Interplay of driving and frequency noise in the spectra of vibrational systems

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We study the spectral effect of the fluctuations of the vibration frequency, which are known to play a major role in nanomechanical and other mesoscopic vibrational systems. We find that, for periodically modulated systems, the interplay of the driving and frequency fluctuations results in specific features of the spectra. These features provide a sensitive means to characterize the fluctuations. The theory is corroborated by experimental observations on carbon nanotube vibrations. The results bear on identifying and suppressing decoherence of mesoscopic oscillators and, more generally, on resonance fluorescence and light scattering by charged oscillators.

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The spectrum of response and the power spectrum of an oscillator is a textbook problem that goes back to Lorentz and Einstein [1,2]. It has attracted much attention recently in the context of nanomechanical systems. Here, the spectra are a major source of information about the classical and quantum dynamics [3,4,11]. This is the case also for mesoscopic oscillators of different nature, such as superconducting cavity modes [12,13] and optomechanical systems [16]. Mesoscopic oscillators experience comparatively large fluctuations. Along with dissipation, these fluctuations determine the shape of the vibrational spectra.

A well-understood and most frequently considered source of fluctuations is thermal noise that comes from the coupling of an oscillator (vibrational system) to a thermal reservoir and is related to dissipation by the fluctuation-dissipation theorem. Dissipation leads to the broadening of the oscillator power spectrum and the spectrum of the response to external driving. However, spectral broadening can also come from fluctuations of the oscillator frequency. Identifying different broadening mechanisms is a delicate task that has been attracting much attention [4,11,12,17,18].

In this paper we study the effect of periodic driving on the power spectra of nanomechanical vibrational systems. For a linear oscillator with no frequency fluctuations, driving leads to a δ-like peak at the driving frequency ω_F, because here the only effect of the driving is forced vibrations linearly superimposed on thermal motion. Frequency fluctuations make forced vibrations random. This qualitatively changes the spectrum leading, as we show, to extra spectral peaks with a characteristic shape. We also study fluctuating nonlinear vibrational systems. Here, too, even weak driving leads to a very specific extra spectral structure.

For a linear oscillator, the interplay of driving and frequency noise leads to the spectral features depicted in Fig. 1. The two limiting cases sketched in Fig. 1 correspond to the long and short correlation time of the frequency noise t_c compared to the oscillator relaxation (decay) time t_r. For t_c ≫ t_r (the left panel), slow fluctuations of the oscillator frequency ω_osc(t) cause fluctuations of the oscillator susceptibility χ, which depends on the detuning of the driving frequency from resonance ω_F − ω_osc(t). The associated slow fluctuations of the amplitude and phase of forced vibrations at frequency ω_F lead to a finite-width spectral peak centered at ω_F. This is a frequency-domain analog of the Einstein light scattering due to susceptibility fluctuations [19].

For t_c ≪ t_r (the right panel), driving-induced random vibrations quickly lose the memory of the driving frequency. They become similar to thermal vibrations. However, their amplitude is determined by the driving, not the temperature. This leads to a spectral peak centered at the oscillator eigenfrequency ω_0 = ⟨ω_osc(t)⟩, with the height quadratic in the driving amplitude.
In the quantum picture, one can think that, as a result of pumping by a driving field, the oscillator emits energy quanta. For the familiar example of an oscillating charge driven by an electromagnetic field these quanta can be photons, and one can speak of light scattering and fluorescence by an oscillator. For $t_c \ll t_r$ the frequency of a quantum, which is emitted over time $\sim t_r$ after an absorption event, is uncorrelated with the excitation frequency $\omega_F$. This is a fluorescence-type process. The energy difference $h(\omega_F - \omega_0)$ comes from the frequency noise. For $t_c \gg t_r$ emission occurs at frequencies close to $\omega_F$. In both cases the spectrum is qualitatively different from just a $\delta$-like peak in the absence of frequency fluctuations [3].

Even to the first order in the driving intensity, the analysis of the spectrum requires going beyond the approximation, see Appendix, the fluctuating linear susceptibility is

\[ \chi_1(t, t') = \frac{i}{2 \omega_0} e^{-\left(\Gamma + i \omega_0\right)(t-t')} \exp\left[-i \int_{t'}^{t} dt'' \xi(t'')\right] + \text{c.c.} \quad (3) \]

Equation [3] often applies even where the oscillator dynamics in the lab frame is non-Markovian.

Explicit expressions for $\chi$ and $\Phi_F(\omega)$ can be obtained from Eq. (3) in the limiting cases. The analysis is straightforward. Some auxiliary expressions are provided in the Appendix. For weak noise $\xi(t)$, the spectrum $\Phi_F$ is proportional to the noise power spectrum $\Xi(\Omega) = \int_{-\infty}^{\infty} dt \langle \xi(t)\xi(0) \rangle \exp(i\Omega t)$,

\[ \Phi_F(\omega) \approx \frac{1}{16 \omega_0^2 [\Gamma^2 + (\omega_F - \omega_0)^2]} \Xi(\omega - \omega_F)^2. \quad (4) \]

This expression already shows the peculiar features qualitatively discussed above. If $\Xi(\Omega)$ is a narrow-band noise with a peak at zero frequency (as for 1/f-type noise, for example), $\Phi_F(\omega)$ has a peak at $\omega_F$, cf. Fig. I(a). The shape of this peak coincides with that of $\Xi(\Omega)$. The peak corresponds to quasi-elastic “scattering” of the driving field. If, on the other hand, $\Xi(\Omega)$ is almost flat on the frequency scale $\Gamma$, $|\delta \omega_F|$ (broad-band noise), $\Phi_F(\omega)$ has a Lorentzian peak at $\omega_0$, cf. Fig. I(b).

