Polygonal inclusions with nonuniform eigenstrains in an isotropic half plane

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Abstract. Polygonal inclusion problem in an isotropic half plane is investigated in this paper. The eigenstrains prescribed in the inclusion are assumed to be characterized by polynomials of arbitrary order in the Cartesian coordinate system. Based on a novel superposition method, the solution of the inclusion problem in a half plane is decomposed into two subproblems: the inclusion problem in a full plane and the auxiliary boundary problem in the half plane. Furthermore, the Kolosov-Muskhelishvili (K-M) potentials for the full plane and the auxiliary potentials for the half plane along with their derivatives are expressed into two sets of basic functions, which involve the boundary integrals of the inclusion domain. For polygonal inclusions, exact explicit expressions for both basic functions are explicitly derived, which leads to those for the induced displacement, strain and stress fields.

1. Introduction
In the context of micromechanics, an inclusion means a subdomain in the body where eigenstrains (stress-free transformation strains, such as thermal strains, phase transformations, and the like) are prescribed. The problem to determine the disturbed elastic fields inside and outside the inclusion is usually called Eshelby problem or inclusion problem. The solution of Eshelby problem of ellipsoidal inclusion with uniform eigenstrains and the well-known equivalent inclusion method (EIM) for a treatment of inhomogeneity problem have got numerous engineering applications in heterogeneous materials, after the seminal work of Eshelby [1-3].

An inclusion under nonuniform eigenstrains is very common in engineering applications. Lots of inclusion problems have been solved for nonuniform eigenstrains, for example, eigenstrain problem of polynomial distribution in isotropic material [4], the inclusion problem for ellipsoids with nonuniform dilatational Gaussian and exponential eigenstrains [5], the isotropic ellipsoidal inclusion with polynomial eigenstrains [6-8], elliptic inhomogeneity problem due to linear and polynomial distributions of eigenstrain [9, 10] among others. Based on conformal mapping technique, Chen [11] derived a closed form solution for the elliptic inclusion with eigenstrains of polynomial distribution. Sun et al [12] derived an explicit closed-form solution for a polygonal inclusion with a linear eigenstrain in the anisotropic piezoelectric full plane and verified the Eshelby's polynomial conservation theorem numerically. The solution was later extended to the corresponding half-plane domain by Chen et al [13]. More recently, we have derived a closed-form solution for the polygonal inclusion problem with polynomial eigenstrains in a magneto-electro-elastic full plane[14].

For 2D isotropic elastic problems in a half plane, the reported works concerned with polygonal inclusions are handful. Chiu [15] formulated the internal stresses of a rectangular inclusion with the
Airy stress function. Glas [16] constructed a closed-form solution of a polygonal inclusion with a uniform hydrostatic eigenstrain by superposition of the solutions of a set of trapezoidal inclusions. Ru [17] obtained a closed-form solution to the problem of an inclusion of arbitrary shape in a full and half plane by means of the techniques of analytical continuation and conformal mapping. Sun and Peng [18] extended Ru's solution and obtained analytic solutions for the cases of fixed and rigid, frictionless boundaries. Zou et al [19] proposed a superposition framework to study the inclusion problem in a bounded space.

However, the polygonal inclusion problem with nonuniform eigenstrains in a half plane is still less touched. In this paper, we present an exact closed-form solution for an arbitrary polygonal inclusion with polynomial eigenstrains in an isotropic elastic half plane.

