Singular Reduction Modules of Differential Equations

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The notion of singular reduction modules, i.e., of singular modules of nonclassical (conditional) symmetry, of differential equations is introduced. It is shown that the derivation of nonclassical symmetries for differential equations can be favourably enhanced by an in-depth prior study of the associated singular modules of vector fields. The form of differential functions and differential equations possessing parameterized families of singular modules is described up to point transformations. Singular cases of finding reduction modules are related to lowering the order of the corresponding reduced equations. As examples, singular reduction modules of evolution equations and second-order quasi-linear equations are studied. Reductions of differential equations to algebraic equations and first-order ordinary differential equations are considered in detail within the framework proposed.

1 Introduction

The “nonclassical” approach to finding solutions of differential equations in closed form was proposed in [4] using the particular example of the (1+1)-dimensional heat equation in order to extend the range of applicability of symmetry methods. Since the end of the 1980s this method was applied to many particular differential equations modeling real-world phenomena, see, e.g., examples in [3, 6, 7, 26] and reviews in [9, 21]. Related objects, which are similar to subalgebras of Lie symmetry algebras, are named in the literature in different ways: nonclassical [16], Q-conditional [9], conditional [12], partial [29] symmetries for short, or involutive families/modules of nonclassical/conditional symmetry operators in a more complete form. The main feature which is inherited by nonclassical symmetries from Lie symmetries is that they allow to construct ansatzes for the unknown function which reduce the corresponding differential equation to differential equations with a smaller number of independent variables [2, 19, 27, 29, 32]. This feature relates nonclassical symmetries to the direct method by Clarkson and Kruskal [5] and the general ansatz method [9]. In fact, however, properties of nonclassical symmetries are more closely connected with theories of differential constraints and formal compatibility of systems of differential equations [14, 19, 27]. This is why we mostly use the term “reduction modules” (of vector fields) instead of “involutive families of conditional symmetry operators” and say that an involutive module of vector fields reduces a differential equation if the equation is reduced by the corresponding ansatz.

Involving the associated invariant surface condition in the conditional invariance criterion gives rise to a few significant complications of nonclassical symmetries in comparison with Lie symmetries. Given a differential equation $\mathcal{L}$, elements of its different reduction modules form no objects of a nice algebraic or differential structure. Hence it is not possible to compose single reduction operators in reduction modules as this is done for the maximal Lie invariance algebra of $\mathcal{L}$ and its subalgebras, which consist of vector fields generating one-parametric Lie symmetry pseudogroups of $\mathcal{L}$. Whereas the system of determining equations for Lie symmetries is linear, similar systems for reduction modules are nonlinear and should additionally be supplemented, in the course of considering modules of dimension greater than one, by the condition of involutivity, i.e., the closure of modules with respect to commutation of vector fields. Moreover, there is
no single system of determining equations even for reduction modules of a fixed dimension. Instead, the entire set of such modules is partitioned into subsets associated with systems of determining equations which are rather different from each other. Solving some of these systems may be equivalent to solving the initial equation, which gives no-go cases of looking for reduction modules. Such no-go cases were known for a number of particular (1+1)-dimensional evolution equations including the linear heat equation \[8, 10, 17, 30\], the Burgers equation \[1, 17\], linear second-order evolution equations \[23, 24\] as well as for the entire classes of (1+1)-dimensional evolution equations \[31\], multi-dimensional evolution equations \[24\] and even systems of such equations \[28\]. Note that in the course of the study of Lie symmetries a similar no-go situation arises for first-order ordinary differential equations \[13\, Section 2.5\]. In fact, all the above no-go cases of reduction operators are occurrences of a no-go case common for evolution equations and one more no-go case specific for linear second-order evolution equations. The causes giving rise to the partition of the module set and to no-go cases for nonclassical symmetries had not been investigated in the literature until recently. It was not understood in what way results on no-go cases can be extended to reduction modules of other, non-evolution, equations.

In \[15\] the partition of the set of reduction modules of a differential equation was related with lowering the order of this equation on the manifolds determined by the associated invariant surface conditions in the appropriate jet space. As a result, studying singular modules of vector fields which lower the order of the equation was included as the initial step in the procedure of finding nonclassical symmetries. In order to explicate the main ideas of the framework proposed, we considered only the case of single partial differential equations in two independent and one dependent variables and single reduction operators. The notion of singular reduction operators was introduced. The weak singularity co-order of a reduction operator \(Q\) were shown to be equal to the essential order of the corresponding reduced equation and the number of essential parameters in the family of \(Q\)-invariant solutions. No-go assertions on singular reduction operators of (1+1)-dimensional evolution and wave equations were derived and then generalized to parameterized families of vector fields which reduce partial differential equations in two independent variables to first-order ordinary differential equations.

In the present paper we extend results of \[15\] to the case of a greater number of independent variables. After giving the basic notions and statements on nonclassical symmetries in Section 2 in Section 3 we introduce the concepts of singular and meta-singular modules of vector fields for differential functions. Any meta-singular module of dimension greater than two proves to be necessarily involutive, in contrast to two-dimensional meta-singular modules. The main result of this section is Theorem 1 which describes, up to point transformations, differential functions possessing meta-singular modules. The analogous notions of weakly singular and meta-singular modules for differential equations are introduced in Section 4 Theorem 2 which characterizes differential equations admitting weakly meta-singular modules, implies that instead of such modules it suffices to study meta-singular modules of the corresponding differential functions. A connection between the weak singularity co-order of reduction operators, the essential order of the corresponding reduced equations and, in the case of reduction to ordinary differential equations, the number of parameters in the corresponding families of invariant solutions is established in Section 5. Revisiting results of \[13\] within the framework of singular reduction modules, in Section 6 we consider the specific case of reduction modules of dimension equal to the number of independent variables, which results in the reduction to algebraic equations. In Section 7 we reformulate and extend no-go results from \[21\] on modules reducing evolution equations to ordinary differential equations with time as the single independent variable. This motivates the consideration of reduction modules of singularity co-order one in Section 8. Supposing that a differential equation \(L\) admits an \(n\)-dimensional meta-singular module \(M\) of singularity co-order one, where \(n\) is the number of independent variables in \(L\), we prove no-go assertions establishing connections between \((n-1)\)-dimensional reduction modules of \(L\) contained in \(M\) and solutions of \(L\). In particular, it is shown that the determining equations for such modules are reduced to
the initial equation $\mathcal{L}$ by a composition of a differential substitution and a hodograph transformation. The final Section 9 is devoted to singular modules for quasi-linear second-order PDEs. Thus, elliptic equations possess no singular modules. Any evolution equation whose matrix of coefficients of second-order derivatives is non-degenerate possesses only singular modules as considered in Section 17 for general evolution equations. Generalized wave equations are much more complicated from this point of view. In particular, they may admit families of singular modules which have no interpretation in terms of meta-singular modules, which makes a further development of the framework of singular modules desirable.

2 Reduction modules of differential equations

In this section, based on [11][12][21][26][32], we briefly recall the required notions and results on nonclassical (conditional) symmetries of differential equations. After substantiating our choice with different arguments, we use the name “reduction modules” instead of “involutive families of nonclassical (conditional) symmetry operators”.

Given a foliated space of $n$ independent variables $x = (x_1, \ldots, x_n)$ and a single dependent variable $u$, consider a finite-dimensional involutive module $Q$ of vector fields in this space, and suppose that the module dimension $p$ of $Q$ (over the ring of smooth functions of $(x, u)$) is not greater than $n$, $0 < p \leq n$. We additionally assume that the module $Q$ satisfies the rank condition, i.e., for each fixed value of $(x, u)$ the projection of $Q$ to the space of $x$ is $p$-dimensional. The attribute ‘involutive’ means that the commutator of any two vector fields from $Q$ belongs to $Q$. It is obvious that any one-dimensional module is involutive. Therefore, in the case $p = 1$ we can omit the attribute ‘involutive’ and talk only about modules.

In what follows the indices $i$ and $j$ run from 1 to $n$, the index $s$ runs from 1 to $p$, and we use the summation convention for repeated indices. Angular brackets $\langle \ldots \rangle$ are used for denoting linear spans over the ring of smooth functions of $(x, u)$. Subscripts of functions denote differentiation with respect to the corresponding variables, $\partial_i = \partial/\partial x_i$ and $\partial_u = \partial/\partial u$. Any function is considered as its zero-order derivative. All considerations are in the local setting.

Suppose that the vector fields $Q_s = \xi^i(x, u)\partial_i + \eta^i(x, u)\partial_u$ form a basis of $Q$, i.e., $Q = \langle Q_1, \ldots, Q_p \rangle$. Then the rank condition is equivalent to the equality $\text{rank}(\xi^i) = p$. The condition that the commutator of any pair of basis elements belongs to $Q$, $[Q_s, Q_{s'}] \in Q$, suffices for the module $Q$ to be involutive. If the vector fields $\bar{Q}_1, \ldots, \bar{Q}_p$ form another basis of $Q$ then there exists a nondegenerate $p \times p$ matrix-function $(\lambda^{ss'}(x, u))$ such that $\bar{Q}_s = \lambda^{ss'}Q_{s'}$.

The first-order differential function $Q_s[u] := \eta^i(x, u) - \xi^i(x, u)u_i$ is called the characteristic of the vector field $Q_s$. In view of the Frobenius theorem, involutivity of $Q$ is equivalent to the fact that the characteristic system $Q[u] = 0$ of PDEs $Q_s[u] = 0$ (also called the invariant surface condition) has $n + 1 - p$ functionally independent integrals $\omega^0(x, u), \ldots, \omega^{n-p}(x, u)$. Therefore, the general solution of this system can implicitly be represented in the form $F(\omega^0, \ldots, \omega^{n-p}) = 0$, where $F$ is an arbitrary smooth function of its arguments.

A differential function $\Theta = \Theta[z]$ of the dependent variables $z = (z^1, \ldots, z^m)$ which in turn are functions of a tuple of independent variables $y = (y_1, \ldots, y_n)$ will be viewed as a smooth function of $y$ and derivatives of $z$ with respect to $y$. The order $r = \text{ord} \Theta$ of the differential function $\Theta$ equals the maximal order of derivatives involved in $\Theta$. The differential function $\Theta$ is defined as a function on a subset of the jet space $J^r = J^r(y[z])$ with independent variables $y$ and dependent variables $z$ [18].

Using another basis of $Q$ gives just another representation of the characteristic system $Q[u] = 0$ with the same set of solutions. This is why the characteristic system $Q[u] = 0$ is associated rather with the module $Q$ than with a fixed basis of $Q$. And vice versa, any family of $n + 1 - p$ functionally independent functions of $x$ and $u$ is a complete set of integrals of the characteristic system of an involutive $p$-dimensional module. Therefore, there exists a one-to-one correspondence between the set of involutive $p$-dimensional modules and the set of families of
n + 1 − p functionally independent functions of x and u, which is factorized with respect to the corresponding equivalence. (We consider two families of the same number of functionally independent functions of the same arguments as equivalent if any function from one of the families is functionally dependent on functions from the other family.)

A function $u = f(x)$ is called invariant with respect to the involutive module $Q$ (or, briefly, $Q$-invariant) if it is a solution of the characteristic system $Q[u] = 0$. This notion is justified by the following facts. In view of the rank condition, we can choose a basis of $Q$ over the underlying field, a $p$-dimensional (Abelian) Lie algebra $g$ of vector fields in the space $(x, u)$. The graph of each solution of the characteristic system $Q[u] = 0$ is obviously invariant with respect to the $p$-parametric local transformation group generated by the algebra $g$.

In view of the rank condition we can assume without loss of generality that $\omega^0 ≠ 0$ and $F_\omega ≠ 0$ and resolve the equation $F = 0$ with respect to $ω^0$: $\omega^0 = ϕ(ω^1, ..., ω^{n−p})$. This representation of the function $u$ is called an ansatz corresponding to the module $Q$.

Next, consider an $r$th order differential equation $L$ of the form $L(x, u(r)) = 0$ for the unknown function $u$ of the independent variables $x = (x_1, ..., x_n)$. Here, $u(r)$ denotes the set of all the derivatives of the function $u$ with respect to $x$ of order not greater than $r$, including $u$ as the derivative of order zero. In the local approach the equation $L$ can be viewed as an algebraic equation in the $r$th order jet space $J^r = J^r(x|u)$ and is identified with the manifold of its solutions in $J^r$:

$$L = \{(x, u(r)) ∈ J^r | L(x, u(r)) = 0]\text{.}$$

We use the same symbol $L$ for this manifold and write $Q_{(r)}$ for the manifold defined by the set of all the differential consequences of the characteristic system $Q[u] = 0$ in $J^r$, i.e.,

$$Q_{(r)} = \{(x, u(r)) ∈ J^r | D^\alpha_1 ... D^\alpha_n Q_s[u] = 0, \ α_i ∈ \mathbb{N} \cup \{0\}, \ |α| = α_1 + ... + α_n < r\},$$

where $D_i = \partial_{x_i} + u_{α+δ_i} \partial_{u_α}$ is the operator of total differentiation with respect to the variable $x_i$, $α = (α_1, ..., α_n)$ is an arbitrary multi-index, and $δ_i$ is the multi-index whose $i$th entry equals 1 and whose other entries are zero. The variable $u_α$ of the jet space $J^r$ corresponds to the derivative $\partial^\alpha u/\partial x_1^{α_1} ... \partial x_n^{α_n}$, and $u_i ≡ u_{δ_i}$.

**Definition 1.** The differential equation $L$ is called conditionally invariant with respect to the involutive module $Q$ if the relation $V_{(r)} L(x, u(r))|_{L∩Q_{(r)}} = 0$ holds for any $V ∈ Q$. This relation is called the conditional invariance criterion. Then $Q$ is called an involutive module of conditional symmetry (or $Q$-conditional symmetry, or nonclassical symmetry, etc.) operators of the equation $L$.

In this definition, $V_{(r)}$ denotes the standard $r$th prolongation of a vector field $V = ξ^i(x, u) \partial_i + η(x, u) \partial_u \text{ [22]}. \quad V_{(r)} = V + \sum_{0<|α|<r} η^α \partial_{u_α}$, where $η^α = D^α_1 ... D^α_n V[u] + ξ^i u_{α+δ_i}$ and $V[u] = η - ξ^i u_i$. It suffices to check the conditional invariance criterion for $V$ running through a basis of $Q$. What basis is chosen for representing the characteristic system $Q[u] = 0$ and checking the conditional invariance criterion is not essential [12, 32].

The equation $L$ is conditionally invariant with respect to the module $Q$ if and only if an ansatz constructed with this module reduces $L$ to a differential equation with $n−p$ independent variables (resp. to an algebraic equation if $n = p$) [22]. Thus, we will briefly call involutive modules of conditional symmetry operators of $L$ reduction modules of $L$. The set of $p$-dimensional reduction modules of the equation $L$ will be denoted by $R_p(L)$.

An alternative approach to conditional invariance is to demand that the joint system of $L$ and $Q_{(r)}$ is formally compatible in the sense of the absence of nontrivial differential consequences [19, 21]. If the conditional invariance criterion is not satisfied but nevertheless the equation $L$ has $Q$-invariant solutions then one talks about weak invariance of the equation $L$ with respect to the module $Q$ [20, 21, 27].
There are reduction modules related to classical Lie symmetries. Let \( g \) be a \( p \)-dimensional Lie invariance algebra of the equation \( \mathcal{L} \), whose basis operators \( Q_1, \ldots, Q_p \) satisfy the condition \( \text{rank}\{\xi^i, \eta^j\} = \text{rank}\{\xi^i, \eta^j\} (= p' \leq p) \). Then the span of \( Q_1, \ldots, Q_p \) over the ring of smooth functions of \((x, u)\) is a \( p'\)-dimensional involutive module which belongs to \( \mathcal{R}^p(\mathcal{L}) \). Modules of this kind are called \textit{Lie reduction modules}. Other reduction modules are called \textit{non-Lie}.

A useful alternative formulation of the conditional invariance criterion is as follows (cf. [32]).

\begin{lemma}
Given an \( r \)-th order differential equation \( \mathcal{L}: L[u] = 0 \), a \( p \)-dimensional (\( 0 < p < n \)) involutive module \( Q \) satisfying the rank condition and differential functions \( \tilde{L}[u] \) and \( \lambda[u] \neq 0 \) of an order not greater than \( r \) such that \( L|_{Q(r)} = \lambda \tilde{L}|_{Q(r)} \), the module \( Q \) is a reduction module of \( \mathcal{L} \) if and only if it is a reduction module of the equation \( \tilde{L}: \tilde{L}[u] = 0 \).
\end{lemma}

The classification of reduction modules can be notably enhanced and simplified by involving Lie symmetry and equivalence transformations of (classes of) differential equations. By \( \mathcal{V}^p \) we denote the set of \( p \)-dimensional modules of vector fields in the space of \((x, u)\). Any point transformation of \((x, u)\) induces a one-to-one mapping of \( \mathcal{V}^p \) into itself via push-forward of vector fields. Namely, the transformation \( g: \tilde{x} = X^i(x, u), \tilde{u} = U(x, u) \) generates \( g_*: \mathcal{V}^p \rightarrow \mathcal{V}^p \) such that for any \( Q \in \mathcal{V}^p \) and \( V \in Q \) the vector field \( V = \xi^i(x, u)\partial_i + \eta(x, u)\partial_u \) is mapped to the vector field \( g_*V = \tilde{\xi}^i(x, u)\partial_i + \tilde{\eta}(x, u)\partial_u \), where \( \xi^i(x, u) = VX^i(x, u), \tilde{\eta}(x, u) = VU(x, u) \).

