ELEMENTARY PARTICLE PHYSICS AND FIELD THEORY

ON GEOMETRIC INTERPRETATION OF THE BERRY PHASE

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UDC 530.145

A geometric interpretation of the Berry phase and its Wilczek–Zee non-Abelian generalization are given in terms of connections on principal fiber bundles. It is demonstrated that a principal fiber bundle can be trivial in all cases, while the connection and its holonomy group are nontrivial. Therefore, the main role is played by geometric rather than topological effects.

Keywords: Berry’s phase, differential geometry.

INTRODUCTION

The Berry phase [1] attracts much interest of theoreticians and experimenters for a long time. The interest is due to two circumstances. First, the nontrivial geometric object – the $U(1)$-connection – arises naturally when solving the Schrödinger equation in nonrelativistic quantum mechanics. Second, there is a widespread opinion in gauge field theory that only gauge field strength rather than gauge potentials themselves that are not gauge-invariant can produce the observable effects. Contrary to this opinion, M. Berry demonstrated that the integral of a gauge field along a closed loop could produce observable effects. This conclusion was soon confirmed experimentally.

The notion of the Berry phase was generalized to the non-Abelian case corresponding to degenerate energy levels of a Hamiltonian by Wilczek and Zee [2]. In this case, non-Abelian $U(r)$-gauge fields naturally arise when solving the Schrödinger equation.

In all cases mentioned above, the observable effects are produced by elements of the holonomy group of corresponding connections. There is no disagreement at this point. However, there is no common opinion on the geometric interpretation. B. Simon [3] and other authors considered the gauge field as a connection on an associated fiber bundle. Since in general the typical fiber of the associated bundle is an infinite dimensional Hilbert space, specific difficulties arose. Important and interesting constructions connected to characteristic classes are related to the existence of global sections of associated fiber bundles rather than to the Berry phase itself. Moreover, definite topological obstructions arose for the existence of global sections. Therefore, the judgment that the Berry phase has its origin in topology is widespread in the literature.

In the present paper, a geometric interpretation of the Berry phase in terms of the connection theory on a principal fiber bundle is given. There are no difficulties related to infinite dimensional manifolds in this approach, because typical fibers are $U(1)$ or $SU(r)$ groups which are finite-dimensional Lie groups. It is demonstrated that the principal fiber bundle can be trivial while the connection arising on it has generally a nontrivial holonomy group and therefore leads to observable effects. As a consequence, the Berry phase has its origin in geometry rather than in topology. Moreover, the existence of global sections on associated fiber bundles is not a necessary condition. If a global section is absent, then the local connection forms are defined on a coordinate covering of the base of the principal fiber bundle defining the unique connection on the principal fiber bundle up to an isomorphism.
ABELIAN CASE: A NONDEGENERATE STATE

We describe the problem considered by M. Berry [1] in its simplest case. Let the Hilbert space ℋ of a quantum mechanical system be finite dimensional and the Hamiltonian \( H = H(\lambda) \) depend sufficiently smoothly on a point of a manifold \( \lambda \in \mathbb{M} \) of dimension \( \text{dim}\mathbb{M} = n \). If we choose a coordinate neighborhood \( U \subset \mathbb{M} \) on \( \mathbb{M} \), then the Hamiltonian will depend on \( n \) parameters \( \lambda^k, k = 1, \ldots, n \) (coordinates of point \( \lambda \)). Assume that the position of point \( \lambda \) on \( \mathbb{M} \) depends on time \( t \) according to a given prescription, i.e., the Hamiltonian depends on a curve \( \lambda(t), \ t \in [0, \infty] \). Parameterization of the curve by a semi-infinite interval corresponds to the adiabatic limit [4–6] where \( t \to \infty \). We assume also that the Hamiltonian depends on time only through the point \( \lambda(t) \in \mathbb{M} \).

