A_{p-2}^k, A_{k-1}^k, 1, t) \right].
\]
Since the process visits first $A^p_{p-1}$ and then $A^k_{k-1}$, we have $\sigma^k_{k-1} > \sigma^p_{p-1}$. By the strong Markov property, the above expression becomes

$$\sum_{p=k+2}^{n-2} E^x \left[ e^{-\int_0^{\sigma^p_{p-1}} V(X_s) \, ds} \times\right.$$  

$$\times E^{X_{\sigma^p_{p-1}}} \left[ e^{-\int_0^{\sigma^k_{k-1}} V(X_s) \, ds} W(X_{\sigma^k_{k-1}}); S(A^\infty_{p-2}, A^k_{k-1}, 1, t-r) \right] \left[ S(A^\infty_{n-2}, A^p_{p-1}, l-1, t) \right].$$

Using the induction hypothesis with $l = 1$ for the inner expectation, we see that the above sum is less than

$$\sum_{p=k+2}^{n-2} E^x \left[ e^{-\int_0^{\sigma^p_{p-1}} V(X_s) \, ds} \left( \frac{C_{5.9}}{g(p-2)} \int_{A^k_{k-1}} f(|X_{\sigma^p_{p-1}} - z|) W(z) \, dz; S(A^\infty_{n-2}, A^p_{p-1}, l-1, t) \right) \right],$$

which is, by Fubini’s theorem, equal to

$$\int_{A^k_{k-1}} \sum_{p=k+2}^{n-2} \frac{C_{5.9}}{g(p-2)} E^x \left[ e^{-\int_0^{\sigma^p_{p-1}} V(X_s) \, ds} f(|X_{\sigma^p_{p-1}} - z|); S(A^\infty_{n-2}, A^p_{p-1}, l-1, t) \right] W(z) \, dz.$$  

Now we estimate the expectation under the sum. Since $\text{dist}(A^p_{p-1}, A^k_{k-1}) \geq 1$, we have $f(|X_{\sigma^p_{p-1}} - z|) \leq (1 + f(1)) f_1(|X_{\sigma^p_{p-1}} - z|)$, cf. (3.5). Using the induction hypothesis for the functions $W_z(w) = f_1(|w - z|), z \in A^k_{k-1}$, we get

$$E^x \left[ e^{-\int_0^{\sigma^p_{p-1}} V(X_s) \, ds} f(|X_{\sigma^p_{p-1}} - z|); S(A^\infty_{n-2}, A^p_{p-1}, l-1, t) \right] \leq \frac{C_{5.9}(1 + f(1))}{2^l g(n-2)} \int_{A^p_{p-1}} f(|x - w|) f_1(|w - z|) \, dw.$$ 

Plugging this estimate into the above expression, we finally get that the initial expectation is bounded by

$$\frac{C_{5.9}(1 + f(1))}{2^l g(n_0 - 2) g(n-2)} \int_{A^k_{k-1}} \left( \int_{k+2 < |w| < n-2} f(|x - w|) f_1(|w - z|) \, dw \right) W(z) \, dz.$$  

Using that $f_1 \leq f$, the convolution condition (A1d) and the fact that $2C_3C_6\sigma_0(1 + f(1)) \leq g(n_0 - 2)$, the last expression is bounded by

$$\frac{C_{5.9}}{2^l g(n-2)} \int_{A^k_{k-1}} f(|x - z|) W(z) \, dz,$$

and we are done. \(\square\)

In the sequel we will often use the following estimate which is based on the decomposition of paths introduced above. For every $x, y \in \mathbb{R}^d$ and $t > t_0 > 0$ we have

$$u_t(x, y) = E^x \left[ e^{-\int_0^{t-t_0} V(X_s) \, ds} u_{t_0}(X_{t-t_0}, y) \right] \leq I + \sum_{k=t_0}^{n-2} \sum_{l=1}^{\infty} I_{k,l},$$

where

$$I := E^x \left[ e^{-\int_0^{t-t_0} V(X_s) \, ds} u_{t_0}(X_{t-t_0}, y); t-t_0 < \tau_{n-2}^\infty \right],$$

$$I_{k,l} := E^x \left[ e^{-\int_0^{t-t_0} V(X_s) \, ds} u_{t_0}(X_{t-t_0}, y); S(A^\infty_{n-2}, A^k_{k-1}, l, t-t_0), t-t_0 < \tau_{A^k_{k-2}} \right].$$

The estimates from Lemma 3.5 will be essential for proving sharp bounds for the terms $I_{k,l}$.  

\[ \text{FIG. 1: Tree Diagram for Path Decomposition} \]
Because of (3.13) we have to estimate \( u_t(x,y) \) as well as the monotonicity (A3.b) of \( u_t \) in (3.7). In the last estimate we use Lemma 3.5 with the function \( A_3 \). Assume Lemma 4.1. 

4.1. The upper bound.

**Lemma 4.1.** Assume (A1), (A2) and (A3.a,b) with \( t_b > 0 \) and \( R_0 > 0 \). Let \( n_0 \in \mathbb{N} \) be as in Lemma 3.3. There exists a constant \( C > 0 \) such that for every \( |x| > n_0 + 3 \), \( y \in \mathbb{R}^d \) and \( t > t_b \) we have

\[
 u_t(x,y) \leq Ce^{(C_\varepsilon + C_\sigma_0)t}g(|x|-2)f_1(|x-y|) + \int_{n_0+2 < |z| < |x|-1} f(|x-z|)f_1(|z-y|)e^{-\frac{t_b}{C_\varepsilon}g(|z|-2)}dz + f(|x|)f_1(|y|).
\]

**Proof.** Let \( |x| > n_0 + 3 \), \( y \in \mathbb{R}^d \) and \( t > t_b \). Pick \( n \in \mathbb{N} \), \( n \geq n_0 + 4 \) such that \( n - 1 < |x| \leq n \).

By (A3.b), (2.1), the Chapman–Kolmogorov equation for \( p_t \) and (A2.a), we get

\[
 I \leq e^{-\frac{t_b}{C_\varepsilon}g(n-2)}e^{C_\varepsilon t}E^x[p_{t_b}(X_{t-t_b} - y)] \leq e^{C_\varepsilon t}e^{-\frac{t_b}{C_\varepsilon}g(|x|-2)}p_t(x-y) \leq Ce^{C_\varepsilon t}e^{C_\sigma_0 t}e^{-\frac{t_b}{C_\varepsilon}g(|x|-2)}f_1(|x-y|).
\]

Now we turn to \( I_{k,l} \). First, assume that \( k \geq n_0 + 2 \). To keep notation simple, we set \( r := t - t_b \). By the fact that \( \sigma_{k-1} < r \) on \( S(A_{n_0-2}^\infty, A_{k-1}^\infty, l, r) \), and (A3.b), (2.1) we have

\[
 I_{k,l} \leq E^x \left[ e^{-\int_0^s V(X_v)dv} u_{t_b}(X_r, y); S(A_{n_0-2}^\infty, A_{k-1}^\infty, l, r), r < \tau_{A_{k-1}^\infty} \right] \leq e^{C_\varepsilon t}e^{-\frac{t_b}{C_\varepsilon}g(k-2)}e^{-\int_0^s V(X_v)dv} u_{t_b}(X_r, y); S(A_{n_0-2}^\infty, A_{k-1}^\infty, l, r)] .
\]

By the strong Markov property, the Chapman–Kolmogorov equation for \( p_t \), (A2) and (A3),

\[
 I_{k,l} \leq e^{C_\varepsilon t}e^{-\frac{t_b}{C_\varepsilon}g(k-2)}e^{-\int_0^{s_{k-1}} V(X_v)dv} E^{X_{s_{k-1}}} p_{t_{s_{k-1}}}(X_{s_{k-1}} - y); S(A_{n_0-2}^\infty, A_{k-1}^\infty, l, r) \]

In the last estimate we use Lemma 3.5 with the function \( W = f_1 \) (which is in the class \( C \) defined in (3.7)) as well as the monotonicity (A3.b) of \( g \).
We still have to estimate $I_{k,l}$ for $k = n_0, n_0 + 1$. Recall that $A_{n_0-1}^\infty \equiv A_{n_0-2}^\infty = \mathbb{R}^d$ and $\tau_{A_{n_0-1}^\infty} = \tau_{A_{n_0-2}^\infty} = \infty$. This is the most critical situation and we cannot use the above arguments. By the strong Markov property and (the analogue of) the Chapman–Kolmogorov equations for the kernel $u_t$, we have

$$I_{k,l} = E^x [e^{-\int_0^t V(X_s) ds} u_t(X_t, y); S(A_{n_0-2}^\infty, A_{k-1}^k, l, r)]$$

$$= E^x [e^{-\int_0^{\tau_{n-1}} V(X_s) ds} E^{X_{n-1}} X_{n-1} \left[ e^{-\int_0^{\tau_{n-2}} V(X_s) ds} u_t(X_{n-2}, y) \right]_{v=\sigma_{n-1}^k} ; S(A_{n_0-2}^\infty, A_{k-1}^k, l, r)]$$

(4.1)

$$= E^x [e^{-\int_0^{\tau_{n-1}} V(X_s) ds} u_{t-\sigma_{n-1}^k} (X_{n-1}, y); S(A_{n_0-2}^\infty, A_{k-1}^k, l, r)].$$

Applying (2.1) and (A2) as before, we get that the above expectation is not greater than

$$C_4 e(C_5 + \rho C_2) E^x [e^{-\int_0^{\tau_{n-1}} V(X_s) ds} W_y (X_{n-1}, y); S(A_{n_0-2}^\infty, A_{k-1}^k, l, r)].$$

Observe that

$$[e^{C_5 t} f(|w|)] \wedge 1 \leq e^{C_5 t} f_1(|w|), \quad w \in \mathbb{R}^d,$$

and recall that for $y \in \mathbb{R}^d$ the function $W_y(w) = f_1(|w - y|)$ is also in the class $C$ defined in (3.7). From Lemma 3.5 we see

$$I_{k,l} \leq C_4 e(C_5 + \rho C_2) E^x \left[ e^{-\int_0^{\tau_{n-1}} V(X_s) ds} W_y(X_{n-1}) ; S(A_{n_0-2}^\infty, A_{k-1}^k, l, r) \right]$$

$$\leq C_4 e(C_5 + \rho C_2) \frac{2g(n-2)}{n-2} \int_{A_{k-1}^k} f(|x - z|) f_1(|z - y|) dz$$

and with (A1b) and (A1c) we conclude that

$$I_{k,l} \leq C_3 e(C_5 + \rho C_2) \frac{2g(n-2)}{n-2} f(|x|) f_1(|y|),$$

for $k = n_0, n_0 + 1$. Combining all inequalities, we finally obtain

$$u_t(x, y) \leq e^{(C_5 + \rho C_2) t} \left( c_1 e^{-\frac{t}{4} g(|x| - 2)} g(|x| - 2) f_1(|x - y|) + c_2 \int_{n_0+2 < |z| < |x| - 1} f(|x - z|) f_1(|z - y|) e^{\frac{t}{2} g(|z| - 2)} dz + c_3 f(|x|) f_1(|y|) \right). \quad \square$$

Lemma 4.2. Assume (A1), (A2) and (A3a,b) with $t_b > 0$ and $R_0 > 0$. For $n_0 \in \mathbb{N}$ as in Lemma 4.1 (and Lemma 3.5) there exists a constant $C > 0$ such that for $t > 4t_b$ the following assertions hold.

a) If $|x|, |y| \leq n_0 + 3$, then

$$u_t(x, y) \leq C e^{(C_5 + \rho C_2) t}.$$

b) If $|x| > n_0 + 3$ and $|y| \leq n_0 + 3$, then

$$u_t(x, y) \leq C e^{(C_5 + \rho C_2) t} \frac{f(|x|)}{g(|x| - 2)};$$

if $|x| \leq n_0 + 3$ and $|y| > n_0 + 3$, then, by symmetry,

$$u_t(x, y) \leq C e^{(C_5 + \rho C_2) t} \frac{f(|y|)}{g(|y| - 2)}.$$

c) If $|x|, |y| > n_0 + 3$, then

$$u_t(x, y) \leq C e^{(C_5+C_6)t} \left( \frac{e^{-\frac{(n-3)t}{4C_0}} g(|x|-2) f_1(|x-y|)}{g(|y|-2)} + \frac{1}{g(|x|-2)g(|y|-2)} \int_{n_0+2 < |z| < |x|-1} f(|x-z|) f_1(|z-y|) e^{-\frac{t}{4C_0} g(|z|-2)} dz \right. $$

$$\left. + \frac{1}{g(|x|-2)g(|y|-2)} f(|x|) f(|y|) \right).$$

In particular, the following symmetrized estimate holds

$$u_t(x, y) \leq C e^{(C_5+C_6)t} \left[ \left( \frac{e^{-\frac{(n-3)t}{4C_0}} g(|x|-3)}{g(|y|-2)} \wedge \frac{e^{-\frac{(n-3)t}{4C_0}} g(|y|-3)}{g(|x|-2)} \right) f_1(|x-y|) \right. $$

$$\left. + \frac{1}{g(|x|-2)g(|y|-2)} \int_{n_0+2 < |z| < |x|-1} f_1(|x-z|) f_1(|z-y|) e^{-\frac{t}{4C_0} g(|z|-2)} dz \right. $$

$$\left. + \frac{1}{g(|x|-2)g(|y|-2)} f(|x|) f(|y|) \right].$$

Proof. a) This follows directly from (2.1) and (A2).

b) By symmetry it is enough to consider the case $|x| > n_0 + 3$ and $|y| \leq n_0 + 3$. Since $|y| \leq n_0 + 3$, the estimate in Lemma 4.1 and (A3a,b) show for every $t > 4t_b$

$$u_t(x, y) \leq C e^{(C_5+C_6)t} \left( c_1 f(|x|) + c_2 \int_{n_0+2 < |z| < |x|-1} f(|x-z|) f(|z|) dz + c_3 f(|x|) \right)$$

and, by (A1d), we conclude that

$$u_t(x, y) \leq \frac{c_4 e^{(C_5+C_6)t}}{g(|x|-2)} f(|x|).$$

c) Let $|x|, |y| > n_0 + 3$ and $t > 4t_b$. In view of (A1d), (A3a,b), (3.4) and (3.5), we get with Lemma 4.1 that

$$u_{2t_b}(z, y) = u_{2t_b}(y, z) \leq C_5 f_1(|y-z|) \frac{1}{g(|y|-2)}, \quad z \in \mathbb{R}^d.$$
By Fubini’s theorem and (3.2), we finally arrive at
\[
    u_t(x, y) \leq C e^{\phi(x, y) t} \left( e^{-\frac{t-\rho_0}{\rho_0} \rho_0 \rho_2 |y|^2} f_1(|x-y|) \right) + \frac{1}{g(|x| - 2) g(|y| - 2)} \int_{|x| - 2 < |z| < |x| - 1} f(|x-z|) f_1(|z-y|) e^{-\frac{t-\rho_0}{\rho_0} \rho_0 |z|^2} dz
\]

This is the first claimed bound. The second one easily follows from the first by symmetry. \(\square\)

4.2. The lower bound. We begin with an auxiliary result. Recall that \(\rho_0(r) = 0\) and \(\psi_{0_H}\) are the ground state eigenvalue and eigenfunction for the process killed on leaving a ball \(B_r(0)\), \(r > 0\).

