The Two-dimensional Nonlinear Burridge-Knopoff Model of Earthquakes

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Abstract
We present a two-dimensional spring-block model of earthquakes with the full nonlinear equations for the forces in terms of displacements. Correspondence of our linearized version to earlier models reveals in them inherently asymmetric elastic properties. Our model generalizes those models and allows an investigation of the effects of internal strains and vectorial forces. The former is found to be relevant to critical properties such as that described by the Gutenberg-Richter law, but the latter as well as nonlinearities up to second order in displacements are irrelevant.

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In 1967, Burridge and Knopoff\cite{1} introduced a one-dimensional (1D) system of springs and blocks to study the role of friction along a fault in the propagation of an earthquake. Since then, many other researchers investigated similar dynamical models of many-body systems with friction, ranging from propagation and rupture in earthquakes\cite{2-10} to the fracture of overlayers on a rough substrate\cite{11}.

Among these developments, a deterministic version of the 1D Burridge-Knopoff (BK) model was analyzed by Carlson and Langer\cite{2} and the same model but in a quasi-static limit was studied by Nakanishi\cite{3}. A 2D quasi-static variant was first simulated by Otsuka\cite{4} and later by Brown, Scholz and Rundle\cite{5}, who introduced a discrete version that was formulated as a cellular automaton. A similar model with continuous local variables, generalizing the model of Bak, Tang and Wiesenfeld\cite{6}, was studied by Olami, Feder and Christensen\cite{7} (OFC). Contrary to previous models, the model is non-conservative and derivable from a 2D BK model (under certain limit, see below). It produces robust critical behavior which depends on the conservation level. OFC argued that such dependence explains the variance of the exponent in the Gutenberg-Richter\cite{12} law observed in real earthquakes\cite{13}.

Model. — As before, our model consists of a 2D array of interconnected blocks in contact with a rough surface. Each block is also connected to a rigid driving plate by a leaf spring whose spring constant is $K_L$. The plate moves at constant, infinitesimal speed. The coordinates for the attachment of the leaf springs on the moving plate, labeled by $(i, j)$ where $1 \leq i, j \leq L$, form a square lattice with lattice constants $a_1, a_2$ (hereafter the subscripts ‘1’ and ‘2’ refer to properties in the $x$- and $y$-direction, respectively). Driving the plate thereby induces stress between the array and the plate. Each block in the bulk is connected to four nearest neighbors via coil springs whose spring constants are $K_1$ and $K_2$ and unstretched lengths $l_1$ and $l_2$. We restrict ourselves to the situation where $a > l$, and the displacements $x_{i,j}, y_{i,j}$ measured from $(i, j)$ fulfill $x_{i,j} \ll a_1$ and $y_{i,j} \ll a_2$, so that Hooke’s law applies. When the net force on a block exceeds a static frictional threshold $F_{th}$, the block slips instantaneously to a new equilibrium position.
While this is an obvious extension of the 1D BK model to two dimensions, equations from previous models fail to describe it correctly. The reason is that elements in those models have been restricted to move in one direction only. To our knowledge, this restriction has not been justified beyond the reason of simplicity. T o address its relevance, we lift this limitation and start with the proper nonlinear equations for forces as functions of the displacements of blocks. The net force $\vec{F}_{i,j}$ (which equals the friction due to the rough surface) on the block at $(i, j)$ is a vector:

$$\vec{F}_{i,j} = \vec{f}_L + \vec{f}_{(i+1,j)-(i,j)} + \vec{f}_{(i-1,j)-(i,j)} + \vec{f}_{(i,j+1)-(i,j)} + \vec{f}_{(i,j-1)-(i,j)}$$  \hspace{1cm} (1)

where $\vec{f}_{(i\pm1,j\pm1)-(i,j)}$ is the force exerted by a neighboring block and $\vec{f}_L$ the loading force by the driving plate. For example, we have

$$f^x_{(i+1,j)-(i,j)} = K_1[a_1 + x_{i+1,j} - x_{i,j} - \frac{(a_1 + x_{i+1,j} - x_{i,j})l_1}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}}]$$ \hspace{1cm} (2)

$$f^y_{(i+1,j)-(i,j)} = K_2[y_{i+1,j} - y_{i,j} - \frac{(y_{i+1,j} - y_{i,j})l_2}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}}]$$ \hspace{1cm} (3)

