COMPATIBLE ACTIONS AND NON-ABELIAN TENSOR PRODUCTS

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Abstract. For a pair of groups $G, H$ we study pairs of actions $G$ on $H$ and $H$ on $G$ such that these pairs are compatible and non-abelian tensor products $G \otimes H$ are defined.

1. Introduction

R. Brown and J.-L. Loday [1,2] introduced the non-abelian tensor product $G \otimes H$ for a pair of groups $G$ and $H$ following works of C. Miller [6], and A. S.-T. Lue [5]. The investigation of the non-abelian tensor product from a group theoretical point of view started with a paper by R. Brown, D. L. Johnson, and E. F. Robertson [3].

The non-abelian tensor product $G \otimes H$ depends not only on the groups $G$ and $H$ but also on the action of $G$ on $H$ and on the action of $H$ on $G$. Moreover these actions must be compatible (see the definition in Section 2). In the present paper we study the following question: what actions are compatible?

The paper is organized as follows. In Section 2, we recall a definition of non-abelian tensor product, formulate some its properties and give an answer on a question of V. Thomas, proving that there are nilpotent group $G$ and some group $H$ such that in $G \otimes H$ the derivative subgroup $[G, H]$ is equal to $G$. In the Section 3 we study the following question: Let a group $H$ acts on a group $G$ by automorphisms, is it possible to define an action of $G$ on $H$ such that this pair of actions are compatible? Some necessary conditions for compatibility of actions will be given and in some cases will be prove a formula for the second action if the first one is given. In the Section 4 we construct pairs compatible actions for arbitrary groups and for 2-step nilpotent groups give a particular answer on the question from Section 3. In Section 5 we study groups of the form $G \otimes \mathbb{Z}_2$ and describe compatible actions.

2. Preliminaries

In this article we will use the following notations. For elements $x, y$ in a group $G$, the conjugation of $x$ by $y$ is $x^y = y^{-1}xy$; and the commutator of $x$ and $y$ is $[x, y] = x^{-1}x^y = y^{-1}y^{-1}xy$. We write $G'$ for the derived subgroup of $G$, i.e. $G' = [G, G]$; $G^{ab}$ for the abelianized group $G/G'$; the second hypercenter $\zeta_2G$ of $G$ is the subgroup of $G$ such that

$$\zeta_2G/\zeta_1G = \zeta_1(G/\zeta_1G),$$

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where \( \zeta G = Z(G) \) is the center of a group \( G \).

Recall the definition of the non-abelian tensor product \( G \otimes H \) of groups \( G \) and \( H \) (see [1][2]). It is defined for a pair of groups \( G \) and \( H \) where each one acts on the other (on right)

\[
G \times H \rightarrow G, \ (g, h) \mapsto g^h; \quad H \times G \rightarrow H, \ (h, g) \mapsto h^g
\]

and on itself by conjugation, in such a way that for all \( g, g_1 \in G \) and \( h, h_1 \in H \),

\[
g^{(h^{g_1})} = \left( (g^{g_1^{-1}})^h \right)^{g_1} \quad \text{and} \quad h^{(g^{h_1})} = \left( (h^{h_1^{-1}}) g \right)^{h_1}.
\]

In this situation we say that \( G \) and \( H \) act compatibly on each other. The non-abelian tensor product \( G \otimes H \) is the group generated by all symbols \( g \otimes h, \ g \in G, \ h \in H \), subject to the relations

\[
(g_1 \otimes h) (g \otimes h) = (g_1 g \otimes h) \quad \text{and} \quad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h_1)
\]

for all \( g, g_1 \in G \), \( h, h_1 \in H \).

In particular, as the conjugation action of a group \( G \) on itself is compatible, then the tensor square \( G \otimes G \) of a group \( G \) may always be defined. Also, the tensor product \( G \otimes H \) is defined if \( G \) and \( H \) are two normal subgroups of some group \( M \) and actions are conjugations in \( M \).

The following proposition is well known. We give a proof only for fullness.

**Proposition 2.1.** 1) Let \( G \) and \( H \) be abelian groups. Independently on the action of \( G \) on \( H \) and \( H \) on \( G \), the group \( G \otimes H \) is abelian.

