Black hole solutions in the warped DGP braneworld

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Abstract
We study the static, analytical solution of black holes in the warped DGP braneworld scenario. We show that the linearized field equations and matching conditions lead to solutions that are not compatible with Schwarzschild-(A)dS$_4$ solutions on the brane. This incompatibility is similar to vDVZ discontinuity in massive gravity theory. Following the standard procedure to remove this discontinuity, which was firstly proposed by Vainshtein, we keep some appropriate nonlinear terms in the field equations. This strategy has its origin in the fact that the spatial extrinsic curvature of the brane plays a crucial role in the nonlinear nature of the solutions and also in recovering the well-measured predictions of general relativity (GR) at small scales. Using this feature, we obtained an interesting black string solution in the bulk when it is compatible with 4D GR solutions on the brane.

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1. Introduction

A model universe with possible extra dimensions has attracted a lot of attention in recent years. Questions such as ‘are we living really in a four-dimensional universe?’ or ‘are there extra dimensions we cannot see?’ have not been addressed precisely yet. However, theoretically there are a lot of interesting outcomes by incorporating some extra dimensions in theories such as gravitation. In fact, extra dimensions are an integral part of fundamental theories of physics such as superstring/M-theories that need more than four spacetime dimensions. The idea of adding an extra dimension to general relativity’s (GR’s) four-dimensional spacetime was first introduced by Kaluza and Klein to unify electromagnetism and gravity. Now $p$-branes and $D$-branes are well known in the context of string theory where branes are solitonic solutions of ten-dimensional string theories. $p$-branes are extended objects of higher dimension than strings (1-branes) and the $D$-branes are kind of $p$-branes on which open strings can end [1]. In braneworld scenarios, we assume that our $(1+3)$-dimensional spacetime is a domain wall
embedded in a five-dimensional spacetime called the bulk [1]. All matter fields and other gauge bosons live on the brane but gravitons, which carry the gravitational interaction, can travel into the extra dimension. There are at least two basic reasons for investigating braneworld models: firstly for solving hierarchy problem, that is, a large difference between Planck and Electroweak scales, secondly for proposing a unified theory of all fundamental interactions in the nature.

In the last two decades, different kinds of braneworld scenarios have been proposed such as the Horava–Witten model [2], Arkani Hamed–Dimopoulos–Dvali (ADD) model [3], Randall–Sundrum (RS) models [4] and the Dvali–Gabadadze–Porrati (DGP) model [5]. Among these models, one of which has attracted most regards in last decade is the RS one-brane (RSII) model. In this model it is assumed that our universe is a 3-brane with positive tension, the extra dimension is large and the bulk has a negative cosmological constant leading to a warped geometry. Another model that has attractive results from cosmological viewpoint is the DGP model with one large extra dimension but the bulk is Minkowski. In this model, there is a new term in the total action that comes from quantum interaction between matter confined on the brane and the bulk gravitons which induces gravity on the brane. In this model, gravity leaks off the 4D brane into the bulk and becomes five-dimensional at large scales, $r \gg r_c$, but on small scales, gravity is effectively bounded to the brane and 4D dynamics is regained. The transition from 4D to 5D behavior is governed by the crossover scale $r_c$. An interesting property of this model is that it contains a self-accelerating branch of the solutions which can explain late time acceleration due to a weakening of gravity at low energies [8].

We note that a basic requirement for any alternative theory of gravity, such as braneworld scenarios, is that they should reproduce general relativity predictions in the appropriate limit to be phenomenologically viable. Two of the most important applications of GR are cosmology and black holes. Here, our concentration is on the issue of braneworld black holes, i.e. finding the bulk and the brane metric when a spherically symmetric energy–momentum distribution is localized on the brane. To obtain black hole solutions in a braneworld scenario, we note that generally braneworld solutions can be obtained by following two different approaches. In the first approach, dynamics and geometry of the whole bulk spacetime are primarily considered; then the dynamics on the brane is extracted by using the Darmois–Israel matching conditions. The second approach is to obtain effective four-dimensional field equations on the brane firstly and then try to extend these solutions to the bulk. This approach has been applied to braneworlds with induced gravity in [6]. The main difficulty in the latter approach is that it is not always possible to obtain a closed set of equations on the brane. So we are not able to predict the behavior of the fields on the brane just with data on the brane.

