Observing Dynamical Currents in a Non-Hermitian Momentum Lattice

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We report on the experimental realization and detection of dynamical currents in a spin-textured lattice in momentum space. Collective tunneling is implemented via cavity-assisted Raman scattering of photons by a spinor Bose-Einstein condensate into an optical cavity. The photon field inducing the tunneling processes is subject to cavity dissipation, resulting in effective directional dynamics in a non-Hermitian setting. We observe that the individual tunneling events are superradiant in nature and locally resolve them in the lattice by performing real-time, frequency-resolved measurements of the leaking cavity field. The results can be extended to a regime exhibiting a cascade of currents and simultaneous coherences between multiple lattice sites, where numerical simulations provide further understanding of the dynamics. Our observations showcase dynamical tunneling in momentum-space lattices and provide prospects to realize dynamical gauge fields in driven-dissipative settings.

Experiments with quantum degenerate atomic gases have successfully realized a wide variety of many-body lattice models, facilitating the exploration of complex out-of-equilibrium phenomena in highly controlled settings [1–3]. Engineering lattice bonds that dynamically depend on the local particle configuration is essential for simulating lattice gauge theories [4–6] and electron-phonon coupling [7, 8]. Specifically, systems exhibiting density-dependent tunneling hold the potential to realize correlated many-body phenomena, such as pair superfluidity [9–11] and quantum scars [12, 13]. So far, density-dependent tunneling in optical lattices has been implemented via periodic driving [14–19] or dipolar interactions [20], yet solely inferred from spectroscopic measurements or by comparison to theory. Here, we realize a complementary experimental scheme that allows us to engineer dynamical tunneling events in a momentum-space lattice and directly measure them in real-time.

Our implementation employs a spinor Bose-Einstein condensate (BEC) coupled to the fundamental mode of a high-finesse optical cavity by two transverse laser beams, see Fig. 1(a). Cavity-assisted Raman scattering transfers atoms between two spin levels (|0⟩, |1⟩), while imparting momentum to the BEC in multiples of the photon recoil [Fig. 1(b,c)]. This engenders spin and particle dynamics in a two-dimensional momentum grid, which we interpret as photon-assisted tunneling events in a synthetic lattice [21]. These events are mediated by an emergent cavity field, which self-consistently evolves with the atomic spin and density configuration. Hence, the tunneling rate dynamically depends on the buildup of coherences between neighboring sites, in contrast to experiments employing Bragg scattering from classical drives to control single-particle hopping rates in momentum lattices [9, 22, 24]. The underlying process is superradiant Raman scattering in an optical cavity [25, 26], which is collectively enhanced by the number of participating emitters [13, 27, 29–32]. Since the resonator linewidth significantly exceeds site-to-site energy offsets, the cavity mode can accept different spectral components and mediate tunneling in a large momentum grid. The inherent dissipation due to cavity losses stimulates the superradiant transfers and renders the dynamics non-Hermitian. Making use of the cavity leakage, we gain nondestructive, real-time access to the atomic currents, which is often challenging in analog quantum simulations [33–35]. By performing frequency-resolved heterodyne measurements of the cavity field, we locally resolve the tunneling events in the momentum grid. A key feature of this implementation is that tunneling processes in opposite directions occur via different quantum paths and are independently controlled by the two drives. Our system constitutes a flexible platform to explore nonequilibrium lattice physics, thanks to the possibility to optically engineer dynamical currents and resolve them via the cavity field.

In our experiments, we prepare a BEC formed by \(N \approx 10^5\) \(^{87}\)Rb atoms in the \(m_F = -1\) magnetic sublevel of the \(F = 1\) hyperfine manifold. A magnetic field along the \(z\)-direction generates a Zeeman splitting of \(\omega_z = 2\pi \cdot 48\) MHz between the sublevels \(m_F = -1\) and \(m_F = 0\), which we label as \(|0\rangle\) and \(|1\rangle\), respectively. The atomic cloud is prepared inside an ultrahigh-finesse optical cavity with decay rate \(\kappa = 2\pi \cdot 1.25\) MHz, and illuminated by two retro-reflected laser fields far red-detuned from the atomic resonance. Their wavelength \(\lambda = 784.7\) nm is associated with a recoil frequency of \(\omega_{\text{rec}} = 2\pi \cdot 3.73\) kHz. The frequencies of the drives \(\omega_{r,b}\) lie on opposite sides of the dispersively shifted cavity resonance \(\omega_c\), with \(\omega_b - \omega_c \approx \omega_z\) and \(\omega_r - \omega_c \approx -\omega_z\). As shown in Fig. 1(a), their standing-wave modulations are shifted by \(\lambda/4\) at the position of the atoms, such that the combined static lattice potential is fully erased for balanced laser powers [36].
κ. (a) Experimental setup. A BEC inside an optical cavity with decay rate κ is illuminated by two x-polarized, retroreflected Raman drives (red and blue) with frequencies ω_r,b and coupling strengths γ_r,b. Their standing-wave modulations are shifted by λ/ω. Raman scattering, involving absorption from the drives (solid arrows) and net emission of light (purple) and [1] (orange circles) differing by ±k_{rec} in x and z-direction, giving rise to dynamical tunneling (red and blue arrows) with rate \( t_{SR} \propto \langle a^\dagger a \rangle \).

This scheme realizes cavity-assisted Raman transitions between the spin states |0⟩ and |1⟩ with two-photon coupling rates γ_r,b, cf. Fig. 1(b). We map the system to a tight-binding model in a rotating frame defined by the intermediate frequency of the two drives \( \bar{\omega} = (\omega_0 + \omega_r) / 2 \) [36], where the cavity detuning is defined as \( \Delta_c = \bar{\omega} - \omega_c \). For balanced two-photon couplings \( \eta = \eta_r = \eta_b \), the many-body Hamiltonian in momentum space \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{SR}} \) contains a diagonal contribution

\[
\hat{H}_0 = -\hbar \Delta_c \hat{a}^\dagger \hat{a} + \sum_{\{j,k\} \in Z, \sigma \in \{0,1\}} \hbar [\sigma \omega_0 + \omega_{\text{kin}}^{\text{kin}}(\omega_{j,k} + \sigma \cdot 2k + \sigma)] c_{\sigma j}^\dagger c_{\sigma k} + c_{\sigma k}^\dagger c_{\sigma j}
\]

and a light-assisted tunneling term

\[
\hat{H}_{\text{SR}} = -\frac{\hbar n}{\sqrt{8}} \sum_{\{j,k\} \in Z, \sigma \in \{0,1\}} \left[ i m_{l,m} \sum_{s_1,s_2 = 1}^1 c_{\sigma j}^\dagger (2j + s_1, 2k + s_2) c_{\sigma k}^\dagger (2j + s_2, 2k) - i n_{s_1} c_{\sigma j}^\dagger (2j + s_1, 2k) c_{\sigma k}^\dagger (2j + s_2, 2k + s_2) \right] + \text{H.C.}
\]

The bosonic operators \( \hat{a}^\dagger \) and \( c_{\sigma (l,m)}^\dagger \) create photons in the fundamental mode of the cavity field and atoms in \( |\sigma\rangle \) with \( (l,m) \) units of recoil momentum \( k_{\text{rec}} \) along \( (x,z) \)-directions. We indicate the corresponding atomic modes in the momentum grid as \( |l,m\rangle \), with \( l,m \) being an even (odd) number for \( \sigma = 0 \) (1). The site-to-site energy offset results from a kinetic contribution \( \omega_{\text{kin}}^{\text{kin}}(l^2 + m^2) \) and a global splitting between the spin manifolds \( \omega_0 = (\omega_r - \omega_\sigma) / 2 - \omega_z \). This key feature allows us to resolve the emerging currents in the lattice by measuring the frequency of the corresponding cavity field. The Hamiltonian in Eq. (S18) describes photon-mediated tunneling between next neighbors in the momentum grid and self-consistent rates \( t_{SR}(t) = -\eta \langle a^\dagger a \rangle / \sqrt{8} \). The components of the atomic state tunneling in ±z-direction acquire a phase of \( \mp i \) when scattering from the drive at \( \omega_0 \), as depicted in Fig. 1(c). This is due to the relative spatial phase between the two standing-wave drives, which is also crucial for suppressing Bragg scattering within a spin sector along z-direction [21], e.g., between \( |0,0\rangle_0 \) and \( |0, \pm 2\rangle_0 \).

The system described by Eq. (S18) is also a multilevel Tavis-Cummings model with collective coupling \( \eta \sqrt{N}/8 \). Since the experiment operates in an overdamped regime \( \kappa \gg \eta \sqrt{N}/8 \), the system is strongly dissipative and decays through superradiant scattering when initialized in \( |0,0\rangle_0 \) [36–38]. In an illustrative picture, the evolution is primarily determined by Raman processes creating cavity photons \( (\propto \hat{a}^\dagger \hat{a}) \), as the opposite process of absorbing photons \( (\propto \hat{a}) \) is hindered by cavity losses. As a consequence, the non-Hermitian dynamics in the momentum lattice are directional, with preferred tunneling directions illustrated by the arrows in Fig. 1(c). The arising superradiant transfers are collectively enhanced, which results in tunneling rates evolving self-consistently with the coherences between the sites involved in each hopping process. This behavior is fundamentally different from the one observed in related experiments employing standing-wave Raman drives with equal spatial phase at the position of the BEC, where a low-momentum stable superradiant phase is created above a critical driving strength [1, 39, 41]. The large cavity linewidth \( \kappa \gg \omega_0, \omega_{\text{rec}} \) facilitates tunneling in a large momentum grid, in contrast to potential implementations involving solely classical drives [21] or subrecoil cavities [42], where multiple lasers would be required.

