On P-Essential Submodules

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Abstract

Let $R$ be a commutative ring with unity and let $A$ be an $R$-module. We call an $R$-submodule $H$ of $A$ as P-essential if $H \cap L \neq 0$ for each nonzero prime submodule $P$ of $A$ and $0 \neq L \subseteq P$. Also, we call an $R$-module $A$ as P-uniform if every non-zero submodule $H$ of $A$ is P-essential. We give some properties of P-essential and introduce many properties to P-uniform $R$-module. Also, we give conditions under which a submodule $H$ of a multiplication $R$-module $A$ becomes P-essential. Moreover, various properties of P-essential submodules are considered.

Keywords: Essential submodules, Uniform modules, Fully prime modules, multiplications modules.

1- Introduction

Let $R$ be a commutative ring with unity and let $A$ be a unitary $R$-module. A non-zero submodule $H$ of $A$ is called essential if $H \cap L \neq 0$ for each non-zero submodule $L$ of $A$ [1]. $A$ is called uniform if every non-zero submodule $H$ of $A$ is essential [1]. In (2019), Ahmad and Ibrahiem studied a new concept, which is named $H$-essential submodules [2]. Ali and Nada [3] introduced the concept of semi-essential submodules as a generalization of the class of essential submodules. They stated that a nonzero submodule $H$ of $A$ is called semi-essential, if $H \cap P \neq 0$ for each nonzero prime submodule $P$ of $A$. In section two, we introduce a P-essential submodule concept as a generalization of the essential submodule concept. We call an $R$-submodule $H$ of $A$ as P-essential if $H \cap L \neq 0$ for each nonzero prime submodule $P$ of $A$ and $0 \neq L \subseteq P$. Our main concerns in this section are to give characterizations for P-

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essential submodules and generalize some known properties of essential submodules to P-essential submodules. In section three, we give conditions under which a submodule \( H \) of a faithful multiplication \( R \)-module \( A \) becomes P-essential. In section four, we present the P-uniform module concept as a generalization of the uniform concept. We also generalize a characterization and some properties of uniform modules to P-uniform modules.

2- P-Essential Submodules

Recall that a non-zero submodule \( H \) of an \( R \)-module \( A \) is called essential if \( H \cap L \neq 0 \) for each submodule \( L \) of \( A \) [1].

Definition(2-1)

Let \( A \) be an \( R \)-module and \( P \) be a non-zero prime submodule of \( A \). A submodule \( H \) of \( A \) is said to be P-essential, written as \( \leq_{pe} A \), if for every proper submodule \( L \) of \( P \), then \( H \cap L = 0 \), which implies that \( L = 0 \).

Or, a non-zero submodule \( H \) of \( A \) is called P-essential, if \( H \cap L \neq 0 \) \( \forall \neq L \subseteq P \).

Remarks and Examples(2-2)

1- Every essential submodule is P-essential submodule, but the converse is not true in general.

For example, consider \( A = \mathbb{Z}_{24} \) as \( \mathbb{Z} \)-module, \( \mathbb{Z}_{24} \) is essential submodule of \( A \), but \( \mathbb{Z}_{24} \), since \( \mathbb{Z}_{24} \) are proper submodules of \( \mathbb{Z}_{24} \), but \( \mathbb{Z}_{24} \) are P-essential in \( \mathbb{Z}_{24} \), since \( \mathbb{Z}_{24} \) are not essential in \( \mathbb{Z}_{24} \), since \( \mathbb{Z}_{24} \) are P-essential in \( \mathbb{Z}_{24} \), but \( \mathbb{Z}_{24} \) are not P-essential in \( \mathbb{Z}_{24} \).

2- A submodule of a P-essential submodule needs not to be P-essential.

For example, let \( A = \mathbb{Z}_{24} \) be a \( Z \)-module, \( \mathbb{Z}_{24} \) is P-essential submodule of \( \mathbb{Z}_{24} \), but \( \mathbb{Z}_{24} \) and \( \mathbb{Z}_{24} \) are not P-essential in \( \mathbb{Z}_{24} \).

