LAPLACE TRANSFORM AND UNIVERSAL $sl_2$ INVARIANTS

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Abstract. We develop a Laplace transform method for constructing universal invariants of 3–manifolds. As an application, we recover Habiro’s theory of integer homology 3–spheres and extend it to some classes of rational homology 3–spheres with cyclic homology. If $|H_1| = 2$, we give explicit formulas for universal invariants dominating the $sl_2$ and $SO(3)$ Witten–Reshetikhin–Turaev invariants, as well as their spin and cohomological refinements at all roots of unity. New results on the Ohtsuki series and the integrality of quantum invariants are the main applications of our construction.

Introduction

For a simple Lie algebra $\mathfrak{g}$, the quantum invariants of 3–manifolds (the Witten–Reshetikhin–Turaev invariants, see e.g. [18]) are defined only when the quantum parameter $q$ is a certain root of unity. Habiro [5] proposed a construction of a universal $sl_2$ invariant of integer homology 3–spheres ($\mathbb{Z}_{HS}$), dominating all the quantum invariants. The results have many important applications, among them are the integrality of quantum invariants at all roots of unity, the recovery of quantum invariants from the LMO invariant, and the possible applications to the integral Topological Quantum Field Theory. All the results were later extended to all simple Lie algebras by Habiro and the third author [7].

In this paper we extend Habiro’s theory to some classes of rational homology 3–spheres and refined quantum invariants – invariants of spin structures and cohomological classes.

0.1. Universal quantum invariants. The universal invariant of a $\mathbb{Z}_{HS}$ is an element of the Habiro ring

$$\mathbb{Z}[q] := \lim_{n \to \infty} \frac{\mathbb{Z}[q]}{(1 - q)(1 - q^2)\ldots(1 - q^n)}.$$
Every element \( f \in \hat{\mathbb{Z}}[q] \) can be written as an infinite sum
\[
f(q) = \sum_{k \geq 0} f_k(q) (1 - q)(1 - q^2) \cdots (1 - q^n),
\]
with \( f_k(q) \in \mathbb{Z}[q] \). If \( \xi \) is a root of unity, then \( f(\xi) \) is well–defined, since the summands become zero if \( k \) is bigger than the order of \( \xi \). The Habiro ring has remarkable properties and is very suitable for the study of quantum invariants. The result of Habiro and Habiro–Le mentioned above is

**Theorem 1.** (Habiro, Habiro–Le) For every simple Lie algebra \( \mathfrak{g} \) and an integral homology 3–sphere \( M \), there exists an invariant \( I^\mathfrak{g}_M(q) \in \hat{\mathbb{Z}}[q] \), such that if \( \xi \) is a root of unity, then \( I^\mathfrak{g}_M(\xi) \) is the quantum invariant at \( \xi \).

0.1.1. **Applications.** Let us mention the most important consequences of the Habiro’s construction. First of all, each product \( (1 - q)(1 - q^2) \cdots (1 - q^n) \) is divisible by \( (1 - q)^n \), hence it is easy to expand every \( f(q) \in \hat{\mathbb{Z}}[q] \) into formal power series in \( (q - 1) \), denoted by \( T(f) \) and called the Taylor series of \( f(q) \) at \( q = 1 \). One important property of \( \hat{\mathbb{Z}}[q] \) is that \( f \in \hat{\mathbb{Z}}[q] \) is uniquely determined by its Taylor series. In other words, the map \( T : \hat{\mathbb{Z}}[q] \to \mathbb{Z}[[q - 1]] \) is injective. In particular, \( \hat{\mathbb{Z}}[q] \) is an integral domain. Another important property is that every \( f \in \hat{\mathbb{Z}}[q] \) is determined by the values of \( f \) at any infinite set of roots of unity of prime power order. From the existence of \( I^\mathfrak{g}_M \) one can derive the following consequences for \( \mathbb{Z}HS \):

- The quantum invariants at all roots of unity are algebraic integers.
- The quantum invariants at any infinite set of roots of unity of prime power order determine the whole set of quantum invariants.
- Ohtsuki series (see [17, 11]) have integer coefficients and determines the whole set of quantum invariants.
- The Le–Murakami–Ohtsuki invariant (see [12]) totally determines the quantum invariants.

The integrality of quantum invariants was established earlier only at roots of unity of prime order (see [14, 10]). The integrality of the Ohtsuki series for \( \mathfrak{g} = sl_2 \) was proved by Rozansky, using quite a different method.

0.2. **Results.** In this paper we extend Habiro’s construction to some classes of rational homology 3–spheres with cyclic \( H_1(M, \mathbb{Z}) \) and with spin/cohomological structures. Our results show that one can expect to generalize Habiro’s theory to rational homology spheres. For the case \( H_1(M, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \), we get the fullest results, when all aspects of Habiro’s theory are generalized.
We will use a new construction of the universal invariant based on the Laplace transform. This method originates from the paper of the third author [9]. For ZHS, this method reproduces Habiro’s results.

0.2.1. The case $H_1(M,\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Let $\mathcal{M}_b$ be the set of all oriented closed 3-manifolds $M$ with $H_1(M,\mathbb{Z}) = \mathbb{Z}/b\mathbb{Z}$. If $M \in \mathcal{M}_2$, the quantum invariant $\tau_M$ depends on a square root $v$ of $q$.

When $q$ is an even root of unity, then the order of $v$ is divisible by 4. In this case, we put $\tau'_M = \tau_M / \tau_{L(2,1)}$, i.e., we renormalize the $sl_2$ quantum invariant to be 1 for the lens space $L(2,1)$.

When $q$ is an odd root of unity, then $\tau_{L(2,1)} = 0$ but the refined $SO(3)$ version $\tau_{L(2,1)}^{SO(3)} \neq 0$. We choose $v$ to be the root of $q$ which has the same order as $q$ does (i.e., also of odd order). Here we put $\tau'_M = \tau_M^{SO(3)} / \tau_{L(2,1)}^{SO(3)}$.

The role of $\hat{\mathbb{Z}}[q]$ will be replaced by

$$\hat{\mathbb{Z}}[v^2] := \lim_{n \to \infty} \frac{\mathbb{Z}[v^{\pm 1}]}{(-v^2; -v)_{2n}},$$

where

$$(-v^2; -v)_{2n} := \prod_{i=2}^{2n+1} (1+(-v)^i) = (1-v^3)(1-v^5) \ldots (1-v^{2n+1}) \times (1+q)(1+q^2) \ldots (1+q^n).$$

Every $f(v) \in \hat{\mathbb{Z}}[v^2]$ can be written as, with $f_n(v) \in \mathbb{Z}[v^{\pm 1}]$,

$$f(v) = \sum_{n=0}^{\infty} f_n(v) (-v^2; -v)_{2n},$$

If

(1) $v$ is a root of unity of order either odd or divisible by 4

then $f(v)$ is well-defined. For every root $q$ of unity, one can choose a square root $v$ of $q$ satisfying (1). The first main result is

**Theorem 2.** For every closed oriented manifold $M \in \mathcal{M}_2$, there exists an invariant $I_M(v) \in \hat{\mathbb{Z}}[v^2]$, such that if $v$ is a root of unity satisfying (1), then $I_M(v) = \tau'_M(v)$.

