Semiparametric posterior limits
under local asymptotic exponentiality

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Abstract

Consider semiparametric models that display local asymptotic exponentiality (Ibragimov and Has’minskii (1981) [18]), an asymptotic property of the likelihood associated with discontinuities of densities. Our interest goes to estimation of the location of such discontinuities while other aspects of the density form a nuisance parameter. It is shown that under certain conditions on model and prior, the posterior distribution displays Bernstein–von Mises-type asymptotic behaviour, with exponential distributions as the limiting sequence. In contrast to regular settings, the maximum likelihood estimator is inefficient under this form of irregularity. However, Bayesian point estimators based on the limiting posterior distribution attain the minimax risk. Therefore, the limiting behaviour of the posterior is used to advocate efficiency of Bayesian point estimation rather than compare it to frequentist estimation procedures based on the maximum likelihood estimator. Results are applied to semiparametric LAE location and scaling examples.

Keywords: Asymptotic posterior exponentiality; Posterior limit distribution; Local asymptotic exponentiality; Semiparametric statistics; Irregular estimation; Bernstein–von Mises; Densities with jumps.

1 Introduction

In recent years, asymptotic efficiency of Bayesian semiparametric methods has enjoyed much attention. The general question concerns a non-parametric model $\mathcal{P}$ in which exclusive interest goes to the estimation of a sufficiently smooth, finite-dimensional functional of interest. Asymptotically, regularity of the estimator combined with the Cramér-Rao bound in the Gaussian location model that forms the limit experiment [30] fixes the rate of convergence to $n^{-1/2}$ and poses a bound to the accuracy of regular estimators expressed, e.g. through Hajék’s convolution [14] and asymptotic minimax theorems [15]. In regular Bayesian context, efficiency of estimation is best captured by a so-called Bernstein–von Mises limit (see, e.g. Le Cam and Yang (1990) [33]). It should be noted here that efficiency of Bayesian estimation in regular models is closely related to asymptotic normality. Since the limit is Gaussian, hence symmetric and unimodal, the location of the limit, which is any best-regular estimator sequence, is directly linked to Bayesian point estimators for bowl-shaped loss functions.
Just like frequentist parametric theory for regular estimates extends quite effortlessly to regular semi-parametric problems, semi-parametric extensions of Bernstein–von Mises-type asymptotic behaviour of posteriors proceeds without essential problems. Although far from developed fully, some general considerations of Bayesian semiparametric efficiency are found in [1, 4, 8, 36, 38] (model- and/or prior-specific derivations of the Bernstein–von Mises limit are many, e.g. [3, 5, 6, 19, 20, 26, 27, 28] (of which most are formulated in (the conjugacy class of) Gaussian white-noise with Gaussian priors)). Limits of posteriors on sieves are considered in Ghosal (1999, 2000) [11, 12] and Bontemps (2011) [2]. Kim and Lee (2004) [21], Kim (2006, 2009) [22, 23] and, more recently, Castillo and Nickl (2013) [7] even consider infinite-dimensional limiting posteriors (notwithstanding the objections raised in Freedman (1999) [10]).

However, not all estimators are regular. The quintessential example calls for estimation of a point of discontinuity of a density: to be a bit more specific, consider an almost-everywhere differentiable Lebesgue density on \( \mathbb{R} \) that displays a jump at some point \( \theta \in \mathbb{R} \); estimators for \( \theta \) exist that converge at rate \( n^{-1} \) with exponential limit distributions [13]. To illustrate the form that this conclusion takes in Bayesian context, consider the following example. For \( \theta \in \mathbb{R} \), let \( F_\theta(x) = (1 - e^{-\lambda(x-\theta)}) \vee 0 \), where \( \lambda > 0 \) is fixed and known. Let \( X_1, X_2, \ldots \) form an i.i.d. sample from \( F_{\theta_0} \), for some \( \theta_0 \). It is easy to see that the maximum likelihood estimator \( \hat{\theta}_n \) is equal to the minimum of the sample \( X_{(1)} \). Moreover, \( n(X_{(1)} - \theta_0) \) is exponentially distributed with rate \( \lambda \) for every \( n \geq 1 \). Therefore, the maximum likelihood estimator is consistent and asymptotically unbiased with the bias equal to \( 1/ (n\lambda) \). However,

\[
P_0 \left( n(X_{(1)} - \theta_0) \right)^2 = \frac{2}{\lambda^2}, \quad P_0 \left( n \left( X_{(1)} - \frac{1}{n\lambda} - \theta_0 \right) \right)^2 = \frac{1}{\lambda^2},
\]

where \( P_0 f \) denotes the expectation of a random variable \( f \) under \( \theta_0 \). Therefore, the maximum likelihood estimator is inefficient.

On the other hand consider the following theorem.

**Theorem 1.1.** Assume that \( X_1, X_2, \ldots \) form an i.i.d. sample from \( F_{\theta_0} \), for some \( \theta_0 \). Let \( \pi : \mathbb{R} \to (0, \infty) \) be a continuous Lebesgue probability density. Then the associated posterior distribution satisfies,

\[
\sup_A \left| \Pi_n( \theta \in A \mid X_1, \ldots, X_n ) - \text{Exp}_{\frac{1}{n\lambda}}(A) \right| \xrightarrow{\theta_0} 0,
\]

where \( \text{Exp}_{\frac{1}{n\lambda}} \) is a negative exponential distribution with rate \( n\lambda \) supported on \((-\infty, X_{(1)}]\).

The proof of this Bernstein–von Mises limit is elementary and does not depend in any crucial way on the particular parametric family of distributions that we chose (c.f. also the proof of Theorem [1.3]).

Consider now the mean \( \tilde{\theta}_n = X_{(1)} - 1/(n\lambda) \) of the limiting exponential distribution. As seen in [11] its squared risk is smaller than the risk of the maximum likelihood estimator. As a matter of fact, \( 1/\lambda^2 \) is the lower bound for the (localized) risk in the exponential experiment. This suggests that Bayesian point estimators based on the posterior distribution for a wide class of loss functions will be asymptotically minimax.
As a frequentist semi-parametric problem, estimation of a support boundary point is a well-understood problem (see Ibragimov and Has’minskii (1981) [18]): assuming that the distribution \( P_{\theta} \) of \( X \) is supported on the half-line \( \theta, \infty \) and an i.i.d. sample \( X_1, X_2, \ldots, X_n \) is given, we follow [18] and estimate \( \theta \) with the first order statistic \( X^{(1)} = \min_i \{X_i\} \).

If \( P_{\theta} \) has an absolutely continuous Lebesgue density of the form \( p_{\theta}(x) = \eta(x - \theta) \mathbf{1}\{x \geq \theta\} \), its rate of convergence is determined by the behaviour of the quantity \( \epsilon \mapsto \int_0^\epsilon \eta(x) \, dx \) for small values of \( \epsilon \). If \( \eta(x) > 0 \) for \( x \) in a right neighbourhood of 0, then,

\[
n \big( X^{(1)} - \theta \big) = O_{P_{\theta}}(1).
\]

For densities of this form, for any sequence \( \theta_n \) that converges to \( \theta \) at rate \( n^{-1} \), Hellinger distances obey (see Theorem VI.1.1 in [18]):

\[
n^{1/2} H(P_{\theta_n}, P_{\theta}) = O(1) \quad (2)
\]

If we substitute the estimators \( \hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n) = X^{(1)} \), uniform tightness of the sequence in the above display signifies rate optimality of the estimator (e.g., Le Cam (1973, 1986) [31, 32]). Regarding asymptotic efficiency beyond rate-optimality, e.g. in the sense of minimal asymptotic variance (or other measures of dispersion of the limit distribution), we have already noticed in [18], in a specific parametric example of shifted exponential distributions, that the (one-sided) limit distributions one obtains for \( X^{(1)} \) can always be improved upon by de-biasing (see Section VI.6, examples 1–3 in [18] and Le Cam (1990) [34]).

As a semi-parametric Bayesian question, the matter of estimating support boundaries is not settled by the above: for the posterior, it is the local limiting behaviour of the likelihood around the point of convergence (see, e.g., Theorems VI.2.1–VI.2.3 in [18]) that determines convergence rather than the behaviour of any particular statistic. The goal of this paper is to shed some light on the behaviour of marginal posteriors for the parameter of interest in semi-parametric, irregular estimation problems, through a study of the Bernstein–von Mises phenomenon. Only the prototypical case of a density of bounded variation, supported on the half-line \( [\theta, \infty) \) or on the interval \( [0, \theta] \), with a jump at \( \theta \), is analysed in detail. We offer a slight abstraction from the prototypical case, by considering the class of models that exhibit a weakly converging expansion of the likelihood called local asymptotic exponentiality (LAE) [18], to be compared with local asymptotic normality [29] in regular problems. Like in the parametric case of Theorem 1.1, this type of asymptotic behaviour of the likelihood is expected to give rise to a (negative-)exponential marginal posterior satisfying the irregular Bernstein–von Mises limit:

\[
\sup_A \left| \Pi_n\left( h \in A \mid X_1, \ldots, X_n \right) - \text{Exp}_0\left( A \right) \right| \xrightarrow{P_0} 0, \quad (3)
\]

where \( h = n(\theta - \theta_0) \) and the random sequence \( \Delta_n \) converges weakly to exponentiality (see Definition 2.1). Like argued already in the parametric case, the limit (3) allows for the asymptotic identification of Bayesian point estimators based on the posterior distribution with the point estimators based on the limiting exponential distribution. The constant \( 1/\gamma_{\theta_0, \eta_0} \)
determines the scale in the limiting exponential distribution and, as such, is related to the asymptotic bound for estimators of $\theta$ (for the quadratic loss the bound is exactly given by the scale). In this paper, we explore general sufficient conditions on model and prior to conclude that the limit (3) obtains.

The main theorem is applied in two semi-parametric LAE example models, one for a shift parameter and one for a scale parameter (compare with the two regular semiparametric questions in Stein (1956) [39]). The former one is an extension of the setting considered in Theorem 1.1 and is closely related to regression problems with one-sided errors, often arising in economics. The later includes a problem of estimation of the scale parameter in the family of uniform distributions $[0, \lambda]$ ($\lambda > 0$).

The paper is structured as follows: in Section 2 we first introduce the notion of local asymptotic exponentiality and then present two semiparametric LAE models satisfying the exponential Bernstein–von Mises property (3) asymptotically. In Section 3 we give the main theorem and a corollary that simplifies the formulation. In Section 4, the proof of the main theorem is built up in several steps, from a particular type of posterior convergence, to an LAE expansion for integrated likelihoods and on to posterior exponentiality of the type described by (3). Section 5 contains the proofs of auxiliary results needed in the proof of the main theorem, as well as verification of the conditions of the simplified corollary for the two models presented in Section 2.

### Notation and conventions

The (frequentist) true distribution of each of the data points in the $i.i.d.$ sample $X_n = (X_1, \ldots, X_n)$ is denoted $P_0$ and assumed to lie in the model $\mathcal{P}$. Associated order statistics are denoted $X_{(1)}, X_{(2)}, \ldots$. The location-scale family associated with the exponential distribution is denoted $\text{Exp}_{\Delta, \lambda}$ and its negative version by $\text{Exp}_{-\Delta, \lambda}$. We localise $\theta$ by introducing $h = n(\theta - \theta_0)$ with inverse $\theta_n(h) = \theta_0 + n^{-1}h$. The expectation of a random variable $f$ with respect to a probability measure $P$ is denoted $P f$; the sample average of $g(X)$ is denoted $P_n g(X) = (1/n) \sum_{i=1}^n g(X_i)$ and $G_n g(X) = n^{1/2}(P_n g(X) - Pg(X))$. If $h_n$ is stochastic, $P_{\theta_n(h_n), \eta} f$ denotes the integral $\int f(\omega) (dP_{\theta_n(h_n), \eta} / dP_0)(\omega) dP_0(\omega)$. The Hellinger distance between $P$ and $P'$ is denoted $H(P, P')$ and induces a metric $d_H$ on the space of nuisance parameters $H$ by $d_H(\eta, \eta') = H(P_{\theta_0, \eta}, P_{\theta_0, \eta'})$, for all $\eta, \eta' \in H$. A prior on (a subset $\Theta$ of) $\mathbb{R}^k$ is said to be thick (at $\theta \in \Theta$) if it is Lebesgue absolutely continuous with a density that is continuous and strictly positive (at $\theta$).

### 2 Local asymptotic exponentiality and estimation of support boundary points

Throughout this paper we consider estimation of a functional $\theta : \mathcal{P} \to \mathbb{R}$ on a nonparametric model $\mathcal{P}$ based on a sample $X_1, X_2, \ldots$, distributed $i.i.d.$ according to some unknown $P_0 \in \mathcal{P}$. We assume that $\mathcal{P}$ is parametrized in terms of a one-dimensional parameter of interest $\theta \in \Theta$.
and a nuisance parameter \( \eta \in H \) so that we can write \( \mathcal{P} = \{ P_{\theta, \eta} : \theta \in \Theta, \eta \in H \} \), and that \( \mathcal{P} \) is dominated by a \( \sigma \)-finite measure on the sample space with densities \( p_{\theta, \eta} \). The set \( \Theta \) is open in \( \mathbb{R} \), and \((H, d_H)\) is an infinite dimensional metric space (to be specified further at later stages). Assuming identifiability, there exist unique \((\theta_0, \eta_0) \in \Theta \times H\) such that \( P_0 = P_{\theta_0, \eta_0} \). Assuming measurability of the map \( (\theta, \eta) \mapsto P_{\theta, \eta} \) and priors \( \Pi_{\Theta} \) on \( \Theta \) and \( \Pi_{H} \) on \( H \), the prior \( \Pi \) is defined as the product prior \( \Pi_{\Theta} \times \Pi_{H} \) on \( \Theta \times H \), lifted to \( \mathcal{P} \). The subsequent sequence of posteriors \( \Pi_n \) takes the form,

\[
\Pi_n(A | X_1, \ldots, X_n) = \int_A \prod_{i=1}^{n} p(X_i) \, d\Pi(P) / \int_{\mathcal{P}} \prod_{i=1}^{n} p(X_i) \, d\Pi(P),
\]

where \( A \) is any measurable model subset.

