PROBABILISTIC MODEL ASSOCIATED WITH THE PRESSURELESS GAS DYNAMICS

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Abstract. Using a method of stochastic perturbation of a Langevin system associated with the non-viscous Burgers equation we construct a solution to the Riemann problem for the pressureless gas dynamics describing sticky particles. As a bridging step we consider a medium consisting of noninteracting particles. We analyze the difference in the behavior of discontinuous solutions for these two models and the relations between them. In our framework we obtain a unique entropy solution to the Riemann problem in 1D case. Moreover, we describe how starting from smooth data a $\delta$ - singularity arises in one component of the solution.

Introduction

We propose a method for solving the Riemann problem as well as for describing the formation of singularities for the pressureless gas dynamics system and a natural extension of it. The system of pressureless gas dynamics is very important since it is believed to be the simplest model describing the formation of structures in the universe (e.g. [27]) and plays a significant role in the theory of cooling gases and granular materials [7]. It is a system consisting of two equations for the components of the density $f$ and velocity $u$ expressing the conservation of mass and momentum

$$\partial_t f + \text{div}_x(fu) = 0,$$

$$\partial_t (fu) + \nabla_x(fu \otimes u) = 0.$$ 

It first appears to be very simple, however a closer analysis reveals that it has some peculiar features due to its non strict hyperbolicity. The system has attracted a significant interest in the last decades and has been investigated quite intensively. In particular, it is well known that the arising in the velocity component of unbounded space derivatives implies the generation of a $\delta$ - singularity in the component of the density. Therefore for this system one needs to define a generalized or measure-valued solution of a special kind. This was done in [20], [6], [16], [15], [29], [24], [8], where the authors used different techniques (vanishing viscosity, weak asymptotics, variational principle, duality) to define the solution and prove its existence. Further, the Riemann problem for the pressureless gas dynamics was studied (e.g. [29], [16], [28]), including a singular Riemann problem with a $\delta$ - singularity concentrated...
at the jump at the initial moment. Nevertheless, even in the 1D case there are certain open problems, not to mention those present in the higher dimensional situation. In particular, there is a problem concerning the uniqueness of solutions. Both in the case of rarefaction and contraction it is possible to construct a whole family of solutions to the Riemann problem satisfying the integral identities and entropy conditions that are used to single out the unique solution in the strictly hyperbolic case (see [12],[19] for details). Further, the process of singularity formation was described up to now only in a very special situation ([13],[14]).

Our method allows to find a unique solution to the Riemann problem with arbitrary smooth left and right states. The most clear and explicit results we get concern the case of constant left and right states. We restrict ourselves to the 1D case, however the formulas that we use are written in a similar way in any dimensions, and our technique can be straightforwardly extended to the case of higher dimensions. Of course the situation in higher dimension is much more complicated, nevertheless, there are no fundamental obstacles to be faced when applying our method.

Further, our method is constructive. It allows to describe the behavior of the system starting from any smooth initial data. In particular, it is possible to describe the singularity formation including the time, position and value of the amplitude of the $\delta$ - function in the component of the density basing on the initial data. Moreover, it is possible to describe the behavior of a solution after the critical time, in particular, the position and the amplitude of the $\delta$ - singularity.

Let us describe shortly the method in the 1D case. First of all it is evident that till the solution to the system of pressureless gas dynamics keeps its $C^1$ - smoothness, the velocity satisfies the non-viscous Burgers (or free transport) equation. We write it in the Langevin form and introduce a stochastic perturbation along the trajectories of the particles. Further, we consider the position $x$ and velocity $u$ of the particles as random variables and find the common probability density $P(t, x, u)$ in the space of positions and velocities as a solution to the corresponding Fokker-Planck equation. Then we introduce a pair of variables: the density of particles $\rho(t, x)$ and the conditional expectation of velocities $\hat{u}(t, x)$ at fixed coordinates (and time).

We define a generalized solution in the sense of free particles (FP), a pair $((f_{FP}(t, x), u_{FP}(t, x)))$, as a special double limit of $(\rho, \hat{u})$ as the parameter $\sigma$ of stochastic perturbation and the parameter $\varepsilon$ of the approximation of initial data go to zero.

We prove that the pair $(\rho, \hat{u})$ satisfies a gas dynamics system with viscous and integral terms. The viscous term is the usual one for the viscous approximation of the solution and it vanishes as the solution and it vanishes as the parameter of stochastic perturbation $\sigma$ goes to zero. The integral term vanishes if $u_{FP}$, a part of the FP - generalized solution, is continuous and persists otherwise. Thus, the FP - generalized solution to the pressureless gas dynamics system solves in fact the extended gas dynamics system with an integral term in the usual sense of integral identities. This integral term can be considered as a spurious pressure, it is equal to the dispersion of $u$ with respect to $P_x(t, x, u)$ ($P_x$ denoting the derivation with respect to $x$).

The FP - solution corresponds to the model describing the behavior of a medium consisting in a micro-level of non-interacting particles. In the component of density of the FP-solution the $\delta$ - singularity can arise only from the domain where of the
initial velocity has a special form (see Sec.6). The \( \delta \) - singularity does not arise in the FP- solution to the Riemann problem with constant left and right states provided the \( \delta \) - singularity was not concentrated at the jump initially: instead of the \( \delta \) - singularity we have an overlapping domain of a non-zero measure (a spurious pressure arises in this overlapping domain). The FP-solution is interesting in itself, moreover, it can be used to construct a solution to the sticky particles models, where the particles are assumed to move together when they meet (the name of sticky particles model is used as a synonym of the pressureless gas dynamics). We propose a method based on the conservation of the mass and momentum that allows to reduce the FP-solution to the solution of the sticky particles model and find the position and the amplitude of the \( \delta \) - singularity in the density component. The spurious pressure degenerates in the process of above reduction. It fact, the problem of collapsing the overlapping domain into a point only arises for the compression waves, since for the rarefaction waves the initial jump in the velocity decays into a smooth profile, such that the FP-solution and the solution in the sense of integral identities coincide.

The paper is organized as follows. In Sec.1 we consider the model of motion for free particles perturbed along their trajectories and introduce the integral characteristics of the medium consisting of these particles, in particular, the mean velocity at a fixed coordinate \( \hat{u} \). In Sec.2 we study the properties of \( \hat{u} \) and prove that the limit of this value as the parameter of the stochastic perturbation go to zero (provided it is smooth) takes part of solution to the pressureless gas dynamics system. In Sec.3 we give a notion of generalized solution in the sense of free particles (FP) to the Cauchy problem for the pressureless gas dynamics system. In Secs.4 and 5 in 1D case we find the FP-solution for the classical and singular Riemann problems, respectively. In Sec.6 we describe the arising of singularity from smooth initial data for the pressureless gas dynamics system. In Sec.7 we discus the difference between the FP-solution and the generalized solution in the sense of integral identities. In Sec.8 we propose a method of changing the FP-solution to the solution in the sense of integral identities, and discuss the solution to the Riemann problem with non-constant states and the evolution of the singularity arising from smooth data. In Sec.9 we extend the method to a class of systems that can be obtained from the scalar conservation law with a convex flux. Sec.10 is a conclusion where we discuss the related approaches and methods.

1. Stochastic perturbation of the Burgers equation

Let us consider the Cauchy problem for the non-viscous Burgers equation:

\[
\partial_t u + (u, \nabla) u = 0, \quad t > 0, \quad u(x, 0) = u_0(x) \in C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n). \tag{3}
\]

Here \( u(x, t) = (u_1, ..., u_n)(x, t) \) is a vector-function \( \mathbb{R}^{n+1} \to \mathbb{R}^n \).

It is well known that solving this equation in the smooth setting is equivalent to solving the system of ODEs

\[
\dot{x}(t) = u(t, x(t)), \quad t > 0, \quad \dot{u}(t, x(t)) = 0 \tag{4}
\]

for the characteristics \( x = x(t) \).

We associate with (4) the following system of stochastic differential equations:

\[
dX_k(t) = U_k(t)dt + \sigma d(W_k)_t, \\
dU_k(t) = 0, \quad k = 1, ..., n, \tag{5}
\]
$X(0) = x, \quad U(0) = u,$

where $X(t)$ and $U(t)$ are considered as random variables with given initial distributions, $(X(t), U(t))$ runs in the phase space $\mathbb{R}^n \times \mathbb{R}^n$, $\sigma$ is a real strictly positive constant and $(W_t) = (W_k)_t, k = 1, \ldots, n$ is an $n$-dimensional Brownian motion, $t > 0$.

System (3) describes the free motion of a particle that "does not feel" the other particles. We assume that initially at time $t = 0$ these non-interacting particles are distributed with a density $f(x)$ and denote by $P(t, x, u)$ the probability density in position and velocity space for the solutions of (5), $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ at time $t$.

The stochastic system described by (5) is associated with the deterministic non-viscous Burgers equation (3) in the sense that the deterministic characteristics (4) are replaced by a stochastic perturbation of the characteristics as described by (5). System (5) with this interpretation is what we understand as "stochastic perturbation of the Burgers equation."

Let us introduce the function

$$\hat{u}(t, x) = \frac{\int_{\mathbb{R}^n} u P(t, x, u) du}{\int_{\mathbb{R}^n} P(t, x, u) du}, \quad t \geq 0, \quad x \in \mathbb{R}^n. \quad (6)$$

This value (6) can be interpreted as the conditional expectation of $U$ for fixed position $X$ [10]. If we choose as initial distribution

$$P_0(x, u) = \delta(u - u_0(x))f_0(x) = \prod_{k=1}^n \delta(u_k - (u_0(x))_k)f_0(x), \quad (7)$$

where $f_0$ is an arbitrary sufficiently regular nonnegative function such that $\int_{\mathbb{R}^n} f_0(x)dx = 1$, then $\hat{u}(0, x) = u_0(x)$. Certain properties of $\hat{u}(t, x)$ have been established in [1] (see also [2] for another type of stochastic perturbation).

The density $P = P(t, x, u)$ obeys the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \left[ -\sum_{k=1}^n u_k \frac{\partial}{\partial x_k} + \sum_{k=1}^n \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x_k^2} \right] P, \quad (8)$$

subject to the initial data (7).

We apply the Fourier transform to $P(t, x, u)$ in (8), (7) with respect to the variables $x$ and $u$ and obtain the Cauchy problem for the Fourier transform $\tilde{P} = \tilde{P}(t, \lambda, \xi)$ of $P(t, x, u)$:

$$\frac{\partial \tilde{P}}{\partial t} = -\frac{1}{2} \sigma^2 |\lambda|^2 \tilde{P} + \lambda \frac{\partial \tilde{P}}{\partial \xi}, \quad (9)$$

$$\tilde{P}(0, \lambda, \xi) = \int_{\mathbb{R}^n} e^{-i(\lambda,s)} e^{-i(\xi,u_0(s))} f_0(s)ds, \quad \lambda, \xi \in \mathbb{R}^n. \quad (10)$$

Equation (9) can easily be integrated and we obtain the solution given by the following formula:

$$\tilde{P}(t, \lambda, \xi) = \tilde{P}(0, \lambda, \xi + \lambda t) e^{-\frac{1}{2} \sigma^2 |\lambda|^2 t}. \quad (11)$$
The inverse Fourier transform (in the distributional sense) allows to find the density $P(t, x, u)$, $t > 0$:

$$P(t, x, u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\lambda, x)} e^{i(\xi, u)} \tilde{P}(t, \lambda, \xi) d\lambda d\xi =$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\lambda, x)} e^{-i(\lambda, s)} f_0(s) ds e^{-\frac{1}{2} \sigma^2 |\lambda|^2 t} d\lambda d\xi =$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f_0(s) \int_{\mathbb{R}^n} e^{i(\xi, u - u_0(s))} d\xi \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sigma^2 t \left( \lambda^2 - \frac{2(\xi - u_0(s))}{\sigma^2} \right)} e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} d\lambda ds =$$

$$= \frac{1}{(\sqrt{2\pi} \sigma)^n} \int_{\mathbb{R}^n} \delta(u - u_0(s)) f_0(s) e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} ds, \quad t \geq 0, x \in \mathbb{R}^n. \quad (12)$$

Then we substitute $P(t, x, u)$ in (13) and get the following expression for $\dot{u}(t, x)$ (sometimes we insert a label $\sigma$ to stress the dependence on this parameter):

$$\dot{u}(t, x) = \dot{u}_\sigma(t, x) = \frac{\int_{\mathbb{R}^n} u_0(s) f_0(s) e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} ds}{\int_{\mathbb{R}^n} f_0(s) e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} ds}. \quad (13)$$

**Remark 1.** The integrals in (13) are defined also for a wider class of $f_0$ than the probability density of the particle positions in the space at the initial moment of time. If the integral $\int f_0(x) dx$ diverges (for example, for $f_0(x) = \text{const}$), we can consider the domain $[-L, L]^n$, where $L > 0$ and use another definition of $\dot{u}_\sigma(t, x)$:

$$\dot{u}_\sigma(t, x) = \lim_{L \to +\infty} \frac{\int_{[-L,L]^n} u_0(s) f_0(s) e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} ds}{\int_{[-L,L]^n} f_0(s) e^{-\frac{|u_0(s) + x - s|^2}{2\sigma^2}} ds}. \quad (14)$$

(provided the limit exists). Evidently, this definition coincides with (13) for $f_0 \in L^1(\mathbb{R}^n)$.

