Shape Invariant Potentials in Supersymmetric Quantum Cosmology

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Abstract

In this brief review, we comment on the concept of shape invariant potentials, which is an essential feature in many settings of \( N = 2 \) supersymmetric quantum mechanics. To motivate its application within supersymmetric quantum cosmology, we present a case study to illustrate the value of this promising tool. Concretely, we take a spatially flat FRW model in the presence of a single scalar field, minimally coupled to gravity. Then, we extract the associated Schrödinger–Wheeler–DeWitt equation, allowing for a particular scope of factor ordering. Subsequently, we compute the corresponding supersymmetric partner Hamiltonians, \( H_1 \) and \( H_2 \). Moreover, we point out how the shape invariance property can be employed to bring a relation among several factor orderings choices for our Schrödinger–Wheeler–DeWitt equation. The ground state is retrieved, and the excited states easily written. Finally, the Hamiltonians, \( H_1 \) and \( H_2 \) are explicitly presented within a \( N = 2 \) supersymmetric quantum mechanics framework.

1 Introduction

Shape Invariant Potentials (SIP) constitute one of the hallmarks of supersymmetric quantum mechanics (SQM), in the sense that it enables a prolific framework to be elaborated. Being more specific, the presence of SIP allows to easily obtain the set of states for a class of quantum systems, suitably based on an elegant algebraic construction. Hence, let us begin by mentioning that there is an algebraic structure associated with the SIP framework. It has gradually been acquiring a twofold relevance and within most of the exactly solvable problems in quantum mechanics [1, 2, 3, 4, 5, 6, 7].

On the one hand, such a structure has provided a method to determine eigenvalues and eigenfunctions, by means of which a spectrum is generated. More specifically, a broad set of those exactly solvable cases can be assembled and assigned within concrete classes; very few exceptions are known [1, 2, 3, 4, 5, 6, 7]. The distinguishing feature of any of such classes is that any exactly solvable case bears a shape invariant potential: supersymmetric partners are of the same shape, and their spectra can be determined entirely by an algebraic procedure comparable to that of the harmonic oscillator. In other words, operators can be defined, namely \( A := \frac{d}{dx} + W(x) \), and its Hermitian conjugate \( A^\dagger := -\frac{d}{dx} + W(x) \), Hamiltonians \( H_1 \) and its superpartner \( H_2 \) being expressed as \( A^\dagger A \) and \( AA^\dagger \), respectively. From this, we can produce and operate with other (more adequate) ladder operators for correspondingly appropriate quantum numbers. These can be maneuvered within a \( J_+, J_3 \) algebra, with comparable features to textbook ladder operators of angular momentum within either \( SU(2) \) or \( SO(3) \); please see [1, 2, 3, 4, 5, 6, 7] for relevant details.

On the other hand, several of these exactly solvable systems also possess a potential algebra: the corresponding Hamiltonian can be written as a Casimir operator of an underlying algebra, which in

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particular cases is of a $SO(2,1)$ nature [2]. Remarkably, there is a close correspondence: shape invariance can be expressed as constraint, which assists in establishing the spectrum; moreover, this shape invariance constraint can be written as an algebraic condition. For a specific set of SIP, the algebraic condition corresponds to the mentioned $SO(2,1)$ potential algebra, where unitary representations become of crucial use. Interestingly, these can be related to that of $SO(3)$; several SIP’s are as such [1, 2, 3, 4, 5, 6, 7]. On the whole, a connection between SIP and potential algebra was attained. Nevertheless, it is also clear that in spite of the structural similarity between $SO(2,1)$ and $SO(3)$ algebras, there are caveats to be aware, related to the differences between those unitary representations.

There are also a couple of additional points that we would like to emphasize. To start with, some quantum states can be retrieved by group theoretical methods. This is further endorsed from the connections between shape invariance and potential algebra, wherein the former is translated into a concrete formulation within the latter [2]. As a result, the scope of the class of potentials where that could be applied was made more prominent [1]. Furthermore, other classes have been explored, related to harmonic oscillator induced second order differential equations, bearing group and algebra features, which subsequently allowed more SIP to be found [8, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12]. In particular, this was further extended toward graded algebras in [8].

Secondly, these algebraic/group theory procedures (within the concrete use of algebras such as $SO(2,1)$ or $SO(3)$), have a striking resemblance to the approach and descriptive language used in [13, 14, 15, 16]; a review of this idea is found in [17]. Therein, it was pointed that intertwining boundary conditions, the algebra of constraints and hidden symmetries in quantum cosmology could be quite fruitful. Specifically, group/algebraic properties within ladder operators, either from angular momentum or from within the explicit presence of specific matter fields (and their properties), determined a partition of wave functions and boundary conditions, according to the Bargmann index [13, 14, 15, 16]. Moreover, we proposed in [17] to extend this framework towards SIP, which could include well known analytically solvable cosmological cases. Being more clear, provided we identify integrability in terms of the shape invariance conditions, we could eventually import those specific features of SQM towards quantum cosmology [18, 19]. That was the challenge we laid out in [17], which is still to be addressed: we hope our review paper herein can further enthuse someone to pick this up. A somewhat related and interesting direction to explore is to also consider an elaboration following [20, 21]. Specifically, accommodating the lines in [13, 14, 15, 16, 17] plus supersymmetry (SUSY) [18, 19].

And thirdly, the interest in the above elements notwithstanding, there are still obstacles that ought to be frankly mentioned. Namely, about the scope of the usefulness of SIP in quantum cosmology. In fact, the list of SIP is quite restrictive, and most potentials therein do not emerge naturally within a minisuperspace. A few do, but for very particular case studies [22]. The classification of spatial geometries upon the Bianchi method implies that the potentials extracted from the gravitational degrees of freedom are very specific. Any 'broadness' can be introduced by inserting (i) very specific matter fields into the minisuperspace (and therein we ought to be using realistic potentials (as indicated by particle physics)) or instead, (ii) try more SIP fitted choices but at the price of being very much ad hoc i.e., an artificial selection. Nevertheless, the list of SIP and similar cases where the algebraic tools could be adopted, has been extended. Although not a strong positive endorsement, there is work [23, 24, 25, 26, 27, 28, 29, 30, 31, 32] that allow us to consider that eventually an extended notion of SIP may be soundly established, such that more cosmological minisuperspaces can be discussed within (see e.g., [22]). For the moment, this is a purpose set in construction and where this review paper aims to enthuse and promote towards.

Upon this introductory section, this review is structured as follows. In section 2, we summarize features of SQM that we will be employing. In particular, a few technicalities about SIP will be presented in subsection 2.2. Then, in section 3, we take a case study, typically a toy model, by means of which we aim to promote work in SUSY quantum cosmology with the novel perspective of SIP. We emphasize that this is a line of investigation that has not yet been attempted before (see subsection 3.3 for details). Section 4 conveys a Discussion and suggestions of Outlook for future research work.

2 Supersymmetric Quantum Mechanics

In this section, let us present a brief review of some of the pillars that characterize SQM. Then, we
proceed to add a summary of the shape invariance concept. This section contains neither new results nor any innovated procedure, but only a very short overview of the results presented within seminal papers, e.g., [33, 34, 35].

2.1 Hamiltonian formulation of supersymmetric quantum mechanics

In order to describe SQM, let us start with the Schrödinger equation:

\[ H \Psi_n(x) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \Psi_n(x) = E_n \Psi_n(x). \]  

We assume that the ground state wave function \( \Psi_0(x) \) (that has no nodes\(^1\)) associated with a potential \( V_1(x) \) is known. Then, by assuming that the ground state energy \( E_0 \) to be zero, the Schrödinger equation for this ground state reduces to

\[ H_1 \Psi_0(x) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x) \right] \Psi_0(x) = 0, \]  

which leads to construct the potential \( V_1(x) \):

\[ V_1(x) = \frac{\hbar^2}{2m} \frac{\Psi_0''(x)}{\Psi_0(x)}, \]  

where a prime denotes differentiation with respect to \( x \). It is straightforward to factorize the Hamiltonian from the operators as follows,

\[ A = \frac{\hbar}{\sqrt{2m}} \left[ \frac{d}{dx} - \frac{\Psi_0'(x)}{\Psi_0(x)} \right], \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \left[ \frac{d}{dx} + \frac{\Psi_0'(x)}{\Psi_0(x)} \right], \]  

through the equation

\[ H_1 = A^\dagger A. \]  

In order to construct the SUSY theory related to the original Hamiltonian \( H_1 \), the next step is to define another operator by reversing the order of \( A \) and \( A^\dagger \), i.e., \( H_2 \equiv AA^\dagger \), by which we indeed get the Hamiltonian corresponding to a new potential \( V_2(x) \):

\[ H_2 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2(x), \]  

where

\[ V_2(x) = -V_1(x) + \frac{\hbar}{m} \left[ \frac{\Psi_0'(x)}{\Psi_0(x)} \right]^2. \]  

The potentials \( V_1(x) \) and \( V_2(x) \) have been referred to as the supersymmetric partner potentials. It should be noted that \( H_2 \), as the partner Hamiltonian corresponding to \( H_1 \), is in general not unique, but there is a class of Hamiltonians \( H^{(m)} \) that can be partner Hamiltonians\(^2\). This point has been specified [36, 33, 37], conveying to a better understanding of the relationship between SQM and the inverse scattering method, established by Gelfand and Levitan [38, 39].

