SPREADING SPEED AND PERIODIC TRAVELING WAVES OF
A TIME PERIODIC AND DIFFUSIVE SI EPIDEMIC MODEL
WITH DEMOGRAPHIC STRUCTURE

SHUANG-MING WANG
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China
and
School of Information Engineering, Lanzhou University of Finance and Economics
Lanzhou, Gansu 730020, China

ZHAOSHENG FENG
School of Mathematical and Statistical Sciences, University of Texas
Edinburg, Texas 78539, USA

ZHI-CHENG WANG* AND LIANG ZHANG
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, China

ABSTRACT. We study the asymptotic spreading properties and periodic traveling wave solutions of a time periodic and diffusive SI epidemic model with demographic structure (follows the logistic growth). Since the comparison principle is not applicable to the full system, we analyze the asymptotic spreading phenomena for susceptible class and infectious class by comparing with respective relevant periodic equations with KPP-type. By applying fixed point theorem to a truncated problem on a finite interval, combining with limit idea, the existence of periodic traveling wave solutions are derived. The results show that the minimal wave speed exactly equals to the spreading speed of infectious class when susceptible class is abundant.

1. Introduction. Epidemiological models described by reaction diffusion systems have been extensively used to investigate the spatial-temporal dynamics of infectious diseases in population level. In regard to epidemiological models posed on a unbounded domain, the asymptotic spreading property in terms of asymptotic speed of spread and traveling wave solutions play a crucial role for understanding the geographical spread phenomenon arising in the outbreak of the epidemic [18].

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* Corresponding author.
In general, the spreading speed describes the speed at which the geographic range of the disease expands. Since 1970s, spreading speed theory was widely applied in population dynamics, ecology and epidemiology, we can refer to Aronson [3], Diekmann [6], Weinberger-Kawasaki-Shigesada [24], and the references. When it comes to epidemiological systems, the investigations for $S$-$I$ epidemic systems become relatively difficult due to the lack of comparison principle. This means that the abstract theory established in [15] can not be applied. Recently, Ducrot [8] considered an $S$-$I$ diffusive epidemic model with external supply. Ducrot-Giletti-Matano [10] studied the spreading properties of a class of two component reaction-diffusion systems with prey-predator type, which covers a large class of $S$-$I$ epidemic models as special cases. The methodologies developed in [8, 10] are mainly based on comparison arguments (applying to each component of the system separately) combining the uniform persistent idea. And in particular, [10] provides an available technical framework for the investigation of the spreading properties of a class of epidemic models with $S$-$I$ type. However, there are very few works involving spreading speed for time dependent reaction-diffusion epidemiological systems with $S$-$I$ type.

In the context of epidemiological models, the existence of a traveling wave solution implies a moving zone of transition from an infective state to the disease-free state [18]. We can refer to Kendall [14] for the pioneer works about the application of traveling wave solutions in epidemiology. When it comes to a reaction-diffusion epidemic model with $S$-$I$ type, the investigation on traveling wave solutions is challenging as a result of lacking in monotonic properties. Hosono-Ilyas [12] studied the existence of traveling wave solutions for two diffusive Kermack-Mckendric epidemic models by using shooting method. Huang [13] developed a geometrical approach for a class of non-monotone reaction-diffusion systems, which covers considerable part of $S$-$I$ epidemic systems. In recent decades, fixed point theorem was used to study the existence of traveling wave solutions for various generalized epidemic systems with $S$-$I$ type. Ducrot-Magal [9] considered a class of age-structured SIR epidemic model by using fixed point method on finite intervals combining with limit arguments. Wang-Wu [22] studied a generalized Kermack-Mckendric epidemic model with non-local delayed interaction by applying Schauders fixed point theorem. The method presented in [22] was widely used in various $S$-$I$ epidemic models, we can refer for instant to Wang-Wang-Wu [21] and Xu [25]. For more results on non-monotone systems covering $S$-$I$ epidemic models, we can see Zhang-Wang-Wang [29]. In recent years, the study of traveling wave solutions involving time dependent (including time periodic) reaction-diffusion systems which has applications in epidemiology also attracted attention of scholars gradually, we can refer to Wang-Zhang-Zhao [23], Zhang-Wang-Zhao [27], Zhang-Wang [28], Wang-Feng-Wang-Zhang [20] and Ambrosio-Ducrot-Ruan [2]. For more generalized results involving periodic Lotka-Volterra systems, we can see Bo-Lin-Ruan [7] and the references. In particular, [23, 27] studied a class of time periodic and diffusive $S$-$I$ epidemic models without demographical structure. The basic idea in [23, 27] is to construct an appropriate convex set consisting of periodic functions and a non-monotone operator, and further to reduce the existence of periodic traveling waves to a fixed point problem of the operator on the set.

In reality, the infectious rates and the recovery rates, etc of many infectious diseases are significantly impacted by seasonality (see, e.g., [1]). This leads us to consider an epidemic model in a long propagation time span (e.g., several periods).
Then the vital dynamics should not be ignored, unlike for an epidemic whose time scale is relatively short. Based on these considerations and above discussions, this paper will consider the following time periodic reaction-diffusive $S$-$I$ epidemic model with demographical structure:

$$\begin{align*}
\frac{\partial S(t,x)}{\partial t} &= d_1 \partial_{xx} S(t,x) + r(t) S(t,x)(1-S(t,x)-I(t,x)) - \frac{\beta(t) S(t,x) I(t,x)}{1+\alpha(t) I(t,x)}, \\
\frac{\partial I(t,x)}{\partial t} &= d_2 \partial_{xx} I(t,x) + \frac{\beta(t) S(t,x) I(t,x)}{1+\alpha(t) I(t,x)} - \gamma(t) I(t,x), \quad t > 0, x \in \mathbb{R}.
\end{align*}$$

(1.1)

In model (1.1), we assume that the host population consists of two sub-populations: susceptible and infectious class. $S(t,x)$ and $I(t,x)$ present the densities of susceptible class and infectious class in time $t$ and location $x$, respectively. $d_1$ and $d_2$ are positive constants denoting the diffusion rates of susceptible class and infectious class, respectively.

We further make some explanations for model (1.1) as follows: (1) In the absence of the disease, we assume that the population density follows the logistic growth with an intrinsic growth rate $r$ and a normalized carrying capacity 1, where $r = \lambda - \mu$, $\lambda$ and $\mu$ is natural birth rate and natural death rate of the host population, respectively. Moreover, we assume that the infected individuals are not capable of reproducing. Although the major purpose of this assumption is to keep the model simple, it has the certain rationality, see Seleem-Boyle-Sriranganathan [19] for the spread of brucellosis. We also point out that the transmission of brucellosis is significantly influenced by seasonal variations [4]. Based on this assumption, then the recruitment rate of the host population is given by $r(t) S(t,x)(1 - S(t,x) - I(t,x))$. (2) $\beta$ and $\gamma$ are the infectious coefficient and the death rate, where $\gamma = \mu + \rho$, $\rho$ denotes the disease-induced mortality rate. The incidence is given by $\frac{\beta(t) S(t,x) I(t,x)}{1+\alpha(t) I(t,x)}$ (saturated incidence rate), we can refer to Capasso-Serio [5] for the detailed epidemiological consideration of the corresponding homogeneous model. (3) $r$, $\beta$ and $\gamma$ are all assumed to be nonnegative periodic functions to reflect the time-varying environment.

This work aims at studying the asymptotic spreading properties and traveling wave solutions of system (1.1). Specific to asymptotic spreading properties, despite system (1.1) possesses somewhat advantageous properties compared to general case in [10], there are a lot of additional difficulties because (1.1) is a non-autonomous system. Above all, to investigate the spreading behaviour behind the propagation front, some entire solutions following from corresponding sequences of solutions of (1.1) are needed. However, these entire solutions may satisfy some time translation systems derived from (1.1) (see (2.8) below) rather than (1.1) itself only. As a result, we have to show the point-wise weak spreading properties (see Lemma 2.5) hold for all derived systems, not just for (1.1). For every derived system, the key to achieve this is finding a sequence of solutions and applying appropriate spatiotemporal translation to this sequence such that the first component of which sufficiently closes to 1 (see Claim 1). It is nontrivial and far more technical compared to that in autonomous case.

When it comes to the existence of periodic traveling wave solutions for system (1.1), we can draw on the methodologies developed in [23, 27]. For this we need to construct a suitable ordered pair of super- and sub-solutions. Nevertheless, it is not always easy especially for time-periodic case, since a complicated construction
usually requires to impose additional restrictions without clear epidemiological significance on the coefficients, while a simplified one cannot guarantee the asymptotic boundary conditions. Combining the dynamics of the kinetic system corresponding to (1.1) (see Lemma 3.2 in Section 3) and some recent works involving the connection between the asymptotic spreading properties and traveling waves for non-monotone equation (system) (Zhang-Wang-Zhao [26]), we are expected to use a relatively simpler construction of super- and sub-solutions to obtain the existence of a periodic traveling wave solution and further verify the asymptotic boundary behaviors with the aid of asymptotic spreading properties.

Before introducing the main arrangements of this paper, we first provide some preliminaries. For the sake of more simplified expressions, the following notations will be used throughout of this paper:

(i). For any $T$-periodic function $g \in C(\mathbb{R})$, let $\tilde{g} := \frac{1}{T} \int_0^T g(t)dt$. For any function $g \in C(\mathbb{R})$ and any constant $\theta \in \mathbb{R}$, we define $g_\theta(t) := g(t + \theta)$. Obviously, $\tilde{g}_\theta = \tilde{g}$ for all $\theta \in \mathbb{R}$.

(ii). For any positive constant $q \in \mathbb{R}$, denote $\lceil q \rceil = q - l_q T$, where $l_q := \max\{n \in \mathbb{N} \text{ and } n \geq 0 : nT \leq q < (n + 1)T\}$.

Throughout of this paper, we make the following assumptions on the coefficients of system (1.1):

**Assumption 1.1.** The coefficients $r, \beta, \gamma, \alpha \in C^\iota(\mathbb{R})$ with some $\iota \in (0, 1)$ are all $T$-periodic functions satisfying $r(t) > 0$, $\beta(t) > 0$, $\gamma(t) > 0$ and $\alpha(t) > 0$, $\forall t \in \mathbb{R}$, wherein $T > 0$ is a constant.

We now give the corresponding kinetic system with respect to (1.1):

\[
\begin{cases}
\frac{dS(t)}{dt} = r(t)S(t)(1 - S(t) - I(t)) - \frac{\beta(t)S(t)I(t)}{1 + \alpha(t)I(t)}, \\
\frac{dI(t)}{dt} = \frac{\beta(t)S(t)I(t)}{1 + \alpha(t)I(t)} - \gamma(t)I(t), \quad t > 0.
\end{cases}
\tag{1.2}
\]

Denote $R_0 := \tilde{\beta}$ as the basic reproduction number of (1.2). In order to derive the spreading properties and the existence of periodic traveling waves, we need the following assumption:

**Assumption 1.2.** $R_0 > 1$.

Let $X = BUC(\mathbb{R}, \mathbb{R}^2)$ be the Banach space of $\mathbb{R}^2$-valued bounded and uniformly continuous functions on $\mathbb{R}$ endowed with the usual sup-norm. Under the condition Assumption 1.2 holds, we define a set $C \subset X$ by

\[
C = \{ (\varphi, \psi) \in X : 0 \leq \varphi \leq 1 \text{ and } 0 \leq \psi \leq G \},
\]

where $G := \inf\{g > 0 : \max_{t \in [0, T]} |\frac{\beta(t)}{1 + \alpha(t)} - \gamma(t)| \leq 0\}$. Then it is easy to check that $C$ is actually positively invariant under system (1.1).

It is evidently that Assumption 1.2 is equivalent to $\bar{\beta} > \bar{\gamma}$. Under the condition Assumption 1.2 holds, we further define $c^* := 2\sqrt{d_1 r}$, $c^{**} := 2\sqrt{d_2 (\bar{\beta} - \bar{\gamma})}$. We shall show in the next section that $c^*$ represents the spreading speed of susceptible class when the infectious class is absent (the spreading speed of susceptible class, for short) while $c^{**}$ denotes the speed of infectious class when the Susceptible individuals is abundant (the spreading speed of infectious class, for short). We further let $c^\Delta = \min\{c^*, c^{**}\}$ throughout this paper.
where $u(x, t)$ is a solution of periodic initial problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d\frac{\partial^2 u}{\partial x^2} + u(t, x) \left[ a(t) - b(t)u(t, x) \right], \\
u(0, x) &= \phi(x) \geq 0, \quad x \in \mathbb{R},
\end{align*}
\]

(2.3)

The remainder of this paper is arranged as follows. In Section 2, we derive the asymptotic spreading properties of system (1.1) under the condition $c^{**} < c^*$ and $c^{**} \geq c^*$, respectively. Section 3 is devoted to establish the existence of periodic traveling wave solutions connecting the disease-free steady state to the endemic state for system (1.1), wherein the spreading speed of infectious class $c^{**}$ is characterised as the minimum wave speed. In Section 4, the nonexistence of periodic traveling waves are investigated for two cases: (i) $R_0 \leq 1$, (ii) $R_0 > 1$ and $c < c^{**}$, respectively. Section 5 is a brief discussion.

2. The asymptotic speed of spread. In this section, we investigate the asymptotic spreading properties of system (1.1) under the conditions $c^{**} < c^*$ and $c^{**} \geq c^*$, respectively. Before proceeding to state the main results, we first provide two important lemmas which will be repeatedly used in this section.

**Lemma 2.1.** Assume $d > 0$ and $c \geq 0$ are two given constants. Let $h \in C(\mathbb{R})$ be a $T$-periodic function with $\bar{h} > \frac{c^*}{4d}$. There exits an $R^0 > 0$ independent of $\theta$ such that, for any $R > R^0$, the parabolic and $T$-periodic eigenvalue problem

\[
\begin{align*}
\frac{\partial \phi}{\partial t} - d\frac{\partial^2 \phi}{\partial x^2} - c\frac{\partial \phi}{\partial x} - h(t)\phi &= \mu \phi, \quad x \in (-R, R), \\
\phi(0, x) &= \phi(T, x), \quad x \in (-R, R), \\
\phi(t, x) &= 0, \quad x = \pm R,
\end{align*}
\]

(2.1)

admits a negative eigenvalue.