2. Plane elasticity description using K-M potentials

Let $\Omega$ be a 2D unbounded space and introduce the complex variable $z = x + i \eta x$ in a 2D Cartesian coordinate system $(x, \eta x)$ to specify a point in $\Omega$, where $i = \sqrt{-1}$. The material forming $\Omega$ is linearly elastic and isotropic, characterized by two constants, Young's modulus $E$ and Poisson's ratio $\nu$. In terms of the Kolosov-Muskhelishvili (K-M) potentials $\gamma$ and $\psi$, which are arbitrary analytic functions in $\Omega$, the displacement, stress and strain components at a point $z$ can be formulated in an elegant form [20-22],

$$u_1 + i u_2 = \frac{1}{2\mu} \left[ k \gamma(z) - z \overline{\gamma(z)} - \overline{\psi(z)} \right],$$

$$\begin{align*}
\sigma_{11} + \sigma_{22} &= 2 \left[ \gamma'(z) + \overline{\gamma(z)} \right], \\
\sigma_{22} - \sigma_{11} + 2\sigma_{12} &= 2 \left[ \overline{\gamma'(z)} + \psi(z) \right], \\
\epsilon_{11} + \epsilon_{22} &= \frac{\kappa-1}{2\mu} \left[ \gamma'(z) + \overline{\gamma(z)} \right], \\
\epsilon_{22} - \epsilon_{11} + 2\epsilon_{12} &= \frac{1}{\mu} \left[ \overline{\gamma''(z)} + \psi'(z) \right],
\end{align*}$$

where an overbar means the conjugate of a complex variable or function, a prime symbol (') and a double-prime symbol (''') mean the first and second order derivatives of a function, respectively, and

$$\mu = \frac{E}{2(1+\nu)}, \quad \kappa = \frac{(3-\nu)/(1+\nu)}{(3-4\nu)} \quad \text{plane stress,}$$

$$\mu = \frac{E}{2(1+\nu)}, \quad \kappa = \frac{(3-\nu)/(1+\nu)}{(3-4\nu)} \quad \text{plane strain}.$$
are characterized by polynomials of one order higher. We assume the eigen-displacements are in the following form

\[ u^\ast_1 + u^\ast_2 = \alpha_1^\ast + \psi_1^\ast \]

where \( \alpha_1^\ast \) is a group of complex coefficients with the same dimension as potentials \( \gamma \) and \( \psi \), \( a \) is a scale length, and \( M \) and \( N \) are integers. It is convenient to recast the eigen-displacements in terms of \( z \) and \( \bar{z} \) through the relations

\[ x_1 = \frac{1}{2}(z + \bar{z}) \]
\[ x_2 = \frac{1}{2}(z - \bar{z}) \]

i.e.,

\[ u^\ast_1 + u^\ast_2 = (\kappa + 1) \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} C_{pq}^{MN} z^M \bar{z}^N \]

with \( \bar{M} = M - p + q \) and \( \bar{N} = N + p - q \) (8)

where

\[ C_{pq}^{MN} = \frac{(-1)^{p-q} \alpha_{MN}}{2^{p+q} (M-p)! (N-q)!} \]

is a set of complex coefficients related to the property of the material, the polynomial order of the eigenstrains and geometry scale of the inclusion.

Denote by \( \gamma_\infty(z) \) and \( \psi_\infty(z) \) the complex potentials which are sectional analytic in the complex plane. \( \gamma_\infty(y) \) and \( \psi_\infty(y) \) \( (y \in \partial \omega) \) indicate the boundary values of \( \gamma_\infty(z) \) and \( \psi_\infty(z) \) in \( \omega \), while \( \gamma_{\infty,-}(y) \) and \( \psi_{\infty,-}(y) \) \( (y \in \partial \omega) \) stand for those in the matrix. The displacement and traction continuities across the boundary \( \partial \omega \) imply that

\[ \kappa \gamma_{\infty,-}(y) - y \gamma_{\infty,+}(y) = \kappa \gamma_{\infty,+}(y) - y \gamma_{\infty,-}(y) \]
\[ \gamma_{\infty,-}(y) + y \gamma_{\infty,+}(y) = \gamma_{\infty,+}(y) + y \gamma_{\infty,-}(y) \]

where use has been made of equations (1), (5) and (7). Combining (10) and (11) yields

\[ \gamma_{\infty,-}(y) = \gamma_{\infty,+}(y) + \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{\gamma_{\infty,-}(y)}{2^{p+q} (M-p)! (N-q)!} \]

