THE RUMOR PERCOLATION MODEL AND ITS VARIATIONS

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Abstract. The study of rumor models from a percolation theory point of view has gained a few adepts in the last few years. The persistence of a rumor, which may consistently spread out throughout a population can be associated to the existence of a giant component containing the origin of a graph. That is one of the main interest in percolation theory. In this paper we present a quick review of recent results on rumor models of this type.

Contents

1. Introduction and basic definitions 2
2. Random sets on $\mathbb{R}^d$ and $\mathbb{R}^{d^+}$ 4
3. Disk percolation 8
  3.1. Disk percolation on trees 9
4. Fireworks on $\mathbb{N}$ 11
  4.1. Fireworks 11
  4.2. Reverse Fireworks 15
5. Cone percolation on $\mathbb{T}_d$ 18
6. Random environments 22
  6.1. Fireworks 23
  6.2. Reverse Fireworks 24

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1. Introduction and basic definitions

We are interested in a long-range percolation model on infinite graphs which we call the *Rumor Percolation Model*. Such models have recently been studied by a few authors in a series of papers. The dynamics of the model describes the spreading of a rumor on a graph in the following way. We assign independent random *radius of influence* $R_v$ to each vertex $v$ of an infinite, locally finite, connected graph $G$. Then we define a chain reaction on $G$ according to the following simple rules: (1) at time zero, only the root (a fixed vertex of $G$) hears the rumor, (2) at time $n \geq 1$, a new vertex hears the rumor if it is a distance at most $R_v$ of some vertex $v$ that previously heard the rumor. We point out that similar models, are of interest in Computer Science, in particular in the area of distributed networks. One of the problems of interest is the broadcasting problem where one node has some information which it wants to pass on to other nodes. Questions of optimal algorithm for achieving this goal are of interest. This question was considered for the case where the nodes are uniformly randomly distributed on an interval $[0, L]$ and the nodes had a transmission radius of one. In [14] asymptotically (in $L$) optimal algorithm was obtained.

**Definition 1.1.** The *Rumor Percolation Model* on $G$.

Let $G = (\mathcal{V}, \mathcal{E})$ be an infinite, locally finite, connected graph and let
\{R_v\}_{v \in \mathcal{V}} \) be a set of independent and identically distributed random variables. Furthermore, for each \( u \in \mathcal{V} \), we define the random sets
\[
B_u = \{ v \in \mathcal{V} : d(u, v) \leq R_u \}. \tag{1.1}
\]
onumber
or
\[
B_u = \{ v \in \mathcal{V}, u \leq v : d(u, v) \leq R_u \}. \tag{1.2}
\]

With these sets we define the Rumor Percolation Model on \( G \), the non-decreasing sequence of random sets \( I_0 \subset I_1 \subset \cdots \) defined as \( I_0 = \{\emptyset\} \) and inductively \( I_{n+1} = \bigcup_{u \in I_n} B_u \) for all \( n \geq 0 \).

**Definition 1.2.** The Rumor Percolation Model survival.

Consider \( I = \bigcup_{n \geq 0} I_n \) be the connected component of the origin of \( G \). Under the rumor process interpretation, \( I \) is the set of vertices which heard the rumor. We say that the process survives (dies out) if \( |I| = \infty \) (\( |I| < \infty \)), referring to the surviving event as \( V \).

In section 2 we review the paper of Athreya et al [1]. Instead of considering a graph structure they consider a homogeneous Poisson point process on \( \mathbb{R}^d \) and \( \mathbb{R}^{d+} \) with \( \{R_v\} \), the box of influence, starting from every point \( v \) of the point process in the sense of (1.2). They work with the concept of the coverage of a set \((t, \infty)^d\) for some \( t > 0 \), the eventual coverage. In section 3 we review the paper of Lebenstayn and Rodriguez [12] where authors consider the Disk Percolation Model. While the set of radius of influence, \( \{R_v\}_{v \in \mathcal{V}} \), has a geometric distribution, the graph \( G \) is quite general. In their version the radius of influence of a vertex \( v \in G \) goes in every possible direction as in (1.1). In section 4 we review the papers of Junior et al [9] and Gallo et al [5]. They work with a processes that they made known as Fireworks on \( \mathbb{N} \) (direct and
They studied an homogeneous version, where there is one informant per vertex and the \textit{radius of influence} are independent and have the same distribution, and a heterogeneous version, where one of these conditions fail. In their models the \textit{radius of influence} goes like in (1.2). In section 5 the papers of Junior \textit{et al} [9], [10] and [11] are briefly reviewed. They work with the Cone Percolation model, a Fireworks model in a tree (homogeneous, spherically symmetric, periodic or Galton Watson). In all these models the \textit{radius of influence} goes like in (1.2). In section 6 we review the paper of Bertacchi and Zucca [3]. They consider a type of random environment in the sense that the number of informants in each vertex of are random.

2. Random sets on $\mathbb{R}^d$ and $\mathbb{R}^{d+}$

The theory of coverage processes was introduced by P. Hall [8] in 1988. He developed a class of stochastic processes intended to be used as a model for \textit{binary images}, that is, images which partition $\mathbb{R}^d$ into two regions, $\mathcal{C}$ and its complement, representing the “black” and “white” parts of an image. In its basic version the process consists of a point process $P = \{\xi_1, \xi_2, \ldots\}$ and a collection of random sets $\{S_1, S_2, \ldots\}$. The “black” region $\mathcal{C}$ is then defined to be $\mathcal{C} = \bigcup_{i=1}^{\infty} (\xi_i + S_i)$. P. Hall [8] developed probabilistic results on geometrical properties of $\mathcal{C}$, such as the size-distribution of its connected subsets. In that work the main assumptions needed to obtain explicit results is that $P$ is an homogeneous Poisson process and the $S_i$ are independent copies of a random closed set. This version is known as the Poisson Boolean model.
Athreya et al [1] considered two different models, both related to rumor percolation. For the first model, arising for genome analysis, they consider \( \{X_i\}_{i \in \mathbb{N}} \) be a \( \{0,1\} \)-valued time-homogeneous Markov chain and \( \{\rho_i\}_{i \in \mathbb{N}} \) an independent and identically distributed sequence of random variables assuming values on \( \mathbb{N} \), independent of the Markov chain. Let \( S_i = [i, i + \rho_i] \) whenever \( X_i = 1 \) (\( \emptyset \) otherwise) and \( C = \bigcup_{i=1}^{\infty} S_i \).

**Definition 2.1.** We say that \( \mathbb{N} \) is eventually covered by \( C \) (or \( C \) eventually covers \( \mathbb{N} \)) if there exists a \( t \geq 1 \) such that \([t, \infty) \subseteq C\).

