APPLICATION OF TOPOLOGICAL RADICALS TO CALCULATION OF JOINT SPECTRAL RADI

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To the memory of our fathers Semen Moiseevich Shulman and Vladimir Vasilyevich Turovskii, the Soviet Army officers who fought against fascism in the World War II

ABSTRACT. It is shown that the joint spectral radius $\rho(M)$ of a precompact family $M$ of operators on a Banach space $X$ is equal to the maximum of two numbers: the joint spectral radius $\rho_e(M)$ of the image of $M$ in the Calkin algebra and the Berger-Wang radius $r(M)$ defined by the formula

$$r(M) = \limsup_{n \to \infty} \left( \sup \{ \rho(a) : a \in M^n \} \right)^{1/n}.$$

Some more general Banach-algebraic results of this kind are also proved. The proofs are based on the study of special radicals on the class of Banach algebras.

1. Introduction and preliminaries

In 1960 Rota and Strang [6] introduced the notion of spectral radius for sets of operators or, more generally, of elements of a Banach algebra. Namely, if $M$ is a bounded subset of a Banach algebra $A$, the joint spectral radius $\rho(M)$ is defined by

$$(1.1) \quad \rho(M) = \lim_{n \to \infty} \|M^n\|^{1/n} = \inf_n \|M^n\|^{1/n},$$

where the norm of a bounded set is the supremum of the norms of its elements, and the products of sets are defined elementwise:

$$M_1 M_2 = \{ab : a \in M_1, b \in M_2\}.$$

The notion turned out to be useful in various branches of mathematics: wavelets, evolution dynamics, difference equations and the operator theory itself. In [7] the joint spectral radius was applied to show the existence of hyperinvariant subspace for every operator algebra whose Jacobson radical contains non-zero compact operators. This stimulates the interest in convenient ways for the calculation of $\rho(M)$.

The map $M \mapsto \rho(M)$ has many convenient algebraic and analytic properties, in particular it is subharmonic [8, Theorem 3.5] (for finite $M$, see also [12, Theorem 3.8]). The latter means that if $M = M(\lambda)$ analytically depends on a complex parameter $\lambda$ under natural conditions then $\lambda \mapsto \rho(M(\lambda))$ is a subharmonic function. The following property [8, Corollary 2.10] is quite important (see also a stronger result in [12, Proposition 3.5]).

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Lemma 1.1. If $\rho(M) = 0$ then all elements in the subalgebra generated by $M$ are quasinilpotent.

A very useful formula for the joint spectral radius of a bounded set of matrices was found in 1992 by M. A. Berger and Y. Wang [1]. To formulate it we introduce, following to [1], another spectral characteristics of a bounded subset $M$ of a Banach algebra:

$$r(M) = \limsup_{n \to \infty} \left( \sup \left\{ \rho(a) : a \in M^n \right\} \right)^{1/n}.$$ 

It is clear that $r(M) \leq \rho(M)$.

It was proved in [1] that

$$(1.2) \quad r(M) = \rho(M)$$

for any bounded set of matrices. This equality (the Berger-Wang formula) was extended in [8] to arbitrary precompact sets of compact operators on Banach spaces.

It should be noted that both restrictions of compactness are essential. It is proved in [3] that there are two non-compact operators $a, b$ such that $r(\{a, b\}) = 0 \neq \rho(\{a, b\})$.

There are also bounded families of compact operators for which the equality (1.2) fails (see for instance [5]).

The Berger-Wang formula for compact operators was applied in [8] to the study of operator semigroups and Lie algebras. As a simplest example of its application, note that it implies immediately the existence of a nontrivial closed invariant subspace for a semigroup of Volterra (i.e. compact quasinilpotent) operators, established in [13].

Indeed, if $G$ is a semigroup of Volterra operators then for each finite subset $M$ of $G$, every power $M^n$ consists of quasinilpotent operators, whence $r(M) = 0$. By (1.2), $\rho(M) = 0$, whence the linear span of $M$ consists of Volterra operators by Lemma 1.1. Thus the linear span of $G$ is an algebra of Volterra operators. By the Lomonosov Theorem [4], it has a nontrivial closed invariant subspace.

Our main aim here is to prove that, for arbitrary precompact set $M$ of operators on $X$,

$$(1.3) \quad \rho(M) = \max \{r(M), \rho_e(M)\},$$

where $\rho_e(M) = \rho(\pi(M))$, the joint spectral radius of the image of $M$ in the Calkin algebra $B(X)/K(X)$ under the canonical homomorphism $\pi$.

We call (1.3) the generalized Berger-Wang formula. This formula not only extends (1.2) to arbitrary operators, but also gives many additional possibilities for applications. Note for example that it implies immediately that (1.2) holds for precompact families of operators of the form $\lambda I + K$, where $\lambda \in \mathbb{C}$ and $K$ is a compact operator.

Indeed, in this case $\pi(M)$ consists of scalar multiples of the unit in the Calkin algebra $B(X)/K(X)$, whence

$$\rho_e(M) = r(\pi(M)) \leq r(M),$$

and (1.3) shows that

$$\rho(M) \leq r(M).$$
In its turn the equality (1.2) for ‘scalar plus compact’ operators was a main ingredient of the proof (in [8]) that any Lie algebra of compact quasinilpotent operators has a nontrivial closed invariant subspace.

Let us recall the proof of this result. Let $L$ be a Lie subalgebra in $\mathcal{B}(\mathcal{X})$, that is a subspace of $\mathcal{B}(\mathcal{X})$ such that

$$TS - ST \in L$$

for all $T, S \in L$. If $L$ consists of quasinilpotent operators then

$$G = \{e^T : T \in L\}$$

is a group (Wojtyński’s version of the Baker-Campbell-Hausdorff Theorem [14]), and all operators in $G$ have spectrum $\{1\}$. It follows that

$$r(M) = 1$$

for each finite set $M \subset G$. If $L$ consists of compact operators then $G$ consists of operators of the form $1 + K$ with compact $K$. By the above,

$$\rho(M) = 1,$$

for each finite subset $M \subset G$. Choose $T \in M$ and define a function $f(z)$ on $\mathbb{C}$ by

$$f(z) = \rho(M(e^{iT} - 1)/z).$$

It is subharmonic and tends to zero on infinity because

$$\rho(M(e^{iT} - 1)) \leq 2$$

(use the fact that the joint spectral radius of a bounded set is not changed if pass to closed convex hull of this set). Hence we have

$$f(z) = 0,$$

for every $z \in \mathbb{C}$. In particular, $f(0) = 0$ and

$$\rho(TM) = 0.$$

Now, by Lemma 1.1,

$$\rho(TS) = 0$$

for any $S \in A$, where $A$ is the linear span of $G$. Since $A$ is an algebra, it has an invariant subspace by Lomonosov’s Lemma. But it is straightforward that an invariant subspace for $G$ is invariant for $L$.

