Crossing Numbers and Combinatorial Characterization of Monotone Drawings of $K_n$

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Abstract In 1958, Hill conjectured that the minimum number of crossings in a drawing of $K_n$ is exactly $Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$. Generalizing the result by Ábrego et al. for 2-page book drawings, we prove this conjecture for plane drawings in which edges are represented by $x$-monotone curves. In fact, our proof shows that the conjecture remains true for $x$-monotone drawings of $K_n$ in which adjacent edges may cross an even number of times, and instead of the crossing number we count the pairs of edges which cross an odd number of times. We further discuss a generalization of this result to shellable drawings, a notion introduced by Ábrego et al. We also give a combinatorial characterization of several classes of $x$-monotone drawings of complete graphs using a small set of forbidden configurations. For a similar local characterization of shellable drawings, we generalize Carathéodory’s theorem to simple drawings of complete graphs.
1 Introduction

Let $G$ be a graph with no loops or multiple edges. In a drawing $D$ of a graph $G$ in the plane, the vertices are represented by distinct points and each edge is represented by a simple continuous arc connecting the images of its endpoints. As usual, we identify the vertices and their images, as well as the edges and the arcs representing them. We require that the edges pass through no vertices other than their endpoints. We also assume for simplicity that any two edges have only finitely many points in common, no two edges touch at an interior point and no three edges meet at a common interior point.

A crossing in $D$ is a common interior point of two edges where they properly cross. The crossing number $\text{cr}(D)$ of a drawing $D$ is the number of crossings in $D$. The crossing number $\text{cr}(G)$ of a graph $G$ is the minimum of $\text{cr}(D)$, taken over all drawings $D$ of $G$. A drawing $D$ is called simple if no two adjacent edges cross and no two edges have more than one common crossing. It is well known and easy to see that every drawing of $G$ which minimizes the crossing number is simple.

According to the famous conjecture of Hill [23,25] (also known as Guy’s conjecture), the crossing number of the complete graph $K_n$ on $n$ vertices satisfies $\text{cr}(K_n) = Z(n)$, where

$$Z(n) = \frac{1}{4} \binom{n}{2} \binom{n-1}{2} \binom{n-2}{2} \binom{n-3}{2}.$$  

This conjecture has been verified for $n \leq 10$ by Guy [24] and recently for $n \leq 12$ by Pan and Richter [32]. Moreover for each $n$, there are drawings of $K_n$ with exactly $Z(n)$ crossings [12,23,25,26]. Current best asymptotic lower bound, $\text{cr}(K_n) \geq 0.8594Z(n)$, follows from the lower bound on the crossing number of the complete bipartite graph $K_{n,n}$ by an elementary double-counting argument [36].

A curve $\alpha$ in the plane is $x$-monotone if every vertical line intersects $\alpha$ in at most one point. A drawing of a graph $G$ in which every edge is represented by an $x$-monotone curve and no two vertices share the same $x$-coordinate is called $x$-monotone (or monotone, for short). The monotone crossing number $\text{mon-cr}(G)$ of a graph $G$ is the minimum of $\text{cr}(D)$, taken over all monotone drawings $D$ of $G$.

The rectilinear crossing number $\overline{\text{cr}}(G)$ of a graph $G$ is the smallest number of crossings in a drawing of $G$ where every edge is represented by a straight-line segment. Since every rectilinear drawing of $G$ in which no two vertices share the same $x$-coordinate is $x$-monotone, we have $\text{cr}(G) \leq \text{mon-cr}(G) \leq \overline{\text{cr}}(G)$ for every graph $G$.

The odd crossing number $\text{ocr}(G)$ of a graph $G$ is the minimum number of pairs of edges crossing an odd number of times in a drawing of $G$ in the plane. The monotone odd crossing number, $\text{mon-ocr}(G)$, is the minimum number of pairs of edges crossing an odd number of times in a monotone drawing of $G$. For these two notions of the
crossing number, optimal drawings do not have to be simple. Moreover, there are
graphs $G$ with $ocr(G) < cr(G)$ [33,42], and for every $n$, there is a graph $G$ with
$\text{mon-ocr}(G) = 1$ and $\text{mon-cr}(G) \geq n$ [19].

We call a drawing of a graph semisimple if adjacent edges do not cross but inde-
dependent edges may cross more than once. The monotone semisimple odd crossing
number of $G$ (called monotone odd $+$ by Schaefer [38]), denoted by $\text{mon-ocr}_+(G)$,
is the smallest number of pairs of edges that cross an odd number of times in a
monotone semisimple drawing of $G$. We call a drawing of a graph weakly semi-
simple if every pair of adjacent edges cross an even number of times; independent
edges may cross arbitrarily. The monotone weakly semisimple odd crossing number
of $G$, denoted by $\text{mon-ocr}_\pm(G)$, is the smallest number of pairs of edges that cross
an odd number of times in a monotone weakly semisimple drawing of $G$. Clearly,
$\text{mon-ocr}(G) \leq \text{mon-ocr}_\pm(G) \leq \text{mon-ocr}_+(G) \leq \text{mon-cr}(G)$.

The monotone crossing number has been introduced by Valtr [43] and recently
further investigated by Pach and Tóth [31], who showed that $\text{mon-cr}(G) < 2\text{cr}(G)^2$
holds for every graph $G$. On the other hand, they showed that the monotone crossing
number and the crossing number are not always the same: there are graphs $G$ with
arbitrarily large crossing numbers such that $\text{mon-cr}(G) \geq 7\text{cr}(G) - 6$.

We study the monotone crossing numbers of complete graphs. The drawings of com-
plete graphs with $Z(n)$ crossings obtained by Blažek and Koman [12] (see also [26])
are 2-page book drawings. In such drawings the vertices are placed on a line $l$ and
each edge is fully contained in one of the half-planes determined by $l$. Since 2-page
drawings may be considered as a strict subset of $x$-monotone drawings, we have
$\text{mon-cr}(K_n) \leq Z(n)$ (Fig. 1).

