Abstract. For an odd prime $p$ and a number field $F$ containing a $p$th root of unity, we study generalised Tate kernels, $D^{[i,n]}_F$, for $i \in \mathbb{Z}$ and $n \geq 1$, having the properties that if $i \geq 2$ and if either $p$ does not divide $i$ or $\mu_p \subset F$ then there are natural isomorphisms $D^{[i,n]}_F \cong K^{(i-1)}_{2i-1}(\mathcal{O}_F^S)/p^n$, and that they are periodic modulo a power of $p$ which depends on $F$ and $n$. Our main result is that if the Gross-Jaulent conjecture holds for $(F,p)$ then there is a natural isomorphism $D^{[1,n]}_F \cong \tilde{E}_F/p^n$ where $\tilde{E}_F$ is the Gross kernel. We apply this result to compute lower bounds for capitulation kernels in even étale $K$-theory.

1. Introduction

Let $F$ be an algebraic number field containing the $p$-th roots of unity $\mu_p$ for some odd prime number $p$. Let $S$ be the set of $p$-adic primes of $F$ and let $\mathcal{O}_F^S$ be the ring of $S$-integers in $F$.

Let $E/F$ be a Galois extension with group $G$. The goal of this paper is to better understand and to calculate lower bounds for the $K$-theory capitulation kernels $\text{Ker}(f_i)$ where $f_i : K^{(i-2)}_{2i-2}(\mathcal{O}_F^S) \to K^{(i-2)}_{2i-2}(\mathcal{O}_E^S)^G$ is the natural functorial map. Our main results apply to the case where $G$ is cyclic of degree $p^n$.

We follow the strategy established by Assim and Movahhedi in [1]. Generalising a result of Kahn, [13], they observe that, when $G$ is a cyclic $p$-group, $\text{Ker}(f_i)$ and $\text{Coker}(f_i)$ have the same order and, in the case where $|G| = p$, they describe this cokernel in terms of certain ‘generalised Tate kernels’ $D^{(i)}_F$ which are subgroups of $F^\times$. $D^{(2)}_F$ is the classical Tate kernel; i.e the group $C_F = \{a \in F^\times | \{a, \zeta_p\} = 0 \text{ in } K_2(F)\}$. Furthermore, Greenberg has shown that if $\tilde{F}$ denotes the compositum of the $\mathbb{Z}_p$-extensions of $F$ and if $A_F = \{a \in F^\times | \sqrt[p]{a} \in \tilde{F}\}$, then $D^{(0)}_F \subset A_F$ with equality if and only if Leopoldt’s conjecture holds for $(F,p)$.
In this paper we describe ‘Tate kernels’ $D_F^{[i,n]}$ for all $i \in \mathbb{Z}$ with the following
four properties:

1. $D_F^{[i,1]} = D_F^{(i)}(F^\times)^p(i - 1)$ where $M(j)$ denotes the $j$th Tate twist of
the Galois module $M$.

2. There are natural isomorphisms

   $D_F^{[i,n]} \cong K^{\text{ét}}_{2i-1}(O_F^S)/p^n$

   for $i \geq 2$ whenever either $p$ does not divide $i$ or $F$ contains the $p^n$th
roots of unity.

3. If the Gross conjecture holds for $(F,p)$ then

   $D_F^{[1,n]} \cong \tilde{E}_F/p^n$

   where $\tilde{E}_F$ is the Gross kernel of $F$.

4. The Tate kernels have a natural periodicity property:

   There exists a number $m = m_n(F) \geq 0$ such that

   $D_F^{[j,n]} = D_F^{[i,n]}(j - i)$

   whenever $i \equiv j \pmod{p^m}$. Furthermore $m = 0$ if $\mu_{p^r} \subset F$ for suffi-
ciently large $r$.

These groups, and variations on them, have been studied in a number of places. The central ideas go back to Tate, [20], and property [2] (at least for $i = 2$) is implicit there. The periodicity property is essentially found in Greenberg’s paper, [8], but see also Assim and Movahhedi, [1], Lemma 2.1 and Vauclair, [21], section 4. The main contribution of the present paper is property [3], although it should be noted that a key step in the proof is Theorem 2.3 of
Kolster, [15]. In particular, [3] implies that if $F$ has only one $p$-adic prime then
$D_F^{[1]} = U_F^S \cdot (F^\times)^p$. This, in conjunction with property [4], explains the theorem of Kersten, [14], that if $F = \mathbb{Q}(\zeta_{p^r})$ for sufficiently large $r$, then $A_F = U_F^S \cdot (F^\times)^p = C_F$. Our result in this paper grew out of the attempt to understand this theorem of Kersten.

The layout of the paper is as follows: In section 2 we introduce the Tate
kernels and establish properties [1], [3] and [4]. In section 3 we give a co-
homological description of the Tate kernels and use this to establish property
[2], as well as to prove certain basic algebraic properties which have already
been used section 2.

In section 4 we show how to describe the groups $\text{Coker}(f_i)$, when $E/F$ is a
cyclic $p$-extension, in terms of the Tate kernels and in the last two sections we
apply these results to give lower bounds for $\text{Ker}(f_i)$. In particular, in section 6
we deal with cyclic $p$-extensions in which there are no tamely-ramified primes.
We exploit the relationship between the Gross kernel and the logarithmic class
group to show (Theorem 6.8) that if $E/F$ is a finite layer of the $p$-cyclotomic
extension of $F$ and if the Gross conjecture holds for $(E,p)$, then for infinitely
many $i$, $|\text{Ker}(f_i)| = |\text{Ker}(f_1)|$ where $f_1$ is the natural functorial map
$\text{Cl}_F \rightarrow$
$\tilde{\text{Cl}}^G_F$ of logarithmic class groups. We give examples both of cyclotomic and of non-cyclotomic extensions for which the $\text{Ker}(f_i)$ are all non-trivial.

Notation: For an abelian group $A$, $A[n]$ and $A/n$ denote the kernel and cokernel respectively of multiplication by $n \in \mathbb{N}$. $\text{Div}(A)$ denotes the maximal divisible subgroup of $A$.

If $R$ is an integral domain and $M$ is an $R$-module, then $\text{tor}_R(M)$ is the torsion submodule of $M$ and $\text{Fr}_R(M) = M/\text{tor}_R(M)$.

For a field $F$, $G_F$ denotes its Galois group. The Tate module for $(F, p)$ is the $\mathbb{Z}_p[G_F]$-module $\mathbb{Z}_p(1) := \lim_i \mu_{p^i}$, the (inverse) limit being taken over the natural surjections $\mu_{p^{i+1}} \to \mu_{p^i}$, $\zeta \mapsto \zeta^p$. More generally, for any $i \in \mathbb{Z}$, we have $\mathbb{Z}_p(i) := \mathbb{Z}_p^{\otimes i}$ for $i \geq 0$, $\mathbb{Z}_p(0) := \mathbb{Z}_p$ and $\mathbb{Z}_p(-i) := \text{Hom}_{\mathbb{Z}_p}((\mathbb{Z}_p(i), \mathbb{Z}_p)$ for $i \geq 1$. For any $\mathbb{Z}_p$-$G_F$-module $M$, the $i$th Tate twist is the $\mathbb{Z}_p$-$G_F$-module $M(i) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ (with diagonal Galois action). Since $\mathbb{Z}_p(i)$ is isomorphic to $\mathbb{Z}_p$ as a $\mathbb{Z}_p$-module, $M(i)$ is isomorphic to $M$ with a twisted Galois-action. Observe that if $M = M[p^n]$, then the natural surjection $\mathbb{Z}_p(i) \to \mu_{p^{i+1}}$ induces an isomorphism $M(i) \cong M \otimes \mu_{p^i}$.

By the Pontryagin dual of the $\mathbb{Z}_p$-$G_F$-module $M$ we will mean the module $M^\ast = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$.

2. Generalised Tate Kernels

Everywhere below $p$ is an odd prime, $F$ is a number field containing the $p$-th roots of unity, $\mu_p$, and $S = S_p(F)$ is the set of prime ideals lying over $p$. Let $F_n = F(\mu_{p^n})$ and let $F_\infty = \bigcup_{n=1}^\infty F_n$. Let $\Gamma = \text{Gal}(F_\infty/F)$. Let $\text{Cl}^S(F)$ denote the $S$-classgroup of $F$ and let $A^S(F)$ denote the $p$-Sylow subgroup $\text{Cl}^S(F)\{p\}$. Let also $\mu_\infty(F) = F \cap \mu_\infty = \mu_{p^{\infty}}$ be the group of $p$-power roots of unity in $F$. We will let $F_S/F$ be the maximal algebraic extension of $F$ unramified outside the $S$ and we will let $G_F = \text{Gal}(F_S/F)$.

As usual, $\Lambda$ will denote the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]] := \lim_n \mathbb{Z}_p[\Gamma/p^n]$. If we fix a topological generator, $\gamma_0$, of $\Gamma$, then we get an isomorphism of topological rings $\Lambda \cong \mathbb{Z}_p[[T]], \gamma_0 \leftrightarrow 1 + T$.

Let $K$ be the maximal abelian pro-$p$ extension of $F_\infty$ and let $\mathcal{K} = F_\infty^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p$. Kummer Theory gives a perfect duality pairing

$$\langle \cdot, \cdot \rangle : \mathcal{K} \times \text{Gal}(K/F_\infty) \to \mu_p\infty$$

determined by the formula

$$g(\sqrt[p^n]{a}) = \left< a \otimes \frac{1}{p^n}, g \right> \sqrt[p^n]{a}$$

for $g \in \text{Gal}(K/F_\infty)$, $a \in F_\infty^\times$ and $n \geq 1$. Furthermore, each term is naturally a $\Gamma$-module and this pairing is compatible with the $\Gamma$-actions in the sense that $(\gamma(\alpha), \gamma(g)) = \gamma((\alpha, g))$ for all $\gamma \in \Gamma$, $\alpha \in \mathcal{K}$ and $g \in \text{Gal}(K/F_\infty)$.
Observe that for all \(i \in \mathbb{Z}\),
\[
\mathcal{K}(i) = F_\infty^\times \otimes_{\mathbb{Q}_p/\mathbb{Z}_p} (i) = \bigcup_n F_\infty^\times \otimes \mu_{p^n}^\oplus = \bigcup_n F_\infty^\times / (F_\infty^\times)^{p^n} \otimes \mu_{p^n}^\oplus.
\]

Now let \(M\) be the maximal abelian pro-
\(p\) extension of \(F_\infty\) which is unramified outside \(p\). Let \(X = \text{Gal}(M/F_\infty)\) and let \(\mathcal{M}\) be the subgroup of \(\mathcal{K}\) corresponding to \(X\) under the pairing \([\ ]\) i.e. \(\mathcal{M}\) is the orthogonal complement of \(\text{Gal}(K/M)\) with respect to the pairing. There is thus an induced perfect pairing \(\mathcal{M} \times X \to \mu_{p^\infty}\).