In a first set of experiments, we prepare \( N \) atoms in the central lattice site \( |0,0\rangle_0 \) and characterize the first tunneling event, which populates a symmetric superposition of nearest-neighbor sites \( |\pm 1, \pm 1\rangle_1 = 1/\sqrt{2} \sum_{l,m=\pm 1} |l,m\rangle_1 \). The strength of the drives is increased to \( \eta = 2 \pi \cdot 0.35(1) \) kHz within 10 ms and population transfer is signaled by a single pulse of the cavity field, cf. Fig. 2(a). We verify the superradiant nature of the scattered field by increasing the initial atom number \( N \) [43–45]. As a result, the light pulse occurs at shorter times and its amplitude increases superlinearly, cf. Fig. 2(b) [13, 14]. We infer peak tunneling strengths...
FIG. 2. Superradiant tunneling in the momentum lattice. (a) Representative photon pulses for different atom numbers \(N = (8.1, 6.6, 4.9, 2.9) \times 10^4\) (darker to lighter green curves), together with a typical fit (red) [36] and coupling ramp \(\eta\) (black line). (b) Pulse amplitude versus \(N\), with a power-law fit (black line) yielding an exponent of 1.8(3). (c) Transfer efficiency as a function of \(N\). Dashed line: mean-field simulations [36]. Inset: sketch of the observed \(|0, 0\rangle \rightarrow |\pm 1, \pm 1\rangle\) hopping. (d) One-to-one relation between the number of atoms \(N_1\) in \(|\pm 1, \pm 1\rangle\) and the photon pulse area \(A\), obtained by fitting the photon traces. A linear fit (solid) yields a slope of 1.09(2), compatible to the expected slope of 1 (dashed line) within the combined systematic uncertainty due to photon (0.07) and atom number calibrations (0.04). Here, \(\Delta_\omega = -2\pi \cdot 1.4(1)\) MHz and \(\omega_0 = 2\pi \cdot 26(1)\) kHz. Throughout this work, the error bars represent the standard error of the mean.

ranging between \(\max(|t_{SR}| / 2\pi) = 0.2(1)-0.4(1)\) kHz for different atom numbers. Tunneling occurs only above a finite atom number, which we attribute to the combined influence of residual spin dephasing and self-trapping due to finite contact interactions [9, 36]. Accordingly, we reach transfer efficiencies of up to 0.8 [Fig. 2(c)], which are well captured by few-mode mean-field simulations [36]. The conservation of total angular momentum in the light-matter system leads to a one-to-one correspondence between the number of transferred atoms \(N_1\) and the photon pulse area \(A = 2\kappa \int_0^\infty n_{ph} dt\). We experimentally verify the relation \(N_1 = A\) [36] in Fig. 2(d).

Next, we further leverage on the real-time readout of the cavity field and demonstrate how frequency-resolved measurements allow us to localize the currents in the momentum grid. Around its peak value, the frequency of the cavity field \(\omega_{01}\) (\(\omega_{10}\)) associated with transitions from \(|0\rangle\) to \(|1\rangle\) (\(|1\rangle\) to \(|0\rangle\)) follows from energy conservation

\[
\omega_{01} - \bar{\omega} = \mp \omega_0 - [\omega_{\text{kin}}(l_f, m_f) - \omega_{\text{kin}}(l_i, m_i)],
\]

where the indices \(l_i, m_i\) (\(l_f, m_f\)) label the initial (final) state of a given tunneling process [36]. In the rotating frame at \(\bar{\omega}\), this corresponds to phase-modulated tunneling rates \(t_{SR} \propto \sqrt{n_{ph}(t)} \exp(i\omega t)\) which remove \((\omega = \omega_{01} - \bar{\omega})\) or provide \((\omega = \omega_{10} - \bar{\omega})\) energy to reach different atomic configurations [47].

To assess this frequency dependence, we expose the system to stronger drives which results in several superradiant tunneling events connecting multiple lattice sites. In Fig. 3(a), we present a representative spectrogram of the cavity field displaying three superradiant pulses, which we attribute to specific tunneling events in the momentum lattice. These are \(|0, 0\rangle \rightarrow |\pm 1, \pm 1\rangle \rightarrow |0, \mp 2\rangle\) or \(|0, \mp 2\rangle \rightarrow |\pm 1, \mp 1\rangle\), with \(|0, \mp 2\rangle = i\sqrt{2}(|0, -2\rangle - |0, 2\rangle\) and \(|\pm 1, \mp 1\rangle = -i/2 \sum_{l,m = \pm 1} m|l, m\rangle\), with the corresponding creation operators defined as \(\psi_j^\dagger = \{j = \{0, 1, 3, 2\}\} [36]\). The observed frequencies of emission agree with Eq. (3) and with the results obtained from few-mode numerical simulations, see Fig. 3(e). We verify the involvement of the aforementioned states by performing spin-resolved measurements of the momentum distribution at different stages of the evolution [Fig. 3(b–d)]. The observed population imbalance between states with \(k_2 > 0\) and \(k_3 < 0\) is attributed to spurious optical losses in the retroreflected path of the standing-wave drives (\(-z\)-direction). We further benchmark the dynamics with ab initio Gross-Pitaevskii simulations (GPS) including the effects of the harmonic confinement and contact interactions [36]. The presence of tunneling terms with opposite signs (\(\pm i t_{SR}\)) can give rise to destructive path interference when hopping towards inner sites in \(-z\)-direction. In particular, this effect is reflected in the suppressed hopping \(|\pm 1, \pm 1\rangle \neq |0, 0\rangle\) [see Fig. 3(d)]. The emerging cavity field and the overall tunneling strength depend, in principle, on the sum of two-site coherences in Eq. (S18) [36]. However, each tunneling event is associated with a well-defined spectral component of the cavity field fulfilling energy conservation. For sufficiently small tunneling rates (\(|t_{SR}| \ll \omega_{rec}\)), this field solely induces coherences between the corresponding adjacent lattice sites and the system exhibits local dynamical tunneling. This is reflected by the simulations of the coherences [Fig. 3(g)], which are compatible with the experimentally determined tunneling amplitudes \(|t_{SR}|\) associated with each of the superradiant pulses [see Fig. 3(f)].

The number of tunneling events can be extended by further increasing the coupling strength. As shown in Fig. 3(h,i), we observe up to seven superradiant transfers involving outer lattice sites, such as \(|2, 2\rangle\), which we identify by reading out the frequency of the cavity field and employing Eq. (3). The tunneling events are not restricted to the shown processes as they arise from multiple competing quantum paths. A quantitative prediction in this regime goes beyond the scope of this work. We identify the following fundamental limitations to the number of tunneling events. First, in the absence of confining lattice potentials, the momentum states move out of the grid nodes due to oscillatory mo-
The observations discussed so far involve independent tunneling events occurring sequentially in time. Our scheme can be extended to generate cascaded dynamics, where the tunneling events between different sites stimulate each other. We reduce the offset between the two spin manifolds to values comparable to the recoil frequency, shifting multiple states in the momentum lattice close to degeneracy [Fig. 4(a)]. In Fig. 4(b), we observe a single strong emission in the cavity field that is accompanied by several tunneling events within the pulse duration. Different from the results in Fig. 2(d), we observe an excess of detected photons in comparison to the population in $|\pm 1, \pm 1\rangle_1$, see Fig. 4(c). This effect is amplified as the emission frequency $\omega_0$ approaches the two-photon resonance ($\omega_p - \bar{\omega} \rightarrow 0$). Concurrently, we observe states with up to $10\hbar v_{\text{rec}}$ kinetic energy in the time-of-flight images [see Fig. 4(d,e)]. The GPS reproduce these results, helping to discern a cascade of hopping events towards outer lattice sites, in which the next tunneling starts before the previous one finishes [36]. These findings indicate that, within the duration of the cavity pulse, finite coherences between multiple lattice sites are simultaneously established, in contrast to the subsequent tunneling events at larger $\omega_0$.

We experimentally demonstrated a scheme giving rise to self-consistent tunneling in a non-Hermitian momentum grid, engineered by superradiant Raman scattering of a spinor BEC coupled to an optical cavity. In particular, the tunneling rates evolve together with the coherences between the sites participating in each hopping event, which we locally resolve via frequency-selective measurements of the leaking cavity field. As an extension, the combination of real-time probing and continuous feedback acting on the phase of the drives [49] could facilitate the realization of nontrivial tunneling phases and pave the way to observe synthetic magnetic fields and topologically protected states in non-Hermitian settings [24, 50, 51]. In particular, an extension to running-wave Raman drives [49] could result in emergent spin-orbit coupling [52–54]. Finally, exploring the interplay between cavity-assisted tunneling and Bose-Hubbard physics [55] holds the potential to realize unconventional strongly-correlated phases and dynamics [56–58].

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FIG. 4. Cascaded lattice dynamics. (a) Coupling scheme. The global splitting $\omega_0$ is reduced below $\omega_{rec}$, shifting several lattice sites close to degeneracy in the rotating frame. (b) Representative photon pulse (green). The couplings $\eta_{p,b}$ (red, blue curves) are increased with a small imbalance $\eta_0/(\eta_{p} + \eta_b) = 0.034(3)$. (c) Photon excess measurement. Pulse area $A$ (circles) and final number of atoms $N_f$ in $|\pm 1, \pm 1\rangle_1$ (diamonds) as functions of $\omega_{p,b}$. (d–e) Representative TOF images, with the white (orange) cross denoting the position of $|0, 0\rangle_0 (|0, 0\rangle_1)$, which are separated by a Stern-Gerlach gradient along $z$. The purple (orange) color map indicates regions solely populated by atoms in $|0\rangle$ ($|1\rangle$). The small distance between the cavity mirrors $L \approx 175 \mu m$ limits the both field of view along $z$-direction to $k_s \lesssim k_{ac}$. The square labels in (c) indicate the data points corresponding to panels (b), (d) and (e). For these measurements, $\Delta_x = 2\pi \cdot 3.42(2) MHz$ and $N = 9.1(1) \cdot 10^4$.

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EXPERIMENTAL DETAILS

In this section, we discuss the relevant experimental protocols and their calibration. In particular, we characterize the erased lattice configuration due to the two phase-shifted drives, and provide details on the heterodyne detection of the cavity field and the absorption imaging.

BEC preparation, B-field and transverse pumps characterization

We prepare a Bose-Einstein condensate (BEC) of $^{87}$Rb atoms in the $|F = 1, m_F = -1 \rangle$ magnetic sublevel of the $5^2S_{1/2}$ electronic ground state. The BEC is confined in the cavity mode by a crossed-beam optical dipole trap with frequencies $[\omega_{hx}, \omega_{hy}, \omega_{hz}] = 2\pi \cdot [175(4), 29(1), 172(1)]$ Hz. We apply a magnetic field along the z-direction $B = B_z e_z$ ($B_z < 0$) and characterize the Zeeman splitting $\omega_z$ between the $m_F = -1$ and $m_F = 0$ sublevels using cavity-assisted Raman transitions [1].

We derive the two transverse pump (TP) drives from the same laser and adjust their frequencies ($\omega_{r,b}$) via independent double-pass acousto optical modulators. A small fraction of each TP is split before recombining both beams, in order to regulate the intensity in each path. The lattice depth associated to each drive is calibrated by means of Kapitza-Dirac diffraction [2]. For all measurements discussed in this work, we increase the pump powers via s-shaped ramps of the form $V_{r,b}(t) = \tilde{V}_{r,b} [3(t/t_r)^2 - 2(t/t_r)^3]$, with $t_r$ and $\tilde{V}_{r,b}$ being the ramp duration and the final power of each drive, respectively. The frequencies of the two drives differ by $\omega_b - \omega_r \approx 2\omega_z = 2\pi \cdot 96$ MHz; their wavelength is $\lambda = 784.7$ nm which is associated to a recoil frequency of $\omega_{rec} = 2\pi \cdot 3.73$ kHz for $^{87}$Rb atoms. Since the drives are Gaussian beams which are red detuned with respect to the effective atomic resonance, they increase the harmonic
confinement in the xy-plane. For representative experimental lattice depths of $\sim 15 \hbar \omega_{\text{rec}}$ per drive, we measure the trap frequencies $[\omega_{x}, \omega_{y}] = 2\pi \cdot [218(4), 165(2)]$ Hz. More details on the experimental setup can be found in Ref. [1].

