Counterexamples of the Conjecture on Roots of Ehrhart Polynomials

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Abstract On roots of Ehrhart polynomials, Beck et al. conjecture that all roots $\alpha$ of the Ehrhart polynomial of an integral convex polytope of dimension $d$ satisfy $-d \leq \Re(\alpha) \leq d - 1$. In this paper, we provide counterexamples for this conjecture.

Keywords Integral convex polytope · Ehrhart polynomial · $\delta$-vector

1 Introduction

Recently, in many research papers on convex polytopes, e.g., [1, 3–5, 9, 10], roots of Ehrhart polynomials have been studied. One of the most important problems is to solve the conjecture given in [1, Conjecture 1.4]. In this paper, we disprove this conjecture.

First of all, we review what the Ehrhart polynomial is. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension $d$ and $\partial \mathcal{P}$ its boundary. Here an integral convex polytope is a convex polytope all of whose vertices have integer coordinates. Given a positive integer $n$, we write

\[ i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N| \quad \text{and} \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial \mathcal{P}) \cap \mathbb{Z}^N|, \]

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ denotes the cardinality of a finite set $X$. The systematic studies of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [6], who established the following fundamental properties:

- $i(\mathcal{P}, n)$ is a polynomial in $n$ of degree $d$.
- $i(\mathcal{P}, 0) = 1$.
We call this polynomial \( i(P, n) \) the \textit{Ehrhart polynomial} of \( P \). We refer the reader to [2, Chap. 3], [8, Part II], or [13, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence \( \delta_0, \delta_1, \delta_2, \ldots \) of integers by the formula

\[
(1 - \lambda)^{d+1} \sum_{n=0}^{\infty} i(P, n) \lambda^n = \sum_{j=0}^{\infty} \delta_j \lambda^j.
\]  

(1)

Since \( i(P, n) \) is a polynomial in \( n \) of degree \( d \) with \( i(P, 0) = 1 \), a fundamental fact on generating functions (see [13, Corollary 4.3.1]) guarantees that \( \delta_j = 0 \) for every \( j > d \). The sequence \( \delta(P) = (\delta_0, \delta_1, \ldots, \delta_d) \) is called the \textit{\( \delta \)-vector} of \( P \). Alternate names of \( \delta \)-vectors are, for example, \textit{Ehrhart} \( h \)-vector, \textit{Ehrhart} \( \delta \)-vector or \textit{h*-vector}.

By the reciprocity law, one has

\[
\sum_{n=1}^{\infty} i^*(P, n) \lambda^n = \frac{\sum_{i=0}^{d} \delta_{d-i} \lambda^{i+1}}{(1 - \lambda)^{d+1}}.
\]  

(2)

The following properties on \( \delta \)-vectors are well known:

- By (1), one has \( \delta_0 = i(P, 0) = 1 \) and \( \delta_1 = i(P, 1) - (d + 1) = |P \cap \mathbb{Z}^N| - (d + 1) \).
- By (2), one has \( \delta_d = i^*(P, 1) = |(P \setminus \partial P) \cap \mathbb{Z}^N| \). In particular, we have \( \delta_1 \geq \delta_d \).
- Each \( \delta_i \) is nonnegative [12].
- When \( d = N \), the leading coefficient of \( i(P, n) \), which coincides with \( \sum_{i=0}^{d} \delta_i / d! \), is equal to the usual volume of \( P \) (see [13, Proposition 4.6.30]). In general, the positive integer \( \text{vol}(P) = \sum_{i=0}^{d} \delta_i \) is called the \textit{normalized volume} of \( P \).

For a complex number \( a \in \mathbb{C} \), let \( \Re(a) \) denote the real part of \( a \). Beck et al. [1] suggest the following

\textbf{Conjecture 1.1} [1, Conjecture 1.4] All roots \( \alpha \) of the Ehrhart polynomial of an integral convex polytope of dimension \( d \) satisfy

\[
-d \leq \Re(\alpha) \leq d - 1.
\]  

(3)

It is proved in [1] and [5] that this conjecture is true when \( d \leq 5 \) or roots are real numbers. Moreover, the norm bound of roots of Ehrhart polynomials is given with \( O(d^2) \) (see [4]), and it is known that this bound is best possible [3, Theorem 1.7]. In addition, in order to provide evidence for this conjecture, roots of the Ehrhart polynomials of several integral convex polytopes arising from finite graphs are discussed in [9].

In this paper, we disprove Conjecture 1.1 (see Example 3.1). We obtain many possible counterexamples by Theorem 2.1, and we find a certain counterexample when \( d = 15 \).

\textbf{Remark 1.2} In a recent paper [10], another counterexample is provided. On the one hand, integral convex polytopes given there are so-called \textit{smooth Fano polytopes} and
a certain counterexample is found when the dimension is 124. On the other hand, our counterexamples are simpler polytopes, which are integral simplices, and we obtain them of small dimension 15.

