Energy Magnetization and Thermal Hall Effect

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We obtain a set of general formulae for determining magnetizations, including the usual electromagnetic magnetization as well as the gravitomagnetic energy magnetization. The magnetization corrections to the thermal transport coefficients are explicitly demonstrated. Our theory provides a systematic approach for properly evaluating the thermal transport coefficients of magnetic systems, eliminating the unphysical divergence from the direct application of the Kubo formula. For a non-interacting anomalous Hall system, the corrected thermal Hall conductivity obeys the Wiedemann-Franz law.

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Thermal Hall effect, or Righi-Leduc effect, is the thermal analogue of the Hall effect [1]. It gives rise to a transverse heat flow when a temperature gradient is applied. There are recent surging experimental interests in studying the thermal Hall effect in various systems, revealing such as the phonon Hall effect [2, 3], magnon Hall effect [4, 5], and so on. There are also many theoretical efforts to study these phenomena [6–8]. However, most of these theoretical studies face a fundamental issue: direct application of the Kubo formula, when done correctly without questionable “tricks” [9], always yields unphysical divergence at the zero temperature [3, 10]. Such an issue is actually a major obstacle in the theoretical studies of the thermal Hall effect.

The underlying reason of the issue had been previously identified [11, 12]: in a system breaking the time-reversal symmetry, either by applying an external magnetic field or due to the spontaneous magnetization, the temperature gradient not only drives the transport (heat) current, but also drives the circulating (heat) current that is not observable in the transport experiment. Both contributions are present in the microscopic current density calculated by the standard linear response theory, and a proper subtraction of non-observable circulating component is necessary. For the electric transport, such subtraction involves the electromagnetic orbital magnetization density, while the subtraction of the energy current will involve the gravitomagnetic energy magnetization density [13], which characterizes the circulating energy flow. However, the previous theoretical discussions do not clarify what the transport current and the magnetizations are, and how the magnetizations can be evaluated for a general extended system. The issue becomes more fundamental because the magnetizations are gauge-dependent quantities [14], and it is not a-priori clear what the proper gauges of the magnetizations should be when calculating the transport coefficients .

In this Letter, we attempt to build the theory of thermal transport of magnetic systems on a firmer basis. We obtain a set of general formulae for determining the magnetizations, including the usual electromagnetic orbital magnetization as well as the gravitomagnetic energy magnetization [Eqs. (7–10)]. We further show that these magnetizations naturally emerge as corrections to the thermal transport coefficients, recovering the Onsager relations and Einstein relations [Eq. (20)], and eliminating the unphysical divergence. The result is a complete set of general formulae for calculating the transport thermal Hall conductivity, as well as the other thermal-electric responses such as Nernst effect and Ettingshausen effect [15]. The formula also clarify what the gravitomagnetic energy magnetization is and how it can be calculated, and its thermodynamics is determined. We test our theory by calculating the thermal Hall coefficient of a non-interacting anomalous Hall system, and observe the emergence of Wiedemann-Franz law, consistent to the recent experimental observation [16].

Preliminaries: To make our discussion specific, we consider a general electronic system. We should note that the formulae we will develop are general, applicable to the other systems such as the phonon and spin systems. We assume that the total Hamiltonian of the unperturbed system can be written as $\hat{H} = \int d\mathbf{r} \hat{h}(\mathbf{r})$, where $\mathbf{r}$ denotes the spatial coordinate, and $\hat{h}(\mathbf{r})$ is the local energy density operator. To study the electric and thermal responses, we introduce the external mechanic fields: the potential $\phi(\mathbf{r})$ and the gravitational field $\psi(\mathbf{r})$, where the gravitational field is introduced as the mechanic counterpart of the temperature gradient, following Luttinger [17]. In the presence of these fields, the local energy density operator of the system is modified to [12]:

$$\hat{h}_{\phi,\psi}(\mathbf{r}) = [1 + \psi(\mathbf{r})] \left[ \hat{h}(\mathbf{r}) + \phi(\mathbf{r}) \hat{n}(\mathbf{r}) \right], \quad (1)$$

where $\hat{n}(\mathbf{r})$ is the local density operator, and the Hamiltonian of the system is $\hat{H}_{\phi,\psi} = \int d\mathbf{r} \hat{h}_{\phi,\psi}(\mathbf{r})$.

The particle and energy current operators of the sys-
tem are defined by the conservation equations [2]:

\[
\frac{\partial \hat{n}(r)}{\partial t} = \frac{1}{i\hbar} \left[ \hat{n}(r), \hat{H}_{\phi,\psi} \right] = -\nabla \cdot \hat{J}_{N}^{\phi,\psi}(r),
\]

(2)

\[
\frac{\partial \hat{J}^{\phi,\psi}(r)}{\partial t} = \frac{1}{i\hbar} \left[ \hat{J}^{\phi,\psi}(r), \hat{H}_{\phi,\psi} \right] = -\nabla \cdot \hat{J}_{E}^{\phi,\psi}(r),
\]

(3)

where \( \hat{J}_{N}^{\phi,\psi} \) and \( \hat{J}_{E}^{\phi,\psi} \) are particle and energy current operators, respectively.

We further require that the current operators in the presence of the external fields can be related to the zero-field current operators \( \hat{J}_{N} \) and \( \hat{J}_{E} \) by [2];

\[
\hat{J}_{N}^{\phi,\psi}(r) = [1 + \psi(r)] \hat{J}_{N}(r),
\]

(4)

\[
\hat{J}_{E}^{\phi,\psi}(r) = [1 + \psi(r)]^{2} \left[ \hat{J}_{E}(r) + \phi(r) \hat{J}_{N}(r) \right].
\]

(5)

We note that the current operator is only defined up to a curl by Eqs. [2][3]. As we will show later [See Eq. [22]], one may use this freedom to find appropriate forms of current operators that do satisfy these scaling relations [22].

When the system is in equilibrium and in the absence of the external fields, we have \( \nabla \cdot \hat{J}_{N}^{eq} = \nabla \cdot \hat{J}_{E}^{eq} = 0 \), where \( \hat{J}_{N(\omega)}^{eq}(r) = \langle \hat{J}_{N(\omega)}(r) \rangle_{0} \) is the expectation value of the particle (current) energy for the equilibrium density matrix \( \hat{\rho}_{0} = (1/Z_{0}) \exp \left[ -\hat{K} / k_{B} T_{0} \right] \), where \( \hat{K} = \int dr \hat{K}(r) \) and \( \hat{K}(r) \equiv \hat{h}(r) - \mu_{0} \hat{n}(r). \) As a result, we can introduce the zero field particle magnetization density \( M_{N}(r) \) and the energy magnetization density \( M_{E}(r) \) so that:

\[
\hat{J}_{N(\omega)}^{eq}(r) = \nabla \times M_{N(\omega)}^{eq}(r).
\]

(6)

The equation can also be considered as the (incomplete) definitions of the magnetizations. To make the so-defined magnetizations physically meaningful, one needs to further require the magnetizations being the properties of material, i.e., they should be well-behaved functions of \( r \), and vanish outside of the sample. We also introduce the zero-field heat magnetization:

\[
M_{Q}(r) = M_{E}(r) - \mu_{0} M_{N}(r).
\]

Magnetizations: We rigorously prove that, with the appropriate current operators that follow the scaling laws Eqs. [4][5], the total magnetizations can be calculated from the following set of equations:

\[
-\frac{\partial M_{N}}{\partial \mu_{0}} = \frac{\beta_{0}}{21} \nabla_{q} \times \left[ \langle \hat{n}_{q}, \hat{J}_{N,q} \rangle \right]_{q \rightarrow 0},
\]

(7)

\[
M_{N} - T_{0} \frac{\partial M_{N}}{\partial T_{0}} = \frac{\beta_{0}}{21} \nabla_{q} \times \left[ \langle \hat{K}_{q}, \hat{J}_{N,q} \rangle \right]_{q \rightarrow 0},
\]

(8)

\[
-\frac{\partial M_{Q}}{\partial \mu_{0}} = \frac{\beta_{0}}{21} \nabla_{q} \times \left[ \langle \hat{n}_{q}, \hat{J}_{Q,q} \rangle \right]_{q \rightarrow 0},
\]

