We propose a new system suitable for studying analogue gravity effects, consisting of a gas flowing in a duct with a compliant wall. Effective transonic flows are obtained from uniform, low Mach number flows through the reduction of the one-dimensional speed of sound induced by the wall compliance. We show that the modified equation for linear perturbations can be written in a Hamiltonian form. We perform a one-dimensional reduction consistent with the canonical formulation, and deduce the analogue metric along with the first dispersive term. In a weak dispersive regime, the spectrum emitted from a sonic horizon is numerically shown to be Planckian, and with a temperature fixed by the analogue surface gravity.

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norm which establishes the anomalous character of the mode mixing.

The model — We consider the propagation of sound waves in a 2-dimensional horizontal channel of uniform height $H$. $x^*$ denotes the cartesian horizontal coordinate, $y^*$ the vertical one, and $t^*$ the time. We assume the flow of air is uniform, with a horizontal velocity $U_0$. Denoting by $c_0$ the sound speed, $\rho_0$ the air density, $v^*$ the velocity perturbation, and $p^*$ the pressure perturbation, the time evolution is given by

$$c_0^{-1}D_t^*p^* = -\rho_0\nabla^* \cdot v^*, \quad \rho_0D_t^*v^* = -\nabla^* p^*, \quad (1)$$

where $D_t^* = \partial_{t^*} + U_0\partial_{x^*}$ is the convective derivative. We define dimensionless quantities as $x = x^*/H$, $y = y^*/H$, $t = t^*/c_0$, $v = v^*/c_0$, $p = p^*/(\rho_0c_0^2)$, and $M = U_0/c_0$. The potential $\phi$ gives the velocity by $v = \nabla\phi$, and the pressure by $p = -D_t\phi$. It obeys

$$D_t^2\phi - (\partial_x^2 + \partial_y^2)\phi = 0. \quad (2)$$

At the lower wall, the impenetrability condition is simply $\partial_y\phi(t, x, y = 0) = 0$, see Fig. 1. At the upper wall $y = 1$, the continuity of the displacement and pressure gives rise to a non local expression in time, see [20, 21] for details. However, for near-critical flows and small frequencies, it can be written as

$$\partial_y\phi + D_t(b(x)D_x\phi) = 0 \text{ at } y = 1, \quad (3)$$

which is second-order in $\partial_t$. For a homogeneous stationary flow, we can look for solutions of the form $\phi_k \propto \cosh(\alpha_k y)e^{i(kx - \omega_k t)}$. Eq. (2) and the two boundary conditions respectively give

$$\alpha_k^2 = k^2 - \Omega_k^2, \quad \text{and } \alpha_k \tanh(\alpha_k) = b\Omega_k^2, \quad (4)$$

where $\Omega_k \equiv \omega_k - Mk$ is the frequency in the co-moving frame. At low wave number, the dispersion relation reads

$$\Omega_k^2 = c_s^2(b)k^2 - k^4/\Lambda_b^2 + O(k^6), \quad (5)$$

where

$$c_s^2(b) = \frac{1}{1 + b}, \quad \Lambda_b^3 = \frac{3(1 + b)^3}{b^2} \quad (6)$$

One sees the important effect of the boundary condition of Eq. (3): the low frequency group velocity with respect to the fluid (= $\partial_k\Omega$) is reduced by the compliant wall. One also sees that the dispersive length $1/\Lambda_b$ given by the quartic term vanishes in the limit $b \rightarrow 0$.

To obtain flows crossing the effective sound speed, we make $b$ vary with $x$, see Fig. 1. We call $b_1 > b_2$ its asymptotic values, and $d_b$ its typical variation length. We choose the following form for $b(x)$:

$$b(x) = \frac{b_1 + b_2}{2} + \frac{b_2 - b_1}{2}\tanh\left(\frac{x}{d_b}\right). \quad (7)$$

We then adjust the flow speed $M$ to obtain

$$c_s(b_1) = \frac{1}{\sqrt{1 + b_1}} < |M| < \frac{1}{\sqrt{1 + b_2}} = c_s(b_2). \quad (8)$$

Since the background flow is stationary, we shall work with (complex) stationary waves:

$$\tilde{\phi}_\omega(x, y) = \int_{-\infty}^{+\infty} \frac{dt}{2\pi} e^{i\omega t}\phi(t, x, y). \quad (9)$$

