A NOTE ON COMPUTING RANGE SPACE BASES OF RATIONAL MATRICES

ANDREAS VARGA

Abstract. We discuss computational procedures based on descriptor state-space realizations to compute proper range space bases of rational matrices. The main computation is the orthogonal reduction of the system matrix pencil to a special Kronecker-like form, which allows to extract a full column rank factor, whose columns form a proper rational basis of the range space. The computation of several types of bases can be easily accommodated, such as minimum-degree bases, stable inner minimum-degree bases, etc. Several straightforward applications of the range space basis computation are discussed, such as, the computation of full rank factorizations, normalized coprime factorizations, pseudo-inverses, and inner-outer factorizations.

Key words. rational matrices, full-rank factorizations, computational methods, descriptor systems

AMS subject classifications. 26C15, 93B40, 93C05, 93B55, 93D15

1. Introduction. For any $p \times m$ real rational matrix $G(\lambda)$ of normal rank $r$, there exists a full-rank factorization of $G(\lambda)$ of the form

$$G(\lambda) = R(\lambda)X(\lambda),$$

where $R(\lambda)$ is a $p \times r$ full column rank rational matrix and $X(\lambda)$ is a $r \times m$ full row rank rational matrix. This factorization generalizes the full-rank factorization of constant matrices, and, similarly to the constant case, it is not unique. Indeed, for any $r \times r$ invertible rational matrix $M(\lambda)$, $G(\lambda) = \tilde{R}(\lambda)\tilde{X}(\lambda)$, with $\tilde{R}(\lambda) = R(\lambda)M^{-1}(\lambda)$ and $\tilde{X} = M(\lambda)X(\lambda)$, is also a full-rank factorization of $G(\lambda)$.

The existence of the full-rank factorization (1) can be inferred from the Smith-McMillan form of $G(\lambda)$ [4], which also indicates that both the poles as well as the zeros of $R(\lambda)$ can be arbitrarily chosen. In particular, the zeros of $G(\lambda)$ can be split between the two factors in (1), such that $R(\lambda)$ only includes a selected set of zeros, while $X(\lambda)$ includes the rest of zeros. A minimum-degree $\tilde{R}(\lambda)$ corresponds to the complete absence of zeros in $R(\lambda)$.

Using (1), it is straightforward to show that $G(\lambda)$ and $R(\lambda)$ have the same range space over the rational functions, i.e.

$$\mathcal{R}(G(\lambda)) = \mathcal{R}(R(\lambda)).$$

For this reason, with a little abuse of language, we will call $R(\lambda)$ the range (or image) matrix of $G(\lambda)$ (or simply the range of $G(\lambda)$). It follows, that for each rational column vector $y(\lambda) \in \mathcal{R}(G(\lambda))$, there exists $x(\lambda) \in \mathcal{R}(R(\lambda))$ such that $R(\lambda)x(\lambda) = y(\lambda)$. Since $R(\lambda)$ has full column rank $r$, its columns form a set of $r$ basis vectors of $\mathcal{R}(G(\lambda))$.

In this note, we describe a general computational approach based on a descriptor state-space realization of the rational matrix $G(\lambda)$ to determine a full column rank $R(\lambda)$, representing a proper range space basis of $G(\lambda)$. The zeros of $R(\lambda)$ can be enforced to lie in a specified domain of the complex plane $C_b$. The main computation is the orthogonal reduction of the corresponding system matrix pencil to a special Kronecker-like form (already employed in [7] and [5]), which allows to immediately extract a full column rank factor $R(\lambda)$, which includes all zeros of $G(\lambda)$ lying in $C_b$. Straightforward applications of the range computation techniques are mentioned and numerical examples are given.

