Four-Qubit Monogamy and Four-Way Entanglement

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Abstract

We examine the various properties of the three four-qubit monogamy relations, all of which introduce the power factors in the three-way entanglement to reduce the tripartite contributions. On the analytic ground as much as possible we try to find the minimal power factors, which make the monogamy relations hold if the power factors are larger than the minimal powers. Motivated to the three-qubit monogamy inequality we also examine whether those four-qubit monogamy relations provide the SLOCC-invariant four-way entanglement measures or not. Our analysis indicate that this is impossible provided that the monogamy inequalities are derived merely by introducing weighting power factors.
I. INTRODUCTION

Recently, quantum technology, i.e. technology based on quantum mechanics, attracts much attention to overcome various limitations of classical technology such as computational speed of computer and insecurity of cryptography. Quantum entanglement\[^1, 2\] is the most important physical resource to develop the quantum technology because it plays a crucial role in the various quantum information processing. In fact, it is used in quantum teleportation\[^3\], superdense coding\[^4\], quantum cloning\[^5\], and quantum cryptography\[^6, 7\]. It is also quantum entanglement, which makes the quantum computer\[^1\] outperform the classical one\[^9\]. Thus, it is very important to understand how to quantify and how to characterize the entanglement.

One of the surprising property of the quantum entanglement arises in its distribution in the multipartite system. It is usually called the monogamy property. For example, let us consider the tripartite quantum state $|\psi\rangle_{ABC}$ in the qubit system. Authors in Ref. \[^10\] have shown the inequality

$$C^2_{A|(BC)} \geq C^2_{A|B} + C^2_{A|C} \quad (1.1)$$

where $C$ is concurrence\[^11\], one of the entanglement measure for bipartite system. This inequality, usually called CKW inequality, implies that the entanglement (measured by the squared concurrence) between $A$ and the remaining parties always exceeds entanglement between $A$ and $B$ plus entanglement between $A$ and $C$. This means that the more $A$ and $B$ are entangled, the lesser $A$ and $C$ are entangled. This is why the quantum cryptography is more secure than classical one. The inequality (1.1) is strong in a sense that the three-qubit W-state\[^12\]

$$|W_3\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) \quad (1.2)$$

saturates the inequality.

Another surprising property of Eq. (1.1) is the fact that the leftover in the inequality

$$\tau_{ABC} = C^2_{A|(BC)} - (C^2_{A|B} + C^2_{A|C}) \quad (1.3)$$

quantifies the true three-way entanglement. For general three-qubit pure state $|\psi\rangle = \sum_{i,j,k=0}^{1} \psi_{ijk} |ijk\rangle_{ABC}$ the leftover $\tau_{ABC}$, which is called the residual entanglement$^2$, reduces

\[^1\] The current status of quantum computer technology was reviewed in Ref.[8].
\[^2\] In this paper $\sqrt{\tau_{ABC}}$ is called the three-tangle.
to
\[ \tau_{ABC} = \left| 2\epsilon_{i_1i_2} \epsilon_{i_3i_4} \epsilon_{j_1j_2} \epsilon_{j_3j_4} \epsilon_{k_1k_3} \epsilon_{k_2k_4} \psi_{i_1j_1k_1} \psi_{i_2j_2k_2} \psi_{i_3j_3k_3} \psi_{i_4j_4k_4} \right|. \] (1.4)

From this expression one can show that \( \tau_{ABC} \) is invariant under a stochastic local operation and classical communication (SLOCC)\[13\].

Then, it is natural to ask whether or not such surprising properties are maintained in the monogamy relation of multipartite system. Subsequently, the generalization of Eq. (1.1) was discussed in Ref. [14]. As Ref. [14] has shown analytically, the following monogamy relation
\[ C^2_{q_1|q_2\cdots q_n} \geq C^2_{q_1|q_2} + C^2_{q_1|q_3} + \cdots + C^2_{q_1|q_n} \] (1.5)
holds in the \( n \)-qubit pure-state system. However, it is shown that the leftover of Eq. (1.5) is not entanglement monotone. In order to remove this unsatisfactory feature the authors in Ref. [15, 16] considered the average leftover of the monogamy relation (1.5). For example, they conjectured that in four-qubit system the following average leftover
\[ \theta_{ABCD} = \pi_A + \pi_B + \pi_C + \pi_D/4 \] (1.6)
is a monotone, where \( \pi_A = C^2_{A(BCD)} - (C^2_{A|B} + C^2_{A|C} + C^2_{A|D}) \) and other ones are derived by changing the focusing qubit. Even though \( \theta_{ABCD} \) might be an entanglement monotone, it is obvious that it cannot quantify a true four-way entanglement because it detects the partial entanglement. For example, \( \theta_{ABCD}(g_3) = 3/4 \), where \( |g_3\rangle = |\text{GHZ}_3\rangle \otimes |0\rangle \) and \( |\text{GHZ}_3\rangle \) is a three-qubit Greenberger-Horne-Zeilinger (GHZ) state defined as
\[ |\text{GHZ}_3\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \] (1.7)