2. **Properties of velocity averaged at a fixed coordinate**

The following property of $\dot{u}(t, x)$ holds:

**Proposition 1.** Let $u_0$ and $f_0 > 0$ be functions of class $C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n)$. If $t_*(u_0) > 0$ is a moment of time such that the solution to the Cauchy problem (3) with the initial condition $u_0$ keeps this smoothness for $0 < t < t_*(u_0) \leq +\infty$, then $\dot{u}_\sigma(t, x)$ tends to a solution of problem (3) as $\sigma \to 0$ for any fixed $(t, x) \in \mathbb{R}^{n+1}$, $0 < t < t_*(u_0)$.

**Proof.** Let us denote by $J(u_0(x))$ the Jacobian matrix of the map $x \mapsto u_0(x)$. As it was shown in [25] (Theorem 1), if $J(u_0(x))$ has at least one eigenvalue which is negative for a certain point $x \in \mathbb{R}^n$, then the classical solution to (3) fails to exist beyond a positive time $t_*(u_0)$. Otherwise, $t_*(u_0) = \infty$. The matrix $C(t, x) = (I + tJ(u_0(x)))$, where $I$ is the identity matrix, fails to be invertible for $t = t_*(u_0)$.
The formula (13) (or (14)) implies, using the weak convergence of measures and the fact that \( f_0 \) and \( u_0 \) are continuous and bounded

\[
\lim_{\sigma \to 0} \hat{u}_\sigma (t, x) = \frac{1}{\sqrt{2\pi \sigma}} \int_{\mathbb{R}^n} u_0(s) f_0(s) e^{-\frac{|u_0(s)+t-x|^2}{2\sigma}} ds = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f_0(s) e^{-\frac{|s|^2}{2}} ds = \int_{\mathbb{R}^n} u_0(s) f_0(s) \delta_{p(t, x, s)} ds
\]

with \( p(t, x, s) = u_0(s)t + s - x \), where \( \delta_y \) is the Dirac measure as \( y \in \mathbb{R}^n \). We can then on the basis of the invertibility of \( C(t, x) \) use locally the implicit function theorem and find \( s = s_{t, x}(p) \). Therefore,

\[
\lim_{\sigma \to 0} \hat{u}_\sigma (t, x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} f_0(s) \det(C(t, s_{t, x}(p)))^{-1} \delta_{p(t, x, s)} ds = u_0(s_{t, x}(0)).
\]

Let us introduce the new notation \( s_0(t, x) \equiv s_{t, x}(0) \). Then the following vectorial equation holds:

\[
u_0(s_0(t, x)) t + s_0(t, x) - x = 0. \tag{15}
\]

Let us show that \( u(t, x) = u_0(s_0(t, x)) \) satisfies the Burgers equation, that is

\[
\sum_{j=1}^n \partial_j (u_{0,i})(s_{0,j})t + \sum_{j,k=1}^n u_{0,j} \partial_k (u_{0,i})(s_{0,k})x_j = 0, \quad i = 1, \ldots, n, \tag{16}
\]

and \( u_0(s_0(0, x)) = u_0(x) \). Here we denote by \( u_{0,i} \) and \( s_{0,i} \) the \( i \)-th components of vectors \( u_0 \) and \( s_0 \), respectively.

We differentiate (15) with respect to \( t \) and \( x_j \) to get the matrix equations:

\[
\sum_{j=1}^n C_{ij} (s_{0,j})t + u_{0,i} = 0, \quad i = 1, \ldots, n,
\]

and

\[
\sum_{k=1}^n C_{ik} (s_{0,k})x_j + \delta_{ij} = 0, \quad i, j = 1, \ldots, n,
\]

where \( \delta_{ij} \) is the Kronecker symbol. The equations imply

\[
(s_{0,j})t = -\sum_{i=1}^n (C^{-1})_{ij} u_{0,i}, \quad (s_{0,k})x_j = -(C^{-1})_{jk}.
\]

It remains now only to substitute (17) into (16).

Further, (15) implies \( u_0(s_0(0, x)) = u_0(x) \). \( \square \)

It is important to note that \( s_0(t, x) \) is unique for all \( t \) for which the solution to the Burgers equation \( u(t, x) \) is smooth.

**Remark 2.** Proposition 1 can naturally be extended to the class of functions \( f_0 \) such that there exists a sequence \( f_\varepsilon^0 \in C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n) \) converging to \( f_0 \) as \( \varepsilon \to 0 \) almost everywhere. In this case \( \hat{u}_\varepsilon(t, x) \) tends to a solution of problem (3) as \( \sigma \to 0 \) almost everywhere on \( (t, x) \in \mathbb{R}^{n+1}, 0 < t < t_\varepsilon(u_0) \).
Let us set \( \rho(t,x) = \int_{\mathbb{R}^n} P(t,x,u)du, \ t \geq 0, \ x \in \mathbb{R}^n \). From (12) we have
\[
\rho(t,x) = \rho_\sigma(t,x) = \frac{1}{(\sqrt{2\pi \sigma t})^n} \int_{\mathbb{R}^n} f_0(s) e^{-\frac{|u(s')|^2}{2\sigma t}} ds, \ t \geq 0, \ x \in \mathbb{R}^n. \tag{18}
\]

**Proposition 2.** The scalar function \( \rho(t,x) \) and the vector-function \( \hat{u}(t,x) = (\hat{u}_1, \ldots, \hat{u}_n) \) defined in (10) solve the following system:
\[
\frac{\partial \rho}{\partial t} + \text{div}_x(\rho \hat{u}) = \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial x_k^2}, \tag{19}
\]
\[
\frac{\partial (\rho \hat{u}_i)}{\partial t} + \nabla(\rho \hat{u}_i \hat{u}_i) = \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 (\rho \hat{u}_i)}{\partial x_k^2} - \int_{\mathbb{R}^n} (u_i - \hat{u}_i)(u - \hat{u}) \nabla_x P(t,x,u)) du, \tag{20}
\]
i = 1, \ldots, n, \ t \geq 0.

**Proof.** The equation (19) follows from the Fokker-Planck equation (8) directly. To prove (20) we note that the definitions of \( \hat{u}(t,x) \) and \( \rho(t,x) \) imply
\[
\frac{\partial (\rho \hat{u})}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^n} uP(t,x,u)du = \int_{\mathbb{R}^n} uP(t,x,u)du = \int_{\mathbb{R}^n} u(x, \nabla_x P(t,x,u))du + \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial x_k^2}, \tag{21}
\]
where \( P_t \equiv \frac{\partial}{\partial t} P. \)

Further, for \( i = 1, \ldots, n \) we have
\[
\frac{\partial (\rho \hat{u}_k \hat{u}_i)}{\partial x_k} = \hat{u}_i \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} u_k P du \right) + \int_{\mathbb{R}^n} u_k P du \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} u_k P du \right) = \int_{\mathbb{R}^n} \hat{u}_i u_k P_{x_k} du + \int_{\mathbb{R}^n} u_k P du \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} P du \right) = \int_{\mathbb{R}^n} u_k \hat{u}_i \hat{u}_k + u_i \hat{u}_k - \hat{u}_k \hat{u}_i) P_{x_k} du, \quad i, k = 1, \ldots, n, \tag{22}
\]
with \( P_{x_k} \equiv \frac{\partial}{\partial x_k} P. \)

Equation (20) follows immediately from (21) and (22). \( \square \)

In the one-dimensional case (19)-(20) has the form
\[
\partial_t \rho + \partial_x (\rho \hat{u}) = \frac{1}{2} \sigma^2 \partial_{xx}^2 \rho, \tag{23}
\]
\[
\partial_t (\rho \hat{u}) + \partial_x (\rho \hat{u}^2) = \frac{1}{2} \sigma^2 \partial_{xx}^2 (\rho \hat{u}) - \int_{\mathbb{R}} (u - \hat{u})^2 P_x(t,x,u)du. \tag{24}
\]

Let us set \( f(t,x) = \lim_{\sigma \to 0} \rho(t,x) \) and \( \hat{u}(t,x) = \lim_{\sigma \to 0} \hat{u}(t,x). \)
Proposition 3. Assume that \((f(t, x), \bar{u}(t, x))\), the limits of \((\rho, \bar{u})\) as \(\sigma \to 0\), are \(C^1\) smooth bounded functions for \((t, x) \in \Omega := [0, t_s(u_0)] \times \mathbb{R}^n, t_s(u_0) \leq \infty\). Then they solve in \(\Omega\) the pressureless gas dynamics system \((\ref{eq:1}), (\ref{eq:2})\).

Proof. As follows from Proposition \((\ref{prop:1})\) the function \(\bar{u}(t, x)\) is a \(C^1\) solution of the non-viscous Burgers equation. Further, \((\ref{eq:23})\) is a linear parabolic equation with respect to \(\rho\), hence the limit as \(\sigma \to 0\) reduces it to the continuity equation \((\ref{eq:1})\). Equation \((\ref{eq:2})\) is a corollary of the non-viscous Burgers equation and \((\ref{eq:1})\). \(\square\)

Remark 3. Proposition \((\ref{prop:3})\) implies that the integral term on the right-hand side of \((\ref{eq:20})\) vanishes as \(\sigma \to 0\) in the case of smooth limit functions \(f\) and \(\bar{u}\). Let us prove this fact alternatively. Indeed, as follows from \((\ref{eq:12})\), we have as \(\sigma \to 0\)

\[
\int_{\mathbb{R}^n} (u_i - \hat{u}_i) \left( (u - \bar{u}), \nabla_x P(t, x, u) \right) \, du =
\]

\[
\frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{\mathbb{R}^n} f_0(s) (u_0(s) - \hat{u}_i) \left( (u_0(s) - \bar{u}), \nabla_x e^{-\frac{|u_0(s) + x - \bar{u}|^2}{2\sigma^2}} \right) \, ds =
\]

\[
\frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{U_{s_0(t, x)}(\varepsilon)} e^{-\frac{|u_0(s) + x - \bar{u}|^2}{2\sigma^2}} f_0(s) (u_0(s) - \hat{u}_i) \left( (u_0(s) - \bar{u}), u_0(s)t + s - x \right) \, ds + o(\sigma),
\]

where \(U_{s_0(t, x)}(\varepsilon)\) is an \(\varepsilon\)-neighborhood of the point \(s_0(t, x)\) (see the proof of Proposition \((\ref{prop:7})\), \(i = 1, \ldots, n\). Further,

\[
|u_0(s) - \hat{u}_i(t, x)| = |(u_0(s) - \bar{u}_i(t, x)) + (\bar{u}_i(t, x) - \hat{u}_i(t, x))| =
\]

\[
|(u_0(s) - \bar{u}_0(s_0(t, x))) + (\bar{u}_0(s_0(t, x)) - \hat{u}_i(t, x))|.
\]

For every fixed \(x\) and \(t \in (0, t_s)\) and for every \(\sigma > 0\) there exists \(\varepsilon(\sigma) > 0\) such that if \(s \in U_{s_0(t, x)}(\varepsilon)\), then \(|(u_0(s) - \bar{u}_0(s_0(t, x)))| < \sigma\). Moreover, since \(\bar{u}_0(t, x) \to \bar{u}(t, x)\) as \(\sigma \to 0\) (we have renamed the parameter), then for every \(\sigma > 0\) there exists \(\sigma_1(\sigma) > 0\) such that for sufficiently small \(\sigma_1\) we have \(|\hat{u}_i(t, x) - \bar{u}_0(s_0(t, x))| < \sigma\).

