DISPLACEMENT ENERGY OF COISOTROPIC SUBMANIFOLDS AND HOFER'S GEOMETRY

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Abstract. We prove that the displacement energy of a stable coisotropic submanifold is bounded away from zero if the ambient symplectic manifold is closed, rational and satisfies a mild topological condition.

1. Introduction and Results

There is positive lower bound for the amount of energy it takes to displace a closed Lagrangian submanifold of a tame symplectic manifold. In particular, every time-dependent function on a symplectic manifold determines a unique Hamiltonian diffeomorphism, and if this diffeomorphism displaces a closed Lagrangian submanifold, then the Hofer norm of the function is bounded away from zero by a constant which depends only on the Lagrangian submanifold and the ambient symplectic manifold. This fundamental fact in symplectic topology was first established for rational Lagrangian submanifolds by Polterovich in [Po], and was later extended to general Lagrangians by Chekanov in [Ch]. Among other things, it implies the nondegeneracy of the Hofer metric on the group of Hamiltonian diffeomorphisms of a tame symplectic manifold.

Recently, Ginzburg proved that there is also a positive lower bound for the amount of energy required to displace certain coisotropic submanifolds. More precisely, in [Gi] it is shown that the displacement energy of a stable coisotropic submanifold of a tame, wide and symplectically aspherical symplectic manifold is bounded away from zero. In the present paper, we extend this coisotropic intersection phenomenon to symplectic manifolds which admit symplectic spheres. The proof utilizes the Floer theoretic methods developed in [Ke], as well as the applications of these methods to the study of Hamiltonian paths which are length minimizing with respect to the Hofer metric.

There is currently no version of Floer theory for the intersection theory of a general coisotropic submanifold and its image under a Hamiltonian diffeomorphism. However, one can study the symplectic properties of a coisotropic submanifold using the Hamiltonian Floer homology of functions which are

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supported in (normal) neighborhoods of it. This indirect approach, which goes back to the pioneering work of Viterbo from [Vi], requires a few compromises.

The first compromise involves the submanifolds. To get useful normal neighborhoods, we restrict our attention to \textit{stable} coisotropic submanifolds. This notion was introduced by Bolle in [Bo1, Bo2] and is defined as follows.

Let \((M, \omega)\) be a symplectic manifold of dimension \(2m\) and let \(N\) be a closed coisotropic submanifold of \(M\) with codimension \(k\). Then \(N\) is said to be \textbf{stable} if there are one-forms \(\alpha_1, \ldots, \alpha_k\) on \(N\) such that the form

\[
\alpha_1 \wedge \cdots \wedge \alpha_k \wedge (\omega|_N)^{m-k}
\]

does not vanish on \(N\), and \(\ker d\alpha_j \supset \ker \omega|_N\) for \(j = 1, \ldots, k\). Examples of stable coisotropic submanifolds include Lagrangian tori and contact hypersurfaces. The stability condition is also closed under products. For more details, the reader is referred to [Bo1, Bo2, Gi].

In using Hamiltonian Floer homology to study the symplectic topology of a coisotropic submanifold, one also needs to recognize nontrivial 1-periodic orbits using only the symplectic action and/or the Conley-Zehnder index. This requires further compromise concerning the ambient symplectic manifolds, \((M, \omega)\), we consider. In [Gi], the symplectic manifolds are assumed to be symplectically aspherical. That is, for every class \(A \in \pi_2(M)\) it is assumed that \(\omega(A) = 0 = c_1(A)\), where the notations \(\omega(A)\) and \(c_1(A)\) refer to the evaluations of the cohomology classes on the elements of \(H_2(M; \mathbb{R})\) and \(H_2(M; \mathbb{Z})\) determined by \(A\). With this assumption, the action and index of a periodic orbit are single-valued and any 1-periodic orbit with sufficiently large action (greater than \(\|H\|^{\pm}\) as defined below) must be nonconstant.

Here, we allow for the existence of nontrivial symplectic spheres and so the action and index may be multi-valued. To distinguish nonconstant periodic orbits we will assume that the quantity

\[
r(M, \omega) = \inf_{A \in \pi_2(M)} \{|\omega(A)| \mid |\omega(A)| > 0\}
\]

is positive.\(^1\) A symplectic manifold with \(r(M, \omega) > 0\) is said to be \textbf{rational}. We will also assume that \((M, \omega)\) satisfies the topological assumption

\[
\omega(A) = 0 \implies c_1(A) \geq 0 \text{ for all } A \text{ in } \pi_2(M).
\]

Finally, we restrict ourselves, in this work, to the case when \(M\) is closed. We expect that the methods developed here are also applicable to symplectic manifolds which are open or have convex boundaries.

Before stating the main result, we first recall the definition of the displacement energy. Let \(C^\infty(S^1 \times M)\) be the space of smooth time-periodic functions on \(M\), where \(S^1 = \mathbb{R}/\mathbb{Z}\) is the circle parameterized by \(t \in [0, 1]\).\(^2\)

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1Bolle refers to such manifolds as being of \textit{almost contact type}.

2We use the convention that the infimum over the empty set is equal to \(\infty\).
The Hofer norm of a function $H$ in $C^\infty(S^1 \times M)$ is defined as

$$\|H\| = \int_0^1 \max_{p \in M} H_t(p) \, dt - \int_0^1 \min_{p \in M} H_t(p) \, dt,$$

where $H_t(\cdot) = H(t, \cdot)$. One can also associate to $H$ its Hamiltonian vector $X_H$ via the equation

$$\omega(X_H, \cdot) = -dH_t(\cdot).$$

The time-$t$ flow of this vector field, also referred to as the Hamiltonian flow of $H$, is denoted by $\phi^t_H$ and is defined for all $t \in [0,1]$. The group of Hamiltonian diffeomorphisms consists of all the time-1 maps, $\phi^1_H$, obtained in this way.

The **displacement energy** of a subset $U$ of $M$ is defined as

$$e(U) = \inf_{H \in C^\infty(S^1 \times M)} \{\|H\| \mid \phi^1_H(U) \cap U = \emptyset\},$$

the minimum variation of a function which generates a Hamiltonian diffeomorphism that moves $U$ off of itself.

Our main result is the following:

**Theorem 1.1.** Let $N$ be a stable coisotropic submanifold of a closed and rational symplectic manifold satisfying (1). There is a positive constant $\Delta > 0$ such that $e(N) \geq \Delta$.

Of course, one starts with the assumption that $N$ can be displaced by some Hamiltonian diffeomorphism, i.e., $e(N) < \infty$. This has deep implications for the Hamiltonian flows supported near $N$. In turn, these flows can be used to probe the geometry of $N$. It is this interaction between the displacability of $N$ and its geometry, which leads to the proof of Theorem 1.1.

The primary difference between the proof of Theorem 1.1 and the proof of the main result in [Gi] is the contribution coming from Floer theory. In [Gi], both the action filtration and action selector are used to prove the existence of a Floer trajectory whose energy yields the crucial estimate for the displacement energy, (Proposition 5.1 of [Gi]). For a rational symplectic manifold, the action filtration and selector cannot be used in the same manner. Instead we use the Floer theoretic techniques which were developed in [Ke] to study the length minimizing properties of Hamiltonian paths. These tools allow us to detect a perturbed holomorphic cylinder in Proposition 2.5 whose energy recovers the crucial estimate.

**Remark 1.2.** Another approach to studying coisotropic intersections is to consider general leaf-wise intersections under Hamiltonian diffeomorphisms, [Mo, EH, Ho]. The most recent work in this direction is [Dr], where Dragnev establishes the existence of leaf-wise intersections for a stable coisotropic submanifold $N$ of $\mathbb{R}^{2n}$, and its image under any Hamiltonian diffeomorphism with energy less than the Floer-Hofer capacity of $N$. 
1.1. **Organization.** The proof of Theorem 1.1 is described in the next section, assuming the contribution from Floer theory, Proposition 2.5. In the third section, we recall the required Floer theory methods and applications from [Ke]. The proof of Proposition 2.5 is then contained in the final section of the paper.

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2. **The proof of Theorem 1.1 (modulo Proposition 2.5).**

Before presenting the proof of Theorem 1.1 in §2.4, we discuss some preliminary notions and results.

2.1. **Properties of stable coisotropic submanifolds.** We begin by recalling some useful implications of the stability assumption. The proofs of these results can be found in [Bo1, Bo2, Gi].

Let $N$ be a stable coisotropic submanifold of codimension $k$ in a symplectic manifold $(M,\omega)$ of dimension $2m$. We then have the following normal neighborhood result.

**Proposition 2.1** ([Bo1, Bo2]). For sufficiently small $r > 0$ there is a neighborhood of $N$ in $(M,\omega)$ which is symplectomorphic to

$$U_r = \{ (q,p) \in N \times \mathbb{R}^k \mid |p| < r \}$$

equipped with the symplectic form

$$\omega|_N + \sum_{j=1}^k d(p_j \pi^* \alpha_j).$$

Here, $|p|$ denotes the standard norm of $p = (p_1, \ldots, p_k) \in \mathbb{R}^k$, and $\pi: U_r \to N$ is the obvious projection.

Recall that the characteristic foliation $\mathcal{F}$ of $N$ is determined by the integrable distribution $\ker \omega|_N$. The normal form above implies that for each manifold $N_p = N \times p$ with $|p| < r$ we have $\omega|_{N_p} = \omega|_N$. Hence, each of the $N_p$ in the tubular neighborhood $U_r$ is a coisotropic submanifold with the same characteristic foliation.

The relevant Hamiltonian dynamical system is the following leaf-wise geodesic flow on the tubular neighborhood $U_r$ of $N$.

**Proposition 2.2** ([Bo1, Bo2, Gi]). The Hamiltonian flow of the function $\frac{1}{2}|p|^2$ on the normal neighborhood $U_r$ is the leaf-wise geodesic flow of the leaf-wise metric $\sum_{j=1}^k (\alpha_j)^2$ on $\mathcal{F}$. Moreover, this metric is leaf-wise flat.

This implies that a nonconstant periodic orbit $x$ of the flow of $\frac{1}{2}|p|^2$ corresponds to a closed geodesic $\gamma$ contained on a leaf of $\mathcal{F}$ in $N$. The fact that the leaf-wise metric is flat implies that this geodesic is noncontractible within its leaf.
For any closed curve $\gamma$ contained in a leaf of $\mathcal{F}$, set

$$\delta(\gamma) = \sum_{j=1}^{k} \left| \int_{\gamma} \alpha_j \right|.$$ 

**Lemma 2.3** ([Bo1, Bo2, Gi]). There is a constant $\delta_N > 0$ such that

$$\delta(\gamma) \geq \delta_N$$

for every nontrivial closed geodesic $\gamma$ of the leaf-wise metric $\sum_{j=1}^{k}(\alpha_j)^2$.

