On finite-dimensional representations of two-parameter quantum affine algebras

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Abstract. We introduce the Drinfeld polynomial for each weight of two-parameter quantum affine algebras and establish a one-to-one correspondence between finite irreducible representations and sets of l-tuples of pairs of polynomials with certain conditions.

1. Introduction

Quantum groups were introduced independently by V. G. Drinfeld and M. Jimbo in 1985 using Chevalley generators and Serre relations. A few years later, Drinfeld found the second definition of quantum affine algebras and Yangians in terms of root vectors and introduced certain polynomials to characterize finite dimensional irreducible representations of Yangians. He showed that the Yangian representations are finite dimensional if and only if their Drinfeld polynomials are of certain form. Later Chari and Pressley generalized the notion of Drinfeld polynomials to quantum affine algebras and proved similar results for finite dimensional representations of both untwisted and twisted quantum enveloping algebras. Drinfeld polynomials are also related to Frenkel-Rechetikhin characters of finite dimensional irreducible representations.

Two-parameter quantum enveloping algebras are generalization of (one-parameter) quantum enveloping algebras, originally introduced as generalization of Hopf algebras and have close connections with Yang-Baxter equations. They were first defined for finite types using specific forms of Cartan matrices and have close connections with Yang-Baxter equations. As in the one-parameter cases various combinatorial realizations for 2-parameter quantum affine algebras were given in the context of McKay correspondence. As in the one-parameter cases various combinatorial realizations for 2-parameter quantum affine algebras were given in the context of McKay correspondence.

Motivated by Drinfeld’s and Chari-Pressley’s work, we study finite dimensional representations of two-parameter quantum affine algebras in this paper. We introduce Drinfeld polynomials for each finite dimensional irreducible representations and show that they are again characterized by Drinfeld polynomials. Although the
theory is much expected, there are some new features in two-parameter situation. The most notable one is that there are in fact a pair of related two-parameter Drinfeld polynomials for each finite dimensional representation instead of one for the positive root vector and negative root vector. For completeness, we provide all proofs in the case of $\mathfrak{sl}_2$, in particular we elaborate more for the cases when two-parameter quantum enveloping algebras have distinct features. Just as in the usual case, the two-parameter cases are essentially in one-to-one correspondence with the one-parameter quantum groups.

This article is organized as follows. After a quick introduction of two-parameter quantum affine algebras in Section 2, we define the category of finite dimensional representations of 2-parameter quantum enveloping algebras in Section 3 and introduce the notion of Drinfeld polynomials. A general result is shown as in the one-parameter case. In Section 4 we study the case of $U_{r,s}(\mathfrak{sl}_2)$ in details and prove the existence of a pair of Drinfeld polynomials for each finite dimensional representation. Finally in Section 5 we discuss some specializations of two parameters.

### 2. Two-parameter quantum affine algebras

2.1 In this subsection, we recall the definition of two-parameter quantum algebras $U_{r,s}(\hat{\mathfrak{g}})$, developed in [HRZ], [HZ1] and [HZ2]. Here we let $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra.

Let $\mathbb{K} = \mathbb{Q}(r, s)$ be a field of rational functions with two indeterminates $r, s$. Let $\Phi$ be the finite root system of $\mathfrak{g}$ with $\Pi$, a base of simple roots, which is a subset of a Euclidean space $\mathbb{R}^n$ with an inner product $(\ , \ )$. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ denote an orthonormal basis of $\mathbb{R}^n$. Let $\delta$ denote the primitive imaginary root of the affine Lie algebra $\hat{\mathfrak{g}}$, and let $\theta$ be the highest root of the simple Lie algebra $\mathfrak{g}$. Define $\alpha_0 = \delta - \theta$, then $\Pi' = \{\alpha_i \mid i \in I_0\}$ is a basis of simple roots of the affine Lie algebra $\hat{\mathfrak{g}}$.

We recall that the quantum number in two parameters is defined by

\[ [n] = \frac{r^n - s^n}{r - s} = r^{n-1} + r^{n-2}s + \cdots + rs^{n-2} + s^{n-1}, \]

and the Guassian numbers are defined similarly.

**Definition 2.1.** ([HRZ], [HZ1], [HZ2]) Two-parameter quantum affine algebra $U_{r,s}(\hat{\mathfrak{g}})$ is the unital associative algebra over $\mathbb{K}$ generated by the elements $e_j, f_j, \omega_j^{\pm 1}, \omega_j'^{\pm 1} \ (j \in I_0 = \{0, \cdots, n\})$, $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}$, satisfying the following relations:

(R1) $\gamma^{\pm \frac{1}{2}}, \gamma'^{\pm \frac{1}{2}}$ are central with $\gamma \gamma' = (rs)^n$ such that $\omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1$, and $[\omega_i^{\pm 1}, \omega_j^{\pm 1}] = [\omega_i^{\pm 1}, \omega_j'^{\pm 1}] = [\omega_i'^{\pm 1}, \omega_j'^{\pm 1}] = 0$.

