IDENTITIES FOR A PARAMETRIC WEYL ALGEBRA OVER A RING

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Abstract. In 2013 Benkart, Lopes and Ondrus introduced and studied in a series of papers the infinite-dimensional unital associative algebra $\mathbb{A}_h$ generated by elements $x,y$, which satisfy the relation $yx-xy = h$ for some $0 \neq h \in \mathbb{F}[x]$. We generalize this construction to $\mathbb{A}_h(\mathbb{B})$ by working over the fixed $\mathbb{F}$-algebra $\mathbb{B}$ instead of $\mathbb{F}$. We describe the polynomial identities for $\mathbb{A}_h(\mathbb{B})$ over the infinite field $\mathbb{F}$ in case $h \in \mathbb{B}[x]$ satisfies certain restrictions.

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1. Introduction

Assume that $\mathbb{F}$ is a field of arbitrary characteristic $p = \text{char}\mathbb{F} \geq 0$. All vector spaces and algebras are over $\mathbb{F}$ and all algebras are associative, unless otherwise is stated. For the fixed $\mathbb{F}$-algebra $\mathbb{B}$ with unity we write $\mathbb{B}(x_1, \ldots, x_m)$ for the $\mathbb{F}$-algebra of non-commutative $\mathbb{B}$-polynomials in variables $x_1, \ldots, x_m$, i.e., $\mathbb{B}(x_1, \ldots, x_m)$ is a free left (and a free right) $\mathbb{B}$-module with the basis given by the set of all non-commutative monomials in $x_1, \ldots, x_m$, where we assume that $\beta x_i = x_i \beta$ for all $\beta \in \mathbb{B}$ and $1 \leq i \leq m$. The unity 1 of $\mathbb{B}(x_1, \ldots, x_m)$ corresponds to the empty monomial. In case the variables are $x_1, x_2, \ldots$ the algebra of non-commutative $\mathbb{B}$-polynomials is denoted by $\mathbb{B}(X)$. Similarly, we define the algebra of commutative $\mathbb{B}$-polynomials $\mathbb{B}[x_1, \ldots, x_m]$ as a free left (and a free right) $\mathbb{B}$-module with the basis given by the set of all monomials in $x_1^{i_1} \cdots x_m^{i_m}$ with $i_1, \ldots, i_m \geq 0$, where we assume that $\beta x_i = x_i \beta$ and $x_i x_j = x_j x_i$ for all $\beta \in \mathbb{B}$ and $1 \leq i, j \leq m$. Note that $\mathbb{B}(x) = \mathbb{B}[x]$.

1.1. Parametric Weyl algebra $\mathbb{A}_h(\mathbb{B})$ as the Ore extension. We study the polynomial identities for the following family of infinite-dimensional unital algebras $\mathbb{A}_h(\mathbb{B})$, which are parametrized by a polynomial $h$ from the center of $\mathbb{B}[x]$:

Definition 1.1. For $h \in \mathbb{Z}(\mathbb{B})[x]$, the parametric Weyl algebra $\mathbb{A}_h(\mathbb{B})$ over the ring $\mathbb{B}$ is the unital associative algebra over $\mathbb{F}$ generated by $\mathbb{B}$ and letters $x, y$ commuting with $\mathbb{B}$ subject to the defining relation $yx = xy + h$ (equivalently, $[y, x] = h$, where $[y, x] = yx - xy$), i.e.,

$$\mathbb{A}_h(\mathbb{B}) = \mathbb{B}(x, y)/id\{yx - xy - h\}.$$ 

For short, we denote $\mathbb{A}_h = \mathbb{A}_h(\mathbb{F})$. The partial cases of the given construction are the Weyl algebra $\mathbb{A}_1$, the polynomial algebra $\mathbb{A}_0 = \mathbb{F}[x, y]$, and the universal

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enveloping algebra $A_h$ of the two-dimensional nonabelian Lie algebra. For $h \in \mathbb{F}[x]$, the following isomorphism of $\mathbb{F}$-algebras holds:

$$A_h(\mathcal{B}) \simeq \mathcal{B} \otimes \mathbb{R} A_h.$$  

(1)

Note that in general the isomorphism (1) does not hold because $A_h$ is not well-defined in case $h \notin \mathbb{F}[x]$. Given a polynomial $f = \eta_d x^d + \eta_{d-1} x^{d-1} + \cdots + \eta_0$ of $\mathcal{B}[x]$ with $d \geq 0$, we say that $\eta_d$ is the leading coefficient of $f$ and the product $\eta_d x^d$ is the leading term of $f$.

The algebra $A_h$ was introduced and studied by Benkart, Lopes, Ondrus [9] [8] [10] as a natural object in the theory of Ore extensions. In particular, they determined automorphisms of $A_h$ over an arbitrary field $\mathbb{F}$ and the invariants of $A_h$ under the automorphisms, completely described the simple modules and derivations of $A_h$ over any field. Then Lopes and Solotar [20] described the Hochschild cohomology $\text{HH}^*(A_h)$ over a field of arbitrary characteristic. Over an algebraically closed field of zero characteristic simple $A_h$-modules were independently classified by Bavula [5]. In recent preprints [6], [7] Bavula continued the study of the automorphism group of $A_h$.

Let us recall that an Ore extension of $\mathcal{R}$ (or, equivalently, a skew polynomial ring over $\mathcal{R}$) $A = \mathcal{R}[y, \sigma, \delta]$ is given by a unital associative (not necessarily commutative) algebra $\mathcal{R}$ over a field $\mathbb{F}$, an $\mathbb{F}$-algebra endomorphism $\sigma : \mathcal{R} \to \mathcal{R}$, and a $\sigma$-derivation $\delta : \mathcal{R} \to \mathcal{R}$, i.e., $\delta$ is $\mathbb{F}$-linear map and $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{R}$. Then $A = \mathcal{R}[y, \sigma, \delta]$ is the unital algebra generated by $y$ over $\mathcal{R}$ subject to the relation

$$ya = \sigma(a)y + \delta(a) \quad \text{for all } a \in \mathcal{R}.$$  

Assume that $\mathcal{R} = \mathcal{B}[x]$, $\sigma = \text{id}_\mathcal{R}$ is the identity automorphism on $\mathcal{R}$, and $\delta : \mathcal{R} \to \mathcal{R}$ is given by $\delta(a) = a'h$ for all $a \in \mathcal{R}$, where $a'$ stands for the usual derivative of a $\mathcal{B}$-polynomial $a$ with respect to the variable $x$. Since $h \in Z(\mathcal{R})$, $\delta$ is a derivation of $\mathcal{R}$. Using the linearity of derivative and induction on the degree of $a \in \mathcal{B}[x]$ it is easy to see that

$$[y, a] = a'h \text{ holds in } A_h(\mathcal{B}) \text{ for all } a \in \mathcal{B}[x].$$  

(2)

Hence $A_h(\mathcal{B}) = \mathcal{R}[y, \sigma, \delta]$ is an Ore extension. The following lemma is a corollary of Observation 2.1 from [8] proven by Awami, Van den Bergh and Van Oystaeyen (see also Proposition 3.2 of [11] and Lemma 2.2 of [5]).

Lemma 1.2. Assume that $A = \mathcal{R}[y, \sigma, \delta]$ is an Ore extension of $\mathcal{R} = \mathbb{F}[x]$, where $\sigma$ is an automorphism of $\mathcal{R}$. Then $A$ is isomorphic to one of the following algebras:

- a quantum plane, i.e., $A \simeq \mathbb{F}(x, y)/\text{id}\{yx - qxy\}$ for some $q \in \mathbb{F}^* = \mathbb{F}\backslash\{0\}$;
- a quantized Weyl algebra, i.e., $A \simeq \mathbb{F}(x, y)/\text{id}\{yx - qxy - 1\}$ for some $q \in \mathbb{F}^*$;
- an algebra $A_h$ for some $h \in \mathbb{F}[x]$.

