Modular symmetry anomaly  
in magnetic flux compactification

Yuki Kariyazono, Tatsuo Kobayashi, Shintaro Takada, Shio Tamba, and Hikaru Uchida  
Department of Physics, Hokkaido University, Sapporo 060-0810, Japan

Abstract

We study modular symmetry anomalies in four-dimensional low-energy effective field theory derived from six-dimensional super $U(N)$ Yang-Mills theory with magnetic flux compactification. The gauge symmetry $U(N)$ breaks to $U(N_a) \times U(N_b)$ by magnetic fluxes. It is found that Abelian subgroup of the modular symmetry corresponding to discrete part of $U(1)$ can be anomalous, but other elements independent of $U(1)$ in the modular symmetry are always anomaly-free.
I. INTRODUCTION

The modular symmetry is a geometrical feature, which torus compactification as well as orbifold compactification has. Furthermore, the modular symmetry plays an important role in four-dimensional (4D) low-energy effective field theory derived from higher dimensional field theory and superstring theory.

The modular symmetry in string-induced supergravity theory was studied in Ref. [1] and also its anomaly was studied in Ref. [2, 3]. (See also for anomalies in explicit heterotic orbifold models Ref. [4].) Recently, these studies were extended to supergravity theory induced by magnetized and intersecting D-brane models [5]. Furthermore, their anomalies are also interesting from the phenomenological viewpoint [3, 6, 7].

Also it was studied how massless modes transform under modular symmetry in heterotic orbifold models [8–10]. Recently, modular transformation behavior of massless modes was studied in magnetized D-brane models as well as intersecting D-brane models [11–14]. Then, it was found that the modular symmetry transforms massless modes each other, and that is a sort of flavor symmetries. On the other hand, it was shown that non-Abelian discrete flavor symmetries appear in heterotic orbifold models [15–20] and magnetized/intersecting D-brane models [21–26] through analysis independent of modular symmetry. Indeed, a relation between modular symmetry and non-Abelian discrete flavor symmetry was also studied [13]. (See also Ref. [27].)

Non-Abelian discrete flavor symmetries are interesting from the phenomenological viewpoints [28–30]. Various finite groups have been utilized such as $S_3, A_4, S_4, A_5$, etc. For 4D field-theoretical model building, many models have been proposed in order to realize quark and lepton masses and their mixing angles and CP phases. The modular group includes $S_3, A_4, S_4, A_5$ as its finite subgroups [31]. This aspect in addition to the above string compactification inspired a new approach of 4D field-theoretical model building [32], where finite subgroups of the modular symmetry are used as non-Abelian discrete flavor symmetries and also couplings and masses are assumed to transform non-trivially under such finite subgroups. Such a new approach has been applied to models with $S_3, A_4, S_4, A_5$ modular symmetries [33–41].

Thus, the modular symmetry is important from both theoretical and phenomenological viewpoints. In general, continuous and discrete symmetries can be anomalous. (See
Anomalous symmetries can be broken by non-perturbative effects. That is, breaking terms induced by non-perturbative effects appear in Lagrangian. Such breaking terms may have important implications. The purpose of this paper is to study the anomaly structure of the modular symmetry in 4D low-energy effective field theory derived from magnetic flux compactification of higher dimensional super Yang-Mills theory, which is effective field theory of magnetized D-brane models.

This paper is organized as follows. In Sec. II, we present our setup and give a brief review on magnetic flux compactification and the modular transformation of zero-modes. In Sec. III, we study the anomaly structure of the modular symmetry. Sec. IV is our conclusion.

II. MODULAR TRANSFORMATION OF MAGNETIC FLUX COMPACTIFICATION

A. Setup and wavefunctions

Here, we present our setup and give a brief review on magnetic flux compactification. We start with six-dimensional \( U(N) \) super Yang-Mills theory on two-dimensional torus \( T^2 \), which can be derived from D-brane system. Similarly, we can study higher dimensional theory such as ten-dimensional super Yang-Mills theory on \( T^2 \times T^2 \times T^2 \), which can also be derived from D-brane system.

