Higher order statistics for three-dimensional shear and flexion

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ABSTRACT
We introduce a collection of statistics appropriate for the study of spinorial quantities defined in three dimensions, focusing on applications to cosmological weak gravitational lensing studies in three dimensions. In particular, we concentrate on power spectra associated with three- and four-point statistics, which have the advantage of compressing a large number of typically very noisy modes into a convenient data set. It has been shown previously by Munshi & Heavens that, for non-Gaussianity studies in the microwave background, such compression can be lossless for certain purposes, so we expect the statistics we define here to capture the bulk of the cosmological information available in these higher order statistics. We consider the effects of a sky mask and noise, and use Limber’s approximation to show how, for high-frequency angular modes, confrontation of the statistics with theory can be achieved efficiently and accurately. We focus on scalar and spinorial fields including convergence, shear and flexion of three-dimensional weak lensing, but many of the results apply for general spin fields.

Key words: gravitational lensing: weak – methods: analytical – methods: numerical – methods: statistical – large-scale structure of Universe.

1 INTRODUCTION
In this paper, we consider a set of new statistics designed to encapsulate much of the information content of third-order and higher statistics for spinorial fields defined in three dimensions. In cosmological applications such higher order statistics can be very noisy, and the dimensionality of the space may also lead to a very large number of data to consider. Thus, some form of data compression is attractive, but preferably in a way which does not reduce the cosmological information content inherent in the original statistics. In this paper, we build on the ideas originally presented in Munshi & Heavens (2010), where it was shown that one statistic, the power spectrum associated with the bispectrum, could be used very effectively to estimate non-Gaussianity in the microwave background, as it was a lossless compression for this purpose, and also had the important added benefit of being able to provide evidence that a non-Gaussianity is primordial. In this paper, we extend the ideas to cover spin-weighted fields which are defined in three dimensions, with particular emphasis on weak lensing fields convergence, shear and flexion.

Weak gravitational lensing of background source galaxies is caused by fluctuations in the intervening mass distribution. It manifests itself in a number of ways, most notably as distortions in their images. This effect arises due to the fluctuations of the gravitational potential and consequent deflection of light by gravity.

Despite being a relatively young subject weak gravitational lensing (Munshi et al. 2008) has made major progress within the last decade, since the first measurements were published (Bacon, Refregier & Ellis 2000; Wittman et al. 2000; Kaiser, Wilson & Luppino 2000; Waerbeke et al. 2000). There has been considerable progress in analytical modelling, technical specification and the control of systematics. By its dependence on the mass power spectrum at lower redshifts, weak lensing surveys play a complementary role to the studies based on large-scale galaxy surveys and cosmic microwave background (CMB) observations. Ongoing and future weak lensing surveys such as the Canada–France–Hawaii Telescope (CFHT) legacy survey,1 Pan-STARRS,2 the Dark Energy Survey, and, further in the future, the Large

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1 http://www.cfht.hawaii.edu/Sciences/CFHTLS/
2 http://pan-starrs.ifa.hawaii.edu/

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1630  D. Munshi et al.

Synoptic Survey Telescope, Widefield Infrared Survey Telescope (WFIRST) and Euclid will provide a wealth of information in terms of mapping the distribution of mass and energy in the universe.

Owing to the lack of photometric redshift information the traditional approach to weak lensing has largely adopted a 2D approach, analysing correlations of the shapes of galaxy images on the sky only. However, the availability of photometric redshifts allows a three-dimensional (3D) weak lensing analysis, which was introduced by Heavens (2003). Later developments by various authors (Heavens, Refregier & Heymans 2000; Castro, Heavens & Kitching 2005; Heavens, Kitching & Taylor 2006; Heavens, Kitching & Verde 2007) have shown that it can play a vital role in constraining the dark energy equation of state (Heavens et al. 2006) and the neutrino mass (Kitching et al. 2008). This has led to recent progress in modelling weak lensing observables in three dimensions extending results previously obtained in projection or using tomographic techniques (Munshi, Coles & Heavens 2011b).

Early results on analytical modelling typically assumed a small survey size and adopted a 2D approach that uses a flat-sky formalism. This is related to the fact that first generation of surveys typically covered a small portion of the sky and lacked any redshift information (Jain, Seljak & White 2000). Indeed, such analytical modelling was very successful in predicting lower order statistical properties of weak lensing convergence and shear very accurately (Munshi 2000; Munshi & Jain 2000, 2001). These results depend on analytical modelling of underlying density perturbations using perturbative and empirical methods (Valageas 2000; Munshi & Jain 2001; Valageas, Barber & Munshi 2004; Munshi & Valageas 2005; Valageas, Munshi & Barber 2005). A tomographic step was next advocated to tighten the cosmological constraints. The tomographic studies typically divide the sources into a few redshift slices (Hu 1999; Takada & White 2004; Takada & Jain 2004; Massey et al. 2007; Schrabback et al. 2009). These slices are then analysed essentially using a 2D approach but including the correlation between different redshift slices. A notable exception to the 2D analysis was Stebbins (1996) who developed an all-sky formalism for weak lensing surveys. The techniques developed in Stebbins (1996) rely on a tensorial formalism, whereas we will be using an equivalent treatment based on spin-weight spherical harmonics. Extending previous studies by Heavens (2003) and Castro et al. (2005), Munshi et al. (2011c) extended the all-sky formalism to three dimensions to take into account the photometric redshift information, as well as extending to higher order statistics. However, they focused on the convergence field which, being a spin-0 field, is relatively easier to analyse. The main motivation behind this work is to extend previous results to arbitrary spinorial fields such as shear and their derivatives flexion.

Weak lensing at small angular scales probes the non-linear regime of gravitational clustering, and the extra modes there can lift degeneracies about background cosmology present in studies involving the power spectrum alone (see e.g. Bernardeau, Van Waerbeke & Mellier 1997; Jain & Seljak 1997; Schneider et al. 1998; Hui 1999; Cooray 2001; Takada & Jain 2003; Valageas & Munshi 2004). The non-linear regime is characterized by gravity-induced non-Gaussianity, and detailed studies that employ the Fisher matrix formalism have already demonstrated the potential of using higher order non-Gaussianity information to lift cosmological degeneracies. Higher order studies are also important in evaluating the variance of lower order statistics, e.g. a proper knowledge of the trispectrum is essential for computing the error bars in the power spectrum (Takada & Jain 2009). The modelling of higher order statistics typically involves either perturbative techniques or empirical modelling of the underlying matter clustering (Fry 1984; Schaeffer 1984; Bernardeau & Schaeffer 1992; Szapudi & Szalay 1993, 1997; Munshi et al. 1999a; Munshi, Coles & Melott 1999a,b; Munshi, Melott & Coles 1999c; Coles, Melott & Munshi 1999; Munshi & Coles 2000, 2002, 2003; Cooray 2001). Using such prescriptions and their extensions, studies involving non-Gaussianity have also been performed in projection (2D) as well as using tomographic information (Hu 1999; Takada & Jain 2003, 2004; Semboloni et al. 2008) with a remarkable success.

Studies involving higher order correlation functions have been performed using observational data (Bernardeau, Van Waerbeke & Mellier 1997; Bernardeau, Mellier & Van Waerbeke 2002b; Pen et al. 2003; Jarvis, Bernstein & Jain 2004). Most of these studies involve one-point moments (cumulants) which collapse the entire correlation function into a single number. Mode-by-mode estimates of higher order correlation functions or multi-spectra though far more interesting is difficult given the low signal-to-noise ratio of current observational data. Current studies by Munshi & Heavens (2010) defined power spectra associated with each multi-spectrum that uses an intermediate option in data compression (Munshi et al. 2011a). While initially this concept was applied to CMB studies, recent work by Munshi et al. (2011b) extended this concept to weak lensing. This initial work focused on convergence \( \kappa \). Being a spin-0 (scalar) object, the analysis of convergence statistics is relatively simple. In their analysis, Munshi et al. (2011c) used the similar statistics for shear and flexion fields but in projection (2D). The main motivation for the present study is to use the full 3D information (available from photometric redshift surveys) in analysing the non-Gaussianity not only in the convergence field but also in shear and flexion. This is particularly interesting as current photometric redshift surveys with good image quality will provide a wealth of data for the analysis of weak lensing which can be used to probe cosmological information. For our study, we combine well-motivated ansatze in modelling the gravitational clustering with the Limber approximation. The results that we derive here are generic and will be useful in other areas of cosmology where integration along line of sight is involved. To keep the results simpler we will ignore the fact that in a realistic survey, the average density of sources will decline with distance, and the distance estimated from photometry will also include error, but these are evidently important ingredients in a practical implementation of these statistics.

3 http://www.lsst.org/lsst_home.shtml
4 http://wfirst.gsfc.nasa.gov/
5 http://sci.esa.int/euclid

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The expressions for higher order multi-spectra generically include multi-dimensional integrals involving multiple spherical Bessel functions. We will be using Limber approximation to simplify these results. We will show that, at each order, we can reduce the dimensionality of these integrals to unity by using Limber approximation. This will considerably simplify the numerical evaluations of such integrals.

This paper is arranged as follows. In Section 2, we discuss the basic formalism of 3D weak lensing. The formalism presented here is a generalization of Munshi et al. (2011b,c) and can analyze higher order statistics of spinorial fields in three dimensions. In Munshi et al. (2011b) results were derived for higher order statistics for the convergence and in Munshi et al. (2011c) the focus was on higher order statistics of spinorial objects but in projection (2D). The notations for 3D harmonic decomposition, which will be used in the following sections, are also introduced here. In Section 3, we introduce the models describing higher order clustering of underlying matter which are then used to construct models for the bispectrum and trispectrum in the non-linear regime. The results obtained are generic and can describe higher order statistics of weak lensing convergence, shear and flexions. In Section 4, we focus on power spectra associated with higher order multi-spectra. Results presented in this section correspond to both all-sky and patch-sky coverage. In Section 5, we focus on error analysis and derive results for scatter (or variance) of various estimators in the presence of observational noise and mask. Finally, Section 6 is devoted to discussion of the results. Though we have mainly focused on weak lensing, the general formalism developed in the paper will have wider applicability. We will use the Hierarchical ansatz to model clustering of underlying mass distribution, but the treatment can also be adopted in the context of more elaborate scenarios of clustering, e.g. halo model.

2 NOTATION

This section is devoted to introducing the basic notation and formalism of 3D weak lensing. We will follow the notation used in Munshi et al. (2011b) which is based mainly on Heavens (2003) and further developed by Castro et al. (2005). The results of Castro et al. (2005) were generalized by Munshi et al. (2011b) to take into account higher order correlations. The aim of this paper is to extend Munshi et al. (2011b,c) to the analysis of shear using full 3D information.

2.1 A tale of two potentials

Linking the 3D lensing potential \(\phi\) and the 3D gravitational potential \(\Phi\) is crucial in connecting lensing observables to theory. In this section we will consider the harmonic decomposition of 3D scalar (spin-0) fields which is a step towards making this connection because examples of such fields include the scalar potentials and the convergence field \(\kappa\) that we encounter in weak lensing. The harmonic decomposition is most naturally done using eigenfunctions that can be constructed using ordinary spherical harmonics and spherical Bessel functions. In the next section we will generalize them to the case of spinorial fields.

The statistics of shear and convergence can be expressed in a natural way through their relation to \(\Phi(r, \theta, \phi)\) the 3D gravitational potential at a 3D position \(r, \theta, \phi\) and \(\phi(r)\) the lensing potential. The density contrast \(\delta\) is directly related to the potential through the Poisson equation. This allows us to link directly the statistics of the weak lensing observables to the underlying statistics of the mass distribution, and hence to cosmological parameters. The radial distance \(r(t)\) is related to the Hubble expansion parameter \(H(t) = \dot{a}/a\) by \(z = c \int_0^t \frac{dt'}{H(t')}\). The Hubble parameter is sensitive to the contents of the Universe thereby making weak lensing a useful probe to study dark energy. The line-of-sight integral relating the two potentials can be written as (Kaiser 1992)

\[
\phi(r) \equiv \phi(r, \hat{\Omega}) = \frac{2}{c^2} \int_0^t \frac{dr'}{r'} F_K(r, r') \Phi(r', \hat{\Omega}); \quad F_K(r, r') \equiv \frac{f_K(r - r')}{f_K(r) f_K(r')} \tag{1}
\]

The Born approximation was used to derive the above expression (Bernardeau et al. 1997; Schneider et al. 1998; Waerbeke et al. 2002). The lensing potential \(\phi(r, \hat{\Omega})\) has a radial dependence and is a 3D quantity. In our notation \(r = r(t)\) is the comoving distance to the source whose observed light was emitted at a given instance of time \(t\). The observer is situated at the origin. The function \(F_K(r, r')\) depends on the background cosmology through the function \(f_K(r)\); \(f_K(r) = \sin r, \sinh r\) for a closed (K = 1), flat (K = 0) or open (K = -1) universes, respectively. Our convention for the Fourier transform for the 3D fields is as in Munshi et al. (2011b). The eigenfunctions of the Laplacian operator in flat space when expressed in spherical coordinates turn out to be a product of spherical Bessel functions when the curvature is small. The eigendecomposition and its inverse transformation can be expressed as (Munshi et al. 2011b,c)

\[
\Phi_{lm}(k) = \int d^3r \Phi(r) Z_{\text{kin}}(r) \tag{2}
\]

and

\[
\Phi(r) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int dk \Phi_{lm}(k) Z_{\text{kin}}(r). \tag{3}
\]

The specific choice of eigenfunctions allows us to easily express the expansion of the density field through the Poisson equation (Heavens 2003) \(\Delta \Phi(r) = 3 \Omega_m H_0^2 \delta(r)/2a\). In the harmonic domain, this can be expressed as \(\Phi_{lm}(r, k) = A \delta_{lm}(k, r)/a(r) k^2\) with \(A \equiv -3 \Omega_m H_0^2/2\). Here, \(\Phi_{lm}(k)\) is the spherical harmonic decomposition of \(\Phi(r)\), and similarly for \(\phi(r)\). In our notations, \(a(z) = 1/(1 + z)\) is the scalefactor at redshift \(z\), \(\Omega_m\) is the total matter density at \(z = 0\) and \(H_0\) is the
Table 1. Notations.

| Power spectrum (R/F) | $\mathcal{D}(r_1, r_2), \mathcal{G}(k_1, k_2)$ | Equation (12) |
|----------------------|---------------------------------|---------------|
| Variance (R/F)       | $\Sigma(r_1, r_2), \Sigma(k_1, k_2)$ | Equation (13) |
| Bispectrum           | $B_{ij(lj)}^k, B_{ij(lj)}^{k\prime}, B_{ij(lj)}^{k\prime\prime}$ | Equations (20) and (24) |
| Skew spectrum (R/F)  | $\mathcal{D}_{l^r,l^r'}(r_1, r_2), \mathcal{G}_{l^r,l^r'}(k_1, k_2)$ | Equations (35) and (37) |
| Skewness (R/F)       | $S_{l^r,l^r'}(r_1, r_2), S_{l^r,l^r'}(k_1, k_2)$ | Equations (43) and (44) |
| Trispectrum          | $T_{l^r,l^r'}^{klm}, T_{l^r,l^r'}^{klm}$ | Equation (25) |
| Kurt spectrum (R/F)  | $\mathcal{D}_{l^r,l^r'}^{r''}(r_1, r_2), \mathcal{G}_{l^r,l^r'}^{r''}(k_1, k_2)$ | Equation (48) |
| Kurtosis (R/F)       | $K_4^{r'',r''}(r_1, r_2), K_4^{r'',r''}(k_1, k_2)$ | Equation (52) |
| Deconvolution, mixing matrix | $\hat{C}_l, \hat{C}_l, M_{l^r}, G_{l^r}$ | Equations (17), (19), (41), (54) |

$^a$(Real/Fourier).