Erased lattice configuration

We adjust the distance between the retro-reflecting mirror and the atomic cloud, such that the standing-wave modulations of the two laser drives are opposite. In this configuration, the corresponding lattice potentials are erased if the power of the two drives are balanced. To obtain the optimal distance, we consider the two \textit{z}-polarized transverse drives as classical standing waves with field amplitudes $E_{r,b}$, frequencies $\omega_{r,b}$ and wavevectors $k_{r,b} = \omega_{r,b}/c$ propagating in $z$-direction. The spatial phase reference for both fields is given by the position of the retro-reflecting mirror. The negative part of the combined electric field $E^(-)$ is given by

$$E^(-) = \frac{E_r}{2} \cos(k_r z)e^{-i\omega_r t} + \frac{E_b}{2} \cos(k_b z)e^{-i\omega_b t}.$$  \hspace{1cm} (S1)

As derived below in Eq. (S11), this electric field results in an optical lattice potential with a maximal amplitude modulated in space by the beat-note between the drives

$$V_{\text{tot}}(z) = \frac{\alpha_s}{4} \left[ E_r^2 \cos^2 \left( \frac{\omega_r}{c} z \right) + E_b^2 \cos^2 \left( \frac{\omega_b}{c} z \right) \right] = \underbrace{V \cos \left( \frac{\omega_b - \omega_r}{c} z \right) \cos \left( \frac{\omega_b + \omega_r}{c} z \right)}_{V_{\text{env}}} + V,$$  \hspace{1cm} (S2)

where $\alpha_s$ is the scalar polarizability at the frequency of the driving lasers [3, 4] and $V = (\alpha_s/4)E^2$ is the maximal lattice depth per drive in a balanced configuration, i.e., $E_r^2 = E_b^2 = E^2$. The expression in Eq. (S2) comprises a rapidly varying $\lambda/2$-periodic lattice potential, with $\lambda = 2\pi c/(\omega_b + \omega_r)$, and a slowly changing envelope $V_{\text{env}}$. In Fig. S1(a), we plot the resulting potential as a function of the distance $z$ between the retro-reflecting mirror and the atomic cloud for the experimentally relevant frequency difference between the drives $\omega_b - \omega_r = 2\pi \cdot 96$ MHz.

![FIG. S1. Principle and parameter dependence of the erased lattice configuration. (a) Combined lattice generated by both transverse pumps (light green) and the lattice envelope (dark green) at a frequency difference between the drives of $\omega_b - \omega_r = 2\pi \cdot 96$ MHz and $V = 12 E_{\text{rec}}$. The physically irrelevant offset $V$ in Eq. (S2) is removed and the combined lattice is undersampled for better visibility. For a distance of $z \approx 0.78$ m the envelope vanishes and the combined lattice is constant over the extent of the atomic cloud. (b) Lattice envelope as a function of mirror distance and frequency difference (shifted by an offset of $V$). The gray line marks the experimentally relevant cut shown in (a).](image)

For an optimal distance of $z_{\text{opt}} = 0.78$ m, the envelope vanishes and the lattice potential is suppressed. Since variations of the envelope are negligible within the extent of the atomic cloud ($\sim 10$ $\mu$m), the potential can be considered to be constant within this scale. The resulting erased lattice motivates the definition of spatial mode profiles for both pumps at the position of the atoms $f_r(x) = \cos(kz)$ and $f_b(x) = \sin(kz)$, with $k = (k_r + k_b)/2$, such that $f_r(x)^2 + f_b(x)^2 = 1$. For completeness, we show the full parameter dependence of the lattice envelope on the frequency difference between the drives and the distance of the atoms from the mirror in Fig. S1(b).

To experimentally assess the quality of the erased lattice configuration at the optimal distance of $z_{\text{opt}} \approx 0.78$ m, we ramp up both drives to a total lattice of $24.2(1) E_{\text{rec}}$ within 20 ms and vary the imbalance between the drives. We measure the population in the momentum states $k_z = \pm 2k_{\text{rec}}$ after a sudden switch-off of all confining potentials, with
$k_{rec} := k$ being the recoil momentum. By performing an analogous protocol with a single drive, we can convert the measured populations into equivalent lattice depths. The equivalent lattice depth for different nominal imbalances is shown in Fig. S2. For optimally balanced drives with a small nominal imbalance of $0.5(4) E_{rec}$, we measure a small residual lattice depth of $0.8(4) E_{rec}$ which indicates a suppression of the lattice modulation by a factor $>30$. The nominal and measured lattice imbalances deviate more at larger values, which we attribute to non-perfect atom counting in increasingly heated clouds.

FIG. S2. (a) Equivalent lattice depth in the erased lattice configuration as a function of pump imbalance. The vertical error bars include the standard error of the mean and a systematic uncertainty of the absorption imaging, while the horizontal ones represent the estimated accuracy for setting the lattice imbalance ($0.5 E_{rec}$). (b-d) Representative time-of-flight images of the atomic cloud showing population in the momentum states $k_z = 0$, $-2k_{rec}$ and $+2k_{rec}$ of a BEC in $m_F = -1$. The corresponding imbalances are marked with gray lines in (a).

Heterodyne detection and photon field spectrograms

The leaking cavity field is monitored by two polarization-selective heterodyne setups, which measure the y- and z-polarization modes, respectively. While the former is employed to probe the cavity resonance after each experimental run, the latter is used to measure the cavity-mediated Raman transfers discussed in this work. Its high bandwidth (250 MS/s) allows for an all-digital demodulation of the beat-note between the cavity field and an optical local oscillator. We estimate a systematic uncertainty in the photon number calibration on the order $\Delta n_{ph,s}/n_{ph} = 0.068$, which arises from the relative fluctuations in the heterodyne detector (3.8%) and the nominal accuracy of the power sensor used for the calibration (5.0%).

The intra-cavity field $\alpha(t) = X(t) - iY(t)$ is obtained from the real $X(t)$ and imaginary quadrature $Y(t)$ after digital demodulation. We compute spectrograms of the cavity field using fast Fourier-transforms $\text{FFT}[\alpha](f) = dt/\sqrt{N} \sum_i \alpha^*(t_i)e^{-i2\pi ft_i}$, [5], where $t_i$ is the time of the $i^{th}$ step and $N$ is the total number of steps in an integration window. The traces are divided in time intervals of $T = 150\mu s$ with 50% overlap to subsequent intervals. Finally, the photon number spectrograms are calculated as $\tilde{n}_{ph}(f) = |\text{FFT}[\alpha](f)|^2/T$.

Absorption imaging

We measure the momentum space distribution of the atoms by imaging the cloud after 8 ms of free time-of-flight expansion (TOF). In order to spatially resolve atoms populating different magnetic sublevels in $F = 1$, we apply a
magnetic field gradient along z-direction during TOF (Stern-Gerlach separation). We perform high intensity absorption imaging [6] and estimate a systematic uncertainty in the atom number calibration on the order $\Delta N/N \approx 0.043$. This arises from uncertainties in the magnification of the imaging setup (1.0%) and in the determination of the effective saturation intensity (4.1%).

**DERIVATION OF THE HAMILTONIAN**

In this section, we derive both the single particle and many-body Hamiltonian. The latter constitutes a tight-binding description of the system in momentum space. We discuss the role of contact interactions within this framework.

**Single particle Hamiltonian**

The Hamiltonian of a single atom coupled to the cavity mode reads

$$\hat{H}_t' = \hat{H}'_{at} + \hat{H}'_{cav} + \hat{H}'_{int},$$

with the bare cavity Hamiltonian described by

$$\hat{H}'_{cav} = \hbar \omega_z \hat{a}^\dagger \hat{a},$$

where the operator $\hat{a}^\dagger$ creates photons in the TEM$_{00}$ z-polarized cavity mode with resonance frequency $\omega_z$. Since the experiment operates in the dispersive regime [7], we adiabatically eliminate the excited electronic states of the atoms. The atoms are initialized in $|F = 1, m_F = -1\rangle$ and coupled to $|F = 1, m_F = 0\rangle$ by near-resonant cavity-assisted Raman transitions. We neglect transitions to the $F = 2$ manifold as they are detuned by the hyperfine splitting $\omega_{HF} = 2\pi \cdot 6.834$ GHz. Thus, we can write the atomic Hamiltonian in terms of the spin operator $\hat{F} = (\hat{F}_x, \hat{F}_y, \hat{F}_z)^T$ in $F = 1$

$$\hat{H}'_{at} = \frac{\hbar^2}{2M} + \hbar \omega_{z(1)} \hat{F}_z + \hbar \omega_{z(2)} \hat{F}_{z^2}.$$

The energy difference between the different Zeeman sublevels is determined by the first- and second-order Zeeman shifts $\hbar \omega_{z(1)} < 0$ and $\hbar \omega_{z(2)} > 0$ [8]. Moreover, we neglect the effect of external confining potentials at this point.

The atom-light interactions in the dispersive regime are given by

$$\hat{H}'_{int} = \alpha_s \hat{E}^{(+)} : \hat{E}^{(-)} - \frac{i \alpha_s}{2F} \left[ \hat{E}^{(+)} \times \hat{E}^{(-)} \right] \cdot \hat{F},$$

where $\hat{E}^{(+)}$ are the positive and negative components of the electric field at the position of the atoms and $\alpha_s$ is the vector (vectorial) polarizability of the atoms at the frequency of the driving lasers [1, 3, 4, 7]. We consider classical fields for the standing-wave transverse pumps and a quantized field for the cavity mode. The negative part of the total electric field $\hat{E}^{(-)}$ is given by

$$\hat{E}^{(-)} = \frac{E_{f_r}}{2} f_r(x) e^{-i \omega_r t} + \frac{E_b}{2} f_b(x) e^{-i \omega_b t} + E_0 g(x) \hat{a} e_z,$$

with unit vectors $e_j$ ($j \in \{x, y, z\}$) and spatial mode profiles $f_r(x)$, $f_b(x)$, $g(x)$. The two laser drives with electric field amplitudes $E_{f_r}$ and $E_b$ give rise to standing wave modulations which are phase shifted by $\lambda/4$ at the position of the atoms, as discussed in the subsection *Erased lattice configuration*. Given their small frequency difference $\omega_b - \omega_r = 2\pi \cdot 96$ MHz, we consider the same wavevector $k = \omega/c$ for the two drives, with $\omega = (\omega_b + \omega_r)/2$. By neglecting the transverse Gaussian modulation of the optical beams, we assume that the mode profile of the drive at frequency $\omega_r$ is given by $f_r(x) = \cos(kz)$ $f_b(x) = \sin(kz)$, while the cavity mode profile is $g(x) = \cos(kx)$. The cavity field amplitude per photon $E_0$ is determined by the resonance frequency and the volume of the mode.