Thus,

\[
\frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{U_{s_0(t, x)}(\varepsilon)} e^{-\frac{|u_0(s) + x - \bar{u}|^2}{2\sigma^2}} f_0(s) (u_0(s) - \hat{u}_i) \left( (u_0(s) - \bar{u}), u_0(s)t + s - x \right) \, ds \leq
\]

\[
\text{Const} \cdot \frac{1}{(\sqrt{2\pi}\sigma)^n} \int_{U_{s_0(t, x)}(\varepsilon)} e^{-\frac{|u_0(s) + x - \bar{u}|^2}{2\sigma^2}} f_0(s) |u_0(s) + s - x| \, ds,
\]

where the constant does not depend of \(\sigma\). The latter integral tends to zero as \(\sigma \to 0\).

In fact, to prove that the integral term vanishes as \(\sigma \to 0\), we have used only the continuity of \(\bar{u}\) and the boundedness of \(f_0\).

However, as we will show in Sec \((\ref{sec:7})\) if we put instead of \(\bar{u}\) a discontinuous function, this integral term does not vanish.
3. Generalized solution in the sense of free particles

Being inspired by the fact that the formula \[13\] makes sense also for discontinuous initial data \((f_0(x), u_0(x))\), we give the following definition for any dimension.

**Definition 1.** We call the couple of functions \((f_{FP}(t, x), u_{FP}(t, x))\) a generalized solution to the Cauchy problem \[1\], \[2\] in the sense of free particles (FP-generalized solution) subject to initial data \((f_0(x), u_0(x)) \in L^2_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), if for almost all \((t, x) \in \mathbb{R}^+_0 \times \mathbb{R}^n\)

\[
\begin{align*}
    f_{FP}(t, x) &= \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho_\varepsilon(t, x), \\
    u_{FP}(t, x) &= \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \tilde{u}_\varepsilon(t, x) \right),
\end{align*}
\]

where \((\rho_\varepsilon(t, x), \tilde{u}_\varepsilon(t, x))\) satisfy the system \[23\], \[24\] with initial data

\[
\begin{align*}
    f_0^\varepsilon &= \eta_\varepsilon \ast f_0, \\
    \bar{u}_0^\varepsilon &= \eta_\varepsilon \ast u_0
\end{align*}
\]

where \(\eta_\varepsilon(x)\) is the standard averaging kernel.

**Remark 4.** The properties of the standard averaging kernel \[21\] imply that \(f_0^\varepsilon\) and \(u_0^\varepsilon\) belong to the class \(C^\infty\) and

\[
\begin{align*}
    \lim_{\varepsilon \to 0} \tilde{f}_0^\varepsilon(x) &= f_0(x), \\
    \lim_{\varepsilon \to 0} \tilde{u}_0^\varepsilon(x) &= u_0(x),
\end{align*}
\]

for almost all fixed \(x \in \mathbb{R}^n\).

**Remark 5.** As we will see below, if \(u_{FP}\) is continuous, the FP-generalized solution is a solution to \[1\], \[2\], for example, in the sense of integral identities. However, if \(u_{FP}\) is discontinuous, the FP-solution solves a different system, namely one that differs from \[1\], \[2\] by an integral term corresponding to a spurious pressure. Nevertheless, using the FP-solution we can solve \[1\], \[2\] itself.

**Definition 2.** We call the pair \((f_0^\varepsilon, u_0^\varepsilon)\) a monotonic approximation of initial data \((f_0(x), u_0(x)) \in L^2_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), if

- \(f_0^\varepsilon\) and \(u_0^\varepsilon\) are from the class \(C_0(\mathbb{R}^n)\), moreover, they are from \(C^1(\mathbb{R}^n)\) almost everywhere;
- \(\lim_{\varepsilon \to 0} f_0^\varepsilon(x) = f_0(x), \quad \lim_{\varepsilon \to 0} u_0^\varepsilon(x) = u_0(x), \quad \text{for almost all fixed } x \in \mathbb{R}^n;\)
- for sufficiently small \(\varepsilon\) and almost all fixed \((x, t) \in \mathbb{R}^{n+1}\) every root \(s_k\) of the equation \(u_0^\varepsilon(s) t + s - x = 0\) belongs to the neighborhood \(U_{\bar{s}_k}(\varepsilon)\) of the root \(\bar{s}_k\) of the equation \(\bar{u}_0^\varepsilon(s) t + s - x = 0\).

**Proposition 4.** The FP-solution \((f_{FP}(t, x), u_{FP}(t, x))\) to the Cauchy problem \[1\], \[2\] with initial data \((f_0(x), u_0(x)) \in L^2_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) does not depend of the choice of the monotonic approximation \((f_0^\varepsilon, u_0^\varepsilon)\).

**Proof** Let us choose two the monotonic approximations \((f_0^{\varepsilon_1}, u_0^{\varepsilon_1})\) and \((f_0^{\varepsilon_2}, u_0^{\varepsilon_2})\) such that

\[
\begin{align*}
    \lim_{\varepsilon \to 0} f_0^{\varepsilon_1}(x) &= \lim_{\varepsilon \to 0} f_0^{\varepsilon_2}(x) = f_0(x), \\
    \lim_{\varepsilon \to 0} u_0^{\varepsilon_1}(x) &= \lim_{\varepsilon \to 0} u_0^{\varepsilon_2}(x) = u_0(x)
\end{align*}
\]

for almost all fixed \(x \in \mathbb{R}^n\). Then the couple

\[
(f_0^\varepsilon, u_0^\varepsilon) = (f_0^{\varepsilon_1} - f_0^{\varepsilon_2}, u_0^{\varepsilon_1} - u_0^{\varepsilon_2})
\]
can be considered as initial data for the problem (11)-(12). To prove the proposition we have to show that the respective solution is identically zero almost everywhere.

Indeed, from (15) we have for any \( t \geq 0 \), and almost all \( x \in \mathbb{R}^n \)

\[
f_{FP}(t, x) = \sum_k \left[ \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^n} f_0^\varepsilon(s) \delta(s - s_{1,k}^\varepsilon(t, x)) ds \right) + \right]
\]

\[
\lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^n} f_{02}^\varepsilon(s) \left( \delta(s - s_{1,k}^\varepsilon(t, x)) - \delta(s - s_{2,k}^\varepsilon(t, x)) \right) ds \right) \right] = \]

\[
\sum_k \left[ \lim_{\varepsilon \to 0} (f_{01}^\varepsilon(s_{1,k}(t, x)) - f_{02}^\varepsilon(s_{1,k}(t, x))) - \lim_{\varepsilon \to 0} (f_{01}^\varepsilon(s_{2,k}(t, x)) - f_{02}^\varepsilon(s_{2,k}(t, x))) \right] = 0.
\]

Here \( s_{k,k}(t, x) \) is the \( k \)-th solution (\( k = 1, 2, ..., K \), \( i = 1, 2 \)) of equation

\[
u_0^\varepsilon(s) t + s - x = 0, \quad i = 1, 2.
\]

We have used the fact that \( |s_{1,k}(t, x) - s_{2,k}(t, x)| \to 0 \), \( k = 1, 2, ..., K \) as \( \varepsilon \to 0 \).

Analogously proceeding from (19), we prove that \( u_{FP}(t, x) \equiv 0 \) for almost all \( t \geq 0, x \in \mathbb{R}^n \). \( \square \)

4. The classical Riemann problem in the FP sense for the 1D case

For the sake of simplicity we restrict ourselves to the one-dimensional case and consider the following initial data:

\[
f_0(x) = f_1(x) + \theta(x)f_2(x), \quad (25)
\]

\[
u_0(x) = u_1(x) + \theta(x)u_2(x), \quad (26)
\]

where \( \theta \) is the Heaviside function with jump at \( x_0 \) (without loss of generality we shall assume \( x_0 = 0 \)), \( u_1, u_2, f_1, f_2 \) are continuous differentiable functions. We shall dwell first on the case where \( f_1, f_2, u_1, u_2 \) are real constants.

According to Definition (11) we must consider the smoothed initial data instead of (25) and (26). It follows from Proposition (10) that we can choose any couple of smoothed initial data. It will be convenient to consider the piecewise linear monotonic approximation of initial data of the form

\[
f_0^\varepsilon(x) = \begin{cases} f_1, & x \leq -\varepsilon, \\ \frac{f_2}{2\varepsilon} x + f_1 + \frac{f_2}{2}, & -\varepsilon < x < \varepsilon, \\ f_1 + f_2, & x \geq \varepsilon, \end{cases} \quad (27)
\]

\[
u_0^\varepsilon(x) = \begin{cases} u_1, & x \leq -\varepsilon, \\ \frac{u_2}{2\varepsilon} x + u_1 + \frac{u_2}{2}, & -\varepsilon < x < \varepsilon, \\ u_1 + u_2, & x \geq \varepsilon, \end{cases} \quad (28)
\]

where \( f_1, f_2, u_1 \) and \( u_2 \) are real constants.

From (15) we can find the density \( \rho_0^\varepsilon(t, x) \) corresponding to the smoothed initial data \( (f_0^\varepsilon(x), u_0^\varepsilon(x)) \) (below we shall omit the index \( \sigma \) for short).

Let us set:

\[
\hat{E}_1(t, x, s) \equiv \exp \left[ -\frac{(s - x + ut)^2}{2\sigma^2t} \right], \quad (29)
\]
\[ \hat{E}_2(t, x, s) = \exp \left[ -\frac{(s - x + (u_1 + u_2)t)^2}{2\sigma^2 t} \right], \quad (30) \]

\[ \hat{E}_3(t, x, s) = \exp \left[ -\frac{(s - x + \frac{u_2}{2\varepsilon} s + u_1 + \frac{u_2}{2} t)^2}{2\sigma^2 t} \right], \quad (31) \]

Then

\[
\rho^\varepsilon(t, x) = \frac{1}{\sqrt{2\pi t} \sigma} \left( \int_{-\infty}^{-\varepsilon} f_1 \hat{E}_1(t, x, s) ds + \int_{\varepsilon}^{\infty} (f_1 + f_2) \hat{E}_2(t, x, s) ds + \int_{-\varepsilon}^{\varepsilon} \left( \frac{f_2}{2\varepsilon} s + f_1 + \frac{f_2}{2} \right) \hat{E}_3(t, x, s) ds \right). 
\]

We denote the third integral by \( I_1^\varepsilon \sigma \) and set

\[
p = \frac{\left( \frac{u_2}{2\varepsilon} t + 1 \right) s + \left( u_1 + \frac{u_2}{2} \right) t - x}{\sigma \sqrt{t}}.
\]

We then have:

\[
I_1^\varepsilon \sigma = L^\varepsilon(t, x) \sigma \int_{-\infty}^{\infty} pe^{-\frac{p^2}{2\sigma^2}} dp + F^\varepsilon(t, x) \sigma \int_{-\infty}^{\infty} e^{-\frac{p^2}{2\sigma^2}} dp =
\]

\[
= L^\varepsilon(t, x) \sigma \left( e^{-\frac{(C^\varepsilon)^2}{2\sigma^2}} - e^{-\frac{(C^\varepsilon)^2}{2\sigma^2}} \right) + F^\varepsilon(t, x) \left[ \Phi \left( \frac{C^\varepsilon}{\sigma} \right) - \Phi \left( \frac{C^\varepsilon}{\sigma} \right) \right], \quad (32)
\]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{p^2}{2}} dp \) is the Gauss function, \( C^\varepsilon_z = u_1 t - x - \varepsilon, C^\varepsilon_2 = (u_1 + u_2) t - x + \varepsilon \), and

\[
L^\varepsilon(t, x) = -\frac{\sqrt{2\pi} f_2 \varepsilon}{\sqrt{\pi} (u_2 t + 2\varepsilon)^2},
\]

\[
F^\varepsilon(t, x) = \frac{2\varepsilon}{u_2 t + 2\varepsilon} \left( f_1 + \frac{f_2}{2} + \frac{f_2}{u_2 t + 2\varepsilon} \right) (x - (u_1 + \frac{u_2}{2} t)). \quad (33)
\]

It can easily be seen that \( \lim_{\varepsilon \to 0} F^\varepsilon(t, x) = 0 \).

