Piecewise Adiabatic Following: Models, General Analysis, and Results

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Abstract. The dynamics of a periodically driven system whose time evolution is governed by the Schrödinger equation with non-Hermitian Hamiltonians can be perfectly stable. This finding was only obtained very recently and will be enhanced by many exact solutions discovered in this work. The main concern of this study is to investigate the adiabatic following dynamics in such non-Hermitian systems stabilized by periodic driving. We focus on the peculiar behaviour of stable cyclic (Floquet) states in the slow-driving limit. It is found that the stable cyclic states can either behave as intuitively expected by following instantaneous eigenstates, or exhibit piecewise adiabatic following by sudden-switching between instantaneous eigenstates. We aim to cover broad categories of non-Hermitian systems under a variety of different driving scenarios. We systematically analyse the sudden-switch behaviour by a universal route. That is, the sign change of the critical exponent in our asymptotic analysis of the solutions is always found to be the underlying mechanism to determine if the adiabatic following dynamics is trivial or piecewise. This work thus considerably extends our early study on the same topic [Gong and Wang, Phys. Rev. A 97, 052126 (2018)] and shall motivate more interests in non-Hermitian systems.
1. Introduction

Recent theoretical and experimental studies have touched upon many physically interesting cases in which the time evolution of a system is governed by the linear Schrödinger equation but with non-Hermitian Hamiltonians [1, 2]. This class of dynamical systems is simply termed non-Hermitian systems below. The dynamics of non-Hermitian systems can be a fruitful and useful topic considering its potential applications in understanding and controlling wave propagation with gain and loss. From the fundamental point of view, the time evolution governed by non-Hermitian Hamiltonians constitutes an interesting extension [3, 4] of the conventional quantum mechanics and prompts us to have a new look at a variety of quantum dynamical phenomena.

Our focus is on the general time-evolution features of non-Hermitian systems in the presence of periodic time modulation. According to the Floquet theorem, all important dynamical aspects of such systems with a time-periodic (though non-Hermitian) Hamiltonian can be captured by the so-called Floquet states or cyclic states. These states are simply the eigenstates of the one-period time evolution operator and hence simply acquire an overall phase factor after one period, with the phase, called Floquet eigenphase, being complex in general. Interestingly, it was earlier shown that by tuning the system parameters, some peculiar dynamical regime can emerge, where periodic time modulation may help to stabilize non-Hermitian systems because all cyclic states can be made to possess real eigenphases [5, 6]. Indeed, under such circumstances, the stroboscopic time evolution becomes unitary up to a similarity transformation [6] and hence resembles much to the familiar quantum evolution governed by Hermitian Hamiltonians.

To gain more insights into non-Hermitian systems stabilized by periodic time modulation, we aim to carry out a rather systematic and technical study of these systems in the limit of slow time modulation. In particular, can such systems just comfortably adapt to the slowly-time-varying Hamiltonian, just as what is naively suggested by the adiabatic theorem from the conventional quantum mechanics [7] and classical mechanics? This issue is also of general interest because in Nature, slow and almost periodic modulation is often naturally introduced to a broad class of systems around us, by the slow periodic change of the four seasons. The common wisdom would say Yes to the question here, but caution must be taken because even in the domain of conventional quantum mechanics, new understandings of the physics of adiabatic following are still emerging [7, 8, 9, 10, 11, 12]. Indeed, as shown by a recent study by us [13], the concept of adiabatic following with the instantaneous eigenstates of the system Hamiltonian is actually not necessarily true. Instead, a non-Hermitian system can display unexpected behaviour of sudden switching between different instantaneous eigenstates of the system Hamiltonian. That is, contrary to our naïve expectations, the adiabatic following dynamics is piecewise. For several two-level non-Hermitian systems subject to a harmonic driving, the system’s time-evolution was analysed from a
geometrical point of view, via the projected Hilbert space depicted by the Bloch sphere [13]. During the time evolution, the system’s trajectory on the Bloch sphere may display drastic changes, a phenomenon unique in non-Hermitian systems. To demonstrate this, we actually went a long way by adopting a physical and highly useful geometrical phase concept [14, 15, 16] to characterize the exotic dynamical behaviour [13]. This geometrical approach has recently led to a new scheme to characterize the so-called dynamical phase transitions in non-Hermitian systems [17].

The task of this work is to lay a solid theoretical foundation, as well as a framework, for understanding the above-mentioned sudden-switch behaviour in the adiabatic following dynamics of non-Hermitian systems. To that end, we treat a broad class of non-Hermitian two-level systems periodically modulated by one, two or even three harmonic driving fields. We shall analyse in detail when and how the sudden-switch phenomenon occurs, thus demonstrating that the intriguing sudden-switching behaviour in the adiabatic following dynamics can be typical. How different driving schemes impact on the adiabatic following dynamics will become clear from this work. More than one sudden-switches (sometimes as many as four) within one driving period are also found to be possible. Our careful theoretical calculations can identify clearly the underlying phase boundaries in the parameter space of such systems. Indeed, having and not having sudden-switch behaviour in the adiabatic following dynamics represent two distinctively different dynamical features, and transitions between them represent a new type of phase transitions unknown in the literature. As seen below, even when the dynamics is not exactly solvable, there is still a powerful technique that allows us to carry out necessary asymptotic analysis in the slow-driving limit. The rather general treatment in these systems not only extend the models we studied before [6, 13], but can also cover interesting models studied by others [18, 19]. Our comprehensive results shall become a useful reference for any future theoretical and experimental study of adiabatic following dynamics in periodically driven and non-Hermitian systems.

Section 2 outlines some general treatments and our notation for $2 \times 2$ non-Hermitian Hamiltonians periodically modulated in time. Based on the nature of the periodic time dependence of the Hamiltonian we divide our model systems into four categories, each of them treated in one of the following sections. The last section concludes this work.

2. Generic $2 \times 2$ time-dependent Hamiltonians

Consider the time-dependent Schrödinger equation

$$i\hbar \dot{|\Psi(t)}\rangle = H(t)|\Psi(t)\rangle,$$  \hspace{1cm} (1)

where the overhead dot denotes the time derivative, $\dot{f}(t) = \frac{df(t)}{dt}$. The time-dependent Hamiltonian is assumed to be a $2 \times 2$ matrix, whose most generic form is given by

$$H(t) = \begin{pmatrix}
 f_0(t) + f_3(t) & f_1(t) - if_2(t) \\
 f_1(t) + if_2(t) & f_0(t) - f_3(t)
\end{pmatrix}. \hspace{1cm} (2)$$
with all the components $f_\mu(t)$ with $\mu = 0, 1, 2, 3$ being complex-valued functions of time in general.

2.1. Trace can always be gauged away

Consider a gauge transformation
\[
|\Psi(t)| \equiv \eta(t)|\psi(t)|, \tag{3}
\]
where
\[
\eta(t) \equiv \exp \left[ \frac{1}{i\hbar} \int_0^t d\tau f_0(\tau) \right]. \tag{4}
\]
The Schrödinger equation satisfied by $|\psi(t)|$ is
\[
i\hbar |\dot{\psi}(t)| = h(t)|\psi(t)|, \tag{5}
\]
where
\[
h(t) \equiv H(t) - i\hbar \eta^{-1}(t)\dot{\eta}(t)\sigma_0
= H(t) - f_0(t)\sigma_0
= \begin{pmatrix}
f_3(t) & f_1(t) - if_2(t) \\
f_1(t) + if_2(t) & -f_3(t)
\end{pmatrix}
\equiv \mathbf{f}(t) \cdot \sigma, \tag{6}
\]
with $\sigma_0$ to be the unity matrix, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ to be Pauli matrices, and the dot product is defined as
\[
\mathbf{A} \cdot \mathbf{B} \equiv A_1B_1 + A_2B_2 + A_3B_3. \tag{7}
\]
Certainly, the dot product between two complex vectors is in general complex valued as well. Both its real and imaginary parts are invariant under rotations of the vectors. Upon this representation change, the transformed Hamiltonian $h(t)$ becomes traceless. For this reason, in the following we will only consider the traceless Hamiltonians without loss of generality.

2.2. Instantaneous eigenstates

The instantaneous eigenstates of the traceless Hamiltonian in Eq. (6) are found to be
\[
|E_{\pm}(t)\rangle = \begin{pmatrix}
f_1(t) - if_2(t) \\
f_3(t) \pm \sqrt{\mathbf{f}(t) \cdot \mathbf{f}(t)}
\end{pmatrix}, \tag{8}
\]
with the corresponding eigenvalues
\[
E_{\pm}(t) = \pm \sqrt{\mathbf{f}(t) \cdot \mathbf{f}(t)}. \tag{9}
\]
2.3. Differential equations

Let us rewrite the two-component wavefunction as follows:

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$  \hfill (10)

Then the Schrödinger equation yields (in $\hbar = 1$ unit)

$$i\dot{a}(t) = f_3(t)a(t) + [f_1(t) - if_2(t)]b(t),$$  \hfill (11)

$$i\dot{b}(t) = [f_1(t) + if_2(t)]a(t) - f_3(t)b(t).$$  \hfill (12)

If $f_1(t) - if_2(t)$ is non-zero or only vanishes at isolated points, we may cancel $b(t)$ and obtain a second order differential equation for $a(t)$,

$$\ddot{a} - \frac{\dot{f}_1 - if_2}{f_1 - if_2}\dot{a} + \left(\mathbf{f} \cdot \mathbf{f} - i\frac{\dot{f}_1 - if_2}{f_1 - if_2}f_3 + if_3\right)a = 0.$$  \hfill (13)

Here we have suppressed the time variable $t$. Similarly, if $f_1(t) + if_2(t)$ is not vanishing over a time interval, a very similar equation of $b(t)$ can be obtained,

$$\ddot{b} - \frac{\dot{f}_1 + if_2}{f_1 + if_2}b + \left(\mathbf{f} \cdot \mathbf{f} + i\frac{\dot{f}_1 + if_2}{f_1 + if_2}f_3 - if_3\right)b = 0.$$  \hfill (14)

From now on, we assume that $f_1(t) - if_2(t)$ vanishes at most at isolated points and solve for Eq. (13). In the case that $H(t)$ has a lower triangular form with $f_1(t) - if_2(t) = 0$, one simply needs to solve Eq. (14) first and then follow a similar procedure described below. In the very special case with both $f_1(t) - if_2(t) = 0$ and $f_1(t) + if_2(t) = 0$, we have $f_1(t) = f_2(t) = 0$ over a time interval and then $H(t)$ reduces to a diagonal form. In that almost trivial case $a(t)$ and $b(t)$ can be easily found by solving the two first-order differential equations as suggested by Eqs. (11) and (12).

