A new algorithm to determine the creation or depletion term of parabolic equations from boundary measurements

Loc Hoang Nguyen

Department of Mathematics and Statistics, University of North Carolina at Charlotte, Charlotte, NC, 28223, USA

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Abstract

We propose a robust numerical method to find the coefficient of the creation or depletion term of parabolic equations from the measurement of the lateral Cauchy information of their solutions. Most papers in the field study this nonlinear and severely ill-posed problem using optimal control. The main drawback of this widely used approach is the need of some advanced knowledge of the true solution. In this paper, we propose a new method that opens a door to solve nonlinear inverse problems for parabolic equations without any initial guess of the true coefficient. This claim is confirmed numerically. The key point of the method is to derive a system of nonlinear elliptic equations for the Fourier coefficients of the solution to the governing equation with respect to a special basis of $L^2$. We then solve this system by a predictor-corrector process, in which our computation to obtain the first and second predictors is effective. The desired solution to the inverse problem under consideration follows.

1. Introduction

Let $\Omega$ be a cube $(-R, R)^d \subset \mathbb{R}^d$, $d \geq 2$, where $R$ is a positive number. Introduce a $d \times d$ matrix valued function $A$ with entries in the class $C^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$. Assume throughout the paper that

1. $A$ is symmetric; i.e., $A^T = A$,
2. the matrix $A$ is uniformly elliptic; i.e., there exists a positive number $\mu$ such that
   \[ A(x)\xi \cdot \xi \geq \mu |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^d, \]
3. for all $x \in \mathbb{R}^d \setminus \Omega$, $A(x) = \text{Id}$ where $\text{Id}$ is the identity matrix.

Let $b$ be a $d$-dimensional vector valued function in $C^1(\mathbb{R}^d, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d})$. Define the operator

\[ Lu(x) = \text{div}(A(x)\nabla u(x)) + b \cdot \nabla u(x) \quad \text{for all } u \in H^2(\mathbb{R}^d), x \in \mathbb{R}^d. \]

Consider the solution $u$ to the following initial value problem for the following parabolic equation

\[
\begin{cases}
  u_t(x, t) = Lu(x, t) + b \cdot \nabla u(x, t) & x \in \mathbb{R}^d, t \in [0, \infty), \\
  u(x, 0) = f(x) & x \in \mathbb{R}^d.
\end{cases}
\] (1)

The second order term $\text{div}(A(x)\nabla u(x, t))$ describes the diffusion, the first order term $b(x) \cdot \nabla u(x, t)$ describes the transport and the zero$^{th}$ order term $c(x)u(x, t)$ describes creation or depletion. In this paper, we numerically solve the problem of reconstructing the coefficient $c(x)$ of the creation or depletion term. More precisely, we propose a method to solve the highly nonlinear and severely ill-posed coefficient inverse problem.

Problem 1.1 (Coefficient Inverse Problem (CIP)). Assume that $f(x) \neq 0$ for all $x \in \Omega$. Given a time $T > 0$ and the lateral Cauchy data

\[
F(x, t) = u(x, t) \quad \text{and} \quad G(x, t) = \partial_n u(x, t)
\]

for all $x \in \partial \Omega$ and $t \in [0, T]$, determine the coefficient $c(x), x \in \Omega$. Here $\nu$ is the outward normal vector of $\partial \Omega$.

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loc.nguyen@uncc.edu (L.H. Nguyen)

ORCID(s):
Problem 1.1 has uncountable applications in the reality. In fact, suppose that interior points of a medium are not accessible. In this case, by measuring both the function \( u \) and the flux at the boundary of that medium for a certain period of time and by solving Problem 1.1, one can determine the coefficient \( c(x) \) of the governing equation in (1), which enables us to inspect that medium without destructing it. We recall here a specific example in bioheat transfer. In this field, the coefficient \( c(x) \) represents the blood perfusion. The knowledge of this coefficient plays a crucial role in calculating the temperature of the blood flowing through the tissue, see [1]. The uniqueness of Problem 1.1 is still open and considered as an assumption of in this paper. One can find the uniqueness of other versions of Problem 1.1 in [2, 3, 4] when some internal data are assumed to be known. When the Dirichlet to Neumann map is given, the reader can find the uniqueness in [5]. Another related problem is the inverse problem of recovery from the measurement of the final time for parabolic equations. This problem is very important and interesting, see [6, 7, 8, 9, 10] for theoretical results and numerical methods.

Coefficient inverse problems for parabolic equations were studied intensively. Up to the knowledge of the author, the widely used method to solve this problem is the optimal control approach, see e.g., [11, 12, 1, 13, 14] and references therein. The authors of [11] applied the optimal control method involving a preconditioner to numerically compute the heat conductivity with high quality. The main drawback of this method is the need of a good initial guess for the true solution while a good initial guess is not always available. On the other hand, we specially draw the reader’s attention to the convexification method, see [15, 16], which can overcome the difficulty about the availability of the initial guess. In those papers [15, 16], the authors introduce a convex functional whose minimizer yields the solution of the problem under consideration, by combining the quasi-reversibility method and the Carleman weight functions. Great 1D numerical examples, illustrating the role of Carleman weight functions in convexifying the cost functionals, are presented in [15]. It is valuable to numerically test this convexification method in higher dimensions. We also cite to [17] for another method to solve Problem 1.1 by repeatedly solving its linearization. In the current paper, we propose a novel method in which no advanced knowledge about the true coefficient is required. This claim is numerically confirmed even in the case when the contrast is high. We therefore called the proposed method “global”.

Our method to solve Problem 1.1 consists of two stages. In the first stage, we eliminate the function \( c(x) \) from (1). The resulting equation obtained in this stage is not a standard equation. A numerical method to solve it is not available yet. We approximate it a coupled system of elliptic partial differential equations. This system is derived based on a truncation of the Fourier series, with respect to a special basis originally introduced in [18]. We apply a predictor-corrector procedure, in which the first approximation of the true solution is computed without any of its advance knowledge, to solve this system. The solution of Problem 1.1 follows.