Remark 2.1. In \([1]\), Assim and Movahhedi introduce the groups \(D_F^{(i)}\), \(i \geq 2\), but define them using \(K\) instead of \(N\). However, as they observe, when \(i \geq 2\), this defines the same group. While the \(\Lambda\)-module \(K\) is, at first glance, a more natural object, there are advantages to using \(N\) in the definition of the groups \(D_F^{(i)}\). In the first place, Iwasawa has proved some very strong results on the \(\Lambda\)-module structure of the dual group of \(N\). Furthermore, using \(N\) instead of \(K\), we get a sequence of groups with analogous properties defined for all \(i \in \mathbb{Z}\), including \(i = 1\), which is the main focus of this paper.

Regarding the structure of \(N\), the following is known:

Let \(\hat{X} = \text{Fr}_\Lambda(X) = X/\text{tor}_\Lambda(X)\) be the dual of \(N\) with respect to the Kummer pairing. Iwasawa, \([9]\) Theorem 17, proved that there is a short exact sequence of \(\Lambda\)-modules
\[
0 \to \hat{X} \to \Lambda^{r_2} \to \hat{H}_F \to 0
\]
with \(\hat{H}_F\) finite. Greenberg, \([8]\), pointed out that the arguments of Iwasawa show that \(\hat{H}_F\) is abstractly isomorphic to (in fact, Kummer dual to ) the group \(\text{Ker}(A^S(F_n) \to A^S(F_\infty))\) for all sufficiently large \(n\). We will use this fact below. (Coates, \([4]\), also showed that \(\hat{H}_F\) is abstractly isomorphic to \(\text{Ker}(K_2(O_{F^\infty}) \to K_2(O_{F^\infty}^\times))\) for all sufficiently large \(n\). For a more precise assertion about limits, see Kahn, \([13]\), Théorème 6.2.)

We deduce the following well-known fact:

Lemma 2.2. For all \(i \in \mathbb{Z}\),
\[
\text{Div}(N(i)^\Gamma) \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{r_2}.
\]
Proof. Taking the $\Gamma$-action into account, the Kummer pairing gives an isomorphism of $\Lambda$-modules $\mathcal{N}(i) \cong \tilde{X}(-1-i)^*$. It follows that

$$\mathcal{N}(i)^{\Gamma} \cong (\tilde{X}(-1-i)_{\Gamma})^* = (\tilde{X}(-1-i)/T\tilde{X}(-1-i))^*.$$ 

Twisting the exact sequence of Iwasawa by $-\Lambda(t)$, and using the fact that $\Lambda(t) \cong \Lambda$ for all $t \in \mathbb{Z}$, gives the exact sequence

$$0 \to \tilde{X}(-1-i) \to \Lambda^{r_2} \to \bar{H}_F(-1-i) \to 0.$$ 

Thus the natural map $\bar{X}(-1-i)/T\bar{X}(-1-i) \to \Lambda^{r_2}/T\Lambda^{r_2} = \mathbb{Z}_p^{r_2}$ has finite kernel and cokernel. Taking Pontryagin duals again, it follows that there is a map $(\mathbb{Q}_p/\mathbb{Z}_p)^{r_2} \to \mathcal{N}(i)^{\Gamma}$ with finite kernel and cokernel. \qed

Greenberg ([8], p1238) conjectured that $\text{Div}(\mathcal{K}(t)^\Gamma) = \text{Div}(\mathcal{N}(t)^\Gamma)$ for all $t \in \mathbb{Z} \setminus \{0\}$. As he pointed out, the analogous statement for $t = 0$ cannot possibly hold since $\text{Div}(\mathcal{K}^\Gamma) \supset F^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p$ which is a countable direct sum of copies of $\mathbb{Q}_p/\mathbb{Z}_p$. Note that if the conjecture is true for any given $t \in \mathbb{Z}$, then it follows that $D_F(t+1) = \{a \in F^\times \mid a \otimes \chi \in \text{Div}(\mathcal{K}(t)^\Gamma) \text{ for all } \chi \in \mu_p^{\otimes t}\}$. If we interpret the groups $\mathcal{K}(t)^\Gamma$ in terms of Galois cohomology (see section 3 below), Greenberg’s conjecture is equivalent to Schneider’s conjecture in [18] that the groups $\mathcal{H}^2(G_F^S, \mathbb{Q}_p/\mathbb{Z}_p(i))$ vanish for all $i \neq 1$. This latter conjecture was proved by Soulé ([19]) when $i \geq 2$ using Borel’s theorem ([24]) on the finiteness of the groups $K_{2i}(\mathcal{O}_F^S)$. It follows, as remarked above, that for $i \geq 2$ we have

$$D_F(i) = \{a \in F^\times \mid a \otimes \chi \in \text{Div}(\mathcal{K}(i-1)^\Gamma) \text{ for all } \chi \in \mu_p^{\otimes (i-1)}\}.$$ 

Greenberg showed in [8] that $D_F^{(2)}$ is the classical Tate Kernel of $F$; the group

$$C_F := \{a \in F^\times \mid \{a, \zeta_p\} = 0 \in K_2(F)\}.$$ 

Furthermore, when $i = 0$, Greenberg showed that

$$\{a \in F^\times \mid a \otimes \chi \in \text{Div}(\mathcal{K}(-1)^\Gamma) \text{ for all } \chi \in \mu_p^{\otimes (-1)}\} = A_F := \{a \in F^\times \mid \sqrt[p]{a} \in \bar{F}_1\}$$

where $\bar{F}_1$ is the compositum of the first layers of the $\mathbb{Z}_p$-extensions of $F$. Thus, $D_F^{(0)} \subset A_F$ and Greenberg’s conjecture in this case is that equality holds here. This is equivalent to Leopoldt’s conjecture for the pair $(F, p)$ since $\dim_{\mathbb{F}_p}(D_F^{(i)}/(F^\times)^p) = 1 + r_2$ (see below).

Our main interest in the groups $D_F^{(i)}$ in this paper stems from the work of Assim and Movahhedi ([1]) who show that for $i \geq 2$ that there is a natural isomorphism

$$D_F^{(i)}/(F^\times)^p(i-1) \cong K^{\text{ét}}_{2i-1}(\mathcal{O}_F^S)/p$$

(for a generalisation, see Corollary 3.8 below).

Before proceeding, we will introduce some more general ‘Tate kernels’:

For each $n \geq 1$, let $\Gamma_n = \text{Gal}(F_\infty/F_n)$ and let $G_n = \text{Gal}(F_n/F) = \Gamma/\Gamma_n$. Let $Q_{i,n}$ be the natural map

$$\left(F_n^\times/(F_n^\times)^{p^n}(i-1)\right)^{G_n} \to \mathcal{K}(i-1).$$
We define
\[ D_F^{[i,n]} := (Q_{i,n})^{-1}(\text{Div}((\mathcal{N}(i - 1)^\Gamma))). \]

Thus if \( F = F_n \) (i.e., if \( \mu_{p^n} \subset F^\times \)) then \( D_F^{[i,n]} \subset F^\times/(F^\times)^{p^n}(i - 1) \) and there exists a subgroup \( D_F^{[i,n]} \) of \( F^\times \) and containing \( (F^\times)^{p^n} \) with the property that \( D_F^{[i,n]} = (D_F^{[i,n]}/(F^\times)^{p^n})(i - 1) \). In particular, \( D_F^{[i,1]} = D_F^{[i]} \) for all \( i \in \mathbb{Z} \).

We will prove below (see [3.8]) that there is a natural isomorphism
\[ D_F^{[i,n]} \cong K_{2i-1}^\text{et}(\mathcal{O}_F)/p^n \]
when \( i \geq 2 \) and when either \( F = F_n \) or \( i \neq 0 \) (mod \( p \)).

Note that when \( i = 1 \), the natural map \( F^\times/(F^\times)^{p^n} \to (F^\times/(F^\times)^{p^n})^G_n \) is an isomorphism. To see this, one can use the Kummer isomorphism \( F^\times/(F^\times)^{p^n} \cong H^1(F,\mu_{p^n}) \) and the fact that the restriction map \( H^1(F,\mu_{p^n}) \to H^1(F_n,\mu_{p^n})^G_n \) is an isomorphism since \( H^1(G_n,\mu_{p^n}) = H^2(G_n,\mu_{p^n}) = 0 \). Thus, for all \( n \geq 1 \)
\[ D_F^{[1,n]} \subset F^\times/(F^\times)^{p^n} \]
and if we let
\[ D_F^{[1,n]} := \{ a \in F^\times | a \otimes 1/p^n \in \text{Div}(\mathcal{N}^\Gamma) \} \]
then
\[ D_F^{[1,n]} = D_F^{[1,n]}/(F^\times)^{p^n}. \]

We let \( p^e = p^{e_F} \) be the exponent of the group \( \tilde{H}_F \). For all \( n \geq 1 \), we will let \( m_n = m_n(F) = \max(0, e_F + n - n_F) \). Observe that \( m_n = 0 \) if there is an \( e \geq 0 \) with \( p^e \tilde{H}_F = 0 \) and \( \mu_{p^{e+n}} \subset F \).

Observe that \( m_n(F_m) = 0 \) whenever \( m \) is sufficiently large. Below we will also let \( m_F \) denote \( m_1(F) \).

Let \( \kappa : \Gamma \to 1 + p\mathbb{Z}_p \) be the Teichmüller character of \( F_\infty/F \) and observe that, by definition of \( m_n = m_n(F) \), for all \( \gamma \in \Gamma \), \( \kappa(\gamma)^{p^m} \equiv 1 \) (mod \( p^{e+n} \)).

The following theorem is essentially due to Greenberg (see also [1], Lemma 2.1):

**Theorem 2.3.** For any number field \( F \) containing \( \mu_p \),
\[ \text{Div}(\mathcal{N}(t)^\Gamma)[p^n] = (\text{Div}(\mathcal{N}(t')^\Gamma)[p^n]) \ (t - t') \text{ whenever } t \equiv t' \pmod{p^{m_n}}. \]

**Proof.** The argument we give here is just an adaptation of the proof of [1], Lemma 2.1, which treats the case \( n = 1 \) and \( m_1 = 0 \). That argument in turn is just a more explicit version of the sketch given by Greenberg in [3].

Observe that since \( \Gamma \) acts trivially on \( \text{Div}(\mathcal{N}(t)^\Gamma)[p^n] \), the assertion simply concerns the equality of two subgroups of \( \mathcal{N}(t) \).

