Abstract

We study multiple harmonic sums as functions of their upper bound, with $p$-adic values; this study is done in relation with the action of Frobenius on the de Rham fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We obtain results of "decomposition", which can be proven both in a geometric way and in a more elementary way. The comparison of the two methods provides a new way to compute explicitly $p$-adic multiple zeta values, which does not refer directly to iterated integration, and which involves in a natural way the version of $p$-adic multiple zeta values which expresses the Frobenius-invariant path. The results also lead to the definition of a geometric notion of finite multiple zeta values.

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1 Introduction

1.1 Definitions and notations

1.1.1 Multiple harmonic sums and finite multiple zeta values

Multiple harmonic sums are the following rational numbers: for \( N \in \mathbb{N}^* \), \( d \in \mathbb{N}^* \), \((s_1, \ldots, s_d) \in (\mathbb{N}^*)^d:\)

\[
H_N(s_d, \ldots, s_1) = \sum_{0 < n_1 < \ldots < n_d < N} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}
\]

(1)

Their properties have been studied in particular by analogy with multiple zeta values. A special role is played by the multiple harmonic sums \( H_p(s_d, \ldots, s_1) \) taken modulo \( p \), where \( p \) is a prime number. More generally, let us denote, for \( M, N \in \mathbb{Z} \) such that \( 0 \leq M < N \) or \( M < N \leq 0 \):

\[
H_{M<N}(s_d, \ldots, s_1) = \sum_{M<n_1<\ldots<n_d<N} \frac{1}{n_1^{s_1} \ldots n_d^{s_d}}
\]

Finite multiple zeta values have been defined more recently by Zagier. Let \( \mathcal{A} \) be the ring of integers modulo infinitely large primes:

\[
\mathcal{A} = \left( \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right) \otimes_{\mathbb{Z}} \mathbb{Q} = \left( \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right) / \left( \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \right)
\]

(2)

It is also the quotient of the adelic ring \( \prod_p \mathbb{Q}_p = \{(x_p) \in \prod_p \mathbb{Q}_p \mid v_p(x_p) \geq 0 \text{ for } p \text{ large}\} \) by the ideal \( \{(x_p) \in \prod_p \mathbb{Q}_p \mid v_p(x_p) \geq 1 \text{ for } p \text{ large}\} \).

Finite multiple zeta values are the numbers

\[
\zeta_A(s_d, \ldots, s_1) = (H_p(s_d, \ldots, s_1))_{p \text{ prime}} \in \mathcal{A}
\]

(3)

We introduce:

**Definition 1.1.** For each \( k \in \mathbb{N}^* \), we call finite multiple zeta values the following numbers, for \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), and \( a \in \mathbb{Z} \):

\[
\zeta_{f_k,a}(s_d, \ldots, s_1) = \left( (p^k)^{s_d+\ldots+s_1} H_{ap^k < (a+1)p^k}(s_d, \ldots, s_1) \right) \in \prod_{p \text{ prime}} \mathbb{Q}_p
\]

(4)

The results of this paper are the reason for this terminology; further justification is the paper \cite{J3}, in which we study algebraically finite multiple zeta values as variants of multiple zeta values.

For all \( k \in \mathbb{N}^* \), \( a \in \mathbb{Z} \), \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), we have:

\[
\zeta_{f_k,a}(s_d, \ldots, s_1) - \zeta_{f_1,a}(s_d, \ldots, s_1) \in \prod_{p \text{ prime}} p^{s_d+\ldots+s_1+1}\mathbb{Z}_p
\]

(5)

In particular, the \( p \)-adic valuations of the components of a finite multiple zeta values \( \zeta_{f_k,a}(s_d, \ldots, s_1) \) are lower bounded by its weight \( s_d + \ldots + s_1 \), and we have:

\[
\zeta_A(s_d, \ldots, s_1) = \left( p^{-s_d-\ldots-s_1} \zeta_{f_k,a}(s_d, \ldots, s_1) \mod p \right)_{p \text{ prime}}
\]

(6)
1.1.2 de Rham realization of the fundamental group of \( \mathbb{P}^1 - \{0, 1, \infty\} \)

Let \( X \) be the variety \( \mathbb{P}^1 - \{0, 1, \infty\} \) over \( \mathbb{Q} \). Let \( \left(\pi^d_{\text{dR}}\right)_x \) be the de Rham realization of the motivic fundamental groupoid of \( X \) (defined in \([D]\)). It is a groupoid of pro-unipotent schemes over \( X \), with base points the rational points of \( X \) and the rational points of \( \mathbb{P}^1 \setminus \{0\} \) with \( x \in \{0, 1, \infty\} \); it has also a canonical base point ([D] §12), which we denote by \( \eta \). There are canonical isomorphisms \( \pi^d_{\eta} \simeq \pi^d_{x} \), \( f \mapsto yf_x \), for all base-points \( x, y \), which are compatible to the groupoid structure. We will consider in particular the tangential base points 1 at 0 and \(-1\) at 1, denoted by \( \tilde{1}_0 \) and \( -\tilde{1}_1 \), or, for simplicity, 0 and 1.

The Lie algebra of \( \eta \pi^d_{\eta} \) is the pro-nilpotent free Lie algebra with two generators \( e_0 \) and \( e_1 \). The group \( \eta \pi^d_{\eta} \) is thus pro-unipotent and is its exponential. It is equal to \( \text{Spec}(\mathcal{H}_m) \) where \( \mathcal{H}_m \) is the shuffle Hopf algebra over \( \mathbb{Q} \) in the variables \( e_0, e_1 \).

For \( R \), a \( \mathbb{Q} \)-algebra let \( R(\langle e_0, e_1 \rangle) \) be the non-commutative algebra of power series in \( e_0, e_1 \); let \( \epsilon : R(\langle e_0, e_1 \rangle) \rightarrow R \) be its augmentation morphism; and let \( \Delta_\epsilon \) be the unique linear multiplicative map, continuous for the \( \ker(\epsilon) \)-adic topology, satisfying \( \Delta_\epsilon(e_i) = e_i \otimes 1 + 1 \otimes e_i \), \( i = 0, 1 \). We have

\[
\text{Lie}(\eta \pi^d_{\eta}) : R \mapsto \{ f \in R(\langle e_0, e_1 \rangle) | \Delta_\epsilon(f) = f \otimes 1 + 1 \otimes f \}
\]

\[
\eta \pi^d_{\eta} : R \mapsto \{ f \in R(\langle e_0, e_1 \rangle) | \Delta_\epsilon(f) = f \otimes f, \epsilon(f) = 1 \}
\]

For \( f \in R(\langle e_0, e_1 \rangle) \) and \( w \) a word in \( e_0, e_1 \), the coefficient of \( w \) in \( f \) is denoted by \( f[w] \), i.e. we have : \( f = \sum_w \text{word } f[w] w \). This notation extends to the whole of \( \mathbb{Q}(\langle e_0, e_1 \rangle) \) by linearity.

For simplicity, the \( y(e_z)z \)'s are usually denoted by \( e_z \).

The groupoid \( \pi^d_{\eta}(X) \) is equipped with the universal nilpotent "KZ" (Knizhnik-Zamolodchikov) connection. On the torsor \( (\pi^d_{\eta}/\tilde{1}_0) \), of paths starting at the tangential base-point \( \tilde{1}_0 \), trivialized at \( \tilde{1}_0 \), the connection is \( \nabla_{\text{KZ}} = \partial - e_0 \frac{dz}{d} - e_1 \frac{dx}{dz} - 1 \), and its horizontal sections are called multiple polylogarithms.

For a word \( w \) in \( e_0, e_1 \), its total number of letters is called its weight and its number of letters \( e_1 \) is called its depth.

Notation : for all \( d \in \mathbb{N} \), and \((s_d, \ldots, s_0) \in (\mathbb{N}^+)^{d+1} \), we will sometimes denote the word \( e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1 e_0^{s_0-1} \) by \( e_0^{s_d-1} \ldots s_0-1 \) and also \( e_0^{s_d-1} \ldots s_0-1, 0 = e_0^{s_d-1} \ldots s_1-1 e_1 \).

1.1.3 Formula for the motivic Galois action on \( 1 \pi^d_{0} \)

Details for this paragraph can be found in \([DG]\). Let \( \text{MT}(\mathbb{Z}) \) be the rigid abelian tensor category of mixed Tate motives over \( \mathbb{Z} \). Let \( \omega : \text{MT}(\mathbb{Z}) \rightarrow \text{Vec}_\mathbb{Q} \) be its de Rham realization functor, and let \( U^\omega = \text{Aut}\mathbb{Q}^\omega \) be its associated Tannakian group. It can be written as a semi-direct product \( U^\omega = \mathbb{G}_m \ltimes U^\omega \), where \( U^\omega \) is a pro-unipotent group scheme. It acts on \( 1 \pi^d_{0} \), as follows : the action of \( U^\omega \) factorizes by a morphism \( \circ : 1 \pi^d_{0} \times 1 \pi^d_{0} \rightarrow 1 \pi^d_{0} \), given by

\[
g \circ f = g.f(e_0^{-1} e_1 g)
\]

The operation \( \circ \) defines a group law on \( 1 \pi^d_{0} \) known as the Ihara group law.

The action of \( \mathbb{G}_m \) is given by \( \lambda \mapsto \) (multiplication by \( \lambda^{\text{weight}} \)). Following \([DG]\), we will denote it
by \( \lambda \mapsto \tau(\lambda) \). It can be seen as a map:

\[
T. : \mathcal{O}(1^{dR}) \twoheadrightarrow \mathcal{O}(1^{dR})[T]
\]
defined by \( w \mapsto T^{\text{weight}(w)}w \) where \( T \) is a formal variable. We will sometimes use the notation \( w \mapsto T.w \). This action fits into a natural commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}(1^{dR}) & \twoheadrightarrow & \mathcal{O}(1^{dR})[T] \\
\downarrow & & \downarrow \\
\mathcal{O}(1^{dR}) & \twoheadrightarrow & \mathcal{O}(1^{dR})[[T]]
\end{array}
\]

Finally, we have of course: \( T.(g \circ f) = (T.g \circ T.f) \).

1.1.4 Frobenius action on the fundamental group; \( p \)-adic multiple zeta values and \( p \)-adic multiple polylogarithms

There exists a Frobenius action \( F^{\ast} \) on \( (\pi^{dR}(X \otimes \mathbb{Q}_p), \nabla_{KZ}) \) ([D], §11), and we have an horizontal isomorphism ([D], §11.11, §11.12):

\[
F^{\ast} : (\pi^{dR}(X \otimes \mathbb{Q}_p), \nabla_{KZ}) \overset{\sim}{\longrightarrow} F^{\ast}(\pi^{dR}(X \otimes \mathbb{Q}_p), \nabla_{KZ})
\]

It gives rise to a notion of \( p \)-adic multiple zeta values ([DG], §5.28). Two different conventions exist, depending on whether one considers the Frobenius action or the "Verschiebung" action, i.e. \( V_{\ast} = \tau(p)F_{\ast}^{-1} \). We will consider, more generally, all possible iterates of Frobenius and Verschiebung.

**Notation 1.2.** For \( k \in \mathbb{N}^* \), let:

\[
\Phi_{p^k} = F_{\ast}^k(1_{10}) \in 1^{dR}(X)(\mathbb{Q}_p)
\]

\[
\Phi_{p^{-k}} = \tau(p^k)F_{\ast}^{-k}(1_{10}) \in 1^{dR}(X)(\mathbb{Q}_p)
\]

Considering iterates of \( pF_{\ast}^{-1} \) and not \( F_{\ast}^{-1} \) is the most convenient choice for our computations.

Let, for \( k \in \mathbb{Z}\setminus\{0\}, \ d \in \mathbb{N}^*, \ (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d, \)

\[
\zeta_{p^k}(s_d, \ldots, s_1) = (-1)^d\Phi_{p^k}[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1]
\]

The horizontality of Frobenius also gives rise to a notion of \( p \)-adic multiple polylogarithms. Following [D], §19.6, let us consider the formal completion of \( \mathbb{P}^1 \) along \( \mathbb{P}^1\setminus\{1\} \) in characteristic \( p \); its generic fiber is the rigid analytic space \( \mathbb{P}^1\setminus\{1\} \); it is endowed with the lift \( z \mapsto z^p \) of Frobenius.

**Notation 1.3.** For \( k \in \mathbb{N}^* \), let the \( p \)-adic multiple polylogarithms be:

\[
\kappa_{p^k}(z) = 01_{F_{\ast}^{k}(z)}F_{\ast}^{k}(1_{10}) \in 0^{dR}(X)(\mathbb{Q}_p)
\]

\[
\kappa_{p^{-k}}(z) = \tau(p^k)(01_{F_{\ast}^{-k}(z)}F_{\ast}^{-k}(1_{10})) \in 0^{dR}(X)(\mathbb{Q}_p)
\]

The horizontality of the Frobenius \( F_{\ast} \) reformulates as a differential equation for \( \kappa_{p^k} \), for all \( k \in \mathbb{Z}\setminus\{0\} \), which is variant of the KZ equation; this justifies the terminology "multiple polylogarithms". We will use it in §3.

Another notion of \( p \)-adic multiple zeta values, and of \( p \)-adic multiple polylogarithms is defined
Notation 1.4. Let the Frobenius invariant path and its Ihara inverse will be denoted by

\[ \Phi_p^\infty = \frac{1}{\pi_d^R}(X)(\mathbb{Q}_p) \quad (14) \]

\[ \Phi_{p^\infty} = (\Phi_p^\infty)^{-1} \quad (15) \]

Let, for \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), respectively,

\[ \zeta_{p^\infty}(s_d, \ldots, s_1) = (-1)^d \Phi_{p^\infty}[e_0^{s_d-1}e_1 \ldots e_0^{s_1-1}e_1] \quad (16) \]

And

\[ \text{Li}_{p^\infty}(z) = a_1(zc_0) \in a_0\pi_d^R(X)(\mathbb{Q}_p) \quad (17) \]

\( \Phi_{p^\infty} \) does not depends on a choice of branch of the logarithm ([F1], theorem 3.10). The reason for the notation \( p^{\pm \infty} \) will appear in §2.4.

The Frobenius and Verschiebung actions on \( \pi_0^d(X)(\mathbb{Q}_p) \) is given as follows: for all \( k \in \mathbb{N}^* \),

\[ F_k^*: u \mapsto \Phi_{p^k} \circ \tau(p^k)u \]

\[ V_k^* = \tau(p^k)F_{-k}^*: u \mapsto \Phi_{p^{-k}} \circ u \]

In particular, \( \Phi_{p^k} \) and \( \Phi_{p^{-k}} \) are inverses of each other for the Ihara action. A similar relation exists between \( \text{Li}_{p^k} \) and \( \text{Li}_{p^{-k}} \), by considering the Ihara and Frobenius action with variable base-points \( z \). The explanation of the notation \( \pm \infty \) will appear in §2.4. We have, for all \( k \in \mathbb{N}^* \):

\[ \Phi_{p^k} = \Phi_{p} \circ \tau(p)\Phi_{p} \circ \ldots \circ \tau(p^{k-1})\Phi_{p} \]

\[ \Phi_{p^{-k}} = \tau(p^{k-1})\Phi_{p^{-1}} \circ \tau(p^{k-2})\Phi_{p^{-1}} \circ \ldots \circ \Phi_{p^{-1}} \]

1.1.5 Brief review of part I

In the first part of this work ([H1]), we have compute explicitly \( p \)-adic multiple polylogarithms \( \text{Li}_{p^k} \), and thus \( p \)-adic multiple zeta values \( \zeta_{p^k} \), for all \( k \in \mathbb{Z}\setminus\{0\} \).

There are formulas for the \( \zeta_{p^k} \)'s as absolutely convergent sums of \( p \)-adic series; the terms of these series are \( \mathbb{Q} \)-linear combinations of "variants" of multiple harmonic sums, whose upper bound is of the form \( p^{|k|M} \), \( M \in \mathbb{N}^* \), multiplied by the power of \( p^{k|M} \) which ensures the homogeneity for the weight; by "variants", we mean that these are finite iterated sums as in equation (1) but with additional conditions of congruences modulo \( p^k \), among the indices \( n_i \)'s. For \( M = 1 \), these conditions are of course empty, and we retrieve the usual multiple harmonic sums \( H_{p^k} \)'s.

The function \( n \in \mathbb{N}^* \mapsto \) (Taylor coefficient of \( \text{Li}_{p^k} \) of order \( n \) at \( z = 0 \)) \( \in \mathbb{Q}_p \) extends continuously to \( \mathbb{Z}_p \) and its extension is locally analytic. Its Taylor coefficients admit, more generally, series expansions of a slightly more general type than the ones of the \( p \)-adic multiple zeta values.

The form of the explicit series implies (non-optimal) bounds on their valuation.

More details about these facts are given throughout the paper, in particular §3.2.
1.2 Main results

The first result is the motivic interpretation of the multiplication by (a power of) \( p \) of the upper bound of a multiple harmonic sum; and, at the same time, the inversion of the series expansion \( p \)-adic multiple zeta values.