(R2) For $i, j \in I_0$,

\[
\omega_j e_i \omega_j^{-1} = (i,j) e_i, \quad \omega_j f_i \omega_j^{-1} = (i,j)^{-1} f_i, \quad \omega_j f_i \omega_j'^{-1} = (j,i)^{-1} f_i, \quad \omega_j' f_i \omega_j'^{-1} = (j,i) f_i.
\]

(R3) For $i, j \in I_0$, we have

\[ [e_i, f_j] = \frac{\delta_{ij}}{r - s} (\omega_i - \omega_j). \]
For any \( i \neq j \), we have the \((r, s)\)-Serre relations:

\[
(ad_{e_i})^{1-a_{ij}}(e_j) = 0,
\]
\[
(ad_{f_i})^{1-a_{ij}}(f_j) = 0,
\]

where the definitions of the left-adjoint action \( ad_{l_i} \) and the right-adjoint action \( ad_{r_i} \) are given in the following sense:

\[
ad_{l_i}(b) = \sum_{(a)} a_{(1)} b S(a_{(2)}), \quad \text{ad}_{r_i}(b) = \sum_{(a)} S(a_{(1)}) b a_{(2)}, \quad \forall a, b \in U_{r,s}(\hat{g}),
\]

where \( \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \) is given by proposition 2.3 below, and the structural constant \( \langle i, j \rangle \) is the \((i, j)\)-entry of the two-parameter quantum Cartan matrix \( J \), given as follows respectively.

For type \( A_n^{(1)} \) with \( n > 1 \),

\[
J = \begin{pmatrix}
rs^{-1} & r^{-1} & 1 & \cdots & 1 & s \\
s & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & rs^{-1} & r^{-1} \\
rs^{-1} & 1 & 1 & \cdots & s & rs^{-1}
\end{pmatrix}
\]

and \( n = 1 \),

\[
J = \begin{pmatrix}
rs^{-1} & r^{-1}s \\
r^{-1}s & rs^{-1}
\end{pmatrix}
\]

For type \( B_n^{(1)} \),

\[
J = \begin{pmatrix}
rs^{-1} & (rs)^{-1} & r^{-1} & \cdots & 1 & rs \\
rs & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & rs^{-1} & r^{-1} \\
(rs)^{-1} & 1 & 1 & \cdots & s & r^\frac{1}{2}s^{-\frac{1}{2}}
\end{pmatrix}
\]

For type \( C_n^{(1)} \),

\[
J = \begin{pmatrix}
rs^{-1} & r^{-1} & 1 & \cdots & 1 & rs \\
s & r^\frac{1}{2}s^{-\frac{1}{2}} & r^{-\frac{1}{2}} & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & r^\frac{1}{2}s^{-\frac{1}{2}} & r^{-1} \\
(rs)^{-1} & 1 & 1 & \cdots & s & rs^{-1}
\end{pmatrix}
\]

For type \( D_n^{(1)} \),

\[
J = \begin{pmatrix}
rs^{-1} & (rs)^{-1} & r^{-1} & \cdots & 1 & (rs)^2 \\
rs & rs^{-1} & r^{-1} & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & rs^{-1} & (rs)^{-1} \\
(rs)^{-2} & 1 & 1 & \cdots & rs & rs^{-1}
\end{pmatrix}
\]
For type $E_6^{(1)}$

$$J = \begin{pmatrix} rs^{-1} & (rs)^{-1} & r^{-2}s^{-1} & (rs)^{-1} & rs & rs & rs \\ rs & rs^{-1} & 1 & r^{-1} & 1 & 1 & 1 \\ rs^2 & 1 & rs^{-1} & 1 & r^{-1} & 1 & 1 \\ rs & s & 1 & rs^{-1} & r^{-1} & 1 & 1 \\ (rs)^{-1} & 1 & s & s & rs^{-1} & r^{-1} & 1 \\ (rs)^{-1} & 1 & 1 & 1 & s & rs^{-1} & r^{-1} \\ (rs)^{-1} & 1 & 1 & 1 & 1 & s & rs^{-1} \end{pmatrix}$$

For type $F_4^{(1)}$

$$J = \begin{pmatrix} rs^{-1} & r^{-2}s^{-1} & (rs)^{-1} & rs & rs \\ rs^2 & rs^{-1} & r^{-1} & 1 & 1 \\ rs & s & rs^{-1} & r^{-1} & 1 \\ (rs)^{-1} & 1 & s & r^2s^{-1} & r^{-\frac{1}{2}} \\ (rs)^{-1} & 1 & 1 & s^\frac{1}{2} & r^\frac{1}{2}s^{-\frac{1}{2}} \end{pmatrix}$$

For type $G_2^{(1)}$

$$J = \begin{pmatrix} rs^{-1} & r^{-2}s^{-1} & rs \\ rs^2 & rs^{-1} & r^{-1} \\ (rs)^{-1} & s & r^\frac{1}{2}s^{-\frac{1}{2}} \end{pmatrix}$$

Remark 2.2. In [JZ1, JZ2] the Fock space realization of two-parametr quantum affine algebras of types $A$ and $C$ were given in terms of Young tableaux. The combinatorial model shows that the parameters $r$ and $s$ correspond roughly the scalars in the insertion and removing operators. The following fact is straightforward.

Proposition 2.3. (HRZ, HZ1, HZ2) Two-parameter quantum affine algebra $U = U_{r,s}(\widehat{g})$ is a Hopf algebra with the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ defined below: for $i \in I_0$, we have

$$\Delta(\omega_i^\pm) = \omega_i^\pm \otimes \omega_i^\pm, \quad \Delta(\omega_i'^\pm) = \omega'_i^\pm \otimes \omega'_i^\pm,$$