**Proof.** For any $R > 0$, let $\sigma_R > 0$ be the principal eigenvalue of the following eigenvalue problem:

\[
\begin{align*}
-d\frac{\partial^2 \psi}{\partial x^2} &= \sigma \psi, \quad x \in (-R, R), \\
\psi(x) &= 0, \quad x \in (-R, R), \\
\psi(x) &= 0, \quad x = \pm R.
\end{align*}
\]

(2.2)

Then we take $\psi_R$ as an eigenfunction of (2.2) corresponding to eigenvalue $\sigma_R$ and further denote

\[
\phi_R(t, x) := e^{\int_0^t h(s)ds - \bar{h}t}e^{-\frac{\sigma_R}{d}x}\psi_R(x).
\]

After some conventional calculations, we can verify that $\phi_R$ is a eigenfunction of (2.1) associated with the principal eigenvalue $\mu_R := \sigma_R - (\bar{h} - \frac{c^*}{4d})$. Obviously, $\mu_R$ is not dependent on $\theta$. Owning to $\sigma_R = \frac{\bar{h}}{d}$, we can find a sufficiently large $R^0 > 0$ such that $\mu_R < 0$ for any $R > R^0$. \hfill \Box

**Lemma 2.2.** ([15]) Consider the following periodic initial problem:

\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= d\frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) \left[ a(t) - b(t)u(t, x) \right], \quad (t, x) \in (0, \infty) \times \mathbb{R}, \\
u(0, x) &= \phi(x) \geq 0, \quad x \in \mathbb{R},
\end{align*}
\]

(2.3)

where $d > 0$ is a constant, $a(t)$ and $b(t)$ are $T$-periodic continuous functions defined on $\mathbb{R}$ with $\bar{a} > 0$ and $b(t) > 0$. Assume $\phi \in C(\mathbb{R})$ and $\phi \neq 0$, then for each $c \in [0, 2\sqrt{d\bar{a}})$, we have

\[
\lim_{t \to \infty} \sup_{|x| < ct} |u(t, x) - u^*(t)| = 0,
\]

where $u^*(t)$ is the unique positive $T$-periodic solution (attracts all solutions with positive initial values) of the corresponding kinetic equation of (2.3). Moreover, if
Theorem 2.3. Let Assumptions 1.1 and 1.2 be satisfied. For any \( \phi \) admits a nonempty compact support in \( \mathbb{R} \), for each \( c \in (2\sqrt{da}, \infty) \), we have

\[
\limsup_{t \to \infty} \sup_{|x| > ct} |u(t, x)| = 0.
\]

Proof. The conclusion can be directly followed from [15, Theorem 4.1] by letting \( \tau = 0 \) and \( f(t, u, v) = v[a(t) - b(t)u] \) in equation (4.1) of [15]. \( \square \)

Theorem 2.4. Let Assumptions 1.1 and 1.2 be satisfied. For any \((S_0, I_0) \in C\) with \( S_0 \neq 0 \) and \( I_0 \neq 0 \), let \((S(t, x), I(t, x))\) be the solution of (1.1) with initial data \((S_0, I_0)\). Suppose \( c^* < c^* \), then we have:

(i) if \( S_0 \) is compactly supported, then

\[
\lim_{t \to \infty} \sup_{|x| \leq ct} S(t, x) = 0 \quad \text{for all } c > c^*;
\]

(ii) if \( I_0 \) is compactly supported, then

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} I(t, x) = 0 \quad \text{for all } c > c^*; \tag{2.5}
\]

(iii) \( \lim_{t \to \infty} \sup_{|x| \leq ct} |S(t, x) - 1| = 0 \) for all \( c_1, c_2 \in (c^*, c^*) \) with \( c_1 < c_2 \);

(iv) there exists \( \varepsilon > 0 \) such that

\[
\liminf_{t \to \infty} \inf_{|x| \leq ct} I(t, x) \geq \varepsilon, \quad \limsup_{t \to \infty} \sup_{|x| \leq ct} S(t, x) \leq 1 - \varepsilon \quad \text{and} \quad \liminf_{t \to \infty} \inf_{|x| \leq ct} S(t, x) \geq \varepsilon
\]

hold for each \( c \in [0, c^*] \).

Theorem 2.4. Let Assumptions 1.1 and 1.2 be satisfied. For any \((S_0, I_0) \in C\) with \( S_0 \neq 0 \) and \( I_0 \neq 0 \), let \((S(t, x), I(t, x))\) be the solution of (1.1) with initial data \((S_0, I_0)\). Suppose \( c^* \geq c^* \), then we have:

(i) if \( S_0 \) is compactly supported, then

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} S(t, x) = 0 \quad \text{for all } c > c^*; \tag{2.6}
\]

Further, if both \( S_0 \) and \( I_0 \) are all compactly supported, then

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} I(t, x) = 0 \quad \text{for all } c > c^*; \tag{2.7}
\]

(ii) there exists \( \varepsilon > 0 \) such that

\[
\liminf_{t \to \infty} \inf_{|x| \leq ct} I(t, x) \geq \varepsilon, \quad \limsup_{t \to \infty} \sup_{|x| \leq ct} S(t, x) \leq 1 - \varepsilon \quad \text{and} \quad \liminf_{t \to \infty} \inf_{|x| \leq ct} S(t, x) \geq \varepsilon
\]

hold for each \( c \in [0, c^*] \).

Proof of Theorem 2.3 (i). Since \( S \) satisfies

\[
\frac{\partial S(t, x)}{\partial t} \leq d_1 \partial_{xx} S(t, x) + r(t) S(t, x)(1 - S(t, x)), \quad \forall x \in \mathbb{R}, t > 0,
\]

then (2.4) can be directly obtained by Lemma 2.2. Thus, we need only to verify (2.5).

According to the definition of \( C \), we see that \( S \leq 1 \) and \( I \leq G \) for any \((S_0, I_0) \in C\). Then we have

\[
\frac{\partial I(t, x)}{\partial t} - d_2 \partial_{xx} I(t, x) \leq [\beta(t) - \gamma(t)] I(t, x) - \frac{\beta(t) \alpha(t)}{1 + \alpha(t) G} I^2(t, x), \quad \forall x \in \mathbb{R}, t > 0.
\]
Hence \( I \) is a sub-solution of

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= d_1 \partial_{xx} S(t, x) + r_\varrho(t) S(t, x)(1 - S(t, x) - I(t, x)) \\
- \frac{\partial \varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)}, \\
\frac{\partial I(t, x)}{\partial t} &= d_2 \partial_{xx} I(t, x) + \frac{\partial \varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)} - \gamma_\varrho(t) I(t, x),
\end{align*}
\]

\( \text{for } t > 0, x \in \mathbb{R}. \) According to Lemma 2.2, with the condition \( c > c^{**} \), we have

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} I(t, x) \leq \lim_{t \to \infty} \sup_{|x| \geq ct} v(t, x) = 0.
\]

This completes the proof.

Note that both Theorem 2.3(iii) and Theorem 2.4(ii) can be restated as one proposition. Before presenting it, we assume that \( c^0 \in [0, c^{**}] \) is an arbitrarily fixed constant throughout the rest of this section. Here we recall that \( c^{**} = \min\{c^*, c^{**}\} \).

**Proposition 1.** (inner spreading properties) Let Assumptions 1.1 and 1.2 be satisfied. For any \((S_0, I_0) \in C \) with \( S_0 \not\equiv 0 \) and \( I_0 \not\equiv 0 \), let \((S(t, x), I(t, x))\) be the solution of (1.1) with initial data \((S_0, I_0)\). Then there exists \( \varepsilon > 0 \) such that for each \( c \in [0, c^0] \), one has

\[
\begin{align*}
\liminf_{t \to \infty} \inf_{|x| \leq ct} I(t, x) &\geq \varepsilon; \\
\limsup_{t \to \infty} \sup_{|x| \leq ct} S(t, x) &\leq 1 - \varepsilon \text{ and } \liminf_{t \to \infty} \inf_{|x| \leq ct} S(t, x) \geq \varepsilon.
\end{align*}
\]

The main ideas and technical framework to prove this proposition follows from [8, 10]. Inspired by [10], we need to verify two crucial weaker spreading properties of solutions of system (1.1) than those in Proposition 1. In order to clearly show these spreading properties, it is important to point out that a class of time translation system derived from (1.1) is needed here. Precisely, for any \( \vartheta \in [0, T) \), replacing \( i(t) \) in system (1.1) by \( i_\varrho(t), \varrho = r, \beta, \gamma, \alpha \), then we obtain the following system with a parameter \( \vartheta \in [0, T) \):

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= d_1 \partial_{xx} S(t, x) + r_\varrho(t) S(t, x)(1 - S(t, x) - I(t, x)) \\
- \frac{\partial \varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)}, \\
\frac{\partial I(t, x)}{\partial t} &= d_2 \partial_{xx} I(t, x) + \frac{\partial \varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)} - \gamma_\varrho(t) I(t, x),
\end{align*}
\]

\( \text{for } t > 0, x \in \mathbb{R}. \)

Besides, the following system with advection and parameter \( \vartheta \in [0, T) \) will also be used many times:

\[
\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= d_1 \partial_{xx} S(t, x) + c_\varrho(t) S(t, x)(1 - S(t, x) - I(t, x)) \\
- \frac{\varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)}, \\
\frac{\partial I(t, x)}{\partial t} &= d_2 \partial_{xx} I(t, x) + c_\varrho(t) I(t, x) + \frac{\varrho(t) S(t, x) I(t, x)}{1 + \alpha_\varrho(t)} - \gamma_\varrho(t) I(t, x),
\end{align*}
\]

\( \text{for } t > 0, x \in \mathbb{R}. \)

It is clear that system (1.1) is a special case of (2.8) with \( \vartheta = 0 \). Obviously, \( C \) is also positively invariant under system (2.8) and (2.9) for each \( \vartheta \in [0, T) \).

Let \((S(t, x; \vartheta), I(t, x; \vartheta))\) be the solution of (2.8) with initial data \((S_0, I_0) \in C \) and \( S_0 \not\equiv 0 \) and \( I_0 \not\equiv 0 \). Let us now give the first weaker property:

**Lemma 2.5.** (point-wise weak spreading) Let Assumptions 1.1 and 1.2 be satisfied. Then there exists an \( \varepsilon > 0 \), such that for any \((S_0, I_0) \in C \) with \( S_0 \not\equiv 0 \) and \( I_0 \not\equiv 0 \),

\[
\text{there exists an } \varepsilon > 0 \text{ such that for any } (S_0, I_0) \in C \text{ with } S_0 \not\equiv 0 \text{ and } I_0 \not\equiv 0,
\]
for each $c \in [0, e^0]$, $e = \pm 1$ and all $x \in \mathbb{R}$, one has

$$\limsup_{t \to \infty} S(t, x + cte; \vartheta) \geq \varepsilon;$$

$$\liminf_{t \to \infty} S(t, x + cte; \vartheta) \leq 1 - \varepsilon \quad \text{and} \quad \limsup_{t \to \infty} I(t, x + cte; \vartheta) \geq \varepsilon$$

uniformly for all $\vartheta \in [0, T]$.

**Proof.** Owing to either $e = 1$ or $e = -1$, without loss of generality, we argue by contradiction by assuming that there exist sequences $\{(S_{0,n}, I_{0,n})\}_{n \geq 1} \subset C$, $\{t_n, x_n\}_{n \geq 1}$ with $\lim_{n \to \infty} t_n = \infty$, $\{c_n\}_{n \geq 1} \subset [0, e^0]$ and $\{\vartheta_n\}_{n \geq 1} \subset [0, T]$ such that at least one of the following items holds true:

$$S_n(t, x_n + c_n t; \vartheta_n) \geq 1 - \frac{1}{n}, \forall t \geq t_n, \quad (2.10)$$

$$S_n(t, x_n + c_n t; \vartheta_n) \leq \frac{1}{n}, \forall t \geq t_n, \quad (2.11)$$

$$I_n(t, x_n + c_n t; \vartheta_n) \leq \frac{1}{n}, \forall t \geq t_n. \quad (2.12)$$

In what follows, we shall attempt to derive some contradictions from (2.10)-(2.12). In addition, as similar as the proof of [10, Lemma 5.2], we can check that (2.11) implies (2.12). In the case when either (2.10) or (2.12) holds, we have the following claim:

**Claim 1** There exists sequences $\{t''_n\}_{n \geq 1}$ with $t''_n \geq t_n$, $\{c_n\}_{n \geq 1} \subset [0, e^0]$ and $\{\vartheta_n\}_{n \geq 1} \subset [0, T]$ such that for any $R > 0$,

$$\lim_{n \to \infty} \sup_{(t,x) \in [0,\infty) \times [-R,R]} |S_n(t''_n + t, x_n + c_n(t''_n + t) + x; \vartheta_n) - 1| = 0. \quad (2.13)$$

**Proof of Claim 1.** In the case when (2.10) is valid, we can easily check Claim 1 as similar as that in Claim 5.4 of [10]. Here we need only to consider the case when (2.12) holds true. From the boundedness of $S_n$ and $I_n$ and standard parabolic estimates, there exists a subsequence of $\{(S_n, I_n)\}$, still denoted it by $\{(S_n, I_n)\}$, such that

$$(S_n, I_n)(t_n + t, x_n + c_n(t_n + t) + x; \vartheta_n) \to (S^{\infty}, I^{\infty})(t, x)$$

locally uniformly for $(t, x) \in \mathbb{R}^2$, $c_n \to c_{\infty}$, $[t_n] \to t_{\infty} \in [0, T)$ and $\vartheta_n \to \vartheta_{\infty} \in [0, T)$, which implies

$$i_{\partial_n}(t_n + t) \to i_{\theta}(t) \quad \text{uniformly in} \quad t \in \mathbb{R}, \quad i = r, \beta, \gamma, \alpha,$$

where $\theta := [t_{\infty} + \vartheta_{\infty}]$. In addition, $(S^{\infty}, I^{\infty})$ is an entire solution of system (2.9) with $c = c_{\infty}$ and $\vartheta = \theta$.

It is clear that $I^{\infty} \geq 0$. Moreover, by (2.12) we have $I^{\infty}(0,0) = 0$. Then it follows from the strong maximum principle that $I^{\infty} \equiv 0$, which implies that for any $R > 0$,

$$\lim_{n \to \infty} I_n(t + t_n, x_n + c_n(t + t_n) + x; \vartheta_n) = 0 \quad \text{uniformly on} \quad [0, \infty) \times [-R, R]. \quad (2.13)$$

Otherwise, we assume by contradiction there exist $R' > 0$, $\delta' > 0$, $t'_n \geq t_n$, and $x'_n \in [-R', R']$ such that

$$I_n(t'_n, x_n + c_n t'_n + x'_n; \vartheta_n) \geq \delta'. \quad (2.14)$$

By the boundedness of $S_n$ and $I_n$ and standard parabolic estimates, we can extract a converging subsequence of $\{(S_n, I_n)\}$, still denoted it by $\{(S_n, I_n)\}$, such that

$$(S_n, I_n)(t'_n + t, x_n + c_n(t'_n + t) + x; \vartheta_n) \to (S^{\infty}, I^{\infty})(t, x)$$
locally uniformly for \((t, x) \in \mathbb{R}^2\), wherein \((S'^{\infty}, I^{\infty})\) is another entire solution of system (2.9) with \(c = c_{\infty}\) and \(\vartheta = [t'_{\infty} + \vartheta_{\infty}]\), wherein \(t'_{\infty}\) is an accumulation point of \([\{t'_n\}\}].\) Following from (2.12) and maximum principle, we have \(I^{\infty} = 0.\) On the other hand, by (2.14), we get \(I^{\infty}(0, x'_0) \geq \delta',\) where \(x'_0\) is an accumulation point of \([x'_n]\). This contradiction confirms the validity of (2.13).