Theorem 1

i) With the notations of §2.5 for the generalized motivic Galois action, we have, for all \( N \in \mathbb{N}^* \), for all \( k \in \mathbb{N}^* \):

\[
\Phi_{p^{-k}}(N e_0, Ne_1) \circ N^{\text{weight}} H_N = (p^k N)^{\text{weight}} H_{p^k N} \tag{18}
\]

In particular, for all \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), \( k \in \mathbb{N}^* \), \( a \in \mathbb{Z} \) we have:

\[
\zeta_{f_{\mathbb{Q}}} (s_d, \ldots, s_1) = \left( \Phi_{p^{-k}}^{-1} e_1 \Phi_{p^{-k}} \left[ \frac{1}{1 - e_0} e_1 \frac{e_0^{s_d - 1}}{(1 + e_0 a)^{s_d}} e_1 \cdots \frac{e_0^{s_1 - 1}}{(1 + e_0 a)^{s_1}} e_1 \right] \right) \in \prod_p \mathbb{Q}_p \tag{19}
\]

ii) By a direct computation, one can express \((p^k N)^{\text{weight}} H_{p^k N}\) in terms of \( N^{\text{weight}} H_N \) and explicit sums of series in \( \mathbb{Q}_p \), whose terms are \( \mathbb{Q} \)-linear combinations of \((p^k)^{\text{weight}} H_{p^k \cdot s}\), or, more generally, sums of series depending on additional parameters.

The formula (18) enables to think of finite multiple zeta values as a variant of multiple zeta values. It must be thought of as an analog of the iterated integral formula of multiple zeta values, i.e. \( \zeta (s_d, \ldots, s_1) = \Phi [e_0^{s_d - 1} e_1 \cdots e_1^{s_1 - 1} e_1] \), where \( \Phi \) is the Drinfeld associator. An algebraic study of finite multiple zeta values partly based on (18) has been initiated in \( [13] \).

The second step is to study finite multiple zeta values as functions of \( p^k \).

Theorem 2

i) Let \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), \( a \in \mathbb{Z} \). Then the finite multiple zeta value \( \zeta_{f_{\mathbb{Q}}} (s_d, \ldots, s_1) \) is an "analytic" function of \( p^k \), given as follows, with the notations of §2 :

\[
\Phi_{p^{-k}}^{-1} e_1 \Phi_{p^{-k}} \left[ \frac{1}{1 - e_0} e_1 \frac{e_0^{s_d - 1}}{(1 + e_0 a)^{s_d}} e_1 \cdots \frac{e_0^{s_1 - 1}}{(1 + e_0 a)^{s_1}} e_1 \right] = \sum_{s \geq 0} (p^k)^s \left( \text{sym} \Phi_{p^s} \circ \text{pr}_{s+1} \right) \left[ \frac{1}{1 - e_0} e_1 \frac{e_0^{s_d - 1}}{(1 + e_0 a)^{s_d}} e_1 \cdots \frac{e_0^{s_1 - 1}}{(1 + e_0 a)^{s_1}} e_1 \right] \tag{20}
\]

ii) By a direct computation, one show that \( \zeta_{f_{\mathbb{Q}}} \) admits an absolutely convergent Taylor expansion with respect to \( p^k \), whose Taylor coefficients are explicit sums of series whose terms are \( \mathbb{Q} \)-linear combinations of multiple harmonic sums.

In both results, the geometric and elementary computations can be identified, and this yields a computation of \( p \)-adic multiple zeta values. Theorem 1 computes Deligne’s \( p \)-adic multiple zeta values and theorem 2 computes Furusho’s \( p \)-adic multiple zeta values (those two variants can also be computed in terms of each other directly).

The identification of the two computations reduces (by induction on the depth) to the identification of two absolutely convergent Taylor expansions.

This way to compute \( p \)-adic multiple zeta values is indirect; and it does not refer to iterated
integration. We use indirectly iterated integration in the parts i) of the theorems, which require
the lower bounds on the valuations of \( p \)-adic multiple zeta values, which have been established in
part I.

The formula i) of theorem 2 can also be combined to the algebraic properties of finite multiple
zeta values \([J3]\) to obtain equations among \( p \)-adic multiple zeta values. First of all, the series
shuffle equation for finite multiple zeta values, which follow s directly from their definition, implies
a version of the series shuffle equation for \( p \)-adic multiple zeta values. This will be detailed in the
part II of the work on algebraic relations initiated in \([J3]\). The fact that \( p \)-adic multiple zeta values
satisfy the series shuffle relation, and thus the double shuffle relations has been proven by Besser
and Furusho \([BF]\) and Furusho and Jafari \([FJ]\) using Besser’s reformulation and generalization of
Coleman integration \([Bes]\).

The present version of this paper is preliminary.

**Related work.** The formula (19) of i) of theorem 1, for \( a = 0, d = 1 \) and \( k = 1 \) concerns
classical \( p \)-adic zeta values and harmonic sums, and is known. For \( a = 0, d = 2 \) and \( k = 1 \), it has
been proved by M. Hirose \([Y]\) ; the general \( a = 0 \) and \( k = 1 \) case has been conjectured by Y asuda
and Hirose \([Y]\), under this form : for all \( d \in \mathbb{N}^* \), \((s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \), and all primes \( p \):

\[
(-1)^d \sum_{k=0}^{d} (-1)^{\sum_{j=k+1}^d s_j} \left( \sum \prod_{i \geq k+1} p_i^{l_i} \left( l_i + s_i - 1 \right) \zeta_{p-1}(s_{k+1} + l_{k+1}, \ldots, s_d + l_d) \right) 
\times \zeta_{p-1}(s_k, \ldots, s_1) = H_p(s_d, \ldots, s_1)
\]

Finally, the principle of the proof of theorem 1 has been explained in the introduction of \([J3]\).

2 Properties of the motivic Galois action and Frobenius on \( 1\pi^d_{0}(\mathbb{Q}_p) \)

2.1 Symmetrization map

2.1.1 Definition and first properties

For details about these facts, see \([J3]\) §2.4 and §5.1.

**Definition 2.1.** One has a map :

\[
sym: 1\pi^d_0 \rightarrow \text{Lie}(0\pi^d_0) \ , \ f \mapsto f^{-1}e_1f
\]

**Definition 2.2.** Let \( \tilde{1}\pi^d_0 \) be the sub-group scheme of \( 1\pi^d_0 \) defined by points \( f \) which are zero in
weight one, i.e. such that \( f[e_0] = f[e_1] = 0 \).

**Lemma 2.3.** (properties of \( \tilde{1}\pi^d_0 \)) \( \tilde{1}\pi^d_0 \) is stable by the action of \( G^w \) ; it is in particular also a sub-group scheme of \((1\pi^d_0, o)\).
Lemma 2.4. (properties of sym)

i) (injectivity) For each Q-algebra $R$, the map sym defines an injective map on $\overline{\pi}_{0}^{dR}(R)$.

ii) (comparison of coefficients) Let $f$ a point of $\overline{\pi}_{0}^{dR}$; let $n,d \in \mathbb{N}^*$; the coefficients of $f$ of weight $n$ and depth $\leq d$, resp. of $\text{sym}(f)$ of weight $n+1$ and depth $\leq d+1$, can be expressed in terms of each other by polynomial expressions with coefficients in $\mathbb{Z}$.

iii) (motivicity) sym is compatible to the action of $G^\omega$, in the following sense:

- We have
  \[ \text{sym}(T.f) = T^{-1} \times T.\text{sym}(f) \]
- For $h_2,h_1$ points of $\text{Lie}(\overline{\pi}_{0}^{dR})$, denoting by
  \[ h_2 \circ h_1 = h_1(e_0,h_2) \]
  we have a commutative diagram:

\[
\begin{array}{ccc}
1\pi_{0}^{dR} \times 1\pi_{0}^{dR} & \xrightarrow{\text{sym} \times \text{sym}} & \text{Lie}(\overline{\pi}_{0}^{dR}) \times \text{Lie}(\overline{\pi}_{0}^{dR}) \\
\downarrow \circ & & \downarrow \circ \\
\pi_{0}^{dR} & \xrightarrow{\text{sym}} & \text{Lie}(\overline{\pi}_{0}^{dR})
\end{array}
\]

2.1.2 Extension to the whole of $R\langle\langle e_0, e_1 \rangle\rangle$

It is convenient to consider the dual maps

\[ \Delta_\circ, \Delta^\text{sym}_\circ : \mathcal{O}(\pi_{0}^{dR}) \to \mathcal{O}(\pi_{0}^{dR}) \otimes \mathcal{O}(\pi_{0}^{dR}) \]

\[ \text{sym}^\circ : \mathcal{O}(\pi_{0}^{dR}) \to \mathcal{O}(\pi_{0}^{dR}) \]

Corollary 2.5. The map sym and the two versions of $\circ$ above extend respectively to maps $R\langle\langle e_0, e_1 \rangle\rangle \to R\langle\langle e_0, e_1 \rangle\rangle$ and $R\langle\langle e_0, e_1 \rangle\rangle \times R\langle\langle e_0, e_1 \rangle\rangle \to R\langle\langle e_0, e_1 \rangle\rangle$, for $R$ a Q-algebra, satisfying the similar commutative diagram:

\[
\begin{array}{ccc}
R\langle\langle e_0, e_1 \rangle\rangle \times R\langle\langle e_0, e_1 \rangle\rangle & \xrightarrow{\text{sym} \times \text{sym}} & R\langle\langle e_0, e_1 \rangle\rangle \\
\downarrow \circ & & \downarrow \circ \\
R\langle\langle e_0, e_1 \rangle\rangle & \xrightarrow{\text{sym}} & R\langle\langle e_0, e_1 \rangle\rangle
\end{array}
\]

Proof. The commutativity of the previous diagram amounts to say that two maps $\mathcal{O}(\pi_{0}^{dR}) \to \mathcal{O}(\pi_{0}^{dR}) \otimes \mathcal{O}(\pi_{0}^{dR})$ are equal after composition by any $m(f \otimes g)$ with $f, g \in \pi_{0}^{dR}(R)$, and $m$ the multiplication in $R$, for all Q-algebra $R$; thus they are equal, because the shuffle algebra is a free polynomial algebra (this is a theorem of Radford). This enables to define the extension of sym and the two versions of $\circ$, which by definition fit into the same commutative diagram with $\pi_{0}^{dR}$.

2.1.3 Combinatorics of the Ihara action and its symmetrized version

Notations 2.6. (combinatorics of the composition of series / of the symmetrized Ihara action)

Let $w$ a word of $\mathcal{H}_w$; let $S_1(w)$ be the set of subwords of $w$ which contains all its letters $e_1$. An element of $S_1(w)$ can be written in a unique way as the collection of its "connected components" $(w_1)$, namely, its subwords $w_1$, maximal for the inclusion, which are made of consecutive letters of $w$.

For $w$ a word of $\mathcal{H}_w$ and $(w_i)$, an element of $S_1(w)$, the quotient word $w/(w_i)$, is the word obtained
by replacing each $w_i$ by $e_1$.

If $f, g$ are elements of $R((e_0,e_1))$ with $R$ a $\mathbb{Q}$-algebra, we have, for all words $w \in H_m$:

$$(f \circ \text{sym}\ g)[w] = \sum_{(w_i) \in S_1(w)} f[w/(w_i)] \times \prod_i g[w_i] \tag{27}$$

**Fact 2.7.** (weight and depth of the various terms)

Let $n, d \in \mathbb{N}^*$; we denote by $\mathcal{O}(1\pi_0^{dR})_{n,d}$ the sub-$\mathbb{Z}$-module of $\mathcal{O}(1\pi_0^{dR})$ generated by the words of weight $n$ and depth $d$. Then we have more precisely:

$$\Delta_{s}(w) : \mathcal{O}(1\pi_0^{dR})_{n,d} \to \sum_{\begin{subarray}{c} n_1+n_2=n \\ d_1+d_2=d \end{subarray}} \mathcal{O}(1\pi_0^{dR})_{n_1,d_1} \otimes \mathcal{O}(1\pi_0^{dR})_{n_2,d_2}$$

$$\Delta_{\text{sym}} : \mathcal{O}(1\pi_0^{dR})_{n,d} \to \sum_{\begin{subarray}{c} n_1+n_2=n-1 \\ d_1+d_2=d-1 \end{subarray}} \mathcal{O}(1\pi_0^{dR})_{n_1,d_1} \otimes \mathcal{O}(1\pi_0^{dR})_{n_2,d_2}$$

### 2.2 Projection onto the parts of given weight

**Definition 2.8.** For $s \in \mathbb{N}$, let $\text{pr}_s : \mathcal{O}(1\pi_0^{dR}) \to \mathcal{O}(1\pi_0^{dR})$ the projection onto the weight $s$ part, i.e. the unique linear map satisfying: for all $w \in \mathcal{O}(1\pi_0^{dR})$:

$$T.w = \sum_{s \geq 0} \text{pr}_s(w)T^s$$

The formula (23) reformulates into

$$\text{pr}_{s+1} \circ \text{sym} = \text{sym} \circ \text{pr}_s$$

With the notation for the extended Ihara action of §2.1.2 we have:

**Fact 2.9.** For all $f, g$, points of $1\pi_0^{dR}$, we have:

$$g \circ (T.f) = \sum_{s \geq 0} g \circ (f \text{pr}_s) T^s \tag{28}$$

$$(T.g) \circ f = \sum_{s \geq 0} (g \text{pr}_s) \circ f T^s \tag{29}$$

Thus, because of the commutativity of (28),

$$\text{sym} g \circ \text{sym}(T.f) = \sum_{s \geq 0} (\text{sym} g) \text{pr}_{s+1} \circ (\text{sym} f) T^s$$

$$\text{sym}(T.g) \circ \text{sym} f = \sum_{s \geq 0} (\text{sym} g) \circ (\text{sym} f) \text{pr}_{s+1} T^s$$

### 2.3 Topologies on $(1\pi_0^{dR}(\mathbb{Q}_p),\circ)$

The following notions are convenient to express the bounds on the valuation of $p$-adic multiple zeta values and several aspects of sums of series whose terms are $p$-adic multiple zeta values.
2.3.1 Formal power series with coefficients in \( \mathbb{R}_+ \)

Let \( \mathbb{R}_+[T] \) be the set of formal power series with coefficients in \( \mathbb{R}_+ \); let us equip it with the product topology associated to the natural identification \( \mathbb{R}_+[T] \cong \mathbb{R}_+^N \). We denote by \( \times \) the usual multiplication of formal power series. A partial order on \( \mathbb{R}_+[T] \) is defined by: \( S = \sum_{n \geq 0} a_n T^n \leq S' = \sum_{n \geq 0} a'_n T^n \) if, for all \( n \in \mathbb{N} \), \( a_n \leq a'_n \).

If \( S \leq S' \) then \( S \times R \leq S' \times R \) for all power series \( R \in \mathbb{R}_+[T] \).

These definitions extend in a natural way to the set \( \mathbb{R}_+[T,U] \) of formal power series in two variables.

2.3.2 Two topologies on \( \pi_0^{dR}(\mathbb{Q}_p) \)

For the rest of this subsection we fix a ultrametric normed field \((K,|.|_K)\).

We will consider two topologies on \( \pi_0^{dR}(K) \):

- The topology of pointwise convergence of functions \( \{\text{words of } \mathcal{H}_m\} \rightarrow K \).

It makes \( \pi_0^{dR}(K) \) into a topological group, both for the law group arising from multiplication of power series and for the motivic law group \( \circ \) which are both polynomial.

- In order to deal with sums of series depending on parameters : the topology of uniform convergence on each subset \( \{w \text{ word } \in \mathcal{H}_m \mid \text{depth}(w) = d\} \), \( d \in \mathbb{N}^* \) of functions \( \{\text{words of } \mathcal{H}_m\} \rightarrow K \).

The subset of \( \pi_0^{dR}(K) \) made of functions whose restriction to each \( \{w \text{ word } \in \mathcal{H}_m \mid \text{depth}(w) = d\} \), \( d \in \mathbb{N}^* \) is bounded is, for both group laws, a subgroup of \( \pi_0^{dR}(K) \), and is a topological group for this topology.

2.3.3 Functions inducing the pointwise convergence topology

**Definition 2.10.** i) Let \( ||.|| : \pi_0^{dR}(K) \rightarrow \mathbb{R}_+[T] \) be the map defined by

\[
f \mapsto ||f|| = \sum_{n \geq 1, \text{weight}(w_n) = n} \max_{w_n \in \mathcal{H}_m} |f[w_n]|_K T^n
\]

ii) Let \( ||.||' : \pi_0^{dR}(K) \rightarrow \mathbb{R}_+[T,U] \) be defined by

\[
f \mapsto ||f||' = \sum_{n,d \geq 1, \text{weight}(w_{n,d}) = n, \text{depth}(w_{n,d}) \leq d} \max_{w_{n,d} \in \mathcal{H}_m} |f[w_{n,d}]|_K T^n U^d
\]

The initial topology on \( \pi_0^{dR}(K) \) associated to, \( ||.|| \), respectively \( ||.||' \) and the product topology on \( \mathbb{R}_+[T] \) resp \( \mathbb{R}_+[T,U] \) of §2.3.1 is the pointwise convergence topology of functions \( \{\text{words of } \mathcal{H}_m\} \rightarrow K \).