Ábrego et al. [1] recently proved that Hill’s conjecture holds for 2-page book draw-
ings of complete graphs. We generalize their techniques and show that Hill’s conjecture
holds for all $x$-monotone drawings of complete graphs, and even for the monotone
weakly semisimple odd crossing number.

**Theorem 1.1** For every $n \in \mathbb{N}$, we have

$$
\text{mon-ocr}_\pm(K_n) = \text{mon-ocr}_+(K_n) = \text{mon-cr}(K_n) = Z(n).
$$
The rectilinear crossing number of $K_n$ is known to be asymptotically larger than $Z(n)$: this follows from the best current lower bound $\text{cr}(K_n) \geq (277/729)(\binom{n}{4}) - O(n^3)$ [5,7] and from the simple upper bound $Z(n) \leq \frac{3}{8}\binom{n}{4} + O(n^3)$.

See a recent survey by Schaefer [38] for an encyclopedic treatment of all known variants of crossing numbers.

During the preparation of this paper, we were informed that the authors of [1] achieved the result $\text{mon-cr}(K_n) = Z(n)$ already during discussions after their presentation at SoCG 2012 and that Silvia Fernandez-Merchant was going to present it in her keynote talk at LAGOS 2013. The proceedings of the conference were recently published [2]. Pedro Ramos [35] then presented the results and some further developments at the XV Spanish Meeting on Computational Geometry (ECG 2013) in his invited talk. Very recently, Ábrego et al. [3] made their paper containing a more general result publicly available.

In Sect. 2, we first prove Theorem 1.1 for semisimple monotone drawings. Then we extend the result to weakly semisimple monotone drawings, by showing that even crossings of adjacent edges can be easily eliminated in such drawings.

In Sect. 3 we introduce a combinatorial characterization of $x$-monotone drawings of $K_n$. We show that there is a one-to-one correspondence between semisimple, simple or pseudolinear $x$-monotone drawings of $K_n$ and mappings $\binom{[n]}{3} \rightarrow \{+, -\}$, called signature functions, avoiding a finite number of certain sub-configurations. The signature functions were introduced by Peters and Szekeres [41] as a generalization of order types of planar points sets.

In Sect. 4 we show a further generalization of Theorem 1.1 to shellable drawings and weakly shellable drawings; we define these notions in the beginning of Sect. 4. We show a local characterization of shellable drawings, for which we generalize Caratheodory’s theorem to simple drawings of complete graphs. We also show that shellable drawings form a more general class than monotone drawings. Finally, we further generalize a key lemma from [1], which implies a generalization of the main result of [3] to weakly semisimple drawings.

In the last section we state our stronger version of Hill’s conjecture.

2 Monotone Crossing Number of the Complete Graph

Let $P$ denote a set of $n$ points in the plane in general position and let $k$ be an integer satisfying $0 \leq k \leq n$. The line segment joining a pair of points $p$ and $q$ in $P$ is a $k$-edge ($\leq k$-edge) if there are exactly (at most, respectively) $k$ points of $P$ in one of the open half-planes defined by the line $pq$.

Ábrego and Fernández-Merchant [6] and Lovász et al. [29] discovered a relation between the numbers of $k$-edges (or $\leq k$-edges) in $P$ and the number of convex 4-tuples of points in $P$, which is equal to the number of crossings of the complete geometric graph with vertex set $P$. This relation transforms every lower bound on the number of $\leq k$-edges to a lower bound on the number of crossings. Using this method, many incremental improvements on the rectilinear and pseudolinear crossing number of $K_n$ have been achieved [4–6,8,11,29].
To prove the lower bound on the 2-page crossing number of $K_n$, Ábrego et al. [1] generalized the notion of $k$-edges to arbitrary simple drawings of complete graphs. They also introduced the notion of $\leq k$-edges, which capture the essential properties of 2-page book drawings better than $\leq k$-edges. We show that the approach using $\leq k$-edges can be generalized to arbitrary semisimple $x$-monotone drawings.

For a semisimple drawing $D$ of $K_n$ and distinct vertices $u$ and $v$ of $K_n$, let $\gamma$ be the oriented arc representing the edge $\{u, v\}$. If $w$ is a vertex of $K_n$ different from $u$ and $v$, then we say that $w$ is on the left (right) side of $\gamma$ if the topological triangle $uvw$ with vertices $u$, $v$ and $w$ traced in this order is oriented counter-clockwise (clockwise, respectively). This generalizes the definition introduced by Ábrego et al. [1] for simple drawings. Further generalization is possible for weakly semisimple drawings, where every two edges of the triangle $uvw$ cross an even number of times; see Sect. 4. However, we were not able to find a meaningful generalization of this notion to arbitrary drawings, where the edges of the triangle $uvw$ can cross an odd number of times.

A $k$-edge in $D$ is an edge $\{u, v\}$ of $D$ that has exactly $k$ vertices on the same side (left or right). Since every $k$-edge has $n - 2 - k$ vertices on the other side, every $k$-edge is also an $(n - 2 - k)$-edge and so every edge of $D$ is a $k$-edge for some integer $k$ where $0 \leq k \leq \lfloor n/2 \rfloor - 1$.

Analogously to the case of point sets, an $i$-edge in $D$ with $i \leq k$ is called a $\leq k$-edge. Let $E_i(D)$ be the number of $i$-edges and $E_{\leq k}(D)$ the number of $\leq k$-edges of $D$. Clearly, $E_{\leq k}(D) = \sum_{i=0}^{k} E_i(D)$. Similarly, the number $E_{\leq \leq k}(D)$ of $\leq \leq k$-edges of $D$ is defined by the following identity.

$$E_{\leq \leq k}(D) = \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{i=0}^{k} (k + 1 - i) E_i(D).$$ (1)

Considering the only three different simple drawings of $K_4$ up to a homeomorphism of the plane, Ábrego et al. [1] showed that the number of crossings in a simple drawing $D$ of $K_n$ can be expressed in terms of the number of $k$-edges in the following way.