If Eq. (13) is solvable, with the two linearly independent special solutions given by $y_1(t)$ and $y_2(t)$, then the general solution $a(t)$ has the form

$$a(t) = C_1y_1(t) + C_2y_2(t).$$  \hfill (15)

Equation (11) then directly gives $b(t)$:

$$b(t) = \alpha(t)a(t) + \beta(t)\dot{a}(t),$$  \hfill (16)

with

$$\alpha(t) \equiv -\frac{f_3}{f_1 - if_2} \quad \text{and} \quad \beta(t) \equiv \frac{i}{f_1 - if_2}.$$  \hfill (17)

Needless to say, only for some very special forms of $\mathbf{f}(t)$, the explicit solutions to Eq. (13) can be indeed found. In the following sections, we will discuss some of these special cases. However, even without obtaining the explicit solution, we can still proceed with our discussions assuming their existence.
2.4. Time evolution operator

For a given initial state, one can match it with the general solution \( \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \) to find \( C_1 \) and \( C_2 \). That is, using

\[
\begin{pmatrix} a(t_0) \\ b(t_0) \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix},
\]

we find the two constants of integrations to be

\[
C_1 = \frac{a_0 [\alpha(t_0)y_2(t_0) + \beta(t_0)y'_2(t_0)] - b_0y_2(t_0)}{\beta(t_0)W(t_0)},
\]

\[
C_2 = -\frac{a_0 [\alpha(t_0)y_1(t_0) + \beta(t_0)y'_1(t_0)] - b_0y_1(t_0)}{\beta(t_0)W(t_0)},
\]

where \( \alpha(t) \) and \( \beta(t) \) are defined above in Eq. \( (17) \) and we have introduced the Wronskian of the two special solutions,

\[
W(t) \equiv y_1(t)y'_2(t) - y'_1(t)y_2(t).
\]

Let \( U(t) \) be the time evolution operator. By definition we have

\[
\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \equiv U(t) \begin{pmatrix} a(t_0) \\ b(t_0) \end{pmatrix}.
\]

Using the previous expressions for \( a(t) \) and \( b(t) \), we arrive at

\[
U(t) = \frac{1}{\beta(t_0)W(t_0)} \begin{pmatrix} \tilde{U}_{11}(t) & \tilde{U}_{12}(t) \\ \tilde{U}_{21}(t) & \tilde{U}_{22}(t) \end{pmatrix},
\]

where

\[
\tilde{U}_{11}(t) = [\alpha(t_0)y_2(t_0) + \beta(t_0)y'_2(t_0)] y_1(t) - [\alpha(t_0)y_1(t_0) + \beta(t_0)y'_1(t_0)] y_2(t),
\]

\[
\tilde{U}_{12}(t) = -y_2(t_0)y_1(t) + y_1(t_0)y_2(t),
\]

\[
\tilde{U}_{21}(t) = [\alpha(t_0)y_2(t_0) + \beta(t_0)y'_2(t_0)] [\alpha(t)y_1(t) + \beta(t)y'_1(t)]
\]
\[
- [\alpha(t_0)y_1(t_0) + \beta(t_0)y'_1(t_0)] [\alpha(t)y_2(t) + \beta(t)y'_2(t)],
\]

\[
\tilde{U}_{22}(t) = -y_2(t_0) [\alpha(t)y_1(t) + \beta(t)y'_1(t)] + y_1(t_0) [\alpha(t)y_2(t) + \beta(t)y'_2(t)].
\]

Interestingly, if we rewrite the arbitrary initial state in terms of the following superposition,

\[
\begin{pmatrix} a(t_0) \\ b(t_0) \end{pmatrix} = C_1 \begin{pmatrix} y_1(t_0) \\ \alpha(t_0)y_1(t_0) + \beta(t_0)y'_1(t_0) \end{pmatrix} + C_2 \begin{pmatrix} y_2(t_0) \\ \alpha(t_0)y_2(t_0) + \beta(t_0)y'_2(t_0) \end{pmatrix},
\]

then the time-evolved state at time \( t \) does maintain this form, namely,

\[
\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = U(t) \begin{pmatrix} a(t_0) \\ b(t_0) \end{pmatrix}
\]
\[ = C_1 \left( \frac{y_1(t)}{\alpha(t)y_1(t) + \beta(t)y'_1(t)} \right) + C_2 \left( \frac{y_2(t)}{\alpha(t)y_2(t) + \beta(t)y'_2(t)} \right) , \]  

(25)

with all time-dependent functions updated but with precisely the same \( C_1 \) and \( C_2 \) as two integration constants. This result can be regarded as one direct outcome of the linearity of the Schrödinger equation, a feature useful for our analysis below.

In the slow-driving limit, if one of the two special solutions used above is much greater than the other, then the wavefunction approximately parallel to the dominant term on the right-hand side of Eq. (25). For example, suppose \( y_1(t) \gg y_2(t) \) for some \( t \), then the term proportional to \( C_2 \) in Eq. (25) is negligible. During the evolution, this dominance relation may change due to the well-known Stokes phenomenon. Say, for some other \( t \), \( y_1(t) \ll y_2(t) \), then the \( C_2 \) term in Eq. (25) becomes dominant. This can then lead to a drastic change in the time-evolving state when it is projected onto smoothly changing basis states, such as the instantaneous eigenstate representation. This qualitative understanding will be important when we analyse different models in the following sections.

Figure 1 shows two distinct behaviours for time-evolving states under the slow-driving limit. The top two panels show that one state follows one of the instantaneous energy eigenstates. The bottom two panels present a drastically different feature. In most of the time, the time-evolving state follows the instantaneous energy eigenstates in a piecewise fashion. During some relative short time windows, the time-evolving state “hops” from one instantaneous eigenstate to the other. This profound breakdown of the adiabatic theorem in non-Hermitian systems was first reported by us [13].

2.5. Time-Periodic systems

Consider now time-periodic Hamiltonians with,

\[ H(t + T) = H(t). \]

(26)

According to the Floquet theorem, the eigenstates of the time evolution operator \( U(T) \) of one period are cyclic states (also called Floquet states), with

\[ U(T)|F^\pm(0)\rangle = e^{\pm i\phi}|F^\pm(0)\rangle. \]

(27)

The zero sum of the two phases is a result of the traceless feature of the Hamiltonian we can always assume. As such, if \( \phi \neq 0 \) and \( \phi \neq \pm \pi \), then we get two non-degenerate cyclic states whose time-dependence is determined by \( |F^\pm(t)\rangle = U(t)|F^\pm(0)\rangle \). Note that \( \phi \) is complex valued in general. When \( \phi \) is real and nonzero, the system has long-term stability insofar the system can only acquire a phase factor on the unit circle after

\[ \dagger \text{Note that such a sudden change will never occur in a unitary evolution. The effective Hamiltonian therein accounting for the time evolution from the start to arbitrary later times is always Hermitian. Thus the critical exponents there are always purely imaginary. The Stokes phenomenon requires a switch in the signs of the real part of the critical exponent. Therefore there is no Stokes phenomenon in a unitary evolution.} \]
an arbitrary number of driving periods. In this sense, the system possesses “extended unitarity” according to Ref. [6]. The special situation with degenerated eigenphases must be treated carefully. If $U(T)$ has two distinct eigenstates, the system is still stable. If $U(T)$ is not diagonalizable, only one cyclic state may be obtained. To form a complete set, one must find a generalised eigenstate in the Jordan chain. A generic state is a linear combination of the Floquet state and the generalised eigenstate, which leads to a linear growth in time [20]. To avoid such a complication, we only consider non-degenerated Floquet states in the rest of the paper.

Note also that the Floquet states can always be expanded in the manner depicted by Eq. (24) with time-independent coefficients $C_1$ and $C_2$.

3. Models with a single Fourier component

All components $f_\mu(t)$ parametrizing a periodic Hamiltonian $H(t)$ are complex-valued functions with the same period,

$$T \equiv \frac{2\pi}{\omega}.$$  

(28)
From the Fourier theorem, we can write \( f(t) \) in terms of its Fourier components, i.e.,

\[
f(t) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{i n \omega t}.
\]

For completeness, we start from the simplest cases with just one nonzero Fourier component.

### 3.1. Time-independent Hamiltonian

When the Hamiltonian is time-independent, say

\[
f(t) = p,
\]

we have

\[
H_0 = \begin{pmatrix}
p_3 & p_1 - ip_2 \\
p_1 + ip_2 & -p_3
\end{pmatrix}.
\]

If \( p_1 - ip_2 \neq 0 \), the solution is simply

\[
a(t) = C_1 e^{i \sqrt{P \cdot P} t} + C_2 e^{-i \sqrt{P \cdot P} t},
\]

\[
b(t) = -C_1 \frac{p_3 + \sqrt{P \cdot P}}{p_1 - ip_2} e^{i \sqrt{P \cdot P} t} - C_2 \frac{p_3 - \sqrt{P \cdot P}}{p_1 - ip_2} e^{-i \sqrt{P \cdot P} t}.
\]

Fitting the initial conditions in Eq. (24) gives

\[
C_1 = \frac{1}{2 \sqrt{P \cdot P}} \left[ (\sqrt{P \cdot P} - p_3) a_0 - (p_1 - ip_2) b_0 \right],
\]

\[
C_2 = \frac{1}{2 \sqrt{P \cdot P}} \left[ (\sqrt{P \cdot P} + p_3) a_0 + (p_1 - ip_2) b_0 \right].
\]

Since the Hamiltonian is time-independent, the cyclic states coincide with the energy eigenstates in Eq. (8),

\[
|F^\pm(0)\rangle = |E^\pm\rangle = \begin{pmatrix} p_1 - ip_2 \\ -p_3 \pm \sqrt{P \cdot P} \end{pmatrix}.
\]

The overall phases of the time-evolving state \( |F^\pm(t)\rangle \) are simply the dynamical phases (time integration of the eigenvalue of the Hamiltonian),

\[
|F^\pm(t)\rangle = e^{\mp i \sqrt{P \cdot P} t} |E^\pm\rangle.
\]

The system is stable if the eigenenergy is real. Note that real energy eigenvalues do not necessarily mean that the Hamiltonian is Hermitian. Rather, such kind of real eigenvalues can be due to the \( \mathcal{PT} \) symmetry [1, 2, 21, 22]. The Floquet states \( |F^\pm(t)\rangle \) are always parallel with the corresponding energy eigenstates \( |E^\pm\rangle \). One can safely say that not much is interesting in the dynamics.
3.2. Single-frequency driving

When the Hamiltonian contains only one Fourier component, say
\[ f(t) = pe^{i\omega t}. \]  
(36)

For any nonzero \( n \), we can always rescale \( \omega \to \omega/n \) to absorb the parameter \( n \). Thus, without loss of generality, consider a Hamiltonian with the following single frequency driving,
\[ H_1 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} e^{i\omega t}. \]  
(37)

If \( p_1 - ip_2 \neq 0 \), the solution is
\[
\begin{align*}
  a(t) &= C_1 \exp \left( \frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} e^{i\omega t} \right) + C_2 \exp \left( -\frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} e^{i\omega t} \right), \\
  b(t) &= -C_1 \frac{p_3 + \sqrt{\mathbf{p} \cdot \mathbf{P}}}{p_1 - ip_2} \exp \left( \frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} e^{i\omega t} \right) - C_2 \frac{p_3 - \sqrt{\mathbf{p} \cdot \mathbf{P}}}{p_1 - ip_2} \exp \left( -\frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} e^{i\omega t} \right).
\end{align*}
\]  
(38)