Given a group \( G \) of point transformations in the space of \((x, u)\), the modules \( Q \) and \( \tilde{Q} \) \((of the same dimension)\) are called \textit{equivalent} with respect to \( G \) if there exists some \( g \in G \) such that \( \tilde{Q} = g_*Q \).

\begin{lemma}
Suppose that \( Q \in \mathcal{R}^p(\mathcal{L}) \), a point transformation \( g \) maps an equation \( \mathcal{L} \) to an equation \( \tilde{\mathcal{L}} \) and the image \( g_*Q \) satisfies the rank condition. Then \( g_*Q \in \mathcal{R}^p(\tilde{\mathcal{L}}) \).
\end{lemma}

\begin{corollary}
Let \( G \) be the point symmetry group of an equation \( \mathcal{L} \). Then the equivalence of \( p \)-dimensional modules of vector fields with respect to the group \( G \) generates an equivalence relation in \( \mathcal{R}^p(\mathcal{L}) \).
\end{corollary}

Next, we consider a class \( \mathcal{L}|_S \) of equations \( \mathcal{L}_\theta: L(x, u(r), \theta) = 0 \) parameterized by the parameter-functions \( \theta = \theta(x, u(r)) \). Here \( L \) is a fixed function of \( x, u(r) \) and \( \theta \). By \( \theta \) we denote the tuple of arbitrary (parametric) differential functions \( \theta(x, u(r)) = (\theta^1(x, u(r)), \ldots, \theta^k(x, u(r))) \) traversing the set \( \mathcal{S} \) of solutions of a system. This system consists of differential equations \( S(x, u(r), \theta(q)(x, u(r))) = 0 \) and differential inequalities \( \Sigma(x, u(r), \theta(q)(x, u(r))) \neq 0 \) \((> 0, < 0, \ldots)\) on \( \theta \), where \( x \) and \( u(r) \) play the role of independent variables and \( \theta(q) \) stands for the set of all the derivatives of \( \theta \) of order not greater than \( q \). Henceforth we call the functions \( \theta \) arbitrary elements. We write \( G^\sim \) for the point transformation group preserving the form of the equations from \( \mathcal{L}|_S \).

By \( P \) we denote the set of all pairs consisting of an equation \( \mathcal{L}_\theta \) from \( \mathcal{L}|_S \) and a module \( Q \) from \( \mathcal{R}^p(\mathcal{L}_\theta) \). It follows from Lemma \ref{lemma2} above that the action of transformations from the equivalence group \( G^\sim \) on \( \mathcal{L}|_S \) and \( \{\mathcal{R}^p | \theta \in \mathcal{S}\} \) induces an equivalence relation on \( P \) [25].

\begin{definition}
Let \( \theta, \theta' \in \mathcal{S}, Q \in \mathcal{R}^p(\mathcal{L}_\theta), Q' \in \mathcal{R}^p(\mathcal{L}_{\theta'}) \). The pairs \( (\mathcal{L}_\theta, Q) \) and \( (\mathcal{L}_{\theta'}, Q') \) are called \textit{\( G^\sim \)-equivalent} if there exists \( g \in G^\sim \) such that \( g \) transforms the equation \( \mathcal{L}_\theta \) to the equation \( \mathcal{L}_{\theta'} \), and \( Q' = g_*Q \).
\end{definition}

We will interpret the classification of reduction operators with respect to \( G^\sim \) as the classification in \( P \) with respect to this equivalence relation, a problem which can be investigated similarly to the usual group classification in classes of differential equations. Namely, at first we construct the reduction modules that are defined for all values of \( \theta \). Then we classify, with respect to \( G^\sim \), the values of \( \theta \) for which the equation \( \mathcal{L}_\theta \) admits additional reduction modules.
3 Singular modules of vector fields for differential functions

Let $L = L[u]$ be a differential function of order $\text{ord} L = r$ (i.e., a smooth function of the independent variables $x = (x_1, \ldots, x_n)$ and derivatives of $u$ with respect to $x$ up to order $r$) and let $Q$ be a $p$-dimensional ($0 < p < n$) involutive module which is generated by the vector fields $Q_s = \xi^i(x,u)\partial_i + \eta^i(x,u)\partial_u$ defined in the space of $(x,u)$ and satisfying the rank condition $\text{rank}(\xi^i) = p$.

**Proposition 1.** Suppose that $L = L[u]$ is a differential function of $u = u(x)$, $x = (x_1, \ldots, x_n)$, and let $Q$ be an involutive module of vector fields defined in the space of $(x,u)$, which is of dimension less than $n$ and satisfies the rank condition. Then $\text{soc}_L Q \leq \text{soc}_L Q$ for any involutive submodule $Q$ of $Q$. In particular, the module $Q$ is singular for $L$ if it contains a submodule singular for $L$.
Proof. If $\hat{Q}$ is an involutive submodule of $Q$ then it necessarily satisfies the rank condition and $Q_{(r)} \subseteq \hat{Q}_{(r)}$, where $r = \text{ord} \ L$. If the differential function $L$ coincides with a differential function $\hat{L}$ on the manifold $\hat{Q}_{(r)}$, the same is true on the manifold $Q_{(r)}$. Therefore, $\text{sco}_{L} \ Q \leq \text{sco}_{L} \ \hat{Q}$. □

**Definition 4.** A $(p+1)$-dimensional module $M$ is called *meta-singular* for the differential function $L$ if any $p$-dimensional involutive submodule of $M$ which satisfies the rank condition is singular for $L$ and the module $M$ contains a family $\mathfrak{M} = \{Q^\Phi\}$ of such submodules parameterized by an arbitrary function $\Phi = \Phi(x, u)$ of all independent and dependent variables. The *singularity co-order* of the meta-singular module $M$ is the maximum of the singularity co-orders of its involutive $p$-dimensional submodules satisfying the rank condition.

Here and in what follows the parameterization by an arbitrary function is understood as a parameterization modulo functions with a smaller number of arguments. The *singularity co-order* of the family $\mathfrak{M} = \{Q^\Phi\}$ is also defined as the maximum of the singularity co-orders of its elements.

The condition that the module $M$ is involutive is not explicitly included in Definition 4. By this definition, the meta-singular module $M$ should only contain a family of $p$-dimensional involutive submodules parameterized by an arbitrary function of all variables. At the same time, in the case $p \geq 2$ this is equivalent to the fact that the module $M$ is involutive.

**Proposition 2.** A $(p+1)$-dimensional module $M$, where $p \geq 2$, contains a family of $p$-dimensional involutive submodules parameterized by an arbitrary function of all independent and dependent variables if and only if the module $M$ is involutive.

Proof. Suppose that the module $M$ contains a family $\mathfrak{M}$ of $p$-dimensional involutive submodules parameterized by an arbitrary function of all variables modulo functions with a smaller number of arguments. Then we can choose a basis $\{Q_0, \ldots, Q_p\}$ of $M$ in such a way that the vector fields $Q_1, \ldots, Q_p$ generate an involutive submodule of $M$ from $\mathfrak{M}$ and, moreover, commute. By a change of variables $(x, u)$ these vector fields are reduced to shift operators, $Q_s = \partial_s$. Up to combining with $Q_s$ and multiplying by a nonvanishing function, the vector field $Q_0$ can be chosen in the form $Q_0 = \xi_0(x, u)\partial_0 + \eta_0(x, u)\partial_u$, where one of the coefficients $\xi_0, \ i = p + 1, \ldots, n$, or $\eta_0$ is equal to 1. The entire set of $p$-dimensional submodules of $M$ is partitioned into the subsets

$$S_0 = \{Q^\theta = (Q_0 + \theta^s Q_0, \ s = 1, \ldots, p)\},$$

$$S_s = \{Q^{\theta^s} = (Q_0, \ldots, Q_{s-1}, Q_{s} + \theta^s Q_s, \ s = s + 1, \ldots, p)\},$$

where $\theta = (\theta^1, \ldots, \theta^p)$, $\theta^s = (\theta^{s+1}, \ldots, \theta^p)$, and all $\theta$’s run through the set of smooth functions of $(x, u)$. By construction, if $Q^{\theta^s}$ is an involutive module then the chosen basis elements commute. This implies that the components of $\theta^s$ satisfy the equations $Q_0 \theta^{s'} = 0, \ldots, Q_{s-1} \theta^{s'} = 0$, $s' = s + 1, \ldots, p$, and can hence be expressed via functions of at most $n + 1 - s$ arguments. Therefore, the module $M$ contains a family $\mathfrak{M}$ of $p$-dimensional involutive submodules parameterized by an arbitrary function of all variables if and only if such a family is a subset of $S_0$.

If a module $Q^{\theta}$ from $S_0$ is involutive, by the Frobenius theorem the overdetermined system of the first-order linear partial differential equations $Q_s \Phi + \theta^s Q_0 \Phi = 0$ with respect to the unknown function $\Phi = \Phi(x, u)$ has a solution with $Q_0 \Phi \neq 0$, i.e., the parameter-functions $\theta^s$ can

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1 This can be seen as follows: Suppose that $R = \langle R_1, \ldots, R_p \rangle$ is a $p$-dimensional submodule of $M$ and write $R_s = \sum_{k=0}^{p} \lambda^{sk} Q_s$, where $\lambda^{sk}$ are smooth functions of $(x, u)$. Set $\Lambda := (\lambda^{sk})_{k=0}^{1, \ldots, p}$ and $\Lambda' := (\lambda^{sk})_{k=1}^{1, \ldots, p}$. We have rank $\Lambda = p$. If rank $\Lambda' = p$, by elementary row operations we may generate a basis of $R$ that identifies it as an element of $S_0$. Otherwise we may transform $\Lambda$ into a form with $\lambda^{10} = 1$ and all other entries of the first column and first row vanishing. Applying the same reasoning as before to the resulting lower right submatrix we obtain either a basis of type $S_1$ or we may again simplify the first column and first row of this submatrix as before. Thus the claim follows by induction.
be represented in the form $\theta^s = -\Phi_s/Q_0 \Phi$. As the basis elements $Q_s + \theta^s Q_0$ of the module $Q^\beta$ should commute, we have

$$[Q_s + \theta^s Q_0, Q_{s'} + \theta^{s'} Q_0] = (\theta_s' + \theta^s Q_0 \theta^{s'} - \theta_{s'} - \theta^s \theta^{s'} Q_0) Q_0 + \theta^s Q_0 s - \theta^{s'} Q_0 s' = 0,$$

$$\Phi_s' Q_0 s \Phi - \Phi_s Q_0 s' \Phi = 0, \quad \lambda^{s \bar{s}} \Phi_s = \lambda^{s \bar{s}} \Phi_{s'}, \quad \bar{s} = 1, \ldots, \bar{p}.$$

As some of the coefficients $\lambda^{s \bar{s}}$ are necessarily nonzero, in this case the function $\Phi$ runs at most through the solution set of a system of first-order linear partial differential equations and, therefore, the number of its arguments is in fact less than $n + 1$. This contradicts the existence of a family of $p$-dimensional involutive submodules of $M$ parameterized by an arbitrary function of all variables modulo functions with a smaller number of arguments.

As a result, all the vector fields $Q_{0,s}$ are zero. Let vector fields $\bar{Q}_1, \ldots, \bar{Q}_{\bar{p}}$ form a basis of the module $\langle Q_{0,s} \rangle$. Then the vector fields $Q_0, \bar{Q}_1, \ldots, \bar{Q}_{\bar{p}}$ are linearly independent, the vector fields $Q_{0,s}$ are represented as $\sum_{s=1}^{\bar{p}} \lambda^{s \bar{s}} \bar{Q}_s$ for some smooth functions $\lambda^{s \bar{s}} = \lambda^{s \bar{s}}(x,u)$ and the above commutation relations imply the following system for the function $\Phi$:

$$\Phi_s Q_0 s \Phi = \Phi_s Q_0 s' \Phi = 0, \quad \lambda^{s \bar{s}} \Phi_s = \lambda^{s \bar{s}} \Phi_{s'}, \quad \bar{s} = 1, \ldots, \bar{p}.$$

Conversely, if the module $M$ is involutive, we choose a basis which consists of commuting vector fields $Q_0, \ldots, Q_p$. Each submodule $Q^\beta = \langle Q_s \rangle/(Q_0 \Phi)$ is involutive and of dimension $p$. Here $\Phi = \Phi(x,u)$ runs through the set of smooth functions of $(x,u)$ with $Q_0 \Phi \neq 0$, i.e., the complement of the solution set of the equation $Q_0 \Phi = 0$. The general solution of this equation is parameterized by a single function of $n$ arguments. The modules $Q^\Phi$ and $Q^{\bar{\Phi}}$ associated with the functions $\Phi$ and $\bar{\Phi}$ coincide if and only if $(Q_0 \Phi) Q_{s} \Phi = (Q_s \Phi) Q_0 \Phi$. We will call such functions $\Phi$ and $\bar{\Phi}$ equivalent. Thus, the set traversed by $\Phi$ should additionally be factorized with respect to this equivalence relation. A necessary and sufficient condition for the equivalence of functions $\Phi$ and $\bar{\Phi}$ is that there exists a smooth function $F$ of $n - p + 1$ arguments for which $\Phi = F(\omega^1, \ldots, \omega^{n-p}, \Phi)$, where $\omega^1, \ldots, \omega^{n-p}$ form a complete set of functionally independent solutions of the system $Q_0 \omega = 0, Q_s \omega = 0$. As the number of arguments of $F$ is less then $n + 1$, the factorization does not affect the degree of arbitrariness of $\Phi$.

**Remark 1.** Families of involutive submodules belonging to the set $S_0$ can be parameterized in different ways. Thus, in the above proof these submodules are parameterized via the representation of the coefficients $\theta^s$ in the form $\theta^s = -Q_s \Phi/Q_0 \Phi$, and the set traversed by $\Phi = \Phi(x,u)$ should be factorized with respect to an equivalence relation. Alternatively, we can choose, e.g., any of the $\theta^s$ instead of $\Phi$ as a parameterized function. After $\Psi^1 = \theta^1$ is chosen as such a function, the condition $[Q_1 + \theta^1 Q_0, Q_2 + \theta^2 Q_0] = 0$ implies the first-order linear partial differential equation $Q_1 \theta^2 + \theta^1 Q_0 \theta^2 = Q_2 \theta^1 + \theta^2 Q_0 \theta^1$ with respect to $\theta^2$, whose general solution is parameterized by an arbitrary function $\Psi^2$ of $n$ arguments. In the same way, the condition $[Q_i + \theta^i Q_0, Q_s + \theta^s Q_0] = 0, i = 1, 2$, gives the system of two first-order linear partial differential equations $Q_i \theta^3 + \theta^i Q_0 \theta^3 = Q_3 \theta^i + \theta^3 Q_0 \theta^i$. This system is associated with the module generated by the vector fields $Q_i + \theta^i Q_0 + (Q_s \theta^3 + \theta^3 Q_0 \theta^3) \partial_{\theta^3}$, which is involutive in view of the equation for $\theta^3$. It follows from the Frobenius theorem that the general solution of the system is parameterized by an arbitrary function $\Psi^3$ of $n - 1$ arguments. Iterating the procedure for each $\theta^s$, we precisely parameterize involutive modules from the submodule set $S_0$ by the tuple $(\Psi^1, \ldots, \Psi^p)$, where $\Psi^s$ runs through the set of smooth functions of $n + 2 - s$ arguments.

**Corollary 2.** Any $(p+1)$-dimensional meta-singular module $M$ for a differential function $L$, where $p \geq 2$, is involutive.
Proposition 3. A \((p+1)\)-dimensional module \(M\), where \(p \geq 2\), is meta-singular for a differential function \(L\) if and only if it contains, up to point transformations in the space of \((x,u)\), a family \(\mathfrak{M}\) of \(p\)-dimensional involutive submodules singular for \(L\) of the form

\[ Q^\Phi = \langle \partial_s - (\Phi_s/\Phi_u)\partial_u, \ s = 1, \ldots, p \rangle \]

parameterized by an arbitrary smooth function \(\Phi = \Phi(x,u)\) of all independent and dependent variables with \(\Phi_u \neq 0\). The singularity co-order of the family \(\mathfrak{M}\) for the differential function \(L\) coincides with that of the entire module \(M\), \(\text{soc}_{L} \mathfrak{M} = \text{soc}_{L} M\).

**Proof.** Suppose that the module \(M\) is meta-singular for a differential function \(L\). As the module \(M\) is involutive in view of Corollary 2, we can choose a basis of \(M\) consisting of commuting vector fields \(Q_0, \ldots, Q_p\) such that the vector fields \(Q_1, \ldots, Q_p\) satisfy the rank condition \(\text{rank}(\xi^s) = p\). A change of the variables \((x,u)\) reduces these vector fields to shift operators, \(Q_s = \partial_s\) and \(Q_0 = \partial_u\). A submodule of the form \(Q^\phi = \langle Q_s + \theta^sQ_0, \ s = 1, \ldots, p \rangle\) of \(M\) is involutive if and only if the coefficients \(\theta^s\) can be represented in the form \(\theta^s = -\Phi_s/\Phi_u\) for some function \(\Phi = \Phi(x,u)\) with \(\Phi_u \neq 0\), cf. the proof of Proposition 2. Thus, the family \(\mathfrak{M} = \{Q^\phi = \langle \partial_s - (\Phi_s/\Phi_u)\partial_u, \ s = 1, \ldots, p \rangle\}\) parameterized by the arbitrary function \(\Phi\) with \(\Phi_u \neq 0\) is of the required form.