Finally, we get

\[
\rho^\varepsilon(t, x) = f_1 \Phi \left( \frac{C^\varepsilon}{\sigma \sqrt{t}} \right) + (f_1 + f_2) \Phi \left( \frac{C^\varepsilon}{\sigma \sqrt{t}} \right) + I_1^\varepsilon \sigma, \quad (34)
\]

To find \( \hat{u}(t, x) \) we compute the numerator in formula \([33]\):

\[
\frac{1}{\sqrt{2\pi t} \sigma} \int_{\mathbb{R}} u_0^\varepsilon(s) f_0^\varepsilon(s) e^{-\frac{(u_0^\varepsilon(s) u_0^\varepsilon(s) + \varepsilon)^2}{2\sigma^2 t}} ds = u_1 \rho^\varepsilon(t, x) + u_2 (f_1 + f_2) \Phi \left( \frac{C^\varepsilon}{\sigma \sqrt{t}} \right) + I_2^\varepsilon \sigma,
\]

where

\[
I_2^\varepsilon \sigma = G^\varepsilon \sigma(t, x) + K^\varepsilon(t, x) \sigma \left( e^{-\frac{(C^\varepsilon)^2}{2\sigma^2}} - e^{-\frac{(C^\varepsilon)^2}{2\sigma^2}} \right), \quad (35)
\]
\[ N^\varepsilon(t, x) \left[ \Phi \left( \frac{C^+}{\sigma \sqrt{t}} \right) - \Phi \left( \frac{C^-}{\sigma \sqrt{t}} \right) \right]. \]

Here we used the notation
\[
K^\varepsilon(t, x) = \frac{u_2 \sqrt{t}}{2 \pi (u_2 t + 2 \varepsilon)} F^\varepsilon(t, x) + \frac{f_2 \sqrt{t}}{2 \sqrt{2} \pi \varepsilon} U^\varepsilon(t, x),
\]
\[
N^\varepsilon(t, x) = \left( \frac{u_2}{u_2 t + 2 \varepsilon} \right)(x - (u_1 + u_2/2)t) + \frac{u_2}{2} F^\varepsilon(t, x).
\]

We recall that \(F^\varepsilon(t, x)\) was determined by (33).

It is easy to deduce that \(\lim_{\varepsilon \to 0} N^\varepsilon(t, x) = 0\).

Thus, we have the following result:
\[
\hat{u}^\varepsilon(t, x) = u_1 + \frac{u_2(f_1 + f_2) \Phi \left( \frac{C^+}{\sigma \sqrt{t}} \right) + I_2^\varepsilon}{f_1 \Phi \left( \frac{C^+}{\sigma \sqrt{t}} \right) + (f_1 + f_2) \Phi \left( \frac{C^-}{\sigma \sqrt{t}} \right) + I_1^\varepsilon}, \tag{36}
\]
where \(I_1^\varepsilon\) and \(I_2^\varepsilon\) are given by (32) and (35), respectively. Note that \(\frac{C^\pm}{\sigma} \to \pm \infty\) as \(\sigma \to 0\).

It can be checked that the initial conditions are satisfied, namely \(\rho(0, x) = f_0^\varepsilon(x)\) and \(\hat{u}^\varepsilon(0, x) = \hat{u}_0^\varepsilon(x)\).

Now we can find the generalized solution to the Riemann problem in the form:
\[
f_{FP}(t, x) = \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho^\varepsilon(t, x) \right),
\]
\[
u_{FP}(t, x) = \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \hat{u}^\varepsilon(t, x) \right).
\]

Let us introduce the points \(\hat{x}_1^\varepsilon = u_1 t - \varepsilon\) and \(\hat{x}_2^\varepsilon = (u_1 + u_2)t + \varepsilon\). Their velocities are \(u_1\) and \(u_1 + u_2\), respectively.

We consider two cases:

1. \(u_2 > 0\) (velocity of the point \(\hat{x}_2^\varepsilon\) is higher than the velocity of the point \(\hat{x}_1^\varepsilon\)).

At first, we can find \(f^\varepsilon(t, x) = \lim_{\sigma \to 0} \rho^\varepsilon(t, x)\) from (33). It is easy to see that
\[
f^\varepsilon(t, x) = \begin{cases} f_1, & x < \hat{x}_1^\varepsilon, \\ \frac{f_1}{2} - \frac{1}{2} F^\varepsilon(t, x), & x = \hat{x}_1^\varepsilon, \\ F^\varepsilon(t, x), & \hat{x}_1^\varepsilon < x < \hat{x}_2^\varepsilon, \\ \frac{f_1}{2} + \frac{f_2}{2} + \frac{1}{2} F^\varepsilon(t, x), & x = \hat{x}_2^\varepsilon, \\ f_1 + f_2, & x > \hat{x}_2^\varepsilon, \end{cases}
\]

Let us note that this formula contains \(F^\varepsilon(t, x)\) and \(\lim_{\varepsilon \to 0} F^\varepsilon(t, x) = 0\). Thus, we obtain the following result for \(f_{FP}(t, x) = \lim_{\varepsilon \to 0} f^\varepsilon(t, x)\):
Further, from (36) we find the solution of the gas dynamic system with smooth initial data \( u_\varepsilon(t, x) = \lim_{\varepsilon \to 0} \hat{u}_\varepsilon(t, x) \) as follows:

\[
\begin{align*}
    u_\varepsilon(t, x) = \begin{cases} 
        u_1, & x < \hat{x}_1^\varepsilon, \\
        u_1 + \frac{\varepsilon}{2} (x - u_1 t), & \hat{x}_1^\varepsilon \leq x \leq \hat{x}_2^\varepsilon, \\
        u_1 + u_2, & x > \hat{x}_2^\varepsilon.
    \end{cases}
\end{align*}
\]

It can be shown that

\[
\lim_{\varepsilon \to 0} \frac{N_\varepsilon(t, x)}{F_\varepsilon(t, x)} = \lim_{\varepsilon \to 0} \frac{u_2}{u_2 t + 2\varepsilon} x + \frac{u_2}{2} = \frac{x}{t} - u_1.
\]

Thus, we get the following solution for \( u(t, x) = \lim_{\varepsilon \to 0} u_\varepsilon(t, x) \):

\[
\begin{align*}
    u_{FP}(t, x) = \begin{cases} 
        u_1, & x < u_1 t, \\
        \frac{x}{t}, & u_1 t \leq x \leq (u_1 + u_2)t, \\
        u_1 + u_2, & x > (u_1 + u_2)t.
    \end{cases}
\end{align*}
\]

We can see that the velocity includes the rarefaction wave (see Fig.1). This is a well known self-similar solution to the Riemann problem with constant left-hand and right-hand states for the Burgers equation ([29]).

---

**Figure 1.** Density and velocity, \( u_2 > 0 \).
It is interesting to note that if we first compute the limit in $\varepsilon$ we get the solution

$$u(t, x) = u_1 + u_2 \theta(x - (u_1 + \frac{u_2}{2})t),$$

which is unstable with respect to small perturbations.

2. $u_2 < 0$ (the velocity of $\dot{x}_2$ is higher than the velocity of $\dot{x}_1$). From (34) and (36) we find as before (see Fig.2)

$$f_{FP}(t, x) = \begin{cases} 
  f_1, & x < (u_1 + u_2)t, \\
  \frac{3f_1 + f_2}{2}, & x = (u_1 + u_2)t, \\
  2f_1 + f_2, & (u_1 + u_2)t < x < u_1t, \\
  \frac{3f_1 + 2f_2}{2}, & x = u_1t, \\
  f_1 + f_2, & x > u_1t,
\end{cases}$$

$$u_{FP}(t, x) = \begin{cases} 
  u_1, & x < (u_1 + u_2)t, \\
  u_1 + \frac{f_1 + f_2}{2}u_2, & (u_1 + u_2)t \leq x \leq u_1t, \\
  u_1 + u_2, & x > u_1t.
\end{cases}$$

Figure 2. Density and velocity, $u_2 < 0$.

5. The singular Riemann problem in the FP sense

By the singular Riemann problem we mean the Cauchy problem with the following data:

$$f_0(x) = f_1 + f_2 \theta(x - x_0) + f_3 \delta(x - x_0) = f_{reg}^0 + f_{sing}^0,$$

$$u_0(x) = u_1 + u_2 \theta(x - x_0),$$

which differs from the classical Riemann problem (25), (26) by a singular part $f_{sing}^0$, the $\delta$ - function of constant amplitude $f_3$ concentrated at the jump of the density. We again set $x_0$ equal to zero (without loss of generality).

Thus, we should adapt the definition of the generalized solution to the case of a singular density. As before we want to approximate initial data by smooth
functions, the $\delta$-function will be instead naturally approximated in the space $\mathcal{D}'(\mathbb{R})$ of distributions.

**Definition 3.** We call the couple of functions $(f_{FP}(t, x), u_{FP}(t, x))$ a generalized solution to the problem (1), (2), (37), (38) in the sense of free particles (FP), if

\[
\begin{align*}
    f_{FP}(t, x) &= \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho_{\varepsilon,\sigma,\text{reg}}^0(t, x) \right) + \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho_{\varepsilon,\sigma,\text{sing}}^0(t, x) \right) = f_{\text{reg}}(t, x) + f_{\text{sing}}(t, x), \\
    u_{FP}(t, x) &= \lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} u_{\varepsilon,\sigma}(t, x) \right),
\end{align*}
\]

where the limits are meant as pointwise at almost all points except for the singular part

\[
\lim_{\varepsilon \to 0} \left( \lim_{\sigma \to 0} \rho_{\varepsilon,\sigma,\text{sing}}^0(t, x) \right),
\]

where the outer limit in $\varepsilon$ is meant in the sense of distributions.

Here $(\rho_{\sigma,\text{reg}}^0(t, x), \rho_{\sigma,\text{sing}}^0(t, x), u^\varepsilon(t, x))$ satisfy the system (23), (24) with initial data $(f_{0,\text{reg}}^0(x) + f_{0,\text{sing}}^0(x), u_0^\varepsilon(x))$ from the class $C^1(\mathbb{R}^n)$ such that

\[
\begin{align*}
    \lim_{\varepsilon \to 0} f_{0,\text{reg}}^0(x) &= f_{0,\text{reg}}^\varepsilon(x), \\
    \lim_{\varepsilon \to 0} u_0^\varepsilon(x) &= u_0(x)
\end{align*}
\]

for almost all fixed $x \in \mathbb{R}^n$ and

\[
\lim_{\varepsilon \to 0} f_{0,\text{sing}}^\varepsilon(x) = f_{0,\text{sing}}^0(x),
\]

the limit in $\varepsilon$ being in the sense of distributions.