**Lemma 4.3.** Assume (A1.a,b,c), (A2) and (A3.a,b) with \(t_b > 0\) and \(R_0 > 0\). For every \(n_0 \geq R_0\) there exist the constants \(C, C > 0\) such that we have for \(t > 2t_b\)

\[
u_t(x, y) \geq C e^{-t \rho_0(\frac{1}{2})} f_1(|x-y|) \times \begin{cases} \frac{e^{-\frac{t-\rho_0}{\rho_0} \rho_0 \rho_2 |y|^2}}{g(||y|+1/2|)} & \text{if } |x| > n_0 + 2, |y| \geq n_0 + 2, \\ \frac{e^{-\frac{t-\rho_0}{\rho_0} \rho_0 \rho_2 |y|^2}}{g(||y|+1/2|)} & \text{if } |x| > n_0 + 2, |y| \leq n_0 + 2, \\ \frac{e^{-\frac{t-\rho_0}{\rho_0} \rho_0 \rho_2 |y|^2}}{g(||y|+1/2|)} & \text{if } |x| \leq n_0 + 2, |y| > n_0 + 2, \\ e^{-Ct} & \text{if } |x| \leq n_0 + 2, |y| \geq n_0 + 2. \end{cases}
\]

**Proof.** Set \(T_b := \frac{3}{2} t_b\). We distinguish between two cases: \(|x-y| < \frac{5}{4}\) and \(|x-y| \geq \frac{5}{4}\).

**Case 1:** Assume that \(t > T_b\) and \(x, y \in \mathbb{R}^d\) satisfy \(|x-y| < \frac{5}{4}\). By (the analogue of) the Chapman–Kolmogorov equations for \(u_t\),

\[
u_t(x, y) = \mathbb{E}^x \left[ e^{-\int_0^t \rho_b V(X_s) \, ds} \, u_{T_b}(X_{\tau_{T_b}}, y) \right]
\]

\[
\geq \mathbb{E}^x \left[ e^{-\int_0^t \rho_b V(X_s) \, ds} \, u_{T_b}(X_{\tau_{T_b}}, y); t - T_b < \tau_{B_{1/2}}(x) \right]
\]

\[
\geq e^{-(t-T_b) \sup_{x \in B_{1/2}(x)} \rho_0(t) V(x) F^0 \left( t - T_b < \tau_{B_{1/2}}(0) \right) \inf_{z \in B_{1/2}(x)} u_{T_b}(z, y) \right]
\]

Moreover,

\[
e^{-((t-T_b) \rho_0(\frac{1}{2}))} \psi_{0, \frac{1}{2}}(0) = \int_{B_{\frac{1}{2}}(0)} p_{B_{\frac{1}{2}}(0)}(t-T_b, 0, y) \psi_{0, \frac{1}{2}}(y) \, dy \leq \| \psi_{0, \frac{1}{2}} \|_{\infty} F^0 \left( t - T_b < \tau_{B_{1/2}}(0) \right)
\]

and, since \(\psi_{0, 1/2}(0) > 0\), we have for some \(c_1 > 0\)

\[
F^0 \left( t - T_b < \tau_{B_{1/2}}(0) \right) \geq c_1 e^{-((t-T_b) \rho_0(\frac{1}{2}))}, \quad t > T_b.
\]

Suppose first that \(|x|, |y| > n_0 + 2\). By (4.3), (A3.a,b) and our assumption \(|x-y| < \frac{5}{4}\), we get

\[
u_t(x, y) \geq c_1 e^{-t \rho_0(\frac{1}{2})} e^{-C(t-T_b) \rho_0 |y|^2} \inf_{x \in B_{1/2}(x)} u_{T_b}(z, y).
\]

So it is enough to estimate the infimum. Let \(z\) be such that \(|x-z| < \frac{1}{2}\). Since \(|y-z| < \frac{7}{4}\), we derive from (2.3) that for all \(s \in (0, T_b)\)

\[
F^0 [p_{B_2}(y)(T_b - s, X_s, y); s < \tau_{B_2}(y)] = \int_{B_2(y)} p_{B_2}(s, z, w) p_{B_2}(y)(T_b - s, w, y) \, dw = p_{B_2}(y)(T_b, z, y).
\]
If we take \( s < \frac{1}{4} t_b \), we have \( T_b - s > t_b \) and \( p_{B_2(y)}(T_b - s, x, y) \leq p(T_b - s, x, y) \) is bounded because of (A2). Using [22] Proof of Th. 3.4, Claim 1) we get from the above equality that the left-hand side, hence \( z \mapsto p_{B_2(y)}(T_b, z, y) \) is continuous.

Moreover, by (2.3), the symmetry and the spatial homogeneity of the process \((X_t)_{t \geq 0}\), we get for every nonnegative function \( h \) supported in \( B_{7/4}(y) \)

\[
\int_{B_2(y)} h(z) p_{B_2(y)}(T_b, z, y) \, dz = \int_{B_2(y)} h(z) p_{B_2(y)}(T_b, y, z) \, dz
\]

\[
= \mathbb{E}^y [h(X_{T_b}); \, T_b < \tau_{B_2(y)}] \]

\[
= \mathbb{E}^0 [h(X_{T_b} - y); \, T_b < \tau_{B_2(0)}] \]

\[
= \int_{B_2(0)} h(z - y) p_{B_2(0)}(T_b, 0, z) \, dz
\]

\[
\geq \inf_{w \in B_{7/4}(0)} p_{B_2(0)}(T_b, 0, w) \int_{B_2(y)} h(z) \, dz.
\]

Inserting for \( h \) a sequence of type delta centered at \( y \), and using the continuity (in the variable \( z \)) of the kernel of the killed process, we get

\[
(4.6) \quad p_{B_2(y)}(T_b, z, y) \geq \inf_{w \in B_{7/4}(0)} p_{B_2(0)}(T_b, 0, w).
\]

By Lemma (2.3a), (4.5), (A3a,b) and (4.6), we have

\[
w_{T_b}(z, y) = \lim_{s \uparrow T_b} \mathbb{E}^z \left[ e^{-\int_0^s V(X_u) \, du} \right] p_{T_b - s}(y - X_s) \]

\[
\geq \liminf_{s \uparrow T_b} \mathbb{E}^z \left[ e^{-\int_0^s V(X_u) \, du} \right] p_{T_b - s}(y - X_s); \ s < \tau_{B_2(y)} \]

\[
\geq e^{-C_0 T_b g(|y| + 2)} \sup_{z \in B_2(y)} V(z) \liminf_{s \uparrow T_b} \mathbb{E}^z \left[ p_{B_2(y)}(T_b - s, X_s, y); \ s < \tau_{B_2(y)} \right]
\]

\[
\geq e^{-C_0 T_b g(|y| + 2)} \inf_{w \in B_{7/4}(0)} p_{B_2(0)}(T_b, 0, w),
\]

and returning to (4.4), we conclude that

\[
u_{t}(x, y) \geq c_2 e^{-\lambda_0(\frac{t}{2})} e^{-C_0 t g(|y| + 2)}.
\]

Set \( \tilde{C} := \sup_{z \in B(0,n_0+6)} \mathcal{V}_z(z) \) and observe that if \( |x| \leq n_0 + 2 \) or \( |y| \leq n_0 + 2 \) — we still assume \( |x - y| < \frac{5}{4} \) —, we get with a similar argument

\[
u_{t}(x, y) \geq c_1 e^{-\lambda_0(\frac{t}{2}) - (t - T_b) \tilde{C}} \inf_{z \in B_{1/2}(x)} w_{T_b}(z, y)
\]

and

\[
\inf_{z \in B_{1/2}(x)} w_{T_b}(z, y) \geq c_2 e^{-\tilde{C} T_b}.
\]

Together with the symmetry of the kernel \( u_{t}(x, y) \), this gives

\[
u_{t}(x, y) \geq c_3 e^{-\lambda_0(\frac{t}{2})} \begin{cases} e^{-C_0 t g(|x| + |y| + 2)} , & \text{if } |x| > n_0 + 2 \text{ and } |y| > n_0 + 2, \\ e^{-\tilde{C} t} , & \text{if } |x| \leq n_0 + 2 \text{ or } |y| \leq n_0 + 2, \end{cases}
\]

as long as \( |x - y| < \frac{5}{4} \) and \( t > T_b \).
Case 2: Assume that \( t > 2t_b, x, y \in \mathbb{R}^d \) satisfy \( |x - y| \geq \frac{5}{4} \). A further application of (the analogue of) the Chapman–Kolmogorov equations for the kernel \( u_t \) and the strong Markov property yields

\[
\begin{align*}
\lim_{t \to \infty} u_t(x, y) & \geq \mathbb{E}^x \left[ e^{-\int_0^{T-b} V(X_s) \, ds} u_{T-b}(X_{T-b}, y) \right] \\
& = \mathbb{E}^x \left[ e^{-\int_0^{T-b} V(X_s) \, ds} u_{T-b}(X_{T-b}, y) \right] = \mathbb{E}^x \left[ e^{-\int_0^{T-b} V(X_s) \, ds} u_{T-b} \right].
\end{align*}
\]

By the Ikeda-Watanabe formula \( (2.5) \), the last expectation is greater than or equal to

\[
\int_0^{t-T_b} \int_{B_{1/2}(x)} e^{-s \sup_{t \leq \xi \leq \inf \xi \in B_{1/2}(x)} V(\xi)} P_{B_{1/2}(x)}(s, x, z) \left( \int_{|w| < 1/2 |z - w|} \nu(z - w) \, dw \right) \, dz \, ds.
\]

and, because of \( (A1) \) and \( |x - y| \geq \frac{5}{4} \), this expression can be estimated from below by

\[
c_d f(|x - y|) \int_0^{t-T_b} e^{-s \sup_{t \leq \xi \leq \inf \xi \in B_{1/2}(x)} V(\xi)} \mathbb{E}^0 \left( s < \tau_{B_{1/2}(0)} \right) \left( \int_{|w| < 1/2 |z - w|} u_{s-t}(w, y) \, dw \right) \, ds.
\]

Suppose first that \( |x|, |y| > n_0 + 2 \). The restriction \( |w| \geq |y| \) in the domain of integration above guarantees that \( |w| > n_0 + 2 \). By \( (A3 a, b), (4.3), (4.7) \) and the fact that \( t - T_b \geq \frac{1}{2} t_b \), we finally get

\[
\begin{align*}
\lim_{t \to \infty} u_t(x, y) & \geq c_5 e^{-\mu_0(\frac{1}{2}) f(|x - y|)} \left( \int_0^{t-T_b} e^{-C_6 s g(|x| + 1/2) - s \mu_0(\frac{1}{2})} \, ds \right) \, e^{-C_6 t |y| + 1/2} f_1(|x - y|).
\end{align*}
\]

For the proof of the first inequality above we use the fact that \( \{ w : |w| < 1/2, |w| \geq |y| \} \geq \frac{1}{2} [B_{1/2}(y)] = \frac{1}{2} \mathbb{P}_{B_{1/2}(0)} \).

If \( |x| > n_0 + 2 \) and \( |y| \leq n_0 + 2 \), a similar reasoning shows

\[
\begin{align*}
\lim_{t \to \infty} u_t(x, y) & \geq c_5 f(|x - y|) \left( \int_0^{t-T_b} e^{-C_6 s g(|x| + 1/2) - s \mu_0(\frac{1}{2})} \, ds \right) \, e^{-\mu_0(\frac{1}{2}) e^{-C t}} f_1(|x - y|).
\end{align*}
\]

Also for \( |x|, |y| \leq n_0 + 2 \), we obtain

\[
\begin{align*}
\lim_{t \to \infty} u_t(x, y) & \geq c_5 f(|x - y|) \left( \int_0^{t-T_b} e^{-C s - s \mu_0(\frac{1}{2})} \, ds \right) \, e^{-\mu_0(\frac{1}{2}) e^{-C t}} f_1(|x - y|).
\end{align*}
\]

Because of the symmetry of \( u_t(x, y) \), this gives the required bound for \( |x - y| \geq \frac{5}{4} \).

**Lemma 4.4.** Assume \( (A1 a, b, c), (A2) \) and \( (A3 a, b) \) with \( t_b > 0 \) and \( R_0 > 0 \). For every \( n_0 \geq R_0 \) there exist constants \( C, \bar{C} > 0 \) such that for any \( t > 4t_b \), the following estimates hold.

a) If \( |x|, |y| \leq n_0 + 2 \), then

\[
\lim_{t \to \infty} u_t(x, y) \geq C e^{-t \mu_0(\frac{1}{2}) + \bar{C}}.
\]

b) If \( |x| > n_0 + 2 \) and \( |y| \leq n_0 + 2 \), then

\[
\lim_{t \to \infty} u_t(x, y) \geq C e^{-t \mu_0(\frac{1}{2}) + \bar{C}} \frac{f(|x|)}{g(|x| + \frac{1}{2})},
\]

if \( |x| \leq n_0 + 2 \) and \( |y| > n_0 + 2 \), then, by symmetry,

\[
\lim_{t \to \infty} u_t(x, y) \geq C e^{-t \mu_0(\frac{1}{2}) + \bar{C}} \frac{f(|y|)}{g(|y| + \frac{1}{2})}.
\]
c) If \(|x|, |y| > n_0 + 2\), then
\[
u_t(x, y) \geq C \frac{e^{-t\mu_0(\frac{1}{2})}}{g(|y| + \frac{1}{2})g(|x| + \frac{1}{2})} \int_{|z| > 1} f(|x - z|)f(|z - y|)e^{-C_0g(|z| + 2)} dz.
\]