2.2. **Hofer’s length functional.** A function $H$ in $C^\infty(S^1 \times M)$ is said to be normalized if

$$\int_{M} H_t \omega^m = 0$$

for all $t$ in $[0, 1]$. The space of normalized functions is denoted by $C_0^\infty(S^1 \times M)$. For every path of Hamiltonian diffeomorphisms, $\psi_t$, there is a unique $H$ in $C_0^\infty(S^1 \times M)$ such that $\phi_{H} \circ \psi_0 = \psi_t$. Following [Ho], this time-dependent generating function is used to define the Hofer length of the path $\psi_t$ by

$$\text{length}(\psi_t) = \|H\| = \int_{0}^{1} \max_{M} H_t \ dt - \int_{0}^{1} \min_{M} H_t \ dt$$

The quantities $\|H\|^+\|$ and $\|H\|^-$ provide different measures of $\psi_t$ called the positive and negative Hofer lengths, respectively. The positive Hofer length will play an important role in the proof of Theorem 1.1.

2.3. **Right asymptotic spanning discs.** A spanning disc for a loop $y: S^1 \rightarrow M$ is a smooth map $w$ from the unit disc in $\mathbb{C}$ to $M$ such that $w(e^{2\pi i t}) = y(t)$. A right asymptotic spanning disc for the loop $y$ is a smooth map $v: \mathbb{R} \times S^1 \rightarrow M$ such that

- there is a sequence $s_j^- \rightarrow -\infty$ for which

$$\lim_{j \rightarrow \infty} v(s_j^-, t) = y(t);$$

- there is a sequence $s_j^+ \rightarrow +\infty$ for which $v(s_j^+, t)$ converges to a constant map $t \mapsto p$ for some point $p \in M$.

Here, convergence is with respect to the smooth topology on $C^\infty(S^1, M)$.

We will detect right asymptotic spanning discs for 1-periodic orbits of the Hamiltonian flow of a function $H$ in $C^\infty(S^1 \times M)$. They will be constructed using a smooth $(\mathbb{R} \times S^1)$-family of $\omega$-compatible almost complex structures, $J_s$, which is independent of $s \in \mathbb{R}$ for $|s|$ sufficiently large. This last auxiliary structure is used to define the energy of $v$ by

$$E(v) = \int_{0}^{1} \int_{-\infty}^{+\infty} \omega(\partial_s v, J_s(v)\partial_s v) \, ds \, dt.$$
For each integer \( j \), we will also consider the quantities
\[
E^j(v) = \int_0^1 \int_{s_j^-}^{s_j^+} \omega(\partial_s v, J_s(v)\partial_s v) \, ds \, dt.
\]
and
\[
A^j_H(v) = \int_0^1 H(t, v(s_j^-, t)) \, dt + \int_0^1 \int_{s_j^-}^{s_j^+} \omega(\partial_s v, \partial_t v) \, ds \, dt.
\]

2.4. The proof of Theorem \[1.1\] We may assume that
\[
3e(N) < r(M, \omega),
\]
otherwise we are done. We will prove the following result which clearly implies Theorem \[1.1\].

**Theorem 2.4.** For sufficiently small values of \( r > 0 \), there is a positive constant \( \Delta > 0 \), independent of \( r \), such that \( e(U_r) > \Delta \).

By \[2\], for all sufficiently small values of \( r > 0 \) we have
\[
3e(U_r) < r(M, \omega)
\]

Fix an \( R > 0 \) for which this inequality holds. Henceforth, we will consider only neighborhoods \( U_r \) for \( 0 < r < R/2 \).

In order to relate the assumption that \( N \) can be displaced by a Hamiltonian diffeomorphism to the properties of the flow from Proposition \[2.2\], we reparameterize this flow so that it extends to a global flow on \( M \) which is supported in \( U_r \).

Let \( \nu: [0, +\infty) \to [0, +\infty) \) be a smooth function with the following properties
- \( \nu(0) = A \) on \( [0, r/3] \);
- \( \nu' < 0 \) on \( (r/3, 2r/3] \);
- \( \nu = -B \) on \( [2r/3, +\infty) \).

Here, \( A \) and \( B \) are positive constants. We then define the function
\[
H_r(q, p) = \begin{cases} \nu(|p|) & \text{when } (q, p) \text{ is in } U_r, \\ -B & \text{otherwise.} \end{cases}
\]

The Hamiltonian flow of \( H_r \) is trivial way from \( U_r \), and inside of \( U_r \) it is a reparameterization of the geodesic flow from Proposition \[2.2\]. Clearly, \( \|H_r\|^+ = A \) and \( \|H_r\|^− = B \). We choose the constant \( A \) so that
\[
2e(U_r) < A < 3e(U_r).
\]

We then choose a constant \( B \) satisfying
\[
0 < B < A \frac{\text{Vol}(U_r)}{\text{Vol}(M \setminus U_r)}.
\]
so that $H_r$ is normalized. Further restricting $R$, if necessary, we may also assume that

$$2e(U_r) < A + B = \|H_r\| < 3e(U_r).$$

The following technical result is proved in the final section of the paper using the methods developed in \[Ke\]. The existence of the map $v$ described below is implied by the fact that the Hamiltonian path generated by $H_r$ does not minimize the positive Hofer length in its homotopy class (see \[12\]).

**Proposition 2.5.** For the function $H_r$ above, there is an $\epsilon > 0$, a family of almost complex structures $J_s$ as in \[2.3\] and a nonconstant 1-periodic orbit $y$ of $H_r$ with a right asymptotic spanning disc $v$ such that

$$-B + E^j(v) \leq A^j_{H_r}(v) \leq A - \epsilon$$

for all $j$. Moreover, $v$ is a solution of

$$\partial_s v + J_s(v)(\partial_t v - X_{\tilde{H}_s}(v)) = 0,$$

where $\tilde{H}_s$ is either the function $H_r$ or the function $(\eta(-s) - 1)B + \eta(-s)H_r$ for a smooth nondecreasing function $\eta(s)$ which equals zero for $s \leq -1$ and equals one for $s \geq 1$.

The following inequality for the energy of the map $v$ detected in Proposition 2.5, is easily derived from the work of Bolle and Ginzburg. We include a proof for the sake of completeness.

**Lemma 2.6** (\[Bo1, Bo2, Gi\]). There is a constant $c_R > 0$ such that for the periodic orbit $y$ and the asymptotic right spanning disc $v$ of Proposition 2.5, we have

$$E(v) > c_R \cdot \delta(\pi(y)).$$

**Proof.** Let $\hat{f}: [0, R) \rightarrow \mathbb{R}$ be a smooth nonincreasing function which is equal to one on $[0, R/2)$ and is equal to zero near $R$. Let $f$ be the function which equals $\hat{f}(|p|)$ in $U_R$ and vanishes outside of $U_R$.

For the one-forms $\sigma_i = f\pi^*\alpha_i$, we have

$$i_{X_{H_r}}d\sigma_i = 0.$$

In particular, away from $U_r$ we have $X_{H_r} = 0$. Within $U_r$, $\sigma_i = \pi^*\alpha_i$ and $X_{H_r}$ is a reparameterization of the leaf-wise geodesic flow from Proposition 2.2. Hence,

$$i_{X_{H_r}}d\sigma_i = i_{X_{H_r}}\pi^*d\alpha_i = 0$$

since the forms $d\alpha_i$ vanish on the leaves of $\mathcal{F}_-$. It follows from \ref{eq:7} that $i_{X_{\tilde{H}_s}}d\sigma_i = 0$ for both of the possible functions $\tilde{H}_s$ from Proposition 2.5.

The $s$-norm of a tangent vector $X \in T_pM$ is defined to be $\|X\|_s = \omega(X, J_sX)$. Since $J_s$ does not depend on $s$ when $|s|$ is large, we can find constants $c_i > 0$ such that

$$|d\sigma_i(X, Y)| \leq c_i \|X\|_s \cdot \|Y\|_s$$
for any pair of tangent vectors \( X, Y \in T_pM \) and every \( s \in \mathbb{R} \).

For the asymptotic right spanning disc \( v \) detected in Proposition 2.5, we then have

\[
E(v) = \int_0^1 \int_{-\infty}^{+\infty} \omega \left( \partial_s v, J_s(v) \partial_s v \right) \, ds \, dt
\]

\[
= \int_0^1 \int_{-\infty}^{+\infty} \| \partial_s v \|_s \cdot \| \partial_t v - X_{\delta_{H_s}} \|_s \, ds \, dt
\]

\[
\geq c_i^{-1} \int_0^1 \int_{-\infty}^{+\infty} |d\sigma_i \left( \partial_s v, \partial_t v - X_{\delta_{H_s}} \right) | \, ds \, dt
\]

\[
\geq c_i^{-1} \lim_{j \to \infty} \int_0^1 \int_{s_j^-}^{s_j^+} |d\sigma_i \left( \partial_s v, \partial_t v \right) | \, ds \, dt
\]

\[
= c_i^{-1} \lim_{j \to \infty} \left| \int_{v(s_j^+)} \sigma_i - \int_{v(s_j^-)} \sigma_i \right|
\]

Since \( y \) is nonconstant, it is contained in \( U_r \) where \( \sigma_i = \pi^* \alpha_i \). The inequality above then implies that

\[
E(v) \geq c_i^{-1} \left| \int_{v} \sigma_i \right| = c_i^{-1} \left| \int_{\pi(y)} \alpha_i \right|
\]

Setting \( c_R = \frac{1}{k} \min \{ c_i^{-1} \} \), we are done. \( \square \)

We can now complete the proof of Theorem 1.1. By Proposition 2.5, we have

\[
-B + E^j(v) \leq A^j_{H_r}(v) \leq A - \epsilon.
\]

for all \( j \). Taking the limit as \( j \to \infty \), yields

(8) \[
\| H_r \| = A + B > E(v).
\]

Together with inequality (1) and Lemmas 2.6 and 2.4, this implies that

\[
e(U_r) > \| H_r \| / 3 > E(v) / 3 > c_R \cdot \delta(\pi(y)) / 3 \geq c_R \cdot \delta_N / 3.
\]

Setting \( \Delta = c_R \cdot \delta_N / 3 \), the proof of Theorem 2.4 and hence Theorem 1.1 will be complete once we prove Proposition 2.5.
3. Floer caps and chain isomorphisms in Morse homology

Throughout this section $H$ will be a normalized function in $C^\infty_0(S^1 \times M)$ whose contractible periodic orbits with period equal to one are nondegenerate. This (finite) set of 1-periodic orbits will be denoted by $\mathcal{P}(H)$.

Let $\mathcal{J}(M,\omega)$ be the space of smooth almost complex structures on $M$ which are compatible with $\omega$, and let $\mathcal{J}_S^1(M,\omega)$ denote the space of smooth $S^1$-families of elements in $\mathcal{J}(M,\omega)$. Fixing a $J$ in $\mathcal{J}_S^1(M,\omega)$, we refer to $(H,J)$ as our Hamiltonian data.

3.1. Homotopy triples and Floer caps. A smooth $\mathbb{R}$-family, $F_s$, of functions in $C^\infty(S^1 \times M)$ or elements of $\mathcal{J}_S^1(M,\omega)$ is called a compact homotopy from $F^{-}$ to $F^{+}$, if there is a constant $\lambda > 0$ such that $F_s = F^{-}$ for $s \leq -\lambda$, and $F_s = F^{+}$ for $s \geq \lambda$. Any such $\lambda$ will be referred to as a horizon of the compact homotopy.

