Numerical Computation of \( \prod_{n=1}^{\infty} (1 - tx^n) \)

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Abstract

I present and analyze a quadratically convergent algorithm for computing the infinite product \( \prod_{n=1}^{\infty} (1 - tx^n) \) for arbitrary complex \( t \) and \( x \) satisfying \(|x| < 1\), based on the identity

\[
\prod_{n=1}^{\infty} (1 - tx^n) = \sum_{m=0}^{\infty} \frac{(-t)^m x^{m(m+1)/2}}{(1-x)(1-x^2) \cdots (1-x^m)}
\]

due to Euler. The efficiency of the algorithm deteriorates as \(|x| \uparrow 1\), but much more slowly than in previous algorithms. The key lemma is a two-sided bound on the Dedekind eta function at pure imaginary argument, \( \eta(iy) \), that is sharp at the two endpoints \( y = 0, \infty \) and is accurate to within 9.1% over the entire interval \( 0 < y < \infty \).

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1 Introduction

The function

\[ R(t, x) = \prod_{n=1}^{\infty} (1 - tx^n), \]  

(1.1)
defined for complex \( t \) and \( x \) satisfying \( |x| < 1 \), was first studied by Euler [13] and has numerous applications in combinatorics, number theory, analytic-function theory and statistical mechanics. The case \( t = 1 \) is equivalent to the Dedekind eta function

\[ \eta(\tau) = \frac{e^{\pi i/12} R(1, e^{2\pi i \tau})}{R(1, x)}, \]  

(1.2)
which is a modular form [3, 21] and plays a central role in the enumeration of partitions [3, 21] and sums of squares [21]. The case \( t = -1 \) is related to \( t = 1 \) via the trivial identity

\[ R(-1, x) = \frac{R(1, x^2)}{R(1, x)}. \]  

(1.3)
Both of these cases are related to theta functions [4, 10, 11, 31] via the identities

\[ R(1, x) \equiv \prod_{n=1}^{\infty} (1 - x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{(3m+1)/2} \]  

(1.4)
\[ R(1, x^3) \equiv \prod_{n=1}^{\infty} (1 - x^{3n}) = \sum_{m=0}^{\infty} (-1)^m (2m+1) x^{(m+1)/2} \]  

(1.5)
\[ \frac{R(1, x^2)}{R(1, x)} \equiv \prod_{n=1}^{\infty} \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_{m=0}^{\infty} x^{m(m+1)/2} \]  

(1.6)
\[ \frac{R(1, x^2)^5}{R(1, x)^2 R(1, x^4)^2} \equiv \prod_{n=1}^{\infty} (1 - x^{2n}) (1 + x^{2n-1})^2 = \sum_{m=-\infty}^{\infty} x^{m^2} \]  

(1.7)
\[ \frac{R(1, x^2)^2}{R(1, x^2)} \equiv \prod_{n=1}^{\infty} \frac{1 - x^n}{1 + x^n} = \sum_{m=-\infty}^{\infty} (-1)^m x^{m^2} \]  

(1.8)
due to Euler, Jacobi and Gauss, which have spawned a plethora of modern extensions [16, 26, 23, 19, 20, 29, 22]. Additional cases of the function \( R(t, x) \) arise in the celebrated Rogers–Ramanujan identities [4, 3]

\[ \prod_{n=0}^{\infty} \frac{1}{(1 - x^{5n+1})(1 - x^{5n+4})} = \sum_{m=0}^{\infty} \frac{x^{m^2}}{(1 - x)(1 - x^2) \cdots (1 - x^m)} \]  

(1.9)
\[ \prod_{n=0}^{\infty} \frac{1}{(1 - x^{5n+2})(1 - x^{5n+3})} = \sum_{m=0}^{\infty} \frac{x^{m(m+1)}}{(1 - x)(1 - x^2) \cdots (1 - x^m)} \]  

(1.10)
which have numerous combinatorial consequences \[3\] and which play a key role in Baxter’s solution of the hard-hexagon problem in statistical mechanics \[6, 7\]. (See also \[33, 25, 14, 2, 3\] for many related identities.) Finally — and this was the initial motivation for the current work — the cases \(t = \pm 1\) and \(t = \pm \omega\), where \(\omega\) is a cube root of unity, arise in Baxter’s solution for the chromatic polynomials of large triangular lattices \[8, 9\]. To determine the limiting curves of chromatic roots for these lattices, it is necessary to compute \(R(t, x)\) to high precision for complex \(x\), including points \(x\) very near the unit circle \[18\].

Surprisingly, there seem to be very few treatments of this problem in the literature \[34, 35, 15, 1\], and the algorithms employed there are only linearly convergent; moreover, these authors (with the exception of Gatteschi \[15\]) considered almost exclusively the case of real \(t\) and \(x\). My purpose here is to propose and analyze a quadratically convergent algorithm for computing \(R(t, x)\) for arbitrary complex \(t\) and \(x\) satisfying \(|x| < 1\), based on the identity

\[
R(t, x) \equiv \prod_{n=1}^{\infty} (1 - tx^n) = \sum_{m=0}^{\infty} \frac{(-t)^m x^{m(m+1)/2}}{(1 - x)(1 - x^2) \cdots (1 - x^m)} \tag{1.11}
\]

due to Euler. In the course of this analysis, I will obtain (Corollary 2.4) a two-sided bound on \(R(1, x)\) for \(0 < x < 1\) (and thus on \(\eta(iy)\) for \(0 < y < \infty\)) that is sharp at the two endpoints \(x = 0, 1\) and is accurate to within 9.1% over the entire interval; this bound is perhaps of some modest independent interest.