Proof. Set \(\tilde{C} := \sup_{z \in B(0, n_0 + 6)} V_z(z)\) and let \(t > 4t_b\). The estimates in [a] and [b] have already been established in Lemma 4.3. The remaining assertion [c] can be shown by the same method which was used in the second part of the proof of Lemma 4.3. By (the analogue of) the Chapman–Kolmogorov equations, the strong Markov properties, and the Ikeda-Watanabe formula we get
\[
u_t(x, y) \geq \mathbb{E}^x \left[ e^{-t\int_{B_1/2} V(X_s) ds} u_{2t_b}(X_{t-2t_b}, y); t - 2t_b > \tau_{B_1/2}(x) \right]
\]
\[= \mathbb{E}^x \left[ e^{-t\int_{B_1/2} V(X_s) ds} u_{t-\tau_{B_1/2}(x)}(X_{\tau_{B_1/2}(x)}, y); t - 2t_b > \tau_{B_1/2}(x) \right]
\]
\[\geq \int_0^{t-2t_b} \int_{B_1/2} e^{-s\sup_{z \in B_1/2} V(z)} p_{B_1/2}(s, x, z) \left( \int_{|w| > 1} \nu(z - w) u_{t-s}(w, y) dw \right) dz ds.
\]
Since \(z \in B_{1/2}(x)\) and \(|x - w| > 1\), we can use (A3a,b) to see \(\nu(z - w) \leq cf(|x - w|)\); this means that we can estimate the previous expression by
\[c_1 \int_0^{t-2t_b} e^{-s\sup_{z \in B_1/2} V(z)} \mathbb{P}^0(s < \tau_{B_1/2}(0)) \int_{|w| > 1} f(|x - w|) u_{t-s}(w, y) dw ds.
\]
If \(|x|, |y| > n_0 + 2\), we use (A3a,b) to see that the last expression is greater than or equal to
\[c_1 \int_0^{t-2t_b} e^{-C_0g(|x| + 1/2)} \mathbb{P}^0(s < \tau_{B_1/2}(0)) \int_{|w| > 1} f(|x - w|) u_{t-s}(w, y) dw ds.
\]

From Lemma 4.3 we know
\[u_{t-s}(w, y) \geq c_2 e^{-(t-s)\mu_0(\frac{1}{2})} \frac{e^{-C_0g(|w| + 2)}}{g(|y| + \frac{1}{2})} f(|y - w|), \quad |y - w| > 1, \quad t - s > 2t_b.
\]
Together with (4.3) this finally gives
\[
u_t(x, y) \geq c_3 \frac{e^{-t\mu_0(\frac{1}{2})}}{g(|y| + \frac{1}{2})} \left( \int_0^{t-2t_b} e^{-C_0g(|x| + \frac{1}{2})} ds \right) \int_{|w| > 1} f(|x - w|) f(|w - y|) e^{-C_0g(|w| + 2)} dw
\]
\[\geq c_4 \frac{e^{-t\mu_0(\frac{1}{2})}}{g(|x| + \frac{1}{2})g(|y| + \frac{1}{2})} \int_{|w| > 1} f(|x - w|) f(|w - y|) e^{-C_0g(|w| + 2)} dw.
\]
This completes the proof of [c]. \(\square\)

4.3 Sharp general two-sided estimates. We are now going to show that the estimates from the two previous sections are sharp in the spatial variable if we assume (A3c) in addition to (A3a,b). These estimates lead to a considerable improvement in \(t\), too. Recall that \(\lambda_0 := \inf \text{spec}(H)\) and \(\varphi_0 \in L^2(\mathbb{R}^d)\) denote the ground state eigenvalue and eigenfunction. Under (A3) it follows from [25] Cor. 2.2 that for every \(R > 0\) there exist constants \(C, C' > 0\) such that
\[C f(|z|) \leq \varphi_0(z) \leq C' f(|z|), \quad |z| \geq R.
\]
Since we always deal with a strictly positive and continuous version of \(\varphi_0\), there are (possibly different) constants \(C, C' > 0\) such that
\[C \leq \varphi_0(z) \leq C', \quad |z| \leq R.
\]
The following lemma will be used to improve the last term of the estimate in Lemma 4.2(c).

**Lemma 4.5.** Assume \([A1], A3\) with \(t_h > 0\) and \(R_0 > 0\). Let \(n_0 \in \mathbb{N}\) be as in Lemma 4.4. If \(\lambda_0 > 0\), then we require, additionally, that \(n_0\) is so large that

\[
g(n_0 - 2) \geq 2C_6\lambda_0. \tag{4.10}
\]

If there is a constant \(C > 0\) such that

\[
u_t(x, y) \leq Ce^{-\lambda_0 t}f_1(|y|), \quad t > 5t_h, \ |x| < n_0 + 2, \ y \in \mathbb{R}^d, \tag{4.11}
\]

then there exists a constant \(\tilde{C} > 0\) such that for every \(t > 10t_h\) and all \(|x|, |y| > n_0 + 3\) we have

\[
u_t(x, y) \leq \tilde{C} e^{(C_5 + C_2) t} e^{-\frac{t}{2C_6} g(|x|)} f_1(|x - y|) \tag{4.12}
\]

\[
+ \tilde{C} e^{(C_5 + C_2) t} \int_{n_0 + 2 < |x| < |y| - 1} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t}{2C_6} g(|z| - 1)} dz.
\]

**Proof.** First we prove that there exists a constant \(\tilde{C} > 0\) such that for every \(|x| > n_0 + 3\), \(y \in \mathbb{R}^d\) and \(t > 5t_h\) we have

\[
u_t(x, y) \leq \tilde{C} e^{(C_5 + C_2) t} e^{-\frac{t}{2C_6} g(|x|)} f_1(|x - y|) \tag{4.12}
\]

\[
+ \int_{n_0 + 2 < |x| < |y| - 1} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t}{2C_6} g(|z| - 1)} dz.
\]

This can be shown with the argument used in Lemma 4.1. In fact, only the estimates of the last two terms \(I_{k,t}\) for \(k = n_0, n_0 + 1\) in the proof of that lemma require a modification. From now on, let \(k = n_0\) or \(k = n_0 + 1\). Recall from (4.1) that

\[
I_{k,t} = \mathbb{E}^x \left[ e^{-\int_0^{t_h - 1} V(X_s) ds} u_{t - \sigma_{k - 1}}(X_{\sigma_{k - 1}}); S(A_{n-2}^{\infty}, A_{k-1}^{\infty}, l, t - 5t_h) \right].
\]

Since \(\sigma_{k - 1} < t - 5t_h\) on the set \(S(A_{n-2}^{\infty}, A_{k-1}^{\infty}, l, t - 5t_h)\) and \(X_{\sigma_{k - 1}} \in B_{n_0 + 2}(0)\), we have by (4.11) that

\[
u_t(x, y) \leq Ce^{\lambda_0 \sigma_{k - 1}} e^{-\lambda_0 t} f_1(|y|); \tag{4.11}
\]

Consequently,

\[
I_{k,t} \leq Ce^{-\lambda_0 t}f_1(|y|) \mathbb{E}^x \left[ e^{-\int_0^{t_h - 1} V(X_s) - \lambda_0 ds} S(A_{n-2}^{\infty}, A_{k-1}^{\infty}, l, t - 5t_h) \right].
\]

The condition (4.10) ensures that the shifted potential \(\tilde{V} := V - \lambda_0\) appearing above also satisfies (A3a,b) with the radius \(\tilde{R}_0 = n_0\), the profile \(\tilde{g}\) such that \(g|_{[0, \tilde{R}_0]} \equiv 1\), \(g|_{\tilde{R}_0, \infty} = g\), and the constant \(\tilde{C}_6 := 2C_6\). Applying Lemma 3.5 with \(W \equiv 1\) to the latter expectation, finally gives

\[
I_{k,i} \leq c_1 e^{-\lambda_0 t} f_1(|y|) \int_{|z| \leq n_0 + 1} f(|x - z|) dz.
\]

With (A1c) we conclude that

\[
I_{k,l} \leq c_2 e^{-\lambda_0 t} f(|x|) f_1(|y|)
\]

for \(k = n_0, n_0 + 1\) and any \(l \in \mathbb{N}\). This gives the required last term of the estimate (4.12).
In order to get the bound in a symmetric form for every $|x|, |y| > n_0 + 3$ and $t > 10t_b$, it is now enough to write $u_t(x, y) = \int_{R^d} u_{t-5t_b}(x, z)u_{5t_b}(y, z) \, dz$ and to repeat the symmetrization argument from the proof of Lemma 4.2.a). Here we use (4.12) to estimate $u_{t-5t_b}(x, z)$. The uniform growth condition (A3.c) is used to replace $1/(g(|x| - 2)g(|y| - 2))$ with $1/(g(|x|)g(|y|))$. □

We are now in a position to prove the main result of this section. Recall that $\mu_0(r) > 0$ and $\psi_{0,r}$ are the ground state eigenvalue and eigenfunction for the process killed on leaving a ball $B_r(0), r > 0$.

**Theorem 4.6** (Sharp two-sided bounds). *Let $H = -L + V$ be the Schrödinger operator with confining potential $V$ such that [A1] - [A3] hold with $t_b > 0$ and $R_0 > 0$. Denote by $\lambda_0$ and $\varphi_0$ the ground-state eigenvalue and eigenfunction, and by $u_t(x, y)$ the density of the semigroup $U_t = e^{-tH}$. Let $n_0 \in \mathbb{N}$ be as in Lemma 4.1 and 4.5 and so large that*

\[
(4.13) \quad g(n_0 - 2) \geq 12(C_5 + C_6 C_7 C_8^2) + \frac{\mu_0(\frac{1}{2})}{C_6 C_7^2}.
\]

*There exists a constant $C \geq 1$ such that for every $t > 30t_b$ we have the following estimates.*

a) If $|x|, |y| \leq n_0 + 3$, then

\[
\frac{1}{C} e^{-\lambda_0 t} \leq u_t(x, y) \leq C e^{-\lambda_0 t}.
\]

b) If $|x| > n_0 + 3$ and $|y| \leq n_0 + 3$, then

\[
\frac{1}{C} e^{-\lambda_0 t} \frac{f(|x|)}{g(|x|)} \leq u_t(x, y) \leq C e^{-\lambda_0 t} \frac{f(|x|)}{g(|x|)}.
\]

By symmetry, if $|x| \leq n_0 + 3$ and $|y| > n_0 + 3$,

\[
\frac{1}{C} e^{-\lambda_0 t} \frac{f(|y|)}{g(|y|)} \leq u_t(x, y) \leq C e^{-\lambda_0 t} \frac{f(|y|)}{g(|y|)}.
\]

c) If $|x|, |y| > n_0 + 3$, then

\[
\frac{1}{C} \frac{F(K_t, x, y) \vee e^{-\lambda_0 t} f(|x|) f(|y|)}{g(|x|)g(|y|)} \leq u_t(x, y) \leq C \frac{F(K_t, x, y) \vee e^{-\lambda_0 t} f(|x|) f(|y|)}{g(|x|)g(|y|)},
\]

where $K = 4C_6 C_7^2$ and

\[
(4.14) \quad F(\tau, x, y) := \int_{n_0 + 2 < |z| < |x| \vee |y|} f_1(|x - z|) f_1(|z - y|) e^{-\tau g(|z|)} \, dz.
\]

**Remark 4.7.** Since $x \mapsto f(x)/g(x)$ is bounded from above and below (away from 0) on compact intervals, it is possible to combine [a] and [b] in a single estimate:

\[
u_t(x, y) \asymp e^{-\lambda_0 t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \left( 1 \wedge \frac{f(|y|)}{g(|y|)} \right) \quad \text{for all} \quad |x| \wedge |y| \leq n_0 + 3.
\]

Moreover, due to (4.8), (4.9), the above two-sided estimates are equivalent to

\[
u_t(x, y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad \text{for all} \quad |x| \wedge |y| \leq n_0 + 3, \quad t > 30t_b.
\]

**Proof of Theorem 4.6.** a) Fix $t > 5t_b$ and let $|x|, |y| \leq n_0 + 3$. It follows from (4.8), (4.9), (A3.c) and the estimates in Lemma 4.2.a) applied to $u_{5t_b}(y, z)$ that

\[
u_t(x, y) = \int_{|z| < n_0 + 3} \nu_{t-5t_b}(x, z) \nu_{5t_b}(y, z) \, dz + \int_{|z| > n_0 + 3} \nu_{t-5t_b}(x, z) \nu_{5t_b}(y, z) \, dz
\]

\[
\leq \sup_{z, w \in B(0, n_0 + 3)} \nu_{5t_b}(z, w) \int_{|z| < n_0 + 3} \nu_{t-5t_b}(x, z) \nu_{5t_b}(y, z) \, dz + c_1 \int_{|z| > n_0 + 3} \nu_{t-5t_b}(x, z) \varphi_0(z) \, dz
\]

\[
\leq c_3 \int_{R^d} \nu_{t-5t_b}(x, z) \varphi_0(z) \, dz
\]

\[
\leq c_4 e^{-\lambda_0 t} \varphi_0(x).
\]
Since $|x| \leq n_0 + 3$, a further application of (4.9) gives the upper estimate. The lower estimate follows from similar arguments based on the lower estimates in Lemma 4.4(a,b) and the two-sided bounds (4.8), (4.9).

[3] Fix $t > 5t_b$. By symmetry, it is enough to assume that $|x| > n_0 + 3$ and $|y| \leq n_0 + 3$. Exactly the same argument as in part [2] shows

$$c_5 e^{-\lambda_0 t} \varphi_0(x) \leq u_t(x, y) \leq c_6 e^{-\lambda_0 t} \varphi_0(x).$$

Since now $|x| > n_0 + 3$, the claimed bound follows immediately from (4.8), (4.9).