In 1999, Chamblin, Hawking and Reall found the Schwarzschild black string solution for the RSII model as follows [10]:

\[
\begin{align*}
\mathsf{d}s^2 &= e^{\frac{2a}{\ell}} \mathsf{g}_{\mu\nu} \, \mathsf{d}x^\mu \, \mathsf{d}x^\nu + \mathsf{d}y^2 \\
\mathsf{g}_{\mu\nu} &= e^{\frac{2a}{\ell}} \mathsf{g}_{\mu\nu} = -\left(1 - \frac{2GM}{r}\right) \mathsf{d}t^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 \mathsf{d}\Omega^2.
\end{align*}
\]

In this solution, there is a line singularity along $r = 0$ for all $y$, but the black string is unstable in response to large-scale perturbations because the 5D curvature is unbounded at the Cauchy horizon, as $y \to \infty$. Another solution for the static uncharged black hole was found by Dadhich et al from the induced field equations on the brane [11]. In this work, a Reissner–Nordström-type correction to the Schwarzschild solution was found and 5D gravitational effects, which are impressed by the bulk Weyl tensor, induce a tidal charge parameter $Q$. In
this approach the bulk metric has not been found. Braneworld black holes have been studied extensively in recent years, some of these studies can be found in [10–17]. An extension of the DGP setup is the warped DGP scenario, which is a hybrid braneworld model containing both RSII and the DGP models as its limits [6, 7]. This model is constructed by considering the effect of an induced gravity term in the RSII model. Black hole solutions in a warped DGP braneworld have been studied recently in the case of a conformally flat bulk for spherically symmetric vacuum on the brane [18]. This study confirms the idea that an extra term in the effective vacuum field equations on the brane can play the role of a positive cosmological constant.

Within this streamline, in this paper we present an analytical solution of a static black hole in the warped DGP braneworld. Firstly, we consider pure (A)dS\(_5\) bulk field equations with a general spherically symmetric 5D metric to achieve independent field equations that should be held in the bulk. We take into account the 3-brane effect through appropriate matching conditions that act as boundary conditions on 3-brane universe in order to constraint general bulk solutions. In this way, we obtain a complex system of equations which should be solved to reach the warped DGP braneworld, static, black hole solutions. Due to difficulties in solving the field equations analytically, we consider the weak-field limit of the scenario to find those solutions of the field equations that describe the gravitational interactions in regions far enough from the gravitational source. We demand these solutions to support 4D GR solutions for \(r \ll r_c\) on the brane. However, the full linearized theory that we have adopted does not lead to a correct 4D Schwarzschild-(A)dS\(_4\) solution on the brane. The reason for this incompatibility and its solution are summarized in which follows. The linear analysis of the DGP model shows that the tensor structure of the induced metric perturbations takes the five-dimensional form even at short distances, i.e. in the limit \(r_c \to \infty\), we have an incompatibility with Einstein gravity [5, 8, 9]. The situation is analogous to the case of models with massive gravitons in which we have a deviation from 4D GR (which is a massless theory), even in the massless limit. This discontinuity in graviton mass is known as van Dam–Veltman–Zakharov (vDVZ) discontinuity [19]. Vainshtein claimed that the vDVZ discontinuity is an artifact of the Pauli–Fierz Lagrangian, i.e. of the linearization of the true, covariant, nonlinear equations of massive gravity, and the 4D GR is recovered by the nonlinear effect [20]. There have been many discussions about this issue, see for instance [21]. Another possibility that can make the mentioned discontinuity disappear is introducing a cosmological constant [22]. In this way, in order to 4D GR be recovered, the cosmological constant should be large enough as far as the graviton mass is not completely negligibly small. The paradox in DGP gravity seems to be that while it is clear that a perturbative, vDVZ-like discontinuity occurs in the potential somewhere, no such discontinuity appears in the cosmological solutions; at large Hubble scales, the theory on the brane appears safely like GR [8, 9]. Indeed, the cosmological solutions at large Hubble scales are extremely nonlinear, and that perhaps, just as Vainshtein suggested for massive gravity, nonlinear effects become important in resolving the DGP version of the vDVZ discontinuity. In the context of the DGP model, there are various works that indicate that 4D GR is recovered at short distances [23, 24]. It was shown that the brane bending becomes nonlinear at a scale shorter than \(r_* \equiv (r_c^2 r_g)^{1/3}\); thus, the linear analysis breaks down there. Also the leading-order correction to 4D GR has been obtained in this regime [24]. In [25], an alternative formalism which can handle general perturbations in weak gravity regime was proposed. The author of [25] has generalized the DGP model by taking into account the bulk cosmological constant and the brane tension balanced with it, so the background metric is AdS\(_5\) in the bulk and Minkowski on the brane. By perturbing the background metric and following the prior approach that had been proposed for the RS scenario [26], they confirmed the recovery of 4D GR at short distances and re-derived the
leading-order correction to it. They have done this work by taking into account the nonlinear brane bending for weak gravity at a small scale \( r < r_* \).

Here, in order to recover the 4D Schwarzschild-(A)dS(4) solution on the brane for \( r < r_* \) or in the limit \( r_c \to \infty \), with regard to previous works, we take into account some nonlinear terms in the field equations. In fact, these nonlinear terms become so large in those regions that one cannot neglect them relative to other nonlinear terms. We show that nonlinear effects become important in resolving the ‘Warped DGP’ version of the vDVZ discontinuity too. By considering these nonlinear terms, we have found interesting solutions for static black holes on the warped DGP brane.