2. Range computation. Let $G(\lambda)$ be a $p \times m$ real rational matrix. We can associate $G(\lambda)$ with the transfer function matrix (TFM) of a generalized linear time-invariant system (or descriptor system), where, for a continuous-time system, the frequency variable has the significance $\lambda = s$, the complex variable in the Laplace-transform, and, for a discrete-time system, the frequency variable has the significance $\lambda = z$, the complex variable in the Z-transform. The underlying descriptor system has a generalized state-space representation of
the form

\[
\begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^m\) is the input vector, and \(y(t) \in \mathbb{R}^p\) is the output vector, and where \(\lambda\) is either the differential operator \(\lambda x(t) = \frac{dx(t)}{dt}\) for a continuous-time system or the advance operator \(\lambda x(t) = x(t+1)\) for a discrete-time system. In all what follows, we assume \(E\) is square and possibly singular, and the pencil \(A - \lambda E\) is regular (i.e., \(\det(A - \lambda E) \neq 0\)). The descriptor system (2) represents a state-space realization of the TFM \(G(\lambda)\) if

\[
G(\lambda) = C(\lambda E - A)^{-1}B + D.
\]

We will also use the equivalent notation for the TFM in (3)

\[
G(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}.
\]

The descriptor system (2) will be alternatively denoted by the quadruple \((A - \lambda E, B, C, D)\).

We recall from [11, 10] some basic notions related to descriptor system realizations. A realization \((A - \lambda E, B, C, D)\) is minimal if it is controllable, observable and has no non-dynamic modes. A controllable and observable realization is called irreducible. The poles of \(G(\lambda)\) are related to \(\Lambda(A - \lambda E)\), the eigenvalues of the pencil \(A - \lambda E\) (also known as the generalized eigenvalues of the pair \((A, E)\)). For a minimal realization, the finite poles of \(G(\lambda)\) are the finite eigenvalues of \(A - \lambda E\), while the multiplicities of the infinite poles of \(G(\lambda)\) are defined by the multiplicities of the infinite eigenvalues of \(A - \lambda E\) minus one. A finite eigenvalue \(\lambda_f \in \Lambda(A - \lambda E)\) is controllable if \(\text{rank } [A - \lambda_f E, B] = n\), otherwise is uncontrollable. Similarly, a finite eigenvalue \(\lambda_f \in \Lambda(A - \lambda E)\) is observable if \(\text{rank } [A^T - \lambda_f E^T C^T] = n\), otherwise is unobservable. Infinite controllability requires that \(\text{rank } [E, B] = n\), while infinite observability requires that \(\text{rank } [E^T, C^T] = n\). The lack of non-dynamic modes can be equivalently expressed as \(\mathcal{AN}(E) \subseteq \mathcal{R}(E)\), where \(\mathcal{N}(E)\) denotes the right nullspace of \(E\). The zeros of \(G(\lambda)\) are related to the eigenvalues of the system matrix pencil

\[
S(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}.
\]

For a minimal realization, the finite zeros of \(G(\lambda)\) are the finite eigenvalues of \(S(\lambda)\), while the multiplicities of the infinite zeros of \(G(\lambda)\) are defined by the multiplicities of the infinite eigenvalues of \(S(\lambda)\) minus one.

Consider a disjunct partition of the complex plane \(C\) as

\[
C = C_g \cup C_b, \quad C_g \cap C_b = \emptyset,
\]

where \(C_g\) and \(C_b\) are symmetric with respect to the real axis. \(C_g\) and \(C_b\) are usually associated with the “good” and “bad” domains of the complex plane \(C\) for the poles and zeros of \(G(\lambda)\). We say the descriptor system (4) is proper \(C_g\)-stable if all finite eigenvalues of \(A - \lambda E\) belong to \(C_g\) and all infinite eigenvalues of \(A - \lambda E\) are simple. The descriptor system (4) (or equivalently the pair \((A - \lambda E, B)\)) is \(C_b\)-stabilizable if \(\text{rank } [A - \lambda E, B] = n\) for all finite \(\lambda \in C_b\) and \(\text{rank } [E, B] = n\). The descriptor system (4) (or equivalently the pair \((A - \lambda E, C)\)) is \(C_b\)-detectable if \(\text{rank } [A - \lambda E C] = n\) for all finite \(\lambda \in C_b\) and \(\text{rank } [E^T, C^T] = n\).