Matching this solution to the initial condition as in Eq. (24), one obtains
\[
\begin{align*}
  C_1 &= \frac{(p_3 + \sqrt{\mathbf{p} \cdot \mathbf{P}}) a_0 + (p_1 - ip_2) b_0}{2\sqrt{\mathbf{p} \cdot \mathbf{P}}} \exp \left( \frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} \right), \\
  C_2 &= -\frac{(p_3 - \sqrt{\mathbf{p} \cdot \mathbf{P}}) a_0 + (p_1 - ip_2) b_0}{2\sqrt{\mathbf{p} \cdot \mathbf{P}}} \exp \left( -\frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} \right). \quad \text{(39)}
\end{align*}
\]

For this time-dependent Hamiltonian subject to an overall time-dependent factor, it is straightforward to find that the Floquet states again coincide with the instantaneous energy eigenstates, with
\[ |F^\pm(0)\rangle = |E^\pm(0)\rangle = \begin{pmatrix} p_1 - ip_2 \\ -p_3 \pm \sqrt{\mathbf{p} \cdot \mathbf{P}} \end{pmatrix}, \]  
(40)

and
\[ |F^\pm(t)\rangle = U(t)|F^\pm(0)\rangle = \exp \left[ \pm \frac{\sqrt{\mathbf{p} \cdot \mathbf{P}}}{\omega} (1 - e^{i\omega t}) \right] \begin{pmatrix} p_1 - ip_2 \\ -p_3 \pm \sqrt{\mathbf{p} \cdot \mathbf{P}} \end{pmatrix}. \]  
(41)

That is, as time evolves the state only acquires some overall complex-valued phases. After one period, each cyclic state returns to itself with total phase zero. The system is hence stable. Interestingly, even though the Hamiltonian is time-dependent, the cyclic state always aligns with the eigenstates of the Hamiltonian, no matter how fast or slow the driving is. Once again, we see no rich dynamics here.

4. Models with one Fourier component as well as a zero-frequency term

4.1. General considerations

In this section, we consider models with two Fourier components with one of the two being a constant. For example,
\[ f(t) = p + q e^{i\omega t}. \]  
(42)
The two components satisfy the differential equations

\[ H_{01} = \mathbf{p} \cdot \mathbf{\sigma} + \mathbf{q} \cdot \mathbf{\sigma} e^{i \omega t} \]

\[ = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} + \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix} e^{i \omega t}. \tag{43} \]

The two components satisfy the differential equations

\[ i \dot{a}(t) = \left( p_3 + q_3 e^{i \omega t} \right) a(t) + \left[ (p_1 - ip_2) + (q_1 - iq_2) e^{i \omega t} \right] b(t), \tag{44} \]

\[ i \dot{b}(t) = \left[ (p_1 + ip_2) + (q_1 + iq_2) e^{i \omega t} \right] a(t) - \left( p_3 + q_3 e^{i \omega t} \right) b(t). \tag{45} \]

If one of \((p_1 - ip_2)\) and \((q_1 - iq_2)\) does not vanish, we may cancel \(b(t)\) and obtain a second order differential equation for \(a(t)\).

\[ \left[ (p_1 - ip_2) + (q_1 - iq_2) e^{i \omega t} \right] \dot{a}(t) - i \omega (q_1 - iq_2) e^{i \omega t} \dot{a}(t) = - \left\{ (p_1 - ip_2) \mathbf{p} \cdot \mathbf{p} + [(p_1 - ip_2)(2 \mathbf{p} \cdot \mathbf{q} - \omega q_3) + (q_1 - iq_2)(\mathbf{p} \cdot \mathbf{p} + \omega p_3)] e^{i \omega t} + [(p_1 - ip_2) \mathbf{q} \cdot \mathbf{q} + 2(q_1 - iq_2) \mathbf{p} \cdot \mathbf{q}] e^{2i \omega t} + (q_1 - iq_2) \mathbf{q} \cdot \mathbf{q} e^{3i \omega t} \right\} a(t). \tag{46} \]

This equation can be separated into two parts, one is proportional to \((p_1 - ip_2)\), and the other is proportional to \((q_1 - iq_2) e^{i \omega t}\).

\[ - (p_1 - ip_2) \{ \dot{a}(t) + [\mathbf{p} \cdot \mathbf{p} + (2 \mathbf{p} \cdot \mathbf{q} - \omega q_3) e^{i \omega t} + \mathbf{q} \cdot \mathbf{q} e^{2i \omega t}] a(t) \} = (q_1 - iq_2) e^{i \omega t} \{ \dot{a}(t) - i \omega \dot{a}(t) + [(\mathbf{p} \cdot \mathbf{p} + \omega p_3) + 2 \mathbf{p} \cdot \mathbf{q} e^{i \omega t} + \mathbf{q} \cdot \mathbf{q} e^{2i \omega t}] a(t) \}. \tag{47} \]

This “master” equation is solvable if either the left or the right side vanishes.

4.2. Asymptotic analysis

To proceed we now introduce a change of variable (which will be used in other following sections as well). In particular, we define

\[ Z(t) \equiv e^{i \omega t} \quad \text{and} \quad a(Z) = a(t). \tag{48} \]

Equation \ref{eq:47} then becomes

\[ - (p_1 - ip_2) \left\{ \omega^2 Z^2 a''(Z) + \omega^2 Z a'(Z) + \left[ \mathbf{p} \cdot \mathbf{p} + (2 \mathbf{p} \cdot \mathbf{q} - \omega q_3) Z + \mathbf{q} \cdot \mathbf{q} Z^2 \right] a(Z) \right\} = (q_1 - iq_2) Z \left\{ Z^2 \omega^2 a''(Z) + \left[ \mathbf{p} \cdot \mathbf{p} + \omega p_3 + 2 \mathbf{p} \cdot \mathbf{q} Z + \mathbf{q} \cdot \mathbf{q} Z^2 \right] a(Z) \right\}. \tag{49} \]

Of our central interest is always the slow-driving limit. However, one must be careful because \(\omega \to 0\) is a singular limit of Eq. \ref{eq:49}. Naïvely setting \(\omega = 0\) will produce only one but not two solutions for \(a(Z)\). The proper procedure is to first let \[ a(Z) \equiv e^{m(Z)}, \tag{50} \]
Then $m(Z)$ satisfies
\[
[(p_1 - ip_2) + (q_1 - iq_2)Z] Z^2 \left\{ \omega m''(Z) + [m'(Z)]^2 \right\} + \omega (p_1 - ip_2) Z m'(Z) \\
= [(p_1 - ip_2) + (q_1 - iq_2)Z] \left( p \cdot p + 2p \cdot q \cdot Z + q \cdot q \cdot Z^2 \right) \\
- \omega [(p_1 - ip_2)q_3 - (q_1 - iq_2)p_3] Z.
\]
This equation has a smooth slow-driving limit. Namely, there are two distinct solutions. To the leading order,
\[
Z^2 [m'(Z)]^2 \sim p \cdot p + 2p \cdot q \cdot Z + q \cdot q \cdot Z^2, \quad \omega \to 0.
\]
Thus,
\[
m(Z) \sim \pm \int^Z dx \sqrt{\frac{p \cdot p + 2p \cdot q \cdot x + q \cdot q \cdot x^2}{x}}.
\]
This indefinite integral has a closed form (see 2.261, 2.266, and 2.267 of Ref. [24]),
\[
\int dx \sqrt{\frac{a + bx + cx^2}{x}} = \sqrt{a + bx + cx^2} - \frac{\sqrt{a}}{2} \ln \frac{2a + bx + 2\sqrt{a} \sqrt{a + bx + cx^2}}{2a + bx - 2\sqrt{a} \sqrt{a + bx + cx^2}} \\
+ \frac{b}{4\sqrt{c}} \ln \frac{b + 2cx + 2\sqrt{c} \sqrt{a + bx + cx^2}}{b + 2cx - 2\sqrt{c} \sqrt{a + bx + cx^2}}.
\]
The integration constant is chosen such that the integral vanishes when $\sqrt{a + bx + cx^2} = 0$.

We are now ready to analyse what happens to $a(Z) = e^{\frac{m(Z)}{\omega}}$ in the slow-driving limit. Since $m(Z)$ is obtained to the leading order of $\omega$, the behaviour of $a(Z)$ is now determined by $m(Z)$ obtained in Eq. (53). For convenience in referring to the solution of $m(Z)$, we define the following function (with $\theta \equiv \omega t$):
\[
g(\theta) = \sqrt{\frac{p \cdot p + 2p \cdot q \cdot e^{i\theta} + q \cdot q \cdot e^{2i\theta}}{\theta}} \\
- \frac{\sqrt{p \cdot p}}{2} \ln \frac{p \cdot p + p \cdot q \cdot e^{i\theta} + \sqrt{p \cdot p} \sqrt{p \cdot p + 2p \cdot q \cdot e^{i\theta} + q \cdot q \cdot e^{2i\theta}}}{p \cdot p + p \cdot q \cdot e^{i\theta} - \sqrt{p \cdot p} \sqrt{p \cdot p + 2p \cdot q \cdot e^{i\theta} + q \cdot q \cdot e^{2i\theta}}} \\
+ \frac{p \cdot q}{2\sqrt{q \cdot q}} \ln \frac{p \cdot q + q \cdot q \cdot e^{i\theta} + \sqrt{q \cdot q} \sqrt{p \cdot p + 2p \cdot q \cdot e^{i\theta} + q \cdot q \cdot e^{2i\theta}}}{p \cdot q + q \cdot q \cdot e^{i\theta} - \sqrt{q \cdot q} \sqrt{p \cdot p + 2p \cdot q \cdot e^{i\theta} + q \cdot q \cdot e^{2i\theta}}}.
\]
Because $m(Z)$ defined above has $\pm$ solutions, asymptotically, $a(t)$ must be a linear combination of the two following terms,
\[
a(t) \sim D_1 v(\omega t)e^{\frac{g(\omega t)}{\omega}} + D_2 v(\omega t)e^{-\frac{g(\omega t)}{\omega}}, \quad \omega \to 0,
\]
where $v(\omega t)$ is an unimportant pre-factor which can be found by calculating the next order correction for $m(Z)$ in Eq. (53). Note that the constant pair $D_1$ and $D_2$ may not be the same as the pair $C_1$ and $C_2$ in Eq. (24). Suppose that the real part of $g(\omega t)$ flips
its sign at $t = t_0$. For example, $\text{Re} [g(\omega t)] > 0$ when $t < t_0$ and $\text{Re} [g(\omega t)] < 0$ when $t > t_0$. As long as neither $D_1$ nor $D_2$ vanishes, then

$$a(t) \sim \begin{cases} D_1 v(\omega t) e^{\frac{g(\omega t)}{\omega}}, & t < t_0, \\ D_2 v(\omega t) e^{-\frac{g(\omega t)}{\omega}}, & t > t_0. \end{cases}$$

(57)

Because of the large factor of $\frac{1}{\omega}$ in the exponent in the slow driving limit, $\omega \to 0$, the flip of this asymptotic behaviour can occur in a relatively short time window. This observation makes it intriguing to understand the existence and features of such a sudden-switch or hopping phenomenon.

