Directed Formation Control of \( n \) Planar Agents with Distance and Area Constraints

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Abstract—In this paper, we take a first step towards generalizing a recently proposed method for dealing with the problem of convergence to incorrect equilibrium points of distance-based formation controllers. Specifically, we introduce a distance and area-based scheme for the formation control of \( n \)-agent systems in two dimensions using directed graphs and the single-integrator model. We show that under certain conditions on the edge lengths of the triangulated desired formation, the control ensures almost-global convergence to the correct formation.

I. INTRODUCTION

Formation control is an important problem in multi-agent coordination and cooperation where the objective is for agents to form a prescribed geometric shape in space. This requirement is intrinsic to tasks such as area coverage, perimeter protection, and co-transportation of large objects.

One of two methods are typically used in formation control: i) regulate the relative position of certain agent pairs to prescribed values [1], [2], or ii) regulate a set of inter-agent distances (magnitude of the relative position vector) to prescribed values [3], [4]. The first method requires the agents to have a common global coordinate frame or that their local coordinate frames be aligned which may not be feasible in practice. On the other hand, the feedback variables in the second method can be calculated in each agent’s local coordinate frame, which do not have to be aligned with a global coordinate frame or with each other. As a result, the desired formation is, at best, only acquired up to translation and rotation; i.e., the agents can converge to any formation that is isomorphic to the desired one.

An important consideration in the distance-based method is how to prevent agents from converging to a formation that is equivalent but noncongruent to the desired one (see Section II-A for the formal definitions of equivalency and congruency). Such formations are undesirable because they do not have the same shape or orientation as the prescribed formation, although they satisfy the set of distance constraints. In other words, the distance constraints do not uniquely define the relative positions of the agents and lead to positional ambiguities [5]. Rigid graph theory provides a partial solution to this problem by requiring the formation graph to be rigid [6], [7]. Specifically, imposing a minimum number of distances to be controlled reduces the undesirable “equilibrium points” to formations that are flipped/ reflected versions of the desired one [5]. Then, the determining factor whether convergence is to a congruent formation or a flipped formation is the initial condition of the rigid formation. That is, rigidity distance-based formation controllers only have local stability properties.

A few approaches have been recently proposed to address the aforementioned issues with distance-based controllers. In [8], a combination of inter-agent distance and angular constraints was used to reduce the likelihood of convergence to noncongruent formations in two dimensions (2D). Although the region of attraction of the desired equilibrium can be somewhat enlarged by a proper choice of control gains, the stability of the control proposed in [8] is still local in nature.

An extension of this work to 3D appeared in [9] by using area and volume constraints. The control method avoids flipped formations but introduces other undesired equilibrium points due to the multiple local minima of the proposed potential function. A related approach was introduced in [10] for the single-integrator agent model where the signed area of a triangle was employed as a controlled variable to prevent flipped formations. That is, the sign of the area enclosed by the formation along with the inter-agent distances were used to uniquely define the correct formation up to translation and rotation. The formation control law in [10] was based on the gradient of a potential function that incorporates distance error and signed area error terms and on the use of undirected graphs (i.e., bidirectional sensing and control). Convergence analyses were conducted for special cases of 3- and 4-agent planar formations.

The purpose of this paper is to explore the approach introduced in [10] further. Specifically, we aim to generalize the approach to systems of \( n \) agents while introducing explicit, sufficient conditions for convergence to the 2D desired formation. The key to our solution is triangulating the directed formation graph to facilitate the use of interconnected system theory. The use of a directed graph has the added benefit of leading to a unidirectional formation controller. Under our solution, mild conditions are imposed on the edge lengths of the interconnected triangles, and the overall formation graph is required to be a Leader-First-Follower (LFF) type of minimally persistent directed graph [11]. We show that our gradient-type control law ensures convergence to the desired formation as long as the leader and first follower are not initially collocated. That is, no restrictions are placed on the initial conditions of the ordinary followers. The closed-loop system is proven to have an almost-global asymptotic equilibrium point corresponding to the desired formation. Thus,
the collinear invariant set and flipped formation problems are hereby solved by the proposed control scheme. We note that our result is not a straightforward extension of [10] since the interconnected triangles form coupled nonlinear subsystems, which complicates the stability analysis of the overall system.

II. BACKGROUND MATERIAL

A. Undirected Graphs

An undirected graph $G$ is represented by a pair $(V, E)$, where $V = \{1, 2, ..., n\}$ is the set of vertices and $E = \{(i, j) | i, j \in V, i \neq j\} \subset V \times V$ is a set of undirected edges. The total number of edges in $E$ is denoted by $a \in \{1, ..., n(n - 1)/2\}$. The set of neighbors of vertex $i \in V$ is represented by

$$N_i(E) = \{j \in V | (i, j) \in E\}.$$  \hspace{1cm} (1)

If $p = [p_1, ..., p_n] \in \mathbb{R}^{2n}$ where $p_i \in \mathbb{R}^2$ is the coordinate of the $i$th vertex, then a framework $F$ is defined as the pair $(G, p)$.

The edge function $\gamma : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ is defined as

$$\gamma(p) = [\ldots, ||p_i - p_j||^2, \ldots] \in \mathbb{R}^n, (i, j) \in E$$  \hspace{1cm} (2)

such that its $m$th component, $||p_i - p_j||$, relates to the $m$th edge of $E$ connecting the $i$th and $j$th vertices. The rigidity matrix $R : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n \times 2n}$ is given by

$$R(p) = \frac{1}{2} \frac{\partial \gamma(p)}{\partial p}$$  \hspace{1cm} (3)

where we have that $\text{rank}[R(p)] \leq 2n - 3$ [6]. Frameworks $(G, p)$ and $(G, \hat{p})$ are equivalent if $\gamma(p) = \gamma(\hat{p})$, and are congruent if $||p_i - p_j|| = ||\hat{p}_i - \hat{p}_j||, \forall i, j \in V$ [12].