Two important steps in our method require us to find vector valued functions satisfying a system of elliptic partial differential equations and both Dirichlet and Neumann boundary conditions. We employ the quasi-reversibility method for this purpose and we also prove the convergence of the quasi-reversibility method in our context, using a new Carleman estimate in [19]. The quasi-reversibility method was first introduced by Lattès and Lions in [20] for numerical solutions of ill-posed problems for partial differential equations. It has been studied intensively since then, see e.g., [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 19]. A survey on this method can be found in [31].

In Section 2, we derive the system mentioned above. In Section 3, we propose a numerical method to solve that system. Also in Section 3, we study the quasi-reversibility method that can be applied in our context. In Section 4, we describe the implementation using the finite difference method. In Section 5, we present some numerical results. Section 6 is for the concluding remarks.

2. A nonlinear coupled system of elliptic equations

From now on, we denote by \( \Omega_T \) the set \( \Omega \times [0, T] \). Define the function
\[
v(x, t) = u_t(x, t) \quad \text{for all } (x, t) \in \Omega_T. \tag{3}
\]
It follows from (1) that
\[
v_t(x, t) = Lv(x, t) + c(x)v(x, t) \quad \text{for all } (x, t) \in \Omega_T. \tag{4}
\]
On the other hand, for all \( x \in \Omega \),
\[
v(x, 0) = u_t(x, 0) = Lf(x) + c(x)f(x).
\]
Therefore,
\[
c(x) = \frac{v(x, 0) - Lf(x)}{f(x)} \quad \text{for all } x \in \Omega.
\] (5)

Plugging (5) into (4), we obtain the following equation
\[
v_t(x, t) = Lv(x, t) - \frac{L f(x)}{f(x)} v(x, t) - \frac{v(x, 0)}{f(x)} f(x) v(x, t)
\] (6)
for all \((x, t) \in \Omega_T\).

**Remark 2.1.** Solving the nonlinear equation (6) is challenging due to the presence of the initial condition \(v(x, 0)\). A theoretical result to solve it is not yet available. We employ the technique of truncating the Fourier series, see [18], to solve (6).

Recall a special orthonormal basis of \(L^2(0, T)\) originally introduced by Klibanov [18] in 2017. This basis plays a crucial role in deriving an approximate model whose solution will be used to directly compute the solution of Problem 1.1. For each \(n \geq 1\), define the function \(\phi_n(t) = (t - T/2)^{n-1} \exp(t - T/2)\). It is well-known that the set \(\{\phi_n\}_{n=1}^\infty\) is complete in \(L^2(0, T)\). Employing the Gram-Schmidt orthonormalization process on this set, we obtain an orthonormal basis of \(L^2(0, T)\). We denote this basis by \(\{\Psi_n\}_{n=1}^\infty\).

We have the proposition

**Proposition 2.1** (see [18]). The basis \(\{\Psi_n\}_{n=1}^\infty\) satisfies the following properties:

1. \(\Psi_n\) is not identically zero for all \(n \geq 1\).
2. For all \(m, n \geq 1\)
\[
\int_0^T \Psi'_n(t) \Psi_m(t) dt = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } n < m. \end{cases}
\]

As a result, for all integer \(N > 0\), the matrix \(S = (s_{mn})_{m,n=1}^N\) is invertible.

Recall the Fourier coefficients of the function \(v(x, t)\)
\[
v_n(x) = \int_0^T v(x, t) \Psi'_n(t) dt \quad \text{for all } x \in \Omega.
\] (7)

We have
\[
v(x, t) = \sum_{n=1}^\infty v_n(x) \Psi_n(t) \quad \text{for all } (x, t) \in \Omega_T.
\]

Fix a number \(N > 0\). We approximate the function \(v(x, t)\) by the partial sum
\[
v(x, t) = \sum_{n=1}^N v_n(x) \Psi_n(t) \quad \text{for all } (x, t) \in \Omega_T.
\] (8)

In this approximation context,
\[
v(x, 0) = \sum_{n=1}^N v_n(x) \Psi_n(0) \quad \text{for all } x \in \Omega
\] (9)

and
\[
v_t(x, t) = \sum_{n=1}^N v_n(x) \Psi'_n(t) \quad \text{for all } x \in \Omega.
\] (10)
Plugging (8), (9) and (10) into (6), we have

$$\sum_{n=1}^{N} u_n(x)\Psi_n(t) = \sum_{n=1}^{N} L u_n(x)\Psi_n(t) - \frac{L f(x)}{f(x)} \sum_{n=1}^{N} u_n(x)\Psi_n(t) + \sum_{n,l=1}^{N} \frac{u_n(x)\Psi_n(0)}{f(x)} v_l(x)\Psi_l(t)$$

for all $$(x, t) \in \Omega_T$$. For each $$m$$ in $$\{1, 2, \ldots, N\}$$, multiply both sides of the equation above by $$\Psi_m(t)$$ and the integrate the resulting equation with respect to $$t$$. Noting that

$$\int_0^T \Psi_n(t)\Psi_m(t)dt = \delta_{m-n},$$

we have for each $$m \in \{1, \ldots, N\},$$

$$Lv_m(x) - \sum_{n=1}^{N} s_{mn} v_n(x) - \frac{L f(x)}{f(x)} v_m(x) + \sum_{n=1}^{N} \frac{\Psi_n(0)}{f(x)} v_n(x)v_m(x) = 0 \quad (11)$$

for all $$x \in \Omega$$. On the other hand, for each $$m \in \{1, \ldots, N\}$$, the function $$v_m(x)$$ satisfies the following constraints

$$v_m(x) = F_m(x) = \int_0^T F_t(x, t)\Psi_m(t)dt,$$
$$\partial_t v_m(x) = G_m(x) = \int_0^T G_t(x, t)\Psi_m(t)dt \quad (12)$$

for all $$x \in \partial \Omega, m \in \{1, \ldots, N\}.$$ 

**Remark 2.2.** From now on, we consider $$F_m$$ and $$G_m$$ as the indirect given data. Let $$F_m^\delta$$ and $$G_m^\delta$$ be the data without noise. The corresponding noisy data, with noise level $$\delta > 0$$, is given by

$$F_m^\delta(x) = F_m^+(x)(1 + \delta \text{rand}(x)), \quad G_m^\delta(x) = G_m^+(x)(1 + \delta \text{rand}(x)) \quad (13)$$

for each $$m \in \{1, \ldots, N\}$$ where rand is the function of uniformly distributed random numbers in the range $$[-1, 1]$$. In this paper, we test our numerical method from simulated data with 10% noise.

