Gauge invariances vis-à-vis Diffeomorphisms in second order metric gravity: A new Hamiltonian approach

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A new analysis of the gauge invariances and their unity with diffeomorphism invariances in second order metric gravity is presented which strictly follows Dirac’s constrained Hamiltonian approach.

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I. INTRODUCTION

Einstein’s General theory of relativity (GTR) stands as a successful theory of classical gravity which is also unique in the sense that here spacetime manifold itself acquires dynamics. The metric tensor $g_{\mu\nu}$ which is a measure of invariant distance between spacetime points constitute the dynamical fields of the theory. As is well known, this feature presents great difficulties in the quantization of gravity. Many variants and extensions of GTR have been proposed which have been argued to be more suitable from one or other points of view. However, a successful theory of Quantum Gravity still eludes us.

It is therefore all the more relevant to understand the classical foundations of the theories of Gravitations from different angles.

The theories of gravitation are distinguished by a common feature which is general covariance. From the active point of view this is the invariance of the spacetime manifold labelled by the coordinates $x^\mu$ under the transformations

$$x^\mu \rightarrow x'^\mu = x^\mu - \Lambda^\mu (x)$$

(1)

where $\Lambda^\mu (x)$ are arbitrary infinitesimal functions of $x^\mu$. This is an automorphism $M \rightarrow M$ that moves points within the manifold. Consequently there arises a certain arbitrariness of description of the gravitational field by the metric tensor $g_{\mu\nu}$ which can be obtained from their transformations under (1). Looking from the Hamiltonian (canonical) point of view this arbitrariness is reflected in the transformations generated by the first class constraints of the theory i.e. the gauge transformations. Stated otherwise, there should exist the right number of gauge invariances corresponding to the invariances (1). The connection is however non-trivial and therefore has been a topic of continuing interest in the literature.

The equivalence between the diffeomorphism (diff.) and gauge invariances is completely established when one can prescribe an exact mapping between the two sets of independent transformation parameters. While on the diff. side the independent parameters are intuitively clear, the same can not be said about the gauge parameters. Thus different works related to the subject vary not only in their interpretation of gauge transformation but also in their approach of abstracting the independent gauge parameters. As a concrete example we may consider the problem in connection with the second order metric gravity theory. In this the gauge transformations are viewed as mapping solutions to solutions and independent gauge generators are obtained following a “more Lagrangean” approach which makes use of the Lagrange equations of motion. Gauge transformations can on the otherhand be considered as mapping field configurations to field configurations. In fact this is the essence of Dirac’s point of view. In this point of view is adopted. They find the connection between the diffeomorphism group and the gauge group by a certain projection technique from the configuration-velocity space to the phase space. Though the approaches in these works differ, they share the following common features:

1. All these works utilise a combination of Lagrangian and Hamiltonian methods. They cannot be identified as strict Hamiltonian approaches.

2. In one way or other these works make use of the Lagranges equations of motion.

In the present paper these aspects will precisely be our points of departure, i.e. our purpose here will be

1. the construction of a dedicated Hamiltonian approach ala Dirac which will lead to the equivalence between the diffeomorphism and gauge transformations.

2. to derive the most general gauge transformation generator without taking recourse to the velocity-space approach.

As concrete example we will also consider the second order metric gravity theory though our approach will be easily applicable to other theories of gravitation as well.

In the Canonical approach to the metric gravity a time parameter needs to be identified. This is attained...
by dividing space-time in to a collection of space-like
three-surfaces with a time-like direction of evolution. This
is the famous Arnowitt–Deser–Misner (A–D–M) decomposition
where the arbitrariness of the foliation is reflected by one ‘lapse’
and three ‘shift’ variables. One can cast the original Einstein–Hilbert action mod-
ulo boundary terms in a form where no time derivative of these variables appear. As a consequence the corresponding
momenta vanish imposing four primary constraints. Conservation of these constraints gives rise to four sec-
ondary constraints. All these constraints are first-class. Since the Hamiltonian is a linear combination of these
constraints no further constraints appear. According to
the Dirac conjecture the gauge generator is a linear com-
bination of all the first-class constraints. There are thus
eight gauge parameters appearing in the generator. How-
ever, only four of them are independent since the number
must be equal to the number of primary first-class con-
straints. As has been pointed out in the above, the crucial
first step in establishing a one-to-one correspondence be-
tween the diffeomorphisms and the gauge variations is to
identify the independent gauge parameters. For the suc-
cess of our programme (1) we need a strictly Hamiltonian
method to achieve this.

