Regular and Anomalous Quantum Diffusion in the Fibonacci Kicked Rotator.

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We study the dynamics of a quantum rotator kicked according to the almost-periodic Fibonacci sequence. A special numerical technique allows us to carry on this investigation for as many as $10^{12}$ kicks. It is shown that above a critical kick strength the excitation of the system is well described by regular diffusion, while below this border it becomes anomalous, and sub-diffusive. A law for the dependence of the exponent of anomalous sub-diffusion on the system parameters is established numerically. The analogy between these results and quantum diffusion in models of quasi-crystal and in the kicked Harper system is discussed.

I. INTRODUCTION

It can be reasonably argued that the quantum kicked rotator (1) is the oldest, and more thoroughly investigated system in the realm of the so-called quantum chaos. Yet, we pretend in this paper that recourse to this system is still justified, and can provide new dynamical phenomena worth of investigation. Consider in fact the problem of transport in quantum extended systems: for instance, a wave-packet is initially localized on a few states of a quantum lattice, and is then left free to evolve. A deep analysis carried out on model systems (2,3) with nearest neighbours interactions has shown that this dynamics is either localized—this is typically the case of random couplings, or, in the case of almost periodic couplings, anomalous (i.e. the diffusion coefficient is either zero or infinity—almost never finite) and intermittent (4,5).

It is well known that the kicked rotator for typical values of the parameters is akin to the first example just presented: its quantum dynamics is (eventually) localized in the action lattice. Therefore, it seems to feature a far simpler dynamics than the second class of systems mentioned above, or than its not-too-close relative, the quantum kicked Harper model (5), in which anomalous diffusion has been documented. A way to cure this simplicity is offered by an idea prof-

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two-dimensional lattice is diffusive, while, in the opposite case, spreading is characterized by anomalous sub-diffusion.

This paper is organized as follows: in the next section we define the model. In Sect. III we introduce growth indicators and their power-law exponents to gauge anomalous diffusion in the dynamics of an initially localized wave-packet. The numerical algorithm and the numerically computed long time dynamics of the system are presented in Sect. IV. Finally, we allow a random element in the definition of the dynamical system to verify the robustness of the obtained results, and to support the conjecture that anomalous diffusion in this model is induced by the almost-periodicity of the sequence of unitary operators.

II. THE DYNAMICAL SYSTEM

The basis of our investigation is the quantum kicked rotator \([1]\): this is the unitary evolution generated in the usual Hilbert space \(L_2(S_{2\pi})\) by the operators

\[
U(k, T) = e^{-ik \cos \theta} e^{-\frac{2 \pi}{\lambda} \hat{n}^2}
\]

which depend on the two parameters \(k, T\) and where the momentum operator is \(\hat{n} = -i \partial_\theta\). The corresponding classical system is the well known Chirikov standard map \([23]\), which is obtained letting \(k\) grow, and \(T\) go to zero, while keeping their product fixed. This product defines the classical chaos parameter, \(K = kT\). In the classical case the system becomes globally diffusive for \(K > 0.9716\).

Now, it is well known that for large values of \(K\), and typical irrational values of \(\frac{K}{T}\), classical and quantum dynamics of the kicked rotator are profoundly different: this is the content of the quantum phenomenon of dynamical localization, which has been well described in the literature, and which predicts that the quantum evolution of an initially focused wave-function is almost periodic in time, and localized on the momentum space of the unperturbed system at \(k = 0\) \([1]\).

In this paper, we study a variation on this theme: while in the usual approach the unitary evolution of eq. (1) is acted repeatedly, at equally spaced times \(t_n = nT\), in our model the length of the time intervals in between kicks will not be constant. Equivalently, we can say that the operators \(U\) act now in Fibonacci sequence to produce the full quantum evolution \(U\):

\[
U(t_n) = U(k, \tau_n) \circ U(k, \tau_{n-1}) \circ \ldots \circ U(k, \tau_1),
\]

where \(t_n = \sum_{j=1}^n \tau_j\) is time, and where \(\circ\) denotes operator composition. As is immediately observed, operators in eq. (2) are characterised by the same value of the quantum kick amplitude \(k\), and differ solely by the value of \(T\). The sequence of values of this latter is determined by letting \(\tau_j\) to take either one of two possible values, which for convenience we denote by \(A\) and \(B\). We choose for \(A\) and \(B\) the values

\[
A = \frac{2\pi}{\lambda}, \quad B = \frac{2\pi}{\lambda^2},
\]

where \(\lambda = 1.3247\ldots\) is an algebraic number, that solves the equation \(\lambda^3 - \lambda - 1 = 0\). In a sense, \(A, B\) is the most irrational couple of numbers.

