ON VORTEX ALIGNMENT AND BOUNDEDNESS OF $L^q$ NORM OF VORTICITY

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Abstract. We show that the spatial $L^q$ ($q > 5/3$) norm of the vorticity of an incompressible viscous fluid in $\mathbb{R}^3$ or $\mathbb{T}^3$ remains bounded uniformly in time, provided that the direction of vorticity is Hölder continuous in the space variable, and that the space-time $L^q$ norm of the vorticity is finite. The Hölder index depends only on $q$. This serves as a variant of the classical result by P. Constantin and Ch. Fefferman (Direction of vorticity and the problem of global regularity for the Navier–Stokes equations, *Indiana Univ. J. Math.*, 42 (1993), 775–789).

1. Introduction

In this paper we consider the Cauchy problem of incompressible Navier–Stokes equations:

$$
\begin{align*}
\partial_t u + \text{div} \, u \otimes u - \nu \Delta u + \nabla p &= 0 \quad \text{in} \, [0, T] \times \Omega, \\
\text{div} \, u &= 0 \quad \text{in} \, [0, T] \times \Omega, \\
u u|_{t=0} &= u_0 \quad \text{on} \, \{0\} \times \Omega,
\end{align*}
$$

(1.1)

where $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. The constant $\nu > 0$ is the viscosity, $u : \Omega \to \mathbb{R}^3$ the velocity, and $p : \Omega \to \mathbb{R}$ the pressure of the fluid. The existence, uniqueness and regularity of Eqs. (1.1)–(1.3) has been a central research topic of nonlinear PDEs; see Fefferman [12], Constantin–Foias [11], Seregin [19] and many other references.

The vorticity $\omega := \nabla \times u$ is an important quantity for the fluid motion. Its time evolution is determined by the vorticity equation, which can be obtained by taking the curl of Eq. (1.1):

$$
\begin{align*}
\partial_t \omega + (u \cdot \nabla) \omega - \nu \Delta \omega &= S \cdot \omega,
\end{align*}
$$

(1.4)

where $S$ is the $3 \times 3$ matrix

$$
S := \frac{\nabla u + \nabla^\top u}{2}.
$$

(1.5)

The alignment of the vorticity is closely related to the regularity of the weak solutions to the Navier–Stokes equations. A celebrated result by Constantin–Fefferman ([9]) shows that, if the vorticity direction does not change too rapidly in the regions with high vorticity magnitude, then a weak solution is automatically strong. More precisely, denote by

$$
\varphi(t, x, y) := \angle (\omega(t, x), \omega(t, y)).
$$

(1.6)

If there exist $\Lambda, \rho > 0$ such that

$$
|\sin \varphi(t, x, y)| \leq \frac{|x - y|}{\rho}
$$

(1.7)

whenever $|\omega(t, x)|, |\omega(t, y)| \geq \Lambda$, then a weak solution $u$ on $[0, T]$ must be a classical solution on $[0, T]$. Here, weak solutions are defined in the Leray–Hopf sense: $u \in L^\infty(0, T; L^2(\Omega)) \cap$
\[ L^2(0, T; H^1(\Omega)) \] with the energy inequality
\[ \frac{1}{2} \int |u(t, x)|^2 \, dx + \nu \int_0^T \int |\nabla u(s, x)|^2 \, ds \, dx \leq \frac{1}{2} \int |u_0(t)|^2 \, dx. \] (1.8)
Throughout the paper, \( \int \) without subscripts denotes the integration over \( \Omega \), and \( \| \cdot \|_{L^q} \equiv \| \cdot \|_{L^q(\Omega)} \).

The above result (Constantin–Fefferman [9]) is established by showing
\[ \omega \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \] (1.9)
which together with Eq. (1.2) implies that \( u \) is classical. Using more refined estimates, Beirão da Veiga and Berselli improved the Lipschitz condition (1.7) in [3] to a Hölder condition:
\[ |\sin \varphi(t, x, y)| \leq \frac{|x - y|^\beta}{\rho}, \qquad \text{where } \beta \in \left[ \frac{1}{2}, 1 \right]. \] (1.10)
The Hölder exponent \( \beta = 1/2 \) is the best up to date. There is an extensive literature on the geometric regularity conditions à la Constantin–Fefferman; see Beirão da Veiga–Berselli [1, 2], Beirão da Veiga [3, 4, 5], Berselli [6], Chae [7], Chae–Kang–Li [8], Giga–Miura [13], Grujić [14], Vasseur [20] and Zhou [21], as well as the references cited therein. Similar conditions for the Euler equations have also been studied; cf. Constantin–Fefferman–Majda [10].