We consider the eigenvalue problem

\[
H\phi = E\phi, \quad E = \text{const},
\]

where \( \phi \in \mathbb{H} \) for all \( \lambda \in \mathbb{M} \). Suppose there exists nondegenerate energy eigenvalue \( E \) which depends sufficiently smoothly on \( \lambda \in \mathbb{M} \). The eigenfunction \( \phi(\lambda) \) is also assumed to be a sufficiently smooth function of \( \lambda \). Without loss of generality, we suppose that the eigenfunction \( \phi \) is normalized to unity, \( \langle \phi, \phi \rangle = 1 \). Then it is unique up to multiplication on a phase factor which can be \( \lambda \)-dependent. We fix somehow this phase factor.

Then we solve the Cauchy problem for the Schrödinger equation [7, 8]

\[
i\hbar \frac{\partial \psi}{\partial t} = H\psi, \tag{1}
\]

where \( \hbar \) is the Planck constant, with the initial condition

\[
\psi \mid_{t=0} = \phi_0, \tag{2}
\]

where \( \phi_0 := \phi(\lambda(0)) \). We set \( \hbar = 1 \) and denote the partial derivative with respect to time by \( \dot{\psi} \), \( \psi := \dot{\psi} \).

In the adiabatic approximation, a quantum system during evolution remains in the eigenstate corresponding to energy level \( E(\lambda) \). Therefore, we seek solution in the form

\[
\psi = e^{i\Theta} \phi,
\]

where \( \Theta = \Theta(t) \) is some unknown function of time. Then the Schrödinger equation implies equation for the phase

\[
\dot{\Theta} = i\langle \phi, \dot{\phi} \rangle - E
\]

with the initial condition \( \Theta \mid_{t=0} = 0 \). Since \( \dot{\phi} = \dot{\lambda}^k \partial_k \phi \), the phase of the solution to the Cauchy problem for Eq. (1) is

\[
\Theta = \int_0^t dt \int_0^s dE(\lambda(s)) = \int_0^{\lambda(t)} d\lambda^k A_k - \int_0^t dE(\lambda(s)), \tag{3}
\]

where we have introduced notation

\[
A_k(\lambda) = i\langle \phi, \dot{\phi} \rangle, \tag{4}
\]

and the integral over \( \lambda \) is taken along the curve \( \lambda(t) \).
Thus, integral (3) in the adiabatic approximation yields the solution of the Cauchy problem for Schrödinger equation (1) with initial condition (2). The first term in Eq. (3) is called the geometric or Berry phase, and the second term is called the dynamic phase.

Note that components (4) are real because of normalization of the wave function. Indeed, differentiation of the normalization condition \( \phi_\lambda \phi = 1 \) yields the equality

\[
(\partial_t \phi, \phi) + (\phi, \partial_t \phi) = (\phi, \partial_t \phi)^\dagger + (\phi, \partial_t \phi) = 0.
\]

It implies that components (4) and hence the Berry phase are real.

We consider now a set of closed curves \( \lambda \in \Omega(M, \lambda_0) \) on a parameter manifold \( M \) with the beginning and end at the point \( \lambda_0 \in M \). Then the total change of in the phase of the wave function is equal to the integral

\[
\Theta = \Theta_B - \int_0^\infty dt E(\lambda(t)),
\]

where

\[
\Theta_B = \oint d\lambda^k A_k.
\]

The dynamic part of the wave function phase diverges. However, we observed in experiments a difference in phases of two eigenvectors with the same dynamic phase which is determined by the Berry phase. Therefore, we consider the Berry phase in more detail.

The Berry phase (5) has simple geometric interpretation, namely, we have a principal fiber bundle \( \mathbb{P}(M, \pi, U(1)) \) whose base is the parameter manifold \( \lambda \in M \), the structure group is \( U(1) \) (phase of the state vector \( e^{i\theta} \)), and \( \pi : \mathbb{P} \to M \) is the projection [9]. The vector in the Hilbert space \( \phi \in H \) represents a local cross section of the associated fiber bundle \( \mathbb{E}(M, \pi_B, H, U(1), \mathbb{P}) \) whose typical fiber is the Hilbert space \( H \) and \( \pi_B : \mathbb{E} \to M \) is the projection.