Figure 2 illustrates the real part of the critical exponent $g(\theta)$ for the same model used in Fig. 1. In this particular example, $\text{Re} [g(\theta)]$ changes signs four times. As a result, one of the Floquet states hops four times during one period in the slow-driving limit, as shown in Fig. 1.

4.3. Exactly solvable cases after a time-independent rotation

Equation (49) can be simplified dramatically and becomes solvable if either $(p_1 - ip_2)$ or $(q_1 - iq_2)$ vanishes. Remarkably, this requirement can be always fulfilled by a time-independent rotation. To see this, let us first write a complex vector as

$$p = A + iB.$$ \hspace{1cm} (58)

Then the relation $(p_1 - ip_2) = 0$ leads to

$$A_1 = -B_2, \quad \text{and} \quad A_2 = B_1.$$ \hspace{1cm} (59)
which are equivalent to the condition that the vector \((A_1, A_2, 0)\) is perpendicular to \((B_1, B_2, 0)\). This condition is always achievable by rotating the coordinate system (see Appendix A for the details). Mathematically, an arbitrary three-dimensional rotation can be parametrized as

\[
R(\alpha, \beta, \gamma) = e^{i\alpha \sigma_3/2} e^{i\beta \sigma_1/2} e^{i\gamma \sigma_3/2}.
\] (60)

The upper-right corner of the rotated matrix \(R^{-1} p \cdot \sigma R\) has the form

\[
(p_1 - i p_2) \rightarrow [(p_1 - i p_2 \cos \beta) \cos \alpha - i(p_1 \cos \beta - i p_2) \sin \alpha + ip_3 \sin \beta] e^{-i\gamma}.
\] (61)

The complex equation

\[
(p_1 - i p_2 \cos \beta) \cos \alpha - i(p_1 \cos \beta - i p_2) \sin \alpha + ip_3 \sin \beta = 0
\] (62)

can always be solved by proper choices of the two real angle \(\alpha\) and \(\beta\). Similarly, one can choose a different pair of angles \(\alpha\) and \(\beta\) such that \((q_1 - iq_2) = 0\).

4.4. Solvable case A with \(p_1 = ip_2\)

In this case,

\[
p \cdot p = p_3^2.
\] (63)

The Hamiltonian has the form

\[
H_{01A}(t) = \begin{pmatrix}
p_3 + q_3 e^{i\omega t} & (q_1 - iq_2) e^{i\omega t} \\
2p_1 + (q_1 + iq_2) e^{i\omega t} & -p_3 - q_3 e^{i\omega t}
\end{pmatrix}.
\] (64)

Equation (49) then reduces to

\[
\omega^2 Z^2 a''(Z) = \left[(p_3 + \omega)p_3 + 2p \cdot q \ Z + q \cdot q \ Z^2\right] a(Z).
\] (65)

4.4.1. \(q \cdot q \neq 0\) If \(q \cdot q\) does not vanish, by changing variables

\[
z(t) \equiv 2 \sqrt{q \cdot q} \omega e^{i\omega t} \quad \text{and} \quad a(z) \equiv a(t),
\] (66)

we get a Whittaker’s equation (see 13.14.1 of Ref. [25]),

\[
a''(z) + \left(-\frac{1}{4} - \frac{p \cdot q}{\omega z \sqrt{q \cdot q}} - \frac{p_3(p_3 + \omega)}{\omega^2 z^2}\right) a(z) = 0.
\] (67)

The solutions are

\[
a(z) = C_1 W_{\kappa, \mu}(z) + C_1 M_{\kappa, \mu}(z)
\] (68)

with

\[
\kappa \equiv -\frac{p \cdot q}{\omega \sqrt{q \cdot q}}, \quad \mu \equiv \frac{1}{2} + \frac{p_3}{\omega}.
\] (69)

In terms of the new variable \(z\), we have

\[
b(z) = a(z) + \beta a'(z) = C_1 \left[\alpha(z) W_{\kappa, \mu}(z) + \beta W_{\kappa, \mu}'(z)\right] + C_2 \left[\alpha(z) M_{\kappa, \mu}(z) + \beta M_{\kappa, \mu}'(z)\right],
\] (70)
where we have introduced short-handed notations
\[
\alpha(z) \equiv -\frac{2p_\alpha \sqrt{q \cdot \bar{q}}}{\omega z(q_1 - iq_2)} - \frac{q_3}{q_1 - iq_2}, \quad \beta \equiv -2\frac{\sqrt{q \cdot \bar{q}}}{q_1 - iq_2}.
\] (71)

At \( t = T \), \( z = z_0 e^{2\pi i} \). Both Whittaker functions have branch-cuts (see 13.14.11 and 13.14.12 of Ref. [25]),
\[
W_\kappa,\mu(z_0 e^{2\pi i}) = -e^{-2\pi i \mu}W_\kappa,\mu(z_0) + \frac{2\pi i}{\Gamma(1 + 2\mu)}M_\kappa,\mu(z_0),
\]
\[
M_\kappa,\mu(z_0 e^{2\pi i}) = -e^{-2\pi i \mu}M_\kappa,\mu(z_0).
\] (72)

Using the above explicit results, we find the eigenphases of the Floquet operator \( U(T) \) to be the following:
\[
U(T)|F^{\pm}(0)\rangle = \exp \left( \pm 2\pi i \frac{p_\alpha}{\omega} \right) |F^{\pm}(0)\rangle.
\] (73)

The system is stable, i.e., having two different real eigenphases if the parameter \( p_\alpha \) is real and nonzero, with two different cyclic states explicitly obtained as follows:
\[
|F^{+}(t)\rangle = \begin{pmatrix} M_{\kappa,\mu}(z) \\ \alpha(z)M_{\kappa,\mu}(z) + \beta M'_{\kappa,\mu}(z) \end{pmatrix},
\] (74)
\[
|F^{-}(t)\rangle = \frac{\pi}{\sin(2\pi \mu)\Gamma\left(\frac{1}{2} - \mu - \kappa\right)\Gamma(1 + 2\mu)}|F^{+}(t)\rangle
+ \begin{pmatrix} W_{\kappa,\mu}(z) \\ \alpha(z)W_{\kappa,\mu}(z) + \beta W'_{\kappa,\mu}(z) \end{pmatrix}.
\] (75)

A few observations are in order. Firstly, the state \( |F^{+}(t)\rangle \) only contains one special function \( M_{\kappa,\mu}(z) \) and its derivative \( M'_{\kappa,\mu}(z) \). Thus it is possible that the small-\( \omega \) behaviour of this cyclic state is governed by one common dominating exponential function⁸. In this case, the adiabatic following dynamics of \( |F^{+}(t)\rangle \) will be smooth and it can be expected to be close to the smooth behaviour of one instantaneous energy eigenstates. Secondly and by contrast, the state \( |F^{-}(t)\rangle \) involves a sum of two different special functions \( M_{\kappa,\mu}(z) \) and \( W_{\kappa,\mu}(z) \) of different exponential behaviour. Therefore, due to the Stokes phenomenon, the relative importance of \( W_{\kappa,\mu}(z) \) and \( M_{\kappa,\mu}(z) \) in this solution can swap during a time window that is rather short as compared with \( T \). This being the case, when analysing \( |F^{-}(t)\rangle \) using smoothly-changing states such as two instantaneous energy eigenstates, a hopping behaviour can emerge. Thus, the Floquet states of this model can behave identically with those states shown in Fig. 1 for a different model. One should not be surprised by such a similarity because the two models are related by a time independent rotation. Note that here we do not need to perform an explicit asymptotic analysis to the Whittaker functions as \( \omega \to 0 \) considering that

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⁸ Another possible subtle complication is that a single special function may still leads to hopping behaviour near the Stokes lines, see Sec. 6.1.2 for more details.

|| Actually, such explicit asymptotic analysis would be very difficult because it involves the dedicated limit of the Whittaker functions as \( \mu, \kappa, \) and \( z \) go to \( \infty \) in a proportional manner. A similar analysis for Bessel functions was performed by us in Ref. [13]. As a matter of fact, such kind of asymptotic expansion formulas for the Whittaker functions are not even available in the well-known comprehensive handbook [25].
Sec. 4.2 already outlined a general method without referring to exact solutions.

4.4.2. \( p \cdot q = 0 \) and \( q \cdot q \neq 0 \) If one of \( p \cdot q \) and \( q \cdot q \) vanishes, but not both, then the solutions are Bessel functions. For example, if \( p \cdot q = 0 \) and \( q \cdot q \neq 0 \), by the following changing variables,

\[
z(t) \equiv \frac{i\sqrt{q \cdot q}}{\omega} e^{i\omega t} \quad \text{and} \quad a(z) \equiv a(t),
\]

Eq. (65) can be transformed into a Bessel equation (see 10.13.1 of Ref. 25),

\[
a''(z) + \left[ 1 - \frac{p_3(p_3 + \omega)}{z^2\omega^2} \right] a(z) = 0.
\]

The solution of \( a(t) \) is

\[
a(t) = C_1 \sqrt{z} J_{\nu}(z) + C_2 \sqrt{z} Y_{\nu}(z)
\]

with

\[
\nu \equiv \frac{1}{2} + \frac{p_3}{\omega}.
\]

The lower component \( b(t) \) is determined by Eq. (17) with

\[
\alpha(z) = - \frac{q_3}{q_1 - iq_2} \quad \text{and} \quad \beta = -i \frac{\sqrt{q \cdot q}}{q_1 - iq_2}.
\]

At \( t = T, z = z_0 e^{2\pi i} \). Both Bessel functions have branch-cuts (see 10.11.1 and 10.11.2 of Ref. 25),

\[
J_{\nu}(z_0 e^{2\pi i}) = e^{2\pi i\nu} J_{\nu}(z_0),
\]

\[
Y_{\nu}(z_0 e^{2\pi i}) = e^{-2\pi i\nu} Y_{\nu}(z_0) + 2i \cos^2(\nu \pi) J_{\nu}(z_0).
\]

The eigenphases of the Floquet operator \( U(T) \) are

\[
U(T)|F^{\pm}(0)\rangle = \exp \left( \pm 2\pi i \frac{P_3}{\omega} \right) |F^{\pm}(0)\rangle.
\]

The system is generically stable (i.e., stable regardless of the actual value of small \( \omega \)) if the parameter \( p_3 \) is real and nonzero.