Throughout most of this paper, the parameter of interest \( \theta \) is represented in localised form, by centering on \( \theta_0 \) and rescaling: \( h = n(\theta - \theta_0) \in \mathbb{R} \). (We also make use of the inverse \( \theta_n(h) = \theta_0 + n^{-1}h \).) The following (irregular) local expansion of the likelihood is due to Ibragimov and Has’minskii (1981) [18].

**Definition 2.1** (Local asymptotic exponentiality). A one-dimensional parametric model \( \theta \mapsto P_{\theta} \) is said to be locally asymptotically exponential (LAE) at \( \theta_0 \in \Theta \) if there exists a sequence of random variables \( (\Delta_n) \) and a positive constant \( \gamma_{\theta_0} \) such that for all \( (h_n) \), \( h_n \to h \),

\[
\prod_{i=1}^{n} \frac{p_{\theta_0 + n^{-1}h_n}(X_i)}{p_{\theta_0}} = \exp(h_n \gamma_{\theta_0} + o_{P_{\theta_0}}(1)) \mathbf{1}_{\{h_n \leq \Delta_n\}},
\]

with \( \Delta_n \) converging weakly to \( \text{Exp}^{+}_{\theta_0, \gamma_{\theta_0}} \).

In many examples, e.g. that of Subsection 2.1, \( \Delta_n \) and its weak limit are independent of \( \theta_0 \). This definition should be viewed as an irregular variation on the one-dimensional version of Le Cam’s local asymptotic normality (LAN) [29], which forms the smoothness requirement in the context of the Bernstein–von Mises theorem (see, e.g. van der Vaart (1998) [40]). Therefore, an LAE expansion is expected to give rise to a one-sided, exponential marginal posterior limit, c.f. [4].

In the main result of the paper we use a slightly stronger version of local asymptotic exponentiality. We say that the model is stochastically LAE if the LAE property holds for every random sequence \( (h_n) \) that is bounded in probability. Therefore, \( h \) in the expansion is also replaced with \( h_n \).

We now turn to two examples of support boundary estimation for which the likelihood displays an LAE expansion. In Subsection 2.1 the parameter of interest is a shift parameter, while in Subsection 2.2 we consider a semiparametric scaling family.

### 2.1 Semiparametric shifts

The so-called location problem is one of the classical problems in statistical inference: let \( X_1, X_2, \ldots \) be i.i.d. real-valued random variables, each with marginal \( F_{\mu} : \mathbb{R} \to [0,1] \), where
\( \mu \in \mathbb{R} \) is the location, i.e. the distribution function \( F_\mu \) is some fixed distribution \( F \) shifted over \( \mu \): \( F_\mu(x) = F(x - \mu) \).

Depending on the nature of \( F \), the corresponding location estimation problem can take various forms: for instance, in case \( F \) possesses a density \( f : \mathbb{R} \to [0, \infty) \) that is symmetric around 0 (and satisfies the regularity condition \( \int (f'/f)^2(x) dF(x) < \infty \)), the location \( \mu \) is estimated at rate \( n^{-1/2} \) (equally well whether we know \( f \) or not \([39]\)). If \( F \) has a support that is contained in a half-line in \( \mathbb{R} \) (i.e. if there is a domain boundary), the problem of estimating the location might become easier, as noticed in the example given in the introduction.

The problem of estimating the boundary of a distribution has important practical motivations, arising in certain auction models, search models, production frontier models, as well as truncated- or censored-regression models. For instance, assume the data \((X_1, Y_1), (X_2, Y_2), \ldots \) are generated by the model

\[ Y_i = f(X_i) + \mu + \epsilon_i, \]

where \( f \) denotes a smooth function satisfying \( f(0) = 0 \), and for simplicity both \( X_i \) and \( Y_i \) are scalars. Moreover, we suppose that the density of the error \( \epsilon_i \), conditional on \( X_i = x \), is supported on \([0, \infty)\). Therefore, the quantities \( f \) and \( \mu \) represent a boundary, and we are interested in a certain aspect of it, namely \( \mu \) itself. For more details and more examples we refer the reader to Hirano and Porter (2003) \([17]\), Chernozhukov and Hong (2004) \([9]\), Hall and van Keilegom (2009) \([16]\).

In this subsection we consider a model of densities with a discontinuity at \( \mu \): we assume that \( p(x) = 0 \) for \( x < \mu \) and \( p(\mu) > 0 \) while \( p : \mathbb{R} \to [0, \infty) \) is continuous at all \( x \geq \mu \).

Observe that an i.i.d. sample \( X_1, X_2, \ldots \) with marginal \( P_0 \). The distribution \( P_0 \) is assumed to have a density of above form, i.e. with unknown location \( \theta \) for a nuisance density \( \eta \) in some space \( H \). Model distributions \( P_{\theta, \eta} \) are then described by densities,

\[ p_{\theta, \eta} : [\theta, \infty) \to [0, \infty) : x \mapsto \eta(x - \theta), \]

for \( \eta \in H \) and \( \theta \in \Theta \subset \mathbb{R} \). As for the family \( H \) of nuisance densities, our interest does not lie in modelling of the tail, we concentrate on specifying the behaviour at the discontinuity. For that reason (and in order to connect with Theorem \([3.1]\)), we impose some conditions on the nuisance space \( H \): assume that \( \eta : [0, \infty) \to [0, \infty) \) is differentiable and that \( \ell(t) = \eta'(t)/\eta(t) + \alpha \) is a bounded continuous function with a limit at infinity. For given \( S > 0 \), let \( \mathcal{L} \) denote the ball of radius \( S \) in the space \((C[0, \infty], \| \cdot \|_\infty)\) of continuous functions from the extended half-line to \( \mathbb{R} \) with uniform norm. An Esscher transform of the form

\[ \eta_i(x) = \frac{e^{-\alpha x + \int_0^x \ell(t) \, dt}}{\int_0^\infty e^{-\alpha y + \int_0^y \ell(t) \, dt} \, dy}, \quad (5) \]

for \( x \geq 0 \), maps \( \mathcal{L} \) to the space \( H \) which we choose to model the nuisance.

Properties of this mapping (c.f. Lemma \([5.1]\)) guarantee that \( H \) consists of functions of bounded variation, hence Theorem V.2.2 in Ibragimov and Has’minskii (1981) \([18]\) confirms that the model exhibits local asymptotic exponentiality in the \( \theta \)-direction for every fixed \( \eta \). In the notation of Definition \([2.1]\) \( \gamma_{\theta_0, \eta} = \eta(0) \), i.e. the size of the discontinuity at zero. Since it is
not difficult to find a prior on a space of bounded continuous functions (see, e.g. Lemma 5.6 below), (Borel) measurability of the Esscher transform as a map between $\mathcal{L}$ and $H$ enables a push-forward prior on $H$.

**Theorem 2.2.** Let $X_1, X_2, \ldots$ be an i.i.d. sample from the location model introduced above with $P_0 = P_{\theta_0, \eta_0}$ for some $\theta_0 \in \Theta$, $\eta_0 \in H$. Endow $\Theta$ with a prior that is thick at $\theta_0$ and $\mathcal{L}$ with a prior $\Pi_\mathcal{L}$ such that $\mathcal{L} \subseteq \text{supp}(\Pi_\mathcal{L})$. Then the marginal posterior for $\theta$ satisfies,

$$\sup_A |\Pi(n(\theta - \theta_0) \in A | X_1, \ldots, X_n) - \text{Exp}_{\Delta_n, \gamma_{\theta_0, \eta_0}}(A)\big| P_0 \to 0, \quad (6)$$

where $\Delta_n$ is exponentially distributed with rate $\gamma_{\theta_0, \eta_0} = \eta_0(0)$.

Details of the proof of Theorem 2.2 can be found in Subsection 5.3

### 2.2 Semiparametric scaling

Another important statistical problem is related to the *scale* or *dispersion* of the probability distribution: let $X_1, X_2, \ldots$ be i.i.d. real-valued random variables, each with marginal $F_\lambda : \mathbb{R} \to [0, 1]$, where $\lambda \in (0, \infty)$ is the *scale*, i.e. the distribution function $F_\lambda$ is some fixed distribution $F$ scaled by $\lambda$: $F_\lambda(x) = F(x/\lambda)$.

Again, depending on the nature of $F$, the corresponding scale estimation problem can take various forms: for instance, in case $F$ possesses a density $f : \mathbb{R} \to [0, \infty)$ with support $\mathbb{R}$ that is absolutely continuous (and satisfies the regularity condition $\int_0^x (1 + t^2)(f'/f)^2(t) \, dt < \infty$), the scale $\lambda$ is estimated at rate $n^{-1/2}$ (equally well whether we know $f$ or not, as conjectured in [39], and studied later in [42] and [35]). If $F$ is supported on $[0, \infty)$ (or $(-\infty, 0]$), the problem can be reparametrized and viewed as a regular location problem. When $F$ has a support that is a closed interval with one non-zero endpoint (i.e. only one point of the support varies with scale), the problem of estimating the scale might become easier. Probably the best known example of this type is estimation of the scale parameter in the family of the uniform distributions $[0, \lambda]$, ($\lambda > 0$).

In this subsection we consider an extension of this uniform example: we assume that $p(x) > 0$ for $x \in [0, \lambda]$ and 0 otherwise while $p : [0, \lambda] \to [0, \infty)$ is continuous at all $x \in (0, \lambda)$. Observed is an i.i.d. sample $X_1, X_2, \ldots$ with marginal $P_0$. The distribution $P_0$ is assumed to have a density of above form, i.e. with unknown scale $\theta$ for a nuisance density $\eta$ in some space $H$. Model distributions $P_{\theta, \eta}$ are then described by densities,

$$p_{\theta, \eta} : [0, \theta] \to [0, \infty) : x \mapsto \frac{1}{\theta} \eta\left(\frac{x}{\theta}\right), \quad (7)$$

for $\eta \in H$ and $\theta \in \Theta \subset (0, \infty)$. Fix $S > 0$ and assume that $\eta : [0, 1] \to [0, \infty)$ is monotone increasing, differentiable and bounded, and that $\dot{\eta}(t) = \eta'(t)/\eta(t) - S$ is a bounded continuous function. For given $S > 0$, let $\mathcal{L}$ denote the ball of radius $S$ in the normed space $(C[0, 1], \|\cdot\|_\infty)$ of continuous functions from the unit interval to $\mathbb{R}$ with uniform norm. The following Esscher transform maps $\mathcal{L}$ to the space $H$ with which we choose to model the nuisance:

$$\eta_\ell(x) = \frac{e^{Sx + \int_0^x \dot{\ell}(t) \, dt}}{\int_0^1 e^{Sy + \int_0^y \dot{\ell}(t) \, dt} \, dy}, \quad (8)$$
Theorem V.2.2 in [18] verifies local asymptotic exponentiality in the $\theta$-direction for every fixed $\eta$, although in its positive version. This does not pose problems in applying results of this paper: we maintain the sign for $h$ and write $\Delta_n = -\nabla_n$, where $\nabla_n = n(\theta_0 - X(n))$. In the notation of Definition 2.1, $\gamma_{\theta_0,\eta} = \eta(1)/\theta_0$, i.e. the rate of the limiting exponential distribution is the size of the discontinuity at the varying endpoint of the support. Again, we use a push-forward prior on $H$ based on a prior for $L$.

As already noted, our scaling and location problems are both LAE and the parametrizations and solutions we formulate are closely related. However, the nuisance parametrizations are quite different and the relation between the models is a subtle one. Therefore the location theorem of the previous subsection and the scaling theorem that follows are very similar in appearance, but form the answers to quite distinct questions.

**Theorem 2.3.** Let $X_1, X_2, \ldots$ be an i.i.d. sample from the scale model introduced above with $P_0 = P_{\theta_0,\eta_0}$ for some $\theta_0 \in \Theta$, $\eta_0 \in H$. Endow $\Theta$ with a prior that is thick at $\theta_0$, and $L$ with a prior $\Pi_L$ such that $L \subseteq \text{supp}(\Pi_L)$. Then the marginal posterior for $\theta$ satisfies,

$$\sup_A \left| \Pi(n(\theta - \theta_0) \in A \mid X_1, \ldots, X_n) - \exp\left[ -\nabla_n(\gamma_{\theta_0,\eta_0})(A) \right] \right| \xrightarrow{P_0} 0,$$

where $\nabla_n$ is exponentially distributed with rate $\gamma_{\theta_0,\eta_0} = \eta_0(1)/\theta_0$.

Details of the proof of Theorem 2.3 can be found in Subsection 5.4.

## 3 General results

In order to establish the limit (9) (also (6) and (9)), we study posterior convergence of a particular type, termed *consistency under perturbation* in [1]. One can compare this type of consistency with ordinary posterior consistency in non-parametric models, except here the non-parametric component is the nuisance parameter $\eta$ and we allow for (stochastic) perturbation by (local) deformations of the parameter of interest $\theta_n(h_n) = \theta_0 + n^{-1}h_n$. In regular situations, this gives rise to accumulation of posterior mass around so-called least-favourable submodels, but here the parameter of interest is irregular and the situation is less involved: accumulation of posterior mass occurs around $(\theta_n(h_n), \eta_0)$. Therefore, posterior consistency under perturbation describes concentration in $d_H$-neighbourhoods of the form, $(\rho > 0)$,

$$D(\rho) = \{ \eta \in H : d_H(\eta, \eta_0) < \rho \}.$$  

To guarantee sufficiency of prior mass around the point of convergence, we use Kullback–Leibler-type neighbourhoods of the form,

$$K_n(\rho, M) = \left\{ \eta \in H : P_0\left( \sup_{|h| \leq M} -1_{A_{\theta_0,\eta_0}} \log \frac{p_{\theta_n(h),\eta}}{p_{\theta_0,\eta_0}} \right) \leq \rho^2, \quad P_0\left( \sup_{|h| \leq M} -1_{A_{\theta_0,\eta_0}} \log \frac{p_{\theta_n(h),\eta}}{p_{\theta_0,\eta_0}} \right)^2 \leq \rho^2 \right\},$$

where $A_{\theta_0,\eta_0}$. For $x \in [0, 1]$.
where, in the present LAE setting,

$$A_{\theta_n(h),\eta} = \left\{ x : \frac{p_{\theta_n(h),\eta}}{p_{\theta_0,\eta_0}}(x) > 0 \right\}.$$ 

Note that $$\prod_{i=1}^n 1_{A_{\theta_n(h),\eta}}(X_i) = 1_{\{h \leq \Delta_n\}}$$, as in the LAE expansion.

