Abstract—In this paper, we develop a Topological Approximate Dynamic Programming (TADP) method for planning in stochastic systems modeled as Markov Decision Processes to maximize the probability of satisfying high-level system specifications expressed in Linear Temporal Logic (LTL). Our method includes two steps: First, we propose to decompose the planning problem into a sequence of sub-problems based on the topological property of the task automaton which is translated from the LTL constraints. Second, we extend a model-free approximate dynamic programming method for value iteration to solve, in an order reverse to a causal dependency of value functions, one for each state in the task automaton. Particularly, we show that the complexity of the TADP does not grow polynomially with the size of the product Markov Decision Process (MDP). The correctness and efficiency of the algorithm are demonstrated using a robotic motion planning example.

I. INTRODUCTION

Temporal logic is an expressive language to describe desired system properties: safety, reachability, obligation, stability, and liveness [1]. This paper introduces a model-free planning method for stochastic systems modeled as MDPs, where the planning objective is to maximize the (discounted) probability of satisfying constraints in a subclass of temporal logic—syntactically co-safe LTL (sc-LTL) formulas [2].

Our goal is to address two major problems in temporal logic-constrained probabilistic synthesis: First, various model checking and probabilistic verification methods for MDPs are model-based [3], [4]. For systems without a model but with a blackbox simulator, Reinforcement Learning (RL) methods for LTL constraints have been developed with both model-based and model-free methods [5], [6], [7], [8]: A model-based RL learns a model and a near-optimal policy simultaneously. A model-free RL learns only the near-optimal policy from sampled trajectories in the stochastic systems. However, model-free reinforcement learning methods, such as policy gradient and actor-critic methods [9], [10], [11], face challenges when being used for planning with temporal logic constraints: First, LTL specification translates to a sparse reward signal: The learner receives reward of one if the constraint is satisfied. This sparse reward provides little gradient information in the policy/value function search. The problem is more severe when complex specifications are involved. Taking the following task as an example, a robot needs to visit regions A, B, and C, but if it visits D, then it must visit B before C. If the robot only visits A or B, it will receive zero reward. When the state space of the MDP is large, a learner receiving zero reward has no way to improve its current policy. To address reward sparsity, reward shaping [12] has been developed: Shaping introduces additional reward signals while guaranteeing the policy invariance—the optimal policy remains the same with/without shaping. However, this method has strict requirements for the range of shaping potentials, which is hard to define in most of the cases when LTL constraints are considered.

In this work, we investigate a different approach besides reward shaping to mitigate the challenges in RL under sparse reward signals in LTL. Our approach is inspired by an idea to make value iteration efficient: In an acyclic MDP, there exists an optimal backup order; such that each state in the MDP only needs to perform one-step backup operation in value iteration [13]. In [14], the authors generalize this optimal backup order for acyclic MDPs to general MDPs. They develop a Topological Value Iteration (TVI) method that divides an MDP into Strongly Connected Components (SCCs) and then solves each component sequentially in the topological order. Though it seems straightforward to apply TVI to the product MDP, which is obtained by augmenting the original MDP with a finite set of memory states related to the task, we are interested in developing Approximate Dynamic Programming (ADP) methods to mitigate the scalability problem when planning with large MDPs. To this end, we propose a Topological Approximate Dynamic Programming (TADP) method that includes two stages: Firstly, we translate the task formula into a Deterministic Finite Automaton (DFA) referred as the task DFA, and then exploit the graphical structure in the automaton to determine a topological optimal backup order for a set of value functions—one for each discrete state in the task DFA. The set of value functions are related by the transitions in the task DFA and jointly determine the optimal policy based on the Bellman equation. Secondly, we introduce function approximations for the set of value functions to reduce the number of decision variables—the number of states in the product MDP—to a number $M$ of weights, with $M \ll N$. Finally, we integrate a model-free ADP with value iteration and combine it with the backup ordering to solve the set of value function approximations, one for each task state, in an optimal order. By doing this, the sparse reward, received when the task is completed, is propagated back to earlier stages of task completion to provide meaningful gradient information for the learning algorithm.

Exploiting the structure of task DFAs for planning has been considered in [15] where the authors partition the task DFA into SCCs and then define progress levels towards satis-
faction of the specification. In this work, we formally define a topological backup order based on a causal dependency among states in task DFA. We prove the optimality in this backup order. Further, this backup order can be integrated with the actor-critic method for LTL constrained planning in [8], or other ADP methods that solve value function approximations, to address the sparse reward problem.

The rest of the paper is structured as follows. Section II provides some preliminaries. Section III contains the main results of the paper, including computing the topological order, proof of optimality in this order, and the TADP algorithm. The correctness and effectiveness of the proposed method are experimentally validated in the Section IV with robot motion planning. Section V summarizes.

II. PRELIMINARIES

Notation: Given a finite set \( X \), let \( \Delta(X) \) be the set of probability distributions over \( X \). The size of the set \( X \) is denoted as \( \text{card}(X) \). Let \( \Sigma \) be an alphabet (a finite set of symbols). Given \( k \in \mathbb{Z} \), \( \Sigma^k \) indicates a set of words with length \( k \), \( \Sigma^{\leq k} \) indicates a set of finite words with length smaller or equal to \( k \), and \( \Sigma^0 = \lambda \) is the empty word. \( \Sigma^* \) is the set of all finite words (also known as Kleene closure of \( \Sigma \)), and \( \Sigma^\omega \) is the set of all infinite words. \( 1_X \) is the indicator function with \( 1_X(x) = 1 \), if \( x \in X \), or 0, otherwise.