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega'_i,$$

$$\Delta(\gamma^\pm) = \gamma^\pm \otimes \gamma^\pm, \quad \Delta(\gamma'^\pm) = \gamma'^\pm \otimes \gamma'^\pm,$$

$$\varepsilon(\omega_i^\pm) = \varepsilon(\omega'_i^\pm) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$\varepsilon(\gamma^\pm) = \varepsilon(\gamma'^\pm) = 1,$$

$$S(\gamma^\pm) = \gamma'^\mp, \quad S(\gamma'^\pm) = \gamma^\mp,$$

$$S(\omega_i^\pm) = \omega_i'^\mp, \quad S(\omega'_i^\pm) = \omega_i^\mp,$$

$$S(e_i) = -\omega_i'^{-1}e_i, \quad S(f_i) = -f_i \omega_i'^{-1}.$$
Definition 2.4. ([HRZ], [HZ1], [HZ2]) The unital associative algebra \( U_{r,s}(\hat{g}) \) over \( K \) is generated by the elements \( x^\pm_j(k), a_i(\ell), \omega_i^{\pm 1}, \omega_i', \gamma^\pm \), \( (i \in I = \{1, 2, \ldots, n\}), k, k' \in \mathbb{Z}, \ell, \ell' \in \mathbb{Z} \setminus \{0\} \), subject to the following defining relations:

(D1) \( \gamma^\pm, \gamma'^\pm \) are central such that \( \gamma' = (rs) \gamma \), \( \omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1 \), and for \( i, j \in I \), one has

\[
[\omega_i^{\pm 1}, \omega_j^{\pm 1}] = [\omega_i^{\prime \pm 1}, \omega_j^{\prime \pm 1}] = [\omega_i^{\prime \prime \pm 1}, \omega_j^{\prime \prime \pm 1}] = 0.
\]

(D2) \( [a_i(\ell), a_j(\ell')] = \delta_{\ell+\ell',0} \frac{(rs)^{ij} (r_i s_i)^{ij} [\ell a_{ij}]}{[\ell]} [\gamma^{|\ell|} - \gamma'^{|\ell|}] 
\)

(D3) \( [a_i(\ell), \omega_j^{\pm 1}] = [a_i(\ell), \omega_j'^{\pm 1}] = 0. \)

(D4) \( \omega_i x_j^+(k) \omega_i^{-1} = (\omega_j', \omega_i)^{\pm 1} x_j^+(k), \quad \omega_j' x_j^+(k) \omega_j'^{-1} = (\omega_i', \omega_j')^{\mp 1} x_j^+(k). \)

(D51) \( [a_i(\ell), x_j^{\pm}(k)] = \pm (rs)^{ji} \frac{a_{ij}}{\ell (r_i - s_i)} \omega^{\pm \ell} x_j^{\pm}(k+\ell), \quad \text{for } \ell > 0, \)

(D52) \( [a_i(\ell), x_j^{\pm}(k)] = \pm (rs)^{ji} \frac{a_{ij}}{\ell (r_i - s_i)} \omega^{\pm \ell} x_j^{\pm}(k-\ell), \quad \text{for } \ell < 0, \)

\[
x_i^{\pm}(k+1) x_j^{\pm}(k') - (j, i)^{\pm 1} x_j^{\pm}(k') x_i^{\pm}(k+1) = - \left( (j, i)^{(i, j)^{-1}} \right)^{\pm \frac{1}{2}} (x_j^{\pm}(k'+1) x_i^{\pm}(k) - (i, j)^{\pm 1} x_i^{\pm}(k) x_j^{\pm}(k'+1)).
\]

(D7) \( [x_i^{\pm}(k), x_j^{\pm}(k')] = \delta_{ij} \frac{\gamma^{i-k} \gamma^{-k+i'} \omega_i(k+k') - \gamma^{k-i'} \gamma^{-i+k'} \omega_i'(k+k')}{r_i - s_i} 
\)

where \( \omega_i(m), \omega'_i(-m) (m \in \mathbb{Z}_{\geq 0}) \) such that \( \omega_i(0) = \omega_i \) and \( \omega'_i(0) = \omega'_i \) are defined as below:

\[
\sum_{m=0}^{\infty} \omega_i(m) z^{-m} = \omega_i \exp \left( (r_i - s_i) \sum_{\ell=1}^{\infty} a_i(\ell) z^{-\ell} \right), \quad (\omega_i(-m) = 0, \forall m > 0); \\
\sum_{m=0}^{\infty} \omega'_i(-m) z^{-m} = \omega'_i \exp \left( -(r_i - s_i) \sum_{\ell=1}^{\infty} a_i(\ell) z^{\ell} \right), \quad (\omega'_i(m) = 0, \forall m > 0).
\]

(D81) \( x_i^{\pm}(m) x_j^{\pm}(k) = (j, i)^{\pm 1} x_j^{\pm}(k) x_i^{\pm}(m), \quad \text{for } a_{ij} = 0, \)

(D82) \( Sym_{m_1, \ldots, m_n} \sum_{k=0}^{n-1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}} \left[ x_i^{\pm}(m_1) \ldots x_i^{\pm}(m_k) x_j^{\pm}(\ell) \right] x_i^{\pm}(m_{k+1}) \ldots x_i^{\pm}(m_n) = 0, \quad \text{for } a_{ij} \neq 0, \quad 1 \leq j < i < n, \)

(D83) \( Sym_{m_1, \ldots, m_n} \sum_{k=0}^{n-1-a_{ij}} (-1)^k (r_i s_i)^{\frac{1}{2}} \left[ x_i^{\pm}(m_1) \ldots x_i^{\pm}(m_k) x_j^{\pm}(\ell) \right] x_i^{\pm}(m_{k+1}) \ldots x_i^{\pm}(m_n) = 0, \quad \text{for } a_{ij} \neq 0, \quad 1 \leq i < j < n, \)

where \([l]_{\pm i}, [l]_{\mp i}\) are defined the same as before by replacing \( r, s \) by \( r_i, s_i \) respectively, \( Sym_{m_1, \ldots, m_n} \) denotes symmetrization w.r.t. the indices \( (m_1, \ldots, m_n) \).
Similar to the classical case the Poincare- Birkhoff-Witt theorem also holds [HRZ].