Consider a change in the local cross section of the associated bundle which is produced by multiplication of a vector in the Hilbert space on a phase factor (vertical automorphism) \( \phi' = e^{ia} \phi \),

where \( a = a(\lambda) \in C^2(M) \) is an arbitrary doubly differentiable function. Then components (4) are transformed according to the rule

\[
A'_k = A_k - \partial_k a.
\]

Comparing this rule with the transformation of components of a local connection form [9], we see that the fields \( A_k(\lambda) \) can be interpreted as components of a local connection form for the \( U(1) \) group. In other words, \( A_k(\lambda) \) is a gauge field for the one-dimensional unitary group \( U(1) \). If the base of the associated fiber bundle \( \mathbb{E}(M, \pi_B, H, U(1), \mathbb{P}) \) is covered by some set of coordinate charts, \( M = \bigcup_j U_j \), then a set of sections given on each coordinate chart \( U_j \) defines a family of local connection forms on the principal fiber bundle \( \mathbb{P}(M, \pi, U(1)) \). A family of local connection forms \( d\lambda^k A_k \) defines the unique connection on \( \mathbb{P} \) up to an isomorphism [9].

Let us recall the expression for an element of the holonomy group in terms of the ordered \( P \)-exponent [10]. In the present case, the group \( U(1) \) is Abelian, and the \( P \)-exponent coincides with the conventional exponent. Therefore,
the Berry phase (5) defines the element $e^{iB_0}$ of the holonomy group $\Phi(\lambda_0,e) \subset U(1)$ of the principal fiber bundle at the point $(\lambda_0,e) \in \mathbb{P}$ corresponding to zero cross section $\mathbf{M} \ni \lambda \rightarrow (\lambda,e) \in \mathbb{P}$, where $\lambda_0 := \lambda(0)$ and $e$ is the unit of the structure group $U(1)$. The section is zero because at the initial moment of time, the Berry phase vanishes, $B_0|_{t=0} = 0$.

The local connection form $d\lambda^k A_k$ corresponds also to zero cross section.

If the base $\mathbf{M}$ is simply connected, then expression (5) for the Berry phase can be rewritten as a surface integral of the local curvature form. Using the Stokes formula, we obtain the following expression:

$$\Theta_B = \frac{1}{2} \int d\lambda^k \wedge d\lambda^l F_{kl}, \quad (6)$$

where $S$ is a surface in $\mathbf{M}$ with the boundary $\gamma \in \Omega(\mathbf{M},\lambda_0)$ and $F_{kl} = \partial_k A_l - \partial_l A_k$ are components of the local curvature form (gauge field strength). If the base $\mathbf{M}$ is not simply connected, then the expression for the Berry phase as surface integral (6) is valid only for those curves which are contractible to a point.

**A SPIN 1/2 PARTICLE IN A MAGNETIC FIELD**

As an example, we calculate the Berry phase for a spin 1/2 particle in an external homogeneous magnetic field. This example is a particular case of a particle of arbitrary spin in an external homogeneous magnetic field [1]. In nonrelativistic quantum mechanics, the 1/2 spin particle is described by a two-component wave function

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$  

We assume that it is located in the Euclidean space $\mathbb{R}^3$ with a given homogeneous magnetic field. Let the strength of the magnetic field $H^k(t)$, $k = 1, 2, 3$, do not depend on the space point but change in time $t$ according to some prescribed fashion. For simplicity, we disregard also the kinetic energy of the particle and assume that other fields are absent. In this case, the Hilbert space $\mathbb{H}$ is two-dimensional, and the Hamiltonian of the particle consists of one term that describes the interactions of the magnetic momentum of the particle with the external magnetic field (for example, see [5, 11]):

$$H = -\mu H^k \sigma_k,$$

where $\sigma_k$ are the Pauli matrices, $\mu$ is the magneton (dimensional constant equal to the ratio of the magnetic momentum of the particle to its spin). To write the Hamiltonian in the form considered above, we introduce new variables $\lambda^k = -\mu H^k$. Then the Hamiltonian is

$$H = \lambda^k \sigma_k = \begin{pmatrix} \lambda^3 & \lambda^- \\ \lambda^+ & -\lambda^3 \end{pmatrix}, \quad (7)$$

where $\lambda^\pm = \lambda^1 \pm i\lambda^2$.