**Remark 2.11.** Let \( f \in \pi_0^{dR}(K) \), and let \( n,d \in \mathbb{N}^* \). We have:

\[
\max_{\text{weight}(w) = n} |f[w]|_K = \max_{l \in \mathbb{N}^*} \max_{\sum_{i=1}^l \text{weight}(w_i) = n} \left| \prod_{i=1}^l f[w_i] \right|_K
\]
\[
\max_{\text{weight}(w) = n, \text{depth}(w) \leq d} \left| f[w] \right|_K = \max_{l \in \mathbb{N}^*} \max_{w_1, \ldots, w_l \in \mathcal{H}_n} \left| \prod_{i=1}^l f[w_i] \right|_K
\]

**Proof.** This follows from the comparison of coefficients of \( F \) or all Lemma 2.14.

ii) Let \( \max \text{weight}(w_i) = n \) and \( \max \text{depth}(w_i) \leq d \) ; similarly for \( \| f \| \).

**Lemma 2.12.** (Axioms of \( || \cdot || \) and \( || \cdot ||' \))

For all \( g, f \in \mathcal{F}_0(K), \lambda \in K, s \in \mathbb{N} \), we have:

i) \( \| f \| = 0 \iff \| f \|' = 0 \iff f = 1 \)

ii) \( \| \lambda \cdot f \|((T)) = \| f \|((\lambda T)) \) and \( \| \lambda \cdot f \|'((T, U)) = \| f \|'((\lambda T, U)) \)

iii) \( \| g \cdot f \| \leq \| g \| \times \| f \| \) ; similarly for \( \| \cdot \|' \)

iv) \( \| f \| \) or \( \| f \|' \) is clear.

For convenience we also introduce:

**Definition 2.13.** i) Let \( \| \cdot \| : \mathcal{F}_0(K) \rightarrow \mathbb{R}_+[[T]] \) be the map defined by

\[
\text{sym } f \mapsto \| \text{sym } f \|' = \sum_{n \geq 1} \max_{\text{weight}(w) = n} \left| \text{sym } f[w_n] \right|_K T^{n-1}
\]

ii) Let \( \| \cdot \|' : \mathcal{F}_0(K) \rightarrow \mathbb{R}_+[[T, U]] \) be the map defined by

\[
\text{sym } f \mapsto \| \text{sym } f \|' = \sum_{n, d \geq 1} \max_{\text{weight}(w) = n, \text{depth}(w) \leq d} \left| \text{sym } f[w_n, d] \right|_K T^{n-1}U^{d-1}
\]

**Lemma 2.14.** For all \( f \in \mathcal{F}_0(K) \), we have \( \| \text{sym } f \| \leq \| f \| \) ; similarly for \( \| \cdot \|' \).

**Proof.** Follows from the comparison of coefficients of \( f \) and \( \text{sym}(f) \) (§2.1) and the shuffle equation for \( f \).

The previous facts imply, in particular, the following property which we will use specifically:

**Corollary 2.15.** For all \( f, g \in \mathcal{F}_0(K) \) and \( s \in \mathbb{N} \) we have:

\[
\| \text{sym}(g \text{ pr}_s \circ f) \|' \leq \| g \| \times \| f \|
\]

2.3.4 A subgroup of \( \mathcal{F}_0(K) \) adapted to sums of series of \( p \)-adic multiple zeta values

**Definition 2.16.** Let \( \mathcal{F}_0(K) \) be the subset of \( \mathcal{F}_0(K) \) made of the points such that for all \( d \in \mathbb{N}^* \),

\[
\sup_{w_n, d \in \mathcal{H}_n} \left| f[w_n, d] \right|_K \rightarrow 0
\]

We recall that \( K \) is ultrametric ; \( \mathcal{F}_0(K) \) is thus also the subset made of the points \( f \) such that, for every sequence \( (w_m) \) of \( \mathcal{H}_m \) such that \( \lim \sup \text{depth}(w_m) < +\infty \) and \( \text{weight}(w_m) \rightarrow m \rightarrow +\infty +\infty \),
we have
\[ \sum_{m \geq 0} |f[w_m]|_K < +\infty \]

One defines in a similar way Lie(\(a\pi^R_0\))(K)_\Sigma.

**Lemma 2.17.** (stability by algebraic operations)
i) \(1\pi^R_0\)(K)_\Sigma is a subgroup of \(1\pi^R_0(K)\) for the usual multiplication of grouplike power series.
ii) Sym maps \(1\pi^R_0(K)_\Sigma\) into Lie(\(a\pi^R_0\))(K)_\Sigma.
iii) \(1\pi^R_0\)(K)_\Sigma and its image by sym are, respectively, subgroups for \(\circ\) and its symmetrized version.
iv) They are stable by \(\tau(\lambda)\) for \(\lambda \in K\) such that \(|\lambda|_K \leq 1\).
v) They are stable by \(pr_s\), \(s \in \mathbb{N}\).

**Proof.** The set of series \(\sum_{n,d \geq 1} a_{n,d}T^nU^d \in \mathbb{R}^+[T,U]\) such that, for all \(d \in \mathbb{N}^+, a_{n,d} \to_{n \to +\infty} 0\), is stable by the multiplication of formal power series, and by taking elements which are inferior for the partial order defined in §2.3.1. Thus its preimage by \(|.|''\) is stable by the operations that are mentioned because of lemmas [2.12 and 2.14]

**Lemma 2.18.** (uniform topology on \(1\pi^R_0(K)_\Sigma\))

Let us equip \(1\pi^R_0(K)_\Sigma\) and its image by sym with the topology of uniform convergence on words of each depth (§2.3.2).
i) \(1\pi^R_0(K)_\Sigma\) and its image by sym are closed inside \(1\pi^R_0\) and sym \(1\pi^R_0\).
ii) For each sequence \((w_m)_{m \in \mathbb{N}}\) of \(H_m\) such that \(\lim\sup depth(w_m) < +\infty\) and weight\((w_m) \to_{m \to +\infty} +\infty\), the maps "sum of series" \(f \mapsto \sum_{m \geq 0} f[w_m]\) with values in \(K\) are continuous.

### 2.3.5 Topology on the torsor \((\pi^R_0)_z\) on the analytic space \(\mathbb{P}^1[1]\)

Let \(A[\mathbb{P}^1[1]\) be the space rigid analytic functions over \(\mathbb{P}^1[1[\). It is endowed with the standard topology of uniform convergence over \(\mathbb{P}^1[1[\), which is also the topology of uniform convergence of the sequences of Taylor coefficients at \(z = 0\).

Let us consider the rigid analytic sections of the fundamental torsor of all paths \((\pi^R_0)_z, z\), starting at 0, on \(\mathbb{P}^1[1]\) : an example of it are the multiple polylogarithms \(Li_{pk}, k \in \mathbb{Z}\setminus\{0\}\). Because of the trivialization of this torsor at 0, we can define topologies on it as follows : replace in the previous paragraphs the norm of \(K\) by the norm of \(A[\mathbb{P}^1[1]\) evoked above, which is still ultrametric (and replace \(1\pi^R_0\) by \(a\pi^R_0\)). In particular, we can still define \(A[\mathbb{P}^1[1][\Sigma\).

### 2.4 Role of the Frobenius invariant path of \(1\pi^R_0(K)_p\)

#### 2.4.1 Limit \(k \to +\infty\)

Let us recall that, for a point of \(1\pi^R_0\), we have \(F_\ast(f) = F_\ast(1) \circ \tau(p)f\); this implies, for all \(f, g\), that
\[ F_\ast(g)^{-1} \circ F_\ast(f) = \tau(p)(g^{-1} \circ f) \]  
(30)

In particular,
\[ ||F_\ast(g)^{-1} \circ F_\ast(f)||(T) \leq ||(g^{-1} \circ f)||(pT) \]  
(31)

Thus, the Frobenius is a "contractant" map, and :
**Corollary 2.19.** \( F \) on \( \pi_{0}^{dR}(X)(\mathbb{Q}_p) \) has an unique fixed point, equal to the limit for the pointwise convergence topology of each sequence of the form \( (F_{k}^{n}(f))_{k \in \mathbb{N}} \) for \( k \to +\infty \).

**Proof.** Formally equivalent to a standard proof \( \square \)

Taking \( f = 1 \), and by continuity of \( \circ \) (see also [F2], proof of proposition 3.1) :

\[
\Phi_{p}^{k} \longrightarrow_{k \to +\infty} \Phi_{p}^{\infty} \quad \text{and} \quad \Phi_{p}^{-k} \longrightarrow_{k \to +\infty} \Phi_{p}^{-\infty}
\]

### 2.4.2 Taylor expansion of Frobenius

The invariance of \( \Phi_{p}^{\infty} \) can be written as

\[
\Phi_{p}^{-k} \circ \Phi_{p}^{\infty} = \tau(p^{k}) \Phi_{p}^{\infty}
\]

This gives an expression of \( \Phi_{p}^{k} \) in terms of \( \Phi_{p}^{\infty} \) and \( p^{k} \). More precisely :

**Lemma 2.20.** i) For all \( k \in \mathbb{N}^{*} \) we have :

\[
\Phi_{p}^{k} = \sum_{s \geq 0} \Phi_{p}^{\infty} \circ (\Phi_{p}^{-\infty} \circ pr_{s}) \cdot (p^{k})^{s}
\]

\[
\Phi_{p}^{-k} = \sum_{s \geq 0} (\Phi_{p}^{\infty} \circ pr_{s}) \cdot \Phi_{p}^{-\infty} \cdot (p^{k})^{s}
\]

ii) More generally, for each \( f \in \pi_{0}^{dR}(\mathbb{Q}_p) \), and for each \( k \in \mathbb{N}^{*} \) :

\[
F_{*}^{k}(f) = \sum_{s \geq 0} \Phi_{p}^{\infty} \circ ((\Phi_{p}^{-\infty} \circ f) \circ pr_{s}) \cdot (p^{k})^{s}
\]

\[
F_{*}^{-k}(f) = \sum_{s \geq 0} \Phi_{p}^{\infty} \circ ((\Phi_{p}^{-\infty} \circ f) \circ pr_{s}) \cdot (p^{-k})^{s}
\]

**Proof.** This is the invariance of \( \Phi_{p}^{\infty} \) combined to Fact 2.9 \( \square \)

### 2.5 Action of the motivic group \( ((\pi_{0}^{dR})_{\Sigma}, \circ) \) on elementary functions

#### 2.5.1 Definition

**Notation 2.21.** Let \( Har \) (as harmonic sums) be the scheme \( \mathbb{A}^{1}_{d \in \mathbb{N}^{*}}(\mathbb{N}^{*})^{d} \) over \( \mathbb{Q} \).

By convention, we associate to a point of \( Har \) an additional coordinate, attached to \( d = 0 \), equal to 1. The coordinates of a point \( h \) of \( Har \) are denoted by \( h(s_{d}, \ldots, s_{1}) \), and \( h(\emptyset) \) for the \( d = 0 \) component.

For all \( N \in \mathbb{N}^{*} \), one has a point \( H_{N} \in Har(\mathbb{Q}) \) defined by multiple harmonic sums of upper bound \( N \). We denote by \( 1 \in Har(\mathbb{Q}) \) the point whose all \( d \geq 1 \) components are 0 ; we remark that it is the multiple harmonic sum \( H_{1} \).

One has a map

\[
\Sigma : ((\pi_{0}^{dR})_{\Sigma}(\mathbb{Q}_p) \to Har(\mathbb{Q}_p) \ , \ f \mapsto \Sigma f
\]

defined by

\[
\Sigma f(s_{d}, \ldots, s_{1}) = \sum_{L=0}^{\infty} \text{sym} f[e_{0}^{L}e_{1}^{e_{0}^{s_{d}-1}}e_{1}^{s_{1}-1}e_{1}]
\]
Because of the assumption on $f$, the formula is also coherent with the convention for the $d = 0$ component, i.e. we have $\sum_{L=0}^{+\infty} (\operatorname{sym} f)[e^L_0 e_1] = 1$.

**Lemma 2.22.** For $f, g$ points of $(\tilde{\pi}_0^{dR})_{\Sigma}$, $\Sigma_{\operatorname{sym}(g \circ f)}$ can be expressed in terms of $\Sigma_{\operatorname{sym}(f)}$ and $\operatorname{sym} g$. This defines a group scheme action of $((\tilde{\pi}_0^{dR})_{\Sigma}, \circ)$ on $\operatorname{Har}(Q_p)$ which fits into the commutative diagram

$$
(\tilde{\pi}_0^{dR})_{\Sigma}(Q_p) \times (\pi_0^{dR})_{\Sigma}(Q_p) \xrightarrow{\circ} (\pi_0^{dR})_{\Sigma}(Q_p) \\
\downarrow \operatorname{id} \times \Sigma \\
(\pi_0^{dR})_{\Sigma}(Q_p) \times \operatorname{Har}(Q_p) \xrightarrow{\circ} \operatorname{Har}(Q_p)
$$

**Proof.** The fact that $\Sigma_{f \circ g}$ can be expressed in terms of $\Sigma_f$ and $g$ follows from the equation (27). Then, the fact that it defines a group action follows from that $\circ_{\operatorname{sym}}$ is a group action. \qed

**Definition 2.23.** Let this action be denoted by :

$\circ : (\tilde{\pi}_0^{dR})_{\Sigma} \times \operatorname{Har} \rightarrow \operatorname{Har}$

**Remark 2.24.** The action on 1 is described by : for each point $g$ of $(\tilde{\pi}_0^{dR})_{\Sigma}$, $\Sigma_{\operatorname{sym} g} \circ 1 = \Sigma_{\operatorname{sym} g}$.

**Proof.** Follows from equation (27). \qed

### 2.5.2 Alternative definition

Let a series $f \in R((e_0, e_1))$ (not necessarily grouplike) such that for all $L \in \mathbb{N}$, $d \in \mathbb{N}^*$, $s_d, \ldots, s_1 \in \mathbb{N}^*$, $f[\epsilon_0^L e_1 e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1]$ depends only on $(s_d, \ldots, s_1)$.

With this assumption, one can define a point $h_f$ of $\operatorname{Har}$ by :

$$
h_f(s_d, \ldots, s_1) = f[\epsilon_0^{L-1} e_1 e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1]
$$

**Lemma 2.25.** If $h \in \operatorname{Har}(Q_p)$ is of the form $h_f$, then, for all $g \in (\pi_0^{dR})_{\Sigma}(Q_p)$, for all $d \in \mathbb{N}^*$ and $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$ :

$$(g \circ h_f)(s_d, \ldots, s_1) = (g \circ f)[\lim_{L \to +\infty} \epsilon_0^{L-1} e_1 e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1]$$

where $g \circ f$ refers to the symmetrized Ihara action of §2.1.2.

**Proof.** Follows from equation (27). \qed

### 2.5.3 Parts of given weight

The weight homogeneity action $T: (\pi_0^{dR})_{\Sigma}(Q_p) \rightarrow (\pi_0^{dR})_{\Sigma}(Q_p[T])$ enables to extend the previous definition to an action

$$
\circ : T(\pi_0^{dR})_{\Sigma}(Q_p) \times \operatorname{Har}(Q_p) \rightarrow \operatorname{Har}(Q_p[[T]])
$$

**Definition 2.26.** Let $s \in \mathbb{N}^*$. Let

$$
\circ_s : (\pi_0^{dR})_{\Sigma} \times \operatorname{Har} \rightarrow \operatorname{Har}
$$
be defined by, for all \( f \in (\pi^d)_0 \mathcal{Q}(\mathcal{Q}_p), h \in Har(\mathcal{Q}_p), \)

\[
(T.f) \circ h = \sum_{s \geq 1} T^s(f \circ_s h)
\]

We have the similar commutative diagram as (33) but with the Ihara product on \( \pi^d \) replaced by \( g \circ f \mapsto g \text{pr}_s \circ f \).

The formula for the action \( \circ \) defined in §2.5.1 involves a sum over sets of the form \( S_1(\sum_{i=0}^{\infty} e_0^i e_1 e_i^{s_i-1} e_1 \ldots e_0^{s_1-1} e_1) \); the action \( \circ_s \) involves the subsum over the subsets made of the \((w_i)_i^s \)'s such that \( \sum_i (\text{weight}(w_i) - 1) = s \).

**Lemma 2.27.** Let \( s \in \mathbb{N}^s, d \in \mathbb{N}^s, (s_d, \ldots, s_1) \in (\mathbb{N}^s)^d \). Then

\[
f \circ_s h(s_d, \ldots, s_1) = \mathbb{1}_{s \geq s_d + \ldots + s_1} f[e_0^{s_{-s_1}} e_1 e_0^{s_{-1}} e_1] + h(s_d, \ldots, s_1) + \text{products of (coefficients of } f \text{ of depth } \leq d) \text{ and (coefficients of } h \text{ of depth } \leq d - 1)
\]

We have \( c_0 = \text{id} \).