The cyclic states are

\[
|F^{+}(t)\rangle = \begin{pmatrix} 2z J_{\nu}(z) \\ [2z\alpha(z) + \beta] J_{\nu}(z) + 2z\beta J'_{\nu}(z) \end{pmatrix},
\]

\[
|F^{-}(t)\rangle = \cos(\nu \pi)|F^{+}(t)\rangle - \sin(\nu \pi) \begin{pmatrix} 2z Y_{\nu}(z) \\ [2z\alpha(z) + \beta] Y_{\nu}(z) + 2z\beta Y'_{\nu}(z) \end{pmatrix}.
\]

Again, because the state \( |F^{-}(t)\rangle \) involves a sum of two different special functions, it may show the hopping behaviour in the slow-driving limit, very much similar to the Berry-Uzdin model studied earlier [13, 18].
Similarly, if \( \mathbf{q} \cdot \mathbf{q} = 0 \) and \( \mathbf{p} \cdot \mathbf{q} \neq 0 \), Eq. (65) can be again transformed into a Bessel equation (see 10.13.5 of Ref. [25]),

\[
z^2 a''(z) - za'(z) + \left[ z^2 - \frac{4p_3(p_3 + \omega)}{\omega^2} \right] a(z) = 0
\]

by the following changing variables,

\[
z(t) \equiv \frac{2i\sqrt{2p \cdot q}}{\omega} e^{\frac{\omega t}{2}} \quad \text{and} \quad a(z) \equiv a(t).
\]

the solution of \( a(t) \) is

\[
a(t) = C_1 zJ_\nu(z) + C_2 zY_\nu(z)
\]

with

\[
\nu \equiv 1 + 2 \frac{p_3}{\omega}.
\]

The lower component \( b(t) \) is determined by Eq. (17) with

\[
\alpha(z) = -\frac{q_3}{q_1 - iq_2} + \frac{8p_3}{q_1 - iq_2} \frac{\mathbf{p} \cdot \mathbf{q}}{\omega^2 z^2} \quad \text{and} \quad \beta(z) = \frac{4\mathbf{p} \cdot \mathbf{q}}{(q_1 - iq_2)\omega z}.
\]

The eigenphases of the Floquet operator \( U(T) \) are

\[
U(T)|F^\pm(0)\rangle = \exp\left( \pm 4\pi i \frac{p_3}{\omega} \right) |F^\pm(0)\rangle.
\]

The system is generically stable if the parameter \( p_3 \) is real and nonzero. There is not much interesting in this simple case in the slow-driving limit.

If both \( \mathbf{p} \cdot \mathbf{q} \) and \( \mathbf{q} \cdot \mathbf{q} \) vanish then the case becomes trivial. The solutions are simply exponential functions,

\[
a(t) = C_1 e^{-ip_3 t} + C_2 e^{i\omega t + ip_3 t},
\]

\[
b(t) = -C_1 \frac{q_3}{q_1 - iq_2} e^{-ip_3 t} - C_2 \frac{2p_3 + \omega + q_3 e^{i\omega t}}{q_1 - iq_2} e^{ip_3 t}.
\]
4.5. Solvable case B with $q_1 = i q_2$

If it is more convenient to choose a base such that $q_1 = i q_2$, then we get another solvable model with five complex parameters and one frequency. The Hamiltonian has the form

$$H_{01B}(t) = \begin{pmatrix} p_3 + q_3 e^{i \omega t} & p_1 - i p_2 \\ p_1 + i p_2 + 2 q_1 e^{i \omega t} & -p_3 - q_3 e^{i \omega t} \end{pmatrix}. \quad (94)$$

In this model, $q \cdot q = q_3^2$.

4.5.1. $q_3 \neq 0$ If $q_3 \neq 0$, then by changing variables,

$$z(t) \equiv \frac{2 q_3}{\omega} e^{i \omega t} \quad \text{and} \quad w(z) \equiv z^{-c} e^{z/2} a(t), \quad (95)$$

with

$$c \equiv \sqrt{\mathbf{p} \cdot \mathbf{p}} / \omega, \quad (96)$$

we get a confluent hypergeometric equation (see 13.2.1 of Ref. [25]),

$$z w''(z) + (1 + 2c - z) w'(z) - \left( \frac{\mathbf{p} \cdot \mathbf{q}}{\omega q_3} + c \right) w(z) = 0. \quad (97)$$

The solutions are

$$a(t) = e^{-\frac{1}{2} z^c} [C_1 V(z) + C_2 M(z)] \quad (98)$$

$$b(t) = -\frac{\omega}{p_1 - i p_2} e^{-\frac{1}{2} z^c} \left\{ C_1 [a V(z) + z V'(z)] + C_2 [a M(z) + z M'(z)] \right\}, \quad (99)$$

where we introduced a parameter

$$\alpha \equiv \frac{p_3}{\omega} + c = \frac{p_3 + \sqrt{\mathbf{p} \cdot \mathbf{p}}}{\omega}, \quad (100)$$

and short-handed notations, $U$ and $M$ for the confluent hypergeometric functions, using notations in Ref. [25],

$$V(z) \equiv U(a, b, z), \quad (101)$$

$$M(z) \equiv M(a, b, z) = \frac{1}{\Gamma(b)} \mathbf{1}_F(a, b, z), \quad (102)$$

with the parameters

$$a \equiv \frac{\mathbf{p} \cdot \mathbf{q}}{\omega q_3} + c = \frac{1}{\omega} \left( \frac{\mathbf{p} \cdot \mathbf{q}}{q_3} + \sqrt{\mathbf{p} \cdot \mathbf{p}} \right), \quad (103)$$

$$b \equiv 1 + 2c = 1 + 2 \frac{\sqrt{\mathbf{p} \cdot \mathbf{p}}}{\omega}. \quad (104)$$

At $t = T$, $z = z_0 e^{2 \pi i}$, $z^c$ acquires a phase,

$$z^c = z_0^c e^{2 \pi i c}. \quad (105)$$

$M(a, b, z)$ is an entire function,

$$M(a, b, z_0 e^{2 \pi i}) = M(a, b, z_0). \quad (106)$$
But \( U(a, b, z) \) has a branch-cut,
\[
U(a, b, z_0 e^{2\pi i}) = e^{-2\pi ib} U(a, b, z_0) + \frac{2\pi i e^{-\pi ib}}{\Gamma(1 + a - b)} M(a, b, z_0). \quad (107)
\]
After plugging parameters into this model, we have
\[
V(z_0 e^{2\pi i}) = e^{-4\pi ic} V(z_0) - \frac{2\pi i e^{-2\pi ic}}{\Gamma(a - 2c)} M(z_0). \quad (108)
\]
The eigensphases of the Floquet operator are then found from
\[
U(T)|F^{\pm}(0)\rangle = e^{\pm 2\pi ic} |F^{\pm}(0)\rangle. \quad (109)
\]
The system is clearly stable for real and nonvanishing \( c \), with one cyclic state given by
\[
|F^{+}(t)\rangle = z^c e^{-\frac{1}{2}z^2} \left( \begin{array}{c} - (p_1 - ip_2) M(z) \\ \omega [\alpha M(z) + z M'(z)] \end{array} \right),
\]
and the other cyclic state given by
\[
|F^{-}(t)\rangle = \frac{\pi}{\sin(2\pi c) \Gamma(a - 2c)} |F^{+}(t)\rangle + z^c e^{-\frac{1}{2}z^2} \left( \begin{array}{c} - (p_1 - ip_2) V(z) \\ \omega [\alpha V(z) + z V'(z)] \end{array} \right). \quad (111)
\]
In the slow-driving limit, we again expect \( |F^{-}(t)\rangle \) (but not \( |F^{+}(t)\rangle \)) to hop between two instantaneous energy eigenstates. The hopping occurs when the relative importance of \( V(z) \) and \( M(z) \) swaps due to the Stokes phenomenon. In principle we could perform an asymptotic analysis to the confluent hypergeometric functions as \( \omega \to 0 \). But again, it is much easier to analyse the differential equation directly as we did in Sec. 4.2.

Before ending this subsection, we would like to mention an interesting model studied by a group from the University of Central Florida (UCF) \[19\]. It is actually a special case here with
\[
p_1 = -1, \quad p_2 = 0, \quad p_3 = ir, \\
q_1 = 0, \quad q_2 = 0, \quad q_3 = -i\rho. \quad (112)
\]
With these choice of the parameters, the Hamiltonian has the form
\[
H_{UCF} = \begin{pmatrix} ir - i\rho e^{i\omega t} & -1 \\ -1 & -ir + i\rho e^{i\omega t} \end{pmatrix}. \quad (113)
\]
For convenience, we rename the parameter \( g_0 \) in Ref. \[19\] as \( r \), and their \( \gamma \) as \( \omega \). In this case,
\[
\mathbf{p} \cdot \mathbf{p} = 1 - r^2, \quad \mathbf{p} \cdot \mathbf{q} = r\rho, \quad \mathbf{q} \cdot \mathbf{q} = -\rho^2. \quad (114)
\]
Both absolute adiabatic following and piecewise adiabatic following with hopping are possible in this model, as shown in Fig. \[1\]. The real part of the critical exponent is also presented in Fig. \[2\]. Remarkably, it is seen that the hopping timing is fully consistent with the locations where \( \text{Re}[g(\omega t)] \) changes its sign.

The model in Eq. \[113\] is parameterised by two complex parameters, \( r \) and \( \rho \). Each parameter is determined by two real numbers. Therefore, the parameter space of the
model has four dimensions. In this four dimensional space, there exists a critical surface. On one side of it, both Floquet states always follow the instantaneous energy eigenstates in the adiabatic limit, i.e., the central clear region in the left panel of Fig. 3. On the other side of it, one Floquet state hops as \( \omega \to 0 \), for example, the shaded regions in Fig. 3. To illustrate, in Fig. 3 we plot the phase diagram of the hopping behaviour, i.e., the intersection of the critical surface on the \( \text{Re}[r]\text{-Re}[\rho] \) plane. In the hopping region, there are two possibilities for this model. The cyclic state may hop twice in one period (grey-shaded), or it may hop four times (blue-shaded).

