Scattering Theory for the Schrödinger Equation in some External Time Dependent Magnetic Fields

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Abstract

We study the theory of scattering for a Schrödinger equation in an external time dependent magnetic field in the Coulomb gauge, in space dimension 3. The magnetic vector potential is assumed to satisfy decay properties in time that are typical of solutions of the free wave equation, and even in some cases to be actually a solution of that equation. That problem appears as an intermediate step in the theory of scattering for the Maxwell-Schrödinger (MS) system. We prove in particular the existence of wave operators and their asymptotic completeness in spaces of relatively low regularity. We also prove their existence or at least asymptotic results going in that direction in spaces of higher regularity. The latter results are relevant for the MS system. As a preliminary step, we study the Cauchy problem for the original equation by energy methods, using as far as possible time derivatives instead of space derivatives.

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1 Introduction

This paper is devoted to the theory of scattering and in particular to the construction of the wave operators for a Schrödinger equation minimally coupled to an external time dependent magnetic field in the Coulomb gauge, namely

\[ i\partial_t u = -(1/2)\Delta_A u \]  

in space dimension 3. Here \( u \) is a complex function defined in space time \( \mathbb{R}^{3+1} \),

\[ \Delta_A \equiv \nabla^2_A \equiv (\nabla - iA)^2 \]  

and the magnetic potential \( A \) is an \( \mathbb{R}^3 \) vector valued function defined in \( \mathbb{R}^{3+1} \) and satisfying the condition \( \nabla \cdot A = 0 \), which is the Coulomb gauge condition. The magnetic potential will be assumed to be sufficiently smooth and to satisfy a number of decay estimates in time that are satisfied by sufficiently regular solutions of the free wave equation \( \square A = 0 \), where \( \square \) is the d’Alembertian operator. At some places \( A \) will even be assumed to be a solution of that equation.

The present problem arises, actually is an intermediate step, in the theory of scattering for the Maxwell-Schrödinger (MS) system. In the Coulomb gauge \( \nabla \cdot A = 0 \), that system takes the form

\[
\begin{aligned}
& i\partial_t u = -(1/2)\Delta_A u + A_0 u \\
& \Delta A_0 = -|u|^2 \\
& \square A + \nabla (\partial_t A_0) = \text{Im} \, \bar{u}\nabla_A u \equiv J(u, A) .
\end{aligned}
\]  

(1.3)

Using the second equation in (1.3) to eliminate \( A_0 \), one can recast that system into the formally equivalent form

\[
\begin{aligned}
& i\partial_t u = -(1/2)\Delta_A u + g(u) \, u \\
& \square A = P \, \text{Im} \, \bar{u}\nabla_A u = P \, J(u, A)
\end{aligned}
\]  

(1.4)

where \( g(u) \) is the Coulomb interaction term

\[ g(u) = (4\pi|x|)^{-1} \ast |u|^2 \]

and \( P = 1 - \nabla \Delta^{-1} \nabla \) is the projector on divergence free vector fields.

The theory of scattering for the MS system (1.4) has been studied and in particular the construction of modified wave operators has been performed in [13] [16] in
the case of small asymptotic states and solutions and in [7] for asymptotic states and solutions of arbitrary size for the Schrödinger function, but only in the special case of vanishing asymptotic magnetic field, namely in the case where the asymptotic state for the magnetic potential is zero. More precisely the method used in [7] starts with the replacement of the Maxwell equation by the associated integral equation with infinite initial time, namely

\[ A = A_0 + A_1 = \dot{K}(t)A_+ + K(t)\dot{A}_+ - \int_t^{\infty} dt'K(t-t')PJ(u, A)(t') \]  

(1.5)

where

\[ K(t) = \omega^{-1} \sin \omega t, \quad \dot{K}(t) = \cos \omega t, \quad \omega = (-\Delta)^{1/2}. \]  

(1.6)

The case treated in [7] is that where the asymptotic state \((A_+, \dot{A}_+)\) for the magnetic potential is zero. As an intermediate step towards the treatment of the general case, it is useful to consider the complementary case where only the free part \(A_0\) is kept in the equation (1.5). If in addition one omits the now well controlled Hartree interaction \(g(u)\), one is led to study the theory of scattering for (1.1) where now \(\Box A = 0\). This is the purpose of the present paper.

The variables \((u, A)\) are not convenient to study the asymptotic behaviour in time and the theory of scattering for (1.1). The present paper will rely in an essential way on the use of new variables defined as follows. The unitary group which solves the free Schrödinger equation can be written as

\[ U(t) = \exp(i(t/2)\Delta) = M(t) \ D(t) \ F \ M(t) \]  

(1.7)

where \(M(t)\) is the operator of multiplication by the function

\[ M(t) = \exp\left(\frac{ix^2}{2t}\right), \]  

(1.8)

\(F\) is the Fourier transform and \(D(t)\) is the dilation operator defined by

\[ D(t) = (it)^{-3/2} \ D_0(t), \]  

(1.9)

\[(D_0(t)f)(x) = f(x/t). \]  

(1.10)

We replace the variables \((u, A)\) by new variables \((w, B)\) defined by

\[ \begin{align*} 
  u(t) &= M(t) \ D(t) \ w(1/t) \\
  A(t) &= -t^{-1} \ D_0(t) \ B(1/t). 
\end{align*} \]

(1.11)
Actually the change of variables from $u$ to $w$ is simply the pseudoconformal inversion. In terms of the new variables $(w, B)$ the original equation, namely (1.1), is easily seen to become

$$i\partial_t w = -(1/2)\Delta_B w - \tilde{B}w$$

(1.12)

where

$$\tilde{B}(t) = t^{-1}(x \cdot B(t)).$$

(1.13)

The study of the asymptotic behaviour in time and of the theory of scattering for (1.1) is then reduced to the study of (1.12) near $t = 0$ and will be performed by studying that equation in the interval $[0, 1]$.

In [7] we performed a different change of variables and used new variables called $(w, B)$ in [7] and which we now denote $(w_*, B_*)$. They are defined by

$$\left\{ \begin{array}{l}
u(t) = M(t) D(t) w_*(t) \\
A(t) = t^{-1} D_0(t) B_*(t) \end{array} \right.$$

(1.14)

so that

$$w(t) = \overline{w_*(1/t)}, \quad B(t) = -B_*(1/t).$$

(1.15)

In terms of the variables $(w_*, B_*)$, (1.1) becomes

$$i\partial_t w_* = -\left(2t^2\right)^{-1} \Delta_{B_*} w_* - \tilde{B}_* w_*$$

(1.16)

where

$$\tilde{B}_*(t) = t^{-1}(x \cdot B_*(t)) = -t^{-1}(x \cdot B(1/t)).$$

(1.17)

The study of the asymptotic behaviour in time and the theory of scattering for (1.1) is then reduced to the same problem for (1.16). The change of variables from $u$ to $w$ or $w_*$ can be rewritten in a slightly different way by introducing

$$\bar{u}(t) = U(-t) u(t) \quad \bar{w}(t) = U(-t) w(t).$$

(1.18)

Then

$$F\bar{u}(t) = \overline{w(1/t)} = U(1/t) w_*(t).$$

(1.19)

Scattering theory is the asymptotic study at infinity in time of an evolution, in the present case that defined by (1.1) or (1.16), by comparison with a simpler reference evolution. In the present case the latter is inspired by the free Schrödinger equation and will be chosen to drive $\bar{u}$ or $w_*$ to a limit as $t \to \infty$, or equivalently to drive
$w$ to a limit as $t \to 0$. The simplest evolution of this type is the free Schrödinger evolution itself for $u$, which is equivalent to taking $\tilde{u}$ constant or equivalently $\tilde{w}$ constant. One is then led to study the convergence properties of $\tilde{u}$, $\tilde{w}$ or possibly of $w$, $w_*$ in a suitable function space $X$ in the appropriate time limit.

The theory of scattering thereby obtained will in general depend significantly on the choice of the space $X$. Let $X$ and $Y$ be two Banach spaces with $Y \subset X$. When comparing the theories of scattering in $X$ and $Y$, one may encounter the following situations. On the one hand, one can in some cases use the theory of scattering in $Y$ together with uniform bounds on the evolution in $X$ to construct the theory of scattering in $X$. This possibility will be exploited in Section 4 below with $X = L^2$ and $Y = H^2$. On the other hand, when restricting the theory of scattering from $X$ to $Y$, one may eliminate some (insufficiently regular) solutions of the original equation, thereby restricting the possible set of asymptotic behaviours. However since one is then interested in convergence properties in a stronger sense, namely in the norm of $Y$ instead of that of $X$, it may also happen that the asymptotic behaviours that were sufficiently accurate in $X$ norm are no longer so in $Y$ norm and have to be replaced by more accurate ones. This last possibility will appear in Sections 6 and 7 below when going from $X = H^2$ to $Y = H^k$ for $3 \leq k \leq 4$.

We now comment on the methods used in the present paper. In order to treat the full nonlinear MS system (1.4) we need to use spaces of sufficiently regular functions, more precisely in [7] we needed to take $w$ or $w_*$ in $H^k$ for $k > 5/2$. Furthermore we have to use methods that are sufficiently simple and robust to accommodate the non linearities. This is the case of the energy methods used in [7] and we shall confine our attention to such methods in this paper. The Schrödinger equation with time dependent magnetic fields has been studied in the literature by more sophisticated methods using the abstract theory of evolution equations or semi classical approximations. We refer to [17] and references therein quoted. Those methods do not seem to be readily extendable to the MS system, and we shall not use them in this paper.

We now turn to an important feature of the MS system, namely the fact that it couples two equations with different scaling properties. In fact one time derivative is homogeneous to two space derivatives for the Schrödinger equation but only to one space derivative for the wave equation. When treating the MS system with $u \in H^{2k}$ by using space derivatives, one needs to consider $\Delta^k u$ and correspondingly $\partial_\alpha^* A$ for multiindices $\alpha$ with $|\alpha| = 2k$. If instead one uses time derivatives, one needs to
consider $\partial_t^k u$ and correspondingly $\partial_t^k A$ which is homogeneous to $\partial_\nu^\alpha A$ with $|\alpha| = k$ only. When considering time decay properties, this fact makes little difference in the case of the variables $(u, A)$ because for solutions of the free wave equation $\Box A = 0$, one can ensure that $\partial_t^j \partial_\nu^\alpha A$ has the same time decay as $A$ for any $j$ and $\alpha$ by considering sufficiently regular solutions. However the same phenomenon occurs when studying the MS system in the transformed variables $(w, B)$. In that case it follows from (1.11) that derivatives $\partial_t^j \partial_\nu^\alpha$ applied to $B$ generate a factor $t^{-j-|\alpha|}$ no matter how regular $A$ is, which is a disaster as regards the behaviour at $t \to 0$. It is therefore important to apply as few derivatives as possible to $B$ and for that purpose it is advantageous to use time derivatives instead of space derivatives of $w$ in order to control the regularity in spaces $H^{2k}$. In this paper we shall therefore use time derivatives as much as possible in the treatment of (1.12). On the other hand, since as mentioned above we need simple methods that can be extended to the full MS system, we shall use only time (and also space) derivatives of integer order.

We now turn to a description of the contents of this paper and to a statement of a representative sample of the results. After a preliminary section containing notation and estimates of a general nature (Section 2), we begin with the study of the Cauchy problem at finite times for a class of equations (see (3.1) (3.2) below) that generalize slightly both (1.1) and (1.12) (Section 3). The main result is that the Cauchy problem is well posed in $L^2$ (Proposition 3.1), in $H^2$ (Proposition 3.2), in $H^3$ (Proposition 3.3) and at the level of $H^4$ (Proposition 3.4) under assumptions on $A$ (or $B$) of a general nature. Let $v$ denote the unknown function. Following the remarks made above, the method is based on energy estimates of $v$, of $\partial_t v$, of $\nabla \partial_t v$ and of $\partial_t^2 v$ respectively, and the number of derivatives on $A$ or $B$ in the assumptions is kept to a minimum, namely 0 or 1, 1, 2 and 2 respectively.

We next turn to the theory of scattering for (1.1) and (1.16) via the study of (1.12) in $[0,1]$, in the more particular case where $A$ satisfies some of the natural decay properties associated with the free wave equation. In Section 4, we exploit Propositions 3.1 and 3.2 to prove the existence and the asymptotic completeness of the wave operators for the equation (1.1) with $u$ or rather $\tilde{u}$ in $L^2$ and in $FH^2$, as compared with the free Schrödinger evolution, or equivalently by (1.19), for the equation (1.16) with $w_*$ in $L^2$ and in $H^2$, as compared with the constant evolution. The main result can be stated as follows (see Propositions 4.1 and 4.2 below for more details).
Proposition 1.1. Let \( A \) satisfy
\[
\| P^j \partial_x^\alpha A \|_r \vee \| P^j (x \cdot A) \|_r \leq C t^{-1+2/r} \tag{1.20}
\]
where \( P = t \partial_t + x \cdot \nabla \), for \( 0 \leq j + |\alpha| \leq 1 \), \( 2 \leq r \leq \infty \) and for all \( t \in [1, \infty) \).

(1) Let \( X = L^2 \) or \( FH^2 \). Then for any \( u_+ \in X \), there exists a unique solution \( u \) of (1.1) such that \( \bar{u} \in \mathcal{C}([1, \infty), X) \) and such that
\[
\| \bar{u}(t) - u_+ \|_X \to 0 \quad \text{when } t \to \infty. \tag{1.21}
\]
Conversely for any solution \( u \) of (1.1) such that \( \bar{u} \in \mathcal{C}([1, \infty), X) \), there exists \( u_+ \in X \) such that (1.21) holds.

(2) Let \( X = L^2 \) or \( H^2 \). Then for any \( w_+ \in X \), there exists a unique solution \( w_* \in \mathcal{C}([1, \infty), X) \) of (1.16) such that
\[
\| w_*(t) - w_+ \|_X \to 0 \quad \text{when } t \to \infty. \tag{1.22}
\]
Conversely for any solution \( w_* \in \mathcal{C}([1, \infty), X) \) of (1.16), there exists \( w_+ \in X \) such that (1.22) holds.

Note that the time decay that occurs in (1.20) is the optimal time decay that can be obtained for the relevant norms of \( A \) if \( A \) is a solution of the wave equation \( \square A = 0 \) satisfying the Coulomb gauge condition \( \nabla \cdot A = 0 \). In that case, that decay can be easily ensured by making appropriate assumptions on the Cauchy data for \( A \) (see Section 2 below, especially Lemma 2.4).

If one makes stronger assumptions on the asymptotic state \( u_+ \) or \( w_+ \), one obtains stronger convergence properties than (1.21) (1.22) for the solutions \( u \) or \( w_* \) constructed in Proposition 1.1 as \( t \to \infty \). The following typical result is extracted from a special case of Proposition 4.3 below, to which we refer for a slightly more general result.

Proposition 1.2. Let \( A \) satisfy the assumptions of Proposition 1.1.

(1) Let \( u_+ \in FH^2 \) and let \( u \) be the solution of (1.1) with \( \bar{u} \in \mathcal{C}([1, \infty), FH^2) \) obtained in Proposition 1.1, part (1). Then the following estimate holds for all \( t \geq 1 \):
\[
\| \bar{u}(t) - u_+ \|_2 \leq C t^{-1}. \tag{1.23}
\]
Let in addition \( u_+ \in FH^3 \). Then the following estimates hold for all \( t \geq 1 \):
\[
\| \bar{u}(t) - u_+ \|_2 \leq C t^{-3/2}, \tag{1.24}
\]
$\| t^2 \partial_t (\tilde{u}(t) - u_+) \|_2 \vee \| x^2 (\tilde{u}(t) - u_+) \|_2 \leq C t^{-1/2}.$ \hspace{1cm} (1.25)

(2) Let $w_+ \in H^2$ and let $w_\ast \in C([1, \infty), H^2)$ be the solution of (1.16) obtained in Proposition 1.1, part (2). Then the following estimate holds for all $t \geq 1$:

$$\| w_\ast(t) - w_+ \|_2 \leq C t^{-1}.$$ \hspace{1cm} (1.26)

Let in addition $w_+ \in H^3$. Then the following estimates hold for all $t \geq 1$:

$$\| w_\ast(t) - U^\ast(1/t)w_+ \|_2 \leq C t^{-3/2},$$ \hspace{1cm} (1.27)

$$t^2 \| \partial_t (w_\ast(t) - U^\ast(1/t)w_+) \|_2 \vee \| \Delta (w_\ast(t) - U^\ast(1/t)w_+) \|_2 \leq C t^{-1/2}.$$ \hspace{1cm} (1.28)

We next study the $H^k$ regularity of the $L^2$ wave operators for $u$ constructed in Section 4, as given in particular by Proposition 1.1 part (1) above with $X = L^2$. As mentioned above, time decay is not impaired by derivatives for sufficiently regular solutions of the free wave equation. Making decay assumptions of that type on $A$ at a sufficient level of regularity, we prove in Section 5 that the $L^2$ wave operators for $u$ essentially preserve $H^k$ regularity for arbitrarily high $k$. The following typical result is a special case of Proposition 5.2, to which we refer for a slightly more general statement.

**Proposition 1.3.** Let $j \geq 0$ be an integer and let $A$ satisfy

$$\| \partial_t^l \partial_x^\alpha A \|_r \vee \| \partial_t^l \partial_x^\alpha (x \cdot A) \|_r \leq C t^{-1+2/r}$$ \hspace{1cm} (1.29)

for all $r$, $2 \leq r \leq \infty$, for $0 \leq l+|\alpha|/2 \leq j$ and for all $t \geq 1$. Let $u_+, xu_+ \in H^{2j} \cap H^{2j}_1$. Then there exists a unique solution $u \in \bigcap_{0 \leq l \leq j} C^{j-1}([1, \infty), H^{2j}_1)$ of (1.1) satisfying the estimates

$$\| \partial_t^{j-l} \Delta^l (u(t) - U(t)u_+) \|_2 \leq C t^{-3/2}$$ \hspace{1cm} (1.30)

for $0 \leq l \leq j$ and for all $t \geq 1$. The solution is actually unique in $C([1, \infty), L^2)$ under the condition (1.30) for $j = 0$.

(See Section 2 below for the definition of $H^{2j}_1$). At the level of regularity of $H^k$ for $u$, however, we lose control of asymptotic completeness since the method does not even allow to prove that generic solutions of (1.1) remain bounded in $H^k$ for $k > 0$ under assumptions on $A$ of the same type (see Proposition 5.1 for a partial result in that direction).
We next turn to the theory of scattering at the level of regularity of $H^k$ with $k \geq 3$ for $w_*$ or $w$, which is of primary interest for subsequent application to the MS system, by studying again (1.12) at that level of regularity. Now however Propositions 3.3 and 3.4 do not apply with initial time $t_0 = 0$ because the relevant norms of $B$ blow up too fast as $t \to 0$. As a consequence we cannot prove that the generic solutions of (1.12) (resp. (1.16)) remain bounded as $t \to 0$ (resp. ($t \to \infty$)) and we lose control of asymptotic completeness. Furthermore, in order to construct the wave operators, which in that case amounts to solving the Cauchy problem for (1.12) with initial time zero, we have to resort to the same indirect method that was used in [7]. We give ourselves a presumed asymptotic behaviour of the solution $w$ of (1.12) near zero in the form of a function $W$ with prescribed value at $t_0 = 0$ and we look for $w$ in the form $w = W + q$ with $q$ tending to zero as $t \to 0$. The evolution equation for $q$ is obviously

$$i\partial_t q = -(1/2)\Delta_B q - \check{B} q - R(W)$$  \tag{1.31}$$

where

$$R(W) = i\partial_t W + (1/2)\Delta_B W + \check{B} W .$$  \tag{1.32}$$

We take $t_0 > 0$, we apply Proposition 3.3 or 3.4 to define a solution $q_{t_0}$ of (1.31) in $(0, 1]$ with suitably small initial condition $q_{t_0}(t_0) = q_0$ at $t_0$, and we take the limit of $q_{t_0}$ as $t_0 \to 0$. We carry out this program in two steps. In Section 6, we construct solutions $q$ of (1.31) tending to zero at $t \to 0$ under general assumptions on $W$, taking mainly the form of time decay estimates of $R(W)$. This is done in $H^3$ in Proposition 6.1 and at the level of $H^4$ in Proposition 6.2. Actually in the latter case, although $\partial^2_t q$ tends to zero in $L^2$ as $t \to 0$, $q$ itself tends to zero in $H^k$ only for some $k$ with $3 < k < 4$, depending on the decay assumptions on $R(W)$, but in general not in $H^4$. In Section 7 we construct asymptotic functions $W$ satisfying the assumptions required in Section 6, restricting our attention to the case where $A$ is a solution of the free wave equation. The simplest way to do that, in keeping with the fact that we compare (1.1) with the free Schrödinger equation, consists in taking $W(t) = U(t)w_+$ where $w_+ = Fu_+$, the Fourier transform of the Schrödinger asymptotic state $u_+$. One then encounters a standard difficulty in that problem, namely the difference of propagation properties of the wave equation and of the Schrödinger equation [5] [16]. That difficulty can be circumvented by imposing a support condition on $w_+$ saying in effect that $w_+$ vanishes in a neighborhood of the unit sphere, so that $u_+$ generates a solution of the free Schrödinger equation which is asymptotically small.
in a neighborhood of the light cone. In Section 7 we first produce an appropriate $W$ under such a support condition. In Proposition 7.1 we reduce the problem to a joint condition on $B$ and on the support of $w_+$ and in Proposition 7.2 we prove that such a condition can be ensured under a suitable support condition on $w_+$ and suitable decay assumptions at infinity in space on the asymptotic state $(A_+, \dot{A}_+)$ of $A$ (see (1.5)). Collecting the implications of Section 6 and of that first part of Section 7 on the theory of scattering for (1.1) at the level of regularity of $FH^4$ for $\tilde{u}$, we construct solutions of (1.1) with prescribed asymptotic state $u_+$ at that level of regularity. The main result is stated in Section 8 as Proposition 8.1 to which we refer for a full mathematical statement. Since that statement is too complicated to be presented at the level of this introduction, we give here only a heuristic preview thereof, stripped from most technicalities.

**Proposition 1.4** Let $A$ be a solution of the free wave equation satisfying suitable conditions of regularity and of decay at infinity in space. Let $u_+ \in FH^5$ and let $w_+ \equiv Fu_+ (\in H^5)$ satisfy the support condition

$$\text{Supp } w_+ \subset \{ x : |x| - 1 | \geq \eta \}$$

for some $\eta$, $0 < \eta < 1$. Then there exists a unique solution $u$ of (1.1) such that $\tilde{u} \in C([1, \infty), FH^4) \cap C^1([1, \infty), FH^2) \cap C^2([1, \infty), L^2)$ and such that $\tilde{u} - u_+$ tends to zero as $t \to \infty$ in suitable norms (related to the previous space) with power law decay in time, with exponents depending on the assumptions made on $A$.

We next try to eliminate the support condition (1.33). This has been done in [12] in the case of the Wave-Schrödinger system (see also [6]) and we try to apply the same method in the present case. We are only partly successful, namely we succeed in controlling the $B \cdot \nabla W$ and $\dot{B}W$ terms in $R(W)$, but not the $B^2W$ term. The results appear in Propositions 7.3 to 7.5 as regards the construction of the asymptotic $W$, but the situation is too intricate to be described here whereas the result is hopefully not final and we refer to the discussion in Section 7 for details. Collecting the implications of Section 6 and of that second part of Section 7, we finally construct solutions of (1.1) at the level of regularity of $FH^4$ for $\tilde{u}$ with prescribed asymptotic behaviour in time given by an asymptotic function $\tilde{u}_a$ constructed from the asymptotic $W$ mentioned above according to (8.7). The result is stated in Section 8 as Proposition 8.2 to which we refer for a full mathematical statement. A
heuristic overview of that proposition can be obtained from Proposition 1.4 above by having the support condition (1.33) removed and by having $u_+$ replaced by the asymptotic function $\tilde{u}_a$ mentioned above in the final decay estimates. In addition, the assumptions on $A$ are slightly different and include some moment conditions.

