Cluster Estimates and Analytic Wavefunctions

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Abstract

The Tomita-Takesaki modular theory is used to establish a cluster estimate extending and modifying that of Thomas and Wichmann [11], so as to extend it to regions within which the relevant observables are not necessarily spacelike separated. This sort of estimate is then applied to the case of a massive free field, to show that wavefunctions localized in a certain sense are analytic functions of momentum.

I Introduction

In [11] Thomas and Wichmann used the Tomita-Takesaki modular theory to demonstrate one version of the well-known [1,6] exponential decay of matrix elements of products of spacelike separated operators, or equivalently of matrix elements of spatial translation operators between localized vectors. These sorts of results are called cluster estimates because they express the decay of correlations between clusters of observables as the spatial separation between clusters increases. Here we present an extended version of their estimate, one which extends to the region within which the observables are not strictly spacelike separated. This version uses localized vectors that have their high-energy behavior tamed by multiplication by an exponential cutoff $e^{-\rho H}$, where $H$ is the Hamiltonian and $\rho$ is a constant representing the size of their region of...
localization. Once we have established this sort of estimate, we will present an application: a proof that certain localized free-field wavefunctions are analytic functions of momentum.

As in [11], the general framework is either that of a relativistic quantum field theory [10], or else that of a relativistic quantum system of von Neumann algebras of local observables [7]. In both cases a central feature of the theory is a map from certain subsets \( \mathcal{O} \) of Minkowski space to algebras of operators on a Hilbert space \( \mathcal{H} \): in the first case, algebras \( P(\mathcal{O}) \) of unbounded averaged field operators defined on a common domain; in the second, von Neumann algebras \( \mathcal{B}(\mathcal{O}) \) of bounded operators considered to be observable within the region \( \mathcal{O} \). This map is such that \( P(\mathcal{O}_1) \subset P(\mathcal{O}_2) \), or \( \mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2) \), whenever \( \mathcal{O}_1 \subset \mathcal{O}_2 \).

In both cases the Hilbert space \( \mathcal{H} \) carries a strongly continuous unitary representation \( U_\lambda \) of the universal covering group \( \hat{\mathcal{P}} \) of the Poincaré group \( \mathcal{P} \), such that if \( \lambda \) is any Poincaré transformation, and \( \lambda \mathcal{O} \) is the image of \( \mathcal{O} \) under \( \lambda \), then \( U_\lambda P(\mathcal{O}) U_\lambda^{-1} = P(\lambda \mathcal{O}) \), or \( U_\lambda \mathcal{B}(\mathcal{O}) U_\lambda^{-1} = \mathcal{B}(\lambda \mathcal{O}) \). The spectral condition requires that the representation be such that the spectrum of the translations is confined to the forward light cone. For the purposes of these estimates, it will be necessary to assume that there is a mass gap: apart from the unique Poincaré-invariant vacuum vector \( \Omega \), the spectrum of the translations is supported above the mass hyperboloid with mass \( m_0 \).

In both frameworks it is possible to define vectors localized in \( \mathcal{O} \) in a certain sense, namely that they are produced by the application of self-adjoint local (bounded or unbounded) operators to the vacuum; in other words, \( P(\mathcal{O}) \text{sa} \Omega \) or \( \mathcal{B}(\mathcal{O}) \text{sa} \Omega \). If the quantum field theory and the system of local algebras are locally associated in an appropriate sense [5] then these two sets will have the same closure, a closed real-linear manifold \( R(\mathcal{O}) = \overline{P(\mathcal{O}) \text{sa} \Omega} = \overline{\mathcal{B}(\mathcal{O}) \text{sa} \Omega} \). The Reeh-Schlieder principle implies that \( R(\mathcal{O}) + iR(\mathcal{O}) \) is dense in \( \mathcal{H} \); we will consider the vectors in \( R(\mathcal{O}) + iR(\mathcal{O}) \) to be localized in \( \mathcal{O} \), and it is with these localized vectors that we will primarily be concerned. They are not, of course, strictly localized [8]; if they were, we would not be discussing the decay of their inner products at spacelike separations. However, they are natural analogues of the strictly localized wavefunctions of non-relativistic quantum mechanics, and they are natural objects of study within these frameworks.

