A NEW CHARACTERISATION OF CONVEX ORDER THROUGH THE 2-WASSERSTEIN DISTANCE

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Abstract. We give a new characterisation of convex order using the 2-Wasserstein distance $W_2$: we show that two probability measures $\mu$ and $\nu$ on $\mathbb{R}^d$ with finite second moments are in convex order (i.e. $\mu \preceq_{c} \nu$) iff

$$W_2(\nu, \rho)^2 - W_2(\mu, \rho)^2 \leq \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx)$$

holds for all probability measures $\rho$ on $\mathbb{R}^d$ with bounded support. Our proof of this result relies on a quantitative bound for the infimum of $\int f d\nu - \int f d\mu$ over all 1-Lipschitz functions $f$, which is obtained through optimal transport duality and Brenier’s theorem. We use our new characterisation to derive new proofs of well-known one-dimensional characterisations of convex order as well as new computational methods for investigating convex order.

1. Introduction and main results

Fix two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ with

$$\int |x|^2 \mu(dx) < \infty, \quad \int |y|^2 \nu(dy) < \infty.$$ 

Recall that $\mu$ and $\nu$ are in convex order (denoted by $\mu \preceq_{c} \nu$) iff

$$\int f d\mu \leq \int f d\nu \quad \text{for all convex functions } f : \mathbb{R}^d \to \mathbb{R}.$$ 

As any convex function is bounded from below by an affine function, the above integrals take values in $(-\infty, \infty]$. The notion of convex order is very well studied, see e.g. [Ross et al. 1996], [Müller and Stoyan 2002], [Shaked and Shanthikumar 2007], [Arnold 2012] and the references therein for an overview. It plays a pivotal role in mathematical finance since Strassen [1965] established that $\mu \preceq_{c} \nu$ if and only if $\mathcal{M}(\mu, \nu)$ — the set of martingale laws on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$ — is non-empty. This result is also the reason why convex order has taken the center stage in the field of martingale optimal transport, see e.g. [Galichon et al. 2014], [Beiglböck et al. 2013, 2015, De March and Touzi 2019], [Obłój and Siorpaes 2017, 2019, Alfonsi et al. 2019, 2020, Alfonsi and Jourdain 2020, Jourdain and Margheriti 2022, Massa and Siorpaes 2022] and the references therein. Furthermore, convex order plays a pivotal role in dependence modelling.

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and risk aggregation, see e.g. [Tchen 1980, Rüschendorf and Uckelmann 2002, Wang and Wang 2011, Embrechts et al. 2013, Bernard et al. 2017]. While there is an abundance of explicit characterisations of convex order available in one dimension (i.e. \(d = 1\)) — see e.g. [Shaked and Shanthikumar 2007, Chapter 3] — the case \(d > 1\) seems to be less studied to the best of our knowledge. The main goal of this article is to fill this gap: we offer a new characterisation of convex order, that holds in general dimensions, and is based on the theory of optimal transport (OT). Optimal transport goes back to the seminal works of Monge [1781] and Kantorovich [1958]. It is concerned with the problem of transporting probability distributions in a cost-optimal way. We refer to Rachev and Rüschendorf [1998] and Villani [2003, 2008] for an overview. For this paper we only need a few basic concepts from OT. Most importantly we will make use of the \(p\)-Wasserstein metric, which is defined as

\[
W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int |x - y|^p \pi(dx, dy),
\]

where \(p \geq 1\). Here \(\Pi(\mu, \nu)\) denotes the set of probability measures on \(\mathbb{R}^d \times \mathbb{R}^d\) with marginals \(\mu\) and \(\nu\). Furthermore we denote the set of probability measures on \(\mathbb{R}^d\) with finite \(p\)-th moment by \(\mathcal{P}_p(\mathbb{R}^d)\). In this paper we will mainly focus on the cases \(p = 1, 2\). Our first main result is the following:

**Theorem 1.1.** The probability measures \(\mu\) and \(\nu\) on \(\mathbb{R}^d\) with finite second moment are in convex order if and only if

\[
W_2(\nu, \rho)^2 - W_2(\mu, \rho)^2 \leq \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx)
\]

holds for all probability measures \(\rho\) on \(\mathbb{R}^d\) with bounded support.

Theorem 1.1 states that convex order of \(\mu\) and \(\nu\) is equivalent to an order relation for \(W_2\) on the space of probability measures. The “only if” direction was first shown in [Alfonsi and Jourdain 2020, Equation (2.2)]. Our proof of Theorem 1.1 is independent and exploits a well-known connection between convex functions and OT for the squared Euclidean distance called Brenier’s theorem, see Brenier [1991], Rüschendorf and Rachev [1990]. We emphasize however, that contrary to the setting of Brenier’s theorem, no assumptions on the probability measures \(\mu\) and \(\nu\) except for square integrability need to be made; in particular we do not need to assume that these are absolutely continuous wrt. the Lebesgue measure. In fact, our proof establishes a finer relationship between convex order and differences of OT functionals. In order to detail this relationship, we first define

\[
C(\mu, \rho) := \sup_{\pi \in \Pi(\mu, \rho)} \int \langle x, y \rangle \pi(dx, dy), \quad C(\nu, \rho) := \sup_{\pi \in \Pi(\nu, \rho)} \int \langle x, y \rangle \pi(dx, dy).
\]