In Ref. [17] another following multipartite monogamy relation is examined:
\[ C^2_{q_1|q_2\cdots q_n} \geq \sum_{j=2}^{n} C^2_{q_1|q_j} + \sum_{k>j=2}^{n} \left[ t_{q_1|q_j|q_k} \right]^{\mu_3} + \cdots + \sum_{\ell=2}^{n} \left[ t_{q_1|q_2|\cdots|q_{\ell-1}|q_{\ell+1}|\cdots|q_n} \right]^{\mu_{n-1}}. \] (1.8)
In Eq. (1.8) the power factors \( \mu_m = 3 \) are included to regulate the weight assigned to the different \( m \)-partite contributions. If all power factors \( \mu_m \) go to infinity, Eq. (1.8) reduces to Eq. (1.5). As a tripartite entanglement measure the residual entanglement or three-tangle can be used independently. Thus, in four-qubit system Eq. (1.8) reduces to following two different expressions:
\[ \Delta_j = t_{1|234} - (t_{1|2} + t_{1|3} + t_{1|4}) - \left( t^{(j)}_{1|2|3} + t^{(j)}_{1|2|4} + t^{(j)}_{1|3|4} \right) \geq 0 \quad (j = 1, 2, 3) \] (1.9)
where

\[ t_{1|234} = C_{1|234}^2 = 4 \det \rho_1 \]
\[ t_{ij} = C^2(\rho_{ij}) \quad (1.10) \]
\[ t_{ij|k}^{(1)} = \left[ \min_{(p_n, |\psi_n\rangle)} \sum_n p_n \sqrt{\tau_{ijk}(\psi_n)} \right]^\mu_1 \]
\[ t_{ij|k}^{(2)} = \left[ \min_{(p_n, |\psi_n\rangle)} \sum_n p_n \tau_{ijk}(\psi_n) \right]^\mu_2. \]

In Eq. (1.10) the tripartite entanglements \( t_{ij|k}^{(j)} \) are expressed explicitly as a convex roof\[18, 19\] for mixed states derived by the partial trace of the four-qubit pure states. In particular, the authors of Ref. [17] conjectured that all four-qubit pure states holds \( \Delta_1 \geq 0 \) when \( \mu_1 \geq 3 \). Different expression of the monogamy relation was introduced in Ref. [20], which is Eq. (1.9) with \( j = 3 \), where

\[ t_{ij|k}^{(3)} = \left[ \min_{(p_n, |\psi_n\rangle)} \sum_n p_n \tau_{ijk}^{1/q}(\psi_n) \right]^q. \quad (1.11) \]

The authors of Ref. [20] conjectured that \( \Delta_3 \) with \( q = 4 \) might be nonnegative for all four-qubit pure states. They also conjectured by making use of their extensive numerical tests that all possible second class states\[3\]

\[ |G\rangle = \mathcal{N} \left[ \frac{a+b}{2} (|0000\rangle + |1111\rangle) + \frac{a-b}{2} (|0011\rangle + |1100\rangle) + c (|0101\rangle + |1010\rangle) + |0110\rangle \right] \]

and their SLOCC transformation hold \( \Delta_3 \geq 0 \) when \( q \geq 2.42 \), where the parameters \( a, b, \) and \( c \) are generally complex, and \( \mathcal{N} \) is a normalization constant given by

\[ \mathcal{N} = \frac{1}{\sqrt{1 + |a|^2 + |b|^2 + 2|c|^2}}. \quad (1.13) \]

The purpose of this paper is two kinds. First one is to find the minimal powers \( (\mu_1)_{\text{min}}, (\mu_2)_{\text{min}}, \) and \( (q)_{\text{min}} \) which make \( \Delta_j \geq 0 \) when the corresponding powers are larger than the minimal powers. Second one is to examine whether or not the leftovers \( \Delta_j \quad (j = 1, 2, 3) \) can be true four-way SLOCC-invariant entanglement measures like the CKW inequality in three-qubit case. In order to explore these issues on the analytical ground as much as possible we confine ourselves into the second class state \( |G\rangle \). In Sec. II and Sec. III various tangles are computed analytically. In fact, the three-tangle of \( |G\rangle \) was computed in Ref.[22]. Since, however, there is some mistake in Ref.[22], we compute \( \tau_{ABC}^{1/q} \quad (q = 1, 2, \cdots) \)

\[3\text{ This is classified as } L_{abc} \text{ in Ref. [21].} \]
of $|G\rangle$ analytically in Sec. III. In Sec. IV we compute $\Delta_j$ analytically for few special cases. Exploiting the numerical results we compute the minimal powers for the cases. In Sec. V we compute the minimal powers for more general cases. In Sec. VI we examine whether or not $\Delta_j$ with particular powers can be SLOCC-invariant four-way entanglement measures. Our analysis indicate that this is impossible provided that the monogamy inequalities are derived merely by introducing weighting power factors. In Sec. VII a brief conclusion is given.