Hubble constant today, $\delta_{lm}(k, r)$ is the eigendecomposition of $\delta(r)$. When appearing after the semi-colon, the $r$ dependence [e.g. of $\Phi_{lm}(k; r)$] is really an expression of the time-dependence of the potentials, which translates to a dependence on $r$, as $r$ depends on look-back time. Using these decompositions, the harmonic decomposition of the lensing potential $\phi_{lm}(k)$ and the 3D gravitational potential $\Phi_{lm}(k, r)$ are related by the following expression (Castro et al. 2005):

$$\phi_{lm}(k) = \frac{4k}{\pi c^2} \int_0^\infty dk' k' \int_0^\infty r^2 dr j_l(kr) \frac{d}{dk} \Phi_{lm}(kr) \Phi_{lm}(kr').$$

(4)

The basis functions for the harmonic decomposition of the spinorial fields such as flexion and shear will involve spin-weight spherical harmonics which we will introduce next. The 3D power spectra for the gravitational potentials $\Phi$ and the lensing potential $\phi$ are defined through the following expressions:

$$\langle \Phi_{lm}(k) \Phi_{lm'}'(k') \rangle = C_{lm}^{\Phi\Phi}(k, k') \delta_{mm'}'$$

$$\langle \phi_{lm}(k) \phi_{lm'}'(k') \rangle = C_{lm}^{\phi\phi}(k, k') \delta_{mm'}'.$$

(5)

See Table 1 for notational details.

2.2 3D eigendecomposition of spinorial functions

In this section we will introduce the generic spin-weight functions and their eigendecomposition. Specific cases that are of interest here include shear and flexions. This will generalize the spin-0 results discussed above for the convergence field. We can expand the fields such as shear $\gamma_{lm}(r)$, flexions $F(r)$, $G(r)$ in 3D basis functions that are constructed out of spin-weight spherical harmonics $Y_{lm}(\hat{\Omega})$ on the celestial sphere and spherical Bessel functions $j_l(kr)$ in the radial direction. Expansion in such bases provides a very simple relationship between harmonic coefficients of the shear, flexion and convergence on the one hand, and the lensing potentials on the other. Moreover, spherical coordinates are the natural choice for eigendecomposition as this provides a clear separation in terms of radial modes and the modes on the surface of the sky, and the ubiquitous presence of a sky mask induces mixing of modes only on the surface of the sphere, and the use of photometric redshift estimates only introduces error in the radial direction without altering the angular position. The choice of eigenfunction is also motivated by the Poisson equation which relates the 3D potential $\Phi(r)$ to the density distribution $\delta(r)$, whose statistical property we will model to predict the statistics of shear, convergence or flexion. Extending the definition of spin-0 eigenvectors $Z_{lm}(r)$ we will denote the spin-$s$ eigenfunctions as $Z_{lm}(r)$, which is defined as $Z_{lm}(r) = \sqrt{\frac{2}{\pi} k_j(kr), Y_{lm}(\hat{\Omega})}$. The spin-weight spherical harmonics are defined in terms of D-matrices (Varshalovich, Moskalev & Khersonskii 1988; Penrose & Rindler 1987, 1988). They satisfy an orthogonality relationship similar to ordinary spherical harmonics. Spin-weight harmonics with the same or different spin indices are orthogonal on the surface of sky. This generalizes the orthogonality relationship of ordinary spin-0 spherical harmonics: $\langle Y_{lm}^{\phi}(\hat{\Omega}) Y_{lm'}^{\phi}(\hat{\Omega}) \rangle = \delta_{ll'} \delta_{mm'}$. We will be using the following overlap integral in our derivations:

$$\int d\hat{\Omega}, Y_{lm}(\hat{\Omega}) Y_{lm'}^{\phi}(\hat{\Omega}) = \frac{(2l+1)(2l'+1)(2m+1)(2m'+1)}{4\pi} \left( \begin{array}{ccc} l & l' & l'' \\ m & m' & m'' \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ -s & -s' & -s'' \end{array} \right).$$

(6)

Next, to express the product of two spin-harmonics we multiply both the sides of the above equation with $\gamma_{lm'}^{\prime\prime}(\hat{\Omega})$ and sum over $l''m''$ and use the completeness relation for the spherical harmonics $\sum_{lm} Y_{lm}(\hat{\Omega}) Y_{lm}(\hat{\Omega}) = \delta_{\hat{\Omega} - \hat{\Omega}'}$. The integrals over angles $d\hat{\Omega}$ collapse because of the delta function. After renaming the variables ($s', l', m'$) as $(S, L, M)$ we arrive at the following expression:

$$\gamma_{lm}(\hat{\Omega}) Y_{lm}(\hat{\Omega}) = \sum_{S LM} \gamma_{S LM}^{\phi} \frac{(2l+1)(2l'+1)(2m+1)}{4\pi} \left( \begin{array}{ccc} L & l & l' \\ M & m & m' \end{array} \right) \left( \begin{array}{ccc} L & l & l' \\ -S & -s & -s' \end{array} \right).$$

(7)

Note that the $3J$ symbols will only be non-vanishing if $S = -s + s'$ and $M = -m + m'$. Hence the sum is effectively only over $L$. It is also useful to point out here that we have the following relationship for complex conjugation of spin-harmonics, $Y_{lm}^{\phi\phi}(\hat{\Omega}) = (-1)^{s+m} Y_{lm}^{\phi\phi}(\hat{\Omega})$. 

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Alternative expansion schemes are indeed possible such as using tensor spherical harmonics but they are perhaps more difficult to work with. It is worth noting here that the formalism of spin-harmonics is extensively used in studies involving CMB polarization (Bunn et al. 2003). The forward and inverse transform of an arbitrary spin function \( f(r) \) from real space to harmonic space links it with its harmonic components \( f_{lm}(k) \) that can be expressed as: \( f(r) = \int \frac{dk}{k} \sum_{lm} k^{\ell} s_{lm}(k), f_{lm}(k) \) and \( f_{lm}(k) = \int \frac{d^2r}{r^2} f(r) Z_{lm}(r) \). The orthogonality relationship satisfied by the 3D spherical basis functions \( Z_{lm}(r) \) depends on the orthogonality of spin-weight harmonics \( Y_{lm}(\Omega) \) and that of the spherical Bessel functions \( j(kr) \). It generalizes a similar relation for the scalar harmonics. For arbitrary spinorial fields with spins \( s \) it reads: \( f^{(s)}(r) = \int d^3r \left[ f_{lm}(r) Z_{lm}(r) \right] \). The inverse transforms are used to define the harmonic components of generic spinorial fields \( \eta(r) \) and \( \eta^*(r) \). The results that we will derive in our later sections are expressed most naturally in the harmonic domain using these components \( \eta_{lm}(k) \) and \( -\eta^*_{lm}(k) \) which can be expressed as: \( \eta_{lm}(k) = \int d^3r \eta(r) Z_{lm}(r) \) and \( -\eta^*_{lm}(k) = \int d^3r \eta^*(r) Z_{lm}(r) \). It is indeed possible to work with \( \eta_{lm}(k) \) and \( -\eta^*_{lm}(k) \) as well as the harmonics \( E_{lm}(k) \) and \( B_{lm}(k) \) that can be constructed from them. Though they contain the same information, the electric or \( E \) and magnetic \( B \) modes provide a rotationally invariant description in full sky. The expansion coefficients \( E_{lm}(k) \) has a parity \((-1)^{l+1}\) while \( B_{lm}(k) \) has a parity \((-1)^{l+1}\). The clear separation of modes with different parity gives a clear mathematical advantage in the case of weak lensing, as it can be shown that, at the first order, weak lensing from gravitational clustering can only generate \( E \) modes, whereas systematics are mostly responsible for the generation of any \( B \)-mode contribution.

The explicit expressions for the electric \( E_{lm}(k) \) and magnetic \( B_{lm}(k) \) components, constructed from these harmonic transforms, are: \( E_{lm}(k) = -\frac{1}{2}[\eta_{lm}(k) + \eta^*_{lm}(k)]; B_{lm}(k) = \frac{1}{2}([\eta_{lm}(k) - \eta^*_{lm}(k)], \) and \( -\eta^*_{lm}(k) = -E_{lm}(k) \pm iB_{lm}(k) \). The individual components of the field \( \eta(r), \eta^*(r) \) and \( \eta_{lm}(k) \) are expressed in terms of eigenfunctions \( Z_{lm}(r) \) that can be constructed from linear combinations of \( Z_{\pm2,lm}(r) \) introduced before. The formalism used here is very similar to Munshi et al. (2011c). The emphasis here, however, is not just on 2D decomposition on the surface of the celestial sphere but rather on a 3D decomposition which relies on the photometric redshift to estimate radial distance.

It is worth mentioning here that unique decomposition of a function into modes \( E \) and \( B \) on the celestial sphere is possible only with complete sky coverage. In the presence of a boundary, which is often the case owing to the presence of masks, the decomposition is ambiguous. For the case of weak lensing shear these equations can be specialized further by ignoring the magnetic contribution which is zero for shear generated purely by gravitational lensing in the absence of any systematics. Indeed, higher order lensing corrections can generate lensing \( B \) mode too (Cooray & Hu 2002) but are sub-dominant.

### 2.3 Harmonic decomposition of convergence, shear and flexion

The results derived in previous section can directly be applied to the case of shear, convergence, and shear. Most of the generic results are applicable to the analysis of shear \( \gamma \) if we specialize the field \( \eta \) with a spin-2 object and identify with 3D shear \( \gamma \). Complex shear \( \gamma \) constructed from its individual components \( \gamma_1(r) = \gamma'(r) \pm i\gamma^*(r) \) acts as a spin-2 object and can be expressed in terms of the lensing potential \( \phi \) using spin-derivatives [see Munshi et al. (2011c) and Castro et al. (2005) for more discussion on spin-derivatives] which are used to construct spinorial fields with different spin-weights. The lensing potential plays the role of the generic scalar field introduced earlier to express arbitrary spin functions. We will use the same symbol \( \phi \) for both. We will use the generalized symbol \( \Gamma \) for general spin fields which will include products of shear fields as well as higher derivative spin objects such as flexions. In our current notation \( 2\Gamma = \gamma + \gamma^* \Rightarrow 2\Gamma(r) = 2\Re[\phi] \) and \( -2\Gamma(r) = \Im[\phi] \). In general, the scalar potential \( \phi(r) \) will have both electric \( \phi_{el}(r) \) and magnetic \( \phi_{mag}(r) \) components: \( \phi(r) = \phi_{el}(r) + i\phi_{mag}(r) \). We would like to point out here that the definition of the \( E \) and \( B \) modes follows the one used in the CMB context but it is not a priori unique. The individual shear components \( \gamma_1(r) \) and \( \gamma^*(r) \) and the convergence \( \kappa(r) \) can be expressed in terms of a complex lensing potential \( \phi(r) = \phi_{el}(r) + i\phi_{mag}(r) \). As pointed out before, the magnetic part of the potential \( \phi_{mag}(r) \) will take contribution mainly from systematics and the electric part corresponds largely to pure lensing contribution \( \gamma_1(r) = \frac{1}{2}\partial \partial \partial \bar{\phi} \). \( \gamma_2(r) = \frac{1}{4}(\partial \partial \partial - \bar{\partial} \bar{\partial}) \bar{\phi} \) and \( \kappa(r) = \frac{1}{4}(\partial \partial \partial + \bar{\partial} \bar{\partial}) \bar{\phi} \) or \( \kappa(r) = (1/2)\bar{\partial} \bar{\partial} \bar{\phi} \). We neglect two terms in \( \kappa \) (Bernardeau, Bonvin & Vernizzi 2010) which are negligible on the small angular scales where most of the cosmological signal resides. Derivatives of the shears are higher spin objects. Using these derivatives quantities such as \( f_{shears} \) are constructed, which are also often used in the context of weak lensing studies (Goldberg & Natarajan 2002; Goldberg & Bacon 2005; Bacon et al. 2006; Bacon & Goldberg 2005; Schneider & Er 2008). The two flexions that are most commonly used are also known as the first flexion \( F \) (spin-1) and \( G \) which is also known as the second flexion (spin-3). These two flexions in combination can specify distortion beyond what is described by shear. The flexions can be used to describe weak ‘arcsines’ in images of lensed galaxies and their relationship with the shapelet formalism is well documented (Bernstein & Jarvis 2002; Refregier 2003; Refregier & Bacon 2003). The flexions \( F(r) \) and \( G(r) \) are both used in the literature mainly for individual halo profiles and also for the study of substructures (Bacon et al. 2006). We are mainly interested, however, in higher order statistics of these objects for generic underlying cosmological clustering. This is done by linking the 3D harmonic decompositions of the flexions to that of the lensing potential \( \phi(r) \): \( F(r) = \frac{1}{4} (\partial \partial \partial + \partial \partial \partial + \bar{\partial} \bar{\partial} \bar{\partial}) \bar{\phi} \) and \( G(r) = \frac{i}{4}\bar{\partial} \bar{\partial} \bar{\partial} \bar{\phi} \). Flexions have been used primarily to measure the galaxy–galaxy lensing to probe the galaxy halo density profiles. Their cosmological use will depend on an accurate understanding of gravitational clustering at small angular scales.

In Fourier space, the harmonics of \( \gamma(r) \) and \( \gamma^*(r) \) can be expressed in terms of the harmonic coefficients of \( \Phi_{el}(r) \) and \( \Phi_{mag}(r) \) denoted by \( E_{lm}(k) \) and \( B_{lm}(k) \), respectively: \( \pm \frac{1}{2}\gamma_{lm}(k) = -[E_{lm}(k) \pm iB_{lm}(k)] \). Analogously, the harmonics of \( F \) and \( G \) denoted by \( F_{lm}(k) \) and \( G_{lm}(k) \) can also be expressed in terms of the \( \phi_{lm}(k) \). In the absence of \( B \) modes the harmonics of the shear components are directly related to the harmonic component of the electric field \( E_{lm}(k) \). The harmonic transforms of the shear components and convergence can also be expressed in terms of the
lensing potential $\phi$ as follows: $\kappa_{lm}(k) = -\frac{b_1 m}{a_1} \phi_{lm}(k)$; $E_{lm}(k) = -\frac{1}{2} \sqrt{\frac{(l+2)(l+1)}{2l+2}} \phi_{lm}(k)$; $F_{lm}(k) = \frac{1}{2} \sqrt{l+1/2(l+1/2)(3l^2+3l-2)} \phi_{lm}(k)$; $G_{lm}(k) = \frac{1}{2} \left( \frac{2l+7}{2l+3} \right) \phi_{lm}(k)$.