Of course, for the special case \(t = 1\) one may employ an even faster algorithm based on using the modular transformation law for the Dedekind eta function \[3, 4, 11, 21, 31\] to move \(x\) away from the unit circle, followed by evaluation of the quadratically convergent sum (1.4). Moreover, the case \(t = -1\) can be reduced to \(t = 1\) via (1.3). But for \(t \neq \pm 1\) no such identities are known.

The plan of this paper is as follows: In Section 2 I formulate and prove the properties of the function \(R(t, x)\) that will be needed in the sequel. In Section 3 I obtain bounds (both \(a \text{ priori}\) and \(a \text{ posteriori}\)) on the rate of convergence of the algorithm defined by (1.11). Finally, in Section 4 I briefly compare this algorithm to other algorithms that have been proposed \[32, 33, 13, 1\] for computing \(R(t, x)\).

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1 For a proof of (1.11), see e.g. \[3, p. 19, Corollary 2.2\], \[21, p. 34, Lemma 4(a)\] or \[32, pp. 22–23\].

2 Surprisingly, I have been unable to find in the literature any discussion of such an algorithm. The details of its implementation — in particular, how to find an appropriate modular transformation — may not be entirely trivial.
2 Properties of $R(t, x)$

We shall assume throughout this note that $|x| < 1$, even if it is not explicitly stated. Likewise, when we write $x = e^{-\gamma}$, we shall assume that $\text{Re}\gamma > 0$.

2.1 Elementary properties

We begin by noting some elementary properties of $R(t, x)$:

1) $R(t, x)$ is a jointly analytic function of $t$ and $x$ for $(t, x) \in \mathbb{C} \times \mathbb{D}$, where $\mathbb{D}$ is the open unit disc. For fixed $x \in \mathbb{D}$, $R(t, x)$ is an entire function of $t$ of order 0, with simple zeros (for $x \neq 0$) at $t = x^{-1}, x^{-2}, \ldots$.

2) By splitting off the first term in the product (1.1), we obtain the functional equation

$$R(t, x) = (1 - tx) R(tx, x), \quad (2.1)$$

from which Euler’s formula (1.11) can easily be derived by comparing coefficients of powers of $t$.

3) Let $\omega$ be a primitive $m$th root of unity, and use the identity $\prod_{j=0}^{m-1} (1 - \omega^j z) = 1 - z^m$; we find

$$\prod_{j=0}^{m-1} R(\omega^j t, x) = R(t^m, x^m). \quad (2.2)$$

The special case $m = 2, t = 1$ is (1.3).

4) By splitting the product (1.1) according to residue classes modulo $m$, we obtain

$$R(t, x) = \prod_{j=1}^{m} R(t x^j, x^m). \quad (2.3)$$

This formula permits the determination of the asymptotic behavior of $R(t, x)$ as $x$ approaches an $m$th root of unity, once the asymptotic behavior as $x \to 1$ is known.

5) We have the trivial upper bound

$$|R(t, x)| \leq R(-T, |x|) \quad (2.4)$$

whenever $|t| \leq T$.

6) We have the trivial lower bound

$$|R(t, x)| \geq R(T, |x|) \quad (2.5)$$

whenever $|t| \leq T \leq |x|^{-1}$. (The condition $T \leq |x|^{-1}$ is of course best possible, since $R(t, x)$ vanishes at $t = x^{-1}$.)

7) Finally (and most importantly), let us take the logarithm (principal branch) of the defining equation (1.1), expand $\log(1 - tx^n)$ in Taylor series, and interchange the absolutely convergent summations; this yields the useful representation as a Lambert series

$$\log R(t, x) = -\sum_{k=1}^{\infty} \frac{t^k}{k} \frac{x^k}{1-x^k}, \quad (2.6)$$

valid whenever $|x| < 1$ and $|tx| < 1$. We shall use this representation repeatedly.
2.2 Elementary bounds

Bounding the denominator of (2.6) using $|1-x^k| \geq 1-|x|^k \geq 1-|x|$, we obtain:

Lemma 2.1 Whenever $|x| < 1$ and $|tx| < 1$, we have

$$|\log R(t, x)| \leq \frac{-\log(1-|tx|)}{1-|x|}$$

(where the principal branch of the logarithm is taken) and hence

$$(1-|tx|)^{1/(1-|x|)} \leq |R(t, x)| \leq (1-|tx|)^{-1/(1-|x|)}.$$  

(2.8)

This is a crude bound that does not exhibit the correct behavior as $|x| \to 1$, but we shall use it as a starting point for further refinements.

