The CS decomposition and conditional negative correlation inequalities for determinantal processes.

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Abstract

For a conditional process of the form \((\phi|A, \not\subset \phi)\) where \(\phi\) is a determinantal process we obtain a new negative correlation inequalities. Our approach relies upon the underlying geometric structure of the elementary discrete determinantal processes by using the canonical representation of a pair of subspaces in terms of principal vectors and angles, as well as the classical CS decomposition.

1 Introduction and statement of results.

Determinantal point processes (DPP) are probabilistic models that arise in a wide range of theoretical and applied areas. Let us cite, to mention a few recent applied works, [2], [10], [11], [18], [23], [24]. From the theoretical point of view, determinantal point processes could be defined in the general locally compact Polish spaces setting, as point processes associated to some locally square integrable, Hermitian, positive semidefinite, locally trace-class operators; a good overview of the main conceptual basis and properties can be found in [20] and in the bibliography therein; see also [15].

There exists also various extensions and variants as for example L-ensemble [7], extended L-ensembles [27], strongly Rayleigh point processes [8] and so on. A common feature of all these processes is to exhibit repulsion between points and to offer, among others, efficient algorithms for sampling and conditioning. Despite of their widespread use the study of their properties is, in general, not easy and several questions and conjectures are still unsolved (even in the most elementary setting). Notice also that some results for the most general processes can be inferred [12], [20], [21], from the corresponding results of the basic processes.

Recall that the basic elementary determinantal point process can be described via the exterior product concept, as follows.

Fix \(1 < p < N\) and let \(\mathcal{A} = \{a^1, \ldots, a^p\}\), \(1 < p < N\), be a set of orthonormal vectors in \(\mathbb{R}^N\). Denote

\[ a^i = (a^i_1, \ldots, a^i_N)^t, \quad i = 1, \ldots, p \]
and 

\[ a_i = (a_1^i, \ldots, a_p^i), i = 1, \ldots, N. \]

The associated determinantal process \( \phi(\mathfrak{A}) \) is a point process, view as a random subset of \( \mathcal{N} = \{1, \ldots, N\} \) of cardinality \( |\phi(\mathfrak{A})| = p \), characterized [19], [20] by the formulas

\[
\mathbb{P}\{\{i_1, \ldots, i_p\} = \phi\} = |(\prod_{i=1}^{p} a^i_{(i_1, \ldots, i_p)})|^2 = [\det((a_k^j)_{k,j=1,\ldots,p})]^2 \quad (1)
\]

for all subsets \( \{i_1, \ldots, i_p\} \subset \mathcal{N} \). Note also that (1) implies

\[
\mathbb{P}\{\{i_1, \ldots, i_k\} \subset \phi\} = \| \prod_{j=1}^{k} a_{i_j} \|^2 \quad (2)
\]

for all \( 1 \leq k \leq p \).

Let \( E = E(\mathfrak{A}) \subset \mathbb{R}^N \) be the vector space spanned by \( \mathfrak{A} \). For all sets of linearly independent vectors \( v^i \in E \), \( i = 1, \ldots, p \), we have \( \sum_{i=1}^{p} v^i = a \sum_{i=1}^{p} a^i \) with \( a \neq 0 \) thus, in particular, if \( \mathfrak{A} = \{\tilde{a}_1^1, \ldots, \tilde{a}_p^p\} \) is another orthonormal basis of \( E = E(\mathfrak{A}) \) then

\[
|(\prod_{i=1}^{p} a^i)_{(i_1, \ldots, i_p)}| = |(\prod_{i=1}^{p} \tilde{a}^i)_{(i_1, \ldots, i_p)}| \quad (3)
\]

for every \( \{i_1, \ldots, i_p\} \subset \mathcal{N} \) and consequently \( \phi(\mathfrak{A}) = \phi(\tilde{\mathfrak{A}}) \).

Remark also that if \( \mathfrak{A}^\perp = \{a_{p+1}, \ldots, a_N\} \) is an orthonormal basis of the orthogonal complement \( E(\mathfrak{A})^\perp \) of \( E(\mathfrak{A}) \) in \( \mathbb{R}^N \) then obviously

\[
\phi(\mathfrak{A}^\perp) = \mathcal{N} \setminus \phi(\mathfrak{A}).
\]

An example of a non-trivial basic determinantal process is given by uniform spanning tree measure on a finite connected graph \( G \). Roughly speaking, if \( G \) fixed, is arbitrary edge-oriented and \( M \) is the vertex-edge incidence matrix (the columns been indexed by vertexes) then the determinantal process associated to the vector space spanned by column vectors but one, provides a uniform probability on spanning trees. This result due [8] is called the Transfer Current Theorem. For more details see [20]. Some extensions of this result are given in [3] with a serie of open questions and conjectures.

The repulsion between points of the process \( \phi(\mathfrak{A}) \) is reflected by its negative dependence properties [2], [17], [23], [26], . The simplest one is the negative correlation inequality

\[
P\{A \subset \phi | B \subset \phi\} \leq P\{A \subset \phi\} \quad (3)
\]

where \( A \) and \( B \) are disjoint subsets of \( \mathcal{N} \) such that \( P\{B \subset \phi\} > 0 \). Note that by formula (2) the inequality (3) can be rephrased as

\[
\| \prod_{i \in A \cup B} a_i \|^2 \leq \| \prod_{i \in A} a_i \|^2 \times \| \prod_{i \in B} a_i \|^2 \quad (4)
\]

which is the well-known Hadamard-Whithney inequality.
Remark 1 For a finite set $E \subset \mathbb{R}^p$ of vectors the formula $\bigwedge_{x \in E} x$ designates exterior product of vectors $x \in E$ when these are arbitrarily numbered. For different numberings the associated exterior products may differ (only) by sign which in our context doesn’t matter.

A strengthening of (3) is the (less obvious) association inequality obtained by R.Lyons in [19]

$$P \{ \phi \in \mathcal{A} | \phi \in \mathcal{B} \} \leq P \{ \phi \in \mathcal{A} \} \tag{5}$$

where $\mathcal{A}$ and $\mathcal{B}$ are increasing events which are measurable with complementary subsets and $P(\phi \in \mathcal{B}) > 0$. An event $\mathcal{A} \subset 2^\mathcal{N}$, is called increasing if whenever $\mathcal{A} \in \mathcal{A}$ and $n \in \mathcal{N}$, we have also $\mathcal{A} \cup \{n\} \in \mathcal{A}$.

The next step is the BK inequality. For a pair $\mathcal{A}, \mathcal{B} \subset 2^\mathcal{N}$ of increasing events the disjoint intersection $\mathcal{A} \circ \mathcal{B}$ is defined [5] by

$$\mathcal{A} \circ \mathcal{B} = \{ K \subset \mathcal{N} : \exists \ L \in \mathcal{A}, \ M \in \mathcal{B}, \ L, M \neq \emptyset, \ L \cap M = \emptyset, K \supset L \cup M \} \tag{6}$$

A point process $\psi$ on $\mathcal{N}$ is said to have the van den Berg - Kesten property (in short the BK property) if

$$P \{ \psi \in \mathcal{A} \circ \mathcal{B} | \psi \in \mathcal{B} \} \leq P \{ \psi \in \mathcal{A} \} \tag{7}$$

for every pair of increasing events such that $P \{ \psi \in \mathcal{B} \} > 0$. In [5] (see also [4]) J. van den Berg and H.Kesten proved that inequality (7) is satisfied when $\psi$ is related to a product probability on $2^\mathcal{N}$. In the basic determinantal process setting, the Conjecture 4.6. of [3] which states that the same is true for the spanning trees determinantal point processes is still unsolved. The question of whether general determinantal processes have the BK property was raised in [19]. Note also that for increasing events $\mathcal{A}$ and $\mathcal{B}$ which are measurable with complementary subsets we have $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \cap \mathcal{B}$ and thus (7) implies (5).

In [13] it was conjectured that for all basic determinantal processes the following conditional inequalities hold:

$$P \{ A \not\subset \phi | A_i \not\subset \phi, \forall i = 1, \ldots, n \} \leq P \{ A \not\subset \phi | A_i \not\subset \phi, \forall i = 1, \ldots, n - 1 \} \tag{8}$$

for all disjoint subsets $A, A_i, i = 1, \ldots, n$, of $\mathcal{N}$ such that $P \{ A_i \not\subset \phi, \forall i = 1, \ldots, n \} > 0$. The inequalities (8) are closely related to the BK inequality namely they imply (theorem 2 in [13]) that the BK inequality is satisfied for increasing events $\mathcal{A}$ and $\mathcal{B}$ which are generated by disjoint sets. When all the sets above are reduced to being simple points then the inequalities (8) are satisfied. This follows easily from the fact that in this particular case the conditioned processes $(\phi | x_i \notin \phi \ \ i = 1, \ldots, n)$ are still determinantal (see for example point 4 of proposition 1 in [13]). However, this later doesn’t hold for general sets as the following example shows.

Example 1 Let $\mathfrak{A} = \{ a^1, a^2 \}$ be orthonormal vectors in $\mathbb{R}^4$ and $\phi = \phi(\mathfrak{A})$ the associated determinantal process. We suppose that

1. $0 < P \{ \phi \neq \{1,2\} \} = \kappa < 1$
2. $P \{ \phi = \{1,4\} \} > 0$
3. \( P\{\phi = \{2, 3\}\} > 0 \),
then the process \( \psi = (\phi\{1, 2\} \not\subset \phi) \), \( |\psi| = 2 \) satisfies

4. \( P\{\psi = \{1, i\}\} = P\{\phi = \{1, i\}\}/\kappa \) \( i = 3, 4 \)

5. \( P\{\psi = \{2, i\}\} = P\{\phi = \{2, i\}\}/\kappa \) \( i = 3, 4 \).

It follows from 2-3 that \( P\{i \in \psi\} > 0 \), \( i = 1, 2, 3, 4 \). Now suppose that \( \psi \) is determinantal. Then there exist orthonormal vectors \( \tilde{\mathbf{A}} = (\tilde{\mathbf{a}}^1, \tilde{\mathbf{a}}^2) \) in \( \mathbb{R}^4 \) such that \( \psi = \psi(\tilde{\mathbf{A}}) \).