Lemma 2.1 ([1]) For every simple drawing $D$ of $K_n$ we have

$$\text{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n - 2 - k) E_k(D),$$ (2)

which can be equivalently rewritten as

$$\text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq \leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n - 2}{2} \right\rfloor - \frac{1}{2} (1 + (-1)^n) E_{\leq \leq \lfloor n/2 \rfloor - 2}(D).$$

Lemma 2.1 generalizes the relation found by Ábrego and Fernández-Merchant [6]. We further generalize it to semisimple drawings of $K_n$ where $\text{cr}(D)$ is replaced by
Fig. 2 The three homeomorphism classes of simple drawings of $K_4$. The fat edges are 1-edges.

ocr($D$), which counts the number of pairs of edges that cross an odd number of times in $D$.

**Lemma 2.2** For every semisimple drawing $D$ of $K_n$ we have

$$\text{ocr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \lfloor \frac{n-2}{2} \rfloor - \frac{1}{2} \left(1 + (-1)^n\right) E_{\leq \lfloor n/2 \rfloor - 2}(D).$$

We recall that a face of a drawing $D$ in the plane is a connected component of the complement of all the edges and vertices of $D$ in $\mathbb{R}^2$. The outer face of $D$ is the unbounded face of $D$.

**Proof** (sketch) We just sketch the main idea, which is common with the proof of Lemma 2.1, and then explain the generalization to semisimple drawings. For the details, we refer the reader to [1, Theorem 1 and Proposition 1].

Let $D$ be a semisimple drawing of $K_n$. A separation in $D$ is an unordered triple $\{ab, c, d\}$, where $ab$ is an edge of $D$, $c, d$ are vertices of $D$ distinct from $a, b$, and the orientations of the two triangles $abc$ and $abd$ are opposite. Observe that $\{ab, c, d\}$ is a separation in $D$ if and only if $ab$ is a 1-edge (and also a halving edge) in the complete subgraph of $D$ induced by the vertices $a, b, c, d$. The total number of separations in $D$ relates to both the crossing number and the numbers of $k$-edges in the following way.

(i) Every $k$-edge belongs to exactly $k(n - k - 2)$ separations.

(ii) Every 4-tuple of vertices inducing a crossing contributes two separations, and every 4-tuple of vertices inducing a planar drawing of $K_4$ contributes three separations. In particular, for every complete subgraph $D$ with 4 vertices we have the equality $\text{cr}(D) + E_1(D) = 3$.

Fact (i) is a direct consequence of the definitions. Fact (ii) is easily seen by inspecting all three homeomorphism classes of simple drawings of $K_4$ in the plane: there is one class with no crossing, and two classes with one crossing, which would form just one class on the sphere; see Fig. 2. Lemma 2.1 follows from the facts (i) and (ii) by elementary computations.

To generalize Lemma 2.1 to semisimple drawings, we observe that semisimple drawings of $K_4$ can be classified analogously as the simple drawings of $K_4$. In particular, the following claim implies that the equality $\text{ocr}(D) + E_1(D) = 3$ is still satisfied for every semisimple drawing $D$ of $K_4$.

**Claim** A semisimple drawing $D$ of $K_4$ has at most one pair of edges crossing an odd number of times. Moreover, $D$ has three separations if $\text{ocr}(D) = 0$ and two separations if $\text{ocr}(D) = 1$. 
In the rest of the proof we prove the claim. Let $D$ be a semisimple drawing of $K_4$. Suppose that $\text{ocr}(D) = 0$. Let $abc$ be a triangle in $D$ and let $d$ be the fourth vertex of $D$. See Fig. 3, left. If the edge $da$ crosses $bc$, then either $d$ and $b$ share no face in the drawing of the subgraph with edges $ab, bc, ad$, or $d$ and $c$ share no face in the drawing of the subgraph with edges $ac, bc, ad$. This means that one of the edges $bd$ or $cd$ either crosses an adjacent edge or crosses another edge an odd number of times. Therefore, the edge $da$ has no crossing with the triangle $abc$. Analogous argument for the edges $db$ and $dc$ shows that $D$ has no crossings at all. In particular, $D$ has three separations; see Fig. 2, left.

Now suppose that $\text{ocr}(D) \geq 1$ and let $ac$ and $bd$ be two edges that cross an odd number of times. Since all the other edges are adjacent to both $ac$ and $bd$, the vertices $a, b, c, d$ share a common face $F$ in the drawing of the subgraph with edges $ac, bd$. Moreover, the cyclic order of the vertices along the boundary of $F$ is $a, b, c, d$, either clockwise or counter-clockwise. See Fig. 3, right.

We show that at most one more pair of edges can cross, either $ab$ and $cd$, or $ad$ and $bc$, but only an even number of times. For example, in the drawing of the subgraph with edges $ac, bd, ab$, the vertices $c$ and $d$ belong to the same face, and the edge $cd$ is allowed to cross only the edge $ab$, each time switching faces. If $ab$ and $cd$ cross, then $a$ and $d$ share a unique face $F_{a,d}$ in the drawing of the graph $K$ with edges $ac, bd, ab, cd$, and $c$ and $b$ share a unique face $F_{b,c}$ different from $F_{a,d}$. Since the edges $ad$ and $bc$ are adjacent to all edges of $K$, the edge $ad$ lies completely in $F_{a,d}$ and thus $ad$ and $bc$ cannot cross. A symmetric argument shows that if $ab$ and $cd$ are disjoint, then $ad$ and $bc$ are either disjoint or cross an even number of times. In any case, we have $\text{ocr}(D) \leq 1$ (and the pair crossing number of $D$ is at most 2).

It remains to show that every semisimple drawing $D$ of $K_4$ with $\text{ocr}(D) = 1$ has exactly two 1-edges. More precisely, we show that the two 1-edges always form a perfect matching.

Let $e$ be an edge in $D$ incident with the outer face. An edge flip is an operation where the portion of $e$ incident with the outer face is redrawn along the other side of the drawing; see Fig. 4. For drawings on the sphere, the edge flip is just a homeomorphism of the sphere. For every bounded face $F$ of $D$, there is a sequence of edge flips that makes $F$ the outer face.

