Bogoliubov Transformations for Amplitudes in Black-Hole Evaporation

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Abstract

In a previous Letter, we outlined an approach to the calculation of quantum amplitudes appropriate for studying the black-hole radiation which follows gravitational collapse. This formulation must be different from the familiar one (which is normally carried out by considering Bogoliubov transformations), since it yields quantum amplitudes relating to the final state, and not just the usual probabilities for outcomes at a late time and large radius. Our approach simply follows Feynman's $+i\epsilon$ prescription. Suppose that, in specifying the quantum amplitude to be calculated, initial data for Einstein gravity and (say) a massless scalar field are specified on an asymptotically-flat space-like hypersurface $\Sigma_I$, and final data similarly specified on a hypersurface $\Sigma_F$, where both $\Sigma_I$ and $\Sigma_F$ are diffeomorphic to $\mathbb{R}^3$. Denote by $T$ the (real) Lorentzian proper-time interval between $\Sigma_I$ and $\Sigma_F$, as measured at spatial infinity. Then rotate: $T \rightarrow |T| \exp(-i\theta)$, $0 < \theta \leq \pi/2$. The classical boundary-value problem is then expected to become well-posed on a region of topology $I \times \mathbb{R}^3$, where $I$ is the interval $[0, |T|]$. For a locally-supersymmetric theory, the quantum amplitude is expected to be dominated by the semi-classical expression $\exp(iS_{\text{class}})$, where $S_{\text{class}}$ is the classical action. Hence, one can find the Lorentzian quantum amplitude from consideration of the limit $\theta \rightarrow 0_+$. In the usual approach, the only possible such final surfaces $\Sigma_F$ are in the strong-field region shortly before the curvature singularity; that is, one cannot have a Bogoliubov transformation to a smooth surface 'after the singularity'. In our complex approach, however, one can put arbitrary smooth gravitational data on $\Sigma_F$, provided that it has the correct mass $M$; thus we do have Bogoliubov transformations to surfaces 'after the singularity in the Lorentzian-signature geometry' — the singularity is simply by-passed in the analytic continuation (see below). In this Letter, we consider Bogoliubov transformations in our approach, and their possible relation to the probability distribution and density matrix in the traditional approach. In particular, we find that our probability distribution for configurations of the final scalar field cannot be expressed in terms of the diagonal elements of some density-matrix distribution.

1. Introduction

In [1-4], based on the thesis [5], we described a calculation of the quantum amplitude for an initial spherically-symmetric configuration of gravity and a massless scalar field $\phi$ to make a transition to another nearly-spherically-symmetric configuration at a very late Lorentzian time $T$, apart from some weak perturbations $(h_{ij}^{(1)}, \phi^{(1)})_F$. The boundary data are specified on an initial and a final asymptotically-flat hypersurface $\Sigma_{I,F}$, and the time $T$ is measured at spatial infinity. The Lorentzian-signature space-time metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) has spatial components $h_{ij} = g_{ij} (i, j = 1, 2, 3)$ on surfaces with $t = x^0 = \text{const.},$
such as $\Sigma_I$ and $\Sigma_F$. This transition could refer to the quantum evaporation of a black hole, following gravitational collapse from a slowly-moving, diffuse, spherical configuration. The final configuration could refer to the nearly-spherical stream of (massless) radiation, and its corresponding Vaidya-like gravitational field \cite{6,7}, as viewed on a space-like hypersurface $\Sigma_F$, of topology $\mathbb{R}^3$, which cuts through all the radiation. The weak perturbations on $\Sigma_F$, in the present field language, correspond to emitted particles.

The calculation proceeds following Feynman’s $+i\epsilon$ prescription \cite{8}. If $T$ is rotated into the complex: $T \rightarrow |T|\exp(-i\theta)$, for $0 < \theta \leq \pi/2$, then the classical boundary-value problem, given the above nearly-spherically-symmetric boundary data, is expected to become well-posed \cite{1}.

For a simple example (cf. \cite{10}), consider the 2-dimensional Euclidean-space Laplace equation
\[
\frac{\partial^2 \phi}{\partial \tau^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 ; \quad 0 < \tau < T, \quad -\infty < x < +\infty , \quad (1.1)
\]
subject to boundary data of the form (say)
\[
\phi(\tau = 0, x) = 0 , \quad \phi(\tau = T, x) = \phi_1(x) , \quad (1.2)
\]
on the 'initial and final surfaces'. Here, the boundary data given by $\phi_1(x)$ should be thought of as a $C^\infty$ function, of rapid decrease as $|x| \rightarrow \infty$.