Equation (3) provides a direct means for measuring the frequency noise spectrum. The weak-noise condition $\Xi \ll [\Gamma^2 + (\omega_F - \omega_0)^2]^{1/2}$ can be achieved by tuning the driving somewhat away from $\omega_0$.

The case of flat $\Xi(\Omega)$, i.e., of $\xi(t)$ being $\delta$-correlated on time scale $t_r$, can be analyzed for an arbitrary noise strength using that the characteristic functional of a $\delta$-correlated noise is $\mathcal{P} [k(t)] = \langle \exp[i \int dt k(t) \xi(t)] \rangle = \exp[-\int dt \mu(k(t)) dt]$, where function $\mu(k)$ is determined by the noise statistics. A straightforward integration (see Appendix) gives

\[ \Phi_F(\omega) = \frac{\text{Re} \mu(1)}{8 \omega_0^2 [\Gamma^2 + (\omega_F - \omega_0)^2] \Gamma^2 + (\omega - \omega_0)^2}. \quad (5) \]

The spectrum [4] has the same shape as the spectrum $\Phi_0(\omega)$ in the absence of periodic driving: it is a Lorentzian centered at the noise-renormalized oscillator eigenfrequency $\tilde{\omega}_0 = \omega_0 - \text{Im} \mu(1)$ with halfwidth $\tilde{\Gamma} = \Gamma + \text{Re} \mu(1)$. The area of the driving-induced term $F^2 \Phi_F(\omega)$ is independent of the intensity ($\propto k_B T$) of the dissipation-related noise. Instead it is proportional to the frequency-noise characteristic $\text{Re} \mu(1)$. Somewhat counterintuitively, it is proportional not to the $\omega_0$ and that its correlation time $t_c \gg \omega_0^{-1}$, so that it does not cause parametric excitation of the oscillator [21,22].

The most simple model of the oscillator dynamics is described by equation $\ddot{q} + 2 \Gamma \dot{q} + [\omega_0^2 + 2 \omega_0 \xi(t)] q = F \cos \omega_F t + f(t)$, where $f(t)$ is thermal noise and $\Gamma = t_c^{-1}$ is the relaxation rate. In the standard rotating wave approximation, see Appendix, the fluctuating linear susceptibility is

\[ \chi_1(t, t') = \frac{i}{2 \omega_0} e^{-\left(\Gamma + i \omega_0\right)(t-t')} \exp\left[-i \int_{t'}^{t} dt'' \xi(t'')\right] + \text{c.c.} \quad (3) \]
power absorbed by the oscillator $\propto F^2 \Im \chi(\omega_F)$, but to $F^2 |\chi(\omega_F)|^2$. Equation (5) suggests how to separate the noise-induced broadening of the oscillator spectrum from the decay-induced broadening.

In the opposite limit of a narrow-band frequency noise on the scale $\Gamma$, one can replace $\xi(t')$ in Eq. (3) with $\xi(t)$. This corresponds to the “instantaneous” slowly fluctuating susceptibility $i/2\omega_0 \Gamma - i(\omega_F - \omega_0 - \xi(t))$. The resulting spectrum $\Phi_F(\omega)$ has a narrow peak at $\omega_F$ with the shape determined by the spectrum and statistics of the frequency noise.

Function $\Phi_F$ can be found in a closed form for a Gaussian noise $\xi(t)$, see Appendix. The results are shown in Fig. 2 for the noise power spectrum with bandwidth $\lambda$, $\Xi(\Omega) = 2D\lambda^2/(\lambda^2 + \Omega^2)$. They illustrate how the shape of $\Phi_F(\omega)$ changes from a peak at $\omega_F$ for a narrow-band noise ($\lambda \ll \Gamma$) to a peak at $\omega_0$ for a broadband noise ($\lambda \gg \Gamma$). The overall area of the spectrum $\Phi_F$ nonmonotonically depends on the frequency noise intensity: it is linear in the noise intensity for weak noise, cf. Eq. (4), but for a large noise intensity it decreases, since the decoherence rate of the oscillator increases.

FIG. 2. The power spectrum of the oscillator with a Gaussian frequency noise with the spectrum $\Xi(\Omega) = 2D\lambda^2/(\lambda^2 + \Omega^2)$. The noise intensity is $D/\Gamma = 2$. Left panel: the full spectrum. The color coding is the same as in Fig. 1 $F^2/16T^2 = 20k_B T$. Right panel: the driving-induced term. The solid lines and dots show the analytic theory and simulations; the consecutive curves are shifted by 0.25 along the ordinate.

