Existence local and global solution of multipoint Cauchy problem for nonlocal nonlinear equations

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Abstract
In this paper, the multipoint Cauchy problem for nonlocal nonlinear wave type equations are studied. The equation involves a convolution integral operator with a general kernel function whose Fourier transform is nonnegative. We establish local and global existence and uniqueness of solutions assuming enough smoothness on the initial data together with some growth conditions on the nonlinear term.

Key Word: Boussinesq equations, Hyperbolic equations, differential operators, Fourier multipliers

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1. Introduction

The aim in this paper is to study the existence and uniqueness of solution of the multipoint initial value problem (IVP) for nonlocal nonlinear wave equation

\[ u_{tt} - a\Delta u + b * u = \Delta \left( g * f(u) \right), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty), \]

\[ u(x, 0) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(x, \lambda_k), \quad \text{for a.e.} \quad x \in \mathbb{R}^n, \]

\[ u_t(x, 0) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(x, \lambda_k), \quad \text{for a.e.} \quad x \in \mathbb{R}^n, \]

where \( m \) is an integer, \( \lambda_k \in (0, \infty) \), \( \alpha_k, \beta_k \) are complex numbers, \( g(x) \), \( b(x) \) are measurable functions on \((0, \infty)\); \( a \geq 0 \), \( \Delta \) denotes the Laplace operator in \( \mathbb{R}^n \), \( f(u) \) is the given nonlinear function, \( \varphi(x) \) and \( \psi(x) \) are the given initial value functions. Note that for \( \alpha_k = \beta_k = 0 \) we obtain classical Cauchy problem for nonlocal equation

\[ u_{tt} - a\Delta u + b * u = \Delta \left( g * f(u) \right), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty), \]

\[ u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n, \]

The predictions of classical (local) elasticity theory become inaccurate when the characteristic length of an elasticity problem is comparable to the atomic
length scale. To solution this situation, a nonlocal theory of elasticity was introduced (see [1–3] and the references cited therein) and the main feature of the new theory is the fact that its predictions were more down to earth than those of the classical theory. For other generalizations of elasticity we refer the reader to [4–6]. The global existence of the classical Cauchy problem for Boussinesq type nonlocal equations has been studied by many authors (see [7–11]). Note that, the existence of solutions and regularity properties for different type Boussinesq equations are considered e.g. in [8–15]. Boussinesq type equations occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describe spinodal decomposition and in the absence of mechanical stresses (see [16–19]).

The $L^p$-well-posedness of the classical Cauchy problem (1.3) depends crucially on the presence of a suitable kernel. Then the question that naturally arises is which of the possible forms of the kernel functions are relevant for the global well-posedness of the multipoint initial-value problem (IVP) (1.1)–(1.2). In this study, as a partial answer to this question, we consider multipoint IVP (1.1) – (1.2) with a general class of kernel functions and provide local, global existence and blow-up results for the solutions of the problem (1.1) – (1.2) in frame of $L^p$ spaces. The kernel functions most frequently used in the literature are particular cases of this general class of kernel functions. Note that nonlocal Cauchy problem for wave equations were studied e.g. in [20, 21].

The strategy is to express the equation (1.1) as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on $L^p$ norms of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate of the solutions of the linearized Boussinesq equation. The methods of harmonic analysis, operator theory, interpolation of Banach Spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

In order to state our results precisely, we introduce some notations and some function spaces.

**Definitions and Background**

Let $E$ be a Banach space. $L^p(\Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$
\|f\|_p = \|f\|_{L^p(\Omega; E)} = \left( \int_{\Omega} \|f(x)\|^p_E \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

$$
\|f\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} \|f(x)\|_E.
$$

Let $\mathbb{C}$ denote the set of complex numbers. For $E = \mathbb{C}$ the $L^p(\Omega; E)$ denotes by $L^p(\Omega)$. 

\[ \text{Page 2} \]
Let $E_1$ and $E_2$ be two Banach spaces. $(E_1, E_2)_{θ,p}$ for $θ ∈ (0, 1)$, $p ∈ [1, ∞]$ denotes the interpolation spaces defined by K-method [22, §1.3.2].

Let $m$ be a positive integer. $W^{m,p}(Ω)$ denotes the Sobolev space, i.e. space of all functions $u ∈ L^p(Ω)$ that have the generalized derivatives $\frac{∂^m u}{∂x_k^n} ∈ L^p(Ω)$, $1 ≤ p ≤ ∞$ with the norm

$$
∥u∥_{W^{m,p}(Ω)} = ∥u∥_{L^p(Ω)} + \sum_{k=1}^{n} \left\| \frac{∂^m u}{∂x_k^n} \right\|_{L^p(Ω)} < ∞.
$$

Let $H^{s,p}(R^n)$, $∞ < s < ∞$ denotes fractional Sobolev space of order $s$ which is defined as:

$$
H^{s,p} = H^{s,p}(R^n) = (I - Δ)^{-\frac{s}{2}} L^p(R^n)
$$

with the norm

$$
∥u∥_{H^{s,p}} = ∥(I - Δ)^{\frac{s}{2}} u∥_{L^p(R^n)} < ∞.
$$

It clear that $H^{0,p}(R^n) = L^p(R^n)$. It is known that $H^{m,p}(R^n) = W^{m,p}(R^n)$ for the positive integer $m$ (see e.g. [23, §15]). For $p = 2$, the space $H^{s,p}(R^n)$ will be denoted by $H^s(ℝ^n)$. Let $S(ℝ^n)$ denote Schwartz class, i.e., the space of rapidly decreasing smooth functions on $R^n$, equipped with its usual topology generated by seminorms. Let $S^′(ℝ^n)$ denote the space of all continuous linear operators $L : S(ℝ^n) → C$, equipped with the bounded convergence topology. Recall $S(ℝ^n)$ is norm dense in $L^p(ℝ^n)$ when $1 ≤ p < ∞$.

Let $1 ≤ p ≤ q < ∞$. A function $Ψ ∈ L^∞(ℝ^n)$ is called a Fourier multiplier from $L_p(ℝ^n)$ to $L_q(ℝ^n)$ if the map $B : u → F^{-1}Ψ(ξ)Fu$ for $u ∈ S(ℝ^n)$ is well defined and extends to a bounded linear operator

$$
B : L_p(ℝ^n) → L_q(ℝ^n).
$$

Let $L_q^\prime(E)$ denote the space of all $E$-valued function space such that

$$
∥u∥_{L_q^\prime(E)} = \left( \int_0^∞ \|u(t)\|_{E}^q \frac{dt}{t} \right)^\frac{1}{q} < ∞, \quad 1 ≤ q < ∞, \quad ∥u∥_{L_q^\prime(ℝ^n)} = \sup_{t ∈ (0,∞)} ∥u(t)∥_E.
$$

Here, $F$ denote the Fourier transform. Fourier-analytic representation of Besov spaces on $R^n$ is defined as:

$$
B^{s}_{p,q}(R^n) = \left\{ u ∈ S^′(R^n), \right\}
$$

$$
∥u∥_{B^{s}_{p,q}(R^n)} = ∥F^{-1} t^{s-q} \left( 1 + |ξ|^2 \right) t^{-|ξ|^2} Fu∥_{L_q^\prime(L^p(ℝ^n))},
$$

$$
|ξ|^2 = \sum_{k=1}^{n} ξ_k^2, ξ = (ξ_1, ξ_2, ..., ξ_n), p ∈ (1, ∞), q ∈ [1, ∞], s > s_0\right\}.
$$
It should be noted that, the norm of Besov space does not depend on $\kappa$ (see e.g. [22, § 2.3]). For $p = q$ the space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ will be denoted by $\dot{B}^s_p(\mathbb{R}^n)$.

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_\alpha$. Moreover, for $u, \upsilon > 0$ the relation $u \lesssim \upsilon$ means that there exists a constant $C > 0$ independent on $u$ and $\upsilon$ such that

$$u \leq C\upsilon.$$ 

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priori estimates for solution of the linearized problem (1.1) − (1.2). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1) − (1.2). In the Section 4 we show the same applications of the problem (1.1) − (1.2).

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $h$, we write $C_h$.

2. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of the Cauchy problem

$$u_{tt} - a\Delta u + b\ast u = g(x,t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \quad T \in (0,\infty),$$

(2.1)

$$u(x,0) = \phi(x) + \sum_{k=1}^{m} \alpha_k u(x,\lambda_k), \quad \text{for a.e. } x \in \mathbb{R}^n,$$

(2.2)

$$u_t(x,0) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(x,\lambda_k), \quad \text{for a.e. } x \in \mathbb{R}^n,$$

Here,

$$X_p = L^p(\mathbb{R}^n), \quad 1 \leq p \leq \infty, \quad Y^{s,p} = H^{s,p}(\mathbb{R}^n), \quad Y^{s,p}_1 = H^{s,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

$$H^{s,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad Y^{s,p}_\infty = H^{s,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad Y^{s,p}_p = H^{s,p}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),$$

$$\|u\|_{Y^{s,p}_2} = \|u\|_{H^{s,p}(\mathbb{R}^n)} + \|u\|_{L^p(\mathbb{R}^n)} < \infty, \quad 1 \leq p \leq \infty.$$

Let $\hat{b}(\xi)$ is the Fourier transformation of $b(x)$, i.e. $\hat{b}(\xi) = Fb$ and let

$$\eta = \eta(\xi) = \left[a|\xi|^2 + \hat{b}(\xi)\right]^{\frac{1}{2}}.$$
Here,

\[ \tau_k = \tau_k (\xi) = \lambda_k \eta (\xi), \quad k = 1, 2, \ldots, m. \]

**Condition 2.1.** Assume \( a \geq 0 \), \( b \) is an integrable function whose \( \hat{b} (\xi) \geq 0 \) and \( a + \hat{b} (\xi) > 0 \) for all \( \xi \in \mathbb{R}^n \). Moreover, let

\[ D_0 (\xi) = 1 - \sum_{k=1}^{m} (\alpha_k + \beta_k) \cos \tau_k + \sum_{i,j=1}^{m} \alpha_i \beta_j \cos (\tau_i - \tau_j) \neq 0 \]

for all \( \xi \in \mathbb{R}^n \) with \( \xi \neq 0 \).

