AN INTRODUCTION TO $b$-MINIMALITY

by

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Abstract. — We give a survey with some explanations but no proofs of the new notion of $b$-minimality by the author and F. Loeser [$b$-minimality, J. Math. Log., 7 no. 2 (2007), 195–227, math.LO/0610183]. We compare this notion with other notions like $o$-minimality, $C$-minimality, $p$-minimality, and so on.

1. Introduction

As van den Dries notes in his book [10], Grothendieck’s dream of tame geometries found a certain realization in model theory, at first by the study of the geometric properties of definable sets for some nice structure like the field of real numbers, and then by axiomatizing these properties by notions of $o$-minimality, minimality, $C$-minimality, $p$-minimality, $v$-minimality, $t$-minimality, $b$-minimality, and so on. Although there is a joke speaking of $x$-minimality with $x = a, b, c, d, \ldots$, these notions are useful and needed in different contexts for different kinds of structures, for example, $o$-minimality is for ordered structures, and $v$-minimality is for algebraically closed valued fields.

In recent work with F. Loeser [4], we tried to unify some of the notions of $x$-minimality for different $x$, for certain $x$ only under extra conditions, to a very basic notion of $b$-minimality. At the same time, we tried to keep this notion very flexible, very tame with many nice properties, and able to describe complicated behavior.

An observation of Grothendieck’s is that instead of looking at objects, it is often better to look at morphisms and study the fibers of the morphisms. In one word, that is what $b$-minimality does: while most notions of $x$-minimality focus on sets and axiomatize subsets of the line to be simple (or tame), $b$-minimality focuses on definable functions and gives axioms on the existence of definable functions with nice fibers.

We give a survey on the new notion of $b$-minimality and put it in context, without giving proofs, and refer to [4] for the proofs.
2. A context

There is a plentitude of notions of tame geometries, even just looking at variants of \( o \)-minimality like quasi-\( o \)-minimality, \( d \)-minimality, and so on. Hence there is a need for unifying notions. Very recently, A. Wilkie [18] expanded the real field with entire analytic functions other than exp, where the zeros are like the set of integer powers of 2, and he shows this structure still has a very tame geometry. Since an \( o \)-minimal structure only allows finite discrete subsets of the line, Wilkie’s structure is not \( o \)-minimal, but it still probably is \( d \)-minimal [14], where an expansion of the field \( \mathbb{R} \) is called \( d \)-minimal if for every \( m \) and definable \( A \subseteq \mathbb{R}^{m+1} \) there is some \( N \) such that for all \( x \in \mathbb{R}^m \), the fiber \( A_x := \{ y \in \mathbb{R} \mid (x, y) \in A \} \) of \( A \) above \( x \) either has nonempty interior or is a union of \( N \) discrete sets. A similar problem exists on algebraically closed valued fields: if one expands them with a nontrivial entire analytic function on the line, one gets infinitely many zeros, and thus such a structure cannot be \( C \)-minimal nor \( v \)-minimal. This shows there is a need for flexible notions of tame geometry.

In [5], a general theory of motivic integration is developed, where dependence on parameters is made possible. It is only developed for the Denef-Pas language for Henselian valued fields (a semi-algebraic language), although only a limited number of properties of this language are used. Hence, one needs a notion of tame geometry for Henselian valued fields that is suitable for motivic integration. In this paper, we give an introduction to a notion satisfying to some extent these requirements, named \( b \)-minimality, developed in detail in [4].

3. \( b \)-minimality

This section is intended to sharpen the reader’s intuition before we give the formal definitions, by giving some informal explanations on \( b \)-minimality. The reader who wants to see formal definitions first, can proceed directly with section [4] or go back and forth between this and the subsequent section.

In a \( b \)-minimal set-up, there are two basic kinds of sets: balls and points. The balls are subsets of the main sort and are given by the fibers of a single predicate \( B \) in many variables, under some coordinate projection. There are also two kinds of sorts: there is a unique main sort and all other sorts are called auxiliary sorts (hence there is a partitioning of the sorts in one main sort and some auxiliary sorts). The points are just singletons. The \( b \) from \( b \)-minimality refers to the word balls.