Figure 4 depicts the timings of the hopping, as determined from \( \text{Re}[g(\theta_{\text{crit}})] = 0 \) for \( \rho = \rho_{\text{crit}} \). For \( |r| \lesssim 0.361 \), there are two \( \theta_{\text{crit}} \) for each \( \rho_{\text{crit}} \), which means that hopping four times within one period is possible. For \( 0.361 \lesssim |r| < 1 \), only hopping twice is possible. For \( |r| > 1 \), the system is unstable because \( c = \sqrt{1 - r^2} \) is complex.

\begin{align*}
\text{Figure 3. (colour online) Phase diagram of the hopping behaviour on the } \text{Re}[r]\text{-Re}[\rho] \\
\text{plane in the model depicted by Eq. (113). In the non-shaded area, both cyclic states follow instantaneous energy eigenstates in the adiabatic limit. In the shaded area, one cyclic state exhibits piecewise following in the slow driving limit. States in the gray-shaded region may hop twice in one period, whereas states in the blue-shaded region may hop four times. The right panel is a zoom-in view for the top region in the left panel.}
\end{align*}

4.6. \( q_3 = 0 \)

If \( q_3 = 0 \),

\[ p \cdot q = (p_1 - ip_2) q_1. \]  

Further assuming that \( p \cdot q \neq 0 \), then Eq. (49) can be transformed into a Bessel equation. The solutions of \( a(t) \) is

\[ a(t) = C_1 J_\nu \left( \frac{2i \sqrt{2p \cdot q}}{\omega} e^{\frac{i \omega t}{2}} \right) + C_2 Y_\nu \left( \frac{2i \sqrt{2p \cdot q}}{\omega} e^{\frac{i \omega t}{2}} \right) \]
Figure 4. (colour online) Left panel: Real part of the critical exponent \( g \) as a function of \( \theta = \omega t \) when \( \rho = \rho_{\text{crit}} \) in the model in Eq. (113). In this plot we choose \( r = 0.1 \). \( \theta_{\text{crit}} \) are defined as the locations where the curves are tangent to the horizontal axis. From the plot, we can obtain that \( \theta_{\text{crit}} = 1.395884, 4.886075 \) for \( \rho_{\text{crit}} = 0.5766416 \) (the red line) and \( \theta_{\text{crit}} = 1.744482, 4.538703 \) for \( \rho_{\text{crit}} = -0.5766416 \) (the blue line). \( \theta_{\text{crit}} \) for the case that hops four times is not plotted because it is always at \( \pi \). Right panel: \( \theta_{\text{crit}} \) as a function of \( r \) in the same model. Colours of lines in the two panels here match those in Fig. 3. The bifurcation/merging of \( \theta_{\text{crit}} \) occurs around \( r \approx \pm 0.361 \).

with

\[
\nu \equiv \frac{2\sqrt{\mathbf{P} \cdot \mathbf{P}}}{\omega}.
\]

(117)

The lower component \( b(t) \) is determined by Eq. (17).

If both \( q_3 = 0 \) and \( \mathbf{p} \cdot \mathbf{q} = 0 \), then \( p_1 = ip_2 \). Combining with the choice \( q_1 = iq_2 \) in this subsection, we get a lower triangular form of matrix Hamiltonian. It is better studied by solving \( b(t) \) first, which will not be elaborated here.

4.6.1. Berry-Uzdin model The Berry-Uzdin model is a special case with [18]

\[
p_1 = \frac{1}{2}i(1 - r), \quad p_2 = -\frac{1}{2}(1 + r), \quad p_3 = 0,
\]

\[
q_1 = \frac{1}{2}i\rho, \quad q_2 = \frac{1}{2}\rho, \quad q_3 = 0.
\]

(118)

With these choice of the parameters, the Hamiltonian has the form

\[
H_{\text{BU}} = i \begin{pmatrix}
0 & 1 \\
-r + \rho e^{i\omega t} & 0
\end{pmatrix}.
\]

(119)

In this case, \( \mathbf{p} \cdot \mathbf{p} = r, \quad \mathbf{p} \cdot \mathbf{q} = -\frac{1}{2}\rho, \quad \mathbf{q} \cdot \mathbf{q} = 0. \)

(120)

The differential equation in Eq. (49) becomes

\[
\omega^2 Z^2 a''(Z) + \omega^2 Z a'(Z) - (r - \rho Z)a(Z) = 0,
\]

(121)

which can be converted to a Bessel equation. The solutions are

\[
a(Z) = C_1 J_\nu \left( 2\sqrt{\frac{\rho Z}{\omega}} \right) + C_2 Y_\nu \left( 2\sqrt{\frac{\rho Z}{\omega}} \right)
\]

(122)
with
\[ \nu \equiv 2 \sqrt{r} \omega. \]  
(123)

In term of the original variable,
\[ a(t) = C_1 J_\nu \left( \frac{2 \sqrt{\rho} e^{i\omega t}}{\omega} \right) + C_2 Y_\nu \left( \frac{2 \sqrt{\rho} e^{i\omega t}}{\omega} \right), \]  
(124)
and
\[ b(t) = \dot{a}(t). \]  
(125)

The constants \( C_1 \) and \( C_2 \) are determined by the initial state. We do not discuss this model further as it was studied in detail in our previous work [13].

5. Two-frequency models

In this section, we consider models with two Fourier components,
\[ f(t) = p e^{i m \omega t} + q e^{i n \omega t}, \]  
(126)
with \( m \neq 0 \) and \( n \neq 0 \). The Hamiltonian
\[ H_{mn} = p \cdot \sigma e^{i m \omega t} + q \cdot \sigma e^{i n \omega t}, \]  
(127)
We extend the technique used in the first part of this work by first changing variables,
\[ Z \equiv e^{i \omega t} \text{ and } a(Z) = a(t). \]  
(128)
We get a more complicated “master equation” in variable \( Z \) as
\[ (p_1 - i p_2) Z^m \{ \text{eqn}_P \} + (q_1 - i q_2) Z^n \{ \text{eqn}_Q \} = 0, \]  
(129)
with
\[ \text{eqn}_P = \omega^2 Z^2 a''(Z) - \omega^2 (m - 1) Z a'(Z) \]
\[ - \left[ p \cdot p Z^{2m} + 2 p \cdot q Z^{m+n} + q \cdot q Z^{2n} + \omega (m-n) q_3 Z^n \right] a(Z), \]
\[ \text{eqn}_Q = \omega^2 Z^2 a''(Z) - \omega^2 (n - 1) Z a'(Z) \]
\[ - \left[ p \cdot p Z^{2m} + 2 p \cdot q Z^{m+n} + q \cdot q Z^{2n} + \omega (n-m) p_3 Z^m \right] a(Z). \]
Again, let
\[ a(Z) \equiv e^{\frac{m(Z)}{\omega}}. \]  
(130)
Then \( m(Z) \) is found to have a smooth slow driving limit. To the leading order, we arrive at
\[ m(Z) \sim \pm \int Z^2 dx \sqrt{\frac{p \cdot p x^{2m} + 2 p \cdot q x^{m+n} + q \cdot q x^{2n}}{x}}. \]  
(131)
In general, neither the “master” equation in Eq. (129) nor this critical exponent is analytically solvable.

To look into some special solvable cases, we may rescale \( \omega \rightarrow \omega/m \) to absorb the parameter \( m \). Thus, without loss of generality, let us consider
\[ H_{1r} = p \cdot \sigma e^{i \omega t} + q \cdot \sigma e^{i r \omega t}, \]  
(132)
where \( r = n/m \) is a rational number.
5.1. Solvable model with \( r = 2 \)

The simplest Hamiltonian \( H_{12} \) is the case with \( r = 2 \).

\[
H_{12} = p \cdot \sigma e^{i\omega t} + q \cdot \sigma e^{2i\omega t} = \left( \begin{array}{cc} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{array} \right) e^{i\omega t} + \left( \begin{array}{cc} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{array} \right) e^{2i\omega t}. \tag{133}
\]

For this model, the critical exponent in Eq. (131) has a closed form. Using the integration formula 2.261 and 2.262 in Ref. [24],

\[
\int dx \sqrt{a + bx + cx^2} = \frac{b + 2cx}{4c} \sqrt{a + bx + cx^2} - \frac{b^2}{16c^3} \ln \left( \frac{b + 2cx + 2\sqrt{c} \sqrt{a + bx + cx^2}}{b + 2cx - 2\sqrt{c} \sqrt{a + bx + cx^2}} \right). \tag{134}
\]

The integration constant is chosen such that the integral vanishes when \( \sqrt{a + bx + cx^2} = 0 \). Define the exponent function as

\[
g(\theta) = \frac{p \cdot q + q \cdot q e^{i\theta}}{2q \cdot q} \sqrt{p \cdot p + 2p \cdot q e^{i\theta} + q \cdot q e^{2i\theta}} + \frac{(p \cdot q)^2 - (p \cdot p)(q \cdot q)}{4(q \cdot q)^{3/2}} \times \ln \frac{p \cdot q + q \cdot q e^{i\theta} + \sqrt{q \cdot q \sqrt{p \cdot p + 2p \cdot q e^{i\theta} + q \cdot q e^{2i\theta}}}}{p \cdot q + q \cdot q e^{i\theta} - \sqrt{q \cdot q \sqrt{p \cdot p + 2p \cdot q e^{i\theta} + q \cdot q e^{2i\theta}}}}. \tag{135}
\]

A sudden-switch behaviour is then expected in the adiabatic following dynamics if the real part of \( g(\omega t) \) obtained above flips signs during the time evolution.

The model \( H_{12} \) becomes exactly solvable if one rotates the coordinates such that \( q_3 \neq 0 \). As we argued earlier, this is always doable. In this new coordinate system, \( q \cdot q = q_3^2 \).

5.1.1. \( q_3 \neq 0 \). If \( q_3 \neq 0 \), the solutions are parabolic cylinder functions. To see this, let us change variables

\[
z(t) \equiv \sqrt{\frac{2}{\omega}} \left( \frac{p \cdot q}{q_3^{3/2}} + \sqrt{q_3} e^{i\omega t} \right), \quad a(z) \equiv a(t). \tag{136}
\]

Then \( a(z) \) satisfies a parabolic cylinder equation,

\[
a''(z) + \left( \nu + \frac{1}{2} - \frac{1}{4} z^2 \right) a(z) = 0, \tag{137}
\]

with

\[
\nu \equiv \frac{(p \cdot q)^2}{2\omega q_3^2} - \frac{p \cdot p}{2\omega q_3}. \tag{138}
\]

The solutions are

\[
a(z) = C_1 D_{\nu}(z) + C_2 D_{\nu}(-z), \quad b(z) = a(z) + \beta a'(z), \tag{139}
\]
with
\[ \alpha(z) = \frac{q_1}{q_3} - \frac{z}{p_1 - ip_2} \sqrt{\frac{\omega q_3}{2}} \quad \text{and} \quad \beta \equiv -\frac{\sqrt{2\omega q_3}}{p_1 - ip_2}. \quad (140) \]

Because \( D_\nu(z) \) and \( \alpha(z) \) are entire functions in the complex \( z \)-plane, the time evolution is periodic. That is
\[ U(T) = U(0) = 1. \quad (141) \]

The eigenvalue of the Floquet operator is simply unity and any state would be cyclic in this model. The system is stable for arbitrary choice of parameters. However, piecewise adiabatic following still presents, i.e., if the real part of the critical exponent \( g(\theta) \) flips its sign due to the underlying Stokes phenomenon.