If $D$ is a semisimple drawing of $K_4$, then every edge flip of an edge $e$ changes the orientation of the two triangles adjacent to $e$. Consequently, exactly the four edges
adjacent to $e$, forming a 4-cycle, change from 1-edges to 0-edges or vice versa. Also observe that the edge flip of $e$ can be performed only if $e$ is a 0-edge. It follows that 1-edges form a perfect matching in $D$ if and only if they form a perfect matching in the drawing obtained by the edge flip.

Let $D$ be a semisimple drawing of $K_4$ with $\text{ocr}(D) = 1$. Let $ac$ and $bd$ be the two edges that cross an odd number of times. By performing edge flips, we may assume that all the vertices are adjacent to the outer face of the drawing of the subgraph $H$ with edges $ac$ and $bd$. Each edge $e$ of the remaining four edges can be drawn in two essentially different ways with respect to $H$, which differ just by an edge flip of $e$ in $H + e$; see Fig. 4. In total, there are 16 possible combinations. We cannot, however, assume any particular combination, since not all edge flips are always available. Observe that the orientations of all triangles are determined by the four binary choices for the edges $ab, bc, cd, ad$. Also, changing the choice for one edge $e$ has the same effect on the orientations of the triangles as the edge flip of $e$. For one particular choice, for example the one yielding the middle drawing in Fig. 2, the 1-edges form a perfect matching. Changing the choice for a subset of edges yields either a perfect matching of 1-edges or a complete graph of 1-edges. However, the latter option is excluded by the fact that in every semisimple drawing the edges incident with the outer face are 0-edges. This finishes the proof of the claim and the lemma.

Considering $\leq k$-edges, Ábrego and Fernández-Merchant [6] and Lovász et al. [29] proved that for rectilinear drawings of $K_n$, the inequality $E_{\leq k} \geq 3\binom{k+2}{2}$ together with (2) gives $\text{cr}(G) \geq Z(n)$. However, there are simple $x$-monotone (even 2-page) drawings of $K_n$ where $E_{\leq k} < 3\binom{k+2}{2}$ for $k = 1$ [1]. Ábrego et al. [1] showed that the inequality $E_{\leq k} \geq 3\binom{k+3}{3}$, which is implied by inequalities $E_{\leq j} \geq 3\binom{j+2}{2}$ for $j \leq k$, is satisfied by all 2-page book drawings. We show that the same inequality is satisfied by all $x$-monotone semisimple drawings of $K_n$.

Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of $K_n$. Note that we can assume that all vertices in an $x$-monotone drawing lie on the $x$-axis. We also assume that the $x$-coordinates of the vertices satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$.

The following observation describes the structure of $k$-edges incident to vertices on the outer face in semisimple drawings of complete graphs. See Fig. 5, left.

**Observation 2.3** Let $D$ be a semisimple drawing of $K_n$, not necessarily $x$-monotone. Let $v$ be a vertex incident to the outer face of $D$ and let $e_i$ be the $i$th edge incident...
Lemma 2.4 Let $D$ be a semisimple $x$-monotone drawing of $K_n$ and let $k$ be a fixed integer such that $0 \leq k \leq (n - 3)/2$. For every $i \in \{1, 2, \ldots, k + 1\}$, the $k + 2 - i$ bottommost and the $k + 2 - i$ topmost right edges at $v_i$ are $k$-edges in $D$. Moreover, at least $k + 2 - i$ of these $k$-edges are $(D, D')$-invariant $\leq k$-edges.

Proof See Fig. 5, right. The first part of the lemma follows directly from Observation 2.3. If the edge $v_i v_n$ is one of the $k + 2 - i$ topmost right edges at $v_i$, then the $k + 2 - i$ bottommost right edges at $v_i$ are $(D, D')$-invariant $\leq k$-edges. Otherwise the $k + 2 - i$ topmost right edges at $v_i$ are $(D, D')$-invariant $\leq k$-edges. □

Corollary 2.5 We have

$$E_{\leq k}(D, D') \geq \sum_{i=1}^{k+1} (k + 2 - i) = \binom{k + 2}{2}.$$
The following theorem gives a lower bound on the number of $\leq k$-edges. The proof is essentially the same as in [1], we only extracted Lemma 2.4, which needed to be generalized. Together with Lemma 2.2, Theorem 2.6 yields the second and the third equality in Theorem 1.1, by the same computation as in [1].

**Theorem 2.6** Let $n \geq 3$ and let $D$ be a semisimple $x$-monotone drawing of $K_n$. Then for every $k$ satisfying $0 \leq k < n/2 - 1$, we have $E_{\leq k}(D) \geq 3\left(\frac{k+3}{3}\right)$.

**Proof** The proof proceeds by induction on $n$ and $k$ starting at $n = 3$ and $k = -1$. The case $n = 3$ is trivially true, and the case $k = -1$ is taken care of by setting $E_{\leq -1}(D) = 0$ for every drawing $D$. Let $n \geq 4$ and let $D$ be a semisimple $x$-monotone drawing of $K_n$. For the induction step we remove the point $v_n$ together with its adjacent edges to obtain a drawing $D'$ of $K_{n-1}$, which is also semisimple and $x$-monotone.

Using Observation 2.3 we see that, for $0 \leq i \leq k < n/2 - 1$, there are two $i$-edges adjacent to $v_n$ in $D$ and together they contribute with $2 \sum_{i=0}^{k} (k+1-i) = 2\left(\frac{k+2}{2}\right)$ to $E_{\leq k}(D)$ by (1).