2. The field equations

We consider a 3-brane \( \Sigma \) embedded in a five-dimensional bulk \( M \). The total action for the system is

\[
S = \frac{1}{2\kappa_5^2} \int_M d^5X \sqrt{-g} \left( (^{(5)}R - 2\Lambda_5) + \int_M d^5x \sqrt{-g} \ L_{\text{bulk}} + S_{\text{brane}} \right),
\]

where \( S_{\text{brane}} \) is the 3-brane action defined as

\[
S_{\text{brane}} = \frac{1}{2\kappa_4^2} \int_{\Sigma} d^4x \sqrt{-q} \left( (^{(4)}R - 2\Lambda_4) + \int_{\Sigma} d^4x \sqrt{-q} \ L_{\text{brane}} + \int_{\Sigma} d^4x \sqrt{-q} \frac{K}{\kappa_4^2} \right).
\]

\( g_{AB} \) is the 5D bulk metric with the corresponding Ricci tensor given by \( (^{(5)}R_{AB}) \). The brane has induced the metric \( q_{\mu\nu} \) with the corresponding Ricci tensor \( (^{(4)}R_{\mu\nu}) \). The boundary Gibbons–Hawking term is implied to yield the correct Einstein equations in the bulk. \( \Lambda_5 \) and \( \Lambda_4 \) are bulk and brane cosmological constants, respectively. \( \frac{1}{\kappa_4^2} \) can be interpreted as the brane tension of the standard Dirac–Nambu–Goto action and can include quantum contributions to the four-dimensional cosmological constant \[12\]. \( L_{\text{bulk}} \) and \( L_{\text{brane}} \) are the bulk and brane matter Lagrangians, respectively.

We chose a coordinate \( y \) for extra dimension so that our 3-brane in the Gaussian normal coordinates is localized at \( y = 0 \). By variation of the action with respect to the bulk metric, 5D field equations would be obtained as follows:

\[
(^{(5)}G_{AB}) = -\Lambda_5 g_{AB} + \kappa_5^2 (^{(\text{bulk})}T_{AB}) + \delta(y)\kappa_5^2 (^{(\text{loc})}T_{AB}),
\]

where

\[
(^{(\text{loc})}T_{AB}) = \frac{1}{\kappa_4^2} \int_{\Sigma} d^4x \sqrt{-q} \left( (^{(4)}G_{\mu\nu}) + \Lambda_4 q_{\mu\nu} - \kappa_4^2 (^{(\text{brane})}T_{\mu\nu}) \right),
\]

is the localized energy–momentum tensor on the brane and

\[
(^{(\text{bulk})}T_{AB}) = -2 \frac{\delta L_{\text{bulk}}}{\delta g^{AB}} + g_{AB} L_{\text{bulk}}.
\]

\( (^{(5)}G_{AB}) \) and \( (^{(4)}G_{\mu\nu}) \) denote the Einstein tensors constructed from the bulk and the brane metrics, respectively. The tensor \( q_{\mu\nu} \) is the induced metric on the brane \( \Sigma \). The field equations in the bulk take the following form:

\[
(^{(5)}G_{AB}) = (^{(5)}R_{AB}) - \frac{1}{2} (^{(5)}R g_{AB}) = -\Lambda_5 g_{AB} + \kappa_5^2 (^{(\text{bulk})}T_{AB}),
\]

or

\[
(^{(5)}R^B_A) - \frac{1}{2} (^{(5)}R g^B_A) = -\Lambda_5 g^B_A + \kappa_5^2 (^{(\text{bulk})}T^B_A).
\]
Moreover, the following modified (due to the presence of induced gravity on the brane) Israel–Darmois junction conditions would be obtained from the distributional character of the brane content:

\[ [K_{\mu\nu}] - q_{\mu\nu}[K] = -\kappa_5^2 \Theta_{\mu\nu} = \frac{\kappa_5^2}{\kappa_4^2} (\nabla^4 G_{\mu\nu} + \Lambda_4 q_{\mu\nu}) - \kappa_5^2 (\text{brane}) T_{\mu\nu}, \]  

(8)

where \( K_{\mu\nu} = \frac{1}{2} \partial_{(\mu} g_{\nu)} \) is the extrinsic curvature of the brane and brackets denote jump across the brane \((y = 0)\).

Now we consider a general, five-dimensional, static metric with spherical symmetry on the brane as follows:

\[ ds_5^2 = -e^{\phi(r,y)} dr^2 + e^{\delta(r,y)} dr^2 + r^2 e^{\mu(r,y)} (d\theta^2 + \sin^2 \theta d\phi^2) + dy^2. \]  

(9)

Also we assume that the matter content of the bulk is just a cosmological constant \( \Lambda_4 \) and the matter content of the 3-brane universe is considered to be a cosmological constant \( \rho \).