In SQM, instead of the ground state wave function \( \Psi_0(x) \) associated with \( H_1 \), the superpotential \( W(x) \) is introduced, being related to \( \Psi_0(x) \) and its first derivative (with respect to \( x \)) by means of

\[ W(x) = -\frac{\hbar}{\sqrt{2m}} \left( \frac{\Psi_0'(x)}{\Psi_0(x)} \right). \]  

\(^1\)In [35], the excited wave functions have also been studied. More concretely, instead of the choice of a nonsingular superpotential that is based on the ground state wave function \( \Psi_0(x) \), a generalized procedure was presented to construct all possible superpotentials.

\(^2\)Cf. next subsection, concretely about Eq.(28).
At this stage, it is worthy to express the operators $A$ and $A^\dagger$ and the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ in terms of the superpotential. Therefore, using (8), equations (3), (4) and (7) are rewritten as

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad \quad (9)$$

$$V_1(x) = -\frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x), \quad V_2(x) = \frac{\hbar}{\sqrt{2m}} W'(x) + W^2(x), \quad \quad (10)$$

where the expressions for $V_1$ and $V_2$ in (10) constitute Riccati equations. Equations (9) and (10) imply that $W'(x)$ is proportional to the commutator of the operators $A$ and $A^\dagger$, and $W^2(x)$ is the average of the partner potentials.

It is easy to show that the wave functions, the energy eigenvalues and the S-matrices of both the Hamiltonians $H_1$ and $H_2$ are related. Let us merely outline the results and abstain from proving them. In this regard, we take $\Psi_n^{(1)}$ and $\Psi_n^{(2)}$ as the eigenfunctions of $H_1$ and $H_2$, respectively. Moreover, we denote their corresponding energy eigenvalues with $E_n^{(1)} \geq 0$ and $E_n^{(2)} \geq 0$ where $n = 0, 1, 2, 3, ..., n$ is the number of the nodes in the wave function. It is straightforward to show that supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ possess the same energy spectrum. However, we should note that for the ground state energy $E_0^{(1)} = 0$ associated with the potential $V_1(x)$, there is no corresponding level for its partner $V_2(x)$. Concretely, it has been shown that

$$E_n^{(2)} = E_n^{(1)}, \quad E_0^{(1)} = 0, \quad \quad (11)$$

$$\Psi_n^{(2)} = \left[ E_{n+1}^{(1)} \right]^{-\frac{1}{2}} A\Psi_n^{(1)}, \quad \quad (12)$$

$$\Psi_{n+1}^{(1)} = \left[ E_n^{(2)} \right]^{-\frac{1}{2}} A^\dagger\Psi_n^{(2)}. \quad \quad (13)$$

In what follows, let us express some facts. (i) If the ground-state wave function $\Psi_0^{(1)}$, which is given by $(A\Psi_0^{(1)} = 0)$

$$\Psi_0^{(1)} = N_0 \exp \left[ -\int^x W(x') dx' \right], \quad \quad (14)$$

is square integrable, then the ground state of $H_1$ has zero energy $\langle E_0 = 0 \rangle$ [33]. For this case, it can be shown that the SUSY is unbroken. (ii) if the eigenfunction $\Psi_{n+1}^{(1)}$ of $H_1$ ($\Psi_n^{(2)}$ of $H_2$) is normalized, then the $\Psi_n^{(2)}$ ($\Psi_n^{(1)}$), will be also normalized. (iii) Assuming the eigenfunction $\Psi_0^{(1)}$ with eigenvalue $E_0^{(1)}$ ($\Psi_n^{(2)}$ with eigenvalue $E_n^{(2)}$) corresponds to the Hamiltonian $H_1$ ($H_2$), it is easy to show that $A\Psi_n^{(1)}$ ($A^\dagger\Psi_n^{(2)}$) will be an eigenfunction of $H_2$ ($H_1$) with the same eigenvalue. (iv) In order to destroy (create) an extra node in the eigenfunction as well as convert an eigenfunction of $H_1$ ($H_2$) into an eigenfunction of $H_2$ ($H_1$) with the same energy, we apply the operator $A$ ($A^\dagger$). (v) The ground state wave function of $H_1$ has no SUSY partner. (v) Applying the operator $A$ ($A^\dagger$), all the eigenfunctions of $H_2$ ($H_1$, except for the ground state) can be reconstructed from those of $H_1$ ($H_2$); please see Fig.(1).

It has been believed that this fascinating procedure, which leads to understand the degeneracy of the spectra of $H_1$ and $H_2$, can be provided from applying the properties of the SUSY algebra. Therefore, let us consider a matrix SUSY Hamiltonian (which is part of a closed algebra including both bosonic and fermionic operators with commutation and anti-commutation relations) containing both the Hamiltonians $H_1$ and $H_2$ [35]:

$$H = \left[ \begin{array}{cc} H_1 & 0 \\ 0 & H_2 \end{array} \right]. \quad \quad (15)$$

Supersymmetric quantum mechanics begins with a set of two matrix, $Q$ and $Q^\dagger$, operators, known as supercharges

$$Q = \left[ \begin{array}{cc} 0 & 0 \\ A & 0 \end{array} \right], \quad \quad (16)$$
Figure 1: The energy levels of $V_1(x)$ and $V_2(x)$ as two supersymmetric partner potentials. The figure is associated with unbroken SUSY. It is seen that, except an extra state $E^{(1)}_0 = 0$, the other energy levels are degenerate. Moreover, in this figure, it is shown that how the operators $A$ and $A^\dagger$ connect eigenfunctions.

\[
Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}.
\]  

(17)

The matrix $H$ is part of a closed algebra in which both bosonic and fermionic operators with commutation and anti-commutation relations are included, such that the bosonic degrees of freedom are changed into the fermionic ones and vice versa by the supercharges.

It is straightforward to show that

\[
\{Q, Q^\dagger\} = H, \quad \{Q, Q\} = 2Q^2 = \{Q^\dagger, Q^\dagger\} = 2 (Q^\dagger)^2 = 0,
\]  

(18)

(19)

(20)

by which the closed superalgebra $\mathfrak{sl}(1,1)$ is described [40] (see also subsection 3.3). Note that the relations (18) are responsible for the degeneracy.

In SQM, when the two partner potentials have continuum spectra, it is possible to relate the reflection and transmission coefficients. A necessary condition for providing scattering in both of the partner potentials is that they must be finite when $x \to -\infty$ or $x \to \infty$.

2.2 Shape invariance and solvable potentials

In the context of the non-relativistic quantum mechanics, there are a number of known potentials (e.g., Coulomb, harmonic oscillator, Eckart, Morse, and Pöschl–Teller) for which we can solve the corresponding Schrödinger equation analytically and determine all the energy eigenvalues and eigenfunctions explicitly. In this regard, the following questions naturally arise: Why are just some potentials solvable? Is there any underlying symmetry property? What is this symmetry?

Gendenshtein was the first to answer these questions by introducing the shape invariance concept [41]. In fact, for such potentials, all the bound state energy eigenvalues, eigenfunctions and the scattering matrix can be retrieved from applying the generalized operator method, which is essentially equivalent to the Schrödinger’s method of factorization [42, 43].