**Theorem 2.2** (Athreya et al [1]). Let \( p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i) \). Assume that \( 0 < p_{00}, p_{10} < 1 \),

(i) If

\[
l = \liminf_{j \to \infty} j \mathbb{P}(\rho_1 > j) > 1,
\]

then

\[
\mathbb{P}(C \text{ eventually covers } \mathbb{N}) = 1
\]

whenever

\[
\frac{p_{01}}{p_{10} + p_{01}} > \frac{1}{L}.
\]

(ii) If

\[
L = \limsup_{j \to \infty} j \mathbb{P}(\rho_1 > j) < \infty,
\]

then

\[
\mathbb{P}(C \text{ eventually covers } \mathbb{N}) = 0
\]

whenever

\[
\frac{p_{01}}{p_{10} + p_{01}} < \frac{1}{L}.
\]
Their second model aims to complement known results on complete coverage in stochastic geometry. For \( B(0, \rho) \) the closed \( d \)-dimensional ball of radius \( \rho \) centered at the origin, some important previous results for the random covered region \( \bigcup_{i=1}^{\infty} (\xi_i + B(0, \rho_i)) \) are presented in the next two theorems.

**Theorem 2.3** (Hall [8]). *For the Poisson Boolean model on \( \mathbb{R}^d \) the space is fully covered by \( \bigcup_{i=1}^{\infty} (\xi_i + B(0, \rho_i)) \) almost surely if and only if \( \mathbb{E}(\rho^d) = \infty \).*

If instead of a Poisson point process one considers an arbitrary ergodic process, there is the following result

**Theorem 2.4** (Meester and Roy [13]). *For the Boolean model on \( \mathbb{R}^d \) the space is fully covered by \( \bigcup_{i=1}^{\infty} (\xi_i + B(0, \rho_i)) \) almost surely if \( \mathbb{E}(\rho^d) = \infty \).*

Athreya *et al* [1] take \( \mathbb{R}^d_{+} \) and the random covered region

\[
C = \bigcup_{\{i: \xi_i \in \mathbb{R}^d_{+}\}} (\xi_i + [0, \rho_i]^d).
\]

Guided by the fact that \( C \) will never completely cover \( \mathbb{R}^d_{+} \) because, for any \( \epsilon > 0 \), \( [0, \epsilon]^d \) will not be covered by \( C \) with positive probability, they work with the notion of *eventual coverage* for the orthant \( \mathbb{R}^d_{+} \).

**Definition 2.5.** We say that \( \mathbb{R}^d_{+} \) is eventually covered by the Poisson Boolean model if there exists a \( t \in (0, \infty) \) such that \( [t, \infty)^d \subseteq C \).

With this notion Athreya *et al* [1] are able to present the following result, considering a Poisson Boolean model on \( \mathbb{R}^d_{+} \). They show that eventual coverage depends on the growth rate of the distribution function of \( \rho \) (even when \( \mathbb{E}(\rho) = \infty \)) as well as on whether \( d = 1 \) or \( d \geq 2 \).
Theorem 2.6 (Athreya et al \[\text{[1]}\]). Assume $d = 1$.

(i) If
\[
0 < l := \liminf_{x \to \infty} x \mathbb{P}(\rho > x) < \infty,
\]
then there exists a $\lambda_0$ such that $0 < \lambda_0 \leq 1/l < \infty$ and
\[
\mathbb{P}_\lambda(\mathbb{R}_+ \text{ is eventually covered by } C) = \begin{cases} 
0 & \text{if } \lambda < \lambda_0, \\
1 & \text{if } \lambda > \lambda_0; 
\end{cases}
\]

(ii) If
\[
0 < L := \limsup_{x \to \infty} x \mathbb{P}(\rho > x) < \infty,
\]
then there exists a $\lambda_1$ such that $0 < 1/L \leq \lambda_1 < \infty$ and
\[
\mathbb{P}_\lambda(\mathbb{R}_+ \text{ is eventually covered by } C) = \begin{cases} 
0 & \text{if } \lambda < \lambda_1, \\
1 & \text{if } \lambda > \lambda_1; 
\end{cases}
\]

(iii) If
\[
\lim_{x \to \infty} x \mathbb{P}(\rho > x) = \infty,
\]
then for all $\lambda > 0$, $\mathbb{R}_+$ is eventually covered by $C$ ($\mathbb{P}_\lambda$-a.s.);

(iv) If
\[
\lim_{x \to \infty} x \mathbb{P}(\rho > x) = 0,
\]
then for any $\lambda > 0$, $\mathbb{R}_+$ is eventually covered by $C$ ($\mathbb{P}_\lambda$-a.s.).

Theorem 2.7 (Athreya et al \[\text{[1]}\]). Let $d \geq 2$. For all $\lambda > 0$,

(i) If
\[
\liminf_{x \to \infty} x \mathbb{P}(\rho > x) > 0,
\]
then
\[
\mathbb{P}_\lambda(\mathbb{R}_+^d \text{ is eventually covered by } C) = 1;
\]

(ii) If
\[
\lim_{x \to \infty} x \mathbb{P}(\rho > x) = 0,
\]
then
\[
\mathbb{P}_\lambda(\mathbb{R}_+^d \text{ is eventually covered by } C) = 0.
\]
It is interesting to observe that while $E(\rho^d) = \infty$ guarantees complete coverage of $\mathbb{R}^d$ by $C$, it is not sufficient to guarantee eventual coverage for $\mathbb{R}^d_+$. This is due to the fact that a boundary effect is present in the orthant $\mathbb{R}^d_+$ but absent in the whole space $\mathbb{R}^d$.

3. Disk percolation

Lebensztayn and Rodriguez studied a long-range percolation model on infinite graphs, the Disk Percolation Model. They assign a random radius of influence $R_v$ to each vertex $v$ of an infinite, locally finite, connected graph $G$, so that all the assigned radii are independent and identically distributed random variables with geometric distribution with parameter $(1 - p)$, which means, satisfying

$$P(R = k) = (1 - p)p^k, k = 0, 1, 2, \ldots$$

Then they defined a growing process on $G$ according to the following rules: (1) at time zero, only the root (a fixed vertex of $G$) is declared infected, (2) at time $n \geq 1$, a new vertex is infected if it is at graph distance at most $R_v$ of some vertex $v$ previously infected, and (3) infected vertices remain infected forever. They investigated the critical value $p_c(G)$ above which this process spreads indefinitely through the graph with positive probability.

