A Mathematica™ Package for Computing $N = 2$ Superfield Operator Product Expansions

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Abstract

We describe a general purpose Mathematica™ package for computing Superfield Operator Product Expansions in meromorphic $N = 2$ superconformal field theory. Given the SOPEs for a set of “basic” superfields, SOPEs of arbitrarily complicated composites can be computed automatically. Normal ordered products are always reduced to a standard form. It is possible to check the Jacobi identities, and to compute Poisson brackets (“classical SOPEs”). We present two explicit examples: a construction of the “small” $N = 4$ superconformal algebra in terms of $N = 2$ superfields, and a realisation of the $N = 2$ superconformal algebra in terms of chiral and antichiral fermionic superfields.

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1 Introduction

It is well known that Operator Product Expansions (OPEs) form a powerful tool in two–dimensional Conformal Field Theory (CFT). Once the OPEs for the currents are known, it is possible to compute their correlation functions algebraically. One also derives differential equations for the other correlation functions in the theory [1]. In gauge theories, one can compute the anomaly in the BRST nilpotency, find the spectrum, etc. Although the calculations are in principal straightforward. In practice, they can become quite cumbersome to do by hand. In the case of a meromorphic CFT, these calculations can be performed with the OPEdefs package [3, 4], implemented in Mathematica\textsuperscript{TM} [1], an interactive environment for performing symbolic computations.

In the case of supersymmetric conformal theories, it is often useful to work in a superspace formalism. This drastically reduces the number of fields one needs to consider, and also the amount of algebraic manipulations. For example, a general $N = 2$ superfield contains 4 components fields (two bosonic and two fermionic), therefore one SOPE for the $N = 2$ superfield defines 10 OPEs for the components. Moreover, the checking of one Jacobi identity in terms of $N = 2$ superfields is equivalent to 64 identities in terms of components.

Unfortunately, although the formulas to deal with super OPEs are similar to the ones for ordinary OPEs, they are more complicated and calculations are more error–prone. This paper describes a Mathematica package that enables one to perform computations automatically in the case of $N = 2$ supersymmetry.

Of course, the simplest supersymmetric extension of the conformal field theories possesses $N = 1$ supersymmetry. But an $N = 1$ superfield contains only one additional fermionic component with respect to the bosonic case and so it is possible to use the OPEdefs package. On the other hand, when dealing with higher supersymmetry $N \geq 3$, most of the superfields are constrained, but they can be nicely represented in terms of $N = 2$ ones. In the case of $N = 2$, the constraints (chirality or antichirality) can be easily taken into account in SOPEN2defs. This is why we decided to limit ourselves for the time being to the case of $N = 2$ superfields.

The package is of a similar nature to OPEdefs. One needs to declare the superfields which are used, and give their respective SOPEs. Nonlinear algebras can be handled by using a point–splitting definition for composites. It is then possible to check the Jacobi identities for the SOPEs. If one introduced arbitrary constants in the SOPEs, it is possible in this way to find the values for which the algebra is associative, and hence to construct a super $W$–algebra. In this way, the superfield version of $N = 2$ $W_3^{(2)}$ was constructed in [5]. Once the basic SOPEs are given, it is in principle possible to compute any OPE (or a particular pole) of any composite. Of course, this

\[1\] Mathematica is a trademark of Wolfram Research Inc. For details, see [4].
is restricted by the available memory and CPU–time. The package is able to compute SOPEs, but also Poisson brackets (“classical” SOPEs). It has been tested by a number of people, already resulting in many publications.

The package can be extended to perform calculations using dummy indices. This is still a limited facility. Not all possible simplifications are performed when tensors appear which have symmetries when interchanging indices. However, for many cases this possibility is quite useful.

This paper is setup as follows. In a first section we explain our conventions for $N = 2$ superspace. Then we derive the formulas for SOPEs. The next section consists of a user’s guide for the package. We end with two explicit examples.

**Notation**

Input for and output from *Mathematica* is written in *typeset* font. Input lines are preceded by “In[n] :=”, and corresponding output statements by “Out[n] =”, as in *Mathematica*.

**2 $N = 2$ superspace. Notations and conventions**

In this section we will introduce our conventions concerning the $N = 2$ superfield formalism.