Let $\Phi_1(k)$ denote the Fourier transform of $\phi_1(x)$:
\[
\Phi_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi_1(x) \, dx . \quad (1.3)
\]
Then the (unique) solution of the real elliptic Dirichlet boundary-value problem (1.1,2) above is given by
\[
\phi(\tau, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\sinh(k\tau)}{\sinh(kT)} \Phi_1(k) \, dk . \quad (1.4)
\]
In common with any solution of an elliptic partial-differential equation with analytic coefficients, $\phi$ is automatically a (real- or complex-) analytic function of both arguments $\tau$ and $x$ \cite{9}.

One can then rotate the Euclidean proper-distance-at-infinity into the complex: $T \rightarrow T\exp(i\theta)$, where $0 \leq \theta < \pi/2$. The Laplace equation (1.1) still holds, although it is perhaps more natural to view it with respect to coordinates $(y_\theta, x)$, where $y_\theta = \tau \exp(i\theta)$, in which case the potential $\phi$ is, for each fixed $\theta$, a (complex) solution of the complexified Laplace equation
\[
e^{2i\theta} \frac{\partial^2 \phi}{\partial y_\theta^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 . \quad (1.5)
\]
Provided that $0 \leq \theta < \pi/2$, this differential equation is strongly elliptic in the language of \cite{11}. We continue to pose the same boundary data (1.2), with $\phi = 0$ on the initial surface $y_\theta = 0$, and with
\[
\phi(y_\theta = T, x) = \phi_1(x) \quad (1.6)
\]
on the final surface \( y_\theta = T \) [equivalently given by \( \tau = T \exp(-i\theta) \)]. The corresponding complexified solution \( \phi(y_\theta, x) \) of this boundary-value problem still exists and is unique, provided always that \( 0 \leq \theta < \pi/2 \), being given by the analytic continuation of Eq.(1.4):

\[
\phi\left(\tau = y_\theta \exp(-i\theta), x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\sinh(k y_\theta \exp(-i\theta))}{\sinh(k T \exp(-i\theta))} \Phi_1(k) \, dk .
\] (1.7)

Provided that \( 0 \leq \theta < \pi/2 \), the denominator in the integrand of Eq.(1.7), namely

\[
\sinh(kT e^{-i\theta}) = \cos(kT \sin \theta) \sinh(kT \cos \theta) - i \sin(kT \sin \theta) \cosh(kT \cos \theta) ,
\] (1.8)

is non-zero for all \( k \neq 0 \). By examining a small neighbourhood of \( k = 0 \), one sees that both numerator and denominator have simple zeros at \( k = 0 \), whence the integrand in Eq.(1.7) is well-behaved for all \( k \). In this way, it can be shown that the integral (1.7) provides the (unique) solution of the boundary-value problem, for \( 0 \leq \theta < \pi/2 \).

On the other hand, for the exactly Lorentzian case with \( \theta = \pi/2 \), one can see that no solution exists for the Dirichlet boundary-value problem, for typical choices of the final boundary data \( \phi_1(x) \), or equivalently of its Fourier transform \( \Phi_1(k) \). This is because, following the representation (1.4), one now has \( \sin(kT) \) for the denominator, rather than \( \sinh(kT) \). The existence of a (smooth) Lorentzian solution \( \phi(t, x) \) would then imply that \( \Phi_1(k) \) vanishes for all those values of \( k \), namely, for all \( k = n\pi/T \) (\( n \) a positive or negative integer, excluding the case \( n = 0 \)), such that \( \sin(kT) \) vanishes. A typical \( \phi_1(x) \) will not have \( \Phi_1(n\pi/T) = 0 \) for a single value of \( n \) (integer, \( \neq 0 \)). This argument can be refined, to show rigorously the non-existence of a solution to the above (non-trivial) Dirichlet problem for the flat wave equation. Thus, in this example, the Dirichlet boundary-value problem for the wave equation is badly posed.