Then the $x$ component of the force in Eq. (1) takes the form:

$$F^x_{i,j} = f^x_L - K_1[2x_{i,j} - x_{i+1,j} - x_{i-1,j} + \frac{(a_1 + x_{i+1,j} - x_{i,j})l_1}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}}] - \frac{(a_1 - x_{i-1,j} + x_{i,j})l_1}{\sqrt{(a_1 - x_{i-1,j} + x_{i,j})^2 + (y_{i,j} - y_{i-1,j})^2}} - K_2[2x_{i,j} - x_{i,j+1} - x_{i,j-1} + \frac{(x_{i,j+1} - x_{i,j})l_2}{\sqrt{(a_2 + y_{i,j+1} - y_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2}}] + \frac{(x_{i,j-1} - x_{i,j})l_2}{\sqrt{(a_2 - y_{i,j-1} + y_{i,j})^2 + (x_{i,j} - x_{i,j-1})^2}}]$$  \hspace{1cm} (4)

and, by symmetry, $F^y_{i,j}$ follows by switching $x \leftrightarrow y$, $i \leftrightarrow j$, $l_1 \leftrightarrow l_2$, $a_1 \leftrightarrow a_2$ and $K_1 \leftrightarrow K_2$.

**Linear version.** — In order to compare with the OFC model, we expand $\vec{F}_{i,j}$ to first order in $x$’s and $y$’s. Specifying $\vec{f}_L = (-K_Lx_{i,j}, 0)$ as in \cite{3, 7}, one can easily show that a slip of the block at $(i, j)$ results in the following force redistribution\cite{13, 10}:

$$F^x_{i\pm1,j} \rightarrow F^x_{i\pm1,j} + \alpha_1 F^x_{i,j},$$

$$F^x_{i,j\pm1} \rightarrow F^x_{i,j\pm1} + S_2 \sigma_1 F^x_{i,j},$$  \hspace{1cm} (5)
\[ F_{i \pm 1, j}^y \to F_{i \pm 1, j}^y + \frac{S_1}{\sigma} \alpha_2 F_{i, j}^y, \]
\[ F_{i, j \pm 1}^y \to F_{i, j \pm 1}^y + \alpha_2 F_{i, j}^y, \]
\[ \vec{F}_{i, j} \to 0, \]

where \( S_1 \equiv (a_1 - l_1)/a_1 \) is the internal strain in the \( x \) direction, and similarly for \( S_2 \).
\( \sigma \equiv K_2/K_1 \) and \( \kappa \equiv K_L/K_1 \) are measures of anisotropies in the couplings. In the bulk, \( \alpha_1 \) and \( \alpha_2 \) are given by:
\[ \alpha_1 = \frac{1}{2(1 + S_2 \sigma) + \kappa}; \quad \alpha_2 = \frac{1}{2(1 + S_1/\sigma)}. \]

Their values at boundary sites depend on the boundary conditions (see below).