**Proof.** Follows from the formula for the Ihara action

\[
\square
\]

**2.5.4 Examples**

**Examples 2.28.** i) \( d = 1 \)

\[
(f \circ h)(s_1) = h(s_1) + \sum_{L \geq 0} \text{sym } f[e_0^L e_1 e_0^{s_1-1} e_1] \quad (34)
\]

ii) \( d = 2 \)

\[
(f \circ h)(s_2, s_1) = h(s_2, s_1) + \sum_{L \geq 0} \text{sym } f[e_0^L e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] + \sum_{r_2=0}^{s_2-1} h(s_2 - r_2) \text{sym } f[e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] + \sum_{r_1=0}^{s_1-1} h(s_1 - r_1) \sum_{L=0}^{\infty} \text{sym } f[e_0^L e_1 e_0^{s_2-1} e_1 e_0^{r_1}] \quad (35)
\]

**Examples 2.29.** i) \( d = 1 \)

\[
(f \circ_s h)(s_1) = h(s_1) + \mathbb{1}_{s \geq s_1} \text{sym } f[e_0^{s_{-s_1}} e_1 e_0^{s_{-1}} e_1] \quad (36)
\]

ii) \( d = 2 \)

\[
(f \circ_s h)(s_2, s_1) = h(s_2, s_1) + \mathbb{1}_{s \geq s_1 + s_2} \text{sym } f[e_0^{s_{-s_1-s_2}} e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] + \mathbb{1}_{s_1 + s_2-1 \geq s_2 + s_1 - s} h(s_2 + s_1 - s) \text{sym } f[e_0^{s_{-s_1}} e_1 e_0^{s_{-1}} e_1] + \sum_{r_1=0}^{\min(s_1-1, s-s_2)} h(s_1 - r_1) \text{sym } f[e_0^{s_{-s_2-r_1}} e_1 e_0^{s_2-1} e_1 e_0^{r_1}] \quad (37)
\]
3 Geometric proofs

3.1 Horizontality of Frobenius and Taylor coefficients at \( z = 0 \)

3.1.1 Preliminaries

The horizontality of the Frobenius of \( \pi^{dR} \) is stated in a general context in [D] §11, then specifically for \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) in [D] §19.6. Under a form similar to [D] §19.6, it also appears in [U1]. The form that we write here is the one appearing in [F2], theorem 2.14 (for \( k = -1 \)), which is the most adapted to our purposes.

**Theorem 3.1.** (horizontality of Frobenius)

For all \( k \in \mathbb{N}^* \) we have (with our convention for \( \text{Li}_{p-k} \) and \( \Phi_{p-k} \)):

\[
\text{Li}_{p-k}(z) \cdot \text{Li}_{p-k}(z) \left( e_0, \Phi_{p-k}^{-1} e_1 \Phi_{p-k} \right) = \text{Li}_{p-k}(z) \left( p^k e_0, p^k e_1 \right) \quad (38)
\]

\[
\text{Li}_{p-k}(z) \cdot \text{Li}_{p-k}(z) \left( p^k e_0, p^k \Phi_{p-k}^{-1} e_1 \Phi_{p-k} \right) = \text{Li}_{p-k}(z) \left( e_0, e_1 \right) \quad (39)
\]

3.1.2 Horizontality of Frobenius and Taylor coefficients at \( z = 0 \)

Let \( S \) a function of \( z \) which is locally analytic around \( z = 0 \); we denote its Taylor coefficient of order \( n \in \mathbb{N} \) at \( 0 \) by \( S[z^n] \).

**Facts 3.2.** (Taylor coefficients of multiple polylogarithms)

Let \( d \in \mathbb{N}^* \), \( (L, s_d, \ldots, s_1) \in \mathbb{N} \times (\mathbb{N}^*)^d \) and \( N \in \mathbb{N}^* \).

i) We have:

\[
(\tau(N) \text{Li}_{p-k}z)(e_0^s e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1)(z^N) = N^{s_d+\ldots+s_1} H_N(s_d, \ldots, s_1) \quad (40)
\]

ii) Provided the limit exists, we have in the sense of §2.5:

\[
\lim_{L \to +\infty} \tau(N)(\text{Li}_{p-k}z\left( e_0, \Phi_{p-k}^{-1} e_1 \Phi_{p-k} \right)) e_0^{L} e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1 \quad (41)
\]

\[
= (\tau(N)(\Phi_{p-k}) \circ N^{\text{weight}}H_N)(s_d, \ldots, s_1)
\]

**Proof.** i) follows from the KZ equation and ii) follows from i) and lemma [U2].

**Facts 3.3.** (vanishing of some coefficients)

Let \( k \in \mathbb{Z} \setminus \{0\} \).

i) For every word \( w \neq 0 \) in \( \mathcal{H}_m \), we have \( \text{Li}_{p-k}[w](z = 0) = 0 \)

i') For every word \( w \neq 0 \) in \( \mathcal{H}_m e_1 \), we have \( \text{Li}_{p-k}[w](z = 0) = 0 \).

ii) For all \( m \in \mathbb{N}^* \), we have \( \text{Li}_{p-k}[e_0^m] = 0 \).

**Proof.** i) follows from the definition of \( \text{Li}_{p-k} \) in terms of \( F \).

i') follows from the KZ equation.

ii) We have \( \text{Li}_{p-k}[e_0](z) = \log(z) \); thus, the theorem 3.1 implies, for all \( k \in \mathbb{N}^* \), \( -\text{Li}_{p-k}[e_0] = \text{Li}_{p-k}[e_0] = \log(z^p) - p^k \log(z) = 0 \).
Corollary 3.4. Let \(d \in \mathbb{N}^*\), \((L, s_d, \ldots, s_1) \in \mathbb{N} \times (\mathbb{N}^*)^d\) and \(N \in \mathbb{N}^*\). We have:

\[
\tau(N) \text{Li}_{p^{-1}}[e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1][z^{pN}]
\]

\[
+ \tau(N) \left( \text{Li}_{pkz} (e_0, \Phi_p^{-1} e_1 \Phi_{p^{-1}}) \right) [e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1][z^N]
\]

\[
+ \sum_{1 \leq n \leq pN-1} \sum_{\substack{w_1^{(L)} \text{ words in } \mathcal{H}_n \\text{ with } e_0^{k}, e_0^{k-1} e_1 \ldots e_0^{k-1} e_1 = w_1^{(L)} w_2 \\text{ depth}(w_1^{(L)}) \geq 1 \\text{ and } w_2 \neq \emptyset}} \tau(N) \text{Li}_{p^{-1}}[w_1^{(L)}][z^n]. \text{Li}_{pkz} (e_0, \Phi_p^{-1} e_1 \Phi_{p^{-1}})[w_2][z^{N-\frac{1}{p}}]
\]

\[
= (pN)^{s_d+\ldots+s_1} H_{pN}(s_d, \ldots, s_1) \quad (42)
\]

**Proof.** This is \(33\) for \(k = 1\), for the coefficient \([e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1][z^{pN}]\), to which we have applied \(\tau(N)\). The previous facts imply that the sum over \(n\) restricts to \(n \in \{1, \ldots, pN - 1\}\) and the sum over \(w_1\) restricts to depth\((w_1)\) \(\geq 1\). \(\square\)

### 3.1.3 Role of the multiple polylogarithms expressing the iterates of Frobenius

We want to emphasize the role of the \(\text{Li}_{p^{-k}}\) for all \(k \in \mathbb{N}^*\) and not only \(k = 1\).

**Corollary 3.5.** Let \(k \in \mathbb{N}^*\); let \(d \in \mathbb{N}^*\), \((L, s_d, \ldots, s_1) \in \mathbb{N} \times (\mathbb{N}^*)^d\). The difference

\[
(p^k)^{s_d+\ldots+s_1} H_{p^k}(s_d, \ldots, s_1) - \sum_{b=0}^{L} \Phi_p^{-1} e_1 \Phi_{p^{-k}}[e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1]
\]

admits the simple geometric expression

\[
\text{Li}_{p^{-k}}[e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1][z^{p^k}]
\]

**Proof.** We use equation \(33\) of theorem 3.1 for all \(k \in \mathbb{N}^*\), translated on the Taylor coefficient or order \(p^k\) at \(z = 0\). If \(S\) is a power series, we have \(S(z^{p^k})[z^n] = 0\) for all \(n \in \{1, \ldots, p^k - 1\}\). Thus the analog of the sum over \(n \in \{1, \ldots, pN - 1\}\) in the previous lemma \(33\) vanishes. We have a term

\[
\tau(1) \text{Li}_{pkz} (z^{p^k})(e_0, \Phi_p^{-1} e_1 \Phi_{p^{-k}})[e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1][z^{p^k}]
\]

\[
\text{Li}_{pkz}(z^{p^k})[z^k] = \text{Li}_{pkz}(z)[z] \text{ is equal to } 1 \text{ in depth one, and to } 0 \text{ in all depths } \geq 2.
\]

Thus, the only the sequences of subwords as in formula \(27\) that give a non-zero contribution are the ones having a quotient of depth one, and this contribution is \(\sum_{b=0}^{L-1} \Phi_p^{-1} e_1 \Phi_{p^{-k}}[e_0^L e_1^{s_d-1} e_1 \ldots e_0^{s_1-1} e_1]\). \(\square\)

### 3.1.4 Remarks

**Remark 3.6.** Analogous equalities can be written using the horizontal equation for positive powers of Frobenius \(39\). Whereas the equation for negative powers will give directly that \(\zeta_{p^k,0} = \Sigma_{\text{sym}} \Phi_{p^{-k}}\) in the sense of §2.5, the equation for positive powers would give the equivalent fact that \(\Sigma_{\text{sym}} \Phi_{p^k} \circ \zeta_{p^k,0} = 1\) in the sense of §2.5.

**Remark 3.7.** One can define multiple harmonic sums associated to words \(w = e_0^L e_1^{r_d-1} e_1 \ldots e_0^{r_1-1} e_1 e_0^r\), with possibly \(r \neq 0\), as equal to \(H_N(w) = \text{Li}_{pkz}[w][z^N]\), where we view \(\text{Li}_{pkz}[w]\) as an element of \(\mathbb{Q}[[z]][\log(z)]\); they lie in \(\mathbb{Q}[[z]]\). Their degree with respect to \(\log(z)\) depends on \(r\), and they also depend polynomially in \(l\). Their coefficients with respect to \(l\) and \(\log(z)\) are linear combinations of the usual multiple harmonic sums \(N^\text{weight\ }H_N\). The previous facts can also be written in terms
of those more general multiple harmonic sums, by considering all coefficients of theorem 3.1, and not only the ones of the words of the form \(ue_1\).

### 3.2 Consequences of the bounds on the valuation of \(p\)-adic multiple zeta values

#### 3.2.1 Explicit computations and lower bounds of valuation of \(p\)-adic multiple zeta values ([J1])

What follows is detailed in [J1], §4. Let \(w \in H_x\) and \(k \in \mathbb{Z}\setminus\{0\}\). The function \(n \in \mathbb{N}^* \mapsto \text{Li}_{p,k}[w][z^n] \in \mathbb{Q}_p\) extends continuously to \(\mathbb{Z}_p\) by a classical lemma of \(p\)-adic interpolation. By induction on the depth of \(w\), using that \(\text{Li}_{p,k}\) is made of overconvergent analytic iterated integrals, we can show that this extension is locally analytic, with Taylor coefficients having valuations satisfying specific bounds and admitting expressions as explicit sums of series, involving variants of multiple harmonic sums.

The form of these series imply more precise bounds on their valuation, then, bounds on the valuation of \(p\)-adic multiple zeta values and of the values of \(n \in \mathbb{N}^* \mapsto \text{Li}_{p,k}[w][z^n]\).

These bounds are however non optimal. The one that we reproduce here is the roughest possible one. We express it in terms of the definitions of §2.3. For all words, we have:

**Fact 3.8.** (bounds of valuations)

i) For all \(k \in \mathbb{Z}\setminus\{0\}\), we have:

\[
\|\text{Li}_{p,k}\|' \leq \sum_{n,d \geq 1} p^{-\inf_{l \in \mathbb{N}} \left(l + n - (2d \log (2d) - d + 1) - 2d \log (l + n)\right)} T^n U^d
\]

In particular, \(\text{Li}_{p,k} \in A(\mathbb{P}^1 - [1])_{\Sigma}\) in the sense of §2.3.5.

ii) For all \(k \in \mathbb{Z}\setminus\{0\}\), we have:

\[
\|\Phi_{p,k}\|' \leq \sum_{n,d \geq 1} p^{-\left(n - (2d \log (2dn) - d + 1)\right)} T^n U^d
\]

In particular, \(\Phi_{p,k} \in 1_{\Sigma_0}^{dR}(\mathbb{Q}_p)_{\Sigma}\) in the sense of §2.3.4.

The fact that (43) and (44) imply respectively \(\text{Li}_{p,k} \in A(\mathbb{P}^1 - [1])_{\Sigma}\) and \(\Phi_{p,k} \in 1_{\Sigma_0}^{dR}(\mathbb{Q}_p)_{\Sigma}\) follows from that the function \(l \mapsto l + n - 2d \log (l + n)\) is increasing on \([2d - n, \infty]\), thus, for \(n > 2d\), the inf of the formula (43) is \(\geq n - 1 - 2d \log (2dn)\), which is also equal to the bound of (44), and which tends to \(+\infty\) when \(n \to +\infty\).

**Remark 3.9.** For \(p\) large enough in function of \(s_1 + \ldots + s_d\) and \(d\), these bounds can be improved significantly ([J1]).
3.2.2 Functions of $L$

Corollary 3.10. For all $d \in \mathbb{N}^*$, $(s_d, \ldots, s_1) \in (\mathbb{N}^*)^d$, $L \in \mathbb{N}$, the quantity

$$
\tau(N) \operatorname{Li}_{p-1}(e_0^L e_1 e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1)[z^{pN}]
$$

$$
+ \sum_{1 \leq n \leq pN-1, p\mid n} \tau(N) \operatorname{Li}_{p-1}(w_1^{(L)}, w_2) \sum_{w_2 \neq \emptyset, \text{depth}(w_1^{(L)}) \geq 1} e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1 = w_1^{(L)} w_2
$$

(45)

tends to 0 when $L \to +\infty$.

Proof. The indexation of $w_2$ doesn’t depend on $L$ ; thus, the $w_2$ factors in the second line are contained in a bounded subset of $\mathbb{Q}_p$ depending only on $s_d, \ldots, s_1$. Moreover, each sequence $(w_1^{(L)})$ is determined by the corresponding $w_2$ and satisfies : $\limsup \text{depth} w_1^{(L)} < +\infty$ and weight $w_1^{(L)} \to +\infty$, as does the sequence $(e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1)_L$. Thus, the corollary follows from the fact 3.8.

This, combined with fact 3.2(ii), finishes the proof of i) theorem 1.

3.2.3 Functions of $k$

Corollary 3.11. The Taylor coefficients of $\Phi_{p^{-1}}^{-1} e_1 \Phi_{p^{-1}}$ and $\Phi_{p^{-1}}^{-1} e_1 \Phi_{p^{-k}}$ with respect to $(p^k)_{k \in \mathbb{N}}$, in the sense of §2.4.2 have norms bounded by

$$
\left( \sum_{n, d \geq 1} p^{-\inf_{l \in \mathbb{N}} \left( l+n-(2d \log(2d)-d+1)-2d \log(l+n) \right)} \right)^2 T^n U^d
$$

and lie also in $\mathbb{Q}_p((e_0, e_1))$. 

Proof. For all $k \in \mathbb{Z} \setminus \{0\}$, we have $||\text{sym } \Phi_{p^k}||' \leq ||\Phi_{p^k}||'$ (lemma 2.14 (2.3.3)) and, by taking limits $k \to \pm \infty$, because of §2.4.1, the bounds of valuation of fact 3.8 give

$$
||\text{sym } \Phi_{p\infty}||' \leq \sum_{n, d \geq 1} p^{-\inf_{l \in \mathbb{N}} \left( l+n-(2d \log(2d)-d+1)-2d \log(l+n) \right)} T^n U^d
$$

The Taylor coefficients of $\Phi_{p^k}$ and $\Phi_{p^{-k}}$ are given in terms of $\Phi_{p\infty}$ and $\Phi_{p^{-\infty}}$ by §2.4.2 ; this gives the Taylor coefficients of $\text{sym } \Phi_{p^k}$ and $\text{sym } \Phi_{p^{-k}}$ by the fact 2.9 of §2.2. Then the corollary 2.15 of §2.3.3 and the lemma 2.17 of §2.3.4 imply the result.

The theorem 1 combined to the Taylor expansion of $\Phi_{p^{-1}}^{-1} e_1 \Phi_{p^{-k}}$ implies the formula involving an absolutely convergent double sum

$$
\zeta_{p,k} = \sum_{L \geq 0} \sum_{s \geq 0} (p^k)^s (\text{sym } \Phi_{p\infty} \circ \Phi_{p^{-k}})[e_0^L e_1 e_0^{s_d-1} e_1 \cdots e_0^{s_1-1} e_1]
$$

Corollary 3.12. This double sum can be inverted.

This finishes the proof of i) of theorem 2.