A. Syntactically co-safe Linear Temporal Logic

Syntactically co-safe LTL formulas [16] are a well-defined subclass of LTL formulas. Formally, given a set \( \mathcal{AP} \) of atomic propositions, the syntax of sc-LTL formulas over \( \mathcal{AP} \) is defined as follows:

\[
\varphi := \text{true} \mid p \mid \neg p \mid \varphi_1 \land \varphi_2 \mid \varphi_1 \lor \varphi_2 \mid \Box \varphi \mid \varphi_1 \U \varphi_2,
\]

where \( \varphi, \varphi_1 \) and \( \varphi_2 \) are sc-LTL formulas, \( \text{true} \) is the unconditional true, and \( p \) is an atomic proposition. Negation (\( \neg \)), conjunction (\( \land \)), and disjunction (\( \lor \)) are defined the same as the standard Boolean operators. Syntactically co-safe LTL formulas only contain temporal operators "Next" (\( \Box \)), "Until" (\( \U \)), and "Eventually" (\( \Diamond \)). However, temporal operator "Always" (\( \forall \)) cannot be expressed in sc-LTL.

An infinite word with alphabet \( 2^{\mathcal{AP}} \) satisfying a sc-LTL formula always has a finite-length good prefix [16]. Formally, given a sc-LTL formula \( \varphi \) and an infinite word \( w = r_0r_1\ldots \) over alphabet \( 2^{\mathcal{AP}} \), \( w \models \varphi \) if there exists \( n \in \mathbb{N} \), \( w|_{[0,n]} \models \varphi \), where \( w|_{[0,n]} = r_0r_1\ldots r_n \) is the length \( n + 1 \) prefix of \( w \).

Thus, a sc-LTL formula \( \varphi \) over \( 2^{\mathcal{AP}} \) can be translated to an DFA \( \mathcal{A}_\varphi = (Q, \Sigma, \delta, q_0, F) \), where \( Q \) is a finite set of states, \( \Sigma = 2^{\mathcal{AP}} \) is a finite set of input symbols called the alphabet, \( \delta : Q \times \Sigma \to Q \) is a transition function, \( q_0 \in Q \) is an initial state, and \( F \subseteq Q \) is a set of accept states. A transition function is recursively extended in the general way: \( \delta(q, aw) = \delta(\delta(q, a), w) \) for given \( a \in \Sigma \) and \( w \in \Sigma^* \). A word is accepting if and only if \( \delta(q, u) \in F \) and \( u \) is a prefix of \( w \), i.e., \( w = uv \) for \( u, v \in \Sigma^* \). DFA \( \mathcal{A}_\varphi \) accepts the set of words satisfying \( \varphi \).

We consider stochastic systems modeled by MDPs. By introducing the labeling function, we relate the paths in a MDP \( M \) to a given specification described by a sc-LTL formula.

Definition II.1 (Labeled MDP). A labeled MDP is a tuple \( M = (S, A, s_0, P, \mathcal{AP}, L) \), where \( S \) and \( A \) are finite state and action sets, \( s_0 \) is the initial state, the transition probability function \( P(\cdot \mid s, a) \in \Delta(S) \) is defined as a probability distribution over the next state \( s' \) given action \( a \) is taken at the current state, \( \mathcal{AP} \) denotes a finite set of atomic propositions, and \( L : S \to 2^{\mathcal{AP}} \) is a labeling function which assigns each state \( s \in S \) to a set of atomic propositions \( L(s) \subseteq \mathcal{AP} \) that are valid at the state \( s \).

A finite-memory stochastic policy in the MDP is a function \( \pi : S^* \to \Delta(A) \) that maps a history of state sequence into a distribution over actions. A Markovian stochastic policy in the MDP is a function \( \pi : s \to \Delta(A) \) that maps the current state into a distribution over actions. Given an MDP \( M \) and a policy \( \pi \), the policy induces a Markov chain \( M^\pi = \{ s_k \mid t = 1, \ldots, \infty \} \) where \( s_k \) is the random variable for the \( k \)-th state in the Markov chain \( M^\pi \), and it holds that \( s_{k+1} \sim P(\cdot \mid s_k, a_k) \) and \( a_k \sim \pi(\cdot \mid s_0s_1\ldots s_k) \).

Given a finite (resp. infinite) path \( \rho = s_0s_1\ldots s_N \in S^* \) (resp. \( \rho \in S^\omega \)), we obtain a sequence of labels \( L(\rho) = L(s_0)L(s_1)\ldots L(s_N) \in \Sigma^* \) (resp. \( L(\rho) \in \Sigma^\omega \)). A path \( \rho \) satisfies the formula \( \varphi \), denoted as \( \rho \models \varphi \), if and only if \( L(\rho) \) is accepted by \( \mathcal{A}_\varphi \). Given a Markov chain induced by policy \( \pi \), the probability of satisfying the specification, denoted as \( P(M^\pi \models \varphi) \), is the expected sum of the probabilities of paths satisfying the specification.

\[
P(M^\pi \models \varphi) := \mathbb{E}\left[ \sum_{t=0}^{\infty} \mathbb{1}(p_t \models \varphi) \right],
\]

where \( p_t = s_0s_1\ldots s_t \) is a path of length \( t \) in \( M^\pi \).