Note that $\hat{\mathbb{Z}}[v^2]$ embeds in $\mathbb{Z}[[v-1]]$, via Taylor series. As a consequence, we will prove

**Corollary 3.** For $M \in \mathcal{M}_2$ and the quantum invariants normalized so that the projective space takes value 1, one has

(a) The quantum invariants at all roots of unity are algebraic integers.
(b) The quantum invariants at any infinite set \( \{ v \} \) of roots of unity of odd prime power order determine the whole set of quantum invariants.

(c) The Ohtsuki series, a formal power series in \( q - 1 \), has coefficients in \( \mathbb{Z}[1/2] \), since it is equal to a formal power series in \( v \), and determines the whole set of quantum invariants. If \( v \) is a root of unity of order \( p^d \) with \( p \) an odd prime, then the Ohtsuki series at \( v \) converges \( p \)-adically to the quantum invariant at \( v \).

(d) The Le–Murakami–Ohtsuki invariant determines the quantum invariants at odd roots of 1.

0.2.2. Spin structure and cohomological classes. Suppose that the order of \( v \) is divisible by 4, i.e. the order of \( q \) is even. There are refined quantum invariants \( \tau_{M,\sigma} \), defined in \[8\], where \( \sigma \) is a spin structure or a cohomological class in \( H^1(M, \mathbb{Z}/2) \), depending on whether the order of \( v \) is equal to 0 (mod 8) or 4 (mod 8). We will renormalize \( \tau_{M,\sigma} \) by dividing by the non–refined invariant of the projective space, i.e \( \tau'_{M,\sigma} := \tau_{M,\sigma}/\tau_{L(2,1)} \). Then we have \( \tau'_M = \sum_{\sigma} \tau'_{M,\sigma} \).

Let
\[
\widehat{\mathbb{Z}[v]} := \lim_{n \to -} \frac{\mathbb{Z}[v]}{(1+q)(1+q^2)\ldots(1+q^n)}.
\]

If \( v \) is a root of unity of order divisible by 4, then \( f(v) \) is well–defined for \( f \in \widehat{\mathbb{Z}[v]}_s \). For fixed \( k \), if \( n \geq 2k \) then \((1+q)(1+q^2)\ldots(1+q^n)\) is divisible by \((1+q)^k\), hence there is a natural map, the Taylor series at \( q = -1 \), sending \( f \in \widehat{\mathbb{Z}[v]}_s \) to \( T_{-1} f \in \mathbb{Z}[I][1/2][[q+1]] \), where \( I \) is the unit complex number. Habiro’s theory \[6\] shows that the map \( T_{-1} \) is an embedding.

**Theorem 4.** For \( M \in \mathcal{M}_2 \) and a spin structure (respectively, a cohomological class) \( \sigma \), there exists an invariant \( I_{M,\sigma}(v) \in \frac{1}{1-v}\widehat{\mathbb{Z}[v]}_s \), such that if \( v \) is a root of unity of order divisible by 8 (respectively, equal to 4 (mod 8)), then \( I_{M,\sigma}(v) \) is the quantum invariant of \( (M, \sigma) \) at \( v \).

The integrality of \( \tau_{M,\sigma} \) for \( \mathbb{Z}/p\mathbb{Z} \)-homology spheres at roots of order \( 2p \), where \( p \) is an odd prime and \( \sigma \) is a cohomological class, was studied by Murakami in \[15, 16\]. Theorem \[4\] shows that \((1-v)I_{M,\sigma}(v)\) is always an algebraic integer for all odd \( p \) and that the quantum invariant \( I_{M,\sigma} \) has an expansion as formal power series in \((1+q)\). (The factor \((1-v)\) appears because we use the normalization for which the projective space takes value 1). Theorems \[4, 5\] give partial answers to Conjecture 5.3 and Remark 5.2 in \[15\].
Examples. Suppose $M$ is obtained by surgery on the figure 8 knot with framing 2. Then

$$I_M(v) = \sum_{n=0}^{\infty} v^{-n(n+2)}(-v, -v)_2^n$$

Suppose that the order of $v$ is 0 (mod 8), and this order divided by 8 is $\zeta$ (mod 2). Let $\sigma_0$ be the characteristic spin structure on $M$, and $\sigma_1$ the other one. Then

$$I_{M,\sigma_\epsilon}(v) = \frac{1}{2(1-v)} \sum_{n=0}^{\infty} v^{-n(n+2)} \prod_{i=1}^{n}(1+q^i) \left[ \prod_{i=0}^{n-1} (1-v^{2i+1}) - (-1)^{\zeta+\epsilon} \prod_{i=0}^{n}(1+v^{2i+1}) \right].$$

Assume that $v$ is a $4p$-th root of unity with odd $p$ and $v^{p^2} = \zeta I$, where $I$ is the unit complex number and $\zeta = \pm 1$. Let $\sigma_\epsilon \in H^1(M, \mathbb{Z}/2\mathbb{Z})$, and $\sigma_1$ be trivial. Then

$$I_{M,\sigma_\epsilon}(v) = \frac{1}{2(1-v)} \sum_{n=0}^{\infty} v^{-n(n+2)} \prod_{i=1}^{n}(1+q^i) \left[ \prod_{i=0}^{n-1} (1-v^{2i+1}) + (-1)^{\zeta} \zeta \prod_{i=0}^{n}(1+v^{2i+1}) \right].$$

0.2.3. The case $H_1(M, \mathbb{Z}) = \mathbb{Z}/b\mathbb{Z}$. Let $M \in \mathcal{M}_b$. Assume that the greatest common divisor of $b$ and the order $r$ of $q$ is a power of two. More precisely, we suppose that $b = 2^t c$ and $r = 2^s d$ with odd $c, d$ and $\gcd(c, d) = 1$. If $r$ is even and $t \neq s + 1$, then $\tau_{L(b,1)} \neq 0$ and we can renormalize $\tau'_M = \tau_M/\tau_{L(b,1)}$ and $\tau'_{M,\sigma} = \tau_{M,\sigma}/\tau_{L(b,1)}$. For odd $r$ ($s = 0$), we put $\tau'_M = \tau'_M^{SO(3)}/\tau'_{L(b,1)}$. We show that the Laplace transform method works and leads to formulas for universal quantum invariants and their refinements. As a consequence, we have

**Theorem 5.** Let $b = 2^t c$ with odd $c$. Let $S = \{ 2^s d \in \mathbb{N} : \gcd(c, d) = 1, d \text{ odd}, s \neq t - 1 \}$. Let $M \in \mathcal{M}_b$. The quantum invariants $\tau'_M$ at roots of unity of order $r \in S$ are algebraic integers. If $t > 1$, then also the refined quantum invariants $\tau'_{M,\sigma}$ at even roots of unity of order $r \in S$ are algebraic integers.

0.3. Plan of the paper. The paper is organized as follows. After introducing the Laplace transform method, we apply it to ZHS and get precise formulas for Habiro’s universal invariants. Then we apply this method to QHS with $|H_1| = 2$. Here again the exact formula for the Laplace transform implies various above mentioned results. After that, refinements of quantum invariants are considered. In Section 4, we derive explicit formulas for the spin and cohomological refinements of universal invariants assuming $|H_1| = 2$. In Section 5, we construct refined universal invariants in the case when $H_1 = \mathbb{Z}/b\mathbb{Z}$ and the greatest common divisor of $r$ and $b$ is a power of two.

