Minimum detection efficiency for a loophole-free violation of the Braunstein-Caves chained Bell inequalities

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The chained Bell inequalities of Braunstein and Caves involving N settings per observer have some interesting applications. Here we obtain the minimum detection efficiency required for a loophole-free violation of the Braunstein-Caves inequalities for any N ≥ 2. We discuss both the case in which both particles are detected with the same efficiency and the case in which the particles are detected with different efficiencies.

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I. INTRODUCTION

Soon after the Clauser-Horne-Shimony-Holt (CHSH) generalization [1] of the original Bell inequality [2], Wigner [3] and Pearle [4] realized that it is possible to make a local hidden variable (LHV) model which produces predictions in agreement with the predictions of quantum mechanics (QM) for a maximal violation of the CHSH Bell inequality, if each particle has not two, but three possible responses to the local measurements: being detected by the detector labeled −1, being detected by the detector labeled +1, or being undetected. “Then instead of four possible outcomes (..), there are nine possible outcomes. In one of these outcomes, neither particle is detected (…). In four of these outcomes one of the particles is not detected. If the experimenter rejects these data (in the belief that the apparatus is not functioning properly and that if it had been functioning properly, the data recorded would have been representative of the accepted data) (…), it is possible to produce a local hidden variable theory [which gives] predictions in agreement with the predictions of quantum theory” [4].

This is the origin of the so-called detection loophole of experimental tests of the violation of Bell inequalities. In most experimental “violations” of Bell inequalities, the overall detector efficiency (defined as the ratio of detected to emitted-particle ratio such that, if there is a close relation between the efficiency of the detectors been perfect (this is the so-called fair-sampling assumption). This is an auxiliary assumption that restricts the studied class of LHV models considerably.

This paper focuses on the question of what minimum overall detection efficiency ηcrit is required to escape from the detection loophole. In other words, how good our detectors need to be to give a conclusive experimental violation of a Bell inequality without the fair-sampling assumption. The bound ηcrit is the value of the detected-to-emitted-particle ratio such that, if η ≤ ηcrit, there is an LHV model reproducing the predictions of QM, but no such LHV models exist if η > ηcrit.

For the CHSH Bell inequality, and assuming a perfect preparation, ηcrit = 2(√2 − 1) ≈ 0.83 if all particles have detected with the same efficiency [7, 8], and ηcrit = 1/√2 ≈ 0.71 if one of the particles is always detected [3, 10].

Mermin proposed an n-party two-setting generalization of the (two-party two-setting) CHSH Bell inequality [11]. For the Mermin Bell inequalities, it has been recently proven that ηcrit(n) = n/(2N − 2) [12]. The amount of violation D (defined, for Bell inequalities involving only averages of products of local operators, as the ratio between the quantum prediction and the bound of the Bell inequality) grows with the number of parties n as D(n) = 2(N−1)/2. Therefore, for the Mermin Bell inequalities, ηcrit = [2 + (log 2/ log D)]/4; it seems likely that there is a close relation between ηcrit and D for other generalizations as well.

Braunstein and Caves (BC) proposed a two-party N-setting generalization of the CHSH Bell inequality [13, 14], in which the first observer can choose one out of N alternative experiments A1, A3, . . . , A2N−1, and the second observer one out of N alternative experiments B2, B4, . . . , B2N, each of them having only outcomes +1 or −1. The BC chained Bell inequalities (in the case of ideal detectors) are

\[ |E(A_1 B_2) + E(A_3 B_4) + E(A_3 B_4) + E(A_5 B_4) + \cdots + E(A_{2N-1} B_{2N}) - E(A_1 B_{2N})| ≤ 2N - 2 \] (1)

These inequalities are violated by correlations (A,B) obtained from QM. For instance [13], for the state

\[ |\psi^-⟩ = \frac{1}{\sqrt{2}} (|01⟩ - |10⟩) , \] (2)


Therefore, the violation is

\[ D(N) = \frac{2N \cos(\pi/2N)}{2N - 2}. \] (5)

That is, \( D(2) = \sqrt{2} \approx 1.414 \) (which is the maximum possible violation of the CHSH Bell inequality in QM [16]) and \( D(3) = 3\sqrt{3}/4 \approx 1.299 \). The violation decreases with \( n \). Indeed, Eq. 5 gives the maximum possible violation of the BC chained Bell inequalities [1] in QM [17].

Violations of the BC chained Bell inequalities have been observed (under the fair-sampling assumption) using pairs of photons entangled in polarization, with \( N = 3, 4 \) [18], and even \( N = 21 \) settings per observer [19].