2 A significant family of integral simplices

This section is devoted to proving the following

**Theorem 2.1** Let \( m, d, k \in \mathbb{Z}_{>0} \) be arbitrary positive integers satisfying

\[
m \geq 1, \quad d \geq 2 \quad \text{and} \quad 1 \leq k \leq \lfloor (d + 1)/2 \rfloor.
\]

Then there exists an integral convex polytope whose Ehrhart polynomial coincides with

\[
\binom{d+n}{d} + m \binom{d+n-k}{d}.
\]

Before proving the theorem, we recall the well-known combinatorial technique to compute the \( \delta \)-vector of an integral simplex.

Given an integral simplex \( \mathcal{F} \subset \mathbb{R}^N \) of dimension \( d \) with the vertices \( v_0, v_1, \ldots, v_d \in \mathbb{Z}^N \), we set

\[
S = \left\{ \sum_{i=0}^{d} r_i (v_i, 1) \in \mathbb{R}^{N+1} : 0 \leq r_i < 1 \right\} \cap \mathbb{Z}^{N+1}.
\]

We define the degree of an integer point \((\alpha, n) \in S\) with \( \deg(\alpha, n) = n \), where \( \alpha \in \mathbb{Z}^N \) and \( n \in \mathbb{Z}_{\geq 0} \). Let \( \delta_i = |\{\alpha \in S : \deg \alpha = i\}| \). Then we have

\[
\delta(\mathcal{F}) = (\delta_0, \delta_1, \ldots, \delta_d).
\]

We also recall the following

**Lemma 2.2** [2, Theorem 2.4] Suppose that \((\delta_0, \delta_1, \ldots, \delta_d)\) is the \( \delta \)-vector of an integral convex polytope of dimension \( d \). Then there exists an integral convex polytope of dimension \( d + 1 \) whose \( \delta \)-vector is \((\delta_0, \delta_1, \ldots, \delta_d, 0)\).

Note that the required \( \delta \)-vector is obtained by forming the pyramid over the integral convex polytope.

**Proof of Theorem 2.1** We show that there exists an integral convex polytope of dimension \( d \) whose \( \delta \)-vector is

\[
\delta_i = \begin{cases} 
1 & \text{if } i = 0, \\
 m & \text{if } i = k, \\
0 & \text{otherwise}.
\end{cases}
\]

\( \square \) Springer
When \( k = 1 \), it is obvious that \((1, m, 0, \ldots, 0)\) is a \( \delta \)-vector. Thus, we assume that \( k \geq 2 \). By Lemma 2.2, it is enough to construct an integral convex polytope of dimension \( d \) with its \( \delta \)-vector

\[
\delta_i = \begin{cases} 
1 & \text{if } i = 0, \\
m & \text{if } i = (d + 1)/2, \\
0 & \text{otherwise,}
\end{cases}
\]

for any positive integer \( m \) and any odd number \( d \) with \( d \geq 3 \).

Let \( d \geq 3 \) be an odd number and \( c = (d - 1)/2 \). We define the integral simplex \( P \subset \mathbb{R}^d \) of dimension \( d \) by setting the convex hull of the integer points \( v_0, v_1, \ldots, v_d \in \mathbb{Z}^d \), which are of the form

\[
v_i = \begin{cases} 
e_i & \text{if } i = 1, \ldots, d - 1, \\
\sum_{j=1}^c e_j + \sum_{j=c+1}^{2c} m e_j + (m + 1) e_d & \text{if } i = d, \\
(0, 0, \ldots, 0) & \text{if } i = 0,
\end{cases}
\]

where \( e_1, e_2, \ldots, e_d \) denote the unit coordinate vectors of \( \mathbb{R}^d \). In other words, for \( i = 1, \ldots, d \), \( v_i \) is equal to the \( i \)th row vector of the \( d \times d \) lower triangular integer matrix

\[
\begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 1 & \ddots & & & \\
& \ddots & \ddots & \ddots & & \\
& & \ddots & \ddots & \ddots & \\
& & & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & 1 & 0 \\
1 & \cdots & 1 & m & \cdots & m + 1
\end{pmatrix}
\] (6)

where there are \( c \) 1’s and \( c \) \( m \)’s in the \( d \)th row. Then we notice that \( \text{vol}(P) = m + 1 \), which coincides with the determinant of (6).