There are several facts available to us about these localized vectors. First, we have the information provided by the principle of locality, that observables localized in spacelike separated regions commute. Since we wish to allow for the possibility of
fermionic fields and local operators, which are not strictly observable, we must generalize this to include anticommutation. For this we will use the device of Bisognano and Wichmann [2], and define \( Y^z = Z Y Z^{-1} \), where \( Z = (I + iU_0)/(1+i) \), and \( U_0 \) represents the rotation by \( 2\pi \) about any axis. Then the condition of locality simply states that \([X,Y^z] = 0\) whenever \( X \) and \( Y \) are localized in spacelike separated regions. If \( X \) and \( Y \) are self-adjoint, this implies that the inner product \( \langle X \Omega | Y^z \Omega \rangle = \langle X \Omega | ZY \Omega \rangle \) is purely real. If \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are spacelike separated, it follows that \( \langle \psi | \phi \rangle \) must be real whenever \( \psi \in R(\mathcal{O}_1) \) and \( \phi \in ZR(\mathcal{O}_2) = R(\mathcal{O}_2) \). The principle of duality goes further, and requires that \( R(\mathcal{O}^c) \) consist precisely of those vectors whose inner products with all vectors in \( R(\mathcal{O}) \) are real, where \( \mathcal{O}^c \) is the spacelike complement of the region \( \mathcal{O} \). This can be (and usually is) regarded as a maximality requirement on the local algebras, and we will hereafter assume it. (Rieffel [9] shows that the version stated here implies the usual one.)

The principle of special duality [2] goes further still, and requires that the real-linear manifolds \( R(W_R) \) and \( R(W_L) \) for the wedge regions \( W_R = \{x_3 > |t|\} \) and \( W_L = \{x_3 < -|t|\} \) be specifically described in a certain way. It is known from the Tomita-Takesaki modular theory that they can always be given as \( R(W_R) = \{\psi \mid \psi = J \Delta^{1/2}\psi\} \) and \( R(W_L) = \{\psi \mid \psi = J \Delta^{-1/2}\psi\} \) for some modular operators \( J, \Delta^{1/2} \), where \( \Delta \) is a positive (unbounded) operator and \( J \) is an antilinear involution. Special duality specifies the form of these modular operators as follows. Let \( V_3(t) = V(t, \hat{x}_3) \) be the representatives of the velocity transformations in the \( \hat{x}_3 \) direction, whose natural action on Minkowski space is given by the matrix

\[
M(t) = \begin{pmatrix}
\cosh t & 0 & 0 & \sinh t \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\sinh t & 0 & 0 & \cosh t
\end{pmatrix}.
\]

To produce the modular operator \( \Delta^{1/2} \) for the right wedge, it is necessary to perform an analytic continuation: let the complex variable \( t \) now be \( \sigma + i\tau \), and let \( D(i\tau) \) be the domain of \( V_3(t) \), which depends only on \( \tau \), and is such that \( D(i\tau') \supset D(i\tau) \) whenever \( 0 \leq \tau' \leq \tau \). Then the modular operator for the right wedge is \( \Delta^{1/2} = V_3(i\pi) \), with domain \( D(i\pi) = R(W_R) + iR(W_R) \), and the modular operator for the left wedge is \( \Delta^{-1/2} = V_3(-i\pi) \), with domain \( D(-i\pi) = R(W_L) + iR(W_L) \). The modular conjugation \( J \) is given by \( J = ZU\Theta \), where \( Z \) is as above, \( U \) represents a rotation by angle \( \pi \) about the 3-axis, and \( \Theta \) is the TCP operator. The result of
Bisognano and Wichmann [2] is that this relation holds in the case of algebras produced from Wightman fields, and thus also in the case of local algebras locally associated to Wightman fields. We will assume hereafter that special duality does indeed hold.

