Coupling parameters and the form of the potential via Noether symmetry

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Abstract

We explore the conditions for the existence of Noether symmetries in the dynamics of FRW metric, non minimally coupled with a scalar field, in the most general situation, and with nonzero spatial curvature. When such symmetries are present we find general exact solution for the Einstein equations. We also show that non Noether symmetries can be found. Finally, we present an extension of the procedure to the Kantowski- Sachs metric which is particularly interesting in the case of degenerate Lagrangian.

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1 Introduction

The importance of ‘Scalar Tensor Theory of Gravity’ is that both the gravitational and the cosmological constants result from a single scalar field which is somehow coupled with the curvature scalar. Both these so-called constants are time dependent, dynamical quantities in the theory. Further, different unification schemes of fundamental interactions based on supergravity and superstring theories lead to scalar tensor theory of gravity in the weak energy limit. The theory is also supposed to produce successful phase transition and help to solve the problems regarding graceful exit and the density perturbations.

Brans-Dicke, being motivated to incorporate Mach’s Principle in general theory of relativity, were the first to present a simple form of the scalar tensor theory of gravity by means of a constant coupling parameter $w$. However, the experimental constraints on $w$, resulting from some classical tests of gravitation, naturally requires that $w$ should be a function of the scalar field $\phi$, rather than a constant. The theory was thereafter generalized by Nordtvedt and Wagoner to entertain arbitrary self interaction of the scalar field in addition to the dynamical coupling with gravity. The importance of the theory was increased by the work of Mathiazhagan and Johri, where they proposed a revised model of the inflationary Universe under the framework of Brans-Dicke theory. A different form of scalar tensor theory was proposed by Zee incorporating the concept of spontaneous symmetry breaking. In that theory, which is known as the ‘Induced Theory of Gravity’, the form of the coupling with the scalar field with gravity is chosen as $\epsilon \phi^2$, where $\epsilon$ is a dimensionless constant. Though it was suggested that the same symmetry breaking mechanism is responsible for breaking a unified gauge theory into the strong, weak and electromagnetic interactions, yet such theory never goes over asymptotically to the standard FRW model. Rather, it is much better to choose the non-minimally coupled theory in its standard form, which again

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arises as the low energy effective action of different string theories [1]. In such theory, the coupling of the scalar field with gravity is taken as \((1 - \zeta \phi^2)\), where \(\zeta\) is a dimensionless coupling constant. It is apparent that if \(\phi\) admits

a solution that dies out with the cosmological evolution, the theory leads to the standard model with a minimally coupled scalar field.

All these Brans-Dicke, induced and non-minimally coupled theories of gravity are special cases of ‘Scalar Tensor Theory of Gravity’, which can be cast in such a general form that it even includes the Einstein-Hilbert action for a minimally coupled scalar field as a special case. The most general form of such a scalar tensor theory of gravity is

\[
A = \int d^4x \sqrt{-g} \left( f(\phi)R - \frac{w(\phi)}{\phi} \phi_{\mu} \phi^\mu - V(\phi) \right). \tag{1}
\]

This form is the most general one, since for \(f(\phi) = \phi\), it reduces to the Brans-Dicke form, for \(f(\phi) = \epsilon \phi^2\), it takes the form of the induced theory of gravity, for \(f(\phi) = 1 - \zeta \phi^2\) and \(\frac{w}{\phi} = \frac{1}{2}\), it is of the form of standard non-minimally coupled scalar field theory, for \(f(\phi) = \frac{\phi^2}{6}\) and \(w(\phi) = \frac{\phi}{2}\), the conformally coupled theory can be obtained. Finally, for \(f(\phi) = 1/2\) and \(\frac{w(\phi)}{\phi} = \frac{1}{2}\), it reduces to the form of Einstein-Hilbert action minimally coupled with a scalar field.

Essentially the form of the coupling parameters \(f(\phi), w(\phi)\)

and the form of the potential \(V(\phi)\) are not known a-priori and can not be obtained from the field equations. A new approach was initiated by de Ritis and coworkers ([1], [3] and the references therein) to find the forms of these parameters by demanding

that the Lagrangian admits Noether symmetry. As well known, the Noether theorem states that if there exists a vector field \(X\), such that the Lie derivative of a given Lagrangian vanishes, then \(X\) is a symmetry for the dynamics and generates a conserved current. The scheme was later carried out by Modak, Kamila and Biswas [14] to find the form of \(w(\phi)\). However, until very recently, none considered the most general form of the action (1) that we are talking

of to get solution of the Einstein field Eqns. In [16] Fay studies the existence of symmetries for this action. Our results are more general, as we find exact solutions when possible. Moreover, in the following paragraphs we discuss the importance of considering such

general form of the action .

It was a general belief that all the dynamical symmetries of a Lagrangian can be extracted by the application of Noether theorem. That it is not so has only recently been shown in a couple of publications by Sanyal and Modak [11] and Sanyal [12] (for a general discussion see [14]). It has been observed in [11] that in the Robertson-Walker metric the Noether symmetry makes the Lagrangian density \([1]\) degenerate, if one looks for \(f(\phi)\) in closed form. Moreover, the forms of \(f(\phi)\) and \(V(\phi)\) thus obtained do not satisfy the field equations for \(k = \pm 1\). This very strange situation that Noether symmetry does not satisfy the field equations has never been experienced before. However, if \(k = 0\), there is no such trouble. Field equations are well satisfied, degeneracy only leads to a constraint that has been analyzed and solutions to the field equations are found. The same work in the Kantowski-Sachs metric again reveals that the Lagrangian has to be degenerate, but here also the field equations are satisfied. Degeneracy has it’s usual feature that leads to a constraint, which can be analyzed, and solutions are obtained. It is thus clear that the very strange situation that we came across in the FRW metric for \(k = \pm 1\) has nothing to do with degeneracy, but the reason of such outcome is still not known and

should be studied in order to understand Noether symmetry better. Moreover such degeneracy leads to the very unpleasant consequence of leading to
	negative Newton’s gravitational constant.