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1. Laplace transform

In this section we introduce the Laplace transform method.

1.1. Cyclotomic expansion of the colored Jones polynomial. Let $K$ be a knot with framing zero. We denote by $J'_K(\lambda)$ the Jones polynomial of $K$ colored by the $\lambda$–dimensional irreducible representation of $sl_2$, and normalized at one for the unknot. Note that $J'_K(\lambda) \in \mathbb{Z}[q^{\pm 1}]$. In [5], Habiro announced that there exist $C_{K,k} \in \mathbb{Z}[q^{\pm 1}]$ such that

\begin{equation}
J'_K(\lambda) = \sum_{k=0}^{\infty} C_{K,k} (q^{1+\lambda})_k (q^{1-\lambda})_k.
\end{equation}

Here we use the standard notation $(a)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$. The sum in (2) is finite, because the summands with $k \geq \lambda$ are zero. This expansion is called the cyclotomic expansion of the colored Jones polynomial. The nontrivial part here is that $C_{K,k}$’s are Laurent polynomials in $q$ with integer coefficients.

Examples. For the right–, left–handed trefoil and the figure 8 knot, we have

\begin{align*}
J'_{3_1}(\lambda) &= \sum_{k=0}^{\infty} q^{-k(k+2)} (q^{1+\lambda})_k (q^{1-\lambda})_k \\
J'_{\overline{3}_1}(\lambda) &= \sum_{k=0}^{\infty} q^k (q^{1+\lambda})_k (q^{1-\lambda})_k \\
J'_{4_1}(\lambda) &= \sum_{k=0}^{\infty} (-1)^k q^{-\frac{k(k+1)}{2}} (q^{1+\lambda})_k (q^{1-\lambda})_k.
\end{align*}

Note. The coefficients $C_{K,k}$ are computed for all twist knots in [13].

1.2. Quantum invariants for knot surgeries. Let $M = S^3(K_b)$ be a QHS obtained by surgery on $K$ with nonzero framing $b$. Assume that $q$ is a primitive $r$–th root of unity and $r$ is even. The quantum $sl_2$ invariant of $M$ is defined as follows, see [18].

\begin{equation}
\tau_M(q) = \frac{\sum_{\lambda=0}^{r-1} q^{\frac{\lambda(\lambda^2-1)}{4}} (1-q^{\lambda})(1-q^{-\lambda})J'_K(\lambda)}{\sum_{\lambda=0}^{r-1} q^{sn(b)(\lambda^2-1)} (1-q^{\lambda})(1-q^{-\lambda})},
\end{equation}

where $sn(b)$ is the sign of $b$. To be precise, one needs to fix a $4$–th root of $q$. Note that when computing the Jones polynomial of a knot (or a link) in this paper, we
always assume that its framing is zero. However in the formula for the quantum invariant, framing is taken into account by means of the factor $q^{b(\lambda^2-1)/4}$.

Substituting Habiro’s formula (2) into (3) we get

$$
\tau_M(q) = \frac{\sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4} \sum_{n=0}^{\infty} C_{K,n} F_n(q^\lambda, q)}{\sum_{\lambda=0}^{r-1} q^{2n(0)(\lambda^2-1)/4} F_0(q^\lambda, q)},
$$

where $F_n(q^\lambda, q) = (q^\lambda)_{n+1}(q^{-\lambda})_{n+1}$.

Suppose $r$ is odd. Then, taking the sum over odd $\lambda$ in the numerator and the denominator of (3) we get the $SO(3)$ invariant of $M$. In this case, there is no need to fix 4–th root of $q$.

1.3. Laplace transform method. The main idea behind the Laplace transform method is to interchange the sums over $\lambda$ and $n$ in (4) and regard $\sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4}$ as an operator (called Laplace transform) acting on $F_n(q^\lambda, q)$.

More precisely, after interchanging the sums in the numerator of (4) we get

$$
\sum_{n=0}^{r-1} C_{K,n} \sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4} F_n(q^\lambda, q).
$$

Now observe, that $F_n(q^\lambda, q) = (q^\lambda)_{n+1}(q^{-\lambda})_{n+1}$ is a polynomial in two variables $q^\lambda$ and $q$. The Laplace transform does not affect $q$, and we only need to compute the action of Laplace on $q^{a\lambda}$.

Suppose the greatest common divisor of $b$ and $r$ is 1 or 2, and $r$ is even. A simple square completion argument shows that

$$
\sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4} q^{a\lambda} = q^{-\frac{a^2 b^*}{gcd(b, r)}} \gamma_{b,r}
$$

where $b^*$ is an integer such that $b^*b = gcd(b, r)$ (mod $r$), and

$$
\gamma_{b,r} := \sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4}.
$$

Summarizing the previous discussion, we get

$$
\sum_{\lambda=0}^{r-1} q^{b(\lambda^2-1)/4} F_n(q^\lambda, q) = ev_r(L_b(F_n(q^\lambda, q))) \gamma_{b,r}.
$$

Here $L_b(F)$ is the Laplace transform of $F$, which is defined as follows. Suppose $F$ is a formal power series in $q^{\pm 1}$ and $q^{\pm \lambda}$. Then $L_b(F)$ is obtained from $F$ by replacing every $q^{a\lambda}$ by $q^{-a^2/b}$. The evaluation map $ev_r$ converts $q^{1/b}$ to $(q^{1/gcd(b, r)})^{b^*}$. Note that
while $\text{ev}_r$ might depend on $r$, the Laplace transform $L_b$ does not. And also if $b = 1$ or $b = 2$, then $\text{ev}_r$ does not depend on $r$: In these cases, $\text{ev}_r(q^{1/b}) = q^{1/b}$.

If $r$ is odd and $\gcd(b, r) = 1, 2$, we can define the Laplace transform by the same formula (i.e. $q^{a\lambda} \mapsto q^{-a^2/b}$). In this case, we have

$$\sum_{\lambda=1 \text{ odd}}^{r-2} q^{\frac{b(\lambda^2 - 1)}{4}} F_n(q^\lambda, q) = \text{ev}_r(L_b(F_n(q^\lambda, q))) \gamma_{b,r}^1,$$

where

$$\gamma_{b,r}^1 := \sum_{\lambda=1 \text{ odd}}^{r-2} q^{\frac{b(\lambda^2 - 1)}{4}}.$$

As a result, we have closed formulas for quantum invariants in terms of the Laplace transform.

**Theorem 1.1.** Let $M = S^3(K_b)$ and $\gcd(b, r)$ divide 2. Then

$$\tau_M(q) = \frac{1}{2(1 - q^{-sn(b)})} \sum_{n=0}^{\infty} C_{K,n} \text{ev}_r(L_b(F_n)),$$

$$\tau_{SO(3)}^M(q) = \frac{1}{2(1 - q^{-sn(b)})} \sum_{n=0}^{\infty} C_{K,n} \text{ev}_r(L_b(F_n)).$$

2. **Habiro theory**

In this section we show how Theorem 1.1 can be used to compute Habiro’s universal invariants of $\mathbb{Z}_{HS}$.

2.1. **Knot surgeries.** Any knot surgery with framing $b = \pm 1$ yields a $\mathbb{Z}_{HS}$. Combining Theorem 1.1 with Lemma 2.2 below we get the following theorem.

