DIRECT LIMIT CLOSURE OF INDUCED QUIVER REPRESENTATIONS

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Abstract. In 2004 and 2005 Enochs et al. characterized the flat and projective quiver-representations of left rooted quivers. The proofs can be understood as filtering the classes $\Phi(\text{Add } X)$ and $\Phi(\text{lim } X)$ when $X$ is the finitely generated projective modules over a ring. In this paper we generalize the above and show that $\Phi(X)$ can always be filtered for any class $X$ in any AB5-abelian category. With an emphasis on $\Phi(\text{lim } X)$ we investigate the Gorenstein homological situation. Using an abstract version of Pontryagin duals in abelian categories we give a more general characterization of the flat representations and end up by describing the Gorenstein flat quiver representations over right coherent rings.

Introduction

Let $Q$ be a quiver (i.e. a directed graph) and consider for a class $\mathcal{X}$ of objects in an abelian category $\mathcal{A}$ the class $\Phi(\mathcal{X}) \subseteq \text{Rep}(Q, \mathcal{A})$ of quiver representations. This is the class containing all representations, $F$, s.t. the canonical map $\bigoplus_{w \to v} F(w) \to F(v)$ is monic and has cokernel in $\mathcal{X}$ for all vertices $v$ - the sum being over all arrows to $v$. When $Q$ is left-rooted (i.e $Q$ has no infinite sequence of composable arrows of the form $\cdots \to \bullet \to \bullet \to \bullet$) it was observed by Enochs, Oyonarte and Torrecillas in [10] and Enochs and Estrada in [7] that when $\mathcal{A}$ is the category of modules over a ring, $\Phi(\text{Proj}(\mathcal{A})) = \text{Proj}(\text{Rep}(Q, \mathcal{A}))$, and $\Phi(\text{Flat}(\mathcal{A})) = \text{Flat}(\text{Rep}(Q, \mathcal{A}))$.

Here the flat objects are precisely the direct limit closure of the finitely generated projective objects. This was done by showing, that if $\mathcal{X}$ is the finitely generated projective modules over a ring we can filter the classes $\Phi(\text{Add } \mathcal{X})$ and $\Phi(\text{lim } \mathcal{X})$ by sums of objects of the form $f_\nu(\mathcal{X})$ where $f_\nu: \mathcal{A} \to \text{Rep}(Q, \mathcal{A})$ is the left-adjoint of the evaluation functor $e_\nu: \text{Rep}(Q, \mathcal{A}) \to \mathcal{A}$ at the vertex $\nu$. They show $\Phi(\text{Add } \mathcal{X}) = \text{Add } f_\nu(\mathcal{X})$ and $\Phi(\text{lim } \mathcal{X}) = \text{lim } \text{add } f_\nu(\mathcal{X})$.

In 2014 Holm and Jørgensen [14] generalized [10] to abelian categories with enough projective objects, and combining [14, Thm. 7.4a and 7.9a] with Štovíček [20, Prop. 1.7] we get the following generalization of [3]. If $\mathcal{X}$ is a generating set of objects in a Grothendieck abelian category, then $\Phi(\text{sFilt } \mathcal{X}) = \text{sFilt } f_\nu(\mathcal{X})$,
where sFilt \( \mathcal{X} \) consists of all summands of \( \mathcal{X} \)-filtered objects. In this paper we show that \( \Phi(\mathcal{X}) \) can always be filtered by \( \bigoplus f_*(\mathcal{X}) \) in the following sense.

**Theorem A.** Let \( \mathcal{A} \) be an AB5-abelian category, let \( \mathcal{X} \subset \mathcal{A} \) and let \( Q \) be a left-rooted quiver. Then

i) Any \( F \in \Phi(\mathcal{X}) \) is \( \bigoplus f_*(\mathcal{X}) \)-filtered.

If \( \mathcal{X} \) is closed under filtrations, then

ii) \( \Phi(\mathcal{X}) = \text{Filt} \bigoplus f_*(\mathcal{X}) \)

In particular we have the following. If \( \mathcal{X} \) is a set, then

iii) \( \Phi(\text{Filt } \mathcal{X}) = \text{Filt } f_*(\mathcal{X}) = \text{Filt } \Phi(\mathcal{X}) \)

iv) \( \Phi(\text{sFilt } \mathcal{X}) = \text{sFilt } f_*(\mathcal{X}) = \text{sFilt } \Phi(\mathcal{X}) \)

If \( X \subseteq FP_{2.5}(\mathcal{A}) \) and \( \mathcal{A} \) is locally finitely presented, then

v) \( \Phi(\lim_{\to} Gproj(\mathcal{A})) = \lim_{\to} \text{ext } f_*(\mathcal{X}) = \lim_{\to} \Phi(\mathcal{X}) \)

Here \( FP_{2.5}(\mathcal{A}) \) is a certain class of objects which sits between \( FP_2(\mathcal{A}) \) and \( FP_3(\mathcal{A}) \) with the property that it is always closed under extensions. In many situations (e.g. \( \mathcal{A} = \text{R-Mod} \)) \( FP_{2.5}(\mathcal{A}) = FP_2(\mathcal{A}) \) (Lemma 1.4).

We note that \( \lim \text{ext } \mathcal{X} = \lim \text{add } \mathcal{X} \) and \( \text{Add } \mathcal{X} = \text{sFilt } \mathcal{X} \) when \( \mathcal{X} \) consists of projective objects and that the finitely generated projective objects are \( FP_n \) for any \( n \). Theorem A is thus a generalization of (3) and (4). It also generalizes (5) to not necessarily generating sets in arbitrary AB5-abelian category. We show how to use this to reprove (1) in abelian categories with enough projective objects. We also show (2) (Lemma 2.12) when the category is generated by finitely generated projective objects and flat is understood as their direct limit closure (see Theorem C however for a more general version).

We then apply Theorem A v) to the Gorenstein homological situation. We let \( GProj(\mathcal{A}) \) be the Gorenstein projective objects, let \( Gproj(\mathcal{A}) = GProj(\mathcal{A}) \cap FP_{2.5}(\mathcal{A}) \) and immediately get \( \Phi(\lim_{\to} Gproj(\mathcal{A})) = \lim_{\to} \text{ext } f_*(Gproj(\mathcal{A})) \). Contrary to the case for ordinary projective objects, it is not clear, that this equals \( \lim_{\to} Gproj(\text{Rep}(Q, \mathcal{A})) \) without some restrictions on \( Q \). In the following target-finite means that there are only finitely many arrows with a given target and locally path-finite means that there are only finitely many paths between two given vertices. We have

**Theorem B.** Let \( \mathcal{A} \) be a locally finitely presented category with enough projective objects, let \( Q \) be a left-rooted quiver and assume that either

- \( Q \) is target-finite and locally path-finite, or
- \( \lim Gproj(\mathcal{A}) = \lim GProj(\mathcal{A}) \) (e.g if \( \mathcal{A} = \text{R-Mod} \) and \( R \) is Iwanaga-Gorenstein).

Then

\[
\Phi(\lim_{\to} Gproj(\mathcal{A})) = \lim_{\to} Gproj(\text{Rep}(Q, \mathcal{A})) = \lim_{\to} \Phi(\text{Gproj}(\mathcal{A})).
\]

In the latter case, this equals \( \lim_{\to} GProj(\text{Rep}(Q, \mathcal{A})) \).

Again contrary to the ordinary projective objects even for \( \mathcal{A} = \text{R-Mod} \) it is not true in general that \( \lim Gproj(\mathcal{A}) \) is all the Gorenstein Flat objects, \( GFlat(\mathcal{A}) \), nor those objects with Gorenstein injective Pontryagin dual, \( wGFlat(\mathcal{A}) \). In the rest of the paper we study these classes in \( \text{Rep}(Q, \mathcal{A}) \). First we must explain what we mean by an abstract Pontryagin dual and we show how these arise naturally and
agree with the standard notion in well-known abelian categories. We go on and characterize those objects with injective (or Gorenstein injective) Pontryagin dual as follows.

**Theorem C.** Let $\mathcal{A}$ be an abelian category with a Pontryagin dual to a category with enough injective objects and let $Q$ be a left-rooted quiver. Then

$$\text{Flat}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{Flat}(\mathcal{A}))$$

$$\text{wGFlat}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{wGFlat}(\mathcal{A}))$$

Here $\text{Flat}(\mathcal{A})$ is those objects with injective Pontryagin dual so this result reproves [2] using the simpler characterization of injective representations in Enochs, Estrada and García Rozas [8, Prop 2.1] instead of going through the proof of [4] as in [10]. Theorem C tells us that, under the conditions of Theorem B, if $\lim \rightarrow \text{Gproj}(\mathcal{A}) = \text{wGFlat}(\mathcal{A})$ then also $\lim \rightarrow \text{Gproj}(\text{Rep}(Q, \mathcal{A})) = \text{wGFlat}(\text{Rep}(Q, \mathcal{A}))$. (Corollary 4.7)

In [8] it is proved that $\text{wGFlat}(\text{Rep}(Q, \mathcal{A})) = \text{GFlat}(\text{Rep}(Q, \mathcal{A}))$ when $\mathcal{A} = \text{R-Mod}$ and $R$ is Gorenstein. We end this paper by showing that this also hold if $R$ is just assumed to be coherent if we impose proper finiteness conditions on $Q$.

