Abstract

We define the scattered subsets of a group as asymptotic counterparts of scattered subspaces of a topological space, and prove that a subset $A$ of a group $G$ is scattered if and only if $A$ contains no piecewise shifted $IP$-subsets. For an amenable group $G$ and a scattered subspace $A$ of $G$, we show that $\mu(A) = 0$ for each left invariant Banach measure $\mu$ on $G$.

1 Introduction

Given a discrete space $X$, we take the points of $\beta X$, the Stone-Čech compactification of $X$, to be the ultrafilters on $X$, with the points of $X$ identified with the principal ultrafilters on $X$. The topology on $\beta X$ can be defined by stating that the sets of the form $A = \{ p \in \beta X : A \in p \}$, where $A$ is a subset of $X$, form a base for the open sets. We note that the sets of this form are clopen and that, for any $p \in \beta X$ and $A \subseteq X$, $A \in p$ if and only if $p \in \overline{A}$. For any $A \subseteq X$ we denote $A^* = \overline{A} \cap G^*$, where $G^* = \beta G \setminus G$. The universal property of $\beta G$ states that every mapping $f : X \to Y$, where $Y$ is a compact Hausdorff space, can be extended to the continuous mapping $f^\beta : \beta X \to X$.

Now let $G$ be a discrete group. Using the universal property of $\beta G$, we can extend the group multiplication from $G$ to $\beta G$ in two steps. Given $g \in G$, the mapping

$$x \mapsto gx : G \to \beta G$$

extends to the continuous mapping

$$q \mapsto gq : \beta G \to \beta G.$$
Then, for each $q \in \beta G$, we extend the mapping $g \mapsto gq$ defined from $G$ into $\beta G$ to the continuous mapping

$$p \mapsto pq : \beta G \to \beta G.$$  

The product $pq$ of the ultrafilters $p, q$ can also be defined by the rule: given a subset $A \subseteq G$,

$$A \in pq \iff \{ g \in G : g^{-1}A \in q \} \in p.$$

To describe the base for $pq$, we take any element $P \in p$ and, for every $x \in P$, choose some element $Q_x \in q$. Then $\cup_{x \in P} xQ_x \in pq$, and the family of subsets of this form is a base for the ultrafilter $pq$.

By the construction, the binary operation $(p, q) \mapsto pq$ is associative, so $\beta G$ is a semigroup, and $G^*$ is a subsemigroup of $\beta G$. For each $q \in \beta G$, the right shift $x \mapsto xq$ is continuous, and the left shift $x \mapsto xq$ is continuous for each $g \in G$.

For the structure of a compact right topological semigroup $\beta G$ and plenty of its applications to combinatorics, topological algebra and functional analysis see [1], [2], [3], [4], [5].

Given a subset $A$ of a group $G$ and an ultrafilter $p \in G^*$, we define a $p$-companion of $A$ by

$$\Delta_p(A) = A^* \cap gp = \{gp : g \in G, A \in gp\},$$

and say that a subset $S$ of $G^*$ is an ultracompanion of $A$ if $S = \Delta_p(A)$ for some $p \in G^*$. For ultracompanions of subsets of groups and metric spaces see [6], [7].

Clearly, $A$ is finite if and only if $\Delta_p(A) = \varnothing$ for each $p \in G^*$.

We say that a subset $A$ of a group $G$ is

- thin if $|\Delta_p(A)| \leq 1$ for each $p \in G^*$;
- $n$-thin, $n \in \mathbb{N}$ if $|\Delta_p(A)| \leq n$ for each $p \in G^*$;
- sparse if each ultracompanion of $A$ is finite;
- disperse if each ultracompanion of $A$ is discrete;
- scattered if, for each infinite subset $Y$ of $A$, there is $p \in Y^*$ such that $\Delta_p(Y)$ is finite.
We denote by \([G]^\omega\) the family of all finite subsets of \(G\). Given any \(F \in [G]^\omega\) and \(g \in G\), we put
\[
B(g, F) = Fg \cup \{g\}
\]
and, following [8], say that \(B(g, F)\) is a ball of radius \(F\) around \(g\). For a subset \(Y\) of \(G\), we put \(B_Y(g, F) = Y \cap B(g, F)\). By [6, Proposition 4], \(Y\) is \(n\)-thin if and only if for every \(F \in [G]^\omega\), there exists \(H \in [G]^\omega\) such that \(|B_Y(y, F)| \leq n\) for each \(y \in Y \setminus H\). For thin subsets of a group, their applications and modifications see [9]–[19].