Surprisingly enough, it was possible to explore some other type of symmetry in both the situations [11] and [12], which has nothing to do with Noether symmetry, and which does not make the Lagrangian to be degenerate. Such a situation when the symmetries are hidden and can not be explored by Noether theorem is rather new and demands that symmetries of a system should be studied thoroughly.

In the following section, we find the field equations for action (1). We shall also find a couple of equations from the field equations, one of

which is found to be elegant in finding the Noether conserved current. The importance of the other equation is to study other forms of symmetry,

which can not be found by the application of Noether theorem. In Sec. 3, we apply Noether theorem and find the solutions of the equations thus produced. As already mentioned, it requires to make additional assumptions
to obtain explicit solutions of the equations, which we do in Sec. 4. In this section we study the situation case by case, find the conserved current, express the Lagrangian in terms of the cyclic co-ordinates and find explicit solutions of the field equations. Sec. 5, is devoted to explore other forms of symmetries. In Sec. 6, we produce corresponding results in the homogeneous, but anisotropic cosmological model, taking Kantowski-Sachs metric as our starting point. Concluding remarks are made in Sec. 7.

2 Action and Field Equations

Our starting point, as already mentioned in the introduction, is the most general form of the scalar tensor theory of gravity, given by the action \( L \). In \( \Sigma \) it is shown that, by a suitable redefinition of the scalar field, it is possible to bring this action in the form of a general nonminimally coupled action. Thus it may appear that this kind of generalization is useless. The simplest way to justify it is by means of an example, which will be discussed in Sec. 4, Case 2.

In the Robertson-Walker metric the Ricci scalar is \( R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \). In view of which the action takes the following form

\[
A = \int \left(-6a^2\dot{\phi} f' - 6faa^2 + 6kfa + \frac{w}{\dot{\phi}}a^3\dot{\phi}^2 - a^3V\right) dt \tag{2}
\]

It should be noted that the Lagrangian turns out to be degenerate if the Hessian determinant \( W = \Sigma \frac{\partial^2 L}{\partial a \partial \dot{\phi}} = 0 \). For the action \( \Sigma \) it is \( W = -12a^4(3f'^2 + 2\frac{w f}{\dot{\phi}}) \). Therefore the Lagrangian under consideration is degenerate under the condition

\[
3f'^2 + 2\frac{w f}{\dot{\phi}} = 0. \tag{3}
\]

The field equations are

\[
2\frac{\ddot{a}}{a} + \frac{f'}{f} \dot{\phi} + \frac{f''}{f} \phi^2 + \frac{w \phi^2}{2f\dot{\phi}} + 2\frac{\ddot{a} \dot{\phi}}{a f} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{V}{2f} = 0 \tag{4}
\]

\[
\frac{\ddot{a}}{a} - \frac{w \phi}{3f\dot{\phi}} + \left(\frac{w}{\phi^2} - \frac{w'}{\phi}\right) \frac{\dot{\phi}^2}{6f} + \frac{a^2}{a^2} - \frac{w \ddot{\phi}}{f a \dot{\phi}} + \frac{k}{a^2} - \frac{V'}{6f} = 0 \tag{5}
\]

\[
\frac{a^2}{a^2} - \frac{w \ddot{\phi}}{6f} \phi^2 + \frac{f' \dot{\phi}}{f a} \phi + \frac{k}{a^2} - \frac{V}{6f} = 0, \tag{6}
\]

where, dot stands for derivative with respect to time and prime represents derivative with respect to \( \phi \). Eqn. \( \Sigma \) is a first integral, which is obtained by the so-called “Energy function” of Lagrangian (2). As we are considering the case of pure scalar field, without matter, we have to set \( E_L = 0 \). The case with matter can be treated also, but we will do this in the future. Interesting results are given in \( \Sigma \).

By simple algebraic manipulations, it is possible to recast these equations in a form which allows to find more general symmetries, discussed in Sec. 5.

\[
\sqrt{3f'^2 + 2\frac{w f}{\dot{\phi}}} \frac{d}{dt} \left(\sqrt{3f'^2 + 2\frac{w f}{\dot{\phi}}} \dot{a} + a^3f' \right) + a^3f \frac{V}{f^2} = 0 \tag{7}
\]

\[
\frac{\ddot{a}}{a} + \left(\frac{f'}{f} + \frac{w}{3f'\phi}\right) \dot{\phi} + \left(\frac{f''}{f} \frac{w}{2f'\phi} - \frac{w'}{6f'\phi^2} + \frac{w'}{6f'\phi}\right) \phi^2 + \left(2\frac{f'}{f} + \frac{w}{f'\phi}\right) \frac{\dot{a}}{a} + \frac{V'}{6f} - \frac{V}{2f} = 0. \tag{8}
\]

3 Application of Noether theorem

As already mentioned in Sec. 1, Noether theorem states that, if there exists a vector field \( X \), for which the Lie derivative of a given Lagrangian \( L \) vanishes i.e. \( L_X L = 0 \), the Lagrangian admits a Noether symmetry and thus yields a conserved current. In the Lagrangian under consideration the configuration space is \( M = (a, \phi) \) and
the corresponding tangent space is $TM = (a, \phi, \dot{a}, \dot{\phi})$. Hence the generic infinitesimal generator of the Noether symmetry is