Note that the definition of $\rho_e(M)$ can be rewritten as follows

$$\rho_e(M) = \limsup_{n \to \infty} \left(\sup\{\|T\|_e : T \in M^n\}\right)^{1/n},$$

where $\|T\|_e = \|\pi(T)\|$ is the essential norm of $T$ ($T \mapsto \|T\|_e$ is a seminorm on $\mathcal{B}(\mathcal{X})$). It is somewhat more convenient to use

$$\rho_\chi(M) = \limsup_{n \to \infty} \left(\sup\{\|T\|_\chi : T \in M^n\}\right)^{1/n}.$$

Here $\|T\|_\chi$ is the Hausdorff measure of non-compactness for $T\mathcal{X}_1$, where

$$\mathcal{X}_1 = \{x \in \mathcal{X} : \|x\| \leq 1\}.$$
In other words, $\|T\|_\chi$ is the infimum of all such $\varepsilon > 0$ that $T \mathcal{X}_1$ can be covered by a finite number of balls of radius $\varepsilon$. The advantage of the work with this seminorm is that

$$\|T\|_{\chi} \leq \|T\|_\chi$$

and

$$\|T\|_{\chi}(\mathcal{X}/\mathcal{Y}) \leq \|T\|_\chi,$$

where $T\vert\mathcal{Y}$ is the restriction of $T$ to an invariant subspace $\mathcal{Y}$ and $T\vert(\mathcal{X}/\mathcal{Y})$ is the operator induced by $T$ on the quotient space $\mathcal{X}/\mathcal{Y}$.

So we will prove that

$$\rho(M) = \max\{r(M), \rho_\chi(M)\}$$

for any precompact set $M$ of operators. Since

$$\rho_\chi(M) \leq \rho_c(M) \leq \rho(M),$$

this is a more strong result than (1.3).

The formula (1.5) was proved in [10] for operators on reflexive spaces and, more generally, for weakly compact operators (see a stronger result in [10, Theorem 4.4]). In the present work we are able to remove these restrictions by applying the theory of topological radicals initiated by P. G. Dixon [2] and further developed in [11].

We heavily use the properties of the radical $R_{eq}$ defined in [11] in terms of the joint spectral radius, and introduce a new topological radical, $R_{hc}$, related to the compactness conditions.

We combine Banach algebraic and operator theoretic approaches. The first one makes the subject more flexible and allows us to approximate to a needed result step by step, ‘removing’ more and more large ideals and passing to the quotients (this process is simplified by means of the theory of radicals). The second one enjoys the possibility to pass to the restrictions of operators to invariant subspaces and to quotient spaces (which is especially important for calculation of spectral radii). Our main results also have both Banach algebraic and operator theoretic nature. We prove first a formula which expresses the joint spectral radius of a family of elements of a Banach algebra via the Hausdorff radius of a related family of multiplication operators (we call this formula the mixed GBWF). It is used to deduce (1.5) (the operator GBWF). Then using (1.5) we obtain an extension of (1.3) to general Banach algebras (the Banach algebraic GBWF). The latter formula is of the same form as (1.3), but the ideal $\mathcal{K}(\mathcal{X})$ is changed by the hypocompact radical $R_{hc}(A)$ of a Banach algebra $A$. It should be noted that, for $A = \mathcal{B}(\mathcal{X})$,

$$R_{hc}(A) \supset \mathcal{K}(\mathcal{X})$$

and the inclusion is proper for some Banach spaces. Hence the Banach-algebraic formula not only extends (1.3) but also strengthens it.

We denote by $A^1$ the Banach algebra obtained by adjoining a unit to $A$ (if $A$ is unital then we put $A^1 = A$). The term ideal always means a closed two-sided ideal, and the term operator always means a bounded linear operator. If $J$ is an ideal of $A$ then by $q_J$ we denote the quotient map from $A$ to $A/J$. The image of a set $M \subset A$ under $q_J$ is denoted by either $q_J(M)$ or, simply, $M/J$. All spaces are assumed to be over the field $\mathbb{C}$. If $M$ is a subset in a Banach space then span $M$ denotes its closed linear span.
2. Auxiliary results

A natural way from the Banach algebraic setting to the operator one is to use multiplication operators. As usual, for an element \(a\) of a Banach algebra \(A\), we denote by \(L_a\) and \(R_a\) the operators of the left and right multiplications by \(a\) on \(A\) defined by

\[ L_a x = ax, \quad R_a x = xa. \]

Furthermore, for \(M, N \subset A\), let

\[ L_M = \{ L_a : a \in M \} \]

and, similarly,

\[ R_N = \{ R_a : a \in N \}. \]

Multiplying these sets of operators, we define \(L_M R_N\) by

\[ L_M R_N = \{ L_a R_b : a \in M, b \in N \}. \]

The joint spectral properties of \(M\) are reflected in the properties of the family \(L_M R_M\).

Lemma 2.1. Let \(M\) be a bounded set \(M\) in a Banach algebra \(A\). Then

(i) \(\rho(L_M R_M) = \rho(M)^2\).
(ii) \(r(L_M R_M) = r(M)^2\).

Proof. (i) Note first of all that

\[
\rho(M^k) = \rho(M)^k
\]

for every bounded subset \(M\) of \(A\) and integer \(k\). This follows from the facts that

\[ M^{km} = (M^k)^m \]

and that in (1.1) one can pass to a subsequence under \(n = mk\).

It is clear that

\[
\|L_M R_N\| \leq \|L_M\| \|R_N\| \leq \|M\| \|N\|
\]

for bounded subsets \(M\) and \(N\) of \(A\). Also the formula

\[
(L_M R_N)^n = L_M^n R_N^n
\]

is evident. It follows from this that

\[
\rho(L_M R_M) \leq \rho(M)^2.
\]

To show the converse, we note that

\[
\|M^3\| = \|L_M R_M(M)\| \leq \|L_M R_M\| \|M\|.
\]

Changing \(M\) by \(M^n\), taking \(n\)-roots and limits, one obtains that

\[
\rho(M)^3 = \rho(M^3) \leq \rho(L_M R_M) \rho(M),
\]

whence

\[
\rho(M)^2 \leq \rho(L_M R_M).
\]

(ii) Since \(L_M\) commutes with \(R_M\), we have that

\[
\rho(L_a R_b) \leq \rho(L_a) \rho(R_b) \leq \rho(a) \rho(b)
\]

for every \(a, b \in M^n\). This shows that

\[
r(L_M R_M) \leq r(M)^2.
\]
On the other hand, for any $a \in M^k$, we obtain that
\[\rho(a)^2 = \lim_{n \to \infty} \|a^{2n+1}\|^{1/n} = \lim_{n \to \infty} \|(L_a R_a)^n(a)\|^{1/n} \leq \rho(L_a R_a)\]
\[\leq \sup_{x,y \in M^k} \rho(L_x R_y).\]

Taking supremum over all choices of $a_i$, one gets that
\[\sup_{a \in M^k} \rho(a)^2 \leq \sup_{x,y \in M^k} \rho(L_x R_y).\]

It remains to take $k$-roots and pass to the upper limit to obtain that
\[r(M)^2 \leq r(L_M R_M).\]

The following result was proved in [8, Corollary 6.5].