2 Notation and preliminary estimates

In this section we introduce some notation and collect a number of estimates which will be used freely throughout this paper. We denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$, $1 \leq r \leq \infty$. For any nonnegative integer $k$ and for $1 \leq r \leq \infty$, we denote by $H^k_r$ the Sobolev spaces

$$H^k_r = \left\{ u : \| u ; H^k_r \| = \sum_{\alpha:0 \leq |\alpha| \leq k} \| \partial^{\alpha}_x u \|_r < \infty \right\}$$

where $\alpha$ is a multiindex. For $1 < r < \infty$, those spaces can be defined equivalently (with equivalent norms) by

$$H^k_r = \left\{ u : \| u ; H^k_r \| = \| <\omega >^k u \|_r < \infty \right\}$$

where $\omega = (-\Delta)^{1/2}$ and $<\cdot >= (1 + |\cdot|^2)^{1/2}$. The latter definition extends immediately to any $k \in \mathbb{R}$ and we shall occasionally use such spaces. The subscript $r$ in $H^k_r$ will be omitted in the case $r = 2$. For any interval $I$ and for any Banach space $X$ we denote by $\mathcal{C}(I, X)$ (resp. $\mathcal{C}_w(I, X)$) the space of strongly (resp. weakly) continuous functions from $I$ to $X$. For any positive integer $k$, we denote by $\mathcal{C}^k(I, X)$ the space of $k$ times differentiable functions from $I$ to $X$. For any integer $k$, $0 < k < \infty$, for any $r$, $1 \leq r \leq \infty$, we denote by $L^r(I, X)$ (resp. $L^r_{loc}(I, X)$) the space of $L^r$ integrable (resp. locally $L^r$ integrable) functions from $I$ to $X$ if $r < \infty$ and the space of measurable essentially bounded (resp. locally essentially bounded) functions from $I$ to $X$ if $r = \infty$. For $I$ an open interval we denote by $D'(I, X)$ the space of vector valued distributions from $I$ to $X$ [9]. For any integer $k$, $0 < k < \infty$, for any $r$, $1 \leq r \leq \infty$, we denote by $H^k_r(I, X)$ (resp. $H^k_{r,loc}(I, X)$) the space of functions from $I$ to $X$ whose derivatives up to order $k$, taken in $D'(I_0, X)$ with $I_0$ the interior of $I$, are in $L^r(I, X)$ (resp. $L^r_{loc}(I, X)$). In the same vein we say that an evolution equation (like (1.1)) has a solution in $I$ with values in $X$ if the equation is satisfied in $D'(I_0, X)$.

We shall use extensively the following Sobolev inequalities, stated here in $\mathbb{R}^n$.
Lemma 2.1. Let $1 < q, r < \infty$, $1 < p \leq \infty$ and $0 \leq j < n$. If $p = \infty$, assume that $k - j > n/r$. Let $\sigma$ satisfy $j/k \leq \sigma \leq 1$ and

$$n/p - j = (1 - \sigma)n/q + \sigma(n/r - k).$$

(2.1)

Then the following inequality holds:

$$\| \omega^j u \|_p \leq C \| u \|_q^{1-\sigma} \| \omega^k u \|_r^\sigma.$$  

(2.2)

The proof follows from the Hardy-Littlewood-Sobolev inequality [8] (from the Young inequality if $p = \infty$), from Paley-Littlewood theory and interpolation.

Occasionally a special case of (2.2) will be used with the ordinary derivative $\nabla$ replaced by the covariant derivative $\nabla_A = \nabla - iA$, where $A$ is a real vector-valued function, namely

$$\| u \|_p \leq C \| u \|_q^{1-\sigma} \| \nabla_A u \|_r^\sigma$$

(2.3)

which holds under the assumptions of Lemma 2.1 with $j = 0, k = 1$. The proof of (2.3) is an immediate consequence of (2.2) with $j = 0, k = 1$ applied to $|u|$ and of the inequality $|\nabla|u|| \leq |\nabla_A u|$.  

We shall also make use of the following two lemmas.

Lemma 2.2. Let $0 \leq \alpha_j < 1$, $a_j \in IR^+$, $1 \leq j \leq n$ and let $y \in IR^+$ satisfy

$$y \leq \sum_{1 \leq j \leq n} a_j y^{\alpha_j}.$$  

(2.4)

Then

$$y \leq C \sum_{1 \leq j \leq n} (a_j)^{1/(1-\alpha_j)}$$

(2.5)

where

$$C = \text{Max}_{1 \leq j \leq n} n^{1/(1-\alpha_j)}.$$  

Proof.

$$\left\{ y : y \leq \sum_{1 \leq j \leq n} a_j y^{\alpha_j} \right\} \subset \bigcup_{1 \leq j \leq n} \left\{ y : y \leq n a_j y^{\alpha_j} \right\} =$$

$$\bigcup_{1 \leq j \leq n} \left\{ y : y \leq (na_j)^{1/(1-\alpha_j)} \right\} \subset \left\{ y : y \leq \sum_{1 \leq j \leq n} (na_j)^{1/(1-\alpha_j)} \right\}$$
which implies (2.5).

Lemma 2.3. Let \( \alpha_j \) satisfy \( 0 \leq \alpha_j < \alpha_1 < 1, \ 2 \leq j \leq n \). Let \( I \) be an interval, let \( a_j \in C(I, IR^+) \), \( 0 \leq j \leq n \) and let \( y \in C(I, IR^+) \) be absolutely continuous and satisfy

\[
|\partial_t y| \leq a_0 y + \sum_{1 \leq j \leq n} a_j y^{\alpha_j}
\]  

(2.6)

for all \( t \in I \). Let \( t_0 \in I, \ y(t_0) = y_0 \) and define

\[
A_0(t) = \left| \int_{t_0}^{t} dt' \ a_0(t') \right|, \quad A_j(t) = (1 - \alpha_j) \left| \int_{t_0}^{t} dt' \ a_j(t') \right|, \quad 1 \leq j \leq n.
\]

Then the following inequality holds

\[
y(t) \leq \exp \left( A_0(t) \right) \left\{ y_0^{1 - \alpha_1} + \sum_{1 \leq j \leq n} A_j(t)^{(1 - \alpha_1)/(1 - \alpha_j)} \right\}^{1/(1 - \alpha_1)}
\]  

(2.7)

for all \( t \in I \).

Proof. We consider only the case \( t \geq t_0 \). By exponentiating the term with \( a_0 \) in (2.6) and by passing to the rescaled variable

\[
y' = \exp(-A_0) y
\]

we can rewrite (2.6) and the initial condition at \( t_0 \) in the form

\[
\begin{cases}
\partial_t y' \leq \sum_{1 \leq j \leq n} a_j \exp((\alpha_j - 1)A_0) y^{\alpha_j} \\
y'(t_0) = y_0
\end{cases}
\]

or equivalently

\[
\begin{cases}
\partial_t y' \leq \sum_{1 \leq j \leq n} (1 - \alpha_j)^{-1} \left( \partial_t A_j' \right) y^{\alpha_j} \\
y'(t_0) = y_0
\end{cases}
\]

where

\[
A_j'(t) \equiv (1 - \alpha_j) \int_{t_0}^{t} dt' \ a_j(t') \exp((\alpha_j - 1)A_0(t')).
\]  

(2.8)

On the other hand it is well known that \( y'(t) \leq z(t) \) where

\[
\begin{cases}
\partial_t z = \sum_{1 \leq j \leq n} (1 - \alpha_j)^{-1} \left( \partial_t A_j' \right) z^{\alpha_j} \\
z(t_0) = y_0
\end{cases}
\]  

(2.9)
From (2.9) we obtain
\[ \partial_t z \geq (1 - \alpha_j)^{-1} \left( \partial_t A'_j \right) z^{\alpha_j}, \quad 1 \leq j \leq n \]
so that
\[ z \geq (A'_j)^{1/(1 - \alpha_j)}, \quad 1 \leq j \leq n. \tag{2.10} \]
Inserting (2.10) into the differential equation in (2.9) yields
\[ \partial_t \left( z^{1 - \alpha_1} \right) \leq \sum_{1 \leq j \leq n} (1 - \alpha_1)(1 - \alpha_j)^{-1} \left( \partial_t A'_j \right) A^{(\alpha_j - \alpha_1)/(1 - \alpha_j)} \]
which implies by integration
\[ z(t)^{1 - \alpha_1} \leq y_0^{1 - \alpha_1} + \sum_{1 \leq j \leq n} \left( A'_j(t) \right)^{(1 - \alpha_1)/(1 - \alpha_j)}. \tag{2.11} \]
Now (2.7) follows from \( y'(t) \leq z(t) \) and from (2.11) by coming back to the original variable \( y \) and by replacing the exponential inside the integral of (2.8) by 1.

We now collect some properties of the solutions of the wave equation \( \Box A = 0 \). The general solution can be written as
\[ A(t) = \cos \omega t \ A_+ + \omega^{-1} \sin \omega t \ \dot{A}_+. \tag{2.12} \]
If \( A \) is vector valued and divergence free (\( \nabla \cdot A = 0 \)) then also \( \Box (x \cdot A) = 0 \) and \( x \cdot A \) can be written as
\[ x \cdot A(t) = \cos \omega t \ (x \cdot A_+) + \omega^{-1} \sin \omega t \ (x \cdot \dot{A}_+) \tag{2.13} \]
We shall need the dilation generator
\[ P = t \partial_t + x \cdot \nabla. \tag{2.14} \]
The operator \( P \) satisfies the following commutation relations
\[ [P, \exp(i\omega t)] = 0 \tag{2.15} \]
\[ PD_0(t) = D_0(t)t\partial_t \tag{2.16} \]
\[ P\omega^{-j} = \omega^{-j}(P + j) \tag{2.17} \]
\[ P\partial^\alpha = \partial^\alpha(P - |\alpha|) \tag{2.18} \]
for any integer \( j \) and for any multiindex \( \alpha \).

If \( \Box A = 0 \) then also \( \Box PA = 0 \) and \( PA \) can be written as

\[
(PA)(t) = \cos \omega t \left( x \cdot \nabla A_+ \right) + \omega^{-1} \sin \omega t \left( (1 + x \cdot \nabla) \dot{A}_+ \right).
\]  

(2.19)

When changing variables from \( A \) to \( B \) according to (1.11) we obtain

\[
x \cdot A(t) = -t^{-1} D_0(t) \tilde{B}(1/t)
\]  

(2.20)

and

\[
\left( (P + 1)^j A \right)(t) = (-1)^j t^{-1} D_0(t) \left( (t \partial_t)^j B \right)(1/t)
\]  

(2.21)

for any integer \( j \geq 0 \).

We finally collect some estimates of divergence free vector solutions of the wave equation.

**Lemma 2.4.** Let \( j \geq 0 \) be an integer and let \( \alpha \) be a multiindex. Assume that \((A_+, \dot{A}_+)\) satisfy the conditions

\[
A \in L^2, \quad \nabla^2 A \in L^1 \quad \text{(2.22)}
\]

\[
\omega^{-1} \dot{A} \in L^2, \quad \nabla \dot{A} \in L^1 \quad \text{(2.23)}
\]

for 

\[
A = (x \cdot \nabla)^j \partial^\alpha_x A_+, \quad \dot{A} = (x \cdot \nabla)^{j'} \partial^\alpha_x A_+ \quad \text{(2.24)}
\]

\[
\dot{A} = (x \cdot \nabla)^j \partial^\alpha_x \dot{A}_+, \quad \ddot{A} = (x \cdot \nabla)^{j'} \partial^\alpha_x (x \cdot \dot{A}_+) \quad \text{(2.25)}
\]

for \( 0 \leq j' \leq j \). Then \( A \) satisfies the following estimates :

\[
\| (P + 1)^j \partial^\alpha_x A(t) \|_r \vee \| (P + 1)^j \partial^\alpha_x (x \cdot A(t)) \|_r \leq b \ t^{-1+2/r} \quad \text{(2.26)}
\]

for \( 0 \leq j' \leq j \), for \( 2 \leq r \leq \infty \) and for all \( t > 0 \).

Let \( B \) and \( \tilde{B} \) be defined by (1.11) and (1.13). Then \( B \) and \( \tilde{B} \) satisfy the following estimates :

\[
\| \partial^\beta_t \partial^\alpha_x B(t) \|_r \vee \| \partial^\beta_t \partial^\alpha_x \tilde{B}(t) \|_r \leq b \ t^{-j-|\alpha|+1/r} \quad \text{(2.27)}
\]

**Proof.** The estimate (2.26) is standard for \( j = 0 \) [14]. For \( j \neq 0 \) it is a consequence of the case \( j = 0 \) and of the commutation relations satisfied by \( P \). The estimate (2.27) follows from (2.26), (1.11), (2.20) and (2.21).
3 The Cauchy problem at finite time

In this section we study the Cauchy problem for the equations (1.1) and (1.12) and for related nonautonomous equations that will appear in Sections 5 and 6, for finite initial time. We shall write those equations in the general form

\[ i\partial_t v = K v + f \]  

(3.1)

where

\[ K = -(1/2)\Delta_A + V . \]  

(3.2)

The equations (1.1) and (1.12) are obtained by replacing \((v, A, V, f)\) by \((u, A, 0, 0)\) and by \((w, B, -\bar{B}, 0)\) respectively. We shall need a parabolically regularized version of (3.1) of the form

\[ i\partial_t v = K_\eta v + f \]  

(3.3)

where

\[ K_\eta = -(1/2)(1 - i\eta)\Delta_A + V \]  

(3.4)

for some \(\eta > 0\). Furthermore we shall regularize \(A, V\) and \(f\).

We shall use extensively the conservation law and/or estimates of the \(L^2\) norm which are formally associated with (3.1) and (3.3) and which actually hold for sufficiently regular \(A, V\) and \(f\). We state them in the following two Lemmas.

**Lemma 3.1.** Let \(I\) be an interval, let \(A \in L^2_{\text{loc}}(I, L^\infty), \nabla A \in L^1_{\text{loc}}(I, L^\infty), V \in L^1_{\text{loc}}(I, L^\infty)\) and \(f \in L^1_{\text{loc}}(I, L^2)\). Let \(v \in (L^\infty_{\text{loc}} \cap C_w)(I, L^2)\).

(1) Let \(\eta > 0\) and let \(v\) satisfy (3.3) in \(I\). Then \(v \in L^2_{\text{loc}}(I, H^1)\) and \(v\) satisfies the following inequality for all \(t_1, t_2 \in I, t_1 \leq t_2\):

\[
\| v(t_2) \|_2^2 - \| v(t_1) \|_2^2 + \eta \int_{t_1}^{t_2} dt \| \nabla A v(t) \|_2^2 \\
\leq \int_{t_1}^{t_2} dt \text{ Im } < v, f > (t) . \]  

(3.5)

(2) Let \(v\) satisfy (3.1) in \(I\). Then \(v \in C(I, L^2)\) and \(v\) satisfies the following equality for all \(t_1, t_2 \in I\):

\[
\| v(t_2) \|_2^2 - \| v(t_1) \|_2^2 = \int_{t_1}^{t_2} dt \text{ Im } < v, f > (t) . \]  

(3.6)
There is a large number of results of the type of Lemma 3.1 in the literature. The proof relies on commutator estimates of vector fields with standard mollifiers. For instance for the proof of (3.6), one starts from the regularized identity
\[
\| \varphi \ast v(t_2) \|_2^2 - \| \varphi \ast v(t_1) \|_2^2 = 2 \text{Re} \int_{t_1}^{t_2} dt < \varphi \ast v, [\varphi \ast, A \cdot \nabla]v > (t) + 2 \text{Im} \int_{t_1}^{t_2} dt \left( < \varphi \ast v, [\varphi \ast, A^2/2 + V]v > + < \varphi \ast v, \varphi \ast f > \right) (t)
\] (3.7)
where \( \varphi \) is a standard mollifier in the space variables. The only delicate term in the elimination of the mollifier is that containing the commutator \([\varphi \ast, A \cdot \nabla] \). We refer to [2] [3] for detailed proofs.

If \( v \) is more regular, similar results hold under weaker assumptions on \( A, V, f \). We state only one such result related to (3.1).

**Lemma 3.2.** Let \( I \) be an interval. Let \( A \in L^2_{\text{loc}}(I, L^3 + L^\infty) \), \( V \in L^1_{\text{loc}}(I, L^{3/2} + L^\infty) \) and \( f \in L^1_{\text{loc}}(I, H^{-1}) \). Let \( v \in (L^\infty_{\text{loc}} \cap C_w)(I, H^1) \) satisfy (3.1) in \( I \). Then \( v \) satisfies (3.6) for all \( t_1, t_2 \in I \).

The proof of Lemma 3.2 is an elementary variant of that of Lemma 3.1. One starts again from (3.7), where now the elimination of the mollifier is elementary under the assumptions made on \( A, V, f \).

**Remark 3.1.** The assumption \( v \in (L^\infty_{\text{loc}} \cap C_w)(I, L^2) \) in Lemma 3.1 is partly redundant. On the one hand, if one is given a solution \( v \in L^\infty_{\text{loc}}(I, L^2) \) of (3.1) or (3.3) in \( I \), that solution has a representative \( v' \in C(I, H^{-2}) \). By a standard compactness argument \( v' \in C_w(I, L^2) \) and therefore \( v' \) satisfies the assumptions of the Lemma (see for instance [9]). On the other hand, if \( v \in C_w(I, L^2) \) then by the uniform boundedness principle, \( v \in L^\infty_{\text{loc}}(I, L^2) \). In practice, we shall use Lemma 3.1 mostly for \( \eta > 0 \) and solutions \( v \in C(I, L^2) \) or for weak* limits of such solutions in \( L^\infty(I, L^2) \) which will be extended to \( C_w(I, L^2) \) as explained above.

We now begin the study of the Cauchy problem for the equation (3.1). As mentioned in the introduction, we shall study that problem successively at the level of regularity of \( L^2, H^2, H^3 \) and \( H^4 \) using time derivatives as much as possible to control the regularity.

We first state the result at the level of \( L^2 \).
Proposition 3.1. Let $I$ be an interval, let $A \in L^2_{\text{loc}}(I, L^4 + L^\infty)$, $V \in L^1_{\text{loc}}(I, L^2 + L^\infty)$ and $f \in L^1_{\text{loc}}(I, L^2)$. Let $t_0 \in I$ and $v_0 \in L^2$. Then

$(1)$ There exists a solution $v \in (L^\infty_{\text{loc}} \cap C_w)(I, L^2)$ of (3.1) in $I$ with $v(t_0) = v_0$. That solution satisfies
\[
\| v(t) \|_2^2 - \| v_0 \|_2^2 \leq \int_{t_0}^t \| f(t') \|_2 \text{ Im } v(t') dt' \quad (3.8)
\]
for all $t \in I$.

$(2)$ Let in addition $A \in L^2_{\text{loc}}(I, L^\infty)$, $V \in L^1_{\text{loc}}(I, L^\infty)$ and $\nabla A \in L^1_{\text{loc}}(I, L^\infty)$. Then the previous solution $v$ is unique in $(L^\infty_{\text{loc}} \cap C_w)(I, L^2)$, $v \in C(I, L^2)$ and $v$ satisfies (3.6) for all $t_1, t_2 \in I$.

Proof.

Part (1). The proof proceeds by a parabolic regularization and a limiting procedure. We consider separately the cases $t \geq t_0$ and $t \leq t_0$ and we begin with $t \geq t_0$. We replace (3.1) by (3.3) with $0 < \eta \leq 1$, where in addition we regularize $A$ in space and time and $V$ in space by the use of standard mollifiers parametrized by $\eta$, so that the regularized $A$ and $V$ belong to $C^\infty(I, H^{N}_N)$ and to $L^1(I, H^{N}_N)$ respectively for any $N \geq 0$. We recast the Cauchy problem for the regularized equation in the form of the integral equation
\[
v(t) = (\phi(v))(t) \equiv U_\eta(t-t_0)v_0 - i \int_{t_0}^t dt' U_\eta(t-t')F_\eta(t') \quad (3.9)
\]
where
\[
U_\eta(t) = \exp(i(t/2)(1 - i\eta)\Delta),
\]
\[
F_\eta = (1 - i\eta) \left( iA \cdot \nabla v + (1/2)A^2 v \right) + Vv + f. \quad (3.11)
\]

We first solve (3.9) locally in time by contraction in $C([t_0, t_0 + T], L^2)$ for some $T > 0$. The semigroup $U_\eta$ satisfies the estimate
\[
\| U_\eta(t) \|_{\mathcal{B}(L^2)} \leq C_\alpha(\eta t)^{-|\alpha|/2} \quad \text{for } t > 0 \quad (3.12)
\]
so that
\[
\| U_\eta(t-t') \|_2 \leq C \left\{ (\eta(t-t'))^{-1/2} \| A(t') \|_\infty + \| A(t') \|_\infty^2 + \| V(t') \|_\infty \right\} \\
\times \| v(t') \|_2 + \| f(t') \|_2. \quad (3.13)
\]
The RHS of (3.13) is in $L^1$ in the variable $t'$. It then follows from (3.13) that (3.9) can be solved by contraction in $C([t_0, t_0 + T], L^2)$ for $T$ sufficiently small. By a standard argument using the fact that the equation is linear, one can extend the solution to $I_+ = I \cap \{ t : t \geq t_0 \}$. Let $v_\eta$ be that solution.

We next take the limit where $\eta$ tends to zero. For that purpose we use the fact that $v_\eta$ satisfies

$$\| v_\eta(t) \|_2^2 \leq \| v_0 \|_2^2 + \int_{t_0}^t dt' \, 2 \, \text{Im} < v_\eta, f > (t')$$

for all $t \in I_+$ by Lemma 3.1, part (1), so that

$$\| v_\eta(t) \|_2 \leq \| v_0 \|_2 + \int_{t_0}^t dt' \, \| f(t') \|_2,$$  

and $v_\eta$ is uniformly bounded with respect to $\eta$ in $L^\infty_{\text{loc}}(I_+, L^2)$. We take the limit $\eta \to 0$ by a compactness argument. We can extract a subsequence of $v_\eta$ which converges to some $v \in L^\infty_{\text{loc}}(I_+, L^2)$ in the weak$^*$ sense in $L^\infty(J, L^2)$ for all $J \subset I_+$. One can see that $v$ satisfies the equation (3.1). In particular $v \in C(I_+, H^{-2})$ and $v$ can therefore be taken in $C_w(I_+, L^2)$ (see Remark 2.1). Furthermore $v_\eta$ converges to $v$ weakly in $L^2$ pointwise in $t$. This can be seen easily from the identity

$$v_\eta(t) = \theta^{-1} \int_t^{t+\theta} dt' \, v_\eta(t') - \theta^{-1} \int_t^{t+\theta} dt' (t + \theta - t') \partial_t v_\eta(t')$$

and from the previous convergence of $v_\eta$ to $v$. This allows to prove that $v$ satisfies the initial condition $v(t_0) = v_0$ and to take the limit $\eta \to 0$ in (3.14), thereby obtaining (3.8). This completes the proof for $t \geq t_0$. A similar proof applies to the case $t \leq t_0$.

**Part (2).** It follows from Lemma 3.1, part (2) that $v$ satisfies (3.6) and that $v \in C(I, L^2)$. Uniqueness follows from (3.6) with $f = 0$ applied to the difference of two solutions.

We now turn to the study of the Cauchy problem for the equation (3.1) at the level of regularity of $H^2$. As mentioned previously, we shall control that regularity through the use of the time derivative $\partial_t v$. That derivative satisfies the equation obtained by taking the time derivative of (3.1), namely

$$i \partial_t \partial_t v = K \partial_t v + f_1$$

(3.17)
\[ f_1 = (\partial_t K) v + \partial_t f = i (\partial_t A) \cdot \nabla_A v + (\partial_t V) v + \partial_t f . \]  
(3.18)

We state the result as Proposition 3.2 below. That proposition is a minor extension of Lemma 3.1 in [10] since we have in addition a linear potential \( V \) and an inhomogeneous term \( f \). In [10] the authors use the covariant space derivatives, which is equivalent to using the time derivative for \( V = f = 0 \), but not for nonzero \( V \) and \( f \).

**Proposition 3.2.** Let \( I \) be an interval, let \( A \in L^\infty_{\text{loc}}(I, L^6 + L^\infty) \cap C(I, L^3 + L^\infty) \), \( \partial_t A \in L^1_{\text{loc}}(I, L^3 + L^\infty) \), \( V \in C(I, L^2 + L^\infty) \), \( \partial_t V \in L^1_{\text{loc}}(I, L^2 + L^\infty) \), \( f \in C(I, L^2) \) and \( \partial_t f \in L^1(I, L^2) \). Let \( t_0 \in I \) and \( v_0 \in H^2 \).

1. There exists a unique solution \( v \in C(I, H^2) \cap C^1(I, L^2) \) of (3.1) in \( I \) with \( v(t_0) = v_0 \). That solution satisfies (3.6) for all \( t_1, t_2 \in I \). That solution is actually unique in \((L^\infty_{\text{loc}} \cap C_w)(I, H^1)\).