5.1.2. \( q_3 = 0 \) The conditions \( q_3 = 0 \) and \( q_1 = iq_2 \) means that \( \mathbf{q} \cdot \mathbf{q} = 0 \). The Hamiltonian is parameterised by four complex parameters,
\[ H_{\text{Airy}} = \mathbf{p} \cdot \boldsymbol{\sigma} e^{i\omega t} + \mathbf{q} \cdot \boldsymbol{\sigma} e^{2i\omega t} \]
\[ = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} e^{i\omega t} + \begin{pmatrix} 0 & 0 \\ 2q_1 & 0 \end{pmatrix} e^{2i\omega t}. \quad (142) \]

In this case, the solutions are Airy functions. To see this, let us change variables
\[ z(t) \equiv \frac{1}{\omega^{2/3}} \left[ \frac{\mathbf{p} \cdot \mathbf{p}}{(2\mathbf{p} \cdot \mathbf{q})^{2/3}} + (2\mathbf{p} \cdot \mathbf{q})^{1/3} e^{i\omega t} \right], \quad a(z) \equiv a(t). \quad (143) \]

Then \( a(z) \) satisfies the Airy equation,
\[ a''(z) = za(z). \quad (144) \]

The solutions are
\[ a(z) = C_1 \text{Ai}(z) + C_2 \text{Bi}(z), \]
\[ b(z) = \alpha a(z) + \beta a'(z), \quad (145) \]

with
\[ \alpha = -\frac{p_3}{p_1 - ip_2} \quad \text{and} \quad \beta \equiv -\frac{(2\omega q_1)^{1/3}}{(p_1 - ip_2)^{2/3}}. \quad (146) \]

Because \( \text{Ai}(z) \) and \( \text{Bi}(z) \) are entire functions in the complex \( z \)-plane, the time evolution is periodic. That is
\[ U(T) = U(0) = 1. \quad (147) \]

The eigenvalue of the Floquet operator is simply unity again. So any state would be cyclic and stable. Consider next specifically the critical exponent \( g(\theta) \):
\[ g(\theta) = \int Z dx \sqrt{\mathbf{p} \cdot \mathbf{p} + 2\mathbf{p} \cdot \mathbf{q} x} \]
\[ = \frac{1}{3\mathbf{p} \cdot \mathbf{q}} \left( \mathbf{p} \cdot \mathbf{p} + 2\mathbf{p} \cdot \mathbf{q} e^{i\theta} \right)^{3/2}. \quad (148) \]
We now exploit this model to elaborate how an expected sudden-switch in the adiabatic following dynamics may be suppressed. As shown in Figs. 5 and 6, the critical exponent \( \text{Re} [g(\theta)] \) changes its sign three times within one driving period. As expected, the solution with \( a(z) = \text{Bi}(z) \) (the red lines in Fig. 6) hops every time when \( \text{Re} [g(\theta)] = 0 \). However, the solution with \( a(z) = \text{Ai}(z) \) (the purple lines in Fig. 6) only hops once around \( t = T/2 \). To develop more understandings, note that the Airy function \( \text{Ai}(z) \) is also the so-called recessive (subdominant) solution, whose asymptotic expansion has only one term with a negative real part in the critical exponent for positive real \( z \). That is, in that regime there is no dominating exponential term \(^{[23]}\).

Consider next what happens after crossing the Stokes line at \( \arg(z) = \pi/3 \), i.e., where \( \text{Re} [g(\theta)] = 0 \). Then the real part of the critical exponent of \( \text{Ai}(z) \) changes from negative to positive. That is, a dominant exponential term emerges only after crossing the Stokes line. Since there is no switch from one dominating exponential term to a different one upon crossing the Stokes line, such a recessive solution does not display any sudden change when projected onto smooth basis states. Therefore, no hopping at \( \arg(z) = \pi/3 \).

By symmetry, there is no hopping for \( \text{Ai}(z) \) at \( \arg(z) = -\pi/3 \) either. Finally, at the next Stokes line at \( \arg(z) = \pi \), a previously subdominant (dominant) term becomes dominant (subdominant), and a sudden-switch behavior emerges. By contrast, for a dominant solution like \( \text{Bi}(z) \), as soon as the first Stokes line at \( \arg(z) = \pi/3 \) is crossed, the previously dominant (predominant) term becomes subdominant (dominant), and therefore hopping occurs there. To end the discussions here, we provide the asymptotic expansions of \( \text{Ai}(z) \) and \( \text{Bi}(z) \) in connection with their respective Stokes lines:

\[
\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-\frac{2}{3} z^{3/2}}, \quad |z| \to \infty
\]

is valid for \( |\arg(z)| < \pi \), but

\[
\text{Bi}(z) \sim \frac{1}{\sqrt{\pi}z^{1/4}} e^{\frac{2}{3} z^{3/2}}, \quad |z| \to \infty
\]

is only valid for \( |\arg(z)| < \pi/3 \) \(^{[23]}\).

5.2. Solvable model with \( r = -1 \)

For completeness let us consider the Hamiltonians \( H_{1r} \) with \( r = -1 \). For convenience, we put \( p \) in front of the lower frequency term. That is,

\[
H_{1-1} = p \cdot \sigma e^{-i\omega t} + q \cdot \sigma e^{i\omega t}
\]

\[
= \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} e^{-i\omega t} + \begin{pmatrix} q_3 & q_1 - iq_2 \\ q_1 + iq_2 & -q_3 \end{pmatrix} e^{i\omega t}.
\]

This Hamiltonian has a symmetry under \( p \leftrightarrow q \) and \( \omega \leftrightarrow -\omega \). Thus all the following discussion in this subsection can be easily extended to \( p \leftrightarrow q \) and \( Z \leftrightarrow 1/Z \).

For this model, the critical exponent in Eq. (131) has the form,

\[
m(Z) \sim \pm \int Z \, dx \sqrt{\frac{p \cdot p + 2p \cdot q \cdot x^2 + q \cdot q \cdot x^4}{x^2}},
\]

(152)
Figure 5. (color online) Left panel: Real part of the critical exponent $g$ as a function of $\theta$ in the model depicted in Eq. (142). The parameters are $p_1 = 1$, $p_2 = 0$, $p_3 = 0.5i$, and $q_1 = 0.6$. The blue line corresponds to $a(z) = \text{Ai}(z)$ and the red line corresponds to $a(z) = \text{Bi}(z)$. Right panel: Density plot of $\text{Re}[g(\theta)]$. The blue lines are the Stokes lines, $\text{arg}(z) = \pm \frac{\pi}{3}, \pi$. The orange line is a unit circle in $Z$. The hopping occurs when the two lines intersect for the solution $\text{Bi}(z)$, whereas the solution $\text{Ai}(z)$ only hops once at the intersection with $\text{arg}(z) = \pi$.

Figure 6. (color online) Real and imaginary parts of $\psi = b(t)/a(t)$ during the time evolution of cyclic states (solid lines) and of the instantaneous energy eigenstates (dashed lines) for the model depicted in Fig. 5 with $\omega = 2\pi/50$. Note that the recessive solution $a(z) = \text{Ai}(z)$ (purple lines) hops only near $t = T/2$, but the dominant solution $a(z) = \text{Bi}(z)$ (red lines) hops three times within one period.

For the model $H_{1-1}$ to be solvable, either $p \cdot p = 0$ or $q \cdot q = 0$. Since there is the internal symmetry between two frequency components, we consider $p \cdot p = 0$ without loss of generality. If we separate $p$ into its real and imaginary parts as

$$p = A + iB,$$

then

$$p \cdot p = A \cdot A - B \cdot B + 2iA \cdot B.$$

(154)
Thus, \( p \cdot p = 0 \) means that
\[
\begin{align*}
A \cdot A &= B \cdot B, \\
A \cdot B &= 0.
\end{align*}
\]
(155)

The first line of the above equation means that vectors \( A \) and \( B \) have the same length; whereas the second line requires that they are perpendicular to each other. We may always rotate the frame such that \( A \) and \( B \) are in the \( xy \)-plane. That is
\[
p_3 = 0.
\]
(156)

In this frame, \( A \perp B \) means that
\[
A_1 = \mp B_2, \quad \text{and} \quad A_2 = \pm B_1,
\]
which is equivalent to
\[
p_1 = \pm ip_2
\]
(158)
in terms of the original complex variables. Without loss of generality, let us choose the upper sign, \( p_1 = ip_2 \). (If the lower sign is more convenient, one simply solves \( b(t) \) first.)

5.2.1. \( q \cdot q \neq 0 \) If \( q \cdot q \neq 0 \), the solutions are Bessel functions. To see this, let us change variables
\[
z(t) \equiv \frac{i}{\omega} \sqrt{q \cdot q} e^{i\omega t}, \quad a(z) \equiv a(t).
\]
(159)
Then \( a(z) \) satisfies a Bessel equation (see 10.13.1 of Ref. [25]),
\[
a''(z) + \left( 1 - \frac{2p \cdot q}{z^2 \omega^2} \right) a(z) = 0.
\]
(160)
The solutions are
\[
a(t) = C_1 \sqrt{z} J_\nu (z) + C_2 \sqrt{z} Y_\nu (z),
\]
(161)
with
\[
\nu \equiv \sqrt{\frac{2p \cdot q}{\omega^2} + \frac{1}{4}}.
\]
(162)
One can then use this explicit solution to investigate the hopping. One may also analyse this by use of the critical exponent in Eq. [152]. We will not repeat the details here as they are much similar to our discussions in previous sections.

5.2.2. \( q \cdot q = 0 \) If both \( p \cdot p = 0 \) and \( q \cdot q = 0 \), then the solutions are simple power functions.
\[
a(t) = C_1 e^{\frac{i\omega t}{2}} + C_2 \exp \left( \frac{i\omega t}{2} - i\sqrt{\frac{2q \cdot q + \omega^2}{4}} \right).
\]
(163)
Evidently, this special solution has no sudden-switch behaviour.

\( \dagger \) This means that the matrix \( p \cdot \sigma \) is \( \mathcal{PT} \)-symmetric [22].
6. Models with two different frequencies and a constant term

In this section we aim to use the same philosophy to treat the most complicated situations with essentially three frequencies (one of them being zero). In particular, we may model a traceless model with double nonzero frequencies as

\[ H_{0mn}(t) = p \cdot \sigma + q \cdot \sigma e^{i\omega t} + s \cdot \sigma e^{i\omega t} \]  

(164)

where 18 real parameters of the model are organized into three complex vectors, \( p \), \( q \), and \( s \).

6.1. \( n = -m \)

In this subclass, we first rescale \( \omega \rightarrow \omega/m \). Then, for convenience, we define

\[ H_{-101}(t) = s \cdot \sigma e^{-i\omega t} + p \cdot \sigma + q \cdot \sigma e^{i\omega t} \]  

(165)

We change variables as before, \( Z = e^{i\omega t} \) and let \( a(Z) = a(t) = e^{m(Z)} \). In the long time limit \( \omega \rightarrow 0 \), Eq. (53) becomes

\[ m(Z) \sim \pm \int dZ \frac{\sqrt{R(x)}}{x^2}, \]  

(166)

where \( R(x) \) is quartic instead of quadratic in general,

\[ R(x) = s \cdot s + 2p \cdot s x + (p \cdot p + 2q \cdot s) x^2 + 2p \cdot q x^3 + q \cdot q x^4. \]  

(167)

This integral can be easily evaluated if \( s \cdot s = 0 \) and \( p \cdot s = 0 \).