One sees that in any of the notions \( x \)-minimal with \( x = o, v, C, p \), a ball makes sense (for example, open intervals in \( o \)-minimal structures), and thus \( b \)-minimality a priori can make sense.

The formal definition of \( b \)-minimality will be given in section [4], but here we describe some reasonable and desirable properties, of which the axioms will be an
abstraction.

To be in a $b$-minimal setup, a definable subset $X$ of the line $M$ in the main sort should be a disjoint union of balls and points. Such unions might be finite, but can as well be infinite, as long as they are “tame” in some sense. Namely, by a tame union, we mean that this union is the union of the fibers of a definable family, parameterized by auxiliary parameters. Hence, infinite unions are “allowed” as long as they are tame in this sense. To force cell decomposition to hold, such family should be $A$-definable as soon as $X$ is $A$-definable, with $A$ some parameters. This is the content of the first axiom for $b$-minimality. Thus so to speak, there is a notion of “allowed” infinite (disjoint) union of balls points, and any subset of the line should be such a union.

Secondly, we really want balls to be different from points, and the auxiliary sorts to be really different from the main sort. A ball should not be a union of points (that is to say, an “allowed” union of points). This is captured in the second axiom for $b$-minimality.

For the third axiom, the idea of a “tame” disjoint union in a $b$-minimal structure is needed to formulate piecewise properties. In the third axiom, we assume a tameness property on definable functions from the line $M$ in the main sort to $M$. Roughly, a definable function $f : M \to M$ should be piecewise constant or injective, where the pieces are forming a tame disjoint union, that is, there exists a definable family whose fibers form a disjoint union of $M$, and whose parameters are auxiliary, and on the fibers of this family the function $f$ is constant or injective.

One more word on tame disjoint unions partitioning a set $X$. Instead of speaking of “a” definable family whose fibers form a partition of $X$ and whose parameters are auxiliary, we will just speak of a definable function

$$f : X \to S$$

with $S$ auxiliary, and the fibers of $f$ then form such a tame union.

4. $b$-minimality: the definition

4.1. Some conventions. — All languages will have a unique main sort, the other sorts are auxiliary sorts. An expansion of a language may introduce new auxiliary sorts. If a model is named $\mathcal{M}$, then the main sort of $\mathcal{M}$ is denoted by $M$.

By definable we shall always mean definable with parameters, as opposed to $\mathcal{L}(A)$-definable or $A$-definable, which means definable with parameters in $A$. By a point we mean a singleton. A definable set is called auxiliary if it is a subset of a finite Cartesian product of (the universes of) auxiliary sorts.

If $S$ is a sort, then its Cartesian power $S^0$ is considered to be a point and to be $\emptyset$-definable.
Recall that \( o \)-minimality is about expansions of the language \( \mathcal{L}_< \) with one predicate \(<\), with the requirement that the predicate \(<\) defines a dense linear order without endpoints. In the present setting we shall study expansions of a language \( \mathcal{L}_B \) consisting of one predicate \( B \), which is nonempty and which has fibers in the \( M \)-sort (by definition called balls). In both instances of tame geometry, the expansion has to satisfy extra properties. A priori, it is not determined to which product of sorts the predicate \( B \) corresponds; this will always be fixed by the context, or it will be supposed to be fixed later on by some context, when it needs to be fixed.

4.2. — Let \( \mathcal{L}_B \) be the language with one predicate \( B \). We require that \( B \) is interpreted in any \( \mathcal{L}_B \)-model \( \mathcal{M} \) with main sort \( M \) as a nonempty set \( B(\mathcal{M}) \) with
\[
B(\mathcal{M}) \subset A_B \times M
\]
where \( A_B \) is a finite Cartesian product of (the universes of) some of the sorts of \( \mathcal{M} \).