2. Let in addition \( A \in L^2_{\text{loc}}(I, L^\infty) \), \( \nabla A \in L^1_{\text{loc}}(I, L^\infty) \) and \( V \in L^1_{\text{loc}}(I, L^\infty) \). Then the previous solution \( v \) is actually unique in \((L^\infty_{\text{loc}} \cap C_w)(I, L^2)\) and \( v \) satisfies

\[
\| \partial_t v(t_2) \|_2^2 - \| \partial_t v(t_1) \|_2^2 = \int_{t_1}^{t_2} dt \, 2 \, \text{Im} < \partial_t v, f_1 > (t) \tag{3.19}
\]

for all \( t_1, t_2 \in I \), where \( f_1 \) is defined by (3.18).

**Remark 3.2.** Assumptions of the type \( \partial_t A \in L^1_{\text{loc}}(I, X) \) almost imply that \( A \in C(I, X) \). The latter condition serves simply to exclude that \( A \) contains a constant term in time which does not belong to \( X \). A similar remark will apply to more complicated assumptions of the same type made in Propositions 3.3 and 3.4 below.

**Proof.** Part (1). The proof proceeds by a parabolic regularization and a limiting procedure as in the case of Proposition 3.1. We consider first the case \( t \geq t_0 \). We replace (3.1) by (3.3) with \( 0 \leq \eta \leq 1 \), where in addition we regularize \( A, V \) and \( f \) by standard mollifiers in space parametrized by \( \eta \), in such a way that the regularization decreases the relevant \( L^r \) norms. We shall not indicate the regularization in the notation for \( A \) and \( V \). As regards \( f \), we shall in general not indicate it either, except in cases of doubt where we shall use the notation \( f_\eta \). The regularized \( A \), \( V \) and \( f \) satisfy conditions obtained from the assumptions of the proposition by replacing \( L^r + L^\infty \) by \( H^N_\infty \) for \( A \) and \( V \) and \( L^2 \) by \( H^N \) for \( f \), for arbitrary \( N \geq 0 \). We consider again the integral equation, namely (3.9). We shall need the time
derivative of $\phi(v)$. Integration by parts yields
\[(i\partial_t \phi(v))(t) = U_\eta(t-t_0) (K_nv_0 + f)(t_0) + \int_{t_0}^t dt' U_\eta(t-t') \partial_t F_\eta(t')\]  
(3.20)

while (3.11) yields
\[\partial_t F_\eta = (1 - i\eta) \left(iA \cdot \nabla \partial_t v + (1/2)A^2 \partial_t v\right) + V \partial_t v + f_{1\eta}\]  
(3.21)

where
\[f_{1\eta} = (\partial_t K_n) v + \partial_f f = i(1 - i\eta) (\partial_t A) \cdot \nabla_A v + (\partial_t V) v + \partial_t f.\]  
(3.22)

We first solve (3.9) locally in time by contraction in $C([t_0, t_0 + T], H^2) \cap C^1([t_0, t_0 + T], L^2)$ for some $T > 0$. For that purpose we estimate
\[\| U_\eta(t-t') F_\eta(t') \|_2 \leq \left\{ \| A \|_\infty \| \nabla v \|_2 + (\| A \|_\infty^2 + \| V \|_\infty) \| v \|_2 \right\}(t')\]  
(3.23)

\[\| U_\eta(t-t') \Delta F_\eta(t') \|_2 \leq C(\eta(t-t'))^{-1/2} \| \nabla (F_\eta - f)(t') \|_2 + \| \Delta f(t') \|_2\]  
\[\leq C(\eta(t-t'))^{-1/2}\left\{ \| A \|_\infty \| \Delta v \|_2 + (\| \nabla A \|_\infty + \| A \|_\infty^2 + \| V \|_\infty) \| \nabla v \|_2 + (\| A \|_\infty \| \nabla A \|_\infty + \| \nabla V \|_\infty) \| v \|_2 \right\}(t') + \| \Delta f(t') \|_2\]  
(3.24)

\[\| U_\eta(t-t') \partial_t F_\eta(t') \|_2 \leq C(\eta(t-t'))^{-1/2} \| A(t') \|_\infty \| \partial_t v(t') \|_2\]  
\[+ C\left\{ (\| A \|_\infty^2 + \| V \|_\infty) \| \partial_t v \|_2 + \| \partial_t A \|_\infty \| \nabla v \|_2\right.\]  
\[\left. + (\| A \|_\infty \| \partial_t A \|_\infty + \| \partial_t V \|_\infty) \| v \|_2 \right\}(t') + \| \partial_t f(t') \|_2.\]  
(3.25)

The RHS of (3.23)-(3.25) are in $L^1$ of the variable $t'$. By the same argument as in Proposition 3.1, one obtains a solution $v_\eta \in C(I_+, H^2) \cap C^1(I_+, L^2)$ with $I_+ = I \cap \{ t : t \geq t_0 \}$.

We next take the limit where $\eta$ tends to zero and for that purpose we need estimates of $v_\eta$ uniform in $\eta$ in the relevant space. We remark that $v_\eta$ satisfies the equation (compare (3.17) (3.18) with (3.22))
\[i\partial_t \partial_t v_\eta = K_n \partial_t v_\eta + f_{1\eta}.\]  
(3.26)

Let $y_0 = \| v_\eta \|_2$ and $y_1 = \| \partial_t v_\eta \|_2$. By Lemma 3.1, part (1), $\partial_t v_\eta$ satisfies
\[y_1(t_2)^2 - y_1(t_1)^2 \leq \int_{t_1}^{t_2} dt \ 2 \ \Im < \partial_t v_\eta, f_{1\eta} > (t)\]  
(3.27)
for all \( t_1, t_2 \in I_+ \), \( t_1 \leq t_2 \), so that
\[
\partial_t y_1 \leq \| f_{1\eta} \|_2 . \tag{3.28}
\]

We have already estimated \( y_0 \) by (3.15) uniformly in \( \eta \). For brevity we continue the estimates by omitting the index \( \eta \) in \( v_\eta \). Furthermore, we keep only the most dangerous, namely the most singular parts of \( A \) and \( V \) by keeping only the \( L^p \) component in all the \( L^p + L^\infty \) spaces that occur in the assumptions of the proposition. The more regular \( L^\infty \) components yield contributions that can be estimated similarly and involve lower norms of \( v \). From (3.22) we obtain
\[
\| f_{1\eta} \|_2 \leq (1 + \eta^2)^{1/2} \| \partial_t A \|_3 \left( \| \nabla v \|_6 + \| A \|_6 \| v \|_\infty \right) + \| \partial_t V \|_2 \| v \|_\infty \\
+ \| \partial_t f \|_2 . \tag{3.29}
\]

On the other hand from (3.3) we obtain by direct estimation
\[
\| \Delta v \|_2 \leq 2 \left\{ y_1 + \| A \|_6 \| \nabla v \|_3 + \| A \|_6^2 \| v \|_6 + \| V \|_2 \| v \|_\infty + \| f \|_2 \right\} . \tag{3.30}
\]

By Lemma 2.1, we estimate
\[
\begin{cases}
\| \nabla v \|_3 \lor \| v \|_\infty \leq C \| v \|_2^{1/4} \| \Delta v \|_2^{3/4} , \\
\| v \|_6 \leq C \| v \|_2^{1/2} \| \Delta v \|_2^{1/2} . \tag{3.31}
\end{cases}
\]

Substituting (3.31) into (3.30) and using Lemma 2.2, we obtain
\[
\| \Delta v \|_2 \leq C \left( y_1 + \| f \|_2 + m^4 y_0 \right) \tag{3.32}
\]

where
\[
 m = \| A \|_6 + \| V \|_2 . \tag{3.33}
\]

From (3.29) and Lemma 2.1, we obtain
\[
\| f_{1\eta} \|_2 \leq C \left\{ \| \partial_t A \|_3 \| \Delta v \|_2 + \left( \| \partial_t A \|_3 \| A \|_6 + \| \partial_t V \|_2 \right) y_0^{1/4} \| \Delta v \|_2^{3/4} \right\} \\
+ \| \partial_t f \|_2 \equiv M_0 \left( \| \Delta v \|_2 , y_0 \right) + \| \partial_t f \|_2 \tag{3.34}
\]

so that by (3.28) (3.32)
\[
\partial_t y_1 \leq \| f_{1\eta} \|_2 \leq M \left( y_1 + \| f \|_2 , y_0 \right) + \| \partial_t f \|_2 \tag{3.35}
\]
\[
M(z, y_0) = C \left\{ \| \partial_t A \|_3 (z + m^4 y_0) + (\| \partial_t A \|_3 \| A \|_6 + \| \partial_t V \|_2) \times (y_0^{1/4} z^{3/4} + m^4 y_0) \right\}.
\]  (3.36)

It then follows from Lemma 2.3 that \( y_1 \) is estimated by
\[
y_1(t) \leq \exp \left( C \int_{t_0}^{t} dt' \| \partial_t A(t') \|_3 \right) \left\{ y_1(t_0)^{1/4} + \int_{t_0}^{t} dt' \left( \| \partial_t V \|_2 \bar{y}_0^{1/4} \right) (t') \right. \\
+ \left( \int_{t_0}^{t} dt' \left( \| \partial_t A \|_3 (\| f \|_2 + m^4 \bar{y}_0) + \| \partial_t V \|_2 \left( \| f \|_2^{3/4} \bar{y}_0^{1/4} + m^4 \bar{y}_0 \right) \right) \right)^{1/4} \right\}^{1/4} \right\}^{1/4}.
\]  (3.37)

where \( \bar{y}_0 \) is an estimate of \( y_0 \) uniform in \( \eta \) as obtained previously from (3.15). Substituting (3.37) into (3.32) finally yields an a priori estimate of \( \| \Delta v \|_2 \). The estimates thereby obtained are uniform in \( \eta \). In fact
\[
y_1(t_0) = \| (K_\eta v_0 + f)(t_0) \|_2 \\
\leq \| \Delta v_0 \|_2 + 2 \| A \|_3 \| \nabla v_0 \|_6 + \left( \| A \|_4^2 + \| V \|_2 \right) \| v_0 \|_\infty + \| f \|_2. \]  (3.38)

The norms of \( A, V \) and \( f \) that occur in (3.32), (3.37) and (3.38) are controlled by the assumptions of the Proposition, so that the norms of the regularized quantities are bounded uniformly with respect to the regularization, since the regularisation is taken such as to decrease those norms.

We can now take the limit \( \eta \to 0 \). The solution \( v_\eta \) is uniformly bounded in \( L^\infty_{\text{loc}}(I_+, H^2) \cap H^1_{\text{loc}}(I_+, L^2) \). By compactness we can extract a subsequence which converges in the weak sense to some \( v \in L^\infty_{\text{loc}}(I_+, H^2) \cap H^1_{\text{loc}}(I_+, L^2) \). One can see that \( v \) satisfies (3.1) and therefore can be chosen in \( C_w(I_+, H^2) \cap C_w^1(I_+, L^2) \). Furthermore \( v_\eta \) converges pointwise to \( v \) weakly in \( H^2 \) and \( \partial_t v_\eta \) converges pointwise to \( \partial_t v \) weakly in \( L^2 \). Together with the fact that \( (K_\eta v_0 + f_\eta)(t_0) \) converges to \( (K v_0 + f)(t_0) \) strongly in \( L^2 \), this allows to prove that \( v \) satisfies the initial conditions \( v(t_0) = v_0 \) and \( i \partial_t v(t_0) = (K v_0 + f)(t_0) \). By Lemma 3.2, the solution \( v \) satisfies (3.6) and therefore is unique in \( (L^\infty_{\text{loc}} \cap C_w)(I_+, H^1) \).

We now turn to the strong continuity of \( \partial_t v \) in \( L^2 \) and of \( v \) in \( H^2 \). When \( \eta \to 0 \), \( y_1(t_0) \) converges to its value for \( \eta = 0 \) so that in the limit \( \eta \to 0 \) the RHS of (3.37) is bounded by its value for \( \eta = 0 \). On the other hand \( y_1(t) \) is non increasing in that limit under pointwise weak convergence of \( \partial_t v_\eta \) in \( L^2 \). Therefore (3.37) also holds in

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the limit $\eta \to 0$. Since $\partial_t v$ is weakly continuous in $L^2$ and since the RHS of (3.37) tends to $y(t_0)$ when $t$ decreases to $t_0$, $\partial_t v$ is strongly continuous from the right at $t_0$.

This completes the proof for $t \geq t_0$, except for strong continuity. A similar proof yields the corresponding results for $t \leq t_0$. In particular it yields strong continuity of $\partial_t v$ from the left at $t_0$, which together with the previous result yields strong continuity of $\partial_t v$ at $t_0$. Strong continuity of $\partial_t v$ for any $t \in I$ now follows from strong continuity at $t_0$ and from uniqueness by varying $t_0$ for a given solution $v$.

Finally strong continuity of $v$ in $H^2$ follows from the strong continuity of $\partial_t v$ and from (3.1) under the available continuity assumptions on $A$, $V$ and $f$.

**Part (2).** Uniqueness is a rewriting of the uniqueness part of Proposition 3.1, part (2). The equality (3.19) follows from Lemma 3.1, part (2) applied to $\partial_t v$ as a solution of (3.17).

We now turn to the study of the Cauchy problem for (3.1) at the level of regularity of $H^3$. In addition to the space derivatives of $v$ and to the time derivative $\partial_t v$ which satisfies (3.17) (3.18), we shall use the mixed space time derivative $\nabla_A \partial_t v$. That derivative satisfies an evolution equation obtained by applying $\nabla_A$ to (3.17), namely

\[
    i\partial_t \nabla_A \partial_t v = -\frac{1}{2} \nabla_A \Delta_A \partial_t v + V \nabla_A \partial_t v + g
\]

where

\[
    g = (\partial_t A + \nabla V) \partial_t v + \nabla_A f_1
\]

\[= (\partial_t A + \nabla V) \partial_t v + i(\partial_t A) \cdot \nabla_A \nabla A v + i(\nabla \partial_t A) \cdot \nabla_A v
\]

\[+ (\partial_t V) \nabla_A v + (\nabla \partial_t V) v + \nabla_A \partial_t f.
\]

(3.40)

The result can be stated as follows.

**Proposition 3.3.** Let $I$ be an interval. Let $A$, $V$ and $f$ satisfy $A \in C(I, L^s + L^\infty)$, $\partial_t A \in L^1_{loc}(I, L^s + L^\infty)$, $\nabla A \in C(I, L^r + L\infty)$, $\partial_t \nabla A \in L^1_{loc}(I, L^r + L^\infty)$ for some $r$, $s$, with $1/r + 1/s = 1/2$, $2 \leq r \leq 3$, $V \in C(I, L^6 + L\infty)$, $\partial_t V \in L^1_{loc}(I, L^6 + L\infty)$, $\nabla V \in C(I, L^2 + L\infty)$, $\partial_t \nabla V \in L^1_{loc}(I, L^2 + L\infty)$, $f \in C(I, H^1)$, $\partial_t f \in L^1_{loc}(I, H^1)$. Let $t_0 \in I$ and $v_0 \in H^3$. Then
(1) There exists a unique solution \( v \in \mathcal{C}(I, H^3) \cap \mathcal{C}^1(I, H^1) \) of (3.1) in \( I \) with \( v(t_0) = v_0 \). That solution satisfies (3.6) and (3.19) for all \( t_1, t_2 \in I \). That solution is actually unique in \((L^\infty_{loc} \cap C_w)(I, H^1)\).

(2) Let in addition \( A \in L^2_{loc}(I, L^\infty) \), \( \nabla A \in L^1_{loc}(I, L^\infty) \) and \( V \in L^1_{loc}(I, L^\infty) \). Then the previous solution \( v \) is actually unique in \((L^\infty_{loc} \cap C_w)(I, L^2)\) and \( v \) satisfies

\[
\| \nabla_A \partial_t v(t_2) \|^2_2 - \| \nabla_A \partial_t v(t_1) \|^2_2 = \int_{t_1}^{t_2} dt \ 2 \ \text{Im} < \nabla_A \partial_t v, g > (t) \tag{3.41}
\]

for all \( t_1, t_2 \in I \), where \( g \) is defined by (3.40).

**Remark 3.3.** We recall that assumptions on the time derivative of a function and on that function are related as explained in Remark 2.2. On the other hand the assumptions on \((A, \partial_t A)\) follow from those on \((\nabla A, \partial \nabla A)\) by Lemma 2.1 for \( r \geq 12/5 \), \( s \leq 12 \). We have written both assumptions explicitly in order to avoid that restriction. Similarly the assumptions on \((V, \partial_t V)\) follow from those on \((\nabla V, \partial_t \nabla V)\).

**Proof.** The proof proceeds by a parabolic regularization and a limiting procedure as in the case of Proposition 3.2. We consider first the case \( t \geq t_0 \). We replace (3.1) by (3.3) with \( 0 < \eta \leq 1 \) and with \( A, V \) and \( f \) regularized in space as in Proposition 3.2. We use again the integral equation for \( v \), namely (3.9), and we solve that equation locally in time by contraction in \( \mathcal{C}([t_0, t_0 + T], H^2) \cap \mathcal{C}^1([t_0, t_0 + T], H^1) \) for some \( T > 0 \). For that purpose we use again the estimates (3.23) (3.24) (3.25), supplemented by an additional estimate for \( U_\eta(t - t') \nabla \partial_t F_\eta(t') \), namely

\[
\| U_\eta(t - t') \nabla \partial_t F_\eta(t') \|_2 \leq C(\eta(t - t'))^{-1/2} \| A(t') \|_\infty \| \nabla \partial_t v(t') \|_2 \\
+ C\bigg\{ \| \nabla A \|_\infty + \| A \|_{2, \infty}^2 + \| V \|_\infty \bigg\} \| \nabla \partial_t v \|_2 \\
+ \| \partial_t A \|_\infty \| \Delta v \|_2 + (\| A \|_\infty \| \nabla A \|_\infty + \| \nabla V \|_\infty) \| \partial_t v \|_2 \\
+ (\| \nabla \partial_t A \|_\infty + \| A \|_\infty \| \partial_t A \|_\infty + \| \partial_t V \|_\infty) \| \nabla v \|_2 \\
+ \big( \| \nabla \partial_t A^2 \|_\infty + \| \nabla \partial_t V \|_\infty \big) \| v \|_2 \bigg\}(t') + \| \nabla \partial_t f(t') \|_2. \tag{3.42}
\]

The RHS of (3.42) as well as those of (3.23)-(3.25) is in \( L^1 \) of the variable \( t' \). By the same argument as in Propositions 3.1 and 3.2, one obtains a solution \( v_\eta \in \mathcal{C}(I_+, H^2) \cap \mathcal{C}^1(I_+, H^1) \), with \( I_+ = I \cap \{ t : t \geq t_0 \} \).

We next take the limit where \( \eta \) tends to zero, and for that purpose we need estimates of \( v_\eta \) uniform in \( \eta \) in the relevant space. We shall need the evolution
equation for $\nabla_A \partial_t v_\eta$, namely

$$i \partial_t \nabla_A \partial_t v_\eta = -(1/2)(1-i\eta)\nabla_A \Delta_A \partial_t v_\eta + V \nabla_A \partial_t v_\eta + g_\eta$$ \hspace{1cm} (3.43)

where (see (3.40))

$$g_\eta = (\partial_t A + \nabla V) \partial_t v_\eta + i(1-i\eta) ((\partial_t A) \cdot \nabla_A \otimes \nabla_A v_\eta + (\nabla A) \cdot \nabla_A v_\eta)$$

$$+ (\partial_t V) \nabla_A v_\eta + (\nabla \partial_t V) v_\eta + \nabla_A \partial_t f.$$ \hspace{1cm} (3.44)

Let $y_0 = \| v_\eta \|_2$, $y_1 = \| \partial_t v_\eta \|_2$ and $y = \| \nabla_A \partial_t v_\eta \|_2$. By a minor variation of Lemma 3.1, part (1), $\nabla_A \partial_t v_\eta$ satisfies

$$y(t_2)^2 - y(t_1)^2 \leq \int_{t_1}^{t_2} dt \Im < \nabla_A \partial_t v_\eta, g_\eta > (t)$$ \hspace{1cm} (3.45)

for all $t_1, t_2 \in I_+$, $t_1 \leq t_2$, so that

$$\partial_t y \leq \| g_\eta \|_2.$$ \hspace{1cm} (3.46)

We have already estimated $y_0$ by (3.15), $y_1$ by (3.37) and $\| \Delta v_\eta \|_2$ by (3.32) uniformly in $\eta$. We now estimate $y$ and at the same time $\| \Delta v_\eta \|_2$ (which is not part of the norm of the space of resolution). As in the proof of Proposition 3.2, we omit the index $\eta$ in $v_\eta$ and we keep only the most dangerous parts of $A$ and $V$ by dropping the $L^\infty$ components allowed by the assumptions on $A$, $V$. From (3.44) (3.46) we obtain

$$\partial_t y \leq (\| \partial_t A \|_3 + \| \nabla V \|_3) \| \partial_t v \|_6 + (1+\eta^2)^{1/2} \left( \| \partial_t A \|_s \| \nabla^2 v \|_r \right.$$

$$+ \| \nabla \partial_t A \|_r \| \nabla_A v \|_s \big) + \| \nabla \partial_t V \|_2 \| v \|_\infty + \| \partial_t V \|_6 \| \nabla_A v \|_3 + \| \nabla_A \partial_t f \|_2.$$ \hspace{1cm} (3.47)

We then estimate

$$\| \nabla^2 v \|_r \leq \| \nabla^2 v \|_r + 2 \| A \|_s \| \nabla v \|_r1 + \| A \|_s^2 \| v \|_r2 + \| \nabla A \|_r \| v \|_\infty$$ \hspace{1cm} (3.48)

$$\| \nabla_A v \|_s \leq \| \nabla v \|_s + \| A \|_s \| v \|_\infty$$ \hspace{1cm} (3.49)

$$\| \nabla_A v \|_3 \leq \| \nabla v \|_3 + \| A \|_s \| v \|_r3$$ \hspace{1cm} (3.50)
where \( 1/r_1 = 1/2 - 2/s, 1/r_2 = 1/2 - 3/s, 1/r_3 = 1/3 - 1/s \), so that \( r_1 \leq r_3 \leq 6 \leq s \).

We next estimate by (2.3) and Lemma 2.1

\[
\| \partial_t v \|_6 \leq C \| \nabla_A \partial_t v \|_2 = Cy
\]  

(3.51)

\[
\| \nabla^2 v \|_r \leq C \| \nabla \Delta v \|_2^\delta \| \Delta v \|_2^{1-\delta}
\]  

(3.52)

\[
\| \nabla v \|_s \leq C \| \nabla \Delta v \|_2^{1/2-\delta} \| \Delta v \|_2^{1/2+\delta}
\]  

(3.53)

where \( 0 \leq \delta = \delta(r) \equiv \delta(3/2 - 3/r) \leq 1/2 \). Furthermore

\[
\| \nabla \Delta v \|_2 \leq \| \nabla \Delta A v \|_2 + \| A \|_s \| \nabla^2 v \|_r + 2 \| \nabla A \|_r \| \nabla v \|_s + 2 \| \nabla A \|_r \| A \|_s \| v \|_\infty + \| A \|_s^2 \| \nabla v \|_{r_1}
\]  

(3.54)

so that by (3.52) (3.53) and Lemma 2.2

\[
\| \nabla \Delta v \|_2 \leq C \left\{ \| \nabla \Delta A v \|_2 + \left( \| A \|_s^{1(1-\delta)} + \| \nabla A \|_r^{1/(2+\delta)} \right)\| \Delta v \|_2 + 2 \| \nabla A \|_r \| A \|_s \| v \|_\infty + \| A \|_s^2 \| \nabla v \|_{r_1} \right\}.
\]  

(3.55)

On the other hand, by a direct estimate of (3.3), we obtain

\[
\| \nabla \Delta A v \|_2 \leq 2 \left\{ y + C \| A \|_s \| y \|_1^{1-\delta} y^\delta + \| \nabla V \|_2 \| v \|_\infty + \| V \|_6 \| \nabla v \|_3 + \| \nabla f \|_2 \right\}
\]  

(3.56)

where we have used (3.51).