The BC chained Bell inequalities have some interesting applications in situations where the CHSH Bell inequality is inadequate. For instance, the use of a BC inequality with \( N = 3 \) solves a problem in Franson’s CHSH Bell experiment [20], and reduces the number of trials needed to rule out local realism in experiments with perfect detection efficiency [21]. Moreover, the use of BC inequalities with higher values of \( N \) improves the security of quantum key distribution protocols [22] and has been also used to investigate nonlocal theories [23, 24].

The aim of this paper is to calculate \( \eta_{\text{crit}}(N) \) for the maximum possible violation of the BC chained Bell inequalities [1] assuming a perfect preparation.

In Sec. II A we introduce some definitions. In Sec. II B we state the main result. The necessary condition is proven in Sec. II C. Both the case with equal (symmetric) and unequal (asymmetric) efficiencies for both particles are discussed. To prove the sufficient condition, explicit LHV models are built for both cases. The sufficient conditions for symmetric and asymmetric efficiencies are developed in Secs. II D and II E respectively.

In Sec. III we present the conclusions and discuss the relation between the amount of violation \( D \) and \( \eta_{\text{crit}} \) for the BC inequalities and the effect of non-perfect visibilities in the state preparation.

II. DETECTION EFFICIENCY FOR THE BRAUNSTEIN-CAVES CHAINED BELL INEQUALITY

A. Basic definitions

In an LHV model, the result of a measurement of \( A_j \) on particle 1 and \( B_k \) on particle 2 is predetermined. This information can be summarized in the state of LHV of an individual pair of particles (hereafter simply called state), which is a list \( \{A_1, A_3, \ldots, A_{2N-1}; B_2, B_3, \ldots, B_{2N}\} \) of 2N instructions. For a given measurement \( A_j \) (or \( B_k \)), the possible instructions are: “give a detection in the detector \(-1\)”, “give a detection in the detector \(+1\)”, and “do not give a detection.” We will denote these instructions as \(-1, +1, 0\), respectively. Therefore, each state is represented by a list of 2N values in \{-1, +1, 0\}.

Because of the special status of the value 0 (“no detection”) it is not easy to estimate \( E(A_jB_k) \) from experiment. An estimate would need counting the number of “no detection” events that has occurred, and this is a nontrivial exercise. The usual approach is to delete (or rather, disregard) the “no detection” events and calculate the conditional correlation, given that a coincidence has occurred. We will use the notation \( \Lambda_{A_jB_k} \) for the ensemble of pairs that give rise to a coincidence, i.e., the ensemble where \( A_j \neq 0 \) and \( B_k \neq 0 \).

Using this notation, the averages easily obtainable from experiments are conditional averages on the form \( E(A_j|A_j) \), and similarly conditional correlations on the form \( E(A_jB_k|A_{A_j}B_{B_k}) \), both averages over observed data. In general, given an ensemble \( \Lambda \) of pairs, \( E(A_j|\Lambda) \) will denote the average restricted to \( \Lambda \). If we divide the ensemble \( \Lambda \) into disjoint subensembles \( \Lambda_i \),

\[ E(A_jB_k|\Lambda) = \sum_i E(A_jB_k|\Lambda_i)P(\Lambda_i|\Lambda). \] (6)

An LHV model for a given Bell experiment is an ensemble of pairs, each of them with its own state, which satisfies the predictions of QM for that experiment and reproduces the behavior of actual detectors. For example, in order to reproduce the predictions of QM for state [2] and local observables [3], the LHV model must satisfy

\[ E(A_j|A_j) = 0, \forall j \in \{1, 3, \ldots, 2N-1\}, \] (7a)
\[ E(B_k|B_k) = 0, \forall k \in \{2, 4, \ldots, 2N\}, \] (7b)

and also must satisfy [from Eqs. 11]

\[ E(A_1B_2|A_{A_1}B_{B_2}) = E(A_3B_2|A_{A_3}B_{B_2}) = E(A_5B_4|A_{A_5}B_{B_4}) = \cdots = E(A_{2N-1}B_{2N}|A_{A_{2N-1}}B_{B_{2N}}) \] (8)
\[ = -E(A_1B_{2N}|A_{A_1}B_{B_{2N}}) = \cos(\pi/2N). \]