When \( a \in A_B \) we write \( B(a) \) for
\[
B(a) := \{ m \in M \mid (a, m) \in B(\mathcal{M}) \},
\]
and if \( B(a) \) is nonempty, we call it a ball (in the structure \( \mathcal{M} \)), or \( B \)-ball when useful.

4.2.1. Definition (\( b \)-minimality). — Let \( \mathcal{L} \) be any expansion of \( \mathcal{L}_B \). We call an \( \mathcal{L} \)-model \( \mathcal{M} \) \( b \)-minimal when the following three conditions are satisfied for every set of parameters \( A \) (the elements of \( A \) can belong to any of the sorts), for every \( A \)-definable subset \( X \) of \( M \), and for every \( A \)-definable function \( F : X \to M \).

(b1) There exists a \( A \)-definable function \( f : X \to S \) with \( S \) an auxiliary set such that for each \( s \in f(X) \) the fiber \( f^{-1}(s) \) is a point or a ball.

(b2) If \( g \) is a definable function from an auxiliary set to a ball, then \( g \) is not surjective.

(b3) There exists a \( A \)-definable function \( f : X \to S \) with \( S \) an auxiliary set such that for each \( s \in f(X) \) the restriction \( F|_{f^{-1}(s)} \) is either injective or constant.

We call an \( \mathcal{L} \)-theory \( b \)-minimal if all its models are \( b \)-minimal.

5. Cell decomposition

In his paper on decision procedures, Cohen [6] develops cell decomposition techniques for real and \( p \)-adic fields, by a kind of Taylor approximation of roots of polynomials. At that time, the writing was rather complicated and it was only through the work by Denef [7][8] that some concrete notion of \( p \)-adic cells became apparent. One should keep in mind that there was no ideological framework of \( o \)-minimality which later on formed intuition of what cells should be and what they should do. An example of a fracture with actual \( o \)-minimal intuition about cells was that these original \( p \)-adic cells were not literally designed to partition definable sets into cells, they merely helped to partition into some nice pieces. On these nice pieces, one could get good properties of functions defined on them, which helped to
calculate $p$-adic integrals \cite{7,8,15,16}.

Another aspect of $o$-minimal intuition is that cells in one variable should be simple and defined by induction on the variables, both aspects were not so clear for the original $p$-adic cells and became even more complicated in the Pas-framework. Also cell decomposition for $C$-minimal structures \cite{11} is somehow complicated. In $v$-minimality \cite{13}, cell decomposition appears mainly implicitly.

The notion of $b$-minimality is intended to give a blueprint for a versatile kind of cell decomposition for tame geometries that is simple in one variable and defined by induction on the variables. A $(1)$-cell is a tame union of balls, and a $(0)$-cell is a tame union of points. Then one builds further with more variables.

6. Cell decomposition: the definitions

Let $\mathcal{L}$ be any expansion of $\mathcal{L}_B$, as before, and let $\mathcal{M}$ be an $\mathcal{L}$-model.

6.0.2. Definition (Cells). — If all fibers of some $f : X \to S$ as in (b1) are balls, then call $(X, f)$ a $(1)$-cell with presentation $f$. If all fibers of $f$ as in (b1) are points, then call $(X, f)$ a $(0)$-cell with presentation $f$. For short, call such $X$ a cell.

Let $X \subset M^n$ be definable and let $(j_1, \ldots, j_n)$ be in $\{0, 1\}^n$. Let $p_n : X \to M^{n-1}$ be the projection. Call $X$ a $(j_1, \ldots, j_n)$-cell with presentation

$$f : X \to S$$

for some auxiliary $S$, when for each $\hat{x} := (x_1, \ldots, x_{n-1}) \in p_n(X)$, the set $p_n^{-1}(\hat{x}) \subset M$ is a $(j_n)$-cell with presentation

$$p_n^{-1}(\hat{x}) \to S : x_n \mapsto f(\hat{x}, x_n)$$

and $p_n(X)$ is a $(j_1, \ldots, j_{n-1})$-cell with presentation

$$f' : p_n(X) \to S'$$

for some $f'$ satisfying $f' \circ p_n = p \circ f$ for some $p : S \to S'$.