An isometry of $\mathbb{R}^2$ is a map $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying [7]

$$||w - z|| = ||T(w) - T(z)||, \forall w, z \in \mathbb{R}^2.$$  \hspace{1cm} (4)

This map includes rotation and translation of the vector $w - z$. Two frameworks are isomorphic if they are correlated via an isometry. It is obvious that (2) is invariant under isomorphic motions of the framework. A framework $F = (G, p)$ is rigid in $\mathbb{R}^2$ if all of its motions satisfy $p_i(t) = T(p_i), \forall i \in V$ and $\forall t \in [0, 1]$; i.e., the family of frameworks $F(t)$ is isomorphic [6, 7]. Some related notions of rigidity are the following. A generic framework $(G, p)$ is infinitesimally rigid if and only if $\text{rank}[R(p)] = 2n - 3$ [7]. A rigid framework is said to be minimally rigid if and only if $a = 2n - 3$ [5]. If the infinitesimally rigid frameworks $(G, p)$ and $(G, \hat{p})$ are equivalent but not congruent, then they are referred to as ambiguous frameworks. A Henneberg framework is persistent if and only if it is constraint consistent and its underlying undirected graph is infinitesimally rigid (see Theorem 3 of [14]). A persistent graph in $\mathbb{R}^2$ is said to be minimally persistent if no single edge can be removed without losing persistence. A necessary condition for a persistent graph in $\mathbb{R}^2$ to be minimally persistent is out$(i) \leq 2$ for all $i \in V$, while a sufficient condition is minimal rigidity [14]. Starting from two vertices with an edge, a minimally persistent (resp., rigid) graph can be constructed by the Henneberg insertion of type I [15], i.e., iteratively adding a vertex with two outgoing (resp., undirected) edges. Henceforth, we refer to a graph constructed in this manner as a Henneberg graph.

B. Directed Graphs

A directed graph $G$ is a pair $(V, E^d)$ where the edge set $E^d$ is directed in the sense that if $(i, j) \in E^d$ then $i$ is the source vertex of the edge and $j$ is the sink vertex. For $i \in V$, the out-degree of $i$ (denoted by out$(i)$) is the number of edges in $E^d$ whose source is vertex $i$ and sinks are in $V - \{i\}$.

For directed graphs, the notion of rigidity defined in Section II-A is not enough to maintain the formation structure (see [13] for an example), and two additional concepts are needed. The first one is the notion of a constraint consistent graph. As explained in [13], the intuitive meaning of constraint consistency is that every agent is able to satisfy all its distance constraints when all the others are trying to do the same. A sufficient condition for a directed graph $(V, E^d)$ in $\mathbb{R}^2$ to be constraint consistent is that $\text{out}(i) \leq 2$ for all $i \in V$ (see Lemma 5 of [14]). The second concept is graph persistency, which has the meaning that, provided all agents are trying to satisfy their distance constraints, the structure of the agent formation is preserved [13]. A directed graph is persistent if and only if it is constraint consistent and its underlying undirected graph is infinitesimally rigid (see Theorem 3 of [14]). A persistent graph in $\mathbb{R}^2$ is said to be minimally persistent if no single edge can be removed without losing persistence. A necessary condition for a persistent graph in $\mathbb{R}^2$ to be minimally persistent is out$(i) \leq 2$ for all $i \in V$, while a sufficient condition is minimal rigidity [14]. Starting from two vertices with an edge, a minimally persistent (resp., rigid) graph can be constructed by the Henneberg insertion of type I [15], i.e., iteratively adding a vertex with two outgoing (resp., undirected) edges. Henceforth, we refer to a graph constructed in this manner as a Henneberg graph.

C. Signed Area

The signed area of a triangular framework, $S : \mathbb{R}^6 \rightarrow \mathbb{R}$, is defined as [10]

$$S(p) = \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\
1 & p_2 & p_3 \end{bmatrix} = \frac{1}{2} (p_3 - p_1)^T J (p_3 - p_2)$$  \hspace{1cm} (5)

where

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  \hspace{1cm} (6)

This quantity is positive (resp., negative) if the vertices are ordered counterclockwise (resp., clockwise). Further, (5) is zero if any two vertices are collocated or the three vertices are collinear.

A Henneberg framework can be divided into triangular sub-frameworks. Therefore, the signed area of a Henneberg framework with $n$ vertices and directed edge set $E^d$, $\chi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n-2}$, is defined as

$$\chi(p) = \left[ \begin{array}{c} \frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\
p_i & p_j & p_k \end{bmatrix} \\ \vdots \end{array} \right]$$  \hspace{1cm} (7)

for $(k, i), (k, j) \in E^d - \{(2, 1)\}$.
such that its $m$th component is related to the signed area of the $m$th triangle constructed with vertices $i$, $j$, and $k$. For example, the signed area of the framework in Figure 2a is given by
\[
\chi(p) = \frac{1}{2} (p_3 - p_1)^T J (p_3 - p_2),
\]
where the three elements of $\chi(p)$ are positive, negative, and positive, respectively. For the framework in Figure 2b, it would be
\[
\chi(p) = \frac{1}{2} (p_4 - p_1)^T J (p_4 - p_2),
\]
where
\[
\Lambda = \begin{pmatrix} a & b & 1 \end{pmatrix}, \quad \chi(p) = \begin{pmatrix} \frac{1}{2} (p_3 - p_1)^T J (p_3 - p_2), \\ \frac{1}{2} (p_4 - p_1)^T J (p_4 - p_2), \\ \frac{1}{2} (p_5 - p_1)^T J (p_5 - p_2) \end{pmatrix}.
\]
(8)

**Fig. 2.** Signed area examples.

We introduce next is an extension of the concept of congruency that includes the signed area.