But \( \|\tilde{\mathbf{a}}_1 \wedge \tilde{\mathbf{a}}_2\|^2 = P\{\psi = \{1, 2\}\} = 0 \) and \( \|\mathbf{a}_i\|^2 = P\{i \in \psi\} > 0 \), \( i = 1, 2 \), therefore there exist \( \alpha \neq 0 \) such that \( \mathbf{a}_1 = \alpha \mathbf{a}_2 \) which implies that

\[
P\{\psi = \{1, i\}\} = \|\tilde{\mathbf{a}}_1 \wedge \mathbf{a}_i\|^2 = \alpha^2 \|\tilde{\mathbf{a}}_2 \wedge \mathbf{a}_i\|^2 = \alpha^2 P\{\psi = \{2, i\}\}, \quad i = 3, 4
\]

and from 4-5

\[
P\{\phi = \{1, i\}\} = \alpha^2 P\{\phi = \{2, i\}\}, \quad i = 3, 4. \tag{9}
\]

In general this is not possible. It suffices to take \( \alpha^1 = (1/\sqrt{2}, 0, 1/\sqrt{2}, 0)^t \) and \( \alpha^2 = (0, 1/\sqrt{2}, 0, 1/\sqrt{2})^t \) to see that the associated process \( \phi_0 \) does not fulfill \( \psi \).

**Remark 2** A generalization of determinantal point processes is given by strongly Rayleigh point processes which are introduced in the work of [6]. A point process \( \psi \) is said strongly Rayleigh if its generating polynomial

\[
g_\psi(z_1, \ldots, z_N) = \sum_{S \subseteq \mathbb{N}} P\{\psi = S\} \prod_{i \in S} z_i, \quad z_i \in \mathbb{C}, \quad i = 1, \ldots, N,
\]

is stable, that is \( g_\psi(z_1, \ldots, z_N) \neq 0 \) whenever \( \text{Im}(z_i) > 0 \) for all \( 1 \leq i \leq N \).

It was proved in [6] that strongly Rayleigh processes has negative association and that determinantal processes are strongly Rayleigh. As regards to example [7] note that the generating polynomial \( g_\psi(z_1, \ldots, z_4) = \frac{1}{4}(z_1z_4 + z_2z_3 + z_3z_4) \) of the the conditioned process \( \psi_0 = (\phi_0\{1, 2\} \not\subset \phi_0) \) is not stable and consequently \( \psi_0 \) is not strongly Rayleigh.

The purpose of this work is to shown that the inequalities \( \mathbf{8} \) are satisfied for \( n = 2 \) and \( n = 3 \). To this aim we reformulate \( \mathbf{8} \) (which is an elementary exercice) in the equivalent form:

\[
P\{A \subseteq \phi, A_i \not\subset \phi, \forall i = 1, \ldots, n\} \times P\{B \subset \phi | A_i \not\subset \phi, \forall i = 1, \ldots, n\} - P\{A \subseteq \phi, B \subset \phi | A_i \not\subset \phi, \forall i = 1, \ldots, n\} \geq 0. \tag{10}
\]

Next, we will shown in Section 2 that in order to obtain \( \mathbf{10} \) it suffices to consider the case when \( A \) and \( B \) are simple points. In this case we are able to compute for \( n = 1 \) and \( n = 2 \) (which corresponds to \( n = 2 \) and \( n = 3 \) for \( \mathbf{8} \)) the exact value of the difference appearing in \( \mathbf{10} \) in terms of some geometric characteristics of the vector space \( E(\mathbf{A}) \). The proof relies upon the classical canonical representation of a pair of subspaces in terms of principal vectors and angles [1], [13], [16] and the link established in [13] between the classical CS decomposition (CSD) [25] of partitioned unitary matrix and the basic determinantal processes which allows to obtain a pertinent bases of the
vector spaces $E = E(\mathfrak{A}) \subset \mathbb{R}^N$ and $E^\perp = (\mathfrak{A})^\perp$. To clarify the latter: if we choose $N - p - n + 2 > 0$, $p = |\phi(\mathfrak{A})|$, and reorder the coordinates of the vectors in $\mathbb{R}^N$ with the order $(2, \ldots, n - 1, n, n + 1, \ldots, N)$ then the first $p$ columns (resp. the last $N-p$ columns) of the matrix below is an orthonormal basis of $E$ (resp. $E^\perp$).

\[
\begin{pmatrix}
cos \theta_1 u_1 \cdots cos \theta_{n-2} u_1^{n-2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
n-1 & \cos \theta_1 u_{n-2} \cdots \cos \theta_{n-2} u_{n-2}^{n-2} & 0 & \cdots & 0 \\
1 & \sin \theta_1 V_1 \cdots \sin \theta_{n-2} V_1^{n-2} & W_1^1 \cdots W_1^{p-n+2} \\
n & \sin \theta_1 V_2 \cdots \sin \theta_{n-2} V_2^{n-2} & W_2^1 \cdots W_2^{p-n+2} \\
\vdots & \vdots & \ddots & \vdots \\
N & \sin \theta_1 V_N \cdots \sin \theta_{n-2} V_N^{n-2} & W_N^1 \cdots W_N^{p-n+2} \\
\end{pmatrix}
\]

Here, $u^1, \ldots, u^n$ are orthonormal vectors in $\mathbb{R}^{n-2}$; $\mathfrak{U} = \{V^1, \ldots, V^n\}$, $\mathfrak{W} = \{W^1, \ldots, W^{p-n}\}$, $\mathfrak{U} = \{W^1, \ldots, W^{N-p-n}\}$, are mutually orthonormal vectors in $\mathbb{R}^{N-n+2}$; the angles appearing in the matrix above are principal Jordan angles between the space $E$ and the basic subspace $\mathbb{R}^N_{\{2, \ldots, n-2\}} = \{x = (x_k) \in \mathbb{R}^N; x_k = 0 \text{ if } k \notin \{2, \ldots, n-2\}\}$.

for more details see Section 2 of [13].

By making use of these particular bases it is shown (Proposition 2 in [13] that

\[
P\{1 \in \psi\} \times P\{n \in \psi\} - P\{1 \in \psi, n \in \psi\}
= \lambda \times \left( (\langle V_1, W_1 \rangle, \langle V_2, W_2 \rangle) \right)^2
= \lambda \times \left( \sum_{k=1}^{n-1} (-1)^k \sum_{2 \leq i_1 < \cdots < i_k \leq n-1} \langle a_1 \wedge \bigwedge_{j=1}^k a_{i_j} \wedge \bigwedge_{j=1}^k a_{i_j} \rangle \right)^2
\]

(12)

where $\psi = (\phi | i \notin \phi, i = 2, \ldots, n - 1)$, and $P\{i \notin \phi, i = 2, \ldots, n - 1\} = \prod_{i=1}^{n-2} \sin^2 \theta_i = \lambda^{-\frac{2}{3}} > 0$.

**Remark 3** Note that the values of the right and left sides of (12) do not depend on the choice of the basis $\mathfrak{A} = \{a^1, \ldots, a^p\}$ and that by reordering the indices the procedure described above works for every subset $\{i_2, \ldots, i_{n-1}\} \subset N$. 

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1. If $|A| = n - 2 = p$ then
   \[ P\{i \in \phi | A \not\subset \phi\} \times P\{j \in \phi | A \not\subset \phi\} - P\{i \in \phi, j \in \phi | A \not\subset \phi\} = (1 - \kappa)^{-2} \times (\langle a_i, a_j \rangle^2 + \kappa \times (a_i \wedge a_j)^2). \]  

(16)

2. If $|A| = n - 2 < p$ then
   \[ P\{i \in \phi | A \not\subset \phi\} \times P\{j \in \phi | A \not\subset \phi\} - P\{i \in \phi, j \in \phi | A \not\subset \phi\} = (1 - \kappa)^{-2} \times \left( (\tilde{v}_1, \tilde{v}_2) + (1 - \kappa)(W_1, W_2) \right)^2 + \kappa \| \tilde{v}_1 \wedge \tilde{v}_2 \|^2 \]  

(17)

where $\kappa = P\{A \subset \phi\} = (\prod_{i=1}^{p} \cos \theta_i)^2$. Moreover we have
   \[ (\tilde{v}_1, \tilde{v}_2) + (1 - \kappa)(W_1, W_2) = \langle a_i, a_j \rangle - \langle a_i \wedge (\bigwedge_{k=1}^{n-2} a_{i_k}), a_j \wedge (\bigwedge_{k=1}^{n-2} a_{i_k}) \rangle. \]  

(18)

Note that the right sides of (16) and (17) do not depend of the choice of the basis.

As a consequence, the inequality
   \[ P\{A \not\subset \phi | A_i \not\subset \phi, i = 1, 2\} \leq P\{A \not\subset \phi | A_1 \not\subset \phi\} \]  

(19)

holds for all disjoint subsets $A, A_1, A_2$ of $\mathcal{N}$ such that $P\{A_1 \not\subset \phi, A_2 \not\subset \phi\} > 0$.

To get a similar result for two disjoint sets $A_1, A_2$ of $\mathcal{N}$, that is to prove that for $i_0, j_0 \not\in A_1 \cup A_2$ we have
   \[ P\{i_0 \in \phi | A_1 \not\subset \phi, A_2 \not\subset \phi\} \times P\{j_0 \in \phi | A_1 \not\subset \phi, A_2 \not\subset \phi\} - P\{i_0 \in \phi, j_0 \in \phi | A_1 \not\subset \phi, A_2 \not\subset \phi\} \geq 0 \]  

(20)

is a more difficult task. Our proof rely upon a geometric/algebraic result of independent interest. To explain this, consider the subsets $\{a_i, i \in A_1\}$ and $\{a_i, i \in A_2\}$ of $\mathbb{R}^p$ where $\mathcal{A} = \{a^1, \ldots, a^p\}, 1 < p < \mathcal{N}, 1 < p < \mathcal{N}$, is a set of orthonormal vectors associated to the determinantal proces $\phi$. Without loss of
generality we may suppose that: $|A_l| < p$, $0 < P\{A_l \subset \phi\} = \| \bigwedge_{i \in A_l} a_i \|^2 < 1$, $l = 1, 2$ and obviously

$$0 < P\{A_1 \not\subset \phi, A_2 \not\subset \phi\} = 1 - \| \bigwedge_{i \in A_1} a_i \|^2 - \| \bigwedge_{i \in A_2} a_i \|^2 + \| \bigwedge_{i \in A_1 \cup A_2} a_i \|^2 < 1.$$  

With these properties in mind, we shall describe the algebraic result of interest which requires a thorough explanations.

