A GENERAL APPROACH TO
JET CROSS SECTIONS IN QCD

Stefano FRIXIONE
Theoretical Physics, ETH, Zurich, Switzerland

Abstract

I illustrate a general formalism based upon the subtraction method for the
calculation of next-to-leading order QCD cross sections for any number of jets
in any type of hard collisions. I discuss the implementation of this formalism in
a numerical program which generates partonic kinematical configurations with
an appropriate weight, thus allowing the definition of arbitrary jet algorithms
and cuts matching the experimental setup at the last step of the computation.
I present results obtained with computer codes which calculate one-jet and
two-jet inclusive quantities in photon-hadron and hadron-hadron collisions.

1Work supported by the Swiss National Foundation.
1. Introduction

The study of jet production provides one of the most fundamental tests for the predictions of perturbative QCD. The jet rates are rather large, and allow for some of the best precision measurements in hadron physics. One-jet and two-jet inclusive distributions have been thoroughly measured in the past few years, and an impressive amount of data for even more exclusive quantities, like three or more jet cross sections, has been collected in $e^+e^-$ and hadron-hadron collisions. The high transverse momenta of the jets set the scale for QCD calculations. The coupling constant is therefore small enough to result in a reliable perturbative series. Next-to-leading order predictions for one-jet and two-jet inclusive quantities in hadronic collisions [1-3] have been available for some time. The uncertainty affecting these results is in general smaller than the corresponding experimental errors, and a detailed comparison between theory and experiments has been carried out. Although some issues need to be clarified (like, for example, the tail of the $E_T$ distribution measured by CDF [4], which appears to be higher than the theory), the overall agreement is quite satisfactory. In the near future, the increased luminosity at HERA will also allow a statistically significant study of large transverse momentum phenomena in photon-hadron collisions and DIS, thus giving another handle to test next-to-leading order QCD predictions [5-7].

The calculation of jet cross sections at next-to-leading order is rather complicated. A large number of infrared divergencies is found in the computation of virtual and real diagram contributions, due to the large number of colour-interacting, massless partons involved in the hard scattering processes. It is then necessary to devise a procedure which allows the analytic calculation of the divergent parts and shows their cancellation in the sum which defines any infrared-safe physical observable. This task has been accomplished in the past by using the slicing [8] and the subtraction methods. In ref. [9] a formalism adopting the subtraction method has been introduced in order to calculate one-jet and two-jet inclusive quantities in hadronic collisions. Although possible, the extension of this formalism to other type of processes or more exclusive observables is not straightforward. For this reason, a fully general formalism based on the subtraction method has been proposed in ref. [10], where the results were reported in a form motivated by the study of three-jet-like quantities in hadronic collisions. The formalism has been subsequently applied to the production of four
jets in $e^+e^-$ collisions in ref. [11], and the same calculation techniques have been used to study the hadroproduction of jets containing heavy quarks in ref. [12] (for other applications of the subtraction method, see refs. [13-17]). Afterwards, a general formalism has been presented in ref. [18], which uses the subtraction method to cancel the infrared divergencies and a new method (called dipole method) to perform the analytic treatment of the divergent terms. See also ref. [19] for another approach.

The aim of this paper is to illustrate some further improvements of the formalism of ref. [10], especially relevant for its implementation in a computer code. I begin by writing the formulae of ref. [10] in a form such that their generality is apparent, and the calculation of any infrared-safe cross section in an arbitrary hard collision is straightforward. I then show how to construct a computer code (which I will call parton generator) that generates partonic events, which are eventually used to plot infrared-safe quantities. In this way, the jet reconstruction algorithm and the definition of cuts matching the experimental conditions are inserted at the last step of the computation. During the same computer run one can therefore obtain cross sections for several, different, jet definitions, as well as predictions for other infrared-safe quantities, like for example shape variables. I point out that, in spite of these features, such a parton generator is not equivalent to the usual Monte Carlo parton shower programs, since it is the result of a fixed-order QCD calculation. Finally, I discuss the special case of processes with two or three partons in the final state, and I present few results obtained with parton generator codes written for photon-hadron and hadron-hadron collisions, suited for applications to HERA physics.

The paper is organized as follows: after presenting the formalism in section 2, in section 3 I show how to write a parton generator code, and I present numerical results. I report my conclusions in section 4. Technical details concerning the formalism adopted are collected in appendix A and appendix B.

2. Formalism

The goal is to calculate the cross section for some infrared-safe quantity in a given hard scattering process. To be specific, I start with the production of $N-1$ jets in hadronic collisions. According to the factorization theorem in QCD [20], any

\footnote{The codes are available upon request.}
differential cross section can be written as

\[ d\sigma^{(H_1H_2)}(K_1, K_2) = \sum_{a_1a_2} \int dx_1 dx_2 f_{\alpha_1}^{(H_1)}(x_1)f_{\alpha_2}^{(H_2)}(x_2)d\hat{\sigma}_{a_1a_2}(x_1K_1, x_2K_2), \tag{2.1} \]

where \( H_1 \) and \( H_2 \) are the incoming hadrons, with momenta \( K_1 \) and \( K_2 \) respectively, \( f_{\alpha_i}^{(H_i)} \) is the non-calculable but universal distribution function for the parton \( a_i \) in the hadron \( H_i \), and \( d\hat{\sigma}_{a_1a_2} \) are the (subtracted) short-distance partonic cross sections. As shown in ref. [10], the cancellation of the infrared divergencies arising in the intermediate steps of the calculation at next-to-leading order can be fully performed at the level of the partonic cross sections. To prove this issue, the fact that initial state partons \( a_1 \) and \( a_2 \) are quarks and gluons is not crucial. This implies that the same proof holds true for quantities like \( d\hat{\sigma}_{\gamma a_2} \) (entering photon-hadron cross sections), \( d\hat{\sigma}_{ea_2} \) (entering DIS cross sections), as well as for the \( e^+e^- \) cross section \( d\hat{\sigma}_{e^+e^-} \). For this reason, in the following I will only deal with partonic cross sections; it is understood that initial state partons will have to be interpreted in a broad sense (that is, they can be quarks, gluons, photons and electrons, depending upon the type of physical hard scattering process one is interested to study. On the other hand, with final state partons I will always mean quarks and gluons). We will get the \((N-1)\)-jet cross section in the collision of particles \( A \) and \( B \) by using the equation

\[ d\sigma^{(AB)}(K_1, K_2) = \sum_{a_1a_2} \int dx_1 dx_2 \tilde{L}_{a_1a_2}^{(AB)}(x_1, x_2)d\hat{\sigma}_{a_1a_2}(x_1K_1, x_2K_2), \tag{2.2} \]

where \( \tilde{L}_{a_1a_2}^{(AB)}(x_1, x_2) \) is a suitable luminosity function. Eq. (2.2) reduces to eq. (2.1) with \( \tilde{L}_{a_1a_2}^{(H_1H_2)}(x_1, x_2) = f_{\alpha_1}^{(H_1)}(x_1)f_{\alpha_2}^{(H_2)}(x_2) \). In the very same way, from eq. (2.2) we get the photon-hadron cross section if we put

\[ \tilde{L}_{a_1a_2}^{(\gamma H_2)}(x_1, x_2) = \delta_{\gamma a_1}\delta(1-x_1)f_{\alpha_2}^{(H_2)}(x_2), \tag{2.3} \]

the electron-hadron cross section in the Weizsäcker-Williams approximation with

\[ \tilde{L}_{a_1a_2}^{(eH_2)}(x_1, x_2) = \delta_{e a_1}\delta(1-x_1)f_{\alpha_2}^{(H_2)}(x_2) \tag{2.4} \]