In Fig. 3a we show the measured spectrum of a modulated carbon nanotube resonator. For low temperature and weak driving, low-lying flexural modes of such resonators are well described by harmonic oscillators. The driving was applied electrostatically as an ac voltage $\delta V_g$ on the gate electrode, and the power spectrum of the (downconverted) modulated current through the nanotube was measured at 1.2 K [23]. The measurements reveal the presence of both broad-band and narrow-band frequency noise. The areas of these two contributions scale as $\delta V_g^2$, in agreement with Eq. (4). The narrow-band noise is of the $1/f^\alpha$ type with $\alpha < 1$. Further details are provided in Appendix; in particular, we show that the change of the spectrum is not a heating effect.

Various types of mesoscopic oscillators often display nonlinearity even in the absence of driving [24]. Because the vibration frequency of a nonlinear oscillator depends on the amplitude, driving-induced vibrations cause a frequency change, and thus shift the power spectrum compared to the spectrum without driving. On the other hand, thermal fluctuations of the vibration amplitude lead to frequency fluctuations. Even in the absence of periodic driving, this makes the power spectrum non-Lorentzian and in general asymmetric [25]. The shape of the spectrum is determined by the interrelation between the frequency uncertainty $\Gamma$ due to the oscillator decay and the width $\Delta \omega$ of the distribution of the vibration frequencies due to thermal distribution of the vibration amplitude.

A model that captures the above effects and describes
a large number of mesoscopic vibrational systems is the Duffing oscillator, with the nonlinear term in the Hamiltonian of the form of $\gamma q^4/4$. When this term is small compared to the harmonic term $\omega_0^2 q^2/2$, the oscillator dynamics can be described in the rotating wave approximation. The oscillator frequency is then $\omega_{osc}(t) \approx \omega_0 + (3\gamma/8\omega_0)A^2(t)$, where $A(t)$ is the vibration amplitude, cf. [6]. The distribution of $A^2$ in the absence of driving is of the Boltzmann form, $\exp(-\omega_0^2 A^2/2k_BT)$, and therefore $\Delta \omega \approx 3|\gamma|k_BT/8\omega_0^2$.

Frequency fluctuations of a nonlinear oscillator result in the change of its power spectrum in the presence of driving. The explicit results for the limiting cases of large and small $\Delta \omega/\Gamma$ are given in Appendix. Fluctuations of the nonlinear susceptibility make a major contribution to the power spectrum in this case.

The strong dependence of the driving-induced spectral change on the ratio $\Delta \omega/\Gamma$ is illustrated in Fig. 4. For small $\Delta \omega/\Gamma$ the major effect of the driving is the shift of the spectrum; then $\Phi_D(\omega) \propto \Phi_0(\omega) \propto (\omega - \omega_0)\Phi_0(\omega)$ has a characteristic dispersive shape, changing sign at $\omega_0$. With increasing $\Delta \omega/\Gamma$ the shape of $\Phi_D(\omega)$ becomes more complicated. Generally it still has positive and negative parts, in dramatic difference from the case of a harmonic oscillator with fluctuating frequency. Also, in contrast to a harmonic oscillator, keeping terms $\propto F^2$ in the power spectrum of a nonlinear oscillator is justified only for weak modulating fields.

The above results show that the interplay of frequency fluctuations and driving qualitatively changes oscillator spectra compared to the spectra with no frequency fluctuations [3]. The change sensitively depends on the frequency fluctuation intensity and their power spectrum. An important characteristic of oscillators is their quality factor, often measured as the ratio of the oscillator frequency to the width of the spectral peak at half-height. This ratio has contributions from both oscillator decay and frequency fluctuations. Our results indicate how to separate these contributions without ring-down measurements, which are often complicated in nanomechanical systems, and moreover, how to discriminate between frequency noise and frequency fluctuations due to oscillator nonlinearity. The above analysis of driven linear oscillators with a fluctuating frequency immediately extends to the quantum regime, which is attracting much interest in nanomechanics [13, 18, 20]. For nonlinear oscillators, on the other hand, the nonequidistance of the energy levels can bring in new features compared to the considered here classical limit.

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Appendix A: General expression for the power spectrum

The explicit expression for the driving-induced term in the power spectrum of fluctuations of the oscillator reads

$$\Phi_F(\omega) = \frac{1}{2} \text{Re} \int_0^\infty dt e^{i(\omega - \omega_F)t} \int_{-\infty}^0 d\tau d\tau' e^{i\omega_F(\tau - \tau')} \times \langle \chi_1(t, t + \tau) [\chi_1(0, \tau') - \langle \chi_1(0, \tau') \rangle] \rangle + \Phi_F^{(2)}(\omega). \tag{A1}$$

This expression follows from Eqs. (1) and (2) of the main text. The first term gives the contribution of the fluctuations of the linear susceptibility. The second term gives the contribution from the nonlinear susceptibility,

$$\Phi_F^{(2)}(\omega) = \text{Re} \int_0^\infty dt e^{i\omega t} \int_{-\infty}^0 d\tau d\tau' \cos[\omega_F(\tau - \tau')] \times \langle \chi_2(t, t + \tau, t + \tau') \rangle. \tag{A2}$$

