The existence of incompatible observables constitutes one of the most prominent characteristics of quantum mechanics (QM) and can be revealed and formalized through uncertainty relations. The Heisenberg-Robertson-Schrödinger uncertainty relation (HRSUR) was proved at the dawn of quantum formalism and is ever-present in the teaching and research on QM. Notwithstanding, the HRSUR possess the so-called triviality problem. That is to say, the HRSUR yields no information about the possible incompatibility between two observables if the system was prepared in a state which is an eigenvector of one of them. After about 85 years of existence of the HRSUR, this problem was solved recently by Lorenzo Maccone and Arun K. Pati. In this article, we start doing a brief discussion of general aspects of the uncertainty principle in QM and recapitulating the proof of HRSUR. Afterwards we present in simple terms the proof of the Maccone-Pati uncertainty relation, which can be obtained basically via the application of the parallelogram law and Cauchy-Schwarz inequality.

Keywords: Quantum mechanics, uncertainty relations

I. INTRODUCTION

One can say that uncertainty is an integral part of our lives [1]. However, the uncertainties we face in our daily lives are frequently something associated more with our ignorance as observers than a characteristic property of the physical entities with which we interact. This scenario changes completely in situations where quantum effects are observationally important. For these systems uncertainty is a fundamental character. That is to say, we just cannot, in general, foretell what is going to happen in the future, even if we have all the information we can have about the history of the object we are describing [2, 3].

For systems whose description requires the use of quantum mechanics (QM) [7–10], we can only calculate probabilities (chances or relative frequencies) for an event to occur. This fact can be attributed to the existence, in QM, of incompatible observables (IO). Once these observables are represented by non-commuting Hermitian matrices, which as a consequence cannot share all eigenvectors, we can, via the measurement of one of them, prepare a state that is a superposition of the eigenvectors of the other observable. In this case, the uncertainty about this last observable is necessarily non-null. This is associated with a positive “width” (measured using e.g. the standard deviation or variance) of the probability distribution (PD) for its eigenvalues.

If we prepare a physical system in state $|\xi\rangle$ via the measurement of an observable $\hat{C}$ [9], we can utilize the kinematic structure of QM to derive restrictions on how small can be the product or sum of the uncertainties associated with other two observables $\hat{A}$ and $\hat{B}$ [11–14]. This kind of inequality, which is dubbed preparation uncertainty relation (PUR), depends on $|\xi\rangle$ and on the regarded observables and is the main theme of this article.

The goal of a PUR is to identify (and somehow quantify) the state-dependent incompatibility of two observables via the general impossibility of preparing the physical system of interest in a state for which both probability distributions (for the eigenvalues of these observables) have null variance. The frequent presence of this kind of uncertainty relation (UR) in QM textbooks points towards its didactic importance concerning the learning of the fundamentals of this theory. Moreover, UR have diverse practical applications, going from the justification for the use of a complex field in QM [15] to areas such as quantum cryptography [16] and entanglement witness [17, 18].

There are several other relevant aspects of the uncertainty principle of QM [19], and we shall mention some of them in this paragraph. In Quantum Information Science [20], especially in Quantum Cryptography [21], error-disturbance UR are particularly important because they impose limits on the amount of information we can obtain by making measurements in a system and the consequent disturbance which will be impinged on its state [22–24]. It is worth mentioning that, as in measuring an observable to extract information about the system we shall generally modify the PD of another observable, the error-disturbance UR are closely related to the UR for joint measurement of these observables. On the other hand, the recognition that quantum correlations, such as entanglement [25–28] and discord [27, 28], can be utilized as a resource for the more efficient manipulation of information motivated the proposal and analysis of UR with quantum memories [29, 30]. Here it has been shown that the constraints on the variances of IO of a system can be weakened if the observer is quantumly correlated with it. Besides, entropic UR, independent of state, can be obtained which constrain the “entropies” of the PD of IO [31]. Another important kind of UR are those involving parameters which are not represented by Hermitian operators, such as time or phase [32].

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of this kind of UR is the energy-time UR, which has a fundamental role for proving limits on how fast quantum states can change with time; which by its turn can be utilized to limit the efficiency of quantum information processing devices. It is worthwhile observing that as the majority of the UR mentioned above involve the measurement of the average of the product of two IO, which is not an Hermitian operator, they are not amenable for experimental tests. Recently a scheme has been proposed which can, in principle, turn possible the experimental verification of UR involving the average value of \( \hat{A}\hat{B} \) but such a technique has not been put to work yet.