3.3 Sketch of a generalization to multiple polylogarithms

Here is a first remark in another direction.
Lemma 3.13. Let \( w \in \mathcal{H}_n \) a word.

i) The sequence \( (p^{-k}\text{weight } \text{Li}_{p^{-k}}[w][z^n])_{n \in \mathbb{N}} \) converges pointwise, when \( k \to +\infty \), to the sequence \( (\text{Li}_{p^{k}\text{weight }}[w][z^n])_{n \in \mathbb{N}} \). 

ii) For \( z \in \mathbb{C}_p \) such that \( |z|_p < 1 \), we have:

\[
 (p^{-k}\text{weight } \text{Li}_{p^{-k}}[w](z) \to \text{Li}_{p^{k}\text{weight }}[w](z), \text{ and this convergence is uniform on each closed disk } \{ z \in \mathbb{C}_p \mid |z|_p \leq \epsilon \}, 0 < \epsilon < 1. 
\]

Proof. i) Theorem 3.1 implies : for \( n < p^k \), \( p^{-k}\text{weight } \text{Li}_{p^{-k}}[w][z^n] = \text{Li}_{p^{k}\text{weight }}[w][z^n] \).

ii) For \( |z|_p < 1 \) we have :

\[
 p^{-k}\text{weight } \text{Li}_{p^{-k}}[w](z) = \sum_{0 < n < p^k} p^{-k}\text{weight } \text{Li}_{p^{-k}}[w][z^n]z^n + \sum_{n \geq p^k} p^{-k}\text{weight } \text{Li}_{p^{-k}}[w][z^n]z^n 
\]

The first term is equal to \( \sum_{0 < n < p^k} p^{-k}\text{weight } \text{Li}_{p^{k}\text{weight }}[w][z^n]z^n \) and tends to \( \text{Li}_{p^{k}\text{weight }}(z) \), uniformly on each disk \( \{ z \in \mathbb{C}_p \mid |z|_p \leq \epsilon \} \). The second term has its valuation lower bounded by the following quantity, for a certain function \( f \), since \( n \geq p^k \) and because of fact 3.8 :

\[
 -k\text{ weight}(w) - f(\text{weight}(w), \text{depth}(w)) + p^k \nu_p(z)
\]

Remark 3.14. The convergence of \( (p^{-k}\text{weight } \text{Li}_{p^{-k}}[w][z^n])_{n \in \mathbb{N}} \) is not uniform. Otherwise, the function \( N \in \mathbb{N}^* \subset \mathbb{Z}_p \to H_N(w) \in \mathbb{Q}_p \) would be continuous, which is false according to the next part.

3.4 Finite multiple zeta values for all \( a \in \mathbb{Z} \) and for \( a = 0 \)

Corollary 3.15. Let \( a \in \mathbb{Z} ; k \in \mathbb{N}^* ; d \in \mathbb{N}^* \), \( (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \). The following series is well-defined

\[
 \Phi_{p^{-k}} e_1 \Phi_{p^{-k}} \left[ \frac{1}{1 - e_0} e_1 \frac{e_0^{s_d} - 1}{(1 + e_0 a)^{s_d}} e_1 \cdots \frac{e_0^{s_1} - 1}{(1 + e_0 a)^{s_1}} e_1 \right] = \sum_{i_d, \ldots, i_1 \geq 1} \prod_{i=1}^{d} \left( -s_i \right) \Phi_{p^{-k}} e_1 \Phi_{p^{-k}} \left[ \frac{1}{1 - e_0} e_1^{s_0 + i_0 - 1} e_1 \cdots e_1^{s_1 + i_1 - 1} e_1 \right] 
\]

Proof. This follows from fact 3.8.

4 Multiple harmonic sums attached to locally analytic characters

The following definitions permit to clarify the computations of the next part.

4.1 Definitions

We will consider below locally analytic characters \( \chi : \mathbb{Z}_p^* \to \mathbb{Z}_p^* \) which are analytic on the closed disks radius \( \frac{1}{p} \), with Taylor coefficients in \( \mathbb{Z}_p \). They form a group which we denote by \( G \). We denote their Taylor expansions by \( \chi(a) = \sum_{i \geq 0} \chi^{(i)}(a)x^i \). The hypothesis on the characters is not optimal, and similar computations could be derived for more general locally analytic characters.
Characters defined on $\mathbb{Z}_p^*$ can be extended by multiplicativity to characters $\chi : \mathbb{Q}_p \setminus \{0\} \to \mathbb{Q}_p$; for some computations, we will consider the case where $u_p(\chi(p^{-1})) \geq 1$.

**Definition 4.1.** To characters $\chi_d, \ldots, \chi_1$ as above we attach a multiple harmonic sum: for $N \in \mathbb{N}^*$, let:

$$H_N(\chi_d, \ldots, \chi_1) = \sum_{0 < n_1 < \ldots < n_d < N} \chi_1(n_1) \ldots \chi_d(n_d)$$

We define similarly $H_{M<N}(\chi_d, \ldots, \chi_1)$, for $M, N \in \mathbb{Z}$ such that $0 \leq M < N$ or $M < N \leq 0$, as in §1.1.1. We also define, for $N_0 \in \mathbb{N}$:

$$H_N^{\text{not}}(\chi_d, \ldots, \chi_1) = \sum_{0 < n_1 < \ldots < n_d < N} \chi_1(n_1) \ldots \chi_d(n_d)$$

$$H_{M<N}^{\text{not}}(\chi_d, \ldots, \chi_1) = \sum_{M < n_1 < \ldots < n_d < N} \chi_1(n_1) \ldots \chi_d(n_d)$$

**Notation 4.2.** For $s \in \mathbb{Z}$, let $\psi_s$ the character $n \mapsto n^s$.

### 4.2 The case of positive powers of the indices

**Notation 4.3.** i) For all $d \in \mathbb{N}^*$ and $(s_d, \ldots, s_1)$ in $\mathbb{Z}^d$, not necessarily in $\mathbb{N}^*$, let:

$$H_N(s_d, \ldots, s_1) = H_N(\psi_{-s_d}, \ldots, \psi_{-s_1})$$

(47) When $s_d, \ldots, s_1 \in \mathbb{N}^*$, this is coherent with the previous notations.

ii) In the case of $s_d = -l_d, \ldots, s_1 = -l_1 \in \mathbb{N}^*$, let:

$$H_N(-l_d, \ldots, -l_1) = \sum_{0 \leq n_1 < \ldots < n_d < N} n_1^{l_1} \ldots n_d^{l_d}$$

We have:

$$H_N(-l_d, \ldots, -l_1) = H_N(-l_d, \ldots, -l_1) + \chi_1(l_1 \neq 0)$$

In other words, for $l_k \neq 0$:

$$H_N(-l_d, \ldots, -l_k, 0, \ldots, 0) = \sum_{j=0}^{k-1} (-1)^j H_N(-l_d, \ldots, -l_j)$$

It is classical that, for $l \in \mathbb{N}^*$, $H_1(l)$ is the Bernoulli polynomial:

$$H_1(l) = \sum_{u=0}^{l+1} B_u T^u \in \mathbb{Q}[T]$$

where we denote by

$$B_u = \frac{1}{l+1} \left( \frac{l+1}{u} \right) B_{l+1-u}$$

More generally:

**Corollary 4.4.** For $l_d, \ldots, l_1 \in \mathbb{N}^*$, $H_N(-l_d, \ldots, -l_1)$ is a polynomial function of $N$ of degree $\leq l_d + \ldots + l_1 + d$, and with zero constant coefficient. For $1 \leq k_d \leq l_1 + \ldots + l_d + d$, the coefficient of order $k_d$ is

$$\mathbb{B}_{l_d \ldots l_1}^{k_d} = \sum_{k_1=1}^{l_1+1} \ldots \sum_{k_d=1}^{l_d+1} \mathbb{B}_{k_1}^{l_1} \mathbb{B}_{k_2}^{l_2} \ldots \mathbb{B}_{k_d}^{l_d+k_d}$$

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4.3 Variant

**Definition 4.5.** Let, with the same hypothesis,

\[ H'_N(\chi_d, \ldots, \chi_1) = \sum_{0 < n_1 < \ldots < n_d < N} \chi_d(\frac{n_d}{N}) \ldots \chi_1(\frac{n_1}{N}) \]

and similarly

\[ H'_{M<N}(\chi_d, \ldots, \chi_1) = \sum_{M < n_1 < \ldots < n_d < N} \chi_d(\frac{n_d}{N-M}) \ldots \chi_1(\frac{n_1}{N-M}) \]

The finite multiple zeta values are

\[ \zeta_{f,a}(s_d, \ldots, s_1) = H'_{ap^k < (a+1)p^k}(\psi_{-s_d}, \ldots, \psi_{-s_1}) \]

4.4 Weight

**Definition 4.6.** i) Let weight : \( G \rightarrow \mathbb{Z} \) the map \( \chi \mapsto v_p(\chi(p))^{-1} \).

It satisfies weight(\( \chi_1 \chi_2 \)) = weight(\( \chi_1 \)) + weight(\( \chi_2 \)).

ii) Let the weight of a function of the type

\[ (N, M) \mapsto \chi'_1(N)\chi'_2(M)\chi'_3(M-N) \times H'_{N,M}(\chi_d, \ldots, \chi_1) \]

be weight(\( \chi'_1 \)) + weight(\( \chi'_2 \)) + weight(\( \chi'_3 \)) + \sum_{j=1}^{d} \text{weight}(\chi_1) .

In particular, each finite multiple zeta value \( \zeta_{f,a}(s_d, \ldots, s_1) \) is of weight \( s_d + \ldots + s_1 \) in this sense.

Equalities in the next part will be homogeneous for the weight in the sense of this definition.

5 Arithmetic proofs

5.1 Symmetry and translation of upper bounds

**Fact 5.1.** (multiplication by -1 of upper bounds)

Let \( M, N \in \mathbb{Z} \) such that \( 0 \leq M < N \) and \( \chi_d, \ldots, \chi_1 \) characters. We have :

\[ H_{M<N}(\chi_d, \ldots, \chi_1) = \left( \prod_{i=1}^{d} \chi_i(-1) \right) H_{-M-N}(\chi_1, \ldots, \chi_d) \]

This enables to restrict the study of finite multiple zeta values \( \zeta_{f,a} \) to the ones such that \( a \geq 0 \).

**Fact 5.2.** (translation of upper bounds)

Let \( M \in \mathbb{Z}, N \in \mathbb{N} \) such that \( 0 \leq M \) or \( M + N \leq 0 \), and \( \chi_d, \ldots, \chi_1 \) characters.

i) We have :

\[ H_{M<N}(\chi_d, \ldots, \chi_1) = \sum_{0 < n_1 < \ldots < n_d < N} \chi_1(n_1 + M) \ldots \chi_d(n_d + M) \]
ii) If moreover, for all \( n \in [1, N - 1] \), we have \( v_p(n) < v_p(M) \), then:

\[
H_{M<M+N}(\chi_d, \ldots, \chi_1) = \sum_{l_1, \ldots, l_d \geq 0} \prod_{i=1}^d M_{l_i}^{(l_i)}(1) H_N(\chi_d^{l_d} \psi^{-l_d}, \ldots, \chi_1^{l_1} \psi^{-l_1})
\]

**Corollary 5.3.** Let \( a \in \mathbb{Z}, k, d \in \mathbb{N}^*, (s_d, \ldots, s_1) \in (\mathbb{N}^*)^d \). Then

\[
\zeta_{f_k,a}(s_d, \ldots, s_1) = \sum_{l_d, \ldots, l_1 \geq 0} \prod_{i=1}^d (-s_i) \zeta_{f_k,a}(s_d + l_d, \ldots, s_1 + l_1)
\]

This is the analogue of the statement of §3.3. It enables us to restrict the study to finite multiple zeta values of the form \( \zeta_{f_k,a} \).

### 5.2 Addition of upper bounds

#### 5.2.1 General statement

**Lemma 5.4.** (addition of upper bounds, first version)

Let \( N_1, N_2 \in \mathbb{N}^* \), and \( \chi_d, \ldots, \chi_1 \) characters; we have:

\[
H_{N_1+N_2}(\chi_d, \ldots, \chi_1) = \sum_{k=0}^d H_{N_1+N_2}(\chi_d, \ldots, \chi_{k+1}) H_{N_1}(\chi_k, \ldots, \chi_1) + \sum_{k=0}^d H_{N_1+N_2}(\chi_d, \ldots, \chi_{k+1}) H_{N_1}(\chi_{k-1}, \ldots, \chi_1) \quad (48)
\]

It is convenient to state it also in another way.

**Notation 5.5.** Let \( N \in \mathbb{N}^* \), let \( \mathcal{Q} \subset [1, N - 1] \). For each \( d \in \mathbb{N}^* \), \( (n_1, \ldots, n_d) \in (\mathbb{N}^*)^d \), such that \( 0 < n_1 < \ldots < n_d < N \), let \( E_{n_1, \ldots, n_d}(\mathcal{Q}) = \{ i \in [1, \ldots, d] \mid n_i \in \mathcal{Q} \} \).

**Lemma 5.6.** (addition of upper bounds, second version)

Let \( N \in \mathbb{N}^* \), \( \mathcal{Q} = \{ q_1, \ldots, q_{d'} \} \subset [1, N - 1] \) with \( q_1 < \ldots < q_{d'} \). We also denote by \( q_0 = 0 \) and \( q_{d'+1} = N \). Then, for \( \chi_d, \ldots, \chi_1 \) characters, we have

\[
H_N(\chi_d, \ldots, \chi_1) = \sum_{0 = i_0 < i_1 < \ldots < i_{d'} < i_{d'+1} = d} \prod_{j=1}^{d'} \chi_{i_j}(q_j) \prod_{j=0}^{d} H_{q_j < q_{j+1}}(\chi_{i_{j+1}-1}, \ldots, \chi_{i_{j}+1})
\]

#### 5.2.2 \( p \)-adic consequences

Let \( N \in \mathbb{N}^* \), and \( p \) a prime number. Let the \( p \)-adic expansion of \( N \) be \( N = a_{y_1} p^{y_1} + a_{y_2-1} p^{y_2-1} + \ldots + a_{y_l} p^{y_l} \) with \( y_1 > \ldots > y_l \), and \( a_{y_i} \in \{1, \ldots, p - 1\} \) for all \( i \). Let

\[
\mathcal{Q}(N, p) = \{ p^{y_1}, \ldots, a_{y_1} p^{y_1}, a_{y_2} p^{y_2} + p^{y_2-1}, \ldots, a_{y_2} p^{y_2} + a_{y_2-1} p^{y_2-1}, \ldots, a_{y_l} p^{y_l} + \ldots + (a_{y_1}-1) p^{y_1}\}
\]

**Corollary 5.7.** All multiple harmonic sums \( H_N(\chi_d, \ldots, \chi_1) \) can be expressed in \( \mathcal{Q}_p \) as absolutely convergent sums whose terms are \( \mathcal{Q} \)-linear combinations (with coefficients depending on the \( p \)-adic expansion of \( N \)) of multiple harmonic sums \( H_{p,k} \) multiplied by powers of \( p \), and multiple harmonic sums \( H_r \), with \( 1 \leq r \leq p - 1 \).
5.3 Multiplication of upper bounds

5.3.1 General statement

Let \( N, M, d \in \mathbb{N}^* \).

Definition 5.8. i) Let :

\[
\varphi : \{(n_1, \ldots, n_d) \in (\mathbb{N}^*)^d \mid 0 < n_1 < \ldots < n_d < MN\} \\
\rightarrow \{(E, \mathcal{P}) \mid E \text{ subset of } \{1, \ldots, d\}, \mathcal{P} \text{ partition of } \{1, \ldots, d\} \setminus E \text{ into segments} \}
\]

defined by :

- \( E_{n_1, \ldots, n_d} = \{i \in \{1, \ldots, d\} \mid M \text{ divides } n_i\} \).
- \( \mathcal{P}_{n_1, \ldots, n_d} \) is the partition of \( \{1, \ldots, d\} \setminus E_{n_1, \ldots, n_d} \) as \( \Pi_0 \leq a \leq N-1 f^{-1}(\{a\}) \), where \( f : \{1, \ldots, d\} \setminus E_{n_1, \ldots, n_d} \rightarrow \mathbb{N}, i \mapsto [\frac{n_i}{M}] \).

Notations 5.9. Let a couple \((E, \mathcal{P})\) where \( E \) is a subset of \( \{1, \ldots, d\} \) and \( \mathcal{P} \) is a partition of \( \{1, \ldots, d\} \setminus E \) into segments of \( \mathbb{N} \). We will denote below such a \((E, \mathcal{P})\) as :

i) \( E = (i_1, \ldots, i_d') \)

ii) \( \mathcal{P} = (\Pi_{x=1}^{i_1-1} \mathcal{P}^{x_1}_0 = [1, i_1 - 1], \Pi_{x=1}^{i_2-1} \mathcal{P}^{x_1}_1 = [i_1 + 1, i_2 - 1], \ldots, \Pi_{x=d'-1}^{i_d'} \mathcal{P}^{x_1}_{d'} = [i_d' + 1, d]) \)

where \( u_i^{x_j} < u_i^{x_j+1} \) for all elements \((u_i^{x_j}, u_i^{x_j+1})\) of \( \mathcal{P}_i^{x_j} \times \mathcal{P}_i^{x_j+1} \).

Moreover, we will denote by \((\chi_{d}, \ldots, \chi_1)|_{\mathcal{P}_i}^{x_j} \) the subsequence of \((\chi_{d}, \ldots, \chi_1)\) corresponding to indices in \( \mathcal{P}_i^{x_j} \).

Lemma 5.10. (multiplication of upper bounds)

We have, with the previous notations for each element \((E, \mathcal{P})\) :

\[
H_{NM}(\chi_d, \ldots, \chi_1) = \sum_{(E, \mathcal{P})} (E, \mathcal{P}) = \sum_{i=1}^{d'} \prod_{t=1}^{d'} \chi_t(u_i(M) \prod_{t=0}^{d'} H_{q_{x_t} M < (q_{x_t} + 1) M}((\chi_d, \ldots, \chi_1)|_{\mathcal{P}_i})
\]

(49)

Proof. We regroup the indices \( n_1, \ldots, n_d \) in function of the value of \((E_{n_1, \ldots, n_d}, \mathcal{P}_{n_1, \ldots, n_d})\).