In [41], the relationship between SUSY, the hierarchy of Hamiltonians, and solvable potentials has been investigated from an interesting perspective (for detailed discussions see, for instance, [33, 35]). In what follows, let us describe briefly the shape invariance concept. “If the pair of SUSY partner potentials $V_{1,2}(x, b)$ are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant” [44]. Let us be more precise. Consider a pair of SUSY partner potentials, $V_{1,2}(x)$, as defined in (10). If the profiles of these potentials are such that they satisfy the relationship

\[
V_2(x, b) = V_1(x, b_1) + R(b_1),
\]  

(21)

where the parameter $b_1$ is some function of $b$, say given by $b_1 = f(b)$, the potentials $V_{1,2}(x)$ are said to bear shape invariance. In other words, to be associated within shape invariance the potentials $V_{1,2}$,
while sharing a similar coordinate dependence, can at most differ in the presence of some parameters. To make the definition of shape invariance clear, consider for example

\[ W = b \tanh \left( \frac{\sqrt{2m}}{\hbar} x \right). \]  

(22)

Then, inserting this superpotential into (10) gives us

\[
V_1(x, b) = -\frac{b(b + 1)}{\cosh^2 \left( \frac{\sqrt{2m}}{\hbar} x \right)} + b^2,
\]

\[
V_2(x, b) = -\frac{b(b - 1)}{\cosh^2 \left( \frac{\sqrt{2m}}{\hbar} x \right)} + b^2
\]

(23)

The above expressions show that one can rewrite \( V_2 \) in terms of \( V_1 \), as expressed in (10) where, in this example, \( b_1 = b - 1 \), and \( R(b_1) := 2b_1 + 1 \). Thus, the potentials \( V_{1,2} \) bear shape invariance in accordance with the definition (10). Then, to use the shape invariance condition, let us assume that (10) holds for a sequence of parameters, \( \{b_k\}_{k=0}^{\infty} \), where

\[
b_k = f \circ f \circ f \circ \ldots \circ f(b) = f^k(b), \quad k = 0, 1, 2, ..., \quad b_0 := b.
\]

(24)

Consequently,

\[
H_2(x, b_k) = H_1(x, b_{k+1}) + R(b_k). \quad (25)
\]

Now, we write \( H^{(0)} = H_1(x, b), \ H^{(1)} = H_2(x, b), \) and we define \( H^{(m)} \) as

\[
H^{(m)} := -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1(x, b_m) + \sum_{k=1}^{m} R(b_k) = H_1(x, b_m) + \sum_{k=1}^{m} R(b_k).
\]

(26)

Using (25), we can extract \( H^{(m+1)} \) as

\[
H^{(m+1)} = H_2(x, b_m) + \sum_{k=1}^{m} R(b_k). \quad (27)
\]

Therefore, in this way, we are able to set up a hierarchy of Hamiltonians \( H^{(k)} \) for various \( k \) values.

Employing condition (21) and the hierarchy of Hamiltonians [35], the energy eigenvalues and eigenfunctions have been obtained for any shape invariant potential when SUSY is unbroken. It should be noted that there is a correspondence between the condition (21) (associated with SQM) and the required mathematical condition applied in the method of the factorization of the Hamiltonian [45]. Although the terminology and ideas associated with these methods are different, they can be considered as the special cases of the procedure employed to handle second-order linear differential equations [46, 47]. Notwithstanding the above, it has been believed that a better understanding of analytically solvable potentials could be achieved by SUSY and shape invariance. Let us elaborate more on this aspect.

\( H_2 \) contains the lowest state with a zero energy eigenvalue, according to the SQM concepts discussed in subsection (2). As a result of (11), the lowest energy level of \( H^{(m)} \) has the value of

\[
E_0^{(m)} = \sum_{k=1}^{m} R(b_k). \quad (28)
\]

Therefore, it is simple to realize that because of the chain \( H^{(m)} \rightarrow H^{(m-1)} \rightarrow \ldots \rightarrow H^{(1)} := H_2 \rightarrow H^{(0)} := H_1 \), the \( n \)th member in this sequence carries the \( n \)th level of the energy spectra of \( H^{(0)} \) (or \( H_1 \)), namely [33]

\[
E_n^{(0)} = \sum_{k=1}^{n} R(b_k), \quad E_0^{(0)} = 0. \quad (29)
\]
Let us now return to the example (22). We rewrite (21) as
\[ V_1(x, b) = V_2(x, b - 1) + b^2 - (b - 1)^2. \] (30)

We can generate \( b_k \) from \( b_0 = b \) as \( b_k = b - k \). Hence, the energy spectrum from \( V_1(x, b) \) yields
\[ E_n^{(0)} = \sum_{k=1}^{n} R(b_k) = \sum_{k=1}^{n} (b^2 - b_k^2) = b^2 - b_n^2 = b^2 - (b - n)^2. \] (31)

It is worthy to note that, according to the requirement (21), the well-known solvable potentials (such as those were listed in the first paragraph of this subsection) are all shape invariant, and, therefore, their energy eigenvalue spectra is given by (29). “In [41], Gendenshtein then conjectured that shape invariance is not only sufficient but may even be necessary for a potential to be solvable [33].” Moreover, by applying SUSY, it is also possible to retrieve the bound-state energy eigenfunctions of \( H_1 \) for shape invariant potentials [34]. In particular, in the same paper, by taking \( \Psi_0(x, b) \) as the ground-state wave function of \( H_1 \) (which is given by (14)), and employing relation (13), a relation is obtained for \( n \)-th state eigenfunctions \( \Psi_n(x, b) \) as
\[ \Psi_n^1(x, b) = A^n(x, b)A^n(x, b_1)\ldots A^n(x, b_{n-1})\Psi_0^1(x, b). \] (32)

For later convenience, let us concentrate on a specific shape invariance that only involving translation of the parameter \( b_0 \) with a translation step \( \eta \) [48] (for other kind of relations between the parameters, see, for instance, [35]):
\[ b_1 = b_0 + \eta. \] (33)

It is feasible to introduce a translation operator as
\[ T(b_0) = \exp \left( \eta \frac{\partial}{\partial b_0} \right), \quad T^{-1}(b_0) = T^\dagger(b_0) = \exp \left( -\eta \frac{\partial}{\partial b_0} \right), \] (34)
which act merely on objects defined on the parameter space.

By composing the translation and bosonic operators, we can construct the following generalized creation and annihilation operators:
\[ B_1(b_0) = A^\dagger(b_0)T(b_0), \] (35)
\[ B_2(b_0) = T^\dagger(b_0)A(b_0). \] (36)

Applying the shape invariant potentials to solve the Schrödinger equation is similar to the factorization method employed to the case of the harmonic oscillator potential [48]. Therefore, we have
\[ B_2(b_0)A(b_0)\Psi_0(x; b_0) = A(b_0)\Psi_0(x; b_0) = 0. \] (37)

In order to obtain the excited states, the creation operator should repeatedly act on \( \Psi_0(x; b_0) \):
\[ \Psi_n(x; b_0) = [B_1(b_0)]^n \Psi_0(x; b_0). \] (38)

We should note that the translation operators \( T \) and \( T^\dagger \) and ladder operators, \( A \) and \( A^\dagger \) do not commute with any \( b_k \)-dependent and any \( x \)-dependent objects, respectively. Therefore, the generalized creation and annihilation operators \( (B_1 \) and \( B_2) \) act on the objects defined on the dynamical variable space and the objects defined on parameter space via the bosonic operators and the translation operators, respectively. According to (37), we have
\[ \Psi_0(x; b_0) \propto \exp \left( -\int^x W(\tilde{x}; b_0)d\tilde{x} \right), \] (39)
which is transformed by a normalization constant (that should, in general, depend on parameters \( b \)) into a relation of equality. Concretely, the action of the generalized operators affects in determining such a normalization constant; for more details, see, [49].
Now let us obtain the relations of the energy eigenvalues and energy spectrum. From using (34), we can write

\[ R(b_n) = T(b_0)R(b_{n-1})T^*(b_0), \]  

(40)

where

\[ b_n = b_0 + n\eta \]  

(41)

is a generalized version of (33). Employing (40), we get

\[ R(b_n)B_1(b_0) = B_1(b_0)R(b_{n-1}). \]  

(42)

Equations (40) and (42) yield a commutation relation as

\[ [H_1, (B_1)^n] = \left( \sum_{k=1}^{n} R(b_k) \right) (B_1)^n. \]  

(43)

Applying (43) on the ground state of \( H_1 \), i.e., \( \Psi_0(x; b_0) \), it is seen that \( (B_1(b_0))^n \Psi_0(x; b_0) \) is also an eigenfunction of \( H_1 \) with eigenvalue \( E^{(1)}_n \) given by (29). Therefore, the energy spectrum is:

\[ E_n = E_0 + E^{(1)}_n, \]  

(44)

where the ground state energy \( E_0 \) is obtained from either

\[ H_1 = H - E_0, \]  

(45)

or, equivalently,

\[ W(x; b) - W'(x; b) = V(x) - E_0 = V_1(x). \]  

(46)

Finally, it should be noted that the above established algebraic approach is self-consistent. More concretely, by considering supersymmetric and shape invariance properties of the system, it can be applied as an appropriate method for obtaining not only the eigenvalues and eigenfunctions of the bound state of a Schrödinger equation, but also exact resolutions for this equation [48].

3 SUSY Quantum Cosmology

In order to apply the formalism presented in the previous section, let us investigate a homogeneous and isotropic cosmology, in the context of General Relativity (GR) together with a single scalar field, \( \phi \), minimally coupled to gravity.

