On the 4-girth-thickness of the line graph of the complete graph

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Abstract

The $g$-girth-thickness $\theta(g, G)$ of a graph $G$ is the minimum number of planar subgraphs of girth at least $g$ whose union is $G$. In this note, we give the 4-girth-thickness $\theta(4, L(K_n))$ of the line graph of the complete graph $L(K_n)$ when $n$ is even. We also give the minimum number of subgraphs of $L(K_n)$, which are of girth at least 4 and embeddable on the projective plane, whose union is $L(K_n)$.

Keywords: girth-thickness, $S$-thickness, planar decomposition, line graph, token graph.

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1 Introduction

The thickness $\theta(G)$ of a graph $G$ is the minimum number of elements in any partition of $E(G)$ such that the induced subgraph of each part is a planar graph. Equivalently, $\theta(G)$ is defined as the minimum number of planar subgraphs whose union is $G$.

The thickness has drawn the attention of several researchers since its introduction in the 60s [20] because it is an NP-hard problem [16] and it has many applications, for instance, in the design of circuits [1], in the Ringel’s earth-moon problem [14] and to bound the achromatic numbers of planar graphs [3], see the survey [17].

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Only some exact results are known, for example, when $G$ is a complete graph \[2, 5, 6\], a hypercube \[15\], or a complete multipartite graph \[7, 13, 21, 22\]. And some generalizations of the thickness also have been studied such that the outerthickness $\theta_o$, defined similarly but with outerplanar instead of planar \[12\], and the $S$-thickness $\theta_S$, considering the thickness on a surfaces $S$ instead of the plane \[4\].

The $g$-girth-thickness $\theta(g, G)$ of a graph $G$, introduced in \[18\], is the minimum number of elements in any partition of $E(G)$ such that the induced subgraphs of each part is a planar graph of girth at least $g$. The $g$-girth-thickness is the usual thickness when $g = 3$ and it is the *arboricity number* when $g = \infty$. Recall that the *girth* of a graph is the size of its shortest cycle or $\infty$ if it is acyclic.

Exact results also are known when $g > 3$ and finite, for instance, the $4$-girth-thickness of the complete graph \[9, 11, 18\], the $4$-girth-thickness of the complete multipartite graph \[11, 19\] and the $6$-girth-thickness of the complete graph \[9\]. Owing to the fact that the hypercube and the complete bipartite are triangle-free graphs, their thickness equal their $4$-girth-thickness which were calculate in \[15\] and partially calculate in \[7, 13\], respectively.

We define the $S$-$g$-girth-thickness $\theta_S(g, G)$ of a graph $G$ as the minimum number of subgraphs embeddable on a surface $S$ of girth at least $g$ whose union is $G$. Of course, if $G$ has girth $g$ then $\theta_S(g, G)$ is $\theta_S(G)$ as in the case of $K_{n,n}$ for $g = 4$, see \[4\].

In this note, we obtain the 4-girth-thickness $\theta(4, L(K_n))$ of the line graph $L(K_n)$ of the complete graph $K_n$ when $n$ is even. To achieve this, in Section 2 we recall some properties about token graphs $F_k(G)$. In Section 3, we determine $\theta(4, F_2(G))$ when $G$ contains a factorization into Hamiltonian paths, in particular

$$\theta(4, L(K_n)) = \frac{n}{2} \text{ and } \theta(4, F_2(K_{n-1,n})) = \frac{n}{2}$$

for $n$ even. Finally, in Section 4 we determine $\theta_S(4, F_2(G))$ when $S$ is the projective plane and $G$ contains a Hamiltonian-factorization, in consequence

$$\theta_S(4, L(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } \theta(4, F_2(K_{2n,2n})) = n$$

for all $n$.

## 2 Token graphs

Consider the following graph $F_k(G)$ called the *$k$-token graph* introduced in \[10\], for given an integer $k \geq 1$ and a graph $G$ of order $n$. The vertex set $V(F_k(G))$ is the family of $k$ subsets of $V(G)$, therefore $|V(F_k(G))| = \binom{n}{k}$. Two such $k$-subsets $X$ and $Y$ are adjacent if its symmetric difference $X \Delta Y = \{x, y\}$ such that $x \in X$, $y \in Y$ and $xy \in E(G)$. The size of
Figure 1: The 2-token graph of the path of order 6.