By [6, Proposition 5] and [20, Theorems 3 and 10], for a subset \(A\) of a group \(G\), the following statements are equivalent

1. \(A\) is sparse;
2. for every infinite subset \(X\) of \(G\), there exists finite subset \(F \subset G\) such that \(\bigcap_{g \in F} gA\) is finite;
3. for every infinite subset \(Y\) of \(A\), there exists \(F \in [G]^\omega\) such that, for every \(H \in [G]^\omega\), we have
   \[
   \{y \in Y : B_A(y, H) \setminus B_A(y, F) = \emptyset\} \neq \emptyset;
   \]
4. \(A\) has no subsets asymorphic to the subset \(W_2 = \{g \in \oplus_\omega \mathbb{Z}_2 : \text{sup}^g \leq 2\}\) of the group \(\oplus_\omega \mathbb{Z}_2\), where \(\text{sup}^g\) is the member of non-zero coordinates of \(g\).

The notion of asymorphisms and coarse equivalence will be defined in the next section. The sparse sets were introduced in [21] in order to characterise strongly prime ultrafilters in \(G^*\), the ultrafilters from \(G^* \setminus \overline{G^*G^*}\). More on sparse subsets can be find in [10], [11], [16], [22].

In this paper, answering Question 4 from [6], we prove that a subset \(A\) of a group \(G\) is scattered if and only if \(A\) is disparse, and characterize the scattered subsets in terms of prohibited subsets. We answer also Question 2 from [6] proving that each scattered subset of an amenable group is absolute null. The results are exposed in section 2, their proofs in section 3.
2 Results

Our first statement shows that, from the asymptotic point of view \[23\], the scattered subsets of a group can be considered as the counterparts of the scattered subspaces of a topological space.

Proposition 1. For a subset $A$ of a group $G$, the following two statements are equivalent

(i) $A$ is scattered;

(ii) for every infinite subset $Y$ of $A$, there exists $F \in [G]^{<\omega}$ such that, for every $H \in [G]^{<\omega}$, we have

$$\{y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \emptyset\} \neq \emptyset.$$  

Proposition 2. A subset $A$ of a group $G$ is scattered if and only if, for every countable subgroup $H$ of $G$, $A \cap H$ is scattered in $H$.

Let $A$ be a subset of a group $G$, $K \in [G]^{<\omega}$. A sequence $a_0, ..., a_n$ in $A$ is called $K$-chain from $a_0$ to $a_n$ if $a_{i+1} \in B(a_i, K)$ for each $i \in \{0, ..., n-1\}$. For every $a \in A$, we denote

$$B_A^\sqsubset(a, K) = \{b \in A : \text{there is a } K\text{-chain from } a \text{ to } b\}$$

and, following [24] Chapter 3], say that $A$ is cellular (or asymptotically zero-dimentional) if, for every $K \in [G]^{<\omega}$, there exists $K' \in [G]^{<\omega}$ such that, for each $a \in A$,

$$B_A^\sqsubset(a, K) \subseteq B_A(a, K').$$

Now we need some more asymptology (see [24] Chapter 1]). Let $G, H$ be groups, $X \subseteq G, Y \subseteq H$. A mapping $f : X \rightarrow Y$ is called a $\prec$-mapping if, for every $F \in [G]^{<\omega}$, there exists $K \in [G]^{<\omega}$ such that, for every $x \in X$,

$$f(B_X(x, F)) \subseteq B_Y(f(x), K).$$

If $f$ is a bijection such that $f$ and $f^{-1}$ are $\prec$-mapping, we say that $f$ is an asymorphism. The subsets $X$ and $Y$ are called coarse equivalent if there exist asymorphic subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that $X \subseteq B_X(X', F)$, $Y \subseteq B_Y(Y', K)$ for some $F \in [G]^{<\omega}$ and $K \in [H]^{<\omega}$. 

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Following [23], we say that the set $Y$ of $G$ has no asymptotically isolated balls if $Y$ does not satisfy Proposition [1](ii): for every $F \in [G]^{<\omega}$, there exists $H \in [G]^{<\omega}$ such that $B_Y(y, H) \setminus B_Y(y, F) \neq \emptyset$ for each $y \in Y$.

By [23], a countable cellular subset $Y$ of $G$ with no asymptotically isolated balls is coarsely equivalent to the group $\oplus \omega \mathbb{Z}_2$.

**Proposition 3.** Let $X$ be a countable subset of a group $G$. If $X$ is not cellular then $X$ contains a subset $Y$ coarsely equivalent to $\oplus \omega \mathbb{Z}_2$.

