We make some comments on the renormalization of Wilson operators (not just vacuum-expectation values of Wilson operators), and the features which arise in Minkowski space. If the Wilson loop contains a straight light-like segment, charge renormalization does not work in a simple graph-by-graph way; but does work when certain graphs are added together. We also verify that, in a simple example of a smooth loop in Minkowski space, the existence of pairs of points which are light-like separated does not cause any extra divergences.
1. Introduction

There has been some discussion in the literature [1-8] of the renormalization of Wilson operators, but some of this has been done in Euclidean space [1,3,6], and most has been restricted to the vacuum-expectation value of the Wilson operator rather than the operator itself. In [2], we have previously noted some complications which appear in the renormalization of the Wilson operator in Minkowski space, when the Wilson loop contains a light-like segment. In particular, we noted that individual graphs do not always have divergent parts which are local. Here we show that renormalization nevertheless works provided that suitable sets of graphs are taken together.

Let the operator be
\[ W = trP \exp g \int_C A.d\mathbf{x} \]  
where \( C \) is a closed curve, \( P \) denotes operator and matrix ordering along \( C \), and the nonabelian gauge field \( A_\mu \) is a matrix in some representation \( R \) of the gauge group \( G \). Sometimes one is interested just in the vacuum-expectation value
\[ \langle W \rangle. \]

\( W \) is an example of a composite operator, and one hopes that it is multiplicatively renormalizable, in the sense that
\[ W_R(A_R;g_R) = Z(\epsilon)W_B(A_B;g_B,\epsilon), \]
where suffices \( R \) and \( B \) denote renormalized and bare quantities. Dimensional regularization is used, with \( d = 4 - \epsilon \), and the presence of \( \epsilon \) in some terms in (3) signals that they are divergent. The relationship between \( g_B \) and \( g_R \) and between \( A_R \) and \( A_B \) should be the same as in ordinary perturbation theory.

If we take the vacuum-expectation value of (3), we get
\[ \langle W_R(g_R) \rangle = Z(\epsilon)\langle W_B(g_B,\epsilon) \rangle. \]

Without any further restriction, (4) is not very restrictive, since it serves merely to define \( Z \). However, it is known [6,7,8] that the contributions to \( Z \) are associated with geometrical features of \( C \), especially corners and cusps. But, once \( Z \) is determined from (4), condition (3) is certainly nontrivial.

The heuristic reason one expects (3) to hold is that the divergences are expected to be short distance features, and therefore local ones on the curve \( C \). For example, the graphs in Fig 1(a) and (b) look like ordinary charge-renormalization graphs, and one expects the divergences to come from the regions in which the points \( x, y, z \) are close together; and so these divergences can be cancelled by the charge-renormalization counter-term graph (c). In fact the divergences from the subgraphs in (a) and (b) are the same as those in the ordinary Feynman graphs which they superficially resemble (with the line denoting the loop being replaced by, say, a quark line). But if the loop \( C \) contains a straight lightlike segment this simple expectation is not fulfilled, as we discuss in the next section.
2. Straight lightlike segments

Take a Wilson loop which contains a straight lightlike segment $AB$, as in Fig.2, where the vector $V$ represents that segment. $A'A$ and $B'B'$ are neighbouring portions of the loop.

We use light-cone coordinates defined as follows. Let $V'$ be another lightlike vector satisfying $V.V' = 1$ and with the momenta $k$ and $p$ in Fig.2. Let

$$V.k = u, \quad V'.k = v, \quad V.p = w, \quad V.K = V'.K = 0,$$

so that

$$k_\mu = vV_\mu + uV'_\mu + K_\mu. \tag{6}$$

We will be concerned with divergences coming from the region where $u$ is finite and $v$ and $K$ are large with $K^2$ scaling like $v$. We will assume that this is the case, and verify for in an Appendix that it is in a special case. In this spirit, to identify divergences, we neglect all components of $p$ except $w$ (since this appears along with $u$). Then Fig.2(b) gives (we use the Feynman gauge)

$$\frac{1}{2}i g^3 C_G t_a V_\mu \int_0^1 ds \int_s^1 dt \int \frac{d^2 K dudv}{(2\pi)^4} e^{-i[(w-u)s+ut]} \frac{[(w-u)-u]}{(uv-K^2+i\eta)(v(u-w)-K^2+i\eta)},$$

where $C_G$ is the gluon colour Casimir and $t_a$ is the colour matrix in the representation used in the definition of the Wilson loop. We will treat separately the two terms $(w-u)$ and $-u$ in the square bracket in the numerator, taking the $-u$ term first. Then, carrying out the $t$-integration, we get a factor