**Theorem 2.1.** (Habiro) For $M_{\pm} = S^3(K_{\pm 1})$, we have

$$\tau_{M_+}(q) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(n+3)}{2}} C_{K,n}(q) \frac{(q^{n+1})_{n+1}}{1 - q},$$

$$\tau_{M_-}(q) = \sum_{n=0}^{\infty} C_{K,n}(q) \frac{(q^{n+1})_{n+1}}{1 - q}.$$

**Remark.** The formulas in Theorem 2.1 do not depend on the order of the root of unity $q$, and, in fact, define elements of the Habiro’s ring which dominate quantum invariants at all roots of unity and, therefore, have to coincide with the Habiro’s universal $sl_2$ invariants of $M_{\pm}$. 
Examples. Denote by $3_{1}$ and $4_{1}$ the Poincare sphere and the 3–manifold obtained by framing 1 surgery on figure 8 knot. By Theorem 2.1, we have

$$
\tau_{3_{1}}(q) = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-(k+2)(3k+1)/2} (q^{k+1})_{k+1}
$$

$$
\tau_{4_{1}}(q) = \frac{q}{1-q} \sum_{k=0}^{\infty} (-1)^k q^{-(k+1)^2} (q^{k+1})_{k+1}
$$

Lemma 2.2.

(5) \hspace{1cm} L_{-1}((q^{\lambda})_{k+1}(q^{-\lambda})_{k+1}) = 2(q^{k+1})_{k+1}.

(6) \hspace{1cm} L_{1}((q^{\lambda})_{k+1}(q^{-\lambda})_{k+1}) = 2(-1)^k q^{-(k+2)(k+1)/2} (q^{k+1})_{k+1}.

Proof. First, note that (6) follows from (5) and

$$
L_{-b}(F_n(q^{\lambda}, q)) = q^{k(k+1)} L_b(F_n(q^{\lambda}, q^{-1})).
$$

Let us prove (5). For this, we split

$$
S_k(q^{\lambda}, q) = \sum_{j=0}^{2k+1} \binom{n}{k} q^{j} q^{-jm-j\lambda} = (-1)^k q^{-k\lambda} q^{k(k+1)/2} (q^{-k})_{2k+1} =
$$

$$
(-1)^k q^{\frac{k(k+1)}{2}} \sum_{j=0}^{2k+1} (-1)^j \binom{2k+1}{j} q^{\frac{j(j-1)}{2}} q^{-k-j}(q^{-j})\lambda
$$

where

$$
\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}.
$$

Taking the Laplace transform we have

$$
L_{-1}(S_k(q^{\lambda}, q)) = (-1)^k q^{\frac{3k^2+1}{2}} \sum_{j=0}^{2k+1} \frac{(q^{-2k+1})_j}{(q)_j} q^{j/2+j-j\lambda}.
$$

The result follows now by applying the Sears–Carlitz transformation (eq. (III.14) in [4]) for terminating $3\phi_2$ series with specializations $a = q^{-2k-1}$, $b, c \to \infty$, $z \to q^{k+2}$. \hfill \square
2.2. Link surgeries. Analogous to the case of knots, Habiro gave an expression for the colored Jones function of links. To introduce his formula we need some notation.

Let \( L \) be an algebraically split framed link of \( l \) components in \( S^3 \) with all framings zero. Let \( n = \{n_1, n_2, \ldots, n_l\} \) be a coloring of \( L \) by \( n \)-dimensional irreducible representations of \( sl_2 \). We denote by \( J_L(n) \) the \( n \)-colored Jones polynomial of \( L \).

We put
\[
J'_L(n) = \frac{J_L(n)}{[n]},
\]
where \([n] = \prod_i [n_i] \) with \([i] = (v^i - v^{-i})/(v - v^{-1})\), and \( v^2 = q \). Theorem 3.3 in [5] implies then the following. More details are given in Appendix.

Proposition 2.3. (Habiro) There exist \( C_{L,k}(v) \in \mathbb{Z}[v^\pm 1] \) such that
\[
J'_L(n) = \sum_{k=0}^{\infty} \left( \sum_{\max k_i = k} C_{L,k}(v) (1 - q)^l \prod_{i=1}^l \frac{(q^{1+n_i})_{k_i}(q^{1-n_i})_{k_i}}{(q^{k+1})_{k_i+1}} \right) \frac{(q^{k+1})_{k+1}}{(1 - q)}
\]

Example. Let \( L \) be the 0–framed Whitehead link.
\[
J'_L(\lambda, \mu) = \sum_{k=0}^{\infty} (-1)^k v^{-k(k+1)}(1 - q)^l \frac{(q^{1+\mu})_{k}(q^{1-\mu})_{k}}{(q^{k+1})_{k+1}} \frac{(q^{1+\lambda})_{k}(q^{1-\lambda})_{k}}{(q^{k+1})_{k+1}}
\]

Let \( M = S^3(L) \) be obtained by surgery on the framed link \( L \) of \( l \) components in \( S^3 \). We denote by \( b_i \) the framing of the \( i \)–th component of \( L \). Let \( \sigma_+ \) (respectively, \( \sigma_- \)) be the number of positive (respectively, negative) eigenvalues of the linking matrix for \( L \). We put
\[
Q_L(n) := J_L(n) \times [n].
\]

By definition, the quantum invariant of \( M \) is
\[
\tau_M(q) = \frac{\sum_{n=0}^{r-1} \prod_{i=1}^l q^{b_i(n^2_i - 1)/4} Q_L(n)}{(\sum_{n=0}^{r-1} q^{(n^2_i - 1)/4[n]^2})^{\sigma_+} (\sum_{n=0}^{r-1} q^{-(n^2_i - 1)/4[n]^2})^{\sigma_-}}
\]

Suppose \( M \) be a \( \mathbb{Z}HS \). Without loss of generality, we can assume that \( L \) is an algebraically split link with framings \( \pm 1 \). Suppose that the first \( \sigma_+ \) components have framing +1, and the others -1. Substituting cyclotomic expansion of the colored Jones polynomial (given in Proposition 2.3) into (7) and applying the Laplace transform method to each component of \( L \), we derive the following formula for the universal \( sl_2 \) invariant of \( M \).
Theorem 2.4. (Habiro) For $M$ as above, we have

$$
\tau_M(q) = \sum_{k=0}^{\infty} \left( \sum_{\max k_i = k} C_{L,k}(v) \prod_{i=1}^{\sigma_+} (1 + (-1)^k q^{k_i} \frac{k_i}{2}) \right) \left( \frac{q^{k+1} + (1 - q)}{} \right).
$$

Again, the right hand side belongs to $\mathbb{Z}[q]$ and defines the universal invariant of Habiro. Note that the $SO(3)$ invariant of $M$ is also given by (8).

3. Rational homology 3–spheres with $|H_1(M)| = 2$

In this section we define universal invariants of $\mathbb{Q}HS$ with $|H_1| = 2$.