**Definition 1:** Henneberg frameworks $F = (G, p)$ and $\hat{F} = (\hat{G}, \hat{p})$ where $G = (V, E)$ are said to be strongly congruent if they are congruent and $\chi(p) = \chi(\hat{p})$.

We represent the set of all frameworks that are strongly congruent to $F$ by $SCgt(F)$. It is obvious that frameworks that are congruent but not strongly congruent are reflected frameworks. Note that if $F$ is a reflected version of $F$, then $\chi(p) = -\chi(\hat{p})$. In summary, the signed area function will be used to rule out the the occurrence of framework ambiguities, especially reflections.

**Lemma 1:** Henneberg frameworks $F = (G, p)$ and $\hat{F} = (\hat{G}, \hat{p})$ are strongly congruent if and only if they are equivalent and $\chi(p) = \chi(\hat{p})$.

**Proof:** See Appendix VI-A.

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**D. Quartic Polynomials**

**Lemma 2:** [16], [17], [18], [19] For any quartic polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where $a \neq 0$, $\Lambda = [16a^3, 12b^2, 4ac, 2ad, b^2c - 3bd]$.

The equation has no real solution if $\Lambda > 0$ and $P > 0$, or $\Lambda > 0$ and $D > 0$.

**Corollary 1:** Consider the equation
\[
a x^4 + b x^3 + c x^2 + d x + e = 0 \tag{10}
\]
where
\[
a = -2 \delta_1^2 (\gamma - 2)^2
\]
\[
b = \delta_1 (\gamma - 4) \sqrt{2 \delta_1^2 \delta_2^2 - \delta_1^4 + 2 \delta_1^2 \delta_3^2 - \delta_2^2 + 2 \delta_2^2 \delta_3^2 - \delta_3^2}
\]
\[
c = -\frac{1}{2} \delta_1^3 \gamma + \delta_1^2 \left( \frac{3}{2} \delta_2^2 + \delta_2^2 + \delta_3^2 \right) \gamma^2 - 4 \delta_1^2 (\delta_2^2 + \delta_3^2) \gamma
\]
\[
d = \frac{1}{4} \delta_1 \gamma^2 (2 \delta_2^2 \gamma - 3 \delta_2^2 - 2 \delta_2^2 - 2 \delta_2^2) \times
\]
\[
\sqrt{2 \delta_1^2 \delta_2^2 - \delta_1^4 + 2 \delta_1^2 \delta_3^2 - \delta_2^2 + 2 \delta_2^2 \delta_3^2 - \delta_3^2}
\]
\[
e = -\frac{1}{8} \delta_1^2 \gamma^3 (2 \delta_2^2 \delta_2^2 - \delta_1^4 + 2 \delta_1^2 \delta_3^2 - \delta_2^4 + 2 \delta_2^2 \delta_3^2 - \delta_3^4),
\]
$\gamma$ is a positive constant, and $\delta_1, \delta_2, \delta_3$ are the lengths of the edges of a triangle. If $\delta_2 \neq \delta_3$ and
\[
|\delta_3^2 - \delta_2^2| < 2 \sqrt{2},
\]
then there exists a $\gamma > 0$ such that (10) has no real solution for $\gamma > \max\{\gamma_2\}$.

**Proof:** See Appendix VI-B.

**Remark 1:** The geometric meaning of condition (11) is discussed in Remark [3]. Although the existence of the lower bound $\gamma$ in Corollary 1 is guaranteed, a closed-form expression for $\gamma$ does not exist in general. However, $\gamma$ can be easily determined by numerical means once $\delta_1, \delta_2,$ and $\delta_3$ are selected.

**E. Stability Results**

**Lemma 3:** [20] Consider the system $\dot{x} = f(x, u)$ where $x$ is the state, $u$ is the control input, and $f(x, u)$ is locally Lipschitz in $(x, u)$ in some neighborhood of $(x = 0, u = 0)$. Then, the system is locally input-to-state stable if and only if the unforced system $\dot{x} = f(x, 0)$ has a locally asymptotically stable equilibrium point at the origin.

**Lemma 4:** [21] Consider the interconnected system
\[
\Sigma_1: \quad \dot{x} = f(x, y)
\]
\[
\Sigma_2: \quad \dot{y} = g(y).
\]
(12)

If subsystem $\Sigma_1$ with input $y$ is locally input-to-state stable and $y = 0$ is a locally asymptotically stable equilibrium point of subsystem $\Sigma_2$, then $|x, y| = 0$ is a locally asymptotically stable equilibrium point of the interconnected system.

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**III. PROBLEM STATEMENT**

Consider a system of $N \geq 2$ mobile agents governed by the kinematic equation
\[
\dot{p}_i = u_i, \quad i = 1, ..., N \tag{13}
\]
where $p_i \in \mathbb{R}^2$ is the position of the $i$th agent relative to an Earth-fixed coordinate frame, and $u_i \in \mathbb{R}^2$ is the velocity-level control input.

The desired formation of agents is modeled by the directed framework $F^* = (G^*, p^*)$ where $G^* = (V^*, E^*)$, $dim(E^*) = \alpha$, $p^* = [p^*_1, ..., p^*_N]$, and $p^*_i \in \mathbb{R}^2$ denotes the desired position of the $i$th agent. The fixed desired distance separating the $i$th and $j$th agents is defined as
\[
d_{ij} = ||p^*_j - p^*_i|| > 0, \quad i, j \in V^* \tag{14}
\]
We assume $F^*$ is constructed to satisfy the following conditions:

**Condition 1** out(1) = 0, out(2) = 1, and out($i$) = 2, $\forall i \geq 3$.

**Condition 2** If there is an edge between agents $i$ and $j$, the direction must be $i \rightarrow j$ if $i < j$.

The above conditions imply that $F^*$ should be a LFF-type minimally persistent formation [11], where agent 1 is the leader, agent 2 is the first follower, and agents $i$ for $i \geq 3$ are called ordinary followers.