Eigenvalues of Hamiltonian (7) are found from the equation

$$\det(H - E) = 0,$$

which has two real roots.
where

\[ |\lambda| = \sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)} \]

is the length of the vector \( \lambda = \{\lambda^k\} \in \mathbb{R}^3 \). It can be easily demonstrated that the equation for eigenfunctions

\[ H\phi_{\pm} = E_{\pm}\phi_{\pm} \]

has two solutions

\[
\phi_{\pm} = \frac{1}{\sqrt{2|\lambda|}} \left\{ \pm \frac{\lambda^-}{\sqrt{|\lambda| \pm \lambda^3}} \right\}.
\]

The factor in the expression derived is chosen in such a way that the eigenfunctions are normalized on unity:

\[ (\phi_{\pm}, \phi_{\pm}) = 1. \]

Thus, Hamiltonian (7) for the 1/2 spin particle in the external homogeneous magnetic field has two nondegenerate eigenstates (9) corresponding to energy levels (8).

For further calculations in the parameter space \( \lambda \in \mathbb{R}^3 \), it is convenient to introduce spherical coordinates \( |\lambda|, \theta, \varphi \):

\[
\lambda_1 = |\lambda| \sin \theta \cos \varphi, \\
\lambda_2 = |\lambda| \sin \theta \sin \varphi, \\
\lambda_3 = |\lambda| \cos \theta.
\]

Then the eigenfunctions assume the form

\[
\phi_+ = \left( \begin{array}{c} \cos \frac{\theta}{2} e^{-i\varphi} \\ \sin \frac{\theta}{2} \end{array} \right), \quad \phi_- = \left( \begin{array}{c} -\sin \frac{\theta}{2} e^{-i\varphi} \\ \cos \frac{\theta}{2} \end{array} \right).
\]

Admit that the experimenter observing the particle varies differentiably the homogeneous magnetic field with time. That is, the parameters \( \lambda^k(t) \) in the Hamiltonian depend differentiably on time. Assume also that the particle was in the state \( \phi_+ \) at the initial moment of time \( t = 0 \). The corresponding solution of Schrödinger equation (1) in the adiabatic approximation is

\[ \psi = e^{i\Theta} \phi_+, \]

where the phase \( \Theta \) is defined by Eq. (3). Components of the local connection form \( A_k = i(\phi_+, \partial_k \phi_+) \) for the eigenstate \( \phi_+ \) are easily calculated.
\[ A_{\lambda^1} = 0, \quad A_0 = 0, \quad A_\theta = \cos^2 \frac{\theta}{2}. \] 

The respective local curvature form has only two nonzero components:

\[ F_{0\theta} = -F_{\phi\theta} = -\frac{1}{2}\sin\theta. \]

Now we calculate the Berry phase for a closed curve in the parameter space \( \gamma = \lambda(t) \in \mathcal{M} \):

\[
\Theta_B = \oint_{\gamma} d\lambda^k A_k = \frac{1}{2} \int_S d\lambda^k \wedge d\lambda^l F_{kl} \\
= \frac{1}{2} \int_S d\theta \wedge d\phi F_{0\phi} = -\frac{1}{2} \int_S d\theta \wedge d\phi \sin\theta = -\frac{1}{2} \Omega(\gamma),
\]

where \( S \) is a surface in \( \mathbb{R}^3 \) with the boundary \( \gamma \) and \( \Omega(\gamma) \) is the solid angle under which the surface \( S \) is seen from the origin of the coordinate system.

If the particle is in the state \( \phi^- \) at the initial moment of time, the calculations are similar. In this case,

\[ A_{\lambda^1} = 0, \quad A_0 = 0, \quad A_\theta = \sin^2 \frac{\theta}{2}, \]

and components of the local curvature form differ by the sign:

\[ F_{0\phi} = -F_{\phi\theta} = \frac{1}{2}\sin\theta. \]

Therefore, the Berry phase differs also only by the sign.