The following result slightly extends [5, Theorem 2.2] and is instrumental for the suggested computational approach of proper range space bases.

**Lemma 2.1.** Let \(G(\lambda)\) be a \(p \times m\) real rational matrix of normal rank \(r\), with a \(C_b\)-stabilizable descriptor system realization \((A - \lambda E, B, C, D)\) satisfying (3). Then, there exist two orthogonal matrices \(U\) and \(Z\) such that

\[
\begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} Z = \begin{bmatrix} A_{rg} - \lambda E_{rg} & * & * \\ 0 & A_{bd} - \lambda E_{bd} & B_{bd} & * \\ 0 & 0 & 0 & B_n \\ 0 & C_{bd} & D_{bd} & * \end{bmatrix},
\]
where
(a) The pencil \( A_{rg} - \lambda E_{rg} \) has full row rank for \( \lambda \in \mathbb{C}_g \) and \( E_{rg} \) has full row rank.
(b) \( E_b\ell \) and \( B_n \) are invertible, the pencil

\[
\begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} \\
C_{b\ell} & D_{b\ell}
\end{bmatrix}
\]

has full column rank \( n_{b\ell} + r \) in \( \mathbb{C}_g \) and the pair \((A_{b\ell} - \lambda E_{b\ell}, B_{b\ell})\) is \( C_b\)-stabilizable.

This lemma allows to construct the range of \( G(\lambda) \) using the following result.

**Theorem 2.2.** Let \( G(\lambda) \) be a \( p \times m \) real rational matrix of normal rank \( r \), with the \( C_b\)-stabilizable descriptor system realization \( (A - \lambda E, B, C, D) \) satisfying (3). Let \( U \) and \( Z \) be the orthogonal matrices used in Lemma 2.1 to obtain the system matrix pencil in the special Kronecker-like form (7). Then, the range matrix of \( G(\lambda) \) which includes the zeros of \( G(\lambda) \) in \( \mathbb{C}_b \) has the proper descriptor system realization

\[
R(\lambda) = \begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} \\
C_{b\ell} & D_{b\ell}
\end{bmatrix}
\]

**Proof.** Since, by construction, \( R(\lambda) \) has full column rank and contains all zeros of \( G(\lambda) \) in \( \mathbb{C}_b \), we have only to show that there exists \( X(\lambda) \) which satisfies the linear rational matrix equation (1). This comes down to show that the compatibility condition

\[
\text{rank } R(\lambda) = \text{rank } [R(\lambda) G(\lambda)] = r
\]

is fulfilled. A descriptor system realization of \([R(\lambda) G(\lambda)]\) is

\[
[R(\lambda) G(\lambda)] = \begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & 0 & B_{b\ell} & 0 \\
0 & A - \lambda E & 0 & B \\
C_{b\ell} & D_{b\ell} & C & D
\end{bmatrix}
\]

and the rank condition (10) is equivalent to

\[
\text{rank } \begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} & 0 & 0 \\
0 & 0 & A - \lambda E & B \\
C_{b\ell} & D_{b\ell} & C & D
\end{bmatrix} = n_{b\ell} + n + r.
\]