The actual formation of agents is modeled by $F(t) = (G^*, p(t))$ where $p = [p_1, ..., p_N]$. Note that $F$ and $F^*$ share the same directed graph, which remains unchanged for all time. The physical meaning of $(j, i) \in E^*$ in the actual formation is that agent $j$ can measure its relative position to agent $i$, $p_j - p_i$, but not vice versa.

The control objective of this paper is to ensure

$$F(t) \rightarrow SCgt(F^*)$$

as $t \rightarrow \infty$, (15)

which is equivalent to saying

$$\|p_j(t) - p_i(t)\| \rightarrow d_{ji}$$

as $t \rightarrow \infty$, $i, j \in V^*$ and (16a)

$$\chi(p(t)) \rightarrow \chi(p^*)$$

as $t \rightarrow \infty$. (16b)

The control objective will be quantified by two types of error variables. If the relative position of two agents is defined as $\hat{p}_{ji} = p_j - p_i$, the distance error is given by [4]

$$z_{ji} = ||\hat{p}_{ji}||^2 - d_{ji}^2.$$ (17)

The stacked vector of all distances errors is defined as $z = [z_{21}, ..., z_{ji}, ..., \forall (j, i) \in E^*$. The area error is defined as [10]

$$\hat{S}_{ijk} = S_{ijk} - S_{i'jk}, (k, i), (k, j) \in E^*$$ (18)

where $S_{ijk} = S(p)$ with $p = [p_1, p_2, p_k]$ and $S_{i'jk} = S(p^*)$ with $p^* = [p_1^*, p_2^*, p_k^*]$. The stacked vector of all area errors is given by $\hat{S} = [\hat{S}_{123}, ..., \hat{S}_{ijk}, ..., \forall (k, i), (k, j) \in E^*$. Since $F^*$ is typically specified in the desired inter-agent distances, a useful formula for calculating $S_{ijk}$ is given by [22]

$$S_{ijk}^* = \pm \sqrt{d_{ijk}(d_{ijk} - d_{ji})(d_{ijk} - d_{ki})(d_{ijk} - d_{kj})}$$ (19)

where

$$d_{ijk} = \frac{d_{ji} + d_{ki} + d_{kj}}{2}.$$ (20)

Note that if the order of agents $i$, $j$, $k$ is counterclockwise (resp., clockwise), then (19) takes the positive (resp., negative) sign.

**IV. CONTROL LAW FORMULATION**

The control law will be dictated by the choice of potential function associated with the error variables (17) and (18). To this end, we consider the Lyapunov function candidate [10]

$$V_k = \begin{cases} \alpha_k z_{21}, & \text{if } k = 2 \\ \alpha_k (z_{21}^2 + z_{21}^2) + \beta_k \hat{S}_{ijk}^2, & \text{if } 2 < k \leq N \end{cases}$$ (21)

where $\alpha_k$ and $\beta_k$ are positive constants, $i < j < k$, and $(k, i), (k, j) \in E^*$. Based on (21) and its time derivative, we propose the following control law

$$u_1 = 0$$ (22a)

$$u_2 = -\alpha_2 z_{21} \hat{p}_{21}$$ (22b)

$$u_k = -\alpha_k (z_{ki} \hat{p}_{ki} + z_{kj} \hat{p}_{kj}) - \beta_k \hat{S}_{ijk} J^T(\hat{p}_{ki} - \hat{p}_{kj})$$ (22c)

for $2 < k \leq N$, $i < j < k$, and $(k, i), (k, j) \in E^*$. The control law is only a function of $\hat{p}_{ki}$, $\hat{p}_{kj}$, $d_{ki}$, $d_{kj}$ and $d_{ji}$ for $i, j \in N_k(E^*)$. Thus, the control law is distributed since it only requires the $k$th agent to measure its relative position to neighboring agents in the directed graph.

The following theorem states our main result.

**Theorem 1:** Let the initial conditions of the formation $F(t) = (G^*, p(t))$ be such that $p_1(0) \neq p_2(0)$, and let $F^*$ satisfy

$$\frac{d^2_1 - d^2_2}{d^2_1} < 2\sqrt{2},$$ (23)

for all $i, j, k$ such that $2 < k \leq N$, $i < j < k$, and $(k, i), (k, j) \in E^*$. Then, the control (22) with

$$\frac{\beta_k}{\alpha_k} > \left( \frac{d_{ji} - d_{kj}^2/4}{d_{ji}^2}, \text{ if } d_{ki} = d_{kj} \right)$$ (24)

where $\gamma$ is determined from Corollary 1 renders $[z, \hat{S}] = 0$ asymptotically stable and ensures $F(t) \rightarrow SCgt(F^*)$ as $t \rightarrow \infty$.

**Proof:** The open-loop dynamics for (17) and (18) are given by

$$\dot{z}_{ji} = 2\hat{p}_{ji}^T(u_j - u_i)$$ (25)

and

$$\dot{\hat{S}}_{ijk} = \frac{1}{2} \left[(u_k - u_i)\hat{p}_{kj} + \hat{p}_{ji}^T J(u_k - u_j)\right],$$ (26)

where (13) was used. Therefore, the time derivative of (21) becomes

$$\dot{V}_k = \begin{cases} \alpha_2 z_{21} \hat{p}_{21}^T (u_2 - u_1), & \text{if } k = 2 \\ \alpha_k [z_{ki} \hat{p}_{ki}^T (u_k - u_i) + z_{kj} \hat{p}_{kj}^T (u_k - u_j)] + \beta_k \hat{S}_{ijk} J^T (u_k - u_j), & \text{if } 2 < k \leq N \end{cases}$$ (27)

**Step 1:** Consider the subsystem composed of agents 1 and 2 only. Substituting (22a) and (22b) into (25) yields

$$\dot{z}_{21} = -2\alpha_2 z_{21}|p_{21}|^2 = -2\alpha_2 z_{21} (z_{21} + d^2_{21})$$ (28)

where (17) was used. The solution to the above nonlinear ODE is

$$z_{21}(t) = \frac{d_{21}^2}{d_{21}^2} \exp(2d^2_{21} \alpha_2 t) (z_{21}(0) + d^2_{21}) - z_{21}(0).$$ (29)