II. ONE- AND TWO-TANGLES

In order to compute the one-tangle we derive the state of the first qubit $\rho_1$:

$$\rho_1 = \text{tr}_{234} |G\rangle \langle G| = \frac{N_2^2}{4N_1^2} |0\rangle \langle 0| + \frac{N_1^2}{4N_1^2} |1\rangle \langle 1|$$  (2.1)

where $N_1$ and $N_2$ are

$$N_1 = \frac{1}{\sqrt{2(|a|^2 + |b|^2 + 2|c|^2)}} \quad N_2 = \frac{1}{\sqrt{2(2 + |a|^2 + |b|^2 + 2|c|^2)}}.$$  (2.2)

It is easy to show the equality $1/N_1^2 + 1/N_2^2 = 4/N^2$, which guarantees the normalization of $\rho_1$. Thus, the one-tangle of $\rho_1$ is given by

$$t_{1|234} \equiv 4\text{det}\rho_1 = \frac{N^4}{4N_1^2N_2^2}.\quad (2.3)$$

In fact, one can show $t_{1|234} = t_{2|134} = t_{3|124} = t_{4|123}$.

In order to compute the two-tangles we derive the two-qubit states, which are obtained by taking the partial trace over the remaining qubits. The final results can be represented as the following matrices in the computational basis:

$$\rho_{12} = N^2 \begin{pmatrix} \frac{|a|^2 + |b|^2}{2} & 0 & 0 & \frac{|a|^2 - |b|^2}{2} \\ 0 & 1 + |c|^2 & c^* & 0 \\ 0 & c & |c|^2 & 0 \\ \frac{|a|^2 - |b|^2}{2} & 0 & 0 & \frac{|a|^2 + |b|^2}{2} \end{pmatrix}, \quad \rho_{13} = N^2 \begin{pmatrix} \beta & 0 & 0 & \alpha \\ 0 & \gamma + 1 & \delta^* & 0 \\ 0 & \delta & 0 \\ \alpha & 0 & 0 & \beta \end{pmatrix},$$

$$\rho_{14} = N^2 \begin{pmatrix} \gamma' + 1 & 0 & 0 & \delta' \\ 0 & \beta' & \alpha' & 0 \\ 0 & \alpha' & \beta' & 0 \\ (\delta')^* & 0 & 0 & \gamma' \end{pmatrix}\quad (2.4)$$
Following the Wootters procedure\cite{11} one can compute the two-tangles of the two-qubit reduced states \( t_{ij} = C^2(\rho_{ij}) \) straightforwardly. The final expressions of the concurrences can be written as follows:

\[
\begin{align*}
C(\rho_{12}) &= \left\{
\begin{array}{l}
\mathcal{N}^2 \max \{2 |c| - (|a|^2 + |b|^2), 0\}, \\
\mathcal{N}^2 \max \{|a|^2 - |b|^2 - 2 |c| \sqrt{1 + |c|^2}, 0\}, \\
\mathcal{N}^2 \max \{|b|^2 - |a|^2 - 2 |c| \sqrt{1 + |c|^2}, 0\},
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
&|c| \left[\sqrt{1 + |c|^2} + 1\right] \geq \{|a|^2, |b|^2\} \\
&|a|^2 \geq \{|b|^2, |c| \left[\sqrt{1 + |c|^2} + 1\right]\} \\
&|b|^2 \geq \{|a|^2, |c| \left[\sqrt{1 + |c|^2} + 1\right]\}
\end{align*}
\]

\[
\begin{align*}
C(\rho_{13}) &= \left\{
\begin{array}{l}
2 \mathcal{N}^2 \max \{\delta - \beta, 0\}, \\
2 \mathcal{N}^2 \max \{\alpha - \sqrt{\gamma (\gamma + 1), 0\}}, \\
2 \mathcal{N}^2 \max \{-\alpha - \sqrt{\gamma (\gamma + 1), 0\}},
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
&\sqrt{\gamma (\gamma + 1) + |\delta|} \geq \{\beta + \alpha, \beta - \alpha\} \\
&\beta + \alpha \geq \{\beta - \alpha, \sqrt{\gamma (\gamma + 1) + |\delta|}\} \\
&\beta - \alpha \geq \{\beta + \alpha, \sqrt{\gamma (\gamma + 1) + |\delta|}\}
\end{align*}
\]

\[
\begin{align*}
C(\rho_{14}) &= \left\{
\begin{array}{l}
2 \mathcal{N}^2 \max \{\delta' - \beta', 0\}, \\
2 \mathcal{N}^2 \max \{\alpha' - \sqrt{\gamma' (\gamma' + 1), 0\}}, \\
2 \mathcal{N}^2 \max \{-\alpha' - \sqrt{\gamma' (\gamma' + 1), 0\}},
\end{array}
\right.
\end{align*}
\]

\[
\begin{align*}
&\sqrt{\gamma' (\gamma' + 1) + |\delta'|} \geq \{\beta' + \alpha', \beta' - \alpha'\} \\
&\beta' + \alpha' \geq \{\beta' - \alpha', \sqrt{\gamma' (\gamma' + 1) + |\delta'|}\} \\
&\beta' - \alpha' \geq \{\beta' + \alpha', \sqrt{\gamma' (\gamma' + 1) + |\delta'|}\}
\end{align*}
\]

where \( a \geq \{b, c\} \) means \( a \geq b \) and \( a \geq c \).

