AN EXPLICIT EXAMPLE OF POLYNOMIALS ORTHOGONAL ON THE UNIT CIRCLE WITH A DENSE POINT SPECTRUM GENERATED BY A GEOMETRIC DISTRIBUTION

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Abstract. We present a new explicit family of polynomials orthogonal on the unit circle with a dense point spectrum. This family is expressed in terms of q-hypergeometric function of type 2Φ1. The orthogonality measure is the wrapped geometric distribution. Some "classical" properties of the above polynomials are presented.

1. Introduction

The monic polynomials orthogonal on the unit circle \( \Phi_n(z) = z^n + O(z^{n-1}) \) (OPUC) are defined through the recurrence relation [16]

\[
\Phi_{n+1}(z) = z\Phi_n(z) - a_n \Phi_n^*(z),
\]

where

\[
\Phi_n^*(z) = z^n \tilde{\Phi}_n(1/z)
\]

and \( \tilde{\Phi}_n(z) \) means complex conjugation of expansion coefficients of the polynomial \( \Phi_n(z) \). The recursion parameters

\[
a_n = -\tilde{\Phi}_n(0)
\]

are called the Verblunsky (sometimes also reflection, Schur etc.) parameters [16]. Under the condition

\[
|a_n| < 1, \quad n = 0, 1, 2, \ldots
\]

the polynomials \( \Phi_n(z) \) are orthogonal on the unit circle with respect to a nondecreasing positive function \( \sigma(\theta) \)

\[
\int_0^{2\pi} \Phi_n(e^{i\theta}) \tilde{\Phi}_m(e^{-i\theta}) d\sigma(\theta) = h_n \delta_{nm}
\]

where

\[
h_n = (1 - |a_0|^2)(1 - |a_1|^2)\ldots(1 - |a_{n-1}|^2)
\]

are normalization constants (which are nonzero due to condition (1.2)). If the function \( \sigma(\theta) \) is normalized as \( \sigma(0) = 0, \sigma(2\pi) = 1 \) then it is determined uniquely through the coefficients \( a_n \).

Note that orthogonality relation (1.3) is equivalent to conditions [16]

\[
\int_0^{2\pi} \Phi_n(e^{i\theta}) e^{-ijn\theta} d\sigma(\theta) = h_n \delta_{nj}, \quad j = 0, 1, 2, \ldots, n
\]

Equivalently, OPUC \( \Phi_n(z) \) can be constructed in terms of trigonometric moments \( \sigma_n \). The latter are defined as

\[
\sigma_n = \int_0^{2\pi} e^{in\theta} d\sigma(\theta), \quad n = 0, \pm 1, \pm 2, \ldots
\]

Then polynomials \( \Phi_n(z) \) have the explicit expression

\[
\Phi_n(z) = (\Delta_n)^{-1}
\]

\[
\begin{vmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_n \\
\sigma_1 & \sigma_0 & \ldots & \sigma_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1-n} & \sigma_{2-n} & \ldots & \sigma_1 \\
1 & z & \ldots & z^n
\end{vmatrix}
\]

where

\[
\Delta_n =
\begin{vmatrix}
\sigma_0 & \sigma_1 & \ldots & \sigma_{n-1} \\
\sigma_1 & \sigma_0 & \ldots & \sigma_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1-n} & \sigma_{2-n} & \ldots & \sigma_0 \\
1 & 1 & \ldots & \ldots
\end{vmatrix}
\]

(1.7)
are Toeplitz determinants which are all positive $\Delta_n > 0$, $n = 0, 1, 2, \ldots$. Note the symmetry property of the trigonometric moments

$$\sigma_{-n} = \bar{\sigma}_n$$

Explicit examples of polynomials orthogonal on unit circle are very interesting from different point view. By ”explicit examples” we mean that all main objects: the parameters $a_n$, the moments $\sigma_n$, the measure $\sigma(\theta)$ and the polynomials themselves $\Phi_n(z)$ have explicit expressions in terms of special functions. Usually, in most known explicit examples the parameters $a_n$ are given by elementary functions of $n$ while the OPUC $\Phi_n(z)$ are expressed in terms of hypergeometric functions (either ordinary or basic). A list of known explicit examples can be found e.g. in Simon’s monograph [16].