Thus, if the particle was not in one of the states \( \phi^\pm \) at the initial moment of time, then after variation of the homogeneous magnetic field along closed curve \( \lambda(t) \), its wave function acquires the phase factor whose geometric part is

\[ \Theta_B^{\pm} = \mp \frac{1}{2} \Omega(\gamma), \]

where \( \Omega(\gamma) \) is the solid angle under which the closed contour \( \gamma \) is seen from the origin of coordinates. This result is valid in the adiabatic approximation when parameters \( \lambda(t) \) change slowly with time. Expression (12) for the Berry phase was confirmed experimentally [12] for the scattering of polarized neutrons in a spiral magnetic field.

The homogeneous magnetic field in the above-considered example can have an arbitrary direction and magnitude. Therefore, the base \( \mathcal{M} \) of the principal fiber bundle \( \mathcal{P}(\mathcal{M}, \pi, \mathbb{U}(1)) \) coincides with the Euclidean space \( \mathbb{R}^3 \). Hence, the principal fiber bundle \( \mathcal{P} \) is trivial, \( \mathcal{P} \approx \mathbb{R}^3 \times \mathbb{U}(1) \). For the Berry phase, the connection on this fiber bundle is given by the section of the associated fiber bundle, for example, \( \phi_+ \) which is obtained by solving the Schrödinger equation. It is easily checked that this section (9) has a singularity on the positive half-axis \( \lambda^3 \geq 0 \). The components of local connection form (10) in the Cartesian system of coordinates have the form
\[ A_i = \frac{\partial \phi}{\partial \lambda^i}, A_\phi = -\frac{\sin \phi \cos \theta}{2|\lambda| \sin \frac{\theta}{2}}, \]
\[ A_2 = \frac{\partial \phi}{\partial \lambda^2}, A_\phi = \frac{\cos \phi \cos \theta}{2|\lambda| \sin \frac{\theta}{2}}, \]
\[ A_3 = \frac{\partial \phi}{\partial \lambda^3}, A_\phi = 0. \]  

(13)

At this point, we are obliged to use the Cartesian system of coordinates, because the spherical system of coordinates is singular on the \( \lambda^3 \) axis and is unsuitable for an analysis of singularities located here. We see that the components of the local connection form are singular on the positive half-axis \( \lambda^3 \geq 0 \) together with the vector \( \phi_+ \). Now we calculate the components of the local form of the curvature tensor. All its components are nonzero:
\[ F_{12} = -F_{21} = -\frac{\cos \theta}{2|\lambda|^2}, \]
\[ F_{13} = -F_{31} = \frac{\sin \theta \sin \phi}{2|\lambda|^2}, \]
\[ F_{23} = -F_{32} = -\frac{\sin \theta \cos \phi}{2|\lambda|^2}. \]

Finally, we calculate the square of the curvature tensor which is the geometric invariant:
\[ F^2 = 2(F_{12}^2 + F_{13}^2 + F_{23}^2) = \frac{1}{2|\lambda|^4}. \]

Thus, the curvature form is singular only at the origin of coordinates.

Let us return to our principal fiber bundle \( \mathbb{R}^3 \times \mathbb{U}(1) \). The local connection form (13) is not defined on it, because it is singular on the half-axis \( \lambda^3 \geq 0 \) which we denote \( \{\lambda^3_+\} \). Hence, to construct a principal fiber bundle with the given connection, we have to remove the inverse image \( \pi^{-1}(\{\lambda^3_+\}) \), where \( \pi: \mathbb{R}^3 \times \mathbb{U}(1) \to \mathbb{R}^3 \) is the natural projection. As a result, we get the trivial principal fiber bundle \( \mathbb{R}^3 \times \{\lambda^3_+\} \times \mathbb{U}(1) \) which is a subbundle on the initial one. Local connection form (13) is smooth on this principal fiber bundle.