In [1] and in this paper, we study the field-theoretic analogue of the boundary-value problem above, with data specified on an initial and a final spacelike hypersurface, with either a (Riemannian) proper distance at spatial infinity, \( T \), between the surfaces, or a complex-rotated separation \( T \exp(i\theta) \), where \( 0 \leq \theta < \pi/2 \). For simplicity, we consider Einstein gravity coupled to a massless scalar field \( \phi \), and first restrict attention to spherically-symmetric configurations, as giving a large number of 'background' or 'reference' spacetimes describing gravitational collapse to a Schwarzschild black hole in Lorentzian signature [12,13]. The metric is taken (without loss) in the form (2.1) below; the corresponding scalar field is of the form \( \phi(t, r) \). In Riemannian signature, the Einstein field equations are given in [14].

As yet, very little is known rigorously in the theory of partial differential equations about existence and uniqueness for this boundary-value problem – even for the purely real Riemannian case with a spherically-symmetric 3-metric \((h_{ij})_{I,F}\) and scalar field \((\phi)_{I,F}\) specified on the initial and final hypersurfaces, and a real proper-distance separation \( T \) given at spatial infinity. In the limit of weak boundary data, and hence, via the classical Riemannian field equations, also of weak fields in the interior, one would expect fixed-point methods, analogous to those used in [15], to establish existence and uniqueness for the Riemannian boundary-value problem. This weak-field Riemannian problem has also
been investigated numerically [14]. For weak scalar boundary data, global quantities such as the mass $M$ and Euclidean action $I$ appear to scale quadratically, in accordance with analytic weak-field estimates [16].

Considering still the real Riemannian case, there does not ‘experimentally’ appear to be any obstacle to extending the above (good) results numerically to strong boundary data, except for the ever-larger amounts of computer time required. It may be that a typical pattern will emerge numerically for the general ‘shape’ of the classical gravitational and scalar fields, in the limit of strong-field boundary data. In that case, it might be possible to find analytic approximations for the strong-field limit (quite different from those valid in the weak-field case), which could provide a large amount of further analytical insight into the solutions of the strong-field Riemannian boundary-value problem. It is, of course, such strong-field boundary-value problems which are of most interest, when we consider rotating: $T \to T \exp(i\theta)$, with $0 \leq \theta < \pi/2$, towards a Lorentzian time-interval.

The strong-field limit for this boundary-value problem is expected to correspond, in the Lorentzian limit in which $\theta$ approaches $\pi/2$ from below (although $\theta$ never attains the value $\pi/2$), to classical spherically-symmetric Einstein/scalar solutions which form a singularity, surrounded by a black hole. As described in the Abstract above, one expects that arbitrary smooth gravitational and scalar data can be posed on the final spacelike hypersurface $\Sigma_F$, allowing for a rotation into the complex of the Riemannian distance-separation $T (>0)$ between $\Sigma_I$ and $\Sigma_F$, of the form $T \to T \exp(i\theta)$, with $0 \leq \theta < \pi/2$, and provided that $\Sigma_F$ has the correct mass $M$. The (real) magnitude $T$ could then be chosen arbitrarily large, corresponding to a non-singular complex solution of the Einstein/scalar field equations between $\Sigma_I$ and a final hypersurface $\Sigma_F$ at late time. In this sense, in the complexified solution, one expects to have Bogoliubov transformations to surfaces $\Sigma_F$ which are ‘after the singularity in the Lorentzian-signature geometry’, provided always that one does not take the limit $\theta = \pi/2$. In the same sense, one may say that ‘the singularity is simply by-passed in the analytic continuation’; by analogy with the scalar-field example of Eqs.(1.1-8) above, there should be an analytic complexified classical solution to the boundary-value problem, which only reaches a singular boundary precisely at Lorentzian signature ($\theta = \pi/2$). Provided that one retains the freedom to rotate $T$ into the complex (as was insisted upon also by Feynman [8] in his $+i\epsilon$ prescription), one has the possibility of circumventing Lorentzian singularities by deforming time-intervals suitably into the complex, much as one avoids singularities of functions $f(z)$ in the ordinary complex $z$-plane.

For the more general Einstein/scalar boundary-value problem, in which one allows for weak non-spherical perturbations in the boundary data $(h_{ij}, \phi)_{I,F}$, the (complex) classical action $S_{class}$ can be studied as a functional of the perturbative final data $(h_{ij}^{(1)}, \phi^{(1)})_F$ and a function of the complex variable $T$, if for simplicity we regard the spherically-symmetric initial data $(h_{ij}, \phi)_I$ as fixed. The classical action $S_{class}$ can be calculated, and thence the semi-classical amplitude, proportional to $\exp(i S_{class})$. Finally, the limit $\theta \to 0_+$ can be taken, yielding the Lorentzian amplitude.