Hereafter, we consider a \( Z_2 \)-symmetry on reflection across the brane; thus, at \( r > R \) the Israel matching conditions become

\[ \bar{K}^\mu_\nu - \bar{\delta}^\mu_\nu = r_e \left( (4) G^\mu_\nu + \Lambda_4 \delta^\mu_\nu \right), \]  

(11)

where \( r_e = \frac{k_4^2}{2 \kappa_4^2} = \frac{M_4^2}{2 \kappa_4^2} \) is the DGP crossover distance, and by definition \( \bar{K}^\mu_\nu = K^\mu_\nu(y = 0^+) = -K^\mu_\nu(y = 0^-) \). Now by putting metric (9) into the bulk field equations (7), we obtain the \((tt)\), \((rr)\), \((\theta\theta)\) and \((r\phi)\) components of the bulk field equations, respectively, as follows:

\[ 4r \mu_{tt} + 3r \mu_{rr}^2 + 12 \mu_r - 4 \lambda_r - 2r \lambda_r \mu_r + \frac{4}{r} (1 - e^{\phi - \mu}) + 2r \lambda_{yy} e^\lambda + 4r \mu_{yy} e^\lambda + r \lambda_r^2 e^\lambda + 3r \mu_r^2 e^\lambda + 2r \lambda_r \mu_r e^\lambda = 4r \Lambda_5 e^\lambda, \]  

(12)

\[ r \mu_r^2 + 2r v_r \mu_r + 4v_r + 4 \mu_r + \frac{4}{r} (1 - e^{\phi - \mu}) + 2r \lambda_{yy} e^\lambda + 4r \mu_{yy} e^\lambda + 3r \mu_r^2 e^\lambda + 2r v_r \mu_r e^\lambda = 4r \Lambda_5 e^\lambda, \]  

(13)

\[ 2r v_r + 2r \mu_r + r v_r^2 + r \mu_r^2 - r v_r \lambda_r + r v_r \mu_r - r \lambda_r \mu_r + 2v_r - 2 \lambda_r + 4 \mu_r + 2r v_r + 4r \lambda_{yy} + \mu_{yy} e^\lambda + r \lambda_r v_r + \lambda_r \mu_r + v_r \mu_r e^\lambda + r [\lambda_r^2 + \nu_r^2 + \mu_r^2] e^\lambda = 4r \Lambda_5 e^\lambda, \]  

(14)

\[ 4r \mu_r + 2r v_r + 3r \mu_r^2 + r v_r^2 + 2r v_r \mu_r - 2r \lambda_r \mu_r - r v_r \lambda_r + 4v_r - 4 \lambda_r + 12 \mu_r + \frac{4}{r} (1 - e^{\phi - \mu}) + r \mu_r^2 e^\lambda + r [\lambda_r v_r + 2 \lambda_r \mu_r + 4v_r \mu_r] e^\lambda = 4r \Lambda_5 e^\lambda, \]  

(15)

\[ 2v_r + 4r \lambda_{yy} + v_r \lambda_r + 2r \lambda_r \mu_r - 2 \lambda_r \mu_r + \frac{4}{r} (\mu_r - \lambda_r) = 0. \]  

(16)
By using the Israel matching conditions (11), we obtain the following conditions on the brane:

\[ -\frac{1}{2} (2\mu + \lambda y) |_{y=0^+} = r_c \{ \Lambda_4 - \frac{e^{-\lambda}}{4r^2} (4 - 4 e^{\lambda} - 4r\mu + 4r \lambda + 4r\lambda + 12r \mu - 2r^2 \lambda \mu) \}, \]

(17)

\[ -\frac{1}{2} (2\mu + \lambda y) |_{y=0^-} = r_c \{ \Lambda_4 - \frac{e^{-\lambda}}{4r^2} (4 - 4 e^{\lambda} - 4r\mu + 4r \lambda + 4r\lambda + 12r \mu - 2r^2 \lambda \mu) \}, \]

(18)

\[ -\frac{1}{2} (v_y + \lambda \gamma + \mu y) |_{y=0^+} = r_c \{ \Lambda_4 - \frac{e^{-\lambda}}{4r^2} (2v \mu + 2\mu r + \mu \lambda + r \lambda - 4\mu r + r \lambda \mu + \lambda \mu) \}. \]

(19)

The subscripts \( y \) and \( r \) in these relations represent partial differentiation with respect to \( y \) and \( r \), respectively. Note that these equations are held on the brane outside our spherical object.