2) (See [2] Proposition 2.4) Let \( G \) and \( H \) be arbitrary groups. If the actions of \( G \) on \( H \) and \( H \) on \( G \) are trivial, then the group \( G \otimes H \cong G_{\text{ab}} \otimes Z H_{\text{ab}} \) is the abelian tensor product.

**Proof.** 1) We have the equality

\[
(g \otimes h)^{g_1 \otimes h_1} = g^{[g_1,h_1]} \otimes h^{[g_1,h_1]},
\]

where \( g^{[g_1,h_1]} \) is the action of the commutator \([g_1, h_1] \in G\) by conjugation on \( g \), but \( G \) is abelian and \( g^{[g_1,h_1]} = g \). Analogously, \( h^{[g_1,h_1]} = h \). Hence, \( G \otimes H \) is abelian.

2) From the previous formula and trivial actions we have

\[
g^{[g_1,h_1]} = g_{g_1^{-1} h_1^{-1} g_{g_1}^{-1}} = \left( g_{g_1^{-1}} \right)^{h_1^{-1} g_{g_1}^{-1}} = \left( g_{g_1^{-1}} \right)^{g_{g_1} h_1} = \left( g_{g_1^{-1}} \right)^{g_{g_1} h_1} = g^h = g.
\]

Analogously, \( h^{[g_1,h_1]} = h \). Hence, \( G \otimes H \) is abelian.

Remind presentation of non-abelian tensor product as a central extension (see [4]).

The **derivative subgroup** of \( G \) by \( H \) is called the following subgroup

\[
D_H(G) = [G,H] = \langle g^{-1}g^h \mid g \in G, h \in H \rangle.
\]

The map \( \kappa : G \otimes H \rightarrow D_H(G) \) defined by \( \kappa(g \otimes h) = g^{-1}g^h \) is a homomorphism, its kernel \( A = \ker(\kappa) \) is the central subgroup of \( G \otimes H \) and \( G \) acts on \( G \otimes H \) by the rule \( (g \otimes h)x = g^x \otimes h^x, \ x \in G \), i.e. there exists the short exact sequence

\[
1 \rightarrow A \rightarrow G \otimes H \rightarrow D_H(G) \rightarrow 1.
\]
In this case $A$ can be viewed as $\mathbb{Z}[D_H(G)]$-module via conjugation in $G \otimes H$, i.e. under the action induced by setting

$$a \cdot g = x^{-1}ax, \ a \in A, x \in G \otimes H, \kappa(x) = g.$$ 

The following proposition gives an answer on the following question: is there non-abelian tensor product $G \otimes H$ such that $[G,H] = G$? which of V. Thomas formulated in some letter to the authors.

**Proposition 2.2.** Let $G = F_n/\gamma_kF_n$, $k \geq 2$, be a free nilpotent group of rank $n \geq 2$ and $H = \text{Aut}(G)$ is its automorphism group. Then $D_H(G) = [G,H] = G$.

**Proof.** Let $F_n$ be a free group of rank $n \geq 2$ with the basis $x_1, \ldots, x_n$, $G = F_n/\gamma_kF_n$ be a free $k-1$-step nilpotent group for $k \geq 2$. Let $G$ acts trivially on $H$ and elements of $H$ act by automorphisms on $G$. It is easy to see that these actions are compatible.

Let us show that in this case $[G,H] = G$. To do it, let us prove that $x_1$ lies in $[G,H]$. Take $\varphi_1 \in H = \text{Aut}(G)$, which acts on the generators of $G$ by the rules:

$$x_1^{\varphi_1} = x_1, \ x_2^{\varphi_1} = x_2x_1, \ x_3^{\varphi_1} = x_3, \ldots, \ x_n^{\varphi_1} = x_n.$$ 

Then

$$x_1^{-1}x_1^{\varphi_1} = 1, \ x_2^{-1}x_2^{\varphi_1} = x_1, \ x_3^{-1}x_3^{\varphi_1} = 1, \ldots, \ x_n^{-1}x_n^{\varphi_1} = 1.$$ 

Hence the generator $x_1$ lies in $[G,H]$. Analogously, $x_2, x_3, \ldots, x_n$ lie in $[G,H]$. This completes the proof. \hfill \Box

### 3. What actions are compatible?

In this section we study

**Question 1.** Let a group $H$ acts on a group $G$ by automorphisms. Is it possible to define an action of $G$ on $H$ such that this pair of actions are compatible?

Consider some examples.