As noted above, \( \mathcal{N}(t + 1) \) is Pontryagin-dual to \( \tilde{X}(-t) \), so that \( \mathcal{N}(t + 1)^\Gamma \) is dual to \( \left( \tilde{X}(-t) \right)^\Gamma = \tilde{X}(-t)/T\tilde{X}(-t) \), which is a finitely-generated \( \mathbb{Z}_p \)-module.
Hence Div(\(N(t^2)^F\)) is dual to Fr_{Z_p} \left( \tilde{X}(t)/T \tilde{X}(t) \right) and, finally, Div(\(N(t + 1)^F\))[p^n] is dual to Fr_{Z_p} \left( \tilde{X}(t)/T \tilde{X}(t) \right)/p^n. Therefore, we must prove that

\[
\text{Fr}_{Z_p} \left( \tilde{X}(i)/T \tilde{X}(i) \right)/p^n = \text{Fr}_{Z_p} \left( \tilde{X}(i)/T \tilde{X}(j) \right)/p^n(i-j)
\]
as quotient groups of \(\tilde{X}(i)\) whenever \(i \equiv j \pmod{p^n}\).

Now, by the result of Iwasawa mentioned above there is a short exact sequence of \(\Lambda\)-modules

\[
0 \to \tilde{X}(i) \to \Lambda(i)^{r_2} \to \tilde{H}_F(i) \to 0
\]
for all \(i \in \mathbb{Z}\).

From the commutative exact diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{X}(i) & \longrightarrow & \Lambda(i)^{r_2} & \longrightarrow & \tilde{H}_F(i) & \longrightarrow & 0 \\
\downarrow T & & \downarrow T & & \downarrow T & & \\
0 & \longrightarrow & \tilde{X}(i) & \longrightarrow & \Lambda(i)^{r_2} & \longrightarrow & \tilde{H}_F(i) & \longrightarrow & 0
\end{array}
\]

we obtain the exact sequence

\[
0 \longrightarrow \tilde{H}_F(i)[T] \overset{\delta}{\longrightarrow} \tilde{X}(i)/T \tilde{X}(i) \longrightarrow \Lambda(i)^{r_2}/T\Lambda(i)^{r_2} \longrightarrow \tilde{H}_F(i)/T\tilde{H}_F(i) \longrightarrow 0.
\]

The image of \(\delta\) is the group

\[
\tilde{X}(i) \cap T\Lambda(i)^{r_2}/T\tilde{X}(i).
\]

But this is precisely the \(\mathbb{Z}_p\)-torsion subgroup of \(\tilde{X}(i)/T \tilde{X}(i)\), since \(\Lambda(i)^{r_2}/T\Lambda(i)^{r_2} \cong (\Lambda/T\Lambda)^{r_2} \cong \mathbb{Z}_p^{r_2}\). Thus

\[
\text{Fr}_{Z_p} \left( \tilde{X}(i)/T \tilde{X}(i) \right)/p^n = \tilde{X}(i)/Y_i \text{ where } Y_i = \tilde{X}(i) \cap T\Lambda(i)^{r_2} + p^n \tilde{X}(i).
\]

Therefore we must prove that whenever \(i \equiv j \pmod{p^{m_n}}\), then \(Y_j(i-j) = Y_i\) as subgroups of \(\tilde{X}(i)\). Note that \(p^{e+n}\Lambda(i)^{r_2} \subset Y_i\) since \(p^{e}\) annihilates \(\tilde{H}_F(i)\) and \(\tilde{X}(i)/Y_i\) is annihilated by \(p^n\).

Fix a topological generator, \(\gamma_0\), of \(\Gamma\), so that the action of \(\Lambda = \mathbb{Z}_p[[T]]\) on a \(\Gamma\)-module is given by \(Tm := (\gamma_0 - 1)m\).

Suppose now that \(y \in Y_i\) and that \(i \equiv j \pmod{p^{m_n}}\). Let \(\zeta \in \mathbb{Z}_p(i-j)\). We must show that \(y \otimes \zeta \in Y_i\). By definition of \(Y_j\), \(y = T\lambda + p^n x\) where \(\lambda \in \Lambda(j)^{r_2}\).
and \( T_{\lambda, i} \in \tilde{X}(j) \). Let \( \zeta \in \mathbb{Z}_p(i - j) \). Thus
\[
(1 + T)(\lambda \otimes \zeta) = \gamma_0(\lambda \otimes \zeta) = \kappa(\gamma_0)^{i-j}(\gamma_0 \lambda \otimes \zeta)
\]

\[
= \gamma_0 \lambda \otimes \zeta + (\kappa(\gamma_0)^{i-j} - 1)(\gamma_0 \lambda \otimes \zeta)
\]

\[
= \gamma_0 \lambda \otimes \zeta + y'
\]

\[
= (1 + T)\lambda \otimes \zeta + y'
\]

where \( y' \in p^{e+n}\Lambda(i)^\gamma_2 \subset Y_i \). Thus
\[
T(\lambda \otimes \zeta) = T\lambda \otimes \zeta + y'
\]

and hence
\[
y \otimes \zeta = T\lambda \otimes \zeta + p^n(x \otimes \zeta)
\]

\[
= T(\lambda \otimes \zeta) + p^n(x \otimes \zeta) - y'
\]

which belongs to \( Y_i \) since \( y' \in Y_i \) and \( y \otimes \zeta, y', p^n(x \otimes \zeta) \in \tilde{X}(i) \implies T(\lambda \otimes \zeta) \in \tilde{X}(i) \).

Thus, \( Y_j(i - j) \subset Y_i \) and similarly \( Y_i \subset Y_j(i - j) \), proving the result. \( \square \)

**Corollary 2.4.** \( D_F^{[i,n]}(j - i) = D_F^{[j,n]} \) whenever \( i \equiv j \pmod{p^{m_n}} \).

**Proof.** Note first that the conditions on \( i \) and \( j \) guarantee that \( G_n \) acts trivially on \( \mu_{p^n}^{\otimes(j-i)} \) since \( p^{m_n} \geq p^{n-n_F} = |G_n| \). The result now follows from the commutative diagram

\[
\begin{array}{ccc}
\left( F_n^\times / (F_n^\times)^{p^n} \right) \left( \begin{array}{c}
(i - 1) \\
(j - 1)
\end{array} \right) & \overset{Q_{i,n}(j-i)}{\longrightarrow} & \text{Div}(\mathcal{N}(i-1)^\Gamma)[p^n](j-i) \\
\downarrow & & \downarrow \\
\left( F_n^\times / (F_n^\times)^{p^n} \right) \left( \begin{array}{c}
(i - 1) \\
(j - 1)
\end{array} \right) & \overset{Q_{j,n}}{\longrightarrow} & \text{Div}(\mathcal{N}(j-1)^\Gamma)[p^n]
\end{array}
\]

\( \square \)

**Corollary 2.5.** If \( p^e \tilde{H}_F = 0 \) and \( \mu_{p^{e+n}} \subset F \), then \( D_F^{[i,n]}(j - i) = D_F^{[j,n]} \) for all \( i, j \in \mathbb{Z} \).

In section 3 below we will see that if \( p \) does not divide \( i \) or if \( \mu_{p^n} \subset F \) there is a short exact sequence
\[
0 \to \mu \to D_F^{[i,n]} \to \text{Div}(\mathcal{N}(i-1)^\Gamma)[p^n] \to 0.
\]

where \( \mu = \mu_{p^e}(F_i)^{\otimes i}/p^n \) is a nontrivial cyclic group of order dividing \( p^n \).

In particular, \( \dim_{\mathbb{F}_p} \left( D_F^{(i)} / (F_x^\times)^p \right) = 1 + r_2 \) for all \( i \in \mathbb{Z} \).

We now consider the structure of \( D_F^{[1,n]} \).

For general number fields, the identification of \( D_F^{[1,n]} \) is related to the Gross conjecture (as extended by Jaulent).
Let $U^S_F$ be the group of $S$-units of $F$. The homomorphism

$$G_F : U^S_F \to \prod_{v|p} \mathbb{Z}_p \cdot v, \ \epsilon \mapsto \sum_{v|p} \log_p [N_{F_v/Q_p} \epsilon] \cdot v$$

extends linearly to a $\mathbb{Z}_p$-module homomorphism

$$g_F : U^S_F \otimes \mathbb{Z}_p \to \prod_{v|p} \mathbb{Z}_p \cdot v.$$

The image of $g_F$ is free $\mathbb{Z}_p$-module of rank at most $|S_p(F)| - 1$. Thus the rank of $\text{Im}(g_F)$ is $|S_p(F)| - 1 - \delta_F$ for some $\delta_F \geq 0$. Since the $\mathbb{Z}_p$-rank of $U^S_F \otimes \mathbb{Z}_p$ is $r_2 + |S_p(F)| - 1$, it follows that $\text{Ker}(g_F)$ has rank $r_2 + \delta_F$. For more details see Sinnott, [6] and Kuzmin, [17].

The conjecture of Gross (as extended by Jaulent) is:

$$\delta_F = 0.$$
Lemma 2.7. Let \( \tilde{\mathcal{E}}_{F_n} = \text{colim}_n \tilde{\mathcal{E}}_{F_n} \). Then there is a short exact sequence
\[
0 \to \tilde{\mathcal{E}}_F \otimes_{Z_p} Q_p/Z_p \to \left( \tilde{\mathcal{E}}_{F_n} \otimes_{Z_p} Q_p/Z_p \right)^\Gamma \to H^1(\Gamma, \tilde{\mathcal{E}}_{F_n}) \to 0
\]
where \( H^1(\Gamma, \tilde{\mathcal{E}}_{F_n}) \) is finite.

Proof. We begin by observing that \( (\tilde{\mathcal{E}}_{F_n})^\Gamma = \tilde{\mathcal{E}}_F \) for all \( n \). This can be seen by taking \( \Gamma \)-invariants of the commutative diagram with exact rows
\[
\begin{array}{cccc}
0 & \longrightarrow & \tilde{\mathcal{E}}_F & \longrightarrow & U_F^S \otimes Z_p & \overset{g_F}{\longrightarrow} & \oplus_v \Gamma Z_p \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tilde{\mathcal{E}}_{F_n} & \longrightarrow & U_{F_n}^S \otimes Z_p & \overset{g_{F_n}}{\longrightarrow} & \oplus_v (\oplus_{w|v} Z_p)
\end{array}
\]
(where \( \Delta \) is the map \((a_v)_v \mapsto ([F_n : F_v(a_v)]_v)\)) and using the fact that \( (U_{F_n}^S \otimes Z_p)^\Gamma = U_F^S \otimes Z_p \) and that the vertical arrows are all injective.

It follows, on taking limits, that \( (\tilde{\mathcal{E}}_{F_n})^\Gamma = \tilde{\mathcal{E}}_F \).