In [26] and [19] new explicit examples of OPUC were presented. In these examples polynomials $\Phi_n(z)$ are expressed in terms of elliptic hypergeometric function $\phi_2(z)$ while the moments $\sigma_n$ and the recurrence parameters $a_n$ have simple expressions in terms of elliptic functions. The most interesting property of the OPUC of these examples is that they are orthogonal on the unit circle with respect to a dense point measure. This means that the function $\sigma(\theta)$ is a step function with infinitely many points $\theta_s$ of jumps, and these points are dense on the interval $[0, 2\pi]$. In terms of the distribution function this can be presented as

$$\rho(\theta) = \sum_{s=-\infty}^{\infty} M_s \delta(\theta - \theta_s),$$

where $\rho(\theta)$ is a distribution defined as $d\sigma(\theta) = \rho(\theta)d\theta$, $\delta(\theta)$ is the Dirac delta function and $M_s$ are concentrated masses located at points of jumps $\theta_s$. The spectral points $z_s = \exp(i\theta_s)$ are dense on the unit circle.

Then orthogonality relation (1.3) can then be presented as

$$\sum_{s=-\infty}^{\infty} M_s \Phi_n(e^{i\theta_s}) \Phi_m(e^{-i\theta_s}) = h_n \delta_{nm}$$

From general considerations (see, e.g. [16]) it follows that polynomials orthogonal with respect to such dense point measures are rather generic if one assume some natural restrictions upon behavior of the recurrence parameters $a_n$. On the other hand, such measures are very important from physical point of view, because they correspond to the phenomenon of the Anderson localization [11], [16].

Usually examples of OPUC with dense point spectrum are related to sequences of the parameters $a_n$ which behave (quasi) stochastically inside the interval $|a_n| < 1$ [16]. OPUC in [26] and [19] provide perhaps the first known examples of a pure point dense measure on the unit circle where both the coefficients $a_n$ and the moments $\sigma_n$ are given explicitly by analytic functions in $n$.

In this paper we propose a much simpler explicit example of polynomials orthogonal on the unit circle with respect to a (wrapped) geometric distribution which is dense on the unit circle. Polynomials themselves are expressed in terms of basic hypergeometric function $\phi_2(q; z)$ with $|q| = 1$.

2. WRAPPED GEOMETRIC DISTRIBUTION AND CORRESPONDING OPUC

Let $q$ be a fixed point belonging to the unit circle $|q| = 1$ and not a root of unity, i.e. we demand that $q^n \neq 1$ for all natural integer $n = 0, 1, \ldots$. Choose a real parameter $p$ within the unit interval $0 < p < 1$. Define the measure on the unit circle as

$$\rho(\theta) = \sum_{s=0}^{\infty} p^s \delta(\theta - s\chi),$$

where $\chi$ is a fixed irrational parameter $0 < \chi < 1$ such that

$$q = \exp(2\pi i\chi).$$

Irrationality of $\chi$ means that the set of points $z_s = q^s$, $s = 0, 1, \ldots$ (i.e. the location of jumps of the measure) is dense on the unit circle. The weights (i.e. the concentrated masses) $w_s$ at the points $z_s$ form the geometric sequence: $w_s = p^s$, $s = 0, 1, 2, \ldots$.

Corresponding trigonometric moments are

$$\sigma_n = \sum_{s=0}^{\infty} z_s^n w_s = (1 - p) \sum_{s=0}^{\infty} q^{sn} p^s = \frac{1 - p}{1 - pq^n}, \quad n = 0, 1, 2, \ldots$$
Note that the measure (2.1) can be interpreted as the wrapped geometric distribution on the unit circle (see, e.g. [14], [9] for definition and discussion of wrapped distributions on the unit circle).

One can present expression (2.3) as

$$\sigma_n = \frac{(p; q)_n}{(pq; q)_n},$$  \hspace{1cm} (2.4)

where the q-shifted factorial (q-Pochhammer symbol) is defined as [4]

$$(a; q)_0 = 1, \quad (a; q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$$

for $n = 1, 2, 3, \ldots$ and as

$$(a; q)_{-n} = \frac{1}{(aq^{-n}; q)_n}$$

for $n = -1, -2, \ldots$.