From (17), it is clear that $z_{21} \in [-d^2_{21}, \infty)$ where $z_{21} = -d^2_{21}$ corresponds to agents 1 and 2 being colocated. If $z_{21}(0) > -d_{21}$, we can show from (29) that $z_{21}(t) > -d^2_{21}$ for all $t > 0$ as follows:
$z_{21}(t) > -d_{21}^2$
\[< \frac{z_{21}(0)}{d_{21}^2 \exp(2d_{21}^2 \alpha_2 t)} - z_{21}(0) > -1 \]
\[< z_{21}(0) - d_{21}^2 \exp(2d_{21}^2 \alpha_2 t) (z_{21}(0) + d_{21}^2) \]
\[< d_{21}^2 \exp(2d_{21}^2 \alpha_2 t) (z_{21}(0) + d_{21}^2) > 0 \]
\[< z_{21}(0) > -d_{21}^2. \]

(30)

Now, after substituting (32a) and (32b) into (27), we obtain
\[V_2 = -\alpha_2^2 \|z_{21}\|^2 \leq 0. \]

(31)

Since $z_{21}(t) > -d_{21}^2$ implies $\|z_{21}(t)\| > 0$, we can see that $V_2 = 0$ only at $z_{21} = 0$. Therefore, $V_2$ is negative definite and $z_{21} = 0$ is asymptotically stable for $z_{21}(0) > -d_{21}^2$ (or equivalently, $p_1(0) \neq p_2(0)$).

**Step 2:** Consider that a third agent is added to the previous subsystem as shown in Figure 3. We can view this new system as the interconnected system

\[\xi_3 = f_3(\xi_3, \Xi_2) \]
\[\Xi_3 = g_3(\Xi_2) \]

where $\xi_3 := [z_{31}, z_{32}, S_{123}]$ is the state of the error dynamics of agent 3 and $\Xi_3 = \xi_2 := z_{21}$.

![Fig. 3. Three-agent system.](image)

We seek to establish the input-to-state stability of (32a) with respect to input $\Xi_3$. When $\Xi_3 = 0$, we can see from (22b) that $u_2 = 0$. Therefore, from (27) with $k = 3$ under the condition that $\Xi_2 = 0$, we have that
\[\dot{V}_3 = \alpha_3 (z_{31}\hat{p}_{31} + z_{32}\hat{p}_{32})^T + \beta_3 S_{123} (\hat{p}_{31} - \hat{p}_{32})^T J \]
\[u_3. \]

(33)

Substituting (22c) with $k = 3$ in (33) gives
\[\dot{V}_3 = -\alpha_3 (z_{31}\hat{p}_{31} + z_{32}\hat{p}_{32}) + \beta_3 S_{123} J (\hat{p}_{31} - \hat{p}_{32})^2. \]

(34)

If $\xi_3 = 0$ is the only value at which $V_3 = 0$, then (33) is negative definite and (32a) is input-to-state stable. It then follows that the origin of (32), i.e., $\Xi_3, \xi_3 = 0$, is asymptotically stable according to Lemma 4. To this end, note that translational and rotational motions of the triangle will not change the value of $V_3$ since it is a function of the relative position of agents and the triangle area [23], [24]. Thus, without the loss of generality, let $p_1 = [-d_{21}/2, 0]$, $p_2 = [d_{21}/2, 0]$, and $p_3 = [x, y]$ for simplicity. Then, $V_3 = 0$ is equivalent to
\[(2x^2 + 2y^2 + \frac{d_{31}}{2} - d_{32}^2 - d_{31}^2)x + \frac{d_{31}}{2} (2d_{21}x - d_{31}^2 + d_{32}^2) = 0 \]

(35)

and
\[(2x^2 + 2y^2 + \frac{d_{31}}{2} - d_{32}^2 - d_{31}^2)y + \frac{\beta_3}{\alpha_3} \left(\frac{d_{31}}{2} y - d_{21}^2 S_{123}^* \right) = 0. \]

One solution to (35) is
\[x = \frac{d_{31}^2 - d_{32}^2}{(2d_{21})} \quad \text{and} \quad y = \frac{2S_{123}^*}{d_{21}}. \]

(36)

which corresponds to $\xi_3 = 0$. We will show next that $\beta_3/\alpha_3$ can be selected such that this is the only solution to (35). This proof will be conducted for two distinct cases: an isosceles triangle and the non-isosceles case.

**Case 2a** Consider that the triangle is such that $d_{32} = d_{31}$.

From (35), we get
\[(2x^2 + 2y^2 + \frac{3d_{31}^2}{2} - 2d_{32}^2) x = 0 \]

and
\[(2x^2 + 2y^2 + \frac{1}{2}d_{32}^2 - 2d_{32}^2) y + \frac{\beta_3}{\alpha_3} \frac{d_{31}^2}{2} \left( y - \frac{1}{2} \sqrt{4d_{32}^2 - d_{31}^2} \right) = 0. \]

(37)

The first equation of (37) implies $x = 0$ or $x^2 + y^2 = \frac{3}{4}d_{31}^2$. Substituting $x = 0$ into the second equation of (37) yields
\[(y - \frac{1}{2} \sqrt{4d_{32}^2 - d_{31}^2}) \times \]
\[8y^2 + 4\sqrt{4d_{32}^2 - d_{31}^2} y + \frac{2\beta_3}{\alpha_3} d_{31}^2 = 0. \]

(38)

It is easy to show that when
\[\frac{\beta_3}{\alpha_3} > \frac{d_{32}^2 - \frac{1}{4}d_{31}^2}{d_{31}^2}, \]

(39)

the discriminant of $8y^2 + 4\sqrt{4d_{32}^2 - d_{31}^2} y + \frac{2\beta_3}{\alpha_3} d_{31}^2$ is less than 0. That is, inequality (39) will lead to $x = 0$ and $y = \frac{1}{2} \sqrt{4d_{32}^2 - d_{31}^2}$ being the only solution to (38).