[4] Let $|x|, |y| > n_0 + 3$ and $t > 30t_b$. We begin with the upper bound. From the already established parts [a] and [b] and (A3.b) we have

$$u_t(x, y) \leq c_7 e^{-\lambda_0 t} f_1(|y|), \quad |x| < n_0 + 3, \quad y \in \mathbb{R}^d, \quad t > 5t_b,$$

which is exactly (4.11). Thus, we can use Lemma 4.5 and get

$$u_t(x, y) \leq c_8 e^{(C_5 + C_{2b} t) t} e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|x|)} f_1(|x - y|)$$

$$+ c_8 e^{(C_5 + C_{2b} t) t} e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|y|)} \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} dz$$

$$+ c_8 e^{-\lambda_0 t} f_1(|x|) f_1(|y|) g(|x|) g(|y|).$$

Without loss of generality we may assume that $|y| \leq |x|$; the case $|y| > |x|$ follows from the fact that $x$ and $y$ play symmetric roles; set $y_0 := (|y| - 1/2)|y|^{-1} y$. We have

$$\int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} dz$$

$$\geq \int_{B_{1/2}(y_0)} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} dz$$

$$\geq c_9 e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|y|)} f_1(|x - y|)$$

which gives the estimate

$$\int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} dz \geq c_9 e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z| \wedge |y|)} f_1(|x - y|)$$

$$\geq c_9 e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} f_1(|x - y|).$$

From this we get at once

$$u_t(x, y) \leq c_{10} e^{(C_5 + C_{2b} t) t} e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f_1(|z - y|) e^{-\frac{t - 10t_b}{4C_6 C_7^2} g(|z|)} dz$$

$$+ c_8 e^{-\lambda_0 t} f_1(|x|) f_1(|y|) g(|x|) g(|y|).$$

In order to complete the proof of the upper bound, it suffices to observe that for every $|z| \geq n_0 + 2$ and $t > 30t_b$ we have

$$(C_5 + C_{2b} t) t - \frac{t - 10t_b}{2C_6 C_7^2} g(|z|) \leq (C_5 + C_{2b} t) t - \frac{t}{3C_6 C_7^2} g(|z|) + \frac{t}{4C_6 C_7^2} g(|z|) - \frac{t}{4C_6 C_7^2} g(|z|)$$

$$= \left(C_5 + C_{2b} t - \frac{1}{12C_6 C_7^2} g(|z|)\right) t - \frac{t}{4C_6 C_7^2} g(|z|)$$

$$\leq -\frac{t}{4C_6 C_7^2} g(|z|);$$

The last inequality requires (4.13). This completes the proof of the upper estimate.
Now we turn to the lower estimate. Let $|x|, |y| > n_0 + 3$ and $t > 30b$. Recall that we have by Lemma 4.4 c) and assumption (A3 c)

$$u_t(x, y) \geq \frac{c_{11}}{g(|x|)g(|y|)} \int_{|x-z| > 1, |y-z| > 1} f(|x-z|)f(|y-z|) e^{-2C_6C^2_7g(|z|)} e^{(C_6C^2_7g(|z|)-\mu_0(\frac{1}{2}))t} \, dz.$$ 

From (4.13) we see that $C_6C^2_7g(|z|) - \mu_0(\frac{1}{2}) \geq C_6C^2_7g(n_0) - \mu_0(\frac{1}{2}) \geq 0$ for $|z| \geq n_0$, and so

$$u_t(x, y) \geq \frac{c_{11}}{g(|x|)g(|y|)} \int_{|x-z| > 1, |y-z| > 1} f(|x-z|)f(|y-z|) e^{-2C_6C^2_7g(|z|)} \, dz.$$ 

As before, we may assume that $|x| \leq |y|$. (4.15), (A3 b), (3.5) and (3.1) yield

$$u_t(x, y) \geq \frac{c_{11}}{g(|x|)g(|y|)} \int_{n_0 \leq |z| \leq |x|-1, |y-z| > 1} f(|x-z|)f(|y-z|) e^{-2C_6C^2_7g(|z|)} \, dz$$

$$\geq \frac{c_{11}}{g(|x|)g(|y|)} e^{-2C_6C^2_7g(|x|-1)} \int_{n_0 \leq |z| \leq |x|-1, |y-z| > 1} f_1(|x-z|)f_1(|y-z|) \, dz$$

$$\geq \frac{c_{12}c_{13}}{g(|x|)g(|y|)} e^{-2C_6C^2_7g(|x|-1)} f_1(|x-y|),$$

where $c_{13} := \inf_{|x|, |y| \geq n_0 + 3} \int_{2|x-z| > 1, |y-z| > 1} f_1(|x-z|) |dz| > 0$. Since, by (3.2),

$$f_1(|x-y|) \geq c_{14} \int_{n_0 + 2 \leq |z| \leq |x| \wedge |y|, |w-z| \leq 1} f_1(|x-z|)f_1(|y-z|) \, dz, \quad w \in \mathbb{R}^d,$$

we finally get from (4.16) and the monotonicity of $g$ that for $w = x$ or $w = y$

$$u_t(x, y) \geq \frac{c_{15}}{g(|x|)g(|y|)} \int_{n_0 + 2 \leq |z| \leq |x| \wedge |y|, |w-z| \leq 1} f_1(|x-z|)f_1(|y-z|) e^{-2C_6C^2_7g(|z|)} \, dz.$$ 

Together with the estimate

$$u_t(x, y) \geq \frac{c_{11}}{g(|x|)g(|y|)} \int_{n_0 + 2 \leq |z| \leq |x| \wedge |y|, |x-z| > 1, |y-z| > 1} f_1(|x-z|)f_1(|y-z|) e^{-2C_6C^2_7g(|z|)} \, dz$$

which comes from (4.15), we obtain the bound

$$u_t(x, y) \geq \frac{c_{16}}{g(|x|)g(|y|)} F(\frac{1}{2}Kt, x, y) \geq \frac{c_{16}}{g(|x|)g(|y|)} F(Kt, x, y).$$

It is now enough to show that

$$u_t(x, y) \geq c_{17}e^{-\lambda_0 t} \frac{f(|x|)f(|y|)}{g(|x|)g(|y|)}.$$ 

From Lemma 4.4 c) (A1 c) and (A3 c), we get for $x_0 := (R_0 + 3, 0, ..., 0)$

$$u_{5t_0}(w, y) \geq \frac{c_{18}}{g(|w|)g(|y|)} \int_{|x|-z| \leq 1} f(|w-z|)f(|z-y|) \, dz \geq \frac{c_{19}}{g(|w|)g(|y|)} f(|w|)f(|y|), \quad |w| > n_0 + 3.$$ 

In the following calculation we estimate the first half of the integral using the lower estimate in Lemma 4.4 b) — in this Lemma $n_0$ was arbitrary, so we may increase it to $n_0 + 1$ — (A3 c) and
Proof. Since the kernel $F$ provides the following estimates.

\[ u_t(x, y) \geq \int_{|w|\leq n_0 + 3} u_{t-5t_b}(x, w)u_{5t_b}(w, y) dw + \int_{|w|>n_0 + 3} u_{t-5t_b}(x, w)u_{5t_b}(w, y) dw \]
\[ \geq c_{20} \frac{f(|y|)}{g(|y|)} \int_{|w|\leq n_0 + 3} u_{t-5t_b}(x, w)\varphi_0(w) dw + c_{21} \frac{f(|y|)}{g(|y|)} \int_{|w|>n_0 + 3} u_{t-5t_b}(x, w)\varphi_0(w) dw \]
\[ \geq c_{22} \frac{f(|y|)}{g(|y|)} \int_{\mathbb{R}^d} u_{t-5t_b}(x, w)\varphi_0(w) dw \]
\[ = c_{22} \frac{f(|y|)}{g(|y|)} U_{t-5t_b}\varphi_0(x) \]
\[ \geq c_{24} e^{-\lambda_0 t} \left( \frac{f(|x|)}{g(|x|)} \right) \frac{f(|y|)}{g(|y|)}. \]

In the last inequality we use (4.8) once again. This completes the proof of the lower bound in $\mathbb{R}$.

\[ \square \]

4.4. Sharp two-sided estimates of $U_t \mathcal{I}(x)$. In this section we apply Theorem 4.6 to obtain two-sided large-time estimates for the functions $U_t \mathcal{I}(x)$. Recall that $\{U_t : t \geq 0\}$ is the Schrödinger semigroup with kernel $u_t(x, y)$.

**Theorem 4.8.** Let $H = -L + V$ be the Schrödinger operator with confining potential $V$ such that (A1)–(A3) hold with $t_0 > 0$ and $R_0 > 0$. Denote by $\lambda_0$ and $\varphi_0$ the ground-state eigenvalue and eigenfunction, and by $u_t(x, y)$ the density of the operator $U_t = e^{-iH}\mathcal{I}$. For $n_0 \in \mathbb{N}$ large enough (as in Theorem 4.6), there exists a constant $C \geq 1$ such that for every $t > 30t_b$ we have the following estimates.

a) If $|x| \leq n_0 + 3$, then
\[ \frac{1}{C} e^{-\lambda_0 t} \leq U_t \mathcal{I}(x) \leq Ce^{-\lambda_0 t}. \]

b) If $|x| > n_0 + 3$, then
\[ \frac{1}{C} \frac{G(Kt, x)}{g(|x|)} \leq U_t \mathcal{I}(x) \leq C \frac{G(K^{-1}t, x)}{g(|x|)}, \]
where $K = 4C_0C_2^2$ and
\[ (4.20) \quad G(r, x) := \int_{n_0 + 2 < |z| \leq |x|} f_1(|x - z|)e^{-r\theta(|z|)} dz. \]

**Proof.** Since $U_t \mathcal{I}(x) = \int_{\mathbb{R}^d} u_t(x, y) dy$, $x \in \mathbb{R}^d$, $t > 0$, all estimates follow from the estimates of the kernel $u_t(x, y)$, cf. Theorem 4.6. Recall that the lower bound of Theorem 4.6 actually holds with $F(\frac{1}{2}Kt, x, y)$, see (4.17). We will use this fact in the following calculation. If $|x| > n_0 + 3$, the key step is to observe that by Tonelli’s theorem
\[ \int_{|y|>n_0+3} \frac{F(\frac{1}{2}Kt, x, y)}{g(|y|)} dy = \int_{|y|>n_0+3} \int_{n_0 + 2 < |z| < |y|} f_1(|x - z|) \frac{f_1(|z - y|)}{g(|y|)} e^{-\frac{1}{2}Kt\theta(|z|)} dz dy \]
\[ \geq \int_{|z|>n_0+2} \int_{|y|>(n_0+3)\vee|z|} f_1(|x - z|) \frac{f_1(|z - y|)}{g(|y|)} e^{-\frac{1}{2}Kt\theta(|z|)} dy dz \]
\[ \geq c_1 \int_{|z|>n_0+2} \int_{|y|<(n_0+3)\vee|z|} f_1(|x - z|) \frac{e^{-\frac{1}{2}Kt\theta(|z|)}}{g(|z|)} dy dz \]
\[ \geq c_2 \int_{|z|>n_0+2} f_1(|x - z|) e^{-Kt\theta(|z|)} dz \]
\[ \geq c_2 G(Kt, x). \]
Similarly, \((4.14)\), the monotonicity of \(g\) and one more use of Tonelli’s theorem, imply
\[
\int_{|y|>n_0+3} \frac{F(K^{-1}t,x,y)}{g(|y|)} \, dy \leq \frac{\|f_1\|_1}{g(n_0+3)} \int_{|z|>n_0+2} f_1(|x-z|)e^{-\frac{1}{5}g(|z|)} \, dz \leq c_3 G(K^{-1}t,x).
\]
The last estimate is a consequence of the fact that for \(|x| > n_0 + 3\) we have
\[
\int_{|z|>n_0+2} f_1(|x-z|)e^{-\frac{1}{5}g(|z|)} \, dz = G(K^{-1}t,x) + \int_{|z|>|x|} f_1(|x-z|)e^{-\frac{1}{5}g(|z|)} \, dz
\]
and, by the monotonicity of \(g\),
\[
\int_{|z|>|x|} f_1(|x-z|)e^{-\frac{1}{5}g(|z|)} \, dz \leq c_4 e^{-\frac{1}{5}g(|z|)} \leq c_5 \int_{n_0+2<|z|<|x|} f_1(|x-z|)e^{-\frac{1}{5}g(|z|)} \, dz. \quad \square
\]

4.5. **Applications to asymptotic intrinsic ultracontractivity.** Under the assumption \((A3)\) some of the aUC results of [37, Corollary 3.3] (see also [35, Corollary 2.3 (2)]) can be recovered from our two-sided estimates of the kernels \(u_r(x,y)\). We continue to use the functions \(F(\tau,x,y)\) and \(G(\tau,x)\) introduced in \((4.14)\) and \((4.20)\).

**Lemma 4.9.** Let \(f : (0,\infty) \to (0,\infty)\) and \(g : [0,\infty) \to (0,\infty)\) be profile functions as in \((A1)\) and \((A3)\), and let \(n_0 \geq R_0 + 2\). Suppose that there exist \(C > 0\) and \(R_1 \geq R_0\) such that \(Cg(r) \geq |\log f(r)|, r \geq R_1\). We have the following estimates.

(a) There is a constant \(\tilde{C} \geq 1\) such that for every \(\tau \geq 3C\) and \(|x|,|y| \geq n_0 + 3\) we have
\[
\frac{1}{\tilde{C}} e^{-\tau g(n_0+3)} f(|x|) f(|y|) \leq F(\tau,x,y) \leq C e^{-\frac{1}{2}g(n_0+2)} f(|x|) f(|y|).
\]

(b) There is a constant \(\tilde{C} \geq 1\) such that for every \(\tau \geq 2C\) and \(|x| \geq n_0 + 3\) we have
\[
\frac{1}{\tilde{C}} e^{-\tau g(n_0+3)} f(|x|) \leq G(\tau,x) \leq \tilde{C} e^{-\frac{1}{2}g(n_0+2)} f(|x|).
\]

**Proof.** The lower bound follows from
\[
F(\tau,x,y) = \int_{n_0+2<|z|<|x|\lor|y|} f_1(|x-z|) f_1(|z-y|) e^{-\tau g(|z|)} \, dz \geq \int_{n_0+2<|z|<n_0+3} f_1(|x-z|) f_1(|z-y|) e^{-\tau g(|z|)} \, dz \\
\geq c_1 e^{-\tau g(n_0+3)} f(|x|) f(|y|).
\]

In a similar way we get
\[
G(\tau,x) = \int_{n_0+2<|z|<|x|} f_1(|x-z|) e^{-\tau g(|z|)} \, dz \geq c_2 e^{-\tau g(n_0+3)} f(|x|).
\]

Let us establish the upper bounds. We give only details for \(F(\tau,x,y)\), since \(G(\tau,x)\) can be dealt with in a similar fashion. We set
\[
F(\tau,x,y) = \left( \int_{n_0+2<|z|<|x|\lor|y|} + \int_{n_0+2<|z|<|x|\lor|y|} \right) f_1(|x-z|) f_1(|z-y|) e^{-\tau g(|z|)} \, dz
\]
and denote the two integrals by I and II. Clearly, \(I \leq c_3 e^{-\tau g(n_0+2)} f(|x|) f(|y|)\). By assumption, \(e^{-\frac{1}{2}g(r)} \leq f_1(r)\), for every \(\tau \geq 3C\) and \(r \geq R_1\). Hence,
\[
II \leq c_4 e^{-\frac{1}{2}g(n_0+2)} \int_{\mathbb{R}^d} f_1(|x-z|) f_1(|z|) f_1(|z-y|) f_1(|z|) \, dz.
\]
From \((3.3)\) and \((3.2)\) we easily get \(II \leq c_5 e^{-\frac{1}{2}g(n_0+2)} f(|x|) f(|y|)\). This completes the proof. \(\square\)
The next corollary contains equivalent conditions for the aIUC property of the semigroup \( \{U_t : t \geq 0\} \). Due to [37, Corollary 3.3] these are in fact also equivalent conditions for the intrinsic hypercontractivity. That means, in particular, that aIUC and intrinsic hypercontractivity coincide in this setting. Recall that every aIUC semigroup is automatically pIUC with the threshold function \( r \equiv \infty \), cf. Definition 1.2.