It is known that the Laurent biorthogonal Pastro polynomials $P(z; a, b)$ [15] depending on two arbitrary parameters $a, b$ can be uniquely defined through their moments [21]

$$\sigma_n = \frac{(a; q)_n}{(b; q)_n}, \quad n = 0, \pm 1, \pm 2, \ldots$$ \hspace{1cm} (2.5)

Explicitly, these polynomials are given by [15], [21]

$$P(z; a, b) = \mu_n 2\phi_1\left(q^{-n}, b \mid aq^{-n}; q^{-1}z\right),$$  \hspace{1cm} (2.6)

where $\mu_n$ is an appropriate normalization factor to fulfill the condition $P_n(z) = z^n + O(z^{n-1})$.

Note the Laurent biorthogonal polynomials (LBP) can be considered as a generalization of the OPUC. Their main distinction from OPUC is that the moments $\sigma_n$ do not satisfy, in general, the symmetry condition (1.8). The LBP can also be characterized by the three-term recurrence relation of $R_I$ type [7], [24]

$$P_{n+1}(z) + g_n P_n(z) = z\left(P_n(z) + d_nP_{n-1}(z)\right), \quad P_0 = 1, \ P_{-1} = 0$$ \hspace{1cm} (2.7)

with some recurrence coefficients $g_n, d_n$.

In contrast to the case of orthogonal polynomials, the recurrence relation (2.7) can be presented in the form of the generalized eigenvalue problem [23]

$$J_1 P(z) = zJ_2 P(z),$$  \hspace{1cm} (2.8)

where $J_1, J_2$ are upper and lower bidiagonal matrices acting on the vector $P(z) = (P_0(z), P_1(z), \ldots)$.

Comparing expressions (2.5) and (2.4) we can conclude that the OPUC corresponding to the wrapped geometric distribution are special case of the Pastro polynomials with $a = p, b = qp$.

This allows us to present the main result

**Theorem 2.1.** The polynomials $\Phi_n(z)$ orthogonal on the unit circle with respect to the wrapped geometric distribution (2.1) have the explicit expression

$$\Phi_n(z) = \mu_n 2\phi_1\left(q^{-n}, pq^{-1}; zq\right),$$  \hspace{1cm} (2.9)

where

$$\mu_n = q^{-n}\frac{(q; q)_n(pq^{-1}; q)_n}{(q^{-n}; q)_n(pq^{-1}; q)_n} = p^n\frac{(p^{-1}; q)_n}{(qp; q)_n}.$$  \hspace{1cm} (2.10)

(The standard notation $2\phi_1(z)$ for the basic hypergeometric function as in [4] is used in (2.9)).

One can directly check that the polynomials (2.9) satisfy orthogonality relations (1.5). Indeed, one has

$$I_{n_j} = \sum_{s=0}^{n} \Phi_n(q^s)q^{-nj}p^s = \mu_n \sum_{s=0}^{n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k(pq^{-1}; q)_k}{(q; q)_k(pq^{-1}; q)_k}q^{(1+s)k}q^{js}p^s.$$  \hspace{1cm} (2.11)

The above expression is reduced to $3\phi_2(1)$ function. It can further be simplified using q-Saalschützian formula [4]. The resulting relation is $I_{n_j} = 0$ for $j = 0, 1, \ldots, n-1$ which is equivalent to orthogonality condition (1.5).

Explicit expression for the the recurrence parameters $a_n$ follows from (2.9) and (2.10):

$$a_{n-1} = -\Phi_n(0) = -\mu_n = -p^n\frac{(p^{-1}; q)_n}{(qp; q)_n}.$$  \hspace{1cm} (2.12)
For the square of absolute values we have rather simple expression

$$|a_{n-1}|^2 = \tilde{a}_{n-1}a_{n-1} = \frac{(1-p)^2}{1+p^2 - p(q^n + q^{-n})} = \frac{1}{1 + \beta \sin^2(\chi \pi n)},$$  

(2.13)

where

$$\beta = \frac{4p}{(1-p)^2},$$  

(2.14)

and where the parameter $\chi$ is the same as in (2.2).