We substitute (3.56) into (3.55), we substitute the result into (3.52) (3.53), we substitute the result into (3.48) (3.49), and we substitute the result and (3.50) (3.51) into (3.47). Using the fact that the remaining norms of \( v \) in (3.48)-(3.50) and in (3.55) (3.56) are controlled by \( \| v; H^2 \| \) and using (3.32), we finally obtain an estimate of the form

\[
\partial_t y \leq N(y, y_1, y_0)
\]  

(3.57)

where \( N \) depends in addition on \( A, V \) and \( f \) through the norms associated with the assumptions of the proposition, \( N \) is homogeneous of degree 1 in \( y, y_0, y_1 \) and \( f \), and \( N \) as a function of \( y \) is the sum of a finite number of powers between 0 and 1. The estimate (3.57) plays the same role in the proof of this proposition as (3.35) in the proof of Proposition 3.2. Using the fact that \( y_0 \) and \( y_1 \) have already been
estimated uniformly in $\eta$ in the proof of Proposition 3.2 and applying Lemma 2.3, we obtain an estimate of the form

$$y(t) \leq P(y(t_0), t)$$

(3.58)

where $P(z, t)$ is uniform in $\eta$, increasing in $z$ and continuous and increasing in $t$ for $t \geq t_0$ with $P(z, t_0) = z$. The estimate (3.58) plays the same role in the proof of this proposition as (3.37) in the proof of Proposition 3.2. Now $y(t_0)$ is estimated uniformly in $\eta$ for $v_0 \in H^3$. This follows from the estimate

$$y(t_0) = \| \nabla_A(K_\eta v_0 + f)(t_0) \|_2 \leq C \left\{ \| \nabla \Delta v_0 \|_2 + \| A \|_s \| \nabla^2 v_0 \|_r 
+ \| A \|_r \| \nabla v_0 \|_s + \| A \|_s^2 \| \nabla v_0 \|_{r_1} + \| V \|_6 \| \nabla v_0 \|_3 
+ (\| A \|_s \| \nabla A \|_r + \| \nabla V \|_2) \| v_0 \|_\infty + \| A \|_s^3 \| v_0 \|_{r_2} 
+ \| A \|_s \| V \|_6 \| v_0 \|_{r_3} + \| \nabla f \|_2 + \| A \|_s \| f \|_r \right\}.$$  

(3.59)

The estimates (3.58) (3.59) provide an estimate of $y$ uniform in $\eta$. Together with the estimates of $\partial_t v$ in $L^2$ and of $v$ in $H^2$ that follow from (3.32) (3.37) and with (3.55) (3.56), they provide an a priori estimate of $v$ in $L^\infty_{\text{loc}}(I_+, H^3) \cap H^1_{\text{loc}}(I_+, H^1)$, uniformly in $\eta$.

We can now take the limit $\eta \to 0$. The end of the proof is the same as in Proposition 3.2 and will be omitted.

**Part (2)** is proved in the same way as Part (2) of Proposition 3.2.

\[\square\]

We now turn to the study of the Cauchy problem for (3.1) at the level of regularity of $H^4$. We shall control the regularity of $v$ at that level through the use of the second time derivative $\partial_t^2 v$. That derivative satisfies an evolution equation obtained by applying $\partial_t^2$ to (3.1), namely

$$i \partial_t \partial_t^2 v = K \partial_t^2 v + f_2$$

(3.60)

where

$$f_2 = 2 (\partial_t K) \partial_t v + f_3$$

(3.61)

$$f_3 = (\partial_t^2 K) v + \partial_t^2 f = i (\partial_t^2 A) \cdot \nabla_A v + (\partial_t^2 A^2 + \partial_t^2 V) v + \partial_t^2 f$$

(3.62)

and $\partial_t K$ can be read from (3.18).

In the present case however there arises a difficulty with the initial condition. We shall perform the same regularization as in the proof of Proposition 3.2, replacing
(3.1) by (3.3) and in addition regularizing $A$, $V$ and $f$ in space. We shall again use the integral equation associated with (3.3), namely (3.9), and we shall need the second time derivative of $\phi(v)$. By integration by parts, that derivative is seen to be

\[
\left( \partial_t^2 \phi(v) \right) (t) = -U_\eta(t-t_0) \left( K_\eta(K_\eta v_0 + f_\eta) + i (\partial_t K_\eta) v_0 + i \partial_t f_\eta \right) (t_0) \\
- i \int_{t_0}^{t} dt' U_\eta(t-t') \partial_t^2 F_\eta(t').
\]

(3.63)

For smooth regularized $A$ and $V$, the domain of $K^2_\eta$ is $H^4$, thereby suggesting to take $v_0 \in H^4$. However we shall make only weak assumptions on the space regularity of $A$ and $V$, namely assumptions of the $L^r$ type, but no assumptions on the space derivatives. Under such assumptions, whereas the domain of $K$ for $\eta = 0$ is easily seen to remain $H^2$, the domain of $K^2$ can be very complicated and completely different from $H^4$. As a consequence, for fixed $v_0 \in H^1$, $K^2_\eta v_0$ may very well blow up in the limit $\eta \to 0$. This has two consequences. First the initial condition $v_0$ should be chosen in a way adapted to $K$, namely such that $K(Kv_0 + f) \in L^2$ for $t = t_0$ and $\eta = 0$. Such a $v_0$ will in general not be in $H^4$. Second we need to regularize the initial $v_0$ to some $v_0_\eta$ for $\eta > 0$. This is most simply done by imposing the condition that $K_\eta(K_\eta v_0 + f_\eta)$ be independent of $\eta$. Actually for technical reasons it is convenient to replace $K_\eta$ in that condition by $(\rho + K_\eta)$ for some sufficiently large positive $\rho$, such that $\rho$ belongs to the resolvent set of $K_\eta$ and that $(\rho + K_\eta)$ be invertible for all $\eta$. Thus we choose $v_0$ and we regularize it in such a way that at $t = t_0$

\[
(\rho + K_\eta)^2 v_0_\eta + (\rho + K_\eta) f_\eta = (\rho + K)^2 v_0 + (\rho + K) f = z \in L^2,
\]

namely such that the LHS of (3.64) be independent of $\eta$ and be a fixed $z \in L^2$. In other words $v_0$ should be chosen as

\[
v_0 = - (\rho + K)^{-1} f + (\rho + K)^{-2} z
\]

(3.65)

namely $v_0$ should be a given vector of $\mathcal{D}(K)$ modulo $\mathcal{D}(K^2)$, and $v_0$ should be regularized to

\[
v_0_\eta = - (\rho + K_\eta)^{-1} f_\eta + (\rho + K_\eta)^{-2} z.
\]

(3.66)

As in Propositions 3.2 and 3.3, we shall need that all the initial conditions for the relevant norms converge when $\eta \to 0$ namely that $(\phi(v))t_0 = v_0_\eta$ and $i(\partial_t \phi(v))(t_0) = (K_\eta v_0 + f_\eta)(t_0)$ converge in $H^2$ and that

\[- \left( \partial_t^2 \phi(v) \right) (t_0) = (K_\eta(K_\eta v_0 + f_\eta) + i (\partial_t K_\eta) v_0_\eta + i \partial_t f_\eta) (t_0) \]

(3.67)
converge in $L^2$ as $\eta \to 0$. This will require some information on the operator $K_\eta$ and in particular the convergence of $K_\eta$ to $K$ in the strong resolvent sense [11].

**Lemma 3.3.** Let $I$ be an interval, let $A \in \mathcal{C}(I, L^6 + L^\infty) \cap \mathcal{C}^1(I, L^3 + L^\infty)$, $V \in \mathcal{C}^1(I, L^2 + L^\infty)$. Let $0 \leq \eta \leq 1$. Let $K$ be defined by (3.2) and let $K_\eta$ be defined by (3.4) with $A$ and $V$ regularized as in the proof of Proposition 3.2. Then, for fixed $t = t_0 \in I$

1. There exists $\rho > 0$ independent of $\eta$ such that $\rho + K$ and $\rho + K_\eta$ have bounded inverses from $L^2$ to $H^2$, with $(\rho + K_\eta)^{-1}$ uniformly bounded in $\eta$ as an operator from $L^2$ to $H^2$.

2. When $\eta \to 0$, $(\rho + K_\eta)^{-1}$ converges strongly to $(\rho + K)^{-1}$ and $(\partial_t K_\eta)(\rho + K_\eta)^{-1}$ converges strongly to $(\partial_t K)(\rho + K)^{-1}$ in $L^2$.

**Proof.** Part (1) follows by standard arguments from the fact that the $A$ and $V$ dependent parts in $K_\eta$ are a Kato small perturbation of the Laplacian uniformly with respect to $\eta \in [0, 1]$. In fact let $A = A_6 + A_\infty$ with $A_6 \in L^6$ and $A_\infty \in L^\infty$, and similarly $V = V_2 + V_\infty$. Then by Lemma 2.1

\[
\| iA \cdot \nabla v + (A^2/2)v \|_2 \leq \| A_6 \|_6 \| \nabla v \|_3 + \| A_6 \|_6^2 \| v \|_6 \\
+ \| A_\infty \|_\infty \| \nabla v \|_2 + \| A_\infty \|_\infty^2 \| v \|_2 \\
\leq C \left( \| A_6 \|_6 \| v \|^{1/4}_6 \| \nabla v \|^{3/4}_2 + \left( \| A_6 \|_6^2 + \| A_\infty \|_\infty \right) \| v \|^{1/2}_2 \| \nabla v \|^{1/2}_2 \right) \\
+ \| A_\infty \|_\infty^2 \| v \|_2 \\
\leq \mu \| \Delta v \|_2 + C \left\{ \mu^{-3} + \mu^{-1} \right\} \| A_6 \|_6^4 + \left( \mu^{-1} + 1 \right) \| A_\infty \|_\infty^2 \| v \|_2 \quad (3.68)
\]

for all $\mu > 0$, and similarly

\[
\| Vv \|_2 \leq \mu \| \Delta v \|_2 + \left( C \mu^{-3} \| V_2 \|_2^4 + \| V_\infty \|_\infty \right) \| v \|_2 \quad . \quad (3.69)
\]

The uniformity in $\eta$ follows from the fact that the regularization does not increase the norms of $A$ and $V$ that appear in (3.68) (3.69).

**Part (2).** We first remark that $K_\eta - K$ and $\partial_t K_\eta - \partial_t K$ converge strongly to zero from $H^2$ to $L^2$ when $\eta \to 0$. For $K_\eta - K$ and for the $A_6$ and $V_2$ components, this follows from the estimates (3.68) (3.69) applied to the differences and from the fact that the regularization tends strongly to the identity in $L^r$ for $1 \leq r < \infty$. For the $L^\infty$ components, the result follows from the pointwise almost everywhere
convergence of the regularized quantities to the unregularized ones and from the dominated convergence theorem. For $\partial_t K_\eta - \partial_t K$ the same argument applies with the only difference that now $\partial_t A$ is decomposed as $(\partial_t A)_3 + (\partial_t A)_\infty$ and that the contribution of $(\partial_t A)_3$ is estimated by

$$
\| (\partial_t A)_3 \cdot \nabla_A v \|_2 \leq \| (\partial_t A)_3 \|_3 \left( \| \nabla v \|_6 + \| A_6 \|_6 \| v \|_\infty + \| A_\infty \|_\infty \| v \|_6 \right)
\leq C \| (\partial_t A)_3 \|_3 \left( \| \Delta v \|_2 + \| A_6 \|_6 \| v \|_{2/4} \| \Delta v \|_{3/4}^{2/4}
\right.
\left. + \| A_\infty \|_\infty \| v \|_2^{1/2} \| \Delta v \|_2^{1/2} \right).
$$

(3.70)

We now turn to the proof of strong resolvent convergence. Let $v \in L^2$. Then

$$
\left( (\rho + K)^{-1} - (\rho + K_\eta)^{-1} \right) v = (\rho + K_\eta)^{-1} (K_\eta - K) (\rho + K)^{-1} v
$$

(3.71)

which tends to zero strongly in $L^\infty$ by the previous convergence of $K_\eta - K$ since $(\rho + K)^{-1} v$ is a fixed vector in $H^2$ and $(\rho + K_\eta)^{-1}$ is uniformly bounded in $\eta$ as an operator in $L^2$. Finally

$$
\begin{align*}
\{ (\partial_t K_\eta) (\rho + K_\eta)^{-1} - (\partial_t K) (\rho + K)^{-1} \} v &= (\partial_t K_\eta - \partial_t K) (\rho + K)^{-1} v \\
- (\partial_t K_\eta) (\rho + K_\eta)^{-1} (K_\eta - K) (\rho + K)^{-1} v.
\end{align*}
$$

(3.72)

The first term in the RHS converges to zero strongly in $L^2$ by the previous strong convergence of $\partial_t K_\eta$ to $\partial_t K$ as an operator from $H^2$ to $L^2$. The second term converges to zero by the previous convergence of $K_\eta$ to $K$ and from the fact that $(\partial_t K_\eta)(\rho + K_\eta)^{-1}$ is uniformly bounded in $\eta$ as an operator in $L^2$.

\[ \square \]

**Remark 3.4.** If it were not for the fact that the regularization does not converge to the identity strongly in $L^\infty$, and in particular if $A$, $V$ and $\partial_t A$, $\partial_t V$ did not have $L^\infty$ components, we would obtain norm convergence instead of strong convergence in Part (2) of the Lemma.

We can now state the result for the Cauchy problem for (3.1) at the level of regularity of $H^4$.

**Proposition 3.4.** Let $I$ be an interval. Let $A$, $V$ and $f$ satisfy $A \in C(I, L^6 + L^\infty) \cap C^1(I, L^3 + L^\infty)$, $\partial_t A \in L^2_{loc}(I, L^4 + L^\infty)$, $\partial_t^2 A \in L^1_{loc}(I, L^3 + L^\infty)$, $V \in C^1(I, L^2 + L^\infty)$, $\partial_t^2 V \in L^1_{loc}(I, L^2 + L^\infty)$, $f \in C^1(I, L^2)$, $\partial_t^2 f \in L^1_{loc}(I, L^2)$. Let $t_0 \in I$ and let $v_0 \in H^2$.
be such that \( v_0 + (\rho + K(t_0))^{-1}f(t_0) \in \mathcal{D}(K(t_0)^2) \) for some \( \rho > 0 \) sufficiently large. Then

(1) There exists a unique solution \( v \in \mathcal{C}^1(I, H^2) \cap \mathcal{C}^2(I, L^2) \) of (3.1) in \( I \) with \( v(t_0) = v_0 \). That solution satisfies (3.6) and (3.19), namely

\[
\| \partial_t^j v(t_2) \|_2 \leq \| \partial_t^j v(t_1) \|_2 = \int_{t_1}^{t_2} dt \, 2 \text{ Im} < \partial_t^j v, f_j > (t) \tag{3.73}
\]

for \( j = 0, 1 \) and for all \( t_1, t_2 \in I \), where \( f_0 = f \) and \( f_1 \) is defined by (3.18). That solution is actually unique in \((L^2_{\text{loc}} \cap C_w)(I, H^1)\). Furthermore \( iK\partial_t v = K(Kv + f) \in \mathcal{C}(I, L^2) \).

(2) Let in addition \( A \in L^2_{\text{loc}}(I, L^\infty) \), \( V \in L^1_{\text{loc}}(I, L^\infty) \) and \( \nabla A \in L^1_{\text{loc}}(I, L^\infty) \). Then the previous solution \( v \) is actually unique in \((L^\infty \cap C_w)(I, L^2)\) and \( v \) satisfies (3.73) for \( j = 2 \) and for all \( t_1, t_2 \in I \), where \( f_2 \) is defined by (3.61) (3.62).

**Remark 3.5.** If \( A, V \) and \( f \) are sufficiently regular in the space variable, the condition \( K(Kv + f) \in \mathcal{C}(I, L^2) \) is equivalent to the condition \( v \in \mathcal{C}(I, H^1) \).

**Proof.** The proof proceeds by a parabolic regularization and a limiting procedure as in the case of Proposition 3.2. We consider first the case \( t \geq t_0 \). We replace (3.1) by (3.3) with \( 0 < \eta \leq 1 \) and with \( A, V \) and \( f \) regularized as in Proposition 3.2. We use again the integral equation for \( v \), namely (3.9), now however with the initial data \( v_0 \) regularized to \( v_{0\eta} \) according to (3.66) with \( z \) defined by (3.65), or equivalently by (3.64), and we solve that equation locally in time by contraction in \( \mathcal{C}([t_0, t_0 + T], H^2) \cap \mathcal{C}^1([t_0, t_0 + T], H^1) \cap \mathcal{C}^2([t_0, t_0 + T], L^2) \) for some \( T > 0 \). For that purpose we use again the estimates (3.23) (3.24) (3.25) and (3.42), supplemented by an additional estimate for \( U_\eta(t - t')\partial_t^2 F_\eta(t') \), namely

\[
\| U_\eta(t - t')\partial_t^2 F_\eta(t') \|_2 \leq C(\eta(t - t'))^{-1/2} \| A(t') \|_{\infty} \| \partial_t^2 v(t') \|_2 + C \left\{ \| \partial_t A \|_{\infty} \| \nabla \partial_t v \|_2 + \| \partial_t^2 A \|_{\infty} \| \nabla v \|_2 \right. \\
+ \left( \| A \|_{L^2_{\infty}} + \| V \|_{L^\infty} \right) \| \partial_t^2 v \|_2 + (\| A \|_{L^1_{\infty}} \| \partial_t A \|_{L^\infty} + \| \partial_t V \|_{L^\infty} ) \| \partial_t v \|_2 \\
+ \left( \| \partial_t^2 A \|_{L^1_{\infty}} \| A \|_{L^\infty} + \| \partial_t A \|_{L^2_{\infty}} + \| \partial_t^2 V \|_{L^\infty} \right) \| v \|_2 \right\} + \| \partial_t^2 f \|_2. \tag{3.74}
\]

The RHS of (3.74) as well as those of (3.23)-(3.25), (3.42) is in \( L^1 \) of the variable \( t' \). On the other hand, the choice (3.66) of the regularized \( (\phi(v))(t_0) \) yields (see (3.67))

\[
(\phi(v))(t_0) = v_{0\eta} \in \mathcal{D}(K_\eta) = H^2, \tag{3.75}
\]
Using those properties and the previous estimates, one obtains a solution \( v_\eta \in C(I_+, H^2) \cap C^1(I_+, H^1) \cap C^2(I_+, L^2) \) of (3.1) with \( I_+ = I \cap \{ t : t \geq t_0 \} \) by the same argument as in the proof of Proposition 3.1. Furthermore it follows from (3.3), more precisely from (3.26), that \( \Delta \partial_t v_\eta \in C(I_+, L^2) \) so that \( v_\eta \in C^1(I_+, H^2) \).

We next take the limit where \( \eta \) tends to zero and for that purpose we need estimates of \( v_\eta \) uniform in \( \eta \) in the relevant spaces. We remark that \( v_\eta \) satisfies the equation (compare with (3.60)-(3.62))

\[
(i \partial_t \phi(v))(t_0) = (K_\eta v_{0\eta} + f_\eta) = -\rho v_{0\eta} + (\rho + K_\eta)^{-1} z \in D(K_\eta) = H^2, \quad (3.76)
\]

\[
- \left( \partial_t^2 \phi(v) \right)(t_0) = K_\eta (K_\eta v_{0\eta} + f_\eta) + i (\partial_t K_\eta) v_{0\eta} + i \partial_t f_\eta
\]

\[
= -\rho K_\eta v_{0\eta} + K_\eta (\rho + K_\eta)^{-1} z + i (\partial_t K_\eta) v_{0\eta} + i \partial_t f_\eta \in L^2. \quad (3.77)
\]

and \( \partial_t K_\eta \) can be read from (3.22).

Let \( y_j = \| \partial_t^j v_\eta \|_2, j = 0, 1, 2 \). By Lemma 3.1, part (1), \( \partial_t^2 v_\eta \) satisfies

\[
y_2(t_2)^2 - y_2(t_1)^2 \leq \int_{t_1}^{t_2} dt \ 2 \ \text{Im} \ < \partial_t^2 v_\eta, f_{2\eta} > (t) \quad (3.81)
\]

for all \( t_1, t_2 \in I_+, t_1 \leq t_2 \), so that

\[
\partial_t y_2 \leq \| f_{2\eta} \|_2 . \quad (3.82)
\]

We have already estimated \( y_0 \) by (3.15) and \( y_1 \) in the proof of Proposition 3.2. Now however the initial values of \( y_0 \) and \( y_1 \) are

\[
y_0(t_0) = \| v_{0\eta} \|_2, \quad y_1(t_0) = \| (K_\eta v_{0\eta} + f_\eta)(t_0) \|_2
\]

by (3.75) (3.76). It follows from (3.15) and from Lemma 3.3 that \( y_0(t_0) \) and \( y_1(t_0) \) are uniformly bounded in \( \eta \), so that the estimates of \( y_0 \) and \( y_1 \) are also uniform in \( \eta \). (See especially (3.35) (3.37)). We next estimate \( y_2 \), omitting again the index \( \eta \) in
\( \nu_\eta \) for brevity and keeping only the most dangerous parts of \( A \) and \( V \) by dropping again the \( L^\infty \) components allowed by the assumptions on \( A \) and \( V \). From (3.79) (3.80), by exactly the same estimates as in the proof of Proposition 3.2 (see (3.29) (3.34)), we obtain

\[
\| f_{2\eta} \| \leq 2M_0 (\| \Delta \partial_t v \|_2, y_1) + \| f_{3\eta} \|_2
\]

where \( M_0 \) is defined in (3.34). On the other hand from (3.26) and again by exactly the same estimates as in the proof of Proposition 3.2 (see (3.30) (3.32)) we obtain

\[
\| \Delta \partial_t v \|_2 \leq C \left( y_2 + \| f_{1\eta} \|_2 + m^4 y_1 \right)
\]

with the same constant \( C \) as in (3.32). From (3.82)-(3.84) we then obtain

\[
\partial_t y_2 \leq \| f_{2\eta} \|_2 \leq 2M_0 (\| f_{2\eta} \|_2, y_1) + \| f_{3\eta} \|_2
\]

where \( M \) is defined by (3.36). The \( L^2 \) norm of \( f_{1\eta} \) is already estimated by (3.34) and it remains only to estimate \( f_{3\eta} \). We obtain

\[
\| f_{3\eta} \|_2 \leq \| \partial_t^2 A \|_3 (\| \nabla v \|_6 + \| A \|_6 \| v \|_\infty) \\
+ \left( \| \partial_t A \|_4^2 + \| \partial_t^2 V \|_2 \right) \| v \|_\infty + \| \partial_t^2 f \|_2 \\
\leq M_1 (y_1 + \| f \|_2, y_0) + \| \partial_t^2 f \|_2
\]

by Lemma 2.1 and (3.32), where

\[
M_1(z, y_0) = C \left\{ \| \partial_t^2 A \|_3 \left( z + m^4 y_0 \right) \\
+ \left( \| \partial_t^2 A \|_3 \| A \|_6 + \| \partial_t A \|_4^2 + \| \partial_t^2 V \|_2 \right) \left( y_0^{1/4} z^{3/4} + m^3 y_0 \right) \right\}.
\]

Note in particular that the assumptions on the time derivatives of \( A, V \) and \( f \) made in the proposition are tailored to ensure that the estimate (3.86) is integrable in time uniformly with respect to the regularization under the already available estimates on \( y_0, y_1 \). From (3.85) (3.86), from the previous estimates of \( y_0 \) and \( y_1 \) and from Lemma 2.1, it follows that \( y_2 \) satisfies an estimate of the form

\[
y_2(t) \leq P_2(y_2(t_0), t)
\]

where \( P_2(z,t) \) is uniform in \( \eta \), increasing in \( z \), continuous and increasing in \( t \) for \( t \geq t_0 \), and satisfies \( P(z,t_0) = z \). Actually that estimate is obtained from (3.37) by replacing \( y_0, y_1, f, \partial_t f \) by \( y_1, y_2, f_{1\eta}, f_{3\eta} \) and using the available estimates for \( y_1, \)
$f_{1\eta}$ and $f_{3\eta}$. Substituting (3.88) into (3.84) then yields an estimate of $\| \Delta \partial_t v \|_2$. Furthermore the initial value of $y_2$, namely

$$y_2(t_0) = \| (K_\eta (K_\eta v_{0\eta} + f_\eta) + i(\partial_t K_\eta)v_{0\eta} + i\partial_t f_\eta)(t_0) \|_2$$

(3.89)

is uniformly bounded in $\eta$ by (3.66) (3.77) and Lemma 3.3. Therefore the estimates (3.88) of $y_2$ and (3.84) of $\| \Delta \partial_t v \|_2$ are also uniform in $\eta$, so that $v$ is estimated in $H^2_{x,loc}(I_+, H^2) \cap H^2_{x,loc}(I_+, L^2)$ uniformly in $\eta$.