First we need a slight extension of Lemma 2.1 for the special case $t = 1$. For $0 \leq x < 1$, define

$$S(x) = -\log R(1, x) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{x^{k}}{1-x^{k}},$$

so that

$$S'(x) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(1-x^{k})^{2}}$$

(2.10)

$$S''(x) = \sum_{k=1}^{\infty} \frac{(k-1)x^{k-2} + (k+1)x^{2k-2}}{(1-x^{k})^{3}}$$

(2.11)

By using $1 \geq 1-x^k \geq 1-x$ in the denominator, we obtain the trivial bounds:

Lemma 2.2 For $0 \leq x < 1$, we have

$$-\log(1-x) \leq S(x) \leq \frac{-\log(1-x)}{1-x} \leq \frac{x}{(1-x)^{2}}$$

(2.12)

$$1 \leq \frac{1}{1-x} \leq S'(x) \leq \frac{1}{(1-x)^{3}}$$

(2.13)

$$3 \leq \frac{1}{(1-x)^{2}} + \frac{2-x^2}{(1-x^2)^{2}} \leq S''(x) \leq \frac{1}{(1-x)^{5}} + \frac{2-x^2}{(1-x^3(1-x^2)^{2}} \leq \frac{3}{(1-x)^{5}}$$

(2.14)

2.3 Case $t = 1$

Now we improve these bounds by using a deep fact: the transformation properties of the Dedekind eta function under the modular group $[3, 5, 11, 21, 31]$. All we
need, in fact, is a special case of the modular transformation law, namely the one for inversion \( \tau \to -1/\tau \):

\[
R(1, e^{-2\pi z}) = z^{-1/2} \exp \left( \frac{\pi z}{12} - \frac{\pi}{12z} \right) R(1, e^{-2\pi/z}) \tag{2.15}
\]

for \( \Re z > 0 \). This allows us to control the behavior near \( x = 1 \) (\( z \to 0 \)) in terms of the (trivial) behavior near \( x = 0 \) (\( z \to +\infty \)). Indeed, from (2.15) and the regularity of \( R(1, x) \) near \( x = 0 \), one immediately deduces the sharp asymptotic formula

\[
\log R(1, e^{-\gamma}) = -\frac{\pi^2}{6\gamma} - \frac{1}{2} \log \gamma + \frac{1}{2} \log(2\pi) + \frac{\gamma}{24} + O(e^{-4\pi^2/\gamma}) \tag{2.16}
\]

as \( \gamma \to 0 \); moreover, an explicit quantitative bound on the \( O(e^{-4\pi^2/\gamma}) \) term can easily be extracted from Lemma 2.1.

For later applications we need also a quantitative error bound valid for real \( \gamma \) in the entire interval \( 0 < \gamma < \infty \). Let us define

\[
f(z) = \log R(1, e^{-2\pi z}) + \frac{\pi z}{12z} - \frac{1}{4} \log \left( 1 + \frac{1}{z^2} \right), \tag{2.17}
\]

so that

\[
f(1/z) = \log R(1, e^{-2\pi/z}) + \frac{\pi z}{12z} - \frac{1}{4} \log(1 + z^2). \tag{2.18}
\]

Then the transformation law (2.15) tells us immediately that \( f(z) = f(1/z) \), and indeed we have:

**Proposition 2.3** For \( 0 < z < \infty \), we have:

(a) \( f(z) = f(1/z) \)

(b) \( \lim_{z \to 0} f(z) = 0 \) and \( \lim_{z \to +\infty} f(z) = 0 \)

(c) \( 0 < f(z) \leq f(1) = \frac{\pi}{6} - \frac{1}{4} \log 2 + \log \eta(i) \approx 0.0866399 \)

(d) \( f'(0) = \pi/12, f'(z) > 0 \) for \( 0 < z < 1 \), \( f'(1) = 0 \), and \( f'(z) < 0 \) for \( z > 1 \)

(e) \( f''(0) = -1/2, \) and there exists \( z_* > 1 \) such that \( f''(z) < 0 \) for \( 0 < z < z_* \) and \( f''(z_*) = 0. \)

\[3\] There are a number of proofs of (2.15). The simplest uses the Poisson summation formula applied to Euler's pentagonal number theorem (1.4) [21, Section 3.3]. Another proof, due to Siegel, uses the Cauchy integral formula [5, Section 3.2] [11, Section VIII.3]. Proofs of the full modular transformation law are given in [5, Sections 3.3–3.6 and pp. 190–195], [21, Sections 3.1–3.3 and 4.1–4.2], [31, Chapter 9], and [3, pp. 82–85].

\[4\] Using the full modular transformation law, one can control in an analogous way the behavior of \( R(1, x) \) near any point \( x = e^{2\pi ih/k} \) (\( h, k \in \mathbb{Z} \)) of the unit circle: see e.g. [5, Chapter 5].
Proof. We have already proven that \( f(z) = f(1/z) \), so we can use (2.17) and (2.18) interchangeably as formulae for \( f(z) \). The limiting values of \( f \) and its derivatives at \( z = 0 \) can be read off (2.18).

