Commutative algebras
and representations of the category of finite sets

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Abstract

We prove that two finite-dimensional commutative algebras over an algebraically closed field are isomorphic if and only if they give rise to isomorphic representations of the category of finite sets and surjective maps.

Let $\Omega$ be the category whose objects are the sets $\underline{n} = \{1, \ldots, n\}$, $n \in \mathbb{N}$ ($= \{0, 1, \ldots\}$), and whose morphisms are surjective maps. Let $k$ be a field. For a commutative algebra $A$ without unity over $k$, let us define a functor $L_A: \Omega \rightarrow k\text{-Mod}$ (a “representation of $\Omega$”). For $n \in \mathbb{N}$, set $L_A(n) = A \otimes k^n$. For a morphism $h: \underline{m} \rightarrow \underline{n}$, set $L_A(h): x_1 \otimes \ldots \otimes x_m \mapsto y_1 \otimes \ldots \otimes y_n$, where $y_j = \prod_{i \in h^{-1}(j)} x_i$.

The functor $L_A$ is a variant of the Loday functor [1].

Theorem. Let the field $k$ be algebraically closed. Let $A$ and $B$ be finite-dimensional algebras without unity. Suppose that the functors $L_A$ and $L_B$ are isomorphic. Then the algebras $A$ and $B$ are isomorphic.

We do not know whether the assertion is true for infinite-dimensional algebras. It is false for the field $\mathbb{R}$ (which is not algebraically closed). Indeed, take the non-isomorphic algebras $A = \mathbb{R}[X]/(X^2 - 1)$ and $B = \mathbb{R}[Y]/(Y^2 + 1)$. We have the bases $\{X^e\}_{e=0,1} (= \{1, X\})$ in $A$ and $\{Y^e\}_{e=0,1}$ in $B$. The linear maps $s_n: A \otimes k^n \rightarrow B \otimes k^n$,

$$X^{e_1} \otimes \ldots \otimes X^{e_n} \mapsto k_{e_1 + \ldots + e_n} Y^{e_1} \otimes \ldots \otimes Y^{e_n}, \quad e_1, \ldots, e_n = 0, 1,$$

where $k_m = (-1)^{[m/2]}$, form a functor isomorphism $s: L_A \rightarrow L_B$.

1. Preliminaries

Algebra of polynomials. If $V$ is a vector space [over $k$], then the symmetric group $\Sigma_n = \text{Aut} \underline{n}$ acts [from the left] on $V \otimes k^n$ by the rule $g(v_1 \otimes \ldots \otimes v_n) = v_{g^{-1}(1)} \otimes \ldots \otimes v_{g^{-1}(n)}$. The symmetric powers $S^n(V) = (V \otimes k^n)_{\Sigma_n}$ form the symmetric algebra

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V)$$
with the multiplication induced by the tensor one: \( \overline{xy} = x \otimes y, \quad x \in V \otimes m, \quad y \in V \otimes n \) (the bar denotes the projection \( V \otimes n \rightarrow S^n(V) \)).

For a vector space \( U \), put \( k[U] = S(U^*) \). For \( u \in U \), there is the evaluation map \( k[U] \rightarrow k, \quad f \mapsto f(u) \), which is the algebra homomorphism defined by the condition \( v(u) = \langle v, u \rangle \) for \( v \in U^* = S^1(U^*) \subseteq k[U] \). For a polynomial \( f \in k[U] \) and a set \( X \subseteq U \), there is the function \( f \mid X: X \rightarrow k, \quad u \mapsto f(u) \). An ideal \( P \subseteq k[U] \) determines the set

\[
Z(P) = \{ u : f(u) = 0 \text{ for all } f \in P \} \subseteq U.
\]

**Symmetric tensors, isomorphism \( \theta \).** Put \( D^n(U) = (U \otimes n)^\Sigma_n \),

\[
\hat{D}(U) = \prod_{n=0}^{\infty} D^n(U).
\]

The pairing

\[
\langle -,- \rangle: (U^*)^\otimes n \times U^\otimes n \rightarrow k,
\]

\[
\langle v_1 \otimes \ldots \otimes v_n, u_1 \otimes \ldots \otimes u_n \rangle = \langle v_1, u_1 \rangle \ldots \langle v_n, u_n \rangle, \quad \text{induces the pairing}
\]

\[
\langle -,- \rangle: S^n(U^*) \times D^n(U) \rightarrow k,
\]

\[
\langle z,w \rangle = \langle z, w \rangle, \quad \text{where } w \in D^n(U) \subseteq U \otimes n, \quad z \in (U^*)^\otimes n. \quad \text{Summing over } n \in \mathbb{N},
\]

we get a pairing

\[
\langle -,- \rangle: k[U] \times \hat{D}(U) \rightarrow k.
\]

We have the linear map

\[
\theta: \hat{D}(U) \rightarrow k[U]^* \quad \langle \theta(W), f \rangle = \langle f, W \rangle.
\]

If \( U \) is finite-dimensional, then the pairings (1) and (2) are perfect and \( \theta \) is an isomorphism.