Let $\gamma$ be an $i$-edge in $D'$. If $i \leq k$, then $\gamma$ contributes with $(k-i)$ to the sum

$$E_{\leq k-1}(D') = \sum_{i=0}^{k-1} (k-i)E_i(D').$$

We already observed that $\gamma$ is either an $i$-edge or an $(i+1)$-edge in $D$. If $\gamma$ is also an $i$-edge in $D$ (that is, $\gamma$ is a $(D, D')$-invariant $i$-edge), then it contributes with $(k+1-i)$ to $E_{\leq k}(D)$. This is a gain of $+1$ towards $E_{\leq k-1}(D')$. If $\gamma$ is an $(i+1)$-edge in $D$, then it contributes only with $(k-i)$ to $E_{\leq k}(D)$. Therefore we have

$$E_{\leq k}(D) = 2\left(\frac{k+2}{2}\right) + E_{\leq k-1}(D') + E_{\leq k}(D, D').$$

By the induction hypothesis we know that $E_{\leq k-1}(D') \geq 3\left(\frac{k+2}{3}\right)$ and thus we obtain

$$E_{\leq k}(D) \geq 3\left(\frac{k+3}{3}\right) - \left(\frac{k+2}{2}\right) + E_{\leq k}(D, D').$$

The theorem follows by plugging the lower bound from Corollary 2.5.

2.1 Removing Even Adjacent Crossings

Here we finish the proof of Theorem 1.1 by showing that allowing adjacent edges to cross evenly yields no substantially new monotone drawings of $K_n$.

The *rotation* at a vertex $v$ in a drawing is the clockwise cyclic order of the neighbors of $v$ in which the corresponding edges appear around $v$. The *rotation system* of a drawing is the set of rotations of all its vertices.
Proposition 2.7 Let $D$ be a weakly semisimple monotone drawing of $K_n$. Then there is a semisimple monotone drawing $D'$ of $K_n$ such that for every two edges $e$, $f$ of $K_n$, the parity of the number of crossings between $e$ and $f$ in $D'$ is the same as in $D$. Moreover, $D'$ and $D$ have the same rotation system and the same above/below relations of vertices and edges.

Proof Let $O(D)$ be the set of pairs of edges of $K_n$ that cross an odd number of times in $D$. Let $D'$ be a weakly semisimple monotone drawing of $K_n$ with minimum total number of crossings such that $D'$ is strongly equivalent to $D'$, that is, $D'$ and $D$ have the same rotation system, the same above/below relations of vertices and edges and $O(D') = O(D)$. We show that $D'$ is semisimple.

Suppose for contrary that $D'$ has two adjacent edges $e$, $f$ that cross. Since $D'$ is weakly semisimple, $e$ and $f$ cross at least twice. Let $v$ be the common vertex of $e$ and $f$ and suppose that $e$ is above $f$ in the neighborhood of $v$. Let $x_1$ and $x_2$ be the two crossings of $e$ and $f$ closest to $v$. See Fig. 6, left. Let $B$ be the closed topological disc bounded by the two portions of $e$ and $f$ between $x_1$ and $x_2$. Clearly, $B$ has no vertex on its boundary. Moreover, we claim that $B$ has no vertex in its interior. For if $B$ contains a vertex $w$ in its interior, then $w$ is below $f$ and above $e$. This implies that the edge $vw$ is below $f$ and above $e$ in the neighborhood of $v$, which is absurd.

Since $B$ contains no vertices, every edge other than $e$ and $f$ crosses the boundary of $B$ an even number of times. Therefore, by redrawing an open segment of $e$ or $f$ containing $x_1$ and $x_2$ along the other side of $B$, we obtain a drawing strongly equivalent to $D'$ with at most $cr(D') - 2$ crossings. See Fig. 6, right. \qed

We note that using slightly more careful redrawing operations (such as those in the proof of Theorem 3.2 [10]), we may obtain a semisimple monotone drawing $D''$ strongly equivalent to $D$ such that for every two edges, the number of their common crossings in $D''$ is not larger than in $D$.

By Proposition 2.7, the odd crossing number of a weakly semisimple monotone drawing of $K_n$ is equal to the odd crossing number of some semisimple monotone drawing of $K_n$. This proves the first equality in Theorem 1.1.

3 Combinatorial Description of Monotone Drawings

In this section we develop a combinatorial characterization of $x$-monotone drawings based on the signature functions introduced by Peters and Szekeres [41] as generalizations of order types of planar point sets. Let $\mathcal{T}_n$ be the set of ordered triples $(i, j, k)$ with $i < j < k$, of the set $[n] = \{1, 2, \ldots, n\}$ and let $\Sigma_n$ be the set of signature functions
The negative and the positive signature $\sigma(i, j, k)$

$$\sigma(i, j, k) = - \quad \sigma(i, j, k) = +$$

$\sigma : T_n \to \{-, +\}$. The set $T_n$ may be also regarded as the set $\binom{[n]}{3}$ of all unordered triples, since we write all the triples in the increasing order of their elements.

Let $D$ be an $x$-monotone drawing of the complete graph $K_n = (V, E)$ with vertices $v_1, v_2, \ldots, v_n$ such that their $x$-coordinates satisfy $x(v_1) < x(v_2) < \cdots < x(v_n)$. We assign a signature function $\sigma \in \Sigma_n$ to the drawing $D$ according to the following rule.

For every edge $e = v_i v_k \in E$ and every integer $j \in (i, k)$, let $\sigma(i, j, k) = -$ if the point $v_j$ lies above the arc representing the edge $e$ and $\sigma(i, j, k) = +$ otherwise. See Fig. 7. Note that if the drawing $D$ is also semisimple, then a triangle $v_i v_j v_k$, with $j \in (i, k)$, is oriented clockwise (counter-clockwise) if and only if $\sigma(i, j, k) = -$ ($\sigma(i, j, k) = +$, respectively).

It is easy to see that, for every signature function $\sigma \in \Sigma_n$, there exists an $x$-monotone drawing $D$ which induces $\sigma$. However, some signature functions are induced only by drawings that are not semisimple. We show a characterization of simple and semisimple $x$-monotone drawings by small forbidden configurations in the signature functions.

For integers $a, b, c, d \in [n]$ with $a < b < c < d$, signs $\xi_1, \xi_2, \xi_3, \xi_4 \in \{-, +\}$ and a signature function $\sigma \in \Sigma_n$, we say that the 4-tuple $(a, b, c, d)$ is of the form $\xi_1\xi_2\xi_3\xi_4$ in $\sigma$ if

$$\sigma(a, b, c) = \xi_1, \quad \sigma(a, b, d) = \xi_2, \quad \sigma(a, c, d) = \xi_3, \quad \text{and} \quad \sigma(b, c, d) = \xi_4.$$ 

Alternatively, we write $\sigma([\pi(a), \pi(b), \pi(c), \pi(d)]) = \xi_1\xi_2\xi_3\xi_4$ for any permutation $\pi$ of the set $\{a, b, c, d\}$.