3.1 A case study: classical setting

By considering the Friedmann–Lemaître–Robertson–Walker (FLRW) line element\(^3\)

\[ ds^2 = N(t)dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right\}, \]  

(47)

the ADM\(^4\) Lagrangian will be \(^5\)

\[ L_{\text{ADM}} = -\frac{3}{N}a\dot{a}^2 + 3kaN + a^3 \left( \frac{\dot{\phi}^2}{2N} - NV(\phi) \right), \]  

(48)

\(^3\)Throughout this paper we work in natural units where \( \hbar = c = k_B = 1. \)

\(^4\)Adler-Deser-Misner (ADM); see [50] for more details.

\(^5\)In this work, we consider a framework in which the scalar field is minimally coupled to gravity, see also [51, 52, 53]. Instead, one can choose other interesting gravitational models where \( \phi \) is non-minimally coupled, see for instance, [54, 55, 56, 57, 58].
where an over-dot denotes a differentiation with respect to the cosmic time \(t\); \(N(t)\) is a lapse function, \(a(t)\) is the scale factor, \(V(\phi)\) is a scalar potential and \(k = \{-1, 0, 1\}\) is the spatial curvature constant associated with open, flat and closed universes, respectively.

In this work, let us consider the scalar potential \(V(\phi)\) to be in the form \([59, 60]\)
\[
V(\phi) = \lambda + \frac{m^2}{2\alpha^2} \sinh^2(\alpha\phi) + \frac{\vartheta}{2\alpha^2} \sinh(2\alpha\phi),
\]
(49)
where \(\lambda\) may be related to the cosmological constant; \(m^2 = \partial^2 V/\partial \phi^2|_{\phi=0}\) is a mass squared parameter; \(\alpha^2 = 3/8\) and \(\vartheta\) is a coupling parameter. Moreover, we will now investigate only the spatially flat FLRW universe. For this case, it has been shown that an oscillator–ghost–oscillator system is produced \([60, 61, 62, 63, 64, 65]\). More concretely, by applying the following transformations \([66, 67, 68, 69]\)
\[
X = \frac{a^2}{\alpha} \cosh(\alpha\phi), \quad Y = \frac{a^2}{\alpha} \sinh(\alpha\phi),
\]
(50)
the Lagrangian (48) transform into
\[
L_{\text{ADM}} = -\frac{1}{2N} \xi^\top J \xi + \frac{N}{2} \xi^\top MJ \xi,
\]
(51)
where
\[
\xi := \begin{pmatrix} X \\ Y \end{pmatrix}, \quad M := \begin{pmatrix} 2\lambda\alpha^2 & -\vartheta \\ \vartheta & 2\lambda\alpha^2 - m^2 \end{pmatrix}, \quad J := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
(52)
It is straightforward to decouple (51) into normal modes \(\gamma := \Sigma^{-1} \xi\) by means of
\[
\gamma := \begin{pmatrix} u \\ v \end{pmatrix}, \quad \Sigma := \begin{pmatrix} -m - \sqrt{m^2 - 4\vartheta^2} & m + \sqrt{m^2 - 4\vartheta^2} \\ \frac{2\vartheta}{1} & \frac{2\vartheta}{1} \end{pmatrix},
\]
(53)
which diagonalize the matrix \(M\) as follows
\[
\Sigma^{-1} M \Sigma = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \quad \omega_{1,2} = \frac{3\lambda}{4} + \frac{m^2}{2} \pm \frac{\sqrt{m^4 - 4\vartheta^2}}{2}.
\]
(54)
Thus, we retrieve the Lagrangian associated with a 2D oscillator–ghost–oscillator:
\[
L_{\text{ADM}}(u, v) = -\frac{1}{2N} \dot{\gamma}^\top \mathcal{I} \dot{\gamma} + \frac{N}{2} \gamma^\top \mathcal{J} \gamma
= -\frac{1}{2} \left\{ \frac{1}{N} \dot{u}^2 - \omega_1^2 Nu^2 \right\} - \left\{ \frac{1}{N} \dot{v}^2 - \omega_2^2 Nv^2 \right\},
\]
(55)
where \(\mathcal{I} := \Sigma^\top J \Sigma\), and \(\mathcal{J} := \Sigma^\top MJ \Sigma\). The conjugate momenta corresponding to \(u\) and \(v\) are:
\[
p_u = \frac{\dot{u}}{N}, \quad p_v = -\frac{\dot{v}}{N}.
\]
(56)
Moreover, the classical Euler–Lagrange equations are given by
\[
\frac{d}{dt} \left( \frac{\dot{u}}{N} \right) + N\omega_1^2 u = 0, \quad \frac{d}{dt} \left( \frac{\dot{v}}{N} \right) + N\omega_2^2 v = 0.
\]
(57)
It is straightforward to show that the Hamiltonian corresponding to the ADM Lagrangian (55) is:
\[
H_{\text{ADM}} = -\frac{N}{2} \left\{ (p_u^2 + \omega_1^2 u^2) - (p_v^2 + \omega_2^2 v^2) \right\},
\]
(58)
which, for the gauge \(N = 1\), yields
\[
u(t) = u_0 \sin(\omega_1 t - \theta), \quad v(t) = v_0 \sin(\omega_2 t).
\]
(59)
In (59), \(\theta\) is an arbitrary phase factor. From using the Hamiltonian constraint, we obtain \(\omega_1 u_0 = \omega_2 v_0\). It is also seen that the classical paths corresponding to solutions (59), in the configuration space \((u, v)\), are the generalized Lissajous ellipses.
3.2 Quantization

In order to establish a quantum cosmological model corresponding to our herein model, let us proceed with the Wheeler–DeWitt equation. The canonical quantization of (58) gives

$$H\Psi(u, v) = \left(- \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \omega_1^2 u^2 - \omega_2^2 v^2 \right) \Psi(u, v) = 0.$$  (60)

Equation (60) is separable and we can obtain a solution as

$$\Psi_{n_1, n_2}(u, v) = \alpha_{n_1}(u)\beta_{n_2}(v),$$  (61)

where

$$\alpha_n(u) = \left(\frac{\omega_1}{\pi}\right)^{1/4} \frac{H_n(\sqrt{\omega_1} u)}{\sqrt{2^{n!}}} e^{-\omega_1 u^2/2},$$  (62)

$$\beta_n(v) = \left(\frac{\omega_2}{\pi}\right)^{1/4} \frac{H_n(\sqrt{\omega_2} v)}{\sqrt{2^{n!}}} e^{-\omega_2 v^2/2}.$$  (63)

In relations (62) and (63), $H_n(x)$ stands for the Hermite polynomials. Moreover, we should note that the Hamiltonian constrain relates the parameters of the model as

$$\left(n_1 + \frac{1}{2}\right) \omega_1 = \left(n_2 + \frac{1}{2}\right) \omega_2, \quad n_1, n_2 = 0, 1, 2, \ldots$$  (64)

The recovery of classical solutions from the corresponding quantum model is one of the essential elements of quantum cosmology. For this aim, a coherent wave packet with reasonable asymptotic behavior in the minisuperspace is often constructed, peaking near the classical trajectory. We can herewith produce a widespread wave packet solution

$$\Psi(u, v) = \sum_{n_1, n_2} C_{n_1, n_2} \alpha_{n_1}(u)\beta_{n_2}(v),$$  (65)

where the summing is restricted to overall values of $n_1$ and $n_2$ satisfying the relation (64). Let us consider the simplest case which is when $\omega_1 = \omega_2 = \omega$, which means $m^2 = 2\vartheta$ in definition of scalar field potential (49). Then, the wave packet will be

$$\Psi(u, v) = \sqrt{\frac{\omega}{\pi}} \sum_{n=0}^{\infty} \frac{C_n}{n! 2^n} \exp\left\{\left(\frac{\omega}{2}(u^2 + v^2)\right)\right\} H_n(\sqrt{\omega} u) H_n(\sqrt{\omega} v),$$  (66)

where $C_n$ is a complex constant. We apply the following identity to create a coherent wave packet with suitable asymptotic behavior in the minisuperspace, peaking around the classical trajectory

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) H_n(y) = \frac{1}{\sqrt{1-t^2}} \exp\left\{\left(\frac{2txy - t^2(x^2 + y^2)}{2(1-t^2)}\right)\right\}.$$  (67)

Using this identity and choosing the coefficients $C_n$ in (66) to be $C_n = B2^n \tanh \xi$, with $B$ and $\xi$ are arbitrary complex constants, we obtain

$$\Psi(u, v) = C \exp\left\{-\frac{\omega}{4} \cos(2\beta_2) \cosh(2\beta_1)(u^2 + v^2 - 2\eta \tanh(2\beta_1)uv)\right\}$$

$$\times \exp\left\{-\frac{i\omega}{4} \sinh(2\beta_1) \sin(2\beta - 2)(u^2 + v^2 - 2\eta \coth(2\beta_1)uv)\right\},$$  (68)

where $\beta_1$ and $\beta_2$ are the real and imaginary parts of $\xi = \beta_1 + i\beta_2$, respectively, $\eta = \pm 1$, and $N$ is a normalization factor. Fig.(2.a) shows the density plot, and Fig.(2.b) illustrates the contour plot of the wave function for typical values of $\beta_1$, $\beta_2$, and $\eta = 1$ for the following combination of the solutions

$$\Psi(u, v) = \Psi_{\beta_1, \beta_2}(u, v) - \Psi_{\beta_1 + \delta \beta_1, \beta_2 + \delta \beta_2}(u, v).$$  (69)
The classical solutions (59) can easily be represented as the following trajectories (for $\omega_1 = \omega_2 = \omega$)

$$u^2 + v^2 - 2\eta uv \cos(\theta) - \nu_0^2 \sin^2(\theta) = 0. \quad (70)$$

This equation describes ellipses whose major axes make angle $\pi/4$ with the positive/negative $u$ axis according to the choices $\pm 1$ for $\eta$. Also, each trajectory’s eccentricity and size are determined by $\theta$ and $u_0$, respectively. It can be seen that the quantum pattern in Fig.(2) and the classical paths (70) in configuration space, $(u, v)$, have a high correlation.