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}},$$

(9)

where $\alpha$ and $\beta$ are both functions of $a$ and $\phi$ and

$$\dot{\alpha} \equiv \frac{\partial \alpha}{\partial \dot{a}} \dot{a} + \frac{\partial \alpha}{\partial \phi} \dot{\phi} ; \quad \dot{\beta} \equiv \frac{\partial \beta}{\partial \dot{a}} \dot{a} + \frac{\partial \beta}{\partial \phi} \dot{\phi}.$$  

(10)

The Cartan one form is

$$\theta_L = \frac{\partial L}{\partial \dot{a}} da + \frac{\partial L}{\partial \dot{\phi}} d\phi.$$  

(11)

The constant of motion $Q = i_X \theta_L$ is given by

$$Q = \alpha(a, \phi) \frac{\partial L}{\partial \dot{a}} + \beta(a, \phi) \frac{\partial L}{\partial \dot{\phi}}.$$  

(12)

If $X$ is found, it is then possible to find a change of variables $u(a, \phi), v(a, \phi)$, such that

$$i_X du = 1 ; \quad i_X dv = 0.$$  

(13)

When the Lagrangian is expressed in the new variables, $u$ turns out to be cyclic. The conserved current assumes a very simple form and we obtain often exact integration.

If we demand the existence of Noether symmetry $L_X L = 0$, we get the following equations,

$$\alpha + 2a \frac{\partial \alpha}{\partial a} + a^2 \frac{\partial \beta}{\partial a} \frac{f'}{f} + a \beta \frac{f'}{f} = 0$$  

(14)

$$3a\phi - a \beta + 2a \frac{\partial \beta}{\partial \phi} \frac{w'}{w} - 6 \phi \frac{\partial \alpha}{\partial \phi} = 0$$  

(15)

$$\left(2\alpha + a \frac{\partial \alpha}{\partial a} + a \frac{\partial \beta}{\partial \phi}\right) f' + a f'' \beta + 2 \frac{\partial \alpha}{\partial \phi} - \frac{w}{3} \frac{\partial \beta}{\partial a} = 0$$  

(16)

$$6k f \left(\alpha + a \beta \frac{f'}{f}\right) = a^2 V \left(3a + \frac{V'}{a} \beta\right).$$  

(17)

We have now to look for conditions on the integrability of this set of equations. We obtain restrictions on the form of $f$, $w$ and $V$, but large freedom of choice will be left, so that all the interesting cases are captured. Due to the very complicate situation we limit to the case when $\alpha$ and $\beta$ are separable (and non null), i.e.

$$\alpha(a, \phi) = A_1(a) B_1(\phi), \quad \beta(a, \phi) = A_2(a) B_2(\phi).$$  

(18)

We are thus well aware of the possibility of losing some solutions (see Sec. 5). We decided also to consider only the case $k \neq 0$, which is not well treated in the literature, with the exception of case 6 in Sec. 5. With these assumptions, we show in Appendix that the conditions for integration are

$$A_1 = -\frac{c}{a} ; \quad A_2 = \frac{l}{a^2} ; \quad V = V_0 f^3 ; \quad \frac{B_1'}{B_1} f' = \frac{2w}{3\phi} ;$$  

(19)

$$\frac{B_2 f'}{B_1 f} = c ; \quad 3f'^2 + \frac{2w}{\phi} f = n \frac{wf^3}{\phi},$$

where, $c, l, n, V_0$ are all arbitrary constants.
4 Solutions under different assumptions

In this section, we make reasonable assumptions on the form of \( f \) or \( w \), in order to find the solutions of Eqns. (14-17). The transformation of variables (13) and the new form of the Lagrangian, which turns out to be surprisingly simple. The conserved current and the \( E_L \) function give now first order differential equations, which can be exactly solved, in principle.

**Case 1.** Let us consider the a general nonminimally coupled case (that is: \( w = \frac{\phi^2}{2} \)), like in [8]. We get in view, of equation (20),

\[
3f'^2 + f = \frac{n}{2}f^3.
\]  

(20)

This gives an elliptic integral, which can be solved for \( f \) in closed form only under the assumption \( n = 0 \), for which the Lagrangian turns out to be degenerate. The solution is \( f = -\frac{(\phi - \phi_0)^2}{12} \), which means that the Newtonian gravitational constant turns out to be negative. More strikingly, the solutions obtained do not satisfy the field equations, as one can easily verify from Eqn. (7). It is still not clear how to interpret such a situation. It is definitely not due to the degeneracy of the Lagrangian, since the similar situation in the Kantowsky-Sachs metric [12] is found to behave properly. This is the situation we came across in [11] and so we leave this case at this stage.

**Case 2.**

Let us now consider a general Brans-Dicke theory (that is \( f = \phi \)).

Under this condition and in view of Eqn. (20), we obtain the following solutions

\[
V = V_0\phi^3, \quad w = \frac{3}{n\phi^2 - 2}, \quad n \neq 0,
\]  

(21)

together with

\[
B_1 = B_0 \frac{\sqrt{n\phi^2 - 2}}{\phi}, \quad B_2 = cB_0 \sqrt{n\phi^2 - 2},
\]  

(22)

where, \( B_0 \) is a constant. Hence \( \alpha \) and \( \beta \) are obtained as,

\[
\alpha = -C \frac{\sqrt{n\phi^2 - 2}}{a\phi}, \quad \beta = C \frac{\sqrt{n\phi^2 - 2}}{a^2},
\]  

(23)

where \( C = c \sqrt{B_0} \) is yet another constant. The conserved current turns out to be,

\[
Q = a\sqrt{n\phi^2 - 2} \left( \frac{\dot{\phi}}{a} + \frac{n\phi^2 - 1}{n\phi^2 - 2} \right).
\]  

(24)

It can be verified that such a conserved current follows from Eqn. (9), upon substituting the values of \( f \), \( w \) and \( V \). The form of \( w \) we have obtained in this process has an excellent feature. Initially when \( \phi \) is large, as should be the case, \( w \) is small, finally when \( \phi \) falls off, \( w \) becomes large enough leading to ordinary G.R.T.