We introduce the unitary transformation $\hat{U} = \exp \left( \frac{i}{\hbar} \hat{H}_{rot} t \right)$ with $\hat{H}_{rot} = \hbar \omega \hat{a}^\dagger \hat{a} - \hbar \omega' \hat{F}_z$ and $\omega' = (\omega_b - \omega_r)/2$. By employing the rotating wave approximation, we obtain a time independent single particle Hamiltonian

$$\hat{H}_1 = \hat{H}_{at} + \hat{H}_{cav} + \hat{H}_s + \hat{H}_v,$$
where
\[
\hat{H}_{\text{at}} = \frac{\hat{p}^2}{2M} + \hbar \delta_z \hat{F}_z + \hbar \omega_z^{(2)} \hat{F}_z^2,
\] (S9)
\[
\hat{H}_{\text{cav}} = -\hbar \Delta_c \hat{a}^\dagger \hat{a},
\] (S10)
with cavity detuning \(\Delta_c = \bar{\omega} - \omega_c\) and effective linear Zeeman shift \(\delta_z = \omega_z^{(1)} + \omega_z^{(2)}\). The light-matter interactions have a scalar (vectorial) contribution \(\hat{H}_{\text{int}}(\hat{H}_{\text{cav}})\) given by
\[
\hat{H}_{\text{int}} = \frac{\alpha_s}{4} \left[ E_z^2 \hat{f}_0(x)^2 + E_z^2 \hat{f}_r(x)^2 \right] + \alpha_s E_0^2 \hat{a}^\dagger \hat{a} \hat{g}(x)^2,
\] (S11)
\[
\hat{H}_{\nu} = -\frac{\alpha_v}{8} \frac{E_0}{g(x)} f_r(x) \left[ \hat{a} \hat{F}_- + \hat{a}^\dagger \hat{F}_+ \right] + \alpha_v \frac{E_0}{8} g(x) f_r(x) \left[ \hat{a} \hat{F}_+ + \hat{a}^\dagger \hat{F}_- \right]
\]
\[
= -\hbar \eta_{r} \cos(kx) \cos(kz) \left[ \hat{a} \hat{F}_- + \hat{a}^\dagger \hat{F}_+ \right] + \hbar \eta_{b} \cos(kx) \sin(kz) \left[ \hat{a} \hat{F}_+ + \hat{a}^\dagger \hat{F}_- \right].
\] (S12)

In the limit of balanced drives \((E_z^2 = E_z^0)\), their combined static lattice is erased at the position of the atoms since \(f_r(x)^2 + f_b(x)^2 = 1\). In Eq. (S12), we introduce the Raman couplings \(\eta_{r,b}\) arising from the photons scattered from the drive at frequency \(\omega_{r,b}\) into the cavity. We relate these couplings to the experimentally measured lattice depth \(V_{r,b} = -\alpha_s E_z^2 / 4\) of the drives at frequency \(\omega_{r,b}\), and find
\[
\eta_{r,b} = \frac{\alpha_v E_0}{8 \hbar} E_{r,b} = \frac{\alpha_v}{4 \text{sgn} [\alpha_s]} \cdot \frac{\sqrt{-U_0 V_{r,b}}}{\hbar}.
\] (S13)

The maximal dispersive shift per cavity photon is \(U_0 = \alpha_e E_z^2 / \hbar = -2\pi \cdot 56.3\) Hz. For the typical photon numbers \(n_{ph} = \langle \hat{a}^\dagger \hat{a} \rangle \lesssim 20\) observed in this work, we estimate a small residual intracavity lattice depth \(V_c = hU_0 n_{ph} \lesssim 0.3 \hbar \omega_{rec}\) which has negligible influence on the dynamics in the momentum lattice.

**Many-body Hamiltonian in momentum space**

We derive a tight-binding description of the system in momentum space. We expand the spinor BEC in a discrete set of two-dimensional plane waves
\[
\psi_{(l,m)}^\sigma(\hat{x}) = \frac{k}{2\pi} e^{ik(lx+mx)} \otimes |\sigma = m_F + 1\rangle,
\] (S14)
with the average wavevector of the two drives \(k\) determining the recoil momentum of the atoms \(k_{rec}(k = k_{rec})\). The indices \(l, m \in \mathbb{Z}\) and \(\sigma \in \{0, 1\}\) label a discrete set of plane waves and the spin state associated to the Zeeman sublevel \(m_F \in \{-1, 0\}\) of \(F = 1\), respectively. The states are normalized within a unit cell \((x, z) \in [-\pi/k, \pi/k] \otimes [-\pi/k, \pi/k] = R\) in real space. As in the main text, we refer to these single particle states as \((l, m)\). We neglect cavity-assisted Raman transitions to \(m_F = +1\), since they are detuned by \(\Delta_{+1} \approx 2\omega_z^{(2)} = 2\pi \cdot 0.7\) MHz at the magnetic field we operate. The plane waves in Eq. (S14) are eigenstates of the kinetic energy operator with \((\hat{p}^2/2M) \psi_{(l,m)}^\sigma = (l^2 + m^2)\hbar \omega_{rec} \psi_{(l,m)}^\sigma\), where \(\omega_{rec} = 2\pi \cdot 3.73\) kHz is the recoil frequency.

In order to obtain a many-body description of the light-matter system, we exploit the fact that the atoms are initialized in a BEC in \(m_F = -1\), i.e., \(\psi_{(0,0)}^0\), and that each cavity-assisted two-photon scattering event simultaneously changes the spin \(\sigma\) and motional states of the atoms \((l, m)\) by \(\pm 1\), cf. Hamiltonian in Eq. (S12). Resorting to second quantization, we expand the atomic field operator as
\[
\hat{\Psi}(\hat{x}) = \sum_{(j,k) \in \mathbb{Z}} \psi_{(2j,2k)}^0(\hat{x}) \hat{c}^0_{(2j,2k)} + \psi^1_{(2j+1,2k+1)}(\hat{x}) \hat{c}^1_{(2j+1,2k+1)},
\] (S15)
with the bosonic operators \(\hat{c}_i^{(2j+\sigma,2k+\sigma)}\) annihilating particles in the mode \(\psi_{(2j+\sigma,2k+\sigma)}^\sigma\). We consider the limit of balanced Raman couplings, \(\eta := \eta_r = \eta_b\), and derive the following many-body Hamiltonian
\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}.
\] (S16)
The diagonal term is given by

\[
\hat{H}_0 = \hat{H}_{\text{cav}} + \int_R \hat{\Psi}^\dagger(\hat{x}) \left[ \hat{H}_s + \hat{H}_a \right] \hat{\Psi}(\hat{x}) d\hat{x} = -\hbar \Delta_c \hat{a}^\dagger \hat{a} + \sum_{\{j,k\} \in \mathbb{Z}, \sigma \in \{0, 1\}} \hbar [\sigma \omega_0 + \omega_{(2j + \sigma, 2k + \sigma)}^\text{kin}] \hat{c}_{(2j + \sigma, 2k + \sigma)}^\dagger \hat{c}_{(2j + \sigma, 2k + \sigma)},
\]

where we introduce the dispersively shifted cavity detuning \( \Delta_c = \Delta_c - NU_0/2 \), where \( U_0 \) is the average dispersive shift per atom. Since the cavity detuning \( \Delta_c \) is large enough, we neglect the dependence of the dispersive shift on the momentum distribution. In the main text, we redefine the detuning as \( \Delta_c \rightarrow \Delta_c \), for better clarity. The energy offset between the different atomic modes arises from a kinetic contribution \( \omega_{(2j + \sigma, 2k + \sigma)}^\text{kin} \) and a global splitting between the two spin manifolds \( \omega_0 = \omega_{1} + \omega_{2} \), which is on the order of \( \sim 2 \pi \cdot 100 \text{ kHz} \). Moreover, the light-matter interactions

\[
\hat{H}_{\text{SR}} = \int_R \hat{\Psi}^\dagger(\hat{x}) \hat{H}_s \hat{\Psi}(\hat{x}) d\hat{x} = -\frac{\hbar \eta}{\sqrt{8}} \sum_{\{j,k\} \in \mathbb{Z}, \sigma_1, \sigma_2 = \pm 1} \left\{ \hat{a}^\dagger \left[ \hat{c}_{(2j + \sigma_1, 2k + \sigma_2)}^\dagger \hat{c}_{(2j, 2k)} + i \hat{c}_{(2j, 2k)} \hat{c}_{(2j + \sigma_1, 2k + \sigma_2)}^\dagger \right] \right\} + \text{h.c.}
\]

(S18)
give rise to Raman-assisted tunnelings between neighboring states in a spin-textured two-dimensional momentum grid. Hence, even (odd) sites in the momentum lattice are exclusively populated by atoms in the spin state \( |0\rangle \) (\( |1\rangle \)). Each tunneling process changes the spin state of the atoms and is associated with the creation \( \propto \hat{a}^\dagger \) or annihilation of cavity photons \( \propto \hat{a} \). This motivates the introduction of a time-dependent self-consistent tunneling amplitude \( t_{\text{SR}}(t) = -\eta/\sqrt{8} \langle \hat{a}^\dagger(t) \rangle \).

