The rate of $L^p$-convergence for the Euler-Maruyama method of the stochastic differential equations with Markovian switching *

Minghui Song∗, Yuhang Zhang, Mingzhu Liu

School of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

Abstract

This work deals with the Euler-Maruyama (EM) scheme for stochastic differential equations with Markovian switching (SDEwMSs). We focus on the $L^p$-convergence rate ($p \geq 2$) of the EM method given in this paper. As far as we know, the skeleton process of the Markov chain is used in the continuous numerical methods in most papers. By contrast, the continuous EM method in this paper is to use the Markov chain directly. To the best of our knowledge, there are only two papers that consider the rate of $L^p$-convergence, which is no more than $1/p$ ($p \geq 2$) in these papers. The contribution of this paper is that the rate of $L^p$-convergence of the EM method can reach $1/2$. We believe that the technique used in this paper to construct the EM method can also be used to construct other methods for SDEwMSs.

Keywords: stochastic differential equations, Markov chain, Euler-Maruyama method, $L^p$-convergence, convergence rate

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1. Introduction

Stochastic differential equations with Markovian switching (SDEwMSs), also known as hybrid stochastic differential equations, play an important role in stochastic theory and have been used in various fields, such as the theory of control and neural networks (1,3). Most of SDEwMSs do not have explicit solutions so it is important to have numerical solutions (4,15). (4) is the first research that developed the Euler-Maruyama (EM) scheme for SDEwMSs with the global Lipschitz continuous coefficients and considered the $L^2$-convergence rate for EM solutions. (6) designed approximation methods of Milstein type for

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*Corresponding author

Email addresses: songmh@hit.edu.cn (Minghui Song), 19b912028@stu.hit.edu.cn (Yuhang Zhang), mzliu@hit.edu.cn (Mingzhu Liu)
SDEwMSs and proved the convergence rate is better than the generally adopted EM procedures.

The primary motivation for this work came from the following observation: to our knowledge, there are plenty of papers on the convergence of numerical algorithms for hybrid systems, most of them showed the convergence (without order) (e.g., [10–16]) or the rate of convergence in the sense of pathwise or mean square (e.g., [4–9, 17, 18]). However, there are only a few papers revealed the $L^p$-convergence order of numerical methods for hybrid systems ([19, 20]). Not only that, the order of $L^p$-convergence for Euler type numerical algorithms proved in these papers are no more than $1/p$ ($p \geq 2$), instead of the well known $1/2$.

To be specific, the main result in [19] (Theorem 3) shows

$$
\mathbb{E} \sup_{t \in [0,T]} |y(t) - x(t)|^p \leq C_5 \Delta(1 + \mathbb{E}|x_0|^p),
$$

where $x(t)$ is the exact solution of the stochastic delay differential equation with phase semi-Markovian switching and Poisson jumps, $y(t)$ denotes the numerical approximation using the continuous $\theta$ method, $\Delta$ is the given step-size, $C_5$ denotes a generic constant that independent of $\Delta$, $x_0$ is the initial data. By analyzing the details in this paper, we find that the problem first appears in the estimations of

$$
\mathbb{E} \int_0^T |f(Z_1(s), Z_1(s-\tau), r_1(s)) - f(Z_1(s), Z_1(s-\tau), r(s))|^p ds,
$$

and

$$
\mathbb{E} \int_0^T |f(Z_2(s), Z_2(s-\tau), r_2(s)) - f(Z_2(s), Z_2(s-\tau), r(s))|^p ds
$$

(Lemma 4 in [19]), we think these two terms can be seen as the errors in approximating $r(s)$ by $r_1(s)$ and $r_2(s)$, where $r(s)$ is the given continuous-time Markov chain. Similar estimations also exist in many works aforementioned, for example, Eq.(3.7) in [4], Lemma 3 in [10], Eq.(3.6) in [11], as well as Corollary 3.1 in [20], etc. Based on this fact, the main idea of this work is to use $r(s)$ itself to construct a numerical scheme, rather than its approximation. Therefore, the innovations of this paper are as follows:

- We will use the continuous-time Markov chain itself to develop the numerical scheme, instead of its approximation.
- The order of $L^p$-convergence for the EM method given in this work to SDEwMSs can reach $1/2$.