Let \( U_{r,s}(n) \) denote the subalgebra of \( U_{r,s}(\mathfrak{n}) \), generated by \( x_i^+(0) \) (\( i \in I \)). By definition, it is clear that \( U_{r,s}(n) \cong U_{r,s}(n) \), the subalgebra of \( U_{r,s}(\mathfrak{g}) \) generated by \( e_i \) (\( i \in I \)).

The algebra \( U_{r,s}(\mathfrak{g}) \) has a triangular decomposition:

\[
U_{r,s}(\mathfrak{g}) = U_{r,s}(\mathfrak{n}^-) \oplus U_{r,s}^0(\mathfrak{g}) \oplus U_{r,s}(\mathfrak{n}^+),
\]

where \( U_{r,s}(\mathfrak{n}^\pm) = \bigoplus_{\alpha \in Q^\pm} U_{r,s}(\mathfrak{n}^\pm)_\alpha \) is generated respectively by \( x_i^\pm(k) \) (\( i \in I \)), and \( U_{r,s}^0(\mathfrak{g}) \) is the subalgebra generated by \( \omega_i^{\pm1}, \omega_i'^{\pm1}, \gamma_i^{\pm\frac{1}{2}}, \gamma_i'^{\pm\frac{1}{2}}, D^{\pm1}, D'^{\pm1} \) and \( a_i(\pm\ell) \) for \( i \in I, \ell \in \mathbb{N} \). Namely, \( U_{r,s}^0(\mathfrak{g}) \) is generated by the toral subalgebra \( U_{r,s}(\mathfrak{g})^0 \) and the quantum Heisenberg subalgebra \( \mathcal{H}_{r,s}(\mathfrak{g}) \) generated by the quantum imaginary root vectors \( a_i(\pm\ell) \) (\( i \in I, \ell \in \mathbb{N} \)).

2.3 In this subsection, we give some automorphisms of the two-parameter quantum affine algebras \( U_{r,s}(\mathfrak{g}) \). Their proofs are direct verification.

**Proposition 2.5.** For \( I \)-tuples \( \sigma = (\sigma_0, \ldots, \sigma_n) \in \{\pm1\}^{n+1} \), there exists an unique automorphism \( a_\sigma \) of two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{g}) \) such that

\[
\begin{align*}
  a_\sigma(\omega_i) &= \sigma_i \omega_i, \\
  a_\sigma(\omega_i') &= \sigma_i \omega_i', \\
  a_\sigma(e_i) &= \sigma_i e_i, \\
  a_\sigma(f_i) &= f_i.
\end{align*}
\]

**Proposition 2.6.** There exists an automorphism \( \Gamma_1 \) of two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{g}) \) such that

\[
\begin{align*}
  \Gamma_1(\gamma^{\pm\frac{1}{2}}) &= -\gamma^{\pm\frac{1}{2}}, \\
  \Gamma_1(\omega_i^+(k)) &= (-1)^k \omega_i^+(k), \\
  \Gamma_1(\omega_i) &= \omega_i, \\
  \Gamma_1(\omega_i') &= \omega_i'.
\end{align*}
\]

**Proposition 2.7.** For \( a \in \mathbb{C}^* \), there exists an unique automorphism \( \Gamma_2 \) of two-parameter quantum affine algebra \( U_{r,s}(\mathfrak{g}) \) such that

\[
\begin{align*}
  \Gamma_2(\gamma^{\pm\frac{1}{2}}) &= \gamma^{\pm\frac{1}{2}}, \\
  \Gamma_2(\omega_i^+(k)) &= a^k \omega_i^+(k), \\
  \Gamma_2(\omega_i) &= a^k \omega_i, \\
  \Gamma_2(\omega_i') &= a^k \omega_i'.
\end{align*}
\]

3. Finite-dimensional representations of \( U_{r,s}(\mathfrak{g}) \)

3.1 In this subsection, we study finite-dimensional representation theory of \( U_{r,s}(\mathfrak{g}) \) analogue to the one-parameter situation (\( [\text{CPT}], [\text{CP3}] \)). For finite dimensional \( \mathfrak{g} \) the highest weight representations of \( U_{r,s}(\mathfrak{g}) \) have been discussed in [BW2] and [BGH2].

Let \( P \) be the weight lattice of the simple Lie algebra \( \mathfrak{g} \), and \( W \) be a representation of \( U_{r,s}(\mathfrak{g}) \). We say \( \lambda \in P \) is a weight of \( W \), if the weight space

\[
W_\lambda = \{ w \in W | \omega_i w = \langle \omega_i, \omega_i \rangle w, \omega_i w = \langle \omega_i, \omega_i \rangle^{-1} w, \forall i \neq 0 \}
\]

Here we have extended the definition of \( \langle \lambda, \rho \rangle \) from \( \lambda \in Q \) to \( \lambda \in P \) via appropriate half-integer powers when necessary. A representation \( W \) of \( U_{r,s}(\mathfrak{g}) \) is said to be of type 1, if it is the direct sum of its weight spaces, that is,

\[
W = \bigoplus_{\lambda \in \text{wt}(W)} W_\lambda.
\]
where \( wt(W) \) is the set of weight of \( W \).