**Lemma 2.2.** $\|L_M R_M\|_\chi \leq 16 \|M\|_\chi \|M\|$ for each precompact set of operators.

Let us define $\rho^\chi(M)$ by
\[\rho^\chi(M) = \rho^\chi(L_M R_M)^{1/2},\]
for a bounded set $M$ of elements of a Banach algebra.

It is easy to check, using (1.4), that
\[\rho^\chi(M/J) \leq \rho^\chi(M),\]
for any closed ideal $J$ (see [10]).

An element $a \in A$ is called **compact** if the operator
\[W_a = L_a R_a\]
is compact on $A$. More generally, a set $M \subset A$ consists of mutually compact elements if all operators in $L_M R_M$ are compact. We will need the following extension of the main result of [13].

**Lemma 2.3.** If $G$ is a semigroup of quasinilpotent mutually compact elements of a Banach algebra $A$ then $\text{span} G$ consists of quasinilpotent elements.

**Proof.** Note that $L_G R_G$ is a semigroup of compact quasinilpotent operators on the Banach space $A$. By [13], $\text{span} L_G R_G$ consists of quasinilpotent elements. Note that $L_b R_c \in \text{span} L_G R_G$ for every $b, c \in \text{span} G$. Hence $L_{\text{span} G} R_{\text{span} G}$ consists of quasinilpotents and the same is true for $\text{span} G$. \qed

### 3. Hereditary topological radicals

We deal here with a class of topological radicals that have especially convenient properties. A **hereditary topological radical** on the class of all Banach algebras is a map $P$ which assigns to each Banach algebra $A$ its ideal $P(A)$ and satisfies the following conditions:

1. **(R1)** $P(A/P(A)) = 0$.
2. **(R2)** $P(J) = J \cap P(A)$ for any ideal $J$ of $A$.
3. **(R3)** $f(P(A)) \subset P(B)$ for continuous surjective homomorphism $f : A \to B$. 
It follows immediately from (R2) that

\[ P(P(A)) = P(A). \]

An algebra is called \( P \)-radical if

\[ A = P(A). \]

It can be proved (see [11]) that ideals and quotients of \( P \)-radical algebras are \( P \)-radical and that the class of all \( P \)-radical algebras is stable with respect to extensions (if \( J \) and \( A/J \) are \( P \)-radical then \( A \) is \( P \)-radical). We will need a more general result, also proved in [11].

Let us call a transfinite increasing sequence \((J_\alpha)_{\alpha \leq \gamma}\) of ideals in a Banach algebra \( A \) a transfinite increasing chain of ideals if \( J_\beta = \bigcup_{\alpha < \beta} J_\alpha \) for any limit ordinal \( \beta \leq \gamma \).

**Lemma 3.1.** Let \( P \) be a hereditary topological radical. If in a transfinite increasing chain of ideals \((J_\alpha)_{\alpha \leq \gamma}\) of a Banach algebra \( A \) the first element \( J_1 \) and all quotients \( J_{\alpha+1}/J_\alpha \) are \( P \)-radical then \( J_\gamma \) is \( P \)-radical.

The most popular example of a hereditary topological radical is the Jacobson radical \( \text{Rad} \). We need some other examples.

### 3.1. The radical \( R_{cq} \)

All definitions and results of this subsection are from [11].

The topological radical we are going to treat now is related to the joint spectral radius. Let us call a Banach algebra \( A \) compactly quasinilpotent if \( \rho(M) = 0 \) for any precompact subset \( M \subset A \).

**Theorem 3.2.** In any Banach algebra \( A \) there is a largest compactly quasinilpotent ideal \( R_{cq}(A) \). The map \( A \rightarrow R_{cq}(A) \) is a hereditary topological radical.

It is possible to give an individual test for an element to belong to \( R_{cq}(A) \). It is formally similar to the known characterization of the elements in the Jacobson radical.

**Theorem 3.3.** An element \( a \in A \) belongs to \( R_{cq}(A) \) if and only if \( \rho(aM) = 0 \) for any precompact set \( M \subset A \).

The following result shows that \( R_{cq}(A) \) can be considered as inessential when one calculates the joint spectral radius.

**Theorem 3.4.** Let \( q = q_{R_{cq}(A)} \) be the quotient map \( A \rightarrow A/R_{cq}(A) \). Then \( \rho(M) = \rho(q(M)) \) for each precompact set \( M \subset A \).

### 3.2. The hypocompact radical

We denote the set of all compact elements of a Banach algebra \( A \) by \( C(A) \). Note that \( C(A) \) is a semigroup ideal of \( A \) not closed, in general, under addition, even for semisimple Banach algebras. The following result follows easily from the Open Mapping Theorem.

**Lemma 3.5.** If \( f : A \rightarrow B \) is a continuous surjective homomorphism of Banach algebras then \( f(C(A)) \subset C(B) \).

**Lemma 3.6.** Let \( J \) be an ideal of \( A \). If \( C(J) \neq 0 \) then \( J \cap C(A) \neq 0 \).

**Proof.** It is easy to see that

\[ W_{ba} = L_a W_a R_b = R_a W_b L_a \]
for all $a, b \in A$. Hence if $a \in C(J)$ then for any $b \in J$, the operator $W_{ba}$ is compact on $A$. So

$$C(J)J \subset J \cap C(A)$$

and we are done if $C(J)J \neq 0$. On the other hand, if $C(J)J = 0$ then

$$C(J) \subset C(A)$$

because $W_a(x) = a(xa) = 0$ for any $a \in C(J)$ and $x \in A$. \hfill \Box

A Banach algebra $A$ is called \textit{bicom pact} if all operators $L_aR_b$ ($a, b \in A$) are compact (in other words, $A$ is bicom pact if it consists of mutually compact elements). An ideal of $A$ is called \textit{bicom pact} if it is bicom pact as a Banach algebra.

\textbf{Lemma 3.7.} Let $A$ be a Banach algebra. Then

- If $a \in C(A)$ then the ideal $\mathcal{J}(a)$ generated by $a$ is bicom pact.
- If $J$ is a bicom pact ideal of $A$ then the operator $L_aR_b$ is compact for every $a, b \in \text{span}(JA)$.

\textit{Proof.} Both statements follow easily from (3.1). \hfill \Box

A Banach algebra $A$ is called \textit{hypocom pact} if every non-zero quotient $A/J$ has a non-zero compact element. An ideal is \textit{hypocom pact} if it is hypocom pact as a Banach algebra.