**Role of contact interactions**

Here, we discuss the role of contact interactions within the momentum grid picture. We assume that all atoms in a given momentum state occupy the same spatial mode and resort to the mode expansion of Eq. (S15). We obtain a momentum space representation of the Hamiltonian describing contact interactions

\[
\hat{H}_u = g \int_R \hat{\Psi}^\dagger(\hat{x}) \hat{\Psi}(\hat{x}) \hat{\Psi}(\hat{x}) d\hat{x} = u \sum_{j_1, j_2, j_3, j_4} \hat{c}_{j_1}^\dagger \hat{c}_{j_2}^\dagger \hat{c}_{j_3} \hat{c}_{j_4},
\]

where we introduce the short-hand notation \( \hat{c}_j = \hat{c}_{(2j + \sigma, 2k + \sigma)} \), with \( \{j, k\} \in \mathbb{Z} \) and \( \sigma \in \{0, 1\} \) for the operators in the momentum grid. The strength of the contact interactions \( g = 4\pi \hbar^2 a_s/m \) depends on the s-wave scattering length \( a_s \), while \( u = g\rho/N \) also depends on the average atomic density \( \rho \). Closely following the approach of previous works on momentum-space lattices [9–11], we neglect four-wave mixing processes [12] and retain only mode-conserving contributions of the form \( \hat{c}_{j_1}^\dagger \hat{c}_{j_2}^\dagger \hat{c}_{j_3} \hat{c}_{j_4} \) and \( \hat{c}_{j_1}^\dagger \hat{c}_{j_2}^\dagger \hat{c}_{j_3} \hat{c}_{j_4} \). By employing the standard bosonic commutation relations, we can obtain a simplified Hamiltonian

\[
\hat{H}_u \approx u \sum_j \left[ \hat{n}_j (\hat{n}_j - 1)/2 + \right. \sum_{k \neq j} \hat{n}_k \hat{n}_j = u \hat{N}(\hat{N} - 1)/2 - u/2 \sum_j \hat{n}_j^2,
\]

(S20)

where we introduce the density operator \( \hat{n}_j = \hat{c}_{j}^\dagger \hat{c}_j \) and enforce particle number conservation \( \hat{N} = \sum_j \hat{n}_j \). For repulsive contact interactions \( u > 0 \), as it is the case for the \( F = 1 \) manifold of \(^{87}\text{Rb} \), the Hamiltonian of Eq. (S20) yields effective on-site attractive interactions in the momentum grid. This term can induce dephasing of the population dynamics [10] or give rise to self-trapping in the initial state of the momentum lattice [9, 11], if it becomes dominant over the effective tunneling strength \( J \) of the system \( (uN > 4J) \). In our experiment, we estimate the contact interactions to be on the order of \( uN/h = g\rho/h \approx 0.8 \text{ kHz} \), by assuming an average number density of \( \rho = 2.1 \times 10^{20} \text{ m}^{-3} \) and a scattering length of \( a_s \approx 100a_0 \) for \(^{87}\text{Rb} \) atoms [8], with \( a_0 \) being the Bohr radius. As the self-consistent tunneling rates in Eq. (S6) can reach larger values, i.e., \( \hbar \cdot \max(|t_{\text{SR}}|) > uN/4 \), we expect that the BEC does not remain self-trapped in the initial momentum state, which is consistent with the experimental observations.
NON-HERMITIAN DYNAMICS IN THE MOMENTUM LATTICE

This section is dedicated to the theoretical description of the non-Hermitian dynamics in the momentum lattice. First, we present a two-mode analytical model which captures the main features of superradiant Raman scattering. Moreover, we derive few-mode equations of motion which are employed in the simulations in the main text.

Time and frequency characteristics of superradiant pulses

We gain insights into the mechanism and characteristics of the cavity-assisted superradiant transfers by studying a simplified scheme involving only two atomic modes and a single Raman drive. They are coupled by a Raman process driven by a single classical field with frequency $\omega_0$ and a quantized cavity mode with resonance frequency $\omega_c$. This scheme provides a good description for the dynamics of our system in the regime where the superradiant transfers driven by each of the two pump lasers at $\omega_{r,b}$ are well separated in time. In the next section, we will describe the numerical solutions for a more elaborated model including both drives and several atomic modes.

The two atomic modes $|\uparrow\rangle$ (initial) and $|\downarrow\rangle$ (final state) are separated by an energy offset $\omega_A$ that includes both Zeeman and kinetic contributions. The bosonic operator $\hat{c}_\uparrow \hat{c}_\downarrow$ describes the annihilation of a particle in mode $|\uparrow\rangle \langle \downarrow|$). Moreover, we enforce particle number conservation, i.e., $(\hat{c}_\uparrow^\dagger \hat{c}_\uparrow + \hat{c}_\downarrow^\dagger \hat{c}_\downarrow) = N_\uparrow + N_\downarrow = N$. In presence of a single-frequency drive, it is convenient to describe the dynamics in the rotating frame defined by the auxiliary Hamiltonian $H_{\text{rot}} = \hbar \omega_0 \hat{a}^\dagger \hat{a} - \hbar \omega_A \hat{F}_z$, with

$$\tilde{\omega} = \omega_0 + \omega_A \quad (S21)$$

being the frequency of the photon field which ensures energy conservation in a Raman transfer from $|\uparrow\rangle$ to $|\downarrow\rangle$. The effective Hamiltonian of the closed system is then derived in complete analogy to the results presented in the section Derivation of the Hamiltonian. It reads

$$\hat{H} = -\hbar \Delta_x \hat{a}^\dagger \hat{a} + \hbar \eta_p \left( \hat{a}^\dagger \hat{J}_- + \hat{a} \hat{J}_+ \right), \quad (S22)$$

where we introduced a pseudo-spin $N/2$ operator $\hat{J}$, with $\hat{J}_x = (\hat{c}_\uparrow^\dagger \hat{c}_\downarrow - \hat{c}_\downarrow^\dagger \hat{c}_\uparrow)/2$, $\hat{J}_+ = \hat{c}_\uparrow^\dagger \hat{c}_\downarrow$ and $\hat{J}_- = \hat{c}_\downarrow^\dagger \hat{c}_\uparrow$. The detuning between the expected frequency of emission $\omega$ and the dispersively-shifted cavity resonance is $\hat{\Delta}_c$, and the effective coupling $\eta_p$ can be found in the same way as Eq. (S13). The Hamiltonian in Eq. (S22) is an effective Tavis-Cummings model with degenerate atomic levels.

We study the mean-field dynamics of the system by deriving semiclassical equations of motion for the cavity field and the collective spin, including the cavity decay $\kappa$. Since cavity dissipation dominates over the coherent coupling ($\kappa \gg \sqrt{N}\eta_p$), we study the effective dynamics of the collective spin $\mathbf{j} = [j_x, j_y, j_z]^T = (\hat{J})$ after adiabatic elimination of the cavity field. We first consider the simplest case of zero cavity detuning $\hat{\Delta}_c = 0$. The system is prepared in mode $|\uparrow\rangle$ at $t = 0$. This is an unstable steady state of the system: even in presence of infinitesimally small fluctuations, i.e., $\mathbf{j}(t = 0) = N/2 \mathbf{e} \cos \theta, \mathbf{e} \sin \theta, \sqrt{1 - \mathbf{e}^2} \mathbf{e}^T$, with $\epsilon \ll 1$ and $\theta \in [0, 2\pi)$, the spin spontaneously evolves towards mode $|\downarrow\rangle$. The dynamics follows the equation of motion

$$\frac{d}{dt} j_z = -\frac{2\eta_p^2}{\kappa} \left( \frac{N^2}{4} - j_z^2 \right), \quad (S23)$$

where we made use of the total spin conservation $|\mathbf{j}(t)| = N/2$. This equation can be solved analytically, and gives the following solution:

$$j_z(t) = -\frac{N}{2} \tanh \left( \frac{t - t_0}{\tau} \right), \quad j_\perp(t) = \frac{N}{2} \text{sech} \left( \frac{t - t_0}{\tau} \right), \quad (S24)$$

where $j_\perp$ is the transverse spin projection $j_\perp = \sqrt{j_x^2 + j_y^2}$. From the associated amplitude of the cavity field $\alpha = \langle \hat{a} \rangle$, we derive the average photon number $n_{ph} = |\alpha|^2$ as

$$n_{ph}(t) = \left( \frac{\eta_p}{\kappa} j_\perp \right)^2 = \frac{N^2\eta_p^2}{4\kappa^2} \text{sech}^2 \left( \frac{t - t_0}{\tau} \right). \quad (S25)$$
Eqs. (S24) and (S25) describe a superradiant decay where the initial spin $|\uparrow\rangle$ fully inverts to $|\downarrow\rangle$ in the presence of a collectively-enhanced pulse of the cavity field. The duration of the superradiant transfer is determined by $\tau = n_c/(\kappa N \eta_b^2)$, while the time $t_0$ at which the pulse reaches its maximum depends on the initial fluctuations $t_0 = -\tau/2 \log(e^2/4)$. Moreover, the super-linear scaling of the maximal photon number with the atom number $\max(n_{ph}) = \eta_b^2 N^2/4\kappa^2$ is a hallmark of a collectively-enhanced superradiant decay [13, 14]. The total number of photons scattered into the cavity is independent of the timescales of the transfer, and is determined by the number of atoms participating in the dynamics

$$2\kappa \int_0^\infty n_{ph}(t)dt = N.$$  \hspace{0.5cm} (S26)

This one-to-one correspondence is a direct consequence of total angular momentum conservation in the system: while a complete transfer from $|\uparrow\rangle$ to $|\downarrow\rangle$ changes the atomic angular momentum by $\pm \hbar N/2$, this can be compensated by absorbing $N \sigma_z$-polarized pump photons and re-emitting them into the $\pi$-polarized cavity field. The scattered cavity field is stationary in the rotating frame, with a frequency of $\omega_{\pm} = \omega$ in the lab frame. The phase of the cavity field is imprinted by the initial phase of the fluctuations. This phase also determines the azimuthal angle $\phi = \arctan(j_y/j_x)$ at which the spin transfer occurs, which stays constant during the dynamics.

We now turn to the more general case of finite cavity detuning $\tilde{\Delta}_c \neq 0$. By following an analogous procedure, we find that the spin dynamics can be decomposed in two parts. First, a spin transfer from $|\uparrow\rangle$ to $|\downarrow\rangle$ occurs, as described in Eqs. (S24) and (S25). The duration $\tau$ of the corresponding superradiant pulse is now given by $\tau = (\Delta_c^2 + \kappa^2)/(N \eta_b^2 \kappa)$. Concurrently, the collective spin precesses about $j_z$ at a variable rate $\omega_{\text{rot}}(t) = \dot{\phi}(t) \propto j_z(t)$. This behavior is signaled by a varying frequency of the cavity field $\omega_{\mp}(t) = \omega + \omega_{\text{rot}}(t)$ in the lab frame. Around the maximum of the field amplitude, i.e., for $j_{\perp} = N/2$ and $j_z = 0$, the frequency is $\omega_{\mp}(t = t_0) = \tilde{\omega}$ and coincides with the result in the resonant case ($\Delta_c = 0$). By combining this finding with the definition of $\tilde{\omega}$ in Eq. (S21), we obtain

$$\omega_{\mp}(t = t_0) = \omega_p + \omega_A.$$  \hspace{0.5cm} (S27)

Thus, near the maximum of the superradiant pulse, the frequency of the scattered field is the one expected from the conservation of energy in a two-photon process connecting the initial and final bare atomic states.

