A Convergence Proof
for
Linked Cluster Expansions

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Abstract

We prove that for a general $N$-component model on a $d$-dimensional lattice $\mathbb{Z}^d$ with pairwise nearest-neighbor coupling and general local interaction obeying a stability bound the linked cluster expansion has a finite radius of convergence. The proof uses Mayer Montroll equations for connected Green functions.

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1 Introduction

Linked cluster expansion is a useful tool to describe models in the massive phase region \[1\]. Under the supposition that the first singularity of this expansion is real and equal to the critical point \(\kappa_{\text{crit}}\) one can study the critical behaviour of the model \[2\] and \[3\]. The high-order coefficients of the series expansion contains information that enables one to compute critical exponents. These methods were applied to study the ultraviolet resp. infrared behaviour of Euclidean field theories \[2\]. We will here not prove that the radius of convergence is equal to \(\kappa_{\text{crit}}\). In this paper it will be shown that the linked cluster expansion has a finite radius of convergence.

Let us consider a model with \(N\)-components on the lattice \(\Lambda = \mathbb{Z}^d\) described by the partition function

\[
Z(J, \kappa) = \int \prod_{x \in \Lambda} d^N \Phi(x) \exp\{-S(\Phi, \kappa) + \sum_{x \in \Lambda} J(x) \cdot \Phi(x)\}. \tag{1}
\]

We have used the scalar product

\[
J(x) \cdot \Phi(x) := \sum_{a=1}^N J_a(x) \Phi_a(x)
\]

for \(J, \Phi \in \mathcal{H}(\Lambda)\), the Hilbert space of square summable functions on \(\Lambda\). The action \(S\) contains a local part \(V\) and a non-local pair interaction

\[
S(\Phi, \kappa) = \sum_{x \in \Lambda} V(\Phi(x)) - \frac{1}{2} \sum_{x,y \in \Lambda} \sum_{a,b=1}^N \Phi_a(x) v_{ab}(x,y) \Phi_b(y).
\]

The nearest-neighbor interaction is defined by

\[
v_{ab}(x, y) = \begin{cases} 
2\kappa \delta_{a,b} : & x, y \text{ nearest-neighbors} \\
0 : & \text{otherwise}. \tag{2}
\end{cases}
\]

\(\kappa\) is called the hopping parameter. The partition function in eq. (1) is the generating function of the Green functions. The generating function of the connected Green functions reads

\[
W(J, \kappa) = \ln Z(J, \kappa).
\]

The connected Green functions are

\[
G^c_{a_1, \ldots, a_n}(x_1, \ldots, x_n) := \frac{\partial^n}{\partial J_{a_1}(x_1) \cdots \partial J_{a_n}(x_n)} W(J, \kappa)|_{J=0},
\]

for all \(a_1, \ldots, a_n \in \{1, \ldots, N\}\) and \(x_1, \ldots, x_n \in \Lambda\). The linked cluster expansion is a Taylor expansion around the local part of the action \[1\]. It uses successively

\[
\frac{\partial}{\partial v_{ab}(x, y)} W = \frac{1}{2} \left( \frac{\partial^2 W}{\partial J_a(x) \partial J_b(y)} + \frac{\partial W}{\partial J_a(x)} \frac{\partial W}{\partial J_b(y)} \right).
\]

We give the convergence proof only for the connected 2-point Green function. Generalization to connected \(n\)-point Green functions are straightforward.
Furthermore, we will suppose translation invariance and $O(N)$-symmetry. These assumptions are not essential for the proof and generalization to more general cases are simply made.