Using Eq. (5), our model can be described as a coupled map lattice as was done in [7]. If we impose \( y_{i, j} \equiv 0 \) at all sites as in [3, 7], so that (to first order) all \( F_{i, j}^y = 0 \), we readily recover the OFC model in the limit \( S_2 \to 1 \), i.e., for maximal internal strain along the \( y \) direction. This also follows from the general relation Eq. (4), as \( F_x \) becomes linear in \( x \) for \( S_2 = 1 \) and \( y = 0 \). The physical rationale behind this correspondence is that in [4, 5, 7], both the loading springs and those in the array along \( y \) were chosen as leaf springs, which by definition cannot be extended along their length but have a restoring force linear in the transverse displacement. They act like fully stretched coil springs. With leaf springs, the forces become scalar and the equations linear. However, it introduces an artificial asymmetry into the system, which to our knowledge has no analog in nature. This manifests itself most clearly in the elastic moduli [4, 15]: For the OFC model and the like, \( C_{1111} = K_1 a_1/a_2 \), \( C_{2222} = \infty \), \( C_{1122} = 0 \), and \( C_{1212} \) (shearmodulus) = \( K_2 a_2/a_1 \). Clearly, setting \( K_1 \equiv K_2 \) does not render the model symmetric. For our model, the asymmetry is absent: \( C_{1111} = K_1 a_1/a_2 \), \( C_{2222} = K_2 a_2/a_1 \), \( C_{1122} = 0 \), and \( C_{1212} = (K_1 K_2 S_1 S_2 a_1 a_2)/(K_1 S_1 a_1^2 + K_2 S_2 a_2^2) \).

For general strain \( 0 \leq S_2 < 1 \) and \( F^y \equiv 0 \), the linear version of our model coincides with the anisotropic OFC model [7] with our \( S_2 \sigma \) corresponding to OFC’s \( K_2/K_1 \). Consequently, except in the unphysical limit \( S_2 \to 1 \), asymmetric loading on a symmetric model \((\kappa = \sigma = 1, S_2 = S_1)\) gives rise to intrinsically asymmetric force redistributions. The statement [7]
regarding the variance in the Gutenberg-Richter law as a result of different elastic parameters
has to be reinterpreted, in the present context, as a result of variances in both internal strains
and elastic parameters.

Allowing for \( y_{i,j} \neq 0 \), Eqs.(5) yield the total change of force after a block in the bulk at
\((i,j)\) slips by a distance \((\delta x, \delta y)\):

\[
\delta F^x = -K_L \delta x = -\kappa \alpha_{i,j}^x ; \quad \delta F^y = 0.
\] (7)

The change on the boundary depends on the boundary conditions, which are defined by
the number of neighbors a boundary block is attached, and the way it is loaded. Following
OFC, in “free” boundary conditions (FBC) the sites on edges have three neighbors while
those at corners have two. In “open” boundary conditions (OBC) all sites has four neighbors.
Both have uniform loading throughout the system by an amount \( \Delta F^x = K_L \Delta x \), where \( \Delta x \)
denotes the displacement of the driving plate relative to the array.

For FBC, the \( \alpha \)'s on the boundary differ from the bulk values in Eq. (6):

\[
\alpha_{1}^{(x=1,L)} = \frac{1}{1 + 2S_2 \sigma + \kappa}, \quad \alpha_{1}^{(y=1,L)} = \frac{1}{2 + S_1 \sigma + \kappa}, \quad \alpha_{1}^{(\text{corner})} = \frac{1}{1 + S_2 \sigma + \kappa},
\]
\[
\alpha_{2}^{(x=1,L)} = \frac{1}{2 + S_2 \sigma + \kappa}, \quad \alpha_{2}^{(y=1,L)} = \frac{1}{1 + 2S_1 \sigma + \kappa}, \quad \alpha_{2}^{(\text{corner})} = \frac{1}{1 + S_1 \sigma + \kappa}.
\] (8)

It turns out then that Eq. (7) holds at all sites. While the change of \( F^x \) is balanced on the
average by loading, the spatial mean \( \bar{F}^y \) is always zero due to Newton’s third law and the
fact that the system is isolated for FBC. Since a slip is decided by \( \sqrt{(F^x)^2 + (F^y)^2 - F_{th}} \), the
two components are locally coupled in the dynamics. We find in the steady state that the
fluctuation of \( F^y \) tends to zero, so that this coupling to a conservative field is irrelevant[17],
i.e., the OFC limit is recovered.