It is seen from (2.13) that the values $|a_n|$ oscillate inside the interval

$$\frac{1-p}{1+p} < |a_n| < 1, \quad n = 0, 1, \ldots$$

(2.15)

Because of irrationality of $\chi$ the absolute value $|a_n|$ achieves the boundaries of this interval with any prescribed accuracy (never achieving exact boundary values). Note that $a_{-1} = -1$ which corresponds to the standard initial conditions for OPUC [5], [16].

The OPUC (2.9) can be considered as $|g| = 1$ analogs of the OPUC introduced by Askey in [1] (see also [3] for more general OPUC of Askey’s type).

3. "Classicality" of the Polynomials $\Phi_n(z)$

The OPUC (2.9) possess "classical" properties which make them similar to classical orthogonal polynomials.

First of all, they satisfy the three-term recurrence relation (2.7) where the recurrence coefficients are

$$g_n = \frac{q^n - p}{1 - pq^{n+1}}, \quad d_n = -\frac{p(1-q^n)^2}{(1-pq^n)(1-pq^{n+1})}. $$

(3.1)

Moreover, the polynomial $\Phi_n(z)$ possess a remarkable duality property. Indeed, one can rewrite polynomials $\Phi_n(z)$ in a different form

$$\Phi_n(z) = p^n \frac{(q;\beta)_n}{(pq;\beta)_n} z^n \phi_2\left(\beta^{-n}, p^{-1}, z^{-1}; q, 0 \right)$$

(3.2)

which can be obtained from (2.9) by standard transformation formulas [4].

From this formula the duality property

$$A_s \Phi_s(q^n) = A_n \Phi_n(q^s)$$

(3.3)

follows, where

$$A_n = \frac{(pq;\beta)_n}{(q;\beta)_n} p^{-n}. $$

(3.4)

This property resembles corresponding duality properties for the classical orthogonal polynomials from the Askey scheme [12], [2], [18]. The main difference is that the polynomials $\Phi_n(z)$ satisfy the generalized eigenvalue problem (2.8) instead of the ordinary eigenvalue problem for orthogonal polynomials.

From the duality property one can derive the second-order q-difference equation

$$B_{s+1} \Phi_n(q^{s+1}) + g_s \Phi_n(q^s) = q^n \left( \Phi_n(q^s) + B_s^{-1} d_s \Phi_n(q^{s-1}) \right), $$

(3.5)

where

$$B_s = \frac{A_{s+1}}{A_s} = \frac{p(1-q^n)}{1-pq^s}. $$

Equation (3.5) can also be presented in the form of the generalized eigenvalue problem

$$L \Phi_n(z) = q^n M \Phi_n(z),$$

(3.6)

where the first-order q-difference operators $L, M$ act on the argument $z$ of the polynomials.

Relations (2.8) and (3.6) mean that the polynomials possess the bispectrality property: they satisfy simultaneously two GEVP: Concerning definition and general theory of bispectrality see e.g. [6]. For orthogonal polynomials from the Askey scheme this property is well known [10]. For biorthogonal polynomials and rational functions the bispectrality is known for some special families. The most general from them are elliptic biorthogonal functions [17]. However the general theory of bispectrality for systems satisfying GEVP is not yet developed (see e.g. [20], [22] for algebraic description of bispectrality on the "lowest" level of hypergeometric functions $\phi_2(1)$ ).
The duality property implies that for \( z = q^s, s = 0, 1, 2, \ldots \), the hypergeometric function in (3.2) reduces to a polynomial of degree \( s \) of the argument \( q^{-n} \).

It is well known (see, e.g. [16]) that if \( z_0 \) is a point on the unit circle corresponding to a concentrated mass \( M_0 \) then the relation

\[
\sum_{n=0}^{\infty} \frac{\Phi_n(z_0)^2}{h_n} = 1/M_0,
\]

(3.7)

holds, where the normalization coefficient \( h_n \) is defined in (1.4).

In our case this means that for every spectral point \( z_s = q^s, s = 0, 1, \ldots \) there exists the identity

\[
\sum_{n=0}^{\infty} \frac{\Phi_n(q^s)^2}{h_n} = M_s^{-1} = \frac{p^s}{1-p}.
\]

(3.8)

Identity (3.8) follows easily from the duality property (3.3) and from orthogonality relation.