3- If \( H_1 \) and \( H_2 \) are P-essential submodules of \( A \), then \( H_1 \cap H_2 \) needs not be to P-essential of \( A \).

For example, let \( A = \mathbb{Z}_{24} \) and \( \mathbb{Z}_{24} \) be \( Z \)-module, \( \mathbb{Z}_{24} \) is P-essential submodule of \( \mathbb{Z}_{24} \), but \( \mathbb{Z}_{24} \) is not P-essential of \( \mathbb{Z}_{24} \).

4- The sum of two P-essential submodules of an \( R \)-module \( A \) is also P-essential submodule.

Proof: Let \( A \) be an \( R \)-module and \( L \) and \( K \) be two P-essential submodules of \( A \). Note that \( L \subseteq K \), since \( L \leq_{pe} A \), implies that \( L \cap K \leq_{pe} A \).

6- A semi-essential submodule needs not to be P-essential submodule, as we see in the following example:

Consider \( \mathbb{Z}_{12} \) as \( \mathbb{Z} \)-module, \( \mathbb{Z}_{12} \) is semi-essential [3], but it is not P-essential where \( P = \mathbb{Z}_{12} \) and \( \mathbb{Z}_{12} \cap \mathbb{Z}_{12} = \mathbb{Z}_{12} \), but \( \mathbb{Z}_{12} \cap \mathbb{Z}_{12} = \mathbb{Z}_{12} \).

Proposition (2-3)

Let \( A \) be an \( R \)-module, \( P \) be a prime submodule of \( A \), and \( K \) be any submodule of \( A \). If \( \leq_{pe} A \), then \( K \leq_{pe} A \) if and only if \( K \leq_{e} A \).

Proof: Suppose that \( K \leq_{pe} A \). Let \( P \) be a prime submodule of \( A \) and let \( L \leq P \) such that \( K \cap L = \mathbb{Z} \), implies that \( K \cap (P \cap L) = \mathbb{Z} \). Since \( P \cap L \leq P \) and \( K \leq_{pe} A \), then \( P \cap L = \mathbb{Z} \).

By hypothesis, \( P \leq_{e} A \), thus \( L = \mathbb{Z} \) which implies that \( K \leq_{e} A \). The converse is obvious.

Proposition (2-4)

A non-zero submodule \( K \) of \( A \) is P-essential if and only if for each non-zero submodule \( L \) of a submodule \( P \), \( \exists x \in L \) and \( r \in R \) such that \( 0 \neq rx \in K \), where \( P \) is a prime submodule of \( A \). The proof is easy and hence is omitted.

Proposition(2-5)
Let $A$ be an $R$-module and let $H_1, H_2$ be submodules of $A$ such that $H_1 \leq H_2$. If $H_1$ is $P$-essential submodule of $A$, then $H_2$ is a $P$-essential submodule of $A$.

**Proof**

Let $P$ be a prime submodule of $A$, $0 \neq L \leq P$. By using proposition (2-4), $x \in L, r \in R$. Since $H_1 \leq_{pe} A$, then $0 \neq r x \in H_1 \leq H_2$, implies that $H_2 \leq_{pe} A$.

The converse of prop.(2-5) is not true in general; for example:

Consider $\mathbb{Z}_{24}$ as a $\mathbb{Z}$-module and $\bar{4}$ is a submodule of $\bar{6}$. By remarks and example (2-2)(3), $\bar{4} \leq_{pe} \mathbb{Z}_{24}$, but $\bar{4} \notin_{pe} \bar{6}$, since $\bar{6} \cap \bar{6} = \{0\}$ and $\bar{4} \neq \{0\}$.

**Corollary(2-6)**

Let $H_1$ and $H_2$ be submodules of $A$. If $H_1 \cap H_2$ is $P$-essential submodule of $A$, then $H_1$ and $H_2$ are $P$-essential.

**Proof**

By using proposition (2-5), since $H_1 \cap H_2 \leq H_1$ and $H_1 \cap H_2 \leq_{pe} A$, so $H_1 \leq_{pe} A$. In the same way, $H_2 \leq_{pe} A$.