A homotopy triple for the pair $(H,J)$ is a collection $\mathcal{H} = (H_s, K_s, J_s)$, where $H_s$ is a compact homotopy from a constant function to $H$, $K_s$ is a compact homotopy from the zero function to itself, and $J_s$ is a compact homotopy in $\mathcal{J}_S^1(M,\omega)$ from some $J^{-}$ to $J$. Although the constant $c$ is an important part of the homotopy triple, it will be suppressed from the notation for simplicity.

For a homotopy triple $\mathcal{H} = (H_s, K_s, J_s)$, we consider smooth maps $u$ from the infinite cylinder $\mathbb{R} \times S^1$ to $M$ which satisfy the following equation
\[
\partial_s u - X_{K_s}(u) + J_s(u)(\partial_t u - X_{H_s}(u)) = 0.
\]

The energy of a solution $u$ of (9) is defined as
\[
E(u) = \int_0^1 \int_{-\infty}^{+\infty} \omega(\partial_s u - X_{K_s}(u), J_s(u)(\partial_t u - X_{H_s}(u))) \, ds \, dt.
\]

If the energy of $u$ is finite, then it follows from standard arguments that $u(+\infty) := \lim_{s \to +\infty} u(s,t) = x(t)$ for some 1-periodic orbit $x \in \mathcal{P}(H)$. The assumption of finite energy also implies that $u(-\infty) := \lim_{s \to -\infty} u(s,t) = p$ for some point $p \in M$.

The set of left Floer caps of $x \in \mathcal{P}(H)$ with respect to $\mathcal{H}$ is $\mathcal{L}(x;\mathcal{H}) = \{ u \in C^\infty(\mathbb{R} \times S^1, M) \mid u \text{ satisfies (9)}, E(u) < \infty, u(\infty) = x \}$.

It is clear from the asymptotic behavior described above, that each left Floer cap $u$ in $\mathcal{L}(x;\mathcal{H})$ determines a unique homotopy class of spanning discs for $x$ and hence a well-defined Conley-Zehnder index $\mu_{CZ}(x,u)$. This index is normalized here so that if $x(t) = p$ is a constant 1-periodic orbit of a $C^2$-small Morse function and $u(z) = p$ is the constant spanning disc, then
\[\mu_{CZ}(x, u) = \text{ind}(p) - m,\] where \(\text{ind}(p)\) is the Morse index of \(p\). The action of \(x\) with respect to \(u\) is defined by

\[\mathcal{A}_H(x, u) = \int_0^1 H(t, x(t)) \, dt - \int_0^1 \int_{-\infty}^{+\infty} \omega(\partial_s u, \partial_t u) \, ds \, dt.\]

Given any map of the form \(F(s, \cdot)\) for \(s \in \mathbb{R}\), we set

\[\overrightarrow{F}(s, \cdot) = F(-s, \cdot).\]

For a homotopy triple \(\mathcal{H} = (H_s, K_s, J_s)\) we will also consider maps \(v: \mathbb{R} \times S^1 \to M\) which satisfy the equation

\[\partial_s v + X_{K_s}(v) + J_s(v)(\partial_t v - X_{H_s}(v)) = 0.\]

The energy of such a map is defined by

\[E(v) = \int_0^1 \int_{-\infty}^{+\infty} \omega\left(\partial_s v + X_{K_s}(v), \overrightarrow{J_s(v)}(\partial_t v + X_{H_s}(v))\right) \, ds \, dt.\]

If a map \(v\) satisfies (10) and has finite energy, then \(v(+\infty)\) is a point in \(M\) and \(v(-\infty)\) is a 1-periodic orbit of \(H\). In particular, such a \(v\) is a right asymptotic spanning disc, as defined in (2.3).

The space of right Floer caps for \(x \in \mathcal{P}(H)\) is defined by

\[\mathcal{R}(x; \mathcal{H}) = \{v \in C^\infty(\mathbb{R} \times S^1, M) \mid v\text{ satisfies } (10), E(v) < \infty, v(-\infty) = x\}.\]

For each \(v\) in \(\mathcal{R}(x; \mathcal{H})\), one can consider the map

\[\overrightarrow{\nu}(s, t) = v(-s, t)\]

and define the Conley-Zehnder index \(\mu_{CZ}(x, \overrightarrow{\nu})\) and the action

\[\mathcal{A}_H(x, \overrightarrow{\nu}) = \int_0^1 H(t, x(t)) \, dt - \int_0^1 \int_{-\infty}^{+\infty} \omega(\partial_s \overrightarrow{\nu}, \partial_t \overrightarrow{\nu}) \, ds \, dt.\]

### 3.2. Curvature.

The curvature of a homotopy triple \(\mathcal{H} = (H_s, K_s, J_s)\) is the function on \(\mathbb{R} \times S^1 \times M\) defined by

\[\kappa(\mathcal{H}) = \partial_s H_s - \partial_t K_s + \{H_s, K_s\}.\]

Here we use the convention \(\{H, K\} = \omega(X_K, X_H)\). The curvature relates the energy and action of solutions of equations (9) and (10) as follows.

Given a solution \(u\) of (9), set

\[E^b_a(u) = \int_a^b \int_0^1 \omega(\partial_s u - X_{K_s}(u), J_s(u)(\partial_t u - X_{H_s}(u))) \, ds \, dt\]

and

\[A^b_a(u) = \int_0^1 H_b(t, u(b, t)) \, dt - \int_a^b \int_0^1 \omega(\partial_s u, \partial_t u) \, ds \, dt.\]

We then derive the following identity from (9)

\[(11) \quad E^b_a(u) = \int_0^1 H_a(t, u(a, t))) \, dt - A^b_a(u) + \int_0^1 \kappa(\mathcal{H})(s, t, v(s, t)) \, ds \, dt.\]
For a solution $v$ of (10), the corresponding map $\vec{v}(s, t) = v(-s, t)$ satisfies

\begin{equation}
- \partial_s \vec{v} + X_{J_s}(\vec{v}) + J_s(\vec{v})(\partial_t \vec{v} - X_{H_s}(\vec{v})) = 0.
\end{equation}

Setting

\begin{align*}
\vec{E}_a^b(v) &= \int_0^1 \int_a^b \omega \left( \partial_s \vec{v} - X_{J_s}(\vec{v}), J_s(\vec{v})(\partial_t \vec{v} - X_{H_s}(\vec{v}) \right) ds \, dt \\
\vec{A}_a^b(v) &= \int_0^1 H_b(t, \vec{v}(b, t)) dt - \int_0^1 \int_a^b \omega(\partial_s \vec{v}, \partial_t \vec{v}) ds \, dt
\end{align*}

equation (12) then yields

\begin{equation}
\vec{E}_a^b(v) = \vec{A}_a^b(v) - \int_0^1 H_a(t, \vec{v}(a, t)) dt - \int_0^1 \int_a^b \kappa(\mathcal{H})(s, t, \vec{v}(s, t)) ds \, dt.
\end{equation}

Finally, we note that if $\lambda$ is a horizon for $H_s$ and if $A \leq a \leq -\lambda$ and $B \geq b$, then

\begin{equation}
\vec{A}_a^b(v) - \vec{E}_a^b(v) \geq \int_0^1 \int_{[A, a] \cup [b, B]} \kappa(\mathcal{H})(s, t, \vec{v}) ds \, dt.
\end{equation}

**Remark 3.1.** In this notation, we have $E(v) = \vec{E}^{+\infty}_{-\infty}(v)$ and $A_H(x, \vec{v}) = \vec{A}^{+\infty}_{-\infty}(v)$.

Moreover, for $j$ sufficiently large, we have

\begin{align*}
E_j^j(v) &= \vec{E}^{-s_j}_{s_j}(v) \\
A_j^j_H(v) &= \vec{A}^{-s_j}_{s_j}(v)
\end{align*}

for the quantities appearing in Proposition 2.5.

The positive and negative norms of the curvature are defined by

\[ |||\kappa(\mathcal{H})|||_+ = \int_{\mathbb{R} \times S^1} \max_{p \in M} \kappa(\mathcal{H}) \, ds \, dt, \]

and

\[ |||\kappa(\mathcal{H})|||_- = -\int_{\mathbb{R} \times S^1} \min_{p \in M} \kappa(\mathcal{H}) \, ds \, dt. \]

**Example 3.2.** Let $\eta: \mathbb{R} \to [0, 1]$ be a smooth nondecreasing function such that $\eta(s) = 0$ for $s \leq -1$ and $\eta(s) = 1$ for $s \geq 1$. This function will be fixed throughout this paper. A **linear homotopy triple** for $(H, J)$ is a homotopy triple of the form $\mathcal{H} = (\mathcal{H}_s, 0, J_s)$ where

\[ \mathcal{H}_s = (\eta(s) - 1)H ||| - \eta(s)H. \]

The constant for $\mathcal{H}$ is $c = -\|H\|^-$. The curvature of $\mathcal{H}$ is

\[ \kappa(\mathcal{H}) = \hat{\eta}(s)(H + \|H\|^-), \]

which is clearly negative. We also have

\[ |||\kappa(\mathcal{H})|||_+ = \int_{S^1} \max_{p \in M}(\|H\|^- + H(t, p)) \, dt = \|H\|, \]
and
\[ |||\kappa(H)\||| = -\int_{S^1} \min_{p \in M}(\|H\| + H(t,p)) \, dt = 0.\]

Any function \( G \) which generates a path that is homotopic to \( \phi^t_H \), relative its endpoints, can be used to construct a useful homotopy triple for \( H \). The following basic result in this direction is a simple consequence of Propositions 2.6 and 2.7 from [Ke].

**Proposition 3.3.** Let \( H \) be function in \( C^\infty_0(S^1 \times M) \). For any \( G \) in \( C^\infty_0(S^1 \times M) \) whose Hamiltonian path \( \phi^t_G \) is homotopic to \( \phi^t_H \) relative its endpoints, there is a family of almost complex structures \( J \) in \( J_{S^1}(M,\omega) \) and a homotopy triple \( H_G \) for \( (H,J) \) such that
\[ |||\kappa(H_G)||| + c \leq \|G\|^+.\]

Here, \( c \) is the constant appearing in the homotopy triple \( H_G \).

### 3.3. Cap data and central orbits.

For the pair \( (H,J) \), we fix a pair of homotopy triples \( H = (H_L,H_R) \). This will be referred to as a choice of **cap data**. The norm of the curvature of the cap data \( H \) is defined as
\[ |||\kappa(H)||| = |||\kappa(H_R)||| - |||\kappa(H_L)|||.\]

A periodic orbit \( x \in P(H) \) is said to be **central** for the cap data \( H \), if there is a pair \( (u,v) \in L(x;H_L) \times R(x;H_R) \) such that
\[ [u\#v] = 0.\]

Here, \( u\#v \) denotes the obvious concatenation of the maps, and \([u\#v] \) is the element of \( \pi_2(M) \) determined by \( u\#v \). We will refer to \( (u,v) \) above as a **central pair** of Floer caps for \( x \).