Let $E_l, |E_l| = k_l < p, l = 1, 2$ be two subsets of $\mathbb{R}^p$ who fullfil the conditions above, that is:

$$0 < \kappa_l = \| \bigwedge_{x \in E_l} x \|^2 < 1, l = 1, 2$$  

and

$$0 < 1 - \| \bigwedge_{x \in E_1} x \|^2 - \| \bigwedge_{x \in E_2} x \|^2 + \| \bigwedge_{x \in E_1 \cup E_2} x \|^2 < 1.$$  

Conditions (21) implie that the subspaces $E_l$ spanned by the subsets $E_l, l = 1, 2$ are respectively of dimension $k_l$. Suppose $k_1 \leq k_2$. There are two possible alternatives:

Case I:

$$\| \bigwedge_{x \in E_1} x \|^2 = 0.$$  

In this case we have $E_1 \cap E_2 \neq \{0\}$ and it is well-known [1], [14], [22] that there exist a set of orthonormal vectors $u_{11}, \ldots, u_{1k_1}, u_{21}, \ldots, u_{2k_2}$ $u_{31}, \ldots, u_{3k_3}, u_{41}, \ldots, u_{4k_4}, u_{51}, \ldots, u_{5k_5}$

( which is a basis of $\mathbb{R}^p$) with

$$k + k_2 = k_1, \quad k + k_3 + k_4 = k_2, \quad 2k + k_3 + k_4 + k_5 = p,$$

as well as Jordan principal angles $0 < \alpha_i < \pi/2, i = 1, \ldots, k$

so that:

(i) the vectors $u_{11}, \ldots, u_{1k_1}, u_{31}, \ldots, u_{3k_3}$ are a basis of $E_1$

(ii) Putting $v_i = u_{1i} \cos \alpha_i + u_{2i} \sin \alpha_i, i = 1, \ldots, k$, the vectors

$$v_1, \ldots, v_k, u_{31}, \ldots, u_{3k_3}, u_{41}, \ldots, u_{4k_4}$$

are a basis of $E_2$.

This is the most general situation subject to the condition (24). If $k_1 = k_2$ then (obviously) the sequence $u_{4i}, i = 1, \ldots, k_4$ above must be taken out and if $2k + k_3 + k_4 = p$ then the same applies to the sequences $u_{5i}, i = 1, \ldots, k_5$.

Case II:

$$\| \bigwedge_{x \in E_1 \cup E_2} x \|^2 \neq 0.$$  

In this case we have $E_1 \cap E_2 = \{0\}$ and consequently the description above works provided the sequence $u_{3i}, i = 1, \ldots, k_3$ is taken out.

Consider now two vectors $y, y' \in \mathbb{R}^p, y \neq y'$. There exists $a = (a_1, \ldots, a_k)$,
\[ b = (b_1, \ldots, b_k), \quad c = (c_1, \ldots, c_k), \quad d = (d_1, \ldots, d_k) \text{ and } e = (e_1, \ldots, e_k) \text{ such that writing} \]

\[
y_0 = \sum_{i=1}^{k} (a_i + b_i \cos \alpha_i) u_{1i} + \sum_{i=1}^{k} b_i (\sin \alpha_i) u_{2i}
\]

\[ y_0' = \sum_{i=1}^{k} (a_i' + b_i' \cos \alpha_i) u_{1i} + \sum_{i=1}^{k} b_i' (\sin \alpha_i) u_{2i} \quad (25)\]

we have

Case I.

\[
y = y_0 + \sum_{i=1}^{k_3} c_i u_{3i} + \sum_{i=1}^{k_4} d_i u_{4i} + \sum_{i=1}^{k_5} e_i u_{5i},
\]

\[ y' = y_0' + \sum_{i=1}^{k_3} c_i' u_{3i} + \sum_{i=1}^{k_4} d_i' u_{4i} + \sum_{i=1}^{k_5} e_i' u_{5i} \quad (26)\]

or

\[ y = (y_0, c, d, e), \quad y' = (y_0', c', d', e') \quad (27)\]

in coordinate representation for the ordered basis

\[ u_{11}, \ldots, u_{1k}, u_{21}, \ldots, u_{2k}, u_{31}, \ldots, u_{3k_3}, u_{41}, \ldots, u_{4k_4}, u_{51}, \ldots, u_{5k_5}. \]

Case II.

\[
y = y_0 + \sum_{i=1}^{k_4} d_i u_{4i} + \sum_{i=1}^{k_5} e_i u_{5i},
\]

\[ y' = y_0' + \sum_{i=1}^{k_4} d_i' u_{4i} + \sum_{i=1}^{k_5} e_i' u_{5i} \quad (28)\]

or

\[ y = (y_0, d, e), \quad y' = (y_0', d', e') \quad (29)\]

in coordinate representation for the ordered basis

\[ u_{11}, \ldots, u_{1k}, u_{21}, \ldots, u_{2k}, u_{41}, \ldots, u_{4k_4}, u_{51}, \ldots, u_{5k_5}. \]

Consider now

Case I.

\[
\langle y, y' \rangle - \kappa_1 \langle y \wedge \bigwedge_{i=1}^{k} u_{1i} \bigwedge_{i=1}^{k_3} u_{3i}, y' \wedge \bigwedge_{i=1}^{k} u_{1i} \bigwedge_{i=1}^{k_3} u_{3i} \rangle \\
- \kappa_2 \langle y \wedge \bigwedge_{i=1}^{k_3} v_i \bigwedge_{i=1}^{k_4} u_{4i}, y' \wedge \bigwedge_{i=1}^{k_3} v_i \bigwedge_{i=1}^{k_4} u_{4i} \rangle. \quad (30)\]

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Case II.

\[
\langle y, y' \rangle - \kappa_1 \langle y \land \bigwedge_{i=1}^{k} u_{i1}, y' \land \bigwedge_{i=1}^{k} u_{i1} \rangle \\
- \kappa_2 \langle y \land \bigwedge_{i=1}^{k} v_{i}, y' \land \bigwedge_{i=1}^{k} v_{i} \rangle \\
+ \kappa_1 \kappa_2 \prod_{i=1}^{k} (\sin \alpha_i)^2 \langle y \land \bigwedge_{i=1}^{k} u_{i1}, y' \land \bigwedge_{i=1}^{k} u_{i1} \rangle.
\]

(31)

The key point is the fact - as previously seen in the special cases (12) and (18) - that the expressions (30) and (31) are scalar products of two vectors \(\tilde{y}, \tilde{y}' \in \mathbb{R}^p\) : lemmas 3, 8, 10 in section 4. Furthermore we will prove in Section 4 that:

**Theorem 2** Define

Case I.

\[
\Lambda_{g1}(y, y') = \| \tilde{y} \land \tilde{y}' \|^2 \\
- (1 - \kappa_1 - \kappa_2) \times \left[ \| y \land y' \|^2 - \kappa_1 \| y \land y' \land \bigwedge_{i=1}^{k} u_{i1} \|^2 \right] \\
- \kappa_2 \| y \land y' \land \bigwedge_{i=1}^{k} v_{i} \|^2.
\]

(32)

Case II.

\[
\Lambda_{g2}(y, y') = \| \tilde{y} \land \tilde{y}' \|^2 \\
- (1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} (\sin \alpha_i)^2) \times \left[ \| y \land y' \|^2 - \kappa_1 \| y \land y' \land \bigwedge_{i=1}^{k} u_{i1} \|^2 \right] \\
- \kappa_2 \| y \land y' \land \bigwedge_{i=1}^{k} v_{i} \|^2 \\
+ \kappa_1 \kappa_2 \prod_{i=1}^{k} (\sin \alpha_i)^2 \| y \land y' \land \bigwedge_{i=1}^{k} u_{i1} \|^2 \right].
\]

(33)

then

\(\Lambda_{g1}(y, y') \geq 0\) and \(\Lambda_{g2}(y, y') \geq 0\). (34)

This result provides a proof of the inequality (20) from which, by Proposition 1 the conditional inequality (8) for \(n = 3\) follows. Indeed, with the choice \(y = a_{i_0}, y' = a_{j_0}, E_1 = \{a_i, i \in A_1\}, E_2 = \{a_i, i \in A_2\}\) we obtain:

(i) \(\| \tilde{y} \|^2 = P\{i_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi\}, \| \tilde{y}' \|^2 = P\{j_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi\}\) (35)
\[ \| \tilde{y} \wedge \tilde{y}' \|^2 = P\{i_0 \in \phi, j_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi \} \]  

(iii) Case I:

\[ P\{A_1 \not\subset \phi, A_2 \not\subset \phi \} = 1 - \kappa_1 - \kappa_2 \]  

(iii') Case II:

\[ P\{A_1 \not\subset \phi, A_2 \not\subset \phi \} = 1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin^2 \alpha_i \]  

Thus, in virtue of (34), substituting these probabilistic expressions into (32) and (33) and applying the well-known formula

\[ \| \tilde{y} \wedge \tilde{y}' \|^2 = \| \tilde{y} \|^2 \times \| \tilde{y}' \|^2 - \langle \tilde{y}, \tilde{y}' \rangle^2 \]  

we obtain:

**Theorem 3** For disjoint subsets \( \{i_0\}, \{j_0\}, A_1 \) and \( A_2 \) of \( \mathcal{N} \) such that \( P\{A_i \not\subset \phi, i = 1, 2 \} > 0 \) we have:

\[ P\{i_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi \} \times P\{j_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi \} - P\{i_0 \in \phi, j_0 \in \phi, A_1 \not\subset \phi, A_2 \not\subset \phi \} = P\{A_1 \not\subset \phi, A_2 \not\subset \phi \}^2 \times (\langle \tilde{y}, \tilde{y}' \rangle)^2 + \Lambda_{i_0}(y, y') \geq 0, i = 1, 2 \]

where \( P\{A_1 \not\subset \phi, A_2 \not\subset \phi \} \) is given by (37) and (38). As a consequence the inequality

\[ P\{A \not\subset \phi \mid A_i \not\subset \phi, \forall i = 1, 2, 3 \} \leq P\{A \not\subset \phi \mid A_i \not\subset \phi, \forall i = 1, 2 \} \]  

holds for all disjoint subsets \( A, A_i, i = 1, 2, 3 \), of \( \mathcal{N} \) such that \( P\{A_i \not\subset \phi, \forall i = 1, 2, 3 \} > 0 \).
2 The reduction principle

Notation 1 For disjoints sets $A_i$, $i = 1 \ldots n$, such that $P\{A_i \not\subset \phi, i = 1 \ldots n\} > 0$ we will note

\[(\phi|A_i \not\subset \phi, i = 1 \ldots n) = \psi. \tag{42}\]

Proposition 1 (reduction principle) Suppose that the inequality

\[P\{x \in \psi, x' \in \psi\} \leq P\{x \in \psi\} \times P\{x' \in \psi\} \tag{43}\]

is satisfied for every choice of disjoint sets \(\{x\}, \{x'\}, A_i, i = 1 \ldots n\) and every basic discrete determinantal process \(\phi\), associated to a subspace of \(\mathbb{R}^N\), such that $P\{A_i \not\subset \phi, i = 1 \ldots n\} > 0$. Then we have:

\[P\{B_1 \subset \psi, B_2 \subset \psi\} \leq P\{B_1 \subset \psi\} \times P\{B_2 \subset \psi\} \tag{44}\]

for every choice of disjoint sets $B_1, B_2, A_i, i = 1 \ldots n$ and every discrete determinantal process \(\phi\) such that $P\{A_i \not\subset \phi, i = 1 \ldots n\} > 0$.