The converse of the previous corollary is not true in general, as shown in remarks and examples(2-2)(5).

**Proposition(2-7)**

Let $A$ be an $R$-module and let $H_1$ and $H_2$ be submodules of $A$. If $H_1$ is an essential submodule of $A$ and $H_2$ is a $P$-essential submodule of $A$, then $H_1 \cap H_2$ is also $P$-essential submodule of $A$.

**Proof**

Let $P$ be prime submodule of $A$ and let $0 \neq L$ submodule of $P$. Since $H_2$ is $P$-essential submodule of $A$, thus $H_2 \cap L \neq \{0\}$. And since $H_1$ is an essential submodule of $A$, then $H_1 \cap (H_2 \cap L) \neq \{0\}$, so $(H_1 \cap H_2) \cap L \neq \{0\}$. This implies that $H_1 \cap H_2$ is $P$-essential submodule of $A$.

**Proposition(2-8)**

Let $A$ and $B$ be $R$-modules and let $f: A \to B$ be an epimorphism. If $K$ is a $P$-essential submodule of $A$, then $f^{-1}(K)$ is a $f^{-1}(P)$-essential of $A$.

**Proof**

We know that if $P$ is a prime submodule of $B$ then $f^{-1}(P)$ is a prime submodule of $A$ [4]. Let $0 \neq L \leq f^{-1}(P)$ and $f^{-1}(K) \cap L \neq \{0\}$. To prove that $L = 0$, then $K \cap f(L) \neq \{0\}$.

Since $K$ is $P$-essential in $B$ and $f(L) \leq P$, then $f(L) = 0$, implies $L \leq f^{-1}(0) = \ker f \leq f^{-1}(K)$. But $f^{-1}(K) \cap L \neq \{0\}$, that is $L \neq 0$. Thus $f^{-1}(K)$ is a $f^{-1}(P)$-essential submodule of $A$.

**Remark(2-9):** Let $f: A \to \hat{A}$ be an isomorphism. If $H \leq_{pe} A$, then $f(H) \leq_{pe} \hat{A}$.

**Proof:** Let $P$ be a prime submodule of $A$ and let $L$ be a non-zero submodule of $P$. Since $f$ is an epimorphism, then $f^{-1}(L)$ is a submodule of $f^{-1}(P)$ which is prime submodule of $A$ by [4]. But $\leq_{pe} A$, then $H \cap f^{-1}(L) \neq \{0\}$. On the other hand, $f$ is a monomorphism, thus $f(H) \cap L \neq \{0\}$. This completes the proof.

**Proposition(2-10)**

If $K$ is a submodule of an $R$-module $A$ and $P_1, P_2$ are prime submodules of $A$, such that $0 \leq P_1 \leq P_2$. If $K \leq_{pe} P_1$, then $K \leq_{pe} P_2$.

**Proof:** Let $L_2 \leq P_2$ such that $K \cap L_2 = \{0\}$. To prove that $L_2 = 0$. $\exists i : P_1 \to P_2$, since $L_2 \leq P_2$, hence $i^{-1}(L_2) \leq P_1$. $i^{-1}(K \cap L_2) = i^{-1}(L_2) = \{0\}$ , implies that $\cap i^{-1}(L_2) = \{0\}$.

Since $\leq_{pe} A$, hence $i^{-1}(L_2) = L_2 = \{0\}$.

**Proposition(2-11)**

Let $C, K, P$ be submodules of an $R$-module $A$ and $P$ is prime submodule of $A, K \leq C$. $K \leq_{pe} A$ if and only if $K \leq_{(P \cap C)e} A$ and $C \leq_{pe} A$. 

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Proof: (⇒) Since $P$ is prime in $A$, $C \leq A$, then $(P \cap C)$ is prime in $C$ [4]. Let $L \leq (P \cap C)$ with $\cap L = \langle 0 \rangle$. To prove that $L = \langle 0 \rangle$, since $L \leq P$, $K \leq pe A$, hence $L = \langle 0 \rangle$. Let $T \leq P$ with $\cap C = \langle 0 \rangle$, implies that $T \cap K = \langle 0 \rangle$ (the hypothesis has been modified in the proposition). Since $\leq pe A$, then $T = 0$.