They worked in a few settings including locally finite graphs in the sense that

$$\Delta = \sup_{v \in G}\{d(v)\} < \infty$$

where $d(v)$ is the number of neighbors (or degree) of a vertex $v$. 
An interesting question is whether such a model presents \textit{phase transition} in the sense that for \( p_c(G) := \inf \{ p : \mathbb{P}(|I| = \infty) \} \) we have that \( 0 < p_c(G) < 1 \).

They provided an answer which relies on a comparison between the \textit{Disk Percolation Model} and the independent site percolation model. To understand this, consider \( p_{c_{\text{site}}}(G) \) the critical probability for the independent site percolation model on \( G \).

\textbf{Theorem 3.1} (Lebenstayn and Rodriguez [12]). \textit{Let} \( G \) \textit{be of bounded degree} \((\Delta < \infty)\) \textit{and be such that} \( p_{c_{\text{site}}}(G) < 1 \). \textit{Then}

\[ 0 < p_c(G) < 1. \]

The proof they presented relies on the following two propositions, the first one is a comparison which gives an upper bound to \( p_c(G) \).

\textbf{Proposition 3.2} (Lebenstayn and Rodriguez [12]).

\[ p_c(G) \leq p_{c_{\text{site}}}(G) \]

while the second one gives a lower bound for the case that \( G \) is of bounded degree.

\textbf{Proposition 3.3} (Lebenstayn and Rodriguez [12]). \textit{Suppose that} \( G \) \textit{is a graph of bounded degree. Then}

\[ p_c(G) \geq -1 + \left(1 + \frac{1}{\Delta - 1}\right)^{1/2}. \]

3.1. \textbf{Disk percolation on trees.}

Consider a tree \( T \) (a connected graph with no cycles) and its set of vertices \( V(T) \). We say that a tree, \( T_d \), is \textit{homogeneous}, if each one of its vertices has degree (number of neighbours) \( d + 1 \).
**Theorem 3.4** (Lebenstayn and Rodriguez [12]). For any $d \geq 2$

$$-1 + \left(1 - \frac{1}{d}\right)^{1/2} \leq p_c(T_d) \leq 1 - \left(1 - \frac{1}{d}\right)^{1/2}.$$

**Corollary 3.5** (Lebenstayn and Rodriguez [12]). For any $d \geq 2$

$$p_c(T_d) = 1/(2d) + O(1/d^2) \text{ as } d \to \infty.$$ 

Single out one vertex from $\mathcal{V}(\mathbb{T})$ and call this $O$, the origin of $\mathcal{V}(\mathbb{T})$. For each two vertices $u, v \in \mathcal{V}(\mathbb{T})$, consider that $u \leq v$ if $u$ belongs to the path connecting $O$ to $v$.

For a tree $\mathbb{T}$ and $n \geq 1$ we define

$$T^u := \{v \in \mathcal{V} : u \leq v\},$$

$$T^u_n := \{v \in T^u : d(v, O) \leq d(u, O) + n\}$$

and

$$M_n(u) := |\partial T^u_n| := |\{v \in T^u : d(v, O) = d(u, O) + n\}|.$$

**Definition 3.6.** Let us define for a tree $\mathbb{T}$

$$\dim \inf \partial \mathbb{T} := \lim_{n \to \infty} \min_{v \in \mathcal{V}} \frac{1}{n} \ln M_n(v).$$

Observe that

$$\dim \inf \partial T_d = \ln d.$$ 

**Definition 3.7.** We say that a tree, $\mathbb{T}_S$, is *spherically symmetric*, if any pair of vertices at the same distance from the origin, have the same degree.

**Theorem 3.8** (Lebenstayn and Rodriguez [12]). For any spherically symmetric tree $\mathbb{T}_S$

$$p_c(\mathbb{T}_S) \leq 1 - \left(1 - e^{-\dim \inf \partial \mathbb{T}_S}\right)^{1/2}.$$
4. Fireworks on $\mathbb{N}$

The Fireworks processes are another interesting version of the Rumor Percolation Model. Junior et al [9] and Gallo et al [5] recently studied discrete time stochastic systems on $\mathbb{N}$ modeling processes of rumor spreading. In their models the involved individuals can either have an active role, working as spreaders and transmitting the information within a random distance to their right, or a passive role, hearing the information from spreaders within a random distance to their left. The appetite in spreading or hearing the rumor is represented by a set of random variables whose distributions may depend on the individuals positions on $\mathbb{N}$. Their main goal is to understand - based on the distribution of those random variables - whether the probability of having an infinite set of individuals knowing the rumor is positive or not.

Junior et al [9] manage to write the survival event as a limit of an increasing sequence of events whose probability can be bounded by a nice use of FKG inequality. The use of a non-standard version of Borel-Cantelli lemma helped in the task of finding conditions for the processes to die out. Gallo et al [5] based the proofs of their results on a clever relationship between the rumor processes and a specific discrete time renewal process. With this technique they were able to obtain more precise results for homogeneous versions of the processes.

Consider $\{u_i\}_{i \in \mathbb{N}}$ a set of vertices of $\mathbb{N}$ such that $0 < u_1 < u_2 < \cdots$ and a set of independent random variables $\{R_i\}_{i \in \mathbb{N}}$ assuming values in $\mathbb{Z}_+$.  

4.1. Fireworks. At time 0, information travels a distance $R_0$ towards the right side of the origin, in such a way that all vertices $u_i \leq R_0$ get
informed. In general, at every discrete time $t$ a vertex $u_j$ informed at time $t-1$ passes the information on (within $R_j$, its radius of influence) and they do this just once, informing the vertices $u_i$ (only those vertices which have not been informed before) $u_j < u_i \leq u_j + R_j$. Observe that, except for the set of vertices $\{u_i\}$, all other vertices are nonactionable, meaning that their radius of influence equals 0 almost surely.

4.1.1. Homogeneous Fireworks. Consider all the $R_i \sim R$ (having the same distribution) and $u_i = i$ for all $i$.

**Theorem 4.1** (Junior et al [9]). Consider in the Homogeneous Fireworks Process

$$a_n = \prod_{i=0}^{n} \mathbb{P}(R \leq i).$$

Then

$$\sum_{n=1}^{\infty} a_n = \infty \text{ if and only if } \mathbb{P}[V] = 0.$$

**Theorem 4.2** (Gallo et al [5]). For the Homogeneous Fireworks Process,

$$\mathbb{P}(V) = \left[1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} \mathbb{P}(R \leq i) \right]^{-1}.$$

Observe that the result presented in Theorem 4.1 is nicely generalized in Theorem 4.2.

