Generalised periodic solutions
to a forced Kepler problem in the plane

D. Papini
joint work with A. Boscaggin & W. Dambrosio

Dip. di Scienze Matematiche, Informatiche e Fisiche, Università di Udine

Topological Methods in Differential Equations
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Periodically forced Kepler problem

Find $T$-periodic solutions to:

$$(FKP_\epsilon) \quad \ddot{x} = -\frac{x}{|x|^3} + \epsilon \nabla_x U(t, x) \quad x \in \mathbb{R}^2 \setminus \{O\}$$

where:

- $U : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ smooth enough;
- $U(t + T, x) = U(t, x)$ for all $(t, x) \in \mathbb{R}^{1+2}$ and some $T > 0$. 

As critical points in $H^1_T := \{x \in H^1(0, T) : x(0) = x(T)\}$ of

$$A_T(x) = \int_0^T \left( \frac{|\dot{x}(t)|^2}{2} + \frac{1}{|x(t)|} + \epsilon U(t, x(t)) \right) dt$$

There are $x \in H^1_T$ such that $O \in x([0, T])$ and $A_T(x) < +\infty$. 
Related papers: collisionless solutions

For \((FKP\epsilon)\):
- Ambrosetti & Coti Zelati 1989: \(U\) even and \(T/2\) periodic;
- Cabral & Vidal, 2000: \(U\) symmetric under rotation and reflection;
- Fonda & Toader & Torres, 2012;
- Fonda & Gallo, 2017: radial perturbation, 2018: symmetry under a rotation;
- Boscaggin & Ortega, 2016: averaging technique;
- Amster & Haddad & Ortega & Ureña 2011: large perturbations;

For \((FKP)\):
- Serra & Terracini 1994: \(U(t, x) = p(t)\) ruled out.
Related papers: generalised solutions

- Solutions attaining $O$ on a zero-measure set: Ambrosetti & Coti Zelati 1993, Bahri & Rabinowitz, Tanaka 1993;
- regularised equations in dimension 1: Ortega 2011, Zhao 2016, Rebelo & Simões 2018;
- regularised equations in higher dimension: Boscaggin & Ortega & Zhao 2019;
- regularised functionals in general setting: Barutello & Ortega & Verzini 2021.
Generalised solutions

A generalised $T$-periodic solution to (FKP) is a $T$-periodic function $x \in C(\mathbb{R})$ that satisfies the following:

1. the collision set $E_x := x^{-1}(O) = \{ t \in [0, T] : x(t) = O \}$ is discrete;
2. $x \in C^2(I)$ and satisfies equation (FKP) in $I$, for each interval $I \subset \mathbb{R} \setminus E_x$;
3. the limits:

$$
\lim_{t \to t_0} \frac{x(t)}{|x(t)|} \quad \text{and} \quad \lim_{t \to t_0} \left( \frac{|\dot{x}(t)|^2}{2} - \frac{1}{|x(t)|} \right)
$$

exist and are finite at every $t_0 \in E_x$. 
Theorem

If \( U(t, x) \) is \( C^1(\mathbb{R}^{1+2}) \), \( T \)-periodic w.r.t. \( t \) and satisfies:

\[
|U(t, x)| \leq C(1 + |x|^\alpha) \quad \forall (t, x) \in \mathbb{R}^{1+2}
\]

for some \( C > 0 \) and \( \alpha \in ]0, 2[ \), then (FKP) has at least one \( T \)-periodic generalised solution.

Candidates are chosen among the local minimisers of the action functional

\[
\mathcal{A}_T(x) = \int_0^T \left( \frac{\dot{x}(t)^2}{2} + \frac{1}{|x(t)|^\alpha} + U(t, x(t)) \right) dt
\]

which lacks coercivity on \( \mathcal{H}_T^1 := \{ x \in \mathcal{H}^1(0, T) : x(0) = x(T) \} \).
Minimisation

We consider $\mathcal{X} := \mathcal{X}_c \cup \mathcal{X}_r$ where:

- $\mathcal{X}_c := \{ x \in H^1_T : O \in x([0, T]) \}$;
- $\mathcal{X}_r := \{ x \in H^1_T : O \not\in x([0, T]) \text{ and } x \text{ is not null-homotopic in } \mathbb{R}^2 \setminus \{ O \} \}$.
- $\mathcal{X}$ is sequentially weakly closed in $H^1_T$.
- A Poincaré-type inequality holds in $\mathcal{X}$:
  \[ \int_0^T |x|^2 \leq K \int_0^T |\dot{x}|^2 \quad \forall x \in \mathcal{X} \implies \mathcal{A}_T(x) \geq \int_0^T \left( \frac{|\dot{x}|^2}{4} + \frac{|x|^2}{8K} \right) - K' \quad \forall x \in \mathcal{X}. \]

Proposition

There exists $x \in \mathcal{X}$ such that $\mathcal{A}_T(x) = \inf_{y \in \mathcal{X}} \mathcal{A}_T(y)$.