This term describes the correlation between fluctuations of the second-order susceptibility and thermal fluctuations in the absence of periodic driving. We emphasize that, for a resonantly modulated underdamped oscillator, it is pronounced at frequencies $\omega$ close to the driving frequency $\omega_F$, not $2\omega_F$. Equation (A2) describes, in particular, the contribution to the spectrum from the nonlinear susceptibility of a nonlinear oscillator. Is is especially convenient in the case of weak nonlinearity, where the oscillator spectrum $\Phi_0(\omega)$ is broadened primarily by the decay rather than by frequency fluctuations due to
the interplay of the nonlinearity and the amplitude fluctuations. In this case \( \chi_2 \) can be easily found from the equation of the complex amplitude of a nonlinear oscillator by perturbation theory in \( F \). The term \( \Phi_F^{(2)} \) gives the main contribution to \( \Phi_F \). The result coincides with Eq. (B2) obtained by a different method.

Appendix B: Averaging over frequency fluctuations for a linear oscillator

Equation (3) of the main text for the susceptibility of a linear underdamped oscillator with fluctuating frequency can be found in a standard way by changing from the fast oscillating variables \( q, \dot{q} \) to slow complex oscillator amplitude \( u(t) = [q(t) + (i\omega_F)^{-1} \dot{q}(t)] \exp(-i\omega_F t)/2 \). If the equation of motion in the lab frame is Markovian, \( \dot{u} + 2\Gamma \ddot{u} + (\omega_0^2 + 2\omega_0 \xi(t)) u = F \cos\omega_F t + f(t) \), where \( f(t) \) is the dissipation-related thermal noise, as in the example discussed in the main text, the equation for \( u(t) \) in the rotating wave approximation reads

\[
\dot{u} = -[\Gamma + i\delta\omega_F - i\xi(t)]u - i\frac{F}{4\omega_0} + f_u(t) \quad (B1)
\]

Here, \( \delta\omega_F = \omega_F - \omega_0 \) is the detuning of the driving frequency from the oscillator eigenfrequency; \( f_u(t) = [f(t)/2\omega_0] \exp(-i\omega_0 t) \). Equation (B1) applies on the time scale that largely exceeds \( \omega_0^{-1} \). On this scale \( f_u(t) \) is \( \delta \)-correlated even where in the lab frame the oscillator dynamics is non-Markovian, cf. \textsuperscript{23}. By integrating Eq. (B1) over time one immediately obtains Eq. (3) of the main text for the oscillator susceptibility \( \chi_1(t, t') \).

We disregard corrections \( \sim |\delta\omega_F|/\omega_F \); in particular in Eq. (B1) for convenience we replaced \( F/\omega_F \) with \( F/\omega_0 \); similarly, in the expression for \( f_u \) we replaced \( f/\omega_F \) with \( f/\omega_0 \).

We note that the noise \( f_u(t) \) drops out from the moments \( \langle u^n(t) \rangle \). This can be used to characterize the statistics of the frequency noise. In this paper we consider the change of the conventionally measured characteristic, the power spectrum, and the extra spectral features related to the interplay of the driving and frequency noise. It is convenient to rewrite Eq. (A1) for the spectrum \( \Phi_F(\omega) \) near its maximum in the form that explicitly takes into account that, when the expression for the susceptibility is substituted into Eq. (A1), the fast-oscillating terms in the integrands can be disregarded. This gives

\[
\Phi_F(\omega) = (8\omega_0^2)^{-1} \text{Re} \int_0^\infty dt \exp[i(\omega - \omega_F)t]
\times \int_{-\infty}^t dt' \int_{-\infty}^0 dt'' \langle \chi_{sl}(t, t') \chi_{sl}(0, t_0') - \langle \chi_{sl}(0, t_1') \rangle \rangle,
\]

\[
\chi_{sl}(t, t') = e^{-(\Gamma - i\omega F)(t-t')} \exp \left[-i \int_{t_1}^t dt'' \xi(t'') \right] \quad (B2)
\]

Here, function \( \chi_{sl}(t, t') \) gives the slowly varying factor in the fast-oscillating time-dependent oscillator susceptibility \( \chi_1(t, t') \). Function \( \langle \chi_{sl}(0, t) \rangle \equiv \langle \chi_{sl}(-t, 0) \rangle \) gives the standard (average) susceptibility

\[
\chi(\omega_F) = \int_0^\infty dt e^{i\omega_F t} \langle \chi_1(t, 0) \rangle = \frac{i}{2\omega_0} \int_0^\infty dt \langle \chi_{sl}(t, 0) \rangle. \quad (B3)
\]

The mean forced displacement of the oscillator in the linear response theory is \( \langle q(t) \rangle = \frac{1}{2} F e^{-i\omega_F t} \chi(\omega_F) + c.c. \).