Note that the torsion submodule of \( \tilde{\mathcal{E}}_F \) is \( \mu_{p^\infty}(F) = \mu_{p^\infty}(F) \otimes Z_p \). Let \( \mathcal{G}_F = \tilde{\mathcal{E}}_F / \mu_{p^\infty}(F) \). Then \( \mathcal{G}_F \) is a free \( Z_p \)-module and hence \( \mathcal{G}_{F_{F_n}} = \text{colim}_n (\mathcal{G}_{F_n}) \) is \( Z_p \)-flat (in fact, it is easy to verify that it is a free \( Z_p \)-module).

Now apply the exact functor \( \mathcal{G}_{F_{F_n}} \otimes_{Z_p} \) to the short exact sequence
\[
0 \to Z_p \to Q_p \to Q_p/Z_p \to 0
\]
to get the short exact sequence of \( \Gamma \)-modules
\[
0 \to \mathcal{G}_{F_{F_n}} \to \mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p \to \mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p/Z_p \to 0.
\]
Taking \( \Gamma \)-invariants gives the long exact sequence
\[
0 \to \mathcal{G}_F \to \mathcal{G}_F \otimes_{Z_p} Q_p \to (\mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p/Z_p)^\Gamma \to H^1(\Gamma, \mathcal{G}_{F_{F_n}}) \to H^1(\Gamma, \mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p) \cdots
\]
But \( H^1(\Gamma, \mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p) = 0 \) since \( (\mathcal{G}_{F_{F_n}} \otimes_{Z_p} Q_p)^\Gamma = \mathcal{G}_{F_n} \otimes_{Z_p} Q_p \) and \( H^1(\Gamma, \mathcal{G}_{F_n} \otimes_{Z_p} Q_p) = 0 \) for all \( n \) (where \( \Gamma_n = \text{Gal}(F_{F_n}/F_n) \) and \( G_n = \Gamma/\Gamma_n \)).

Now observing that \( \mathcal{G}_{F_n} \otimes_{Z_p} Q_p/Z_p = \tilde{\mathcal{E}}_{F_n} \otimes_{Z_p} Q_p/Z_p \) for all \( n \) and that \( H^1(\Gamma, \mathcal{G}_{F_{F_n}}) = H^1(\Gamma, \tilde{\mathcal{E}}_{F_{F_n}}) \) (since \( H^1(\Gamma, \mu_{p^\infty}) = H^2(\Gamma, \mu_{p^\infty}) = 0 \)) gives the short exact sequence we require.

Finally, we must show that \( H^1(\Gamma, \tilde{\mathcal{E}}_{F_{F_n}}) \) is finite. Of course, \( H^1(\Gamma, \tilde{\mathcal{E}}_{F_{F_n}}) = \text{colim}_n H^1(G_n, \tilde{\mathcal{E}}_{F_n}) \) since \( \tilde{\mathcal{E}}_{F_n} = (\tilde{\mathcal{E}}_{F_{F_n}})^\Gamma \). However, by Theorem 2.6
\[
H^1(G_n, \tilde{\mathcal{E}}_{F_n}) = \text{Ker}(\tilde{Cl}_F \to \tilde{Cl}_{F_n}).
\]
Thus \( H^1(\Gamma, \tilde{\mathcal{E}}_{F_{F_n}}) \subset \tilde{Cl}_F \). Since \( H^1(\Gamma, \tilde{\mathcal{E}}_{F_{F_n}}) \) is a torsion \( Z_p \)-module and \( \tilde{Cl}_F \) is a finitely-generated \( Z_p \)-module, the result follows. \( \square \)

As an immediate corollary, we have:
Corollary 2.8.  
\[ \text{Div}\left(\tilde{E}_{F,\infty} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p\right)^\Gamma = \tilde{E}_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p. \]

Theorem 2.9.  
\[ \text{Div}(\mathcal{N}^T) \subset \tilde{E}_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \text{ with equality if and only if } \delta_F = 0. \]

Proof. M. Kolster has proven ([15], Theorem 2.3) that \( \mathcal{N} \subset \tilde{E}_{F,\infty} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \) and thus taking \( \Gamma \)-invariants and then maximal divisible subgroups and using the last corollary gives the result. The statement about equality follows from the fact that \( \tilde{E}_F \otimes \mathbb{Q}_p/\mathbb{Z}_p \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{2+\delta_F}. \)

Recall that \( D_F^{(1,n)} = \{ a \in F^\times | a \otimes 1/p^n \in \text{Div}(\mathcal{N}^T) \}. \)

Corollary 2.10. For any number field \( F \) containing \( \mu_p \), \( D_F^{(1,n)} \subset U_F^S \cdot (F^\times)^p^n \) with equality if and only if \( F \) has exactly one \( p \)-adic prime.

Proof. Since \( \tilde{E}_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \subset U_F^S \otimes \mathbb{Q}_p/\mathbb{Z}_p \) with equality if \( F \) has one \( p \)-adic prime, it follows that
\[ D_F^{(1,n)} \subset \{ a \in F^\times | a \otimes 1/p^n \in U_F^S \otimes \mathbb{Q}_p/\mathbb{Z}_p \} = U_F^S \cdot (F^\times)^p^n. \]

with equality when \( F \) has one \( p \)-adic prime.

Remark 2.11. In fact, Theorem 17 Iwasawa [9] implies that \( \mathcal{N} \subset U_{F,\infty}^S \otimes \mathbb{Q}_p/\mathbb{Z}_p \) and Lemma 7 of that paper implies that \( \text{Div}((U_{F,\infty}^S \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma) = U_{F,\infty}^S \otimes \mathbb{Q}_p/\mathbb{Z}_p \), so that this corollary can also be derived directly from the results of Iwasawa.

Corollary 2.12. If the Gross conjecture holds for \( (F,p) \), then there is a natural isomorphism
\[ D_F^{[1,n]} \cong \tilde{E}_F/p^n. \]

Proof. Since \( \delta_F = 0 \), we have
\[ \text{Div}(\mathcal{N}^T)[p^n] = (\tilde{E}_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)[p^n] = (\mathcal{G}_F \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)[p^n] = \mathcal{G}_F \otimes_{\mathbb{Z}_p} \left( \frac{1}{p^n} \mathbb{Z}_p/\mathbb{Z}_p \right) = \mathcal{G}_F/p^n \]
since \( \mathcal{G}_F \) is a free \( \mathbb{Z}_p \)-module. Thus there is a natural short exact sequence
\[ 0 \to \mu_{p^n}(F)/p^n \to D_F^{(1,n)}/(F^\times)^p^n \to \mathcal{G}_F/p^n \to 0. \]

On the other hand, since \( (F^\times)^p^n \subset D_F^{(1,n)} \subset U_F^S(F^\times)^p^n \), there is a natural isomorphism
\[ D_F^{[1,n]} = D_F^{(1,n)}/(F^\times)^p^n \cong (U_F^S \cap D_F^{(1,n)})/(U_F^S)^p^n. \]

Now let \( \epsilon \in U_F^S \cap D_F^{(1,n)}. \) Let \( \tilde{U}_F = (U_F^S \otimes \mathbb{Z}_p)/\mu_{p^n}(F) \) so that \( \mathcal{G}_F \subset \tilde{U}_F \) as a \( \mathbb{Z}_p \) direct summand. Then
\[ \epsilon \otimes \frac{1}{p^n} \in \mathcal{G}_F \otimes_{\mathbb{Z}_p} \left( \frac{1}{p^n} \mathbb{Z}_p/\mathbb{Z}_p \right) \subset \tilde{U}_F \otimes_{\mathbb{Z}_p} \left( \frac{1}{p^n} \mathbb{Z}_p/\mathbb{Z}_p \right) \]
so that the image of $\epsilon \otimes 1$ in $\tilde{U}_F/p^n$ lies in $G_F/p^n$. It follows that $\epsilon \otimes 1 \in \tilde{E}_F$ since $\mu_{p^n}(F) \otimes \mathbb{Z}_p \subset \tilde{E}_F$. Thus we obtain a natural well-defined homomorphism $\mathcal{D}_{F}^{[1,n]} \to \tilde{E}_F/p^n$ which is an isomorphism in view of the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \mu_{p^n}(F)/p^n & \to & \mathcal{D}_{F}^{[1,n]} & \to & G_F/p^n & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mu_{p^n}(F)/p^n & \to & \tilde{E}_F/p^n & \to & G_F/p^n & \to & 0 \\
\end{array}
\]

\[\square\]

**Remark 2.13.** This result (together with Corollary 2.14) implies and clarifies the result of Brauckmann, [3] (see also Kolster and Movahhedi, [16], Theorem 2.15 and Corollary 2.16) that $\mathcal{D}_{F}^{(i)}/(F\times)^p \cong \tilde{E}_{F\setminus p}$ for all $i \geq 2$ when $m$ is sufficiently large and assuming the Gross conjecture for $(F\setminus p)$.

**Corollary 2.14.** Suppose that $F$ has exactly one $p$-adic prime. Then $\mathcal{D}_{F}^{(i)} = U_F^S(F\times)^p$ for all $i \equiv 1 \pmod{p^{m_{n}(F)}}$.

If furthermore $m_{F} = 0$, then $U_F^S(F\times)^p \subset A_F$ and Leopoldt’s conjecture holds for the field $F$ if and only if there is equality.

**Remark 2.15.** Suppose that $F$ has only one prime dividing $p$. Then $\tilde{H}_F = 0$ if $\text{Cl}(F) \{p\} = 0$, since under these hypotheses, $F_n/F$ is (totally) ramified at the unique $p$-adic prime and hence $\text{Cl}(F_n) \{p\} = 0$ for all $n$.

It can be shown furthermore (see Greenberg, [8], p1241) that if $F$ is a CM-field with only one prime dividing $p$ and if $\text{Cl}(F_+) \{p\} = 0$, then $\tilde{H}_F = 0$. If Vandiver’s conjecture holds for the prime $p$, then this latter condition holds for the cyclotomic field $F = \mathbb{Q}(\zeta_p)$.

**Example 2.16.** Thus, for example, we can deduce the following theorem of Kersten: ([14])

If $n$ is sufficiently large and $F = \mathbb{Q}(\zeta_{p^n})$, then

$$A_F = U_F^S(F\times)^p = C_F.$$  

(These fields have only one $p$-adic prime, so that $D_{F}^{(1)} = U_F^S \cdot (F\times)^p$ for all $n$. They are abelian fields and thus the Leopoldt conjecture hold for these fields and $D_{F}^{(0)} = A_F$ for all $n$. Finally, $m_{F} = 0$ for $n$ sufficiently large.)

**Example 2.17.** Greenberg ([8]) proves that if $|\tilde{H}_F| = p$ and $\mu_{p^2} \not\subset F\times$ (so that $m = 1$), then $D_{F}^{(i)} = D_{F}^{(j)}$ if and only if $i \equiv j \pmod{p}$.