Suppose that $$A$$ in (4) is of the form $$A = B \times H$$ for some measurable $$B \subset \Theta$$. Since we use a product prior $$\prod_{\Theta} \times \prod_{H}$$, the marginal posterior of the parameter $$\theta \in \Theta$$ depends on the nuisance factor only through the integrated likelihood,

$$S_n : \Theta \rightarrow \mathbb{R} : \theta \mapsto \int H \prod_{i=1}^n \frac{p_{\theta,\eta}}{p_{\theta_0,\eta_0}}(X_i) d\Pi_H(\eta),$$

and its localised version, $$h \mapsto s_n(h) = S_n(\theta_0 + n^{-1}h)$$. One of the conditions of the subsequent theorem is a domination condition based on the quantities,

$$U_n(\rho, h_n) = \sup_{\eta \in D(\rho)} P_{\theta_0,\eta}^n \left( \prod_{i=1}^n \frac{p_{\theta_n(h_n),\eta}}{p_{\theta_0,\eta}}(X_i) \right),$$

Another condition required in the irregular version of the semiparametric Bernstein–von Mises theorem is one-sided contiguity (c.f. condition (iv) of Theorem 3.1 below). Lemma 4.2 shows that such one-sided contiguity and domination as in (13) are closely related and provides two different sufficient conditions for both to hold in general. The log-Lipschitz construction is used in the examples of Section 2; in other applications of the theorem it may be more convenient to by-pass Lemma 4.2 and prove (13) and contiguity directly from the model definition.

**Theorem 3.1** (Irregular semiparametric Bernstein–von Mises). Let $$X_1, X_2, \ldots$$ be distributed i.i.d.-$$P_0$$, with $$P_0 \in P$$. Let $$\Pi_H$$ and $$\Pi_{\Theta}$$ be priors on $$H$$ and $$\Theta$$ and assume that $$\Pi_{\Theta}$$ is thick at $$\theta_0$$. Suppose that $$\theta \mapsto P_{\theta,\eta}$$ is stochastically LAE in the $$\theta$$-direction, for all $$\eta$$ in a $$d_H$$-neighbourhood of $$\eta_0$$ and that $$\gamma_{\theta_0,\eta_0} > 0$$. Assume also that for large enough $$n$$, the map $$h \mapsto s_n(h)$$ is continuous on $$(-\infty, \Delta_n]$$, $$P_0^n$$-almost-surely. Furthermore, assume that there exists a sequence $$(\rho_n)$$ with $$\rho_n \downarrow 0$$, $$n\rho_n^2 \rightarrow \infty$$ such that,

(i) for all $$M > 0$$, there exists a $$K > 0$$ such that for large enough $$n$$,

$$\Pi_H(K_n(\rho_n, M)) \geq e^{-Kn\rho_n^2},$$

(ii) for all $$n$$ large enough, the Hellinger metric entropy satisfies,

$$N(\rho_n, H, d_H) \leq e^{n\rho_n^2},$$

and, for every bounded, stochastic $$(h_n)$$,

(iii) the model satisfies the domination condition,

$$U_n(\rho_n, h_n) = O(1),$$

(13)
(iv) for every $\eta \in D(\rho)$ for $\rho > 0$ small enough, the sequence $P^n_{h_n,\eta}$ is contiguous with respect to the sequence $P^n_{0,\eta}$.

(v) and for all $L > 0$, Hellinger distances satisfy the uniform bound,

$$\sup_{\eta \in D(L\rho)} \frac{H(P_{h_n,\eta}, P_{\theta_0,\eta})}{H(P_{0,\eta}, P_0)} = o(1).$$

Finally, suppose that,

(vi) for every $(M_n)$, $M_n \to \infty$, the posterior satisfies

$$\Pi_n(|h| \leq M_n|X_1,\ldots,X_n) \xrightarrow{P_0} 1.$$ 

Then the sequence of marginal posteriors for $\theta$ converges in total variation to a negative exponential distribution,

$$\sup_A \left| \Pi_n(h \in A|X_1,\ldots,X_n) - \text{Exp}_{\Delta_n,\gamma_{\theta_0,\eta_0}}(A) \right| \xrightarrow{P_0} 0. \quad (14)$$

Regarding the nuisance rate of convergence $\rho_n$, conditions (i) and (ii) are expected in some form or other in order to achieve consistency under perturbation. As stated, they almost coincide with requirements for non-parametric convergence at rate $(\rho_n)$ without a parameter of interest [13]. A simplified version of Theorem 3.1 that does not refer to any specific nuisance $\rho_n$ is stated as Corollary 3.1. In the rate-free case of Corollary 3.1, conditions on prior mass and entropy numbers ((i) and (ii)) essentially require nuisance consistency (at some rate rather than a specific one), thus weakening requirements on model and prior. Concerning conditions (iii)–(v), note that, typically, the numerator in condition (v) converges to zero at rate $O(n^{-1/2})$, c.f. (2), while the denominator goes to zero at slower, non-parametric rate. As such, condition (v) is to be viewed as a weak condition that rarely poses a true restriction on the applicability of the theorem. Furthermore, Lemma 4.2 formulates two slightly stronger conditions to validate both (iii) and (iv) above for any rate $(\rho_n)$.

Condition (vi) of Theorem 3.1 appears to be the hardest to verify in applications. On the other hand it cannot be weakened since (vi) also follows from (14). Besides condition (i), only condition (vi) implies a requirement on the nuisance prior $\Pi_H$. Experience with the examples of Section 2 suggests that conditions (i)–(v) are relatively weak in applications, while (vi) harbours the potential for negative surprises, mainly due to semiparametric bias leading to sub-optimal asymptotic variance, sub-optimal marginal rate or even marginal inconsistency. On the other hand, there are conditions under which condition (vi) is easily seen to be valid: in Section 4.3 we present a model condition that guarantees marginal posterior convergence according to (vi) for any choice of the nuisance prior $\Pi_H$.

As discussed already after Theorem 3.1 in many situations the domination condition holds for any rate $(\rho_n)$. This circumstance simplifies the result substantially, leading to the conditions that are comparable to those of Schwartz’ consistency theorem (see Schwartz (1965) [37]).
Corollary 3.1 (Rate-free irregular semiparametric Bernstein–von Mises). Let $X_1, X_2, \ldots$ be distributed i.i.d. $P_0$, with $P_0 \in \mathcal{P}$ and let $\Pi_\Theta$ be thick at $\theta_0$. Suppose that $\theta \mapsto P_{\theta, \eta}$ is stochastically LAE in the $\theta$-direction, for all $\eta$ in a $d_H$-neighbourhood of $\eta_0$ and that $\gamma_{\theta_0, \eta_0}$ is strictly positive. Also assume that for large enough $n$, the map $h \mapsto s_n(h)$ is continuous on $(-\infty, \Delta_n] P_0^n$-almost-surely. Furthermore, assume that,

(i) for all $\rho > 0$, the Hellinger metric entropy satisfies $N(\rho, H, d_H) < \infty$, and the nuisance prior satisfies $\Pi_H(K(\rho)) > 0$,

(ii) for every $M > 0$, there exists an $L > 0$ such that for all $\rho > 0$ and large enough $n$ $K(\rho) \subset K_n(L\rho, M)$,

and that for every bounded, stochastic $(h_n)$,

(iii) there exists an $r > 0$ such that $U_n(r, h_n) = O(1)$,

(iv) for every $\eta \in D(r)$ the sequence $P^n_{\theta_n(h_n), \eta}$ is contiguous to the sequence $P^n_{\theta_0, \eta}$,

(v) and that Hellinger distances satisfy, $\sup_{\eta \in H} H(P_{\theta_n(h_n), \eta}, P_{\theta_0, \eta}) = O(n^{-1/2})$.

Finally, assume that,

(vi) for every $(M_n)$, $M_n \to \infty$, the posterior satisfies,

$$\Pi_n(\{|h| \leq M_n|X_1, \ldots, X_n\}) \xrightarrow{n} 1.$$ 

Then marginal posteriors for $\theta$ converge in total variation to a negative exponential distribution,

$$\sup_A \left| \Pi_n(h \in A|X_1, \ldots, X_n) - \text{Exp}_{\Delta_n, \gamma_{\theta_0, \eta}}(A) \right| \xrightarrow{P_0^n} 0.$$ 

**Proof** Under conditions (i), (ii), (v), and the stochastic LAE assumption, the assertion of Corollary 4.1 holds. Due to conditions (iii) (and (iv)), conditions (iii) (respectively (iv)) in Theorem 3.1 are satisfied for large enough $n$. Condition (vi) then suffices for the assertion of Theorem 4.3.

4 Asymptotic posterior exponentiality

In this section we give the proof of Theorem 3.1 in several steps: the first step (Subsection 4.1) is a proof of consistency under perturbation under a condition on the nuisance prior $\Pi_H$ and a testing condition. In Subsection 4.2 we show that the integral of the likelihood with respect to the nuisance prior displays an LAE-expansion, if consistency under perturbation obtains and contiguity/domination conditions are satisfied. In the third step, also discussed in Subsection 4.2 we show that an LAE-expansion of the integrated likelihood gives rise to a semiparametric exponential limit for the posterior in total variation, if the marginal posterior for the parameter of interest converges at $n^{-1}$-rate. The rate of marginal convergence depends on the control of likelihood ratios, which is discussed in Subsection 4.3. Put together, the results constitute a proof of Theorem 3.1. Stated conditions are verified in Section 5 for the two examples of Section 2.
4.1 Posterior convergence under perturbation

Given a rate sequence \((\rho_n), \rho_n \downarrow 0\), we say that the conditioned nuisance posterior is **consistent under \(n^{-1}\)-perturbation at rate \(\rho_n\)**, if, for all bounded, stochastic sequences \((h_n)\),

\[
\Pi_n \left( D^c(\rho) \mid \theta = \theta_0 + n^{-1}h_n, X_1, \ldots, X_n \right) \xrightarrow{P_0} 0,
\]

For a more elaborate discussion of this property, the reader is referred to Bickel and Kleijn (2012) [1].

**Theorem 4.1** (Posterior convergence under perturbation). Assume there is a sequence \((\rho_n), \rho_n \downarrow 0, n\rho_n^2 \to \infty\) with the property that for all \(M > 0\) there exist a \(K > 0\) such that,

\[
\Pi_H(K_n(\rho_n, M)) \geq e^{-K\rho_n^2}, \quad N(\rho_n, H, d_H) \leq e^{n\rho_n^2},
\]

for large enough \(n\). Assume also that for all \(L > 0\) and all bounded, stochastic \((h_n)\),

\[
\sup_{\eta \in D^c(L\rho_n)} \frac{H(P_{\theta_n(h_n), \eta}, P_{\theta_0, \eta})}{H(P_{\theta_0, \eta}, P_0)} = o(1).
\]  \((15)\)

Then, for every bounded, stochastic \((h_n)\) there exists an \(L > 0\) such that,

\[
\Pi_n \left( D^c(L\rho_n) \mid \theta = \theta_0 + n^{-1}h_n, X_1, \ldots, X_n \right) = o_{P_0}(1).
\]

The proof of this theorem can be broken down into two separate steps, with the following testing condition in between: for every bounded, stochastic \((h_n)\) and all \(L > 0\) large enough, a test sequence \((\phi_n)\) and constant \(C > 0\) must exist, such that,

\[
P_0^\phi \phi_n \to 0, \quad \sup_{\eta \in D^c(L\rho_n)} P_{\theta_n(h_n), \eta}^n (1 - \phi_n) \leq e^{-CL^2n\rho_n^2},
\]  \((16)\)

for large enough \(n\). According to Lemma 3.2 in [1], the metric entropy condition and “cone condition” \((13)\) suffice for the existence of such a test sequence. While the above testing argument is instrumental in the control of the numerator of \((4)\), the denominator of the posterior is lower-bounded with the help of the following lemma, which adapts Lemma 8.1 in [13] to \(n^{-1}\)-perturbed, irregular setting. The proof of Theorem 4.1 follows then the proof of Theorem 3.1 in [1].

**Lemma 4.1.** Let \((h_n)\) be stochastic and bounded by some \(M > 0\). Then

\[
P_0^n \left( \left\{ \int_H \prod_{i=1}^n \frac{p_{\theta_n(h_n), \eta}(X_i)}{p_0} d\Pi_H(\eta) < e^{-(1+C)n\rho^2} \Pi_H(K_n(\rho, M)) \right\} \cap \{ h_n \leq \Delta_n \} \right) \leq \frac{1}{C^2n\rho^2},
\]

for all \(C > 0, \rho > 0\) and \(n \geq 1\), where \(\theta_n(h_n) = \theta_0 + n^{-1}h_n\).

The proof of this lemma can be found in Section 5.

In many applications, \((\rho_n)\) does not play an explicit role because consistency at some rate is sufficient. The following provides a possible formulation of weakened conditions guaranteeing consistency under perturbation. Corollary [4] is based on the family of Kullback–Leibler
neighbourhoods that would also play a role for marginal posterior consistency of the nuisance
with known $\theta_0$ (as in [13]):

$$K(\rho) = \left\{ \eta \in H : -P_0 \log \frac{p_{\theta_0,\eta}}{p_0} \leq \rho^2, P_0 \left( \log \frac{p_{\theta_0,\eta}}{p_0} \right)^2 \leq \rho^2 \right\},$$

for $\rho > 0$.

**Corollary 4.1.** Assume that for all $\rho > 0$, $N(\rho, H, d_H) < \infty$ and $\Pi_H(K(\rho)) > 0$. Furthermore, assume that for every stochastic, bounded $(h_n)$,

(i) for every $M > 0$, there exists an $L > 0$ such that for all $\rho > 0$ and large enough $n$, $K(\rho) \subset K_n(L\rho, M)$.