So far, we have considered the case when \( q \) is not a root of unity. If, otherwise, \( q \) is a primitive root of unity

\[ q = \exp\left(\frac{2\pi i M}{N}\right) \]

(3.9)

with coprime integers \( M, N \), then there are only \( N \) distinct mass points on the unit circle located at \( z_s = q^s, s = 0, 1, 2, \ldots, N - 1 \). In this case the polynomials \( \Phi_n(z) \) are orthogonal on vertices of a regular \( N \)-gon with respect to the finite wrapped geometric distribution:

\[
\sum_{n=0}^{N-1} \Phi_n(q^s)\Phi_m(q^{-s})(1-p^N)p^s = h_n \delta_{nm}, \quad n, m = 0, 1, \ldots, N - 1.
\]

(3.10)

See [25] for other explicit examples of polynomials orthogonal on the vertices of regular polygons.

4. Concluding remarks

In contrast to examples of OPUC obtained in [26], the polynomials (2.6) have non-real moments \( \sigma_n \) and hence the coefficients \( a_n \) non-real as well. This means that it is impossible to associate with OPUC (2.6) polynomials orthogonal on an interval of the real line. In [26] explicit examples of polynomials orthogonal with dense point spectrum on an interval were presented using standard Szegö mapping from OPUC to an interval of the real line.

The OPUC (2.6) allow a trivial generalization which shifts all spectral points on the unit circle on the same constant angle \( \varphi \), i.e. we can consider the same weights \( w_s = p^s(1-p) \) located at the points

\[ \theta_s = 2\pi \chi s + \varphi. \]

Equivalently, this means that the new spectral points will be \( z_s = e^{i\varphi}q^s, s = 0, 1, 2, \ldots. \)

Such transformation is equivalent to a simple rotation of the argument of OPUC [5], [16]:

\[ \tilde{\Phi}_n(z) = e^{-i\varphi n} \Phi(e^{i\varphi}z). \]

Another generalization of the OPUC (2.6) is more substantial. It leads to Laurent biorthogonal polynomials orthogonal on the unit circle with dense point measure.

Indeed, assume that the spectral points on the unit circle are the same: \( z_s = q^s, s = 0, 1, 2, \ldots. \)

Take slightly more general weights:

\[
w_s = p^s(\frac{q^k}{q};q)_s(\frac{q}{q};q)_s, \quad 0 < p < 1, \quad k = 1, 2, 3, \ldots
\]

(4.1)

For \( k = 1 \) we return to the case of the wrapped geometric distribution. For \( k > 1 \) we have for the moments

\[
\sigma_n = \sum_{s=0}^{\infty} \frac{(q^k; q)_s}{(q; q)_s} p^s q^{sn}.
\]

(4.2)

By q-binomial theorem [4], the above sum is simplified to

\[
\sigma_n = \frac{(pq^k; q)_{\infty}}{(pq^k; q)_{\infty}} \frac{1}{(pq^{k+1}; q)_{k}} = \frac{(p; q)_n}{(p; q)_k (pq^k; q)_n},
\]

(4.3)

Remark. Usually, the convergence problem for q-series like (4.2) with \( |q| = 1 \) is highly nontrivial (see, e.g. [13]). In our case however this problem does not appear because for integer \( k \) there is
cancellation of almost all terms (apart of a finite number of initial ones) in denominators of the coefficients in (4.2). Hence the convergence for $0 < p < 1$ still takes place.

For fixed $k$ the moments $\sigma_n$ coincide (up to the constant factor) with the moments (2.5) for the Pastro polynomials with $a = p, b = pq^k$. Hence from (2.6) we have explicit expression for them

$$P(z) = \mu_n 2\Phi_1 \left( \begin{array}{c} q^{-n}, pq^k \\ pq^{k-n} \end{array} ; qz \right),$$

(4.4)

where

$$\mu_n = \frac{p^n(p^{-1};q)_n}{(pq^k;q)_n}.$$  

These polynomials are NOT OPUC (apart from the already considered case $k = 1$) because the weights (4.1) are not positive and hence the moments $\sigma_n$ do not satisfy symmetry property (1.8).

Existence of other explicit examples of OPUC with dense point spectrum is an interesting open problem.

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