There exists a Hamiltonian approach in the literature
which provides a general algorithm for abstracting the
independent gauge parameters in any gauge theory. This method was applied to analyze the gauge invari-
ances in various field and string theoretic models in the
literature. We like to use the same algo-

rithm here. This approach of analyzing the gauge invari-
ances is a novel one which can be contrasted with the
approach of analyzing the gauge invari-
ances. Hence the Lagrangean (3) is suitable for canoni-
ocal analysis because it does not contain time derivatives
of the system are defined whereas the shift variables
represent variations along the three-surface. They are
distinguished from their analogue defined on the
three-hypersurface which are written without any such
pre-superscript. Note that

\[ N_{\mu} = \dot{g}_{0\mu} - g_{00} \dot{N} - g^{ij} N_{ij} \]

where \( K = K_i^i = g_{ij} K^{ij} \) and \( R \) is the Ricci scalar on the
three-surface. The lapse variable \( N^\perp \) represents arbitrary
variation normal to the three-surface on which the state
of the system are defined whereas the shift variables \( N^i \)
represent variations along the three-surface. They are
defined by

| \( N^i \) | \( = g^{ij} g_{0j} \) |
|---|---|

Note that \( N^i \) is contained in the Lagrangean through the
definition of \( K_{ij} \) given by

\[ K_{ij} = \frac{1}{2N^\perp} \left( \dot{g}_{ij} + N_{ij} + N_{ji} \right) \]

where the \( \mid \) indicates covariant derivative on the three-
surface. Since the lapse and shift variables represent arbitrary deformations of the hypersurface one can ex-
pect them not to be restricted by the Hamiltonian equa-
tions. Hence the Lagrangean is suitable for canonical
analysis because it does not contain time derivatives
of \( N^\mu (N^\perp, N^i) \). One can immediately write down the
primary constraints following from the definition of the
conjugate momenta of \( N^\mu \)

\[ \pi_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{N^\mu}} = 0 \]
The second fundamental form of the three-surface $K_{ij}, (i, j = 1, 2, 3)$ contains the velocities $\dot{g}_{ij}$ and therefore related to the momenta canonical to $g_{ij}$ by

$$\pi^{ij} = \frac{\partial L}{\partial \dot{g}_{ij}} = - (g)^{1/2} (K^{ij} - Kg^{ij})$$

(8)

The inverse relation expresses $K_{ij}$ in terms of the dynamical variables of the theory

$$K^{ij} = - (g)^{-1/2} \left( \pi^{ij} - \frac{1}{2} \pi g^{ij} \right)$$

(9)

where $\pi = g_{ij} \pi^{ij}$. The non-trivial Poisson Brackets (PB) between the pair of conjugate variables of the theory are

$$\{ g_{ij} (x), \pi^{kl} (x') \} = \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^j_k \delta^i_l \right) \delta^{(3)} (x - x')$$

$$\{ N^\mu (x), \pi_\nu (x') \} = \delta^\mu_\nu \delta^{(3)} (x - x')$$

(10)

Using equations (3), (5), (7) and (9) the canonical Hamiltonian can be worked out as

$$H_c = \int d^3 x \left( \pi_\mu \dot{N}^\mu + \pi^{ij} \dot{g}_{ij} - L \right)$$

$$= \int d^3 x (N^{-1} \mathcal{H}_+ + N^{i} \mathcal{H}_i)$$

(11)

where,

$$\mathcal{H}_+ = g^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2 \right) - (g)^{1/2} R$$

$$\mathcal{H}_i = -2 \pi_{ij} \mathcal{H}_j$$

(12)

We denote the primary constraints as

$$\Omega_\mu = \pi_\mu \approx 0$$

(14)

and they are conserved with the Hamiltonian (11) using the basic brackets (10) to generate the secondary constraints given by

$$\Omega_4 = \mathcal{H}_+ \approx 0$$

$$\Omega_{4+i} = \mathcal{H}_i \approx 0$$

(15)

(16)

Using the basic PBs the constraint algebra becomes

$$\{ \Omega_4 (x), \Omega_4 (x') \} = g^{ij} \left[ \Omega_{4+i} (x) + \Omega_{4+i} (x') \right]$$

$$\times \delta_i (x - x')$$

$$\{ \Omega_{4+i} (x), \Omega_{4+j} (x') \} = \Omega_{4+i} \delta_j (x - x')$$

$$\{ \Omega_{4+i} (x), \Omega_{4+j} (x') \} = \Omega_{4+i} \delta_j (x - x')$$

$$+ \Omega_{4+j} (x) \delta_i (x - x')$$

(17)

This weakly involutive algebra signifies that the set (14) - (16) are first-class constraints. This concludes our review of the canonical formulation of meric gravity. In the next section we will analyze the gauge symmetry and establish its underlying unity with the reparametrization invariance of the theory in an explicit manner.