One proves that if $X$ is a $(i_1, \ldots, i_n)$-cell, then $X$ is not a $(i'_1, \ldots, i'_n)$-cell, for the same ordering of the factors of $M^n$, for any tuple $(i'_1, \ldots, i'_n)$ different from $(i_1, \ldots, i_n)$. Thus $(i_1, \ldots, i_n)$ can be called the type of the $(i_1, \ldots, i_n)$-cell $X$.

One proves the cell decomposition theorem by compactness.

6.1. Theorem (Cell decomposition). — Let $\mathcal{M}$ be a model of a $b$-minimal theory. Let $X \subset M^n$ be a definable set. Then there exists a finite partition of $X$ into cells.
7. Refinements

Often, one has a cell decomposition of $X$, but one needs a finer cell decomposition, such that more properties hold on the parts. Here, it is not only the cells $X_i$ that should be partitioned further into cells, but each $X_i$ is already written as a union of fibers which resemble products of balls and points, and all these fibers should be partitioned into finer parts to speak of a genuine refinement.

7.0.1. Definition. — Let $\mathcal{P}$ and $\mathcal{P}'$ be two finite partitions of $X$ into cells $(X_i, f_i)$, resp. $(Y_j, g_j)$. Call $\mathcal{P}'$ a refinement of $\mathcal{P}$ when for each $i$ there exists $j$ such that $Y_j \subset X_i$ and such that $g_j$ is a refinement of $f_{ij} := f_i|Y_j$, that is, for each $a \in g_j(Y_j)$, there exists a (necessarily unique) $b \in f_{ij}(Y_j)$ such that $g_j^{-1}(a) \subset f_{ij}^{-1}(b)$.

One proves by compactness that refinements exist.

8. Relative cells

Cells use an order of the variables, so they are very well suited to work relatively over some of the variables.

Since in a $b$-minimal set-up there are many sorts, not all definable sets are subsets of $M^n$, with $M$ the main sort. Still, we want most notions to make sense for the main sort, and not to bother about the auxiliary sorts, as long as (b1), (b2) and (b3) are not violated. So there is a need to define all the concepts for definable subsets of $S \times M^n$ with $S$ auxiliary, or more generally, for definable subsets $X$ of $Y \times M^n$ for any definable $Y$. That way, one defines relative dimension over $Y$, cells over $Y$, a presentation over $Y$, and so on.

We just define a $(i)$-cell over $Y$ with $i = 0,1$. A definable set $X \subset Y \times M$ is called a $i$-cell with presentation

$$f : X \to Y \times S$$

with $S$ auxiliary if $f$ commutes with the projections $Y \times S \to Y$ and $p : X \to Y$ to $Y$, and for each $y$, the set $p^{-1}(y)$ is a $(i)$-cell with presentation

$$p^{-1}(y) \to S : m \mapsto f(y, m),$$

where we have identified $\{y\} \times S$ with $S$ and $p^{-1}(y)$ with a subset of $M$.

9. Dimension theory

Very similar to the $o$-minimal dimension as in [10], a dimension theory for $b$-minimal structures unfolds.

There are many sorts, but we want the dimension to live in the main sort.
9.0.2. Definition. — The dimension of a nonempty definable set $X \subset M^n$ is defined as the maximum of all sums

$$i_1 + \ldots + i_n$$

where $(i_1, \ldots, i_n)$ runs over the types of all cells contained in $X$, for all orderings of the $n$ factors of $M^n$. To the empty set we assign the dimension $-\infty$.

If $X \subset S \times M^n$ is definable with $S$ auxiliary, the dimension of $X$ is defined as the dimension of $p(X)$ with $p : S \times M^n \to M^n$ the projection.