From now on, we assume that $x \in \mathcal{X}_c$. 
Exploring collisions

- The collision set \( E_x = x^{-1}(O) \subset [0, T] \) has measure 0 since \( \mathcal{A}_T(x) \in \mathbb{R} \);
- \( \mathbb{R} \setminus \bigcup_{k \in \mathbb{Z}} (E_x + kT) \) is the (at most) countable union of pairwise disjoint open intervals \( ]a_n, b_n[ \) where \( x \) is \( C^2(]a_n, b_n[) \) and satisfies (FKP) \( (n \in \mathbb{N}) \).
- If we let
  \[
  h_x(t) = \frac{\dot{x}(t)^2}{2} - \frac{1}{|x(t)|} \quad t \in [0, T] \setminus E_x,
  \]
  we have that
  \[
  \int_0^T |h_x(t)| dt \leq \mathcal{A}_T(x) - \int_0^T U(t, x(t)) dt \implies h_x \in L^1(0, T).
  \]
Exploring collisions: the energy

**Proposition**

\[ h_x \in W^{1,1}_{\text{loc}} \text{ and, therefore, the energy can be extended to a continuous function.} \]

- choose any \( \phi \in C^\infty_c(0, T) \) and define \( \psi_\lambda(t) = t + \lambda \phi(t) \) and let \( x_\lambda = x \circ \psi_\lambda \);
- if \( \lambda \) is small enough, \( \phi_\lambda \) is a diffeomorphism, \( x_\lambda([0, T]) = x([0, T]) \) and, in particular, \( x_\lambda \in \mathcal{X}_c \);
- if \( a(\lambda) := A_T(x_\lambda) \), then \( a(\lambda) \geq a(0) = A_T(x) \) for each \( \lambda \) in a neighborhood of 0 and, thus, \( a'(0) = 0 \);
- more precisely:

\[
\int_0^T \left[ h_x(t) \dot{\phi}(t) + \langle \nabla_x U(t, x(t)), \dot{x}(t) \rangle \phi(t) \right] dt = 0 \quad \forall \phi \in C^\infty_c
\]

and, hence, \( h_x \in W^{1,1}_{\text{loc}} \).
Exploring collisions: the collision set $E_x$

**Proposition**

*The collision set $E_x = x^{-1}(O) \subset [0, T]$ is finite.*

- Letting $l_x(t) := \frac{|x(t)|^2}{2}$, we have the virial identity:

  $$l_x''(t) = \frac{1}{|x(t)|} + \langle \nabla_x U(t, x(t)), x(t) \rangle + 2h_x(t), \quad t \in [0, T] \setminus E_x.$$

- $l_x''(t) \to +\infty$ as $t$ approaches a collision time, therefore $t \mapsto |x(t)|^2$ is strictly convex in a neighborhood of collision times.

- Collision times are isolated.
Exploring collisions: asymptotic directions at a collision time $t_0$

- For each small $\delta > 0$ there exist $t_\delta^-, t_\delta^+ > 0$ such that
  
  $$|x(t_0 \pm t_\delta^\pm)| = \delta$$

  $$|x(t)| < \delta \quad \forall t \in \left[t_0 - t_\delta^-, t_0 + t_\delta^+\right]$$

  and $t \mapsto |x(t)|^2$ is (strictly) convex in $[t_0 - t_\delta^-, t_0 + t_\delta^+]$.

- Sperling’s asymptotics (Celestial Mech. 1969/70) at an isolated collision time $t_0$: there are two versors $x_0^+$ and $x_0^-$ such that:

  $$x(t) = \sqrt{\frac{9}{2}} |t - t_0|^{2/3} x_0^\pm + o\left(|t - t_0|^{2/3}\right)$$

  $$\dot{x}(t) = \frac{2}{3} \sqrt{\frac{9}{2}} (t - t_0)^{-1/3} x_0^\pm + o\left(|t - t_0|^{-1/3}\right)$$

  as $t \to t_0^\pm$

Goal: $x_0^+ = x_0^-$. 

Exploring collisions: blow-up analysis at $t_0$

Rescaling:

$z_\delta(s) := \frac{1}{\delta}x(\delta^{3/2}s + t_0)$ for $s \in [-\sigma_{\delta}, \sigma_{\delta}]$

$\sigma_{\delta}^\pm := t_\delta^{\pm}/\delta^{3/2}$, $|z_\delta(\sigma_{\delta}^\pm)| = 1$, $z_\delta(0) = O$, $|z_\delta(t)| < 1 \forall t \in [-\sigma_{\delta}, \sigma_{\delta}]$. 