We are going to solve the singular Riemann problem with constant left and right states. To approximate the $\delta$-function we use the well known fact that

\[
\frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}} \to \delta(x), \quad \varepsilon \to 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).
\]

The part of the solution that relates to the regular part of the density ($f_3 = 0$) is found in Sec.4. Now we have to calculate the singular part of the density $\rho_{\varepsilon,\sigma,\text{sing}}^0(t, x)$ for $f_{0,\text{sing}}^\varepsilon(x) = f_3 \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$. As before we omit the index $\sigma$.

From (18) we obtain:

\[
\begin{align*}
    \rho_{\varepsilon,\sigma,\text{sing}}^0(t, x) &= \frac{f_3}{\sqrt{2\pi(\sigma^2t + \varepsilon)}} \left( \exp \left[ -\frac{(u_1t - x)^2}{2(\sigma^2t + \varepsilon)} \right] \Phi \left( \frac{D_1^{\sigma,\varepsilon}}{\sigma\sqrt{t}} \right) + \exp \left[ -\frac{(u_1 + u_2)t - x)^2}{2(\sigma^2t + \varepsilon)} \right] \Phi \left( \frac{-D_2^{\sigma,\varepsilon}}{\sigma\sqrt{t}} \right) \right) + J_2^{\varepsilon,\sigma},
\end{align*}
\]

where

\[
\begin{align*}
    D_1^{\sigma,\varepsilon} &= \frac{(u_1t - x)\sqrt{\varepsilon}}{\sqrt{\sigma^2t + \varepsilon}} - \sqrt{\varepsilon(\sigma^2t + \varepsilon)}, \\
    D_2^{\sigma,\varepsilon} &= \frac{(u_1 + u_2)t - x)\sqrt{\varepsilon}}{\sqrt{\sigma^2t + \varepsilon}} + \sqrt{\varepsilon(\sigma^2t + \varepsilon)},
\end{align*}
\]

and

\[
J_2^{\varepsilon,\sigma} = \int_{-\varepsilon}^{\varepsilon} \exp \left[ -\frac{s^2}{2\varepsilon} - (s - x + \frac{u_2}{2\varepsilon}s + u_1 + \frac{u_2}{2})^2}{2\sigma^2t} \right] ds.
\]
It can be checked that

\[ J_2^{\epsilon, \sigma} = \frac{f_3}{\sqrt{2\pi} \sqrt{A^{\sigma, \epsilon}}} \exp \left[ -\frac{(u_1 + u_2)}{2\epsilon} t - x \right]^2 \left( \frac{K_{\epsilon}^{\sigma, \epsilon}}{\sigma\sqrt{t}} - \frac{K_{\epsilon}^{\sigma, \epsilon}}{\sigma\sqrt{t}} \right), \]

where

\[ A^{\sigma, \epsilon} = \sigma^2 t + \epsilon \left( \frac{u_2 t}{2\epsilon} + 1 \right)^2, \quad B^{\sigma, \epsilon} = \frac{\epsilon \left( \frac{u_2 t}{2\epsilon} + 1 \right) \left( (u_1 + \frac{u_2}{2}) t - x \right)}{\sqrt{A^{\sigma, \epsilon}}}, \]

\[ K_{\epsilon}^{\sigma, \epsilon} = \frac{1}{\sqrt{\epsilon}} \left( \sqrt{A^{\sigma, \epsilon}(\pm \epsilon)} + B^{\sigma, \epsilon} \right). \]

It is easy to see that

\[ I_1 = \lim_{\sigma \to 0} D_1^{\sigma, \epsilon}(t, x) = u_1 t - x - \epsilon, \quad I_2 = \lim_{\sigma \to 0} D_2^{\sigma, \epsilon}(t, x) = (u_1 + u_2) t - x - \epsilon, \]

\[ K_\pm = \lim_{\sigma \to 0} K_{\epsilon}^{\sigma, \epsilon}(\epsilon) = \left( \frac{u_2 t}{2 \epsilon} + 1 \right) (\pm \epsilon) + (u_1 + \frac{u_2}{2}) t - x, \]

i.e. \( K_\pm = D_1^\epsilon = C_-^\epsilon, \quad K_\pm = D_2^\epsilon = C_+^\epsilon \). Therefore, in this case we also have two jump points \( \hat{x}_1^\epsilon = u_1 t - \epsilon \) and \( \hat{x}_2^\epsilon = (u_1 + u_2) t + \epsilon \).

We have to consider two cases \( u_2 > 0 \) and \( u_2 < 0 \), as before. But for \( f_0^{\text{sing}}(x) = f_3 \delta(x) \) we obtain the same result:

\[ f^{\text{sing}}(t, x) = \lim_{\epsilon \to 0} \lim_{\sigma \to 0} \rho^{\epsilon, \sigma}(t, x) = \frac{1}{2} f_3 (\delta(x - u_1 t) + \delta(x - (u_1 + u_2) t)). \]

Thus, we have two \( \delta \)-functions with equal amplitudes \( \frac{1}{2} F(t) \), which move with the jump points.

Further, we can find the regular part of the density \( f_{\text{reg}}(t, x) = \lim_{\epsilon \to 0} \lim_{\sigma \to 0} \rho^{\epsilon, \sigma}(t, x) \), where

\[ \rho_{\text{reg}}(t, x) = f_1 \Phi \left( \frac{C_-^\epsilon}{\sigma\sqrt{t}} \right) + (f_1 + f_2) \Phi \left( -\frac{C_+^\epsilon}{\sigma\sqrt{t}} \right) + I_1. \]

Analogously, from \( (13) \) we can calculate the velocity \( \hat{u}(t, x) \):

\[ \hat{u}(t, x) = u_1 + \frac{f_3 u_2}{\sqrt{2\pi} \sqrt{A^{\sigma, \epsilon}}} \frac{G_{\text{reg}}^{\sigma, \epsilon}(t, x) \Phi \left( \frac{K_{\epsilon}^{\sigma, \epsilon}}{\sigma\sqrt{t}} \right) + J_3^{\epsilon, \sigma}}{\rho^{\epsilon, \sigma}(t, x)}, \]

where

\[ J_3^{\epsilon, \sigma} = \frac{f_3}{\sqrt{2\pi} \sqrt{A^{\sigma, \epsilon}}} G^{\sigma, \epsilon}(t, x) \left( \frac{u_2 \sqrt{t} \epsilon}{2 \epsilon A^{\sigma, \epsilon}} \left[ e^{-\frac{|u_{\epsilon}^{\sigma, \epsilon}|^2}{2 \epsilon A^{\sigma, \epsilon}}} - e^{-\frac{|K_{\epsilon}^{\sigma, \epsilon}|^2}{2 \epsilon A^{\sigma, \epsilon}}} \right] + \right. \]

\[ + \left. \frac{1}{\sqrt{A^{\sigma, \epsilon}}} \left( \frac{u_2}{2} - \frac{u_2 B^{\sigma, \epsilon}}{2 \epsilon A^{\sigma, \epsilon}} \right) \left[ \Phi \left( \frac{K_{\epsilon}^{\sigma, \epsilon}}{\sigma\sqrt{t}} \right) - \Phi \left( \frac{K_{\epsilon}^{\sigma, \epsilon}}{\sigma\sqrt{t}} \right) \right] \right), \]

\[ G^{\sigma, \epsilon}(t, x) = \exp \left[ -\frac{(u_1 + u_2)}{2 A^{\sigma, \epsilon}} \right], \]

\[ G_{2}^{\sigma, \epsilon}(t, x) = \exp \left[ -\frac{(u_1 + u_2)}{2 (\sigma^2 t + \epsilon)} \right]. \]
The expressions for $\rho_{reg}^\varepsilon(t, x) = \rho_{reg}(t, x) + \rho_{sing}^\varepsilon(t, x)$ and $I_x^\varepsilon$ have been given in (43), (39) and (55), respectively.

1. If $u_2 > 0$, then:

$$u^\varepsilon(t, x) = \begin{cases} u_1, & x < \hat{x}_1^\varepsilon, \\ u_1 + \frac{f_3}{2\pi} \frac{T^\varepsilon G^\varepsilon(t, x) + N^\varepsilon}{\sqrt{2\pi \varepsilon}}, & \hat{x}_1^\varepsilon \leq x \leq \hat{x}_2^\varepsilon, \\ u_1 + u_2, & x > \hat{x}_2^\varepsilon, \end{cases}$$

where

$$T^\varepsilon = \lim_{\sigma \to 0} \frac{1}{A^{\sigma, \varepsilon}} \left( \frac{w_2}{2} - \frac{w_2 B^{\sigma, \varepsilon}}{2\varepsilon A^{\sigma, \varepsilon}} \right) \equiv \frac{u_2}{2} - \frac{2\varepsilon((u_1 + \frac{u_2}{2})t - x)u_2}{(2\varepsilon + u_2 t)^2 \sqrt{\varepsilon}}.$$ 

Therefore

$$u_{FP}(t, x) = \lim_{\varepsilon \to 0} u^\varepsilon(t, x) = \begin{cases} u_1, & x < u_1t, \\ \frac{x}{t}, & u_1t \leq x \leq (u_1 + u_2)t, \\ u_1 + u_2, & x > \hat{x}(u_1 + u_2)t, \end{cases}$$

Further, from (43), (39) we find the density as follows:

$$f_{FP}(t, x) = \frac{1}{2} f_3 (\delta(x - u_1t) + \delta(x - (u_1 + u_2)t)) + \begin{cases} f_1, & x < u_1t, \\ \frac{f_1}{2}, & x = u_1t, \\ 0, & u_1t < x < (u_1 + u_2)t, \\ \frac{f_1 + f_2}{2}, & x = (u_1 + u_2)t, \\ f_1 + f_2, & x > (u_1 + u_2)t, \end{cases}$$

2. If $u_2 < 0$, we have

$$u^\varepsilon(t, x) = \begin{cases} u_1, & x < \hat{x}_1^\varepsilon, \\ u_1 + \frac{u_2 (f_1 + f_2) - N^\varepsilon + \frac{f_3}{\sqrt{2\pi \varepsilon}} (G_2^\varepsilon - \varepsilon T^\varepsilon G^\varepsilon)}{2f_1 + f_2 + F^\varepsilon + \frac{f_3}{\sqrt{2\pi \varepsilon}} (G_1^\varepsilon + G_2^\varepsilon - \frac{2\varepsilon}{2\varepsilon + u_2 t} G^\varepsilon)}, & \hat{x}_1^\varepsilon \leq x \leq \hat{x}_2^\varepsilon, \\ u_1 + u_2, & x > \hat{x}_2^\varepsilon, \end{cases}$$

where $G_1^\varepsilon \equiv \exp \left[ -\frac{(u_1 t - x)^2}{2\varepsilon} \right]$, $G_2^\varepsilon \equiv \exp \left[ -\frac{(u_1 + u_2)t - x)^2}{2\varepsilon} \right].$

Thus,

$$u_{FP}(t, x) = \begin{cases} u_1, & x < (u_1 + u_2)t, \\ u_1 + \frac{(f_1 + f_2)}{2f_1 + f_2} u_2, & (u_1 + u_2)t \leq x \leq u_1t, \\ u_1 + u_2, & x > u_1t. \end{cases}$$
As a consequence, for the velocity we obtain the same result as in the non-singular case. Analogously,

\[ f_{FP}(t, x) = \frac{1}{2} f_3(\delta(x-u_1 t)+\delta(x-(u_1+u_2) t)) + \begin{cases} 
    f_1, & x < (u_1 + u_2) t, \\
    \frac{3f_1 + f_2}{2}, & x = (u_1 + u_2) t, \\
    2f_1 + f_2, & (u_1 + u_2) t < x < u_1 t, \\
    \frac{3f_1 + 2f_2}{2}, & x = u_1 t, \\
    f_1 + f_2, & x > u_1 t.
\end{cases} \]

6. Singularity arising from smooth data

We are going to show that at the point of formation of a singularity from smooth initial data in the solution to the pressureless gas dynamics model a \( \delta \) - function appears in the density component. For the sake of simplicity we again restrict ourselves to the 1D case.

