HIGHER REGULARITY FOR THE SIGNORINI PROBLEM FOR THE
HOMOGENEOUS, ISOTROPIC LAMÉ SYSTEM

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Abstract. In this note we discuss the (higher) regularity properties of the Signorini problem for the homogeneous, isotropic Lamé system. Relying on an observation by Schumann [35], we reduce the question of the solution’s and the free boundary regularity for the homogeneous, isotropic Lamé system to the corresponding regularity properties of the obstacle problem for the half-Laplacian.

1. The Lamé Problem for Homogeneous, Isotropic Materials

1.1. Set-up. The Signorini problem consists in finding the equilibrium position of an elastic body resting on a rigid surface [37, 38]. When the body is isotropic, homogeneous and when the surface is flat, the problem can be formulated locally as finding local minimizers \( u = (u^1, \ldots, u^n) \), \( n \geq 2 \), to the functional

\[
J(u) = \int_{B^+_1} \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\text{div } u)^2 + Fu \, dx
\]

over the closed convex set

\[
K_0 := \{ u = (u^1, \ldots, u^n) \in W^{1,2}(B^+_1; \mathbb{R}^n) : u^n \geq 0 \text{ on } B^+_1 \},
\]

i.e. \( J(u) \leq J(v) \) for all \( v \in K_0 \) with \( v - u = 0 \) on \( \partial B_1 \cap \{ x_n > 0 \} \). Here \( \lambda, \mu > 0 \) are the Lamé constants, \( F \in L^p(B^+_1; \mathbb{R}^n) \) with \( p > 2n \) is a given inhomogeneity, \( F_u := \sum_{j=1}^n F^j u^j \), \( B_1 = \{ x \in \mathbb{R}^n : |x| < 1 \}, B_1^+ = B_1 \cap \{ x_n > 0 \} \) and \( B'_1 = B_1 \cap \{ x_n = 0 \} \). It was shown in [35] that local minimizers are \( C^{1,\alpha}_{\text{loc}}(B^+_1 \cup B'_1) \) regular for some \( \alpha \in (0,1) \). Furthermore, they satisfy the Euler-Lagrange equations

\[
\mu \Delta u^j + (\mu + \lambda) \partial_j (\text{div } u) = F^j \text{ in } B^+_1, j \in \{ 1, \cdots, n \},
\]

\[
\partial_n u^j + \partial_j u^n = 0 \text{ on } B'_1, j \in \{ 1, \cdots, n-1 \},
\]

and the Signorini boundary condition

\[
\begin{align*}
0 \leq u^n, & \quad 2\mu \partial_n u^n + \lambda \text{div } u \leq 0, & & u^n(2\mu \partial_n u^n + \lambda \text{div } u) = 0 \text{ on } B'_1.
\end{align*}
\]

The set \( \Lambda_u := \{ x \in B'_1 : u^n(x) = 0 \} \) is called the contact set and \( \Gamma_u := \partial \{ x \in B'_1 : u^n(x) > 0 \} \cap B'_1 \) is the free boundary.

In this article we are interested in the optimal regularity of the solution and the so-called regular and singular free boundaries, cf. Definition 4.1 and (5) for the precise definitions. Before stating our results, we give a brief overview of the literature. There has been an extensive study of the Signorini problem for scalar equations, for a (non-exhaustive) overview we refer to [4, 5, 16, 24, 11, 15, 9] for the Laplace operator, to [19, 18, 17, 25, 27, 34] for variable coefficient second order elliptic operators as well as to [33, 32] for fully nonlinear elliptic operators. For further background and references we point to the survey articles [10, 13] and the book [31]. It is worth mentioning that under a decay assumption on the solution, the Signorini
problem for Laplacian in the upper half space $\mathbb{R}^n_+ := \mathbb{R}^n \cap \{ x_n > 0 \}$ is equivalent to the obstacle problem for the half-Laplacian $(-\Delta)_{\mathbb{H}}^+ \frac{1}{2}$ on $\mathbb{R}^{n-1} \times \{ 0 \}$ via the Dirichlet-to-Neumann mapping $u \mapsto -\partial_n \bar{u}|_{\mathbb{H}^{-1} \times \{ 0 \}}$, where $\bar{u}$ is the solution to the Poisson equation $\Delta \bar{u} = 0$ in $\mathbb{R}^n_+$ with the Dirichlet boundary value $\bar{u} = u$ on $\mathbb{R}^{n-1} \times \{ 0 \}$, cf. [7, 8]. The obstacle problem for the fractional Laplacian $\frac{1}{2}$ with $s \in (0, 1)$ and more general integro-differential operators were studied in [39, 7, 6, 1, 21, 28].

Compared with the scalar case, there are fewer results concerning the regularity properties for the original Signorini problem of elasticity. One of the main reasons is that the common techniques for scalar equations such as comparison principles and monotonicity formulas, in general, do not apply to non-diagonal systems. Existence of weak solutions for general non-isotropic, inhomogeneous bodies was shown by Fichera [14] via variational inequalities (cf. also [12]). Concerning the Signorini problem for the Lamé system (3)–(4), Schumann proved the $C^1_{\text{loc}}$ regularity of the solution by transforming the problem into a scalar obstacle problem for a pseudo-differential operator on the boundary [35]. In [3] Andersson showed that when $F = 0$ the optimal regularity of the solution is $C^{1,\frac{1}{2}}_{\text{loc}}$ and the regular free boundary is $C^{1,\gamma}$ for some $\gamma \in (0, 1)$. Instead of reducing the vectorial problem to a scalar problem, Andersson developed a linearization technique which is directly applicable to the vectorial problem and which, in particular, does not rely on any comparison principle.

1.2. Main results. Following the approach of Schumann [35], we will use the “Dirichlet-to-Neumann map” of a so-called boundary contact problem, cf. (6), to reduce the vectorial problem to a scalar problem on the boundary. However, working with constant coefficient operators and seeking to derive higher regularity results, different from [35], we locally obtain the full symbol of the “Dirichlet-to-Neumann map” (instead of the principal symbol only). This is achieved by first extending the problem to the whole upper half space, then computing the explicit solution to an ODE system originating from the Fourier transform of (6) in tangential directions and finally relating this to a corresponding obstacle problem for the half-Laplacian. In particular, it turns out that the full symbol of this “Dirichlet-to-Neumann map” is, up to a constant, the half-Laplacian (see Propositions 2.1 and 3.2). Therefore, by invoking the known regularity results for the obstacle problem for the half-Laplacian we first obtain the optimal regularity of the solution to the Signorini problem.

**Theorem 1.** Let $u \in H^1(B_1^+; \mathbb{R}^n)$ be a solution to (3)–(4) with $F \in L^p(B_1^+; \mathbb{R}^n)$, $p > 2n$. Then $u \in C^{1,\frac{1}{2}}_{\text{loc}}(B_1^+ \cup B_1')$.

Building on this, our next result concerns the higher regularity of the regular free boundary with a rather general obstacle $\varphi$ and inhomogeneity $F$. A point $x_0 \in \Gamma_u$ is regular if and only if, after subtracting an affine solution to the Signorini problem, the vanishing order of $u - \varphi$ at $x_0$ is (strictly) less than 2, cf. Lemma 4.2.

**Theorem 2.** Let $u \in H^1(B_1^+; \mathbb{R}^n)$ be a solution to the thin obstacle problem for the Lamé system (17) with obstacle $\varphi$ and inhomogeneity $F$.

(i) Suppose that $\varphi \in C^{\theta+\frac{1}{2}}(B_1')$ and $F \in C^{\theta-\frac{1}{2}}(B_1^+ \cup B_1'; \mathbb{R}^n)$ with $\theta > 2$ and $\theta \notin \mathbb{N}$, $\theta + \frac{1}{2} \notin \mathbb{N}$. Let $x_0 \in B_1'$ be a regular free boundary point. Then the free boundary is $C^\theta$ in a neighbourhood of $x_0$.

(ii) Suppose that $\varphi \in C^\omega(B_1')$ and $F \in C^\omega(B_1^+ \cup B_1'; \mathbb{R}^n)$. Then for any regular free boundary point $x_0 \in B_1'$ the free boundary is $C^\omega$ in a neighbourhood of $x_0$.

