Shape Invariance in the Calogero and Calogero-Sutherland Models

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Abstract
We show that the Calogero and Calogero-Sutherland models possess an \( N \)-body generalization of shape invariance. We obtain the operator representation that gives rise to this result, and discuss the implications of this result, including the possibility of solving these models using algebraic methods based on this shape invariance. Our representation gives us a natural way to construct supersymmetric generalizations of these models, which are interesting both in their own right and for the insights they offer in connection with the exact solubility of these models.

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1 Introduction

Exactly soluble models have long provided a foundation for the study of quantum mechanical systems. In the case of quantum systems with one degree of freedom, shape invariance (see [1] and the references therein) has proven to be perhaps the most compelling technique for proving exact solubility and for obtaining the exact solution. In this paper, we demonstrate that the Calogero [2] and Calogero-Sutherland [3] models — two of the best known $N$-body exactly soluble quantum models [4, 5] — exhibit an $N$-body generalization of shape invariance, and we use this as a tool to analyze some features of these models. We will not be able to use this algebraic result to solve completely the models in question, but we will outline how this might be done, and make the preliminary steps in that direction. Clearly, an algebraic explanation of exact solubility in these models would be a significant improvement in our understanding of these models, and so our analysis of the shape invariance structure of these models is valuable.

To lay the appropriate groundwork for our paper, we quickly review shape invariance with one degree of freedom in the following section. Then we turn to the heart of the matter, obtaining a representation for the Calogero and Calogero-Sutherland Hamiltonians in terms of a set of operators similar to raising and lowering operators, and we use this representation to demonstrate the $N$-body shape invariance of these models. The power of shape invariance in the one-dimensional case makes clear the potential significance of this result. With this in mind, we then establish a variety of connections, quantitative and qualitative, between the Calogero and Calogero-Sutherland models on the one hand, and the shape invariant one-dimensional models on the other hand, which provides additional evidence that the $N$-body shape invariance and one-variable shape invariance are indicators of common features in these various models. Our operator representation is also valuable as it leads to a natural construction of the supersymmetric extensions of the $N$-body models we are studying; indeed, we believe this is the first construction of the supersymmetric Calogero-Sutherland model. An interesting consequence that we find of the $N$-body shape invariance is that the supersymmetric extension in each of these models is not unique. Note that we examine these supersymmetric extensions, not simply because they are interesting in their own right, but because their analysis may well lead to the solution of the original non-supersymmetric models, an approach we explore in the penultimate section of this paper.

Readers familiar with shape invariance may be familiar with the result that, for example, the solution of the hydrogen atom may be obtained via shape invariance. Does this not represent the application of shape invariance to a problem with three degrees of freedom? The answer is no; shape invariance in this case is actually only applied to the effective one-dimensional radial problem. As another example, the multidimensional harmonic oscillator completely factorizes into one-dimensional harmonic oscillators, and so can be solved by applying shape invariance for a single degree of freedom repeatedly. In this paper, we are examining an intrinsically $N$-body property, one that does not trivially reduce to a one-dimensional problem.

Recently, Ruhl and Turbiner [6] have explained the integrability of the Calogero and
Calogero-Sutherland models using Turbiner's definition of exact solubility \[^7\]. In this approach, given an infinite set of finite dimensional spaces \(V_0, V_1, V_2, \ldots\), forming an infinite flag \(V_0 \subset V_1 \subset V_2 \subset \ldots\), an exactly soluble operator \(h\) is described as a linear operator that preserves the infinite flag of the spaces. With this definition and a series of lengthy calculations, they show that the Calogero and Calogero-Sutherland models are exactly soluble. Our goal is something different. In the same way that shape invariance has illuminated our understanding of exact solubility in one-dimensional quantum mechanics (even in those models already known to be exactly soluble), demonstrating that an apparent analytic coincidence actually arises from an elegant and simple algebraic structure, so too do we expect that the algebraic relationships we have uncovered will ultimately lead to a ready and efficient determination not only of the exact solubility of these models, but indeed of the solutions themselves. Thus we are ultimately interested in the algebraic understanding of these models that shape invariance can provide, not simply in the ability to solve the models \textit{per se}.

\section{Shape Invariance for Systems with a Single Degree of Freedom}

Why is the condition of shape invariance significant? In the case of systems with one degree of freedom, shape invariance leads not only to exact solubility, but indeed to the exact solution. In fact, the algebraic methods of shape invariance probably give the clearest understanding of why these models are soluble and certainly give the easiest computational methods for obtaining the energy eigenvalues and wavefunctions. We are hopeful that our \(N\)-body generalization will be similarly powerful. In order to consider how to proceed in the \(N\)-body case, we here summarize the salient features of the one-dimensional case.

Consider a Hamiltonian \(H\) which is a function of some parameter \(\alpha\). One wishes to find the eigenfunctions and eigenstates of \(H(\alpha)\). Suppose one can write \(H(\alpha) = A(\alpha)^\dagger A(\alpha)\). As is well known from studies of supersymmetry, the partner Hamiltonian \(\tilde{H}(\alpha) = A(\alpha)A(\alpha)^\dagger\) has essentially the same spectrum. To be more precise, if

\[ H\psi = E\psi , \]

then

\[ \tilde{H}(A\psi) = A(H\psi) = E(A\psi) , \]

and likewise, if

\[ \tilde{H}\tilde{\psi} = \tilde{E}\tilde{\psi} , \]

then

\[ H(A^\dagger\tilde{\psi}) = \tilde{E}(A^\dagger\tilde{\psi}) . \]

Thus, the spectrum of non-zero energies is the same for the two Hamiltonians, and the corresponding states that have the same energy are easily constructed from each other by

\[^1\alpha\ may\ as\ well\ stand\ as\ a\ collective\ symbol\ to\ indicate\ dependence\ of\ H\ on\ many\ parameters.\]
acting with $A$ or $A^\dagger$. The only place they may differ in their spectra is that, for example, $H$ may have a zero-energy eigenstate while $\tilde{H}$ does not. Note too that the representation in terms of $A$ and $A^\dagger$ ensures that the two Hamiltonians are positive semi-definite, which means a zero energy state, if it exists, is the ground state. Furthermore, such a zero energy eigenstate is especially easy to construct, as one need only solve $A\psi = 0$, a first-order rather than a second-order differential equation.