3. The weak-field solutions

The nonlinear, second-order partial differential equations (12)–(16) should be solved and the corresponding solutions should satisfy junction conditions (17), (18) and (19) across the \( \mathbb{Z}_2 \)-symmetric brane. This is not an easy task at all! To find some analytical solutions, we consider only the weak-field regime (i.e. far enough from the source localized on the brane). In this respect, we adopt the assumption that \(|v|, |\lambda|, |\mu|\) are small quantities compared to unity; that is, \(|v|, |\lambda|, |\mu| \ll 1 \). By adopting this assumption, we linearize our field equations and also brane boundary conditions. By keeping only the leading-order terms, the bulk equations (12)–(16) reduce to the following equations:

\[ 2r \mu_{rr} + 4 \mu_r - 2 \lambda_r - v_r - \frac{c}{r^2} + r (\lambda_{yy} + 2 \mu_{yy}) = \frac{4}{3} r \Lambda_5, \]

(20)

\[ v_r = \frac{c}{r^2} + r (v_{yy} + 2 \mu_{yy}) = \frac{4}{3} r \Lambda_5, \]

(21)

\[ r (v_{rr} + \mu_{rr}) + v_r + 2 \mu_r - \lambda_r + r (v_{yy} + \lambda_{yy} + \mu_{yy}) = 2r \Lambda_5, \]

(22)

\[ \frac{2}{r} (\mu - \mu) = v_r + 2 \mu_r + \frac{c}{r^2} - \frac{2}{3} r \Lambda_5, \]

(23)

where \( c \) is an integration constant to be fixed later. In the same procedure applied to equations (17), (18) and (19), we achieve the following linearized junction conditions:

\[ -\frac{1}{2} (\lambda_r + 2 \mu_y) |_{y=0^+} = r_c \{ \Lambda_4 - \frac{1}{r^2} (\mu - \lambda + 3 r \mu_r + r^2 \mu_{rr} - r \lambda_r) \}, \]

(24)

\[ -\frac{1}{2} (v_r + 2 \mu_y) |_{y=0^-} = r_c \{ \Lambda_4 - \frac{1}{r^2} (\mu - \lambda + r \mu_r + r v_r) \}. \]

(25)

\[ -\frac{1}{2} (v_r + \lambda_y + \mu_r) |_{y=0^+} = r_c \{ \Lambda_4 - \frac{1}{2r} (r v_{rr} + r \mu_{rr} + 2 \mu_r + v_r - \lambda_r) \}. \]

(26)

By solving the bulk field equations (20)–(23), we obtain \( \mu \) and \( \lambda \) in terms of \( v \) as follows:

\[ \mu(r, y) = -\frac{1}{2r} \int \int v_y \, dy \, dr = \frac{1}{2} \int v_y \, dy \, dr \cdot \frac{c}{4r^2} y^2 + \frac{1}{3} \Lambda_5 y^2 + (F_1(r) + H(r)) |y| + F_2(r) + G(r), \]

(27)
\[ \lambda(r, y) = -2 \int \int v_r \, dy \, dy - \frac{3}{r} \int \int v_r \, dy \, dy - 2v + \frac{c}{2r} + \frac{2}{3} \Lambda_5 r^2 + r|y|F'_1(r) \]
\[ + |y|F_1(r) + rF'_2(r) + F_2(r) + \frac{4}{3} \Lambda_5 y^2 - \frac{c}{2r^3} y^2 + A(r)|y| + B(r), \tag{28} \]

where \( A(r) \), \( B(r) \), \( F'_1(r) \), \( F_2(r) \), \( G(r) \) and \( H(r) \) are arbitrary functions of \( r \), and a prime denotes derivative with respect to \( r \). Also \( v(r, y) \) should satisfy the following differential equation:
\[ v_{yy} = -v_{yyy} - \frac{2}{r^3} v_r - \frac{1}{r} v_{yy}y + \frac{2}{r^2} v_r - \frac{2}{r} v_{rr}. \tag{29} \]

Therefore, for a given \( v(r, y) \) that satisfies this equation, \( \mu(r, y) \) and \( \lambda(r, y) \) can be obtained via (27) and (28), respectively. Equation (29) has the following solution for \( v(r, y) \):
\[ v(r, y) = \frac{a}{r} + br^2 + ey^2 + d |y| + \frac{f}{r} |y| + qr^2 |y|. \tag{30} \]

Now by using equations (27) and (28), we calculate \( \lambda(r, y) \) and \( \mu(r, y) \) to obtain
\[ \lambda(r, y) = \left( -\frac{f}{6r^3} - \frac{5}{3} g \right) |y|^3 - \left( \frac{a + c}{2r^2} + \frac{b}{2} + 2e - \frac{4}{3} \Lambda_5 \right) y^2 \]
\[ - \left( 2qr^2 + \frac{1}{r} f + 2d - rF'_1 - F_1 - P(r) \right) |y| \]
\[ + \frac{c - 4a}{2r} - 2br^2 + \frac{2}{3} \Lambda_5 r^2 + rF'_2 + F_2 + Q(r), \tag{31} \]
\[ \mu(r, y) = \left( \frac{f}{12r^3} - \frac{g}{6} \right) |y|^3 + \left( \frac{a + c}{4r^3} + \frac{b}{4} - \frac{e}{2} \right) y^2 \]
\[ - \left( \frac{f}{2r} + \frac{a + c}{2} r^2 + d - F_1 - M(r) \right) |y| - \frac{a}{2r} - \frac{b}{2} r^2 + F_2 + N(r), \tag{32} \]