\( (f_{\gamma}^{(e)} \) is the Weizsäcker-Williams function), the DIS cross section with

\[ \tilde{L}_{a_1a_2}^{(eH_2)}(x_1, x_2) = \delta_{ea_1}\delta(1-x_1)f_{\alpha_2}^{(H_2)}(x_2), \tag{2.5} \]

\[^{3}\text{In this paper, I will never deal with the problem of ultraviolet divergencies. I assume that they are renormalized in a proper way.}\]
and the $e^+e^-$ cross section with
\begin{equation}
L_{e_1e_2}^{(e^+e^-)}(x_1, x_2) = \delta_{e^+a_1}\delta(1-x_1)\delta_{e^-a_2}\delta(1-x_2).
\end{equation}

At the next-to-leading order in QCD, I write the partonic cross sections for the production of $(N-1)$-jets as
\begin{equation}
\hat{d}\sigma_{a_1a_2} = \hat{d}\sigma_{a_1a_2}^{(0)} + \hat{d}\sigma_{a_1a_2}^{(1)}.
\end{equation}
The leading order term $\hat{d}\sigma_{a_1a_2}^{(0)}$ gets contributions from the processes
\begin{equation}
\begin{aligned}
a_1(k_1) + a_2(k_2) &\rightarrow a_3(k_3) + \cdots + a_{N+1}(k_{N+1}) \left( + e(k_e) \right),
\end{aligned}
\end{equation}
where the final state partons $a_i$, $i = 3, \ldots, N + 1$, are quarks and gluons, the electron in the final state is present only in the case of DIS (in which case, $a_1 = e$, $a_2 = g, q, \bar{q}$). I will denote the processes of eq. (2.8) as $(N-1)$-parton processes, apart from the presence of the electron in the final state. We can write
\begin{equation}
\hat{d}\sigma_{a_1a_2}^{(0)} = \frac{1}{(N-1)!} \sum_{\{a_i\}_{1}^{N+1}} \mathcal{M}^{(N-1)}(\{a_i\}_{1}^{N+1})\mathcal{S}_{N-1}d\phi_{N-1}.
\end{equation}
In this equation, $\mathcal{M}^{(N-1)}(\{a_i\}_{1}^{N+1})$ is the invariant amplitude for the process of eq. (2.8), squared, summed over final state and averaged over initial state colour and spin degrees of freedom and multiplied by the flux factor, and $d\phi_{N-1}$ is the phase-space for $N-1$ massless partons (plus the electron in DIS). In order to include the contribution of all the partonic processes initiated by $a_1 + a_2$, a sum over the flavours $g, u, \bar{u}, \ldots$ of final state partons has been performed; the statistical factor $1/(N-1)!$ has therefore to be inserted to avoid double counting. The quantity $\mathcal{S}_{N-1}$, called measurement function, embeds the definition of the momenta of the $N - 1$ jets in terms of the momenta of the $N - 1$ partons. I will discuss its properties at length in the following.

The next-to-leading order term is
\begin{equation}
\hat{d}\sigma_{a_1a_2}^{(1)} = d\sigma_{a_1a_2}^{(v)} + d\sigma_{a_1a_2}^{(r)} + d\sigma_{a_1a_2}^{(c)},
\end{equation}
where
\[ d\sigma_{a_1a_2}^{(v)} = \frac{1}{(N-1)!} \sum_{\{a_l\}_{1}^{N+1}} \mathcal{M}^{(N-1,v)}(\{a_l\}_{1}^{N+1}) \mathcal{S}_{N-1} d\phi_{N-1} \] (2.11)
is the contribution of the QCD loop corrections to processes in eq. (2.8),
\[ d\sigma_{a_1a_2}^{(r)} = \frac{1}{N!} \sum_{\{a_l\}_{3}^{N+2}} \mathcal{M}^{(N)}(\{a_l\}_{1}^{N+2}) \mathcal{S}_{N} d\phi_{N} \] (2.12)
is the contribution of the tree amplitude of \(N\)-parton processes, and \(d\sigma_{a_1a_2}^{(c)}\) is the contribution of the initial state collinear counterterms (if needed). Eq. (2.12) is analogous to eq. (2.9). The \(N-1\) jets have now to be defined in terms of the \(N\)-body partonic kinematics: this is accomplished by the measurement function \(S_N\).

Although the quantities defined in eqs. (2.11) and (2.12) and the collinear counterterms are infrared divergent, their sum in eq. (2.10) is finite provided that the corresponding observable is infrared safe. As it was shown in refs. [9,10], this requirement can be easily expressed in terms of the measurement functions. Explicitly, the conditions (infrared limits)
\[ \lim_{k_{i}^{0} \to 0} S_{N} = S_{N-1} , \qquad \lim_{k_{i} \parallel k_{j}} S_{N} = S_{N-1} , \] (2.13)
\[ \lim_{k_{i} \parallel k_{1}} S_{N} = S_{N-1} , \qquad \lim_{k_{i} \parallel k_{2}} S_{N} = S_{N-1} , \] (2.14)
with \(3 \leq i \leq N + 2, 3 \leq j \leq N + 2, i \neq j\), guarantee that the next-to-leading order contribution \(d\hat{\sigma}_{a_1a_2}^{(1)}\) is finite. In eqs. (2.13) and (2.14), \(S_{N-1}\) is constructed with the \((N-1)\)-parton kinematics obtained from the \(N\)-parton kinematics in the limits indicated. I point out that the proof of ref. [10] has been carried out for \(N = 4\) (which corresponds to three-jet production); nevertheless, the fact that \(N = 4\) has never been explicitly used, and therefore the proof is valid for an arbitrary \(N\). Also, the proof of ref. [10] exploits eqs. (2.13) and (2.14), which in that case follow from the specific jet definition adopted. Here I do not fix a jet definition, and therefore eqs. (2.13) and (2.14) play the role of conditions which must be fulfilled by the jet definition chosen to induce an infrared-safe cross section. Finally, notice that eqs. (2.13) and (2.14) imply that most of the multiple infrared limits of \(S_N\) vanish (by definition, in these limits two or more infrared divergencies overlap). This is because in these limits the \(N\)-parton kinematics reduces to a configuration where only \(N-m\)
partons \((m \geq 2)\) have non-vanishing transverse momentum, and it is not possible to define \(N-1\) jets with less than \(N-1\) hard transverse partons. The only non-vanishing multiple infrared limits are the soft-collinear ones

\[
\lim_{k_1^0 \to 0, \vec{k}_i \parallel \vec{k}_j} S_N = S_{N-1}, \quad \lim_{k_1^0 \to 0, \vec{k}_i \parallel \vec{k}_1} S_N = S_{N-1}, \quad \lim_{k_1^0 \to 0, \vec{k}_i \parallel \vec{k}_2} S_N = S_{N-1}.
\]

(2.15)

I remark that multiple infrared configurations which are not soft-collinear can contribute to QCD cross sections beyond next-to-leading order.

In order to prove that eq. (2.10) is finite, one has to evaluate analytically, with some suitable regularization, eq. (2.11), eq. (2.12) and the collinear counterterms and then show that the divergent terms mutually cancel. The structure of the divergencies in eq. (2.11) naturally arises from the calculation of loop integrals. The case of eq. (2.12) is more involved: in fact, the divergencies are due to the integration over the regions of the phase space where one parton is soft, or two partons are collinear (which I will call infrared singular regions). Due to the complexity of the \(N\)-body kinematics, one can not perform the analytic integration over the whole phase space. The best one can do is to deal with one soft-collinear singularity at a time. To achieve this goal in the framework of the subtraction method, in ref. [3] the \(N\)-body matrix elements squared (in that case, \(N = 3\)) were decomposed into single-singular terms (having, by definition, one soft-collinear singularity at most); each term was then integrated over the relevant infrared singular region. Although in principle this method can be extended to larger values of \(N\), the amount of algebraic calculations and analytic integrations required grows very rapidly, and poses serious difficulties already with \(N = 4\). In ref. [10] a different approach was proposed to overcome this problem. The key idea is to use the properties of the measurement functions to integrate over the infrared singular regions. In particular, the following decomposition can be exploited

\[
S_N = \sum_{i=3}^{N+2} S_i^{(0)} + \sum_{j=3}^{N+2} S_i^{(1)} \theta(k_{ij}^2 - k_{i\tau}^2).
\]