Then for any \(R > 0\) and \(\delta > 0,\) we can choose a \(N_1(\delta, R) \in \mathbb{N}_+\) large enough such that

\[
I_n(t + t_n, x_n + c_n(t + t_n) + x; \vartheta_n) \leq \delta, \quad \forall (t, x) \in [0, \infty) \times [-R, R] \tag{2.15}
\]

for all \(n \geq N_1.\) Label \((U_n, V_n)(t, x) := (S_n, I_n)(t + t_n, x_n + c_n(t + t_n) + x; \vartheta_n), \forall n \geq N_1.\) Then by (2.15), we have

\[
\partial_t U_n(t, x) - d_1 \partial_{xx} U_n(t, x) - c_n \partial_x U_n(t, x)
\]

\[
= r_{\vartheta_n + t_n}(t) U_n(t, x)(1 - U_n(t, x) - V_n(t, x)) - \beta_{\vartheta_n + t_n}(t) U_n(t, x) V_n(t, x) \tag{2.16}
\]

\[
\geq r_{\vartheta_n + t_n}(t) U_n(t, x)(1 - U_n(t, x) - \delta(r_{\vartheta_n + t_n}(t) + \beta_{\vartheta_n + t_n}(t)) U_n(t, x),
\]

for all \((t, x) \in [0, \infty) \times [-R, R].\) Thus \(U_n\) is a super-solution of the following initial-boundary problem with parameter \(\delta:\)

\[
\begin{cases}
\partial_t \phi_n^{\delta, R}(t, x) - d_1 \partial_{xx} \phi_n^{\delta, R}(t, x) - c_n \partial_x \phi_n^{\delta, R}(t, x) \\
= r_{\vartheta_n + t_n}(t) \phi_n^{\delta, R}(t, x)(1 - \phi_n^{\delta, R}(t, x)) \\
- \beta_{\vartheta_n + t_n}(t) \phi_n^{\delta, R}(t, x), \quad t > 0, x \in (-R, R),
\end{cases}
\]

\[
u_n^{\delta, R}(0, x) = \varphi_n^{\delta, R}(x), \quad x \in (-R, R),
\]

\[
u_n^{\delta, R}(t, \pm R) = 0, \quad t \geq 0,
\]

where the initial function \(\varphi_n^{\delta, R} \in C([-R, R])\) satisfies \(0 \leq \varphi_n^{\delta, R} \leq S_n(t_n, x_n + c_n t_n + x; \vartheta_n), \varphi_n^{\delta, R}(x) \neq 0, \forall x \in (-R, R),\) and \(\varphi_n^{\delta, R}(\pm R) = 0.\)

Linearizing (2.16) at \(u_n^{\delta, R} = 0,\) then we get the corresponding linear periodic eigenvalue problem:

\[
\begin{cases}
\partial_t \phi_n^{\delta, R}(t, x) - d_1 \partial_{xx} \phi_n^{\delta, R}(t, x) - c_n \partial_x \phi_n^{\delta, R}(t, x) = \\
(1 - \delta) r_{\vartheta_n + t_n}(t) - \beta_{\vartheta_n + t_n}(t) \phi_n^{\delta, R}(t, x) = \mu_n^{\delta, R} \phi_n^{\delta, R}(t, x), \quad t > 0, x \in (-R, R),
\end{cases}
\]

\[
\phi_n^{\delta, R}(t + T, x) = \phi_n^{\delta, R}(t, x), \quad x \in [-R, R],
\]

\[
\phi_n^{\delta, R}(t, \pm R) = 0, \quad t \geq 0
\]

(2.17)

We choose a \(\delta \in \left(0, \frac{1}{r + c_n} \right)\), then (2.17) admits a negative eigenvalue for sufficiently large \(n\) and \(R\) by using Lemma 2.1 with \(d = d_1, c = c_n, \) and \(h(t) = (1 - \delta) r_{\vartheta_n + t_n}(t) - \beta_{\vartheta_n + t_n}(t).\) It then follows from [11, Thorem 28.1] that (2.16) admits a positive \(T-\)periodic solution \(u_n^{\delta, R_*}\), which is unique and globally asymptotically stable (with respect to nonnegative and nontrivial initial data). Moreover, we can show \(u_n^{\delta, R_*}(0, 0)\) is bounded from below uniformly for sufficiently large \(n\) by some positive constant only relying on \(\delta\) and \(R.\)

Indeed, we can choose a sufficiently small \(\omega(\delta, R) > 0\) such that

\[
\Upsilon_n^{\delta, R}(t, x) := \omega e^{\int_0^t ((1 - \delta) r_{\vartheta_n + t_n}(s) + \beta_{\vartheta_n + t_n}(s)) ds - \frac{(1 - \delta) r_{\vartheta_n + t_n}(s) + \beta_{\vartheta_n + t_n}(s) }{2} t^2} x \varphi_R(x) \] satisfies

\[
\partial_t \Upsilon_n^{\delta, R}(t, x) - d_1 \partial_{xx} \Upsilon_n^{\delta, R}(t, x) - c_n \partial_x \Upsilon_n^{\delta, R}(t, x) \\
\leq r_{\vartheta_n + t_n}(t) \Upsilon_n^{\delta, R}(t, x)(1 - \Upsilon_n^{\delta, R}(t, x)) - \delta(r_{\vartheta_n + t_n}(t) + \beta_{\vartheta_n + t_n}(t)) \Upsilon_n^{\delta, R}(t, x)
\]
for all \((t, x) \in (0, \infty) \times [-R, R]\). In addition, it is apparent that \(\Upsilon_n^\delta,R(0, x) = v e^{-\frac{\delta t}{2}} \psi_R(x), \forall x \in [-R, R]\), and \(\Upsilon_n^\delta,R(t, \pm R) = 0, \forall t \geq 0\). Let \(w_n^{\delta,R}\) be the solution of

\[
\begin{align*}
\partial_t w_n^{\delta,R}(t, x) - d_1 \partial_{xx} w_n^{\delta,R}(t, x) - c_n \partial_x w_n^{\delta,R}(t, x) \\
= r \vartheta_{n+\tau_n}(t) w_n^{\delta,R}(t, x)(1 - w_n^{\delta,R}(t, x)) \\
- \delta (r \vartheta_{n+\tau_n}(t) + \beta) w_n^{\delta,R}(t, x), \ t > 0, x \in (-R, R), \\
\end{align*}
\]

\[
\begin{align*}
w_n^{\delta,R}(0, x) = v e^{-\frac{\delta t}{2}} \psi_R(x), \ x \in (-R, R), \\
w_n^{\delta,R}(t, \pm R) = 0, \ t \geq 0.
\end{align*}
\]

Then it follows from the comparison principle that \(w_n^{\delta,R}(t, x) \geq \Upsilon_n^{\delta,R}(t, x)\) for all \((t, x) \in (0, \infty) \times [-R, R]\). In particular, we have \(w_n^{\delta,R}(kT, 0) \geq \Upsilon_n^{\delta,R}(kT, 0)\) for all \(k \in \mathbb{N}_+\). On the other hand, the definition of \(w_n^{\delta,R}\) implies that \(w_n^{\delta,R}(kT, 0) - w_n^{\delta,R}(kT, 0)\) tended to 0 as \(k\) approaches infinity. By (2.16) combining the periodicity of \(u_n^{\delta,R}(t, \cdot)\) and \(\Upsilon_n^{\delta,R}(t, \cdot)\), it follows from the comparison argument that \(w_n^{\delta,R}(0, 0) \geq \frac{1}{2} \Upsilon_n^{\delta,R}(0, 0) = \frac{\varpi \psi_R(0)}{2} > 0\) for all \(n \geq N_1\).

Next we show that there exists sequences \(\{s_n\}\) such that \(S_n(s_n, x_n + c_n s_n; \vartheta_n)\) converges to some positive constant for sufficiently large \(n\). Fixing a \(\delta > 0\) small enough and a sufficiently large \(R > 0\), by the comparison argument, we can find a \(N_2 > N_1(\delta, R)\) such that

\[
S_n(t + t_n, x_n + c_n(t + t_n); \vartheta_n) = U_n(t, 0) \geq \frac{1}{2} w_n^{\delta,R}(t, 0)
\]

for all \(n \geq N_2\) and sufficiently large \(t\). Then there exists \(\{\bar{t}_n\}_{n \geq N_2}\) with \(\bar{t}_n = l_n T\) such that

\[
S_n(\bar{t}_n + t_n, x_n + c_n(\bar{t}_n + t_n); \vartheta_n) > \frac{w_n^{\delta,R}(\bar{t}_n, 0)}{2} = \frac{w_n^{\delta,R}(0, 0)}{2} > 0, \tag{2.18}
\]

where \(\{t_n\}_{n \geq N_2} \subset \mathbb{N}_+\). Label \(s_n := \bar{t}_n + t_n, \forall n \geq N_2\). Without loss of generality, we assume \(S_n(s_n, x_n + c_n s_n; \vartheta_n)\) converges to some \(S^0 \in [0, 1]\) as \(n\) approaches infinity. Besides, if \(S^0 = 0\), it follows from (2.18) that \(\varpi = 0\), a contradiction occurs. Then we have \(S^0 > 0\).

Moreover, by the boundedness of \(S_n\) and \(I_n\) and standard parabolic estimates, we can find a converging subsequence of \(\{(S_n, I_n)\}\), still denoted it by \(\{(S_n, I_n)\}\), such that

\[
(S_n, I_n)(s_n + t, x_n + c_n(s_n + t) + x; \vartheta_n) \longrightarrow (S_\infty, I_\infty)(t, x), \tag{2.19}
\]

locally uniformly for \((t, x) \in \mathbb{R}^2\) as \(n\) tends to infinity, and further \(S_\infty\) satisfies

\[
\partial_t S_\infty(t, x) - d_1 \partial_{xx} S_\infty(t, x) - c_\infty \partial_x S_\infty(t, x) = r \vartheta(t) S_\infty(t, x)(1 - S_\infty(t, x))
\]

for all \(t > 0\) and \(x \in \mathbb{R}\), wherein \(\vartheta(t)\) is an accumulation point of \(\{s_n + \vartheta_n\}\). In addition, \(S_\infty(0, 0) = S^0 > 0\).

Let \(U(t, x) := S_\infty(t, x - c_\infty t)\), then we have

\[
\begin{align*}
\frac{\partial U(t, x)}{\partial t} = d_1 \partial_{xx} U(t, x) + r \vartheta(t) U(t, x)(1 - U(t, x)), \ x \in \mathbb{R}, t > 0
\end{align*}
\]

Owing to \(c_\infty < c^* = 2\sqrt{d_1 r}\) and \(U(0, 0) = S_\infty(0, 0) > 0\), then Lemma 2.2 implies that

\[
\lim_{t \to \infty} |U(t, c_\infty t) - 1| = 0.
\]
Owing to $U(t, c_\infty t) = S_\infty(t, 0)$, we have $\lim_{t \to \infty} S_\infty(t, 0) = 1$. Then according to (2.19), there exists subsequence $\{n_k\}_{k \geq N_2}$ and $\{k\}_{k \geq N_2}$ such that

$$S_{n_k}(s_{n_k} + \hat{t}_k, x_n + c_n(s_{n_k} + \hat{t}_k); \vartheta_n) > 1 - \frac{1}{k}.$$  (2.20)

Denoting $t_n'' := s_{n_k} + \hat{t}_k$. To ease the notations, we can assume without loss of generality that $n_k = k$ for all $k \geq N_2$ and further replace the subscript ’$k$’ with ’$n$’.

Recalling again (2.13) and taking into account the fact $t_n'' \geq t_n$ for all $n \geq N_2$, for any $R > 0$ and $\delta > 0$, we can choose a $N_3(\delta, R) > N_2$ large enough such that

$$I_n(t + t_n'', x_n + c_n(t + t_n'') + x; \vartheta_n) \leq \delta, \forall (t, x) \in [0, \infty) \times [-R, R]$$

for all $n \geq N_3$. Let $W_n^{\delta, R}(t, x)$, $\forall n \geq N_3$ be the solution of the following initial-boundary problem with parameter $\delta$:

$$\begin{cases}
\partial_t W_n^{\delta, R}(t, x) - d_1 \partial_{xx} W_n^{\delta, R}(t, x) - c_n \partial_x W_n^{\delta, R}(t, x) \\
= r_{\vartheta_n + t_n''}(t) W_n^{\delta, R}(t, x) (1 - W_n^{\delta, R}(t, x)) - \delta (r_{\vartheta_n + t_n''}(t) + \beta_{\vartheta_n + t_n''}(t)) W_n^{\delta, R}(t, x), \\
W_n^{\delta, R}(0, x) = \lambda_n^{\delta, R}(x), \quad x \in (-R, R), \\
W_n^{\delta, R}(t, \pm R) = 0, \quad t \geq 0,
\end{cases}$$

(2.21)

where the initial function $\chi_n^{\delta, R} \in C([-R, R])$ satisfies

$$\chi_n^{\delta, R}(x) = S_n(t_n'', x_n + c_n t_n'' + x; \vartheta_n), \forall x \in [-\frac{R}{2}, \frac{R}{2}],$$

$$0 \leq \chi_n^{\delta, R}(x) \leq S_n(t_n'', x_n + c_n t_n'' + x; \vartheta_n), \forall x \in (-R, -\frac{R}{2}) \cup (\frac{R}{2}, R)$$

and $\chi_n(\pm R) = 0$. It is easy to see that $S_n(t + t_n', x_n + c_n(t + t_n') + x; \vartheta_n)$ is a super-solution of (2.21) for all $n \geq N_3$. Therefore, we have

$$S_n(t + t_n', x_n + c_n(t + t_n') + x; \vartheta_n) \geq W_n^{\delta, R}(t, x), \forall (t, x) \in [0, \infty) \times [-R, R]$$

(2.22)

for all $n \geq N_3$.

Moreover, owning to the boundedness of $W_n^{\delta, R}(t, x)$ and standard parabolic estimates, we can extract a converging subsequence of $\{W_n^{\delta, R}(t, x)\}$, still denoted it by $\{W_n^{\delta, R}(t, x)\}$, such that

$$\lim_{\delta \to 0, R \to \infty} \lim_{n \to \infty} W_n^{\delta, R}(t, x) \to W_{\infty}^{0, \infty}(t, x)$$

(2.23)

locally uniformly for $(t, x) \in \mathbb{R}^2$, and further $W_{\infty}^{0, \infty}$ satisfies

$$\begin{align*}
\partial_t W_{\infty}^{0, \infty}(t, x) - d_1 \partial_{xx} W_{\infty}^{0, \infty}(t, x) - c_\infty \partial_x W_{\infty}^{0, \infty}(t, x) \\
= r_{\tilde{\theta}}(t) W_{\infty}^{0, \infty}(t, x) (1 - W_{\infty}^{0, \infty}(t, x))
\end{align*}$$

for all $t > 0$ and $x \in \mathbb{R}$, wherein $\tilde{\theta}$ is an accumulation point of $\{[t_n'' + \vartheta_n]\}$. In addition, it follows that $W_{\infty}^{0, \infty}(0, 0) = 1$ from (2.20). Consequently, thanks to the strong maximal principle, we have $W_{\infty}^{0, \infty}(t, x) \equiv 1$ for all $(t, x) \in \mathbb{R}^2$, which combining with (2.22) and (2.23) completes the proof of Claim 1.

We now proceed to prove Lemma 2.5. By Claim 1, for any $\eta \in (0, 1 - \frac{\gamma + \tau_0}{\beta})$, $R > \frac{\sigma_1}{(1 - \eta)^{\beta - \gamma + \frac{\sigma_1}{\beta}}}$, there exists a $N_4(\eta, R)$ large enough such that

$$S_n(t + t_n', x_n + c_n(t + t_n') + x; \vartheta_n) > 1 - \eta, \forall (t, x) \in [0, \infty) \times [-R, R]$$
for all \( n \geq N_5 \), where \( \sigma_1 \) is the constant given in the proof of Lemma 2.1.