We want to prove the following theorem

**Theorem 1.1** Let an $N$-component model in $d$ dimensions be defined by the partition function in eq. (4). Suppose that there exists a positive $c > 0$ and real $\delta$ such that for all $\Phi \in \mathbb{R}^N$

$$V(\Phi) \geq c\Phi^2 - \delta.$$ (3)

Then there exists a positive constant $\kappa_*$ such that the 2-point Green function $G_{aa}(x, y)$ is an analytic function in $K(\kappa_*):= \{\kappa \in \mathbb{C}| |\kappa| < \kappa_*\}$ and there exist for all $\kappa \in K(\kappa_*)$ positive $\alpha > 0$ and $m > 0$ such that for all $a \in \{1, \ldots, N\}$, $x, y \in \Lambda$

$$|G_{aa}^c(x, y)| \leq e^\alpha q_{V}^{(2)} \delta_{xy} + \alpha \exp\{-m||x - y||\},$$ (4)

where

$$q_{V}^{(2)} := \frac{\int d^N \Phi \Phi_a \exp\{-V(\Phi)\}}{\int d^N \Phi \exp\{-V(\Phi)\}}.$$ (5)

Furthermore, for $\kappa \to 0$,

$$m(\kappa) = O(\ln |\kappa|) > 0.$$ (6)

Thus, using a well-known theorem for analytic functions, the above theorem implies that for $m > 0$ the series in the definition of the 2-point susceptibility

$$\chi_2 := \sum_{y \in \Lambda} G_{aa}^c(x, y)$$

exists and is an analytic function in $K(\kappa_*)$ if $\kappa_*$ is small enough.

In the remainder of this section we give an outline of the organization of the proof.

The existence of the Green functions in the infinite volume $\Lambda$ is not a priori assured. One can easily see that in a finite volume (e.g. on a torus) the derivatives of partition functions (not-normalized Green functions) are well-defined quantities. Furthermore, by the dominated convergence theorem, they are entire functions in $\kappa$. We only have to show the existence of the integral in the partition function. This is assured by the stability bound (3) for the local action.

Thus the first step in our convergence proof is to define the model on finite subsets of the lattice and state the stability bound for the local interaction $V$. This restriction defines a polymer system and polymer activities. Then we are in the position to express the Green functions in terms of derivatives of polymer activities.

In a second step we formulate the thermodynamic limit of the Green functions in terms of a series expansion (Mayer Montroll equations) and show the convergence under certain assumptions. This leads to a uniform bound of the linked cluster expansion for small $\kappa$. By a standard theorem for series of analytic functions we can show that the linked cluster expansion has a finite radius of convergence. The Mayer Montroll equations avoid the problem of zeroes in the partition function.
2 Polymer system and stability

In this section we derive the Mayer Montroll equations for the connected 2-point Green function (cp. [5]) and show that the partition functions for finite subsystems are well-defined if the local interaction fulfills the stability bound.

Polymers are certain finite nonempty subsets of the lattice $\Lambda$. The exact definition of the polymers is given below. For a polymer $X \subset \Lambda$ define the partition function

$$Z(X|J, \kappa) = \int \prod_{x \in X} d\Phi(x) \exp\{-S(X|\Phi, \kappa) + \sum_{x \in X} J(x) \cdot \Phi(x)\}, \quad (7)$$

where the action is

$$S(X|\Phi, \kappa) = \sum_{x \in X} V(\Phi(x)) - \frac{1}{2} \sum_{x,y \in X} \sum_{a,b=1}^{N} \Phi_a(x)v_{ab}(x,y)\Phi_b(y).$$

The propagator kernel $v_{ab}(x,y)$ is zero unless $x, y$ are nearest-neighbors and $a = b$ (cf. [2]).

By the dominated convergence theorem we may conclude that $Z(X|J, \kappa)$ is analytic for all $\kappa \in \mathbb{C}$ if the integral in eq. (7) exists. The existence of the integral can be shown if the local interaction satisfies the stability bound given by (3). This is easily seen by using the estimate

$$\frac{1}{2} \sum_{x,y \in X} \sum_{a,b=1}^{N} \Phi_a(x)v_{ab}(x,y)\Phi_b(y) = \sum_{<xy>: <xy> \in \mathcal{L}(X)} (2\kappa) \Phi(x) \cdot \Phi(y) \leq 4d|\kappa| \sum_{x \in X} \Phi(x)^2.$$

$\mathcal{L}(X)$ denotes the set of all links $<xy>$ ($x$ and $y$ are nearest-neighbors), $x, y \in X$, in the lattice $\Lambda$. The links have no directions, i.e. $<xy> \equiv <yx>$. 