We use the complex coordinate \( z = x^1 + \tau x^2 \) on \( T^2 \), where \( \tau \) is the complex modular parameter, and \( x^1 \) and \( x^2 \) are real coordinates. The metric on \( T^2 \) is given by

\[
g_{\alpha\beta} = \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix} = (2\pi R)^2 \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}. \tag{1} \]

We identify \( z \sim z + 1 \) and \( z \sim z + \tau \) on \( T^2 \).

We introduce the following magnetic flux along the diagonal direction,

\[
F = i \frac{\pi}{\text{Im} \tau} (dz \wedge d\bar{z}) \begin{pmatrix} M_a \mathbb{I}_{N_a \times N_a} & 0 \\ 0 & M_b \mathbb{I}_{N_b \times N_b} \end{pmatrix}, \tag{2} \]

where \( N_a + N_b = N \), \( \mathbb{I}_{N_a,b \times N_a,b} \) denotes the \((N_{a,b} \times N_{a,b})\) identity matrix and \( M_{a,b} \) must be...
integer. This form of magnetic flux corresponds to the vector potential,

\[ A(z) = \frac{\pi}{\text{Im } \tau} \text{Im} (\bar{z} dz) \left( \begin{array}{cc} M_a I_{N_a \times N_a} & 0 \\ 0 & M_b I_{N_b \times N_b} \end{array} \right). \] (3)

Because of this gauge background, the \( U(N) \) gauge symmetry breaks to \( U(N_a) \times U(N_b) \).

Now let us study the gaugino sector. On \( T^2 \), the spinor has two components, \( \lambda_{\pm} \). They are decomposed to

\[
\begin{pmatrix}
\lambda_{\pm}^{aa} \\
\lambda_{\pm}^{ab}
\end{pmatrix}
\]

Here \( \lambda^{aa} \) and \( \lambda^{bb} \) correspond to the gaugino fields of unbroken gauge groups, \( U(N_a) \) and \( U(N_b) \), respectively, while \( \lambda^{ab} \) and \( \lambda^{ba} \) correspond to \( (N_a, \bar{N}_b) \) and \( (\bar{N}_a, N_b) \) under \( U(N_a) \times U(N_b) \).

The zero-mode equation with the above gauge background (3),

\[ i D \lambda_{\pm} = 0, \] (5)

has chiral solutions. When \( M = M_a - M_b \) is positive, \( \lambda_{\pm}^{ab} \) and \( \lambda_{\pm}^{ba} \) have \( M \) degenerate zero-modes, whose profiles are written by \[11\]

\[
\psi_{T^2}^{j, M}(z) = \mathcal{N} e^{\pi M z \frac{\text{Im } \tau}{\text{Im } \tau}} \cdot \vartheta \begin{pmatrix} \nu \\ \frac{M}{M} \end{pmatrix} (Mz, M\tau),
\]

with \( j = 0, 1, \cdots, (M - 1) \), where \( \vartheta \) denotes the Jacobi theta function,

\[
\vartheta \begin{pmatrix} a \\ b \end{pmatrix} (\nu, \tau) = \sum_{l \in \mathbb{Z}} e^{\pi i (a+l)^2 \tau} e^{2\pi i (a+l)(\nu+b)}.
\] (7)

Here, \( \mathcal{N} \) denotes the normalization factor given by

\[
\mathcal{N} = \left( \frac{2\text{Im } \tau M}{\mathcal{A}^2} \right)^{1/4},
\] (8)

with \( \mathcal{A} = 4\pi^2 R^2 \text{Im } \tau \). On the other hand, for \( M \) is negative, \( \lambda_{\pm}^{ab} \) and \( \lambda_{\pm}^{ba} \) have \( |M| \) degenerate zero-modes, whose profiles are the same as \( \psi_{T^2}^{j, M}(z) \) except replacing \( M \) by \( |M| \). Hereafter, we set \( M \) to be positive.