The sequence of this article is organized as follows. In Sec. II we discuss the Cauchy-Schwarz inequality and its use for obtaining the UR of Heisenberg, Robertson, and Schrödinger (HRSUR). Afterwards we discuss the triviality problem of the HRSUR and prove, in Sec. III the UR of Maccone and Pati (MPUR). In contrast to the HRSUR, the MPUR leads to non-zero lower bounds for the sum of the variances of two observables whenever the system state is not an eigenvector of both corresponding Hermitian operators; therefore the MPUR can be seen as an improvement for the HRSUR. At last, after presenting an example of application of these uncertainty relations in Sec. IV some final remarks are included in Sec. V.

II. HEISENBERG-ROBERTSON-SCHRÖDINGER UNCERTAINTY RELATION AND ITS TRIVIALITY PROBLEM

In view of its importance for proving the results we discuss in this article, we shall begin recapitulating the Cauchy-Schwarz inequality (CSI). The CSI states that for any pair of non-null vectors \( |\psi\rangle \) and \( |\phi\rangle \) in a Hilbert space \( \mathcal{H} \), it follows that

\[
\langle \psi | \psi \rangle \langle \phi | \phi \rangle \geq | \langle \psi | \phi \rangle |^2,
\]

(1)

with equality obtained if and only if \( |\psi\rangle \) and \( |\phi\rangle \) are colinear. Let us recall that, for the state spaces we deal with here, the inner product between two vectors \( |\psi\rangle \) and \( |\phi\rangle \) is defined as: \( \langle \psi | \phi \rangle = |\psi|^\dagger \phi \), with \( x^\dagger \) being the conjugate transpose of the vector (or matrix) \( x \). We observe that a simple manner to prove the CSI is by applying the positivity of the norm,

\[
|||\xi||| = \sqrt{\langle \xi | \xi \rangle} \geq 0,
\]

(2)

to the vector \( |\xi\rangle = |\psi\rangle - (\langle \phi | \psi \rangle / \langle \phi | \phi \rangle) |\phi\rangle \). The condition for equality in the CSI can be inferred from the fact that \( |||\xi||| = 0 \) if and only if \( |\xi\rangle \) is the null vector.

Let us see how the CSI can be used for deriving the Heisenberg-Robertson-Schrödinger uncertainty relation (HRSUR). Let \( \hat{A} \) and \( \hat{B} \) be two observables of a physical system prepared in state \( |\xi\rangle \). Let \( \langle \hat{X} \rangle = \langle \xi | \hat{X} |\xi\rangle \) denote the average value of any operator \( \hat{X} \), and we use \( I \) for the identity operator in \( \mathcal{H} \). Then we define the vectors

\[
|\psi\rangle = (\hat{A} - \langle \hat{A} \rangle I) |\xi\rangle \quad \text{and} \quad |\phi\rangle = (\hat{B} - \langle \hat{B} \rangle I) |\xi\rangle
\]

(3)

and substitute them in the CSI. Firstly we notice that

\[
(\psi | \psi \rangle = \text{Var}(\hat{A}) \quad \text{and} \quad \langle \phi | \phi \rangle = \text{Var}(\hat{B}),
\]

(4)

with \( \text{Var}(\hat{X}) = \langle (\hat{X} - \langle \hat{X} \rangle)^2 \rangle \) being the variance of \( \hat{X} \). We can also verify that

\[
(\psi | \phi \rangle = (\hat{A}\hat{B} - \langle \hat{A} \rangle \langle \hat{B} \rangle I)
\]

(5)

\[
= 2^{-1} \langle [\hat{A}, \hat{B}] - 2\langle \hat{A} \rangle \langle \hat{B} \rangle I \rangle
\]

where

\[
[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad \text{and} \quad [\hat{A}, \hat{B}] = \hat{A}\hat{B} + \hat{B}\hat{A}
\]

(6)

are, respectively, the commutator and anti-commutator of \( \hat{A} \) and \( \hat{B} \). As \( \{\hat{A}, \hat{B}\} \) and \( [\hat{A}, \hat{B}] \) are, respectively, Hermitian and anti-Hermitian operators, their mean values are, respectively, purely real and purely imaginary numbers. So, considering that

\[
| \langle \psi | \phi \rangle |^2 = (\text{Re}(\psi | \phi \rangle)^2 + (\text{Im}(\psi | \phi \rangle)^2,
\]

(7)

after some manipulations we obtain the HRSUR [11],[33]:

\[
\text{Var}(\hat{A}) \text{Var}(\hat{B}) \geq (\text{CovQ}(\hat{A}, \hat{B}))^2 + 2^{-2} |\langle [\hat{A}, \hat{B}] \rangle|^2 = T_1,
\]