3.1. Normalization. Suppose that the order of $v$ is divisible by 4. The projective space, or the lens space $L(2,1)$ should be considered as the unit in this class. It’s easy to show that the quantum invariant of $L(2,1)$ is given by

$$
\tau_{L(2,1)}(v) = \frac{\gamma_{2,r}}{(1 + v^{-1}) \gamma_{1,r}^{1}} = \frac{\gamma_{-2,r}}{(1 + v) \gamma_{-1,r}^{1}}.
$$

For $M \in \mathcal{M}_2$, we will use a normalization such that the projective space $L(2,1)$ takes value 1:

$$
\tau'_M := \frac{\tau_M}{\tau_{L(2,1)}}.
$$

If $v$ is an odd root of unity, we put

$$
\tau'_M := \frac{\tau_M^{SO(3)}}{\tau_{L(2,1)}^{SO(3)}}.
$$

where

$$
\tau_{L(2,1)}^{SO(3)} = \frac{\gamma_{2,r}}{(1 + v^{-1}) \gamma_{1,r}^{1}} = \frac{\gamma_{-2,r}}{(1 + v) \gamma_{-1,r}^{1}}.
$$

3.2. Universal invariants. Let $M_{\pm} = S^3(L)$, where $L$ is an $(l+1)$–component link numbered by $0, 1, \ldots, l$. Assume that the 0–th component has framing $\pm2$, the next $s$ components have framing 1, and the remaining ones have framing $-1$.

Proposition 3.1. For $M_{\pm}$ as above, we have

$$
\tau'_{M_+}(v) = \sum_{k=0}^{\infty} \left( \sum_{\max k_i = k} C_{L,k}(v) (-v)^{-k_0} \prod_{i=k_0+1}^{k} (1 + v^{2i+1}) \prod_{i=1}^{s} (-1)^{k_i} q^{\frac{k_i(k_i+3)}{2}} \right) (v^2; -v)^{2k}.
$$

$$
\tau'_{M_-}(v) = \sum_{k=0}^{\infty} \left( \sum_{\max k_i = k} C_{L,k}(v) \prod_{i=k_0+1}^{k} (1 + v^{2i+1}) \prod_{i=1}^{s} (-1)^{k_i} q^{\frac{k_i(k_i+3)}{2}} \right) (-v^2; -v)^{2k}.
$$
Note that $\tau'_{M_{\pm}} \in \widehat{\mathbb{Z}[v]}_2$. Theorem 2 follows from Proposition 3.1 and Lemma 3.3 below, which states that every $M \in \mathcal{M}_2$ can be obtained from $S^3$ by surgery along a link as described.

**Example.** Let $L$ be the Whitehead link with framings 2 and $-1$. Let $M = S^3(L)$.

$$\tau'_{M}(v) = \sum_{k} v^{-k(2k+2)}(-v^2; -v)_{2k}$$

**Proof.** The proof is again an application of the Laplace transform method, and Lemma 3.2 below. In addition, we use the following identity

$$\frac{(-v^2; -v)_{2k_0}}{(q^{k_0+1})_{k_0+1}} (q^{k_1+1})_{k_1+1} = (-v^2; -v)_{2k} \prod_{i=k_0+1}^{k} (1 + v^{2i+1}),$$

whose proof is left to the reader. Clearly, the formulas in Lemma 3.2 remain true after replacing $\gamma_{b,r}$ with $\gamma^1_{b,r}$ and $\tau_{L(2,1)}(v)$ with $\tau_{S^3(3)}(L(2,1))(v)$.

**Lemma 3.2.**

$$L_{2}[(q^{\lambda})_{k+1}(q^{-\lambda})_{k+1}] \frac{\gamma_{2,r}}{2(1 - q^{-1})\gamma_{1,r}} = (-v)^{-k} (-v^2; -v)_{2k} \tau_{L(2,1)}(v)$$

$$L_{-2}[(q^{\lambda})_{k+1}(q^{-\lambda})_{k+1}] \frac{\gamma_{-2,r}}{2(1 - q)\gamma_{-1,r}} = (-v^2; -v)_{2k} \tau_{L(2,1)}(v)$$

(9)

**Proof.** We proceed by proving (9). By the $q$–binomial theorem we get

$$L_{-2}(F_k(q^\lambda, q)) = 2(-1)^k q^{k^2+k/2} \sum_{j=0}^{2k+1} \frac{(q^{-2k-1})_j}{(q)_j} q^{j^2+j^2/2}.$$ 

The Sears–Carlitz transformation (eq. (III.14) in [4]) with $a = q^{-2k-1}$, $c = -q^{-k}$, $z = q^{k+3/2}$ and $b \to \infty$ reduce this sum to $2\phi_1(-q^{-k-1/2}, q^{-k}; q^{-k+1/2}, q)$ which can be computed by the $q$–Vandermode formula (eq. (II.6) in [4]). As a result, we get

$$L_{-2}(F_k) = (1 - v)(-v^2; -v)_{2k}, \quad L_{2}(F_k) = (-1)^{k+1} v^{-k-1}(1 - v)(-v^2; -v)_{2k}.$$

**Lemma 3.3.** Any $M \in \mathcal{M}_2$ can be obtained from $S^3$ by surgery on an algebraically split link with framing $\pm 2$ on one component and framings $\pm 1$ on the others.
Proof. Choose a loop $K$ representing the nontrivial homology class of $M$. Then $M \setminus K$ has homology of a solid torus. By doing an integral surgery on $K$, we get a $\mathbb{Z}HS$ $M'$. In $M'$, $K$ spans a surface. We shrink the surface to its core, i.e. a 1–dimensional complex. Now $M'$ can be obtained from $S^3$ by surgery on an algebraically split link $L$. Furthermore, $L$ can be isotoped to miss the core of the spanning surface. Hence, $L \cup K$ is an algebraically split surgery link for $M$ satisfying the required conditions. \hfill \square

Proof of Corollary 3. By Theorem 5.4 in [6], there exists an injective homomorphism $\mathbb{Z}_2[v] \to \mathbb{Z}_2[[1-v]]$ generating Ohtsuki series. More details will be given in [2].

4. Refinements

In this section we show that the Laplace transform method can effectively be used also to define refined universal invariants.

Suppose $\sigma$ is a spin structure (respectively, a cohomological class in $H^1(M, \mathbb{Z}/2)$) and the order of $v$ is divisible by 8 (respectively, is equal to 4 (mod 8)), then there is defined the refined invariant $\tau_{M, \sigma}(v)$. We will use the normalization

$$\tau'_{M, \sigma}(v) = \frac{\tau_{M, \sigma}(v)}{\tau_{L(2,1)}(v)}, \quad \tau'_{M} = \sum_{\sigma} \tau'_{M, \sigma}.$$ 

4.1. Spin refinements for $M \in \mathcal{M}_2$. Without loss of generality, we will assume that $M$ is obtained by surgery along the link $L$ of $(l+1)$ components, as described in the previous section. The framing of the 0–th component is $\eta 2$, where $\eta = \pm 1$. Then $M$ has 2 spin structure $\sigma_0$ and $\sigma_1$, corresponding to the two characteristic sublinks: one is the whole $L$ and the other is $L$ with the 0–th component removed.

In this subsection we suppose that $q$ is an $r$–th root of unity of order divisible by 4 and $v^2 = q$. By definition,

$$(10) \quad \tau_{M, \sigma_\varepsilon}(v) = \frac{\sum_{n_0 \equiv \varepsilon (\text{mod } 2)} \sum_{n_1, n_2, \ldots, n_l \text{ even}} q^{\eta(n_0^2-1)/2} \prod_{i=1}^{s} q^{(n_i^2-1)/4} \prod_{i=s+1}^{l} q^{-(n_i^2-1)/4} Q_L(n)}{(\sum_{n=0}^{r-1} q^{(n^2-1)/4})^s + \eta (\sum_{n=0}^{r-1} q^{-(n^2-1)/4})^{l+1-s-\eta}}$$

The next lemma is well–known (compare [8], [1]).