Wave equation — Since the height $H$ of the duct is much smaller than typical longitudinal wavelengths, we expect that stationary waves obey an effective 1 dimensional equation in $x$, as it is the case in elongated atomic Bose condensates [22], and in flumes [14, 23]. To obtain such a reduction is non trivial as the $x$-dependence of $b(x)$ prevents us from factorizing out a $y$-dependent factor. To proceed, and to make contact with the above Refs., it is useful to exploit the fact that Eq. (2) and Eq. (3) can be derived from the following action

$$S = \frac{1}{2}\int dt\int dx\int_0^1 dy\mathcal{L} \quad \mathcal{L} = (D_t\phi)^2 - (\nabla\phi)^2 + \delta(y - 1)b(x)(D_x\phi)^2. \quad (10)$$

We notice that the non-trivial condition of Eq. (3) is incorporated by the above boundary term. Introducing the conjugate momentum $\pi(t, x, y) = \partial\mathcal{L}/\partial(\partial_t\phi)$ one obtains the Hamiltonian $H$ by the usual Legendre transformation. In addition, the scalar product $\langle \cdot \rangle$ also takes the standard form: [4, 23]

$$\langle \phi_1, \phi_2 \rangle = i\int_{-\infty}^{+\infty} dx\int_0^1 dy(\pi^*_1\pi_2 - \phi^*_1\pi^*_2), \quad (11)$$

where $\phi_1, \phi_2$ are two complex solutions of Eq. (2, 3), and $\pi_1, \pi_2$ their associated momenta. Eq. (11) is conserved in virtue of Hamilton’s equations. As for sound waves in other media, it identically vanishes for all real solutions. However it provides a key information when studying stationary modes, namely the sign of their norm ($\phi_\omega, \tilde{\phi}_\omega$). Indeed, as we shall see, for a fixed $\omega > 0$, there will be both positive and negative norm modes. When the flow is stationary, for any complex solution $\phi$, the wave energy is conserved and related to Eq. (11) by

$$H[2\text{Re}(\phi)] = \langle \phi, i\partial_t\phi \rangle. \quad (12)$$

Moreover, when the flow is also asymptotically homogeneous, for every asymptotic plane wave $\phi_k \propto \cosh(\alpha_k y)e^{i(kx - \omega_k t)}$, the sign of $H$ is that of $\omega_k\Omega_k$. This relation will allow us to identify the negative energy waves without ambiguity.

We can now proceed following the hydrodynamic treatment of [23]. As a first step, it is useful to derive the effective $(1+1)$-dimensional equation from which an effective space-time metric can be read out. When the (adimensional) wavelength in the $x$ direction is much larger than 1, we can assume that $\partial_y\phi$ is independent of $y$. As $\partial_y\phi = 0$ at $y = 0$, we write the field as

$$\phi(x, y, t) \approx \Phi(x, t) + y^2\Psi(x, t). \quad (13)$$
Plugging this into the action Eq. (10) and varying it with respect to $\Phi$ and $\Psi$, we get two coupled equations. Combining them, we obtain $(\hat{\mathcal{O}}_2 + \mathcal{O}_1) \Phi = 0$, where $\mathcal{O}_n$ is a $n^{th}$-order operator in $\partial_t$ and $\partial_x$. The quadratic term is

$$\hat{\mathcal{O}}_2 = \partial_{\mu} F^{\mu\nu}(x) \partial_{\nu},$$

where

$$F^{\mu\nu}(x) = \left( c_2^2(b(x)) - M^2 \frac{M}{1} \right),$$

and $c_2^2(b(x))$ is given in Eq. (6). Up to a conformal factor, we obtain the d’Alembert equation in a two dimensional space-time with metric $g^{\mu\nu} \propto F^{\mu\nu}$. This metric has a Killing horizon where $c_2^2 - M^2$ vanishes [24]. This correspondence with gravity relates the anomalous scattering described below to the Hawking effect.