Note that by Theorem 9.3 of [11] the algebra $A_h$ is not a generalized Weyl algebra over $\mathbb{F}[x]$ in the sense of Bavula [4] in case $h \notin \mathbb{F}$.

Since the algebra of $\mathcal{B}$-polynomials $\mathcal{B}[x, y]$ is well studied, in what follows we assume that $h$ is non-zero. Moreover, we assume that the following restriction holds:

**Convention 1.3.** The leading coefficient of $h \in Z(\mathcal{B})[x]$ is not a zero divisor.
1.2. Polynomial identities. A polynomial identity for a unital \( F \)-algebra \( A \) is an element \( f(x_1, \ldots, x_m) \) of \( F(X) \) such that \( f(a_1, \ldots, a_m) = 0 \) in \( A \) for all \( a_1, \ldots, a_m \in A \). The set \( \text{Id}(A) \) of all polynomial identities for \( A \) is a T-ideal, i.e., \( \text{Id}(A) \) is an ideal of \( F(X) \) such that \( \phi(\text{Id}(A)) \subset \text{Id}(A) \) for every endomorphism \( \phi \) of \( F(X) \). An algebra that satisfies a nontrivial polynomial identity is called a PI-algebra. A T-ideal \( I \) of \( F(X) \) generated by \( f_1, \ldots, f_k \) is the minimal T-ideal of \( F(X) \) that contains \( f_1, \ldots, f_k \). We say that \( f \in F(X) \) follows from \( f_1, \ldots, f_k \) if \( f \in I \). Given a monomial \( w \in F(x_1, \ldots, x_m) \), we write \( \deg_x(w) \) for the number of letters \( x_i \) in \( w \) and \( \ndeg(w) \) for the multidegree \( (\deg_{x_1}(w), \ldots, \deg_{x_m}(w)) \) of \( w \). An element \( f \in F(X) \) is called multihomogeneous if it is a linear combination of monomials of the same multidegree. We say that algebras \( A, B \) are called PI-equivalent and write \( A \sim_{PI} B \) if \( \text{Id}(A) = \text{Id}(B) \).

Denote the \( n \)th Weyl algebra by
\[
A_n = F(x_1, \ldots, x_n, y_1, \ldots, y_n)/I,
\]
where the ideal \( I \) is generated by \([y_i, x_j] - \delta_{ij}, [x_i, x_j] = 0, [y_i, y_j] = 0\) for all \( 1 \leq i, j \leq n \). Note that \( A_1 = A \).

Assume that \( p = 0 \). It is well-known that the algebra \( A_n \) does not have nontrivial polynomial identities. Nevertheless, some subspaces of \( A_n \) satisfy certain polynomial identities. Namely, denote by \( A_n^{(1,1)} \) the \( F \)-span of \( x_i y_j \) in \( A_n \) for all \( 1 \leq i, j \leq n \) and by \( A_n^{(-r)} \) the \( F \)-span of \( a y_{j_1} \cdots y_{j_r} \) in \( A_n \) for all \( 1 \leq j_1, \ldots, j_r \leq n \) and \( a \in F[x_1, \ldots, x_n] \). Dzhumadil’daev and Yeliussizov \cite{14} studied the standard polynomial
\[
\text{St}_N(t_1, \ldots, t_N) = \sum_{\sigma \in S_N} (-1)^n t_{\sigma(1)} \cdots t_{\sigma(N)}
\]
over some subspaces of \( A_n \). Namely, he showed that
- \( \text{St}_N \) is a polynomial identity for \( A_n^{(-1)} \) in case \( N \geq n^2 + 2n \);
- \( \text{St}_N \) is not a polynomial identity for \( A_n^{(-1)} \) in case \( N < n^2 + 2n - 1 \);
- \( \text{St}_N \) is a polynomial identity for \( A_1^{(-r)} \) if and only if \( N > 2r \);
- the minimal degree of nontrivial identity in \( A_1^{(-r)} \) is \( 2r + 1 \).

Using graph–theoretic combinatorial approach Dzhumadil’daev and Yeliussizov \cite{14} established that
- \( \text{St}_{2n} \) is a polynomial identity for \( A_n^{(1,1)} \) if and only if \( n = 1, 2, 3 \).

Note that the space \( A_n^{(-1)} \) together with the multiplication given by the Lie bracket is the \( n \)th Witt algebra \( W_n \), which is a simple infinitely dimensional Lie algebra. The polynomial identities for the Lie algebra \( W_n \) were studied by Mishchenko \cite{21}, Razmyslov \cite{22} and others. The well-known open conjecture claims that all polynomial identities for \( W_1 \) follow from the standard Lie identity
\[
\sum_{\sigma \in S_4} (-1)^n \left[ \left[ \left[ t_0, t_{\sigma(1)} \right], t_{\sigma(2)} \right], t_{\sigma(3)} \right], t_{\sigma(4)} \right].
\]
\( \mathbb{Z} \)-graded identities for \( W_1 \) were described by Freitas, Koshlukov and Krasilnikov \cite{15}.

1.3. Results. In Theorem \cite{19} we prove that over an infinite field \( F \) of positive characteristic \( p \) the algebra \( A_h(B) \) is PI-equivalent to the algebra of \( p \times p \) matrices over \( B \) in case \( h(\alpha) \) is not a zero divisor for some \( \alpha \in \mathbb{Z}(B) \). On the other hand, over a finite field the similar result does not hold in case \( B = F \) (see Theorem \cite{11}.
As about the case of zero characteristic, in Theorem 3.2 we prove that similarly to $A_1$, the algebra $A_h(B)$ does not have nontrivial polynomial identities.

In Section 4 we consider the algebra $A_h(B) = A_C(F^2)$ such that $h = \zeta$ does not satisfy Convention 1.3 and the statements of Theorems 5.2 and 4.9 do not hold for $A_C(F^2)$. We describe polynomial identities for $A_C(F^2)$ and compare them with the polynomial identities for the Grassmann unital algebra of finite rank.

2. Properties of $A_h(B)$

Many properties of an Ore extension $A = R[y, \sigma, \delta]$ are inherited from an underlying algebra $R$. Namely, it is well-known that when $\sigma$ is an automorphism, then:

- $A$ is a free right and a free left $R$-module with the basis $\{y^i \mid i \geq 0\}$ (see Proposition 2.3 of [19]);
- in case $R$ is left (right, respectively) noetherian we have that $A$ is left (right, respectively) noetherian (see Theorem 2.6 of [19]);
- in case $R$ is a domain we have that $A$ is a domain (see Exercise 2.6 of [19]).

In case $B = F$ the algebra $A_h = A_h(B)$ is a noetherian domain, but in general case $A_h(B)$ lacks these properties, since $B \subset A_h(B)$ (see also Example 2.6 below).

In order to distinguish the generators for the algebras $A_h(B)$ and $A_1(B)$, we will use the following

Con convention 2.1. The generators of $A_h(B)$ are denoted by $x, y, 1$ and the generators of $A_1(B)$ are denoted by $x, y, 1$.

Lemma 2.2. The sets $\{x^i y^j \mid i, j \geq 0\}$ and $\{y^j x^i \mid i, j \geq 0\}$ are $B$-bases for $A_h(B)$.