($\Leftarrow$) Let $L \leq P$ such that $L \cap K = \langle 0 \rangle$, then $(L \cap K) \cap C = \langle 0 \rangle$, implies that $(L \cap C) \cap K = \langle 0 \rangle$, $L \cap C \leq P \cap C$ and $K \leq (P \cap C) pe A$, hence $L \cap C = \langle 0 \rangle$. Since $\leq pe A$, then $L = \langle 0 \rangle$, thus $K \leq pe A$.

In the following proposition, we give the transitive property for non-zero P-essential submodules.

Proposition (2-12)

Let $A$, $B$, $C$ be $R$ –modules such that $A \leq B \leq C$. If $A \leq pe B$ and $B \leq pe C$, then $A \leq pe C$.

Proof: Let $P$ be a prime submodule of $C$ and let $L$ be a submodule of $P$ such that $A \cap L = 0$. Note that $0 = A \cap L = (A \cap L) \cap B = A \cap (L \cap B)$. If $B \leq L$ then $0 = A \cap (L \cap B) = A \cap B$, hence $A \cap B = 0$, but $A \leq B$, so $A \cap B = A$, which implies that $A=0$. But this is a contradiction. Thus $B \not\leq L$ and $L \cap B \leq P$. But $A \leq pe B$, therefore $L \cap B = 0$, and since $B \leq pe C$, then $L = 0$, that is $A \leq pe C$.

The converse of proposition (2-12) is not true in general, as the following example shows:

Consider $Z_{24}$ as $Z$-module, the submodule $\langle 6 \rangle$ is P-essential of $Z_{24}$, by remarks and examples (2-2). But $\langle 6 \rangle > \langle 2 \rangle$ is not P-essential submodule of $\langle 2 \rangle$ where $\langle 2 \rangle \leq Z_{24}$.

Recall that an R-module $A$ is fully prime, if every proper submodule of $A$ is a prime submodule [2].

Proposition (2-13)

Let $A = A_1 \oplus A_2$ be a fully prime $R$-module where $A_1$ and $A_2$ are submodules of $A$, and let $0 \neq K_1 \leq A_1$ and $0 \neq K_2 \leq A_2$. Then $K_1 \oplus K_2$ is P-essential of $A_1 \oplus A_2$ if and only if $K_1$ is a P-essential submodule of $A_1$ and $K_2$ is a P-essential submodule of $A_2$.

Proof

($\Rightarrow$) Since $A$ is a fully prime module, then by [5], $K_1 \oplus K_2$ is an essential submodule of $A_1 \oplus A_2$ and by [6, proposition (5-20)], $K_1$ is an essential submodule $A_1$ and $K_2$ is an essential submodule of $A_2$. But since every essential submodule is a P-essential, so we are done.

($\Leftarrow$) It follows similarly.

Proposition (2-14)

Let $A$ be an R-module and let $H_1$ and $H_2$ be P-essential submodules of $A$ such that $H_1 \cap H_2 \neq 0$, then $H_1 \cap H_2$ is P-essential submodule of $A$.

Proof

Let $P$ be a prime submodule of $A$ and let $L \leq P$ such that $(H_1 \cap H_2) \cap L = 0$. This implies that $H_2 \cap (H_1 \cap L) = 0$. If $H_1 \leq L$, then we have a contradiction with the assumption, thus $H_1 \not\leq L$. This implies that $H_1 \cap L$ is a submodule of $A$ [5]. Since $H_2$ is P-essential submodule of $A$ and, by our assumption, $H_1 \cap L$ is a submodule of $A$, then $H_1 \cap L = 0$. But $H_1$ is P-essential submodule of $A$, therefore $L = 0$, hence $H_1 \cap H_2$ is P-essential submodule of $A$.