**Theorem D.** Let $R$ be a right coherent ring and let $Q$ be a left-rooted and target-finite quiver. Then

$$\text{wGFlat}(\text{Rep}(Q, \text{R-Mod})) = \text{GFlat}(\text{Rep}(Q, \text{R-Mod}))$$

See also Proposition 5.6 for a version for abelian categories. If $Q$ is further locally path-finite (or $R$ is Gorenstein and $Q$ is just assumed to be left-rooted) the conditions for Theorem B and Theorem C are satisfied as well, so in this case (Corollary 5.8) if $\lim \rightarrow \text{Gproj}(\mathcal{A}) = \text{GFlat}(\mathcal{A})$ then

$$\lim \rightarrow \text{Gproj}(\text{Rep}(Q, \mathcal{A})) = \text{GFlat}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{GFlat}(\mathcal{A}))$$

The equality $\lim \rightarrow \text{Gproj}(\text{R-Mod}) = \text{GFlat}(\text{R-Mod})$ is known to hold when $R$ is an Iwanaga-Gorenstein ring (Enochs and Jenda [9, Thm. 10.3.8]) or if $R$ is an Artin algebra which is virtually Gorenstein (Beligiannis and Krause [3, Thm. 5]). In general $\lim \rightarrow \text{Gproj}(\text{R-Mod})$ and $\text{GFlat}(\text{R-Mod})$ are different (Holm and Jørgensen [13, Thm. A]).

1. **Locally finitely presented categories**

In the following let $\mathcal{A}$ be an abelian category. First we recall some basic notions.

We say $\mathcal{A}$ is (AB4) if $\mathcal{A}$ is cocomplete and forming coproducts is exact, (AB4*) if $\mathcal{A}$ is complete and forming products is exact, (AB5) if filtered colimits are exact, Grothendieck if it is (AB5) and has a generator (i.e. a generating object or equivalently a generating set). Here a class $\mathcal{I} \subseteq \mathcal{A}$ is said to generate $\mathcal{A}$ if it detects zero-morphisms i.e. a morphism $X \xrightarrow{f} Y$ is zero iff $S \xrightarrow{g} X \xrightarrow{f} Y$ is zero for all $g$ with $S \in \mathcal{I}$.

We write $X \in \lim \rightarrow \mathcal{I}$ if $X = \lim X_i$ for some filtered system $\{X_i\} \subseteq \mathcal{I}$. We write $X \in \text{Filt} \mathcal{I}$ if there is a chain $X_0 \subseteq \ldots \subseteq X_\lambda = X$ for some ordinal $\lambda$ s.t. $X_{\alpha+1}/X_\alpha \in \mathcal{I}$ for all $\alpha < \lambda$ and $\lim \rightarrow_{\alpha<\alpha_0} X_\alpha = X_{\alpha_0}$, for any limit ordinals $\alpha_0 \leq \lambda$. We say $X \in \text{Filt} \mathcal{I}$ is $\mathcal{I}$-filtered. When $\lambda$ is finite, we say $X$ is a finite extension of (objects of) $\mathcal{I}$, and we let $\text{ext}(\mathcal{I})$ denote the class of finite extensions of $\mathcal{I}$. This is also the extension closure of $\mathcal{I}$ i.e. the smallest subcategory of $\mathcal{A}$ containing $\mathcal{I}$.
and closed under extensions. For example the class $\bigoplus X$ is the class of all (infinite) sums of elements of $X$. Such a sum, $\bigoplus_{i=1}^{\lambda} X_i$ is a colimit of a diagram with no arrows, and as such is neither a direct limit nor a filtration. It can however be realized as a filtration by $\{\bigoplus_{i=1}^{\alpha} X_i\}$, for $\alpha < \lambda$ and as a direct limit as $\{\bigoplus_{i \in I} X_i\}$, for $I$ finite, with arrows the inclusions. In fact $\bigoplus X = \text{Filt } X$ when $X$ consists of projective objects. We say that $X \in \mathcal{A}$ is $FP_n$ if the canonical map $\lim\rightarrow \text{Ext}^k(X, Y_i) \to \text{Ext}^k(X, \lim\rightarrow Y_i)$ is an isomorphism for every $0 \leq k < n$. The objects $FP_1(\mathcal{A})$ are called finitely presented, and the objects s.t the above map is injective for $k = 0$ is called finitely generated and denoted $FP_0(\mathcal{A})$.

The objects $FP_1(\mathcal{A})$ are called locally finitely presented if it satisfies one (and therefore all) of the following equivalent conditions:

(i) $FP_1(\mathcal{A})$ is skeletally small (i.e. the isomorphism classes form a set) and $\lim\rightarrow FP_1(\mathcal{A}) = \mathcal{A}$ (Crawley-Boevey [6])

(ii) $\mathcal{A}$ is Grothendieck and $FP_1(\mathcal{A})$ generate $\mathcal{A}$. (Breitsprecher [4])

(iii) $\mathcal{A}$ is Grothendieck and $\lim\rightarrow FP_1(\mathcal{A}) = \mathcal{A}$ ([4]).

The direct limit is very well-behaved in locally finitely presented categories. In particular we have that if $X \subseteq FP_1(\mathcal{A})$ is closed under direct sums, then $\lim\rightarrow X$ is closed under direct limits, and is thus the direct limit closure of $X$ [6, Lemma p. 1664]. We also have the following. The proof was communicated to me by Jan Šťovíček (any mistakes are mine).

**Proposition 1.1.** Let $\mathcal{A}$ be a locally finitely presented abelian category. If $X \subseteq FP_2(\mathcal{A})$ is closed under extensions then so is $\lim\rightarrow X$. It is thus closed under filtrations.

**Proof.** Let $\{S_i\}, \{T_j\} \subseteq \mathcal{X}$ be directed systems and let

$$0 \to \lim\rightarrow S_i \to E \to \lim\rightarrow T_j \to 0$$

be an exact sequence. We want to show that $E \in \lim\rightarrow \mathcal{X}$. First by forming the pullback

$$
\begin{array}{cccccc}
0 & \rightarrow & \lim\rightarrow S_i & \rightarrow & E_j & \rightarrow & T_j & \rightarrow & 0 \\
\downarrow & & \downarrow r & & \downarrow & & \downarrow & \\
0 & \rightarrow & \lim\rightarrow S_i & \rightarrow & E & \rightarrow & \lim\rightarrow T_j & \rightarrow & 0
\end{array}
$$

we see that $E = \lim\rightarrow E_j$ since $\mathcal{A}$ is AB5 as it is locally finitely presented abelian, hence Grothendieck. Now since $T_j$ is in $FP_2(\mathcal{A})$ for every $j$ we have that

$$[0 \to \lim\rightarrow S_i \to E_j \to T_j \to 0] \in \text{Ext}^1(T_j, \lim\rightarrow S_i)$$

is in the image of the canonical map from $\lim\rightarrow \text{Ext}^1(T_j, E_i)$, that is, it is a pushout

$$
\begin{array}{cccccc}
0 & \rightarrow & S_i & \rightarrow & E_{ij} & \rightarrow & T_j & \rightarrow & 0 \\
\downarrow & & \downarrow j & & \downarrow & & \downarrow & \\
0 & \rightarrow & \lim\rightarrow S_i & \rightarrow & E_j & \rightarrow & \lim\rightarrow T_j & \rightarrow & 0
\end{array}
$$

for some $i$ and some extension $E_{ij} \in \mathcal{A}$. 
Now construct for every $k \geq i$ the pushout

$$
\begin{array}{c}
0 & \longrightarrow & S_i & \longrightarrow & E_{ij} & \longrightarrow & T_j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & S_k & \longrightarrow & E_{kj} & \longrightarrow & T_j & \longrightarrow & 0
\end{array}
$$

Then $\lim_k E_{kj} = E_j$ so $E_j \in \lim \mathcal{X}$ as $E_{kj} \in \mathcal{X}$ when $\mathcal{X}$ is closed under extensions.

Finally $E = \lim E_j \in \lim \mathcal{X}$ as $\lim \mathcal{X}$ is closed under direct limits when $\mathcal{X} \subset FP_1(\mathcal{A})$.

The classes $FP_n(\mathcal{A})$ are all closed under finite sums (as in [4, Lem. 1.3]). They are not necessarily closed under extensions, but the following subclasses are:

**Definition 1.2.** Let $\mathcal{A}$ be an abelian category. We say $X \in \mathcal{A}$ is $FP_{n,5}$ if $X$ is $FP_n$ and furthermore, that the natural map $\lim \ Ext^n(X,Y_i) \to \ Ext^n(X, \lim Y_i)$ is monic for every filtered system $\{Y_i\} \subseteq \mathcal{A}$. We let $FP_*$ stand for an unspecified (but fixed) $FP_n$ or $FP_{n,5}$.

Note that by definition $FP_0(\mathcal{A}) = FP_{0,5}(\mathcal{A})$ and also $FP_1(\mathcal{A}) = FP_{1,5}(\mathcal{A})$ by Stenström [19] Prop. 2.1 when $\mathcal{A}$ is AB5. We have the following generalization of [4, Lem. 1.9] for $n, * = 1$ and $\mathcal{A}$ Grothendieck.

**Lemma 1.3.** Let $\mathcal{A}$ be an AB5-abelian category and let $0 \to A \to B \to C \to 0$ be an exact sequence. Then

(i) If $A$ and $C$ are $FP_{n,5}$, then so is $B$.

(ii) If $B$ is $FP_*$ then $A$ is $FP_{* - 1}$ iff $C$ is $FP_*$. 