Let us now perform the change of variables to obtain the corresponding cyclic coordinates. We need a particular solution of Eqn. (13). We find

\[
u = \frac{a\phi^2}{2} \sqrt{n\phi^2 - 2}; \quad v = a\phi,
\]  

(25)

which can be inverted by

\[
a^2 = \frac{nv^4 - 4uv^2}{2v^2}; \quad \phi^2 = \frac{2v^4}{nv^4 - 4uv^2}.
\]  

(26)

Being always \( a > 0 \), the Jacobian of transformation does not give any problems, and the same will be for all the cases below. Under this transformation, the Lagrangian takes the nice form

\[
L = \frac{3\dot{u}^2}{v} - 3nv^2 + 6kv - V_0v^3.
\]  

(27)

As announced, \( u \) is cyclic. The conserved current gives

\[
Q = \frac{\partial L}{\partial \dot{u}} = \frac{6\dot{u}}{v}.
\]  

(28)
We use now the condition $E_L = 0$ to find $v$

$\left( \frac{Q^2}{12} - 6k \right) + V_0v^2 = 3nv^2$, \hspace{1cm} (29)

which can be integrated. Setting $F = 6k - Q^2/12$, we get

$v = \frac{-e\sqrt{\frac{\alpha}{V_0}} t + 4F e^{-\sqrt{\frac{\alpha}{V_0}} t}}{4\sqrt{V_0}}$, \hspace{1cm} (30)

and

$u = \frac{\sqrt{3}Q}{24V_0} \left( e^{\sqrt{\frac{\alpha}{V_0}} t} 4 Fe^{-\sqrt{\frac{\alpha}{V_0}} t} \right) + u_0$. \hspace{1cm} (31)

Here and below we set to zero the integration constant for the origin of time. The condition $E_L = 0$ fixes another one, so that we are left with two ($A$ and $u_0$). The expression of $a$ and $\phi$ is rather involved and we do not write it explicitly. It is interesting anyway to show the behavior at large times. It is

$a(t \rightarrow \infty) \propto e^{\sqrt{\frac{\alpha}{V_0}} t}; \quad \phi(t \rightarrow \infty) = \text{const.}$ \hspace{1cm} (32)

We see that we have inflationary asymptotic behavior and that $\phi$ (and then $w$) goes to a constant.

Let us observe finally that, as said before, it would be possible, with a transformation of $\phi$, to bring this case to the above one. Being $n \neq 0$, we arrive to the nondegenerate case, so that $f$ is not in closed form. Actually the transformation itself is not obtained in closed form. We see thus that, although the two situations are mathematically equivalent, we would pass from a solvable and physically significant situation to a totally unmanageable one, from both points of view.

**Case 3.**

Let us consider the induced theory of gravity by choosing $f = \epsilon \phi^2$, $\epsilon$ being the coupling constant. Under this choice, Eqn. (22) gives

$V = V_0\phi^6$; $\quad w = \frac{12\epsilon \phi}{n\epsilon^2 \phi^4 - 2}$ \hspace{1cm} (33)

along with

$B_1 = \frac{q\sqrt{nc^2 \phi^4 - 2}}{\epsilon \sqrt{\phi^2}}$; $\quad B_2 = \frac{cq\sqrt{nc^2 \phi^4 - 2}}{2\epsilon \sqrt{\phi^2}}$, \hspace{1cm} (34)

where, $c, q$ are constants. As a result we find

$\alpha = -\frac{N\sqrt{nc^2 \phi^4 - 2}}{a\phi^2}$; $\quad \beta = \frac{N\sqrt{nc^2 \phi^4 - 2}}{2a^2 \phi}$, \hspace{1cm} (35)

where, $N = \frac{c \ell q}{\epsilon \sqrt{n}}$ is a constant,

which we can set to unity. The conserved current turns out to be

$Q = a\sqrt{nc^2 \phi^4 - 2} \left( \frac{\dot{a}}{a} + 2 \right) \frac{\phi}{\phi (nc^2 \phi^4 - 2)} \left( \frac{\phi (nc^2 \phi^4 - 1)}{\phi (nc^2 \phi^4 - 2)} \right)$. \hspace{1cm} (36)

Here again we observe that the conserved current follows from equation (3), upon substituting the forms of $f, w$ and $V$ in it. Further the form of $w$ here also has got the same excellent feature as in the previous case.

We find now the transformed Lagrangian. As the procedure strictly follows the above one, we make here and below some shortening. The transformation is

$a^2 = \frac{\epsilon^2 n v^6 - 4u^2}{2v^2}; \quad \phi^2 = \frac{\sqrt{2} v^4}{\epsilon^2 \epsilon^2 n v^6 - 4u^2}$, \hspace{1cm} (37)

and the transformed Lagrangian

$L = \frac{3\epsilon \dot{u}^2}{v^2} - 12\epsilon^3 n v^4 \dot{v}^2 + 6\epsilon k v^2 - V_0 v^6$. \hspace{1cm} (38)
Let us set \( F = Q^2 - 72c^2k, \ G = 12cV_0, \ H = 12c^2\sqrt{n}. \) We get

\[
v = \frac{\sqrt{e^{2\phi^2} - 4F e^{-2\phi^2}}}{2G^{1/4}}; \quad u = \frac{Q}{48cG} \left(e^{\frac{2\phi^2}{H}} + 4Fe^{-\frac{2\phi^2}{H}}\right). \tag{39}
\]

Again we obtain, for large \( t, \) that \( a \) approaches to an exponential and \( \phi \) to a constant.