3- P-Essential Submodules in Multiplication Modules

An R-module $A$ is called multiplication if every submodule $H$ of $A$ is of the form $IA$ for some ideal $I$ of $R$ [7] and an R-module $A$ is called faithfull if $ann(A) = 0$. In this section, we give a condition under which a submodule $H$ of $A$ is a faithful multiplication R-module that becomes P-essential.

Theorem (3-1)

Let $A$ be a faithful multiplication R-module and $H$ be a submodule of $A$. Then $H$ is P-essential of $A$ if and only if $I$ is P-essential of $R$. 

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Assume that $H$ is $P$-essential submodule of $A$, let $P$ be a prime ideal of $R$ and $L \leq P$ such that $I \cap L = 0$. Since $A$ is a faithful multiplication $R$-module, then $(I \cap L)A = IA \cap LA = 0$. Now, $PA$ is a prime submodule of $A$, $LA \leq PA$ and $(IA = H$ is $P$-essential submodule of $A$), implies that $LA = 0$. Since $A$ is finitely generated faithful multiplication $R$-module, then $L = 0$. Therefore, $I$ is a $P$-essential submodule. Conversely, let $P$ be a prime submodule of $A$ and $L$ be a submodule of $P$ such that $H \cap L = 0$. Since $A$ is multiplication, then there exists an ideal $B$ of $R$ such that $L = BA$ [8]. Hence $H \cap L = IA \cap BA = (I \cap BA)A = 0$. But $A$ is faithful, so $I \cap B = 0$. Since $I$ is a $P$-essential ideal of $R$, then $B = 0$, therefore $L = BA = 0$, thus $H$ is a $P$-essential submodule of $A$.

**Theorem (3-2)**

Let $A$ be a faithful multiplication $R$-module. Then $H$ is a $P$-essential submodule of $A$ if and only if $[H: < x >]$ is a $P$-essential ideal of $R$ for each $x \in A$.

**Proof**

Assume that $H$ is $P$-essential submodule. Since $A$ is faithful multiplication $R$-module, then $[H: A]$ is a $P$-essential of $R$, by Theo.(3-1). But $[H: A] \subseteq [H: < x >]$ for each $x \in A$, so $H = [H: A]A \subseteq [H: < x >]A$, [7]. Hence $[H: < x >]A$ is $P$-essential by Proposition (2-5), hence $[H: < x >]$ is a $P$-essential ideal of $R$ by Theorem (3-1).

**Proposition (3-3)**

Let $A$ be a finitely generated, faithful and multiplication $R$-module. If $I \leq_{pe} J$, then $IA \leq_{pe} JA$ for every ideals $I$ and $J$ of $R$.

**Proof**

Let $P$ be a prime submodule of $JA$ such that $P = KA$ for some prime ideal $K$ of $R$ and $K \subseteq J$, [8] and let $L$ be a submodule of $P$ such that $LA \cap L = 0$. Since $A$ is a multiplication module, then $L = EA$ for some ideal $E$ of $R$. So $E \cap EA = 0$, implies that $(I \cap E)A = 0$. Since $A$ is a faithfull module, then $E = 0$. Since $EA \leq KA$ and $A$ is finitely generated, faithful and multiplication, so by [8], $E \leq K$. Since $I$ is a $P$-essential ideal of $J$, then $E = 0$ and hence $L = 0$. That is, $IA \leq_{pe} JA$.

**Proposition (3-4)**

Let $A$ be a non-zero multiplication $R$-module with only one maximal submodule $H$. If $H \neq 0$, then $H$ is an essential (hence $P$-essential) submodule of $A$.

**Proof**

Let $L$ be a submodule of $A$ with $L \cap H = 0$. If $L = A$, then $H \cap A = 0$, hence $H = 0$, which is a contradiction. Thus $L$ is a proper submodule of $A$, and since $A$ is a non-zero multiplication module, so by [8], $L$ is contained in some maximal submodule of $A$. But $A$ has only one maximal submodule, which is $H$. Thus $L \subseteq H$, implies that $L = 0$, that is $H$ is an essential (hence $P$-essential) submodule of $A$.

Recall that a non-zero $R$-module $A$ is called fully essential if every non-zero semi-essential submodule of $A$ is an essential submodule of $A$, [5].