The $N = 2$ superspace $Z = \{z, \theta, \bar{\theta}\}$ can be described by one real bosonic coordinate $z$ and pair of two conjugate Grassman coordinates $\theta, \bar{\theta}$:

$$\theta^2 = \bar{\theta}^2 = 0, \quad \theta \bar{\theta} = -\bar{\theta} \theta.$$

(1)

The integration measure in $N = 2$ superspace is $dZ \equiv dz d\theta d\bar{\theta}$ and the standard convention for the integration over $\theta, \bar{\theta}$ is assumed.

To deal with $N = 2$ superfields $\Phi(Z)$, it is useful to introduce chiral and antichiral spinor derivatives $\mathcal{D}, \mathcal{\overline{D}}$

$$\mathcal{D} = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \mathcal{\overline{D}} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z},$$

(2)

which obey the following relations

$$\mathcal{D}^2 = \mathcal{\overline{D}}^2 = 0, \quad \{\mathcal{D}, \mathcal{\overline{D}}\} = -\partial_z.$$

(3)

Now we can define chiral $F$ and antichiral $\bar{F}$ superfields which are subjected to the following constraints

$$\mathcal{D}F = \mathcal{\overline{D}}\bar{F} = 0,$$

(4)

respectively.
It is also instructive to write down the Taylor expansion for a general superfield $\Phi(Z)$

$$
\Phi(Z_1) = \sum_{n \geq 0} \frac{Z_1^n}{n!} \partial^n \left\{ 1 + \theta_{12} D_2 + \bar{\theta}_{12} \bar{D}_2 - \theta_{12} \bar{\theta}_{12} \frac{1}{2} \left[ D_2, \bar{D}_2 \right] \right\} \Phi(Z_2),
$$

(5)

where

$$
\theta_{12} = \theta_1 - \theta_2, \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2, \quad Z_{12} = z_1 - z_2 + \frac{1}{2} (\theta \bar{\theta} - \bar{\theta} \theta) .
$$

(6)

We give the SOPE for the simplest case of $N = 2$ superconformal algebra (SCA):

$$
J(Z_1)J(Z_2) = \frac{c/4}{Z_{12}^2} + \frac{\theta_{12} \bar{\theta}_{12} J(Z_2)}{Z_{12}^2} + \frac{\bar{\theta}_{12} \bar{D}_2 J(Z_2)}{Z_{12}} - \frac{\theta_{12} D_2 J(Z_2)}{Z_{12}} + \frac{\theta_{12} \bar{\theta}_{12} \partial J(Z_2)}{Z_{12}},
$$

(7)

where $J(Z)$ is a general bosonic superfield.\[3]

### 3 Formulas for SOPEs

In principle, the formulas to work with SOPEs follow by expanding everything in components, applying the standard OPE formulas, and reassembling again into superfields. However, this approach is impractical. It is easier (and conceptually clearer) to construct the formulas by using Taylor expansions, contour integration . . . in superspace. We will discuss SOPEs here and briefly comment on which changes are needed for the Poisson bracket case.

In fact, it will be useful to introduce a notation valid for any number of supersymmetries $N$. For a list of numbers $[i]$, we will write $\theta^{[i]}$ for $\theta^{i_1} \theta^{i_2} \ldots$ and $\theta^{[N]}$ is used for $\theta^{i_1} \ldots \theta^{i_N}$. The “complement” $N - [i]$ is defined such that $\theta^{[N]} = \theta^{N - [i]} \theta^{[i]}$.

We also introduce projectors $P$ to extract a component of superfield or a SOPE. $P_{\emptyset}$ projects on the term without $\theta$’s, $P_{[i]}$ recovers the term proportional to $\theta^1$ and so on:

$$
P_{[i]} = \int d\theta^{[N]} \theta^{N - [i]}
$$

(8)

We will use the following notation for SOPEs:

$$
A(Z_1)B(Z_2) = \sum_{n \leq h(A,B)} \frac{[AB]_n(Z_2, \theta_{12}^{[i]})}{(Z_{12})^n},
$$

(9)

where $h(A, B)$ is some finite number, and is usually given in terms of the conformal dimensions of $A$ and $B$ as $h(A, B) = [d_A + d_B + N/2]$. Note that
with this definition (9), \([AB]_n\) depends on the \(\theta_{12}^i\), but not on \(z_1\). The term in \([AB]_0\) which has no \(\theta_{12}^i\) dependence is called the normal ordered product of \(A\) and \(B\):

\[
:AB: = \mathcal{P}_0[AB]_0.
\] (10)

With the definition (9), we have:

\[
\mathcal{P}_i[AB]_n(Z_2, \theta_{12}^i) = \oint_{C_2} \frac{dz_1}{2\pi i} \int d\theta_1 \theta_1^{N-i}(Z_2)_{n-1} A(Z_1)B(Z_2),
\] (11)

where the contour \(C_2\) encircles \(z_2\). If we replace the SOPE of \(A\) and \(B\) by the Poisson bracket \(\{A(Z_1), B(Z_2)\}_{PB}\), we adopt the above formula as the definition for \([AB]_n\) with \(n > 0\) in the Poisson brackets ("classical" SOPEs).