Lemma 4.1. 

$$\sum_{n_1, n_2, \ldots, n_l \text{ even}} \prod_{i=1}^{s} q^{(n_i^2-1)/4} \prod_{i=s+1}^{l} q^{-(n_i^2-1)/4} Q_L(n) =$$
\[
\sum_{n_1, n_2, \ldots, n_l = 0}^{r-1} \prod_{i=1}^{s} q^{(n_i^2-1)/4} \prod_{i=s+1}^{t} q^{-(n_i^2-1)/4} Q_L(n).
\]

**Theorem 6.** Suppose the order \(2r\) of \(v\) is divisible by 8. Let \(r/4 \equiv \zeta \pmod{2}\). Then

\[
\tau'_{M,\sigma_\varepsilon}(v) = \frac{1}{2} \left[ \tau'_{M}(v) - \eta(-1)^{\varepsilon+1} \frac{(1+v)}{(1-v)} \tilde{\tau}_{M}(-v) \right],
\]

where \(\tilde{\tau}_{M}(-v)\) is obtained from \(\tau'_{M}(-v)\) given in Proposition 3.1 by replacing \(C_{L,k}(-v)\) with \(C_{L,k}(v)\).

The proof will be given in the next subsection. It’s easy to see that the right hand side of (11) belongs to \(1_{1-v} \tilde{Z}[v]_{s}\), and define an invariant of 3–manifold \(M \in \mathcal{M}_2\) with a fixed spin structure. This proves the part of Theorem 4 concerning the spin structure.

### 4.2. Proof of Theorem 6

Let us first introduce the odd and even Laplace transforms as follows. For \(\varepsilon = 0, 1\), we put

\[
\gamma^\varepsilon_{b,r} = \sum_{\lambda = \varepsilon \pmod{2}}^{r-1} q^b(\lambda^2-1)/4.
\]

We set

\[
L^\varepsilon_b(P(q^\lambda, v)) := \frac{1}{\gamma_{b,r}} \sum_{\lambda = \varepsilon \pmod{2}}^{r-1} q^b(\lambda^2-1)/4 P(q^\lambda, v)
\]

where \(P(q^\lambda, v)\) is a Laurent polynomial in \(q^\lambda\) and \(v\).

Let us prove Theorem 6 assuming Lemma 4.2 below. To compute the invariant, we need to insert the cyclotomic expansion of the colored Jones polynomial (given in Proposition 2.3) into (10) and use the Laplace transform method. By Lemma 4.1, we need to apply \(L_{\pm 1}\) to all components except of the 0–th one, and \(L_{\pm 2}\) to the 0–th component. From \(L_{\pm 2}^0 + L_{\pm 2}^1 = L_{\pm 2}\) and c) of the lemma we get

\[
L^\varepsilon_{\pm 2} = \frac{1}{2} \left( L_{\pm 2} + (-1)^{\varepsilon+1} (c_1 - c_0) L_{\pm 2}|_{v \rightarrow -v} \right).
\]

The constants \(c_\varepsilon\) are given in the proof of Lemma 4.2. The result follows now from the next two formulas:

\[
\frac{L_2(F_k)|_{v \rightarrow -v}}{2(1-q^{-1})} \frac{\gamma_{2,r}}{\gamma_{1,r}} = -\frac{v^{-k}(1+v)}{1-v} (-v^2; v)_{2k} \tau_{L(2,1)}(v)
\]
\[
\frac{L_{-2}(F_k)|_{v \to -v}}{2(1-q)} \gamma_{-2,r} = \frac{1 + v}{1 - v} (-v^2; v)_{2k} \tau_{L(2,1)}(v)
\]

Lemma 4.2. There exist constants \( c_ε \), independent on \( r \), such that \( c_0 + c_1 = 1 \), and

a) \( \gamma_{\pm 2,r} = c_ε \gamma_{\pm 2,r} \);

b) \( L_{\pm 2}(q^{a\lambda}) = c_{\varepsilon + a} L_{\pm 2}(q^{a\lambda}) \), where \( \varepsilon + a \) is taken modulo 2;

c) \( (L_{\pm 2}^1 - L_{\pm 2}^0)(q^{a\lambda}) = (c_1 - c_0)L_{\pm 2}(q^{a\lambda})|_{v \to -v} \).

Proof. a) By shifting \( \lambda \to \lambda + r/2 \), we see that \( \gamma^0 = 0 \) if \( r = 4p \) (\( p \) odd) or \( \varepsilon = 1 \), and \( \gamma^1 = 0 \) if \( r \) is divisible by 8 or \( \varepsilon = 0 \). This implies \( c_0 = 0 \) in the first case, and \( c_1 = 0 \) in the second one.

b) If \( r = 4p \) (\( p \) odd), we have (compare with the case \( s = t + 1 \) in the next section)

\[
L_{\pm 2}^0(q^{a\lambda}) = \begin{cases} 
    v^{\mp a^2} & \text{for } a = 2k + 1, \ k \in \mathbb{Z} \\
    0 & \text{otherwise}
\end{cases}
\]

\[
L_{\pm 2}^1(q^{a\lambda}) = \begin{cases} 
    v^{\mp a^2} & \text{for } a = 2k, \ k \in \mathbb{Z} \\
    0 & \text{otherwise}
\end{cases}
\]

This proves b) for \( r = 4p \). The other case is similar.

c) From b) we have

\[
(L_{\pm 2}^1 - L_{\pm 2}^0)(q^{a\lambda}) = (-1)^a (c_1 - c_0) L_{\pm 2} = (-1)^a (c_1 - c_0) v^{\mp a^2} = \\
(c_1 - c_0)(-v)^{\mp a^2} = (c_1 - c_0)L_{\pm 2}|_{v \to -v}.
\]

\[\square\]

4.3. Cohomological classes. If \( r = 2 \mod 4 \), then the formula (10) defines cohomological refinements of the quantum invariant. Here \( \sigma_0 \) is the nontrivial cohomological class and \( \sigma_1 \) is the other one. We assume throughout this subsection that \( r = 2p \), with odd \( p \). Then \( v^{p^2} = \zeta I \), where \( I \) is the complex unit and \( \zeta = \pm 1 \).

Theorem 7. Suppose \( v \) is a \( 4p \)-root of unity with \( p \) odd and \( v^{p^2} = \zeta I \).

\[
\tau'_{M,\pm,\sigma_\varepsilon}(v) = \frac{1}{2} \left[ \tau'_{M}(v) + (-1)^\varepsilon \zeta I \frac{1 + v}{1 - v} \right] \tau_{M}(-v)
\]

Proof. The proof is analogous to the proof of Theorem 6 replacing Lemma 4.2 with Lemma 4.3. \[\square\]
It’s easy to see that the right hand side of \( (13) \) belongs to \( \frac{1}{1-v}Z[v]_s \), and define an invariant of 3–manifold \( \mathcal{M} \in \mathcal{M}_b \) with a fixed homological structure. This proves the part Theorem 4 concerning cohomological structures.