The diagrams illustrate the blow-up analysis at $t_0$, showing the rescaling function $z_\delta(s)$ and the regions of interest for $s$. The diagrams also depict the behavior of $x(t)$ at $t_0$, with arrows indicating the direction of the blow-up at the critical points $x_0^-$ and $x_0^+$. The diagrams are annotated with labels such as $x(t_0-t_\delta^-)$ and $x(t_0+t_\delta^+)$, highlighting the paths of interest in the blow-up analysis.
Exploring collisions: blow-up analysis at $t_0$

$$z_\delta(s) := \frac{1}{\delta^3/2} x(\delta^{3/2} s + t_0), \quad s \in [-\sigma^-_\delta, \sigma^+_\delta] \quad (\sigma^\pm_\delta := t^\pm_\delta / \delta^{3/2})$$

$$|z_\delta(\sigma^\pm_\delta)| = 1, \quad z_\delta(0) = O, \quad |z_\delta(t)| < 1 \quad \forall t \in ]-\sigma^-_\delta, \sigma^+_\delta[$$

A straightforward computation gives:

$$A_{[t_0-t^-_\delta,t_0+t^+_\delta]}(x) = \int_{-\sigma^-_\delta}^{\sigma^+_\delta} \left( \frac{|\dot{z}_\delta|^2}{2} + \frac{1}{|z_\delta|} \right) + \delta^2 \int_{-\sigma^-_\delta}^{\sigma^+_\delta} U \left( t_0 + \delta^{3/2} s, z_\delta(s) \right) ds$$
Exploring collisions: blow-up analysis at \( t_0 \)

Sperling’s asymptotics as \( \delta \to 0^+ \) give that \( \sigma_\delta^\pm \to s_0 \), \( z_\delta(s) \to \zeta(t; x_0^-, x_0^+) \) and \( \dot{z}_\delta(s) \to \dot{\zeta}(t; x_0^-, x_0^+) \) \( \forall 0 < |s| < s_0 \), where:

\[
\zeta(t; x_0^-, x_0^+) := \begin{cases} 
\frac{3}{\sqrt{2}} \left| s \right|^{2/3} x_0^- & \text{if } -s_0 \leq s \leq 0, \\
\frac{3}{\sqrt{2}} \left| s \right|^{2/3} x_0^+ & \text{if } 0 \leq s \leq s_0,
\end{cases}
\]

(b.t.w. \( s_0 = \sqrt{2}/3 \)).

is the parabolic collision-ejection solution of the following two-point bvp:

\[
(2PK) \quad \begin{cases} 
\ddot{z} = -\frac{z}{|z|^3} & s \in [-s_0, s_0] \\
z(\pm s_0) = x_0^\pm
\end{cases}
\]

Moreover:

\[
\liminf_{\delta \to 0^+} \frac{A_{[t_0-t_\delta^-, t_0+t_\delta^+]}(x)}{\delta^{1/2}} \geq \psi_0 := \int_{-s_0}^{s_0} \left( \frac{\dot{\zeta}^2}{2} + \frac{1}{|\zeta|} \right) = 4 \frac{3}{\sqrt{2}} \sqrt{2}.
\]
Exploring collisions: alternative routes

If $x_0^- \neq x_0^+$ it is known that $\zeta(\cdot; x_0^-, x_0^+)$ does not minimise the Keplerian action over the paths joining $x_0^-$ to $x_0^+$ in the time interval $[-s_0, s_0]$.

Lemma [Fusco & Gronchi & Negrini, 2011]

If $x_0^- \neq x_0^+$ then there are exactly two classical solutions $\xi_i = \xi_i(\cdot; x_0^-, x_0^+)$ of (2PK) (for $i = 1, 2$) such that:

1. $\phi^i(x_0^-, x_0^+) := \int_{-s_0}^{s_0} \left( \frac{|\dot{\xi}_i|^2}{2} + \frac{1}{|\xi_i|} \right) < \psi_0$ for $i = 1, 2$;
2. they are not homotopic to each other in $\mathbb{R}^2 \setminus \{O\}$;
3. they depend smoothly on the data of the problem.

See also: Albouy, Lecture notes on the two-body problem (2002).

If we have $x_0^- \neq x_0^+$, we can use these $\xi_i$ to modify $x$ in a neighborhood of $t_0$ and decrease its action.
Exploring collisions: cut-and-paste near $t_0$

and $x$ wouldn’t anymore be minimal for $A_T$ on $X$. 