**Example 3.1.** Let us take $G = \{1,a,a^2\} \cong \mathbb{Z}_3$, $H = \{1,b,b^2\} \cong \mathbb{Z}_3$. In dependence on actions we have three cases.

1) If the action of $H$ on $G$ and the action of $G$ on $H$ are trivial, then by the second part of Proposition $[G \otimes H] = \mathbb{Z}_3 \otimes_{\mathbb{Z}_3} \mathbb{Z}_3 \cong \mathbb{Z}_3$ is abelian tensor product.

2) Let $H$ acts non-trivially on $G$, i.e. $a^b = a^2$ and the action $G$ on $H$ is trivial. It is not difficult to check that $G$ and $H$ act compatibly on each other. To find $D_H(G) = [G,H]$ we calculate

$$[a,b] = a^{-1}a^b = a^2a^2 = a.$$ 

Hence, $D_H(G) = G$. But $D_G(H) = 1$.

By the definition, $G \otimes H$ is generated by elements

$$a \otimes b, a^2 \otimes b, a \otimes b^2, a^2 \otimes b^2.$$ 

Using the defining relations:

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h), \ g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}),$$

$$g \otimes h = g^{(1,0,0)} \otimes h^{(1,0,0)}.$$
we find
\[ a^2 \otimes b = (a^a \otimes b^a)(a \otimes b) = (a \otimes b)^2, \]
\[ a \otimes b^2 = (a \otimes b)(a^b \otimes b^b) = (a \otimes b)(a^2 \otimes b) = (a \otimes b)^3. \]

On the other side
\[ 1 = a^2 a \otimes b = (a^2 \otimes b^a)(a \otimes b) = (a \otimes b)^3. \]

Hence,
\[ a \otimes b^2 = a^2 \otimes b^2 = 1 \]
and in this case we have the same result: \( Z_3 \otimes Z_3 = Z_3. \)

3) Let \( H \) acts non-trivially on \( G \), i.e. \( a^b = a^2 \) and \( G \) acts non-trivially on \( H \). In this case \( G \) and \( H \) act non-compatibly on each other. Indeed,
\[ a^{(b^a)} = a^{b^2} = (a^2)^b = a, \]
but
\[ \left( \left( a^{a^{-1}} \right)^b \right)^a = (a^b)^a = (a^2)^2 = a^2. \]

Hence, the equality
\[ a^{(b^a)} = \left( \left( a^{a^{-1}} \right)^b \right)^a \]
does not hold.

Let \( G, H \) be some groups. Actions of \( G \) on \( H \) and \( H \) on \( G \) are defined by homomorphisms
\[
\begin{align*}
\beta : G & \to \text{Aut}(H), \quad \alpha : H \to \text{Aut}(G),
\end{align*}
\]
and by definition
\[ g^h = g^{\alpha(h)}, \quad h^g = h^{\beta(g)}, \quad g \in G, h \in H. \]

The actions \((\alpha, \beta)\) are compatible, if
\[ g^{\alpha(h^{(g_{1})})} = \left( \left( g^{g_{1}^{-1}} \right)^{\alpha(h)} \right)^{g_{1}} \]
and
\[ h^{\beta(g^{(h_{1})})} = \left( \left( h^{h_{1}^{-1}} \right)^{\beta(g)} \right)^{h_{1}} \]
for all \( g, g_{1} \in G, h, h_{1} \in H \). In this case we will say that the pair \((\alpha, \beta)\) is compatible.

Rewrite these equalities in the form
\[ \alpha \left( h^{\beta(g_{1})} \right) = \widehat{g}_{1}^{-1} \alpha(h) \widehat{g}_{1} \]
(1)
and
\[ \beta \left( g^{\alpha(h_{1})} \right) = \widehat{h}_{1}^{-1} \beta(g) \widehat{h}_{1}, \]
(2)
where \( \widehat{g} \) is the inner automorphism of \( G \) which is induced by conjugation of \( g \), i.e.
\[ \widehat{g} : g_{1} \mapsto g_{1}^{-1} g_{1} g, \quad g, g_{1} \in G, \]
and analogously, \( \widehat{h} \) is the inner automorphism of \( H \) which is induced by the conjugation of \( h \), i.e.
\[ \widehat{h} : h_{1} \mapsto h_{1}^{-1} h_{1} h, \quad h, h_{1} \in H. \]
Theorem 3.2. 1) If the pair \((\alpha, \beta)\) defines compatible actions of \(H\) on \(G\) and \(G\) on \(H\), then the following inclusions hold

\[ N_{\mathrm{Aut}(G)}(\alpha(H)) \geq \mathrm{Inn}(G), \quad N_{\mathrm{Aut}(H)}(\beta(G)) \geq \mathrm{Inn}(H). \]

Here \(\mathrm{Inn}(G)\) and \(\mathrm{Inn}(H)\) are the subgroups of inner automorphisms.