These harmonic expressions can be used to reconstitute the real-space spinorial fields: $\psi_{2l}(r, \Omega) = \int k dk \sum_{m=-l}^{l} \left( \frac{2l+7}{2l+3} \right) \phi_{lm}(k) Z_{klm}(r)$ and $\psi(r) = \int k dk \sum_{m=-l}^{l} \sum_{m=0}^{l} \phi_{lm}(k) Z_{klm}(r)$. These results derived above are useful in linking the statistics of weak lensing fields $\kappa(r) \gamma(r)$, $F(r)$ and $\phi(r)$ with those of the underlying density field $\delta(r)$ responsible for generation of the lensing potential $\phi(r)$ with the help of equation (4).

The theoretical modelling of the underlying mass distribution that we employ in our study is based on the hierarchical ansatz. The hierarchical ansatz is more suited to model gravitational clustering at smaller scales, which makes it particularly suitable for modelling the flexion statistics which put more weight on smaller scales. A comment about noise contribution due to intrinsic flexions of source galaxies is in order. While it is relatively easy to model the intrinsic ellipticity of source galaxies, detailed modelling of intrinsic flexion of source galaxies is much more complicated and depends heavily on modelling of galaxy shapes beyond the simplest description. This uncertainty is also expected to increase with survey depth.

In addition to shear, convergence and flexion which are used in weak lensing studies, we can also consider a generic scalar tracer field $\Psi$ in our study. Such fields can represent a suitable large-scale tracers which are sometimes used for cross-correlation studies or studies involving weak lensing magnification.

The statistics of shear and flexions can be best related to that of convergence with certain $l$ dependent multiplicative factors that we will call form factors. In later sections $F_i^j$ will denote the form factor associated with a generic spin-weight field $\Gamma$. So the form factor for the shear $\gamma_\pm$ will be denoted by $F_i^\pm$.

### 3D Weak Lensing Statistics: Power Spectrum and Beyond

For the study of non-Gaussianity we need to go beyond the study of power spectra. In this section we will present results for two-, three- and four-point statistics, and show the relation between observables and theory. The various multi-spectra involve multi-dimensional integrals, which we simplify by employing various levels of approximations involving the high $l$ behaviour of $j_i(x)$.

#### 3.1 Power spectrum

We will start by deriving the power spectrum $C_i^{\phi\phi}(k, l)$ for the 3D weak lensing fields. Our derivation of the 3D convergence power spectrum is based on expressing its harmonic coefficients $\kappa_{lm}(k)$ in terms of the 3D density field $\delta$ with the help of Poisson’s equation and using the definition of the convergence field in terms of the projected lensing potential $\phi$ gives

$$\kappa_{lm}(k) = \frac{2kA}{\pi^2 c^2} (l+1) \int_0^{\infty} dk' k' \int_0^{\infty} r^2 dr j_i(kr) \int_0^{r'} dr' F_k(r, r') \frac{\delta_{lm}(k', r')}{k'^2 u(t' r')}.$$  

We will use a short-hand notation $I_i(k, l)$ (defined below) useful for simplification of our results. We will approximate the cross-spectra at two different epochs using the approximation $P^{\phi\phi}(k, r, r') = \sqrt{P^{\phi\phi}(k, r) P^{\phi\phi}(k, r')}$ (Castro et al. 2005). Use of this approximation leads to separation of respective integrals. As we will see below the use of the extended Limber approximation, which is valid at high $l$, implies that dominant contribution will come from single time slices $r = r'$ and this approximation is not detrimental to any of the final results which are quite generic. The 3D power spectrum can be expressed in terms of $I_i(k, l)$ as (Castro et al. 2005)

$$I_i(k, l) = k \int_0^{\infty} dr^2 j_i(kr) \int_0^{r'} dr' F_k(r, r') \frac{\delta_{lm}(k', r')}{k'^2 u(t' r')}.$$  

$$C_i^{\phi\phi}(k_1, k_2) = \frac{16}{\pi^2 c^4} \int_0^{\infty} k^2 I_i(k_1, k) I_i(k_2, k) dk; \quad C_i^{\phi\phi}(k_1, k_2) = \frac{16}{\pi^2 c^4} \int_0^{\infty} k^2 I_i(k_1, k) I_i(k_2, k) dk; \quad C_i^{F_i F_i^j}(k_1, k_2) = F_i^j F_i^j C_i^{\phi\phi}(k_1, k_2).$$  

Clearly, the above expression is quite generic and contains all the weak lensing information at the second-order level. This expression is, however, quite cumbersome for any numerical implementation as it involves 3D integral which are computationally quite demanding. We will be using extended Limber approximation valid at high $l$ to simplify the above expression. Using this approximation we can reduce the integrals to 1D integrals. In any case we will quote the generic result that is valid without any approximation. Note that the following approximation is also independent of the factorization of the power spectrum introduced before.

$$C_i^{\phi\phi}(k_1, k_2) = \frac{16}{\pi^2 c^4} \int_0^{\infty} k^2 dk k_i(k_1, k) k_i(k_2, k)$$  

$$= \frac{16}{\pi^2 c^4} \int_0^{\infty} k^2 dk k_i(k_1, k) k_i(k_2, k) \int_0^{\infty} dr r^2 F_k(r, r') \int_0^{\infty} dr' r'^2 F_k(r, r')$$  

$\int_0^{\infty} dr r^2 F_k(r, r') \int_0^{\infty} dr' r'^2 F_k(r, r')$.
We will next use the Limber approximation equation (A2) to simplify the $k$ integral which produces a $\delta_{1D}(r_a - r_b)$ function. Integrating out $r_a$ with the help of the delta function and renaming the dummy variable $r_a'$ to $r'$ we can finally write

$$C^e_{ij}(k_1, k_2) = \frac{2}{\pi} k_1 k_2 \int_0^{r_{\text{min}}} r_1^2 dr_1 j_i(k_1 r_1) \int_0^{r_{\text{min}}} r_2^2 dr_2 j_j(k_2 r_2) D^e_{ij}(r_1, r_2);$$

$$D^e_{ij}(r_1, r_2) = \frac{A^2}{c^2} \int_0^{r_{\text{min}}} r^2 \frac{dr'}{a'(r')} F_k(r_1, r') F_k(r_2, r') P_2 \left( \frac{l^2}{r_1^2}; r' \right); \quad r_{\text{min}} = \min(r_1, r_2). \quad (11)$$

Use of the Limber approximation projects multi-time correlators to a single time correlator. Going one step further, if we use the high $l$ approximation to the spherical Bessel function equation (A3) to reduce the dimensionality of the above integrals involving the spherical Bessel functions $j_i$, we arrive at the following simpler approximate equation. Use of equation (A3) allows us to replace $r_1$ and $r_2$ in terms of $k_1, k_2$ and $l$.

$$C^e_{ij}(k_1, k_2) = [Ae^{-2}]^2 \int_0^{r_{\text{min}}} r^2 \frac{dr'}{a'(r')} F_k(r_1, r') F_k(r_2, r') P_2 \left( \frac{l^2}{r_1^2}; r' \right). \quad (12)$$

We can define a statistic $\Sigma(k_1, k_2)$ which will include all available information from individual harmonics, as a function of $k_1, k_2$.

$$\Sigma^e_{ij}(k_1, k_2) = \sum_l (2l + 1) C^e_{ij}(k_1, k_2); \quad \Sigma^e_{ij}(r_1, r_2) = \sum_l (2l + 1) C^e_{ij}(r_1, r_2). \quad (13)$$

We have ignored angular smoothing in our derivation. Typically, observations will involve a smoothing filter. Top-hat and compensated filters are the ones that are most commonly used that can be incorporated in equation (12). As pointed out before, the above equation is derived using very general arguments. It is valid at high $l$ as the derivation is based only on high $l$ approximation to the spherical Bessel function $j_i(x)$. Nevertheless, the derivation of a 3D skew spectrum has wider applicability in cosmology. The technique can be applied to compute 3D power spectrum in other context (e.g. integrated Sachs–Wolfe effect or Kinetic Sunyaev–Zeldovich effect). A detailed analysis for such cases will be presented elsewhere.

We have used Limber approximation to simplify results (LoVerde & Afsbordi 2008). It was pointed out by LoVerde & Afsbordi (2008) that using $l$ instead of $l + \frac{1}{2}$, as is often done in the literature, spoils the accuracy of Limber approximation to $O(\frac{1}{l})$. In general, the error in Limber approximation will scale as $O(l)$.

A series expansion of the spherical Bessel functions can also be constructed to estimate the next as the derivation is based only on high $l$ approximation to the spherical Bessel function $j_i(x)$.

While the results derived above are valid for all-sky surveys, observations invariably will introduce mask. We will next analyse the case of 3D power spectrum estimation in the presence of a general mask. The results that we derive will have general applicability and will be valid for span near all-sky coverage.

### 3.1.1 The effect of an angular sky mask

We start by 3D decomposition of an arbitrary spinorial field $\Gamma$ in the presence of a sky mask $w(\hat{\Omega})$, which is equal to 0 or 1 in simple cases. The decomposition into radial harmonics, using spherical Bessel functions $j_i(kr)$, can be performed independent of the mask. The harmonic decomposition on the surface of the celestial sphere involves spin-weighted spherical harmonics, $Y_{lm}^{\gamma}(\hat{\Omega})$. The following expression relates the masked observed harmonics $[\Gamma w]_{lm}(k)$ and the unmasked $\Gamma_{lm}(k)$. We will leave the spinorial field $\Gamma$ arbitrary and derive the expression of the cross-correlation power spectrum, in the presence of a mask, with another arbitrary spinorial field $\gamma$, $\Gamma'(\hat{\Omega})$. The results are generic and do not depend on any specific assumption that is used to model the all-sky power spectrum itself.

$$[\Gamma w]_{lm}(k) = [\Gamma]_{lm}(k) \equiv \frac{\sqrt{2}}{\pi} \int d^3 \mathbf{k} j_i(kr)[Y_{lm}^{\gamma}(\hat{\Omega})][\Gamma(r) w(\hat{\Omega})] = \frac{\sqrt{2}}{\pi} \int kr^2 dk j_i(kr) \int d\hat{\Omega}[\Gamma(r) w(\hat{\Omega})][Y_{lm}^{\gamma}(\hat{\Omega})]$$

$$= \sum_{l' m'} \left[ \frac{\sqrt{2}}{\pi} \int dk k r^2 j_i(kr) \Gamma_{l' m'}(r) \right] w_{l' m'} \int [1, Y_{lm}^{\gamma}(\hat{\Omega})][Y_{lm}^{\gamma}(\hat{\Omega})][1, Y_{lm}^{\gamma}(\hat{\Omega})]d\hat{\Omega}$$

$$= \frac{2}{\pi} \sum_{l' m'} \sum_{l'' m''} (-1)^{l''+m''} \int dr \int dk' k' r^2 j_i(kr) j_i(k'r') \Gamma_{l' m' l'' m''}(k) w_{l'' m''} J_{l'' m'' l'' m''} \left( \frac{l'}{l''}, \frac{l''}{m''}, \frac{l''}{m''}, \frac{l''}{m''} \right); \quad (14)$$

$$\approx \sum_{l' m' l'' m''} (-1)^{l''+m''} (\frac{2l' + 2l'' + 1}{2l' + 2l'' + 1})^2 \int \left[ \frac{2l' + 2l'' + 1}{2l' + 2l'' + 1} \right] w_{l'' m''} J_{l'' m''} \left( \frac{l'}{l''}, \frac{l''}{m''}, \frac{l''}{m''}, \frac{l''}{m''} \right). \quad (15)$$

The masked cross-spectrum $C_{ij}^{\gamma \Gamma'}(k, k')$ involving $\Gamma(\hat{\Omega})$ and $\Gamma'(\hat{\Omega})$ can be described in terms of the all-sky cross-spectra $C_{ij}^{\gamma \Gamma'}(k, k')$ and a mode-mixing matrix $G$ that describes the effect of mode–mode coupling resulting from the presence of the mask. The mode-mixing matrix
depends on the spin-weight of the respective fields $s$ and $s'$ and also on the power spectrum of the mask $w_l$.

\[
C_{ij}^{\Gamma\Gamma}(k, k') = \frac{1}{2l+1} \sum_{m} \left[ \tilde{\Gamma}_{lm}(k) \tilde{\Gamma}^{\ast}_{lm}(k') \right] = \frac{1}{2l+1} \sum_{m} \left[ \tilde{\Gamma}_k w_{lm}(k) \tilde{\Gamma}^\ast_{lm}(k') \right] \approx \sum_{l'} G_{l'l'}^{\Gamma\Gamma} \left[ \frac{2l+1}{2l' + 1} \right] \left( \frac{2l' + 1}{2l' + 1} \right) ;
\]

(16)

\[
G_{ll} = \frac{1}{4\pi} \sum_{l''} \frac{(2l'' + 1)^2}{(2l'' + 1)} \left( \begin{array}{ccc} l & l' & l'' \\ s & s' & 0 \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ s & s' & 0 \end{array} \right) |w_{l''}|^2 .
\]

(17)

The radial direction remains unaffected by the mask which introduces mode mixing only on the surface of the celestial sphere hence the matrix $M$ is independent of radial wavenumber $k$. For a given pair of radial wavenumbers $k, k'$ the all-sky cross-spectra $C_{ij}^{\Gamma\Gamma}(k, k')$ can in principle be recovered by inverting the above expression (equation 17). In general, there will be contributions from noise which can be from intrinsic ellipticity or flexion distribution of galaxies in case of shear or flexion. Such contributions need to be subtracted to make any estimation unbiased, and there may be standard issues with inversion which may require regularization.

To recover the power spectrum of $E$ and $B$ modes of a spin $\pm 2$ fields, commonly used in the context of analysis, e.g. as in Brown, Castro & Taylor (2005) we have to express angular harmonics of $\ell = \ell', \Gamma$ and $\ell = \ell', \Gamma' = -2$, $\Gamma = \pm 2$ in terms of their electric (E) and magnetic (B) components $\Gamma_{lm} = E_{lm} + iB_{lm}$. Then, using equation (17) we can relate the cut-sky power spectra $C_{ij}^{EE} = \frac{1}{2\pi} \sum_{l''} E_{lm} E_{l''m}$ and $C_{ij}^{BB} = \frac{1}{2\pi} \sum_{l''} B_{lm} B_{l''m}$ in terms of their all-sky counterparts $C_{ij}^{EE}$ and $C_{ij}^{BB}$. However equation (17) generalizes such results to generic spin functions with arbitrary spin-weights. For generic spin-weight functions the cut-sky and all-sky relations are

\[
C_{ij}^{EE} = G_{ll}^{EE} C_{ij}^{EE} + G_{ll}^{EB} C_{ij}^{EB} ;
\]

\[
C_{ij}^{BB} = G_{ll}^{EE} C_{ij}^{EE} + G_{ll}^{BB} C_{ij}^{BB} .
\]

A sum over repeated indices is assumed in each of these equations. The matrices $G_{ll}^{EE}$, $G_{ll}^{BB}$ and $G_{ll}^{EB}$ are defined through the following expressions:

\[
G_{ll}^{EE} = \frac{1}{8\pi} \sum_{l''} \frac{(2l'' + 1)^2}{(2l'' + 1)} \left( \begin{array}{ccc} l & l' & l'' \\ s & 0 & s \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ s & 0 & s \end{array} \right) ,
\]

(18)

\[
G_{ll}^{EB} = \frac{1}{8\pi} \sum_{l''} \frac{(2l'' + 1)^2}{(2l'' + 1)} \left( \begin{array}{ccc} l & l' & l'' \\ s & 0 & s \end{array} \right) \left( \begin{array}{ccc} l & l' & l'' \\ s' & 0 & s' \end{array} \right) ,
\]

(19)

It is interesting to note here that instead of $C_{ij}(k_1, k_2)$ if we study $\Sigma(k_1, k_2) = \sum_{l}(2l + 1)C_{ij}(k_1, k_2)$ they will have exactly similar mixing properties as the ordinary 2D fields, modulo the re-mapping of the radial harmonics, as the usual 3D power spectrum $C_l$ when a mask is applied, i.e. $C^{\ast\ast}(k_1, k_2) = \sum_{l} M_{ll} \Sigma_{l}(2l + 1)k_1, k_2)$, where $G_{ll} = \frac{2l'' + 1}{2l'' + 1} M_{ll}$. This property will be valid not just as the level of power spectrum but also for skew and kurt spectra as well as for multi-spectra of arbitrary order.