When $p = 3$, he shows that the field $F = \mathbb{Q}(\sqrt{257}, \sqrt{-3})$ satisfies this condition. There is a unique 3-adic prime in this field. Thus $D_{F}^{(1)} = U_F^S(F\times)^p$. Since $F$ is an abelian number field, Leopoldt’s Conjecture holds for $F$ and thus $A_F = D_{F}^{(0)}$. As observed above, $D_{F}^{(2)} = C_F$, the classical Tate kernel. Thus in
this case we can conclude that the groups $A_F$, $U^\otimes_F(F^\times)^p$ and $C_F$ are pairwise distinct.

3. COHOMOLOGICAL INTERPRETATION OF THE GENERALISED TATE KERNELS

Let $E$ be any subfield of $K$ which is Galois over $F$. (We are primarily interested in the two cases $E = K$ and $E = \tilde{N}$.) Let $E \subset K$ be its dual with respect to the Kummer pairing. Let $G = \text{Gal}(E/F)$. Let $Y = \text{Gal}(E/F\infty)$ so that there is a group extension

$$1 \to Y \to G \to \Gamma \to 1. \quad (2)$$

Thus there is a perfect duality pairing $E \times Y \to \mu_{p\infty}$. Taking, the $\Gamma$-module structure into account, this gives a perfect pairing

$$E(i - 1) \times Y(-i) \to \mathbb{Q}_p/\mathbb{Z}_p$$

for all $i \in \mathbb{Z}$.

**Lemma 3.1.** For all $i \in \mathbb{Z}$, there are natural isomorphisms of $\Gamma$-modules

$$H^1(Y, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong E(i - 1).$$

**Proof.** Since $Y$ acts trivially on $\mu_{p^n} \subset F^\times_\infty$ for all $n$, we have

$$H^1(Y, \mathbb{Q}_p/\mathbb{Z}_p(i)) = \text{Hom}(Y, \mathbb{Q}_p/\mathbb{Z}_p(i))$$

$$\cong \text{Hom}(Y(-i), \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\cong E(i - 1).$$

□

**Lemma 3.2.** (a) For $i \neq 0$, there is a natural isomorphism

$$H^1(G, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong E(i - 1)^\Gamma.$$

(b) There is a (split) short exact sequence

$$0 \to H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \to E(-1)^\Gamma \to 0.$$

**Proof.** (a) We begin with the observation (the ‘Lemma of Tate’), that when $i \neq 0$, $H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^2(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0$. Thus, the sequence of terms of low degree of the spectral sequence associated to the extension (2) yields a natural isomorphism

$$H^1(G, \mathbb{Q}_p/\mathbb{Z}_p(i)) \cong H^1(Y, \mathbb{Q}_p/\mathbb{Z}_p(i))^\Gamma \cong E(i - 1)^\Gamma.$$

(b) When $i = 0$, we have $H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \cong \mathbb{Q}_p/\mathbb{Z}_p$ and $H^2(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Thus the sequence of terms of low degrees gives the short exact sequence

$$0 \to H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(G, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(Y, \mathbb{Q}_p/\mathbb{Z}_p)^\Gamma \to 0$$

which is split since $H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p)$ is divisible. □
Consider now the sequences of coefficient modules

\[
0 \rightarrow \mu_{p^n} \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(i) \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(i) \rightarrow 0
\]

and

\[
0 \rightarrow \mathbb{Z}_p(i) \rightarrow \mathbb{Z}_p(i) \rightarrow \mu_{p^n} \rightarrow 0.
\]

Let \(\alpha^n_G\) be the composite homomorphism

\[
H^1(G, \mu_{p^n}^\otimes i) \rightarrow H^1(G, \mathbb{Q}_p / \mathbb{Z}_p(i))[p^n] \rightarrow \mathcal{E}(i - 1)^\Gamma[p^n]
\]

associated to the sequence (3). Thus \(\alpha^n_G\) is surjective.

Let \(j_n\) be the injective homomorphism

\[
H^1(G, \mathbb{Z}_p(i))/p^n \rightarrow H^1(G, \mu_{p^n}^\otimes i)
\]

associated to the sequence (4).

**Theorem 3.3.** The image of \(j_n\) is \((\alpha^n_G)^{-1}(\text{Div}(\mathcal{E}(i - 1)^\Gamma)[p^n])\) and there is a natural short exact sequence

\[
0 \rightarrow \mu_{p^\infty}(F_i)^\otimes i/p^n \rightarrow H^0(G, \mathbb{Q}_p / \mathbb{Z}_p(i))/p^n \rightarrow \text{Div}(\mathcal{E}(i - 1)^\Gamma)[p^n] \rightarrow 0
\]

where we define \(F_{-i} = F_i\) for \(i > 0\) and \(\mu_{p^\infty}(F_0)^\otimes 0/p^n := \mathbb{Z}_p/p^n\).

**Proof.** Bringing the coefficient sequence

\[
0 \rightarrow \mathbb{Z}_p(i) \rightarrow \mathbb{Q}_p(i) \rightarrow \mathbb{Q}_p / \mathbb{Z}_p(i) \rightarrow 0
\]

into play gives us the following commutative diagram with exact rows and columns:

The statement about the image of \(j_n\) follows by diagram-chasing.

For the second statement, observe first that for \(i \neq 0\), \(H^0(G, \mathbb{Q}_p / \mathbb{Z}_p(i)) = \mu_{p^\infty}(F_i)^\otimes i\).
When $i = 0$, $H^0(G, \mathbb{Q}_p/\mathbb{Z}_p)/p^n = 0$ and $H^1(G, \mathbb{Z}/p^n) \cong H^1(G, \mathbb{Q}_p/\mathbb{Z}_p)[p^n]$. However, taking the maximal divisible subgroups and then $p^n$-torsion subgroups of the split-exact sequence of Lemma 3.2(b), now gives a short exact sequence

$$0 \to \mathbb{Z}/p^n \to H^1(G, \mathbb{Z}/p^n) \to \mathcal{E}(-1)^n[p^n] \to 0.$$  

\[\square\]

**Lemma 3.4.** If either $\mu_{p^n} \subset F$ or $i \not\equiv 0 \pmod{p}$, the restriction map induces an isomorphism

$$H^1(F, \mu_{p^n}^\otimes i) \cong H^1(F_n, \mu_{p^n}^\otimes i)^G_n$$

**Proof.** If $\mu_{p^n} \subset F$, then $F = F_n$ and $G_n = 0$ and the statement is trivial. Otherwise $G_n \neq 0$.

Let $G_F$ denote the absolute Galois group of $F$. The sequence of terms of low degree associated to the extension

$$0 \to G_{F_n} \to G_F \to G_n \to 0$$

has the form

$$0 \to H^1(G_n, \mu_{p^n}^\otimes i) \to H^1(F, \mu_{p^n}^\otimes i) \to H^1(F_n, \mu_{p^n}^\otimes i)^G_n \to H^2(G_n, \mu_{p^n}^\otimes i) \ldots$$

Because $i \not\equiv 0 \pmod{p}$ the map $G_n \to \text{Aut}(\mu_{p^n}^\otimes i) = (\mathbb{Z}/p^n)^\times$ is injective. Thus the proof concludes with the observation that if $M$ is cyclic of order $p^n$ and if $H$ is a nonzero subgroup of $(\mathbb{Z}/p^n)^\times = \text{Aut}(M)$ then $H^1(H, M) = H^2(H, M) = 0$. (On the other hand, if $p|i$ then the map $G_n \to \text{Aut}(\mu_{p^n}^\otimes i)$ has a nonzero kernel and it is then easy to see that $H^1(G_n, \mu_{p^n}^\otimes i) \neq 0$.)

Recall that Kummer Theory gives a natural isomorphism $\delta : F^\times/(F^\times)^{p^n} \cong H^1(F, \mu_{p^n})$ and hence for all $i \in \mathbb{Z}$ there are natural isomorphisms $H^1(F_n, \mu_{p^n}^\otimes i) \cong H^1(F_n, \mu_{p^n}) \otimes \mu_{p^n}^\otimes (i - 1) \cong F_n^\times / (F_n^\times)^{p^n} (i - 1)$.

**Corollary 3.5.** Let $\tilde{H} = \text{Gal}(\tilde{N}/F)$. If either $p \nmid i$ or $\mu_{p^n} \subset F$ the image of the natural injective map

$$H^1(\tilde{H}, \mathbb{Z}_p(i))/p^n \to H^1(\tilde{H}, \mu_{p^n}^\otimes i) \to H^1(F, \mu_{p^n}^\otimes i) \to (F_n^\times / (F_n^\times)^{p^n} (i - 1)$$

is $D_{F, n}^{i,n}$.

**Proof.** Let $H = \text{Gal}(K/F)$. We begin with the observation that the natural map induces an isomorphism $H^1(H, \mu_{p^n}^\otimes i) \cong H^1(F, \mu_{p^n}^\otimes i) = H^1(G_F, \mu_{p^n}^\otimes i)$. To see this, first note that $H^1(F, \mu_{p^n}^\otimes i) = H^1(F_n, \mu_{p^n}^\otimes i)^G_n$ and $H^1(H_n, \mu_{p^n}^\otimes i) = H^1(H, \mu_{p^n}^\otimes i)^G_n$ (where $H_n = \text{Gal}(K/F_n)$) since $H^1(G_n, \mu_{p^n}^\otimes i) = H^1(G_n, \mu_{p^n}^\otimes i) = 0$. But we have $H_n^\ab/p^n = G_n^\ab/p^n$ and thus $H^1(H_n, \mu_{p^n}^\otimes i) = \text{Hom}(H_n, \mu_{p^n}) = \text{Hom}(H_n^\ab/p^n, \mu_{p^n}^\otimes i) = \text{Hom}(G_n^\ab/p^n, \mu_{p^n}^\otimes i) = H^1(F_n, \mu_{p^n}^\otimes i)$. 