To design a policy \( \pi \) that maximizes the probability of satisfying the specification, the planning is performed in a MDP with an augmented state space.

A problem in probabilistic planning under temporal logic constraints is to maximize the probability of constraints. A formal problem statement is in the following.

Problem 1. Given a labeled MDP \( M \) and an sc-LTL formula \( \varphi \), the MaxProb problem is to synthesize a policy \( \pi \) that maximizes the probability of satisfying \( \varphi \), that is

\[
\pi^* = \arg \max_\pi \Pr(M^\pi \models \varphi).
\]

The MaxProb problem can be in the product MDP defined as follows.

Definition II.2 (Product MDP). Given a labeled MDP \( M = (S, A, s_0, P, \mathcal{AP}, L) \), sc-LTL formula \( \varphi \), and a corresponding DFA \( \mathcal{A}_\varphi = (Q, \Sigma, \delta, q_0, F) \), the product of M and \( \mathcal{A}_\varphi \), denoted as \( M \times \mathcal{A}_\varphi = (S \times Q, (s_0, q_0), S \times F, A, \delta, R) \) with \( S \times Q \) is the set of states, \( A \) an initial state \( (s_0, q_0) \),\( S \times F \) is the set of accepting states, \( R \) the transition function is defined by \( P((s', q'), a') \mid (s, q), a) = P(s' \mid s, a)\mathbf{1}_{q'}(\delta(q, L(s))) \), where \( R: S \times Q \times A \to [0, 1] \) to be defined.
such that
\[
R((s, q), a) = \sum_{(s', q')} P((s', q') \mid (s, q), a) \cdot 1_F(q').
\]

We make all states in \( S \times F \) sink/absorbing states, i.e., for any \((s, q) \in S \times F\), for any \( a \in A\), \( P((s, q) \mid (s, q), a) = 1\) and define \( R((s, q), a) = 0\) for all states \((s, q) \in S \times F\) and \( a \in A\). For clarity, we denote this product MDP by \( M_{\phi}\), i.e., \( M_{\phi} = M \otimes A_{\phi}\). When the specification \( \phi \) is clear from the context, we denote the product MDP by \( M\).

By definition, the path will receive a reward one if it ends in the set of accepting states \( S \times F\). The total expected reward given a policy \( \pi \) is the probability of satisfying the formula \( \phi\). By maximizing the total reward we find an optimal policy for the MaxProb problem. In practice, we are often interested in maximizing a discounted total reward, which is the discounted probability of satisfying \( \phi\). Let \( \gamma \) be a discounting factor. The planning problem is to solve the optimal value function and policy function that satisfy
\[
V((s, q)) = \tau \log \sum_a \exp(R((s, q), a))
\]
\[
+ \gamma \sum_{s', q'} P((s', q') \mid (s, q), a) V((s', q'))/\tau,
\]
\[
Q((s, q), a) = R((s, q), a) + \gamma \sum_{s', q'} P((s', q') \mid (s, q), a) V((s', q'))/\tau,
\]
\[
\pi(a \mid (s, q)) = \exp((Q((s, q), a) - V((s, q)))/\tau),
\]
where \( \tau \) is the user-specified temperature parameter, and we use softmax Bellman operator instead of hardmax Bellman operator. Value Iteration (VI) can solve the optimal value function in the product MDP and converges in the polynomial time of the size of the state space, i.e., card\((S \times Q)\). However, VI is model-based and also difficult to scale to large planning problems with a complex specification.

III. MAIN RESULT

In this work, we are interested in developing model-free reinforcement learning algorithms for solving the MaxProb problem. However, if we directly solve for approximate optimal policies in the product MDP using the method in Section III-B then as the reward is rather sparse, it is possible that a path satisfying the specification is a rare event to be sampled. As a consequence, the estimate of the gradient in [17] has a high variance with finite samples. To address this problem, we develop Topological Approximate Dynamic Programming that leverages the structure property in the specification automaton to improve the convergence due to sparse and temporally extended rewards with LTL specifications.

A. Hierarchical decomposition and causal dependency

First, it is observed that given temporally extended goals, it is possible to partition the product state space based on the discrete automaton states, also referred as discrete modes. The following definitions are generalized from almost-sure invariant set [18] in Markov chains to MDPs.

Definition III.1 (Invariant set and guard set). Given a DFA state \( q \in Q \) and an MDP \( M\), the invariant set of \( q \) with respect to \( M\), denoted as \( \text{Inv}(q, M)\), is a set of MDP states such that no matter which action is selected, the system has probability one to stay within the set \( q\). Formally,
\[
\text{Inv}(q, M) = \{ s \in S \mid \forall a \in A, \exists s' \in S, P(s' \mid s, a) > 0 \implies \delta(q, L(s')) = q\}. \tag{2}
\]
where \( \implies \) means implication.

Given a pair \((q, q')\) of DFA states, where \( q \neq q'\), the guard states of the transition from \( q \) to \( q'\), denoted as \( \text{Guard}(q, q', M)\), is a subset of \( S \) in which a transition from \( q \) to \( q'\) may occur. Formally,
\[
\text{Guard}(q, q', M) = \{ s \in S \mid \exists a \in A, \exists s' \in S, P(s' \mid s, a) > 0 \land \delta(q, L(s')) = q', \text{ where } q \neq q'\}. \tag{3}
\]

When the MDP \( M\) is clear from the context, we refer \( \text{Inv}(q, M)\) to \( \text{Inv}(q)\) and \( \text{Guard}(q, q', M)\) to \( \text{Guard}(q, q')\). Next, we define causal dependency between modes: In the product MDP \( M\), a state \((s_1, q_1)\) is causally dependent on state \((s_2, q_2)\), denoted as \((s_1, q_1) \rightarrow (s_2, q_2)\), if there exists an action \( a \in A\) such that \( P((s_2, q_2) \mid (s_1, q_1), a) > 0\). This causal dependency is initially introduced in [14] and generalized to the state space of the product MDP.