By premultiplying the pencil

\[
S(\lambda) := \begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} & 0 & 0 \\
0 & 0 & A - \lambda E & B \\
C_{b\ell} & D_{b\ell} & C & D
\end{bmatrix}
\]

with \( \tilde{U} = \text{diag}(I_{n_{b\ell}}, U, I) \) and postmultiplying it with \( \tilde{Z} = \text{diag}(I_{n_{b\ell}+r}, Z) \) we obtain

\[
\tilde{S}(\lambda) := \tilde{U} S(\lambda) \tilde{Z} = \begin{bmatrix}
A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{rg} - \lambda E_{rg} & * & * & * \\
0 & 0 & 0 & A_{b\ell} - \lambda E_{b\ell} & B_{b\ell} & * \\
0 & 0 & 0 & 0 & B_n & * \\
C_{b\ell} & D_{b\ell} & 0 & C_{b\ell} & D_{b\ell} & *
\end{bmatrix}
\]

To prove (11), we show that rank \( \tilde{S}(\lambda) = n_{b\ell} + n + r \), by performing successive block row and block column operations which preserve its rank. The first three block operations are:

1) subtract the first block column multiplied from right with \((A_{b\ell} - \lambda E_{b\ell})^{-1} B_{b\ell}\) from the second block column;
2) subtract the resulting first block row multiplied from left with \(C_{b\ell}(A_{b\ell} - \lambda E_{b\ell})^{-1}\) from the last block row;
3) subtract the third block row multiplied from left with \( C_{bl}(A_{bl} - \lambda E_{bl})^{-1} \) from the last block row.

After performing these operations, we obtain

\[
\text{rank } \tilde{S}(\lambda) = \text{rank } \begin{bmatrix} A_{bl} - \lambda E_{bl} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{rg} - \lambda E_{rg} & * & * & * \\ 0 & 0 & 0 & A_{bl} - \lambda E_{bl} & B_{bl} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R(\lambda) & 0 & 0 & R(\lambda) & D_n(\lambda) \end{bmatrix}
\]

where \( R(\lambda) = C_{bl}(\mu E_{bl} - A_{bl})^{-1} B_{bl} + D_{bl} \) and \( D_n(\lambda) \) denotes the resulting rational matrix in the last block of the last block row. We continue the reduction of the resulted rational matrix by performing two additional operations:

4) subtract the fourth block row multiplied from left with \( B^{r - 1} D_n(\lambda) \) from the last block row;

5) subtract the second block column from the fifth block column.

We finally obtain

\[
\text{rank } \tilde{S}(\lambda) = \text{rank } \begin{bmatrix} A_{bl} - \lambda E_{bl} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{rg} - \lambda E_{rg} & * & * & * \\ 0 & 0 & 0 & A_{bl} - \lambda E_{bl} & B_{bl} & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R(\lambda) & 0 & 0 & R(\lambda) & D_n(\lambda) \end{bmatrix}
\]

from which we immediately have the desired result by observing that

\[
\text{rank } \tilde{S}(\lambda) = \text{rank}(A_{bl} - \lambda E_{bl}) + \text{rank } R(\lambda) + \text{rank } \begin{bmatrix} A_{rg} - \lambda E_{rg} & * & * & * \\ 0 & A_{bl} - \lambda E_{bl} & B_{bl} & * \\ 0 & 0 & 0 & 0 \\ 0 & R(\lambda) & 0 & 0 \end{bmatrix} = n_{bl} + r + n.
\]

For the computation of the descriptor realization (9) of the range \( R(\lambda) \), a numerically stable algorithm can be devised, which exclusively uses orthogonal transformations to reduce the system matrix pencil to the special form (7). The main steps of such an algorithm are given in the (constructive) proof of Theorem 2.2 in [5]. The basic ingredients of such an algorithm are: (a) column and row compressions to full column rank or full row rank matrices, respectively, performed via QR-factorizations with column pivoting, or, more reliably, using singular value decompositions; (b) reduction of a linear pencil to a Kronecker-like staircase form using orthogonal similarity transformations, such that the right, regular and left Kronecker structures are separated; (c) reordering of the eigenvalues of the regular part using orthogonal similarity transformations via the QZ-algorithm. Suitable computational algorithms are described in [3] for (a) and (c), and in [6] for (b) (see also [9, Chapter 10] for an overview of these techniques).