Let us point how we can introduce a time-evolving wave-function. By employing a canonical transformation on the $(v, p_v)$ sector of the Hamiltonian (58), we observe that in the total Hamiltonian, the momentum associated with the new canonical variable appears linearly. Let us be more precise.

Consider the following canonical transformation $(v, p_v) \rightarrow (T, p_T)$ given by

$$v = \sqrt{\frac{2p_T}{\omega^2}} \sin(\omega_2 T), \quad p_v = \sqrt{\frac{2p_T}{\omega^2}} \cos(\omega_2 T), \quad \{T, p_T\} = 1. \quad (71)$$

It is easy to check out that the inverse map is given by following relations

$$p_T = \frac{1}{2} p_v^2 + \frac{1}{2} \omega_1^2 v^2, \quad T = \frac{1}{\omega_2} \tan^{-1}\left(\frac{\omega_2}{p_v} \frac{v}{p_v}\right). \quad (72)$$

In fact, the new set of phase space coordinates $(T, p_T)$ is related to the harmonic oscillator’s action-angle variables, $(\varphi, p_\varphi)$, by [70, 71]

$$\varphi = \omega_2 T, \quad p_\varphi = \frac{p_T}{\omega_2}. \quad (73)$$

The ADM Hamiltonian (58) simply takes the form

$$H_{ADM} = N \left(\frac{1}{2} p_u^2 + \frac{1}{2} \omega_1^2 u^2 - p_T\right). \quad (74)$$

The classical field equations corresponding to (74) are

$$\begin{cases} \dot{u} = N p_u, & p_u = -N \omega_1^2 u, \\ \dot{T} = -N, & \dot{p}_T = 0. \end{cases} \quad (75)$$

For $N = 1$ we find

$$T = -t, \quad p_T = \text{const}. \quad (76)$$

Thus, the motion in 2D phase space $(T, p_T)$ becomes trivial, i.e., flow paths are straight lines with constant $p_T$. As seen, the second set of solutions for (75) implies that $T$ plays the role of the time parameter. Consequently, the Poisson bracket of the time parameter and super-Hamiltonian does not vanish but instead we have $\{T, H\} = 1 = \{T, p_T\}$, which implies that $T$ is not a Dirac observable, and therefore, we may consider it as a time variable; see for instance, [70] and references therein.
3.3 Supersymmetric quantization

Employing the Hamiltonian constraint upon (74), and then substituting \( p_u = -i \frac{d}{du} \) and \( p_T = -i \frac{\partial}{\partial T} \), we get a Schrödinger–Wheeler–WeDitt equation:

\[
i \frac{\partial}{\partial t} \Psi(u,t) = \left[ -\frac{1}{2} \frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2 \right] \Psi(u,t). \tag{77}\]

In the process of obtaining equation (77), we have further used the following factor ordering procedure:

\[
p_n^2 = -\frac{1}{3} \left( u^\alpha \frac{d}{du} u^\beta \frac{d}{du} u^\gamma + u^\gamma \frac{d}{du} u^\alpha \frac{d}{du} x^\beta + u^\beta \frac{d}{du} u^\gamma \frac{d}{du} u^\alpha \right), \tag{78}\]

where the parameters \( \alpha, \beta, \) and \( \gamma \) satisfy the requirement \( \alpha + \beta + \gamma = 0 \), and we have set \( \frac{1}{3} (\beta^2 + \gamma^2 + \beta \gamma) := l(l + 1) \).

The time independent sector of equation (77) reads

\[
H_l \Psi_n^l(u) = E_n^l \Psi_n^l(u), \tag{79}\]

where

\[
H_l := -\frac{1}{2} \frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2. \tag{80}\]

According to the Eq.(10), we now introduce first-order differential operators

\[
\begin{align*}
A_l &:= \frac{1}{\sqrt{2\omega_1}} \frac{d}{du} + \sqrt{\frac{l+1}{2\omega_1}} u, \\
A_l^\dagger &:= -\frac{1}{\sqrt{2\omega_1}} \frac{d}{du} + \sqrt{\frac{l+1}{2\omega_1}} u.
\end{align*} \tag{81}\]

For \( l \in \mathbb{N} \), we correspondingly obtain the following supersymmetric partner Hamiltonians

\[
\begin{align*}
H_1 &= \omega_1 A_l^\dagger A_l = -\frac{1}{2} \frac{d^2}{du^2} + \frac{(l+1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2 + \omega_1 (l - \frac{3}{2}) = H_l - \omega_1 (l + \frac{3}{2}), \\
H_2 &= \omega_1 A_l A_l^\dagger = -\frac{1}{2} \frac{d^2}{du^2} + \frac{(l+1)(l+2)}{2u^2} + \frac{3}{2} \omega_1^2 u^2 - \omega_1 (l + \frac{1}{2}) = H_{l+1} - \omega_1 (l + \frac{1}{2}).
\end{align*} \tag{82}\]

These two Hamiltonians have the same energy spectrum except the ground state of \( H_2 \)

\[
\begin{align*}
H_l \Psi_n^l &= \left[ E_n^l + \omega_1 - (l + \frac{3}{2}) \right] \Psi_n^l, \\
H_2 \Psi_{n+1}^{(l+1)} &= \left[ E_{n+1}^{(l+1)} - \omega_1 (l + \frac{1}{2}) \right] \Psi_{n+1}^{(l+1)} = \left[ E_n^l - \omega_1 (l + \frac{1}{2}) \right] \Psi_{n+1}^{(l+1)}.
\end{align*} \tag{83}\]

It is seen that these equations refer to the shape-invariance condition, by which we, equivalently, can write

\[
A_{l-1}^\dagger A_{l-1} - A_l A_l^\dagger = \frac{2}{\omega_1}, \tag{84}\]

Thus, altering the sequence of operators \( A_l \) and \( A_l^\dagger \) causes the value of \( l \) to change. This demonstrates how shape-invariance properties link the various factor orderings of the Schrödinger–Wheeler–DeWitt equation (79). Shape-invariant potentials are well recognized for being simple to deal with when using lowering and raising operators, similar to the harmonic oscillator. However, we should note that the commutator of \( A_l \) and \( A_l^\dagger \) does not provide a constant value. Namely,

\[
[A_l, A_l^\dagger] = 1 + \frac{l + 1}{\omega_1 u^2}, \tag{85}\]

which implies that these operators are not suitable to proceed with. As the eigenvalue relation (82) shows, the potentials \( V_1(u; b_0) \) and \( V_2(u; b_1) \) introduced in (9) are given by

\[
\begin{align*}
V_1(u; b_0) &= \frac{b_0 (b_0 + 1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2, \\
V_2(u; b_1) &= \frac{b_1 (b_1 + 1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2. \tag{86}\]
\]

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where \( b_1 = l + 1 \) and \( b_0 = l \). Therefore, in relation (33) this corresponds to \( \eta = 1 \). According to the sub-section 2.2, we presume that replacing \( l + 1 \) by \( l \) in a given operator can be accomplished via a similarity transformation, (34), and so we build an appropriate algebraic structure via translation operator

\[
T(l) = \exp \left( \frac{\partial}{\partial l} \right), \quad T^{-1}(l) = T^\dagger(l) = \exp \left( - \frac{\partial}{\partial l} \right). \tag{87}
\]