(ii) Hellinger distances satisfy $\sup_{\eta \in H} H(P_{\theta_0, h_n}(\eta), P_{\theta_0, \eta}) = O(n^{-1/2})$.

Then there exists a sequence $(\rho_n)$, $\rho_n \downarrow 0$, $n\rho_n^2 \to \infty$, such that the conditional nuisance posterior converges under $n^{-1}$-perturbation at rate $(\rho_n)$.

**Proof** See the proof of Corollary 3.3 in Bickel and Kleijn (2012) [1]. □

4.2 Marginal posterior asymptotic exponentiality

To see how the irregular Bernstein–von Mises assertion (3) arises, we note the following: the marginal posterior density $\pi_n : \Theta \to \mathbb{R}$ for the parameter of interest with respect to the prior $\Pi_\Theta$ is given by,

$$\pi_n(\theta) = \frac{\int_H \prod_{i=1}^n \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}}(X_i) \, d\Pi_H(\eta) / \int_{\Theta} \int_H \prod_{i=1}^n \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}}(X_i) \, d\Pi_H(\eta) \, d\Pi_\Theta(\theta),}{P_0^n}$$

$P_0^n$-almost-surely. This form resembles that of a parametric posterior density on $\Theta$ if one replaces the ordinary, parametric likelihood by the integral of the semiparametric likelihood with respect to the nuisance prior, c.f. $S_n(\theta)$ in (12). If $S_n(\theta)$ displays properties similar to those that lead to posterior asymptotic normality in the smooth parametric case, we may hope that in the irregular, semiparametric setting the classical proof can be largely maintained. More specifically, we shall replace the LAN expansion of the parametric likelihood by a stochastic LAE expansion of the likelihood integrated over the nuisance as in (12). Theorem 4.3 uses this observation to reduce the proof of the main theorem of this paper to a strictly parametric discussion.

In this subsection, we prove marginal posterior asymptotic exponentiality in two parts: first we show that $S_n(\theta)$ satisfies an LAE expansion of its own, and second, we use this to obtain Bernstein–von Mises assertion (3), proceeding along the lines of proofs presented in Le Cam and Yang (1990) [33], Kleijn and van der Vaart (2012) [25] and Kleijn (2003) [24].

We restrict attention to the case in which the model itself is stochastically LAE and the posterior is consistent under $n^{-1}$-perturbation (although other, less stringent formulations are conceivable).
Theorem 4.2 (Integrated Local Asymptotic Exponentiality). Suppose that the model is stochastically locally asymptotically exponential in the $\theta$-direction at all points $(\theta_0, \eta)$, $(\eta \in H)$ and that conditions (iii) and (iv) of Theorem 3.1 are satisfied. Furthermore, assume that model and prior $\Pi_H$ are such that for some rate $(\rho_n)$ and every bounded, stochastic $(h_n)$,

$$
\Pi_n(D^c(\rho_n) \mid \theta = \theta_0 + n^{-1}h_n; X_1, \ldots, X_n) \xrightarrow{P_0} 0.
$$

Then the integral LAE-expansion holds, i.e.,

$$
\int_{H} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_0}(X_i) \, d\Pi_H(\eta) = \int_{H} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_0}(X_i) \, d\Pi_H(\eta) \exp(h_n\gamma_{\theta_0, \eta_0} + o_{P_0}(1))1_{\{h_n \leq \Delta_n\}},
$$

for any stochastic sequence $(h_n) \subset \mathbb{R}$ that is bounded in $P_0$-probability.

The following theorem uses the above integrated LAE expansion in conjunction with a marginal posterior convergence condition to derive the exponential Bernstein–von Mises assertion. Marginal posterior convergence forms the subject of the next subsection.

Theorem 4.3 (Posterior asymptotic exponentiality). Let $\Theta$ be open in $\mathbb{R}$ with thick prior $\Pi_\Theta$. Suppose that for every $n \geq 1$, $h \mapsto s_n(h)$ is continuous on $(-\infty, \Delta_n]$, $P_0$-almost-surely. Assume that for every stochastic sequence $(h_n) \subset \mathbb{R}$ that is bounded in probability,

$$
s_n(h_n) = \exp(h_n\gamma_{\theta_0, \eta_0} + o_{P_0}(1))1_{\{h_n \leq \Delta_n\}},
$$

for some positive constant $\gamma_{\theta_0, \eta_0}$. Suppose that for every $M_n \to \infty$, we have,

$$
\Pi_n(\{|h| \leq M_n \mid X_1, \ldots, X_n\}) \xrightarrow{P_0} 1.
$$

Then the sequence of marginal posteriors for $\theta$ is asymptotically exponential in $P_0$-probability, converging in total variation to a negative exponential distribution,

$$
\sup_A \left| \Pi_n(\{h \in A \mid X_1, \ldots, X_n\}) - \text{Exp}_{-\Delta_n, \gamma_{\theta_0, \eta_0}}(A) \right| \xrightarrow{P_0} 0.
$$

Conditions (iii) and (iv) of Theorem 3.1 are crucial in the derivation of the two theorems presented above. In the following lemma we present two sufficient conditions for both the domination and the one-sided contiguity condition to hold. The first method poses the domination condition in slightly stronger form (see “$q$-domination” below); the second relies on a log-Lipschitz condition for model densities and uniform finiteness of exponential moments of the Lipschitz constant.

Lemma 4.2. Suppose that the model satisfies at least one of the following two conditions:

(i) (“$q$-domination” condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(ii) (Uniform $\gamma$-contiguity)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(iii) (Uniform Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(iv) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(v) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(vi) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(vii) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(viii) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(ix) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(x) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,

$$
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_0, \eta}(h_n)}{p_{\theta_0, \eta}}(X_i) \right)^q = O(1),
$$

(xi) (Uniform $\gamma$-contiguity and Lipschitz condition)

for every bounded, stochastic $(h_n)$, small enough $\rho > 0$, and some $q > 1$,
(ii) *(log-Lipschitz condition)*

or, for all \( \eta \in H \) there exists a measurable \( m_{\theta_0, \eta} > 0 \) such that for every \( x \in \mathcal{A}_{\theta_0, \eta} \) and for every \( \theta \) in a neighbourhood of \( \theta_0 \),

\[
\frac{p_{\theta, \eta}(x)}{p_{\theta_0, \eta}(x)} \leq e^{m_{\theta_0, \eta}(x) |\theta - \theta_0|},
\]

and for small enough \( \rho > 0 \) and all \( K > 0 \), \( \sup_{\eta \in D(\rho)} P_{\theta_0, \eta} e^{Km_{\theta_0, \eta}} < \infty \).

Then, for fixed \( \rho > 0 \) small enough,

(i) the model satisfies the domination condition

\[
\sup_{\eta \in D(\rho)} P^n_{\theta_0, \eta} \left( \prod_{i=1}^{n} \frac{p_{\theta_n(h_n), \eta}(X_i)}{p_{\theta_0, \eta}(X_i)} \right) = O(1),
\]

(ii) and, for every \( \eta \in D(\rho) \), the \( (P^n_{\theta_n(h_n), \eta}) \) is contiguous with respect to the \( (P^n_{\theta_0, \eta}) \).

The log-Lipschitz version of this lemma is used in both examples of Section 2 to satisfy conditions (iii) and (iv) of Theorem 3.1.

### 4.3 Marginal posterior convergence at \( n^{-1} \)-rate

One of the conditions in the main theorem is marginal consistency at rate \( n^{-1} \), so that the posterior measure of a sequence of model subsets of the form

\[
\Theta_n \times H = \{ (\theta, \eta) \in \Theta \times H : n|\theta - \theta_0| \leq M_n \},
\]

converge to one in \( P_0 \)-probability, for every sequence \( (M_n) \) such that \( M_n \to \infty \). Marginal (semiparametric) posteriors have not been studied extensively or systematically in the literature. As a result fundamental questions (e.g. semiparametric bias) concerning marginal posterior consistency have not yet received the attention they deserve. Here, we present a straightforward formulation of sufficient conditions, based solely on bounded likelihood ratios. This has the advantage of leaving the nuisance prior completely unrestricted but may prove to be too stringent a condition on the model in some applications. Conceivably \[6\], the nuisance prior has a much more significant role to play in questions on marginal consistency. The inadequacy of Lemma 4.3 manifests itself primarily through the occurrence of a supremum over the nuisance space \( H \) in condition (22), a uniformity that is too coarse. It can be refined somewhat by requiring uniform bound on the likelihood ratios on a sequence of model subsets, capturing the most of the full nonparametric posterior mass. Reservations aside, it appears from the examples of Section 2 that the lemma is also useful in the form stated.

**Lemma 4.3.** Let the sequence of maps \( \theta \mapsto S_n(\theta) \) be \( P_0 \)-almost surely continuous on \( (-\infty, \Delta_n] \) and exhibit the stochastic integral LAE property. Furthermore, assume that there exists a constant \( C > 0 \) such that for any \( (M_n) \), \( M_n \to \infty \), \( M_n \leq n \) for \( n \geq 1 \), and \( M_n = o(n) \),

\[
P_0 \left( \sup_{\eta \in H} \sup_{\theta \in \Theta_n} \frac{P_0^{n, \eta}}{P_0^{\theta_0, \eta}} \log \frac{p_{\theta, \eta}}{p_{\theta_0, \eta}} \leq -\frac{CM_n}{n} \right) \to 1.
\]
Then, for any nuisance prior \( \Pi_H \) and \( \Pi_\Theta \) that is thick at \( \theta_0 \),

\[
\Pi_n \left( n|\theta - \theta_0| > M_n \mid X_1, \ldots, X_n \right) \xrightarrow{P_0} 0,
\]

for any \( (M_n) \), \( M_n \to \infty \).

**Proof** Let us first note, that if marginal consistency holds for a sequence \( M_n \), then it also holds for any sequence \( M'_n \) that diverges faster (\textit{i.e.} if \( M_n = O(M'_n) \)). Without loss of generality, we therefore assume that \( M_n \) diverges more slowly than \( n \), \textit{i.e.} \( M_n = o(n) \). We can also assume \( M_n \leq n \) for \( n \geq 1 \). Define \( F_n \) to be the events in (22) so that \( P_0^n(F_n^c) = o(1) \) by assumption. In addition, let

\[
G_n = \left\{ \left( X_1, \ldots, X_n \right) : \int_\Theta S_n(\theta) \, d\Pi_\Theta(\theta) \geq e^{-CM_n/2} S_n(\theta_0) \right\}.
\]

By Lemma 4.3 \( P_0^n(G_n^c) = o(1) \) as well. Hence,

\[
P_0^n\Pi_n \left( n|\theta - \theta_0| > M_n \mid X_1, \ldots, X_n \right)
\]

\[
\leq P_0^n\Pi_n \left( n|\theta - \theta_0| > M_n \mid X_n \right) \int_{F_n \cap G_n} S_n(\theta_0) \, d\Pi_\Theta(\theta) + o(1)
\]

\[
\leq e^{CM_n/2} P_0^n \left( \frac{1}{S_n(\theta_0)} \int_H \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \, d\Pi_\Theta \, d\Pi_H \, 1_{F_n}(X_n) \right)
\]

\[+ o(1).\]

On the events \( F_n \) we have

\[
\int_H \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \, d\Pi_\Theta \, d\Pi_H
\]

\[
= \int_H \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \int_{\Theta_\eta} \exp \left( n P_n \log \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}} \right) \, d\Pi_\Theta \, d\Pi_H
\]

\[
\leq \int_H \prod_{i=1}^n \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}}(X_i) \, d\Pi_H \sup_{\eta \in H} \sup_{\theta \in \Theta_\eta^c} \exp \left( n P_n \log \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}} \right)
\]

\[
\leq S_n(\theta_0) \exp \left( \sup_{\eta \in H} \sup_{\theta \in \Theta_\eta^c} n P_n \log \frac{P_{\theta_i,\eta}}{P_{\theta_i,\theta_0}} \right),
\]

which ultimately proves marginal consistency at rate \( n^{-1} \). □

In the proof of Lemma 4.3 the lower bound for the denominator of the marginal posterior comes from the following lemma. (Let \( \Pi_n \) denote the prior \( \Pi_\Theta \) in the local parametrization in terms of \( h = n(\theta - \theta_0) \).)

**Lemma 4.4.** Let the sequence of maps \( \theta \mapsto s_n(\theta) \) exhibit the LAE property of (17). Assume that the prior \( \Pi_\Theta \) is thick at \( \theta_0 \) (and denoted by \( \Pi_n \) in the local parametrization in terms of \( h \)). Then

\[
P_0^n \left( \int s_n(h) \, d\Pi_n(h) < a_n s_n(0) \right) \to 0,
\]

for every sequence \( (a_n) \), \( a_n \downarrow 0 \).
5 Proofs

In this section, several longer proofs of theorems and lemmas in the main text have been collected.