Clearly each bicom pact algebra is hypocom pact. We will see that all hypocom pact algebras can be obtained by subsequent extensions of bicom pact ones. Let us prove first that the class of all hypocom pact algebras is stable under extensions.

\textbf{Lemma 3.8.} Let $J$ be an ideal of $A$. If $J$ and $A/J$ are hypocom pact then $A$ is hypocom pact.

\textit{Proof.} Let $I$ be an ideal of $A$. If $J \subset I$ then $A/I$ can be identified with $(A/J)/(I/J)$, the quotient of a hypocom pact algebra. By definition, the latter contains non-zero compact elements, so does $A/I$.

Suppose now that $I$ does not contain $J$. Setting $K = J \cap I$ we have that

$$C(J/K) \neq 0.$$ 

By Lemma 3.6,

$$J/K \cap C(A/K) \neq 0.$$ 

Let $0 \neq q_K(a) \in J/K \cap C(A/K)$. Then

$$a \notin I$$

and

$$q_I(a) \in C(A/I)$$

by Lemma 3.5. Thus $A/I$ contains non-zero compact elements. \hfill \Box

\textbf{Proposition 3.9.} Let $J$ be an ideal of a Banach algebra $A$. Then the following conditions are equivalent.

- (i) $J$ is hypocom pact.
- (ii) For each continuous surjective homomorphism $f : A \rightarrow B$, either $f(J) = 0$ or $f(J) \cap C(B) \neq 0$.
- (iii) There is a transfinite increasing chain of ideals $(J_\alpha)_{\alpha \leq \gamma}$ in $A$ such that $J_1 = 0$, $J_\gamma = J$, and all $J_{\alpha+1}/J_\alpha$ are bicom pact.
Proof. (i)⇒(ii) Let $I = \ker f$ and $K = I \cap J$. If $J \subset I$ then

$$f(J) = 0.$$ 

Otherwise there is a non-zero $a \in J/K \cap \mathcal{C}(A/K)$ by Lemma 3.6. Let $g : A/K \to B$ be defined by

$$g(q_K(b)) = f(b)$$

for every $b \in A$. Then $g$ is a continuous surjective homomorphism. Take $b \in J$ such that

$$a = q_K(b).$$

Then $f(b) = g(a)$ is a non-zero compact element of $B$.

(ii)⇒(iii) Let us consider all transfinite increasing chains $(J_\alpha)_{\alpha \leq \beta}$ such that $J_\alpha \subset J$, $J_{\alpha+1}/J_\alpha$ is bicom pact and $J_\alpha \neq J_{\alpha+1}$ for any $\alpha < \beta$. Clearly these chains form a set because $A$ is a set. Order the set of chains by

$$(J_\alpha)_{\alpha \leq \beta_1} \prec (J_\alpha)_{\alpha \leq \beta_2}$$

if $\beta_1 \leq \beta_2$ and $J_\alpha = I_\alpha$ for $\alpha \leq \beta_1$.

It follows from the Zorn Lemma that there is a maximal element $(J_\alpha)_{\alpha \leq \gamma}$ in this set. If $J_\gamma \neq J$ then there is a bicom pact ideal $I$ of $J/J_\gamma$. Put

$$J_{\gamma+1} = \{ x \in J : q_{J_\gamma}(x) \in I \}.$$ 

Then one can add $J_{\gamma+1}$ to the chain, in contradiction with its maximality.

(iii)⇒(i) Let $I$ be an ideal of $J$. Let $\alpha$ be the first ordinal for which $J_\alpha$ is not contained in $I$. Then

$$q_I(J_\alpha) \subset \mathcal{C}(J/I).$$

It follows that

$$\mathcal{C}(J/I) \neq 0.$$ 

□

Corollary 3.10. An ideal of a hypocompact Banach algebra is hypocompact.

Proof. Let $A$ be hypocompact and $J$ an ideal of $A$. Let $f : A \to B$ be a continuous surjective homomorphism, $I = \ker f$ and $K = I \cap J$. Assuming $f(J) \neq 0$, we get that

$$\mathcal{C}(J/I) \neq 0,$$

whence there is $a \in J$ such that

$$0 \neq q_K(a) \in \mathcal{C}(A/K).$$

It follows from Lemma 3.5 that

$$0 \neq q_I(a) \in \mathcal{C}(A/I),$$

so $f(a)$ is a non-zero compact element of $B$ in $f(J)$.

□

It follows easily from the definition that a quotient of a hypocompact algebra by an ideal is hypocompact. So Lemma 3.8 can be stated in the classical form of Three Subspaces Theorem:

Let $A$ be a Banach algebra. The following are equivalent.

- $A$ is hypocompact.
- $J$ and $A/J$ are hypocompact for every ideal $J$ of $A$.
- $J$ and $A/J$ are hypocompact for some ideal $J$ of $A$.

As a consequence, we have the following.

Corollary 3.11. In any Banach algebra there is the largest hypocompact ideal.
Proof. Let $J$ be the closed linear span of the union of all hypocompact ideals of a Banach algebra $A$. We have to prove that the ideal $J$ is hypocompact.

By Proposition 3.9, it suffices to prove that if $f : A \to B$ is a continuous surjective homomorphism with $f(J) \neq 0$ then

$$f(J) \cap \mathcal{C}(B) \neq 0.$$ 

But if $f(J) \cap \mathcal{C}(B) = 0$ then

$$f(I) = 0$$

for each hypocompact ideal $I$ of $A$. Hence

$$f(J) = 0.$$ 

□

The largest hypocompact ideal of a Banach algebra $A$ will be denoted by $\mathcal{R}_{hc}(A)$.

**Lemma 3.12.** If $J$ is an ideal of $A$ then $\mathcal{R}_{hc}(J) = J \cap \mathcal{R}_{hc}(A)$.

**Proof.** By Corollary 3.10, the ideal $J \cap \mathcal{R}_{hc}(A)$ of $J$ is hypocompact so it is contained in $\mathcal{R}_{hc}(J)$. We have to prove the converse inclusion.

Let $I = \text{span}(A^1 \mathcal{R}_{hc}(J)A^1)$ be the ideal of $A$ generated by $\mathcal{R}_{hc}(J)$. Then

$$I^3 = \text{span} \left( (A^1 \mathcal{R}_{hc}(J)A^1) \mathcal{R}_{hc}(J) (A^1 A^1 \mathcal{R}_{hc}(J)A^1) \right)$$

$$\subset \text{span} \left( J \mathcal{R}_{hc}(J)J \right) \subset \mathcal{R}_{hc}(J).$$

Hence $I^3$ is a hypocompact ideal. But $I/I^3$ is bicom pact because $L_a R_b = 0$ for every $a, b \in I/I^3$. By Lemma 3.8, $I$ is hypocompact. Then

$$I \subset \mathcal{R}_{hc}(A)$$

and

$$\mathcal{R}_{hc}(J) \subset \mathcal{R}_{hc}(A).$$

□

**Lemma 3.13.** The algebra $A/\mathcal{R}_{hc}(A)$ has no hypocompact ideals and compact elements.

**Proof.** If $J$ is a hypocompact ideal of $A/\mathcal{R}_{hc}(A)$ then, by Lemma 3.8, its preimage

$$J_1 = \{ x \in A : q_{\mathcal{R}_{hc}(A)}(x) \in J \}$$

is a hypocompact ideal of $A$ strictly containing $\mathcal{R}_{hc}(A)$, a contradiction.