Shape invariance exists when the two partner Hamiltonians describe the same physical model at different values of the coupling constant(s), i.e., when the $\tilde{H}$ are related in the following way,

$$\tilde{H}(\alpha) = H(\alpha_1) + R(\alpha_1), \quad (1)$$

where $\alpha_1 = f(\alpha)$ and $R(\alpha)$ are both $\alpha$-dependent constants. When this occurs, one can obtain the entire spectrum of $H$ and all the wavefunctions once one knows the ground state. Here is how it works.

The ground state of $H(\alpha)$ we find by $A(\alpha)\psi_0(\alpha) = 0$. This has zero energy, and hence no “tilde” partner. The first excited state of $H$ is degenerate with the ground state of $\tilde{H}$. However, $\tilde{H}(\alpha) = H(\alpha_1) + R(\alpha_1)$, and so its ground state is $\psi_0(\alpha_1)$, with energy equal to $E_1 = R(\alpha_1)$. Then the first excited state of $H(\alpha)$ is $A^\dagger(\alpha)\psi_0(\alpha_1)$ and also has energy $E_1 = R(\alpha_1)$. Iterating this process, one obtains the exact energies

$$E_k = \begin{cases} 0, & \text{if } k = 0, \\ \sum_{j=1}^{k} R(\alpha_j), & \text{if } k > 0. \end{cases} \quad (2)$$

and corresponding energy eigenfunctions

$$\psi_n(x; \alpha_0) = A^\dagger(\alpha_0) A^\dagger(\alpha_1) \ldots A^\dagger(\alpha_{n-1}) \psi_0(x; \alpha_n), \quad (3)$$

where $\alpha_{j+1} = f(\alpha_j)$.

An example of a shape invariant model is given by the trigonometric Rosen-Morse model, with Hamiltonian

$$H = p^2 + \frac{b(b-a)}{\sin^2(ax)} - b^2. \quad (4)$$

The parameters of this potential are $\alpha = (b,a)$. Define

$$A = \frac{d}{dx} - b \cot(ax), \quad (5)$$

$$A^\dagger = -\frac{d}{dx} - b \cot(ax). \quad (6)$$

Then $A^\dagger A = H(b,a)$ and $AA^\dagger = H(b_1,a_1) + R(b_1,a_1)$ where $b_1 = b + a$, $a_1 = a$ and

$$R(b_1,a_1) = (b + a)^2 - b^2. \quad (7)$$

The spectrum of the model is thus given by

$$E_n = (b + na)^2 - b^2, \quad (8)$$

where $n$ is a natural number $n = 0, 1, \ldots$. It is worth noting that at $b = a$, the Hamiltonian $H$ is that of a free particle in a box.
3 $N$-Body Models: Exact Solubility and Shape Invariance

The use of shape invariance, or some suitable generalization thereof, to solve systems with multiple degrees of freedom remains an unrealized goal. Where shape invariance has been applied to such systems, for example the 3-dimensional hydrogen atom \cite{8}, it has actually been applied to the effective one-dimensional radial problem that arises in a sector of fixed angular momentum.

Nonetheless, the idea that there should be an algebraic explanation for the exact solubility of systems with multiple degrees of freedom remains compelling. Indeed, it seems unlikely, given our experience with ordinary shape invariance, that exact solubility should arise simply as an analytic accident.

Two of the most studied $N$-body integrable models are the Calogero and Calogero-Sutherland models. In this section, we demonstrate that these models are shape invariant, or, to be more precise, satisfy an algebraic identity that is the natural $N$-body generalization of shape invariance. We expect that the algebraic results we have found will lead, ultimately, to a ready derivation of the integrability of these models, and to the efficient algebraic calculation of the actual wavefunctions and energy levels of these systems. What we will do in this paper is to obtain these identities, and then discuss the analysis of these models that our representation permits.

The generalization of shape invariance we obtain is as follows. In the models in question, we find we can define a set of operators $A_i$ and $A_i^\dagger$ such that the Hamiltonian can be written as

$$ H(\alpha) = A_1^\dagger A_1 + \cdots + A_N^\dagger A_N . $$

However, the operators $A_i$ and $A_j^\dagger$ do not commute. Thus, this identity provides an elegant representation for the Hamiltonian, but nonetheless one that generalizes, but falls short of, separability.

The compelling result, however, is the following algebraic structure which turns out to be present in these models. If we interchange the $A_i$’s and $A_i^\dagger$’s, we obtain the associated Hamiltonian

$$ \tilde{H}(\alpha) = A_1 A_1^\dagger + \cdots + A_N A_N^\dagger . $$

We are able to identify these models as shape invariant because, it turns out that

$$ \tilde{H}(\alpha) = H(\alpha_1) + R(a_1) . $$

This identity generalizes the notion of shape invariance familiar from models with one degree of freedom. We conjecture that the ability to write the Hamiltonian in these two ways underlies the exact solubility of these models.

In the rest of this section, we will present the specific forms of these representations. Then, in this and later sections, we will discuss the applications of this representation of the Hamiltonians, providing support for our conjecture, keeping an eye toward how this $N$-body shape invariance should be related to the exact solubility of the models in...
3.1 The Calogero Model

The Calogero model is an \(N\)-body non-relativistic quantum mechanical model given by the Hamiltonian

\[
H_{C} = -\sum_{i} \partial_{i}^{2} + \sum_{i} \sum_{j}^{'} \frac{g/2}{(x_{i} - x_{j})^2},
\]

where the prime in the summation symbol means ‘restricted sum’, i.e. summation over the dummy variable \(j\) excluding the value \(i\) for which the denominator vanishes. We can obtain a useful representation of this Hamiltonian as follows.

Let us define the prepotential\(^2\)

\[
W_{i} = \sum_{j}^{'} \frac{-\alpha}{x_{i} - x_{j}},
\]

in terms of which we define the operators

\[
A_{i} = \partial_{i} + W_{i}, \quad A_{i}^{\dagger} = -\partial_{i} + W_{i}.
\]

These operators obey the following algebra:

\[
[A_{i}, A_{j}] = 0, \quad [A_{i}^{\dagger}, A_{j}^{\dagger}] = 0,
\]

\[
[A_{i}^{\dagger}, A_{j}] = -2 \partial_{i}W_{j} = \begin{cases} \sum_{k} \frac{-2\alpha}{(x_{i} - x_{j})^2}, & \text{if } i = j, \\ \frac{2\alpha}{(x_{i} - x_{j})^2}, & \text{if } i \neq j. \end{cases}
\]

Two further relevant identities are that

\[
\sum_{i} W_{i} = 0
\]

and that the curvature associated with the prepotential vanishes, i.e.,

\[
\partial_{i}W_{j} = \partial_{j}W_{i}.
\]