**Formulas**

With all these definitions, we can start to derive how we have to compute with SOPEs. First, there are some formulas for computing an SOPE of two superfields when one of them is a derivative. These follow simply by taking derivatives of the SOPE. We give only one example (in \(N = 2\)):

\[
[DA B]_n(Z_2, \theta_{12}^i) = \frac{n-1}{2} \bar{\theta}_{12} [A B]_{n-1}(Z_2, \theta_{12}^i) + \mathcal{D}_1 [A B]_n(Z_2, \theta_{12}^i),
\] (12)

where the subindex on \(\mathcal{D}_1\) indicates it acts on coordinates \(Z_1\). This implies

\[
\mathcal{P}_\theta[A B]_n = \mathcal{P}_0[DA B]_n.
\] (13)

For a term in the regular part of an OPE, we find:

\[
[A B]_{-n} = \frac{1}{n!} \theta_0 A B_0.
\] (14)

The SOPE \(B(Z_1)A(Z_2)\) is by analytic continuation equal to \(A(Z_2)B(Z_1)\). It remains then to apply a Taylor expansion (9) to the fields in the latter. The result is straightforward. We list only the special case for normal ordering, where drastic simplifications occurs because of the projection \(\mathcal{P}_\theta\) in (10):

\[
:BA: = (-1)^{|A||B|} \cdot AB + (-1)^{|A||B|} \sum_{l \geq 1} \frac{(-1)^l}{l!} \partial^l \mathcal{P}_\theta[AB]_l
\] (15)

Applying this formula for \(A = B\) a fermionic superfield, one gets an expression for \(AA\): in terms of the poles in the SOPE \(A(Z_1)A(Z_2)\).

Associativity of the SOPE means that SOPEs can be computed in any order (inside correlation functions). This can be used to derive the Jacobi identities. As a first step, we select a particular term in a double SOPE by using contour integrals:

\[
\mathcal{P}_i[AP_j][BC]_q(Z_3) = \oint_{C_3} \frac{dZ_1}{2\pi i} \theta_1^{N-i} Z_1^{q-1} \oint_{C_3} \frac{dZ_2}{2\pi i} \theta_2^{N-j} Z_2^{p-1} A(Z_1)B(Z_2)C(Z_3),
\] (16)
where $C_2$ denotes a contour which encircles $z_2$ once anti-clockwise. We can now use a contour deformation argument relating the contour integral in eq. (16) to a contour integral where the integration over $Z_2$ is performed last. In fact, the contour deformation works only for the $\theta$–independent part of the integral. For the Berezin integral over $\theta$, we just interchange the order of integration (taking signs into account). The resulting integral has two terms: one where the $z_1$ contour is around $z_3$, and one where it is around $z_2$. We find:

$$P_{[i]}[A \mathcal{P}_{[j]}[BC]_{[p]}q = (-1)^{|A|+|i|+|j|}P_{[i]}[B \mathcal{P}_{[j]}[AC]_{[q]}p +
\sum_{l>0} \left( \frac{q-1}{l-1} \right) P_{[j]}[\mathcal{P}_{[i]}[AB]_l C]_{p+q-l}. \quad (17)$$

This equation has to hold (inside correlators) for consistency of the OPE-formalism. It is valid for any integers $p, q$, i.e. also negative numbers. However, in practice we use the Jacobi identities eq. (17) for positive $p, q$ as equations for the singular part of the SOPEs, while for $p$ or $q$ zero, they define how one should calculate with composites. For example, to compute an SOPE with a composite, we insert $p = 0, [i] = \emptyset$ into eq. (17). When $q = 0, [i] = \emptyset$ as well, we find:

$$:A :BC:: = (-1)^{|A||B|} :B :AC:: + :\left( :AB: - (-1)^{|A||B|} :BA: \right) C: , \quad (18)$$

where we used eq. (15).