From our LHV model, we can now obtain probabilities like \( P(\Lambda_{A_j}) \), the probability that \( A_j \) is nonzero,
the probability that both \( A_j \) and \( B_k \) are nonzero, and \( P(\Lambda_{A_j}|\Lambda_{B_k}) \), the conditional probability that \( A_j \neq 0 \) given that \( B_k \neq 0 \). Note that the last probability is simple to extract from an experiment while the former two are more difficult to get at. Also, \( P(\Lambda_{A_j}B_k) = P(\Lambda_{A_j}|\Lambda_{B_k})P(\Lambda_{B_k}) \). We will use the minimum conditional detection probability as an efficiency measure of our setups. In general, this means that the two detection sites can have individual efficiency measures,

\[
\eta_A \equiv \min_{j,k} P(\Lambda_{A_j}|\Lambda_{B_k}), \\
\eta_B \equiv \min_{j,k} P(\Lambda_{B_k}|\Lambda_{A_j}).
\]

The efficiency of the whole setup can be measured as

\[
\eta \equiv \min \eta_A, \eta_B.
\]

In order to reproduce the behavior of actual detectors, we will construct the LHV model to give nondetections satisfying equiprobably that both \( A_j \) and \( B_k \) are not experimentally accessible, we need to relate the existence of an LHV model giving the quantum violation probabilities consistent on the probability space defined just a particular kind of ensemble), and, by definition, any of its subensembles (a specific value of the LHV is even with this restriction. In the model, this corresponds to the probabilities must satisfy

\[
P(\Lambda_{A_j}) = \eta_A, \\
P(\Lambda_{B_k}) = \eta_B, \\
P(\Lambda_{A_j}|\Lambda_{B_k}) = \eta_A, \\
P(\Lambda_{B_k}|\Lambda_{A_j}) = \eta_B,
\]

for the relevant combinations of \( j \in \{1,3,\ldots,2N-1\} \), and \( k \in \{2,4,\ldots,2N\} \).

For our purposes, it is also useful to realize that any LHV model can also be defined as a set of states and their probabilities of appearance. Clearly, the same applies to any of its subensembles (a specific value of the LHV is just a particular kind of ensemble), and, by definition, those probabilities will always be relative to the whole LHV model. This choice makes their interpretation as probabilities consistent on the probability space defined by the LHV model.

### B. Main results

In what follows, we will prove the following theorem:

**Theorem 1.** The BC inequality (11) has a well-defined critical efficiency. That is, an efficiency below or equal to this critical value is necessary and sufficient for the existence of an LHV model giving the quantum violation of the inequality. Moreover, the value in the symmetric case \( \eta_A = \eta_B = \eta \) is

\[
\eta_{\text{crit}}(N) = \frac{2}{N \cos \left( \frac{\pi}{2N} \right) + 1},
\]

and, when \( \eta_A \neq \eta_B \), the relation between \( \eta_{\text{crit}}(N) \) and \( \eta_{\text{crit}} \) is

\[
\eta_{\text{crit}}(N) = \frac{1}{N \cos \left( \frac{\pi}{2N} \right) + 1 - \frac{1}{\eta_{\text{crit}}}}.
\]

### C. Necessary condition

We now prove that the right-hand sides of Eqs. (12) and (13) are indeed upper bounds. The following proof does not need to assume independent errors [e.g., that \( P(\Lambda_{A_j}|\Lambda_{B_k}) = P(\Lambda_{A_j}) \)] or constant error rates [e.g., that \( P(\Lambda_{A_j}) = P(\Lambda_{A_k}) \)], hinted at above.

In the ideal case, the BC inequalities assert

\[
\begin{align*}
|E(A_1B_2) + E(A_3B_2) + |E(A_3B_4) + E(A_5B_4)| + \cdots & + |E(A_{2N-3}B_{2N-2}) + E(A_{2N-1}B_{2N-2})| \\
& + |E(A_{2N-1}B_{2N}) - E(A_1B_{2N})| \\
& \leq 2N - 2,
\end{align*}
\]

This inequality applies on the ensemble on which all experimental setups would give results, i.e., \( A_j, B_k \neq 0, \forall j,k \). We would like an inequality that applies on correlations we can obtain from experiment, such as \( E(A_1B_2|A_1B_2) \). To do this, we note that the above inequality can be written

\[
\begin{align*}
|E(A_1B_2|A_0) + |E(A_3B_2|A_0) + |E(A_3B_4|A_0) + |E(A_5B_4|A_0)| & + \cdots + |E(A_{2N-3}B_{2N-2}|A_0) + E(A_{2N-1}B_{2N-2}|A_0)| \\
& + |E(A_{2N-1}B_{2N}|A_0) - E(A_1B_{2N}|A_0)| \\
& \leq 2N - 2,
\end{align*}
\]

where \( A_0 = A_{A_1}B_2A_3B_4\ldots A_{2N-1}B_{2N} \) denotes the ensemble where all measurements give results. Since \( E(A_jB_k|A_0) \) are not experimentally accessible, we need to relate the ensemble \( A_0 \) to the ensembles \( A_{A_j}B_k \), and we do that by formally defining

\[
\delta_{2N,2} = \min_{\text{settings}} P(\Lambda_0|A_{A_j}B_k).
\]