A non-zero vector \( w \in W_\lambda \) is called a highest weight vector if \( e_i \cdot w = 0 \) for all \( i \in I \), and \( W \) is a highest weight representation with highest weight \( \lambda \) if \( W = U_{r,s}(\hat{g})w \) for some highest weight vector \( w \in W_\lambda \). Any highest weight representation is of type 1.

We now turn to the representation theory of \( U_{r,s}(\hat{g}) \). A representation \( V \) of \( U_{r,s}(\hat{g}) \) is of type 1 if \( \gamma^{1/2} \) and \( \gamma^{-1/2} \) act as the identity on \( V \), and if \( V \) is of type 1 as a representation of \( U_{r,s}(\hat{g}) \). A vector \( v \in V \) is a highest weight vector if

\[
x_i^+(k) \cdot v = 0, \omega_i(m) \cdot v = \Phi_i^+ m v, \omega_i(-m) \cdot v = \Phi_i^- m v, \gamma^{1/2} \cdot v = v, \gamma^{-1/2} \cdot v = v,
\]

for some complex number \( \Phi_i^\pm m \). A type 1 representation \( V \) is a highest weight representation if \( V = U_{r,s}(\hat{g})v \) for some highest weight vector \( v \), and let \( \epsilon = \pm = \pm 1 \), the pair of \( (I \times \mathbb{Z}) \)-tuples \((\Phi_i^\pm m)_{i \in I, m \in \mathbb{Z}^\geq 0}\) is called the highest weight of \( V \).

The following result can be proved similarly as in the one-parameter case. In fact they are exactly proved as in the classical cases (cf. [CP1, CP3]).

**Proposition 3.1.** If \( V \) is a finite dimensional irreducible representation of \( U_{r,s}(\hat{g}) \), then

1. \( V \) can be obtained from a type 1 representation by twisting with a product of an automorphism \( a_m \).
2. If \( V \) is of type 1, then \( V \) is a highest weight module.

### 3.2 In this subsection, we give the the main results on the finite-dimensional representation of \( U_{r,s}(\hat{g}) \). Though the results are similar to the one-parameter case, they also carry some different aspects for the two-parameter case.

If \( \lambda \in P^+ \), let \( P^\lambda \) be the set of \( I \)-tuples \((P_i)_{i \in I}\) of polynomials \( P_i \in \mathbb{C}[u] \), with constant term 1, such that \( deg(P_i) = \lambda(i) \) for all \( i \in I \). Set \( \mathcal{P} = \bigcup_{\lambda \in P^+} \mathcal{P}^\lambda \).

**Theorem 3.2.** Let \( \Phi^\epsilon = (\Phi_i^\epsilon m)_{i \in I, m \in \mathbb{Z}} \) be a pair of \( (I \times \mathbb{Z}) \)-tuples of complex numbers, then the irreducible representation \( V(\Phi, \Psi) \) of \( U_{r,s}(\hat{g}) \) is finite-dimensional if and only if there exists \( \mathcal{P} = (P_i)_{i \in I} \in \mathcal{P} \) such that

\[
\sum_{m=0}^{\infty} \Phi_i^\epsilon m z^m = r_1^{deg(P_i)} P_i((rs)^{-deg(P_i)} s_{1,2}) P_i((rs)^{-deg(P_i)} r_{1,2})
\]

in the sense that the positive and the negative terms are the Laurent expansions of the middle term about 0 and \( \infty \), respectively.

**Proof.** We will discuss the case of \( U_{r,s}(\hat{s}l_2) \) in next section. Assuming the result is true for \( U_{r,s}(\hat{s}l_2) \), we can argue the general case easily. First of all, the necessary condition (the “only if” part) is trivial. Now suppose that there exists a family of polynomials \( \mathcal{P} = (P_i)_{i \in I} \in \mathcal{P} \) satisfying the condition in the theorem.

For each \( i \in I \), as a module for the \( i \)th copy \( U_{r,s}(\hat{s}l_2) \), the existence of Drinfeld polynomials implies that for each \( i \in I \) the span of irreducible \( i \)th \( U_{r,s}(\hat{s}l_2) \)-module is finite dimensional. However, as in the case of quantum affine algebras, the union of these spans are exactly the whole module for \( U_{r,s}(\hat{g}) \), so it is also finite dimensional.

Assigning to \( V \) the \( n \)-tuple \( P \) defines a bijection between the set of isomorphism classes of finite dimensional irreducible representation of \( U_{r,s}(\hat{g}) \) of type 1 and \( P \).
We denote the finite-dimensional irreducible representation of $U_r,s(\mathfrak{g})$ associated to $P$ by $V(P)$, and we will simply say that $P$ its highest weight.

**Remark 3.3.** For each $i \in I$ and $a \in \mathbb{C}^*$, we can define the irreducible representation $V_{a_i}(a)$ as $V(P_a^{(i)})$, where $P_a^{(i)} = (P_1, \ldots, P_n)$ is the $n$-tuple of polynomials, such that $P_i(u) = 1 - au, P_j(u) = 1,$ for all $j \neq i$. We call $V_{a_i}(a)$ the $i$th fundamental representation of $U_{r,s}(\mathfrak{g})$. As we will see in the following they constitute the building blocks for general finite dimensional modules.

**Theorem 3.4.** Let $P, Q \in \mathcal{P}$ be as above, Let $v_P$ and $v_Q$ be highest weight vector of $V(P)$ and $V(Q)$, respectively. If $P \otimes Q$ denotes the $I$-tuple $(P_iQ_i)_{i \in I}$, Then $V(P \otimes Q)$ is isomorphic to a quotient of the submodule of $V(P) \otimes V(Q)$ generated by the tensor product of the highest weight vectors $v_P$ and $v_Q$.