Many properties as in [10] follows, for example, a $(i_1, \ldots, i_n)$-cell has dimension $\sum_j i_j$, and if $f : X \to Y$ is a definable functions, then $\dim(X) \geq \dim(f(X))$.

10. Preservation of balls

For o-minimal structures, piecewise monotonicity of definable functions plays a key role. On a general b-minimal structure, there is no order $<$, so functions cannot be called monotone. Nevertheless, the Monotonicity Theorem for o-minimal structures does have an analogue for b-minimal structures. It is not a consequence of b-minimality but has to be required as an extra property, named preservation of (all) balls. When we look at an o-minimal structure as a b-minimal structure as we do below, preservation of all balls is a consequence of the Monotonicity Theorem. The notion is especially useful for Henselian valued fields in the context of motivic integration [5], where it is used for the change of variables in one variable, see below.

10.1. Definition (Preservation of balls). — Let $\mathcal{M}$ be a b-minimal $\mathcal{L}$-model. We say that $\mathcal{M}$ preserves balls if for every set of parameters $A$ and $A$-definable function

$$F : X \subset M \to M$$

there is a $A$-definable function

$$f : X \to S$$

as in (b1) such that for each $s \in S$

$$F(f^{-1}(s))$$

is either a ball or a point.

If moreover there exists such $f$ such that for every map $f_1 : X \to S_1$ as in (b1) refining $f$ (in the sense that the fibers of $f_1$ partition the fibers of $f$) the set

$$F(f_1^{-1}(s_1))$$

is also either a ball or a point for each $s_1 \in S_1$, then say that $\mathcal{M}$ preserves all balls.

We say that a b-minimal theory preserves balls (resp. preserves all balls) when all its models do.
10.2. — Let’s give an example of $p$-adic integration and its change of variables formula in one variable, using preservation of balls.

If one integrates $|f(x)|$ over $\mathbb{Z}_p$ with $f$, say, a semi-algebraic function $\mathbb{Z}_p \to \mathbb{Z}_p$, and $|\cdot|$ the $p$-adic norm, it is useful to know a $b$-minimal cell decomposition of $\mathbb{Z}_p$ relative to $\text{ord}(f)$. That is, one takes for $X$ the definable subset of $\mathbb{Z}_p \times (\mathbb{Z} \cup \{+\infty\})$ given by $\text{ord}(f(x)) = a$ for $x$ in $\mathbb{Z}_p$ and $a$ in $\mathbb{Z} \cup \{+\infty\}$ and one takes a $b$-minimal cell decomposition of $X$ over $\mathbb{Z} \cup \{+\infty\}$ to find cells $X_j$ over $\mathbb{Z} \cup \{+\infty\}$ with presentation $f_j : X_j \to (\mathbb{Z} \cup \{+\infty\}) \times \mathbb{Z}^m$ for some $m$. The fibers of $f_j$ are either balls or points, depending on $j$ only, and since points have zero measure we can focus on 1-cells. Then

(10.2.1) \[
\int_{\mathbb{Z}_p} |f(x)||dx|,
\]

with $|dx|$ the Haar measure, is easily integrated, since the volume of a ball is an easy function of its size, and since $\text{ord}(f(x))$ by construction is constant on the fibers of the $f_j$. Since the measure of a ball is of the form $p^b$ for some $b \in \mathbb{Z}$, and since $|f(x)|$ for any $x$ is of the form $p^{-b'}$ for some $b' \in \mathbb{Z} \cup \{+\infty\}$, the integral (10.2.1) equals a converging sum

(10.2.2) \[
\sum_{a \in S} p^{-b(a)},
\]

with $S$ a Presburger set, and $b : S \to \mathbb{Z} \cup \{+\infty\}$ a Presburger function. Indeed, one rewrites $\mathbb{Z}_p$ as the “tame” disjoint union of the balls occurring in the 1-cells (these balls are parameterized by a single Presburger set $S$), and on each such ball, say parameterized by $a \in S$, one multiplies the volume of the ball, $p^v(a)$, with the value of $|f(x)| = p^{-w(a)}$, where $v$ and $w$ are Presburger functions, to obtain $p^{-b(a)} = p^{v(a)-w(a)}$, and one then sums $p^{-b(a)}$ over $S$.