**Theorem 1.** (Asymptotics of the approximating solution) Let the initial data \((f_0, u_0)\) for the system \([7], [4]\) be at least \( C^m \) - smooth and bounded, \( m \geq 2 \). Assume that there exists an instant \( 0 < t_\ast < \infty \), such that \( t_\ast = \inf_{x \in \mathbb{R}} \left( -\frac{1}{u_0(x)} \right) \) and \( u^{(k)}(s_\ast) = 0, k = 1, \ldots, m - 1 \), however \( u^{(m)}(s_\ast) \) does not vanish at the point \( s_\ast(t_\ast, x_\ast) \), where \( s_\ast \) is a solution to the equation \( u_0(s) t_\ast = x_\ast - s \). Here \( x_\ast \) is such that the line \( y = \frac{x_\ast}{t_\ast} \) intersects the graph of the initial velocity \( y = u_0(s) \) at a unique point and it is tangent to the graph.

Then at the moment \( t = t_\ast \) the following properties of the function \( \rho_\sigma(t, x) \), entering the solution \((\rho_\sigma, \tilde{u}_\sigma)\) to the system \([23], [24]\), hold:

1. \( \rho_\sigma(t_\ast, x_\ast) \sim B(x_\ast, t_\ast) f_0(s_\ast) \sigma^{-\frac{m-1}{m}}, \quad \sigma \to 0, \tag{44} \)

   where 
   \[ B(x_\ast, t_\ast) = K_m \left\{ \frac{|u'_0(s_\ast)|^{\frac{m}{m-1}}}{|u_0(s_\ast)|^{\frac{1}{m-1}}} \right\}, \quad K_m = \frac{1}{2^{\frac{m}{2m} - 1}} \frac{1}{m!} \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2m} \right), \]

   \( \sim \) means that the quotient of the left-hand side by the right-hand side converges to 1 as \( \sigma \to 0 \);

2. \( \rho_\sigma(t_\ast, x) \to f_0(s_0(t_\ast, x)), \quad \sigma \to 0, \quad \text{for } x \neq x_\ast, \tag{45} \)

   where the function \( s_0(t, x) \) has been introduced in the proof of Proposition \([4]\).

**Proof.** Proceeding as in the proof of Proposition \([13]\) we can readily obtain the property \([15]\). Thus, let us dwell on the property \([14]\).

Let us analyze the formula \([18]\) at the point \((t_\ast, x_\ast)\). To that end we note that since \( u_0(s_\ast) t_\ast = x_\ast - s_\ast \) and \( t_\ast = -\frac{1}{u_0(s_\ast)} \), for \( s \) belonging to the \( \varepsilon \) - neighborhood \( U_{s_\ast}(\varepsilon) \) of the point \( s_\ast \) we have

\[ u_0(s) t_\ast + s - x = (u_0(s_\ast) + u'_0(s_\ast)(x-s_\ast) + \frac{1}{m!} u_0^{(m)}(s_\ast)(s-s_\ast)^m) t_\ast + s - x = \]

\[ = \frac{1}{m!} u_0^{(m)}(s_\ast)(s-s_\ast)^m t_\ast - (x-x_\ast), \tag{46} \]
Then from [18] we have
\[
\rho(t_*, x_*) = \frac{1}{\sqrt{2\pi t_* \sigma}} \int_{\mathbb{R}} f_0(s) e^{-\frac{(s - \varepsilon x_*)^2}{2\sigma^2 t_*}} \, ds =
\]
\[
\frac{1}{\sqrt{2\pi t_* \sigma}} \int_{\mathbb{R}} f_0(s) e^{-\frac{\sigma^2 (s - x_*)^2}{2\sigma^2 t_*}} \, ds +
\]
\[
\frac{1}{\sqrt{2\pi t_* \sigma}} \int_{\mathbb{R}} f_0(s) \left( e^{-\frac{(s - \varepsilon x_*)^2}{2\sigma^2 t_*}} - e^{-\frac{\sigma^2 (s - x_*)^2}{2\sigma^2 t_*}} \right) \, ds =
\]
\[I_1 + I_2.
\]

The first integral \(I_1\) is equal to
\[
\sigma^{-\frac{m+1}{m}} I_\sigma(t_*, x_*),
\]
where \(I_\sigma(t_*, x_*) = B(x_*, t_*) f_0(x_*)\) as \(\sigma \to 0\), where \(B(x_*, t_*)\) is specified in the statement of Theorem 1. To evaluate \(I_\sigma\) we have used the formula
\[
\int_{\mathbb{R}} e^{-a^2 s^{2m}} \, ds = \frac{\Gamma \left( \frac{1}{2m} \right)}{|a|^{\frac{1}{m}}} a = \text{const} \neq 0.
\]

Now let us prove that \(I_2 \to 0\) as \(\sigma \to 0\).

Let us choose \(\varepsilon > 0\) so small that for all \(s \in U_{\sigma}(\varepsilon)\)
\[
\left| e^{-\frac{(s - \varepsilon x_*)^2}{2\sigma^2 t_*}} - e^{-\frac{\sigma^2 (s - x_*)^2}{2\sigma^2 t_*}} \right| < \sigma.
\]

Then
\[
|I_2| \leq \frac{1}{\sqrt{2\pi t_* \sigma}} \int_{U_{\sigma}(\varepsilon)} |f_0(s)| \left| e^{-\frac{(s - \varepsilon x_*)^2}{2\sigma^2 t_*}} - e^{-\frac{\sigma^2 (s - x_*)^2}{2\sigma^2 t_*}} \right| \, ds +
\]
\[
\frac{1}{\sqrt{2\pi t_* \sigma}} \int_{\mathbb{R} \setminus U_{\sigma}(\varepsilon)} |f_0(s)| \left( e^{-\frac{(s - \varepsilon x_*)^2}{2\sigma^2 t_*}} + e^{-\frac{\sigma^2 (s - x_*)^2}{2\sigma^2 t_*}} \right) \, ds.
\]

The first part in the right-hand side of the inequality due to the boundedness of \(f_0\) is less than \(\sigma \cdot \varepsilon\), the second part tends to zero as \(\sigma \to 0\) due to the boundedness of \(f_0\). Since \(\varepsilon\) can be chosen arbitrarily small, the statement is proved. \(\square\)

**Remark 6.** The following asymptotics of \(K_m\) holds:
\[
K_m = \frac{\sqrt{\pi}}{\varepsilon m} + O(\ln m) \quad m \to \infty.
\]

**Theorem 2.** (Amplitude of the \(\delta-\) function) Let the initial data \((f_0, u_0)\) for the system [1], [2] be \(C^1\) smooth and bounded and let the critical instant \(t_* = \inf_{x \in \mathbb{R}} \left( -\frac{1}{u_0(x)} \right) \) be positive and finite. Assume that the initial datum \(u_0\) is linear on the segment \(\Omega = [x_1, x_2]\), moreover, the second left-hand derivative \(u_0''(x_1 - 0)\) at the point \(x_1\) and the second right-hand derivative \(u_0''(x_2 + 0)\) at the point \(x_2\) do not
vanish. Let \( x_* \) be the unique point such that the line \( y = \frac{x-x_*}{t_*} \) and the graph of the initial velocity \( y = u_0(s) \) have a common linear segment \( \Omega = [s_1, s_2] \).

Then at the moment \( t = t_* \) at the point \( x = x_* \) the component of the density develops a \( \delta \) - singularity of amplitude

\[
A(x_*) = \int_{s_1}^{s_2} f(s) \, ds. \quad (47)
\]

**Proof.** We are going to prove that

\[
\int_{\mathbb{R}} \rho_{\sigma}(t_*, x) \phi(x) \, dx \to A(x_*) \phi(x_*), \quad \sigma \to 0.
\]

From (15) we have

\[
\int_{\mathbb{R}} \rho(t_*, x) \phi(x) \, dx = \frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R}} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx =
\]

\[
\frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R}} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx +
\]

\[
\frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R} \setminus \Omega} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx = I_1 + I_2.
\]

First we analyze \( I_1 \).

\[
I_1 = \phi(x_*) \int_{\Omega} f_0(s) \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t_*}} e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx +
\]

\[
\frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R} \setminus \Omega} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) - \phi(x_*) \, ds \, dx = I_{11} + I_{12} + I_{13},
\]

where \( U_{x_*}(\varepsilon) \) is an \( \varepsilon \) - neighborhood of \( x_* \). The integral \( I_{11} \) is equal to \( A(x_*) \phi(x_*) \), with \( A(x_*) \) specified in the statement of Theorem \( 2 \) \( I_{12} \) and \( I_{13} \) tend to zero as \( \sigma \to 0 \), as can be shown in a standard way.

Further,

\[
I_2 = \frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R} \setminus \Omega} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx +
\]

\[
\frac{1}{\sqrt{2\pi t_*}} \int_{\mathbb{R}} f_0(s) e^{-\frac{(u_0(s)x + x_*)^2}{2\sigma^2 t_*}} \phi(x) \, ds \, dx = I_{21} + I_{22}.
\]

Let us prove that \( I_2 \) vanishes as \( \sigma \to 0 \). Since \( I_{21} \) and \( I_{22} \) can be analyzed similarly, we consider only \( I_{21} \).

For \( s \in U_{x_*}(0), \varepsilon > 0 \) from (16), \( m = 2 \), we have

\[
u_0(s)t_* + s - x = \frac{1}{2} u_0^{(2)}(s_*) (s - s_*)^2 t_* - (x - x_*),
\]
with \( s_{\ast} \in U_{s_{\ast} - 0}(\varepsilon) \).

Thus,

\[
I_{21} = \frac{1}{\sqrt{2\pi} t_{s_{\ast}}} f_0(s_1) \phi(x_s) \int \int_{\mathbb{R}} e^{-\frac{(u(s_1) - 0)t_{s_{\ast}}}{2\pi t_{s_{\ast}}} (x_{s_{\ast}} - x)^2} ds \, dx + \frac{1}{\sqrt{2\pi} t_{s_{\ast}}} \int_{-\infty}^{s_1} e^{-\frac{u''(s_1)U_{s_{\ast}}^2}{2\pi t_{s_{\ast}}}} (f_0(s) \phi(x) - f_0(s_1) \phi(x_s)) \, ds \, dx + \frac{1}{\sqrt{2\pi} t_{s_{\ast}}} \int_{-\infty}^{s_1} \left( e^{-\frac{(u(s)(x_{s_{\ast}} - x))^2}{2\pi t_{s_{\ast}}}} - e^{-\frac{u''(s_1)U_{s_{\ast}}^2}{2\pi t_{s_{\ast}}}} (x_{s_{\ast}} - x)^2 \right) f_0(s) \phi(x) \, ds \, dx = I_{211} + I_{212} + I_{213},
\]

To evaluate \( I_{211} \) we use the formula

\[
\int e^{-\left(\frac{a x^2 - x^2}{2}\right)^2} ds = \frac{\sqrt{\pi}}{2} e^{-\frac{x^2}{2a}} \sqrt{\frac{\pi}{a}} B\left(\frac{1}{2}, \frac{1}{2}\right),
\]

where \( a \neq 0, b \neq 0 \) are constants, \( B\left(\frac{1}{2}, \frac{1}{2}\right) \) is the Bessel function of the second type \( (30) \). Thus,

\[
I_{211} = \sigma^{\frac{1}{2}} f_0(s_1) \phi(x_s) M(t_{s_{\ast}}, x_s), \tag{48}
\]

where

\[
M(t_{s_{\ast}}, x_s) = \frac{1}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} |u''(s_1)|^{\frac{1}{2}}} L, \quad L = \int_{\mathbb{R}} e^{-\left(\frac{x^2}{2}\right)} \sqrt{\frac{\pi}{a}} B\left(\frac{1}{2}, \frac{1}{2}\right) \left( -\frac{x^2}{2a} \right) dx.
\]

The integral in the expression of \( L \) converges, since the integrand is finite at \( x = 0 \) and decays exponentially at infinity.