It is well-known that \(C(\mu, \rho)\) (resp. \(C(\nu, \rho)\)) has the same optimisers as \(W_2(\mu, \rho)\) (resp. \(W_2(\nu, \rho)\)), if \(\mu, \nu\) are square integrable — however \(C(\mu, \rho)\) and \(C(\nu, \rho)\) are still well-defined when we merely require that \(\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)\) and that \(\rho\) is compactly supported. Our second main result is the following:

**Theorem 1.2.** Assume that \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d)\) have finite first moments. Then

\[
\inf_{f \in C^1(\mathbb{R}^d)} \left( \int f d\nu - \int f d\mu \right) = \inf_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \left( C(\nu, \rho) - C(\mu, \rho) \right),
\]
where
\[ P^1(\mathbb{R}^d) := \{ \rho \in \mathcal{P}(\mathbb{R}^d) : \text{supp}(\rho) \subseteq B_1(0) \} \]
and
\[ C^1(\mathbb{R}^d) := \{ f : \mathbb{R}^d \rightarrow \mathbb{R} \text{ convex, } 1\text{-Lipschitz} \}. \]

As the convex order is classically embedded in \( P^1(\mathbb{R}^d) \), see e.g. Nendel [2020],
Theorem 1.2 is not just more concise than Theorem 1.1 but arguably also a very
natural generalisation of Theorem 1.1.

In this paper we then focus on two implications of our new characterisations
given in Theorems 1.1 and 1.2: we first use Theorem 1.1 to give a new proof of a
characterisation of convex order in one dimension through quantile functions. Then
we use Theorem 1.2 to derive new computational methods for testing convex order
between \( \mu \) and \( \nu \). For the computation we exploit state of the art computational
OT methods, which are efficient for potentially high-dimensional problems. These
have recently seen a spike in research activity. We refer to Peyré and Cuturi [2019]
for an overview.

Contrary to standard characterisations of convex order using potential functions
or cdfs, Theorems 1.1 and 1.2 hold in any dimension. To the best of our knowledge,
our results are one of the first attempts to characterise convex order via classical
OT theory. Our results are probably closest in spirit to Gozlan and Juillet [2020],
who investigate specific couplings, that are solutions of a weak optimal transport
problem, and derive a combination of Brenier’s and Strassen’s theorems.

The remainder of this article is structured as follows: In Section 2 we state
examples and consequences of Theorems 1.1 and 1.2. In particular we connect them
to some well-known results in the theory of convex order. The proof of the main
results is given in Section 3. Section 4 discusses numerical examples. Remaining
proofs are collected in Section 5.

2. Discussion and consequences of main results

To sharpen intuition, let us first discuss the case \( d = 1 \). By Theorem 1.1
we can obtain a new proof of a well-known representation of convex order on the real
line, see e.g. Shaked and Shanthikumar [2007] Theorem 3.A.5. Here we denote
the quantile function of a probability measure \( \mu \) by
\[ F_{\mu}^{-1}(x) := \inf \{ y \in \mathbb{R} : \mu((-\infty, y]) \geq x \}. \]

Corollary 2.1. For \( d = 1 \) we have
\[ \mu \preceq_c \nu \iff \int_0^x [F_{\mu}^{-1}(y) - F_{\nu}^{-1}(y)] dy \geq 0. \]
for all \( x \in [0, 1] \), with equality for \( x = 1 \).

The proofs of all results of this section are collected in Section 5. We continue
with general \( d \in \mathbb{N} \) and give a geometric interpretation of Theorem 1.1 by restating
it as follows: \( \mu \preceq_c \nu \) holds if
\[ W_2(\nu, \rho)^2 - W_2(\mu, \rho)^2 \leq W_2(\nu, \delta_z)^2 - W(\mu, \delta_z)^2 \]
for all \( \rho \in \mathcal{P}(\mathbb{R}^d) \) with bounded support, where \( \delta_z, z \in \mathbb{R}^d \) is a Dirac measure.
Indeed, varying \( \rho \) over Dirac measures in (1) implies that the means of \( \mu \) and \( \nu \)
have to be equal; equation (3) then follows from simple algebra. This implies in
particular that the difference between squared Wasserstein cost from $\nu$ and $\mu$ to $\rho$ is maximised at the point masses. Lastly, (1) can also be reformulated as: $\mu \preceq_c \nu$ iff

$$\sup_{\pi \in \Pi(\mu, \rho)} \int \langle x, z \rangle \pi(dx, dz) \leq \sup_{\pi \in \Pi(\nu, \rho)} \int \langle y, z \rangle \pi(dy, dz),$$

i.e. for any $\rho \in \mathcal{P}(\mathbb{R}^d)$ with bounded support, the maximal covariance between $\mu$ and $\rho$ is less than the one between $\nu$ and $\rho$. This provides a natural intuition for a classical pedestrian description of convex order, namely that “$\nu$ being more spread out than $\mu$”.