With an additional similarity transformation of the form

\[
Q = \text{diag} \left( I, \begin{bmatrix} I_{n_{bl}} & 0 \\ F & I_r \end{bmatrix} \right)
\]

we achieve

\[
(12) \quad \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} ZQ = \begin{bmatrix} A_{rg} - \lambda E_{rg} & * & * & * \\ 0 & A_{bl} + B_{bl} F - \lambda E_{bl} & B_{bl} & * \\ 0 & 0 & 0 & 0 \\ 0 & C_{bl} + D_{bl} F & D_{bl} & * \end{bmatrix}.
\]

It follows that, with an arbitrary invertible \( W \),

\[
(13) \quad R(\lambda) = \begin{bmatrix} A_{bl} + B_{bl} F - \lambda E_{bl} & B_{bl} W \\ C_{bl} + D_{bl} F & D_{bl} W \end{bmatrix}.
\]
is also a proper range of $G(\lambda)$. Since the pair $(A_{bf} - \lambda E_{bf}, B_{bf})$ is $C_b$-stabilizable, we can choose $F$ such that all eigenvalues of $A_{bf} - \lambda E_{bf}$ lying in $C_b$, can be moved to arbitrary locations in an appropriate stability domain of $C$.

3. Some applications.

3.1. Full rank factorizations. A full rank factorization of an arbitrary rational matrix $G(\lambda)$ of normal rank $r$ of the form (1) can be determined with a proper $R(\lambda)$ of the form (13) and $X(\lambda)$ of the form

\begin{equation}
X(\lambda) = \begin{bmatrix}
A - \lambda E & B \\
C & D
\end{bmatrix}
\end{equation}

where $[\tilde{C} \; \tilde{D}] = W^{-1}[0 \; -F \; I_r \; 0]Z^T$. The expression of $X(\lambda)$ can be verified by explicitly computing the descriptor realization of $R(\lambda)X(\lambda)$ (see the proof of Theorem 3.1 in [5]).

A dual full rank factorization of $G(\lambda)$ is

\begin{equation}
G(\lambda) = \tilde{X}(\lambda)\tilde{R}(\lambda),
\end{equation}

where $\tilde{R}(\lambda)$ is a full row rank coimage of $G(\lambda)$ (i.e., $\tilde{R}^T(\lambda)$ is the range of $G^T(\lambda)$) and $\tilde{X}(\lambda)$ is a full column rank rational matrix. The dual factorization (15) can be computed, by determining the full rank factorization of $G^T(\lambda)$, in the form $G^T(\lambda) = \tilde{R}^T(\lambda)\tilde{X}(\lambda)$.

3.2. Minimum proper bases of the range space. The columns of the range matrix $R(\lambda)$ form a rational basis of $R(G(\lambda))$. This basis is called minimal if the McMillan degree of $R(\lambda)$ (i.e., the number of poles of $R(\lambda)$) is the least achievable one. To determine a minimal proper basis, we choose $C_g = C \cup \{\infty\}$ and $C_b = \emptyset$, in which case, $R(\lambda)$ has no zeros. A stable minimal basis can be constructed in the form (13). A minimal inner basis, satisfying $R^-(\lambda)R(\lambda) = I_r$, can be computed for a suitable choice of $F$ and $W$ in (13) (see [7] for the continuous-time case, and [5] for the discrete-time case).\(^1\)