$$e^{-iws} - e^{-iu+i(u-w)s}. \tag{8}$$

The contributions from these two terms have the structures shown in Fig.3(a) and (b). Fig.3(a) contributes half of the usual nonabelian part of the quark vertex part renormalization constant (with dimensional regularization with $d = 4 - \epsilon$),

$$Z_{1f} = 1 - \frac{2}{\epsilon} C_G \frac{g^2}{16\pi^2} \tag{9}$$

(in the notation of Itzykson and Zuber eq.(12-122), also $Z_3$ is defined in eq.(12-144)), the other half coming from the $(w-u)$ term in the numerator of (7), as we shall discuss below. Together with the gluon vacuum polarization graph renormalization constant $Z_3$ this is absorbed by coupling constant renormalization. But the second contribution from (8) inserted into (7), denoted by Fig.3(b), is also divergent. We will show that its divergence is cancelled by the contributions from Fig.2(c) and (d). Take first Fig.2(c). The numerator contains the factor

$$2V.dxk_\mu - k.dxV_\mu - V.k dx_\mu, \tag{10}$$

where factors containing $p$ have been neglected and $dx_\mu = \frac{dx_\mu}{dr}dr$ is an element of the portion $BB'$ of the loop (parametrized by $r$ with $r = 0$ at $B$). The last term in (10) may also be neglected since $V.k = u$ is not large.
In the contribution from the second term in (10), the integral along $BB'$ has the form
\[
\int_0^1 dr \, k \cdot \frac{dx}{dr} e^{-ik \cdot x(r)},
\] (11)
and the contribution from the lower limit is the relevant one (any other contribution being finite because of the oscillating exponential). This contribution has the same structure as Fig.3(b), and it is easily verified to cancel that term (i.e. the contribution to (7) from the $(-u)$ in the numerator).

Next take Fig.2(d), which we shall treat along with the contribution to Fig. 2(c) from the first term in (10). Fig.2(d) gives
\[
\frac{1}{2} g^3 C_G t_a V(2\pi)^{-4} \int dudv dt \int dr \, k \cdot \frac{dx}{dr} e^{-ik \cdot x(r)} \int_0^1 ds \int_s^1 \frac{dt}{uv - K^2 + i\eta} e^{-iw\pm ius}.
\] (12)
The contribution to Fig.2(c) from the first term in (10) has an $r$-integral of the same form as in (12).

In order to proceed we need to discuss the curve $BB'$ in more detail. The factor $e^{-ik \cdot x}$ will make (12) converge except infinitesimally close to $r = 0$. Assume that
\[
x_{\mu}(r) = V_{\mu} + c_\mu r + O(r^2).
\] (13)
We will consider two possibilities:-(i) That $c_\mu \propto V_{\mu}$ so that the tangent to the curve $C$ varies continuously at $B$. (ii) That $c_\mu$ is not proportional to $V_{\mu}$, so that there is a discontinuity in the tangent to $C$ at $B$.

In case (ii), we may use in (12) the approximation $c.k \approx vc.V$ (from (6)); so that
\[
V \cdot \frac{dx}{dr} \approx v^{-1}k \cdot \frac{dx}{dr},
\] (14)
with neglect of convergent terms. Then the $r$-integration in (12) is exact, and the contribution from the lower limit $r = 0$ is
\[
-\frac{1}{2} g^3 C_G t_a V(2\pi)^{-4} \int dudv dt \int dr \, k \cdot \frac{dx}{dr} e^{-ik \cdot x(r)} \int_0^1 ds \frac{e^{-i(1-s)u}(e^{-iws} - e^{-iw})}{wv(u - K^2 + i\eta)}.
\] (15)
In the contribution from the second term in the numerator (15) we make the change of variables $u \to (u - w)$. Then (15) becomes
\[
\frac{1}{2} g^3 C_G t_a V \int \frac{dudv dt}{(2\pi)^4} \int_0^1 ds \frac{e^{-iu - is(w - u)}}{v(w - K^2 + i\eta)} \left[ \frac{1}{v(u - w) - K^2 + i\eta} - \frac{1}{u - K^2 + i\eta} \right].
\] (16)
This cancels the contribution to Fig. 2(c) from the first term in (10) (since the two denominators in (16) are the same as the Feynman denominators from Fig. 2(c)).