We next give a simple example for Theorem 1.1.

**Example 2.2.** Let us take $\mu = \delta_0$ and $\nu$ with mean zero. Now, recalling (4) and bounding $W_2(\nu, \rho)$ from above by choosing the product coupling, we obtain that for any $\rho$ with finite second moment

$$W_2(\nu, \rho)^2 - W_2(\mu, \rho)^2 = W_2^2(\nu, \rho) - \int |x|^2 \rho(dx) \leq \int |y|^2 \nu(dy) - 2 \int (x, y) \nu(dx) \rho(dy) = \int |y|^2 \nu(dy) \leq \int |x|^2 \mu(dx).$$

In conclusion we recover the well-known fact $\delta_0 \preceq_c \nu$.

We now state two direct corollaries of Theorem 1.1. We consider the cost $c(x, y) := |x - y|^2/2$ and recall that a function $f$ is $c$-concave, if

$$f(x) = \inf_{y \in \mathbb{R}^d} (g(y) - c(x, y))$$

for some function $g : \mathbb{R}^d \to \mathbb{R}$. We then have the following:

**Corollary 2.3.** We have

$$\int g \, d\nu \leq \int g \, d\mu \quad \text{for all } c\text{-concave functions } g : \mathbb{R}^d \to \mathbb{R}$$

if and only if

$$W_2(\nu, \rho)^2 \leq W_2(\mu, \rho)^2 \quad \text{for all } \rho \in \mathcal{P}(\mathbb{R}^d) \text{ with compact support}.$$

Theorem 1.1 also directly implies the following well-known result:

**Corollary 2.4.** If $\mu \preceq_c \nu$ then

$$W_2(\mu, \nu)^2 \leq \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx).$$

In particular $\mu \preceq_c \nu$ implies

$$\sup_{\pi \in \Pi(\mu, \nu)} \int \langle x, y \rangle \pi(dx, dy) \geq \int |x|^2 \mu(dx).$$
3. Proof of main results

Let us start by setting up some notation. We denote the scalar product on \(\mathbb{R}^d\) by \(\langle \cdot, \cdot \rangle\). We write \(|\cdot|\) for the Euclidean norm on \(\mathbb{R}^d\). The ball in \(\mathbb{R}^d\) around \(x\) of radius \(r > 0\) will be denoted by \(B_r(x)\). We write \(\nabla f(x)\) for the derivative of a function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) at a point \(x \in \mathbb{R}^d\). We denote the \(d\)-dimensional Lebesgue measure by \(\lambda\).

In order to keep this article self-contained, we summarise some properties of optimal transport at the beginning of this section, and refer to [Villani, 2003, Chapter 2.1] for a more detailed treatment.

By definition we have for any \(\rho \in \mathcal{P}_2(\mathbb{R}^d)\) that
\[
W_2(\mu, \rho)^2 = \int |x|^2 \mu(dx) + \int |y|^2 \rho(dy) - 2 \sup_{\pi \in \Pi(\mu, \rho)} \int \langle x, y \rangle \pi(dx, dy).
\]
(4)

In this section we thus (re-)define the cost function \(c(x, y) := \langle x, y \rangle\) and recall that the convex conjugate \(f^* : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}\) of a function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\) is given by
\[
f^*(y) := \sup_{x \in \mathbb{R}^d} (\langle y, x \rangle - f(x)).
\]

The subdifferential of a proper convex function \(f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}\) is defined as
\[
\partial f(x) := \{y \in \mathbb{R}^d : f(x') - f(x) \geq \langle y, x' - x \rangle \text{ for all } x' \in \mathbb{R}^d\}.
\]

It is non-empty if \(x\) belongs to the interior of the domain of \(f\). We have
\[
f(x) + f^*(y) - \langle x, y \rangle = 0 \iff y \in \partial f(x).
\]
(5)

Lastly we recall the duality
\[
C(\mu, \rho) = \sup_{\pi \in \Pi(\mu, \rho)} \int \langle x, y \rangle \pi(dx, dy)
\]
\[
= \inf_{f \geq c, f \text{ proper, convex}} \int f \, d\mu + \int g \, d\rho
\]
(6)

and the existence of an optimal pair \((f, f^*)\) of (lower semicontinuous, proper) convex conjugate functions. Replacing \(\mu\) by \(\nu\) in the display above, we obtain a similar duality for \(C(\nu, \rho)\).