(9)

\[
2 M_{Q} - T_{0} \frac{\partial M_{Q}}{\partial T_{0}} = \frac{\beta_{0}}{21} \nabla_{q} \times \left[ \langle \hat{K}_{q}, \hat{J}_{Q,q} \rangle \right]_{q \rightarrow 0},
\]

(10)

where \( \langle \hat{a}; \hat{b} \rangle_{0} \equiv (1 / \beta_{0}) \int d\lambda \text{Tr} \left[ \hat{\rho}_{0} \hat{a}(-i\hbar) \hat{b} \right] \) is the Kubo canonical correlation function \( \beta_{0} = 1 / k_{B} T_{0} \), \( M_{N(q)} \equiv \int dr M_{N(q)}(r) \), \( J_{Q}(r) \equiv J_{E}(r) - \mu_{0} M_{N}(r) \), and \( \hat{n}_{q}, \hat{K}_{q}, \hat{J}_{N,q}, \hat{J}_{Q,q} \) are the Fourier transform of \( \hat{n}(r), \hat{K}(r), \hat{J}_{N}(r), \hat{J}_{Q}(r) \). Equations [4][5][10] are the central results of this Letter. The total magnetizations can be obtained by integrating over either the chemical potential \( \mu_{0} \) [Eqs. [7][3]] or the temperature \( T_{0} \) [Eqs. [8][10]]. The corresponding boundary conditions are that at \( \mu_{0} \rightarrow -\infty, M_{N(q)} \rightarrow 0 \) and at \( T_{0} \rightarrow 0, M_{N}(2M_{Q}) \) coincides with right hand side (RHS) of Eq. [8][10], respectively. For electronic system, the two approaches are equivalent. On the other hand, for systems without the chemical potential, such as the phonon and magnon systems, Equation [10] is the only option for calculating the heat (energy) magnetization.

In Ref. [19], a similar formula for the electromagnetic orbital magnetization \( M \equiv -e M_{N} \) was derived from its thermodynamic definition \( M = -(\partial \Omega / \partial B)_{\mu_{0},T_{0}} \), where \( \Omega \) is the grand thermodynamic potential, and \( B \) is the magnetic field. It is easy to identify that RHS of Eq. [5] is nothing but \( -(\partial \Omega / \partial B)_{\mu_{0},T_{0}} \), where \( K \equiv \Omega + T_{0} S \) and \( S \) is the entropy of the system. Similarly, Eq. [7] is just the Maxwell relation between \( \partial M / \partial \mu \) and \( \partial N / \partial B \), where \( N \) is the total particle number of the system.

One can develop a similar thermodynamic interpretation for the heat magnetization as well. For this purpose, it is necessary to introduce a fictitious “magnetic field” \( B_{s} \) which couples to \( M_{s} \equiv M_{Q}/T_{0} \) so that \( M_{s} = -(\partial \Omega / \partial B_{s})_{\mu_{0},T_{0}} \). \( B_{s} \) can be related to the physical gravitational field \( B_{g} \) [20] by \( B_{s} \equiv -(T_{0} / c^{2}) B_{g} \). In analogy to the particle magnetization, RHSs of Eqs. [9][10] are \( -T_{0} (\partial N / \partial B_{s})_{\mu_{0},T_{0}} \) and \( -T_{0} (\partial K / \partial B_{s})_{\mu_{0},T_{0}} \), respectively, and these equations are nothing but the thermodynamic relations. It is important to note that the particular way to introduce the thermodynamic quantities (e.g., \( M_{s} \) instead of \( M_{Q} \)) is necessary for accounting for the extra factor of 2 in front of \( M_{Q} \) in Eq. [10].

We sketch the proof of Eqs. [4][10] in the following [21]. We introduce the static response functions:

\[
\chi_{ij}(r, r') = \beta_{0} \langle \Delta \hat{n}_{i}(r') \Delta \hat{J}_{j}(r) \rangle_{0}, \quad i, j = 1, 2,
\]

(11)

where \( \hat{n}_{1}(r) \equiv \hat{n}(r), \hat{n}_{2}(r) \equiv \hat{K}(r), \hat{J}_{1}(r) \equiv \hat{J}_{N}(r), \hat{J}_{2}(r) \equiv \hat{J}_{Q}(r) \), and \( \Delta \hat{a} \equiv \hat{a} - \langle \hat{a} \rangle_{0} \). Applying Eqs. [2][3], we obtain \( \nabla \cdot \chi_{ij}(r, r') = (1 / i \hbar) \langle \langle \hat{n}_{i}(r'), \hat{n}_{j}(r) \rangle \rangle_{0} \), which implies:

\[
\nabla \cdot \chi_{ij}^{eq}(r) + i q \cdot \left[ \chi_{ij}^{eq}(r) - \nabla \times M_{ij}(r) \right] = 0,
\]

(12)
commutators $[\hat{n}_i(r'), \hat{n}_i(r)]$. Equation (3) is then used to determine the equilibrium expectation values of the resulting commutators.

Therefore, $\chi_{ij}^q(r)$ must have the decomposition:

$$\chi_{ij}^q(r) = -iq \times M_{ij}(r) + e^{-iq \cdot r} \nabla \cdot \kappa_{ij}^q(r). \quad (13)$$

Because both $M_{ij}(r)$ and $\chi_{ij}^q(r)$ are properties of material, $\kappa_{ij}^q(r)$ must also be well-behaved, i.e., it should be bounded and vanish outside of the sample. Moreover, the value of $\kappa_{ij}^q(r)$ at the long wave limit ($q = 0$) can be related to the macroscopic thermodynamic quantities $[\partial M_{N(Q)}(r)/\partial \mu_0]_{T_0}$ and $[\partial M_{N(Q)}(r)/\partial T_0]_{\mu_0}$ [22]. Applying $(i/2)\nabla_q \times$ to both sides of Eq. (13), taking limit $q \to 0$, substituting $M_{ij}$ and $\kappa_{ij}^q=0$ and integrating over $r$, we obtain Eqs. (7–10).

**Thermal transport coefficients:** We can show that the magnetizations determined by Eqs. (7–10) will emerge naturally as corrections to the thermal transport coefficients. To see this, we calculate the full response of the currents to small deviation from the global equilibrium. In this case, the system can be approximately described by the density matrix:

$$\hat{\rho} \approx \hat{\rho}_{\text{eq}} + \hat{\rho}_1, \quad (14)$$

where $\hat{\rho}_{\text{eq}}$ is the local equilibrium density matrix characterized by the local chemical potential $\mu(r)$ and local temperature $T(r)$:

$$\hat{\rho}_{\text{eq}} = \frac{1}{Z} \exp \left[ -\int dr \frac{\hat{h}(r) - \mu(r) \hat{n}(r)}{k_B T(r)} \right]. \quad (15)$$

$\hat{\rho}_1$ is the linear response correction to the local equilibrium density matrix, determined by the Liouville equation $i\hbar \hat{\rho}/\partial t + [\hat{H}, \hat{\rho}] = 0$ [22]. We define $\alpha(r) \equiv [1 + \psi(r)]/[\phi(r) + \mu(r)]$, $\beta(r) \equiv 1/\beta [1 + \psi(r)] T(r)$. It is easy to see that when $\alpha(r)$ and $\beta(r)$ are spatially uniform, $\hat{\rho}_{\text{eq}}$ becomes the exact global equilibrium density matrix corresponding to the Hamiltonian $\hat{H}_{\phi,\psi}$, and $\hat{\rho}_1 = 0$. Therefore, the conditions of the global equilibrium are $\nabla \alpha(r) = 0$ and $\nabla \beta(r) = 0$ [17].