For Poisson brackets, the normal ordered product of two operators is simply replaced by (noncommutative) multiplication. In particular, this means that double contractions ($l < q$) in eq. (17) for $p = 0, [i] = \emptyset$ drop out. However, the Jacobi identities eq. (17) for positive $p, q$ remain exactly the same.

**Improvements**

The rules given in this section up to now are sufficient to compute any SOPE, and to reorder any composite into a standard form. For example, to compute an SOPE where the first operator is a composite, we could use the fact that $A(Z_1)B(Z_2) = B(Z_2)A(Z_1)$ and (17). However, it is possible to construct a rule by using contour integrals to do this in one step. We find:

$$[\mathcal{P}_0[AB]_{[0]} C]_q = \sum_{l \geq 0} \frac{1}{l!} \partial^l A [BC]_{l+q}_0 + (-1)^{|A||B|} \sum_{l \geq 0} \frac{1}{l!} \partial^l B [AC]_{l+q}_0$$

$$+ (-1)^{|A||B|} \sum_{l=1}^{q-1} [B [AC]_{q-l}]_{l} , \quad (19)$$

where $q \geq 1$ and we used eq. (14). To have the formula in this simple form, we used the convention that any terms with $\theta_{12}, \bar{\theta}_{12}$ move to the left.
of the OPE bracket. For instance, if \([BC]_m = \theta_{12}D\), then 
\([A[BC]_m]_n = (-1)^{|A|\theta_{12}}[AD]_n\).

For composites, we find:

\[
\therefore AB \colon C: = :A :BC: + \sum_{l>0} \frac{1}{l!} \partial^l A \mathcal{P}_l [BC]_l : + (-1)^{|A||B|} \sum_{l>0} \frac{1}{l!} :\partial^l B \mathcal{P}_l [AC]_l : \tag{20}
\]

4 User’s Guide

This section is intended as a user’s guide to the package \textit{SOPEN2defs}. Explicit examples are given for most operations. \textit{SOPEN2defs} is based on \textit{OPEdefs} 3.1 \cite{4} and most commands are exactly the same. \textit{SOPEN2defs} requires \textit{Mathematica} 2.0 or later.

As \textit{SOPEN2defs} is implemented as a \textit{Mathematica} package, it has to be loaded \textit{before} any of its global symbols are used. Loading the package a second time will clear all previous definitions of operators and OPEs, as well as all stored intermediate results. Assuming that the package is located in the \textit{Mathematica}-path, \textit{e.g.} in your current directory\textsuperscript{3}, issue:

\texttt{In[1] := <<SOPEN2defs.m}

After loading \textit{SOPEN2defs} into \textit{Mathematica}, help for all the global symbols is provided using the standard help-mechanism, \textit{e.g.} \texttt{?OPE}.

Now, you need to declare the operators that will be used. If you want to define bosonic (resp. fermionic) operators, use \texttt{Bosonic} (resp. \texttt{Fermionic}). For chiral operators, prefix with \texttt{Chiral}, for antichiral ones, use \texttt{AChiral}. For instance, to define a bosonic field \(J\) and an antichiral bosonic field \(W\), the statements are:

\texttt{In[2] := Bosonic[J]}
\texttt{In[3] := AChiralBosonic[W]}

For derivatives we use the following notation:

\[
\partial_z J \rightarrow J' , \quad \mathcal{D} J \rightarrow \mathcal{D}T[J] , \quad \mathcal{D} \mathcal{J} \rightarrow \mathcal{D}BT[J] , \quad [\mathcal{D}, \mathcal{D}]J \rightarrow \mathcal{D}DB[J] \tag{21}
\]

The order of the declarations fixes also the ordering of operators used by the program:

\[
J' < \mathcal{D}BT[J] < \mathcal{D}DB[J] < \mathcal{D}T[J] < J < W \tag{22}
\]

By default, spatial derivatives of an operator are considered “smaller” than the operator itself. This can be reversed using the global option \texttt{NOOrdering} (see below).