Notice that \( R(W_R^R) \) and \( R(W_R^L) \) depend on the TCP operator \( \Theta \), but if we are interested only in \( R(W_R^R) + iR(W_R^L) \) and \( R(W_L^R) + iR(W_L^L) \), these depend only on the representation of the Lorentz group. If two systems—for example, an interacting field theory and its asymptotic free theory—share a common representation of \( \hat{P} \), then they have the same localized vectors for any wedge region. This does not necessarily imply anything about the localized vectors for other regions, such as the double-cone regions; we know that \( R(K) \subset R(W) \) if \( K \subset W \), but there is no reason to think that we can compute \( R(K) \) even given all of the \( R(W) \). In particular, it is not the case that \( R(K) = \text{intersection of all } R(W) \) for all \( W \supset K \). This will become clearer in the third section, when we discuss \( R(K) \) in the particular case of the free fields.

II A Cluster Estimate

The form of the cluster estimate due to Fredenhagen [6] was originally stated in terms of bounded operators \( A \) and \( B \), which we can take to be localized in regions separated by a spacelike distance \( s \); then his estimate was

\[
|\langle \Omega | AB\Omega \rangle - \langle \Omega | A\Omega \rangle \langle \Omega | B\Omega \rangle| \leq e^{-s \text{smo}} \sqrt{\|A^*\Omega\| \|A\Omega\| \|B^*\Omega\| \|B\Omega\|}. \tag{2}
\]

This form explicitly exhibits the cluster decomposition nature of the estimate, but it will be convenient hereafter to use a slightly different form. Let \( F \) be the projection onto \( \{\Omega\}^\perp \), the orthogonal complement of the vacuum; \( F \) will commute with \( \hat{P} \), and \( F\psi \) will be a localized vector if and only if \( \psi \) is. Then we may write the left-hand side of (2) as \( |\langle A^*\Omega \mid FB\Omega \rangle| \), and the estimate will appear as a bound on the matrix elements of inner products of the localized vectors \( FA^*\Omega \) and \( FB\Omega \). Thomas and Wichmann [11] established the following estimate: let \( \psi \in D(-i\pi/2) \) and \( \phi \in D(i\pi/2) \); then for \( s \geq 0 \),

\[
|\langle \psi | T(s\hat{x}_3)F\phi \rangle| \leq e^{-s \text{smo}} \|FV_3(-i\pi/2)\psi\| \|FV_3(i\pi/2)\phi\|, \tag{3}
\]

where \( T(x) \) is the representative of the translation by \( x \). This can be placed in the form of (2) by noting that if \( \phi = B\Omega \), where \( B \) is local to the right wedge, then \( \phi \in D(i\pi) \supset D(i\pi/2) \), and

\[
\|FV(i\pi/2)\phi\|^2 = \langle FB\Omega \mid FV(i\pi)B\Omega \rangle = \langle FB\Omega \mid FJB^*\Omega \rangle \leq \|B\Omega\| \|B^*\Omega\|. \tag{4}
\]
likewise if \( \psi = A^\ast \Omega \), where \( A \) is local to the left wedge, then \( \psi \in D(-i\pi) \supset D(-i\pi/2) \), and

\[
\|FV(-i\pi/2)\psi\|^2 = \langle F A^\ast \Omega | FV(-i\pi) A^\ast \Omega \rangle = \langle F A^\ast \Omega | FJ A \Omega \rangle \leq \|A\Omega\| \|A^\ast \Omega\| .
\] (5)

These estimates are useful in case the two regions of localization are separated by some positive distance \( s \), but in some cases one wishes to have an estimate for matrix elements like those of (3) that will also cover the case in which the regions of localization overlap. That is what we will provide here.

We find that if we multiply \( \psi \) and \( \phi \) by an exponential cutoff in energy \( e^{-\rho H} \), a modification which presumably improves these vectors’ high-energy behavior, we can in fact produce estimates that do not depend on strict spacelike separation. Cutoffs of this sort have been considered in connection with nuclearity requirements [3], in an attempt to characterize models with reasonable particle interpretations, but it is not immediately clear why they should appear in cluster estimates. Nevertheless, we will see that just such a cutoff provides the essential element in eliminating the requirement of spacelike separation.