**Case 4.**

Let us now consider the theory of a scalar field being nonminimally coupled with gravity, by choosing \( f = 1 - \zeta\phi^2. \) As a result, we get,

\[
V = V_0(1 - \zeta\phi^2)^3, \quad w = \frac{12\zeta^2\phi^3}{(1 - \zeta\phi^2)[n(1 - \zeta\phi^2)^2 - 2]}, \tag{40}
\]

and

\[
B_1 = \sqrt{n(1 - \zeta\phi^2)^2 - 2} \quad \text{and} \quad B_2 = -\frac{c}{2\sqrt{2}\zeta} \sqrt{n(1 - \zeta\phi^2)^2 - 2}. \tag{41}\]

Hence we obtain

\[
\alpha = -\frac{cl}{\sqrt{2}} \frac{\sqrt{n(1 - \zeta\phi^2)^2 - 2}}{a(1 - \zeta\phi^2)}, \quad \beta = -\frac{cl}{a^2}\frac{\sqrt{n(1 - \zeta\phi^2)^2 - 2}}{\phi}. \tag{42}\]

Finally, we obtain the conserved current as,

\[
Q = \sqrt{n(1 - \zeta\phi^2)^2 - 2} \left( \frac{n(1 - \zeta\phi^2)^2 - 1}{(1 - \zeta\phi^2)(n(1 - \zeta\phi^2)^2 - 2)} a\phi^2 \right). \tag{43}\]

A little algebraic calculations shows that here also one can generate the above conserved current simply from the first integral of equation (3), upon substituting \( f, f, f'', w, w', \) and \( V \) from solutions (23).

Again, we can find a transformation

\[
a^2 = \frac{ne^{2v} - 4ue^{-2v}}{2}; \quad \phi^2 = \frac{1}{\zeta} \left( 1 - \sqrt{\frac{2}{ne^{2v} - 4ue^{-2v}}} \right), \tag{44}\]

with a new Lagrangian

\[
L = 3e^{-v}u^2 - 3e^{3v}\phi^2 + 6kv - V_0e^{3v}, \tag{45}\]

and solutions

\[
v = \log(e^{2\lambda t} - 4F) - \lambda t; \quad u = -\frac{Q\sqrt{3n}}{4V_0}(e^{\lambda t} + 4Fe^{-\lambda t}) + u_0 \tag{46}\]

where \( B \) is the conserved current, \( u_0 \) is an integration constant and \( F = 3B^2 - 6k, \lambda = \sqrt{V_0/3n}. \)

Asymptotic behaviors of \( a \) and \( \phi \) are the same as before.

**Case 5.**

Let \( \frac{wn}{\phi}f^3 = \text{constant} = q. \) Upon imposing this assumption, following results emerge in view of equation (24).

\[
f = m\sqrt{1 + \epsilon\phi^2}; \quad V = V_0(1 + \epsilon\phi^2)^3, \quad w = \frac{3}{2}\epsilon m\phi \left( 1 + \epsilon\phi^2 \right)^2; \tag{47}\]

where, \( m = \sqrt{\frac{2}{n}} \) and \( \epsilon = \frac{2n}{6}. \) \( B_1 \) and \( B_2 \) are given by,

\[
B_1 = \frac{\phi}{\sqrt{1 + \epsilon\phi^2}}, \quad B_2 = \frac{c}{\epsilon a^2}\sqrt{1 + \epsilon\phi^2}. \tag{48}\]

Hence, \( \alpha \) and \( \beta \) take the following form,

\[
\alpha = -\frac{cl}{a} \left( \frac{\phi}{\sqrt{1 + \epsilon\phi^2}} \right), \quad \beta = -\frac{cl}{\epsilon a^2} \sqrt{1 + \epsilon\phi^2}. \tag{49}\]
Finally, the conserved current turns out to be,

\[ Q = a\phi \left( \frac{\dot{a}}{a} + \frac{2\epsilon\dot{\phi}^2 + 1}{2(1 + \epsilon\phi^2)} \dot{\phi} \right). \]  

As before it is quite trivial to show that the first integral of equation (9), upon substituting \( f, \ f', \ f'', \ V, \ w, \ w' \) form solution (47) in it, leads to the above conserved current.

The transformation is

\[ a^2 = e^{2\epsilon v} - \frac{u^2}{v} e^{-2\epsilon v}; \quad \phi^2 = \frac{u^2}{\epsilon^2 \left( e^{4\epsilon v} - \frac{u^2}{\epsilon} \right)}; \]  

so that the Lagrangian is

\[ L = \frac{3m}{2\epsilon} e^{-\epsilon v} a^2 - 6m e^{3\epsilon v} v^2 + 6 k m e^v - V_0 e^{3\epsilon v}, \]  

with solutions

\[ v = \frac{1}{\epsilon} \log(e^{2\lambda t} - 4F); \quad u = \frac{HQ \lambda t}{3m\sqrt{G}} (e^{\lambda t} + 4Fe^{-\lambda t}) + u_0, \]  

where \( u_0 \) is an integration constant and \( F = \epsilon Q^2 - 36km^2, \ G = 6mV_0, \ H = 6em. \)

Case 6

Let, \( f = constant = f_0 > 0 \) (in order to have positive Newton constant.)