We line up \(A\)'s and \(B\)'s according to the well known Fibonacci sequence: this is obtained by an accretion rule on the initial finite pieces of the infinite sequence, \(\sigma_1 = A\) and \(\sigma_2 = A, B\). According to this rule, the next finite piece is obtained by joining these two sequences, the first at the end of the second: \(\sigma_3 = A, B, A\). In general,

\[
\sigma_{n+1} = \sigma_n \cup \sigma_{n-1},
\]

where the symbol \(\cup\) means here joining the right and left sequences in the specified order, so that \(\sigma_1 = A, B, A, A, B\), and so on. It is then easy to see that the length of \(\sigma_n\) is equal to the \(n\)-th Fibonacci number, \(F_n\). Eq. (3) expresses a property that will be instrumental in the following to obtain the quantum evolution up to large times.

III. GROWTH EXPONENTS OF AN INITIALLY LOCALIZED WAVE-FUNCTION

We now let \(\psi(0)\) be the initial state of the evolution, and we compute the time-dynamics \(\psi(t_n) = U(t_n)\psi(0)\). Typically, we choose for initial state \(\psi(0)\) the usual \(\psi_0\) momentum basis state of \(L_2(S_{2\pi})\), the full basis being \(\psi_n = e^{i n \theta}\), with \(n \in \mathbb{Z}\).

As we have already remarked, the quantum kicked rotator, for typical irrational values of the kicking period \(T\), displays the phenomenon of quantum localization: action in the system eventually saturates in time, and quantum diffusion stops. On the other hand, for kick sequences with random time intervals after each kick, the increase in momentum is diffusive and never comes to an end \([24]\). Of particular interest is then to study the kicking sequence defined in the previous section, which is neither periodic, nor random: it belongs to the family of almost-periodic sequences. We shall see that under these circumstances dynamical localization leaves places to a delocalization which shares many characteristics with that taking place in almost-periodic lattice systems.

To study the dynamics of the quantum wave-function, it is convenient to introduce a number of
quantities suitable to define its spreading: we shall call them collectively growth indicators. The first obvious choice are moments. Let \( \nu_\alpha(t) \) be the moment of index \( \alpha \) of the probability distribution over the lattice \( \{e_n\} \) given by \( |(e_n, \psi(t))|^2 \), where obviously \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( L_2(S_{2\pi}) \):

\[
\nu_\alpha(t) := \sum_n |(e_n, \psi(t))|^2 |n|^\alpha.
\]

For instance, \( \nu_2 \) is the usual second moment—the dispersion—which, in the case of regular diffusion, grows linearly in time. In general, we cannot expect \( \nu_2 \) to behave as \( t \), but super-diffusive and sub-diffusive growth will be the rule. For this, we also define the growth exponent function \( \beta \) by the asymptotic relation

\[
\nu_\alpha(t) \sim t^{\alpha \beta(\alpha)} \quad \text{for} \quad t \to \infty,
\]

where the asymptotic behaviour is to be understood in a suitable sense. The function \( \beta(\alpha) \) is the so-called quantum intermittency function, described at length in [8–10].

When \( \alpha \) tends to zero, the moment \( \nu_\alpha \) tends to the constant \( \nu_0 = 1 \). Nonetheless, one can define the limit of the function \( \beta \) for \( \alpha \) tending to zero, \( \beta(0) \), a number which also appears in the scaling of the logarithmic moment:

\[
\sum_{n \neq 0} |(e_n, \psi(t))|^2 \log(|n|) \sim \beta(0) \log(t).
\]

In the quantities [8] and [9] the ordering of the basis is relevant: a function of \( n \) appears in the summations. We can nonetheless define and compute quantities which do not make reference to this characteristics. The first of these is the entropy:

\[
-\sum_n |(e_n, \psi(t))|^2 \log((|e_n, \psi(t)|)^2) \sim \gamma + \beta_e \log(t),
\]

whose asymptotic behaviour (here and in the following to be understood in the infinite time limit) defines the exponent \( \beta_e \). In the case of localization, \( \beta_e \) is null, and \( e^{\gamma} \) (the constant contribution in eq. [9]) gives a measure of the localization length. Another basis independent quantity is the inverse participation ratio,

\[
\sum_n |(e_n, \psi(t))|^4 \sim t^{-\beta_p},
\]

with its exponent \( \beta_p \).