We remark that while Theorem 1 for $F = 0$ was already proved in [3], the higher regularity result of Theorem 2 is completely new, even for $F = 0$. It relies on an analogous result for the thin obstacle problem for the half Laplacian, see [1] and [26].
Similarly as in our study of the regular free boundary, also a one-to-one correspondence between the singular free boundary can be established for the Signorini problems for the Lamé operator and the Laplacian. Here, for \( u \) a solution to (1)-(2) we say that \( x_0 \in \Sigma(u) \cap B'_1 \) is an element of the singular free boundary for the Signorini problem for the Lamé operator if \( x_0 \in \Gamma_u \cap B'_1 \) and

\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(\Lambda_u \cap B'_r(x_0))}{\mathcal{H}^{n-1}(B'_r(x_0))} = 0, \quad B'_r(x_0) \subset B'_1.
\]  

(5)

Working with this definition, the structure results for the Signorini problem for the Laplacian can be exploited further, thus allowing to transfer more results from the scalar to this vectorial problem. As a sample result, using the results from [9], we for instance obtain the following stratification of the singular set:

**Theorem 3.** Let \( u \in H^1(B^+_1; \mathbb{R}^n) \) be a solution to the thin obstacle problem for the Lamé system (1)–(2). Then,

\[
\Sigma(u) = \bigcup_{d=0}^{n-2} \Sigma_d(u),
\]

where the sets \( \Sigma_d(u) \subset B'_1 \) are contained in a countable union of \( d \)-dimensional \( C^{1,\log} \) manifolds.

We remark that lacking direct monotonicity formulae for the Signorini problem for the Lamé operator, the singular free boundary had not been studied earlier.

We expect that the equivalence between the Laplacian and the full Lamé Signorini problems can be exploited further, thus allowing to transfer more results from the scalar to this vectorial problem.

1.3. Outline. The remainder of the article is organized as follows: In Section 2 we introduce the auxiliary boundary contact problem and compute its symbol in the half-space setting. Exploiting this, in Section 3 we show that free boundary regularity for the isotropic, inhomogeneous Lamé problem can be reduced to that of the half-Laplacian. Finally, in Section 4 we invoke the regularity results for the half-Laplacian to prove the analogous statements (Theorems 1–3) for the Lamé problem.

1.4. Notation. In the following sections we will mainly use rather standard notation. We however point out the following conventions:

- For \( \mathbb{R}^+_n := \{ x \in \mathbb{R}^n : x_n > 0 \} \) we set \( B^+_1(x_0) := \{ x \in \mathbb{R}^+_n : |x - x_0| < 1 \} \); for \( x_0 \in \mathbb{R}^{n-1} \times \{ 0 \} \), \( B^+_1(x_0) := \{ x \in \mathbb{R}^{n-1} \times \{ 0 \} : |x - x_0| < 1 \} \). For convenience of notation, we also identify \( \mathbb{R}^{n-1} \times \{ 0 \} \) with \( \mathbb{R}^{n-1} \) and write \( B^+_1 := B^+_1(0) \) as well as \( B'_1 := B'_1(0) \).
- We use both the notation \( C^{k,\alpha} \) and \( C^{k+\alpha} \) for the Hölder spaces with \( k \in \mathbb{N}_0 \) and \( \alpha \in (0, 1] \). With slight abuse of notation, in our reference to function spaces, we mostly suppress the image spaces. For instance we use the notation \( H^1(B^+_1) \) both for vector and scalar valued function of the corresponding regularity if there is no confusion possible.
- The Fourier transform of a function \( u : \mathbb{R}^n \to \mathbb{R} \) is denoted by \( \hat{u} \). With slight abuse of notation, if there is no possibility of misunderstanding, we do not distinguish between the tangential and full, tangential and normal Fourier transforms in our notation.
- We recall that a weak solution \( u : B^+_1 \to \mathbb{R}^n \) to (1)–(2) refers to a solution of the corresponding variational inequality [12, 35], i.e. \( u \in H^1(B^+_1; \mathbb{R}^n) \) is such that for all \( \zeta \in \mathcal{K}_0 \) with \( \zeta - u = 0 \) on \( \partial B_1 \cap \{ x_n > 0 \} \) it holds

\[
a(u, \zeta - u) \geq (F, \zeta - u)_{L^2(B^+_1)},
\]
where for $u, v \in H^1(B_1^\circ; \mathbb{R}^n)$

$$a(u, v) := \int_{B_1^\circ} 4ue(u) : e(v) + \lambda \text{div}(u) \text{div}(v) \, dx,$$

$$e(u) := \frac{1}{2}((\nabla u + (\nabla u)^T), (A : B)_{ij} = A_{ij}B_{ij} \text{ for } A, B \in \mathbb{R}^{n \times n},$$

and $(u, v)_{L^2(B_1^\circ)} := \sum_{j=1}^n \int_{B_1^\circ} u^jv^j \, dx$.

- Similarly, a weak solution to obstacle problem for the half-Laplacian in $\mathbb{R}^{n-1}$ with obstacle $\psi \in L^2(\mathbb{R}^{n-1})$ is defined as a function $\tilde{v} \in H^\frac{1}{2}(\mathbb{R}^n)$ such that for all $\tilde{\zeta} \in H^\frac{1}{2}(\mathbb{R}^{n-1})$ with $\tilde{\zeta} \geq \psi$ a.e. in $\mathbb{R}^{n-1}$ it holds

$$\tilde{a}(\tilde{v}, \tilde{\zeta} - \tilde{v}) \geq 0,$$

where $\tilde{a}(\tilde{u}_1, \tilde{u}_2) := \int_{\mathbb{R}^{n-1}} (-\Delta)^\frac{1}{2}\tilde{u}_1(-\Delta)^\frac{1}{2}\tilde{u}_2 \, dx$.

### 2. Computation of the Auxiliary Operator

We consider the following boundary contact problem

$$\begin{align*}
\mu \Delta u^j + (\mu + \lambda)\partial_j(\text{div} u) &= 0 \text{ in } \mathbb{R}^n_+, \ j \in \{1, \ldots, n\}, \\
\partial_n u^j + \partial_j u^n &= 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}, \ j \in \{1, \ldots, n-1\}, \\
u^n &= \varphi \text{ on } \mathbb{R}^{n-1} \times \{0\}, \\
u(x) &\to 0 \text{ as } |x| \to \infty.
\end{align*}\tag{6}$$

Following Schumann [36], we study an associated ΨDO given by

$$P\varphi := (-2\mu\partial_u^nu^n - \lambda \text{div} u)|_{\mathbb{R}^{n-1} \times \{0\}}.\tag{7}$$

This problem should be viewed in analogy to the Caffarelli-Silvestre extension [8]. It is a constant coefficient problem, so the full symbol of the operator $P$ can be computed by reducing the problem to a system of second order ODEs after Fourier transforming in the tangential directions. We present the details of this in the remainder of this section, following the computations in [36].

We denote the bulk operator $\mu \Delta + (\mu + \lambda)\nabla$ div by $a(D', D_j)$, where $a(\xi) = \mu|\xi|_1^2I_n + (\mu + \lambda)\xi \otimes \xi$ for $\xi \in \mathbb{R}^n$, $D_j = \frac{\lambda}{\mu}D_j$, $D' = \frac{\lambda}{\mu}\nabla'$ and $t := x_n$. Applying the Fourier transform in the tangential directions to (6), one obtains the following ODE system

$$\begin{align*}
a(\xi', D_j)\tilde{u}(\xi', t) &= 0 \text{ for } t > 0, \\
D_j\tilde{u}(\xi', 0) + \xi_j\tilde{u}^n(\xi', 0) &= 0, \quad \tilde{u}^n(\xi', 0) = \hat{\varphi}(\xi') \text{ for } t = 0.
\end{align*}\tag{8}$$

We first compute the fundamental solution $W = (W_{\xi})_{n \times n}$ of the ODE system

$$a(\xi', D_t)W_{\xi}(\xi', t) = 0 \text{ for } t > 0, \quad \xi' \neq 0\tag{9}$$

with prescribed Dirichlet data on $t = 0$ and decay at $t \to \infty$, following the ideas of Schumann [36]. Here $W_{\xi} = (W_{\xi}^1, \ldots, W_{\xi}^n)^T$ for $\xi \in \{1, \ldots, n\}$.

To this end, we begin by considering the whole space fundamental solution to the (whole space) Lamé system. In complete (i.e. in tangential and normal) Fourier variables it is given by

$$F^j(\xi) = \frac{\delta_{j\ell}}{\mu|\xi|^2} - \frac{(\lambda + \mu)\xi_j\xi_\ell}{(2\mu + \lambda)\mu|\xi|^4}, \ j, \ell \in \{1, \ldots, n\},$$

where for $u, v \in H^1(B_1^\circ; \mathbb{R}^n)$

$$a(u, v) := \int_{B_1^\circ} 4ue(u) : e(v) + \lambda \text{div}(u) \text{div}(v) \, dx,$$

$$e(u) := \frac{1}{2}((\nabla u + (\nabla u)^T), (A : B)_{ij} = A_{ij}B_{ij} \text{ for } A, B \in \mathbb{R}^{n \times n},$$

and $(u, v)_{L^2(B_1^\circ)} := \sum_{j=1}^n \int_{B_1^\circ} u^jv^j \, dx$.
cf. [29, Proposition 10.14]. We seek to transform this back in the normal direction \( t > 0 \).