3.3. Normalized coprime factorizations. A straightforward application of the minimal inner range computation is the determination of a normalized right coprime factorization of an arbitrary $p \times m$ rational matrix $G(\lambda)$ as

\begin{equation}
G(\lambda) = N(\lambda)M^{-1}(\lambda),
\end{equation}

such that $N(\lambda)$ and $M(\lambda)$ are stable and $\begin{bmatrix} N(\lambda) \\ M(\lambda) \end{bmatrix}$ is inner (i.e., $N^{-}(\lambda)N(\lambda) + M^{-}(\lambda)M(\lambda) = I$). The factors $N(\lambda)$ and $M(\lambda)$ can be computed from a minimal inner basis $R(\lambda)$ of the range of $\begin{bmatrix} G(\lambda) \\ I_m \end{bmatrix}$ satisfying

\begin{equation}
\begin{bmatrix} G(\lambda) \\ I_m \end{bmatrix} = R(\lambda)X(\lambda),
\end{equation}

with

\begin{equation}
R(\lambda) = \begin{bmatrix}
N(\lambda) \\ M(\lambda)
\end{bmatrix},
\end{equation}

\begin{equation}
X(\lambda) = M^{-1}(\lambda).
\end{equation}

3.4. Moore-Penrose pseudo-inverse. Another straightforward application of inner minimal bases of range spaces is the computation of the Moore-Penrose pseudo-inverse $G^\#(\lambda)$ of a rational matrix $G(\lambda)$. This computation can be performed in three steps, using a simplified version of the approach described in [7]:

1. Compute a full-rank factorization

\begin{equation}
G(\lambda) = U(\lambda)G_1(\lambda),
\end{equation}

with $U(\lambda)$, a minimal inner range matrix, and $G_1(\lambda)$ full row rank.

\(^{1}\)For a TFM $G(s)$ of a continuous-time system, the conjugate (or adjoint) is defined as $G^-(s) := G^T(-s)$, while for the TFM $G(z)$ of a discrete-time system $G^-(z) := G^T(1/z)$.
2. Compute the dual full-rank factorization
\[ G_1(\lambda) = G_2(\lambda)V(\lambda), \]
with \( V(\lambda) \), a minimal co-inner coimage (i.e., \( V(\lambda)V^\sim(\lambda) = I \)), and \( G_2(\lambda) \) invertible.

3. Compute
\[ G^\#(\lambda) = V^\sim(\lambda)G_2^{-1}(\lambda)U^\sim(\lambda). \]
The dual full-rank factorization at Step 2 can be simply determined by computing the full-rank factorization \( G^T_1(\lambda) = V^T(\lambda)G^T_2(\lambda) \), with \( V^T(\lambda) \), minimal inner range matrix.

3.5. Inner-outer factorization. Let \( C_g \) be the appropriate “stability” domain representing the closed left half complex plane, including infinity, for a continuous-time system, or the closed unit disc centered in the origin, for a discrete-time system, and define \( C_b \) as its complement \( C_b = C \setminus C_g \). The generalized inner–quasi-outer factorization of a \( p \times m \) rational matrix \( G(\lambda) \), with normal rank \( r \), is a special full rank factorization
\[ G(\lambda) = G_i(\lambda)G_o(\lambda), \]
where \( G_i(\lambda) \) is a \( p \times r \) stable inner factor (i.e., \( G_i^\sim(\lambda)G_i(\lambda) = I_p \)) and \( G_o(\lambda) \) is quasi-outer, having full row rank and only zeros in \( C_g \).

To compute the factorization (17), we can determine the range matrix \( R(\lambda) \) of \( G(\lambda) \) in (13), by choosing \( F \) and \( W \) such that \( R(\lambda) \) is inner (see [7] for the continuous-time case, and [5] for the discrete-time case). Once the inner factor \( G_i(\lambda) := R(\lambda) \) is determined, the quasi-outer factor results as \( G_o(\lambda) := X(\lambda) \), where \( X(\lambda) \) has the form (14).

4. Examples. The described range computation approach has been implemented as a MATLAB function \texttt{grange}, which belongs to the free software collection of \textit{Descriptor Systems Tools (DSTOOLS)} [8]. This function also allows to compute inner range bases, including minimal inner range bases. For the computation of the special Kronecker-like form (7) of a system matrix pencil, the function \texttt{gsklf} has been implemented, which allows several choices of \( C_b \). For the computation of the involved Kronecker-like form, the function \texttt{gsklf} is available, which is based on the Algorithm 3.2.1 of [1]. This algorithm underlies the implementation available in the SLICOT library [2], which served as basis for the mex-function \texttt{slgklf}, which has been used to implement \texttt{gsklf}.