Equation (12) is a particular case of the Privalov or Riemann-Hilbert problem [23, 24]. So, assuming that the inclusion boundary \( \partial \omega \) is simple, closed, regular and positively oriented, and the jumping term \( \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{\gamma_{\infty,-}(y)}{2^{p+q} (M-p)! (N-q)!} \) in (12) satisfies the Hölder condition on \( \partial \omega \), the solution of equation (12) can be expressed by the following Cauchy type integral

\[ \gamma_{\infty}(z) = -\frac{1}{2\pi i} \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{C_{pq}^{MN}}{2^{p+q} (M-p)! (N-q)!} \int_{\partial \omega} \frac{dy}{y-z} \]

The derivative of \( \gamma_{\infty}(z) \) is derived as

\[ \gamma'_{\infty}(z) = -\frac{1}{2\pi i} \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{C_{pq}^{MN}}{2^{p+q} (M-p)! (N-q)!} \frac{dy}{(y-z)^2} \]

or

\[ \gamma'_{\infty}(z) = -\frac{1}{2\pi i} \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{C_{pq}^{MN}}{2^{p+q} (M-p)! (N-q)!} \frac{dy}{(y-z)^2} \]

Because the term \( d(y_\infty y_{\infty})/dy \) satisfies the Hölder condition, based on the Plemelj formula, for \( \forall y_0 \in \partial \omega \), the inner and outer boundary values of \( \gamma_{\infty}(z) \) are

\[ \gamma_{\infty,-}(y_0) = \frac{1}{2} \left[ \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} C_{pq}^{MN} \frac{d(y_\infty y_{\infty})}{dy} \right]_{y=y_0} + \frac{1}{2\pi i} \int_{\partial \omega} \frac{dy}{y-y_0} \left[ \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} C_{pq}^{MN} \frac{d(y_\infty y_{\infty})}{dy} \right]_{y=y_0} \]

\[ \gamma_{\infty,+}(y_0) = -\frac{1}{2} \left[ \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} C_{pq}^{MN} \frac{d(y_\infty y_{\infty})}{dy} \right]_{y=y_0} + \frac{1}{2\pi i} \int_{\partial \omega} \frac{dy}{y-y_0} \left[ \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} C_{pq}^{MN} \frac{d(y_\infty y_{\infty})}{dy} \right]_{y=y_0} \]

Combining (16) and (17), replacing \( y_0 \) back to \( y \), the following jump relation is delivered

\[ \gamma_{\infty,-}(y) - \gamma_{\infty,+}(y) = \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{C_{pq}^{MN}}{2^{p+q} (M-p)! (N-q)!} \frac{dy}{y-z} \]

Substituting equations (12) and (18) into (11) yields

\[ \psi_{\infty,-}(y) = \psi_{\infty,+}(y) - \sum_{p=0}^{\bar{M}} \sum_{q=0}^{\bar{N}} \frac{C_{pq}^{MN}}{2^{p+q} (M-p)! (N-q)!} \frac{dy}{y-z} \]
It is shown that equation (19) also possesses the structure of the Privalov or Riemann-Hilbert problem. So, the potential $\psi_\infty(z)$ can be achieved as

$$\psi_\infty(z) = \frac{1}{2\pi i} \sum_{p=0}^{M} \sum_{q=0}^{N} \left[ \frac{c_{pq}^{MN}}{2\pi i} \oint_{\partial \omega} \frac{y^{N} dy}{y-z} + \bar{M} c_{pq}^{MN} \oint_{\partial \omega} \frac{y^{N+1} dy}{y-z} + \bar{N} c_{pq}^{MN} \oint_{\partial \omega} \frac{y^{N+2} dy}{y-z} \right].$$

Equations (13) and (20) are the potentials for solving Eshelby's problem of an arbitrarily shaped inclusion in an infinite plane, concerning with polynomial eigenstrains of arbitrary order.