3.1. Proof of Theorem 1.2: the equivalent case.

We first prove Theorem 1.2 for measures \(\mu, \nu\), which are equivalent to the \(d\)-dimensional Lebesgue measure \(\lambda\), i.e. \(\mu, \nu \sim \lambda\). As \(\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)\), the domain of the optimising potential \(f\) for \(C(\mu, \rho)\) (resp. \(C(\nu, \rho)\)) is \(\mathbb{R}^d\) in this case. Recall furthermore that
\[
\partial f(x) = \overline{\operatorname{Conv} \left( \lim_{x_k \rightarrow x} \nabla f(x_k) \right)},
\]
(7)
see e.g. [Villani 2003, 2.1.3.3]. We write
\[
\|\partial f\|_\infty := \sup_{x \in \mathbb{R}^d} \sup_{y \in \partial f(x)} |y|.
\]

We now prove Theorem 1.2 when \(\mu, \nu \sim \lambda\):
Proposition 3.1. Assume $\mu, \nu \in P_1(\mathbb{R}^d)$, $\mu, \nu \sim \lambda$. Recall

$$P_1(\mathbb{R}^d) = \{\rho \in P(\mathbb{R}^d) : \text{supp}(\rho) \subseteq B_1(0)\}$$

as well as the 1-Lipschitz convex functions

$$C^1(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \text{ convex}, \|\partial f\|_{\infty} \leq 1\}.$$

Then we have

$$\inf_{f \in C^1(\mathbb{R}^d)} \left( \int f \, d\nu - \int f \, d\mu \right) = \inf_{\rho \in P_1(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)).$$

Proof. As $\mu, \nu$ have finite first moment and $\rho$ is compactly supported, $|C(\mu, \rho)|, |C(\nu, \rho)| < \infty$ follows from Hölder’s inequality. We now fix $\rho \in P_1(\mathbb{R}^d)$ and take an optimal convex pair $(\hat{f}, \hat{g})$ in (6) for $C(\nu, \rho)$. Next we apply Brenier’s theorem in the form of [Villani, 2003, Theorem 2.12], which states that $\rho = \nabla \hat{f} \ast \nu$. Furthermore, as $\text{supp}(\rho) \subseteq B_1(0)$ we conclude $\|\partial \hat{f}\|_{\infty} \leq 1$ by (7) and $C(\nu, \rho) - C(\mu, \rho) \geq \int \hat{f} \, d\nu - \int \hat{f} \, d\mu$.

Taking the infimum over $\rho \in P_1(\mathbb{R}^d)$ shows that

$$\inf_{\rho \in P_1(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)) \geq \inf_{f \in C^1(\mathbb{R}^d)} (\int f \, d\nu - \int f \, d\mu).$$

On the other hand, fix $f \in C^1(\mathbb{R}^d)$ and set $g := f^*$. Define $\hat{\rho} := \nabla f \ast \mu$ and note that $\hat{\rho} \in P_1(\mathbb{R}^d)$. Then again by Brenier’s theorem we obtain optimality of the pair $(f, g)$ for $C(\mu, \hat{\rho})$, and thus

$$\int f \, d\nu - \int f \, d\mu = \left( \int f \, d\nu + \int g \, d\hat{\rho} \right) - \left( \int g \, d\hat{\rho} + \int f \, d\mu \right) \geq C(\nu, \hat{\rho}) - C(\mu, \hat{\rho}) \geq \inf_{\rho \in P_1(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)).$$

Taking the infimum over $f \in C^1(\mathbb{R}^d)$ shows

$$\inf_{f \in C^1(\mathbb{R}^d)} \left( \int f \, d\nu - \int f \, d\mu \right) \geq \inf_{\rho \in P_1(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)).$$

This concludes the proof. □

\footnote{We note that the result is stated under the additional requirement that $\nu, \rho \in P_2(\mathbb{R}^d)$. However as $\rho$ is supported on the unit ball, it can be checked that the arguments of [Villani, 2003, proof of Theorem 2.9] (in particular boundedness from below) carry over, when simply adding $|x|^2/2$ to $f$ and $|y|^2/2$ to $\hat{g}$.}
3.2. Proof of Theorem 1.2: the general case. We now prove Theorem 1.2 for general measures $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$ through approximation in the 1-Wasserstein sense.

