On a J-polar decomposition of a bounded operator and matrix representations of J-symmetric, J-skew-symmetric operators.

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Introduction.
Complex symmetric, skew-symmetric and orthogonal matrices are classical objects of the finite-dimensional linear analysis [1]. In particular, the canonical spectral forms are known for them. Certainly, they have a more complicated structures as for Hermitian matrices. However, in a certain sense complex symmetric matrices are more universal. Namely, an arbitrary square complex matrix is similar to a symmetric matrix. If one introduces a J-form and write conditions for a symmetric, skew-symmetric and orthogonal matrix (continued by zeros to the right and to the bottom to obtain a semi-infinite matrix) in its terms, one arrives to the well-known J-symmetric, J-skew-symmetric and J-isometric operators.

A general definition of a J-symmetric operator was given by I.M. Glazman in his paper [2]. A study of these operators had been continued in papers of N.A. Zhyhar and A. Galindo (see the references in a monograph [3]). Later, an investigation of these operators had been performed by A.D. Makarova, L.A. Kamerina, V.P. Li, T.B. Kalinina, A.N. Kochubey, B.G. Mironov (a seria of papers by these authors appeared in 70-th, 80-th of the 20-th century in Ulyanovskiy sbornik "Funkcionalniy analiz"), L.M. Rayh, E.R. Tsekanovskiy and others. Most of these papers were devoted to the questions of extensions of J-symmetric operators to J-self-adjoint operators and to a description of all such extensions. At the present time, J-self-adjoint operators are studied by S.R. Garcia, M. Putinar, E. Prodan (see the paper [4] and References therein).

A definition of a bounded J-skew-symmetric operator was given by Sh. Asadi and I.E. Lutsenko in the paper [5]. A general definition appeared in a paper of T.B. Kalinina [6], she continued to study these operators in papers [7], [8]. J-symmetric and J-skew-symmetric operators also appeared in a book [9] in a study of Volterra operators context.

In papers of L.A. Kamerina J-isometric and quasi-unitary operators and a notion of quasi-unitary equivalence were introduced [10], [11].

Consider a separable Hilbert space $H$. Recall that a conjugation (involution) in $H$ is an operator $J$, defined on the whole $H$ and satisfying the following properties [12], [13]

$$J^2 = E, \quad (Jx, Jy) = \overline{(x, y)}, \quad x, y \in H,$$

(1)
where $E$ is the identity operator in $H$, and $(\cdot, \cdot)$ is a scalar product in $H$.

For each conjugation there exists an orthonormal basis $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}_+}$ in $H$ such that
\[ Jx = \sum_{k=0}^{\infty} \overline{x_k} f_k, \quad x = \sum_{k=0}^{\infty} x_k f_k \in H. \] (2)

This basis is not uniquely determined, it is determined up to a unitary transformation which commutes with $J$ (J-real). An arbitrary such a basis $\mathcal{F}$ we shall call **corresponding** to the involution $J$. Define the following linear with respect to the both arguments functional (J-form):
\[ [x, y]_J := (x, Jy), \quad x, y \in H. \] (3)

A linear operator $A$ in $H$ is said to be J-symmetric, if
\[ [Ax, y]_J = [x, Ay]_J, \quad x, y \in D(A), \] (4)

and is said to be J-skew-symmetric if
\[ [Ax, y]_J = -[x, Ay]_J, \quad x, y \in D(A). \] (5)

If the following condition is true:
\[ [Ax, Ay]_J = [x, y]_J, \quad x, y \in D(A), \] (6)

then the operator is said to be J-isometric.

Let the domain of $A$ is dense in $H$. The operator $A$ is said to be J-self-adjoint if
\[ A = JA^* J, \] (7)

and is said to be J-skew-self-adjoint if
\[ A = -JA^* J. \] (8)

If
\[ A^{-1} = JA^* J, \] (9)

then the operator $A$ we shall call a J-unitary. Notice that the operator $A^T = JA^* J$ in [12] was called **transposed** (later, in some papers it was also called J-adjoint, but we shall use the latter word for the operator $\tilde{A} = JAJ$).

For non-densely defined operators, one can also introduce a notion of J-symmetric and J-skew-symmetric linear relations, see, e.g., [14].

Let $A$ be a linear bounded operator in $H$. In this case, conditions (4),(5), (6) mean that the matrix of the operator in an arbitrary basis $\mathcal{F}$, which is...
coresponding to $J$, will be symmetric, skew-symmetric and orthogonal, respectively. This remark and some properties of the $J$-form allow to obtain some simple properties of eigenvalues and eigenvectors of such matrices.

In this work we obtain a $J$-polar decomposition for bounded operators (under some conditions). This decomposition is analogous to the polar decomposition of a bounded operator and to the $J$-polar decomposition in $J$-spaces [15]. Also, we obtain other decompositions which are analogous to decompositions for finite-dimensional matrices in [1]. A possibility of the matrix representation for $J$-symmetric and $J$-skew-symmetric operators and its properties are studied. A structure of the following null set

$$H_{J,0} = \{ x \in H : [x, x]_J = 0 \},$$

is studied, as well.

**Notations.** As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}^+$ the sets of real numbers, complex numbers, positive integers, non-negative integers and the real plane, respectively. Everywhere in this paper, all Hilbert spaces are assumed to be separable, $(\cdot, \cdot)$ and $\| \cdot \|$ denote the scalar product and the norm in a Hilbert space, respectively.

For a set $M$ in a Hilbert space $H$, by $\overline{M}$ we mean a closure of $M$ in the norm $\| \cdot \|$. For $\{x_k\}_{k \in \mathbb{Z}^+}$, $x_k \in H$, we write $\text{Lin} \{x_k\}_{k \in \mathbb{Z}^+} := \{ y \in H : y = \sum_{j=0}^{n} \alpha_j x_j, \alpha_j \in \mathbb{C}, n \in \mathbb{Z}_+ \}$; $\text{span} \{x_k\}_{k \in \mathbb{Z}^+} := \overline{\text{Lin} \{x_k\}_{k \in \mathbb{Z}^+}}$.

The identity operator in a Hilbert space $H$ is denoted by $E$. For an arbitrary linear operator $A$ in $H$, the operators $A^*, \overline{A}, A^{-1}$ mean its adjoint operator, its closure and its inverse (if they exist). By $D(A)$ and $R(A)$ we mean the domain and the range of the operator $A$, and by $\text{Ker} A$ we mean the kernel of the operator $A$. By $\sigma(A)$, $\rho(A)$ we denote the spectrum of $A$ and the resolvent set of $A$, respectively. The resolvent function of $A$ we denote by $R_A(\lambda), \lambda \in \rho(A)$. Also, we denote $\Delta_A(\lambda) = (A - \lambda E)D(A)$. The norm of a bounded operator $A$ is denoted by $\|A\|$.

By $l_2$ we denote the space of complex sequences $x = (x_0, x_1, x_2, ...)^T, x_k \in \mathbb{C}, k \in \mathbb{Z}_+$, with a finite norm $\|x\| = (\sum_{k=0}^{\infty} |x_k|^2)^{\frac{1}{2}}$ (the superscript $T$ stands for the transposition).

1 **Properties of eigenvalues and eigenvectors.**

We shall begin with some simple properties of $J$-symmetric, $J$-skew-symmetric and $J$-orthogonal operators which, in particular, lead to some new properties of finite-dimensional complex symmetric, skew-symmetric and orthogonal matrices. Let $J$ be a conjugation in a Hilbert space $H$.

Vectors $x$ and $y$ are said to be **$J$-orthogonal**, if $[x, y]_J = 0$. The following proposition is true (concerning statement (i) of the Proposition see. The-
Proposition 1.1 Let $A$ be a $J$-symmetric operator in a Hilbert space $H$. The following statements are true:
(i) Eigenvectors of the operator $A$ which correspond to different eigenvalues are $J$-orthogonal;
(ii) If vectors $x$ and $Jx$, $x \in D(A)$, are eigenvectors of the operator $A$, then they correspond to the same eigenvalue.

Proof. In fact, we can write
\[ \lambda_x [x, y]_J = [Ax, y]_J = [x, Ay]_J = \lambda_y [x, y]_J, \]
and therefore
\[ (\lambda_x - \lambda_y) [x, y]_J = 0. \] (10)
Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator $A$, which correspond to eigenvalues $\lambda_x$ and $\lambda_{\overline{x}}$, respectively. Write (10) with $y = \overline{x}, \lambda_y = \lambda_{\overline{x}}$:
\[ (\lambda_x - \lambda_{\overline{x}}) [x, \overline{x}]_J = 0. \]
Since $[x, \overline{x}]_J = \|x\|^2 > 0$, we get $\lambda_x = \lambda_{\overline{x}}$. $\square$

Define the following set:
\[ H_{J,0} := \{ x \in H : [x, x]_J = 0 \}. \] (11)

In a similar to the latter proof manner the validity of the following two propositions is established.