Let $(g_n)_{n<\omega}$ be an injective sequence in a group $G$. The set

$$\{g_{i_1}g_{i_2}...g_{i_n} : 0 \leq i_1 < i_2 < ... < i_n < \omega\}$$

is called an IP-set [1, p. 406], the abbreviation for "infinite dimensional parallelepiped".

Given a sequence $(b_n)_{n<\omega}$ in $G$, we say that the set

$$\{g_{i_1}g_{i_2}...g_{i_n}b_{i_n} : 0 \leq i_1 < i_2 < ... < i_n < \omega\}$$

is a piecewise shifted IP-set.

**Theorem 1.** For a subset $A$ of a group $G$, the following statements are equivalent

(i) $A$ is scattered;

(ii) $A$ is disparse;

(iii) $A$ contains no subsets coarsely equivalent to the group $\oplus \omega \mathbb{Z}_2$;

(iv) $A$ contains no piecewise shifted IP-sets.

By the equivalence $(i) \Leftrightarrow (ii)$ and Propositions 10 and 12 from [6], the family of all scattered subsets of an infinite group $G$ is a translation invariant ideal in the Boolean algebra of all subsets of $G$ strictly contained in the ideal of all small subsets.

Now we describe some relationships between the left invariant ideals $Sp_G$, $Sc_G$ of all sparse and scattered subsets of a group $G$ on one hand, and closed left ideals of the semigroup $\beta G$.

Let $J$ be a left invariant ideal in the Boolean algebra $P_G$ of all subsets of a group $G$. We set

$$\hat{J} = \{p \in \beta G : G \setminus A \in p \text{ for each } A \in J\}$$
and note that $\hat{J}$ is a closed left ideal of the semigroup $\beta G$. On the other hand, for a closed left ideal $L$ of $\beta G$, we set

$$\hat{L} = \{ A \subseteq G : A \notin p \text{ for each } p \in L \}$$

and note that $\hat{L}$ is a left invariant ideal in $\mathcal{P}_G$. Moreover, $\hat{J} = J$ and $\hat{L} = L$.

Clearly, $[G]^{<\omega} = G^*$ and by Theorem 1

$$(\star) \quad \hat{S}_c G = \text{cl}\{ p \in \beta G : Gp \text{ is discrete in } \beta G \} = \text{cl}\{ p \in \beta G : p = \varepsilon p \text{ for some idempotent } \varepsilon \in G^* \}.$$

Given a left invariant ideal $J$ in $\mathcal{P}_G$ and following [11], we define a left invariant ideal $\sigma(J)$ by the rule:

$$A \in \sigma(J) \text{ if and only if } \Delta_p(A) \text{ is finite for every } p \in \hat{J}.$$

Equivalently, $\sigma(J) = \text{cl}(G^* \hat{J})$. Thus, we have

$$\hat{S}_p G = \text{cl}(G^* G^*).$$

We say that a left invariant ideal $J$ in $\mathcal{P}_G$ is sparse-complete if $\sigma(J) = J$ and denote by $\sigma^*(J)$ the intersection of all sparse-complete ideals containing $J$. Clearly, the sparse-completion $\sigma^*(J)$ is the smallest sparse-complete ideal such that $J \subseteq \sigma^*(J)$. By [11] Theorem 4(1), $\sigma^*(J) = \bigcup_{n \in \omega} \sigma^n(J)$, where $\sigma^0(J) = J$ and $\sigma^{n+1}(J) = \sigma(\sigma^n(J))$. We can prove that $A \in \sigma^n([G]^{<\omega})$ if and only if $A$ has no subsets asymorphic to $W_n = \{ g \in \oplus_\omega \mathbb{Z}_2 : \text{supp}g \leq n \}$.

By [11] Theorem 4(2), the ideal $S_p G$ is not sparse complete. By $(\star)$, the ideal $S_c G$ is sparse-complete. Hence $\sigma^*([G]^{<\omega}) \subseteq S_c G$ but $\sigma^*([G]^{<\omega}) \neq S_c G$.

Recall that a subset $A$ of an amenable group $G$ is absolute null if $\mu(A) = 0$ for each left invariant Banach measure $\mu$ on $G$. For sparse subsets, the following theorem was proved in [10, Theorem 5.1].