Proof of Theorem 1.2. Let us take sequences of $(\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}$ in $\mathcal{P}_1(\mathbb{R}^d)$ satisfying
\[
\lim_{n \to \infty} W_1(\mu, \mu_n) = 0 = \lim_{n \to \infty} W_1(\nu, \nu_n), \quad \mu_n, \nu_n \sim \lambda \text{ for all } n \in \mathbb{N}.
\]
Recall that $C^1(\mathbb{R}^d)$ denotes the set of convex 1-Lipschitz functions. Thus, e.g. by the Kantorovich-Rubinstein formula ([Villani 2008, (5.11)]),
\[
\lim_{n \to \infty} \sup_{f \in C^1(\mathbb{R}^d)} \left| \int f \, d\mu - \int f \, d\mu_n \right| \leq \lim_{n \to \infty} W_1(\mu, \mu_n) = 0.
\]
(8)
The same holds for $(\nu_n)_{n \in \mathbb{N}}$ and $\nu$. Next, take an optimal coupling $\pi = \pi(x, dy)$ for $C(\nu, \rho)$ and an optimal coupling $\hat{\pi}^n = \hat{\pi}^n(x, dz)$ for $W_1(\nu, \nu_n)$. Then $\hat{\pi}^n(dy, dz) := \int \pi^n(dx, dz) \pi_x(dy)$ is a coupling of $\rho$ and $\nu_n$. Furthermore, as $|y| \leq 1 \rho$-a.s. we have
\[
C(\nu, \rho) - C(\nu_n, \rho) \leq \left| \int y(x-z) \pi^n(dx, dz) \pi_x(dy) \right| \\
\leq \int |x-z| \pi^n(dx, dz) \\
\leq W_1(\nu_n, \nu).
\]
Exchanging the roles of $\nu$ and $\nu_n$ then yields
\[
|C(\nu, \rho) - C(\nu_n, \rho)| \leq W_1(\nu_n, \nu).
\]
As the rhs is independent of $\rho \in \mathcal{P}_1(\mathbb{R}^d)$ this shows
\[
\lim_{n \to \infty} \sup_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} |C(\nu, \rho) - C(\nu_n, \rho)| = 0.
\]
(9) A similar argument holds for $(\mu_n)_{n \in \mathbb{N}}$ and $\mu$. We can now write
\[
\inf_{f \in C^1(\mathbb{R}^d)} \left( \int f \, d\nu - \int f \, d\mu \right) = \inf_{f \in C^1(\mathbb{R}^d)} \left[ \left( \int f \, d\nu_n - \int f \, d\mu_n \right) + \left( \int f \, d\nu - \int f \, d\nu_n \right) - \left( \int f \, d\mu - \int f \, d\mu_n \right) \right]
\]
and
\[
\inf_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \left( C(\nu, \rho) - C(\mu, \rho) \right) = \inf_{\rho \in \mathcal{P}_1(\mathbb{R}^d)} \left[ \left( C(\nu_n, \rho) - C(\mu_n, \rho) \right) + \left( C(\nu, \rho) - C(\nu_n, \rho) \right) - \left( C(\mu, \rho) - C(\mu_n, \rho) \right) \right].
\]
Applying Proposition 3.1, taking $n \to \infty$ and using (8), (9) then concludes the proof.
3.3. Proof of Theorem 1.1. We now detail the proof of Theorem 1.1. We start with a preliminary result, which is an immediately corollary of Theorem 1.2.

**Corollary 3.2.** Assume $\mu, \nu \in P_1(\mathbb{R}^d)$. Then we have

$$\inf_{f \text{ convex}} \int f \, d\nu - \int f \, d\mu = \inf_{\rho \in P_\infty(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)),$$

where $P_\infty(\mathbb{R}^d)$ denotes the set of probability measures with bounded support. In particular

$$\int d\mu \leq \int d\nu \quad \text{for all convex functions } f : \mathbb{R}^d \to \mathbb{R}$$

if and only if

$$C(\mu, \rho) \leq C(\nu, \rho) \quad \text{for all } \rho \in P_\infty(\mathbb{R}^d).$$

**Proof.** Multiplying both sides of (2) by $k > 0$ yields

$$\inf_{f \in C^k(\mathbb{R}^d)} \int f \, d\nu - \int f \, d\mu = \inf_{\rho \in P^k(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho))$$

with the definitions

$$P^k(\mathbb{R}^d) = \{\rho \in P(\mathbb{R}^d) : \text{supp}(\rho) \subseteq B_k(0)\}$$

and

$$C^k(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \text{ convex, } \|\partial f\|_\infty \leq k\}.$$

Taking $k \to \infty$ we obtain

$$\inf_{f \text{ convex, Lipschitz}} \int f \, d\nu - \int f \, d\mu = \inf_{\rho \in P_\infty(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)).$$

Lastly, any convex function $f : \mathbb{R}^d \to \mathbb{R}$ can be approximated pointwise from below by convex Lipschitz functions. Thus

$$\inf_{f \text{ convex}} \int f \, d\nu - \int f \, d\mu = \inf_{\rho \in P_\infty(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)).$$

The claim thus follows. \qed

**Remark 3.3.** If $\mu, \nu \in P_p(\mathbb{R}^d)$ for some $p \geq 1$, then by Hölder’s inequality and density of finitely supported measures in the $q$-Wasserstein space we also obtain

$$\inf_{f \text{ convex}} \int f \, d\nu - \int f \, d\mu = \inf_{\rho \in P_q(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)),$$

where $1/p + 1/q = 1$.