The rest of the paper is arranged as follows. In Section 2, we present some notations and fundamental assumptions, moreover, we further introduce the classical EM method for SDEwMSs which is often used in literatures. Then we develop a different EM scheme in Section 3. The rate of $L^p$-convergence for the EM method be proved in Section 4. Finally, we give the conclusion of this paper in Section 5.
2. Notations and preliminaries

In the rest of this work, except as otherwise noted, we let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) which satisfies the general conditions (namely, it is right continuous and \(\mathcal{F}_0\) involves all \(\mathbb{P}\)-null sets). The transpose of \(A\) is denoted by \(A^T\) when \(A\) is a vector or matrix. \(B(t) = (B_1(t), \ldots, B_d(t))^T\) represents a \(d\)-dimensional Brownian motion defined on the \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). If \(x\) is a vector, \(|x|\) denotes its Euclidean norm.

\[
|A| = \sqrt{\text{trace}(A^TA)}
\]
denotes the trace norm of a matrix \(A\). If \(u\) and \(v\) are two real numbers, let \(u \lor v\) and \(u \land v\) be \(\max\{u, v\}\) and \(\min\{u, v\}\), respectively. Let \(L^p([a,b]; \mathbb{R}^n)\) be the family of \(\mathbb{R}^n\)-valued processes \(\{f(t)\}_{a \leq t \leq b}\) which satisfies \(\mathcal{F}_t\)-adapted and \(\int_a^b |f(t)|^p dt < \infty, \text{a.s.}\). Let \(L^p(\mathbb{R}_+; \mathbb{R}^n)\) denotes the family of processes \(\{f(t)\}_{t \geq 0}\) such that \(\{f(t)\}_{0 \leq t \leq T} \in L^p([0,T]; \mathbb{R}^n)\) for any \(T > 0\). In this paper, we use \(C\) represents a common positive number independent of \(\Delta\), its value may vary with each appearance.

Suppose \(\alpha(t), t \geq 0\), is a right-continuous Markov chain taking values in \(S = \{1, 2, \ldots, N\}\) with generator \(\Gamma = (\gamma_{ij})_{N \times N}\), \(\gamma_{ij} \geq 0\) denotes the transition rate from \(i\) to \(j\) when \(i \neq j\), and \(\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}\).

Assuming that \(\alpha(\cdot)\) is independent of the Brownian motion \(B(\cdot)\).

Let \(T > 0\), consider the following SDEwMS

\[
\begin{aligned}
\text{d}z(t) &= f(z(t), \alpha(t))\text{d}t + g(z(t), \alpha(t))\text{d}B(t), \\
z(0) &= z_0 \in \mathbb{R}^n, \alpha(0) = i_0 \in S,
\end{aligned}
\]

on \(t \in [0,T]\), where

\(f : \mathbb{R}^n \times S \to \mathbb{R}^n\) and \(g : \mathbb{R}^n \times S \to \mathbb{R}^{n \times d}\).

We impose the following conditions:

**Assumption 2.1.** There is a number \(L > 0\) such that

\[
|f(z, i) - f(\bar{z}, i)| \lor |g(z, i) - g(\bar{z}, i)| \leq L|z - \bar{z}|,
\]

for all \(i \in S\) and \(z, \bar{z} \in \mathbb{R}^n\).

**Assumption 2.2.** There is a number \(K > 0\) such that

\[
|f(0, i)| \lor |g(0, i)| \leq K, \quad \forall i \in S.
\]

**Remark 1.** By Assumptions 2.1 and 2.2 we can easily arrive at

\[
|f(z, i)| \lor |g(z, i)| \leq (K + L)(1 + |z|),
\]

for all \(i \in S\) and \(z \in \mathbb{R}^n\).
Lemma 2.3 (Lemma 2.2 in [4]). If Assumptions 2.1 and 2.2 hold, then for arbitrary $p \geq 2$, there is a unique solution for Eq. (2) with the initial data $z_0$. In addition, the solution satisfies
\[
E \left( \sup_{0 \leq t \leq T} |z(t)|^p \right) \leq C.
\]

In the following, we will first introduce the classical methods for simulating discrete-time Markov chain that have been used in many papers, and further present the well known EM method. In the next section, we will introduce the method used to simulate Markov chain in this paper, and further construct another type of EM method for SDEwMS which is different from the one given in the Ref. [4].