We define $J_1^{\phi,\psi} = J_{N}^{\phi,\psi}$ and $J_2^{\phi,\psi} = J_{Q}^{\phi,\psi} \equiv J_{E}^{\phi,\psi} - \alpha(r)J_{N}^{\phi,\psi}$. The forces conjugate to these currents are $X_1 = -\beta(r) \nabla \alpha(r)$ and $X_2 = \nabla \beta(r)$, respectively, so that the entropy generation is $\delta s/\partial t + \nabla \cdot (\beta J_{Q}^{\phi,\psi}) = \sum_j J_{j}^{\phi,\psi} \cdot X_j [1]$. The expectation values of the currents have two parts of contributions:

$$J_{j}^{\phi,\psi} = J_{j}^{\text{eq}} + J_{j}^{\text{Kubo}}, \quad (16)$$

where $J_{j}^{\text{Kubo}} \equiv \text{Tr} \hat{\rho}_1 J_{j}^{\phi,\psi}$ is just the usual linear response contribution with the response coefficients determinable by the Kubo formula [22]. Besides this, there is an extra contribution $J_{j}^{\text{eq}} = \text{Tr} \hat{\rho}_{\text{eq}} J_{j}^{\phi,\psi}$, which is due to the inhomogeneous local chemical potential and temperature field. We assume that the deviation from the homogeneity is small so that $\mu(r) \approx \mu_0 + \delta \mu(r)$, $1/T(r) \approx (1/T_0) \pm \delta (1/T(r))$. By applying the static response theory [19], we obtain, to the linear order of $\delta x_1(r) = \delta \mu(r)$ and $\delta x_2(r) = -T_0 \delta (1/T(r))$:

$$J_{j}^{\text{eq}}(r) \approx J_{j}^{\text{eq}}(r) + \sum_{j=1}^{2} \int dr' \chi_{ij}(r, r') x_j(r'), \quad (17)$$

where $\chi_{ij}(r, r')$ is the static response function defined in Eq. (11), and $J_{j}^{\text{eq}}(r) \equiv \langle J_{j}^{\phi,\psi}(r) \rangle_{0}$, which can be determined by Eq. (6) and Eqs. (4–5). Substituting Eq. (13) into Eq. (17), and after some algebra, we obtain, to the linear order of $\phi$, $\psi$, $\delta \mu$ and $\delta (1/T)$ [21],

$$J_{1}^{\text{eq}}(r) \approx \nabla \times M_{N}^{\phi,\psi}(r) - \frac{1}{\beta} M_{N}(r) \times X_2, \quad (18)$$

$$J_{2}^{\text{eq}}(r) \approx \nabla \times M_{E}^{\phi,\psi}(r) - \alpha(r) \nabla \times M_{N}^{\phi,\psi}(r) - \frac{1}{\beta} M_{N}(r) \times X_1 - \frac{2}{\beta} M_{Q}(r) \times X_2, \quad (19)$$

where $M_{N}^{\phi,\psi}(r) \equiv [1 + \psi(r)] M_{N}(r) + \delta M_{N}(r)$, $M_{E}^{\phi,\psi}(r) \equiv [1 + \psi(r)]^2 [M_{E}(r) + \phi(r) M_{N}(r)] + \delta M_{E}(r)$, and $\delta M_{N(E)}(r)$ is the correction to the particle (energy) magnetization due to the spatial gradients of the chemical potential and temperature, determinable by $\kappa_{ij}^q(r)$.

Applying Eqs. (10, 18, 19), we can obtain the total currents responding to the non-equilibrium forces. However, due to the presence of $J_{j}^{\text{eq}}$, such responses break the fundamental non-equilibrium thermodynamic relations [1]: (1) Onsager reciprocal relations; (2) Einstein relations, i.e., the currents should only be proportional to $\nabla \alpha$ and $\nabla \beta$, and vanish when the system is in the global equilibrium. The problem can be remedied by defining the transport currents as $J_{N(E)}^{\phi,\psi} = J_{N(E)}^{\phi,\psi} - \nabla \times M_{N(E)}^{\phi,\psi}$, and the corresponding transport responses then become:

$$\begin{bmatrix} J_{1}^{\phi,\psi} \\ J_{2}^{\phi,\psi} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{(11)}^{\phi,\psi} \\ \mathcal{L}_{(21)}^{\phi,\psi} - \frac{M_{N}^{\phi,\psi}}{\beta \alpha} \times \mathcal{L}_{(22)}^{\phi,\psi} - \frac{2 M_{Q}^{\phi,\psi}}{\beta \alpha} \times X_1 \\ X_2 \end{bmatrix}, \quad (20)$$

where $J_{j}^{\phi,\psi} \equiv (1/V) \int dr J_{j}^{\phi,\psi,\alpha \gamma}(r)$, and $V$ is the total volume of the system. $\mathcal{L}_{(ij)}^{\phi,\psi}$ is a tensor of rank two with the component $L^{\phi,\psi}_{ij} = \int_0^\infty dt e^{-\gamma t} \langle J_{j}\gamma; J_{i}\alpha(t) \rangle_0$ ($\alpha, \gamma = x, y, z$), which is the usual response coefficient determined by the Kubo formula [22]. It is easy to verify that both the Onsager relations and the Einstein relations are recovered. The magnetizations determined in Eqs. (7–10) naturally emerge as the corrections to the thermal transport coefficients.

**Application:** We can apply these general results to study the thermal Hall coefficient of a non-interacting anomalous Hall system [15, 23], and show how the unphysical
the zero temperature.

\[ \hat{h} \phi \psi (r) = [1 + \psi (r)] \left\{ \frac{m}{2} [\hat{\psi} \cdot \nabla \hat{\psi}] \cdot [\hat{\psi} \cdot \nabla \hat{\phi}] \right\} + \hat{\phi} (V(r) + \phi (r)) \hat{\phi} (r), \tag{21} \]

where \( \hat{\psi} (r) \) is the electron annihilation (creation) field operator with the two spin components, \( \hat{\psi} \equiv (1/m) [-i \hbar \nabla + A_{\psi} (r)] \) is the velocity operator with \( A_{\psi} (r) \) being the non-abelian gauge potential characterizing the spin-orbit coupling, and \( V(r) \) is the periodic potential. The field operator \( \hat{\phi} (r) \) satisfies the Schrödinger equation: 

\[ i \hbar \partial \phi / \partial t = \hat{H}_{\phi, \psi} \phi \]

with \( \hat{H}_{\phi, \psi} = (m/2) \hat{\phi} \cdot [1 + \psi (r)] \hat{\phi} + [1 + \psi (r)] V(r) + \phi (r) \). An appropriate energy current operator that does satisfy both Eq. (3) and the scaling law Eq. (5) is,

\[ J^0_{\phi} (r) = \frac{1 + \psi}{2} \left[ \left( \hat{\psi} \hat{\phi} \right)^{\dagger} \left( \hat{H}_{\phi, \psi} \phi \right) + \left( \hat{H}_{\phi, \psi} \phi \right)^{\dagger} \left( \hat{\psi} \phi \right) \right] + \frac{i \hbar}{8} \nabla \times \left[ (1 + \psi)^2 \left( \hat{\psi} \phi \right)^{\dagger} \times \left( \hat{\psi} \phi \right) \right]. \tag{22} \]

The presence of the last term is essential for satisfying the scaling law Eq. (5).

With the appropriate energy current operator at hand, we calculate the thermal Hall coefficient. The usual Kubo formula yields,

\[ \kappa_{xy} \equiv \frac{L_{xy} (22)}{k_B T_0^2} = \frac{1}{2T_0 \hbar V} \sum \Pi_{nk} f_{nk}, \tag{23} \]

where \( \Pi_{nk} = \text{Im} \left\langle \frac{\partial u_n}{\partial k_y} \left( \hat{H}_n + e \kappa_n - 2 \mu \right)^2 \frac{\partial u_k}{\partial k_y} \right\rangle \), \( u_n \) is the periodic part of Bloch wave function for band \( n \) and quasi-momentum \( k \), \( f_{nk} = f (\epsilon_{nk}) \) is the Fermi distribution function, \( \hat{H}_n = (1/2m) [-i \hbar \nabla + A_{\psi} (r) + \hbar \kappa^2] \hat{\psi} \hat{\psi} + V(r) \), and \( e \kappa_n \) is the electron dispersion \( \kappa \). It is easy to see that the coefficient diverges at the zero temperature.