In summary, we have proved the following proposition.

**Proposition 2.2.** Fix $$N > 0$$. Assume that the function $$v(x, t), (x, t) \in \Omega_T$$, can be well-approximated by the expression in (8) with the function $$v_n(x), n \in \{1, 2, \ldots, N\}$$, given in (7). Then the Fourier coefficients $$v_n$$ satisfies the over-determined system of partial differential equations (11)–(12).

### 3. The method to solve Problem 1.1

Solving Problem 1.1 is reduced to solve the system of nonlinear system of partial differential equation (11)–(12).

#### 3.1. An iterative process

Solving the nonlinear system (11)–(12) is challenging. We propose the following iterative method, in which a predictor-corrector procedure is applied. The first predictor, named as $$V^{(0)}$$, is set to be the solution of the linear system obtained by removing from (11) the nonlinear term. More precisely, we set $$V^{(0)} = (v_1^{(0)}, \ldots, v_N^{(0)})^T$$ as the solution of

$$Lv_m^{(0)}(x) - \sum_{n=1}^{N} s_{mn} v_n^{(0)}(x) - \frac{L f(x)}{f(x)} v_m^{(0)}(x) = 0 \quad \text{for all} \ x \in \Omega \quad (14)$$

and

$$v_m^{(0)}(x) = F_m(x), \partial_t v_m^{(0)}(x) = G_m(x) \quad \text{for all} \ x \in \partial \Omega \quad (15)$$
for \( m \in \{1, \ldots, N\} \). Next, by induction, assume that \( V^{(p)} \) is known for some positive integer \( p \), we find \( V^{(p+1)} \) by solving the equation obtained from (11) by replacing \( v_m \) in the nonlinear term by its approximation \( v_m^{(p)} \). That means, \( V^{(p+1)} = (v_1^{(p+1)}, \ldots, v_N^{(p+1)})^T \) is set to be the solution of

\[
L_m^{(p+1)}(x) - \sum_{n=1}^{N} s_{mn} v_n^{(p+1)}(x) - \frac{L_f(x)}{f(x)} v_m^{(p+1)}(x) + \sum_{n=1}^{N} w_n(0) v_n^{(p+1)}(x)v_m^{(p)}(x) = 0 \quad \text{for all } x \in \Omega \tag{16}
\]

and

\[
v_m^{(p+1)}(x) = F_m(x), \quad \partial_n v_m^{(p+1)}(x) = G_m(x) \quad \text{for all } x \in \partial\Omega \tag{17}
\]

for each \( m \in \{1, \ldots, N\} \).

Due to the presence of the lateral Cauchy data, both problems (14)–(15) and (16)–(17) are over-determined. We employ the quasi-reversibility method to solve them.

### 3.2. The quasi-reversibility method

We next recall the quasi-reversibility method to solve systems of partial differential equations with Cauchy boundary data. The two systems of elliptic partial differential equations (14)–(15) and (16)–(17) are over-determined due to both Dirichlet and Neumann boundary conditions imposed. We use the quasi-reversibility method to solve them. A general form of these systems can be read as

\[
\begin{cases}
\text{div}(AV) + BV = 0 & x \in \Omega, \\
V = F & x \in \partial\Omega, \\
\partial_n V = G & x \in \partial\Omega
\end{cases} \tag{18}
\]

where \( A \) is introduced in Section 1 and \( B \) is a \( N \times N \) matrix valued function in \( L^\infty(\Omega) \). We have the proposition whose proof closely follows that of Theorem 3.1 in [19].

**Proposition 3.1.** Fix \( \varepsilon > 0 \). Then, the functional

\[
J_\varepsilon(V) = \int_\Omega |\text{div}(AV) + BV|^2 \, dx + \int_\Omega |V - F|^2 \, d\sigma(x) + \int_{\partial\Omega} |\partial_n V - G|^2 \, d\sigma + C\|V\|^2_{H^2(\Omega)}
\]

has a unique minimizer on \( H^2(\Omega) \). This minimizer \( V_\varepsilon \) is called the regularized solution of (18).

In summary, we propose Algorithm 1 to solve Problem 1.1 via solving (11)–(12) by the quasi-reversibility method. The following inequality plays an important role for the convergence of the quasi-reversibility method.

**Lemma 3.1 (Carleman estimate).** Let the number \( b > R \). Then there exist numbers \( p_0 \geq 1 \) and \( \lambda \geq 1 \) depending only on \( \mu, b, d, R, \|A\|_{L^\infty(\Omega)^{d\times d}} \) such that the following Carleman estimate holds:

\[
\int_\Omega |\text{div}(AV)|^2 \, dx \leq 2\lambda(x_d + b)^p \int_\Omega [\lambda^2 |\nabla u|^2 + \lambda^2 u^2] \exp \left[ 2\lambda(x_d + b)^p \right] \, dx,
\]

for all \( \lambda \geq \lambda_0, p \geq p_0 \) and \( u \in H^2(\Omega) \) with \( u = \partial_0 = 0 \) on \( \partial\Omega \). Here, the constant \( C \) depends only on \( \mu, b, d, R, \|A\|_{L^\infty(\Omega)^{d\times d}} \).

Lemma 3.1 is a direct consequence of [19, Theorem 4.1]. We do not repeat the proof in this paper.