**Example 4.3.** Consider the Homogeneous Fireworks Process such that

$$\mathbb{P}(R = k) = \frac{2}{(k + 2)(k + 3)} \text{ for } k \in \mathbb{N}^*.$$

Then $\mathbb{P}[V] = \frac{1}{2}$. 
Corollary 4.4 (Junior et al [9]). For the Homogeneous Fireworks Process, consider
\[ L = \lim_{n \to \infty} n\mathbb{P}(R \geq n). \]
We have that
(i) If \( L > 1 \) then \( \mathbb{P}[V] > 0 \).
(ii) If \( L < 1 \) then \( \mathbb{P}[V] = 0 \).
(iii) If \( L = 1 \) and there exists \( N \) such that for all \( n \geq N \)
\[ \mathbb{P}(R \geq n) \leq \frac{1}{n-1}, \] then \( \mathbb{P}[V] = 0 \).

Let \( M \) be the final number of spreaders.

Theorem 4.5 (Gallo et al [5]). If \( \mathbb{E}(R) < \infty \) then the random variable \( M \) has finite expectation. Besides, \( M \) has exponential tail distribution when \( \mathbb{P}(R \leq n) \) increases exponentially fast to 1.

Under more specific assumptions, it is possible to obtain more precise information on the tail distribution. Items (i) and (iii) of next proposition follows from Proposition B.2 of Gallo et al. [6], item (ii) is due to Remark 5 from Bressaud et al. [4] and item (iv) follows from Theorem 1.1 of Garsia and Lamperti [7].

Proposition 4.6 (Gallo et al [5]). We have the following explicit bounds for the tail distributions.

(i) If \( \mathbb{P}(R > k) \leq C_r r^k, k \geq 1, \) for some \( r \in (0, 1) \) and a constant \( C_r \in (0, \log \frac{1}{r}) \) then
\[ \mathbb{P}(M \geq k) \leq \frac{1}{C_r}(e^{C_r})^k. \]
(ii) If \( \mathbb{P}(R > k) \sim \log(k)^{\beta} k^{-\alpha}, \beta \in \mathbb{R}, \alpha > 1 \), then there exists \( C > 0 \) such that, for large \( k \)'s, we have \( \mathbb{P}(M \geq k) \leq C(\log k)^{\beta} k^{-\alpha} \).
(iii) If \( P(R > k) = \frac{r}{k}, k \geq 1 \) where \( r \in (0,1) \), there exists \( C > 0 \) such that, for large \( k \), we have
\[
P(M \geq k) \leq C \frac{(\ln k)^{3+r}}{(k)^{2-(1+r)^2}}.
\]
(iv) If \( P(R > k) \sim (\frac{k+1}{k+2})^\alpha \), \( \alpha \in (1/2, 1) \), then there exists \( C = C(\alpha) > 0 \) such that, for large \( k \), we have
\[
P(M \geq k) \leq \frac{C}{k^{1-\alpha}}.
\]

4.1.2. Heterogeneous Fireworks.

Remark 4.7. Consider the Heterogeneous Fireworks Process. One can get a sufficient condition for \( P[V] = 0 \) (\( P[V] > 0 \)) by a coupling argument. Consider \( P(R_i \geq k) \leq P(R \geq k) \) \((P(R_i \geq k) \geq P(R \geq k))\) for some random variable \( R \) whose distribution \( P \) satisfies \( \lim_{n \to \infty} n P(R \geq n) < 1 \) \((\lim_{n \to \infty} n P(R \geq n) > 1) \). Finally use item (ii) (item (i)) of Corollary 4.4.

Theorem 4.8 (Junior et al [9]). Consider a Heterogeneous Fireworks Process for which actionable vertices are at integer positions \( u_0 = 0 < u_1 < u_2 < \ldots \) such that \( u_{n+1} - u_n \leq m \), for \( m \geq 1 \). Besides, let us assume \( P(R_n < m) \in (0,1) \) for all \( n \). Then

(i) If \( \sum_{n=0}^{\infty} [P(R_n < tm)]^t < \infty \) for some \( t \geq 1 \) then \( P[V] > 0 \).

(ii) If for some random variable \( R \), with distribution \( P \), the following conditions hold
\[
\bullet \ P(R \geq k) - P(R_n \geq k) \leq b_k \text{ for all } k \geq 0 \text{ and all } n \geq 0,
\]
\[
\bullet \ \lim_{n \to \infty} n[P(R \geq n) - b_n] > m,
\]
\[
\bullet \ \lim_{n \to \infty} b_n = 0,
\]
then \( P[V] > 0 \).

(iii) \( P(V) \geq \prod_{j=0}^{\infty} \left[ 1 - \prod_{i=0}^{j} P(R_{j-i} < (i+1)m) \right] \).
4.2. **Reverse Fireworks.** At time 0, only the origin has the information. At time 1, individuals placed at vertices \( u_i \) such that \( u_i \leq R_i \) get the information from the origin. At time \( t \in \mathbb{N} \) the set of vertices \( u_j \) which can find an informed individual at time \( t - 1 \) within a distance \( R_j \) to its left, get the information. Let us call this set \( A_t \). If for some \( t \), \( A_t \) is empty the process stops. If the process never stops we say it survives.

Let \( S \) be the event “the reverse process survives”. Besides, we denote by \( Z \) the final number of spreaders.

4.2.1. **Homogeneous Reverse Fireworks.** Consider all the \( R_i \) having the same distribution and \( u_i = i \) for all \( i \).

**Theorem 4.9** (Junior et al [9]). Consider the Homogeneous Reverse Fireworks Process. We have that

(i) If \( E(R) = \infty \) then \( P(S) = 1 \).

(ii) If \( E(R) < \infty \) then \( P(S) = 0 \).

**Theorem 4.10** (Gallo et al [5]). Consider the Homogeneous Reverse Fireworks Process. If \( E(R) < \infty \) then \( Z \sim G \left( \prod_{k=0}^{\infty} P(R \leq k) \right) \) in the sense that for \( p = \prod_{k=0}^{\infty} P(R \leq k) \) we have

\[
P(Z = k) = p(1 - p)^k \text{ for all } k.
\]

For any \( n \geq 1 \), let \( Z(n) \) be the number of spreaders in \( \{1, \ldots, n\} \). We will now state limit theorems for the proportion of spreaders within \( \{1, \ldots, n\} \), \( Z(n)/n \), when \( n \) tends to \( \infty \).
Let

\[ \mu := 1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} \mathbb{P}(R \leq i) \] and

\[ \sigma^2 := \sum_{k=1}^{\infty} k^2 \mathbb{P}(R > k - 1) \prod_{i=0}^{k-2} \mathbb{P}(R \leq i) - \mu^2. \]

Notice that \( \mu < \infty \) implies that \( \prod_{k=0}^{\infty} \mathbb{P}(R \leq k) = 0 \) (this implies \( \mathbb{E}(R) = \infty \)).