To prove \( f(z) > 0 \) for \( 0 < z < \infty \), it suffices to prove it for \( 0 < z \leq 1 \). Using (2.18), we make the following crude bounds:

\[
\begin{align*}
  f(z) &= -S(e^{-2\pi/z}) + \frac{\pi z}{12} - \frac{1}{4} \log(1 + z^2) \quad (2.19a) \\
  &\geq -\frac{e^{-2\pi/z}}{(1-e^{-2\pi/z})^2} + \frac{\pi z}{12} - \frac{1}{4} \log(1 + z^2) \quad (2.19b) \\
  &\geq -\frac{e^{-2\pi/z}}{(1-e^{-2\pi/z})^2} + \left( \frac{\pi}{12} - \frac{1}{4} \right) z \quad (2.19c)
\end{align*}
\]

where we have used (2.12) and the fact that \( 0 < z \leq 1 \). So we need only show that

\[
\frac{x}{(1-x)^2} < \left( \frac{\pi}{12} - \frac{1}{4} \right) \left( -\frac{2\pi}{\log x} \right) \quad (2.20)
\]

for \( 0 < x \leq e^{-2\pi} \). But \(-x \log x/(1-x)^2\) is an increasing function of \( x \) for \( 0 < x \leq 1 \), and its value at \( x = e^{-2\pi} \) is \( 2\pi e^{-2\pi}/(1-e^{-2\pi})^2 \approx 0.011777 < (\pi/12 - 1/4)(2\pi) \approx 0.074138 \).

Next let us prove that there exists \( \epsilon > 0 \) such that \( f''(z) < 0 \) for \( 0 < z < 1 + \epsilon \). Differentiating (2.18) twice with respect to \( z \), we obtain

\[
f''(z) = \left( -\frac{4\pi^2}{z^4} + \frac{4\pi}{z^3} \right) e^{-2\pi/z} S'(e^{-2\pi/z}) - \frac{4\pi^2}{z^4} e^{-4\pi/z} S''(e^{-2\pi/z}) - \frac{1-z^2}{2(1+z^2)}.
\]

From (2.13)/(2.14) we have \( S'(x) \geq 1 \) and \( S''(x) \geq 3 \), so the first two terms in (2.21) are \( < 0 \) for \( 0 < z < \pi \), and the third term is \( \leq 0 \) for \( 0 < z \leq 1 \). This proves the claim.

We have just proven that \( f'(z) \) is a strictly decreasing function of \( z \) on \( 0 < z < 1+\epsilon \). From \( f(z) = f(1/z) \) it follows that \( f'(1) = 0 \). Therefore \( f'(z) > 0 \) for \( 0 < z < 1 \); by \( f(z) = f(1/z) \) it follows that \( f'(z) < 0 \) for \( z > 1 \); and thus \( f(z) \leq f(1) \) for all \( z \).

Finally, it is not possible that \( f''(z) < 0 \) for all \( z \), as this would imply that \( f(z) < 0 \) for some \( z \in (1, \infty) \). So we can define \( z_* > 1 \) to be the smallest \( z \) such that \( f''(z) = 0 \).

\[ \square \]

Remark. Numerical calculations show that \( f'' \) has a unique zero, which is located at \( z_* \approx 1.974174 \). But we shall not bother to prove this. Graphs of \( f(z) \) versus \( z \) and \( \log z \) are shown in Figure [1]; the latter shows the \( z \leftrightarrow 1/z \) symmetry more clearly.

Proposition 2.3 can be rephrased by defining

\[
R_0(1, x) = e^{\pi^2/(6\log x)} \left( 1 + \frac{4\pi^2}{(\log x)^2} \right)^{1/4}, \quad (2.22)
\]

which we interpret as an “approximate” version of \( R(1, x) \). We then have:
Corollary 2.4 For $0 < x < 1$,
\[ e^{\pi^2/(6\log x)} < R_0(1, x) < R(1, x) \leq CR_0(1, x) \]  
(2.23)
where $C = e^{f(1)} = e^{\pi/6 \cdot 2^{-1/4} \eta(i)} \approx 1.09054$.

In other words, we have a two-sided bound on $R(1, x)$, in which the lower bound is sharp at the two endpoints $x = 0, 1$ and is accurate to within 9.1% over the entire interval $0 < x < 1$. We shall frequently use the lower bound of Corollary 2.4 in the form

\[ R(1, e^{-\gamma}) \geq e^{-\pi^2/6\gamma} \left( 1 + \frac{4\pi^2}{\gamma^2} \right)^{1/4} \geq e^{-\pi^2/6\gamma} \]  
(2.24)
for $\gamma > 0$.

2.4 Case $t = -1$

We can now handle the case $t = -1$ by using (1.3) to relate it to $t = 1$. From (2.16) and (1.3) we obtain the sharp asymptotic formula

\[ \log R(-1, e^{-\gamma}) = \frac{\pi^2}{12\gamma} - \frac{1}{2} \log 2 + \frac{\gamma}{24} + O(e^{-\pi^2/\gamma}) \]  
(2.25)

as $\gamma \to 0$, where again a quantitative bound on the $O(e^{-\pi^2/\gamma})$ term can easily be extracted from Lemma 2.1. Moreover, we can obtain a quantitative error bound valid for real $z$ in the entire interval $0 < z < \infty$. Let us define

\begin{align*}
g(z) &= f(\sqrt{2}z) - f(z/\sqrt{2}) \\
&= \log R(-1, e^{-\sqrt{2}\pi z}) - \frac{\pi}{12\sqrt{2}z} + \frac{1}{4} \log \left( \frac{z^2 + 2}{z + \frac{1}{2}} \right). \tag{2.26a,2.26b}
\end{align*}

It follows immediately from Proposition 2.3 that:
Proposition 2.5 For $0 < z < \infty$, we have:

(a) $g(z) = -g(1/z)$

(b) $\lim_{z \to 0} g(z) = 0$ and $\lim_{z \to +\infty} g(z) = 0$

(c) $g'(z) < 0$ for $1/\sqrt{2} \leq z \leq \sqrt{2}$

(d) $g(z) > 0$ for $1/\sqrt{2} \leq z < 1$, $g(1) = 0$, and $g(z) > 0$ for $1 < z \leq \sqrt{2}$