Using the residue theorem, we obtain that the corresponding solution for \( t > 0 \) and \( \xi' \neq 0 \) is given by

\[
W'_\ell(\xi', t) = \int_{\mathbb{R}} e^{it\xi_n} \hat{F}'_\ell(\xi', \xi_n) \, d\xi_n = \int_{\Gamma^+(\xi')} e^{it\xi_n} \hat{F}'_\ell(\xi', \xi_n) \, d\xi_n
= 2\pi i \text{Res}_{\xi_n = i|\xi'|} \left( e^{it\xi_n} \hat{F}'_\ell(\xi', \xi_n) \right).
\]  

(10)

Here \( \Gamma^+(\xi') \) is a positively oriented contour in the complex upper half-plane \( \{\xi_n \in \mathbb{C} : \text{Im } \xi_n > 0\} \) which encloses the point \( \xi_n = i|\xi'| \) in which the complex continuation of \( \hat{F}'_\ell(\xi) \) ceases to be holomorphic as a function of \( \xi_n \in \mathbb{C} \). Indeed, this contour is obtained as a deformation of the real axis by using the fact that due to \( t > 0 \) and the decay of \( \hat{F}'_\ell(\xi) \) in \( |\xi| \to \infty \) the integral over the boundary of an upper half-ball with increasing radius vanishes as the radius tends to infinity.

Invoking the residue theorem to compute this integral (10), i.e.

\[
\frac{1}{2\pi i} W'_\ell(\xi', t) = \frac{1}{\mu} \text{Res}_{\xi_n = i|\xi'|} \left[ \frac{\delta_{\ell t}}{|\xi'|^{2-\ell t\xi_n}} \right] - \frac{\lambda + \mu}{(2\mu + \lambda)\mu} \text{Res}_{\xi_n = i|\xi'|} \left[ \frac{\xi_n \xi_n'}{|\xi'|^4} e^{it\xi_n} \right]
= \frac{1}{\mu} \lim_{\xi_n \to i|\xi'|} (\xi_n - i|\xi'|) \frac{\delta_{\ell t}}{|\xi'|^{2-\ell t\xi_n}} - \frac{\lambda + \mu}{(2\mu + \lambda)\mu} \lim_{\xi_n \to i|\xi'|} \frac{d}{d\xi_n} \left( (\xi_n - i|\xi'|)^2 \frac{\xi_n \xi_n'}{|\xi'|^4} e^{it\xi_n} \right),
\]

we obtain

\[
W(\xi', t) = 2\pi e^{-|\xi'|t} \begin{pmatrix}
\frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_1}{|\xi'|^2} & -\frac{\kappa \xi_2}{|\xi'|^2} & \cdots & -\frac{\kappa \xi_{\ell t}}{|\xi'|^2} \\
-\frac{\kappa \xi_1}{|\xi'|^2} & \frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_2}{|\xi'|^2} & \cdots & -\frac{\kappa \xi_{\ell t}}{|\xi'|^2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{\kappa \xi_{\ell t}}{|\xi'|^2} & -\frac{\kappa \xi_{\ell t}}{|\xi'|^2} & \cdots & \frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_1}{|\xi'|^2} \\
0 & 0 & \cdots & -\frac{\kappa \xi_1}{|\xi'|^2} & \frac{\nu}{|\xi'|^2} + \kappa t
\end{pmatrix},
\]

(11)

where \( \kappa = \frac{\lambda + \mu}{4\mu(2\mu + \lambda)} \) and \( \nu = \frac{1}{2\mu} - \kappa \).

We claim that there exist constants \( C_1(\xi'), \ldots, C_n(\xi') \) such that (8) is satisfied for \( \hat{u}(\xi', t) := W(\xi', t)(C_1, \ldots, C_n)^T \). To see that this is the case, we first compute

\[
\hat{u}(\xi', 0) = \frac{2\pi}{|\xi'|} \begin{pmatrix}
\frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_1}{|\xi'|^2} & -\frac{\kappa \xi_2}{|\xi'|^2} & \cdots & 0 \\
-\frac{\kappa \xi_1}{|\xi'|^2} & \frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_2}{|\xi'|^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{\kappa \xi_{\ell t}}{|\xi'|^2} & -\frac{\kappa \xi_{\ell t}}{|\xi'|^2} & \cdots & \frac{1}{2\mu|\xi'|} - \frac{\kappa^2}{|\xi'|^2} & -\frac{\kappa \xi_1}{|\xi'|^2} \\
0 & 0 & \cdots & -\frac{\kappa \xi_1}{|\xi'|^2} & \frac{\nu}{|\xi'|^2} + \kappa t
\end{pmatrix} \begin{pmatrix}
C_1(\xi') \\
C_2(\xi') \\
\vdots \\
C_n(\xi')
\end{pmatrix},
\]
and

\[ \partial_1 \hat{u}(\xi', 0) = -2\pi \left( \begin{array}{cccc} \frac{1}{2\mu} - \frac{\kappa \xi_j^2}{|\xi'|^2} & -\frac{\kappa \xi_j \xi_k}{|\xi'|^2} & \cdots & 0 \\ \frac{\kappa \xi_j \xi_k}{|\xi'|^2} & \frac{1}{2\mu} - \frac{\kappa \xi_j \xi_k}{|\xi'|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \nu \end{array} \right) \left( \begin{array}{c} C_1(\xi') \\ C_2(\xi') \\ \vdots \\ C_n(\xi') \end{array} \right) \]

\[ + 2\pi \left( \begin{array}{cccc} \frac{1}{2\mu} & 0 & \cdots & \frac{\nu \xi_j}{|\xi'|} \\ -\frac{\nu \xi_j}{|\xi'|} & \frac{1}{2\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\nu \xi_j}{|\xi'|} & -\frac{\nu \xi_j}{|\xi'|} & \cdots & \kappa \end{array} \right) \left( \begin{array}{c} C_1(\xi') \\ C_2(\xi') \\ \vdots \\ C_n(\xi') \end{array} \right) \]

\[ = -2\pi \left( \begin{array}{cccc} \frac{1}{2\mu} & 0 & \cdots & \frac{\nu \xi_j}{|\xi'|} \\ -\frac{\nu \xi_j}{|\xi'|} & \frac{1}{2\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\nu \xi_j}{|\xi'|} & -\frac{\nu \xi_j}{|\xi'|} & \cdots & \nu - \kappa \end{array} \right) \left( \begin{array}{c} C_1(\xi') \\ C_2(\xi') \\ \vdots \\ C_n(\xi') \end{array} \right) \]

From the equation \( u^n = \phi \) on \( \mathbb{R}^{n-1} \times \{0\} \) we thus infer that

\[ C_n(\xi') = \frac{|\xi'|}{2\pi \nu} \hat{\phi}(\xi'). \]

The equations

\[ \partial_n u^j + \partial_j u^n = 0 \] on \( \mathbb{R}^{n-1} \times \{0\} \)

turn into

\[ -\frac{1}{i} \left[ \frac{1}{2\mu} C_j(\xi') + \frac{i \kappa \xi_j}{|\xi'|} C_n(\xi') \right] + \frac{\nu \xi_j}{|\xi'|} C_n(\xi') = 0. \]

This yields

\[ C_j(\xi') = \frac{2\mu (\kappa - \nu) \xi_j}{i |\xi'|} C_n(\xi') = \frac{2\mu (\kappa - \nu) \xi_j}{2\pi \nu} \hat{\phi}(\xi') \text{ for } j \in \{1, \ldots, n-1\}. \]

Using these observations, we seek to compute

\[ P \phi = -2\mu \partial_n u^n - \lambda \text{div}(u). \]