Lemma 2.1 Suppose that there exists a positive $c > 0$ and real $\delta$ such that the stability bound for the local interaction eq. (3) is valid. Then we have, for all $\kappa \in \mathbb{C}$ and $\epsilon > 0$ obeying $4d|\kappa| + \epsilon < c$,

$$S(\Phi, \kappa) \geq \epsilon \sum_{x} \Phi^2(x) - \delta$$

and the integral on the right hand side of eq. (7) is convergent.

Define polymer activities $A(Q|J, \kappa)$ for all polymers $Q$ of $\Lambda$ such that the polymer representation of the partition functions hold

$$Z(X|J, \kappa) = \sum_{X=\sum Q} \prod Q A(Q|J, \kappa) \quad (8)$$

for all polymers $X$ of $\Lambda$. The sum goes over all partitions of $X$ into disjoint polymers. Eq. (8) defines the activities uniquely. The reverse relation is

$$A(X|J, \kappa) = \sum_{n \geq 1} (-1)^{n-1}(n-1)! \sum_{X=\sum_{i=1}^{n}Q_i} \prod_{i=1}^{n} Z(Q_i|J, \kappa). \quad (9)$$
Since $Z(X|J, \kappa)$ is analytic in $\kappa \in \mathbb{C}$ under the supposition of lemma 2.1 we see by eq. (9) that the activities $A(X|J, \kappa)$ are also analytic functions.

The condition of nearest-neighbor couplings for the non-local part of the action implies that polymer activities are zero for non-connected polymers. We say that a polymer $Q$ is connected if any $x, y \in Q$ can be connected by links $l = \langle xy \rangle \in \mathcal{L}(X)$. Suppose that $X = X_1 + X_2$ is the disjoint partition of $X$ into connected polymers $X_1$ and $X_2$. Then the partition function factorizes

$$Z(X|J, \kappa) = Z(X_1|J, \kappa) Z(X_2|J, \kappa).$$

This implies that $A(Q|J, \kappa) = 0$ if $Q$ is not connected. In the following we call finite nonempty subsets of $\Lambda$ polymers if they are connected.

A polymer $X \subset \Lambda$ with only one element, $|X| = 1$, is called monomer. The corresponding monomer activity is special. It is equal to the monomer partition function

$$A(\{x\}|J, \kappa) = Z(\{x\}|J, \kappa) = \int d^N \Phi(x) \exp\{-V(\Phi(x)) + J(x) \cdot \Phi(x)\}. \quad (10)$$

For later convenience we will replace the partition functions

$$Z(X|J, \kappa) \longrightarrow \frac{Z(X|J, \kappa)}{\prod_{x \in X} Z(\{x\}|J = 0, \kappa)}.$$

Obviously, this is equivalent to the replacement of activities

$$A(X|J, \kappa) \longrightarrow \frac{A(X|J, \kappa)}{\prod_{x \in X} A(\{x\}|J = 0, \kappa)}.$$

This replacement only changes the normalization factor and will not change the Green functions. After replacement we have the normalization condition

$$A(\{x\}|J = 0, \kappa) = 1 \quad (11)$$

for all $x \in \Lambda$.