Contrary to what happens for sound waves in atomic BECs, or water waves in the incompressible limit, $\mathcal{O}_4$ also contains third and fourth derivatives in time. This prevents to apply the standard treatment on the sole field $\Phi$. However, the set of two coupled equations on $(\Phi, \Psi)$ is Hamiltonian and can be used to study the scattering. Alternatively, one can work with the original model in $2+1$ dimensions based on Eq. (10). We performed numerical simulations with that model and found similar results.

**Anomalous mode mixing** — Since stationary waves with different frequencies $\omega$ do not mix, the scattering only concerns the discrete set of modes with the same $\omega$. To characterize it, we identify its dimensionality and the norms of the various asymptotic modes for $x \to \pm \infty$. Fig. 2 shows the dispersion relation and the roots at fixed stream. It thus corresponds to a white-hole flow, as those studied in [17, 23, 25]. 1 For definitens, we discuss only the case $\omega > 0$. The same results are directly applicable to $\omega < 0$ after complex conjugation. In the subsonic region, on the right of the horizon, there exists a critical frequency $\omega_{\text{max}}$ (close to 0.42 in the Figure) at which two roots merge. For $\omega < \omega_{\text{max}}$, the dispersion relation has 4 real roots (as well as an infinite number of complex ones). 2 Following [23], we call their wave-vectors $k_\omega$, $k_\omega^\text{in}$, $k_\omega^\text{out}$, and $-k_\omega$. The corresponding asymptotic modes are, respectively, $\phi_\omega^\text{in}$, $\phi_\omega^\text{out}$, $\phi_\omega^\text{in}$, and $(\phi_\omega^\text{in})^\ast$. They are characterized by three important properties:

1) co- or counter-propagating nature: $\phi_\omega^\text{co}$ is co-propagating (its group velocity in the frame of the fluid is positive) while the three others are counter-propagating;

2) in- or out-character: $\phi_\omega^\text{in}$ is incoming (it moves towards the horizon) while the three other modes are outgoing (they move away from the horizon);

3) energy sign: $(\phi_\omega^\text{out})^\ast$ carries a negative energy and a negative norm, see Eq. (12). (It has been complex conjugated so that $\phi_\omega^\text{out}$ is a positive norm mode). The three other modes have positive energy and norm.

It is useful to separate co- and counter-propagating modes in the fluid frame because only the latter are significantly mixed in a transonic flow [9, 10]. In effect, the co-propagating mode acts essentially as a spectator.

To get the $S$ matrix, we need to identify the bases of incoming and outgoing modes. The incoming (resp. outgoing) modes $\phi_\omega^\text{in}$ (out) are those which contain only one incoming (resp. outgoing) asymptotic plane wave.

For $\omega < \omega_{\text{max}}$, there are three of them, so the scattering matrix has a size $3 \times 3$ [9]. In this paper we will be primarily interested in the counter-propagating incoming $\phi_\omega^\text{in}$ mode for a white hole flow. This mode has been a focus of interest in hydrodynamic flows [16, 17, 25, 26] and is a good candidate to probe the analogue Hawking effect. For $x \to \infty$, it is a sum of four asymptotic modes

$$\phi_\omega^\text{in} \approx \phi_\omega^\text{in} + \alpha_\omega \phi_\omega^\text{out} + \beta_\omega (\phi_\omega^\text{out})^\ast + A_\omega \phi_\omega^\text{co,out}.$$

In transonic flows, there is no transmitted wave [25]. $\phi_\omega^\text{in}$ thus vanishes for $x \to -\infty$. When working with asymptotic modes of unit norm, the norm of $\phi_\omega^\text{in}$ evaluated at late time (in the sense of a broad wave packet)

$$(N_\omega^\text{out})^2 = |\alpha_\omega|^2 - |\beta_\omega|^2 + |A_\omega|^2,$$

must be exactly 1 because of the conservation of Eq. (11). (Since we work in a stationary flow, $(N_\omega^\text{out})^2 = 1$ also expresses the conservation of the wave energy, see Eq. (12).) The minus sign in front of $|\beta_\omega|^2$ is the signature of an

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1 For $M < 0$, one would describe a black hole flow. The forthcoming analysis applies by reversing the sign of velocities and the “in” or “out” character of the modes, see [9].

2 In the supersonic region, only two real wave-vectors remain: $k_\omega^\text{co}$ and $-k_\omega^\text{in}$. They both describe incoming modes.