To this end, we first note that for \( j \in \{1, \ldots, n-1\} \)

\[ \hat{u}^j(\xi', 0) = \frac{2\pi}{|\xi'|} \left( \frac{1}{2\mu} C_j(\xi') - \frac{\kappa}{|\xi'|^2} \sum_{\ell=1}^{n-1} \xi_j \xi_\ell C_\ell(\xi') \right) \]

\[ = \frac{2\pi}{|\xi'|} \left( \frac{1}{2\mu} C_j(\xi') - \frac{\kappa (\kappa - \nu)}{2\pi \nu} \xi_j \hat{\phi}(\xi') \sum_{\ell=1}^{n-1} \xi_\ell \right) \]

\[ = \frac{2\pi}{|\xi'|} \left( \frac{1}{2\mu} C_j(\xi') - \frac{\kappa (\kappa - \nu)}{2\pi \nu} \xi_j \hat{\phi}(\xi') \right) \]

\[ = \frac{2\pi}{|\xi'|} \left( \frac{\kappa - \nu}{2\pi \nu} \xi_j \hat{\phi}(\xi') \right) \]

\[ = \frac{\xi_j}{i |\xi'|} \left( \frac{\kappa - \nu}{\nu} \right) (1 - 2\mu \nu) \hat{\phi}(\xi'). \]
As a consequence,
\[ \sum_{j=1}^{n-1} i\xi_j \hat{u}^j(\xi', 0) = \sum_{j=1}^{n-1} \frac{\xi_j^2}{|\xi'|} (\kappa - \nu) (1 - 2\mu\nu) \hat{\varphi}(\xi') = |\xi'| \frac{(\kappa - \nu)}{\nu} (1 - 2\mu\nu) \hat{\varphi}(\xi'). \] (12)

Moreover,
\[ \partial_n \hat{u}^n(\xi', 0) = -2\pi \sum_{j=1}^{n-1} i \xi_j \sum_{j=1}^{n-1} \frac{\xi_j^2}{|\xi'|} C_j(\xi') + (\nu - \kappa) C_n(\xi') \]
\[= -2\pi \sum_{j=1}^{n-1} i \xi_j \frac{2\mu}{|\xi'|} \hat{\varphi}(\kappa - \nu) + (\nu - \kappa) \frac{|\xi'|}{2\pi\nu} \hat{\varphi}(\xi') \]
\[= - \frac{(\kappa - \nu)(2\mu\kappa - 1)}{\nu} |\xi'| \hat{\varphi}(\xi'). \] (13)

With the expressions (12) and (13) in hand, we return to the computation of the operator \( P \) for which we thus obtain
\[ \hat{P} \varphi(\xi') = - (2\mu + \lambda) \partial_n \hat{u}^n(\xi', 0) - \lambda \sum_{j=1}^{n-1} i \xi_j \hat{u}^j(\xi', 0) \]
\[= (2\mu + \lambda) |\xi'| \hat{\varphi}(\xi') (2\mu\kappa - 1) - \lambda \frac{(\kappa - \nu)(2\mu\kappa - 1)}{\nu} (1 - 2\mu\nu) |\xi'| \hat{\varphi}(\xi') \]
\[= \frac{\kappa - \nu}{\nu} [(2\mu + \lambda)(2\mu\kappa - 1) - \lambda(1 - 2\mu\nu)] |\xi'| \hat{\varphi}(\xi') \]
\[= \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu} |\xi'| \hat{\varphi}(\xi'). \]

We summarize the discussion from above in the following Proposition.

**Proposition 2.1.**
(i) Given \( \varphi \in H^2(\mathbb{R}^{n-1}) \), there is a unique solution \( u \in C_0^\infty(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \), \( u(\cdot, x_n) \in H^\frac{n}{2}(\mathbb{R}^{n-1}) \) for each \( x_n > 0 \) to the boundary contact problem (6) with \( u^\alpha(x', x_n) \to \varphi(x') \) in \( H^\frac{n}{2}(\mathbb{R}^{n-1}) \) as \( x_n \to 0_+ \). If additionally \( \varphi \in C_0^k(\mathbb{R}^{n-1}) \) for \( k \geq 0 \) and \( \alpha \in (0, 1) \), then \( u \in C_0^k(\mathbb{R}^n) \cup (\mathbb{R}^{n-1} \times \{0\}) \) up to the boundary.

(ii) The Dirichlet-to-Neumann map \( P \) as given in (7) is well-defined from \( H^\frac{n}{2}(\mathbb{R}^{n-1}) \) to \( H^{-\frac{n}{2}}(\mathbb{R}^{n-1}) \). The symbol of \( P \) is
\[ \sigma(P) = c_{\lambda, \mu} |\xi'|, \quad c_{\lambda, \mu} = \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu}. \]

**Remark 2.2.** In the following, we will refer to the function \( u \) from Proposition 2.1 as the Lamé extension of \( \varphi \).

**Proof.** From the discussion above, we obtain that
\[ u^j(x) = \frac{1}{(2\pi)^{n-j}} \int_{\mathbb{R}^{n-1}} e^{i\xi' \cdot x'} W_j^j(\xi', x_n) C_j(\xi') \ d\xi', \quad j \in \{1, \ldots, n\}, \]
is a solution to (6). From the expression for \( W \) in (11) and the observation that
\[ C(\xi') = \left( \frac{2\mu(\kappa - \nu)\xi'}{2\pi i} \right)^T \hat{\varphi}(\xi'), \]
we deduce that
\[ W(\xi', x_n)C(\xi') = e^{-|\xi'|x_n} \left( \frac{-(\lambda+i\mu)\xi' x_n}{2\mu+i\lambda} + \frac{i\mu \xi'}{2\mu+i\lambda} |\xi'| \right) \phi(\xi'). \]
The integral converges absolutely for \( x_n > 0 \). Moreover, by Cauchy-Schwarz and Plancherel one infers
\[
\|u(\cdot, x_n)\|_{H^1(\mathbb{R}^{n-1})} \leq C\|\varphi\|_{H^1(\mathbb{R}^{n-1})}; \quad |u(x)| \leq C\|\varphi\|_{L^2(\mathbb{R}^{n-1})} |x_n|^{-\frac{1}{2}}, \quad x_n > 0,
\]
for some \( C = C(\lambda, \mu, n) \). Convergence of \( u^n \) to the initial data \( \varphi \) in \( H^1(\mathbb{R}^{n-1}) \) as \( x_n \to 0_+ \) is a consequence of the Plancherel identity. Note that the set of rigid displacements which satisfy the boundary conditions consists of
\[
\mathcal{R} = \{ v : v = a + b \wedge x, \ a, b \in \mathbb{R}^n; \ v^n = 0 \text{ on } \mathbb{R}^{n-1} \times \{0\}, \ |v(x)| \to 0 \text{ as } |x| \to \infty \} = \{0\}.
\]
Hence, by [12, Chapter III, Theorem 3.3] one can conclude that \( u \) is the unique solution to the boundary contact problem (6). The up-to-the-boundary-regularity of \( u \) in Hölder spaces follows from [2]. The remaining statements of the proposition follow from the previous computations. \( \square \)

For later purposes we also need the up to the boundary regularity properties for solutions to the following problem
\[
\begin{align*}
\mu \Delta w^j + (\mu + \lambda) \partial_j \text{div } w &= f^j \text{ in } \mathbb{R}^n_+, \quad j \in \{1, \ldots, n\}, \\
\partial_n w^j + \partial_j w^n &= g^j \text{ on } \mathbb{R}^{n-1} \times \{0\}, \quad j \in \{1, \ldots, n-1\}, \quad (14)
\end{align*}
\]

\noindent \textbf{Proposition 2.3.} Given \( f \in L^2(\mathbb{R}^n) \) and \( g \in H^2(\mathbb{R}^{n-1}) \), there exists a unique solution \( w \in \dot{H}^2(\mathbb{R}^n) \) to (14). Moreover, if \( f \in C^{k-1,\alpha}(B_1^+) \) and \( g \in C^{k,\alpha}(B_1^+) \) for \( k \in \mathbb{N}, \ k \geq 1 \), then \( w \in C^{k+1,\alpha}_{loc}(B_1^+ \cup B_1) \). If \( f \) is analytic in \( B_1^+ \cup B_1 \) and \( g \) is analytic in \( B_1 \), then \( w \) is analytic in any (relatively) open set in \( B_1^+ \cup B_1 \).