This paper serves as a variant of the above results in [3, 1]. In comparison with Eq. (1.9), concerning the growth of the \( L^2 \) norm of vorticity \( \omega \), we shall study the growth of the \( L^q \) norm of \( \omega \) under assumptions of the form Eq. (1.10), in which the Hölder exponent depends on \( q \). More precisely, the main result of the paper is as follows:

**Theorem 1.1.** Let \( u : \Omega \times [0, T] \to \mathbb{R}^3 \) be a weak solution to Eqs. (1.11)–(1.13), \( \Omega = \mathbb{R}^3 \) or \( \mathbb{T}^3 \). Assume that, for \( q > 5/3 \), there exist \( \Lambda, \rho > 0 \) such that
\[ |\sin \varphi(t, x, y)| \leq \frac{|x - y|^\beta}{\rho} \quad \text{where } \beta \in \left[ \max \left\{ 0, \frac{5}{q} - 2 \right\}, 1 \right] \] (1.11)
whenever \( |\omega(t, x)|, |\omega(t, y)| \geq \Lambda \); the angle \( \varphi \) is as in Eq. (1.6). In addition, suppose that \( \omega \in L^q(\Omega \times [0, T]) \). Then
\[ \omega \in L^\infty(0, T; L^q(\Omega)) \quad \text{and} \quad |\omega|^{q/2} \in L^2(0, T; H^1(\Omega)). \] (1.12)

In particular, for \( q = 2 \), \( \beta = 1 \) Theorem 1.1 recovers the result by Constantin–Fefferman [9]; and for \( q = 2 \), \( \beta = 1/2 \) the result by Beirão da Veiga–Berselli [1]. Indeed, when \( q = 2 \) the assumption \( \omega \in L^q(\Omega \times [0, T]) \) is automatically verified by the energy inequality (1.3).

Theorem 1.1 provides a new characterisation for the control of vorticity under suitable alignment of the vortex structures in 3D incompressible fluids. Roughly speaking, it suggests a self-improvement property from the average-in-time bound for the (spatial) \( L^q \) norm of \( \omega \) to the uniform-in-time bound, provided that the vorticity does not change its directions too sharply wherever its magnitude is large.

Moreover, we remark that regularity conditions for vorticity have also been established under space-time integrability conditions on the vorticity magnitude. For example, Grujić–Ruzmaikina [15] proved that for \( \beta \in [1/q, 1] \) and \( \int_0^T \left( \int |\omega(t, x)|^q \, dx \right)^{1/(q-1)} \, dt < \infty \), the \( L^q \) norm of \( \omega \) remains bounded as \( t \to T^- \). The special case \( q = 2 \) also coincides with the result by Beirão da Veiga–Berselli [1].
2. Preliminary Identities and Estimates

In this section we summarise several identities and inequalities that shall be used in the subsequent development.

First of all, we recall the singular integral representation of the rate-of-strain tensor $S$ in terms of $\omega$, which is crucial to the arguments in Constantin–Fefferman [9]. Denoting by $\hat{a} := a/|a|$ for three-vectors $a \in \mathbb{R}^3$, there holds (Eq. (4) in [9]):

$$S(t, x) = \frac{3}{8\pi} \text{p.v.} \int \left\{ \frac{x - y \otimes (x - y \times \omega(t, x)) + (x - y \times \omega(t, x)) \otimes (x - y)}{|x - y|^3} \right\} dy. \quad (2.1)$$

The symbol p.v. denotes the principal value of the integral. Thus, the normalised vortex stretching term $S : (\hat{\omega} \otimes \hat{\omega})$ can be expressed as follows:

$$S : (\hat{\omega} \otimes \hat{\omega})(t, x) = \frac{3}{4\pi} \text{p.v.} \int \left\{ \frac{D(\hat{x} - y, \hat{\omega}(t, x), \hat{\omega}(t, x - y))|\omega(t, x)|}{|x - y|^3} \right\} dy, \quad (2.2)$$

where

$$D(e_1, e_2, e_3) := (e_1 \cdot e_3) \det(e_1, e_2, e_3), \quad (2.3)$$

and $e_i$ are three-vectors (column vectors) for $i = 1, 2, 3$. As shown on pp.778–780 in [9], the bound for the angle $\varphi$ can be translated to a bound for the $D$ term:

**Lemma 2.1.** Under the assumptions of Theorem 1.1 we have

$$\left| D(\hat{x} - y, \hat{\omega}(t, x), \hat{\omega}(t, x - y)) \right| \leq \frac{|x - y|^\beta}{\rho}. \quad (2.4)$$

Next, the time-evolution of the $L^q$ norm of $\omega$ (for any $q \geq 1$) has been derived by Qian in [18]; see the proof of Lemma 2 therein:

**Lemma 2.2.** Let $u$ be a weak solution to Eqs. (1.1)–(1.3). Then, for $q \geq 1$, there holds

$$\frac{d}{dt} \left( \int |\omega(t, x)|^q \, dx \right) + \frac{4(q - 1)}{q} \left( \int |\nabla (|\omega(t, x)|^{q/2})|^2 \, dx \right) \leq q \int |\omega(t, x)|^{q-2} S(t, x) : (\omega(t, x) \otimes \omega(t, x)) \, dx. \quad (2.5)$$

Finally, in Sect. 3 we shall make crucial use of the Hardy–Littlewood–Sobolev interpolation inequality (cf. p106, Lieb–Loss [16]), with $n = 3$, $\lambda = 2 + \delta$ and $f, h$ supported in $\Omega$:

**Lemma 2.3.** Let $1 < p, r < \infty$, $0 < \lambda < n$ satisfy $1/p + \lambda/n + 1/r = 2$. Let $f \in L^p(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. Then there exists $C = C(n, \lambda, p)$ such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x - y|^\lambda} \, dx \, dy \right| \leq C \|f\|_{L^p(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}. \quad (2.6)$$

3. Proof of Theorem 1.1

Equipped with Lemmas 2.1, 2.2, and 2.3 above, we are ready to prove Theorem 1.1.

As in Constantin–Fefferman [9], let us decompose the vorticity into “big” and “small” parts, with respect to the (large) constant $\Lambda > 0$ in Theorem 1.1. To this end, taking $\chi \in C^\infty([0, \infty])$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $[0, 1]$ and $\chi \equiv 0$ on $[2, \infty]$, we define

$$\omega(t, x) := \omega^{(<)}(t, x) + \omega^{(>)}(t, x)$$

where
where
\[ \omega(x) = \chi \left( \frac{|\omega(t, x)|}{\Lambda} \right) \omega(t, x) + \left\{ 1 - \chi \left( \frac{|\omega(t, x)|}{\Lambda} \right) \right\} \omega(t, x). \]

We also write
\[ S^{(i)}(t, x) = \frac{3}{8\pi} p.v. \int \left\{ \frac{x - y \otimes (x - y \times \omega^{(i)}(t, x)) + (x - y \times \omega^{(i)}(t, x)) \otimes x - y}{|x - y|^3} \right\} \, dy \tag{3.2} \]
for \( i \in \{<, >\} \), namely the corresponding singular integral with input \( \omega^{(i)} \).

Now, in view of Lemma 2.2 our goal is to estimate
\[ q \int |\omega(t, x)|^{q-2} S(t, x) : (\omega(t, x) \otimes \omega(t, x)) \, dx =: q \int K(t, x) \, dx. \tag{3.3} \]

Following the notations in Constantin–Fefferman [9], there holds
\[ |K(t, x)| \leq |X(t, x)| + |Y(t, x)| + |Z(t, x)|, \tag{3.4} \]
where
\[ X(t, x) := \sum_{(i,j) \neq (>,<)} |\omega(t, x)|^{q-2} \left\{ S(t, x) : (\omega^{(i)}(t, x) \otimes \omega^{(j)}(t, x)) \right\}, \]
\[ Y(t, x) := |\omega(t, x)|^{q-2} \left\{ S^{(<)}(t, x) : (\omega^{(>)}(t, x) \otimes \omega^{(<)}(t, x)) \right\}, \]
\[ Z(t, x) := |\omega(t, x)|^{q-2} \left\{ S^{(>)}(t, x) : (\omega^{(>)}(t, x) \otimes \omega^{(>)}(t, x)) \right\}. \]

We shall estimate these three terms in order in Subsections 3.1–3.3 below.