2) If \(\alpha : H \to \mathrm{Aut}(G)\) is an embedding and \(N_{\mathrm{Aut}(G)}(\alpha(H)) \geq \mathrm{Inn}(G)\), then defining \(\beta : G \to \mathrm{Aut}(H)\) by the formula

\[ \beta(g) : h \mapsto \alpha^{-1}(\hat{g}^{-1}\alpha(h)\hat{g}), \quad h \in H, \]

we get the compatible actions \((\alpha, \beta)\).

Proof. The first claim immediately follows from the relations (1), (2).

To prove the second claim it is enough to check (2), or that is equivalent, the equality

\[ h^\beta(g^{\alpha(h_1)}) = \left( h^{h_1^{-1}} \right)^{\beta(g)} h_1. \]  \hspace{1cm} (3)

Using the definition \(\beta\), rewrite the left side of (3):

\[ h^\beta(g^{\alpha(h_1)}) = \alpha^{-1}\left( g^{\hat{\alpha}(h_1)} \right)^{-1} \alpha(h)g^{\alpha(h_1)}. \]  \hspace{1cm} (4)

Rewrite the right side of (3):

\[ \left( h^{h_1^{-1}} \right)^{\beta(g)} h_1 = h_1^{-1}(h_1hh_1^{-1})^{\beta(g)} h_1 = h_1^{-1}\alpha^{-1}(\hat{g}^{-1}\alpha(h_1hh_1^{-1})\hat{g}) h_1. \]  \hspace{1cm} (5)

From (4) and (5):

\[ \alpha^{-1}\left( g^{\hat{\alpha}(h_1)} \right)^{-1} \alpha(h)g^{\alpha(h_1)} = h_1^{-1}\alpha^{-1}(\hat{g}^{-1}\alpha(h_1hh_1^{-1})\hat{g}) h_1. \]

Using the homomorphism \(\alpha\):

\[ g^{\hat{\alpha}(h_1)}^{-1} \alpha(h)g^{\alpha(h_1)} = \alpha(h_1^{-1}\alpha^{-1}(\hat{g}^{-1}\alpha(h_1hh_1^{-1})\hat{g}) h_1) = \]

\[ = \alpha(h_1^{-1}\hat{g}^{-1}\alpha(h_1hh_1^{-1})\hat{g}) = \alpha(h_1^{-1}\hat{g}^{-1}\alpha(h_1)\alpha(h_1^{-1}\alpha(h_1^{-1}))\hat{g}) = g^{\hat{\alpha}(h_1)}^{-1} \alpha(h)g^{\alpha(h_1)}. \]

In the last equality we used the formula

\[ \alpha(h_1^{-1}\hat{g}) = g^{\hat{\alpha}(h_1)}. \]

Hence, the equality (3) holds. \qed

Question 2. Are the inclusions

\[ N_{\mathrm{Aut}(G)}(\alpha(H)) \geq \mathrm{Inn}(G), \quad N_{\mathrm{Aut}(H)}(\beta(G)) \geq \mathrm{Inn}(H) \]

sufficient for compatibility of the pare \((\alpha, \beta)\)?
4. Compatible actions for nilpotent groups

At first, recall the following definition.

**Definition 4.1.** Let $G$ and $H$ be groups and $G_1 \leq G$, $H_1 \leq H$ are their normal subgroups. We will say that $G$ is comparable with $H$ with respect to the pair $(G_1, H_1)$, if there are homomorphisms

$$\varphi : G \to H, \quad \psi : H \to G,$$

such that

$$x \equiv \psi \varphi (x) \text{ (mod } G_1), \quad y \equiv \varphi \psi (y) \text{ (mod } H_1)$$
for all $x \in G$, $y \in H$, i.e.

$$x^{-1} \cdot \psi \varphi (x) \in G_1, \quad y^{-1} \cdot \varphi \psi (y) \in H_1.$$

Note that if $G_1 = 1$, $H_1 = 1$, then $\varphi$, $\psi$ are mutually inverse isomorphisms. The following theorem holds.