2.1. The classical EM method

The most commonly used method to generate the discrete Markov chain $\{\alpha^\Delta_k, k = 0, 1, 2, \ldots \}$ is based on the properties of embedded discrete Markov chain: For any given step size $\Delta > 0$, let $\alpha^\Delta_k = \alpha(k\Delta)$ for $k \geq 0$. Then $\{\alpha^\Delta_k\}$ is a discrete Markov chain with the one-step transition probability matrix
\[
P(\Delta) = (P_{ij}(\Delta))_{N \times N} = e^{\Gamma \Delta}.
\]
Hence, the discrete Markov chain $\{\alpha^\Delta_k, k = 0, 1, 2, \ldots \}$ can be generated as follows: Let $\alpha^\Delta_0 = i_0$ and compute a pseudo-random number $\zeta_1$ from the uniform $[0, 1]$ distribution. Define
\[
\alpha^\Delta_{k+1} = \begin{cases} i_1, & \text{if } i_1 \in S - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} P_{i_0,j}(\Delta) \leq \zeta_1 < \sum_{j=1}^{i_1} P_{i_0,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{i_0,j}(\Delta) \leq \zeta_1, \end{cases}
\]
where we set $\sum_{j=1}^{0} P_{i_0,j}(\Delta) = 0$ as usual. Generally, having calculated $\alpha^\Delta_0, \alpha^\Delta_1, \ldots, \alpha^\Delta_k$, we compute $\alpha^\Delta_{k+1}$ by drawing a uniform $[0, 1]$ pseudo-random number $\zeta_{k+1}$ and setting
\[
\alpha^\Delta_{k+1} = \begin{cases} i_{k+1}, & \text{if } i_{k+1} \in S - \{N\} \text{ such that } \\
\sum_{j=1}^{i_{k+1}-1} P_{\alpha^\Delta_k,j}(\Delta) \leq \zeta_{k+1} < \sum_{j=1}^{i_{k+1}} P_{\alpha^\Delta_k,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} P_{\alpha^\Delta_k,j}(\Delta) \leq \zeta_{k+1}. \end{cases}
\]
After explaining how to get the Markov chain $\{\alpha^\Delta_k, k = 0, 1, 2, \ldots \}$, we can now give the classical EM method for the SDEwMS (2). Given a step size $\Delta > 0$, let $t_k = k\Delta$ for $k \geq 0$, setting $X_0 = z_0, \alpha^\Delta_0 = i_0$ and forming
\[
X_{k+1} = X_k + f(X_k, \alpha^\Delta_k) \Delta + g(X_k, \alpha^\Delta_k) \Delta B_k, 
\]
where $\Delta B_k = B(t_{k+1}) - B(t_k)$, $X_k$ is the approximation of $z(t_k)$. Let
\[
\bar{X}(t) = X_k, \quad \bar{\alpha}(t) = \alpha^\Delta_k \quad \text{for } t \in [t_k, t_{k+1}),
\]
and the continuous EM method is defined by

\[ X(t) = X_0 + \int_0^t f(\bar{X}(s), \bar{\alpha}(s))ds + \int_0^t g(\bar{X}(s), \bar{\alpha}(s))dB(s). \]  

(4)

It can be verified that \( X(t_k) = \bar{X}(t_k) = X_k \).

**Remark 2.** As we said in the Section 1, there are only a few papers that estimates the error between the numerical approximation and the exact solution for hybrid systems in the sense of \( p \)-th moment. Inequality (1) is equivalent to

\[ \left( \mathbb{E} \sup_{t \in [0,T]} |y(t) - x(t)|^p \right)^{1/p} \leq C \Delta^{1/p}, \]

where \( C = (C_5(1 + \mathbb{E}|x_0|^p))^{1/p} \), this implies the \( L^p \)-convergence order for the \( \theta \) method to the hybrid system is \( 1/p \), instead of \( 1/2 \), which is the convergence order of \( \theta \) method for stochastic systems without Markov chain ([21]). Main result in [20] is similar to the Theorem 3 in [19].

In the next section, we will give a different EM scheme using another method to formulate the Markov chain \( \alpha(t) \), and we will prove that the EM method given in this paper will converge to Eq. (2) in the sense of \( L^p \) (\( p \geq 2 \)) with the order \( 1/2 \).