### III. THREE-TANGLE

In order to compute the three-tangles we should derive the three-qubit states by taking partial trace over the remaining qubit. For example, \( \rho_{123} \) can be written as

\[
\rho_{123} \equiv \text{tr}_4 |G\rangle \langle G| = p |\psi_1\rangle \langle \psi_1| + (1 - p) |\psi_2\rangle \langle \psi_2|,
\]

where \( p = \mathcal{N}^2/(4\mathcal{N}_1^2) \) and

\[
|\psi_1\rangle = \mathcal{N}_1 \left[(a - b)|001\rangle + 2c|010\rangle + (a + b)|111\rangle\right]
\]

\[
|\psi_2\rangle = \mathcal{N}_2 \left[(a + b)|000\rangle + 2|011\rangle + 2c|101\rangle + (a - b)|110\rangle\right].
\]

The residual entanglements of \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are

\[
\tau_3(\psi_1) = 0 \quad \tau_3(\psi_2) = 64\mathcal{N}_2^4 |(a^2 - b^2)c|.
\]
In order to compute the three-way entanglements of $\rho_{123}$ we consider the superposed state

$$|\Psi(p, \varphi)\rangle = \sqrt{p}|\psi_1\rangle + e^{i\varphi}\sqrt{1-p}|\psi_2\rangle. \quad (3.4)$$

If the phase factor $\varphi$ is chosen as

$$\varphi = \varphi_{\pm} = -\frac{\theta_1 - \theta_2}{2} \pm \frac{\pi}{2} \quad (3.5)$$

with $\theta_1 = \text{Arg}[(a^2 - b^2)c]$ and $\theta_2 = \text{Arg}[(a^2 - c^2)(b^2 - c^2)]$, the residual entanglement of $|\Psi(p, \varphi)\rangle$ becomes

$$\tau_3(\Psi(p, \varphi_{\pm})) = 64N_1^2N_2^2(1-z)|(a^2 - c^2)(b^2 - c^2)|(1-p)|p-p_0| \quad (3.6)$$

where

$$z = -\frac{N_2^2}{N_1^2} \left| \frac{(a^2 - b^2)c}{(a^2 - c^2)(b^2 - c^2)} \right| \quad p_0 = \frac{z}{z-1}. \quad (3.7)$$

Since $z \leq 0$, we get $0 \leq p_0 \leq 1$. Thus, $\tau_3(\Psi(p, \varphi_{\pm}))$ becomes zero at $p = p_0$. It is easy to show that at the region $0 \leq p \leq p_0$ the sign of the second derivative of $\tau_3(\Psi(p, \varphi_{\pm}))$ becomes

$$\frac{d^2}{dp^2}\tau_3(\Psi(p, \varphi_{\pm})) \geq 0 \quad \frac{d^2}{dp^2}\frac{1}{2}\tau_3(\Psi(p, \varphi_{\pm})) \leq 0 \quad (q = 2, 3, 4, \cdots). \quad (3.8)$$

Since the three-way entanglement $t_{123}$ should be convex in the entire range of $p$, we have to adopt an appropriate convexification procedure appropriately. For, example, the optimal decomposition of $t_{123}^{(1)}$ is

$$\rho_{123}(p) = \begin{cases} \frac{p}{2p_0} \left[ |\Psi(p_0, \varphi_+\rangle\langle\Psi(p_0, \varphi_+)| + |\Psi(p_0, \varphi_-)\rangle\langle\Psi(p_0, \varphi_-)| \right] + \frac{p-p_0}{p_0} |\psi_2\rangle\langle\psi_2| & (0 \leq p \leq p_0) \\ \frac{1-p}{2(1-p_0)} \left[ |\Psi(p_0, \varphi_+\rangle\langle\Psi(p_0, \varphi_+)| + |\Psi(p_0, \varphi_-)\rangle\langle\Psi(p_0, \varphi_-)| \right] + \frac{p-p_0}{p_0} |\psi_1\rangle\langle\psi_1| & (p_0 \leq p \leq 1) \end{cases} \quad (3.9)$$

The resulting $t_{123}^{(1)}$ is

$$t_{123}^{(1)} = \left[ 8N_2^2 \sqrt{|(a^2 - b^2)c|} \left( 1 - \frac{p}{p_0} \right) \right]^{\mu_1} \quad (0 \leq p \leq p_0) \quad (3.10)$$

$$= 0 \quad (p_0 \leq p \leq 1).$$

The optimal decomposition for $t_{123}^{(2)}$ at the region $0 \leq p \leq p_0$ is different from Eq. (3.9) as

$$\rho_{123}(p) = \frac{1}{2} \left[ |\Psi(p, \varphi_+\rangle\langle\Psi(p, \varphi_+)| + |\Psi(p, \varphi_-)\rangle\langle\Psi(p, \varphi_-)| \right] \quad (3.11)$$
and the resulting $t^{(2)}_{1|2|3}$ becomes

$$ t^{(2)}_{1|2|3} = \begin{cases} 
64N_1^2N_2^2(1-z)|(a^2-c^2)(b^2-c^2)|(1-p)(p_0-p) \mu^2 & (0 \leq p \leq p_0) \\
0 & (p_0 \leq p \leq 1).
\end{cases} \tag{3.12} $$

The optimal decomposition for $t^{(3)}_{1|2|3}$ is exactly the same with that of $t^{(1)}_{1|2|3}$ and the resulting $t^{(3)}_{1|2|3}$ is

$$ t^{(3)}_{1|2|3} = \begin{cases} 
64N_2^2|a^2-b^2)c| \left(1 - \frac{p}{p_0}\right)^q & (0 \leq p \leq p_0) \\
0 & (p_0 \leq p \leq 1).
\end{cases} \tag{3.13} $$

One can show straightforwardly that $t^{(a)}_{i|j|k}$ ($a = 1, 2, 3$) of other three parties are the same with $t^{(a)}_{1|2|3}$ in the second class $|G\rangle$.