(e) $|g(z)| \leq f(1) \approx 0.0866399$ for $0 < z < \infty$

Remark. Numerical calculations show that $g'$ vanishes when (and only when) $\pm \log z \approx 1.180158$, i.e. $z$ or $1/z \approx 3.254889$, and that the maximum value of $|g(z)|$ is $\approx 0.0251707$. It follows that $R(1, x)$ differs from

\[ R_0(-1, x) \equiv \frac{R_0(1, x^2)}{R_0(1, x)} = e^{\pi^2/(12 \log x)} \left( \frac{1 + \frac{\pi^2}{(\log x)^2}}{1 + \frac{4 \pi^2}{(\log x)^2}} \right)^{1/4} \]  

(2.27)

by less than 2.6% over the entire interval $0 < x < 1$. Graphs of $g(z)$ versus $z$ and $\log z$ are shown in Figure 2.

2.5 Asymptotics of $R(t, x)$ for General $t$

Finally, let us discuss briefly the asymptotics of $R(t, x)$ as $x \to 1$ when $t$ is fixed with $|t| < 1$ (or more generally varies within a compact subset of the open unit disc). Let us write $x = e^{-\gamma}$ with $\Re \gamma > 0$ and study the behavior as $\gamma \to 0$, using the representation (2.6). We have

\[ \frac{x^k}{1-x^k} = \frac{1}{e^{k\gamma} - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!}(k\gamma)^{m-1} \]  

(2.28)
where $B_m$ is the $m$th Bernoulli number; this series is absolutely convergent for $|\gamma| < 2\pi/k$.

Inserting this into (2.23) and formally interchanging the order of summation, we obtain

$$- \log R(t, e^{-\gamma}) \sim \sum_{m=0}^{\infty} \frac{B_m}{m!} \text{Li}_{2-m}(t) \gamma^{m-1}$$

(2.29)

where

$$\text{Li}_p(t) \equiv \sum_{k=1}^{\infty} \frac{t^k}{k^p}$$

(2.30)

is the polylogarithm function [24]. However, because the radius of convergence of (2.28) is nonuniform in $k$ and tends to zero as $k \to \infty$, it is reasonable to expect that the series (2.29) is not convergent but is only asymptotic. One further expects that this asymptotic expansion should hold uniformly as $t$ varies within a compact subset of the open unit disc. All these expectations are true [36]. What is perhaps more surprising is that the expansion (2.29) holds also for $t$ on the unit circle, except at the point $t = 1$. Indeed, under suitable restrictions on $\arg \gamma$ it holds in a much larger domain of the complex $t$-plane, which in the most favorable case ($\gamma$ real and positive) encompasses the entire complex $t$-plane except for a cut along $[1, \infty)$. These results will be reported elsewhere [36]. For real $\gamma > 0$ and $0 < t < 1$, the expansion (2.29) was proven some years ago by Moak [28, Theorem 3]. For real $\gamma > 0$ and $t \leq 1$, the expansion (2.29) and some generalizations thereof have recently been proven by McIntosh [27]. For real $\gamma > 0$ and $t \in \mathbb{C} \setminus [1, \infty)$, the expansion (2.29) has been proven by Prellberg [30, Lemma 3.2]. All these works use the Euler–Maclaurin sum formula. Our approach [36], by contrast, uses complex integration.

For real $\gamma > 0$ and $0 \leq t \leq 1$, we can use the method just sketched to obtain a two-sided bound on $R(t, e^{-\gamma})$ that incorporates the first two terms of the expansion (2.29). For $z > 0$ we have the elementary inequalities

$$\frac{1}{z} - \frac{1}{2} \leq e^{-z} \left( \frac{1}{z} + \frac{1}{2} \right) \leq \frac{1}{e^z - 1} \leq \frac{e^{-z/2}}{z} \leq \frac{1}{z}.$$ 

(2.31)

Setting $z = k\gamma$ and inserting these bounds into (2.24), we obtain:

**Proposition 2.6** For $0 \leq t \leq 1$ and $\gamma > 0$, we have

$$- \log R(t, e^{-\gamma}) \leq \gamma^{-1} \text{Li}_2(te^{-\gamma/2})$$

(2.32a)

$$\leq \gamma^{-1} \text{Li}_2(t)$$

(2.32b)

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5 See e.g. [17, equation (6.81)].

6 See also [12, p. 58, exercise 2] and [27, Theorem 4] for this formula.

7 Equation (4.2) of [28] contains a misprint: there should be a minus sign before the integral. Correspondingly, in equation (4.3), the minus sign before the integral should be a plus sign.

8 The first two inequalities can be derived from $\tanh(z/2) \leq z/2$; the third can be derived from $z/2 \leq \sinh(z/2)$; and the fourth is trivial. Note that all of these bounds, except the last, capture the first two terms of the Laurent series for $1/(e^z - 1)$ around $z = 0$. 