### III. GAUGE SYMMETRY AND DIFFEOMORPHISM

We will now proceed to find the desired mapping between the independent gauge parameters and the reparametrization parameters. As mentioned in the introduction, the algorithm of G (18) - (10) will be followed to find the independent gauge parameters. It will thus be convenient to begin with a summary of the useful results of G (18) - (10).

Consider a theory with first class constraints only. The set of constraints $\Omega_\mu$ is assumed to be classified as

$$[\Omega_{a_1}, \Omega_{a_2}] = 0$$

(18)

where $a_1$ belong to the set of primary and $a_2$ to the set of secondary constraints. The total Hamiltonian is

$$H_T = H_c + \Sigma \lambda^{a_1} \Omega_{a_1}$$

(19)

where $H_c$ is the canonical Hamiltonian and $\lambda^{a_1}$ are Lagrange multipliers enforcing the primary constraints. The most general expression for the generator of gauge transformations is obtained according to the Dirac conjecture (18) as

$$G = \Sigma \epsilon^{a} \Omega_{a}$$

(20)

where $\epsilon^{a}$ are the gauge parameters. Note that all the first-class constraints appear in $G$. However, only $a_1$ of the parameters $\epsilon^{a}$ are independent, the number being equal to the number of primary first-class constraints (20). By demanding the commutation of an arbitrary gauge variation with the total time derivative, (i.e. $\frac{d}{dt} (\delta q) = \delta (\frac{d}{dt} q)$) we arrive at the following equations (9) - (10)

$$\delta \lambda^{a_1} = \frac{d\epsilon^{a_1}}{dt} - \epsilon^{a} \left( V_{a_1}^{\lambda_1} + \lambda^{b_1} C^{a_1}_{b_1} \right)$$

(21)

$$0 = \frac{d\epsilon^{a_2}}{dt} - \epsilon^{a} \left( V_{a_2}^{\lambda_2} + \lambda^{b_2} C^{a_2}_{b_2} \right)$$

(22)

Here the coefficients $V_{a_1}^{\lambda_1}$ and $C^{a_1}_{b_1}$ are the structure functions of the involutive algebra, defined as

$$\{ H_c, \Omega_a \} = V_{a}^{b_1} \Omega_b$$

$$\{ \Omega_a, \Omega_b \} = C_{ab} \Omega_c$$

(23)

Solving (22), it is possible to choose $a_1$ independent gauge parameters from the set $\epsilon^{a}$ and express $G$ of (20) entirely terms of them. The other set (21) gives the gauge variations of the Lagrange multipliers. It can be shown that these equations are not independent conditions but appear as internal consistency conditions. In fact the conditions (21) follow from (22) (9) - (10).

Before proceeding further let us note the following point:

The assumption on which (22) is based only involves the
The equations in (29) suggest that the set \( \{ \epsilon^0, \epsilon^1 \} \) will be the appropriate choice of the dependent gauge parameters. We can immediately express them in terms of remaining parameters \( \{ \epsilon^2, \epsilon^{i+} \} \) as

\[
\begin{align*}
\epsilon^0 (x) &= \left[ \epsilon^4 + \epsilon^{i+} \partial_i N^\perp - N^i \partial_i \epsilon^4 \right] (x) \\
\epsilon^1 (x) &= \left[ \epsilon^{i+} + \epsilon^i \partial_i N^\perp - N^i \partial_i \epsilon^{i+} \right. \\
&\quad \left. - N \perp g^{ri} \partial_r \epsilon^4 + \epsilon^4 g^{ri} \partial_r N^\perp \right] (x)
\end{align*}
\]

(30) (31)

Substituting the above expressions in (23) we obtain the gauge generator solely in terms of the independent gauge parameters the number of which matches with the number of independent first-class constraints, as it should be.20 Also note that the most general form of the gauge generator contains time derivatives of the independent gauge parameters. It is remarkable that in our approach this feature follows naturally from the formalism and needs no special treatment.

After identifying the most general gauge generator of the theory we now proceed to derive the desired mapping between the gauge and the reparametrization parameters. This is conveniently obtained from the gauge variations of \( N^i \), comparing them with the corresponding variations due to reparametrization.