5.3.2 \( p \)-adic consequences

Corollary 5.11. Let \( k \in \mathbb{N}^* \) and \( \chi_d, \ldots, \chi_1 \) characters.

i) Then the function \( N \mapsto H'_{N p^k}(\chi_d, \ldots, \chi_1) \) can be expressed in \( \mathbb{Q}_p \) as an absolutely convergent series whose terms are \( \mathbb{Q} \)-linear combinations of products of \( H'_{p^k} \), and \( H_N \) involving the characters
\[\chi_d, \ldots, \chi_1\] and the \(\psi_i, l \in \mathbb{N}\).

ii) Assume that \(\chi_i = \psi_{s_i}\), where \(s_i \in \mathbb{N}^+\). Then \(N \mapsto H_{N|p^k}\) can be written as a linear combination of \(H_N(t_{1d}, \ldots, t_{1d})\) with \(d' \geq d\), and \(t_{1d} \leq s_{1d}, \ldots, t_{1d} \leq s_{1d}\), whose coefficients are absolutely convergent infinite sums whose terms are multiple harmonic sums \(H'_{p^k}\) multiplied by powers of \(N\).

**Proof.** We apply the lemma \(5.10\) taken in the case where \(N = p^k\), \(k \in \mathbb{N}^+\). The result involves the factors

\[
\prod_{0 \leq i \leq d' \atop 1 \leq i \leq j} H_{q_i, p}^{q_i} \left( (\chi_d, \ldots, \chi_1) \right)_{p^i}
\]

which are equal, because of fact \(5.2\) ii), to:

\[
\sum_{l_i \geq 0} \sum_{i \in P_{l_i}^{1}, P_{l_i}^{1}, \ldots, P_{l_i}^{1}} \sum_{m_i \leq q_{i1} \ldots q_{ij} \leq m_{i+1} - 1} \prod_{i=1}^{j} (p_k)^{m_i} \prod_{i \in P_{l_i}^{1}} (\chi_d^{r_i}, \ldots, \chi_1^{r_i})_{p^i} \prod_{i \in P_{l_i}^{1}} (\chi_d^{r_i})_{p^i} (1) (50)
\]

This gives i). We obtain ii) by expressing

\[
\sum_{m_i \leq q_{i1} \ldots q_{ij} \leq m_{i+1} - 1} \sum_{i \in P_{l_i}^{1}} \prod_{i \in P_{l_i}^{1}} = H_{m_{i+1}}^{m_{i+1}} = \sum_{i \in P_{l_i}^{1}} \sum_{i \in P_{l_i}^{1}} l_i
\]

as a polynomial of \((m_{i+1}, m_{i+1})\).

This finishes the proof of ii) of theorem 1.

### 5.4 Taylor expansion with respect to \(p^k\)

#### 5.4.1 Introduction

Let \(p\) be a prime number. Multiple harmonic sums can be re-indexed in a way which takes into account the \(p\)-adic valuation of their indices; and one can also add information on the Euclidean division of the \(n, p^{-v_p(n)}\)'s by \(p\) (or, more generally, a power of \(p\)).

**Notations 5.12.** Let \(d, N \in \mathbb{N}^+\), and :

\[
((v_1, \ldots, v_d), (q_1, \ldots, q_d), (r_1, \ldots, r_d)) \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^d.
\]

We denote the condition \(0 < p^{v_1}(pq_1 + r_1) < \ldots < p^{v_d}(pq_d + r_d) < N\) by \(*((v_1), (q_1), (r_1), N)*\).

**Lemma 5.13.** Let \(\chi_1, \ldots, \chi_d\) characters,

i) Any multiple harmonic sum can be re-indexed and expanded \(p\)-adically as

\[
H'_{N}(\chi_d, \ldots, \chi_1) = \sum_{(v_1), (q_1), (r_1)} \prod_{i=1}^{d} \chi_i \left( \frac{p_{v_i}}{N} \right) (\chi_i^{r_i}) \chi_i^{(v_1)} (1) (p_{q_i})^{l_i}
\]

In the case where \(N = p^k\), \(k \in \mathbb{N}^+\), the sum can be restricted to \(*((v_1), (q_1), (r_1))\) \(i=1,d\) \(\in [0, k-1]^d \times [0, p^{k-1} - 1]^d \times [1, p-1]\), and for such indices the last inequality \((\leq p^k)\) is redundant.

Because of §5.2.2, the study of the case where \(N\) is a power of \(p\) implies facts for the general case. In the particular case of finite multiple zeta values we have:

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\[
\zeta_{f^{s_0}}(s_d, \ldots, s_1) = \sum_{(v_0), (q_i), (r_i), (l_i)} \prod_{i=1}^{d} (p^{k-v_i})^{s_i} \sum_{l_i \geq 0} (pq_i)^{l_i} \frac{1}{r_i^{s_i+l_i}} (-s_i) \quad (51)
\]

**Remark 5.14.** We could also derive the computation by using that: for all \( r \in \mathbb{N}^* \):

\[
\zeta_{f^{s_0}}(s_d, \ldots, s_1) \equiv \sum_{(v_0), (q_i), (r_i), (l_i)} \prod_{i=1}^{d} (p^{k-v_i})^{s_i} (pq_i + r_i)^{p^{r-1}(p-1)-s_i} \mod p^r \quad (52)
\]

### 5.4.2 Suppression of the additional parameters

**Remark 5.15.** Let \( \{(v_1, \ldots, v_d), (q_1, \ldots, q_d), (r_1, \ldots, r_d)\} \in \mathbb{N}^d \times \mathbb{N}^d \times [1, p-1]^d \) satisfying \(*\{ (v_i), (q_i), (r_i), N \} \). Then, for all \( i \in \{1, \ldots, d-1\} \), we have:

\[
p^{v_i}(pq_i + r_i) < p^{v_{i+1}}(pq_{i+1} + r_{i+1}) \Leftrightarrow \begin{cases} q_i \leq p^{v_{i+1} - v_i - 1}(pq_{i+1} + r_{i+1}) - 1 & \text{if } v_i < v_{i+1} \\ q_i \leq q_{i+1} & \text{if } v_i = v_{i+1} \text{ and } r_i < r_{i+1} \\ q_i \leq q_{i+1} - 1 & \text{if } v_i = v_{i+1} \text{ and } r_i \geq r_{i+1} \\ p^{v_{i+1} - v_i - 1}(pq_i + r_i) \leq q_{i+1} & \text{if } v_i > v_{i+1} \end{cases}
\]

**Lemma 5.16.** Let \( d \in \mathbb{N}^* ; k \in \mathbb{N}^* \); let \( (v_1, \ldots, v_d) \in \{0, \ldots, k-1\}^d \); let \( (r_1, \ldots, r_d) \in [1, p-1]^d \). Then each

\[
\sum_{(q_i)} \prod_{i=1}^{d} q_i^{l_i}
\]

can be expressed as a polynomial with coefficients in \( \mathbb{Z} \) of \( p^k \), the \( r_i \)'s, \( p^{v_i-1}v_i-1 \) for \( v_i \neq v_{i+1} \), \( p \), with degree at most \( l_1 + \ldots + l_d + d \) in each of these variables, and \( \mathbb{N}^k \) with total degree at most \( d \), and \( l \leq l_1 + \ldots + l_d + d \).

**Proof.** Induction on \( d \). For \( d = 1 \) we have \( \sum_{0 < p^{v_i} < p^k} q_i^l = \sum_{q_i = 0}^{\frac{p^k - v_i - 1}{v_i}} q_i^l = Q_i(p^{k-v_i}). \) For \( d > 1 \), let \( j \in \{1, \ldots, d\} \) such that \( v_j = \min(v_1, \ldots, v_d) \). Then the sum is equal to:

\[
\sum_{(q_i)} \prod_{i=1}^{d} q_i^{l_i}
\]

and, because of Remark 5.15, the right factor can be indexed under the form \( q_j \in [A, B] \) where \( A \) and \( B \) are polynomial expressions of: \( q_{j-1}, r_{j-1}, q_j, r_j, p^{v_{j-1}-v_j-1} \) if \( v_{j-1} > v_j \), and \( p^{v_{j+1}-v_j-1} \) if \( v_{j+1} > v_j \).

**Lemma 5.17.** Let \( d, k \in \mathbb{N}^* \); let \( (v_1, \ldots, v_d) \in \{0, \ldots, k-1\}^d \). Then

\[
\sum_{(q_i), (r_i)} \prod_{i=1}^{l} q_i^{l_i} (\chi_i \psi_i - l_i)(r_i)
\]

can be expressed under the form \( \sum_i P_i(p^{k-v_i-1}, p^{v_i+1-v_i-1}) \) for \( v_i 
eq v_{i+1} \) \( H(p, \chi_{d'}, \ldots, \chi_1) \), where
\( P_i \) are polynomials with coefficients in \( \mathbb{Z} \) of \( p^k \), the \( r_i \)'s, \( p^{\mid v_i+1-v_i \mid -1} \) for \( v_i \neq v_{i+1} \), with degree at most \( l_1 + \ldots + l_d + d \) in each of these variables, and \( B^d_k \) with total degree at most \( d \).

**Proof.** Follows from the previous lemma and from that the conditions of Remark 5.4 depend only on inequalities of the form \( r_i > r_{i+1} \) or \( r_i \leq r_{i+1} \).

The set \( \{0,k-1\}^d \) admits a partition indexed by the set of couples \((P,\sigma)\), where \( P \) is a partition of \( \{1, \ldots, d\} \) and \( \sigma \) is a permutation of \( \{1, \ldots, n\} \) where \( n \) is the number of elements of \( P \). Indeed, for each \((v_1, \ldots, v_d) \in \{0,k-1\}^d \), and \((P,\sigma)\) as above, we say that \((v_1, \ldots, v_d) \in (P,\sigma)\) if and only if, for all, \( i, i', \alpha : \{ v_i = v_{i'} \) for \( i, i' \in P_{\sigma(a)} \)
\( v_i < v_{i'} \) for \( i \in P_{\sigma(a)}, i' \in P_{\sigma(a)+1} \)

The sum over \((v_1, \ldots, v_d)\) can be reindexed according to this partition. We obtain functions of the following type, with powers of \( p \) for the variables \( X_i \).

**Lemma 5.18.** Let \( d', k \in \mathbb{N}^* \); let \( I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, d\} \) with \( i_1 < \ldots < i_r \), and \( X_{i_1}, \ldots, X_{i_r} \) be formal variables. We have:

\[
\sum_{0 \leq w_1 < \ldots < w_{d'} \leq k-1} X_{i_1}^{w_1} \ldots X_{i_r}^{w_r} = \sum_{0 = w_0 \leq w_1 < \ldots < w_{d'} \leq k-1} X_{i_1}^{w_1} \ldots X_{i_r}^{w_r} \cdot H(0, \ldots, 0) \prod_{j=1}^{r-1} H_{w_{i_j} < w_{i_{j+1}}}(0, \ldots, 0) \cdot H_{w_{i_r} \leq k-1}(0, \ldots, 0) \quad (53)
\]

**Definition 5.19.** Let \( d, k \in \mathbb{N}^* \); let \( A_1, \ldots, A_d \in \mathbb{Q}[T] \) be polynomials, \( X_1, \ldots, X_d \) formal variables. Then:

\[
F_k(A_1, \ldots, A_d)(X_1, \ldots, X_d) = \sum_{0 \leq w_1 < \ldots < w_d \leq k-1} X_1^{w_1} \ldots X_d^{w_d} A_1(w_1) \ldots A_d(w_d)
\]

**Notation 5.20.** Let \( E(X_1, \ldots, X_d) \) be the finite set \( \{\{U_1, \ldots, U_d\} \in \mathbb{Q}(X_1, \ldots, X_d)^d \mid U_1 = X_1 \) and, for all \( k \in \{1, \ldots, d-1\}, U_{k+1} \in \{U_kX_{k+1} - X_{k+1}\}\).

**Lemma 5.21.** i) For \( d = 1 \), we have:

\[
\sum_{0 \leq w \leq W-1} X^{w} = \sum_{l=0}^{\alpha} \left( -1 \right)^l \frac{X^l}{(X - 1)^{l+1}} (X^{w} - 1) \sum_{\beta_0 + \ldots + \beta_l = \alpha - l} \frac{w^{\beta_0}}{\beta_0!} (w + 1)^{\beta_1} \ldots (w + l)^{\beta_l}
\]

ii) We have more generally

\[
F_k(A_1, \ldots, A_d)(X_1, \ldots, X_d) = \sum_{(U_1, \ldots, U_d) \in E(X_1, \ldots, X_d)} \sum_{l_1, \ldots, l_d \geq 0} \frac{C_{l_1, \ldots, l_d}(k)}{l_1! \ldots l_d!} \frac{U_{l_1}^{l_1} \ldots U_{l_d}^{l_d}}{U_1^{l_1+1} \ldots U_d^{l_d+1}}
\]

with \( C_{l_1, \ldots, l_d} \in \mathbb{Z}[\text{coefficients of } A_1, \ldots, A_d][T] \).

**Proof.** i) We have \( \sum_{0 \leq w \leq W-1} X^{w} = \left( \frac{X}{\partial X} \right)^\alpha \left( \sum_{0 \leq w \leq W-1} X^w \right) = \left( \frac{X}{\partial X} \right)^\alpha \left( \frac{X^W - 1}{X - 1} \right) \).

On the other hand, for all \( n \in \mathbb{N}^*, m \in \mathbb{N} \), we have \( \frac{X^m}{\partial X} (X^{n+1}) = m \frac{n}{(X - 1)^{n+1}} \).

So, by induction on \( \alpha \), for all \( \alpha \in \mathbb{N}^*, \left( X^{\frac{\partial}{\partial X}} \right)^\alpha \frac{X^m}{(X - 1)^{n+1}} = \sum_{l=0}^{\alpha} \left( -n \right) \left( -n-1 \right) \ldots \left( -n-l+1 \right) \frac{X^{m+l}}{(X - 1)^{n+l+1}} \).
ii) Induction on \( d \): assume that \( A_1 = \sum_{\alpha_1=0}^{\deg A_1} u_{\alpha_1} X^{\alpha_1} \) with \( u_{\alpha_1} \in \mathbb{Q} \). We have, for all \( \alpha_1 \in \{0, \ldots, \deg A_1\} \):

\[
F_k(X^{\alpha_1}, A_2, \ldots, A_d) = \sum_{l=0}^{a_1} (-1)^l l! \frac{X^l}{(X-1)^{l+1}} \left[ F_k(B_{2,l}, A_3, \ldots, A_d) - F_k(B_{2,l}, A_3, \ldots, A_d)(X_2, \ldots, X_d) \right]
\]

with \( B_{2,l} = \left( \sum_{\beta_0 + \ldots + \beta_l = \alpha_1 - l} X^{\beta_0} \ldots (X + l)^{\beta_l} \right) A_2. \)

\[\square\]

### 5.4.3 Inversion of a sum of series

This paragraph is only sketched in this first version. As in \( \text{[J1]} \) we need the following lower bounds of valuations:

**Fact 5.22.** i) For any \( n \in \mathbb{N}^* \), it follows from \( p^{v_p(n)} \leq n \) that \( v_p(\frac{1}{n}) \geq -\frac{\log(\alpha)}{\log(p)} \).

ii) \( \text{(Von-Staudt Clausen theorem)} \) For any \( n \in \mathbb{N}^* \) \( v_p(B_n) \geq -1 \).

Separately, we have:

**Fact 5.23.** i) If \( X, Y \) are formal variables and \( k \in \mathbb{N}^* \), we have \( \frac{X^k-Y}{X^k-1} = \frac{X^k-X}{X^k-1} - \frac{X^k-Y}{X^k-1} \).

ii) Let \( z \in \mathbb{Q}_p \) with \( v_p(z) \neq 0 \). Then we have \( v_p(\frac{1}{z-1}) > 0 \) if \( v_p(z) > 0 \), and \( v_p(\frac{1}{z-1}) > v_p(z^{-1}) \) if \( v_p(z) < 0 \).

This combined with the previous lemmas roughly justifies that we obtain sums of series of a similar type with part I; thus those series can be inverted in order to have an absolutely convergent Taylor expansion in terms of \( p^k \); which finishes the proof of ii) of theorem 2.

### 6 Variant of the application to \( p \)-adic multiple zeta values

Here is a slightly different way to retrieve the series expansion of \( p \)-adic multiple zeta values using the computations of the previous parts. Instead of sticking to the multiple harmonic sums themselves, we consider their variants "twisted by Frobenius" as in part I; we have showed in part I that they extend to locally analytic functions on \( \mathbb{Z}_p \). What we have shown in this paper is that the multiple harmonic sums themselves satisfy a weak version of this local analyticity; namely, the Taylor expansion along powers of \( p \). Actually, this property is almost sufficient to retrieve the series expansions of \( p \)-adic multiple zeta values, as we explain now.

**Fact 6.1.** \( \text{[J1]} \) Because of a lemma of Mahler on \( p \)-adic interpolation, we have, for all \( k_0 \in \mathbb{Z} \setminus \{0\} \),

\[
L_{p^{k_0}}[w][z^N] \xrightarrow{|N|_p \to 0} \Phi_{p^{k_0}}(e_0, -e_0 - e_1)[w].
\]

In particular, we have \( L_{p^{k_0}}[w][z^{p^m}] \to_{m \to +\infty} \Phi_{p^{k_0}}(e_0, -e_0 - e_1)[w] \).