We can get another way out. Since the magnetic field is external, then we can assume that it varies, for example, in the half-space \( \mathbb{R}^3_+ \) defined by the inequality \( \lambda_1 > 0 \). The corresponding principal fiber bundle is trivial \( \mathbb{P} \approx \mathbb{R}^3 \times \mathbb{U}(1) \), because the half-space \( \mathbb{R}^3_+ \) is diffeomorphic to all Euclidean space \( \mathbb{R}^3 \). In this case, no problem arises with the definition of the connection, because local connection form (13) is smooth. At the same time, previous expression (12) for the Berry phase is valid.

Thus, the Berry phase is the geometric rather than topological notion, because the topology of the principal fiber bundle is trivial. It arises due to nontrivial connection defined by cross sections of the associated fiber bundle.
The notion of the Berry phase was generalized to the case when energy levels of the Hamiltonian are degenerate [2]. In this case, the principal fiber bundle $P(M, π, U(r))$ with the structure group $U(r)$, where $r$ is the number of independent eigenfunctions corresponding to the degenerate energy level $E$, appears when solving the Schrödinger equation. Here we describe this construction in detail.

We suppose that the Hamiltonian of a quantum system depends on a point of some manifold $λ(t) ∈ M$ as was assumed earlier. Let $E$ be a degenerate eigenvalue of the Hamiltonian $H$ with $r$ independent eigenfunctions $φ^a$, $a = 1, ..., r$,

$$Hφ^a = Eφ^a$$

for all moments of time. We assume that $E(λ)$ and $φ^a(λ)$ are differentiable functions at a point $λ$ of the manifold, and the number of eigenfunctions $r$ does not change in time.

The eigenfunctions can be chosen orthonormalized

$$(φ^a, φ^b) = δ^a_b,$$

where $δ^a_b$ is the Kronecker symbol and $φ_b = φ^aδ_{ab}$. We search for solution $ψ^a$ of the Cauchy problem for Schrödinger equation (1) with the initial condition

$$ψ^a(0) = ψ^a_0 = φ^a(λ(0)).$$

That is, the system is in one of the eigenstates $φ^a$ at the initial moment of time. In the adiabatic approximation, solution $ψ^a$ is the eigenstate of the Hamiltonian $H(λ)$ corresponding to the energy value $E(λ)$ for all moments of time. Therefore, it can be decomposed with respect to eigenfunctions of the degenerate state

$$ψ^a = U^{-1a}_bφ^b,$$

(14)

where $U(λ) ∈ U(r)$ is some unitary matrix which depends differentiably on the point $λ ∈ M$.

The unitarity of the matrix $U$ is dictated by the following circumstance. Consider solutions $ψ^a$ for all values of index $a = 1, ..., r$. Differentiating the scalar product $(ψ^a, ψ_b)$ with respect to time and using the Schrödinger equation, we obtain equation

$$\frac{∂}{∂t}(ψ^a, ψ_b) = -i(ψ^a H, ψ_b) + i(ψ^a, Hψ_b) = 0.$$

The last equality follows from the self-adjointness of the Hamiltonian. As a consequence, if vectors $ψ^a_0 = φ^a(λ(0))$ are orthonormalized at the initial moment of time, then the corresponding solutions of the Schrödinger equation remain orthonormalized for all subsequent moments of time. Hence the matrix $U$ in decomposition (14) is unitary.

The Schrödinger equation for solution (14) is reduced to equation

$$iU^{-1b}_cφ^c + iU^{-1b}_cφ^c = HU^{-1b}_cφ^c.$$

Let us take the scalar product of the left and right hand sides with $φ_a$. As a result, we derive equation for the unitary matrix
\[ U_{a}^{-1b} = \lambda^{k} U_{c}^{-1b} A_{ka}^{c} - i EU_{a}^{-1b}, \]  

(15)

where we have introduced the following notation:

\[ A_{ka}^{c} := - (\partial_{k} \phi^{b}, \phi_{a}). \]  

(16)