We notice that, in principle, the exponents \( \beta \) of all the quantities so far defined are different. Yet, when finite, they are all consistent with the fact that the spreading of the wave-packet over the basis takes place in a power-law fashion. This is indeed the case: in fig. [3], we plot the \( t^2 \) power of a set of moments \( \nu_\alpha \) versus time, in doubly logarithmic scale, together with one over the inverse participation ratio, and with the exponential of the entropy and of the logarithmic moment. It is immediately appreciated that quantum diffusion in this case is regular: these curves are parallel to the reference line \( \sqrt{T} \), so that all growth exponents are equal to one half.

To render even more evident the fact that we are in the presence of regular diffusion we plot in fig. [4] a snapshot of the wave-packet at a large time, for the same case of fig. [1]: The fitting line is a normal distribution.

**IV. DETECTING ANOMALOUS DIFFUSION IN LARGE TIME DYNAMICS**

Let us now investigate the effect of varying the kick strength \( k \): in particular, let us decrease it from the value \( k = 10 \) at which we observed regular diffusion. Two difficulties arise when performing this analysis: since we observe a reduced delocalization of the wave-packet, very large time-scale computations are required to say something reliable about the asymptotic behaviour. Secondly, large systematic oscillations in the behaviour of the moments (and of the other quantities) tend to overwhelm and submerge the power-law increase, the more the slower this increase.

Let us first present the cure to the second difficulty: we can introduce a magnetic flux \( \alpha \) in the evolution \( e^{i\alpha} \), which is nothing else than replacing the operator \( \hat{n} + \phi \): \( \phi \) is a real number, so that for \( \phi = 0 \) we obtain the usual kicked system. By performing the evolution with a set of values of \( \phi \), and by averaging the results with respect to this sample, we can reduce the systematic fluctuations super-imposed to the leading power-law behaviour. Please observe that the classical dynamics is left invariant by the introduction of this flux.

For the first difficulty, a more complex strategy is required. We can push the evolution to extremely large times by resorting to a matrix algorithm. Let \( U_{F_n} \) be the unitary evolution operator acting from time zero up to the time of the kick number \( F_n \), the \( n \)-th Fibonacci number. Then, eq. [8] implies that these operators are simply related by

\[
U_{F_{n+1}} = U_{F_{n-1}} \circ U_{F_n},
\]

where \( \circ \) is the usual operator multiply, and where the order of operators is essential, as it was in eq. [3].
This relation must be initialised by setting
\[ U^{F_1} = U(k, A), \]
\[ U^{F_2} = U(k, B) \circ U(k, A), \]
so to produce the full sequence \( \{U^{F_n}\} \). In fact, the matrix elements of \( U(K, \tau) \) are explicitly known, and eq. (11) involves then only a matrix-to-matrix multiplication. Of course, the cost of this operation scales as the cube of the size of the matrix, and this is certainly more expensive than the usual evolution effected via a sequence of fast Fourier transforms. Nonetheless, this technique becomes advantageous when dealing with extremely large times, for in \( n \) steps one reaches up to \( F_n \) map iterations, and it is well known that Fibonacci numbers are geometrically increasing in \( n \).

We therefore effect the matrix multiplication for \( n = 1, \ldots, N \) and then look at the zeroth column of the resulting operator \( U^{F_N} \): this is the evolved of the initial state \( e_0 \). In our numerical experiments, the initial state of the evolution will always be \( e_0 \). Suitable combinations of the columns can provide the evolved of any arbitrary initial state, if desired.

A question of numerical concern is to be discussed prior to showing the results of these calculations: when dealing with equations (11) a finite truncation of the operators involved is necessary. Now, for any practical purpose \( U(k, \tau) \) is a banded matrix, and we must obviously choose the size of the numerical truncation much larger than the band size. Yet this is not enough: appropriate boundary conditions at the basis edges must be imposed. Two choices are at hand: Dirichlet, and Neumann. In the latter case, we are producing the unitary evolution on the torus, while in the first case we end up with a non-unitary evolution.