5.1 Proof of Lemma 4.1

Proof (of Lemma 4.1)

Let \( C > 0, \rho > 0, \) and \( n \geq 1 \) be given. If \( \Pi_H(K_n(\rho, M)) = 0 \), the assertion holds trivially, so we assume \( \Pi_H(K_n(\rho, M)) > 0 \) without loss of generality and consider the conditional prior \( \Pi_n(A) = \Pi_H(A|K_n(\rho, M)) \) (for measurable \( A \subset H \)). Since,

\[
\int_H \prod_{i=1}^n \frac{p_{\theta_n(h_n), \eta}}{p_0}(X_i) d\Pi_H(\eta) \geq \Pi_H(K_n(\rho, M)) \int_H \prod_{i=1}^n \frac{p_{\theta_n(h_n), \eta}}{p_0}(X_i) d\Pi_n(\eta),
\]

we may choose to consider only the neighbourhoods \( K_n \). Restricting attention to the event \( \{h_n \leq \Delta_n\} \), we obtain,

\[
\log \int \prod_{i=1}^n \frac{p_{\theta_n(h_n), \eta}}{p_0}(X_i) d\Pi_n(\eta) \geq \int n\mathbb{P}_n \log \frac{1_{A_{\theta_n(h_n), \eta}}}{p_0} \frac{p_{\theta_n(h_n), \eta}}{p_0} d\Pi_n(\eta)
\]

\[
\geq \int \inf_{|h| \leq M} n\mathbb{P}_n 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} d\Pi_n(\eta) \geq \int n\mathbb{P}_n \sup_{|h| \leq M} 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} d\Pi_n(\eta)
\]

\[
\geq \sqrt{n} \int -\mathcal{G}_n \left( \sup_{|h| \leq M} 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} \right) d\Pi_n(\eta) - n\rho^2,
\]

using the definition of \( K_n \) in the last step (see (11)). Then,

\[
P^n_0 \left( \left\{ \int \prod_{i=1}^n \frac{p_{\theta_n(h_n), \eta}}{p_0}(X_i) d\Pi_n(\eta) < e^{-(1+C)n\rho^2} \right\} \cap \{h_n \leq \Delta_n\} \right)
\]

\[
\leq P^n_0 \left( \int -\mathcal{G}_n \left( \sup_{|h| \leq M} 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} \right) d\Pi_n(\eta) < -\sqrt{nC}\rho^2 \right).
\]

By Chebyshev’s inequality, Jensen’s inequality, Fubini’s theorem and the fact that for any \( P_0 \)-square-integrable random variables \( Z_n \), \( P^n_0(\mathcal{G}_n Z_n^2) \leq P^n_0 Z_n^2 \),

\[
P^n_0 \left( \int -\mathcal{G}_n \left( \sup_{|h| \leq M} 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} \right) d\Pi_n(\eta) < -\sqrt{nC}\rho^2 \right)
\]

\[
\leq \frac{1}{nC^2\rho^4} \int P^n_0 \left( \mathcal{G}_n \sup_{|h| \leq M} 1_{A_{\theta_n(h), \eta}} \log \frac{p_{\theta_n(h), \eta}}{p_0} \right)^2 d\Pi_n(\eta) \leq \frac{1}{nC^2\rho^4},
\]

where the last step follows again from definition (11). \( \square \)

5.2 Proof of Theorems 4.2 and 4.3, and Lemmas 4.2 and 4.4

Proof (of Theorem 4.2)

Let \( (h_n) \) be bounded in \( P_0 \)-probability. Throughout this proof we write \( \theta_n(h_n) = \theta_0 + n^{-1}h_n \). Let \( \delta, \epsilon > 0 \) be given. There exists a constant \( M > 0 \) such that \( P^n_0(|h_n| > M) < \delta/2 \) for all \( n \geq 1 \). By the consistency assumption, for large enough \( n \),

\[
P^n_0 \left( \log \Pi_n(D(\rho_n) \mid \theta = \theta_n; X_1, \ldots, X_n) \geq -\epsilon \right) > 1 - \frac{\delta}{2}.
\]
This implies that the posterior’s numerator and denominator are related through,

\[
P_0^n \left( \int_H \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) d\Pi_H(\eta) \right) \leq \epsilon^c 1_{|h_n| \leq M} \int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) d\Pi_H(\eta) > 1 - \delta,
\]

for this \( M \) and all \( n \) large enough. We continue with the integral over \( D(\rho_n) \) under the restriction \( |h_n| \leq M \). By stochastic local asymptotic exponentiality for every fixed \( \eta \), we have,

\[
\prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) = \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) \exp(h_n \gamma_{\theta_0, \eta} + R_n(h_n, \eta; X_n)),
\]

where the rest-term \( R_n(h_n, \eta; X_n) \) converges to zero in \( P_{\theta_0, \eta} \)-probability. Define for all \( \epsilon > 0 \) the events,

\[
F_n(\eta, \epsilon) = \left\{ X_n : \sup_{|h| \leq M} |h \gamma_{\theta_0, \eta} - h \gamma_{\theta_0, \eta_0}| \leq \epsilon \right\},
\]

and note that \( F_n^c(0, \epsilon) = \emptyset \). With the domination condition (iii) of Theorem 3.1 Fatou’s lemma yields:

\[
\limsup_{n \to \infty} \int_{D(\rho_n)} P_{\theta_0(\eta), h_n}(F_n^c(\eta, \epsilon)) d\Pi_H(\eta) \leq \int \limsup_{n \to \infty} 1_{D(\rho_n) \setminus \{0\}} F_n^c(\eta, \epsilon) d\Pi_H(\eta) = 0.
\]

Combined with Fubini’s theorem, this suffices to conclude that

\[
\int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) d\Pi_H(\eta) = \int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) 1_{F_n(\eta, \epsilon)}(X_n) d\Pi_H(\eta) + o_{P_0}(1), \quad (23)
\]

and we continue with the first term on the r.h.s. For every \( \eta \in H \), define the events,

\[
G_n(\eta, \epsilon) = \left\{ X_n : \sup_{|h| \leq M} |R_n(h, \eta; X_n)| \leq \epsilon/2 \right\},
\]

and note that \( P_n^{\theta_0, \eta}(G_n^c(\eta, \epsilon)) \to 0 \). By the contiguity condition (iv) of Theorem 3.1 the probabilities \( P_n^{\theta_0(\eta), h_n}(G_n^c(\eta, \epsilon)) \) converge to zero as well. Reasoning as with the events \( F_n(\eta, \epsilon) \), we conclude that,

\[
\int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) 1_{F_n(\eta, \epsilon)}(X_n) d\Pi_H(\eta) = \int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) 1_{G_n(\eta, \epsilon) \cap F_n(\eta, \epsilon)}(X_n) d\Pi_H(\eta) + o_{P_0}(1).
\]

For fixed \( n \) and \( \eta \) and for all \( X_n \in G_n(\eta, \epsilon) \cap F_n(\eta, \epsilon) \), and by stochastic local asymptotic exponentiality,

\[
\left| \log \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) - \log \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) - h_n \gamma_{\theta_0, \eta_0} \right| \leq |R_n(h_n, \eta; X_n)| + |h_n(\gamma_{\theta_0, \eta_0} - \gamma_{\theta_0, \eta})| \leq 2\epsilon,
\]

and by the restriction \( |h_n| \leq M \), the integral \( \int_{D(\rho_n)} \prod_{i=1}^n \frac{p_{\theta_i}(h_n), \eta}{p_0}(X_i) d\Pi_H(\eta) \to 1 \).
from which it follows that,

\[
\exp(h_n \gamma_{\theta_0, \eta_0} - 2\epsilon) \int_{D(\rho_0)} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0} 1_{G_n(\eta, \epsilon) \cap F_n(\eta, \epsilon)}(X_n) \, d\Pi_H(\eta) \\
\leq \int_{D(\rho_0)} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0} 1_{G_n(\eta, \epsilon) \cap F_n(\eta, \epsilon)}(X_n) \, d\Pi_H(\eta) \\
\leq \exp(h_n \gamma_{\theta_0, \eta_0} + 2\epsilon) \int_{D(\rho_0)} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0} 1_{G_n(\eta, \epsilon) \cap F_n(\eta, \epsilon)}(X_n) \, d\Pi_H(\eta).
\]

The integrals can be relieved of indicators for \(G_n \cap F_n\) by reversing preceding arguments (with \(\theta_0\) replacing \(\theta_n\), at the expense of an \(\exp(o(\rho_0(1)))\)-factor, leading to,

\[
\exp(h_n \gamma_{\theta_0, \eta_0} - 3\epsilon + o\rho_0(1)) \int_{H} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0}(X_i) \, d\Pi_H(\eta) \\
\leq \int_{H} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0}(X_i) \, d\Pi_H(\eta) \\
\leq \exp(h_n \gamma_{\theta_0, \eta_0} + 3\epsilon + o\rho_0(1)) \int_{H} \prod_{i=1}^{n} \frac{p_{\theta_0, \eta_0}(X_i)}{p_0}(X_i) \, d\Pi_H(\eta).
\]

for all \(h_n \leq \Delta_n\). Since this holds for arbitrarily small \(\epsilon > 0\), it proves desired result. \(\square\)

**Proof** (of Theorem 4.3)

Let \(C\) be an arbitrary compact subset of \(\mathbb{R}\) containing an open neighbourhood of the origin. Denote the (randomly located) distribution \(\text{Exp}_{\Delta_n, \gamma_{\theta_0, \eta_0}}\) by \(\Xi_n\). The prior and marginal posterior for the local parameter \(h\) are denoted \(\Pi_n\) and \(\Pi_n(\cdot | X_n)\). Conditioned on \(C \subseteq \mathbb{R}\), these measures are denoted \(\Xi_n^C, \Pi_n^C\) and \(\Pi_n^C(\cdot | X_n)\) respectively. Define the functions \(\xi_n^*, \xi_n : \mathbb{R} \to \mathbb{R}\) as,

\[
\xi_n^*(x) = \gamma_{\theta_0, \eta_0} e^{\gamma_{\theta_0, \eta_0}(x - \Delta_n)}, \quad \xi_n(x) = \xi_n^*(x) 1_{\{x \leq \Delta_n\}}.
\]

noting that \(\xi_n\) is the Lebesgue density for \(\Xi_n\). Also define \(s_n^*(h) = s_n(h)\) on \((\infty, \Delta_n]\) and \(s_n^*(h) = s_n(0) \exp(h \gamma_{\theta_0, \eta_0} + d_n)\) elsewhere. Finally, define, for every \(g, h \in C\) and large enough \(n\),

\[
f_n(g, h) = \left(1 - \frac{\xi_n(h) s_n(g) \pi_n(g)}{\xi_n(g) s_n(h) \pi_n(h)}\right) 1_{\{g \leq \Delta_n\}} 1_{\{h \leq \Delta_n\}},
\]

and

\[
f_n^*(g, h) = \left(1 - \frac{\xi_n^*(h) s_n^*(g) \pi_n(g)}{\xi_n^*(g) s_n^*(h) \pi_n(h)}\right) 1_{\{g \leq \Delta_n\}} 1_{\{h \leq \Delta_n\}}.
\]

By (17) we know that \(d_n = \log s_n(\Delta_n) - \log s_n(0) - \Delta_n \gamma_{\theta_0, \eta_0} = o\rho_0(1)\). Furthermore, for every stochastic sequence \((h_n)\) in \(C\),

\[
\log s_n^*(h_n) = \log s_n^*(0) + h_n \gamma_{\theta_0, \eta_0} + o\rho_0(1), \quad \log \xi_n^*(h_n) = (h_n - \Delta_n) \gamma_{\theta_0, \eta_0} + \log \gamma_{\theta_0, \eta_0}.
\]

Since \(\xi_n^*(h)\) and \(\xi_n(h)\) \((s_n^*(h)\) and \(s_n(h)\), respectively\) coincide on \(\{h \leq \Delta_n\}\), \(f_n(g, h) \leq f_n^*(g, h)\). For any two stochastic sequences \((h_n), (g_n)\) in \(C\), \(\pi_n(g_n)/\pi_n(h_n) \to 1\) as \(n \to \infty\) since \(\pi\) is continuous and non-zero at \(\theta_0\). Combination with the above display leads to,

\[
\frac{\xi_n^*(h) s_n^*(g) \pi_n(g)}{\xi_n^*(g) s_n^*(h) \pi_n(h)} = (h_n - \Delta_n) \gamma_{\theta_0, \eta_0} - (g_n - \Delta_n) \gamma_{\theta_0, \eta_0} + g_n \gamma_{\theta_0, \eta_0} - h_n \gamma_{\theta_0, \eta_0} + o\rho_0(1) = o\rho_0(1).
\]
Since \( x \mapsto (1 - e^x) \) is continuous on \((-\infty, \infty)\), we conclude that for any stochastic sequence \((g_n, h_n)\) in \(C \times C\), \(f_n^*(g_n, h_n) \overset{P_0}{\to} 0\). To render this limit uniform over \(C \times C\), continuity is enough: \((g, h) \mapsto \pi_n(g)/\pi_n(h)\) is continuous since the prior is thick. Note that \(\xi_n^*(h)/s_n^*(h)\) is of the form \(\gamma_{\theta_0, \theta_n}(\Delta_n + R_n(h))\) for all \(n\), \(n \geq 1\), and \(R_n(h) = o_{\theta_0}(1)\). Tightness of \(\Delta_n\) and \(R_n\) implies that \(\xi_n^*(h)/s_n^*(h) \in (0, \infty)\), \((P_0^n - a.s.)\). Continuity of \(h \mapsto s_n(h)\) and \(h \mapsto \xi_n^*(h)\) then implies continuity of \((g, h) \mapsto (\xi_n^*(h)s_n^*(g))/(\xi_n^*(g)s_n^*(h))\), \((P_0^n - a.s.)\). Hence we conclude that,

\[
\sup_{(g,h)\in C \times C} f_n(g,h) \leq \sup_{(g,h)\in C \times C} f_n^*(g,h) \overset{P_0}{\to} 0.
\] (24)

Since \(s_n(h)\) is supported on \((-\infty, \Delta_n]\), since \(C\) contains a neighbourhood of the origin and since \(\Delta_n\) is tight and positive, \(\Xi_n(C) > 0\) and \(\Pi_n(C^1_{X_n}) > 0\), \((P_0^n - a.s.)\). So conditioning on \(C\) is well-defined (for the relevant cases where \(h \leq \Delta_n\)). Let \(\delta > 0\) be given and define events,

\[
\Omega_n = \left\{ X_n : \sup_{(g,h)\in C \times C} f_n(g,h) \leq \delta \right\}.
\]

Based on \(\Omega_n\) and (24), write,

\[
P_0^n \sup_A \left| \Pi_n^C(h \in A| X_n) - \Xi_n^C(A) \right| \leq P_0^n \sup_A \left| \Pi_n^C(h \in A| X_n) - \Xi_n^C(A) \right| 1_{\Omega_n} + o(1).
\]