We can now take the limit $\eta \to 0$. For that purpose, we need the strong convergence of the initial conditions as $\eta \to 0$, more precisely the convergence of $(\phi(v))(t_0)$ and $(\partial_t \phi(v))(t_0)$ in $H^2$ and the convergence of $(\partial^2_t \phi(v))(t_0)$ in $L^2$. Those convergences follow from (3.75)-(3.77), from (3.66) and from Lemma 3.3. With that information available, the end of the proof is the same as that of Proposition 3.2 and will be omitted.

Part (2) is proved in the same way as Part (2) of Proposition 3.2.

\[\square\]

### 4 Scattering theory at the level of $L^2$ and $H^2$ for $w$

In this section we begin the study of scattering theory for (1.1) with a potential $A$ satisfying conditions of the type (2.26), or equivalently for (1.16) with a potential $B$ satisfying conditions of the type (2.27). Here we study that theory for (1.1) in the spaces $L^2$ and $FH^2$, or equivalently for (1.16) in the spaces $L^2$ and $H^2$. This will be done by studying (1.12) in $L^2$ and $H^2$ and will rely on Propositions 3.1 and 3.2.

The main result of this section has been stated as Proposition 1.1 in the introduction and is repeated here as the following proposition.

**Proposition 4.1.** Let $A$ satisfy

$$\| P^j \partial_x^\alpha A \|_r \vee \| P^j (x \cdot A) \|_r \leq C t^{-1+2/r}$$

(1.20) $\equiv$ (4.1)

where $P = t\partial_t + x \cdot \nabla$, for $0 \leq j + |\alpha| \leq 1$, $2 \leq r \leq \infty$ and for all $t \in [1, \infty)$.

1. Let $X = L^2$ or $FH^2$. Then for any $u_+ \in X$, there exists a unique solution $u$ of (1.1) such that $\bar{u} \in C([1, \infty), X)$ and such that

$$\| \bar{u}(t) - u_+; X \| \to 0 \quad \text{when} \ t \to \infty.$$  

(1.21) $\equiv$ (4.2)
Conversely for any solution $u$ of (1.1) such that $\tilde{u} \in C([1, \infty), X)$, there exists $u_+ \in X$ such that (4.2) holds.

(2) Let $X = L^2$ or $H^2$. Then for any $w_+ \in X$, there exists a unique solution $w_* \in C([1, \infty), X)$ of (1.16) such that

$$\| w_*(t) - w_+; X \| \to 0 \quad \text{when} \ t \to \infty . \quad (1.22) \equiv (4.3)$$

Conversely for any solution $w_* \in C([1, \infty), X)$ of (1.16), there exists $w_+ \in X$ such that (4.3) holds.

**Proof.** By (1.19), Parts (1) and (2) are equivalent, with $w_+ = Fu_+$. We concentrate on Part (2). By (1.11) (2.20) (2.21), the assumption (4.1) on $A$ can be rewritten in terms of $B$ as

$$\| \partial_t^j \partial_x^2 B \|_r \leq C t^{-j-|\alpha|+1/r} , \quad \| \partial_t^j \tilde{B} \|_r \leq C t^{-j+1/r} .$$

For $X = H^2$, Part (2) is then obtained as an immediate consequence of Proposition 3.2, part (1) applied with $(v, A, V, f)$ replaced by $(w, B, -\tilde{B}, 0)$ with $w(t) = w_*(1/t)$. In fact, from the previous assumption on $B$,

$$\| B(t) \|_6 \leq C t^{1/6} , \quad \| \partial_t B(t) \|_3 \leq C t^{-2/3} , \quad \| \partial_t \tilde{B}(t) \|_2 \leq C t^{-1/2}$$

so that the assumptions of Proposition 3.2, part (1) are satisfied.

For $X = L^2$, the situation is slightly more delicate. For the same choice of $(v, A, V, f)$, the assumptions of Proposition 3.1, part (1) are satisfied in $I = [0, 1]$, but the assumptions of Proposition 3.1, part (2) are satisfied only in $(0, 1]$ because

$$\| \nabla B \|_\infty \leq C t^{-1}$$

is not integrable at $t = 0$. We shall therefore combine Proposition 3.1 in $(0, 1]$ with an approximation argument using Proposition 3.2, part (1). Let first $t_0 \in [0, 1]$ and $w_0 \in L^2$. We approximate $w_0$ by a sequence $\{w_{0j}\}$ in $H^2$ converging strongly to $w_0$ in $L^2$. By Proposition 3.2, part (1), each $w_{0j}$ generates a solution $w_j \in C([0, 1], H^2)$ of (1.12) with $w_j(t_0) = w_{0j}$. Furthermore $L^2$ norm conservation holds for those solutions so that

$$\| w_j(t) - w_l(t) \|_2 = \| w_{0j} - w_{0l} \|_2 \quad \text{for all} \ t \in [0, 1] .$$

Therefore $w_j$ converges in norm in $L^\infty([0, 1], L^2)$ to a solution $w \in C([0, 1], L^2)$, with constant $L^2$ norm. Using that result with $t_0 = 0$ and $w_0 = \overline{w}_+$ yields the
existence part of the first statement of Part (2). We next prove uniqueness. Let $w_1, w_2 \in C([0, 1], L^2)$ be two solutions of (1.12) in $[0, 1]$ with $w_1(0) = w_2(0) = \overline{w_+}$.

By Proposition 3.1 applied in $(0, 1]$, $w_1 - w_2$ satisfies $L^2$ norm conservation, namely

$$\| w_1(t) - w_2(t) \|_2 = C \text{ for } t \in (0, 1] .$$

Since $w_1$ and $w_2$ have the same strong $L^2$ limit as $t \to 0$, the last constant is zero and therefore $w_1 = w_2$.

We finally prove the second statement of Part (2).

Let $w \in C((0, 1], L^2)$ be a solution of (1.12). By Proposition 3.1 applied in $(0, 1]$, $w$ is uniquely determined in $C((0, 1], L^2)$ (actually in $(C_w \cap L^\infty_{lo}(0, 1], L^2)$) by its value $w_0 = w(t_0)$ for some $t_0 > 0$. By the previous $H^2$ approximation method, we can construct a solution $w' \in C([0, 1]; L^2)$ of (1.12) with $w'(t_0) = w_0$. By uniqueness, $w = w'$ in $(0, 1]$ and therefore $w$ has a strong limit in $L^2$ as $t \to 0$. This proves the second statement of Part (2).

\[
\square
\]

In the language of scattering theory, Proposition 4.1 essentially expresses the existence of the wave operators and their asymptotic completeness for the equation (1.1) in $L^2$ and in $FH^2$, as compared with the free Schrödinger evolution, and for the equation (1.16) in $L^2$ and in $H^2$, as compared with the constant evolution.

In Proposition 4.1, we have obtained the existence and asymptotic completeness of the wave operators by using Propositions 3.1 and 3.2. If one is only interested in the existence of the wave operators in $L^2$, one can avoid using Proposition 3.2 and use only Proposition 3.1. As a consequence no assumption is needed on the time derivative of $B$ or equivalently on $PA$.

**Proposition 4.2.** Let $A$ satisfy

$$\| \partial^\alpha x A \|_r \lor \| x \cdot A \|_r \leq C t^{-1+2/r} \quad (4.4)$$

for $0 \leq |\alpha| \leq 1$, $2 \leq r \leq \infty$ and for all $t \in [1, \infty)$. Then

1. For any $u_+ \in L^2$, there exists a unique solution $u \in C([1, \infty), L^2)$ of (1.1) such that

$$\| \tilde{u}(t) - u_+ \|_2 \to 0 \quad \text{when } t \to \infty . \quad (4.5)$$

2. For any $w_+ \in L^2$, there exists a unique solution $w_* \in C([1, \infty), L^2)$ of (1.16) such that

$$\| w_*(t) - w_+ \|_2 \to 0 \quad \text{when } t \to \infty . \quad (4.6)$$
Proof. Again Parts (1) and (2) are equivalent by (1.19) with \( w_+ = Fu_+ \) and can be rephrased in an obvious way in terms of \( w \) and of (1.12). The assumption (4.4) on \( A \) can be rewritten in terms of \( B \) as

\[
\| \partial_x^2 B \|_r \leq C t^{-|\alpha|+1/r}, \quad \| \bar{B} \|_r \leq C t^{1/r}.
\]

Under that assumption Proposition 3.1 holds in \((0, 1]\). Let \( w_+ \in H^2 \). Let \( t_0 > 0 \). By Proposition 3.1, there exists a (unique) solution \( w_{t_0} \in C((0, 1], L^2) \) of (1.12) such that \( w_{t_0}(t_0) = \overline{w_+} \). Furthermore for \( t_0 > t_1 > 0 \), \( w_{t_0} - w_{t_1} \) satisfies \( L^2 \) norm conservation in \((0, 1]\), while \( w_{t_0} - \overline{w_+} \) satisfies the equation

\[
i\partial_t (w_{t_0} - \overline{w_+}) = \left(-\frac{1}{2}\Delta_B - \bar{B}\right) (w_{t_0} - \overline{w_+}) - R(\overline{w_+}) \tag{4.7}
\]

where \( R(\cdot) \) is defined by (1.32), so that

\[
R(\overline{w_+}) = \left((1/2)\Delta_B + \bar{B}\right) \overline{w_+}. \tag{4.8}
\]

Therefore, for all \( t \in (0, 1] \)

\[
\| w_{t_0}(t) - w_{t_1}(t) \|_2 = \| w_{t_0}(t_1) - \overline{w_+} \|_2 \leq \int_{t_1}^{t_0} dt \| R(\overline{w_+}) \|_2.
\]

Now

\[
\| R(\overline{w_+}) \| \leq \| \Delta w_+ \|_2 + \| B \|_3 \| \nabla w_+ \|_6 + \left(\| B \|_4^2 + \| \bar{B} \|_2\right) \| w_+ \|_\infty \leq C \tag{4.9}
\]

so that

\[
\| w_{t_0}(t) - w_{t_1}(t) \|_2 \leq C|t_1 - t_0| \quad \text{for all } t \in (0, 1]. \tag{4.10}
\]

It follows from (4.10) that \( w_{t_0} \) converges in norm in \( L^\infty((0, 1], L^2) \) to a limit \( w \in C((0, 1], L^2) \) which is also a solution of (1.12) when \( t_0 \to 0 \). Furthermore by taking the limit \( t_1 \to 0 \) in (4.10), we obtain

\[
\| w_{t_0}(t) - w(t) \|_2 \leq C t_0 \quad \text{for all } t \in (0, 1] \tag{4.11}
\]

and in particular

\[
\| w(t_0) - \overline{w_+} \|_2 \leq C t_0 \tag{4.12}
\]

so that \( w \) can be extended to \( C([0, 1], L^2) \) with \( w(0) = \overline{w_+} \).

This proves the existence part in the proposition in the special case where \( w_+ \in H^2 \). The proof for general \( w_+ \in L^2 \) follows therefrom by the same approximation argument as in Proposition 4.1.
The proof of the uniqueness part is the same as in Proposition 4.1 since it does not use Proposition 3.2.

In the framework of Proposition 4.1, if we assume additional regularity properties of $w_+$ and $u_+$, we obtain stronger convergence properties than (4.2) (4.3), in the form of time decay as powers of $t$. We have stated typical results of this type, expressed in terms of $u$ and of $w_+$, as Proposition 1.2 in the introduction. That proposition follows as a special case (namely with $m = 1$) of the following proposition, where the corresponding results are stated in terms of $w$.

**Proposition 4.3.** Let $A$ satisfy the assumptions of Proposition 4.1. Let $w_+ \in H^2$ and let $w \in C([0, 1], H^2)$ be the solution of (1.12) with $w(0) = \overline{w_+}$ obtained by Proposition 3.2. Then the following estimates hold:

1. \[ \| w(t) - \overline{w_+} \|_2 \leq C t. \tag{4.13} \]

2. Let in addition $w_+ \in H^{2+m}$ for $m \geq 0$. Then

\[ \| w(t) - U(t)\overline{w_+} \|_2 \leq \begin{cases} 
C \frac{t^{(4+m)/3}}{t^{1/2}} & \text{for } m < 1/2 \\
C \frac{t^{(3-\varepsilon)/2}}{t^{1/2}} & \text{for } m = 1/2 \\
C t^{3/2} & \text{for } m > 1/2.
\end{cases} \tag{4.14} \]

3. Let $w_+ \in H^3$. Then

\[ \| \partial_t (w(t) - U(t)\overline{w_+}) \|_2 \vee \| \Delta (w(t) - U(t)\overline{w_+}) \|_2 \leq C t^{1/2}. \tag{4.15} \]

**Proof of Proposition 1.2.** Part (2) of that proposition follows from the special case $m = 1$ of Proposition 4.3 and from (1.15). Part (1) follows from Part (2) and from (1.19).

**Proof of Proposition 4.3.** Part (1). In the same way as in the proof of Proposition 4.2, we estimate

\[ |\partial_t \| w(t) - \overline{w_+} \|_2| \leq \| R(\overline{w_+}) \|_2 \leq C \tag{4.16} \]

from which (4.13) follows by integration.
Part (2). We estimate similarly
\[\left| \partial_t \left\| w(t) - U(t)\overrightarrow{w}_+ \right\|_2 \right| \leq \left\| R(U(t)\overrightarrow{w}_+) \right\|_2\] (4.17)
where \(R(\cdot)\) is defined by (1.32). We compute
\[R(U(t)\overrightarrow{w}_+) = -iB \cdot \nabla U(t)\overrightarrow{w}_+ - \left( B^2/2 - \tilde{B} \right) U(t)\overrightarrow{w}_+ \] (4.18)
and we estimate
\[\left\| B \cdot \nabla U(t)\overrightarrow{w}_+ \right\|_2 \leq\left\{\begin{array}{ll}
C \parallel B \parallel^{3/(1+m)} \parallel \omega^{m+2} w_+ \parallel_2 & \text{for } m < 1/2 \\
C \parallel B \parallel^{2/(1-\varepsilon)} \parallel w_+; H^{3/2} \parallel & \text{for } m = 1/2 \\
C \parallel B \parallel \parallel w_+; H^{2+m} \parallel & \text{for } m > 1/2 ,
\end{array}\right.\] (4.19)
\[\left\| (B^2/2 - \tilde{B})U(t)\overrightarrow{w}_+ \right\|_2 \leq C \left( \parallel B \parallel^2_3 + \parallel \tilde{B} \parallel_2 \right) \parallel w_+; H^2 \parallel \leq C t^{1/2} \] (4.20)
by (4.1) and Lemma 2.1. The result follows from (4.17) (4.19) (4.20) by integration on time.

Part (3). We use the estimates in the proof of Proposition 3.2 applied with \((v, A, V, f)\) replaced by \((w - U(t)\overrightarrow{w}_+, B, -\tilde{B}, -R(U(t)\overrightarrow{w}_+))\). In addition to the estimate
\[\left\| R(U(t)\overrightarrow{w}_+) \right\|_2 \leq C t^{1/2} \] (4.21)
which follows from (4.19) (4.20), we need the estimate
\[\left\| \partial_t R(U(t)\overrightarrow{w}_+) \right\|_2 \leq \left\| \partial_t B \right\|_2 \left\| \nabla U(t)\overrightarrow{w}_+ \right\|_\infty + \left\| B \right\|_\infty \left\| \nabla \Delta w_+ \right\|_2
\]
\[\quad + \left\| B\partial_t B - \partial_t \tilde{B} \right\|_2 \left\| U(t)\overrightarrow{w}_+ \right\|_\infty + \left( \left\| B \right\|_3^2 + \left\| \tilde{B} \right\|_\infty \right) \left\| \Delta w_+ \right\|_2
\]
\[\leq C \parallel w_+; H^3 \parallel t^{-1/2} \leq C t^{-1/2} \] (4.22)
by (4.1) and Lemma 2.1.

From (4.21) it follows immediately by integration that
\[\left\| w - U(t)\overrightarrow{w}_+ \right\|_2 \leq C t^{3/2} . \] (4.23)
From the estimate (3.37), it follows by substituting (4.21)-(4.23) that
\[\left\| \partial_t (w - U(t)\overrightarrow{w}_+) \right\|_2 \leq C t^{1/2} \] (4.24)
which yields the estimate of the first term in (4.15). Substituting (4.24) (4.21) (4.23) into (3.32) yields the estimate of the second term in (4.15). \(\square\)
5 \( H^k \) regularity of the wave operators for \( u \)

In this section we study the theory of scattering for (1.1) in spaces \( H^k \) for \( k > 0 \) for sufficiently smooth \( A \) satisfying conditions of the type (2.26). We have already obtained \( L^2 \) wave operators satisfying asymptotic completeness in \( L^2 \) for (1.1) in Proposition 4.1, part (1), and the problem is that of additional regularity for those wave operators. We restrict our attention to the case where \( k \) is an even integer. For brevity in all this section we shall take for granted the existence of solutions of (1.1) at the required level of regularity and we shall concentrate on the derivation of higher norm estimates. The existence results follow from Propositions 3.1-3.4 for \( k \leq 4 \) and can be proved in the same way for higher \( k \). We first derive bounds for higher norms of generic solutions of (1.1). Those bounds are unfortunately not uniform in \( t \).

**Proposition 5.1.** Let \( j \geq 1 \) be an integer and let \( A \) satisfy the estimates

\[
\| \partial_t^l A \|_\infty \leq C t^{-1} \quad \text{for } 0 \leq l \leq j .
\]

Let \( u \in C^j([1, \infty), L^2) \cap C^{j-1}([1, \infty), H^2) \) be a solution of (1.1). Then \( u \) satisfies the estimates

\[
\| \partial_t^j u \|_2 \vee \| \partial_t^{j-1} \Delta u \|_2 \leq C (1 + \ell n t)^{2j} .
\]

Let in addition \( A \) satisfy the estimates

\[
\| \partial_t^l \partial_x^\alpha A \|_\infty \leq C t^{-1} \quad \text{for } 0 \leq |\alpha|/2 + l \leq j - 1
\]

and let \( u \in \bigcap_{0 \leq l \leq j} C^{j-l}([1, \infty), H^{2l}) \). Then \( u \) satisfies the estimates

\[
\| \partial_t^{j-l} \Delta^l u \|_2 \leq C (1 + \ell n t)^{2j} \quad \text{for } 0 \leq l \leq j .
\]

**Proof.** The proof proceeds by induction on \( j \) and possibly \( l \), starting from (1.1). Since for the proof of Proposition 5.2 below, we shall need a similar induction for a slightly more general equation, we shall replace (1.1) by the more general inhomogeneous equation

\[
i \partial_t u = -(1/2) \Delta_A u + f .
\]

For the needs of the present proof we shall take \( f = 0 \) at the end. Taking the \( j \)-th time derivative of (5.5), we obtain

\[
i \partial_t^{j+1} u = -(1/2) \Delta_A \partial_t^j u + \sum_{0 \leq l < j} i C^l \left\{ \partial_t^{j-l-1} \left( (\partial_x A) \cdot \nabla A \right) \right\} \partial_t^l u + \partial_t^j f
\]
\[ -(1/2) \Delta_A \partial_t^i u + \sum_{0 \leq l < j} C^j_l \{ i \left( \partial_t^{j-l} A \right) \cdot \nabla_A \partial_t^l u \]
\[ + \sum_{0 < m < j-l} C_{j-l-1}^m \left( \partial_t^{j-l-m} A \right) \left( \partial_t^m A \right) \partial_t^l u \}
\[ + \partial_t^j f \]  
(5.6)

where we have used the relation (see (3.18))

\[ - (1/2) \left( \partial_t \Delta_A \right) = i \left( \partial_t A \right) \cdot \nabla_A . \]  
(5.7)

We now define

\[ y_l = \| \partial_t^l u \|_2 \text{ for } 0 \leq l \leq j , \quad z_l = \| \Delta_A \partial_t^{j-1} u \|_2 \text{ for } 1 \leq l \leq j , \quad z_0 = 0 . \]

In the same way as in Section 3 and by using (5.1), we estimate

\[ |\partial_t y_j | \leq C \ t^{-1} \sum_{0 \leq l < j} \left( \| \nabla_A \partial_t^l u \|_2 + t^{-1} y_l \right) + \| \partial_t^j f \|_2 . \]  
(5.8)

Now

\[ \| \nabla_A \partial_t^l u \|_2 \leq (y_l z_{l+1})^{1/2} \leq (1/2) (y_l + z_{l+1}) . \]  
(5.9)

We substitute (5.9) into (5.8), using the middle bound of (5.9) for \( l = j-1 \) and the last bound of (5.9) for \( l < j-1 \). We obtain

\[ |\partial_t y_j | \leq C \ t^{-1} \left\{ (y_{j-1} z_j)^{1/2} + \sum_{0 \leq l < j} (y_l + z_l) \right\} + \| \partial_t^j f \|_2 . \]  
(5.10)

On the other hand by a similar estimate, we obtain

\[ z_j \leq 2y_j + C \ t^{-1} \sum_{0 \leq l < j-1} \left( \| \nabla_A \partial_t^l u \|_2 + t^{-1} y_l \right) + 2 \| \partial_t^{j-1} f \|_2 \]  
(5.11)

so that by (5.9) again

\[ z_j \leq 2y_j + C \ t^{-1} \sum_{0 \leq l < j} (y_l + z_l) + 2 \| \partial_t^{j-1} f \|_2 . \]  
(5.12)

We now prove the estimate (5.2) by induction on \( j \) through the use of (5.10) (5.12) with \( f = 0 \). The starting point is \( y_0 = C \) by \( L^2 \) norm conservation. We next assume that

\[ y_l \vee z_l \leq C (1 + \ell n \ t)^{2l} \text{ for } 0 \leq l < j . \]

Substituting that assumption into (5.10) (5.12) yields

\[ |\partial_t y_j | \leq C \ t^{-1} \left\{ (1 + \ell n \ t)^{j-1} z_j^{1/2} + (1 + \ell n \ t)^{2(j-1)} \right\} \]  
(5.13)
\[ z_j \leq 2y_j + C \, t^{-1}(1 + \ell n t)^{2(j-1)} \] 

so that

\[ |\partial_t y_j| \leq C \, t^{-1}\left\{ (1 + \ell n t)^{j-1} y_j^{1/2} + (1 + \ell n t)^{2(j-1)} \right\}. \] 

Integrating over time by Lemma 2.3 yields the first estimate of (5.2), from which the second estimate follows by (5.14) and (5.1).

We now turn to the proof of (5.4). For that purpose we define

\[ z_l^j = \| \partial_j^{l-l} \Delta^l u \|_2 \quad \text{for } 0 \leq l \leq j \]

so that \( z_0^j = y_j \) is already estimated by (5.2). Now for \( 1 \leq l \leq j \),

\[
\begin{align*}
    z_l^j & = \| \partial_l^{j-l} \Delta^{j-l} \left( (-2i \partial_t + 2i A \cdot \nabla + A^2)u + 2f \right) \|_2 \\
    & \leq 2z_{j-l}^j + C \sum_{0 \leq m \leq j-1} \sum_{|\alpha + \beta| = 2(l-1)} \left\{ \| \partial_t^m \partial^{\alpha}_x A \|_\infty \| \partial_l^{j-l-m} \partial^{\beta}_x \nabla u \|_2 \right. \\
    & \quad + \left. \| \partial_t^m \partial^{\alpha}_x (A^2) \|_\infty \| \partial_l^{j-l-m} \partial^{\beta}_x u \|_2 \right\} + 2 \| \partial_l^{j-l} \Delta^{l-1} f \|_2 \\
    & \leq 2z_{j-l}^j + C \, t^{-1} \left\{ \left( z_{j-l}^{j-l} \right)^{1/2} + \sum_{m \leq k \leq j-1} z_k^m \right\} + 2 \| \partial_l^{j-l} \Delta^{j-l} f \|_2 \\
    & \leq 2z_{j-l}^j + C \, t^{-1} \left\{ \left( z_{j-l}^{j-l} \right)^{1/2} \right. \\
    & \quad \left. + \frac{1}{2} \left( z_j^j + z_{j-1}^{j-1} \right) \right\} \\
    & \leq 2z_{j-l}^j + C \, t^{-1} \left\{ \left( z_{j-l}^{j-l} \right)^{1/2} \right. \\
    & \quad \left. + \frac{1}{2} \left( z_j^j + z_{j-1}^{j-1} \right) \right\} \leq \frac{1}{2} \left( z_j^j + z_{j-1}^{j-1} \right). \tag{5.17}
\end{align*}
\]

We now prove the result by induction on \( j \) and for each \( j \) by induction on \( l \) starting from \( z_0^0 = y_0 = C \). Thus we assume that \( z_m^m \leq C(1 + \ell n t)^{2k} \) for \( 0 \leq m \leq k \leq j-1 \) and for \( k = j \), \( 0 \leq m \leq l-1 \). The estimate (5.4) follows immediately from (5.16) with \( f = 0 \) and from that assumption.

\[ \square \]

Although we are unable to prove the \( H^k \) boundedness of general \( H^k \) solutions of (1.1), we can nevertheless construct densely defined \( H^k \) wave operators for that equation, namely construct solutions that are \( H^k \) bounded and that are asymptotic in the \( H^k \) sense to prescribed model asymptotics of the type \( u_a = U(t)u_+ \) with asymptotic state \( u_+ \) in a dense subspace of \( H^k \). The method of construction is that sketched in the introduction, adapted to the fact that we are now working directly with \( t \to \infty \). One looks for \( u \) in the form \( u = u_a + v \). The evolution equation for \( v \) becomes obviously

\[ i\partial_t v = -(1/2)\Delta_A v - \tilde{R}(u_a) \tag{5.18} \]
where
\[ \tilde{R}(u_a) = (i\partial_t + (1/2)\Delta) u_a \]
\[ = -iA \cdot \nabla u_a - (1/2)A^2 u_a. \]  
(5.19)

One takes a large finite \( t_0 \), one constructs a solution \( v_{t_0} \) of (5.18) with suitably small initial data \( v_0 = v_{t_0}(t_0) \) at \( t_0 \), and one takes the limit of \( v_{t_0} \) as \( t_0 \to \infty \). The key of the proof consists in estimating \( v_{t_0} \) for \( t \leq t_0 \) uniformly in \( t_0 \) at the required level of regularity. The estimates thereby obtained remain true in the limit and provide asymptotic estimates for the solution \( u \) at that level. The method will be presented in detail in Section 6 below in the more interesting case of the equation (1.12) for \( w \) at the level of regularity of \( H^3 \) and \( H^4 \). Here we only provide the basic step thereof, namely we derive the estimates on \( v \). For simplicity we derive them directly on the limiting \( v \), assuming that it tends to zero at infinity in the relevant norms. As a preliminary step, we need decay estimates on \( \tilde{R}(u_a) \).