Finally, the nonregular OPEs between the basic operators have to be given by listing the operators that occur at the poles, the first operator in

\textsuperscript{2}Use the \textit{Mathematica} command \texttt{SetDirectory} for this.
the list is the one at the highest non-zero pole, the last operator has to be
the one at the first order pole. For example, to give \( J \) the OPE (7), and \( \mathcal{W} \)
of a conformal dimension 1 primary antichiral superfield with \( U(1) \)-charge
\(-2\):

\[
\text{In}[4] := \text{OPE}[J,J] = \text{MakeOPE}[\{c/4 \text{ One} + t12**tb12**J,}
\quad
\text{tb12***DBT}[J] - t12**DT[J] + t12**tb12**J']\}];
\]

\[
\text{In}[5] := \text{OPE}[J,\mathcal{W}] = \text{MakeOPE}[\{t12**tb12**\mathcal{W},}
\quad
-2 \mathcal{W} - t12**DT[\mathcal{W}] + t12**tb12**\mathcal{W}'\}];
\]

Note the operator \( \text{One} \) which specifies the unit-operator. The symbols \( t12 \),
\( \text{tb12} \) denote \( \theta_{12}, \bar{\theta}_{12} \). In an OPE, they should always be multiplied with an
operator using **. In particular, one could have \( t12**\text{One} \).

**Warning:** it is important that the operators occurring as arguments of OPE in a *definition* should be given in the order the operators are declared \([22]\), otherwise wrong results will be generated.

A normal ordered product : \( AB : \) \([10]\) is entered in the form \( \text{NO}[A,B] \). Multiple composites can be entered using only one \( \text{NO} \) head, e.g. \( \text{NO}[A,B,C] \). This input is effectively translated into \( \text{NO}[A, \text{NO}[B, C]] \). All output is
normal ordered with the same convention, i.e. from right to left (input can
be in any order). Also, the operators in composites will always be ordered
according to the standard order \([22]\).

\[
\text{In}[6] := \text{OPE}[J,\text{NO}[J,J]]
\]

\[
\text{Out}[6] = \ll 4|| c t12**tb12**\text{One}/4 ||3|| 0 ||2|| c J/2 +
\quad
\text{tb12***DBT}[J] + t12**DT[J] + 2 t12**tb12**\text{NO}[J, J]
\quad
||1|| \text{tb12***DBT}[J'] + 2 \text{tb12**NO}[J, \text{DBT}[J]] +
\quad
\text{t12**DT}[J'] - 2 t12**\text{NO}[J, \text{DT}[J]] +
\quad
2 t12**tb12**\text{NO}[J', J] >>
\]

This also shows what the output for an OPE looks like. For an OPE with
\( n \) poles, the output is \( \ll n|| \ldots ||n-1|| \ldots \ldots ||1|| \ldots > >. \)

**Warning:** when computing OPEs with composites, or when reordering composites, \( \text{SOPEN2defs} \) remembers by default some intermediate results. Thus, it is dangerous to change the definition of the basic OPEs after some calculations have been performed. For example, consider a constant \( a \) in an OPE. If calculations are performed after assigning a value to \( a \), the intermediate results are stored with this value. Changing \( a \) afterwards will give wrong results.

The other globally defined functions available from the package are:

- \( \text{OPEOperator[operator\_, parity\_]} \) provides a more general way to declare an operator than \( \text{Bosonic} \) and \( \text{Fermionic} \). The second argu-
ment is the parity of the operator such that \((-1)^{\text{parity}}\) is +1 for a boson, and -1 for a fermion. It can be a symbolic constant. This can be used to declare a \(bc\)-system of unspecified parity. In such cases, the operator can contain a named pattern:

\[
\text{In}[7]:= \quad \text{OPEOperator}[J[i_], \text{parity}[i]]
\]

If one wants to declare more operators, one can group each operator and its parity in a list:

\[
\text{In}[8]:= \quad \text{OPEOperator}[\{b[i_], \text{parity}[i]\}, \{c[i_], \text{parity}[i]\}]
\]

See also \text{SetOPEOptions}[ParityMethod, _].

- \text{ChiralQ}[\text{operator}_] tests if \text{operator} is chiral. One can use \text{ChiralQ}[A] = \text{True}.

- \text{AChiralQ}[\text{operator}_] is the same for antichiral operators.

- \text{OPEPole}[n_][\text{ope}_] gets a single pole of an OPE:
  \[
  \text{In}[9]:= \quad \text{OPEPole}[2][\text{Out}[6]]
  \]

\text{Out}[9] = c J/2 + tb12**\text{DBT}[J] + t12**\text{DT}[J] + 2 t12**tb12**\text{NO}[J, J]

\text{OPEPole}[n_][A_, B_] can also be used to compute only one pole term of an OPE:

\[
\text{In}[10]:= \quad % - \text{OPEPole}[2][J, J]/\text{Expand}
\]

\text{Out}[10] = 0

\text{OPEPole} can also give terms in the regular part of the OPE:

\[
\text{In}[11]:= \quad \text{OPEPole}[-1][J, J]
\]

\text{Out}[11] = \text{NO}[J', J] + tb12**\text{NO}[\text{DBT}[J'], J] + t12**\text{NO}[\text{DT}[J'], J] - t12**tb12**\text{NO}[\text{DDB}[J'], J]/2

- \text{MaxPole}[\text{ope}_] gives the order of the highest pole in the OPE.