Defining \((\bar{U}_n, \bar{V}_n)(t,x) := (S_n, I_n)(t + \nu''_n, x_n + c_n(t + \nu''_n) + x; \vartheta_n)\), we have

\[
\begin{align*}
\partial_t \bar{V}_n(t,x) - d_2 \partial_x \bar{V}_n(t,x) - c_n \partial_x \bar{V}_n(t,x) \\
= \beta_{\vartheta_n + \nu''_n}(t)\bar{U}_n(t,x)\bar{V}_n(t,x) \\
+ \frac{1}{1 + \alpha_{\vartheta_n + \nu''_n}(t)} V_n(t,x) \\
\geq [(1 - \eta)\beta_{\vartheta_n + \nu''_n}(t) - \gamma_{\vartheta_n + \nu''_n}(t)]\bar{V}_n(t,x) - \beta_{\vartheta_n + \nu''_n}(t)a_{\vartheta_n + \nu''_n}(t)\bar{V}_n^2(t,x),
\end{align*}
\]

for all \( t > 0, \ x \in [-R,R] \). We consider the following parabolic problem with parameter \( \eta \):

\[
\begin{cases}
\partial_t v_n(t,x) - d_2 \partial_x v_n(t,x) - c_n \partial_x v_n(t,x) \\
= [(1 - \eta)\beta_{\vartheta_n + \nu''_n}(t) - \gamma_{\vartheta_n + \nu''_n}(t)]v_n(t,x) - \beta_{\vartheta_n + \nu''_n}(t)a_{\vartheta_n + \nu''_n}(t)v_n^2(t,x), & t > 0, x \in (-R,R), \ (2.24) \\
v_n(0,x) = \psi_n(x), & x \in (-R,R), \\
v_n(t,\pm R) = 0, & t \geq 0,
\end{cases}
\]

where \( \psi_n \in C([-R,R]) \) satisfies \( 0 \leq \psi_n(x) \leq I_n(t''_n, x_n + c_n(t''_n) + x; \vartheta_n), \psi_n(x) \neq 0, \forall x \in (-R,R) \), and \( \psi_n(\pm R) = 0 \). Clearly, \( \bar{V}_n(t,x) \) is a super-solution of (2.24).

By virtue of Lemma 2.1 with \( d = d_2, c = c_n, \) and \( h(t) = (1 - \eta)\beta_{\vartheta_n + \nu''_n}(t) - \gamma_{\vartheta_n + \nu''_n}(t) \), and similar arguments as before, we can reach the conclusion that \( \bar{I}_n(t + \nu''_n, x_n + c_n(t + \nu''_n) + x; \vartheta_n) = \bar{V}_n(t,x) \geq \frac{1}{2}v^*_n(t,x) \) for all sufficiently large \( t \), all \( x \in (-R,R) \) and all \( n \geq N_4 \), here \( v^*_n \) is the unique and globally asymptotically stable (with respect to nonnegative and nontrivial initial data) positive \( T \)-periodic solution of (2.24). In addition, by similar arguments as before, fixing a \( \tilde{\eta} > 0 \) small enough and a sufficiently large \( \tilde{R} > 0 \), we can verify that there exists a sufficiently small constant \( \xi > 0 \) such that \( v^*_n(t,x) \geq \xi, \forall (t,x) \in (t_0,\bar{T}) \times [0,\tilde{R}/2] \) for \( n \geq N_5 \), where \( N_5 > N_4(\tilde{\eta}, \tilde{R}) \) is a constant determined by \( \tilde{\eta} \) and \( \tilde{R} \). To ease the notations, we still label \( \tilde{\eta} \) and \( \tilde{R} \) by \( \eta \) and \( R \), respectively. As a consequence, there exists a sequence \( \{s'_n\}_{n \geq N_5} \) with every \( s'_n > 0 \) sufficiently large such that

\[
\bar{V}_n(t,x) \geq \frac{1}{2}v^*_n(t,x) \geq \frac{\xi}{2} > 0, \ \forall (t,x) \in [s'_n, \infty) \times \left[-\frac{R}{2}, \frac{R}{2}\right].
\]

Denoting \((\bar{U}_n, \bar{V}_n)(t,x) := (\bar{U}_n, \bar{V}_n)(t + s'_n, x)\), we have

\[
\begin{align*}
\partial_t \bar{U}_n(t,x) - d_1 \partial_x \bar{U}_n(t,x) - c_n \partial_x \bar{U}_n(t,x) \\
\leq \left(1 - \xi\right)r_{\vartheta_n + \nu''_n + s'_n}(t) - \frac{\beta_{\vartheta_n + \nu''_n + s'_n}(t)\xi}{1 + \alpha_{\vartheta_n + \nu''_n + s'_n}(t)\xi} \bar{U}_n(t,x) \\
- r_{\vartheta_n + \nu''_n + s'_n}(t)\bar{U}_n^2(t,x), & (t,x) \in (0,\infty) \times \left[-\frac{R}{2}, \frac{R}{2}\right]
\end{align*}
\]

for all \( n \geq N_4 \). Then by the comparison principle, we have

\[
\bar{U}_n(t,x) \leq \bar{U}_n(t,x), \ \forall (t,x) \in (0,\infty) \times \left[-\frac{R}{2}, \frac{R}{2}\right].
\]
wherein \( \hat{U}_n \) is the solution of
\[
\begin{align*}
\partial_t \hat{U}_n(t, x) - d_1 \partial_{xx} \hat{U}_n(t, x) - c_n \partial_x \hat{U}_n(t, x) \\
= \left[ (1 - \xi) r_{\alpha_n + \alpha_n' + s_n'}(t) - \frac{\beta_{\alpha_n + \alpha_n' + s_n'}(t) \xi}{1 + \alpha_n + \alpha_n' + s_n'(t)} \right],
\end{align*}
\]
\[
\begin{align*}
\hat{U}_n(t, x) - r_{\alpha_n + \alpha_n' + s_n'}(t) \hat{U}_n^2(t, x), & \quad (t, x) \in (0, \infty) \times (-\frac{R}{2}, \frac{R}{2}), \\
\hat{U}_n(0, x) = 1, & \quad x \in (-\frac{R}{2}, \frac{R}{2}), \\
\hat{U}_n(t, \pm \frac{R}{2}) = 1, & \quad t \in [0, \infty).
\end{align*}
\]
By the boundedness of \( \hat{U}_n \) and standard parabolic estimates, possibly along a subsequence, we can extract a converging subsequence of \( \hat{U}_n \), still denoted by \( \hat{U}_n \), converges to \( \hat{U}_\infty \) locally uniformly on \([0, \infty) \times [-\frac{R}{2}, \frac{R}{2}]\), and \( \hat{U}_\infty \) satisfies
\[
\begin{align*}
\partial_t \hat{U}_\infty(t, x) - d_1 \partial_{xx} \hat{U}_\infty(t, x) - c_\infty \partial_x \hat{U}_\infty(t, x) = r_{\theta'}(t) \hat{U}_\infty^2(t, x),
\end{align*}
\]
\[
\begin{align*}
\hat{U}_\infty(0, x) = 1, & \quad x \in (-\frac{R}{2}, \frac{R}{2}), \\
\hat{U}_\infty(t, \pm \frac{R}{2}) = 1, & \quad t \in [0, \infty),
\end{align*}
\]
wherein \( \theta' = \lceil s_\infty' + \theta \rceil \) and \( s_\infty' \) is an accumulation point of \( \{ s_n' \} \). Since \( \hat{U}_\infty \equiv 1 \) is a strict super-solution of the above equation, then we have \( \hat{U}_\infty(t_0, 0) < 1 \) for some \( t_0 > 0 \). As a consequence,
\[
\limsup_{n \to \infty} \hat{U}_n(t_0, 0) \leq \limsup_{n \to \infty} \hat{U}_n(t_0, 0) < 1,
\]
that is,
\[
\limsup_{n \to \infty} S_n(t_0 + t_{\infty}'' + s_n', x_n) + c_n(t_0 + t_{\infty}'' + s_n'); \partial_n) < 1,
\]
which contradicts with Claim 1. The proof is completed.

Based on Lemma 2.5, we further to prove the second weaker property. Unlike the first one, the second one is only involving system (1.1). So we let \((S(t, x), I(t, x))\) be the solution of (1.1) with initial data \((S_0, I_0) \in C \) and \( S_0 \neq 0, I_0 \neq 0 \). The second weaker property reads as follows:

**Lemma 2.6.** (point-wise spreading) Let Assumptions 1.1 and 1.2 hold. There exists an \( \epsilon > 0 \) such that, for any \( (S_0, I_0) \in C \) for each \( c \in [0, c^0] \), \( e = \pm 1 \) and \( x \in \mathbb{R} \), one has
\[
\liminf_{t \to \infty} I(t, x + c t e) \geq \epsilon;
\]
\[
\limsup_{t \to \infty} S(t, x + c t e) \leq 1 - \epsilon \text{ and } \liminf_{t \to \infty} S(t, x + c t e) \geq \epsilon.
\]

**Proof.** We only show the first assertion is valid and last two can be verified by similar arguments. Since either \( e = 1 \) or \( e = -1 \), without loss of generality, we assume by contradiction that there exists sequences \( \{(S_0, n, I_0, n)\}_{n \geq 1} \subset C \), \( \{x_n\}_{n \geq 1} \) and \( \{c_n\}_{n \geq 1} \) with \( 0 \leq c_n < c^0 \) such that
\[
\liminf_{t \to \infty} I_n(t, x_n + c_n t) < \frac{1}{n}.
\]
By Lemma 2.5, we can find sequences \( \{t_n\}_{n \geq 1} \) and \( \{s_n\}_{n \geq 1} \) with \( t_n, s_n \geq 0 \) and \( \lim_{n \to \infty} t_n = \infty \) such that
\[
I_n(t_n, x_n + c_n t_n) = \frac{\xi}{2}, \quad (2.25)
\]
\[
I_n(t_n, x_n + c_n t) \leq \frac{\xi}{2}, \quad \forall t \in [t_n, t_n + s_n], \quad (2.26)
\]
\[ I_n(t_n + s_n, x_n + c_n(t_n + s_n)) = \frac{1}{n}, \tag{2.27} \]

where \( \varepsilon \) is the positive constant provided in Lemma 2.5.

In view of parabolic estimates, there exists a \( \tilde{\vartheta} \in [0, T) \) such that \( c_n \), possibly along a subsequence, converges to \( c_\infty \in [0, \varepsilon^0] \) and

\[ (S_n, I_n)(t + t_n + s_n, x_n + c_n(t_n + s_n) + x) \text{ (possibly along a subsequence)} \]

\[ \rightarrow (\tilde{S}_\infty, \tilde{I}_\infty)(t, x; \tilde{\vartheta}) \]

locally uniformly in \( \mathbb{R}^2 \) as \( n \) tends to infinity, wherein \((\tilde{S}_\infty, \tilde{I}_\infty)\) is an entire solution of system \((2.8)\) with \( \vartheta = \tilde{\vartheta} \) and \( \tilde{\vartheta} \) is an accumulation point of \([0, \varepsilon^0] \). Moreover, it follows from \((2.27)\) that \( \tilde{I}_\infty(0, 0; \tilde{\vartheta}) = 0 \), which implies \( \tilde{I}_\infty(\cdot, \cdot; \tilde{\vartheta}) \equiv 0 \) by the strong maximum principle. Let \( l = \lim_{n \to \infty} s_n \), then we have \( l = \infty \). Otherwise, if \( l < \infty \), then the definition of \((\tilde{S}_\infty, \tilde{I}_\infty)\) implies that \( \tilde{I}_\infty(\cdot, -lc_\infty; \tilde{\vartheta}) = \lim_{n \to \infty} I_n(t_n, x_n + c_n t_n) = \frac{\varepsilon}{2} \), which is a contradiction.

Similarly, we can find a \( \tilde{\vartheta} \in [0, T) \) such that \( c_n \), possibly along a subsequence, converges to \( c_\infty \in [0, \varepsilon^0] \), and

\[ (S_n, I_n)(t + t_n, x_n + c_n t_n + x) \text{ (possibly along a subsequence)} \]

\[ \rightarrow (\tilde{S}_\infty, \tilde{I}_\infty)(t, x; \tilde{\vartheta}) \]

locally uniformly in \( \mathbb{R}^2 \) as \( n \to \infty \), wherein \((\tilde{S}_\infty, \tilde{I}_\infty)\) is an entire solution of system \((2.8)\) with \( \vartheta = \tilde{\vartheta} \) and \( \tilde{\vartheta} \) is an accumulation point of \([0, \varepsilon^0] \). By \((2.25)\), we have \( \tilde{I}_\infty(0, 0; \tilde{\vartheta}) = \frac{\varepsilon}{2} \), which implies \( \tilde{S}_\infty(0, \cdot; \tilde{\vartheta}) \neq 0 \). Consequently, it follows from Lemma 2.5 that

\[ \lim_{t \to \infty} \tilde{I}_\infty(t, x + ct; \tilde{\vartheta}) \geq \varepsilon, \forall x \in \mathbb{R}, c \in [0, \varepsilon^0]. \tag{2.28} \]

On the other hand, by \((2.26)\), we get \( I_n(t + t_n, x_n + c_n(t + t_n)) \leq \frac{\varepsilon}{2}, \forall t \in [0, s_n) \). Besides, by \( l = \infty \), we have \( I_n(t + t_n, x_n + c_n(t + t_n)) \to \tilde{I}_\infty(t, c_\infty t; \tilde{\vartheta}) \)

locally uniformly for all \( t \in [0, \infty) \) as \( n \to \infty \). And hence,

\[ \lim_{t \to \infty} \tilde{I}_\infty(t, c_\infty t; \tilde{\vartheta}) \leq \frac{\varepsilon}{2}, \]

which contradicts with \((2.28)\). The proof is completed. \( \square \)

In view of Lemma 2.5 and Lemma 2.6, the proof of Proposition 1, namely, Theorem 2.3(ii) and Theorem 2.4(ii), can be completed by a similar argument to the proof of [10, Lemma 5.7]. The remaining task in this section is verify Theorem 2.3(ii) and Theorem 2.4(i).

**Proof of Theorem 2.3(ii).** We first show the following claim:

**Claim 2** For any \( c \in [c_1, c_2] \), one has \( \lim_{n \to \infty} S(t, ct) = 1 \).

**Proof of Claim 2.** Fix a \( c \in [c_1, c_2] \). Since \( S(t, ct) \leq 1 \) for all \( t > 0 \), suppose by contradiction that there exist a sufficiently small constant \( c \in (0, 1) \) and a sequence \( \{t_n\}_{n \geq 1} \) with \( \lim_{n \to \infty} t_n = \infty \) such that

\[ S(t_n, ct_n) \leq 1 - \epsilon \quad \text{as} \quad n \to \infty. \tag{2.29} \]

It follows from Theorem 2.3(i) that \( I(t_n, ct_n) \to 0 \) as \( n \to \infty \), then by similar arguments to the proof of Claim 1, there exist a \( N_5 \in \mathbb{N}_+ \) and a sequence \( \{s''_n\}_{n \geq N_5} \)

such that
with $s''_n \geq 0$ such that

$$S(t + t_n + s''_n, c(t + t_n + s''_n)) > 1 - \frac{\epsilon}{2}$$

is uniformly for all $t > 0$ and $n \geq N_6$. Then we can find a subsequence of $\{t_n\}_{n \geq N_6}$, labeled it by $\{t_{n_m}\}$, such that

$$S(t_{n_m}, c t_{n_m}) > 1 - \frac{\epsilon}{2}. \tag{2.30}$$

In fact, we need only to let $n_1 = N_5$ and $n_{m+1} = \min\{l \in \mathbb{N}_+ : t_l \geq t_{n_m} + s''_{n_m}\}$ for $m \geq 1$. Then we see that (2.30) contradicts (2.29) and the proof of Claim 2 is completed.