Proof. Obviously, $A_h(B)$ is the $B$-span of each of the sets from the lemma. On the other hand, $B$-linear independence of these sets follows from the fact that $A_h(B)$ is a free right and a free left $B[x]$-module with the basis $\{y^i \mid i \geq 0\}$. \qed

Introduce the following lexicographical order on $\mathbb{Z}^2$: $(i, j) < (r, s)$ in case $j < s$ or $j = s$, $i < r$. Denote the multidegree of a monomial $w = x^i y^j$ of $A_h(B)$ by $\text{mdeg}(w) = (i, j)$. Given an arbitrary element $a = \sum_{i,j \geq 0} \beta_{ij} x^i y^j$ of $A_h(B)$, where only finitely many $\beta_{ij} \in B$ are non-zero, define its multidegree $\text{mdeg}(a) = (d_x, d_y)$ as the maximal multidegree of its monomials, i.e. as the maximal element of the set $\{(i, j) \mid \beta_{ij} \neq 0\}$. By Lemma 2.2 the multidegree is well-defined. As above, the coefficient $\beta_{d_x, d_y}$ is called the leading coefficient of $a$ and the product $\beta_{d_x, d_y} x^{d_x} y^{d_y}$ is called the leading term of $a$. In case $a \in B$ we set $\text{mdeg}(a) = (0, 0)$ and the leading coefficient as well as the leading term of $a$ is $a$.

Lemma 2.3. Assume $i, j, r, s \geq 0$. Then

(a) the leading term of $x^i y^j \cdot x^r y^s$ is $x^{i+r} y^{j+s}$;
(b) in case $h \in B$ we have

\[
x^i y^j \cdot x^r y^s = \sum_{k=0}^{\min\{j, r\}} k! \binom{j}{k} \binom{r}{k} x^{i+r-k} h^k y^{j+s-k}.
\]
**Proof.** Recall that \( \delta(a) = a'h \) for each \( a \in B[x] \). Since \([\gamma, a] = \delta(a)\), the induction on \( j \) implies that
\[
\gamma^j x^r = \sum_{k=0}^{j} \binom{j}{k} \delta^k(x^r) \gamma^{j-k}
\] (cf. Lemma 5.2 of [9]). Taking \( k = 0 \) in equality (3), we obtain that the leading term of \( \gamma^j x^r \) is \( x^r \gamma^j \). Similarly we conclude the proof of part (a). Part (b) follows from (3) and
\[
\delta^k(x^r) = \begin{cases} \frac{r!}{(r-k)!} x^{r-k} h^k, & \text{if } k \leq r \\ 0, & \text{if } k > r \end{cases}
\]
\( \square \)

**Lemma 2.4.** If the leading coefficient of one of non-zero elements \( a, b \in A_h(B) \) is not a zero divisor, then \( \text{mdeg}(ab) = \text{mdeg}(a) + \text{mdeg}(b) \). In particular, \( ab \) is not zero.

**Proof.** Consider \( a = \sum_{i=1}^{m} \beta_i x^{r_i} \gamma^{s_i} \) and \( b = \sum_{j=1}^{n} \gamma_j x^{k_j} \gamma^{l_j} \) for \( m, n \geq 1 \) and non-zero \( \beta_i, \gamma_j \in B \), where we assume that elements of each of the sets \( \{(r_i, s_i) \mid 1 \leq i \leq m\} \) and \( \{(k_j, l_j) \mid 1 \leq j \leq n\} \) are pairwise different. Assume that \( \text{mdeg}(a) = (r_1, s_1) \) and \( \text{mdeg}(b) = (k_1, l_1) \). Part (a) of Lemma 2.3 implies that
\[
\text{mdeg}(x^{r_1} \gamma^{s_1} x^{k_1} \gamma^{l_1}) = (r_1 + k_1, s_1 + l_1) \quad \text{and} \quad \text{mdeg}(x^{r_j} \gamma^{s_j} x^{k_j} \gamma^{l_j}) < (r_1 + k_1, s_1 + l_1)
\]
if \( (i, j) \neq (1, 1) \). Since \( \beta \gamma_1 \neq 0 \), we obtain \( \text{mdeg}(ab) = (r_1 + k_1, s_1 + l_1) \) and the proof is concluded. \( \square \)

**Lemma 2.5.**

(a) The \( B \)-linear homomorphism of \( A \)-algebras \( \psi : A_h(B) \to A_1(B) \), defined by
\[
1 \to 1, \quad x \to x, \quad \gamma \to y h,
\]
is an embedding \( A_h(B) \subset A_1(B) \).

(b) \( \{ x^i h^j y^l \mid i, j, l \geq 0 \} \) and \( \{ y^l h^j x^i \mid i, j, l \geq 0 \} \) are \( B \)-bases for \( A_h(B) \subset A_1(B) \).

**Proof.** (a) Since \( \psi([\gamma, x] - h) = ([y, x] - 1)h = 0 \) in \( A_1(B) \), the homomorphism \( \psi \) is well-defined. Assume that \( \psi \) is not an embedding, i.e., there exists non-zero finite sum \( a = \sum_{i,j \geq 0} \beta_{ij} x^i y^j \) with \( \beta_{ij} \in B \) such that
\[
\psi(a) = \sum_{i,j \geq 0} \beta_{ij} x^i y^j h = 0 \quad \text{in} \quad A_1(B).
\]
Denote \( \text{mdeg}(a) = (r, s) \). If \( (r, s) = (0, 0) \), then \( a \in B \) and \( \psi(a) = a \) is not zero; a contradiction. Therefore, \( (r, s) \neq (0, 0) \). Since \( \text{mdeg}(x^i y^j h) = (i + j \deg(h), j) \) by Lemma 2.4 and Conventions 1.3 we obtain that \( \text{mdeg}(\psi(a)) = (r + s \deg(h), s) \) is not zero; a contradiction.

(b) Since \( h \) lies in the center of \( B[x] \), repeating the proof of Lemma 3.4 from [9] for \( A_h(B) \) we can see that
\[
A_h(B) = \bigoplus_{j \geq 0} B[x] h^j y^j = \bigoplus_{j \geq 0} y^l h^j B[x].
\]
Similarly to part (a), we conclude the proof by the reasoning with multidegree. \( \square \)
Example 2.6. Assume $B$ is the $\mathbb{F}$-algebra of double numbers, i.e., $B$ has an $\mathbb{F}$-basis $\{1, \zeta\}$ with $\zeta^2 = 0$. Then the ideal $I = \mathbb{F}$-span $\{\zeta^i\zeta^j \mid i, j \geq 0\}$ is a proper nilpotent ideal of $A_h(B)$. In particular, the algebra $A_h(B)$ is not semi-prime.

3. $A_h(B)$ as the algebra of differential operators

Denote by $\text{Map}(B[z])$ the algebra of all $\mathbb{F}$-linear maps over $B[z]$ with respect to composition. Assume that $\mathcal{D}_h(B[z])$ is the subalgebra of $\text{Map}(B[z])$ generated by the following maps: the multiplication $\beta$ by an element $\beta$ of $B$, i.e., $(\beta \text{Id})(f) = \beta f$, the multiplication $\chi$ by $z$, i.e., $\chi(f) = zf$, and the derivation $\partial$ given by $\delta(f) = f' h(z)$ for all $f \in B[z]$. Note that $\delta = h(\chi) \partial$, where $\partial$ stands for the operator of the usual derivative. Obviously, maps $\chi, h(\chi)$ and $\partial$ are $B$-linear. For short, we write $\chi^0$ for $\text{Id}$.

Proposition 3.1.

(a) $\{\chi^i \chi^j \partial^j \mid i, j \geq 0\}$ is an $B$-basis for $\mathcal{D}_h(B[z])$ in case $p = 0$.

(b) $\{\chi^i \chi^j \partial^j \mid 0 \leq i, 0 \leq j < p\}$ is an $B$-basis for $\mathcal{D}_h(B[z])$ in case $p > 0$.

(c) $A_h(B)/\text{id}(h^p \mathbb{F}) \simeq \mathcal{D}_h(B[z])$ for each $p \geq 0$.