Because of the chiral spectrum, \( U(1)_a \) and \( U(1)_b \) are anomalous in 4D low-energy effective field theory. For example, both the mixed anomalies, \( U(1)_a - SU(N_b)^2 \) and \( U(1)_b - SU(N_a)^2 \)
are proportional to $M$. When we embed this super Yang-Mills theory into D-brane system, such anomalies can be canceled by the Green-Schwarz mechanism. The Green-Schwarz mechanism cancels anomalies by the shift of axions $\chi_{a,b}$,

$$
\chi_{a,b} \rightarrow \chi_{a,b} + \alpha_{a,b},
$$

(9)

under $U(1)_{a,b}$ transformation, where $\alpha_{a,b}$ are $U(1)_{a,b}$ gauge transformation parameters \[46\]. Those axions are eaten by $U(1)_{a,b}$ gauge bosons and then $U(1)_{a,b}$ gauge bosons become massive.

In the next section, we will study the $T^2/Z_2$ orbifold background. The zero-mode wavefunctions on $T^2/Z_2$ are obtained from the above wavefunctions \[47\]. The above wavefunctions have the following property:

$$
\psi_{j,M}^{T^2}(-z) = \psi_{M-j,M}^{T^2}(z).
$$

(10)

Thus, the $T^2$ wavefunction with $j = 0$ is still the $Z_2$-even zero-mode on $T^2/Z_2$. Also, when $M = \text{even}$, the $T^2$ wavefunction with $j = M/2$ is still the $Z_2$-even zero-mode on $T^2/Z_2$. That is, we obtain

$$
\psi_{j,M}^{T^2/Z_2^+}(z) = \psi_{j,M}^{T^2}(z),
$$

(11)

for $j = 0, M/2$. For the other, the $Z_2$-even and odd zero-modes can be written by

$$
\psi_{j,M}^{T^2/Z_2^+}(z) = \frac{1}{\sqrt{2}} \left( \psi_{j,M}^{T^2}(z) \pm \psi_{M-j,M}^{T^2}(z) \right).
$$

(12)

When $M = \text{even}$, totally the numbers of $Z_2$-even and odd zero-modes are equal to $(M/2+1)$ and $(M/2-1)$, respectively. When $M = \text{odd}$, the numbers of $Z_2$-even and odd zero-modes are equal to $((M-1)/2+1)$ and $((M-1)/2)$, respectively.

The anomalies of $U(1)_a$ and $U(1)_b$ on the $T^2/Z_2$ orbifold, e.g. for the $Z_2$-even modes $\psi_{j,M}^{T^2/Z_2^+}(z)$, can be studied in the same way as on the torus. Those anomalies can also be canceled by the Green-Schwarz mechanism.

### B. Modular transformation

Here, we give a brief review on modular transformation of zero-mode wavefunctions \[11-14\]. Following \[12\], we restrict ourselves to even magnetic fluxes $M$.

Under the modular transformation, the modulus $\tau$ transforms as

$$
\tau \rightarrow \frac{a\tau + b}{c\tau + d},
$$

(13)

where $a, b, c, d$ are integers satisfying $ad - bc = \pm 1$.
This group includes two important generators, $S$ and $T$,

\begin{align}
S : \tau &\rightarrow -\frac{1}{\tau}, \\
T : \tau &\rightarrow \tau + 1.
\end{align}

(14) \hspace{1cm} (15)

Under $S$, the zero-mode wavefunctions transform as

$$\psi^{j,M} \rightarrow \frac{1}{\sqrt{M}} \sum_{k} e^{2\pi ijk/M} \psi^{k,M}.$$  

(16)

On the other hand, the zero-mode wavefunctions transform as

$$\psi^{j,M} \rightarrow e^{\pi ij^2/M} \psi^{j,M},$$

(17)

under $T$. Generically, the $T$-transformation satisfies

$$T^{2^M} = \mathbb{I}_{M \times M},$$

(18)

on the zero-modes, $\psi^{j,M}$. Furthermore, in Ref. [12] it is shown that

$$(ST)^3 = e^{\pi i/4} \mathbb{I}_{M \times M},$$

(19)

on the zero-modes, $\psi^{j,M}$. Hence, on $\psi^{j,M}$, $T$ and $(ST)^3$ are represented by diagonal matrices, and they are $Z_{2^M}$ and $Z_{8}$ symmetries, respectively.

