A nonuniform Littlewood-Offord inequality for all norms

Kyle Luh *    David Xiang †

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Abstract

Let \( v_i \) be vectors in \( \mathbb{R}^d \) and \( \{\varepsilon_i\} \) be independent Rademacher random variables. Then the Littlewood-Offord problem entails finding the best upper bound for \( \sup_{x \in \mathbb{R}^d} \mathbb{P}(\sum \varepsilon_i v_i = x) \).

Generalizing the uniform bounds of Littlewood-Offord, Erdős and Kleitman, a recent result of Dzindzalieta and Juškevičius provides a non-uniform bound that is optimal in its dependence on \( \|x\|_2 \). In this short note, we provide a simple alternative proof of their result. Furthermore, our proof demonstrates that the bound applies to any norm on \( \mathbb{R}^d \), not just the \( \ell_2 \) norm. This resolves a conjecture of Dzindzalieta and Juškevičius.

1 Introduction

Let \( \{\varepsilon_k\}_{k=1}^n \) be independent Rademacher random variables (i.e. \( \mathbb{P}(\varepsilon_k = 1) = 1/2 \) and \( \mathbb{P}(\varepsilon_k = -1) = 1/2 \)). We let \( R_n \) denote the sum of these random variables. In their study of random polynomials, Littlewood and Offord [8] encountered the following problem. What is the best bound on \( \mathbb{P}(\sum_{i=1}^n a_i \varepsilon_i = x) \) with \( |a_i| \leq 1 \). Littlewood and Offord established that \( \max_x \mathbb{P}(\sum a_i \varepsilon_i = x) = O(\log n/n^{1/2}) \) for all \( a_i \) such that \( |a_i| \leq 1 \). [8]. With a short, insightful argument, Erdős [3] established the optimal bound

\[
\rho(a) := \max_x \mathbb{P}\left( \sum_{i=1}^n a_i \varepsilon_i = x \right) \leq \frac{(\lfloor n/2 \rfloor)^2}{2n} = O(n^{-1/2}).
\]

(1)

The results of Littlewood, Offord and Erdős attracted the attention of many researchers and numerous variants of the Littlewood-Offord problem have been proposed and investigated. Erdős and Moser showed that an improved bound held when all the \( a_i \) are distinct [4]. Later, Sárközy and Szemerédi obtained the optimal bound for distinct \( a_i \). Many more results were obtained when considering more complex arithmetic structure of the \( a_i \)'s [12 6 11]. In a different direction, Erdős conjectured that a result analogous to (1) should hold in higher dimensions. This extension was non-trivial and it took two decades before such a result was verified by Kleitman [7].

**Theorem 1.1.** Let \( d \in \mathbb{N} \) and \( v_i \in \mathbb{R}^d \) with \( \|v_i\|_2 \leq 1 \) and \( v_i \neq 0 \). Then,

\[
\rho(v_1, \ldots, v_n) := \sup_{x \in \mathbb{R}^d} \mathbb{P}\left( \sum_{i=1}^n \varepsilon_i v_i = x \right) \leq \frac{(\lfloor n/2 \rfloor)^2}{2n}.
\]

*Department of Mathematics, University of Colorado Boulder. Email: kyle.luh@colorado.edu.
†Harvard University. Email: davidxiang@college.harvard.edu.
Inspired by the inverse problems of additive combinatorics, Tao and Vu began a line of work known as inverse Littlewood-Offord theorems which attempt to explain when \( \rho(a) \) is large \[13\]. Essentially, they showed that \( \rho(a) \) is large only when the entries of \( a \) reside in a generalized arithmetic progression. Many results in this direction followed and culminated in the optimal inverse Littlewood-Offord theorems of Nguyen and Vu \[10\]. This theory and its variants played an important role in estimating the singularity probability of random matrices (see \[13, 11, 5, 15\] and the references therein).

In another vein of work, Tiep and Vu \[14\] obtained a Littlewood-Offord-type inequality in the setting of non-commutative groups and Juškevičius and Šematulskis obtained optimal bounds for arbitrary groups. Bandeira, Ferber and Kwan proposed a new perspective and investigated a resilience version of the Littlewood-Offord problem, namely the number of coefficients in \( a \) that an adversary can change to force \( \rho(a) \) to be large \[1\].

Recently, Dzindzalieta and Juškevičius established a non-uniform Littlewood-Offord inequality in all dimensions. The bound is non-uniform in that it incorporates information about the vector \( x \).

**Theorem 1.2.** Let \( \mathbf{v}_i \in \mathbb{R}^d \) with \( \|\mathbf{v}_i\|_2 \leq 1 \) and \( \mathbf{v}_i \neq 0 \) for all \( i \in [n] \). Then,

\[
P\left( \sum_{i=1}^{n} \varepsilon_i \mathbf{v}_i = \mathbf{x} \right) \leq P(R_n = k + \delta_{n,k}) = \frac{\left\lceil \frac{n+k}{2} \right\rceil}{2^n}.
\]

where \( k = \lceil \|x\|_2 \rceil \) and \( \delta_{n,k} \) is defined as follows:

\[
\delta_{n,k} = \begin{cases} 
1 & \text{if } n + k \text{ is even} \\
0 & \text{otherwise}
\end{cases}
\]

This result is optimal in \( n \) and \( \|x\|_2 \) as can be seen by setting \( \mathbf{v}_i = (\frac{\|x\|_2}{k + \delta_{n,k}}, 0, \ldots, 0) \). In \[2\], it was conjectured that the result should hold for any norm on \( \mathbb{R}^d \), not just the \( \ell_2 \) norm.