Conversely, let the module \(M\) contain, up to point transformations in the space of \((x,u)\), such a family \(\mathfrak{M} = \{Q^\phi\}\) and let \(\text{soc}_{L} \mathfrak{M} = k < r = \text{ord} L\). Hence we have that \(\text{soc}_{L} Q^\phi \leq k\) for any allowed value of the parameter-function \(\Phi\). In the initial coordinates, the basis elements of a submodule \(Q^\phi\) take the form \(Q_s - (Q_s/\Phi_u)Q_0\), where \(Q_0, \ldots, Q_p\) are commuting vector fields. It suffices to prove that any \(p\)-dimensional involutive submodule \(P\) of \(M\), which satisfies the rank condition and does not belong to the family \(\{Q^\phi\}\), is singular for \(L\). Up to a permutation of the vector fields \(Q_1, \ldots, Q_p\), a basis of \(P\) consists of the vector fields \(Q_0\) and \(Q_s' + \theta^{s'}Q_1, \ s' = 2, \ldots, p\), where the coefficients \(\theta^{s'}\) are smooth functions of \((x,u)\). For convenience, by a point transformation of the variables \((x,u)\) we reduce the vector fields \(Q_0, Q_2, \ldots, Q_p\) and \(Q_1\) to the shifts operators \(\partial_1, \partial_2, \ldots, \partial_p\) and \(\partial_u\), respectively. As the submodule \(P\) is involutive, the coefficients \(\theta^{s'}\) possess the representation \(\theta^{s'} = -\Psi_{s'}/\Psi_u\), where \(\Psi\) is a smooth function of \(x_2, \ldots, x_n\) and \(u\) with \(\Psi_u \neq 0\). Consider the family of involutive modules of the form

\[ Q^\varepsilon = (Q_0 + \varepsilon Q_1, Q_{s'} + \theta^{s'}\varepsilon Q_1, \ s' = 2, \ldots, p) \]

parameterized by a constant \(\varepsilon\) running through a neighborhood of zero. Here the coefficients \(\theta^{s'}\varepsilon\) are obtained via replacing the argument \(u\) in \(\theta^{s'}\) by \(u - \varepsilon x_1\), \(\theta^{s'} = \theta^{s'}(x_2, \ldots, x_n, u - \varepsilon x_1)\).

For each nonzero value of \(\varepsilon\) the module \(Q^\varepsilon\) belongs to the family \(\{Q^\phi\}\). This is obvious after the transition from the chosen basis to the basis \(\{Q_1 + \varepsilon^{-1}Q_0, Q_{s'} - \varepsilon^{-1}\theta^{s'}\varepsilon Q_0\}\). Therefore, any module \(Q^\varepsilon\) with \(\varepsilon \neq 0\) is singular for the differential function \(L\). Consider the differential function \(L^\varepsilon\) which is associated with \(L\) on the manifold \(Q^r\) via the exclusion of the derivatives \(u_{\alpha_1} + \cdots + u_{\alpha_p} > 0\) from \(L\) using the equations \(u_1 = \varepsilon, \ u_{s'} = \theta^{s'}\varepsilon\) and their differential consequences. The function \(L^\varepsilon\) is smooth in the totality of the parameter \(\varepsilon\), the variables \(x\) and derivatives of \(u\) with respect to \(x_{p+1}, \ldots, x_n\). As ord \(L^\varepsilon \leq k\) for any \(\varepsilon \neq 0\), the same statement is true for \(\varepsilon = 0\) through continuity. This means that the submodule \(P = Q^\varepsilon|_{\varepsilon=0}\) is singular for \(L\), and \(\text{soc}_L P \leq k = \text{soc}_L \mathfrak{M}\).

It is obvious that \(\text{soc}_L M \leq \text{soc}_L \mathfrak{M}\). As any \(p\)-dimensional involutive submodule \(Q\) of \(M\) satisfies the inequality \(\text{soc}_L Q \leq \text{soc}_L \mathfrak{M}\), we obtain that \(\text{soc}_L M = \text{soc}_L \mathfrak{M}\).

The case of two-dimensional meta-singular modules is special. As any one-dimensional module of vector fields is involutive, the fact that a two-dimensional module is meta-singular for a differential function does not imply that this module is involutive. More specifically, the conclusion of Proposition 2 is not true if \(p = 1\). This is why the reduced form of submodules of a two-dimensional meta-singular module depends on whether this module is involutive or not.
Proposition 4. A two-dimensional module \( M \) is meta-singular for a differential function \( L \) if and only if it contains, up to point transformations in the space of \((x,u)\), a family \( \mathfrak{M} \) of one-dimensional submodules singular for \( L \) with basis vector fields in a reduced form parameterized by an arbitrary smooth function \( \theta = \theta(x,u) \). The reduced form is

\[
\partial_1 + \theta \partial_u \quad \text{or} \quad \partial_1 + u \partial_2 + \xi^3 \partial_3 + \cdots + \xi^n \partial_n + \theta \partial_u
\]

if the module \( M \) is involutive or not involutive, respectively. Here \( \xi^i = \xi^i(x,u), \ i = 3, \ldots, n \) are fixed smooth functions. The singularity co-order of the family \( \mathfrak{M} \) for the differential function \( L \) coincides with that of the entire module \( M \), \( \text{sco}_L \mathfrak{M} = \text{sco}_L M \).

In order to make the further consideration for \( p = 1 \) consistent with the case of \( p \geq 2 \), the parameter-function \( \theta \) can be represented in the form \( \theta = -\Phi_1/\Phi_n \), where \( \Phi = \Phi(x,u) \) is an arbitrary smooth function with \( \Phi_u \neq 0 \).

Proof. If the module \( M \) is involutive, the proof is similar to that of Proposition 2. We therefore only consider the case when the module \( M \) is not involutive.

Let the module \( M \) be meta-singular for a differential function \( L \). We choose a basis \( \{Q_0, Q_1\} \) of \( M \) such that the vector field \( Q_1 \) satisfies the rank condition \( \text{rank}(\xi^1) = 1 \) and reduce the vector field \( Q_0 \) by a change of the variables \((x,u)\) to the shift operator with respect to \( u \). \( Q_0 = \partial_u \). Up to permutation of the variables \( x_1, \ldots, x_p \), we can assume that \( \xi^1 \neq 0 \). Then we replace the basis element \( Q_1 \) by \( (\xi^1)^{-1}(Q_1 - \eta^1 Q_0) \) in order to set \( \eta^1 = 0 \) and \( \xi^1 = 1 \). As the module \( M \) is not involutive, the commutator \([Q_0, Q_1] = \xi^1 u \partial_2 + \cdots + \xi^n \partial_n \) does not vanish. Hence we can assume up to permutation of the variables \( x_2, \ldots, x_p \) that \( \xi^2 \neq 0 \). The change of variables \( x^s = x^s \) and \( u^s = \xi^1(x,u) \) with the simultaneous replacement of \( Q_0 \) by \( (\xi^1)^{-1}Q_0 \) reduces the basis elements of \( M \) to the form \( Q_0 = \partial_u \) and \( Q_1 = \partial_1 + u \partial_2 + \xi^2 \partial_3 + \cdots + \xi^n \partial_n \). Then the family \( \mathfrak{M} = \{Q_1 + \theta Q_0\} \) of one-dimensional submodules of \( M \) singular for \( L \), where the parameter \( \theta = \theta(x,u) \) runs through the set of smooth functions of all independent and dependent variables, is of the required form.

Conversely, let the module \( M \) contain, up to point transformations in the space of \((x,u)\), such a family \( \mathfrak{M} \) with \( \text{sco}_L \mathfrak{M} = k < r = \text{ord} L \). Hence we have that \( \text{sco}_L \langle Q_1 + \theta Q_0 \rangle \leq k \) for any value of the parameter-function \( \theta \). After returning to the initial coordinates, it suffices to prove that the submodule \( \langle Q_0 \rangle \) of \( M \) is singular for \( L \) if the vector field \( Q_0 \) also satisfies the rank condition in these coordinates. For convenience we reduce \( Q_0 \) by a change of coordinates to the shift operator \( \partial_1 \). Consider the family of modules of the form \( Q^\varepsilon = \langle Q_0 + \varepsilon(Q_1 - \xi^1 Q_0) \rangle \) parameterized by a constant \( \varepsilon \) running through a neighborhood of zero. For each nonzero value of \( \varepsilon \) the module \( Q^\varepsilon \) belongs to the family \( \mathfrak{M} \) as in this case we have \( Q^\varepsilon = (Q_1 + (\varepsilon^{-1} - \xi^1)Q_0) \). Therefore, any module \( Q^\varepsilon \) with \( \varepsilon \neq 0 \) is singular for the differential function \( L \). Consider the differential function \( \tilde{L}^\varepsilon \) which is associated with \( L \) on the manifold \( Q^\varepsilon(r) \) via the exclusion of the derivatives \( u \) with \( \alpha_1 > 0 \) from \( L \) using the equation \( u_1 = \varepsilon(\eta^{11} - \xi^{12} u_2 - \cdots - \xi^{1n} u_n) \) and its differential consequences. The function \( \tilde{L}^\varepsilon \) is smooth in the totality of the parameter \( \varepsilon \), the variables \( x \) and derivatives of \( u \) with respect to \( x_2, \ldots, x_n \). As the order of \( \tilde{L}^\varepsilon \) is not greater than \( k \) for any nonzero \( \varepsilon \), the same statement is true for \( \varepsilon = 0 \) by continuity. This means that the submodule \( \langle Q_0 \rangle = \tilde{Q}^0 |_{\varepsilon=0} \) is singular for \( L \), and \( \text{sco}_L \langle Q_0 \rangle = k = \text{sco}_L \mathfrak{M} \).

The set of one-dimensional submodules of \( M \) is exhausted by \( Q_0 \) and the elements of the family \( \mathfrak{M} \). Hence \( \text{sco}_L \mathfrak{M} = \text{sco}_L M \). \( \square \)

Theorem 1. An \( r \)th order differential function \( L \) with one dependent and \( n \) independent variables possesses a \( k \)th co-order meta-singular \((p+1)\)-dimensional module of vector fields if and only if it can be represented, up to point transformations, in the form

\[
L = \tilde{L}(x, \Omega_{r,k,p}),
\]
where $\Omega_{r,k,p} = (\omega_{\alpha}, |\alpha| \leq r, \alpha_{p+1} + \cdots + \alpha_n \leq k)$, $\tilde{L}_{\omega_\alpha} \neq 0$ for some $\omega_{\alpha}$ with $\alpha_{p+1} + \cdots + \alpha_n = k$, and $\omega_{\alpha} = u_\alpha$ or, only for the case $p = 1$, $\omega_{\alpha} = \partial_1^2 \cdots \partial_n^\alpha (D_1 + uD_2 + \xi^3 \partial_3 + \cdots + \xi^n \partial_n)^{\alpha_1} u$ for some fixed smooth functions $\xi^i$.

### Proof

Suppose that a differential function $\tilde{L}$ possesses a $k$th co-order $(p+1)$-dimensional metasingular module $M$. Up to combining basis elements and change of variables, a basis of the module $M$ consists of either the vector fields $Q_s = \partial_s$ and $Q_0 = \partial_u$ or, if $p = 1$ and the module $M$ is not involutive, the vector fields $Q_s = \partial_1 + u\partial_2 + \xi^3 \partial_3 + \cdots + \xi^n \partial_n$ and $Q_0 = \partial_u$, cf. Propositions 3 and 4. Although the form of the initial differential function $\tilde{L}$ will also be transformed by the change of variables, for simplicity we will continue to use the old notations for all new values.

We choose the family $\mathfrak{M} = \{Q^\Phi = (Q_s - (\Phi_s/\Phi_u)\partial_u, s = 1, \ldots, p)\}$ of $p$-dimensional involutive submodules of $M$ which are parameterized by an arbitrary function $\Phi = \Phi(x, u)$. Then we fix an arbitrary point $z^0 = (x^0, u_0^0) \in J^r$ and consider the $p$-dimensional involutive modules from $\mathfrak{M}$ for which $z^0 \in Q^\Phi_{|r}$, where $Q^\Phi_{|r}$ denotes the manifold determined by the module $Q^\Phi$ in the jet space $J^r$. This condition implies that the values of the derivatives of $\Phi$ with respect to $u$ only $x_1$, $x_0$ at the point $(x^0, u^0)$, which contain differentiation with respect to $x$, are expressed via $u_0^0$ and values of derivatives of $\Phi$ in $(x^0, u_0^0)$, containing differentiation with respect to $u$. The latter values are not constrained. For instance, if the module $M$ is involutive, we get

\[
\Phi_s(x^0, u^0) = -u_s^0 \Phi_u(x^0, u^0), \\
\Phi_{si}(x^0, u^0) = -u_{si}^0 \Phi_u(x^0, u^0) - u_i^0 \Phi_{su}(x^0, u^0) - u_s^0 \Phi_{iu}(x^0, u^0) - u_s^0 u_i^0 \Phi_{uu}(x^0, u^0), \ldots.
\]

We introduce the new coordinates $\{x_i, \omega_\alpha, |\alpha| \leq r\}$ in $J^r$ instead of the standard ones $\{x_1, u_0, |\alpha| \leq r\}$. If the module $M$ is involutive, this change of coordinates is just a re-notation of variables in order to guarantee consistency with the special case of non-involutive modules for $p = 1$. In the latter case, this is a valid change of coordinates since the Jacobian matrix $(\partial \omega_\alpha / \partial u_{\alpha'})$ is nondegenerate: it is a triangular matrix with all diagonal entries equal to 1 if we implement the following order of multi-indices:

\[
\alpha < \beta \iff |\alpha| < |\beta| \lor (|\alpha| = |\beta| \land \alpha_1 < \beta_1) \lor (|\alpha| = |\beta| \land \alpha_1 = \beta_1 \land \alpha_2 < \beta_2) \lor \cdots.
\]

Denote by $\tilde{L}$ the differential function obtained from $\tilde{L}$ by the procedure of excluding, on the manifold $Q^\Phi_{|r}$, the derivatives of $u$ involving differentiations with respect to $x_0$ (see $\Phi$). For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we set $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_p)$ and $\tilde{\alpha} = (\alpha_{p+1}, \ldots, \alpha_n)$, i.e., $\tilde{\alpha} = (\tilde{\alpha}, \tilde{\alpha})$. The symbol $\tilde{0}$ denotes the tuple of $p$ zeros. As $Q^\Phi$ is a $k$th co-order singular module for $L$, the function $\tilde{L}$ does not depend on the derivatives $u_0(\tilde{\alpha})$, i.e., $|\tilde{\alpha}| = k + 1, \ldots, r$. We use this condition step-by-step, starting from the greatest value of $|\tilde{\alpha}|$ and re-writing the derivatives in the new coordinates of $J^r$ and in terms of $L$. In the course of this procedure, we take into account the equality $\omega_\tilde{\beta} = \psi^\beta[u]$ satisfied on the manifold $Q^\Phi_{|r}$ for each multi-index $\tilde{\beta}$ with $|\tilde{\beta}| \leq r$. Here the differential function $\psi^\beta = \psi^\beta[u]$ is defined by the equality $\psi^\beta = D_{\beta_{p+1}}^\alpha \cdots D_{\beta_n}^\alpha (Q^\Phi)^{\beta_1} \cdots (Q^\Phi)^{\beta_p} u$, and hence it is of $|\tilde{\beta}|$th order and possesses the representation

\[
\psi^\beta = (\partial_u(Q^\Phi)^{\beta_1} \cdots (Q^\Phi)^{\beta_p} u)_{(0, \beta)} + \hat{\psi}^\beta[u]
\]

with some differential function $\hat{\psi}^\beta = \hat{\psi}^\beta[u]$ of order less than $|\tilde{\beta}|$. Therefore, for each $\alpha$ with $\tilde{\alpha} = \tilde{0}$ and $|\tilde{\alpha}| \leq r$ the chain rule implies that

\[
\tilde{L}_{u_\alpha}(z^0) = \sum_{\beta: |\beta| \leq r, |\tilde{\beta}| \geq |\tilde{\alpha}|} L_{\omega_\beta}(z^0) \psi^\beta(u^0)(z^0).
\]
Thus, in the new coordinates for each \( \alpha \) with \( \hat{\alpha} = 0 \) and \( |\hat{\alpha}| = r \) the equation \( \hat{L}_{\omega_\alpha}(z^0) = 0 \) can be written in the form \( L_{\omega_\alpha}(z^0) = 0 \). Indeed, in this case we have that \( \psi_{\alpha, \alpha}^0 = 1 \) if \( \beta = \alpha \) and \( \psi_{\alpha, \beta}^0 = 0 \) otherwise. This completes the first step.

In the second step we fix a value of \( \alpha \) with \( \hat{\alpha} = 0 \) and \( |\hat{\alpha}| = r - 1 \). As \( L_{\omega_{\hat{\beta}}}(z^0) = 0 \) if \( \hat{\beta} = \hat{\alpha} = 0 \) and \( |\hat{\beta}| = r \), the summation multi-index in \( \hat{3} \) with the fixed \( \alpha \) can be assumed to run through the set \( B_1 = \{ \beta \mid |\hat{\beta}| = 1, |\hat{\beta}| = r - 1 \} \). The derivative \( \psi_{\alpha, \alpha}^0 \) is equal to 1, \( \partial_u Q_{\alpha}^0 u \) and 0 for \( \beta = (0, \hat{\alpha}), \beta = (\hat{\alpha}, \hat{\alpha}) \) and all other values of \( \beta \) from \( B_1 \), respectively. Here \( \delta_x \) is the \( p \)-tuple with the \( x \)th entry equal to 1 and the other entries equal to 0. Therefore, the equation \( \hat{L}_{u_{(\hat{\beta}, \alpha)}}(z^0) = 0 \) implies that

\[
L_{\omega_{(0, \alpha)}}(z^0) + L_{\omega_{(\delta_x, \alpha)}}(z^0) (\partial_u Q_{\alpha}^0 u) |_{(x,u)=(z^0,w^0)} = 0.
\]

Note that \( \partial_u Q_{\alpha}^0 u = - (\Phi_\alpha / \Phi_u) u \). We split with respect to the value \( \Phi_\alpha u(z^0, w^0) \) as it is unconstrained. Thereby we arrive at the equations \( L_{\omega_{(0, \alpha)}}(z^0) = 0 \) and \( L_{\omega_{(\delta_x, \alpha)}}(z^0) = 0 \).