The first important observation is that the Hamiltonian can be written in a simple way using the \(A_{i}\)’s and \(A_{i}^{\dagger}\)’s. In particular, one finds that

\[
H_{C}(\alpha) = \sum_{i} A_{i}^{\dagger}(\alpha)A_{i}(\alpha)
\]

\(^2\)The quantity we term the prepotential is sometimes referred to as the “superpotential” in the literature. We prefer to use the term “prepotential,” reserving “superpotential” for the quantity whose gradient gives \(W_{i}\), in order to be consistent with the terminology of supersymmetric field theory.
at coupling constant 

\[ g = 2\alpha(\alpha - 1) \, . \]

In order to verify this result, note that

\[ A_i^\dagger A_i = -\partial_i^2 + W_i^2 - \partial_i W_i \, . \]  

(19)

To see that this expression yields the Calogero Hamiltonian, it is necessary to verify that the three-body interactions contained in the \( W_i^2 \) term cancel. (No other term contributes to a direct three-body potential.) To see this, we consider all terms in \( A_i^\dagger A_i \) that involve \( x_i, x_j, \) and \( x_k \), where these three coordinates are distinct. The terms are

\[
\frac{1}{(x_i - x_j)(x_i - x_k)} + \frac{1}{(x_j - x_i)(x_j - x_k)} + \frac{1}{(x_k - x_i)(x_k - x_j)},
\]

which it is easy to verify is zero by putting everything over a common denominator.

This representation for the Hamiltonian in terms of \( A_i \)'s and \( A_i^\dagger \)'s is potentially rather powerful. Unfortunately, these operators do not diagonalize the Hamiltonian, due to the non-commutation contained in (16). Nonetheless, the vanishing of the three-body interactions, the representation of the Hamiltonian as a sum of quadratic terms, and the pairwise commutation of the \( A_i \)'s (respectively, the \( A_i^\dagger \)'s) does at least give us a structure akin to separation of variables. It is the rare Hamiltonian that takes this form.

There is however a more significant algebraic identity at work here. If we switch \( A_i \) and \( A_i^\dagger \), we obtain

\[ \tilde{H}_C(\alpha) = \sum_i A_i(\alpha)A_i^\dagger(\alpha) = H_C(\alpha - 1) \, . \]  

(20)

In other words,

\[ \sum_i A_i(\alpha)A_i^\dagger(\alpha) = \sum_i A_i^\dagger(\alpha + 1)A_i(\alpha + 1) \, . \]  

(21)

Interchange of the \( A_i \)'s and \( A_i^\dagger \)'s has the same effect as shifting the coupling constant! This is the hallmark of shape invariance, and in the later sections of this paper, we consider the possible connection of this identity to the exact solubility of the model. We have thus have demonstrated that the Calogero model exhibits an \( N \)-body generalization of shape invariance, the first time it has been found in such a model.

These operators can be used to represent physically relevant quantities. An elementary but useful result is that the total momentum can be written as

\[ P_{\text{TOT}} = -i\sum_i A_i = i\sum_i A_i^\dagger, \]

since \( \sum_i W_i = 0 \). The quantity \( P_{\text{TOT}} \) commutes with each of the \( A_i \) and \( A_i^\dagger \) individually, and hence we can classify solutions in terms of their total momentum (not surprisingly) in a way that does not interfere with our operator representation of the Hamiltonian.

The Calogero model has an important variation which we now describe. One can introduce additional pairwise interactions among the particles quadratic in the coordinates:

\[ H_{hC} = -\sum_i \partial_i^2 + \sum_i \sum_j' \frac{g/2}{(x_i - x_j)^2} + \sum_i \sum_j' \frac{1}{4} \omega^2 (x_i - x_j)^2 + c \, . \]  

(23)
In order to distinguish this from the model we discussed initially, we shall coin the term “harmonic-Calogero model” to describe the theory with this quadratic interaction. The discussion presented above in the case of the ordinary Calogero model can be repeated nearly verbatim, with only minor changes. For simplicity, we here highlight only the important details.

The operators we must define are

\[
A_i = \partial_i - \sum_{j}^{'} \frac{\alpha}{x_i - x_j} + \beta \sum_{j}^{'} (x_i - x_j),
\]

\[
A_i^{\dagger} = -\partial_i - \sum_{j}^{'} \frac{\alpha}{x_i - x_j} + \beta \sum_{j}^{'} (x_i - x_j),
\]

where

\[
2 \alpha (\alpha - 1) = g, \quad \beta = \frac{\omega}{2\sqrt{N}}.
\]

With these operators, we have both the generalizations of separation of variables and of shape invariance that we found in the ordinary Calogero model. To be precise,

\[
H_{hC}(\alpha) = \sum_i A_i^{\dagger}(\alpha) A_i(\alpha),
\]

and the constant \(c\) in equation (23) is

\[
c = -\frac{\omega}{\sqrt{2}} \sqrt{N} (N - 1) (\alpha N + 1).
\]

Then

\[
\sum_i A_i(\alpha) A_i^{\dagger}(\alpha) = \sum_i A_i^{\dagger}(\alpha_1) A_i(\alpha_1) + R(\alpha_1),
\]

where

\[
\alpha_1 = \alpha + 1, \quad R(\alpha_1) = \frac{\omega}{\sqrt{2}} \sqrt{N} (N - 1) (\alpha_1 - \alpha) N.
\]

Further analysis of this model can be performed exactly as for the ordinary Calogero model; we leave the particular calculations to the interested reader.

### 3.2 The Calogero-Sutherland Model

Our formulation of the Calogero-Sutherland model, and the operators we use to represent the Hamiltonian, can be developed very similarly to the Calogero model. The Calogero-Sutherland model is given by an \(N\)-body, non-relativistic quantum mechanical Hamiltonian,

\[
H_{CS}(g) = -\sum_i \partial_i^2 + \sum_i \sum_{j}^{'} \frac{g/2}{\sin^2[a(x_i - x_j)]} + c.
\]

In the following, we shall set \(a = 1\). We now define a prepotential

\[
W_i = -\alpha \sum_{j}^{'} \cot(x_i - x_j).
\]
This leads us to introduce the operators

\[ A_i = \partial_i + W_i , \quad A_i^\dagger = -\partial_i + W_i . \]  

(33)