**Lemma 5.1.** Let \( j \geq 0 \) be an integer, let \( \alpha \) be a multiindex and let \( A \) satisfy the estimates
\[ \| \partial_t^l \partial_x^\beta A \|_r \vee \| \partial_t^l \partial_x^\beta (x \cdot A) \|_r \leq C t^{-1+2/r} \]  
(5.20)
for all \( r, 2 \leq r \leq \infty \), for \( 0 \leq l \leq j \), \( 0 \leq \beta \leq \alpha \) and for all \( t \geq 1 \). Let \( u_+ \), \( xu_+ \in H^{2j+|\alpha|}_\bar{r} \) for some \( \bar{r} \), \( 1 \leq \bar{r} \leq 2 \).

Then \( \tilde{R} \equiv \tilde{R}(U(t)u_+) \) satisfies the estimate
\[ \| \partial_t^j \partial_x^\alpha \tilde{R} \|_2 \leq C t^{-5/2+1/r} \]  
(5.21)
where \( 1/r + 1/\bar{r} = 1 \), for all \( t \geq 1 \).

**Proof.** Using the generator of Galilei transformations
\[ J \equiv J(t) = x + it\nabla = U(t)xU(-t) \]  
(5.22)
we rewrite \( \tilde{R} \) as
\[ \tilde{R} = -t^{-1}A \cdot (J - x)u_a - (1/2)A^2 u_a \]
\[ = -t^{-1}A \cdot U(t)xu_+ + t^{-1}(x \cdot A)U(t)u_+ - (1/2)A^2 U(t)u_+ . \]  
(5.23)
Using the basic estimate of the Schrödinger group
\[ \| U(t)f \|_r \leq (2\pi|t|)^{-\delta(r)} \| f \|_{\bar{r}} \]  
(5.24)
where \(2 \leq r \leq \infty, 1/r + 1/r = 1\) and \(\delta(r) = 3/2 - 3/r\), we estimate

\[
\| \partial_t^j \partial_x^\alpha \tilde{R} \|_2 \leq C t^{-\delta(r)} \sum_{0 \leq l \leq j} \sum_{\beta \leq \alpha} \{ t^{-1} \| \partial_t^l \partial_x^\beta A \|_s \| \Delta^{j-l} \partial_x^{\alpha-\beta} (xu_+) \|_r \\
+ \left( t^{-1} \| \partial_t^l \partial_x^\beta (x \cdot A) \|_s + \| \partial_t^l \partial_x^\beta A^2 \|_s \right) \| \Delta^{j-l} \partial_x^{\alpha-\beta} u_+ \|_r \}
\]

where \(1/s = 1/2 - 1/r\),

\[
\cdots \leq C t^{-5/2 + 1/r}
\]

by (5.20).

We can now derive asymptotic estimates of higher norms for the solutions of (1.1) with prescribed asymptotic behaviour \(U(t)u_+\) for sufficiently regular \(u_+\).

**Proposition 5.2.** Let \(j \geq 0\) be an integer and let \(A\) satisfy

\[
\| \partial_t^j \partial_x^\alpha A \|_r \vee \| \partial_t^l \partial_x^\beta (x \cdot A) \|_r \leq C t^{-1+2/r}
\]

(1.29) \(\equiv\) (5.25) for all \(r, 2 \leq r \leq \infty, \) for \(0 \leq l+|\alpha|/2 \leq j\) and for all \(t \geq 1\). Let \(u_+, xu_+ \in H^{2j} \cap H^{2j}_r\) for some \(\bar{r}, 1 \leq \bar{r} \leq 2\). Then there exists a unique solution \(u \in \cap_{0 \leq l \leq j} C^{j-l}([1, \infty), H^2)\) of (1.1) satisfying the estimates

\[
\| \partial_t^{j-l} \Delta^l (u(t) - U(t)u_+) \|_2 \leq C t^{-3/2 + 1/r}
\]

(5.26) where \(1/r + 1/\bar{r} = 1\), for \(0 \leq l \leq j\) and for all \(t \geq 1\). The solution is actually unique in \(C([1, \infty), L^2)\) under the condition (5.26) for \(j = 0\).

**Proof.** As mentioned above, we concentrate on the derivation of the estimates. We first prove (5.26) for \(l = 0, 1\). For that purpose we estimate \(v = u - U(t)u_+\) by the same method as in the proof of Proposition 5.1, starting from (5.18) instead of (5.5), namely with \((u, f)\) replaced by \((v, -\tilde{R})\). Defining

\[
y_l = \| \partial_t^l v \|_2 \text{ for } 0 \leq l \leq j, \quad z_l = \| \Delta^{j-l} v \|_2 \text{ for } 1 \leq l \leq j, \quad z_0 = 0,
\]

and using the fact that

\[
\| \partial_t^l \tilde{R} \|_2 \leq C t^{-1-\lambda} \text{ for } 0 \leq l \leq j
\]

(5.27) with \(\lambda = 3/2 - 1/r\), we obtain (see (5.10) (5.12))

\[
|\partial_t y_j| \leq C t^{-1} \left\{ (y_{j-1} z_j)^{1/2} + \sum_{0 \leq l < j} (y_l + z_l) \right\} + C t^{-1-\lambda}
\]

(5.28)
We now prove the estimate (5.26) for $l = 0$ by induction on $j$. The starting point $y_0 \leq C t^{-\lambda}$ is obtained by integrating (5.28) with $j = 0$. We next assume that

$$y_l \vee z_l \leq C t^{-\lambda} \quad \text{for } 0 \leq l < j .$$

(5.30)

Substituting (5.30) into (5.28) (5.29) yields

$$|\partial_t y_j| \leq C t^{-1-\lambda/2} z_j^{1/2} + C t^{-1-\lambda}$$

(5.31)

$$z_j \leq 2y_j + C t^{-1-\lambda}$$

(5.32)

from which the result follows by integration over time by the use of Lemma 2.3.

We next prove (5.26) for general $l$, $0 \leq l \leq j$. In the same way as in the proof of Proposition 5.1, we define

$$z^l_j = \| \partial_t^{j-l} \Delta^l v \|_2 \quad \text{for } 0 \leq l \leq j$$

and by the same computation, we estimate (see (5.16))

$$z^l_j \leq 2z^{l-1}_j + C t^{-1} \left\{ (z^l_j z^{l-1}_j)^{1/2} + \sum_{m \leq k \leq j-1} z^m_k \right\} + C t^{-1-\lambda}$$

(5.33)

from which the result follows as before by induction on $l$ and $j$.

6 Scattering theory at the level of $H^k$ with $k \geq 3$ for $w$

In this section, we begin the construction of the wave operators for $u$ at the level of regularity corresponding to $w_*$ or $w$ in $H^k$ for $k \geq 3$ by studying the Cauchy problem for $w$, namely for (1.12), with initial time zero, at that level of regularity. We apply the indirect method sketched in the introduction in order to circumvent the fact that Propositions 3.3 and 3.4 do not apply with initial time zero under the assumptions made on $B$. More precisely, we give ourselves an asymptotic behaviour for $w$ at $t = 0$ in the form of a model $W$ defined in $(0, 1]$ and we look for $w$ in the
form $w = W + q$ with $q$ tending to zero as $t \to 0$. The evolution equation for $q$ becomes (1.31) with $R(\cdot)$ defined by (1.32). That equation is of the form (3.1) with $(v, A, V, f)$ replaced by $(q, B, -\mathring{B}, -R(W))$. In this section, we prove the existence of $w$ at the relevant level of regularity under general assumptions on $W$, the most important of which are decay properties of $R(W)$. In the next section, we shall construct $W$ satisfying those properties.

We first state the relevant result at the level of regularity of $H^3$.

**Proposition 6.1.** Let $I = (0, 1]$ and let $B$ satisfy

$$
\| \partial_t^j \partial_x^\alpha B \|_r \vee \| \partial_t^j \partial_x^\alpha \mathring{B} \|_r \leq b t^{-j-|\alpha|+1/r} 
$$

for some constant $b$, for $2 \leq r \leq \infty$, $0 \leq j \leq 1$, $0 \leq |\alpha| \leq 1$ and for all $t \in I$. Let $\lambda_0$, $\lambda_1$ and $\lambda$ satisfy

$$
\begin{cases}
\lambda_1 \leq \lambda_0 - 1/2 \\
0 < \lambda \leq \lambda_1 \land (\lambda_0 - 3/2) \land ((\lambda_0 + \lambda_1)/2 - 1)
\end{cases}
$$

and let $\lambda'_1 = \lambda_1 \land (\lambda_0 - 1)$.

Let $W \in C(I, H^3) \cap C^1(I, H^1)$ be such that $R \equiv R(W) \in (C \cap H^1_\infty)(I, H^3)$ and that $R$ satisfies the estimates

$$
\| \partial_t^j R \|_2 \leq C t^{\lambda_j-1} \quad \text{for } j = 0, 1 ,
$$

$$
\| \nabla \partial_t^j R \|_2 \leq C t^{\lambda_j-1} \quad \text{for } j = 0, 1 ,
$$

for all $t \in I$.

Then there exists a unique solution $w \in C(I, H^3) \cap C^1(I, H^1)$ of (1.12) in $I$ satisfying the estimates

$$
\| \partial_t^j (w - W) \|_2 \leq C t^{\lambda_j} \quad \text{for } j = 0, 1 ,
$$

$$
\| \Delta (w - W) \|_2 \vee \| \Delta_B (w - W) \|_2 \leq C t^{\lambda'},
$$

$$
\| \nabla \partial_t (w - W) \|_2 \vee \| \nabla \Delta_B (w - W) \|_2 \vee \| \nabla \Delta (w - W) \|_2 \leq C t^\lambda
$$

for all $t \in I$. The solution is actually unique in $(L^\infty \cap C_w)(I, L^2)$ under the condition (6.5) for $j = 0$. 

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Remark 6.1. The condition (6.2) is satisfied in particular by the linear scale
\[ 0 < \lambda = \lambda_1 - 1 = \lambda_0 - 2 \] which will occur in a natural way for the available \( W \). (See Section 7 below). In that case \( \lambda'_1 = \lambda_1 \).

Proof. By (6.1) (6.3) (6.4), the assumptions of Proposition 3.3 are satisfied in \( I = (0,1] \) (but not in \([0,1]\)) for the equation (1.31), namely for \((v,A,V,f)\) replaced by \((q,B,-\tilde{B},-R)\). Let \( 0 < t_0 \leq 1 \) and let \( q_{t_0} \in C(I,H^3) \cap C^1(I,H^1) \) be the solution of (1.31) with \( q_{t_0}(t_0) = 0 \) obtained from Proposition 3.3. The proof will consist in taking the limit \( t_0 \to 0 \) of \( q_{t_0} \). For that purpose we shall estimate \( q_{t_0} \) in \( H^3 \) and \( \partial_t q_{t_0} \) in \( H^1 \) uniformly in \( t_0 \) for \( t_0 \leq t \leq 1 \). Those estimates will rely on the identities (3.6) (3.19) and (3.41) satisfied by \( q_{t_0} \) in \( I \). We now estimate \( q_{t_0} \), omitting the subscript \( t_0 \) for brevity. We define

\[ y_j = \| \partial^j t q \|_2, \quad j = 0,1, \quad y = \| \nabla_B \partial_t q \|_2. \]  (6.8)

We first estimate \( y_0 \). From (3.6), we obtain

\[ |\partial_t y_0| \leq \| R \|_2 \]  (6.9)

and therefore by integration and by (6.3) with \( j = 0 \)

\[ y_0 \leq C t^{\lambda_0} \quad \text{for } t_0 \leq t \leq 1. \]  (6.10)

We next estimate \( y_1 \). From (3.19), in the same way as in Proposition 3.2, we obtain (see (3.28))

\[ |\partial_t y_1| \leq \| f_1 \|_2 \]  (6.11)

where now (see (3.18))

\[ f_1 = i (\partial_t B) \cdot \nabla_B q - (\partial_t \tilde{B}) q - \partial_t R \]  (6.12)

so that by (6.1)

\[ |\partial_t y_1| \leq t^{-1} b (\| \nabla_B q \|_2 + y_0) + \| \partial_t R \|_2. \]  (6.13)

Now by direct estimate of (1.31)

\[ \| \nabla_B q \|_2^2 \leq \| q \|_2 \| \Delta_B q \|_2 \leq 2y_0 (y_1 + b y_0 + \| R \|_2) \]  (6.14)

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so that for \( t \geq t_0 \)

\[
|\partial_t y_1| \leq C \, t^{-1} \left\{ (y_0 \, y_1)^{1/2} + y_0 + (y_0 \, \|R\|_2)^{1/2} \right\} + \| \partial_t R \|_2
\leq C \left( t^{-1+\lambda_0/2} \, y_1^{1/2} + t^{\lambda_0-3/2} + t^{\lambda_1-1} \right)
\]  

(6.15)

by (6.3) (6.10). Integrating (6.15) by Lemma 2.3 with \( y_1(t_0) = \| R(t_0) \|_2 \) satisfying (6.3) yields

\[
y_1 \leq C \left( t^{\lambda_1} + t_0^{\lambda_0-1} \right) \leq C \, t^{\lambda_1} \quad \text{for} \quad t_0 \leq t \leq 1
\]  

(6.16)

and in particular

\[
y_1 \leq C \, t^{\lambda_1} \quad \text{for} \quad t_0 \lor t_0^{(\lambda_0-1)/\lambda_1} \leq t \leq 1.
\]  

(6.17)

Furthermore it follows from (6.14) (6.10) (6.16) (6.3) that

\[
\| \Delta Bq \|_2 \leq C \, t^{\lambda_1},
\]  

(6.18)

\[
\| \nabla Bq \|_2 \leq C \, t^{(\lambda_0 + \lambda_1)/2},
\]  

(6.19)

\[
\| \Delta q \|_2 \leq \| \Delta Bq \|_2 + 2b \| \nabla Bq \|_2 + b^2 \| q \|_2 \leq C \, t^{\lambda_1}.
\]  

(6.20)

We shall also need an estimate of \( \| \nabla_B \otimes \nabla_B q \|_2 \). Now

\[
\| \nabla_B \otimes \nabla_B q \|_2^2 = -\sum_j <q, \nabla_{Bj} \Delta B \nabla_{Bj} q> ,
\]

\[
\nabla_{Bj} \Delta B \nabla_{Bj} = \Delta_B^2 - iG_{jl} \nabla_B l \nabla_B j - i\nabla_B l G_{jl} \nabla_B j
\]

where

\[
G_{jl} = i \left[ \nabla_{Bj}, \nabla_B l \right] = \nabla_j B_l - \nabla_l B_j
\]  

(6.21)

so that

\[
\| \nabla_B \otimes \nabla_B q \|_2^2 \leq \| \Delta Bq \|_2^2 + \| G \|_{\infty} \| q \|_2 \left( \| \Delta Bq \|_2 + \| \nabla_B \otimes \nabla_B q \|_2 \right)
\]

and therefore

\[
\| \nabla_B \otimes \nabla_B q \|_2 \leq \| \Delta Bq \|_2 + \| G \|_{\infty} \| q \|_2 \leq C \, t^{\lambda_1}
\]  

(6.22)

by (6.10) (6.18).

We now turn to the estimate of \( y \) defined by (6.8). From (3.41), in the same way as in the proof of Proposition 3.3, we obtain (see (3.46))

\[
[\partial_t y] \leq \| g \|_2
\]  

(6.23)
where now (see (3.40))

\[ g = (\partial_t B - \nabla \tilde{B}) \partial_t q + i (\partial_t B) \cdot \nabla B \otimes B q + i (\nabla \partial_t B) \cdot B q - (\partial_t B) \nabla B q - (\nabla \partial_t B) q - \nabla B \partial_t R \]  

(6.24)

so that

\[
|\partial_t y| \leq \left( \| \partial_t B \|_\infty + \| \nabla \tilde{B} \|_\infty \right) y_1 + \| \partial_t B \|_\infty \| \nabla B \otimes B q \|_2 \\
+ \left( \| \nabla \partial_t B \|_\infty + \| \partial_t \tilde{B} \|_\infty \right) \| \nabla B q \|_2 + \| \nabla \partial_t \tilde{B} \|_\infty \ y_0 + \| \nabla B \partial_t R \|_2 \\
\leq b t^{-1} \left\{ 2y_1 + \| \nabla B \otimes B q \|_2 + \left( t^{-1} + 1 \right) \| \nabla B q \|_2 + t^{-1} y_0 \right\} \\
+ \| \nabla \partial_t R \|_2 + b \| \partial_t R \|_2 \\
\leq C \left( t^{-1+\lambda'_1} + t^{-2+2(\lambda_0+\lambda'_1)/2} + t^{-2+\lambda_0} + t^{-1+\lambda} + t^{-1+\lambda'_1} \right) \leq C t^{-1+\lambda} 
\]  

(6.25)

by (6.3) (6.4) (6.10) (6.16) (6.19) (6.22) and (6.2). Integrating (6.25) with initial condition

\[ y(t_0) = \| \nabla B R(t_0) \|_2 \leq C t_0^\lambda \]

yields

\[ y \leq C \ t^\lambda \ \text{for} \ t_0 \leq t \leq 1. \]  

(6.26)

It follows therefrom and from (1.31) (6.4) (6.19) that

\[ \| \nabla B \Delta B q \|_2 \leq 2 \left( y + \| \nabla B \tilde{B} q \|_2 + \| \nabla B R \|_2 \right) \leq C t^\lambda. \]  

(6.27)

Furthermore by similar elementary estimates and by (6.2)

\[ \| \nabla \Delta B q \|_2 \lor \| \Delta B \nabla q \|_2 \lor \| \nabla \Delta q \|_2 \leq C \ t^\lambda. \]  

(6.28)

We can now take the limit \( t_0 \to 0 \). We come back to the original notation \( q_{t_0} \) for that part of the argument. The solution \( q_{t_0} \) and its time derivative are estimated in \( H^3 \) and \( H^1 \) respectively by (6.10) (6.20) (6.28) and by (6.16) (6.26) uniformly in \( t_0 \) for \( t_0 \leq t \leq 1 \). Let \( 0 < t_1 \leq t_0 \leq 1 \). Then the \( L^2 \) norm of the difference \( q_{t_1} - q_{t_0} \) is conserved so that for all \( t \in [t_0, 1] \)

\[ \| q_{t_1}(t) - q_{t_0}(t) \|_2 = \| q_{t_1}(t_0) \|_2 \leq C \ t_0^{\lambda_0} \]  

(6.29)

by (6.10). Therefore \( q_{t_0} \) converges in norm in \( L^\infty([T, 1], L^2) \) for all \( T > 0 \) to some \( q \in C(I, L^2) \). By the previous uniform estimates and a standard compactness argument, \( q \in L^\infty(I, H^3) \cap H^1_{\infty}(I, H^1) \) and \( q \) satisfies the same estimates. Furthermore \( q \) also
satisfies (1.31) and therefore \( q \) can be chosen in \( C_w(I, H^3) \cap C^1_w(I, H^1) \). By Proposition 3.3, actually \( q \in C(I, H^3) \cap C^1(I, H^1) \). Together with the estimates, this proves that \( q \in C([0, 1], H^3) \cap C^1([0, 1], H^1) \) with \( q(0) = \partial_t q(0) = 0 \). Returning to the variables \( w \) proves the existence part of the proposition and the estimates (6.5)-(6.7), except for the fact that the estimate (6.15) for \( y_1 \) used so far for \( t_0 \leq t \leq 1 \) has \( t^\lambda_1 \) instead of \( t^{\lambda_1} \). However the final estimate (6.5) for \( j = 1 \) follows from (6.17) in the limit \( t_0 \to 0 \).

The uniqueness statement in the proposition follows from that of Proposition 3.3 in \( I \) and from (6.5) for \( j = 0 \).

We now state the corresponding result at the level of regularity of \( H^4 \).

**Proposition 6.2.** Let \( I = (0, 1] \) and let \( B \) satisfy (6.1) for \( 2 \leq r \leq \infty \), \( 0 \leq j + |\alpha| \leq 2 \) and for all \( t \in I \). Let \( \lambda_j, j = 0, 1, 2 \) satisfy

\[
\begin{cases} 
\lambda_0 > 2 , & 1 < \lambda_1 \leq \lambda_0 - 1/2 \\
0 < \lambda_2 \leq \lambda_2' - 1/2 \equiv (\lambda_1 - 1/2) \wedge (\lambda_0 - 3/2) 
\end{cases}
\]  

(6.30)

and let \( \lambda_2' = \lambda_2 \wedge (\lambda_1 - 1) \).

Let \( W \in C^1(I, H^2) \cap C^2(I, L^2) \) be such that \( R \equiv R(W) \in (C^1 \cap H^2_\infty)(I, L^2) \) and that \( R \) satisfies the estimates

\[ \| \partial^j_t R \|_2 \leq C t^{\lambda_j - 1} \quad \text{for } j = 0, 1, 2 \]  

(6.31)

for all \( t \in I \). Then

1. There exists a unique solution \( w \in C^1(I, H^2) \cap C^2(I, L^2) \) of (1.12) in \( I \) satisfying the estimates (6.6) and

\[ \| \partial^j_t (w - W) \|_2 \leq C t^{\lambda_j} \quad \text{for } j = 0, 1, 2 , \]  

(6.32)

\[ \| \partial_t \Delta_B (w - W) \|_2 \vee \| \Delta_B \partial_t (w - W) \|_2 \vee \| \Delta \partial_t (w - W) \|_2 \leq C t^{\tilde{\lambda}_2} \]  

(6.33)

for all \( t \in I \). The solution is actually unique in \( (L^\infty \cap C_w)(I, L^2) \) under the condition (6.32) for \( j = 0 \).

2. Assume in addition that \( B, \tilde{B}, R \in C(I, H^2) \) and that \( R \) satisfies the estimate

\[ \| \Delta R \|_2 \leq C \ t^{\tilde{\lambda}_1 - 1} \]  

(6.34)
for some \( \tilde{\lambda}_2 \) satisfying
\[
\tilde{\lambda}_2 \leq (\lambda_0 - 3/2) \land ((\lambda_0 + \lambda_1)/2 - 1) \land \lambda_1 \land (\lambda_2 + 1)
\] (6.35)
and for all \( t \in I \). Then \( w - W \in C(I, H^1) \) and \( w \) satisfies
\[
\| \Delta^2 (w - W) \|_2 \leq C \ t^{\tilde{\lambda}_2 - 1}
\] (6.36)
for all \( t \in I \). In particular \( w - W \in C([0, 1], H^{2(1+\theta)}) \) for \( 0 \leq \theta \leq 1, \ \theta(1 + \lambda_1' - \tilde{\lambda}_2) < \lambda_1' \).