For a period orbit \( x \in P(H) \) and Floer caps \( u \in L(x;H_L) \) and \( v \in R(x;H_R) \), equations (11) and (13) yield
\[ 0 \leq E(u) = c_L - A_H(x,u) + \int_0^1 \int_{-\infty}^{+\infty} \kappa(H_L)(s,t,u(s,t)) \, ds \, dt,\]
and
\[ 0 \leq E(v) = A_H(x,v) - c_R - \int_0^1 \int_{-\infty}^{+\infty} \kappa(H_R)(s,t,v(s,t)) \, ds \, dt,\]
where \( c_L \) and \( c_R \) are the constants for \( H_L \) and \( H_R \), respectively.

If \( (u,v) \) is a central pair of Floer caps for \( x \) with respect to \( H \), then \( A_H(x,u) = A_H(x,v) \) and (15) and (16) imply that
\[ -|||\kappa(H_R)||| + c_R \leq A_H(x,v) = A_H(x,u) \leq |||\kappa(H_L)||| + c_L.\]

For a central pair \( (u,v) \) one also obtains from (15), (16) and (17), the uniform energy bounds
\[ E(u), E(v) \leq |||\kappa(H)||| + c_L - c_R.\]
3.4. **Small cap data and a chain isomorphism in Morse homology.**

For an almost complex structure $J$ in $\mathcal{J}(M, \omega)$, let $h(J)$ be the infimum over the symplectic areas of all nonconstant $J$-holomorphic spheres in $M$. We then set

$$ h = \sup_{J \in \mathcal{J}(M, \omega)} h(J). $$

This constant is positive and greater than or equal to $r(M, \omega)$.

We now describe a chain isomorphism in Morse homology which can be constructed using cap data $H$ that satisfies

$$ |||\kappa(H)||| + c_L - c_R < h. \tag{19} $$

This chain map will be used in §4.3 to find central periodic orbits whose right Floer caps will, in turn, be used to detect the right asymptotic spanning disc of Proposition 2.5 in §4.4.

Let $f$ be a Morse function on $M$ and $g$ a metric on $M$ such that the Morse complex, $(\text{CM}(f), \partial_g)$, is well-defined. Here $\text{CM}(f)$ is the $\mathbb{Z}$-module generated by the critical points of $f$. The $\mathbb{Z}$-module generated by the elements of $\mathcal{P}(H)$ is denoted by $\text{CF}(H)$.

**Proposition 3.4.** Let $H$ be a generic choice of cap data for $(H, J)$. If $|||\kappa(H)||| + c_L - c_R < h$, then there are two $\mathbb{Z}$-module homomorphisms $\Phi_L : \text{CM}(f) \to \text{CF}(H)$ and $\Phi_R : \text{CF}(H) \to \text{CM}(f)$ whose composition

$$ \Phi_H = \Phi_R \circ \Phi_L : \text{CM}(f) \to \text{CM}(f) $$

is a chain map which is chain homotopic to the identity.

This result is strongly motivated by the work of Chekanov in [Ch]. The proof of Proposition 3.4 is contained in [Ke] where it appears as Proposition 2.4. While it is assumed there that $c_L = c_R = 0$, the proof from [Ke] extends easily to the present setting. The genericity assumption of this result concerns the almost complex structure $J$ as well as the families of almost complex structures appearing in the cap data $H$. As usual, this assumption is included to ensure that the moduli spaces used to construct the maps are regular. These almost complex structures should also be chosen to lie in specific open sets of $\mathcal{J}(M, \omega)$ so that inequality (19) can be used to avoid bubbling. These technical details, which are discussed in detail in [Ke], can be safely ignored in the present discussion.

Since the maps $\Phi_R$, $\Phi_L$ and $\Phi_H$ play important roles in the proof of Proposition 2.5, we will recall the relevant aspects of their constructions. We begin with the map $\Phi_L$. 


A left or right Floer cap will be called **short** if its energy is less than \( h \). The subset of short elements in \( \mathcal{L}(x; \mathcal{H}_L) \) will be denoted by \( \mathcal{L}'(x; \mathcal{H}_L) \). Consider the space of left-half gradient trajectories:

\[
\ell(p) = \{ \alpha: (-\infty, 0] \to M \mid \dot{\alpha} = -\nabla_g f(\alpha), \alpha(-\infty) = p \}.
\]

For a critical point \( p \) of \( f \) and an orbit \( x \) in \( \mathcal{P}(H) \), set

\[
\mathcal{L}(p, x; f, \mathcal{H}_L) = \{ (\alpha, u) \in \ell(p) \times \mathcal{L}'(x; \mathcal{H}_L) \mid \alpha(0) = u(-\infty) \}.
\]

For generic data, \( \mathcal{L}(p, x; f, \mathcal{H}_L) \) is a smooth manifold and the local dimension of the component containing \((\alpha, u)\) is

\[
\text{ind}(p) - n - \mu_{cz}(x, u),
\]

**PSS.** The homomorphism \( \Phi_L: \text{CM}(f) \to \text{CF}(H) \) is defined on each generator \( p \) of \( \text{CM}(f) \) by

\[
\Phi_L(p) = \sum_{x \in \mathcal{P}(H)} \# \mathcal{L}_0(p, x; f, \mathcal{H}_L)x,
\]

where \( \# \mathcal{L}_0(p, x; f, \mathcal{H}_L) \) is the number of zero-dimensional components in \( \mathcal{L}(p, x; f, \mathcal{H}_L) \) counted with signs determined by a fixed coherent orientation. The shortness assumption implies that \( \mathcal{L}_0(p, x; f, \mathcal{H}_L) \) is compact and so the map \( \Phi_L \) is well-defined.

Next we consider the space of right-half gradient trajectories

\[
r(q) = \{ \beta: [0, +\infty) \to M \mid \dot{\beta} = -\nabla_g f(\beta), \beta(+\infty) = q \},
\]

and define

\[
\mathcal{R}(x, q; \mathcal{H}_R, f) = \{ (v, \beta) \in \mathcal{R}'(x; \mathcal{H}_R) \times r(q) \mid v(+\infty) = \beta(0) \}.
\]

Here, \( \mathcal{R}'(x; \mathcal{H}_R) \) is the set of short right Floer caps of \( x \). For generic data, each \( \mathcal{R}(x, q; \mathcal{H}_R, f) \) is a smooth manifold, and the dimension of the component containing \((v, \beta)\) is \( \mu_{cz}(x, \overline{v}) - \text{ind}(q) + m \). Let \( \mathcal{R}_0(x, q; \mathcal{H}_L, f) \) be the set of zero-dimensional components in \( \mathcal{R}(x, q; \mathcal{H}_L, f) \). The map \( \Phi_H \) is then defined by setting the coefficient of \( q \) in \( \Phi_H(p) \) to be the integer

\[
\sum_{x \in \mathcal{P}(H)} \# \big\{ ((\alpha, u), (v, \beta)) \in \mathcal{L}_0(p, x; \mathcal{H}_R, f) \times \mathcal{R}_0(x, q; \mathcal{H}_R, f) \mid [u\#v] = 0 \big\}.
\]

The map \( \Phi_R: \text{CF}(H) \to \text{CM}(f) \) is determined by \( \Phi_L \) and \( \Phi_H \) as follows. Let \( V_L \) be the submodule of \( \text{CF}(H) \) generated by the orbits in \( \mathcal{P}(H) \) which appear in an element in the image of \( \Phi_L \) with a nonzero coefficient. The maps \( \Phi_L \) and \( \Phi_H \) uniquely determine the restriction of \( \Phi_R \) to \( V_L \). Setting \( \Phi_R = 0 \) on the complement of \( V_L \) we obtain the full map. In particular, the coefficient of \( q \) in \( \Phi_R(x) \) is the signed count of elements \((v, \beta) \in \mathcal{R}_0(x, q; \mathcal{H}_R, f) \) for which there is an element \((u, \alpha)\) in some \( \mathcal{L}_0(p, x; \mathcal{H}_R, f) \) such that \([u\#v] = 0\).
4. Proof of Proposition 2.5

4.1. Step 1: approximating \(H_r\) by generic functions. The results of the previous section cannot be applied directly to \(H_r\) because the elements of \(\mathcal{P}(H_r)\) are degenerate. To overcome this, we now approximate \(H_r\) by a sequence of functions \(H_k\) whose 1-periodic orbits are nondegenerate. These functions are constructed explicitly in order to retain suitable control over their periodic orbits.

Let \(F^0: M \to \mathbb{R}\) be a Morse-Bott function with the following properties:

- The submanifold \(N\) is a critical submanifold with index equal to \(\text{codim}(N) = k\).
- All other critical submanifolds are isolated critical points of Morse index less than \(\text{dim}(M) = 2m\).
- On \(U_r\), \(F^0 = f^0(|p|)\) for some decreasing function \(f^0: [0, r] \to \mathbb{R}\).

Let \(f_N: N \to \mathbb{R}\) be a Morse function with a unique local maximum at a point \(Q\) in \(N\). Choose a bump function \(\hat{\sigma}: [0, +\infty) \to \mathbb{R}\) such that \(\hat{\sigma}(s) = 1\) for \(s\) near zero and \(\hat{\sigma}(s) = 0\) for \(s \geq r/4\). Let \(\sigma = \hat{\sigma}(|p|)\) be the corresponding function on \(M\) with support in \(U_{r/4}\) and set

\[
F = F^0 + \epsilon_N \cdot \sigma \cdot f_N.
\]

For a sufficiently small choice of \(\epsilon_N > 0\), \(F\) is a Morse function whose critical points away from \(U_{r/4}\) agree with those of \(F^0\) and whose critical points in \(U_{r/4}\) are precisely the critical points of \(f_N\) on \(N \subset M\).

Now let

\[
H^0_k = H_r + \frac{1}{k} F.
\]

Each \(H^0_k\) is also a Morse function with \(\text{Crit}(H^0_k) = \text{Crit}(F)\). As well, \(Q\) is the only critical point of \(H^0_k\) with Morse index equal to \(2m\). For an interval \(I \subset [0, R/2]\), we introduce the notation \(U_I = \{(q, p) \in U_R \mid |p| \in I\}\). When \(k\) is sufficiently large, the 1-periodic orbits of \(H^0_k\) are either critical points or nonconstant orbits contained in \(U_{(r/3, 2r/3)}\). In fact, these nonconstant orbits lie in \(U_{(r/3 + \delta, 2r/3 - \delta)}\) for some \(\delta > 0\). This follows from the fact that \(dH^0_k\) converges to zero in the \(C^\infty\)-topology along the boundary of \(U_{(r/3, 2r/3)}\).

Perturbing each \(H^0_k\) away from \(\text{Crit}(F)\), one obtains a sequence of functions \(H_k\) with the following properties

- \(H_k \to H_r\) in \(C^\infty(S^1 \times M)\).
- The orbits in \(\mathcal{P}(H_k)\) are nondegenerate and are of two types: constant orbits which coincide with the critical points of \(F\), and nonconstant orbits in \(U_{(r/3 + \delta, 2r/3 - \delta)}\) for some \(\delta > 0\).
- The constant periodic orbits equipped with their constant spanning discs have Conley-Zehnder indices less than \(m\) except for the constant orbit at the point \(Q \in N\), which has Conley-Zehnder index equal to \(m\).