Proposition 1.2 If $A$ is a $J$-skew-symmetric operator in a Hilbert space $H$, then the following is true:
(i) Eigenvectors of the operator $A$, which correspond to non-zero eigenvalues, belong to the set $H_{J,0}$;
(ii) If $\lambda_x, \lambda_y$ are eigenvalues of the operator $A$ such that $\lambda_x \neq -\lambda_y$, then the corresponding to them eigenvectors are $J$-orthogonal;
(iii) Suppose that $x, \overline{x} := Jx \in D(A)$ are eigenvectors of the operator $A$, corresponding to the eigenvalues $\lambda_x$ and $\lambda_{\overline{x}}$, respectively. Then $\lambda_x = -\lambda_{\overline{x}}$.

Proposition 1.3 Let $A$ be a $J$-isometric operator in a Hilbert space $H$. Then the following statements are true:
(i) Eigenvectors of the operator $A$, which correspond to different from $\pm 1$ eigenvalues belong to the set $H_{J,0}$;
(ii) If \( \lambda_x, \lambda_y \) are eigenvalues of the operator \( A \) such that \( \lambda_x \neq \frac{1}{\lambda_y} \), then the corresponding to them eigenvectors are \( J \)-orthogonal;

(iii) Suppose that \( x, \overline{x} := Jx \in D(A) \) are eigenvectors of the operator \( A \), corresponding to the eigenvalues \( \lambda_x \) and \( \lambda_{\overline{x}} \), respectively. Then \( \lambda_x = \frac{1}{\overline{\lambda_x}} \).

It is interesting to notice that in the finite-dimensional case the point 0 for a skew-symmetric matrix and points \( \pm 1 \) for an orthogonal matrix are distinguished in a special manner in the spectrum, as well.

In the case of a unitary space \( U^n \) with a dimension \( n, n \in \mathbb{Z}_+ \), in an analogous manner, a conjugation \( J \), a \( J \)-form, and \( J \)-orthogonality are defined. So, the latter statements are true for complex symmetric, skew-symmetric and orthogonal matrices.

**Example 1.1.** Consider a numerical matrix \( A = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix} \). Its eigenvalues are \( \lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \), \( \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i \), and the corresponding to them normed eigenvectors are \( f_1 = \frac{1}{\sqrt{2}} \left( \sqrt{3} - i \right) \), \( f_2 = \frac{1}{\sqrt{2}} \left( -\sqrt{3} - i \right) \). Vectors \( f_1, f_2 \) are not orthogonal. However, they are \( J \)-orthogonal.

Let \( J \) be a conjugation in a Hilbert space \( H \) and \( A \) be a bounded linear operator in \( H \). The norm of \( A \), as it can be easily seen from the properties of the involution, can be calculated by the following formula

\[
\|A\| = \sup_{x,y \in H: \|x\| = \|y\| = 1} |[Ax, y]_J|.
\]

The following statement is true:

**Proposition 1.4** If \( A \) is a bounded \( J \)-symmetric operator in a Hilbert space \( H \), then its norm can be calculated as

\[
\|A\| = \sup_{x \in H: \|x\| = 1} |[Ax, x]_J|.
\]

**Proof.** Consider an operator \( A \) such as in the statement of the Proposition. Set \( C := \sup_{x \in H: \|x\| = 1} |[Ax, x]_J| \). For arbitrary elements \( x, y \in H : x \neq \pm y \) we can write

\[
[A(x + y), x + y]_J - [A(x - y), x - y]_J = 4[Ax, y]_J;
\]

\[
|[Ax, y]_J| \leq \frac{1}{4} (|[A(x + y), x + y]_J| + |[A(x - y), x - y]_J|) =
\]

\[
= \frac{1}{4} \left( \left| A\left( \frac{x + y}{\|x + y\|} \right), \frac{x + y}{\|x + y\|} \right|_J \right) \|x + y\|^2 + \left| A\left( \frac{x - y}{\|x - y\|} \right), \frac{x - y}{\|x - y\|} \right|_J \right)^*.
\]

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\[ \|x - y\|^2 \leq \frac{1}{4} C (\|x + y\|^2 + \|x - y\|^2) = \frac{1}{2} C (\|x\|^2 + \|y\|^2). \tag{14} \]

Thus, by using (12) and (14) we get
\[ \|A\| = \sup_{x, y \in H: \|x\| = \|y\| = 1} |\langle Ax, y \rangle_J| \leq C. \]

On the other hand, we can write
\[ C = \sup_{x, y \in H: \|x\| = 1} \|Ax\| \leq \sup_{x, y \in H: \|x\| = \|y\| = 1} \|Ax, y\|_J = \|A\|. \]

Therefore \( C = \|A\| \). \( \Box \)

For a J-skew-symmetric operator \( A \), its norm can not be calculated by the formula (13). Moreover, the following characteristic property of J-skew-symmetric operators is true.

**Proposition 1.5** A linear operator \( A \) in a Hilbert space \( H \) is J-skew-symmetric if and only if the following equality is true
\[ \langle Ax, x \rangle_J = 0, \quad x \in D(A). \tag{15} \]

**Proof.** We first notice that from the properties of an involution it follows that \( \langle x, y \rangle_J = \langle y, x \rangle_J, \quad x, y \in H \). Let us check the necessity. From relation (5) it follows that
\[ \langle Ax, x \rangle_J = -\langle x, Ax \rangle_J = -\langle Ax, x \rangle_J, \]
and therefore (15) holds true.

Let us check the sufficiency. By using (15) we write
\[ 0 = \langle A(x + y), x + y \rangle_J = \langle Ax, x \rangle_J + \langle Ax, y \rangle_J + \langle Ay, x \rangle_J + \langle Ay, y \rangle_J = \]
\[ = \langle Ax, y \rangle_J + \langle Ay, x \rangle_J, \quad x, y \in D(A). \]

From this relation we obtain that \( \langle Ax, y \rangle_J = -\langle Ay, x \rangle_J = -\langle x, Ay \rangle_J \). \( \Box \)

Let \( J \) be a conjugation in a Hilbert space \( H \) and \( A \) be an arbitrary linear operator in \( H \). The operator \( \tilde{A} := (A)_J := JAJ \) we shall call J-adjoint to the operator \( A \). We first note that \( \tilde{A} = A \) and the following easy to check lemma is true.

**Lemma 1.1** For a linear operator \( A \) in a Hilbert space \( H \), equalities \( \overline{D(A)} = H \) and \( D(\tilde{A}) = H \) are true or false simultaneously. The same can be said about equalities \( \overline{R(A)} = H \) and \( \overline{R(\tilde{A})} = H \).
An operation of the construction of the J-adjoint operator commutes with basic operations on operators. Let us formulate the necessary for us properties as propositions.

**Proposition 1.6** Let $A$ be a linear operator in a Hilbert space $H$ such that $D(A) = H$ and $J$ be a conjugation in $H$. Then the following relation is true

$$\tilde{A}^* = (\tilde{A})^*.$$  (16)

**Proof.** Choose an arbitrary element $g \in D((\tilde{A})^*)$. On one hand, it is true

$$(\tilde{A}x, g) = (x, (\tilde{A})^*g) = (JJx, J(\tilde{A})^*g) = (Jx, J(\tilde{A})^*g) =$$

$$= (J(\tilde{A})^*g, Jx), \quad x \in D(\tilde{A}).$$

On the other hand, we can write

$$(\tilde{A}x, g) = (JAJx, JJg) = (AJx, Jg) = (Jg, AJx), \quad x \in D(\tilde{A}).$$

Comparing right hand sides we obtain that

$$(AJx, Jg) = (Jx, J(\tilde{A})^*g),$$

and therefore $Jg \in D(A^*), A^*Jg = J(\tilde{A})^*g$. Multiplying by $J$ both sides of the latter equality we get $\tilde{A}^*g = (\hat{A})^*g$. Therefore

$$(\hat{A})^* \subseteq \tilde{A}^*.$$  (17)

In order to obtain the inverse inclusion one should write the inclusion (17) with the operator $\hat{A}$, and then to take J-adjoint operators for the both sides (the inclusion under the last operation will stay true). $\square$

**Proposition 1.7** Let $A$ be a linear operator in a Hilbert space $H$ and $J$ be a conjugation in $H$. Suppose that operators $A$ and $\tilde{A}$ admit closures. Then the following equality is true

$$\tilde{A} = \widehat{\tilde{A}}.$$  (18)

**Proof.** Choose an arbitrary element $g \in D(\tilde{A})$. Then there exists a sequence $x_n \in D(\tilde{A}), n \in \mathbb{Z}_+$, such that $x_n \rightarrow x, \tilde{A}x_n = JAJx_n \rightarrow \tilde{A}x$ when $n \rightarrow \infty$. By continuity of the operator $J$ from this it follows that

$$Jx_n \rightarrow Jx, \quad AJx_n \rightarrow J\tilde{A}x.$$
Consequently, we obtain, that \( Jx \in D(\overline{A}) \) \( \overline{A}x = JAx. \) Therefore \( x \in D(\overline{A}) \) and \( \overline{A}x = Ax. \) From this relation we conclude that
\[
\overline{A} \subseteq \overline{A}.
\] (19)