**IV. FEW SPECIAL CASES**

In this section we examine the minimal power which makes $\Delta_j$ to be positive when the power factors are larger than the corresponding minimal powers.

**A. special case I: $b = c = ia$**

In this subsection we examine the minimal powers $(\mu_1)_{\text{min}}, (\mu_2)_{\text{min}}, (q)_{\text{min}}$, which make $\Delta_j$ positive when $a$ is positive and $b = c = ia$. In this case the normalization constants given in Eqs. (1.13) and (2.2) become

$$ \mathcal{N} = \frac{1}{\sqrt{1+4a^2}}, \quad \mathcal{N}_1 = \frac{1}{\sqrt{8a}}, \quad \mathcal{N}_2 = \frac{1}{2\sqrt{1+2a^2}}. \tag{4.1} $$

Thus, the one-tangle $t_{1|234}$ simply reduces to

$$ t_{1|234} = \frac{8a^2(1+2a^2)}{(1+4a^2)^2}. \tag{4.2} $$

Since Eq. (2.5) yields

$$ \alpha = -\alpha' = a^2, \quad \beta = \beta' = \frac{3}{2}a^2, \quad \gamma = \gamma' = \frac{a^2}{2}, \quad \delta^* = \delta' = \frac{1+i}{2}a, \tag{4.3} $$

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FIG. 1: (Color online) The power factor dependence of the regions $\Delta_j > 0$ with varying $a$. From Fig. 1(a) $\Delta_1$ and $\Delta_2$ become positive regardless of $a$ if $\mu_1 = 2\mu_2 \geq (\mu_1)_{\min}$, where $(\mu_1)_{\min} = 2.152$. Fig. 1(b) shows that $\Delta_3$ becomes positive regardless of $a$ if $q \geq (q)_{\min}$, where $(q)_{\min} = 2.305$.

The concurrences given in Eq. (2.6) become

$$C(\rho_{12}) = \begin{cases} \frac{2a(1-a)}{1+4a^2} & 0 \leq a \leq 1 \\ 0 & 1 \leq a \end{cases}$$

(4.4)

$$C(\rho_{13}) = C(\rho_{14}) = \begin{cases} \frac{a}{1+4a^2}(\sqrt{2} - 3a) & 0 \leq a \leq \frac{\sqrt{2}}{3} \\ 0 & \frac{\sqrt{2}}{3} \leq a \leq \sqrt{\frac{2}{3}} \\ \frac{a}{1+4a^2}(2a - \sqrt{2 + a^2}) & \sqrt{\frac{2}{3}} \leq a \end{cases}$$

(4.5)

The parameters $p$, $z$, and $p_0$ defined in the previous section are given by $p = 2a^2/(1 + 4a^2)$, $z = -\infty$, and $p_0 = 1$ in this special case. Using these the various three-way entanglements become

$$t^{(1)}_{1|2|3} = \left( \frac{\sqrt{8a^3}}{1 + 4a^2} \right)^{\mu_1} t^{(2)}_{1|2|3} = \left( \frac{8a^3}{(1 + 4a^2)^2} \right)^{\mu_2} t^{(3)}_{1|2|3} = \frac{8a^3(1 + 2a^2)q^{-2}}{(1 + 4a^2)^q}.$$  

(4.6)

In this special case $t^{(1)}_{1|2|3} = t^{(2)}_{1|2|3}$ when $\mu_1 = 2\mu_2$. However this relation does not hold generally. Combining Eqs. (4.2), (4.4), and (4.6), one can compute $\Delta_j$ defined in Eq. (1.9), whose expressions are

$$\Delta_j = -3t^{(j)}_{1|2|3} + \frac{2a^2}{1 + 4a^2}f(a)$$

(4.7)
where

\[
f(a) = \begin{cases} 
  a(4 + 6\sqrt{2} - 3a) & 0 \leq a \leq \frac{\sqrt{2}}{3} \\
  2(1 + 2a + 3a^2) & \frac{\sqrt{2}}{3} \leq a \leq \sqrt{\frac{2}{3}} \\
  a(4 + a + 4\sqrt{2 + a^2}) & \sqrt{\frac{2}{3}} \leq a \leq 1 \\
  2 + 3a^2 + 4a\sqrt{2 + a^2} & 1 \leq a
\end{cases}
\] (4.8)

In Fig. 1 we plot the \(\mu_1\)-dependence of \(\Delta_1 > 0\) region and \(q\)-dependence of \(\Delta_3 > 0\) region with varying \(a\). From Fig. 1(a) \(\Delta_1\) and \(\Delta_2\) become positive regardless of \(a\) if \(\mu_1 = 2\mu_2 \geq (\mu_1)_{\text{min}}\), where \((\mu_1)_{\text{min}} = 2.152\). Fig. 1(b) shows that \(\Delta_3\) becomes positive regardless of \(a\) if \(q \geq (q)_{\text{min}}\), where \((q)_{\text{min}} = 2.305\).