This description is clearly very different from that usually adopted for black-hole evaporation [17-20], which involves the study of Bogoliubov transformations between an
early- and a late-time surface, where the late-time surface passes inside the black-hole event horizon, close to the curvature singularity. The standard approach yields probabilities for configurations of outgoing particles, and a late-time density matrix, but not quantum amplitudes for states defined typically by weak data over an $\mathbb{R}^3$ surface at late times. as in the present case. It is intended, in this Letter, to make some contact between these two approaches, even though our approach addresses a wider class of questions than does the usual approach. In particular, we show how the Bogoliubov transformation between our two $\mathbb{R}^3$-surfaces $\Sigma_I$ and $\Sigma_F$ can be described in terms of the quantities appearing in [2,3] (see also [1]). The Bogoliubov quantities $|\beta_{\omega\ell m}|^2$ for the original calculation of black-hole radiance [17] agree with those found by the above boundary-value approach, to high accuracy (Sec.3). In Sec.4, we consider the resulting probability distribution for configurations of the perturbative scalar field $\phi^{(1)}$ on the final surface $\Sigma_F$. It is found that this probability distribution cannot be expressed in terms of the diagonal elements of some density-matrix distribution; thus, the correspondence between our approach and the density-matrix approach of [18] is limited.

2. Bogoliubov transformations between $R^3$ surfaces

As in [1-4], we write the Lorentzian-signature classical ‘background’ metric in the form

$$ds^2 = -e^{b(t,r)} dt^2 + e^{a(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.1)$$

For later reference, we also define the ‘mass function’ $m(t,r)$ by

$$\exp(-a(t,r)) = 1 - \frac{2m(t,r)}{r}. \quad (2.2)$$

In the boundary-value problem outlined above, let us write $(\gamma_{\mu\nu}, \Phi)$ for the ‘background’ spherically-symmetric metric and scalar field. Writing also $\nabla_\mu$ for the background covariant derivative, consider a linearised classical solution $\phi^{(1)}$ of the scalar wave equation

$$\nabla^\mu \nabla_\mu \phi^{(1)} = 0, \quad (2.3)$$

Consider a basis $\{f_{\omega'\ell m}(x)\}$ of (nearly-) separated mode solutions adapted to $\Sigma_I$ and another such basis $\{p_{\omega\ell m}(x)\}$ adapted to $\Sigma_F$. On $\Sigma_I$, the $\{f_{\omega'\ell m}\}$ are chosen to be an orthonormal, complete family of complex solutions of the wave equations, which contain only positive frequencies ($\omega' > 0$). Correspondingly for the $\{p_{\omega\ell m}\}$ on $\Sigma_F$. In a standard fashion [19], one expands out $\phi^{(1)}(x)$ in terms of one or the other basis, as

$$\phi^{(1)}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} d\omega' \left[c_{\omega'\ell m} f_{\omega'\ell m}(x) + \text{c.c.}\right], \quad (2.4)$$

$$\phi^{(1)}(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^{\infty} d\omega \left[b_{\omega\ell m} p_{\omega\ell m}(x) + \text{c.c.}\right]. \quad (2.5)$$
Note that we shall assume throughout that the asymptotic time interval $T = |T| \exp(-i\theta)$ has negative imaginary part. For convenience of visualisation, one might wish to regard the angle $\theta$ as being extremely small. Nevertheless, one only takes $\theta \to 0_+$ at the end of the calculation. Thus, this Section deals with a larger class of Bogoliubov transformations than those appearing in the usual formulation.

The bases above may be chosen to have the form

$$
\phi_{\omega \ell m}(x) = N(\omega) \frac{R_{\omega \ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\Omega),
$$

where $N(\omega)$ is a normalisation factor. On $\Sigma_I$, we require $\phi_{\omega \ell m} \propto e^{-i\omega v}$ at large $r$, where $v = t + r^*$ and where $r^* \sim r^*_s$, with

$$
r^*_s = r + 2M \ln(\left(r/2M\right) - 1)
$$

denoting the Regge-Wheeler radial coordinate \[21,22\] in an exactly Schwarzschild geometry of mass $M$. On $\Sigma_F$, we require $\phi_{\omega \ell m} \propto e^{-i\omega u}$ at large $r$, where $u = t - r^*$. Thus, the $\{f_{\omega \ell m}\}$ are purely ingoing at infinity, while the $\{p_{\omega \ell m}\}$ are purely outgoing.