**Functor \( T_A \).** Let \( \Sigma \subseteq \Omega \) be the subcategory of isomorphisms. We have \( \Sigma = \Sigma_0 \sqcup \Sigma_1 \sqcup \ldots \). For a vector space \( A \), we have the functor \( T_A: \Sigma \rightarrow k\text{-Mod}, \quad T_A(y) = A^\otimes n \) (with the ordinary action of \( \Sigma_n \)). If \( A \) is an algebra without unity, then \( T_A = L_A[\Sigma] \).

**Kronecker product, isomorphism \( \kappa \).** If a group \( G \) acts on vector spaces \( X \) and \( Y \), then it acts on \( \text{Hom}(X,Y) \) by the rule \( g(t)(x) = g(t(y^{-1}x)) \). We have \( \text{Hom}(X,Y)^G = \text{Hom}_G(X,Y) \).

Let \( A \) and \( B \) be vector spaces. The Kronecker product \( \text{Hom}(A,B)^\otimes n \rightarrow \text{Hom}(A^\otimes n, B^\otimes n), \quad w \mapsto [w] \) (a notation), preserves the action of \( \Sigma_n \) and thus induces a linear map \( D^n(B^A) \rightarrow \text{Hom}_{\Sigma_n}(A^\otimes n, B^\otimes n) \) (from now on, \( B^A = \text{Hom}(A,B) \)). Since

\[
\text{Hom}_\Sigma(T_A, T_B) = \prod_{n=0}^{\infty} \text{Hom}_{\Sigma_n}(A^\otimes n, B^\otimes n),
\]
these maps form a linear map

\[ \kappa : \hat{D}(B^A) \to \text{Hom}_\Sigma(T_A, T_B). \]

If \( A \) and \( B \) are finite-dimensional, then \( \kappa \) is an isomorphism.

**Morphisms** \( T_A \to T_B \) and **functionals on** \( k[B^A] \), **isomorphism** \( \xi \). For finite-dimensional vector spaces \( A \) and \( B \) we have the isomorphism \( \xi \) that fits in the commutative diagram

\[ \begin{array}{ccc}
\text{Hom}_\Sigma(T_A, T_B) & \xrightarrow{\xi} & k[B^A]^* \\
\hat{D}(B^A) & \xrightarrow{\theta} & \\
\end{array} \]

**Example.** A linear map \( u : A \to B \) induces the functor morphism \( T_u : T_A \to T_B \), \((T_u)_n = u^\otimes n\). Then \( \langle \xi(T_u), f \rangle = f(u), f \in k[B^A] \).

**Antisymmetrization.** For a vector space \( V \), we have the operator \( \text{alt}_n : V^\otimes n \to V^\otimes n \),

\[ \text{alt}_n(w) = \sum_{g \in \Sigma_n} \text{sgn } g \ g w. \]

2. **The determinant**

Let \( A \) and \( B \) be vector spaces of equal finite dimension \( m \). Put \( U = B^A \). Choose bases \( e_1, \ldots, e_m \in A \) and \( f_1, \ldots, f_m \in B \). Put

\[ E = \text{alt}_m(e_1 \otimes \ldots \otimes e_m) \in A^{\otimes m}, \quad F = \text{alt}_m(f_1 \otimes \ldots \otimes f_m) \in B^{\otimes m}. \]

We have the bases \( \hat{f}^1, \ldots, \hat{f}^m \in B^* \), \( \langle \hat{f}^j, f_i \rangle = \delta^j_i \) (\( \delta^j_i \) is the Kronecker delta) and \( l^j_i \in U^*, i, j = 1, \ldots, m, \langle l^j_i, u \rangle = \langle \hat{f}^j, u(e_i) \rangle \). Put

\[ H = \sum_{g \in \Sigma_m} \text{sgn } g \ l^1_{g^{-1}(1)} \otimes \ldots \otimes l^m_{g^{-1}(m)} \in (U^*)^{\otimes m}. \]