From the definition of the Poincaré group we find that \( V_3(t)T(x) V_3(t)^{-1} = T(M(t)x) \) for real \( t \) and \( x \), and this relation can be extended by analytic continuation on the appropriate domains, bearing in mind that \( V_3(t)^{-1} = V_3(-t) = V_3(t^\ast)^\dagger \), and that \( T(i\rho \hat{x}_0) = e^{-\rho H} \) is bounded for \( \rho \geq 0 \). We can then establish the following:

**Theorem 1:** Let \( \psi \) and \( \phi \) be two vectors, and let \( \rho > 0 \) be such that \( T(\rho \hat{x}_3) \phi \in D(i\pi/4) \) and \( T(-\rho \hat{x}_3) \psi \in D(-i\pi/4) \). Then for all \( s \geq 0 \),

\[
\left| \langle e^{-\rho H} \psi | T(s \hat{x}_3) F e^{-\rho H} \phi \rangle \right| \leq e^{-s\rho_0/\sqrt{2}} \| FV_3(-i\pi/4) e^{-\rho H} \psi \| \| FV_3(i\pi/4) e^{-\rho H} \phi \|. \] (6)

**Proof:** Notice that

\[
V_3(\pm i\tau)e^{-\rho H}T(\mp \rho \hat{x}_3)V_3(\mp i\tau) = T\left( \begin{array}{cccc}
\cos \tau & 0 & 0 & \pm i \sin \tau \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\pm i \sin \tau & 0 & 0 & \cos \tau \\
\end{array} \right) \left( \begin{array}{c}
i \rho \\
0 \\
0 \\
\mp \rho \\
\end{array} \right) \] (7)

\[
= e^{-\rho (\cos \tau \mp i \sin \tau)} T(\mp \rho (\cos \tau + \sin \tau) \hat{x}_3)
\]
gives a bounded operator for $0 \leq \tau \leq \pi/4$. Thus
\begin{align*}
V_3(-i\tau)e^{-\rho H}\psi &= \left(V_3(-i\tau)e^{-\rho H}T(\rho \hat{x}_3)V_3(i\tau)\right) \left(V_3(-i\tau)T(-\rho \hat{x}_3)\psi\right) \\
\text{and} \\
V_3(i\tau)e^{-\rho H}\phi &= \left(V_3(i\tau)e^{-\rho H}T(-\rho \hat{x}_3)V_3(-i\tau)\right) \left(V_3(i\tau)T(\rho \hat{x}_3)\phi\right)
\end{align*}
(8)