Proof.– By induction and the following lemma.

Lemma 2 Fix $1 \leq k$ and disjoint sets $B_2, A_i, i = 1, \ldots, n$. Suppose (the induction hypothesis) that the inequality \[44\] holds for each choice of the set $B_1$ disjoint from $B_2 \cup (\bigcup_{i=1}^{n} A_i)$, and satisfying $|B_1| \leq k$. Then, \[44\] holds equally under the condition $|B_1| \leq k + 1$.

Proof.– Suppose that $|B_1| = k$ and let $y \notin B_1 \cup B_2 \cup \bigcup_{i=1}^{n} A_i$. We want to show that

\[P\{B_1 \cup \{y\} \subset \psi, B_2 \subset \psi\} \leq P\{B_1 \cup \{y\} \subset \psi\} \times P\{B_2 \subset \psi\}. \tag{45}\]

We may suppose that $P\{B_1 \cup \{y\} \subset \phi, B_2 \subset \phi, A_i \not\subset \phi, i = 1 \ldots n\} > 0$ (otherwise there is nothing to prove) which implies that $P\{y \in \phi\} > 0$ and $P\{y \in \phi, A_i \not\subset \phi, i = 1 \ldots n\} > 0$. Furthermore, it is well known that

\[\phi_y = (\phi|y \in \phi)\backslash\{y\} \tag{46}\]

is a basic determinantal process (see, for example, property 3 of the proposition 1 in \[43\])

Consequently we have

\[P\{A_i \not\subset \phi_y, i = 1 \ldots n\} > 0.\]

Induction hypothesis implies that

\[P\{B_1 \subset \phi_y, B_2 \subset \phi_y|A_i \not\subset \phi_y, i = 1 \ldots n\} \leq P\{B_1 \subset \phi_y|A_i \not\subset \phi_y, i = 1 \ldots n\} \times P\{B_2 \subset \phi_y|A_i \not\subset \phi_y, i = 1 \ldots n\} \tag{47}\]

which by \[46\] can be reformulated as follows:

\[P\{y \in \phi, B_1 \subset \phi, B_2 \subset \phi, A_i \not\subset \phi, i = 1 \ldots n\} \leq P\{y \in \phi, B_1 \subset \phi, B_2 \subset \phi, A_i \not\subset \phi, i = 1 \ldots n\} \times P\{y \in \phi, B_2 \subset \phi, A_i \not\subset \phi, i = 1 \ldots n\} \tag{48}\]
\[ \iff \]
\[ P\{y \in \psi, B_1 \subset \psi, B_2 \subset \psi\} \leq P\{y \in \psi, B_1 \subset \psi\} \times \frac{P\{y \in \psi, B_2 \subset \psi\}}{P\{y \in \psi\}}. \tag{49} \]

From the induction hypothesis we get also
\[ P\{y \in \psi, B_2 \subset \psi\} \leq P\{y \in \psi\} \times P\{B_2 \subset \psi\}. \tag{50} \]

This and formulas (49) provides (15) as desired.

3 Proof of theorem [1]

1 For \(|A| = p\) such that \(0 < \kappa = P\{A \subset \phi\} < 1\) we have trivially
\[ P\{(A \setminus A) \cap \phi \neq \emptyset \text{ and } A \subset \phi\} = 0. \]

Consequently,
\[
P\{i \in \phi|A \not\subset \phi\} \times P\{j \in \phi|A \not\subset \phi\} - P\{i \in \phi, j \in \phi|A \not\subset \phi\}
= P\{i \in \phi\} \times P\{j \in \phi\} - P\{A \not\subset \phi\} P\{i \in \phi, j \in \phi\}
= \| a_i \|^2 \times \| a_j \|^2 - P\{A \not\subset \phi\} \times \| a_i \wedge a_j \|^2
= \frac{(1 - \kappa)^2}{(1 - \kappa)^2} \]
where the last equality follows from the formula
\[ \| a_i \wedge a_j \|^2 = \| a_i \|^2 \times \| a_j \|^2 - \langle a_i, a_j \rangle^2. \]

2 For \(A = \{i_1, \ldots, i_{n-2}\} \subset N, |A| = n - 2 < p, 0 < \kappa = P\{A \subset \phi\} < 1,\) choose a basis as described in section [4] after the remark [3]. Then, with corresponding notations, namely (13), (14) and (15) we obtain
\[
P\{i \in \phi, A \not\subset \phi\} = P\{i \in \phi\} - P\{i \in \phi, A \subset \phi\}
= \| a_i \|^2 - \| a_i \wedge \bigwedge_{k=1}^{n-2} a_{i_k} \|^2 \tag{52}
= \| \tilde{V}_1 \|^2 + \| W_1 \|^2 - \| \bigwedge_{k=1}^{n-2} \cos \theta_k u^k \|^2 \times \| W_1 \|^2
= \| \tilde{V}_1 \|^2 + \| W_1 \|^2 \times (1 - \kappa)
\]

and similarly
\[ P\{j \in \phi, A \not\subset \phi\} = \| \tilde{V}_2 \|^2 + \| W_2 \|^2 \times (1 - \kappa). \tag{53} \]

Moreover, we have
\[
P\{i \in \phi, j \in \phi, A \not\subset \phi\} = P\{i \in \phi, j \in \phi\} - P\{i \in \phi, j \in \phi, A \subset \phi\}
= \| a_i \wedge a_j \|^2 - \| a_i \wedge a_j \wedge \bigwedge_{k=1}^{n-2} a_{i_k} \|^2 \tag{54}
= \| a_i \wedge a_j \|^2 - \kappa \| W_1 \wedge W_2 \|^2 \times \| a_i \|^2 \times \| a_j \|^2 - \langle a_i, a_j \rangle^2
- \kappa (\| W_1 \|^2 \times \| W_2 \|^2 - (W_1, W_2)^2). \]
Now, recall that \( \| a_i \|^2 = \| \tilde{v}_i \|^2 + \| W_1 \|^2 \), \( \| a_j \|^2 = \| \tilde{v}_j \|^2 + \| W_2 \|^2 \), and \( \langle a_i, a_j \rangle = \langle \tilde{v}_i, \tilde{v}_j \rangle + \langle W_1, W_2 \rangle \). From this and \((52)-(54)\) a simple calculation gives formula \((57)\). Besides, the obvious equality \( \langle a_i \wedge (\bigwedge_{k=1}^{n-2} a_{i_k}), a_j \wedge (\bigwedge_{k=1}^{n-2} a_{i_k}) \rangle = \kappa \langle W_1, W_2 \rangle \) implies \((58)\).

### 4 Proof of Theorem 2

#### 4.1 The case \( k = |E_1| = |E_2| \) and \( p = 2k \).

Recall that \( E_1, |E_1| = k < p, l = 1, 2 \) are two subsets of \( \mathbb{R}^p \) who fullfil the conditions \((21)\) and \((22)\). In addition we suppose that \( k = |E_1| = |E_2| \) and \( p = 2k \). This and \((21)\) implie that the subspaces \( E_i \) respectively spanned by the subsets \( E_i, i = 1, 2 \) are of dimension \( k \) and that there exist a set of orthonormal vectors \( u_1, \ldots, u_k, w_1, \ldots, w_k \) in \( \mathbb{R}^p \) as well as Jordan principal angles \( 0 < \alpha_i < \pi/2, \ i = 1, \ldots, k \) so that the vectors \( u_1, \ldots, u_k \) are a basis of \( E_1 \); the vectors \( v_i = u_i \cos \alpha_i + w_i \sin \alpha_i, \ i = 1, \ldots, k \) are a basis of \( E_2 \); hence we can simplify the notations we write \( u_i, \ i = 1, \ldots, k \) (resp. \( w_i, \ i = 1, \ldots, k \)) instead \( u_{1i}, \ i = 1, \ldots, k \) (resp. \( w_{2i}, \ i = 1, \ldots, k \)). Hence, we have:

\[
\bigwedge_{x \in E_1} x = \pm \sqrt{\kappa_1} \bigwedge_{i=1}^{k} u_i, \quad \bigwedge_{x \in E_2} x = \pm \sqrt{\kappa_2} \bigwedge_{i=1}^{k} v_i, \tag{55}
\]

and

\[
1 - \| \bigwedge_{x \in E_1} x \|^2 - \| \bigwedge_{x \in E_2} x \|^2 - \| \bigwedge_{i \in E_1 \cup E_2} x \|^2 = 1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin^2 \alpha_i. \tag{56}
\]

Introduce the notations

\[
\delta_{1,i} = 1 - \kappa_1 \sin^2 \alpha_i \quad \text{and} \quad \delta_{2,i} = 1 - \kappa_2 \sin^2 \alpha_i, \ i = 1, \ldots, k, \tag{57}
\]

and observe that by \((21)\) and \((22)\) we have, for all \( i = 1, \ldots, k, 0 < \delta_{1,i} < 1, 0 < \delta_{2,i} < 1 \) and \( 0 < 1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \sin^2 \alpha_i \). Consequently, we can define the angles \( 0 < \alpha_i' < \pi/2, \ i = 1, \ldots, k \), via the formulas

\[
\cos \alpha_i' = \frac{\cos \alpha_i}{\sqrt{\delta_{1,i} \delta_{2,i}}}, \tag{58}
\]

and

\[
\sin \alpha_i' = \frac{\sin \alpha_i \sqrt{1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \sin^2 \alpha_i}}{\sqrt{\delta_{1,i} \delta_{2,i}}}. \tag{59}
\]
Now, fix two vectors \( y, y' \) in \( \mathbb{R}^p \). They are of the form

\[
y = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{k} b_i v_i = \sum_{i=1}^{k} (a_i + b_i \cos \alpha_i) u_i + \sum_{i=1}^{k} b_i \sin \alpha_i w_i \tag{60}
\]

\[
y' = \sum_{i=1}^{k} a'_i u_i + \sum_{i=1}^{k} b'_i v_i = \sum_{i=1}^{k} (a'_i + b'_i \cos \alpha_i) u_i + \sum_{i=1}^{k} b'_i \sin \alpha_i w_i \tag{61}
\]

or, in coordinate representation (for the ordered basis \( u_1, \ldots, u_k, w_1, \ldots, w_k \))

\[
y = (a_1, \ldots, a_k, b_1, \ldots, b_k)H, \quad y' = (a'_1, \ldots, a'_k, b'_1, \ldots, b'_k)H \tag{62}
\]

with the matrix

\[
H = \begin{pmatrix}
I_k & 0 \\
\Delta & \Delta_{22}
\end{pmatrix}
\]

where \( I_k \) is the unit matrix of size \( k \) and

\[
\Delta = \begin{pmatrix}
\cos \alpha_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \cos \alpha_k
\end{pmatrix}, \quad \Delta_{22} = \begin{pmatrix}
\sin \alpha_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sin \alpha_k
\end{pmatrix}
\]

are diagonal matrices.