These operators satisfy a variety of algebraic relations, and are useful in analyzing the Calogero-Sutherland model. For these new \( A_i \) and \( A_i^\dagger \) operators, we have

\[
\begin{align*}
[A_i, A_j] &= 0 , \quad [A_i^\dagger, A_j^\dagger] = 0 , \\
[A_i^\dagger, A_j] &= -2 \partial_i W_j = \begin{cases} \\
\sum_k \frac{2\alpha}{\sin^2(x_i-x_k)} , & \text{if } i = j , \\
\frac{2\alpha}{\sin^2(x_i-x_j)} , & \text{if } i \neq j .
\end{cases}
\end{align*}
\]  

(34)

(35)

It is worth noting again here, as in the previous model, that

\[ \sum_i W_i = 0 \]

and that

\[ \partial_i W_j = \partial_j W_i . \]

Setting

\[ g = 2\alpha (\alpha - 1) \]

and

\[ c = -\alpha^2 \frac{N(N^2 - 1)}{3} , \]

(36)

the Hamiltonian can be written as

\[ H_{CS}(\alpha) = \sum_i A_i^\dagger(\alpha)A_i(\alpha) , \]

(37)

Just as with the Calogero model, the three-body interactions automatically cancel. As before, the only source of three-body interactions is the \( W^2 \) contribution to \( A_i^\dagger A_i \). Let us consider all the terms that include \( x_i, x_j, \) and \( x_k \). For simplicity, let \( a, b, \) and \( c \) represent the differences \( x_i - x_j, x_j - x_k, \) and \( x_k - x_i, \) respectively. Then we have terms

\[ -\cot a \cot b - \cot b \cot c - \cot c \cot a . \]

Now, since \( a + b + c = 0 \), we see

\[ \cot a \cot b + \cot a \cot c + \cot b \cot c = 1 . \]

(38)

Thus we can write the Hamiltonian in terms of the \( A_i \)'s and \( A_i^\dagger \)'s in a form that generalizes separation of variables.

And, again as before, there is more. If we switch the hermitian conjugates, we get the partner Hamiltonian

\[ \tilde{H}_{CS}(\alpha) = \sum_i A_i(\alpha)A_i^\dagger(\alpha) . \]

(39)
Remarkably, as with the previous model, we see that interchanging the hermitian conjugates is equivalent to shifting the coupling constant! To be precise, we find that

\[ \tilde{H}_{CS}(\alpha) = H_{CS}(\alpha_1) + R(\alpha_1) , \]  

(40)

with

\[ \alpha_1 = \alpha + 1 , \quad R(\alpha + 1) = (\alpha_1^2 - \alpha^2) \frac{N(N^2 - 1)}{3} . \]  

(41)

Put it another way,

\[ \sum_i A_i(\alpha)A_i^\dagger(\alpha) = \sum_i A_i^\dagger(\alpha_1)A_i(\alpha_1) + R(\alpha_1) . \]  

(42)

We have therefore obtained a generalized shape invariance in another \( N \)-body model.

Note, too, that we again find

\[ P_{TOT} = -i \sum_i A_i = +i \sum_i A_i^\dagger , \]

where of course \( P_{TOT} \) commutes with all the individual \( A_i \)'s and \( A_i^\dagger \)'s.

Notice that in both these models, the key to shape invariance is that, up to an additive constant, \( \partial_i W_i \) and \( W_i W_i \) have the same functional dependence, which is the same as that of the potential. This relationship, we expect, and the generalized shape invariance it implies, truly underlie the solubility of the system.

### 4 Generalizations of 1-body Shape Invariance

The Calogero and Calogero-Sutherland models share a number of features in common with familiar shape invariant models of one degree of freedom. These similarities go beyond the algebraic identities above, and relate to the features of the model. We review here some of these similarities, laying the groundwork for a future solution of these models by means of shape invariance.

1. Interchanging \( A_i \) and \( A_i^\dagger \) has the same effect on the Hamiltonian as shifting the coupling constant. This is the fundamental hallmark of shape invariance in the one-body problem, and arises as well in the \( N \)-body Calogero and Calogero-Sutherland models.

2. These \( N \)-body models are known to be exactly soluble. Likewise, the shape invariant models with one degree of freedom are known to be exactly soluble (indeed, were generally known to be so before shape invariance was discovered).

3. The \( N \)-body models in question are the obvious multiparticle generalizations of known shape invariant models with one degree of freedom. More precisely, the \( N \)-body models have potentials which involve only pairwise interactions; the pairwise potential in each case is given in terms of a single relative coordinate; and this two-body potential is just one of the shape invariant, exactly soluble potentials from ordinary quantum mechanics with one degree of freedom.
4. The partition function computed semiclassically is exact in these theories. The evidence for that has been presented in \[9\]. This is also a feature exhibited by shape invariant models with one degree of freedom \[1\].

5. For the case \(N=2\), i.e., the two-body problem, we actually directly recover the conventional case of shape invariance with one degree of freedom, once we separate out the total momentum \(P_{TOT}\).

6. The ground state is easy to construct, by requiring it to be a state annihilated by all the \(A_i\)'s. This gives us a zero energy state, but not for the partner Hamiltonian. Shape invariance, however lets us get the ground state for the partner.

It is worth turning our attention to the last two of these items in particular. We turn first to construction of the ground state.

To find the Calogero or Calogero-Sutherland ground state, we use the representation of the Hamiltonian as \(H = \sum_i A_i^\dagger A_i\). Since this expression is positive semi-definite, the states of the theory must have non-negative energy. Therefore, if we can find a zero-energy state, this will of necessity be the ground state.

In fact, since the Hamiltonian is the sum of \(N\) positive semi-definite terms, in order for a state \(\Phi_0\) to have zero energy, it must satisfy

\[
A_i \Phi_0 = 0 , \quad \forall \ i .
\]

It is easy enough to solve this, obtaining

\[
\Phi_0(x_1, \ldots, x_n) = \phi_0 e^{-\sum_i \int x_i W_i dx_i} .
\]

As long as this function is normalizable, it yields a sensible state, and then this is the ground state wavefunction of this system.

Notice that normalizability forces us to recognize that if there are solutions to the equation \(A_i(\alpha) \Phi_0(\alpha) = 0\), there are no corresponding solutions \(\Phi_0(\alpha)\) to \(A_i^\dagger(\alpha) \Phi_0(\alpha) = 0\). Nonetheless, we can find the ground state \(\Phi'(\alpha)\) of \(\sum_i A_i(\alpha)A_i^\dagger(\alpha)\) by exploiting the shape invariance of the model. In particular, since \(\sum_i A_i(\alpha)A_i^\dagger(\alpha) = \sum_i A_i^\dagger(\alpha + 1)A_i(\alpha + 1) + R(\alpha + 1)\), the ground state \(\Phi'(\alpha) \propto \Phi(\alpha + 1)\). The energy is given by the constant \(R(\alpha + 1)\). Thus shape invariance allows us immediately to obtain the ground state energy of \(\sum_i A_iA_i^\dagger\), despite the absence of a physical solution to \(A_i^\dagger \Phi_0 = 0\).