**Definition (3-5):** A non-zero $R$-module $A$ is called fully $P$-essential if every non-zero $P$-essential submodule of $A$ is an essential submodule of $A$. A ring $R$ is called fully $P$-essential if every non-zero $P$-essential ideal $I$ of $R$ is essential ideal of $R$.

**Examples (3-6):**

1. $Z_9$ as a $Z$-module is fully $P$-essential $Z$-module.
2. $Z_{12}$ as a $Z$-module is not fully $P$-essential, since the submodule $<3>$ of $Z_{12}$ is $P_2$-essential where $P_2 = <3>$, but not essential since $<3> \cap <4> = <0>$ but $<4> \neq <0>$.
3. Every fully essential is fully $P$-essential.

The following theorem gives the hereditary of fully $P$-essential property between $R$-module $A$ and the ring $R$. 

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Theorem (3-7)

Let $A$ be a non-zero faithful and multiplication R-module, then $A$ is a fully P-essential module if and only if $R$ is a fully P-essential ring.

Proof

($\Rightarrow$) Assume that $A$ is a fully P-essential module and let $I$ be a non-zero P-essential ideal of $R$, then $IA$ is a submodule of $A$, say $H$. This implies that $H$ is a P-essential submodule of $A$. Since $I \neq 0$ and $A$ is faithful module, then $H \neq 0$. But $A$ is a fully P-essential module, thus $H$ is an essential submodule of $A$. Since $A$ is a faithful and multiplication module, therefore $I$ is an essential ideal of $R$ [8], that is $R$ is a fully P-essential ring.

($\Leftarrow$) Suppose that $R$ is a fully P-essential ring and let $0 \neq H \subseteq_{pe} A$. Since $A$ is a multiplication module, then $H = IA$ for some P-essential ideal of $R$. By assumption, $I$ is an essential ideal of $R$. But $A$ is faithful and multiplication module, then $H$ is an essential submodule of $A$ [8]. Thus $A$ is fully P-essential module.

4- P-Uniform Modules

Recall that a non-zero R-module $A$ is called uniform if every non-zero submodule of $A$ is essential [9]. Recall that a non-zero R-module $A$ is called semi-uniform if every non-zero submodule of $A$ is semi-essential [10]. In this section, we give a P-uniform module concept as a generalization of the uniform module concept. We also generalize some properties of uniform modules to P-uniform modules.

Definition (4-1)

A non-zero R-module $A$ is called P-uniform if every non-zero submodule of $A$ is P-essential. A ring $R$ is called P-uniform if $A$ is a P-uniform R-module.

Remarks (4-2)

1- Each uniform R-module is P-uniform, but the converse is not true in general. For example, $Z_{15}$ as a $Z$-module is P-uniform but not uniform since $<3> \cap <5> = <0>$, while $<5> \neq <0>$; see remarks and examples (2,2), (2).

2- Each simple R-module $A$ is P-uniform. But the converse is not true in general. For example, $Z_9$ is a P-uniform $Z$-module where $= <3>$, but not simple $Z$-module.

3- $Z_{12}$ as a $Z$-module is not P-uniform, where $P = <2>$ is prime submodule of $Z_{12}$, $<3> \cap <4> = <0>$ and $<4> \subseteq_{pe} <2>$.

4- We can note that a semi-uniform R-module needs not to be P-uniform, as shown in the following example:

The $Z$-module $Z_{36}$ is semi-uniform [3], but not $P_1$-uniform and not $P_2$-uniform, where $P_1 = <2>$, $P_2 = <3>$, since $<18> \cap <12> = <0>$, but $<12> \neq <0>$, as in the following table:

| $H \subseteq A$ | $e^{ss}$ | $p_2 - e^{ss}$ | $p_2 - e^{ss}$ | Semi-e$^{ss}$ |
|-----------------|---------|----------------|----------------|-------------|
| $Z_{26}$        | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| (2)             | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| (3)             | $\times$   | $\checkmark$ | $\times$   | $\checkmark$ |
| (4)             | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| (6)             | $\checkmark$ | $\times$   | $\checkmark$ | $\checkmark$ |
| (9)             | $\times$   | $\times$   | $\times$   | $\checkmark$ |
| (12)            | $\times$   | $\times$   | $\times$   | $\checkmark$ |
| (18)            | $\times$   | $\times$   | $\times$   | $\checkmark$ |

Proposition (4-3)

Let $A$ be an R-module, then $A$ is uniform if and only if $A$ is P-uniform and fully P-essential.