**Proof of Theorem 1.1.** Recall from (4) that

$$C(\mu, \rho) = \frac{1}{2} \left( \int |x|^2 \, \mu(dx) + \int |z|^2 \, \rho(dz) - \mathcal{W}_2(\mu, \rho)^2 \right),$$

$$C(\nu, \rho) = \frac{1}{2} \left( \int |y|^2 \, \nu(dy) + \int |z|^2 \, \rho(dz) - \mathcal{W}_2(\nu, \rho)^2 \right).$$
Combining this with (10) from Corollary 3.2 yields
\[ \inf_{f \text{ convex}} \left( \int f \, d\nu - \int f \, d\mu \right) = \inf_{\rho \in \mathcal{P}_{\infty}(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)) \]
\[ = \frac{1}{2} \inf_{\rho \in \mathcal{P}_{\infty}(\mathbb{R}^d)} \left( \int |y|^2 \nu(dy) + \int |z|^2 \rho(dz) - W_2(\nu, \rho)^2 \right. \]
\[ \left. - \int |x|^2 \mu(dx) - \int |z|^2 \rho(dz) + W_2(\mu, \rho)^2 \right) \]
\[ = \frac{1}{2} \inf_{\rho \in \mathcal{P}_{\infty}(\mathbb{R}^d)} \left( W_2(\mu, \rho)^2 - W_2(\nu, \rho)^2 + \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx) \right). \]

Thus
\[ \int f \, d\mu \leq \int f \, d\nu \quad \text{for all convex functions } f : \mathbb{R}^d \to \mathbb{R} \]
\[ \iff \inf_{f \text{ convex}} \left( \int f \, d\nu - \int f \, d\mu \right) \geq 0 \]
\[ \iff \inf_{\rho \in \mathcal{P}_{\infty}(\mathbb{R}^d)} \left( W_2(\mu, \rho)^2 - W_2(\nu, \rho)^2 \right) \geq \int |x|^2 \mu(dx) - \int |y|^2 \nu(dy) \]
\[ \iff \sup_{\rho \in \mathcal{P}_{\infty}(\mathbb{R}^d)} \left( W_2(\nu, \rho)^2 - W_2(\mu, \rho)^2 \right) \leq \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx). \]

The claim follows. \( \square \)

4. NUMERICAL EXAMPLES

In this section we illustrate Theorem 1.1 numerically. We focus on the following toy examples, where convex order or its absence is easy to establish:

**Example 4.1.** \( \mu = N(0, \sigma^2 I) \) and \( \nu = N(0, I) \) for \( \sigma^2 \in [0, 2] \) for \( d = 1, 2 \).

**Example 4.2.** \( \mu = \frac{1}{2} \left( \delta_{-1-s} + \delta_{1+s} \right) \) and \( \nu = \frac{1}{2} \left( \delta_{-1} + \delta_1 \right) \) for \( s \in [-1, 1] \).

**Example 4.3.**
\[ \mu = \frac{1}{4} \left( \delta_{-1-s,0} + \delta_{1+s,0} + \delta_{0,1+s} + \delta_{0,-1-s} \right) \]
and
\[ \nu = \frac{1}{4} \left( \delta_{-1,0} + \delta_{1,0} + \delta_{0,1} + \delta_{0,-1} \right) \]
for \( s \in [-1, 1] \).

A general numerical implementation for testing convex order of the two measures \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \) in general dimensions and the examples discussed here can be found in the Github repository [https://github.com/johanneswiesel/Convex-Order](https://github.com/johanneswiesel/Convex-Order). In the implementation we use the POT package [https://pythonot.github.io](https://pythonot.github.io) to compute optimal transport distances.

We set
\[ V(\mu, \nu) := \inf_{\rho \in \mathcal{P}^1(\mathbb{R}^d)} (C(\nu, \rho) - C(\mu, \rho)) \]
and note that by Theorem 1.2 we have the relationship
\[ \mu \leq_c \nu \iff V(\mu, \nu) \leq 0. \]
Clearly the numerical performance of $V(\mu, \nu)$ hinges on the numerical exploration of the convex set $\mathcal{P}^1(\mathbb{R}^d)$. We offer four different methods for this:

- **Indirect Dirichlet with histograms**: draw a Dirichlet random variable to select one probability measure $\rho \in \mathcal{P}^1(\mathbb{R}^d)$ supported on a grid of $B_1(0)$. Use the histograms of $\mu, \nu, \rho$ to compute $V(\mu, \nu)$.
- **Indirect Dirichlet with samples**: draw a Dirichlet random variable to select one probability measure $\rho \in \mathcal{P}^1(\mathbb{R}^d)$ supported on a grid of $B_1(0)$. Use samples of $\mu, \nu, \rho$ to compute $V(\mu, \nu)$.
- **Direct Dirichlet**: draw (signed) Dirichlet samples. Use these and samples of $\mu, \nu$ to compute $V(\mu, \nu)$.
- **Direct randomized Dirichlet**: draw (signed) Dirichlet samples and randomize. Use these and samples of $\mu, \nu$ to compute $V(\mu, \nu)$.

We refer to the github repository for a more detailed discussion. For each example, and each pair $(\mu, \nu)$ we plot $V(\mu, \nu)$ for these four methods, see Figures 1 and 2.

**Figure 1.** Values of different estimators of $V(\mu, \nu)$ for Example 4.1. Both plots use $N = 100$ samples and show the average over 10 different draws (solid line) together with 95% confidence intervals (light areas).