**Corollary 4.10.** Assume \( \{A1 - A3\} \) with \( t_0 > 0 \) and \( R_0 > 0 \). The following statements are equivalent.

a) There exist \( C > 0 \) and \( R_1 > 0 \) such that \( V(x) \geq C |\log v(x)| \), for \( |x| \geq R_1 \).

b) There exist \( \tilde{C} > 0 \) and \( \tilde{R}_1 > 0 \) such that \( g(r) \geq \tilde{C} |\log f(r)| \), for \( r \geq \tilde{R}_1 \).

c) There exists some \( t_0 > 0 \) such that

\[
U_t(x,y) \asymp e^{-\lambda_0 t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \left( 1 \wedge \frac{f(|y|)}{g(|y|)} \right), \quad x,y \in \mathbb{R}^d, \quad t \geq t_0,
\]

or, equivalently,

\[
u_t(x,y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad x,y \in \mathbb{R}^d, \quad t \geq t_0,
\]

i.e. the semigroup \( \{U_t : t \geq 0\} \) is asymptotically intrinsic ultracontractive (aIUC).

d) There exists some \( t_0 > 0 \) such that

\[
U_t 1(x) \asymp e^{-\lambda_0 t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right), \quad x \in \mathbb{R}^d, \quad t \geq t_0.
\]

or, equivalently,

\[
U_t 1(x) \asymp e^{-\lambda_0 t} \varphi_0(x), \quad x \in \mathbb{R}^d, \quad t \geq t_0.
\]

i.e., the semigroup \( \{U_t : t \geq 0\} \) is asymptotically ground state dominated.

More precisely, if [\textbf{b}] is true for some \( \tilde{C} > 0 \), then \textbf{c} and \textbf{d} hold with \( t_0 = 30t_0 + 3\tilde{C}K \).

**Proof.** The statements [\textbf{a}] and [\textbf{b}] are equivalent because of (A1.a) and (A3.a); [\textbf{b}] implies \textbf{c} because of the estimates in Theorem 4.6 (see also Remark 4.7 and Lemma 4.9) [\textbf{c}] implies \textbf{d} by integration; [\textbf{b}] follows from [\textbf{d}] with the estimates from Theorem 4.8. Indeed, with the upper bound in [\textbf{d}], the lower bound in Theorem 4.8 b) the definition (4.20) of \( G \) and (A3.b), we get for \( |x| \) large enough

\[
\frac{c_1 e^{-K_0 g(|x|)}}{g(|x|)} \leq \frac{c_2}{g(|x|)} G(K_{t_0}, x) \leq U_{t_0} 1(x) \leq c_3 e^{-\lambda_0 t_0} \frac{f(|x|)}{g(|x|)},
\]

which implies [\textbf{b}]. Alternatively, we can use the argument from the proof of [35, Theorem 2.6 (2)] to see that [\textbf{d}] gives [\textbf{b}]. \( \square \)

### 4.6. Applications to spectral functions.

A further application of our results is the study of spectral regularity of compact semigroups, e.g. the heat trace or the heat content. We are not aware of such results in the literature. Recall that \( U_t \) is said to be a/have a

- (TC) **trace class operator** if \( \int_{\mathbb{R}^d} u_t(x,x) \, dx < \infty \);

- (HS) **Hilbert-Schmidt operator** if \( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_t^2(x,y) \, dx \, dy < \infty \);

- (fHC) **finite heat content** if \( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_t(x,y) \, dx \, dy < \infty \).

By (the analogue of) the Chapman–Kolmogorov equations, if the integrals in (TC), (HS) and (fHC) are finite for some \( t_0 > 0 \), then they are finite for all \( t \geq t_0 \).

Using our bounds on \( u_t(x,y) \), we can give a necessary and sufficient condition for the spectral properties (TC), (HS) and (fHC). Recall that \( K = 4C_0 C^{2} \).
Corollary 4.11. Assume (A1)–(A3) with \( t_b > 0 \) and \( R_0 > 0 \). For large times the properties (TC), (HS) and (HC) coincide and they are equivalent to the condition

\[
\text{there exists } s > 0 \text{ such that } \int_{|x| > R_0} e^{-sV(x)} \, dx < \infty.
\]

More precisely, the following assertions hold.

\(\text{(4.21)}\)

\[\begin{align*}
a) & \text{ If } \text{(4.21)} \text{ is true with some } s > 0, \text{ then the integrals in } (\text{TC}), (\text{HS}) \text{ and } (\text{HC}) \text{ are finite for all } t \geq 30t_b + Ks. \\
b) & \text{ If the integral in } (\text{TC}) \text{ or } (\text{HC}) \text{ is finite for some } t > 0, \text{ then } \text{(4.21)} \text{ holds for all } s \geq K(30t_b + t). \\
c) & \text{ If the integral in } (\text{HS}) \text{ is finite for some } t > 0, \text{ then } \text{(4.21)} \text{ holds for all } s \geq 2K(30t_b + t).
\end{align*}\]

5. Applications to specific classes of nonlocal Schrödinger operators

5.1. Potentials as functions of \(|\log \nu|\). aIUC- and non-aIUC-regime. Up to now we have studied Schrödinger operators whose Lévy density \( \nu \) and potential \( V \) are controlled by profiles \( f \) and \( g \) which satisfy the assumptions (A1) and (A3) with some \( R_0 > 0 \). From now on we assume, in addition, (A4) which says that \( f \) satisfies \( f(R_0) < 1 \) and \( f \) and \( g \) are connected via

\[ g(r) = h(|\log f(r)|), \quad r \geq R_0; \]

\( h : [\log f(R_0)], \infty) \to (0, 0) \) is an increasing function such that \( h(s)/s \) is monotone (increasing or decreasing). We will always assume that monotone functions are right-continuous.

Remark 5.1. If (A4) holds, the Schrödinger operators with profiles \( f \) and \( g \) are divided into two distinct classes:

\(\text{a) (aIUC-regime): there exist } C \text{ and } R \geq R_0 \text{ such that } C g(r) \geq |\log f(r)|, \quad r \geq R.\)

This is equivalent to the property that \( h(s)/s \) is, outside some compact set, bounded below by a strictly positive constant. Since \( h(s)/s \) is monotone this is the same as to say that \( \lim_{s \to \infty} h(s)/s \in (0, \infty) \). If necessary, we may increase the constant \( C \) to ensure that \( C g(r) \geq |\log f(r)|, \quad r \geq R_0; \) this is equivalent to \( e^{-\gamma g(r)} \leq f(r) \), for every \( r \geq R_0 \) and \( \tau \geq \tau_0 := C \).

\(\text{b) (non-aIUC-regime): } \lim_{r \to \infty} g(r)/|\log f(r)| = 0.\)

This is equivalent to \( \lim_{s \to \infty} h(s)/s = 0 \). Since \( h(s)/s \) is monotone, the limit is actually an infimum. This class will be discussed in Lemma 5.2 below.

We will use Remark 5.1 as definition of the aIUC- and non-aIUC-regimes. Since \( h(s)/s \) is a monotone function, the two cases in Remark 5.1 are indeed complementary and exhaustive classes.

The following fact will be crucial for our further investigations. It explains the relation between the profile functions \( f \) and \( g \) in the non-aIUC-regime.

Lemma 5.2. Let \( f : (0, \infty) \to (0, \infty) \) be a decreasing function such that \( \lim_{r \to \infty} f(r) = 0 \), pick \( R_0 > 0 \) such that \( f(R_0) < 1 \) and let \( g(r) = h(|\log f(r)|), \quad r \geq R_0, \) with an increasing function \( h : [\log f(R_0)], \infty) \to (0, 0) \) such that \( h(s)/s \) is decreasing and \( \lim_{s \to \infty} h(s)/s = 0 \). Define

\[
\Lambda(r) := \frac{|\log f(r)|}{h(|\log f(r)|)}, \quad r \geq R_0,
\]

and its right-continuous generalized inverse

\[
\rho(\tau) := \Lambda^{-1}(\tau) := \inf \{ r \geq R_0 : \Lambda(r) > \tau \}, \quad \tau \geq \Lambda(R_0).
\]
Lemma 5.3. Let $G$ and $\tau, x, y > 0$. Then, for every $\tau > \Lambda(R_0)$,
\[ e^{-\tau g(r)} \leq f(r), \quad r \in [R_0, \rho(\tau)), \]
resp.,
\[ e^{-\tau g(r)} \geq f(r), \quad r \in [\rho(\tau), \infty). \]
Moreover, the function $r \mapsto e^{-\tau g(r)/f(r)}$ is increasing on $[\rho(\tau), \infty)$.

Proof. Since $h(s)/s$ decreases to zero as $s \to \infty$ and $|\log f(r)|$ increases to $\infty$ as $r \to \infty$, we see that $\rho(\tau)$ also increases to $\infty$ as $\tau \to \infty$. Moreover, we have
\[ \frac{e^{-\tau g(r)}}{f(r)} = e^{(\log f(r) - \tau h(\log f(r)))} = e^{\log f(r)(1 - \frac{h(\log f(r))}{\log f(r)})}, \quad r \geq R_0, \tau > \Lambda(R_0). \]

From $h(|\log f(r)|)/|\log f(r)| \geq 1/\tau$ for $r \in [R_0, \rho(\tau))$, we get $e^{-\tau g(r)/f(r)} \leq 1$, and the first inequality holds.

Similarly, $h(|\log f(r)|)/|\log f(r)| \leq 1/\tau$ for $r \geq \rho(\tau)$, which gives the second estimate.

The last assertion follows from the fact that $|\log f(r)|$ is an increasing function on $[R_0, \infty)$ and that the function $s \mapsto s(1 - \tau(h(s)/s))$ is positive and increasing on $[|\log f(\rho(\tau))|, \infty)$. \(\square\)

From now on we will choose $R_0$ and $n_0$, depending on $g$ and $f$ in such a way that all results from Section 3 and Section 4 become available. This means, in particular, that $R_0$ and $n_0$ are so large that

\[ f(R_0) < 1 \]
\[ n_0 \geq R_0 + 2 \]
\[ g(n_0 - 2) \geq 2C_6[\vartheta_0 + 2C_3C_6(1 + f(1))] \]
\[ g(n_0 - 2) \geq 2C_6\lambda_0 \]
\[ g(n_0 - 2) \geq 12(C_5 + C_7)C_6C_7^2 + \frac{\mu_0(1)}{C_6C_7^2} \quad \text{(cf. Theorem 4.6).} \]

Throughout, we will also use the function $\Lambda(\cdot)$ and its generalized inverse $\rho(\cdot)$ introduced in Lemma 5.2. Note that $\Lambda(\rho(\tau)) \geq \tau$ and $\rho(\Lambda(r)) \geq r$. This means, in particular, that
\[ \tau \geq 2\rho(n_0 + 4) \implies \Lambda(\frac{1}{2}\tau) \geq \Lambda(\rho(n_0 + 4)) \geq n_0 + 4 > n_0 + 3. \]

5.2. Progressive intrinsic ultracontractivity. Our results indicate that it makes sense to consider a new type of intrinsic contractivity property for Schrödinger semigroups which is essentially weaker than aUc. In this section we identify and discuss the concept of progressive intrinsic ultracontractivity (pIUC), cf. Definition 1.2.

Recall that our bounds for $u_t(x, y)$ and $U_t \mathbb{I}(x)$ are given in terms of the functions $F(\tau, x, y)$ and $G(\tau, x)$ defined in (4.14) and (4.20).

Lemma 5.3. Let $f : (0, \infty) \to (0, \infty)$ be a profile function satisfying (A1b) and (A1d). Moreover, let $g(r) = h(|\log f(r)|), \quad r \geq R_0$, with an increasing function $h : [|\log f(R_0)|, \infty) \to (0, \infty)$ such that $h(s)/s$ is increasing and $\lim_{s \to \infty} h(s)/s = 0$.

a) There are constants $C', C'' > 0$ such that for every $\tau \geq 2\Lambda(n_0 + 4)$ and $n_0 + 3 < |x| < \rho(\tau/2)$
\[ C'e^{-\tau g(n_0 + 3)} f(|x|) \leq G(\tau, x) \leq C''e^{-\frac{1}{2}\tau g(n_0 + 3)} f(|x|). \]

b) There are constants $C', C'' > 0$ such that for every $\tau \geq 3\Lambda(n_0 + 4)$ and $|x|, |y| > n_0 + 3,
\[ C'e^{-\tau g(n_0 + 3)} f(|x|) f(|y|) \leq F(\tau, x, y) \leq C''e^{-\frac{1}{2}\tau g(n_0 + 3)} f(|x|) f(|y|). \]
Proof. a) We have $\Lambda(n_0 + 4) \geq \Lambda(R_0)$ and $\rho(\frac{1}{2} \tau) > n_0 + 3$ for $\tau \geq 2\Lambda(n_0 + 4)$. If $|x| < \rho(\frac{1}{2} \tau)$, then by Lemma 5.2 we have $e^{-\frac{1}{2} \tau g(|z|)} \leq f(|z|)$, for $n_0 + 2 \leq |z| \leq |x|$ and $\tau \geq 2\Lambda(n_0 + 4)$. From the definition of $G(\tau, x)$ we get

$$G(\tau, x) \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) e^{-\frac{1}{2} \tau g(|z|)} dz \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f(|z|) dz,$$

and from (A1.d)

$$G(\tau, x) \leq c_1 e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|),$$

for all $n_0 + 3 < |x| < \rho(\frac{1}{2} \tau)$ and $\tau \geq 2\Lambda(n_0 + 4)$. The lower bound is easier:

$$G(\tau, x) = \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) e^{-\tau g(|z|)} dz \geq \int_{n_0 + 2 < |z| < n_0 + 3} f_1(|x - z|) e^{-\tau g(|z|)} dz \geq c_2 e^{-\tau g(n_0 + 3)} f(|x|).$$

b) Without loss of generality we may assume that $n_0 + 3 < |y| < \rho(\frac{1}{3} \tau)$, $\tau \geq 3\Lambda(n_0 + 4)$ and $|y| \leq |x|$. From the definition of $F(\tau, x, y)$ we get

$$F(\tau, x, y) = \left( \int_{n_0 + 2 < |z| < |y|} + \int_{|y| < |z| < |x|} \right) f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz =: I + II.$$

The monotonicity of $g$, (3.2), the inequality $e^{-\frac{1}{2} \tau g(|y|)} \leq f(|y|)$ and (3.3) imply

$$II \leq c_3 e^{-\tau g(y)} f_1(|x - y|) \leq c_3 e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|y|) f_1(|x - y|) \leq c_4 e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|) f(|y|).$$

We still need to estimate $I$. Since $e^{-\frac{1}{2} \tau g(|z|)} \leq f(|z|)$, for $|z| < |y| < \rho(\frac{1}{3} \tau)$, we get with (3.2) and (3.3),

$$I \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{n_0 + 2 < |z| < |y|} f_1(|x - z|) f_1(|y - z|) e^{-\frac{3}{2} \tau g(|z|)} dz \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{n_0 + 2 < |z| < |y|} f_1(|x - z|) f(|z|) f_1(|y - z|) f(|z|) dz \leq c_5 e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|) \int_{n_0 + 2 < |z| < |y|} f(|z|) f_1(|y - z|) f(|z|) dz \leq c_6 e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|) f(|y|),$$

which yields the upper bound in b). The lower bound is again simpler:

$$F(\tau, x, y) = \int_{n_0 + 2 < |z| < |x|} f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz \geq \int_{n_0 + 2 < |z| < n_0 + 3} f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz \geq c_7 e^{-\tau g(n_0 + 3)} f(|x|) f(|y|).$$