$F_k(G)$ is $(\frac{n-2}{k-1})|E(G)|$, see [10]. An example of a 2-token graph is showed in Figure 1, which is the $F_2(P_6)$.

The 2-token graph $F_2(K_n)$ of the complete graph $K_n$ is the line graph $L(K_n)$ of the complete graph $K_n$ because each pair of incident edges $xz$ and $zy$ has symmetric difference the set $\{x,y\}$ which is the edge $xy$ of the complete graph. In general, the Johnson graph $J(n,k) \cong F_k(K_n)$ owing to the fact that it is the graph whose vertices are the $k$-subsets of an $n$-set, where two such subsets $X$ and $Y$ are adjacent whenever $|X \cap Y| = k - 1$.

In [8], the authors remark that the 2-token graph $F_2(P_n)$ of the path graph $P_n$ of $n$ vertices is planar of girth at least 4 for every $n$.

Now, we prove that an edge-partition of a graph $G$ induces an edge partition of $F_k(G)$.

**Lemma 2.1.** Let $G$ be a non empty graph and $P = \{E_1, \ldots, E_l\}$ an edge-partition of $G$. Then the set $\{E'_1, \ldots, E'_l\}$ is an edge-partition of $F_k(G)$ where $E'_i = E(F_k(G[E_i]))$ for all $i \in \{1, \ldots, l\}$.

**Proof.** Let $XY$ be an edge of $F_k(G)$, that is, $X$ and $Y$ are $k$-subsets of $V(G)$ such that for some $x \in X$ and some $y \in Y$, the symmetric difference of $X$ and $Y$ is $\{x,y\}$, and $xy$ is an edge of $G$. Let $j$ the unique index in the set $\{1, \ldots, l\}$ such that $xy \in E_j$. Thus $xy$ is an edge of $G[E_j]$ and in consequence $XY$ is an edge of $F_k(G[E_j]) = E'_j$. Then $XY \in E'_j$. Moreover, if $XY \in E_i$ for some $i \in \{1, \ldots, l\}$, then $xy \in E(G[E_i]) = E_i$. But $P = \{E_1, \ldots, E_l\}$ is an edge-partition of $G$, and $xy \in E_j$, so $i = j$. Therefore each edge of $F_k(G)$ is in a unique element of $\{E'_1, \ldots, E'_l\}$. In order to guarantee that every $E'_i$ is a non empty set, we need that $G$ has order at least $k + 1$. In that case, if $xy \in E_i$ and $U = \{g_1, \ldots, g_{k-1}\} \subseteq V(G) \setminus \{x,y\}$, then $X' = U \cup \{x\}$ and $Y' = U \cup \{y\}$ are two $k$-subsets of $G[E_i]$ such that its symmetric difference is $\{x,y\}$, and then $E'_i \neq \emptyset$, because $X'Y' \in E'_i$. $\square$
\section{Determining $\theta(4, L(K_n))$ for $n$ even}

A planar graph of $n$ vertices and girth at least 4 has at most $2(n - 2)$ edges for $n \geq 4$ and at most $n - 1$, otherwise. In consequence, the 4-girth-thickness $\theta(4, G)$ of a graph $G$ is at least $\lceil \frac{|E(G)|}{2(n-2)} \rceil$ for $n \geq 4$ and at least $\lceil \frac{|E(G)|}{n-1} \rceil$, otherwise.

Therefore we have the following theorem.

\textbf{Theorem 3.1.} If $G$ contains a factorization into $k$ Hamiltonian paths, then $\theta(4, F_2(G)) = k$.