The final detail to account for is the normalization condition. If we add the function \(-\int_0^1 H_k(t, \cdot) \omega^m\) to \(H_k\), the resulting function is normalized and
retains the properties described above. In particular, it determines the same Hamiltonian vector field. Abusing notation, this new normalized function will still be denoted by $H_k$.

The following lemma provides a simple criteria for detecting nonconstant periodic orbits of $H_k$.

**Lemma 4.1.** If $x(t)$ is a 1-periodic orbit of $H_k$ which admits a spanning disk $w$ such that $-\|H_k\|^- \leq A_{H_k}(x, w) < \|H_k\|^+$ and $\mu_{CZ}(x, w) = m$, then $x(t)$ is nonconstant.

**Proof.** Arguing by contradiction, we assume that $x(t) = P$ for some point $P$ in $M$. The spanning disk $w$ then represents an element $[w]$ in $\pi_2(M)$, and we have

$$A_{H_k}(x, w) = \int_0^1 H_k(t, P) dt - \omega([w]).$$

Moreover, the point $P$ corresponds to a critical point of $F$ and $H_k$ is $C^2$-small near $P$, so our normalization of the Conley-Zehnder index yields

(21) $$\mu_{CZ}(x, w) = \text{ind}(P) - m - 2c_1([w]).$$

If $\omega([w]) = 0$, then assumption [1] implies that $c_1([w]) \geq 0$. It then follows from (21) that the Morse index of $P$ must be $2m$. This implies that $P = Q$, since $Q$ is the unique fixed local maximum of $H_k$. However, the action of $Q$ with respect to a spanning disc $w$ with $\omega([w]) = 0$ is equal to $\|H_k\|^+$. This is outside the assumed action range and hence a contradiction.

We must therefore have $\omega([w]) \neq 0$ and thus

$$|\omega([w])| \geq r(M, \omega) > \|H_k\|.$$

For the case $\omega([w]) < 0$, this implies that

$$A_{H_k}(x, w) \geq \int_0^1 H_k(t, P) dt + \|H_k\| \geq \|H_k\|^+$$

which is a contradiction, as above. If $\omega([w]) > 0$, then

$$A_{H_k}(x, w) \leq \int_0^1 H_k(t, P) dt - \|H_k\| = \int_0^1 H_k(t, P) dt - \|H_k\|^+ - \|H_k\|^-.$$

So, either $A_{H_k}(P, w) < -\|H_k\|^-$ or $P = Q$. Both of these conclusions again contradict our hypotheses. Therefore $x(t)$ must be nonconstant. \qed

**4.2. Step 2: curve shortening.** We now prove that the Hamiltonian path $\phi^t_{H_k}$ does not minimize the positive Hofer length in its homotopy class. We also show that the same is true of the paths $\phi^t_{H_k}$ when $k$ is sufficiently large.

For a Hamiltonian path $\psi_t$, let $[\psi_t]$ be the class of Hamiltonian paths which are homotopic to $\psi_t$ relative to its endpoints. Denote the set of normalized functions which generate the paths in $[\psi_t]$ by

$$C_0^\infty([\psi_t]) = \{H \in C_0^\infty(S^1 \times M) \mid [\phi^t_H \circ \psi_0] = [\psi_t]\}.$$
The Hofer semi-norm of $[\psi_t]$ is then defined by
\[ \rho([\psi_t]) = \inf_{H \in C^\infty_0([\psi_t])} \{ \|H\| \}. \]
The positive and negative Hofer semi-norms of $[\psi_t]$ are defined similarly as
\[ \rho^\pm([\psi_t]) = \inf_{H \in C^\infty_0([\psi_t])} \{ \|H\|^\pm \}. \]
Clearly
\[ \rho([\psi_t]) \geq \rho^+([\psi_t]) + \rho^-([\psi_t]). \]
In these terms, the displacement energy of a subset $U \subset M$ is equal to
\[ e(U) = \inf_{\psi_t} \{ \rho([\psi_t]) \mid \psi_0 = id \text{ and } \psi_1(U) \cap U = \emptyset \}. \]

The following result is an easy application of Sikorav’s curve shortening procedure. The proof follows very closely the proof of Proposition 2.1 in [Sc].

**Lemma 4.2.** Let $H$ be an autonomous normalized Hamiltonian that is constant and equal to its minimal value on the complement of an open set $U \subset M$ which has finite displacement energy. If $\|H\|^+ > 2e(U)$, then
\[ \|H\|^+ > \rho^+([\phi^t_H]) + \frac{1}{2}\|H\|^- . \]
In other words, $\phi^t_H$ does not minimize the positive Hofer semi-norm in its homotopy class.

**Proof.** Let $\phi_t$ and $\psi_t$ be Hamiltonian paths and let $\varphi$ be a symplectomorphism. The following properties of the positive and negative Hofer semi-norms are easily checked.
- $\rho^\pm([\phi_t \circ \psi_t]) \leq \rho^\pm([\phi_t]) + \rho^\pm([\psi_t])$
- $\rho^\pm([\phi_t \circ \psi_t]) = \rho^\pm([\phi_t])$
- $\rho^\pm([\psi_t^{-1} \circ \phi_t \circ \psi_t]) = \rho^\pm([\phi_t])$
- $\rho^\pm([\phi^{-1}_t]) = \rho^-([\phi_t])$.

Now choose a Hamiltonian path $\psi_t$ starting at the identity such that $\psi_1(U) \cap U = \emptyset$.

The path $\phi^t_H$ can then be factored as follows.
\[ \phi^t_H = \left( \phi^{t/2}_H \circ \psi_t \circ \phi^{t/2}_H \circ \psi_t^{-1} \right) \circ \left( \psi_t \circ (\phi^{t/2}_H)^{-1} \circ \psi_t^{-1} \circ \phi^{t/2}_H \right) \]
\[ = b_t \circ a_t. \]

Hence,
\[ \rho^+([\phi^t_H]) \leq \rho^+([a_t]) + \rho^+([b_t]). \]
For the first summand on the right, we have
\[ \rho^+([a_t]) = \rho^+([\psi_t \circ (\phi_H^{t/2})^{-1} \circ \psi_t^{-1} \circ \phi_H^{t/2}]) \]
\[ \leq \rho^+([\psi_t]) + \rho^+([(\phi_H^{t/2})^{-1} \circ \psi_t^{-1} \circ \phi_H^{t/2}]) \]
\[ = \rho^+([\psi_t]) + \rho^+([(\phi_H^{t/2})^{-1} \circ \psi_t^{-1} \circ \phi_H^{t/2}]) \]
\[ = \rho^+([\psi_t]) + \rho^+([\psi_t^{-1}]) \]
\[ \leq \rho([\psi_t]), \]
and for the second summand we have
\[ \rho^+([b_t]) = \rho^+([\phi_H^{t/2} \circ \psi_t \circ \phi_H^{t/2} \circ \psi_t^{-1}]) \]
\[ = \rho^+([\phi_H^{t/2} \circ \psi_t \circ \phi_H^{t/2} \circ \psi_t^{-1}]) \]
\[ \leq \left\| \frac{1}{2} H + \frac{1}{2} H \circ \psi_t^{-1} \circ (\phi_H^{t/2})^{-1} \right\|^+ \]
\[ = \left\| \frac{1}{2} H \circ \phi_H^{t/2} + \frac{1}{2} H \circ \psi_t^{-1} \right\|^+ \]
\[ = \max_{p \in M} \left( \frac{1}{2} H(p) + \frac{1}{2} H \circ \psi_t^{-1}(p) \right) \]
\[ = \frac{1}{2} \left\| H \right\|^+ - \frac{1}{2} \left\| H \right\|^-. \]
Together, these inequalities imply that for any Hamiltonian path \( \psi_t \) which displaces \( U \) we have
\[ \rho^+([\phi_H]) \leq \rho([\psi_t]) + \frac{1}{2} \left\| H \right\|^+ - \frac{1}{2} \left\| H \right\|^-. \]
Taking the infimum over all such paths we get
\[ \rho^+([\phi_H]) \leq e(U) + \frac{1}{2} \left\| H \right\|^+ - \frac{1}{2} \left\| H \right\|^-. \]
\[ < \left\| H \right\|^+ - \frac{1}{2} \left\| H \right\|^-. \]
\[ \square \]
By construction, we have \( \left\| H_r \right\|^+ = A > 2e(U_r) \). Together with Lemma 4.2 this implies that \( \phi_H^t \) does not minimize the positive Hofer length in its homotopy class. In other words, there is a function \( G \) in \( C_0^\infty([\phi_H^t]) \) such that
\[ (22) \quad \left\| H \right\|^+ = \left\| G \right\|^+ + 2\epsilon \]
for some \( \epsilon > 0 \). We now show that for sufficiently large \( k \), the paths \( \phi_H^k \) can be shortened in their respective homotopy classes by at least \( \epsilon \).
To see this, consider the Hamiltonian path \( \phi^t_{H_k} \circ (\phi^t_{H_r})^{-1} \) which is generated by the function

\[
F_k = H_k - H_r \circ \phi^t_{H_r} \circ (\phi^t_{H_k})^{-1}.
\]

These functions clearly converge to zero in the \( C^\infty \)-topology. The path

\[
\phi^t_{H_k} \circ (\phi^t_{H_r})^{-1} \circ \phi^t_G
\]
is homotopic to \( \phi^t_{H_k} \) and is generated by the function

\[
G_k = F_k + G \circ (\phi^t_{F_k})^{-1}.
\]

Hence, we have functions \( G_k \) in \( C^\infty_0([\phi^t_{H_k}]) \) such that \( \|G_k\|^+ = \|F_k\|^+ + \|G\|^+ \). For large enough \( k \) we then have

\[
\|G_k\|^+ \leq \|H_k\|^+ - \epsilon. \tag{23}
\]

4.3. Step 3: nontrivial linear right Floer caps. Fix a family of almost complex structures \( J_k \) for each \( H_k \) such that the \( J_k \) converge to \( J \) in \( J_{S^1}(M, \omega) \). As in Example 3.2 set

\[
\overline{\mathcal{H}}_k = ((\eta(s) - 1)\|H_k\|^+ + \eta(s)H_k, 0, J_k, s),
\]

where the \((\mathbb{R} \times S^1)\)-families of almost complex structures \( J_{k,s} \) converge to a compact homotopy \( J_s \) from some \( J^- \) to \( J \). The linear homotopy triples \( \overline{\mathcal{H}}_k \) then converge to the linear homotopy triple \( \overline{\mathcal{H}}_r = (\overline{\mathcal{H}}_s, 0, J_s) \) for \((H_r, J)\).