Next we take the alternative (i) above. In this case we can suppose that

\[ x = V + \alpha r V + \frac{1}{2} r^2 V' + O(r^3), \]  

(17)

where \( \alpha \) is a constant. Then

\[ k.x = (1 + \alpha r)u + \frac{1}{2} v r^2 + O(r^3), \quad V.dx = r dr + O(r^2). \]  

(18)

The relevant range of the \( r \)-integration is \( r \simeq v^{-\frac{1}{2}} \), so we may make the approximation

\[ k.\frac{dx}{dr} \approx \alpha u + rv + O(r^2), \]  

(19)

with neglect of terms down by a factor \( v^{-1/2} \), such terms being therefore not divergent. With the use of (19), the \( r \)-integration again is exact and the result reduces to case (ii) previously considered.

The contribution to Fig. 2(b) from the term \( w - u \) in the numerator of (7) is similarly cancelled by graphs like Fig. 2(c) and (d) but with the gluon \( k \) attached to \( AA' \) instead of \( BB' \).

3. An example of a Wilson loop in Minkowski space

The unexpected divergences encountered in Section 2 come from points on the curve \( C \) which have a null interval between them. For any closed smooth curve, at a point where the tangent is light-like, the light cone intersects the curve again in at least two other points. Are there similar unexpected divergences connected with all pairs of points with a null interval between them? In order to investigate this question, we take the simple case where \( C \) is the ellipse

\[ x = (a \cos u; 0, 0, b \sin u) \quad (0 \leq u < 2\pi). \]  

(20)

In this case, the second order contribution to \( \langle W \rangle \) is

\[
2g^2 C_R (2\pi)^{2-\epsilon/2} e^{i\pi\epsilon} \Gamma(1 - \frac{1}{2} \epsilon) \int_0^{2\pi} du \int_0^{2\pi} dv (a^2 \sin u \sin v - b^2 \cos u \cos v) \\
\times [a^2 (\cos u - \cos v)^2 - b^2 (\sin u - \sin v)^2 - i\eta]^{-1+\epsilon/2} \\
= 2g^2 C_R (2\pi)^{2-\epsilon/2} e^{i\pi\epsilon} \int_0^{\pi} dy (\sin^2 y)^{-1+\epsilon/2} \int_y^{2\pi-y} dx [a^2 \sin^2 x - b^2 \cos^2 x + i\eta]^{1+\epsilon/2} \\
+ 2g^2 C_R (2\pi)^{2-\epsilon/2} e^{i\pi\epsilon} (a^2 - b^2) \int_0^{\pi} dx [a^2 \sin^2 x - b^2 \cos^2 x + i\eta]^{-1+\epsilon/2}
\]
\[ x \cdot \left\{ \int_0^x dy (\sin^2 y)^{\frac{1}{2}} + \int_0^{\pi-x} dy (\sin^2 y)^{\frac{1}{2}} \right\}, \]  

where \( u - v = 2y, \ u + v = 2x \) The first integrand in (21) is singular at \( y = 0 \), where the two points \( u \) and \( v \) coincide. This singularity is expected and occurs also in Euclidean space. It is a ‘linear divergence’. With dimensional regularization, it gives rise to no pole at \( \epsilon = 0 \). The second integral in (21) is singular at \( \tan^2 x = \frac{b^2}{a^2} \). This is the case when the points \( u \) and \( v \) are on each others light cone, and is the possible divergence we wish to investigate. We will see that (in dimensional regularization at least) this singularity does not give rise to a pole at \( \epsilon = 0 \).

Let us consider the two lines in (21), starting with the first. If we changed the order of integration and did the \( y \)-integral first, we would get no pole at \( \epsilon = 0 \). Therefore we may set \( \epsilon = 0 \) in the \( x \)-integral. This integral is then trivial, and, after symmetrization, we obtain just the integral

\[ \int_0^\pi dy (\sin^2 y)^{-1+\epsilon/2} = \frac{\Gamma(\frac{\epsilon-1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\epsilon}{2})}, \]  

which gives zero in the limit \( \epsilon \to 0 \).

Now turn to the second line in (21). Once again, we set \( \epsilon = 0 \) in the nonsingular \( y \)-integral. In this case, the \( x \)-integral too is well-defined for \( \epsilon = 0 \), and is then just

\[ (a^2-b^2) \int_0^\pi dx [a^2 \sin^2 x - b^2 \cos^2 x + i\eta]^{-1} = (a^2-b^2) \int_\infty^- \frac{dz}{a^2 z^2 - b^2 + i\eta} = -i\pi \frac{(a^2-b^2)}{ab}. \]  

Thus the exitence of pairs of points with null interval causes no divergence. In fact the Minkowski space result (23) is obtainable from the Euclidean space one simply by the continuation \( b \to ib \). It is amusing that for the case of a circular Wilson loop \( a = b \) the Minkowski result (23) vanishes.