10
\[ -\log R(t, e^{-\gamma}) \geq \gamma^{-1} \text{Li}_2(te^{-\gamma}) - \frac{1}{2} \log(1 - te^{-\gamma}) \geq \gamma^{-1} \text{Li}_2(t) + \frac{1}{2} \log(1 - t) \] (2.33a)

(2.33b)

It is worth remarking that, even for \( t = 1 \), the bounds (2.32a) and (2.33a) capture the first two terms of the asymptotic expansion (2.16), i.e. they get the correct \( \log \gamma \) term.

One application of Proposition 2.6 is to bounding the partial product

\[ \prod_{k=1}^{n} (1 - x^k) = \frac{R(1, x)}{R(x^n, x)} \] (2.34)

when \( 0 < x = e^{-\gamma} < 1 \) (and we will usually take \( n \) to be of order \( 1/\gamma \)). Inserting the lower bound (2.24) on \( R(1, x) \) and the upper bound (2.33b) on \( R(x^n, x) \), we obtain:

**Corollary 2.7** Let \( \gamma > 0 \). Then

\[ \prod_{k=1}^{n} (1 - e^{-k\gamma}) \geq \exp \left[ \frac{\text{Li}_2(e^{-n\gamma}) - \pi^2/6}{\gamma} \right] (1 - e^{-n\gamma})^{1/2}. \] (2.35)

In particular, for \( n \leq (\log 2)/\gamma \) we have

\[ \prod_{k=1}^{n} (1 - e^{-k\gamma}) \geq \exp \left[ \frac{-(\log 2)^2/2 - \pi^2/12}{\gamma} \right] (1 - e^{-n\gamma})^{1/2}. \] (2.36)

Here (2.36) follows from (2.35) and the well-known fact [24, 25]

\[ \text{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}. \] (2.37)

McIntosh [27] has recently obtained a complete asymptotic expansion of the partial product \( \prod_{k=1}^{n} (1 - te^{-k\gamma}) \) for \( n = \mu/\gamma \) (\( \mu \) fixed, real \( \gamma \downarrow 0 \)) and either \( t = 1 \) or \( t < 1 \).

### 3 Numerical Computation of \( R(t, x) \)

In this section we discuss the use of Euler’s formula

\[ R(t, x) = \sum_{n=0}^{\infty} \frac{(-t)^n x^{n(n+1)/2}}{(1-x)(1-x^2) \cdots (1-x^n)} \] (3.1)

to compute \( R(t, x) \) for complex \( t \) and \( x \) satisfying \( |x| < 1 \). We shall give two types of bounds on the error committed by truncating the series (3.1):

(a) an \textit{a priori} bound in terms of \( |t| \) and \( |x| \) alone; and
We shall also give some guidance about the needed numerical precision in intermediate stages of the calculation, by comparing the largest term in the sum to the final answer.

We use the following definitions:

- The $n$th term: $a_n = \frac{(-t)^n x^{n(n+1)/2}}{(1 - x)(1 - x^2) \cdots (1 - x^n)}$
- The partial sum after $N - 1$ terms: $S_N = \sum_{n=0}^{N-1} a_n$
- The remainder after $N - 1$ terms: $R_N = \sum_{n=N}^{\infty} a_n$
- The absolute error after $N - 1$ terms: $\Delta_N = |R_N|$
- The relative error after $N - 1$ terms: $\delta_N = \frac{|R_N|}{R(t, x)}$
- The modified relative error after $N - 1$ terms: $\delta'_N = \frac{|R_N|}{S_N}$

Clearly $\delta'_N/(1 + \delta'_N) \leq \delta_N \leq \delta'_N/(1 - \delta'_N)$, so the two types of relative error are essentially indistinguishable when $\delta_N, \delta'_N \ll 1$.

**Lemma 3.1** If $|x| < 1$ and $|t| |x|^{N+1} < 1$, then

$$\sum_{n=N}^{\infty} |t^n x^{n(n+1)/2}| \leq \frac{|t|^N |x|^{N(N+1)/2}}{1 - |t| |x|^{N+1}}.$$  \hspace{1cm} (3.2)

**Proof.** Bound the sum by a geometric series, using

$$\left| \frac{t^{n+1} x^{(n+1)(n+2)/2}}{t^n x^{n(n+1)/2}} \right| = |t| |x^{n+1}| \leq |t| |x|^{N+1}$$  \hspace{1cm} (3.3)

for $n \geq N$. \hfill \blacksquare

**Lemma 3.2** If $|x| \leq e^{-\gamma}$ ($\gamma > 0$), then

$$\left| \prod_{k=1}^{n} (1 - x^k) \right| \geq \prod_{k=1}^{n} (1 - |x|^k) \geq \prod_{k=1}^{\infty} (1 - |x|^k) \equiv R(1, |x|) \geq e^{-\pi^2/6\gamma}.$$  \hspace{1cm} (3.4)

**Proof.** An immediate consequence of Corollary 2.4. \hfill \blacksquare

**Remark.** An improved bound on the partial product $\prod_{k=1}^{n} (1 - x^k)$ can be obtained from Corollary 2.4; it is advantageous when $n \gamma \gg 1$.  

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Proposition 3.3 Suppose that $|x| \leq e^{-\gamma}$ with $\gamma > 0$.