According to the definition of Bellman equation (4), if there exists a probabilistic transition from \((s_1, q_1)\) to \((s_2, q_2)\) in the product MDP, then the Bellman equation will have the optimal value \( V(s_1, q_1)\) depending on the value \( V(s_2, q_2)\).

Two states can be causally dependent on each other. If that is the case, we say that these two states are mutually causally dependent. Next, we lift the causal dependency from product MDP to the specification DFA, by introducing casually dependent modes.

Definition III.2 (Causally dependent modes). A mode \( q_1\) is causally dependent on mode \( q_2\) if and only if \( \text{Guard}(q_1, q_2) \neq \emptyset\), where \( q_1 \neq q_2\). That is, there exists a transition in the product MDP from a state in mode \( q_1\) to a state in mode \( q_2\), where \( q_1 \neq q_2\). A pair of modes \((q_1, q_2)\) is mutually causally dependent if and only if \( q_1\) is causally dependent on \( q_2\) and \( q_2\) is causally dependent on \( q_1\).

Definition III.3 (Meta-mode). A meta mode \( X \subseteq Q\) is a subset of modes that are mutually causally dependent on each other. If a mode \( q\) is not mutually causally dependent on any other modes, then the set \( \{q\}\) itself is a meta mode. A meta mode \( X\) is maximal if there is no other state in \( Q \setminus X\) that is mutually causally dependent on a state in \( X\).

Definition III.4 (The maximal set of Meta-modes). \( X\) be the set of maximal meta modes in the product MDP if and only if it satisfies: i) any set \( X \in X\) is a maximally meta mode, ii) the union of sets in \( X\) yields the set \( Q\), i.e., \( \cup_{X \in X} X = Q\).

Lemma III.1. The maximal set \( X\) of meta modes is a partition of \( Q\).
Given two meta-modes $X \to X'$ if a mode $q \in X$ is causally dependent on mode $q' \in X'$ by the transitivity property, then the causal dependency of $X_1 \to X_2$ and $X_2 \to X_3$, then we represent the causal dependency of $X_1$ on $X_3$ by $X_1 \to \Rightarrow X_3$. The following lemma states that two states in the product MDP are causally dependent if their discrete modes are causally dependent.

**Lemma III.2.** Given two meta-modes $X, X' \in \mathcal{X}$, if $X \Rightarrow X'$ but not $X' \Rightarrow X$, then for any state $(s, q) \in S \times X$ and $(s', q') \in S \times X'$, the state that either $(s, q) \Rightarrow (s', q')$ or $s' \Rightarrow q$ or these two states are causally independent.

**Proof.** By way of contradiction, if $(s, q)$ and $(s', q')$ are causally dependent and $(s', q') \Rightarrow (s, q)$, there must exist a state $(s'', q'')$ such that $(s', q') \Rightarrow (s'', q'')$ and $(s'', q'') \Rightarrow (s, q)$.

Relating the causal dependency of states in the product MDP and the definition of guard set, we have $s'' \in \text{Guard}(q', q)$ and $q'' \Rightarrow q$. Further $q' \Rightarrow q$, therefore $X' \Rightarrow X$, which is a contradiction as $X' \not\Rightarrow X$. □

**Lemma III.2** provides structural information about topology value iteration. If $X \Rightarrow X'$ and $X' \not\Rightarrow X$, then based on TVI, we shall update the value functions for states in the set $\{(s, q) \mid q \in X'\}$ before updating the values for states in the set $\{(s, q) \mid q \in X\}$.

However, the causal dependency in meta-modes does not provide us with a total order over the set of maximally meta modes because two meta modes can be causally independent. A total order is needed for deciding the order in which the optimal value functions for modes are computed.

To obtain a total order, we construct a total ordered sequence of sets of maximal meta modes. Given $\mathcal{X}$ the set of maximal meta-modes, $L_0 = \{X \in \mathcal{X} \mid X \cap F \neq \emptyset\}$ and $i = 1$. Move to the next step.

1. Let $L_i = \{X \in \mathcal{X} \mid \exists_{k=0}^{i-1} L_k \in L_{i-1} \text{ such that } X \Rightarrow X', \text{ and increase } i \text{ by 1, until } i = n\}$ and $L_{n+1} = \emptyset$.

We refer the set $\{L_i, i = 0, \ldots, n\}$ as the level sets over meta modes. Based on the definition of set $\{L_i, i = 0, \ldots, n\}$, we define an ordering $\Rightarrow$ as follows: $L_i \Rightarrow L_{i-1}, i = 1, \ldots, n$.

The following two statements can be proven.

**Lemma III.3.** If there exists $X \in \mathcal{X}$ such that $X \notin L_i$ for any $i = 0, \ldots, n$, then the set of states in $X$ is not coaccessible from the final set $F$ of states in DFA $A_{x_i}$.