In order to obtain the inverse inclusion, we write the inclusion \[19\] for the operator \( \overline{A}, \) and then to take J-adjoint operators for the both sides. □

**Proposition 1.8** Let \( A \) be a linear invertible operator in a Hilbert space \( H \) and \( J \) be a conjugation in \( H. \) Then the operator \( \overline{A} \) is also invertible and the following equality is true
\[
\overline{A}^{-1} = (\overline{A})^{-1}.
\] (20)

**Proof.** Since \( \overline{A}^{-1}\overline{A} = E|_{D(\overline{A})}, \) and \( D(\overline{A}^{-1}) = JD(\overline{A}^{-1}) = JR(A) = R(\overline{A}), \) the operator \( \overline{A} \) is invertible and relation \[20\] is true. □

Notice that a condition of a \( J \)-symmetric operator \[4\] with the help of J-adjoint operator will be written as follows:
\[
(Ax, y) = (x, \tilde{A}y), \quad x \in D(A), \ y \in D(\tilde{A}).
\] (21)

Conditions of a \( J \)-skew-symmetric operator \[5\] and \( J \)-isometric operator \[6\] will be written as
\[
(Ax, y) = - (x, \tilde{A}y), \quad x \in D(A), \ y \in D(\tilde{A}),
\] (22)
and
\[
(Ax, \tilde{A}y) = (x, y), \quad x \in D(A), \ y \in D(\tilde{A}),
\] (23)
respectively.

Now we shall assume that the operator \( A \) is densely defined in \( H. \) Notice that in this case from condition \[23\] it follows that the operator \( A \) is invertible. In fact, equality \( Ax = 0 \) implies the equality \( (x, y) = 0 \) on a dense in \( H \) set \( D(\tilde{A}). \) Thus, a densely defined \( J \)-isometric operator is always invertible.

Note that in the case of a densely defined operator \( A, \) conditions \[21, 22, 23\] are equivalent to the following conditions
\[
A \subseteq (\tilde{A})^*,
\] (24)
\[
A \subseteq -(\tilde{A})^*,
\] (25)
and
\[
A^{-1} \subseteq (\tilde{A})^*,
\] (26)
respectively. From these relations, in particular, it immediately follows that densely defined J-symmetric and J-skew-symmetric operators admit closures. As it is seen from relations (1)–(5), their closures will also be J-symmetric or J-skew-symmetric operators, respectively. For a densely defined J-isometric operator one can only state that its inverse operator admits a closure. However, from relation (6) it is easily seen that the inverse operator to a J-isometric is also J-isometric. Consequently, if the range of the original J-isometric operator (the domain of the inverse operator) is also dense, then it admits a closure. In this case, also from the relation (6), it is seen that this closure will be a J-isometric operator.

Note that the operation of the construction of a J-adjoint operator does not change the defined above by us types of operators. Namely, the following proposition is true:

**Proposition 1.9** Let $A$ be a linear operator in a Hilbert space $H$ and $J$ be a conjugation in $H$. If the operator $A$ is J-symmetric, J-skew-symmetric or J-isometric, then the same is the operator $\tilde{A} = JAJ$, as well.

**Proof.** The statement about a J-symmetric (J-skew-symmetric, J-isometric) operator follows from relation (21) (22), (23), respectively, taking into account that $A = \tilde{A}$. □

For an element $x \in H$ and a set $M \subseteq H$ we write $x \perp_J M$, if $x \perp_J y$, for all $y \in M$. For a set $M \subseteq H$ we denote $M^\perp_J = \{ x \in H : x \perp_J y, y \in M \}$.

It is known that the residual spectrum of a J-self-adjoint operator is empty. It follows from the theorem below.

**Theorem 1.1** ([16, Theorem 4, p.87]) Let $A$ be a J-self-adjoint operator in a Hilbert space $H$. A complex number $\lambda$ is an eigenvalue of $A$ if and only if

$$\Delta_A(\lambda) \neq H.$$  \hspace{1cm} (27)

In this case, $(\Delta_A(\lambda))^\perp_J$ will be an eigen-subspace which corresponds to $\lambda$.

We shall obtain analogous results for J-skew-symmetric and J-isometric operators. The following theorem is true:

**Theorem 1.2** Let $A$ be a J-skew-self-adjoint operator in a Hilbert space $H$. A complex value $\lambda$ is an eigenvalue of $A$ if and only if

$$\Delta_A(-\lambda) \neq H.$$  \hspace{1cm} (28)

In this case, $(\Delta_A(-\lambda))^\perp_J$ will be an eigen-subspace which corresponds to $\lambda$. 

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Proof. Necessity. Let $x$ be an arbitrary eigenvector of the operator $A$ which corresponds to an eigenvalue $\lambda$. Since $A$, in particular, is skew-symmetric, then we can write for an arbitrary $y \in D(A)$

$$0 = [(A - \lambda E)x, y]_J = -[x, (A + \lambda E)y]_J.$$  \hspace{1cm} (29)

Therefore $x \perp_J \Delta_A(-\lambda)$ and by the continuity of $[,]_J$ we get

$$x \perp_J \overline{\Delta_A(-\lambda)}.$$  \hspace{1cm} (30)

Since $[x, Jx] = \|x\|^2 > 0$, then $Jx \notin \Delta_A(-\lambda)$ and therefore $\Delta_A(\lambda) \neq H$.

Sufficiency. Suppose that equality (28) is true. Then there exists $0 \neq y \in H$ such that

$$(z, y) = 0, \quad z \in \Delta_A(-\lambda).$$  \hspace{1cm} (31)

Therefore $((A + \lambda E)x, y) = 0$, and from this relation we get $(Ax, y) = (x, (\overline{-\lambda})y)$, $x \in D(A)$. Thus, we have $y \in D(A^*)$ and

$$A^*y = -\overline{\lambda}y.$$  \hspace{1cm} (32)

But since $A$ is $J$-skew-self-adjoint, then $A^* = -\overline{A}$, and we obtain

$$\overline{A}y = \overline{\lambda}y.$$  \hspace{1cm} (32)

From this relation it follows that $Jy \neq 0$ is an eigenvector of the operator $A$ with an eigenvalue $\lambda$.

Let us show that the following set

$$V(\lambda) := (\Delta_A(-\lambda))^\perp \setminus \{0\},$$  \hspace{1cm} (33)

is a set of eigenvectors of the operator $A$, corresponding to a eigenvalue $\lambda$. Denote the latter set by $S(\lambda)$. By the proven property (30), the inclusion $S(\lambda) \subseteq V(\lambda)$ is true. On the other hand, if $x \in V(\lambda)$, then for $y := Jx$ relation (31) is true. Repeating arguments which follow after this formula we come to a conclusion that $x$ is an eigenvector of the operator $A$, corresponding to $\lambda$. Thus, the inverse inclusion is also true.

Finally, since $A = (\overline{A})^*$, then $A$ is closed. Therefore $(\Delta_A(-\lambda))^\perp_J$ is an eigen-subspace of the operator $A$, which corresponds to $\lambda$. \qed

Corollary 1.1 The point $0$ can not belong to the residual spectrum of a $J$-skew-self-adjoint operator.

In an analogous manner, the following result for $J$-unitary operators is established.
Theorem 1.3 Let \( A \) be a J-unitary operator in a Hilbert space \( H \). A complex number \( \lambda \) is an eigenvalue of \( A \) if and only if
\[
\Delta_A \left( \frac{1}{\lambda} \right) \neq H.
\]
In this case, \( \left( \Delta_A \left( \frac{1}{\lambda} \right) \right)^\perp \) is an eigen-subspace, which corresponds to \( \lambda \).

Corollary 1.2 Points \( \pm 1 \) can not belong to the residual spectrum of a J-unitary operator.

From relations (21),(22) it is seen that a defined in the whole \( H \) J-symmetric (J-skew-symmetric) operator is a bounded J-self-adjoint (respectively J-skew-self-adjoint) operator. The following statements are also true.

Proposition 1.10 Let \( A \) be a linear densely defined operator in a Hilbert space \( H \), which is J-symmetric (J-skew-symmetric). Suppose that \( R(A) = H \). Then the operator \( A \) is a J-self-adjoint (respectively J-skew-self-adjoint) operator.

Proposition 1.11 Let \( A \) be a linear densely defined operator in a Hilbert space \( H \), which is J-symmetric (J-skew-symmetric). Suppose that \( R(A) = H \). Then the operator \( A \) is invertible and the operator \( A^{-1} \) is also a J-symmetric (respectively J-skew-symmetric) operator.