(2.16)

The terms in the RHS of this equation are defined by their behaviour close to the infrared singular regions. In particular

\[
S_i^{(0)} \neq 0 \quad \text{only if} \quad k_i^0 \to 0, \quad \vec{k}_i \parallel \vec{k}_1, \quad \vec{k}_i \parallel \vec{k}_2,
\]

(2.17)
I stress that $S^{(0)}_i$ and $S^{(1)}_{ij}$ vanish in the infrared limits not explicitly indicated in eqs. (2.17) and (2.18). Two remarks are in order here. Firstly, eqs. (2.17) and (2.18) only constrain the infrared limits of $S^{(0)}_i$ and $S^{(1)}_{ij}$. Therefore, these quantities can be redefined up to terms which vanish in these limits. This is the case of the functions $S^{(fin)}_i$ introduced in ref. [10], which have been re-absorbed in the present paper into $S^{(0)}_i$ and $S^{(1)}_{ij}$ (I will show in appendix A that this can be consistently accomplished). Secondly, in ref. [10], eqs. (2.16), (2.17) and (2.18) have been derived using a given jet definition. Nevertheless, as already observed there, it is easy to understand that $S^{(0)}_i$ and $S^{(1)}_{ij}$ fulfilling these equations can be defined starting from any infrared-safe prescription, since they are directly induced by eqs. (2.13) and (2.14). I will give explicit examples in the following.

Inserting eq. (2.16) into eq. (2.12), and exploiting eqs. (2.17) and (2.18), we see that $d\sigma^{(r)}_{a_1a_2}$ is split into a sum of terms each of which has one soft-collinear singularity at most. Therefore, this procedure is equivalent to a single-singular decomposition of the matrix elements squared, without requiring any algebraic computation. In the end, following ref. [10], the result for the next-to-leading order term is arranged as the sum of a $N$-parton contribution and of a $(N-1)$-parton contribution

$$d\hat{\sigma}^{(1)}_{a_1a_2} = d\hat{\sigma}^{(1,N)}_{a_1a_2} + d\hat{\sigma}^{(1,N-1)}_{a_1a_2}. \tag{2.19}$$

I report in appendix A and appendix B the explicit form of the quantities in the RHS of eq. (2.19). Each term contributing to eq. (2.19) is finite, and therefore we have an operational prescription for the numerical evaluation of an arbitrary jet cross section in the framework of the subtraction method.

## 3. Numerical calculations

I now turn to the problem of implementing the formalism of the previous section in a computer code. As a benchmark example, I use two-jet production in hadronic collisions, and define the jets through the algorithm introduced by Ellis and Soper in ref. [21] and formulated in terms of the quantities

$$d_i = k^2_{i\perp} \tag{3.1}$$
\[ R^2_{ij} = (\eta_i - \eta_j)^2 + (\varphi_i - \varphi_j)^2, \quad (3.2) \]

\[ d_{ij} = \min(k^2_{T_i}, k^2_{T_j}) \frac{R^2_{ij}}{D^2}. \quad (3.3) \]

The constant \( D \) is the jet-resolution parameter. We have (\( p_{J_i} \) are the jet momenta)

\[ S^{(0)}_i = \sum_{\sigma(J)} \delta (\vec{p}_{J_1} - \vec{k}_i) \delta (\vec{p}_{J_2} - \vec{k}_l) \times \theta(\min([d_i]) - d_i) \theta \left( p_{J_1T} - p_{1T}^{\text{min}} \right) \theta \left( p_{J_2T} - p_{2T}^{\text{min}} \right), \quad (3.4) \]

\[ S^{(1)}_{ij} = \sum_{\sigma(J)} \delta (\vec{p}_{J_1} - \vec{k}_i - \vec{k}_j) \delta (\vec{p}_{J_2} - \vec{k}_l) \times \theta(\min([d_{ij}]) - d_{ij}) \theta \left( p_{J_1T} - p_{1T}^{\text{min}} \right) \theta \left( p_{J_2T} - p_{2T}^{\text{min}} \right), \quad (3.5) \]

and \( S_3 \) is defined through eq. (2.10). Here, \( \{i, j, l\} = \{3, 4, 5\} \); \( \sum_{\sigma(J)} \) denotes the sum over the permutations of jet labels, with a normalization factor \( 1/2 \) inserted. \( \min([d_i]) \) \( (\min([d_{ij}])) \) is the minimum of the quantities \( d_\alpha, d_{\alpha\beta} \) with \( d_i \) \( (d_{ij}) \) excluded. \( p_{1T}^{\text{min}} \) and \( p_{2T}^{\text{min}} \) are the minimum observable transverse energies of the two jets (they are fixed, input parameters). Finally, when two partons are merged into a jet, eq. (3.5), the jet three-momentum is defined as the sum of the parton three-momenta; other definitions would only imply changing the argument of the first \( \delta \) in eq. (3.5). When only two partons are present in the final state, we have

\[ S_2 = \sum_{\sigma(J)} \delta (\vec{p}_{J_1} - \vec{k}_3) \delta (\vec{p}_{J_2} - \vec{k}_4) \theta \left( p_{J_1T} - p_{1T}^{\text{min}} \right) \theta \left( p_{J_2T} - p_{2T}^{\text{min}} \right). \quad (3.6) \]

It is very easy to check explicitly that eqs. (3.4), (3.5) and (3.6) fulfill the conditions on the measurement functions discussed in the previous section.

It is now possible to evaluate the quantity

\[ < H > = \sum_{a_1a_2} \int H L_{a_1a_2} \left( d\hat{\delta}^{(0)}_{a_1a_2} + d\hat{\delta}^{(1)}_{a_1a_2} \right), \quad (3.7) \]

where \( L_{a_1a_2} \) is the relevant parton luminosity and \( H \) is any function of the jet momenta. We may think of \( H \) as a product of \( \theta \) functions, implementing experimental cuts and selecting a bin of a given histogram. \( < H > \) will therefore be interpreted as the next-to-leading order QCD prediction for the cross section in that bin. The rules
for the numerical evaluation of eq. (3.7) can be read from eqs. (2.9), (A.3), (A.11), (A.16) and (A.26). Schematically, one has to perform the following operations:

1. generate the Bjorken $x$’s and the partonic kinematics;

2. evaluate the weight and the jet momenta for this kinematical configuration, as specified by eq. (3.7) and by the expressions of the partonic cross sections in eqs. (2.9), (A.3), (A.11), (A.16) and (A.26). Call an output routine with the weight and the jet momenta as entries;

3. in the output routine, put the weight in the histogram bin selected by the jet momenta.

I stress that the real computation is actually more complicated than the one I outlined above, since eqs. (A.3), (A.11) and (A.26) require subtractions. I will not discuss here the Monte Carlo calculation of a subtracted quantity, which is by now a standard procedure. The interested reader can find a thorough discussion in ref. [15]. Following the prescriptions implicit in eq. (3.7), it is possible to write a computer code which can calculate $\langle H \rangle$ for any well-defined quantity $H$. I call this code a *jet generator*. For a given choice of input parameters, it returns event by event the jet momenta (as defined by the measurement functions), which are eventually put by the user in some histogram bin.

By using the formulae collected in appendix A and B, the construction of a jet generator is therefore fairly straightforward, and allows the computation of the cross section for the production of any number of jets in any hard collision in the framework of the subtraction method, without requiring algebraic manipulations of the partonic transition amplitudes, as in ref. [9]. The measurement functions, embedding the jet definition, can be written by the user in his own computer routine.