We calculate \( \frac{L_{xy}^{(22)}}{k_B T_0^2} \)

\[ \frac{L_{xy}^{(22)}}{k_B T_0^2} = (\beta_0 / 2i) \nabla \cdot \left( \hat{K}_{-q} J_q \right) \bigg|_{q \to 0}, \]

and obtain:

\[ \hat{M}_{\hat{Q}} = - \frac{1}{4 \hbar} \sum \Pi_{nk} \left[ 2 f_{nk} + (\epsilon_n - \mu) f_{nk} \right] \]

\[ + 2 \Omega_{nk} (\epsilon_n - \mu)^3 f_{nk} \bigg|_{q \to 0}, \tag{24} \]

where \( \Omega_{nk} = -2 \text{Im} \left( \frac{\partial u_n}{\partial k_x} \frac{\partial u_k}{\partial k_y} \right) \). \( \hat{M}_{\hat{Q}} \) is obtained by integrating Eq. (22). After some algebra, we obtain \( \kappa_{xy}^{t} \equiv \kappa_{xy}^{Kubo} + (2 \hat{M}_{\hat{Q}} / T_0 V) \):

\[ \kappa_{xy}^{t} = - \frac{1}{e^2 T_0} \int \text{d} \epsilon (\epsilon - \mu)^2 \sigma_{xy} (\epsilon) f (\epsilon), \tag{25} \]

where \( \sigma_{xy} (\epsilon) = -(e^2 / \hbar) \sum \Omega_{nk} \) is the zero temperature anomalous Hall coefficient for a system with the chemical potential \( \epsilon \). It recovers the Wiedemann-Franz law at the low temperature \( k_B T_0 \ll \mu \) and the unphysical divergence is eliminated.

In summary, we have developed a systematic approach for calculating the particle and heat (energy) magnetizations. We also explicitly show that these magnetizations naturally emerge as the corrections to the thermal transport coefficients, recovering the Onsager and Einstein relations, and eliminating the unphysical divergences. Our approach make no assumption on the nature of the system, so is equally applicable to fermionic (e.g., electron) or bosonic (e.g., phonon, magnon) systems, either non-interacting or interacting. The approach does not involve the ill-defined spatially extended operators, so is usable in practical calculations.

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[25] A common trick used in the bosonic calculation is to ignore the terms involving $aa$ and $a^\dagger a^\dagger$ in the energy current operator, where $a$ ($a^\dagger$) is the bosonic annihilation (creation) operator, as pointed out in Ref [7]. This will result in an apparently converging albeit incorrect thermal Hall coefficient.
[26] The deviations from the scaling laws of the second or higher order gradients of $\phi$ and $\psi$ will not affect our results. The first gradient deviation can be eliminated by redefining the current operators, as shown in Eq. (22).
Supplementary information for “Energy Magnetization and Thermal Hall Effect”

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I. KUBO’S CANONICAL CORRELATION FUNCTION

Kubo’s canonical correlation function is defined as [1]:

\[
\langle \hat{B}; \hat{A} \rangle_0 = \frac{1}{\beta_0} \text{Tr} \left[ \hat{\rho}_\text{eq} \int_0^{\beta_0} d\lambda e^{\lambda \hat{H}} \hat{B} e^{-\lambda \hat{H}} \right],
\]

(S1)

where \( \hat{H} \) is the Hamiltonian of the system, and \( \hat{\rho}_\text{eq} = (1/Z) \exp(-\beta_0 \hat{H}) \). Some of its properties used in the main text are [1]:

\[
\langle \Delta \hat{A}; \Delta \hat{B} \rangle_0 = \langle \Delta \hat{B}; \Delta \hat{A} \rangle_0,
\]

(S2)

and:

\[
\beta_0 \langle \Delta \hat{A}; \Delta \hat{B} \rangle_0 = \beta_0 \langle \Delta \hat{B}; \Delta \hat{A} \rangle_0 = \frac{1}{i\hbar} \langle [\Delta \hat{A}, \Delta \hat{B}] \rangle_0,
\]

(S3)

where \( \Delta \hat{A} \equiv \hat{A} - \langle \hat{A} \rangle_0 \) and \( \Delta \hat{B} \equiv (1/i\hbar)[\hat{A}, \hat{H}] \).

For a system perturbed by a static external force:

\[
\hat{H}' = \hat{H} - \hat{A}\delta x.
\]

(S4)

The change of the expectation value of an operator \( \hat{B} \) can be calculated to the linear order [1]:

\[
\delta \langle \hat{B} \rangle = \langle \Delta \hat{B} \rangle_{\delta x} - \langle \hat{B} \rangle_0 \equiv \chi_{BA} \delta x,
\]

(S5)

where:

\[
\chi_{BA} \equiv \beta_0 \langle \Delta \hat{B}; \Delta \hat{A} \rangle_0.
\]

(S6)
II. DETAILS OF DERIVATION FOR EQS. (7-10), MAGNETIZATION FORMULA

The derivation of Eqs. (7-10) is detailed in the following:

(1) We introduce \( \chi_{ij}(r, r') \equiv \beta_0 \left< \Delta \hat{n}_j (r') : \Delta \hat{J}_i (r) \right>_0 \) with \( i, j = 1, 2 \), where \( \hat{n}_1 (r) \equiv \hat{n}_N (r) \), \( \hat{n}_2 (r) \equiv \hat{K} (r) \), \( \hat{J}_1 (r) \equiv \hat{J}_N (r) \), \( \hat{J}_2 (r) \equiv \hat{J}_Q (r) \), \( \Delta \hat{n} \equiv \hat{n} - \langle \hat{n} \rangle_0 \) and \( \langle \cdots \rangle_0 \equiv Tr [\rho_0 \cdots] \); we have:

\[
\nabla \cdot \chi_{ij} (r, r') = \beta_0 \left< \Delta \hat{n}_j (r') : \nabla \cdot \Delta \hat{J}_i (r) \right>_0 ,
\]

\[
= - \beta_0 \left< \hat{n}_j (r') : \hat{n}_i (r) \right>_0 .
\]

From Eq. (S7) to Eq. (S8), we have used \( \nabla \cdot \hat{J}_i (r) = - \hat{n}_i \) and \( \nabla \cdot \hat{J}_i^{eq} = 0 \). Using Eq. (S3), we obtain:

\[
\nabla \cdot \chi_{ij} (r, r') = \frac{1}{i \hbar} \left[ \langle \hat{n}_j (r') , \hat{n}_i (r) \rangle \right]_0 .
\]

We define \( \chi_{ij}^q (r) \equiv \int dr' \chi_{ij} (r, r') e^{-iq \cdot (r-r')} \), and have:

\[
\nabla \cdot \chi_{ij}^q (r) + i q \cdot \chi_{ij}^q (r) = \frac{1}{i \hbar} \int dr' \left[ \langle \hat{n}_j (r') , \hat{n}_i (r) \rangle \right]_0 e^{-iq \cdot (r-r')} ,
\]

where \( i, j = 1, 2 \).

(2) We can obtain the right hand side of Eq. (S10) from the definitions of currents and their scaling laws. Basically, we have:

\[
\frac{1}{i \hbar} \left[ \langle \hat{n} (r) , \hat{H}_{\phi, \psi} \rangle \right] = \nabla \cdot \hat{J}_{N}^{\phi, \psi} (r) ,
\]

and:

\[
\hat{J}_{N}^{\phi, \psi} (r) = [1 + \psi (r)] \hat{J}_N (r) ,
\]

\[
\hat{J}_{E}^{\phi, \psi} (r) = [1 + \psi (r)]^2 \left[ \hat{J}_E (r) + \phi (r) \hat{J}_N (r) \right] .
\]

If we set \( \phi (r) = 0 \) and \( 1 + \psi (r) = e^{i q \cdot r} \), Eq. (S11) becomes:

\[
\frac{1}{i \hbar} \int dr' \left[ e^{i q \cdot r'} \hat{h} (r'), \hat{n} (r) \right] = \nabla \cdot \left[ e^{i q \cdot r} \hat{J}_N (r') \right] ,
\]

\[
= e^{i q \cdot r} i q \cdot \hat{J}_N (r) + e^{i q \cdot r} \nabla \cdot \hat{J}_N (r) ,
\]

and we obtain:

\[
\frac{1}{i \hbar} \int dr' \left[ \left< e^{i q \cdot (r'-r)} \hat{h} (r') , \hat{n} (r) \right> \right]_0 = i q \cdot \nabla \times M_N (r) ,
\]

where we have used \( \nabla \cdot \hat{J}_N^{eq} (r) = 0 \) and \( \hat{J}_N^{eq} (r) = \nabla \times M_N (r) \). This is exactly the right hand side of Eq. (S10) for \( i = 1, j = 2 \).