**Proof.** This proof is almost the same as one-parameter case. By abuse of notation, if $P = (P_i)_{i \in I}, Q = (Q_i)_{i \in I}$, we denote $P \otimes Q \in \mathcal{P}$ be the $I$-tuple $(P_iQ_i)_{i \in I}$. Then in $V(P) \otimes V(Q)$, for all $i \in I, k, m \in \mathbb{Z}$, we have

\[
\begin{aligned}
  s_i^+(k) \cdot (v_P \otimes v_Q) &= 0, \\
  \omega_i(m) \cdot (v_P \otimes v_Q) = \Phi_i^{+}(v_P \otimes v_Q), \\
  \omega_i^*(m) \cdot (v_P \otimes v_Q) = \Phi_i^{-}(v_P \otimes v_Q).
\end{aligned}
\]

Thus it is an direct consequence of the above arguments. \hfill \Box

The following result is an immediate consequence of Theorem 3.3.

**Corollary 3.5.** Any finite dimensional irreducible module of $U_{r,s}(\mathfrak{g})$ of type 1 is isomorphic to an quotient of the submodule of a tensor product of fundamental representations.

**3.3** Similar to the one-parameter case, for any representation $V$ of $U_{r,s}(\mathfrak{g})$, we can decompose $V$ into a direct sum $V = \oplus V_{\gamma_{i,m}}$, where

\[
V_{\gamma_{i,m}} = \{ x \in V | (\Phi_i^{+} - \gamma_{i,m})^p \cdot x = 0, \text{ for some } p, \forall i, m \}.
\]

Given a collection $(\gamma_{i,m}^{\pm})$ of eigenvalues, we form the generating functions

\[
\gamma_{i,m}^{\pm}(u) = \sum_{m>0} \gamma_{i,m}^{\pm} u^{\pm m}.
\]

The following result generalizes the corresponding result for one-parameter cases [FR] and also generalizes Theorem 3.2.

**Proposition 3.6.** The generating functions $\gamma_{i,m}^{\pm}(u)$ of eigenvalues on any finite dimensional representation of $U_{r,s}(\mathfrak{g})$ have the form

\[
\gamma_{i,m}^{\pm}(u) = \frac{deg R_i - deg Q_{i,m}}{\delta_i} \frac{deg Q_i}{\delta_i} \frac{R_i(u_{r_i})Q_i(u_{r_i})}{R_i(u_{r_i})Q_i(u_{r_i})} u^{\pm m}
\]

as elements of $\mathbb{C}[[u]]$ and $\mathbb{C}[[u^{-1}]]$, respectively, where $Q_i(u), R_i(u)$ are polynomials in $u$.

**Proof.** The case of $U_{r,s}(\mathfrak{sl}_2)$ will be discussed in next section. For the general case, one can view the module as a finite dimensional module for each $U_{r,s}(\mathfrak{sl}_2)$, the $i$th copy of the subalgebra of $U_{r,s}(\mathfrak{g})$. One sees that the generating function $\gamma_{i,m}^{\pm}(u)$ must be of the stated form. \hfill \Box
4. The case of \( U_{r,s}(\widehat{sl}_2) \)

4.1 We recall the evaluation map of the two-parameter quantum affine algebra \( U_{r,s}(\widehat{sl}_2) \) in this subsection and give necessary computations (cf. [ZP]). As in the one-parameter case, \( U_{r,s}(\widehat{sl}_2) \) plays a similar role for the general theory. This is in alignment with the general principle that \( sl(2) \) effectively controls the general Lie theory. For this reason, we will give detailed computation for the case of \( sl(2) \). Our computation reconfirms this principle for the general two-parameter quantum affine algebras.

**Proposition 4.1.** For any \( a \in \mathbb{C}^\ast \), there exists an algebra morphism \( ev_a \) from \( U_{r,s}(\widehat{sl}_2) \) to \( U_{r,s}(\widehat{sl}_2) \) defined as follows:

\[
ev_a(e_0) = r^{-1}a f, \quad ev_a(f_0) = rs^{-1}a^{-1}e, \quad ev_a(e_1) = e, \quad ev_a(f_1) = f,
\]

\[
ev_a(\omega_0) = \omega', \quad ev_a(\omega) = \omega, \quad ev_a(\omega''_0) = \omega, \quad ev_a(\omega'') = \omega'.
\]

Using the Drinfeld isomorphism theorem, we can lift the evaluation morphism \( ev_a \) to its Drinfeld realization \( U_{r,s}(\widehat{sl}_2) \), we also denote it by \( ev_a \) without ambiguity.

**Proposition 4.2.** There exists an algebra morphism \( ev_a \) from \( U_{r,s}(\widehat{sl}_2) \) to \( U_{r,s}(\widehat{sl}_2) \) defined as follows:

\[
ev_a(\gamma) = 1 = ev_a(\gamma'), \quad ev_a(\omega) = \omega, \quad ev_a(\omega') = \omega',
\]

\[
ev_a(x^+(k)) = r^{-k}s^ka^k\omega'u^{-k}e, \quad ev_a(x^-(k)) = e^{-k}s^ka^k\omega'v^{-k}e.
\]

Similar to the one-parameter case, we can define the evaluation representation of \( U_{r,s}(\widehat{sl}_2) \) as follows:

**Definition 4.3.** For \( a \in \mathbb{C}^\ast \) and \( n \in \mathbb{N} \), we call these \( V_n(a) \) evaluation representations of quantum affine algebra \( U_{r,s}(\widehat{sl}_2) \).