**Lemma 3** Introduce

1. The matrix

\[
H_1 = \begin{pmatrix}
I_k & 0 \\
\Delta' & \Delta'_{22}
\end{pmatrix}
\]

where \( I_k \) is the unit matrix of size \( k \) and

\[
\Delta' = \begin{pmatrix}
\cos \alpha'_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \cos \alpha'_k
\end{pmatrix}, \quad \Delta'_{22} = \begin{pmatrix}
\sin \alpha'_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sin \alpha'_k
\end{pmatrix}
\]

are diagonal matrices.

2. The diagonal matrix

\[
D = \begin{pmatrix}
\sqrt{\delta_{2,1}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sqrt{\delta_{1,k}}
\end{pmatrix}
\]

Then, the vectors

\[
\bar{y} = (a_1, \ldots, a_k, b_1, \ldots, b_k)D H_1
\]

14
\( \tilde{g}' = (a'_1, \ldots, a'_k, b'_1, \ldots, b'_k)DH_1 \) \hspace{1cm} (65)

or, in terms of the basis \( u_1, \ldots, u_k, w_1, \ldots, w_k \)

\[
\tilde{g} = \sum_{i=1}^{k} (a_i \sqrt{\delta_{2i}} + b_i \sqrt{\delta_{i1}} \cos \alpha_i)u_i + \sum_{i=1}^{k} b_i \sqrt{\delta_{i1}} \sin \alpha_i w_i
\]

\[
\tilde{g}' = \sum_{i=1}^{k} (a'_i \sqrt{\delta_{2i}} + b'_i \sqrt{\delta_{i1}} \cos \alpha'_i)u_i + \sum_{i=1}^{k} b'_i \sqrt{\delta_{i1}} \sin \alpha'_i w_i
\] \hspace{1cm} (66)

satisfy the equality

\[
\langle \tilde{g}, \tilde{g}' \rangle = \langle y, y' \rangle - \langle y \wedge \bigwedge_{x \in E_1} x, y' \wedge \bigwedge_{x \in E_1} x \rangle - \langle y \wedge \bigwedge_{x \in E_2} x, y' \wedge \bigwedge_{x \in E_2} x \rangle.
\]

\hspace{1cm} (67)

\begin{proof}

According to (65), (69), (61) and noting that the multivectors \( w_i \wedge \bigwedge_{i=1}^{k} u_i, i = 1, \ldots, k \), are orthonormal we get

\[
\langle y \wedge \bigwedge_{x \in E_1} x, y' \wedge \bigwedge_{x \in E_1} x \rangle = \kappa_1 \langle y \wedge \bigwedge_{i=1}^{k} u_i, y' \wedge \bigwedge_{i=1}^{k} u_i \rangle
\]

\[
= \kappa_1 \left( \sum_{i=1}^{k} b_i \sin \alpha_i u_i \right) \wedge \bigwedge_{i=1}^{k} u_i \wedge \left( \sum_{i=1}^{k} b_i \sin \alpha_i u_i \right) \wedge \bigwedge_{i=1}^{k} u_i
\]

\[
= \kappa_1 \sum_{i=1}^{k} b_i b'_i \sin^2 \alpha_i.
\] \hspace{1cm} (68)

By the same argument we obtain

\[
\langle y \wedge \bigwedge_{x \in E_2} x, y' \wedge \bigwedge_{x \in E_2} x \rangle = \kappa_2 \sum_{i=1}^{k} a_i a'_i \sin^2 \alpha_i.
\] \hspace{1cm} (69)

From (69), (61), (68) and (69) we get

\[
\langle y, y' \rangle - \langle y \wedge \bigwedge_{x \in E_1} x, y' \wedge \bigwedge_{x \in E_1} x \rangle - \langle y \wedge \bigwedge_{x \in E_2} x, y' \wedge \bigwedge_{x \in E_2} x \rangle
\]

\[
= \sum_{i=1}^{k} \left[ a_i a'_i (1 - \kappa_2 \sin^2 \alpha_i) + b_i b'_i (1 - \kappa_1 \sin^2 \alpha_i) + (a_i b'_i + b_i a'_i) \cos \alpha_i \right]
\] \hspace{1cm} (70)

which is exactly what a direct computation derived from the formulas (66) gives.

\( \square \)

Our goal now is to compute

\[
\Lambda(y, y') = \| \tilde{g} \wedge \tilde{g}' \|^2 - (1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin^2 \alpha_i)
\]

\[
\times \left[ \| y \wedge y' \|^2 - \| y \wedge y' \wedge \bigwedge_{x \in E_1} x \|^2 - \| y \wedge y' \wedge \bigwedge_{x \in E_2} x \|^2 \right]
\] \hspace{1cm} (71)
and prove that $\Lambda(y, y') \geq 0$. This requires several stages. First, consider the exterior product

$$(a_1, \ldots, a_k, b_1, \ldots, b_k) \wedge (a'_1, \ldots, a'_k, b'_1, \ldots, b'_k) = \tilde{t} = (t(1, 2), \ldots, t(p - 1, p))$$

(72)

the components of $\tilde{t}$ being indexed in the lexicographic order. It is (certainly) well-known that we have

$$y \wedge y' = ((a_1, \ldots, a_k, b_1, \ldots, b_k)H) \wedge ((a'_1, \ldots, a'_k, b'_1, \ldots, b'_k)H) = \tilde{t} \tilde{H}$$

(73)

where $\tilde{H}$ is the compound matrix of order 2. Properties of compound matrices (see [9] p.19) imply that

$$\| y \wedge y' \|^2 = \| \tilde{t} \tilde{H} \|^2 = \langle \tilde{t}, \tilde{t} \tilde{HH}^t \rangle.$$ (74)

Straightforwardly

$$HH^t = \left( \begin{array}{c|c} I_k & \Delta \\ \hline \Delta & I_k \end{array} \right)$$

(75)

and thus the compound matrix $\tilde{HH}^t$ has a simple structure. Precisely; for $1 \leq i < j \leq k$ denote

$$\tilde{t}(i, j) = (t(i, j), t(i, k + j), t(j, k + i), t(k + i, k + j))$$

(76)

and accordingly consider the following order

$$(1, 2)(1, 2 + k)(2, 1 + k)(1 + k, 2 + k) \ldots (i, j)(i, j + k)(j, i + k)(i + k, j + k) \ldots (k - 1, k)(k - 1, 2k)(k, 2k - 1)(2k - 1, 2k)(1, 1 + k)(2, 2 + k) \ldots (k, 2k).$$

(77)

obtained by concatenating the $\tilde{t}(i, j) \ 1 \leq i < j \leq k$ in the lexicographic order followed by the sequence $(1, 1 + k)(2, 2 + k) \ldots (k, 2k)$. Then we have:

**Lemma 4.** By reordering the row and columns of $\tilde{HH}^t$ according to the order given above we obtain the following block-diagonal matrix

$$\begin{pmatrix}
A(1, 2) & \ldots & \\
\vdots & \ddots & \\
\vdots & & A(k - 1, k) \\
& \sin^2 \alpha_1 & \\
& \vdots & \\
& \sin^2 \alpha_k & \\
\end{pmatrix}$$

whith

$$A(i, j) = \begin{pmatrix}
1 & \cos \alpha_j & -\cos \alpha_i & \cos \alpha_j \\
\cos \alpha_j & 1 & -\cos \alpha_i \cos \alpha_j & \cos \alpha_i \\
-\cos \alpha_i & -\cos \alpha_i \cos \alpha_j & 1 & -\cos \alpha_j \\
\cos \alpha_i \cos \alpha_j & -\cos \alpha_i & -\cos \alpha_j & 1
\end{pmatrix}.$$ (79)
From (74) and (76) - (78) we get

**Proof:**

With the notations of [9] denote by \((j_1, j_2)\) the minors of order 2 of the matrix \(HH^t\) and define for \(1 \leq i < j \leq k\) the matrices

\[
A(i, j) = \begin{bmatrix}
(i \ j) & (i \ j) & (i \ j) \\
(i \ j) & (i \ k+j) & (i \ k+j) \\
(i \ k+j) & (i \ k+j) & (i \ k+j) \\
(j \ k+i) & (j \ k+i) & (j \ k+i) \\
(i \ j) & (i \ k+j) & (i \ k+j) \\
(i \ j) & (i \ k+i) & (i \ k+i) \\
(i \ k+i) & (i \ k+i) & (i \ k+i) \\
(i \ j) & (i \ k+j) & (i \ k+j) \\
(i \ j) & (i \ k+j) & (i \ k+j) \\
\end{bmatrix}
\]

It is obvious that the not null minors of the matrix \(HH^t\) are given either by the elements of matrices \(A(i, j)\), \(1 \leq i < j \leq k\) or are of the form

\[
(i \ k+i) = \det \begin{bmatrix}
1 & \cos \alpha_i \\
\cos \alpha_i & 1
\end{bmatrix} = \sin^2 \alpha_i.
\]

Therefore, we obtain (78), and computing the elements of the matrices \(A(i, j)\), \(1 \leq i < j \leq k\), (79) follows. \(\square\)

By means of lemma [11] and (74) we deduce:

**Lemma 5** We have:

\[
\| y \wedge y' \|^2 - \| y \wedge y' \wedge \bigwedge_{x \in E_1} x \|^2 - \| y \wedge y' \wedge \bigwedge_{x \in E_2} x \|^2 = \sum_{1 \leq i < j \leq k} (\tilde{\ell}(i, j), \tilde{\ell}(i, j)B(i, j)) + \sum_{1 \leq i \leq k} t(i, k + i)^2 \sin^2 \alpha_i
\]

where the symmetric matrices \(B(i, j) =

\[
\begin{pmatrix}
1 - \kappa_2(\sin \alpha_i \sin \alpha_j)^2 & \cos \alpha_i & -\cos \alpha_i & \cos \alpha_i & \cos \alpha_i \\
\cos \alpha_i & 1 & -\cos \alpha_i & \cos \alpha_i & \cos \alpha_i \\
-\cos \alpha_i & -\cos \alpha_i & 1 & -\cos \alpha_i & \cos \alpha_i \\
\cos \alpha_i & \cos \alpha_i & -\cos \alpha_i & 1 & -\kappa_1(\sin \alpha_i \sin \alpha_j)^2
\end{pmatrix}
\]

are positive-definite.