The main drawback of such a jet generator is the following: the jet definition is used, through the measurement functions, to disentangle the singularities appearing in the real contribution. Therefore, to get the jet cross section for several, different, jet definitions, one has to perform several computer runs. Furthermore, a jet generator outputs only jet momenta, which implies that it is not possible to calculate non-jet-like infrared-safe observables, as for example shape variables. I will now show that it is not difficult to overcome these problems using the formalism of ref. [10]. To be specific, I start by considering again two-jet production is hadronic collisions. With
three partons in the final state, I introduce the following quantities, defined in terms of the partonic momenta

\[ P_i^{(0)} = \Theta_i^{(0)} \theta(k_{3T} + k_{4T} + k_{5T} - E_T^{\text{min}}), \quad (3.8) \]

\[ P_{ij}^{(1)} = \Theta_{ij}^{(1)} \theta(k_{3T} + k_{4T} + k_{5T} - E_T^{\text{min}}), \quad (3.9) \]

\[ P_3 = \sum_{i=3}^{5} \left( P_i^{(0)} + \sum_{j=3}^{5} \Theta_{ij}^{(1)} \theta(k_j^2 - k_{iT}^2) \right), \quad (3.10) \]

where \( \Theta_i^{(0)} \) and \( \Theta_{ij}^{(1)} \) are suitable products of \( \theta \) functions. With two partons in the final state, I define

\[ P_2 = 1 \cdot \theta(k_{3T} + k_{4T} - E_T^{\text{min}}). \quad (3.11) \]

I require that, close to the infrared singular regions, the quantities defined in eqs. (3.8) and (3.9) have the following properties

\[ P_i^{(0)} \neq 0 \quad \text{only if} \quad k_i^0 \to 0, \quad \bar{k}_i \parallel \bar{k}_1, \quad \bar{k}_i \parallel \bar{k}_2, \quad (3.12) \]

\[ P_{ij}^{(1)} \neq 0 \quad \text{only if} \quad k_i^0 \to 0, \quad k_j^0 \to 0, \quad \bar{k}_i \parallel \bar{k}_j \quad (3.13) \]

(which means that they vanish in the infrared limits not explicitly indicated), and that they are such that (a relabeling of the partons is understood, if necessary)

\[ \lim_{k_i^0 \to 0} P_3 = P_2, \quad \lim_{\bar{k}_i \parallel \bar{k}_j} P_3 = P_2, \quad (3.14) \]

\[ \lim_{\bar{k}_i \parallel k_1} P_3 = P_2, \quad \lim_{\bar{k}_i \parallel k_2} P_3 = P_2. \quad (3.15) \]

Furthermore, the following equation must be fulfilled

\[ \sum_{i=3}^{5} \left( \Theta_i^{(0)} + \sum_{j=3}^{5} \Theta_{ij}^{(1)} \theta(k_j^2 - k_{iT}^2) \right) \equiv 1. \quad (3.16) \]

If we choose the free parameter \( E_T^{\text{min}} \) such that \( E_T^{\text{min}} < 2 \min(p_{1T}^{\text{min}}, p_{2T}^{\text{min}}) \), eq. (3.10) implies that

\[ S_3 \equiv S_3 P_3, \quad S_2 \equiv S_2 P_2, \quad (3.17) \]

where the \( S \) functions were defined at the beginning of this section (notice that it is always possible to choose \( E_T^{\text{min}} \) without any reference to a specific jet definition. For
example, $E_{T}^{\min}$ can be less than twice the minimum transverse momentum observable by the detector). I now define quantities analogous to the partonic cross sections appearing in eqs. (2.9) and (2.19) (see also appendix A), by formally substituting the $S$ functions with the $P$ functions. I adopt the following notation

\begin{align*}
&d\hat{\sigma}^{(0)}_{a_1a_2}(P_2) = d\hat{\sigma}^{(0)}_{a_1a_2} |_{S_2 \to P_2}, \\
&d\hat{\sigma}^{(1,2)}_{a_1a_2}(P_2) = d\hat{\sigma}^{(1,2)}_{a_1a_2} |_{S_2 \to P_2}, \\
&d\sigma^{(in,f)}_{a_1a_2,i}(P^{(0)}_i) = d\sigma^{(in,f)}_{a_1a_2,i} |_{S^{(0)}_i \to P^{(0)}_i}, \\
&d\sigma^{(out,f)}_{a_1a_2,ij}(P^{(1)}_{ij}) = d\sigma^{(out,f)}_{a_1a_2,ij} |_{S^{(1)}_{ij} \to P^{(1)}_{ij}}, \\
&d\hat{\sigma}^{(1,3)}_{a_1a_2}(P_3) = \sum_{i=3}^{5} \left( d\sigma^{(in,f)}_{a_1a_2,i}(P^{(0)}_i) + \sum_{j=3 \atop j \neq i}^{5} d\sigma^{(out,f)}_{a_1a_2,ij}(P^{(1)}_{ij}) \right). 
\end{align*}

(3.18)

(3.19)

(3.20)

(3.21)

(3.22)

The quantity

\begin{equation}
[HS] = \sum_{a_1a_2} \int L_{a_1a_2} \left( HS_2 d\hat{\sigma}^{(0)}_{a_1a_2}(P_2) + HS_2 d\hat{\sigma}^{(1,2)}_{a_1a_2}(P_2) + HS_3 d\hat{\sigma}^{(1,3)}_{a_1a_2}(P_3) \right) 
\end{equation}

(3.23)

can be numerically evaluated, being finite. This can be easily understood by observing that eqs. (3.18)-(3.22) are finite thanks to eqs. (3.12)-(3.15); the proof is identical to the proof of ref. [10], which showed that eqs. (2.13), (2.14), (2.17) and (2.18) guarantee the finiteness of the next-to-leading order cross section. Furthermore, thanks to eq. (3.17), we have

\begin{equation}
<H> = [HS],
\end{equation}

(3.24)

where $<H>$ was evaluated in eq. (3.7). Therefore, every quantity which can be calculated with eq. (3.7) can also be calculated with eq. (3.23). Nevertheless, eq. (3.23) is much more flexible. In fact, its numerical evaluation requires the following steps

1. generate the Bjorken $x$’s and the partonic kinematics;

2. evaluate the weight for this kinematical configuration, as specified by eq. (3.23) and by the expressions of the partonic cross sections in eqs. (3.18)-(3.22). Call an output routine with the weight and the parton momenta as entries;
3. in the output routine, define the jet momenta as specified by the jet-finding algorithm embedded in the $S$ functions, and put the weight in the histogram bin selected by the jet momenta.

Therefore, the code implementing eq. (3.23) returns event by event the parton momenta, which are eventually manipulated by the user. I call such a code a parton generator. The main difference between a jet generator and a parton generator can be read from eqs. (3.7) and (3.23); while in eq. (3.7) the measurement functions $S$ enter the partonic cross sections and render them finite, in eq. (3.23) they have the same rôle of the function $H$. In eq. (3.23) the partonic cross sections are finite thanks to the $P$ functions, which can be chosen once and forever without any reference to a specific jet definition. For this reason, such a parton generator is able to plot, on an event-by-event basis, any infrared-safe quantity defined with two or three partons in the final state. Namely, in a single run it can produce one-jet and two-jet observables, with the jets defined by several algorithms, as well as various shape variables. Notice that eqs. (3.11) and (3.16) guarantee that the events are generated in the whole phase space. Some of them are eventually rejected, namely those having a kinematics which is not fulfilling

$$\theta(k_{3T} + k_{4T} + k_{5T} - E_{T}^{\text{min}}),$$

$$\theta(k_{3T} + k_{4T} - E_{T}^{\text{min}}).$$

Nevertheless, $E_{T}^{\text{min}}$ can always be chosen in such a way that eqs. (3.25) and (3.26) are less stringent than the physical cuts which are required to define any infrared-safe quantity. Technically, eqs. (3.25) and (3.26) are inserted to avoid multiple non soft-collinear infrared singularities, which do not contribute to the cross section at next-to-leading order. In the soft-collinear limits we have, analogously to eq. (2.15),

$$\lim_{k_0 \to 0, k_i \parallel k_j} P_3 = P_2, \quad \lim_{k_0 \to 0, k_i \parallel k_1} P_3 = P_2, \quad \lim_{k_0 \to 0, k_i \parallel k_2} P_3 = P_2.$$