**Theorem 4.11** (Gallo et al [5]). If \( \mu < \infty \) then

\[ \frac{Z(n)}{n} \xrightarrow{a.s.} \mu^{-1}, \]

and thus, with probability one, \( \mu^{-1} \) is the final proportion of spreaders.

Moreover, if \( \sigma^2 \in (0, \infty) \), then

\[ \sqrt{n} \left( \frac{Z(n)}{n} - \mu^{-1} \right) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\sigma^2}{\mu^3} \right). \]

Otherwise, \( Z(n)/n \to 0 \).

In particular, observe that if the \( \mathbb{P}(R \leq k) \)'s satisfy at the same time

\( \prod_{k=0}^{\infty} \mathbb{P}(R \leq k) = 0 \) and \( \mu = \infty \) (for instance, if they are as in items (iii) and (iv) of Proposition 4.6), then the information reaches infinitely many individuals, but the final proportion of informed individuals is zero.

**4.2.2. Heterogeneous Reverse Fireworks.**

**Theorem 4.12** (Junior et al [9]). Consider the Heterogeneous Reverse Fireworks Process. It holds that

(i) \( \sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty \) for all \( n \) if and only if \( \mathbb{P}(S) = 1 \).

(ii) If \( \sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k) < \infty \) then \( \mathbb{P}(S) > 0 \).
Remark 4.13. By a coupling argument and Theorem 4.9 one can see that if there is a random variable $R$, whose distribution is $P$, with $\mathbb{E}[R] < \infty$ ($\mathbb{E}[R] = \infty$), such that $\mathbb{P}(R_n \geq k) \leq \mathbb{P}(R \geq k)$ ($\mathbb{P}(R_n \geq k) \geq \mathbb{P}(R \geq k)$) for all $k$ then $\mathbb{P}(S) = 0$ ($\mathbb{P}(S) = 1$).

Example 4.14. It is possible to have in the Heterogeneous Fireworks Process the expectation of the radius of influence infinite for all vertices altogether and the process dies out almost surely.

Let \( \{b_n\}_{n \in \mathbb{N}} \) be a non-increasing sequence convergent to 0 and such that $b_0 < 1$.

(i) $\mathbb{P}(R_n = 0) = 1 - b_n$ and $\mathbb{P}(R_n = k) = b_{n+k-1} - b_{n+k}$ for $k \geq 1$.

(ii) $\sum_{n=0}^{\infty} b_n = \infty$.

(iii) $\lim_{n \to \infty} nb_n = 0$.

Observe that $\mathbb{E}(R_n) = \infty$ for all $n$ from (ii). Besides $\mathbb{P}[V] = 0$ from (iii), because for

\[ V_n = \{ \text{The individual at vertex } u_n \text{ gets the information} \}, \]

\[ \mathbb{P}(V_n) \leq \sum_{k=0}^{n-1} \mathbb{P}(R_k \geq n - k) = \sum_{k=0}^{n-1} b_{n-1} = (n - 1)b_n. \] (4.1)

and the fact that $V = \lim_{n \to \infty} V_n$.

Example 4.15. It is possible to have in the Heterogeneous Fireworks Process the expectation of the radius of influence finite for all vertices and the process survives with positive probability. Assume that $\sum_{n=0}^{\infty} b_n < \infty$, while

(i) $\mathbb{P}(R_n = 0) = b_n$

(ii) $\mathbb{P}(R_n = 1) = 1 - b_n$

Then $\mathbb{E}(R_n) < 1$ for all $n$ and $\mathbb{P}(V) > 0$ by item (i) of Theorem 4.8 with $m = t = 1$. 
Example 4.16. Next we present an example where $\mathbb{P}[S] = 1$ for a Heterogeneous Reverse Fireworks Process while $\mathbb{P}[V] = 0$ for a Heterogeneous Fireworks Process. For this aim consider

(i) $\mathbb{P}(R_n = 0) = 1 - b_n$ and $\mathbb{P}(R_n = n) = b_n$.
(ii) $\sum_{n=0}^{\infty} b_n = \infty$.
(iii) $\lim_{n \to \infty} nb_n = 0$.

Observe that even though $\lim_{n \to \infty} \mathbb{E}[R_n] = 0$ and $\lim_{n \to \infty} \mathbb{P}(R_n = 0) = 1$, from Theorem 4.12 and (ii) it is true for the Heterogeneous Reverse Fireworks Process that $\mathbb{P}(S) = 1$. In the opposite direction, by (4.1) and (iii) one have that $\mathbb{P}[V] = 0$ for the Heterogeneous Fireworks Process.

5. Cone percolation on $\mathbb{T}_d$

Junior et al [10] consider a process which allows us to associate the dynamic activation on the set of vertices to a discrete rumor process. Individuals become spreaders as soon as they hear the rumor. Next time, they propagate the rumor within their radius of influence and immediately become stiflers. Junior et al [10] establish whether the process has positive probability of involving an infinite set of individuals. Besides, they present sharp lower and upper bounds for the probability of that event, depending on the general distribution of the random variables that define the radius of influence of each individual. Their proofs are based on comparisons with branching processes.

Pick a $v \in \mathcal{V}(\mathbb{T}_d)$ such that $d(\mathcal{O}, v) = 1$ and consider $\mathbb{T}_d^+ = \mathbb{T}_d \setminus \mathbb{T}_d^+(v)$. Consider $\mathbb{P}_+$ and $\mathbb{P}$ the probability measures associated to the processes on $\mathbb{T}_d^+$ and $\mathbb{T}_d$ (we do not mention the random variable $R$ unless absolutely necessary). By a coupling argument one can see that for a fixed
distribution of $R$

$$\mathbb{P}_{+}[V] \leq \mathbb{P}[V].$$

Furthermore, by the definition of $T^+_d$ and its relation with $T_d$ we have that for a fixed distribution of $R$

$$\mathbb{P}_{+}[V] = 0 \text{ if and only if } \mathbb{P}[V] = 0.$$ 

Let $p_0 = \mathbb{P}(R = 0)$.

**Theorem 5.1** (Junior et al [10]). Consider the Cone Percolation Model on $T^+_d$ with radius of influence $R$.

(i) If $(1 - p_0)d > 1$, then $\mathbb{P}_{+}[V] > 0$,

(ii) If $(1 - p_0)d \leq 1$ and $E(d^R) > 1 + p_0$, then $\mathbb{P}_{+}[V] > 0$,

(iii) If $E(d^R) \leq 2 - \frac{1}{d}$, then $\mathbb{P}_{+}[V] = 0$.