\textbf{Proof.} We first consider solutions to
\[
\begin{align*}
\mu \Delta w^j + (\mu + \lambda) \partial_j \text{div } w &= 0 \text{ in } \mathbb{R}^n_+, \quad j \in \{1, \ldots, n\}, \\
\partial_n w^j + \partial_j w^n &= g^j \text{ on } \mathbb{R}^{n-1} \times \{0\}, \quad j \in \{1, \ldots, n-1\}, \quad (15)
\end{align*}
\]
By a similar argument as for Proposition 2.1 solutions to (15) can be expressed in the form
\[ w^j(\xi', t) = W(\xi', t)(C_1, \ldots, C_n) \tau, \]
with \( W(\xi', t) \) as above but where now for \( j \in \{1, \ldots, n-1\} \) the functions \( C_j(\xi') = -\frac{\xi_j}{r} g^j(\xi') \) and \( C_n(\xi') = 0 \). As above this provides the unique solvability of this problem with
\[ w^j(x) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i \xi' x} K^j_x(\xi', x_n) g^j(\xi') \, d\xi', \]
where \( K^j_x(\xi', x_n) = -\frac{\xi_j}{r} W^j(\xi', x_n) \). Noting that the mixed Dirichlet-Neumann boundary condition in (14) is elliptic in the sense of [2], one further obtains Schauder estimates up to the boundary. In particular, we thus infer that if \( g \in C^{k,\alpha}(B_1^+) \), then \( w \in C^{k+1,\alpha}(B_1^+ \cup B_1') \) for any \( r \in (0, 1) \).

Moreover, if the data are analytic near and up to the boundary, solutions are (locally) analytic. Indeed, as the system is elliptic, it suffices to show \( w^j|_{B_1} \) is real analytic and then invoke [30]. This can be directly read off from the expression of \( \hat{w}^j(\xi', 0) \) and the characterization of analyticity.
through the Fourier transform (see for instance [20, Section 8.4]). This also persists for data which are locally analytic.

Finally, we remark that solutions to
\begin{align}
\mu \Delta w^j + (\mu + \lambda) \partial_j \text{div} w &= f^j \text{ in } \mathbb{R}^n_+ , \quad j \in \{1, \cdots, n\}, \\
\partial_n w^j + \partial_j w^n &= 0 \text{ on } \mathbb{R}^{n-1} \times \{0\} , \quad j \in \{1, \cdots, n-1\}, \\
w^n &= 0 \text{ on } \mathbb{R}^{n-1} \times \{0\},
\end{align}

(16)
can be reduced to solutions of the whole space Lamé problem by a reflection argument (reflecting \(w^n\) oddly and \(w^j\) for \(j \in \{1, \ldots, n-1\}\) evenly). In particular, again elliptic regularity estimates can be invoked implying that if \(f^j \in C^{k,\alpha}(\mathbb{R}^n_+ \cup (\mathbb{R}^{n-1} \times \{0\}))\), then \(w^j \in C^{k+2,\alpha}(\mathbb{R}^n_+ \cup (\mathbb{R}^{n-1} \times \{0\}))\). Similar observations hold for the propagation of analyticity. We remark that for the problem (16) this can be even reduced to the propagation of analyticity for the Laplacian: After the extension of the problem to the whole space Lamé problem, the function \(w^j\) is even analytic in the whole space with data \(\text{div}(f^j)\). Hence \(p\) is analytic. Then solving the problem for \(\Delta w^j\) and putting the divergence contribution onto the right hand side implies the desired regularity also for the functions \(w^j, j \in \{1, \ldots, n\}\). \(\square\)

**Remark 2.4.** An alternative argument yielding analyticity of the functions \(w^j|_{B^+_1}, \partial_n w|_{B^+_1}\) and \(\text{div}(w)|_{B^+_1}\) from (15) in the tangential directions (which suffices for our regularity discussion in Section 4 below) can also be obtained through an application of the analytic implicit function theorem as in [28, 23]. Yet another argument proceeds by bootstrapping regularity estimates as in [22].

### 3. Reductions

In this section, we relate the thin obstacle problem for the Lamé system on \(B^+_1 \subset \mathbb{R}^n\) and the obstacle problem for the half-Laplacian on \(\mathbb{R}^{n-1}\). Here, on the one hand, the (strong form of the) thin obstacle problem for the isotropic, homogeneous Lamé problem on the unit ball reads

\begin{align}
\mu \Delta u^j + (\mu + \lambda) \partial_j \text{div}(u) &= F^j \text{ in } \mathbb{R}^n_+, \quad j \in \{1, \cdots, n\}, \\
\partial_n u^j + \partial_j u^n &= 0 \text{ on } \mathbb{R}^{n-1}_+ , \quad j \in \{1, \cdots, n-1\}, \\
u^n &\geq \varphi \text{ on } \mathbb{R}^n_+, \quad (17)
\end{align}

On the other hand, for a sufficiently regular obstacle \(\psi\) with suitably decaying as \(|x| \to \infty\), the (strong form of the) obstacle problem for the half-Laplacian is given by

\[
\min\{(-\Delta)^{\frac{n}{2}} u(x), u(x) - \psi(x)\} = 0 \text{ for } x \in \mathbb{R}^{n-1}.
\]

(18)

We begin by showing that a solution to (18) can be recast as a solution to (17):

**Proposition 3.1.** Let \(u : \mathbb{R}^{n-1} \to \mathbb{R}\) with \(u \in H^\frac{n}{2}(\mathbb{R}^{n-1})\) be a solution to (18) with the obstacle \(\psi \in H^\frac{n}{2}(\mathbb{R}^{n-1})\). Let \(w : \mathbb{R}^n_+ \to \mathbb{R}^n\) with \(w \in H^1(\mathbb{R}^n_+\)) be the Lamé extension of \(u\) as in Proposition 2.1. Then, the function \(w\) weakly (i.e. in the sense of a variational inequality, cf. [12]) satisfies (17) in \(B^+_1\) with \(\varphi = \psi\) and \(F^j = 0\).
Moreover, solves the equations function which is equal to one in $B$ extension.

The proof directly follows from the results from the previous section. By these, the Lamé proof.

Linear problem with better regularity, it is possible to reduce the Lamé problem to the obstacle problem for the half Laplacian. It is this direction which we mainly employ in our analysis of the free boundary problem for the isotropic, homogeneous Lamé system.

Proposition 3.2. Let $u : B^+_1 \to \mathbb{R}^n$ with $u \in H^1(B^+_1)$ be a solution to (17) with the obstacle $\varphi = \psi$ in $B^+_1$ as claimed. Conversely, if $u : B^+_1 \to \mathbb{R}^n$ with $u \in H^1(B^+_1)$ is a solution to (17), then up to solving a linear problem with better regularity, it is possible to reduce the Lamé problem to the obstacle problem for the half Laplacian. It is this direction which we mainly employ in our analysis of the free boundary problem for the isotropic, homogeneous Lamé system.

Proposition 3.2. Let $u \in H^1(B^+_1)$ be a solution to (17) with the obstacle $\varphi \in H^1(B^+_1)$ and inhomogeneity $F \in L^2(B^+_1)$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$, $\eta(x) = \eta(|x|) \geq 0$ be a smooth cut-off function with $\eta = 1$ in $B_r(x_0) \subset B_1$ for some $r \in (0, 1)$ and $x_0 \in B^*_1$, and $\eta = 0$ outside $B_1$. Then there exists a function $\bar{w} \in H^1(B^+_1) \cap \{ \eta \geq 0 \}$ such that $\bar{w}(x') := u^\eta(x', 0)\eta(x', 0) - \bar{w}(x') \in H^1(B^+_1)$ is a weak solution to the obstacle problem for $(-\Delta)^{1/2}$ with the obstacle $\psi := \varphi \eta - \bar{w} \in H^1(B^+_1)$, i.e.

\[
\min \{ (-\Delta)^{1/2} \bar{v}(x'), \varphi - \psi(x') \} \geq 0, \quad x' \in \mathbb{R}^{n-1}.
\]

In particular, $\Lambda_\psi \big|_{\text{supp}(\eta)} = \Lambda_\varphi \big|_{\text{supp}(\eta)}$, where $\Lambda_\psi := \{ x' \in \mathbb{R}^{n-1} : \bar{v}(x') = \psi(x') \}$. Proof. (i) Extension. We begin by defining $\bar{u} := \eta \hat{u}$, where $\eta(x) = \eta(|x|)$ is a smooth cut-off function which is equal to one in $B_r(x_0) \subset B_1$ for some $r \in (0, 1)$ and vanishes outside $B_1$. Then, $\bar{u}$ solves the equations

\[
\begin{align*}
\mu \Delta \bar{u}^j + (\mu + \lambda) \partial_j \div \bar{u} &= f^j \text{ in } \mathbb{R}^n, \quad j \in \{ 1, \ldots, n \}, \\
\partial_n \bar{u}^j + \partial_j \bar{u}^n &= g^j \text{ on } \mathbb{R}^{n-1} \times \{ 0 \}, \quad j \in \{ 1, \ldots, n-1 \}, \\
\bar{u}^n &= u^\eta \text{ on } \mathbb{R}^{n-1} \times \{ 0 \},
\end{align*}
\]

where

\[
\begin{align*}
f^j &= F^j\eta + 2\mu \nabla u^j \cdot \nabla \eta + \mu \nabla^2 \eta \eta + (\mu + \lambda) \sum_{k=1}^n (u^k \partial_j \partial_k \eta + \partial_j u^k \partial_k \eta + \partial_j \partial_k u^k), \\
g^j &= u^\eta \partial_j \eta.
\end{align*}
\]