Since the $\{f_{\omega \ell m}\}$ give a complete orthonormal set on $\Sigma_I$, a typical basis function (solution) $p_{\omega \ell m}$ on $\Sigma_I$ may be expanded as

$$
p_{\omega \ell m} = \int_0^\infty d\omega' \left( \alpha_{\omega' \omega \ell m} f_{\omega' \ell m} + \beta_{\omega' \omega \ell m} f^*_{\omega' \ell, -m} \right),
$$

The sets of complex numbers $\{\alpha_{\omega' \omega \ell m}\}$ and $\{\beta_{\omega' \omega \ell m}\}$ are the Bogoliubov coefficients, and the standard properties of Bogoliubov transformations are, for example, given in [13].

In our treatment [1-5] of linearised scalar perturbations $\phi^{(1)}(x)$, propagating at late times through the Vaidya-like region containing outgoing black-hole radiation, almost all modes are adiabatic. That is, a general solution of Eq.(2.1) in this region can be built up from separated solutions of the form (2.6), with $\omega = -k$ and $R_{\omega \ell}(r) \sim \xi_{k\ell}(r)$. Here, $\xi_{k\ell}(r)$ obeys

$$
\frac{\partial^2 \xi_{k\ell}}{\partial r^{*2}} + (k^2 - V_{\ell}) \xi_{k\ell} = 0,
$$

where, as usual,

$$
\frac{\partial}{\partial r^*} = e^{(b-a)/2} \frac{\partial}{\partial r}, \quad e^{-a} \simeq e^b \simeq 1 - \frac{2m(t,r)}{r},
$$

and

$$
V_{\ell}(t,r) = \frac{e^{b(t,r)}}{r^2} \left( \ell(\ell + 1) + \frac{2m(t,r)}{r} \right).
$$

The ‘left’ boundary condition on $\xi_{k\ell}$ as $r \to 0$ is

$$
\xi_{k\ell}(0) = 0.
$$
The 'right' boundary condition, as \( r \to \infty \), is
\[
\xi_{k\ell}(r) \sim z_{k\ell} \exp(i k r^*) + z_{k\ell}^* \exp(-i k r^*),
\]
where, for each \((k, \ell)\), \(z_{k\ell}\) is a dimensionless complex constant. We then wrote out \(\phi^{(1)}\) in terms of 'coordinates' \(\{a_{k\ell m}\}\) on the final surface \(\Sigma_F\),
\[
\phi^{(1)}(x) \bigg|_{\Sigma_F} = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} dk \ a_{k\ell m} \xi_{k\ell}(r) \ Y_{\ell m}(\Omega).
\]

In the nearly-Lorentzian case, the late-time behaviour of quantum amplitudes \([1-5]\) will be dominated by the eigen-frequencies \(k_n = \frac{n \pi}{T}\), \(n = 1, 2, \ldots\). Correspondingly, one discretises the frequency integral (2.11). It will also be convenient to define real functions \(\{f_{n\ell}(r)\}\) for \(n = 1, 2, \ldots; \ \ell = 0, 1, 2, \ldots\), such that
\[
\xi_{n\ell}(r) = (z_{n\ell} e^{ik_n r^*} + z_{n\ell}^* e^{-ik_n r^*}) f_{n\ell}(r),
\]
where \(f_{n\ell}(r) \to 1\) as \(r \to \infty\), \(f_{n\ell}(r) \to 0\) as \(r \to 0\).

It is now possible to make a comparison of this treatment of the quantum amplitude from initial data on \(\mathbb{R}^3\) to final data on \(\mathbb{R}^3\), with the corresponding description in terms of Bogoliubov transformations (again, in our formulation, not in the standard formulation). One takes a discretised version of Eq.(2.5), appropriate to the final surface \(\Sigma_F\), and compares the expansions of \(\phi^{(1)}\) in basis functions on \(\Sigma_F\). In particular, one finds that
\[
|b_{n\ell m}|^2 = 2 \pi k_n |z_{n\ell}|^2 |a_{n\ell m} + a_{-n\ell m}|^2 = 2 f_{\ell m}(k_n),
\]
where \(\{b_{n\ell m}\}\) are the coefficients appearing in Eq.(2.5). We also find an expression for \(|b_{\omega\ell m}|^2\) in terms of Bogoliubov coefficients. Including an implicit summation over the index \(m\):
\[
|b_{\omega\ell m}|^2 = \int_0^{\infty} d\omega' \int_0^{\infty} d\omega'' \left( \alpha_{\omega'\ell}^* \alpha_{\omega''\ell} + \beta_{\omega'\ell}^* \beta_{\omega''\ell} \right) c_{\omega'\ell m} c_{\omega''\ell m}^* - 2 \int_0^{\infty} d\omega' \int_0^{\infty} d\omega'' \Re \left( \beta_{\omega''\ell}^* \alpha_{\omega'\ell} c_{\omega'\ell m} c_{\omega''\ell, -m} \right).
\]