**Theorem 3.1 (The convergence of the quasi-reversibility method for (18)).** Assume that there uniquely exists a true solution to (18) with the boundary data \( F \) and \( G \) replaced by the corresponding noiseless ones, denoted by \( F^* \) and \( G^* \) respectively. Let \( F^\delta \) and \( G^\delta \) be the corresponding noisy data for some \( \delta > 0 \). Assume that there exists an “error” vector valued function \( E \) such that

\[
E = F^\delta - F^* \quad \text{and} \quad \partial_n E = G^\delta - G^*
\]

(21)
on \( \partial \Omega \) and assume that
\[
\| E \|_{H^2(\Omega)} \leq \delta
\] (22)

Then, \( V_\varepsilon^\delta \), the regularized solution to (18), satisfies the estimate
\[
\| V_\varepsilon^\delta - V^* \|_{H^1(\Omega)}^2 \leq C(\delta^2 + \varepsilon \| V^* \|_{H^2(\Omega)}^2).
\] (23)

**Remark 3.1.** The convergence for the quasi-reversibility method guaranteed by Theorem 3.1 is similar to that in [19, Theorem 5.1]. The main difference of two results is in the objective functional is minimized subject to some boundary constraints while in the current paper, such constraints are relaxed by adding the two boundary integrals in (19).

**Proof of Theorem 3.1.** Since \( V_\varepsilon^\delta \) is the regularized solution to (18), it is the minimizer of \( J_\varepsilon \), defined in (19). Hence, for all \( \phi \in H^2(\Omega)^N \), we have
\[
\langle \text{div}(AVV_\varepsilon^\delta) + BV_\varepsilon^\delta, \text{div}(AV\phi) + B\phi \rangle_{L^2(\Omega)^N} + \langle V_\varepsilon^\delta - V^\delta, \phi \rangle_{L^2(\partial\Omega)^N} + \langle \partial_\nu V_\varepsilon^\delta - \mathcal{G}^\delta, \partial_\nu \phi \rangle_{L^2(\partial\Omega)^N} + \varepsilon \langle V_\varepsilon^\delta, \phi \rangle_{H^2(\Omega)} = 0
\] (24)
for all \( \phi \in H^2(\Omega)^N \). On the other hand, since \( V^* \) is the true solution to (18)
\[
\langle \text{div}(AVV^*), \text{div}(AV\phi) + B\phi \rangle_{L^2(\Omega)^N} + \langle V^* - V^\delta, \phi \rangle_{L^2(\partial\Omega)^N} + \langle \partial_\nu V^* - \mathcal{G}^*, \partial_\nu \phi \rangle_{L^2(\partial\Omega)^N} + \varepsilon \langle V^*, \phi \rangle_{H^2(\Omega)} = \varepsilon \langle V^*, \phi \rangle_{H^2(\Omega)}
\] (25)
for all \( \phi \in H^2(\Omega)^N \). Taking the difference of (24) and (25), we have
\[
\langle \text{div}(AVW), \text{div}(AV\phi) + B\phi \rangle_{L^2(\Omega)^N} + \langle W - (F^\delta - F^*), \phi \rangle_{L^2(\partial\Omega)^N} + \langle \partial_\nu W - (\mathcal{G}^\delta - \mathcal{G}^*), \partial_\nu \phi \rangle_{L^2(\partial\Omega)^N} + \varepsilon \langle W, \phi \rangle_{H^1(\Omega)} = -\varepsilon \langle V^*, \phi \rangle_{H^2(\Omega)}
\] (26)
where \( W = V_\varepsilon^\delta - V^* \) for all \( \phi \in H^2(\Omega)^N \). Using
\[
\phi = W - E = V_\varepsilon^\delta - V^* - E
\] (27)
as a test function in (26) and using (21), we have
\[
\| \text{div}(AV\phi) + B\phi \|_{L^2(\Omega)^N}^2 + \langle \text{div}(AVE), \text{div}(AV\phi) + B\phi \rangle_{L^2(\Omega)^N} + \| \phi \|_{L^2(\partial\Omega)^N}^2 + \| \partial_\nu \phi \|_{L^2(\partial\Omega)^N}^2 + \varepsilon \| \phi \|_{H^2(\Omega)}^2 + \varepsilon \| E \|_{H^2(\Omega)}^2 = -\varepsilon \langle V^*, \phi \rangle_{H^2(\Omega)}.
\]
Applying the inequality \( |\langle u, v \rangle| \leq 1/2(\| u \|^2 + \| v \|^2) \), (22) and the trace theory, we have
\[
\| \text{div}(AV\phi) + B\phi \|_{L^2(\Omega)^N}^2 \leq C(\delta^2 + \varepsilon \| V^* \|_{H^2(\Omega)}^2).
\] (28)
Here, \( C \) is a generic constant that might change from estimate to estimate. Choose \( b > R, \lambda > \lambda_0, p \geq p_0 \) where \( b, \lambda_0 \) and \( p_0 \) are as in Lemma 3.1. It is not hard to verify that the function \( \phi \) satisfies the homogeneous boundary conditions \( \phi = \partial_\nu \phi = 0 \) on \( \partial\Omega \times [0, T] \). Using the Carleman estimate in Lemma 3.1, we can bound the left hand side of (28) as follows
\[
\int_{\Omega} |\text{div}(AV\phi) + B\phi|^2 dx \geq \exp(-2\lambda(R + b)) \int_{\Omega} \exp(2\lambda(x_d + b)) |\text{div}(AV\phi) + B\phi|^2 dx
\]
\[
\geq C \int_{\Omega} [\exp(2\lambda(x_d + b)) |\text{div}(AV\phi)|^2 - \exp(2\lambda(x_d + b)) |B\phi|^2] d\Omega
\]
\[
\geq C \int_{\Omega} [\exp(2\lambda(x_d + b))(\lambda^2 |\phi|^2 + \lambda |\nabla \phi|^2) - \exp(2\lambda(x_d + b)) |B\phi|^2] d\Omega.
\]
Choosing $\lambda$ sufficiently large, since $B \in L^\infty(\Omega)$, we have
\[
\|\text{div}(AV\phi) + B\phi\|^2_{L^2(\Omega)^N} \geq C\|\phi\|^2_{H^1(\Omega)^N}.
\]
This, together with (27) and (28), implies
\[
\|V^\delta - V^*\|^2_{H^1(\Omega)^N} \leq C(\delta^2 + \epsilon\|V^*\|^2_{H^2(\Omega)^N}).
\]
The theorem is proved.