The fact that \( I_{212} \) and \( I_{213} \) vanish as \( \sigma \to 0 \) can be proved routinely. □

**Remark 7.** Under the assumptions of Theorem 4 one can similarly show (analogously to (18)) that

\[
\rho_\sigma(t_{s_{\ast}}, x_s) \sim \sigma^{\frac{1}{2}} M(x_{s_{\ast}}, t_{s_{\ast}}, m) f_0(s_1) R_\sigma(t_{s_{\ast}}, x_s), \quad \sigma \to 0,
\]

where \( M(x_{s_{\ast}}, t_{s_{\ast}}, m) \) is a constant depending only on the properties of \( u_0 \) and \( R_\sigma(t_{s_{\ast}}, x_s) \) tends to \( \delta(x - x_s) \) as \( \sigma \to 0 \) in \( \mathcal{D}'(\mathbb{R}) \). Thus, the \( \delta \) - function, arising from smooth initial data without linear segments, has initially a zero amplitude.

7. The Hugoniot conditions and the spurious pressure

As follows from the results of Sec.2, if \( f_{FP} \) and \( u_{FP} \) are smooth, they solve the pressureless gas dynamics system. Now we ask the question which system satisfies the FP-generalized solution with jumps obtained in Sec.4.

The system of conservation laws \( (1), (2) \) implies two Hugoniot conditions that should be held on the jumps of the solution \( (28) \). This signifies that the solution satisfies the system in the sense of integral identities. If we denote by \( \mathcal{D} \) the velocity of the jump and by \( [h(y)] = h(y + 0) - h(y - 0) \) the value of the jump, then the continuity equation and the momentum conservation give \( [f] \mathcal{D} = [fu] \) and \( [fu] \mathcal{D} = [fu^2] \), respectively.
In the case $u_2 > 0$ the velocity is continuous, therefore the Hugoniot conditions hold trivially.

We should check these conditions for the jumps in the case $u_2 < 0$. An easy computation shows that the first one is satisfied: for the point $\hat{x}_1 = u_1 t$ we have:

$$D = \frac{(f_1 + f_2)(u_1 + u_2) - (2f_1 + f_2)u_1 - (f_1 + f_2)u_2}{f_1} = u_1,$$

and for $\hat{x}_2 = (u_1 + u_2)t$:

$$D = \frac{(2f_1 + f_2)u_1 + (f_1 + f_2)u_2 - f_1 u_1}{f_1 + f_2} = u_1 + u_2.$$

However, the second Hugoniot condition does not hold. To understand the reason for this let us estimate the integral term in $[20]$ in the case $u_2 < 0$ as $\sigma \to 0$:

$$\int_\mathbb{R} (u - \hat{u}(t, x))^2 P_\sigma(t, x, u) \, du =$$

$$\frac{1}{\sqrt{2\pi t\sigma}} \int \frac{f_0(s)(u_0(s) - \hat{u}(t, x))^2 \left(e^{-\frac{(u_0(s) + s - x)^2}{2\sigma^2}}\right)}{s} \, ds =$$

$$- \frac{1}{\sqrt{2\pi t\sigma}} \int f_0(s)((u_0(s) - u_{FP}(t, x)) + (u_{FP}(t, x) - \hat{u}(t, x))^2 \left(e^{-\frac{(u_0(s) + s - x)^2}{2\sigma^2}}\right)}{s} \, ds -$$

$$\frac{2}{\sqrt{2\pi t\sigma}} \int f_0(s)(u_0(s) - u_{FP}(t, x)) \left(e^{-\frac{(u_0(s) + s - x)^2}{2\sigma^2}}\right) \, ds +$$

$$\left(u_{FP}(t, x) - \hat{u}(t, x)\right)^2 \int f_0(s) \left(e^{-\frac{(u_0(s) + s - x)^2}{2\sigma^2}}\right) \, ds =$$

$$I_1 + I_2 + I_3.$$  

The integrals $I_2$ and $I_3$ tend to zero as $\sigma \to 0$ due to properties of the Riemann data since $\hat{u}(t, x) \to u_{FP}(t, x)$ for almost all $x \in \mathbb{R}$. Let us estimate $I_1$.

$$I_1 = - \frac{1}{\sqrt{2\pi t\sigma}} \int_{u_2 t}^0 f_1(u_1 - u_{FP}(t, x))^2 \left(e^{-\frac{(u_1 + s - x)^2}{2\sigma^2}}\right) \, ds -$$

$$\frac{1}{\sqrt{2\pi t\sigma}} \int_0^{-u_2 t} (f_1 + f_2)(u_1 + u_2) - u_{FP}(t, x))^2 \left(e^{-\frac{(u_1 + s - x)^2}{2\sigma^2}}\right) \, ds =$$

$$- \frac{1}{\sqrt{2\pi t\sigma}} \frac{f_1(f_1 + f_2)^2 u_2^2}{(2f_1 + f_2)^2} \left(e^{-\frac{(u_1 + s - x)^2}{2\sigma^2}}\right) \left(e^{-\frac{(u_1 + u_2)(s - x)^2}{2\sigma^2}}\right) -$$

$$\frac{1}{\sqrt{2\pi t\sigma}} \frac{f_2(f_1 + f_2)^2 u_2^2}{(2f_1 + f_2)^2} \left(e^{-\frac{(u_1 + s - x)^2}{2\sigma^2}}\right) \left(e^{-\frac{(u_1 + u_2)(s - x)^2}{2\sigma^2}}\right).$$

Thus,

$$I_1 \to - \frac{f_1(f_1 + f_2)u_2^2}{(2f_1 + f_2)} (\delta(x - (u_1 + u_2)t) - \delta(x - u_1 t)), \quad \sigma \to 0,$$
in the distributional sense.

Thus, the integral term corresponds to a spurious pressure \( p(t, x) \) between the jumps \( x = (u_1 + u_2) t \) and \( x = u_1 t \), namely,

\[
p(t, x) = \frac{f_1 (f_1 + f_2) u_2^2}{(2f_1 + f_2)} (\theta(x - (u_1 + u_2) t) - \theta(x - u_1 t)),
\]

(49)

see Fig. 3.

**Figure 3.** Spurious pressure, \( u_2 < 0 \).

The Hugoniot condition \( [fu] D = [fu^2 + p] \) is satisfied with this kind of pressure. Thus, we get the following theorem.

**Theorem 3.** The generalized FP-solution to the Riemann problem (25), (26) with constant left-hand and right-hand initial states for the pressureless gas dynamics system in the case of a discontinuous velocity \( u_2 < 0 \) solves in fact the gas dynamics system with a pressure defined by (49).

The analogous calculations for the case of rarefaction \( u_2 < 0 \) show that

\[
I_1 = -\frac{1}{\sqrt{2\pi t\sigma}} \int_{-\infty}^{0} f_1 (u_1 - u_{FP}(t, x))^2 \left( e^{-\frac{(u_1 t + x - s)^2}{2\sigma^2 t}} \right) ds - \frac{1}{\sqrt{2\pi t\sigma}} \int_{0}^{+\infty} (f_1 + f_2) ((u_1 + u_2) - u_{FP}(t, x))^2 \left( e^{-\frac{(u_1 + u_2)(t + x - s)^2}{2\sigma^2 t}} \right) ds = \frac{1}{\sqrt{2\pi t\sigma}} \left( (u_1 - \frac{x}{t})^2 f_1 e^{-\frac{(u_1 t + x - s)^2}{2\sigma^2 t}} - ((u_1 + u_2) - \frac{x}{t})^2 f_1 e^{-\frac{(u_1 + u_2)(t + x - s)^2}{2\sigma^2 t}} \right).
\]

Here we use the FP-solution \( (f_{FP}, u_{FP}) \), obtained in Sec. 4.

Thus, \( I_1 \rightarrow 0 \) as \( \sigma \rightarrow 0 \), therefore the integral term vanishes in the case \( u_2 < 0 \).
8. Sticky particles model vs noninteracting particles

In our model the particles are allowed to go through the discontinuity as one particle does not feel the others. However, in the frame of the sticky particles model the particles meeting each other are assumed to stick together on the jump \[20\]. The noninteracting particles model and the sticky particles model are equivalent to the same system \[1, 2\] for smooth densities and velocities, however, if the initial data have a jump, the solutions behavior differs drastically.

8.1. Riemann problem with constant states. Nevertheless, we can study the solution to the Riemann problem in the case of \(u_2 < 0\) for the sticky particles model, too, using the solution obtained above for the non-interacting model. Indeed, the jump position \(x_j(t)\) is a point between \(x_1(t) = (u_1 + u_2) t\) and \(x_2(t) = u_1 t\). The mass \(m(t)\) accumulates in the jump due to the impenetrability of the discontinuity as

\[
m(t) = (x_j(t) - (u_1 + u_2)t)(f_1 + f_2) + (u_1 t - x_j(t))f_1 + f_3 = -((u_1 + u_2)(f_1 + f_2) - u_1 f_1) t + x(t)f_2 + f_3 = -[uf] t + [f] x_j(t) + f_3,
\]

where \([\ ]\) stands for the jump value, and \(m(0) = f_3, x_j(0) = 0\). Further, if we change heuristically the overlapped mass between \(x_1\) and \(x_2\) to the mass concentrated at a point (see Fig[3]), then from the condition of equality of momenta in both cases we can find the velocity of the point singularity:

\[
(u_1 + u_2)(f_1 + f_2)(x_j(t) - (u_1 + u_2)t) + u_1 f_1 (u_1 t - x_j(t)) = -[u^2 f] t + [uf] x_j(t) = m(t) \dot{x}_j(t).
\]

Thus, to find the position of the point singularity we get the equation

\[
([f] x_j(t) - [uf] t + f_3) \dot{x}_j(t) = [uf] x_j(t) - [u^2 f] t,
\]

subject to the initial data \(x_j(0) = 0\). The respective solution is

\[
x_j(t) = \frac{1}{[f]} \left( [uf] t - f_3 + \sqrt{f_3^2 - 2[uf] f_3 t + ([uf]^2 - [uf][u^2 f]) t^2} \right), \text{ if } [f] \neq 0,
\]

and

\[
x_j(t) = \frac{[u^2] f t^2}{2([uf] f t - f_3)}, \text{ if } [f] = 0.
\]

In particular, from the latter formula in the case \(f_3 = 0\) we get the known expression for the velocity of the jump \[20\]:

\[
\dot{x}_j(t) = \frac{2u_1 + u_2}{2} = \frac{u_{left} + u_{right}}{2}.
\]

It can be checked that \(x_1(t) < x_j(t) < x_2(t)\). The condition expressed by these inequalities is equivalent to the Lax stability condition \(u_1 < \dot{x}_j(t) < u_1 + u_2\).

The formulas describing the amplitude of the delta-function in the density component and the singularity position obtained earlier in \[19, 16, 12\] give the same result. Moreover, our method allows to find the jump position in a unique way (in contrast to the method used in \[12\]).
It is worth mentioning that the spurious pressure (49) does not arise in the sticky particles model.

**Remark 8.** The case of the singular Riemann problem with $f_3 \neq 0$ is in fact more different from the case of the regular Riemann problem than it seems at first glance. Let us begin with the constant initial density ($f = 0$), when the trajectory satisfies (52). It is natural to set the amplitude of the initial $\delta$–function $f_3$ greater or equal than zero. Then the trajectory $x(t)$ is continuous since the denominator in (52) does not vanish. However, if we take $f_3 < 0$, then the trajectory goes to infinity at the finite moment where $m(t)$ vanishes, and then the trajectory jumps to infinity of the other sign.