**Lemma 4.3.**

\( a) \) \( \gamma^\varepsilon_{2,r} = c_\varepsilon \gamma_{2,r}, \quad \gamma^\varepsilon_{-2,r} = c_{\varepsilon + 1} \gamma_{-2,r}, \quad c_0 + c_1 = 1; \)

\( b) \) \( L^{0}_{\pm 2}(q^{a\lambda}) = \frac{1 \pm \zeta(-1)^a I}{2} L_{\pm 2}(q^{a\lambda}), \)

\( L^{1}_{\pm 2}(q^a) = \frac{1 \pm \zeta(-1)^a I}{2} L_{\pm 2}(q^a); \)

\( c) \) \( L^{1}_{\pm 2}(q^{a\lambda}) - L^{0}_{\pm 2}(q^{a\lambda}) = \pm \zeta I L_{\pm 2}(q^{a\lambda}|_{v \rightarrow -v}). \)

**Proof.** \( a) \) By shifting \( \lambda \rightarrow \lambda + r/2 \), we see that \( \gamma^1_{\pm 2,r} = \pm \zeta I \gamma^0_{\pm 2,r} \). This shows that \( c_0 = (1 - \zeta I)/2 \) and \( c_1 = (1 + \zeta I)/2 \).

\( b) \) From the definition of the odd and even Laplace transforms we have

\[
L^\varepsilon_{\pm 2}(q^{a\lambda}) = \frac{1}{\gamma_{\pm 2,r}} \sum_{\lambda \equiv \varepsilon \mod 2} q^{\pm \left(\frac{\lambda^2 - 1}{2}\right)} q^{a\lambda}.
\]

If \( a \) is even,

\[
L^\varepsilon_{\pm 2}(q^{a\lambda}) = \frac{\gamma^\varepsilon_{\pm 2,r}}{\gamma_{\pm 2,r}} v^{\mp a^2}.
\]

For odd \( a \),

\[
L^\varepsilon_{\pm 2}(q^{a\lambda}) = \frac{\gamma^{\varepsilon + 1}_{\pm 2,r}}{\gamma_{\pm 2,r}} v^{\mp a^2}.
\]

This implies the result.

\( c) \) Follows from \( b) \) analogously to \( c) \) Lemma 4.2.

\( \square \)

5. **Quantum invariants for \( QHS \) with \( H_1 = \mathbb{Z}/b\mathbb{Z} \)**

For \( M \in \mathcal{M}_b \), we show that the Laplace transform method applies in the case, when \( \text{gcd}(b, r) \) is a power of two. This leads to a construction of universal invariants dominating quantum invariants and their refinements at roots of unity of order \( r \) with \( \text{gcd}(b, r) = 2^n \). As a application, we derive new integrality properties of quantum invariants.
5.1. Laplace transforms. We define the odd and even Laplace transforms as follows.

\[
\ev_r(L_b^\epsilon(F_n(q^\lambda, q))) := \frac{1}{\gamma_{b,r}} \sum_{\lambda=\epsilon \bmod 2}^{r-1} q^{\delta(b^2-1)} F_n(q^\lambda, q).
\]

**Proposition 5.1.** For \( b = 2^t c \) and \( r = 2^s d \) with \((c,d) = 1\), and \( t \neq s + 1\), the odd and even Laplace transforms are well-defined.

**Proof.** The proof is by case by case checking.

First assume \( s \geq t + 2 \). Then, by shifting \( \lambda \to \lambda + 2s-td \), we see that \( \gamma_{b,r}^1 = 0 \). Furthermore,

\[
\sum_{\lambda=1 \text{ odd}}^{r-1} q^{b(\lambda^2-1)/4} q^{a\lambda} = \begin{cases} q^{-a^2/b} \gamma_{b,r} & \text{for } a = 2^{t-1} (2k+1), \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]

Indeed,

\[
\sum_{\lambda=1 \text{ odd}}^{r-1} q^{b(\lambda^2-1)/4} q^{a\lambda} = q^{-a^2/b} q^{-b/4} \sum q^{(b\lambda+2a)^2/4b}.
\]

For \( a = 2^l a' \), \( 0 \leq l < s - 1 \), \( a' \) odd, and \( l = s - 1 \), \( a' \) even, it is easy to see that the summands for \( \lambda \) and \( \lambda + r/2^{t+1} \) cancel with each other. For \( a = 2^{t-1} (2k+1) \), the sum is equal to \( q^{-a^2/b} \gamma_{b,r} \).

Analogously, the following formulas define the even Laplace transform.

\[
\sum_{\lambda=0 \text{ even}}^{2r-1} q^{b(\lambda^2-1)/4} q^{a\lambda} = \begin{cases} q^{-a^2/b} \gamma_{b,r} & \text{for } a = 2^t k, \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]

For \( s = t + 1 \), we see that \( \gamma_{b,r}^0 = 0 \) by shifting \( \lambda \to \lambda + 2d \). Moreover,

\[
L_b^0(q^{a\lambda}) = \begin{cases} q^{-a^2/b} & \text{for } a = 2^{t-1} (2k+1), \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]

\[
L_b^1(q^{a\lambda}) = \begin{cases} q^{-a^2/b} & \text{for } a = 2^t k, \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]

If \( s = t \), \( \gamma_{b,r}^1 = \pm I \gamma_{b,r}^0 \) (by shifting \( \lambda \to \lambda + d \)) and

\[
L_b^0(q^{a\lambda}) = \begin{cases} (1 \pm I)/2 q^{-a^2/b} & \text{for } a = 2^{t-1} (2k+1), \ k \in \mathbb{Z} \\ (1 \mp I)/2 q^{-a^2/b} & \text{for } a = 2^t k, \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]

\[
L_b^1(q^{a\lambda}) = \begin{cases} (1 \pm I)/2 q^{-a^2/b} & \text{for } a = 2^{t-1} (2k+1), \ k \in \mathbb{Z} \\ (1 \pm I)/2 q^{-a^2/b} & \text{for } a = 2^t k, \ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}
\]
Finally, for \( s \leq t - 2 \), \( \gamma_{b,r}^0 = -\gamma_{b,r}^1 \) and
\[
L_b^0(q^a\lambda) = L_b^1(q^a\lambda) = \begin{cases} 
1/2 q^{-a^2/b} & \text{for } a = 2^k, \ k \in \mathbb{Z} \\
0 & \text{otherwise} 
\end{cases}
\]
\[
\square
\]

**Note.** If \( t = s + 1 \), we have \( \gamma_{b,r}^0 = -\gamma_{b,r}^1 \) and \( \gamma_{b,r} = 0 \). Hence, the Laplace transform cannot be defined by (14) in this case. But at least if \( d = 1 \), the method applies, if we normalize the Laplace transform by \( \gamma_{b,r}^0 \) instead of \( \gamma_{b,r} \).

### 5.2. Refined universal invariants

Let \( M \in \mathcal{M}_b \). Let \( \gcd(b, r) = 2^n \), \( n \in \mathbb{N} \), i.e., we assume \( b = 2^c \) and \( r = 2^d \) with odd \( c, d \), and \( \gcd(c, d) = 1 \). If \( t \neq s + 1 \), and \( s > 0 \), then
\[
\tau_{L(b,1)} = \frac{L_b(F_0) \gamma_{b,r}}{2(1 - q^{-s(n(b))}) \gamma_{s(n(b)),r}}
\]
is nonzero and we can renormalize
\[
\tau'_M = \frac{\tau_M}{\tau_{L(b,1)}}, \quad \tau'_{M,\sigma_\epsilon} = \frac{\tau_{M,\sigma_\epsilon}}{\tau_{L(b,1)}}.
\]
If \( s = 0 \), \( \tau'_{\text{SO}(3)} \) is always nonzero. In this case, we put
\[
\tau'_M = \frac{\tau'_{\text{SO}(3)}}{\tau_{L(b,1)}}.
\]

Without loss of generality, we assume that \( M = S^3(L) \), where \( L \) is an algebraically split link of \((l + 1)\) components, the framing of the \( 0 \)-th component is \( b \), the next \( p \) components have framing 1, and the remaining ones have framing \(-1\).