First we need the following lemmas:

**Lemma 2.1.** Let the Condition 2.1. holds. Then, the problem (2.1) – (2.2) has a unique solution.

**Proof.** By using of the Fourier transform, we get from (2.1) – (2.2):

\[ \hat{u}_{tt} (\xi, t) + \eta^2 (\xi) \hat{u} (\xi, t) = \hat{g} (\xi, t), \quad (2.3) \]

\[ \hat{u} (\xi, 0) = \hat{\varphi} (\xi) + \sum_{k=1}^{m} \alpha_k \hat{u} (\xi, \lambda_k), \quad \hat{u}_t (\xi, 0) = \hat{\psi} (\xi) + \sum_{k=1}^{m} \beta_k \hat{u} (\xi, \lambda_k), \quad (2.4) \]

where \( \hat{u} (\xi, t) \) is a Fourier transform of \( u (x, t) \) with respect to \( x \) and \( \hat{\varphi} (\xi), \hat{\psi} (\xi) \) are Fourier transform of \( \varphi, \psi \), respectively.

By using the variation of constants it is easy to see that the general solution of (2.3) is represented as

\[ \hat{u} (\xi, t) = g_1 (\xi) \cos \eta t + g_2 (\xi) \cos \eta t + \frac{1}{2\eta} \int_{0}^{t} \sin \eta (t - \tau) \hat{g} (\xi, \tau) d\tau, \quad (2.5) \]

where \( g_1, g_2 \) are general continuous differentiable functions. By taking the multipoint condition (2.4) from (2.5) we get that (2.3) – (2.4) has a solution for \( \xi \in \mathbb{R}^n \), when \( g_1, g_2 \) are solution of the following system

\[ g_1 - \sum_{k=1}^{m} \alpha_k \gamma_k + g_1 \cos \tau_k + g_2 \sin \tau_k = \hat{\varphi} (\xi), \quad (2.6) \]

\[ \eta g_2 - \sum_{k=1}^{m} \beta_k [\eta_k + \eta (-g_1 \sin \tau_k + g_2 \cos \tau_k)] = \hat{\psi} (\xi), \]

where

\[ \gamma_k = \gamma_k (\xi) = \frac{1}{2\eta} \int_{0}^{\lambda_k} \sin \eta (\lambda_k - \tau) \hat{g} (\xi, \tau) d\tau, \]

\[ \mu_k = \mu_k (\xi) = -\frac{1}{2} \int_{0}^{\lambda_k} \cos \eta (\lambda_k - \tau) \hat{g} (\xi, \tau) d\tau. \]
By Condition 2.1 we get

\[ D(\xi) = \eta(\xi) D_0(\xi) \neq 0 \]

for all \( \xi \neq 0 \). Solving the system (2.6), we obtain

\[ g_1 = \frac{D_1(\xi)}{D(\xi)}, \quad g_2 = \frac{D_1(\xi)}{D(\xi)}, \quad (2.7) \]

where

\[
D(\xi) = \begin{vmatrix}
1 & -\sum_{k=1}^{m} \alpha_k \sin \omega_k \\
\eta & 1 - \sum_{k=1}^{m} \beta_k \cos \omega_k
\end{vmatrix},
\]

\[
D_1(\xi) = \begin{vmatrix}
\hat{\phi}(\xi) + \sum_{k=1}^{m} \alpha_k \gamma_k & -\sum_{k=1}^{m} \alpha_k \sin \omega_k \\
\hat{\psi}(\xi) + \sum_{k=1}^{m} \beta_k \mu_k & 1 - \sum_{k=1}^{m} \beta_k \cos \omega_k
\end{vmatrix},
\]

\[
D_2(\xi) = \begin{vmatrix}
1 & \sum_{k=1}^{m} \alpha_k \cos \omega_k \\
-\sum_{k=1}^{m} \beta_k \sin \omega_k & \hat{\phi}(\xi) + \sum_{k=1}^{m} \alpha_k \gamma_k
\end{vmatrix}.
\]

Here,

\[
\gamma_k(\xi) = \frac{\lambda_k}{2\eta} \int_0^\lambda \sin \eta (\lambda_k - \tau) \hat{g}(\xi, \tau) d\tau,
\]

\[
\mu_k(\xi) = -\frac{\lambda_k}{2} \int_0^\lambda \cos \eta (\lambda_k - \tau) \hat{g}(\xi, \tau) d\tau.
\]

Hence, problem (2.3) – (2.4) has a unique solution expressed as (2.5), where \( g_1 \) and \( g_2 \) are defined by (2.7), i.e. problem (2.1) – (2.2) has a unique solution

\[
u(x, t) = F^{-1} [C(\xi, t) g_1(\xi)] + [F^{-1} S(\xi, t) g_2(\xi)] + \frac{1}{2\eta} \int_0^t F^{-1} [\sin \eta (t - \tau) \hat{g}(\xi, \tau)] d\tau.
\]

(2.8)

**Theorem 2.1.** Let the Condition 2.1 holds and \( s > \frac{n}{2} \). Then for \( \varphi, \psi, g(x, t) \in Y_{t,q}^{s,p} \) the solution (2.1) – (2.2) satisfies the following uniformly in \( t \in [0, T] \) estimate

\[
\|u\|_{X_{t,q}} + \|u_t\|_{X_{t,q}} \leq C_0 \left[ \|\varphi\|_{Y_{t,q}^{s,p}} + \right] \]

(2.9)
\[
\|\psi\|_{Y_{t,p}} + \int_0^t (\|g(.,\tau)\|_{Y_{t,p}} + \|g(.,\tau)\|_{X_1}) \, d\tau,
\]

where the positive constant \(C\) depends only on initial data.

**Proof.** From (2.7) we deduced that

\[
g_1 (\xi) = \eta^{-1}(\xi) \, D_0^{-1}(\xi) \left[ \eta \left( 1 - \sum_{k=1}^{m} \beta_k \cos \xi_k \right) \dot{\phi}(\xi) + \right.
\]
\[
 \left. \eta \sum_{k=1}^{m} \alpha_k \sin \xi_k \right] + \left( \sum_{k=1}^{m} \beta_k \mu_k \right) \left( \sum_{k=1}^{m} \alpha_k \sin \xi_k \right)
\]
\[
= \eta \sum_{k=1}^{m} \alpha_k \gamma_k \left( 1 - \sum_{k=1}^{m} \beta_k \cos \xi_k \right),
\]

(2.10)

\[
g_2 (\xi) = \eta^{-1}(\xi) \, D_0^{-1}(\xi) \left[ \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \xi_k \right) \dot{\psi}(\xi) + \right.
\]
\[
 \sum_{k=1}^{m} \beta_k \mu_k \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \xi_k \right) + \dot{\phi}(\xi) \sum_{k=1}^{m} \beta_k \sin \xi_k + \eta \sum_{k=1}^{m} \beta_k \gamma_k \left( \sum_{k=1}^{m} \alpha_k \sin \xi_k \right).
\]

Then, from (2.5), (2.8) and (2.10) we obtain that the solution (2.1) – (2.2) can be expressed as

\[
u (x, t) = S_1 (x, t) \phi + S_2 (x, t) \psi + \Phi (g) (x, t) + \frac{1}{2\eta} \int_0^t F^{-1} \left[ \sin \eta (t - \tau) \hat{g}(\xi, \tau) \right] \, d\tau,
\]

(2.11)

where

\[
S_1 (x, t) \phi = F^{-1} \left\{ D_0^{-1}(\xi) \left[ \left( 1 - \sum_{k=1}^{m} \beta_k \cos \xi_k \right) \sin (\eta t) \right] \right. + \]
\[
\left. \eta^{-1}(\xi) \sum_{k=1}^{m} \beta_k \sin \xi_k \cos (\eta t) \right\} \dot{\phi}(\xi),
\]

\[
S_2 (x, t) \psi = F^{-1} \left\{ \eta^{-1}(\xi) D_0^{-1}(\xi) \left( \sum_{k=1}^{m} \alpha_k \sin \xi_k \right) \sin (\eta t) \right. + \]
\[
\eta^{-1}(\xi) \left. \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \xi_k \right) \cos (\eta t) \right\} \dot{\psi}(\xi),
\]

7
Φ (x; t) = Φ (g) (x; t) = \left[ \sum_{j=1}^{n} \sum_{k=1}^{m} F^{-1} F_{jk} (ξ; t) \right],

where

Φ_{1k} (ξ; t) = \frac{1}{2} D_{0}^{-1} (ξ) \beta_k A_1 \cos (\eta t) \int_{0}^{\lambda k} \cos (\lambda k - \tau) \hat{g} (ξ, \tau) ,

Φ_{2k} (ξ; t) = \frac{1}{2} D_{0}^{-1} (ξ) \alpha_k A_2 \cos (\eta t) \int_{0}^{\lambda k} \sin (\lambda k - \tau) \hat{g} (ξ, \tau) ,

Φ_{3k} (ξ; t) = \frac{1}{2} D_{0}^{-1} (ξ) \beta_k B_1 \sin (\eta t) \int_{0}^{\lambda k} \cos (\lambda k - \tau) \hat{g} (ξ, \tau) ,

Φ_{4k} (ξ; t) = \frac{1}{2} D_{0}^{-1} (ξ) \alpha_k B_2 \sin (\eta t) \int_{0}^{\lambda k} \sin (\lambda k - \tau) \hat{g} (ξ, \tau) ,

dear,

A_1 = \sum_{k=1}^{m} \alpha_k \sin \varphi_k , \quad A_2 = 1 - \sum_{k=1}^{m} \beta_k \cos \varphi_k ,

B_1 = \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \varphi_k \right) , \quad B_2 = \sum_{k=1}^{m} \beta_k \cos \varphi_k .