**Proof.** By construction, this meta mode $X$ is not causally dependent on any meta mode that contains $F$, thus, it is not coaccessible in the task DFA $A_{x_i}$, i.e., there does not exist a word $w$ such that $\delta(q, w) \in F$ for some $q \in X$. □

If a DFA is coaccessible, we have $\cup_{i=0}^{n} L_i = \mathcal{X}$. A state $q$ that is not coaccessible from the final set $F$ should be trimmed before planning because the value $V(s, q)$ for any $s \in S$ will not be used for optimal planning in the product MDP to reach $F$.

**Lemma III.4.** The ordering $\Rightarrow$ is a total order:

$L_n \Rightarrow L_{n-1} \ldots \Rightarrow L_0$.

**Proof.** By definition. □

**Theorem III.5** (Optimal Backup Order [13]). If a MDP is acyclic, there exists an optimal backup order. By applying the optimal order, the optimal value function can be found with any state needing only one backup.

We generalize the Optimal Backup Order on an acyclic MDP to the product MDP as the following:

**Theorem III.6** (Generalized optimal backup order for hierarchical planning). Given the optimal planning problem in the product MDP and the causal ordering of meta modes, then by updating the value functions of all meta-modes in the same level set, in a sequence reverse to the causally ordering $\Rightarrow$, then the optimal value function for each meta mode can be found with only one backup, i.e., solving the value function of that meta mode using value iteration or an ADP method that solves the value function approximation.

**Proof.** We show this by induction. Suppose there exists only one level set, the problem is reduced to optimal planning in a product MDP with only one update for value functions of meta-modes in this level set. When there are multiple level sets, each time the optimal planning performs value function update for one level set. The value $V(s, q)$ for $q \in X$ only depends on the values of its descents states, that is, $\{V(s', q') \mid (s, q) \Rightarrow (s', q')\}$. It is noted that the mode $q'$ of any descendant $(s', q')$ must belong to either meta-modes $X$, or some $X' \in \mathcal{X}$ such that $X \Rightarrow X'$. By definition of level sets, if $X \in L_i$, then $X' \in L_k$ for some $k \leq i$. It means the value $V(s', q')$ for any descendant $(s', q')$ is either updated in level $L_k$, $k < i$ or along with the value $V(s, q)$ when $k = i$. As a result, after the value functions $\{V(\cdot, q) \mid q \in X, X \in L_i\}$ converge, the Bellman residuals of states in $\{(s, q) \mid q \in X, X \in L_k, k \leq i\}$ remain unchanged, while the value functions of meta-modes in other level sets with higher levels are updated. Thus, each mode only needs to be updated once. □

**Example III.1.** We use a simple example to illustrate. Given a system-level specification: $\Diamond \lnot (a \lor \Diamond \circ b) \land \Diamond \circ \circ c$, the corresponding DFA is shown in Fig. [1] In this DFA, each state is its own meta mode $X_k = \{q_{i+1}\}, i = 0, \ldots, 8$. Different level sets $L_i, i = 0 \ldots 4$ are contained in different shaped ellipses, as shown in this figure.

**B. Mode-free ADP for planning with temporal logic constraints**

ADP refers to a class of methods to find an approximately optimal policy for Problem [1]. First, let’s define the softmax
Bellman operator [19] by

\[ BV(s, q) = \tau \log \sum_a \exp \{ (R((s, q), a) \} + \gamma \sum_{(s', q')} \mathbb{P}((s', q') \mid (s, q), a) V(s', q')) / \tau, \] \hspace{1cm} (4)

where \( \tau > 0 \) is a predefined temperature parameter.

We introduce mode-dependent value function approximation as follows: For each \( q \in Q \), the value function is approximated by \( V(\cdot; \theta_q) : S \rightarrow \mathbb{R} \) where \( \theta_q \in \mathbb{R}^{\alpha_q} \) is a parameter vector of length \( \alpha_q \). A linear function approximation of \( V(\cdot; \theta_q) = \sum_{k=1}^{\alpha_q} \phi_k(s) \theta_q[k] = \Phi_q \theta_q \), where \( \phi_k(s) : S \rightarrow \mathbb{R}, k = 1, \ldots, \alpha_q \) are pre-selected basis functions.

We first define two sets: For meta-modes \( X, X' \in \mathcal{X} \), let

\[ \mathbf{Inv}(X) = \bigcup_{q \in X} \mathbf{Inv}(q), \] \hspace{1cm} and 

\[ \mathbf{Guard}(X, X') = \bigcup_{q \in X, q' \in X'} \mathbf{Guard}(q, q'). \]

Given the level sets \( \mathcal{L}_i, i = 0, \ldots, n \), the computation of value function approximations for each DFA state carries in the order of level sets.

1) Starting with level 0, let \( V((s, q); \theta_q) = 1 \) for all \( s \in S \) and \( q \in F \). For each \( X \in \mathcal{L}_0 \), solve the ADP problem:

\[ \min_{\{\theta_q \mid q \in X \}} \sum_{(s, q) \in S \times X} c(s, q) V((s, q); \theta_q), \] \hspace{1cm} (5)

subject to: \( BV((s, q); \theta_q) - V((s, q); \theta_q) \leq 0 \),

\[ \forall s \in \mathbf{Inv}(X) \bigcup (\bigcup_{X' \in \mathcal{X}} \mathbf{Guard}(X, X')), \]

where parameters \( c(s, q) \) are state relevant weights. All states \( \{s, q \mid q \in F \} \) are absorbing with values of 1.