In a semi-algebraic setup, preservation of balls holds such that moreover the size of the balls is changed in a way compatible with the Jacobian. If $g : A \subset \mathbb{Z}_p \to \mathbb{Z}_p$ is a semi-algebraic bijection, then

\[
\int_{\mathbb{Z}_p} |f(x)||dx| = \int_{A} |f \circ g(y)||J\text{ac}(g)(y)||dy|,
\]

by the change of variables formula. This change of variables formula holds here by general theory of the Haar measure on $p$-adic fields, but such arguments fail for motivic integrals because they involve much more general valued fields, like $k((t))$ with $k$ of characteristic zero. However, if one takes the above cell decomposition such that balls are preserved through $g^{-1}$ and such that their sizes change as predicted by the Jacobian, then we can translate both integrals into Presburger sums as (10.2.2) which one sees are exactly the same Presburger sums. Indeed, the norm of the Jacobian makes up for the difference in size of a ball $B_a$ and its inverse image $g^{-1}(B_a)$. Thus one finds an alternative proof of the change of variables formula in one variable. In [4] the motivic case is worked out. That the preservation of balls also changes the size of the balls w.r.t. the Jacobian, is a corollary of Weierstrass.
division and thus also holds in a motivic setting and even in subanalytic motivic settings, as long as the Henselian valued field has characteristic zero.

11. Some examples of $b$-minimal structures

11.1. $o$-minimal structures and non $o$-minimal expansions. — Any $o$-minimal structure $R$ admits a natural $b$-minimal expansion by taking as main sort $R$ with the induced structure, the two point set $\{0, 1\}$ as auxiliary sort and two constant symbols to denote these auxiliary points. A possible interpretation for $B$ is clear, for example,

$$B = \{(x, y, m) \in R^2 \times R \mid x < m < y \text{ when } x < y, \quad x < m \text{ when } x = y, \quad \text{and } m < y \text{ when } x > y\},$$

so that in the $m$ variable one gets all open intervals as fibers of $B$ above $R^2$. Property (b3) and preservation of all balls is in this case a corollary of the Monotonicity Theorem for $o$-minimal structures.

The notion of $b$-minimality leaves much more room for expansions than the notion of $o$-minimality: some structures on the real numbers are not $o$-minimal but are naturally $b$-minimal, for example, the field of real numbers with a predicate for the integer powers of 2 are $b$-minimal by [9] when adding to the above language the set of integer powers of 2 as auxiliary sort and the natural inclusion of it into $\mathbb{R}$ as function symbol.

Recently [18], Wilkie extended van den Dries’s construction to polynomially bounded structures, hence finding new entire analytic functions on the reals (other than exp) with tame geometry. These structures seem to be $b$-minimal as well, w.r.t. similar auxiliary sorts as for van den Dries’s structure $\mathbb{R}, 2^\mathbb{Z}$.

11.2. Henselian valued fields of characteristic zero. — In [4] is proved that the theory of Henselian valued fields of characteristic zero is $b$-minimal, in a natural definitional expansion of the valued field language, by adapting the Cohen - Denef proof. As far as we know, this is the first written instance of cell decomposition in mixed characteristic for unbounded ramification.

Let Hen denote the collection of all Henselian valued fields of characteristic zero (hence mixed characteristic is allowed).

For $K$ in Hen, write $K^\circ$ for the valuation ring and $M_K$ for the maximal ideal of $K^\circ$.