Averaging over \( \xi(t) \) in Eqs. (B2) and (B3) can be done using the noise characteristic functional (cf. \textsuperscript{31}),

\[
\mathcal{P}[k(t)] = \exp \left[i \int dt k(t)\xi(t) \right].
\]

As seen from Eq. (B2), function \( \langle \chi_{sl}(t, t') \rangle \) is determined by \( \mathcal{P}[k(t'')] \) with \( k(t'') = -1 \) if \( t' < t'' < t \) and \( k(t) = 0 \) otherwise. For \( \delta \)-correlated noise, where \( \mathcal{P}[k(t)] = \exp[-i \int dt \mu(k(t))] \), taking into account that \( \mu(0) = 0 \) and \( \mu(-k) = \mu^*(k) \), we obtain

\[
\langle \chi_{sl}(t, t') \rangle = \exp[-(\Gamma - i\delta\omega_F + \mu^*(1))(t - t')]. \quad (B4)
\]

Thus, frequency noise leads to both decay of the conventional susceptibility with decrement Re \( \mu(1) \) and effective shift of the oscillator eigenfrequency by \( \text{Im} \mu(1) \). We note that the noise can be considered \( \delta \)-correlated when its spectrum is flat not just on the scale \( \gtrsim \Gamma \), but on the scale \( \gtrsim \Gamma + \text{Re} \mu(1) \), which itself depends on the noise intensity.

Averaging the term \( \langle \chi_{sl}(t, t') \chi_{sl}(0, t_1') \rangle \) in Eq. (B2) comes to calculating

\[
\left\langle \exp \left[-i \int_{t_1'}^{t''} dt'' \xi(t'') + i \int_{t_1'}^{0} dt'' \xi(t_1'') \right] \right\rangle
\equiv \left\langle \exp \left[i \int_{-\infty}^\infty dt_2 \xi(t_2) \xi(t_2) \right] \right\rangle. \quad (B5)
\]

Here \( t > 0 \) and \( -\infty < t' < t, -\infty < t_1' < 0 \). Clearly, in this equation \( k(t_2) = 0, \pm 1 \). For \( t' < 0 \) we have \( k(t_2) = \text{sgn}(t'' - t_1') \) if \( \min(t'' - t_1') < t_2 = \max(t_1, t_1') \) and \( k(t_2) = -1 \) if \( 0 < t_2 < t' \); for \( t' > 0 \) we have \( k(t_2) = 1, \text{if} t_1' < t_2 < 0 \) and \( k(t_2) = -1, \text{if} t_2 < t' < t \); otherwise \( k(t_2) = 0 \). For a \( \delta \)-correlated noise the averaging using the explicit form of \( \mathcal{P}[k(t)] \) and integration over \( t', t_1' \) gives Eq. (5) of the main text.

For a stationary Gaussian noise the characteristic functional is expressed in terms of the noise correlator \textsuperscript{31},

\[
\mathcal{P}[k(t)] = \exp \left[-\frac{1}{2} \int dt dt' \langle \xi(t)\xi(t') \rangle k(t)k(t') \right].
\]

If the correlator \( \langle \xi(t)\xi(t') \rangle \) or equivalently, the power spectrum \( \Xi(\Omega) \), are known, using the values of \( k(t) \) given below Eq. (B5) one can perform the averaging in Eq. (B2) and then perform integration over time to find the power spectrum \( \Phi_F \). This was done to obtain the results shown in Fig. 2 of the main text.
For slowly varying frequency noise on the scale of the oscillator relaxation time $\Gamma^{-1}$, the evaluation of the susceptibility following the prescription given in the main text leads to expression

$$\Phi_F(\omega) \approx \frac{1}{\delta \omega_0} \operatorname{Re} \int_0^\infty dt e^{i(\omega - \omega_F)t} \langle X(t)|X^*(0) \rangle,$$

$$- \langle X^*(0) \rangle, \quad X(t) = [\Gamma - i \delta \omega_F + i \xi(t)]^{-1}. \quad (B6)$$

It can be used for numerical calculations if the statistics of the noise $\xi(t)$ is known.

We note that both the standard susceptibility $\chi(\omega)$ and the power spectrum in the absence of driving $\Phi_0(\omega)$ are affected by frequency noise. In the considered case they are related by the fluctuation-dissipation relation, $\Phi_0(\omega) = (2k_B T/\omega) \operatorname{Im} \chi(\omega)$. For a $\delta$-correlated frequency noise, the shift -Im $\mu(1)$ and broadening Re $\mu(1)$ of the power spectrum $\Phi_0(\omega)$ were discussed in the main text. For a non-white frequency noise the spectrum becomes non-Lorentzian. For example, for weak noise we have for the susceptibility

$$\chi(\omega) \approx \frac{i}{2 \omega_0 (\Gamma - i \delta \omega)} \left[ 1 - \int \frac{d\Omega}{2(\Gamma - i \omega_0) \Gamma - i \delta \omega - i \Omega} \right], \quad (B7)$$

where $\Xi(\Omega)$ is the frequency noise power spectrum and $\delta \omega = \omega - \omega_0$. Importantly, this correction just slightly distorts the susceptibility. For example, a sharp low-frequency peak of $\Xi(\Omega)$ does not lead to a narrow peak in $\chi(\omega)$ and, respectively, in the power spectrum $\Phi_0(\omega)$. This should be contrasted with the narrow peak in $\Phi_F(\omega)$, which emerges in this case.