Remark 6.2. Under the condition (6.30), the condition (6.35) is satisfied for \( \tilde{\lambda}_2 = \lambda_2 \). Furthermore, the conditions (6.30) (6.35) are satisfied in particular by the linear scale \( 0 < \tilde{\lambda}_2 = \lambda_2 = \lambda_1 - 1 = \lambda_0 - 2 \) (see Remark 6.1). In that case \( \lambda_1' = \lambda_1 \) and \( \lambda_2' = \lambda_2 \).

Proof. Part (1). It follows from (6.1) (6.31) that the assumptions of Proposition 3.4 are satisfied in \( I = (0, 1] \) (but not in \([0, 1]\)) for the equation (1.31), namely for (3.1) with \((v, A, V, f)\) replaced by \((q, B, -\hat{B}, -R)\). The proof proceeds again by taking the limit \( t_0 \to 0 \) of a solution \( q_{t_0} \) of (1.31) with suitable data at \( t_0 \) for some \( t_0 \in I \).

We choose the initial data at \( t_0 \) in the same way as in Proposition 3.4, namely
\[
q_{t_0}(t_0) = q_0 = \left( t_0^{-1} + b + K(t_0) \right)^{-1} R(t_0)
\] (6.37)
where now \( K = -(1/2) \Delta_B - \hat{B} \) and \( b \) is the constant occurring in (6.1), so that \( K + b \) is a positive operator. The choice (6.37) is the special case of (3.65) where \((v_0, \rho, f, z) = (q_0, t_0^{-1} + b, -R, 0)\). With that choice
\[
\| q_{t_0}(t_0) \|_2 \leq t_0 \| R(t_0) \|_2 \leq C \ t_0^{\lambda_0},
\] (6.38)
\[
i \left( \partial_t q_{t_0} \right) (t_0) = - \left( t_0^{-1} + b \right) q_0 = - \left( t_0^{-1} + b \right) \left( t_0^{-1} + b + K(t_0) \right)^{-1} R(t_0)
\] (6.39)
so that
\[
\| \partial_t q_{t_0}(t_0) \|_2 \leq (1 + b t_0) \| R(t_0) \|_2 \leq C \ t_0^{\lambda_0 - 1}
\] (6.40)
and
\[
- \left( \partial^2_t q_{t_0} \right) (t_0) = (K_i \partial_t q_{t_0} + i (\partial_t K) q_{t_0} - i \partial_t R)(t_0)
\]
\[
= \left( - \left( t_0^{-1} + b \right) K + i \partial_t K \right) \left( t_0^{-1} + b + K \right)^{-1} R - i \partial_t R \right) (t_0).
\] (6.41)
We need to estimate the last quantity in $L^2$. Since $K + b$ is a positive operator, we have
\[-b t_0 \leq K \left( t_0^{-1} + b + K \right)^{-1} \leq 1\]
so that
\[\| (t_0^{-1} + b) K \left( t_0^{-1} + b + K \right)^{-1} R(t_0) \|_2 \leq t_0^{-1} (1 + b t_0)^2 \| R(t_0) \|_2 . \tag{6.42}\]

On the other hand, from (6.1) and from
\[\partial_t K = i (\partial_t B) \cdot \nabla B - \partial_t \bar{B} \tag{6.43}\]
we obtain
\[\| (\partial_t K) q_0 \|_2 \leq t_0^{-1} b \left( \| \nabla B q_0 \|_2 + \| q_0 \|_2 \right) . \tag{6.44}\]
Now
\[\| \nabla B q_0 \|_2^2 = 2 < q_0, (K + \bar{B}) q_0 > \leq 2 < q_0, (K + b) q_0 > = 2 < R(t_0), (t_0^{-1} + b + K)^{-2} (K + b) R(t_0) > \leq t_0 \| R(t_0) \|_2^2 . \tag{6.45}\]

Collecting (6.41)-(6.45) and using (6.31) (6.38) yields
\[\| \partial_t^2 q_{t_0}(t_0) \|_2 \leq \left( t_0^{-1} (1 + b t_0)^2 + b t_0^{-1/2} + b \right) \| R(t_0) \|_2 + \| \partial_t R(t_0) \|_2 \leq C \left( t_0^{\lambda_0 - 2} + t_0 \lambda_1^{-1} \right) \leq C t_0^{\lambda_1 - 1} . \tag{6.46}\]

Let $q_{t_0} \in C^1(I, H^2) \cap C^2(I, L^2)$ be the solution of (1.31) with initial data (6.37) at $t_0$ obtained from Proposition 3.4. In order to take the limit $t_0 \to 0$, we need to estimate $q_{t_0}$ and $\partial_t q_{t_0}$ in $H^2$ and $\partial_t^2 q_{t_0}$ in $L^2$ uniformly in $t_0$ for $t_0 \leq t \leq 1$. We again omit the subscript $t_0$ on $q$ and we define
\[y_j = \| \partial_t^j q \|_2 . \tag{6.47}\]

The estimates (6.38) (6.40) (6.46) yield
\[y_0(t_0) \leq C t_0^{\lambda_0} , \quad y_1(t_0) \leq C t_0^{\lambda_0 - 1} , \tag{6.48}\]
\[y_2(t_0) \leq C t_0^{\lambda_1 - 1} . \tag{6.49}\]

We have already estimated $y_0, y_1$ and $\| \Delta q \|_2, \| \Delta_B q \|_2$, in the proof of Proposition 3.3 by using only (6.1) and (6.3) for $j = 0, 1$. Here the initial condition at $t_0$ is different, but because of (6.48) it makes no difference in the basic estimates (6.10).
By a direct estimate of the time derivative of (1.31), we obtain

$$|\partial_t y_2| \leq \| f_2 \|_2$$

(6.50)

where now (see (3.61) (3.62) and (6.43))

$$f_2 = 2i (\partial_t \lambda) \cdot \nabla_B \partial_t q - 2 (\partial_t \lambda) \partial_t q + i \left( \partial_t t \right) \cdot \nabla_B q + \left( (\partial_t \lambda)^2 - (\partial_t \lambda) \right) q - \partial_t^2 R$$

(6.51)

so that by (6.1)

$$|\partial_t y_2| \leq 2bt^{-1} (\| \nabla_B \partial_t q \|_2 + y_1) + bt^{-2} (\| \nabla_B q \|_2 + (1 + b)y_0) + \| \partial_t^2 R \|_2.$$

(6.52)

On the other hand

$$\| \nabla_B \partial_t q \|_2 \leq y_1 \| \Delta_B \partial_t q \|_2 \leq y_1 \left( \| \partial_t \Delta_B q \|_2 + 2bt^{-1} \| \nabla_B q \|_2 \right).$$

(6.53)

By a direct estimate of the time derivative of (1.31), we obtain

$$\| \partial_t \Delta_B q \|_2 \leq 2 \left( y_2 + b \left( y_1 + t^{-1}y_0 \right) + \| \partial_t R \|_2 \right)$$

(6.54)

and therefore

$$\| \nabla_B \partial_t q \|_2 \leq 2y_1 \left( y_2 + b \left( t^{-1} \| \nabla_B q \|_2 + y_1 + t^{-1}y_0 \right) + \| \partial_t R \|_2 \right).$$

(6.55)

Substituting (6.55) into (6.52) yields

$$|\partial_t y_2| \leq C t^{-1} \left\{ (y_1y_2)^{1/2} + \left( y_1 \ t^{-1} \| \nabla_B q \|_2 \right)^{1/2} + y_1 + \left( t^{-1}y_1 \ y_0 \right)^{1/2} + (y_1 \ | \partial_t R \|_2)^{1/2} + t^{-1} \| \nabla_B q \|_2 + t^{-1}y_0 \right) + \| \partial_t^2 R \|_2$$

(6.56)

so that by (3.31) (6.10) (6.16) (6.19)

$$|\partial_t y_2| \leq C \left\{ t^{-1+\lambda'_1/2} \ y_2^{1/2} + t^{-1+\lambda'_1} + t^{-3/2+\left( \lambda_1+\lambda'_1 \right)/2} + t^{-2+\left( \lambda_0+\lambda'_1 \right)/2} + t^{-2+\lambda_0} + t^{-1+\lambda_2} \right\}$$

$$\leq C \left( t^{-1+\lambda'_1/2} \ y_2^{1/2} + t^{-1+\lambda_2} \right)$$

(6.57)

provided

$$\lambda_2 \leq \lambda'_1 \wedge \left( \left( \lambda_1 + \lambda'_1 \right)/2 - 1/2 \right) \wedge \left( \lambda_0 + \lambda'_1 \right)/2 - 1$$

(6.58)

which reduces to the last inequality in (6.30) by an elementary computation. Integrating (6.57) by Lemma 2.3 with initial condition satisfying (6.49) yields

$$y_2 \leq C \left( t^{\lambda_2 + t_0^{\lambda'_1-1}} \right) \quad \text{for } t_0 \leq t \leq 1$$

(6.59)
and therefore
\[ y_2 \leq C \ t^{\lambda_2} \quad \text{for } t_0 \vee t_0^{(\lambda_1^{-1})/\lambda_2} \equiv \bar{t}_0 \leq t \leq 1. \quad (6.60) \]

It then follows from (6.54) (6.10) (6.16) (6.31) (6.60) that
\[ \| \partial_t \Delta Bq \|_2 \leq C \ t^{\lambda_2} \quad \text{for } \bar{t}_0 \leq t \leq 1 \quad (6.61) \]
and from (6.53) (6.19) (6.61) that
\[ \| \Delta B \partial_t q \|_2 \leq C \ t^{\lambda_2} \quad \text{for } \bar{t}_0 \leq t \leq 1. \quad (6.62) \]

Finally by (6.16) (6.62)
\[ \| \Delta \partial_t q \|_2 \leq \| \Delta B \partial_t q \|_2 + 2b(y_1 \| \Delta B \partial_t q \|_2)^{1/2} + b^2 y_1 \leq C \ t^{\lambda_2} \quad (6.63) \]
for \( \bar{t}_0 \leq t \leq 1 \). The estimates (6.10) (6.16) (6.20) (6.60) (6.63) provide uniform estimates in \( t_0 \) of \( q, \partial_t q \) in \( H^1 \) and of \( \partial_t^2 q \) in \( L^2 \) for \( \bar{t}_0 \leq t \leq 1 \). We can now take the limit \( t_0 \to 0 \). The argument is the same as in the proof of Proposition 6.1 and will be omitted.

**Part (2).** The continuity of \( q = w - W \) in \( H^4 \) follows from (1.31), from the continuity of \( B \tilde{B}, R \) in \( H^2 \) and from Part (1). In order to prove the estimate (6.36), it is sufficient to estimate \( \Delta^2 q_{t_0} \) in \( L^2 \) uniformly in \( t_0 \) for \( \bar{t}_0 \leq t \leq 1 \). We omit again the subscript \( t_0 \). We know already from (6.63) that
\[ \| \Delta(Kq - R) \|_2 \leq C \ t^{\lambda_2} . \quad (6.64) \]

We next estimate
\[
\| \Delta^2 q \|_2 \leq 2 \| \Delta Kq \|_2 + 2 \| \Delta (B \cdot \nabla q) \|_2 + \| \Delta ((B^2 - 2\tilde{B})q) \|_2 \\
\leq 2 \| \Delta Kq \|_2 + 2b \left( \| \Delta q \|_2 \| \Delta^2 q \|_2 \right)^{1/2} + \\
+ C \left( t^{-2} (\| \nabla_B q \|_2 + \| q \|_2) + t^{-1} \| \Delta q \|_2 \right) . \quad (6.65)\]

By Lemma 2.2 and (6.19) (6.20) (6.64), this implies
\[ \| \Delta^2 q \|_2 \leq C \left( \| R \|_2 + t^{\lambda_2} + t^{-2+(\lambda_0+\lambda_1)/2} + t^{-1+\lambda_1} \right) \leq C \ t^{\tilde{\lambda}_2-1} \quad (6.66) \]
under the assumption (6.34), provided
\[ \tilde{\lambda}_2 \leq (\lambda_2 + 1) \wedge ((\lambda_0 + \lambda_1) / 2 - 1) \wedge \lambda_1 , \quad (6.67) \]
a condition which reduces to (6.35). The estimate (6.66) holds for \( \bar{t}_0 \leq t \leq 1 \) uniformly in \( t_0 \).

The last statement of Part (2) follows from (6.6) and (6.66) by interpolation.
7 Choice of $W$ and remainder estimates

In this section we continue the program started in Section 6 by constructing model functions $W$ satisfying the assumptions of Propositions 6.1 and 6.2. In all this section we assume $A$ to satisfy the free wave equation and therefore to be given by (2.12) for suitable $(A_+, \dot{A}_+)$. We recall that $A$ and $B$ are related by (1.11). We first choose $W$ in the form $W = U(t)w$ and we obtain sufficient conditions on $(w_+, A_+, \dot{A}_+)$ to ensure the required assumptions. Those conditions will require a support condition on $w_+$. We begin by deriving sufficient conditions in terms of $\chi$, $B$ and $\dot{B}$, where $\chi$ is the characteristic function of the support of $w_+$.

**Proposition 7.1.** Let $0 < \lambda \leq 1$ and let $w_+ \in H^5$. Let $B$ satisfy (2.27) (equivalently (6.1)) for $2 \leq r \leq \infty$, $0 \leq j + |\alpha| \leq 2$ and in addition
\[
\| \chi \partial_t^j \partial_x^\alpha B(t) \|_2 \lor \| \chi \partial_t^j \partial_x^\alpha \dot{B}(t) \|_2 \leq C t^{1+\lambda-j-|\alpha|} \tag{7.1}
\]
for $0 \leq j + |\alpha| \leq 2$ and for all $t \in (0, 1]$. Then the following inequality holds
\[
\| \partial_t^j \partial_x^\alpha R(U(t)w_+) \|_2 \leq C t^{1+\lambda-j-|\alpha|} \tag{7.2}
\]
for $0 \leq j + |\alpha| \leq 2$ and for all $t \in (0, 1]$.

**Proof.** From (1.32) with $W = U(t)w_+$, we obtain
\[
R(W) = -iB \cdot \nabla W + \left(\dot{B} - B^2/2\right) W \tag{7.3}
\]
It will be sufficient to prove an estimate of the type
\[
\| \partial_t^j \partial_x^\alpha BW \|_2 \leq C t^{1+\lambda-j-|\alpha|} \tag{7.4}
\]
for the relevant $j, \alpha$, for $w_+$ in $H^4$ and under the assumptions made on $B$. The final estimate (7.2) will then be obtained by applying that special case with $(B, w_+)$ replaced by $(B, \nabla w_+)$, by $(\dot{B}, w_+)$ and by $(B^2, w_+)$, given the fact that the estimates available for $B$ imply the same estimates for $B^2$. By a Taylor expansion of $U(t)$ to second order
\[
BW = Bw_+ + i(t/2)B\Delta w_+ - (1/4)B \int_0^t dt'(t - t')U(t')\Delta^2 w_+, \tag{7.5}
\]
we estimate
\begin{equation}
\| BW \|_2 \leq \| \chi B \|_2 \| w_+ \|_\infty + t \| \chi B \|_2 \| \Delta w_+ \|_\infty + t^2 \| B \|_\infty \| \Delta^2 w_+ \|_2
\end{equation}

which yields (7.4) for \( j = |\alpha| = 0 \).

In the case \( j + |\alpha| \neq 0 \), we obtain various terms which we estimate differently. The terms with all derivatives on \( B \) are estimated exactly as in (7.5) (7.6), with an extra power \( t^{-j-|\alpha|} \) coming from the assumptions on \( B \). The terms with one derivative on \( W \) are estimated more simply by a Taylor expansion of \( U(t) \) to first order so that

\begin{align*}
\| B \nabla W \|_2 & \leq \| \chi B \|_2 \| \nabla w_+ \|_\infty + t \| B \|_\infty \| \nabla \Delta w_+ \|_2 \leq C t \\
\| B \partial_t W \|_2 & \leq \| \chi B \|_2 \| \Delta w_+ \|_\infty + t \| B \|_\infty \| \Delta^2 w_+ \|_2 \leq C t
\end{align*}

in the cases \( j = 0, |\alpha| = 1 \) and \( j = 1, \alpha = 0 \) respectively, which completes the proof of (7.4) for \( j + |\alpha| = 1 \). Similar estimates hold and take care of the terms with one derivative on \( W \) if \( j + |\alpha| = 2 \). Finally, the terms with two derivatives on \( W \) in the case \( j + |\alpha| = 2 \) are estimated simply by

\begin{equation}
\| B \partial_t^j \partial_x^\alpha W \|_2 \leq \| B \|_\infty \| \Delta^j \partial_x^\alpha w_+ \|_2 \leq C
\end{equation}

thereby completing the proof of (7.4) in that case.

\[ \square \]

We next complete the argument by giving sufficient conditions on \( w_+ \) and on \((A_+, \dot{A}_+)\) that ensure (7.1). The following proposition is a slight extension of Lemma 5.2, part (2) in [5].

**Proposition 7.2.** Let \( \lambda \geq 0 \) and let \( w_+ \) satisfy the support condition

\begin{equation}
\text{Supp } w_+ \subset \{ x : |x| - 1 \geq \eta \} \quad (1.33) \equiv (7.7)
\end{equation}

for some \( \eta, 0 < \eta < 1 \). Let \( j \geq 0 \) be an integer, let \( \alpha \) be a multiindex and let \( \chi_R \) be the characteristic function of the set \( \{ x : |x| \geq R \} \). Let \((A_+, \dot{A}_+)\) satisfy

\begin{align}
\| \chi_R \partial_x^\alpha (x \cdot \nabla)^j A_+ \|_2 & \vee \| \chi_R \partial_x^\alpha (x \cdot \nabla)^j (x \cdot A_+) \|_2 \leq C R^{-\lambda - 1/2} \quad (7.8) \\
\| \chi_R (x \cdot \nabla)^j \dot{A}_+ \|_{6/5} & \vee \| \chi_R (x \cdot \nabla)^j (x \cdot \dot{A}_+) \|_{6/5} \leq C R^{-\lambda - 1/2} \quad (7.9)
\end{align}
\[
\| \chi_R \partial_x^{\alpha'} (x \cdot \nabla)^j \hat{A}_+ \|_2 \leq C \frac{R^{-\lambda-1/2}}{R_0} \quad (7.10)
\]

for some $\alpha' \leq \alpha$ with $|\alpha'| = |\alpha| - 1$ if $\alpha \neq 0$, for all $j'$, $0 \leq j' \leq j$ and for all $R \geq R_0$ for some $R_0 > 0$. Then (7.1) holds for all $t \in (0, 1]$.

**Proof.** In the special case $j = 0$ and as regards $B$, the result is that of Lemma 5.2, part (2) of [5] to which we refer for the proof. That proof is a simple application of the finite propagation speed for the wave equation. The case of general $j \geq 0$ follows therefrom and from (2.21) (2.19). Finally the result for $\tilde{B}$ follows from that for $B$ and from (2.20) (2.13).

We next try to eliminate the support condition (7.7) on $w_+$. For that purpose we choose a more complicated $W$. We take $W$ in the following form

\[
W = (1 - i h \cdot \nabla + \hat{h}) W_0 \quad (7.11)
\]

where $h$ and $\hat{h}$ are defined by

\[
\Delta h = -2B, \quad \Delta \hat{h} = -2\hat{B} \quad (7.12)
\]

and where $W_0$ is a solution of the equation

\[
\left( i \partial_t + (1/2) \Delta - B^2/2 \right) W_0 = 0 . \quad (7.13)
\]

From (7.12) it follows that $h$ and $\hat{h}$ can be made to satisfy estimates similar to the estimates (2.27) (or (6.1)) for $B$ and $\hat{B}$ improved by a factor $t^2$. This will be proved in Proposition 7.4 below. Anticipating on that fact and on suitable estimates of $W_0$ which will be proved in Proposition 7.5 below, we now show that $R(W)$ satisfies the estimates required in Propositions 6.1 and 6.2.

**Proposition 7.3.** Let $I = (0, 1]$. Let $B$ and $h$, $\hat{h}$ defined by (7.12) satisfy the estimates (2.27) (or (6.1)) and

\[
\| \partial_t^j \partial_x^\alpha h \|_r \vee \| \partial_t^j \partial_x^\alpha \hat{h} \|_r \leq C t^{2 - j - |\alpha| + 1/r} \quad (7.14)
\]

for $2 \leq r \leq \infty$, $0 \leq j + |\alpha| \leq 3$ and for all $t \in I$. Let $r_0 > 3$ and let $W_0 \in C(I, H^4) \cap C^1(I, H^3) \cap C^2(I, H^2)$ be a solution of (7.13) in $I$ satisfying the estimates
\[ \| \partial_t W_0 \|_2 \vee \| W_0 \|_H^2 \vee \| \nabla^2 W_0 \|_{r_0} \leq C, \quad (7.15) \]
\[ \| \partial^2_t W_0 \|_2 \vee \| \partial_t \Delta W_0 \|_2 \vee \| \nabla \Delta W_0 \|_2 \leq C t^{-1/2}, \quad (7.16) \]
\[ \| \partial^2_t \Delta W_0 \|_2 \vee \| \partial_t \nabla \Delta W_0 \|_2 \vee \| \Delta^2 W_0 \|_2 \leq C t^{-3/2} \quad (7.17) \]

for all \( t \in I \). Then \( R(W) \) satisfies the estimates
\[ \| \partial^j_t R(W) \|_2 \leq C t^{1+\lambda-j} \quad \text{for } 0 \leq j \leq 2 \quad (7.18) \]
\[ \| \partial^j_t \nabla R(W) \|_2 \leq C t^{\lambda-j} \quad \text{for } j = 0, 1 \quad (7.19) \]
\[ \| \Delta R(W) \|_2 \leq C t^{-1+\lambda} \quad (7.20) \]

with \( \lambda = 1/2 - 1/r_0 \), for all \( t \in I \). In particular \( R(W) \) satisfies the estimates \((6.3) (6.4) \) and \((6.31) (6.34) \) of Propositions 6.1 and 6.2 with \( \lambda_j = \lambda + 2 - j \) and \( \tilde{\lambda}_2 = \lambda \).

**Proof.** Substituting \((7.11)\) into the definition \((1.32)\) of \( R(W) \) we obtain
\[
R(W) = \left( 1 - i h \cdot \nabla + \hat{h} \right) \left( i \partial_t + (1/2) \Delta - B^2/2 \right) W_0 - i h B \cdot (\nabla B) W_0
\]
\[
- i h \left( -i B \cdot \nabla + \hat{B} \right) \nabla W_0 + \left( 1 + \hat{h} \right) \left( -i B \cdot \nabla + \hat{B} \right) W_0
\]
\[
- i \left( i \partial_t h + (1/2) \Delta h - i B \cdot \nabla h + \nabla h \nabla \right) \cdot \nabla W_0
\]
\[
+ \left( i \partial_t \hat{h} + (1/2) \Delta \hat{h} - i B \cdot \nabla \hat{h} + \nabla \hat{h} \cdot \nabla \right) W_0 . \quad (7.21)
\]

By \((7.12) \) \((7.13)\), \( R(W) \) reduces to
\[
R(W) = \left( i \partial_t \hat{h} - i B \cdot \nabla \hat{h} - i h B \cdot \nabla B + \hat{h} \hat{B} \right) W_0
\]
\[
+ \left( \partial_t h - B \cdot \nabla h + \nabla \hat{h} - i (h \hat{B} + \hat{h} B) \right) \cdot \nabla W_0 - (i \nabla h + h B) \cdot \nabla^2 W_0
\]
\[
\equiv N_0 W_0 + N_1 \nabla W_0 + N_2 \nabla^2 W_0 . \quad (7.22)
\]

The contractions in \((7.21) \) \((7.22)\) have been left partly unspecified since they will disappear in the estimates.

We first estimate
\[ \| R(W) \|_2 \leq \| N_0 \|_2 \| W_0 \|_\infty + \| N_1 \|_2 \| \nabla W_0 \|_\infty + \| N_2 \|_s \| \nabla^2 W_0 \|_{r_0} \]
\[ \leq C \left( t^{3/2} + t^{1+\lambda} \right) \leq C t^{1+\lambda} = M_0 \quad (7.23) \]

where \( \lambda = 1/s = 1/2 - 1/r_0 \), by \((2.27) \) \((7.14) \) \((7.15)\).
We next estimate $\partial_t R(W)$. The terms with $\partial_t$ applied to $B$, $\tilde{B}$ or to $h$, $\tilde{h}$ are estimated in a way similar to (7.23), thereby yielding a contribution $C t^{-1} M_0$, so that

$$
\| \partial_t R(W) \|_2 \leq C t^{-1} M_0 + \| N_0 \|_\infty \| \partial_t W_0 \|_2 + \| N_1 \|_\infty \| \partial_t \nabla W_0 \|^{1/2}_2
$$

$$
+ \| N_2 \|_\infty \| \partial_t \nabla^2 W_0 \|_2 \leq C \left( t^{-1} M_0 + t^{1/2} \right) \leq C t^\lambda = M_1
$$

(7.24)

by (2.27) (7.14) (7.15) (7.16) and by using in particular the fact that $\| N_j \|_\infty \leq C t$ for $j = 0, 1, 2$.