where \( a, b, d, e, f \) and \( g \) are constants, whereas \( P(r) \), \( Q(r) \), \( M(r) \) and \( N(r) \) are arbitrary functions of \( r \). Equations (20)–(26) can help us to determine the arbitrary parameters and functions appeared in \( v(r, y) \), \( \lambda(r, y) \) and \( \mu(r, y) \). By this way, we found that \( b, d, f \) and \( g \) should be zero and also \( \Lambda_5 = -\Lambda_4 \) and \( e = \frac{1}{2} \Lambda_5 \) in order for equations (20)–(26) to be satisfied. In this situation, the solutions simplify to the following equations:
\[ v(r, y) = \frac{a}{r} + \frac{2}{3} \Lambda_5 y^2, \tag{33} \]
\[ \lambda(r, y) = -\left( \frac{a + c}{2r^3} \right) y^2 + r_2 \left( \frac{4}{3} \Lambda_5 + \frac{a + c}{r^3} \right) |y| + \frac{c - a}{2r} - \frac{1}{3} \Lambda_5 r^2 + rW' + W(r), \tag{34} \]
\[ \mu(r, y) = \left( \frac{a + c}{4r^4} \right) y^2 + r_2 \left( \frac{4}{3} \Lambda_5 - \frac{a + c}{2r^3} \right) |y| + W(r) - \frac{a}{2r}, \tag{35} \]

where \( W(r) = N(r) + F_2(r) \). The corresponding quantities on the brane \( y = 0 \) take the following forms:
\[ v(r, y = 0) = \frac{a}{r}, \tag{36} \]
\[ \lambda(r, y = 0) = \frac{c - a}{2r} + \frac{1}{3} \Lambda_4 r^2 + r W' + W(r), \tag{37} \]

\[ \mu(r, y = 0) = W(r) - \frac{a}{2r}. \tag{38} \]

Since we expect \( \mu(r, y = 0) = 0 \), we set \( W(r) = \frac{a}{2r} \). So we find
\[ \lambda(r, y = 0) = \frac{c - a}{2r} + \frac{1}{3} \Lambda_4 r^2. \tag{39} \]

c is an integration constant which we set to be zero in order to find more familiar results as follows:
\[ \nu(r, y) = \frac{a}{r} + \frac{2}{3} \Lambda_5 y^2, \tag{40} \]

\[ \lambda(r, y) = -\frac{a}{2r} + \frac{1}{3} \Lambda_4 r^2 - \frac{a}{2r^3} y^2 + r c \left( \frac{4}{3} \Lambda_5 + \frac{a}{r^3} \right) |y|, \tag{41} \]

\[ \mu(r, y) = \frac{a}{4r^3} y^2 + r c \left( \frac{4}{3} \Lambda_5 - \frac{a}{2r^3} \right) |y|. \tag{42} \]

Note that in these relations we have used \( \Lambda_5 = -\Lambda_4 \) interchangeably. To find the Schwarzschild solution in appropriate limit, we set \( a = -2m \), where \( m \) is the Schwarzschild geometric mass. It is clear that in our linearized warped DGP setup, \( \lambda(r, y = 0) \neq -\nu(r, y = 0) \) on the 3-brane. We expect that in the limit of \( r_c \to \infty \), the solutions reduce to the 4D general relativistic solutions on the brane, that is, to a Schwarzschild-(A)dS\(_4\) metric. However, this has not occurred here. We note that the reason behind having a non-Schwarzschild-(A)dS\(_4\) solution can be explained through the fact that the curvatures of the bulk and brane spacetimes should not exactly cancel each other. In the following section, we focus on this issue.

4. Recovering the Schwarzschild solution on the brane

To recover the Schwarzschild-(A)dS\(_4\) solution on the brane, we note that all metric components should be much smaller than unity as a result of the weak-field prescription. But this is violated by the linearized solutions we obtained in the last section. Nevertheless, if we look at the 5D bulk solutions (40), (41) and (42), we will see that in 5D, \( \lambda \) and \( \mu \) are large for large \( r_c \) (see for instance [24], [23] and [9]). Although we are interested only in 4D solutions, the linearized theory is not applicable at large \( r_c \). These features show that the completely linearized theory does not result in correct 4D Schwarzschild-(A)dS\(_4\) solutions (which are given by \( \mu(r) = 0, \lambda(r) = -\nu(r) \) and \( \nu(r) = -\frac{2m}{r} - \frac{1}{3} \Lambda_4 r^2 \)). In fact, in the limit \( r_c \to \infty \), but yet in the weak-field regime, we cannot neglect all nonlinear terms in some subspaces. For instance, if \( r_c^2 |a| \gg r^3 \), quantities such as \( \lambda_y \) and \( \mu_y \) are large enough that we have to save their squares in the field equations. Therefore, for \( |a| \ll r \ll r_s \equiv (r_c^2 |a|)^{1/2} \), the nonlinear field equations that should be solved are as follows:
\[ \frac{2}{r} (\mu - \lambda) = -\nu_r - 2 \mu_r + U(r), \tag{43} \]