We arrive at the following result:

**Lemma 1.** Relation (16) between the subensemble that obeys the BC inequality and the subensemble we see in experiment enables the inequality

\[
\begin{align*}
|E(A_1B_2|A_{A_1}B_2) + |E(A_3B_2|A_{A_1}B_2) + |E(A_3B_4|A_{A_1}B_4) & + \cdots + |E(A_{2N-3}B_{2N-2}|A_{A2N-1}B_{2N-2}) \\
& - E(A_1B_{2N}|A_{A1}B_{2N})| \\
& \leq 2N - 2\delta_{2N,2}.
\end{align*}
\]

**Proof.** Clearly, \( A_0 \subset A_{A_j}B_k \), so we can split \( A_{A_j}B_k \) into two subensembles \( A_0 \) (where all measurement settings
give detections), and \( \Lambda_s = \Lambda_{A_1B_k} \setminus \Lambda_0 \) (where \( A_1B_k \) give detections but one or more of the others do not). Note that \( \Lambda_0 \cup \Lambda_s = \Lambda_{A_1B_k} \). We can write

\[
\left| E(A_jB_k|A_{A_1B_k}) - \delta_{2N,2}E(A_jB_k|\Lambda_0) \right|
\]

\[
\leq \left| P(A_s|A_{A_1B_k})E(A_jB_k|\Lambda_s) \right| + \left| P(\Lambda_0|A_{A_1B_k})E(A_jB_k|\Lambda_0) \right|
\]

\[
- \delta_{2N,2}E(A_jB_k|\Lambda_0)
\]

\[
= P(A_s|A_{A_1B_k})E(A_jB_k|\Lambda_s)
\]

\[
+ \left[ P(\Lambda_0|A_{A_1B_k}) - \delta_{2N,2} \right] E(A_jB_k|\Lambda_0)
\]

\[
\leq P(A_s|A_{A_1B_k})E\left(|A_jB_k|\Lambda_s\right)
\]

\[
+ \left[ P(\Lambda_0|A_{A_1B_k}) - \delta_{2N,2} \right] E\left(|A_jB_k|\Lambda_0\right)
\]

\[
= 1 - \delta_{2N,2}.
\]

(18)

Now,

\[
\left| E(A_jB_k|A_{A_1B_k}) + E(A_jB_k|A_{A_1B_k}) \right|
\]

\[
\leq \delta_{2N,2}E(A_jB_k|\Lambda_0) + \left| E(A_jB_k|\Lambda_0) \right| + \left| E(A_jB_k|\Lambda_{A_1B_k}) \right|
\]

\[
- \delta_{2N,2}E(A_jB_k|\Lambda_0) + \delta_{2N,2}E(A_jB_k|\Lambda_0)
\]

\[
\leq \delta_{2N,2}E(A_jB_k|\Lambda_0) + \left| E(A_jB_k|\Lambda_0) \right| + \delta_{2N,2}E(A_jB_k|\Lambda_0)
\]

\[
= 2N - 2\delta_{2N,2}.
\]

(19)

so that finally,

\[
\left| E(A_1B_2|A_{A_1B_2}) + E(A_3B_2|A_{A_3B_2}) + E(A_5B_2|A_{A_5B_2}) + \cdots + E(A_{2N-1}B_2|A_{A_{2N-1}B_2}) \right|
\]

\[
- E(A_1B_2|\Lambda_0) + \left( E(A_3B_2|\Lambda_0) + E(A_5B_2|\Lambda_0) + \cdots + E(A_{2N-1}B_2|\Lambda_0) \right)
\]

\[
\leq 2N - 2\left( 2N - 2\delta_{2N,2} \right)
\]

\[
= 2N - 2\delta_{2N,2}.
\]

(20)

The following lemma gives the relation between the constant \( \delta_{2N,2} \) and the efficiency in the symmetric case.