3.1. The \( X(t, x) \) Term. To estimate \( X(t, x) \), recall that \( \omega^{(i)} \mapsto S^{(i)} \) is a Calderon–Zygmund singular integral; hence, for some \( C = C(r, \Omega) \) we have
\[ \|S^{(i)}\|_{L^r} \leq C\|\omega^{(i)}\|_{L^r} \quad \text{for each } r \in [1, \infty] \quad \text{and } i \in \{<, >\}. \tag{3.5} \]
As \( |\omega^{(<)}| \leq \Lambda \), for \( q > 1 \) we can bound by Hölder’s inequality and Eq. (3.5):
\[ \left| \int X(t, x) \, dx \right| \leq \Lambda \int |\omega(t, x)|^{q-1} |S| \, dx \]
\[ \leq \Lambda \left\| |\omega(t, \cdot)|^{q-1} \right\|_{L^{\frac{q}{q-1}}} \|S\|_{L^q} \]
\[ \leq CA \left\| |\omega(t, \cdot)|^{q-1} \right\|_{L^{\frac{q}{q-1}}} \|\omega\|_{L^q} \leq CA \|\omega(t, \cdot)\|_{L^q}^2, \tag{3.6} \]
where \( C = C(q, \Omega) \).

3.2. The \( Y(t, x) \) Term. For this purpose, let us denote by \( \|\omega\|_{L^{\infty}(0, T; L^1(\Omega))} \leq \Gamma \). Indeed, \( \Gamma \) is finite for any weak solution to Eqs. (1.1)–(1.3); see [18, 9]. Then we estimate
\[ \left| \int Y(t, x) \, dx \right| \leq \int |S^{(<)}(t, x)| |\omega(t, x)|^q \, dx \]
\[ \leq \left( \int (|\omega(t, x)|^{q/2})^4 \, dx \right)^{\frac{1}{4}} \left( \int |S^{(<)}(t, x)|^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int \left| \nabla (|\omega(t, x)|^{q/2}) \right|^2 \, dx \right)^{\frac{1}{4}} \left( \int |\omega(t, x)|^q \, dx \right)^{\frac{1}{4}} \left( \int |\omega^{(<)}(t, x)|^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq C \sqrt{\Lambda} \Gamma \left( \int \left| \nabla (|\omega(t, x)|^{q/2}) \right|^2 \, dx \right)^{\frac{1}{4}} \left( \int |\omega(t, x)|^q \, dx \right)^{\frac{1}{4}} \]
\[
\leq \frac{2(q-1)}{q} \nu \left( \int \left| \nabla \left( |\omega(t,x)|^{q/2} \right) \right|^2 \, dx \right) + C \nu^{-3}(\Lambda)^2 \left( \int |\omega(t,x)|^q \, dx \right). \tag{3.7}
\]

In the second line we use the Cauchy–Schwarz inequality; in the third line the Gagliardo–Nirenberg–Sobolev interpolation inequality (indeed, the special case known as the Ladyzhenskaya inequality) and Eq. (3.3); in the fourth line \( |\omega^{(\cdot)}| \leq \Lambda \) and \( \|\omega\|_{L^\infty(0,T;L^1(\Omega))} \leq \Gamma \), and in the final line the Young’s inequality \( ab \leq a^4/4 + 3b^{4/3}/4 \) for suitable \( a, b \geq 0 \).

### 3.3. The \( Z(t,x) \) Term

\( Z \) is the difficult term. To control it, we observe that \( \hat{\omega}^{(\cdot)} \) is the direction of vorticity on the region with large vorticity magnitude. Thanks to Eq. (2.2), Lemma 3.1 and the assumptions in Theorem 1.1, we have

\[ \left| \int Z(t,x) \, dx \right| \leq qJ(t), \tag{3.8} \]

where

\[ J(t) := \rho^{-1} \int |\omega(t,x)|^q I(t,x) \, dx, \tag{3.9} \]

\[ I(t,x) := \int |\omega(t,y)|^{\lambda} \, dy, \tag{3.10} \]

\[ \lambda = 3 - \beta := 2 + \delta. \tag{3.11} \]

The bound for \( J(t) \) is achieved by the lemma below. The parameters \( \theta, \alpha \) involved therein will be carefully chosen later.