Proof. (i) Let $\{X_i\} \subseteq \mathcal{A}$ be a filtered system. From the long exact sequence in homology we get for all $k < n$:

$$
\begin{array}{c}
\lim_k \ Ext^{k-1}(A, X_i) \to \lim_k \ Ext^k(C, X_i) \to \lim_k \ Ext^k(B, X_i) \to \lim_k \ Ext^{k+1}(A, X_i) \\
\downarrow \cong \hspace{1cm} \downarrow \cong \hspace{1cm} \downarrow \cong \hspace{1cm} \downarrow \cong \\
\Ext^{k-1}(A, \lim X_i) \to \Ext^k(C, \lim X_i) \to \Ext^k(B, \lim X_i) \to \Ext^{k+1}(A, \lim X_i)
\end{array}
$$

and

$$
\begin{array}{c}
\lim_k \ Ext^{n-1}(A, X_i) \longrightarrow \lim_k \ Ext^n(C, X_i) \longrightarrow \lim_k \ Ext^n(B, X_i) \longrightarrow \lim_k \ Ext^n(A, X_i) \\
\downarrow \cong \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
\Ext^{n-1}(A, \lim X_i) \longrightarrow \Ext^n(C, \lim X_i) \longrightarrow \Ext^n(B, \lim X_i) \longrightarrow \Ext^n(A, \lim X_i)
\end{array}
$$

And the result follows by the 5-lemma. (ii) is proved similarly. Note that when $* = 1$ we must use that $FP_1 = FP_{1,5}$ because $FP_0 = FP_{0,5}$.

**Lemma 1.4.** Let $\mathcal{A}$ be an AB5-abelian category generated by a set of $FP_{n,5}$-objects. Then

(i) If $X \in FP_0(\mathcal{A})$ there exists an epi $X_0 \to X$ with $X_0 \in FP_{n,5}(\mathcal{A})$.

(ii) $FP_k(\mathcal{A}) = FP_{k,5}(\mathcal{A})$ for all $k \leq n$.
Proof. For (i) notice that by \[\text{[4, satz 1.6]}\] if \(\mathcal{A}\) is generated by \(X \subseteq \text{FP}_1(\mathcal{A})\) and \(C \in \text{FP}_0(\mathcal{A})\) then we have an epi from a finite sum of elements of \(X\) to \(C\). But \(\text{FP}_n\) (and \(\text{FP}_{n,5}\)) are all closed under finite sums. The proof of (ii) goes by induction. The case \(n = 0\) is true by definition, so assume \(\mathcal{A}\) is generated by a set of \(\text{FP}_{n,5}\)-objects and that \(X \in \text{FP}_n(\mathcal{A})\). By (i) we get an exact sequence

\[0 \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0\]

with \(X_0 \in \text{FP}_{n,5}(\mathcal{A})\). By Lemma \[\text{[1.3 (ii)]}\] \(X_1 \in \text{FP}_{n-1}(\mathcal{A})\) which by induction hypothesis equals \(\text{FP}_{n-1,5}(\mathcal{A})\) so \(X \in \text{FP}_{n,5}(\mathcal{A})\) again by Lemma \[\text{[1.3 (ii)]}\]. \(\square\)

In particular \(\text{FP}_{n,5}(\text{R-Mod}) = \text{FP}_n(\text{R-Mod})\) is closed under extensions for any \(n\) and any ring \(R\). We think of the objects of \(\text{FP}_n(\mathcal{A})\) as being small.

2. Quiver representations

Let \(Q\) be a quiver, i.e. a directed graph. We denote the vertices by \(Q_0\) and we denote an arrow (resp. a path) from \(w\) to \(v\) by \(w \rightarrow v\) (resp. \(w \leadsto v\)). A quiver may have infinitely many vertices and arrows, but we will need the following finiteness conditions.

**Definition 2.1.** Let \(Q\) be a quiver. We say \(Q\) is target-finite (resp. source-finite) if there are only finitely many arrows with a given target (resp. source). We say \(Q\) is left-rooted (resp. right-rooted) if there is no infinite sequence of composable arrows \(\cdots \rightarrow \bullet \rightarrow \bullet\) (resp. \(\bullet \rightarrow \bullet \rightarrow \cdots\)). Finally we say \(Q\) is locally path-finite if there is only finitely many paths between any two given vertices.

**Remark 2.2.** Notice that \(Q\) is target-finite (resp. left-rooted) iff \(Q^\text{op}\) is source-finite (resp. right-rooted) and that left/right-rooted quivers are necessarily acyclic (i.e. have no cycles or loops). Locally path-finite is self-dual. Even if a quiver satisfies all of the above finiteness conditions, it can still have infinitely many vertices and arrows, e.g. the quiver \(\cdots \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \cdots\)

When the quiver is left-rooted we can use the following sets for inductive arguments. Let \(V_0 = \emptyset\) and define for any ordinal \(\lambda\), \(V_{\lambda+1} = \{v \in Q_0 | w \rightarrow v \Rightarrow w \in V_{\lambda}\}\) and for limit ordinals \(V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha\). Notice that \(V_1\) is precisely the sources of \(Q\).

As noted in \[\text{[10 Prop. 3.6]}\] a quiver is left-rooted precisely when \(Q_0 = V_\lambda\) for some \(\lambda\).

**Example 2.3.** Let \(Q\) be the (left-rooted) quiver:

```
1
\(\uparrow\)
2
\(\downarrow\)
3
\(\uparrow\)
4
\(\uparrow\)
5
```

For this quiver, the transfinite sequence \( \{V_\alpha\} \) looks like this:

\[
\begin{array}{cccccc}
0 & 5 & 0 & 5 & 0 & \bullet \\
5 & 0 & 5 & 0 & 5 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 \\
1 & 2 & 1 & 2 & 1 & 2 \\
\end{array}
\]

\( V_0 = \emptyset \) \( V_1 = \{1, 2\} \) \( V_2 = \{1, 2, 3\} \) \( V_3 = \{1, 2, 3, 4\} \) \( V_4 = Q_0 \)

Let now \( \mathcal{A} \) be an abelian category. A quiver \( Q \) generates a category \( \overline{Q} \), called the path category, with objects \( Q_0 \) and morphisms the paths in \( Q \). We define \( \text{Rep}(Q, \mathcal{A}) = \text{Fun}(\overline{Q}, \mathcal{A}) \). Note that \( F \in \text{Rep}(Q, A) \) is given by its values on vertices and arrows and we picture \( F \) as a \( Q \)-shaped diagram in \( \mathcal{A} \).

For \( v \in Q_0 \) the evaluation functor \( e_v : \text{Rep}(Q, \mathcal{A}) \to \mathcal{A} \) is given by \( e_v(F) = F(v) \) for \( v \in Q_0 \) and \( e_v(\eta) = \eta_0 \) for \( \eta : F \to G \). If \( \mathcal{A} \) has coproducts (or \( Q \) is locally path-finite) this has a left-adjoint \( f_v : \mathcal{A} \to \text{Rep}(Q, \mathcal{A}) \) given by

\[
f_v(X)(w) = \bigoplus_{v \to w} X
\]

where the sum is over all paths from \( v \) to \( w \) and \( f_v(X)(w \to w') \) is the natural inclusion. For \( \mathcal{X} \subseteq \mathcal{A} \) we define

\[
f_*(\mathcal{X}) = \{ f_v(X) \mid v \in Q_0, X \in \mathcal{X} \} \subseteq \text{Rep}(Q, \mathcal{A}).
\]

See [10] or [14] for details.

**Remark 2.4.** Limits and colimits are point-wise in \( \text{Rep}(Q, \mathcal{A}) \), so \( e_v \) preserves them and is in particular exact. Thus its left-adjoint \( f_v \) preserves projective objects.

**Definition 2.5.** For any quiver \( Q \), any abelian category \( \mathcal{A} \), any \( F \in \text{Rep}(Q, \mathcal{A}) \) and any \( v \in Q_0 \) we have a canonical map \( \varphi^F_v = \bigoplus_{w \to v} F(w) \to F(v) \) and we set

\[
\Phi(\mathcal{X}) = \{ F \in \text{Rep}(Q, \mathcal{A}) \mid \forall v \in Q_0 : \varphi^F_v \text{ is monic and } \text{coker } \varphi^F_v \in \mathcal{X} \}.
\]

**Remark 2.6.** Observe that \( f_v(\mathcal{X}) \subseteq \Phi(\mathcal{X}) \). In fact for any \( v \in Q_0 \), \( \varphi^{f_v(X)}_w \) is an isomorphism, unless \( w = v \) in which case it is monic (in fact zero if \( Q \) is acyclic) with cokernel \( X \). As in [14] Prop. 7.3 if \( Q \) is left-rooted then \( \Phi(\mathcal{X}) \subseteq \text{Rep}(Q, \mathcal{X}) \) if \( \mathcal{X} \) is closed under arbitrary sums or \( Q \) is locally path-finite and \( \mathcal{X} \) is closed under finite sums.

The aim of this section is to show that sums of objects of \( f_*(\mathcal{X}) \) filter \( \Phi(\mathcal{X}) \). Let us first see how \( f \) and \( \Phi \) play together with various categorical constructions.

**Lemma 2.7.** Let \( Q \) be a quiver, \( \mathcal{A} \) an abelian category satisfying AB4, and \( \mathcal{X} \subseteq \mathcal{A} \) arbitrary. Then

(i) \( f_*(\text{extensions of } \mathcal{X}) \subseteq \text{extensions of } f_*(\mathcal{X}) \),
(ii) \( f_*(\text{summands of } \mathcal{X}) \subseteq \text{summands of } f_*(\mathcal{X}) \),
(iii) \( f_*(\text{lim } \mathcal{X}) \subseteq \text{lim } f_*(\mathcal{X}) \),
(iv) \( f_*(\text{Filt } \mathcal{X}) \subseteq \text{Filt } f_*(\mathcal{X}) \).
Proof. (i) follows since \( f_v \) is exact when \( \mathcal{A} \) is AB4 and (iii) since \( f_v \) is a left adjoint. (ii) is clear and (iv) follows from (i) and (iii).