Proof. Consider the $B$-linear homomorphism of $\mathbb{F}$-algebras $\Phi : A_1(B) \to \text{Map}(B[z])$ given by $1 \to \text{Id}, x \to \chi, y \to \partial$. Since $\Phi((y, x) - 1)(f) = (\partial \chi - \chi \partial - \text{Id})(f) = f' + zf' - zf - f = 0$ for all $f \in B[z]$, the map $\Phi$ is a well-defined. Applying $\Phi$ to parts (a) and (b) of Lemma 2.5, we obtain that $\mathcal{D}_h(B[z]) = \Phi(A_h(B))$ is an $B$-span of $\{\chi^i \chi^j \partial^j \mid i, j \geq 0\}$.

Let $p = 0$. Assume that some non-zero finite sum $\pi = \sum_{i, j \geq 0} \beta_{ij} \chi^i \chi^j \partial^j$ with $\beta_{ij} \in B$ belongs to the kernel of $\Phi$. Denote by $j_0$ the minimal $j \geq 0$ with $\beta_{ij} \neq 0$ for some $i$ and denote by $i_0$ the maximal $i$ with $\beta_{i_0 j_0} \neq 0$. Then $\pi(z^n) = j_0! h(z)^{j_0} \sum_{0 \leq i \leq i_0} \beta_{i_0 j_0} z^i = 0$ in $B[z]$. Thus Convention 1.3 together with $j_0! \beta_{i_0 j_0} \neq 0$ implies a contradiction. Part (a) is proven.

Assume that $p > 0$ and some non-trivial finite sum

$$\pi = \sum_{0 \leq i, 0 \leq j < p} \beta_{ij} \chi^i \chi^j \partial^j$$

with $\beta_{ij} \in B$ belongs to the kernel of $\Phi$. As above, we obtain a contradiction. Since $\partial^p = 0$, we conclude the proof of part (b).

Parts (a) and (b) together with part (b) of Lemma 2.5 conclude the proof of part (c).

Theorem 3.2. In case $p = 0$ the algebra $A_h(B)$ does not have nontrivial polynomial identities.

Proof. Assume that $B$-algebra $A_h(B)$ has a nontrivial polynomial identity. Since $p = 0$, there exists $N > 0$ such that $A_h(B)$ satisfies a nontrivial multilinear identity $f(x_1, \ldots, x_N) = \sum_{\sigma \in S_n} \alpha_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(N)}$ with $\alpha_{\sigma} \in \mathbb{F}$. Moreover, we can assume that $\alpha_{\text{Id}} \neq 0$ for the identity permutation $\text{Id}$. Given $j > 0$, we write $F_j$ for a $B$-linear map $\chi^j h(\chi)^j \partial^j$ from $\mathcal{D}_h(B[z])$. Denote by $d \geq 0$ the degree of $h$ and we write $\eta$ for the leading coefficient of $h$. Recall that $\eta$ is not a zero divisor by Convention 1.3.

Note that

$$F_j(z^n) = \begin{cases} m! z^m & \text{in case } m < j \\ 0 & \text{in case } m = j \\ z^{m+j} h(z)^j & \text{in case } m \geq j. \end{cases}$$

(4)
In particular, \( \text{deg}(F_j(z^l)) = j(d + 2) \) and the leading coefficient of \( F_j(z^l) \) is \( j! \eta^l \), which is not a zero divisor.

By parts (a) and (c) of Proposition 3.1, the equality \( f(F_1, \ldots, F_N) = 0 \) holds in \( \mathfrak{D}_h(B[z]) \) for all \( j_1, \ldots, j_N > 0 \). Consider \( j_k = (d + 2)^{N-k} \) for all \( 0 \leq k \leq N \). Note that \( 1 = j_N < j_{N-1} < \cdots < j_0 \). We claim that for any \( \sigma \in S_N \) we have

\[
F_{j_1(z_1)} \circ \cdots \circ F_{j_N(z_N)}(z) \neq 0 \text{ if and only if } \sigma = \text{Id}.
\]

The leading term of \( F_{j_1(z_1)} \circ \cdots \circ F_{j_N(z_N)}(z) \) is \( j_1! \cdots j_N! \eta^{j_1 + \cdots + j_N} z^{j_0} \).

Let us prove these claims. Assume that \( F_{j_1(z_1)} \circ \cdots \circ F_{j_N(z_N)}(z) \neq 0 \) for some \( \sigma \in S_N \).

Since \( F_{j_1(z_1)}(z) \neq 0 \), then equality (4) implies that \( \sigma(N) = N \), \( j_{\sigma(N)} = 1 \), \( \text{deg}(F_{j_{\sigma(N)}}(z)) = d + 2 = j_{N-1} \) and the leading coefficient of \( F_{j_{\sigma(N)}}(z) \) is \( \eta \), which is not a zero divisor.

Similarly, assume that for \( 1 \leq l < N \) with \( \sigma(l) \leq l \) the inequality \( F_{j_{\sigma(l)}}(g) \neq 0 \) holds for some \( g \in B[z] \) with the leading term \( j_{l+1}! \cdots j_N! \eta^{j_{l+1} + \cdots + j_N} z^{j_l} \). Then equality (1) implies that \( \sigma(l) = l \) and \( \text{deg}(F_{j_{\sigma(l)}}(g)) = j_{l-1} \). Moreover, the leading term of \( F_{j_{\sigma(l)}}(g) \) is \( j_l! \cdots j_N! \eta^{j_{l+1} + \cdots + j_N} z^{j_{l-1}} \). Consequently applying this reasoning to \( l = N-1, l = N-2, \ldots, l = 1 \), we conclude the proof of claims (5) and (6).

Claims (5) and (6) imply that \( 0 = f(F_1, \ldots, F_N)(z) = \alpha_{\text{Id}} F_{j_1(z_1)} \circ \cdots \circ F_{j_N(z_N)}(z) \neq 0 \) by Convention 1.3. a contradiction.

4. POLYNOMIAL IDENTITIES FOR \( A_h(B) \) IN POSITIVE CHARACTERISTIC

We write \( M_n = M_n(\mathbb{F}) \) for the algebra of all \( n \times n \) matrices over \( \mathbb{F} \) and denote by \( \tilde{M}_n \) the algebra of all \( n \times n \) matrices over \( B[x, y] \). Denote by \( I_n \) the identity \( n \times n \) matrix and by \( E_{ij} \in M_n \) the matrix such that the \( (i, j) \)th entry is equal to one and the rest of entries are zeros. Consider the properties of the next two matrices of \( M_p \):

\[
A_0 = \sum_{i=1}^{p-1} E_{i+1,i} \quad \text{and} \quad B_0 = \sum_{i=1}^{p-1} E_{i,i+1}.
\]

Lemma 4.1.

(a) For all \( 0 \leq k < p \) we have that

\[
A_0^k = \sum_{i=1}^{p-k} E_{k+i,i} \quad \text{and} \quad B_0^k = \sum_{i=1}^{p-k} \frac{(k+i-1)!}{(i-1)!} E_{i,k+i},
\]

where \( A_0^0 \) and \( B_0^0 \) stand for \( I_p \).

(b) \( B_0 A_0 - A_0 B_0 = I_p \).

Proof. The formula for \( A_0^k \) is trivial. We prove the formula for \( B_0^k \) by induction on \( k \). For \( k = 1 \) the claim holds. Assume that the claim is valid for some \( k < p - 1 \). Then

\[
B_0^{k+1} = \sum_{i=1}^{p-k} \frac{(k+i-1)!}{(i-1)!} E_{i,k+i} \left( \sum_{r=1}^{p-1} r \cdot E_{r,r+1} \right) = \sum_{i=1}^{p-k} \frac{(i+k)!}{(i-1)!} E_{i,k+1+i}
\]

and the required is proven. Part (b) is straightforward. \( \square \)
Define the \( B \)-linear homomorphism \( \varphi : B(x, y) \to \widetilde{M}_p \) of algebras by
\[
x \mapsto A, \quad y \mapsto B, \quad 1 \mapsto I_p,
\]
where \( A = xI_p + A_0 \) and \( B = yI_p + B_0 \). Since \( \beta A = A\beta \) and \( \beta B = B\beta \) for each \( \beta \in B \), the homomorphism \( \varphi \) is well-defined.

