ESSENTIAL DIMENSION OF CENTRAL SIMPLE ALGEBRAS
OF DEGREE 8 AND EXPONENT 2 IN CHARACTERISTIC 2

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Abstract. The goal of this note is to reduce the existing upper bound
for the essential dimension of central simple algebras of degree 8 and
exponent 2 over fields of characteristic 2 from 10 to 9.

1. Introduction

Fixing a field $k$ in the background, we consider the covariant functor
$\text{Alg}_{d,e}$ sending any field $F$ containing $k$ to the set of all (isomorphism classes
of) central simple algebras of degree $d$ and exponent dividing $e$ over $F$. The
essential dimension of an algebra $A$ in $\text{Alg}_{d,e}(F)$ is the minimal possible
transcendence degree of a field $E$ over $k$ with $k \subseteq E \subseteq F$ for which there ex-
ists a central simple algebra $A_0$ in $\text{Alg}_{d,e}(E)$ with $A_0 \otimes F = A$. The essential
dimension of $\text{Alg}_{d,e}$, denoted $\text{ed}(\text{Alg}_{d,e})$, is thus the supremum on the essen-
tial dimensions of $A$ where $A$ ranges over all algebras in $\text{Alg}_{d,e}(F)$ and $F$ ranges
over all fields containing $k$. The question of computing $\text{ed}(\text{Alg}_{d,e})$ is
a very difficult one (see [Mer13, Section 10]) and the exact value is known
only in specific cases: it is known that $\text{ed}(\text{Alg}_{2,2}) = \text{ed}(\text{Alg}_{3,3}) = 2$, but
for any prime $p > 3$ it is still not known what $\text{ed}(\text{Alg}_{p,p})$ is. The value of
$\text{ed}(\text{Alg}_{4,2})$ is 4 when $\text{char}(k) \neq 2$ and 3 when $\text{char}(k) = 2$. The value of
$\text{ed}(\text{Alg}_{8,2})$ is 8 when $\text{char}(k) \neq 2$ ([BM12, Corollary 1.4]). The exact value
of $\text{ed}(\text{Alg}_{8,2})$ when $\text{char}(k) = 2$ is not known, but was bounded from above
by 10 and below by 3 in [Bae11]. The lower bound was improved to 4 in
[McK17]. The goal of this short note is to prove the following theorem:

Theorem 1.1 (Main Theorem). When $\text{char}(k) = 2$, we have $\text{ed}(\text{Alg}_{8,2}) \leq 9$.

This is proved in Section 3 and Section 2 describes the main ingredients.

2. Preliminaries

When $F$ is a field of $\text{char}(F) = 2$, a quaternion algebra over $F$ is of
the form $\langle \alpha, \beta \rangle_F = F \langle i, j : i^2 + i = \alpha, j^2 = \beta, ijj^{-1} = i + 1 \rangle$ for some

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\( \alpha \in F \) and \( \beta \in F^\times \). We write \( \wp(t) \) for \( t^2 + t \) and \( F[\wp^{-1}(\alpha)] \) thus stands for \( F[i : i^2 + i = \alpha] \). By [GS17 Theorem 9.1.4] (credited to Teichmüller), every class in \( 2Br(F) \) is represented by a tensor product of quaternion algebras. In particular, when \( A \) is a central simple algebra of degree 4 and exponent 2, it decomposes as a tensor product of two quaternion algebras ([Rac74]).

A tool that makes the study of the essential dimension of central simple algebras of degree 8 and exponent 2 easier is the chain lemma for quaternion algebras ([Dra83, Chapter 14, Theorem 7]): If \( [\alpha, \beta, \gamma] \) is a central simple algebra of degree 4 and exponent 2, it decomposes as a tensor product of two quaternion algebras ([Rac74]).

Another important tool is the fact that the essential dimension of central simple algebras of degree 8 and exponent 2 is 1 (see [Led04, Lemma 2] and [BF03, Remark 3.8] for reference, and for an explicit construction; despite the condition \( |k| \geq 2^n \) appearing in [Led04, Lemma 2], the proof of the latter goes through under the weaker assumption that \( |F| \geq 2^n \), and for our purpose this always applies, for in order for noncommutative division algebras to exist over \( F \), the cardinality of \( F \) must be infinite). The meaning is that if \( K = F[\wp^{-1}(\alpha), \wp^{-1}(\beta), \wp^{-1}(\gamma)] \) is a compositum of three cyclic quadratic extensions of a field \( F \) containing \( k \), then there exists a field \( L \supseteq k \) transcendent degree 1 over \( k \) containing \( a, b, c \) such that the tensor product \( L[\wp^{-1}(\alpha), \wp^{-1}(b), \wp^{-1}(c)] \otimes_L F \) is \( K \). In other words, the elements \( \alpha, \beta, \gamma \) can be chosen to come from a single field \( L \) of transcendence degree at most 1 over \( k \).

The last tool is the following lemma:

**Lemma 2.1** ([Bae11 Lemma 3.3]). Given a field \( F \) of characteristic 2 and a field extension \( E = F[i : i^2 + i = \alpha] \) and \( \beta \in F \), if \( x^2 + xy + y^2 \alpha = u^2 + uv + v^2 \beta \) for some \( x, y, u, v \in F \), then \( [\beta, x + yi)_E = [\beta, y + v]_E \).