Theorem 3.1 guarantees that Steps 2 and 4 in Algorithm 1 provide good approximations of the sequence $\{c_p^\epsilon\}_{p=1}^\infty$ in comparison to the sequence $\{c_p^*\}_{p=1}^\infty$ with the Lipschitz rate provided that $\epsilon = O(\delta^2)$ as $\delta \to 0^+$. If the sequence $\{c_p^*\}_{p=1}^\infty$ converges to the solution of Problem 1.1, Algorithm 1 yields a numerical procedure to solve it. This convergence is verified numerically in Section 5.

3.3. The procedure to solve the coefficient inverse problem for parabolic equations

By Proposition 2.2, the strategy to solve (11)–(12) described in Section 3.1 and the convergence of the quasi-reversibility method, see Theorem 3.1, we propose Algorithm 1 to reconstruct the coefficient $c(x), x \in \Omega$.

**Algorithm 1** The procedure to solve Problem 1.1

1. Choose a number $N$. Construct the functions $\Psi_m, 1 \leq m \leq N$, and compute the matrix $S$ as in Proposition 2.1. Fix $\epsilon > 0$.
2. Find the regularized solution $V^{(0)}$ of (14)–(15).
3. Compute $c^{(0)}$ via (8) and $c^{(0)}$ via (5).
4. Assume that we know $V^{(p)}$ and $c^{(p)}$. Set $V^{(p+1)}$ as the regularized solution to (16)–(17).
5. Compute $c^{(p+1)}$ via (8) and $c^{(p+1)}$ via (5).
6. Define
   \[
   \mathcal{E}(p) = \frac{\|c^{(p)} - c^{(p+1)}\|_{L^\infty(\Omega)}}{\|c^{(p+1)}\|_{L^\infty(\Omega)}}.
   \]
   Choose $\epsilon = c^{(p^*)}$ for $p^*$ such that $\mathcal{E}(p^*)$ is sufficiently small.

**Remark 3.2.** Unlike the widely used numerical method to solve ill-posed inverse problems, we do not require a good initial guess for the true coefficient $c(x)$. Our first approximation is computed in Step 2 and Step 3 of Algorithm 1. It is shown in Section 5 that the functions $c^{(0)}$ are acceptable in Tests 1, 3 and 4. In contrast, our computed $c^{(0)}$ is poor in Test 2. However, the error is automatically corrected when we find $c^{(1)}$ in Step 4 and Step 5.

4. The implementation using the finite difference method

We test our method in the simple case when $d = 2, \Omega = (-1, 1)^2, L = \Delta$ is the Laplacian and $f(x) = 1$.

4.1. The forward problem

To generate the simulated data, we solve the forward problem of Problem 1.1. That means we compute the solution $u(x, t)$ to (1) on the whole plane $\mathbb{R}^2$, given $c(x)$. Instead of doing so, we solve an analog of (1) on a domain $\Omega_1 = (R_1, R_1)^2$ where $R_1 = 3 > R = 1$. To guarantee the correctness of this domain approximation, we take $T$ small, $T = 0.3$, so that the heat generated by $c(x)$ does not have enough time to hit the boundary of $\Omega_1$ as $t \in [0, T]$. In other words, we solve the equation

\[
\begin{align*}
&\begin{cases}
  u_t(x, t) = \Delta u(x, t) + c(x)u(x, t) & x \in \Omega_1, t \in [0, T], \\
  u(x, 0) = f(x) & x \in \Omega_1,
  
end{cases} \\
&\begin{cases}
  u(x, t) = f(x) & x \in \partial\Omega_1, t \in [0, T].
end{cases}
\end{align*}
\]

(29)
The function \( f(x, t) \) is the solution of the forward problem (29).

Here, we choose the time-independent Dirichlet boundary data for the simplicity. In this paper, we solve problem (29) by the implicit method using finite difference. In the finite difference scheme, we find the function \( u(x, t) \) on the grid of points

\[
\left\{ (x_i = -R + (i - 1)d_x, y_j = -R + (j - 1)d_y, (l - 1)d_l) : 1 \leq i, j \leq N_x, 1 \leq l \leq N_l \right\} \subset \overline{\Omega} \times [0, T]
\]

where \( N_x \) and \( N_l \) are two large integers, \( d_x = 2R/(N_x - 1) \) and \( d_l = T/(N_l - 1) \). In our computational program, \( N_x \) is set to be 240 and \( N_l = 100 \). Having the function \( u(x, t) \) for all \( x \in \Omega \) and \( t \in [0, T] \), we can extract the data \( F(x, t) = u(x, t) \) and \( G(x, t) = \partial_t u(x, t) \) on \( \partial \Omega \times [0, T] \).

4.2. The inverse problem

In this section, we present how to implement Algorithm 1 in the finite difference scheme. Similarly to the previous section, we define and then compute the function \( c \) on a uniform grid of points

\[
\left\{ (x_i = -R + (i - 1)d_x, y_j = -R + (j - 1)d_y, (l - 1)d_l) : 1 \leq i, j \leq N_x, 1 \leq l \leq N_l \right\} \subset \overline{\Omega} \times [0, T]
\]

where \( N_x \) and \( N_l \) are two large integers, \( d_x = 2R/(N_x - 1) \) and \( d_l = T/(N_l - 1) \). In all numerical tests in Section 5, we take \( N_x = 80 \) and \( N_l = 100 \).