□

It is clear from Definition 1.2 that every aIUC-semigroup $\{U_t : t \geq 0\}$ is also pIUC for the threshold function $\rho \equiv \infty$. We will now show that under the assumptions (A1)–(A4) every non-aIUC semigroup $\{U_t : t \geq 0\}$ is still pIUC. This is the main result of this section. Recall that $\lambda_0$ and $\varphi_0$ denote the ground state eigenvalue and eigenfunction.
Corollary 5.4. Assume \((A1)-(A4)\) with \(t_b > 0\). Set \(K_1 := 2K = 8C_0C^2\) and \(K_2 := 3K = 12C_0C^2\). Let \(\lim_{r \to \infty} g(r)/|\log f(r)| = 0\), i.e. we are in the non-aIUC-regime.

a) For every \(t > \max \{30t_b, K_1\Lambda(n_0 + 4)\}\)
\[
U_t \mathbb{1}(x) \asymp e^{-\lambda_0 t} \left(1 \wedge \frac{f(|x|)}{g(|x|)}\right), \quad |x| < \rho(t/K_1).
\]
Equivalently, for \(t > \max \{30t_b, K_1\Lambda(n_0 + 4)\}\)
\[
U_t \mathbb{1}(x) \asymp e^{-\lambda_0 t} \varphi_0(x), \quad |x| < \rho(t/K_1).
\]

b) \((pIUC)\) For every \(t > \max \{30t_b, K_2\Lambda(n_0 + 4)\}\)
\[
u_t(x,y) \asymp e^{-\lambda_0 t} \left(1 \wedge \frac{f(|x|)}{g(|x|)}\right) \left(1 \wedge \frac{f(|y|)}{g(|y|)}\right), \quad |x| \land |y| < \rho(t/K_2).
\]
Equivalently, for \(t > \max \{30t_b, K_2\Lambda(n_0 + 4)\}\)
\[
u_t(x,y) \asymp e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad |x| \land |y| < \rho(t/K_2).
\]

Proof. The estimates for the functions \(G(\tau, x)\) and \(F(\tau, x, y)\) from Lemma 5.3 allow us to simplify the bounds in Theorems 4.6(c) and 4.8(b). Since \(\lim_{r \to \infty} g(r) = \infty\), there is some large \(n_0\) such that \(e^{-K_1 t g(n_0 + 3)} \vee e^{-(t/K_1) g(n_0 + 2)} \leq e^{-\lambda_0 t}\) and \(e^{-K_2 t g(n_0 + 3)} \vee e^{-(t/K_2) g(n_0 + 2)} \leq e^{-\lambda_0 t}\), for every \(t > 0\).

The lower bounds are obtained directly by taking \(\tau = K_1 t\), while the upper bounds follow by taking \(\tau = K_2 t\).

The alternative equivalent statements are a consequence of the two-sided bound \(\varphi_0(x) \asymp 1 \wedge \frac{f(|x|)}{g(|x|)}\) which is valid for all \(x \in \mathbb{R}^d\).

5.3. Explicit estimates of \(U_t \mathbb{1}(x)\). Under the assumption \((A4)\) we can find explicit two-sided estimates for the function \(G(\tau, x)\) (defined in (4.20)) for the full range of \((\tau, x)\). Throughout we use \(\Lambda(\cdot)\) and \(\rho(\cdot)\) from Lemma 5.2 and choose \(K_0\) and \(n_0\) according to (5.3).

Lemma 5.5. Let \(f : (0, \infty) \to (0, \infty)\) be a profile function satisfying \((A1)\b)\) and \((A1)\d). Assume that \(g(r) = h(|\log f(r)|), r \geq R_0\), with an increasing function \(h : [|\log f(R_0)|, \infty) \to (0, \infty)\) such that \(h(s)/s\) is monotone.

a) If \(C > 0\) is such that \(C g(r) \geq |\log f(r)|, r \geq R_0\), then there are constants \(C', C'' > 0\) such that for every \(|x| > n_0 + 3\) and \(\tau \geq 2n_0 := C\)
\[
C' e^{-\frac{1}{2} \tau g(n_0 + 3)} f(|x|) \leq G(\tau, x) \leq C'' e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|).
\]

b) If \(\lim_{r \to \infty} g(r)/|\log f(r)| = 0\), then there are constants \(C', C'' > 0\) such that for every \(\tau \geq 2\Lambda(n_0 + 4)\)

b1) \(C' e^{-\frac{1}{2} \tau g(n_0 + 3)} f(|x|) \leq G(\tau, x) \leq C'' e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|)\) for \(n_0 + 3 < |x| < \rho(\tau/2)\).

b2) \(C' e^{-\tau g(|x|)} \leq G(\tau, x) \leq C'' e^{-\frac{1}{2} \tau g(n_0 + 2)} e^{-\frac{1}{2} \tau g(|x|)}\) for \(|x| \geq \rho(\tau/2)\).

Proof. Part a) follows directly from Lemma 4.9(b) and part b1) is exactly Lemma 5.3(a). We only need to show b2). From the definition (4.20) of \(G\) we get
\[
G(\tau, x) = \int_{n_0 + 2 < |x| < \rho(\tau/2)} f_1(|x - z|) e^{-\tau g(|z|)} \, dz + \int_{\rho(\tau/2) < |z| < |x|} f_1(|x - z|) e^{-\tau g(|z|)} \, dz =: I + II.
\]

Exactly the same argument as in Lemma 5.3(a) yields
\[
I \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{n_0 + 2 < |z| < \rho(\tau/2)} f_1(|x - z|) f(|z|) \, dz \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} f(|x|).
\]
Since \(|x| \geq \rho(\tau/2)|\), the second inequality in Lemma 5.2 shows \(I \leq c_2 e^{-\frac{1}{2} \tau g(n_0 + 2)} e^{-\frac{1}{2} \tau g(|z|)}\). In order to estimate II we write
\[
II \leq e^{-\frac{1}{2} \tau g(n_0 + 2)} \int_{\rho(\tau/2) < |z| < |x|} f_1(|x - z|) f(|z|) e^{-\frac{1}{2} \tau g(|z|)} \frac{e^{-\frac{1}{2} \tau g(|z|)} d|z|}{f(|z|)} \, dz.
\]
which is less than or equal to
\[ e^{-\frac{1}{2} g(n_0+2) \frac{e^{-\frac{1}{2} g(|x|)}}{f(|x|)}} \int_{n_0+2<|z|\leq |x|} f_1(|x-z|) f(|z|) \, dz \leq c_3 e^{-\frac{1}{2} g(n_0+2) e^{-\frac{1}{2} g(|x|)}}, \]
by the last (monotonicity) assertion in Lemma 5.2 and (3.2). This gives the upper bound in b. The corresponding lower estimate follows directly:
\[ G(\tau, x) \geq e^{-\tau g(|x|)} \int_{n_0+2<|z|\leq |x|} f_1(|x-z|) \, dz \geq c_4 e^{-\tau g(|x|)}. \]

We are now ready to state the following corollary to Lemma 5.3 which simplifies the estimates of the function \( U_t \mathbb{I}(x) \) in Theorem 4.8.5.b.

**Corollary 5.6.** Assume (A1)-(A4) with \( t_b > 0 \) and set \( K_1 := 2K = 8C_0C^2 \).

a) **(aIUC-regime)** If \( Cg(r) \geq \log f(r), \ r \geq R_0, \) for some \( C > 0, \) then for every \( t > \max\{30t_b, K_1\tau_0\} \) with \( \tau_0 := C \)
\[ U_t \mathbb{I}(x) \asymp e^{-\lambda t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \quad \text{for all} \ x \in \mathbb{R}^d. \]

b) **(non-aIUC-regime)** If \( \lim_{r \to \infty} g(r)/\log f(r) = 0, \) then there are constants \( C', C'' > 0 \) such that for every \( t > \max\{t_b, K_1\Lambda(n_0+4)\} \)
\[ C' e^{-\lambda t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \leq U_t \mathbb{I}(x) \leq C'' e^{-\lambda t} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \quad \text{for} \ |x| < \rho(t/K_1), \]
\[ C' e^{-K_1\log(|x|)} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \leq U_t \mathbb{I}(x) \leq C'' e^{-K_1\log(|x|)} \left( 1 \wedge \frac{f(|x|)}{g(|x|)} \right) \quad \text{for} \ |x| \geq \rho(t/K_1). \]

**Proof.** Part a and the first formula in b are already stated in Corollaries 4.10 and 5.4, the second set of estimates in b follows by arguments similar to those in Lemma 5.5.2. \( \square \)

5.4. Doubling Lévy measures. In this section we discuss profiles \( f \) which enjoy the doubling property. This means that \( f \) is a decreasing function such that
\[ \text{there exists} \ C \geq 1 \ \text{such that} \ f(r) \leq Cf(2r) \ \text{for every} \ r > 0. \]
Throughout, we choose \( n_0 \) according to (5.3).

**Example 5.7** (Fractional and layered fractional Schrödinger operators). Let
\[ \nu(x) = \sigma \left( \frac{x}{|x|} \right) f(|x|) \quad \text{with} \ f(r) = r^{-d-\alpha} (e \lor r)^{-\gamma}, \]
where \( \alpha \in (0, 2) \) and \( \gamma \geq 0, \) and \( \sigma : S^{d-1} \to (0, \infty) \) is a function on the unit sphere \( S^{d-1} \subset \mathbb{R}^d \) such that \( \sigma(-\theta) = \sigma(\theta) \) and \( 0 < \inf_{\theta \in S^{d-1}} \sigma(\theta) \leq \sup_{\theta \in S^{d-1}} \sigma(\theta) < \infty. \) Moreover, we assume that there is no Gaussian part in the Lévy–Khintchine formula (1.1), i.e. \( A \equiv 0. \) Recall that this setup covers the following two important classes of Lévy processes.

a) Symmetric \( \alpha \)-stable processes (if \( \gamma = 0), \) see [20];
b) Layered symmetric \( \alpha \)-stable processes (if \( \gamma > 2-\alpha), \) see [22].

The assumptions (A1,a,b) are clearly satisfied and (A1,c,d) follow from the doubling property (5.4) (if (A1,d) is checked in Lemma 3.2.4). Moreover, for both classes of processes [a] and [b] the conditions in the assumption (A2) follows directly from the available estimates of the corresponding transition densities, see e.g. [33, Theorem 2]. In fact, they hold true for every fixed \( t_b > 0 \) with appropriate constants \( C_4, C_5, \) depending on \( t_b. \)

Let
\[ V(x) = (1 \lor \log |x|)^\beta, \]
for some \( \beta > 0. \) Let us check (A3) and (A4). We take \( g(r) = (1 \lor \log r)^\beta \) and \( R_0 = e \) in (A3); obviously, \( C_6 = 1 \) and \( C_7 = \log(1+e)^\beta. \) Since \( \log f(r) = (d+\alpha+\gamma)\log r \) for \( r \geq e, \)
\( (A4) \) is satisfied with \( h(r) = (r/(d+\alpha+\gamma))^\beta. \) We then have \( A(r) = (d+\alpha+\gamma)(\log r)^{1-\beta} \) and
\[ \rho(\tau) = \exp \left( \frac{\tau}{d + \alpha + \gamma} \right)^{\frac{1}{1 - \beta}}, \] for \( \beta \in (0, 1) \). Set \( K_2 := 3K = 12C_6C_7^2 = 12(\log(1 + e))^{2\beta} \) and 
\( K_3 := 4K = 16C_6C_7^2 = 16(\log(1 + e))^{2\beta} \).

We have the following large time estimates.

a) If \( \beta \geq 1 \), then we are in the aIUC-regime, and we have for \( t > 30t_b + (d + \alpha + \gamma)K_2 \) and all \( x, y \in \mathbb{R}^d \)
\[ u_t(x, y) \asymp e^{-\lambda_{0}t} \]
\[ (1 + |x|)^{d + \alpha + \gamma}(1 + |y|)^{d + \alpha + \gamma}(1 + \log |y|)^{\beta}. \]

b) If \( \beta \in (0, 1) \), then we are in the non-aIUC-regime, and there exists some \( C \geq 1 \) such that

b1) for all \( t > \max\{30t_b, K_2\Lambda(n_0 + 4)\} \) and \( |x| \wedge |y| < \exp \left( \frac{t}{K_2(d + \alpha + \gamma)} \right)^{\frac{1}{1 - \beta}} \), one has
\[ u_t(x, y) \leq C e^{-\lambda_{0}t} \]
\[ (1 + |x|)^{d + \alpha + \gamma}(1 + |y|)^{d + \alpha + \gamma}(1 + \log |y|)^{\beta}. \]

In particular, the semigroup \( \{U_t : t \geq 0\} \) is pIUC.

b2) for all \( t > \max\{30t_b, K_2\Lambda(n_0 + 4), (d + \alpha + \gamma)K_2^{1/\beta}(4|\lambda_0|)^{(1 - \beta)/\beta})\} \) and \( |x|, |y| \geq \exp \left( \frac{t}{K_2(d + \alpha + \gamma)} \right)^{\frac{1}{1 - \beta}} \), one has
\[ C^{-1} e^{-K_3t(\log |x| \wedge |y|)^\beta} \]
\[ (1 + |x - y|)^{d + \alpha + \gamma}(1 + \log |x|)^{\beta}(1 + \log |y|)^{\beta} \leq u_t(x, y) \]
\[ \leq C e^{-K_3t(\log |x| \wedge |y|)^\beta} \]
\[ (1 + |x - y|)^{d + \alpha + \gamma}(1 + \log |x|)^{\beta}(1 + \log |y|)^{\beta}. \]

The estimates in a) and b1) follow directly from Corollaries 4.10 and 5.4; part b2) is a consequence of the Corollary 5.9 stated below.

Clearly, if the growth order of the potential \( V \) at infinity is slower than that in (5.5), e.g. like \((\log \log |x|)^\beta\), then the corresponding Schrödinger heat kernels enjoy two-sided estimates as in part b) with appropriate threshold functions \( \rho(t/K_2) \).

The next lemma is needed in the proof of Corollary 5.9. Recall that \( \Lambda(r) \) and \( \rho(\tau) \) are defined in (5.1) and (5.2).