3. Particle emission rates for a final \(R^3\) surface

The total energy of the late-time Einstein/scalar system must be equal to \(M\), the ADM mass [22] or energy of the original configuration. The 'second-variation' contribution \(E^{(2)}\) to \(M\), namely the contribution from all the particles produced, is
\[
E^{(2)} = \sum_{\ell m} \int_0^{\infty} d\omega \ \omega |b_{\omega\ell m}|^2.
\]
This can also be written in terms of the ‘harmonic-oscillator coordinates’ \( \{c_{\omega \ell m}\} \) of Eq.(2.4), on using Eq.(2.18). Further, the expected number \( < n_{\omega \ell m} > \) of particles emitted \([5]\) is

\[
<n_{\omega \ell m} > = |\beta_{\omega \ell}|^2 ,
\]

where \( |\beta_{\omega \ell}|^2 \) is defined by

\[
\int_0^\infty \! d\omega' \beta_{\omega' \omega \ell} \beta^{*}_{\omega' \omega'' \ell} = |\beta_{\omega \ell}|^2 \delta(\omega, \omega'').
\]

In the original, standard, calculation of Bogoliubov coefficients and of probabilities for particle emission by a non-rotating black hole \([17]\), neglecting back-reaction, it was found that

\[
|\beta_{s\omega \ell m}|^2 = \Gamma_{s\omega \ell m}(\tilde{m}) \left(e^{4\pi \tilde{m}} - (-1)^{2s}\right)^{-1} ,
\]

where \( \Gamma_{s\omega \ell m}(\tilde{m}) \) is the transmission probability over the centrifugal barrier \([23]\) and \( \tilde{m} = 2M\omega \). This calculation, of course, referred to the case where one did not have a final surface \( \Sigma_F \) of topology \( \mathbb{R}^3 \), with a freely-chosen metric on it, subject to the mass being \( M \). But, because of the very-high-frequency (adiabatic) method in which this expression for \( |\beta_{s\omega \ell m}|^2 \) was calculated, it should still be valid (up to tiny corrections) in our present \( \mathbb{R}^3 \) case, noting that the calculation of scattering over the centrifugal barrier should allow for the extremely slow time- and radial-dependence of the mass function \( m(t,r) \) in Eq.(2.9-11).

4. Probabilities and density matrices

Following from the Lorentzian quantum amplitude calculated in \([1-5]\), for perturbative spin-0 data given on the final surface \( \Sigma_F \), while taking (for simplicity) exactly spherically-symmetric ‘background’ data on \( \Sigma_I \), one finds the corresponding conditional probability density over the final perturbative scalar boundary data \( \phi^{(1)}|_{\Sigma_F} \), given that \( \phi^{(1)}|_{\Sigma_I} = 0 \). For a large (real) Lorentzian time-interval \( T \), this is

\[
P\left[\{a_{k\ell m}\}; |T|\right] = \hat{N} e^{-\delta|T|M I} \exp\left(-2 \text{Im} S_{\text{class}}^{(2)} \left[\{a_{k\ell m}\}; |T|\right]\right) ,
\]

where \( \hat{N} \) is a suitable normalisation factor. The quantity \( S_{\text{class}}^{(2)} \left[\{a_{k\ell m}\}; |T|\right] \) is calculated in \([3]\).

The probability distribution (4.1) arises, of course, from squaring a quantum amplitude. But, given the thermal nature of black-hole evaporation in the usual description, one can still ask whether the probability distribution can be viewed in terms of the diagonal elements of some density matrix in a suitable basis \([18-20]\). This is simpler to understand if one assumes the relations

\[
\int_0^\infty \! d\omega' \beta^{*}_{\omega'' \omega \ell} \beta_{\omega' \omega \ell} = |\beta_{\omega' \ell}|^2 \delta(\omega', \omega'') ,
\]
\[ \int_0^\infty d\omega \beta_{\omega'\omega} \alpha^*_{\omega'\omega} = 0. \] (4.3)

These conditions (4.2,3) do hold for the steady-state Bogoliubov coefficients in the calculation which neglects back-reaction on the metric [17]. One needs to check whether this diagonal form persists under adiabatic propagation through a slowly-varying potential. The slow variation of the potential with \( t \) and \( r \) occurs, as above, because, in the Vaidya metric [6] describing the radiative part of the space-time, the Schwarzschild mass \( M \) is replaced by the mass function \( m(t, r) \) of Eq.(2.2,10).