In the remainder of this section we want to express the connected Green functions by polymer activities. We discuss this at the example of the 2-point Green function. It is given by the (up to now formal) thermodynamic limit

$$G_{aa}(x, y) = \lim_{X \searrow \Lambda} \frac{\partial^2 Z(X|J, \kappa)}{\partial J_a(x) \partial J_a(y)}|_{J = 0}$$

$$= \sum_{Q\subseteq X} \frac{\partial^2 A(Q|J, \kappa)}{\partial J_a(x) \partial J_a(y)}|_{J = 0} \rho_{\Lambda}(Q|J = 0, \kappa), \quad (12)$$

where the reduced correlation function $\rho_X$ is defined by

$$\rho_X(Q|J, \kappa) := \frac{Z(X - Q|J, \kappa)}{Z(X|J, \kappa)},$$

for all polymers $Q, X, Q \subseteq X$. Eq. (12) can be derived by using the Kirkwood-Salsburg equation (cp. [6])

$$Z(X|J, \kappa) = \sum_{Q: x \in Q \subseteq X} A(Q|J, \kappa) Z(X - Q|J, \kappa)$$

for later convenience we will replace the partition functions

$$Z(X|J, \kappa) \longrightarrow \frac{Z(X|J, \kappa)}{\prod_{x \in X} Z(\{x\}|J = 0, \kappa)}.$$
for all $x \in X$. This follows easily from the polymer representation eq. (8).

Let us define Mayer activities $M$ by

$$M(X|J, \kappa) = -\delta_{1,|X|} + A(X|J, \kappa).$$

The normalization condition eq. (11) is equivalent to

$$M(\{x\}|J = 0, \kappa) = 0$$

for all $x \in \Lambda$.

The Mayer Montroll equations express the reduced correlation functions in terms of Mayer activities. For the formulation of these equations we need some notations and definitions (see also [5]). Two polymers $P_1$ and $P_2$ are called compatible, $P_1 \sim P_2$, iff they are disjoint, $P_1 \cap P_2 = \emptyset$. A finite set $\mathcal{P} = \{P_1, \ldots, P_n\}$ consisting of polymers is called admissible iff $P \sim P'$ for all $P, P' \in \mathcal{P}$, $P \neq P'$. $K(X)$ denotes the set of all admissible $\mathcal{P}$ which consists of polymers $P \subseteq X$

$$K(X) := \{\mathcal{P} = \{P_1, \ldots, P_n\}| \mathcal{P} \text{ admissible, } P_i \subseteq X\}.$$

Let $\Pi(Y)$ be the set of all partitions of $Y$ into nonempty disjoint subsets. Define $\Pi(\emptyset) := \emptyset$. Then we have

$$K(X) = \bigcup_{Y: Y \subseteq X} \Pi(Y).$$

Two admissible sets $\mathcal{P}^{(1)} = \{P_1^{(1)}, \ldots, P_n^{(1)}\}$ and $\mathcal{P}^{(2)} = \{P_1^{(2)}, \ldots, P_n^{(2)}\}$ are called compatible, $\mathcal{P}^{(1)} \sim \mathcal{P}^{(2)}$ iff $P \sim P'$ for all $P \in \mathcal{P}^{(1)}$, $P' \in \mathcal{P}^{(2)}$.

Denote by $\text{Conn}_X(\mathcal{P})$ the set which consists of admissible sets $\mathcal{P}' \in K(X)$ which contain polymers that are incompatible with at least one polymer $P$ in $\mathcal{P}$, $P \not\sim P'$,

$$\text{Conn}_X(\mathcal{P}) := \{\mathcal{P}' \in K(X)| \mathcal{P}' \not\sim \mathcal{P} \forall \mathcal{P}' \in \mathcal{P}'\}.$$

We will use the notation $\text{Conn}(\mathcal{P}) \equiv \text{Conn}_\Lambda(\mathcal{P})$. For a finite set $\mathcal{P} = \{P_1, P_2, \ldots\}$ define

$$M^\mathcal{P} := \prod_{P \in \mathcal{P}} M(P|J, \kappa).$$

The polymer representation eq. (8) implies

$$Z(X|J, \kappa) = \sum_{\mathcal{P}: \mathcal{P} \in K(X)} M^\mathcal{P}.$$
3 Thermodynamic Limit

In this section the thermodynamic limit of the Mayer Montroll equation for the 2-point Green function is proven for small complex $\kappa$. This finishes the proof of the theorem 1.1.