Appendix A

In this Appendix we quote from [2] the result for the integral appearing in Fig.2(b), thus confirming the validity of the approximation made in (7) for identifying the divergence.

The relevant integral over \( k \), holding \( u \) fixed, is

\[ \int d^2 K dv [uv - K^2]^{-1} [(u - w)(v - V'.p) - (K - P)^2]^{-1} \]

\[ = i\pi^{-\epsilon/2} \Gamma(\epsilon/2) w^{-1} ( -p^2 - i\eta )^{-\epsilon/2} [a(1-a)]^{-\epsilon/2} \theta(a) \theta(1-a), \]  

where \( a = u/w \). Thus there is indeed a pole, coming from the integral with \( u \) held fixed, and it is the same as would come from the approximate form in (7).
Appendix B

In this Appendix we treat particular examples of the graphs in Fig.2(b) and (c), for the special case where the segment $BB'$ becomes a straight line. For definiteness, we have completed the loop with two further straight sides to make a parallelogram, and inserted an extra gluon $q$ (so that the trace $(1)$ is nonzero), as in Fig.4 and Fig.5. Let the top and bottom sides of these parallelograms be represented by the vector $N$. For any vector $k$, let

$$k = xV + yN + K_T, \quad V.K_T = N.K_T = 0.$$  \hfill (B1)

Let us denote

$$u = V.k, \quad v = N.k, \quad d^4k = d^2K_Tdudv \mid (V,N) \mid^{-1}.$$  \hfill (B2)

The graph in Fig.4 is

$$W_4 = -ig^4C_GTr(t_at_d)V_\mu N_\rho \int \frac{d^n k}{(2\pi)^n} \frac{(p-2k).V}{k^2(p-k)^2} \left\{ \frac{1}{k.V} [e^{i(p-k).V} - 1] - \frac{1}{p.V} [e^{ip.V} - 1]\right\} \frac{1}{q.V} (1-e^{iq.V})e^{iq.N}$$  \hfill (B3)

Using (A1) and performing also the $\alpha$ integration we obtain

$$W_4 = \frac{g^4}{16\pi^2} C_G Tr(t_at_d)V_\mu N_\rho \Gamma(\frac{\epsilon}{2}) (-p^2 - i\eta)^{-\frac{\epsilon}{2}} \frac{1}{q.V} (1-e^{iq.V})e^{iq.N}$$

$$\times \frac{1}{p.V} \{(e^{ip.V} - 1)[ci(p.V) - \ln(p.V) - C + 2] - i(e^{ip.V} + 1)[si(p.V) + \frac{\pi}{2}]\}. \hfill (B4)$$

where $C$ is the Euler’s constant and $ci(p.V)$ and $si(p.V)$ are integrated cosine and sine functions. The graph in Fig.5 gives

$$W_5 = -\frac{g^4}{2\pi^2}(C_R - \frac{1}{2}C_G) Tr(t_at_d)V_\mu N_\rho e^{i\pi\frac{\epsilon}{2}} (|N^2|)^{-\frac{\epsilon}{2}} (V,N)^\epsilon (V,p)^{-\epsilon-1}\Gamma(\frac{\epsilon}{2})$$

$$\times e^{iV.p}\{ci(V.p) + isi(V.p) - C + \frac{i\pi}{2} - \ln(V.p)\} \times \frac{1}{q.V} (1-e^{iq.V})e^{iq.N}.$$  \hfill (B5)
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Figure Captions

Fig.1. Graphs contributing to the charge renormalization for general Wilson loops.

Fig.2. A portion of the Wilson loop which contains a straight lightlike segment $AB$ and the neighbouring portions $A'A$ and $BB'$ of the loop. Besides the graph in Fig.2(b), the graphs where a gluon is attached to the neighbouring portion $BB'$ contribute to the charge renormalization. The remainder of the Wilson loop is not drawn. There may be additional gluons attached to it.

Fig.3. The graphs to which a part of Fig.(2b) reduces after one parameter integration.

Figs.4 and 5. Examples of complete Wilson loop with two sides along the lightlike vector $V$ and two sides along vector $N$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9601122v1
This figure "fig1-2.png" is available in "png" format from:

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