Proof. In a view of analogous considerations, we shall check the validity of this Proposition only for the case of a J-skew-symmetric operator \( A \). Notice that \( \text{Ker} A^* = H \ominus \overline{R(A)} = \{0\} \). Thus, the operator \( A^* \) is invertible. Since \( A \) is J-skew-symmetric, the following inclusion is true \( \tilde{A} \subseteq -A^* \) and therefore \( \tilde{A} \) is invertible, as well. By Proposition 1.8 we conclude that the operator \( A \) has an inverse operator. From the inclusion \( \tilde{A} \subseteq -A^* \) it follows the following inclusion
\[
(\tilde{A})^{-1} \subseteq -(A^*)^{-1}.
\]
Notice that \( \overline{D(A^{-1})} = \overline{R(A)} = H \). Thus, we can state that \( (A^*)^{-1} = (A^{-1})^* \). Using this equality and using Proposition 1.8 from relation (35) we obtain the following inclusion
\[
\tilde{A}^{-1} \subseteq -(A^{-1})^*.
\]
And this means that the operator \( A^{-1} \) is J-skew-symmetric. □
Proposition 1.12 Let $A$ be a J-self-adjoint (J-skew-self-adjoint) operator in a Hilbert space $H$. Suppose that $\overline{R(A)} = H$. Then the operator $A$ is invertible and the operator $A^{-1}$ is also J-self-adjoint (respectively J-skew-self-adjoint) operator.

Proof. In view of analogous considerations, we shall give the proof only for the case of J-self-adjoint operator $A$. By Proposition 1.11 the operator $A$ is invertible. By Proposition 1.8 the operator $\tilde{A}$ is invertible, as well. From Lemma 1.1 it follows that $R(\tilde{A}) = H$, $D(\tilde{A}) = H$. Thus, we have $D((A)^{-1}) = H$. Consequently, the following equality is true $((\tilde{A})^*)^{-1} = ((A)^{-1})^*$. Since the operator $A$ is J-self-adjoint, the last equality can be written as $A^{-1} = ((\tilde{A})^{-1})^*$. Using Proposition 1.8 we obtain the following equality $A^{-1} = (A^{-1})^*$, which shows that the operator $A^{-1}$ is J-self-adjoint. □

2 A J-polar decomposition of bounded operators.

We shall extend in the case of J-symmetric, J-skew-symmetric and J-isometric operators a seria of properties of finite-dimensional complex symmetric, skew-symmetric and orthogonal matrices (see [1]).

The following lemma is true:

Lemma 2.1 Let $A$ be a bounded self-adjoint and J-isometric operator in a Hilbert space $H$. Then the operator $A$ admits the following representation:

$$A = Ie^{iK},$$

(36)

where $I$ is a bounded self-adjoint J-real involutory ($I^2 = E$) operator in $H$, and $K$ is a commuting with $I$ bounded skew-self-adjoint J-real operator in $H$.

If additionally it is known that the operator $A$ is positive, $A \geq 0$, then one can choose $I = E$.

Proof. Consider an operator $A$ such as in the statement of the Lemma. Since the operator $A$ is J-isometric and bounded, then from (6) we obtain $A^*JA = J$, or $A^*\tilde{A} = E$. Since $A$ is self-adjoint, then

$$A\tilde{A} = E.$$  

(37)

For the operator $A$ we can write the following representation

$$A = S + iT,$$  

(38)
where \( S = \frac{1}{2}(A + \tilde{A}) \), \( T = \frac{1}{2i}(A - \tilde{A}) \). By this, operators \( S \) and \( T \) are \( J \)-real, the operator \( S \) is self-adjoint and \( J \)-self-adjoint, and the operator \( T \) is skew-self-adjoint and \( J \)-skew-self-adjoint. Since \( \tilde{A} = S - iT \), then from relation (37) we get

\[
E = A\tilde{A} = (S + iT)(S - iT) = S^2 + T^2 + i(TS - ST).
\]

From this relation it follows that operators \( T \) and \( S \) commute and

\[
S^2 + T^2 = E. \tag{39}
\]

Since operators \( S \) and \( iT \) are commuting bounded self-adjoint operators, then they admit spectral representations

\[
S = \int_L \lambda dE_\lambda, \quad iT = \int_L z dF_z, \tag{40}
\]

where \( E_\lambda, F_z \) are commuting resolutions of unity of the operators, and \( L = (l_1, l_2), l_1, l_2 \in \mathbb{R} \), is a finite interval of the real line which contains the spectra of operators. From equality (39), by using spectral resolutions we get

\[
\int_L \int_L (\lambda^2 - z^2 - 1)dE_\lambda dF_z = 0, \tag{41}
\]

where the integral means a limit in the norm of \( H \) of the corresponding Riemann-Stieltjes type sums (in the plane).

A point \( (\lambda_0, z_0) \in \mathbb{R}^2 \) we call a point of increase for the measure \( dE_\lambda dF_z \), if for an arbitrary number \( \varepsilon > 0 \), there exists an element \( x \in H \) such that

\[
(E_{\lambda_0 + \varepsilon} - E_{\lambda_0 - \varepsilon})(F_{z_0 + \varepsilon} - F_{z_0 - \varepsilon})x \neq 0, \tag{42}
\]

or, equivalently,

\[
((E_{\lambda_0 + \varepsilon} - E_{\lambda_0 - \varepsilon})(F_{z_0 + \varepsilon} - F_{z_0 - \varepsilon})x, x) > 0. \tag{43}
\]

For an arbitrary point of increase \( (\lambda_0, z_0) \in \mathbb{R}^2 \) of the measure \( dE_\lambda dF_z \) it is true

\[
\lambda_0^2 - z_0^2 - 1 = 0. \tag{44}
\]

In fact, if the latter equality is not true for a point of increase \( u_0 = (\lambda_0, z_0) \in \mathbb{R}^2 \), then \( |\lambda^2 - z^2 - 1| \geq a, a > 0, \) in a neighborhood \( U = U(\lambda_0, z_0; \varepsilon) = \{(\lambda, z) \in \mathbb{R}^2 : \lambda_0 - \varepsilon < \lambda < \lambda_0 + \varepsilon, z_0 - \varepsilon < z < z_0 + \varepsilon, \varepsilon > 0, \) of the point
For this number \( \varepsilon \), there exists an element \( x \in H \) such that (43) is true. But

\[
0 = \| \int_L \int_L (\lambda^2 - z^2 - 1) dE_\lambda dF_z x \|^2 = \int_L \int_L |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x) \geq \int \int_U |\lambda^2 - z^2 - 1|^2 (dE_\lambda dF_z x, x) \geq a^2 ((E_{\lambda_0 + \varepsilon} - E_{\lambda_0 - \varepsilon})(E_{z_0 + \varepsilon} - E_{z_0 - \varepsilon}) x, x) > 0.
\]

If two continuous functions \( \varphi(\lambda, z) \) \( \hat{\varphi}(\lambda, z) \) on \( L^2 = \{(\lambda, z) \in \mathbb{R}^2 : \lambda, z \in L\} \) coincide in the points of increase of the measure \( dE_\lambda dF_z \), then

\[
\int_L \int_L \varphi(\lambda, z) dE_\lambda dF_z = \int_L \int_L \hat{\varphi}(\lambda, z) dE_\lambda dF_z. \quad (45)
\]

In fact,

\[
\| \int_L \int_L (\varphi(\lambda, z) - \hat{\varphi}(\lambda, z)) dE_\lambda dF_z x \|^2 = \int_L \int_L |\varphi(\lambda, z) - \hat{\varphi}(\lambda, z)|^2 (dE_\lambda dF_z x, x),
\]

and it remains to notice that \( (dE_\lambda dF_z x, x) \) is a positive measure on \( L^2 \), and the function under the integral is equal to zero in all points of increase of this measure.

Consider a set \( \Gamma \subset \mathbb{R}^2 \), which consists of points \( (\lambda, z) \in \mathbb{R}^2 \), such that

\[
\lambda^2 - z^2 - 1 = 0. \quad (46)
\]

From (46) it follows that for all points of the set \( \Gamma \) it is true \( |\lambda| = \sqrt{1 + z^2} \) (where we mean the arithmetic value of the root). Hence, for all points of \( \Gamma \)

\[
\lambda = \text{sgn}(\lambda) \sqrt{1 + z^2}, \quad (47)
\]

where

\[
\text{sgn}(\lambda) = \begin{cases} 
1, & \lambda > 0, \\
-1, & \lambda \leq 0
\end{cases} \quad (48)
\]

By the identity \( z = \text{sh } \text{arcsh } z \), the equality (47) can be rewritten in the following form

\[
\lambda = \text{sgn}(\lambda) \sqrt{\text{ch}^2(\text{arcsh } z)} = \text{sgn}(\lambda) \text{ch}(\text{arcsh } z), \quad (49)
\]

in a view of positivity of the hyperbolic cosine function. By this representation we can write

\[
A = S + iT = \int_L \int_L (\lambda + z) dE_\lambda dF_z = \int_L \int_L (\text{sgn}(\lambda) \text{ch}(\text{arcsh } z) + z) dE_\lambda dF_z =
\]
\[
\int_{L} \int_{L_{+}} e^{\text{arcsh} z} dE dF_z + \int_{L} \int_{L_{-}} (-e^{-\text{arcsh} z}) dE dF_z, \quad (50)
\]
where \( L_{+} = (0, \infty) \cap L, \ L_{-} = (-\infty, 0] \cap L. \)

Define the following operator

\[
V = \int_{L} \int_{L} \text{sgn}(\lambda) \text{arcsh} z dE dF_z = \int_{L} \text{sgn}(\lambda) dE \int_{L} \text{arcsh} z dF_z. \quad (51)
\]

The operator \( V \) is bounded self-adjoint and J-imaginary. In fact, since the operator \( S \) is J-real, then its resolution of unity \( E_\lambda \) commutes with \( J \) (see [13]). Therefore the operator

\[
I := \int_{L} \text{sgn}(\lambda) dE_\lambda, \quad (52)
\]
is a bounded J-real self-adjoint involutory operator. On the other hand, \( \text{arcsh}(iT) = \sum_{k=0}^{\infty} a_{2k+1}(iT)^{2k+1}, \ a_{2k+1} \in \mathbb{R}, \) is a J-imaginary, as a limit of J-imaginary operators (here the convergence is understood in the norm of \( H \)).