By Condition 2.1,

D_{0}^{-1} (ξ) \eta^{-1} (ξ) , \quad \eta^{-1} \left( \sum_{k=1}^{m} \alpha_k \sin \varphi_k \right) , \quad \eta^{-1} \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \varphi_k \right)

are uniformly bounded. From (2.11) and (2.8) we obtain

|g_1 (ξ)| \leq |\hat{g} (ξ)| + |\hat{\psi} (ξ)| + |Φ (ξ)| ,

(2.12)

|g_2 (ξ)| \leq |\hat{g} (ξ)| + |\hat{\psi} (ξ)| + |Φ (ξ)| ,

where

Φ (ξ) = \sum_{k=1}^{m} \int_{0}^{\lambda k} \hat{g} (ξ, \tau) d\tau ,

Let N ∈ \mathbb{N} and

Π_N = \{ ξ : ξ \in \mathbb{R}^n , |ξ| \leq N \} , \quad \Pi'_N = \{ ξ : ξ \in \mathbb{R}^n , |ξ| \geq N \} .
From (2.8) we deduced that
\[
\|S_1 (x, t) g_1\|_{X_\infty} + \|S_2 (x, t) g_2\|_{X_\infty} \lesssim \]
\[
\|F^{-1} C (\xi, t) g_1 (\xi)\|_{L^\infty (\Pi_N')} + \|F^{-1} S (\xi, t) g_2 (\xi)\|_{L^\infty (\Pi_N')} + \]
\[
\|F^{-1} C (\xi, t) g_1 (\xi)\|_{L^\infty (\Pi_N')} + \|F^{-1} S (\xi, t) g_2 (\xi)\|_{L^\infty (\Pi_N')} .
\]

From (2.10) – (2.13) due to uniform boundedness of \(D_0^{-1} (\xi)\) and \(C (\xi, t), S (\xi, t)\) we have
\[
\|S_1 (x, t) g_1\|_{X_\infty} + \|S_2 (x, t) g_2\|_{X_\infty} \lesssim \|F^{-1} \hat{\varphi} (\xi)\|_{X_\infty} + \|F^{-1} \hat{\psi} (\xi)\|_{X_\infty} +
\]
\[
\left\| F^{-1} \sum_{k=1}^m \Phi (\xi) \right\|_{X_\infty},
\]
\[
\|S_1 (x, t) g_1\|_{X_\infty} + \|S_2 (x, t) g_2\|_{X_\infty} \lesssim \|F^{-1} \hat{\varphi} (\xi)\|_{X_\infty} + \|F^{-1} \hat{\psi} (\xi)\|_{X_\infty} +
\]
\[
\|F^{-1} \Phi (\xi)\|_{X_\infty},
\]

In view of (2.12), by using the Minkowski’s inequality for integrals from above we get
\[
\|F^{-1} C (\xi, t) g_1 (\xi)\|_{L^\infty (\Pi_N')} + \|F^{-1} S (\xi, t) g_2 (\xi)\|_{L^\infty (\Pi_N')} \lesssim \quad (2.14)
\]

\[
\left[\|\varphi\|_{X_1} + \|\psi\|_{X_1} + \|g\|_{X_1}\right].
\]

Moreover, by (2.11) and (2.12) we have
\[
\|F^{-1} C (\xi, t) g_1 (\xi)\|_{L^\infty (\Pi_N')} + \|F^{-1} S (\xi, t) g_2 (\xi)\|_{L^\infty (\Pi_N')} \lesssim
\]
\[
\|F^{-1} C (\xi, t) \hat{\varphi} (\xi)\|_{L^\infty (\Pi_N')} + \|F^{-1} S (\xi, t) \hat{\psi} (\xi)\|_{L^\infty (\Pi_N')} +
\]
\[
\|F^{-1} \Phi (\xi)\|_{L^\infty (\Pi_N')} \lesssim
\]
\[
\left| F^{-1} \left( 1 + |\xi|^2 \right)^{-\frac{7}{2}} C (\xi, t) \left( 1 + |\xi|^2 \right)^{\frac{7}{2}} \hat{\varphi} (\xi) \right|_{L^\infty (\Pi_N')} + \quad (2.15)
\]
\[
\left| F^{-1} \left( 1 + |\xi|^2 \right)^{-s} S (\xi, t) \left( 1 + |\xi| \right)^{\frac{7}{2}} \hat{\psi} (\xi) \right|_{L^\infty (\Pi_N')} +
\]
\[
\left| F^{-1} \left( 1 + |\xi|^2 \right)^{-s} S (\xi, t) \left( 1 + |\xi| \right)^{\frac{7}{2}} \Phi (\xi) \right|_{L^\infty (\Pi_N')} +
\]

9
By using (2.5), (2.7) and (12.2) we get

\[
\sup_{\xi \in \mathbb{R}^n, t \in [0, T]} |\xi|^{\alpha + \frac{n}{p}} D^\alpha \left[ \left(1 + |\xi|^2\right)^{-\frac{s}{2}} C(\xi, t) \right] \leq C_2,
\]

\[
\sup_{\xi \in \mathbb{R}^n, t \in [0, T]} |\xi|^{\alpha + \frac{n}{p}} D^\alpha \left[ \left(1 + |\xi|^2\right)^{-\frac{s}{2}} S(\xi, t) \right] \leq C_2
\]

(2.16)

for \( s > \frac{n}{p} \), \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \), \( \alpha_k \in \{0, 1\}, \xi \in \mathbb{R}^n \) and uniformly in \( t \in [0, T] \). By multiplier theorems (see e.g. [24]) from (2.16) we get that the functions \( \left(1 + |\xi|^2\right)^{-\frac{s}{2}} C(\xi, t), \left(1 + |\xi|^2\right)^{-\frac{s}{2}} S(\xi, t) \) are \( \mathcal{L}^p(R^n) \rightarrow \mathcal{L}^\infty(R^n) \) Fourier multipliers. Then by Minkowski’s inequality for integrals, from (2.11) and (2.14) − (2.16) we obtain

\[
\|F^{-1}C(\xi, t) g_1(\xi)\|_{\mathcal{L}^\infty(\mathbb{R}^n)} + \|F^{-1}S(\xi, t) g_2(\xi)\|_{\mathcal{L}^\infty(\mathbb{R}^n)} \lesssim \|\phi\|_{\mathcal{Y}^{s,p}} + \|\psi\|_{\mathcal{Y}^{s,p}} + \|g\|_{\mathcal{Y}^{s,p}}.
\]

(2.17)

By reasoning as the above we have

\[
\|F^{-1}\Phi(\xi)\|_{X_p} \leq C \int_0^t \left(\|g(\cdot, \tau)\|_{\mathcal{Y}^{s,p}} + \|g(\cdot, \tau)\|_{X_1}\right) d\tau.
\]

(2.18)

Thus, from (2.8) and (2.15) we obtain

\[
\|u\|_{X_p} \leq C \left[ \|\phi\|_{\mathcal{Y}^{s,p}} + \|\psi\|_{X_1} + \|\psi\|_{\mathcal{Y}^{s,p}} + \|\psi\|_{X_1} + \int_0^t \left(\|g(\cdot, \tau)\|_{\mathcal{Y}^{s,p}} + \|g(\cdot, \tau)\|_{X_1}\right) d\tau \right].
\]

(2.19)

By using (2.5), (2.7) and in view of (2.17) in similar way, we deduced the estimate of type (2.19) for \( u_t \), i.e. we obtain the assertion.