The maps \( N \mapsto L_{p^{k_0}}[w][z^N] \) are the multiple harmonic sums twisted by Frobenius. They are described as follows.

Let the three differential forms \( \omega_0 = \frac{dz}{z}, \omega_1 = \frac{dz}{z+1}, \omega_{p^{k_0}} = \frac{z^{p^{k_0}-1} dz}{1-z^{p^{k_0}}} \). The words in the alphabet \( \{\omega_0, \omega_1, \omega_{p^{k_0}}\} \) of the form \( \omega_0^{s_0-1} \omega_{s_1} \ldots \omega_{x_1}^{-1} \omega_{x_2} \ldots \omega_{x_d} \), with \( x_0, \ldots, x_1 \in \{1, p^k\} \), define iterated
integrals which are analytic on \( \{ z \in \mathbb{C} \mid |z|_p < 1 \} \), with Taylor coefficients at \( z = 0 \) equal to

\[
H_N(\omega_0^{n_0-1}\omega_x \cdots \omega_0^{n_d-1}\omega_x) = \sum_{\substack{0=n_0<\ldots<n_d<n_{d+1}=N \\ n_i\equiv n_{i-1} \mod p^k \text{ for } i \text{ s.t. } x_i=p^{n_0}}} \frac{1}{n_1^{n_1} \ldots n_d^{n_d}}
\]

(54)

**Fact 6.2.** \((\square)\) The map \( N \mapsto \text{Li}_{p(k_0)}(w)[z^N] \) is a \( \mathbb{Q}_p \)-linear combination of such variants of multiple harmonic sums where, if \( w \) is a word of depth \( d \), the coefficients are made of coefficients of \( \Phi_{p^{k_0}}(e_0,-e_0-e_1) \) of depth \( \leq d - 1 \).

Because of the way we have derived §5.4, we restrict to \( k_0 = 1 \). See §7.3 for a discussion of the general case.

**Lemma 6.3.** The computations of §5.4 remain true for the multiple harmonic sums of the form \( (54) \).

**Proof.** Let us consider the variables \((v_i),(q_i),(r_i)\) as in §5.4. The additional conditions of congruence modulo \( p \) which appear concern only the variables \( r_i \) and amounts to say that some of the \( r_i \)'s are equal. This doesn’t change the bounds of the valuation, nor the type of sums of series which still involve multiple harmonic sums \( H_p \).

As a corollary, theorem 2 also provides a natural way to compute Deligne’s \( p \)-adic multiple zeta values, and not only Furusho’s \( p \)-adic multiple zeta values.

**Remark 6.4.** In part I, the computation of \( p \)-adic multiple zeta values was direct, since we considered directly regularized iterated integrals and their special values. In the present paper, we have indirect computations of \( p \)-adic multiple zeta values, obtained by identification of two decomposition. This last variant is a mix between the two types of computations.

**Remark 6.5.** In part I, we have dealt with regularization of iterated integrals associated to multiple harmonic sums of the form \( (54) \) depending on the choice of an algebraic differential form of residue 1 at \( \infty \). Two natural choices were \( \omega_1 \) and \( \omega_{p^{k_0}} \), which had different advantages and drawbacks.

Here, this construction gives rise to an "absolute" regularization operator on the same iterated integrals.

### 7 Final remarks

#### 7.1 Generation of all \( p \)-adic multiple zeta values by the computations of theorem 1 and 2

**Remark 7.1.** Because of the form of the Ihara action (lemma \( 2.27 \) of §2.5.3), and because the comparison, for a series \( f \in \tilde{\pi}^H_0(\mathbb{Q}_p) \), of the coefficients of \( f \) and \( \text{sym} f \) (lemma \( 2.4 \) of §2.1.1) the combinations of \( p \)-adic multiple zeta values computed by theorem 1 and 2 generate all \( p \)-adic multiple zeta values (respectively, Deligne’s version and Furusho’s version) by induction on the depth.
Remark 7.2. Combining theorems 1 and 2, we get that, for all \( N \in \mathbb{N}^* \), and all \( s_d, \ldots, s_1 \in \mathbb{N}^* \), the map \( p^k \mapsto (p^k N)^{\text{weight}}H_{p^k N}(s_d, \ldots, s_1) \) is still given by an absolutely convergent Taylor expansion.

7.3 Computations for more general locally analytic characters

Remark 7.3. Let \( k_0 \in \mathbb{N}^* \), and let \( \chi_d, \ldots, \chi_1 \) locally analytic characters which are analytic on the disk of radius \( p^{-k_0} \) around 1. Then, given a multiple harmonic sum \( H_{p^k}(\chi_d, \ldots, \chi_1) = \sum_{0<n_1<\ldots<n_d<p^k} \chi_1(n_1) \ldots \chi_d(n_d) \) with \( k \geq k_0 \), we can a priori generalize the \( p \)-adic computations of §5. Indeed, the expression of \( H_{Np^k} \) in terms of \( H_N \) and \( H_{p^k} \) (§5.3) still works for \( k \geq k_0 \). Moreover, let us write each index \( n_i \) of \( H_{p^k} \) as \( p^{n_i(q_i)}(q_i r_i) \) with \( q_i \in \mathbb{N} \) and \( r_i \in [1, p^{k_0} - 1] \). A priori, the proof of the Taylor expansion in terms of \( p^k \) (§5.4) could be generalized with this starting point.

7.4 On the type of sums of series expressing \( p \)-adic multiple zeta values

Remark 7.4. The computations of [13] gave rise to sums of series, with coefficients built out of two types of rational numbers: the coefficients \( B_k \) involving Bernoulli numbers, and, on the other hand, the binomial coefficients \( \binom{s}{r} \), \( s \in \mathbb{N}^* \), \( l \in \mathbb{N}^* \).

The elementary proof of theorem 1 gives an expression of \( H_{p^k N} \) involving multiple harmonic sums \( H_{p^k} \), \( H_N \) and the variants \( \tilde{H}_N(-l_d, \ldots, -l_1) \), for \( l_d, \ldots, l_1 \in \mathbb{N}^* \). In this expression, the coefficients of the type \( B_k \) are included in the \( \tilde{H}_N \), and it is only when the \( \tilde{H}_N \) are expanded as polynomials of \( N \) that those coefficients appear in the formulas.

Thus, the elementary proof of theorem 1 gives a way to separate the two types of rational coefficients that arise.

7.5 Finite multiple zeta values which are not multiple harmonic sums

Definition 7.5. We can define as well, for \( k \in \mathbb{N}^* \) and all possible sets of indices:

\[
\zeta_{f-e,a}(s_d, \ldots, s_1) = \Phi_{p^k e_1} \Phi_{p^k} \left[ \frac{1}{1 - e_0^{e_1}} \frac{e_0^{s_0 - 1}}{1 + e_0 a^{s_1}} e_1 \ldots \frac{e_0^{d_1 - 1}}{1 + e_0 a^{d_1}} e_1 \right]
\]

These are no more multiple harmonic sums; but they satisfy the same algebraic properties (described in [13]) with the finite multiple zeta values \( \zeta_{f-e,a} \) (in the first version of [13], only the \( a = 0 \) case appears).

7.6 Ihara action of multiple harmonic sums on multiple harmonic sums

Let \( \overline{\text{Har}}(\mathbb{Q}_p) \) be the subset of \( \text{Har}(\mathbb{Q}_p) \) made of the points \( h \) such that the valuation of each \( h(s_d, \ldots, s_1) \geq s_d + \ldots + s_1 \). The \( p \)-adic expression of \( (p^k N)^{\text{weight}}H_{p^k N} \) in terms of \( N^{\text{weight}}H_N \) and \( (p^k)^{\text{weight}}H_{p^k} \) gives rise to a function \( \overline{\text{Har}}(\mathbb{Q}_p) \times \text{Har}(\mathbb{Q}_p) \rightarrow \text{Har}(\mathbb{Q}_p) \).

We will define and study in part III a different action, adapted to the "formal symmetrized multiple zeta values" defined in [13]. The equality among the two actions will follow from the standard families of algebraic relations of \( p \)-adic multiple zeta values.
A Details in low depth

A.1 Depth one

Let \( N, k, s \in \mathbb{N}^* \), \( u \in \mathbb{N} \).

A.1.1 Horizontality of Frobenius

Lemma A.1.

\[
(p^k N)^s H_{p^k N}(s) = N^s H_N(s) + \sum_{L \geq 0} N^{s+L} (\Phi_{p^{-s}}^{-1} e_1) \{ e_1^L e_1 e_0^{s-1} e_1 \}
\]

\[
(p^k N)^s \sum_{0 < n < p^k N \atop p^k | n} \frac{1}{n^s} = \sum_{L \geq 1} N^{s+L} \binom{L + s - 1}{L} (-1)^s \zeta_{p^{-s}}(s + L)
\]

Proof. The shape of the Ihara action on multiple harmonic sums of depth one has been described in §2.5.4. This gives the first formula. The second one follows then by applying the shuffle equation, and that \((1 + (-1)^s) \zeta_{p^{-s}}(s) = \zeta_{p^{-s}}(s) = 0\).

Lemma A.2.

\[
(p^k N)^s H_{p^k N}(s) = N^s H_N(s) + \sum_{l \geq 0} N^{s+L} \sum_{l \geq L-1} \mathbb{B}_L^l (p^k)^{s+l} H_{p^k}(s + l) \binom{-s}{l}
\]

Proof. \((p^k N)^s H_{p^k N}(s)\) is the sum of two terms :

\[
(p^k N)^s \sum_{0 < n < p^k N \atop p^k | n} \frac{1}{n^s} = N^s H_N(s)
\]

\[
(p^k N)^s \sum_{0 < n < p^k N \atop p^k | n} \frac{1}{n^s} = (p^k N)^s \sum_{q=0}^{N-1} \sum_{r=1}^{p^k - 1} \frac{1}{(r + p^k q)^s} = (p^k N)^s \sum_{q=0}^{N-1} \sum_{r=1}^{p^k - 1} \sum_{l \geq 0} \binom{-s}{l} \frac{(p^k q)^l}{p^k l}
\]

A.1.2 Taylor expansion along powers of \( p \)

Lemma A.3.

\[
\frac{1}{u!} \frac{\partial^u \zeta_{p^{k}(s, u)}(s)}{\partial p^k u}(0) = \begin{cases} (-1)^s \sum_{L=1}^{+\infty} \binom{L+s-1}{L} \zeta_{p^k}(s + L) & \text{if } u = 0 \\ 0 & \text{if } 1 \leq u \leq s \\ (-1)^{s(u-1)} \zeta_{p^k}(u) & \text{if } u \geq s + 1 \end{cases}
\]

Proof. We have :

\[
\text{sym} \left( \Phi_{p^k} \circ \tau(p^k) \Phi_{p^{-k}} \right) \left[ \sum_{L=0}^{+\infty} e_1^L e_1 e_0^{s-1} e_1 \right]
\]
Let us write

\[ \Phi_p \approx \left[ \sum_{L=0}^{+\infty} e_0^L e_1 e_0^{s-1} e_1 \right] + \sum_{L=0}^{+\infty} \left( p^L \right) \Phi_{p-\infty} \left[ \sum_{L=0}^{+\infty} e_0^L e_1 e_0^{s-1} e_1 \right] \]

We use finally the shuffle equation and \( \zeta_p(s) + (-1)^s \zeta_p(-s) = 0. \)

**Lemma A.4.**

\[
\frac{1}{u!} \frac{\partial^u \zeta_{f,s}(s)}{(\partial p)^u} (0) = \begin{cases} 
- \sum_{l \geq 0} p^{s+l} \binom{s}{l} H_p(l+s) \sum_{u=1}^{l+1} B^l_{u-1} \frac{1}{p^{u+l}} & \text{if } u = 0 \\
0 & \text{if } 1 \leq u \leq s \\
- \frac{1}{p^s} \sum_{l \geq 0} p^{s+l} \binom{s}{l} H_p(l+s) B^l_{u-s} & \text{if } u \geq s+1
\end{cases}
\]

**Proof.** We have:

\[
\zeta_{f,v}(s) = \sum_{v=0}^{k-1} \sum_{q=0}^{p-1} \sum_{r=1}^{(p-k-v)} \sum_{v=0}^{k-1} \sum_{q=0}^{p-1} \sum_{r=1}^{(p-k-v)} \sum_{l=0}^{s+1} \left( \frac{pq}{l} \right) \binom{s}{l} H_p(l+s) \sum_{u=1}^{l+1} B^l_{u} (p^{k-v-1})^u
\]

\[
= p^s \sum_{l \geq 0} p^l H_p(l+s) \sum_{u=1}^{l+1} \frac{B^l_{u}}{l} \left( \frac{s}{l} \right) \left( \frac{p^{k-v-1}}{l} \right) \sum_{u=1}^{l+1} B^l_{u} \left( \frac{s}{l} \right)
\]

**A.1.3 Expansion of \( H_N \) in terms of \( H_p \)'s and \( H_r \), \( 1 \leq r \leq p-1 \)**

Let us write \( N = a_{y_U} p^{y_U} + a_{y_{U-1}} p^{y_{U-1}} + \ldots + a_{y_1} p^{y_1} \) with \( y_U > \ldots > y_1 \), and \( a_{y_i} \in \{1, \ldots, p-1\} \).

**Lemma A.5.** We have:

\[
H_N(s) = \frac{H_{a_{y_U+1}}(s)}{(p^{y_U})^s} + \sum_{i=1}^{U-1} \sum_{l \geq 0} H_{a_{y_i+1}}(s+l) \left( \frac{s}{l} \right) \left( \frac{a_{y_i} p^{y_U} + \ldots + a_{y_{i+1}} p^{y_{i+1}}}{l} \right) + \sum_{1 \leq j \leq U} \sum_{0 \leq u'_{i} \leq a_{y_j} - 1} \left( \frac{s}{l} \right) H_{y_j}(s+l) \left( \sum_{m=j+1}^{U} a_{y_m} p^{y_m} + a_{y_j}' p^{y_j} \right)
\]

**Proof.** Let \( Q(N, p) = \{ p^{y_U}, \ldots, a_{y_U} p^{y_U}, a_{y_{U-1}} p^{y_{U-1}}, \ldots, a_{y_1} p^{y_1} \} \). We have:

\[
H_N(s) = \sum_{0 < u < N} \frac{1}{n^s} + \sum_{0 < u < N} \frac{1}{n^s}
\]

\[
= \sum_{1 \leq j \leq U} \sum_{0 \leq u'_{i} \leq a_{y_j} - 1} \left( \frac{s}{l} \right) H_{y_j}(s+l) \left( \sum_{m=j+1}^{U} a_{y_m} p^{y_m} + a_{y_j}' p^{y_j} \right) + \sum_{1 \leq j \leq U} \sum_{0 \leq u'_{i} \leq a_{y_j} - 1} \left( \frac{s}{l} \right) H_{y_j}(s+l) \left( \sum_{m=j+1}^{U} a_{y_m} p^{y_m} + a_{y_j}' p^{y_j} \right)
\]
A.2 Depth two

Let $N, k, s_1, s_2 \in \mathbb{N}^*$, $u \in \mathbb{N}$.

A.2.1 Multiplication of upper bounds

Lemma A.6.

$(p^k N)^{s_1 + s_2} H_{p^k N}(s_2, s_1) = N^{s_1 + s_2} H_N(s_2, s_1)$

$$+ \sum_{L \geq 0} N^{L + s_1 + s_2} (\Phi_{p^k}^{-1} e_1 \Phi_{p^{-k}})[e_0^L e_1 e_0^{s_1 + 1} e_1]$$

$$+ \sum_{r_2 = 0}^{s_2 - 1} N^{s_1 - r_2} H_N(s_2 - r_2, N^{r_2 + s_1} (\Phi_{p^{-k}}^{-1} e_1 \Phi_{p^{-k}})[e_0^{r_2} e_1 e_0^{s_1 + 1} e_1]$$

$$+ \sum_{r_1 = 0}^{s_1 - 1} N^{s_1 - r_1} H_N(s_1 - r_1, \sum_{L = 0}^{\infty} N^{L + s_2 + r_2} (\Phi_{p^{-k}}^{-1} e_1 \Phi_{p^{-k}})[e_0^L e_1 e_0^{s_2 + 1} e_1 e_0^{r_2} (58)$$

Proof. The shape of the Ihara action on multiple harmonic sums of depth two has been described in §2.5.4.