We now proceed to the proof of Theorem 2.3(ii). For any $t > 0$, set $Q_t := \{(t, x) \in \mathbb{R}^2 : c_1 t \leq x \leq c_2 t\}$. Choose $\zeta > 0$ such that $c_\zeta^* := 2\sqrt{d_1((1 - \zeta) r - \zeta \beta)} \in (c_2, c^*)$.

With the aid of (2.5), there exists a $T_1 > 0$ sufficiently large such that $I(t, x) < \zeta$ in $Q_t$ for all $t \geq T_1$. Then we have

$$\frac{\partial S(t, x)}{\partial t} \geq d_1 \partial_{xx} S(t, x) + [(1 - \zeta) r(t) - \zeta \beta(t)] S(t, x) - r(t) S^2(t, x),$$

$$\forall (t, x) \in Q_t, t \geq T_1.$$

Let $S_\zeta^*$ be the globally asymptotically stable positive $T$-periodic solution of

$$\frac{d S_\zeta(t)}{dt} = [(1 - \zeta) r(t) - \zeta \beta(t)] S_\zeta(t) - r(t) S_\zeta^2(t), t > 0.$$

Then we have $\max_{t \in [0, T]} S_\zeta^*(t) < 1$ and

$$\max_{t \in [0, T]} | S_\zeta^*(t) - 1 | \to 0 \text{ as } \zeta \to 0. \tag{2.31}$$

Then it follows from Claim 2 that we can find a sufficiently large constant $T_2 \geq T_1$ such that $S(t, c_i t) \geq S_\zeta^*(t), \forall t \geq T_2, i = 1, 2$.

Let $S_\zeta$ be the solution of

$$\begin{cases}
\frac{\partial S_\zeta(t, x)}{\partial t} = d_1 \partial_{xx} S_\zeta(t, x) + [(1 - \zeta) r(t) - \zeta \beta(t)] S_\zeta(t, x) - r(t) S_\zeta^2(t, x), \\
0 < S_\zeta(T_2, x) \leq S(T_2, x), S_\zeta(T_2, x) \in C(Q_{T_2}), c_1 T_2 < x < c_2 T_2, \\
S_\zeta(t, c_i t) = S_\zeta^*(t), t \geq T_2, i = 1, 2.
\end{cases}$$

As a consequence, by comparison principle, we have

$$S(t, x) \geq S_\zeta(t, x), \forall (t, x) \in Q_t \text{ with } t \geq T_2. \tag{2.32}$$

Define $\tilde{S}_\zeta$ by

$$\tilde{S}_\zeta(t, x) := \begin{cases}
S_\zeta(t, x), & (t, x) \in Q_t, \\
S_\zeta^*(t), & (t, x) \in [T_2, \infty) \times \mathbb{R} \setminus Q_t,
\end{cases} \quad t \geq T_2,$$

then $\tilde{S}_\zeta$ satisfies

$$\frac{\partial \tilde{S}_\zeta(t, x)}{\partial t} = d_1 \partial_{xx} \tilde{S}_\zeta(t, x) + [(1 - \zeta) r(t) - \zeta \beta(t)] \tilde{S}_\zeta(t, x) - r(t) \tilde{S}_\zeta^2(t, x), \quad x \in \mathbb{R}, t > T_2$$

and $\tilde{S}_\zeta(T_2, x) > 0, \forall x \in \mathbb{R}$. As a consequence, for any $c \in [c_1, c_2]$, it follows from Lemma 2.2 that

$$\lim_{t \to \infty} \sup_{|x| \leq ct} | \tilde{S}_\zeta(t, x) - S_\zeta^*(t) | = 0,$$
that is,
\[
\lim_{t \to \infty} \sup_{|x| \leq ct} |S_\zeta(t, x) - S_\zeta^*(t)| = 0,
\]
which combining (2.31) imply that
\[
\lim_{\zeta \to 0} \lim_{t \to \infty} \sup_{(t,x) \in Q_t} |S_\zeta(t, x) - 1| = 0. \tag{2.33}
\]
Let \(\zeta \to 0\), it then follows from (2.32) and (2.33) that \(\lim_{t \to \infty} \inf_{|x| \leq ct} S(t, x) = 1\) and the proof is completed. \(\square\)

**Proof of Theorem 2.4(i).** Noting that the proof for (2.6) is same as that in Theorem 2.3(i), we need only to check (2.7).

We assume by contradiction that there exits a constant \(\zeta > 0\) and a sequence \(\{(t_n, x_n)\}_{n \geq 1}\) with \(|x_n| > ct_n\) such that \(|I(t_n, x_n)| > \zeta\). Denote \((S_n, I_n)(t, x) := (S, I)(t + t_n, x_n + x), \forall (t, x) \in \mathbb{R}\). Following the boundedness of \(S_n\) and \(I_n\) and standard parabolic estimates, without loss of generality, we can assume that \((S_n, I_n)\) converges to \((S_\infty, I_\infty)\) locally and uniformly in \(\mathbb{R}^2\). Then \((S_\infty, I_\infty)\) is an entire solution of system (2.8) with parameter \(\vartheta = \theta\), which is an accumulation point of \(\{[t_n]\}\). It is easy to see that \(I_\infty(0,0) \geq \zeta\). Moreover, it follows from (2.6) that \(S_\infty(0,0) = 0\), which implies that \(S_\infty \equiv 0\). Then we have
\[
\frac{\partial I_\infty}{\partial t} = d_2 \partial_{zz} I_\infty(t, x) - \gamma \vartheta I_\infty(t, x), \quad (t, x) \in \mathbb{R}^2.
\]
With the help of the boundedness of \(I_\infty\), we obtain \(I_\infty \equiv 0\), which contradicts with \(I_\infty(0,0) \geq \zeta\). This completes the proof. \(\square\)

3. **Periodic traveling wave solutions.** The aim of this section is to derive the existence of periodic traveling wave solutions of system (1.1) under the condition \(R_0 > 1\). The technology framework we used is mainly inspired by [26, 27]. Moreover, with the help of spreading properties obtained in Proposition 1, we can provide a quite simpler construction of the sub- and super-solutions and a more concise way to show the asymptotic boundary behavior of the periodic traveling wave solutions than that in [27]. Throughout 3.1-3.3, we always let Assumptions 1.1 and 1.2 hold.

It is easy to see that the kinetic system (1.2) has a unique disease-free equilibrium \((1, 0)\). We now introduce the concept of periodic traveling wave solution for (1.1).

**Definition 3.1.** A solution admitting special form
\[
(S(t, x), I(t, x)) := (\phi(t, z), \psi(t, z)) \text{ with } z = x + ct
\]
is called a periodic traveling wave solution of system (1.1) if it satisfies
\[
\begin{align*}
\phi_t(t, z) &= d_1 \phi_{zz}(t, z) - c \phi_z(t, z) + r(t) \phi(t, z)(1 - \phi(t, z) - \psi(t, z)) \\
&\quad - \frac{\beta(t) \phi(t, z) \psi(t, z)}{1 + \alpha(t) \psi(t, z)}, \quad (t, z) \in \mathbb{R}^2, \\
\psi_t(t, z) &= d_2 \psi_{zz}(t, z) - c \psi_z(t, z) + \frac{\beta(t) \phi(t, z) \psi(t, z)}{1 + \alpha(t) \psi(t, z)} - \gamma(t) \psi(t, z), \quad (t, z) \in \mathbb{R}^2,
\end{align*}
\tag{3.1}
\]
and
\[
\lim_{z \to -\infty} (\phi(t, z), \psi(t, z)) = (1, 0) \text{ uniformly for } t \in \mathbb{R}, \tag{3.2}
\]
where $c > 0$ is the wave speed, $z = x + ct$ presents the moving coordinate and $(\phi, \psi)$ denotes the wave profile. Further, we say a periodic traveling wave solution $(\phi, \psi)$ is persistent if it satisfies
\[
\limsup_{z \to \infty} \phi(t, z) < \infty, \quad \limsup_{z \to \infty} \psi(t, z) < \infty, \quad \liminf_{z \to \infty} \phi(t, z) > 0, \quad \liminf_{z \to \infty} \psi(t, z) > 0 \quad (3.3)
\]
uniformly for $t \in \mathbb{R}$.

To determine the asymptotic behaviour of a periodic traveling wave solution of system (1.1) as $z \to \infty$, we need to establish the dynamics of the kinetic system (1.2).

**Lemma 3.2.** Let Assumption 1.1 be satisfied, then we have:

(i) If $R_0 < 1$, then the disease-free state $(1, 0)$ is globally attractive for system (1.2);

(ii) If $R_0 > 1$, then system (1.2) at least admits a positive $T$-periodic solution $(S^*(t), I^*(t))$. Moreover, there exists a positive constant $\eta > 0$ such that
\[
\liminf_{t \to \infty} (S(t), I(t)) > (\eta, \eta)
\]
for all $(S, I)$ with initial data in $(0,1] \times (0, G]$.

This lemma can be proved by a similar argument as [30, Theorem 3.1], so we omit it.

Inspired by Lemma 3.2, we shall establish the existence of time periodic traveling wave solutions of (1.1). We start it with the construction of suitable sub- and super-solutions.

### 3.1. The construction of sub- and super-solutions

Suppose $R_0 > 1$. Linearizing the second equation of system (3.1) at the disease-free steady state $(1, 0)$, we have
\[
\tilde{I}_t(t, x) = d_2 \tilde{I}_{zz} - c \tilde{I}_z(t, z) + (\beta(t) - \gamma(t)) \tilde{I}(t, x).
\]
Define
\[
\Theta^c(\lambda) = d_2 \lambda^2 - c \lambda + \bar{\beta} - \bar{\gamma}, \quad c \in \mathbb{R}, \lambda \in \mathbb{R}.
\]
For any fixed $c > c^*$, denoting
\[
\lambda_1 = \frac{c - \sqrt{c^2 - 4d_2(\beta - \gamma)}}{2d_2}, \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4d_2(\beta - \gamma)}}{2d_2},
\]
we have $\Theta^{c}(\lambda_1) = \Theta^{c}(\lambda_2) = 0$ and $\Theta^{c}(\lambda) < 0, \forall \lambda \in (\lambda_1, \lambda_2)$. Labeling
\[
K(t) := \exp \left( \int_0^t [d_2 \lambda_1^2 - c \lambda_1 + (\beta(s) - \gamma(s))] ds \right)
\]
\[
= \exp \left( \int_0^t [\beta(s) - \gamma(s) - (\bar{\beta} - \bar{\gamma})] ds \right),
\]
we further define four functions:
\[
\phi^+(t, z) := 1, \quad \phi^-(t, z) := \max\{1 - ke^{\eta z}, 0\},
\]
\[
\psi^+(t, z) := \min\{e^{\lambda_1 z} K(t), G\}, \quad \psi^-(t, z) := \max\{e^{\lambda_1 z} - M e^{\mu z} K(t), 0\},
\]
where the constant $G$ has been defined in Section 1 and $k, \eta, \mu, M$ are all positive constants being assumed to satisfy the following options:
A1. let \( \eta < \lambda_1 \) sufficiently small and \( k > 1 \) large enough such that
\[
k(d_1 \eta^2 - c\eta) + (r(t) + \beta(t))K(t) < 0, \quad \forall t \in \mathbb{R},
\] and
\[
\frac{1}{\eta} - \frac{1}{k} < \min_{t \in [0, T]} \frac{1}{\lambda_1} \ln \frac{G(t)}{K(t)};
\] (3.5)

A2. let \( \mu \in (1, \min\{2, \frac{\lambda_2}{\lambda_1}\}) \) such that
\[
\mu < 1 + \frac{\eta}{\lambda_1};
\]

A3.
\[
M \geq \max \left\{ \max_{t \in [0, T]} \left( k + 4\alpha(t)K(t)\beta(t) - \Theta_\sigma(\mu \lambda_1) \right), k^{\frac{1}{\kappa}((\mu - 1)\lambda_1)} \right\}.
\]

Here we stress that the choice of these parameters obey the following sequence:
\( \eta, \mu \) or \( k, M \).

Lemma 3.3. The functions \( \phi^+ \) and \( \psi^- \) satisfy
\[
\phi_i^+(t, z) - d_1 \phi_z^+(t, z) + c\phi_z^+(t, z) - r(t)\phi^+(t, z)(1 - \phi^+(t, z) - \psi^-(t, z))
+ \frac{\beta(t)\phi^+(t, z)\psi^-(t, z)}{1 + \alpha\psi^-(t, z)} \geq 0
\]
for all \( (t, x) \in \mathbb{R} \).

Lemma 3.4. The functions \( \phi^+ \) and \( \psi^+ \) satisfy
\[
\psi_i^+(t, z) - d_2 \psi_z^+(t, z) + c\psi_z^+(t, z) - \beta(t)\phi^+(t, z)\psi^+(t, z)
+ \frac{\gamma(t)\psi^+(t, z)}{1 + \alpha\psi^+(t, z)} \geq 0
\]
for all \( (t, x) \in \mathbb{R} \).

We remark that Lemma 3.3 is self-evident and Lemma 3.4 can be easily checked by the definition of \( \lambda_1, K(t) \) and \( G \), so we omit the proofs for them.

Lemma 3.5. The functions \( \phi^- \) and \( \psi^+ \) satisfy
\[
\phi_i^-(t, z) - d_1 \phi_z^-(t, z) + c\phi_z^-(t, z) - r(t)\phi^-(t, z)(1 - \phi^-(t, z) - \psi^+(t, z))
+ \frac{\beta(t)\phi^-(t, z)\psi^+(t, z)}{1 + \alpha\psi^+(t, z)} \leq 0
\] (3.5)
for any \( z \neq z_1 := \frac{1}{\eta} \ln \frac{G(t)}{k} < 0 \).

Proof. If \( z > z_1 \), then \( \phi^- (t, z) = 0 \), the conclusion is self-evident. Under the condition \( z < z_1 \), we have \( \phi^- (t, z) = 1 - ke^{\eta z} \) and \( \psi^+ (t, z) = e^{\lambda_1 z}K(t) \) by (3.5). It then follows from (3.4) that
\[
\phi_i^-(t, z) - d_1 \phi_z^-(t, z) + c\phi_z^-(t, z) - r(t)\phi^- (t, z)(1 - \phi^- (t, z) - \psi^+(t, z))
+ \frac{\beta(t)\phi^- (t, z)\psi^+(t, z)}{1 + \alpha\psi^+(t, z)}
= k(d_1 \eta^2 - c\eta)e^{\eta z} - r(t)(1 - ke^{\eta z})ke^{\eta z} + r(t)(1 - ke^{\eta z})e^{\lambda_1 z}K(t)
+ \frac{\beta(t)(1 - ke^{\eta z})e^{\lambda_1 z}K(t)}{1 + \alpha(t)e^{\lambda_1 z}K(t)}
\leq k(d_1 \eta^2 - c\eta)e^{\eta z} + (r(t) + \beta(t))e^{\lambda_1 z}K(t)
\]

for any \( z \neq z_1 \).
Reduction to a fixed point problem on finite intervals.

Lemma 3.6. The functions $\phi^-$ and $\psi^-$ satisfy

$$
\psi_i^-(t, z) - d_2 \psi_{zz}^-(t, z) + c \psi_z^-(t, z) - \frac{\beta(t) \phi^-(t, z) \psi^-(t, z)}{1 + \alpha(t)} + \gamma(t) \psi^-(t, z) \leq 0 \quad (3.6)
$$

for any $z \neq z_2 := \frac{\ln M}{(\mu - 1)\lambda_1}$.