Under this situation Eqns. (14-17) can be solved only for the vanishing curvature constant, i.e., for \( k = 0 \). The method of separation of variables yields,

\[ A_1 = \frac{c_1}{\sqrt{a}}; \quad A_2 = -\frac{2c_2}{3a^{3/2}}, \]  

where \( c_1 \) and \( c_2 \) are constants. Further, following differential equations are obtained, viz.

\[ \frac{V'}{V} = \frac{9c_1 B_1}{2c_2 B_2}; \quad \frac{B'_1}{B_2} = \frac{c_2 w}{2f c_1 \phi}; \quad B_2 - 2\phi B'_2 - \frac{w'}{w} \phi B_2 + \frac{9c_1}{2c_2} B_1 \phi = 0. \]  

One equation is lost in the process and we have thus to impose yet another assumption.

Subcase 1:

Let \( V = m^2 \phi^2 \).

Under this assumption we get

\[ w = \frac{8f_0 \phi}{3\phi^2 + n}. \]  

It is better to treat separately the cases \( n = 0 \) and \( n \neq 0 \).

Subcase 1a: \( n = 0 \)

In this case we have

\[ \alpha = \frac{\phi}{\sqrt{a}}; \quad \beta = -\frac{3\phi^2}{2a^{3/2}}; \]  

and the transformation is

\[ a^3 = 3uw; \quad \phi^2 = \frac{v}{3u}. \]  

The transformed Lagrangian is

\[ L = m^2 v^2 - 8f_0 \dot{u} \dot{v}. \]  

The solutions are very simple and we can give directly \( a \) and \( \phi \)

\[ a = a_0 t^{4/3}; \quad \phi = \frac{2\sqrt{2}}{mt}. \]  

Subcase 1b: \( n \neq 0 \)

We get now

\[ \alpha = \sqrt{\frac{n + 3\phi^2}{a}}; \quad \beta = -\frac{3\phi \sqrt{n + 3\phi^2}}{2a^{3/2}}. \]
with transformation

\[ a^3 = \frac{9n^2u^2 - 12v^2}{4n} ; \quad \phi^2 = \frac{4nv^2}{9n^2u^2 - 12v^2}, \] (62)

and new Lagrangian

\[ L = 6f_0n\dot{u}^2 - \frac{8f_0}{n}v^2 + m^2v^2, \] (63)

with solutions

\[ a^3 = a_0\left(t^2 - \frac{8f_0}{nm^2} \sin^2 \omega t \right) ; \quad \phi^2 = \frac{8f_0}{3m^2n t^2 - \frac{8f_0}{m^2n} \sin^2 \omega t}, \] (64)

where \( a_0 \) is an integration constant and \( \omega = \frac{8f_0}{m^2n} \). This case differs from the others, as the asymptotic behaviour of \( a(t) \) for \( t \to \infty \) is not inflationary; in fact we have that \( a(t \to \infty) \propto t^{2/3} \), and \( \phi(t \to \infty) \propto \frac{\sin (\omega t)}{t} \).

**Subcase 2:**

\[ V = V_0 \left( A \exp \left( \frac{\lambda \phi}{2} \right) + B \exp \left( -\frac{\lambda \phi}{2} \right) \right)^2 , \] with \( \lambda = \frac{1}{\sqrt{3}} \sqrt{\frac{2}{2}} \), and \( M^2 = (8\pi G)^{-1} \).

This case has been extensively treated in ([8], [14], [12]), so that we refer to them for the details.

**Subcase 3:** \( V = V_0 \) This case has been treated in ([15]) and admit as conserved current:

\[ Q = a^3 \phi. \] (65)

## 5 Existence of other symmetries

In the previous section it has been observed that for the existence of Noether symmetry of the action (2) in the Robertson-Walker space time for \( k = \pm 1 \), \( V \) is proportional to \( f^3 \). In this section we shall explore other possible symmetries of the theory, which could not be obtained by the application of Noether theorem. For this purpose we call upon equation (7) and note that for \( V(\phi) \) proportional to \( f^2(\phi) \), i.e.

\[ V(\phi) = V_0 f^2(\phi) \] (66)

where, \( V_0 \) is a constant, we obtain two possibilities. The first one is,

\[ Q = (3f'^2 + \frac{2w}{\phi} f) \dot{a}^3 \phi = \text{constant}. \] (67)

This is a very interesting result. It shows that symmetry exists even for \( V \) proportional to \( f^2 \), and holds for arbitrary \( f \) (hence \( V \) and \( \phi \)), which could not be explored by previous treatment. This interesting result was pointed out in a couple of recent publications [11] and [12]. This conserved charge exists even for \( V = 0 \).

There of course exists yet another possibility, viz.

\[ 3f'^2 + \frac{2wf}{\phi} = 0 \] (68)

It has been pointed out right at the beginning that, under this condition, the Hessian determinant vanishes; as a result the Lagrangian becomes degenerate. Further, this equation is satisfied at the cost of making either \( f \) (and hence Newton’s gravitational constant) negative, which is of course unphysical, or the Brans-Dicke coupling parameter \( w \) negative, which leads to all time acceleration of the Universe. Further, in view of equations (66) and (67) we find that equation (8) leads to,

\[ \frac{d}{dt} \left( 2 \dot{a} + f' \frac{a \dot{\phi}}{f} \right) = \frac{1}{3} V_0 a f. \] (69)

This means that there exists no conserved current in general in such a situation other than for \( V = 0 \).