Appendix C: Power-law noise in carbon nanotube resonators

The device consists of a carbon nanotube contacted by source and drain electrodes and suspended over a gate electrode. Details of the fabrication and the geometry of the device can be found in Ref. \cite{22}. We measure power spectra of displacement fluctuations using the experimental setup sketched in Fig. 5b. Displacement fluctuations induce conductance fluctuations. We parametrically down-convert these conductance fluctuations by applying an AC voltage $\delta V_{sd}(t)$ between source and drain at a non-resonant frequency $\omega_{sd}$, resulting in current fluctuations at frequencies $|\omega_0 - \omega_{sd}| \sim 2\pi \times 10 \text{ kHz}.$

The spectrum shown in Fig. 3a of the main text is obtained in the presence of a near resonant oscillating electrostatic force $\delta F(t)$. This force is created by applying an oscillating voltage $\delta V_g(t) = \delta V_{AC}^g \cos \omega_F t$ at a frequency $\omega_F = \omega_0 - 2\pi \times 102 \text{ Hz}$, with $\omega_0/(2\pi) = 6.3 \times 10^6 \text{ Hz}$, and an amplitude $\delta V_{AC}^g = 4.9 \times 10^{-7} \text{ V}$. In this experiment, a DC gate voltage $V_{g}^{DC} = 1.454 \text{ V}$ and an AC source-drain voltage of amplitude $\delta V_{sd}^{AC} = 89 \times 10^{-6} \text{ V}$ are used. The amplitude $\delta V_{sd}^{AC}$ is kept below the threshold beyond which the variance of displacement of the nanotube increases with $\delta V_{sd}^{AC}$ (as in Ref. \cite{23}). The mode temperature is 1.2 K. The integration time is 32 s.

It is important to verify that applying $\delta V_g(t)$ does not result in an increase in the mode temperature. We consider the case $\omega_F = \omega_0$ where an increase of temperature, if any, should be most pronounced. Two mechanisms are liable to increase the mode temperature: (i) dissipated power related to the work done by the oscillating resonant force from the gate electrode $\delta F(t)$, and (ii) Joule heating related to the current, flowing through the nanotube, that is induced by the time-varying capacitance between the nanotube and the gate electrode. We now discuss the effects of these mechanisms.

(i) From the work of a resonator subject to an oscillating force $\delta F$, the time average power reads:

$$\langle P_{\delta F} \rangle = \frac{\delta F^2 Q}{2 M \omega_0}, \quad (C1)$$

where $Q$ is the quality factor and $M$ is the effective mass of the mode. The amplitude of the oscillating force is $\delta F = C_{g'} V_{g}^{DC} \delta V_{g}$, where $C_{g'}$ is the derivative of the gate capacitance with respect to a small displacement (we assume that the whole length of the nanotube is at a single, well-defined potential). From Coulomb blockade measurements, we estimate that $C_{g'} = 1.2 \times 10^{-12} \text{ F/m}$ as detailed in Ref. \cite{22}.

We estimate the mass $M = 9.8 \times 10^{-21} \text{ kg}$ from the diameter and the length of the nanotube. In Figs. 3b, c of the main text, the maximum amplitude $\delta V_g$ is $\sim 6.4 \times 10^{-7} \text{ V}$. Using $Q = 1.2 \times 10^4$, $V_{g}^{DC} = 1.454 \text{ V}$, and $\omega_0/(2\pi) = 6.3 \times 10^6 \text{ Hz}$, we find that the maximum dissipated power is $\langle P_{\delta F} \rangle_{\text{max}} \approx 2 \times 10^{-20} \text{ W}$. This is a minuscule power.

Using a thermal conductance of $10^{-12} \text{ W/K}$, this dissipated power translates into a temperature increase $\Delta T \sim 10^{-8} \text{ K}$, a truly insignificant increase. This thermal conductance is inferred from two published measurements at liquid helium temperature. The thermal conductance for a multi-wall carbon nanotube with a length of 2.5 $\mu$m and a diameter of 14 nm was measured to be $\sim 10^{-10} \text{ W/K}$ \cite{32}. The thermal conductivity of aligned single-wall nanotubes was measured to be $\sim 1 \text{ Wm}^{-1}\text{K}^{-1}$ \cite{33}. These two measurements indicate that the thermal conductance is in the range $10^{-12} - 10^{-11} \text{ W/K}$ for a nanotube with a diameter of 1 nm and a length of 2 $\mu$m.