Example 1. This is Example 1 from [7] of the transfer function matrix of a continuous-time proper system:
\[ G(s) = \begin{bmatrix} \frac{s - 1}{s + 2} & \frac{s}{s + 2} & \frac{1}{s + 2} \\ 0 & \frac{s}{s - 2} & \frac{s}{s - 2} \\ \frac{s - 1}{s + 2} & \frac{s^2 + 2s - 2}{(s + 1)(s + 2)} & \frac{2s - 1}{(s + 1)(s + 2)} \end{bmatrix}. \]

\( G(s) \) has zeros at \( \{1, 2, \infty\} \), poles at \( \{-1, -1, -2, 2\} \), and normal rank \( r = 2 \).
A minimum proper basis of the range of \( G(s) \), computed with \texttt{grange}, is
\[ R(s) = \frac{1}{s + 1.374} \begin{bmatrix} 1.552s + 2.124 & 1.314s + 1.817 \\ 0.5931s + 1.186 & -0.758s - 1.516 \\ 2.145s + 2.717 & 0.555s + 1.059 \end{bmatrix}, \]
has McMillan-degree 1 and no zeros. The full row rank factor \( X(s) \), satisfying \( G(s) = R(s)X(s) \), has McMillan degree 4, and zeros at \( \{1, 2, 1.374, \infty\} \). The zero at \(-1.374\) is equal to the pole of \( R(s) \).
If we include in the computed range \( R(s) \), both finite unstable zeros of \( G(s) \), then \( R(s) \) has precisely only these (unstable) zeros at \( \{1, 2\} \) and has McMillan degree 3, with poles at
If we determine an inner range $R(s)$, then the unstable zeros of $G(s)$ are reflected to symmetric positions in $\{-1, -2\}$ as zeros of $R(s)$ and the poles of $R(s)$ are at $\{-1, -1.732, -2\}$. Since $R(s)$ is the inner factor of an inner quasi-outer factorization of $G(s)$, it follows that the full row rank factor $X(s)$, satisfying $G(s) = R(s)X(s)$, is the quasi-outer factor. For reference purposes, we give the resulting inner factor

$$R(s) = \begin{bmatrix}
\frac{-0.6078s^3 - 1.944s^2 - 1.501s + 0.6181}{(s + 2)(s + 1.732)(s + 1)} & \frac{0.5452s^3 - 0.2354s^2 - 2.743s - 2.507}{(s + 2)(s + 1.732)(s + 1)} \\
\frac{-0.1683s^3 - 0.6903s^2 + 0.673s + 2.761}{(s + 2)(s + 1.732)(s + 1)} & \frac{-0.799s^3 - 0.4361s^2 + 3.196s + 1.744}{(s + 2)(s + 1.732)(s + 1)} \\
\frac{-0.7761s^3 - 2.466s^2 - 0.474s + 1.999}{(s + 2)(s + 1.732)(s + 1)} & \frac{-0.2538s^3 + 0.1274s^2 - 0.7092s - 1.635}{(s + 2)(s + 1.732)(s + 1)}
\end{bmatrix}.$$ 

Example 2. This is Example 2 from [5] of the transfer function matrix of a discrete-time polynomial system:

$$(19) \quad G(z) = \begin{bmatrix}
z^2 + z + 1 & 4z^2 + 3z + 2 & 2z^2 - 2 \\
z & 4z - 1 & 2z - 2 \\
z^2 & 4z^2 - z & 2z^2 - 2z
\end{bmatrix},$$

which has two infinite poles (i.e., McMillan-degree of $G(z)$ is equal to 2), a zero at 1, and has a minimal descriptor state-space realization of order 4. 