By Lemma 3.6, if $A/\mathcal{R}_{hc}(A)$ has compact elements then it has bicom pact ideals. □

**Theorem 3.14.** The map $A \mapsto \mathcal{R}_{hc}(A)$ is a hereditary topological radical.

**Proof.** (R1) Since $A/\mathcal{R}_{hc}(A)$ has no hypocompact ideals, we have that

$$\mathcal{R}_{hc}(A/\mathcal{R}_{hc}(A)) = 0.$$ 

(R2) Let $f : A \to B$ be a continuous surjective homomorphism. Denote $q_{\mathcal{R}_{hc}(B)}$ by $q$, for brevity. Clearly $q \circ f$ is a continuous surjective homomorphism of $A$ to $B/\mathcal{R}_{hc}(B)$. Since $\mathcal{R}_{hc}(A)$ is hypocompact, $q \circ f(\mathcal{R}_{hc}(A))$ is either zero or contains a compact element of $B$. But the latter is impossible by Lemma 3.13. Hence

$$q \circ f(\mathcal{R}_{hc}(A)) = 0$$

and

$$f(\mathcal{R}_{hc}(A)) \subset \mathcal{R}_{hc}(B).$$
3.3. The radical $\text{Rad} \wedge \mathcal{R}_{hc}$. Starting with a family of radicals, one can obtain some new ones. The following construction has a well known analogue in the theory of algebraic radicals on rings. Let $P_1$ and $P_2$ be hereditary topological radicals. For any Banach algebra $A$, put

$$P_0(A) = P_1(A) \cap P_2(A).$$

**Theorem 3.15.** The map $A \mapsto P_0(A)$ is a hereditary topological radical.

**Proof.** Set $D = A/P_0(A)$. There is a natural surjective homomorphism $q_1 : D \to A/P_1(A)$ defined by

$$q_1(a/P_0(A)) = a/P_1(A).$$

Since $P_1(A/P_1(A)) = 0$, we have that $q_1(P_1(D)) = 0$ and $P_1(D) \subset \ker(q_1)$. Therefore, if an element $a/P_0(A)$ of $D$ belongs to the kernel of the homomorphism $q_1$, then $a \in P_1(A)$. We get that

$$P_0(D) \subset \{a/P_0(A) : a \in P_1(A)\}.$$

Similarly,

$$P_2(D) \subset \{a/P_0(A) : a \in P_2(A)\}.$$

Hence

$$P_0(D) \subset \{a/P_0(A) : a \in P_1(A) \cap P_2(A)\} = P_0(A)/P_0(A).$$

In other words, $P_0(D) = 0$. We proved that $P_0$ satisfies condition (R1).

If $f : A \to B$ is a continuous surjective homomorphism then

$$f(P_0(A)) \subset f(P_1(A)) \subset P_1(B)$$

and, similarly,

$$f(P_0(A)) \subset P_2(B).$$

Hence

$$f(P_0(A)) \subset P_0(B),$$

therefore $P_0$ satisfies (R2).

For an ideal $J \subset A$, one has

$$P_0(J) = P_1(J) \cap P_2(J) = (P_1(A) \cap J) \cap (P_2(A) \cap J) = P_0(A) \cap J.$$

We proved (R3).}

The radical $P_0$ constructed in the previous theorem is denoted by $P_1 \wedge P_2$. We will consider the radical $\text{Rad} \wedge \mathcal{R}_{hc}$.

Let us introduce a standard order in the class of all topological radicals by the rule

$$P_1 \leq P_2$$

if $P_1(A) \subset P_2(A)$ for every Banach algebra $A$. One can show that $P_1 \wedge P_2$ is the largest hereditary topological radical $P$ having the property $P \leq P_1$ and $P \leq P_2$, but we needn’t it here.
The following result shows in particular that for compact (more generally, for hypocompact) algebras the radical $R_{cq}$ coincides with the Jacobson radical.

**Theorem 3.16.** $\text{Rad} \cap R_{hc} \leq R_{cq}$.

**Proof.** Let us first show that each bicompact Jacobson radical algebra $A$ is compactly quasinilpotent. Indeed, if $M$ is a precompact family in $A$ then $r(M) = 0$ because $A$ consists of quasinilpotent elements. For $N = L_M R_M$, we see from Lemma 2.1 that $r(N) = 0$. Since $N$ is a precompact family of compact operators, $\rho(N) = 0$ by the Berger-Wang formula. Again, we obtain from Lemma 2.1 that $\rho(M) = 0$.

Let now $A$ be an arbitrary Banach algebra and $J = \text{Rad}(A) \cap R_{hc}(A)$. Since $J$ is a hypocompact ideal then, by Proposition 3.9, there is a transfinite increasing chain $(J_\alpha)_{\alpha \leq \gamma}$ of ideals such that $J_\gamma = J$ and all $J_{\gamma+1}/J_\gamma$ are bicom pact. Since all bicom pact Jacobson radical ideals are compactly quasinilpotent, all $J_{\gamma+1}/J_\gamma$ are $R_{cq}$-radical. By the transfinite extension property (see Lemma 3.1), $J$ is $R_{cq}$-radical. \hfill $\square$

4. **Main results**

4.1. **Mixed GBWF.** We will prove for any precompact set $M$ of elements in a Banach algebra $A$ that

$$(4.1) \quad \rho(M) = \max\{\rho^\lambda(M), r(M)\},$$

where as above we set

$$\rho^\lambda(M) = \rho(\chi(L_M R_M))^{1/2}.$$

Note that it suffices to prove this result under the assumption that $A$ is generated by $M$ as a Banach algebra. Indeed, $\rho(M)$ and $r(M)$ do not change if calculated in the closed subalgebra $B = A(M)$ generated by $M$. The value $\rho^\lambda(M)$ in this case cannot increase because the multiplication operators act on a smaller space. But the nontrivial inequality in (4.1) is only $\leq$.

So we may assume in what follows that $A = A(M)$. A semigroup $G$ of elements of a Banach algebra is called a Radjavi semigroup ($R$-semigroup for brevity) if $\lambda a \in G$ for every $a \in G$ and $\lambda \geq 0$.