Lemma 2.8. Let again \( Q \) be a quiver, \( \mathcal{A} \) an abelian category satisfying AB4, and \( \mathcal{X} \subseteq \mathcal{A} \) arbitrary. Then

(i) \( \Phi(\text{extensions of } \mathcal{X}) \subseteq \text{ extensions of } \Phi(\mathcal{X}) \),
(ii) \( \text{summands of } \Phi(\mathcal{X}) \subseteq \Phi(\text{summands of } \mathcal{X}) \).

When \( \mathcal{A} \) is AB5 we further have

(iii) \( \lim \Phi(\mathcal{X}) \subseteq \Phi(\lim \mathcal{X}) \),
(iv) \( \text{Filt } \Phi(\mathcal{X}) \subseteq \Phi(\text{Filt } \mathcal{X}) \).

When \( \mathcal{A} \) is AB4* and \( Q \) is target-finite we have

(v) \( \prod \Phi(\mathcal{X}) \subseteq \Phi(\prod \mathcal{X}) \).

Proof. (ii) follows as retracts respects kernels and cokernels, (iii) is clear when \( \mathcal{A} \) satisfies AB5. For (i) let \( 0 \rightarrow F \rightarrow F'' \rightarrow F' \rightarrow 0 \), be an exact sequence with \( F,F' \in \Phi(\mathcal{X}) \). For every \( v \in Q_0 \) we have that

\[
\begin{array}{cccccc}
0 & & & & 0 \\
\downarrow & & & & \downarrow \\
0 & \rightarrow & \oplus_{w \rightarrow v} F(w) & \rightarrow & \oplus_{w \rightarrow v} F''(w) & \rightarrow & \oplus_{w \rightarrow v} F'(w) & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \rightarrow & F(v) & \rightarrow & F''(v) & \rightarrow & F'(v) & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
C & & & & C' & & & & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & & & & 0
\end{array}
\]

has exact rows since \( \mathcal{A} \) is AB4 and \( e_v \) is exact. The condition follows from the snake lemma, since \( C,C' \in \mathcal{X} \).

Again (iv) follows from (i) and (iii). For (v) we notice that for any \( \{F_i\} \subset \mathcal{A} \) and vertex \( v \) we have \( \prod_i \phi_{F_i}^v = \phi_{\prod F} \) since the sum in the definition of \( \phi \) is finite, hence a product, when \( Q \) is target-finite. \( \square \)

As for smallness we have the following

Lemma 2.9. Let \( \mathcal{A} \) be an abelian category.

(i) If \( \mathcal{A} \) satisfies AB5 then \( f_v \) preserves \( FP \)

(ii) If \( Q \) is locally path-finite, then \( e_v(-) \) preserves \( FP \).

(iii) If \( Q \) is target-finite and locally path-finite then

\( \Phi(\mathcal{X}) \cap FP(\text{Rep}(Q,\mathcal{A})) \subseteq \Phi(\mathcal{X} \cap FP(\mathcal{A})) \).

Proof. (i) This follows from the natural isomorphism ( [14 prop 5.2])

\[ \text{Ext}^i(f_v(X),-) \cong \text{Ext}^i(X,e_v(-)) \]

and the fact that \( e_v \) preserves filtered colimits (Remark 2.4).

(ii) In this case \( e_v \) has a right adjoint \( g_v(X)(w) = \prod_{w \rightarrow v} X \) (see [14 3.6]) which is a finite product, hence a sum, as \( Q \) is locally path-finite. So \( g_v(-) \) preserves
filtered colimits. Thus \( e_v \) preserves \( FP_* \), by the natural isomorphism ( [13] prop 5.2))

\[
\text{Ext}^1(e_v(X), -) \cong \text{Ext}^1(X, g_v(-))
\]

(iii) Let \( F \in \Phi(\mathcal{X}) \) be \( FP_* \). Given \( v \in Q_0 \) we only need to show that \( \text{coker} \phi^e_v \) is \( FP_* \). Since \( Q \) is target-finite, \( \oplus_{w \rightarrow v} F(w) \) is a finite sum of \( FP_* \)-objects by (ii) and since \( FP_* \) is closed under finite sums the result follows from (ii) and Lemma 1.3.

\[ \square \]

The following two lemmas will be used to construct a \( \oplus f_*(\mathcal{X}) \)-filtration for any \( F \in \Phi(\mathcal{X}) \) for suitable \( \mathcal{X} \subset \mathcal{A} \) when \( Q \) is left-rooted. This is the key in proving Theorem A.

**Lemma 2.10.** Let \( Q \) be an acyclic (e.g. left-rooted) quiver and \( \mathcal{A} \) an abelian category satisfying AB4. If \( F \in \Phi(\mathcal{X}) \) there exists a subrepresentation \( F' \subseteq F \) such that

(a) \( F' \in \bigoplus f_*(\mathcal{X}) \),

(b) \( F'(v) = F(v) \quad \forall v \in V^F = \{ v \in Q_0 | w \rightarrow v \Rightarrow F(w) = 0 \} \),

(c) \( F/F' \in \Phi(\mathcal{X}) \), with \( \text{coker} \phi^e_{F/F'} = \text{coker} \phi^F_v \) when \( v \notin V^F \).

**Proof.** Define \( F' = \bigoplus_{v \in V^F} f_v(F(v)) \). We wish to prove that \( F' \) is a subrepresentation and that it satisfies (a)-(c).

Clearly \( F' \) satisfy (a). To see (b) it suffices to prove, that for any non-trivial path \( w \rightarrow v \) with \( v \in V^F \) we have \( F(w) = 0 \) - because then for any \( v \in V^F \) we have \( f_v(F(v))(v) = F(v) \) and \( f_w(F(w))(v) = 0 \). But then also \( F(w') = 0 \) as \( v \in V^F \). To see (c) we use the map \( F' \rightarrow F \) induced by the counits \( f_ec_v(F) \rightarrow F \). If \( v \) is not reachable from \( V^F \) (i.e. there is no path \( w \rightarrow v \) with \( w \in V^F \)) this is trivial since then \( F'(v) = 0 \).

To see that \( F' \) is a subrepresentation satisfying (c) we use the map \( F' \rightarrow F \) induced by the counits \( f_ec_v(F) \rightarrow F \). If \( v \) is not reachable from \( V^F \) (i.e. there is no path \( w \rightarrow v \) with \( w \in V^F \)) this is trivial since then \( F'(v) = 0 \).

So let \( Q' \) be the subquiver consisting of all vertices \( Q'_0 \) reachable from \( V^F \) (i.e. \( Q'_0 = \{ v \in Q_0 | \exists w \rightarrow v, w \in V^F \} \) with arrows \( \{ w \rightarrow v | w \in Q'_0 \} \)). We want for all \( v \in Q'_0 \) that there are exact sequences

(1) \( 0 \rightarrow F'(v) \rightarrow F(v) \)

(2) \( 0 \rightarrow \bigoplus_{w \rightarrow v} F/F'(w) \rightarrow F/F'(v) \rightarrow \text{coker} \phi^e_{F/F'} \rightarrow 0 \) when \( v \notin V^F \).

Since \( Q \) is acyclic, \( Q' \) is left-rooted with sources \( V^F \). We can thus proceed by induction on the sets \( V'_\lambda \). The case \( v \in V'_\lambda = V_F \) is taken care of by (b), so assume (1) for all \( w \in V'_\alpha \) and all \( \alpha < \lambda \neq 1 \), and let \( v \in V'_\lambda \). Then we have the following
commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \downarrow & \\
0 & \rightarrow & \oplus_{w \rightarrow v} F'(w) & \rightarrow & F'(v) & \rightarrow & 0 \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \\
0 & \rightarrow & \oplus_{w \rightarrow v} F(w) & \rightarrow & F(v) & \rightarrow & \text{coker } \phi_v \\
\downarrow & & \downarrow & & & & \\
\oplus_{w \rightarrow v} F/F'(w) & \downarrow & \downarrow & & & & \\
& & & & & & 0
\end{array}
\]

The first row is exact as \( v \notin V^F \) (see Remark 2.6), the second as \( F \in \Phi(\mathcal{X}) \) and the first column by induction hypothesis and the assumption that \( \mathcal{A} \) is \( AB4 \). Now (1) and (2) follows for \( v \in V_1 \) by the snake lemma.  \( \square \)

**Lemma 2.11.** Let \( Q \) be an acyclic quiver, \( \mathcal{A} \) an \( AB5 \)-abelian category, and let \( \mathcal{X} \subseteq \mathcal{A} \). Then for any \( F \in \Phi(\mathcal{X}) \) there exists a chain \( 0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_\lambda \subseteq \ldots \subseteq F \) of subrepresentations of \( F \), such that for all ordinals \( \lambda \)

1. \( F_\lambda / F_\alpha \in \bigoplus f_\alpha(\mathcal{X}) \), if \( \lambda = \alpha + 1 \)
2. \( F_\lambda(v) = F(v) \) for all \( v \in \bigcup_{\alpha < \lambda} V^{F/F_\alpha} \)
3. \( F/F_\lambda \in \Phi(\mathcal{X}) \) with \( \text{coker } \phi_v^{F/F_\lambda} = \text{coker } \phi_v^F \) for all \( v \notin \bigcup_{\alpha < \lambda} V^{F/F_\alpha} \)

Notice that \( \bigcup_{\alpha+1} V^{F/F_\alpha} = V^{F/F_\beta} \)

**Proof.** We will construct such a filtration by transfinite induction. \( 0 = F_0 \) is evident so assume \( F_\alpha \) satisfying (a)-(c) has been constructed for all \( \alpha < \lambda \)

If \( \lambda = \alpha + 1 \) then by Lemma 2.10 we have an \( F' \subseteq F / F_\alpha \) s.t. \( F' \in \bigoplus f_\alpha(\mathcal{X}) \) and s.t. \( F''(v) = 0 \) for all \( v \in V^{F/F_\alpha} = \{ v \in Q_0 | w \rightarrow v \implies F(w) = F_\alpha(w) \} \)

and

\[ \text{coker } \phi_v^{F''} = \text{coker } \phi_v^{F/F_\alpha} = \text{coker } \phi_v^F \text{ for all } v \notin V^{F/F_\alpha} \]

Now let \( F_\lambda \) be the pullback

\[
\begin{array}{ccc}
F_\lambda & \rightarrow & F' \\
\downarrow & & \downarrow \\
F & \rightarrow & F/F_\alpha
\end{array}
\]

Then (a) follows as \( F_\lambda / F_\alpha \cong F' \) and (b) and (c) follows since \( F/F_\lambda \cong F'' \).