Insertion of eq. (13) into eq. (12) yields the Mayer Montroll equation for the 2-point Green function

$$G_{aa}(x, y) = \sum_{Q: \text{polymer}} \sum_{n \geq 0} \sum_{P_1, \ldots, P_n \in K(Q)} \sum_{\emptyset \neq P_i \in \text{Conn}(P_{i-1})} \frac{\partial^2 M(Q|J, \kappa)}{\partial J_a(x) \partial J_a(y)} (-M)^{P_1+\ldots+P_n} |_{J=0}. \quad (14)$$

The series starts with $P_0 = \{\{x\} | x \in Q\}$. The individual terms in this series expansion of $G_{aa}(x, y)$ are analytic in $\kappa$. We will show that this expansion is uniformly bounded for all $|\kappa| \leq \kappa_*$ by a convergent series expansion. Then we may conclude by a standard theorem of complex functions that $G_{aa}(x, y)$ is analytic in $\kappa$, $|\kappa| \leq \kappa_*$.

For estimations we need the following tree graph formula for the activities (cp. [4]). Let $T(X)$ be the set of all tree graphs with lines $(xy)$, $x, y \in X$ and vertex set $X$. In the case of nearest-neighbor interactions lines are links in $L(X)$. The tree graph formula holds also for general pair interactions. For all polymers $X$, $|X| \geq 2$, we can write the polymer activity as a sum over tree graphs $\tau$

$$A(X|J, \kappa) = \sum_{\tau: \tau \in T(X)} (2\kappa)^{|X|-1} \int_{x \in X} q^N \Phi(x) \exp\left\{ \sum_{x \in X} (-V(\Phi(x)) + J(x) \cdot \Phi(x)) \right\} \prod_{(xy) \in \tau} \int_0^1 dt_{xy} \prod_{<xy> \in \tau} (\Phi(x) \cdot \Phi(y)) \exp\left\{ 2\kappa \sum_{<xy> \in L(X)} t_{\tau \min}^{\text{min}}(x, y) \Phi(x) \cdot \Phi(y) \right\}, \quad (15)$$

where

$$t_{\tau \min}^{\text{min}}(x, y) := \min \{ t_l | \text{path connecting } x \text{ and } y \text{ and containing link } l \in \tau \}.$$

For a proof of this formula see [4].

The following lemma shows under which conditions the Mayer Montroll expansion for the connected 2-point Green function is convergent and exponentially bounded. For convergence conditions of cluster expansion and general polymer systems for the free energy cf. [7].

**Lemma 3.1** Suppose that there exists positive constants $\alpha, \kappa_*, m > 0$ such that

$$\sum_{P, y \in P, |P| \geq 2} |M(P|J = 0, \kappa)| \exp\{\alpha|P|\} \leq \frac{\alpha}{2} \quad (16)$$
\[
\sum_{P' : y \in P} \sum_{n \geq 0} \left( \frac{\alpha}{2} \right)^n \frac{n!}{2^{n}} \exp\{\alpha|P'|\} |M(P'|J = 0, \kappa)| \leq \frac{1}{2} \exp\{\alpha|\mathcal{P}|\}.
\]  
(20)