In Sec. 3 we have underlined that, due to the choice (18) there is the possibility that conditions (19) do not cover all the possible Noether symmetries. Therefore we briefly show that the symmetry defined in Eqn. (67) is indeed not of Noether type. To this goal we rewrite the current (67) in the following way:

\[ Q = G(\phi)a^3 \dot{\phi}, \] (70)
where \( G(\phi) = \sqrt{3(f')^2 + \frac{2wf}{\phi}} \). If Eqn. (67) is a Noether current, then there exists a vector field \( X \) (the generator) on \( TQ \)

\[
X = \alpha(a, \phi) \frac{\partial}{\partial a} + \beta(a, \phi) \frac{\partial}{\partial \phi} + \dot{\alpha}(a, \phi) \frac{\partial}{\partial \dot{a}} + \dot{\beta}(a, \phi) \frac{\partial}{\partial \dot{\phi}},
\]

such that \( i_X \theta_L = Q \). Thus

\[
i_X \theta_L = \left( -6a^2 f' \alpha + 2\beta \frac{w}{\phi} a^3 - 12fa\alpha \dot{\alpha} \right) \equiv G(\phi) a^3 \dot{\phi}, \tag{71}\]

which implies \( \alpha = 0 \) and \( \beta = \frac{\phi}{2w} G(\phi) \). Substituting into Eqns (14-17) gives inconsistency, unless \( f = \text{constant} \).

Due to the fact that this symmetry is generally not of Noether type, we cannot find cyclic variables and the treatment is more difficult. We discuss an interesting general feature of the solution and some very particular cases. From the expression of the conserved current, it is possible to extract \( \dot{\phi} \) and insert it into \( E_L \). Now from \( E_L = 0 \) we obtain \( \dot{a} \). If we derive \( E_L \) w.r.t. time and use again the expression of \( \dot{a} \), we obtain an interesting expression for \( \ddot{a} \)

\[
\ddot{a} = \frac{2w(na^6 f^3 + Q^2) + 3na^6 \phi f^2 (f')^2}{6fa^5 (2fw + 3\phi (f'))^2} > 0, \tag{72}\]

which shows that all the evolution is inflationary. In particular, when \( f \) is constant, we get the very simple form

\[
\ddot{a} = \frac{f_0 na}{6} + \frac{N^2}{6f_0^2 a^5}, \tag{73}\]

which shows that there are two regimes, with a late time exponential behavior. As we said this case is indeed a special Noether symmetry. In order to treat it like the ones above, however, we should specify the function \( w \). If \( \frac{w}{\phi} = \frac{1}{2} \), we obtain a minimally coupled field with constant potential and the conserved current is just \( a^3 \dot{\phi} \). As said above, this case was treated in \( [15] \). Other choices of do not seem to be of particular physical interest. It is in any case interesting that Eqn. (72) holds independently of \( w \).

In some special cases it is possible to use the current (67) to find particular solutions of the Einstein equations. As an example we can consider the case \( f = \phi, \ w = \frac{\phi}{2} \). Thus the Eqn. (67) becomes

\[
Q = \sqrt{3 + \phi a^3}. \tag{74}\]

We can obtain a solution imposing that \( \phi = \frac{\phi_0}{a^2} \), so that

\[
\dot{a} = \frac{-|Q|}{2} \frac{a}{\sqrt{3a^2 + 1}}, \tag{75}\]

where \( G = -|N| \).With this choice the Eqn. (72) becomes:

\[
\ddot{a} = \frac{(n\phi_0 + Q^2) + 3na^2 \phi_0^2}{a(3a^2 + 1)}. \tag{76}\]

In order that the Eqs. (73) and (74) are compatible the following condition is required: \( |Q| = -\frac{2(6k - n\phi_0)\phi_0}{\sqrt{2\phi_0}} \).

Then Eqn. (73) can be solved and inverted in terms of elliptical integrals: in the following we show the asymptotic behaviors

\[
a(t \to 0) \propto t \tag{77}
\]

\[
\phi(t \to 0) \propto t^{-2} \tag{78}\]

and

\[
a(t \to \infty) \propto t \tag{79}
\]

\[
\phi(t \to \infty) \propto t^{-2}. \tag{80}\]
6 Corresponding results in Kantowski-Sachs metric

We start with the same action (1). The Ricci scalar is now \( 4R = 2\left(\frac{a}{b} + 2\frac{\dot{a}}{b} + 2\frac{\ddot{a}}{ab} + \frac{b^2}{b^2} + \frac{1}{b^2} - \frac{V}{b}\right) \). Hence the action, apart from a total derivative term, is

\[
A = 4\pi \int \left[-4f' a b \dot{\phi} - 2f' b^2 a \dot{\phi} - 4f b \dot{\phi} - 2f a b^2 + ab^2 \frac{w}{\phi} \dot{\phi}^2 + 2fa - ab^2 V(\phi)\right] dt. \tag{81}
\]

The Hessian determinant is \(-32\pi^2 f a b^4 (3f'^2 + 2f \frac{w}{\phi})\). So the Lagrangian is degenerate under the condition \(3f'^2 + 2f \frac{w}{\phi} = 0\). The field equations are,

\[
\frac{\ddot{b}}{b} + \frac{f'}{f} \dot{\phi} + \frac{f''}{f} \dot{\phi}^2 + 2\frac{f' b}{f b} \dot{\phi} + \frac{b^2}{b^2} + \frac{w}{2f} - \frac{V(\phi)}{2f} = 0 \tag{82}
\]

\[
\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{f'}{f} \dot{\phi} + \frac{\ddot{b}}{ab} + \frac{\dot{a}b}{ab} + \frac{\dot{b}a}{bf} + \left(\frac{f''}{f} + \frac{w}{2f}\right) \dot{\phi}^2 - \frac{V(\phi)}{2f} = 0 \tag{83}
\]

\[
\frac{\ddot{b}}{b} + \frac{2\ddot{b}}{b} + \frac{w}{\phi} + 2\frac{\dot{a}b}{ab} + \frac{\dot{b}a}{ab} + \frac{\dot{b}a}{bf} + \frac{\dot{f} b}{f b} + \frac{w}{2f} \dot{\phi}^2 + \frac{w}{2f} \phi - \frac{V(\phi)}{2f} = 0 \tag{84}
\]