**Lemma 2.** In the symmetric case,

\[
\delta_{2N,2} \geq 2N - 1 - \frac{2N - 2}{\eta}.
\]

(21)

**Proof.** We have

\[
P(A_{A_2}|A_{A_1B_2}) = \frac{P(A_{A_3}|A_{B_2})}{P(A_{A_1}|A_{B_2})}
\]

\[
= \frac{P(A_{A_3}|A_{B_2}) - P(A_{A_1} \cup A_{A_3}|A_{B_2})}{P(A_{A_1}|A_{B_2})}
\]

\[
\geq 1 + \frac{\eta - 1}{P(A_{A_1}|A_{B_2})}
\]

\[
\geq 2 - \frac{1}{\eta},
\]

(22)

which gives

\[
P(A_{A_3}|A_{A_1B_2}) = P(A_{A_3}|A_{A_1B_2}) + P(A_{B_2}|A_{A_1B_2})
\]

\[
- P(A_{A_3} \cup A_{B_2}|A_{A_1B_2})
\]

\[
\geq 2\left( 2 - \frac{1}{\eta} \right) - 1
\]

\[
= 3 - \frac{2}{\eta}.
\]

(23)

Now (Bonferroni),

\[
P(\Lambda_0|A_{A_1B_2}) = P(\Lambda_{A_2B_4} \cap \Lambda_{A_3B_4} \cap \cdots \cap \Lambda_{2N-1}B_{2N} | A_{A_1B_2})
\]

\[
\geq P(\Lambda_{A_2B_4} \cap A_{A_1B_2}) + P(\Lambda_{A_3B_4} \cap A_{A_1B_2}) + \cdots
\]

\[
+ P(\Lambda_{2N-1}B_{2N} \cap A_{A_1B_2}) - (N - 2)
\]

\[
\geq (N - 1) \left( 3 - \frac{2}{\eta} \right) - (N - 2)
\]

\[
= 2N - 1 - \frac{2N - 2}{\eta}.
\]

(24)

Taking the minimum over the possible measurement settings immediately gives the lemma.

These two lemmas give the BC inequality for the symmetric case as

\[
| E(A_1B_2|A_{A_1B_2}) + E(A_3B_2|A_{A_1B_2}) + E(A_5B_2|A_{A_1B_2}) + \cdots + E(A_{2N-1}B_{2N}|A_{A_{2N-1}B_{2N}}) | - E(A_1B_2|\Lambda_0)
\]

\[
\leq 2N - 2\left( 2N - 1 - \frac{2N - 2}{\eta} \right)
\]

\[
= 2(N - 1) \left( \frac{2}{\eta} - 1 \right).
\]

(25)

For a generic value \( \beta \) on the left-hand side,

\[
\beta \leq 2(N - 1) \left( \frac{2}{\eta} - 1 \right),
\]

(26)

which leads to

\[
\eta \leq \frac{2(N - 1)}{N - 1 + \frac{\beta}{2}}.
\]

(27)

Inserting the value of \( \beta = 2N \cos(\pi/2N) \) predicted by QM, we arrive at the right-hand side of Eq. (12).

The relation between the constant \( \delta_{2N,2} \) and the efficiency in the asymmetric case is as follows.

**Lemma 3.** In the asymmetric case,

\[
\delta_{2N,2} \geq 2N - 1 - \frac{N - 1}{\eta_A} - \frac{N - 1}{\eta_B},
\]

(28)
Proof. The above approach gives
\[
P(\Lambda_{A_3B_4}|A_{A_1B_2}) = P(\Lambda_{A_3}|A_{A_1B_2}) + P(\Lambda_{B_4}|A_{A_1B_2}) \\
- P(\Lambda_{A_3 \cup A_4}|A_{A_1B_2}) \\
\geq 2 - 1 \eta_A + 2 - 1 \eta_B - 1 \\
= 3 - 1 \eta_A - 1 \eta_B.
\]
(29)

The proof proceeds as that of Lemma 2.

Lemma 1 and Lemma 3 give the BC inequality for the asymmetric case as
\[
|E(A_1B_2|A_{A_1B_2}) + E(A_3B_4|A_{A_1B_2})| + |E(A_3B_4|A_{A_3B_4})| + \cdots + |E(A_{2N-1}B_{2N}|A_{A_{2N-1}B_{2N}}) \\
- E(A_1B_{2N}|A_{A_1B_{2N}})| \\
\leq 2(N-1)\left(\frac{1}{\eta_A} + \frac{1}{\eta_B} - 1\right),
\]
(30)
and, as before, for a value $\beta$ on the left-hand side
\[
\beta \leq 2(N-1)\left(\frac{1}{\eta_A} + \frac{1}{\eta_B} - 1\right),
\]
(31)
or, equivalently,
\[
\eta_A \leq \frac{\beta}{2(N-1)} + 1 - \frac{1}{\eta_B}.
\]
(32)

Again, for the quantum prediction on $\beta$ we obtain the right-hand side of Eq. (13).