We have plotted the projected power spectrum for convergence $C_{i}^{\ast\ast}$ in Fig. 1 as a function of $l$ (left-hand panel). Two different redshifts were considered $z = 0.5$ and $z = 1.0$. The $\Lambda$CDM background cosmology that we will be using throughout this paper is characterized by the following set of parameters: $\Omega_m = 0.3, \Omega_{\Lambda} = 0.7, \Gamma = 0.21, h = 0.7$ and $\sigma_8 = 0.90$. The power spectra associated with other spinorial fields (shear and flexions) are also plotted (right-hand panel). The 3D power spectrum $C_{ij}^{\ast\ast}$ is plotted in Fig. 4 for the same background cosmology. We plot $C_{ij}^{\ast\ast}(k, k)$ for three different choices of $k$ values as a function of $l$ (left-hand panel) as well as $C_{ij}^{\ast\ast}(k, k)$ for three selection of $l$ values as a function of the radial wavenumber $k$. 

Figure 1. Left-hand panel compares the power spectrum $C_l$ for the convergence $\kappa$ for two different source redshift $z_s = 1$ and $z_s = 0.5$. Right-hand panel shows the power spectrum $C_l$ for the convergence $\kappa$, electric part of the shear $E$, flexions $F$ and $G$ as a function of $l$. The flexion power spectra $C_E^E, C_E^F$ are normalized by $\ell$ for display. The cosmology assumed is a cold dark matter ($\Lambda$CDM) and all sources are assumed to be at the same redshift of $z_s = 1$. The $\Lambda$CDM background cosmology that we have considered is described by the following set of parameters: $\Omega_m = 0.3, \Omega_{\Lambda} = 0.7, \Gamma = 0.21, h = 0.7$ and $\sigma_8 = 0.90$. 

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3.2 Bispectrum

The power spectrum carries the bulk of the information in any cosmological observations. However, often a set of degenerate cosmological scenarios can lead to a very similar power spectrum. Analysing higher order correlation functions can lift this degeneracy to some extent. The non-Gaussianity used can be either due to primordial or secondary effects, and in the case of weak lensing the main source of non-Gaussianity comes from gravitational instability. Note that a non-zero bispectrum signifies the lowest-order departure from Gaussianity, and its detection is generally easier than higher order multi-spectra.

To make contact with the observables we use the fact that the convergence can be related directly to the 3D density field. We will start by linking the 3D convergence bispectrum $B$ and the 3D density bispectrum expressed in harmonic coordinates. In the next section, we will express the bispectrum in spherical coordinate in terms of the bispectrum in rectangular coordinates and use some well-motivated approximations to simplify the results.

Statistical isotropy requires that

$$\langle \kappa_{lm}(k_1; r) \kappa_{lm}(k_2; r) \kappa_{lm}(k_3; r) \rangle = \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \right) B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k; r)$$

and using equation (8) we can write

$$B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k; r) = A^3 \mathcal{L}_1 \mathcal{L}_2 \mathcal{L}_3 \left( \frac{2k_1}{\pi c^2} \right) \left( \frac{2k_2}{\pi c^2} \right) \left( \frac{2k_3}{\pi c^2} \right) \int_0^\infty dk_i \int_0^\infty dr_i j_i(k_i r_i) \int_0^\infty dr_j j_j(k_j r_j) \int_0^\infty dr_k j_k(k_k r_k) \frac{dr_i}{a(r_i)} F_k(r_i, r_j) \times \int_0^\infty dk_i \int_0^\infty dr_i j_i(k_i r_i) j_i(k_i r_i) \int_0^\infty dr_i j_i(k_i r_i),$$

$$\mathcal{L}_1 = l(l+1) = l_i^2.$$

The bispectrum $B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k; r)$ can now be expressed in terms of the underlying matter bispectrum $B^\delta$. The above relation mixes modes only in the radial directions $r$, and on the surface of the sky there is no mixing of angular harmonics if there is no sky mask. While expressing the density harmonics in terms of the 3D potential harmonics, we pick up additional scalefactor $a(r)$ and wavenumber $k_i$ dependence in the denominator.

We have so far ignored the presence of noise. Indeed, because of the limited number of galaxies available it may not be possible to probe individual modes of the bispectrum at high signal-to-noise ratio. In later sections we will be able to address issues related to optimum combinations of individual modes which may be better suited for observational studies.

The convergence bispectrum can be written in terms of the density bispectrum as follows, using the Limber approximation to simplify the results:

$$B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k; r) = H_1 H_2 H_3 \int_0^\infty r_i^2 dr_i j_i(k_i r_i) \int_0^\infty r_j^2 dr_j j_j(k_j r_j) \int_0^\infty r_k^2 dr_k j_k(k_k r_k) \mathcal{L}_3^{(3)}(r_1, r_2, r_3);$$

$$H_i = A \frac{k_i}{c^2} \frac{\sqrt{\pi}}{3^2};$$

$$\mathcal{L}_3^{(3)}(r_1, r_2, r_3) = S_{l_1l_2l_3} b_{l_1l_2l_3} = S_{l_1l_2l_3} \int_0^{r_{\text{min}}} r^2 dr B^\delta \left( \frac{l_1}{r}, \frac{l_2}{r}, \frac{l_3}{r}, r, r, r \right) R_1(r) R_2(r) R_3(r);$$

$$S_{l_1l_2l_3} = \left( \frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi} \right)^{1/2} \left( \frac{l_1}{r}, \frac{l_2}{r}, \frac{l_3}{r} \right).$$

To derive this result we have used the extended Limber approximation equation (A2) to simplify the $k_i^3$ integrals. The integral here extends to the overlapping region, i.e. $r_{\text{min}} = \min(r_1, r_2, r_3)$. In particular we can use the gravity-induced bispectrum here, or include others, such as a primordial bispectrum.

It is possible to further simplify the above expression using equation (A1):

$$B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k_1, k_2, k_3) = [A e^{-2}]^3 \left( \frac{1}{k_1 k_2 k_3} \right) \left( \frac{2l_1 + 1}{2} \right)^{1/2} \left( \frac{2l_2 + 1}{2} \right)^{1/2} \left( \frac{2l_3 + 1}{2} \right)^{1/2} S_{l_1l_2l_3} \mathcal{L}_3^{(3)} \left( \frac{2l_1 + 1}{2}, \frac{2l_2 + 1}{2}, \frac{2l_3 + 1}{2} \right).$$

We will use a generic hierarchical ansatz to model the matter correlation hierarchy. Such ansatz constructs multi-point correlation functions from the products of lower order correlation functions. In the Fourier domain, this will lead to the construction of multi-spectra for products of ordinary power spectra. Such models have been tested against simulations and are routinely used both for projected galaxy surveys and for 2D weak lensing surveys. At the level of bispectrum we have $B(k_1, k_2, k_3) = Q_3[P(k_1, r)P(k_2, r) + P(k_2, r)P(k_3, r) + P(k_3, r)P(k_3, r)]$. More detailed modelling will make $Q_3$ a function of the wave vector triplet $(k_1, k_2, k_3)$, e.g. in the halo model (Cooray & Seth 2002) or in Hyper Extended Perturbation Theory (Scoccimarro et al. 1998). While the expression above is for the convergence field $\kappa$, it can be used to construct the other bispectra involving shear or flexion:

$$B^{\gamma \gamma \gamma}_{l_1l_2l_3}(k_1) = F^{\gamma \gamma \gamma}_{l_1} B^{\gamma \kappa \kappa}_{l_1l_2l_3}(k_1); \quad B^{\gamma \gamma \gamma}_{l_1l_2l_3}(k_1) = F^{\gamma \gamma \gamma}_{l_1} F^{\gamma \gamma \gamma}_{l_2} B^{\kappa \kappa \kappa}_{l_3l_3l_3}(k_1); \quad B^{\gamma \gamma \gamma}_{l_1l_2l_3}(k_1) = F^{\gamma \gamma \gamma}_{l_1} F^{\gamma \gamma \gamma}_{l_2} F^{\gamma \gamma \gamma}_{l_3} B^{\kappa \kappa \kappa}_{l_1l_2l_3}(k_1).$$

Results involving flexion can be constructed replacing the form factor $F^{\gamma \gamma \gamma}$'s with the ones for the flexions, i.e. $F^{\gamma \gamma \gamma}_{l_1} F^{\gamma \gamma \gamma}_{l_2} F^{\gamma \gamma \gamma}_{l_3}$ defined accordingly.

The radial dependence of convergence, shear or flexion harmonics is the same.

Mode coupling is introduced by the presence of an observational mask. It is not possible to deconvolve the effect of a mask while analysing the bispectrum from a realistic survey, as the inversion is typically unstable. However, later we will introduce a power spectrum...
associated with the bispectrum (the skew spectrum), which can be computed from realistic data in the presence of mask and deconvolution can be done in a way very similar to the estimation of power spectrum discussed previously.

If we assume the intrinsic shear and flexion of source galaxies to be distributed according to a Gaussian distribution, then they do not contribute to the estimated bispectrum, but this is, of course, an assumption. However even in this case the scatter or variance in estimation does get a contribution from such a source of noise.

It is important to realize the results derived above are generic. They do not depend on the specific model used as an example (hierarchical ansatz). If we replace the underlying bispectrum with a primordial bispectrum of a specific type (e.g. local), we can still use the formalism developed here to compute various relevant statistics, we will introduce later, e.g. the skew spectrum. Later we will also introduce an optimized estimator for the skew spectrum. This estimator is not only optimized to detect any specific type of non-Gaussianity but it can also give an estimate of leakage from a specific source of non-Gaussianity (e.g. gravity-induced) while estimating another (primordial).

3.3 Trispectrum

The trispectrum or, alternatively, the four-point correlation function can provide an important sanity check (Kamionkowski, Smith & Heavens 2010) to validate lower order detection of non-Gaussianity based solely on the bispectrum. The trispectrum, being a four-point correlation function, is generally harder to probe compared to the bispectrum. However, in cases where the bispectrum vanishes due to symmetry considerations, the trispectrum is the lowest probe to study gravity-induced non-Gaussianity. In addition, while the study of trispectrum is interesting in itself it is also important in the proper characterization of the error in the power spectrum. As in the case of bispectrum we will start by modelling the bispectrum of the convergence field which can then be generalized to model trispectra associated with various spinorial fields. These results will eventually be used to model two different power spectra associated with the trispectrum (the kurt spectra).

The convergence trispectrum \( T^{(l)}_{ijkl}(L, k; r_i) \) is the four-point correlation function in the harmonic domain and can be expressed as

\[
T^{(l)}_{ijkl}(L, k; r_i) = \sum_{LM} (-1)^M \frac{1}{L^4} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} \frac{l_3}{m_3} \frac{l_4}{m_4} \right) T^{(l)}_{ijkl}(L, k; r_i). \tag{25}
\]

The vectors \( l_1, l_2, l_3, l_4 \) represent the sides of a quadrilateral and \( L \) is the length of the diagonal. The matrices as before are the Wigner 3j symbols. The vectors are only non-zero when they satisfy several conditions, which are \( |l_1 - l_2| \leq L \leq l_1 + l_2, |l_3 - l_4| \leq L \leq l_3 + l_4; l_1 + l_2 + L = \text{even}, l_3 + l_4 + L = \text{even} \) and \( m_1 + m_2 = M \) as well as \( m_3 + m_4 = -M \). In our notation for the trispectrum, \( T^{(l)}_{ijkl}(k; r_i; L) \), the indices \( k; r_i \) encode their dependence on various Fourier modes of the density harmonics in the radial direction, used in their construction. No summation will be assumed over these variables unless explicitly specified. To model the trispectrum we will relate it to the underlying trispectra of the density distribution \( T^{(l)}_{ijkl}(k; r_i) \).

\[
T^{(l)}_{ijkl}(k; r_i) = A^4 L^3 \frac{1}{2k^2 \pi^2} \int_i L^2 \frac{d\delta(k_1)}{k_1} \int_i \frac{d\delta(k_2)}{k_2} \int_i \frac{d\delta(k_3)}{k_3} \int_i \frac{d\delta(k_4)}{k_4} F_k(r_1, r_2, r_3, r_4) \tag{26}
\]

The above expression is a direct consequence of equation (4). To make any further progress we need to consider a specific form for the matter trispectrum. There are two generic prescriptions for treating a gravity-induced trispectrum. The halo model is one, and has been developed over the last several years and is very popular for modelling the correlation hierarchy of the underlying mass distribution. Another is that perturbative descriptions can also provide a reasonable description of the onset of non-linearity at comparatively larger length-scales. The hierarchical ansatz on the other hand describes the matter correlation hierarchy in the highly non-linear regime on smaller length-scales. It builds up the higher order correlation hierarchy from the two-point correlation function. All possible diagrams that connect points at which the correlation function is being constructed are considered. These diagrams are attributed various amplitudes according to their topology. Different diagrams with the same topologies are associated with same amplitude. At the level of the four-point correlation function, there are only two different topologies star and snake. We will denote the corresponding amplitudes by \( R_s \) and \( R_n \) and consider each of these contributions separately next.

3.3.1 Star diagrams

The star diagrams are easier to handle because topologically they consist of a single vertex and lack any internal momentum that needs to be integrated out. In generic hierarchical scenarios a star diagram appears at each order in the hierarchy. In the case of projected 2D analysis, it has been found that replacing all diagrams with the same number of star diagrams often is sufficient to reproduce all one-point statistical features of convergence maps (Munshi, Valageas & Barber 2004; Barber, Munshi & Valageas 2004; Valageas et al. 2005). However, the results presented here are generic and includes both contributions.