1 We note that a static source placed on a brane creates a nonzero scalar curvature around it. For a source of the size \( < r_s \), this curvature extends to a distance \( \sim r_s \). More intuitively, a static source distorts a brane medium around it creating a potential well, and the distortion extends to a distance \( r \sim r_s \). This curvature suppresses nonlinear interactions that otherwise would become strong at the scale below \( r_s \), see for instance [14].
\[ 4r \mu r + 8\mu r - 8\lambda r - 2\nu r + 2U(r) + 2r(\lambda_{yy} + 2\mu_{yy}) + r\lambda^2 + 3\mu^2 + 2r\lambda y_{yy} = 4r\Lambda_5, \quad (44) \]

\[ 2\nu r + 2U(r) + 2r v_{yy} + 4r\mu_{yy} + 3r\mu_y^2 = 4r\Lambda_5, \quad (45) \]

\[ 2r v_{yy} + 2r\mu_{yy} + 2r - 2\lambda r + 4\mu r + 2r(v_{yy} + \lambda_{yy} + \mu_{yy}) + r\lambda y_{yy} + r(\lambda_y^2 + \mu_y^2) = 4r\Lambda_5, \quad (46) \]

\[ 4r\mu_{yy} + 2r v_{yy} + 2r - 4\lambda r + 8\mu r + 2U(r) + 2r\lambda_{yy} + r\mu_y^2 = 4r\Lambda_5, \quad (47) \]

where \( U(r) \) is an arbitrary function of \( r \). Junction conditions in this situation are the same as the linearized case given by equations (24), (25) and (26). By solving this set of equations, we find the following class of solutions:

\[ \nu(r, y) = \frac{a}{r} + \frac{1}{9}\Lambda_5 r^2 + \frac{4}{9}r_c\Lambda_5 |y| + \frac{4}{27}r_c^2\Lambda_5^2 y^2, \quad (48) \]

\[ \lambda(r, y) = -\frac{a}{r} - \frac{1}{9}\Lambda_5 r^2 + \frac{4}{9}r_c\Lambda_5 |y|, \quad (49) \]

\[ \mu(r, y) = \frac{4}{9}r_c\Lambda_5 |y|, \quad (50) \]

where now \( \Lambda_4 = -\frac{1}{4}\Lambda_5 \) with \( \Lambda_4 = -\frac{3}{2}r^2 \). Note that in this framework we obtained a geometric interpretation of the brane cosmological constant in terms of the DGP crossover scale. Moreover, as we have mentioned at the end of the previous section, the curvatures of the bulk and brane spacetimes do not cancel each other exactly which is the reason behind having a non-Schwarzschild-(A)dS\(_{(4)}\) solution in the previous section. On the brane with \( y = 0 \), we obtain

\[ \nu(r, y = 0) = \frac{a}{r} - \frac{1}{3}\Lambda_4 r^2, \quad (51) \]

\[ \lambda(r, y = 0) = -\frac{a}{r} + \frac{1}{3}\Lambda_4 r^2, \quad (52) \]

\[ \mu(r, y = 0) = 0, \quad (53) \]

which is compatible with the 4D Schwarzschild-AdS\(_{(4)}\) solution at the leading order if we set \( a \equiv -2m \)

\[ ds^2 = -\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda_4 r^2\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda_4 r^2\right)} + r^2 d\Omega^2. \quad (54) \]

Therefore, the Einstein gravity is recovered and the vDVZ problem is resolved by this strategy. Indeed, the spatial extrinsic curvatures of the brane, i.e. \( \lambda_y|_{y=0} \) and \( \mu_y|_{y=0} \), play a crucial role in the nonlinear nature of the solution and recovering the predictions of GR. In this respect, the nonlinear behavior arises from purely spatial geometric factors (see also [24]). We note also that there is a massless scalar mode, which arises from the decomposition of five degrees of freedom of the bulk graviton in the massless limit (i.e. \( r_c \to \infty \)). This extra scalar mode persists as an extra degree of freedom in all regimes of the theory. The extrinsic curvature suppresses this extra scalar field inside the region \( r \ll r_* \), whereas outside this region, the brane bending mode is free to propagate (see [9] for more details).