Therefore, we introduce the operators

\[
B_l := \frac{1}{\sqrt{2}} T^\dagger(l) A_l, \quad B^\dagger_l := \frac{1}{\sqrt{2}} A_l^\dagger T(l), \quad N^\dagger_B := B^\dagger_l B_l, \tag{88}
\]

which lead us to the simple harmonic oscillator (Heisenberg–Weyl) algebra

\[
[B_l, B^\dagger_l] = 1, \quad [N^\dagger_B, B_l] = -B_l, \quad [N^\dagger_B, B^\dagger_l] = B^\dagger_l. \tag{89}
\]

These commutation relations show that \( B^\dagger_l \) and \( B_l \) are the appropriate creation and annihilation operators for the spectra of our shape-invariant potentials. The action of these operators on normalized eigenfunctions yields

\[
B_l \Psi^l_n = \sqrt{n} \Psi^l_{n-1}, \quad B^\dagger_l \Psi^l_n = \sqrt{n+1} \Psi^l_{n+1}, \quad N^\dagger_B \Psi^l_n = n \Psi^l_n, \quad n = 0, 1, 2, \ldots . \tag{90}
\]

Equation (79) and the last equation of the above set give:

\[
E^l_n = \omega_1 \left( 2n + l + \frac{3}{2} \right). \tag{91}
\]

In addition, the condition \( B_l \Psi^l_0 = 0 \) gives us the ground state of the model universe for factor ordering \( l \) by

\[
\Psi^l_0 = C_l \exp \left\{ -\frac{\omega_1}{2} u^2 - \frac{l+1}{u^2} \right\}, \tag{92}
\]

where \( C_l \) is a normalization constant. The excited states can be easily determined by applying (37).

In what follows, let us complete our procedure by including the Grassmannian variables \( \psi \) and \( \bar{\psi} \), which satisfy

\[
\psi^2 = 0, \quad \bar{\psi}^2 = 0, \quad \psi \bar{\psi} + \bar{\psi} \psi = 1, \tag{93}
\]

involving them into the Hamiltonian (80). By means of such a procedure, we can subsequently construct a supersymmetric extension of our Hamiltonian:

\[
H_{\text{SUSY}} = -\frac{1}{2} \frac{d^2}{du^2} + \frac{l(l+1)}{2u^2} + \frac{1}{2} \omega_1^2 u^2 + \omega_1 \bar{\psi} \psi. \tag{94}
\]

In our herein work, the convention of the left derivative for these variables has been adapted. Up to now, we specified the bosonic creation and annihilation operators \( B \) and \( B^\dagger \) in terms of the dynamical variable \( u \) and its conjugate momenta. Here, we can also introduce fermionic creation and annihilation operators \( C^\dagger = \bar{\psi} \) and \( C = \psi \). Therefore, the Hamiltonian (94) can simply be written as

\[
H_{\text{SUSY}} = 2\omega_1 (B^\dagger B + C^\dagger C). \tag{95}
\]

Adapting the basic commutator and anticommutator brackets

\[
[B, B^\dagger] = 1, \quad \{C, C^\dagger\} = 1, \tag{96}
\]

and considering all the others to be zero, it is easy to show that the operators

\[
N_B = B^\dagger B, \quad N_F = C^\dagger C, \quad Q = B^\dagger C, \quad Q^\dagger = C^\dagger B, \tag{97}
\]

(where the indices \( B \) and \( F \) refer to the bosonic and fermionic quantities, respectively) will be conserved quantities. Namely,

\[
[Q, H_{\text{SUSY}}] = [Q^\dagger, H_{\text{SUSY}}] = 0, \quad [N_B, H_{\text{SUSY}}] = [N_F, H_{\text{SUSY}}] = 0. \tag{98}
\]
Moreover, we have
\[
\begin{align*}
[Q, N_B] &= -Q, \\
[Q^\dagger, N_B] &= Q^\dagger, \\
\omega_1 \{Q, Q^\dagger\} &= H_{\text{SUSY}}, \\
Q^2 &= \frac{1}{2} \{Q, Q\} = 0, \\
(Q^\dagger)^2 &= \frac{1}{2} \{Q^\dagger, Q^\dagger\} = 0.
\end{align*}
\] (99)

From (98) and (99) an explicit algebra is therefore produced, where the Hamiltonian \(H_{\text{SUSY}}\) is a Casimir operator for the whole algebra [72]. If we use a matrix representation, then we can write, alternatively
\[
\psi = C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\psi} = C^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\] (100)

\[
H_{\text{SUSY}} = 2\omega_1 \begin{pmatrix} B_1^1B_1 + 1 & 0 \\ 0 & B_1^1B_1^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},
\] (101)

4 Discussion

This paper is a review that embraces a twofold endorsement. On the one hand, it imports SUSY features (in a quantum mechanical setting). However, the current fact is that SUSY has not (yet...) been found in nature; searches prevail for any evidence, being it directly or indirectly. On the other hand, this review refers to quantum cosmology as a phenomenological domain regarding the full quantization of gravity. Likewise, there is yet no clear-cut observational evidence of such a stage in the very early universe. Whereas the latter is fairly expected as the cosmos is further probed, proceeding gradually to prior times, the former, although alluringly elegant, may just be a formal framework. So, producing a review on a topic involving these two ideas may seem twice likely to raise discomfort. But maybe not; perhaps SUSY quantum cosmology deserves to be kept nearby, just over an arm’s length, so to say\(^6\), if the occasion (or data) emerges to either support it or at least enthuse more research about it. There are still open aspects to appraise and the one brought on this review is among them \([18, 19]\).

In what concerns SUSY quantum cosmology, there have been a few books in the past 30 years or so \([73, 18, 19]\) plus selected reviews on the different procedures that were constructed and subsequently promoted \([74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84]\). Likewise chapters (and sections) about SUSY quantum cosmology in well known textbooks concerning quantum gravity \([85, 86, 87, 88]\). In particular, the direction and extension of \(N = 2\) SQM to SUSY quantum cosmology was led by \([89, 90, 91, 92]\) and subsequently by \([93, 94]\), referring to conformal issues.

Therefore, the opportunity to produce this review enthused us to refer and to explore a specific particular aspect that was indicated as a concrete open problem in SUSY quantum cosmology (Nb. we emphasize many more still remain; cf. in \([18, 19]\)). Concretely, investigating the setting of SIP, fairly present in \(N = 2\) SQM. This framework has been developed previously and independently from SUSY quantum cosmology, to explore issues in supersymmetric quantum field theory, namely SUSY breaking (which the seminal paper \([95]\) has made possible).

Hence, a necessary and yet to be done analysis remained to be elaborated: explicitly considering SIP as we just mentioned within quantum cosmology. I.e., bringing up this possibility and using it intertwined within SUSY quantum cosmology. Thus is a new idea for other researchers to pick up and evolve forward, producing their own assertions. The content of our review is thus very open, conveying a direction for further exploration. It involves algebraic quantum-mechanical aspects that are present when SIP characterizes particular models. It also deals with integrability, which SUSY seems to bring so elegantly.

In this paper, besides contributing with a topical review towards this Special Issue, we also provided a constructive example to illustrate how promising the framework can be. Concretely, we provided a case study, consisting of a spatially flat FRW model in the presence of a single scalar field, minimally

\[^6\]“Ah, but a man’s reach should exceed his grasp, Or what’s a heaven for?”, Robert Browning (in ‘Andrea del Sarto’ l. 97 (1855))
coupled to gravity. We extracted the Schrödinger–Wheeler–DeWitt equation containing a particular set of possible factor ordering. Next, we computed the corresponding supersymmetric partner Hamiltonians. Intriguingly, the shape invariance properties can be related to the several factor orderings of our Schrödinger–Wheeler–DeWitt equation. The ground state was computed and the excited states as well. Consistently, the partner Hamiltonians, were explicitly presented within a $N = 2$ SQM framework.

We made implicitly another suggestion in Section 1 (Introduction). In more detail, we suggested to build a twofold framework. On the one hand, importing the ideas employed in references [13, 14, 15, 16, 17], where the presence of constraints, their algebra plus a natural integrability induces separability in the Hilbert space of solutions for the Wheeler–DeWitt equation. On the other hand, explore, at least to begin with on formal terms, whether any such algebra of the constraints generators for a minisuperspace, would bear any similarity to an algebra of supersymmetry generators. In others words, perhaps producing a sequence of new operators $A_l$ and $A_l^\dagger$, assisting SIP properties but also related to SUSY constraints of a Schrödinger–Wheeler–DeWitt equation similar to (79). SIP are well recognized for being simple to deal. In essence, our suggestion is to explore (i) if there is any relation between SIP and the descriptive report in [13, 14, 15, 16, 17] and, if positive, (ii) apply it within SUSY quantum cosmology.

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References

[1] F Cooper, J N Ginocchio, and A Wipf. Supersymmetry, operator transformations and exactly solvable potentials. *J. Phys. A*, 22(17):3707–3716, sep 1989.