**Theorem 2.** Every scattered subset $A$ of an amenable group $G$ is absolute null.

Let $A$ be a subset of $\mathbb{Z}$. The upper density $\overline{d}(A)$ is denoted by

$$\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{-n, -n+1, ..., n-1, n\}|}{2n+1}.$$

By [25] Theorem 11.11, if $\overline{d}(A) > 0$ then $A$ contains a piecewise shifted IP-set. We note that Theorem 2 generalizes this statement because there exists a Banach measure $\mu$ on $\mathbb{Z}$ such that $\overline{d}(A) = \mu(A)$.

In connection with Theorem 1, one may ask if it possible to replace piecewise shifted IP-sets to (left or right) shifted IP-sets. By Theorem 2 and [25] Theorem 11.6, this is impossible.
3 Proofs

Proof of Proposition 1

(i) ⇒ (ii). We take \( p \in Y^* \) such that \( \Delta_p(Y) \) is finite, so \( \Delta_p(Y) = Fp \) for some \( F \in [G]^{<\omega} \). Given any \( H \in [G]^{\omega} \), we have \( hp \notin \Delta_p(Y) \) for each \( h \in H \setminus F \). Hence \( hP_h \cap Y = \emptyset \) for some \( P_h \in p \). We put \( P = \bigcap_{h \in H \setminus P} P_h \) and note that

\[
P \subseteq \{ y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \emptyset \}.
\]

(ii) ⇒ (i). We take an infinite subset \( Y \) of \( A \), choose corresponding \( F \in [G]^{<\omega} \) and, for each \( H \in [G]^{<\omega} \), denote \( P_H = \{ y \in Y : B_Y(y, H) \setminus B_Y(y, F) = \emptyset \} \). By (ii), the family \( \{ P_H : H \in [G]^{<\omega} \} \) has a finite intersection property and \( \bigcap_{H \in [G]^{<\omega}} P_H = \emptyset \). Hence \( \{ P_H : H \in [G]^{<\omega} \} \) is contained in some ultrafilter \( p \in Y^* \). By the choice of \( p \), we have \( gp \notin \Delta_p(Y) \) for each \( g \in G \setminus (F \cup \{ e \}) \), \( e \) is the identity of \( G \). It follows that \( \Delta_p(Y) \) is finite so \( A \) is scattered.

Proof of Proposition 2

Assume that \( A \) is not scattered and choose a subset \( Y \) of \( A \) which does not satisfy the condition (ii) of Proposition 1. We take an arbitrary \( a \in A \) and put \( F_0 = \{ e, a \} \). Then we choose inductively a sequence \( (F_n)_{n \in \omega} \) in \( [G]^{<\omega} \) such that

1. \( F_n F_n^{-1} \subset F_{n+1} \);
2. \( B_Y(y, F_{n+1}) \setminus B_Y(y, F_n) \neq \emptyset \) for every \( y \in Y \).

After \( \omega \) steps, we put \( H = \bigcup_{n \in \omega} F_n \). By the choice of \( F_0 \), \( Y \cap H \neq \emptyset \). By (1), \( H \) is a subgroup. By (2), \( (Y \cap H) \) is not scattered in \( H \).

Proof of Proposition 3

Replacing \( G \) by by the subgroup generating by \( X \), we assume that \( G \) is countable. We write \( G \) as an union of an increasing chain \( F_n \) of finite subsets such that \( F_0 = \{ e \}, F_n = F_n^{-1} \). In view of \( [G]^{<\omega} \), it suffices to find a cellular subset \( Y \) of \( X \) with no asymptotically isolated balls.

Since \( X \) is not cellular, there exists \( F \in [G]^{<\omega} \) such that
for every \( n \in \mathbb{N} \), there is \( x \in X \) such that
\[
B_X(x, F) \setminus B_X(x, F_n) \neq \emptyset.
\]

We assume that \( G \) is finitely generated and choose a system of generators \( K \in [G]^{< \omega} \) such that \( K = K^{-1} \) and \( F \subseteq K \). Then we consider the Cayley graph \( \Gamma = \text{Cay}(G, K) \) with the set of vertices \( G \) and the set of edges \( \{ \{ g, h \} : g^{-1}h \in K \} \). We endow \( \Gamma \) with the path metric \( d \) and say that a sequence \( a_0, ..., a_n \in G \) is a geodesic path if \( a_0, ..., a_n \) is the shortest path from \( a_0 \) to \( a_n \), in particular, \( d(a_0, a_n) = n \). Using (1), for each \( n \in \mathbb{N} \), we choose a geodesic path \( L_n \) of length \( 3^n \) such that \( L_n \subset X \) and
\[
B_G(L_n, F_n \cap B_G(L_{n+1}, F_{n+1})) = \emptyset \quad \text{for every} \quad n \in \mathbb{N}.
\]