We use eq. (13) and the normalization condition eq. (11) to perform the sums over \(P'_1, \ldots, P'_n\). Let us start with the sum over \(P'_n\). Using eq. (16) yields a factor \((|\mathcal{P}| - n + 1)\frac{\alpha}{2}\) since \(P'_n\) has to be incompatible with \(\mathcal{P} = \{P'_1, \ldots, P'_{n-1}\}\) which consists of at most \(|\mathcal{P}| - n + 1\) elements. Then we sum over \(P'_{n-1}\). This yields a factor \((|\mathcal{P}| - n + 2)\frac{\alpha}{2}\). In the same way we sum over \(P'_{n-2}, \ldots, P'_1\). Thus the term in (20) is bounded by

\[
\sum_{n \geq 1} \left( \frac{|\mathcal{P}|}{n} \right) \left( \frac{\alpha}{2} \right)^n \leq \frac{1}{2} [(1 + \alpha)|\mathcal{P}| - 1] \leq \frac{1}{2} \exp\{\alpha|\mathcal{P}|\}.
\]

This proves eq. (19). Using eq. (19) inductively, we obtain

\[
|G^c_{aa}(x, y)| \leq e^{\alpha q_V^{(2)} \delta_{xy}} + \sum_{P : y \in P} \sum_{n \geq 0} 2^{-n} \left( \frac{\alpha}{2} \right)^n \frac{n!}{2^{n}} |\partial^2 M(P|J, \kappa)|_{J = 0} \exp\{\alpha|P|\}. 
\]  
(21)

The bounds (17) and (21) imply the assertion (18). \(\square\)

It remains to prove the supposition of lemma 3.1.
Lemma 3.2 Suppose that the stability bound \((3)\) holds. For all \(m > 0\) there exists positive constants \(\alpha > 0\) and \(\kappa^* > 0\) such that inequalities \((16)\) and \((17)\) are valid for all \(|\kappa| \leq \kappa^*\). Furthermore, for \(\kappa \to 0\),

\[ m(\kappa) = O(\ln |\kappa|) > 0. \]

Proof: Using the stability bound \((3)\), lemma \(2.1\) and the tree graph formula \((15)\) for the polymer activities \(A\) we obtain for all polymers \(X\), \(|X| \geq 2\),

\[ |M(X|J = 0, \kappa)| \leq \sum_{\tau: \tau \in T(X)} (2\kappa N)^{|X|-1}(e^\delta q_V)^{|X|}\epsilon^{-\frac{\sum_{x \in X}(d_\tau(x)+1)}{2}} \prod_{x \in X} \left( \frac{d_\tau(x)+1}{2} \right)!, \quad (22) \]

where

\[ d_\tau(x) := |\{l \in \tau| l \text{ emerges from vertex } x\}| \]

is the number of lines in the tree graph \(\tau\) which are connected to vertex \(x\). We have used

\[ 2 \int_0^\infty dx \exp\{-ex^2\}x^n = e^{-n+1}(\frac{n+1}{2})! \]

and the definition

\[ q_V := \int d^N \Phi \exp\{-V(\Phi)\}. \]

Since the lines in the tree graph \(\tau\) are links (=pairs of nearest neighbors) the number of lines emerging from a vertex is bounded by

\[ d_\tau(x) \leq 2d. \quad (23) \]

We will use

\[ \sum_{x \in X} d_\tau(x) = 2(|X| - 1). \quad (24) \]

The number of random walks containing \(l\) links starting from a site \(x\) is equal to \((2d)^l\). We can run through a tree graph \(\tau \in T(X)\) by a random walk starting from a vertex \(x \in X\) and visiting each line \(l\) two times and ending in vertex \(x\). Such a vertex contains \(2(|X| - 1)\) lines. Thus the number of all tree graphs with vertex set \(X\) is bounded by

\[ |T(X)| \leq (2d)^{2(|X| - 1)}. \quad (25) \]

By the same argument the number of polymers \(X\) with \(n\) elements containing site \(x \in \Lambda\) is bounded by

\[ |\{X \subset \Lambda| X \text{ polymer, } |X| = n, x \in X\}| \leq (2d)^{2(n-1)}. \quad (26) \]