By defining a set of basic functions $g_{pq}^1(z)$ and $g_{pq}^2(z)$

$$g_{pq}^1(z) = \frac{1}{2\pi} \oint_{\partial \omega} \frac{y^{N} dy}{(y-z)^t}, ~ g_{pq}^2(z) = \frac{1}{2\pi} \oint_{\partial \omega} \frac{y^{N+1} dy}{(y-z)^t},$$

potentials $\gamma_\infty(z), \psi_\infty(z)$ and their derivatives can be simply expressed in a unified form

$$\gamma_\infty(z) = -\sum_{p=0}^{M} \sum_{q=0}^{N} c_{pq}^{MN} g_{pq}^1(z), \quad \gamma_\infty(z) = -\sum_{p=0}^{M} \sum_{q=0}^{N} c_{pq}^{MN} g_{pq}^2(z), \quad \gamma_\infty(z) = -\sum_{p=0}^{M} \sum_{q=0}^{N} 2c_{pq}^{MN} g_{pq}^3(z),$$

and

$$\psi_\infty(z) = \sum_{p=0}^{M} \sum_{q=0}^{N} c_{pq}^{MN} g_{pq}^1(z) + \bar{M} c_{pq}^{MN} g_{pq}^1(z) + \bar{N} c_{pq}^{MN} g_{pq}^2(z),$$

$$\psi_\infty(z) = \sum_{p=0}^{M} \sum_{q=0}^{N} c_{pq}^{MN} g_{pq}^1(z) + \bar{M} c_{pq}^{MN} g_{pq}^1(z) + \bar{N} c_{pq}^{MN} g_{pq}^2(z).$$

4. Solutions of the inclusion problem in a half plane

Assume that an inclusion $\omega$ locates in the lower semi-infinite plane $\Omega$, whose horizontal boundary ($y=0$) is denoted by $\partial \Omega$. In the framework of complex plane elasticity, according to the superposition procedure proposed for the inclusion problem in a finite domain by Zou et al [19], the complex K-M potentials $\gamma$ and $\psi$ for the inclusion problem in a half plane admit the following decompositions

$$\gamma = \gamma_\omega + \gamma_b, \quad \psi = \psi_\omega + \psi_b,$$

where $\gamma_\omega$ and $\psi_\omega$ are the potentials for the problem in the full plane developed above, while $\gamma_b$ and $\psi_b$ are the potentials to be determined for the auxiliary boundary value problem in the lower half plane. In what follows, we will establish the equations governing $\gamma_b$ and $\psi_b$.

First, for the Dirichlet boundary condition on the surface $\partial \Omega$ which means the displacements on the boundary are zero, substitution of equation (27) into (1) leads to

$$\gamma_b(\tau) + \tau \gamma_b'(\tau) + \psi_b(\tau) = -\tau \gamma_\omega(\tau) + \tau \gamma_\omega'(\tau) + \psi_\omega(\tau),$$

where $\tau \in \partial \Omega$ and $\text{Im} \tau = 0$. Next, considering the problem where the tractions $f_b$ on the surface $\partial \Omega$ are equal to the tractions $f_\omega$ derived from $\gamma_\omega$ and $\psi_\omega$, and making use of equation (4), the Neumann boundary condition is in question and can be expressed by

$$\gamma_\omega(\tau) + \tau \gamma_\omega'(\tau) - \psi_\omega(\tau) = \gamma_b(\tau) + \tau \gamma_b'(\tau) - \psi_b(\tau), \quad \tau \in \partial \Omega.$$