If \( \lambda \) is a limit ordinal, we set

\[ F_\lambda = \bigcup_{\alpha < \lambda} F_\alpha \]

so that

\[ F(v) = F_\lambda(v) \text{ when } v \in \bigcup_{\alpha < \lambda} V^{F/F_\alpha} \]
Then \((a)\) is void and we get \((b)\) by noting that when \(v \in V^{F/F_\alpha}\) for some \(\alpha < \lambda\), then \(F_\lambda(v)\) is the limit of a filtration eventually equal to \(F(v)\)

\[
F_\lambda(v) = e_v \left( \bigcup_{\alpha < \lambda} F_\alpha \right) = \bigcup_{\alpha < \lambda} F_\alpha(v) = F(v).
\]

To prove \((c)\) we similarly notice that \(\varphi_v^{F/\varprojlim F_\alpha} = \varprojlim \varphi_v^{F/F_\alpha}\) is monic for any vertex \(v\) as \(\mathcal{A}\) is AB5 and when \(v \notin \bigcup_{\alpha < \lambda} V^{F/F_\alpha}\) then

\[
\text{coker} \varphi_v^{F/\varprojlim F_\alpha} = \varprojlim \text{coker} \varphi_v^{F/F_\alpha} = \varprojlim \text{coker} \varphi_v^F = \text{coker} \varphi_v^F
\]

\(\square\)

The following figure shows an example of this construction.

\[
\begin{align*}
(x \oplus y \oplus z_0)^2 \oplus z_1 & \quad (x \oplus y)^2 \quad z_0^2 \oplus z_1 \quad (x \oplus y \oplus z_0)^2 \quad z_1 \\
(x \oplus y \oplus z_0)^2 \oplus z_1 & \quad (x \oplus y)^2 \quad z_0^2 \oplus z_1 \quad (x \oplus y \oplus z_0)^2 \\
x \oplus y \oplus z_0 & \quad x \oplus y \quad z_0 \quad x \oplus y \oplus z_0 \\
x & \quad y \quad 0 \quad 0 \quad y
\end{align*}
\]

\[
F \quad F_1 \quad F/F_1 \quad F_2 \quad F/F_2
\]

\[
F_3 = F
\]

**Figure 1.** Example of the construction of the subrepresentations \(F_\alpha\)

We can now prove Theorem \(\ref{thm:construction}\) from the introduction.

**Proof of Theorem \(\ref{thm:construction}\)**

\(i)\) Let \(F \in \Phi(\mathcal{A})\) and let \(\{F_\alpha\}\) be the filtration of Lemma \(\ref{lem:filtration}\). First we show that \(F_\lambda(v) = F(v)\) for all \(v \in V_\lambda\). The case \(\lambda = 0\) is trivial, so let \(\lambda = \alpha + 1\), assume \(F_\alpha(v) = F(v)\), and let \(v \in V_\lambda\). Then for paths \(w \to v\) we have \(w \in V_\alpha\) so \(F_\alpha(w) = F(w)\). This precisely says that \(v \in V^{F/F_\alpha}\) i.e \(F_\lambda(v) = F(v)\). If \(\lambda\) is a limit ordinal then \(F_\lambda = \bigcup_{\alpha < \lambda} F_\alpha\) so \(F_\lambda(v) = F(v)\) when \(v \in \bigcup_{\alpha < \lambda} V_\alpha = V_\lambda\). Now since \(Q\) is left-rooted, \(F = F_\lambda\) for some \(\lambda\).

\(ii)\) When \(\mathcal{X}\) is closed under filtrations we have

\[
\text{Filt} \bigoplus f_*(\mathcal{X}) \underset{\text{Rem.}\,\ref{rem:union}}{\subseteq} \text{Filt} \bigoplus \Phi(\mathcal{X}) \underset{\text{Lem.}\,\ref{lem:union}}{\subseteq} \Phi(\text{Filt} \bigoplus \mathcal{X}) \subseteq \Phi(\mathcal{X}).
\]

\(iii)\) When \(\mathcal{X}\) is a set, \(\text{Filt} f_*(\mathcal{X})\) is closed under filtrations \([20]\,\text{Lem.}\,1.6\) hence

\[
\begin{align*}
\text{Filt} \Phi(\mathcal{X}) \underset{\text{Lem.}\,\ref{lem:union}}{\subseteq} \Phi(\text{Filt} \mathcal{X}) \subseteq \text{Filt} \bigoplus f_*(\text{Filt} \mathcal{X}) \\
\subseteq \text{Filt} \bigoplus \text{Filt} f_*(\mathcal{X}) \subseteq \text{Filt} f_*(\mathcal{X})
\end{align*}
\]

\(\text{Filt} \Phi(\mathcal{X}) \)
iv) This is proven similar to iii). Just observe that a filtration of summands is a summand of a filtration.

v) When \( \mathcal{X} \) is \( FP_{2,5} \) then \( f_*(\mathcal{X}) \) is \( FP_{2,5} \) by Lemma 2.14 and so is \( \text{ext} f_*(\mathcal{X}) \) by Lemma 1.3 Hence \( \text{lim ext} f_*(\mathcal{X}) \) is closed under extensions by Proposition 1.1

We now have

\[
\lim_{\to} \Phi(\mathcal{X}) \subseteq \Phi(\lim_{\to} \mathcal{X}) \subseteq \text{Filt} \bigoplus \lim_{\to} f_*(\mathcal{X}) \subseteq \text{Filt} \bigoplus \lim_{\to} \text{ext} f_*(\mathcal{X}) \subseteq \lim_{\to} \text{ext} \Phi(\mathcal{X}) \]  

\[
\lim_{\to} \Phi(\text{ext} \mathcal{X}) \subseteq \lim_{\to} \text{ext} \Phi(\mathcal{X}) \]  

□

As mentioned in the introduction we also get iii) by combining results in [14] and [20] but for a more restrictive class of abelian categories.

As a special case we get the known results from [10] and [7].

Lemma 2.12. Let \( \mathcal{A} \) be an AB5-abelian category, let \( Q \) be a left-rooted quiver and let \( X \subseteq \mathcal{A} \) be a set of projective objects. Then

i) \( \Phi(\bigoplus X) = \bigoplus f_*(\mathcal{X}) = \bigoplus \Phi(\mathcal{X}) \)

ii) \( \Phi(\text{Add} X) = \text{Add} f_*(\mathcal{X}) = \text{Add} \Phi(\mathcal{X}) \)

If \( \mathcal{A} \) has enough projectives, then

iii) \( \Phi(\text{Proj} \mathcal{A}) = \text{Proj}(\text{Rep}(Q, \mathcal{A})) \)

If \( \mathcal{A} \) is locally finitely presented, generated by \( \text{proj}(\mathcal{A}) \) (the finitely generated projective objects) then

iv) \( \Phi(\lim_{\to} \text{proj}(\mathcal{A})) = \lim_{\to} \text{proj}((\text{Rep}(Q, \mathcal{A}))) \)

Proof. For i) and ii) just notice that any filtration is a sum as all extensions of projective objects are split. For iii) and iv) we notice that if \( \mathcal{X} = \text{Proj}(\mathcal{A}) \) (resp. \( \mathcal{X} = \text{proj}(\mathcal{A}) \)) generate \( \mathcal{A} \) then \( f_*(\mathcal{X}) \subseteq \text{Proj}(\text{Rep}(Q, \mathcal{A})) \) (resp. \( f_*(\mathcal{X}) \subseteq \text{proj}(\text{Rep}(Q, \mathcal{A})) \)) generate \( \text{Rep}(Q, \mathcal{A}) \). Hence \( \text{Add} f_*(\mathcal{X}) = \text{Proj}(\text{Rep}(Q, \mathcal{A})) \). (resp. \( \text{add} f_*(\mathcal{X}) = \text{proj}(\text{Rep}(Q, \mathcal{A})) \)). Now use Theorem A (i) (resp. v))

□

As noted in the introduction, iii) can be seen by using cotorsion pairs as in [14].

In the rest of the paper we study the Gorenstein situation.

3. Gorenstein projective objects

We will now define the small (i.e. \( FP_{2,5} \)) Gorenstein projective objects and describe their direct limit closure using Theorem A

Definition 3.1. Let \( \mathcal{A} \) be an abelian category and \( \mathcal{P} \) a class of objects in \( \mathcal{A} \). A complete \( \mathcal{P} \)-resolution is an exact sequence with components in \( \mathcal{P} \) that stays exact after applying \( \text{Hom}(P, -) \) and \( \text{Hom}(-, P) \) for any \( P \in \mathcal{P} \).

We say that \( X \) has a complete \( \mathcal{P} \)-resolution if it is a syzygy in a complete \( \mathcal{P} \)-resolution, i.e. if there exists a complete \( \mathcal{P} \)-resolution

\[
\ldots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \ldots
\]

s.t. \( X = \ker(P_0 \rightarrow P_{-1}) \).
We say that $X \in \mathcal{A}$ is Gorenstein projective (resp. Gorenstein injective) if it has a complete $\mathcal{P}$-resolution where $\mathcal{P}$ is the class of all projective (resp. injective) objects.