The above representations of $S$ and $T$ on $\psi^{j,M}$ are reducible. It is obvious that $\psi^{j,M}$ and $\psi^{M-j,M}$ transform in the same way under both $S$ and $T$. That implies that the orbifold basis $\psi^{j,M}_{T^2/Z_2^\pm}(z)$ corresponds to the irreducible representation. We denote such irreducible representations by $S_{\pm}$ and $T_{\pm}$. Their explicit forms can be read off from the above representations of $S$ and $T$. Note that when $M = \text{even}$, $S_+$ and $T_+$ are $(M/2 + 1) \times (M/2 + 1)$ matrices, and $S_-$ and $T_-$ are $(M/2 - 1) \times (M/2 - 1)$ matrices.

III. MODULAR SYMMETRY ANOMALY

Here, we study the modular symmetry anomaly. Anomalies of non-Abelian discrete symmetries were studied in Ref. [45]. Each element of a non-Abelian discrete group, $g$, generates Abelian discrete symmetry, $Z_K$ i.e. $g^K = 1$. Thus, basically anomalies of non-Abelian discrete group are studied by analyzing Abelian discrete anomalies of each element, $g$. However,
states correspond to a multiplet under a non-Abelian discrete symmetry. That is, \( g \) is represented by a matrix. Suppose that zero-modes correspond to the (anti-)fundamental representation of \( SU(N_b) \). Then, if \( \det g = 1 \), the mixed \( Z_K - SU(N_b)^2 \) anomaly vanishes. Otherwise, the \( Z_K \) symmetry generated by \( g \) can be anomalous. Furthermore, suppose that zero-modes correspond to the bi-fundamental representation \((N_a, \bar{N}_b)\) under \( SU(N_a) \times SU(N_b) \). Then, if \( \det g^{N_a} = 1 \), the mixed \( Z_K - SU(N_b)^2 \) anomaly vanishes. Otherwise, the \( Z_K \) symmetry generated by \( g \) is anomalous. Hence, the quantity \( \det g \) is important to examine whether or not the corresponding discrete symmetry can be anomalous.

A. \( T^2/Z_2 \) orbifold

As mentioned above, the orbifold basis is more fundamental. Thus, we first study anomalies due to the \( Z_2 \)-even modes on the \( T^2/Z_2 \) orbifold. Here, we study anomalies by examining \( \det g \) for smaller \( M \) concretely.

1. \( M = 2 \)

Here, we study the modular symmetry for \( M = 2 \). Note that the zero-modes on \( T^2 \) are the same as the \( Z_2 \)-even zero-modes on \( T^2/Z_2 \). First, we study diagonal elements, \( T \) and \((ST)^3\). Their explicit forms are written as

\[
T_{(2)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad (S_{(2)}T_{(2)})^3 = e^{\pi i/4} \mathbb{I}_{2 \times 2},
\]

where we have omitted vanishing off-diagonal entries. That is the \( Z_4 \times Z_8 \) symmetry, and they satisfy \( \det T_{(2)} \neq 1 \) and \( \det (S_{(2)}T_{(2)})^3 \neq 1 \). Thus, both symmetries can be anomalous. However, their combination,

\[
T'_{(2)} = T_{(2)}(S_{(2)}T_{(2)})^{-3} = \begin{pmatrix} e^{-\pi i/4} \\ 1 \\ e^{\pi i/4} \end{pmatrix},
\]