We recall the representation of \( U_{r,s}(\widehat{sl}_2) \), see [BW2] and [BGH2] for more detail. For the representation \( V_n \) of \( U_{r,s}(\widehat{sl}_2) \) of dimension \( n+1 \), there exists a basis \( v_0, v_1, \cdots, v_n \) of \( V_n \) such that

\[
\omega \cdot v_i = r^n(rs^{-1})^{-i}v_i, \quad \omega' \cdot v_i = s^n(rs^{-1})^i v_i,
\]

\[
e \cdot v_i = [n+1-i]v_{i-1}, \quad f \cdot v_i = [i+1]v_{i+1}.
\]

**Proposition 4.4.** As a vector space, \( V_n(a) \) is equal to \( V_n \), and the action of \( U_{r,s}(\widehat{sl}_2) \) is given by:

\[
x^+(k) \cdot v_i = a^ks^{-nk}(rs^{-1})^{-ki}[n+1-i]v_{i-1},
\]

\[
x^-(k) \cdot v_i = a^kr^{nk}(rs^{-1})^{-k(i+1)}[i+1]v_{i+1}.
\]

A vector \( v \in V \) is a highest weight vector if for \( l \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0} \), we have

\[
x^+(l) \cdot v = 0, \quad a(k) \cdot v = d^+_kv, \quad a(-k) \cdot v = d^-kv, \quad \gamma \cdot v = \gamma' \cdot v = 1
\]

for some complex numbers \( \Phi_0^+ \) and \( \Phi_0^- \). Note that \( \Phi_0^+ \Phi_0^- \) is the powers of \( rs \). We will show that evaluation representations \( V_n(a) \) for \( a \in \mathbb{C}^\ast \) are highest weight representations.
It is easy to see from the above proposition that \( v_0 \) is annihilated by all \( x^+(k) \). The actions of \( \omega_0 \) and \( \omega'_0 \) can be easily computed

\[
\omega_0 \cdot v_0 = r^n v_0, \quad \omega'_0 \cdot v_0 = s^n v_0.
\]

In general, we have

**Proposition 4.5.** All finite dimensional irreducible representations of \( U_{r,s}(\mathfrak{sl}_2) \) are highest weight representations.

**Theorem 4.6.** (1) Let \( V \) be a finite dimensional highest weight representation of \( U_{r,s}(\mathfrak{sl}_2) \), and as an evaluation module for \( U_{r,s}(\widehat{\mathfrak{sl}_2}) \) there exists a polynomial \( P(z) \in \mathbb{C}[z] \) such that \( P(0) \neq 0 \) and

\[
\sum_{k=0}^{\infty} \Phi_+^k z^k = r^{\text{deg} P(z)} P(sz) P(rz),
\]

\[
\sum_{k=0}^{\infty} \Phi_-^k z^{-k} = r^{\text{deg} P(z)} Q(sz) Q(rz),
\]

where \( Q(z) = P((rs)^{\text{deg} P(z)}) \).

(2) For any series of complex number \( \Phi = (\Phi_+^k, \Phi_-^k) \in \mathbb{N} \) such that \( \Phi_+^0 \Phi_-^0 = (rs)^n \) for some integer \( n \), there exists a finite dimensional irreducible highest weight module \( V(\Phi) \).

The polynomial \( P(z) \) in the theorem is called Drinfel’d polynomial. And the following proposition is an example the above theorem.

**Proposition 4.7.** For the evaluation representation \( V_n(a) \) of \( U_{r,s}(\mathfrak{sl}_2) \), we obtain the Drinfel’d polynomial as follows.

\[
P(z) = \sum_{k=1}^{n} (1 - ar^{-k-1}s^{k-n}z).
\]

**Proof.** We first have

\[
\Phi_+^k = (r-s)(ar^{-1}s^{1-n})^k[n],
\]

and

\[
\Phi_-^k = -(r-s)(a^{-1}r^{1-n}s^{-1})^k[n],
\]

where \( [n] = \frac{r^n - s^n}{r-s} \).

It is easy to see that

\[
\sum_{k=0}^{\infty} \Phi_+^k z^k = \frac{r^n + \sum_{k=1}^{n} (r^n - s^n) \frac{ar^{-1}s^{1-n}z}{1 - ar^{-1}s^{1-n}z}}{1 - ar^{-1}s^{1-n}z}.
\]

\[
= r^n \frac{1 - a(r^{-1}s)^{r-n}z}{1 - a(r^{-1}s)s^{-n}z}.
\]

\[
= r^n \frac{(1 - a(r^{-1}s)^2s^{-n}z)(1 - a(r^{-1}s)^3s^{-n}z) \cdots (1 - a(r^{-1}s)^{n+1}s^{-n}z)}{1 - a(r^{-1}s)^{n+1}s^{-n}z}.
\]

\[
= \frac{P(sz)}{P(rz)}.
\]
where \( P(z) = \sum_{k=1}^{n} (1 - a(r^{-1}s)^{k}r^{-1}s^{-n}z) \).

Similarly, we have

\[
\sum_{k=0}^{\infty} \Phi_{-k}z^{-k} = r^{\deg p} \frac{Q(sz)}{Q(rz)},
\]

where \( Q(z) = \sum_{k=1}^{n} (1 - a(r^{-1}s)^{k}r^{-1}z) = P((rs)^{n}z). \)

\(\square\)

The following proposition gives the Drinfeld polynomials for all evaluation representations of two-parameter quantum affine algebra \( \mathcal{U}_{r,s}(\widehat{\mathfrak{sl}_2}) \).