We let $\text{GProj}(\mathcal{A})$ (resp. $\text{GInj}(\mathcal{A})$) denote the Gorenstein projective (resp. Gorenstein injective) objects of $\mathcal{A}$ and let $\text{Gproj}(\mathcal{A}) = \text{GProj}(\mathcal{A}) \cap F_{P2.5}(\mathcal{A})$.

**Remark 3.2.** Notice that the class $\text{GProj}(\mathcal{A})$ is closed under extensions see [12, thm 2.5]. Hence so is $\text{Gproj}(\mathcal{A})$ by Lemma 1.3.

Dually to the already mentioned characterization of the projective representations, we have a characterization of the injective representations. This was first noted in [8] and generalized to abelian categories in [14]. A similar description is possible for Gorenstein projective and Gorenstein injective objects as proven first for modules over Gorenstein rings in [8] and then modules over arbitrary rings in [11, Thm. 3.5.1]. This proof work in any abelian category with enough projective (resp. injective) objects. We collect the results here for ease of reference.

**Theorem 3.3.** Let $Q$ be a left-rooted quiver, $\mathcal{A}$ an abelian category with enough projective objects, and $B$ a category with enough injective objects. Then

$$
\text{Proj}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{Proj}(\mathcal{A}))
$$

$$
\text{GProj}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{GProj}(\mathcal{A}))
$$

$$
\text{Inj}(\text{Rep}(Q^{\text{op}}, B)) = \Psi(\text{Inj}(B))
$$

$$
\text{GInj}(\text{Rep}(Q^{\text{op}}, B)) = \Psi(\text{GInj}(B))
$$

where for $\mathcal{Y} \subseteq B$ we define

$$
\Psi(\mathcal{Y}) = \{ F \in \text{Rep}(Q^{\text{op}}, B) \mid \forall v \in Q_0 : \psi_v^F \text{ epi and ker } \psi_v^F \in \mathcal{X} \}.
$$

and

$$
\psi_v^F = F(v) \rightarrow \prod_{w \to v} F(w).
$$

As mentioned in the proofs, left-rooted is not needed for the inclusions ($\subseteq$) in the non-Gorenstein cases. We note that $f_v$ preserves Gorenstein projectivity:

**Lemma 3.4.** Suppose $\mathcal{A}$ satisfies AB4* or has enough projective objects or $Q$ is locally path-finite. If $X \in \mathcal{A}$ is Gorenstein projective, then so is $f_v(X) \in \text{Rep}(Q, \mathcal{A})$.

**Proof.** Let $P_\bullet$ be a complete projective resolution of $X$. Then $f_v(P_\bullet)$ is exact and has projective components by Remark 2.4.

Obviously $\text{Hom}(P, f_v(P_\bullet))$ is exact for any projective $P$, and $\text{Hom}(f_v(P_\bullet), P) \cong \text{Hom}(P_\bullet, e_v(P))$ is exact if $e_v$ preserves projective objects.

If $\mathcal{A}$ has enough projective objects then $\text{Proj}(\text{Rep}(Q, \mathcal{A})) \subseteq \Phi(\text{Proj}(\mathcal{A}))$. (Theorem 3.3) If $\mathcal{A}$ satisfies AB4* or $Q$ is locally path-finite, then as in the proof of Lemma 2.4, $e_v$ has an exact right-adjoint (see [14, 3.6]). In all cases $e_v$ preserves projective objects. \qed

Using these and Theorem A we have

**Proof of Theorem B.** By Theorem A and Remark 3.2 we have

$$
\lim \Phi(\text{Gproj}(\mathcal{A})) = \Phi(\lim \text{Gproj}(\mathcal{A})) = \lim \text{ext } f_*(\text{Gproj}(\mathcal{A}))
$$
Now $f_\nu$ preserves smallness (Lemma 2.9 (i)) and Gorenstein projectivity (Lemma 3.4), so
\[ \lim \text{ext } f_\nu (\text{Gproj}(\mathcal{A})) \subseteq \lim \text{Gproj}(\text{Rep}(Q, \mathcal{A})). \]
If $Q$ is locally path-finite and target-finite, Theorem 3.3 and Lemma 2.9 (iii) give
\[ \text{Gproj}(\text{Rep}(Q, \mathcal{A})) = \Phi(\text{GProj}(\mathcal{A})) \cap FP_{2.5}(\text{Rep}(Q, \mathcal{A})) \subseteq \Phi(\text{Gproj}(\mathcal{A})). \]
so
\[ \lim \text{Gproj}(\text{Rep}(Q, \mathcal{A})) \subseteq \lim \Phi(\text{Gproj}(\mathcal{A})). \]
If instead $\lim \text{Gproj}(\mathcal{A}) = \lim \Phi(\text{GProj}(\mathcal{A}))$ then by Theorem 3.3
\[ \lim \text{Gproj}(\text{Rep}(Q, \mathcal{A})) \subseteq \lim \Phi(\text{Gproj}(\mathcal{A})). \]

4. Weakly Gorenstein flat objects

In this section we will first explain what we mean by an abstract Pontryagin dual. It mimics the behavior of $\text{Ab}(\mathcal{C})$ for $\mathcal{C}$ an abelian category.

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ creates exactness when $A \to B \to C$ is exact in $\mathcal{C}$ if and only if $FA \to FB \to FC$ is exact in $\mathcal{D}$.

**Definition 4.1.** A Pontryagin dual is a contravariant adjunction between abelian categories that creates exactness. I.e. let $\mathcal{C}, \mathcal{D}$ be abelian categories. A Pontryagin dual between $\mathcal{C}$ and $\mathcal{D}$ consists of two functors $(-)^+ : \mathcal{C}^{\text{op}} \to \mathcal{D}$, $(-)^+ : \mathcal{D}^{\text{op}} \to \mathcal{C}$ that both create exactness together with a natural isomorphism $\mathcal{C}(A, B^+)^+ \cong \mathcal{D}(B, A^+)$. We call it $\otimes$-compatible if there is a continuous bifunctor $\otimes : \mathcal{D} \times \mathcal{C} \to \mathcal{K}$ to some abelian category $\mathcal{K}$ s.t.
\[ \mathcal{C}(A, B^+) \cong \mathcal{K}(B \otimes A, E) \cong \mathcal{D}(B, A^+) \]
for some injective cogenerator $E \in \mathcal{K}$ (i.e. $\mathcal{K}(-, E)$ creates exactness). Here continuous means that it respects direct limits.

Note that $\text{Ab}(-, \mathbb{Q}/\mathbb{Z}) : \mathcal{C}^{\text{op}} \to \text{Ab}$ is a Pontryagin dual compatible with the usual tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with $E = \mathbb{Q}/\mathbb{Z}$.

**Example 4.2.** As the following examples shows, (abstract) Pontryagin duals abound. 1) Let $(\mathcal{C}, [-, -], \otimes, 1)$ be a symmetric monoidal abelian category. Let $E \in \mathcal{C}$ be an injective cogenerator s.t. also $[-, E]$ creates exactness. Then $[-, E]$ is a $\otimes$-compatible Pontryagin dual, and any $\otimes$-compatible Pontryagin dual is of this form. It will thus also satisfy
\[ [A, B^+] \cong (B \otimes A)^+ \cong [B, A^+]. \]
This example includes the motivating example $\mathcal{C} = \text{Ab}, E = \mathbb{Q}/\mathbb{Z}$ as well as $\mathcal{C} = \text{Ch}(-), E = \mathbb{Q}/\mathbb{Z}$ (i.e. $\mathbb{Q}/\mathbb{Z}$ in degree 0 and 0 otherwise).
2) If \((-) \dagger : \mathcal{C}^{\text{op}} \to \mathcal{D}\) is a Pontryagin dual it induces a Pontryagin dual \(\text{Fun}(\mathcal{A}, \mathcal{C})^{\text{op}} \to \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{D})\) for any small category \(\mathcal{A}\) by applying \((-) \dagger\) component-wise.

If \((-) \dagger : \mathcal{C}^{\text{op}} \to \mathcal{D}\) is compatible with \(\otimes : \mathcal{D} \times \mathcal{C} \to \mathcal{K}\), then \((-) \dagger : \text{Fun}(\mathcal{A}, \mathcal{C})^{\text{op}} \to \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{D})\) is compatible with \(\otimes : \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{D}) \times \text{Fun}(\mathcal{A}, \mathcal{C}) \to \mathcal{K}\) where \(G \otimes F\) is the coend of

\[
\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathcal{D} \times \mathcal{C} \to \mathcal{K}
\]

i.e. the coequalizer of the two obvious maps

\[
\bigoplus_{a \to b} G(b) \otimes F(a) \rightrightarrows \bigoplus_{a \in \mathcal{A}} G(a) \otimes F(a)
\]

provided the required colimits exists. (see Oberst and Röhrle [17] or Mac Lane [16, IX.6] for this construction).

This includes the case \(\text{Rep}(Q, \mathcal{C})\) for any quiver \(Q\).

3) As in 2), any Pontryagin dual \((-) \dagger : \mathcal{C}^{\text{op}} \to \mathcal{D}\) gives a component-wise Pontryagin dual \(\text{Ch}(\mathcal{C})^{\text{op}} \to \text{Ch}(\mathcal{D})\) of chain-complexes. If \((-) \dagger : \mathcal{C}^{\text{op}} \to \mathcal{D}\) is compatible with \(\otimes : \mathcal{D} \otimes \mathcal{C} \to \mathcal{K}\) with injective cogenerator \(E \in \mathcal{K}\), then \((-) \dagger : \text{Ch}(\mathcal{C})^{\text{op}} \to \text{Ch}(\mathcal{D})\) is compatible with the total tensor product \(\text{Ch}(\mathcal{D}) \times \text{Ch}(\mathcal{C}) \to \text{Ch}(\mathcal{K})\), the injective cogenerator being \(E\) in degree 0 and 0 otherwise.