For OBC, Eq. (8) holds at all sites but Eq. (7) is modified to \( \delta F^y / F^y < 0 \) at boundary
sites due to their coupling to the external imaginary blocks. This implies that a system with
arbitrary initial spatial mean, \( \bar{F}^y \), will always flow in steady state to the \( \bar{F}^y = 0 \) (stable)
fixed point (i.e., the OFC model). We conclude, for OBC too, that the steady state of our
linearized version is the same as that of the OFC model.
We now remark on the boundary conditions. We first reveal a flaw in previous treat-
ments of OBC[7]. Effectively, the “imaginary layer of blocks” in OBC corresponds to a rigid
frame that attaches to the array by springs. Spatially uniform loading in Ref. [6] implies
no relative displacement between the frame and the array during loading. Since the imag-
inary layer never slips, it is permanently pinned on the rough surface. Although this is an
unphysical setup, it has not been appreciated before in [7, 9] because it is not obvious in
the $F_{i,j}$ configurations. Working with the linearly related $x_{i,j}$ instead, we show in Fig. 1
how the initially square array of blocks is distorted due to the pinning. At long times,
the distortion can be arbitrarily large so that there is no meaningful steady state for this
model. To make physical sense, the frame has to move along with the driving plate. It is
then intuitively clear that loading cannot be uniform: due to pulling and pushing by the
frame, boundary blocks are loaded more than in the bulk (e.g., $\Delta F_{1,j}^x = (1 + \kappa^{-1})\Delta F_{\text{bulk}}^x$).
Incorporating such non-uniformity, we observe numerically[15] that the model becomes non-
critical and reaches periodic states much like with periodic boundary conditions[9]. Second,
for FBC, we implicitly assume in Eqs. (5) and (8) that the internal strains are somehow
sustained. Otherwise, due to missing neighbors at the boundary, the equilibrium positions
of blocks would be shifted inward relative to a regular lattice. In practice, the locations of
loading springs on the driving plate may be adjusted to eliminate the shifts. In any case,
bulk properties are not affected because the shifts decay exponentially into the bulk over a
distance[=1/ln(2+$\kappa$)] of order unity[15]. The use of leaf springs along $y$ direction in [4, 5, 7]
may also be interpreted as another means of imposing the strains.

Nonlinear version. — Eq. (4) allows for an investigation of nonlinear effects which
naturally arise from the full forcing. However, it is no longer possible to describe the model
as a coupled map lattice [cf. Eq. (5)]. Instead, one has to keep track of the displacements in
order to determine whether $|\vec{F}| > F_{th}$ — if so, one has to solve for the displacements $(\tilde{x}, \tilde{y})$
that defines the zero-force position of the block in terms of its environment. To $n^{th}$ order,
this means to solve two coupled equations of the form [for brevity, \( q \equiv (i, j) \)]:

\[
\tilde{F}_q^x \equiv 0 = f_L^x - \sum_p K_{p,q} \{ u_{p,q} - \sum_{m=0}^n \frac{1}{m!} [u_{p,q} \frac{\partial}{\partial x} + v_{p,q} \frac{\partial}{\partial y}]^m G_{p,q}(x, y)|_{(x,y)=(0,0)} \}, \tag{9}
\]

plus a corresponding equation (\( \tilde{F}_q^y \equiv 0 \)) obtained by the transformation after Eq. (4). Here \( p \) denotes the four nearest neighbors \((i \pm 1, j \pm 1)\), and the symbols stand for:

\[
K_{p,q} = K_1, \quad G_{p,q}(x, y) = \frac{(x \mp a_1)l_1}{\sqrt{(x \mp a_1)^2 + y^2}}; \quad \text{for } p = (i \pm 1, j);
\]

\[
K_{p,q} = K_2, \quad G_{p,q}(x, y) = \frac{x l_2}{\sqrt{(y + a_1)^2 + x^2}}; \quad \text{for } p = (i, j \pm 1);
\]

\[
u_{p,q} = \tilde{x}_q - x_p, \quad v_{p,q} = \tilde{y}_q - y_p, \quad \text{for } p = (i \pm 1, j), (i, j \pm 1).
\tag{10}
\]

Note that we have implicitly assumed that no nearest neighbors can be supercritical at the same time, due to infinitesimal loading rate.