(a) If $|t| < e^{(N+1)\gamma}$, then

$$\Delta_N \equiv \left| \sum_{n=N}^{\infty} a_n \right| \leq \frac{|t|^N e^{\pi^2/6(N+1)\gamma/2}}{1 - |t| e^{-(N+1)\gamma}}.$$ 

(b) If $|t| \leq 1$, then

$$\delta_N \equiv \left| \sum_{n=N}^{\infty} a_n \right| \leq \frac{e^{\pi^2/3\gamma - N(N+1)\gamma/2}}{1 - e^{-(N+1)\gamma}}.$$ 

(c) If $|t| < e^{\gamma}$, then

$$\delta_N \equiv \left| \sum_{n=N}^{\infty} a_n \right| \leq \frac{e^{\pi^2/3\gamma - N(N+1)\gamma/2}}{1 - e^{-(N+1)\gamma}} \frac{1}{1 - |t| e^{-\gamma}}.$$ 

Proof. (a) is an immediate consequence of Lemmas 3.1 and 3.2. (b) follows from (a) together with the bound $|R(t, x)| \geq R(1, |x|) \geq e^{-\pi^2/6\gamma}$ from (2.5) and Corollary 2.4. (c) follows from (b) and (2.1). 

Corollary 3.4 Let $K \geq 0$, and suppose that $|t| \leq 1$ and $|x| \leq e^{-\gamma}$ ($\gamma > 0$).

(a) If $N \geq \sqrt{\frac{\pi^2}{3\gamma^2} + \frac{2K}{\gamma}}$, then $\Delta_N \leq e^{-K}$.

(b) If $N \geq \sqrt{\frac{2\pi^2}{3\gamma^2} + \frac{2K}{\gamma}}$, then $\delta_N \leq e^{-K}$.

Proof. Since $K \geq 0$, we have $N\gamma \geq \pi/\sqrt{3}$ and hence

$$\frac{e^{-N\gamma/2}}{1 - e^{-(N+1)\gamma}} \leq \frac{e^{-\pi/2\sqrt{3}}}{1 - e^{-\pi/\sqrt{3}}} \approx 0.482426 < 1. \tag{3.5}$$

Now $\pi^2/6\gamma - N^2\gamma/2 \leq -K$ in case (a), and $\pi^2/3\gamma - N^2\gamma/2 \leq -K$ in case (b). The result then follows from Proposition 3.3(a,b). 

Please note that the bound in Proposition 3.3(a) is asymptotically within 9.1% of being sharp when $0 < x = e^{-\gamma} < 1$ and $N \gg 1/\gamma$ (and in this case is moreover asymptotically sharp as $\gamma \downarrow 0$); but it is overly pessimistic in other cases, because the denominator $(1 - x)(1 - x^2) \cdots (1 - x^n)$ is not really as small as Lemma 3.2 says it could be. Likewise, the bound in Proposition 3.3(b) is asymptotically (almost-)sharp when, in addition to the above conditions, we have $t = 1$; but it is overly pessimistic in other cases, because $|R(t, x)|$ is not really as small as the bound $|R(t, x)| \geq R(1, |x|)$ says it could be.

It is thus of some value to provide an a posteriori bound on the truncation error that is more realistic, when $x \not\in (0, 1)$, than the a priori bound; such a bound can be used a stopping criterion in the numerical algorithm. We need the following elementary observation:
Lemma 3.5 If $|x| \leq e^{-\gamma}$ with $\gamma > 0$, then

$$\left| \frac{a_n}{a_{n-1}} \right| = \left| \frac{t|x^n|}{1 - x^n} \right| \leq \frac{|t||x^n|}{1 - |x^n|} \leq \frac{|t|e^{-n\gamma}}{1 - e^{-n\gamma}}. \quad (3.6)$$

Lemma 3.3 tells us that, at least for $0 < x < 1$, the terms $a_n$ increase in magnitude until $|x|^n \approx 1/(1 + |t|)$, i.e. $n \approx \lceil \log(1 + |t|) \rceil / \gamma$, and then decrease. (For general complex $x$, the terms will sometimes increase up to this point, i.e. for those $n$ for which $\arg x^n \approx 0 \mod 2\pi$. How often this occurs depends on the Diophantine properties of $\arg x$.) We can use Lemma 3.5 to bound the tail of the sum by a geometric series:

Proposition 3.6 Suppose that $|x| \leq e^{-\gamma}$ ($\gamma > 0$) and $N > \lfloor \log(1 + |t|) \rfloor / \gamma$. Then:

(a) $\Delta_N \equiv \left| \sum_{n=N}^{\infty} a_n \right| \leq |a_{N-1}| \frac{|t|e^{-N\gamma}}{1 - (1 + |t|)e^{-N\gamma}}$

(b) $\delta'_N \equiv \left| \sum_{n=N}^{\infty} \frac{a_n}{|S_n|} \right| \leq \left| \frac{a_{N-1}}{|S_N|} \right| \frac{|t|e^{-N\gamma}}{1 - (1 + |t|)e^{-N\gamma}}$

In particular, if $N \geq \lceil \log(1 + 2|t|) \rceil / \gamma$, we have $\Delta_N \leq |a_{N-1}|$ and $\delta'_N \leq |a_{N-1}| / |S_N|$.