Using the similar approach, we can prove that:

\[
\frac{1}{i \hbar} \int dr' \left[ \langle \hat{n}_j (r') , \hat{n}_i (r) \rangle \right] e^{-iq \cdot (r-r')} = i q \cdot \nabla \times M_{ij} (r) ,
\]

where \( M_{11} (r) = 0 \), \( M_{12} (r) = M_N (r) \), \( M_{21} (r) = M_N (r) \), and \( M_{22} (r) = 2M_Q (r) \).

Therefore, \( \chi_{ij}^q (r) \) satisfies the equation:

\[
\nabla \cdot \chi_{ij}^q (r) + i q \cdot \left[ \chi_{ij}^q (r) - \nabla \times M_{ij} (r) \right] = 0 ,
\]

and it has the general solution:

\[
\chi_{ij}^q (r) = -i q \times M_{ij} (r) + e^{-iq \cdot r} \nabla \times \kappa_{ij}^q (r) ,
\]
where $\kappa^q_{ij}(r)$ is an arbitrary function. This equation can be considered as a decomposition of $\chi^q_{ij}(r)$. It is important to note that the decomposition is not necessary to be unique, because the magnetization can only be defined up to a gradient. However, the arbitrariness does not affect our result on the total magnetizations, as long as both $M_{ij}(r)$ and $\kappa^q_{ij}(r)$ are well behaved functions; i.e., they are bounded for all $r$. The constraint is necessary because, first, magnetizations are properties of materials; second, only when these functions are well behaved, can their contributions presented in Eq. (18-19) be well defined.

(4) We can relate $\kappa^q_{ij}(r)$ to the macroscopic thermodynamic quantities. To see this, we use Eq. (17) and see how the equilibrium currents are perturbed by the spatially uniform changes of the chemical potential and the temperature. We have:

\[
\delta J^{eq}_i(r) \approx \int dr' \left[ \chi_{i1}(r, r') \delta \mu_0 - \chi_{i2}(r, r') T_0 \delta (1/T_0) \right],
\]

where $\chi_{i1}(r, r') = \nabla \times \kappa^q_{i1}(r)$, $\chi_{i2}(r, r') = \nabla \times \left[ \kappa^q_{i1}(r) \delta \mu_0 - \kappa^q_{i2}(r) T_0 \delta (1/T_0) \right]$. Note that $\delta J^{eq}_2(r) \equiv \text{Tr} \left[ J^{eq, \psi}_E(r) \delta \rho_{eq} \right] - \rho_0 \left[ J^{eq, \psi}_E(r) \delta \rho_{eq} \right] \approx \delta J^{eq}_E(r) - \mu_0 \delta J^{eq}_N(r)$. On the other hand, $\delta J^{eq}_i$ is, by definition:

\[
\begin{align*}
\delta J^{eq}_1 &= \nabla \times (\delta M_N), \quad (S24) \\
\delta J^{eq}_2 &= \nabla \times (\delta M_E) - \mu_0 \nabla \times (\delta M_N). \quad (S25)
\end{align*}
\]

Comparing the two sides, we obtain:

\[
\begin{align*}
\delta J^{eq}_1 &= \nabla \times \left( \frac{\partial M_N(r)}{\partial \mu_0} \right)_{T_0}, \quad (S26) \\
\kappa^q_{i1}(r) &= \nabla \times \left( \frac{\partial M_N(r)}{\partial \mu_0} \right)_{T_0}, \quad (S27) \\
\kappa^q_{i2}(r) &= \nabla \times \left[ \frac{\partial M_N(r)}{\partial \mu_0} \right]_{T_0} + M_N(r), \quad (S28) \\
\kappa^q_{i2}(r) &= \nabla \times \left( \frac{\partial M_N(r)}{\partial \mu_0} \right)_{T_0}. \quad (S29)
\end{align*}
\]

(5) Equation (S20) can be rewritten as:

\[
\chi^q_{ij}(r) = -iq \times \left[ M_{ij}(r) - e^{-iq \cdot r} \kappa^q_{ij}(r) \right] + \nabla \times \left[ e^{-iq \cdot r} \kappa^q_{ij}(r) \right], \quad (S30)
\]

Applying $\nabla \times$ to the both sides of Eq. (S30) and setting $q \to 0$, we obtain:

\[
\frac{1}{2} \nabla_q \times \chi^q_{ij}(r) \bigg|_{q \to 0} = -M_{ij}(r) + \kappa^q_{ij}(r) - \nabla \times U_{ij}(r),
\]

where $U_{ij}(r) = \frac{i}{2} \nabla_q \times \left( e^{-iq \cdot r} \kappa^q_{ij}(r) \right) \bigg|_{q \to 0}$. After substituting different components of $M_{ij}(r)$ and $\kappa^q_{ij}(r)$ and integrating over $r$ we come to the formulae for the total magnetizations:

\[
\begin{align*}
- \frac{\partial M_N}{\partial \mu_0} &= \frac{\beta_0}{2i} \nabla_q \times \langle \hat{n}_{-q}; \hat{J}_N, q \rangle \bigg|_{q \to 0}, \quad (S32) \\
M_N - T_0 \frac{\partial M_N}{\partial T_0} &= \frac{\beta_0}{2i} \nabla_q \times \langle \hat{K}_{-q}; \hat{J}_N, q \rangle \bigg|_{q \to 0}, \quad (S33) \\
- \frac{\partial M_Q}{\partial \mu_0} &= \frac{\beta_0}{2i} \nabla_q \times \langle \hat{n}_{-q}; \hat{J}_Q, q \rangle \bigg|_{q \to 0}, \quad (S34) \\
2MQ - T_0 \frac{\partial M_Q}{\partial T_0} &= \frac{\beta_0}{2i} \nabla_q \times \langle \hat{K}_{-q}; \hat{J}_Q, q \rangle \bigg|_{q \to 0}. \quad (S35)
\end{align*}
\]

In the derivation, we assume that $\int dr \nabla \times U_{ij}(r) = 0$. This is guaranteed because $\kappa^q_{ij}$ is a well behaved function.
III. DETAILS OF DERIVATION FOR EQS. (18-19), LOCAL EQUILIBRIUM CURRENTS

Inserting Eq. (13) into Eq. (17), we obtain:

\[
J_i^{\text{eq}}(r) \equiv J_i^{\text{eq}}(r) + \sum_{j=1}^{2} \left( M_{ij}(r) \times \nabla x_j(r) + \int \frac{dq}{(2\pi)^3} \nabla \times \kappa_{ij}^q(r) x_{jq} \right). 
\] (S36)

Because \( x_{jq} = \int dr' x_j(r') e^{-iq\cdot r'} \), we have:

\[
J_i^{\text{eq}}(r) \approx J_i^{\text{eq}}(r) + \sum_{j=1}^{2} \left( M_{ij}(r) \times \nabla x_j(r) + \nabla \times \int dr' \kappa_{ij}(r, r') x_j(r') \right), 
\] (S37)

where \( \kappa_{ij}(r, r') = \int dq/ (2\pi)^3 \kappa_{ij}^q(r) e^{-iq\cdot r'} \).