Moreover, we have that

\[
\begin{align*}
\bar{u}^n &\geq \varphi \eta, \\
-2\mu \partial_n \bar{u}^n - \lambda \div \bar{u} &\geq h \text{ on } \mathbb{R}^{n-1} \times \{ 0 \}, \\
-2\mu \partial_n \bar{u}^n - \lambda \div \bar{u} &= h \text{ in } (\mathbb{R}^{n-1} \times \{ 0 \}) \cap \{ \bar{u}^n > \varphi \eta \},
\end{align*}
\]

where $h = -\lambda \sum_{k=1}^n u^k \partial_k \eta \big|_{\mathbb{R}^{n-1} \times \{ 0 \}}$. In the above computations we have used that $\partial_n \eta(x', 0) = 0$. For $F^j \in L^2(B^+_1)$ and $u \in H^1(B^+_1)$ as in (17), we immediately infer that $f^j \in L^2(\mathbb{R}^n)$, $g^j \in H^1(\mathbb{R}^{n-1})$ and $h \in H^{1/2}(\mathbb{R}^{n-1})$. 

(ii) Auxiliary problem. We next consider the following auxiliary problem
\begin{align}
\mu \Delta w^j + (\mu + \lambda) \partial_j \div w = f^j \quad \text{in } \mathbb{R}^n, \quad j \in \{1, \cdots, n\}, \\
\partial_n w^j + \partial_j w^n = g^j \quad \text{on } \mathbb{R}^{n-1} \times \{0\}, \quad j \in \{1, \cdots, n-1\}, \\
w^n = 0 \quad \text{on } \mathbb{R}^{n-1} \times \{0\}.
\end{align}

with decay at \(|x| \to \infty\). By Proposition 2.3, this problem (with decay as \(x_n \to \infty\)) is uniquely solvable yielding solutions \(w \in \dot{H}^2(\mathbb{R}^n_+)\). Further the Fourier characterization from Proposition 2.3 also implies that \(w^j \in H^1(\mathbb{R}^{n-1})\) for \(j \in \{1, \ldots, n-1\}\) and \(\partial_n w^n \in L^2(\mathbb{R}^{n-1})\).

We next observe that, as a consequence, the function \(v := \tilde{u} - w \in \dot{H}^1(\mathbb{R}^n_+)\) satisfies
\begin{align}
\mu \Delta v^j + (\mu + \lambda) \partial_j \div v = 0 \quad \text{in } \mathbb{R}^n, \quad j \in \{1, \cdots, n\}, \\
\partial_n v^j + \partial_j v^n = 0 \quad \text{on } \mathbb{R}^{n-1} \times \{0\}, \quad j \in \{1, \cdots, n-1\} \\
v^n = w^n \geq \varphi \eta \quad \text{on } \mathbb{R}^{n-1} \times \{0\}, \\
-2 \mu \partial_n v^n - \lambda \div v \geq \tilde{h} \quad \text{on } \mathbb{R}^{n-1} \times \{0\}, \\
-2 \mu \partial_n v^n - \lambda \div v = \tilde{h} \quad \text{on } \{v^n > \varphi \eta\} \cap (\mathbb{R}^{n-1} \times \{0\}).
\end{align}

Here \(\tilde{h} = h + 2\mu \partial_n w^n + \lambda \div(w) \in L^2(\mathbb{R}^{n-1})\) where we used that \(w^j \in \dot{H}^1(\mathbb{R}^{n-1})\) for \(j \in \{1, \ldots, n-1\}\) and \(\partial_n w^n \in L^2(\mathbb{R}^{n-1})\). By the results from Section 2 we have that \(\bar{v}(x') := v^n(x', 0)\) is a weak solution to
\begin{align}
\bar{v} \geq \varphi \eta, \quad (-\Delta)^{\frac{\gamma}{2}} \bar{v} \geq \tilde{h} \quad \text{in } \mathbb{R}^{n-1}, \\
(-\Delta)^{\frac{\gamma}{2}} \bar{v} = \tilde{h} \quad \text{in } \mathbb{R}^{n-1} \cap \{x' \in \mathbb{R}^{n-1} : \bar{v}(x') > \varphi \eta\}.
\end{align}

Finally, denoting \(\bar{w}(x') = (-\Delta_{x'})^{\frac{\gamma}{2}} \tilde{h}(x') \in \dot{H}^1(\mathbb{R}^{n-1})\) and setting \(\tilde{v}(x') := \bar{v}(x') - \bar{w}(x')\) and \(\psi := \varphi \eta - \bar{w}\) then concludes the argument. \(\square\)

Remark 3.3. We remark that it is possible to carry out the above reduction by means of slight variations of the auxiliary problem (22) which is used in the proof of Proposition 3.2.

4. Proofs of Theorems 1-3

In this section, we present the proofs of our main results. To this end, we use the reduction arguments from the previous sections to deduce regularity results for the Signorini problem for the Lamé system from corresponding ones for the obstacle problem for the half-Laplacian, after starting with some minimal, non-optimal local Hölder regularity for the Lamé problem. For the convenience of the reader, we first collect these facts which we will then use in the sequel to provide the arguments for Theorem 1-3. Recalling the fact that by virtue of the Caffarelli-Silvestre extension [8], when suitably localized, the obstacle problem for the half-Laplacian is equivalent to the Signorini problem for the Laplacian, we will not distinguish between references for these two problems below.

(i) Solutions \(u\) to the minimization problem (1)-(2) with \(F \in L^p(B^+_1)\) for \(p > 2n\) are \(C^{1,\alpha}_{loc}(B^+_1)\) regular for some \(\alpha > 0\) (see [35]).

(ii) Solutions \(\bar{v}\) to the obstacle problem (18) enjoy the optimal \(C^{1,1/2}_{loc}(B^+_1)\) regularity if \(\psi \in C^{1,\gamma}(B^+_1)\) for some \(\gamma > \frac{1}{2}\) [34, Theorem 4] (see also [28, Section 1.3] relating the nonlocal obstacle problem for the fractional Laplacian to an analogous local Signorini problem for a (degenerate) elliptic operator in the upper-half space). For any \(r \in (0, 1)\) the free boundary \(\Gamma_r \cap B^r_2\) is given by \(\partial \{x \in B^r_2 : \bar{v}(x) = \psi(x)\}\). At a free boundary point \(x_0 \in \Gamma_r \cap B^r_1\), the solution \(\bar{v} - \psi\) has a well-defined order of vanishing. For \(\gamma = 1 - \epsilon\) it is either equal to \(3/2\) or larger than or equal to \(2 - \epsilon\) (see [34, Theorem 5]) for the \(C^{1,\gamma}\)-regular obstacle and [31, Chapter 9] for the flat obstacle. The free boundary \(\Gamma_r \cap B^r_2\)
separates into the regular free boundary and its complement (see [31, Chapter 9]). At the regular free boundary the solution vanishes of order 3/2. For a \( C^\theta_{\text{loc}}(\mathbb{R}^{n-1}) \) obstacle with \( \theta > 0, \theta, \theta + \frac{1}{2} \notin \mathbb{N} \), the regular free boundary is locally a \( C^\theta \) regular manifold [1].

(iii) For a solution \( \tilde{v} \) to the obstacle problem for the fractional Laplacian the singular set \( \Sigma(\tilde{v}) \) is defined in analogy to (5). Indeed, \( x_0 \in \Sigma(\tilde{v}) \), if

\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(\Lambda_r \cap B_r'(x_0))}{\mathcal{H}^{n-1}(B_r'(x_0))} = 0, \quad B'_r(x_0) \subset B'_1(0).
\]

(23)

It is known [16, 9] that if \( \varphi \in C^\infty \), then the blow-up \( p_{x_0} \) of \( \tilde{v} - \varphi \) is a harmonic polynomial of order \( 2m \) with \( m \in \mathbb{N} \). The singular free boundary decomposes as

\[
\Sigma(\tilde{v}) = \bigcup_{d=0}^{n-2} \Sigma_d(\tilde{v}),
\]

where

\[
\Sigma_d(\tilde{v}) = \{ x_0 \in \Sigma(\tilde{v}) : \text{dim}\{ \xi \in \mathbb{R}^{n-1} : \sum_{j=1}^{n-1} \xi_j \partial x_j p_{x_0}(x', 0) = 0 \text{ for all } x' \in \mathbb{R}^{n-1} \} = d \}.
\]

The sets \( \Sigma_d(\tilde{v}) \) are contained in a countable union of \( d \)-dimensional \( C^{1,1,\log} \) manifolds.