We estimate similarly

$$
\| \nabla R(W) \|_2 \leq C t^{-1} M_0 + \| N_0 \|_\infty \| \nabla W_0 \|_2 + \| N_1 \|_\infty \| \nabla^2 W_0 \|_2
$$

$$
+ \| N_2 \|_\infty \| \nabla \Delta W_0 \|_2 \leq C t^\lambda = M_1.
$$

(7.25)

We next estimate the second order derivatives of $R(W)$. The terms with one or two derivatives applied to $B$, $\tilde{B}$ or to $h$, $\tilde{h}$ are estimated in a way similar to (7.24) (7.25), thereby yielding a contribution $C t^{-1} M_1$, and only the terms with two derivatives on $W_0$ need separate consideration. Estimating again the $W_0$ factors in $L^2$ and using again the fact that $\| N_j \|_\infty \leq C t$, we obtain

$$
\| \partial^2_t R(W) \|_2 \leq C t^{-1+\lambda} + C t \left( \| \partial^2_t W_0 \|_2 + \| \partial^2_t \nabla W_0 \|_2 \right)
$$

$$
\leq C t^{-1+\lambda} + C t t^{-1/2} \leq C t^{-1+\lambda}
$$

(7.26)

$$
\| \partial_t \nabla R(W) \|_2 \leq C t^{-1+\lambda} + C t \left( \| \partial_t \nabla W_0 \|_2 + \| \partial_t \Delta W_0 \|_2 \right)
$$

$$
\leq C t^{-1+\lambda}
$$

(7.27)

$$
\| \Delta R(W) \|_2 \leq C t^{-1+\lambda} + C t \left( \| \Delta W_0 \|_2 + \| \nabla \Delta W_0 \|_2 + \| \Delta^2 W_0 \|_2 \right)
$$

$$
\leq C t^{-1+\lambda}
$$

(7.28)

which yields the required second order estimates.

\qed

We next derive the estimates (7.14). This will be done conveniently by using homogeneous Besov spaces [1] [15]. For that purpose we introduce a Paley-Littlewood dyadic decomposition in the following standard way. Let $F \psi \equiv \hat{\psi} \in C_0^\infty(\mathbb{R}^3)$ with $0 \leq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. Let $\hat{\varphi}_0(\xi) = \hat{\psi}(\xi) - \hat{\psi}(2\xi)$
and for any \( j \in \mathbb{Z}, \hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) \) so that \( \hat{\varphi}_j \) is supported in \( \{ \xi : 2^{j-1} \leq |\xi| \leq 2^{j+1} \} \) and for any \( \xi \in \mathbb{R}^3 \setminus \{0\},\)

\[
\sum_j \hat{\varphi}_j(\xi) = 1
\]

with at most two nonvanishing terms in the sum for each \( \xi \). The homogeneous Besov space \( \dot{B}^s_{r,s} \) is then defined for any \( \rho \in \mathbb{R} \) and \( 1 \leq r, s \leq \infty \) by

\[
\dot{B}^s_{r,s} = \{ v : \| v; \dot{B}^s_{r,s} \| = \| 2^{\rho j} \varphi_j * v; l^s_j(L^r) \| < \infty \} \quad (7.29)
\]

where \( F \varphi_j = \hat{\varphi}_j \) and \( * \) denotes the convolution in \( \mathbb{R}^3 \).

We can now state the result as follows.

**Proposition 7.4.** Let \( I = (0,1] \), let \( j \geq 0 \) be an integer and let \( \alpha \) be a multiindex.

1. Assume that \( (A_+, \dot{A}_+) \) satisfies the conditions

\[
A \in \dot{B}^{-2}_{2,2} \cap \dot{B}^0_{1,1} \quad , \quad \dot{A} \in \dot{B}^{-3}_{2,2} \cap \dot{B}^{-1}_{1,1} \quad (7.30)
\]

for

\[
\begin{align*}
A &= \partial_x^\alpha (x \cdot \nabla)^{j'} A_+ \quad , \quad \dot{A} = \partial_x^\alpha (x \cdot \nabla)^{j'} (x \cdot A_+) \\
\dot{A} &= \partial_x^\alpha (x \cdot \nabla)^{j'} \dot{A}_+ \quad , \quad \dot{A} = \partial_x^\alpha (x \cdot \nabla)^{j'} (x \cdot \dot{A}_+) 
\end{align*}
\]

(7.31)

(7.32)

for \( 0 \leq j' \leq j \). Then \( h, \dot{h} \) satisfy the estimates (7.14) for \( 2 \leq r \leq \infty \) and for all \( t \in I \).

2. Let \( A \) and \( \dot{A} \) satisfy

\[
\begin{align*}
\omega^s A &\in L^1 \quad , \quad \# \ A \in L^1 \quad , \quad \int dx \ A(x) = 0 \\
< x >^{1+\theta} \dot{A} &\in L^1 \quad , \quad \int dx \ \dot{A}(x) = \int dx \ x\dot{A}(x) = 0
\end{align*}
\]

(7.33)

(7.34)

for some \( \theta > 1/2 \). Then \( A, \dot{A} \) satisfy (7.30).

**Proof.** From (1.11) and (7.12), it follows that

\[
2\omega^{-2} A = -2t^{-1} \omega^{-2} D_0 B(1/t)
\]

\[
= -t^{-1} D_0 \left( t^{-2} h \right)(1/t) \quad (7.35)
\]

and similarly

\[
2\omega^{-2}(x \cdot A) = -t^{-1} D_0 \left( t^{-2} \dot{h} \right)(1/t) \quad .
\]

(7.36)
We are now in the same situation as in Lemma 2.4, where the estimates (2.27) for $B$, $\tilde{B}$ were obtained from the estimates (2.26) for $A$. In order to prove the estimates (7.14) for $h$, $\tilde{h}$, it is sufficient to show that $\omega^{-2}A$ and $\omega^{-2}(x \cdot A)$ satisfy the same estimates (2.26) as $A$ and $(x \cdot A)$. Since $x \cdot A$ satisfies the wave equation as well as $A$, it is sufficient to prove those estimates for $A$. Furthermore by (2.15)-(2.18), it is sufficient to consider the case $j = 0$, $\alpha = 0$.

Part (1). From the basic estimate (see (3.13) in [4])

$$
\| \exp(i\omega t) \varphi_j * f \|_r \leq C |t|^{-1+2/r} 2^{j(2-4/r)} \| \varphi_j * f \|_r
$$

with $2 \leq r \leq \infty$, $1/r + 1/\bar{r} = 1$, we obtain

$$
\| \varphi_j * \omega^{-2}A \|_r \leq C t^{-1+2/r} \{ 2^{-4j/r} \| \varphi_j * A_+ \|_r + 2^{-j(1+4/r)} \| \varphi_j * \dot{A}_+ \|_r \}.
$$

(7.37)

Taking the $l^2$ norm for $r = 2$ and the $l^1$ norm for $r = \infty$ yields

$$
\| \omega^{-2}A; \dot{B}^0_{2,2} \| \leq C \left( \| A_+; \dot{B}^{-2}_{2,2} \| + \| \dot{A}_+; \dot{B}^{-3}_{2,2} \| \right),
$$

$$
\| \omega^{-2}A; \dot{B}^0_{\infty,1} \| \leq C t^{-1} \left( \| A_+; \dot{B}^0_{1,1} \| + \| \dot{A}_+; \dot{B}^1_{1,1} \| \right).
$$

By interpolation and by the standard embedding properties of Besov spaces, this implies

$$
\| \omega^{-2}A \|_r \leq C t^{-1+2/r}
$$

where the last constant depends on the relevant norms of $(A_+, \dot{A}_+)$. 

Part (2). As in Part (1) it is sufficient to consider the case where $(A, \dot{A}) = (A_+, \dot{A}_+)$. Using the Young inequality and the homogeneity relation

$$
\| \varphi_j \|_r = 2^{3j/r} \| \varphi_0 \|_r
$$

we estimate the bracket in (7.37) by

$$
2^{-4j/r} \| \varphi_j * A_+ \|_r + 2^{-j(1+4/r)} \| \varphi_j * \dot{A}_+ \|_r
$$

$$
\leq C \left\{ 2^{-\varepsilon j-j/r} \| \omega^\varepsilon A_+ \|_1 + 2^{-j(1+1/r)} \| \dot{A}_+ \|_1 \right\}.
$$

(7.38)

Therefore the high frequency part of the Besov norms, more precisely the summation over $j \geq 0$, is controlled by the conditions $\omega^\varepsilon A_+ \in L^1$, $\dot{A}_+ \in L^1$. 

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We now consider the low frequency part of the Besov norms. Using the vanishing integral condition on $A_+$, we rewrite

$$( \varphi_j * A_+ ) (x) = \int dy (\varphi_j(x - y) - \varphi_j(x)) A_+ (y)$$

so that

$$\| \varphi_j * A_+ \|_r \leq \operatorname{Sup}_y |y|^{-\theta} \| \varphi_j(\cdot - y) - \varphi_j \|_r \| |x|^\theta A_+ \|_1$$

$$= 2^{j(\theta + 3/r)} \operatorname{Sup}_y |y|^{-\theta} \| \varphi_0(\cdot - y) - \varphi_0 \|_r \| |x|^\theta A_+ \|_1$$

$$= C \ 2^{j(\theta + 3/r)} \| |x|^\theta A_+ \|_1$$

(7.39)

by homogeneity and the fact that the last Sup is finite for $0 \leq \theta \leq 1$. This implies the summability over $j \leq 0$ of the contribution of $A_+$ to the bracket in (7.37) for $2 \leq r \leq \infty$ and $\theta > 1/2$. Similarly using the vanishing integral conditions on $\dot{A}_+$, we rewrite

$$( \varphi_j * \dot{A}_+ ) (x) = \int dy (\varphi_j(x - y) - \varphi_j(x) + y \cdot \nabla \varphi_j(x)) \dot{A}_+ (y)$$

so that

$$\| \varphi_j * \dot{A}_+ \|_r \leq \operatorname{Sup}_y |y|^{-(1+\theta)} \| \varphi_j(\cdot - y) - \varphi_j + y \cdot \nabla \varphi_j \|_r \| |x|^{1+\theta} \dot{A}_+ \|_1$$

$$= 2^{j(1+\theta + 3/r)} \operatorname{Sup}_y |y|^{-(1+\theta)} \| \varphi_0(\cdot - y) - \varphi_0 + y \cdot \nabla \varphi_0 \|_r \| |x|^{1+\theta} \dot{A}_+ \|_1$$

$$= C \ 2^{j(1+\theta + 3/r)} \| |x|^{1+\theta} \dot{A}_+ \|_1$$

(7.40)

by homogeneity and by the finiteness of the last Sup for $0 \leq \theta \leq 1$. This implies the summability over $j \leq 0$ of the contribution of $\dot{A}_+$ to the bracket in (7.37) for $2 \leq r \leq \infty$ and $\theta > 1/2$.

We now turn to the study of the equation (7.13) and we prove that it admits solutions $W_0$ satisfying the requirements of Proposition 7.3. We rewrite that equation in a form similar to (3.1), namely

$$i \partial_t v = -(1/2) \Delta v + V v$$

(7.41)

with $V = (1/2)B^2$.
Proposition 7.5. Let $I = (0,1]$. Let $V \in C(I, H^4) \cap C^1(I, H^2 \cap L_0^{6/5}) \cap C^3(I, L^2)$ satisfy the estimates

$$
\| \partial_t^j \partial_x^\alpha V \|_r \leq C t^{-j-|\alpha|+1/r}
$$

(7.42)

for $0 \leq j + |\alpha| \leq 1$ and $r = \infty$, for $0 \leq j + |\alpha| \leq 3$ and $r = 2$, and for $\alpha = 0$, $j = 1$ and $6/5 \leq r \leq 2$. Let $v_1 \in H^6$. Then there exists a unique solution $v \in C(I, H^4) \cap C^1(I, H^3) \cap C^2(I, H^2) \cap C^3(I, L^2)$ of (7.41) in $I$ with $v(1) = v_1$, satisfying the following estimates

$$
\| v(t) \|_2 = \| v_1 \|_2
$$

(7.43)

$$
\| \partial_t v \|_2 \vee \| \Delta v \|_2 \leq C
$$

(7.44)

$$
\| \partial_t^2 v \|_2 \vee \| \partial_t \Delta v \|_2 \vee \| \nabla \Delta v \|_2 \leq C t^{-1/2}
$$

(7.45)

$$
\| \Delta v \|_r \leq \begin{cases}
C & \text{for } 2 \leq r < 4 \\
C \ln t & \text{for } r = 4 \\
C t^{-1/2+2/r} & \text{for } 4 < r < 6
\end{cases}
$$

(7.46)

for all $t \in I$. The solution is actually unique in $C(I, L^2)$.

Proof. The existence can be proved by an extension of the method of Proposition 3.4 to the level of $H^6$ using the third derivative $\partial_t^3 v$, simplified by the fact that here $A = 0$ and $f = 0$. For that purpose, one has to ensure in particular that $(\partial_t^3 v)(1) \in L^2$. Now by an easy computation

$$
(-i\partial_t^3 v)(1) = \left( K^3 + 3i (\partial_t V) K - i (\nabla \partial_t V) \cdot \nabla - (i/2) (\Delta \partial_t V) - \partial_t^2 V \right) v_1
$$

(7.47)

where now $K = -(1/2) \Delta + V$. Under the assumptions made on $V$, this belongs to $L^2$ for $v_1 \in H^6$. In particular the condition $V \in C(I, H^4)$ implies that $D(K^3) = H^6$. The solution $v$ comes out with additional regularity properties that are of no interest here and have not been stated. We skip the details and we concentrate on the derivation of the estimates (7.43)-(7.46).

In the same way as in the proof of Proposition 3.2, we first estimate

$$
\| \Delta v \|_2 \leq 2 \left( \| \partial_t v \|_2 + \| V \|_\infty \| v \|_2 \right),
$$

$$
\partial_t \| \partial_t v \|_2 \leq \| \partial_t V \|_2 \| v \|_\infty \leq C \| \partial_t V \|_2 \| v \|^{1/4}_2 \| \Delta v \|^{3/4}_2
$$

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which implies the first inequality in (7.44) by integration. Furthermore

\[ \partial_t \| \partial_t^2 v \|_2 \leq 2 \left\{ \| \partial_t^2 v \|_2 + \| \partial_t V \|_\infty + 2 \| \partial_t V \|_\infty \| \partial_t v \|_2 \right\} \leq C \left( t^{-3/2} + t^{-1} \right) \leq C t^{-3/2} \]

which implies the first inequality in (7.45) by integration. Furthermore

\[ \| \partial_t \Delta v \|_2 \leq 2 \left\{ \| \partial_t^2 v \|_2 + \| \partial_t V \|_\infty \| \partial_t v \|_2 \right\} \leq C \left( t^{-1/2} + 1 \right) \leq C t^{-1/2} \]

\[ \| \nabla \Delta v \|_2 \leq 2 \left\{ \| \nabla \partial_t v \|_2 + \| \nabla V \|_\infty \| \partial_t v \|_2 \right\} \leq C \left( t^{-1/4} + t^{-1/2} + 1 \right) \leq C t^{-1/2} \]

which completes the proof of (7.44).

We next estimate in a similar way

\[ \partial_t \| \partial_t^3 v \|_2 \leq 3 \| \partial_t^2 V \|_2 \| \partial_t v \|_\infty + 3 \| \partial_t V \|_\infty \| \partial_t^2 v \|_2 \leq C \left( t^{-5/2} + t^{-15/8} + t^{-3/2} \right) \leq C t^{-5/2} \]

which implies the first inequality in (7.45) by integration. Furthermore

\[ \| \partial_t \nabla \Delta v \|_2 \leq 2 \left\{ \| \partial_t^2 \nabla v \|_2 + \| \partial_t \nabla V \|_2 \| \partial_t v \|_\infty + \| \partial_t V \|_\infty \| \nabla v \|_2 + \| V \|_\infty \| \partial_t \nabla v \|_2 \right\} \leq C \left( t^{-1} + t^{-3/2} + t^{-1} + t^{-1/4} \right) \leq C t^{-3/2} , \]

\[ \| \Delta^2 v \|_2 \leq 2 \left\{ \| \partial_t \Delta v \|_2 + \| \Delta V \|_2 \| \partial_t v \|_\infty + 2 \| \nabla V \|_\infty \| \nabla v \|_2 + \| V \|_\infty \| \Delta v \|_2 \right\} \leq C \left( t^{-1/2} + t^{-3/2} + t^{-1} + 1 \right) \leq C t^{-3/2} , \]

which completes the proof of (7.45).
We finally prove (7.46). Since
\[ \| \Delta v \|_r \leq 2 (\| \partial_t v \|_r + \| V \|_\infty \| v \|_r) \leq (\| \partial_t v \|_r + C) \] (7.48)
it suffices to estimate \( \partial_t v \in L^r \). We start from the integral relation
\[ i \partial_t v = U(t-1) \left(-\frac{1}{2} \Delta + V(1)\right) v_1 - \int_t^1 dt' U(t-t') (Vv)(t') . \] (7.49)
We estimate for \( 2 \leq r \leq 6 \)
\[ \| U(t-1) \left(-\frac{1}{2} \Delta + V(1)\right) v_1 \|_r \leq C \| \Delta v_1 ; H^1 \| + \| V(1)v_1; H^1 \| \leq C . \] (7.50)
On the other hand from the basic estimate (5.24) of the Schrödinger evolution group with \( 2 \leq r \leq \infty \), \( 1/r + 1/\bar{r} = 1 \) and \( \delta(r) = 3/2 - 3/r \), we obtain
\[ \| \int_t^1 dt' U(t-t') \partial_t (Vv)(t') \|_r \]
\[ \leq C \int_t^1 dt' (t-t')^{-\delta} \left( \| \partial_t V(t') \|_r \| v(t') \|_\infty + \| V(t') \|_{3/\delta} \| \partial_t v(t') \|_2 \right) \]
\[ \leq C \int_t^1 dt' (t-t')^{-\delta} \left( t'^{-1/r} + t'^{1/2 - 1/r} \right) \] (7.51)
\[ \int_t^1 dt' (t-t')^{-\delta} t'^{-1/r} = \int_t^{2t^{1/2}} + \int_t^{2t^{1/2}} \leq C \left\{ t^{-1/2 + 2/r} + \int_{2t^{1/2}}^1 dt' t'^{-3/2 + 2/r} \right\} . \] (7.52)
Integrating (7.52) and collecting the result and (7.48) (7.50) (7.51) yields (7.46).

**Remark 7.1.** The assumptions on \( V \) in Proposition 7.5 are unnecessarily restrictive. Under the condition \( V \in C^2(I, L^2 + L^\infty) \), \( \partial_t^2 V \in L^1_{\text{loc}}(I, L^2 + L^\infty) \), one can prove the existence of a unique solution \( v \in C^3(I, L^2) \cap C^2(I, H^2) \) of (7.41) in \( I \). Furthermore the additional estimates (7.43)-(7.46) on \( v \) can be derived by using only a subset of the assumptions made on \( V \). On the other hand, the assumptions made on \( V \) are easily seen to follow from sufficient regularity assumptions on \( B \) and from estimates of the type (2.27).

We now discuss briefly the situation that arises from Propositions 7.3-7.5 as regards the construction of an asymptotic \( W \) satisfying the assumptions of Propositions 6.1 and 6.2. We have actually constructed a \( W \) satisfying the required assumptions by (7.11) (7.12) and by taking for \( W_0 \) the solution \( v \) of (7.41) obtained in Proposition 7.5. However that \( W \) cannot be parametrized by the asymptotic state
$u_+$ of $u$, and is parametrized instead by $v_1 = W_0(1)$. Now it follows from Proposition 3.2 that the Cauchy problem for (7.41) is well posed in $[0, 1]$ in $H^2$. In particular the previous $W_0$ has an $H^2$ limit $\overline{w_+} \in H^2$ as $t \to 0$, which can be identified with $F\overline{w_+}$, and $W_0$ can be reconstructed from that $\overline{w_+}$ by solving the $H^2$ Cauchy problem with initial time zero. The weak point however is that we are unable to characterize those $w_+ \in H^2$ that arise from solutions of (7.41) with the regularity at the level of $H^6$ which is required for the needs of Propositions 6.1 and 6.2.

8 Wave operators and asymptotics for $u$

In this section we collect the implications of Sections 6 and 7 on the theory of scattering at the level of regularity of $FH^3$ and $FH^4$ for (1.1). The main results consist in obtaining solutions of (1.1) with given asymptotic behaviour at infinity in time, which is essentially equivalent to the construction of the wave operators. We consider separately the simple case of asymptotics provided by Propositions 7.1 and 7.2 and the more complicated case provided by Propositions 7.3-7.5. On the other hand since in both cases we derive all the required estimates on $R(W)$ for Propositions 6.1 and 6.2 together, we also state the implications of those two propositions together. In all this section $A$ is assumed to be a solution of the free wave equation.

We begin with the case of the simple asymptotics provided by $W = U(t)\overline{w_+}$.

**Proposition 8.1.** Let $A$ be a solution of the free wave equation satisfying the estimates (2.26) and the decay assumptions (7.8)-(7.10) for $0 \leq j + |\alpha| \leq 2$ and for some $\lambda \in (0, 1)$. Let $u_+ \in FH^5$ and let $w_+ = Fu_+$ satisfy the support condition (7.7). Then there exists a unique solution $u$ of (1.1) such that $\tilde{u} \in C([1, \infty), FH^4) \cap C^1([1, \infty), FH^2) \cap C^2([1, \infty), L^2)$ and satisfying the estimates

$$\| \tilde{u} - u_+ \|_2 \leq C t^{-2-\lambda}$$  \hspace{1cm} (8.1)

$$\| |x|^{2+j}(\tilde{u} - u_+) \|_2 \leq C t^{-1-\lambda+j} \quad \text{for } 0 \leq j \leq 2$$  \hspace{1cm} (8.2)

$$\| |x|^j \partial_t (\tilde{u} - u_+) \|_2 \leq C t^{-3-\lambda+j} \quad \text{for } 0 \leq j \leq 1$$  \hspace{1cm} (8.3)

$$\| x^2 (it^2 \partial_t + (1/2)x^2) (\tilde{u} - u_+) \|_2 \leq C t^{-\lambda}$$  \hspace{1cm} (8.4)

$$\| \partial_t (it^2 \partial_t + (1/2)x^2) (\tilde{u} - u_+) \|_2 \leq C t^{-2-\lambda}$$  \hspace{1cm} (8.5)
for all $t \geq 1$. The solution is actually unique in $C([1, \infty), L^2)$ under the condition (8.1).

**Proof.** The results follow immediately from Propositions 6.1, 6.2 and 7.1, 7.2 through the change of variables (1.19). The latter implies in particular that

\[
U(-t)i\partial_tw(t) = F((it^2\partial_t + (1/2)x^2)\tilde{u})(1/t) .
\]  

(8.6)

The estimates (8.1)-(8.5) are a rewriting of (6.5)-(6.7) and of (6.32) (6.33) (6.36).

\[ \square \]

We now turn to the more complicated situation covered by Propositions 7.3-7.5.

**Proposition 8.2.** Let $A$ be a solution of the free wave equation with $A \in C(\mathbb{R}^+, H^4)$ satisfying the estimates (2.26) and the conditions (7.33) (7.34) for $0 \leq j + |\alpha| \leq 3$. Let $0 < \lambda < 1/4$. Let $v_1 \in H^6$, let $W_0$ be the solution of (7.13) in $(0, 1]$ with $W_0(1) = v_1$ obtained from Proposition 7.5. Define $W$ by (7.11) (7.12) and $\tilde{u}_a$ by

\[
F\tilde{u}_a = W(1/t) .
\]  

(8.7)

Then the same conclusions as in Proposition 8.1 hold with $u_+$ replaced by $\tilde{u}_a$ in all the estimates.

**Proof.** The results follow immediately from Propositions 6.1, 6.2 and 7.3-7.5 by the change of variables (1.19).

\[ \square \]

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