We can obtain \( J_i^{\text{eq}}(r) \) through the scaling law. Without \( \psi \) and \( \phi \), we have:

\[
J_{N(E)}^{\text{eq}}(r) = \nabla \times M_{N(E)}(r). 
\] (S38)

When \( \psi(r) \) and \( \phi(r) \) are present, according to the scaling law in Eq. (S13) and (S14) we have:

\[
J_1^{\text{eq}}(r) = [1 + \psi(r)] \nabla \times M_N(r), 
\] (S39)

\[
J_2^{\text{eq}}(r) = [1 + \psi(r)]^2 [\nabla \times M_E(r) - \mu(r) \nabla \times M_N(r)]. 
\] (S40)

For \( i = 1 \), inserting Eq. (S39) into Eq. (S37),

\[
J_1^{\text{eq}}(r) \approx [1 + \psi(r)] \nabla \times M_N(r) - M_N(r) \times T_0 \nabla \frac{1}{T} + \nabla \times \int dr' \sum_{j=1}^{2} \kappa_{1j}(r, r') x_j(r'), 
\] (S41)

\[
= \nabla \times ([1 + \psi(r)] M_N(r)) - \frac{1}{\beta} M_N(r) \times X_2 + \nabla \times \int dr' \sum_{j=1}^{2} \kappa_{1j}(r, r') x_j(r'), 
\] (S42)

so we can write:

\[
J_1^{\text{eq}}(r) \approx \nabla \times M_N^{\psi, \psi}(r) - \frac{1}{\beta} M_N(r) \times X_2, 
\] (S43)

where \( M_N^{\psi, \psi}(r) \equiv [1 + \psi(r)] M_N(r, T_0, \mu_0) + \delta M_N(r) \) and \( \delta M_N(r) \equiv \sum_{j=1}^{2} \int dr' \kappa_{1j}(r, r') x_j(r') \).

Similarly, for \( i = 2 \), inserting Eq. (S40) into Eq. (S37),

\[
J_2^{\text{eq}}(r) \approx [1 + \psi(r)]^2 [\nabla \times M_E(r) - \mu(r) \nabla \times M_N(r)] + M_N(r) \times \nabla \mu - 2M_Q \times T_0 \nabla \frac{1}{T} \] (S44)

\[
+ \nabla \times \int dr' \sum_{j=1}^{2} \kappa_{2j}(r, r') x_j(r'), 
\] (S45)

\[
= \nabla \times \left( [1 + \psi(r)]^2 (M_E(r) + \phi(r) M_N(r)) - \alpha(r) \nabla \times ([1 + \psi(r)] M_N(r)) \right) - \frac{1}{\beta} M_N(r) \times X_1 - 2M_Q(r) \times X_2 + \nabla \times \sum_{j=1}^{2} \int dr' \kappa_{2j}(r, r') x_j(r'). 
\] (S46)

Further, by substituting \( M_N^{\psi, \psi}(r) \) into Eq. (S46) we can write \( J_2^{\text{eq}}(r) \) as:

\[
J_2^{\text{eq}}(r) \approx \nabla \times M_N^{\psi, \psi}(r) - \alpha(r) \nabla \times M_N^{\psi, \psi}(r) - \frac{1}{\beta} M_N(r) \times X_1 - \frac{2}{\beta} M_Q(r) \times X_2, 
\] (S48)

where \( M_N^{\psi, \psi}(r) \equiv (1 + \psi(r))^2 (M_E(r, T_0, \mu_0) + \phi(r) M_N(r, T_0, \mu_0)) + \delta M_E(r) \), \( \delta M_E(r) \equiv \sum_{j=1}^{2} \int dr' \kappa_{2j}^\prime (r, r') x_j(r') \), and \( \kappa_{2j}^\prime \equiv \kappa_{2j} + \mu_0 \kappa_{1j} \).
IV. DETAILS OF DERIVATION FOR EQ. (22), DEFINITION OF ENERGY CURRENT

The energy density can be written as:

\[ \hat{h}^{\phi,\psi}(r) = [1 + \psi(r)] \left\{ \frac{m}{2} [\hat{\psi}\hat{\phi}(r)] \cdot [\hat{\psi}\hat{\phi}(r)] + \varphi^\dagger(r) [V(r) + \phi(r)] \hat{\phi}(r) \right\}. \]  \hspace{1cm} (S49)

The Schrödinger equation for the system is \( i\hbar \frac{\partial \hat{\psi}}{\partial t} = \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi} \), where \( \hat{\mathcal{H}}^{\phi,\psi} \equiv \frac{\hbar}{2} [1 + \psi(r)] \hat{v} + [1 + \psi(r)] [V(r) + \phi(r)] \). Therefore, we have:

\[
\frac{\partial \hat{h}^{\phi,\psi}(r)}{\partial t} = \frac{1}{i\hbar} \left[ 1 + \psi(r) \right] \left\{ \frac{m}{2} [\hat{\psi}\hat{\phi}(r)] \cdot [\hat{\psi}\hat{\phi}(r)] - \frac{m}{2} [\hat{\phi}\hat{\phi}(r)] \cdot [\hat{\phi}\hat{\phi}(r)] \right\} 
+ \varphi^\dagger(r) [V(r) + \phi(r)] \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right] - \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right]^\dagger [V(r) + \phi(r)] \hat{\phi}(r) \right\}, \hspace{1cm} (S50)

\[
= -\nabla \cdot \left\{ \frac{1}{2} [1 + \psi(r)] \left[ \hat{\psi}\hat{\phi}(r) \right]^\dagger \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right] + \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right]^\dagger [\hat{\psi}\hat{\phi}(r)] \right\}, \hspace{1cm} (S51)
\]

so we can identify \( \hat{J}_E^{\phi,\psi}(r) \) as:

\[ \hat{J}_E^{\phi,\psi}(r) = \frac{1}{2} [1 + \psi(r)] \left\{ [\hat{\psi}\hat{\phi}(r)]^\dagger \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right] + \left[ \hat{\mathcal{H}}^{\phi,\psi} \hat{\phi}(r) \right]^\dagger [\hat{\psi}\hat{\phi}(r)] \right\}. \]  \hspace{1cm} (S53)

Because:

\[ \hat{\mathcal{H}}^{\phi,\psi} = [1 + \psi(r)] \left[ \hat{h}_0 + \phi(r) \right] - \frac{i\hbar}{2} [\nabla \psi(r)] \cdot \hat{v}, \]  \hspace{1cm} (S54)

where \( \hat{h}_0 \equiv \hat{\mathcal{H}}_{\phi=0,\psi=0} \), we obtain the following scaling equation:

\[ \hat{J}_E^{\phi,\psi}(r) = [1 + \psi(r)]^2 \left[ \hat{J}_E(r) + \phi(r) \hat{J}_N(r) \right] + \nabla \psi(r) \hat{\Lambda}(r), \]  \hspace{1cm} (S55)

where \( \hat{\Lambda}(r) = \frac{\hbar}{8\alpha} \left( \hat{\psi} \right)^{\dagger} \right) \times \left( \hat{\psi} \right). \)

In order to satisfy the scaling law Eq. (5), we redefine the energy current operator as:

\[ \hat{J}_E^{\phi,\psi}(r) \to \hat{J}_E^{\phi,\psi}(r) - \nabla \times \left( (1 + \psi(r))^2 \hat{\Lambda}(r) \right), \hspace{1cm} (S56) \]

\[ \hat{J}_E(r) \to \hat{J}_E(r) - \nabla \times \hat{\Lambda}(r). \hspace{1cm} (S57) \]

This is exactly the energy current definition Eq. (22). It is straightforward to show that modified energy current operator satisfies the scaling law Eq. (5).

The particle current operator is defined as usual. It automatically satisfies the corresponding scaling law Eq. (4).