The reward function \( R((s, q), a) = 0 \) for \( s \in S, q \in X \), and \( a \in A \). After solving the set of value functions \( \{V(s, q; \theta_q) \mid q \in X, X \in \mathcal{L}_0 \} \). The solution of this ADP is proven to be a tight upper bound of the optimal value function [17]. See Appendix for more information about this ADP method.

2) Let \( i = i + 1 \).

3) At the \( i \)-th step, given the value \( \{V(s, q; \theta_q) \mid q \in X \land X \in \mathcal{L}_k, k < i \} \), we solve, for each \( X \in \mathcal{L}_i \), an ADP problem stated as follows:

\[ \min_{\{\theta_q \mid q \in X \}} \sum_{(s, q) \in S \times X} c(s, q) V((s, q); \theta_q), \] \hspace{1cm} (6)

subject to: \( BV((s, q); \theta_q) - V((s, q); \theta_q) \leq 0 \),

\[ \forall s \in \mathbf{Inv}(X) \bigcup (\bigcup_{X' \in \mathcal{X}} \mathbf{Guard}(X, X')), \]

where \( V((s', q'); \theta_{q'}) \) to be solved either has \( q' \in X \) or \( q' \in X' \) for some \( X' \in \mathcal{L}_k, k \leq i \). Note that by Theorem [III.6] the meta-mode \( X' \), for which \( \mathbf{Guard}(X, X') \) is nonempty, cannot be in a level set higher than \( i \). When \( q' \in X' \) and \( X' \in \mathcal{L}_i \), then \( \theta_{q'} \) is a decision variable for this ADP. When \( q' \in X' \) and \( X' \in \mathcal{L}_k \) for some \( k < i \), then the value \( V((s', q'); \theta_{q'}) \) is computed in previous iterations and substituted herein. A state \( (s', q') \) whose values are determined in previous iterations are made absorbing in this iteration. The reward function \( R((s, q), a) = 0 \) for \( s \in S, q \in X \), and \( a \in A \).

4) Repeat step 2 until \( i = n \). Return the set \( \{V(s, q; \theta_q) \mid q \in Q \} \). The policy is computed using the softmax Bellman operator, defined in [1] by substituting the value function \( V(s, q) \) with its approximation \( V((s, q); \theta_q) \).

Remark 1. The problems solved by ADP is essentially a stochastic shortest path problem. For such a problem, two approaches can be used: One is to fix the values of states to be reached and assign the reward to be zero. During value iteration, the value of the states to be reached will be propagated back to the values of other states. The aforementioned reward design and ADP formulation use the first approach. Another way to introduce a reward function defined by \( R((s, q), a) = \sum_{q' \in X} \mathbb{P}((s', q') \mid (s, q), a) r((s', q')) \), where \( r((s', q')) = V((s', q'); \theta_{q'}) \) if \( q' \in X' \) for some \( X' \in \mathcal{L}_k \) and \( k < i \), otherwise, \( r((s', q')) = 0 \).

Note that a value iteration using softmax Bellman operator finds a policy that maximizes a weighted sum of total reward and the entropy of policy (see [20] for more details). When the value/reward is small, the total entropy of policies accumulated with the softmax Bellman operator overshadows the value given the reward function. This is called the value diminishing problem. Thus, for both cases, when the value \( V((s', q'); \theta_{q'}) \) of the state to be reached is small, we scale this value by a constant \( \alpha \) to avoid the value diminishing problem. Given the nature of the MaxProb problem, with a reward of 1 being assigned when the LTL constraint is satisfied, we almost always need to amplify the reward to avoid the value diminishing problem.

IV. CASE STUDY

We validate the algorithm in a motion planing problem modeled as a stochastic 11 \times 11 grid world under sc-LTL shown in Fig. 2 using the specification: \( \Diamond (s \land \Diamond (b U c)) \land (s \land (\neg (a U d))) \land \Diamond \text{goal} \), and the corresponding DFA plotted in Fig. 3 The partitions of meta modes are shown in Fig. 3 with different meta modes are boxed in different styled rectangles. The task automaton is partitioned into 4 meta modes \( X_i, i = \)
0, . . . , 3, and each level set \( L_i \), \( i = 0, . . . , 3 \) contains one meta mode with the same index. The reward is defined as the following: the robot receives a reward of 40 (an amplified reward to avoid value diminishing) if the trajectory satisfies the specification. In each state \( s \in S \) and for robot’s different actions (heading up (‘U’), down (‘D’), left(‘L’), right(‘R’)), the probability of arriving at the correct cell is \( 1 - 0.1 \times \text{card(neighbors)} \). The grid world contains walls which form a narrow passage and surround the grid world. If the system hits a wall, it will be bounced back to its original cell.

![Image of the gridworld](image)

**Fig. 2:** The gridworld with one simulated run using the policy computed by TADP.

The planning objective is to find an approximately optimal policy for satisfying the specification with a maximal probability. After the convergence, we adopt the policy computed by TADP and simulate the system, and one simulation of the system is plotted in Fig. 2. In this case, the system starts at the initial state \( s_{\text{init}} \), then it visits region B and D sequentially. After reaching region B and D, it eventually visits the goal state \( s_{\text{goal}} \).

**Parameters:** The user-specified temperature \( \tau = 0.5 \) and error tolerance \( \epsilon = 10^{-3} \) are shared by TVI, VI, and TADP, where the stopping criterion is \( \| \max(V^j(s) - V^{j-1}(s)) \| \leq \epsilon \) for iterations \( j \) of in TVI and VI and for each inner iteration \( j \) of TADP, respectively.