The result in Eq. (S27) allows us to exploit the readout of the frequency of the cavity field to assess the energy difference $\omega_A$ between the initial and final states of the superradiant transfer, as stated in Eq. (4) in the main text. Specifically, we apply this to a transfer in the momentum space lattice driven by the pump at frequency $\omega_p$, i.e., from $|\uparrow\rangle = |l_i, m_i\rangle_0$ to $|\downarrow\rangle = |l_f, m_f\rangle_1$. We substitute $\omega_p \rightarrow \omega_r$ and $\omega_{\mp} \rightarrow \omega_0$, set the energy splitting between the two states to $\omega_A = \omega_z + \omega_{\text{kin}}(l_i, m_i) - \omega_{\text{kin}}(l_{\mp}, m_{\mp})$, and obtain

$$\omega_0 = \omega_r + \omega_z + \omega_{\text{kin}}(l_i, m_i) - \omega_{\text{kin}}(l_{\mp}, m_{\mp}) = \tilde{\omega} - \omega_0 - \left[\omega_{\text{kin}}(l_i, m_i) - \omega_{\text{kin}}(l_{\mp}, m_{\mp})\right],$$ \hspace{0.5cm} (S28)

which is equivalent to the expression of Eq. (4) in the main text. The derivation of the frequency for the opposite process $\omega_{-0}$ is analogous.

As a final note, we consider in more detail the effect of the dispersive shift. When the atomic mode $|\uparrow\rangle$ ($|\downarrow\rangle$) is populated, the cavity resonance gets dispersively shifted by a frequency $N_{\uparrow}U_{\uparrow} + N_{\downarrow}U_{\downarrow}$, where $U_{\uparrow,\downarrow}$ are the dispersive shift per particle and atom number in the corresponding mode, respectively. We label the average and differential dispersive shift as $U = (U_{\uparrow} + U_{\downarrow})/2$ and $\delta U = (U_{\uparrow} - U_{\downarrow})/2$, respectively. With these definitions, the effective cavity detuning reads $\Delta_c = \Delta_c - N\delta U j_z$, with $\Delta_c = \omega_p - \omega_c$. The differential dispersive shift $\delta U$ is responsible for a dynamical $j_z$-dependent shift of the cavity resonance, which modifies the frequency evolution of the cavity field during the superradiant transfer. However, close to the pulse maximum ($j_z = 0$), the frequency of the field is only slightly affected by this shift. More specifically, the deviation from $\tilde{\omega}$ is $\omega_{\text{diff}}(t = t_0) \approx \delta U N \eta_b^2/[(\Delta_c^2 + \kappa^2)]$. Since for our typical parameters this shift amounts to less than $2\pi \cdot 1$ Hz and is below our frequency resolution, we neglect the effect of the differential dispersive shift in the frequency analysis of the superradiant pulses obtained in the experiment.

**Few-mode expansion and mean-field equations of motion**

In this section, we extend the notion of the superradiant decay to describe the non-Hermitian dynamics in the momentum grid in the presence of cavity dissipation. The open system dynamics is determined by the master equation

$$\dot{\rho} = -i[H, \rho] + \mathcal{L}[\rho],$$
We derive Langevin equations of motion (EOM) for the expectation values of the cavity field $\alpha$ phenomenological spin dephasing rate, which we estimate to be on the order of $\Gamma L \cos(\omega_0)$, capturing the evolution arising from photon loss at a rate $\kappa$. As discussed in previous sections, we expect the superradiant dynamics to be primarily determined by the interplay of cavity leakage and Hamiltonian terms creating cavity photons. In order to efficiently simulate the dynamics and capture the first few superradiant processes, we identify four low-energy atomic modes

\[
\begin{align*}
\dot{\psi}_0 &= \dot{c}^0_{(0,0)}, \\
\dot{\psi}_1 &= (c^1_{(1,1)} + c^1_{(1,-1)} + c^1_{(-1,1)} + c^1_{(-1,-1)})/2, \\
\dot{\psi}_2 &= i(c^2_{(1,-1)} + c^2_{(-1,-1)} - c^2_{(1,1)} - c^2_{(-1,1)})/2, \\
\dot{\psi}_3 &= i(c^3_{(0,-2)} - c^3_{(0,2)})/\sqrt{2},
\end{align*}
\]

which are coupled by terms creating cavity photons in Eq. (S18), if the system is initialized in the mode associated to $\dot{\psi}_0$. The single particle wave functions of these orthonormal modes are $\psi_0 \propto 1$, $\psi_1 \propto \cos(kx)\cos(kz)$, $\psi_2 \propto \cos(kx)\sin(kz)$ and $\psi_3 \propto \sin(2kz)$. Within this expansion, the Hamiltonian in Eq. (S16) can be simplified to

\[
\dot{H} = -\hbar \Delta_c \hat{a}^\dagger \hat{a} + \hbar(\omega_0 + 2\omega_{rec})(\hat{\psi}_1^\dagger \hat{\psi}_1 + \hat{\psi}_2^\dagger \hat{\psi}_2) + 4\hbar \omega_{rec} \hat{\psi}_1^\dagger \hat{\psi}_3 - \frac{\hbar \omega_{rec}}{\sqrt{2}} \left[ \frac{\sqrt{3}}{2} \hat{\psi}_1^\dagger \hat{\psi}_0 - \frac{1}{\sqrt{2}} \hat{\psi}_3^\dagger \hat{\psi}_1 + \frac{1}{\sqrt{2}} \hat{\psi}_2^\dagger \hat{\psi}_3 - \hat{\psi}_0^\dagger \hat{\psi}_2 \right] + \text{h.c.} .
\]

We derive Langevin equations of motion (EOM) for the expectation values of the cavity field $\alpha = \langle \hat{a} \rangle / \sqrt{N}$, atomic populations $\rho_{jj} = \langle \hat{\psi}_j^\dagger \hat{\psi}_j \rangle / N$ and atomic coherences $\rho_{jk} = \langle \hat{\psi}_j^\dagger \hat{\psi}_k \rangle / N$, with $\{j,k\} \in \{0,1,2,3\}$. We obtain a set of eleven complex coupled EOM

\[
\begin{align*}
\frac{d}{dt} \alpha &= - (\kappa - i \Delta_c) \alpha + i \sqrt{N} \eta \left( \frac{1}{\sqrt{2}} \rho^*_{01} - \frac{1}{\sqrt{2}} \rho^*_{02} - \frac{1}{2} \rho^*_{13} + \frac{1}{2} \rho^*_{23} \right), \\
\frac{d}{dt} \rho_{00} &= i \sqrt{\frac{N}{2}} \eta (\alpha \rho_{01} - \alpha^* \rho^*_{01} + \alpha \rho_{02} - \alpha^* \rho^*_{02}), \\
\frac{d}{dt} \rho_{11} &= i \sqrt{N} \eta \left( - \frac{1}{\sqrt{2}} \alpha \rho_{01} + \frac{1}{\sqrt{2}} \alpha^* \rho^*_{01} - \frac{1}{2} \alpha \rho_{13} + \frac{1}{2} \alpha^* \rho^*_{13} \right), \\
\frac{d}{dt} \rho_{22} &= i \sqrt{N} \eta \left( - \frac{1}{\sqrt{2}} \alpha \rho_{02} + \frac{1}{\sqrt{2}} \alpha^* \rho^*_{02} - \frac{1}{2} \alpha \rho^*_{23} + \frac{1}{2} \alpha^* \rho^*_{23} \right), \\
\frac{d}{dt} \rho_{33} &= i \sqrt{\frac{N}{2}} \eta (\alpha \rho_{13} - \alpha^* \rho^*_{13} + \alpha \rho_{23} - \alpha^* \rho^*_{23}), \\
\frac{d}{dt} \rho_{01} &= - [\Gamma + i(\omega_0 + 2\omega_{rec})] \rho_{01} + i \sqrt{N} \eta \left[ \frac{1}{\sqrt{2}} \alpha^* (\rho^*_{00} - \rho_{11}) + \frac{1}{\sqrt{2}} \alpha \rho^*_{12} - \frac{1}{2} \alpha \rho_{03} \right], \\
\frac{d}{dt} \rho_{02} &= - [\Gamma + i(\omega_0 + 2\omega_{rec})] \rho_{02} + i \sqrt{N} \eta \left[ - \frac{1}{\sqrt{2}} \alpha^* (\rho^*_{00} - \rho_{22}) - \frac{1}{\sqrt{2}} \alpha \rho^*_{21} + \frac{1}{2} \alpha^* \rho_{03} \right], \\
\frac{d}{dt} \rho_{03} &= - (\Gamma + i4\omega_{rec}) \rho_{03} + i \sqrt{N} \eta \left( - \frac{1}{2} \alpha^* \rho_{01} + \frac{1}{2} \alpha \rho_{02} - \frac{1}{2} \alpha \rho^*_{13} + \frac{1}{2} \alpha^* \rho^*_{23} \right), \\
\frac{d}{dt} \rho_{12} &= i \sqrt{N} \eta \left( - \frac{1}{2} \alpha \rho_{02} - \frac{1}{2} \alpha \rho^*_{02} - \frac{1}{2} \alpha \rho_{13} + \frac{1}{2} \alpha^* \rho^*_{23} \right), \\
\frac{d}{dt} \rho_{13} &= - [\Gamma + i(2\omega_{rec} - \omega_0)] \rho_{13} + i \sqrt{N} \eta \left( \frac{1}{2} \alpha^* (\rho^*_{33} - \rho_{11}) - \frac{1}{\sqrt{2}} \alpha^* \rho^*_{03} - \frac{1}{2} \alpha \rho_{12} \right), \\
\frac{d}{dt} \rho_{23} &= - [\Gamma + i(2\omega_{rec} - \omega_0)] \rho_{23} + i \sqrt{N} \eta \left( - \frac{1}{2} \alpha^* (\rho_{33} - \rho_{22}) + \frac{1}{\sqrt{2}} \alpha \rho^*_{03} - \frac{1}{2} \alpha^* \rho^*_{12} \right),
\end{align*}
\]

where we employ the mean-field decoupling $\langle \hat{a} \hat{\psi}_j^\dagger \hat{\psi}_k \rangle \approx N^{3/2} \alpha \rho_{jk}$ and set $\rho^*_{jk} = \rho_{kj}$. Moreover, we include a phenomenological spin dephasing rate, which we estimate to be on the order of $\Gamma / 2\pi \approx 200$ Hz in our experiment.
We attribute it to the combined effect of atomic collisions and magnetic field fluctuations [1], which effectively damp atomic coherences between states in different spin manifolds at rate $\Gamma$.