**Theorem 5.2** (Junior et al [11]). Consider a Cone Percolation Model on $T_d$. Then for $E(d^R) < 2 - \frac{1}{d}$, we have

$$\frac{d + E(d^R) - p_0}{d[1 - E(d^R) + p_0]} \leq E(|I|) \leq \frac{E(d^R) + d - 2}{2d - 1 - dE(d^R)}.$$ 

**Example 5.3** (Junior et al [11]). Consider $R \sim G(1 - p)$, a radius of influence satisfying

$$\mathbb{P}(R = k) = (1 - p)p^k, k = 0, 1, 2, \ldots$$

and assume also $pd < \frac{1}{2}$. So we have

$$\frac{1 - dp + p - p^2}{1 - 2dp + dp^2} \leq E(|I|) \leq \frac{1 - dp - p}{1 - 2dp}.$$ 

That gives us a fairly sharp bound even when we pick $p$ and $d$ such that $pd$ is very close to $\frac{1}{2}$ as, for example, $p = 10^{-6}$ and $d = 499,000$. For these parameters we get $250.438 \leq E(|I|) \leq 250.501$. 

Let $\rho$ and $\psi$ be, respectively, the smallest non-negative roots of the equations
\[
\begin{align*}
E(\rho^{d^R}) + (1 - \rho)p_0 &= \rho, \\
E(\psi^{\frac{d}{d-R}(d^R-1)}) &= \psi.
\end{align*}
\]

**Theorem 5.4** (Junior et al [10]). Consider the Cone Percolation Model on $\mathbb{T}_d^+$. Then
\[
1 - \rho \leq \mathbb{P}_+(V) \leq 1 - \psi.
\]

**Theorem 5.5** (Junior et al [10]). For the Cone Percolation Model on $\mathbb{T}_d$ with radius of influence $R$, it holds that
\[
1 - \left(1 - \rho^{\frac{d+1}{d}}\right)p_0 - \mathbb{E}\left(\rho^{\frac{(d+1)}{d}d^R}\right) \leq \mathbb{P}[V] \leq 1 - \mathbb{E}\left(\psi^{\frac{(d+1)}{d}(d^R-1)}\right).
\]

Consider $d = 2$ and $R$ following a Binomial distribution with parameters $4$ and $\frac{1}{2}$ ($R \sim \mathcal{B}(4, \frac{1}{2})$). Therefore $\rho$ and $\psi$ are, respectively, solutions of
\[
\begin{align*}
x^{16} + 4x^8 + 6x^4 + 4x^2 - 16x + 1 &= 0, \\
x^{30} + 4x^{14} + 6x^6 + 4x^2 - 16x + 1 &= 0.
\end{align*}
\]
So $\rho = 0.0635146$ and $\psi = 0.06350850$, which implies that
\[0.937435919 \leq \mathbb{P}[V] \leq 0.937435962.\]

**Theorem 5.6** (Junior et al [10]). The Heterogeneous Cone Percolation Process in $\mathbb{T}_d^+$ has a giant component with positive probability if for some fixed $n$,
\[
\liminf_{j \to \infty} d^n \prod_{k=0}^{n-1} \left[1 - \prod_{i=0}^k \mathbb{P}_+[R_{jn+i} < k + 1 - i]\right] > 1. \tag{5.1}
\]

A consequence of Theorem 5.4 from Bertacchi and Zucca [3] is the following result.
Corollary 5.7. Consider a Homogeneous Reverse Fireworks Process on $\mathbb{T}_d$. Then
\[
\mathbb{P}(S) = 1 \text{ if and only if } \sum_{n=1}^{\infty} d^n \mathbb{P}(R \geq n) = \infty.
\]
\[
\mathbb{P}(S) = 0 \text{ if and only if } \sum_{n=1}^{\infty} d^n \mathbb{P}(R \geq n) \prod_{j=1}^{n-1} [1 - \mathbb{P}(R \geq j)] \leq 1.
\]

Theorem 5.8 (Junior et al [11]). For a Cone Percolation Model in $\mathbb{T}_S$ and $R$, the radius of influence, $\mathbb{P}(V) > 0$ if
\[
\lim_{n \to \infty} \sqrt[n]{\rho_n} > e^{-\dim \inf \partial \mathbb{T}_S}
\]
where
\[
\rho_n := \prod_{k=0}^{n-1} \left[1 - \prod_{i=0}^{k} \mathbb{P}(R < i + 1)\right].
\]

Corollary 5.9 (Junior et al [11]). For a Cone Percolation Model in $\mathbb{T}_S$ and $R$, a radius of influence satisfying $\mathbb{P}(R \leq k) = 1$ for some $k \in \mathbb{N}$, $\mathbb{P}(V) > 0$ if
\[
\dim \inf \partial \mathbb{T}_S > \ln \left[\frac{1}{1 - \prod_{j=1}^{k} \mathbb{P}(R < j)}\right].
\]

Definition 5.10. A $k$-periodic tree with degree $\tilde{d} = (d_1, \cdots, d_k)$, $d_i \geq 2$ for all $i = 1, 2, \cdots, k$, is as tree such that for any vertex whose distance to the origin is $nk + i - 1$ for some $n \in \mathbb{N}$ has degree $d_i + 1$. We refer to this tree as $\mathbb{T}_{\tilde{d}}$.

Example 5.11 (Junior et al [11]). Consider a Cone Percolation Model in $\mathbb{T}_S$ with $R \sim \mathcal{B}(p)$, a radius of influence satisfying
\[
\mathbb{P}(R = 1) = p = 1 - \mathbb{P}(R = 0).
\]
(i) If $\dim \inf \partial \mathbb{T}_S > -\ln p$ then $\mathbb{P}(v) > 0$,
(ii) If $T_S = T_\tilde{d}$ and $\sqrt[\kappa]{\prod_{j=1}^{\kappa} d_j} > \frac{1}{p}$ then $\mathbb{P}(V) > 0$.

6. Random environments

In this section we review the Fireworks and the Reverse Fireworks processes, with a random number of stations at each vertex. Bertacchi and Zucca [3] consider an extra source of randomness: the number of individuals sitting on each vertex. They consider two families of random variables \( \{N_x\}_{x \in \mathcal{G}} \) and \( \{R_{x,i}\}_{i \in \mathbb{N}, x \in \mathcal{G}} \) such that \( \{N_x, R_{x,i}\} \) are independent and \( \{R_{x,i}\}_{i \in \mathbb{N}} \) are identically distributed for all \( x \in \mathcal{G} \) that is \( R_{x,i} \sim R_x \). In their paper \( N_x \) represents the random number of individuals at vertex \( x \) (in particular \( N_\varnothing \) is the number of individuals at the origin) while \( \{R_{x,i}\}_{i=1}^{N_x} \) are their radius of influence. The main question about this model is to understand under which conditions, the signal, starting from one vertex of a graph (\( \mathbb{N} \) or a Galton-Watson tree), will spread indefinitely with positive probability or die out almost surely in a finite number of steps.