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![Fig. 2. Dispersion relation $\omega$ versus $k$ in a homogeneous subsonic flow. The blue, plain curves show the roots with positive comoving frequency $\Omega$, and the red, dashed ones those with $\Omega < 0$. The dotted black line represents $\omega = 0.4$, and the dot-dashed one shows the value $\omega_{\text{max}}$ of $\omega$ at which the two roots with $k < 0$ merge. The parameters are $b = 1$, $M = 0.4$, and the effective sound velocity $c_2$ is equal to $1/\sqrt{2} \approx 0.71$.](image-url)
anomalous scattering. It stems from the negative norm carried by \((\varphi^{\text{out}})\), see the above point 3. The coefficient \(\beta_\omega\) thus mixes modes of opposite norms and energies. In quantum settings, \(|\beta_\omega|^2\) would give the mean number of spontaneously produced particles from amplifying vacuum fluctuations, that is, the Hawking radiation \([2]\). The gravitational analogy \([1, 4]\) indicates that \(|\beta_\omega|^2\) should follow a Planck law when dispersion effects (and grey body factors\([27, 28]\)) are negligible. Moreover, it predicts that the effective temperature \(T\) should be given by \(T_H = \kappa/2\pi\), where \(\kappa\) is the surface gravity obtained from the analogue metric of Eq. (15). (We choose the units so that \(k_B = 1\).) \(T_H\) and \(\kappa\) are thus both frequencies. Using Eq. (6), one gets

\[
\kappa = \partial_x c_S|_{c_S = M} = \frac{M^3}{2}\partial_x b|_{c_S = M}. \tag{18}
\]

**Spectral analysis** – We numerically solved the set of coupled equations on the \((1+1)\)-dimensional fields \(\Phi\) and \(\Psi\), using the method of \([25]\) adapted to the present case. The results concerning the incoming mode of Eq. (16) propagating in a transonic flow described by Eq. (7) are shown in Fig. 3. We stopped the integration for \(\omega\) slightly below the critical frequency \(\omega_{\text{max}}\), where \(\beta_\omega\) and \(A_\omega\) both vanish. We tuned the various parameters (given in the caption of Fig. 3), so that the flow is near-critical: \(M/c_S(b_1) \approx 1.054\) and \(M/c_S(b_2) \approx 0.943\). Using these parameters, one has \(\omega_{\text{max}} \approx 0.0053\), and \(\kappa \approx 0.019\) of the same order as the dispersive frequency scale \(\Lambda_0 c_S^2\) evaluated at the horizon. This means that we worked just outside the weak dispersive regime \([11]\). Yet, for frequencies up to \(\omega_{\text{max}}\), \(|\beta_\omega|^2\) follows rather well the Hawking prediction \(|\beta_\omega|^2 = 1/(e^{\omega/T_H} - 1)\), that is, a Planck law with \(T_H\) given by \(\kappa/(2\pi)\), see Eq. (18). At low frequency, the relative difference \((|\beta_\omega|^2 - |\beta_\omega^H|^2)/|\beta_\omega^H|^2\) is of order 20% (when we used \(d_b = 3\), the difference reduced to about 0.3%, as expected since we were then in a weakly dispersive regime). Moreover, we see that the coefficient \(A_\omega\) involving the co-propagating mode can be safely neglected as \(|A_\omega|^2\) remains smaller that 0.1%. In order to estimate the numerical errors, we also show the quantity \((N^\text{out})^2 - 1\) which must be zero to zero, as explained below Eq. (17).

In brief, the properties we obtain are in close agreement with those found in other media \([9, 12, 22, 29]\).