With \(\mathcal{C} = \text{Ab}, \((-) \dagger = [-, \mathbb{Q}/\mathbb{Z}]\) this construction gives the standard one in \(\text{Ch}(\text{Ab})\) as mentioned in 1).

4) If \(\mathcal{C} = \mathcal{D}\) is symmetric monoidal with a \(\otimes\)-compatible Pontryagin dual as in 1) then the dual of a map \(A \otimes X \xrightarrow{m} X\) gives a map \(X^+ \otimes A \xrightarrow{m^+} X^+\) via the isomorphisms

\[
\text{Hom}(X^+, (A \otimes X)^+) \cong \text{Hom}(X^+, [A, X^+]) \cong \text{Hom}(X^+ \otimes A, X^+).
\]

One can check that if \(A\) is a ring object and \(m\) is a left multiplication then \(m^+\) is a right multiplication and we get a Pontryagin dual \((-) \dagger : (\text{A-Mod})^{\text{op}} \to \text{Mod-A}\) from the category of left \(A\)-modules to the category of right \(A\)-modules.

This is \(\otimes\)-compatible with \(- \otimes_A - : (\text{Mod-A}) \times (\text{A-Mod}) \to \mathcal{C}\), where \(X \otimes_A Y\) is the coequalizer of the two obvious maps

\[
X \otimes A \otimes Y \rightrightarrows X \otimes Y.
\]

(See Pareigis [18] for the details of this construction). This gives the standard Pontryagin dual in \(\text{R-Mod}\) for any ring \(R\) (i.e. a ring object in \(\text{Ab}\)), and by 3) the standard one in \(\text{Ch}(\text{R-Mod})\). It also gives the character module of differential graded \(A\)-modules (DG-\(A\)-Mod) when \(A\) is a differential graded algebra, i.e. a ring object in \(\text{Ch}(\text{Ab})\) (see Avramov, Foxby and Halperin [2]). By 2) we also get the one in [10 Cor 6.7] for \(\text{Rep}(Q, \text{R-Mod})\) for any ring \(R\) and quiver \(Q\).

**Definition 4.3.** Let \((-) \dagger : \mathcal{C}^{\text{op}} \to \mathcal{D}\) be a Pontryagin dual. We say that

- \(X \in \mathcal{C}\) is flat if \(X^+\) is injective in \(\mathcal{D}\),
- \(X \in \mathcal{C}\) is weakly Gorenstein flat (wGFlat) if \(X^+\) is Gorenstein injective.
- \(F \in \text{Ch}(\mathcal{C})\) is a complete flat resolution if \(F^+\) is a complete injective resolution in \(\text{Ch}(\mathcal{D})\),
- \(X \in \mathcal{C}\) is Gorenstein flat (GFlat) if it has a (i.e. is a syzygy in \(a\)) complete flat resolution,
Gorenstein flat always implies weakly Gorenstein flat. The other implication requires one to construct a complete flat resolution when the dual has a complete injective resolution. We will look at when this is possible in the next section.

With $\otimes$-compatibility these notions agree with the standard notions.

**Proposition 4.4.** If $(-)^+ : \mathcal{C}^{\text{op}} \to \mathcal{D}$ is $\otimes$-compatible, then
1) $F \in \mathcal{C}$ is flat if and only if $- \otimes F$ is exact.
2) $F_* \in \text{Ch}(\mathcal{C})$ is a complete flat resolution if and only if $F_i$ is flat for all $i$ and $I \otimes F_*$ is exact for all injective objects $I \in \mathcal{D}$.

**Proof.** 1) We have the following equivalences

$F$ is flat $\iff$ $F^+ \in \text{Inj}(\mathcal{C})$

$\iff \text{Hom}(-, F^+) \text{ is exact}$

$\iff \text{Hom}(- \otimes F, E) \text{ is exact for some injective cogenerator } E$

$\iff - \otimes F \text{ is exact}$

2) Let $F_* \in \text{Ch}(\mathcal{C})$. Then $F^+_i$ is injective iff $F_i$ is flat and $\text{Hom}(I, F^+_*)$ is exact iff $\text{Hom}(I \otimes F_*, E)$ is exact for some injective cogenerator $E$ by $\otimes$-compatibility, see Example 4.2 3). But this happens iff $I \otimes F_*$ is exact. \qed

The following lemma shows how the classes $\Phi(\mathcal{X})$ (Definition 2.5) and $\Psi(\mathcal{X})$ (see Theorem 3.3) behave with respects to the Pontryagin duals. The proofs are straightforward.

**Lemma 4.5.** Let $(-)^+ : \mathcal{A} \to \mathcal{B}$ be a Pontryagin dual between abelian categories, let $Q$ a quiver, let $\mathcal{X} \subseteq \mathcal{A}$ and $\mathcal{Y} \subseteq \mathcal{B}$. Then

$\Phi(\mathcal{X})^+ \subseteq \Psi(\mathcal{Y})$.

In particular if $\mathcal{X} = \{X \in \mathcal{A} \mid X^+ \in \mathcal{Y}\}$ then

$F \in \Phi(\mathcal{X})$ $\iff$ $F^+ \in \Psi(\mathcal{Y})$.

If $Q$ is target-finite then

$\Psi(\mathcal{Y})^+ \subseteq \Phi(\mathcal{X})$.

**Proof.** For the first assertion we must notice, that $(\phi^F)^+ = \psi^F_+$ for all $F \in \text{Rep}(Q, \mathcal{A})$ and all $v \in Q_0$. For the second, that $(\psi^G)^+ = \phi^G_+$ for all $G \in \text{Rep}(Q^{op}, \mathcal{B})$ and all $v \in Q_0$ when $Q$ is target-finite. This is because the product in the definition of $\psi^G_+: G(v) \to \prod_{w \to v} G(w)$ is finite when $Q^{op}$ is source-finite, thus it is a sum and so is the dual. \qed

This immediately gives the following:

**Proof of Theorem C.**

\[
F \in \text{Flat}(\text{Rep}(Q, \mathcal{A})) \overset{\text{Def. 2.5}}{=} F^+ \in \text{Inj}(\text{Rep}(Q^{op}, \mathcal{B})) \overset{\text{Thm. 3.3}}{=} F^+ \in \Psi(\text{Inj}(\mathcal{B})) \overset{\text{Lem. 4.5}}{=} F \in \Phi(\text{Flat}(\mathcal{A}))
\]

The same proof works in the Gorenstein situation. \qed
Remark 4.6. This gives a straightforward proof of [10, Thm 3.7] using the characterization of the injective representations from [8].

Combining this with Theorem 4.3 we get:

**Corollary 4.7.** Let $(\mathcal{A}^+)^\text{op} \to \mathcal{B}$ be a Pontryagin dual, let $Q$ be a left-rooted quiver and assume

- $\mathcal{A}$ has enough projective objects
- $\mathcal{B}$ has enough injective objects
- $Q$ is target-finite and locally path-finite, or $\lim_{\to} \text{Gproj}(\mathcal{A}) = \lim_{\to} \text{GPproj}(\mathcal{A})$.

If $\lim_{\to} \text{Gproj} = \text{wGFlat}$ in $\mathcal{A}$ then the same is true in $\text{Rep}(Q, \mathcal{A})$.

5. **Gorenstein flat objects**

We will now find conditions on the category $\mathcal{A}$ and the quiver $Q$ s.t.

$$\text{wGFlat}(\text{Rep}(Q, \mathcal{A})) = \text{GFlat}(\text{Rep}(Q, \mathcal{A})).$$

Firstly we have the following known result:

**Proposition 5.1.** [12, Prop. 3.6] Let $R$ be a right coherent ring. Then $\text{wGFlat}(R\text{-Mod}) = \text{GFlat}(R\text{-Mod})$.

Looking more closely at the proof (see Christensen [5, Thm. 6.4.2]) we arrive at Lemma 5.3. We include a full proof, as our notions of flatness differ.

**Lemma 5.2.** Let $\mathcal{A}$ be an abelian category. If $0 \to X' \to J \to X \to 0$ is exact and $J$ is injective (or just Gorenstein injective), $X$ is Gorenstein injective and $\text{Ext}^1(I, X') = 0$ for all injective $I \in \mathcal{A}$. Then $X'$ is Gorenstein injective.

*Proof.* This is the dual of [12, 2.11]. The proof is for modules but works in any abelian category. □

Now recall that a class $\mathcal{X} \subseteq \mathcal{C}$ is preenveloping if for every $M \in \mathcal{C}$ there is a map $\phi: M \to X$ called the preenvelope to some $X \in \mathcal{X}$ s.t. every map from $M$ to an object in $\mathcal{X}$ factors through $\phi$. It is monic whenever there exists some monomorphism from $M$ to an object of $\mathcal{X}$.

**Lemma 5.3.** Let $(\mathcal{D})^+ : \mathcal{C}^\text{op} \to \mathcal{D}$ be a Pontryagin dual and assume

1. $\text{Inj}(\mathcal{D})^+ \subseteq \text{Flat}(\mathcal{C})$
2. $\text{Flat}(\mathcal{C})$ is preenveloping.
3. $\mathcal{C}$ has enough flat objects.

Then any weakly Gorenstein flat object of $\mathcal{C}$ is Gorenstein flat.