(9)
can be defined for $0 \leq \tau \leq \pi/4$. If $t = \sigma + i\tau$, then
\begin{align*}
T(M(t)s\hat{x}_3) &= T \left(\begin{bmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh t & 0 & 0 & \cosh t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \right) \\
&= e^{-s \cosh \sigma \sin \tau (H^2 + 4)} e^{-\rho H} \left(\begin{bmatrix} \cosh (t + \pi) & 0 & 0 & \sinh (t + \pi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh (t + \pi) & 0 & 0 & \cosh (t + \pi) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ s \end{bmatrix} \right)
\end{align*}
(10)
is also defined and bounded for $0 \leq \tau \leq \pi/4$. It follows that for any fixed $s \geq 0$ the function
\begin{align*}
\xi(t; s) &= \left\langle V_3(t^*)e^{-\rho H}\psi \middle| T(M(t)s\hat{x}_3)FV_3(t)e^{-\rho H}\phi \right\rangle \\
\xi(0; s) &= \xi(i\tau; s) = \left\langle e^{-\rho H}\psi \middle| T(s\hat{x}_3)F\hat{x}_3)FV_3(i\tau)e^{-\rho H}\phi \right\rangle
\end{align*}
(11)
is defined and continuous as a function of $t$ on the strip $0 \leq \tau \leq \pi/4$, and analytic on its interior. But $\xi(\sigma; s) = \xi(0; s) = \left\langle e^{-\rho H}\psi \middle| T(s\hat{x}_3)F\hat{x}_3)FV_3(i\tau)e^{-\rho H}\phi \right\rangle$ is constant on the real axis, and hence constant throughout the strip. Thus
\begin{align*}
\xi(0; s) &= \xi(i\tau; s) = \left\langle V_3(-i\tau)e^{-\rho H}\psi \middle| e^{-s H \sin \tau}T(s \cos \tau \hat{x}_3)FV_3(i\tau)e^{-\rho H}\phi \right\rangle
\end{align*}
(12)
for any $\tau$ with $0 \leq \tau \leq \pi/4$. But since $H \geq m_0 I$ on the orthogonal complement of the vacuum, $e^{-s H \sin \tau} \leq e^{-s m_0 \sin \tau} I$ on the range of $F$. Thus
\begin{align*}
|\xi(0; s)| &\leq e^{-s m_0 \sin \tau} \left\|FV_3(-i\tau)e^{-\rho H}\psi\right\| \left\|T(s \cos \tau \hat{x}_3)FV_3(i\tau)e^{-\rho H}\phi\right\| \\
&= e^{-s m_0 \sin \tau} \left\|FV_3(-i\tau)e^{-\rho H}\psi\right\| \left\|FV_3(i\tau)e^{-\rho H}\phi\right\|
\end{align*}
(13)
of which (6) is the extreme case $\tau = \pi/4$.

For example, we might choose $\psi, \phi \in R(K) + iR(K)$ for some double-cone $K$, and let $\rho$ be such that $T(\rho \hat{x}_3)K \subset W_R$ and $T(-\rho \hat{x}_3)K \subset W_L$. Then $T(\rho \hat{x}_3)\psi \in D(i\pi) \supset D(i\pi/4)$, and similarly $T(-\rho \hat{x}_3)\psi \in D(-i\pi) \supset D(-i\pi/4)$. If we compare the result (6) with the estimate (3), we see that in both cases there is an exponential decay; however, the coefficient in the exponent in (6) is weaker by a factor of $1/\sqrt{2}$. In addition, the constant prefactor is somewhat different. However, in Theorem 1, the two vectors are
no longer required to be spacelike separated; their regions of localization may overlap
by a distance \( \rho \), so that the two vectors \( \psi \) and \( \phi \) might be localized in the same region. If this is the case, then it is possible to interchange \( \psi \) and \( \phi \), and to obtain an estimate of the form

\[
\left| \left\langle e^{-\rho H} \psi \right| T(s \mathbf{x}_3) F e^{-\rho H} \phi \right\rangle \leq C e^{-|s|m_0/\sqrt{2}} \tag{14}
\]

for all \( s \). We may then go further, and allow the direction to vary, and obtain estimates of the form

\[
\left| \left\langle e^{-\rho H} \psi \right| T(\mathbf{x}) F e^{-\rho H} \phi \right\rangle \leq C e^{-|x|m_0/\sqrt{2}} \tag{15}
\]

for a general spatial translation \( T(\mathbf{x}) \), for some suitable \( \rho \), and some constant \( C \). It is estimates of this sort that will make possible the theorem of the next section, which could not be established with purely spacelike-separated estimates like (3).

### III Free-Field Wavefunctions

We will now specialize to the case of a free-field theory, in particular a theory of a single free particle of mass \( m \) and spin \( s \). The Hilbert space \( \mathcal{H} \) is a symmetric or antisymmetric Fock space based on the one-particle Hilbert space \( \mathfrak{h} \). When there is only a single species of particle, the two-particle states will be symmetric for bosons, or antisymmetric for fermions; but where there are multiplets of particles, the two-particle states may be either spatially symmetric or spatially antisymmetric, and for this reason everything here will be framed so as to include both cases. Therefore we generally suppress indications of symmetrization or antisymmetrization, and use an unsymmetrized tensor product. For real linear manifolds \( r \), tensor products will always be taken to be real, formed by taking limits of real linear combinations.