We next present each step of Algorithm 1.

Step 1. In this step, to choose “truncation” number \( N \). To do so, we take a “reference” function \( v \) in one of the examples in Section 5 and then compute the absolute difference

\[
e_N(x, t) = \left| v(x, t) - \sum_{n=1}^{N} v_n(x)\Psi_n(t) \right| \quad \text{for all } x \in \Omega, t \in [0, T].
\]

We observe that the larger \( N \), the smaller \( \|e_N\|_{L^\infty(\Omega \times [0, T])} \). We examine the function \( e_N \) when \( N = 5 \), \( N = 10 \) and \( N = 25 \), see Figure 1. It is evident from (1c) that when \( N = 25 \), \( \|e_N\|_{L^\infty} \) is sufficiently small, about \( 4 \times 10^{-3} \).

As a result, we choose \( N = 25 \). We use this choice of \( N \) for all numerical tests. We observe that using higher \( N \) does not improve the quality of the reconstructed coefficient \( c(x) \). Also in Step 1 of Algorithm 1, we choose the regularized number \( c = 10^{-9} \).

Step 2. Compute the vector valued function \( V^{(0)} \), which is set to be the minimizer of the functional, due to the quasi-reversibility method,

\[
J^{(0)}_c(V) = \sum_{m=1}^{N} \left[ \int_{\Omega} |\Delta u_m(x) - \sum_{n=1}^{N} s_{mn} v_n(x) - \frac{\Delta f(x)}{f(x)} v_m(x) |^2 dx + \int_{\partial \Omega} |v_m(x) - F_m(x)|^2 d\sigma(x) \\
+ \int_{\partial \Omega} |\partial_n v_m(x) - G_m(x)|^2 d\sigma(x) + c \int_{\Omega} (|v_m(x)|^2 + |\nabla v_m(x)|^2) dx \right].
\]  

\[(30)\]
Here, we replace the term $\|V\|^2_{L^2(\Omega)}$ in (19) by the term $\|V\|^2_{H^1(\Omega)}$. This is because the $H^1(\Omega)$-norm is easier to work with computationally than the $H^2(\Omega)$-norm. On the other hand, we have not observed any instabilities probably because the number $80 \times 80$ of grid points we use is not too large and all norms in finite dimensional spaces are equivalent. We now identify \{v_m(x_i,y_j) : 1 \leq i,j, N_x, 1 \leq m \leq N\} by the $N_x^2 N$ dimensional vector $\mathbf{v}$ whose the $i^{th}$ entry is given by

$$v_i = v_m(x_i,y_j).$$

(31)

Here, $(i,j,m)$ is such that

$$i = (i-1)N_x N + (j-1)N + m.$$  

(32)

Then, by approximate all differential operators in the right hand side of (30) by the corresponding finite difference version, we have

$$J^{(0)}_e(V) = d_x^2 |\mathbf{L} \cdot \mathbf{v}|^2 + d_x |\mathbf{D}_1 \mathbf{v} - \mathbf{G}|^2 + d_x |\mathbf{D}_2 \mathbf{v} - \mathbf{H}|^2 + c d_x^2 |\mathbf{v}|^2 + c d_x^2 |D_x \mathbf{v}|^2 + c d_x^2 |D_y \mathbf{v}|^2$$

(33)

where the matrices $\mathbf{L}$, $\mathbf{D}_1$, $\mathbf{D}_2$, $D_x$ and $D_y$ and the vectors $\mathbf{G}$ and $\mathbf{H}$ are described below. The $N_x^2 N \times N_x^2 N$ matrix $\mathbf{L}$ is given by

1. $L_{ij} = -4/d_x^2 - s_{mn} - \Delta f(x_i,y_j)/f(x_i,y_j)$ if $i = j = (i-1)N_x N + (j-1)N + m$;

2. $L_{ij} = 1/d_x^2$ if $i = (i-1)N_x N + (j-1)N + m$ and $j = (i \pm 1 - 1)N_x N + t(j \pm 1 - 1)N + m$;

3. $L_{ij} = -s_{mn} N_x N + (j-1)N + m$ and $j = (i-1)N_x N + (j-1)N + n, n \neq m$;

4. all other entries of $\mathbf{L}$ are 0;

for all $2 \leq i,j \leq N_x - 1, 1 \leq m \leq N$. The $N_x^2 N \times N_x^2 N$ matrix $\mathbf{D}_1$ is given by

1. $(\mathbf{D}_1)_{ij} = 1$ if $i = j = (i - 1)N_x N + (j - 1)N + m$ for $i \in \{1, N_x\}, 1 \leq j \leq N_x, 1 \leq m \leq N$;

2. $(\mathbf{D}_1)_{ij} = 1$ if $i = j = (i - 1)N_x N + (j - 1)N + m$ for $1 \leq i \leq N_x, j \in \{1, N_x\}, 1 \leq m \leq N$;

3. all other entries of $\mathbf{D}_1$ are 0.

The $N_x^2 N \times N_x^2 N$ matrix $\mathbf{D}_2$ is given by

1. $(\mathbf{D}_2)_{ij} = 1/d_x$ if $i = j = (i - 1)N_x N + (j - 1)N + m$ for $i \in \{1, N_x\}, 1 \leq j \leq N_x, 1 \leq m \leq N$;

2. $(\mathbf{D}_2)_{ij} = -1/d_x$ if $i = (i - 1)N_x N + (j - 1)N + m$ and $j = (i + 1 - 1)N_x N + (j - 1)N + m$ for $i = 1, 1 \leq j \leq N_x, 1 \leq m \leq N$;

3. $(\mathbf{D}_2)_{ij} = -1/d_x$ if $i = (i - 1)N_x N + (j - 1)N + m$ and $j = (i - 1)N_x N + (j - 1)N + m$ for $i = N_x, 1 \leq j \leq N_x, 1 \leq m \leq N$;

4. $(\mathbf{D}_2)_{ij} = 1/d_x$ if $i = j = (i - 1)N_x N + (j - 1)N + m$ for $1 \leq i \leq N_x, j \in \{1, N_x\}, 1 \leq m \leq N$;

5. $(\mathbf{D}_2)_{ij} = -1/d_x$ if $i = (i - 1)N_x N + (j - 1)N + m$ and $j = (i - 1)N_x N + (j + 1 - 1)N + m$ for $1 \leq i \leq N_x, j = 1, 1 \leq m \leq N$;

6. $(\mathbf{D}_2)_{ij} = -1/d_x$ if $i = (i - 1)N_x N + (j - 1)N + m$ and $j = (i - 1)N_x N + (j - 1)N + m$ for $1 \leq i \leq N_x, j = N_x, 1 \leq m \leq N$;

7. all other entries of $\mathbf{D}_2$ are 0.