It should be clear that the method described so far can be used, without any modification, to write a parton generator for photon-hadron or $e^+e^-$ collisions. In the latter case, we can also set $E_{T}^{\text{min}} = 0$, since multiple soft singularities are forbidden by energy conservation and configurations with partons collinear to the incoming leptons are not singular.
The situation is slightly more complicated with more than three particles in the final state in DIS (two or more jet production), and for three or more jet production in hadron-hadron, photon-hadron and $e^+e^-$ collisions. Indeed, it is easy to understand that with a four-body (or more) kinematics, the analogous of eqs. (3.25) and (3.26) are not enough to avoid multiple non soft-collinear infrared singularities. Eq. (3.25) and eq. (3.26) have to be substituted by suitable products of $\theta$ functions which vanish in the multiple non soft-collinear infrared regions. In the end, one should get quantities $P_N$ and $P_{N-1}$, defined analogously to $P_3$ and $P_2$ in eqs. (3.10) and (3.11), which fulfill eqs. (3.12)-(3.16) and (3.27) (with the formal substitutions $2 \rightarrow N-1$, $3 \rightarrow N$). The products of $\theta$ functions which substitute eqs. (3.25) and (3.26) must be chosen in such a way that

$$S_N \equiv S_N P_N, \quad S_{N-1} \equiv S_{N-1} P_{N-1}.$$  \hfill (3.28)

As already observed before, I point out that, in the case of hadron-hadron, photon-hadron and $e^+e^-$ collisions with two or three partons in the final state, with a single choice of the $P$ functions it is possible to fulfill eq. (3.28) for any measurement functions $S$. The same is not true for the other cases. In practice, if one chooses the $P$ functions in such a way that they only vanish extremely close to the multiple non soft-collinear infrared regions, then eq. (3.28) holds for all the physically meaningful choices of the measurement functions $S$. We therefore get again a parton generator which is able to plot, event by event, $(N-1)$-jet inclusive quantities with several different jet definitions, and shape variables which get contributions at next-to-leading order from $(N-1)$- and $N$-parton configurations.

In order to write a code for a parton generator, a definite choice for the $P$ functions has to be made. Although these functions can be rather freely chosen, the only constraints being eqs. (3.12)-(3.16) and (3.27), their properties are motivated by the properties of the measurement functions of a jet algorithm. Therefore, it is quite natural to use a jet algorithm to construct the $P$ functions. Notice that, by definition, this jet algorithm has nothing to do with the jet algorithm(s) eventually used to obtain predictions for physical observables.

I now consider the case of photon-hadron and hadron-hadron collisions with two or three partons in the final state. Using the prescription of ref. [21], we get, as can
be also directly seen from eqs. (3.4) and (3.5),

\[ \Theta^{(0)}_i = \theta(\text{min}([d_i]) - d_i), \]  
\[ \Theta^{(1)}_{ij} = \theta(\text{min}([d_{ij}]) - d_{ij}), \]  

where the relevant quantities have been introduced in eqs. (3.1)-(3.3). In order to fulfill eq. (3.16), we must have \( D < 2\pi/3 \). If we adopt the prescription of the cone algorithm \[22\], we get

\[ \Theta^{(0)}_i = \theta(R_{ij} - g_{ij})\theta(R_{il} - g_{il})\theta(R_{jl} - g_{jl})\theta(\text{min}(k_{jr}, k_{ir}) - k_{ir}), \]  
\[ \Theta^{(1)}_{ij} = \theta(g_{ij} - R_{ij})\theta(R_{il} - g_{il})\theta(R_{jl} - g_{jl})\theta(k_{lr} - \text{min}(k_{ir}, k_{jr})), \]  

where

\[ g_{ij} = \frac{k_{ir} + k_{jr}}{\text{max}(k_{ir}, k_{jr})} R. \]  

Eq. (3.16) requires that \( R < \pi/3 \).

I wrote a fortran code for a parton generator in photon-hadron collisions, and a fortran code for a parton generator in hadron-hadron collisions. The latter can be used also in the case of hadronic photon-hadron collisions, that is for photon-hadron interactions where the photon fluctuates into a hadronic state before undergoing a hard collision. Therefore, the codes are suitable for applications to HERA physics. In order to test the codes, I produced single-inclusive jet and two-jet observables. The results obtained with the photon-hadron (hadron-hadron) code have been compared with the results of ref. \[6\] (refs. \[6,3\]). In both cases, I found nice agreement.

In figures 1 and 2 I show the transverse energy of the single-inclusive jet, the azimuthal distance and the invariant mass of the pair of the two hardest jets, and the transverse thrust distributions for \( ep \) collisions in the HERA energy range, \( E_{CM} = 300 \) GeV, as predicted by the aforementioned parton generators. The Weizsäcker-Williams approximation has been adopted (the form of ref. \[23\], which includes non-logarithmic singular terms, has been used with \( Q^2_{WW} = 4 \) GeV\(^2\) and \( 0.2 < y < 0.8 \)), and therefore we are dealing with a photoproduction process. As such, both the pointlike photon cross section (whose contribution, obtained with the photon-hadron code, is shown in figure 1) and the hadronic photon cross section (figure 2, obtained with the hadron-hadron code) are sizeable. I point out that the curves presented in
Figure 1: Jet observables and transverse thrust in ep collisions (Weizsäcker-Williams approximation) at HERA. Pointlike photon only.
Figure 2: Jet observables and transverse thrust in ep collisions (Weizsäcker-Williams approximation) at HERA. Hadronic photon only.

the figures are not physical quantities, if taken separately, since only their sum is measurable. Nevertheless, here I am only interested in discussing the properties of parton generator codes. Phenomenological results for HERA physics will be reported in a forthcoming publication [24]. The four plots appearing in the same figure have been obtained with a single computer run. The jets have been defined using the cone algorithm [22] with \( R = 1 \) (solid histograms), and the algorithm of ref. [21], with \( D = 1 \) (dashed histograms). In the same run, I also computed the jet cross sections with the cone algorithm and \( R = 0.7 \). I do not show the results, since they are extremely close to the dashed histograms. I used the set MRSA' for the parton densities in the proton, and the set GRV-HO for the parton densities in the
photon. The value of $\Lambda_{QCD}$ was fixed at the value suggested by the MRSA' set, $\Lambda_5 = 152$ MeV. Factorization and renormalization scales have been taken equal to half of the total transverse energy of the event. The figures have been produced using eqs. (3.31) and (3.32), with $R = 1$, to define the $P$ functions; I verified that completely equivalent results can be obtained by using eqs. (3.29) and (3.30) with several values of $D$. This gives a consistency check on the method used to construct the parton generator codes. Also notice that the curves corresponding to the two different jet definitions adopted have comparable quality. Naively, one may think that the curves relevant for the jet definition which matches the prescription for the $P$ functions could converge somewhat faster. This is not true; the convergence properties are almost completely dictated by the behaviour of the $P$ functions and of the jet definitions close to the infrared singular regions, where they are strongly constrained by the infrared safeness conditions, which are definition-independent and must hold in any case.