In view of the above field equations one can construct yet another important equation, viz.,

\[
\sqrt{(3f'^2 + 2\frac{w}{\phi})}\frac{d}{dt} \left(\sqrt{(3f'^2 + 2\frac{w}{\phi})} ab^2 \dot{\phi}\right) + ab^2 f^3 \sqrt{\frac{V}{f^2}} = 0 \tag{86}
\]

Now the detailed calculation reveals that Noether symmetry in this situation exists at the price of making the Lagrangian degenerate. There is nothing wrong for a Lagrangian to be degenerate. We have shown earlier [12], how to deal with such Lagrangians. However, this degeneracy indicates that either of \(f\) or \(w\) has to be negative, which is nasty. It thus appears that Noether theorem does not reveal any physically reasonable form of the coupling parameters. Nevertheless it can be shown as before that there exists other conserved currents which can be explored from the field equations but not from the consideration of Noether theorem.

In equation (81) if one considers \(V(\phi)\) proportional to \((f(\dot{\phi})^2\) then last term vanishes. Hence one can choose either \(\sqrt{(3f'^2 + 2\frac{w}{\phi})} = 0\), which is the outcome of Noether symmetry, or

\[
Q = ab^2 \dot{\phi} \sqrt{3f'^2 + 2\frac{w}{\phi}} = conserved. \tag{87}
\]

This has the same form of the conserved current (66).

7 Concluding remarks

Once again, the Noether symmetry approach revealed a powerful tool in the study of scalar tensor theories.

We have studied the more general possible action in the case of a pure scalar field domination.

The first result is that asymptotic inflationary behaviour is always obtained. A second important result is the possibility of symmetries of more general type; a feature which is, to our knowledge, completely unexplored in this kind of problems. There is of course room for deeper investigation on this point.

A third improvement lies in the consideration that the action (1), more general of the one treated in [16], gives indeed new possibilities, despite the mathematical equivalence of the two cases. Some open problems were also discussed: there is the clearly need of more investigation on degenerate cases; also the treatment in the presence of perfect fluid is important in view of applications to recent observations (for a recent treatment in the minimal coupling case see [13], [17]. But most of all a good understanding of the physical meaning of Noether symmetries in this context would be the greatest hit.
8 Appendix

In this section we want at least sketch the calculation, leading to the Eqs. (20) starting from the factoring hypothesis in Eqn. (18). In order to perform this let us start from the Eqn. (14) and divide by $B_1$, obtaining:

$$\frac{A_1 + 2aA_1aB_1}{a(A_2 + 2aA_2,a)} = -\frac{B_2 f'}{B_1} = -C_1,$$

where we assume $(A_2 + 2aA_2,a) \neq 0$. The case $(A_2 + 2aA_2,a) = 0$ will be discussed later. From the Eqn. (3) we obtain

$$aA_2 = 3\phi \left( \frac{B_1 - 2\phi f' B_2}{B_2 - 2\phi B_2' - \phi \frac{\phi'}{w} B_2} \right) = C_2$$

(89)

where we assume that $B_2 - 2\phi B_2' - \phi \frac{\phi'}{w} B_2 \neq 0$. Using the Eqs. (88) and (89), we can replace $A_2$ and $B_2$ in Eqn. (17), obtaining the following relation:

$$6kf (1 + C_1 C_2) = a^2 V \left( 3 + C_1 C_2 \frac{V' f}{V f} \right),$$

(90)

which, for $k \neq 0$ implies that:

$$C_1 C_2 = -1$$

(91)

$$V = V_0 f^3.$$  

(92)

In order to obtain $A_1$ and $A_2$, let us set $C_1 = c = -\frac{1}{C_2}$, and replace in the Eqn. (88) and (89), so that

$$A_2 = \frac{l}{a^2}$$

(93)

$$A_1 = -\frac{cl}{a}.$$  

(94)

$$B_2 = c \frac{f}{f'} B_1$$

(95)

Finally, using the Eqn. (16) and (88) we obtain the last two equations in (20):

$$B_1' = \frac{2}{3 \phi f} B_1,$$

(96)

$$3f'^2 + 2 \frac{w}{\phi} f = n \frac{wf^3}{\phi}.$$  

(97)

We note that the Eqs. (93–95) imply the following general relation between $\alpha$ and $\beta$:

$$\alpha = -a\beta \frac{f'}{f}.$$  

(98)

Since the Eqn (17) this implies that both the sides in the (17) vanishes separately, as was supposed a priori in (8). Let us go back to dismiss the assumptions put forward above in the calculations, that is $(A_2 + 2aA_2,a) \neq 0$ and $B_2 - 2\phi B_2' - \phi \frac{\phi'}{w} B_2 \neq 0$. If $(A_2 + 2aA_2,a) = 0$, then also $(A_1 + 2aA_1,a) = 0$, so that

$$A_1 = \frac{q}{\sqrt{a}}$$

(99)

$$A_2 = \frac{p}{a}.$$  

(100)

while $B_2 f'$ and $B_2$ remain arbitrary (but $\neq 0$). However such solutions are incompatible with the Eqn. (8) unless $a = 0$, which is a unphysical trivial solution. Consider now the case $B_2 - 2\phi B_2' - \phi \frac{\phi'}{w} B_2 = 0$, which implies that
\[ B_1 - 2\phi \frac{f'}{w} B_1' = 0, \text{ i.e.} \]

\[
\frac{B_1'}{B_1} = \frac{w}{2\phi f'} \\
B_2' = \frac{b\phi}{w}. \tag{101}
\]

Such equations are incompatible with Eqn. (16), unless one reduces to the degenerate case

\[ 3f'^2 + 2\frac{w}{\phi} f' = 0. \]

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