Let $G = S(M)$ be the semigroup generated by a set $M$ of operators. Then

$$G = \cup_{n=1}^{\infty} M^n.$$

An operator $T \in M^n$ is called leading (more precisely, $n$-leading) if

$$\|T\| \geq \|S\|$$

for all $S \in \cup_{k \leq n} M^k$. Note that an operator may be in the different $M^n$’s, this justifies the more precise term ‘$n$-leading operator’, but we write just 'leading'
when it is clear which \( n \) is meant. A leading sequence in \( G \) is a sequence that consists of leading operators \( T_k \in M_{n(k)} \) for \( n(k) \to \infty \).

Let \( \mathcal{S}_+(M) \) be the \( R \)-semigroup generated by \( M \). Clearly

\[
\mathcal{S}_+(M) = \mathbb{R}_+ \mathcal{S}(M),
\]

where \( \mathbb{R}_+ = \{ t \in \mathbb{R} : t \geq 0 \} \).

The following lemma was proved in [8, see Theorem 6.10].

**Lemma 4.1.** Let \( N \) be a precompact set of operators. Suppose that \( \rho_x(N) < \rho(N) = 1 \) and \( \mathcal{S}(N) \) is unbounded. Then there is a sequence \( T_n \in \mathcal{S}_+(N) \), with \( \|T_n\| = 1 \), converging to a compact operator \( T \). Moreover, to obtain such a sequence \( T_n \) it suffices to take any leading sequence \( S_n \) in \( \mathcal{S}(N) \) and choose a convergent subsequence from \( S_n/\|S_n\| \) (it always exists).

**Lemma 4.2.** Let \( A \) be a Banach algebra, \( M \subset A \) be precompact and \( N = L_M R_M \). Suppose that \( A = \mathcal{A}(M) \), \( \rho_x(N) < \rho(N) = 1 \) and \( \mathcal{S}(N) \) is unbounded. Then the closure \( \overline{\mathcal{S}_+(N)} \) contains a non-zero compact operator \( T \) such that

1. \( L_{T_n} R_{T_n} \) is compact for every elements \( h, g \) of \( A \).
2. If also \( r(N) < 1 \) then \( T(A) \subset \text{Rad}(A) \).

**Proof.** All elements in \( \mathcal{S}_+(N) \) are of the form

\[
P = \lambda L_a R_b,
\]

where \( a, b \in \mathcal{S}(M) \) and \( \lambda \geq 0 \). For brevity, we will denote \( P^o \) for any \( P \) by

\[
P^o = \lambda L_a R_b.
\]

Note that \( P^o \) may be not uniquely determined by \( P \), but the equality

\[
L_{P^o} R_{P^o} = \lambda L_{P} R_{P}
\]

holds independently of the choice of \( P^o \), for every \( h, g \in M \).

Let \( (S_n) \) be a leading sequence in \( \mathcal{S}(N) \). For every \( S_n \in N^{m(n)} \), the operator \( S_n^o \) can also be chosen in \( N^{m(n)} \), so we may assume that

\[
\|S_n\| \geq \|S_n^o\|
\]

for all \( n \). By Lemma 4.1, we may choose a sequence of operators

\[
T_n = S_{k_n}/\|S_{k_n}\|
\]

that tends to a compact operator \( T \). Note that all operators

\[
T_n^o = S_{k_n}^o/\|S_{k_n}\|
\]

are contractive by (4.2). Now for any \( h, g \in A \), we have

\[
L_{T_n} R_{T_n} = \lim_{n \to \infty} L_{T_n\cdot h} R_{T_n\cdot g} = \lim_{n \to \infty} T_n L_h R_g T_n^o = \lim_{n \to \infty} T L_h R_g T_n^o.
\]

Hence the operator \( L_{T_n} R_{T_n} \) is a limit of compact operators, so it is compact. Part (i) is proved.

Let \( r(N) < 1 \), and let us now prove that \( u(Tx)v \) is quasinilpotent for every \( u, v, x \in \mathcal{S}(M) \). By our construction,

\[
T = \lim_{n \to \infty} \lambda_{k_n} S_{k_n},
\]

where \( \lambda_{k_n} = \|S_{k_n}\|^{-1} \to 0 \) as \( n \to \infty \), \( S_{k_n} = L_{a_{k_n}} R_{b_{k_n}} \) for some \( a_{k_n}, b_{k_n} \in \mathcal{S}(M) \). Since

\[
L \mathcal{S}(M) R \mathcal{S}(M) = \mathcal{S}(N),
\]
we have that
\[ W_{ua_nxb_nv} = (L_uR_v)(L_{a_n}R_{b_n})(W_{x}(L_bnR_{a_n})(L_vR_u)) \in S(N). \]
Since \( r(N) < 1 \) implies that \( \{\rho(S) : S \in S(N)\} \) is bounded, we obtain that
\[ \rho(u(Tx)v) = \lim_{n \to \infty} \lambda_{k_n} \rho(ua_nxb_nv) = \lim_{n \to \infty} \lambda_{k_n} \rho(W_{ua_nxb_nv})^{1/2} = 0, \]
Thus we see that the set \( S_+(M)(Tx)S_+(M) \) consists of mutually compact quasinilpotent elements of \( A \) for every \( x \in S(M) \). So does the closure \( \overline{S_+(M)(Tx)S_+(M)} \).
Since \( T x \in S_+(N)S(M) = \overline{L_{S_+(M)}R_{S_+(M)}S(M) \subset S_+(M)S(M)S_+(M)} \)
the set \( \overline{S_+(M)(Tx)S_+(M)} \) is a semigroup. By Lemma 2.3, its closed linear span \( J \) also consists of compact quasinilpotent elements. Since \( J \) coincides with the ideal \( A(Tx)A \) of \( A \), we have that
\[ A(Tx)A \subset \text{Rad}(A), \]
whence, by the quasi-regular characterization of the Jacobson radical,
\[ A(Tx) \subset \text{Rad}(A) \]
and also
\[ Tx \in \text{Rad}(A). \]
Since \( A = \text{span} S(M) \), we obtain that
\[ T(A) \subset \text{Rad}(A). \]
\[ \square \]
Let us call any closed bicomponent ideal that consists of quasinilpotent operators a \( qb \)-ideal. The above lemma implies the following result.

**Corollary 4.3.** If \( \max \{\rho^\lambda(M), r(M)\} < \rho(M) = 1 \) and the semigroup \( S(L_MR_M) \) is unbounded then \( A(M) \) has a non-zero \( qb \)-ideal.