**Lemma 4.2.** The homomorphism \( \varphi \) induces the injective \( B \)-linear homomorphism \( \overline{\varphi} : A_1(B) \to \widetilde{M}_p \). In particular, the restriction of \( \overline{\varphi} \) to \( A_h(B) \subset A_1(B) \) is the injective \( B \)-linear homomorphism \( A_h(B) \to \widetilde{M}_p \).

**Proof.** By part (b) of Lemma 1.1 we have that \( \varphi(yx - xy - 1) = BA - AB - I_p = 0 \). Therefore, \( \varphi \) induces a \( B \)-linear homomorphism \( \overline{\varphi} : A_1(B) \to \widetilde{M}_p \) of algebras.

Assume that there exists a nonzero \( a \in A_1(B) \) such that \( \overline{\varphi}(a) = 0 \). Since \( \{ x^iy^j \mid i, j \geq 0 \} \) is an \( B \)-basis for \( A_1(B) \) by Lemma 2.2 we have \( a = \sum_{i,j \geq 0} \beta_{ij} x^i y^j \) for a finite sum with \( \beta_{ij} \in B \). Thus \( 0 = \overline{\varphi}(a) = \sum_{i,j \geq 0} \beta_{ij} A_1^i B^j \). The equalities
\[
A_1^i = \begin{pmatrix} x^i & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & * \end{pmatrix}
\quad \text{and} \quad B^j = \begin{pmatrix} y^j & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & * \end{pmatrix}
\]
imply that the \((1, 1)\)th entry of \( A_1^i B^j \) is \((A_1^i B^j)_{1,1} = x^i y^j \). Hence \( 0 = (\varphi(a))_{1,1} = \sum_{i,j \geq 0} \beta_{ij} x^i y^j \) in \( B[x, y] \). Hence \( \beta_{ij} = 0 \) for all \( i, j \geq 0 \) and \( a = 0 \); a contradiction. Therefore \( \overline{\varphi} \) is injective.

Given \( 1 \leq i, j \leq p \) and \( k \geq 1 \), we write \( z_{ij}(k) \) for \( x_{i+p(j-1)+p^2(k-1)} \in F(X) \). The generic \( p \times p \) matrix \( X_k \) with non-commutative elements is the matrix \( X_k = (z_{ij}(k))_{1 \leq i, j \leq p} \).

**Corollary 4.3.** \( \text{Id}(M_p(B)) \subset \text{Id}(A_h(B)) \), if the field \( F \) is infinite.

**Proof.** Lemma 1.2 implies that \( \text{Id}(\widetilde{M}_p) \subset \text{Id}(A_h(B)) \).

Since \( B \subset B[x, y] \), we have \( \text{Id}(\widetilde{M}_p) \subset \text{Id}(M_p(B)) \). On the other hand, assume that \( f = f(x_1, \ldots, x_n) \) is a polynomial identity for \( M_p(B) \). Then \( f(X_1, \ldots, X_n) = (f_{ij})_{1 \leq i, j \leq n} \) for some \( f_{ij} \in \mathbb{F}(X) \) with \( f_{ij} \in \text{Id}(B) \). It is well-known that for an infinite field \( \mathbb{F} \) and a commutative unital \( \mathbb{F} \)-algebra \( C \) the polynomial identities for a unital \( \mathbb{F} \)-algebra \( B \) and \( C \otimes_\mathbb{F} B \) are the same (for example, see Lemma 1.4.2 of [13]). Since \( B[x, y] = \mathbb{F}[x, y] \otimes_\mathbb{F} B \), we obtain \( f_{ij} \in \text{Id}(B[x, y]) \) and \( f \in \text{Id}(\widetilde{M}_p) \). The required is proven.

For each \( \alpha \in Z(B) \) consider the evaluation \( B \)-linear homomorphism \( \epsilon_\alpha : B[x, y] \to B \) of unital \( \mathbb{F} \)-algebras defined by
\[
x \mapsto \alpha, \quad y \mapsto 0
\]
and extend it to the evaluation homomorphism \( \epsilon_\alpha : \widetilde{M}_p \to M_p(B) \). Since \( A_h(B) \) is a subalgebra of \( \widetilde{M}_p \) by means of embedding \( \overline{\varphi} \) (see Lemma 4.2), we can consider the images of \( x, \tilde{y} \in A_h(B) \) in \( M_p(B) \), which we denote by \( C_\alpha \) and \( D_\alpha \), respectively:
\[
C_\alpha = \epsilon_\alpha(\overline{\varphi}(x)) = \epsilon_\alpha(A) = \alpha I_p + A_0,
D_\alpha = \epsilon_\alpha(\overline{\varphi}(\tilde{y})) = \epsilon_\alpha(Bh(A)) = B_0 \epsilon_\alpha(h(A)).
\]
Obviously, $\beta C_\alpha = C_\alpha \beta$ and $\beta D_\alpha = D_\alpha \beta$ for each $\beta \in B$. To obtain the explicit description of the matrix $D_\alpha$ we calculate $h(A)$. For $r \geq 1$ denote the $r^{th}$ derivative of $h \in B[x]$ by $h^{(r)} = \frac{d^r h}{dx^r}$ and write $h^{(0)}$ for $h$. Note that $h^{(r)}(\alpha)$ lies in the center of $B$.

**Lemma 4.4.**

$$h(A) = \sum_{i=1}^{p} \sum_{j=1}^{i} \frac{1}{(i-j)!} h^{(i-j)} E_{ij}.$$

**Proof.** We start with the case of $h = x^k \in B[x]$ for some $k \geq 0$. Obviously, the claim of the lemma holds for $h = 1$. Therefore, we assume that $k \geq 1$. Since $A_0^r = 0$ for all $r \geq p$, we have

$$h(A) = A^k = (xI_p + A_0)^k = \sum_{r=0}^{\min\{k,p-1\}} \binom{k}{r} x^{k-r} A_0^r.$$

Part (a) of Lemma 4.1 implies

$$h(A) = \sum_{r=0}^{\min\{k,p-1\}} \binom{k}{r} x^{k-r} \left( \sum_{i=1}^{p-r} E_{r+i,i} \right).$$

Regrouping the terms we obtain

$$h(A) = \sum_{i=1}^{p} \sum_{j=\max\{1,i-k\}}^{i} \binom{k}{i-j} x^{k-(i-j)} E_{ij}.$$  \hspace{1cm} (7)

Note that for $0 \leq r < p$ we can rewrite

$$\frac{1}{r!} h^{(r)} = \left\{ \begin{array}{ll} \binom{k}{r} x^{k-r}, & \text{if } r \leq k \\ 0, & \text{otherwise} \end{array} \right.$$  

where $r!$ is not zero in $\mathbb{F}$. Hence equality (7) implies that the claim holds for $h = x^k$.

The general case follows from the proven partial case and the $B$-linearity of derivatives. \hspace{1cm} $\square$

**Lemma 4.4** together with the definition of $D_\alpha$ immediately implies the next corollary.

**Corollary 4.5.**

$$D_\alpha = \sum_{i=1}^{p-1} \sum_{j=1}^{i} \frac{i}{(i-j+1)!} h^{(i-j+1)}(\alpha) E_{ij}.$$

For short, denote the $(i,j)^{th}$ entry of $D_\alpha$ by $\xi_{i,j} \in Z(B)$ and for all $1 \leq k < p$ define

$$D_{\alpha,k} = D_\alpha - \sum_{r=0}^{k-1} \xi_{k,k-r} A^r = D_\alpha - \sum_{r=0}^{k-1} \sum_{i=1}^{p-r} \xi_{k,k-r} E_{r+i,i}.$$  \hspace{1cm} (8)
We apply the following technical lemma in the proof of key Proposition 4.7 (see below).