Now, substituting
\[x^2 + y^2 = \frac{3}{4}d_{31}^2 \]

into the second equation of (37) gives
\[2y \left(\frac{\beta_3}{\alpha_3} - 2 \right) = \frac{\beta_3}{\alpha_3} \sqrt{4d_{32}^2 - d_{31}^2}. \]

(41)

After squaring (41) and using (40) again to eliminate $y$, we obtain
\[2x^2 \left( \frac{\beta_3}{\alpha_3} - 2 \right) = \]
\[- \left( \frac{d_{31}^2}{2} \left( \frac{\beta_3}{\alpha_3} \right)^2 + (8d_{32}^2 - 6d_{31}^2) \frac{\beta_3}{\alpha_3} + 6d_{21}^2 - 8d_{32}^2 \right) \]

(42)
Since the left hand side of \((42)\) is nonnegative for any \(\beta_3/\alpha_3\), this equation has no solution if \(\phi(\beta_3/\alpha_3) > 0\). This means that \(\phi(\beta_3/\alpha_3)\) should have no real roots, or \(\beta_3/\alpha_3\) should be chosen to be greater (resp., smaller) than the largest (resp., smallest) root. The discriminant of \(\phi(\beta_3/\alpha_3)\) is given by

\[
\Delta = 4 \left( 4d_3^2 - 3d_3 \right) \left( 4d_3^2 - d_2^2 \right). \tag{43}
\]

If \(4d_3^2 < d_2^2 < 4d_3^2\), then \(\Delta < 0\) for any \(\beta_3/\alpha_3 > 0\) and \(\phi(\beta_3/\alpha_3) > 0\).

If \(d_2^2 \leq 4d_3^2\), then \(\Delta \geq 0\) and \(\phi(\beta_3/\alpha_3)\) has real roots. Since the smallest root is less than zero, the only option for ensuring \(\phi(\beta_3/\alpha_3) > 0\) is to choose \(\beta_3/\alpha_3\) greater than the largest root, i.e.,

\[
\frac{\beta_3}{\alpha_3} > \frac{3d_3^2 - 4d_3^2 + \sqrt{(4d_3^2 - 3d_3^2)(4d_3^2 - d_2^2)}}{d_3^2}. \tag{44}
\]

Note that the case where \(d_2^2 \geq 4d_3^2\) is not possible since it contradicts the fact that \(d_{21} + d_{31} = 2d_{22} > d_{21}\).

Combining the three cases, we see that \((44)\) will have no solution if \((44)\) holds. Finally, it is not difficult to show that \((39)\) is a sufficient condition for \((44)\). Therefore, for the isosceles triangle, the condition for \(\xi_3 = 0\) to be the only value where \(V_3 = 0\) is given by \((39)\).

**Case 2b** Consider that the triangle is not isosceles \((d_{32} \neq d_{31})\). After substituting the first equation of \((35)\) into the second one, eliminating \(x\), and factoring the resulting polynomial of \(y\), we obtain

\[
\begin{align*}
( y - \frac{2c_4^2}{d_2^2} ) ( c_4 y^4 + c_3 y^3 + c_2 y^2 + c_1 y + c_0 ) &= 0 \tag{45}
\end{align*}
\]

where

\[
\begin{align*}
c_4 &= -2d_2^2 \left( \frac{\beta_3}{\alpha_3} - 2 \right)^2 \tag{46a} \\
c_3 &= d_2 \left( \frac{\beta_3}{\alpha_3}^2 - 4 \right) \times \\
&\sqrt{2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2} \tag{46b} \\
c_2 &= -\frac{1}{2} d_2^2 \left( \frac{\beta_3}{\alpha_3}^3 + d_2^2 \left( \frac{3}{2} d_2^2 + d_3^2 + d_3^2 \right) \left( \frac{\beta_3}{\alpha_3}^2 \right) \right)^2 \\
&- 4d_2^2 \left( d_3^2 + d_3^2 \right) \left( \frac{\beta_3}{\alpha_3} \right) \tag{46c} \\
c_1 &= \frac{1}{4} d_2^2 \left( \frac{\beta_3}{\alpha_3}^2 \right)^2 \left( 2d_2^2 - 3d_2^2 - 2d_3^2 - 2d_2^2 \right) \times \\
&\sqrt{2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2} \tag{46d} \\
c_0 &= -\frac{1}{8} d_2^2 \left( \frac{\beta_3}{\alpha_3} \right)^3 \times \\
(2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2 + 2d_2^2 d_3^2 - d_3^2). \tag{46e}
\end{align*}
\]

Note that the quartic polynomial in \((45)\) is similar to \((10)\). Thus, by Corollary \((1)\) if

\[
\begin{align*}
\left| \frac{d_{31}^2 - d_{32}^2}{d_{31}^2} \right| < 2\sqrt{2} \tag{47}
\end{align*}
\]

and \(\beta_3/\alpha_3 > \max\{\sqrt{2}, 2\}\) (see proof of Corollary \((1)\) for detail of \(\gamma\)), the quartic polynomial has no real solution, and \(y = 2S_{123}/d_{21}\) is the only solution to \((45)\).

**Step k**: The process of adding a vertex \(k\) with two outgoing edges to any two distinct vertices \(i\) and \(j\) of the previous graph can be followed one step at a time, resulting at each step in the interconnected system

\[
\begin{align*}
\dot{\xi}_k &= f_k(\xi_k, \Xi_{k-1}) \tag{48a} \\
\Xi_{k-1} &= g_k(\Xi_{k-1}) \tag{48b}
\end{align*}
\]

where \(\xi_k := [z_{ki}, z_{kj}, S_{ijk}], (k, i), (k, j) \in E^*\) is the state of error dynamics of the \(k\)th agent and \(\Xi_{k-1} := [\xi_2, \ldots, \xi_{k-1}]\).