The $N_x^2 N \times N_x^2 N$ matrix $D_x$ is given by

1. $(D_x)_{ij} = 1/d_x$ if $i = j = (i - 1)N_x N + (j - 1)N + m$ for $1 \leq j \leq N_x - 1, 1 \leq j \leq N_x - 1, 1 \leq m \leq N$;
CIP for parabolic equations

2. \((D_x)_{ij} = -1/d_x\) if \(i = (i-1)N_x N + (j-1)N + m\) and \(j = (i+1)N_x N + (j-1)N + m\) for \(i = 1, 1 \leq j \leq N_x, 1 \leq m \leq N;\)

3. all other entries of \(D_x\) are 0.

The \(N_x^2 N \times N_x^2 N\) matrix \(D_y\) is given by

1. \((D_y)_{ij} = 1/d_x\) if \(i = j = (i-1)N_x N + (j-1)N + m\) for \(1 \leq j \leq N_x - 1, 1 \leq j \leq N_x - 1, 1 \leq m \leq N;\)

2. \((D_y)_{ij} = -1/d_x\) if \(i = (i-1)N_x N + (j-1)N + m\) and \(j = (i+1)N_x N + (j-1)N + m\) for \(i = 1, 1 \leq j \leq N_x, 1 \leq m \leq N;\)

3. all other entries of \(D_y\) are 0.

The vector \(\mathbf{f}\) is defined as

1. \(\mathbf{f}_i = F_m(x_i, y_j)\) if \(i = (i-1)N_x N + (j-1)N + m\) for \(i \in \{1, N_x\}, 1 \leq j \leq N_x, 1 \leq m \leq N;\)

2. all other entries of \(\mathbf{f}\) are 0.

The vector \(\mathbf{g}\) is defined as

1. \(\mathbf{g}_i = G_m(x_i, y_j)\) if \(i = (i-1)N_x N + (j-1)N + m\) for \(i \in \{1, N_x\}, 1 \leq j \leq N_x, 1 \leq m \leq N;\)

2. all other entries of \(\mathbf{g}\) are 0.

Since \(c\) is small (in our computational program \(c = 10^{-9}\)), to find the minimizer \(V^{(0)}\) of the finite difference version of \(J_c\) in (33), we solve the linear system

\[
\begin{bmatrix}
\mathbf{L} \\
\mathbf{D}_1 \\
\mathbf{D}_2
\end{bmatrix}
+ \epsilon(1 + D_x^T D_x + D_y^T D_y) \begin{bmatrix}
\mathbf{L} \\
\mathbf{D}_1 \\
\mathbf{D}_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
\mathbf{g}_1 \\
\mathbf{g}_2
\end{bmatrix}
\]

Having \(b\) in hand, we can compute \(V^{(0)} = (v_1^{(0)}, ..., v_N^{(0)})^T\) using (31) and (32).

**Step 4.** The implementation for this step is similar to that for Step 2. In this step, we minimize

\[
J_c^{(p+1)}(V) = \sum_{m=1}^{N} \int_{\Omega} \left| \Delta u_m(x) - \sum_{n=1}^{N} s_{mn} v_n(x) - \frac{\Delta f(x)}{f(x)} v_m(x) + \sum_{n=1}^{N} \Psi_n(0) f(x) v_m(x) \right| \left| \partial_{x} v_m(x) - G_m(x) \right|^2 d\sigma(x) + e \int_{\Omega} \left| v_m(x) \right|^2 + |\nabla v_m(x)|^2 d\mathbf{x}
\]

(34)

To this end, we identify the vector valued function \(V\) by the vector \(b\) as in (31) and (32) and then solve the linear system

\[
\begin{bmatrix}
\mathbf{L} \\
\mathbf{D}_1 \\
\mathbf{D}_2
\end{bmatrix}
+ \epsilon(1 + D_x^T D_x + D_y^T D_y) \begin{bmatrix}
\mathbf{L} \\
\mathbf{D}_1 \\
\mathbf{D}_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
\mathbf{g}_1 \\
\mathbf{g}_2
\end{bmatrix}
\]

Here, the matrices \(\mathbf{D}_1, \mathbf{D}_2, D_x\) and \(D_y\) and the vectors \(\mathbf{g}\) and \(\mathbf{f}\) are defined in the implementation section for Step 2. The \(N_x^2 N \times N_x^2 N\) matrix \(\mathbf{L}\) is given by

1. \(L_{ij} = -4/d_x^2 - s_{mn} - \Delta f(x_i, y_j)/f(x_i, y_j) + v_m^{(0)}(x_i, y_j)\Psi_n(0)/f(x_i, y_j)\) if \(i = j = (i-1)N_x N + (j-1)N + m;\)

2. \(L_{ij} = 1/d_x^2\) if \(i = (i-1)N_x N + (j-1)N + m\) and \(j = (i \pm 1 - 1)N_x N + (j \pm 1)N + m;\)

3. \(L_{ij} = -s_{mn} + v_m^{(0)}(x_i, y_j)\Psi_n(0)/f(x_i, y_j)\) if \(i = (i-1)N_x N + (j-1)N + m\) and \(j = (i-1)N_x N + (j-1)N + m, n \neq m;\)
CIP for parabolic equations

(a) The function $c_{\text{true}}$
(b) The function $c^{(0)}$
(c) The function $c^{(1)}$
(d) The function $c^{(2)}$
(e) The function $c^{(10)}$
(f) The error function $\mathcal{E}$

Figure 2: Test 1. The true coefficient and computed coefficient $c$. We observe from Figure 2b that $c^{(0)}$, computed by Step 2 of Algorithm 1, has good "image" of the true inclusion. It is evident from the graph of the error function $\mathcal{E}$, see Figure 2f, that the sequence $\{c^{(p)}\}_p$ converges fast.