Iterating this procedure, before the \( \mu \)-th step, \( \mu \in \{3, \ldots, r - k \} \), we derive the equations \( L_{\omega_{\beta}}(z^0) = 0 \), where the multi-index \( \beta \) runs through values for which \( r - \mu = 2 \leq |\hat{\beta}| \leq r \) and \( |\hat{\beta}| \leq r - 1 |\hat{\beta}| \). Then for each fixed value of \( \alpha \) with \( \hat{\alpha} = 0 \) and \( |\hat{\alpha}| = r - \mu + 1 \) the summation multi-index in \( \hat{3} \) can be assumed to run through the set \( B_\mu = \{ \beta \mid |\hat{\beta}| \leq \mu - 1, |\hat{\beta}| = r - \mu + 1 \} \). For \( \beta \in B_\mu \), the derivative \( \psi_{\alpha, \alpha}^0 \) equals \( \partial_u (Q_{\alpha}^1) \beta_1 \cdots (Q_{\alpha}^p) \beta_p u \) if \( \beta = \hat{\alpha} \) and is zero otherwise. Therefore, the equation \( L_{u_{(0, \alpha)}}(z^0) = 0 \) implies the condition

\[
\sum_{|\hat{\beta}| \leq \mu - 1, \hat{\beta} = \hat{\alpha}} L_{\omega_{\beta}}(z^0) (\partial_u (Q_{\alpha}^1) \beta_1 \cdots (Q_{\alpha}^p) \beta_p u) |_{(x,u)=(z^0,w^0)} = 0.
\]

The values \( (\partial_{\alpha}^1 \cdots \partial_{\alpha}^p \Phi_\alpha)(x^0, u^0) \), \( |\hat{\beta}| \leq \mu - 1 \), are unconstrained. Hence by splitting with respect to them or, equivalently, by splitting with respect to \( (\partial_u (Q_{\alpha}^1) \beta_1 \cdots (Q_{\alpha}^p) \beta_p u) |_{(x,u)=(z^0,w^0)} \), \( |\hat{\beta}| \leq \mu - 1 \), we obtain the equations \( L_{\omega_{\beta}}(z^0) = 0 \), \( |\hat{\beta}| = r - \mu + 1 \) and \( |\hat{\beta}| \leq \mu - 1 \).

Finally, after the \( (r - k) \)-th step we derive the system \( L_{\omega_\alpha}(z^0) = 0 \), \( |\hat{\alpha}| > k \) and \( |\alpha| \leq r \), which implies the condition \( \hat{2} \).

Conversely, let an \( r \)-th order differential function \( L \) be of the form \( \hat{2} \) (after a point transformation). For an arbitrary smooth function \( \Phi = \Phi(x, u) \) with \( \Phi_u \neq 0 \), we consider the involutive module \( Q_\Phi \) generated by either the vector fields \( Q_{\phi}^0 = \partial_\phi - (\Phi_\phi / \Phi_u) \partial_u \) in the general case or the vector field \( Q_{\phi}^0 = \partial_\phi + u \partial_\phi + \xi^\alpha \partial_\alpha + \cdots + \xi^\beta \partial_\beta - (\Phi_\phi / \Phi_u) \partial_u \) in the special case with \( p = 1 \). Using \( \hat{L} \) and \( Q_\Phi \), we construct the differential function \( \hat{L}_\Phi = \hat{L}(x, \hat{\Omega}_{r,k,p}) \), where

\[
\hat{\Omega}_{r,k,p} = (\omega_\alpha = D_{p+1}^{\alpha} \cdots D_{n}^{\alpha} (Q_{\phi}^1) \alpha_1 \cdots (Q_{\phi}^p) \alpha_p u, |\hat{\alpha}| \leq k, |\alpha| \leq r).
\]

By construction, \( \hat{L}_\Phi \leq k \) for all values of the parameter-function \( \Phi \) with \( \Phi_u \neq 0 \). Moreover, \( \hat{L}_\Phi = k \) for almost all values of this parameter-function except those which satisfy a system of differential equations. As

\[
L|_{Q_\Phi} = \hat{L}|_{Q_\Phi}.
\]

where \( \Phi \) runs through the set of smooth functions of \( (x, u) \) with nonvanishing derivative with respect to \( u \), the family \( \hat{\mathfrak{M}} = \{ Q_\Phi \} \) is a \( k \)-th order singular family of \( p \)-dimensional involutive modules for the differential function \( L \) in the new coordinates. We return to the old coordinates. In view of Proposition \( \hat{3} \) if \( p \geq 2 \) or Proposition \( \hat{4} \) if \( p = 1 \), the module of vector fields which contains the family \( \hat{\mathfrak{M}} \) is a \( k \)-th order meta-singular \( (p+1) \)-dimensional module for the differential function \( L \).

Excluding the special case of two-dimensional non-involutive meta-singular modules, the result presented in Theorem \( \hat{1} \) can be formulated in the following way: A differential function
with one dependent and \( n \) independent variables admits a \( k \)th co-order meta-singular \((p+1)\)-
dimensional involutive module \( M \) if and only if it can be reduced by a point transformation of
the variables to a differential function in which the total differentiation with respect to \( n - p \)
fixed independent variables in each derivative of the dependent variable is of order not greater
than \( k \).

**Corollary 3.** Any differential function with one dependent and \( n \) independent variables (not
identically vanishing) admits no meta-singular \((p+1)\)-dimensional \((0 < p < n)\) module of singu-
larity co-order \(-1\) (i.e., whose \( p \)-dimensional involutive submodules are ultra-singular for this
differential function).

**Remark 2.** Clearly a \( k \)th co-order meta-singular \((p+1)\)-dimensional module \( M \) may contain
\( p \)-dimensional involutive modules whose singularity co-orders are less than the singularity co-
order of the entire module \( M \). Consider a family \( \mathfrak{M} = \{ Q^\Phi = (Q_s - (\Phi_s/\Phi_u)Q^0, s = 1, \ldots, p) \} \)
of \( p \)-dimensional involutive submodules of \( M \) which are parameterized by an arbitrary function
\( \Phi = \Phi(x,u) \) and assume that \( \mathfrak{M} \) is singular for a differential function \( L \), and \( \text{scor}_L \mathfrak{M} = k \). Here
the vector fields \( Q_s \) and \( Q^0 \) are assumed to be reduced to the form presented in the begin-
ing of the proof of Theorem 1. Then the values of \( \Phi \) for which \( \text{scor}_L Q^\Phi < k \) are solutions of the system

\[
\sum_{|\alpha| \leq r-k} L_{\omega_{\alpha}}(x, \tilde{\Omega}_{r,k,p})(\partial_u(Q^\Phi_1)^{\alpha_1} \cdots (Q^\Phi_p)^{\alpha_p} u) = 0, \quad |\alpha| = k,
\]

where \( \tilde{\Omega} \) and \( \tilde{\Omega}_{r,k,p} \) are defined in Theorem 1 and its proof, respectively. In other words, the regular
values of \( \Phi \) associated with the submodules of the maximal singularity co-order \( k \) in \( \mathfrak{M} \)
satisfy, for some \( \alpha \) with \( |\alpha| = k \), the inequality

\[
\sum_{|\alpha| \leq r-k} L_{\omega_{\alpha}}(x, \tilde{\Omega}_{r,k,p})(\partial_u(Q^\Phi_1)^{\alpha_1} \cdots (Q^\Phi_p)^{\alpha_p} u) \neq 0.
\]

### 4 Singular modules of vector fields for differential equations

An involutive module \( Q \) satisfying the rank condition is called *(strongly) singular for a differ-
etial equation* \( L \) if it is singular for the differential function \( L[u] \) constituting the left hand side
of the canonical representation \( L[u] = 0 \) of the equation \( L \). Usually we will omit the attribute
“strongly”.

As left hand sides of differential equations are defined up to multipliers which are nonvan-
ishing differential functions, the conditions from Definition 3 can be weakened when considering
differential equations.

**Definition 5.** A \( p \)-dimensional \((0 < p < n)\) involutive module \( Q \) which satisfies the rank
condition is called *weakly singular* for the differential equation \( L \): \( L[u] = 0 \) if there exists a
differential function \( \tilde{L} = L[u] \) of an order less than \( r \) and a nonvanishing differential function
\( \lambda = \lambda[u] \) of an order not greater than \( r \) such that \( L|_{Q_{(r)}} = (\lambda \tilde{L})|_{Q_{(r)}} \). Otherwise we call \( Q \)
a *weakly regular* module for the differential equation \( L \). If the minimal order of differential
functions whose restrictions on \( Q_{(r)} \) coincide, up to nonvanishing functional multipliers, with
\( L|_{Q_{(r)}} \) is equal to \( k \) \((k < r)\) then the module \( Q \) is said to be *weakly singular of co-order* \( k \) for
the differential equation \( L \).

An involutive module \( Q \) is considered to be weakly ultra-singular for the differential equation
\( L \): \( L[u] = 0 \) if it is strongly ultra-singular for \( L \). As in the case of strong regularity, weakly regular
modules for the differential equation \( L \) are defined to have weak singularity co-order \( r = \text{ord} L \). We write \( \text{wscor}_L Q \) for the weak singularity co-order of the module \( Q \) for the equation \( L \).

Strong singularity implies weak singularity and consequently weak regularity implies strong
regularity. The weak singularity co-order is always less or equal and may be strictly less than
the strong singularity co-order. Thus strongly regular vector fields may be singular in the weak sense. As a simple example, consider the equation \( x_2 u_{111} + x_1 u_{222} = e^{333}(u_3 + u) \). It possesses the two-dimensional singular module \((\partial_1, \partial_2)\) whose strong and weak singularity co-orders equal 2 and 1, respectively. The same module \((\partial_1, \partial_2)\) is strongly regular and is of weak singularity co-order 1 for the equation \( x_2 u_{111} + x_1 u_{222} = e^{333}(u_3 + u) \).

Let \( \hat{\mathcal{L}} \) be the differential function associated with \( L \) on the manifold \( Q_\pi \) by excluding, in view of equations defining \( Q_\pi \), those derivatives of \( u \) which contain differentiations with respect to \( x_s \).

Suppose that \( L \) is of maximal rank in a derivative \( u_\alpha \) of the highest order \( k \) appearing in this differential function, i.e., \( L_{u(\hat{\alpha},\hat{\alpha})} \neq 0 \) for some \( \hat{\alpha} \) with \( |\hat{\alpha}| = k \) on the solution manifold of the equation \( \hat{\mathcal{L}} = 0 \) (see the notation in the proof of Theorem 1). It then follows that the weak singularity co-order of \( Q \) for the equation \( \mathcal{L} : L = 0 \) equals the order \( k \) of \( \hat{L} \) and, consequently, the strong singularity co-order of \( Q \). It follows that in this case there is an entirely algorithmic procedure for testing whether an involutive module is weakly singular for a partial differential equation.

The following assertion is proven in the same way as Proposition 1.

**Proposition 5.** Suppose that \( \mathcal{L} \) is a differential equation with respect to the unknown function \( u \) of \( n \) independent variables \( x \) and let \( Q \) be an involutive module of vector fields defined in the space of \((x,u)\), which is of dimension less than \( n \) and satisfies the rank condition. Then \( \text{wsco}_\mathcal{L} Q \leq \text{wsco}_\mathcal{L} Q^0 \) for any involutive submodule \( Q \) of \( Q \). In particular, the module \( Q \) is weakly singular for \( \mathcal{L} \) if it contains a submodule weakly singular for \( \mathcal{L} \).

The notion of meta-singular modules for differential equations is defined similarly to the case of differential functions. Thus, we have the following definition of modules which are meta-singular in the weak sense.

**Definition 6.** A \((p+1)\)-dimensional module \( M \) is called meta-singular for the differential equation \( L \) in the weak sense if any \( p \)-dimensional involutive submodule of \( M \) which satisfies the rank condition is weakly singular for \( L \) and the module \( M \) contains a family \( \mathfrak{M} = \{Q^p\} \) of such submodules parameterized by an arbitrary function \( \Phi = \Phi(x,u) \) of all independent and dependent variables. The weak singularity co-order of the meta-singular module \( M \) coincides with the maximum of the weak singularity co-orders of its involutive submodules satisfying the rank condition.

**Theorem 2.** An \( r \)th order differential equation \( \mathcal{L} : L[u] = 0 \) of maximal rank with one dependent and \( n \) independent variables possesses a \( k \)th co-order weakly meta-singular \((p+1)\)-dimensional module of vector fields if and only if the differential function \( L \) can be represented, up to point transformations of variables, in the form

\[
L = \Lambda[u] \hat{\mathcal{L}}(x, \Omega_{r,k,p}),
\]

where \( \Lambda \) is a nonvanishing differential function of order not greater than \( r \), \( \hat{\mathcal{L}} \) is a smooth function of \( x \) and \( \Omega_{r,k,p} = (\omega_{\alpha}, |\alpha| \leq r, \alpha_{p+1} + \cdots + \alpha_n = k) \), \( L_{\omega_{\alpha}} \neq 0 \) for some \( \omega_{\alpha} \) with \( \alpha_{p+1} + \cdots + \alpha_n = k \), and \( \omega_{\alpha} = u_\alpha \) or, only for the case \( p = 1 \), \( \omega_{\alpha} = D_2^2 \cdots D_n^\alpha \xi^1 \xi^2 \cdots \xi^n \) for some fixed smooth functions \( \xi^i = \xi^i(x,u), i = 3, \ldots, n \). The value of \( k \) should be minimal among all possible representations of the differential function \( L \) of the form (4).

**Proof.** We will freely use the notations and definitions from the proof of Theorem 1.

To begin with, assume that the differential equation \( \mathcal{L} : L[u] = 0 \) is of maximal rank and admits a \( k \)th co-order weakly meta-singular \((p+1)\)-dimensional module of vector fields. Up to point transformations and changes of module basis, it suffices to consider only the family \( \mathfrak{M} = \{Q^s = (Q_s - (\Phi_s/\Phi_u)Q^0, s = 1, \ldots, p)\} \) of \( p \)-dimensional involutive submodules of \( M \) which are parameterized by an arbitrary function \( \Phi = \Phi(x,u) \). Here \( Q^0 \) is reduced to the shift operator \( \partial_u \) and the vector fields \( Q^s \) take either the form \( Q_s = \partial_s \) or, if \( p = 1 \) and the module \( M \) is not involutive, the form \( Q_1 = \partial_1 + u \partial_2 + \xi^3 \partial_3 + \cdots + \xi^n \partial_n \).
Corollary 4. A differential equation \( \mathcal{L} \): \( L[u] = 0 \) of maximal rank with one dependent and \( n \) independent variables possesses a \( k \)th co-order weakly singular \( p \)-dimensional module of vector fields \( 0 < p < n \) if and only if this module is \( k \)th co-order strongly singular for \( \mathcal{L} \) (possibly in a representation differing from \( L[u] = 0 \) by multiplication by a nonvanishing differential function of \( u \)).

5 Singularity of reduction modules and parametric families of solutions

Definition 7. An involutive module \( Q \) is called a singular reduction module of a differential equation \( \mathcal{L} \) if \( Q \) is both a reduction module of \( \mathcal{L} \) and a weakly singular module of \( \mathcal{L} \).

The following assertions are obtained as direct extensions of the corresponding results for the case \( n = 2 \) [13].
Proposition 6. Let $Q$ be a $p$-dimensional non-ultra-singular reduction module ($0 < p < n$) of an equation $L$. Then the weak singularity co-order of $Q$ for $L$ coincides with the essential order of a corresponding reduced differential equation.

Proof. By a point transformation we may achieve the situation that in the new variables the module $Q$ has a basis $\{Q_s = \partial_s, s = 1, \ldots, p\}$. (Again we use the same notation for old and new variables). Then an ansatz constructed with $\varphi \in \mathcal{P}_{Q,1}$ and a nonvanishing differential function $\varphi = \varphi(\omega)$, where $\varphi = \varphi(\omega)$ is the new unknown function, $\omega = (\omega_1, \ldots, \omega_{n-p})$ and $\omega_1 = x_{p+1}, \ldots, \omega_{n-p} = x_n$ are the invariant independent variables. The manifold $\mathcal{Q}_{Q,r}$ is defined by the system $u_\alpha = 0$, where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfies the conditions $\alpha_1 + \cdots + \alpha_p > 0$, $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq r = \text{ord } L$.

As $Q \in \mathcal{R}^p(L)$, there are differential functions $\lambda = \lambda[\varphi]$ and $\tilde{L} = \tilde{L}[\varphi]$ of an order not greater than $r$ such that $L|_{x=x(\omega)} = \lambda \tilde{L}$ (cf. [32]). The function $\lambda$ does not vanish and may depend on the variables $x_s$ as parameters. Modulo equivalence generated by nonvanishing multipliers we may assume that the function $\tilde{L}$ is of minimal order $\tilde{r}$. Then the reduced equation $\mathring{L} = 0$ has essential order $\tilde{r}$.