Note that an alternative way to obtain a quadratic \( R(x) \) is to set \( q \cdot q = 0 \) and \( p \cdot q = 0 \). This is actually an equivalent set-up if one adopts \( e^{-i\omega t} = Z \) as a change of the variables.

6.1.1. Solvable Double-frequency models with \( s_1 = is_2 \), \( s_3 = 0 \), and \( p \cdot s = 0 \). Having a quadratic \( R(x) \) not only makes Eq. (166) integrable, but also renders the model solvable. The “master” equation about \( a(Z) = a(t) \) is

\[ \omega^2 Z^2 a''(Z) = [(p_3 + \omega)p_3 + 2q \cdot s + 2p \cdot q Z + q \cdot q Z^2] a(Z). \]  

(169)

The analysis to Eq. (169) is very similar to that to Eq. (65). Depending on whether none of, one of, or both of \( p \cdot q \) and \( q \cdot q \) vanish, the solutions to Eq. (169) are Whittaker functions, Bessel functions, or exponential functions, respectively. For example, if none of the two dot products vanishes, the solution of \( a(t) \) is Whittaker functions,

\[ a(t) = C_1 W_{\kappa,\mu} \left( \frac{2\sqrt{q \cdot q}}{\omega} e^{i\omega t} \right) + C_1 M_{\kappa,\mu} \left( \frac{2\sqrt{q \cdot q}}{\omega} e^{i\omega t} \right), \]  

(170)

with

\[ \kappa \equiv -\frac{p \cdot q}{\omega \sqrt{q \cdot q}}, \quad \mu \equiv \sqrt{(2p_3 + \omega)^2 + 8q \cdot s} \frac{2\omega}{2\omega}. \]  

(171)
The corresponding $b(t)$ is determined by the same equation in Eq. [14].

There is a particular interesting example with

$$
p_1 = \frac{1}{2}u_1, \quad p_2 = -i\frac{1}{2}u_1, \quad p_3 = u_3,
q_1 = \frac{1}{2}v_1, \quad q_2 = i\frac{1}{2}v_1, \quad q_3 = v_3,
s_1 = \frac{1}{2}v_2, \quad s_2 = -i\frac{1}{2}v_2, \quad s_3 = 0.
$$

In the case, the Hamiltonian has the form

$$
h_1(t) = \begin{pmatrix}
    u_3 + v_3 e^{i\omega t} & v_1 e^{i\omega t} \\
    u_1 + v_2 e^{-i\omega t} & -u_3 - v_3 e^{i\omega t}
\end{pmatrix}.
$$

This Hamiltonian is equivalent to the single frequency model in Sec. 4.5 with

$$
p_1 = \frac{1}{2}(v_1 + v_2), \quad p_2 = i\frac{1}{2}(v_1 - v_2), \quad p_3 = u_3,
q_1 = \frac{1}{2}u_1, \quad q_2 = -i\frac{1}{2}u_1, \quad q_3 = v_3.
$$

The two Hamiltonians are linked by a simple gauge transformation,

$$
h_1(t) = e^{i\omega t/2} \sigma_3 h_2(t) e^{-i\omega t/2} \sigma_3.
$$

6.2. $n=2m$

For this subclass, we first rescale $\omega \rightarrow \omega/m$. Then, for convenience, we define

$$
H_{012}(t) = s \cdot \sigma + p \cdot \sigma e^{i\omega t} + q \cdot \sigma e^{2i\omega t}
$$

We change variables as before, $Z = e^{i\omega t}$ and let $a(Z) = a(t) = e^{-m(Z)}$. In the long time limit $\omega \rightarrow 0$, $m(Z)$ becomes

$$
m(Z) \sim \pm \int Z \frac{\sqrt{R(x)}}{x} \, dx,
$$

with $R(x)$ to be identical as in Eq. [167]. This integral can also be handled if

$$
q \cdot q = 0 \quad \text{and} \quad p \cdot q = 0,
$$

which requires

$$
q_1 = \pm iq_2, \quad \text{and} \quad q_3 = 0.
$$

Without loss of generality, let us choose the upper sign, $q_1 = iq_2$. (If the lower sign is more convenient, one simply solves $b(t)$ first.) In this frame, $p \cdot q = 0$ means that

$$
p_1 = ip_2.
$$

Note that an alternative way to obtain a quadratic $R(x)$ is to set $s \cdot s = 0$ and $p \cdot s = 0$. However, in this case, the model is still hard to solve.
6.2.1. Solvable Double Frequency Models with $q_1 = i q_2$, $q_3 = 0$, and $p \cdot q = 0$

Consider one example with $s_1 \neq is_2$ and $p_3^2 + q \cdot s \neq 0$. If $s_1 \neq is_2$, then the previous “master” equation about $a(Z) = a(t)$ with $Z = e^{i \omega t}$ becomes

$$\omega^2 \left[ Z^2 a''(Z) + Z a'(Z) \right] = \left[ s \cdot s + (2p \cdot s - \omega p_3) Z + (p_3^2 + 2q \cdot s) Z^2 \right] a(Z).$$

(182)

If $p_3^2 + q \cdot s \neq 0$, the above equation can be transformed into a confluent hypergeometric equation by changing variables

$$z(t) \equiv \frac{2}{\omega} \sqrt{p_3^2 + 2q \cdot s} e^{i \omega t}, \quad w(z) \equiv e^{\frac{i}{2} e^{i \omega t} - \sqrt{s} \cdot s} a(t). \quad (183)$$

Then $w(z)$ satisfies

$$zw''(z) + \left( 1 + \frac{2}{\omega} \sqrt{s} \cdot s - z \right) w'(z)$$

$$= \left[ \frac{1}{2} - \frac{p_3}{2 \sqrt{p_3^2 + 2q \cdot s}} + \frac{1}{\omega} \left( \sqrt{s} \cdot s + \frac{p \cdot s}{\sqrt{p_3^2 + 2q \cdot s}} \right) \right] w(z). \quad (184)$$

The solutions are the confluent hypergeometric functions

$$a(t) = e^{-\frac{i}{2} e^{i \omega t}} e^{i \sqrt{s} \cdot s} \left[ C_1 U \left( a, b, \frac{2}{\omega} \sqrt{p_3^2 + 2q \cdot s} e^{i \omega t} \right) + C_2 \left( a, b, \frac{2}{\omega} \sqrt{p_3^2 + 2q \cdot s} e^{i \omega t} \right) \right],$$

(185)

with

$$a \equiv \frac{1}{2} - \frac{p_3}{2 \sqrt{p_3^2 + 2q \cdot s}} + \frac{1}{\omega} \left( \sqrt{s} \cdot s + \frac{p \cdot s}{\sqrt{p_3^2 + 2q \cdot s}} \right),$$

$$b \equiv 1 + \frac{2}{\omega} \sqrt{s} \cdot s. \quad (186)$$

As a second example, consider $p_3^2 + q \cdot s = 0$ and $s_1 \neq is_2$. Then if $p_3^2 + q \cdot s = 0$, but $s_1 \neq is_2$, the “master” equation becomes a Bessel equation again. The solutions of $a(t)$ is

$$a(t) = C_1 J_{\nu} \left( \frac{2i \sqrt{2p \cdot s - \omega p_3}}{\omega} e^{i \omega t} \right) + C_2 J_{-\nu} \left( \frac{2i \sqrt{2p \cdot s - \omega p_3}}{\omega} e^{i \omega t} \right) \quad (187)$$

with

$$\nu \equiv \frac{2 \sqrt{s} \cdot s}{\omega}. \quad (188)$$

As a final solvable example, consider $s_1 = is_2$. Then the differential equations reduce to the first-order ones. The solution of $a(t)$ is exponential function and that of $b(t)$ is related to the incomplete Gamma function.

In all these cases, our efforts to explicitly work out the critical exponent help us to find conditions under which the complicated non-Hermitian cycling problem can admit exactly solvable solutions. Many of these exact solutions might not necessarily further enhance our understanding of piecewise adiabatic following. However, they do clearly
indicate that piecewise adiabatic following can occur in multi-frequency driving cases that are even exactly solvable. If the problem is not exactly solvable after all, then we can still resort to the critical exponents emerging from our asymptotic analysis to predict and understand the sudden-switch behaviours.

7. Conclusion

In this work, we have extensively investigated the interesting adiabatic following dynamics in periodically driven non-Hermitian systems. The central concern is the peculiar behaviour of cyclic (Floquet) states in the slow-driving limit. It is found that the cyclic states can either behave as intuitively expected by following instantaneous eigenstates of the non-Hermitian Hamiltonian, or exhibit sudden-switching between the instantaneous eigenstates. As learned from several categories of models under different driving scenarios, the sudden switches from following one instantaneous eigenstate to the other eigenstate can be analysed or predicted by a universal route – the sign change of the critical exponent in our asymptotic analysis, which suggests a switch between two terms in the solution with different exponential behaviour. In doing so, we have also discovered many exact solutions in a great variety of non-Hermitian cycling models. We hope that the many exact solutions found by us can be a useful reference in their own right. Our next task is to find potential applications of piecewise adiabatic following dynamics in non-Hermitian systems.

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Appendix A. Time-independent rotation to achieve $p_1 - ip_2 = 0$

Here is the procedure to find such a new basis. Considering three planes intersected at the origin. Let us label the crossing line between Plane 1 and Plane 2 as Line-12, the crossing line between Plane 1 and Plane 3 as Line-13, and that between Plane 2 and Plane 3 as Line-23. Three crossing lines meet at the origin. First, we request that Planes 1 and 2 be perpendicular to each other. Then the angle between Line-13 and Line-23 is in general less than $\pi/2$. Second, we adjust Plane 3 such that the angle between Line-13 and Line-23 equals the angle between $\mathbf{A}$ and $\mathbf{B}$ if $\mathbf{A} \cdot \mathbf{B} > 0$, or the angle between $\mathbf{A}$ and $-\mathbf{B}$ if $\mathbf{A} \cdot \mathbf{B} < 0$. Finally, we rotate these three planes together to make Line-13 coincide with $\mathbf{A}$ and Line-23 coincide with $\mathbf{B}$ ($-\mathbf{B}$) if $\mathbf{A} \cdot \mathbf{B} > 0$ ($\mathbf{A} \cdot \mathbf{B} < 0$). The final step is to define Line-12 as the new $z$-axis, and the plane normal to this $z$
axis (not Plane 3) as the $xy$-plane. Then both $A$ and $A_\perp = (A_1, A_2, 0)$ are in Plane 1, and both $B$ and $B_\perp = (B_1, B_2, 0)$ are in Plane 2. In this new coordinate system, $(A_1, A_2, 0) \perp (B_1, B_2, 0)$. Fig. A1 illustrates our procedure.

![Figure A1. (color online) Choose a new coordinate system such that $A_\perp = (A_1, A_2, 0)$ is perpendicular to $B_\perp = (B_1, B_2, 0)$.](image)

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