The second point we wish to turn to is the special situation when \(N=2\), i.e., when we have a 2-body Hamiltonian. In our two models in this case, then, the Hamiltonian is

\[
H^{(2)} = A_1^\dagger A_1 + A_2^\dagger A_2 ,
\]  \hfill (43)

where the superscript indicates the number of particles. We can write the Hamiltonian in this case as

\[
H^{(2)} = \frac{1}{2} (A_1^\dagger + A_2^\dagger)(A_1 + A_2) + \frac{1}{2} (A_1^\dagger - A_2^\dagger)(A_1 - A_2) .
\]  \hfill (44)
Recall that the total momentum
\[ P_{TOT} = -i(A_1 + A_2) = i(A_1^\dagger + A_2^\dagger) \]  
(45)
and that both \( A_1 - A_2 \) and \( A_1^\dagger - A_2^\dagger \) commute with \( P_{TOT} \). Consequently, if we define (for simplicity of notation) \( B = A_1 - A_2 \), we can write the Hamiltonian as
\[ H^{(2)} = \frac{1}{2} P_{TOT}^2 + \frac{1}{2} B B^\dagger . \]  
(46)
Since \([P_{TOT}, B] = [P_{TOT}, B^\dagger] = 0\), the solutions of this model are
\[ \psi(x_1, x_2) = e^{ik(x_1 + x_2)} \psi_B(x_1 - x_2) , \]
where \( \psi_B(x_1 - x_2) \) is an eigenstate of \( B B^\dagger \).

What of the partner Hamiltonian, \( \tilde{H}_2 = A_1 A_1^\dagger + A_2 A_2^\dagger \)?

Using the same definitions as above, we see that
\[ \tilde{H}_2 = \frac{1}{2} P_{TOT}^2 + B B^\dagger \]  
(48)
Thus, other than the overall center-of-momentum term \( \exp[ik(x_1 + x_2)] \) in the wavefunction, solving this problem simply involves finding the wavefunctions of \( B B^\dagger \) and \( B^\dagger B \).

To close this section, we note that there is a generalization of our separation in the two-body case into \( P_{TOT} \) and \( B \). While unfortunately this generalization does not lead directly to the exact solution of the problem, it does give a representation of the Hamiltonian in an alternative basis that is superior in certain respects to the representation in terms of the \( A_i \)'s and \( A_i^\dagger \)'s above.

Let us go to a Jacobi\(^3\) basis. We define
\[ B_1 = \frac{1}{\sqrt{2}} (A_1 - A_2) , \]  
(54)
\(^3\)Our terminology comes from the similarity of the above transformation to the choice of Jacobi coordinates. In classical mechanics, the Jacobi coordinates
\[
\begin{align*}
y_1 &= x_1 - x_2 , \\
y_2 &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} - x_3 , \\
\cdots & \cdots \cdots \\
y_{N-1} &= \frac{m_1 x_1 + m_2 x_2 + \cdots + m_{N-1} x_{N-1}}{m_1 + m_2 + \cdots + m_{N-1}} - x_N , \\
Y &= \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} ,
\end{align*}
\]
are used to separate out the center-of-mass motion.
\[ B_2 = \frac{1}{\sqrt{6}}(A_1 + A_2 - 2A_3) , \]  
\[ \ldots \ldots \]  
\[ B_{N-1} = \frac{1}{\sqrt{N(N-1)}} [A_1 + A_2 + \cdots + A_{N-1} - (N-1)A_N] , \]  
\[ B_N = \frac{1}{\sqrt{N}}(A_1 + A_2 + \cdots + A_N) . \]  
\[ \text{(55)} \]
\[ \text{(56)} \]
\[ \text{(57)} \]
\[ \text{(58)} \]

In this operator basis, we still have the generalized separability result,

\[ H = \sum_i B_i^\dagger B_i \]

and we can still see directly the \( N \)-body shape invariance result

\[ \sum_i B_i B_i^\dagger(\alpha) = \sum_i B_i^\dagger B_i(\alpha_1) + R(\alpha_1) . \]

The advantage of this representation is that the commutation relations are, though less symmetric, in one respect, a little simpler. We again have

\[ [B_i, B_j] = [B_i^\dagger, B_j^\dagger] = 0 , \]

for all \( i \) and \( j \). But now the operators \( B_N \) and \( B_N^\dagger \) are proportional to the total momentum, and so we have

\[ [B_N, B_j^\dagger] = [B_N^\dagger, B_j] = 0 , \]

for all \( j \). In addition, this choice automatically pulls out the total momentum, while preserving a raising/lowering operator formalism. In other words, we can write

\[ H = \frac{1}{2} P_{TOT}^2 + \sum_{i=1}^{N-1} B_i^\dagger B_i \]

and

\[ \bar{H} = \frac{1}{2} P_{TOT}^2 + \sum_{i=1}^{N-1} B_i B_i^\dagger . \]

\[ \text{(55)} \]

\[ \text{(56)} \]

\[ \text{(57)} \]

\[ \text{(58)} \]

5 Supersymmetric Construction and Generalizations

For the case of shape invariance with one degree of freedom, the algebraic results are best understood in the context of supersymmetry. Thus we anticipate that the supersymmetric generalizations of the Calogero and Calogero-Sutherland models offer the best possibility for understanding the exact solubility of these models. To this end, we here review the relevance of supersymmetry to shape invariance with one degree of freedom, and then proceed to construct the supersymmetric Calogero and Calogero-Sutherland models. We will see that our representation of the Hamiltonians in these cases leads to an easy and
natural construction of supersymmetric generalizations for these models. In fact, we believe that this is the first appearance of the supersymmetric Calogero-Sutherland model in the literature; the supersymmetric Calogero model has been discussed in [9, 10]. Incidentally, we mention that recently higher dimensional supersymmetric quantum mechanics has been studied by other authors too [11], although with a different perspective.