Now since $\text{wsco}_{L} Q = k$ it follows that there exists a strictly $k$th order differential function $\tilde{L} = \tilde{L}[u]$ and a nonvanishing differential function $\lambda = \lambda[u]$ of an order not greater than $r$, which depend at most on $x$ and derivatives of $u$ with respect to $x_{p+1}, \ldots, x_n$ such that $L|_{Q,r} = \tilde{L}$.

Suppose first that $\tilde{r}$ would be less than $k$. In this case we can use $\lambda_{\text{new}} = \lambda_{|u=\varphi}$ and $\tilde{L}_{\text{new}} = \tilde{L}_{|u=\varphi}$ in the definition of weak singularity to arrive at the contradiction $\text{wsco}_{L} Q \leq \text{ord } \tilde{L}_{\text{new}} = \tilde{r} < k$. Thus $\tilde{r} \geq k$. (Here “$y \rightsquigarrow z$” means that the value $y$ should be substituted instead of the value $z$.)

On the other hand, if $\tilde{r} > k$ we conclude that $\lambda \tilde{L} = (\tilde{L}[u=\varphi(\omega)])_{x=x(\omega)}$, where the variables $x_s$ play the role of parameters. Fixing a value $x_s^0$ of $x_s$ for each $s$, we obtain the representation

$$
\tilde{L} = \frac{\lambda}{\lambda_{|u=\varphi(\omega)}} \bigg|_{x_s=x_s^0} \tilde{L} \bigg|_{u=\varphi(\omega), x_s=x_s^0}
$$

However, as $\text{ord } \tilde{L}_{|u=\varphi(\omega), x_s=x_s^0} \leq k < \tilde{r}$, this representation contradicts the condition that $\tilde{r}$ is the essential order of the reduced equation $\mathring{L}$.

We conclude that $\tilde{r} = k$. The inverse change of variables preserves the claimed property. □

The properties of ultra-singular modules of vector fields as reduction modules are obvious.

Proposition 7. 1) Any $p$-dimensional module $Q$ of vector fields ($0 < p \leq n$) which is ultra-singular for a differential equation $L$ is a reduction module of this equation. An ansatz constructed with $Q$ reduces $L$ to the identity. Therefore, the family of $Q$-invariant solutions of $L$ is parameterized by an arbitrary function of $n - p$ $Q$-invariant variables.

2) If $Q$ is a $p$-dimensional module of vector fields and the family of $Q$-invariant solutions of $L$ is parameterized by an arbitrary function of $n - p$ $Q$-invariant variables then $Q$ is an ultra-singular module of vector fields for $L$.

Given a partial differential equation $L$, consider a reduction module $Q$ for $L$ of dimension $n-1$, i.e., $p = n - 1$. Such a module reduces the equation $L$ to an ordinary differential equation whose essential order, in view of Proposition [6] is equal to $\text{wsco}_{L} Q$. This allows us to relate $\text{wsco}_{L} Q$ with the maximal number of parameters in families of $Q$-invariant solutions of $L$. An exception occurs for the ultra-singular modules which reduce $L$ to identities. In this case we define the order of the reduced equation to be $-1$.

Proposition 8. Suppose that $Q$ is an $(n-1)$-dimensional module of vector fields and a differential function of minimal order, associated with the differential function $L[u]$ on the manifold $Q_{(r)}$, up to a nonvanishing multiplier ($r = \text{ord } L$), is of maximal rank in the highest order derivative
of \( u \) appearing in this differential function. Then any two of the following properties imply the third one.

1) \( Q \) is a reduction module of the equation \( L: L = 0 \).
2) The weak singularity co-order of \( Q \) for \( L \) equals \( l \), where \( 0 \leq l \leq r \).
3) The equation \( L \) possesses an \( l \)-parametric family of \( Q \)-invariant solutions, and any \( Q \)-invariant solution of \( L \) belongs to this family.

Proof. If \( Q \) is an \((n-1)\)-dimensional non-ultra-singular reduction module of an equation \( L \) then the weak singularity co-order of \( Q \) for \( L \) equals the maximal number \( N_{L,Q} \) of essential continuous parameters in families of \( Q \)-invariant solutions of \( L \). Indeed, the weak singularity co-order of \( Q \) for \( L \) coincides with the essential order \( \hat{r} \) of the reduced ordinary differential equation \( \hat{L} \) associated with \( Q \), \( \hat{r} = \text{wsco}_L Q \). The maximal number of essential continuous parameters in solutions of \( \hat{L} \) also equals \( \hat{r} \). The substitution of these solutions into the corresponding ansatz leads to parametric families of \( Q \)-invariant solutions of \( L \), and all \( Q \)-invariant solutions of \( L \) are obtained in this way. Therefore, \( N_{L,Q} = \hat{r} \).

In view of the condition of maximal rank, the equation \( \hat{L} \) can be written in normal form and hence has an \( \hat{r} \)-parametric general solution which contains all solutions of \( \hat{L} \). Substituting it into the corresponding ansatz, this solution gives an \( \hat{r} \)-parametric family of \( Q \)-invariant solutions of \( L \). There are no other \( Q \)-invariant solutions of \( L \). Therefore, conditions 2 and 3 are equivalent if condition 1 holds.

For any \((n-1)\)-dimensional involutive module \( Q \) of vector fields we have \( N_{L,Q} \leq k \), where by \( k \) we denote \( \text{wsco}_L Q \). Let us prove that \( Q \) is a reduction module of \( L \) if \( N_{L,Q} = k \).

Point transformations of the variables do not change the claimed property. We use the variables and notations from the proof of Proposition 6. Consider the differential function \( \hat{L}[\varphi] = \hat{L}_{|u=\varphi(\omega)} \). It depends on the variables \( x_s \) as parameters, and \( \text{ord} \hat{L} = k \). Due to the condition of maximal rank, we can resolve the equation \( \hat{L} = 0 \) with respect to the highest order derivative \( \varphi^{(k)} \): \( \varphi^{(k)} = R[\varphi] \), where \( \text{ord} R < k \).

If \( R_{x_s} \neq 0 \) for some \( s \), splitting with respect to \( x_s \) in the equation \( \hat{L} = 0 \) results in an ordinary differential equation \( \hat{R}[\varphi] = 0 \) of an order lower than \( k \). Any \( Q \)-invariant solution of \( L \) has the form \( u = \varphi(\omega) \), where the function \( \varphi \) satisfies, in particular, the equation \( \hat{R}[\varphi] = 0 \). This contradicts the condition that \( N_{L,Q} = k \).

Therefore, \( R_{x_s} = 0 \), i.e., the equation \( \varphi^{(k)} = R[\varphi] \) is a reduced equation obtained from the equation \( L \) by the substitution of the ansatz \( u = \varphi(\omega) \) constructed with the module \( Q \).

6 Reduction to algebraic equations

As remarked in Section 3, the case when the dimension of modules of vector fields coincides with the number of independent variables, \( p = n \), is special for the singularity of modules for differential functions. This case is also special for reduction of differential equations. In contrast to reduction modules of lower dimensions, \( n \)-dimensional reduction modules of any differential equation \( L \) with \( n \) independent variables reduce this equation to algebraic equations instead of differential equations with a smaller number of independent variables. Moreover, this is the only case when both regular and singular reduction modules can be studied in a uniform way.

Let \( Q \) be an involutive module of dimension \( p = n \) which satisfies the rank condition. Then we can choose the basis of \( Q \) consisting of the vector fields \( Q_s = \partial_s + \eta^s(x,u) \partial_u \). As the module \( Q \) is involutive, the basis elements \( Q_s \) should commute, i.e., the coefficients \( \eta^s = \eta^s(x,u) \) satisfy the system

\[
\eta^s_{s'} + \eta^{s'} \eta^s_u = \eta^s_{s'} + \eta^s \eta^{s'}_u.
\]

The Frobenius theorem implies that the system \( Q_s \Phi = \Phi_s + \eta^s \Phi_u = 0 \) with respect to the function \( \Phi = \Phi(x,u) \) has a single functionally independent solution, which is not constant. In
other words, the coefficients \( \eta^s \) can be represented in the form \( \eta^s = -\Phi_s/\Phi_u \) for some smooth function \( \Phi = \Phi(x, u) \) with \( \Phi_u \neq 0 \). An implicit ansatz constructed for \( u \) with the module \( Q \) is \( \Phi(x, u) = \varphi \), where \( \varphi \) is the new unknown function. It is nullary since the module dimension \( p \) of \( Q \) equals the number \( n \) of independent variables \( x \).

Suppose that \( Q \) is a reduction module of an \( r \)-th order differential equation \( \mathcal{L}: L[u] = 0 \). All the derivatives of \( u \) with respect to \( x \) from order 1 up to order \( r \) are expressed, on the manifold \( Q(r) \), via the variables \( x \) and \( u \):

\[
u_\alpha = h^\alpha(x, u) := (\partial_1 + \eta^1 \partial_u)^{\alpha_1} \cdots (\partial_n + \eta^n \partial_u)^{\alpha_n} u, \quad 1 \leq |\alpha| \leq r.
\]

As the vector fields \( Q_s \) commute, the above representation for derivatives of \( u \) is well defined since it does not depend on the ordering of operators in the right hand side expression. Using these expressions for excluding the appropriate derivatives \( u_\alpha \) from \( L \), we get a differential function \( \bar{L} = \bar{L}[u] \) of order zero, i.e., a function of \((x, u)\). Varying \( \eta^s \), it can also be interpreted as a differential function with the independent variables \( x \) and \( u \) and the dependent variables \( \eta^s \).

Therefore, in view of Lemma 11 the fact that \( Q \) is a reduction module of \( L \) is equivalent to the condition \( Q_s \bar{L}|_{\bar{L}=0} = 0 \), which holds in the zero-order jet space. This condition jointly with equations \( Q_s \) gives the complete system of determining equations for the coefficients \( \eta^s \). Under the representation \( \eta^s = -\Phi_u/\Phi_u \) it can be interpreted as an equation for the function \( \Phi \), where \( L^\Phi = \bar{L}_{|\Phi} = -\Phi_u/\Phi_u \) is considered as a differential function with the independent variables \( x \) and \( u \) and the dependent variable \( \Phi \).

In order to handle the condition \( Q_s \bar{L}|_{\bar{L}=0} = 0 \), we can solve the equation \( \bar{L} = 0 \) with respect to a variable (either one of the \( x \)'s or \( u \)) and excluding this variable from the equation \( Q_s \bar{L} = 0 \) using the obtained expression. As the functions \( \bar{L} \) and \( \eta^s \) (resp. \( \Phi \)) depend on the same arguments, the above procedure does not result in a usual differential equation.

This is why we use another, more convenient, way, which involves the Hadamard lemma. Within the local approach, it suffices to split the further consideration into two cases: either the function \( \bar{L} \) is of maximal rank or it is a constant.

In the first case, for each \( s \) the condition \( Q_s \bar{L}|_{\bar{L}=0} = 0 \) is equivalent, in view of the Hadamard lemma, to the equality \( Q_s \bar{L} = \lambda^s \bar{L} \) for some smooth function \( \lambda^s = \lambda^s(x, u) \). After cross differentiation for each pair \((s, s')\) we obtain

\[
Q_s Q_{s'} \bar{L} - Q_{s'} Q_s \bar{L} = Q_s (\lambda^{s'} \bar{L}) - Q_{s'} (\lambda^s \bar{L}) = (Q_s \lambda^{s'} - Q_{s'} \lambda^s) \bar{L} = 0,
\]

i.e., \( Q_s \lambda^{s'} = Q_{s'} \lambda^s \). Therefore, there exists a smooth function \( \Lambda = \Lambda(x, u) \) such that \( \lambda^s = Q_s \Lambda \).

The system of the equations \( Q_s \bar{L} = \lambda^s \bar{L} \) then implies in view of the representation \( \eta^s = -\Phi_u/\Phi_u \) that \( L^\Phi = \Lambda \zeta(\Phi) \) for some smooth function \( \zeta \) of one argument, where \( \Lambda = e^\Lambda \) is a nonvanishing function of \((x, u)\).

In the second case, the condition \( Q_s \bar{L} = 0 \) is satisfied by each \((x, u)\) as a differential consequence of the current supposition that the function \( \bar{L} \) is a constant. Setting \( \Lambda = 1 \) and \( \zeta(\Phi) = L^\Phi = \text{const} \), we derive the same equation \( L^\Phi = \Lambda \zeta(\Phi) \) with the nonvanishing multiplier \( \Lambda \).

In fact, this equation is precisely the condition of reduction of the equation \( \mathcal{L} \) by the ansatz \( \Phi(x, u) = \varphi \) associated with the module \( Q \). Indeed, if the function \( u = u(x) \) is implicitly defined by this ansatz then the derivatives of \( u \) (of nonzero orders) are found from differential consequences of the equations \( u_s = -\Phi_u/\Phi_u \). Therefore, the substitution of the ansatz to \( \mathcal{L} \) leads to the equation \( L^\Phi|_{\Phi(x, u) = \varphi} = 0 \) which is equivalent, in view of the reduction condition, to the algebraic equation \( \zeta(\varphi) = 0 \) with respect to the constant \( \varphi \).

It is obvious that the module \( Q \) is ultra-singular for \( \mathcal{L} \) if and only if the function \( \zeta \) identically vanishes.

Summing up the above considerations results in the following assertion:
Proposition 9. A module $Q$ is an $n$-dimensional reduction module of a differential equation $\mathcal{L}$: $L[u] = 0$ with $n$ independent variables $x$ if and only if this module is spanned by the vector fields $\partial_s - (\Phi_s/\Phi_u)\partial_u$, $s = 1, \ldots, n$, where the function $\Phi = \Phi(x,u)$ satisfies the equation $L^\Phi = \Lambda(\Phi)$, $\Lambda$ is a nonvanishing function of $(x,u)$ and the differential function $L^\Phi = L^\Phi[\Phi]$ is obtained from $L[u]$ by the exclusion of the derivatives of $u$ using differential consequences of the equations $u_s = -\Phi_s/\Phi_u$. The ansatz $\Phi(x,u) = \varphi$ reduces the equation $\mathcal{L}$ to the algebraic equation $\zeta(\varphi) = 0$ with respect to the constant $\varphi$.

Theorem 3. Up to the equivalence of solution families, for any differential equation $\mathcal{L}$ with respect to a single unknown function of $L$ and $\tilde{\Phi}$ is equivalent to the equality $\Phi = \tilde{\Phi}$ functionally dependent. More precisely, $\Phi = \Phi(x,u)$ defines a solution of the system of the determining equations for coefficients of ultra-singular reduction modules. Namely, each module of the above kind corresponds to the family of solutions which are invariant with respect to this module. The problems of the construction of all one-parametric solution families of the equation $\mathcal{L}$ and the exhaustive description of its $n$-dimensional ultra-singular reduction modules are completely equivalent.

Proof. Let $Q$ be an $n$-dimensional ultra-singular reduction module of the equation $\mathcal{L}$. It follows from Proposition 9 that the ansatz $\Phi(x,u) = \varphi$ constructed with the module $Q$ reduces the equation $\mathcal{L}$ to the identity. In other words, for each value of the constant $\varphi$ this ansatz implicitly defines a solution of $\mathcal{L}$.

Conversely, suppose that $\mathcal{F} = \{u = f(x,\varphi)\}$ is a family of solutions of $\mathcal{L}$ parameterized by the single constant parameter $\varphi$. As this parameter is essential and, therefore, the derivative $f_\varphi$ is nonzero, we can express $\varphi$ from the equality $u = f(x,\varphi)$. As a result, we obtain that $\varphi = \Phi(x,u)$ for some function $\Phi = \Phi(x,u)$ with $\Phi_u \neq 0$. Consider the module $Q = \{Q_1, \ldots, Q_n\}$, where $Q_s = \partial_u + \eta^s\partial_u$ and the coefficients $\eta^s = \eta^s(x,u)$ are defined by $\eta^s = -\Phi_s/\Phi_u$. It is an $n$-dimensional involutive module and $Q[u] = 0$ for any element of $\mathcal{F}$. The ansatz $u = f(x,\varphi)$, where $\varphi$ is the new unknown (nullary) function, is associated with $Q$ and reduces $\mathcal{L}$ to the identity. This means that $Q$ is an ultra-singular reduction module for $\mathcal{L}$.

One-parametric families $\mathcal{F} = \{u = f(x,\varphi)\}$ and $\tilde{\mathcal{F}} = \{u = \tilde{f}(x,\tilde{\varphi})\}$ are defined to be equivalent if they consist of the same functions and differ only by parameterizations, i.e., if there exists a function $\zeta = \zeta(\varphi)$ such that $\zeta_u \neq 0$ and $\tilde{f}(x,\zeta(\varphi)) = f(x,\varphi)$. This is true if and only if the functions $\Phi = \Phi(x,u)$ and $\tilde{\Phi} = \tilde{\Phi}(x,u)$ associated with the families $\mathcal{F}$ and $\tilde{\mathcal{F}}$, respectively, are functionally dependent. More precisely, $\tilde{\Phi} = \zeta(\Phi)$. As $\Phi_u \Phi_u \neq 0$, the functional dependence of $\Phi$ and $\tilde{\Phi}$ is equivalent to the equality $\Phi_s/\Phi_u = \tilde{\Phi}_s/\tilde{\Phi}_u$. Therefore, equivalent one-parametric families of solutions correspond to the same ultra-singular reduction module $Q$ of $\mathcal{L}$ and, conversely, any two one-parametric families of $Q$-invariant solutions are equivalent.