[2] Asim Gangopadhyaya, Jeffry V. Mallow, and Uday P. Sukhatme. Shape invariance and its connection to potential algebra. In Henrik Aratyn, Tom D. Imbo, Wai-Yee Keung, and Uday Sukhatme, editors, *Supersymmetry and Integrable Models*, volume 502, pages 341–350. Springer Berlin Heidelberg, 1998.

[3] A. B. Balantekin. Algebraic approach to shape invariance. *Phys. Rev. A*, 57:4188–4191, Jun 1998.

[4] Asim Gangopadhyaya, Jeffry V. Mallow, and Uday P. Sukhatme. Broken supersymmetric shape invariant systems and their potential algebras. *Phys. Lett. A*, 283:279–284, 2001.

[5] Gang Chen, Zi-Dong Chen, and Pei-Cai Xuan. Exactly solvable potentials of the Klein Gordon equation with the supersymmetry method. *Phys. Lett. A*, 352(4-5):317–320, 2006.

[6] Avinash Khare and Rajat K. Bhaduri. Supersymmetry, shape invariance and exactly solvable noncentral potentials. *Am. J. Phys.*, 62:1008–1014, 1994.

[7] Christiane Quesne. Deformed Shape Invariant Superpotentials in Quantum Mechanics and Expansions in Powers of $\hbar$. *Symmetry*, 12(11):1853, 2020.

[8] V. K. Oikonomou. A relation between $Z_3$-graded symmetry and shape invariant supersymmetric systems. *J. Phys. A*, 47:435304, 2014.

[9] D. Bazeia and Ashok Das. Supersymmetry, shape invariance and the Legendre equations. *Phys. Lett. B*, 715:256–259, 2012.

[10] A. Stahlhofen. Remarks on the equivalence between the shape-invariance condition and the factorisation condition. *J. Phys. A*, 22(8):1053–1058, 1989.

[11] M. A. Jafarizadeh and H. Fakhri. Supersymmetry and shape invariance in differential equations of mathematical physics. *Phys. Lett. A*, 230(3):164–170, 1997.
[12] A.R. Amani and H. Ghorbanpour. Supersymmetry approach and shape invariance for pseudo-harmonic potential. *Acta Phys. Pol. B*, 43(9):1795–1803, 2012.

[13] M. Fathi, S. Jalalzadeh, and P. V. Moniz. Classical Universe emerging from quantum cosmology without horizon and flatness problems. *Eur. Phys. J. C*, 76(10):527, 2016.

[14] S. Jalalzadeh, T. Rostami, and P. V. Moniz. On the relation between boundary proposals and hidden symmetries of the extended pre-big bang quantum cosmology. *Eur. Phys. J. C*, 75(1):38, 2015.

[15] T. Rostami, S. Jalalzadeh, and P. V. Moniz. Quantum cosmological intertwining: Factor ordering and boundary conditions from hidden symmetries. *Phys. Rev. D*, 92(2):023526, 2015.

[16] S. Jalalzadeh and P. V. Moniz. Dirac observables and boundary proposals in quantum cosmology. *Phys. Rev. D*, 89(8):083504, 2014.

[17] S. Jalalzadeh, T. Rostami, and P. V. Moniz. Quantum cosmology: From hidden symmetries towards a new (supersymmetric) perspective. *Int. J. Mod. Phys. D*, 25(3):1630009, 2016.

[18] Paulo Vargas Moniz. *Quantum Cosmology - The Supersymmetric Perspective - Vol. 1*, volume 803 of *Lecture Notes in Physics*. Springer, Berlin, Heidelberg, 2010.

[19] Paulo Vargas Moniz. *Quantum Cosmology - The Supersymmetric Perspective - Vol. 2*, volume 804 of *Lecture Notes in Physics*. Springer, Berlin, Heidelberg, 2010.

[20] Ruben Cordero, Victor D. Granados, and Roberto D. Mota. Novel Complete Non-Compact Symmetries for the Wheeler–DeWitt Equation in a Wormhole Scalar Model and Axion-Dilaton String Cosmology. *Class. Quant. Grav.*, 28:185002, 2011.

[21] Ruben Cordero and Roberto D. Mota. New exact supersymmetric wave functions for a massless scalar field and axion–dilaton string cosmology in a FRWL metric. *Eur. Phys. J. Plus*, 135(1):78, 2020.

[22] José Socorro García Díaz, Marco A Reyes, Carlos Villaseñor Mora, and Edgar Condori Pozo. Supersymmetric quantum mechanics: Two factorization schemes and quasi-exactly solvable potentials. In *Panorama of Contemporary Quantum Mechanics-Concepts and Applications*. IntechOpen, 2019.

[23] Rajat K Bhaduri, Jamal Sakhr, D W L Sprung, Ranabir Dutt, and Akira Suzuki. Shape invariant potentials in susy quantum mechanics and periodic orbit theory. *J. Phys. A*, 38(11):L183–L189, Mar 2005.

[24] Jonathan Bougie, Asim Gangopadhyaya, and Jeffry V Mallow. Method for generating additive shape-invariant potentials from an euler equation. *J. Phys. A*, 44(27):275307, Jun 2011.

[25] Elso Drigo Filho and M A Cândido Ribeiro. Generalized ladder operators for shape-invariant potentials. *Phys. Scripta*, 64(6):548–552, Jan 2001.

[26] Asim Gangopadhyaya, Jeffry V. Mallow, Constantin Rasinariu, and Jonathan Bougie. Exactness of swkb for shape invariant potentials. *Phys. Lett. A*, 384(28):126722, 2020.

[27] Jonathan Bougie, Asim Gangopadhyaya, Jeffry Mallow, and Constantin Rasinariu. Supersymmetric quantum mechanics and solvable models. *Symmetry*, 4(3):452–473, 2012.

[28] José F Caruñena and Arturo Ramos. Shape-invariant potentials depending on a parameters transformed by translation. *J. Phys. A: Mathematical and General*, 33(17):3467–3481, may 2000.

[29] Wang-Chang Su. Shape invariant potentials in second-order supersymmetric quantum mechanics. *Journal of Physical Mathematics*, 3(none):1 – 12, 2011.

[30] Shi-Hai Dong. *Factorization Method in Quantum Mechanics*. Springer, 2007.

[31] Yuta Nasuda and Nobuyuki Sawado. SWKB Quantization Condition for Conditionally Exactly Solvable Systems and the Residual Corrections. *arXiv e-prints*, 2021.
[32] Asim Gangopadhyaya, Jeffry V Mallow, and Constantin Rasinariu. *Supersymmetric quantum mechanics: An introduction*. World Scientific, 2017.

[33] Fred Cooper, Joseph N. Ginocchio, and Avinash Khare. Relationship Between Supersymmetry and Solvable Potentials. *Phys. Rev. D*, 36:2458–2473, 1987.

[34] Ranabir Dutt, Avinash Khare, and Uday P Sukhatme. Supersymmetry, shape invariance, and exactly solvable potentials. *Am. J. Phys.*, 56(2):163–168, 1988.

[35] Fred Cooper, Avinash Khare, and Uday Sukhatme. Supersymmetry and quantum mechanics. *Phys. Rept.*, 251:267–385, 1995.

[36] Michael Martin Nieto. Relationship Between Supersymmetry and the Inverse Method in Quantum Mechanics. *Phys. Lett. B*, 145:208–210, 1984.

[37] D. L. Pursey. Isometric operators, isospectral hamiltonians, and supersymmetric quantum mechanics. *Phys. Rev. D*, 33:2267–2279, Apr 1986.

[38] I. M. Gel’fand and B. M. Levitan. On the determination of a differential equation from its spectral function. *Izv. Akad. Nauk SSSR Ser. Mat.*, 15(4):309–360, 1951.

[39] P. B. Abraham and H. E. Moses. Changes in potentials due to changes in the point spectrum: Anharmonic oscillators with exact solutions. *Phys. Rev. A*, 22:1333–1340, Oct 1980.

[40] V. G. Kac. A sketch of Lie superalgebra theory. *Comm. Math. Phys.*, 53(1):31 – 64, 1977.

[41] L. E. Gendenshtein. Derivation of Exact Spectra of the Schrodinger Equation by Means of Supersymmetry. *JETP Lett.*, 38:356–359, 1983.

[42] E. Schrödinger. A method of determining quantum-mechanical eigenvalues and eigenfunctions. *Proc. R. Ir. Acad., A Math. phys. sci.*, 46:9–16, 1940.

[43] L. Infeld and T. E. Hull. The Factorization Method. *Rev. Mod. Phys.*, 23:21–68, Jan 1951.

[44] Avinash Khare. Supersymmetry in quantum mechanics. *Pramana-J. Phys.*, 49(1):41–64, 1997.

[45] E. Schrödinger. Further studies on solving eigenvalue problems by factorization. *Proc. R. Ir. Acad., A Math. phys. sci.*, 46:183–206, 1940.