Let us first therefore consider quantum mechanics with one degree of freedom. The degeneracy of the spectra of the two Hamiltonians $H$ and $\tilde{H}$ can be understood as a consequence of supersymmetry. One constructs a single supersymmetric system, with Hamiltonian

$$\mathcal{H} = \begin{bmatrix} H & 0 \\ 0 & \frac{1}{H} \end{bmatrix}. \quad (59)$$

Defining the operators

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad (60)$$

$$Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}, \quad (61)$$

we can easily verify the algebra

$$[\mathcal{H}, Q] = [\mathcal{H}, Q^\dagger] = 0, \quad (62)$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad (63)$$

$$\{Q, Q^\dagger\} = \mathcal{H}. \quad (64)$$

The ground state, if it has zero energy, is a supersymmetry singlet; other than that all states must be paired.

Recognizing that $\sigma_3$ measures fermion number in this example, one sees that using shape invariance and supersymmetry to link together wavefunctions at different fermion number is what enables one to solve the theory. Motivated by this one-body result, we now construct the supersymmetric generalization of the $N$-body Calogero and Calogero-Sutherland models, and then study the ways in which the wavefunctions and energy levels at different fermion number are related to each other. We will find that going from fermion number 0 to the maximal fermion number $N$, one travels from the original Calogero or Calogero-Sutherland Hamiltonian to its shape invariance partner. While we will not be able to relate these two fermion number levels directly, we will identify an approach for obtaining such a relationship, and hence solving the original models via shape invariance.

5.1 Supersymmetric Generalizations

One advantageous feature of our representation of the Hamiltonian and related physical quantities in terms of $A_i$’s and $A_i^\dagger$’s is that it makes construction of the supersymmetric generalizations of the Calogero and Calogero-Sutherland models completely straightforward. We thus can and do use the same formalism for both models.
Let us define Grassmann variables $\psi_i$ and $\psi_i^\dagger$ with the anticommutation relations

$$\{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0, \quad \{\psi_i, \psi_j^\dagger\} = \delta_{ij}. \quad (65)$$

We define the supercharges

$$Q = \sum_i A_i \psi_i, \quad Q^\dagger = \sum_i A_i^\dagger \psi_i^\dagger,$$

and the Hamiltonian

$$H_{susy} = \{Q^\dagger, Q\}.$$

It is easy to verify that $Q^2 = (Q^\dagger)^2 = 0$. Thus this theory has an $N = 2$ supersymmetry; both the real and imaginary parts of $Q$ square to the Hamiltonian.

The structure of these supersymmetric theories is related to the generalized shape invariance. We note that the superalgebra determines that all states are annihilated by $Q$ or $Q^\dagger$. The zero energy states are annihilated by both. Of the remaining states, those eigenstates of the Hamiltonian annihilated by $Q$ are also in the range of $Q$; and those annihilated by $Q^\dagger$ are in the range of $Q^\dagger$.

### 5.2 Relating States at Different Fermion Number

Ignoring the zero energy states for simplicity, we consider dividing the space of eigenstates into states of definite fermion number (which can be anything from 0 to $N$) and also according to whether they reside in $\ker Q$ or $\ker Q^\dagger$.

1. The 0-fermion states: These states are automatically in $\ker Q$. Furthermore, they satisfy the equation

$$\sum_i A_i(\alpha)A_i^\dagger(\alpha)\phi = E\phi. \quad (67)$$

In other words, the states of the ordinary bosonic Calogero or Calogero-Sutherland model are the 0-fermion states of the supersymmetric version.

2. The $N$-fermion states: These states are automatically in $\ker Q^\dagger$. Furthermore, they satisfy the equation

$$A_i^\dagger(\alpha)A_i(\alpha)\phi = E\phi. \quad (68)$$

In other words, the $N$-fermion states are also the states of the ordinary bosonic Calogero or Calogero-Sutherland model — except at a shifted value of the coupling constant, due to shape invariance.

3. The 1-fermion states in $\ker Q^\dagger$: These are obtained by applying $Q^\dagger$ to the zero fermion states. Thus, these states are degenerate with and derivable from the 0-fermion states, i.e., from the original Calogero or Calogero-Sutherland model.
4. The 1-fermion states in \( \ker Q \): The best way to understand these is as follows. Let us write them as

\[
\sum_i \phi_i(x_1, \ldots, x_N)\psi_i | \emptyset \rangle.
\]

Then it is straightforward to show that \( A_i \phi_i = 0 \), and that

\[
\sum_i (A_i A_i^\dagger \phi_j - A_i A_i^\dagger \phi_i) = \lambda \phi_j.
\]

Let us define \( \Phi = \sum_i \phi_i \). One immediately determines that since \( \sum_i A_i^\dagger \propto P_{TOT} \) commutes with the \( A_i \),

\[
A_i A_i^\dagger \Phi = \lambda \Phi.
\]

This means that \( \Phi = \sum_i \phi_i \) is either a 0-fermion wavefunction (i.e., a solution of the original Calogero or Calogero-Sutherland model) or is itself vanishing. Thus, as long as the sum does not vanish, the energy of such a wavefunction is degenerate with a state of the zero-fermion spectrum.

5. The 2-fermion states: Those in \( \ker Q^\dagger \) we obtain from the 1-fermion states in \( \ker Q \). The others we construct similarly to what we did above.

6. \((N - 1)\)-fermion states in \( \ker Q \): These are simply obtained by applying \( Q \) to the \( N \)-fermion states, and so we get a spectrum and set of eigenfunctions directly from the ordinary Calogero or Calogero-Sutherland model at shifted coupling constant.

7. \((N - 1)\)-fermion states in \( \ker Q^\dagger \): We can proceed much as we did for the 1-fermion states in \( \ker Q \). We find that the \((N - 1)\)-fermion states in \( \ker Q^\dagger \) can be written as

\[
\sum_i \chi_i \psi_i | N \rangle,
\]

where \( | N \rangle \) is the state with \( N \) fermions in the fermionic subspace (i.e., proportional to \( \psi_1 \cdots \psi_N | \emptyset \rangle \)), and \( \chi(x_1, \ldots, x_n) \) is the bosonic piece of the wavefunction. Then

\[
A_i^\dagger \chi_i = 0
\]

and one finds too that \( X = \sum_i \chi_i \) either vanishes or is itself an eigenstate of \( \sum_i A_i^\dagger A_i \), and so the energy spectrum includes the energy eigenvalues of the bosonic Calogero or Calogero-Sutherland model at shifted coupling constant.

5.3 Comments

We see from this method that we can related the spectra at different fermion number. Ultimately, we expect that refinements in this procedure will enable one to relate the spectra of the 0-fermion and \( N \)-fermion states, which means relating the Hamiltonians \( H \) and \( \hat{H} \), which in turn means relating the Hamiltonians \( H(\alpha) \) and \( \hat{H}(\alpha_1) \). Consequently,
we would expect to be able to solve the model algebraically, just as one does in shape
invariant systems with one degree of freedom.