**Lemma 3.1.** Suppose \( \|\omega(t,\cdot)\|_{L^1(\Omega)} \leq \Gamma \) for all \( t \in [0,T]; \, q > 1 \). Then

\[ J(t) \leq \frac{q-1}{q^2} \nu \left( \int \left| \nabla \left( |\omega(t,x)|^{q/2} \right) \right|^2 \, dx \right) + C \Gamma^{\frac{\theta}{\pi-\theta} - \frac{\alpha}{\pi} - \frac{\beta}{2}} \left( \int |\omega(t,x)|^q \, dx \right)^{1+\frac{1-\theta}{\pi(1-\alpha)}}. \tag{3.12} \]

The parameters \( \theta, \alpha \in [0,1] \) depend on \( q, \lambda, \) and the constant \( C \) depends on \( \lambda, \Omega, q \) and \( \alpha \).

**Proof.** The proof is divided into five steps.

1. By Hölder’s inequality, there holds

\[ J(t) \leq \left( \int |\omega(t,x)|^{pq} \right)^{\frac{1}{2}} \left( \int |I(t,x)|^{pr} \right)^{\frac{1}{2r}} = \left\| |\omega(t,\cdot)|^{\frac{q}{2}} \right\|_{L^{2p}} \left\| I(t,\cdot) \right\|_{L^{r'}}. \tag{3.13} \]

   We write \( r' = \frac{q}{p'} \) for the conjugate exponent of \( p \). For the moment we require no condition on the index \( p \) more than \( p \in [1, \infty[ \).

2. The Gagliardo–Nirenberg–Sobolev interpolation inequality (cf. Nirenberg [17]) yields

\[ \left\| |\omega(t,\cdot)|^{\frac{q}{2}} \right\|_{L^{2p}}^2 \leq C_1 \left( \int \left| \nabla \left( |\omega(t,x)|^{q/2} \right) \right|^2 \, dx \right) \left( \int |\omega(x)|^q \, dx \right)^{1-\alpha}, \tag{3.14} \]

where \( \alpha \in [0,1[ \) is chosen such that

\[ p = \frac{3}{3 - 2\alpha}. \tag{3.15} \]

We notice that \( p \in [1,3[; \, C_1 \) depends only on \( q, \Omega \).
3. For the $J$ term in Eq. (3.13), we apply the Hardy–Littlewood–Sobolev interpolation inequality (Lemma 2.3 plus an elementary duality argument) to find
\[ \|J(t, \cdot)\|_{L^{p'}} \leq C_2\|\omega(t, \cdot)\|_{L^p}, \]  
where $C_2 = C(p, \lambda, \Omega)$. The indices satisfy
\[ \frac{1}{\sigma} + \frac{1}{p} + \frac{\lambda}{3} = 2, \]
with $1 < \sigma, p < \infty$ and $0 < \lambda < 3$. For our purpose we shall specialise to $\lambda \in [2, 3]$, hence $\delta = \lambda - 2 \in [0, 1]$, as well as $\sigma \in [1, q]$.

4. Now let us put together Eqs. (3.13) (3.14) (3.16) and apply Young’s inequality $ab \leq \alpha a^{\frac{1}{\nu}} + (1 - \alpha)b^{\frac{1}{\nu}}$ for suitable $a, b \geq 0$ (recall that $\alpha \in [0, 1]$). This gives us
\[
J(t) \leq \frac{q - 1}{4q^2} \nu \left( \int \left| \nabla (|\omega(t, x)|^{q/2}) \right|^2 \, dx \right)
+ C_3 q^{-1} \nu^{-\frac{\alpha}{q}} \rho^{-\frac{1}{q}} \left( \int |\omega(t, x)|^q \, dx \right)^{\frac{1}{q-1}} \left( \int |\omega(t, x)|^p \, dx \right)^{-\frac{1}{q-1}},
\]
where the constant $C_3$ above depends on $q, p, \lambda, \Omega$ (note that $\alpha$ is determined by $p$).