**Proof.** Indeed, every ideal \( J \) generated by \( Tx \in A(M) \) is a \( qb \)-ideal. \[ \square \]

**Lemma 4.4.** If \( A(M) \) has no non-zero \( qb \)-ideals then the equality (4.1) holds.

**Proof.** Suppose that (4.1) fails. We may assume that
\[ \rho(M) = 1. \]
Let \( N = L_MR_M \). Then we have that
\[ \rho(N) = 1 \]
by Lemma 2.1.
If the semigroup \( S(N) \) is bounded, then
\[ \max \{\rho^\lambda(N), r(N)\} = 1 \]
holds by [8, Proposition 9.6]. If \( \rho^\lambda(N) = 1 \) then \( \rho^\lambda(M) = 1 \). Otherwise \( r(N) = 1 \) and \( r(M) = 1 \) by Lemma 2.1. In both of the cases (4.1) holds, a contradiction.
So \( S(N) \) is unbounded. Then \( A(M) \) has a non-zero \( qb \)-ideal by Corollary 4.3.
This contradicts to our assumptions. \[ \square \]
Theorem 4.5. Let $A$ be a Banach algebra. The equality (4.1) holds for each precompact subset $M$ of $A$.

Proof. Recall that we may assume that $A = \mathcal{A}(M)$. Let $J = \text{Rad}(A) \cap \mathcal{R}_{\text{hc}}(A)$. Since $J \subset \mathcal{R}_{\text{cq}}(A)$, we obtain that

$$
\rho(M) = \frac{1}{2} \rho(M/J)\rho(M).
$$

by Theorem 3.4. Furthermore, the algebra $A/J$ has no $qb$-ideals. Indeed, if $I$ is such an ideal, and $U$ is its preimage in $A$, then it is evident that

$$
U \subset \text{Rad}(A),
$$

and that

$$
U \subset \mathcal{R}_{\text{he}}(A)
$$

by the extension property of radicals. Hence

$$
U \subset J
$$

and, as a consequence,

$$
I = 0.
$$

Taking into account that $A/J = \mathcal{A}(M/J)$, and applying Lemma 4.4, we have that

$$
\rho(M) = \rho(M/J) = \max\{\rho^X(M/J), r(M/J)\} \leq \max\{\rho^X(M), r(M)\}.
$$

The converse inequality is evident. □

4.2. Operator GBWF. Now we can prove (1.5).

Theorem 4.6. If $M \subset B(\mathcal{X})$ is precompact then

$$
\rho(M) = \max\{\rho^X(M), r(M)\}.
$$

Proof. By Lemma 2.2,

$$
\|L_M R_M\|_X \leq 16\|M\|_X\|M\|.
$$

Changing $M$ by $M^n$, taking $n$-th roots and limits as $n \to \infty$, we obtain that

$$
\rho^X(M)^2 = \rho^X(L_M R_M) \leq \rho^X(M)\rho(M).
$$

Applying this in (4.1), we get that

$$
\rho(M) \leq \max\{\rho^X(M)^{1/2}\rho(M)^{1/2}, r(M)\},
$$

whence

$$
\rho(M) \leq \max\{\rho^X(M)^{1/2}\rho(M)^{1/2}, r(M)^{1/2}\rho(M)^{1/2}\}.
$$

It follows from this that

$$
\rho(M)^{1/2} \leq (\max\{\rho^X(M), r(M)\})^{1/2},
$$

and we are done, because the converse is evident. □
4.3. Banach algebraic GBWF. Our next aim is to prove for any Banach algebra $A$ and a precompact subset $M \subset A$, that

$$\rho(M) = \max\{\rho(M/R_{hc}(A)), r(M)\}. \quad (4.3)$$

It will be more convenient for us to prove (4.3) in the following form:

$$\rho(M) = \max\{\rho(M/J), r(M)\} \quad \text{for any hypocompact ideal } J \text{ of } A. \quad (4.4)$$

We will begin with the case that $J$ is bicompact.

**Lemma 4.7.** Let $J$ be a bicompact ideal of $A$. Then

$$\rho_e(L_MR_M) \leq (\rho(M/J)\rho(M))^{1/2} \quad (4.5)$$

**Proof.** Let us prove first the inequality

$$\|L_MR_M\| \leq 3\|M/J\|\|M\| \quad (4.6)$$

Let $a, b \in M$, $\varepsilon > 0$. Choose $u, v \in J$ such that

$$\max\{\|a-u\|, \|b-v\|\} < \|M/J\| + \varepsilon.$$  

In particular, we have that

$$\|u\| < \|a\| + \|a-u\| \leq \|M\| + \|M/J\| + \varepsilon \leq 2\|M\| + \varepsilon.$$  

Then we obtain that

$$\|L_aR_b\| \leq \|L_aR_b - L_uR_v\| = \|L_{a-u}R_b + L_uR_{b-v}\|$$

$$\leq (\|M/J\| + \varepsilon)\|M\| + (2\|M\| + \varepsilon)(\|M/J\| + \varepsilon)$$

$$\leq (\|M/J\| + \varepsilon)(3\|M\| + \varepsilon),$$

and it remains to take $\varepsilon \to 0$ and supremum over all $a, b \in M$.

To obtain (4.5), change in (4.6) $M$ by $M^n$, take $n$-th roots and $n \to \infty$. \hfill \Box

**Corollary 4.8.** The equality (4.4) holds for each bicompact ideal $J$.

**Proof.** It follows from (4.5) that

$$\rho^x(M) \leq \rho_e(L_MR_M)^{1/2} \leq (\rho(M/J)^{1/2}\rho(M))^{1/2},$$

whence by (4.1),

$$\rho(M) = \max\{\rho^x(M), r(M)\} \leq \max\{\rho(M/J)^{1/2}\rho(M)^{1/2}, r(M)^{1/2}\rho(M)^{1/2}\}$$

and (4.4) follows immediately. \hfill \Box

**Lemma 4.9.** Let $I, K$ be ideals of $A$ and $I \subset K$. If $K/I$ is bicompact and (4.4) holds for $J = I$ then it holds for $J = K$.

**Proof.** The isomorphism $A/K \to (A/I)/(K/I)$ implies

$$\rho(M/K) = \rho((M/I)/(K/I)),$$  

whence

$$\rho(M) = \max\{\rho(M/I), r(M)\} = \max\{\max\{\max\{\rho((M/I)/(K/I)), r(M/I)\}, r(M)\}\}$$

$$\leq \max\{\rho(M/K), r(M)\}.$$  

The converse inequality is trivial. \hfill \Box
Lemma 4.10. If $J = \bigcup J_\alpha$, where $(J_\alpha)$ is an increasing net of closed ideals of a Banach algebra $A$, then, for a precompact subset $M \subset A$,

\begin{equation}
\|M/J\| = \lim_\alpha \|M/J_\alpha\| = \inf_\alpha \|M/J_\alpha\|
\end{equation}

and

\begin{equation}
\rho(M/J) = \lim_\alpha \rho(M/J_\alpha) = \inf_\alpha \rho(M/J_\alpha).
\end{equation}