Lemma 5.8. Let \( f : (0, \infty) \to (0, \infty) \) be a decreasing profile with the doubling property (5.4) and \( \lim_{r \to \infty} f(r) = 0 \). Assume that \( g(r) = h(\log f(r)) \), \( r \geq R_0 \), with an increasing function \( h : \| \log f(R_0) \|, \infty \to (0, \infty) \) such that \( h(s)/s \) decreases to 0 as \( r \to \infty \). There are constants \( C', C'' > 0 \) such that for every \( \tau \geq 3\Lambda(n_0 + 4) \) and \( |x|, |y| \geq \rho(1/\tau) \)
\[ C' e^{-\tau g(|x| \wedge |y|)} f_1(|x - y|) \leq F(\tau, x, y) \leq C'' e^{-\frac{1}{2}\tau g(n_0 + 2)} e^{-\frac{1}{3}\tau g(|x| \wedge |y|)} f_1(|x - y|). \]

Proof. The lower bound follows easily:
\[ F(\tau, x, y) \geq \int_{n_0 + 2 < |z| < |y| < 1} f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz \geq c_1 e^{-\tau g(|y|)} f_1(|x - y|). \]

For the proof of the upper bound we assume, without loss of generality, that \( |y| \leq |x| \). Let
\[ F(\tau, x, y) = \left( \int_{n_0 + 2 < |z| < |y| < 1} f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz \right) + \int_{|y| \leq |z| \leq |x|} f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz =: I + II, \]
and observe that, by the monotonicity of \( g \) and (3.2), \( II \leq c_2 e^{-\tau g(|y|)} f_1(|x - y|) \). We only need to estimate I. Write
\[ I = \left( \int_{n_0 + 2 < |z| < \rho(1/\tau)} + \int_{\rho(1/\tau) < |z| \leq |y| \leq 1} \right) f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} dz =: I_1 + I_2. \]
With the argument in the proof of Lemma 5.3(b), we get $I_1 \leq c_3 e^{-\frac{1}{2}r_0 g(n_0+2)} f(|x|) f(|y|)$. Since $|x-y| \leq 2|x|$ and $|y| \geq \rho(\frac{1}{2}r)$, (5.4) together with the inequality $f(|y|) \leq e^{-\frac{1}{2}r g(|y|)}$ yields $f(|x|) f(|y|) \leq c_4 f_1(|x-y|) e^{-\frac{1}{2}r g(|y|)}$. This proves $I_1 \leq c_5 e^{-\frac{1}{2}r g(n_0+2)} f_1(|x-y|) e^{-\frac{1}{2}r g(|y|)}$.

Now we estimate $I_2$. The monotonicity of $g$ and $f_1$ combined with $|z| \geq \rho(\frac{1}{2}r) \geq n_0 + 3$ gives

$$I_2 \leq \left( \int_{\rho(\frac{1}{2}r) \leq |z| \leq |y|} + \int_{\rho(\frac{1}{2}r) \leq |z| \leq |x|} \right) f_1(|x-z|) f_1(|y-z|) e^{-r g(|z|)} \, dz$$

$$\leq f_1(\frac{1}{2}|x-y|) e^{-\frac{1}{2}r g(n_0+2)} \int_{\rho(\frac{1}{2}r) \leq |z| \leq |y|} (f_1(|x-z|) + f_1(|y-z|)) e^{-r g(|z|)} \, f(|z|) \, dz.$$  

Note that $f_1$ inherits the doubling property (5.4) from $f$. Together with the last (monotonicity) assertion in Lemma 5.2 and (A1b,c,d) we get

$$I_2 \leq c_3 f_1(|x-y|) e^{-\frac{1}{2}r g(n_0+2)} e^{-\frac{1}{2}r g(|y|)} \int_{\rho(\frac{1}{2}r) \leq |z| \leq |y|} (f_1(|x-z|) + f_1(|y-z|)) \, f(|z|) \, dz$$

$$\leq c_4 e^{-\frac{1}{2}r g(n_0+2)} f_1(|x-y|) e^{-\frac{1}{2}r g(|y|)}.$$  

This completes the proof.  

We have the following corollary.

**Corollary 5.9.** Assume the doubling condition (5.4), (A1), (A4) with $R_0 > 0$, $t_b > 0$ and $\lim_{r \to \infty} g(r)/|\log f(r)| = 0$. Define $K_2 = 12C_0 C_1^2$ and $K_3 = \frac{1}{2}K_2$. There are constants $C', C'' > 0$ such that for every $t > \max\{30b, K_2\} \{n_0 + 4\}$ and $|x|, |y| \geq \rho(t/K_2)$ satisfying $g(\rho(t/K_2)) \geq 4K_2 |\lambda_0|$

$$C' e^{-K_2 g(|x|) |\lambda|} f_1(|x| - y) \leq u_t(x,y) \leq C'' e^{-\frac{1}{2}r_0 g(|x|) |\lambda|} f_1(|x| - y).$$

Note that for all $t \geq K_2 (g^{-1}(4K_2 |\lambda_0|))$ — here $g^{-1}$ stands for the generalized right-inverse of the increasing function $g$ — the inequality $g(\rho(t/K_2)) \geq 4K_2 |\lambda_0|$ is satisfied. This follows immediately from the fact that $g(g^{-1}(s)) \geq s$ and $A(\rho(s)) \geq s$. In particular, we have a threshold value for $t$ in Corollary 5.9.

**Proof of Corollary 5.9.** The lower bound follows immediately if we insert the lower estimate from Lemma 5.8 into Theorem 4.6(c).

Similarly, we get for the upper estimate

$$u_t(x,y) \leq e^{-\frac{1}{2}r_0 g(|x|) |\lambda|} f_1(|x-y|) \vee e^{-\lambda_0 f(|x|) f(|y|)} g(|x|) g(|y|)$$

$$= e^{-\frac{1}{2}r_0 g(|x|) |\lambda|} f_1(|x-y|) \vee e^{-\lambda_0 f(|x|) |\lambda|} f(|x|) g(|y|)$$

Since $f_1$ is doubling and decreasing,

$$f(|x| \vee |y|) \leq c_2 f_1(|x| \vee |y|) \leq c_3 f_1(2(|x| \vee |y|)) \leq c_3 f_1(|x-y|).$$

Moreover, by Lemma 5.2

$$f(|x| \vee |y|) \leq e^{-\frac{1}{2}r_0 g(|x|) |\lambda|}, \quad |x|, |y| \geq \rho(t/K_2).$$

Thus, for all $|x|, |y| \geq \rho(t/K_2)$

$$u_t(x,y) \leq c_4 e^{-\lambda_0 f} e^{-\frac{1}{2}r_0 g(|x|) |\lambda|} f_1(|x-y|) g(|x|) g(|y|)$$

$$\leq c_4 e^{\left(\left|\lambda\right| - \frac{1}{2}r_0 g(\rho(t/K_2))\right) t} e^{-\frac{1}{2}r_0 g(|x|) |\lambda|} f_1(|x-y|) g(|x|) g(|y|).$$
and the claimed bound follows for \( t > \max\{30t_b, K_2\Lambda(n_0 + 4)\} \) and \( |\lambda_0| \leq g(\rho(t/K_2))/(4K_2) \) hold.

### 5.5. Exponential Lévy measures

Throughout this section we assume that the profile \( f \) of the Lévy density has exponential decay at infinity, i.e. \( f \) is a decreasing function such that

\[
f(r) = e^{-\kappa r - \gamma}, \quad r \geq 1,
\]

for some \( \kappa > 0 \) and \( \gamma \geq 0 \). This setting covers several important examples.

Let us show that such a profile satisfies the direct jump property (A1.d) if \( \gamma > \frac{1}{2}(d+1) \). We can check the criteria from Lemma 3.2: If \( \gamma > d \), we can use Lemma 3.2[b], note that \( d = 1 \) implies \( \frac{1}{2}(d+1) = d \). If \( d > 1 \) and \( d \geq \gamma > \frac{1}{2}(d+1) \), we use Lemma 3.2[c]. For this, we have to check the integrability condition (3.6). Since \( f'(|y|)/f(|y|) = -\kappa - \gamma/|y| \), we have for \( |y| > 1 \)

\[
\int_{|y|>1} e^{-\frac{\kappa}{f'(|y|)}|y|} f(|y|) \, dy = \int_{y_1 \leq 1 \atop |y|>1} e^{\kappa y_1 + \gamma \frac{y_1}{|y|}} f(|y|) \, dy + \int_{y_1 > 1 \atop |y|>1} e^{\kappa y_1 + \gamma \frac{y_1}{|y|}} f(|y|) \, dy.
\]

It is easy to see that the first integral on the right-hand side is finite. The second integral is bounded by \( e^{\gamma} I \), where

\[
I := \int_{y_1 > 1} e^{\kappa y_1} f(|y|) \, dy = \int_{y_1 > 1} e^{\kappa (y_1 - |y|)} |y|^{-\gamma} \, dy.
\]

Now we introduce spherical coordinates for \( (y_2, \ldots, y_d) \in \mathbb{R}^{d-1} \) and observe that \( y_1 - \sqrt{y_1^2 + r^2} = -r^2/(y_1 + \sqrt{y_1^2 + r^2}) \). So,

\[
I = c \int_1^\infty \int_0^\infty e^{-\frac{\kappa}{2(1 + \sqrt{y_1^2 + r^2})^2} r^d - \frac{1}{2}} dr \, dy_1 \leq c \int_1^\infty \int_0^\infty e^{-\frac{\kappa}{2(1 + \sqrt{1 + y_1^2})^2} r^d - \frac{1}{2}} r^{d-2} y_1^{-\delta} dr \, dy_1;
\]

changing variables in the inner integral according to \( r = \sqrt{y_1} u \) yields that the double integral factorizes

\[
\int_0^\infty e^{-\frac{\kappa}{1 + \sqrt{1 + y_1^2}} u^{d-2}} \frac{1}{y_1} \big(1/(d-1) - \delta\big) \, du \cdot \int_1^\infty \frac{1}{y_1} \big(1/(d-1) - \delta\big) \, dy_1.
\]

The first integral is finite and the second is finite if, and only if \( \gamma > \frac{1}{2}(d+1) \). This implies that \( I \) is finite for this range of \( \gamma \).

**Example 5.10** (Quasi-relativistic Schrödinger Operators). Let \( \sigma \) be a function on the unit sphere \( S^{d-1} \subset \mathbb{R}^d \) such that \( \sigma(-\theta) = \sigma(\theta) \) and \( 0 < \inf_{\theta \in S^{d-1}} \sigma(\theta) \leq \sup_{\theta \in S^{d-1}} \sigma(\theta) < \infty \). Moreover, let \( \alpha \in (0, 2) \), \( \kappa > 0 \) and \( \gamma > \frac{1}{2}(d+1) \). Consider

\[
\nu(x) = \sigma \left( \frac{x}{|x|} \right) g(|x|) \quad \text{where} \quad g(r) \asymp f(r) := e^{-\kappa r - \gamma (1 \vee r)^{d+\alpha - \gamma}}.
\]

We assume that \( A \equiv 0 \) in (1.1). This setting covers two important classes of Lévy processes whose Lévy measure decays exponentially.

a) *Relativistic symmetric \( \alpha \)-stable processes* \( (\kappa = m^{1/\alpha}, m > 0, \gamma = \frac{1}{2}(d + \alpha + 1)) \), see [19];

b) *Exponentially* tempered symmetric \( \alpha \)-stable processes \( (\kappa > 0, \gamma = d + \alpha) \), see [38].

Assumption (A1) holds, we have checked (A1.d) above, and (A2) follows from [38] Theorem 2]; as before, it holds true for every fixed \( t_b > 0 \) with appropriate constants \( C_4, C_5 \), depending on \( t_b \). Let

\[
V(x) = (1 \vee |x|)^\beta,
\]

for some \( \beta > 0 \). Since \( |\log f(r)| = \kappa r + \gamma \log r \) for \( r \geq 1 \), we may choose \( h(r) := (r/\kappa)^\beta \) and \( g(r) := ((r \vee 1) + (\gamma/\kappa) \log (r \vee 1))^{\beta} \) so that \( g(r) = h(\log f(r)) \) for \( r \geq 1 \). The assumptions (A3) and (A4) hold for this choice of \( g \) and \( h \) with \( R_0 = 1 \). With a straightforward calculation we
can check that $C_6 = (\kappa e/(\gamma + \kappa e))^{\beta}$ and $C_7 = (2 + (\gamma/\kappa) \log 2)^{\beta}$. In particular, $g(r) \lesssim (1 \vee r)^{\beta}$, for $r \geq 0$. The threshold functions defined in \[(5.1)\] and \[(5.2)\] are given by

$$\Lambda(r) = \kappa^\beta u(r)^{1-\beta} \quad \text{and} \quad \rho(\tau) = u^{-1} \left( \left( \frac{\tau}{\kappa^\beta} \right)^{1-\beta} \right)$$

where $u(r) = |\log f(r)| = \kappa r + \gamma \log r$. Clearly, $\rho(\tau) \approx (\tau/\kappa^\beta)^{1-\beta}$, $\tau \geq \kappa$. Set $K_2 := 3K = 12C_6C_7^2$ and $K_4 := C_6K_2 = 12C_6C_7^2$.

We have the following large time estimates. As before, parts a) and b1) follow immediately from Corollaries 4.10 and 5.4 above, while part b2) will be a consequence of Corollary 5.12 stated below.

a) If $\beta \geq 1$, then we are in the $aIUC$-regime, and for every $t > 30t_b + \kappa K_2$ we have

$$u_t(x, y) \asymp e^{-\lambda_0 t} \frac{e^{-\kappa |x| - \kappa |y|}}{(1 + |x|)^{\gamma + \beta}(1 + |y|)^{\gamma + \beta}}, \quad x, y \in \mathbb{R}^d.$$

b) If $\beta \in (0, 1)$, then we are in the non-$aIUC$-regime, and there exists a constant $C \geq 1$ such that for every $t > \max\{30t_b, K_2\Lambda(n_0 + 4)\}$ we have

b1) for $|x| \wedge |y| < \rho(t/K_2)$,

$$u_t(x, y) \asymp e^{-\lambda_0 t} \frac{e^{-\kappa |x| - \kappa |y|}}{(1 + |x|)^{\gamma + \beta}(1 + |y|)^{\gamma + \beta}}.$$

In particular, the semigroup $\{U_t : t \geq 0\}$ is $pIUC$.

b2) for $|x|, |y| \geq \rho(t/K_2)$,

$$\frac{1}{C} \frac{1}{|x|^\beta |y|^\beta} \left( \frac{e^{-\lambda_0 t - \kappa |x| - \kappa |y|}}{|x|^\gamma |y|^\gamma} \vee \frac{e^{-K_4 t(|x| \wedge |y|)^{\beta} - \kappa |x| - \kappa |y|}}{(1 + |x| - y)^\gamma} \vee H(K_4 t, x, y) \right) \leq u_t(x, y)$$

$$\leq C \frac{1}{|x|^\beta |y|^\beta} \left( \frac{e^{-\lambda_0 t - \kappa |x| - \kappa |y|}}{|x|^\gamma |y|^\gamma} \vee \frac{e^{-\kappa t|z|^{\beta}}(|x| \wedge |y|)^{\beta} - \kappa |x| - \kappa |y|}{(1 + |x| - y)^\gamma} \vee H(t/K_4, x, y) \right),$$

where

$$H(\tau, x, y) := \int_{n_0 + 2 \leq |z| \leq |x| \wedge |y|} \frac{e^{-\kappa |x| - |z| - y}}{(1 \vee |x - z|)^\gamma (1 \vee |z| - y)^\gamma} e^{-\tau |z|^\beta} dz.$$

For $d = 1$ these bounds can be simplified:

$$\frac{1}{C} \frac{1}{|x|^\beta |y|^\beta} \left( \frac{e^{-\lambda_0 t - \kappa |x| - \kappa |y|}}{|x|^\gamma |y|^\gamma} \vee \frac{e^{-K_4 t(|x| \wedge |y|)^{\beta} - \kappa |x| - \kappa |y|}}{(1 + |x| - y)^\gamma} \right) \leq u_t(x, y)$$

$$\leq C \frac{1}{|x|^\beta |y|^\beta} \left( \frac{e^{-\lambda_0 t - \kappa |x| - \kappa |y|}}{|x|^\gamma |y|^\gamma} \vee \frac{e^{-\kappa t|z|^{\beta}}(|x| \wedge |y|)^{\beta} - \kappa |x| - \kappa |y|}{(1 + |x| - y)^\gamma} \right).$$

Also in the multidimensional case one can give (upper) estimates for $H(t/K_4, x, y)$ which will still lead to, say, exponential decay of $u_t(x, y)$, but we may lose the sharp two-sided estimates in b2) Since such estimates depend very much on the particular setting, we do not give further details here. See, however, Corollary 5.12.a).