For a sign $\xi \in \{-, +\}$ we use $\bar{\xi}$ to denote the opposite sign, that is, if $\xi = +$ then $\bar{\xi} = -$ and conversely, if $\xi = -$ then $\bar{\xi} = +$.

3.1 Simple and Semisimple $x$-Monotone Drawings

**Theorem 3.1** A signature function $\sigma \in \Sigma_n$ can be realized by a semisimple $x$-monotone drawing if and only if every 4-tuple of indices from $[n]$ is of one of the forms

$$++++, \quad ------, \quad +++-, \quad --++, \quad ---+, \quad -+++,$$

$$-----+, \quad +++, \quad +---, \quad +++++$$

in $\sigma$. The signature function $\sigma$ can be realized by a simple $x$-monotone drawing if, in addition, there is no 5-tuple $(a, b, c, d, e)$ with $a < b < c < d < e$ such that
σ(a, b, e) = σ(a, d, e) = σ(b, c, d) = ̅σ(a, c, e).

See Figs. 13 and 10 for an illustration of the first and the second part of the theorem.

**Proof** Let σ be a signature function with a forbidden 4-tuple, that is, an ordered 4-tuple \((a, b, c, d)\) whose form is not listed in the statement of the theorem. Such a 4-tuple \((a, b, c, d)\) is one of the forms \(\xi_1\xi_2\xi_1\xi_2\) or \(\xi_2\xi_1\xi_1\xi_2\) where \(\xi_1, \xi_2 \in \{-+, +\}\). If \((a, b, c, d)\) is of the form \(+−+\xi\) where \(\xi \in \{-+, +\}\) is an arbitrary sign, then the edges \(v_a v_e\) and \(v_d v_c\) are forced to cross between the vertical lines going through \(v_b\) and \(v_c\); see Fig. 8. But this is not allowed in a semisimple drawing and we have a contradiction. The other cases are symmetric.

On the other hand, let σ be a signature function such that every 4-tuple is of one of the ten allowed forms in σ. We will construct a semisimple x-monotone drawing \(D\) of \(K_n\) which induces σ. We use the points \(v_i = (i, 0), i \in \{1, n\}\), as vertices and connect consecutive pairs of vertices by straight-line segments.

For \(m \in \{1, n\}\), let \(L_m\) be the vertical line containing \(v_m\). In every x-monotone drawing, the line \(L_m\) intersects every edge \(\{v_i, v_j\}\) with \(1 \leq i < m \leq j \leq n\) exactly once. To draw the edges of \(K_n\), it suffices to specify the positions of their intersections with the lines \(L_m\) and to draw the edges as polygonal lines with bends at these intersections. Instead of the absolute position of these intersections on \(L_m\), we only need to determine their vertical total ordering, which we represent by a total ordering \(\preceq_m\) of the corresponding edges. The edges whose right endpoint is \(v_m\) will be ordered by \(\preceq_m\) according to their vertical order in the left neighborhood of \(v_m\). The edges with left endpoint \(v_m\) are not considered in \(\preceq_m\).

The idea of the construction is to interpret the signature function as the set of above/below relations for vertices and edges and take a set of orderings \(\preceq_m\) that obey these relations and minimize the total number of crossings. In the rest of the proof we show a detailed, explicit construction of the orderings \(\preceq_m\) which induce an x-monotone semisimple drawing.

For \(i \in \{1, n\}\), we define an ordering \(\preceq_i\) of the edges with a common left endpoint \(v_i\) (that is, the right edges at \(v_i\)) in the following way. If \(e = \{v_i, v_j\}\) and \(f = \{v_i, v_k\}\) \(i < j, k\), are two such edges, then we set \(e \preceq_i f\) if either \(j < k\) and \(\sigma(i, j, k) = +\), or \(k < j\) and \(\sigma(i, j, k) = −\). Clearly, the relation \(\preceq_i\) is irreflexive, antisymmetric and for every two right edges \(e, f\) at \(v_i\) either \(e \preceq_i f\) or \(f \preceq_i e\). To show that \(\preceq_i\) is a total ordering, it remains to prove that it is transitive. Suppose for contrary that there are three edges \(e = \{v_i, v_j\}, f = \{v_i, v_k\}\) and \(g = \{v_j, v_l\}\) with \(i < j < k < l\) such that \(e \preceq_i f, f \preceq_i g\) and \(g \preceq_i e\). Then \(\sigma(i, j, k) = +, \sigma(i, j, l) = +\) and \(\sigma(i, j, l) = −\), so the 4-tuple \(i, j, k, l\) is of the form \(+−+\xi\), which is forbidden. Similarly, if \(f \preceq_i e, e \preceq_i g\) and \(g \preceq_i f\), then the 4-tuple \(i, j, k, l\) is of the form \(−+−\xi\), which is forbidden as well.
We proceed by induction on \(m\). In the case \(m = 1\) the ordering \(\prec_1\) is empty. For \(m = 2\) the ordering \(\prec_2\) compares only edges with the common endpoint \(v_1\), so we can set \(\prec_2 = \preceq_1\). Since all the edges are drawn by line segments starting in a common endpoint, no crossings appear between \(L_1\) and \(L_2\).