A particularly interesting case is when $\eta_B = 1$. In terms of a generic $\beta$ we have
\[
\eta_A \leq \frac{2(N-1)}{\beta},
\]
(33)
and, in particular for $\beta = 2N \cos(\pi/2N)$,
\[
\eta_A \leq \frac{N-1}{N \cos(\pi/2N)}.
\]
(34)

### D. Sufficient condition for symmetric efficiencies

To prove sufficiency of the established bounds, it is convenient to go back to our first approach to an LHV model, in terms of ensembles of pairs of particles, with pairs of specified values occurring at a given probability. We will simply build an LHV model with the desired $\beta$ and $\eta$.

We start by splitting the total ensemble into subensembles $\Lambda_i$ that collect states that have exactly $i$ non-detections (zero values) of the constituent $A_j$’s and $B_k$’s. We note that the $\Lambda_0$ so defined coincides with the $\Lambda_0$ defined at inequality [19], and therefore that the BC inequality holds for it. In fact, we have the following lemma:

Lemma 4. It is possible to construct a LHV model so that the results from the subensemble $\Lambda_0$ satisfy $E(A_j|\Lambda_0) = E(B_k|\Lambda_0) = 0$ and saturate the BC inequality.

Proof. Let $\Lambda_0$ consist of $4N$ states ($n = 1, \ldots, 2N$ and $m = \pm 1$), all with equal probability, defined so that
\[
A_j \text{ and } B_j = \begin{cases} 
  m, & j < n \\
  -m, & j \geq n
\end{cases}
\]
(35)

It is immediately obvious that the individual results have equal probability, and it is simple to verify that
\[
E(A_1B_2|\Lambda_0) = E(A_3B_2|\Lambda_0) = \cdots = E(A_{2N-1}B_{2N}|\Lambda_0) = -E(A_1B_{2N}|\Lambda_0) = 1 - \frac{1}{N}.
\]
(36)

Thus, the BC inequality is saturated by this model. ■

The subensembles where one or more non-detections occur are not hindered by the BC inequality. Indeed, for those that give well-defined correlations we have the following result:

Lemma 5. It is possible to construct a LHV model so that the results from the subensembles $\Lambda_i$, $1 \leq i \leq 2N - 2$, satisfy $E(A_j|\Lambda_i) = E(B_k|\Lambda_i) = 0$ and give all correlations the extreme value 1, and therefore maximally violate the BC inequality.

Proof. Let $\Lambda_1$ consist of $4N$ states ($n = 1, \ldots, 2N$ and $m = \pm 1$), all with equal probability, defined so that
\[
A_j \text{ and } B_j = \begin{cases} 
  m, & j < n \\
  0, & j = n \\
  -m, & j > n
\end{cases}
\]
(37)

It is again immediately obvious that the individual results have equal probability, and this time it is also obvious that
\[
E(A_1B_2|\Lambda_1) = E(A_3B_2|\Lambda_1) = \cdots = E(A_{2N-1}B_{2N}|\Lambda_1) = -E(A_1B_{2N}|\Lambda_1) = 1.
\]
(38)

Ensembles $\Lambda_i$ with this property for $i > 1$ can trivially be constructed from $\Lambda_1$ by adding events with additional zeros and thus, the lemma holds for those as well. ■

We are now in a suitable position to build an LHV model for the required values of $\eta$ and $\beta$. The existence is sufficiently important to give the result the status of a theorem.
Theorem 2. Sufficient condition for \( \eta_A = \eta_B = \eta \): When \( 2N - 2 \leq \beta \leq 2N \) we can always build an LHV model with

\[
\eta = \frac{2(N-1)}{N-1 + \frac{\beta}{2}}. \tag{39}
\]

Proof. We use the above ensemble construction of \( \Lambda_0 \) and \( \Lambda_1 \), and also a subensemble with no detections \( \Lambda_{2N} \); we let the other subensembles have probability zero. In this model, under the assumption of independent errors, the probabilities of single detection and coincidence are

\[
P(\Lambda_0) + (1 - \frac{1}{N}) P(\Lambda_1) = \eta, \tag{40a}
\]

\[
P(\Lambda_0) + (1 - \frac{1}{N}) P(\Lambda_1) = \eta^2. \tag{40b}
\]

Solving for the unknown probabilities, we obtain

\[
P(\Lambda_0) = (2N-1)\eta^2 - (2N-2)\eta, \tag{41a}
\]

\[
P(\Lambda_1) = 2N(\eta - \eta^2). \tag{41b}
\]