For \( j = 1, 2, \ldots, m \), since

\[
\sum_{i=0}^c \frac{m + 1 - j}{m + 1} (v_i, 1) + \sum_{i=c+1}^d \frac{j}{m + 1} (v_i, 1) = (1, 1, \ldots, 1, j, j, \ldots, j, c + 1) \in \mathbb{Z}^{d+1}_{c+1}
\]

and

\[
0 \leq \frac{m + 1 - j}{m + 1} < 1, \quad 0 \leq \frac{j}{m + 1} < 1,
\]

we have \( \delta_{c+1} \geq m \). Moreover, from \( \text{vol}(P) = m + 1 \) together with the non-negativity of \( \delta \)-vectors, we obtain \( \delta_{(d+1)/2} = m \). Therefore, we conclude that \( P \) has the required \( \delta \)-vector. \( \square \)
3 Counterexamples of Conjecture 1.1

In this section, we consider the roots of the polynomial (5) given in Theorem 2.1. Let $f(n)$ be the polynomial (5) of degree $d$. Since

$$f(n) = \prod_{j=d-k+1}^{d}(n+j)m\prod_{j=0}^{k-1}(n-j),$$

negative integers $-1, -2, \ldots, -d+k$ are always the roots of $f(n)$. Let $g_{m,d,k}(n) = \prod_{j=d-k+1}^{d}(n+j)m\prod_{j=0}^{k-1}(n-j)$ be the polynomial in $n$ of degree $k$. We consider the roots of $g_{m,d,k}(n)$.

**Example 3.1** Let us consider the polynomial $g_{9,15,8}(n)$. When $1 \leq m \leq 8$, all its roots satisfy (3). On the other hand, when $m = 9$, its eight roots are approximately

$$14.37537447 \pm 25.02096544\sqrt{-1}, \quad -0.77681486 \pm 10.23552765\sqrt{-1},$$
$$-2.56596317 \pm 4.52757516\sqrt{-1}, \quad \text{and} \quad -3.03259644 \pm 1.31223697\sqrt{-1}.$$

By virtue of Theorem 2.1, this implies that there exists a counterexample of Conjecture 1.1. Moreover, it can be verified that for every $15 \leq d \leq 100$, $g_{9,d,\lfloor(d+1)/2\rfloor}(n)$ possesses a root which violates (3), that is, there exists a counterexample of Conjecture 1.1 for each dimension $15 \leq d \leq 100$. It also seems to be true that there exists a counterexample when $d \geq 101$. In addition, we remark that when $d \geq 17$, we can verify that $g_{9,d,\lfloor(d+1)/2\rfloor}(n)$ possesses a root whose real part is greater than $d$. (Those are computed by Maple and Maxima.)

These computational results are also supported theoretically. For example, on the roots of $g_{9,15,8}(n)$, by applying the Routh–Hurwitz criterion (see, e.g., [7, pp. 226–233]), we can check that $g_{9,15,8}(n+14.3)$ possesses a root whose real part is nonnegative but $g_{9,15,8}(n+14.4)$ possesses no root whose real part is nonnegative. Of course, this means that $g_{9,15,8}(n)$ possesses a root $\alpha$ with $14.3 \leq \Re(\alpha) < 14.4$.

**Remark 3.2** On the order of the largest real part of the non-real roots of $g_{9,d,\lfloor(d+1)/2\rfloor}(n)$, the order seems not to be linear in $d$. For example, when $d = 30, 50, 100, \text{and } 200$, the largest real parts of the non-real roots of $g_{9,d,\lfloor(d+1)/2\rfloor}(n)$ are as follows:

| $d$  | approximate real part |
|------|-----------------------|
| 30   | 60                    |
| 50   | 174                   |
| 100  | 722                   |
| 200  | 2940                  |
Thus, it is more natural to claim that the real parts of roots of Ehrhart polynomials are bounded with \( O(d^2) \), which is known as the best possible norm bound of roots of Ehrhart polynomials.

\textbf{Remark 3.3}

(a) When \( m = 1 \), the real parts of all the roots of \( g_{1,d,k}(n) \) are \((-d + k - 1)/2\), which satisfies \(-d < (-d + k - 1)/2 < -1/2\). In fact, since all the roots of \( 1 + \lambda^k \) are on the unit circle in the complex plane, we can apply the theorem of [11] to the polynomial \( \left(\frac{n+d}{d}\right) + \left(\frac{n+d-k}{d}\right) \). On the other hand, when \( m = 2 \), we can obtain other counterexample of Conjecture 1.1 when \( d = 37 \) and \( k = 19 \).

(b) When \( k = 1 \), one has \( g_{m,d,1}(n) = (m + 1)n + d \). Thus, its root is \(-d/(m + 1)\), which satisfies \(-d < -d/(m + 1) < 0\). When \( k = 2 \), one has \( g_{m,d,2}(n) = (m + 1)n^2 + (2d - m - 1)n + d(d - 1) \). If its discriminant is negative, then the real part of its roots is \(-d/(m + 1) + 1/2\), which satisfies \(-d + 1/2 < -d/(m + 1) + 1/2 < 1/2\).

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