Having in mind what we want to obtain (i.e. the evolution of the \( e_0 \) basis state in the full cylinder space) we have adopted Dirichlet conditions, for two reasons. First, when the truncation size is sufficiently large, both Dirichlet and Neumann conditions must produce an excellent approximation of the true \( U^{F_N} e_0 \). Secondly, we can gauge numerically two quantities: the normalization of the zeroth column of \( U^{F_N} \) (which is a common technique) and its effective dimension—which is less common, but more instructive. In fact, this latter was defined above as the number of components of \( U^{F_N} e_0 \) of magnitude larger than a certain threshold: this quantifies the dimension of the wave-packet after \( F_N \) kicks, and can be profitably used to control the optimal size of the truncation. It must be added that the effective dimension in itself has physical and mathematical relevance [1].

In summary, we have been able to reach easily about sixty iterations of the Fibonacci multiplication, eq. (11), which correspond to more than \( 10^{12} \) usual iterations, and this with a basis size of the order of the thousand. Numerical stability, controlled by various techniques, is here the limiting factor.

A set of typical results is shown in fig. 3, where we plot the second moment, \( \nu_2 \), as a function of time, in doubly logarithmic scale, for a set of different values of \( k \). We clearly see that these values decrease with the latter, but, more important, the slopes of the curves (i.e. the growth exponent \( \beta(2) \)) also decrease with \( k \). As a consequence, we can safely conclude that quantum diffusion is anomalous in this model for a wide set of parameters [2].

At this point, a word must be spent about intermittency: this is present when the function \( \beta(\alpha) \) (see eq. (11)) is not constant. In the case at hand, the variation of \( \beta \) with \( \alpha \) is smaller than the uncertainty with which \( \beta \) is determined, both due to numerical effects, and to the super-imposed oscillations of the moments \( \nu_n \). We can therefore only conclude that intermittency, if present, is low.

This fact is not totally negative: by averaging over \( \alpha \), we can define a more reliable growth exponent \( \beta_{av} \) which is now function only of the kick amplitude \( k \). This immediately prompts for the study of this dependence. Fig. 4 plots the numerical results obtained by the procedure exposed above. Two regions clearly emerge from the investigation of this picture: for large values of \( k \) we observe the regular diffusive value \( \beta_{av} = \frac{1}{2} \). Anomalous diffusion is observed for smaller values of \( k \): quite interestingly, we find a power-law behaviour of the growth exponent of the form
\[ \beta_{av}(k) \sim k^\eta, \]
with \( \eta \) very close to the value \( \frac{2}{3} \). The transition from anomalous to regular diffusion takes place for \( k > k_c \); for the set of values we have chosen, this critical value \( k_c \) is approximately equal to two.

Let us now investigate the origin of anomalous diffusion in this model. When acting on the basis set \( e_n = e^{i n \theta} \) the free evolution \( e^{-i \frac{T}{2} n^2} \) produces the phase factor \( e^{-i \frac{2}{3} n^2} \). The arithmetic nature of \( T/2 \pi \) is at the root of the spectral properties of the usual kicked rotator. The numerical studies presented so far have been carried on for the most irrational pair \( A, B \). We have found similar results for other irrationally related pairs. We are so lead to conclude that the nature of the observed unbounded diffusion lies in the almost-periodic arrangement of in-kicks intervals \( A \) and \( B \).

To substantiate this hypothesis we have replaced the phases \( \frac{2}{3} n^2 \) by two sequences of equally distributed pseudo-random numbers, one for the operator \( U(k, A) \) and one for its companion \( U(k, B) \). In so doing, we have obtained quite similar results as
In this way the usual kicked rotator (a solid-state hopping model, similar to the Anderson model, with hopping only to a finite number of nearby sites. On the contrary, the almost-periodic sequence of unitary operators of the Fibonacci kicked rotator renders the situation much richer, and gives rise to a transition from regular to anomalous diffusion.

As a matter of facts, the dynamics of the Fibonacci kicked rotator seems more similar to a wave spreading on a two-dimensional quasi-crystal lattice. Indeed, studies of quantum diffusion over an octagonal quasiperiodic tiling have shown a similar transition from anomalous to regular diffusion. However, in spite of this first glance similarity more detailed studies are required to establish a quantitative relation between these models, and to gain a better theoretical understanding of the results presented in this paper.