We also obtain

\[
E(A_1B_2|\Lambda_{A_1B_2}) = \cdots = E(A_{2N-1}B_{2N}|\Lambda_{A_{2N-1}B_{2N}}) = E(A_1B_2|\Lambda_{A_1B_2}) + (1 - \frac{1}{N}) P(\Lambda_0) + (1 - \frac{1}{N}) P(\Lambda_1)
\]

\[
= \left( 1 - \frac{1}{N} \right) \frac{2N - 2\eta - \eta^2}{\eta^2} P(\Lambda_0) + \left( 1 - \frac{1}{N} \right) \frac{2}{\eta} = \left( 1 - \frac{1}{N} \right) \left( \frac{2}{\eta} - 1 \right) . \tag{42}
\]

This makes the left-hand side of the BC inequality obey

\[
|E(A_1B_2|\Lambda_{A_1B_2}) + E(3B_2|\Lambda_{A_3B_2}) + E(A_3B_4|\Lambda_{A_3B_4}) + E(A_2B_4|\Lambda_{A_2B_4}) + \cdots + E(A_{2N-1}B_{2N}|\Lambda_{A_{2N-1}B_{2N}}) - E(A_1B_2|\Lambda_{A_1B_2})| = (2N-2)\left( \frac{2}{\eta} - 1 \right) = \beta.
\]

Solving for \( \eta \), we arrive at Eq. \( 39 \). \( \blacksquare \)

E. Sufficient condition for the asymmetric case

To complete the sufficiency proof for \( \eta_A \neq \eta_B \), we first need to redefine our subensembles, to reflect the asymmetry of the two detectors. Here, we split the total ensemble into subensembles \( \Lambda_{i,l} \) that collect states that have exactly \( i \) non-detections (zero values) of the constituent \( A_j \)'s and exactly \( l \) non-detections of the constituent \( B_k \)'s. We note that again the \( \Lambda_{0,0} \) so defined coincides with the \( \Lambda_0 \) defined at inequality \( 15 \), the BC inequality holds for it, andLemma 4 gives a LHV model that saturates the BC inequality.

The subensembles where one or more non-detections occur are still not hindered by the BC inequality. Indeed, for those that give well-defined correlations we have the following result:

Lemma 6. It is possible to construct a LHV model so that the results from the subensembles \( \Lambda_{i,l} \) \( 0 \leq i,l \leq N - 1 \) and not both zero, satisfy \( E(A_j|\Lambda_{i,l}) = E(B_k|\Lambda_{i,l}) = 0 \) and give all correlations the extreme value 1, and therefore maximally violate the BC inequality.

Proof. In the case \( i = 0 \), let \( \Lambda_{1,0} \) consist of \( 2N \) states \( (n = 1, \ldots, N \text{ and } m = \pm 1) \), all with equal probability, defined so that

\[
A_j = \begin{cases} m, & j < 2n - 1 \\ 0, & j = 2n - 1 \text{ and } B_j = \begin{cases} m, & j < 2n \\ -m, & j \geq 2n \end{cases} \end{cases}
\]

Once more, it is immediately obvious that the individual results have equal probability; it is also obvious that

\[
E(A_1B_2|\Lambda_{1,0}) = E(A_3B_2|\Lambda_{1,0}) = \cdots = E(A_{2N-1}B_{2N}|\Lambda_{1,0})
\]

\[
= -E(A_1B_2|\Lambda_{1,0}) = 1.
\]

The case \( i = 0 \) is handled similarly, and the cases when both \( i \) and \( l \) are nonzero can trivially be constructed by adding events with additional zeros to, say, \( \Lambda_{1,0} \) and thus, the lemma holds for these cases as well. \( \blacksquare \)

We are now in a suitable position to build an LHV model for the required values of \( \eta \) and \( \beta \).