Finally, I would like to comment on the numerical calculations performed with the subtraction method. The main drawback of the method is that in the subtracted integrals the cancellation takes place between terms which have different kinematics (see for example eq. (A.5)). For this reason, in ref. [10] the arbitrary parameters $\xi_{cut}$, $\delta_I$, $\delta_o$ have been introduced in the calculation (see also refs. [13-16]). To be specific, consider eq. (A.5); we see that the counterevent (that is, the integrand function calculated on the pole, $f(0)$) is subtracted only if $\xi_i < \xi_{cut}$. Therefore, with a suitable choice of $\xi_{cut}$, the counterevent is subtracted only if the event is close to the soft limit $\xi_i = 0$. In this way, the subtraction is performed only in those cases when the kinematics of the event and of the counterevent are quite close to each other, thus resulting in an improved numerical stability of the result. As a bonus, we save computing time, since in most of the cases the calculation of the counterevent is not necessary. It is very important to realize that the parameters $\xi_{cut}$, $\delta_I$, $\delta_o$ have nothing to do with the non-physical parameters which must be introduced in the slicing method. In the present case, the parameters do not have to be small, since no approximation was performed in the intermediate steps of the calculation. Therefore, there is no need to prove that the physical cross section is independent of $\xi_{cut}$, $\delta_I$, $\delta_o$, since this is true by construction. We can exploit this fact as a powerful test of the numerical implementation of the formalism, by verifying that the cross section is a constant with respect to the choice of $\xi_{cut}$, $\delta_I$, $\delta_o$. By choosing $(\xi_{cut},\delta_I,\delta_o)$ close to $(0,0,0)$, we also see that the slicing method can be obtained as a limit of
the subtraction method (for this to be rigorously true, the kinematics of the events close to the infrared limits must be set equal to the kinematics of the corresponding counterevents. For example, this amounts to set \( f(\xi_i) = f(0) \) for \( \xi_i < \xi_{\text{cut}} \) in eq. (A.5)). In this case, as it is apparent from eqs. (A.3), (A.11) and (A.16), cancellations between large numbers take place in the sum which defines the physical cross section.

4. Conclusions

In this paper I have studied the definition of infrared-safe cross sections at next-to-leading order in QCD. The formalism of ref. [10], which is based on the subtraction method, has been written in a form which clearly shows its generality and universality. The formulae can be applied to any hard scattering process, with an arbitrary number of final state partons. I then turned to the problem of the numerical computation of infrared-safe cross sections. It has been shown how to write a computer code which, event by event, outputs the momenta of the final state partons. The momenta are eventually used in an analysis routine to define the physical observables. The special case of infrared-safe quantities defined with two or three partons in the final state has been studied. Two codes, one dealing with photon-hadron collisions and one dealing with hadron-hadron collisions, have been written. Sample numerical results have been presented for the case of \( e\,p \) collisions in the Weizsäcker-Williams approximation. This is relevant for applications to HERA physics, where both the pointlike photon and the hadronic photon give sizeable contributions to the physical cross section. I presented predictions for one-jet and two-jet observables (as an example, I defined the jets using two different prescriptions), and for transverse thrust.

Acknowledgements

It is a pleasure to thank Z. Kunszt, P. Nason and G. Ridolfi for useful discussions. Part of the results presented in this paper are due to a collaboration with Z. Kunszt and G. Ridolfi.
References

[1] F. Aversa, M. Greco, P. Chiappetta and J. P. Guillet, Z. Phys. C46 (1990) 253; Phys. Rev. Lett. 65 (1990) 401.

[2] S. D. Ellis, Z. Kunszt and D. E. Soper, Phys. Rev. Lett. 64 (1990) 2121; Phys. Rev. Lett. 69 (1992) 1496; S. D. Ellis and D. E. Soper, Phys. Rev. Lett. 74 (1995) 5182.

[3] W. T. Giele, E. W. N. Glover and D. A. Kosower, Phys. Rev. Lett. 73 (1994) 2019.

[4] CDF Coll., F. Abe et al., Phys. Rev. Lett. 68 (1992) 1104; Phys. Rev. Lett. 70 (1993) 1376.

[5] L. E. Gordon and J. K. Storrow, Phys. Lett. B291 (1992) 320; G. Kramer and S. G. Salesh, Z. Phys. C61 (1994) 277; D. Bödeker, G. Kramer and S. G. Salesh, Z. Phys. C63 (1994) 471; M. Klasen and G. Kramer, Z. Phys. C72 (1996) 107, hep-ph/9511405; B. W. Harris and J. F. Owens, preprint FSU-HEP-970411, hep-ph/9704324.

[6] M. Klasen and G. Kramer, preprint DESY 96-246, hep-ph/9611450.

[7] E. Mirkes and D. Zeppenfeld, Phys. Lett. B380 (1996) 205, hep-ph/9511448; S. Catani and M. H. Seymour, proceedings of the Workshop Future Physics at HERA, p. 519, hep-ph/9609521.

[8] W. T. Giele and E. W. N. Glover, Phys. Rev. D46 (1992) 1980; W. T. Giele, E. W. N. Glover and D. A. Kosower, Nucl. Phys. B403 (1993) 633.

[9] Z. Kunszt and D. E. Soper, Phys. Rev. D46 (1992) 192.

[10] S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B467 (1996) 399, hep-ph/9512328.

[11] A. Signer and L. Dixon, Phys. Rev. Lett. 78 (1997) 811, hep-ph/9609460; hep-ph/9706285.

[12] S. Frixione and M. Mangano, Nucl. Phys. B483 (1997) 321, hep-ph/9605270.
[13] R. K. Ellis, D. A. Ross and A. E. Terrano, *Nucl. Phys.* **B178**(1981)421;
Z. Kunszt and P. Nason, in *Z Physics at LEP 1*, eds. G. Altarelli, R. Kleiss
and C. Verzegnassi, Geneva, 1989.

[14] B. Mele, P. Nason and G. Ridolfi, *Nucl. Phys.* **B357**(1991)409.

[15] M. Mangano, P. Nason and G. Ridolfi, *Nucl. Phys.* **B373**(1992)295.

[16] S. Frixione, P. Nason and G. Ridolfi, *Nucl. Phys.* **B383**(1992)3;
S. Frixione, *Nucl. Phys.* **B410**(1993)280;
S. Frixione, M. Mangano, P. Nason and G. Ridolfi, *Nucl. Phys.* **B412**(1994)225.

[17] B. W. Harris and J. Smith, *Nucl. Phys.* **B452**(1995)109.

[18] S. Catani and M. H. Seymour, *Phys. Lett.* **B378**(1996)287, hep-ph/9602277;
*Nucl. Phys.* **B485**(1997)291, hep-ph/9605323.

[19] Z. Nagy and Z. Trocsanyi, *Nucl. Phys.* **B486**(1997)189, hep-ph/9610498.

[20] J. C. Collins, D. E. Soper and G. Sterman, in *Perturbative Quantum Chromodynamics*, 1989, ed. Mueller, World Scientific, Singapore, and references therein.

[21] S. D. Ellis and D. E. Soper, *Phys. Rev.* **D48**(1993)3160.

[22] F. Aversa *et al.*, Proceedings of the Summer Study on High Energy Physics,
Research Directions for the Decade, Snowmass, CO, Jun 25 - Jul 13, 1990.

[23] S. Frixione, M. Mangano, P. Nason and G. Ridolfi, *Phys. Lett.* **B319**(1993)339.

[24] S. Frixione and G. Ridolfi, in preparation.

[25] R. K. Ellis and J. Sexton, *Nucl. Phys.* **B269**(1986)445.