We shall need the relation [19]

\[ |\alpha_{\omega'\ell}|^2 - |\beta_{\omega'\ell}|^2 = 1. \] (4.4)

From these equations, one finds

\[ P \left( \{a_{\omega lm}\} \right) = P \left( \{c_{\omega lm}\} \right) = \hat{N} \prod_{n \ell m} \exp \left[ \frac{-2\pi}{T} \left( 1 + 2|\beta_{n \ell}|^2 \right) |c_{n \ell m}|^2 \right]. \] (4.5)

As expected in the linearised theory, the modes evolve independently, corresponding to the product over \( n \ell m \).

In studying the question of whether \( P \left( \{c_{\omega lm}\} \right) \) can be related to a density-matrix distribution, we shall need the relation

\[ \frac{1}{(1-s)^m} \exp\left[ -x s/(1-s) \right] = \sum_{k=0}^{\infty} L_k(x) s^k, \quad |s| < 1, \] (4.6)

which gives the generating function for the Laguerre polynomials \( \{L_k(x)\} \) [24]. We set

\[ s = \left| \frac{\beta_{n \ell}}{\alpha_{n \ell}} \right|^2 < 1 \] (4.7)

and

\[ x = 2 X_{n \ell m} = 4 (\Delta \omega_n) |c_{n \ell m}|^2, \] (4.8)

which is dimensionless. Writing \( j = n \ell m \), we find

\[ P(\{c_j\}) = \hat{N} \prod_j \exp(-X_j) \exp\left( -2|\beta_j|^2 X_j \right) \]

\[ = \hat{N} \prod_j \exp(-X_j) \sum_{k_j=0}^{\infty} P(k_j) L_{k_j}(2X_j) \] (4.9)

\[ = \hat{N} \sum_{k_j} P(k_j) \prod_j \exp(-X_j) L_{k_j}(2X_j), \]
where $P(k_j)$ is defined as

$$P(k_j) = \frac{1}{|\alpha_j|^2} |\beta_j\alpha_j|^{2k_{n\ell m}}, \quad (4.10)$$

giving the (normalised) probability to observe the field in the state $k_j$. Equivalently, $P(k_j)$ is the probability of finding $k_j$ particles outgoing at future null infinity, in the mode $(n\ell m)$. Eq.(4.9) describes the probability distribution $P[\{c_j\}]$ for the final scalar field, with the help of the Bose-Einstein distribution (4.10). From Eq.(4.5), $P[\{c_j\}]$ depends only on the modulus $|c_j|$ of the complex number $c_j$; hence, the sum in Eq.(4.9) is diagonal.

In Eq.(4.9), the Laguerre polynomial $L_k(x)$ has $k$ real, distinct roots. For positive argument $X_j > 0$ and excited states $(k_j > 0)$, the function $\exp(-X_j) L_{k_j}(2X_j)$ takes negative values for certain ranges of $X_j$. Therefore, this function cannot be interpreted as a probability density.

For comparison, consider a density matrix of the form

$$\rho(x, y) = \sum_i w_i \psi_i(x) \psi_i^*(y), \quad (4.11)$$

where the $\{\psi_i(x)\}$ form a complete set and $w_i$ is the probability to be in the state $\psi_i(x)$. If the system is in the single state defined by $\psi_i(x)$, then the probability density for observing $x$ is $|\psi_i(x)|^2$. There is not a probabilistic interpretation for the full $\rho(x, y)$, but the diagonal contribution

$$P(x) = \rho(x, x) = \sum_i w_i |\psi_i(x)|^2 \quad (4.12)$$

is the probability density for observing the coordinate $x$. In our example, if the field is in the $k_j$-th state, then $\hat{N} \exp(-X_j) L_{k_j}(2X_j)$ is the 'probability density' for the 'harmonic oscillator coordinate' $X_j$. We then obtain the statistical 'probability' distribution (4.9) on multiplying by the probability $P(k_j)$ to be in the $k_j$-th state, then summing over all possible states $k_j$. This gives an interesting way of regarding the probability distribution (4.9), but it does not provide a sensible density matrix, based on the quantum amplitude $\exp\left(i S^{(2)}_{\text{class}}[\{a_{k\ell m}\}, T]\right)$ of [3-5]. This shows that the density-matrix approach, based conceptually on linearised quantum field theory around a classical space-time, does not apparently replicate the results of a fully quantum approach (at least, not in the locally-supersymmetric case).