Theorem 3. Sufficient condition for \( \eta_A \neq \eta_B \): When \( 2N - 2 \leq \beta \leq 2N \) we can always build an LHV model with

\[
\eta_A = \frac{1}{2(N-1)} + \frac{1}{\eta_B}. \tag{46}
\]

Proof. We use the above ensemble construction of \( \Lambda_{0,0}, \Lambda_{1,0}, \text{ and } \Lambda_{0,1} \), and also a subensemble with no detections \( \Lambda_{N,N} \); we let the other subensembles have probability zero. In this model, under the assumption of independent errors, the probabilities of single detection and coincidence are

\[
P(\Lambda_{0,0}) + P(\Lambda_{0,1}) + (1 - \frac{1}{N}) P(\Lambda_{1,0}) = \eta_A, \tag{47a}
\]

\[
P(\Lambda_{0,0}) + (1 - \frac{1}{N}) P(\Lambda_{0,1}) + P(\Lambda_{1,0}) = \eta_B, \tag{47b}
\]

\[
P(\Lambda_{0,0}) + (1 - \frac{1}{N}) [P(\Lambda_{0,1}) + P(\Lambda_{1,0})] = \eta_A\eta_B. \tag{47c}
\]

Solving for the unknown probabilities, we obtain

\[
P(\Lambda_{0,0}) = (2N-1)\eta_A\eta_B - (N-1)(\eta_A + \eta_B), \tag{48a}
\]

\[
P(\Lambda_{0,1}) = N(\eta_A - \eta_A\eta_B), \tag{48b}
\]

\[
P(\Lambda_{1,0}) = N(\eta_B - \eta_A\eta_B). \tag{48c}
\]
We also obtain

\[ E(A_1 B_2 | A_{A_1} B_{A_2}) = \cdots = E(A_{2N-1} B_{2N} | A_{A_2N-1} B_{A_{2N}}) \]

\[ = -E(A_1 B_{2N} | A_{A_1} B_{B_2}) \]

\[ = (1 - \frac{1}{N}) P(\Lambda_{0,0}) + P(\Lambda_{1,0}) + P(\Lambda_{0,1}) + P(\Lambda_{1,1}) \]

\[ = \left( 1 - \frac{1}{N} \right) \frac{\eta_A + \eta_B - \eta_A \eta_B}{\eta_A \eta_B} \]

\[ = \left( 1 - \frac{1}{N} \right) \left( \frac{1}{\eta_A} + \frac{1}{\eta_B} - 1 \right). \]

This makes the left-hand side of the BC inequality obey

\[ |E(A_1 B_2 | A_{A_1} B_{A_2}) + E(A_3 B_2 | A_{A_3} B_{A_2}) + \cdots + |E(A_{2N-1} B_{2N} | A_{A_{2N-1}} B_{A_{2N}}) - E(A_1 B_{2N} | A_{A_1} B_{B_2})| \]

\[ = 2N - 2 \left( \frac{1}{\eta_A} + \frac{1}{\eta_B} - 1 \right) \]

\[ = \beta. \]

Solving for \( \eta_A \), we arrive at Eq. (49). □

III. CONCLUSIONS

We have obtained the minimum detection efficiency required for a loophole-free violation of the BC chained Bell inequalities involving \( N \) settings per observer. If both particles are detected with the same efficiency, the minimum detection efficiency is given by Eq. (12). If the particles are detected with different efficiencies the minimum efficiencies are related by Eq. (13).

The required detection efficiency increases with the number of settings and tends to one as \( N \) tends to infinity. This result shows that the BC inequalities are not adequate for closing the detection loophole. At this point, one should note that the BC inequalities are useful in other situations, where other properties than a high detection bound are important [20, 21, 22, 23, 24].

Our results also establish the connection between the amount of violation \( D \) and \( \eta_{\text{crit}} \) for the BC inequalities. From Eqs. (11) and (12), if \( \eta_A = \eta_B \), we obtain

\[ \eta_{\text{crit}} = \frac{2}{D + 1}, \]

which establishes a close relation between \( \eta_{\text{crit}} \) and \( D \), similar to the one already found for the Mermin Bell inequalities [12]. If \( \eta_A \neq \eta_B \), from Eq. (13), we have

\[ \eta_{A, \text{crit}} = \frac{1}{D + 1 - \frac{1}{\eta_{B, \text{crit}}}}. \]

Notice that \( D \) is related to the minimum visibility \( V_{\text{crit}} \) required to violate the BC chained Bell inequalities, when, instead of \( |\psi^-\rangle \), we have \( \rho = V |\psi^-\rangle \langle \psi^-| + (1 - V) \mathbb{I} / 4 \), where \( \mathbb{I} \) is the identity matrix. Specifically, a simple calculation shows that \( V_{\text{crit}} = 1 / D \). Curiously, the same relation between \( V_{\text{crit}} \) and \( D \) is found in stabilizer Bell inequalities for graph states [25].

So far, we have assumed that the prepared states have perfect visibility \( (V = 1) \). The effect of a non-perfect visibility \( V < 1 \) can be easily calculated by replacing \( \beta = 2N \cos(\pi/2N) \) by \( 2VN \cos(\pi/2N) \) in all the previous results.

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