Corollary 5. The system of the determining equations for coefficients of ultra-singular reduction modules of $\mathcal{L}$, which consists of equations in $\Phi$ and the equation $\tilde{\mathcal{L}} = 0$, is reduced by the composition of the nonlinear substitution $\eta^s = -\Phi_s/\Phi_u$, where $\Phi$ is a function of $(x,u)$, and the hodograph transformation

the new independent variables: $\tilde{x}_i = x_i$, $\varphi = \Phi,

the new dependent variable: $\tilde{u} = u$

to the initial equation $\mathcal{L}$ in the function $\tilde{u} = \tilde{u}(\tilde{r},\tilde{x},\varphi)$ with $\varphi$ playing the role of a parameter.

Note that the reduction of differential equations to algebraic ones using nonclassical symmetries was considered in [13]. An assertion similar to Theorem 3 was obtained therein. Using the notion of singular reduction operators makes Theorem 3 more precise.
7 Motivating example: evolution equations

We investigate $n$-dimensional singular reduction modules of $(1+n)$-dimensional evolution equations of the general form

$$u_t = H(t,x,u_{(r,x)})$$

for the unknown function $u$ depending on the variables $t = x_0$ and $x = (x_1, \ldots, x_n)$. (For convenience, in this section we set the number of independent variables to $n+1$ instead of $n$ and additionally single out the variable $x_0$.) Here $u_{(r,x)}$ denotes the set of all derivatives of the functions $u$ with respect to the space variables $x$ of order not greater than $r$, including $u$ as derivative of order zero. Also, $u_t = \partial u/\partial t$ and $r = \max\{|\alpha| \mid H_{\alpha} \neq 0\} \geq 2$, i.e., we assume the order of the equations under consideration to be not less than two. We fix an arbitrary equation $\mathcal{L}$ of the form (6).

To reduce the equation $\mathcal{L}$ to an ordinary differential equation, we should use its $n$-dimensional reduction module. In general, such a module $Q$ is spanned by $n$ vector fields of the form

$$Q_s = \tau^s(t,x,u)\partial_t + \xi^{s1}(t,x,u)\partial_i + \eta^s(t,x,u)\partial_u,$$

with $\text{rank}(\tau^s, \xi^{s1}) = n$. (We use the same convention for the ranges of indices as in the whole paper, i.e., the indices $i$ and $j$ run from 1 to $n$ and the index $s$ runs from 1 to $p$ noting that here $p$ coincides with $n$). In contrast to the specific case $n = 1$, for general values of $n$ we can completely describe singular $n$-dimensional modules of the equation $\mathcal{L}$ only after more precisely knowing the form of $H$. At the same time, by direct generalization of the case $n = 1$ we easily find a family of such modules for any equation of the form (6).

**Proposition 10.** If the coefficient of $\partial_t$ in any vector field from an $n$-dimensional involutive module $Q$ is zero then $Q$ is a singular module for the differential function $L = u_t - H(t,x,u_{(r,x)})$. The co-order of singularity of $Q$ equals one, $\text{soc}_\mathcal{L} Q = 1$.

**Proof.** Suppose that $\tau^s = 0$. Then $\text{rank}(\xi^{s1}) = n$ and up to changing basis of $Q$ we can set

$$Q_s = \partial_s + \eta^s(t,x,u)\partial_u.$$  

(7)

All the derivatives of $u$ with respect to $x$ from order 1 up to order $r$ can be expressed, on the manifold $Q_{(r)}$, via $t$, $x$ and $u$:

$$u_\alpha = h^\alpha(t,x,u) := (\partial_t + \eta^{1}\partial_u)^{\alpha_1} \cdots (\partial_n + \eta^{n}\partial_u)^{\alpha_n} u.$$  

Here and in what follows $1 \leq |\alpha| \leq r$ and $\alpha_0 = 0$. In view of the module $Q$ being involutive, the vector fields $Q_s$ commute. (Moreover, we have $\eta^s = -\Phi_s/\Phi_u$ for some smooth function $\Phi = \Phi(x,u)$ with $\Phi_u \neq 0$.) Therefore, the above representation for $u_\alpha$ is well defined since it does not depend on the ordering of operators in the right hand side expression. Using these expressions for excluding the appropriate derivatives $u_\alpha$ from $L$, we get the differential function

$$\tilde{L} = u_t - \tilde{H}(t,x,u),$$

where $\tilde{H} = H(t,x,u)$ is the function obtained from $H$ by the substitution of $h^{\alpha}$ instead of $u_\alpha$. The order of $\tilde{L}$ equals 1. Hence the vector field $\tilde{Q}$ is singular for the differential function $L$, and its singularity co-order equals 1.

**Corollary 6.** The module $M = \langle \partial_1, \ldots, \partial_n, \partial_u \rangle$ is a first co-order strictly meta-singular $(n+1)$-dimensional module for any $(1+n)$-dimensional evolution equation.

**Proof.** A $p$-dimensional involutive submodule $Q$ of $M$ satisfies the rank condition if and only if it is spanned by the vector fields $\partial_s - (\Phi_s/\Phi_u)\partial_u$ for some smooth function $\Phi = \Phi(x,u)$. It follows from Proposition [10] that the submodule $Q$ is a first co-order strictly singular module for any evolution equation of the form (6). Varying the parameter-function $\Phi$, we obtain a family of such submodules parameterized by an arbitrary function of all independent and dependent variables.
The subsystem (8) implies the existence of a function \( \Phi = \Phi(\eta, t, x, u) \).

Proof. To the initial equation

\[ L = \eta^s u^\kappa \]

versed by the parametric function \( \Phi \) from the above representation for \( u \) in the single function \( \Phi \), that is equivalent to a single equation in \( \Phi \). Indeed, we have

\[ \eta^s + \eta^s \eta_u = \eta^s + \eta^s \eta_u^s \]

follows from the condition that the module \( Q \) is involutive and hence the basis elements \( Q^s \) commute. The second subsystem

\[ \eta^s + H \eta_u = H_s + \eta^s H_u \]

is a consequence of the conditional invariance criterion applied to the equation \( L \) and \( Q \). The function \( H = H(t, x, u) \) is defined in the proof of Proposition 10. It coincides with \( H \) on the manifold \( Q_r \). The total number of equations in the joint system \( DE_0(L) \) of (8) and (9) is \( n(n+1)/2 \) and thereby greater than the number of the unknown functions \( \eta^s \) if \( n > 1 \).

Hence the system \( DE_0(L) \) looks strongly overdetermined in the multidimensional case. In fact, the equations of the subsystems agree well with each other. The subsystem (8) implies the representation \( \eta^s = -\Phi_s/\Phi_u \) of \( \eta^s \) via a single arbitrary function \( \Phi = \Phi(t, x, u) \). Substituting this representation into the subsystem (9), we obtain a system of \( p \) partial differential equations in the single function \( \Phi \), that is equivalent to a single equation in \( \Phi \). Indeed, we have

\[ H_s + \eta^s H_u - \eta^s - \eta^s H = H^\Phi - \frac{\Phi_s}{\Phi_u} H_u + \left( \frac{\Phi_s}{\Phi_u} \right)_n + \left( \frac{\Phi_s}{\Phi_u} \right)_u H^\Phi \]

\[ = \frac{1}{\Phi_u} \left( \partial_s - \frac{\Phi_s}{\Phi_u} \partial_u \right) \left( \Phi_n + \Phi_u H^\Phi \right) = 0, \]

i.e., this system is equivalent to the equation \( \Phi_t + \Phi_u H^\Phi = \chi(t, \Phi) \). Here \( \chi \) is an arbitrary smooth function of its arguments and \( H^\Phi \) coincides with \( H \) under the substitution \( \eta^s = -\Phi_s/\Phi_u \). The expression for \( H^\Phi \) involves derivatives of \( \Phi \) up to order \( r + 1 \). The function \( \Phi \) associated with a fixed module \( Q \) is defined up to the transformation \( \Phi = \theta(t, \Phi) \). As \( \tilde{\eta}^s = -\Phi_s/\Phi_u = -\Phi_s/\Phi_u = \eta^s \), the functions \( H^\Phi \) and \( H^\tilde{\Phi} \) coincide. At the same time, if we choose \( \theta \) satisfying the equation \( \theta_t + \chi \theta = 0 \) then \( \Phi_t + \Phi_u H^\Phi = 0 \). Therefore, up to the equivalence on the set of functions parameterizing singular modules we can assume that the function \( \Phi \) is a solution of the equation \( \Phi_t + \Phi_u H^\Phi = 0 \).

Recalling the above arguments, we derive the following result.

Proposition 11. The system \( DE_0(L) \) of the determining equations (8) and (9) is reduced by the composition of the nonlocal substitution \( \eta^s = -\Phi_s/\Phi_u \), where \( \Phi \) is a function of \( (t, x, u) \), and the hodograph transformation

the new independent variables: \( \tilde{t} = t, \quad \tilde{x}_i = x_i, \quad \kappa = \Phi \),

the new dependent variable: \( \tilde{u} = u \)

to the initial equation \( L \) in the function \( \tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}, \kappa) \) with \( \kappa \) playing the role of a parameter.

Proof. The subsystem (8) implies the existence of a function \( \Phi = \Phi(t, x, u) \) with \( \Phi_u \neq 0 \) such that \( \eta^s = -\Phi_s/\Phi_u \). The entire system \( DE_0(L) \) is reduced, up to equivalence on the set traversed by the parametric function \( \Phi \) from the above representation for \( \eta^s \), to the single equation \( \Phi_t + \Phi_u H^\Phi = 0 \). The last equation is mapped by the hodograph transformation to the initial equation \( L \) for the function \( \tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}, \kappa) \). This directly follows from the definition of the function \( H^\Phi \) and the rule for calculating derivatives under the hodograph transformation, \( \tilde{u}_{\tilde{x}_i} = -\Phi_t/\Phi_u \), etc. \( \square \)
Proposition 12. For any equation \( \mathcal{L} \) of the form (6), the following statements are equivalent:

1) The span \( Q = \langle Q_s = \partial_s + \eta^s(t,x,u)\partial_u, s = 1, \ldots, n \rangle \) is a reduction module of \( \mathcal{L} \).

2) The module \( \tilde{Q} = \langle \tilde{Q}_0, Q_1, \ldots, Q_n \rangle \), where \( \tilde{Q}_0 = \tilde{\partial}_t + \tilde{H}\partial_u \) and the function \( \tilde{H} = \tilde{H}(t,x,u) \) coincides with \( H \) on the manifold \( Q_2 \), is involutive.

3) There exists a vector field \( \tilde{Q}_0 = \partial_t + \eta^0(t,x,u)\partial_u \) such that the module \( \tilde{Q} = \langle \tilde{Q}_0, Q_1, \ldots, Q_n \rangle \) is ultra-singular for \( \mathcal{L} \).

Moreover, under these equivalent conditions the coefficient \( \eta^0 \) is uniquely determined as \( \eta^0 = \tilde{H} \), i.e., we necessarily have \( \tilde{Q}_0 = \tilde{Q}_1 \).

Proof. Both the first and second statement are equivalent to the fact that the coefficients \( \eta^s \) satisfy the system DE1(\( \mathcal{L} \)) consisting of the equations (3) and (4). The module \( \tilde{Q} \) is ultra-singular for \( \mathcal{L} \) if and only if \( \eta^0 = \tilde{H} \), i.e. the module \( \tilde{Q} \) coincides with \( Q \), and this module is involutive.

Theorem 4. Up to the equivalence of solution families, for any equation of the form (6) there exists a one-to-one correspondence between one-parametric families of its solutions and \( n \)-dimensional reduction modules formed by vector fields with zero coefficient of \( \tilde{\partial}_t \). Namely, each module of the above kind corresponds to the family of solutions which are invariant with respect to this module. The problems of the construction of all one-parametric solution families of equation (6) and the exhaustive description of its \( n \)-dimensional reduction modules formed by vector fields with vanishing coefficient of \( \tilde{\partial}_t \) are completely equivalent.

Proof. Let \( \mathcal{L} \) be an equation from class (6) and \( Q = \langle Q_1, \ldots, Q_n \rangle \in R^n_0(\mathcal{L}) \), i.e., \( Q_s = \partial_s + \eta^s(\tilde{t},\tilde{x},\tilde{u})\partial_{\tilde{u}} \), where the coefficients \( \eta^s = \eta^s(t,x,u) \) satisfy the system DE0(\( \mathcal{L} \)). An ansatz constructed with \( Q \) has the form \( u = f(t,x,\varphi(\omega)) \), where \( f = f(t,x,\varphi) \) is a given function, \( f_\varphi \neq 0 \), \( \varphi = \varphi(\omega) \) is the new unknown function and \( \omega = t \) is the invariant independent variable. This ansatz reduces \( \mathcal{L} \) to a first-order ordinary differential equation \( \mathcal{L}' \) in \( \varphi \), solvable with respect to \( \varphi_\omega \).

The general solution of the reduced equation \( \mathcal{L}' \) can be represented in the form \( \varphi = \varphi(\omega,\kappa) \), where \( \varphi_\omega \neq 0 \) and \( \kappa \) is an arbitrary constant. The form of the general solution is defined up to a transformation \( \tilde{\kappa} = \zeta(\kappa) \) of the parameter \( \kappa \). Substituting this solution into the ansatz results in the one-parametric family \( F \) of solutions \( u = \tilde{f}(t,x,\kappa) \) of \( \mathcal{L} \) with \( \tilde{f} = f(t,x,\varphi(t,\kappa)) \), and any \( Q \)-invariant solution of \( \mathcal{L} \) belongs to this family. Expressing the parameter \( \kappa \) from the equality \( u = \tilde{f}(t,x,\kappa) \), we obtain that \( \kappa = \Phi(t,x,u) \), where \( \Phi_\kappa \neq 0 \). Then \( \eta^s = u_\kappa = -\Phi_s/\Phi_\kappa \) for any \( u \in F \), i.e., for any admissible value of \( (t,x) \). This implies that \( \eta^s = -\Phi_s/\Phi_\kappa \) for any admissible value of \( (t,x) \).

The proof of the converse assertion is similar to that of Theorem 3.

Consider a one-parametric family \( F = \{ u = f(t,x,\kappa) \} \) of solutions of \( \mathcal{L} \). The derivative \( f_\kappa \) is nonzero since the parameter \( \kappa \) is essential. We express \( \kappa \) from the equality \( u = f(t,x,\kappa) \) and \( \kappa = \Phi(t,x,u) \) for some function \( \Phi = \Phi(t,x,u) \) with \( \Phi_\kappa \neq 0 \). The span \( Q = \langle Q_1, \ldots, Q_n \rangle \), where \( Q_s = \partial_s + \eta^s(\tilde{t},\tilde{x},\tilde{u})\partial_{\tilde{u}} \) and the coefficients \( \eta^s = \eta^s(t,x,u) \) is defined by \( \eta^s = -\Phi_s/\Phi_\kappa \), is an \( n \)-dimensional involutive module. For any \( u \in F \) we have \( Q[u] = 0 \). The ansatz \( u = f(t,x,\varphi(\omega)) \), where \( \omega = t \), associated with \( Q \), reduces \( \mathcal{L} \) to the equation \( \varphi_\omega = 0 \). Therefore (see (32)), \( Q \in R_0^n(\mathcal{L}) \).

One-parametric families \( F = \{ u = f(t,x,\kappa) \} \) and \( \bar{F} = \{ u = f(t,x,\tilde{\kappa}) \} \) of solutions of \( \mathcal{L} \) are equivalent if and only if the associated functions \( \Phi = \Phi(t,x,u) \) and \( \bar{\Phi} = \bar{\Phi}(t,x,u) \) satisfy the condition \( \Phi_s/\Phi_\kappa = \bar{\Phi}_s/\bar{\Phi}_\kappa \). Therefore, equivalent one-parametric families of solutions correspond to the same module \( Q \) from \( R_0^n(\mathcal{L}) \) and, conversely, any two one-parametric families of \( Q \)-invariant solutions are equivalent.

Remark 3. The triviality of the above ansatz and the reduced equation results from the above special representation for the solutions of the determining equation. In this approach difficulties in the construction of ansatzes and the integration of the reduced equations are replaced by difficulties in obtaining the representation for the coefficients of the reduction modules.
Remark 4. In fact, Theorem 11 is a consequence of Theorem 5 and Proposition 11. This observation provides a uniform background for no-go results on reduction modules. It suffices to note that any \(Q\)-invariant solution of \(L\) is \(Q\)-invariant, where the modules \(Q\) and \(L\) are defined in Proposition 11. We have given a direct proof as it leads to a deeper understanding of reduction of evolution equations to first-order ordinary differential equations.

8 Reduction modules of singularity co-order one

Taking the previous example on evolution equations as a model case, we now proceed to studying \((n-1)\)-dimensional co-order one singular reduction modules of general partial differential equations in one dependent and \(n\) independent variables. In the course of considering single modules of this kind it is only possible to derive an assertion similar to Proposition 12.

**Proposition 13.** Let \(L\): \(L[u] = 0\) be a partial differential equation in one dependent and \(n\) independent variables, let \(Q\) be an \((n-1)\)-dimensional involutive module of vector fields defined in the space of \((x,u)\) and satisfying the rank condition, as well as \(\text{wsco}_{L} Q = 1\). We also suppose that a first-order differential function \(\hat{L}\) associated with \(L\) on the manifold \(Q_{(r)}\) up to nonvanishing multiplier, where \(r = \text{ord} L\), is of maximal rank with respect to the unique first-order derivative of \(u\) appearing in \(\hat{L}\). Then \(Q\) is a reduction module of \(L\) if and only if there exists a (unique) \(n\)-dimensional module \(\hat{Q}\) which is ultra-singular for \(L\) and contains \(Q\).