Introducing an auxiliary function

$$G(\tau; \eta) = -\eta \gamma_\omega(\tau) + \tau \gamma_\omega'(\tau) + \psi_\omega(\tau)$$

with $\eta = \kappa$ for the Dirichlet case and $\eta = -1$ for the Neumann case, then both boundary conditions (28) and (30) can be unified as

$$\eta \gamma_b(\tau) - \tau \gamma_b(\tau) - \psi_b(\tau) = G(\tau; \eta), \tau \in \partial \Omega.$$
It is known from Muskhelishvili [20] and Lu [21] that the solution to equation (32) is given by

\[ \psi_b(z) = \frac{1}{2} \int_0^{\infty} \frac{g(r_\eta) \partial \psi_b(z)}{r_\eta - z} \, dr_\eta, \]

Substituting equation (33) into (34) and (35), and making use of the following integral result

\[ \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{d\tau}{\tau-z} = \begin{cases} \frac{1}{2} \log(\frac{x}{z}), & x < 0 \\ \frac{1}{2} \log(\frac{z}{x}), & x > 0 \end{cases}, \]

the complex potentials for solving the auxiliary boundary value problem associated to the inclusion problem in the lower half plane are arrived,

\[ \psi_b(z) = \sum_{p=0}^M \sum_{q=0}^N \left[ \frac{1}{\eta \rho_{pq}^2} \frac{\eta \rho_{pq}^2}{2 \pi i} \int_{-\infty}^{\infty} \frac{\eta \rho_{pq}^2 \partial \psi_b(z)}{\eta \rho_{pq}^2 - z} \, d\eta \right], \]

Recall the predefined basic functions (21) in Section 3, potentials \( \psi_b(z) \) and their derivatives can be written in the following neat form

\[ \psi_b(z) = \sum_{p=0}^M \sum_{q=0}^N \left[ \frac{1}{\eta \rho_{pq}^2} \frac{\eta \rho_{pq}^2}{2 \pi i} \int_{-\infty}^{\infty} \frac{\eta \rho_{pq}^2 \partial \psi_b(z)}{\eta \rho_{pq}^2 - z} \, d\eta \right]. \]

Combining equations (39)-(43) and (22)-(26) along with (27), we will get the potentials for solving the inclusion problem in a half plane concerning with polynomial eigenstrains. Then, applying formulae (1) and (2), the corresponding displacement, strain and stress fields will be arrived. It should be remarked that the solutions derived above are only in terms of the basic functions \( g_{p,q}(z) \) and \( g^2_{p,q}(z) \) which involve boundary integrals of the inclusion domain. In next section, for polygonal inclusions, we will explicitly derive these boundary integrals.
5. Explicit solutions of polygonal inclusion problems

In this section, based on the solutions derived above, we consider an arbitrary polygonal inclusion with \( N \) sides and carry out the integrals involved in the two functions \( g^1_{pq}(z) \) and \( g^2_{pq}(z) \). We define the points on the \( t \)-th side of the polygon in terms of parameter \( t \) as

\[
y = y_i + s_i t, \quad t \in [0,1] \text{ with } s_i = y_{i+1} - y_i,
\]

where \( y_i \) and \( y_{i+1} \) are the initial and terminal points of the side. We further denote \( w_i = y_i - z_i \) as the relative position vector. Then, we rewrite \( y \) and \( \tilde{y} \) in the following compact form

\[
y = c_1 + c_2 (y - z), \quad c_1 = z, \quad c_2 = 1; \quad \tilde{y} = d_1 + d_2 (y - z), \quad d_1 = \tilde{y}_k + \frac{s_k}{s_k} (z - y_k), \quad d_2 = \frac{s_k}{s_k}.
\]