Then \( \overline{H} \in k[U] \) is the determinant, so

\[ \overline{H}(u) = \det u, \quad u \in U. \] (3)

We have \( \langle \hat{f}^1 \otimes \ldots \otimes \hat{f}^m, [v](E) \rangle = \langle H, v \rangle, v \in U^{\otimes m} \). Hence

\[ \langle \langle \hat{f}^1 \otimes \ldots \otimes \hat{f}^m \rangle^\otimes r, [w](E^\otimes r) \rangle = \langle H^\otimes r, w \rangle, \quad w \in U^{\otimes mr}, \quad r \in \mathbb{N}. \] (4)

For \( w \in D^{mr}(U) \), we have

\[ [w](E^\otimes r) = (\overline{H}_r, w) F^{\otimes r}. \] (5)
Indeed, $E^\otimes r$ belongs to the image of $\text{alt}^\otimes r: A^\otimes mr \to A^\otimes mr$. The image of $\text{alt}^\otimes r: B^\otimes mr \to B^\otimes mr$ is generated by $F^\otimes r$ since the image of $\text{alt}_m: B^\otimes m \to B^\otimes m$ is generated by $F$. The map $[w]: A^\otimes mr \to B^\otimes mr$ preserves the action of $\Sigma_{mr}$ and thus commutes with $\text{alt}_m$. Therefore, $[w](E^\otimes r) = tF^\otimes r$ for some $t \in k$. From (4), we get $t = \langle H^\otimes r, w \rangle = \langle \overline{\mathcal{T}}, w \rangle$.

For a morphism $s: T_A \to T_B$, we have

$$s_{mr}(E^\otimes r) = \langle \xi(s), \overline{\mathcal{T}} \rangle F^\otimes r, \quad r \in \mathbb{N}. \quad (6)$$

This follows from (5): if $s = \kappa(W)$, $W \in \mathcal{D}(U)$, then $s_{mr} = [W_{mr}]$ and $\langle \xi(s), \overline{\mathcal{T}} \rangle = \langle \overline{\mathcal{T}}, W_{mr} \rangle$.

3. Homomorphisms $A \to B$ and morphisms $L_A \to L_B$

Let $A$ and $B$ be finite-dimensional algebras without unity. Put $U = B^A$.

**Multiplicativity ideal.** Take $x, y \in A$ and $p \in B^*$. We have the linear form $I_{x,y}^p \in U^*$,

$$\langle I_{x,y}^p, u \rangle = \langle p, u(xy) \rangle, \quad u \in U$$

(the multiplication in $A$ is used) and the tensor $J_{x,y}^p \in (U^*)^\otimes 2$,

$$\langle J_{x,y}^p, u \otimes v \rangle = \langle p, u(x)v(y) \rangle, \quad u, v \in U$$

(the multiplication in $B$ is used). Put

$$g_{x,y}^p = J_{x,y}^p - I_{x,y}^p \in k[U].$$

We have

$$g_{x,y}^p(u) = \langle p, u(x)u(y) - u(xy) \rangle, \quad u \in U.$$

Let $M \subseteq k[U]$ be the ideal generated by the polynomials $g_{x,y}^p$, $x, y \in A$, $p \in B^*$.

**Lemma 1.** The set $Z(M) \subseteq U$ coincides with the set of algebra homomorphisms $A \to B$.

Note that $\text{Hom}_\Omega(L_A, L_B) \subseteq \text{Hom}_\Sigma(T_A, T_B)$.

**Lemma 2.** Let $s \in \text{Hom}_\Sigma(T_A, T_B)$. Then the conditions $s \in \text{Hom}_\Omega(L_A, L_B)$ and $\xi(s) \perp M$ are equivalent.

Thus we establish an isomorphism $\text{Hom}_\Omega(L_A, L_B) \to (k[U]/M)^*$. 