**Theorem 2.2.** Let the Condition 2.1 holds and \( s > \frac{n}{p} \). Then for \( \varphi, \psi, g(x, t) \in \mathcal{Y}^{s,p} \) the solution of (2.1) − (2.2) satisfies the following uniform estimate

\[
(\|u\|_{\mathcal{Y}^{s,p}} + \|u_t\|_{\mathcal{Y}^{s,p}}) \leq C_0 \left( \|\varphi\|_{\mathcal{Y}^{s,p}} + \|\psi\|_{\mathcal{Y}^{s,p}} + \int_0^t \|g(\cdot, \tau)\|_{\mathcal{Y}^{s,p}} d\tau \right).
\]

(2.20)

**Proof.** From (2.7) and (2.12) we have the following uniform estimate

\[
\left( \left\|F^{-1} \left(1 + |\xi|^2\right)^{s/2} \tilde{u}_t \right\|_{X_p} + \left\|F^{-1} \left(1 + |\xi|^2\right)^{s/2} \tilde{u}_t \right\|_{X_p} \right) \leq (2.21)
\]
\[
C \left\{ \left\| F^{-1} \left( 1 + |\xi| \right)^{s} C (\xi, t) \right\|_{X_p} + \left\| F^{-1} \left( 1 + |\xi| \right)^{s} S (\xi, t) \right\|_{X_p} + \right. \\
\left. \int _{0}^{t} \left\| (1 + |\xi|^{s} g (., \tau) \right\|_{X_p} d\tau \right\}.
\]

By Condition 2.1 and by virtue of Fourier multiplier theorems (see [24, § 2.2]) we get that \( C (\xi, t), S (\xi, t) \) and \( \Phi (\xi) \) are Fourier multipliers in \( L^{p} (\mathbb{R}^{n}) \) uniformly with respect to \( t \in [0, T] \). So, the estimate (2.21) by using the Minkowski’s inequality for integrals implies (2.20).

3. Local well posedness of IVP for nonlinear nonlocal equation

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1) \( - (1.2) \). For the study of the nonlinear problem (1.1) \( - (1.2) \) we need the following lemmas

**Lemma 3.1** (Nirenberg’s inequality) [25]. Assume that \( u \in L^{p} (\Omega), D^{m} u \in L^{q} (\Omega), p, q \in (1, \infty) \). Then for \( i \) with \( 0 \leq i \leq m, m > \frac{n}{q} \) we have

\[
\left\| D^{i} u \right\|_{p} \leq C \left\| u \right\|_{p}^{1 - \mu} \sum_{k=1}^{n} \left\| D_{k}^{m} u \right\|_{q}^{\mu}, \quad (3.1)
\]

where

\[
\frac{1}{r} = \frac{i}{m} + \mu \left( \frac{1}{q} - \frac{m}{n} \right) + (1 - \mu) \frac{1}{p}, \quad \frac{i}{m} \leq \mu \leq 1.
\]

**Lemma 3.2** [26]. Assume that \( u \in W^{m,p} (\Omega) \cap L^{\infty} (\Omega) \) and \( f (u) \) possesses continuous derivatives up to order \( m \geq 1 \). Then \( f (u) - f (0) \in W^{m,p} (\Omega) \) and

\[
\left\| f (u) - f (0) \right\|_{p} \leq \left\| f^{(1)} (u) \right\|_{\infty} \left\| u \right\|_{p},
\]

\[
\left\| D^{k} f (u) \right\|_{p} \leq C_{0} \sum_{j=1}^{k} \left\| f^{(j)} (u) \right\|_{\infty} \left\| u \right\|_{\infty}^{j-1} \left\| D^{k} u \right\|_{p}, \quad 1 \leq k \leq m, \quad (3.2)
\]

where \( C_{0} \geq 1 \) is a constant.

Let

\[
X_{p} = L^{p} (\mathbb{R}^{n}), \quad \left\| u \right\|_{p} = \left\| u \right\|_{X_{p}}, \quad Y = W^{2,p} (\mathbb{R}^{n}), \quad E_{0} = (X_{p}, Y)_{\frac{1}{2}p, p} = B_{p}^{2-\frac{1}{p}} (\mathbb{R}^{n}).
\]

**Remark 3.1.** By using J.Lions-I. Petree result (see e.g. [21, § 1.8.]) we obtain that the map \( u \rightarrow u (t_{0}), t_{0} \in [0, T] \) is continuous and surjective from \( W^{2,p} (0, T) \) onto \( E_{0} \) and there is a constant \( C_{1} \) such that

\[
\left\| u (t_{0}) \right\|_{E_{0}} \leq C_{1} \left\| u \right\|_{W^{2,p} (0, T)}, \quad 1 \leq p \leq \infty.
\]
First all of, we define the space $Y(T) = C([0, T]; Y_{\infty}^{2,p})$ equipped with the norm defined by

$$
\|u\|_{Y(T)} = \max_{t \in [0,T]} \|u\|_{Y_{\infty}^{2,p}} + \max_{t \in [0,T]} \|u\|_{X_{\infty}}, \ u \in Y(T).
$$

It is easy to see that $Y(T)$ is a Banach space. For $\varphi, \psi \in Y_{\infty}^{2,p}$, let

$$
M = \|\varphi\|_{Y_{\infty}^{2,p}} + \|\varphi\|_{X_{\infty}} + \|\psi\|_{Y_{\infty}^{2,p}} + \|\psi\|_{X_{\infty}}.
$$

**Definition 3.1.** For any $T > 0$ if $\varphi, \psi \in Y_{\infty}^{2,p}$ and $u \in C([0, T]; Y_{\infty}^{2,p})$ satisfies the equation (1.1) – (1.2) then $u(x,t)$ is called the continuous solution or the strong solution of the problem (1.1) – (1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem (1.1) – (1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of the problem (1.1) – (1.2).

**Condition 3.1.** Assume:

1. Assume that the kernel $g$ is an integrable function whose Fourier transform satisfies

$$
0 \leq \hat{g}(\xi) \leq \left(1 + |\xi|^2\right)^{-1} \text{ for all } \xi \in \mathbb{R}^n;
$$

2. The Condition 2.1 holds, $\varphi, \psi \in Y_{\infty}^{2,p}$ for $1 < p < \infty$ and $\frac{3}{2} < 2$;

3. the function $u \rightarrow f(x,t,u) : \mathbb{R}^n \times [0, T] \times E_0 \rightarrow E$ is a measurable in $(x,t) \in \mathbb{R}^n \times [0, T]$ for $u \in E_0; f(x,t,u)$. Moreover, $F(x,t,u)$ is continuous in $u \in E_0$ and $f(x,t,u) \in C^{(3)}(E_0; E)$ uniformly with respect to $x \in \mathbb{R}^n$,

Main aim of this section is to prove the following result:

**Theorem 3.1.** Let the Condition 3.1. holds. Then problem (1.1) – (1.2) has a unique local strange solution $u \in C^{(2)}([0, T_0]; Y_{\infty}^{2,p})$, where $T_0$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup_{t \in [0,T_0]} \left(\|u\|_{Y_{\infty}^{2,p}} + \|u_t\|_{Y_{\infty}^{2,p}}\right) < \infty
$$

then $T_0 = \infty$.

**Proof.** First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1) – (1.2) by contraction mapping principle. Consider a map $G$ on $Y(T)$ such that $G(u)$ is the solution of the Cauchy problem

$$
G_t(u) - a \Delta G(u) = \Delta [g * f(G(u))], \ x \in \mathbb{R}^n, \ t \in (0, T),
$$

$$
G(u)(x,0) = \varphi(x) + \sum_{k=1}^{m} \alpha_k G(u)(x, \lambda_k), \ \text{for a.e. } \ x \in \mathbb{R}^n,
$$

$$
G_t(u)(0,x) = \psi(x) + \sum_{k=1}^{m} \beta_k G_t(u)(x, \lambda_k), \ \text{for a.e. } \ x \in \mathbb{R}^n.
$$
From Lemma 3.2 we know that \( F(u) \in L^p(0, T; Y_{\infty}^{2,p}) \) for any \( T > 0 \). Thus, by Lemma 2.1, problem (3.4) has a solution which can be written as

\[
G(u)(x, t) = [S_1(x, t) \varphi + S_2(x, t) \psi + \Phi(g \ast f(G(u)))] + \tag{3.5}
\]

\[
\frac{1}{2\pi} \int_0^t F^{-1} \left[ \sin \eta (t - \tau) |\xi|^2 \hat{g}(\xi) \hat{f}(G(u)(\xi)) \right] d\tau,
\]

where \( S_1(x, t), S_2(x, t), \Phi \) are operator functions defined by (2.10) and (2.11), where \( g \) replaced by \( g \ast f(G(u)) \). From Lemma 3.2 it is easy to see that the map \( G \) is well defined for \( f \in C^{(2)}(X_0; \mathbb{C}) \). We put

\[
Q(M; T) = \{ u \mid u \in Y(T), \|u\|_{Y(T)} \leq M + 1 \}.
\]

First, by reasoning as in [9] let us prove that the map \( G \) has a unique fixed point in \( Q(M; T) \). For this aim, it is sufficient to show that the operator \( G \) maps \( Q(M; T) \) into \( Q(M; T) \) and \( G: Q(M; T) \to Q(M; T) \) is strictly contractive if \( T \) is appropriately small relative to \( M \). Consider the function \( \tilde{f}(\xi): [0, \infty) \to [0, \infty) \) defined by

\[
\tilde{f}(\xi) = \max_{|x| \leq \xi} \left\{ \left\| f^{(1)}(x) \right\|_{C}, \left\| f^{(2)}(x) \right\|_{C} \right\}, \quad \xi \geq 0.
\]

It is clear to see that the function \( \tilde{f}(\xi) \) is continuous and nondecreasing on \([0, \infty)\). From Lemma 3.2 we have

\[
\|f(u)\|_{Y_{2,p}} \leq \left\| f^{(1)}(u) \right\|_{X_{\infty}} \|u\|_{X_p} + \left\| f^{(1)}(u) \right\|_{X_{\infty}} \|Du\|_{X_p} + \nonumber
\]

\[
C_0 \left[ \left\| f^{(1)}(u) \right\|_{X_{\infty}} \|u\|_{X_p} + \left\| f^{(2)}(u) \right\|_{X_{\infty}} \|u\|_{X_p} \right] \leq \|D^2u\|_{X_p} \leq (3.6)
\]

\[
2C_0 \tilde{f}(M + 1)(M + 1) \|u\|_{Y_{2,p}}.
\]