- \text{OPEParity}[A_] returns an even (odd) integer if \(A\) is bosonic (fermionic).

- \text{OPESimplify}[\text{ope}_, \text{Function} \to \text{function}_] “collects” all terms in \text{ope} with the same operator and applies \text{function} on the coefficients. The default setting for the option \text{Function} is \text{Expand}, but this can be changed using \text{SetOptions}.

\text{OPESimplify}[\text{pole}_, \text{Function} \to \text{function}_] does the same simplifications on sums of operators.

The alternative syntax \text{OPESimplify}[\text{ope}_, \text{function}_] is also allowed.

- \text{OPEMap}[\text{function}_, \text{ope}_] maps \text{function} to all poles of \text{ope}.

- \text{OPEMapAt}[\text{function}_, \text{ope}_, \text{position}_] maps \text{function} to a part of \text{ope} specified by \text{position} (see \text{MapAt}). For example,

\[
\text{OPEMapAt}[\text{Expand}, \text{ope}, \{1\}]
\]

maps \text{Expand} on the highest order pole of \text{ope}.
• GetCoefficients[expr_] returns a list of all coefficients of operators in expr which can be OPEs or poles. Applied on a list, it maps on the elements of the list.

• GetOperators[expr_] returns a list of all operators in expr which can be OPEs or poles. Applied on a list, it maps on the elements of the list.

• OPEJacobi[op1_,op2_,op3_] computes the Jacobi-identities \([17]\) for the singular part of the OPEs of the three arguments. In general, all different orderings should be tried to ensure associativity. OPEJacobi returns a list of which all should be zero up to null fields to be associative. OPEJacobi accepts an option Function -> function which it passes to OPESimplify at intermediate steps.

• Delta[i_,j_] is the Kronecker delta symbol \(\delta_{ij}\).

• Epsilon[i_,j__] is the antisymmetric symbol (with any number of arguments).

• N2OPEToComponents[J_] gives the \(N = 0\) components of a superfield \(J\). For \(J = A + \theta B + \bar{\theta} C + \theta \bar{\theta} D\) the components are \(\{A, B, C, D\}\). In SOPEN2defs, the components are represented as

\[
\{J, D J, \bar{D} J, -\frac{1}{2}[D, \bar{D}] J\},
\]

where a projection on the \(\theta\) independent part is understood.

N2OPEToComponents[ope_,J1_,J2_] computes the 16 OPEs of the components of \(J_1\) and \(J_2\). It is a double list where the \((m,n)\)-th element is the \(N = 0\) OPE of the \(m\)-th component of \(J_1\) with the \(n\)-th component of \(J_2\).

• ClearOPESavedValues[] clears all stored intermediate results, but not the definition of the operators and their OPEs. To clear everything, reload the package.

• TeXForm[ope_] gives TeX output for an OPE. The arguments are always \(Z_1, Z_2\).

• TeXFormTD can be assigned a string which will be used by tt TeXForm for the output of DT, DBT, DDB.

• OPESave[filename_] (with filename a string between double quotes) saves the intermediate results that OPEdfs remembers to file (see the option OPESaving below).

• SetOPEOptions is a function to set the global options of the package. The current options are:
- `SetOPEOptions[NOOrdering, n_]`: if n is negative, order higher spatial derivatives to the left (default), if n is positive, order them to the right.

- `SetOPEOptions[ParityMethod, 0|1]`: makes it possible to use operators of an unspecified parity. When the second argument is 0 (default), all operators have to be declared to be bosonic or fermionic. When the argument is 1, `OPEOperator` can be used with a symbolic parity. Note that in this case, powers of \(-1\) are used to compute signs, which is slightly slower than the boolean function which is used by the first method. This option is not normally needed as the use of `OPEOperator` with a non-integer second argument sets this option automatically.

- `SetOPEOptions[OPESaving, boolean_]`: if boolean evaluates to True (default), `OPEdefs` stores the intermediate results when computing OPEs of composites and when reordering composites. This option is useful if Mathematica runs short of memory in a large calculation, or when computing with dummy indices (see section 5.2).