![Plot of \(N^\text{out}/|\beta_\omega|^2\) (blue, plain) and \(|A_\omega|^2\) (orange, dashed) of Eq. (16) as functions of the frequency. The black, dot-dashed curve shows \(|\beta_\omega|^2\) for a Planck law at the Hawking temperature \(T_H = \kappa/(2\pi)\). The parameters are: \(M = 1/3, b_1 = 9, b_2 = 7,\) and \(d_b = 1\). The green, dotted line represents \((N^\text{out})^2 - 1\) where \((N^\text{out})^2\) is given in Eq. (17). Its non-vanishing value quantifies the numerical errors.](image)

**Conclusions** – We showed that a low Mach number uniform flow of air in a tube with a compliant wall can produce a sonic horizon by reducing the local effective one-dimensional speed of sound. Despite the unusual boundary condition at the compliant wall, the problem was phrased in a Hamiltonian formalism. For near-critical flows, a \((1+1)\)-dimensional reduction was performed, exhibiting an analogue metric and the first dispersion terms while retaining the hamiltonian structure. The Hawking spectrum was numerically recovered for sufficiently slowly varying profiles of the compliant wall. In light of these results, this system seems to be a good candidate to probe Hawking’s prediction in the laboratory.

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3 When sending a localized wave-packet on a white hole horizon, we observed at late times the formation of an undulation, see \([21]\).

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Supplementary Materials:
Slow Sound in a duct, effective transonic flows and analogue black holes

Wave equation and small-frequency approximation
When considering a stationary mode $\phi_\omega \propto e^{-i\omega t}$, the boundary condition at the compliant wall $y = 1$ reads
$$\partial_y \phi_\omega = - (i\omega + M\partial_x) \left( \frac{\tan(\omega b(x))}{\omega} \right) (-i\omega + M\partial_x) \phi_\omega. \quad (S1)$$

To avoid having to deal with time derivatives of high orders, we consider a small-frequency limit $\tan(\omega b(x)) / \omega \approx b$, to obtain Eq. (3). It must be noted that this simplification does not affect the quadratic term in the dispersion relation, but changes the quartic term. The relative difference is of order $(1 - M/cS)^2$ when considering the counter-propagating modes. This approximation is thus valid for all the spanned range of $\omega \in [0, \omega_{\text{max}}]$ provided $M/cS(b(x))$ remains everywhere close to unity.

Wave equation and normalization in the (1+1)D model
The two operators $\hat{O}_2$ and $\hat{O}_4$ appearing below Eq. (13) are given by
$$\hat{O}_2 = D_t (1 + b(x)) D_t - \partial^2_x, \quad \hat{O}_4 = \frac{2}{30} \left( D_t^2 - \partial^2_x \right) \left( D_t (1 + 6b(x)) D_t - \partial^2_x \right). \quad (S2)$$

Asymptotic modes are given by two-component plane waves:
$$\Phi(x, t) = u e^{i(kx - \omega t)}, \quad \Psi(x, t) = v e^{i(kx - \omega t)}, \quad (S3)$$

where $\Phi, \Psi$ are defined in Eq. (13). The prefactors $u$ and $v$ are given by
$$\frac{u}{v} = -\frac{1}{3} \frac{(1 + 3b) \Omega^2 - k^2}{(1 + b) \Omega^2 - k^2}, \quad (S4)$$
$$((1 + b) M\Omega + k) |u|^2 + \frac{2}{3} ((1 + 3b) M\Omega + k) \text{Re}(uv^*) + \frac{1}{5} ((1 + 5b) M\Omega + k) |v|^2 = \pm 1. \quad (S5)$$

The first condition comes from the wave equation derived by varying the action Eq. (10) with respect to $\Phi$, while the second one ensures that all the asymptotic modes have a unit norm (up to a minus sign).

Wave-packet and undulation
In Fig. S1 we show a space-time diagram of the perturbation obtained by sending a localized wave-packet on a white hole horizon. The important point is the appearance of a long-lasting undulation, i.e., of a zero-frequency mode with a macroscopic amplitude. Its presence is due to the diverging character of $|\beta_{\omega}|^2 \sim |\alpha_{\omega}|^2$ as $1/\omega$ for $\omega \to 0$ which amplifies the small-frequency components of the incident wave-packet. See Eq. (23) in [23] and Eq. (27) in [30] for studies of the same mechanism in related contexts.
FIG. S1. Space-time plot of the undulation obtained when sending a wave packet initially centered around $x = 50$ in a white hole flow. The sonic horizon is located around $x = 0$. As in the main text, the sign of $M$ is positive and the subsonic region is on the right side of the horizon. The amplitude represented is that of $F = \int_0^1 \phi dy = \Phi + \frac{1}{3} \Psi$. The top plot shows the time-evolution with a linear color scale, and the bottom one shows the late-time configuration.