**Proof:**

From (74) and (76) - (78) we get

\[
\| y \wedge y' \|^2 = \| \tilde{\ell}H \| = \langle \tilde{\ell}, \tilde{\ell}H^t \rangle = \sum_{1 \leq i < j \leq k} (\tilde{\ell}(i, j), \tilde{\ell}(i, j)A(i, j)) + \sum_{1 \leq i \leq k} t(i, k + i)^2 \sin^2 \alpha_i.
\]
Moreover, from (55) (60) (61) we obtain

\[ y \wedge y' \wedge x = \pm \sqrt{\kappa_1} \left( \sum_{i=1}^{k} b_i v_i \right) \wedge \left( \sum_{i=1}^{k} b'_i v_i \right) \wedge \bigwedge_{i=1}^{k} u_i \]

\[ = \pm \sqrt{\kappa_1} \left[ \sum_{i=1}^{k} b_i (u_i \cos \alpha_i + w_i \sin \alpha_i) \right] \wedge \left[ \sum_{i=1}^{k} b'_i (u_i \cos \alpha_i + w_i \sin \alpha_i) \right] \wedge \bigwedge_{i=1}^{k} u_i \]

\[ \wedge \bigwedge_{i=1}^{k} u_i \]

\[ = \pm \sqrt{\kappa_1} \sum_{1 \leq i < j \leq k} (b_i b'_j - b'_i b_j) (\sin \alpha_i \sin \alpha_j)^2 w_i \wedge w_j \wedge \bigwedge_{i=1}^{k} u_i. \]

the multivectors appearing above being orthonormal. Therefore

\[ \| y \wedge y' \wedge x \|^2 = \kappa_1 \sum_{1 \leq i < j \leq k} t(k + i, k + j)^2 (\sin \alpha_i \sin \alpha_j)^2. \quad (84) \]

By a similar computation we have also

\[ \| y \wedge y' \wedge \bigwedge_{x \in E_2} x \|^2 = \kappa_2 \sum_{1 \leq i < j \leq k} t(i, j)^2 (\sin \alpha_i \sin \alpha_j)^2 \quad (85) \]

Clearly, (82), (84) and (85) imply (80) and (81).

In order to see that the matrices \( B(i, j) \) are positive definite it suffices to show that their leading principal minors are positive. Recall that \( 0 < \kappa_1 < 1, i = 1, 2 \) and that \( 0 < 1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin^2 \alpha_i < 1 \). Computing we obtain

(i)

\[ \det B(i, j) = (\sin \alpha_i \sin \alpha_j)^2 \left[ 1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2(1 - (\cos \alpha_i \cos \alpha_j)^2) \right] > 0 \quad (86) \]

(ii)

\[ \det \left( \begin{array}{cc}
1 - \kappa_2(\sin \alpha_i \sin \alpha_j)^2 & \cos \alpha_i \\
\cos \alpha_j & 1 - \kappa_2(\cos \alpha_i \cos \alpha_j)^2
\end{array} \right) = (\sin \alpha_i \sin \alpha_j)^2 \left[ 1 - \kappa_2(1 - (\cos \alpha_i \cos \alpha_j)^2) \right] > 0 \quad (87) \]

(iii)

\[ \det \left( \begin{array}{c}
1 - \kappa_2(\sin \alpha_i \sin \alpha_j)^2 \\
\cos \alpha_j
\end{array} \right) = \sin^2 \alpha_j (1 - \kappa_2 \sin^2 \alpha_i) > 0 \quad (88) \]

(iv)

\[ 1 - \kappa_2(\sin \alpha_i \sin \alpha_j)^2 > 0 \quad (89) \]
as desired. □

Now we proceed to evaluate the norm of the exterior product \( \tilde{y} \wedge \tilde{y}' \). As above we have

\[
\tilde{y} \wedge \tilde{y}' = ((a_1, \ldots, a_k, b_1, \ldots, b_k)DH_1) \wedge ((a_1', \ldots, a_k', b_1', \ldots, b_k')DH_1) = i\tilde{D}H_1
\]

(90)

where \( \tilde{D}H_1 = \tilde{D}H_1 \) is the compound matrix of order 2 and therefore

\[
\| \tilde{y} \wedge \tilde{y}' \|^2 = \| i\tilde{D}H_1 \|^2 = \langle \tilde{t}, i\tilde{D}H_1H_1^\dagger \tilde{D} \rangle.
\]

(91)

The description of the matrices \( H_1 \) and \( D \) given in Lemma 3 shows that the matrix \( \tilde{D}H_1H_1^\dagger \tilde{D} = (DH_1H_1^\dagger D) \) has exactly the same structure as the matrix \( H\tilde{H}^\dagger \). Hence we obtain trivially.

**Lemma 6** Reordering the row and columns of the matrix \( \tilde{D}H_1H_1^\dagger \tilde{D} \) as described in Lemma 3 we obtain mutatis mutandis the following block-diagonal matrix:

\[
\begin{pmatrix}
A'(1,2) & \ldots & \\
\vdots & \ddots & \\
A'(k-1,k) & & s(1) \\
& \ddots & \vdots \\
& & \ldots & s(k)
\end{pmatrix}
\]

with

\[
A'(i,j) = \begin{pmatrix}
\delta_{2i}\delta_{2j} & \delta_{2i}\cos\alpha_j & -\delta_{2j}\cos\alpha_j & \cos\alpha_i\cos\alpha_j \\
\delta_{2i}\cos\alpha_j & \delta_{2i}\delta_{1j} & -\cos\alpha_i\cos\alpha_j & \delta_{1j}\cos\alpha_i \\
-\delta_{2j}\cos\alpha_j & -\cos\alpha_i\cos\alpha_j & \delta_{2j}\delta_{1j} & -\delta_{1j}\cos\alpha_j \\
\cos\alpha_i\cos\alpha_j & \delta_{1j}\cos\alpha_i & -\delta_{1j}\cos\alpha_j & \delta_{1j}\delta_{1j}
\end{pmatrix}
\]

(92)

and

\[
s(i) = (1 - \kappa_1 - \kappa_2 + \kappa_1\kappa_2\sin^2\alpha_i)\sin^2\alpha_i.
\]

(93)

Consequently

\[
\| \tilde{y} \wedge \tilde{y}' \|^2 = \langle \tilde{t}, i\tilde{D}H_1H_1^\dagger \tilde{D} \rangle
\]

\[
= \sum_{1 \leq i < j \leq k} \langle \tilde{t}(i,j), \tilde{t}(i,j)A'(i,j) \rangle
\]

\[
+ \sum_{1 \leq i \leq k} t(i, k + i) \langle 1 - \kappa_1 - \kappa_2 + \kappa_1\kappa_2\sin^2\alpha_i, \sin^2\alpha_i \rangle.
\]

(94)

**Remark 4** Notice that the elements of the matrix \( A'(i,j) \), as well as the \( s(i) \), which are initially expressed in terms of angles \( \alpha'_i \), have been recalculated by means of formulas (26)-(27) in terms of \( \alpha_i \).
Now, we are in position to compute the exact value of $\Lambda(y, y')$.

**Theorem 4** Consider the matrices

$$ M_{ij} = \begin{pmatrix} 1 - \kappa_2 (\sin \alpha_i \sin \alpha_j)^2 & \cos \alpha_i \cos \alpha_j \\ \cos \alpha_j & \cos \alpha_i \\ - \cos \alpha_i & - \cos \alpha_j \\ \cos \alpha_i \cos \alpha_j & 1 - \kappa_1 (\sin \alpha_i \sin \alpha_j)^2 \end{pmatrix} \quad (95) $$

$1 \leq i < j \leq k$ and the diagonal matrix

$$ \Delta_\kappa = \begin{pmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_2} \end{pmatrix}. \quad (96) $$

Then

1. For $k = 2$ we have

$$ \Lambda(y, y') = \| \tilde{t}(1, 2) M_{12} \Delta_\kappa \|^2 + \kappa_1 \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2 \left[ t(1, 3) \sin \alpha_2 + t(2, 4) \sin \alpha_1 \right]^2 \quad (97) $$

2. For $k > 2$ we have

$$ \Lambda(y, y') = \Lambda_1(y, y') + \Lambda_2(y, y') + \Lambda_3(y, y') \quad (98) $$

where

(i) $\Lambda_1(y, y') = \sum_{1 \leq i < j \leq k} \| \tilde{t}(i, j) M_{ij} \Delta_\kappa \|^2 \quad (99)$

(ii) $\Lambda_2(y, y') =$

$$ \kappa_1 \kappa_2 \left[ \sum_{1 \leq i \leq k} t(i, k + i)^2 \left( 1 - \prod_{1 \leq j \leq k, j \neq i} \sin^2 \alpha_j \right) \sin^4 \alpha_i \right] + 2 \sum_{1 \leq i < j \leq k} t(i, k + i) t(j, j + k) (\sin \alpha_i \sin \alpha_j)^2 \cos \alpha_i \cos \alpha_j \quad (100) $$