The parameters used in the TADP algorithm for the \( k \)th problem are the following: the increasing coefficient of the penalty \( b = 1.1 \), the initial learning rate \( \eta = 0.1 \), the initial penalty parameter \( \nu = 1.0 \), and the initial Lagrangian multipliers \( \lambda = 0 \). During each inner iteration \( j \), 30 trajectories of length \( \leq 3 \) are sampled. The value function \( V(\cdot; \theta_q) \) is approximated by a weighted sum of Gaussian

| Algorithm       | VI        | TVI        | TADP       |
|-----------------|-----------|------------|------------|
| 11 × 11         | Bellman Backup Operations (times) | 348966     | 32313      | N/A        |
|                 | Running Time (Seconds) | 15.18      | 10.35      | 93.86      |
| 21 × 21         | Bellman Backup Operations (times) | 131586     | 127106     | N/A        |
|                 | Running Time (Seconds) | 141.68     | 70.09      | 326.91     |

**TABLE I:** Bellman Backup Operations and Running Time between VI, TVI, and TADP.

Kernels: \( V(\cdot; \theta_q) = \Phi_q \beta_q \), where basis functions \( \Phi_q = [\phi_1, \phi_2, \ldots, \phi_{\ell_q}]^T \) are defined as the following: \( \Phi_j(s) = K(s, e^{(j)}) \) and \( K(s, s') = \exp(-SP(s, s')^2/\sigma_j^2) \), where \( \{e^{(j)}, j = 1, \ldots, \ell_q\} \) is a set of pre-selected centers and \( \sigma = 3 \). In this example, we select the centers to be uniformly selected points with interval 2 within the grid world and goals.

**Running Time:** In Table I comparing VI with TVI running time is reduced by 28.52% and 51.23% by exploiting the topological structure, and the total number of Bellman Backup Operations is reduced by 7.32% and 6.40% in different sizes of grid worlds. In terms of the simple specifications, the decomposition occupies major CPU time, but the advantage of exploiting the topological structure will be uncovered if a more complex specification is associated. The TADP converges after 93.86 seconds and 326.91 seconds, respectively for grid worlds of different sizes. The running time of the TVI and VI in a 21 × 21 grid world is 7-10 times their running time in the 11 × 11 grid world. However, the running time of TADP in a 21 × 21 grid world is only 3 times the running time of TADP in the 11 × 11 grid world. TADP is more beneficial in large MDP problems or with more complex specifications. It is noted that though TADP takes in general longer time to converge, it is model-free. TVI and VI are model-based.

**Convergence:** In Fig. 5 we plot the convergence of values for different states in the 11 × 11 grid world and modes in automaton against epochs that is the number of inner iterations in TADP. It indicates that the values initially oscillate, but all values converge after 250 iterations. Especially for (3, 2, 1), the value converges around 150 epochs.

**Statistical Results:** After the convergence, we use the policies computed by TADP and TVI to simulate trajectories. The statistical results for steps of reaching the goal at two different states of 11 × 11 grid world are shown in Fig. 6 using the violin plot that is a combination of the box plot and kernel density plot. The white spots are the means of the steps of reaching the goal and the black boxes show the variances. The black lines fill the gap between the maximum and the minimum and the kernel densities at different values are plotted correspondingly. The plot shows the policy computed by TADP is suboptimal due to the nature of ADP, but the performance gap between two policies is not significant.

**V. CONCLUSION**

We present a topological approximate dynamic programming method to maximize the probability of satisfying high-level system specifications in LTL. We decompose the product MDP and define the topological order for updating value functions at different task states to mitigate the sparse reward
Fig. 4: Comparison between VI, TVI, TADP for different states at $q_3$: (a) (b) (c) are the heat maps of $V(\cdot, q_3)$ obtained with VI on Product MDP $M_\phi$, with TVI, and with TADP, respectively. (d) (e) (f) are the corresponding value surfs of $V(\cdot, q_3)$ obtained with VI, TVI, and TADP, respectively.

Fig. 5: The convergence of values in TADP in the $11 \times 11$ stochastic grid world for different states in the product MDP. A product state $(5, 5, 3)$ means the grid cell $(5, 5)$ and the DFA state $q_3$.

problems in model-free reinforcement learning with LTL objectives. The correctness of the algorithm is demonstrated on a robotic motion planning problem under LTL. It is noted one needs update the value functions for all discrete states in a meta-mode at a time. When the size of meta-mode is large, then the number of parameters in value function approximations to be solved is large, which raises the scalability issue due to the complexity of the specifications.

We will investigate action elimination technique within the framework, not at the low-level actions in the MDP, but at the high-level decisions of transitions in the task DFA. By eliminating transitions in the DFA, it is possible to decompose large meta-mode into a subset of small meta-modes whose value functions can be efficiently solved.

APPENDIX

In this section, we present a model-free ADP method with value iteration. The method has been introduced in our previous work [17] and briefly reviewed here for completeness.

Given an MDP $M = (S, A, P, s_0, \gamma, R)$ with state set $S$, action set $A$, the transition function $P : S \times A \times S \to [0, 1]$, the initial state $s_0$, a discounting factor $\gamma \in (0, 1]$, and a reward function $R : S \times A \to \mathbb{R}$, the objective is to find a policy $\pi$ that maximizes the total discounted return given by $J(s_0) = \max_{\pi} \mathbb{E}_\pi \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t)$, where $s_t, a_t$ is the $t$-th state and action in the chain induced by policy $\pi$.