Proof:

($\Rightarrow$) It is clear.

($\Leftarrow$) Let $H$ be a non-zero submodule of $A$. Since $A$ is P-uniform module, then $H \subseteq_{pe} A$. But $A$ is fully essential module, then $H \subseteq_e A$, implies that $A$ is uniform module.
Theorem(4-4)
Let \( A \) be a faithful multiplication \( R \)-module, then \( A \) is a \( P \)-uniform \( R \)-module if and only if \( R \) is a \( P \)-uniform ring.

Proof
Suppose that \( A \) is \( P \)-uniform and let \( E \) be a non-zero ideal of \( R \). Thus \( EA \) is \( P \)-essential submodule of \( A \). By theorem (3-1), \( E \) is a \( P \)-essential ideal of \( R \). Conversely, assume that \( R \) is \( P \)-uniform and \( H \) is a submodule of \( A \). Since \( A \) is multiplication, then there exists an ideal \( B \) of \( R \) such that \( H = BA \). But \( R \) is \( P \)-uniform, so \( B \) is \( P \)-essential. Thus \( H \) is \( P \)-essential by theorem(3-1).

Proposition(4-5)
Let \( A_1 \) and \( A_2 \) be two \( R \)-modules and let \( f:A_1 \rightarrow A_2 \) be an epimorphism. Then:
1- If \( A_1 \) is \( P \)-uniform \( R \)-module, then \( A_2 \) is also \( P \)-uniform \( R \)-module.
2- If \( A_2 \) is \( P \)-uniform \( R \)-module for each prime submodule \( P \) of \( A_1 \), then \( A_1 \) is \( f^{-1}(P) \)-uniform \( R \)-module.

Proof
1-Let \( H_2 \) be a non-zero submodule of \( A_2 \), then \( f^{-1}(H_2) \) is a non-zero submodule of \( A_1 \). Since \( A_1 \) is \( P \)-uniform \( R \)-module, thus \( f^{-1}(H_2) \) is a \( P \)-essential submodule of \( A_1 \). By remark(2-9), we get \( f(f^{-1}(H_2)) = H_2 \) is a \( P \)-essential submodule of \( A_2 \). Therefore, \( A_2 \) is \( P \)-uniform \( R \)-module.
2- Let \( H_1 \) be a non-zero submodule of \( A_1 \), then \( f(H_1) \) is a non-zero submodule of \( A_2 \). Since \( A_2 \) is \( P \)-uniform \( R \)-module, then \( f(H_1) \) is a \( P \)-essential submodule of \( A_2 \). By proposition(2-8), we get \( f^{-1}(f(H_1)) = H_1 \) is a \( f^{-1}(P) \)-essential submodule of \( A_1 \). Therefore, \( A_1 \) is \( f^{-1}(P) \)-uniform \( R \)-module.

Proposition(4-6)
Let \( A = A_1 \oplus A_2 \) be \( R \)-module, where \( A_1 \) and \( A_2 \) are \( R \)-modules. If \( A \) is \( P \)-uniform, then \( A_1 \) and \( A_2 \) are \( P \)-uniform modules.

Proof
Let \( H_1 \) be non-zero submodule of \( A_1 \), so \( H_1 \leq A \). But \( A \) is \( P \)-uniform, then \( H_1 \) is a \( P \)-essential submodule of \( A \). Thus, \( H_1 \) is a \( P \)-essential submodule of \( A_1 \). Therefore, \( A_1 \) is \( P \)-uniform \( R \)-module. In a similar way, we can proof that \( A_2 \) is a \( P \)-uniform \( R \)-module.

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