(8)

where

\[
\text{CovQ}(\hat{A}, \hat{B}) = 2^{-1} (\text{Cov}(\hat{A}, \hat{B}) + \text{Cov}(\hat{B}, \hat{A}))
\]

(9)

is the quantum covariance, with \( \text{Cov}(\hat{X}, \hat{Y}) = \langle \hat{X}\hat{Y} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle \) being the covariance between the observables \( \hat{X} \) and \( \hat{Y} \). If \( \hat{A} \) and \( \hat{B} \) are compatible observables, i.e., if \( [\hat{A}, \hat{B}] = 0 \), then we shall have \( \text{CovQ}(\hat{A}, \hat{B}) = \text{Cov}(\hat{A}, \hat{B}) \).

Let us look now at the triviality problem of the HRSUR. Without loss of generality, let’s suppose that the system is prepared in a state which coincides with an eigenvector of \( \hat{A} \), that is to say, \( |\xi\rangle = |a_j\rangle \) with \( \hat{A}|a_j\rangle = a_j|a_j\rangle \) and \( a_j \in \mathbb{R} \). In this case it is not difficult verifying that \( \text{Var}(\hat{A}) = \text{CovQ}(\hat{A}, \hat{B}) = \langle [\hat{A}, \hat{B}] \rangle = 0 \). Therefore the HRSUR gives

\[
0 \text{Var}(\hat{B}) \geq 0.
\]

(10)

So, in this case, the HRSUR does not provide any information about the possible incompatibility between the observables \( \hat{A} \) and \( \hat{B} \). In the next section we shall present the proof of an uncertainty relation which avoids the triviality problem, witnessing the incompatibility of two observables even when the system is prepared in one of their eigenvectors.

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1 Even though this inequality is usually dubbed Heisenberg’s uncertainty relation, here we prefer to give credit also for Robertson and Schrödinger, who have obtained it in its more general forms. An alternative proof of HRSUR can be found in Ref. [27].
III. MACCONE-PATI UNCERTAINTY RELATION

In contrast to the HRSUR, the Maccone-Pati uncertainty relation (MPUR), which shall be proved in this section, gives lower bounds for the sum of the variances associated with two observables [14]:

$$\text{Var}(A) + \text{Var}(B) \geq \max(L_1, L_2),$$

with

$$L_1 = 2^{-1}|\langle \xi |(\hat{A} \pm \hat{B})\xi_\perp \rangle|^2,$$

$$L_2 = \pm i|\langle \hat{A}, \hat{B} \rangle + |\langle \xi |(\hat{A} \pm i\hat{B})\xi_\perp \rangle|^2,$$

where $|\xi_\perp \rangle$ is any normalized vector orthogonal to the system state $|\xi \rangle$. The signs in Eqs. [12] and [13] are chosen, respectively, to maximize $L_1$ and $L_2$. Of course, once MPUR holds for any $|\xi_\perp \rangle$, we should search for the $|\xi_\perp \rangle$ yielding the bigger lower bound for the sum of the variances. It is important to note the the lower bounds $L_1$ and $L_2$ will be equal to zero only if the system state, $|\xi \rangle$, is a common eigenvector for both observables $\hat{A}$ and $\hat{B}$. It is worthwhile mentioning that the MPUR was already verified experimentally for the special case of observables represented by unitary operators [38].

It is worthwhile also mentioning that the novelty of MPUR is not simply the use of the sum of variances instead of their product. One can easily obtain a HRSUR involving sum of variances by using $\langle A - B \rangle^2 \geq 0$, with the standard deviation of the observable $X$ defined as $\sigma_X = \sqrt{\text{Var}(X)}$. This inequality leads to

$$\text{Var}(\hat{A}) + \text{Var}(\hat{B}) \geq 2\sigma_A \sigma_B \geq |\langle \hat{A}, \hat{B} \rangle| = T_2,$$

where the last inequality is a particular case of the HR-SUR, Eq. [6]. But one can verify that if the system state is an eigenvector of one of the observables, then the uncertainty relation of Eq. [14] also suffers from the triviality problem.