The derivation of the star contribution to the convergence trispectrum follows a similar technique as the bispectrum. The first step is to express the density convergence in terms of the triplets of matter power spectra. Next, using the expression equation (4) we can relate the star
contribution of convergence trispectra to that of underlying mass distribution:

\[
\delta T_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{star}} = \frac{2\pi^2}{L} k_1 k_2 k_3 k_4 \int_0^\infty dr r_1 j_1(k_1 r_1) \cdots \int_0^\infty dr_4 j_4(k_4 r_4) J_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{star}},
\]

where

\[
J_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{star}} = R_0 S_{i,j,k} S_{i,j,k} \int_0^\infty r^2 dr j_1(r) \cdots j_4(r) \{ P(k_1, r) P(k_2, r) P(k_3, r) + \text{cyc. perm.} \}.
\]

The above equation is for the stellar contribution to the trispectrum of the underlying density distribution. The additional three terms can be recovered by cyclic permutation of the \( k_i \) variables. We can use this result next to express the convergence trispectrum. The simplification relies on the use of the extended Limber approximation to simplify the \( k_i \) integrals. Hierarchical ansatz and Limber approximations are both known to be valid at small scales which justifies their combined use (Munshi et al. 2011b).

\[
\delta T_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{star}} = H_i H_j H_k \int_0^\infty r_1^2 dr j_1(k_1 r_1) \cdots \int_0^\infty r_4^2 dr j_4(k_4 r_4) \mathcal{I}_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{star}}
\]

where

\[
\mathcal{I}_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{star}} = R_0 S_{i,j,k} S_{i,j,k} \int_0^\infty r^2 dr \mathcal{R}_1(r) \cdots \mathcal{R}_4(r) \left\{ P \left( \frac{l_1}{r}, r \right) P \left( \frac{l_2}{r}, r \right) P \left( \frac{l_3}{r}, r \right) + \text{cyc. perm.} \right\}.
\]

In generic hierarchical scenarios the trispectrum is a cubic combination of underlying matter power spectra, just as the convergence bispectrum is a quadratic function of matter power spectra. We will focus on a specific model, the hierarchical ansatz, to model the underlying density distribution. However, as it was pointed out similar techniques can also be applied in the context of more elaborate halo model prescription.

### 3.3.2 Snake diagrams

The analysis of the snake diagrams is more difficult than that of the star diagrams. This is related to the fact that while higher order star diagrams at each order are straightforward generalizations of the lower order star diagrams, the snake diagrams are however constructed using two different lower order star diagrams. The following expression was derived in Munshi et al. (2011b) which relates the star contribution to convergence trispectra. In addition to various form factors, the following expression depends on the cubic product of the underlying matter power spectra. We will focus on a specific model, the hierarchical ansatz, to model the underlying density distribution. However, as it was pointed out similar techniques can also be applied in the context of more elaborate halo model prescription.

\[
Q_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{snake}} = H_i H_j H_k \int_0^\infty r_1^2 dr j_1(k_1 r_1) \cdots \int_0^\infty r_4^2 dr j_4(k_4 r_4) \mathcal{I}_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{snake}}
\]

\[
\mathcal{I}_{i,j,k}^{(4)}(r_1, r_2, r_3, r_4)_{\text{snake}} = R_0 S_{i,j,k} S_{i,j,k} \int_0^\infty r^2 dr \mathcal{R}_1(r) \cdots \mathcal{R}_4(r) \left\{ P \left( \frac{l_1}{r}, r \right) P \left( \frac{l_2}{r}, r \right) P \left( \frac{l_3}{r}, r \right) + \text{cyc. perm.} \right\}.
\]

The cyc. perm. here represents a total of three other terms. These terms can be obtained by rearranging \( l_i \), \( (l_1 \to l_2) \), \( (l_1 \to l_3) \) and \( (l_1 \to l_2, l_3 \to l_1) \). The other terms can be obtained by considering two additional pairings by considering the exchanges \( (l_2 \to l_1) \) and \( (l_1 \to l_3) \). These will lead us to \( Q_{i,j,k}^{(12)} \) and \( Q_{i,j,k}^{(12)} \). The total number of snake terms considering three distinct pairings and permutations within each pairings is 12. The snake contribution to trispectrum can be written as

\[
T_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{snake}} = Q_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{snake}} + (2L + 1) \sum_{L'} (-1)^{L'} \left\{ \begin{array}{ccc} l_1 & l_2 & L \\ l_3 & l_4 & L' \end{array} \right\} Q_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{snake}}
\]

\[
+ (2L + 1) \sum_{L''} (-1)^{L''} \left\{ \begin{array}{ccc} l_1 & l_2 & L \\ l_3 & l_4 & L'' \end{array} \right\} Q_{i,j,k}^{(12)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right)_{\text{snake}}.
\]

The matrices in curly braces are the 6j symbols (Edmonds 1968). The differences in snake and star contributions are also apparent in various choices of permutations of \( (l_1, l_2, l_3, l_4, L) \) associated with individual terms.

The derivations outlined for both bispectrum and trispectrum are quite generic and depend only on the use of extended Limber approximation, and the analysis can be generalized to higher order multi-spectra.

We will use these results to construct the power spectra associated with the trispectrum, or kurt spectra. Construction of trispectra for generic spin-weight functions can be achieved by using the form factors in a manner similar to the one adopted for the bispectrum equation (24).

\[
T_{i,j,k}^{(4)} \left( \frac{L}{r}, \frac{l_i}{r}; r_i \right) = F_{i,j,k}^{(4)} F_{i,j,k}^{(4)} F_{i,j,k}^{(4)} T_{i,j,k}^{(12)}(L, k_i).
\]

The two different kurt spectra that we will construct next will provide independent probes of the underlying mass trispectra, and can play a valuable role in checking any cross-contamination from systematics.
4 POWER SPECTRA ASSOCIATED WITH MULTI-SPECTRA

The multi-spectrum may encode a great deal of information, but there is certain amount of degeneracy involved in it. Owing to the low signal-to-noise ratio associated with the estimation of multi-spectra, it is impossible to estimate multi-spectra mode by mode. Estimation of multi-spectra is also hampered by their complicated response to the survey mask and complicated noise characteristics. Recent studies have shown that degenerate sets of power spectra can be constructed from multi-spectra at a given order. These compress some of the available information in the multi-spectra and can be computed from the observational data with relatively ease. We will construct the power spectra associated with bispectrum and trispectrum in this section. These power spectra were studied in some detail in Munshi et al. (2011c) in projection (2D). We generalize these results here to three dimensions. First we will obtain generic results, applicable irrespective of detailed analytical modelling of underlying multi-spectra. Next, we will further specialize these results for the models outlined above, and use extended Limber approximation to simplify the generic results.

4.1 skew spectrum

4.1.1 All-sky results

We will start by expanding the product of two generic spinorial fields with associated spin indices $s, s' \{\Gamma_\ell, \Gamma'_\ell\}(r)$ in three dimensions in terms of their individual harmonics. The product of two spinorial fields with spins $s$ and $s'$ is a spinorial field of spin $s + s'$. The harmonic expansion therefore will have to be in terms of spin-weighted spherical harmonics with spin index $s + s'$. In addition to expanding on the surface of the celestial sphere, we will also consider the expansion in the radial direction using spherical Bessel functions.

\[
\langle \Gamma_\ell(r), \Gamma'_\ell(r) \rangle_{lm}(k) = \frac{\sqrt{2}}{\pi} \int_0^\infty kr^2 \, dk j_l(kr) \int d^2\Omega, \Gamma_\ell(r), \Gamma'_\ell(r) \rangle_{s+s'} Y_{lm}^*(\hat{\Omega})
\]

\[
= \left( \frac{2}{\pi} \right)^{3/2} \int_0^\infty r^2 dr k j_l(kr) \sum_{l_m} \int_0^\infty k_2 d k_2 j_l(k_2r) \int_0^\infty \Gamma_{l_1 m_1}^l(k_1) \int_0^\infty \Gamma_{l_2 m_2}^l(k_2) \int_0^\infty \Gamma_{l_3 m_3}^l(k_3) \int_0^\infty \int_0^\infty \int_0^\infty \Gamma_{l_1 m_1}(k_1) \Gamma_{l_2 m_2}(k_2) \Gamma_{l_3 m_3}(k_3) \times J_{l_1 l_2 l_3}(l, l, l; m, m, m) \times J_{l_1 l_2 l_3}(l, l, l; m, m, m)
\]

\[
= \left( \frac{2}{\pi} \right)^{3/2} \sum_{l_m} \int_0^\infty dr kr^2 dr j_l(kr) \int_0^\infty k_1 d k_1 j_l(k_1r) \int_0^\infty \Gamma_{l_1 m_1}^l(k_1) \int_0^\infty k_2 d k_2 j_l(k_2r) \int_0^\infty \Gamma_{l_2 m_2}^l(k_2) \int_0^\infty k_3 d k_3 j_l(k_3r) \int_0^\infty \Gamma_{l_3 m_3}^l(k_3) \times J_{l_1 l_2 l_3}(l, l, l; m, m, m) \times J_{l_1 l_2 l_3}(l, l, l; m, m, m)
\]

We have used the Gaunt integral to express integrals involving spherical harmonics in terms of the Wigner 3j symbols. To construct the skew spectrum, next we contract the multi-pole \(\{\Gamma_\ell(r), \Gamma'_\ell(r)\}\) with the multi-pole equation (22). The extended Limber approximation, equation (A2), integrals to 1D Dirac delta functions. These delta functions are used to reduce the integrals involving \(k\) in three dimensions in

\[
\frac{c_{\ell''\ell''\ell''}^{\Gamma_\ell \Gamma'_\ell}(k, k')}{\Omega^2} = \frac{1}{\Omega^2} \sum_m \langle \Gamma_\ell, \Gamma'_\ell, \Gamma''_\ell \rangle_{lm}(k) \langle \Gamma''_\ell \rangle_{lm}(k')
\]

The bispectrum is defined by a triangular configuration in multi-pole space with lengths of side \((l_1, l_2, l_3)\). The power spectrum constructed above will essentially capture information, through a summation, with all triangular configurations with one of these sides kept fixed at length $l$. If we now use the following expression for the 3D bispectrum $B_{l_1 l_2 l_3}^{\Gamma_\ell \Gamma'_\ell}(k_1, k_2, k_3)$ introduced earlier,

\[
B_{l_1 l_2 l_3}^{\Gamma_\ell \Gamma'_\ell}(k_1, k_2, k_3; r) = \sum_{m_1, m_2, m_3} \langle \Gamma_{l_1 m_1}^l(k_1 r), \Gamma_{l_2 m_2}^l(k_2 r), \Gamma_{l_3 m_3}^l(k_3 r) \rangle \chi_{m_1 m_2 m_3}
\]

we can express the two-to-one power spectrum as:

\[
c_{\ell''\ell''\ell''}^{\Gamma_\ell \Gamma'_\ell}(k, k') = \left( \frac{2}{\pi} \right)^{3/2} \int_0^\infty r^2 dr j_l(kr) \sum_{l'_{12}} \int_0^\infty k_1 d k_1 j_l(k_1r) \int_0^\infty k_2 d k_2 j_l(k_2r) \int_0^\infty k_3 d k_3 j_l(k_3r) \times J_{l_1 l_2 l_3}(l, l, l; m, m, m)
\]

Although the above relation is general and contains all the information regarding the evaluation of the skew spectrum for arbitrary spin-weight functions, the presence of multi-dimensional integrals makes it difficult to implement computationally.

We will simplify the results by using the extended Limber approximation to reduce the integrals involving $k_1$ and $k_2$. First we express the $B$ in terms of the underlying matter bispectrum $B$ using the expression equation (22). The extended Limber approximation, equation (A2), collapses the $k$ integrals to 1D Dirac delta functions. These delta functions are next used to reduce the $r_1$ and $r_2$ integrals. These simplifications are known to be valid at high $l$. Finally we are left with two integrals involving $r$ and $r_3$ which can be further simplified by equation (A3). The final expression contains only a 1D integral and directly relates the skew spectrum $C_{\ell''\ell''\ell''}^{\Gamma_\ell \Gamma'_\ell}(k, k')$ to the matter bispectrum:

\[
c_{\ell''\ell''\ell''}^{\Gamma_\ell \Gamma'_\ell}(k, k') = \sum_{l_{12}} \left[ \frac{2}{\ell_1 + 1} \frac{2}{\ell_2 + 1} \frac{2}{\ell_3 + 1} \right]^2 J_{l_{12}} \left[ \begin{array}{ccc} 2\ell_1 + 1 & 2\ell_2 + 1 & 2\ell_3 + 1 \\ 2\ell_1 + 1 & 2\ell_2 + 1 & 2\ell_3 + 1 \\ k & k & k \end{array} \right] B_{l_1 l_2 l_3}^{\Gamma_\ell \Gamma'_\ell} \left[ \begin{array}{ccc} 2\ell_1 + 1 & 2\ell_2 + 1 & 2\ell_3 + 1 \\ 2\ell_1 + 1 & 2\ell_2 + 1 & 2\ell_3 + 1 \\ k & k & k \end{array} \right] \chi_{m_1 m_2 m_3}
\]
The skew spectrum \( C_{l}^{\alpha,\beta,\gamma} \) defined in equation (12) for the convergence \( \kappa \) is plotted as a function of \( l \) for two different source redshifts \( z_s = 1 \) (upper curve) and \( z_s = 0.5 \) (lower curve). A hierarchical form for the bispectrum was assumed and the hierarchical amplitude is set to unity. As \( \Lambda CDM \) background cosmology is assumed. We have not incorporated any smoothing window – calculations are done by introducing a sharp cut-off at \( l_{\max} = 2000 \).

Finally, if we use the expression for the bispectrum derived using the same approximation, we can write

\[
C_I^{\beta\gamma}(k, k') = \left[ A e^{-2} \right] \left[ \frac{2}{2l' + 1} \right] \left[ \frac{2l' + 1}{2k'} \right]^2 \sum_{l'} \int_{l'^3} \left( \begin{array}{ccc} l_1 & l_2 & l \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} l_1 & l_2 & l \\ s & s' & -(s + s') \end{array} \right) \left[ \frac{2l' + 1}{2k'} \right] \left[ \frac{2l' + 1}{2k'} \right] \left[ \frac{2l' + 1}{2k'} \right] I_{i,j,l}^{(3)} \int_0^\infty r^2 dr B(k) \left[ \begin{array}{ccc} l_1 & l_2 & l \\ r & r & r \end{array} \right] R_l(r) R_{l'}(r) R_{l'}(r) ; \quad R_l(r) = \frac{F_k(r, r)}{a(r)}.
\]

The bispectrum \( B(k, k', k'' \rangle \) for convergence \( C_I^{\alpha,\beta,\gamma} \) in Fig. 3. Two different redshifts were considered \( z_s = 0.5 \) and \( z_s = 1.0 \). The \( \Lambda CDM \) background cosmology that we have considered are described by the following set of parameters: \( \Omega_M = 0.3, \Omega_L = 0.7, \Gamma = 0.21, h = 0.7 \) and \( \sigma_8 = 0.90 \). We have taken a hierarchical form for the matter bispectrum and the amplitude \( Q_I \) is set to unity. Changing the amplitude \( Q_I \) only changes the overall normalization of the curve. Individual values of \( C_I^{\beta\gamma}(k, k') \) at a given \( l \) depend on the modelling of the bispectrum for the entire range of \( l \) values being considered. Accurate modelling of the window function will therefore be an important ingredient in such calculation. For our study we have considered a sharp cut-off in the multi-pole space at \( l_{\max} = 2000 \). The 3D power spectrum is plotted in Fig. 4. The 3D skew spectrum \( C_I^{\alpha,\beta,\gamma} \) is plotted in Fig. 5 for the same background cosmology. We plot \( C_I^{\alpha,\beta,\gamma}(k, k) \) for three different choice of \( k \) values as a function of \( l \) (left-hand panel) as well as \( C_I^{\alpha,\beta,\gamma}(k, k) \) for three selection of \( l \) values as a function of the radial wavenumber \( k \).