- `SetOPEOptions[OPEMethod, method_]`: with method set to `QuantumOPEs` enables normal OPE computations (default setting), while `ClassicalOPEs` enables Poisson bracket computations. Using this option implicitly calls the function `ClearOPESavedValues[]`.

- `SetOPEOptions[EnableDummies, True]`: loads the `Dummies` package and sets it up.

5 Examples

5.1 The $N = 4$ superconformal algebra

As a first example of using the `SOPEdefs` package, let us construct the OPEs of $N = 4$ $SU(2)$ SCA in terms of $N = 2$ superfields. The $N = 4$ $SU(2)$ SCA contains the following component currents: the stress-tensor $T(z)$, affine $su(2)$ currents $J^i(z), i = 1, 2, 3$ and two doublets of fermionic spin 3/2 currents $G_\alpha, \bar{G}_\beta, \alpha, \beta = 1, 2$ which form respectively the fundamental and conjugated representations of $su(2)$. Due to the spin structure, the simplest way to put these component currents into $N = 2$ supermultiplets is to start with a general spin 1 $N = 2$ superfield $J(Z)$ which forms the $N = 2$ SCA (spin contents is $\{1, 3/2, 3/2, 2\}$) together with two bosonic spin 1 chiral-antichiral superfields $\bar{W}(Z), W(Z)$, $\bar{D}W = DW = 0$ (spin content $\{1, 3/2, 1, 3/2\}$).

First of all we need to define all superfields we are dealing with:

```
In[1] := <<SOPEN2defs.m
In[2] := Bosonic[J]
```
As the next step we define the OPE between all the super currents:

\begin{verbatim}
In[5] := OPE[J,J] = MakeOPE[{c/4 One + t12**tb12**J,
   tb12**DBT[J]-t12**DT[J]+t12**tb12**J'}];
In[6] := OPE[J,W] = MakeOPE[{t12**tb12**W,
   -2 W - t12**DT[W]+t12**tb12**W'}];
In[7] := OPE[J, WB] = MakeOPE[{t12**tb12**WB,
   2 WB+tb12**DBT[WB]+t12**tb12**WB'}];
In[8] := OPE[W, WB] = MakeOPE[{a1 t12**tb12**One,
   a2 One + t12**tb12**J, a3 J + a4 t12**DT[J]}];
\end{verbatim}

Let us note that due to the first OPE, \( J(Z) \) obeys the \( N=2 \) SCA, while the second and third OPEs define \( W(Z) \) and \( \bar{W}(Z) \) as antichiral-chiral superfields primary spin 1 with respect to \( J(Z) \). As concerning the last OPE we introduce the arbitrary coefficients \( a1 − a4 \) for all the terms because we want to determine the exact form of this OPE. The possible terms and their structure in this OPE are completely fixed by the antichiral-chiral structure of the \( W, \bar{W} \) supercurrents (the coefficient before term \( \theta_{12} \star \bar{\theta}_{12} \star J \) can be chosen to be 1 due to the scaling invariance \( W \rightarrow \alpha W \)).

Now we will fix all coefficients from the Jacobi identities:

\begin{verbatim}
In[9] := OPEJacobi[J,W,WB, Function -> Factor]//Union
Out[9] = {0, ((2-a3)*J)/4, ((2-a3)*J)/2, ((-2+a3)*J)/2,
   (a3-a4)*J, ((a1-a2)*One)/2, (a1-a2)*One,
   (-a1+a2)*One, ((4*a1+c)*One)/4, ((8*a2+a3+c)*One)/4,
   ((8*a2+a4+c)*One)/4, ((-2+a4)*DT[J])/2, (-a3+a4)*DT[J]}
\end{verbatim}

We can solve the system of equation inside Mathematica, e.g.

\begin{verbatim}
In[10] := sol1 = Solve[GetCoefficients[%]==0]
Out[10] = {{a1 -> -c/4, a2 -> -c/4, a3 -> 2, a2 -> 2}}
\end{verbatim}

Now we can check that all other Jacobi identities are satisfied:

\begin{verbatim}
In[11] := OPESimplify[OPEJacobi[W,W,WB] /. First[sol1]]//Union
Out[11] = {0}
\end{verbatim}

By this we finished the construction of the \( N = 4 \) \( SU(2) \) SCA in terms of \( N = 2 \) superfields.