\textbf{Proof.} For $G = K_2$ or $G = P_3$, it is easy to check that $\theta(4, F_2(G)) = 1$. Assume that $G$ is a graph of order $n \geq 4$ containing a factorization into Hamiltonian paths. Then $G$ has size $e = (n - 1)k \leq \binom{n}{2}$, then $k \leq n/2$ and

$$k < \frac{n}{2} + 1 + \frac{1}{n-3}.$$

Since, the 2-token graph $F_2(G)$ has order $\binom{n}{2}$ and size $(n - 2)(n - 1)k$, it follows that

$$\theta(4, F_2(G)) \geq \lceil \frac{(n - 2)(n - 1)k}{2\binom{n}{2}} \rceil = \left\lfloor k - \frac{2nk - 6k}{n^2 - n - 4} \right\rfloor.$$

Because $k < \frac{n}{2} + 1 + \frac{1}{n-3} = \frac{n^2 - n - 4}{2n - 6}$ then

$$0 < \frac{k(2n - 6)}{n^2 - n - 4} < 1$$

and we have

$$\theta(4, F_2(G)) \geq k.$$

By Lemma 2.1, the partition of $k$ Hamiltonian paths $\{G_1, \ldots, G_k\}$ of $G$ induces a partition of $F_2(G)$ into $k$ planar subgraphs of girth at least 4, $\{F_2(G_1), \ldots, F_2(G_k)\}$ and the result follows.

We have the following corollaries.

\textbf{Corollary 3.2.} If $n$ is even then $\theta(4, F_2(K_{n-1,n})) = n/2$.

\textbf{Corollary 3.3.} If $n$ is even then $\theta(4, L(K_n)) = n/2$.

\section{$\theta_S(4, L(K_n))$ when $S$ is the projective plane}

Although the problem of finding the minimum number of planar graphs of girth at least 4 into which the line graph of the complete graph can be decomposed remains partially
solved, the corresponding problem can be solved for the surface called the projective plane. A similar proof provide the solution.

On one hand, a maximal graph of order \( n \) and girth at least 4 embeddable in the projective plane \( S \) has size at most \( 2n - 2 \). On the other hand, since the 2-token graph of a cycle is a graph embeddable in \( S \) with girth 4, see Figure 2 for an example, we can give the following theorem.

![](image)

**Figure 2**: The 2-token graph of the cycle of order 6.

**Theorem 4.1.** If \( G \) is a graph of order \( n \geq 4 \) and contains a factorization into \( k \) Hamiltonian cycles, then \( \theta_S(4, F_2(G)) = k \) when \( S \) is the projective plane.

**Proof.** Let \( G \) be a graph of order \( n \geq 4 \) containing a Hamiltonian-factorization, that is, a factorization into Hamiltonian cycles. Then \( G \) has size \( e = nk \leq \binom{n}{2} \), then \( k \leq (n - 1)/2 \) and

\[
k < n + 1 + 2/(n - 2).
\]

Since, the 2-token graph \( F_2(G) \) has order \( \binom{n}{2} \) and size \( (n - 2)nk \), it follows that

\[
\theta_S(4, F_2(G)) \geq \left\lceil \frac{(n - 2)nk}{2\binom{n}{2} - 2} \right\rceil = \left\lceil k - \frac{nk - 2k}{n^2 - n - 2} \right\rceil.
\]

Because \( k < n + 1 + \frac{2}{n-2} = \frac{n^2-n-2}{n-2} \) then

\[
0 < \frac{k(n-2)}{n^2-n-2} < 1
\]

and we have

\[
\theta_S(4, F_2(G)) \geq k.
\]

By Lemma 2.1, the partition of \( k \) Hamiltonian cycles \( \{G_1, \ldots, G_k\} \) of \( G \) induces a partition of \( F_2(G) \) into \( k \) planar subgraphs of girth at least 4 embeddable in \( S \), \( \{F_2(G_1), \ldots, F(G_k)\} \) and the result follows. \( \square \)
We have the following corollaries.

**Corollary 4.2.** If $n$ is even then $\theta_S(4, F_2(K_n,n)) = n/2$.

**Corollary 4.3.** For all $n$, we have that $\theta_S(4, L(K_n)) = \lfloor \frac{n}{2} \rfloor$.

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