**Proposition 4.3.** For large enough \( k \), there is a nonconstant 1-periodic orbit \( x_k \) of \( H_k \) and a right Floer cap \( v_k \) in \( \mathcal{R}(x_k; \overline{\mathcal{H}}_k) \) such that

\[
E(v_k) < \|H_k\| < r(M, \omega) \tag{24}
\]

and

\[
-\|H_k\|^- \leq A_{H_k}(x_k, \overline{H}_k) < \|G_k\|^+ \leq \|H_k\|^+ - \epsilon. \tag{25}
\]

**Proof.** As shown above, for large enough \( k \) there is a function \( G_k \) such that the Hamiltonian path \( \phi^t_{G_k} \) is homotopic to \( \phi^t_{H_k} \), relative endpoints, and \( \|G_k\|^+ \leq \|H_k\|^+ + \epsilon \). Applying Proposition 4.3 to \( G_k \), we get a \( J_k \) in \( J_{S^1}(M, \omega) \) and a homotopy triple \( \mathcal{H}_k \) for \((H_k, J_k)\) such that

\[
\|\kappa(\mathcal{H}_k)\|^+ + c_{L,k} \leq \|G_k\|^+ \tag{26}
\]

We now consider the following cap data for \((H_k, J_k)\),

\[
H_k = (\mathcal{H}_k, \overline{\mathcal{H}}_k).
\]

By inequalities (23), (26), and the curvature norm estimates for linear homotopy triples derived in Example 3.2, we have

\[
\|\kappa(\mathcal{H}_k)\|^+ + \|\kappa(\overline{\mathcal{H}}_k)\|^+ + c_{L,k} - c_{R,k} \leq \|G_k\|^+ + \|H_k\|^-. \tag{27}
\]

By construction (see inequalities (3) and (4)) we also have \( \|H_r\| < r(M, \omega) \). Hence, for sufficiently large \( k \), (27) implies that

\[
\|\kappa(\mathcal{H}_k)\|^+ + \|\kappa(\overline{\mathcal{H}}_k)\|^+ + c_{L,k} - c_{R,k} < \|H_k\| < r(M, \omega). \tag{28}
\]
From this point on, we will assume that $k$ is large enough for this inequality to hold.

Since $r(M, \omega) \leq \hbar$, inequality \[22\] allows us to apply Proposition 3.4 to the homotopy data $H_k$. In particular, for any Morse-Smale pair $(f, g)$ on $M$ we can construct two $\mathbb{Z}$-module homomorphisms

$$
\Phi_{L,k}: \text{CM}(f) \to \text{CF}(H_k)
$$

and

$$
\Phi_{R,k}: \text{CF}(H_k) \to \text{CM}(f)
$$

whose composition is a chain map

$$
\Phi_{H_k}: \text{CM}(f) \to \text{CM}(f)
$$

which is chain homotopic to the identity.

For simplicity, we choose the Morse-Smale pair $(f, g)$ so that the function $f$ has a unique local (and hence global) maximum at a point $q \in M$. Standard arguments imply that $q$ is the unique nonexact cycle of degree $2m$ in the Morse complex $(\text{CM}(f), \partial_g)$, and so

$$
\Phi_{H_k}(q) = q.
$$

Let $V_{L,k}$ be the submodule of $\text{CF}(H_k)$ generated by 1-periodic orbits of $H_k$ which appear in an element of the image of $\Phi_{L,k}$ with a nonzero coefficient. Let $K_{R,k}$ be the submodule of $\text{CF}(H_k)$ generated by periodic orbits which lie in the kernel of $\Phi_{R,k}$ and let $p_k: V_{L,k} \to V_{L,k}/K_{R,k}$ be the projection map. We then have

$$
\Phi_{H_k} = \Phi_{R,k} \circ p_k \circ \Phi_{L,k}.
$$

It follows from the definitions of these maps that any periodic orbit which appears in the image of $p_k \circ \Phi_{L,k}$ is central with respect to $H_k$.

Let

$$
X_k = p_k \circ \Phi_{L,k}(q).
$$

By the construction of $\Phi_{H_k}$, $X_k$ is a finite sum of the form

$$
X_k = \sum n^j_k x^j_k
$$

where the $n^j_k$ are nonzero integers and the $x^j_k$ are central 1-periodic orbits of $H_k$.

Since $X_k$ gets mapped to $q$ under $\Phi_{L,k}$, the moduli space

$$
\mathcal{R}_0(X_k, q; \mathcal{H}_k, f) = \bigcup_j \mathcal{R}_0(x^j_k, q; \mathcal{H}_k, f),
$$

which determines the image $\Phi_{R,k}(X_k)$, must be nonempty. Choose a $(v_k, \sigma_k)$ in $\mathcal{R}_0(X_k, q; \mathcal{H}_k, f)$ for each $k$. The caps $v_k$ belongs to $\mathcal{R}(x_k; \mathcal{H}_k)$ for some $x_k$ in $\mathcal{P}(H_k)$ which appears in $X_k$ with a nonzero coefficient. Moreover, $v_k$ is part of a central pair for $x_k$ with respect to $H_k$, and so by \[17\], \[23\] and \[26\], we have

$$
-\|H_k\|^- \leq \mathcal{A}_{H_k}(x_k, \tilde{v}_k) \leq \|H_k\|^+ - 2\epsilon.
$$
Inequality (18) together with (28) yields the desired uniform energy bound
\[ E(v_k) \leq \|\kappa(H_k)\| + c_{L,k} - c_{R,k} < \|H_k\| < r(M,\omega). \]

It only remains to show that the orbits \(x_k\) are nonconstant. Each \(x_k\) appears in \(p_k \circ \Phi_{L,k}(q)\) with a nonzero coefficient. Hence, there is a pair of maps \((\alpha_k, u_k)\) in \(\mathcal{L}^0(q, x_k; f, \mathcal{H}_{G_k})\) such that \(u_k\) is part of a central pair for \(x_k\) with respect to \(H_k\). The existence of the regular pair \((\alpha_k, u_k)\) together with the dimension formula for \(\mathcal{L}^0(q, x_k; f, \mathcal{H}_{G_k})\), (20), implies that
\[ \mu_{CZ}(x_k, u_k) = \text{ind}(q) - m = m. \]

Since \(u_k\) is part of a central pair for \(x_k\) the action \(A_{H_k}(x_k, u_k)\) satisfies the same bounds, (25), as \(A_{H_k}(x_k, v_k)\), i.e.,
\[ -\|H_k\| \leq A_{H_k}(x_k, u_k) \leq \|H_k\| - 2\epsilon. \]

Lemma 4.1 then implies that the orbits \(x_k\) are nonconstant and the proof of Proposition 4.3 is complete.

\[ \square \]

4.4. Step 4: A nonconstant limit of linear right Floer caps. Let \(C\) be the closed subset of \(C^\infty(\mathbb{R} \times S^1, M)\) consisting of maps \(v: \mathbb{R} \times S^1 \to M\) such that \(v(0, t)\) is a contractible loop in \(M\). We consider this space as being equipped with the \(C^\infty_{\text{loc}}\)-topology.

By Proposition 4.3 we have a sequence of nonconstant periodic orbits \(x_k \in \mathcal{P}(H_k)\) and a sequence of right Floer caps \(v_k \in \mathcal{R}(x_k, \mathcal{H}_k)\) which satisfy (24) and (25). The linear homotopy triples \(\mathcal{H}_k\) were chosen so that they converge to \(\mathcal{H}_r = (\mathcal{H}_s, 0, J_s)\). Together with uniform energy bound (24), this implies that there is a subsequence of the \(v_k\) which converges in \(C\) to a map \(\tilde{v}\). This limiting map \(\tilde{v}\) is a solution of the equation
\[
\partial_t \tilde{v} + J_s(\tilde{v})(\partial_t \tilde{v} - X_{\mathcal{H}_s}(\tilde{v})) = 0,
\]
for
\[
\mathcal{H}_s = (\eta(-s) - 1)B + \eta(-s)\mathcal{H}_r.
\]

It also satisfies
\[ 0 \leq E(\tilde{v}) < r(M, \omega). \]

The map \(\tilde{v}\) may or not be constant. To find the periodic orbit and the right asymptotic spanning disc of Proposition 2.5, we need to consider both possibilities.

4.4.1. Case 1: a nonconstant limit. We assume here that the subsequence, which we still denote by \(v_k\), converges to a nonconstant solution \(\tilde{v}\) of (29). In this case, the map \(\tilde{v}\) will be the asymptotic right spanning disc of Proposition 2.5 and we will write \(\tilde{v} = v\).

The energy bound (30) implies that the limit \(v(+\infty) = \lim_{s \to +\infty} v(s, t)\) is a point in \(M\). It also implies that there is a sequence \(s_j^- \to -\infty\) such that
Let $v(s_j^- t)$ converge to some $y(t)$ in $P(H_r)$. For simplicity we assume that the sequence $s_j^-$ is monotone decreasing and that $s_1^- < -1$.

It remains for us to show that the limiting periodic orbit $y$ is nonconstant and that there is an $\epsilon > 0$ such that (5) holds for all $j$, i.e.,

$$E^j(v) - B \leq A^j_{H_r}(v) \leq A - \epsilon.$$ 

We begin by proving that (5) holds for $\epsilon = \frac{1}{2}(\|H_r\|^+ - \|G\|^+)$. By Remark 3.1 we have $E^j(v) = E_{s_j^-}(v)$ and $A^j_{H_r}(v) = A^-_{s_j^-}(v)$. Since the $v_k$ converge to $v$ in the $C^\infty_{loc}$-topology, and $H_k$ converges to $H_r$ in the $C^\infty$-topology, it suffices to show that for large enough $k$ we have

$$(31) \quad E_{s_j^-}(v_k) - \|H_k\|^+ \leq A^-_{s_j^-}(v_k) \leq \|H_k\|^+ - \epsilon.$$ 

The first of these inequalities follows immediately from equation (13). In particular, this identity implies that

$$A^-_{s_j^-}(v_k) \geq E_{s_j^-}(v_k) - \|H_k\|^+ + \int_0^1 \int_{s_j^-} v(s, t, v_k(s, t)) ds dt \geq E_{s_j^-}(v_k) - \|H_k\|^+,$$

since $\kappa(\mathcal{H}_k) = \dot{\eta}(s)(\|H_k\|^+ + H_k) \geq 0$.

To prove the second inequality in (31), we first note that for $i > j$ inequality (14) yields

$$A^-_{s_i^-}(v_k) - A^-_{s_j^-}(v_k) \geq \int_0^1 \int_{[s_j^- s_i^-]} v(s, t, v_k(s, t)) ds dt.$$ 

Hence, for each $k$, the sequence $A^-_{s_j^-}(v_k)$ is nondecreasing and

$$A_{H_k}(x_k, v_k) = \lim_{i \to \infty} A^-_{s_i^{-}}(v_k) \geq A^-_{s_j^{-}}(v_k).$$ 

By (25), this yields

$$A^-_{s_j^{-}}(v_k) \leq \|H_k\|^+ - \epsilon.$$ 

Thus, (31) holds and we have established inequality (5).