5. Finally, for $1 \leq \sigma \leq q$ we have the interpolation inequality for Lebesgue spaces:
\[
\left( \int |\omega(t, x)|^\sigma \, dx \right)^{\frac{1}{\sigma(1-\alpha)}} \leq \left( \int |\omega(t, x)|^q \, dx \right)^{\frac{\sigma}{q}} \left( \int |\omega(t, x)|^p \, dx \right)^{\frac{1}{q-1}},
\]
with $\theta \in [0, 1]$ determined by
\[ \frac{1}{\sigma} = \frac{\theta}{1} + \frac{1 - \theta}{q}. \]
In view of Steps 1–5, the proof is complete.

3.4. A Condition on the Indices $\theta, \alpha$. Now, let us single out the following condition
\[ 1 - \theta \leq q(1 - \alpha) \]
on the parameters $\theta, \alpha$ in Lemma 3.1. We then have the following

**Lemma 3.2.** Assume (♣) for the parameters $\theta, \alpha$ in Lemma 3.1. Then, under the assumptions of Theorem 1.1 we have $|\omega|^q/2 \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$.

**Proof.** Collecting the estimates in Sects. 3.1–3.3 (in particular, Eqs. (3.4) (3.6) (3.7) and Lemma 3.1), we can reduce the estimate in Lemma 2.2 to the following:
\[
\frac{dQ}{dt} + \frac{q - 1}{q} \nu \left( \int \left| \nabla (|\omega(t, x)|^{q/2}) \right|^2 \, dx \right) \leq C \left\{ \Lambda + \nu^{-3} \Lambda^2 \Gamma^2 + \Gamma \nu^{-\frac{\alpha}{q}} \nu^{-\frac{\alpha}{q}} \rho^{-\frac{1}{q}} \Omega^q \right\} Q,
\]
where $C$ depends only on $q$ and $\Omega, \gamma := \frac{1 - \theta}{q(1-\alpha)} < 1$ by (♣), and
\[
Q(t) := \int |\omega(t, x)|^q \, dx.
\]
Now, let us invoke the assumption $\omega \in L^q(\Omega \times [0, T])$ in Theorem 1.1 to get
\[ \int_0^T Q^\gamma \, dt \leq T^{1-\gamma} \|\omega\|_{L^q(\Omega \times [0, T])}^q < \infty. \]
Therefore, by the ordinary differential inequality (neglecting the \( \int |\nabla (|\omega(t,x)|^{q/2})|^2 \, dx \) term that has the favourable sign),
\[
Q(t) \leq Q_0 \exp \left\{ C(\Lambda + \nu^{-3}\Lambda^2 \Gamma^2) T + CT^{1-\gamma} \|\omega\|_{L^q(\Omega \times [0,T])}^{1-\theta} \nu^{\frac{\alpha}{1-\alpha}} \rho^{-\frac{1}{\alpha}} \right\}. \tag{3.24}
\]
This proves \(|\omega|^{q/2} \in L^\infty(0,T;L^2(\Omega))\). The other half of the conclusion follows immediately from Eq. (3.21); so the proof is complete. \(\Box\)

3.5. Completion of the Proof of Theorem 1.1. Finally, we observe that Theorem 1.1 follows directly from the conclusion of Lemma 3.2. Therefore, to complete the proof, it remains to specify the indices \(\theta, \alpha\) in Lemma 3.1 that satisfy (♣). Here enters the restriction of the range of \(q\) to \(\left[\frac{5}{3}, \infty\right]\), as well as the choice of \(\theta\) and \(\alpha\), which are dependent on \(q\). In effect, the choice of \(\theta, \alpha\) amounts to the specification of the parameter \(p\) in the proof of Lemma 3.1. To this end, we shall keep track in detail of the range of parameters for all the inequalities involved in the proof of Lemma 3.1:

**Proof of Theorem 1.1.** Recall that \(p \in \left[1, 3\right]\) in the proof of Lemma 3.1 by Eq. (3.17) and \(\lambda = 2+\delta\),
\[
p = \frac{3\sigma}{(4-\delta)\sigma - 3}. \tag{3.25}
\]
Here \(\sigma \in [1,q]\) due to Step 5 in the proof of Lemma 3.1 hence \(p \in \left[\frac{3q}{(4-\delta)q-3}, \frac{3}{1-\delta}\right]\). Moreover, the right endpoint \(\frac{3}{1-\delta}\) is no less than 3. We thus have
\[
p \in \left[1, 3\right] \cap \left[\frac{3q}{(4-\delta)q-3}, \frac{3}{1-\delta}\right]. \tag{3.26}
\]
This condition is non-vacuous, since \(q > \frac{5}{3}\) implies \(\frac{3q}{(4-\delta)q-3} < 3\) for \(\delta \in [0,1]\). In the sequel we shall use it to derive more stringent conditions on \(\delta\).