Let \( A = \{ g_{i_1}g_{i_2}...g_{i_n}b_{i_n} : 0 \leq i_1 < ... < i_n < \omega \} \)
of \( G \) is not scattered. For each \( m \in \omega \), let
\[
A_m = \{ g_{i_1}g_{i_2}...g_{i_n}b_{i_n} : m < i_1 < ... < i_n < \omega \}.
\]

We take an arbitrary \( p \in A^* \) and show that \( \Delta_p(A) \) is infinite.
If \( A_n \in p \) for every \( m \in \omega \) then \( g_n p \in A^* \) for each \( n \in \omega \). Otherwise, there exists \( m \in \omega \) such that
\[
\{g_m g_{i_1} \ldots g_{i_n} b_{i_n} : m < i_1 < \ldots < i_n < \omega\} \subseteq p.
\]
Then \( g_m^{-1} p \in A^* \) and we repeat the arguments for \( g_m^{-1} p \).

\((iv) \Rightarrow (ii)\). Assume that \( A \) is not disperse and take \( p \in A^* \) such that \( p \) is not isolated in \( \Delta_p(A) \). Then \( p = qp \) for some \( q \in G^* \). The set \( \{x \in G^* : xp = p\} \) is a closed subsemigroup of \( G^* \) and, by [11] Theorem 2.5, there is an idempotent \( r \in G^* \) such that \( p = rp \). We take \( R \in r \) and \( P_g \in p \), \( g \in R \) such that \( \bigcup_{g \in R} gp \subseteq A \). Since \( r \) is an idempotent, by [11] Theorem 5.8, there is an injective sequence \( (g_n)_{n \in \omega} \) in \( G \) such that
\[
\{g_{i_1} \ldots g_{i_n} : 0 \leq i_1 < \ldots < i_n < \omega\} \subseteq R
\]. For each \( n \in \omega \), we pick \( b_n \in \bigcap \{P_g : g = g_{i_1} \ldots g_{i_n} : 0 \leq i_1 < \ldots < i_n < \omega\} \) and note that
\[
\{g_{i_1} \ldots g_{i_n} b_{i_n} : 0 \leq i_1 < \ldots < i_n < \omega\} \subseteq A.
\]

\((ii) \Rightarrow (iii)\). We assume that \( A \) contains a subset coarsely equivalent to the group \( B = \bigoplus_{\omega} \mathbb{Z}_2 \). Then there exist a subset \( X \) of \( B , H \in [B]^<\omega \) such that \( B = H + X \), and an injective \( \prec \)-mapping \( f : X \to A \). We take an arbitrary idempotent \( r \in B^* \), pick \( h \in H \) such that \( h + X \in r \) and put \( p = r - h \). Since \( r + p = r \), we see that \( p \) is not isolated in \( \Delta_p(X) \). We denote \( q = f^\beta(p) \). Let \( b \in B , b \neq 0 \) and \( b + p \in X^* \). Since \( f \) is an injective \( \prec \)-mapping, there is \( g \in G \setminus \{e\} \) such that \( f^\beta(b + p) = g + q \). It follows that \( q \) is not isolated in \( \Delta_q(A) \). Hence \( A \) is not disperse.

\((iii) \Rightarrow (i)\). Let \( X \) be a countable subset of \( A \). By Proposition [13] \( X \) is cellular. By [23], \( X \) satisfies Proposition [11] \((ii)\). Hence \( X \) is scattered. By Proposition [12] \( A \) is scattered.

**Proof of Theorem [2]**

We assume that \( \mu(A) > 0 \) for some Banach measure \( \mu \) on \( G \). We use the arguments from [10] p. 506-507 to choose a decreasing sequence \( (A_n)_{n \in \omega} \) of subsets of \( G \) and an injective sequence \( (g_n)_{n \in \omega} \) in \( G \) such that \( A_0 = A \) and \( \mu(A_n) > 0 \) \( g_n A_{n+1} \subseteq A_n \) for each \( n \in \omega \). We pick \( x_n \in A_{n+1} \) and put
\[
X = \{g_0^{\varepsilon_0} \ldots g_n^{\varepsilon_n} x_n : n \in \omega, \varepsilon_i \in \{0,1\}\}.
\]
By the construction \( X \) is a piecewise shifted \( IP \)-sets and \( X \subseteq A \). By Theorem [11] \( X \) is not scattered.

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