Bertacchi and Zucca [3] rely in their analysis on associating the processes with random numbers of stations (fireworks or reverse fireworks), with processes with one station per vertex as in Junior et al [9]. Indeed, they consider processes with one station on each vertex \( x \) and radius of influence \( \tilde{R}_x = 1_{\{N_x \geq 1\}} \max\{R_{x,j} : j = 1, \ldots, N_x\} \). They call this process, the deterministic counterpart or annealed counterpart of the original process. They observe that the annealed counterpart does not retain any information about the environment, nevertheless the probability of survival for the original process and for its annealed counterpart are the same.
6.1. **Fireworks.** For \( x \in G \), let us define

\[
\phi_{N_x}(t) := \mathbb{E}(t^{N_x}) = \sum_{j=0}^{\infty} \mathbb{P}(N_x = j) t^j
\]

6.1.1. **Homogeneous Fireworks.** Consider \( R_i \sim R \) and \( N_x \sim N \) for all \( x \in G \). Let us define

\[
f_{R,N}(n) := n\{1 - \phi_N(\mathbb{P}(R < n))\}.
\]

**Theorem 6.1** (Bertacchi and Zucca [3]).

(i) If \( \limsup_{n \to \infty} f_{R,N}(n) < 1 \) then \( \mathbb{P}(V) = 0 \).

(ii) If \( \liminf_{n \to \infty} f_{R,N}(n) > 1 \) then \( \mathbb{P}(V) > 0 \).

(iii) If \( \mathbb{E}(N) < \infty \) and \( \limsup_{n \to \infty} n\mathbb{P}(R \geq n) < \frac{1}{\mathbb{E}(N)} \) then \( \mathbb{P}(V) = 0 \).

(iv) If \( \mathbb{E}(N) < \infty \) and \( \mathbb{E}(R) < \infty \) then \( \mathbb{P}(V) = 0 \).

(v) If \( \liminf_{n \to \infty} n\mathbb{P}(R \geq n)\phi'_N(\mathbb{P}(R < n)) > 1 \) then \( \mathbb{P}(V) > 0 \).

A consequence of Theorem 1 from Gallo et al [5] is the following result

**Corollary 6.2.**

\[
\mathbb{P}(V) = \left[ 1 + \sum_{j=1}^{\infty} \prod_{i=0}^{j-1} \phi_N(\mathbb{P}(R \leq i)) \right]^{-1}
\]

**Remark 6.3.** It is possible to have \( \mathbb{E}(N) = \infty, \mathbb{E}(R) = \infty \) and \( \mathbb{P}(V) = 0 \). Take \( \mathbb{P}(N \geq n) \sim \frac{1}{n} \) when \( n \to \infty \) and \( \mathbb{P}(R \geq n) = \frac{1}{n \ln n \ln(\ln n)} \)

6.1.2. **Heterogeneous Fireworks.**

**Theorem 6.4** (Bertacchi and Zucca [3]). *In the heterogeneous case, if*

\[
\sum_{n=0}^{\infty} \prod_{i=0}^{n} \phi_{N_i}(\mathbb{P}(R_i < n - i + 1))
\]

*then \( \mathbb{P}(V) > 0 \).
Adapting the proof of Theorem 2.3 from Junior et al. [9] we have

**Theorem 6.5.** In the heterogeneous case, if

(i) \( \varphi_{N_i}(\mathbb{P}(R_i < 1)) \in (0, 1) \).

(ii) \( \lim_{n \to \infty} \prod_{i=0}^{n-1} \varphi_{N_i}(\mathbb{P}(R_i < 2n - 1)) = 1 \).

(iii) \( \lim_{n \to \infty} \prod_{i=n}^{2n-1} \varphi_{N_i}(\mathbb{P}(R_i < 2n - 1)) > 0 \).

then \( \mathbb{P}(V) = 0 \).

6.2. Reverse Fireworks.

6.2.1. **Homogeneous Reverse Fireworks.** Let us define

\[
W = \sum_{n=0}^{\infty} [1 - \varphi_N(\mathbb{P}(R < n))]
\]

**Theorem 6.6** (Bertacchi and Zucca [3]).

(i) If \( W = \infty \) then \( \mathbb{P}(S) = 1 \).

(ii) If \( W < \infty \) then \( \mathbb{P}(S) = 0 \).

Theorem 6.6 can also be obtained as a consequence of Theorem 3.2 from Junior et al. [10] or as a consequence of Theorem 2 from Gallo et al. [5].

**Remark 6.7** (Bertacchi and Zucca [3]). Theorems 6.1 and 6.6 admit a similar corollary

(i) For every unbounded random variable \( R \) there exists a random variable \( N \) such that \( \mathbb{P}(V) > 0 \) \( (\mathbb{P}(S) = 1) \). For \( \epsilon > 0 \) and \( \delta \in (0, 1) \) consider \( N \) satisfying

\[
\mathbb{P}\left( N \geq \frac{\ln(1 - \delta)}{\ln(\mathbb{P}(R < n))} \right) \geq \frac{1 + \epsilon}{n \delta}.
\]
(ii) For every random variable $N$ such that $\mathbb{P}(N = 0) < 1$ there exists a random variable $R$ such that $\mathbb{P}(V) > 0$ ($\mathbb{P}(S) = 1$). Take $R$ satisfying $\mathbb{P}(R \geq n) = p_n$, where $p_n = \inf\{t \geq 0; \varphi_N (1 - t) \leq 1 - \frac{2}{n}\}$.

6.2.2. Heterogeneous Reverse Fireworks.

**Theorem 6.8** (Bertacchi and Zucca [3]). *In the heterogeneous case,*

\[
\sum_{k=0}^{\infty} \left[ 1 - \varphi_{N_{n+k}} (\mathbb{P}(R_{n+k} < k)) \right] = \infty, \text{ if and only if } \mathbb{P}(S) = 1.
\]

*By other hand, if*

\[
\sum_{n=0}^{\infty} \prod_{k=1}^{\infty} \varphi_{N_{n+k}} (\mathbb{P}(R_{n+k} < k)) < \infty, \mathbb{P}(S) > 0.
\]

6.3. Galton Watson. Let us define the space of unlabelled GW-trees (the usual GW-trees). Consider a GW-process, with offspring distribution $\mathbb{P}(D = d), 0 \leq d < \infty$. We assume that $\mathbb{P}(D = 1) < 1$ (otherwise the resulting random tree is $N$) and we suppose that $\mu_D := \sum_{d=0}^{\infty} d \mathbb{P}(D = d) > 1$ (the supercritical case). The underlying random graph will be a GW-tree generated by this process. Let $g(s) := \sum_{d=0}^{\infty} s^d \mathbb{P}(D = d)$ be the generating function of $D$ and let $\pi \in [0, 1]$ be the smallest nonnegative fixed point of $g$. If $\mathbb{P}(D > k) = 0$ for some $k$ we say that the GW-tree has maximum degree $k$ or that it is $k$-bounded.