Now let us express all the other constants — \(\sigma, \theta\) and \(\alpha\) — in terms of \(p\), and then match the condition (♣). Indeed, from Eq. (3.17) we get
\[
\sigma = \frac{3p}{(4-\delta)p - 3}. \tag{3.27}
\]
Together with Eq. (3.20), it leads to
\[
\theta = \frac{(4-\delta)pq - 3(p + q)}{3p(q - 1)}. \tag{3.28}
\]
Also, Eq. (3.15) can be written as
\[
\alpha = \frac{3}{2} \left(1 - \frac{1}{p}\right). \tag{3.29}
\]
Thus, substituting in Eqs. (3.27) (3.28) and (3.29), an elementary computation shows that (♣) is equivalent to \(3q(3-p) + (5-2\delta)p - 15 \geq 0\). This gives us an upper bound for \(\delta\):
\[
\delta \leq \frac{(3q - 5)(3-p)}{2p} =: \mathcal{U}(p,q). \tag{3.30}
\]
Notice that for \(q > \frac{5}{3}\) (by assumption) and \(p < 3\) (by Eq. (3.20)), our condition (3.30) allows for non-trivial \(\delta \in [0,1]\); in addition, \(p \mapsto \mathcal{U}(p,q)\) is decreasing on \([1,3]\).

To conclude the proof, let us observe
\[
\frac{3q}{(4-\delta)q-3} < (=, >) 1 \iff \delta < (=, >) 1 - \frac{3}{q}. \tag{3.31}
\]
Therefore, to ensure $\delta \geq 0$, for $q \in [5/3, 3]$ we must require $\frac{3q}{(4-\delta)q-3} > 1$; in this case, the choice of the index $p$ is restricted to $\left[\frac{3q}{(4-\delta)q-3}, 3\right]$, in view of Eq. (3.20). On the other hand, in $q > 3$, both $\delta < 1 - 3/q$ and $\delta \geq 1 - 3/q$ is allowed.

Case 1: $q \in [5/3, 3]$. As discussed above, one may choose $p$ arbitrarily in $\left[\frac{3q}{(4-\delta)q-3}, 3\right]$. To maximise $U(p, q)$ in Eq. (3.30), let us take $p = \frac{3q}{(4-\delta)q-3}$; so

$$\delta \leq (3q - 5) \left(3 - \frac{3q}{(4-\delta)q-3}\right) = \left(3 - \delta\right)q - 3 \left(3q - 5\right),$$

which is equivalent to $\delta \leq 3 - \frac{5}{q}$. Together with $\delta \in [0, 1]$, we get

$$\delta \in \left[0, \min\left\{1, 3 - \frac{5}{q}\right\}\right] \quad \text{for} \ q \in \left[\frac{5}{3}, 3\right]. \quad (3.32)$$

Case 2: $q \in ]3, \infty[$. In this case, if $\delta \leq 1 - 3/q$, namely $\frac{3q}{(4-\delta)q-3} \leq 1$ by Eq. (3.31), in light of Eq. (3.20) we can take any $p \in [1, 3]$. To maximise $U(p, q)$ we choose $p \to 1^+$; then Eq. (3.30) gives us $\delta < 3q - 5$, which is automatically true for $\delta \in [0, 1]$. On the other hand, if $\delta > 1 - 3/q$, i.e., $\frac{3q}{(4-\delta)q-3} > 1$, then by the computation in Case 1 above, we have $\delta \leq 3 - 5/q$. In summary,

$$\delta \in \left[0, 1 - \frac{3}{q}\right] \cup \left[1 - \frac{3}{q}, \min\left\{1, 3 - \frac{5}{q}\right\}\right] \equiv \left[0, 1\right] \quad \text{for} \ q \in ]3, \infty[. \quad (3.33)$$

Putting together Cases 1–2 and using $3 - \beta = \lambda + 2$, we now complete the proof. \qed

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