has \( \det T'_{(2)} = 1 \) and is always anomaly-free. This is the \( Z_8 \) symmetry. Hence, the \( Z_4 \times Z_8 \) symmetry can be broken to \( Z_8 \) by anomalies. The generator of the could-be anomalous symmetry is \( A_{(2)} = (S_{(2)}T_{(2)})^3 \). Note that \( (A_{(2)})^4 = (T'_{(2)})^4 \). It is obvious that \( A_{(2)} \) is commutable with any element. Therefore, at least the elements \( (A_{(2)})^k g \ (k = 1, 2, 3) \) with
det \, g = 1 \text{ has } det((A_{(2)})^k g) \neq 1 \text{ and can be anomalous among all of the elements, which are generated by } S_{(2)} \text{ and } T_{(2)}. \text{ Indeed, by explicit calculation it is found that the order of the full group generated by } S_{(2)} \text{ and } T_{(2)} \text{ is equal to } 192, \text{ and among them the number of elements with } \det g = 1 \text{ is equal to } 48. \text{ Thus, all of the elements with } \det h \neq 1 \text{ can be written by } h = (A_{(2)})^k g \,(k = 1, 2, 3) \text{ with } \det g = 1. \text{ That is, only the element } A_{(2)} \text{ is important for anomalies.}

The element } A_{(2)} \text{ can be anomalous. For example, it can lead to the mixing anomalies with } SU(N_a) \text{ and } SU(N_b). \text{ However, it is remarkable that the element } A_{(2)} \text{ corresponds to a subgroup of } U(1)_a \text{ as well as } U(1)_b. \text{ Thus, when we embed this system to D-brane models, the discrete anomaly corresponding to } A_{(2)} \text{ can also be canceled by the same Green-Schwarz mechanism as one for } U(1)_a \text{ and } U(1)_b. \text{ The other discrete parts independent of } A_{(2)} \text{ are always anomaly-free.}

As mentioned in the previous section, in the Green-Schwarz mechanism the axion } \chi \text{ shifts under the } U(1) \text{ transformation to cancel anomalies. Such an axion is the pure imaginary part of a complex field, the so-called modulus field } U, \text{ where axionic shift } \alpha \text{ leads to } U \rightarrow U + i\alpha \text{ under the } U(1) \text{ gauge transformation with the transformation parameter } \alpha. \text{ It implies that } e^{-cU} \text{ transforms linearly and it behaves to have the } U(1) \text{ "charge" } -c. \text{ Non-perturbative effects such as D-brane instanton effects induce new terms } e^{-cU}\phi_1 \phi_2 \cdots \text{ in 4D low-energy effective field theory. Such terms are invariant under the anomalous } U(1) \text{ and discrete symmetry with taking into account the transformation of } e^{-cU}. \text{ However, when we replace } U \text{ by its vacuum expectation value, such terms correspond to breaking terms. Thus, breaking terms for anomalous symmetries appear. Similar breaking terms would also appear by field-theoretical instanton effects even if we do not take string non-perturbative effects into account.}

2. \quad M = 4

Similarly, we study the orbifold model with } M = 4, \text{ in particular the } Z_2\text{-even modes. First, we study diagonal elements, } T \text{ and } (ST)^3. \text{ Their explicit forms are written as}

\[
T_{(4)+} = \begin{pmatrix}
1 \\
e^{\pi i/4} \\
-1
\end{pmatrix}, \quad (S_{(4)+}T_{(4)+})^3 = e^{\pi i/4} \mathbb{I}_{3 \times 3}. \tag{22}
\]
They correspond to the $Z_8 \times Z_8$ symmetry. We find that det $T_{(4)+} \neq 1$ and det$(S_{(4)+}T_{(4)+})^3 \neq 1$. They can be anomalous. However, their combination,

$$T'_{(4)+} = T_{(4)+}(S_{(4)+}T_{(4)+})^3 = \begin{pmatrix} e^{\pi i/4} \\ e^{2\pi i/4} \\ e^{5\pi i/4} \end{pmatrix},$$

has det $T'_{(4)+} = 1$, and is always anomaly-free. This is the $Z_8$ symmetry. The $Z_8 \times Z_8$ symmetry can be broken to $Z_8$ by anomalies. The generators of the could-be anomalous symmetry is $A_{(4)} = (S_{(4)+}T_{(4)+})^3 = e^{\pi i/4}\mathbb{I}_{3\times 3}$, again, and this is commutable with any element. At least the elements $(A_{(4)+})^k g$ ($k = 1, \cdots, 7$) with det $g = 1$ has det$((A_{(4)+})^k g) \neq 1$ and can be anomalous among all of the elements, which are generated by $S_{(4)+}$ and $T_{(4)+}$. Indeed, by explicit calculation we find that the order of the full group generated by $S_{(4)+}$ and $T_{(4)+}$ is equal to 768, and among them the number of elements with det $g = 1$ is equal to 96. Thus, all of the elements with det $h \neq 1$ can be written by $h = (A_{(4)+})^k g$ ($k = 1, \cdots, 7$) with det $g = 1$.