The result then follows from the commutative diagram

$$
\begin{array}{cccc}
H^1(\tilde{H}, \mathbb{Z}_p(i))/\rho^n & \xrightarrow{\alpha^\rho_n} & \text{Div}(\mathcal{N}(i - 1)^F)[p^n] \\
\downarrow j_n & & & \downarrow \\
H^1(\tilde{H}, \mu_{p^n}) & \xrightarrow{\alpha^\rho_n} & \mathcal{N}(i - 1) & \downarrow \\
\downarrow & & & \downarrow \\
H^1(H, \mu_{p^n}) & \xrightarrow{\alpha^\rho_n} & K(i - 1) & \downarrow \\
\downarrow \cong & & & \\
H^1(F, \mu_{p^n}) & \xrightarrow{\cong} & (F_n^\times / (F_n^\times)^p(i - 1))^{G_n} \\
\end{array}
$$


Corollary 3.6. If either $i \not\equiv 0 \pmod{p}$ or $\mu_{p^n} \subset F$, there is a natural short exact sequence

$$0 \to \mu_{p^n}(F_i)^\otimes / p^n \to D_F^{[i,n]} \to \text{Div}(\mathcal{N}(i - 1)^F)[p^n] \to 0.$$

Lemma 3.7. Suppose that $\text{Div}(K(i - 1)^F) = \text{Div}(\mathcal{N}(i - 1)^F)$ (i.e. suppose that Schneider’s conjecture holds for $i$). Let $E/F$ be a Galois extension containing $\tilde{N}$ and let $\tilde{G} = \text{Gal}(\tilde{E}/F)$. Suppose that either $\mu_{p^n} \subset F$ or $p \nmid i$. Then the natural restriction map induces an isomorphism

$$H^1(\tilde{H}, \mathbb{Z}_p(i))/\rho^n \cong H^1(\tilde{G}, \mathbb{Z}_p(i))/\rho^n.$$

Proof. Let $E/F$ be the largest sub-extension of $K/F$ which is contained in $\tilde{E}/F$. Let $G = \text{Gal}(E/F)$. I claim that the restriction map induces an isomorphism

$$H^1(G, \mathbb{Z}_p(i))/\rho^n \cong H^1(\tilde{G}, \mathbb{Z}_p(i))/\rho^n.$$

To see this observe that in the commutative exact diagram

$$
\begin{array}{cccc}
0 & \to & H^1(G, \mathbb{Z}_p(i))/\rho^n & \xrightarrow{\rho} H^1(G, \mu_{p^n}) \\
\downarrow \text{res} & & & \downarrow \tau \\
0 & \to & H^1(\tilde{G}, \mathbb{Z}_p(i))/\rho^n & \xrightarrow{\psi} H^1(\tilde{G}, \mu_{p^n}) \\
\end{array}
$$

$\tau$ is injective and it is therefore sufficient to show that $\rho$ is an isomorphism. Let $K_0 = \ker(\tilde{G} \to G)$. Then $\rho$ fits into the exact sequence

$$0 \to H^1(G, \mu_{p^n}) \xrightarrow{\rho} H^1(\tilde{G}, \mu_{p^n}) \xrightarrow{\psi} H^1(K_0, \mu_{p^n})$$

and it suffices to show that $\psi$ is the zero map. Let $G_0 = \ker(\tilde{G} \to \Gamma)$. Then $K_0 \subset G_0$ and $G_0/K_0$ is the largest abelian pro-$p$ quotient of $G_0$. The map $\psi$ factors through $H^1(G_0, \mu_{p^n}) = \text{Hom}(G_0^{ab}/p^n, \mu_{p^n}) \to \text{Hom}(K_0, \mu_{p^n}) = \{0\}.$
H^1(K_0, \mu_p^{\otimes i}) which is the zero map since \(K_0 \subset \text{Ker}(G_0 \to G_0^\ab/p^n)\). This proves the claim.

Let now \(E\) be the dual of \(Y = \text{Gal}(E/F_\infty)\). Thus \(N \subset E \subset K\) and hence \(\text{Div}(\mathcal{N}(i-1)^F) = \text{Div}(\mathcal{E}(i-1)^F) = \text{Div}(\mathcal{N}(i-1)^F)\) so that we get the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & \mu_p^\infty(F_i)^{\otimes i}/p^n & \to & H^1(\tilde{H}, \mathbb{Z}_p(i))/p^n & \to & \text{Div}(\mathcal{N}(i-1)^F)[p^n] & \to & 0 \\
& & \downarrow & & \downarrow\text{res} & & \downarrow\cong & & \\
0 & \to & \mu_p^\infty(F_i)^{\otimes i}/p^n & \to & H^1(G, \mathbb{Z}_p(i))/p^n & \to & \text{Div}(\mathcal{E}(i-1)^F)[p^n] & \to & 0 \\
\end{array}
\]

Now let \(G^S_F\) denote the Galois group of the maximal algebraic extension of \(F\) which is unramified outside \(S\).

**Corollary 3.8.** Let \(i \geq 2\) and suppose that either \(\mu_p^n \subset F\) or \(p \nmid i\). There are natural isomorphisms

\[D_F^{[i:n]} \cong K^{\text{ét}}_{2i-1}(\mathcal{O}_F^S)/p^n.\]

**Proof.** As remarked above, \(\text{Div}(\mathcal{N}(i-1)^F) = \text{Div}(\mathcal{K}(i-1)^F)\) for all \(i \geq 2\). Furthermore, the étale \(K\)-theory groups \(K^{\text{ét}}_{2i-1}(\mathcal{O}_F^S)\) are isomorphic to the groups \(H^1(G_F^S, \mathbb{Z}_p(i))\) (Dwyer and Friedlander, [5], Proposition 5.1).

Compare this last result with the identification of \(D_F^{[1:n]}\) in Corollary 2.12 above.

4. Capitulation Kernels: General Results

Let \(E/F\) be a Galois extension of number fields, with Galois group \(G\). As in [1], \(f_i\) \((i \geq 2)\) denotes the natural functorial homomorphism \(K^{\text{ét}}_{2i-2}(\mathcal{O}_F^S) \to K^{\text{ét}}_{2i-2}(\mathcal{O}_E^S)^G\).

From [1], Propositions 1.1 (which is based on the work of B. Kahn, [13]) and the remarks that follow it, we have:

**Theorem 4.1.** Let \(E/F\) be cyclic of degree \(p^n\). Then \(|\text{Ker}(f_i)| = |\text{Coker}(f_i)|\) and

\[\text{Coker}(f_i) \cong (K^{\text{ét}}_{2i-1}(\mathcal{O}_F^S)/p^n)/N_{E/F}(K^{\text{ét}}_{2i-1}(\mathcal{O}_E^S)/p^n)\] for all \(i \geq 2\).

**Corollary 4.2.** Let \(E/F\) be cyclic of degree \(p^n\). Suppose that \(\mu_p^n \subset F\) or that \(p\) does not divide \(i\). Then

\[|\text{Ker}(f_i)| = |\text{Coker}(f_i)| = [D_F^{[n,i]} : N_{E/F}(D_E^{[n,i]})].\]
Proof. We have a commutative diagram

\[
\begin{array}{c}
K^\text{ét}_{2i-1}(\mathcal{O}_E^S)/p^n \xrightarrow{\cong} D_E^{[i,n]} \\
\downarrow N_{E/F} \quad \downarrow N_{E/F}
\end{array}
\]

In particular, we get the following result of Assim and Movahhedi:

**Corollary 4.3.** Let $E/F$ be cyclic of degree $p^n$. Then for all $i \geq 2$,

\[
\text{Coker}(f_i) \cong D_F(i)/\left(N_{E/F}D_E^i(F^\times)p\right).
\]

Proof. When $n = 1$, $D_F^{[i,1]} = D_F^i/(F^\times)p$. □

**Corollary 4.4.** Let $E/F$ be cyclic of degree $p^n$. Suppose that the Gross conjecture holds for $(E,p)$. Let $m_n = \max(m_n(F), m_n(E))$. Then

\[
|\Ker(f_i)| = |\text{Coker}(f_i)| = [\tilde{E}_F : N_{E/F}(\tilde{E}_E)] \text{ for all } i \equiv 1 \pmod{p^m}.
\]

If $E$ has only one $p$-adic prime, then

\[
|\Ker(f_i)| = |\text{Coker}(f_i)| = [U^S_F : N_{E/F}(U^S_F)] \text{ for all } i \equiv 1 \pmod{p^m}.
\]

Proof. By Corollary 2.12 we have a commutative diagram

\[
\begin{array}{c}
K^\text{ét}_{2i-1}(\mathcal{O}_E^S)/p^n \xrightarrow{\cong} D_E^{[i,n]} \xrightarrow{\cong} D_E^{[i,n]}(i-1) \xrightarrow{\cong} \tilde{E}_E/p^n(i-1) \\
\downarrow N_{E/F} \quad \downarrow N_{E/F} \quad \downarrow N_{E/F}(i-1) \quad \downarrow N_{E/F}(i-1)
\end{array}
\]

and the result follows since $p^n\tilde{E}_F \subset N_{E/F}(\tilde{E}_E)$ and $\mu_{p^n(i-1)}$ is a trivial Galois-module. □

5. **Capitulation Kernels: Tamely Ramified $p$-extensions**

For a number field $F$ (containing $\mu_p$) we will set

\[
B_F := \{a \in F^\times | F(\sqrt[p]{a})/F \text{ is unramified outside } p\}.
\]

Thus $B_F$ is a subgroup of $F^\times$ containing $U_F^S \cdot (F^\times)^p$ and $A_F$. It can also be shown that $C_F \subset B_F$. More generally, we have:

**Lemma 5.1.** $D_F^i \subset B_F$ for all $i \in \mathbb{Z}$.

Proof. Let $a \in D_F^i$. Let $\zeta_p$ be a $p$th root of unity in $F$. Then $a \otimes \zeta_p^{\otimes(i-1)} \in \text{Div}(\mathcal{N}(i-1)^\Gamma) \subset \mathcal{M}(i-1)$. Thus

\[
a \otimes \frac{1}{p} \in \mathcal{M}
\]
and hence $\sqrt{a} \in M$. It follows that $F(\sqrt{a})$ is unramified outside $p$ since $M/F$ is ramified only at $p$-adic primes.

Suppose now that $W$ is any subgroup of $B_F$ containing $(F^\times)^p$. Let $F_W = F(\sqrt[\prime]{W})$ and $G_W = \text{Gal}(F_W/F)$. Thus, Kummer theory gives a perfect pairing of $\mathbb{F}_p$-vectorspaces:

$$G_W \times W/(F^\times)^p \rightarrow \mu_p$$

$$(\sigma, w) \mapsto \sigma(\sqrt{w})/\sqrt{w}.$$

Let $E/F$ be a cyclic degree $p$ extension. Let $S_{\text{ram}}(E/F)$ be the set of primes of $F$ which ramify in this extension.

Let $H_W = H_W(E/F)$ be the subspace of $G_W$ spanned by the set

$$\{\sigma_v | v \in S_{\text{ram}}(E/F) \setminus S_p(F)\}$$

(where $\sigma_v$ denotes the Frobenius of $v$ in $F_W/F$).

Let $t_W = \dim_{\mathbb{F}_p}(H_W)$.

**Theorem 5.2.** Suppose that $E/F$ is a cyclic extension of degree $p$ and that $W$ is a subgroup of $B_F$ containing $(F^\times)^p$. Then

$$[W : W \cap N_{E/F}(E^\times)] \geq p^{t_W}.$$

with equality if $F$ has exactly one $p$-adic prime.