**B. special case II: \(b = c > 0\)**

![Image of Fig. 2](image)

**FIG. 2:** (Color online) The power factor dependence of the regions \(\Delta_j > 0\) with varying \(a\). From Fig. 1(a) \(\Delta_1\) and \(\Delta_2\) become positive regardless of \(a\) and \(b\) if \(\mu_1 = 2\mu_2 \geq (\mu_1)_{\text{min}}\), where \((\mu_1)_{\text{min}} = 2.01\). Fig. 1(b) shows that \(\Delta_3\) becomes positive regardless of \(a\) and \(b\) if \(q \geq (q)_{\text{min}}\), where \((q)_{\text{min}} = 2.00\).

In this subsection we examine the minimal powers \((\mu_1)_{\text{min}}, (\mu_2)_{\text{min}}, (q)_{\text{min}}\), which make \(\Delta_j\) positive when \(a, b,\) and \(c\) are real and positive with \(b = c\). In this case the normalization constants given in Eqs. (1.13) and (2.2) become

\[
\mathcal{N} = \frac{1}{\sqrt{1 + a^2 + 3b^2}}, \quad \mathcal{N}_1 = \frac{1}{\sqrt{2(a^2 + 3b^2)}}, \quad \mathcal{N}_2 = \frac{1}{\sqrt{2(2 + a^2 + 3b^2)}}.
\] (4.9)
Using the various formula presented in the previous section the one-tangle is given by
\[ t_{1|234} = \frac{(a^2 + 3b^2)(2 + a^2 + 3b^2)}{(1 + a^2 + 3b^2)^2} \] (4.10)
and the concurrences are
\[ C(\rho_{12}) = \begin{cases} 
N^2 \max [2b - (a^2 + b^2), 0] & b(\sqrt{1 + b^2} + 1) \geq a^2 \\
N^2 \max [a^2 - b^2 - 2b\sqrt{1 + b^2}, 0] & b(\sqrt{1 + b^2} + 1) \leq a^2 
\end{cases} \] (4.11)
\[ C(\rho_{13}) = \begin{cases} 
2N^2 \max \left[ b(a + b) - \frac{|a-b|}{2} \sqrt{1 + \left(\frac{a-b}{2}\right)^2}, 0 \right] & \frac{(a+3b)^2}{4} \geq \frac{|a-b|}{2} \left( 1 + \sqrt{1 + (\frac{a-b}{2})^2} \right) \\
N^2 \max [2|a - b| - (a^2 + 2ab + 5b^2), 0] & \frac{(a+3b)^2}{4} \leq \frac{|a-b|}{2} \left( 1 + \sqrt{1 + (\frac{a-b}{2})^2} \right) 
\end{cases} \]
\[ C(\rho_{14}) = \begin{cases} 
2N^2 \max \left[ b(b - a) - \frac{a+b}{2} \sqrt{1 + \left(\frac{a+b}{2}\right)^2}, 0 \right] & b \geq a \text{ and } \frac{(a+3b)^2}{4} \geq \frac{a+b}{2} \left( 1 + \sqrt{1 + (\frac{a+b}{2})^2} \right) \\
N^2 \max [2(a + b) - (a^2 - 2ab + 5b^2), 0] & \text{elsewhere} \end{cases} \]
Since \( p_0 = 1 \) in this case too, the \( t_{1|23}^{(j)} \) given in Eqs. (3.10), (3.12), and (3.13) are expressed as
\[
t_{1|23}^{(1)} = \left( \frac{2\sqrt{b}|a^2 - b^2|}{1 + a^2 + 3b^2} \right)^{\mu_1}, \quad t_{1|23}^{(2)} = \left( \frac{4b|a^2 - b^2|}{(1 + a^2 + 3b^2)^2} \right)^{\mu_2}, \quad t_{1|23}^{(3)} = 2^{4-q}b|a^2 - b^2|\left( \frac{2 + a^2 + 3b^2}{1 + a^2 + 3b^2} \right)^{a-2}.
\]
(4.12)
As in the previous special case we have a relation \( t_{1|23}^{(1)} = t_{1|23}^{(2)} \) if \( \mu_1 = 2\mu_2 \).

In Fig. 2 the full parameter space is divided into two regions, i.e. \( \Delta_j > 0 \) and \( \Delta_j \leq 0 \) regions. The division enables us to find the minimal powers \( (\mu_1)_{\min}, (\mu_2)_{\min}, \) and \( q_{\min} \), which makes \( \Delta_j \geq 0 \) \( (j = 1, 2, 3) \) regardless of the parameters. Fig. 2 shows \( (\mu_1)_{\min} = 2(\mu_2)_{\min} = 2.01 \) and \( (q)_{\min} = 2.00 \) in this special case.