The one-particle Hilbert space is \( \mathfrak{h} = \mathcal{L}^2(\mathbb{R}^3)^s \), with the action of \( \hat{\mathcal{P}} \) on it given by

\[
(u(a, \Lambda)f)_i(p) = e^{ia \cdot p} \sqrt{\frac{\omega(p \Lambda^{-1})}{\omega(p)}} D_{ij}^{(s)}(u_w(\Lambda; p)) f_j(p \Lambda^{-1}), \tag{16}
\]

where \( \omega(p) = \sqrt{p^2 + m^2} \), \( \Lambda \) is a Lorentz transform, \( p_\Lambda \) is the spatial part of its action on \( (\omega(p), \mathbf{p}) \), and \( u_w(\Lambda; p) \) is the Wigner rotation corresponding to \( \Lambda \) and \( p \). Furthermore the action of the TCP operator \( \Theta \) on \( \mathfrak{h} \) will be given by \( (\Theta f)_j(p) = e^{-i\pi j} f_{-j}(p) \). For a region \( \mathcal{O} \) of spacetime, we will define a real linear manifold \( r(\mathcal{O}) \) in \( \mathfrak{h} \); then the local algebra \( \mathfrak{B}(\mathcal{O}) \) will be generated by the Weyl operators corresponding to elements of
r(\mathcal{O})$, and the restriction of $R(\mathcal{O})$ to the $n$-particle subspace will be just the $n$-fold real symmetric or antisymmetric tensor product of $r(\mathcal{O})$ with itself.

We will define first of all $r(\mathcal{O})$ for regions that are the causal completion of a base subset $\mathcal{O}_0$ of the plane $t = 0$; then from these the manifolds for other regions can be produced by Poincaré transformation. We can describe explicitly $r(W)$ for the right wedge, whose base is a half-space, according to the Bisognano-Wichmann result: a wavefunction $f$ is in $r(W)$ if and only if $f = ZU\Theta V_3(i\pi)f$, where $U$ is a rotation by angle $\pi$ about the 3-axis. The manifolds for other such wedges can be derived from this one by rotation and spatial translation. If $\mathcal{O}_0$ is, for example, a sphere in 3-space, so that $\mathcal{O}$ is a double cone $K$, then $r(K)$ can be defined as the intersection of the manifolds for the wedges whose bases contain $\mathcal{O}_0$.

From this we can see that, as stated earlier, $R(K)$ is not equal to the intersection of the $R(W)$ for all $W \supset K$. For example, we may consider the two-particle component $R^{(2)}(K) = r(K) \otimes r(K)$ of $R(K)$. Since $r(K)$ is equal to the intersection of the $r(W)$ for all $W \supset K$, it follows that

$$R^{(2)}(K) = \bigcap_{W_1, W_2 \supset K} r(W_1) \otimes r(W_2),$$

which is strictly smaller than the intersection of $R^{(2)}(W) = r(W) \otimes r(W)$ for all $W \supset K$. Similar results hold for the $n$-particle components where $n > 2$. These considerations will play a large role in the proof of the theorem to follow, in which it will be necessary to employ translations and rotations by differing amounts in each variable separately.

Suppose $\mathcal{O}$ is a region with some geometrical symmetries, represented by operators $U_i$; the action of $U_i$ on a free-field theory will be the multiplicative promotion of the restriction $u_i$ of $U_i$ to the one-particle space. The modular operators $J_\mathcal{O}$, $\Delta_\mathcal{O}$ will also be multiplicative promotions of their restrictions $j_\mathcal{O}$, $\delta_\mathcal{O}$ to the one-particle space. But $j_\mathcal{O}$ and $\delta_\mathcal{O}$ will commute with every $u_i$; it follows that on the $n$-particle subspace, $J_\mathcal{O}$ and $\Delta_\mathcal{O}$ will commute not only with each $U_i$, but with every operator of the form $u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$.