**Proof.** We follow closely the argument of Assim and Movahhedi, (11) which deals with the case $W = A_F$.

In fact we will show that the complement, $H_W^\perp$, of $H_W$ with respect to the Kummer pairing contains $(W \cap N_{E/F}(E^\times))/(F^\times)^p$ (and hence the dual of $H_W$ is a quotient of $W/W \cap N_{E/F}(E^\times)$). This is equivalent to the statement

$$\sigma(\sqrt{w}) = \sqrt{w}$$

for all $\sigma \in H_W \implies w \in N_{E/F}(E^\times)$

which, in turn, is equivalent to the statement

$$\exists v \in S_{\text{ram}}(E/F) \setminus S_p(F) \text{ with } \sigma_v(\sqrt{w}) \neq \sqrt{w} \implies w \notin N_{E/F}(E^\times)$$

and hence to the statement

$$\exists v \in S_{\text{ram}}(E/F) \setminus S_p(F) \text{ with } v \text{ inert in } F(\sqrt{w})/F \implies w \notin N_{E/F}(E^\times).$$

Suppose, then, that there is a non-$p$-adic prime $v$ which is inert in $K = F(\sqrt{w})$ and ramified in $E/F$. By Kummer Theory, $E = F(\sqrt{b})$ for some $b \in F^\times$. Then $K_v/F_v$ is an unramified cyclic extension and hence $N_{K_v/F_v}(K_v^\times) = U_{F_v} \cdot (F^\times)^p$. But $b \notin U_{F_v} \cdot (F^\times)^p$ since $F_v(\sqrt{b})/F_v = E_u/F_v$ is ramified. It follows that the Hilbert symbol $(\frac{u,b}{v})_p$ is nontrivial, and hence $w \notin N_{E_u/F_v}(E_v^\times)$.

Conversely, suppose that $|S_p(F)| = 1$ and that $w \notin N_{E/F}(E^\times)$. We must show that there is a non-$p$-adic prime ramifying in $E$ but inert in $F(\sqrt{w}) = K$. 
By Hasse’s norm theorem, there is a finite prime \( v_0 \) such that \( w \not\in N_{E_{w_0}/F_{w_0}}(E_{w_0}^\times). \) Hence
\[
\left( \frac{w,b}{v_0} \right)_p \neq 1.
\]
By Artin’s reciprocity law (and the fact that \( |S_p| = 1 \)) there exists \( v \not\in S_p \) with
\[
\left( \frac{w,b}{v} \right)_p \neq 1.
\]
Thus \( w \not\in N_{E_u/F_v}(E_u^\times). \) In particular, \( v \) does not split in \( E. \)

Let \( K = F(\sqrt[p]{w}). \) Then \( v \) does not split in \( K \) either. So \( K_v/F_v \) is a cyclic degree \( p \) extension. But since \( w \) is a norm from \( K_v, \) we have \( K_v \neq K_u. \) But \( K_u/F_v \) is an unramified extension since \( v \not\in S_p(F) \) and \( w \in B_F, \) and hence is the unique cyclic degree \( p \) unramified extension of \( F_v. \) It follows that \( E_u/F_v \) is ramified; i.e., \( v \) ramifies in \( E \) but is inert in \( K \) as required.

\( \Box \)

**Corollary 5.3.** Suppose that \( E/F \) is a cyclic degree \( p \) extension. Let \( m = \max(m_F, m_E). \) Fix \( j \in \mathbb{Z} \) and let \( t_j := t_{D_F^{(j)}}(E/F). \) Then
\[
|\text{Ker}(f_i)| \geq p^{i} \text{ for all } i \equiv j \pmod{p^m} (i \geq 2).
\]

**Proof.** We have \( D_F^{(i)} = D_F^{(j)} \) for all \( i \) with \( i \equiv j \pmod{p^m}. \) Thus, for \( i \geq 2 \) and \( i \equiv j \pmod{p^m} \) we thus have
\[
|\text{Ker}(f_i)| = [D_F^{(j)} : N_{E/F}(D_E^{(j)})(F^\times)^p] \\
= [D_F^{(j)} : N_{E/F}(D_E^{(j)})(F^\times)] \\
\geq [D_F^{(j)} : D_E^{(j)} \cap N_{E/F}E^\times] \\
\geq p^{i}.
\]

\( \Box \)

**Corollary 5.4.** Suppose that \( E/F \) is a cyclic degree \( p \) extension and that \( F \) has exactly one \( p \)-adic prime. Let \( m = \max(m_F, m_E). \) Then
\[
|\text{Ker}(f_i)| \geq p^{t} \text{ for all } i \equiv 1 \pmod{p^m} (i \geq 2)
\]
where \( t = t_{U_F^S}(E/F). \)

**Remark 5.5.** Observe that given a subgroup, \( W, \) of \( B_F, \) containing \( (F^\times)^p, \)
\( t_W = t_W(E/F) \) is the maximal size of a set \( \{v_1, \ldots, v_t\} \) of non-\( p \)-adic primes ramifying in \( E \) and such that \( \sigma_{v_1}, \ldots, \sigma_{v_t} \) is linearly independent in \( \text{Gal}(F(\sqrt[p]{W})/F). \)
In the case \( W = A_F, F(\sqrt[p]{W}) = \tilde{F}_1, \) the compositum of the first layers of the \( \mathbb{Z}_p \)-extensions of \( F, \) and the set \( \{v_1, \ldots, v_t\} \cup S_p \) is said to be primitive for \( (F, p) \) (see [7]).

For other subgroups \( W \) of \( B_F \) (eg, \( W = U_F^S, W = C_F, W = D_F^{(i)} \) any \( i, \) etc) we can use the term \( W \)-primitive for \( (F, p) \). Thus the number \( t \) in the
last corollary is the maximal number of tamely-ramified primes in $E/F$ which belong to a $U_{S_F}^+$-primitive set for $(F,p)$.

**Corollary 5.6.** Suppose that Leopoldt’s conjecture holds for $(F,p)$. Suppose that $E/F$ is cyclic of degree $p$. Let $m = \max(m_F, m_E)$. Then

$$|\text{Ker}(f_i)| \geq p^t$$

for all $i \equiv 0 \pmod{p^m}$ ($i \geq 2$)

where $t$ denotes the maximal number of tamely-ramified primes in $E/F$ belonging to a primitive set for $(F,p)$.

**Proof.** Since Leopoldt’s conjecture holds, we have $D_F^{(0)} = A_F$. □

6. CAPITULATION KERNELS: WILDLY RAMIFIED EXTENSIONS

The results of the last section give no information about the situation in which there are no tamely-ramified primes. The lower bounds obtained depend on the index $[D_F^{(i)} : D_F^{(i)} \cap N_{E/F}(E^\times)]$. However, when there is no tame ramification we have:

**Lemma 6.1.** Suppose that the field $F$ has exactly one $p$-adic prime and that $E/F$ is a cyclic degree $p$ extension in which all non-$p$-adic primes are unramified. Then $B_F \subset N_{E/F}(E^\times)$.

**Proof.** Taking $W = B_F$ in Theorem 5.2. Then $t_W = 0$ since $S_{\text{ram}}(E/F) \setminus S_p(F) = \emptyset$. Thus $[B_F : B_F \cap N_{E/F}(E^\times)] = 1$. □

**Corollary 6.2.** Suppose that the field $F$ has exactly one $p$-adic prime and that $E/F$ is a cyclic degree $p$ extension in which all non-$p$-adic primes are unramified. Then $D_F^{(i)} \subset N_{E/F}(E^\times)$ for all $i \in \mathbb{Z}$.

Vandiver’s conjecture implies that the groups $K_{2i-2}^G(S_F)$ satisfy Galois descent in the $p$-cyclotomic tower of $Q$:

**Lemma 6.3.** Let $p$ be a prime number for which Vandiver’s conjecture is true. Let $n \geq 1$ and let $F = Q(\zeta_{p^n})$ and $E = Q(\zeta_{p^{n+1}})$. Then

$$\text{Ker}(f_i) = \text{Coker}(f_i) = 1 \text{ for all } i \geq 2.$$ 

**Proof.** The assumption (Vandiver’s conjecture) implies that $p$ does not divide $|\text{Cl}^S(F_+)|$ or $|\text{Cl}^S(E_+)|$. It follows that $\tilde{H}_F = \tilde{H}_E = 0$ (see Remark 2.15). Thus $m = \max(m_F, m_E) = 0$.

However, the fact that $p$ does not divide the $S$-class number of $F_+$ or $E_+$ implies that $U_{E_F}^S/(U_{F_F}^S)^p$ and $U_{E_E}^S/(U_{E_E}^S)^p$ are generated by roots of unity and units of the form $1 - \zeta^a$ where $\zeta$ is a root of unity (this follows from the fact that $[U_{F_+} : C(F_+)] = |\text{Cl}^S(F_+)|$ where $C(F_+)$ is the subgroup of cyclotomic units; see Washington, [22], Theorem 8.2).

Since $N_{E/F}(\zeta_{p^{n+1}}^a) = \zeta_{p^n}^a$ and $N_{E/F}(1 - \zeta_{p^{n+1}}^a) = 1 - \zeta_{p^n}^a$, it follows that $N_{E/F}(U_{E_E}^S) = U_{F_F}^S$. □
On the other hand, Greenberg’s results easily provide examples of $p$-cyclo-
tomic extensions for which the kernel and cokernel of the maps $f_i$ are nontrivial
for all $i$:

**Example 6.4.** Suppose that $|\tilde{H}_F| = p$ and $\mu_{p^2} \not\subset F^\times$. Thus $m_F = 1$ and,
by Greenberg’s work, $D_F^{(i)} = D_F^{(j)}$ if and only if $i \equiv j \pmod{p}$. Note that it
follows, of course, that $D_F^{(i)} \not\subset D_F^{(j)}$ whenever $i \not\equiv j \pmod{p}$, since $(F^\times)^p \subset D_F^{(i)}$
and $\dim_{\mathbb{F}_p} \left(D_F^{(i)}/(F^\times)^p\right) = 1 + r_2$ for all $i$.

Now let $E = F(\mu_{p^2})$. Then $\tilde{H}_E = \tilde{H}_F \implies |\tilde{H}_E| = p$, but $\mu_{p^2} \subset E^\times$. Thus,
$m_E = 0$ and so $D_E^{(i)} = D_E^{(j)}$ for all $i, j \in \mathbb{Z}$.

It follows that, for any $i$, $N_{E/F}(D_E^{(i)}) = N_{E/F}(D_E^{(j)}) \subset D_F^{(j)}$ for all $j$ and hence

$$N_{E/F}(D_E^{(i)}) \subset \cap_{j=0}^{p-1} D_F^{(j)} \neq D_F^{(i)}$$

for all $i$. Thus, for all $i \geq 2$ we have

$$|\text{Ker}(f_i)| = |\text{Coker}(f_i)| \geq [D_F^{(i)} : \cap_{j=0}^{p-1} D_F^{(j)}] \geq p.$$

Let $E/F$ be a cyclic extension of degree $p$ with Galois group $G$. Let $H(E/F) = H_{U_F^S}(E/F)$ be the
subspace of $G_{U_F^S} = \text{Gal}(\sqrt[p]{U_F^S}/F)$ spanned by the Frobeni-
us automorphisms of those non-$p$-adic primes which ramify in $E/F$. Let
$H(E/F)^*$ be the dual vectorspace.