Note that both \(\Xi_n^C\) and \(\Pi_n^C(h| X_n)\) have strictly positive densities on \(C\). Therefore, \(\Xi_n^C\) is dominated by \(\Pi_n^C(h| X_n)\) for all \(n\) large enough. With that observation, the first term on the right-hand side of the above display is calculated to be,

\[
\begin{align*}
\frac{1}{2} P_0^n \sup_A \left| \Pi_n^C(h \in A| X_n) - \Xi_n^C(A) \right| 1_{\Omega_n} (X_n) \\
&= P_0^n \int_C \left(1 - \frac{d\Xi_n^C}{d\Pi_n^C(h| X_n)}\right) + \int_{\{h \leq \Delta_n\}} d\Pi_n^C(h| X_n) 1_{\Omega_n}(X_n) \\
&= P_0^n \int_C \left(1 - \xi_n^C(h) \int_{C} \frac{s_n^*(g)\pi_n(g)}{s_n(h)\pi_n(h)} 1_{\{g \leq \Delta_n\}} dg\right) + \int_{\{h \leq \Delta_n\}} d\Pi_n^C(h| X_n) 1_{\Omega_n}(X_n) \\
&= P_0^n \int_C \left(1 - \int_{C} \frac{s_n(h)\pi_n(h)}{s_n(h)\pi_n(h)\xi_n(h)} 1_{\{g \leq \Delta_n\}} dg\right) + \int_{\{h \leq \Delta_n\}} d\Pi_n^C(h| X_n) 1_{\Omega_n}(X_n),
\end{align*}
\]

for large enough \(n\). Jensen’s inequality leads to

\[
\begin{align*}
\frac{1}{2} P_0^n \sup_A \left| \Pi_n^C(h \in A| X_n) - \Xi_n^C(A) \right| 1_{\Omega_n} (X_n) \\
&\leq P_0^n \int \left(1 - \frac{s_n(h)\pi_n(g)\xi_n(h)}{s_n(h)\pi_n(h)\xi_n(g)}\right) + 1_{\{h \leq \Delta_n\}} \frac{d\Xi_n^C}{d\Pi_n^C(h| X_n)} 1_{\Omega_n}(X_n) \\
&\leq P_0^n \int \sup_{(g,h)\in C \times C} f_n(g,h) d\Xi_n^C(g) d\Pi_n^C(h| X_n) 1_{\Omega_n}(X_n) \leq \delta.
\end{align*}
\]

We conclude that for all compact \(C \subset \mathbb{R}\) containing a neighbourhood of the origin, \(P_0^n ||\Pi_n^C - \Xi_n^C|| \to 0\). To finish the argument, let \((C_m)\) be a sequence of closed balls centred at the origin with radii \(M_m \to \infty\). For each fixed \(m \geq 1\) the above display holds with \(C = C_m\), so if we traverses the sequence \((C_m)\) slowly enough, convergence to zero can still be guaranteed, \(i.e.\) there exist \((M_n)\), \(M_n \to \infty\) such that, \(P_0^n ||\Pi_n^{B_m} - \Xi_n^{B_m}|| \to 0\). Using Lemmas 2.11 and 2.12 in [24] we conclude that (19) holds. □
Proof (of Lemma 4.2)
Assume first that the “$q$-domination” condition is satisfied. Assertion (i) follows from Jensen’s inequality. For the second assertion, fix $\eta \in D(\rho)$ and take a sequence of events $(F_n)$ such that $P_{\theta_0,\eta}^n(F_n) \to 0$. Contiguity now follows from Hölder’s inequality (with $1/p + 1/q = 1$),

$$P_{\theta_0,\eta}^n(F_n) \leq \left( \prod_{i=1}^n \frac{P_{\theta_0,\eta}^n(X_i)}{P_{\theta_0,\eta}^n} \right)^{\frac{q}{q-1}} \leq P_{\theta_0,\eta}(F_n)^{1/p} \to 0.$$ 

Next, assume that the log-Lipschitz condition is satisfied. Let $(\eta_n)$ be a stochastic sequence bounded by $M > 0$. By (21),

$$\prod_{i=1}^n \frac{P_{\theta_0,\eta}^n(X_i)}{P_{\theta_0,\eta}^n} \leq \exp\left( \sum_{i=1}^n m_{\theta_0,\eta}(X_i) \frac{|h_n|}{n} \right) \leq \exp\left( \frac{M}{n} \sum_{i=1}^n m_{\theta_0,\eta}(X_i) \right),$$

for $X_i$ in $A_{\theta_0,\eta}$, which holds with $P_{\theta_0,\eta}$-probability one. Therefore,

$$P_{\theta_0,\eta}^n\left( \prod_{i=1}^n \frac{P_{\theta_0,\eta}^n(X_i)}{P_{\theta_0,\eta}^n} \right) \leq P_{\theta_0,\eta}^n\left( \exp\left( \frac{M}{n} \sum_{i=1}^n m_{\theta_0,\eta}(X_i) \right) \right) \leq P_{\theta_0,\eta}^n \exp(Mm_{\theta_0,\eta}).$$

Due to the uniformity of the assumed bound on $P_{\theta_0,\eta}^n \exp(Km_{\theta_0,\eta})$, this proves (i). For the second assertion, fix $\eta \in D(\rho)$ for some $\rho > 0$ small enough, and take a sequence of events $F_n$ such that $P_{\theta_0,\eta}^n(F_n) \to 0$. Then,

$$P_{\theta_0,\eta}^n(F_n) \leq \int \exp\left( \frac{M}{n} \sum_{i=1}^n m_{\theta_0,\eta}(X_i) \right) 1_{F_n}(X_n) dP_{\theta_0,\eta}^n$$

$$\leq \left( \int \exp\left( \frac{qM}{n} \sum_{i=1}^n m_{\theta_0,\eta}(X_i) \right) dP_{\theta_0,\eta}^n \right)^{1/q} \left( \int 1_{F_n} dP_{\theta_0,\eta}^n \right)^{1/p}$$

$$\leq \left( P_{\theta_0,\eta}^n \exp(qMm_{\theta_0,\eta}) \right)^{1/q} \left( P_{\theta_0,\eta}^n(F_n) \right)^{1/p} \to 0,$$

where we have used Hölder’s inequality (with $1/p + 1/q = 1$) and Jensen’s inequality. The uniform bound on $P_{\theta_0,\eta}^n \exp(Km_{\theta_0,\eta})$ implies that $\left( P_{\theta_0,\eta}^n \exp(qMm_{\theta_0,\eta}) \right)^{1/q}$ is finite for any $\eta \in D(\rho)$ and $q > 1$. \hfill $\square$

Proof (of Lemma 4.4)
Let $M > 0$ be given and define the set $C = \{ h : -M \leq h \leq 0 \}$. Denote the $o_{P_0}(1)$ rest-term in the integral LAE expansion (17) by $h \mapsto R_n(h)$. By continuity of $\theta \mapsto S_\eta(\theta)$, the expansion holds uniformly over compacts for large enough $n$ and in particular, $\sup_{h \in C} |R_n(h)|$ converges to zero in $P_\rho$-probability. Let $(K_n)$, $K_n \to \infty$ be given. The events $B_n = \{ \sup_{C} |R_n(h)| \leq K_n/2 \}$ satisfy $P_0^n(B_n) \to 1$. Since $\Pi_\rho$ is thick at $\theta_0$, there exists a $\pi > 0$ such that $\inf_{h \in C} d\Pi_\eta/dh \geq \pi$, for large enough $n$. Therefore,

$$P_0^n \left( \int_C \frac{s_n(h)}{s_n(0)} d\Pi_\eta(h) \leq e^{-K_n} \right) \leq P_0^n \left( \left\{ \int_C \frac{s_n(h)}{s_n(0)} dh \leq \pi^{-1} e^{-K_n} \right\} \cap B_n \right) + o(1).$$

On $B_n$, the integral LAE expansion is lower bounded so that, for large enough $n$,

$$P_0^n \left( \left\{ \int_C \frac{s_n(h)}{s_n(0)} d\Pi_\eta(h) \leq \pi^{-1} e^{-K_n} \right\} \cap B_n \right) \leq P_0^n \left( \int_C e^{h\gamma_{\theta_0,\eta}} dh \leq \pi^{-1} e^{-\frac{K_n}{\gamma_{\theta_0,\eta}}} \right).$$

Since $\int_C e^{h\gamma_{\theta_0,\eta}} dh \geq M e^{-\gamma_{\theta_0,\eta}}$ and $K_n \to \infty$, $e^{-\frac{K_n}{\gamma_{\theta_0,\eta}}} \leq M e^{-\gamma_{\theta_0,\eta}}$ for large enough $n$. Combination of the above with $K_n = -\log a_n$ proves the desired result. \hfill $\square$
5.3 Proofs of Subsection 2.1

We first present properties of the map defining the nuisance space.

**Lemma 5.1.** Let $\alpha > S$ be fixed. Define $H$ as the image of $\mathcal{L}$ under the map that takes $\hat{\ell} \in \mathcal{L}$ into densities $\eta_i$ defined by (5) for $x \geq 0$. This map is uniform-to-Hellinger continuous and the space $H$ is a collection of probability densities that are (i) monotone decreasing with sub-exponential tails, (ii) continuously differentiable on $[0, \infty)$ and (iii) log-Lipschitz with constant $\alpha + S$.

**Proof** One easily shows that $\hat{\ell} \mapsto \exp(-\alpha x + \int_0^x \hat{\ell})$ is uniform-to-uniform continuous and that $\exp(-\alpha x + \int_0^x \hat{\ell}) > 0$, which implies uniform-to-Hellinger continuity of the Escher transform. For the properties of $\eta_i$, note that $\int_0^x \hat{\ell}(y) dy \leq S x < \alpha x$, so that $x \mapsto \exp(-\alpha x + \int_0^x \hat{\ell}(t) dt)$ is sub-exponential, which implies that $\hat{\ell} \mapsto \eta_i$ gives rise to a probability density. The density $\eta$ is differentiable and monotone decreasing. Furthermore, for all $\theta, \theta_0 \in \Theta$ and all $x \geq \theta_0$,

\[ \frac{\eta(x - \theta)}{\eta_i(x - \theta_0)} \leq \exp\left(\alpha(\theta - \theta_0) + \int_{x-\theta_0}^{x-\theta} \hat{\ell}(t) dt\right) \leq e^{(\alpha+S)|\theta-\theta_0|}, \]

proving the log-Lipschitz property. \( \square \)

The proof of Theorem 2.2 consists of a verification of the conditions of Corollary 3.1. The following lemmas make the most elaborate steps explicit.

**Lemma 5.2.** Hellinger covering numbers for $H$ are finite, i.e. for all $\rho > 0$, $N(\rho, H, d_H) < \infty$.

**Proof** Given $0 < S < \alpha$, we define $\rho_0^2 = \alpha - S > 0$. Consider the distribution $Q$ with Lebesgue density $q > 0$ given by $q(x) = \rho_0^2 e^{-\rho_0^2 x}$ for $x \geq 0$. Then the family $\mathcal{F} = \{x \mapsto \sqrt{\eta_i/q(x)} : \hat{\ell} \in \mathcal{L}\}$ forms a subset of the collection of all monotone functions $\mathbb{R} \mapsto [0, C]$, where $C$ is fixed and depends on $\alpha$ and $S$. Referring to Theorem 2.7.5 in van der Vaart and Wellner (1996) [11], we conclude that the $L_2(Q)$-bracketing entropy $N_1(\epsilon, \mathcal{F}, L_2(Q))$ of $\mathcal{F}$ is finite for all $\epsilon > 0$. Noting that,

\[ d_H(\eta, \eta_0)^2 = d_H(\eta_i, \eta_i_0)^2 = \int_{\mathbb{R}} \left(\sqrt{\eta_i/q(x)} - \sqrt{\eta_i_0/q(x)}\right)^2 dQ(x), \]

it follows that $N(\rho, H, d_H) = N(\rho, \mathcal{F}, L_2(Q)) \leq N_1(2\rho, \mathcal{F}, L_2(Q)) < \infty$. \( \square \)

The following lemma establishes that condition (ii) of Corollary 3.1 is satisfied. Moreover, assuming that the nuisance prior is such that $\mathcal{L} \subset \text{supp}(\Pi_{\mathcal{L}})$, this lemma establishes that $\Pi_H(K(\rho)) > 0$. This, together with the assertion of the previous lemma, verifies condition (i) of Corollary 3.1.

**Lemma 5.3.** For every $M > 0$ there exist constants $L_1, L_2 > 0$ such that for small enough $\rho > 0$, $\{\eta_i \in H : \|\hat{\ell} - \hat{\ell}_0\|_\infty \leq \rho^2\} \subset K(L_1 \rho) \subset K_n(L_2 \rho, M)$.