**Proof.** Under these assumptions, there exists a vector field \(Q_{0}\) such that the equation \(\hat{L} = 0\) is equivalent to the equation \(Q_{0}[u] = 0\), where \(Q_{0}[u]\) is the characteristic of \(Q_{0}\). Consider the module \(Q\) spanned by \(Q\) and \(Q_{0}\). It is \(n\)-dimensional and satisfies the rank condition. Lemma 1 implies that \(Q\) is a reduction module of \(L\) if and only if it is a reduction module of the equation \(\hat{L} = 0\). The last condition is equivalent to the involution of the module \(Q\), and then the equation \(L\) is an identity on the well-defined manifold \(Q_{(r)}\), i.e., the module \(Q\) is ultra-singular for \(L\). The conditions that the \(n\)-dimensional module \(\hat{Q}\) is ultra-singular for \(L\) and contains \(Q\) uniquely determine \(\hat{Q}\). \(\square\)

**Remark 5.** The assumption that the differential function \(\hat{L}\) is of maximal rank with respect to the unique first-order derivative of \(u\) appearing in \(\hat{L}\) can be replaced by the weaker supposition that the equation \(\hat{L} = 0\) is equivalent to the equation \(\hat{L} = 0\), where the differential function \(\hat{L}\) satisfies the above condition of maximal rank.

In order to have a richer theory, we shall consider families of \((n-1)\)-dimensional co-order one singular reduction modules parameterized by arbitrary functions. Thus let \(L\): \(L = 0\) be a differential equation, with \(L = L[u]\) a differential function of order \(r > 1\). We assume that the function \(L\) admits a first co-order \(n\)-dimensional meta-singular module \(M\) of vector fields. (Due to Corollary 4 we may restrict ourselves to considering only strongly singular modules of vector fields for differential equations.) As the special case of two independent variables when metasingular modules may be non-involutive was considered in [15], in what follows we additionally assume that the module \(M\) is involutive.

Up to a change of variables we may suppose that the module \(M\) contains a first co-order singular family \(\mathfrak{M}\) = \(\{Q^{\Phi}\}\) of \((n-1)\)-dimensional modules in reduced form parameterized by an arbitrary smooth function \(\Phi\) of \((x,u)\), i.e., \(Q^{\Phi} = \langle \partial_{s} + \eta^{s}\partial_{u} \rangle\), where \(\eta^{s} = -\Phi_{s}/\Phi_{u}\) and the index \(s\) runs from 1 to \(p = n - 1\). We use the representation for \(\eta^{s}\) from the very beginning.

By Theorem 1 the differential function \(L\) can be written in the form \(L = \hat{L}(x, \Omega_{(r,1,n-1)})\), where \(\Omega_{(r,1,n-1)} = (u_{\alpha}, \alpha_{n} \leq 1, |\alpha| \leq r)\), \(\hat{L}_{u_{\alpha}} = 0\) for some \(u_{\alpha}\) with \(\alpha_{n} = 1\). In this case the restriction of \(L\) to \(Q_{(r)}\) coincides with the restriction, to the same manifold \(Q_{(r)}\), of the function \(\hat{L}^{\Phi} = \hat{L}(x, \Omega_{(r,1,n-1)})\), where

\[
\hat{\Omega}_{(r,1,n-1)} = (D_{n}^{\alpha_{n}}(Q_{1}^{\Phi})^{\alpha_{1}} \cdots (Q_{n-1}^{\Phi})^{\alpha_{n-1}} u_{\alpha}, \alpha_{n} \leq 1, |\alpha| \leq r). 
\]
Consequently, the form of $\hat{L}_\Phi$ is determined by the form of $L$ and a chosen value of the parameter-function $\Phi$. Depending on the value of $\Phi$, the differential function $\hat{L}_\Phi$ may either identically vanish or be of order 0 or 1. Thus either the module $Q^\Phi$ is ultra-singular or $\text{sco}_L Q^\Phi = 0$ or $\text{sco}_L Q^\Phi = 1$. We analyze each of these cases separately. In addition, we suppose that the function $\hat{L}_\Phi$ is of maximal rank with respect to $u$ (resp. $u_n$) if $\text{sco}_L Q^\Phi = 0$ (resp. $\text{sco}_L Q^\Phi = 1$).

The condition $\hat{L}_\Phi = 0$, where $u$ and $u_n$ are considered as independent variables, determines those values of $\Phi$ for which the module $Q^\Phi$ is ultra-singular for $L$. We split this condition with respect to $u_n$, thereby obtaining a system $S_{-1}$ of partial differential equations in $\Phi$ of orders not greater than $r$, which may be incompatible in the general case. Incompatibility here amounts to the family $\mathfrak{M}$ containing no ultra-singular modules. E.g., evolution equations of orders greater than 1 have no ultra-singular modules generated by vector fields of the general form (7), see Section 7. That $\Phi$ satisfies the ultra-singularity condition $S_{-1}$ guarantees that $Q^\Phi \in \mathcal{R}^{n-1}(L)$ and the family of $Q^\Phi$-invariant solutions of $L$ is parameterized by an arbitrary function of the single $Q^\Phi$-invariant variable $x_n$.

Under the assumption $\text{sco}_L Q^\Phi = 0$ it follows that the parameter-function $\Phi$ satisfies the condition $\hat{L}_{u_n} = 0$ with $u$ and $u_n$ viewed as independent variables, which is weaker than the ultra-singularity condition. In this case the corresponding system $S_0$ of partial differential equations in $\Phi$ of orders not greater than $r$, obtained by splitting the zero co-order singularity condition with respect to $u_n$, is more likely to be compatible than $S_{-1}$.

Next we mention certain sufficient conditions for the compatibility of $S_0$. If $\hat{L}_{u_n} = 0$ then the system $S_0$ is compatible since it is satisfied by any $\Phi$ with $(\Phi_s/\Phi_u)_{u_n} = 0$. Up to equivalence of functions parameterizing modules from the family $\mathfrak{M}$, we can assume that $\Phi = u - \zeta(x)$ and $\eta^s = \zeta_s$, i.e., $Q^{u-\zeta} = (\partial_s + \zeta_s \partial_u)$. In other words, we have $\text{sco}_L Q^{u-\zeta} \leq 0$ for any $\zeta = \zeta(x)$.

Assuming $\hat{L}_u = 0$, the condition $\hat{L}^{u-\zeta} = 0$ under the assumption $\zeta = \zeta(x)$ implies only a single partial differential equation with respect to $\zeta$. Any of its solutions is a solution of $S_{-1}$ and hence the corresponding module $Q^{u-\zeta}$ is ultra-singular for $L$.

Otherwise $\text{sco}_L Q^{u-\zeta} = 0$ and we can resolve the equation $L$ with respect to $u$ as a variable of the underlying jet space. In other words, we represent this equation in the form $u = K[u]$. Here $K[u]$ is a differential function which depends at most on the variables $x$ and $u_\alpha$ with $0 < |\alpha| \leq r$, $\alpha_n \leq 1$ and, if $\alpha_n = 1$, $|\alpha| > 1$. The equation $\hat{L}^{u-\zeta} = 0$ is obtained via replacing $u_s$ by $\zeta_s(x)$ in $L$. Therefore, it can be represented in the form $u = G^\zeta(x)$, where $G^\zeta(x) = K[u]|_{u=\zeta(x)}$. Thus, the expression for the function $G^\zeta$ involves derivatives of the parameter-function $\zeta = \zeta(x)$ up to order $r$. The conditional invariance of the equation $L$ with respect to the module $Q^{u-\zeta}$ is equivalent to the compatibility of the system $u = G^\zeta$, $u_s = \zeta_s$ with respect to $u$ and hence leads to the system of $n - 1$ partial differential equations $\zeta_s = G^s$ with respect to $\zeta$. Here $G^s$ is interpreted as a differential function of $\zeta = \zeta(x)$. Then $G^s$ is the total derivative of $G^\zeta$ with respect to $x_s$. As the function-parameter $\zeta$ is defined for a fixed module up to a constant summand, the system is equivalent to the single partial differential equation $\zeta = G^\zeta$. This means that $Q^{u-\zeta}$ is a reduction module of the equation $L$ if and only if up to shifts of the dependent variable the parameter-function $\zeta$ is a solution of the same equation. The ansatz constructed with $Q^{u-\zeta}$ can be taken in the form $u = \varphi(\omega) + \zeta(x)$, where $\varphi = \varphi(\omega)$ is the new unknown function and $\omega = x_n$ is the invariant independent variable. It reduces the initial equation $L$ to the trivial algebraic equation $\varphi = 0$, i.e., the function $u = \zeta(x)$ is the unique $Q^{u-\zeta}$-invariant solution of $L$. Conversely, we fix a solution $u = \zeta(x)$ of the equation $L$. Then $\zeta = G^\zeta(x)$ and hence $\zeta_s = G^s$, which is equivalent to the conditional invariance criterion for the case of the module $Q^{u-\zeta} = (\partial_s + \zeta_s \partial_u)$ and the equation $L$, i.e., $Q^{u-\zeta}$ is a reduction module of $L$, and $\zeta_n = 0$. The solution $u = \zeta(x)$ is invariant with respect to $Q^{u-\zeta}$ by construction. Thus we obtain:

**Theorem 5.** Suppose that an equation $L$: $L = 0$ possesses a first co-order singular family $\mathfrak{M} = \{Q^\Phi\}$ of $(n-1)$-dimensional modules in reduced form $Q^\Phi = \langle \partial_s - (\Phi_s/\Phi_u) \partial_u \rangle$ parameterized by an arbitrary smooth function $\Phi$ of $(x, u)$, i.e., the left hand side $L$ of this equation is represented in
the form \( L = \tilde{L}(x, \Omega; r) \), where \( \Omega = (u_n, \alpha_n \leq 1, |\alpha| \leq r) \), \( \tilde{L}_{u_n} \neq 0 \) for some \( u_n \) with \( \alpha_n = 1 \), and additionally \( L_{u_n} = 0 \) and \( \tilde{L}_u \neq 0 \). Then there exists a one-to-one correspondence between solutions of \( L \) and reduction modules from \( \mathfrak{M} \) with \( \Phi = u - \zeta(x) \). Namely, any such module is of singularity co-order 0 and corresponds to the unique solution which is invariant with respect to this module. The problems of solving the equation \( \mathcal{L} \) and the exhaustive description of its reduction modules of the above form are completely equivalent.

Next we turn to analyzing the regular values of \( \Phi \) for which the singularity co-order of \( Q^\Phi \) coincides with the singularity co-order of the whole family \( \mathfrak{M} \) (and equals 1). If \( scoL Q^\Phi = 1 \), the parameter-function \( \Phi \) satisfies the regularity condition \( \tilde{L}_{u_n} \neq 0 \). Thus the equation \( \tilde{L}^\Phi = 0 \) which is equivalent to \( L \) on the manifold \( Q^\Phi(x) \), can be solved with respect to \( u_n \): \( u_n = G^\Phi(x, u) \).

Here the expression for the function \( G^\Phi \) depends on derivatives of the parameter-function \( \Phi \) up to order \( r \). The conditional invariance criterion, when applied to the equation \( \mathcal{L} \) and the operator \( Q^\Phi \), gives the system

\[
\eta_n^s + \eta_s^u G^\Phi = G_s^\Phi + \eta_s^u G_u^\Phi
\]  

with respect to the function \( \Phi \) (recall that \( \eta^s = -\Phi_s/\Phi_u \)). This system coincides with the formal compatibility condition of the equations \( u_n = G^\Phi \) and \( u_s = \eta^s \) with respect to \( u \). As the function \( G^\Phi \) can be also expressed directly in terms of the coefficients \( \eta^s \), the system (10) supplemented with the involution condition \( \eta_n^s + \eta_s^u u_n^s = \eta_n^s + \eta^s u_n^s \) can be interpreted as a system with respect to \( \eta^s \), cf. Section 7.

In fact, the system (10) is equivalent to a single equation in \( \Phi \). Indeed,

\[
G_s^\Phi + \eta^s G_u^\Phi - \eta_n^s - \eta_s^u G^\Phi = G_s^\Phi - \frac{\Phi_s}{\Phi_u} G_u^\Phi + \left( \frac{\Phi_s}{\Phi_u} \right) u_n^s + \left( \frac{\Phi_s}{\Phi_u} \right) u_s^u G^\Phi = 0,
\]

i.e., system (10) is equivalent to the equation \( \Phi_n + \Phi_u G^\Phi = \chi(x_n, \Phi) \), where \( \chi \) is an arbitrary smooth function of its arguments. The function \( \Phi \) associated with a fixed module \( Q^\Phi \) is defined up to the transformation \( \tilde{\Phi} = \theta(x_n, \Phi) \). As \( \tilde{\eta}^s = -\Phi_s/\Phi_u = -\Phi_s/\Phi_u = \eta^s \), the functions \( G^\Phi \) and \( G^\Phi \) coincide. At the same time, if we choose \( \theta \) satisfying the equation \( \theta_n + \chi \theta = 0 \) then \( \tilde{\Phi}_n + \tilde{\Phi}_u G^\Phi = 0 \). Therefore, up to the equivalence on the set of functions parameterizing singular modules we can assume that the function \( \Phi \) is a solution of the equation \( \Phi_n + \Phi_u G^\Phi = 0 \).

The order of each equation from system (10) with respect to \( \Phi \) equals \( r + 1 \) and hence is greater than the order of the system \( \mathcal{S}_0 \). Under certain smoothness assumptions (e.g., analyticity) this implies that the equation (10) has solutions which are not solutions of \( \mathcal{S}_0 \). Consequently, the equation \( \mathcal{L} \) necessarily possesses first co-order singular reduction modules which belong to \( \mathfrak{M} \).

**Proposition 14.** Suppose that any \((n-1)\)-dimensional involutive module of the reduced form \( Q = (\partial_s + \eta^s \partial_u) \) is a first co-order singular module for an equation \( \mathcal{L} \). Then the determining system for values of \( \eta^s \) associated with reduction modules of \( \mathcal{L} \) is reduced by the composition of the nonlocal substitution \( \eta^s = -\Phi_s/\Phi_u \), where \( \Phi \) is a smooth function of \((x, u)\) with \( \Phi_u \neq 0 \), and the hodograph transformation

- the new independent variables: \( \tilde{x}_s = x_s \), \( \tilde{x} = \Phi \),
- the new dependent variable: \( \tilde{u} = u \)

to the initial equation \( \mathcal{L} \) for the function \( \tilde{u} = \tilde{u}(\tilde{x}, \tilde{x}) \) with \( \tilde{x} \) playing the role of a parameter.

**Proof.** The possibility of representing \( \eta^s \) in the form \( \Phi_s/\Phi_u \) with some function \( \Phi = \Phi(x, u) \) follows from the involution condition \( \eta_n^s + \eta_s^u u_n^s = \eta_n^s + \eta^s u_n^s \) for the module \( Q \). In this representation, the system of determining equations for values of \( \eta^s \) associated with reduction modules
of \(\mathcal{L}\) has the form (10) and is equivalent, up to equivalence on the set traversed by the parametric function \(\Phi\), to the single equation \(\Phi_n + \Phi_u G^\Phi = 0\). In view of the definition of the function \(G^\Phi\) and the expressions for derivatives under the hodograph transformation, \(\tilde{u}_{\tilde{x}_i} = -\Phi_i/\Phi_u\), etc., the hodograph transformation maps the equation \(\Phi_n + \Phi_u G^\Phi = 0\) into to the initial equation \(\mathcal{L}\) for the function \(\tilde{u} = \tilde{u}(\tilde{x}, \kappa)\).

The above connection between the initial equation \(\mathcal{L}\) and the determining equation (10) can be stated as a relation between one-parametric families of solutions and \((n-1)\)-dimensional first co-order singular reduction modules. The results of Section 5 imply that for each \((n-1)\)-dimensional first co-order singular reduction module \(Q\) of the equation \(\mathcal{L}\) there exists a one-parametric family of \(Q\)-invariant solutions of \(\mathcal{L}\). If the equation \(\mathcal{L}\) admits an \(n\)-dimensional meta-singular module of singularity co-order one, the converse statement is true as well. It is convenient to prove this statement without transforming the meta-singular module to the reduced form.

**Theorem 6.** Suppose that an equation \(\mathcal{L}\) possesses an \(n\)-dimensional first co-order meta-singular module \(M\). Then for any one-parametric family \(\mathcal{F}\) of solutions of \(\mathcal{L}\) there exists an \((n-1)\)-dimensional involutive submodule \(Q\) of \(M\) that is a reduction module of \(\mathcal{L}\) and each solution from \(\mathcal{F}\) is \(Q\)-invariant.

**Proof.** Let \(\mathcal{F} = \{ u = f(x, \kappa) \}\) be a one-parametric family of solutions of \(\mathcal{L}\). Here, \(f_\kappa\) is nonzero since the parameter \(\kappa\) is essential. From \(u = f(x, \kappa)\) we conclude that \(\kappa = \Phi(x, u)\) with some function \(\Phi = \Phi(x, u)\) with \(\Phi_u \neq 0\).