Let \(m > 2\). For the inductive step we consider the following sets \(S_1, \ldots, S_6\) of edges which intersect \(L_{m-1}\) and \(L_m\) (see Fig. 9):

\[
\begin{align*}
S_1 &= \{ (v_i, v_j) \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = - \}, \\
S_2 &= \{ (v_{m-1}, v_j) \mid \sigma(m-1, m, j) = - \}, \\
S_3 &= \{ (v_i, v_j) \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = - \text{ or } j = m \}, \\
S_4 &= \{ (v_i, v_j) \mid \sigma(i, m-1, j) = -, \sigma(i, m, j) = + \text{ or } j = m \}, \\
S_5 &= \{ (v_{m-1}, v_j) \mid \sigma(m-1, m, j) = + \}, \\
S_6 &= \{ (v_i, v_j) \mid \sigma(i, m-1, j) = +, \sigma(i, m, j) = + \}.
\end{align*}
\]

The edges within sets \(S_2\) and \(S_5\) are ordered according to \(\preceq_{m-1}\) and the edges in each of the remaining sets \(S_k\) according to \(\prec_{m-1}\). For \(e \in S_k\) and \(f \in S_l\) where \(k < l\), we set \(e \prec_m f\). Observe that \(\prec_m\) is a total ordering.

We show that the drawing \(D\) determined by the orders \(\prec_m\) is semisimple. Suppose for contradiction that two adjacent edges \(e = \{v_i, v_j\}\) and \(f = \{v_i, v_k\}\), with \(i < j, k\) and \(e \prec_i f\), cross. Their leftmost crossing occurs between lines \(L_{m-1}\) and \(L_m\), where \(i < m - 1\) and \(m \leq j, k\). There are three cases:

1. \(e \in S_6\) and \(f \in S_3\),
2. \(e \in S_4\) and \(f \in S_1\), or
3. \(e \in S_4\) and \(f \in S_3\).

We analyze the cases (i) and (iii) together, case (i) and case (ii) are symmetric. If \(j < k\) then \(\sigma(i, m, k) = -\) and by the definition of the relation \(\prec_i\), we have \(\sigma(i, j, k) = +\). This further implies that \(m < j\) and \(\sigma(i, m, j) = +\). Thus \((i, m, j, k)\) forms a forbid-
den 4-tuple. If \( k < j \), then \( \sigma(i, m, j) = +, \sigma(i, k, j) = - \), which implies that \( m < k \) and \( \sigma(i, m, k) = - \), and so we obtain a forbidden 4-tuple \((i, m, k, j)\).

Now suppose that two adjacent edges \( e = \{v_i, v_k\} \) and \( f = \{v_j, v_k\} \), with \( i, j < k \), cross. Their leftmost crossing occurs between lines \( L_{m-1} \) and \( L_m \), where \( i, j \leq m - 1 \) and \( m < k \). We may assume that \( f <_{m} e \) and \( e <_{m-1} f \). There are five cases:

(i) \( e \in S_6 \) and \( f \in S_3 \),
(ii) \( e \in S_4 \) and \( f \in S_1 \),
(iii) \( e \in S_4 \) and \( f \in S_3 \),
(iv) \( e \in S_4 \) and \( f \in S_2 \), or
(v) \( e \in S_5 \) and \( f \in S_3 \).

Case (i) and case (ii) are symmetric, as well as case (iv) and case (v). Therefore it is sufficient to consider cases (i), (iii) and (v). In all these three cases \( \sigma(j, m, k) = - \) and \( \sigma(i, m, k) = + \). If \( j < i \), then \( \sigma(j, i, k) = + \) since \( e <_{m-1} f \) and the edges \( e \) and \( f \) do not cross to the left of \( L_{m-1} \). Hence \((j, i, m, k)\) forms a forbidden 4-tuple. If \( i < j \), then analogously \( \sigma(i, j, k) = - \) and \((i, j, m, k)\) forms a forbidden 4-tuple. This finishes the proof that \( D \) is semisimple.

It remains to show the second part of the theorem. If \( D \) is a drawing with a signature function \( \sigma \) with a forbidden 5-tuple \((a, b, c, d, e)\), then \( D \) is not simple as the edges \( v_a v_e \) and \( v_b v_d \) are forced to cross at least twice; see Fig. 10.

In the rest of the proof we show the second part of the theorem.

Given a signature function \( \sigma \) with no forbidden 4-tuples and 5-tuples we apply the same construction as before to obtain a semisimple \( x \)-monotone drawing \( D \). We show that \( D \) is, in addition, simple. Since \( D \) is semisimple, no two crossing edges have an endpoint in common. By the construction of \( D \), every crossing \( c \) of two edges \( e \) and \( f \) occurs between lines \( L_m \) and \( L_{m+1} \) for some \( m \in \{n - 1\} \) and we say that \( v_{m+1} \) is the right neighbor of \( c \). The right neighbor is either an endpoint of \( e \) or \( f \) or it separates the crossings of \( L_{m+1} \) with \( e \) and \( f \). Suppose that there are edges \( e = v_i v_j \) and \( f = v_k v_l \) with \( i < k < j, l \) that cross at least twice. We show that then there is always a forbidden 4-tuple or a forbidden 5-tuple in \( \sigma \).

Let \( v_m \) be the right neighbor of the leftmost crossing and \( v_m' \) the right neighbor of the second leftmost crossing of \( e \) and \( f \). Observe that \( i, k < m < m' \leq j, l \).

First assume that \( l < j \). Refer to Fig. 11. If \( \sigma(i, k, j) = \sigma(i, l, j) = \xi \) for some \( \xi \in \{-, +\} \), then \( \xi = \sigma(k, m, l) = \sigma(i, m, j) \) and so \((i, k, m, l)\) forms a forbidden 5-tuple. If \( \sigma(i, k, j) = \sigma(i, l, j) = \xi \) for some \( \xi \in \{-, +\} \), then \( e \) and \( f \) cross at least three times and so \( m' < l \) and \( j \). We have \( \xi = \sigma(k, m, l) = \sigma(i, m, j) = \sigma(k, m', l) = \sigma(i, m', j) \). If \( \sigma(k, m, m') = \bar{\xi} \), then \((k, m, m', l)\) forms a forbidden 4-tuple. If \( \sigma(k, m, m') = \xi \), then \((i, k, m, m', j)\) forms a forbidden 5-tuple.
Fig. 11  Edges $v_iv_j$ and $v_kv_l$ crossing twice imply a forbidden 5-tuple or 4-tuple; case $l < j$