Note that the one-particle Hilbert space $\mathfrak{h}$ as a whole contains $2s+1$ vectors linearly independent up to multiplication by an overall function $a(p)$ of momentum—that is, up to translation. But by the Reeh-Schlieder principle, the translations of any manifold $r(\mathcal{O})$ are total in $\mathfrak{h}$, so each such manifold must also contain $2s+1$ vectors linearly independent up to multiplication by a function of momentum.

In non-relativistic quantum mechanics, one obvious feature of a (strictly) localized
wavefunction is that it is real-analytic in every component of every momentum variable: since it is localized in position space, by a Paley-Wiener theorem the Fourier transform, which is the momentum-space wavefunction, is real-analytic. In fact, the Paley-Wiener theorem gives a precise characterization of its analyticity, depending on the region within which it is localized; but for now, let us consider only the fact that it is analytic in some neighborhood of the real axes. Since the localized wavefunctions of relativistic quantum mechanics are not strictly localized, we cannot expect the finer analyticity properties to carry over, but we may well ask whether these wavefunctions are still real-analytic.

For $\mathcal{O}$ a compact region, it is possible to show directly that the real linear manifold $r(\mathcal{O})$ consists entirely of real-analytic functions of $\mathbf{p}$. For the wedge regions, although the wavefunctions are not necessarily real-analytic functions of the momentum, it is possible to characterize them precisely based on their analyticity in certain other variables [4]. Nevertheless, it is not at all evident that analyticity properties of this sort will hold for the two-particle wavefunctions $h_{jk}(\mathbf{p}_a, \mathbf{p}_b) \in r(\mathcal{O}) \otimes r(\mathcal{O})$, or for more general wavefunctions in $R(\mathcal{O}) + iR(\mathcal{O})$.

However, we can then use estimates like those of Theorem 1 to derive the following (wherein we will for the moment ignore symmetrization and antisymmetrization):

**Theorem 2:** If $K$ is any double cone, then for any $h \in r(K) \otimes r(K)$, each component $h_{jk}(\mathbf{p}_a, \mathbf{p}_b)$ is real-analytic in the $\mathbf{p}_a, \mathbf{p}_b$.

**Proof:** Without loss of generality we may, for simplicity, assume that $K$ has its base in the plane $t = 0$ and its center at the origin, with radius $\rho$; the result for arbitrary double cones follows by Poincaré transformation. Let $f = f_j(\mathbf{p}_a)$ and $g = g_k(\mathbf{p}_b)$ be one-particle wavefunctions associated with $r(K)$, and let us write $f_\mathbf{x}$ and $g_\mathbf{y}$ for the exponentially cut off and spatially translated wavefunctions $f_\mathbf{x} = T(0, \mathbf{x})e^{-\rho \mathbf{H}} f$ and $g_\mathbf{y} = T(0, \mathbf{y})e^{-\rho \mathbf{H}} g$. The estimate we will use is that

$$\left| \left< f_\mathbf{x} \otimes g_\mathbf{y} | e^{-\rho \mathbf{H}} h \right> \right| \leq C e^{-m(|\mathbf{x}|+|\mathbf{y}|)/\sqrt{2}}$$

(18)

where $C$ is some positive constant (dependent on the choice of $f$ and $g$). This will follow from Theorem 1, but let us postpone the derivation for a moment and consider the consequences. We can rewrite this as

$$\left| \int d^3 \mathbf{p}_a e^{i\mathbf{x} \cdot \mathbf{p}_a} e^{i\mathbf{y} \cdot \mathbf{p}_b} e^{-2(\omega_\mathbf{a}+\omega_\mathbf{b})} f^*_j(\mathbf{p}_a) g^*_k(\mathbf{p}_b) h_{jk}(\mathbf{p}_a, \mathbf{p}_b) \right| \leq C e^{-m(|\mathbf{x}|+|\mathbf{y}|)/\sqrt{2}}.$$  

(19)