V. NUMERICAL ANALYSIS

In this section we compute the minimal powers \( (\mu_1)_{\min}, (\mu_2)_{\min}, \) and \( (q)_{\min} \) for some more general cases by making use of numerical approach. First, we consider \( b = c = ira \) with \( a > 0 \). Since \( p_0 = 1 \) in this case, \( t_{1|23}^{(j)} \) can be computed directly. One can show easily \( t_{1|1|3}^{(1)} = t_{1|1|3}^{(2)} \) if \( \mu_1 = 2\mu_2 \) in this case too. Thus, we have a constraint \( (\mu_1)_{\min} = 2(\mu_2)_{\min} \).

| \( r \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 10 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|
| \( (\mu_1)_{\min} \) | 1.99 | 2.01 | 2.07 | 2.17 | 2.26 | 2.27 | 2.25 | 2.21 | 2.18 | 2.15 | 2.00 |
| \( (q)_{\min} \) | 1.97 | 2.01 | 2.09 | 2.31 | 3.31 | 13.6 | 2.91 | 2.41 | 2.33 | 2.31 | 2.00 |
Table I: The $r$-dependence of the minimal powers $(\mu_1)_{\text{min}}$ and $(q)_{\text{min}}$ when $b = c = ira$.

The $r$-dependence of the minimal powers $(\mu_1)_{\text{min}}$ and $(q)_{\text{min}}$ is summarized in Table I. Both powers increase with increasing $r$ from 0.1 to 0.6. Both decrease with increasing $r$ from 0.6 and seem to be saturated to $(\mu_1)_{\text{min}} = (q)_{\text{min}} = 2$ at large $r$. At $r = 0.6$ $(q)_{\text{min}}$ becomes very large as $(q)_{\text{min}} = 13.6$ while $(\mu_1)_{\text{min}}$ is not so large as $(\mu_1)_{\text{min}} = 2.27$.

Next, we consider $b = ira$ and $c = nra$, where $r$ and $a$ are real with integer $n$. The minimal powers can be computed by making use of the three-dimensional plot similar to Fig. 2. The results are summarized in Table II.

| $n$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $(\mu_1)_{\text{min}}$ | 2.13 | 2.03 | 2.01 | 2.00 | 2.00 |
| $(\mu_2)_{\text{min}}$ | 1.07 | 1.02 | 1.003 | 1.00 | 0.99 |
| $(q)_{\text{min}}$ | 2.28 | 2.10 | 1.98 | 1.85 | 1.91 |

Table II: The $n$-dependence of minimal powers when $b = ira$ and $c = nra$.

All minimal powers exhibit decreasing behavior with increasing $n$. In this case $(\mu_2)_{\text{min}}$ roughly equals to the half of $(\mu_1)_{\text{min}}$ as in the previous cases.

We also examine the case of $b = ni$ when $a$ and $c$ are real. The minimal powers of this case is summarized in Table III.

| $n$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $(\mu_1)_{\text{min}}$ | 2.28 | 2.03 | 1.99 | 1.99 | 1.99 |
| $(\mu_2)_{\text{min}}$ | 1.14 | 1.02 | 0.98 | 0.98 | 0.98 |
| $(q)_{\text{min}}$ | 2.36 | 2.04 | 1.98 | 1.97 | 1.97 |

Table III: The $n$-dependence of minimal powers when $b = ni$.

Similar to the previous case all minimal powers exhibit decreasing behavior with increasing $n$. In this case also $(\mu_2)_{\text{min}}$ roughly equals to the half of $(\mu_1)_{\text{min}}$.

Finally, we choose $N = 10000000$ second class states randomly with imposing $b = c$ and compute $\Delta_j$ with particular powers. The number of states which give negative $\Delta_1$ or $\Delta_2$ are summarized in Table IV.

| $\mu_1$ or $\mu_2$ | $\mu_1 = 2.0$ | 2.1 | 2.2 | 2.3 | $\mu_2 = 1.00$ | 1.05 | 1.10 | 1.15 |
|---------------------|---------------|-----|-----|-----|---------------|-----|-----|-----|
| No. states          | 1216071       | 191610 | 16818 | 0   | 1213371       | 191002 | 16755 | 0   |
Table IV: Number of states which give negative $\Delta_1$ or $\Delta_2$ for arbitrary chosen 10000000 states.

The number of states which give negative $\Delta_3$ are summarized in Table V.

| $q$  | 2.0 | 2.1 | 2.2 | 2.3 | 2.4 | 2.5 | 2.6 | 2.7 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|
| No. states | 1214527 | 429823 | 170308 | 49247 | 832 | 110 | 35 | 7 |

Table V: Number of states which give negative $\Delta_3$ for arbitrary chosen 10000000 states.

All the results discussed in section II and section III indicate that $(\mu_1)_{\min} \approx 2(\mu_2)_{\min} \geq 2.3$ and $(q)_{\min} \geq 14$, at least in the whole second class. However, as Table I and Table V indicate, the region of negative $\Delta_3$ in the parameter space is extremely small for $2.7 \leq q \leq 13$. Thus, it seems to be highly difficult to find such states in the random number generation.