**Lemma 4.6.** For all $1 \leq r \leq p$ and $1 \leq k < p$ we have

$$E_{rk}D_{\alpha,k} = k \cdot h(\alpha) E_{r,k+1}.$$  

Proof. We have

$$E_{rk}D_{\alpha,k} = \sum_{i=1}^{p-1} \sum_{j=1}^{i+1} \xi_{i,j} E_{rk} E_{ij} - \sum_{j=0}^{k-1} \sum_{i=1}^{p-j} \xi_{k-k-j} E_{rk} E_{i+j,i}$$

$$= \sum_{j=1}^{k+1} \xi_{k,j} E_{rj} - \sum_{j=1}^{k} \xi_{k,j} E_{rj}$$

$$= \xi_{k,k+1} E_{r,k+1}.$$

Equality $\xi_{k,k+1} = k \cdot h(\alpha)$ concludes the proof. 

**Proposition 4.7.**

(a) Assume $\alpha \in Z(B)$. Then $\varepsilon_\alpha(A_0(B))$ contains $h(\alpha)^2(p-1)M_\beta(B)$.

(b) Assume $h(\alpha)$ is invertible in $B$ for some $\alpha \in Z(B)$. Then $\text{Id}(A_\beta(B)) \subset \text{Id}(M_\beta(B))$.

(c) Assume $h(\alpha)$ is not a zero divisor for some $\alpha \in Z(B)$ and $\mathbb{F}$ is infinite. Then $\text{Id}(A_\beta(B)) \subset \text{Id}(M_\beta(B))$.

Proof. For short, we write $\beta$ for $h(\alpha) \in Z(B)$.

(a) Denote by $L = \varepsilon_\alpha(A_0(B))$ the $\mathbb{F}$-algebra generated by $B, I_p, C_\alpha, D_\alpha$. Since $A_0 = C_\alpha - \alpha I_p$, we obtain that

$$A^k_0 = \sum_{i=1}^{p-k} \left\{ E_{k+i,i} \right\} \in L$$

for all $0 \leq k < p$. In particular, $E_{p1} = A^0_{0} \in L$. Equality (8) implies that $D_{\alpha,k} \in L$ for all $1 \leq k < p$.

The statement of part (a) follows from the following claim:

$$\{ \beta^{p+k-r-1} E_{rk} \mid 1 \leq r, k \leq p \} \subset L. \quad (9)$$

To prove the claim we use descending induction on $r$.

Assume $r = p$. We have $E_{p1} \in L$. Lemma 4.6 implies that $E_{p1} D_{\alpha,1} = \beta E_{p2}$. Since $E_{p1}, D_{\alpha,1}$ belong to $L$, we can see that $\beta E_{p2} \in L$. Similarly, the equality $\beta E_{p2} D_{\alpha,2} = \beta^2 E_{p3}$ implies $\beta^2 E_{p3} \in L$. Repeating this reasoning we obtain that $\beta^{k-1} E_{pk} \in L$ for all $1 \leq k < p$.

Assume that for some $1 \leq r < p$ claim (9) holds for all $r' > r$, i.e., for every $1 \leq k \leq p$ we have $\beta^{p+k-r'-1} E_{r'k} \in L$. Since

$$\beta^{p-r} \left( A^r_0 - \sum_{k=2}^{p-r+1} E_{(r-1)+k,k} \right) = \beta^{p-r} E_{r1},$$

we obtain $\beta^{p-r} E_{r1} \in L$. Lemma 4.6 implies that $\beta^{p-r} E_{r1} D_{\alpha,1} = \beta^{p+1-r'} E_{r2}$. Hence $\beta^{p+1-r} E_{r2} \in L$. Repeating this reasoning we obtain that $\beta^{p+k-r-1} E_{rk} \in L$.
for all $1 < k \leq p$, since $\beta^{p+k-r-2}E_{r,k-1}D_{\alpha,k-1} = (k-1)\beta^{p+k-r-1}E_{r,k}$. Claim (9) is proven.

(b) Since $h(\alpha)$ is invertible in $B$, part (a) implies that $\varepsilon_\alpha(A_h(B)) = M_p(B)$. Since $\varepsilon_\alpha$ is a homomorphism of $F$-algebras, the required is proven.

(c) Consider a polynomial identity $f \in F(x_1, \ldots, x_m)$ for $A_0(B)$. Since $F$ is infinite, without loss of generality we can assume that $f$ is homogeneous with respect to the natural grading of $F(x_1, \ldots, x_m)$ by degrees, i.e., each monomial of $f$ has one and the same degree $t > 0$. Part (a) implies that for every $A_1, \ldots, A_m$ from $M_p(B)$ there exist $a_1, \ldots, a_m$ from $A_h(B)$ such that

$$\beta^{2(p-1)t}f(A_1, \ldots, A_m) = f(\beta^{2(p-1)}A_1, \ldots, \beta^{2(p-1)}A_m) = f(\varepsilon_\alpha(a_1), \ldots, \varepsilon_\alpha(a_m)).$$

Since $\varepsilon_\alpha$ is a homomorphism of $F$-algebras, we have $f(\varepsilon_\alpha(a_1), \ldots, \varepsilon_\alpha(a_m)) = 0$. Therefore $f$ is a polynomial identity for $A_h(B)$ because $\beta$ is not a zero divisor.

To illustrate the proof of part (a) of Proposition 4.7 we repeat it in the partial case of $p = 3$ in the following example.

Example 4.8. Assume $p = 3$ and $h(\alpha) \neq 0$ for some $\alpha \in Z(B)$. For short, denote $\beta = h(\alpha)$, $\beta' = h'(\alpha)$ and $\beta'' = h''(\alpha)$. Then $A_0 = E_{21} + E_{32}$,

$$C_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 1 & \alpha \end{pmatrix}, \text{ and } D_\alpha = \begin{pmatrix} \beta' & \beta & 0 \\ \beta'' & 2\beta' & 2\beta \\ 0 & 0 & 0 \end{pmatrix}.$$ 

To show that $L = \text{alg}_B\{I_p, C_\alpha, D_\alpha\}$ contains $\beta^4M_3(B)$, we consider the following elements of $L$:

$$D_{\alpha,1} = D_\alpha - \beta'I_3 = \begin{pmatrix} 0 & \beta & 0 \\ \beta' & \beta' & 2\beta \\ 0 & 0 & -\beta' \end{pmatrix},$$

$$D_{\alpha,2} = D_\alpha - 2\beta'I_3 - \beta''A_0 = \begin{pmatrix} -\beta' & \beta & 0 \\ 0 & 0 & 2\beta \\ 0 & -\beta'' & -2\beta' \end{pmatrix}.$$ 

Note that $A_0 = C_\alpha - \alpha I_3$ and $A_0^2 = E_{31}$ belong to $L$. Since

$$E_{31}D_{\alpha,1} = \beta E_{32} \text{ and } \beta E_{32}D_{\alpha,2} = 2\beta^2E_{33},$$

we obtain that $\beta E_{32}, \beta^2E_{33} \in L$. Thus $\beta(A_0 - E_{32}) = \beta E_{21}$ lies in $L$. Since

$$\beta E_{21}D_{\alpha,1} = \beta^2E_{22} \text{ and } \beta^2E_{22}D_{\alpha,2} = 2\beta^3E_{23},$$

we obtain that $\beta^2E_{22}, \beta^3E_{23} \in L$. Hence $\beta^2(I_3 - E_{22} - E_{33}) = \beta^2E_{11}$ lies in $L$. Since

$$\beta^2E_{11}D_{\alpha,1} = \beta^3E_{12} \text{ and } \beta^3E_{12}D_{\alpha,2} = 2\beta^4E_{13},$$

we obtain that $\beta^3E_{12}, \beta^4E_{13} \in L$. Therefore, $L$ contains $\beta^4M_3(B)$.