4. all other entries of $\mathcal{L}$ are 0;

for all $2 \leq i, j \leq N_x - 1$, $1 \leq m \leq N$. Having $v$ in hand, we can compute $V^{(p+1)} = (v^{(p+1)}_1, \ldots, v^{(p+1)}_N)^T$ using (31) and (32).

Steps 3, 5 and 6. The implementation of these steps is straight forward.

In the next section, we show some numerical results.

5. Numerical examples

The numerical results presented below are computed from the knowledge of $F_m(x, t)$ and $G_m(x, t)$, $m \in \{1, \ldots, N\}$, on $\partial \Omega \times [0, 0.3]$ including 10% of noise where $F_m$ and $G_m$ are the boundary data in Remark 2.2. The number of truncation $N$ is 25. The regularization parameter is $\epsilon = 10^{-9}$. The computational program is implemented by the finite difference method.

1. Test 1. The true function $c_{\text{true}}$ has a smooth inclusion

$$c_{\text{true}} = \begin{cases} 0 & x^2 + (y + 0.3)^2 \geq 0.35^2 \\ 20e^{\frac{x^2 + (y + 0.3)^2}{-0.23^2}} & x^2 + (y + 0.3)^2 \geq 0.35^2. \end{cases}$$

The numerical results for this case is displayed in Figure 2. One can observe in Figures 2b–2e that the circular shape and location of the inclusion can be successfully detected. The true maximal value of the function $c_{\text{true}}$ is 20. The reconstructed maximal value of the function $c_{\text{comp}} = c^{(10)}$ is 19.07. The relative error is 4.65%.

2. Test 2. We test the case when the function $c_{\text{true}}$ is a step function with two rectangular inclusions. This example is interesting since $c_{\text{true}}$ is not smooth and the gap at the boundaries of the inclusions is high. The function $c_{\text{true}}$
Figure 3: Test 2. The true coefficient and computed coefficient $c$. In this test, although the reconstructed coefficient $c^{(0)}$, see Figure 3b, is poor, the coefficient $c^{(1)}$ meets the expectation. It is evident from the graph of the error function $E$, see Figure 3f, that the sequence $\{c^{(p)}\}_p$ converges fast.

is given by

$$c_{\text{true}} = \begin{cases} 10 & |x| < 0.8 \text{ and } |y \pm 0.4| < 0.15, \\ 0 & \text{otherwise}. \end{cases}$$

The numerical results for this case is displayed in Figure 3. One can observe in Figures 3c–3e that the reconstructed rectangular shape and location of the inclusion are satisfactory. The true maximal value of the function $c_{\text{true}}$ is 10. The reconstructed maximal value of the function $c_{\text{comp}}$ is 10.98. The relative error is 9.80%. Similarly to the previous test, it is evident from Figure 3f that our method converges fast.

3. **Test 3.** We test the case of two circular inclusions. In this case, the function $c_{\text{comp}}$ is a step function with high gap at the boundary of the inclusions. The function $c_{\text{true}}$ is given by

$$c_{\text{true}} = \begin{cases} 5 & x^2 + (y + 0.5)^2 < 0.23^2, \\ 8 & x^2 + (y - 0.5)^2 < 0.23^2, \\ 0 & \text{otherwise}. \end{cases}$$

The numerical results for this case is displayed in Figure 4. One can observe in Figure 4b that the circular shape and can be successfully detected at the first step. The true maximal value of the function $c_{\text{true}}$ at the lower inclusion is 8 and the reconstructed one is 8.90. The relative error is 11.25%. The true maximal value of the function $c_{\text{true}}$ at the upper inclusion is 5 and the reconstructed one is 5.24. The relative error is 4.80%. Figure 4f shows the stability of our method.

4. **Test 4.** We test the case when the function $c_{\text{true}}$ is allowed to be negative. In this case, the function $c_{\text{comp}}$ is the
The true maximal positive value of the function $c_{\text{true}}$ is 8 and the reconstructed one is 8.90. The relative error is 11.25%. The true minimal negative value of the function $c_{\text{true}}$ is -8 and the reconstructed one is -7.93. The relative error is 0.88%. Again, Figure 5f shows the stability of our method.

Remark 5.1. It is evident from Figures 2c–5c that Algorithm 1 is robust in the sense that it provides good reconstructed coefficient $c_{\text{comp}}$ after a few iterations. It is remarkable mentioning that, in all tests above, our method provides good numerical results without any advanced knowledge of the true coefficient $c_{\text{true}}$.

6. Concluding remarks

In this paper, we introduced a new approach to numerically compute the perfusion coefficient of a general parabolic equation. The method consists of deriving and solving a nonlinear system of coupled partial differential equations.

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The function $c_{\text{true}}$

(b) The function $c^{(0)}$

(c) The function $c^{(1)}$

(d) The function $c^{(2)}$

(e) The function $c^{(10)}$

(f) The error function $\mathcal{E}$

Figure 5: Test 4. The true coefficient and computed coefficient $c$. We already see the letter “X” in the graph of the first approximation $c^{(0)}$, computed by Step 2 of Algorithm 1, see Figure 5b. It is evident from the graph of the error function $\mathcal{E}$, see Figure 5f, that the sequence $\{c^{(p)}\}_p$ converges fast.

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