5. Conclusion

Here, we have made some connection between the boundary-value approach of [1-5] to calculating quantum amplitudes for generic late-time configurations defined (say) on $\mathbb{R}^3$, which involves a rotation of the time-at-infinity: $T \to |T| \exp(-i\theta)$ into the complex, and the familiar Bogoliubov-coefficient approach to calculating probabilities in black-hole evaporation, with a final hypersurface near the space-time singularity. In the case that both the initial and the final space-like hypersurfaces $\Sigma_I, \Sigma_F$ have the topology of $\mathbb{R}^3$, there
is a relation between the Bogoliubov transformations in the two different approaches. The $|β_{sωℓm}|^2$ quantities for the original (Bogoliubov) calculation of black-hole radiance and for the topologically-$R^3$ approach agree, at least very closely. However, the probability distribution calculated from the $'R^3'$ quantum amplitude cannot be expressed in terms of the diagonal elements of some density-matrix distribution, so supporting the view that the two approaches are genuinely different.

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References

[1] A.N.St.J.Farley and P.D.D’Eath, Phys Lett. B 601 (2004) 184.
[2] A.N.St.J.Farley and P.D.D’Eath, ’Quantum Amplitudes in Black-Hole Evaporation; I. Complex Approach’, submitted for publication (2005).
[3] A.N.St.J.Farley and P.D.D’Eath, ’Quantum Amplitudes in Black-Hole Evaporation: II. Spin-0 Amplitude’, submitted for publication (2005).
[4] A.N.St.J.Farley and P.D.D’Eath, ’Bogoliubov Transformations in Black-Hole Evaporation’, submitted for publication (2005).
[5] A.N.St.J.Farley, ’Quantum Amplitudes in Black-Hole Evaporation’, Cambridge Ph.D. dissertation, approved 2002 (unpublished).
[6] P.C.Vaidya, Proc. Indian Acad. Sci. A33 (1951) 264.
[7] A.N.St.J.Farley and P.D.D’Eath, ’Vaidya Space-Time and Black-Hole Evaporation’, submitted for publication (2005).
[8] R.P.Feynman and A.R.Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, New York, 1965.
[9] P.R.Garabedian, Partial Differential Equations, Wiley, New York, 1964.
[10] P.D.D’Eath, Supersymmetric Quantum Cosmology, Cambridge Univ. Press, Cambridge, 1996.
[11] W.McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge Univ. Press, Cambridge, 2000.
[12] D.Christodoulou, Commun. Math. Phys. 105 (1986) 337; 106 (1986) 587; 109 (1987) 591; 109 (1987) 613.
[13] M.W.Choptuik, Critical behaviour in massless scalar field collapse, in R.d’Inverno (Ed.), Approaches to Numerical Relativity, Cambridge Univ. Press, Cambridge, 1992.
[14] P.D.D’Eath, A.Sornborger, Class. Quantum Grav. 15 (1998) 3435.
[15] P.D.D’Eath, Ann. Phys. (N.Y.), 98 (1976) 237.
[16] P.D.D’Eath, in preparation.
[17] S.W.Hawking, Commun. Math. Phys. 43 (1975) 199.
[18] S.W.Hawking, Phys. Rev. D 14 (1976) 2460.
[19] N.D.Birrell and P.C.W.Davies, Quantum fields in curved space, Cambridge Univ. Press, Cambridge, 1982.
[20] V.P.Frolov and I.D.Novikov, Black Hole Physics, Kluwer Academic, Dordrecht, 1998.
[21] T.Regge and J.A.Wheeler, Phys. Rev. 108 (1957) 1063.
[22] C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation, Freeman, San Francisco, 1973.
[23] J.A.H.Futterman, F.A.Handler and R.A.Matzner, Scattering from Black Holes, Cambridge Univ. Press, Cambridge, 1988.
[24] I.S.Gradshteyn and I.M.Ryzhik, Tables of Integrals, Series and Products, Academic Press, New York, 1965.