**Theorem 5.2.** Suppose \( M \in \mathcal{M}_b \), and \( b = 2^c \), \( r = 2^d \) are as above \((t \neq s + 1)\). If \( s \neq 0 \), the refined quantum invariant of \((M, \sigma_\epsilon)\) is given by the following formula
\[
\tau'_{M,\sigma_\epsilon}(q) = \sum_{k=0}^\infty \left( \sum_{\max k_i = k} C_{L,k}(v) \prod_{i=1}^p (-1)^k i \frac{k_i(k_i+3)}{(q^{k_i+1})_k+1} \frac{L_{b}^\epsilon(F_{k_0})}{L_{b}(F_0)} \right) \left( q^{k+1}\right)_{k+1}
\]
where \( L_{b}^\epsilon \) are defined in the proof of Proposition 5.1. Here \( \sigma_\epsilon \in H^1(M, \mathbb{Z}/2\mathbb{Z}) \) if \( s = 1 \), otherwise \( \sigma_\epsilon \) is a spin structure. If \( s = 0 \),
\[
\tau'_{M}(q) = \sum_{k=0}^\infty \left( \sum_{\max k_i = k} C_{L,k}(v) \prod_{i=1}^p (-1)^k i \frac{k_i(k_i+3)}{2} \frac{L_{b}^\epsilon(F_{k_0})}{L_{b}(F_0)} \right) \left( q^{k+1}\right)_{k+1}
\]

**Example.** Suppose \( L \) is the Whitehead link with framings \(-1\) and \(-4\). Let \( M = S^3(L) \). Then
\[
\tau'_{M,\sigma_\epsilon}(q) = \frac{1}{2} \sum_{k=0}^\infty (-1)^k q^{-k(k+1)/2} L_{-4}^\epsilon(F_k).
\]
5.3. Proof of Theorem Let us first assume that \( s > 0 \). Then \( \tau_M = \tau'_M, \sigma_0 + \tau'_M, \sigma_1 \).

If \( t > 1 \), \( L_b \) sends \( q^{a\lambda} \) to zero if \( a \neq 2t^{-1}k \) with \( k \in \mathbb{Z} \), e.g. if \( a \) is odd. We deduce that \( L_b(F_0) = 2 \). But \( L_b \) is divisible by 2. The result follows.

If \( t = 1 \), \( L_b \) sends \( q^{a\lambda} \) to \( q^{-a^2/b} \) for all \( a \). Then \( L_b(F_0) = 2(1 - x) \) with \( x^{-b} = q \).

We claim that \( L_b(F_k) \) is divisible by \( L_b(F_0) \) for all \( k \in N \). Indeed, \( L_b(F_k) \) can be considered as a polynomial in \( x \). By the \( q \)-binomial formula, we have

\[
L_b(F_k) = 2(-1)^k x^{-bk(k+1)/2} \sum_{j=0}^{2k+1} (-1)^j \left[ \frac{2k+1}{j} \right] x^{-bj(j-1)/2} x^{bkj} x^{j(k-j)^2}.
\]

Then

\[
\lim_{x \to 1} L_b(F_k) = 2(-1)^k \sum_{j=0}^{2k+1} (-1)^j \left[ \frac{2k+1}{j} \right] = 0.
\]

If \( s = 0 \) or \( r \) is odd, then \( \gcd(b, r) = 1 \) and \( L_b^1 \) also sends \( q^{a\lambda} \) to \( q^{-a^2/b} \) for all \( a \).

The same argument shows that \( L_b^1(F_n) \) is divisible by \( 2(1 - x) \).

\[\square\]

Appendix

Here we deduce Proposition 2.3 from the Habiro’s results in [5]. Let \( L \) be an algebraically split link of \( l \) components with all framings zero. Let \( n = \{n_1, n_2, ..., n_l\} \) be a coloring of \( L \) by \( n \)-dimensional irreducible representations of \( sl_2 \).

**Proposition A.** There exist \( C_L,k(v) \in \mathbb{Z}[v^{\pm 1}] \), such that

\[
J_L'(n) = \sum_{k=0}^{\infty} \left( \sum_{\max_k k_i = k} C_L,k(v) (1 - q)^l \prod_{i=1}^l \frac{(q^{1+n_i})_{k_i}(q^{1-n_i})_{k_i}}{(q^{k_i+1})_{k_i+1}} \right) \frac{(q^{k+1})_{k+1}}{(1 - q)}
\]

**Proof.** For a 0–framed link, we have

\[
J_L(n) = (-1)^l - \sum_n \langle L(e_{n-1}) \rangle,
\]

where \( \langle L(e_{n-1}) \rangle \) is the Kauffman bracket of \( L \), where each component is cabled by \( e_{n-1} \) (see [1]). Recall that \( \{e_i\}_{i \geq 0} \) provides a basis for the skein algebra of a solid torus. An other basis is given by elements \( \{R_i\}_{i \geq 0} \)

\[
R_k = \prod_{i=0}^{k-1} (z - \lambda_{2i}), \quad \lambda_i = -v^{i+1} - v^{-i-1},
\]

where \( z \) is the 0–framed closed line \( S^1 \times pt \) in the interior of \( S^1 \times D^2 \), and \( z^i \) means \( i \) parallel copies of \( z \).
The basis change is given by the following formula (compare with \[13\])

\[ (15) \]

\[ e_{n-1} = \sum_{k=0}^{n-1} (-1)^{n-1-k} \left[ \begin{array}{c} n + k \\ n - 1 - k \end{array} \right] R_k, \quad \text{where} \]
\[
\left[ \begin{array}{c} a \\ b \end{array} \right] = \frac{[a]!}{[b]![a-b]!}, \quad [a]! = \prod_{i=1}^{a} \{i\}.
\]

Using (15) we get

\[ J_L(n) = \sum_{k} (-1)^{l-\sum_i(n_i-k_i)} \prod_{i=1}^{l} \left[ \begin{array}{c} n_i + k_i \\ n_i - 1 - k_i \end{array} \right] J_L(R_{k_0}, R_{k_1}, \ldots, R_{k_{l-1}}). \]

The crucial step in the proof is Theorem 3.3 in [5]. The first part of Theorem 3.3 provides the existence of \( c_{L,k} \in \mathbb{Z}[v^{\pm 1}] \) such that

\[ J'_L(n) = \sum_{k} c_{L,k}(1-q)^l \prod_{i=1}^{l} \frac{S(n_i,k_i)}{(q^{k_i+1})_{k_i+1}}, \]

where

\[ S(n,k) = \frac{(n-k)(n-k+1)...(n+k)}{\{n\} \{i\} = v^i - v^{-i}.} \]

The second part of Theorem 3.3 implies the result. \( \square \)

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