If the growth order of the potential $V$ at infinity is slower than that in \[(5.7)\] (e.g. log $|x|^\beta$), then the corresponding Schrödinger heat kernels enjoy two-sided estimates similar to those in part b) above with appropriate threshold functions $\rho(t/K_2)$.

In order to complete Example 5.10, we need Corollary 5.12, which is based on the following two-sided estimates for the function $F(\tau, x, y)$ defined in \[(4.14)\].

**Lemma 5.11.** Let $d = 1$ and let $f : (0, \infty) \to (0, \infty)$ be a decreasing profile such that \[(5.6)\] holds with $\gamma > 1$. Assume that $g(r) = h(\kappa r + \gamma \log r)$, $r \geq R_0$, with an increasing function $h : [\kappa R_0 + \gamma \log R_0, \infty) \to (0, \infty)$ such that $h(s)/s$ decreases to 0 as $s \to \infty$ and $h(\kappa s + \gamma \log s)/\log s$...
is monotone for $s \geq R_0 \vee e$. There are constants $C', C''$ such that for every $\tau \geq 3\Lambda(n_0 + 4)$ and $|x|, |y| \geq \rho(\frac{1}{3}\tau)$

$$C'(e^{-r\tau g(n_0 + 3)} f(|x|) f(|y|) \vee e^{-r\tau g(|x|\vee|y|)}f_1(|x-y|)) \leq F(\tau, x, y)$$

$$\leq C''(e^{-\frac{1}{2}r\tau g(n_0 + 2)} f(|x|) f(|y|) \vee e^{-\frac{1}{4}r\tau g(n_0 + 2)}e^{-\frac{1}{4}r\tau g(|x|\vee|y|)}f_1(|x-y|)) .$$

**Proof.** The lower bound is easy, cf. the last lines of the proof of Lemma 5.3 and the second line of the proof of the proof of Lemma 5.8.

The proof of the upper bound is similar to the proof of Lemma 5.8. The only difference is in the upper estimate of the integral

$$I_2 := \int_{\rho(\frac{1}{3}\tau) \leq |z| < |y|} f_1(|x - z|) f_1(|y - z|) e^{-r\tau g(|z|)} dz$$

in the case when $|y| > \rho(\frac{1}{3}\tau)$. By symmetry, we may assume that $|x| \geq |y|$ and $x > 0$. We have two cases: $\rho(\frac{1}{3}\tau) < y \leq x$ and $-x \leq y < -\rho(\frac{1}{3}\tau)$.

Case 1: $\rho(\frac{1}{3}\tau) < y \leq x$. We have $|x - z| > |x - y|$ and, by monotonicity, $f_1(|x - y|) \geq f_1(|x - z|)$.

Together with the last assertion in Lemma 5.2, which says that the function $r \mapsto e^{-\frac{1}{4}r\tau g(r)}/f(r)$ is increasing on $[\rho(\frac{1}{3}\tau), \infty)$ and (A1.d), this gives

$$I_2 \leq c_1 f_1(|x - y|) e^{-\frac{1}{4}r\tau g(n_0 + 2)} \int_{\rho(\frac{1}{3}\tau) \leq |z| < |y|} f_1(|y - z|) e^{-\frac{1}{4}r\tau g(|z|)} f(|z|) dz$$

$$\leq c_2 f_1(|x - y|) e^{-\frac{1}{4}r\tau g(n_0 + 2)} e^{-\frac{1}{2}r\tau g(|y|)} \int_{\rho(\frac{1}{3}\tau) \leq |z| < |y|} f_1(|y - z|) f(|z|) dz$$

$$\leq c_3 e^{-\frac{1}{2}r\tau g(n_0 + 2)} f_1(|x - y|) e^{-\frac{1}{4}r\tau g(|y|)},$$

which is the required estimate.

Case 2: $-x \leq y < -\rho(\frac{1}{3}\tau)$. For $\rho(\frac{1}{3}\tau) \leq |z| < |y|$ we have $|x - z| + |y - z| = |x| + |y| = |x - y|$, which gives

$$f_1(|x - z|) f_1(|y - z|) \leq c_4 e^{-\kappa(|x - z| + |y - z|)} (1 \wedge |x - z|^{-\gamma})(1 \wedge |z - y|^{-\gamma})$$

$$= c_4 e^{-\kappa(|x + y|)} (1 \wedge |x - z|^{-\gamma})(1 \wedge |z - y|^{-\gamma})$$

$$= c_4 e^{-\kappa(|x - y|)} (1 \wedge |x - z|^{-\gamma})(1 \wedge |z - y|^{-\gamma}).$$

This shows

$$I_2 = \int_{\rho(\frac{1}{3}\tau) \leq |z| < |y|} f_1(|x - z|) f_1(|y - z|) e^{-r\tau g(|z|)} dz$$

$$\leq c_4 e^{-\kappa(|x| + |y|)} \tilde{F}(\tau, x, y)$$

$$= c_4 e^{-\kappa|x - y|} \tilde{F}(\tau, x, y),$$

where

$$\tilde{F}(\tau, x, y) := \int_{n_0 + 2 \leq |z| < |x|} \tilde{f}_1(|x - z|) \tilde{f}_1(|y - z|) e^{-r\tau g(|z|)} dz \quad \text{with} \quad \tilde{f}(r) = r^{-\gamma}. $$

We still have to estimate the function $\tilde{F}(\tau, x, y)$ from above. Define $\tilde{h}(s) := h(\kappa e^{|z| + s} + s)$ and observe that for $r \geq R_0$ we have $g(r) = h(\kappa r + \gamma \log r) = h(|\log \tilde{f}(r)|)$. By assumption, $h(s)$ is increasing and $h(\kappa s + \gamma \log s)/\log s$ is monotone for $s \geq R_0 \vee e$. This shows that $\tilde{h}(s)$ is increasing and $\tilde{h}(s)/s$ is monotone for $s \geq \gamma \log R_0 \vee e$. It implies that the pairs $\tilde{f}$ and $g$ show all possible behaviours which were discussed in Remark 5.1. In particular, the assumptions of
Lemma 4.9(a), 5.3(b) and 5.8 are satisfied for \( \tilde{f} \) and \( g \). If we combine these lemmas, we obtain the upper estimate of the function \( \tilde{F}(\tau, x, y) \) in all possible regions, and so
\[
\tilde{F}(\tau, x, y) \leq c_5 \left( e^{-\frac{1}{2} \tau g(n_0+2)} \tilde{f}(|x|) \tilde{f}(|y|) \vee e^{-\frac{1}{2} \tau g(n_0+2)} e^{-\frac{1}{2} \tau g(|x|)} \tilde{f}_1(|x-y|) \right).
\]
Plugging this into (5.9), gives, in view of (5.6),
\[
I_2 \leq c_6 \left( e^{-\frac{1}{2} \tau g(n_0+2)} f(|x|) f(|y|) \vee e^{-\frac{1}{2} \tau g(n_0+2)} e^{-\frac{1}{2} \tau g(|x|)} f_1(|x-y|) \right),
\]
and the proof is finished. \( \square \)

We are now ready to state our last corollary. Recall that \( \Lambda(r) \) and \( \rho(\tau) \) are defined in (5.1) and (5.2).

**Corollary 5.12.** Assume \( (A_1) - (A_4) \) with \( R_0 > 0, t_0 > 0 \), let \( f \) be of exponential type (5.6) and \( \lim_{\tau \to \infty} g(r)/|\log f(r)| = 0 \).

a) There are constants \( C', C'' > 0 \) such that for every \( t > \max\{30t_0, K_2\Lambda(n_0+4)\} \) and \(|x|, |y| \geq \rho(t/K_2)\)
\[
\frac{C'}{g(|x|)g(|y|)} \left( e^{-\lambda_0 t - \kappa|x-y|} + e^{-K_2\rho g(|x|)|y|-\kappa|x-y|} (1 + |x-y|)^\gamma \vee H(K_2t, x, y) \right) \leq u_t(x, y),
\]
\[
\leq \frac{C''}{g(|x|)g(|y|)} \left( e^{-\lambda_0 t - \kappa|x-y|} + e^{-\Lambda(r)|x-y|} (1 + |x-y|)^\gamma \vee H(t/K_2, x, y) \right),
\]
where
\[
H(\tau, x, y) := \int_{n_0+2 \leq |z| \leq |x| - |y|} e^{-\kappa(|x-z| + |x-y|)} (1 \vee |x-z|)^\gamma (1 \vee |x-y|)^\gamma e^{-\tau g(|z|)} dz.
\]
Moreover, there exists \( C^* > 0 \) such that for every \( \tau > 3\Lambda(n_0+4) \) and \(|x|, |y| \geq \rho(\frac{1}{3}\tau)\)
\[
H(\tau, x, y) \leq C^* \left[ e^{-\frac{1}{3} \tau g(n_0+2) - \kappa|x-y|} + \frac{e^{-\lambda_0 t - \kappa|x-y|}}{(1 \vee |x-y|)^\gamma} \vee \frac{e^{-\kappa|x-y|}}{(1 \vee |x-y|)^\gamma} \right] \cdot \int_{\rho(\frac{1}{3}\tau) \leq |z| \leq |x| - |y|} e^{-\frac{1}{3} \tau g(|z|)} dz.
\]

b) If, in addition, \( \gamma > 1 = d \) and the function \( r \mapsto g(r)/\log r \) is monotone on \([R_0 \vee e, \infty)\), then there exist constants \( C', C'' > 0 \) such that for every \( t > \max\{30t_0, K_2\Lambda(n_0+4)\} \) and \(|x|, |y| \geq \rho(t/K_2)\)
\[
\frac{C'}{g(|x|)g(|y|)} \left( e^{-\lambda_0 t - \kappa|x-y|} + e^{-K_2\rho g(|x|)|y|-\kappa|x-y|} (1 + |x-y|)^\gamma \right) \leq u_t(x, y),
\]
\[
\leq \frac{C''}{g(|x|)g(|y|)} \left( e^{-\lambda_0 t - \kappa|x-y|} + e^{-\Lambda(r)|x-y|} (1 + |x-y|)^\gamma \vee H(t/K_2, x, y) \right),
\]
Proof. We begin with (5.2) and show that for every \( n_0 > R_0 \) there are constants \( c_1, c_2 > 0 \) such that for all \( \tau > 0 \) and \(|x|, |y| \geq n_0 + 3\)
\[
c_1 \left( e^{-\tau g(|x|)} \right) \frac{e^{-\kappa|y|}}{(1 \vee |x-y|)^\gamma} \vee H(\tau, x, y) \leq F(\tau, x, y)
\]
\[
\leq c_2 \left( e^{-\tau g(|x|)} \right) \frac{e^{-\kappa|y|}}{(1 \vee |x-y|)^\gamma} \vee H(\tau, x, y),
\]
where \( F(\tau, x, y) \) is defined in (4.14). By symmetry, we may assume that \(|y| \leq |x|\). Clearly,
\[
F(\tau, x, y) \asymp H(\tau, x, y) + \int_{|y| \leq |z| \leq |x|} f_1(|x-z|) f_1(|y-z|) e^{-\tau g(|z|)} dz.
\]
From this, it is easy to get the lower bound in (5.12), cf. the second line of the proof of Lemma 5.8.

For the proof of the upper bound, we bound the term \( e^{-\tau g(|z|)} \) under the integral by \( e^{-\tau g(|y|)} \), and then we use (3.2).

The estimates of the heat kernel in part [a] follows from a combination of the estimates in Theorem 4.6(c) and (5.12) — note that \( t > \max\{30t_0, K_2 \Lambda(n_0 + 4)\} \) so that \( \rho(t/K_2) > n_0 + 3 \), and take \( \tau = t/K_2 \) in the upper bound and \( \tau = K_2 t \) in the lower bound.

Now we show the estimates (5.11). We have

\[
H(\tau, x, y) \lesssim c_3 \left( \int_{n_0 + 2 < |z| < \rho(\frac{1}{4}\tau)} + \int_{\rho(\frac{1}{4}\tau) < |z| \leq |y|} \right) f_1(|x - z|) f_1(|y - z|) e^{-\tau g(|z|)} \, dz =: I_1 + I_2.
\]

With the argument from the proof of Lemma 5.8 we get

\[
I_1 \lesssim c_4 e^{-\frac{1}{4} \tau g(n_0 + 2)} f(|x|) f(|y|) = e^{-\frac{1}{4} \tau g(n_0 + 2)} e^{-\kappa |x - \gamma| y - \gamma} |y|\gamma
\]

and

\[
I_2 \lesssim c_5 f_1 \left( \frac{|x - y|}{2} \right) e^{-\frac{1}{4} \tau g(|x| / |y|)} \leq c_6 e^{-\frac{1}{4} \tau g(|x| / |y|)} - \frac{1}{2} |x - y| (1 \vee |x - y|)^{-\gamma}.
\]

On the other hand, we can also use (3.3) to get

\[
I_2 \lesssim c_7 \frac{e^{-\kappa |x - y|}}{(1 \vee |x - y|)^\gamma} \int_{\rho(\tau) < |z| \leq |x| / |y|} e^{-\frac{1}{4} \tau g(|z|)} \, dz.
\]

If we combine these estimates, we get the upper bound in (5.11).

Part [b] follows from Theorem 4.6(c) and Lemma 5.11. \( \square \)

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