**Numerical simulations**

In order to model the dynamics of the system, we numerically evaluate the mean-field EOM derived in the previous section. We employ the MATLAB solver ‘ode45’ which is based on a Runge-Kutta (4,5) method [15]. We employ adaptive time steps and constrain the relative error tolerance in each step to $10^{-8}$. In order to seed the mean-field dynamics, we sample small fluctuations on top of the expectation value of the cavity field. This sampling ensures an initial cavity field at $t = 0$ compatible with a coherent vacuum state [1]. For the simulations presented in Fig. 2 and 3 in the main text, we initialize the atoms in a zero-momentum BEC in $m_F = -1$ ($\rho_{00} = 1$) and increase the couplings via s-shaped ramps $\eta(t)$ compatible to the experimental ones. Moreover, the simulated photon number spectrograms $\tilde{n}_{ph}(t,\omega)$ are constructed using the same method as the experimental ones (see section Heterodyne detection and photon field spectrograms). For the simulations presented in Fig. 2 and 3 in the main text, we choose spin dephasing rates of $\Gamma = 2\pi \cdot 150$ Hz and $2\pi \cdot 250$ Hz, respectively.

**GROSS-PITAEVSKII SIMULATIONS**

In this section, we present ab initio Gross-Pitaevskii equation simulations to benchmark the dynamics in the momentum lattice: After deriving the equations of motion, we present simulations for the parameters of Fig. 3 and 4 in the main text. Moreover, we provide evidence for the simultaneous build-up of coherences in the cascaded tunneling regime, and elucidate the lifetime of the momentum lattice due to oscillatory dynamics in the harmonic trap.

**Equations of motion**

We complement our experimental results and few-mode simulations with simulations of the Gross-Pitaevskii equations (GPS) using the Multiconfigurational Time-Dependent Hartree Method for Indistinguishable Particles [16–20], which is implemented in the MCTDH-X software [21]. We solve the time evolution of the two-component mean-field Hamiltonian

$$\hat{H} = N \int \Phi^\dagger \hat{H}^{(1)} \Phi \, dx \, dz + \frac{g_0}{2} N(N-1) \int |\phi_0 + \phi_1|^4 \, dx \, dz$$  (S33)

with $\Phi = (\phi_0, \phi_1)^T$, where $\phi_0(x, z)$ and $\phi_1(x, z)$ are the mean-field wave functions of the two spin levels $|0\rangle$ and $|1\rangle$ with normalization $\int \Phi^\dagger \Phi \, dx \, dz = 1$. The second term describes contact interactions between the atoms, where we assume identical inter- and intra-spin coupling constants $g_0$. This is a good approximation for $^{87}$Rb atoms in the $F = 1$ manifold [8]. Moreover, the first term integrates over the single-particle Hamiltonian, which is given by

$$\hat{H}^{(1)} = \left[ \frac{\hat{p}^2}{2M} + \frac{M}{2} (\omega_{\text{hx}}^2 x^2 + \omega_{\text{hz}}^2 z^2) \right] I - \omega_0 \sigma_z + \eta(\alpha + \alpha^*) \cos(k_{\text{rec}}x) \cos(k_{\text{rec}}z) \sigma_y + i\eta(\alpha - \alpha^*) \cos(k_{\text{rec}}x) \sin(k_{\text{rec}}z) \sigma_x,$$  (S34)

where $\sigma_j$ refer to the Pauli matrices, with $j \in \{x, y, z\}$. This Hamiltonian contains the same contributions as the one presented in Eq. (S8), but further considers the harmonic confinement and omits the constant term arising from scalar light-matter interactions [cf. Eq. (S11)]. More specifically, the first line includes the kinetic term, the harmonic trap with typical experimental trapping frequencies $[\omega_{\text{hx}}, \omega_{\text{hz}}] = 2\pi \cdot [218, 172]$ Hz, and the total splitting between the two levels ($\propto \omega_0$), cf. Eq. (S9). Moreover, the second line describes the cavity-assisted Raman transitions between the two spin levels, and coincides with the Hamiltonian in Eq. (S12) in the limit of balanced drives $\eta = \eta_r = \eta_i$. The contact interaction strength $N g_0 = 1210 \hbar^2 / m$ is chosen such that the initial Thomas-Fermi radii coincide with the experimental values $[r_{\text{TF},x}, r_{\text{TF},z}] = [4.3, 5.5]$ µm. The cavity field is treated as a coherent light field and represented by a complex number $\alpha$, whose evolution follows

$$\partial_t \alpha = [i\Delta_c - \kappa]\alpha - i\eta N \theta,$$  (S35a)

$$\theta = \int \Phi^\dagger [\cos(k_{\text{rec}}x) \cos(k_{\text{rec}}z) \sigma_y + i\cos(k_{\text{rec}}x) \sin(k_{\text{rec}}z) \sigma_x] \Phi \, dx \, dz.$$  (S35b)
Numerical solution of the dynamics

Using MCTDH-X, we employ a variational method and evolve the wave function $\Phi(x, z; t)$ to numerically solve the dynamics. The system is prepared in a slightly perturbed BEC state in a harmonic trap. This perturbation represents the noise in the system, and is empirically set such that the first superradiant pulse occurs at a time comparable to the one observed in the experiment. We then activate the cavity field and evolve the system under two different sets of cavity parameters, each corresponding to the observations reported in Fig. 3 and Fig. 4 in the main text, respectively.

The first simulation is performed with $\tilde{\Delta}_c = -2\pi \cdot 0.7$ MHz and $\omega_0 = 2\pi \cdot 72.5$ kHz, while the coupling is increased to $\eta_{\text{max}} = 2\pi \cdot 0.62$ kHz within $t_r = 1.5$ ms using an s-shaped ramp as in the experiment. The simulation results are presented in Fig. S3. The behavior of the cavity field [Fig. S3(a)] and its spectrogram [Fig. S3(b)] reproduce the experimental results presented in Fig. 3 in the main text. Three strong photon pulses are observed, whose frequencies are determined by the atomic splitting and the recoil frequency according to Eq. (S28). Accompanying each photon pulse, the energy of the atomic state $E = \langle \hat{\mathcal{H}} \rangle$ [Fig. S3(c)] and the atomic occupation of the $|0\rangle$ spin manifold [Fig. S3(d)] $N_0 = N \int |\phi_0_{0,1}(x, z)|^2 dx dz$ change drastically. To better understand the atomic dynamics induced by the emerging cavity field, we look at four representative spin and density distributions taken between the photon bursts. The snapshots of the real-space $\rho_{0,1}(x, z) = |\phi_{0,1}(x, z)|^2$ and momentum-space distributions $\rho_{0,1}(k_x, k_z) = |\phi_{0,1}(k_x, k_z)|^2$ at different points in time are shown in Fig. S3(e-t), where $\phi_{0}(k_x, k_z)$ and $\phi_{1}(k_x, k_z)$ are the Fourier transforms of $\phi_{0}(x, z)$ and $\phi_{1}(x, z)$, respectively. The atomic transfers in the momentum-space lattice are clearly visible, and the momentum space densities Fig. S3(g,h,k,l,o,p) qualitatively reproduce the experimental results as shown in Figs. 3(b-d) in the main text. Through the first and second bursts, the majority of the atoms undergo the transfer $|0, 0\rangle_0 \rightarrow |\pm 1, \pm 1\rangle_1 \rightarrow |0, \pm 2\rangle_0$. This dynamics is also reflected in real space by the formation of the corresponding density waves. At long times $t > 1.83$ ms [Fig. S3(q-t)], the momentum distribution becomes washed out and starts to deviate from a tight-binding description. This is due to the combined effect of the harmonic trap and contact interactions, which induces complex dynamics in momentum space. In the next subsection, we characterize this effect in detail.

The second simulation is performed with $\tilde{\Delta}_c = -2\pi \cdot 1.26$ MHz and $\omega_0 = 2\pi \cdot 3.7$ kHz $\approx \omega_{\text{rec}}$, while the coupling is increased up to $\eta_{\text{max}} = 2\pi \cdot 0.50$ kHz within $t_r = 2$ ms using the experimental protocol. The simulations are shown in Fig. S4 and reproduce the experimental results presented in Fig. 4 in the main text. Compared to the first simulation, here we observe a strong single photon burst lasting for a relatively long time [Fig. S4(a)]. During this pulse, the spectrogram [Fig. S4(b)], system energy [Fig. S4(c)], and atomic occupation of the $\phi_0$ level [Fig. S4(c)] all show complex, rapidly changing behaviors. To better understand the system dynamics, we again choose four representative time points and show the corresponding density distributions in Fig. S4(e-t). From the momentum space densities in Fig. S4(g,h,k,l,o,p), we notice a “cascaded atomic transfer” taking place only between nearest neighboring sites in the momentum lattice. Remarkably, the next tunneling event starts before the previous one is fully completed. This is also evidenced in the population dynamics of $|0\rangle$ [Fig. S4(c)], which keeps decreasing as the mode $|0, \pm 2\rangle_0$ gets populated. As the time elapses, the atoms gradually occupy modes with larger momentum within the same photon burst, giving rise to qualitatively different dynamics from the one presented in Fig. S3. We can thus understand this single pulse as a conjunction of several bursts, where due to the small energy difference between the two spin manifolds ($\omega_0 \approx \omega_{\text{rec}}$), the succeeding pulse is stimulated by the preceding one and starts before the latter finishes. As a result, we show that multiple sites in the momentum lattice can be occupied simultaneously, see Fig. S4(o,p). These features indicate the presence of finite coherences between multiple pairs of adjacent sites during the extension of the photon pulse. Moreover, in Fig. S4(o,p), we show that before oscillatory dynamics in the harmonic trap completely blurs the momentum lattice, the highest order momentum states occupied are $|1, 3\rangle_1, |1, -3\rangle_1, |-1, 3\rangle_1$ and $|-1, -3\rangle_1$. This is compatible to the experimental observations discussed in Fig. S9.

Cascaded tunneling regime: Relation between site coherences and the cavity field

As discussed in the main text, the emerging cavity field and the associated tunneling events dynamically depend on the build-up of coherences between adjacent sites in the momentum grid. In particular, our experiment operates in an overdamped regime ($\kappa \gg \omega_0, \omega_{\text{rec}}$), where the cavity field follows the atomic configuration adiabatically [cf. Eq. (S35a)], resulting in

$$\partial_t \alpha = 0 \quad \Rightarrow \quad \alpha = \frac{\eta N}{\tilde{\Delta}_c + i\kappa} \theta. \quad (S36)$$
FIG. S3. Simulations of Gross-Pitaevskii equations reproducing results from Fig. 3 in the main text. (a-d) Time evolution of the real part [(a), blue], magnitude [(a), orange] and spectrogram (b) of the cavity field, (c) system energy, and (d) occupation of the mode. In panel (b), the thick dashed lines indicate \( \omega = \pm \omega_0 \), whereas the thin dotted lines indicate \( \omega = \pm \omega_0 \pm 2\omega_{\text{rec}} \). (e-t) The real space and momentum space density distributions are shown at four representative time points \( t = \{0.32, 0.88, 1.20, 1.83\} \) ms for atoms in the spin states \( |0\rangle \) (purple) and \( |1\rangle \) (orange colormaps). These four time points are indicated as vertical dashed points in panels (a-d).