Note that the asymptotic stability of \(\Xi_{k-1} = 0\) for \((48b)\) was already established in Step \(k - 1\). Therefore, we only need to check the input-to-state stability of \((48a)\) with respect to input \(\Xi_{k-1}\). To this end, when \(\Xi_{k-1} = 0\), \((27)\) becomes

\[
\begin{align*}
\dot{V}_k &= \alpha_k(z_{ki}\tilde{p}_{ki} + z_{kj}\tilde{p}_{kj})^T + \beta_kS_{ijk}(\tilde{p}_{ki} - \tilde{p}_{kj})^T J u_k. \tag{49}
\end{align*}
\]

Now, substituting \((22c)\) into \((49)\) gives

\[
\begin{align*}
\dot{V}_k &= -\left\| \alpha_k(z_{ki}\tilde{p}_{ki} + z_{kj}\tilde{p}_{kj}) + \beta_kS_{ijk} J (\tilde{p}_{ki} - \tilde{p}_{kj}) \right\|^2. \tag{50}
\end{align*}
\]

Similar to Step 2, we can show that if the gain ratio \(\beta_k/\alpha_k\) is selected according to \((23)\) and the edges of triangle \(\Delta_{ijk}\) satisfy \((23)\), then \((50)\) is negative definite. As a result, \((48a)\) is input-to-state stable and \((\Xi_{k-1}, \xi_k) = 0\) in \((48)\) is asymptotically stable by Lemma \((2)\).

Repeating this process until \(k = N\) leads to the conclusion that \([\xi_2, \ldots, \xi_N] = 0\) is asymptotically stable, which implies \(z(t) \to 0\) and \(\chi(p(t)) \to \chi(p^*)\) as \(t \to \infty\). Given that \(F^*\) and \(F(t)\) have the same edge set and \(F^*\) is minimally persistent by design, then we have that \(F(t) \to SC(GF^*)\) as \(t \to \infty\) from Lemma \((1)\) \(\blacksquare\).

**Remark 2**: Theorem \((1)\) only requires that the leader and first follower not be collocated at \(t = 0\). If agents 1 and 2 were initialized at the same position, then \(u_1 = u_2 = 0\) and they would remain at this position forever. In other words, the condition \(p_1 = p_2\) is an invariant set. As for the ordinary followers, \((22)\) guarantees formation acquisition regardless of their initial conditions. For example, if agents 2, 3, 4, and 5 in Figure \((2)\) are all initially collocated, then \(u_4 = u_5 = 0\) at \(t = 0\) which means agents 4 and 5 will not move at first. However, \(u_3 \neq 0\), so agent 3 will move. This results in \(u_4 \neq 0\), causing agent 4 to move, and finally \(u_5\) becomes nonzero, so agent 5 moves.

**Remark 3**: Condition \((24)\) on the desired formation has the following geometric interpretation. Consider the three vertices in Figure \((4)\) where, for simplicity, \(p^*_i = [-d_{ji}/2, 0]\), \(p^*_j = [d_{ji}/2, 0]\), and \(p^*_k = [x, y]\) \(\square\).
Given that
\[
\left| \frac{d_{ji}^2 - d_{kj}^2}{d_{ji}^2} \right| < 2\sqrt{2}
\]
\[
\left| (x + d_{ji}/2)^2 + y^2 - (x - d_{ji}/2)^2 - y^2 \right|< 2\sqrt{2}
\]
\[
\Rightarrow \frac{2 |xd_{ji}|}{d_{ji}^2} < 2\sqrt{2}
\]
\[
\Rightarrow |x| < \sqrt{2}d_{ji}.
\]
(51)

any point \( p_k \) inside the shaded region in Figure 4 satisfies (23). It is important to point out that (23) is sufficient but not necessary for stability. For example, consider a triangular formation with \( d_{21} = 1 \), \( d_{31} = 2.1 \), and \( d_{32} = 3 \), which does not satisfy (23). If however \( \beta_3/\alpha_3 \) is selected in the range (10.42, 13.55), the stability result of Theorem 1 will hold. In fact, the gain ratio \( \beta_k/\alpha_k \) and (24) impose a lower bound on the relative weight of the distance error and area error in the potential function (21) in order to guarantee stability.

**Remark 4:** Mathematically, the role of the area-based term \( \beta_kS_{ijk}^2 \) is to guarantee the existence of a unique minimum for the potential function (21) in the Euclidean plane, and thus avoid the system from converging to an undesirable local minima. To illustrate this, consider a triangular formation where \( p_1 = [-1, 0], p_2 = [1, 0], p_3 = [x, y] \), and \( d_{21} = d_{31} = d_{32} = 2 \), and let \( W = \frac{1}{2}(z_3^2 + z_{32}^2) \) be the potential function with only the distance error terms of (21) with \( k = 3 \). In Figure 5 we plot \( \ln(W + 1) \) and \( \ln(V_3 + 1) \) versus \( p_3 \) to have a better view of their minima. We can clearly see that \( W(p_3) \) has two minima, one near \( p_3 \), corresponding to the desired position for agent 3 and its reflected position, whereas \( V_3(p_3) \) has a unique minimum.

**V. CONCLUSIONS**

This paper presented a 2D formation control scheme that uses distance and signed area information to guarantee convergence to the desired formation shape. The asymptotic convergence result is valid under mild conditions on the edge lengths of the triangulated-like framework and when the leader agent and the first follower are not collocated at time zero. The scheme is applicable to systems with any number of agents governed by the single-integrator model.