V. DETAILS OF DERIVATION FOR EQ. (23), KUBO CONTRIBUTION

The thermal current operator \( \hat{J}_{Qx}(r) \) is:

\[ \hat{J}_{Qx}(r) = \frac{\left( \hat{K}\hat{\phi}(r) \right)^\dagger \hat{v}_x\hat{\phi}(r) + \left( \hat{v}_x\hat{\phi}(r) \right)^\dagger \hat{K}\hat{\phi}(r)}{2} - \frac{\hbar}{8\gamma} \sum_{\gamma} \nabla_{\gamma} \left( \left( \hat{v}_x\hat{\phi}(r) \right)^\dagger \hat{v}_\gamma\hat{\phi}(r) - \left( \hat{v}_\gamma\hat{\phi}(r) \right)^\dagger \hat{v}_x\hat{\phi}(r) \right), \]  \hspace{1cm} (S58)

where \( \gamma = x, y, z \) and we have set \( \phi(r) = 0 \) and \( \psi(r) = 0 \). According to our definition, we have \( \hat{J}_{Kubo}^{(22)} = \frac{\hat{J}_{(22)}^{(22)}}{k_B T_0} \)

and:

\[ L^{(22)}_{xy} = \frac{1}{V} \int_0^\infty dt e^{-st} \langle \hat{J}_{Qy}; \hat{J}_{Qx}(t) \rangle \right\} \hspace{1cm}, \]  \hspace{1cm} (S59)

\[ = -\frac{\hbar}{\beta_0 V} \sum_{nk, n'k'} \frac{f_{nk} - f_{n'k'}}{i (\epsilon_{nk} - \epsilon_{n'k'})} \langle \psi_{nk} | \hat{J}_{Qy} | \psi_{n'k'} \rangle \langle \psi_{n'k'} | \hat{J}_{Qx} | \psi_{nk} \rangle, \]  \hspace{1cm} (S60)
where $\psi_{nk}$ is the Bloch wave function for band $n$ and quasi-momentum $k$. $f_{nk} \equiv f(\epsilon_{nk})$ is the Fermi distribution function, and $\epsilon_{nk}$ is the electron dispersion. According to our definition Eq. (S58) for $\hat{J}_{Qx}$, we have:

$$
\langle \psi_{nk} | \hat{J}_{Qx} | \psi_{nk} \rangle = \frac{1}{2} \left( \frac{\hbar}{\beta_0 V} \sum_{n \neq n'} \left( \frac{f_{nk} - f_{n'k}}{4} \right) \frac{\epsilon_{nk} + \epsilon_{n'k} - 2\mu_0}{\epsilon_{nk} - \epsilon_{n'k}} \right) \langle u_{nk} | \hat{v}_{kx} | u_{nk} \rangle 
- \langle \nabla \gamma \hat{v}_{nk} \psi_{nk} \rangle + \langle \hat{v}_{nk} \psi_{nk} | \nabla \gamma \psi_{nk} \rangle,
$$

(S61)

Note:

$$
\langle \psi_{nk} | \hat{J}_{Qx} | \psi_{nk} \rangle = \langle \nabla \psi_{nk} | \hat{J}_{Qx} | \psi_{nk} \rangle = 0.
$$

(S62)

with $\hat{v}_{kx} = \partial \hat{H}_k / \partial (\hbar k_x)$, and:

$$
\langle \nabla \gamma \hat{v}_{nk} \psi_{nk} \rangle + \langle \hat{v}_{nk} \psi_{nk} | \nabla \gamma \psi_{nk} \rangle = -\langle \psi_{nk} | \hat{v}_{nk} \psi_{nk} \rangle + \langle \psi_{nk} | \hat{v}_{nk} \psi_{nk} \rangle,
$$

(S63)

and similarly, $\langle \nabla \gamma \hat{v}_{nk} \psi_{nk} \rangle + \langle \hat{v}_{nk} \psi_{nk} | \nabla \gamma \psi_{nk} \rangle = 0$, so we come to:

$$
L_{xy}^{(22)} = -\frac{\hbar}{2\beta_0 V} \sum_{n \neq n'} \frac{f_{nk} \left( \epsilon_{nk} + \epsilon_{n'k} - 2\mu_0 \right)^2}{4 \left( \epsilon_{nk} - \epsilon_{n'k} \right)^2} \Im \left( \langle u_{nk} | \hat{v}_{kx} | u_{nk} \rangle \langle \psi_{nk} | \hat{v}_{kx} | \psi_{nk} \rangle \right).
$$

(S64)

Using the identity:

$$
\langle u_{nk} | \hat{v}_{kx} | u_{nk} \rangle = \frac{1}{\hbar} \frac{\partial u_{nk}}{\partial k_x} \left( \epsilon_k - \epsilon_{nk} \right),
$$

(S65)

we have:

$$
L_{xy}^{(22)} = \frac{1}{2\beta_0 V} \sum_{nk} \Im \left[ \left( \frac{\partial u_{nk}}{\partial k_x} \right) \left( \hat{H}_k + \epsilon_{nk} - 2\mu_0 \right) \frac{\partial u_{nk}}{\partial k_y} \right] f_{nk}.
$$

(S66)

The formula can be rewritten as the alternative form. We introduce the new notations:

$$
m_2(\epsilon) \equiv \frac{1}{\hbar} \Im \sum_{nk} \left( \frac{\partial u_{nk}}{\partial k_x} \right) \left( \hat{H}_k + \epsilon \right) \frac{\partial u_{nk}}{\partial k_y} \delta (\epsilon - \epsilon_{nk}),
$$

(S67)

$$
m_1(\epsilon) \equiv \frac{1}{\hbar} \Im \sum_{nk} \left( \frac{\partial u_{nk}}{\partial k_x} \right) \left( \hat{H}_k + \epsilon \right) \frac{\partial u_{nk}}{\partial k_y} \delta (\epsilon - \epsilon_{nk}),
$$

(S68)

$$
\Omega_z (\epsilon) \equiv -\frac{1}{\hbar} \Im \sum_{nk} \left( \frac{\partial u_{nk}}{\partial k_x} \right) \frac{\partial u_{nk}}{\partial k_y} \delta (\epsilon - \epsilon_{nk}).
$$

(S69)

Therefore, we can express $\kappa_{xy}^{\text{Kubo}}$ as:

$$
\kappa_{xy}^{\text{Kubo}} = \frac{1}{2T_0 V} \int d\epsilon \left[ m_2(\epsilon) + 4 \left( \epsilon - \mu_0 \right) m_1(\epsilon) - 2 \left( \epsilon - \mu_0 \right)^2 \Omega_z (\epsilon) \right] f(\epsilon).
$$

(S70)

It is easy to see $\kappa_{xy}^{\text{Kubo}}$ is divergent in the low temperature limit.

VI. DETAILS OF DERIVATION FOR EQ. (24), ENERGY MAGNETIZATION

To calculate $M_{Q,z}$, we use $2M_Q = T_0 \frac{\partial M_Q}{\partial q} = \frac{\beta_0}{2t_1} \nabla_q \times \left( \hat{K} - q \cdot \hat{J}_{Q,q} \right)_{0|q \to 0}$. We can show:

$$
\tilde{M}_{Q,z} = \frac{\beta_0}{2t_1} \nabla_q \times \left( \hat{K} - q \cdot \hat{J}_{Q,q} \right)_{0|z,q \to 0} = -\beta_0 \frac{\partial}{\partial q_y} \left( \hat{K} - q \cdot \hat{J}_{Q,q} \right)_{0|q \to 0}.
$$

(S71)
So we have:

\[
\tilde{M}_{Q,z} = \frac{\partial}{\partial y_q} \sum_{n,k,k'} f_{nk} - f_{n'k'} \frac{\hat{K} e^{iq \cdot r} + e^{iq \cdot r} \hat{K}}{2} |\psi_{nk}| \langle \psi_{n'k'} | \hat{J}_{Q_x,q} | \psi_{nk} \rangle .
\]  

(S75)

We have a careful calculation of \( \langle \psi_{n'k'} | \hat{J}_{Q_x,q} | \psi_{nk} \rangle \),

\[
\langle \psi_{n'k'} | \hat{J}_{Q_x,q} | \psi_{nk} \rangle = \left\langle \hat{K} \psi_{n'k'} | e^{-iq \cdot r} \hat{v}_x \psi_{nk} \right\rangle + \left\langle \hat{v}_x \psi_{n'k'} | e^{-iq \cdot r} \hat{K} \psi_{nk} \right\rangle \]

\[
= \frac{\hbar}{2i} \sum_{\gamma} \left[ \left\langle \nabla_{\gamma} \hat{v}_x \psi_{n'k'} | e^{-iq \cdot r} \hat{v}_x \psi_{nk} \right\rangle + \left\langle \hat{v}_x \psi_{n'k'} | e^{-iq \cdot r} \nabla_{\gamma} \hat{v}_x \psi_{nk} \right\rangle \right] .
\]  

(S76)