Let us conclude by estimating the size of the largest term $\max_n |a_n|$. Define

$$b_n = \frac{|t|^n |x|^{n(n+1)/2}}{(1 - |x|)(1 - |x|^2) \cdots (1 - |x|^n)}, \quad (3.7)$$

so that $|a_n| \leq b_n$ (with equality if $|t| = 1$ and $0 < x < 1$). Suppose that $|x| = e^{-\gamma}$; it then follows from the computation in (3.6) that $b_n$ attains its maximum value at $n = \lceil \log(1 + |t|) / \gamma \rceil$, and that this maximum value is $\exp[C(|t|) / \gamma + O(1)]$ where

$$C(t) = \frac{1}{2} \log(1 + t) \log \left( \frac{t}{1 + t} \right) - \text{Li}_2 \left( \frac{1}{1 + t} \right) + \frac{\pi^2}{6}. \quad (3.8)$$

In particular, $C(1) = \pi^2/12$ [from (2.37)]. Therefore, for $|t| = 1$ the largest term can be as large in magnitude as $e^{\pi^2/12 \gamma}$ (and is indeed of this order when $0 < x < 1$); while the answer $R(t, x)$ can be as small in magnitude as $e^{-\pi^2/6 \gamma}$ (and is indeed of this order when $t = 1$ and $0 < x < 1$). It is therefore necessary to maintain, in intermediate stages of the calculation, approximately $(\pi^2/4 \gamma)/\log 10 \approx 1.07 / \gamma$ digits of working precision beyond the number of significant digits desired in the final answer.

4 Comparison with other algorithms

Let us conclude by briefly comparing the algorithm based on (1.11) with some alternative algorithms for computing $R(t, x)$.
Direct use of the defining product (1.1) manifestly gives an algorithm that is only linearly convergent, and in which the convergence rate deteriorates linearly as \(|x| \uparrow 1\). Moreover, there is severe loss of numerical precision when multiplying numbers that are very near 1. An alternative approach can be based on the logarithmic variant (2.6); this sum is again only linearly convergent, but the problem of loss of numerical precision is alleviated by use of the logarithm.

A slight improvement to the algorithm based on (1.1) can be obtained by noting that

\[
\prod_{n=N+1}^{\infty} (1 - tx^n) = 1 - \frac{tx^{N+1}}{1 - x} + O(x^{2N}) , \quad (4.1)
\]

so that correcting the product (1.1) by the factor \(1 - tx^{N+1}/(1 - x)\) yields an estimate with error \(O(x^{2N})\) rather than \(O(x^{2N})\). But the basic inefficiencies of the elementary algorithm remain.

Gatteschi \[15\] has proposed the following iterative algorithm for computing \(R(t, x)\):

Choose a complex number \(\sigma \notin \{0, tx\}\) and define

\[
\begin{align*}
\alpha_0 &= 1 & (4.2a) \\
\beta_0 &= \frac{\sigma}{\sigma - tx} & (4.2b) \\
\alpha_{n+1} &= \alpha_n \frac{\sigma \alpha_n + (1 - \sigma) \beta_n}{\beta_n} & (4.2c) \\
\beta_{n+1} &= \alpha_n \frac{\sigma \alpha_n + (1 - \sigma) \beta_n}{x \alpha_n + (1 - x) \beta_n} & (4.2d)
\end{align*}
\]

Gatteschi proves that \(\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = R(t, x)\). In fact, it can easily be shown by induction that

\[
\begin{align*}
\alpha_n &= \prod_{k=1}^{n} (1 - tx^k) & (4.3a) \\
\beta_n &= \frac{\sigma}{\sigma - tx^{n+1}} \alpha_n & (4.3b)
\end{align*}
\]

(though Gatteschi does not note this); so the iteration (4.2) gives simply a disguised way of computing the defining product (1.1) and a slight variant of it. Now, it is easily seen that

\[
\frac{\alpha_n}{R(t, x)} = 1 + \frac{tx^{n+1}}{1 - x} + O(x^{2n}) \quad (4.4a)
\]

\[
\frac{\beta_n}{R(t, x)} = 1 + tx^{n+1} \left( \frac{1}{1 - x} + \frac{1}{\sigma} \right) + O(x^{2n}) \quad (4.4b)
\]

I have altered his notation to conform to that of the present paper: his \(a, q, \xi, x_n, y_n\) correspond to my \(tx, x, 1 - \sigma, \alpha_n, \beta_n\). Gatteschi’s algorithm has been employed by Allasia and Bonardo [4].
Therefore, if we set $\lambda = 1 + \sigma/(1-x)$, the linear combination

$$\hat{\alpha}_n \equiv \lambda \alpha_n + (1-\lambda)\beta_n = \left[1 - \frac{\sigma t x^{N+1}}{(1-x)(\sigma - tx^{N+1})}\right] \alpha_n$$

(4.5)

converges to $R(t, x)$ more rapidly than either $\alpha_n$ or $\beta_n$ does (as Gatteschi observes in a special case): namely, $\hat{\alpha}_n/R(t, x) = 1 + O(x^{2n})$. But this is essentially equivalent (modulo higher-order terms) to the “improved” elementary algorithm based on the correction factor $\lambda$.

Finally, Slater [34, 35] has computed $R(t, x)$ using the “other” Euler formula

$$\frac{1}{R(t, x)} \equiv \prod_{n=1}^{\infty} (1 - tx^n)^{-1} = \sum_{m=0}^{\infty} \frac{t^m x^m}{(1-x)(1-x^2)\cdots(1-x^m)}.$$  

(4.6)

But this algorithm is only linearly convergent; it is no better than the logarithmic sum (2.6), and indeed is somewhat inferior due to the potentially small denominator.

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