**Proof.** For $n \in \mathbb{N}$, define the morphism $\tau_n: n + 2 \to n + 1$ by the rules $1 \mapsto 1$ and $i \mapsto i - 1$, $i > 1$. The category $\Omega$ is obtained from $\Sigma$ by adjunction of the morphisms $\tau_n$. Therefore, the condition $s \in \text{Hom}_\Omega(L_A, L_B)$ is equivalent to commutativity of the diagrams

$$A^{\otimes(n+2)} \xrightarrow{s_{n+2}} B^{\otimes(n+2)} \xrightarrow{L_B(\tau_n)} B^{\otimes(n+1)},$$

$$A^{\otimes(n+1)} \xrightarrow{s_{n+1}} B^{\otimes(n+1)} \xrightarrow{L_B(\tau_n)} \cdots.$$
\( n \in \mathbb{N} \). Consider the discrepancy

\[
  r_n = L_B(\tau_n) \circ s_{n+2} - s_{n+1} \circ L_A(\tau_n) : A^{\otimes (n+2)} \to B^{\otimes (n+1)}.
\]

For \( z \in A, q \in B^* \), we have the linear form \( l^z_q \in U^* \), \( \langle l^z_q, u \rangle = \langle q, u(z) \rangle \), \( u \in U \). These forms generate \( U^* \). For \( n \in \mathbb{N} \), \( x, y, z_1, \ldots, z_n \in A \), \( p, q_1, \ldots, q_n \in B^* \), put

\[
  G_{x,y,z_1,\ldots,z_n}^{p,q_1,\ldots,q_n} = g_{x,y,z_1}^{p,q_1} \cdots g_{z_n}^{p,q_n} \in k[U].
\]

These polynomials linearly generate \( M \). Therefore, it suffices to prove that

\[
  \langle p^\sim, r_n(x^\sim) \rangle = \langle \xi(s), G_{x,y,z_1,\ldots,z_n}^{p,q_1,\ldots,q_n} \rangle,
\]

where \( x^\sim = x \otimes y \otimes z_1 \otimes \cdots \otimes z_n \), \( p^\sim = p \otimes q_1 \otimes \cdots \otimes q_n \).

We have

\[
  \{ p^\sim, [w_1](L_A(\tau_n)(x^\sim)) \} = \{ l^\sim_{x,y} \otimes l^\sim, w_1 \}, \quad w_1 \in U^{\otimes (n+1)},
\]

\[
  \{ p^\sim, L_B(\tau_n)([w_2](x^\sim)) \} = \{ J^p_{x,y} \otimes l^\sim, w_2 \}, \quad w_2 \in U^{\otimes (n+2)},
\]

where \( l^\sim = l^z_{z_1} \otimes \cdots \otimes l^z_{z_n} \) (direct check). By construction,

\[
  G_{x,y,z_1,\ldots,z_n}^{p,q_1,\ldots,q_n} = J_{x,y}^p \otimes l^\sim - I_{x,y}^p \otimes I^\sim.
\]

We have \( s = \kappa(W) \) for some sequence \( W \in \mathcal{D}(U) \), so \( s_n = [W_n] \). We have

\[
  \langle p^\sim, r_n(x^\sim) \rangle = \langle p^\sim, L_B(\tau_n)([W_{n+2}](x^\sim)) \rangle - \langle p^\sim, [W_{n+1}](L_A(\tau_n)(x^\sim)) \rangle =
\]

\[
  = \langle J_{x,y}^p \otimes l^\sim, W_{n+2} \rangle - \langle I_{x,y}^p \otimes I^\sim, W_{n+1} \rangle =
\]

\[
  = \langle \theta(W), G_{x,y,z_1,\ldots,z_n}^{p,q_1,\ldots,q_n} \rangle = \langle \xi(s), G_{x,y,z_1,\ldots,z_n}^{p,q_1,\ldots,q_n} \rangle.
\]

**Proof of Theorem.** Let \( s : L_A \to L_B \) be a functor isomorphism. Then \( s_1 : A \to B \) is an isomorphism of vector spaces. Put \( m = \dim A = \dim B \). Choose bases in \( A \) and \( B \). Let the tensors \( E, F \) and \( H \) be as in \( \S \, 2 \). We seek an algebra homomorphism \( u : A \to B \) with \( \det u \neq 0 \). Assume that there exists no such a homomorphism. Then, by (3) and Lemma 1, \( \mathcal{F} | Z(M) = 0 \). By Hilbert’s Nullstelleinsatz, \( \mathcal{F} \in M \) for some \( r \in \mathbb{N} \). By (6) and Lemma 2, \( s_{mr}(E^{\otimes r}) = \langle \xi(s), \mathcal{F} \rangle F^{\otimes r} = 0 \). This is absurd since \( E^{\otimes r} \neq 0 \) and \( s_{mr} \) is an isomorphism of vector spaces.

**Reference**

[1] Hochschild homology, English Wikipedia entry.

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