**VI. APPENDIX**

A. **Proof of Lemma 7**

(Proof of \( \Rightarrow \)) If \( F \) and \( \tilde{F} \) are strongly congruent, then \( ||p_i - p_j|| = ||\tilde{p}_i - \tilde{p}_j||, \forall i, j \in V \) and \( \chi(p) = \chi(\tilde{p}) \) by definition. Therefore, since \( E \subset V \times V \), we know \( ||p_i - p_j|| = ||\tilde{p}_i - \tilde{p}_j||, \forall (i, j) \in E, \) i.e., \( F \) and \( \tilde{F} \) are equivalent.

(Proof of \( \Leftarrow \)) If \( \dim(V) = 3 \), then framework equivalency and congruency are equivalent, so the conditions for strong congruency are trivially satisfied. If a vertex is added such that \( \dim(V) = 4 \), the resulting framework would have two additional edges and one additional triangle. Consider without loss of generality the framework given by \( S' := S_{123} - S_{234} \). Since \( \chi(p) = \chi(\tilde{p}) \), we know that \( S'(p) = S'(\tilde{p}) \), so it follows from the general quadrilateral area formula [22] that

\[
\frac{1}{4} \left[ 4 ||p_3 - p_2||^2 ||p_4 - p_1||^2 - (||p_2 - p_1||^2 + ||p_4 - p_3||^2)^2 \right]^{\frac{1}{2}}
\]
\[
= \frac{1}{4} \left[ 4 ||\tilde{p}_3 - \tilde{p}_2||^2 ||\tilde{p}_4 - \tilde{p}_1||^2 - (||\tilde{p}_2 - \tilde{p}_1||^2 + ||\tilde{p}_4 - \tilde{p}_3||^2)^2 \right]^{\frac{1}{2}}.
\]  
(52)

Since \( F \) and \( \tilde{F} \) are equivalent, \( ||p_2 - p_1|| = ||\tilde{p}_2 - \tilde{p}_1|| \), \( ||p_3 - p_1|| = ||\tilde{p}_3 - \tilde{p}_1||, ||p_3 - p_2|| = ||\tilde{p}_3 - \tilde{p}_2||, ||p_4 - p_2|| = ||\tilde{p}_4 - \tilde{p}_2||, \) and \( ||p_4 - p_3|| = ||\tilde{p}_4 - \tilde{p}_3||. \) Therefore, we have from (52) that \( ||p_4 - p_1|| = ||\tilde{p}_4 - \tilde{p}_1||, \) so \( F \) and \( \tilde{F} \) are strongly congruent for \( \dim(V) = 4 \). Since the quadrilateral signed area formula in (52) applies to both convex and concave quadrilaterals, a similar analysis exists for all other cases, some of which are shown in Figure 6.
As more vertices are added, each additional vertex will create a quadrilateral, so above process can be repeated to show that \( F \) and \( \hat{F} \) are strongly congruent for \( \dim(V) = n \).

**B. Proof of Corollary**

Based on Lemma 2 (10) has no real solution if the following quantities are positive:

\[
\Lambda = \gamma^6 (\gamma - 2)^2 \left[ -1 \delta_2 \left( \delta_2^2 - \delta_3^2 \right)^2 \left( 8 \delta_4^4 - \delta_2^2 - \delta_3^2 \right)^2 \right] \times \left( \delta_1^4 - 2 \delta_1^2 \left( \delta_2^2 + \delta_3^2 \right) + \left( \delta_2^2 - \delta_3^2 \right)^2 \right) \gamma^7 + f_1(\delta_1, \delta_2, \delta_3, \gamma) \]

\[P = (\gamma - 2)^2 \left[ 8 \delta_1^5 \gamma^3 + f_2(\delta_1, \delta_2, \delta_3, \gamma) \right] \quad (53a)

where \( f_1(\cdot) \) and \( f_2(\cdot) \) are polynomials in \( \gamma \) of at most, degree 6 and 2, respectively.

Given a polynomial \( p(\gamma) = a_n \gamma^n + \sum_{i=0}^{n-1} a_i \gamma^i \) where \( n \geq 3 \) is an odd integer, we can see from Figure 7 that \( p(\gamma) = 0 \) has at least one real root. Denote the largest or the unique real root by \( \gamma^* \). Then,

\[
p(\gamma) > 0 \ \forall \gamma > \gamma^*, \quad \text{if} \ a_n > 0
\]

\[
p(\gamma) < 0 \ \forall \gamma > \gamma^*, \quad \text{if} \ a_n < 0.
\]

**Fig. 7. Odd degree polynomial.**

Consider that \( \gamma \neq 2 \) in \( (53a) \) and \( (53b) \). It follows from the conditions \( \delta_2 \neq \delta_3 \) and \( \left( \delta_2^2 - \delta_3^2 \right) / \delta_1^2 \) < \( 2 \sqrt{2} \) that

\[
\delta_1^4 - 2 \delta_1^2 \left( \delta_2^2 + \delta_3^2 \right) + \left( \delta_2^2 - \delta_3^2 \right)^2 = (\delta_1 + \delta_2 + \delta_3) (\delta_1 - \delta_2 - \delta_3) (\delta_1 + \delta_2 - \delta_3) (\delta_1 - \delta_2 + \delta_3) < 0.
\]

Therefore, the coefficient of \( \gamma^7 \) in \( (53a) \) is positive, and \( \Lambda > 0 \) for \( \gamma > \gamma^*_1 \) where \( \gamma^*_1 \) is the lower bound from \( (54) \). Likewise, the coefficient of \( \gamma^3 \) in \( (53b) \) is positive, and \( P > 0 \) for \( \gamma > \gamma^*_2 \) where \( \gamma^*_2 \) is some lower bound. Thus, the overall sufficient condition for \( \Lambda > 0 \) and \( P > 0 \) is given by \( \gamma > \max\{\gamma^*_1, \gamma^*_2\} \) where \( \gamma := \max\{\gamma^*_1, \gamma^*_2\} \).

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