A brief look at the case \( N = 2 \) is worthwhile. While we discussed the case \( N = 2 \)
without supersymmetry and saw how to use generalized shape invariance to solve such a
system, we see here how supersymmetry organizes the states usefully. In the case \( N = 2 \),
the 1-fermion states in ker \( Q \) for example are degenerate with and can be obtained by
acting on the 2-fermion states with \( Q \); but they may also be related to the 0-fermion
states as described above. The bosonic part of the state has components \( \phi_1 \) and \( \phi_2 \), and
as long as these do not cancel additively, the sum \( \phi_1 + \phi_2 \) is a 0-fermion state that is
degenerate with the 1-fermion state, and hence in turn with a 2-fermion state.

The case of \( N = 3 \) also deserves special study, as everything is still very tightly
constrained. Here, the spectra of the 1-fermion states in ker \( Q \) and the 2-fermion states
in ker \( Q^\dagger \) are degenerate; indeed, using the labels \( \phi_i \) and \( \chi_i \) as above, one finds

\[
\phi_i = \pm c \sum_{i,j} \epsilon_{ijk} A_j \chi_k
\]

and a conjugate expression for \( \chi_i \). Since these spectra are degenerate, and in turn are
related, respectively, to \( H(\alpha) \) and \( H(\alpha_1) \) as discussed in items 4 and 7 in the preceding
subsection, one has nearly all the ingredients in place to relate directly the spectra of
\( H(\alpha) \) and \( H(\alpha_1) \) in the \( N = 3 \) case.

5.4 Non-Uniqueness of Supersymmetric Extensions

Exactly because the Calogero and Calogero-Sutherland models are shape invariant, their
supersymmetric extensions are not unique. This is easy to see, and important to recognize.

Let us consider the pure Hamiltonian

\[
H = -\partial_i^2 + gV + c,
\]

where \( g = 2\alpha(\alpha - 1) \). We can write this Hamiltonian in two distinct ways,

\[
H = \sum_i A_i^\dagger(\alpha)A_i(\alpha)
\]

and

\[
H = \sum_i A_i(\alpha + 1)A_i^\dagger(\alpha + 1).
\]

Up to an additive constant, these Hamiltonians are the same, and thus describe exactly
the same physics. We can take either one of these representations as the starting point
for our supersymmetric construction. However, although both these models will have the
same 0-fermion sector, they will have different states at higher fermion number\(^5\).

\(^4\)In this subsection, we have separated the coupling constant dependence from the potential.

\(^5\)This is in addition to the more obvious observation that one can consider two generalizations of the
original Hamiltonian, one in which it appears in the zero-fermion sector and one in which it appear in
the \( N \)-fermion sector. This, however, is unrelated to shape invariance, and merely involves switching \( \psi_i \)
and \( \psi_i^\dagger \).
Thus, because of the $N$-body shape invariance, we have the two supersymmetric Hamiltonians, which have the same bosonic sector but are otherwise distinct,

$$H_{S1} = \{ A_i^{\dagger}(\alpha)\psi_i^\dagger, A_i(\alpha)\psi_i \} ,$$

and

$$H_{S2} = \{ A_i(\alpha + 1)\psi_i^\dagger, A_i^{\dagger}(\alpha + 1)\psi_i \} .$$

## 6 Ending Remarks

We have identified the algebraic identity that we expect underlies the exact solubility of the Calogero and Calogero-Sutherland models. This identity is a natural $N$-body generalization of shape invariance, and we have explored its implications for the analysis of these models, indicating in particular how it relates to integrability. We have also seen how this identity can be incorporated directly into our understanding of the supersymmetric Calogero and Calogero-Sutherland models; indeed, it appears that the supersymmetric context will be the natural arena for using the generalized shape invariance to solve the original non-supersymmetric models. Thus, the work in this paper should provide a foundation for obtaining a simpler understanding of the solutions of Calogero and Calogero-Sutherland models, and a deeper understanding more generally of exact solubility in the $N$-body case. One particular point to mention is that the eigenfunctions of the Calogero-Sutherland model are deeply related to the so-called Jack polynomials of the mathematics literature. These polynomials are defined in the space of symmetric polynomials obeying certain conditions [12]. Although, they are quite well studied, mathematicians are lacking an explicit representation. Recently, Lapointe and Vinet presented such a formula [13]. Our proposal for an extended shape invariance principle in higher than one dimensions might prove useful to either derive this formula in a compact and natural way or find an alternative simpler one.

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A Theory of Many-Body Shape Invariant Potentials

Here, we systematize the theory of $N$-body shape invariant potentials. We develop the theory in a completely analogous way to the 1-dimensional theory so that the similarities and differences are apparent.

A.1 Factorization of the Schrödinger Equation in the Many-Body Case

Let us consider a potential $V_0(x_1, x_2, \ldots, x_N)$ with ground state wavefunction $\Psi^{(0)}_0(x_1, \ldots, x_N)$ and ground state energy $E^{(0)}_0$. As in the 2-particle case, without loss of generality, we can assume that $E^{(0)}_0 = 0$. Now from the time-independent Schrödinger equation

$$H^{(0)} \Psi^{(0)}_0 = E^{(0)}_0 \Psi^{(0)}_0 = 0,$$

where

$$H^{(0)} = -\sum_{i=1}^{N} \partial_i^2 V_0(x_1, x_2, \ldots, x_N),$$

where $\partial_i \equiv \partial/\partial x_i$.