Finally we must show that the periodic orbit $y$ is nonconstant. This is easily derived from (5) as follows. Set

$$y^{[s]}(t) = v(s, t)$$

and consider the annulus

$$v^{[s]} = v|_{[s, \infty) \times S^1}.$$ 

Since $y^{[s_j]} \to y$ and $y^{[s_j]} \to p$, for large values of $j$ the annuli $v^{[s_j]}$ can be extended and reparameterized to form spanning discs for $y \in P(H)$ in a fixed homotopy class. These extensions can be made arbitrarily small for
sufficiently large $j$. Hence, by inequality (5) and the assumption that $v$ is nonconstant, we can choose such a spanning disc $w$ for $y$ such that

$$-B < A_{H_r}(y, w) < A.$$  

Assume now that $y$ is a constant periodic orbit, i.e., $y(t) = P$ for some critical point $P$ of $H_r$. Then $w$ represents a class $[w] \in \pi_2(M)$ and

$$A_{H_r}(y, w) = H_r(P) - \omega([w]).$$  

If $\omega([w]) = 0$, then (32) and (33) imply that $P$ must be a critical point of $H_r$ with critical value in $(-B, A)$. Since there is no such critical point, we must have $\omega([w]) \neq 0$ and hence

$$|\omega([w])| \geq r(M, \omega) > \|H_r\| = A + B.$$  

However, this implies that $A_{H_r}(y, w)$ fails to lie in the interval $(-B, A)$, which contradicts (32). The orbit $y$ must therefore be nonconstant.

4.4.2. Case 2: a constant limit. We now assume that the maps $v_k$ converge in $C$ to a constant map $\bar{v}(s, t) = \bar{P}$. In this case, we will adapt a topological argument from [Gi] to prove that there is a sequence $\tau_k \to -\infty$, such that $v_k(s + \tau_k, t)$ converges to a nonconstant solution $v$ of the equation

$$\partial_s v + J(v)(\partial_t v - X_{H_r}(v)) = 0.$$  

This will be the right asymptotic spanning disc of Proposition 2.5.

To detect this map, we first pass to a subsequence of the $v_k$ whose negative asymptotic limits converge to a nonconstant element of $\mathcal{P}(H_r)$. Recall that $x_k = v_k(-\infty)$ is a nonconstant 1-periodic of $H_k$. Since the $x_k$ are nonconstant, they are contained in the region $U_{r/3 + \delta}$ by Arzela-Ascoli, there is a convergent subsequence of the $x_k$ that converges to some $x \in \mathcal{P}(H_r)$. Since it is contained in $U_{r/3 + 2\delta}$, the orbit $x$ is also nonconstant. From this point on, we restrict our attention to a subsequence of the $v_k$ for which the $x_k$ converge to $x$. For simplicity, this subsequence will still be denoted by $v_k$.

There is a natural action of $\mathbb{R}$ on $\mathcal{C}$ defined by $\tau \cdot v(s, t) = v(s + \tau, t)$. We set

$$\Gamma(v_k) = \{\tau \cdot v_k \mid \tau \in \mathbb{R}\},$$  

and define $\Sigma$ to be the set of limits of all convergent sequences of the form

$$v = \lim_{k \to -\infty} \tau_k \cdot v_k.$$  

There are two continuous maps on $\Sigma$ which will be useful in what follows. The first is the evaluation map $ev : \Sigma \to M$ defined by

$$ev(v) = v(0, 0).$$  

The second map is the function $\overline{\mathcal{A}}_{-\infty}^0 : \Sigma \to \mathbb{R}$ which is defined, as in (3.2), by

$$\overline{\mathcal{A}}_{-\infty}^0(v) = \int_0^1 H_r(\bar{v}(0, t)) \, dt - \int_0^1 \int_{-\infty}^0 \omega(\partial_s \bar{v}, \partial_t \bar{v}) \, ds \, dt.$$
Lemma 4.4. For every \( v \) in \( \Sigma \),
\[-B \leq \overline{A}_{-\infty}^0(v) \leq A - \epsilon.\]

Proof. By the definition of \( \Sigma \), we have \( v = \lim_{k \to \infty} \tau_k \cdot v_k \) for some sequence \( \tau_k \). Hence,
\[\overline{A}_{-\infty}^0(v) = \lim_{k \to \infty} \overline{A}_{-\infty}^0(\tau_k \cdot v_k)\]
and it suffices to show that for sufficiently large \( k \) we have
\[E_{-\infty}^0(\tau_k \cdot v_k) - ||H_k||^- \leq \overline{A}_{-\infty}^0(\tau_k \cdot v_k) \leq ||H_k||^+ - \epsilon.\]
The proof of these inequalities is entirely similar to the proof of \( (31) \). In particular, \( (13) \) implies that
\[\overline{A}_{-\infty}^0(\tau_k \cdot v_k) \geq E_{-\infty}^0(\tau_k \cdot v_k) - ||H_k||^-.
\]
On the other hand, \( (14) \) yields
\[A_{H_k}(x_k, \frac{\tau_k \cdot v_k}{H_k}) - \overline{A}_{-\infty}^0(\tau_k \cdot v_k) \geq \int_0^1 \int_0^\infty \kappa(H_k)(s, t, \frac{\tau_k \cdot v_k}{H_k}) ds dt,
\]
and by \( (25) \), we then have
\[||H_k||^+ - \epsilon \geq A_{H_k}(x_k, \frac{\tau_k \cdot v_k}{H_k}) = A_{H_k}(x_k, \frac{\tau_k \cdot v_k}{H_k}) \geq \overline{A}_{-\infty}^0(\tau_k \cdot v_k).\]

Lemma 4.5. Every element of \( \Sigma \) is a solution of \( (34) \) with energy less that \( r(M, \omega) \).

Proof. Let \( v = \lim_{k \to \infty} \tau_k \cdot v_k \). By \( (24) \), the energy of each \( v_k \) is less than \( r(M, \omega) \). Since \( E(v_k) = E(\tau_k \cdot v_k) \), the energy of \( v \) is also less than \( r(M, \omega) \). It only remains to show that \( v \) is a solution of \( (34) \).

Recall that, \( \overline{H}_s \) is a compact homotopy from \( H_r \) to \( -B \). If \( \tau_k \to -\infty \), then \( v \) is clearly a solution of \( (34) \). If the sequence of shifts \( \tau_k \) is bounded, then \( v \) is equal to the constant map \( \widehat{v}(s, t) = \tilde{P} \). Since \( \widehat{v} \) is also a solution of \( (29) \), we must have \( X_{\overline{H}_s}(\tilde{P}) = 0 \) for all \( s \in \mathbb{R} \). In other words, \( \tilde{P} \) is a critical point of \( H_r \) and hence must lie in \( U_{r/3} \cup U_{(2r/3, +\infty)} \), where \( U_{(2r/3, +\infty)} \) denotes the complement of \( U_{2r/3} \) in \( M \). Lemma \( 4.4 \) implies that
\[-B \leq \overline{A}_{-\infty}^0(\widehat{v}) = H_r(\tilde{P}) \leq A - \epsilon.\]
Hence, \( \tilde{P} \) belongs to \( U_{(2r/3, +\infty)} \). On this set \( \overline{H}_s = H_r = -B \), and so \( v \) is a trivial solution of \( (34) \).

Finally, when the shifts \( \tau_k \to \infty \), the limit \( v \) is a solution of
\[\partial_s v + J^-(v) \partial_t v = 0\]
with energy less than \( r(M, \omega) \). Any such map can be uniquely extended to a holomorphic sphere with the same energy. Since \( r(M, \omega) < h \), the almost complex structure \( J^- \) can be chosen, at the outset, to satisfy \( h(J^-) > ||H_r|| \).
The map \( v \) must therefore be constant. Lemma 4.4 implies that the constant maps in \( \Sigma \) all lie in \( U_{(2r/3, +\infty)} \). Hence, \( v \) is again a trivial solution of (34).

\[ \quad \]

**Lemma 4.6.** The function \( \tau \mapsto \overline{A}_{-\infty}(\tau \cdot v) \) is nonincreasing. It is strictly decreasing unless \( v \) belongs to \( \mathcal{P}(H_r) \).

Here, the elements of \( \mathcal{P}(H_r) \) are identified with elements of \( \mathcal{C} \) that do not depend on \( s \).

**Proof.** For \( \tau' > \tau \), a simple computation yields

\[
\overline{A}_{-\infty}(\tau' \cdot v) - \overline{A}_{-\infty}(\tau \cdot v) = - \int_0^1 \int_{-\tau'}^{-\tau} \omega(\partial_s \overline{v}, J(\overline{v}) \partial_s \overline{v}) \, ds \, dt.
\]

Since the integrand is nonnegative the function \( \overline{A}_{-\infty}(\tau \cdot v) \) is nonincreasing. If \( \overline{A}_{-\infty}(\tau' \cdot v) = \overline{A}_{-\infty}(\tau \cdot v) \) for \( \tau' > \tau \), then (36) implies that \( \partial_s v = 0 \) for \( s \in (\tau, \tau') \). By Lemma 4.5, \( v \) is a solution of (34), and hence \( v(s, t) = v(t) \) is a 1-periodic orbit of \( H_r \), for \( s \in (\tau, \tau') \). The Unique Continuation Theorem of \( \text{[FHS]} \) then implies that \( v(s, t) = v(t) \) for all values of \( s \).

\[ \quad \]

Following [Gi] we now prove:

**Lemma 4.7.** The set \( \Sigma \) has the following properties.

(i) the point \( \tilde{P} \) and the nonconstant 1-periodic orbit \( x(t) \) belong to \( \Sigma \);

(ii) the subsets \( \Gamma(v_k) \subset \mathcal{C} \) converge to \( \Sigma \) in the Hausdorff topology;

(iii) the set \( \Sigma \) is connected, compact and preserved by the \( \mathbb{R} \)-action on \( \mathcal{C} \);

(iv) The action of \( \mathbb{R} \) on \( \Sigma \) is nontrivial.

**Proof.** The first two properties follow almost immediately from the definition of \( \Sigma \). The same is true of the fact that \( \Sigma \) is invariant under the \( \mathbb{R} \)-action.

The compactness of \( \Sigma \) follows from Lemma 4.5 and the fact that \( \Sigma \) is closed. In particular, the subset of \( \mathcal{C} \) consisting of solutions of (34) with energy less than \( \hbar \) is itself compact by the usual Floer compactness theorem.

To prove that \( \Sigma \) is connected, consider any two disjoint open sets in \( \mathcal{C} \), \( U_1 \) and \( U_2 \), which cover \( \Sigma \). Let \( \Sigma_{\tilde{P}} \) be the component of \( \Sigma \) which contains \( \tilde{P} \) and suppose that \( \Sigma_{\tilde{P}} \subset U_1 \). By (ii), the \( \Gamma(v_k) \) are contained in \( U_1 \cup U_2 \) for all sufficiently large \( k \). Since the \( \Gamma(v_k) \) are connected and \( \tilde{P} \) is a limit point of the \( \Gamma(v_k) \), they must be contained in \( U_1 \) for large \( k \). Thus, \( \Sigma \cap U_2 = \emptyset \) and it follows that \( \Sigma \) must be connected.