V. CONCLUSIONS

We have studied the dynamics of a quantum rotator kicked at times generated by the almost periodic Fibonacci sequence. Contrary to that of the usual kicked rotator, this dynamics never shows quantum localization. We have introduced and computed various indicators of the spreading of an initially localized wave packet, which have permitted to show that the dynamics features regular diffusion for large values of the kick amplitude (k > k_c) and anomalous sub-diffusion for small values (k < k_c). In this latter range, the exponent of this diffusion appears to follow a power-law behaviour with the kick amplitude, \( \beta_{av} \propto k^2 \). This relation, although only numerically established, is quite interesting, and deserves in our opinion further investigation.

Similarities and differences between the Fibonacci kicked rotator and other quasi-periodic models are to be noted. We have already observed that anomalous diffusion is typically found in almost periodic, one-dimensional lattice systems. It also appears in the kicked Harper model. To the contrary, the Fibonacci kicked rotator is qualitatively different from the rotator acted upon by equally spaced kicks, with an amplitude k which is a quasi-periodic function of time. The case when this function contains m incommensurate frequencies has been studied in \([26,28]\). It was shown that this model is a dynamical analog of the Anderson localization in a space of effective dimension \( d = m + 1 \). In this way the usual kicked rotator (\( m = 0 \)) corresponds to \( d = 1 \), and--of course--is always localized. For \( m = 1 \) the excitation is still always localized, but the localization length grows exponentially with \( k \). In analogy with the Anderson localization in \( d = 2 \). Finally, for \( m > 1 \) i.e. \( d > 2 \), a transition from localization to diffusive excitation takes place above some critical kick amplitude, in analogy with the Anderson transition for \( d > 2 \). As we see this behaviour drastically differs from the one we find in this paper for the Fibonacci kicked rotator.

In our opinion this difference might be due to the fact that in \([26,28]\) the kick amplitude k is an analytic function of incommensurate phases (frequencies). For this, such a model can be mapped into an effective solid-state hopping model, similar to the Anderson model, with hopping only to a finite number of nearby sites. On the contrary, the almost-periodic sequence of unitary operators of the Fibonacci kicked rotator renders the situation much richer, and gives rise to a transition from regular to anomalous diffusion.

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\[ Of course, proper diffusive (chaotic) behaviour is not possible in quantum mechanics: we use the term quantum diffusion to signify the stable and time reversible power-law increase of the moments of the position operator.} \]
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Of course, numerical experiments are not a proof that anomalous diffusion holds on indefinitely. Yet, the large time-scale over which this phenomenon is observed in fig. 3 renders this conclusion plausible.

FIG. 1. Regular quantum diffusion. Growth indicators are plotted versus time given in number of kicks (in abscissa) for the case $k = 10$. The lowest curve (dots-dashes) is the reference line $\sqrt{t}$. The other curves are (from bottom to top): the exponential of the entropy (long dashes), the square root of the second moment (short dashes), the sixth root of the sixth moment (dots), one over the inverse participation ratio (dots-dashes), and the exponential of the logarithmic moment (continuous line).

FIG. 2. Wave packet for the case of fig. 1. Plotted are occupation probabilities $p_n = |\langle e_n, \psi(t) \rangle|^2$ versus site number $n$ at time $t = 1.797 \times 10^4$ (thin line). The thick line is an interpolating normal distribution.
FIG. 3. Second moment $\nu_2$ as a function of time, for $k = 0.125, 0.25, 0.5, 1, 2, 4$. (bottom to top diamond curves). Also shown are the fitting lines from which the exponent $\beta(2)$ is extracted.

FIG. 4. Growth exponent $\beta_{av}$ (crosses) obtained as the average of $\beta(\alpha)$, for $\alpha = 2, 4, 6$, versus the kick amplitude $k$. The horizontal line marks the value $\beta_{av} = \frac{1}{2}$; the fitting line for small values of $k$ grows proportional to $k^\eta$, with $\eta = \frac{2}{3}$. Also reported (diamonds) are the result obtained when random rotation phases replace the irrational phases $Tn^2/2$ (see text).