(ii) As the nanotube vibrates, the distance that separates it from the gate electrode is modulated, and so is the gate capacitance $C_g$. The driving of $C_g$ results in a current at the driving frequency that flows through the nanotube. On resonance, this current
reads:
\[
I_{SC}(t) = \omega_0 V_g^{DC} \delta C_g \sin \omega_0 t, \tag{C2}
\]
where \(\delta C_g\) is the driving amplitude of \(C_g\). Note that \(I_{SC}(t)\) also has components proportional to \(C_g \delta V_g\), but these have amplitudes that are several orders of magnitude smaller than \(\omega_0 V_g^{DC} \delta C_g\). The time average dissipated power related to Joule heating reads
\[
\langle P_d \rangle = R_t \langle I_{SC}(t)^2 \rangle = R_t (V_g^{DC} \omega_0 \delta C_g)^2 / 2, \tag{C3}
\]
where \(R_t\) is the resistance of the nanotube. We estimate \(\delta C_g = C_g^0 \delta a_0 \approx 10^{-21} \text{ F}\), using the resonant displacement \(\delta z_0 = QC_g^0 V_g^{DC} \delta V_g^0 / (M \omega_0^2) \approx 0.6 \times 10^{-9} \text{ m}\) as an approximation of the motional amplitude. Hence, the dissipated power is \(\langle P_d \rangle_{\text{max}} \approx 10^{-22} \text{ W}\). Here again, the induced temperature increase can be neglected.

Confirming these estimates, Fig. 5a shows that the inverse of the quality factor \(1/Q\) does not vary as \(\delta V_g^2\) increases. Since an increase in temperature would result in an increase in \(1/Q\), this further indicates that \(\delta V_g(t)\) does not affect the mode temperature.

To highlight spectral features that we associate to frequency noise (light and dark green shaded areas in Fig. 3a of the main text), we exclude the \(\delta\)-peak at driving frequency \(\omega_F\). To this end, we observe that the response of our signal analyzer to a voltage oscillating at a given frequency is a delta peak that consists of 3 points above the background. Similarly, the \(\delta\)-peak at \(\omega_F\) displayed as a black trace in Fig. 3a of the main text consists of 3 data points above the background signal. We remove those 3 points from the measured spectra to estimate the spectral areas plotted in Figs. 3b, c.

The spectral feature at \(\omega_F\), which we associate to a narrow band frequency noise, is not related to the phase noise of the source used to supply \(\delta V_g(t)\). Indeed, the phase noise of our source \(\sim 10 \text{ Hz}\) away from \(\omega_F\) is \(\sim -60 \text{ dBc/Hz}\), which would result in side bands of amplitude \(\sim 10^{-27} \text{ A}^2\). These side bands would then be 4 orders of magnitude smaller than the spectral feature we associate with narrow band frequency noise.

Since the narrow-band frequency noise in the nanotube is comparatively weak, one can interpret the results using the weak-noise expression for the spectrum Eq. (4) of the main text. Then the shape of the resonator spectrum gives the shape of the noise power spectrum. As seen from Fig. 1c, the spectrum is of \(1/f^\alpha\) type. Our data indicate that \(\alpha < 1\).

**Appendix D: The area of the driving-induced spectral peak for a linear oscillator**

We now consider the area \(S_F\) of the driving induced spectral peak for \(\omega\) close to \(\omega_0, \omega_F > 0\); note that this peak may have several maxima, as seen from Fig. 2 of the main text. We will specifically consider the peak at \(\omega > 0\), \(S_F = \int_0^\infty d\omega \Phi_F(\omega)\). Keeping in mind that \(\Phi_F(\omega)\) is small for large \(|\omega - \omega_F| \sim \omega_F\) [in fact, Eq. (B2) does
the noise intensity $\delta\omega_F/\Gamma = 5$

FIG. 6. The scaled area $\tilde{S}_F = 8\Gamma^2 \omega_0^2 S_F$ of the driving-induced peak in the oscillator power spectrum as a function of the frequency noise parameters. The data refer to Gaussian frequency noise with the power spectrum $\Xi(\Omega) = 2D\lambda^2/(\lambda^2 + \Omega^2)$.

**Appendix E: Power spectrum of a driven nonlinear oscillator**

The analysis of the dynamics of a nonlinear oscillator is complicated by the mutual influence of the frequency fluctuations that come from the thermally-induced amplitude fluctuations and the oscillator decay, which determines the correlation time of these fluctuations. The average linear susceptibility could be found for an arbitrary relation between the appropriate parameters, $\Delta\omega$ and $\Gamma$, see [25]. Finding the driving-induced terms in the power spectrum is more complicated, as it requires evaluating the susceptibility correlation function and also the nonlinear susceptibility and its correlator. An alternative approach is to consider first the dynamics of a modulated oscillator without fluctuations and then to take into account fluctuations.

For comparatively weak nonlinearity described by the term $\gamma q^4/4$ in the oscillator potential energy, the oscillator equation of motion in the rotating wave approximation has the form

$$\dot{u} = -(\Gamma + i\delta\omega_F)u + \frac{3i\gamma}{2\omega_0} |u|^2 u - i\frac{F}{4\omega_0} + f_u(t), \quad (E1)$$

where we have disregarded the direct frequency noise.

For $\Delta\omega \equiv 3|\gamma|k_BT/8\omega_0^3 \ll \Gamma$, one can study the dynamics by linearizing equations of motion about the state of forced vibrations, which is given by the stationary solution of Eq. (E1) for $f_u = 0$. The linearized equations of motion take the form

$$\dot{u} = \Phi_F(u), \quad (E2)$$

The spectrum (E2) is proportional to the derivative of the Lorentzian spectrum of the harmonic oscillator $\Phi_0(\omega) \propto 1/[\Gamma^2 + (\omega - \omega_0)^2] \times \omega$. It has a characteristic dispersive shape, being of the opposite signs on the other sides of $\omega_0$. As explained in the main text, this is the result of
the shift of the oscillator vibration frequency $\propto \gamma F^2$ due to the driving. Such shift is the main effect of the driving for small $\Delta \omega / \Gamma$.