[26] M. Glück, E. Reya and A. Vogt, *Phys. Rev.* **D45**(1992)3986;
S. Frixione, M. Mangano, P. Nason and G. Ridolfi, hep-ph/9702287, to appear
in *Heavy Flavours II*, eds. A. J. Buras and M. Lindner, World Scientific.
APPENDIX A: N-parton contribution

The $N$-parton contribution, first term in the RHS of eq. (2.19), is written as the sum of finite terms, whose form is induced by the decomposition of the measurement function of eq. (2.16)

$$d\hat{\sigma}^{(1,N)}_{a_1a_2} = \sum_{i=3}^{N+2} \left( d\sigma^{(in,f)}_{a_1a_2,i} + \sum_{j=3}^{N+2} \right) d\sigma^{(out,f)}_{a_1a_2;ij} . \tag{A.1}$$

The first term in the RHS of this equation is proportional to $S^{(0)}_i$, the second one to $S^{(1)}_{ij}$. By recalling the properties of the measurement functions close to the infrared limits, it is easy to choose a set of variables suitable to perform the calculation with the subtraction method. In the partonic center-of-mass frame, we parametrize the momentum of the final state parton $i$ as

$$k_i = \frac{\sqrt{S}}{2} \xi_i \left( 1, \sqrt{1 - y_i^2} e_i, y_i \right) , \tag{A.2}$$

where $\sqrt{S}$ is the partonic center-of-mass energy. We get [10]

$$d\sigma^{(in,f)}_{a_1a_2,i} = \frac{1}{N!} \frac{1}{2} \frac{1}{\xi_i} \left[ \left( \frac{1}{1 - y_i} \right) \delta_i + \left( \frac{1}{1 + y_i} \right) \delta_i \right] \frac{1}{2(2\pi)^3} \left( \frac{\sqrt{S}}{2} \right)^2$$

$$\times \left( 1 - y_i^2 \right) \xi_i^2 \sum_{\{a_l\}^{N+2}} \mathcal{M}^{(N)}(\{a_l\}^{N+2}) S^{(0)}_i d\phi d\xi d y_i d\varphi_i . \tag{A.3}$$

The measure $d\phi$ is implicitly defined by writing the phase space for $N$ partons (plus electron in DIS) in 4 dimensions as

$$d\phi_N = d\phi \frac{1}{2(2\pi)^3} \left( \frac{\sqrt{S}}{2} \right)^2 \xi_i d\xi d y_i d\varphi_i . \tag{A.4}$$

The statistical factor $1/N!$ is due to the sum over the flavours of final state partons (see eq.(2.12)). The damping factor $(1 - y_i^2) \xi_i^2$ and $S^{(0)}_i$ guarantee that the integrand is finite everywhere in the phase space. We can therefore use

$$\int_0^1 d\xi_i f(\xi_i) \left( \frac{1}{\xi_i} \right) = \int_0^1 d\xi_i \frac{f(\xi_i) - f(0) \theta(\xi_{cut} - \xi_i)}{\xi_i} , \tag{A.5}$$
\[
\int_{-1}^{1} dy_i g(y_i) \left( \frac{1}{1 \mp y_i} \right) \delta_j = \int_{-1}^{1} dy_i \frac{g(y_i) - g(\pm 1)\theta(\pm y_i - 1 + \delta_i)}{1 \mp y_i}, \quad (A.6)
\]

which hold for any smooth functions \( f \) and \( g \). These prescriptions are almost identical to the usual + prescription, except for the fact that the value of the integrand function at the pole is subtracted only if the integration variable satisfies the condition imposed by the \( \theta \) functions. The parameters

\[
0 < \xi_{cut} \leq 1, \quad 0 < \delta_i \leq 2 \quad (A.7)
\]

can be arbitrarily chosen in the indicated range. Finally, notice that if \( S_i^{(0)} \) vanishes in the soft (\( \xi_i \to 0 \)) and collinear (\( y_i \to \pm 1 \)) limits, eq. (A.3) becomes

\[
d\sigma^{(in,f)}_{a_1a_2} = \frac{1}{N!} \sum_{\{a_l\}_{i=1}^{N+2}} \mathcal{M}^{(N)}(\{a_l\}_{i=1}^{N+2}) S_i^{(0)} d\phi_N. \quad (A.8)
\]

This expressions is identical to the one induced by the functions \( S_i^{fin} \) in ref. [10], and therefore proves that the contribution of \( S_i^{fin} \) can be safely re-absorbed into the contribution of \( S_i^{(0)} \).

In order to give the explicit expression for the second term in the RHS of eq. (A.1), we rewrite eq. (A.2) as

\[
k_i = \sqrt{S} \frac{\xi_i}{2} (1, \hat{k}_i), \quad \hat{k}_i = \hat{p}_i R, \quad \hat{p}_i = (\mathbf{0}, 1), \quad (A.9)
\]

where \( R \) is a suitable matrix, and parametrize the momentum of parton \( j \) as

\[
k_j = \sqrt{S} \frac{\xi_j}{2} (1, \hat{k}_j), \quad \hat{k}_j = \hat{p}_j R, \quad \hat{p}_j = \left( \sqrt{1 - y_j^2\hat{e}_{jT}}, y_j \right). \quad (A.10)
\]

Therefore, in the limit \( y_j \to 1 \) the partons \( i \) and \( j \) become collinear to each other. We get [10]

\[
d\sigma^{(out,f)}_{a_1a_2ij} = \frac{1}{N!} \left( \frac{1}{\xi_{i} c} \right) \left( \frac{1}{1 - y_j} \right) \delta_o \left( 1 - y_j \right) \xi_i^2 \xi_j \sum_{\{a_l\}_{i=1}^{N+2}} \mathcal{M}^{(N)}(\{a_l\}_{i=1}^{N+2}) \\
\times S^{(1)}_{ij} (k_{jT}^2 - k_{iT}^2) \left( \frac{1}{2(2\pi)^3} \left( \frac{\sqrt{S}}{2} \right) \right)^2 \tilde{d}\phi d\xi_i d\xi_j dy_i dy_j d\varphi_i d\varphi_j. \quad (A.11)
\]
where
\[
\int_{-1}^{1} dy_j g(y_j) \left( \frac{1}{1 - y_j} \right) = \int_{-1}^{1} dy_j \frac{g(y_j) - g(1)\theta(y_j - 1 + \delta_o)}{1 - y_j} \quad (A.12)
\]
and \(0 < \delta_o \leq 2\). The phase-space for \(N\) partons (plus electron in DIS) in 4 dimensions has been written in the following way, thus implicitly defining \(d\phi\)
\[
d\phi_N = d\tilde{\phi} \left( \frac{1}{2(2\pi)^3} \left( \frac{\sqrt{S}}{2} \right)^2 \right)^2 \xi_i \xi_j d\xi_i d\xi_j dy_i dy_j d\varphi_i d\varphi_j . \quad (A.13)
\]
If \(S_{ij}^{(1)}\) vanishes in the soft (\(\xi_i \to 0\)) and collinear (\(y_j \to 1\)) limits, we get
\[
d\sigma^{(\text{out},f)}_{a_1a_2,ij} = \frac{1}{N!} \sum_{\{a_i\}_3}^{N+2} M^{(N)}(\{a_i\}_1^{N+1}) S_{ij}^{(1)} \theta(k^2_{JT} - k^2_{IT}) d\phi_N . \quad (A.14)
\]
Like eq. (A.8), this shows that the contribution of \(S_{ij}^{\text{fin}}\) can also be re-absorbed into the contribution of \(S_{ij}^{(1)}\).