From relations (50), (51), (52) we conclude that

\[
A = I e^V.
\]

Set \( K = -iV, \) and we obtain the required representation (36).

If it is additionally known that the operator \( A \) is positive, \( A \geq 0, \) then

\[
I = A e^{-V} = (e^{-\frac{V}{2}})^* A e^{-\frac{V}{2}},
\]
is positive, as well. Therefore \( I \) is a positive square root of \( E. \) By the uniqueness of such a root we conclude that \( I = E. \) \( \Box \)

The following theorem is true.

**Theorem 2.1** Let \( A \) be a bounded J-unitary operator in a Hilbert space \( H. \)

The operator \( A \) admits the following representation:

\[
A = R e^{iK}, \quad (53)
\]

where \( R \) is J-real unitary operator in \( H, \) and \( K \) is a bounded J-real skew-self-adjoint operator in \( H. \)

**Proof.** Consider an operator \( A \) such as in the statement of the Theorem. Suppose that representation (53) is true. Then

\[
A^* A = e^{iK} R^* R e^{iK} = e^{2iK}.
\]

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Now we shall do not assume an existence of representation \((53)\) and notice that the operator \(G := A^* A\) is positive self-adjoint and \(J\)-unitary. In fact, since the operator \(A\) is bounded by assumption and \(J\)-unitary, then \(A^*\) is also bounded and \(J\)-unitary. A product of bounded \(J\)-unitary operators is a bounded \(J\)-unitary operator, this is verified directly by the definition. By Lemma 2.1 we find a bounded \(J\)-real skew-self-adjoint operator \(K\) such that

\[
G = e^{2iK}. \tag{54}
\]

Now set by definition

\[
R = Ae^{-iK}. \tag{55}
\]

By equality \((54)\) we can write

\[
R^* R = e^{-iK} A^* A e^{-iK} = E,
\]

and, hence, the operator \(R\) is unitary. Now notice that

\[
J e^{-iK} J = J (\cos(iK) - i \sin(iK)) J = \cos(iK) + i \sin(iK),
\]

since the operator \(iK\) is \(J\)-real and therefore its resolution of unity commutes with \(J\). Consequently, we have

\[
J e^{-iK} J = e^{iK} = (e^{-iK})^{-1}, \tag{56}
\]

and the operator \(e^{-iK}\) is \(J\)-unitary. By \((55), (56)\) and using that the operator \(A\) is \(J\)-unitary we conclude that

\[
R^{-1} = e^{iK} A^{-1} = J e^{-iK} J A^* J = J (A e^{-iK})^* J = (\tilde{R}^*),
\]

and therefore the operator \(R\) is \(J\)-unitary. Then \(R^{-1} = R^* = J R^* J\), and therefore \(R^*\) is a \(J\)-real operator. Using matrix representations of operators \(R^*\) and \(R\) in an arbitrary basis, which corresponds to the involution \(J\), we conclude that the operator \(R\) is \(J\)-real. \(\Box\)

**Lemma 2.2** Let \(A\) be a \(J\)-self-adjoint and unitary operator in a Hilbert space \(H\). The operator \(A\) admits the following representation:

\[
A = e^{iS}, \tag{57}
\]

where \(S\) is a bounded \(J\)-real self-adjoint operator in \(H\).
Proof. Consider an operator $A$ such as in the statement of the Lemma. For the $J$-self-adjoint operator $A$ it is true $A^* = \tilde{A}$, and we can write the following representation

$$A = S + iT,$$

(58)

where $S = \frac{1}{2}(A + \tilde{A}) = \frac{1}{2}(A + A^*)$, $T = \frac{1}{2i}(A - \tilde{A}) = \frac{1}{2i}(A - A^*)$. Here operators $S$ and $T$ are $J$-real and self-adjoint. Since the operator $A$ is unitary, then

$$E = A^*A = (S - iT)(S + iT) = S^2 + T^2 + i(ST - TS).$$

From this relation it follows that operators $T$ and $S$ commute and

$$S^2 + T^2 = E.$$

(59)

Since operators $S$ and $T$ are commuting bounded self-adjoint operators, then they admit the following spectral resolutions

$$S = \int_L \lambda dE_\lambda, \quad T = \int_L z dF_z,$$

(60)

where $E_\lambda$, $F_z$ are commuting resolutions of unity of operators, and $L = (l_1, l_2]$, $l_1, l_2 \in \mathbb{R}$, is a finite interval of the real line, which contains the spectra of operators. Moreover, since operators $S$ and $T$ are $J$-real, then their resolutions of unity commute with $J$. By equality (59) and using spectral resolutions we get

$$\int_L \int_L (\lambda^2 + z^2 - 1) dE_\lambda dF_z = 0,$$

(61)

where the integral means a limit in the norm of $H$ of the corresponding Riemann-Stieltjes type sums. Thus, in all points of increase of the measure $dE_\lambda dF_z$ the following relation is true

$$\lambda^2 + z^2 - 1 = 0.$$

(62)

A circle (62) in the plane $\mathbb{R}^2$ we denote by $\Gamma$. For all points of the circle $\Gamma$ it is true $|z| = \sqrt{1 - \lambda^2}$ (where we mean the arithmetic value of the root). Therefore for all points of $\Gamma$

$$z = \text{sgn}(z) \sqrt{1 - \lambda^2},$$

(63)

where $\text{sgn}(\cdot)$ is from (48). By the identity $\lambda = \cos \arccos \lambda$, $\lambda \in [-1, 1]$, the equality (63) can be rewritten in the following form

$$z = \text{sgn}(z) \sqrt{\sin^2(\arccos \lambda)} = \text{sgn}(z) \sin(\arccos \lambda),$$

(64)
where we have used the positivity of sine function on $[0, \pi]$. By this representation we can write

$$A = S + iT = \int_L \int_L (\lambda + iz) dE_{\lambda} dF_z = \int_L \int_L (\cos \arccos \lambda + i \text{sgn}(z) * \sin(\arccos \lambda)) dE_{\lambda} dF_z$$

\hspace{1cm}

$$\text{sgn}(\lambda) dE_{\lambda} dF_z = \int_L \int_L e^{i \arccos \lambda} dE_{\lambda} dF_z + \int_L \int_L e^{-i \arccos \lambda} dE_{\lambda} dF_z,$$

where $L_+ = (0, \infty) \cap L$, $L_- = (-\infty, 0] \cap L$. Define the following operator

$$S := \int_L \int_L \text{sgn}(z) \arccos \lambda dE_{\lambda} dF_z = \int_L \text{sgn}(z) dF_z \int_L \arccos \lambda dE_{\lambda}.$$  

(66)

It is obvious that $S$ is a J-real self-adjoint operator. From relation (65) it is seen that (57) is true. \[ \square \]

Using the proven lemma we shall establish the following theorem.

**Theorem 2.2** Let $A$ be a unitary operator in a Hilbert space $H$. The operator $A$ admits the following representation:

$$A = Re^{iS},$$

(67)

where $R$ is J-real unitary operator in $H$, and $S$ is a bounded J-real self-adjoint operator in $H$.