Theorem 4.9. Assume that $F$ is an infinite field of characteristic $p > 0$.

(a) If $h(\alpha)$ is not a zero divisor for some $\alpha \in Z(B)$, then $A_h(B) \sim_{p1} M_p(B)$.

(b) If $B = F$, then $A_h \sim_{p1} M_p$. 

Corollary 4.10. Assume that $\mathbb{F}$ is an infinite field of characteristic $p > 0$ and for $h = \eta_d x^d + \eta_d-1 x^{d-1} + \cdots + \eta_0$ from $Z(B)[x]$ we have that $\eta_d$ and $\eta_0$ are not zero divisors. Then $A_h(B) \sim_{P_1} M_p(B)$.

5. $A_h$ over finite fields

In this section we assume that $B = \mathbb{F}$ is the field of finite of order $q = p^k$ and $\mathbb{F} \subset K$ for an infinite field $K$. Since $h \in \mathbb{F}[x]$, Convention 1.3 is equivalent to the inequality $h \neq 0$. As above, we write $M_p$ for $M_p(\mathbb{F})$ and $A_h$ for $A_h(\mathbb{F})$. Given a $K$-algebra $\mathcal{A}$, we write $\text{Id}_K(\mathcal{A})$ for the ideal of $\mathbb{K}(X)$ of polynomial identities for $\mathcal{A}$ over $\mathbb{K}$. In this section we prove the next result.

Theorem 5.1.

(a) $\text{Id}_K(\text{Id}(\mathcal{A}_h(\mathbb{K}))) \cap \mathbb{F}(X) \subset \text{Id}(\mathcal{A}_h)$.

(b) $\text{Id}(\mathcal{A}_h) \subset \text{Id}(M_p)$, if $h(\alpha) \neq 0$ for some $\alpha \in \mathbb{F}$.

(c) $\mathcal{A}_h \not\sim_{P_1} M_p$.

Proof. Since $\mathcal{A}_h \subset \mathcal{A}_h(\mathbb{K}) = \mathcal{A}_h \otimes_{\mathbb{F}} \mathbb{K}$ as $\mathbb{F}$-algebras, we can see that

$$\text{Id}(\mathcal{A}_h) \otimes_{\mathbb{F}} \mathbb{K} = \text{Id}_K(\text{Id}(\mathcal{A}_h(\mathbb{K}))) \cap \mathbb{F}(X) \subset \text{Id}(\mathcal{A}_h).$$

Part (b) of Theorem 5.1 concludes the proof of part (a). Part (b) follows from part (b) of Proposition 4.7.

Consider $F_{p,q}(x,y) = G_{p,q}(x) R_{p,q}(x,y)(y^q - y)$ of $\mathbb{F}(x,y)$, where

$$G_{p,q}(x) = (x^q - x)(x^{q^2} - x)\cdots(x^{q^r} - x),$$

$$R_{p,q}(x,y) = \left(1 - (y \text{ ad } x)^{p-1} y^{-1}\right) \left(1 - (y \text{ ad } x)^{p-2} y^{-1}\right) \cdots \left(1 - (y \text{ ad } x)^{q-1}\right)$$

for $y \text{ ad } x = [y, x]$. Genov [16] proved that $F_{p,q}(x,y)$ is a polynomial identity for $M_p$.

Since $x \text{ ad } x = [x, x] = 0$, for $x \in \mathcal{A}_h$ we have $R_{p,q}(x,x) = 1$ and

$$F_{p,q}(x,x) = (x^q - x)(x^{q^2} - x)\cdots(x^{q^r} - x).$$

By part (b) of Lemma 2.5, elements $x, x^2, x^3, \ldots$ are linearly independent in $\mathcal{A}_h$. Therefore, $F_{p,q}(x,x) \neq 0$ in $\mathcal{A}_h$; part (c) is proven. □

Conjecture 5.2. $\text{Id}(M_p(\mathbb{K})) \cap \mathbb{F}(X) = \text{Id}(\mathcal{A}_h)$. 
6. Counterexample

In this section we consider a counterexample to show that without Convention 1.3 Theorems 3.2 and 4.9 do not hold. Namely, we consider the commutative algebra \( B \cong \mathbb{F}^2 \) of double numbers from Example 2.6, i.e., \( B \) has a \( \mathbb{F} \)-basis \( \{1, \zeta \} \) with \( \zeta^2 = 0 \), and set \( h = \zeta \). Note that Convention 1.3 does not hold for \( h \). Then the statements of Theorems 3.2 and 4.9 are not valid for \( A_\zeta(\mathbb{F}^2) = A_h(B) \) (see Proposition 6.2 below).

**Remark 6.1.** If Convention 1.3 does not hold for \( h \), then Lemmas 2.2 and 2.3 are still valid for \( A_h(B) \).

Part (b) of Lemma 2.3 together with Remark 6.1 implies that for all \( i, j, r, s \geq 0 \) we have
\[
x^{i+j} \hat{y}^r \cdot x^{-r} \hat{y}^s = x^{i+r} \hat{y}^{s+j} + \zeta \, j \cdot x^{i+r-1} \hat{y}^{s+j-1},
\]
where we use conventions that monomials \( \hat{y} \) can be empty and \( \hat{y}^0 = 1 \). Then
\[
[x^{i+j} \hat{y}^r, x^{-r} \hat{y}^s] = \zeta (j \cdot r - is) x^{i+r-1} \hat{y}^{s+j-1} \quad \text{in} \quad A_\zeta(\mathbb{F}^2).
\]

The unital finite dimensional Grassmann algebra \( G_k \) of rank \( k \) has an \( \mathbb{F} \)-basis
\[
\{1, e_{i_1} \cdots e_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq k\}
\]
and satisfies the defining relations \( e_i^2 = 0 \) and \( e_ie_j = -e_je_i \) for all \( 1 \leq i, j \leq k \). The polynomial identities for \( G_k \) were described by Di Vincenzo [11] for \( p = 0 \) and by Giambruno, Koshlukov [17] for any infinite field.

**Proposition 6.2.** Assume that \( \mathbb{F} \) is an infinite field.

(a) The \( T \)-ideal of identities \( \text{Id}(A_\zeta(\mathbb{F}^2)) \) is generated by
\[
f_1 = [[x_1, x_2], x_3], \quad f_2 = [x_1, x_2] [x_3, x_4].
\]
(b) \( A_\zeta(\mathbb{F}^2) \not\subset_{T^t} M_t(\mathbb{C}) \) for every \( t \geq 2 \) and every \( \mathbb{F} \)-algebra \( \mathbb{C} \) with unity.
(c) \( A_\zeta(\mathbb{F}^2) \not\subset_{T^t} G_k \) if and only if \( k \in \{2, 3\} \).

**Proof.** (a) By \( \mathbb{F} \)-linearity formula (11) implies that \( [a, b] \) belongs to \( \zeta A_\zeta(\mathbb{F}^2) \) for all \( a, b \in A_\zeta(\mathbb{F}^2) \). Then \( f_1, f_2 \in \text{Id}(X) \) are nontrivial polynomial identities for \( A_\zeta(\mathbb{F}^2) \), since \( \zeta^2 = 0 \). Note that
\[
f_3 = [x_1, x_2] x_3 [x_4, x_5] = [[x_1, x_2], x_3][x_4, x_5] + x_3[x_1, x_2][x_4, x_5] \in \text{Id}(A_\zeta(\mathbb{F}^2))
\]
follows from \( f_1, f_2 \). Denote by \( I \) the \( T \)-ideal generated by \( f_1, f_2 \).