In view of the assumption (1) and by using Minkowski’s inequality for integrals we obtain from (3.5):

\[
\|G(u)\|_{X_{\infty}} \leq \|\varphi\|_{X_{\infty}} + \|\psi\|_{X_{\infty}} + \int_0^t \|\Delta [g \ast f(G(u))](x, \tau)\|_{X_{\infty}} d\tau, \tag{3.7}
\]

\[
\|G(u)\|_{Y_{2,p}} \leq \|\varphi\|_{Y_{2,p}} + \|\psi\|_{Y_{2,p}} + \int_0^t \|\Delta [g \ast f(G(u))](x, \tau)\|_{Y_{2,p}} d\tau. \tag{3.8}
\]

Thus, from (3.6) – (3.8) and Lemma 3.2 we get

\[
\|G(u)\|_{Y(T)} \leq M + T(M + 1) \left[ 1 + 2C_0(M + 1) \tilde{f}(M + 1) \right] .
\]
If $T$ satisfies
\[ T \leq \left\{ (M + 1) \left[ 1 + 2C_0 (M + 1) \bar{f} (M + 1) \right] \right\}^{-1}, \tag{3.9} \]
then
\[ \|Gu\|_{Y(T)} \leq M + 1. \]
Therefore, if (3.9) holds, then $G$ maps $Q (M; T)$ into $Q (M; T)$. Now, we are going to prove that the map $G$ is strictly contractive. Assume $T > 0$ and $u_1$, $u_2 \in Q (M; T)$ given. We get
\[ G(u_1) - G(u_2) = \int_0^t F^{-1} S (t - \tau, \xi) |\xi|^2 \hat{g} (\xi) \left[ \hat{f} (u_1) (\xi, \tau) - \hat{f} (u_2) (\xi, \tau) \right] d\tau, \; t \in (0, T). \]
By using the assumption (3) and the mean value theorem, we obtain
\[ \hat{f} (u_1) - \hat{f} (u_2) = \hat{f}^{(1)} (u_2 + \eta_1 (u_1 - u_2)) (u_1 - u_2), \]
\[ D_\xi \left[ \hat{f} (u_1) - \hat{f} (u_2) \right] = \hat{f}^{(2)} (u_2 + \eta_2 (u_1 - u_2)) (u_1 - u_2) D_\xi u_1 + \]
\[ \hat{f}^{(1)} (u_2) (D_\xi u_1 - D_\xi u_2), \]
\[ D_\xi^2 \left[ \hat{f} (u_1) - \hat{f} (u_2) \right] = \hat{f}^{(3)} (u_2 + \eta_3 (u_1 - u_2)) (u_1 - u_2) (D_\xi u_1)^2 + \]
\[ \hat{f}^{(2)} (u_2) (D_\xi u_1 - D_\xi u_2) (D_\xi u_1 + D_\xi u_2) + \]
\[ \hat{f}^{(2)} (u_2 + \eta_4 (u_1 - u_2)) (u_1 - u_2) D_\xi^2 u_1 + \hat{f}^{(1)} (u_2) (D_\xi^2 u_1 - D_\xi^2 u_2), \]
where $0 < \eta_i < 1$, $i = 1, 2, 3, 4$. Thus, using Hölder’s and Nirenberg’s inequality, we have
\[ \left\| \hat{f} (u_1) - \hat{f} (u_2) \right\|_{X_p} \leq \hat{f} (M + 1) \left\| u_1 - u_2 \right\|_{X_p}, \tag{3.10} \]
\[ \left\| \hat{f} (u_1) - \hat{f} (u_2) \right\|_{X_p} \leq \hat{f} (M + 1) \left\| u_1 - u_2 \right\|_{X_p}, \tag{3.11} \]
\[ \left\| D_\xi \left[ \hat{f} (u_1) - \hat{f} (u_2) \right] \right\|_{X_p} \leq (M + 1) \hat{f} (M + 1) \left\| u_1 - u_2 \right\|_{X_p} + \hat{f} (M + 1) \left\| D_\xi u_1 \right\|_{Y_{2,p}}^2 + \hat{f} (M + 1) \left\| D_\xi (u_1 - u_2) \right\|_{Y_{2,p}} \left\| D_\xi (u_1 + u_2) \right\|_{Y_{2,p}} + \hat{f} (M + 1) \left\| u_1 - u_2 \right\|_{X_\infty} \left\| D_\xi^2 u_1 \right\|_{X_p} + \hat{f} (M + 1) \left\| D_\xi (u_1 - u_2) \right\|_{X_p} \leq \]
\[ C^2 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|u_1\|_{X_\infty} \|D^2_\xi u_1\|_{X_p} + \]

\[ C^2 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} \|D^2_\xi (u_1 - u_2)\|_{X_p} + \]

\[ + (M + 1) \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} + \bar{f}(M + 1) \|D^2_\xi (u_1 - u_2)\|_{X_p} \leq \]

\[ 3C^2 (M + 1)^2 \bar{f}(M + 1) \|u_1 - u_2\|_{X_\infty} + 2C^2 (M + 1) \bar{f}(M + 1) \|D^2_\xi (u_1 - u_2)\|_{X_p}, \]

where \( C \) is the constant in Lemma 3.1. From (3.10) – (3.11), using Minkowski’s inequality for integrals, Fourier multiplier theorems in \( X_p \) spaces and Young’s inequality, we obtain

\[ \|G(u_1) - G(u_2)\|_{Y(T)} \leq \int_0^t \|u_1 - u_2\|_{X_\infty} \, d\tau + \int_0^t \|u_1 - u_2\|_{Y,2,p} \, d\tau + \]

\[ \int_0^t \|f(u_1) - f(u_2)\|_{X_\infty} \, d\tau + \int_0^t \|f(u_1) - f(u_2)\|_{Y,2,p} \, d\tau \leq \]

\[ T \left[ 1 + C_1(M + 1)^2 \bar{f}(M + 1) \right] \|u_1 - u_2\|_{Y(T)}, \]

where \( C_1 \) is a constant. If \( T \) satisfies (3.9) and the following inequality holds

\[ T \leq \frac{1}{2} \left[ 1 + C_1(M + 1)^2 \bar{f}(M + 1) \right]^{-1}, \]

then

\[ \|Gu_1 - Gu_2\|_{Y(T)} \leq \frac{1}{2} \|u_1 - u_2\|_{Y(T)}. \]

That is, \( G \) is a contractive map. By contraction mapping principle we know that \( G(u) \) has a fixed point \( u(x, t) \in Q(M; T) \) that is a solution of (1.1) – (1.2). From (2.9) – (2.11) we get that \( u \) is a solution of the following integral equation

\[ u(x, t) = S_1(t)g_1 + S_2(t)g_2 - \]

\[ \int_0^t F^{-1}\left[S(t - \tau, \xi) |\xi|^2 \hat{g}(\xi, \bar{f}(u)(\xi, \tau)) \right] \, d\tau, \quad t \in (0, T). \]

Let us show that this solution is a unique in \( Y(T) \). Let \( u_1, u_2 \in Y(T) \) are two solution of the problem (1.1) – (1.2). Then

\[ u_1 - u_2 = \int_0^t F^{-1}\left[S(t - \tau, \xi) |\xi|^2 \hat{g}(\xi, \bar{f}(u_1)(\xi, \tau) - \bar{f}(u_2)(\xi, \tau)) \right] \, d\tau. \quad (3.15) \]

By the definition of the space \( Y(T) \), we can assume that

\[ \|u_1\|_{X_\infty} \leq C_1(T), \quad \|u_1\|_{X_\infty} \leq C_1(T). \]
Hence, by Minkowski’s inequality for integrals and Theorem 2.2 we obtain from (3.15)

$$\|u_1 - u_2\|_{Y^{2,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{2,p}} \, d\tau. \quad (3.16)$$

From (3.16) and Gronwall’s inequality, we have $$\|u_1 - u_2\|_{Y^{2,p}} = 0$$, i.e. problem (1.1)–(1.2) has a unique solution which belongs to $$Y(T)$$. That is, we obtain the first part of the assertion.

Now, let $$[0, T_0)$$ be the maximal time interval of existence for $$u \in Y(T_0)$$. It remains only to show that if (3.3) is satisfied, then $$T_0 = \infty$$. Assume contrary that, (3.3) holds and $$T_0 < \infty$$. For $$T \in [0, T_0)$$, we consider the following integral equation

$$v(x, t) = S_1(t) u(x, T) + S_2(t) u_t(x, T) - \int_0^t F^{-1} \left[ S(t - \tau, \xi) |\xi|^2 \hat{g}(\xi) \hat{f}(v)(\xi, \tau) \right] \, d\tau, \quad t \in (0, T). \quad (3.17)$$

By virtue of (3.3), for $$T' > T$$ we have

$$\sup_{t \in [0, T)} \left( \|u\|_{Y^{2,p}} + \|u\|_{X^\infty} + \|u_t\|_{Y^{2,p}} + \|u_t\|_{X^\infty} \right) < \infty. \quad (3.18)$$

By reasoning as a first part of theorem and by contraction mapping principle, there is a $$T^* \in (0, T_0)$$ such that for each $$T \in [0, T_0)$$, the equation (3.17) has a unique solution $$v \in Y(T^*)$$. The estimates (3.9) and (3.14) imply that $$T^*$$ can be selected independently of $$T \in [0, T_0)$$. Set $$T = T_0 - \frac{T^*}{2}$$ and define

$$\tilde{u}(x, t) = \begin{cases} u(x, t), & t \in [0, T] \\ v(x, t - T), & t \in [T, T_0 + \frac{T^*}{2}] \end{cases}. \quad (3.18)$$

By construction $$\tilde{u}(x, t)$$ is a solution of the problem (1.1)–(1.2) on $$[T, T_0 + \frac{T^*}{2}]$$ and in view of local uniqueness, $$\tilde{u}(x, t)$$ extends u. This is against to the maximality of $$[0, T_0)$$, i.e we obtain $$T_0 = \infty$$.