In the discrete representation of the momentum lattice [cf. Eq. (S18)], the order parameter \( \theta \) can be evaluated as

\[
\theta = \sum_{j,k \in \mathbb{Z}} \sum_{s_1,s_2 = \pm 1} \theta_{j,k,s_1,s_2} \tag{S37a}
\]

\[
N \theta_{j,k,s_1,s_2} = -\frac{1}{\sqrt{8}} \left[ \langle \hat{c}^\dagger_{(2j+s_1,2k+s_2)} \hat{c}^0_{(2j,2k)} \rangle - i s_2 \langle \hat{c}^0_{(2j,2k)} \hat{c}^\dagger_{(2j+s_1,2k+s_2)} \rangle \right] \tag{S37b}
\]
which is a sum of local two-site coherences $\left\langle \hat{c}^{1\dagger}_{(2j+s_1,2k+s_2)}\hat{c}_0^{\dagger}(2j,2k) \right\rangle$. Since the GPS work in the continuum, we approximate them by integrals over the corresponding Brillouin zones:

$$\left\langle \hat{c}^{1\dagger}_{(2j+s_1,2k+s_2)}\hat{c}_0^{\dagger}(2j,2k) \right\rangle \approx \int_{-k_{\text{rec}}}^{k_{\text{rec}}} dk_x dk_z \phi_0^*(k_x - (2j + s_1)k_{\text{rec}}, k_z - (2k + s_2)k_{\text{rec}}) \phi_0(k_x - 2jk_{\text{rec}}, k_z - 2kk_{\text{rec}}).$$

(S38)

Moreover, we scale the coherences by a factor of $\eta N/(\Delta_c + i\kappa)$ and group them according to the associated tunneling
events in the momentum lattice

\[
\begin{align*}
\xi_1 &= \frac{\eta N}{\Delta x + i k} (\theta_{0,0,+1,+1} + \theta_{0,0,+1,-1} + \theta_{0,0,-1,+1} + \theta_{0,0,-1,-1}), \\
\xi_2 &= \frac{\eta N}{\Delta x + i k} (\theta_{0,2,+1,-1} + \theta_{0,2,-1,-1} + \theta_{0,-2,+1,+1} + \theta_{0,-2,-1,+1}), \\
\xi_3 &= \frac{\eta N}{\Delta x + i k} (\theta_{2,2,-1,-1} + \theta_{2,-2,-1,+1} + \theta_{-2,2,+1,-1} + \theta_{-2,-2,+1,+1}), \\
\xi_4 &= \frac{\eta N}{\Delta x + i k} (\theta_{2,0,-1,+1} + \theta_{2,0,-1,-1} + \theta_{-2,0,+1,+1} + \theta_{-2,0,+1,-1}), \\
\xi_5 &= \frac{\eta N}{\Delta x + i k} (\theta_{0,2,+1,+1} + \theta_{0,2,-1,+1} + \theta_{0,-2,-1,-1} + \theta_{0,-2,-1,-1}),
\end{align*}
\]

(S39)

as shown in the schematics in Fig. S5(a).

In Fig. S5(b), we present the dynamics of \(\xi_j \ (j \in \{1,\ldots,5\})\) to elucidate the build-up of simultaneous two-site coherences in the parameter regime of Fig. 4 in the main text. Indeed, we observe a cascaded build-up of coherences between adjacent lattice sites accompanied by a strong photon pulse, indicating that the next tunneling event starts before the previous one is fully finished.

![Diagram](image)

**FIG. S5.** Cascaded currents in the momentum lattice: (a) Schematic representation of the two-site coherences \(\xi_j\) associated with the different tunneling events in the lattice, and (b) GPS comparing the evolution of the cavity field \(|\alpha|^2\) and \(|\xi_j|^2\) for the simulations shown in Fig. S4. We note that the cavity field is related to the coherences as \(\alpha = \sum \xi_i\), such that \(|\alpha|^2 \neq \sum |\xi_i|^2\) due to interference terms.

**Dynamics due to harmonic confinement and contact interactions**

Harmonically confined Bose-Einstein condensates exhibit oscillatory motion when prepared away from their equilibrium configuration [22, 23], for example through excited breathing modes [24]. As the states in the momentum lattice generally differ from the equilibrium Thomas-Fermi distribution, we expect them to oscillate in real space in the trap. This moves the momentum components out of the grid nodes, progressively rendering the tight-binding picture invalid. In particular, we observed in the simulations in Fig. S3, that the lifetime of the momentum lattice (~1 ms) is roughly on the same order as the inverse trap frequency. Since this time scale is on the same order of magnitude as the dynamics in the momentum lattice, it constitutes one of the major limitations of our scheme. Nevertheless, we observe in our simulations that contact interactions can increase this lifetime. To quantify this process, we perform Gross-Pitaevskii simulations for a spinless system, where we prepare an initial wave function

\[
\phi(x, z) = \psi(x, z) \cos(k_{\text{rec}}x) \cos(k_{\text{rec}}z).
\]

(S40)

The envelop function \(\psi\) describes a Thomas-Fermi profile or a Gaussian profile, depending on whether contact interactions are considered or not. This state resembles the atomic state after the first tunneling event in the momentum
lattice, i.e., $|\pm 1, \pm 1\rangle$ [cf. Figs. S3(j,l)]. We then propagate the state freely in the harmonic trap while enforcing the cavity field to be zero. The same simulation is performed for both the experimentally relevant contact interaction strength $N g_0 = 1210 \hbar^2/m$ and for a smaller value $N g_0 = 121 \hbar^2/m$. During the simulation, we measure the overlap between the instantaneous wave function and the initial one

$$\zeta = \int dk_x dk_z \phi^*(k_x, k_z; t = 0) \phi(k_x, k_z; t) = \int dx dz \phi^*(x, z; t = 0) \phi(x, z; t),$$  

and show it in Fig. S6. For the experimentally relevant interaction strength, the lifetime is roughly 1 ms, which is approximately twice longer than for a system with weak contact interactions. The real and momentum space densities of the two simulations are also shown in Fig. S7. We observe that strong contact interactions effectively diffuse the lattice peaks in momentum space, which slows down the evolution of the atomic distribution away from the lattice sites in momentum space. We note that this lifetime is also consistent with the simulation results in Figs. S3 and S4, where oscillatory motion in the trap washes out the momentum lattice roughly 1 ms after the first photon pulse.

As a conclusion, these simulations of the Gross-Pitaevskii equations capture the dynamics observed in the experiment and validate the tight-binding description of the momentum lattice at sufficiently small times. In addition, they allow us to estimate the lifetime due to the dynamics in the harmonic trap in the presence of contact interactions.

![FIG. S6. The overlap $\zeta$ [cf. Eq. (S41)] as a function of time for the evolution of the state Eq. (S40) in a harmonic trap with strong (purple) and weak contact interactions (orange curve).]
COMPLEMENTARY EXPERIMENTAL RESULTS

In this section, we present additional experimental observations. They complement the superradiant tunneling events and the cascaded dynamics discussed in Fig. 2 and Fig. 4 in the main text, respectively.

Delay of superradiant tunneling

Here, we elaborate on the time delay of the photon pulses discussed in Fig. 2 in the main text. From our model in Eq. (S25), we expect the delay $t_0$ to monotonically decrease with increasing atom number ($\propto 1/N$), if the dynamics is collectively enhanced. In Fig. S8(a), we display the average delay time $t_0$ and observe an overall monotonically decreasing trend with $N$. However, large fluctuations are clearly visible, especially at small atom numbers. From our model, we expect indeed a monotonically decreasing $\propto 1/N$ dependence with atom number, i.e., $t_0 = -\left(\kappa/2Nn_0^2\right)\ln(\epsilon^2/4)$. However, the exact value of the delay is directly affected by fluctuations ($\epsilon$) of the occupation of the initial spin state and cavity field, such as vacuum and classical fluctuations on top of the initially empty cavity mode. We attribute the large spread of $t_0$ at small atom numbers to this effect. In contrast, the maximal pulse amplitude is not appreciably affected by these fluctuations [Fig. S8(b) and Fig. 2 (b) in the main text], as it is expected to be independent of $\epsilon$.
i.e., \( \text{max}(n_{\text{ph}}) = N^2 n_{p}^2 / 4\kappa^2 \). Hence, the latter appears to be the more robust observable, which allows to reliably fit the power-law scaling and experimentally diagnose superradiance in our experiment.

![Graphs](image)

**FIG. S8.** Superradiant tunneling in the momentum lattice. (a) Average pulse delay \( t_0 \) versus atom number \( N \). (b) Corresponding photon peak amplitude \( \text{max}(n_{\text{ph}}) \), which is also shown in Fig. 2(b) in the main text. The quantities are fitted using Eq. (S25) and the error bars represent the standard error of the mean. Here, \( \Delta = -2\pi \cdot 1.4(1) \text{ MHz and } \omega_0 = 2\pi \cdot 26(1) \text{ kHz.} \)

**Spectrogram in cascaded dynamics regime**

In Fig. S9(b), we present a typical photon number spectrogram showcasing a single photon pulse in the parameter regime of the cascaded dynamics discussed in Fig. 4 in the main text [Fig. S9(a)]. The spectrogram extends over \( \sim 7 \omega_{\text{rec}} \) in the frequency domain, which is significantly larger than the resolution of the FFT (\( \sim \omega_{\text{rec}} \)) and suggests that multiple Raman transfers between different states of the momentum grid can occur within the duration of a single photon pulse. This is confirmed by the absorption images showing the occupation of the momentum lattice in Fig. 4(e) in the main text.

![Spectrogram](image)

**FIG. S9.** Cavity field spectrogram in the regime exhibiting cascaded dynamics in the momentum lattice. (a) Representative realization, showing the coupling ramps and a single strong photon pulse and (b) corresponding photon number spectrogram. For these measurements, \( N = 9.1(1) \cdot 10^4 \) and \( \Delta = -2\pi \cdot 3.4(2) \text{ MHz, while the peak frequency of the emitted photon field is } \omega_p - \bar{\omega} = -2\pi \cdot 10 \text{ kHz.} \)
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