A minimum proper basis of the range of $G(z)$, computed with \texttt{grange}, is

$$R(z) = \frac{1}{z + 0.3304} \begin{bmatrix}
-1.564z - 0.8277 & 0.06277z + 0.4338 \\
-0.9414 & 1.25 \\
-0.9414z & 1.25z
\end{bmatrix},$$

has McMillan-degree 1 and no zeros. The full row rank factor $X(z)$, satisfying $G(z) = R(z)X(z)$, has McMillan degree 2, and zeros at \{-0.3304, 1\}. Notice that the zero at -0.3304 is equal to the pole of $R(z)$.

An inner range $R(z)$ results as

$$R(z) = \begin{bmatrix}
-0.7614 & -0.6483 \\
-0.4584 & 0.5384 \\
-0.4584 & 0.5384
\end{bmatrix},$$

has McMillan-degree 1 and no zeros. The quasi-outer factor $X(z)$ results as

$$X(z) = \begin{bmatrix}
-1.676z^2 - 0.7614z - 0.7614 & -6.713z^2 - 1.367z - 1.523 & -3.356z^2 + 1.834z + 1.523 \\
0.4285z^2 - 0.6483z - 0.6483 & 1.714z^2 - 3.022z - 1.297 & 0.8571z^2 - 2.154z + 1.297
\end{bmatrix},$$

has McMillan degree 2 and zeros at \{0, 1\}.

5. Conclusions. In this note we described a numerically reliable general approach to compute proper bases for the range space of a rational matrix and, simultaneously, to produce a complete full rank factorization of this matrix. The underlying computational algorithms use descriptor system state-space realizations, for which, the only restriction is a certain stabilizability condition (always fulfilled when using irreducible realizations). The techniques described in this note served for the implementation of robust computational software, which is part of DSTOOLS, a free collection of descriptor systems tools for MATLAB [8].
REFERENCES

[1] T. Beelen and P. Van Dooren, An improved algorithm for the computation of Kronecker’s canonical form of a singular pencil, Linear Algebra Appl., 105 (1988), pp. 9–65.
[2] P. Benner, V. Mehrmann, V. Sima, S. Van Huffel, and A. Varga, SLICOT – a subroutine library in systems and control theory, in Applied and Computational Control, Signals and Circuits, B. N. Datta, ed., vol. 1. Birkhäuser, 1999, pp. 499–539.
[3] G. H. Golub and C. F. Van Loan, Matrix Computations, 4th Edition, John Hopkins University Press, Baltimore, 2013.
[4] T. Kailath, Linear Systems, Prentice Hall, Englewood Cliffs, 1980.
[5] C. Oară, Constructive solutions to spectral and inner-outer factorizations with respect to the disk, Automatica, 41 (2005), pp. 1855–1866.
[6] C. Oară and P. V. Dooren, An improved algorithm for the computation of structural invariants of a system pencil and related geometric aspects, Syst. Control Lett., 30 (1997), pp. 39–48.
[7] C. Oară and A. Varga, Computation of general inner-outer and spectral factorizations, IEEE Trans. Automat. Control, 45 (2000), pp. 2307–2325.
[8] A. Varga, DSTOOLS – The Descriptor System Tools for MATLAB. https://sites.google.com/site/andreasvargacontact/home/software/dstools.
[9] A. Varga, Solving Fault Diagnosis Problems – Linear Synthesis Techniques, vol. 84 of Studies in Systems, Decision and Control, Springer International Publishing, 2017.
[10] G. Verghese, B. Lévy, and T. Kailath, A generalized state-space for singular systems, IEEE Trans. Automat. Control, 26 (1981), pp. 811–831.
[11] G. Verghese, P. Van Dooren, and T. Kailath, Properties of the system matrix of a generalized state-space system, Int. J. Control, 30 (1979), pp. 235–243.