For $n > 0$ an integer, set $nM_K = \{nm \mid m \in M_K\}$ and consider the natural group morphism

$$rv_n : K^\times \to K^\times/1 + nM_K$$

which we extend to $rv_n : K \to (K^\times/1 + nM_K) \cup \{0\}$ by sending 0 to 0.
For every \( n > 0 \) we write \( RV_n(K) \) for

\[
RV_n(K) := (K^\times / 1 + nM_K) \cup \{0\},
\]

\( rv \) for \( rv_1 \) and \( RV \) for \( RV_1 \).

We define the family \( B(K) \) of balls by

\[
B(K) = \{(a, b, x) \in K^\times \times K^2 \mid |x - b| < |a|\}.
\]

Hence, a ball is by definition any set of the form \( B(a, b) = \{x \in K \mid |x - b| < |a|\} \) with \( a \) nonzero.

It is known that \( T_{\text{Hen}} \) allows elimination of valued field quantifiers in the language \( L_{\text{Hen}} \) by results by Scanlon \cite{17}, F.V. Kuhlmann and Basarab \cite{1}.

**11.2.1. Theorem.** — The theory \( T_{\text{Hen}} \) is \( b \)-minimal. Moreover, \( T_{\text{Hen}} \) preserves all balls.

**11.2.2. Remark.** — In fact, a slightly stronger cell decomposition theorem than Theorem 6.1 holds for \( T_{\text{Hen}} \), namely a cell decomposition with centers. We refer to \cite{4} to find back the full statement of cell decomposition with centers and the definition of a center of a cell in a \( b \)-minimal context.

The search for an expansion of \( T_{\text{Hen}} \) with a nontrivial entire analytic function is open and challenging. Nevertheless, in \cite{2} \( b \)-minimality for a broad class of analytic expansions of \( T_{\text{Hen}} \) is proved. This class of analytic expansions is an axiomatization of previous work \cite{3}.

**12. A further study and context**

Among other things, \( b \)-minimality is an attempt to lay the fundamentals of a tame geometry on Henselian valued fields that is suitable for motivic integration, as in \cite{5}. We hope to develop this theory in future work. One goal is to generalize the study in \cite{13} by Hrushovski and Kazhdan on Grothendieck rings in a \( v \)-minimal context to a \( b \)-minimal context.

Theories which are \( v \)-minimal \cite{13}, or \( p \)-minimal \cite{12} plus an extra condition, are \( b \)-minimal, namely, for the \( p \)-minimal case, under the extra condition of existence of definable Skolem functions. Also for \( C \)-minimality, some extra conditions are needed to imply \( b \)-minimality. For \( p \)-minimality, for example, cell decomposition lacks exactly when there are no definable Skolem functions. A possible connection with \( d \)-minimality needs to be investigated further.

For notions of \( x \)-minimality with \( x = p, C, v, o \), an expansion of a field with an entire analytic function (other than \( \exp \) on the real field) is probably impossible, intuitively since such functions have infinitely many zeros. In a \( b \)-minimal context, an infinite discrete set does not pose any problem, see section 11.1 for an example, as long as it is a “tame” union of points. So, one can hope for nontrivial expansions
of $b$-minimal fields by entire analytic functions, as done by Wilkie with exp and other entire functions on the reals, see section 11.1.

We give some open questions to end with:

Does a $b$-minimal $L_{\text{Hen}}$-theory of valued fields imply that the valued fields are Henselian?

As soon as the main sort $M$ is a normed field, is a definable function $f : M^n \to M$ then automatically $C^1$, that is, continuously differentiable?

Is there a weaker condition for expansions of $\mathcal{L}_B$ than preservation of (all) balls that together with (b1), (b2) and (b3) implies preservation of (all) balls?

Acknowledgments. — Many thanks to Anand Pillay for the invitation to write this article and to Denef, Hrushovski, and Pillay for useful advice on $b$-minimality.

During the writing of this paper, the author was a postdoctoral fellow of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.) and was supported by the European Commission - Marie Curie European Individual Fellowship with contract number HPMF CT 2005-007121.

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