Proof. If $z > z_2$, we see that $\psi^-(t, z) = 0$ and then (3.6) holds. In view of A3, we have $M \geq k^{\frac{1}{\mu - 1}\lambda_1}$, which implies $z_2 < z_1 < 0$. Thus, when $z < z_2$, we have $\phi^-(t, z) = 1 - ke^{\eta z}$ and $\psi^-(t, z) = (e^{\lambda_1 z} - Me^{\mu\lambda_1 z})K(t)$. It follows that

$$
\psi_i^-(t, z) - d_2 \psi_{zz}^-(t, z) + c \psi_z^-(t, z) - \frac{\beta(t) \phi^-(t, z) \psi^-(t, z)}{1 + \alpha(t)} + \gamma(t) \psi^-(t, z) = e^{\lambda_1 z}\{K'(t) - d_2 \lambda_1^2 K(t) + c \lambda_1 K(t) - [\beta(t) - \gamma(t)]K(t)\} + \beta(t) e^{\lambda_1 z} K(t) - M e^{\mu\lambda_1 z} K(t) - \frac{\beta(t)(1 - ke^{\eta z})(e^{\lambda_1 z} - Me^{\mu\lambda_1 z})K(t)}{1 + \alpha(t)(e^{\lambda_1 z} - Me^{\mu\lambda_1 z})K(t)}
$$

Then by virtue of A2 and A3, we have

$$
\psi_i^-(t, z) - d_2 \psi_{zz}^-(t, z) + c \psi_z^-(t, z) - \frac{\beta(t) \phi^-(t, z) \psi^-(t, z)}{1 + \alpha(t)} + \gamma(t) \psi^-(t, z) \leq e^{\mu\lambda_1 z}\{M e^{\Theta^+(\mu_1)} + k \beta(t)[e^{(1-\mu)\lambda_1 + \eta z} + 4 \beta(t)e^{\Theta^+(\mu_1)}K(t)]\}K(t) \leq 0,
$$

and the proof is completed.

3.2. Reduction to a fixed point problem on finite intervals. Take $N > -z_2$.

Let $C_N := C(\mathbb{R} \times [-N, N], \mathbb{R}^2)$ and further define

$$
\Gamma_N := \left\{(\phi, \psi) \in C_N : (\tilde{\phi}, \tilde{\psi})(t, z) = (\phi, \psi)(t + T, z), \forall (t, z) \in \mathbb{R} \times [-N, N] : \begin{cases} (\phi^-, \psi^-)(t, z) \leq (\tilde{\phi}, \tilde{\psi})(t, z) \leq (\phi^+, \psi^+)(t, z), \\
(\phi, \psi)(t, \pm N) = (\phi^-, \psi^-)(t, \pm N), \forall t \in \mathbb{R} \end{cases} \right\}.
$$
We define two maps on $\Gamma_N$ by

\[
\begin{align*}
f_1(\tilde{\phi}, \tilde{\psi})(t, z) &:= \alpha_1 \tilde{\phi}(t, z) + r(t) \phi(t, z)(1 - \tilde{\phi}(t, z) - \tilde{\psi}(t, z)) - \frac{\beta(t) \tilde{\phi}(t, z) \tilde{\psi}(t, z)}{1 + \alpha(t) \psi(t, z)}, \\
\end{align*}
\]
and

\[
\begin{align*}
f_2(\tilde{\phi}, \tilde{\psi})(t, z) &:= \alpha_2 \tilde{\psi}(t, z) + \frac{\beta(t) \tilde{\phi}(t, z) \tilde{\psi}(t, z)}{1 + \alpha(t) \psi(t, z)} - \gamma(t) \tilde{\psi}(t, z),
\end{align*}
\]
where $\alpha_1$ and $\alpha_2$ are all positive constants with $\alpha_1 > \max_{t \in [0, T]} r(t) + \max_{t \in [0, T]} \frac{\beta(t)}{\alpha(t)}$ and $\alpha_2 > \max_{t \in [0, T]} \gamma(t)$, respectively.

Let $A_i u = d_i \partial_z u - c \partial_z u - \alpha_i u$, $i = 1, 2$. For any given $(\tilde{\phi}, \tilde{\psi}) \in \Gamma_N$, we construct the following linear parabolic system

\[
\begin{align*}
\begin{cases}
\partial_t \phi(t, z) - A_1 \phi(t, z) = f_1(\tilde{\phi}, \tilde{\psi})(t, z), \\
\partial_t \psi(t, z) - A_2 \psi(t, z) = f_2(\tilde{\phi}, \tilde{\psi})(t, z),
\end{cases}
\end{align*}
\]
with initial condition

\[
\begin{align*}
\phi(0, z) = \phi_0(z), \psi(0, z) = \psi_0(z), \quad \forall z \in [-N, N],
\end{align*}
\]
and boundary condition

\[
\begin{align*}
\phi(t, \pm N) = G_1(t, \pm N), \psi(t, \pm N) = G_2(t, \pm N), \quad \forall t \geq 0,
\end{align*}
\]
where $\phi_0, \psi_0 \in C([-N, N])$ and

\[
\begin{align*}
G_1(t, z) := \frac{1}{2} \psi^-(t, -N) - \frac{z}{2N} \psi^-(t, -N), \quad G_2(t, z) := \frac{1}{2} \psi^-(t, -N) - \frac{z}{2N} \psi^-(t, -N),
\end{align*}
\]
\[
\forall (t, z) \in [0, T] \times [-N, N].
\]
It is easy to see that $G_1(t, \pm N) = \psi^-(t, \pm N), G_2(t, \pm N) = \psi^-(t, \pm N)$ for $t \in \mathbb{R}$, and $G_i \in C^{1,2}(\mathbb{R} \times [-N, N])$ is $T$-periodic on $t$, $i = 1, 2$. Let

\[
\begin{align*}
V_1(t, z) = \phi(t, z) - G_1(t, z), \quad V_2(t, z) = \psi(t, z) - G_2(t, z)
\end{align*}
\]
and $\tilde{G}_i = A_i G_i(t, z) - \partial_z G_i(t, z)$, $i = 1, 2$. Then we can convert system (3.7) with (3.8-3.9) to the following system with homogeneous boundary condition:

\[
\begin{align*}
\begin{cases}
\partial_t V_i(t, z) - A_i V_i(t, z) = f_i(\tilde{\phi}, \tilde{\psi})(t, z) + \tilde{G}_i(t, z), \quad t > 0, \quad z \in [-N, N], \quad i = 1, 2, \\
V_i(0, z) = \phi_0(z) - G_1(0, z), \quad V_i(0, z) = \psi_0(z) - G_2(0, z), \quad z \in [-N, N], \\
V_i(t, \pm N) = 0, \quad t \geq 0, \quad i = 1, 2.
\end{cases}
\end{align*}
\]

Define the realization of $A_i$ in $C([-N, N])$ by

\[
\begin{align*}
D(A_i^0) = \left\{ w \in \bigcap_{p \geq 1} W^2_p((-N, N)) : w, A_i w \in C([-N, N]), w|_{\pm N} = 0 \right\},
\end{align*}
\]
\[
A_i^0 w = A_i w, \quad i = 1, 2.
\]
Let $T_i(t)_{t \geq 0}$ be the strongly continuous analytic semigroup generated by $A_i^0 : D(A_i^0) \subset C([-N, N]) \rightarrow C([-N, N])$ [17]. It is easy to see that

\[
T_i(t) \varphi(x) = e^{-\alpha_i t} \int_{-N}^{N} \Gamma_i(t, x, y) \varphi(y) dy, \quad \varphi \in C([-N, N]),
\]
\[
\varphi \in C([-N, N]),
\]
for all $t > 0, x \in [-N, N]$, where $\Gamma_i$ is the Green function associated with $d_i \partial_{xx} - c_i \partial_x$ and Dirichlet boundary condition, $i = 1, 2$. Based on above discussions, by (3.10), we get the following integral system:

$$
\begin{cases}
\phi(t, z) = T_1(t) (\phi_0 - G_1(0)) (z) + \int_0^t T_1(t-s) \left( f_1 [\tilde{\phi}, \tilde{\psi}] (s) + \tilde{G}_1(s) \right) (z) ds \\
+ G_1(t, z), \\
\psi(t, z) = T_2(t) (\psi_0 - G_2(0)) (z) + \int_0^t T_2(t-s) \left( f_2 [\tilde{\phi}, \tilde{\psi}] (s) + \tilde{G}_2(s) \right) (z) ds \\
+ G_2(t, z)
\end{cases}
$$

(3.11)

for all $t \geq 0$ and $z \in [-N, N]$. It is clear that a solution of (3.11) is a mild solution of (3.7) with (3.8-3.9).

Let $C_N := C([-N, N], \mathbb{R}^2)$ and further define a closed and convex set

$$
\Gamma_N' := \left\{ (\phi_0, \psi_0) \in C_N : \forall z \in [-N, N] : \\
\phi_0(\pm N) = \phi^-(0, \pm N), \psi_0(\pm N) = \psi^-(0, \pm N) \right\}
$$

with the usual supreme norm. For any $(\phi_0, \psi_0) \in \Gamma_N'$, let $(\phi_N, \psi_N) (t, z; \phi_0, \psi_0)$ be the solution of the system (3.11) with the initial value $(\phi_0, \psi_0)$. Based on Lemma 3.3-3.6, using a similar method as the proof of [27, Lemma 2.4], we can obtain

$$
\phi^- (t, z) \leq \phi_N (t, z; \phi_0, \psi_0) \leq \phi^+ (t, z), \quad \psi^- (t, z) \leq \psi_N (t, z; \phi_0, \psi_0) \leq \psi^+ (t, z)
$$

(3.12)

for all $t > 0$ and $z \in [-N, N]$. Then for a given $(\tilde{\phi}, \tilde{\psi}) \in \Gamma_N$, we define a map $F_{(\tilde{\phi}, \tilde{\psi})} : \Gamma_N \to C_N$ by

$$
F_{(\tilde{\phi}, \tilde{\psi})}[\phi_0, \psi_0](\cdot) = (\phi_N, \psi_N)(T, \cdot; \phi_0, \psi_0),
$$

where $(\phi_N, \psi_N) (t, \cdot; \phi_0, \psi_0)$ is the solution of (3.7) with initial value $(\phi_0, \psi_0) \in \Gamma_N'$. By (3.12) and the periodicity of $\phi^\pm (t, \cdot)$ and $\psi^\pm (t, \cdot)$, we have $F_{(\tilde{\phi}, \tilde{\psi})} (\Gamma_N') \subset \Gamma_N'$. It is easy to see that $\Gamma_N'$ is a complete metric space with a distance induced by the supreme norm.

Moreover, we can further check that $F_{(\tilde{\phi}, \tilde{\psi})} : \Gamma_N \to \Gamma_N'$ is a contraction map by some conventional computations. Then according to the Banach fixed point theorem, we come to the conclusion that $F_{(\tilde{\phi}, \tilde{\psi})}$ admits a unique fixed point $(\phi_0^*, \psi_0^*) \in \Gamma_N'$. Denote $(\phi_N^*, \psi_N^*)(t, z) := (\phi_N, \psi_N)(t, z; \phi_0^*, \psi_0^*), \forall (t, x) \in [0, \infty) \times [-N, N]$ be the solution of (3.11) with initial value $(\phi_0^*, \psi_0^*)$. Based on the fact that $(\phi_0^*, \psi_0^*)(\cdot) = (\phi_N^*, \psi_N^*)(T, \cdot)$, the existence interval of $(\phi_N^*, \psi_N^*)(t, \cdot)$ can be extended to $\mathbb{R}$. Besides, $(\phi_N^*, \psi_N^*)$ is $T$-periodic on $t$ for any $z \in [-N, N]$. Combining with (3.12), we have $(\phi_N^*, \psi_N^*) \in \Gamma_N$. As a consequence, for any given $(\tilde{\phi}, \tilde{\psi}) \in \Gamma_N$, there exists a unique $(\phi_N^*, \psi_N^*) \in \Gamma_N$ satisfies

$$
\begin{cases}
\phi_N^*(t) = T_1(t-s) (\phi_N^*(s) - G_1(s)) + \int_s^t T_1(t-r) \left( f_1 [\tilde{\phi}, \tilde{\psi}] (r) + \tilde{G}_1(r) \right) dr \\
+ G_1(t), \\
\psi_N^*(t) = T_2(t-s) (\psi_N^*(s) - G_2(s)) + \int_s^t T_2(t-r) \left( f_2 [\tilde{\phi}, \tilde{\psi}] (r) + \tilde{G}_2(r) \right) dr \\
+ G_2(t)
\end{cases}
$$

for all $t \geq s$.

We further define an operator $\mathcal{F} : \Gamma_N \to \Gamma_N$ by $\mathcal{F}(\tilde{\phi}, \tilde{\psi}) = (\phi_N^*, \psi_N^*)$. In addition, we can use a similar method as the proof for [27, Lemma 3.5] to verify that $\mathcal{F}$ is completely continuous. Then it from the Schauder’s fixed point theorem that $\mathcal{F}$
admits a fixed point in $\Gamma_N$. To ease notations, we still denote this fixed point by $(\phi^*_N, \psi^*_N)$. In particular, $(\phi^*_N, \psi^*_N)(t + T, \cdot)$ = $(\phi^*_N, \psi^*_N)(t, \cdot)$ for all $t \in \mathbb{R}$. Note that $\phi^*_N, \psi^*_N \in C^{\theta/2}(\mathbb{R} \times [-N, N])$ for some $\theta \in (0, 1)$. By [17, Theorem 5.1.18 and 5.1.19], we have that

$$\partial_t \phi^*_N(t, z) = d_1 \partial_z z \phi^*_N(t, z) - c_\partial \partial_z \phi^*_N(t, z) + r(t)\phi^*_N(t, z)(1 - \phi^*_N(t, z) - \psi^*_N(t, z))$$

for all $(t, z) \in \mathbb{R} \times [-N, N]$, where

$$\partial_t \psi^*_N(t, z) = d_2 \partial_z z \psi^*_N(t, z) - c_\partial \partial_z \psi^*_N(t, z) + \frac{\beta(t)\phi^*_N(t, z)\psi^*_N(t, z)}{1 + \alpha(t)\psi^*_N(t, z)} - \gamma(t)\psi^*_N(t, z),$$

for all $(t, z) \in \mathbb{R} \times [-N, N]$, where

$$\phi^*_N(t, \pm N) = \phi^-(t, \pm N), \quad \psi^*_N(t, \pm N) = \psi^-(t, \pm N), \quad t \in \mathbb{R}.$$  

(3.13)

In what follows, we will derive a solution of system (3.1) by limiting arguments applied on $\{ (\phi^*_N, \psi^*_N) \}$, so we first need some local uniform estimates on $\phi^*_N$ and $\psi^*_N$:

**Lemma 3.7.** Let $p \geq 2$. For any given $Z > 0$, there exists a positive constant $B(p, Z)$ such that

$$\|\phi^*_N\|_{W^{1,2}_p([0, T] \times [-Z, Z])} \leq B \quad \text{and} \quad \|\psi^*_N\|_{W^{1,2}_p([0, T] \times [-Z, Z])} \leq B$$

for all $N > \max\{Z, -z_2\}$. Moreover, there exists a positive constant $B'(Z)$ such that for any $z_0 \in \mathbb{R}$,

$$\|\phi^*_N\|_{C^{(1+\theta)/2, 1+\theta}([0, T] \times [z_0 - Z, z_0 + Z])}, \quad \|\psi^*_N\|_{C^{(1+\theta)/2, 1+\theta}([0, T] \times [z_0 - Z, z_0 + Z])} \leq B'$$

for all $N > \max\{Z + |z_0|, -z_2\}$, wherein $\theta \in (0, 1)$.