Lemma A.7. Expression involving $\tilde{H}_N(-l_d, \ldots, -l_1)$ with $l_d, \ldots, l_1 \in \mathbb{N}$:

$$(p^k N)^{s_1 + s_2} H_{p^k N}(s_2, s_1) = N^{s_1 + s_2} H_N(s_2, s_1)$$

$$+ \sum_{l_1, l_2 \geq 0} \prod_{i = 1}^{2} \left(\frac{-s_1}{l_i}\right) N^{s_1 + s_1 + l_1} \times \left[\tilde{H}_N(-l_1 - l_2) H_{p^k}(s_2 + l_2, s_1 + l_1) + \tilde{H}_N(-l_2, -l_1) \prod_{l = 1}^{2} H_{p^k}(s_1 + l_i)\right]$$

$$+ N^{s_1 + s_2} \left[\sum_{l_1 \geq 0} (p^k s_1 + l_1) H_{p^k}(s_1 + l_1) \left(\frac{-s_1}{l_1}\right) \tilde{H}_N(s_2, -l_1) - \sum_{l_2 \geq 0} (p^k s_2 + l_2) H_{p^k}(s_2 + l_2) \left(\frac{-s_2}{l_2}\right) \tilde{H}_N(s_1, -l_2)\right]$$

$$+ N^{s_2} H_N(s_2) \sum_{l_1 \geq 0} (p^k s_1 + l_1) H_{p^k}(s_1 + l_1) \left(\frac{-s_1}{l_1}\right) H_{p^k}(s_1 + l_1) (59)$$
Expression involving only usual multiple harmonic sums:

\[
(p^k N)^s_1s_2 H_{p^k N}(s_2, s_1) = N^{s_1s_2} H_N(s_2, s_1) \\
+ \sum_{t \geq 1} N^{s_1s_2+\tau} \left[ \sum_{l_1, l_2 \geq 0} l_1 \prod_{i=1}^{2} \left( -\frac{s_i}{l_i} \right) (p^k)^{\tau + l_1} H_{p^k} (s_2 + l_2, s_1 + l_1) + \sum_{l_1, l_2 \geq 0} l_1 \prod_{i=1}^{2} \left( -\frac{s_i}{l_i} \right) (p^k)^{s_1 + l_1} H_{p^k} (s_1 + l_1) \right] \\
+N^{s_2s_1} \left[ \sum_{1 \leq t \leq s_2 - 1} H_N (s_2 - t) l_1 \prod_{i=1}^{2} \left( -\frac{s_1}{l_1} \right) (p^k)^{s_1 + l_1} H_{p^k} (s_1 + l_1) + \sum_{1 \leq t \leq s_2 - 1} H_N (s_1 - l) l_1 \prod_{i=1}^{2} \left( -\frac{s_2}{l_2} \right) (p^k)^{s_2 + l_2} H_{p^k} (s_2 + l_2) \right] \\
- N^{s_2s_1} \left[ \sum_{l_1 \geq s_2 - 1} l_1 \prod_{i=1}^{2} \left( -\frac{s_1}{l_1} \right) (p^k)^{s_1 + l_1} H_{p^k} (s_1 + l_1) + \sum_{l_1 \geq s_2 - 1} l_1 \prod_{i=1}^{2} \left( -\frac{s_2}{l_2} \right) (p^k)^{s_2 + l_2} H_{p^k} (s_2 + l_2) \right] \\
+ \sum_{t' \geq 1} N^{t'} \left[ \sum_{1 \leq t' \leq s_2 - 1} l_{t'} \prod_{i=1}^{2} \left( -\frac{s_1}{l_{t'}} \right) (p^k)^{s_1 - t'} l_{t'} \prod_{i=1}^{2} \left( -\frac{s_2}{l_{t'}} \right) (p^k)^{s_2 + l_{t'}} H_{p^k} (s_2 + l_{t'}) \right] \\
\right]
\]

(60)

Proof.

1) The part \(p^k n_1, p^k n_2\):

\[N^{s_1s_2} H_N(s_2, s_1)\]

2) The part \(p^k \uparrow n_1, p^k \uparrow n_2\):

\[
\left( \sum_{0 \leq q_1 \leq r_1 \leq N-1} \sum_{1 \leq r_1 \leq p^k-1} \sum_{1 \leq r_2 \leq p^k-1} \frac{(p^k N)^{s_1s_2}}{(p^k q_1 + r_1)^{s_1}(p^k q_2 + r_2)^{s_2}} \right)
\]

3) The part \(p^k \uparrow n_1, p^k \uparrow n_2\):

\[
(p^k N)^{s_1s_2} \sum_{0 < m_2 < N} \left( \frac{1}{(p^k)^{s_2} m_2^{s_2}} \right) \sum_{0 < m_1 < N} \left( \frac{1}{m_1^{s_1}} \right) \frac{(p^k N)^{s_1s_2}}{(p^k q_1 + r_1)^{s_1}(p^k q_2 + r_2)^{s_2}}
\]

\[
= \frac{1}{(p^k N)^{s_1s_2}} \sum_{0 < m_2 < N} \left( \frac{1}{(p^k)^{s_2} m_2^{s_2}} \right) \sum_{l_1 \geq 0} \left( -\frac{s_1}{l_1} \right) H_{m_2} (-l_1) (p^k)^{l_1} H_{p^k} (s_1 + l_1)
\]

\[
= \frac{1}{(p^k N)^{s_1s_2}} \sum_{0 < m_2 < N} \sum_{l_1 \geq t_1} m_2^{t_2} l_1 \prod_{i=1}^{2} \left( -\frac{s_i}{l_i} \right) (p^k)^{s_1 + l_1} H_{p^k} (s_1 + l_1) \left( -\frac{s_1}{l_1} \right)
\]

4) The part \(p^k \uparrow n_1, p^k \uparrow n_2\):

\[
(p^k N)^{s_1s_2} \sum_{0 < m_1 < N} \left( \frac{1}{(p^k)^{s_1} m_1^{s_1}} \right) \sum_{0 < m_2 < N} \left( \frac{1}{m_2^{s_2}} \right)
\]

34
\[
\sum_{0 \leq n_1, n_2 < p^k} \frac{1}{n_1^{s_1} n_2^{s_2}} = \left( \sum_{0 \leq n_1 < p^k N} \frac{1}{n_1^{s_1}} \right) \left( \sum_{0 < n_2 < p^k N} \frac{1}{n_2^{s_2}} \right) - \left( \sum_{0 < n_2 < n_1 < p^k N} \frac{1}{n_1^{s_1} n_2^{s_2}} \right)
\]

\[\text{Lemma A.8.}\] We have
\[
\frac{1}{u!} \frac{\partial^n \zeta_p \circ (s_2, s_1)}{(dp)^u} (0) =
\]

\[
\left\{ \begin{array}{ll}
\text{sym} \Phi_{p^{-\infty}}[\sum_{L \geq 0} e_0^L e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] & \text{if } u = 0 \\
0 & \text{if } 1 \leq u \leq \min(s_1, s_2) - 1 \\
\mathbb{I}_{u \geq s_1} (-1)^{s_1+s_2} (u-s_1)_L \zeta_p \circ (u) \sum_{L=1}^{+\infty} (-u)_L^{s_1+s_2-u} \zeta_p \circ (s_1 + s_2 + L - u - 1) & \text{if } \min(s_1, s_2) \leq u \leq s_1 + s_2 - 1 \\
\text{sym} \Phi_{p^{\infty}}[e_0^{u-s_1-s_2} e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] + (-1)^{s_1} (u-s_2)_L \zeta_p \circ (u + s_2) \sum_{L=1}^{+\infty} (-s_1)_L \zeta_p \circ (s_1 + L) & \text{if } u \geq s_1 + s_2
\end{array} \right.
\]

(61)

\[\text{Proof.}\] It follows from the formulas of §2.5.4 that :
\[
\left\{ \begin{array}{ll}
\text{sym} \Phi_{p^{-\infty}}[\sum_{L \geq 0} e_0^L e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] & \text{if } u = 0 \\
0 & \text{if } 1 \leq u \leq \min(s_1, s_2) - 1 \\
\mathbb{I}_{u \geq s_1} \text{sym} \Phi_{p^{-\infty}}[\sum_{L=0}^{+\infty} e_0^L e_1 e_0^{s_2+s_1-u-1} e_1], \text{sym} \Phi_{p^{\infty}}[e_0^{u-s_1-s_2} e_1 e_0^{s_2-1} e_1] & \text{if } \min(s_1, s_2) \leq u \leq s_1 + s_2 - 1 \\
+ \mathbb{I}_{u \geq s_2} \sum_{r_1=0}^{u-s_2} \text{sym} \Phi_{p^{\infty}}[\sum_{L=0}^{+\infty} e_0^L e_1 e_0^{s_1-r_1-1} e_1], \text{sym} \Phi_{p^{-\infty}}[e_0^{u-s_2-r_1} e_1 e_0^{s_2-1} e_1 e_0^{r_1}] & \text{if } \min(s_1, s_2) \leq u \leq s_1 + s_2 - 1 \\
\text{sym} \Phi_{p^{\infty}}[e_0^{u-s_1-s_2} e_1 e_0^{s_2-1} e_1 e_0^{s_1-1} e_1] & \text{if } u \geq s_1 + s_2
\end{array} \right.
\]

(62)

The result follows using the shuffle equation, the fact that \(\zeta_{p^{\infty}}(s) + (-1)^s \zeta_{p^{\infty}}(s) = 0\), and the fact that \(\Phi_{p^{\infty}}[e_0] = \Phi_{p^{-\infty}}[e_0] = 0\).

\[\text{Lemma A.9.}\] The Taylor expansion of \(\zeta_{p^{\infty}}(s_2, s_1)\) with respect to \(p^k\) can be computed as follows.
We have

\[(p^k s_2 + s_1 H_{p^k} (s_2, s_1) = \sum_{0 \leq v_1, v_2 \leq k-1} \sum_{0 \leq q_1 < p^{k-v_1-1}} (pq_1 + r_1)^{s_1} (pq_2 + r_2)^{s_2} (p^k - v_1) s_1 (p^k - v_2)^{s_2}) \]

1) The part \( v_1 = v_2 \):

\[
\sum_{0 \leq v \leq k-1} \left( \sum_{0 \leq q_1 < p^{k-v-1} \leq p-1, r_1, r_2 \leq p-1} \frac{(p^k - v)^{s_1 + s_2}}{(pq_1 + r_1)^{s_1} (pq_2 + r_2)^{s_2}} \right) + \sum_{0 \leq q < p^{k-1} \leq p-1} \frac{(p^k - v_1)^{s_1} (p^k - v_2)^{s_2}}{(pq + r_1)^{s_1} (pq + r_2)^{s_2}}
\]

\[
= \sum_{0 \leq v \leq k-1} \left( \prod_{i=1}^{l} (p^k - v)^{s_1} p^l \left( \frac{(-s_i)}{l_i} \right) \right) H_{p^k-v-1} (-l_2, -l_1) \prod_{i=1}^{l} H_p(s_i + l_i) + H_{p^k-v-1} (l_1 + l_2) H_p(s_2 + l_2, s_1 + l_1)
\]

\[
= \sum_{0 \leq v \leq k-1} \left( \prod_{i=1}^{l} (p^k - v)^{s_1} p^l \left( \frac{(-s_i)}{l_i} \right) \right) \prod_{i=1}^{l} H_p(s_i + l_i) + \prod_{i=1}^{l} H_p(s_i + l_i) \prod_{i=1}^{l} H_p(s_2 + l_2, s_1 + l_1)
\]

\[
= p^{s_1 + s_2} \sum_{l_1, l_2 \geq 0} \frac{l_1 + l_2 + 1}{p^u s_1 + s_2 - 1} \sum_{u=1}^{l_1 + l_2 + 1} \prod_{i=1}^{l_1} p_{l_i} \left( \frac{-s_i}{l_i} \right) \prod_{i=1}^{l_1} H_p(s_i + l_i) + \prod_{i=1}^{l_1} H_p(s_2 + l_2, s_1 + l_1)
\]

2) The part \( v_1 < v_2 \):

\[
\sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{0 \leq q_1 < p^{k-v_2-1} \leq p-1} \frac{(p^k - v_1)^{s_1}}{l_1, l_2 \geq 0} \sum_{i=1}^{2} \prod_{i=1}^{l_i} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{0 \leq q_1 < p^{k-v_2-1} \leq p-1} \sum_{l_1, l_2 \geq 0} H_{p^{v_2-v_1-1}} (pq_2 + r_2) (-l_1) q_2^l \sum_{i=1}^{2} \prod_{i=1}^{l_i} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{0 \leq q_1 < p^{k-v_2-1} \leq p-1} \sum_{l_1, l_2 \geq 0} H_{p^{v_2-v_1-1}} (pq_2) (-l_1) q_2^l \sum_{i=1}^{2} \prod_{i=1}^{l_i} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]

\[
= \sum_{0 \leq v_1 < v_2 \leq k-1} \sum_{l_1, l_2 \geq 0} \sum_{t=0}^{l_1 + l_2} \prod_{i=1}^{l_1 + l_2} \left( \frac{p_{l_i} q_i^{s_i}}{r_1 + 1} \right) \frac{(-s_i)}{l_i}
\]
\[
\prod_{i=1}^{2} \left( -\frac{s_i}{l_i} \right) H_p(s_1 + l_1) H_p(s_2 + l_2 - t + j)
\]

We introduce \((M_1, M_2) = (l_1 - t, l_2 + j - u)\).

\[
= p^{s_1 + s_2} \sum_{u \geq 1} \sum_{t \geq 1} \sum_{0 \leq v_2 < v_1 \leq k-1} (p^{k-1-v_1})^{s_1} (p^{k-1-v_2})^{s_2 + u} (p^{v_2 - v_1})^t p^u \sum_{M_1 \geq -1, M_2 \geq -1} p^{M_1 + M_2} \tilde{B}_{v_1}^{M_1 + t} \tilde{B}_{v_2}^{M_2 + u}
\]

\[
\sum_{j=0}^{t} \left( \begin{array}{c} t \\ j \end{array} \right) \left( -\frac{s_1}{M_1 + t} \right) \left( -\frac{s_2}{M_2 + u - j} \right) H_p(s_1 + M_1 + t) H_p(s_2 + M_2 + u - t)
\]

\[
= p^{s_1 + s_2} \sum_{u \geq 1} \sum_{t \geq 1} \sum_{0 \leq v_2 < v_1 \leq k-1} (p^{v_2 - v_1})^{s_1 + t} (p^{v_2})^{s_2 + u - t} p^u \sum_{M_1 \geq -1, M_2 \geq -1} p^{M_1 + M_2} \tilde{B}_{v_1}^{M_1 + t} \tilde{B}_{v_2}^{M_2 + u}
\]

\[
\min(t, M_2 + u) \sum_{j=0}^{t} \left( \begin{array}{c} t \\ j \end{array} \right) \left( -\frac{s_1}{M_1 + t} \right) \left( -\frac{s_2}{M_2 + u - j} \right) H_p(s_1 + M_1 + t) H_p(s_2 + M_2 + u - t)
\]

Assume that \(t \neq s_2 + u\). Then

\[
p^u \sum_{0 \leq v_2 < v_1 \leq k-1} (p^{v_2 - v_1})^{s_1 + t} (p^{v_2})^{s_2 + u - t} = p^u \frac{1}{p^{s_2 + u - t} - 1} \left[ \left( \frac{p^{s_1 + s_2 + u - 1}}{p^{s_2 + s_1 + u} - 1} - \left( \frac{p^{s_1 + t - 1}}{p^{s_1 + t} - 1} \right) \right) \right]
\]

Thus, if \(s_2 + u > t\), the valuation of each of the four terms is \(\geq t + u\).

If \(t > s_2 + u\), this is still true since \(p^u \frac{1}{p^{s_2 + u - t} - 1} = p^{t + s_2} - p^{t + s_2 - 1}\).

Assume that \(t = s_2 + u\). Then

\[
p^u \sum_{0 \leq v_2 < v_1 \leq k-1} (p^{v_2 - v_1})^{s_1 + t} (p^{v_2})^{s_2 + u - t} = p^u \left[ k \left( \frac{1}{p^{s_1 + s_2 + u} - 1} \right)^2 \right] \left( \frac{p^{s_1 + s_2 + u}}{(p^{s_1 + s_2 + u} - 1)^2} \right)
\]

Then, each of the three terms has valuation \(\geq u\).

3) The part \(v_2 > v_1\):

\[
\sum_{0 \leq n_1 < n_2 < p^k \atop v_p(n_1) > v_p(n_2)} \frac{1}{n_1^2 n_2^2} = \sum_{0 \leq n_1 < n_2 < p^k \atop v_p(n_1) > v_p(n_2)} \frac{1}{n_1^2 n_2^2} - \sum_{0 \leq n_2 < n_1 < p^k \atop v_p(n_2) < v_p(n_1)} \frac{1}{n_2^2 n_1^2}
\]

The second term is obtained from the part \(v_1 < v_2\) by exchanging the role of \(s_1\) and \(s_2\). The first term is

\[
\sum_{0 \leq v_2 < v_1 \leq k-1 \atop 0 \leq s_1 \leq k^{v_2 - v_1} - 1 \atop 0 \leq i_1, i_2 \geq 0} \sum_{0 \leq i_1, i_2 \leq k^{v_2 - v_1} - 1} \prod_{i=1}^{2} (p^{k-v_i})^{s_i} \frac{p^{-1}}{r_i^{s_i + i}} \left( -\frac{s_i}{l_i} \right)
\]

\[
= p^{s_1 + s_2} \sum_{0 \leq v_2 < v_1 \leq k-1} \sum_{0 \leq i_1, i_2 \geq 0} \prod_{i=1}^{2} (p^{k-1-v_i})^{s_i} \frac{p^{-1}}{r_i^{s_i + i}} \left( -\frac{s_i}{l_i} \right) H_p(s_i + l_i)
\]

\[
= p^{s_1 + s_2} \sum_{0 \leq v_2 < v_1 \leq k-1} \sum_{0 \leq i_1, i_2 \geq 0} \prod_{i=1}^{2} (p^{k-1-v_i})^{s_i} \frac{p^{-1}}{r_i^{s_i + i}} \left( -\frac{s_i}{l_i} \right) \tilde{B}_{v_1}^{l_i} H_p(s_i + l_i)
\]

[37]
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