The ADP for value iteration is to solve the following optimization problem:

$$\min_{\theta} \sum_{s \in S} c(s; \theta) V(s; \theta),$$

subject to: $BV(s; \theta) - V(s; \theta) \leq 0, \ \forall s \in S,$  \hspace{1cm} (7)

where the state relevant weight $c(s) = c(s; \theta)$ to be the frequency with which different states are expected to be visited in the chain under policy $\pi(\cdot; \theta)$, and $BV(s; \theta) = \tau \log \sum_{a} \exp\{(R(s, a) + \gamma \sum_{s'} P(s' | s, a) V(s'; \theta)/\tau\}$ is the softmax Bellman operator with the temperature $\tau > 0$. The policy $\pi(\cdot; \theta) : S \to \Delta(A)$ is computed from $V(\cdot; \theta)$ using...
It is shown in [17] that a value function $V(\cdot; \theta)$ satisfying the constraint in (7) is an upper bound on the optimal value function. The objective is to minimize this upper bound to approximate the optimal value function.

We introduce a continuous function $B:\mathbb{R} \rightarrow \mathbb{R}_+$ with support equal to $[0, \infty)$. One such function is $B(x) = \max\{x, 0\}$. Let $g(s; \theta) = BV(s; \theta) - V(s; \theta), \forall s \in S$. Using randomized optimization [21], an equivalent representation of the set of constraints is

$$\mathbb{E}_{s \sim \Delta} B(g(s; \theta)) = 0,$$

where $s$ is a random variable with a distribution $\Delta$ whose support is $S$. The augmented Lagrangian function of (7) with constraints replaced by (8) is

$$L_\nu(\theta, \lambda, \xi) = \sum_{s \in S} c(s; \theta) V(s; \theta) + \lambda \cdot \mathbb{E}_{s \sim \Delta} B(g(s; \theta)) + \frac{\nu}{2} \cdot \mathbb{E}_{s \sim \Delta} |B(g(s; \theta))|^2$$

where $\lambda$ and $\xi$ are the Lagrange multipliers and $\nu$ is a large penalty constant. Using the quadratic Penalty function method [22], the solution is found by solving a sequence of inner optimization problems of the form:

$$\min_{\theta \in \mathbb{R}^K} L_{\nu, k}(\theta, \lambda^k, \xi^k),$$

where $\{\lambda^k\}$ and $\{\xi^k\}$ are sequences in $\mathbb{R}$, $\{\nu^k\}$ is a positive penalty parameter sequence, and $K$ is the size of $\theta$. After the inner optimization for (10) converges, we update formula for multipliers $\lambda$ and $\xi$ in the outer optimization as

$$\lambda^{t+1} = \lambda^t + \nu^t \cdot \mathbb{E}_{(s, q) \sim \Delta} B(g((s, q); \theta^t_{Xk})) (11)$$

The outer optimization stops when it reaches the maximum number of iterations or $\|\nabla_{\theta_{Xk}} L_{\nu, t}(\theta^t_{Xk}, \lambda^t)\| \leq 10^{-3} \cdot \epsilon^t$. See [22, Chap. 5.2] for more details about the augmented Lagrangian method.

By selecting $c(s; \theta) = \sum_{t=0}^{\infty} \text{Pr}(X_t = s)$ the state visitation frequency in the Markov chain $M^0$, for an arbitrary function $f : S \rightarrow \mathbb{R}$, it holds that

$$\sum_{s \in S} c(s; \theta) f(s; \theta) = \int p(h; \theta) f(h; \theta) dh,$$

where $p(h; \theta)$ is the probability of path $h$ in the Markov chain $M^0, f(h; \theta) = \sum_{i=1}^{[h]} f(s_i; \theta)$.

By selecting $\Delta \propto c(\cdot; \theta)$ and letting $f(s; \theta) = V(s; \theta) + \lambda^k B(g(s; \theta)) + \frac{\nu^k}{2} |B(g(s; \theta))|^2$, the $k$-th objective function in (10) becomes

$$\min_{\theta \in \mathbb{R}^K} \int p(h; \theta) f(h; \theta) dh.$$ Parameter $\theta$ is updated by $\theta^{t+1} \leftarrow \theta^t - \eta_1 \cdot \nabla_{\theta} F(\theta) - \eta_2 \cdot \nabla_{\theta} m(\Delta; 2; \theta)$, where $j$ represents the $j$-th inner iteration, $\eta_1$ and $\eta_2$ are positive step sizes. $\nabla_{\theta} F(\theta) = \int \nabla_{\theta} p(h; \theta) f(h; \theta) dh$

$$+ \int p(h; \theta) \nabla_{\theta} f(h; \theta) dh,$$

where using Monte-Carlo approximation, we have $1 \approx \frac{1}{N_\delta} \sum_{h \sim p(h; \theta)} \left[ \sum_{t=0}^{[h]} \nabla_{\theta} \log \pi(a_t | s_t; \theta) f(h; \theta) \right]$.

Now, as the gradient of the objective function of the inner optimization problem can be computed from sampled trajectories, we can update the parameter $\theta$ using sampling-based augmented Lagrangian method, and thus have a model-free ADP method. The reader is referred to [17] for the technical details regarding the derivation of the method.

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