**Proposition 4.8.** For the evaluation representation \( W_{n}(a) = V_{n}(rs^{-1}a) \) of two-parameter quantum affine algebra \( \mathcal{U}_{r,s}(\widehat{\mathfrak{sl}_2}) \), we can obtain the eigenvalues \( \Phi_{k,j} \) in the above proposition 3.6. as follows:

\[
\sum_{m>0} \Phi_{i,\pm m} u^{\pm m} = r^{\deg R_{i}} \frac{\frac{d\deg Q_{i}}{s} R_{i}(u) Q_{i}(ur)}{R_{i}(ur) Q_{i}(us)}
\]

where

\[
Q_{i}(z) = \prod_{j=1}^{i} (1 - ar^{-j}s^{j-n-1}u)(1 - ar^{1-j}s^{j-n-2}u),
\]

\[
R_{i}(z) = \prod_{j=1}^{n} (1 - ar^{-j}s^{j-n-1}u).
\]

**Proof.** It is easy to see that \( W_{n}(a) \) admits a linear space decomposed as \( \mathcal{U}_{r,s}(\widehat{\mathfrak{sl}_2}) \)-module as follows:

\( W_{n}(a) = \mathbb{C}v_{0} \oplus \mathbb{C}v_{1} \oplus \cdots \oplus \mathbb{C}v_{n} \).

It follows from proposition 4.4 that

\[
x^{+}(k) \cdot v_{i} = a^{k}s^{-nk}(rs^{-1})^{-ki}[n+1-i] v_{i-1},
\]

\[
x^{-}(k) \cdot v_{i} = a^{k}r^{nk}(rs^{-1})^{-k(i+1)}[i+1] v_{i+1}.
\]

So by direct calculation, we get,

\[
\omega(k) \cdot v_{i} = (r-s)a^{k}s^{-nk}(rs^{-1})^{-ki}[rs^{-1}]^{k}((rs^{-1})^{-k}[i+1][n-i] - [n+1-i][i]) \cdot v_{i}
\]
Then we obtain
\[
\sum_{k=0}^{\infty} \Phi^+_{i,k} u^k = r^n (rs^{-1})^{-i} + \sum_{k=1}^{\infty} (r - s) a^k s^{-nk} (rs^{-1})^{-ki} (rs^{-1})^k
\]
\[
\left((rs^{-1})^{-k}[i + 1][n - i] - [n + 1 - i][i]\right)
\]
\[
= r^n (rs^{-1})^{-i} + (r - s) \left( \frac{ar^{-i}s^{i-n}u}{1 - ar^{-i}s^{i-n}u} [i + 1][n - i]ight.
\]
\[
- \frac{ar^{1-i}s^{1-n-1}u}{1 - ar^{1-i}s^{1-n-1}u} [n + 1 - i][i]\right)
\]
\[
= r^{n-i} s^i \frac{(1 - ar^{s^{-1}}u)(1 - ar^{-n}u)}{(1 - ar^{-1}s^{i-n}u)(1 - ar^{1-i}s^{1-n-1}u)}
\]
\[
\]
where
\[
Q_i(z) = \prod_{j=1}^{i} (1 - ar^{-j}s^{j-n-1}u)(1 - ar^{1-j}s^{j-n-2}u),
\]
\[
R_i(z) = \prod_{j=1}^{n} (1 - ar^{-j}s^{j-n-1}u).
\]

The another relation is similar. Thus we have completed the proof. □

5. Specializations

5.1 By our previous analysis of Drinfeld polynomials, the theory of finite dimensional representations of $U_{r,s}(\widehat{\mathfrak{sl}_2})$ is quite similar to the classical case in the generic case when the parameters $r$ and $s$ are independent. When we consider specializations of the two parameters $r$ and $s$, there are some special phenomena.

(I) If we specialize $s$ to $r^{-1}$, then the quantum Cartan matrix of $U_{r,s}(\widehat{\mathfrak{sl}_2})$ becomes

\[
\left(\begin{array}{cc}
r^2 & r^{-2} \\
r^{-2} & r^2
\end{array}\right),
\]

which is the same to that of the classical case.

(II) If we specialize $s$ to $r$, then we have $[n] = nr^{n-1}$, and the quantum Cartan matrix of $U_{r,s}(\widehat{\mathfrak{sl}_2})$ becomes

\[
\left(\begin{array}{cc}
1 & 1 \\
1 & 1
\end{array}\right),
\]

which implies the group-like elements $\omega$ and $\omega'$ are in the center of $U_{r,s}(\widehat{\mathfrak{sl}_2})$. On the other hand, from the actions of generators $\omega, \omega'$:

\[
\omega \cdot v_i = r^n v_i; \quad \omega' \cdot v_i = r^n v_i; \quad \omega_0 \cdot v_i = r^n v_i; \quad \omega'_0 \cdot v_i = r^n v_i,
\]

they have the same eigenvalue $r^n$, while the finite dimensional representation $V_n$ is still irreducible.
(III) If we specialize \( r \) to \( s^k \), where \( k \in \mathbb{Z}/\{1\} \), or \( r \) and \( s \) are independent, then we just let \( q^2 = rs^{-1} \), quantum Cartan matrix of \( U_{r,s}(\hat{sl}_2) \) becomes

\[
\begin{pmatrix}
q^2 & q^{-2} \\
q^{-2} & q^2
\end{pmatrix}.
\]

Thus the finite dimensional representation theory is similar to that of one-parameter case.

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