Inequalities \((22)\), \((23)\), \((24)\) and eq. \((24)\) imply, for \(|X| \geq 2\),

\[ |M(X|J = 0, \kappa)| \leq (2(2d)^2\kappa N)^{|X|-1}(e^\delta q_V^{-1})^{|X|}\epsilon^{-\frac{3}{2}|X|} \left( d + \frac{1}{2} \right)! |X| \]

\[ = \frac{e^\delta (d + \frac{1}{2})!}{q_V \epsilon^{1/2}} \left( \frac{2(2d)^2\kappa e^\delta N}{q_V \epsilon^{3/2}} \right)^{|X|-1}. \quad (27) \]
Similarly, we derive, for $|X| \geq 2$,

$$
\left| \frac{\partial^2 M(X|J, \kappa)}{\partial J_a(x) \partial J_a(y)} \right|_{J=0} \leq e^{\delta (d + \frac{1}{2})! (1 + \frac{1}{d+1})^2 \frac{2(2d)^2 \kappa e^\delta N(d + \frac{1}{2})!}{q_V e^{3/2}}} |X|^{-1}.
$$

(28)

For the monomer case $X = \{x\}$, $|X| = 1$, we have, using definition (5)

$$
\left| \frac{\partial^2 M(X|J, \kappa)}{\partial J_a(x) \partial J_a(y)} \right|_{J=0} = q_V \delta_{xy}
$$

(29)

for all $y \in \Lambda$, $x \in X$. Eqs. (27) and (28) imply, for $|\kappa| \leq \kappa^*$, $\kappa^*$ small enough,

$$
\sum_{P: x \in P} |M(P|J = 0, \kappa)| \exp \{ \alpha |P| \} \leq \frac{2(d + \frac{1}{2})! e^{\alpha + \delta}}{q_V e^{1/2}} u,
$$

(30)

and

$$
\sum_{P: x \in P \atop |P| \geq 2} |\frac{\partial^2 M(P|J, \kappa)}{\partial J_a(x) \partial J_a(y)}|_{J=0} \leq \frac{2(1 + \frac{1}{2d})^2 (d + \frac{1}{2})! e^{\alpha + \delta}}{q_V e^{3/2}} u \exp \{-m \|x - y\|\},
$$

(31)

where

$$
u := \frac{2(2d)^4 \kappa_* N(d + \frac{1}{2})! e^{\alpha + \delta}}{q_V e^{3/2} e^{-m}} \leq \frac{1}{2}.
$$

The factor in eq. (31) $\exp \{-m \|x - y\|\}$ comes from the fact that the terms on the left hand side of eq. (31) are zero unless $x, y \in P$.

For $\kappa^*$ small enough there exists a positive constant $\alpha > 0$ such that

$$
\max(\frac{2(d + \frac{1}{2})! e^{\alpha + \delta}}{q_V e^{1/2}} u, \frac{2(1 + \frac{1}{2d})^2 (d + \frac{1}{2})! e^{\alpha + \delta}}{q_V e^{3/2}} u) \leq \frac{\alpha}{2}.
$$

(32)

Inequalities (30) and (32) proves the bound (16). Inequalities (29), (31) and (32) imply the bound (17). Furthermore, we can take $m$ very large if $\kappa^* e^m$ remains small. This proves that $m$ is of order $\ln(|\kappa|)$ for small $|\kappa|$, eq. (6). $\square$

4 Summary

The partition functions restricted to finite connected subsets of the lattice are analytic functions in the hopping parameter $\kappa$. By the polymer representation the Mayer activities are also analytic. The Mayer Montroll equations express the connected Green functions by Mayer activities. If the local part of the action obeys a stability condition it turns out that the linked cluster expansion of the Green functions can be uniformly bounded using these Mayer Montroll equations. The result is analyticity of the connected Green functions and susceptibilities for $\kappa$ in a small neighborhood of zero.

The proof presented here is restricted to nearest-neighbor interaction. In a forthcoming paper the convergence proof will be generalized to linked cluster expansions beyond nearest neighbor interactions.
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