Further, if $[f] \neq 0$, then we have to use the formula (51). It can be checked that it is possible to find values $u_1, u_2, f_1, f_2$ (for example, $u_1 = -1, u_2 = -2, f_1 = 2, f_2 = -1.8$) such that the expression under the square root vanishes within a finite time $t^*$. However, as follows from (50), (51),

$$m(t) = \sqrt{f_3^2 - 2[u] f_3 t + ([u]^2 - [f][u^2 f]) t^2}, \quad (53)$$

therefore within the same time $t^*$ the amplitude of the $\delta$–function becomes zero. Thus, at the moment $t^*$ we have to set a new Riemann problem with the jump at the point $x(t^*)$.

**8.2. Riemann problem with non-constant states.** Now we extend formulas (51) and (52) to the case of the Riemann problem with non-constant left and right states:

$$f_0(x) = f_1(x) + \theta(x)f_2(x) + f_3\delta(x), \quad (54)$$

$$u_0(x) = u_1(x) + \theta(x)u_2(x), \quad (55)$$

where $u_1(x), u_2(x), f_1(x), f_2(x)$ are smooth functions, $f_3$ is a real constant. We restrict ourselves to a situation that is quite similar to the case of constant states. Namely, we assume that for every $x \in \mathbb{R}$ and $t > 0$ the straight line $y = \frac{x - s(t)}{t}$ has at most two common points with the graph of the function $y = u_0(x)$, moreover, let $u_2^0 = \lim_{x \to 0^+} u_2(x) < 0$ and assume that the intersection points $s_-(t, x)$ and $s_+(t, x)$ lie on either side of the origin $x = 0$ (see Fig. 5).
Figure 5. The overlapping domain in the Riemann problem

So, every point $x$ at the moment $t > 0$ lies in the overlapping domain

$$D = [x_-(t), x_+(t)], \quad x_-(t) = u_+ t, \quad x_+(t) = u_- t,$$

if the line $y = \frac{x - s}{t}$ and the graph of $y = u_0(s)$ have two points of intersection, $s_-(t, x) < 0$ and $s_+(t, x) > 0$. We set $u_- \equiv \lim_{x \to 0^-} u_1(x)$ and $u_+ \equiv u_- + u_0^2$.

Further, we assume that for any fixed $t$ the number $K$ of points $\bar{x}_k(t) \in D$ such that the straight line and the graph of the initial velocity have a common linear segment is finite. Let us again denote by $x_j(t)$ the position of the singularity that should change the overlapping domain $D$ in the sticky particles model. Then the conservation of mass gives

$$m(t) = \int_{x_j(t)}^{x_+(t)} f(s_+(t, x)) \, dx + \int_{x_j(t)}^{u_- t} f(s_-(t, x)) \, dx + \sum_{k=1}^{K} A_k(t, \bar{x}_k(t)) + f_3, \quad (56)$$

where $A_k(t, \bar{x}_k(t))$ is the amplitude of the $\delta$ – function formed at the point $\bar{x}_k(t)$, calculated by using the formula [37].

Further, from the conservation of momentum we have

$$m(t) \dot{x}_j(t) = \int_{u_+ t}^{x_j(t)} f(s_+(t, x)) u(s_+(t, x)) \, dx + \int_{x_j(t)}^{u_- t} f(s_-(t, x)) u(s_-(t, x)) \, dx + (57)$$

$$+ \sum_{k=1}^{K} A_k(t, \bar{x}_k(t)) \frac{d}{dt} \bar{x}_k(t),$$
moreover, the velocity $\frac{d}{dt} \bar{x}_k(t)$ can be found by the formula

$$\frac{d}{dt} \bar{x}_k(t) = u_0(s_*(t, \bar{x}_k(t))),$$

where $s_*(t, \bar{x}_k(t))$ is the coordinate of the point where the graph of $y = u_0(s)$ and the line $y = \frac{\bar{x}_k}{t}$ have a common linear segment. If the graph of $y = u_0(s)$ does not contain any linear segments, then the respective parts in formulas (56) and (57) vanish. Thus, it is sufficient to substitute (56) into (57) to get the integro-differential equation that governs the singularity position. This equation should be considered together with the initial condition $x_j(0) = 0$.

8.3. Evolution of the singularity formed from smooth data. As we have seen in Sec[6], if at a point $x$ and a moment of time $t_*$ starting from smooth initial data loses its smoothness, then there arises a gradient catastrophe in the velocity component (the derivative becomes unbounded), whereas in the density component there arises a $\delta$–singularity. In the framework of the pressureless gas dynamics for $t > t_*$ the $\delta$–singularities encompass the overlapping domain $D$ and in the overlapping domain the spurious pressure (given by the integral term discussed in Sec[7]) appears. In the sticky particles model we have to collapse the overlapping domain to the one point, where the whole mass of $D$ accumulates. The position of this new singularity should be found based on the conservation of mass and momentum.

For the sake of simplicity we assume that every straight line $y = \frac{s_0}{t}$ intersects the graph of the smooth initial velocity $y = u_0(s)$ at most three times. For every fixed $x$ initially the intersection point is unique $(0 < t < t_*)$, then at the moment $t = t_*$ the straight line becomes tangent to the graph of the initial velocity in a certain point, and for $t > t_*$ we have three intersection points. Our aim is to find the position of the singularity and the amplitude of the $\delta$–function in the density component.

We denote as before by $x_-(t)$ and $x_+(t)$ the endpoints of the domain $D$, $x_j(t)$ – the position of the new singularity, $A(x_-(t))$ and $A(x_+(t))$ – the amplitude of $\delta$–functions produced at the point where the graphs of $y = u_0(s)$ and $y = \frac{s_0}{t}$ have a common linear segment (for fixed $t \geq t_*$). Further, let $s_-(t, x)$, $s_0(t, x)$, $s_+(t, x)$ be the subsequent point of intersection $(s_- < s_0 < s_+)$ Let $m(t)$ be the amplitude of the $\delta$ – function in the density component.

Then, the conservation of mass gives

$$m(t) = A(x_-(t)) + A(x_+(t)) +$$

$$\int_{x_-(t)}^{x_j(t)} (f(s_+(t, x)) + f(s_0(t, x))) dx + \int_{x_j(t)}^{x_+(t)} (f(s_-(t, x)) + f(s_0(t, x))) dx,$$

where $A(x_\pm(t))$ can be found by formula (57).

From the conservation of momentum analogously to (57) we have

$$m(t) \ddot{x}_j(t) = \int_{x_-(t)}^{x_j(t)} \left( f(s_+(t, x)) u(s_+(t, x)) + f(s_0(t, x)) u(s_0(t, x)) \right) dx +$$

$$+ \int_{x_j(t)}^{x_+(t)} \left( f(s_-(t, x)) u(s_-(t, x)) + f(s_0(t, x)) u(s_0(t, x)) \right) dx.$$
Thus, equations (68), (69) and the initial conditions \( m(t_*) = A(x_*(t_*)), x_j(t_*) = x_* \) define the position and the amplitude of the singularity of the \( \delta \) - function in the component of density. Here \( x_* \) is a point such that \( y = \frac{x - s}{t_*} \) and \( y = u_0(s) \) have a common point \( s_* \) or a common linear segment \([s_-, s_+]\) such that the derivatives of both functions are equal at \( s_* \) or on \([s_-, s_+]\), \( A(x_*) \) is defined in the statement of Theorem 2.

Remark 9. If we want to consider the global evolution of the solution to the Burgers equation itself, we should set \( f_0 = \text{const.} \) The continuity equation plays here an auxiliary role. We are not interested in the properties of the density \( f \), which is constant everywhere except for domains of vacuum and except for the points giving rise to the \( \delta \) - singularity.

9. Extension to more general scalar conservation law

Let us consider the following equation

\[
\partial_t v + (G(v), \nabla)v = 0,
\]

subject to initial data \( v(x, 0) = v_0(x) \), where \( v(x, t) = (v_1, ..., v_n) \) is a vector-function \( \mathbb{R}^{n+1} \to \mathbb{R}^n \), \( G(v) \) is a non-degenerate differential mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), such that its Jacobian satisfies \( \frac{\partial G_i(v)}{\partial y_j} \neq 0, \) \( i, j = 1, ..., n \).

We can multiply (60) by \( \nabla G_i(v) \), \( i = 1, ..., n \), to get

\[
\partial_t G(v) + (G(v), \nabla)G(v) = 0.
\]

Thus, we can introduce a new vectorial variable \( u = G(v) \) to reduce the Cauchy problem for (61) to (3) with \( u_0(x) = G(v_0(x)) \). The stochastic perturbation for (61) is (54) with \( U \) replaced by \( G(V) \).

Therefore we find the representation of the solution to the stochastically perturbed along the characteristics of equation (61) using the formula (13) with \( G(v_0(x)) \) instead of \( v_0(x) \).

Thus, we can apply the results obtained in the previous sections to the investigation of the Riemann problem and the arising of singularities for the following analogue of the pressureless gas dynamics system:

\[
g_t + \text{div}_x(gG(v)) = 0, \quad (gG(v))_t + \nabla_x(gv \otimes G(v)) = 0,
\]

where \( g(t, x) \) is a scalar function that can be interpreted as a density.

Thus, just as we relate with the non-viscous Burgers equation the system of pressureless gas dynamics, so also with the equation (61) one can relate the system (62).

To obtain the solution to the Cauchy problem for (60) itself we have to consider the solution to the Cauchy problem for (62) with the data \( (g_0, G(v_0)) \), \( g_0 \equiv \text{const} \) and then perform the inverse transform \( v(t, x) = G^{-1}(u(t, x)) \).
10. Conclusion

1. Let us notice that the solution of the Riemann problem for the pressureless gas dynamics system obtained in this paper for the 1D case satisfies the entropy condition

\[ \frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} \leq \frac{1}{t}, \tag{63} \]

for any sufficiently small \( x_1 \) and \( x_2 \) (e.g. [19]) and the balance relations on the jump that the definition of solution in the sense of integral identity implies ([16]) are satisfied as well. It is known that these conditions do not guarantee uniqueness ([19], [12]). However, our solution is unique both in the case of rarefaction and compression. In the case of rarefaction it is automatically self-similar (we recall that the assumption of the self-similarity implies uniqueness [29]). In the case of contraction the problem of uniqueness was open for the solution to the singular Riemann problem, where a non-zero mass is concentrated on the jump at the initial time. As was noticed in [12], for the uniqueness one must prescribe the derivative of the amplitude of the \( \delta \) - function. In our framework the solution is unique and the value of the derivative of the amplitude of the \( \delta \) - function follows from the expression for the amplitude itself.

2. In [17] an analog of the system (19), (20) (without the integral term) in the 1D case was obtained. Namely, it was proved that for smooth initial data a local in time strong solution \((\rho(t, x), u(t, x))\) to this system can be constructed by means of a nonlinear diffusion process

\[ X_t = X_0 + \int_0^t E[(u_0(X_0))|X_s] \, ds + \sigma W_t, \quad \mathcal{L}(X_0) = \rho(0, x), \]

so that \( \rho(t, x) \) is the probability density of the diffusion process \( X_t \) and \( u(t, x) = E[(u_0(X_0))|X_t = x] \) (with \( E[.,.|.] \) standing for conditional expectation and \( \mathcal{L}(X_0) \) for the probability density of \( X_0 \)). In fact, this result relates to our Proposition 2, since we have shown that the integral term arises only for discontinuous data.

Further, in [18] the system (19), (20) (without the integral term) was considered in any dimension. In this work there was constructed a global weak solution using discrete approximations, and the interaction of particles is given by a sticky particles dynamics.

3. There exist formalisms to represent solutions of parabolic PDE’s as the expected value of functionals of stochastic processes (see e.g. [22], [23], [3], [5] and references therein). In particular, in [11] one can find a recent result concerning the stochastic formulation of the viscous Burgers equation. An alternative approach to the stochastic formulation for a much more wider class of parabolic equations and systems can be found in [4].

4. We would also like to mention the paper [9], where a numerical method of particles for the solution of the pressureless gas dynamics in 1D an 2D case has been developed. The method is mostly inspired by [20] and in fact in this paper the problem of transition from the non-interacting particles to the sticky particle model was solved numerically.

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