### 3. Euler-Maruyama method

For the generation of the Markov chain \( \alpha(t) \), we cite the methodology of formulating the Markov chain from Section 2.4 in [22]. In order to get the sample paths of \( \alpha(t) \), we need to determine the time of residence in each state and the succeeding actions. The chain remains at any given state \( i_0 (i_0 \in S) \) for a random length of time, \( \tau_1 \), which follows an exponential distribution with parameter \(-\gamma_{i_0i_0}\), hence \( \tau_1 \) can be obtained by

\[ \tau_1 = \frac{\log(1 - \zeta_1)}{-\gamma_{i_0i_0}}, \]

where \( \zeta_1 \) is a random variable uniformly distributed in \((0, 1)\). Then, the process will enter another state. In addition, the probability that state \( j \) (with \( j \in S, j \neq i_0 \)) becomes the next residence of the chain is \( \gamma_{i_0j}/(-\gamma_{i_0i_0}) \). The position after the jump is determined by a discrete random variable \( i_1 (i_1 \in S \setminus \{i_0\}) \), namely \( \alpha(\tau_1) = i_1 \). The value of \( i_1 \) is given by

\[
\begin{cases}
1, & \text{if } \xi_1 < \gamma_{i_01}/(-\gamma_{i_0i_0}), \\
2, & \text{if } \gamma_{i_01}/(-\gamma_{i_0i_0}) \leq \xi_1 < (\gamma_{i_01} + \gamma_{i_02})/(-\gamma_{i_0i_0}), \\
\vdots & \\
N, & \text{if } \sum_{j \neq i_0, j \leq N-1} \gamma_{i_0j}/(-\gamma_{i_0i_0}) < \xi_1,
\end{cases}
\]
where $\xi_1$ is a random variable uniformly distributed in $(0,1)$.

The chain remains at state $i_1$ for a random length of time, $\tau_2$, which follows an exponential distribution with parameter $-\gamma_{i_1}i_1$, thus

$$\tau_2 = \frac{\log(1-\xi_2)}{\gamma_{i_1}i_1},$$

where $\xi_2$ is also a random variable uniformly distributed in $(0,1)$. Then, the process will enter another state. The post-jump location is identified by a discrete random variable $i_2(i_2 \in S \setminus \{i_1\})$, which implies $\alpha(\tau_1 + \tau_2) = i_2$. The value of $i_2$ is determined by

$$i_2 = \begin{cases} 1, & \text{if } \xi_2 < \gamma_{i_1}i_1/(-\gamma_{i_1}i_1), \\ 2, & \text{if } \gamma_{i_1}/(-\gamma_{i_1}i_1) \leq \xi_2 < (\gamma_{i_1} + \gamma_{i_2})/(-\gamma_{i_1}i_1), \\ \vdots & \vdots \\ N, & \text{if } \sum_{j \neq i_1,j \leq N-1} \gamma_{i_1j}/(-\gamma_{i_1}i_1) < \xi_2, \end{cases}$$

where $\xi_2$ is a random variable uniformly distributed in $(0,1)$. Therefore, repeating the procedure above, the sampling path of $\alpha(t)$, $t \geq 0$ is composed of exponential random variables and $U(0,1)$ random variables alternately.

Recall that nearly all sample paths of $\alpha(\cdot)$ are right-continuous piecewise constant function with finite sample jumps in $[0,T]$. Thus, there are stopping times $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_N = T$ ($N \in \mathbb{N}_+$), where $\tau_k = \sum_{j=1}^k \tau_j$, $k = 1, 2, \ldots, N - 1$, such that

$$\alpha(t) = \sum_{k=0}^{N-1} i_k I_{[\tau_k,\tau_{k+1})}(t).$$

Now we are in a position to define the EM method to SDEwMS (2). Given a step size $\Delta > 0$, let $t_k = k\Delta$ ($k \in \mathbb{N}$) be the gridpoints.

Define

$$J_i = \begin{cases} 0, & \text{if } i = 0, \\ \inf\{t \in (J_{i-1},T] \mid \alpha(t) \neq \alpha(t^-)\} \land \inf\{t \in (J_{i-1},T] \mid t = t_k, k \in \mathbb{N}\}, & \text{if } i \geq 1. \end{cases}$$

According to the definition of $J_i$, it is easy to know that

$$\hat{N} := \#\{J_i, i = 0, 1, \ldots\} \leq \lfloor T/\Delta \rfloor + \hat{N} + 1,$$

where $\#\{J_i\}$ denotes the number of elements in set $\{J_i\}$. Then we define the EM method to (2) of the following type by setting $Z_0 = z(0) = z_0$,

$$Z_{k+1} = Z_k + \sum_{i=0}^{\hat{N}-1} f(Z_k, \alpha(J_i)) I_{[t_k,t_{k+1})}(J_i) \Delta J_i + \sum_{i=0}^{\hat{N}-1} g(Z_k, \alpha(J_i)) I_{[t_k,t_{k+1})}(J_i) \Delta B_i,$$