Orthonormality of eigenfunctions \( \phi^{a} \) implies antiunitarity of components \( A_{ka}^{b} \) for all \( k = 1, \ldots, n \) if indices \( a \) and \( b \) are considered as matrix ones. Indeed, differentiating the orthonormality condition \( (\phi^{b}, \phi_{a}) = \delta_{a}^{b} \), we obtain equality

\[ (\partial_{k} \phi^{b}, \phi_{a}) + (\phi^{b}, \partial_{k} \phi_{a}) = (\phi^{a}, \partial_{k} \phi_{b}) + (\phi^{b}, \partial_{k} \phi_{a}) = 0. \]

That is, matrices \( A_{k} \) are antiunitary and therefore belong to the Lie algebra \( u(r) \). Consequently, the matrices \( A_{k} \) define 1-forms in some neighborhood \( \mathbb{U} \subset \mathbb{M} \) with values in the Lie algebra, as components of the local connection form.

The initial condition for the unitary matrix has the form

\[ U_{a}^{-1b} \big|_{s=0} = \delta_{a}^{b}. \]

The solution of the Cauchy problem for Eq. (15) can be written as the \( P \)-product

\[
U^{-1}(t) = P \exp \left( \int_{0}^{t} ds \lambda^{k}(s) A_{k}(s) - i \int_{0}^{t} ds E(\lambda(s)) \right).
\]

(17)

where we have omitted the matrix indices for simplicity.

The first factor is the generalization of the Berry phase to the case of degenerate states, and the second factor is the dynamic phase. The dynamic phase has the same form as for the nondegenerate state.

The first factor in solution (17) represents the Wilczek–Zee unitary matrix

\[
U^{-1}_{\text{WZ}} = P \exp \left( \int_{\lambda(0)}^{\lambda(t)} d\lambda^{k} A_{k} \right),
\]

(18)

which can be given the following geometric interpretation. We have the principal fiber bundle \( \mathbb{P}(\mathbb{M}, \pi, \mathbb{U}(r)) \) with the structure group \( \mathbb{U}(r) \) (transformation (14)). The set of eigenfunctions \( \phi^{a} \) is the cross section of the associated fiber bundle \( \mathbb{E}(\mathbb{M}, \pi_{r}, \mathbb{H}^{r}, \mathbb{U}(r), \mathbb{P}) \) with the typical fiber being the tensor product of Hilbert spaces

\[ \mathbb{H}^{r} := \mathbb{H} \otimes \cdots \otimes \mathbb{H}. \]

Under the vertical automorphism given by the unitary matrix \( U(\lambda) \in \mathbb{U}(r) \),

\[ \phi^{a} = U^{-1a}_{\phi} \phi^{b}, \quad \phi_{a} = U_{\phi}^{b} \phi_{b}, \]

fields (16) transform according to the rule
\[ A'_k = U^{-1} A_k U + U^{-1} \partial_k U, \]  

(19)

where we have omitted matrix indices. It implies that the fields \( A_k \) can be interpreted as components of the local connection form or Yang–Mills fields. A set of these components given on a coordinate covering of the base \( \mathcal{M} \) defines uniquely the connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \mathbb{U}(r)) \).

If the path is closed, \( \gamma \in \Omega(\mathcal{M}, \lambda_0) \), then unitary Wilczek–Zee matrix (18) represents the element of the holonomy group \( U^{-1}_{WZ} \in \Phi(\lambda_0, e) \) at the point \( (\lambda_0, e) \in \mathbb{P} \) corresponding to the zero cross section \( \mathcal{M} \ni \lambda \rightarrow (\lambda, e) \in \mathbb{P} \), where \( \lambda_0 := \lambda(0) \) and \( e \) is the unity of the structure group \( \mathbb{U}(r) \).