**Proof.** Fix $Z > 0$ and $z_0 \in \mathbb{R}$. Let $N > \max\{Z + |z_0|, -z_2\}$. Defining

$$\nu^\phi_{N}(t, z) := e^{-\frac{c(z-z_0)}{2\sigma_1}} \phi^*_N(t, z) \quad \text{and} \quad \nu^\psi_{N}(t, z) := e^{-\frac{c(z-z_0)}{2\sigma_2}} \psi^*_N(t, z)$$

for all $(t, z) \in \mathbb{R} \times [-N, N]$, we have

$$\partial_t \nu^\phi_{N}(t, z) = d_1 \partial_z z \nu^\phi_{N}(t, z) - \frac{c^2}{4d_1 - e^{-\frac{c(z-z_0)}{2\sigma_1}}} \phi^*_N(t, z) + r(t)\phi^*_N(t, z)(1 - \phi^*_N(t, z) - \psi^*_N(t, z)) - \frac{\beta(t)\phi^*_N(t, z)\psi^*_N(t, z)}{1 + \alpha(t)\psi^*_N(t, z)}$$

for any $(t, z) \in \mathbb{R} \times (-N, N)$. Define

$$\Lambda^\phi_{N}(t, z) = -\frac{c^2}{4d_1 - e^{-\frac{c(z-z_0)}{2\sigma_1}}} \phi^*_N(t, z) - r(t)\phi^*_N(t, z)(1 - \phi^*_N(t, z) - \psi^*_N(t, z))$$

and

$$\Lambda^\psi_{N}(t, z) = -\frac{c^2}{4d_2 - e^{-\frac{c(z-z_0)}{2\sigma_2}}} \psi^*_N(t, z) + \frac{\beta(t)\phi^*_N(t, z)\psi^*_N(t, z)}{1 + \alpha(t)\psi^*_N(t, z)} - \gamma(t)\psi^*_N(t, z)$$

for all $(t, z) \in \mathbb{R} \times [-N, N]$. Due to $(\phi^*_N, \psi^*_N) \in \Gamma_N$, we have

$$\sup_{(t, z) \in \mathbb{R} \times [-N, N]} \phi^*_N(t, z) \leq 1, \quad \sup_{(t, z) \in \mathbb{R} \times [-N, N]} \psi^*_N(t, z) \leq G.$$  

For any $(t^\circ, z^\circ) \in \mathbb{R}^2$ and $\tau > 0$, define

$$Q((t^\circ, z^\circ), \tau) := \{(t, z) \in \mathbb{R}^2 \mid |z - z^\circ| < \tau, |t - t^\circ| < \tau, t < t^\circ\}.$$
Let $R = \max\{2Z, \sqrt{3T}\}$. Applying [16, Proposition 7.14], for $N > 72R + |z_0|$, we can find a positive constant $B_1(p, R)$ independent of $N$, such that
\[
\|\partial_z \nu^i_N\|_{L^p(Q((2T,z_0),2R))} \leq B_1 \left( \|\nu^i_N\|_{L^p(Q((2T,z_0),72R))} + \|\Lambda^i_N\|_{L^p(Q((2T,z_0),72R))} \right),
\]
where $i = \phi, \psi$. By the definition of $\nu^\phi_N$ and $\nu^\psi_N$, there exists a positive constant $B_2(p, R)$ independent of $N$, such that
\[
\|\partial_z \nu^\phi_N\|_{L^p(Q((2T,z_0),2R))}, \|\partial_z \nu^\psi_N\|_{L^p(Q((2T,z_0),2R))} \leq B_2.
\]
Further, thanks to above estimates, it follows from [16, Proposition 7.18] that there exists a positive constant $B_3(p, R)$, which is independent of $N$, such that
\[
\|\partial_{zz} \phi^*_N\|_{L^p(Q((2T,z_0),R))} + \|\partial_t \phi^*_N\|_{L^p(Q((2T,z_0),R))} \leq B_3
\]
and
\[
\|\partial_{zz} \psi^*_N\|_{L^p(Q((2T,z_0),R))} + \|\partial_t \psi^*_N\|_{L^p(Q((2T,z_0),R))} \leq B_3.
\]

Consequently, we can find a positive constant $B(p, R)$ independent of $N$, such that
\[
\|\phi^*_N\|_{W^2_0(Q((2T,z_0),R))}, \|\psi^*_N\|_{W^2_0(Q((2T,z_0),R))} \leq B.
\]

Because of $[0,T] \times [-Z, Z] \subset Q((2T,0), R)$, we further have
\[
\|\phi^*_N\|_{W^2_0([0,T] \times [-Z,Z])}, \|\psi^*_N\|_{W^2_0([0,T] \times [-Z,Z])} \leq B.
\]
Here $R$ merely depends on $Z$, then $B$ only relies on $Z$ and $p$.

Take $p > 3$. Using the embedding theorem, we have
\[
\phi^*_N, \psi^*_N \in C^{(1+\theta)/2,1+\theta}(0,T] \times [Z_0 - Z, Z_0 + Z]), \text{ for some } \theta \in (0,1)
\]
and
\[
\|\phi^*_N\|_{C^{(1+\theta)/2,1+\theta}(0,T] \times [-Z,Z])}, \|\psi^*_N\|_{C^{(1+\theta)/2,1+\theta}(0,T] \times [-Z,Z])} \leq B',
\]
where $B' > 0$ is a constant depending upon $p$ and $Z$.

3.3. Existence of persistent periodic traveling wave solutions.

Theorem 3.8. Suppose $R_0 > 1$. For any $c > c^{**}$, system (1.1) admits a time periodic traveling wave solution $(\phi^*, \psi^*)$, which satisfies (3.1), (3.2) and (3.3).

Proof. Let $\{N_m\}$ be an increasing sequence with $N_m \geq -z_2$ and $\lim_{m \to \infty} N_m = \infty$. Then we have $(\phi^*_{N_m}, \psi^*_{N_m})$ satisfies Lemma 3.7 and (3.13) with $N = N_m$.

In view of the periodicity of $(\phi^*_{N_m}, \psi^*_{N_m})$ in $t \in \mathbb{R}$, we can extract a subsequence of $(\phi^*_{N_m}, \psi^*_{N_m})$, for notational simplicity, still labeled it by $(\phi^*_{N_m}, \psi^*_{N_m})$, such that

$(\phi^*_{N_m}, \psi^*_{N_m})$ converges to $(\phi^*, \psi^*) \in C(\mathbb{R})$ in $C_{loc}^{1+\beta,2+\nu}(\mathbb{R}^2)$, in $H^1_{loc}(\mathbb{R}^2)$ and in $L^2_{loc}(\mathbb{R}, H^2_{loc}(\mathbb{R}))$ all weakly, where $\beta \in (0,\theta)$ and $\theta \in (0,1)$ is given in Lemma 3.7.

Based on above discussions and the local uniform estimates of $\phi^*_{N_m}$ and $\psi^*_{N_m}$ in Lemma 3.7, we can show that $\phi^*, \psi^* \in C^{1+\frac{\nu}{2}+\nu'}(\mathbb{R}^2)$ for some $\nu \in (0,1)$. Moreover, by similar arguments to those of [27, Theorem 2.9], we can verify $(\phi^*, \psi^*)$ satisfies (3.1), that is
\[
\begin{cases}
\phi^*_t(t,z) = d\phi^*_zzz(t,z) - c\phi^*_z(t,z) + r(t)\phi^*(t,z)(1 - \phi^*(t,z) - \psi^*(t,z)) - \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{1 + \alpha(t)\psi^*(t,z)}, \\
\psi^*_t(t,z) = d\psi^*_zzz(t,z) - c\psi^*_z(t,z) + \frac{\beta(t)\phi^*(t,z)\psi^*(t,z)}{1 + \alpha(t)\psi^*(t,z)} - \gamma(t)\psi^*(t,z), \quad (t,z) \in \mathbb{R}^2.
\end{cases}
\]

Due to $(\phi^*_{N_m}, \psi^*_{N_m}) \in \Gamma_{N_m}$, then we have
\[
\phi^*(t,z) \leq \phi^*_{N_m}(t,z) \leq \phi^+(t,z), \quad \psi^-(t,z) \leq \psi^*_{N_m}(t,z) \leq \psi^+(t,z),
\]
where $\phi^+(t,z) = \phi^*_{N_m}(t,z) + \phi^+(t,z)$, $\psi^-(t,z) = \psi^*_{N_m}(t,z) + \psi^-(t,z)$.
\[ \forall (t, z) \in \mathbb{R} \times [-N_m, N_m], \]

which implies that \((\phi^*, \psi^*)\) satisfies

\[ \phi^-(t, z) \leq \phi^*(t, z) \leq 1, \quad \psi^-(t, z) \leq \psi^*(t, z) \leq \psi^+(t, z), \quad \forall (t, z) \in \mathbb{R}^2. \]

Hence, (3.2) holds. In view of Definition 3.1, the remaining task is to present the asymptotic behavior of \(\phi^*(t, z)\) and \(\psi^*(t, z)\) as \(z \to \infty\). In what follows we only show \(\liminf_{z \to \infty} \psi^*(t, z) > 0\) uniformly for all \(t \in \mathbb{R}\) and that for \(\phi^*\) can be followed by a similar way.

Denote \(I^*(t, x) := \psi^*(t, x+ct)\), then we have \(\psi^*(t, z) = I^*(t, z-ct), \forall (t, x) \in \mathbb{R}^2\). We assume by contradiction that there exists \(\{t_n\}_{n \geq 1} \subset [0, T]\) and \(\{z_n\}_{n \geq 1}\) with \(\lim_{n \to \infty} z_n = \infty\) such that \(\psi^*(t_n, z_n) < \frac{1}{n}\). By the periodicity of \(\psi^*(t, \cdot)\), we have

\[ \psi^*(t_n, z_n) = \psi^*(t_n + k_nT, z_n) = I^*(t_n + k_nT, z_n - c(t_n + k_nT)), \]

where \(\{k_n\} \subset \mathbb{N}_+\) is a strictly increasing sequence to be further determined below.

Fix a \(\hat{c} \in (0, c^\Delta)\). Here we recall that \(c^\Delta = \min\{c^*, c^{**}\}\). Owning to \(z_n \to \infty\) as \(n \to \infty\), we can choose a subsequence of \(\{z_n\}\), still denoted by \(\{z_n\}\), such that

\[ \frac{1}{T}(\frac{z_{n+1}}{c + \hat{c}}) > \frac{1}{T}(\frac{z_n}{c + \hat{c}}) + 3 \quad (3.14) \]

for sufficiently large \(n\). We further let

\[ k_n := \min\{k \in \mathbb{N}_+ : k \geq \frac{1}{T}(\frac{z_n}{c + \hat{c}} - t_n)\} \]

for sufficiently large \(n\). In view of (3.14), we have

\[ \frac{1}{T}(\frac{z_{n+1}}{c + \hat{c}} - t_{n+1}) > \frac{1}{T}(\frac{z_{n+1}}{c + \hat{c}} - T) > \frac{1}{T}(\frac{z_n}{c + \hat{c}} + T) + 1 > \frac{1}{T}(\frac{z_n}{c + \hat{c}} - t_n) + 1, \]

which implies that \(\{k_n\}\) is strictly increasing and \(\lim_{n \to \infty} k_n = \infty\). Moreover, the definition of \(\{k_n\}\) implies that \(0 < \frac{z_n}{c + \hat{c}} - c < \hat{c}\), that is, \(|z_n - c(t_n + k_nT)| < \hat{c}(t_n + k_nT)|\) for all sufficiently large \(n\). Then by Proposition 1, we have

\[ \lim_{n \to \infty} I^*(t_n + k_nT, z_n - c(t_n + k_nT)) > \varepsilon > 0, \]

wherein \(\varepsilon\) is the constant given in Proposition 1. On the other hand,

\[ \lim_{n \to \infty} I^*(t_n + k_nT, z_n - c(t_n + k_nT)) = \lim_{n \to \infty} \psi^*(t_n, z_n) = 0. \]

Then a contradiction occurs and the proof is completed. \(\square\)

**Remark 1.** Despite that we establish the existence of time periodic traveling waves of system (1.1) in the set \(C\), we can show that if (1.1) admits a time periodic and persistent traveling wave solution \((\phi, \psi)\), then there must be \((\phi(t, \cdot), \psi(t, \cdot)) \in C\) for any \(t \in \mathbb{R}\).

**Proposition 2.** If system (1.1) has a time periodic traveling wave solution \((\phi, \psi)\) satisfying (3.3), then \((\phi(t, \cdot), \psi(t, \cdot)) \in C\) for any \(t \in \mathbb{R}\).

**Proof.** According to the definition of \(C\), we need only to prove that \(\phi(t, z) \leq 1\) and \(\psi(t, z) \leq G\) for any \(t \in \mathbb{R}\) and \(z \in \mathbb{R}\). We now first show that \(\phi(t, z) \leq 1\) for any \(t \in \mathbb{R}\) and \(z \in \mathbb{R}\). Otherwise, we assume by contradiction that \(\sup_{(t, z) \in [0, T] \times \mathbb{R}} \phi(t, z) := \ell > 1\). Then there exists \(\{(t_k, z_k)\}_{k \geq 1}\) with \(t_k \in [0, T]\) such that \(\phi(t_k, z_k) \to \ell\) as \(k \to \infty\). Denote \((\phi_k, \psi_k)(t, z) := (\phi, \psi)(t, z + z_k), \forall (t, z) \in \mathbb{R}\), then \(\phi_k(t_k, 0) = \phi(t_k, z_k)\). From the boundedness of \(\phi_k\) and \(\psi_k\) and standard parabolic estimates,
there exists a subsequence of \( \{(\phi_k, \psi_k)\} \), still denoted it by \( \{(\phi_k, \psi_k)\} \), such that \( t_k \to t^* \) and 
\[
(\phi_k, \psi_k)(t, z) \to (\hat{\phi}, \hat{\psi})(t, z)
\]
locally uniformly on \([0, T] \times \mathbb{R}\). Moreover, we have
\[
\frac{\beta(t) \hat{\phi}(t, z) \hat{\psi}(t, z)}{1 + \alpha(t) \psi(t, z)}
\]
for all \((t, z) \in \mathbb{R}^2\). Clearly, the definition of \( \hat{\phi} \) implies that \( \hat{\phi}(t_*, 0) = \ell \) and \( \hat{\phi}(t, z) \leq \ell \), \( \forall (t, z) \in [0, T] \times \mathbb{R} \), that is, \( \hat{\phi} \) attains its maximum at \((t_*, 0)\), which implies
\[
\hat{\phi}(t_*, 0) = 0, \quad \hat{\phi}_z(t_*, 0) = 0, \quad \hat{\phi}_{zz}(t_*, 0) \leq 0.
\]
On the other hand, it is easy to see
\[
\frac{\beta(t_*) \hat{\phi}(t_*, 0) \hat{\psi}(t_*, 0)}{1 + \alpha(t_*) \psi(t_*, 0)} < 0.
\]
Consequently, the left side of (3.15) equals to 0 while the right side of (3.15) is strictly less than 0 at \((t_*, 0)\). This is a contradiction. We have proved that \( \phi(t, z) \leq 1 \) for any \( t \in \mathbb{R} \) and \( z \in \mathbb{R} \).