*Proof.* Let $X$ be weakly Gorenstein flat, i.e. $X^+$ is Gorenstein injective. Our goal is to construct a complete flat resolution for $X$. The left part of such a resolution is easy when $\mathcal{C}$ has enough flat objects. As $X^+$ is Gorenstein injective it has an injective resolution $I_\bullet$ s.t. $\text{Hom}(J, I_\bullet)$ is exact for any injective $J$. But then this holds for any injective resolution of $X^+$. In particular $F_\bullet^+$, where $F_\bullet$ is a flat (left-) resolution of $X$ which exists when $\mathcal{C}$ has enough flats.

For the right part we construct the resolution one piece at a time by constructing for any weakly Gorenstein flat $X \in \mathcal{C}$ a short exact sequence $0 \to X \to F \to X' \to 0$ where $F$ is flat s.t. $\text{Ext}^1(I, X'^+) = 0$ for any injective $I \in \mathcal{D}$. Then $X'^+$ is Gorenstein injective by Proposition 5.1 and this process can be continued to give a flat (right-) resolution $F_\bullet$ of $X$ s.t. $\text{Hom}(I, F_\bullet^+)$ is exact for any injective $I$. 
So let again \( X \in \mathcal{C} \) be weakly Gorenstein flat, and let \( \varphi : X \to F \) be a flat preenvelope. We first show that \( \varphi \) is monic by showing that there exists some monomorphism from \( X \) to a flat object. Since \( X^+ \) is Gorenstein injective there exist an epimorphism \( E \to X^+ \) from some injective \( E \in \mathbb{D} \). But then \( X \to X^{++} \to E^+ \) is monic, since \((-)^+\) creates exactness and \( X^{++} \to X^+ \) is split epi by the unit-counit relation. Thus \( \varphi \) is monic since \( E^+ \) is flat by (1). We thus have a short exact sequence

\[
0 \to X \xrightarrow{\varphi} F \to X' \to 0
\]

inducing for any injective \( I \in \mathbb{D} \) a long exact sequence

\[
0 \to \text{Hom}(I, X') \to \text{Hom}(I, F^+) \xrightarrow{\varphi^*} \text{Ext}^1(I, X') \to \text{Ext}^1(I, F^+).
\]

Now \( \varphi^* \) is epi as \( \varphi^* : \text{Hom}(F, I^+) \to \text{Hom}(X, I^+) \) is epi because \( I^+ \) is flat and \( \varphi \) is a flat preenvelope. Since \( \text{Ext}^1(I, F^+) \) is 0 because \( F^+ \) is injective we must have \( \text{Ext}^1(I, X') = 0 \).

We notice that \( \mathscr{A} = \text{R-Mod} \) satisfies these conditions when \( R \) is right coherent. ((1) is Xu [21, Lem. 3.1.4]) and (2') is [9, Prop. 6.5.1]). Our task is thus to find conditions on \( Q \) s.t. the conditions from Lemma 5.3 lift from \( \mathscr{A} \) to \( \text{Rep}(Q, \mathscr{A}) \).

Lifting the condition that the flat objects are preenveloping is not obvious. But being closed under products is sometimes enough as the next lemma shows. We will reuse standard results on purity and therefore assume our Pontryagin Dual is \( \otimes \)-compatible and \( \mathscr{A} \) to be generated by \( \text{proj}(\mathscr{A}) \).

**Lemma 5.4.** Let \( \mathscr{A} \) be a locally finitely presented abelian category with a Pontryagin dual and assume that

1. The flat objects are closed under products
2. \( \mathscr{A} \) is generated by \( \text{proj}(\mathscr{A}) \)
3. The Pontryagin dual is \( \otimes \)-compatible

Then the flat objects are preenveloping

**Proof.** Let \( X \in \mathscr{A} \). The idea (as in [9] Prop. 6.2.1)) is to find a set of flat objects \( \mathscr{F} \) s.t. every map \( X \to Y \) with \( Y \) flat factors as \( X \to S \hookrightarrow Y \) with \( S \in \mathscr{F} \). Then we can construct a flat preenvelope as

\[
X \to \prod_{S \in \mathscr{F}} S_{\varphi},
\]

with \( S_{\varphi} = S \) because the flat objects are closed under products by (2).

As in the proof of [9] Lemma 5.3.12 there is a set of objects \( \mathscr{F} = \mathscr{A} \) s.t. every map \( X \to Y \) to some \( Y \in \mathbb{D} \) factors as \( X \to S \hookrightarrow Y \) for some \( S \in \mathscr{F} \) with the property that, given a commutative square

\[
\begin{array}{ccc}
L_0 & \rightarrow & L_1 \\
\downarrow & & \downarrow \\
S & \rightarrow & Y
\end{array}
\]

with \( L_0 \) finitely generated and \( L_1 \) finitely presented there is a lift \( L_1 \to S \) s.t. the left triangle commutes. The proof is for modules and bounds size of \( \mathscr{F} \) by some cardinality. If we are not interested in the cardinality, the proof works in any well-powered category, i.e. a category where there is only a set of subobjects of
any given object. As in Adámek and and Rosický [1] any Grothendieck category is well-powered. We are left with proving that if \( Y \) is flat, so is \( S \), i.e. if \( Y^+ \) is injective, so is \( S^+ \). Now Jensen and Lenzing [15, Prop. 7.16] shows (using (3)) that the above lifting property implies (in fact is equivalent to) that \( S \rightarrow Y \) is a direct limit of split monomorphisms. [15] Thm 6.4 then shows (using (4)) that this implies, that \( Y^+ \rightarrow S^+ \) is split epi. (Equivalence of these statements uses that the generators in R-Mod are dualizable). Thus if \( Y^+ \) is injective, so is \( S^+ \).

Lemma 5.5. Let \((\quad)^+ : \mathcal{A}^{\mathsf{op}} \rightarrow \mathcal{B}\) be a Pontryagin dual where \( \mathcal{A} \) is AB4* and \( \mathcal{B} \) has enough injective objects. Let \( Q \) be a left-rooted and target-finite quiver. If \( \mathcal{A} \) satisfies (1)-(4) (from Lemma 5.3 and 5.4) then so does \( \operatorname{Rep}(Q, \mathcal{A}) \).

Proof. For (1) we have

\[
\Psi(\operatorname{Inj}(\mathcal{B}))^{++} \subseteq \Phi(\operatorname{Flat}(\mathcal{A}))^{+} \subseteq \Phi(\operatorname{Flat}(\mathcal{A})).
\]

since \( \mathcal{B} \) has enough injective objects and \( Q \) is left-rooted and target-finite. For (2) we have

\[
\prod \operatorname{Flat}(\operatorname{Rep}(Q, \mathcal{A}))^{\mathsf{op}} \subseteq \Phi(\operatorname{Flat}(\mathcal{A}))^{+} \subseteq \Phi(\operatorname{Flat}(\mathcal{A})).
\]

since \( \mathcal{A} \) is AB4* and \( \mathcal{B} \) has enough injective objects and \( Q \) is left-rooted and target-finite. (3) and (4) lifts without conditions on \( \mathcal{A} \) and \( Q \). For (3), if \( \mathcal{A} \) is generated by a set \( \mathcal{X} \) of finitely generated projective objects then \( f_*(\mathcal{X}) \) is a generating set of finitely generated projective objects by Lem. 2.9(i) and Remark 2.4 (4) is lifted in Example 4.2.

Proposition 5.6. Let \( \mathcal{A} \) be a locally finitely presented abelian AB4* -category. Let \( \mathcal{B} \) be an abelian category with enough injective objects, let \((\quad)^+ : \mathcal{A}^{\mathsf{op}} \rightarrow \mathcal{B}\) be a \( \otimes \)-compatible Pontryagin dual. If

- \( \mathcal{A} \) is generated by \( \operatorname{proj}(\mathcal{A}) \)
- \( \operatorname{Flat}(\mathcal{A}) \) is closed under products
- \( \operatorname{Inj}(\mathcal{B})^{+} \subseteq \operatorname{Flat}(\mathcal{A}) \)

then \( \operatorname{Flat}(\mathcal{A}) \) is preenveloping and \( wG\operatorname{Flat}(\mathcal{A}) = \operatorname{GFlat}(\mathcal{A}) \). Assume further that \( Q \) is a left-rooted and target-finite quiver. Then \( \operatorname{Flat}(\operatorname{Rep}(Q, \mathcal{A})) \) is preenveloping and

\[
wG\operatorname{Flat}(\operatorname{Rep}(Q, \mathcal{A})) = \operatorname{GFlat}(\operatorname{Rep}(Q, \mathcal{A})).
\]

Proof. This follows from Lemma 5.3 and 5.4 and 5.5. We also need (3') to hold and we could lift this directly by noting that \( f_* \) respects flatness, but it also follows from (3).

We can now prove

Proof of Theorem D. Use Proposition 5.6 and the remark above it. □
Remark 5.7. In [8] Lem 6.9 and proof of Thm. 6.11] it is proved that
\[
wGFlat(\text{Rep}(Q, R-\text{Mod})) = GFlat((\text{Rep}(Q, R-\text{Mod}))
\]
when \( R \) is Iwanaga-Gorenstein and \( Q \) is only required to be left-rooted. Theorem D thus weakens the condition of \( R \) but must then strengthen the conditions on \( Q \).

Corollary 5.8. Let \( Q \) be a left-rooted quiver and let \( A \) be as in Proposition 5.6.
If \( A = R-\text{Mod} \) for some Iwanaga-Gorenstein ring \( R \) or \( Q \) is target-finite and locally path-finite
\[
\limlim Gproj(\text{Rep}(Q, R-\text{Mod})) = GFlat(\text{Rep}(Q, R-\text{Mod})) = \Phi(GFlat(R-\text{Mod})).
\]

Proof. Apply Corollary 4.7 and Proposition 5.6 (or Remark 5.7 for the Gorenstein case) to get \( \limlim Gproj = wGFlat = GFlat \) in \( \text{Rep}(Q, A) \). The last equality then follows from Theorem C.

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