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Figure 4. The left-hand panel shows 3D power spectrum $l(l+1)k^2C_{ll}^\epsilon(k, k)$ as a function of $k$ for three different values of $l$, whereas the right-hand panel shows $l(l+1)k^2C_{ll}^\epsilon(k, k)$ as a function of $l$ for a fixed value of $k$. A $\Lambda$CDM background cosmology was assumed with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$, $\sigma_8 = 0.8$, $n_s = 1.0$, $w = -1$; we use an Eisenstein & Hu (1998) linear power spectrum and the Smith et al. (2003) non-linear correction. The sharp cut-off in these plots reflects survey depth through the Bessel inequality. Here $k$ is displayed in $h\text{Mpc}^{-1}$.

Figure 5. The right-hand panel shows 3D power spectrum $l(l+1)k^2C_{ll}^{\epsilon,\epsilon}(k, k)$ as a function of $k$ for three different values of $l$, whereas the left-hand panel shows $l(l+1)k^2C_{ll}^{\epsilon,\epsilon}(k, k)$ as a function of $l$ for a fixed value of $k$. Three different $k$ values were chosen as depicted. A $\Lambda$CDM background cosmology was assumed with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.7$, $\sigma_8 = 0.8$, $n_s = 1.0$; we use an Eisenstein & Hu (1998) linear power spectrum and the Smith et al. (2003) non-linear correction. A hierarchical ansatz was assumed for the bispectrum with the amplitude fixed at $Q = Q_3 = 1$ (see the text for more details). Here, the wave vector $k$ is displayed in units of $h\text{Mpc}^{-1}$.

4.1.2 The effect of a sky mask

A mask on the sky is defined through a generic function $w(\Omega) = \sum_{lm} w_{lm} Y_{lm}(\Omega)$ on the surface of the sky. The mask can be a simple zero and one step function signifying masked and observed part of the sky or it can also be a more complex apodizing function with specific weights attached to the different parts of the sky. We will compute the skew spectrum in the presence of mask and express it in terms of the skew spectrum in the absence of any mask. This will allow us to eventually design an estimator which can estimate the unbiased skew spectrum from the real data in the presence of a mask and noise. We will consider Gaussian noise, which means that the estimation of bispectrum is not affected as the noise has vanishing skew spectrum; however the scatter associated with the estimator will change in the presence of noise. We will denote the masked power spectrum by $\tilde{C}_l$. We will see that convolved or masked skew spectrum is a linear sum of individual all-sky $C_l$s. This is related to the fact that the use of mask introduces mode–mode correlation. A matrix inversion with suitable binning can produce the all-sky $C_l$s from the masked $\tilde{C}_l$s.
Repeated application of Gaunt’s integral allows us to write the product harmonics in terms of the individual harmonics:

\[
\hat{I}_2(r) = \frac{2}{r} \sum_{l_1, l_2, l_3} W_{l_1, l_2, l_3} \Gamma_{l_1 l_2 l_3}(r) \]

The Fourier–Bessel decomposition of the quadric field along the radial direction can be used to relate the 3D harmonics of the product field with that of individual constituent fields. This will eventually allow us to express the 3D skew spectrum in terms of the 3D bispectrum. The 3D skew spectrum presented here is a generalization of our previous work and incorporates full 3D information. This can be viewed also as a natural generalization of the 3D power spectrum presented in Castro et al. (2005). Higher order counterparts at the level of trispectrum will be presented in the next section.

For the 3D harmonic decomposition \( \hat{I}_2(r) \) of individual fields we use the expression derived previously in equation (15). Using these expressions for the 3D power spectrum in the presence of a mask from equations (39) and (15) and in equation (40) we can express the skew spectrum constructed from the masked multi-poles or the pseudo-skew spectrum in terms of a coupling matrix and the all-sky skew spectrum in three dimensions and a mode-coupling matrix. The mode–mode coupling matrix involves only the mask power spectra \( W_{l_1} = \frac{1}{4\pi} \sum_{l_2} W_{l_1, l_2, l_3} \) and depends on the spinorial indices of the respective 3D fields:

\[
G_{l_1 l_2 l_3} = \frac{1}{4\pi} \sum_{i} \left( \frac{2l_1 + 1}{2l_2 + 1} \right) \left( \frac{2l_3 + 1}{2l_2 + 1} \right) \Gamma_{l_1 l_2 l_3}(r) \]

The deconvolution process to recover the all-sky skew spectrum involves inversion of the coupling matrix \( G \) which depends not only on indices but also on spinorial indices associated with the fields. In case of small-sky coverage which might be the case for present generation of surveys, the inversion of the matrix will typically involve binning to avoid any possible numerically singular matrices.

Here, it is worth pointing out that the various spinorial fields that we can use to construct individual skew spectrum do probe the same underlying matter bispectrum. This can be used as a helpful diagnostic to probe possible spurious effects of mask and noise. The power spectrum \( \hat{C}_{l_2}(r) \) reported here is an extension of similar power spectrum introduced in Munshi & Heavens (2010) for CMB studies. Later, it was used in Munshi et al. (2011b) to probe the convergence skew spectrum and was also generalized for projected spin-shear skew spectrum in Munshi et al. (2011c). In this study, we present skew spectrum for spinorial fields using a complete 3D analysis. The power spectrum is specified by multi-pole indices on the surface of the celestial sphere as well as radial harmonics along the line-of-sight direction. Results are generic for fields with arbitrary spin and can be used to probe shear, flexion or convergence maps. Similar statistics in the coordinate space have been reported before. Bernardeau, van Waerbeke & Mellier (2003b) studied \( \langle \gamma^2(\hat{\Omega}) \gamma(\hat{\Omega}) \rangle \) which directly deals with shear maps as opposed to convergence maps. These statistics along with a similar but simpler version which uses \( \langle \gamma^2(\hat{\Omega}) \rangle \) were also studied. The perturbation theory was employed to model the underlying mass distribution, and it used a flat-sky approximation to simplify their calculations. A complementary statistic \( \langle \gamma(\hat{\Omega}) \rangle \) was also considered which relies on more detailed modelling of the bispectrum. These statistics were used by Bernardeau et al. (2002b) later to detect non-Gaussianity from the VIRMOS-DESCART Lensing Survey.

Our results presented here deal with power spectrum associated with the higher order multi-spectra, are derived using generic all-sky treatment and can also handle decomposition into electric and magnetic components in a much more straightforward manner. The results
presented here are not only applicable to shear or convergence but are also applicable for higher order spinorials such as Flexions. To increase the signal-to-noise ratio of the estimates, it is customary to often sum all possible mode of $C_i^{l_1,l_2} (k, k)$ into a single number which is called skewness.

$$S_l^{l_1,l_2} (k, k) = \sum_i (2i + 1) C_i^{l_1,l_2} (k, k). \tag{43}$$

For a concrete expression for $S_l^{l_1,l_2} (k, k)$ we need to replace $C_i^{l_1,l_2} (k, k)$ by the expression derived in equation (37) or equation (35).

To make connection with the real-space statistics we can use the following relation:

$$C_i^{l_1,l_2} (k, k) = \frac{2}{\pi} \int dk \int dk' \int dr \int r^2 \delta (k - k') D_i^{l_1,l_2} (r, r'), \tag{44}$$

and a similar relation can be derived for the skewness $S_l^{l_1,l_2} (r, r')$ and $S_k^{l_1,l_2} (k, k)$ defined above. One point statistics such as $S_3(k, k')$ only contain radial information as all spherical harmonics are already summed over.

If we make the approximation of replacing the spherical Bessel function with a delta-function form we can write

$$C_i^{l_1,l_2} (k, k) = \frac{2}{2l + 1} D_i^{l_1,l_2} \left( \frac{2l + 1}{2k^2} \right). \tag{45}$$

An equivalent expression is valid for $S_k^{l_1,l_2} (k, k')$. In both equations (44) and (45) we use the same notations to define the harmonic space $C_i^{l_1,l_2} (k, k')$ and real-space $C_i^{l_1,l_2} (r, r')$ power spectra. However, their functional dependence on their arguments is different. It is interesting to note that in Fourier domain the $C_i^{l_1,l_2} (k, k')$ probes specific length-scales.

4.2 The kurt spectrum

The four-point correlation function, or its harmonic counterpart the trispectrum, has been extensively studied in the literature. This contains the information about the non-Gaussianity beyond the lowest level (Hu 1999; Okamoto & Hu 2002).

For the case of weak lensing studies, the gravity-induced non-Gaussianity is clearly the main motivation. Studies in trispectrum analysis have also been pursued using various other probes, e.g. using 21-cm surveys (Cooray, Li & Melchiorri 2008) or more extensively in several CMB studies [see Bartolo et al. (2004) for a review]. However, these studies typically probe the trispectrum induced by primordial non-Gaussianity. It is important to note that at the level of four-point, the Gaussian part of the signal and the noise both carry a non-zero (unconnected) trispectrum. This degrades the signal-to-noise ratio for various estimators and clearly needs to be subtracted out before an unbiased comparison with the theoretical predictions can be made. It is obvious that the detection of the trispectrum from noisy data is far more non-trivial than the estimation of the bispectrum. We provide analytical expressions here mainly for completeness and to show that the generic results can be obtained on the basis of very simple hierarchical ansatz.

Previous studies have mainly concentrated on one-point estimators which collapse the data to a single number – known as the kurtosis. We extend studies involving kurtosis $\langle [\Gamma, \Gamma', \Gamma'', \Gamma'''](\hat{\Omega}, \hat{r}) \rangle$ to its two-point counterparts: $\langle [\Gamma, \Gamma', \Gamma''](\hat{\Omega}, \hat{r}) [\Gamma, \Gamma', \Gamma''](\hat{\Omega'}, \hat{r'}) \rangle$ and $\langle \Gamma \Gamma''(\hat{\Omega}, \hat{r}) \Gamma \Gamma''(\hat{\Omega'}, \hat{r'}) \rangle$. In practice however we will consider the Fourier transforms of these objects in three dimensions, the kurt spectra, which are the power spectra associated with the trispectrum, $C_i^{l_1,l_2} (k, k')$ and $C_i^{l_1,l_2} (r, r')$. As was the case for the skew spectrum, we do harmonic decomposition not only on the surface of the celestial sphere but also on the radial direction for the construction of $C_i^{l_1,l_2} (k, k')$.

4.2.1 Two-to-two kurt spectrum

The first of two kurt spectra $C_i^{l_1,l_2} (k, k')$ can be constructed from the harmonic transform of $\langle [\Gamma, \Gamma', \Gamma'', \Gamma'''](\hat{\Omega}, \hat{r}) \rangle$ of the quadratic combination of two arbitrary spin-weight fields discussed previously in the context of the skew spectrum equation (32). The resulting kurt spectrum is generic and can be defined for any given set of four spin-weight functions defined in three dimensions. Unlike the skew spectrum which is zero for Gaussian fields, the kurt spectra are non-zero even in the absence of any non-Gaussianity, which introduces additional complexity. The Gaussian contribution (also known as the disconnected piece) needs to be subtracted out before it is employed for the study of non-Gaussianity. The noise, often assumed Gaussian, can also be subtracted following the same technique. It will contribute only to the disconnected part. Later in this section, we will also consider the effect of a mask as we did for the skew spectrum.

We will use these results to derive expressions for $C_i^{l_1,l_2} (k, k')$ which leads to: $C_i^{l_1,l_2} (k, k') = \frac{1}{\pi^3} \sum_l [\Gamma_m \Gamma_l \Gamma_n \Gamma_r]\langle [\Gamma, \Gamma', \Gamma'', \Gamma'''](\hat{\Omega}, \hat{r}) \rangle$. The power spectra directly probe $T^{l_1,l_2}_i (l, k, k')$. It compresses all the available information in quadruplets of modes specified by $(l_1, l_2, l_3, l_4)$ to a power spectrum. The power spectra $C_i^{l_1,l_2} (k, k')$ and $C_i^{l_1,l_2} (r, r')$ differ in the way they associate weights to various modes and contain complementary information. The reduced trispectrum $T^{l_1,l_2}_i (k; L)$ is defined in terms of $\langle [\Gamma_m (\hat{k}) \Gamma_n (\hat{k}) \Gamma_r (\hat{k}) \Gamma_l (\hat{k})]\rangle$, as follows. We have added the radial distances $\hat{r}$ associated with each spherical harmonic in the argument with $L$, which specifies the diagonal formed by the quadruplet of four quantum numbers $l_1, l_2, l_3, l_4$, $L$ is $\sum \Lambda (1)^4 T^{l_1,l_2}_i (k; L) \langle \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \rangle$. The final expression depends on the spin indices of various fields as

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well as on a Kernel $F$ (defined below) which has angular harmonic numbers $l_i$ and radial wavenumbers $k, k'$ as its arguments:

$$C_i^{(r', r'^*)}(k, k') = \frac{1}{(2l + 1)^2} \sum_{ij} \int \left( l_1 \quad l_2 \quad l \right) \left( l_3 \quad l_4 \quad l \right) F^{(2, 2)}(l_i, l, k, k').$$

(46)

The kernel $F^{(2, 2)}(l_i, l, k, k')$ is defined in terms of the reduced trispectrum $T_{(i,j)}^{(ik)}(k_i; L)$.

$$F^{(2, 2)}(l_i, l, k, k') = \left( \frac{2}{\pi} \right)^3 \int_0^\infty dr \frac{r^4 k_i j_i(k r) j_i(k' r')}{r^2 k^2} \int_0^\infty \frac{dr' r'^4 k^2 j_i(k' r') j_i(k r)}{k'^2} \int_0^\infty dk_i j_i(k r) j_i(k' r') T_{(i,j)}^{(ik)}(k_i; L).$$

(47)

We will consider the two components of the trispectrum ‘snake’ and ‘stars’ separately for each of the two kurt spectra. If we follow the algebra, which is very similar to what was done previously to derive the skew spectrum we arrive at the following expression for the star component of the three-to-one kurt spectrum. The expression reduces to a 1D integral as we use the Limber approximation for simplification.

$$C_i^{(r', r'^*)}(k, k') = \frac{A^4}{c^4} \left[ \frac{2}{2l + 1} \right] \left[ \frac{2l + 1}{2k} \right]^2 \left[ \frac{2l + 1}{2k'} \right]^2 \frac{1}{(2l + 1)^2} \times \sum_{ij} \int \int \int \frac{dr}{r} \frac{r^2 j_i(k r) j_i(k' r')}{r^2 k^2} \int_0^\infty dk_i j_i(k r) j_i(k' r') \frac{1}{(2l + 1)^2}.$$
For the *snake* component, we need to replace the kernel \( I_{\text{stat}}^{(d)} \) with \( I_{\text{snake}}^{(d)} \), which also involves a single integration. The evaluation of *snake* terms are, however, difficult as in addition there will be terms that involve 6j symbols equation (30) because of their permutation symmetry. As was the case with skew spectrum, we can collapse these fourth-order two-point objects to reduce them to one-point numbers, the kurtosis, which will be a function of both radial harmonics \( k_1, k_2 \):

\[
K_{4}^{\Gamma \Gamma', \Gamma', \Gamma'}(k_1, k_2) = \sum_{l} (2l + 1) c_{l}^{|\Gamma\Gamma', \Gamma', \Gamma'}(k_1, k_2); \quad K_{4}^{\Gamma \Gamma', \Gamma', \Gamma'}(k_1, k_2) = \sum_{l} (2l + 1) c_{l}^{|\Gamma \Gamma', \Gamma', \Gamma'}(k_1, k_2).
\]

The corresponding real-space (in radial direction) versions \( K_{4}^{\Gamma \Gamma', \Gamma', \Gamma'}(r_1, r_2) \), and \( K_{4}^{\Gamma \Gamma', \Gamma', \Gamma'}(r_1, r_2) \) can be related using an equation similar to equation (44).