The left-hand side is a Fourier transform of the function

$$h'(\mathbf{p}_a, \mathbf{p}_b) = e^{-2(\omega_\mathbf{a}+\omega_\mathbf{b})} f^*_j(\mathbf{p}_a) g^*_k(\mathbf{p}_b) h_{jk}(\mathbf{p}_a, \mathbf{p}_b);$$

(20)
thus by a Paley-Wiener theorem \( h' \) is real-analytic in \( p_a \) and \( p_b \). But \( f \) and \( g \) were arbitrary, save for the restriction that they be local wavefunctions; since they are local, their components are analytic, and there are \( 2s + 1 \) of them linearly independent up to multiplication by functions of momentum. This implies that each \( h_{jk} \) must be real-analytic.

Let us now derive the estimate. The methods will be those of Theorem 1, but we will also need unitary operators defined only on the two-particle subspace, corresponding to translations and rotations by differing amounts in each variable. These we will denote by \( T(x; y) = T(x) \otimes T(y) \) and \( R(\theta, \hat{x}; \varphi, \hat{y}) = R(\theta, \hat{x}) \otimes R(\varphi, \hat{y}) \); we will generally suppress the axes and simply write \( R(\theta; \varphi) \). Since the rotations are symmetries of \( K \), we have already argued that the individual rotations \( R(\theta; \varphi) \) must commute with \( J_K \) and \( \Delta_K \) on the two-particle subspace, and thus will take \( r(K) \otimes r(K) \) onto itself; in other words, \( R(\theta; \varphi) (r(K) \otimes r(K)) = R(\theta) r(K) \otimes R(\varphi) r(K) = r(K) \otimes r(K) \). For any particular \( x \) and \( y \), we can find an \( R(\theta; \varphi) \) such that \( R(\theta)x \) and \( R(\varphi)y \) both lie in the negative \( \hat{x}_3 \) direction; thus \( R(\theta; \varphi) f_x \otimes g_y = R(\theta) f_x \otimes R(\varphi) g_y \in e^{-\rho H} W_L \). Then as in Theorem 1,

\[
\left| \langle f_x \otimes g_y \mid e^{-\rho H} h \rangle \right| = \left| \langle e^{-\rho H} f \otimes g \mid T(0, -x; 0, -y) e^{-\rho H} h \rangle \right| \\
= \left| \langle R(\theta; \varphi) e^{-\rho H} f \otimes g \mid T(|x| \hat{x}_3; |y| \hat{x}_3) R(\theta; \varphi) e^{-\rho H} h \rangle \right| \\
= \left| \langle V_3(-i\tau) R(\theta; \varphi) e^{-\rho H} f \otimes g \mid V_3(i\tau) R(\theta; \varphi) e^{-\rho H} h \rangle \right| \\
\leq e^{-m(|x| + |y|)} \sqrt{2} \left| V_3(-i\pi/4) R(\theta; \varphi) e^{-\rho H} f \otimes g \right| \left| V_3(i\pi/4) R(\theta; \varphi) e^{-\rho H} h \right| .
\]

Then the constant \( C \) will be the supremum of the product of the two norms above as \( R(\theta; \varphi) \) varies over all individual rotations (in fact, a suitably chosen finite set of rotations would suffice), or equivalently as the directions of \( x \) and \( y \) vary. This establishes the estimate (18), and the theorem.

This theorem clearly generalizes to the case of three or any larger number of particles, although notational difficulties would stand in the way of writing out the proof in the general case. Furthermore this real analyticity is not affected by symmetrization or antisymmetrization, nor by extension to the (open) complex linear span. Thus we may set it out generally, that \( n \)-particle wavefunctions localized in the sense of belonging
to \( R(\mathcal{O}) + iR(\mathcal{O}) \) are real-analytic functions of every component of every momentum variable.

IV Conclusion

We have seen that introducing an exponential cutoff in energy enables us to establish a cluster estimate that covers all spatial translations, not just those which leave the observables concerned strictly spacelike separated. Estimates of this sort have a direct application to free-field theories, in which they can be used to establish the momentum-space analyticity of localized wavefunctions.
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