VI. FOUR-WAY ENTANGLEMENT MEASURE

In this section we discuss a following question: Is it possible that the monogamy relation $\Delta_j(G)$ defined in Eq. (1.9) quantifies the SLOCC-invariant four-way entanglement in particular powers like a leftover of CKW inequality in three-way entanglement? In order to explore this question we note that for $n$-qubit system there are $2(2^n - 1) - 6n$ independent SLOCC-invariant monotones[23]. Thus, in four-qubit system there are six invariant monotones. Among them, it was shown in Ref. [24–26] by making use of the antilinearity[19] that there are following three independent invariant monotones which measure the true four-way entanglement:

\begin{align*}
F_1^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\nu \sigma_\lambda \sigma_2) \\
F_2^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\nu \sigma_\lambda \sigma_2) \cdot (\sigma_2 \sigma_\nu \sigma_\lambda \sigma_2) \\
F_3^{(4)} &= \frac{1}{2} (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \cdot (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \cdot (\sigma_\mu \sigma_2 \sigma_\nu \sigma_\lambda) \cdot (\sigma_\nu \sigma_2 \sigma_\nu \sigma_\lambda)
\end{align*}

where $\sigma_0 = \mathbb{1}$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, and the Einstein convention is understood with a metric $g^{\mu \nu} = \text{diag}\{-1,1,0,1\}$. The solid dot in Eq. (6.1) is defined as follows. Let $|\psi\rangle$ be a four-qubit state. Then, for example, $F_1^{(4)}$ of $|\psi\rangle$ is defined as

\begin{align*}
F_1^{(4)}(\psi) &= \left| \langle \psi^* | \sigma_\mu \otimes \sigma_\nu \otimes \sigma_2 \otimes \sigma_2 | \psi \rangle \langle \psi^* | \sigma^\mu \otimes \sigma_2 \otimes \sigma_\lambda \otimes \sigma_2 | \psi \rangle \langle \psi^* | \sigma_2 \otimes \sigma_\nu \otimes \sigma_\lambda \otimes \sigma_2 | \psi \rangle \right|.
\end{align*
Of course, other measures can be computed similarly. Thus, if $\Delta_j(G)$ properly quantifies the SLOCC-invariant four-way entanglement, it should be represented as a combination of $F_j^{(4)}$. For simplicity, we consider only the second class state $\text{(1.12)}$ with $b = c = ia$. In this case $\Delta_j$ is computed analytically in Eq. $\text{(4.7)}$. In this case $F_j^{(4)}$ becomes

$$F_j^{(4)} = \frac{48a^6}{(1 + 4a^2)^3} \quad F_j^{(4)} = \frac{96a^8}{(1 + 4a^2)^4} \quad F_j^{(4)} = \frac{3456a^{12}}{(1 + 4a^2)^6}. \quad (6.3)$$

These results are plotted in Fig. 3. Thus, all four-way entanglement measures $F_j^{(4)}$ exhibit monotonically increasing behavior with respect to $a$ when the quantum state is chosen as a second class $\text{(1.12)}$ with $b = c = ia$.

It is easy to show that $\Delta_j$ cannot be expressed in terms of $F_j^{(4)}$ because $\Delta_j$ have different expressions in the various range of $a$ as Eq. $\text{(4.8)}$ shows while $F_j^{(4)}$ have same expressions regardless of the range of $a$. For example, one can find a least-square fit of $\Delta_1$ with $\mu_1 = 3$ as

$$\Delta_1(\mu_1 = 3) \approx c_1F_1^{(4)} + c_2F_2^{(4)} + c_3F_3^{(4)} \quad (6.4)$$

where $c_1 = 10.117$, $c_2 = -30.8143$, and $c_3 = 5.7116$. The left- and right-handed sides of Eq. $\text{(6.4)}$ are plotted in Fig. 4 as solid and dashed lines. Although both exhibit similar
behavior, they do not coincide with each other exactly as expected. Same is true for $\Delta_2$ and $\Delta_3$. Thus, the monogamy constraints (1.10) derived by introducing a weighting factor in the power of the three-way entanglement cannot quantify the four-way entanglement properly in the SLOCC-invariant manner.

VII. CONCLUSIONS

In this paper we examine the properties of the three four-qubit monogamy relations presented in Eq. (1.9), all of which introduce the power factors $\mu_1$, $\mu_2$, and $q$ in the three-way entanglement. First, we examine the minimal powers $(\mu_1)_{\text{min}}$, $(\mu_2)_{\text{min}}$, and $(q)_{\text{min}}$, which make $\Delta_j \ (j = 1, 2, 3)$ to be positive when the powers are larger than the minimal powers. In order to explore this problem on the analytic ground as much as possible we confine ourselves into the second-class state $|G\rangle$ defined in Eq. (1.12). Our analysis indicates that $(\mu_1)_{\text{min}} \approx 2(\mu_2)_{\text{min}} \geq 2.3$ and $(q)_{\text{min}} \geq 14$.

Second, we try to provide an answer to a question “can the leftovers of the four-qubit monogamy relations with particular powers be a SLOCC-invariant four-way entanglement
measures like that of CKW inequality in three-qubit system?”. Our analysis indicates that this is impossible in the monogamy relations given in Eq. (1.9). Probably, same is true if monogamy relation is derived by introducing any form of weighting factors. Then, it is natural to ask a following question: Does the monogamy inequality exist in the multipartite qubit system, whose leftover quantifies the SLOCC-invariant entanglement measure? We do not have definite answer to this question.

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