3. **Proof of the main theorem**

Consider a division algebra \( A \) of degree 8 and exponent 2 over a field \( F \) containing a subfield \( k \) of \( \text{char}(k) = 2 \). By [Row84], \( A \) contains a maximal subfield \( K = F[\wp^{-1}(\alpha), \wp^{-1}(\beta), \wp^{-1}(\gamma)] \). By [Led04, Lemma 2], \( \alpha, \beta, \gamma \) can be chosen to come from a single subfield \( k \subseteq L \subseteq F \) of transcendence degree 1 over \( k \). Let \( E = F[\wp^{-1}(\alpha)] = F[\mu : \mu^2 + \mu = \alpha] \). Then \( A_E \) is Brauer equivalent to \( \beta, b)_E \otimes [\gamma, c)_E \) for some \( b, c \in E \). The corestriction back to \( F \) is trivial, and so \( [\beta, N_b)_F = [\gamma, N_c)_F \), where \( N_b = b_0^2 + b_0b_1 + b_1^2 \alpha \) and
$N_c = c_0^2 + c_0c_1 + c_1^2\alpha$, given that $b = b_0 + b_1\mu$ and $c = c_0 + c_1\mu$ for some $b_0, b_1, c_0, c_1 \in F$ (\cite{MM91}). By the chain lemma for quaternion algebras, there exists $\delta \in F$ for which $[\beta, N_b)_F = [\delta, N_c)_F = [\gamma, N_c)_F$. Therefore $\delta = \beta + u^2 + u + x^2N_b = \gamma + y^2 + y + z^2N_c = \lambda^2 + \lambda + t^2N_bN_c$ for some $u, x, y, z, \lambda, t \in F$. By replacing $\delta, y, \lambda$ with $u^2 + u + y + u, \lambda + u$ respectively, we can assume $u = 0$. Note that $y = \varphi^{-1}(\beta + y + x^2N_b + z^2N_c)$ and $\lambda = \varphi^{-1}(\beta + x^2N_b + t^2N_bN_c)$. If $x = 0$ then $\delta = \beta$, which means $[\beta + \delta, b)F$ is split. If $x \neq 0$, then from the equation $\delta = \beta + x^2N_b$ we obtain $(\beta + \delta)^\frac{1}{x} = b_0^2 + b_0b_1 + b_1^2\alpha$. It follows then from Lemma \ref{lem:split} that $[\beta + \delta, b)_E = [\beta + \delta, b_1 + \frac{1}{x}E$. Similarly, if $z = 0$ then $[\gamma + \delta, c)_E$ is split, and otherwise $[\gamma + \delta, c)_E = [\gamma + \delta, c_1 + \frac{1}{x}E$, and if $t = 0$ then $[\delta, bc)_E$ is split and otherwise $[\delta, bc)_E = [\delta, \frac{1}{x} + b_0c_1 + b_1c_1 + b_1c_1)E$. Therefore, of the case of $\mu = 0$. Hence, $A \sim_{Br} [a, a)_F \otimes [\beta + \delta, b_1 + \frac{1}{x}E \otimes [\gamma + \delta, c_1 + \frac{1}{x}E \otimes [\delta, \frac{1}{x} + b_0c_1 + b_1c_1 + b_1c_1)E$.

Consider now the algebra $B = [a, a)_T \otimes [\beta + \delta, b_1 + \frac{1}{x}T \otimes [\gamma + \delta, c_1 + \frac{1}{x}T \otimes \delta, \frac{1}{x} + b_0c_1 + b_1c_1 + b_1c_1)$, where $T = L(a, b_0, b_1, c_0, c_1, x, z, t, y, \lambda)$. Note that $y$ and $\lambda$ are algebraic over $L(a, b_0, b_1, c_0, c_1, x, z, t)$, and thus the transcendence degree of $T$ over $k$ is at most 9. In order to conclude the argument, we need to explain why $B$ is of index 8 rather than 16, which means that it is $M_8(A_0)$ for some division algebra $A_0$ of degree 8 over $T$, and thus $A$ descends to a degree 8 division algebra over a field of transcendence degree at most 9 over $k$. The restriction of $B$ to $R = T[\varphi^{-1}(\alpha)]$ is Brauer equivalent to $[\beta + \delta, b_1 + \frac{1}{x}_R \otimes [\gamma + \delta, c_1 + \frac{1}{x}_R \otimes [\delta, \frac{1}{x} + b_0c_1 + b_1c_1 + b_1c_1)_R$. Since $[\beta + \delta, b_1 + \frac{1}{x}_R = [\beta + \delta, b)_R$ (for the same equality $\delta = \beta + x^2N_b$ holds true as before), $[\gamma + \delta, c_1 + \frac{1}{x}_R = [\gamma + \delta, c)_R$ and $[\delta, \frac{1}{x} + b_0c_1 + b_1c_1 + b_1c_1)_R = [\delta, bc)_R$, we get that $B \otimes R$ is Brauer equivalent to $[\beta, b)_R \otimes [\gamma, c)_R$. Thus, $B$ is split by $T[\varphi^{-1}(\alpha), \varphi^{-1}(\beta), \varphi^{-1}(\gamma)]$, and so its index is 8.

\textbf{Remark 3.1.} It seems that in the argument in \cite{Bae11}, the author is suggesting that one could assume $x = z = 1$ in our computation above. If we were talking only about the quadratic equations $\delta = \beta + x^2N_b = \gamma + y^2 + y + z^2N_c$ this would be correct, because replacing $b_j$ and $c_j$ with $\frac{ab_j}{x}$ and $\frac{ac_j}{x}$ would eliminate their occurrences in these equations. However, that would change...
$[\beta, b]_E$ to $[\beta, \frac{b}{c}]_E$ and $[\gamma, c]_E$ to $[\gamma, \frac{c}{z}]_E$. It is therefore not clear why one can assume $x = z = 1$. If this were true, that would reduce the upper bound of the essential dimension from 9 to 7.

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