Consider the problem (1.1)–(1.2), when $$\varphi, \psi \in H^s$$. We first need two lemmas concerning the behaviour of the nonlinear term [8, 13, 27].

**Lemma 3.3.** Let $$s \geq 0$$, $$f \in C^{[s]+1}(R)$$ with $$f(0) = 0$$. Then for any $$u \in H^s \cap L^\infty$$, we have $$f(u) \in H^s \cap L^\infty$$. Moreover there is some constant $$A(M)$$ depending on $$M$$ such that for all $$u \in H^s \cap L^\infty$$ with $$\|u\|_{L^\infty} \leq M$$,

$$\|f(u)\|_{H^s} \leq A(M) \|u\|_{H^s}. \quad (3.19)$$
Lemma 3.4. Let \( s \geq 0, f \in C^{[s]+1}(R) \). Then for any \( M \) there is some constant \( B(M) \) depending on \( M \) such that for all \( u, v \in H^s \cap L^\infty \) with \( \|u\|_{L^m} \leq M, \|v\|_{L^\infty} \leq M, \|u\|_{H^s} \leq M, \|v\|_{H^s} \leq M \),

\[
\|f(u) - f(v)\|_{H^s} \leq B(M) \|u - v\|_{H^s}, \quad \|f(u) - f(v)\|_{L^\infty} \leq B(M) \|u - v\|_{L^\infty}.
\]

By reasoning as in Theorem 3.1 and [13, Theorem 1.1] we have

**Corollary 3.1.** Let \( s > \frac{n}{2}, f \in C^{[s]+1}(R) \). Then for any \( B \) there is some constant \( B(M) \) depending on \( M \) such that for all \( u, v \in H^s \) with \( \|u\|_{H^s} \leq M, \|v\|_{H^s} \leq M \),

\[
\|f(u) - f(v)\|_{H^s} \leq B(M) \|u - v\|_{H^s}.
\]

**Lemma 3.5.** If \( s > 0 \), then \( Y^s_{\infty,2} \) is an algebra. Moreover, for \( f, g \in Y^s_{\infty,2} \),

\[
\|fg\|_{H^s} \leq C(\|f\|_{H^s} + \|g\|_{H^s} + \|f\| + \|g\|) - \delta.
\]

**Lemma 3.6** [28, Lemma X 4]. Let \( s \geq 0, f \in C^{[s]+1}(R) \) and \( f(u) = O(\|u\|^{\alpha+1}) \) for \( u \to 0, \alpha \geq 1 \) be a positive integer. If \( u \in Y^s_{\infty,2} \) and \( \|u\|_{\infty} \leq M \), then

\[
\|f(u)\|_{H^s} \leq C(M) \|u\|_{H^s} \|u\|_{\infty}^\alpha,
\]

\[
\|f(u)\|_{1} \leq C(M) \|u\|_{2} \|u\|_{\infty}^{\alpha-1}.
\]

**Lemma 3.7** [13, Lemma 3.4]. Let \( s \geq 0, f \in C^{[s]+1}(R) \) and \( f(u) = O(\|u\|^{\alpha+1}) \) for \( u \to 0, \alpha \geq 1 \) be a positive integer. If \( u, v \in Y^s_{\infty,2} \), \( \|u\|_{H^s} \leq M, \|v\|_{H^s} \leq M \) and \( \|u\|_{\infty} \leq M, \|v\|_{\infty} \leq M \), then

\[
\|f(u) - f(v)\|_{H^s} \leq C(M) \|u\|_{\infty} - \|v\|_{\infty} \|u\|_{H^s} + \|v\|_{H^s},
\]

\[
(||u||_{\infty} + ||v||_{\infty})^{\alpha-1},
\]

\[
\|f(u) - f(v)\|_{1} \leq C(M) (||u||_{\infty} + ||v||_{\infty})^{\alpha-1} (||u||_{2} + ||v||_{2}) \|u - v\|_{2}.
\]

By reasoning as in [31, Theorem 1.1] we have:

**Theorem 3.2.** Let the Condition 3.1 hold. Assume \( f \in C^k(R) \), with \( k \) an integer \( k \geq s > \frac{n}{2} \), satisfies \( f(u) = O(\|u\|^{\alpha+1}) \) for \( u \to 0 \). Then there exists a constant \( \delta > 0 \), such that for any \( \varphi, \psi \in Y^s_{1,2} \) satisfying

\[
\|\varphi\|_{Y^s_{1,2}} + \|\psi\|_{Y^s_{1,2}} \leq \delta,
\]

problem (1.1) – (1.2) has a unique local strange solution \( u \in C(2) ([0, \infty); Y^s_{\infty,2}) \). Moreover,

\[
\sup_{0 \leq t < \infty} (||u||_{Y^s_{\infty,2}} + ||u||_{Y^s_{\infty,2}}) \leq C\delta,
\]

where the constant \( C \) only depends on \( f \) and initial data.
Proof. Consider a metric space defined by

\[ W^s = \{ u \in (2) ([0, \infty); Y^{s,2}) , \| u \|_{W^s} \leq 3C_0 \delta \}, \]

equipped with the norm

\[ \| u \|_{W^s} = \sup_{t \geq 0} (\| u \|_{Y^{s,2}} + \| u_t \|_{Y^{s,2}}), \]

where \( \delta > 0 \) satisfies (3.19) and \( C_0 \) is a constant in Theorem 2.1. It is easy to prove that \( W^s \) is a complete metric space. From Sobolev imbedding theorem we know that \( \| u \|_\infty \leq 1 \) if we take that \( \delta \) is enough small. Consider the problem (3.4). From Lemma 3.6 we get that \( f(u) \in L^2(0,T; Y_{1,2}^{s,1}) \) for any \( T > 0 \). Thus the problem (3.4) has a unique solution which can be written as (3.5). We should prove that the operator \( G(u) \) defined by (3.5) is strictly contractive if \( \delta \) is suitable small. In fact, by (2.9) in Theorem 2.1 and Lemma 3.6 we get

\[
\begin{align*}
\| G(u) \|_{W^s} &\leq C_0 \left[ \| \phi \|_{Y^{s,2}} + \| \psi \|_{Y^{s,2}} + \right. \\
&\left. \int_0^t \| K(u)(\cdot, \tau) \|_{Y^{s,2}} \, d\tau \right] \leq C_0 \delta + C_0 \int_0^t \| K(u)(\cdot, \tau) \|_{Y^{s,2}} \, d\tau \leq C_0 \delta + C \int_0^t \left[ \| u(\cdot, \tau) \|_{Y^{s,2}}^2 + \| u(\cdot, \tau) \|_\infty^{\alpha-1} + \| u(\tau) \|_{H^s}^2 \| u(\tau) \|_\infty^\alpha \right] \, d\tau \leq C_0 \delta + C \| u \|_{W^s}^{\alpha+1}. \tag{3.21}
\end{align*}
\]

On the other hand, by (2.20) in Theorem 2.2 and Lemma 3.6 we have

\[
\begin{align*}
\| G(u) \|_{H^s} + \| G_t(u) \|_{H^s} &\leq C_0 \left[ \| \phi \|_{H^s} + \| \psi \|_{H^s} + \right. \\
&\left. \int_0^t \| K(u)(\cdot, \tau) \|_{H^s} \, d\tau \right] \leq C_0 \delta + C_0 \int_0^t \| K(u)(\cdot, \tau) \|_{H^s} \, d\tau \leq C_0 \delta + C \int_0^t \| u(\cdot, \tau) \|_{H^s} \| u(\tau) \|_\infty^\alpha \, d\tau \leq C_0 \delta + C \| u \|_{W^s}^{\alpha+1}. \tag{3.22}
\end{align*}
\]

Therefore, combining (3.21) with (3.22) yields

\[
\| G(u) \|_{W^s} \leq 2C_0 \delta + C \| u \|_{W^s}^{\alpha+1}. \tag{3.23}
\]
Taking that $\delta$ is enough small such that $C (3C_0 \delta)^{\alpha} < 1/3$, from (3.23) and from Theorems 2.1, 2.2 we deduced that $G$ maps $W^s$ into $W^s$. Then, by reasoning as the remaining part of [13, Theorem 1.1] we obtain that $G : W^s \rightarrow W^s$ is strictly contractive. Using the contraction mapping principle, we know that $G(u)$ has a unique fixed point $u(x, t) \in C^2([0, \infty), H^s)$ and $u(x, t)$ is the solution of the problem (1.1) − (1.2).