**Conjecture 1.3.** \[2, Conjecture 2\] Let \( \|\cdot\| \) be an arbitrary norm on \( \mathbb{R}^d \). Let \( \mathbf{v}_i \in \mathbb{R}^d \) be such that \( \|\mathbf{v}_i\| \leq 1 \) and \( \mathbf{v}_i \neq 0 \) for all \( i \in [n] \). Then,

\[
P\left( \sum \varepsilon_i \mathbf{v}_i = \mathbf{x} \right) \leq P(R_n = k + \delta_{n,k}).
\]

In \[2\], they used a rotation argument to reduce the multi-dimensional case to the one dimensional case. However, their rotation only preserves the \( \ell_2 \) norm and so their argument only applies to this norm. In this short note, we provide an alternate proof of the main result in \[2\] and prove Conjecture \[1.3\].

**Theorem 1.4.** Let \( \|\cdot\| \) be an arbitrary norm on \( \mathbb{R}^d \). Let \( \mathbf{v}_i \in \mathbb{R}^d \) be such that \( \|\mathbf{v}_i\| \leq 1 \) and \( \mathbf{v}_i \neq 0 \) for all \( i \in [n] \). Then,

\[
P\left( \sum \varepsilon_i \mathbf{v}_i = \mathbf{x} \right) \leq P(R_n = k + \delta_{n,k}).
\]

where \( k = \lceil \|x\| \rceil \) and \( \delta_{n,k} \) is defined as follows:

\[
\delta_{n,k} = \begin{cases} 
0 & \text{if } n + k \text{ is even} \\
1 & \text{otherwise}
\end{cases}
\]
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2 Auxiliary Results

We will make use of the following one dimensional non-uniform Littlewood-Offord bound.

Proposition 2.1. [2] For non-zero \( a_i \in \mathbb{R} \) such that \( |a_i| \leq 1 \), we have that

\[
\mathbb{P}(\sum_{i=1}^{n} \epsilon_i a_i = x) \leq \mathbb{P}(R_n = k + \delta_{n,k})
\]

where \( k = \lceil |x| \rceil \).

We will also utilize the basic theory of dual norms.

Definition 2.2. Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^d \). Let \( \| \cdot \|_* \) denote the dual norm where for any \( u \in \mathbb{R}^d \),

\[
\| u \|_* = \sup\{ \langle u, x \rangle : \| x \| \leq 1 \}
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^d \).

We then have a basic Cauchy-Schwarz type inequality. We include the elementary proof for the reader’s convenience.

Lemma 2.3. Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^d \) and \( \| \cdot \|_* \) be its dual. Then, for \( x, y \in \mathbb{R}^d \),

\[
|\langle x, y \rangle| \leq \| x \| \| y \|_*
\]

Proof. Let \( v = x/\| x \| \). Then we have

\[
\langle x, y \rangle = \| x \| \langle v, y \rangle \leq \| x \| \| y \|_*. 
\]

To include the absolute value, we apply the same argument to \( -x \). \qed

Additionally, we will make use of the standard fact that in finite-dimensional spaces, the double dual norm is the same as the original norm.

Lemma 2.4. (e.g. [3, Theorem 1.11.9]) Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^d \). Then, for \( x \in \mathbb{R}^d \),

\[
\| x \| = \| x \|_{**}.
\]
3 Proof of Theorem 1.4

Proof. For any $y \in \mathbb{R}^d$, we have that

$$P \left( \sum_{i=1}^{n} \varepsilon_i v_i = x \right) \leq P \left( \left\langle \sum_{i=1}^{n} \varepsilon_i v_i, y \right\rangle = \langle x, y \rangle \right)$$

In particular, if we let $y = \arg\max \|u\| \leq 1 \langle x, u \rangle$, we can conclude that

$$P \left( \sum_{i=1}^{n} \varepsilon_i v_i = x \right) \leq P \left( \left\langle \sum_{i=1}^{n} \varepsilon_i v_i, y \right\rangle = \langle x, y \rangle \right) = P \left( \sum_{i=1}^{n} \langle v_i, y \rangle \varepsilon_i = \|x\|_* \right).$$

Since $\|v\| \leq 1$ by assumption, Lemma 2.3 implies that

$$|\langle v_i, y \rangle| \leq \|v\| \|y\|_* \leq 1.$$

Therefore, we can apply Proposition 2.1 so

$$P \left( \sum_{i=1}^{n} \varepsilon_i v_i = x \right) \leq P(R_n = k + \delta_{n,k})$$

where $k = \lceil \|x\|_* \rceil = \lceil \|x\| \rceil$. This final equality follows from Lemma 2.4. In our application of Proposition 2.1, we implicitly assumed that $\langle v_i, y \rangle \neq 0$. To ensure this, we can simply choose a small perturbation of $y$ such that $\lceil \langle y, x \rangle \rceil = \lceil \|x\|_* \rceil$.


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