The could-be anomalous element $A_{(4)+}$ is a sub-element of $U(1)_a$ as well as $U(1)_b$. Thus, anomalies originated from $A_{(4)+}$ can be canceled by the Green-Schwarz mechanism when we embed this system in D-brane models.

3. $M = 6$

Similarly, we study the orbifold model with $M = 6$, in particular the $Z_2$-even modes. The diagonal elements, $T$ and $(ST)^3$, are explicitly written by

$$T_{(6)+} = \begin{pmatrix} 1 \\ e^{\pi i/6} \\ e^{2\pi i/3} \\ e^{3\pi i/2} \end{pmatrix}, \quad (S_{(6)+}T_{(6)+})^3 = e^{\pi i/4}\mathbb{I}_{4\times 4},$$

where det$(S_{(6)+}T_{(6)+})^3 = -1$. They correspond to the $Z_{12} \times Z_8$ symmetry. They can be anomalous. By their combinations, we can construct the diagonal elements with det $g = 1$. 

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such as

\[(S_{(6)}+T_{(6)+})^6 = iI_{4 \times 4}, \quad (T_{(6)+})^3(S_{(6)}+T_{(6)+})^3 = \begin{pmatrix}
    e^{\pi i/4} & e^{3\pi i/4} \\
    e^{\pi i/4} & e^{3\pi i/4}
\end{pmatrix}, \quad (25)\]

etc. They include \(T_{(6)+}^k\) only for \(k = 3k' + 1\) and \(k = 3k' + 2\) have \(\det g \neq 1\) and can be anomalous. The order of the above group with \(\det g = 1\) in the \(Z_{12} \times Z_8\) symmetry is equal to 16. Thus, its order reduces by the factor 1/6. Indeed, the order of the full group generated by \(S_{(6)+}\) and \(T_{(6)+}\) is equal to 2304, and among them the number of elements with \(\det g = 1\) is equal to 384. That is, the order reduces by the factor 1/6. Here, it seems that the group elements including \(T_{(6)+}^k\) with \(k = 1, 2\) in addition \((S_{(6)}+T_{(6)+})^3\) can be anomalous. That is different from the above cases with \(M = 2\) and 4.

However, \((S_{(6)}+T_{(6)+})^3\) corresponds to the sub-element of \(U(1)_{a,b}\). Let us combine \(T_{(6)+}\) and a discrete transformation of \(U(1)_{a,b}\),

\[T'_{(6)+} = e^{i\alpha}T_{(6)+}. \quad (26)\]

When \(\alpha = -1/12\), we have \(\det T'_{(6)+} = 1\), and \(T'_{(6)+}\) is written explicitly as

\[T'_{(6)+} = \begin{pmatrix}
    e^{-\pi i/12} & e^{\pi i/12} \\
    e^{\pi i/12} & e^{7\pi i/12}
\end{pmatrix}. \quad (27)\]

As a result, in the comprehensive symmetry including the modular symmetry and \(U(1)_{a,b}\), only \(U(1)_{a,b}\) including their discrete symmetries can be anomalous. In this sense, the anomaly structure for \(M = 6\) is the same as the previous examples for \(M = 2\) and \(M = 4\), where only discrete symmetries of \(U(1)_{a,b}\) as well as of course \(U(1)_{a,b}\) themselves can be anomalous.

4. Larger \(M\)

Similarly, we can study anomalies for larger \(M\). The anomaly structure for larger \(M\) is the same as one for \(M = 2, 4, 6\). For \(M \neq 6k\), \(T_{(M)+}\) and \((S_{(M)+}T_{(M)+})^3\), in general,
have \( \det T_{(M)+} \neq 1 \) and \( \det(S_{(M)+}T_{(M)+})^3 \neq 1 \), although in specific values of \( M \) we have \( \det T_{(M)+} = 1 \) for \((M + 1)(M/2 + 1) = 24k\)\(^1\) and \( \det(S_{(M)+}T_{(M)+})^3 = 1 \) for \( M = 16k - 2 \).