To prove (iv), we note that (iii) implies that \( ev(\Sigma) \) is connected. Since \( \tilde{P} \) belongs to \( U_{(2r/3, +\infty)} \) and \( x(t) \) belongs to \( U_{(r/3+\delta, 2r/3-\delta)} \) there must be some \( v \) in \( \Sigma \) for which \( ev(v) \) belongs to \( U_{(2r/3-\delta, 2r/3)} \). There are no 1-periodic orbits on the level sets in \( U_{(2r/3-\delta, 2r/3)} \), so the loop \( v(0, t) \) is not in \( \mathcal{P}(H_r) \). Hence, \( v \) is not a fixed point of the \( \mathbb{R} \)-action by Lemma 4.6.

\[ \quad \]

We now consider the set

\[ \Sigma_{\text{min}} = \{ v \in \Sigma \mid \overline{A}_{-\infty}(v) = -B \} \]
The properties of $\Sigma$ established above imply that $\Sigma_{\text{min}}$ is comprised of elements in $\mathcal{P}(H_r)$. In particular, for $v \in \Sigma_{\text{min}}$ choose a $\tau < 0$. Lemmas 4.4, 4.6 and 4.7 yield

$$-B = \overrightarrow{A}_{-\infty}^0(v) \leq \overrightarrow{A}_{-\infty}^0(\tau \cdot v) \geq -B.$$  

The second statement of Lemma 4.6 then implies that $v$ belongs $\mathcal{P}(H_r)$. Note that the constant elements of $\Sigma_{\text{min}}$ take values in the set $U_{\left(\frac{2r}{3},+\infty\right]}$. Identifying $U_{\left(\frac{2r}{3},+\infty\right]}$ with the space of constant maps in $C$ which take values in $U_{\left(\frac{2r}{3},+\infty\right]}$, we define

$$C = U_{\left[2r/3,\infty\right)} \cap \Sigma_{\text{min}} = U_{\left[2r/3,\infty\right)} \cap \Sigma.$$  

The set $C$ is a compact subset of $\Sigma$. By property (iv) of Lemma 4.7, $C$ is also a proper subset of $\Sigma$. Most importantly, $C$ is nonempty because it contains $\tilde{P}$.

**Lemma 4.8.** The set $C$ is a union of connected components of $\Sigma_{\text{min}}$.

**Proof.** If one assumes the contrary, then there is a sequence of nonconstant periodic orbits $x_k \in \Sigma_{\text{min}} \setminus C$ which converges to a point of $C$. This is a contradiction since the nonconstant orbits are contained in the closure of $U_{2r/3-\delta}$. □

Let $\mathcal{N}_c = \{w \in \Sigma \mid \overrightarrow{A}_{-\infty}^0(v) < -B + c\}$. Fix a connected component $C_0$ of $C$ and let $\mathcal{V}_c$ be the component of $\mathcal{N}_c$ which contains $C_0$.

**Lemma 4.9.** For any open set $\mathcal{V}$ in $\Sigma$ which contains $C_0$, there is a $c > 0$ such that $\mathcal{V}_c \subset \mathcal{V}$.

**Proof.** Assume the contrary. Then there is neighborhood $\mathcal{V} \supset C_0$ and a sequence $c_i \to 0^+$ such that $\mathcal{V}_{c_i}$ is not contained in $\mathcal{V}$. Let $v_i$ be an element in $\mathcal{V}_{c_i} \setminus \mathcal{V}$. Since $\Sigma$ is compact, there is a subsequence of the $v_i$ which converges to an element of $\bigcap_i \mathcal{V}_{c_i} \setminus \mathcal{V}$. On the other hand, $\bigcap_i \mathcal{V}_{c_i}$ is a connected subset of $\Sigma_{\text{min}}$ which contains $C_0$. This contradicts the fact that $C_0$ is a connected component of $\Sigma_{\text{min}}$. □

We can now complete the proof of Proposition 2.5 in the present case. By Lemma 4.8 we can find a constant $c > 0$ such that $\mathcal{V}_c \subset ev^{-1}(U_{2r/3-\delta,+\infty})$. Either $\mathcal{V}_c \cap C = C_0$ or $\mathcal{V}_c \cap C$ is disconnected. In both cases, the fact that $\Sigma$ is connected and contains the nonconstant orbit $x(t)$ implies that

$$\mathcal{V}_c \setminus C \neq \emptyset.$$  

Let $v$ be any map in $\mathcal{V}_c \setminus C$. We will show that $v$ is a right asymptotic spanning disc with the desired properties.

Properties (5) and (6) are easily verified. By Lemma 4.5, $v$ is a solution of (34) and hence (6) for $\bar{H}_s = H_r$. By the definition of $\Sigma$, $v = \lim_{k \to \infty} \tau_k \cdot v_k$ and so

$$\mathcal{A}_{H_r}^j(v) = \overrightarrow{A}_{s_j}^{-s_j'}(v) = \lim_{k \to \infty} \overrightarrow{A}_{s_j}^{-s_j'}(\tau_k \cdot v_k).$$
To prove that \( v \) satisfies (5), it then suffices to show that
\[
\overline{E}_{s_j^+}^{-s_j^-}(\tau_k \cdot v_k) - \|H_k\| \leq \overline{A}_{s_j^+}^{-s_j^-}(\tau_k \cdot v_k) \leq \|H_k\|^+ - \epsilon
\]
for sufficiently large \( k \). The proof of these inequalities is entirely similar to the proofs of (31) and (35) and is left to the reader.

We still must show that \( v \) is a right asymptotic spanning disc for some nonconstant periodic orbit \( y \) in \( \mathcal{P}(H_r) \). Since \( v \) satisfies (31) and has energy less than \( r(M, \omega) \), there is a sequence \( s_j^- \to -\infty \) such that \( \lim_{j \to \infty} v(s_j^-, t) = y(t) \in \mathcal{P}(H_r) \). We now prove that there is a sequence \( s_j^+ \to +\infty \) such that \( v(s_j^+, t) \) converges to a constant map.

**Lemma 4.10.** A fixed point of the \( \mathbb{R} \)-action which belongs to \( \mathcal{V}_c \subset \mathcal{C} \) is a constant periodic orbit of \( H_r \) contained in \( \mathcal{C} \).

**Proof.** A fixed point of the \( \mathbb{R} \)-action in \( \mathcal{V}_c \) is an element of \( \mathcal{P}(H_r) \) which gets mapped by the evaluation to \( U_{(2r/3-\delta, +\infty)} \). Any periodic orbit of \( H_r \) which enters \( U_{(2r/3-\delta, +\infty)} \) must be constant.

Let \( \tau_j \to +\infty \) be a sequence of positive numbers. Passing to a subsequence, if necessary, we may assume that \( \tau_j \cdot v \) converges to a map \( \hat{v} \) in \( \Sigma \). Since \( v \) is not in \( \mathcal{C} \), it is nonconstant by Lemma 4.10. Lemma 4.6 then implies that \( \hat{v} \) belongs to \( \mathcal{V}_c \).

The limit \( \hat{v} \) is also a fixed point of the \( \mathbb{R} \)-action. To prove this, we consider the function \( \tau \mapsto \overline{A}_{-\infty}^0(\tau \cdot v) \). Lemma 4.4 implies that this function is bounded from below by \(-B \). Since \( v \) is nonconstant, Lemma 4.6 implies that it is also strictly decreasing. Hence, the limit
\[
\lim_{\tau \to +\infty} \overline{A}_{-\infty}^0(\tau \cdot v) = d
\]
for some \( d \geq -B \). By continuity, we then have
\[
\overline{A}_{-\infty}^0(\tau \cdot \hat{v}) = \lim_{j \to +\infty} \overline{A}_{-\infty}^0((\tau + \tau_j) \cdot v) = \lim_{j \to +\infty} \overline{A}_{-\infty}^0(\tau_j \cdot v) = \overline{A}_{-\infty}^0(\tau \cdot v)
\]
for every \( \tau \). Thus, \( \hat{v} \) is an element of \( \mathcal{P}(H_r) \) by Lemma 4.6.

It now follows from Lemma 4.10 that \( \hat{v} \) is a constant periodic orbit of \( H_r \) corresponding to some point \( p \) in \( U_{(2r/3-\delta, +\infty)} \). Since the sequence \( \tau_j \cdot v \) converges to the constant map \( p \) in the \( C_{\text{loc}}^\infty \)-topology on \( \mathcal{C} \), the maps \( \tau_j \cdot v(0,t) = v(\tau_j, t) \) converge to \( p \) in \( C^\infty(S^1, M) \). Setting \( s_j^+ = \tau_j \) we have verified that \( v \) is a right asymptotic spanning disc for \( y \).

Finally, as in §4.4.1, the fact that \( y \) is nonconstant follows easily from (5) and the fact that \( v \) is nonconstant.

**References**

[Al] P. Albers, A note on local Floer homology, Preprint 2006; [math.SG/0606600]

[Bo1] P. Bolle, Une condition de contact pour les sous-variétés coisotropes d’une variété symplectique, C. R. Acad. Sci. Paris, Série I, 322 (1996), 83-86.
P. Bolle, A contact condition for p-dimensional submanifolds of a symplectic manifold \((2 \leq p \leq n)\), *Math. Z.*, 227 (1998), 211–230.

Y. Chekanov, Lagrangian intersections, symplectic energy, and areas of holomorphic curves, *Duke Math. J.*, 95 (1998), 213–226.

D. Dragnev, Symplectic rigidity, symplectic fixed points and global perturbations of Hamiltonian systems, Preprint 2005, [math.SG/0512109](http://arxiv.org/abs/math.SG/0512109).

I. Ekeland, H. Hofer, Two symplectic fixed-point theorems with applications to Hamiltonian dynamics, *J. Math. Pures Appl.* 68 (1989), no.4, 467489 (1990).

A. Floer, H. Hofer, D. Salamon, Transversality in elliptic Morse theory for the symplectic action, *Duke Math. J.*, 80 (1995), 251–292.

V.L. Ginzburg, Coisotropic intersections, to appear in *Duke Math. J.*

H. Hofer, On the topological properties of symplectic maps, *Proc. Royal Soc. Edinburgh*, 115 (1990), 25–38.

H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, 1994.

E. Kerman, Hofer’s geometry and Floer theory under the quantum limit, Preprint 2007, [math.SG/0703064](http://arxiv.org/abs/math.SG/0703064).

J. Moser, A fixed point theorem in symplectic geometry, *Acta Math.*, 141 (1978), 1734.

S. Piunikhin, D. Salamon, M. Schwarz, Symplectic Floer–Donaldson theory and quantum cohomology, in *Contact and symplectic geometry (Cambridge, 1994)*, 171–201, Publ. Newton Inst., 8, Cambridge Univ. Press, Cambridge, 1996.

L. Polterovich, Symplectic displacement energy for Lagrangian submanifolds, *Ergod. Th. & Dynam. Sys.*, 13, (1993), pp. 357–367.

D.A. Salamon, Lectures on Floer homology, in *Symplectic Geometry and Topology*, Eds: Y. Eliashberg and L. Traynor, IAS/Park City Mathematics series, 7, 1999, pp. 143–230.

F. Schlenk, Applications of Hofer’s geometry to Hamiltonian dynamics, *Comment. Math. Helv.*, 81 (2006) 105–121.

C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.*, 100 (1990), 301–320.

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