### 5.2 A Miura–like realisation of \( N = 2 \)

As a second example we will demonstrate how to use the `SOPEN2defs` package to find a realisation of the \( N = 2 \) SCA in terms of \( M \) pairs of chiral–antichiral fermionic superfields (Miura type realisation). We also show here our technique to deal with summing over repeated indices, which is implemented in a separate package `Dummies`. This package can also be used together with `OPEdefs`, or even independently. We only explain some basic

\[^3\]This statement takes 5 seconds on a Pentium 90 MHz running Mathematica 2.2 for Windows.
commands of *Dummies* here.

The first step is loading the package.

```
In[1] := <<SOPEN2defs.m
```

Enable working with dummy indices.

```
In[2] := SetOPEOptions[EnableDummies, True]
```

We now define which indices we are going to use. All summation indices will have the form \( i[1], i[2], \) etc. That is, they all have \texttt{Head} \( i \).

```
In[3] := DefineDummy[i]
```

For convenience, we define an abbreviation for the summation range.

```
In[4] := dimension[i] = M;
```

Now we define the superfields we are going to play with. In the case at hand we can define \( M \) pairs of chiral-antichiral fermionic superfields \( F_i, \bar{F}_j \) through two *Mathematica* statements:

```
In[5] := ChiralFermionic[F[i_]]
In[6] := AChiralFermionic[FB[i_]]
```

Using indices the SOPEs between superfields \( F_i \) and \( \bar{F}_j \) can be defined mostly as they are defined in the textbooks:

```
In[7] := OPE[F[i_],FB[j_]] :=
    Delta[i,j] MakeOPE[{-1/2 t12**tb12**One, One}]
```

Now we will introduce a composite superfield \( J(Z) \) which should satisfy the \( N = 2 \) SCA eq. (7):

\[
J = a_1 \sum_i F_i \bar{F}_i + a_2 D F_1 + a_3 D \bar{F}_1
\]  

(23)

where besides the standard term \( F_i \bar{F}_i \) we also introduced the \( N = 2 \) analog of a Feigin-Fuchs term. In *Mathematica* this becomes:

```
In[8] := J := NewDummies[
    a1 NO[F[i[1]],FB[i[1]]] +
    a2 DBT[F[1]] + a3 DT[FB[1]]]
```

Note the \texttt{NewDummies} statement and the assignment using ‘:=’. Together, they make sure that everytime \( J \) is used, the summation indices will have a new number.

After these definitions we are ready to do real calculations. For example, to check for which values of parameters \( a_1, a_2, a_3 \) the superfield \( J(Z) \) spans \( N = 2 \) SCA, let us calculate the SOPE \( J \) with \( J \) and subtract the known result (7):

```
In[9] := OPESimplify[DummySimplify[
    OPE[J,J] =
    MakeOPE[{c/4 One + t12**tb12**J,
    tb12**DBT[J]-t12**DT[J]+t12**tb12**J'}]
    ], Function -> Factor]
```

\[
\text{Out}[9] = << 2 \mid One (-c + 4 M a1^2 - 8 a2 a3)/4 + \]
\[
(-1+a1) a2 t12 ** tb12 ** DBT[F[1]] + \ldots \mid1\mid\]
\[
(-1+a1) a1 tb12 ** NO[DBT[F[i[1]]], FB[i[1]]] + \ldots>>
\]
where \texttt{DummySimplify} renumerates the dummy indices to get a unique form.\footnote{In fact, the result of \texttt{DummySimplify} is not unique when tensors with certain symmetries appear, e.g. $J^3 = J^4$.}

For brevity, we omitted a number of terms (all proportional to $a_1 - 1$) in the output and replaced them by an ellipsis. As can be immediately seen, the coefficient $a_1$ must be equal to 1, while $a_2, a_3$ are arbitrary. Let us note that our result \textit{Out}[9] also gives the expression for the central charge of the $N = 2$ SCA:

$$c = 4M - 8a_2a_3$$

Thus we reproduced the standard Miura realisation for the $N = 2$ SCA.

We hope that these simple examples will help users to use the \texttt{SOPEN2defs} package in more complicated calculations.

6 How to get it, and the future

If you are interested in \texttt{SOPEN2defs}, you can get it by Email from the authors. You’ll also find it at \url{http://euclid.tp.ph.ic.ac.uk/~krthie/} or via anonymous ftp at \url{euclid.tp.ph.ic.ac.uk}. Please put a reference to this paper in your paper when you use it. Questions, remarks and improvements are welcome. The package will be extended to other numbers of supersymmetries $N$.

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