The gauge variations of the shift variables are

\[
\delta N^i (x) = \left\{ N^i (x), G \right\} = \left[ \epsilon^4 + \epsilon^{i+} \partial_i N^\perp - N^i \partial_i \epsilon^4 \right] (x)
\]

(32)

To find the corresponding variations due to reparametrization we have to use the variations of the four-metric \( g_{\mu\nu} \) under the infinitesimal transformation (1)

\[
\delta (4) g_{\mu\nu} = (4) g_{\gamma\nu} \partial_\mu \Lambda^\gamma + (4) g_{\gamma\mu} \partial_\nu \Lambda^\gamma + \Lambda^\gamma \partial_\gamma (4) g_{\mu\nu}
\]

(33)

Using (33) and (1) we can compute the desired variations under the reparametrization (1):

\[
\delta N^i (x) = \left( \frac{d}{dt} - N^k \partial_k \right) \left( \Lambda^i + \Lambda^0 N^k \right)
\]

(34)

where we have also used the inverse of the relations (5), namely

\[
g_{ij} N^j = N^i
\]

(35)

\[
g_{ij} N^i N^j - \left( N^\perp \right)^2 = g_{00}
\]

(36)

Comparing the variations of the shift variable \( N^i \) from (32) and (34) we obtain the sought-for mapping between the reparametrization parameters and the independent gauge parameters

\[
\epsilon^{i+} = \Lambda^i + \Lambda^0 N^i
\]

(37)

\[
\epsilon^4 = N^\perp N^0
\]

(38)
Note that similar mapping between the different sets of parameters were obtained earlier in\textsuperscript{1} and also in\textsuperscript{2}. Observe however that in comparison to these earlier works we follow a strictly Dirac approach of constrained Hamiltonian analysis. Moreover, we provide a structured algorithm for metric gravity where the occurrence of time derivative of the gauge parameter need not be addressed separately\textsuperscript{3}. Though discussed in connection with the second order metric gravity it is apparent that this algorithm is applicable in the same general form to other theories of gravitation as well.

A through consistency check of the whole formalism is now in order. The mapping (38) when used in the gauge variation of the lapse variable $N^\perp$

$$\delta N^\perp (x) = \left[ \epsilon^4 + \epsilon^{4+k} \partial_k N^\perp - N^i \partial_k \epsilon^i \right] (x) \quad (39)$$
gives its variation in terms of the diff. parameters

$$\delta N^\perp (x) = \left( \frac{d}{dt} - N^i \partial_i \right) \Lambda^0 N^\perp + \Lambda^0 N^i \partial_i N^\perp + \Lambda^i \partial_i N^\perp \quad (40)$$

which is identical with the variation calculated from (33). Similarly, we work out the gauge variation of $g_{ij}$ generated by $G$\textsuperscript{25} which gives

$$\delta g_{ij} (x) = \{ g_{ij} (x) , G \}$$

$$= -2\epsilon^4 K_{ij} + \epsilon^{4+k} \partial_k g_{ij}$$

$$+ g_{ki} \partial_j \epsilon^{4+k} + g_{kj} \partial_i \epsilon^{4+k} \quad (41)$$

and use the mapping (38) in it. The resulting expression can be identified with the reparametrization variation of $g_{ij}$ given by

$$\delta g_{ij} (x) = \left( \Lambda^0 \frac{d}{dt} - \Lambda^k \partial_k \right) g_{ij} + N_i \partial_j \Lambda^0$$

$$+ N_j \partial_i \Lambda^0 + g_{ki} \partial_j \Lambda^k + g_{kj} \partial_i \Lambda^k \quad (42)$$

This completes the explicit identification of the gauge invariance and diffeomorphism in second order metric gravity theory.

IV. CONCLUSION

We discussed a novel approach of obtaining the most general gauge invariances of the second order metric gravity theory following the general Hamiltonian method\textsuperscript{21,20} and used this analysis to establish a one-to-one mapping between the gauge and reparametrization parameters. We have performed explicit computation to check the consistency of our method. Though we rederive already available results\textsuperscript{4,5} our method is completely new in the following senses:

1. This is a new dedicated Hamiltonian approach to the problem and does not require to refer to the velocity space at any stage in the calculational algorithm. As far as we know this is the first time such a calculational scheme is advanced in canonical gravity.

2. This approach reveals properly to what extent the mapping between diffeomorphisms and gauge invariances can be considered valid off-shell. Our Hamiltonian method clearly reveals that it is dependent only on the first set of Hamilton’s equations which connects the velocities, momenta and the Lagrange multipliers. In other words the specific phase space structure is only important but not the full dynamics. Note however dynamics must be invoked in establishing the equivalence of transformations of the full set of phase space variables as we have already mentioned.

In addition to these attractive features our method has the advantage of providing a structured algorithm which can easily be applied to other theories of gravitation.

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