**Figure 2.** Values of different estimators of $V(\mu, \nu)$ for Example 4.2 (left) and 4.3 (right). Both plots use $N = 100$ samples and show the average over 10 different draws (solid line) together with 95% confidence intervals (light areas).
Discounting the numerical errors, all estimators seem to detect convex order. The “Direct Dirichlet” method however does not seem to explore the \( \mathcal{P}^1(\mathbb{R}^d) \)-space very well in the case that \( \mu \not\asymp c \nu \), as the value of \( V(\mu, \nu) \) stays close to zero then. This is why we also implemented the “Direct randomized Dirichlet” method, which shows an improvement in this regard. When histograms can be easily computed for \( \mu, \nu \) and \( \rho \), then this seems to reduce the numerical error significantly and yields the most convincing results. However, working on samples directly might more convenient for practical applications on real data.

As can be expected from the numerical implementation, the histogram method consistently yields the lowest runtimes, while runtimes of the other three methods are much higher. Indeed, when working with samples, the weights \( \mu, \rho, \nu \) are constant, while the OT cost matrices \( M_\mu \) and \( M_\nu \) in the implementation have to be recomputed in each iteration and this is very costly; for histograms the weights \( \rho \) change, while the grid and thus \( M_\mu \) and \( M_\nu \) stay constant.

5. Remaining proofs

Proof of Corollary 2.3. Recall that a function \( g : \mathbb{R}^d \to \mathbb{R} \) is \( c \)-concave, iff \( f(x) := |x|^2/2 - g(x) \) is convex. In particular

\[
\int g \, d\mu - \int g \, d\nu = \left( -\int \left[ \frac{|x|^2}{2} - g(x) \right] \mu(dx) + \int \left[ \frac{|y|^2}{2} - g(y) \right] \nu(dy) \right) \\
+ \int \frac{|x|^2}{2} \mu(dx) - \int \frac{|y|^2}{2} \nu(dy) \\
= \int f \, d\nu - \int f \, d\mu + \int \frac{|x|^2}{2} \mu(dx) - \int \frac{|y|^2}{2} \nu(dy).
\]

By \( \text{[10]} \) we obtain

\[
\inf_{g \text{ \text{-concave}}} \left( \int g \, d\mu - \int g \, d\nu \right) \\
= \inf_{f \text{ \text{convex}}} \left( \int f \, d\nu - \int f \, d\mu \right) + \int \frac{|x|^2}{2} \mu(dx) - \int \frac{|y|^2}{2} \nu(dy) \\
= \frac{1}{2} \inf_{\rho \in \mathcal{P}^\infty(\mathbb{R}^d)} \left( W_2^2(\mu, \rho) - W_2^2(\nu, \rho) + \int |y|^2 \nu(dy) - \int |x|^2 \mu(dx) \right) \\
+ \int \frac{|x|^2}{2} \mu(dx) - \int \frac{|y|^2}{2} \nu(dy) \\
= \frac{1}{2} \inf_{\rho \in \mathcal{P}^\infty(\mathbb{R}^d)} \left( W_2(\mu, \rho)^2 - W_2(\nu, \rho)^2 \right).
\]

This concludes the proof. \( \square \)

Proof of Corollary 2.4. The first claim follows from Theorem 1.1 by setting \( \rho = \mu \). By \( \text{[1]} \) the above implies

\[
2 \int |x|^2 \mu(dx) \leq 2 \sup_{\pi \in \Pi(\mu, \nu)} \int (x,y) \pi(dx,dy),
\]

so the second claim follows. \( \square \)
Proof of Corollary 2.7. First, [Wang et al., 2020, Theorem 2 & Lemma 1] show that \( \mu \preceq_c \nu \) iff

\[
\int_0^1 [F_\nu^{-1}(1-u) - F_\mu^{-1}(1-u)] \, dh(u) \geq 0
\]  

(11)

for all concave functions \( h \) such that the above integral is finite. As any concave function is Lebesgue-almost surely differentiable, standard approximation arguments imply that (11) holds iff

\[
\int_0^1 g(u)[F_\nu^{-1}(u) - F_\mu^{-1}(u)] \, du \geq 0
\]

for all bounded increasing left-continuous functions \( g : (0,1) \to \mathbb{R} \). But \( \{F_\rho^{-1} : \rho \in \mathcal{P}(\mathbb{R}) \text{ with bounded support} \} \) is exactly the set of all bounded increasing left-continuous functions on \((0,1)\). Noting that by [Villani, 2003, Equation (2.47)]

\[
\mathcal{W}_2(\nu,\rho)^2 = \int_0^1 (F_\nu^{-1}(x) - F_\rho^{-1}(x))^2 \, dx
\]

we calculate

\[
\mathcal{W}_2(\nu,\rho)^2 - \mathcal{W}_2(\mu,\rho)^2 = \int y^2 \nu(dy) - 2 \int_0^1 F_\nu^{-1}(u)F_\rho^{-1}(u) \, du + \int z^2 \rho(dz)
\]

\[
- \int x^2 \mu(dy) + 2 \int_0^1 F_\rho^{-1}(u)F_\mu^{-1}(u) \, du - \int z^2 \rho(dz)
\]

\[
= 2 \int_0^1 F_\rho^{-1}(u)[F_\mu^{-1}(u) - F_\nu^{-1}(u)] \, du
\]

\[
+ \int y^2 \nu(dy) - \int z^2 \mu(dy).
\]

This concludes the proof. \( \square \)

References

A. Alfonsi and B. Jourdain. Squared quadratic Wasserstein distance: optimal couplings and Lions differentiability. *ESAIM Prob. Stat.*, 24:703–717, 2020.