Let \(\{ Q_0, \ldots, Q_n \}\) be a basis of \(M\). Suppose that there exists \(\sigma \in \{0, \ldots, n\}\) such that \(Q_\sigma \Phi \neq 0\). Up to permutation of basis elements we can assume that \(Q_0 \Phi \neq 0\). Then we set \(Q_\sigma = Q_\sigma - (Q_\sigma \Phi)/(Q_0 \Phi) Q_0\). If \(Q_\sigma \Phi = 0\) for all \(\sigma \in \{0, \ldots, n\}\), we set \(Q_\sigma = Q_1\).

Consider the submodule \(Q\) generated by the vector fields \(Q_\sigma\). This submodule is involutive and of dimension \(n-1\). Hence \(\text{sc}Q \Phi^\sigma \leq 1\). It is also obvious that \(Q_\sigma \Phi = 0\). As \(f_i = -(\Phi_i/\Phi_u)|_{u = f}\), this means that any solution from the family \(\mathcal{F}\) is \(Q\)-invariant. The case \(\text{sc}Q = 0\) is impossible as otherwise the equation \(\mathcal{L}\) could not have a one-parametric family of \(Q\)-invariant solutions. Therefore either \(Q\) is an ultra-singular module for \(\mathcal{L}\) or \(\text{sc}Q = 1\). Any ultra-singular module for \(\mathcal{L}\) is a reduction module of \(\mathcal{L}\). If \(\text{sc}Q = 1\) then \(Q\) is a reduction module of \(\mathcal{L}\) by Proposition 8.

For the correspondence between one-parametric families of solutions and \((n-1)\)-dimensional reduction modules to be one-to-one, the related meta-singular module should satisfy additional restrictions.

**Theorem 7.** Suppose that an equation \(\mathcal{L}\) possesses an \(n\)-dimensional first co-order meta-singular module \(M\) and all \((n-1)\)-dimensional submodules of \(M\) as well as the entire module \(M\) are not ultra-singular for \(\mathcal{L}\). Then up to the equivalence of solution families there exists a bijection between one-parametric families of solutions of \(\mathcal{L}\) and its \((n-1)\)-dimensional reduction modules contained in \(M\). Namely, each module of this kind corresponds to the family of solutions which are invariant with respect to it.

In other words, the problems of the construction of all one-parametric solution families of the equation \(\mathcal{L}\) and the exhaustive description of its reduction modules of the above form are completely equivalent.

**Proof.** If \(Q\) is an \((n-1)\)-dimensional reduction module of \(\mathcal{L}\) contained in \(M\), we have \(\text{sc}Q = 1\). Therefore, in view of Proposition 8 the equation \(\mathcal{L}\) possesses a one-parametric family \(\mathcal{F}\) of \(Q\)-invariant solutions, and any \(Q\)-invariant solution of \(\mathcal{L}\) belongs to this family. Each one-parametric family of \(Q\)-invariant solutions of \(\mathcal{L}\) is obtained from \(\mathcal{F}\) by re-parameterizing.

Conversely, consider a one-parametric family \(\mathcal{F} = \{ u = f(x, \kappa) \}\) of solutions of \(\mathcal{L}\). Theorem 6 implies that there exists \(Q \in \mathcal{R}^{n-1}(\mathcal{L})\) such that \(Q\) is a submodule of \(M\) and any solution
from $\mathcal{F}$ is $Q$-invariant. Let us prove that the module $Q$ is unique. Suppose that there exists an $(n-1)$-dimensional involutive submodule $\tilde{Q}$ of $M$ different from $Q$ and any solution from $\mathcal{F}$ is $Q$-invariant. This implies that the family $\mathcal{F}$ consists of solutions invariant with respect to the entire module $M$.

Therefore, the module $M$ satisfies the rank condition. To show this, we fix a basis \( \{Q_0, \ldots, Q_p\} \) of $M$, where $Q_\sigma = \xi^{\sigma_1}(x, u)\partial_1 + \eta^{\sigma}(x, u)\partial_u$, $\sigma = 0, \ldots, n$. The function $\Phi$ defined in the proof of Theorem \( \square \) is an invariant of all $Q_\sigma$, i.e., $Q_\sigma\Phi = 0$. As $\Phi_u \neq 0$, we have that $\text{rank}(\xi^{\sigma_1}) = n$.

As $M$ is an $n$-dimensional involutive module satisfying the rank condition and the family $\mathcal{F}$ formed by $M$-invariant solutions of the equation $\mathcal{L}$ is one-parametric, in view of Proposition \( \blacksquare \) the module $M$ is ultra-singular for $\mathcal{L}$, thereby contradicting an assumption of the theorem. This means that the module $Q$ is unique.

The bijection discussed in Theorem \( \blacksquare \) is broken in the presence of $(n-1)$-dimensional ultra-singular submodules or ultra-singularity of the entire meta-singular module. Indeed, if an $(n-1)$-dimensional involutive module $Q$ is ultra-singular for the equation $\mathcal{L}$, the family of $Q$-invariant solutions of $\mathcal{L}$ is parameterized by an arbitrary function of a single argument, cf. Proposition \( \blacksquare \)

If the entire $n$-dimensional module $M$ is ultra-singular for the equation $\mathcal{L}$, this equation possesses a one-parametric family $\mathcal{F}$ of $M$-invariant solutions. Then any solution from the family $\mathcal{F}$ is invariant with respect to each $(n-1)$-dimensional involutive submodule of $M$.

9 Singular modules for quasi-linear second-order PDEs

For some classes of differential equations it is possible to exhaustively describe the associated singular modules. We study certain quasi-linear second-order PDEs from this point of view.

It is natural to distinguish elliptic, evolution and generalized wave equations. In all cases, $Q$ is an involutive module of vector fields defined in the corresponding space of independent and dependent variables and satisfies the rank condition, and the dimension of $Q$ is less than the number of independent variables.

**Elliptic equations.** Consider an equation $\mathcal{L}$ for the single unknown function $u$ of the independent variables $x = (x_1, \ldots, x_n)$, having the general form

$$L[u] := a^{ij}(x, u_{(1)})u_{ij} + b(x, u_{(1)}) = 0,$$

where the coefficients $a^{ij}$ and $b$ are defined on the same domain $\Omega$ of the first-order jet space, $a^{ij} = a^{ji}$ and the matrix-function $(a^{ij})$ is positive definite in each point of $\Omega$. We will prove that the equation $\mathcal{L}$ possesses no singular modules of dimensions less than $n$.

Denote $\dim Q$ by $p$, $0 < p < n$. Up to permutation of $x$ we can locally choose a basis of $Q$ which consists of vector fields of the form $Q_\sigma = \partial_\sigma + \xi^{\sigma_1}(x, u)\partial_1 + \eta^{\sigma}(x, u)\partial_u$, cf. Section 8. Here and in what follows the index $\iota$ runs from $p+1$ to $n$. Then any derivative of $u$ of order one or two is expressed, on the manifold $Q_{(2)}$, via derivatives of $u$ with respect to $x_{p+1}, \ldots, x_n$ only and the coefficients $\xi^{\sigma_1}$ and $\eta^{\sigma}$, see equation (1). As only second-order terms in the expressions for second-order derivatives of $u$ are essential here, we use the representations

$$u_{\sigma_1} = -\xi^{\sigma_1'}u_{\iota'^1} + R^{\sigma_1}(x, u_{(1)}), \quad u_{\sigma_1'\iota'} = \xi^{\sigma_1'}\xi^{\iota'^1'}u_{\iota'\iota'} + R^{\sigma_1'\iota'}(x, u_{(1)}),$$

where $R^{\sigma_1}$ denote the terms without second-order derivatives. Substituting the above expressions for the derivatives $u_{\sigma_1}$ into $L[u]$, we obtain the differential function $\tilde{L}[u] := \tilde{a}^{ij}u_{ij} + \tilde{b}$, which is associated with $L[u]$ on the manifold $Q_{(2)}$. Here $\tilde{a}^{ij} = a^{ij} - a^{\sigma_1}\xi^{\sigma_1'} + a^{\sigma_1'\iota'}\xi^{\sigma_1'} + a^{\sigma_1'\iota'}\xi^{\sigma_1'}$ and $\tilde{b} = b + a^{\sigma_1}R^{\sigma_1} + a^{\sigma_1'\iota'}R^{\sigma_1'\iota'}$. In other words, for each fixed $\iota$ and $\iota' = \iota$ the coefficient $\tilde{a}^{\iota'^1}$ coincides with the value of the quadratic form whose matrix is $(a^{ij})$ at the tuple $(z^1, \ldots, z^n)$, where $z^1 = -\xi^{\iota'^1}$, $z^1 = 1$ and the other $z$’s equal zero. As the matrix $(a^{ij})$ is positive definite, the coefficient $(a^{ij})$ is nonvanishing.
This implies that the differential function $L[u]$ cannot coincide on the manifold $Q_{(2)}$ with a differential function of order less than two, not even up to a nonvanishing multiplier. Therefore, \( \text{wsc}_{Q} Q = \text{soc}_{Q} Q = 2 \).

**Evolution equations.** Similarly to Section 7 for this and the next class of equations we set the number of independent variables to $n + 1$ instead of $n$ and additionally single out the variable $t = x_0$, i.e., the unknown function $u$ depends on the variables $t$ and $x = (x_1, \ldots, x_n)$. The general form of a quasi-linear second-order evolution equation $\mathcal{L}$ we intend to study is

$$ u_t = H[u] := a^{ij}(t, x, u_{(1, x)})u_{ij} + b(t, x, u_{(1, x)}), \quad (12) $$

where $u_{(1, x)} = (u, u_1, \ldots, u_n)$, the coefficients $a^{ij}$ and $b$ are defined on the same domain $\Omega$ as traversed by their arguments, and the matrix-function $(a^{ij})$ is symmetric and positive definite in each point of $\Omega$. Up to permutation of the variables $x_i$, any involutive module $Q$ of vector fields defined in the space of $(t, x, u)$ locally assumed to be spanned either by the vector fields $\partial_t + \xi^0 \partial_u, \partial_s + \xi^{*0} \partial_u, \partial_s + \xi^{0*} \partial_u$ with $p = \dim Q - 1$ or by the vector fields $\partial_s + \tau^0 \partial_u, \partial_s + \eta^0 \partial_u$ with $p = \dim Q$. Here $\xi^0, \eta^0, \xi^{*0}$ and $\eta^{*0}$ are smooth functions of $(t, x, u)$. We can show similarly as for elliptic equations that in the second case with $p = n$ and only in this case the module $Q$ is singular for $\mathcal{L}$, and \( \text{wsc}_{Q} Q = \text{soc}_{Q} Q = 1 \). This is the case that has been studied in Section 7.

**Generalized wave equations.** Consider the equation $\mathcal{L}$: $u_{tt} = H[u]$ for the single unknown function $u$ of the independent variables $t$ and $x = (x_1, \ldots, x_n)$, where the differential function $H = H[u]$ is defined analogously to (12) but the coefficients $a^{ij}$ and $b$ may additionally depend on $u_t$. We partition the set of appropriate involutive modules in a way different from that used for evolution equations. Up to permutation of the variables $x$, any involutive module $Q$ of vector fields defined in the space of $(t, x, u)$ and satisfying the rank condition, where $\dim Q < n + 1$, can be locally assumed to be spanned either by the vector fields $\partial_t + \eta^0 \partial_u, \partial_s + \xi^0 \partial_u, \partial_s + \xi^{*0} \partial_u$ with $p = \dim Q - 1$ or by the vector fields $\partial_s + \tau^0 \partial_u, \partial_s + \eta^0 \partial_u$ with $p = \dim Q$. Here $\eta^0$, $\tau^0$, $\xi^{*0}$ and $\eta^{*0}$ are smooth functions of $(t, x, u)$. Only in the second case with $p = n$ the module $Q$ may be singular for $\mathcal{L}$. We consider this case in detail.

As $p = n$, the basis elements of $Q$ take the form $Q^s = \partial_s + \tau^0 \partial_t + \eta^0 \partial_u$. On the manifold $Q_{(2)}$ we have that $u_s = \eta^0 - \xi^{*0} u_t, u_{ss} = \xi^0 \xi^{*0} u_{tt} + R^{ss'}$, where $R^{ss'}$ denotes the corresponding collection of the terms depending at most on $t, x, u, u_t$ and $u_{tt}$. Substituting the above expressions for the derivatives of $u$ with respect to $x$ into the differential function $L[u] = u_{tt} - H[u]$, we obtain the differential function $\hat{L}[u] := (1 - \hat{a}^{ij} \tau^0 \tau^j) u_{tt} + \hat{b}(t, x, u, u_t)$, which is associated with $L[u]$ on the manifold $Q_{(2)}$ and depends at most on $t, x, u$ and $u_t$. Here the coefficient $\hat{a}^{ij} = \tilde{a}^{ij}(t, x, u, u_t)$ is obtained from $a^{ij}$ by the substitution $u_s = \eta^0 - \xi^{*0} u_t$ and the precise form of the coefficient $\hat{b} = \tilde{b}(t, x, u, u_t)$ is not essential. The module $Q$ is singular for $\mathcal{L}$ if and only if ord $\hat{L}[u] < 2$, i.e., the coefficients $\tau^s = \tau^s(t, x, u)$ satisfy the equation $\hat{a}^{ij} \tau^j \tau^j = 1$. If some of the coefficients $\hat{a}^{ij}$ depend on $u_t$, this equation should be split with respect to this derivative and hence it may be inconsistent.

As the module $Q$ is involutive and, therefore, the vector fields $Q^s$ commute, the coefficients $\tau^s$ and $\eta^s$ jointly satisfy the system $Q^s \tau^s = Q^s \tau^s, Q^s \eta^s = Q^s \eta^s$, i.e.,

$$
\tau^s + \tau^s \tau^t = \tau^s + \tau^s \tau^t + \eta^s \tau^u, \\
\eta^s + \tau^s \eta^t = \eta^s + \tau^s \eta^t + \eta^s \eta^u.
$$

In view of the Frobenius theorem this implies that the system $Q^s \Phi = 0$ with respect to the unknown function $\Phi = \Phi(t, x, u)$ admits solutions $\Phi^l, l = 1, 2$, such that $\Phi^1 \Phi^2 - \Phi^2 \Phi^1 \neq 0$. Solving the pair of the equations $\Phi^l_{ss} + \tau^s \Phi^l + \eta^s \Phi^l = 0$ for each fixed $s$ as a system of linear algebraic equations with respect to $($\(\tau^s, \eta^s\)$), we derive the representation

$$
\tau^s = \frac{\Phi^1 \Phi^2 - \Phi^2 \Phi^1}{\Phi^1 \Phi^2 - \Phi^2 \Phi^2}, \quad \eta^s = \frac{-\Phi^1 \Phi^2 - \Phi^2 \Phi^1}{\Phi^1 \Phi^2 - \Phi^2 \Phi^2}. \quad (13)
$$

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Conversely, it can be checked by direct calculation that for arbitrary functions $\Phi^1$ and $\Phi^2$ with $\Phi^1_t\Phi^2_u - \Phi^1_u\Phi^2_t \neq 0$ the module $Q$ spanned by the vector fields $Q^a = \partial_s + \tau^s\partial_t + \eta^s\partial_u$ with the coefficients defined by (13) is involutive. Up to functional dependence of pairs of functions $\Phi^1$ and $\Phi^2$ with $\Phi^1_t\Phi^2_u - \Phi^1_u\Phi^2_t \neq 0$, there exists a bijection between such pairs and involutive modules spanned by $n$ vector fields of the form $Q^a = \partial_s + \tau^s\partial_t + \eta^s\partial_u$.

If the coefficients $\hat{a}^{ij}$ do not depend on $u_t$ (this is the case if, e.g., $a^{ij} = a^{ij}(t,x,u)$), the substitution of the expressions (13) for $\tau^s$ into the equation $\hat{a}^{ij}\tau^i\tau^j = 1$ gives a single equation in the two unknown functions $\Phi^1$ and $\Phi^2$. Each solution of this equation is associated with a singular module $Q$ for the equation $\mathcal{L}$. In general, in the multidimensional case $n > 1$, the coefficients $\tau^s$ and $\eta^s$ of the corresponding basis elements $Q^a$ are coupled in a nonlocal and nonlinear way. This is why it is impossible to exhaustively describe singular modules of multidimensional nonlinear wave equations within the framework of meta-singular modules. At the same time, under additional conditions for the coefficients $a^{ij}$ nonlinear wave equations possess families of singular modules which fit well into the above framework.

Thus, let the coefficients $\hat{a}^{ij}$ only depend on $t$ and $x$. We look for singular modules of $\mathcal{L}$ for which the corresponding coefficients $\tau^s$ also do not depend on $u$. It is then sufficient to consider pairs of functions $\Phi^1$ and $\Phi^2$ with $\Phi^1_u = 0$ and, therefore, $\Phi^1_t\Phi^2_u \neq 0$. Then the expressions (13) for $\tau^s$ are reduced to $\tau^s = -\Phi^1_u/\Phi^1_t$. The equation $\hat{a}^{ij}\tau^i\tau^j = 1$ is equivalent to the equation $(\Phi^1)^2 = a^{ij}\Omega^i\Omega^j$, which is the eiconal equation associated with the wave equation $\mathcal{L}$. We fix a solution $\Psi = \Psi(t,x)$ of the eiconal equation with $\Psi_t \neq 0$ and consider the module $M^\Psi$ spanned by the vector fields $\Psi_t\partial_t - \Psi_x\partial_s$, $s = 1, \ldots, n$, and $\partial_u$. The module $M^\Psi$ is meta-singular for the equation $\mathcal{L}$, and $\text{soc}_\mathcal{L} M^\Psi \leq 1$ for each $\Psi$. Hence, the results of Section 8 are relevant in this case.

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