(iii) $\Lambda_3(y, y') =$

$$ \kappa_1 \kappa_2 \sum_{1 \leq i < j \leq k} (\sin \alpha_i \sin \alpha_j)^2 \left( 1 - \prod_{1 \leq l \leq k, l \neq i, j} \sin^2 \alpha_l \right) \langle \tilde{t}(i, j), \tilde{t}(i, j) B(i, j) \rangle \quad (101) $$

satisfy $\Lambda_i(y, y') \geq 0, i = 1, 2, 3.$
Proof:
The case \( k = 2 \).
From (80) and (94) we obtain
\[
A(2) = \langle \tilde{t}(1, 2), \tilde{t}(1, 2)A'(1, 2) \rangle \\
+ \sum_{i=1}^{2} (t(i, 2 + i))^2 (1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \sin^2 \alpha_i) \sin^2 \alpha_i. \\
- (1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2) \times \left[ \langle \tilde{t}(1, 2), \tilde{t}(1, 2)B(1, 2) \rangle \\
+ \sum_{i=1}^{2} (t(i, 2 + i))^2 \sin^2 \alpha_i \right] \\
= \langle \tilde{t}(1, 2), \tilde{t}(1, 2)C(1, 2) \rangle + \kappa_1 \kappa_2 \left[ t(1, 3)^2 \sin^4 \alpha_1 \cos^2 \alpha_2 \\
+ t(2, 4)^2 \sin^4 \alpha_2 \cos^2 \alpha_1 \right]
\]
where
\[
C(1, 2) = A'(1, 2) - (1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2)B(1, 2).
\]
Now, computing from (81) and (92) the elements of the symmetric matrix \( C(1, 2) = (c_{ij})_{i,j=1,...,4} \) we obtain:
\[
\begin{align*}
  c_1^1 &= \kappa_2 (\cos \alpha_1 \cos \alpha_2)^2 + \kappa_1 \left( 1 - \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2 \right)^2 \\
  c_1^2 &= \cos \alpha_2 \left[ \kappa_2 (\cos \alpha_1)^2 + \kappa_1 (1 - \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2) \right] \\
  c_1^3 &= \cos \alpha_1 \left[ \kappa_2 (\cos \alpha_2)^2 + \kappa_1 (1 - \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2) \right] \\
  c_1^4 &= \kappa_1 (\cos \alpha_2)^2 + \kappa_2 (\cos \alpha_1)^2 \\
  c_2^1 &= \kappa_2 (\cos \alpha_2)^2 + \kappa_1 (1 - \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2) \\
  c_2^2 &= \cos \alpha_1 \left[ \kappa_2 (\cos \alpha_2)^2 + \kappa_1 (1 - \kappa_2 (\sin \alpha_1 \sin \alpha_2)^2) \right] \\
  c_2^3 &= \kappa_1 (\cos \alpha_1)^2 + \kappa_2 (\cos \alpha_2)^2 \\
  c_2^4 &= \kappa_1 (\cos \alpha_1)^2 + \kappa_2 (1 - \kappa_1 (\sin \alpha_1 \sin \alpha_2)^2) \\
  c_3^1 &= \kappa_1 (\cos \alpha_1)^2 + \kappa_2 (1 - \kappa_1 (\sin \alpha_1 \sin \alpha_2)^2) \\
  c_3^2 &= \kappa_1 (\cos \alpha_1)^2 + \kappa_2 (\cos \alpha_2)^2 \\
  c_3^3 &= \kappa_1 (\cos \alpha_1)^2 + \kappa_2 (1 - \kappa_1 (\sin \alpha_1 \sin \alpha_2)^2) \\
  c_3^4 &= \kappa_1 (\cos \alpha_1 \cos \alpha_2)^2 + \kappa_2 (1 - \kappa_1 (\sin \alpha_1 \sin \alpha_2)^2)^2 \\
\end{align*}
\]
Recall (76) that \( \tilde{t}(1, 2) = (t(1, 2), t(1, 4), t(2, 3), t(3, 4)) \). A close look at formulas
above enables us to see that
\[
\langle \tilde{t}(1, 2), \tilde{t}(1, 2)C(1, 2) \rangle = \kappa_2 \left[ t(1, 2) \cos \alpha_1 \cos \alpha_2 + t(1, 4) \cos \alpha_1 - t(2, 3) \cos \alpha_2 \right. \\
+ t(3, 4)(1 - \kappa_1(\sin \alpha_1 \sin \alpha_2)^2) + \kappa_1 \left[ t(1, 2)(1 - \kappa_2(\sin \alpha_1 \sin \alpha_2)^2) \right.
+ t(1, 4) \cos \alpha_2 - t(2, 3) \cos \alpha_1 + t(3, 4) \cos \alpha_1 \cos \alpha_2 \\
+ 2\kappa_1\kappa_2(\sin \alpha_1 \sin \alpha_2)^2 \cos \alpha_1 \cos \alpha_2 \left( t(1, 2)t(3, 4) + t(1, 4)t(2, 3) \right). \\
\tag{104}
\]
Applying now the Plücker relation
\[
t(1, 2)t(3, 4) + t(1, 4)t(2, 3) = t(1, 3)t(2, 4). \\
\tag{105}
\]
to the right side of (104) we obtain from (102), (103) and (105) that
\[
\Lambda(2) = \kappa_2 \left[ t(1, 2) \cos \alpha_1 \cos \alpha_2 + t(1, 4) \cos \alpha_1 - t(2, 3) \cos \alpha_2 \right. \\
+ t(3, 4)(1 - \kappa_1(\sin \alpha_1 \sin \alpha_2)^2) + \kappa_1 \left[ (1 - \kappa_2(\sin \alpha_1 \sin \alpha_2)^2) \right.
+ t(1, 4) \cos \alpha_2 - t(2, 3) \cos \alpha_1 + t(3, 4) \cos \alpha_1 \cos \alpha_2 \\
+ \kappa_1\kappa_2 \left[ t(1, 3) \sin^2 \alpha_2 \cos \alpha_1 + t(2, 4) \sin^2 \alpha_1 \cos \alpha_2 \right]
\right]^{2}
= \| \tilde{t}(1, 2)M_{12}\Delta_k \|^{2} + \\
\kappa_1\kappa_2 [t(1, 3) \sin^2 \alpha_2 \sin \alpha_1 + t(2, 4) \sin^2 \alpha_1 \sin \alpha_2]^{2} \\
\tag{106}
\]
as desired.

Now we will proceed for \( k > 2 \).

From (71), (80) and (94) we obtain
\[
\Lambda(y, y') = \sum_{1 \leq i < j \leq k} (\tilde{t}(i, j), \tilde{t}(i, j)A'(i, j)) \\
+ \sum_{1 \leq i \leq k} t(i, k + i)(1 - \kappa_1 - \kappa_2 + \kappa_1\kappa_2 \sin^2 \alpha_i) \sin^2 \alpha_i \\
- (1 - \kappa_1 - \kappa_2 + \kappa_1\kappa_2 \prod_{1 \leq i \leq k} \sin^2 \alpha_i) \times \left( \sum_{1 \leq i < j \leq k} (\tilde{t}(i, j), \tilde{t}(i, j)B(i, j)) \right) \\
+ \sum_{1 \leq i \leq k} t(i, k + i)^2 \sin^2 \alpha_i. \\
\tag{107}
\]
Denote
\[
C(i, j) = A'(i, j) - (1 - \kappa_1 - \kappa_2 + \kappa_1\kappa_2(\sin \alpha_i \sin \alpha_j)^2)B(i, j).
\]
Then, substituting in (103) the indices \( i, j \) in place of the indices \( 1, 2 \) we obtain the elements of the matrix \( C(i, j) \) and proceeding as for \( C(1, 2) \), using the
Plücker relation \( t(i, j)t(i + k, j + k) + t(i, j + k)t(j, i + k) = t(i, i + k)t(j, j + k) \), this gives (with notation (99))
\[
\sum_{1 \leq i < j \leq k} (\tilde{t}(i, j), \tilde{t}(i, j)C(i, j) = \Lambda_1(y, y') + \sum_{1 \leq i < j \leq k} (\sin \alpha_i \sin \alpha_j)^2 t(i, i + k)t(j, j + k) \cos \alpha_i \cos \alpha_j.
\]
From (107) and (108) we deduce that
\[
\Lambda(y, y') = \Lambda_1(y, y') + \sum_{1 \leq i < j \leq k} (\sin \alpha_i \sin \alpha_j)^2 t(i, i + k)t(j, j + k) \cos \alpha_i \cos \alpha_j.
\]
and the proof of formula (98) is completed.

The fact (see Lemma 5) that the matrices \( B(i, j) \) are positive-definite implies straightforwardly that \( \Lambda_3(y, y') \geq 0 \). It is perhaps a little less obvious that \( \Lambda_2(y, y') \geq 0 \). This is a direct consequence of the lemma which follows. This result is certainly known but not having a reference at hand we give a proof of it in an appendix.

Lemma 7 With the choice \( k \geq 3 \) and \( 0 < \alpha_i < \pi/2, \ i = 1, \ldots, k \), the symmetric matrix
\[
M(k) = (m_{i,j})_{i,j=1,\ldots,k}
\]
with
\[
m_{i}^{j} = \left(1 - \prod_{1 \leq j \leq k,j \neq i} \sin^{2} \alpha_{j}\right) \quad \text{and} \quad m_{i}^{j} = \cos \alpha_{i} \cos \alpha_{j}, \quad \text{for} \quad i \neq j,
\]
is positive-definite.

4.2 The general case.

As announced with respect to the formulas (30) and (31) those expressions are scalar products of two vectors \( \tilde{y}, \tilde{y}' \in \mathbb{R}^p \). The main point which allows to extend results of (4.1) to the general case is the fact that \( \tilde{y} \) and \( \tilde{y}' \) may be expressed themselves via vectors \( \tilde{y}_0 \) and \( \tilde{y}'_0 \) given by formulas (64)-(66).

Consider at first the Case I (23).
Lemma 8 With the notations (26) and (27) the expression (30) coincides with the scalar product of the vectors
\[
\tilde{y} = \left(\tilde{y}_0, c, d\sqrt{1 - \kappa_1}, e\sqrt{1 - \kappa_1 - \kappa_2}\right)
\]
and
\[
\tilde{y}' = \left(\tilde{y}_0', c', d'\sqrt{1 - \kappa_1}, e'\sqrt{1 - \kappa_1 - \kappa_2}\right)
\]
where \(\tilde{y}_0\) and \(\tilde{y}_0'\) are given by formulas (64)-(66) of lemma 3.

Proof:
The proof lies on the obvious fact that the multivectors
\[
\begin{align*}
&\bigwedge_{i=1}^k u_{2i} \bigwedge_{i=1}^{k_3} u_{3i} \quad m = 1, \ldots, k \\
&\bigwedge_{i=1}^k u_{m} \bigwedge_{i=1}^{k_3} u_{3i} \quad l = 4, 5, \quad m = 1, \ldots, k_l
\end{align*}
\]
are mutually orthonormal. This implies that
\[
\begin{align*}
\langle y \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i}, y' \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i} \rangle \\
= \langle y_0 \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i}, y'_0 \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i} \rangle + \langle d, d' \rangle + \langle e, e' \rangle
\end{align*}
\]
and on the same basis
\[
\begin{align*}
\langle y \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i}, y' \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i} \rangle \\
= \langle y_0 \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i}, y'_0 \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i} \rangle + \langle e, e' \rangle.
\end{align*}
\]
From (26), (27) we obtain
\[
\langle y, y' \rangle = \langle y_0, y'_0 \rangle + \langle c, c' \rangle + \langle d, d' \rangle + \langle e, e' \rangle.
\]
Therefore
\[
\begin{align*}
\langle y, y' \rangle - \kappa_1 \langle y \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i}, y' \bigwedge_{i=1}^k u_{1i} \bigwedge_{i=1}^{k_3} u_{3i} \rangle \\
- \kappa_2 \langle y \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i}, y' \bigwedge_{i=1}^k v_i \bigwedge_{i=1}^{k_4} u_{4i} \rangle \\
= \langle \tilde{y}_0, \tilde{y}_0' \rangle + \langle c, c' \rangle + (1 - \kappa_1) \langle d, d' \rangle + (1 - \kappa_1 - \kappa_2) \langle e, e' \rangle
\end{align*}
\]
from which (111) follows.