Assume that \( f = \sum_k \alpha_kw_k \) is a nontrivial identity for \( A_\zeta(\mathbb{F}^2) \), where \( \alpha_k \in \mathbb{F} \) and \( w_k \in \mathbb{F}(x_1, \ldots, x_m) \) is a monomial. Since \( \mathbb{F} \) is infinite, we can assume that \( f \) is multihomogeneous. i.e., there exists \( \underline{d} \in \mathbb{N}^m \) with \( \text{mdeg}(w_k) = \underline{d} \) for each \( k \). We apply equalities
\[
ux_i x_j v = ux_i x_j v - u [x_i, x_j] v, \\
u[x_i, x_j] v = [x_i, x_j] uv - [[x_i, x_j], u] v,
\]
where monomials \( u, v \) can be empty and \( i < j \), to monomials \( \{w_k\} \) and then repeat this procedure. Since \( f, f_3 \in \text{Id}(A_\zeta(\mathbb{F}^2)) \), we finally obtain that there exist \( g \in I \), \( \alpha_0, \alpha_{ij} \in \mathbb{F} \) such that
\[
f = g + \alpha_0 x_{d_1} \cdots x_{d_m} + \sum_{1 \leq i < j \leq m} \alpha_{ij} [x_i, x_j] x_{d_1} \cdots x_{d_{i-1}} x_{d_{j-1}} \cdots x_{d_m} \quad \text{in} \quad \mathbb{F}(X).
\]
Since $f(1, \ldots, 1) = g(1, \ldots, 1) + \alpha_0$, we obtain that $\alpha_0 = 0$. Consider $i < j$ with $d_i, d_j \geq 1$. Making substitutions $x_i \to x$, $x_j \to \hat{y}$, $x_1 \to 1$ for each $l$ different from $i$ and $j$, we can see that $0 = 0 + \alpha_{ij}[x, \hat{y}]x^{d_i - 1}\hat{y}^{d_j - 1}$ in $A_k(F^2)$. Thus $-\alpha_{ij}x^{d_i - 1}\hat{y}^{d_j - 1} = 0$ in $A_k(F^2)$. Lemma 2.2 together with Remark 6.1 implies that $\alpha_{ij} = 0$. Therefore, $f = g$ lies in $I$.

(b) Since $F \subset C$, every polynomial identity for $M_t(C)$ lies in $\text{Id}(M_t(F))$. By Amitsur–Levitzki Theorem [2] the minimal degree of a polynomial identity for $M_t(F)$ is $2t$. In particular, $f_1$ is not an identity for $M_t(C)$.

(c) Since $G_k$ is commutative in case $p = 2$ or $k = 1$, we can assume that $p \neq 2$ and $k \geq 2$. Note that

$$f_2(e_1, e_2, e_3, e_4) = 4e_1e_2e_3e_4 \neq 0 \text{ in } G_k \text{ for } k \geq 4. \hspace{1cm} (12)$$

Thus we can assume that $k \in \{2, 3\}$. The T-ideal $\text{Id}(G_k)$ is generated by

- $f_1, f_2$ in case $p = 0$ or $p = k = 3$.
- $f_1, \text{St}_4$ in case $p > k$, where $k \in \{2, 3\}$.

Since

$$\text{St}_4(x_1, x_2, x_3, x_4) = [x_1, x_2] \circ [x_3, x_4] - [x_1, x_3] \circ [x_2, x_4] + [x_1, x_4] \circ [x_2, x_3],$$

where $u \circ v$ stands for $uv + vu$, part (a) implies that $\text{St}_4$ lies in $\text{Id}(A_k(F^2))$. On the other hand, we can see that $f_2$ is a polynomial identity for $G_k$ when $k \in \{2, 3\}$.

Part (c) is proven. \hfill $\square$

REFERENCES

[1] J. Alev, F. Dumas, *Invariants du corps de Weyl sous l’action de groupes finis* (French, with English summary), Communications in Algebra 25 (1997), no. 5, 1655–1672.

[2] A.S. Amitsur, J. Levitzki, *Minimal identities for algebras*, Proc. Amer. Math. Soc. 1 (1950), 449–463.

[3] M. Awami, M. Van den Bergh, and F. Van Oystaeyen, *Note on derivations of graded rings and classification of differential polynomial rings*, Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 175–183. Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987).

[4] V.V. Bavula, *Generalized Weyl algebras and their representations* (Russian), Algebra i Analiz 4 (1992), no. 1, 75–97. English translation: St. Petersburg Math. J. 4 (1993), no. 1, 71–92.

[5] V.V. Bavula, *Classification of simple modules of the Ore extension $K[X][Y; f(X,Y)]$*, Math. Comput. Sci. 14 (2020), 317–325.

[6] V.V. Bavula, *Isomorphism problems and groups of automorphisms for Ore extensions $K[x][y]; b)$ (zero characteristic),* arXiv: 2107.09401.

[7] V.V. Bavula, *Isomorphism problems and groups of automorphisms for Ore extensions $K[x][y]; f_{x+y}^{-1})$ (prime characteristic),* arXiv: 2107.09977.

[8] G. Benkart, S.A. Lopes, M. Ondrus, *A parametric family of subalgebras of the Weyl algebra II. Irreducible modules*, Recent developments in algebraic and combinatorial aspects of representation theory, 73–98, Contemp. Math., 602, Amer. Math. Soc., Providence, RI, 2013.

[9] G. Benkart, S.A. Lopes, M. Ondrus, *A parametric family of subalgebras of the Weyl algebra I. Structure and automorphisms*, Transactions of the American Mathematical Society 367 (2015), no. 3, 1993–2021.

[10] G. Benkart, S.A. Lopes, M. Ondrus, *Derivations of a parametric family of subalgebras of the Weyl algebra*, Journal of Algebra 424 (2015), 46–97.

[11] O.M. Di Vincenzo, *A note on the identities of the Grassmann algebras*, Unione Matematica Italiana. Bollettino. A. Serie VII, 5 (1991), no. 3, 307–315.

[12] A.S. Dzhumadil’daev, *N-commutators*, Comment. Math. Helv. 79 (2004), no. 3, 516–553.

[13] A.S. Dzhumadil’daev, *2p-commutator on differential operators of order p*, Lett. Math. Phys. 104 (2014), no. 7, 849–869.

[14] A.S. Dzhumadil’daev, D. Yeliussov, *Path decompositions of digraphs and their applications to Weyl algebra*, Adv. in Appl. Math. 67 (2015), 36–54.
[15] J.A. Freitas, P. Koshlukov, A. Krasilnikov, Z-graded identities of the Lie algebra $W_1$, Journal of Algebra 427 (2015), 226–251.
[16] G. Genov, Basis for identities of a third order matrix algebra over a finite field, Algebra Log. 20 (1981), 241–257.
[17] A. Giambruno, P. Koshlukov, On the identities of the Grassmann algebras in characteristic $p > 0$, Israel Journal of Mathematics 122 (2001), 305–316.
[18] A. Giambruno, M. Zaicev, Polynomial identities and asymptotic methods, Math. Surveys Monographs vol. 122, AMS, 2005.
[19] K.R. Goodearl, R.B. Warfield Jr., An Introduction to Noncommutative Noetherian Rings, 2nd edition, Cambridge University Press, 2004.
[20] S.A. Lopes, A. Solotar, Lie structure on the Hochschild cohomology of a family of subalgebras of the Weyl algebra, arXiv: 1903.01226.
[21] S.P. Mishchenko, Solvable subvarieties of a variety generated by a Witt algebra, Math. USSR Sb. 64 (1989), no. 2, 415–426.
[22] Yu. Razmyslov, Identities of algebras and their representations, Transl. Math. Monogr., vol. 138, Amer. Math. Soc., Providence, RI, 1994.

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