We claim that the solution $u(x, t)$ of the problem (1.1)−(1.2) is also unique in $C^2([0, \infty), H^s)$. In fact, let $u_1$ and $u_2$ be two solutions of the problem (1.1)−(1.2) and $u_1, u_2 \in C^2([0, \infty), H^s)$. Let $u = u_1 - u_2$; then

$$u_{tt} - a\Delta u + b * u = \Delta [g * (f(u_1) - f(u_2))].$$

This fact is derived in a similar way as in Theorem 3.2, by using Theorems 2.1, 2.2 and Gronwall’s inequality.

**Condition 3.2.** Let the Condition 2.1 holds. Assume $f \in C^{[s]+1}(R)$ with $f(0) = 0$ for some $s \geq 0$.

(2) Assume that the kernel $g$ is an integrable function whose Fourier transform satisfies

$$0 \leq \hat{g}(\xi) \leq \left(1 + |\xi|^2\right)^{-\frac{\gamma}{2}} \text{ for all } \xi \in R^n \text{ and } r \geq 2.$$

**Theorem 3.3.** Let the Condition 3.2 hold. Moreover, $s \geq 0$ and $r \geq 2$. Then there is some $T > 0$ such that the multipoint IVB (1.1) − (1.2) is well posed with solution in $C^2([0, T]; H^s)$ for initial data $\varphi, \psi \in H^s$.

**Proof.** Consider the convolution operator $u \rightarrow Ku = \Delta [g * f(u)]$. In view of assumptions we have

$$\|\Delta g * v\|_{H^s} \lesssim \left\| (1 + |\xi|^2)^{\frac{\gamma}{2}} |\xi|^2 \hat{g}(\xi) \right\| \lesssim \|v\|_{H^s}, \quad (3.24)$$

i.e. $\Delta g * v$ is a bounded linear operator on $H^s$. Then by Corollary 3.1, $K(u)$ is locally Lipschitz on $H^s$. Then by reasoning as in Theorem 3.2 and [13, Theorem 1.1] we obtain that $G: H^s \rightarrow H^s$ is strictly contractive. Using the contraction mapping principle, we get that the operator $G(u)$ defined by (3.5) has a unique fixed point $u(x, t) \in C^2([0, \infty), H^s)$ and $u(x, t)$ is the solution of the problem (1.1) − (1.2). Moreover, we show that the solution $u(x, t)$ of (1.1) − (1.2) is also unique in $C^2([0, \infty), H^s)$. In fact, let $u_1$ and $u_2$ be two solutions of the problem (1.1) − (1.2) and $u_1, u_2 \in C^2([0, \infty), H^s)$. Let $u = u_1 - u_2$; then

$$u_{tt} - a\Delta u + b * u = \Delta [g * (f(u_1) - f(u_2))].$$

This fact is derived in a similar way as in Theorem 3.2, by using Theorems 2.1, 2.2 and Gronwall’s inequality.

**Theorem 3.4.** Let the Condition 3.2 hold and $r > 2 + \frac{n}{2}$. Then there is some $T > 0$ such that the multipoint IVB (1.1) − (1.2) is well posed with solution in $C^2([0, T]; Y_{\infty, 2}^s)$ for initial data $\varphi, \psi \in Y_{\infty, 2}^s$. 

19
Proof. All we need here, is to show that $K * f(u)$ is Lipschitz on $Y_{\infty}^{s,2}$. Indeed, by reasoning as in Theorem 3.3 we have

$$\| \Delta g \ast v \|_{H^{s+r-2}} \lesssim \left\| \left(1 + \xi \right)^{s+r-2} |\xi|^2 \hat{g}(\xi) \hat{v}(\xi) \right\| \lesssim \| v \|_{H^s},$$

Then $\Delta g \ast v$ is a bounded linear map from $H^{s}$ into $H^{s+r-2}$. Since $s \geq 0$ and $r > 2 + \frac{n}{2}$ we get $s + r - 2 > \frac{n}{2}$. The Sobolev embedding theorem implies that $\Delta g \ast v$ is a bounded linear map from $Y_{\infty}^{s,2}$ into $Y_{\infty}^{s,2}$. Lemma 3.4 implies the Lipschitz condition on $Y_{\infty}^{s,2}$. Then, by reasoning as in Theorem 3.3 we obtain the assertion.

The solution in theorems 3.2-3.4 can be extended to a maximal interval $[0, T_{\text{max}})$, where finite $T_{\text{max}}$ is characterized by the blow-up condition

$$\lim_{T \to T_{\text{max}}} \| u \|_{Y_{\infty}^{s,2}} = \infty.$$ 

Lemma 3.8. Suppose the conditions of theorems 3.4, 3.5 hold and $u$ is the solution of multipoint IVP (1.1) – (1.2). Then there is a global solution if for any $T < \infty$ we have

$$\sup_{t \in [0, T]} \left( \| u \|_{Y_{\infty}^{s,p}} + \| u_t \|_{Y_{\infty}^{s,p}} \right) < \infty. \quad (3.25)$$

Proof. Indeed, by reasoning as in the second part of the proof of Theorem 3.1, by using a continuation of local solution of (1.1) – (1.2) and assuming contrary that, (3.25) holds and $T_0 < \infty$ we obtain contradiction, i.e. we get $T_0 = T_{\text{max}} = \infty$.

4. Conservation of energy and global existence.

In this section, we prove the existence and the uniqueness of the global strong solution and the global classical solution for the problem (1.1) – (1.2). For this purpose, we are going to make a priori estimates of the local strong solution for the problem (1.1) – (1.2).

Condition 4.1. Let the Condition 2.1 holds. Assume that the kernel $g$ is an integrable function whose Fourier transform satisfies

$$0 < \hat{g}(\xi) \leq \left(1 + |\xi|^2 \right)^{-r} \quad \text{for all} \ \xi \in \mathbb{R}^n \ \text{and} \ r \geq 2.$$

Let $F^{-1}$ denote the inverse Fourier transform. We consider the operator $B$ defined by

$$u \in D(B) = H^{s}, \ Bu = F^{-1} \left[ |\xi|^{-1} (\hat{g}(\xi))^{-\frac{1}{r}} \hat{u}(\xi) \right],$$

Then it is clear to see that

$$B^{-2}u = -\Delta g \ast u, \ B^{-1}u = F^{-1} \left[ |\xi| (\hat{g}(\xi))^{-\frac{1}{r}} \hat{u}(\xi) \right]. \quad (4.1)$$
First, we show the following

**Lemma 4.1.** Suppose the conditions of theorems 3.4, 3.5 hold with $\hat{g}(\xi) > 0$ and the solution of multipoint IVP (1.1) $-$ (1.2) exists in $C^2([0, T]; Y_{\infty}^2)$ for some $s \geq 0$. If $B\phi \in L^2$ and $B\psi \in L^2$, then $Bu, Bu_0 \in C^1([0, T); L^2)$.

**Proof.** By Lemma 2.1, problem (1.1) $-$ (1.2) is equivalent to following integral equation,

$$u(x, t) = [S_1(x, t) \varphi + S_2(x, t) \psi + \Phi (g * f(u))] +$$

$$\frac{1}{2\eta} \int_0^t F^{-1} \left[ \sin \eta (t - \tau) |\xi|^2 \hat{g}(\xi) \hat{f}(G(u)(\xi)) \right] d\tau,$$

where $S_1(x, t), S_2(x, t), \Phi$ are operator functions defined by (2.10) and (2.11), where $g$ replaced by $g * f(u)$.

From (4.2) we get that

$$u_t(x, t) = \left[ \frac{d}{dt} S_1(x, t) \varphi + \frac{d}{dt} S_2(x, t) \psi + \frac{d}{dt} \Phi (g * f(u)) \right] +$$

$$\frac{1}{2} \int_0^t F^{-1} Q(\xi, t - \tau) d\tau,$$

where

$$Q(\xi, t - \tau) = \cos \eta (t - \tau) |\xi|^2 \hat{g}(\xi) \hat{f}(u)(\xi),$$

$$S_1(x, t) \varphi = F^{-1} \left\{ D_0^{-1}(\xi) \left[ \left( 1 - \sum_{k=1}^m \beta_k \cos \kappa_k \right) \sin (\eta t) \right] , + \right.$$

$$\left. \left[ \eta^{-1}(\xi) \sum_{k=1}^m \beta_k \sin \kappa_k \cos (\eta t) \right] \hat{\varphi}(\xi) \right\} ,$$

$$\frac{d}{dt} S_1(x, t) \varphi = F^{-1} \left\{ D_0^{-1}(\xi) \eta \left[ \left( 1 - \sum_{k=1}^m \beta_k \cos \kappa_k \right) \cos (\eta t) \right] - \right.$$

$$\left. \sum_{k=1}^m \beta_k \sin \kappa_k \sin (\eta t) \hat{\varphi}(\xi) \right\} ,$$

$$S_2(x, t) \psi = F^{-1} \left\{ \eta^{-1}(\xi) D_0^{-1}(\xi) \left( \sum_{k=1}^m \alpha_k \sin \kappa_k \right) \sin (\eta t) \right\} +$$

$$\eta^{-1}(\xi) D_0^{-1}(\xi) \left[ \left( 1 - \sum_{k=1}^m \alpha_k \cos \kappa_k \right) \cos (\eta t) \right] \hat{\psi}(\xi),$$