**Proof.** Consider an operator $A$ such as in the statement of the Theorem. Suppose that representation (67) is true. Then $A^* = e^{-iS} R^*$

$$\hat{A}^* = e^{-iS} R^* = J (\cos S - i \sin S) J R^* = (\cos S + i \sin S) R^* = e^{iS} R^*,$$

since $S$ and $R$ are J-real. Since $R$ is unitary, we can write

$$\hat{A}^* A = e^{iS} R^* R e^{iS} = e^{2iS}. \hspace{1cm} (68)$$

Now we shall not suppose that representation (67) holds true. Since the operator $A$ is unitary, then operators $A^{-1} = A^*$, $JA^* J$ and $G := \hat{A}^* A$ are unitary, as well. The operator $G$ is J-self-adjoint since $G^* = A^* \hat{A} = JA^* AJ = \hat{G}$. Applying to this operator Lemma 2.2 we find J-real self-adjoint operator $S$ such that

$$G = e^{2iS}.$$ 

(69)

Now we set by definition

$$R = Ae^{-iS}. \hspace{1cm} (70)$$

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The operator $R$ is unitary as a product of two unitary operators. Then we can write $R^* = Je^{iS}A^*J = e^{-iS}A^*$, and therefore
\[
\tilde{R}^* R = e^{-iS}A^*e^{-iS} = e^{-iS}Ge^{-iS} = E.
\]
Since the range of a unitary operator $R$ is the whole $H$, then by the latter equality we get $R^* = R^{-1}$. Thus, the operator $R$ is $J$-unitary. Since the operator $R$ is unitary and $J$-unitary, it is $J$-real. From (70) it follows the representation (67). □

Let $A$ be a linear bounded operator in a Hilbert space $H$ and $J$ be a conjugation in $H$. It is easy to verify that operators $A^T A = JA^*JA$, $AA^T = AJA^*J$ are bounded $J$-self-adjoint operators. If $A^T A = AA^T$, then the operator $A$ we shall call $J$-normal. It is clear that bounded $J$-self-adjoint, $J$-skew-self-adjoint and $J$-unitary operators are $J$-normal. The following theorem is true:

**Theorem 2.3** Let $A$ be a linear bounded operator in a Hilbert space $H$ and $0 \notin \sigma(A)$. Let $J$ be a conjugation in $H$. Suppose that the spectrum of the operator $AA^T$ has an empty intersection with a radial ray $L_\varphi = \{ z \in \mathbb{C} : z = xe^{i\varphi}, \ x \geq 0 \}$ ($\varphi \in [0, 2\pi]$) in the complex plane. Then the operator $A$ admits a representation
\[
A = SU, \quad (71)
\]
where $S$ is a bounded $J$-self-adjoint operator in $H$, and $U$ is a bounded $J$-unitary operator in $H$. Here
\[
S = \sqrt{AA^T}, \quad (72)
\]
where the square root is understood according to the Riss calculus. Operators $U$ and $S$ commute if and only if the operator $A$ is $J$-normal. Moreover, the operator $A$ admits a representation
\[
A = U_1S_1, \quad (73)
\]
where $U_1$ is a bounded $J$-unitary operator in $H$, and $S_1 = \sqrt{A^TA}$ is a bounded $J$-self-adjoint operator in $H$. Operators $U_1$ and $S_1$ commute if and only if $A$ is $J$-normal.

In particular, representations (71) and (73) are true for operators
\[
A = E + K, \quad (74)
\]
where $K$ is a compact operator in $H$, $\|K\| < 1$. 

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Proof. Consider an operator $A$ such as in the statement of the Theorem. We set by definition

$$S = \sqrt{AA^T} = \int_\Gamma \sqrt{\lambda} R_\lambda(AA^T) \, d\lambda. \quad (75)$$

A contour $\Gamma$ is constructed in the following way. Let $T_R = \{ z \in \mathbb{C} : |z| = R \}$ be a circle, which contains $\sigma(AA^T)$ inside, $R > 0$. Let $d > 0$ be a distance between a closed set $\sigma(AA^T)$ and a segment $[0, Re^{i\varphi}]$, where $\varphi$ is from the statement of the Theorem. Consider parallel segments on the distance $\frac{d}{2}$ of the above segment, join them by a half of a circle in a neighborhood of zero and completing the contour with a part of big circle $T_R$, it is not hard to construct a contour $\Gamma$, which contains the spectrum of the operator $AA^T$ inside, but do not contain the ray $L_\varphi$ inside. We choose and fix an arbitrary analytic branch of the root in $\mathbb{C}\setminus L_\varphi$.

A bounded operator $B := AA^T$ is $J$-self-adjoint, as it was noticed above. Consequently, its resolvent is also a $J$-self-adjoint operator. In fact, we can write

$$R_\lambda^*(B) = ((B - \lambda E)^{-1})^* = (B^* - \overline{\lambda}E)^{-1} = (\overline{B} - \overline{\lambda}E)^{-1} = (J(B - \lambda E)J)^{-1} = J(B - \lambda E)^{-1}J = JR_\lambda(B)J, \quad \lambda \in \rho(B).$$

The operator $S$ is $J$-self-adjoint, as a limit of $J$-self-adjoint integral sums. Moreover, there exists an inverse operator $S^{-1}$, which is also $J$-self-adjoint. Set

$$U = S^{-1}A, \quad (76)$$

and notice that $U^{-1} = A^{-1}S$ (recall that $0 \notin \sigma(A)$). Then

$$U\tilde{U}^* = S^{-1}AA^*(S^{-1})^* = S^{-1}S^2S^{-1} = E.$$ 

Multiplying the latter equality from the left side by $U^{-1}$ we get

$$\tilde{U}^* = U^{-1}.$$ 

Thus, the operator $U$ is $J$-unitary.

Suppose now that in representation (71) operators $U$ and $S$ commute. Then

$$AA^T = SU(\tilde{U}^*)(S^*) = S^2,$$

$$A^TA = (\tilde{U}^*)SSU = S^2.$$
Conversely, if operators \( A \) and \( A^T \) commute, then using last relations (without the latter equality) we write:

\[
S^2 = (U^*) S^2 U = U^{-1} S^2 U, \\
US^2 = S^2 U. 
\] (77)

Since \( U \) commutes with \( S^2 \), then it commutes with an arbitrary function of this operator. In particular, \( U \) commutes with \( S \).

We shall now establish a possibility of resolution (73) for the operator \( A \). First of all we notice that for an arbitrary linear bounded operator \( D \) in \( H \) we can write

\[
JR^*_\lambda(D)J = J(D^* - \overline{\lambda}E)^{-1}J = (JD^*J - \lambda E)^{-1} = R_\lambda(D^T), \quad \lambda \in \rho(D).
\]

Therefore

\[
\rho(D) = \rho(D^T), 
\] (78)

for an arbitrary linear bounded operator \( D \) in \( H \). Using this equality for operators \( A \) and \( AA^T \) we conclude that \( 0 \notin \sigma(A^T) \) and the ray \( L_\phi \) does not intersect with the spectrum of the operator \( A^T A \). Applying the proven part of the Theorem with the operator \( A^T \), we shall get a resolution \( A^T = SU \), where \( S = \sqrt{A^T} \) is a bounded \( J \)-self-adjoint operator, \( U \) is a bounded \( J \)-unitary operator. Therefore

\[
A = \bar{U}^* \tilde{S}^* = U^{-1} S,
\]

and it remains to notice that \( U^{-1} \) is a bounded \( J \)-unitary operator.

If the operator \( A \) has the form (74), then \( 0 \notin \sigma(A) \) and

\[
AA^T = (E + K)J(E + K^*)J = E + C, \quad \text{(79)}
\]

where \( C := K + JK^*J + KJK^*J \). Notice that the operator \( C \) is compact as a sum of compact operators. The operator \( J(E + K^*)^{-1}J(E + K)^{-1} \), as it is easy to see, is the inverse operator for the operator \( AA^T \). Therefore \( 0 \notin \sigma(AA^T) \). Since the spectrum of a compact operator \( C \) is discrete, having a unique point of concentration \( 0 \), one can find a ray which is required in the statement of the Theorem. □

3 Matrix representations of \( J \)-symmetric and \( J \)-skew-symmetric operators.

We shall now turn to a study of matrix representations of \( J \)-symmetric and \( J \)-skew-symmetric operators. Properties which are analogous to the properties
of symmetric operators are valid here. Let $J$ be a conjugation in a Hilbert space $H$ and $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}^+}$ be an orthonormal basis in $H$, which corresponds to $J$. Let $A$ be a linear operator in $H$, which is $J$-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$.

Define a matrix of the operator $A$ in the basis $\mathcal{F}$: $A_M := (a_{i,j})_{i,j \in \mathbb{Z}^+}$, $a_{i,j} = (Af_f, f_i)$. It is not hard to verify that this matrix is complex symmetric (skew-symmetric) in the case of $J$-symmetric (respectively J-skew-symmetric) operator $A$. Notice that the columns of this matrix are square summable, i.e. belong to $l^2$.

It is known that for an arbitrary linear operator $A$ in a Hilbert space $H$, in the case when the set $D(A) \cap D(A^*)$ is dense in $H$, the action of the operator $A$ is given by a matrix multiplication [13]. In particular, it is true for symmetric operators. As far as we know, for other classes of operators a possibility to describe the action of the operator as a matrix multiplication was not established earlier. This property possess J-symmetric and J-skew-symmetric operators, as it shows the following theorem.