1.1. (N-1)-parton contribution

We now turn to the second term in the RHS of eq. (2.19). We write it in the following way
\[
d\hat{\sigma}_{N-1}^{(1, N-1)} = d\hat{\sigma}_{a_1a_2}^{(1, N-1)v} + d\hat{\sigma}_{a_1a_2}^{(1, N-1)r} \quad (A.15)
\]
where the first term reads [10]
\[
d\hat{\sigma}_{a_1a_2}^{(1, N-1)v} = \frac{\alpha_s}{2\pi} \sum_{\{a_i\}_3}^{N+1} Q(\{a_i\}_1^{N+1}) d\sigma^{(0)}(\{a_i\}_1^{N+1})
\]
\[
+ \frac{\alpha_s}{2\pi} \sum_{i,j=1}^{N+1} T_{ij}^{(\text{reg})} \sum_{\{a_i\}_3}^{N+1} d\sigma^{(0)}_{ij}(\{a_i\}_1^{N+1})
\]
\[
+ \frac{\alpha_s}{2\pi} \sum_{i,j=1}^{N+1} M^{(N-1,v)}(\{a_i\}_1^{N+1}) S_{N-1} d\phi_{N-1} . \quad (A.16)
\]
Here

\[
\mathcal{I}^{(\text{reg})}_{ij} = \frac{1}{8\pi^2} \left[ \frac{1}{2} \log^2 \frac{\xi^2_{\text{cut}} S}{Q^2} + \log \frac{\xi^2_{\text{cut}} S}{Q^2} \log \frac{k_j \cdot k_i}{2E_j E_i} - \text{Li}_2 \left( \frac{k_j \cdot k_i}{2E_j E_i} \right) \right.
\]

\[
+ \frac{1}{2} \log^2 \frac{2k_j \cdot k_i}{E_j E_i} - \log \left( 4 - \frac{2k_j \cdot k_i}{E_j E_i} \right) \log \frac{k_j \cdot k_i}{2E_j E_i} - 2 \log^2 2 \right]
\]

(A.17)

and

\[
\mathcal{Q}(\{a_i\}_{1}^{N+1}) = \sum_{j=3}^{N+1} \left[ \gamma'(a_j) - \log \frac{S\delta_o}{2Q^2} \left( \gamma(a_j) - 2C(a_j) \log \frac{2E_j}{\xi_{\text{cut}} \sqrt{S}} \right) \right.
\]

\[
+ 2C(a_j) \left( \log^2 \frac{2E_j}{\sqrt{S}} - \log^2 \xi_{\text{cut}} \right) - 2\gamma(a_j) \log \frac{2E_j}{\sqrt{S}} \right]
\]

\[
- \log \frac{\mu^2}{Q^2} \left( \gamma(a_1) + 2C(a_1) \log \xi_{\text{cut}} + \gamma(a_2) + 2C(a_2) \log \xi_{\text{cut}} \right). \] (A.18)

Here \(E_i\) is the energy of parton \(i\) in the partonic center-of-mass frame, \(\mu\) is the factorization scale, and \(Q\) is the Ellis-Sexton scale (see below). The expressions for the colour factors \(C(a)\), \(\gamma(a)\) and \(\gamma'(a)\) are

\[
C(g) = C_A, \quad C(q) = C_F, \quad \gamma(g) = \frac{11C_A - 4T_F N_f}{6}, \quad \gamma(q) = \frac{3}{2} C_F, \quad \gamma'(g) = \frac{67}{9} C_A - \frac{2\pi^2}{3} C_A - \frac{23}{9} T_F N_f, \quad \gamma'(q) = \frac{13}{2} C_F - \frac{2\pi^2}{3} C_F. \] (A.19-21)

Notice that eq. (A.18) depends upon the flavours of the initial state partons. In the case of photon-hadron collisions (DIS), we have \(a_1 = \gamma (a_1 = e)\), and in the case of \(e^+e^-\) collisions we have \(a_1 = e^+, a_2 = e^-\). Eq. (A.18) still holds, and we just define

\[
C(a) = 0, \quad \gamma(a) = 0 \quad \text{if} \quad a = \gamma, e. \] (A.22)

We also defined

\[
d\sigma^{(0)}(\{a_i\}_{1}^{N+1}) = \frac{1}{(N-1)!} \mathcal{M}^{(N-1)}(\{a_i\}_{1}^{N+1}) S_{N-1} d\phi_{N-1}, \] (A.23)
which is identical to eq. (2.9) except for the fact that the sum over the flavour of final state partons is not performed, and

\[
\begin{align*}
\frac{d\sigma_{ij}^{(0)}}{d\phi_N} (\{a_l\}_{1}^{N+1}) &= \frac{1}{(N-1)!} M_{ij}^{(N-1)} (\{a_l\}_{1}^{N+1}) S_{N-1} d\phi_{N-1}. \\
\end{align*}
\]

(A.24)

The functions \(M_{ij}^{(N-1)}\) are usually denoted as colour-linked Born squared amplitudes. They enter the expression of the virtual corrections to the \((N-1)\)-parton processes, eq. (2.11). We adopt the following form [25,9]

\[
M_{ij}^{(N-1,\nu)} = \frac{\alpha_s}{2\pi} \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left[ - \left( \frac{1}{\epsilon^2} \sum_{i=1}^{N+1} C(a_i) + \frac{1}{\epsilon} \sum_{i=1}^{N+1} \gamma(a_i) \right) M^{(N-1)} \right. \\
+ \left. \frac{1}{2\epsilon} \sum_{i,j=1}^{N+1} \log \left( \frac{2k_i \cdot k_j}{Q^2} \right) \frac{1}{8\pi^2} M_{ij}^{(N-1)} + M_{ij}^{(N-1,\nu)} \right].
\]

(A.25)

This equation also defines the finite part of the virtual correction, \(M_{ij}^{(N-1,\nu)}\), which enters eq. (A.16).

The second term in the RHS of eq. (A.13) is

\[
\begin{align*}
\frac{d\hat{\sigma}_{a_1 a_2}^{(1,N-1)}}{d\phi_{a_1 a_2}} &= \frac{\alpha_s}{2\pi} \sum_d \left\{ \xi P_{da1}^< (1 - \xi, 0) \left[ \frac{1}{\xi} \log \frac{S\delta_j}{2\mu^2} + 2 \left( \frac{\log \xi}{\xi} \right) c \right] \\
- \xi P_{da1}^< (1 - \xi, 0) \left( \frac{1}{\xi} \right) c - K_{da1} (1 - \xi) \right\} C_{da1} \\
\times \sum_{\{a_l\}_{3}^{N+1}} d\sigma^{(0)} (d, a_2, \{a_l\}_{3}^{N+1}; (1 - \xi)k_1, k_2) d\xi \\
+ \frac{\alpha_s}{2\pi} \sum_d \left\{ \xi P_{da2}^< (1 - \xi, 0) \left[ \frac{1}{\xi} \log \frac{S\delta_j}{2\mu^2} + 2 \left( \frac{\log \xi}{\xi} \right) c \right] \\
- \xi P_{da1}^< (1 - \xi, 0) \left( \frac{1}{\xi} \right) c - K_{da1} (1 - \xi) \right\} C_{da2} \\
\times \sum_{\{a_l\}_{3}^{N+1}} d\sigma^{(0)} (a_1, d, \{a_l\}_{3}^{N+1}; k_1, (1 - \xi)k_1, (1 - \xi)k_2) d\xi,
\end{align*}
\]

(A.26)

where \(P_{ab}^< (z, 0) + \epsilon P_{ab}^\epsilon (z, 0) + O(\epsilon^2)\) is the Altarelli-Parisi kernel for \(z < 1\) in \(4-2\epsilon\) dimensions, the sum \(\sum_d\) runs over \(g, u, \bar{u}, \ldots\), \(K_{ab}\) define the finite part of the initial state collinear subtraction (in the \(\overline{\text{MS}}\) scheme they are equal to zero), \(C_{ab} = 1\) when
$b$ is a quark or a gluon and, analogously to eq. (A.5),

$$
\int_0^1 d\xi f(\xi) \left( \frac{\log \xi}{\xi} \right)_c = \int_0^1 d\xi \left[ f(\xi) - f(0)\theta(\xi_{\text{cut}} - \xi) \right] \frac{\log \xi}{\xi} .
$$

(A.27)

In the case of photon-hadron collisions, we define

$$
P_{d\gamma}(z) = \delta_{dq} \frac{N_c}{T_F} P_{qq}(z), \quad C_{d\gamma} = \delta_{dq} e_q^2 \frac{\alpha_{em}}{\alpha_s} .
$$

(A.28)

The functions $K_{d\gamma}$ depend upon the scheme adopted for the partonic densities in the photon, entering the hadronic photon-hadron cross section (for details on this issue, see for example ref. [26]). Finally, $P_{de} = K_{de} = 0$, $C_{de} = 1$. This formal statement corresponds to the fact that no collinear singularity is associated with an incoming lepton.