In Eq. (S76), we can show:

\[
\left\langle \hat{K} \psi_{n'k'} | e^{-iq \cdot r} \hat{v}_x \psi_{nk} \right\rangle = \left\langle u_{nk} | e^{-ik' \cdot r} \hat{K} e^{-iq \cdot r} \hat{v}_x e^{ik \cdot r} | u_{nk} \right\rangle ,
\]

\[
= \left\langle u_{nk-k-q} | \hat{K}_{k-q} \hat{v}_{kx} | u_{nk} \right\rangle \delta_{k',k-q} .
\]  

(S77)

In Eq. (S77), similarly:

\[
\langle \nabla_{\gamma} \hat{v}_x \psi_{n'k'} | e^{-iq \cdot r} \hat{v}_x \psi_{nk} \rangle = - \left\langle u_{nk-k-q} | \hat{v}_{k-q} \left( \nabla_{\gamma} + i k_{\gamma} - i q_{\gamma} \right) \hat{v}_{kx} | u_{nk} \right\rangle \delta_{k',k-q} ,
\]

(S80)

and:

\[
\langle \hat{v}_x \psi_{n'k'} | e^{-iq \cdot r} \nabla_{\gamma} \hat{v}_x \psi_{nk} \rangle = - \left\langle u_{nk-k-q} | \hat{v}_{k-q} \left( \nabla_{\gamma} + i k_{\gamma} - i q_{\gamma} \right) \hat{v}_{kx} | u_{nk} \right\rangle \delta_{k',k-q} ,
\]

(S81)

Therefore, \( \langle \psi_{n'k'} | \hat{J}_{Q_x} e^{-iq \cdot r} | \psi_{nk} \rangle \) is,

\[
\langle \psi_{n'k'} | \hat{J}_{Q_x} e^{-iq \cdot r} | \psi_{nk} \rangle = \left( u_{nk-k-q} | \frac{\hat{K}_{k-q} \hat{v}_{kx} + \hat{v}_{k-q} \hat{K}}{2} - \frac{\hbar}{8} \sum_{\gamma} \left( \hat{v}_{k-q} \hat{v}_{k \gamma} - \hat{v}_{k-q} \hat{v}_{k \gamma} \right) \right) \left\langle u_{nk} | \delta_{k',k-q} \right\rangle .
\]  

(S86)

\( \tilde{M}_{Q,z} \) can be simplified as,

\[
\tilde{M}_{Q,z} = \frac{\partial}{\partial y_q} \sum_{n,k,n'} f_{nk} - f_{n'k'} \frac{\hat{K}_{k-q} + \hat{K}_{k-q}}{2} 
\left\langle u_{nk} | \hat{v}_{kx} \right\rangle 
\left\langle u_{n'k} | \hat{v}_{kx} \right\rangle 
\left\langle u_{nk} \right| \hat{v}_{qy} \hat{v}_{kx} \left| u_{n'k} \right\rangle 
\left\langle u_{nk} \right| \hat{v}_{kx} \left| u_{n'k} \right\rangle 
\frac{\hbar}{8} \sum_{\gamma} \left( \hat{v}_{k-q} \hat{v}_{k \gamma} - \hat{v}_{k-q} \hat{v}_{k \gamma} \right) \left\langle u_{nk} \right| \delta_{k',k-q} \right\rangle .
\]  

(S87)

First, we calculate \( \tilde{M}_{Q,z}^{\text{inter}} \) for \( n \neq n' \). When \( q \to 0 \), we have:

\[
\tilde{M}_{Q,z}^{\text{inter}} = \frac{1}{4} \sum_{n \neq n'} \left( \epsilon_{nk} + \epsilon_{n'k} - 2 \mu \right)^2 \left\langle u_{nk} \left| \frac{\partial u_{nk}}{\partial k_y} \right\rangle \right\langle u_{n'k} \left| \hat{v}_{kx} \right\rangle \right\rangle 
\left\langle u_{nk} \right| \frac{f_{nk} - f_{n'k}}{\epsilon_{nk} - \epsilon_{n'k}} \right\rangle .
\]  

(S88)

Using the identity Eq. (S88), we finally come to:

\[
\hat{M}_{Q,z}^{\text{inter}} = \frac{1}{2\hbar} \sum_{nk} \left\langle \hat{u}_{nk} \left| \frac{\partial u_{nk}}{\partial k_x} \right\rangle \right\rangle \left( \hat{H}_{k} + \epsilon_{nk} - 2 \mu \right)^2 \left\langle u_{nk} \right| \frac{f_{nk} - f_{n'k}}{\epsilon_{nk} - \epsilon_{n'k}} \right\rangle .
\]  

(S89)

Next, we calculate \( \tilde{M}_{Q,z}^{\text{intra}} \) for \( n = n' \). When \( q \to 0 \), we have:
\[
\tilde{M}_{\text{intra}}^{Q,z} = \left(-\frac{1}{4}\right) \sum_{n,k} \left(\epsilon_{nk} - \mu_0\right)^2 \text{Im} \left[ \left\langle u_{nk} \right| \frac{\partial u_{nk}}{\partial k_y} \right| u_{nk} \rangle \right] f_{nk}' \tag{S91}
\]

\[
- \left(-\frac{1}{4}\right) \sum_{n,k} 2 \left(\epsilon_{nk} - \mu_0\right) \text{Im} \left[ \left\langle u_{nk} \right| \hat{H}_k \hat{v}_{kx} + \hat{v}_{kx} \hat{H}_k \right| u_{nk} \rangle \right] f_{nk}' \tag{S92}
\]

\[
- \frac{\hbar}{4} \sum_{n,k} \left(\epsilon_{nk} - \mu_0\right) \text{Im} \left[ \left\langle u_{nk} \right| \hat{v}_{kx} \hat{v}_{kx} \right| u_{nk} \rangle \right] f_{nk}'. \tag{S93}
\]

Using the identity Eq. (S68) and after some simple algebra, we obtain:

\[
\tilde{M}_{\text{intra}}^{Q,z} = \left(-\frac{1}{4}\right) \sum_{n,k} \text{Im} \left[ \langle \partial u_{nk} \partial k_x \rangle \right] \left(\epsilon_{nk} - \hat{H}_k\right)^2 - 4 \left(\epsilon_{nk} - \mu_0\right) \left(\epsilon_{nk} - \hat{H}_k\right) \langle \partial u_{nk} \partial k_y \rangle \right] \left(\epsilon_{nk} - \mu_0\right) f_{nk}'. \tag{S94}
\]

Therefore, we have:

\[
\tilde{M}_{Q,z} = -\frac{1}{2\hbar} \sum_{n,k} \text{Im} \left[ \langle \partial u_{nk} \partial k_x \rangle \right] \left(\epsilon_{nk} - \hat{H}_k\right)^2 - 4 \left(\epsilon_{nk} - \mu_0\right) \left(\epsilon_{nk} - \hat{H}_k\right) \langle \partial u_{nk} \partial k_y \rangle \right] \left(\epsilon_{nk} - \mu_0\right) f_{nk}'. \tag{S95}
\]

\[
-\frac{1}{4\hbar} \sum_{n,k} \text{Im} \left[ \langle \partial u_{nk} \partial k_x \rangle \right] \left(\epsilon_{nk} - \hat{H}_k\right)^2 - 4 \left(\epsilon_{nk} - \mu_0\right) \left(\epsilon_{nk} - \hat{H}_k\right) \langle \partial u_{nk} \partial k_y \rangle \right] \left(\epsilon_{nk} - \mu_0\right) f_{nk}'. \tag{S96}
\]

We use \(2M_{Q,z} - T_0(\partial M_{Q,z}/\partial T_0) = \tilde{M}_{Q,z}\) to obtain \(M_{Q,z}\). Using the notations of Eqs. (S70)–(S72), we obtain:

\[
M_{Q,z} = -\frac{1}{2} \int \left[ \frac{1}{2} m_2 (\epsilon) f (\epsilon) + 2 (\epsilon - \mu_0) m_1 (\epsilon) f (\epsilon) - 2 \Omega_z (\epsilon) \int_0^{\epsilon - \mu_0} dx f (x) \right]. \tag{S97}
\]

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[2] G.D. Mahan, Many-Particle Physics, Third Edition, (Kluwer Academic, 2000).