(5)
for \( k \in \mathbb{N} \), where \( \Delta J_i = J_{i+1} - J_i \), \( \Delta B_{J_i} = B(J_{i+1}) - B(J_i) \). \( Z_k \) is the approximate value of \( z(t_k) \). Let

\[
Z(t) = \sum_{k=0}^{\infty} Z_k I_{[t_k, t_{k+1})}(t),
\]

the continuous EM method is given by

\[
Z(t) = Z_0 + \int_0^t f(\bar{Z}(s), \alpha(s))ds + \int_0^t g(\bar{Z}(s), \alpha(s))dB(s). \tag{6}
\]

It can be verified that \( Z(t_k) = \bar{Z}(t_k) = Z_k \), \( \forall k \geq 0 \).

4. Rate of the \( L^p \)-convergence for the EM method

Similar to Lemma 4.1 in \cite{23}, we can easily obtain the following conclusion.

**Lemma 4.1.** Let Assumptions 2.1 and 2.2 hold. Then for any \( \Delta \in (0, 1] \) and \( p \geq 2 \), the EM method \((6)\) has the following property

\[
E \left( \sup_{0 \leq t \leq T} |Z(t)|^p \right) \leq C.
\]

The proof is omitted because it is similar to that for Lemma 4.1 in \cite{23}.

**Lemma 4.2.** Suppose that Assumptions 2.1 and 2.2 hold. Then for any \( p \geq 2 \),

\[
\sup_{0 \leq t \leq T} E|Z(t) - \bar{Z}(t)|^p \leq C \Delta^{p/2}.
\]

**Proof.** For any \( t \in [0, T] \), according to \cite{6} and the basic inequality \(|u| + |v|)^p \leq 2^{p-1}(|u|^p + |v|^p), \ p \geq 2 \), one has

\[
E|Z(t) - \bar{Z}(t)|^p \leq 2^{p-1}E \left| \int_{[t/\Delta]} f(\bar{Z}(s), \alpha(s))ds \right|^p + 2^{p-1}E \left| \int_{[t/\Delta]} g(\bar{Z}(s), \alpha(s))dB(s) \right|^p.
\]

Applying Hölder’s inequality and Theorem 1.7.1 in \cite{24}, one can arrive at

\[
E|Z(t) - \bar{Z}(t)|^p \leq C \Delta^{p-1} \int_{[t/\Delta]} |f(\bar{Z}(s), \alpha(s))|^p ds + C \Delta^{p/2-1} E \int_{[t/\Delta]} |g(\bar{Z}(s), \alpha(s))|^p ds.
\]

On the basis of Remark 1 and Lemma 4.1, one has

\[
E|Z(t) - \bar{Z}(t)|^p \leq C \Delta^{p/2-1} \int_{[t/\Delta]} (1 + |\bar{Z}(s)|^p) ds \leq C \Delta^{p/2-1} \left( 1 + \sup_{0 \leq s \leq t} |Z(s)|^p \right) ds \leq C \Delta^{p/2},
\]

since \( t \in [0, T] \) is arbitrary, the proof is completed. \( \square \)
Theorem 4.3. Under Assumption 2.1 for any $p \geq 2$, the EM method (6) has the property that
\[ E \left( \sup_{0 \leq t \leq T} |z(t) - Z(t)|^p \right) \leq C \Delta^{p/2}. \]

Proof. Recall (2) and (6), for any $t \in [0, T]$, according to Itô’s formula, we have
\[ |z(t) - Z(t)|^2 = \int_0^t 2(z(s) - Z(s))^T (f(z(s), \alpha(s)) - f(\tilde{Z}(s), \alpha(s))) \, ds \]
\[ + \int_0^t |g(z(s), \alpha(s)) - g(\tilde{Z}(s), \alpha(s))|^2 \, ds \]
\[ + \int_0^t 2(z(s) - Z(s))^T (g(z(s), \alpha(s)) - g(\tilde{Z}(s), \alpha(s))) \, dB(s). \]