6.3.1. Homogeneous Fireworks. In this case, the random number of stations are independent and identically distributed $\mathbb{N}$-valued random variables with common law $N$. Analogously, The radii of the stations are independent and identically distributed with distribution $R$ (either discrete or continuous random variable).
Definition 6.9. We define

$$\Phi(t) := \varphi_\infty(\mathbb{P}(R < 1)) + \sum_{n=1}^{\infty} [\varphi_N(\mathbb{P}(R < n + 1)) - \varphi_N(\mathbb{P}(R < n))]t^n. $$

In particular observe that

$$\Phi(0) = \varphi_N(\mathbb{P}(R < 1))$$

and the case $N = 1$ a.s.,

$$\Phi(t) = \sum_{n=0}^{\infty} \mathbb{P}(n \leq R < n + 1)]t^n.$$ 

Theorem 6.10 (Bertacchi and Zucca [3]). Consider a Homogeneous Fireworks Process. We have that

(i) If $\Phi(\mu_D) - 1 > \Phi(0) = \varphi_N(P(R < 1))$ and $\mathbb{P}(N = 0) = 0$ then for the Fireworks process there is survival with positive probability for almost every realization of the environment such that the underlying tree is infinite and there is at least one station at the root.

(ii) If $\Phi(\mu_D) - 1 > \Phi(0) = \varphi_N(P(R < 1))$ and $\mathbb{P}(N = 0) > 0$ then for the Fireworks process $\mathbb{P}(V|\tau = T, N_\mathcal{O} = n) > 0$ for almost every $(T, n)$ such that $T$ is an infinite (unlabelled) tree and $n \geq 1$.

(iii) If the GW-tree is $k$-bounded and $\Phi(k) \leq 2 - \frac{1}{k}$ then the Fireworks process becomes extinct a.s. for almost every realization of the environment.

6.3.2. Homogeneous Reverse Fireworks. In this case, the random number of stations are independent and identically distributed $\mathbb{N}$-valued random variables with common law $N$, except by numbers of station at the root $\mathcal{O}$. For the root, we take $N_\mathcal{O} = \min\{n > 0 : \mathbb{P}(N = n) > 0\}$
Besides, the radii of the stations are independent and identically distributed with distribution $R$ (either discrete or continuous random variable).

**Definition 6.11.** We define

$$\phi_1(t) := \sum_{n=1}^{\infty} [1 - \varphi_N(\mathbb{P}(R < n))] \mu_D^n$$

$$\phi_2(t) := \sum_{n=1}^{\infty} [1 - \varphi_N(\mathbb{P}(R < n))] \mu_D^n \prod_{j=1}^{n-1} \varphi_N(\mathbb{P}(R < j))$$

**Theorem 6.12** (Bertacchi and Zucca [3]). Consider a Homogeneous Reverse Fireworks Process. The following hold

(i) If $\phi_1(\mu_D) = \infty$ then there is survival with probability 1 for the Reverse Fireworks process for almost all realizations of the environment such that the underlying tree is infinite.

(ii) If $\mathbb{P}(N = 0) = 0$, $\phi_1(\mu_D) < \infty$ and $\phi_2(\mu_D) > 1$ then there is survival with positive probability (strictly smaller than 1) for the Reverse Fireworks process for almost all realizations of the environment such that the underlying tree is infinite.

(iii) If $\mathbb{P}(N = 0) > 0$, $\phi_1(\mu_D) < \infty$ and $\phi_2(\mu_D) > 1$ then $\mathbb{P}(S|\tau = T) \in (0, 1)$ for almost every infinite (unlabelled) tree $T$.

(iv) If $\phi_1(\mu_D) < \infty$ and $\phi_2(\mu_D) \leq 1$ then there is a.s. extinction for the Reverse Fireworks process for almost all realizations of the environment.

**Definition 6.13.** We define

$$M_c := \left[ \limsup_{n \to \infty} \mathcal{N} \right]^{-1}.$$ 

**Corollary 6.14** (Bertacchi and Zucca [3]). There exists a critical value $\mu_c \in [1, \infty)$, $\mu_c \leq M_c$ such that
\(i\) \(\mu_D < \mu_c\) implies a.s. extinction for almost all realizations of the environment.

\(ii\) \(\mu_c < \mu_D < \mathcal{M}_c\) and \(\mathbb{P}(N = 0) = 0\) implies survival with positive probability for almost all realizations of the environment such that the underlying tree is infinite.

\(iii\) \(\mu_c < \mu_D < \mathcal{M}_c\) and \(\mathbb{P}(N = 0) > 0\) implies survival with positive probability for almost every infinite (unlabelled) tree.

\(iv\) \(\mathcal{M}_c < \mu_D\) implies survival with probability 1 for almost all realizations of the environment such that the underlying tree is infinite.

\(v\) If \(\mu_D = \mu_c < \mathcal{M}_c\) then there is a.s. extinction for almost all realizations of the environment.

7. Open problems

Some natural extensions for these models are those considering

\(i\) Fireworks processes (direct and reverse) on \(\mathbb{Z}^d\). An especially interesting case is when \(d = 2\) and the boxes of influence are distributed as \([0, R_x) \times [0, R_y)\) with \(R_x\) independent of \(R_y\) and the rumor starting from \((0, 0)\) or from every \((x, y)\) such that \(x = 0\) or \(y = 0\);

\(ii\) Fireworks processes on \(\mathbb{Z}\). Heterogenous versions with radius of influence non i.i.d. and with individuals being initially placed following a renewal process or a Markovian process.

\(iii\) Reverse fireworks processes on \(\mathbb{Z}\). Individuals throw their radius of influence to every direction as in [5] (See Gallo et al. [5]). They believe that conditions for survival will be the same but the final proportion of informed individual will be strictly larger.
(iv) Cone Percolation on Spherically Symmetric and on Galton Watson trees. Lower and upper bounds for the survival probability and for the extinction time.

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