□

As a consequence of (111) and (32) a simple calculation, which we do not detail, gives the exact value of $\Lambda_{g_1}(y, y')$.

Lemma 9 We have

$$\Lambda_{g_1}(y, y') = \Lambda(y_0, y'_0)$$

$$+ (\kappa_1 + \kappa_2) \| c \wedge c' \| ^2 + \kappa_2 (1 - \kappa_1) \| d \wedge d' \| ^2$$

$$+ \kappa_2 \sum_{1 \leq i \leq k_3} \sum_{1 \leq j \leq k_4} (c_i d'_j - c'_i d_j)^2$$

$$+ m(a, b, c) + m(a, b, d)$$

where

$$m(a, b, c) = \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq k_3} \left[ (\kappa_1 + \kappa_2 \cos^2 \alpha_i)(a_i c'_j - a'_i c_j)^2ight.$$

$$+ (\kappa_2 + \kappa_1 \cos^2 \alpha_i)(b_i c'_j - b'_i c_j)^2$$

$$+ 2(\kappa_1 + \kappa_2)(a_i c'_j - a'_i c_j)(b_i c'_j - b'_i c_j) \cos \alpha_i \right] \geq 0, \quad (118)$$

$$m(a, b, d) = \kappa_2 \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq k_4} \left[ (1 - (1 - \kappa_1) \sin^2 \alpha_i)(a_i d'_j - a'_i d_j)^2ight.$$}

$$+ (1 - \kappa_1 \sin^2 \alpha_i)(b_i d'_j - b'_i d_j)^2$$

$$+ 2(a_i d'_j - a'_i d_j)(b_i d'_j - b'_i d_j) \cos \alpha_i \right] \geq 0$$

and $\Lambda(y_0, y'_0) \geq 0$ is given by (97) and (98).

The Case II (24).

Lemma 10 The expression (31) coincides with the scalar product of the vectors

$$\tilde{y} = (\tilde{y}_0, \sqrt{1 - \kappa_1} d, e) \sqrt{1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin \alpha_i}$$

and

$$\tilde{y}' = (\tilde{y}'_0, \sqrt{1 - \kappa_1} d', e') \sqrt{1 - \kappa_1 - \kappa_2 + \kappa_1 \kappa_2 \prod_{i=1}^{k} \sin \alpha_i}$$

where $\tilde{y}_0$ and $\tilde{y}'_0$ are given by formulas (64)-(66) of lemma 3.

Proof:

In this case we have by (25) and (29)

$$\langle y, y' \rangle = \langle y_0, y'_0 \rangle + \langle d, d' \rangle + \langle e, e' \rangle$$

(120)
and straightforwardly:
\[
\langle y \wedge \bigwedge_{i=1}^{k} u_{1i}, y' \wedge \bigwedge_{i=1}^{k} u_{1i} \rangle = \langle y_0 \wedge \bigwedge_{i=1}^{k} u_{1i}, y'_0 \wedge \bigwedge_{i=1}^{k} u_{1i} \rangle + \langle d, d' \rangle + \langle e, e' \rangle, \quad (121)
\]
\[
\langle y \wedge \bigwedge_{i=1}^{k} v_i \bigwedge_{i=1}^{k} u_{4i}, y' \wedge \bigwedge_{i=1}^{k} v_i \bigwedge_{i=1}^{k} u_{4i} \rangle = \langle y_0 \wedge \bigwedge_{i=1}^{k} v_i, y'_0 \wedge \bigwedge_{i=1}^{k} v_i \rangle + \langle e, e' \rangle \quad (122)
\]
and
\[
\langle y \wedge \bigwedge_{i=1}^{k} v_i \bigwedge_{i=1}^{k} u_{4i}, y' \wedge \bigwedge_{i=1}^{k} v_i \bigwedge_{i=1}^{k} u_{4i} \rangle = \langle y_0 \wedge \bigwedge_{i=1}^{k} v_i, y'_0 \wedge \bigwedge_{i=1}^{k} v_i \rangle + \langle e, e' \rangle. \quad (123)
\]
Accordingly, formulas (119) follow directly from (31) and (120) - (123). □

From (119) and (33) we get, by an easy calculus which we do not detail either.

**Lemma 11** We have
\[
\Lambda_{g2}(y', y) = \Lambda(y_0, y'_0) + \kappa_2(1 - \kappa_1)(1 - \kappa_1 \prod_{j=1}^{k} \sin^2 \alpha_j) \| d \wedge d' \|^2 + m(a, b, d) \quad (124)
\]
where
\[
m(a, b, d) = \kappa_2 \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq k_4} \left( \left( \cos^2 \alpha_i + \kappa_1(\sin^2 \alpha_i - \prod_{j=1}^{k} \sin^2 \alpha_j) \right)(a_i d'_j - a'_i d_j)^2 \right.
\]
\[
+ (1 - \kappa_1 \sin^2 \alpha_i)(1 - \kappa_1 \prod_{j=1}^{k} \sin^2 \alpha_j)(b_i d'_j - b'_i d_j)^2
\]
\[
+ 2(1 - \kappa_1 \prod_{j=1}^{k} \sin^2 \alpha_j)(a_i d'_j - a'_i d_j)(b_i d'_j - b'_i d_j) \cos \alpha_i \left. \right] \geq 0
\]
\[
(125)
\]
and \( \Lambda(y_0, y'_0) \geq 0 \) is given by (97) and (98).

**5 Concluding remarks**

Following the argument developed in (section 5) we can see that the inequality (8) is still valid (for \( n = 2 \) and \( n = 3 \)) in the setting of general discrete determinantal processes (finite or infinite), namely to discrete determinantal processes associated to positive contractions. It remains to investigate for the basic determinantal processes the case \( n \geq 4 \) which does not seems very easy to handle.
Appendix: proof of lemma 7

The proof proceeds by induction. First, observe that if in a real symmetric positive-definite matrix we replace the diagonal entries respectively by greater elements then the new matrix obtained in this way remains positive-definite. Now if in a $M(k)$ matrix we delete the first row and the first column then the resulting matrix coincides with the $M(k-1) = (m'_{ij})_{i,j=2,...,k}$ matrix in which the diagonal entries

$$m'_i = \left(1 - \prod_{2 \leq j \leq k, j \neq i} \sin^2 \alpha_j \right), \quad i = 2, \ldots, k$$

are replaced respectively by the greater elements $\left(1 - \sin^2 \alpha_1 \prod_{2 \leq j \leq k, j \neq i} \sin^2 \alpha_j \right)$. Consequently, if the $M(k-1)$ matrix is positive-definite then the leading principal minors of $M(k)$ of orders less or equal to $k-1$ are positive and thus in order to prove that $M(k)$ is positive-definite it is enough to shown that the determinant $\det(M(k))$ is positive. Now, for $k = 3$ an elementary computation gives

$$\det M(3) = \left( \prod_{i=1}^{3} \sin^2 \alpha_i + (\sin \alpha_1 \sin \alpha_2)^2 + (\sin \alpha_1 \sin \alpha_3)^2 + (\sin \alpha_2 \sin \alpha_3)^2 \right)$$

$$\times \prod_{i=1}^{3} \cos^2 \alpha_i > 0.$$  \hspace{1cm} (126)

and it is obvious that the leading principal minors of $M(3)$ are positive.

To see that $\det M(k) > 0$ for $k > 3$, denote $a_i = \cos^2 \alpha_i$ and $b_i = \sin^2 \alpha_i$, $i = 1, \ldots, k$, and suppose without loss of generality, permuting rows and columns (with the same index) of $M(k)$, if necessary, that

$$1 > b_1 \geq b_2 \geq \cdots \geq b_k > 0. \hspace{1cm} (127)$$

We obtain then

$$\det M(k) = \prod_{i=1}^{k} a_i \times \det M_0(k)$$

where

$$M_0(k) = (n'_{ij})_{i,j=1,...,k}$$

such that

$$n'_i = \left(1 - \prod_{1 \leq j \leq k, j \neq i} b_j \right)/a_i \quad \text{and} \quad n'_j = 1, \quad \text{for} \quad i \neq j.$$  \hspace{1cm}

It is well known and easy to see (for example by Sherman-Morrison formula) that

$$\det M_0(k) = \left[ n''_k + (n''_k - 1) \sum_{i=1}^{k-1} \frac{1}{n'_i - 1} \right] \times \prod_{i=1}^{k-1} (n'_i - 1).$$

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Moreover, the inequalities (127) imply that for $i = 1, \ldots, k - 1$ we have

$$n^i - 1 = \frac{b_i - \prod_{1 \leq j \leq k, j \neq i} b_j}{1 - b_i} > 0.$$  

Hence, to finish the proof it suffices to shown that

$$f(b_k) = 1 - \prod_{i=1}^{k-1} b_i + (b_k - \prod_{i=1}^{k-1} b_i) \sum_{i=1}^{k-1} \frac{1 - b_i}{b_i - \prod_{1 \leq j \leq k, j \neq i} b_j} > 0.$$  

Fix $b_1, \ldots, b_{k-1}$ and observe that for all $i = 1, \ldots, k - 1$ the function $x \mapsto x - \prod_{i=1}^{k-1} b_i$ is increasing and consequently

$$f(b_k) \geq f(0) = 1 - \prod_{i=1}^{k-1} b_i - \prod_{i=1}^{k-1} b_i \times \sum_{i=1}^{k-1} \frac{1 - b_i}{b_i}.$$  

Putting $0 < c_i = \frac{1 - b_i}{b_i}$ we obtain

$$f(0) = \prod_{i=1}^{k-1} (1 + c_i)^{-1} \times \left[ \prod_{i=1}^{k-1} (1 + c_i) - 1 - \sum_{i=1}^{k-1} c_i \right] > 0$$  

as desired.

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