A. Proof of the first lower bound in the MPUR

For the sake of proving the MPUR, we will make use of parallelogram law. This rule is depicted in Fig. [1] and states that for any two vectors $|\psi \rangle$ and $|\phi \rangle$ in the Hilbert space $H$, the following equality holds:

$$2(|\psi|^2 + |\phi|^2) = (|\langle \psi | + i|\phi \rangle|^2 + |\langle i\psi | + \phi \rangle|^2).$$

Let us insert the vectors defined in Eq. [3] in the parallelogram law, Eq. [15]. As $|||\psi|||^2 = \text{Var}(A)$ and $|||\phi|||^2 = \text{Var}(B)$,

$$2(|||\psi|||^2 + |||\phi|||^2) = (|||\psi + |\phi \rangle||^2 + |||\psi - |\phi \rangle||^2).$$

Thus, if we utilize $i|\phi \rangle$ in place of $|\phi \rangle$ in the parallelo-
Fig. 2: Lower bounds for the variances for the observables \( \hat{X} \) and \( \hat{Z} \), as imposed by the uncertainty relations of Heisenberg, Robertson, and Schrödinger (\( T_1 \) e \( T_2 \)) and of Maccone and Pati (\( L_1 \) e \( L_2 \)), for a qubit prepared in state \( |\xi\rangle = 2^{-1/2}(|0\rangle + e^{i\alpha}|1\rangle) \).

gram law, as \( \|i|\phi\| = |||\phi|| \), we get

\[
2(\text{Var}(\hat{A}) + \text{Var}(\hat{B})) \geq \text{Var}(\hat{A}) + \text{Var}(\hat{B}) + i \langle [\hat{A}, \hat{B}] \rangle \nonumber + |\langle \xi | (\hat{A} + i\hat{B}) |\xi_\perp\rangle|^2,
\]  

from which we promptly obtain the lower bound \( L_2 \) of Eq. (13). The sign in Eq. (24) is determined by which of the terms \( \|(|\psi\rangle \pm i|\phi\rangle)\|^2 \) in Eq. (15) the inequality (23) is applied to, and is chosen such that \( L_2 \) is maximized.

IV. EXAMPLE: COMPLEMENTARITY FOR A QUBIT

In this section we look at a two-level system, a qubit, prepared in the state

\[
|\xi\rangle = 2^{-1/2}(|0\rangle + e^{i\alpha}|1\rangle),
\]

with \(|0\rangle\) and \(|1\rangle\) being eigenvectors of the Pauli matrix \( \hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1| \) and \( \alpha \in [0, 2\pi] \). Of course, everything we say in this section holds for the popular example of a spin 1/2 particle measured with Stern-Gerlach apparatuses [5]. We regard the application of HRSUR and MPUR to witness the well known incompatibility between the observables \( \hat{Z} \) and \( \hat{X} = |0\rangle\langle 1| + |1\rangle\langle 0| \). One can verify that for the state \( |\xi\rangle: \langle \hat{Z}\rangle = 0, \langle \hat{X}\rangle = \cos \alpha, \) and \( \langle \hat{Z}\hat{X}\rangle = -\langle \hat{X}\hat{Z}\rangle = i\sin \alpha \). We this we have CovQ(\( \hat{X}, \hat{Z}\)) = 0 e \( |\langle [\hat{X}, \hat{Z}] \rangle|^2 = 2^2 \sin^2 \alpha \). The two lower bounds in the HRSUR of Eqs. (8) and (14) are then given by

\[
T_1 = 2^{-2}T_2^2 = \sin^2 \alpha.
\]

Fig. 2

Taking into account that for this example there is only one normalized vector orthogonal to \( |\xi\rangle: |\xi_\perp\rangle = 2^{-1/2}(|0\rangle - e^{i\alpha}|1\rangle) \), after some simple calculations, we obtain the lower bounds for the MPUR, Eqs. (12) e (13):

\[
L_2 = 2L_1 = 1 + \sin^2 \alpha.
\]

These four lower bounds for the variances of \( \hat{X} \) and \( \hat{Z} \) are shown in Fig. 2. We see that even though the qualitative behavior of the curves is generally similar, there are important quantitative differences for the phases \( \alpha = \{0, \pi, 2\pi\} \). For these values of \( \alpha \), the system state, \( |\xi\rangle \), is an eigenvector of \( \hat{X} \) and, in contrast to the MPUR, the HRSUR, due to the triviality problem, is not capable of indicating that the width of the probability distribution for the eigenvalues of \( \hat{Z} \) is non-null.

V. FINAL REMARKS

In this article, after discussing some aspects of the uncertainty principle of quantum mechanics (QM), we presented a didactic proof of the Maccone-Pati uncertainty relation and exemplified its application to a two-level system. It is a curious fact that a relevant restriction within QM (as is the Heisenberg-Robertson-Schrödinger uncertainty relation) has an important problem which, although probably being for long noticed by several teachers and researchers in the area, was solved so much time after its conception. Thus we hope that the simple derivation of the MPUR we presented in this article will further motivate its inclusion in QM courses.

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