**Theorem 4.2.** Let $G$, $H$ be groups and there exist homomorphisms

$$\varphi : G \to H, \quad \psi : H \to G,$$

such that

$$x \equiv \psi \varphi (x) \text{ (mod } \zeta_2 G), \quad y \equiv \varphi \psi (y) \text{ (mod } \zeta_2 H)$$
for all $x \in G$, $y \in H$. Then the action of $G$ on $H$ and the action of $H$ on $G$ by the rules

$$x^y = \psi(y)^{-1} \psi(xy), \quad y^x = \varphi(x)^{-1} \varphi(y), \quad x \in G, \ y \in H,$$

are compatible, i.e. the following equalities hold

$$x^{(y^x)} = ((x^{y^{-1}})^x)^{x_1}, \quad y^{(x^y)} = ((y^{x^{-1}})^y)^{y_1}, \quad x, x_1 \in G, \ y, y_1 \in H.$$

**Proof.** Let us prove that the following relation holds

$$x^{(y^x)} = ((x^{y^{-1}})^y)^{x_1}.$$

For this denote the left hand side of this relation by $L$ and transform it:

$$L = x^{(y^x)} = x^{\varphi(x_1)^{-1} \psi(x_1)} = \psi(\varphi(x_1)^{-1} y^{-1} \varphi(x_1)) \psi(x_1)^{-1} \psi(y) \varphi(x_1) =$$

$$= (\psi \varphi (x_1))^{-1} \psi(y)^{-1} (\psi \varphi (x_1))^x (\psi \varphi (x_1))^{-1} \psi(y) (\psi \varphi (x_1)) =$$

$$= (c(x_1)^{-1} x_1^{-1} \psi(y)^{-1} x_1 c(x_1)) x(x_1)^{-1} x_1^{-1} \psi(y) x_1 c(x_1)).$$

Here $\psi \varphi (x_1) = x_1 c(x_1), \ c(x_1) \in \zeta_2 G$. Since $c(x_1) \in \zeta_2 G$, then the commutator $[x_1^{-1} \psi(y)x_1, c(x_1)]$ lies in the center of $G$. Hence

$$L = x^{x_1^{-1} \psi(y)x_1}.$$

Denote the right hand side of this relation by $R$ and transform it:

$$R = ((x^{y^{-1}})^y)^{x_1} = ((x^{x_1^{-1}})^{x_1} \psi(y))^{x_1} = x^{x_1^{-1} \psi(y)x_1}.$$

We see that $L = R$, i.e. the first relation from the definition of compatible action holds. The checking of the second relation is the similar.  \[\square\]
From this theorem we have particular answer on Question\textsuperscript{4.1} for 2-step nilpotent groups.

**Corollary 4.3.** If $G$, $H$ are 2-step nilpotent groups, then any pair of homomorphisms

$$\varphi : G \rightarrow H, \quad \psi : H \rightarrow G$$

define the compatible action.

**Problem 1.** Let $G$ and $H$ be free 2-step nilpotent groups. By Corollary\textsuperscript{4.3} any pair of homomorphisms $(\varphi, \psi)$, where $\varphi \in \text{Hom}(G, H)$, $\psi \in \text{Hom}(H, G)$ defines a tensor product $M(\varphi, \psi) = G \otimes H$. Give a classification of the groups $M(\varphi, \psi)$.

Note that for arbitrary groups Corollary\textsuperscript{4.3} does not hold. Indeed, let $G = \langle x_1, x_2 \rangle$, $H = \langle y_1, y_2 \rangle$ be free groups of rank 2. Define the homomorphisms

$$\varphi : G \rightarrow H, \quad \psi : H \rightarrow G$$

by the rules

$$\varphi(x_1) = y_1, \quad \varphi(x_2) = y_2, \quad \psi(y_1) = \psi(y_2) = 1.$$  

Then

$$y_2^{x_1} = y_2^{\varphi(x_1)} = y_2^y \neq y_2,$$

i.e. the conditions of compatible actions does not hold.