Let $f_1^S$ denote the natural functorial homomorphism $A^S(F) \to \left(A^S(E)\right)^G$.

**Theorem 6.5.** Suppose that $E$ has exactly one $p$-adic prime. Let $m = \max(m_F, m_E)$.
Let $t$ be the number of non-$p$-adic primes of $F$ which ramify in $E$.

Then for all $i \equiv 1 \pmod{p^m}$ with $i \geq 2$, there is a exact sequence

$$0 \to \text{Ker}(f_1^S) \to H^1(G, U_F^S) \to (\mathbb{Z}/p\mathbb{Z})^t \to \text{Coker}(f_1^S) \to \text{Coker}(f_i) \to H(E/F)^* \to 0.$$

**Proof.** Let $f_1^t$ be the functorial homomorphism $\text{Cl}^S(F) \to \left(\text{Cl}^S(E)\right)^G$. Since
$E/F$ has degree $p$, we have $\text{Ker}(f_1^S) = \text{Ker}(f_1^t)$ and $\text{Coker}(f_1^S) = \text{Coker}(f_1^t)$.

The first part of the sequence is the well-known formula of Chevalley: Let
$E/F$ be a Galois extension of number fields. Let $P_F^S = F^\times/U_F^S$ and let $I_E^S$ be
the group of $S$-fractional ideals of $F$. By considering the natural map from
the sequence $1 \to P_F^S \to I_E^S \to \text{Cl}^S(F) \to 1$ to the corresponding sequence for $E$, by taking $G$-invariants and then
applying the snake lemma, one obtains an exact sequence

$$0 \to \text{Ker}(f_1^S) \to (P_F^S)^G/P_F^S \to \oplus_{v \in S} \mathbb{Z}/e_v \mathbb{Z} \to \text{Coker}(f_1^S) \to H^1(G, P_F^S) \to 0.$$

The surjectivity of the last map follows from the fact that $I_E^S$ is a permutation
$\mathbb{Z}[G]$-module, and thus $H^1(G, I_E^S) = 0$ by Shapiro’s Lemma. Hilbert’s Theorem
90 gives a natural isomorphism $(P_F^S)^G/P_F^S \cong H^1(G, U_F^S)$.

Now, by Hilbert’s Theorem 90,

$$H^1(G, P_F^S) \cong \text{Ker}(H^2(G, U_F^S) \to H^2(G, E^\times)).$$
Since $G$ is cyclic, the right-hand side is just
\[
\text{Ker}(U_S^F/N_{E/F}(U_S^E)) \to F^\times/N_{E/F}(E^\times) = \frac{U_S^F \cap N_{E/F}(E^\times)}{N_{E/F}(U_S^E)}.
\]
However, by our assumptions on $E$, $F$ and $m$, we have $D_F^{(1)} = U_S^F(F^\times)^p$, $D_E^{(i)} = U_S^E(E^\times)^p$ and hence $D_F^{(i)} = U_S^F(F^\times)^p$ and $D_E^{(i)} = U_S^E(E^\times)^p$ whenever $i \equiv 1 \pmod{p^m}$ and thus
\[
\text{Coker}(f_i) \cong D_F^{(i)}/N_{E/F}(D_E^{(i)})(F^\times)^p = U_S^F(F^\times)^p/N_{E/F}(U_S^E)(F^\times)^p \cong U_S^F/N_{E/F}(U_S^E)
\]
whenever $i \equiv 1 \pmod{p^m}$. Finally, by (the proof of) Theorem 5.2 above, the Kummer pairing induces a natural isomorphism
\[
\frac{U_S^F}{U_S^F \cap N_{E/F}(E^\times)} \cong H(E/F)^*.
\]

\[\square\]

**Corollary 6.6.** Let $E/F$ be cyclic of degree $p$. Suppose that $E$ has only one $p$-adic prime and that no non $p$-adic primes ramify in $E/F$. Let $m = \max(m_F, m_E)$. Then
\[
\text{Coker}(f_i) \cong \text{Coker}(f_1^S) \quad \text{for all } i \geq 1, \quad i \equiv 1 \pmod{p^m}.
\]

**Remark 6.7.** If $E$ or $F$ have more than one $p$-adic prime then we need to replace $U_S^F$ by $\tilde{E}_F$, of course, and the $S$-class group $A_S(F)$ should be replaced by the logarithmic class group $\tilde{Cl}_F$. In these circumstances there is an analogous exact sequence relating the cohomology of the logarithmic units to the kernel and cokernel of the natural functorial homomorphism
\[
f_1 : \tilde{Cl}_F \to \left(\tilde{Cl}_E\right)^G.
\]
For details of this sequence and of the logarithmic class group, see the article of Jaulent, [12]. However, the cohomology of the group $\tilde{D}_l_E$ of logarithmic divisors is somewhat more complicated than the cohomology of the group $I_S^E$ of $S$-divisors.

However, in the particular case of a finite layer of the $p$-cyclotomic extension of $F$ we have:

**Theorem 6.8.** Let $E = F_k$ for some $k \geq 2$. Suppose that the Gross conjecture holds for $(E, p)$. Let $p^n = [F_k : F]$ and let $m = \max(m_n(F), m_n(E))$. Then
\[
|\text{Ker}(f_i)| = |\text{Ker}(f_1^S)| \quad \text{for all } i \equiv 1 \pmod{p^m}.
\]

**Proof.** Let $G = \text{Gal}(E/F)$. By Corollary 4.4 we have
\[
\text{Coker}(f_i) \cong \tilde{E}_F/N_{E/F}(\tilde{E}_E) = H^2(G, \tilde{E}_E).
\]
Now use Theorem 2.6 \[\square\]

**Lemma 6.9.** Let $E/F$ be a cyclic extension with Galois group $G$ in which at least one prime ramifies totally. Then $|\text{Coker}(f_1^S)| \geq |\text{Ker}(f_1^S)|$. 

Thus $\mu_p$-ramifies at the unique $E/F$ prime ideal $p$ with one prime ideal dividing $p$.

We consider the following situation: the field $E$ is a CM-field with the property that each $p$-adic prime is stable under complex conjugation $J$. Suppose that $E/F$ is cyclic degree $p$ ramified at the $p$-adic prime and at no other prime. Let $m = \max(m_F, m_E)$. Then

$$|\text{Ker}(f_i)| \geq |\text{Ker}(f_i^S)| = |H^1(G, U_{E}^S)|$$

for all $i \equiv 1 \pmod{p^m}$, $i \geq 2$.

**Proof.** The first inequality follows from Lemma 6.9 and Corollary 6.6 above. The second equality follows from the proof of Theorem 6.5.

**Example 6.11.** We consider the following situation: the field $F$ is a CM-field with one prime ideal dividing $p$. We will assume further that $\text{Cl}(F_+\{p\}) = 0$. Furthermore $E/F$ is cyclic degree $p$ extension of CM-fields which is unramified at all primes not dividing $p$. (It follows therefore that $E/F$ is cyclic degree $p$ ramified at the unique $p$-adic prime.) Finally, we will suppose that $\mu_{p^\infty}(E) = \mu_{p^\infty}(F)$; i.e. $E/F$ is not a $p$-cyclotomic extension.

In this situation $\tilde{H}_F = \tilde{H}_E = 0$, so that $m_F = m_E = 0$ (see Remark 2.15). Thus $|\text{Ker}(f_i)| \geq |\text{Ker}(f_i^S)|$ for all $i \geq 2$.

Let $G = \text{Gal}(E/F)$ and let $H$ be the group of order 2 generated by complex conjugation $J$. If $M$ is a $H$-module on which $2$ is invertible, let

$$e_+ = \frac{1 + J}{2}, \quad e_- = \frac{1 - J}{2} \in \text{End}(M).$$

So $M = e_+(M) \oplus e_-(M)$.

Now $\text{Ker}(f_i^S) \cong H^1(G, U_{E}^S)$. Now $\text{Cl}(F_+\{p\}) = e_+(\text{Cl}(F)\{p\}) = 0$ so that $e_+(A_S(F)) = 0$ and hence $\text{Ker}(f_i^S) = e_-(\text{Ker}(f_i^S))$. It follows that $H^1(G, U_{E}^S) = e_-(H^1(G, U_{E}^S))$.

However, we observe the following: If $E/F$ is an odd-degree Galois extension of CM-fields with the property that each $p$-adic prime is stable under complex conjugation then $e_-(H^1(G, U_{E}^S)) = H^1(G, \mu(E)/\mu_{2\infty}(E))$. **Proof:** (cf. Theorème 6 of [10]) For an abelian group $A$, let $A[1/2] = A \otimes \mathbb{Z}[1/2]$. Thus $e_-(H^1(G, U_{E}^S)) = e_-(H^1(G, U_{E}^S)[1/2]) = e_-(H^1(G, U_{E}^S[1/2])) = H^1(G, e_-(U_{E}^S[1/2]))$ (since the actions of $H$ and $G$ commute). Now if $u \in U_{E}^S[1/2]$, then the hypothesis on the $p$-adic primes ensures that $(1 - J)(u) \in U_{E}[1/2]$. Thus $e_-(U_{E}^S[1/2]) = e_-(U_{E}[1/2])$. If $u \otimes 1/2^n \in e_-(U_{E}[1/2])$, then $u \otimes 1/2^n = 1$.
and hence $|u| = 1$. The same holds for all conjugates of $u$ since $E$ is a CM-field, and thus $u \otimes 1 \in \mu(E)[1/2]$.

Thus, in our situation, $\text{Ker}(f_i^S) \cong H^1(G, \mu_{p^\infty}(E)) = H^1(G, \mu_{p^\infty}(F)) = \text{Hom}(G, \mu_{p^\infty}(F))$ is a group of order $p$ (and, in particular, $A^S(F) \neq 0$). We conclude that $|\text{Ker}(f_i)| \geq p$ for all $i \geq 2$.

**Example 6.12.** A special case of the last example is the following:

Let $p$ be an irregular prime for which Vandiver’s conjecture holds. Let $F = \mathbb{Q}(\zeta_p)$. We are supposing that $\text{Cl}(F_+)[p] = 0$, but $\text{Cl}(F)[p] = A^S(F) \neq 0$. Under these hypotheses, there exists a cyclic degree $p$ extension $E'/F_+$ which is unramified outside $p$ and is not equal to the extension $\mathbb{Q}(\zeta_{p^2})_+/F_+$ (Washington, [22], Proposition 10.13). Let $E = E'(\zeta_p)$ (and thus $E' = E_+$). So the hypotheses of Example 6.11 hold for $E/F$ and $\text{Ker}(f_i) \neq 0$ for all $i \geq 2$.

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