Then for any $T_1 \in [0, T]$, it is easily to get that
\[ E \left( \sup_{0 \leq t \leq T_1} |z(t) - Z(t)|^p \right) \]
\[ \leq 3^{p/2} E \left\{ \sup_{0 \leq t \leq T_1} \left( \int_0^t 2(z(s) - Z(s))^T (f(z(s), \alpha(s)) - f(\tilde{Z}(s), \alpha(s))) \, ds \right)^{p/2} \right\} \]
\[ \quad + 3^{p/2} E \left\{ \sup_{0 \leq t \leq T_1} \left( \int_0^t |g(z(s), \alpha(s)) - g(\tilde{Z}(s), \alpha(s))|^2 \, ds \right)^{p/2} \right\} \]
\[ \quad + 3^{p/2} E \left\{ \sup_{0 \leq t \leq T_1} \left( \int_0^t 2(z(s) - Z(s))^T (g(z(s), \alpha(s)) - g(\tilde{Z}(s), \alpha(s))) \, dB(s) \right)^{p/2} \right\}. \]

Using Hölder’s inequality and Assumption 2.1, one can deduce that
\[ A_1 \leq 2^{p/2} T_1^{p/2-1} E \int_0^{T_1} |z(s) - Z(s)|^{p/2} |f(z(s), \alpha(s)) - f(\tilde{Z}(s), \alpha(s))|^{p/2} \, ds \]
\[ \leq 2^{p/2} T_1^{p/2-1} L^{p/2} E \int_0^{T_1} |z(s) - Z(s)|^{p/2} |z(s) - \tilde{Z}(s)|^{p/2} \, ds \]
\[ \leq C E \int_0^{T_1} |z(s) - Z(s)|^p \, ds + C E \int_0^{T_1} |Z(s) - \tilde{Z}(s)|^p \, ds, \]
and

\[ A_2 \leq T_1^{p/2-1} E \int_0^{T_1} |g(z(s), \alpha(s)) - g(\bar{Z}(s), \alpha(s))|^p ds \]

\[ \leq T_1^{p/2-1} L^p E \int_0^{T_1} |z(s) - \bar{Z}(s)|^p ds \]

\[ \leq C E \int_0^{T_1} |z(s) - Z(s)|^p ds + C E \int_0^{T_1} |Z(s) - \bar{Z}(s)|^p ds. \]  

(9)

Applying Theorem 1.7.2 in [24], together with Assumption 2.1, we can obtain that

\[ A_3 \leq C E \int_0^{T_1} |z(s) - Z(s)|^{p/2} |g(z(s), \alpha(s)) - g(\bar{Z}(s), \alpha(s))|^{p/2} ds \]

\[ \leq C E \int_0^{T_1} |z(s) - Z(s)|^p ds + C E \int_0^{T_1} |Z(s) - \bar{Z}(s)|^p ds. \]  

(10)

Substituting (8)-(10) into (7), yields

\[ E \left( \sup_{0 \leq t \leq T_1} |z(t) - Z(t)|^p \right) \leq C E \int_0^{T_1} |z(t) - Z(t)|^p dt + C E \int_0^{T_1} |Z(t) - \bar{Z}(t)|^p dt. \]

By Lemma 4.2 one can further show that

\[ E \left( \sup_{0 \leq t \leq T_1} |z(t) - Z(t)|^p \right) \leq C \int_0^{T_1} E \left( \sup_{0 \leq s \leq t} |z(s) - Z(s)|^p \right) dt + C \Delta^{p/2}. \]

It therefore follows from the Gronwall inequality that

\[ E \left( \sup_{0 \leq t \leq T_1} |z(t) - Z(t)|^p \right) \leq C e^{C T_1} \Delta^{p/2}, \]

since \( T_1 \in [0,T] \) is arbitrary, hence

\[ E \left( \sup_{0 \leq t \leq T} |z(t) - Z(t)|^p \right) \leq C \Delta^{p/2}. \]

The proof is completed.

5. Conclusions

In this paper, we develop the EM scheme, which is different from the one given in the references (such as [4, 10]), to generate the approximate solutions to a class of SDEwMSs, and analyze the order of the errors in the \( L^p \)-sense. It has been proved that the \( L^p \)-convergence rate of the EM scheme given in this paper for SDEwMSs can reach 1/2. We further point out that the techniques used in this paper to construct the EM method can also be used to construct other schemes for hybrid systems, such as stochastic theta method, tamed EM method, and Milstein method, etc. We believe that this approach will also contribute to the \( L^p \)-convergence order of these numerical methods for hybrid systems.
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