However, we can find the element \( T'_{(M)+} = T_{(M)+}(S_{(M)+}T_{(M)+})^{3m} \) satisfying \( \det T'_{(M)+} = 1 \). Then, only the element \( (S_{(M)+}T_{(M)+})^3 \) can be anomalous. That is, only the discrete symmetry of \( U(1)_{a,b} \) can be anomalous.

For \( M = 6k \), even if we combine \( T_{(M)+}^\ell \) and \( (S_{(M)+}T_{(M)+})^{3m} \), there are elements with \( \det g \neq 1 \) except \( (S_{(M)+}T_{(M)+})^{3m} \). However, we can obtain \( T'_{(M)+} = e^{i\alpha}T_{(M)+} \) with \( \det T'_{(M)+} = 1 \) by combining \( T_{(M)+} \) with a proper discrete element of \( U(1)_{a,b} \).

As a result, it is found that only the \( U(1)_{a,b} \) including their discrete symmetries can be anomalous, but the other symmetry independent of \( U(1)_{a,b} \) is always anomaly-free.

### B. \( T^2 \)

Similarly, we can discuss \( T^2 \) models. The zero-modes of \( T^2 \) are combinations of \( Z_2 \)-even and odd modes on \( T^2/Z_2 \) orbifold. For \( M = 2 \), all of the zero-modes on \( T^2 \) are the \( Z_2 \)-even zero-modes. Thus, \( S \) and \( T \) are represented by \( S_{(2)} \) and \( T_{(2)} \).

For \( M = 4 \), there is one \( Z_2 \)-odd mode. Then, the diagonal elements, \( T \) and \((ST)^3\) are represented by

\[
T_{(4)} = \begin{pmatrix} T_{(4)+} \\ T_{(4)-} \end{pmatrix}, \quad (S_{(4)}T_{(4)})^3 = \begin{pmatrix} (S_{(4)+}T_{(4)+})^3 \\ (S_{(4)-}T_{(4)-})^3 \end{pmatrix},
\]

where \( T_{(4)-} = e^{\pi i/4} \) and \( (S_{(4)-}T_{(4)-})^3 = e^{\pi i/4} \). That is, we have \( (S_{(4)}T_{(4)})^3 = e^{\pi i/4}I_{4 \times 4} \). This element corresponds to the discrete sub-group of \( U(1)_{a,b} \) and can be anomalous. Other elements independent of \( U(1)_{a,b} \) discrete subgroup are always anomaly-free. For example, from \( T_{(4)} \) we can construct \( T'_{(4)} = e^{i\alpha}T_{(4)} \) with \( \det T'_{(4)} = 1 \) by choosing a proper value of \( \alpha \).

### IV. CONCLUSION

We have studied the modular symmetry anomalies in magnetic flux compactification. Our model is six-dimensional super \( U(N) \) Yang-Mills theory, where \( U(N) \) gauge symmetry is

\(^1\) \( M \) is obtained by \( M = 16n - 2 \) with \( n \) satisfying \( n(16n - 1) = 3k \).
broken down to $U(N_a) \times U(N_b)$ by magnetic fluxes in the compact space. Discrete sub-

symmetries of $U(1)_{a,b}$ in the modular symmetry can be anomalous, but other discrete elements

independent of $U(1)_{a,b}$ are always anomaly-free. Anomalies of such discrete symmetries can

be canceled by the Green-Schwarz mechanism when we embed our theory into D-brane sys-

As a result, breaking terms can be induced only for continuous and discrete $U(1)_{a,b}$
symmetries.

Here we have studied super $U(N)$ Yang-Mills theory, which can be derived from D-brane

models. Similar representations of $S$ and $T$ were derived in heterotic orbifold models.

It is interesting to carry out a similar analysis on heterotic orbifold models.

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