Apparently this program is not computationally efficient for large \( n \). But \( n = 2 \) is simple: it is easy to see that all second order terms (\( \tilde{x}^2, \tilde{y}^2, \tilde{x}\tilde{y} \)) cancel in the bulk (i.e., when \((i, j)\) has four nearest neighbors), resulting in two coupled first order equations for \((\tilde{x}, \tilde{y})\) in terms of \( F \) and \((x, y)\) before slipping:

\[
\frac{F_{i,j}^x}{K_1} = (\tilde{x}_{i,j} - x_{i,j})[\alpha^{-1} + \frac{\sigma l_2}{a^2} (y_{i,j+1} - y_{i,j-1})] + (\tilde{y}_{i,j} - y_{i,j})\frac{l_1}{a^2} (y_{i,j+1} - y_{i,j-1}) + \frac{\sigma l_2}{a^2} (x_{i,j+1} - x_{i,j-1}), \tag{11}
\]

plus a corresponding equation for \( F_{i,j}^y \). Note that it yields nonlinear dependence of \((\tilde{x}, \tilde{y})\) on \((x, y)\) via those terms proportional to \( l/a^2 \).

Based on Eq. (11), we have performed simulations for different boundary conditions. Due to missing neighbors of the blocks on the edges for FBC, second order terms in \((\tilde{x}, \tilde{y})\) survive and one has to solve higher order (3rd and 4th) equations for the equilibrium positions of the boundary blocks. Remarkably, even with substantial nonlinearities (determined by \( l/a^2 \)), in none of the cases of FBC, OBC and PBC do we find the critical behavior to deviate from that of the linear cases.

To summarize, we have investigated the critical behavior of a general 2D spring-block model of earthquakes, within the context of the BK model under the quasi-static limit. We
discover that internal strains are important ingredients in the variance of the Gutenberg-Richter law. Nonlinear perturbations, to which the OFC model displays remarkable stability, are not introduced arbitrarily but generated generically at the microscopic level from the elastic spring-block interactions. Among possible extensions, the case of internal compressional strains (i.e., \( S < 0 \)), and of different loadings, including those along both directions [i.e., \( f_L = -(K_x^L x, K_y^L y) \)] and spatially non-uniform ones (e.g., shear applied through the boundary), are being pursued[15].

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In 2D, the equations of motion in the continuum approach of Ref. [2, 10] to first order in the displacements \( \vec{U} \equiv (U_x, U_y) \), including strain \( S \), are similarly modified:

\[
\begin{align*}
\partial_t^2 U_x &= \xi^2 (\partial_x^2 U_x + S \partial_y^2 U_x) - U_x - \partial_t U_x \phi(|\partial_t \vec{U}|) \\
\partial_t^2 U_y &= \xi^2 (S \partial_x^2 U_y + \partial_y^2 U_y) - \partial_t U_y \phi(|\partial_t \vec{U}|),
\end{align*}
\]

where \( \phi(|\partial_t \vec{U}|) \) is a speed-dependent friction term and \( \xi \) a characteristic length. The Eq. (9) of Ref. [10] may be obtained by setting \( S = 1 \) and \( U_y = 0 \).

In analogy to critical dynamics [see, e.g., P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. 49, 435 (1977)], we may consider the model as a generalized coupled map lattice with an auxiliary field \( F^y \) conserved at \( \bar{F}^y \neq 0 \). We found that the value of \( \bar{F}^y \) is relevant, e.g., the exponent \( B \) in Gutenberg-Richter law decreases as \( \bar{F}^y \) increases, implying that \( F^y \equiv 0 \) is special.
Figure captions.

Fig. 1. Snapshot of a configuration of blocks after 2000 avalanches, for $L = 20$, $a_1 = a_2 = 1$, $l_1 = l_2 = 0.1$ and $\sigma = \kappa = K_1 = F_{th} = 1$, showing the unphysical effect of pinned frame (denoted by □) in OBC. The array is pulled to the right. The system evolves according to Eq. (11).