We will deal with the mode-coupling issues arising from the partial sky coverage next. We will show how to deconvolve the effect of mask. It is important to keep in mind that the one-point objects such as \( K_{4}^{\Gamma \Gamma', \Gamma', \Gamma'}(k_1, k_2) \) will have more signal-to-noise ratio and can play important role in studying growth of non-Gaussianity along the radial direction.

A few general comments are in order. At the level of the trispectrum there are two hierarchical amplitudes. If we employ the two sets of kurt spectra, then the amplitudes can be determined. This is very similar to their use in the CMB where the trispectrum can be used to provide independent constraints for parameters \( g_{NL} \) and \( \tau_{NL} \) that describe the Taylor expansion of the inflationary potential (Hu 1999; Okamoto & Hu 2002). Indeed, in some hierarchical models the *snake* amplitude \( R_{s} \) and the bispectrum amplitude are related by \( R_{s} = Q^2 \). Similar relation also exists between \( f_{NL} \) and \( \tau_{NL} \). The study of skew spectrum can provide direct estimates of the parameter \( Q_{s} \). The estimates from skew spectrum is expected to be more significant statistically due to the higher signal-to-noise ratio. An independent estimation using kurt spectra can provide a direct test of various hierarchical ansatz. It is important to realize that the skew spectra as well as the kurt spectra are integrated quantities, i.e. their amplitudes at a specific \( l \) depends on the entire range of \( l \) values being considered. In terms of modelling of these quantities, it means that any successful theoretical prediction will have to correctly model the relevant multi-spectra for the entire range of \( l \) values being probed. The procedure can be extended to even higher order. The number of distinct topological diagrams that are needed to build a correlation function at a given order will be the same as the number of related power spectra, e.g. at the fifth order there are three topological amplitudes and three multi-spectra. The procedure outlined above can be extended in such cases as signal-to-noise ratio of the available data improves.

### 4.2.3 The effect of a mask and subtraction of Gaussian contribution

The partial sky coverage will mean that the measured power spectrum \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) \) is not the same as theoretical expectation, but is related as before by: \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k, k') = G_{k}C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k, k') \). In fact, it can be shown that for arbitrary sky coverage with arbitrary mask the above analysis can be generalized to arbitrary order of correlation hierarchy. If we consider a correlation function at \( p + q \) order, for every possible combination of \( p, q \) we will have associated power spectrum. Using the same expression for the mode-mixing matrix, we can invert the observed \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) \) to \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) \). Hence, for an arbitrary mask with arbitrary weighting functions the deconvolved set of estimators can be written as: \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) = \sum_{r} G_{k}C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(\frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} k_1, \frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} k_2) \). The matrices \( G \) (defined below) depend on the spin indices as well as the power spectrum of the mask.

\[
G_{k}^{s', s, s', s'} = \frac{1}{4\pi} \sum_{l} \frac{(2l + 1)^2}{2l + 1} (2l + 1) \left( \begin{array}{ccc} l & l & l \\ s + s' & 0 & -(s + s') \\ s'' + s''' & 0 & -(s'' + s''') \end{array} \right) |w_k|^2;
\]

\[
G_{k}^{s', s, s', s'} = \frac{1}{4\pi} \sum_{l} \frac{(2l + 1)^2}{2l + 1} (2l + 1) \left( \begin{array}{ccc} l & l & l \\ (s + s' + s'') & 0 & -(s + s' + s''') \end{array} \right) |w_k|^2; \quad s, s', s'', s''' \in 0, 1, \pm 2, 3.
\]

However, as pointed out before, if we define kurt spectra \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) = (2l + 1) C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) \) we can use the same mode-coupling matrices that are used in projection for the purpose of deconvolution. However, the 3D treatment introduces a remapping of the radial mode due to the presence of a mask \( C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2) = \sum_{r} M_{k}C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(\frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} k_1, \frac{2}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} k_2) \).

The Gaussian contribution to the trispectrum \( G \) can be written as

\[
G_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_2, k_3, k_4; L) = (-1)^{i+j+k} \sqrt{(2l + 1)(2l + 1)(2l + 1) C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_4)C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_4)\delta_{k_1k_3} \delta_{k_1k_4}}
\]

\[
+ (2L + 1) (-1)^{i+j+l+L} \delta_{i+j+l} \delta_{k_1k_2} C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_3)C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_2, k_4) + (2L + 1) C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_1, k_3)C_{k}^{(\Gamma \Gamma', \Gamma', \Gamma')}(k_2, k_3) \delta_{i+j+l} \delta_{k_1k_4}.
\]
\begin{align*}
G_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k_1, k_2), \text{ respectively:} \\
G_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k_1, k_2) &= \frac{1}{2L+1} \sum_{l_1, l_2} 1 \sum_{l_1, l_2} S_{l_1, l_2} S_{l_2, l_1} G_l^{\mathcal{T}^{\mathcal{T}}}(l_1, k_1, k_2); \\
G_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k_1, k_2) &= \frac{1}{(2L+1)^2} \sum_{l_1, l_2} S_{l_1, l_2} S_{l_2, l_1} G_l^{\mathcal{T}^{\mathcal{T}}}(l_1, k_1, k_2).
\end{align*}

For realistic surveys with a mask, the unconnected (Gaussian) contributions to the total kurt spectrum, listed above, can be deconvolved in a manner identical to what we have presented before for the connected part of the total trispectrum. The mode-mixing matrix for the Gaussian contribution is identical to what we have introduced in equation (54). From the estimated \( \tilde{D}_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(r_1, r_2) \) and \( \tilde{D}_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(r_1, r_2) \) these contributions need to be subtracted out before comparing them against the theoretical expectations \( C_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k, k) \) and \( C_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k, k) \). An equivalent expression holds for the Gaussian contributions that needs to be subtracted: \( G_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k_1, k_2) \) and \( G_l^{\mathcal{T}^{\mathcal{T}, \mathcal{T}}}(k_1, k_2) \).

We have so far only considered the gravity-induced trispectrum in our discussion. However the kurt spectra for primordial non-Gaussianity can be derived in a very similar manner by replacing the gravity-induced trispectrum with a corresponding model for the primordial trispectrum. However, it is expected that gravity-induced non-Gaussianity will dominate the primordial ones at least at the lower redshift.

5 ERROR ANALYSIS

In the previous section, we have derived the expression for the 3D power spectrum associated with convergence field and indicated how a similar analysis can be performed for other spinorial fields. Estimation of these power spectra from noisy data will, however, always have to deal with issues such as noise and partial sky coverage. An estimator which can deal with such observational constraints was developed for the case of power spectra in equation (17). However, for the estimation of the power spectra to be meaningful, we need an approximate idea of the associated error bars.

The error analysis for the PCL estimator that was introduced can be done using the formalism used in Sun et al. (2013d) which is based on pseudo-C\( \tilde{C} \) formalism developed by various authors in the context of CMB data analysis (Efstathiou 2004, 2006; Brown et al. 2005).

The contributions to the error covariance will have three different components. On large angular scales or small \( l \) the error will be dominated mainly by cosmic variance whereas the high \( l \) or smaller angular scale will mainly be dominated by noise due to the intrinsic ellipticity of galaxies.

5.1 Power spectrum

\( C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \) defines the cross-spectra between two spinorial fields, \( \mathcal{T}(\mathbf{\Omega}) \) and \( \mathcal{T}(\mathbf{\Omega}) \). That is \( C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) = \frac{1}{2\pi^2} \sum_m \text{Re}[C_m^{\mathcal{T}^{\mathcal{T}}}(k, k)] \). It is possible to derive the covariance of estimates under certain simplifying assumptions. The general principles for deriving these results are outlined in Sun et al. (2013d) and will not be repeated here, and we quote the results for the ordinary power spectra here. For simplicity, we will only consider \( s = s' \). In later sections we will also consider covariance matrices for the skew spectrum. The covariance matrix of the estimates is \( \delta C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \), where \( \delta C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \) are the deviations from the ensemble average \( C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \):

\[
\langle \delta C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \delta C_l^{\mathcal{T}^{\mathcal{T}}}(k, k) \rangle \approx \Sigma_l^{ss}(k, k) + \Sigma_l^{SN}(k, k) + \Sigma_l^{NN}(k, k)
\]

\( \Sigma_l^{ss}(k, k) = \frac{1}{4\pi} \left( \sqrt{C_l^{\mathcal{T}^{\mathcal{T}}}(k,k)} C_l^{\mathcal{T}^{\mathcal{T}}}(k,k) \right)^2 \sum_{l_m, m} \left( \begin{array}{cc} l & I \\ s & 0 \end{array} \right)^2 |w_l|^2
\]

\( \Sigma_l^{SS}(k, k) = \frac{1}{4\pi} \sum_{l_m, m} \left( \begin{array}{cc} l & I \\ s & 0 \end{array} \right)^2 \left\{ |\sigma^{\mathcal{T}^{\mathcal{T}}}(k, k)|^2 \left| w_{l_m}(k, k) \right|^2 + \left| \sigma^{\mathcal{T}^{\mathcal{T}}}(k, k) \right|^2 \right\}
\]

\( \Sigma_l^{SN}(k, k) = \frac{1}{4\pi} \sum_{l_m, m} \left( \begin{array}{cc} l & I \\ s & 0 \end{array} \right)^2 \left\{ \left| \sigma^{\mathcal{T}^{\mathcal{T}}}(k, k) w_{l_m}(k, k) \right|^2 \sqrt{C_l^{\mathcal{T}^{\mathcal{T}}}(k,k)} + \text{symm. term} \right\} + 2 \left| \sigma^{\mathcal{T}^{\mathcal{T}}}(k, k) w_{l_m}(k, k) \right|^2 \sqrt{C_l^{\mathcal{T}^{\mathcal{T}}}(k,k)} \right.^2
\]

The symm. term can be constructed by exchanging \( k \) and \( k' \) as well as \( \Gamma \) and \( \Gamma' \). We have divided the total contribution into three different components. The term \( \Sigma_l^{SS}(k, k') \) is the cosmic variance contribution and depends on the target spectra but is independent of noise. The term \( \Sigma_l^{SN}(k, k') \) signifies the noise contribution and finally \( \Sigma_l^{NN}(k, k') \) is a cross-term which gets contributions from both signal and noise. We have assumed that the noise is statistically uncorrelated but it varies with pixel-position in the sky, i.e. \( (\Gamma(\mathbf{\Omega}, r) \Gamma'(\mathbf{\Omega}', r')) = \sigma^r(\mathbf{\Omega}, r, r') \delta_{\mathbf{\Omega}}(\mathbf{\Omega}, -\mathbf{\Omega}') \).

A detailed modelling of source distribution is required for 3D error estimates. The observational mask \( w(\mathbf{\Omega}) \) that we use is completely generic; however our results use completeness and orthogonality of spherical harmonics on the cut-sky. This means results will be accurate only for near-all-sky coverage. The various window functions that we have introduced are constructed from the 3D harmonic transforms such as the 3D gravitational potential, which can be further approximated by the 3D harmonic transforms of higher angular scales.
as \([\sigma^2 w^2]_{LM}(k)\) and \([\sigma w]_{LM}(k)\) of maps constructed from 3D noise maps and the mask (2D). These window functions are assumed to be much sharper than any variation in the power spectra. However, such an assumption is unlikely to pose a problem as the weak lensing power spectrum lacks features unlike that of the CMB. The above expression is expected to be reasonably accurate at high \(l\) regime where the noise dominates. The 3D harmonics that we have used in our derivation are based on the following definitions:

\[
\sigma^{\Gamma,\Gamma'}(\Omega, k, k') = 2 \int \frac{dk}{k} C \delta(\Omega) \delta(\Omega') \frac{2}{2l+1} \sigma^{\Gamma,\Gamma'} \Omega, \frac{2l+1}{k}, \frac{2l+1}{k'}.
\]

In the final step, we have used the Limber approximation. Similar terms such as \([\sigma^{\Gamma,\Gamma'}(\Omega) w(\Omega)]_{LM}(k, k')\) can be dealt with in a similar manner. In practice, the evaluation of these terms will depend on the redshift distribution of galaxies.

The deconvolution of the error-covariance matrix can be performed using a similarity transformation. It involves the mode-coupling matrix introduced before: \(\delta C \delta C_{\Gamma'}(k, k') = M_{L}^{-1}(\delta C \delta C_{\Gamma'}(k, k') M_{L})\). A sum over repeated indices is assumed in this equation. The mode-coupling matrix introduced here depends on the spins of the relevant fields which involved \(s\) and \(s'\) (equation (17)). We will next extend this result to the skew spectrum of arbitrary spinorial fields. For higher order spectra the results are more involved but they can be computed using the same techniques considered here.

### 5.2 Skew spectrum

The expression for the skew spectrum, valid in the high-\(l\) regime using the Limber approximation, is derived in equation (37). The estimator for the skew spectrum quoted in equation (40) depends on cross-correlating an arbitrary product field \([\delta C \delta C_{\Gamma'}(k, k')]\) with arbitrary spin-weight \(s\) and \(s'\) respectively:

\[
\langle \delta C \delta C_{\Gamma'}(k, k') \delta C \delta C_{\Gamma'}(k, k') \rangle = M_{L}^{-1}(\delta C \delta C_{\Gamma'}(k, k') M_{L})\).
\]

The individual terms in terms of noise and signal power spectra are as follows:

\[
\sigma^{\Gamma,\Gamma'}(k, k') = \frac{1}{4\pi} \left\{ C^{\Gamma,\Gamma'}(k, k) C^{\Gamma,\Gamma'}(k', k') \right\}^{\frac{1}{2}} C \delta(\Omega) \delta(\Omega') \left( \Omega, \frac{2l+1}{k}, \frac{2l+1}{k'} \right)
\]

The error covariance depends on noise maps for the product field as well as the individual field. The noise in our analysis is not assumed constant and can vary with position. The terms such as \([w^2 \sigma^{\Gamma,\Gamma'}]_{LM}(k)\) can also be defined using the expression similar to equation (58). It underlines the difficulty associated with an accurate error estimation beyond the power spectrum.

These results are based on various assumptions valid at high-\(l\) regime. However for future, near all-sky surveys for which the harmonic description is more appropriate these results can provide a good handle on estimation errors. Alternatively, the error covariance can be computed using Monte Carlo simulations. Simulating multiple copies of the observed sky, with all observational details, can be computationally expensive. Hence, often simplifying assumptions are employed to compute the covariance. The approach developed here can play a complementary role in cross-checking and validating such results. The lower order covariance such as that we have considered above typically depends on higher order power spectra. It is customary to quote error bars associated with estimated power spectra. However, it is important to note that the error bars for the higher order spectra such as the skew spectra may not be fully informative, as the probability distribution of the skew spectrum for a given \(l\) can be skewed. In such cases, the error bars can still give an idea of statistical scatter around the estimate.