(4.5)
\[
\frac{d}{dt} S_2 (x, t) \psi = F^{-1} \left\{ \left[ D_0^{-1} (\xi) \left( \sum_{k=1}^{m} \alpha_k \sin \kappa_k \right) \cos (\eta t) \right] - D_0^{-1} (\xi) \left[ \left( 1 - \sum_{k=1}^{m} \alpha_k \cos \kappa_k \right) \sin (\eta t) \right] \right\} \hat{\psi} (\xi),
\]

\[
\Phi (x; t) = \Phi (g * f (u)) = \sum_{j=1}^{4} \sum_{k=1}^{m} F^{-1} \Phi_{jk} \left( \|\xi\|^2 \hat{\varphi} (\xi) \hat{f} (u) \right) (\xi; t), \quad (4.6)
\]

Since

\[
D_0^{-1} (\xi), \left( 1 - \sum_{k=1}^{m} \beta_k \cos \kappa_k \right) \sin (\eta t), \eta^{-1} (\xi) \sum_{k=1}^{m} \beta_k \sin \kappa_k \cos (\eta t)
\]

are uniformly bounded for fixed \( t \) by (4.1), (4.4) (4.5) we have

\[
\|B S_1 (x, t) \varphi\|_{L^2} = \left\| F^{-1} \left[ \|\xi\|^{-1} (\hat{\varphi} (\xi))^{-\frac{1}{2}} \hat{\varphi} (\xi) S_1 (x, t) \varphi \right] \right\|_{L^2} \lesssim \|\varphi\|_{H^s} < \infty, \quad (4.7)
\]

\[
\|B S_2 (x, t) \varphi\|_{L^2} = \left\| F^{-1} \left[ \|\xi\|^{-1} (\hat{\varphi} (\xi))^{-\frac{1}{2}} \hat{\varphi} (\xi) S_2 (x, t) \psi \right] \right\|_{L^2} \lesssim \|\psi\|_{H^s} < \infty.
\]

For fixed \( t \), we have \( f (u) \in H^s \). Since \( D_0^{-1} (\xi), \cos (\eta t), \sin (\eta t) \) are uniformly bounded, from (4.1), (2.11) and (4.6) we get

\[
\|B \Phi\|_{L^2} \lesssim \left\| F^{-1} \left[ \|\xi\|^{-1} (\hat{\varphi} (\xi))^{-\frac{1}{2}} \|\xi\|^2 \hat{\varphi} (\xi) \hat{f} (u) \right] \right\|_{L^2} \lesssim \|f (u)\|_{H^s} < \infty. \quad (4.8)
\]

From (4.8) we have

\[
\|B \Phi\|_{L^2} \leq \left\| F^{-1} \left[ \|\xi\|^{-1} (\hat{\varphi} (\xi))^{-\frac{1}{2}} Q (\xi, t - \tau) \right] \right\|_{L^2} \leq \quad (4.9)
\]

\[
\left\| F^{-1} \left[ \|\xi\|^{-1} (\hat{\varphi} (\xi))^{-\frac{1}{2}} |\xi|^2 \hat{\varphi} (\xi) \hat{f} (u) \right] \right\|_{L^2} \lesssim \|f (u)\|_{H^s} < \infty.
\]

Then from (4.2), (4.7), (4.8) and (4.9) we obtain the assertion.

**Remark 4.1.** Due to nonlocality of initial conditions the additional conditions appears in Theorem 4.1. For classical Cauchy problem this extra conditions are not required

**Lemma 4.2.** Assume the conditions of theorems 3.4, 3.5 hold with \( a = 0, \hat{\varphi} (\xi) > 0 \) and

\[
\hat{b} (\xi) = O \left( 1 + |\xi|^2 \right)^{\frac{d+1}{2}}.
\]

Suppose the solution of (1.1) – (1.2) exists in \( C^2 ([0, T]; Y_s^2) \) for some \( s \geq 0 \). If \( B \psi \in L^2, B u (x, \lambda_k) \in L^2, k = 1, 2, \ldots, m \), then \( B u \in C^2 ([0, T]; L^2) \). Moreover, if \( B \varphi \in L^2, B u (x, \lambda_k) \in L^2 \) then \( B u \in C^1 ([0, T]; L^2) \).
Proof. Integrating the equation (1.1) for $a = 0$, twice and calculating the resulting double integral as an iterated integral, we have

\[ u(x, t) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(x, \lambda_k) + t \left[ \psi(x) + \sum_{k=1}^{m} \beta_k u_t(x, \lambda_k) \right] - \]

\[ \int_{0}^{t} (t - \tau) (b * u)(x, \tau) d\tau + \int_{0}^{t} (t - \tau) \Delta (g * f(u))(x, \tau) d\tau, \quad (4.10) \]

\[ u_t(x, t) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(x, \lambda_k) - \]

\[ \int_{0}^{t} (b * u)(x, \tau) d\tau + \int_{0}^{t} \Delta (g * f(u))(x, \tau) d\tau. \quad (4.11) \]

From (4.1) and (4.11) for fixed $t$ and $\tau$ we get

\[ \|Bu_t(x, t)\|_{L^2} = \|B\psi(x)\|_{L^2} + \sum_{k=1}^{m} \beta_k \|Bu_t(x, \lambda_k)\|_{L^2} - \]

\[ \int_{0}^{t} \|B(b * u)(x, \tau)\|_{L^2} d\tau - \int_{0}^{t} \|B^{-1} f(u)(x, \tau)\|_{L^2} d\tau. \quad (4.12) \]

By assumption on $b$, $g$ and by (4.1) for fixed $\tau$ we have

\[ \|B(b * u)(x, \tau)\|_{L^2} \leq \left\| F^{-1} \left[ \xi^{-1} \hat{b}(\xi) (\hat{g}(\xi))^{-\frac{r}{2}} \hat{u}(\xi, \tau) \right] \right\|_{L^2} \lesssim \|u(., \tau)\|_{H^s}. \quad (4.13) \]

Moreover, by Lemma 3.3 for all $t$ we have $f(u) \in H^s$. Also

\[ \|B(f(u))(x, \tau)\|_{L^2} \lesssim \|f(u)(., \tau)\|_{H^s}. \quad (4.14) \]

Then from (4.12) – (4.14) we obtain $Bu_t \in C^2([0, T]; L^2)$. The second statement follows similarly from (4.10).

From Lemma 4.2 we obtain the following result.

Result 4.1. Assume the conditions of theorems 3.4, 3.5 hold with $a = 0$, $\hat{g}(\xi) > 0$, $\alpha_k = \beta_k = 0$ and

\[ \hat{b}(\xi) = O \left( 1 + |\xi|^2 \right)^{s+1}. \]
Suppose the solution of \((1.1) - (1.2)\) exists in \(C^2 ([0, T] ; Y^2_\infty)\) for some \(s \geq 0\). If \(B\psi \in L^2\) then \(Bu_t \in C^2 ([0, T] ; L^2)\). Moreover, if \(B\phi \in L^2\), then \(Bu \in C^1 ([0, T] ; L^2)\).

Here,

\[
G(\tau) = \int_0^{\tau} g(s) \, ds.
\]

**Lemma 4.3.** Assume the all conditions of Lemma 4.2 are satisfied. Let \(B\psi \in L^2\), \(Bu_t (x, \lambda_k) \in L^2\), \(k = 1, 2, ..., m\) and \(G(\phi) \in L^1\). Then for any \(t \in [0, T)\) the energy

\[
E(t) = \|Bu_t\|_{L^2}^2 + a \left\| F^{-1} \hat{g} * u \right\|_{L^2}^2 + \|B (b * u)\|_{L^2}^2 + 2 \int_{\mathbb{R}^n} G(u) \, dx
\]

is constant \([0, T)\).

**Proof.** By use of equation \((1.1)\), it follows from straightforward calculation that

\[
\frac{d}{dt} E(t) = 2 (Bu_{tt}, Bu_t) + 2a \left( F^{-1} \hat{g} * u, (F^{-1} \hat{g} * u) \right) +
\]

\[
2 \left[ B (b * u), B (b * u) u_t(t) \right] + 2 \left( f(u), u_t \right) = 2 \left( B^2 u_{tt}, u_t \right) +
\]

\[
2 \left( B^2 (b * u), (b * u) u_t(t) \right) + 2 \left[ B^2 (b * u), (b * u) u_t(t) \right] =
\]

\[
2B^2 \left[ (u_{tt} - a\Delta u + b * u + \Delta (g * f(u)), u_t) \right] = 0,
\]

where \((u, v)\) denotes the inner product of \(L^2\) space. Integrating the above equality with respect to \(t\), we have \((4.15)\).

By using the above lemmas we obtain the following results

**Theorem 4.1.** Let the Condition 3.2 hold \(a = 0\), \(\hat{g}(\xi) > 0\) and

\[
\hat{b}(\xi) = O \left(1 + |\xi|^2 \right)^{\frac{s+r}{2} + 1}.
\]

Moreover, let \(B\psi \in L^2\), \(Bu_t (x, \lambda_k) \in L^2\), \(k = 1, 2, ..., m\) and \(G(\phi) \in L^1\), \(s \geq 0\), \(r > 3\) and there is some \(k > 0\) so that \(G(r) \geq -kr^2\) for \(r \in \mathbb{R}\). Then there is some \(T > 0\) such that the multipoint IVB \((1.1) - (1.2)\) has a global solution \(u \in C^2 ([0, \infty) ; Y^2_\infty)\).

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