A. Alfonsi, J. Corbetta, and B. Jourdain. Sampling of one-dimensional probability measures in the convex order and computation of robust option price bounds. *Int. J. Theor. Appl. Finance*, 22(3), 2019.

A. Alfonsi, J. Corbetta, and B. Jourdain. Sampling of probability measures in the convex order by Wasserstein projection. *Ann. Henri Poincare*, 56(3):1706–1729, 2020.

B. Arnold. *Majorization and the Lorenz order: A brief introduction*, volume 43. Springer Science & Business Media, 2012.

M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices—a mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.

M. Beiglböck, M. Nutz, and N. Touzi. Complete duality for martingale optimal transport on the line. *Ann. Prob.*, 45(5):3038–3074, 2015.
C. Bernard, L. Rüschendorf, and S. Vanduffel. Value-at-risk bounds with variance constraints. *J. Risk Insur.*, 84(3):923–959, 2017.

Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

H. De March and N. Touzi. Irreducible convex paving for decomposition of multi-dimensional martingale transport plans. *Ann. Prob.*, 47(3):1726–1774, 2019.

P. Embrechts, G. Puccetti, and L. Rüschendorf. Model uncertainty and variance aggregation. *J. Bank. Financ.*, 37(8):2750–2764, 2013.

A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Prob.*, 24(1):312–336, 2014.

N. Gozlan and N. Juillet. On a mixture of Brenier and Strassen theorems. *Proc. Lond. Math. Soc.*, 120(3):434–463, 2020.

Gaoyue Guo and Jan Obłój. Computational methods for martingale optimal transport problems. *The Annals of Applied Probability*, 29(6):3311–3347, 2019.

B. Jourdain and W. Margheriti. Martingale Wasserstein inequality for probability measures in the convex order. *Bernoulli*, 28(2):830–858, 2022.

L. Kantorovich. On the translocation of masses. *Manag. Sci.*, (5):1–4, 1958.

M. Massa and P. Siorpaes. How to quantise probabilities while preserving their convex order. *arXiv preprint arXiv:2206.10514*, 2022.

G. Monge. *Mémoire sur la théorie des déblais et des remblais*. De l’Imprimerie Royale, 1781.

A. Müller and D. Stoyan. *Comparison methods for stochastic models and risks*, volume 389. Wiley, 2002.

M. Nendel. A note on stochastic dominance, uniform integrability and lattice properties. *Bull. Lond. Math. Soc.*, 52(5):907–923, 2020.

J. Obłój and P. Siorpaes. Structure of martingale transports in finite dimensions. *arXiv preprint arXiv:1702.08433*, 2017.

G. Peyré and M. Cuturi. Computational optimal transport: With applications to data science. *Foundations and Trends® in Machine Learning*, 11(5-6):355–607, 2019.

S. Rachev and L. Rüschendorf. *Mass Transportation Problems: Volume I: Theory*, volume 1. Springer Science & Business Media, 1998.

S. M Ross, J. Kelly, R. Sullivan, W. Perry, D. Mercer, R. Davis, T. Washburn, E. Sager, J. Boyce, and V. Bristow. *Stochastic processes*, volume 2. Wiley New York, 1996.

L. Rüschendorf and S. Rachev. A characterization of random variables with minimum L2-distance. *J. Multivariate Anal.*, 32(1):48–54, 1990.

L. Rüschendorf and L. Uckelmann. Variance minimization and random variables with constant sum. In et al. Cuadras, editor, *Distributions with given marginals and statistical modelling*, pages 211–222. Springer, 2002.

M. Shaked and J. Shanthikumar. *Stochastic orders*. Springer, 2007.

V. Strassen. The existence of probability measures with given marginals. *Ann. Math. Statist.*, pages 423–439, 1965.

A Tchen. Inequalities for distributions with given marginals. *Ann. Prob.*, pages 814–827, 1980.

C. Villani. *Topics in optimal transportation*. Number 58. American Mathematical Soc., 2003.
C. Villani. *Optimal transport: old and new*, volume 338. Springer Berlin, 2008.

B. Wang and R. Wang. The complete mixability and convex minimization problems with monotone marginal densities. *J. Multivariate Anal.*, 102(10):1344–1360, 2011.

Q. Wang, R. Wang, and Y. Wei. Distortion riskmetrics on general spaces. *Astin Bull.*, 50(3):827–851, 2020.

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