Thus the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \mathbb{U}(r)) \) arises in the case of a degenerate energy level of the Hamiltonian. In the above-considered case, the base \( \mathcal{M} \) is the parameter manifold \( \lambda \in \mathcal{M} \) with the Hamiltonian being dependent on this point. We suppose that this manifold is finite dimensional. The structure group is the unitary group \( \mathbb{U}(r) \) which is also finite dimensional. The connection on the principal fiber bundle is defined by the cross sections of the associated bundle \( \mathbb{E}(\mathcal{M}, \pi_g, \mathbb{H}_r, \mathbb{U}(r), \mathbb{P}) \). Generally, the typical fiber of the associated fiber bundle can be infinite dimensional Hilbert space \( \mathbb{H}_r \). In the present paper, we do not consider infinite dimensional manifolds to avoid difficulties which can arise [13]. Nevertheless, in our case everything that is needed is the transformation formulas for components of local connection form (19) which can be easily checked in each particular case. If the associated bundle is not diffeomorphic to the direct product \( \mathcal{M} \otimes \mathbb{H}_r \), then the state of a quantum system is given by a family of local cross sections on a coordinate covering of the base \( \mathcal{M} \). It defines a family of local connection forms (16). In its turn, the family of local connection forms defines a connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \mathbb{U}(r)) \) uniquely up to an isomorphism.

We see once again that principal and associated fiber bundles can be trivial or not depending on the problem under consideration. The connection on the principal fiber bundle \( \mathbb{P}(\mathcal{M}, \pi, \mathbb{U}(r)) \) can be nontrivial and imply nontrivial Wilczek–Zee matrix (18) describing parallel transport of fibers along a path on the base \( \lambda(t) \in \mathcal{M} \) even for trivial bundles. This observation confirms its geometric rather than topological origin. For closed paths \( \gamma \in \Omega(\mathcal{M}, \lambda_0) \) with the beginning and end at a point \( \lambda_0 \in \mathcal{M} \), the Wilczek–Zee matrix defines the element of the holonomy group \( U_{WZ} \in \Phi(\lambda_0, e) \subset \mathbb{U}(r) \).

For simplicity, we assumed the Berry phase and the Wilczek–Zee matrix to be finite dimensional. This assumption can be essentially relaxed. The formulas obtained are valid for all levels for which the adiabatic theorem holds, that is, it must be an isolated level with the energy level separated from the rest of the spectrum.

CONCLUSIONS

In this paper, we have considered the Berry phase and its non-Abelian generalization by Wilczek and Zee. These effects are demonstrated to be the consequences of nontrivial connections on the principal fiber bundles which define the nontrivial holonomy group. At the same time, the topology of the principal fiber bundle can be trivial. Therefore, the effects considered above are not topological as they are often called in modern physical literature but rather geometric effects.

The interpretation proposed in the paper contains nothing except differential geometric notions. In the geometric interpretation of mathematical physics models, one has to take into account that a connection exists on any principal fiber bundle independently of the topology of the base [9]. Moreover, if a family of the local connection forms is given on an arbitrary closed submanifold of the base of some principal fiber bundle, then the corresponding connection can always be extended to the whole principal fiber bundle. This can be done in many ways. A connection defines the holonomy group which is nontrivial in the general case.

In experiments on testing the existence of the Berry phase, the observable effects are produced not by the whole holonomy group but a fixed element of the holonomy group which depends on the connection and the closed contour. The topology of the base can be trivial or not, it does not play any role. If the topology is trivial, then the integration
contour can be contracted to a point. The effect disappears in this case, because the corresponding element of the holonomy group tends to the unity element, and this is quite natural from the physical point of view.

A connection on the principal fiber bundle defines connections on all fiber bundles associated with it. In particular, if the typical fiber is an infinitely dimensional Hilbert space, then the connection is also defined. At present, the interpretation of the Berry phase, as a rule, is reduced to consideration of a connection on an associated fiber bundle, and this forces one to consider infinite dimensional manifolds and to take into account the related subtleties. From our point of view, the interpretation of the geometric effects in terms of connections on principal fiber bundles is simpler and more natural.

The author is grateful to I. V. Volovich and D. V. Treshtchev for discussions of the paper and useful comments.

This work was supported in part by the Russian Foundation for Basic Research (grants Nos. 08-01-00727-a and 09-01-12161-ofi_m), the Program for Supporting Leading Scientific Schools (grant NSh-7675.2010.1), and the Program “Modern Problems of Theoretical Mathematics” of the Russian Academy of Sciences.

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