We note that equations (48), (49) and (50) can be rewritten by using \( \Lambda_4 = -\frac{3}{4}r^2 \) as follows:
\[ \nu(r, y) = -\frac{2m}{r} + \frac{1}{4} \frac{r^2}{r_c} + \frac{1}{4} \frac{|y|}{r_c} + \frac{3}{4} \frac{y^2}{r_c^2}, \]  
(55)

\[ \lambda(r, y) = \frac{2m}{r} - \frac{1}{4} \frac{r^2}{r_c} + \frac{1}{4} \frac{|y|}{r_c}, \]  
(56)

\[ \mu(r, y) = \frac{1}{r_c} |y|. \]  
(57)

In the limit \( r_c \to \infty \), we find the Schwarzschild solution on the brane. This means that the effect of the brane cosmological constant is negligible in this limit. Thus, it is essentially impossible to observe a Schwarzschild-(A)dS\(^4\) interaction in our observations, and the Newtonian gravity is dominant at least in our solar system scale. It is important to note that the above solution is actually a black string solution since we find a singularity at \( r = 0 \) for any value of \( y \). This can be shown through singularity of the five-dimensional Kretschmann scalar. The expression of the five-dimensional Kretschmann scalar is too long to be presented here, but our calculation by using the Maple package has shown that the five-dimensional Kretschmann scalar is singular at \( r = 0 \) for all values of \( y \). In fact, in the \( r \to 0 \) limit, it reduces to the following simpler form:

\[ \lim_{r \to 0} K^2 = \frac{32r_c^3 y + 16r_c^2 y^2 + 16r_c^4}{(r_c + y)^4} \]  
(58)

which obviously diverges at \( r = 0 \). This shows that singularity at \( r = 0 \) is an intrinsic singularity, and this is true for all values of \( y \), which confirm that the solutions are actually black string. Moreover, the four-dimensional Kretschmann scalar of metric (54) expressed in terms of \( r_c \) is given by

\[ K^2 \equiv \langle^{(4)}R_{\alpha\beta\gamma\delta}\rangle \langle^{(4)}R_{\alpha\beta\gamma\delta}\rangle = \frac{3}{2} \left( \frac{r_c^6 + 32m^2}{r^6} \right), \]  
(59)

which shows that singularity at \( r = 0 \) on the brane is an intrinsic singularity. In the limit of \( r_c \to \infty \), we have only one horizon which is the Schwarzschild horizon at \( r = 2m \). But for \( m = 0 \) there is no de Sitter horizon on the brane. The stability of black string solutions should be considered in the light of the results of Gregory–Laflamme pioneering work [27].

Another issue which requires special attention here is the fact that a de Sitter bulk invalidates most of the attractive points of the Randall–Sundrum model such as the localization of the zero mode of the graviton near the brane. The question then arises: how is the situation in the warped DGP setup? In DGP-like models that have a 4D Ricci scalar in the bulk action, gravity becomes five-dimensional at \( r \gg r_c \) where 5D Einstein–Hilbert action is dominant. For \( r \ll r_c \) gravity is four-dimensional but not the 4D GR one. This property of these models is definitely different from the RSII model, in which at low energies, \( H\ell \ll 1 \), 4D gravity is recovered to a good approximation. In the RSII model what prevents gravity from leaking into the extra dimension at low energies is the negative bulk cosmological constant. Moreover, in the RSII model, the 4D gravitational constant strongly relies on the presence of the brane tension, and the brane tension should be positive in order to have the correct sign of the effective 4D gravitational constant. This positive tension and the bulk negative cosmological constant are balanced with each other to have a zero 4D cosmological constant on the brane. In our warped DGP setup, we found \( \Lambda_4 < 0 \) and \( \Lambda_5 > 0 \), which is completely different from the RSII case. Here we do not need a negative \( \Lambda_5 \) and positive \( \Lambda_4 \) because the induced gravity term gives us the desired results. Solutions that we have found in this section, equations (55)–(57), are valid only at \( r \ll r_c \) and for \( r \gg r_c \) appropriate solutions should be obtained.
Nevertheless, since a warped DGP scenario is a hybrid braneworld model which contains both the pure DGP and RSII models in appropriate limits [6], it is expected that some of the mentioned shortcomings are still present in this case too. This needs further justification that we are going to study separately.

As the final remark, we note that the Birkhoff’s theorem is absent in the theories of modified gravity such as the DGP braneworld scenario [28]. By calculation of the gravitational force on a test particle due to a spherical mass shell in the DGP setup, one can show that unlike in GR, the force depends on the mass distribution. In particular, the gravitational force within a spherical mass shell depends on the geometric structure of the bulk. In our setup, having the Schwarzschild-(A)dS\((4)\) solution means that Birkhoff’s theorem holds good and this is due to the geometric structure of the bulk manifold in this case which differs from the pure DGP case.

5. Summary

In this paper, we obtained a class of static black hole solutions in the warped DGP braneworld. Firstly, we solved the general bulk field equations in the weak-field limit (by linearizing the field equations and matching conditions). The solutions on the brane then were found by using the Israel matching conditions. However, these brane solutions were not compatible with Schwarzschild-(A)dS\((4)\) solutions on the brane. We adopted a strategy to solve this problem based on keeping appropriate nonlinear terms in the field equations. This strategy has its origin in the fact that the spatial extrinsic curvature of the brane plays a crucial role in the nonlinear nature of the solutions, and also in recovering the predictions of general relativity. In fact, the nonlinear behavior arises from purely spatial geometric factors. Using this feature, we obtained some interesting black sting solutions compatible with well-known black hole solutions on the brane.

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