In the opposite limit $\Delta \omega \gg \Gamma$ the nonlinearity of fluctuations about the state of forced vibrations is substantial. The analysis is simplified if the decay rate is small compared to the detuning of the modulating field from the small-amplitude oscillator frequency, $|\delta \omega_F| \gg \Gamma, \Delta \omega$. In this case the motion in the rotating frame is weakly damped. If we disregard damping and fluctuations, the quantity

$$g_{sl}(u, u^*) \approx \frac{\omega_0}{3 \gamma} \left[ \frac{3 \gamma}{2 \omega_0} |u|^2 - \delta \omega_F \right]^2 - \frac{F}{4 \omega_0}(u + u^*) \quad (E3)$$

is conserved, as seen from Eq. (E1); $g_{sl}$ plays the role of the energy of slow motion in the rotating frame. The dissipation-free motion is described by equation $\dot{u} = \partial_{\omega} g_{sl}$ and corresponds to vibrations in the rotating frame with frequency $\omega_{sl}(g_{sl})$ that can be found from Eq. (E3).

Damping and fluctuations lead to drift and diffusion of $g_{sl}$, forming a quasi-Boltzmann stationary distribution over $g_{sl}$ characterized by an effective temperature $T_e$ [34, 35]. To zeroth order in $\Gamma$, the oscillator power spectrum at frequency $\omega$ is determined by the probability density $\Phi_0(\omega_{sl} + \omega_{sl}(g_{sl}))$. Evaluating to the lowest order in $F^2$ the general implicit expression for the spectrum [37], we find

$$F^2 \Phi_F(\omega) = \beta \left\{ \theta(1 - 6 \beta) \Phi_0(\omega - 2 \beta \delta \omega F; T_e) \right\}_{\beta = 0},$$

$$\Phi_0(\omega; T_e) = \frac{\pi k_B T_e}{2 \omega_0^2 \Delta \omega} x e^{-x} \theta(x), \quad x = \frac{\omega - \omega_0}{2 \Delta \omega} \quad (E4)$$

where $\Phi_0(\omega; T_e)$ is the power spectrum in the absence of driving in the limit $\Gamma / \Delta \omega \to 0$ [29]. In Eq. (E4), $\theta(x)$ is the step function, $\beta = 3 \gamma F^2 / 32 \omega_0^3 (\delta \omega F)^3$ is the scaled driving intensity, and the effective temperature $T_e = T (1 + 6 \beta)$.

Equation (E4) refers to the most interesting case where $\gamma (\omega_F - \omega_0) > 0$, so that the driving frequency is in the region where the power spectrum in the absence of driving can be large; for $\gamma < 0$ one should replace $\Delta \omega \to - \Delta \omega$ in the expression for $x$. The simple asymptotic form of $\Phi_0(\omega; T_e)$ given in Eq. (E4) does not describe the tails of $\Phi_0(\omega; T)$. For $\Gamma, \Delta \omega \ll \delta \omega_F$, Eq. (E4) well describes $\Phi_F$ even where the ratio $\Delta \omega / \Gamma$ is not large, provided one uses for $\Phi_0(\omega)$ the full expression for the power spectrum of a nonlinear oscillator [25], which applies for arbitrary $\Delta \omega / \Gamma$. The corresponding analysis is beyond the scope of this paper.

The analytical results on the spectra of the modulated nonlinear oscillator are compared with the results of numerical simulations in Fig. 7. The simulations were performed in a standard way by integrating the stochastic differential equations [11] using the Heun scheme [38]; the same method was used to obtain the results of simulations in Fig. 2 of the main text for a linear oscillator. For a nonlinear oscillator, we verified that the values of the modulating field amplitude $F$ were in the range where the driving-induced term in the power spectrum was quadratic in $F$. As explained above, in the expression (E4) for $\Phi_F$ we used the full expression for $\Phi_0(\omega; T)$ [29]. As seen from this figure, the simulations are in excellent agreement with analytical results.

![Image of Fig. 7](image-url)

**FIG. 7.** The power spectra of a modulated nonlinear oscillator for large detuning of the driving frequency, $\delta \omega F / \Gamma = 40 \Delta \omega / \Gamma$. The solid curves and the dots show the analytical expression for the spectrum $\Phi_F$, while the full expression for $\Phi_0(\omega)$ was used in the simulations. The inset shows the change of the power spectrum in the absence of driving with varying $\Delta \omega / \Gamma$.

In the intermediate range, where the nonlinearity is not weak and the driving is not too far detuned, i.e., $|\delta \omega F| \sim \max(\Gamma, \Delta \omega)$, we obtained the spectrum $\Phi_F(\omega)$ by running numerical simulations. These results are presented in Fig. 4 of the main text. They show that the general trend seen in Fig. 7 that $\Phi_F(\omega)$ changes signs and is asymmetric for a nonlinear oscillator persists in this case as well.

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