Substituting equations (45) and (46) into (21), the integrals along every rectilinear side involved in (21) will be explicitly carried out, so the analytic functions \( g^1_{pq}(z) \) and \( g^2_{pq}(z) \) can be derived as

\[
g^1_{pq}(z) = \frac{1}{2m} \sum_{k=1}^{N} \sum_{r=0}^{p} \sum_{s=0}^{q} \frac{e^{p-r} c_2^{d_1-r} d_2^{d_1-s} \Lambda(z)}{(p-r)! r! (q-s)! s!}, \tag{47}
\]

\[
g^2_{pq}(z) = \frac{1}{2m} \sum_{k=1}^{N} \sum_{r=0}^{p} \sum_{s=0}^{q} \frac{s_k c_1^{p-r} c_2^{d_1-r} d_2^{d_1-s} \Lambda(z)}{(p-r)! r! (q-s)! s!}, \tag{48}
\]

where

\[
\Lambda(z) = \begin{cases} 
\ln \frac{w_{k+1}}{w_k} & \text{for } r + s - l = -1 \\
\frac{1}{r+s-l+1} \left[ w_{k+1}^{r+s-l+1} - w_k^{r+s-l+1} \right] & \text{for } r + s - l \neq -1.
\end{cases}
\]

It should be mentioned that when we calculate \( g^1_{pq}(\tilde{z}) \) and \( g^2_{pq}(\tilde{z}) \) which involve the variable \( \tilde{z} \) rather than \( z \), the position vector \( w_k \) in equations (47) and (48) should be replaced by the alternative one \( w_k' = y_k - \tilde{z} \).

Substituting equations (47) and (48) into (39)-(42) along with the potentials and their derivatives for solving the polygonal inclusion problem involving polynomial eigenstrains in a half plane will be get explicitly.

6. Numerical examples

In this section, as an numerical example of the above analytical solution, we discuss a square inclusion beneath the horizontal traction-free boundary \( \partial \Omega \). The origin of the square inclusion is locate at point \((0, -2)\) in the Cartesian coordinate system, with both groups of opposite sides parallel to the coordinate axes. The diagonal length of the square is set to 2. When it undergoes a linear eigenstrain \( \varepsilon^{11}_x = x \), the induced stress and displacement fields are plotted in Figure 1 Nondimensional stress fields of the square inclusion prescribed with single linear eigenstrain \( \varepsilon^{11}_x = x \). (a) \( \sigma^{11}_x \); (b) \( \sigma^{22}_x \); (c) \( \sigma^{12}_x \). Figure 1 and Figure 2, respectively. Manifest non-uniform induced fields can be found inside the inclusion domain, both for stresses and displacements. At the vertexes of the inclusion, the stress components \( \sigma_{11} \) and \( \sigma_{22} \) are exhibiting sharply singularity. As for the symmetry feature, normal stress components \( \sigma_{11}, \sigma_{22} \) and displacement \( u_2 \) are anti-symmetric about \( x_2 \) axis, while shear stress component \( \sigma_{12} \) and displacement \( u_1 \) are symmetric. All the stress and displacement components exhibit no symmetry features about the horizontal centroid symmetry axis of the inclusion shape, which shows the influence of the traction-free boundary condition.
Figure 1 Nondimensional stress fields of the square inclusion prescribed with single linear eigenstrain \( \varepsilon_{11}^* = x \). (a) \( \sigma_{11} \); (b) \( \sigma_{22} \); (c) \( \sigma_{12} \).

Figure 2 Nondimensional displacement fields of the square inclusion prescribed with single linear eigenstrain \( \varepsilon_{11}^* = x \). (a) \( u_1 \); (b) \( u_2 \).

7. Concluding Remarks
In this paper, we derived the analytical solutions of the polygonal inclusion problem in an isotropic half plane, in which the prescribed eigenstrains are assumed to be in an arbitrary polynomial form. Both the Dirichlet and Neumann boundary conditions on the horizontal boundary are unified in a compact equation, which is a striking feature of the present study. Based on the analytical solutions derived in this paper, further numerical studies could be put into practice, and some key issues in micromechanics of composite could be deeply analyzed, such as the influences of the inclusion shape, orientation, volume fraction and boundary conditions to the local elastic fields.

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