**Theorem 3.1** Let $J$ be a conjugation in a Hilbert space $H$ and $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}^+}$ be an orthonormal basis in $H$, which corresponds to $J$. Let $A$ be a linear operator in $H$, which is $J$-symmetric (J-skew-symmetric) and such that $\mathcal{F} \subset D(A)$. Let $A_M = (a_{i,j})_{i,j \in \mathbb{Z}^+}$ be a matrix of the operator $A$ in the basis $\mathcal{F}$. Then

\[ Ag = \sum_{i=0}^{\infty} y_i f_i, \quad y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \quad g = \sum_{k=0}^{\infty} g_k f_k \in D(A). \quad (80) \]

**Proof.** Let us verify the validity of the statement of the Theorem for J-skew-symmetric operator. For the case of J-symmetric operator the proof is analogous. Choose an arbitrary element $g = \sum_{k=0}^{\infty} g_k f_k \in D(A)$. Using that the matrix $A_M$ is skew-symmetric and using relation [13] we write

\[ y_i = (Ag, Jf_i) = -(Af_i, Jg) = -\left(\sum_{k=0}^{\infty} (Af_i, f_k) f_k, \sum_{l=0}^{\infty} g_l f_l\right) = \]

\[ = -\sum_{k=0}^{\infty} (Af_i, f_k) g_k = -\sum_{k=0}^{\infty} a_{i,k} g_k = \sum_{k=0}^{\infty} a_{i,k} g_k. \]

\[ \square \]

Let us find out, how strong the matrix $A_M$ of the operator $A$ (considered above) determines the operator $A$. Since J-symmetric and J-skew-symmetric
operators admit closures, which are also J-symmetric (respectively J-skew-
symmetric) operators, we shall already suppose that the operator \( A \) is closed.

By the matrix \( A_M \) one can define, as a matrix multiplication, an operator 
\( T \) on \( L := \text{Lin} \mathcal{F} \). It is easy to check that this operator is J-symmetric (J-
skew-symmetric) in the case of J-symmetric (respectively J-skew-symmetric) 
operator \( A \). This operator admits a closure \( \overline{T} \), which is also a J-symmetric 
(J-skew-symmetric) operator. If \( A = \overline{T} \), then the basis \( \mathcal{F} \) we shall call 
a basis of the matrix representation of the operator \( A \).

A question appears: If for every complex symmetric (skew-symmetric) 
semi-infinite matrix \( B \) with square summable columns there exists a J-
symmetric (respectively J-skew-symmetric) operator \( A \) such that the 
matrix \( B \) will be a matrix of the operator in a corresponding to \( J \) basis \( \mathcal{F} \), and 
also \( \mathcal{F} \) will be a basis of the matrix representation for the operator \( A \)? The 
answer on this question is affirmative.

**Theorem 3.2** Let an arbitrary complex semi-infinite symmetric (skew-symmetric) 
matrix \( M = (m_{i,j})_{i,j \in \mathbb{Z}^+} \) with columns in \( l^2 \) is given. Then there exist a 
Hilbert space \( H \), a conjugation \( J \) in \( H \), a J-symmetric (respectively J-skew-
symmetric) operator in \( H \), a corresponding to \( J \) orthonormal basis \( \mathcal{F} \) in \( H \), 
\( \mathcal{F} \subset D(\tilde{A}) \), such that the matrix \( M \) is a matrix of the operator \( A \) in the 
basis \( \mathcal{F} \) and \( \mathcal{F} \) is a basis of the matrix representation for \( A \).

**Proof.** For an arbitrary complex semi-infinite symmetric (skew-symmetric) 
matrix \( M \) with columns in \( l^2 \) it is enough to choose an arbitrary Hilbert 
space \( H \), an arbitrary orthonormal basis \( \mathcal{F} \) in it and to define a conjugation 
in \( H \) by formula (2). Then, by using the described above procedure, one 
constructs an operator \( \overline{T} \), which is the required operator. \( \square \)

Notice that, if \( \mathcal{F} \) is a basis of the matrix representation for a closed 
J-symmetric (J-skew-symmetric) operator \( A \), then \( \mathcal{F} \) will be a basis of the 
matrix representation for the J-adjoint operator \( \tilde{A} = JA J \), as well. In 
fact, the operator \( \tilde{A} \) is J-symmetric (respectively J-skew-symmetric) by 
Proposition [1.9]. From the continuity of the operator \( J \) it follows that 
\( \tilde{A} \) is closed. Then if we choose an arbitrary element \( x \in D(\tilde{A}) \), then 
\( Jx \in D(A) \) and there exists a sequence \( \tilde{x}_n \in L := \text{Lin}\{f_k\}_{k \in \mathbb{Z}^+}, \ n \in \mathbb{Z}^+: \) 
\( \tilde{x}_n \rightarrow Jx, A\tilde{x}_n \rightarrow AJx, \ n \rightarrow \infty \). But then we have 
\( J\tilde{x}_n \in L, J\tilde{x}_n \rightarrow x, \) 
\( JA\tilde{x}_n \rightarrow \tilde{A}Jx = \tilde{A}x, n \rightarrow \infty \).

The following theorem is true:

**Theorem 3.3** Let \( J \) be a conjugation in a Hilbert space \( H \) and \( \mathcal{F} = \{f_k\}_{k \in \mathbb{Z}^+} \) 
is a corresponding to \( J \) orthonormal basis in \( H \). Suppose that \( A \) is a closed J-
symmetric (J-skew-symmetric) operator in \( H \), \( \mathcal{F} \subset D(A) \), and \( \mathcal{F} \) is a basis of
the matrix representation for the operator $A$. Let $a_{i,j} = (Af_j, f_i), \ i, j \in \mathbb{Z}_+$. Define an operator $B$ in the following way:

$$Bg = \sum_{i=0}^{\infty} y_i f_i, \ y_i = \sum_{k=0}^{\infty} a_{i,k} g_k, \ g = \sum_{k=0}^{\infty} g_k f_k \in DB, \quad (81)$$

on a set $DB = \{g = \sum_{k=0}^{\infty} g_k f_k \in H : \sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} a_{i,k} g_k|^2 < \infty\}$. Then $A \subseteq A^T = B$ (respectively $A \subseteq -A^T = B$).

Without conditions that $A$ is closed and $F$ is a basis of the matrix representation for $A$, one can only state that $A \subseteq A^T \subseteq B$ (respectively $A \subseteq -A^T \subseteq B$).

**Proof.** The proof will be given in the case of J-skew-self-adjoint operator $A$. The case of J-symmetric operator is considered analogously. We first show that $-A^T = -(A)^* \subseteq B$. Choose an arbitrary $g \in D(-(A)^*)$ and set $-(A)^* g = g^*$. Let $g = \sum_{k=0}^{\infty} g_k f_k$, $g^* = \sum_{i=0}^{\infty} \tilde{y}_i f_i$. We can write

$$\tilde{y}_i = (g^*, f_i) = (-\tilde{A}^* g, f_i) = -(g, \tilde{A} f_i) = -(\sum_{k=0}^{\infty} g_k f_k, \sum_{j=0}^{\infty} (\tilde{A} f_i, f_j) f_j) =$$

$$= -\sum_{k=0}^{\infty} g_k (A f_i, f_k) = -\sum_{k=0}^{\infty} a_{i,k} g_k = \sum_{k=0}^{\infty} a_{i,k} g_k, \ i \in \mathbb{Z}_+.$$ 

Therefore $\sum_{i=0}^{\infty} |\sum_{k=0}^{\infty} a_{i,k} g_k|^2 < \infty$ and, hence, we get $g \in DB$. Also we have $-(A)^* g = g^* = Bg$. Thus, we obtain an inclusion $-(A)^* \subseteq B$. Here we did not use that $A$ is closed and that $F$ is a basis of the matrix representation for $A$. The inclusion $A \subseteq -(A)^*$ is obvious.

Let us prove the inclusion $B \subseteq -A^T$. As it was shown above, the operator $\tilde{A}$ is closed and $F$ is a basis of the matrix representation for $\tilde{A}$, as well. Choose an arbitrary $g \in DB, \ g = \sum_{k=0}^{\infty} g_k f_k$. Using the fact that the matrix of the operator $A$ is skew-symmetric, we write

$$(\tilde{A} f_i, g) = (\sum_{j=0}^{\infty} (\tilde{A} f_i, f_j) f_j, \sum_{k=0}^{\infty} g_k f_k) = \sum_{k=0}^{\infty} (\tilde{A} f_i, f_k) \overline{g_k} =$$

$$= \sum_{k=0}^{\infty} a_{i,k} g_k = -\sum_{k=0}^{\infty} a_{i,k} g_k = -\sum_{k=0}^{\infty} a_{i,k} g_k = -y_i, \ i \in \mathbb{Z}_+;$$

$$(Bg, f_i) = y_i, \ i \in \mathbb{Z}_+.$$
Therefore
\[-(\widetilde{A}f_i, g) = (Bg, f_i) = (f_i, Bg),\]
and
\[-(\widetilde{A}f, g) = (f, Bg), \quad f \in \text{Lin}\{f_k\}_{k \in \mathbb{Z}_+} =: L.\]
For an arbitrary \(f \in D(\widetilde{A})\) there exists a sequence \(\{f_k\}_{k \in \mathbb{Z}_+}, f_k \in L: f_k \to f, \ \widetilde{A}f_k \to \widetilde{A}f, \text{ as } k \to \infty.\) Passing to the limit as \(k \to \infty\) in the equality
\[-(\widetilde{A}f_k, g) = (f_k, Bg)\]
and using the continuity of the scalar product, we obtain
\[-(\widetilde{A}f, g) = (f, Bg), \quad f \in D(\widetilde{A}).\]
Thus, we have \(g \in D((\widetilde{A})^*) = (\widetilde{A})^* g = -Bg.\) Therefore we get an inclusion \(B \subseteq -(\widetilde{A})^*. \)