5. **Tensor products $G \otimes \mathbb{Z}_2$**

Note that the group $\text{Aut}(\mathbb{Z}_2)$ is trivial and hence, any group $G$ acts on $\mathbb{Z}_2$ only trivially.

This section is devoted to the answer on the following question.

**Question 3.** Let $G$ be a group and $\psi \in \text{Aut}(G)$ be an automorphism of order 2. Let $\mathbb{Z}_2 = \langle \varphi \rangle$ and $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(G)$ such that $\alpha(\varphi) = \psi$. Under what conditions the pare $(\alpha, 1)$ is compatible?

If $\psi \in \text{Aut}(G)$ is trivial automorphism, then by the second part of Proposition\textsuperscript{2.1} $G \otimes \mathbb{Z}_2 = G^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ is an abelian tensor product. In the general case we have

**Proposition 5.1.** Let

1) $G$ be a group,

2) $\mathbb{Z}_2 = \langle \varphi \rangle$ be a cyclic group of order two with the generator $\varphi$,

3) $\alpha : \mathbb{Z}_2 \rightarrow \text{Aut}(G)$ be a homomorphism, $\beta = 1 : G \rightarrow \text{Aut}(\mathbb{Z}_2)$ be the trivial homomorphism,

Then the pare of actions $(\alpha, \beta)$ is compatible if and only if for any $g \in G$ holds

$$g^{\alpha(\varphi)} = gc(g),$$

where $c(g)$ is a central element of $G$ such that $c(g)^{\alpha(\varphi)} = c(g)^{-1}$. In particular, if the center of $G$ is trivial, then $G \otimes \mathbb{Z}_2 = G^{ab} \otimes_{\mathbb{Z}} \mathbb{Z}_2$.  

Proof. Since $\text{Inn}(G)$ normalizes $\alpha(Z_2)$, then for every $g \in G$ holds

$$\hat{g}^{-1} \alpha(\varphi) \hat{g} = \alpha(\varphi).$$

Using this equality for arbitrary element $x \in G$ we get

$$g^{-1} g^\alpha(\varphi) x^\alpha(\varphi) (g^{-1} g^\alpha(\varphi))^{-1} = x^\alpha(\varphi).$$

Since $x^\alpha(\varphi)$ is an arbitrary element of $G$, then $c(g)$ is a central element of $G$. Applying $\alpha(\varphi)$ to the equality $g^\alpha(\varphi) = gc(g)$ we have

$$g = g^\alpha(\varphi)^2 = g^\alpha(\varphi) c(g)^\alpha(\varphi) = gc(g) c(g)^\alpha(\varphi),$$

that is $c(g)^\alpha(\varphi) = c(g)^{-1}$.

For an arbitrary abelian group $A$ we know that $A \otimes \mathbb{Z} \cong A$. The following proposition is some analog of this property for non-abelian tensor product.

**Proposition 5.2.** Let $A$ be an abelian group, $Z_2 = \langle \varphi \rangle$ is the cyclic group of order 2 and $\varphi$ acts on the elements of $A$ by the following manner

$$a^\varphi = a^{-1}, \quad a \in A.$$

Then the non-abelian tensor product $A \otimes Z_2$ is defined and there is an isomorphism

$$A \otimes Z_2 \cong A.$$

Proof. It is not difficult to check that defined actions are compatible.

Since $A$ acts on $Z_2$ trivially and $A$ is abelian, then the defining relations of the tensor product:

$$a a_1 \otimes h = (a a_1 \otimes h) (a_1 \otimes h), \quad a, a_1 \in A, \quad h \in Z_2,$$

have the form

$$a a_1 \otimes h = (a \otimes h)(a_1 \otimes h) = (a_1 \otimes h)(a \otimes h).$$

(1)

The relations

$$a \otimes h h_1 = (a \otimes h_1)(a h_1 \otimes h_1), \quad a \in A, \quad h, h_1 \in Z_2,$$

give only one non-trivial relation

$$1 = a \otimes \varphi^2 = (a \otimes \varphi)(a^{-1} \otimes \varphi), \quad a \in A,$$

which follows from (1).

Since the set of relations (1) is a full system of relations for $A \otimes Z_2$, then there exists the natural isomorphism of $A \otimes Z_2$ on $A$ that is defined by the formula

$$a \otimes \varphi \mapsto a, \quad a \in A.$$

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