Proof. We have

$$\|M/J\| \leq \|M/J_\alpha\|$$

for every $\alpha$, whence

$$\rho(M/J) = \inf_\alpha \|M^n/J\|^{1/n} \leq \inf_\alpha \inf_\alpha \|M^n/J_\alpha\|^{1/n} = \inf_\alpha \rho(M/J_\alpha).$$

and also

$$\|M/J\| \leq \inf_\alpha \|M/J_\alpha\| \leq \liminf_\alpha \|M/J_\alpha\|.$$  

On the other hand, it is easy to see that for any $a \in M$ and $\varepsilon > 0$ there is $\alpha = \alpha(a, \varepsilon)$ with

\begin{equation}
\|a/J_\alpha\| \leq \|a/J\| + \varepsilon \leq \|M/J\| + \varepsilon.
\end{equation}

Take a finite subset $N$ of $M$ with $\text{dist}(b, N) \leq \varepsilon$ for every $b \in M$. It is clear that, for every $c \in A$,

$$\|c/J_\beta\| \leq \|c/J_\alpha\|$$

if $\alpha < \beta$. So, choosing $\gamma \geq \max \{\alpha(a, \varepsilon) : a \in N\}$, we obtain from (4.9) that

$$\text{dist}(b/J_\gamma, N/J_\gamma) \leq \text{dist}(b, N) \leq \varepsilon$$

for every $b \in M$, and so

$$\|M/J_\gamma\| \leq \|N/J_\gamma\| + \varepsilon \leq \|M/J\| + 2\varepsilon.$$  

Therefore

\begin{equation}
\inf_\alpha \|M/J_\alpha\| \leq \limsup_\alpha \|M/J_\alpha\| \leq \|M/J\|,
\end{equation}

whence (4.7) holds. Take $n \in \mathbb{N}$ such that

$$\|M^n/J\|^{1/n} \leq \rho(M/J) + \varepsilon.$$  

Then, by (4.10) applied to $M^n$,

$$\inf_\alpha \rho(M/J_\alpha) \leq \limsup_\alpha \rho(M/J_\alpha) \leq \limsup_\alpha \|M^n/J_\alpha\|^{1/n} \leq \|M^n/J\|^{1/n} \leq \rho(M/J) + \varepsilon.$$  

Therefore (4.8) holds. \qed

Now we can finish the proof of (4.4).

Theorem 4.11. The equality (4.4) holds for every hypocompact ideal $J$.

Proof. Indeed, there is a transfinite chain $\{J_\alpha\}_{\alpha \leq \beta}$ of closed ideals such that $J_0 = 0$, $J_\beta = J$, and all $J_{\alpha+1}/J_\alpha$ are bicompact. Suppose that $\gamma$ is the least $\alpha$, for which (4.4) fails. It cannot be a limit ordinal because of Lemma 4.10 and cannot have a predecessor because of Lemma 4.9. Therefore (4.4) holds for all $\alpha$. \qed
4.4. Applications to continuity of the joint spectral radius. Since the operator GBWF is now proved in full generality we may remove the restriction of weak compactness in the applications to the continuity of joint spectral radius which were obtained in [10, Corollary 4.6].

Recall [8, Proposition 3.1] that the joint spectral radius is an upper semicontinuous function of a bounded subset $M$ of a Banach algebra. This means that

$$\limsup_{n \to \infty} \rho(M_n) \leq \rho(M)$$

if a sequence $M_n$ of bounded subsets tends to $M$ in the sense that the Hausdorff distance between $M_n$ and $M$ tends to zero.

Indeed, we have that

$$M_m^m \to M^m$$

as $n \to \infty$ for every $m$, whence

$$\|M_m^m\|^{1/m} \to \|M^m\|^{1/m}$$

as $n \to \infty$. Since $\rho(M_n) = \rho(M_m^m)^{1/m} \leq \|M_m^m\|^{1/m}$, we see that

$$\limsup_{n \to \infty} \rho(M_n) \leq \|M^m\|^{1/m}.$$ 

Taking $m \to \infty$, we get that

$$\limsup_{n \to \infty} \rho(M_n) \leq \rho(M).$$

We say that a set $M$ of elements in a Banach algebra $A$ is a point of continuity for the joint spectral radius if $\rho(M_n) \to \rho(M)$ for any sequence $M_n$ of bounded sets tending to $M$.

It is well known that if the norm of an operator $T$ is more than its essential norm then $T$ is a point of continuity of the (usual) spectral radius. The following result establishes the same for precompact families of operators.

**Corollary 4.12.** Let $M$ be a precompact set of operators on a Banach set $X$. If $\rho_x(M) < \rho(M)$ then $M$ is a point of continuity of the joint spectral radius.

**Proof.** Let $M_n$ tend to $M$. Since

$$\limsup_{n \to \infty} \rho(M_n) \leq \rho(M),$$

we have only to prove that

$$\liminf_{n \to \infty} \rho(M_n) \geq \rho(M).$$

Suppose the contrary. Multiplying by a scalar and changing $M_n$ by a subsequence, we may assume that

$$\rho(M_n) \to \alpha < 1 < \rho(M)$$

and

$$\rho_x(M) < 1.$$ 

It follows from the formula (1.5) that

$$\rho(M) = r(M).$$

Hence

$$\sup\{\rho(T) : T \in M^k\} > 1$$
for some $k$. This means that there is an operator $T \in M^k$ with 
$$\rho(T) > 1.$$  
Note that 
$$\rho_\chi(T) \leq 1.$$  
Indeed, since $\rho_\chi(M) < 1$ then 
$$\|M^n\|_\chi < 1$$  
for sufficiently large $n$. Hence 
$$\|T^n\|_\chi \leq \|M^{nk}\|_\chi < 1$$  
and it remains to take the $n$-th roots.

Since for operators (one-element families) the numbers $\rho_\chi$ and $\rho_e$ coincide, we conclude that 
$$\rho_e(T) < \rho(T).$$  
By our assumptions, there are $T_n \in M^k_n$ such that 
$$T_n \to T.$$  
Since $T$ is a point of continuity of the usual spectral radius, 
$$\rho(T_n) \to \rho(T).$$  
But this is impossible because 
$$\rho(T_n) \leq \rho(M^k_n) = \rho(M_n)^k \to \alpha^k < 1.$$  

Acknowledgement. The results of the paper, as well as some other results on topological radicals of Banach algebras, were announced in [9]. At about that time the authors realized that the theory of topological radicals admits a systematic treatment in a much more general setting, not only for Banach algebras, but for non-necessarily complete and non-necessarily associative algebras. Also, the class of morphisms in such a theory may be different. The development of this approach took a lot of time and in the present moment is far from the end (only one paper of the planned series, [11], is published). Thus the proof of the generalized Berger-Wang formula was unpublished for several years. The aim of the present publication is to make the proof available for specialists.

It should be noted that the list of generalized Berger-Wang formulae given here is not complete. The stronger variants of these formulae, as well as an exposition of some topics of topological radicals, will be published elsewhere.

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