Next, we need to show \( \psi(t, z) \leq G \) for any \( t \in \mathbb{R} \) and \( z \in \mathbb{R} \). By a similar way, we assume by contradiction that \( \sup_{(t,z) \in [0, T] \times \mathbb{R}} \psi(t, z) := G^* > G \). Then there exists \( \{(t_m, z_m)\}_{m \geq 1} \) with \( t_m \in [0, T] \) such that \( \psi(t_m, z_m) \to G^* \) as \( m \to \infty \). Define \( (\phi_m, \psi_m)(t, z) := (\phi, \psi)(t, z + z_m) \). In particular, we have \( \psi_m(t_m, 0) = \psi(t_m, z_m) \).

From the boundedness of \( \phi_m \) and \( \psi_m \) and standard parabolic estimates, we can extract a subsequence of \( \{(\phi_m, \psi_m)\} \), still denoted it by \( \{(\phi_m, \psi_m)\} \), such that \( t_m \to t^* \) and 
\[
(\phi_m, \psi_m)(t, z) \to (\hat{\phi}, \hat{\psi})(t, z)
\]
locally uniformly on \([0, T] \times \mathbb{R}\). Moreover, \( (\hat{\phi}, \hat{\psi}) \) satisfies
\[
\hat{\psi}_1(t, z) = d_2 \hat{\psi}_{zz}(t, z) - c \hat{\psi}_z(t, z) + \frac{\beta(t) \hat{\phi}(t, z) \hat{\psi}(t, z)}{1 + \alpha(t) \psi(t, z)} - \gamma(t) \hat{\psi}(t, z), \quad (t, z) \in \mathbb{R}^2. \quad (3.16)
\]

By similar arguments as above discussion, we see that \( \hat{\psi} \) attains its maximum \( G^* \) at \((t^*, 0)\) and
\[
\hat{\psi}_1(t^*, 0) = 0, \quad \hat{\psi}_z(t^*, 0) = 0, \quad \hat{\psi}_{zz}(t^*, 0) \leq 0. \quad (3.17)
\]
On the other hand, recalling the definition of \( G \), we have
\[
\frac{\beta(t^*) \hat{\phi}(t^*, 0)}{1 + \alpha(t^*) \psi(t^*, 0)} - \gamma(t^*) \leq \frac{\beta(t^*)}{1 + \alpha(t^*) G} - \gamma(t^*) < \frac{\beta(t^*)}{1 + \alpha(t^*) G} - \gamma(t^*) \leq 0,
\]
which combines (3.17) deduce that the left side of (3.16) equals to 0 while the right side of (3.16) is strictly less than 0 at \((t^*, 0)\). A contradiction occurs and then we have proved the proposition. \( \square \)
3.4. Nonexistence of persistent periodic traveling wave solution. In this section, our task is to investigate the nonexistence of time periodic traveling wave solutions for two cases: (i) $R_0 \leq 1$; (ii) $R_0 > 1$ and $0 < c < c^{**}$. Throughout this section, we always let Assumption 1.1 hold. We shall first show that there is periodic traveling wave solution in the case where $R_0 \leq 1$.

**Theorem 3.9.** Assume that $R_0 \leq 1$. Then for any $c \geq 0$, there is no nonnegative, nontrivial and time periodic traveling wave solution for system (1.1).

**Proof.** By contradiction, suppose that there exists a nonnegative, nontrivial and time periodic solution $(\phi(t, z), \psi(t, z))$ of (3.1). Since $0 \leq \phi(t, z) \leq 1$, $\forall t \geq 0$, $x \in \mathbb{R}$, we have

$$\psi_t(t, z) = d_2 \psi_{zz}(t, z) - c \psi_z(t, z) + \frac{\beta(t)\phi(t, z)\psi(t, z)}{1 + \alpha(t)\psi(t, z)} - \gamma(t)\psi(t, z)$$

$$\leq d_2 \psi_{zz}(t, z) - c \psi_z(t, z) + \left[\frac{\beta(t)}{1 + \alpha(t)\psi(t, z)} - \gamma(t)\right] \psi(t, z)$$

for any $t > 0$ and $z \in \mathbb{R}$. Let $\varrho := \sup_{z \in \mathbb{R}} \psi(0, z) < \infty$, then $\psi(0, z) \leq \varrho$, $\forall z \in \mathbb{R}$. By the comparison principle, we have $\psi(t, z) \leq v(t; \varrho)$, $\forall t > 0, z \in \mathbb{R}$, where $v(t; \varrho)$ is the solution of the following periodic ordinary differential equation:

$$\begin{cases}
\frac{dv(t)}{dt} = \left[\frac{\beta(t)}{1 + \alpha(t)v(t)} - \gamma(t)\right] v(t), & t > 0, \\
v(0) = \varrho.
\end{cases}$$

Owing to $R_0 \leq 1$, that is, $\beta \leq \gamma$, then we have

$$\int_0^T \left[\frac{\beta(t)}{1 + \alpha(t)v(t)} - \gamma(t)\right] dt \leq \int_0^T \left[\beta(t) - \gamma(t)\right] dt \leq 0.$$ 

As a consequence, it follows that $\lim_{t \to \infty} v(t; \varrho) = 0$. Thus $\lim_{t \to \infty} \psi(t, z) = 0$, $\forall z \in \mathbb{R}$, which contradicts with the time periodicity of $\psi(t, \cdot)$. The proof is completed.

It remains to show the nonexistence of persistent periodic traveling wave solution under the condition $R_0 > 1$ as well as $0 < c < c^{**}$.

**Theorem 3.10.** Assume that $R_0 > 1$. For any $c \in (0, c^{**})$, system (1.1) does not have a periodic traveling wave solution satisfying (3.3) uniformly for $t \in \mathbb{R}$.

**Proof.** We will prove this theorem in two cases: (i) $c^{**} \leq c^*$; (ii) $c^* < c^{**}$.

(i) The case $c^{**} \leq c^*$. We suppose by contradiction that there exists some $c \in (0, c^{**})$ such that

$$(S(t, x), I(t, x)) = (\phi(t, x + ct), \psi(t, x + ct))$$

is a periodic traveling wave solution of system (1.1) with (3.3) uniformly for $t \in \mathbb{R}$.

Let $\bar{c} = \frac{c + c^{**}}{2} \in (c, c^{**})$, then we have

$$I(nT, -\bar{c}nT) = \psi(nT, (\bar{c} - c)nT) = \psi(0, (c - \bar{c})nT), \ \forall n \in \mathbb{N}_+.$$ 

By Proposition 2, we have $0 \leq S(t, x) \leq 1$ and $0 \leq I(t, x) \leq G$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$. Then it follows from Proposition 1 that

$$\lim_{n \to \infty} \inf I(nT, -\bar{c}nT) > 0.$$ 

However, the asymptotic boundary (3.2) requires

$$\lim_{n \to \infty} \psi(0, (c - \bar{c})nT) = \psi(0, -\infty) = 0.$$
This is a contradiction. Therefore, in the case that \( c^{**} < c^* \), the conclusion of this theorem is true.

(ii) The case \( c^* < c^{**} \). Suppose, by contradiction, that there exists periodic traveling wave solution \( (\phi(t,(x + ct)), \psi(t,(x + ct))) \) satisfying (3.3) for some \( c \in (0,c^{**}) \). Owning to \( \bar{R}_0 > 1 \), that is, \( \bar{\beta} > \bar{\gamma} \), then there exists a sufficiently small positive number \( q_0 \) such that \( \beta(1-q_0)-\bar{\gamma} > 0 \). We fix a \( q \in (0,q_0) \) such that \( c < 2\sqrt{R_{2q}} \), here \( \kappa_q := \bar{\beta}(1-q)-\bar{\gamma} \). Since \( \lim_{t \to -\infty}(\phi(t,z), \psi(t,z)) = (1,0) \) uniformly for \( t \in \mathbb{R} \), there exists an \( M_0 > 0 \) sufficiently large such that for any \( z < -M_0 \),

\[
\frac{\phi(t,z)}{1 + \alpha(t)\psi(t,z)} > 1 - q \text{ uniformly for } t \in \mathbb{R}.
\]

(3.18)

Fix a \( c_0 \in (c,2\sqrt{R_{2q}}) \). Define

\[
\Theta^q_\lambda(\lambda) = d_2\lambda^2 - c_0\lambda + \bar{\beta}(1-q) - \bar{\gamma}, \lambda \in \mathbb{R}.
\]

Then we denote \( \lambda_{1,2}^q := A_q \pm iB_q \) be the two roots of the equation \( \Theta^q_\lambda(\lambda) = 0 \), where \( i \) is the imaginary unit. It is easy to see that \( A_q > 0 \) and \( B_q > 0 \). We further define

\[
K_q(t) = \exp \left( \int_0^t \beta(s)(1-q) - \gamma(s) \right) ds - \kappa_q t.
\]

Then we see that \( K_q(t) \) is a positive \( T \)-periodic function and satisfies

\[
\frac{dv(t)}{dt} = \beta(t)(1-q) - \gamma(t)v(t) - \kappa_qv(t), \forall t \in \mathbb{R}.
\]

We now define a function \( \Psi(t,z) := e^{A_q z} \sin(B_q z)K_q(t) \). It is easy to verify that \( \Psi(t,\cdot) \) is \( T \)-periodic and satisfies

\[
\partial_t \Psi(t,z) = d_2\partial_{zz}\Psi(t,z) - c_0\partial_z\Psi(t,z) + [\beta(t)(1-q) - \gamma(t)]\Psi(t,z), \forall t > 0, z \in \mathbb{R}.
\]

Let \( Z_1 = \frac{-2\lambda_1 \pi}{B_q}, Z_2 = \frac{-2\lambda_2 \pi}{B_q} \), where \( k_0 \in \mathbb{N}^+ \) with \( \frac{2k_0 \pi}{B_q} > M_0 \). It is easy to see that \( \sin(B_q Z_1) = \sin(B_q Z_2) = 0 \) and \( \sin(B_q z) > 0, \forall z \in [Z_1, Z_2] \). Owning to \( \psi(0,z) \) is strictly positive on \([Z_1, Z_2]\), then there exists an \( \epsilon > 0 \) such that \( \psi(0,z) \geq \epsilon \psi(0,z), \forall z \in [Z_1, Z_2] \).

Denote \( \tilde{\psi}(t,x) := \psi(t,x + (c - c_0)t) \). Since \( c - c_0 < 0 \) and \( Z_1 \leq Z_2 < -M_0 \), then we have \( x + (c - c_0)t \leq -M_0, \forall t \geq 0, x \in [Z_1, Z_2] \). As a consequence, with the aid of (3.18), we follows that

\[
\partial_t \tilde{\psi}(t,x) = d_2\partial_{xx}\tilde{\psi}(t,x) - c_0\partial_x\tilde{\psi}(t,x) + \beta(t)\phi(t,x + (c - c_0)t)\tilde{\psi}(t,x) - \gamma(t)\psi(t,x) + \frac{\beta(t)\phi(t,x + (c - c_0)t)\tilde{\psi}(t,x)}{1 + \alpha(t)\psi(t,x)} - \gamma(t)\psi(t,x)
\]

for all \( t > 0 \) and \( x \in [Z_1, Z_2] \). Let \( V(t,x) := \tilde{\psi}(t,x + (c - c_0)t) - \epsilon \Psi(t,x), \forall t \geq 0, x \in [Z_1, Z_2] \). Then we have

\[
\begin{cases}
\partial_t V(t,x) \geq d_2\partial_{xx}V(t,x) - c_0\partial_xV(t,x) + [\beta(t)(1-q) - \gamma(t)]V(t,x), \\
V(0,x) \geq 0, x \in [Z_1, Z_2], \\
V(t,Z_j) \geq 0, t \geq 0, j = 1, 2.
\end{cases}
\]

Consequently, the maximum principle implies that \( V(t,x) \geq 0, \forall (t,x) \in (0,\infty) \times [Z_1, Z_2] \), that is,

\[
\psi(t,x + (c - c_0)t) \geq \epsilon \Psi(t,x)
\]
for all \( t > 0 \) and \( x \in [Z_1, Z_2] \). Since \( c - c_0 < 0 \), we have \( \psi(t, x + (c - c_0)t) \to 0 \) as \( t \to \infty \) for all \( x \in [Z_1, Z_2] \). This is a contradiction. We complete the proof.

**Remark 2.** In the case (ii), using similar arguments as the case (i), we only can verify the conclusion for \( c \in (0, c^*) \). Under the condition where \( c \in [c^*, c^{**}) \), the method used for the case (i) would be not applicable due to the different asymptotic spreading property as the case (i).

4. **Discussion.** In this paper, we investigated the asymptotic spreading properties and the periodic traveling wave solutions of system (1.1), which models the transmission of epidemic in time periodic environment. We presented an overall picture for the spreading phenomena in terms of two asymptotic speeds of spread and further established the existence and nonexistence of the persistent periodic traveling wave solutions, respectively. The result indicates that the minimum wave speed of the periodic traveling wave solutions equals to the spreading speed of the infectious class.

Despite we only represent the asymptotic spreading properties on \( \mathbb{R} \), we remark that the conclusions are indeed valid in \( \mathbb{R}^N \) with any \( N \geq 1 \). In addition, with the aid of inner spreading properties, we provided a comparatively convenient method to verify the asymptotic boundary conditions of the periodic traveling wave fronts. Meanwhile, this method implies that the periodic traveling wave solution is certainly being persistent provided that the solutions with positive initial data admit inner spreading properties with a positive spreading speed.

Besides, if the incidence function is replaced by \( \frac{\beta(t)S(t,x)I(t,x)}{S(t,x)+I(t,x)} \) (standard incidence rate) in (1.1), then in the case \( c^{**} < c^* \), we can obtain the same inner asymptotic spreading properties under an additional technical assumption \( 2\sqrt{d_1(\bar{r} - \bar{\beta})} \geq c^{**} \). While our strategy may not work in another case \( c^{**} \geq c^* \). How to verify inner asymptotic spreading properties for system (1.1) with standard incidence rate is one of our future projects.

Moreover, for a more realistic purpose, we can consider a more general case by incorporating the latency of disease into model (1.1), then the nonlocal interaction occurs because of the mobility of individuals during latent period. By using an analogous argument to that of [20], we can derive the following nonlocal system:

\[
\begin{align*}
\frac{\partial S(t,x)}{\partial t} &= d_1 \Delta S(t,x) + r(t)S(t,x)(1 - S(t,x) - I(t,x)) - \frac{\beta(t)S(t,x)I(t,x)}{1 + \alpha(t)I(t,x)}, \\
\frac{\partial I(t,x)}{\partial t} &= d_2 \Delta I(t,x) - \gamma(t)I(t,x) \\
&\quad + \int_\mathbb{R} \Gamma(t, t - \tau; x - y) \frac{\beta(t - \tau)S(t - \tau, y)I(t - \tau, y)}{1 + \alpha(t - \tau)I(t - \tau, y)} \, dy, \quad t > 0, x \in \mathbb{R},
\end{align*}
\]

(4.1)

wherein the kernel function \( \Gamma \) reveals the effects of nonlocal interaction. The nonlocal effect in (4.1) may bring a lot of additional challenges. Most notably, the spreading speed of the infectious class \( c^{**} \) cannot be expressed explicitly in such case. The periodic traveling wave solutions and the asymptotic spreading properties of system (4.1) is also worth to be deeply inquired.

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*E-mail address*: wangshm17@lzu.edu.cn (Shuang-Ming Wang)

*E-mail address*: zhaosheng.feng@utrgv.edu (Zhaosheng Feng)

*E-mail address*: wangzhch@lzu.edu.cn (Zhi-Cheng Wang)

*E-mail address*: lz@lzu.edu.cn (Liang Zhang)