Let \(J\) be a conjugation in a Hilbert space \(H\) and \(F = \{f_k\}_{k \in \mathbb{Z}_+}\) be a corresponding to \(J\) orthonormal basis in \(H\). Let \(A\) be a closed \(J\)-symmetric (\(J\)-skew-symmetric) operator in \(H\) and \(\mathcal{F} \subset D(A).\) Set \(a_{i,j} = (Af_j, f_i), \ i, j \in \mathbb{Z}_+,\) and define an operator \(B\) by formula (81). Is the operator \(B\) \(J\)-symmetric (\(J\)-skew-symmetric)? We first notice that the domain of an operator \(\widetilde{B} = JB\) is a set
\[D(\widetilde{B}) = \{h = \sum_{k=0}^{\infty} h_k f_k \in H : \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |a_{i,k} h_k|^2 < \infty\}.\]
If \(h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B}), \) then
\[\widetilde{B}h = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{\infty} a_{i,k} h_k\right) f_i.\]
Choose an arbitrary elements \(g = \sum_{k=0}^{\infty} g_k f_k \in D_B, \ h = \sum_{k=0}^{\infty} h_k f_k \in D(\widetilde{B}).\) Using relations (21), (22) it is easy to check that the operator \(B\) is \(J\)-symmetric (\(J\)-skew-symmetric), if the following equalities are true (for all \(g \in D_B, h \in D(\widetilde{B})\))
\[\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{i,k} g_k h_i = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} a_{i,k} g_k h_i.\]
In the latter case, the last theorem can be applied with the operator \(B\) to obtain that the operator \(B\) is \(J\)-self-adjoint (\(J\)-skew-self-adjoint).
A question appears about existence of a basis of the matrix representation for a closed J-symmetric (J-skew-symmetric) operator. For an arbitrary closed operator there exists an orthonormal basis in which the operator is a closure of its values on the linear span of the basis (see the proof for symmetric operators in [12], which is valid in the general case, as well). A difficulty in the case of J-symmetric (J-skew-symmetric) operators is that this new basis can be a basis which does not correspond to the conjugation J. So, this question remains open.

4 A structure of the null set.

Consider an arbitrary Hilbert space $H$. Let $J$ be a conjugation in $H$ and $\mathcal{F} = \{f_k\}_{k \in \mathbb{Z}_+}$ be a corresponding to $J$ orthonormal basis in $H$. Let us study the set $H_{J,0}$, which we defined above ($H_{J,0} = \{x \in H : [x, x]^J = 0\}$).

Set

$$H_R := \{x \in H : (x, f_k) \in \mathbb{R}, \ k \in \mathbb{Z}_+\}. \quad (82)$$

Notice that for an arbitrary element $x \in H$ we can write a resolution:

$$x = x_R + ix_I, \quad x_R, x_I \in H_R. \quad (83)$$

Namely, if $x = \sum_{k=0}^{\infty} x_k f_k$, we set $x_R := \sum_{k=0}^{\infty} \text{Re} x_k f_k$, $x_I := \sum_{k=0}^{\infty} \text{Im} x_k f_k$.

It is easy to see that representation (83) is unique.

Define the following vectors:

$$f_{k,l}^+ := \frac{1}{\sqrt{2}}(f_k + if_l), \quad f_{k,l}^- := \frac{1}{\sqrt{2}}(f_k - if_l), \quad k, l \in \mathbb{Z}_+. \quad (84)$$

The following theorem holds true.

**Theorem 4.1** Let $H$ be a Hilbert space and $J$ be a conjugation in $H$. Let $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ be a corresponding to $J$ orthonormal basis in $H$. The set $H_{J,0}$ has the following properties:

1. The set $H_{J,0}$ is closed;
2. $x \in H_{J,0} \Rightarrow Jx \in H_{J,0}, \ \alpha x \in H_{J,0}, \ \alpha \in \mathbb{C};$
3. $x, y \in H_{J,0} : x \perp_J y \Rightarrow \alpha x + \beta y \in H_{J,0}, \ \alpha, \beta \in \mathbb{C};$
4. $H_{J,0} = \{x \in H : x = x_R + ix_I, \ x_R, x_I \in H_R, \ |x_R| = |x_I|, (x_R, x_I) = 0\};$
5. The set $H_{J,0}$ has no inner points;
6. $\text{span} H_{J,0} = H;$
7. A set $\{f_{2k,2k+1}^+, f_{2k+1,2k}^-, f_{2k,2k+1}^-, f_{2k+1,2k}^+\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis in $H$ which elements belong to $H_{J,0}$. 

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Proof. The 1-st statement of the Theorem follows from the continuity of the operator \( J \) and from the continuity of the scalar product in \( H \).

The second and third statements follows from the linearity of the J-form and from the properties of the conjugation \( J \).

The 4-th statement is directly verified.

Suppose that the set \( H_{J;0} \) has an inner point \( x_0 \) such that
\[
x \in H, \quad \|x - x_0\| < \varepsilon \Rightarrow x \in H_{J;0},
\]
for a number \( \varepsilon > 0 \). Let us write for \( x_0 \) the resolution \( (83) \):
\[
x_0 = x_{0,R} + ix_{0,I}, \quad x_{0,R}, x_{0,I} \in H_R.
\]

(86)

Suppose first that \( x_{0,I} \neq 0 \). Set
\[
x_\varepsilon := x_0 + \frac{i \varepsilon}{2\|x_{0,I}\|} x_{0,I} = x_{0,R} + ix_{0,I} \left( 1 + \frac{\varepsilon}{2\|x_{0,I}\|} \right).
\]

(87)

Notice that \( \|x_\varepsilon - x_0\| = \frac{\varepsilon}{2} < \varepsilon \), and, thus, by \( (83) \), we obtain that \( x_\varepsilon \in H_{J;0} \).

Using the proven fourth statement of the Theorem for points \( x_0 \) and \( x_\varepsilon \), we get
\[
\|x_{0,R}\| = \|x_{0,I}\|
\]
and
\[
\|x_{0,R}\| = \|x_{0,I}\| \left( 1 + \frac{\varepsilon}{2\|x_{0,I}\|} \right) \| = \|x_{0,I}\| + \frac{\varepsilon}{2} > \|x_{0,I}\|
\]
respectively. The obtained contradiction proves statement 5 for the case \( x_{0,I} \neq 0 \).

If \( x_{0,I} = 0 \), then by the fourth statement of the Theorem the relation \( (83) \) is true and therefore \( x_0 = 0 \). But if zero is an inner point of the set \( H_{J;0} \), then by the proven second statement of the Theorem we get \( H_{J;0} = H \). But it is a nonsense, since, for example, elements of the basis \( F \) do not belong to the set \( H_{J;0} \).

Let us prove the seventh statement of the Theorem. Using orthonormality of elements \( f_k, k \in \mathbb{Z}_+ \), it is directly verified that elements of the set \( \{f_{2k,2k+1}^+, f_{2k,2k+1}^-\}_{k \in \mathbb{Z}_+} \), are orthonormal. Notice that
\[
f_{2k} = \frac{1}{\sqrt{2}}(f_{2k,2k+1}^+ + f_{2k,2k+1}^-), \quad f_{2k+1} = \frac{1}{\sqrt{2i}}(f_{2k,2k+1}^+ - f_{2k,2k+1}^-),
\]
\[
k \in \mathbb{Z}_+.
\]

(89)

Therefore \( \text{span}\{f_{2k,2k+1}^+, f_{2k,2k+1}^-\}_{k \in \mathbb{Z}_+} = H \) and a set \( \{f_{2k,2k+1}^+, f_{2k,2k+1}^-\}_{k \in \mathbb{Z}_+} \) is an orthonormal basis in \( H \). It remains to notice that
\[
[f_{2k,2k+1}^\pm]_J = \frac{1}{2}[f_{2k} \pm if_{2k+1}, f_{2k} \pm if_{2k+1}]_J = 0,
\]

[27]
and therefore \( f_{2k,2k+1}^\pm \in H_{J,0}, \ k \in \mathbb{Z}_+ \).

The sixth statement of the Theorem follows from the proven seventh statement. \( \square \)

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On a J-polar decomposition of a bounded operator and matrix representations of J-symmetric, J-skew-symmetric operators.

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In this work a possibility of a decomposition of a bounded operator which acts in a Hilbert space $H$ as a product of a J-unitary and a J-self-adjoint operators is studied, $J$ is a conjugation (an antilinear involution). Decompositions of J-unitary and unitary operators which are analogous to decompositions in the finite-dimensional case are obtained. A possibility of a matrix representation for J-symmetric, J-skew-symmetric operators is studied. Also, some simple properties of J-symmetric, J-antisymmetric, J-isometric operators are obtained, a structure of a null set for a J-form is studied.

Key words and phrases: polar decomposition, matrix of an operator, conjugation, J-symmetric operator.

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