The results for deconvolved PCL estimates for the skew spectrum can be computed using a similarity transformation:

\[
\langle \delta C \delta C_{\Gamma'}(k, k') \delta C \delta C_{\Gamma'}(k, k') \rangle = \sum_{LL} G_{LL}^{\Gamma,\Gamma'}(k, k') \delta C \delta C_{\Gamma'}(k, k') M_{L}^{-1}(\delta C \delta C_{\Gamma'}(k, k') M_{L})\).
\]

It is also possible to compute the cross-covariance of these estimators for power spectrum and the skew spectrum which can be used jointly. These can result in tighter cosmological constraints. The results can be derived using the techniques presented above.
matrices are different for different spinorial fields. They depend on the spin indices of the constituent spinorial fields. The spinorial fields considered above are, however, completely generic. If we assume that the magnetic or $B$ mode is absent then further simplification can be achieved.

5.3 Optimal estimators

The estimators that we have constructed can be generalized if we optimally weight the harmonics with an inverse variance weight. The generalized two-to-one power spectrum $S^{TT,TT}$ in this case takes the following form:

$$S^{TT,TT}_l(k,k') = \frac{1}{2l+1} \sum_{l''} \Lambda_{l''} B^{TT,TT}(k,k',k,k');$$

$$\Lambda^{-1} = \left( [C_{ij}^{TT}(k,k)C_{ij}^{TT}(k,k)C_{ij}^{TT}(k,k')]^{-1} + \text{cyc.perm.} \right).$$

(61)

This particular result is valid for all-sky coverage. The denominator $\Lambda$ is the scatter in the estimator in the Gaussian limit. For partial sky coverage a more elaborate treatment in line with Munshi & Heavens (2010) is required. In the absence of spherical symmetry due to lack of all-sky coverage or asymmetric noise, we will have linear terms in the estimator. The estimator constructed in this way can achieve maximum possible signal-to-noise ratio for a given data set. The one-point counterpart for this estimator, denoted as $S_b$, can be recovered by summing over angular harmonics $l$, i.e. $S^{TT,TT}_l(k,k') = \sum_{l'} (2l+1) S^{TT,TT}_l(k,k')$. An interesting point which we note here is that the hierarchical ansatz is factorizable which will allow easy construction of optimal weights. In general, the expressions for gravity-induced the bi- or trispectra are not factorizable. At bispectrum level we can use the skew spectrum estimator to recover the tree amplitude $Q_b$ and similar estimators can be designed for the kurt spectra. The two different kurt spectra will allow independent estimation of topological amplitudes $R_c$ and $R_b$. However, it is expected that signal-to-noise ratio at the level of trispectrum will be low. These optimal estimators can be optimized to detect either the gravity-induced non-Gaussianity or different models of primordial non-Gaussianity. They can also be used to forecast cross-contamination in a specific estimator from various sources of non-Gaussianity.

It is worth repeating here that the next generation of weak lensing surveys will have nearly all-sky coverage. This will probe a wide range of angular scales. The most commonly used technique for statistical characterization of such surveys is real-space analysis using two- or three-point correlation functions. One of the advantages of using the higher order correlation hierarchy is its ability to extract information from a complex survey geometry due to partial sky coverage. However, the real-space analysis introduces highly correlated measurements for various angular scales. These correlations are more dominant at small angular scales where most of the observational information is contained. The alternative is to use the harmonic space representation of the correlation functions, e.g. the multi-spectra that are being studied here. Though mathematically equivalent the power spectra or their higher order generalizations are much less used in the context of weak lensing. However, the theoretical interpretation of multi-spectra is much simpler and different harmonics are much less correlated. The main difficulty in harmonic analysis is related to the partial sky coverage. Typically, the mask consists of bright stars and saturated spikes where no lensing measurements can be performed. The analytical results presented here provide a general analysis of the problem these pose. The results relate the convolved and deconvolved power spectra that can be constructed from the higher order multi-spectra. The deconvolution process consists of simple matrix inversion and can be performed for arbitrary sky coverage. For the case of convergence the matrix representing mode–mode coupling in the presence of mask is independent of the order of the multi-spectra being probed. However, that is not the case for shear or flexions. The formalism developed here also allows for computation of scatter or variance associated with various estimators.

6 CONCLUSIONS

It is now well accepted that the next generation of weak lensing surveys will play an important part in further reducing the uncertainty in fundamental cosmological parameters, including those that describe the evolution of equation of state of dark energy (Refregier et al. 2010). They will also be instrumental in testing various alternative gravity models (e.g. Heavens et al. 2007; Amendola, Kunz & Sapone 2008; Schrabback et al. 2009; Kilbinger et al. 2009; Benyon, Bacon & Koyama 2010). The power of weak lensing surveys largely depends on the fact that they can exploit both the angular diameter distance and the growth of structure to constrain cosmological parameters. It is therefore very important to develop analytical techniques and statistical tools that can fully exploit the potential of future weak lensing surveys.

Typically, without the redshift information the data from weak lensing surveys are analysed in projection for the entire source distribution. However, it was found that by binning sources in a few photometric redshift bins the constraints improve (Hu 1999). In recent years a full 3D formalism which exploits the photometric redshifts of individual sources were developed. Such an approach does not involve any binning (see Heavens 2003; Castro et al. 2005; Heavens et al. 2006). Further studies along these lines demonstrate that 3D lensing can provide more powerful and tighter constraints on the dark energy equation of state parameter, and on neutrino masses (de Bernardis et al. 2009; Jimenez et al. 2010), as well as testing braneworld and other alternative gravity models. These constitute the main science drivers for the future weak lensing surveys. Initial studies in weak lensing focused on two-point correlation functions or the power spectrum mainly due to the low signal-to-noise ratio available for higher order studies from most first-generation surveys. With the availability of modern surveys it is useful to include the non-Gaussianity information in the data analysis pipeline (Takada & Jain 2004; Sembolini et al. 2009) that can help to lift some of the degeneracies in estimation of cosmological parameters involving power spectrum alone.
In their recent work, Munshi et al. (2011b) have explored the possibility of extending the higher order statistics of convergence to three dimensions. The main motivation of this work is to generalize those results to spinorial objects and perform a full 3D analysis for the higher order statistics. In this sense this is also an extension of results derived in Munshi et al. (2011c) which analysed higher order statistics of spinorial fields but only in projection (2D). The results are here valid for all-sky surveys. It depends on full 3D spherical harmonic decomposition on the surface of the sky as well as along the radial directions. Such an approach in analysing the data will be useful for future surveys that will cover a large fraction of the sky.

The higher order statistics of convergence $\kappa$, shear $\gamma_\pm$ or flexions $F$ and $G$ depend on accurate modelling of the underlying density contrast $\delta$. Various models are used, such as the hierarchical ansatz which we use here, known to be valid in the highly non-linear regime. However, the techniques developed here are generic and can also be used in association with other models such as the halo model.

The higher order multi-spectra contain invaluable information. Some of these information is, however, degenerate because of symmetries associated with higher order correlation functions. It is difficult to estimate the higher order multi-spectra mode by mode because of the associated scatter involved in such estimation especially from a noisy data set. In Munshi & Heavens (2010), various power spectra (skew spectrum, kurt spectra) were introduced, which are associated with a multi-spectra of a given order and can be estimated in the presence of mask and noise. These spectra carry some of the information contents of the multi-spectra from which they are constructed. In our present study we express the skew spectrum and two degenerate kurt spectra of generic spinorial fields in terms of the bi- and trispectrum. This extends earlier results for the convergence (spin-0) field. Extending the previously introduced 3D power spectrum $C_i(k_1, k_2)$ to higher order, we introduce a series of power spectra related to multi-spectra at each order. We have introduced the 3D skew spectrum $C_i^{\Gamma, \Gamma'}(k_1, k_2)$ associated with the bispectrum of arbitrary triplets of spinorial fields $\Gamma$, $\Gamma'$, $\Gamma''$. Analogously, at the level of trispectrum we have introduced two 3D kurt spectra $C_i^{\Gamma, \Gamma, \Gamma'}(k_1, k_2)$ and $C_i^{\Gamma, \Gamma, \Gamma''}(k_1, k_2)$ for arbitrary choice of spinorial fields. These extend the skew and kurt spectra defined in Munshi et al. (2011b) where harmonic decomposition was performed only on the surface of the celestial sphere and a real-space analysis was performed on the radial direction leading to a mixed representation of skew spectrum $C_i^{\mu, \mu'}(r_2, r_1)$ as well as their higher order counterparts, i.e. the two kurt spectra $C_i^{\gamma, \gamma'}(r_2, r_1)$ and $C_i^{\gamma', \gamma''}(r_2, r_1)$.

The generic expression for the skew and kurt spectra involves spherical Bessel functions. We simplified these radial integrals by using the Limber approximation, whose accuracy scales as $O(\ell^4)$. We show that at each order the Limber approximation can reduce the dimensionality of the integrals to unity which dramatically reduces computational cost. Both the Limber approximation and the hierarchical ansatz are accurate at smaller scales and their joint use can help us to compute the skew and kurt spectra very efficiently with reasonable accuracy, but the method can accommodate different models for non-linear clustering.

We also present analytical results for dealing with a mask, via a pseudo-$C_i$ approach, encapsulated in a mode-mixing matrix. The estimation of unbiased skew or kurt spectra is done by simple inversion of the mixing matrix $M$, which depends on the spins associated with the spinorial fields. Some regularization will normally be required. The presence of an observational mask typically only induces mode mixing on the celestial sphere and not on the radial direction. We have also showed how our formalism presented here can also be used for the computation of scatter under certain simplifying assumptions in the presence of an observational mask, and we have identified individual terms that correspond to contributions from noise, partial sky coverage (cosmic variance) and cross-terms.

The results presented here will be relevant for the study of cosmic magnification studies in three dimensions as well as in many other contexts where integrated radial information is used. The estimators for skew or kurt spectra that we have described here can be improved by inverse variance weighting of 3D harmonics. Finally, to summarize we can say that in this article:

(i) we have studied higher order multi-spectra in the context of 3D weak lensing surveys.
(ii) we use a full 3D Fourier decomposition which employs spin-weight spherical harmonics.
(iii) our generic results are valid for arbitrary 3D spinorial objects.
(iv) the results are relevant for convergence $\kappa$, magnification $\mu$, shear $\gamma_\pm$ as well as flexions $F$ and $G$ or an arbitrary scalar tracer field $\Phi$.
(v) in our analysis we define power spectra $C_i(k_1, k_2)$ that are related to the bispectrum (skew spectra) and to the trispectrum (kurt spectra).
(vi) we provide both all-sky exact results and corresponding approximate results using the Limber approximation.
(vii) use of Limber’s approximation reduces multi-dimensional integrations along the radial direction to 1D integrals.
(viii) we show how the multi-spectra can be recovered from a masked sky in the presence of noise, and show how the presence of masks mixes modes not only on the surface of the sky but also in the radial direction.
(ix) the modelling was done using the hierarchical ansatz but the formalism can work with any input underlying density multi-spectra.
(x) under certain simplifying approximations, we also obtain expressions for the covariance of our power spectra and skew spectra estimators.
(xi) we outline how inverse variance weights can be introduced and optimal estimators can be defined for the detection of a specific type of non-Gaussianity.
(xii) the formalism can be relevant in many other contexts where line-of-sight integrations of non-Gaussianities are performed or in studies involving cross-spectra or mixed bispectra.

In this paper we have ignored many observational complexities for simplicity, such as that in a realistic survey the lensing potential can only be sampled at the discrete positions of galaxies, and the average number of source galaxies will decline with redshift. We also ignore photometric redshift errors.
In addition, we would like to point out that the skew spectra (or equivalently the kurt spectra) are integrated statistics. This means for a fixed $l$ it will receive contributions from the entire $l$ range being probed. The modelling of the higher order statistics is difficult at high $l$ and so is its measurement from the noisy data, which also require correct modelling of noise at high $l$. However, these problems are also present in the measurement of one-point statistics (skewness or kurtosis). Typically smoothing with an appropriate window (e.g. top-hat or aperture mass) can alleviate such issues and can also be incorporated in the analysis involving skew or kurt spectra. Moreover, inverse variance weighting schemes exist which can be employed to suppress modes with low signal-to-noise ratio or modes that are likely to be contaminated by other systematics.

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APPENDIX A: SPHERICAL BESSEL FUNCTIONS

The orthogonality relationship for the spherical Bessel functions is given by the following expression:

$$\int k^2 j_l(kr_1) j_l(kr_2) dk = \left[ \frac{\pi}{2r_1^2} \right] \delta_{1D}(r_1 - r_2).$$  \hspace{1cm} (A1)

The extended Limber approximation is also implemented through the following approximate relation (LoVerde & Afshordi 2008):

$$\int k^2 F(k) j_l(kr_1) j_l(kr_2) dk \sim \left[ \frac{\pi}{2r_1^2} \right] F \left( \frac{l}{r_1} \right) \delta_{1D}(r_1 - r_2).$$  \hspace{1cm} (A2)

Thus, for high $l$, the spherical Bessel functions can be replaced by a Dirac delta function $\delta_{1D}$:

$$\lim_{x \to \infty} j_l(x) = \sqrt{\frac{\pi}{2l+1}} \delta_{1D} \left( l + \frac{1}{2} - x \right).$$  \hspace{1cm} (A3)
APPENDIX B: 3j SYMBOLS

The following properties of 3j symbols were used to simplify various expressions.

\[
\sum_{l_3m_3}(2l_3 + 1) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} = \delta_{l_3m_3}^{kl_3m'_3}
\]

(B1)

\[
\sum_{m_1m_2}(l_1 l_2 l_3) \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} = \frac{\delta_{l_3m_3}^{kl_3m'_3} \delta_{l_3m_3}^{kl'_3m'_3}}{2l_3 + 1}.
\]

(B2)

APPENDIX C: WIGNER D-MATRIX

Introduced by Eugene Wigner, in quantum mechanics the D-matrix (or its complex conjugate) is an eigenfunction of the Hamiltonian of spherical and symmetric rigid rotors. The matrix was introduced in 1927 by Eugene Wigner. They play an important role in our understanding of irreducible representation of the groups SU(2) and SO(3).

\[
D_{m'm}^j(\alpha, \beta, \gamma) = \exp(-im'\alpha)\exp(-im\gamma).
\]

(C1)

Here angles \(\alpha, \beta\) and \(\gamma\) are the Euler angles and the Wigner’s (small) \(d\) matrix is in turn defined as

\[
d_{m'm}^j = [(j + m)!/(j - m)!]^{1/2}(j + m - s)!/(j - m - s)![(j + m + s)!/s!][(j - m + s)!/s!]^{1/2} \sum_s (-1)^{m' + m + s} \left[ \cos \left( \frac{\beta}{2} \right) \right]^{2j + m - m' - 2s} \left[ \sin \left( \frac{\beta}{2} \right) \right]^{m' - m + 2s}.
\]

(C2)

The sum is taken over such values of the indices that renders the factorials non-negative.

The Wigner’s D matrix elements form a complete set of orthogonal functions of the Euler angles \(\alpha, \beta\) and \(\gamma\):

\[
\int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma D_{m'k}^j(\alpha, \beta, \gamma)D_{m'k'}^j(\alpha, \beta, \gamma) = \frac{8\pi^2}{2j + 1} \delta_{m'm} \delta_{kk'}.
\]

(C3)

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