Using the relation between the Signorini problem for the Lamé system and the obstacle problem for the half-Laplacian (see Proposition 3.2), as well as the observation (i) from above, we first prove Theorem 1, which concerns the optimal regularity of the solution.

**Proof of Theorem 1.** Let \( u \) be a solution to (3)–(4) in \( B_1^+ \) with \( F \in L^p(B_1^+; \mathbb{R}^n) \) and \( p > 2n \). Let \( \tilde{u} := u\eta \), where \( \eta \) is a smooth, radial cut-off function whose support is in \( B_1 \) and which is equal to one in \( B_r \) for some \( r \in (0, 1) \). Then \( \tilde{u} \) solves (20)–(21) with compactly supported data \( f^\prime, g^\prime, h \) as in the proof of Proposition 3.2. Using that by assumption \( F \in L^p \) and that \( u \in C^{1,\alpha}_{\text{loc}}(B_1^+ \cup B_1^-) \) for some \( \alpha \in (0, 1) \) (cf. [35]), we infer that \( f \in L^p(\mathbb{R}^n), g \in C^{1,\alpha}(\mathbb{R}^{n-1}) \) and \( h \in C^1(\mathbb{R}^{n-1}) \). We write \( \tilde{u} = w + v \), where \( w \) solves the auxiliary problem (22). By the \( W^{2,p} \) regularity results for the (linear) boundary value problem (22) (which follows from the characterization of the operator in Proposition 2.3 and for instance [2]), one has that \( w \in W_2^{2,p}(\mathbb{R}^n_+ \cup (\mathbb{R}^{n-1} \times \{0\})) \hookrightarrow C^{1,\gamma}_{\text{loc}}(\mathbb{R}^n_+ \cup (\mathbb{R}^{n-1} \times \{0\})) \) with \( \gamma = 1 - \frac{m}{p} \in (\frac{1}{2}, 1) \). By Proposition 3.2, \( v^n \big|_{\mathbb{R}^{n-1} \times \{0\}} \) solves the obstacle problem for \( (-\Delta)^{\frac{\alpha}{2}} \) with the obstacle \( \psi = -\varphi = -(-\Delta)^{\frac{\alpha}{2}}(-\lambda \sum \kappa w^\kappa \partial \kappa \eta + 2\mu \partial_n w^n + \lambda \text{div } w) \in C^{1,\gamma}_{\text{loc}}(\mathbb{R}^{n-1}) \) (see for instance [41, Chapter 5] for the mapping properties of \( (-\Delta)^{\frac{\alpha}{2}} \) in Hölder-Zygmund spaces or [39, Proposition 2.8]). Then we infer from the known regularity results for the thin obstacle problem with \( C^{1,\gamma} \) obstacle (cf. [34]) that \( v^n(x', 0) \in C^{1,\frac{\gamma}{2}}_{\text{loc}}(\mathbb{R}^{n-1}) \). Thus, \( v \in C^{1,\frac{\gamma}{2}}_{\text{loc}}(\mathbb{R}^n_+ \cup (\mathbb{R}^{n-1} \times \{0\})) \) by the up to the boundary regularity for the boundary contact problem (6), cf. Proposition 2.1. This together with the regularity of \( w \) implies that \( u \) is \( C^{1,\frac{\gamma}{2}}_{\text{loc}} \) up to the boundary. \( \square \)

We next approach the regularity of the regular free boundary for the Signorini problem for the Lamé system. Using the Dirichlet-to-Neumann map associated to the boundary contact problem (6), we reduce the results to the analogous ones for the half-Laplacian.

We begin by recalling the notion of a regular point for the Signorini problem for the Lamé system.

**Definition 4.1.** Let \( e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n \) denote the \( n \)-th unit vector. Let \( u \in H^1(B_1^+) \) be a solution to (3)–(4). A point \( x_0 \in \Gamma_u \) is called a regular free boundary point if for an affine
solution $p_1$ of the Lamé system and for some $\xi \in \mathbb{R}^{n-1} \setminus \{0\}$
\[ \lim_{r \to 0} \frac{(u - p_1)(x_0 + rx)}{r^2} = p_2^\xi(x). \]

Here $p_2^\xi(x) = |\xi|p_2^\xi(\frac{x}{|\xi|} \cdot x', x_n)$, where for $\xi \in \mathbb{S}^{n-1} \cap \{x_n = 0\}$, $p_2^\xi(\hat{e} \cdot x', x_n) : \mathbb{R}^n_+ \to \mathbb{R}^n$ is the $\frac{3}{2}$-homogeneous solution to the Lamé system, which satisfies
\[ p_2^\xi(\hat{e} \cdot x', x_n) \cdot \hat{e} = 0, \]
and in polar coordinates $x' \cdot \hat{e} = r \cos(\theta)$, $x_n = r \sin(\theta)$,
\[
\begin{align*}
p_2^\xi(\hat{e} \cdot x', x_n) \cdot e &= r^\frac{3}{2} \left( \frac{3\mu + \lambda + \frac{3}{2}(\mu + \lambda)}{4\mu(1 + \frac{3}{2})} \cos(\frac{3}{2}\theta) - \frac{\mu + \lambda}{4\mu} \cos(\frac{1}{2}\theta) \right), \\
p_2^\xi(\hat{e} \cdot x', x_n) \cdot e_n &= r^\frac{3}{2} \left( \frac{3\mu + \lambda - \frac{3}{2}(\mu + \lambda)}{4\mu(1 + \frac{3}{2})} \sin(\frac{3}{2}\theta) - \frac{\mu + \lambda}{4\mu} \sin(\frac{1}{2}\theta) \right).
\end{align*}
\]

We emphasize that we allow for affine off-sets in our definition of the regular free boundary. Moreover, if $u \in C_{loc}^1(B_1^+ \cup B_1^+)$ one has $p_1(x) = -u(x_0) - \nabla u(x_0) \cdot x$. In [3] it is shown that when $F = 0$, those free boundary points with vanishing order less than 2 (cf. (24)) are regular. In the next lemma we show that also for a general $F \in L^\infty$ (but for simplicity $\varphi = 0$) the vanishing order condition indeed characterizes the regular free boundary points, as in the obstacle problem for $(-\Delta)\frac{3}{2}$.

**Lemma 4.2.**

(i) Let $u \in H^1(B_1^+)$ be a solution to (17) with $F \in L^\infty(B_1^+)$. Then $x_0 \in \Gamma_u$ is a regular free boundary point if and only if up to an affine function the vanishing order of $u$ at $x_0$ is no larger than 2, i.e.
\[ \limsup_{r \to 0} \frac{\ln(r^{-\frac{3}{2}} \|u - p_1\|_{L^2(B^+_1(x_0))})}{\ln r} < 2, \] (24)
where $p_1(x) = -u(x_0) - \nabla u(x_0) \cdot x$.

(ii) Let $\bar{u}(x')$ be the function from Proposition 3.2, which solves the obstacle problem for $(-\Delta)\frac{3}{2}$ on $\mathbb{R}^{n-1} \times \{0\}$. Then, $x_0 \in \Gamma_\bar{u} \cap B_1^+$ is a regular free boundary point if and only if $x_0 \in \Gamma_u \cap B_1^+$ is a regular free boundary point.

**Proof.** Without loss of generality, we assume $x_0 = 0$ and $u(0) = |Du(0)| = 0$.

Proof for (i). One direction is simple. Assume that $\lim_{r \to 0} \frac{\ln(r^{-\frac{3}{2}} \|u - p_1\|_{L^2(B^+_1(x_0))})}{\ln r} \leq 2$. Then $\bar{u}(x') - \bar{u}(0)$ is a regular free boundary point.