Obviously, the potential $V_0(x_1, x_2, \ldots, x_N)$ is related to the ground state wavefunction $\Psi^{(0)}_0$ by the equation:

$$V_0(x) = \sum_{i=1}^{N} \frac{\partial_i^2 \Psi^{(0)}_0}{\Psi^{(0)}_0}.$$

In this case we introduce, the following creation and annihilation operators:

$$A^\dagger_i \equiv -\partial_i - \frac{\partial_i \Psi^{(0)}_0}{\Psi^{(0)}_0},$$

$$A_i \equiv \partial_i - \frac{\partial_i \Psi^{(0)}_0}{\Psi^{(0)}_0}.$$

We clearly see

$$H^{(0)} = \sum_{i=1}^{N} A^\dagger_i A_i.$$

Finally, in the many-body case we define

$$W_i \equiv -\partial_i \frac{\Psi^{(0)}_0}{\Psi^{(0)}_0}.$$

In terms of the functions $W_i$, the creation and annihilation operators take the form

$$A^\dagger_i = -\partial_i + W_i,$$

$$A_i = \partial_i + W_i.$$
A.2  A Special Class of Potentials

A special class of potentials that has been discussed thoroughly in the literature is the class of potentials for which the ground state can be written in the form

$$\Psi_0^{(0)} = \prod_{i=1}^{N} \prod_{j=i+1}^{N} \psi_0^{(0)}(x_i - x_j).$$  \hspace{1cm} (81)

This demand leads to the condition

$$W_i = -\sum_{j=1}^{N} \frac{\partial_i \psi_0^{(0)}(x_i - x_j)}{\psi_0^{(0)}(x_i - x_j)}.$$  \hspace{1cm} (82)

If $W(x)$ is the corresponding prepotential for the 2-body problem, we see that

$$W_i = \sum_{j=1}^{N} W(x_i - x_j).$$  \hspace{1cm} (83)

Now, the previous choice of the wave function determines uniquely the potential $V_0(x_1, x_2, \ldots, x_N)$. Using relation (74), we find that

$$V_0(x_1, x_2, \ldots, x_N) = \sum_{j \neq i} \frac{\partial^2 \psi_0(x_i - x_j)}{\psi_0(x_i - x_j)} + \sum_{i \neq j \neq k \neq i} \frac{\partial_i \psi_0(x_i - x_j) \partial_j \psi_0(x_i - x_k)}{\psi_0(x_i - x_k)}$$

$$= \sum_{j \neq i} v_0(x_i - x_j) + \sum_{i \neq j \neq k \neq i} W(x_i - x_j) W(x_i - x_k).$$  \hspace{1cm} (84)

We notice that the last term consists of a sum of terms

$$W(a - b)W(a - c) + W(b - a)W(b - c) + W(c - a)W(c - b)$$

Let $A = a - b$, $B = b - c$, and $C = c - a$ for which $A + B + C = 0$. Although we are not restricting ourselves to shape invariant potentials at this point, we make the following observation. For the usual shape invariant potentials, except the Morse potential, $W(x)$ is an odd function; so it is quite useful to restrict ourselves to odd super-potentials. Therefore, the expression written above has the form

$$-W(A)W(C) - W(A)W(B) - W(C)W(B).$$

One now sees that direct 3-body interactions are avoided if there is a function $\tilde{v}_0(x)$ such that

$$-W(A)W(C) - W(A)W(B) - W(C)W(B) = \tilde{v}_0(A) + \tilde{v}_0(B) + \tilde{v}_0(C).$$  \hspace{1cm} (85)

In this case the potential $V_0$ is written

$$V_0(x_1, \ldots, x_N) = \sum_{j \neq i} v_0(x_i - x_j) + \sum_{j \neq i} \tilde{v}_0(x_i - x_j).$$  \hspace{1cm} (86)

There are a few known solutions \cite{14} to equation (85):
In the above table, \( \zeta(x) \) is the Weierstrass zeta function, \( \zeta'(x) = -\mathcal{P}(x) \), and \( \omega \) is the half-period [12].

### A.3 Shape Invariant Many-Body Potentials

We now introduce the dependence of the potential (86) on possible parameters \( A \):

\[
V_0(x_1, \ldots, x_N; \alpha_0) = \sum_{i,j=1}^{N} v_0(x_i - x_j; \alpha_0) + \sum_{i,j=1}^{N} \bar{v}_0(x_i - x_j; \alpha_0),
\]

(87)

As we have discussed, the \( N \)-body Hamiltonian is written as

\[
H^{(0)}(\alpha_0) = \sum_{i=1}^{N} A_i^\dagger(\alpha_0) A_i(\alpha_0).
\]

(88)

We notice now that the operators \( A(\alpha) \) and \( A^\dagger(\alpha) \) satisfy the commutation relations

\[
[A_i(\alpha), A_j(\alpha')] = + W'(x_i - x_j; \alpha) - W'(x_j - x_i; \alpha') ,
\]

(89)

\[
[A_i^\dagger(\alpha), A_j^\dagger(\alpha')] = - W'(x_i - x_j; \alpha) + W'(x_j - x_i; \alpha') ,
\]

(90)

\[
[A_i(\alpha), A_j^\dagger(\alpha')] = - W'(x_i - x_j; \alpha) - W'(x_j - x_i; \alpha') ,
\]

(91)

if \( i \neq j \) and

\[
[A_i(\alpha), A_i(\alpha')] = - \sum_k W'(x_i - x_k; \alpha) + \sum_k W'(x_i - x_k; \alpha') ,
\]

(92)

\[
[A_i^\dagger(\alpha), A_i^\dagger(\alpha')] = + \sum_k W'(x_i - x_k; \alpha) - \sum_k W'(x_i - x_k; \alpha') ,
\]

(93)

\[
[A_i(\alpha), A_i^\dagger(\alpha')] = + \sum_k W'(x_i - x_k; \alpha) + \sum_k W'(x_i - x_k; \alpha') .
\]

(94)

These commutation relations forbid us from repeating the standard discussion of 1-dimensional shape invariance (see section [8]), and thus there is no automatic equivalence of the spectra for the Hamiltonian (88) and

\[
H^{(1)}(\alpha_0) = \sum_{i=1}^{N} A_i(\alpha_0) A_i^\dagger(\alpha_0).
\]

(95)
However, if
\[ \sum_i A_i(\alpha_0)A_i^\dagger(\alpha_0) = \sum_i A_i^\dagger(\alpha_1)A_i(\alpha_1) + R(\alpha_1), \] (96)
the ground states of the two hamiltonians are related by
\[ \Psi_0^{(1)}(\alpha_0) \propto \Psi_0^{(0)}(\alpha_1). \] (97)

Furthermore, the ground state of \( H^{(1)} \) has energy \( R(\alpha_1) \).

Obviously, one can repeat the procedure, exactly as in the 1-dimensional case, and obtain the series of Hamiltonians
\[ H^{(n)}(\alpha_0) = H^{(0)}(\alpha_n) + \sum_{k=1}^n R(\alpha_k). \] (98)

The corresponding ground states are given by
\[ \Psi_0^{(n)}(\alpha_0) \propto \Psi_0^{(0)}(\alpha_n), \] (99)
and have energies
\[ E_0^{(n)} = \sum_{k=1}^n R(\alpha_k). \] (100)

Unfortunately, there is no immediate relationship between these states and the spectrum of the original model.

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