**Proof** Let $\rho$, $0 < \rho < \rho_0$ and $\hat{\ell} \in \mathcal{L}$ such that $\|\hat{\ell} - \hat{\ell}_0\|_\infty \leq \rho^2$ be given. Then,

\[ \left| \log \frac{p_{\theta_0, \eta}(x)}{p_{\theta_0, \eta_0}(x)} - \int_0^{x-\theta_0} (\hat{\ell} - \hat{\ell}_0)(t) dt \right| \leq \rho^2 P_0(X - \theta_0) + O(\rho^4), \] (25)
for all $x \geq \theta_0$. Define, for all $\alpha > S$ and $\ell \in \mathcal{L}$, the logarithm $z$ of the normalising factor in (25). Then the relevant log-density-ratio can be written as,

$$
\log \frac{p_{\theta_0,\eta}(x)}{p_{\theta_0,\eta_0}} = \int_0^{x-\theta_0} (\ell - \hat{\ell}_0)(t) \, dt - z(\alpha, \hat{\ell}) + z(\alpha, \hat{\ell}_0),
$$

where only the first term is $x$-dependent. Assume that $\hat{\ell} \in \mathcal{L}$ is such that $\|\hat{\ell} - \hat{\ell}_0\|_\infty < \rho^2$. Then, $|\int_0^{x-\theta_0} (\ell - \hat{\ell}_0)(t) \, dt| \leq \rho^2 (y - \theta_0)$, so that $z(\alpha - \rho^2, \hat{\ell}_0) \leq z(\alpha, \hat{\ell}) \leq z(\alpha + \rho^2, \hat{\ell}_0)$. Noting that $d^k z/da^k(\alpha, \hat{\ell}_0) = (-1)^k P_0(\theta - \theta_0)^k < 0$ and using the first-order Taylor expansion of $z$ in $\alpha$, we find, $z(\alpha \pm \rho^2, \hat{\ell}_0) = z(\alpha, \hat{\ell}_0) + \rho^2 P_0(\theta - \theta_0) + O(\rho^4)$, and (25) follows. Next note that, for every $k \geq 1$,

$$
\left| P_0 \left( \int_0^{x-\theta_0} (\ell - \hat{\ell}_0)(t) \, dt \right)^k \right| \leq \rho^{2k} \int_0^{\theta_0} \left( \int_0^{x-\theta_0} dy \right)^k \, dP_0 = \rho^{2k} P_0(\theta - \theta_0)^k,
$$

(26)

Using (25) we bound the differences between KL divergences and integrals of scores as follows:

$$
\left| \log \frac{p_{\theta_0,\eta}(x)}{p_{\theta_0,\eta_0}}(x) - \left( \int_0^{x-\theta_0} (\ell - \hat{\ell}_0)(t) \, dt \right) \right| \leq \rho^2 (P_0(\theta - \theta_0) + O(\rho^2)),
$$

$$
\left| \log \frac{p_{\theta_0,\eta}(x)}{p_{\theta_0,\eta_0}}(x) - \left( \int_0^{x-\theta_0} (\ell - \hat{\ell}_0)(t) \, dt \right)^2 \right| \leq \rho^2 (P_0(\theta - \theta_0) + O(\rho^2))
$$

and, combining with the bounds (25), we see that,

$$
P_0 \left( \log \frac{p_{\theta_0,\eta}(x)}{p_{\theta_0,\eta_0}}(x) \right)^2 \leq \rho^4 (P_0(\theta - \theta_0)^2 + 3P_0(\theta - \theta_0) + O(\rho^2)),
$$

which proves the first inclusion. Let $M > 0$. Note that $A_{\theta,\eta} = [\theta, \infty)$ for every $\eta$, and that

$$
\sup_{|\ell| \leq M} -1 A_{\theta_0(h),\eta} \log \frac{p_{\theta_0,\eta}(h)}{p_{\theta_0,\eta_0}} = \sup_{|\ell| \leq M} 1 A_{\theta_0(h),\eta} \log \frac{p_{\theta_0,\eta}(h)}{p_{\theta_0,\eta_0}} = \sup_{|\ell| \leq M} 1 A_{\theta_0(h),\eta} \log \frac{p_{\theta_0,\eta_0}}{p_{\theta_0,\eta}},
$$

so that,

$$
P_0 \left( \sup_{|\ell| \leq M} -1 A_{\theta_0(h),\eta} \log \frac{p_{\theta_0,\eta}(h)}{p_{\theta_0,\eta_0}} \right) \leq -P_0 \log \frac{p_{\theta_0,\eta_0}}{p_{\theta_0,\eta}} + \frac{(\alpha + S)M}{\eta},
$$

$$
P_0 \left( \sup_{|\ell| \leq M} -1 A_{\theta_0(h),\eta} \log \frac{p_{\theta_0,\eta}(h)}{p_{\theta_0,\eta_0}} \right)^2 \leq P_0 \left( \log \frac{p_{\theta_0,\eta_0}}{p_{\theta_0,\eta}} \right)^2 + \frac{2(\alpha + S)M}{\eta} \left[ P_0 \left( \log \frac{p_{\theta_0,\eta_0}}{p_{\theta_0,\eta}} \right) \right]^{1/2} + \frac{(\alpha + S)^2 M^2}{\eta^2},
$$

implying the existence of a constant $L_2$. □

By Lemma 5.1, the log-Lipschitz constant $m_{\theta_0,\eta}$ of Lemma 4.2 equals $\alpha + S$ for every $\eta \in H$, so that the domination condition (iii) and contiguity requirement (iv) of Corollary 3.1 are satisfied. The following lemma shows that condition (v) of Corollary 3.1 is also satisfied.
Lemma 5.4. For all bounded, stochastic sequences \((h_n)\), Hellinger distances between \(P_{\theta_n(h_n),\eta}\) and \(P_{\theta_0,\eta}\) are of order \(n^{1/2}\) uniformly in \(\eta\), i.e. \(\sup_{\eta \in H} n^{1/2} H(P_{\theta_n(h_n),\eta}, P_{\theta_0,\eta}) = O(1)\).

Proof Fix \(n\) and \(\omega\); write \(h_n\) for \(h_n(\omega)\). First we consider the case that \(h_n \geq 0\), for \(x \geq \theta_0\),

\[
(\eta^{1/2}(x-\theta_n(h_n)) - \eta^{1/2}(x-\theta_0))^2 = \eta(x-\theta_0)1_{[\theta_0,\theta_n(h_n)]}(x) + (\eta^{1/2}(x-\theta_n(h_n)) - \eta^{1/2}(x-\theta_0))^21_{[\theta_n(h_n),\infty)}(x)
\]

To upper bound the second term, we use the absolute continuity of \(\eta^{1/2}\),

\[
|\eta^{1/2}(x-\theta_0) - \eta^{1/2}(x-\theta_n(h_n))| = \frac{1}{2} \left| \int_{x-\theta_0}^{h_n \eta(x-\theta_0)} \eta'/\eta^{1/2}(y)\, dy \right| \leq \frac{1}{2} \int_0^M |\eta'/\eta^{1/2}(z + x - \theta_n(h_n))|\, dz,
\]

and then by Jensen’s inequality,

\[
(\eta^{1/2}(x-\theta_0) - \eta^{1/2}(x-\theta_n(h_n)))^2 \leq \frac{M}{4n} \int_0^M (\eta')^2 (z + x - \theta_n(h_n))\, dz.
\]

Similarly for \(h_n < 0\) and \(x \geq \theta_n(h_n)\),

\[
(\eta^{1/2}(x-\theta_0) - \eta^{1/2}(x-\theta_n(h_n)))^2 \leq \eta(x-\theta_n(h_n))1_{[\theta_n(h_n),\theta_0]}(x) - \eta(x-\theta_n(-M))1_{[\theta_n(-M),\theta_0]}(x)
\]

Combining these results, we obtain a bound for the squared Hellinger distance:

\[
H^2(P_{\theta_n(h_n),\eta}, P_{\theta_0,\eta}) \leq \int_{\theta_0}^{\theta_n(M)} \eta(x-\theta_0)\, dx + \int_{\theta_n(-M)}^{\theta_0} \eta(x-\theta_n(-M))\, dx + \int_{\theta_n(h_n)}^{\theta_0} \eta(x-\theta_n(h_n))\, dx - 1_{[\theta_n(h_n),\theta_0]}(x) - \int_{\theta_n(-M)}^{\theta_0} \eta(x-\theta_n(-M))\, dx
\]

As for the first two terms on the right-hand side of (27), we note the following inequality:

\[
\int_{\theta_0}^{\theta_n(M)} \eta(x-\theta_0)\, dx + \int_{\theta_n(-M)}^{\theta_0} \eta(x-\theta_n(-M))\, dx \leq 2\gamma_{\theta_0,\eta} \frac{M}{n} + \frac{M^2}{n^2} \int_0^\infty |\eta'(y)|\, dy,
\]

by Lemma 5.13. Furthermore, by shifting appropriately, we find that the third and fourth term of (27) satisfy the bound,

\[
1_{[\theta_n(h_n),\theta_0]}(x) \left( \int_{\theta_n(h_n)}^{\theta_0} \eta(x-\theta_n(h_n))\, dx - \int_{\theta_n(-M)}^{\theta_0} \eta(x-\theta_n(-M))\, dx \right)
\]

\[
= 1_{[\theta_n(h_n),\theta_0]} \left( \int_0^{-h_n/m} \eta(y)\, dy - \int_0^M \eta(y)\, dy \right) = -1_{[\theta_n(h_n),\theta_0]} \int_{-h_n/m}^M \eta(y)\, dy \leq 0,
\]

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(where it is noted that the \( h_n \) dependent integral in the above display is well defined for any \( h_n \)). Finally, the fifth and sixth term of (27) are bounded by the Fisher information for location associated with \( \eta \):

\[
\int_0^\infty \int_0^M \frac{(\eta')^2}{\eta} (z + x) \, dz \, dx = \int_0^M \int_\eta^\infty \frac{(\eta')^2}{\eta} (x) \, dx \, dz \leq \frac{M}{n} \int_0^\infty \frac{(\eta')^2}{\eta} (x) \, dx;
\]

Combining, we obtain the following upper bound for the relevant Hellinger distance,

\[
H^2(P_{\theta_n(h_n),\eta}, P_{\theta_0,\eta}) \leq 2\gamma_{\theta_0,\eta} \frac{M}{n} + 2\frac{M^2}{n} \left( \int_0^\infty \frac{|\eta'(x)|}{\eta(x)} |\eta(x)| \, dx + \int_0^\infty \left( \frac{\eta'(x)}{\eta(x)} \right)^2 |\eta(x)| \, dx \right).
\]

which proves the lemma upon noting that \(|\eta'(x)| = |\eta(x)|\ell(x) - \alpha| \leq \eta(x)(\alpha - S)\). □

To verify condition (vi) of Corollary 3.1 we now check condition (22) of Lemma 4.3

**Lemma 5.5.** Let \((M_n), M_n \to \infty, M_n \leq n\) for \( n \geq 1\), \( M_n = o(n) \) be given. Then there exists a constant \( C > 0 \) such that the condition of Lemma 4.3 is satisfied.

**Proof.** Note first that for fixed \( x \) and \( \eta \), the map \( \theta \mapsto p_{\theta,\eta}(x) \) is monotone increasing. Therefore

\[
\sup_{\theta \in \Theta_n} \frac{1}{n} \log \prod_{i=1}^n \frac{p_{\theta,\eta}(X_i)}{p_{\theta_0,\eta}(X_i)} \leq \frac{1}{n} \log \prod_{i=1}^n \eta(X_i - \theta^*) \frac{1}{\eta(X_i - \theta_0)} 1 \{X_i \geq \theta^* \}(X_n),
\]

where \( \theta^* = X(1) \) if \( X(1) \geq \theta_0 + M_n/n \), or \( \theta_0 - M_n/n \) otherwise. We first note that \( X(1) < \theta_0 + M_n/n \) with probability tending to one. Indeed, shifting the distribution to \( \theta = 0 \), we calculate,

\[
P_{0,\theta_0}^n \left( X(1) \geq \frac{M_n}{n} \right) = \left( 1 - \int_0^{\frac{M_n}{n}} \eta_0(x) \, dx \right)^n \leq \exp \left( -n \int_0^{\frac{M_n}{n}} \eta_0(x) \, dx \right).
\]

By Lemma 6.13 the right-hand side of the above display is bounded further as follows,

\[
\exp \left( -\gamma_{\theta_0,\theta_0} M_n + M_n \int_0^{\frac{M_n}{n}} |\eta'_0(x)| \, dx \right) \leq \exp \left( -\frac{\gamma_{\theta_0,\theta_0}}{2} M_n \right),
\]

for large enough \( n \). We continue with \( \theta^* = \theta_0 - M_n/n \). By absolute continuity of \( \eta \) we have

\[
\eta(X_i - \theta^*) = \eta(X_i - \theta_0) + \int_{X_i - \theta_0}^{X_i - \theta^*} \eta'(y) \, dy,
\]

and the conditions on the nuisance \( \eta \) yield the following bound,

\[
\int_{X_i - \theta_0}^{X_i - \theta^*} \eta'(y) \, dy \leq (\theta_0 - \theta^*)(S - \alpha) \eta(X_i - \theta_0).
\]

Therefore

\[
\frac{1}{n} \log \prod_{i=1}^n \frac{\eta(X_i - \theta^*)}{\eta(X_i - \theta_0)} 1 \{X_i \geq \theta^* \}(X_n) \leq \frac{1}{n} \log \left( 1 - \frac{(\alpha - S)M_n}{n} \right)^n \leq \frac{(\alpha - S)M_n}{n}.
\]

If \( C < \alpha - S \), the condition of Lemma 4.3 is clearly satisfied. □

To demonstrate that priors exist such that \( \mathcal{L} \subset \text{supp}(\Pi_{\mathcal{L}}) \), an explicit construction based on the distribution of Brownian sample paths is provided in the following lemma.
Corollary 3.1. Denote by \( \eta \) the \( \sqrt{\varphi} \). The proof is similar to the proof of Lemma 5.1 and is therefore omitted.

The following lemmas make the most elaborate steps explicit, as in the proof of Theorem 2.2.

5.4 Proofs of Subsection 2.2

Again we first present properties of the mapping defining the nuisance space.

Lemma 5.6. Let \( S > 0 \) be given. Let \( \{ W_t : t \in [0, 1] \} \) be Brownian motion on \([0, 1]\) and let \( Z \) be independent and distributed \( N(0, 1) \). We define the prior \( \Pi_{\varphi} \) on \( \mathcal{L} \) as the distribution of the process,

\[
\hat{\ell}(t) = S \Psi(Z + W_{\varphi(t)}),
\]

where \( \Psi : [-\infty, \infty] \to [-1, 1] : x \mapsto 2 \arctan(x)/\pi \). Then \( \mathcal{L} \subset \text{supp}(\Pi_{\varphi}) \).