[46] Gaston Darboux. On a proposition relative to linear equations. *C. R. Acad. Sci. Paris*, 94:1456–59, 1882.

[47] Marshall Luban and D. L. Pursey. New schrödinger equations for old: Inequivalence of the darboux and abraham-moses constructions. *Phys. Rev. D*, 33:431–436, Jan 1986.

[48] Elso Drigo Filho and M A Cândido Ribeiro. Generalized ladder operators for shape-invariant potentials. *Phys. Scr.*, 64(6):548–552, jan 2001.

[49] Takahiro Fukui and N. Aizawa. Shape-invariant potentials and an associated coherent state. *Phys. Lett. A*, 180(4–5):308–313, 1993.

[50] Shahram Jalalzadeh and Paulo Vargas Moniz. *Challenging Routes in Quantum Cosmology*. WORLD SCIENTIFIC, 2022.

[51] S.M.M. Rasouli, Nasim Saba, Mehrdad Farhoudi, João Marto, and P.V. Moniz. Inflationary universe in deformed phase space scenario. *Ann. Phys. (N Y)*, 393:288–307, 2018.

[52] S.M.M. Rasouli, R. Pacheco, M. Sakellariadou, and P.V. Moniz. Late time cosmic acceleration in modified sáez–ballester theory. *Phys. Dark Univ.*, 27:100446, 2020.

[53] S. M. M. Rasouli. Noncommutativity, saez-ballester theory and kinetic inflation. *Universe*, 8(3), 2022.

[54] SMM Rasouli, Mehrdad Farhoudi, and Nima Khosravi. Horizon problem remediation via deformed phase space. *Gen. Rel. Grav.*, 43(10):2895–2910, 2011.
S. M. M. Rasouli and Paulo Vargas Moniz. Noncommutative minisuperspace, gravity-driven acceleration, and kinetic inflation. *Phys. Rev. D*, 90:083533, Oct 2014.

S.M.M. Rasouli, A.H. Ziaie, S. Jalalzadeh, and P.V. Moniz. Non-singular brans–dicke collapse in deformed phase space. *Ann. Phys. (N Y)*, 375:154–178, 2016.

S. M. M. Rasouli and Paulo Vargas Moniz. Gravity-Driven Acceleration and Kinetic Inflation in Noncommutative Brans-Dicke Setting. *Odessa Astron. Pub.*, 29:19, 2016.

S.M.M. Rasouli, J. Marto, and P.V. Moniz. Kinetic inflation in deformed phase space brans–dicke cosmology. *Phys. Dark Univ.*, 24:100269, 2019.

T. Dereli and Robin W. Tucker. Signature dynamics in general relativity. *Class. Quant. Grav.*, 10:365–374, 1993.

T. Dereli, M. Onder, and Robin W. Tucker. Signature transitions in quantum cosmology. *Class. Quant. Grav.*, 10:1425–1434, 1993.

S. Jalalzadeh, F. Ahmadi, and H. R. Sepangi. Multidimensional classical and quantum cosmology: Exact solutions, signature transition and stabilization. *JHEP*, 08:012, 2003.

N. Khosravi, S. Jalalzadeh, and H. R. Sepangi. Quantum noncommutative multidimensional cosmology. *Gen. Rel. Grav.*, 39:899–911, 2007.

P. Pedram and S. Jalalzadeh. Quantum cosmology with varying speed of light: Canonical approach. *Phys. Lett. B*, 660:1–6, 2008.

A. Bina, K. Atazadeh, and S. Jalalzadeh. Noncommutativity, generalized uncertainty principle and FRW cosmology. *Int. J. Theor. Phys.*, 47:1354–1362, 2008.

Shahram Jalalzadeh and Babak Vakili. Quantization of the interior Schwarzschild black hole. *Int. J. Theor. Phys.*, 51:263–275, 2012.

N. Khosravi, S. Jalalzadeh, and H. R. Sepangi. Non-commutative multi-dimensional cosmology. *JHEP*, 01:134, 2006.

B. Vakili, P. Pedram, and S. Jalalzadeh. Late time acceleration in a deformed phase space model of dilaton cosmology. *Phys. Lett. B*, 687:119–123, 2010.

Farhad Darabi. Large scale - small scale duality and cosmological constant. *Phys. Lett. A*, 259:97–103, 1999.

F. Darabi and A. Rastkar. A quantum cosmology and discontinuous signature changing classical solutions. *Gen. Rel. Grav.*, 38:1355–1366, 2006.

S. Jalalzadeh, M. Rashki, and S. Abarghouei Nejad. Classical universe arising from quantum cosmology. *Phys. Dark Univ.*, 30:100741, 2020.

H. A. Kastrup. A new look at the quantum mechanics of the harmonic oscillator. *Ann. Phys.*, 519(7-8):439–528, 2007.

R. Kumar and R. P. Malik. Supersymmetric Oscillator: Novel Symmetries. *EPL*, 98(1):11002, 2012.

P. D. D’Eath. *Supersymmetric Quantum Cosmology*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1996.

Paulo Vargas Moniz. Supersymmetric quantum cosmology: shaken, not stirred. *Int. J. Mod. Phys. A*, 11(24):4321–4382, 1996.

Paulo Vargas Moniz. Quantum cosmology: Meeting susy. In Alfonso García-Parrado, Filipe C. Mena, Filipe Moura, and Estelita Vaz, editors, *Progress in Mathematical Relativity, Gravitation and Cosmology*, pages 117–125, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.
[76] Hugo García-Compeán, Octavio Obregón, and Cupatitzio Ramírez. Topics in supersymmetric and noncommutative quantum cosmology. *Universe*, 7(11):434, 2021.

[77] Paulo Vargas Moniz. Supersymmetric quantum cosmology: a ‘socratic’ guide. *Gen. Rel. Grav.*, 46(1):1–35, 2014.

[78] JL López and O Obregón. Supersymmetric quantum matrix cosmology. *Class. quant. grav.*, 32(23):235014, 2015.

[79] O Obregon and C Ramírez. Dirac-like formulation of quantum supersymmetric cosmology. *Phys. Rev. D*, 57(2):1015, 1998.

[80] J Bene and R Graham. Supersymmetric homogeneous quantum cosmologies coupled to a scalar field. *Phys. Rev. D*, 49(2):799, 1994.

[81] Andras Csordas and Robert Graham. Supersymmetric minisuperspace with nonvanishing fermion number. *Phys. Rev. Lett.*, 74(21):4129, 1995.

[82] Axel Kleinschmidt, Michael Koehn, and Hermann Nicolai. Supersymmetric quantum cosmological billiards. *Phys. Rev. D*, 80(6):061701, 2009.

[83] Alfredo Macías, Eckehard W Mielke, and José Socorro. Supersymmetric quantum cosmology for bianchi class a models. *Int. J. Mod. Phys. D*, 7(05):701–712, 1998.

[84] Thibault Damour and Philippe Spindel. Quantum supersymmetric cosmology and its hidden kac–moody structure. *Class. Quant. Grav.*, 30(16):162001, 2013.

[85] Claus Kiefer. *Quantum gravity*, volume 124 of *International Series of Monographs on Physics*. Clarendon, Oxford, 2004.

[86] Giampiero Esposito. *Quantum gravity, quantum cosmology and Lorentzian geometries*, volume 12 of *Lecture Notes in Physics Monographs*. Springer Berlin, Heidelberg, 2009.

[87] Gianluca Calcagni. *Classical and quantum cosmology*. Springer Cham, 2017.

[88] Martin Bojowald. *Quantum cosmology: a fundamental description of the universe*, volume 835 of *Lecture Notes in Physics*. Springer New York, NY, 2011.

[89] James E Lidsey. Quantum cosmology of generalized two-dimensional dilaton-gravity models. *Phys. Rev. D*, 51(12):6829, 1995.

[90] James E Lidsey and P Vargas Moniz. Supersymmetric quantization of anisotropic scalar-tensor cosmologies. *Class. Quant. Grav.*, 17(23):4823, 2000.

[91] Robert Graham. Supersymmetric bianchi type ix cosmology. *Phys. Rev. Lett.*, 67(11):1381, 1991.

[92] James E Lidsey. Scale factor duality and hidden supersymmetry in scalar-tensor cosmology. *Phys. Rev. D*, 52(10):R5407, 1995.

[93] VI Tkach, JJ Rosales, and O Obregón. Supersymmetric action for bianchi type models. *Class. Quant. Grav.*, 13(9):2349, 1996.

[94] O Obregón, JJ Rosales, and VI Tkach. Superfield description of the frw universe. *Phys. Rev. D*, 53(4):R1750, 1996.

[95] Edward Witten. Dynamical breaking of supersymmetry. *Nucl. Phys. B*, 188(3):513–554, 1981.