Assuming (24), then by [3, Proposition 7.1] (which holds for $F \in L^\infty$ as well) there is a sequence $r_j \to 0$ such that after a rotation of coordinates
\[ \frac{u(r_j x)}{r_j^\frac{3}{2} \|u\|_{L^2(B^+_1)}} \to cp_2^\xi(x_{n-1}, x_n) \] (25)
in $H^1_{loc}(\mathbb{R}^{n}_x)$ for some $c = c(\mu, \lambda, n) > 0$. Let $\tilde{u} := u\eta$ and $\tilde{u} = w + \tilde{v}$ as in Proposition 3.2, where $w$ solves the auxiliary problem (22) and $\tilde{v} := v|_{\mathbb{R}^{n-1} \times \{0\}}$ solves the obstacle problem for $(-\Delta)\frac{3}{2}$ with the obstacle $\psi := -\tilde{w}$. Since the inhomogeneity satisfies $F \in L^\infty$, analogously as in the proof of Theorem 1 by the up to the boundary regularity for (22) and the definition of $\tilde{w}$ it follows that $\tilde{v} \in C_{loc}^{1,\gamma}(\mathbb{R}^{n-1})$ for any $\gamma \in (0, 1)$. Then (25) implies that the vanishing order of $\tilde{v} - \psi$ at 0 has to be $\frac{1}{2}$. Indeed, assume this were not true, then by the gap of the vanishing order
integrability, i.e.,
\[
0 \leftarrow \frac{(\tilde{\nu} - \psi)(r_jx')}{r_j^2 \|u\|_{L^2(B_r_j^\pm)}} = \frac{u(x_jx',0)}{r_j^2 \|u\|_{L^2(B_r_j^\pm)}} \in L^2_{\text{loc}}(\mathbb{R}^{n-1} \times \{0\}),
\]
which is a contradiction to (25). Thus \(x_0 = 0\) is a regular free boundary point for \(\tilde{\nu}\) (by the definition of the regular free boundary point for \((-\Delta)^{\frac{\gamma}{2}}\), see [25, Definition 4.1 and Proposition 4.2]). This implies
\[
\lim_{r \to 0} u^\circ(r_jx',0) = \lim_{r \to 0} \frac{(\tilde{\nu} - \psi)(r_jx')}{r_j^{\frac{n}{2}}} = \tilde{\nu}(x_{n-1},0) \cdot e_n
\]
for some \(\tilde{c} > 0\) in \(H^1_{\text{loc}}(\mathbb{R}^n_+)\). Moreover, the free boundary \(\Gamma_{\tilde{\nu}}\) (and thus \(\Gamma_u\)) is a \(C^{1,\beta}\)-graph in a neighborhood of 0. This together with the optimal \(C^{1,\frac{\beta}{2}}_{\text{loc}}\) regularity of the solution implies that along subsequences the limits of \(u(rx)/r^{3/2}\) as \(r \to 0\) exist and solve the global Lamé system with the flat free boundary \(\{x_{n-1} = x_n = 0\}\). Since the restriction of the \(n\)-th-component of the limits to \(\mathbb{R}^{n-1} \times \{0\}\) is unique and equal to \(\tilde{\nu}p_{3/2}(x_{n-1},0) \cdot e_n\), we can conclude that the limit \(u_{r \to 0} u(rx)/r^{3/2}\) exists and is equal to \(\tilde{\nu}p_{3/2}(x_{n-1},x_n)\).

Proof for (ii). First by Proposition 3.2 one has that \(\Gamma_u \cap K = \Gamma_{\tilde{\nu}} \cap K\) for any \(K \subset B'_1\). On the hand, let \(x_0 \in \Gamma_{\tilde{\nu}} \cap B'_1\) be a regular free boundary point, then (26) holds true at \(x_0\) up to a rotation of coordinates. A similar argument as in the proof for (i) yields that \(\lim_{r \to 0} u(rx)/r^{3/2}\) exists and is equal to \(\tilde{\nu}p_{3/2}\) for some \(\tilde{c} > 0\). This means that \(x_0\) is also a regular free boundary point for the Lamé system on \(\Gamma_u\). On the other hand, given \(x_0\) a regular free boundary point on \(\Gamma_u \cap B'_1\), by the definition of \(\tilde{\nu}\) and Definition 4.1 it holds that \(x_0 \in \Gamma_{\tilde{\nu}} \cap B'_1\) is also a regular free boundary point for the obstacle problem for the half-Laplacian.

Using our above established relation between the problems for the Lamé free boundary problem and the half-Laplacian (see Proposition 3.2), the regularity and the higher regularity of the free boundary of the Lamé problem follows immediately from that for the fractional Laplacian (see [1] and [26]) by carrying out a truncation strategy as in [26].

Proof of Theorem 2. We first discuss (i): For simplicity of notation we assume that \(x_0 = 0\). Seeking to obtain a local result, we consider \(\tilde{u} := u_\eta\) for \(\eta\) as in Proposition 3.2 and deduce estimates in any set \(B_r^\pm \subset B_r^\pm\) where \(r = 1\). In the set \(B_r^\pm\), the local regularity of the auxiliary function \(w\) from Step (ii) in the proof of Proposition 3.2 is determined by \(F\) only. In particular, since the inhomogeneity satisfies \(F \in C^{0,\frac{\beta}{2}}(B_r^+ \cup B_r^-)\), then by the up to the boundary Schauder estimate for Neumann problem for the Lamé system (cf. [2]), \(w\) is \(C^{0,\frac{\beta}{2}}(B_r^+ \cup B_r^-)\). Additionally, Proposition 2.3 yields integrability, i.e., \(w \in C^{0,\frac{\beta}{2}}(B_r^+ \cup B_r^-)\). Therefore, by Proposition 3.2, we infer \(\tilde{h} = h + 2\mu \partial_n w^n + \lambda \div w \in C^{0,\frac{\beta}{2}}(B_r^+ \cap L^2(\mathbb{R}^{n-1}_+))\) and, as a consequence of Proposition 3.2, the interaction of pseudodifferential operators and Hölder-Zygmund spaces (see for instance [40, Chapter VI, Section 5.3]), and of the pseudolocality of the fractional Laplacian (as a pseudodifferential operator, see [40, Chapter VI, Section 2]), \(\tilde{w} = (\Delta x')^{\frac{\beta}{2}} \tilde{h} \in C^{0,\frac{\beta}{2}}(B_r^+\)). Hence \(\tilde{w}(x') := w(x',0)\eta(x',0) - \tilde{w}(x')\) solves the obstacle problem for \((-\Delta)^{\frac{\gamma}{2}}\) with the obstacle \(\psi = \phi \eta - \tilde{w} \in C^{0,\frac{\beta}{2}}(B_r^+\)) . By Lemma 4.2 a regular point of the Lamé system is mapped to a regular point for the half-Laplacian and vice-versa. By [1, Theorem 1.2] or [26, Theorem 2], around the regular point \(0 \in \Gamma_{\tilde{\nu}} \cap B'_1\) the free boundary is \(C^0, \beta\) regular. Since restricted to the support of \(\eta\) the free boundary satisfies \(\Gamma_u = \Gamma_{\tilde{\nu}}\), we have that \(\Gamma_u\) is also \(C^0, \beta\) regular in a neighborhood of \(x_0 = 0\).
Finally, (ii) follows analogously by the results of [26] for the analyticity of the regular free boundary for the obstacle problem for half-Laplacian with real analytic obstacle, once the (tangential) analyticity of $w|_{B_r(x_0)}$, $\text{div}(w)|_{B_r(x_0)}$ and $\partial_tw|_{B_r(x_0)}$ (and thus of $\psi$) is established for some $r > 0$. It hence remains to discuss the analyticity of these functions: This however is a consequence the fact that the data in the problem for $w$ (cf. (22)) are analytic in $B_r^+(x_0) \cup B_r^-(x_0)$ (with $\eta$ chosen to be one in $B_r(x_0)$, $r > 0$ small). By Proposition 2.3 we obtain the analyticity of $w$ and thus of $\bar{w}$ and $\psi$ in a neighborhood of $x_0$. □

Last but not least, we turn to the proof of Theorem 3.

Proof of Theorem 3. It suffices to establish that if $x_0 \in \Sigma_u \cap B'_1$, then $x_0 \in \Sigma_0 \cap B'_1$. The remaining arguments then follow from the properties of the singular set of the Signorini problem for the Laplace operator (see (iii) at the beginning of this section) noting that the corresponding obstacle $\psi = -\bar{w}$ is smooth in a neighborhood of $x_0$. Since $\Lambda_0 \cap \tilde{K} = \Lambda_u \cap \tilde{K}$ for any $\tilde{K} \Subset B'_1$, the implication that if $x_0 \in \Sigma_u \cap B'_1$, then $x_0 \in \Sigma_0 \cap B'_1$ is immediate. □

Remark 4.3. We remark that extensions of Theorem 3 to settings with non-zero obstacle are possible but require a more careful investigation of the vanishing order of the blow-ups.

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