Proof Consider \( C[0, 1] \) with the uniform norm and its Borel \( \sigma \)-algebra, equipped with the law \( \Pi \) of \( t \mapsto Z + W_t \), as a probability space. Since \( \Psi \) is Lipschitz, the map \( f \) that takes \( C[0, 1] \) into \( C[0, \infty], Z + W \mapsto Z + W_{\varphi(t)} \) is continuous, norm-preserving, and Borel-to-Borel measurable. This enables the view of \( C[0, \infty] \) with its Borel \( \sigma \)-algebra as a probability space, with probability measure \( \Pi'(B) = \Pi(f^{-1}(B)) \). Similarly, the map \( g \) that takes \( C[0, \infty] \) into \( \mathcal{L}, Z + W_{\varphi(t)} \mapsto S \Psi(Z + W_{\varphi(t)}) \) is continuous and Borel-to-Borel measurable. We view \( \mathcal{L} \) with its Borel \( \sigma \)-algebra as a probability space, with probability measure \( \Pi_{\varphi}(C) = \Pi'(g^{-1}(C)) \). Let \( T \) denote a closed set in \( \mathcal{L} \) such that \( \Pi_{\varphi}(T) = 1 \). Note that \( f^{-1}(g^{-1}(T)) \) is closed and \( \Pi(f^{-1}(g^{-1}(T))) = 1 \), so that \( \text{supp}(\Pi) \subset f^{-1}(g^{-1}(T)) \). Since the support of \( \Pi_{\varphi} \) equals the intersection of all such \( T \), \( \text{supp}(\Pi) \subset \bigcap T f^{-1}(g^{-1}(T)) = f^{-1}(g^{-1}(\text{supp}(\Pi_{\varphi}))) \). Since \( \text{supp}(\Pi) = C[0, 1], \) for every \( y \in C[0, 1], f(g(y)) \in \text{supp}(\Pi_{\varphi}) \). The continuity does not change under \( g \circ f \), so \( \text{supp}(\Pi_{\varphi}) \) includes \( \mathcal{L} \). \( \square \)

Lemma 5.7. Define \( H \) as the image of \( \mathcal{L} \) under the map that takes \( \hat{\ell} \in \mathcal{L} \) into densities \( \eta_\ell \) defined by (3) for \( x \in [0, 1] \). This map is uniform-to-Hellinger continuous and the space \( H \) is a collection of probability densities that are (i) monotone increasing and bounded away from zero and infinity and (ii) continuously differentiable on \([0, 1]\). Moreover, the resulting densities \( p_{\theta, \eta} \) satisfy the log-Lipschitz condition (21) in an \( \epsilon \)-neighbourhood \( \epsilon < \theta_0/2 \) with \( m_{\theta, \eta} = (2 + 8\epsilon)/(\theta_0) \).

Proof The proof is similar to the proof of Lemma 5.1 and is therefore omitted. \( \square \)

The proof of Theorem 2.2 consists of a verification of the conditions of Corollary 3.1 (after the aforementioned modification to comply with the positive version of the LAE expansion). The following lemmas make the most elaborate steps explicit, as in the proof of Theorem 2.2.

Lemma 5.8. Hellinger covering numbers for \( H \) are finite, i.e. for all \( \rho > 0, N(\rho, H, d_H) < \infty. \)

Proof Denote by \( Q \) the distribution with density \( \eta_0 = \eta_{\hat{\ell}_0} \). Then the family \( \mathcal{F} = \{ x \mapsto \sqrt{\eta_{\ell}/\eta_0} : \hat{\ell} \in \mathcal{L} \} \) forms a subset of the collection \( C_1^1([0, 1]) \), where \( M \) is fixed and depends on \( S \). Referring to Corollary 2.7.2 in [11], we conclude that the \( L_2(Q) \)-bracketing entropy \( N_q(\epsilon, \mathcal{F}, L_2(Q)) \) of \( \mathcal{F} \) is finite for all \( \epsilon > 0 \). Similarly as in the proof of Lemma 5.8, it follows that \( N(\rho, H, d_H) = N(\rho, \mathcal{F}, L_2(Q)) \leq N_q(2\rho, \mathcal{F}, L_2(Q)) < \infty. \) \( \square \)

The previous lemma together with the following lemma verify conditions (i) and (ii) of Corollary 3.1.
Lemma 5.9. For every $M > 0$ there exist constants $L_1, L_2 > 0$ such that for small enough $\rho > 0$, $\{\eta \in H : \|\hat{\ell} - \hat{\ell}_0\|_{\infty} \leq \rho^2\} \subset K(L_1\rho) \subset K_n(L_2\rho, M)$.

Proof. Let $\rho > 0$ and $\hat{\ell} \in \mathcal{L}$ such that $\|\hat{\ell} - \hat{\ell}_0\|_{\infty} \leq \rho^2$ be given. Then,

$$\left| \log \frac{P_{\theta_0, \eta}(x)}{P_{\theta_0, \eta_0}(x)} - \int_0^{x/\theta_0} (\hat{\ell} - \hat{\ell}_0)(t) \, dt \right| \leq \rho^2 P_0(X/\theta_0) + O(\rho^4), \quad (28)$$

for all $x \in [0, \theta_0]$. Define, for all $\alpha \in \mathbb{R}$ and $\hat{\ell} \in \mathcal{L}$,

$$z(\alpha, \hat{\ell}) = \log \int_0^1 e^{\alpha y + \int_0^y \hat{\ell}(t) \, dt} \, dy.$$ 

Then the relevant log-density-ratio can be written as,

$$\log \frac{P_{\theta_0, \eta}(x)}{P_{\theta_0, \eta_0}(x)} = \int_0^{x/\theta_0} (\hat{\ell} - \hat{\ell}_0)(t) \, dt - z(S, \hat{\ell}) + z(S, \hat{\ell}_0),$$

where only the first term is $x$-dependent. Assume that $\hat{\ell} \in \mathcal{L}$ is such that $\|\hat{\ell} - \hat{\ell}_0\|_{\infty} < \rho^2$. Then, $\int_0^1 (\hat{\ell} - \hat{\ell}_0)(t) \, dt \leq \rho^2 y$, so that $z(S - \rho^2, \hat{\ell}_0) \leq z(S, \hat{\ell}) \leq z(S + \rho^2, \hat{\ell}_0)$. Noting that $d^k z/d\alpha^k(S, \hat{\ell}_0) = P_0(X/\theta_0) < \infty$ and using the first-order Taylor expansion of $z$ in $\alpha$, we find, $z(S \pm \rho^2, \hat{\ell}_0) = (S, \hat{\ell}_0) \pm \rho^2 P_0(X/\theta_0) + O(\rho^4)$, and (28) follows.

Next note that, for every $k \geq 1$,

$$P_0\left( \int_0^{X/\theta_0} (\hat{\ell} - \hat{\ell}_0)(t) \, dt \right)^k \leq \rho^2 k \int_0^{X/\theta_0} \left( \int_0^{x/\theta_0} dy \right)^k \, dP_0 = \rho^2 k P_0(X/\theta_0)^k, \quad (29)$$

Using (28) we bound the differences between KL divergences and integrals of scores and, combining with the bounds (29), we see that,

$$-P_0 \log \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}} \leq 2\rho^2 \left( P_0(X/\theta_0) + O(\rho^2) \right),$$

$$P_0 \left( \log \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}} \right)^2 \leq \rho^4 \left( P_0(X/\theta_0)^2 + 3P_0(X/\theta_0) + O(\rho^2) \right),$$

which proves the first inclusion. Let $M > 0$. Similarly as in the proof of Lemma 5.3 we can show that

$$P_0 \left( \sup_{|h| \leq M} - 1_{A_{\eta_n(h), \eta}} \log \frac{P_{\theta_n(h), \eta}}{P_{\theta_0, \eta_0}} \right) \leq -P_0 \log \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}} + \frac{2 + 8S M}{\theta_0} \frac{M}{n},$$

$$P_0 \left( \sup_{|h| \leq M} - 1_{A_{\eta_n(h), \eta}} \log \frac{P_{\theta_n(h), \eta}}{P_{\theta_0, \eta_0}} \right)^2 \leq P_0 \left( \log \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}} \right)^2 + \frac{4 + 16S M}{\theta_0} \frac{M}{n} \left[ P_0 \left( \log \frac{P_{\theta_0, \eta}}{P_{\theta_0, \eta_0}} \right) \right]^{1/2} + \frac{(2 + 8S)^2 M^2}{\theta_0^2} \frac{M^2}{n^2},$$

implying the existence of a constant $L_2$. \hfill \Box

By Lemma 5.4 the model satisfies the Lipschitz condition of Lemma 4.2 with the same Lipschitz constant for every $\eta \in H$, so that the domination condition (iii) and contiguity requirement (iv) of Corollary 5.1 are satisfied.

Lemma 5.10. For all bounded, stochastic sequences $(h_n)$, Hellinger distances between $P_{\theta_n(h_n), \eta}$ and $P_{\theta_0, \eta}$ are of order $n^{1/2}$ uniformly in $\eta$, i.e. $\sup_{\eta \in H} n^{1/2} H(P_{\theta_n(h_n), \eta}, P_{\theta_0, \eta}) = O(1)$. 27
\textbf{Proof} Note that the elements of the nuisance space $H$ are uniformly bounded by $e^{2S}$. Fix $n$ and $\omega$; write $h_n$ for $h_n(\omega)$. First we consider the case that $h_n \geq 0$,

$$\left( \frac{\eta^{1/2}(x/\theta_n(h_n))}{\theta_n^{1/2}(h_n)} - \frac{\eta^{1/2}(x/\theta_0)}{\theta_0^{1/2}} \right)^2 = \frac{\eta(x/\theta_n(h_n))}{\theta_n(h_n)} 1_{[\theta_0, \theta_n(h_n)]}(x) + \left( \frac{\eta^{1/2}(x/\theta_n(h_n))}{\theta_n^{1/2}(h_n)} - \frac{\eta^{1/2}(x/\theta_0)}{\theta_0^{1/2}} \right)^2 1_{[0, \theta_0]}(x).$$

Note that the first term is bounded from above by $(e^{2S}/\theta_0) 1_{[\theta_0, \theta_n(M)]}(x)$. To upper bound the second term, we use the absolute continuity of $\eta^{1/2}$. Let $g(y) = \eta(x/y)/y^{1/2}$,

$$\left| \frac{\eta^{1/2}(x/\theta_n(h_n))}{\theta_n^{1/2}(h_n)} - \frac{\eta^{1/2}(x/\theta_0)}{\theta_0^{1/2}} \right| = \int_{\theta_0}^{\theta_n(h_n)} g'(y) dy \leq \int_{\theta_0}^{\theta_n(M)} |g'(y)| dy.$$

By the definition of the nuisance space, for $y \in [\theta_0, \theta_n(M)]$, and $x \leq \theta_0$,

$$|g'(y)| \leq \frac{e^{2S}}{\theta_0^{3/2}}(S + 1),$$

and then,

$$\left( \frac{\eta^{1/2}(x/\theta_n(h_n))}{\theta_n^{1/2}(h_n)} - \frac{\eta^{1/2}(x/\theta_0)}{\theta_0^{1/2}} \right)^2 \leq \frac{M^2 e^{2S}}{n^2 \theta_0^{3/2}}(S + 1)^2.$$

Similarly for $h_n < 0$,

$$\left( \frac{\eta^{1/2}(x/\theta_n(h_n))}{\theta_n^{1/2}(h_n)} - \frac{\eta^{1/2}(x/\theta_0)}{\theta_0^{1/2}} \right)^2 \leq \frac{Me^{2S}}{n\theta_0 - M} 1_{[\theta_0(-M), 0]}(x) + \frac{M^2 e^{2S}}{n^2 \theta_0^{3/2}}(S + 1) \left( \frac{S\theta_0}{\theta_0 - M/n} + 1 \right)^2 1_{[0, \theta_0]}(x).$$

Combining these results, we obtain a bound for the squared Helliger distance:

$$H^2(P_{\theta_n(h_n), \eta}, P_{\theta_0, \eta}) \leq \frac{Me^{2S}}{n\theta_0 - M} + \frac{M^2 e^{2S}}{n^2 \theta_0^{3/2}}(S + 1)^2 + \frac{M^2}{n^2} \left( \frac{e^{2S} \theta_0}{\theta_0 - M/n} + 1 \right)^2 \left( \frac{S\theta_0}{\theta_0 - M/n} + 1 \right)^2.$$

To verify condition (vi) of Corollary 3.1, we now check condition (22) of Lemma 4.3.

\textbf{Lemma 5.11.} Let $(M_n)$, $M_n \to \infty$, $M_n \leq n$ for $n \geq 1$, $M_n = o(n)$ be given. Then there exists a constant $C > 0$ such that the condition of Lemma 4.3 is satisfied.

\textbf{Proof} The proof of this lemma is similar to the proof of Lemma 5.5. Therefore, we only note that by absolute continuity of $\eta$ we have

$$\frac{\eta(X_i/\theta^*)}{\theta^*} = \frac{\eta(X_i/\theta_0)}{\theta_0} + \int_{\theta_0}^{\theta^*} g'(y) dy,$$

where $g(y) = \eta(X_i/y)/y$, and

$$g'(y) = \frac{1}{y} \eta'(X_i/y) \left(-\frac{X_i}{y^2}\right) + \eta(X_i/y) \left(-\frac{1}{y^2}\right) \leq \eta(X_i/y) \left(-\frac{1}{y^2}\right).$$

To demonstrate that priors exist such that $\mathcal{L} \subset \text{supp}(\Pi_{x})$, an explicit construction based on the distribution of Brownian sample paths is provided in the following simplified version of Lemma 5.6.
Lemma 5.12. Let $S > 0$ be given. Let $\{W_t: t \in [0,1]\}$ be Brownian motion on $[0,1]$ and let $Z$ be independent and distributed $N(0,1)$. We define the prior $\Pi_L$ on $L$ as the distribution of the process,

$$\dot{\ell}(t) = S \Psi(Z + W_t),$$

where $\Psi : [-\infty, \infty] \to [-1, 1] : x \mapsto 2 \arctan(x)/\pi$. Then $L \subset \text{supp}(\Pi_L)$.

Lemma 5.13. For every differentiable $\eta$ and $\epsilon > 0$ the following inequalities hold:

$$\eta(0)\epsilon - \epsilon \int_0^\epsilon |\eta'(x)| \, dx \leq \int_0^\epsilon \eta(x) \, dx \leq \eta(0)\epsilon + \epsilon \int_0^\epsilon |\eta'(x)| \, dx.$$

Proof. Integration by parts yields

$$\int_0^\epsilon \eta(x) \, dx = \eta(0)\epsilon + \int_0^\epsilon (\epsilon - x)\eta'(x) \, dx.$$

Since $-\epsilon|\eta'(x)| \leq (\epsilon - x)|\eta'(x)| \leq \epsilon|\eta'(x)|$ for $x \in [0, \epsilon]$, the assertion holds. \qed

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