Conversely let $j < l$. Refer to Fig. 12. Assume that $\sigma(i, k, j) = \bar{\sigma}(k, j, l) = \xi$ for some $\xi \in \{-, +\}$. Then $\xi = \sigma(k, m, l) = \bar{\sigma}(i, m, j)$. If $\sigma(k, m, j) = \xi$, we get a forbidden 4-tuple $(i, k, m, j)$, otherwise $\sigma(k, m, j) = \bar{\xi}$ and we get a forbidden 4-tuple $(k, m, j, l)$. Finally, assume that $\sigma(i, k, j) = \sigma(k, j, l) = \xi$ for some $\xi \in \{-, +\}$. The proof in this case is identical to the proof of the case $l < j$ and $\sigma(i, k, j) = \bar{\sigma}(k, j, l) = \xi$ in the previous paragraph. ⊓⊔

3.2 Pseudolinear $x$-Monotone Drawings

A drawing $D$ of a complete graph $K_n$ is pseudolinear (also pseudogeometric or extendable) if the edges of $D$ can be extended to unbounded simple curves that cross each other exactly once, thus forming an arrangement of pseudolines. The vertices of $D$ together with the $\binom{n}{2}$ pseudolines extending the edges are said to form a pseudoarrangement of points (also generalized configuration of points). Note that the pseudoarrangement of points extending $D$ is usually not unique as there is a certain freedom in choosing where the pseudolines extending disjoint noncrossing edges of $D$ cross.

It is well known that every arrangement of pseudolines can be made $x$-monotone by a suitable isotopy of the plane (this follows, for example, by the duality transform established by Goodman [20,22]). Therefore, every pseudolinear drawing of $K_n$ is isotopic to an $x$-monotone pseudolinear drawing. Every rectilinear drawing of $K_n$ is
\(x\)-monotone and pseudolinear, but there are pseudolinear drawings of \(K_n\) that cannot be “stretched” to rectilinear drawings.

We show that \(x\)-monotone pseudolinear drawings of \(K_n\) can be characterized in a combinatorial way by forbidden 4-tuples in the corresponding signature function, by further restricting the conditions on the signatures in Theorem 3.1. In fact, the conditions in Theorem 3.2 are precisely the geometric constraints that Peters and Szekeres [41] used to restrict the set of signature functions in their investigation of the Erdős–Szekeres problem. Fig. 13 illustrates the classification of 4-tuples from Theorem 3.1 and Theorem 3.2.

**Theorem 3.2**  
A signature function \(\sigma \in \Sigma_n\) can be realized by a pseudolinear \(x\)-monotone drawing if and only if every ordered 4-tuple of indices from \([n]\) is of one of the forms

\[
++++, +++-, +++-, +---, \\
-----, -----+, ----+, ++++
\]

in \(\sigma\).

Pseudolinear drawings of complete graphs are equivalent to CC systems introduced by Knuth [27], although this equivalence is not easily seen. The CC systems are ternary counter-clockwise relations of finite sets satisfying a certain set of five axioms involving triples, 4-tuples or 5-tuples of elements. CC systems generalize the order types of planar point sets in general position: an ordered triple in the counter-clockwise relation is interpreted as a triple of points in the plane placed in the counter-clockwise order, like a triple with signature \(+\) in the signature function. Unlike the signature functions, the CC systems have no fixed ordering of the elements. Therefore, some of the axioms for CC systems involve 5-tuples of elements, whereas 4-tuples are sufficient in the case of signature functions. In fact, the axioms of CC systems specify exactly that every 5-tuple of elements can be realized as a point set in the plane.
Knuth [27] established a correspondence between CC systems and reflection networks (also called wiring diagrams), which are simple arrangements of pseudolines dual to the pseudorarrangements of points extending the pseudolinear drawings of complete graphs. Knuth [27] also showed a two-to-one correspondence between CC systems and uniform acyclic oriented matroids of rank 3 on the same underlying set. Here the CC system is, in fact, the chirotope of the corresponding oriented matroid.

Streinu [40] characterized sets of signed circular permutations (directed clusters of stars) that arise from generalized configurations of n points as circular sequences of pseudolines at each of the n points, and provided an $O(n^2)$ drawing algorithm. It is easy to show that the set of signed circular permutations determines the orientation of all triangles (and thus the corresponding CC system) and vice versa. However, many details are omitted in the extended abstract [40].

Felsner and Weil [16, 17] proved that triangle-sign functions of simple arrangements of n pseudolines are precisely those functions $f : \binom{[n]}{3} \to \{+, -\}$ that are monotone on all 4-tuples. This is the same condition as the condition on signature functions in Theorem 3.2. That is, Theorem 3.2 is a dual analogue of Felsner’s and Weil’s result. Felsner and Weil [16, 17] also introduced r-signotopes, a notion unifying permutations, allowable sequences and monotone triangle-sign functions of simple arrangements. In this notation, the signature functions satisfying the conditions of Theorem 3.2 are 3-signotopes.

Theorem 3.2 can be deduced from each of these previous results. However, we did not find any of these ways particularly easy or straightforward. We provide a direct, self-contained proof of Theorem 3.2 in the extended version of this paper [10].

3.3 A Remark on Rectilinear Drawings

A similar characterization of rectilinear drawings of $K_n$ (equivalently, order types of planar point sets in general position) in terms of signature functions or CC systems with a finite number of forbidden configurations is impossible: for example, Bokowski and Sturmfels [13] constructed infinitely many minimal CC systems (simplicial affine 3-chirotopes) that are not realizable as sets of points in the plane. This and related results were also referred to by the phrase “missing axiom for chirotopes is lost forever”.

Moreover, recognizing signature functions of rectilinear drawings of $K_n$ (or, order types of planar point sets in general position), is polynomially equivalent to rectilinear realizability of complete abstract topological graphs and to stretchability of pseudoline arrangements [28], which is polynomially equivalent to the existential theory of the reals [30]. In the terminology introduced by Schaefer [37], these problems are $∃R$-complete. It is known that $∃R$-complete problems are in PSPACE [14] and NP-hard, but they are not known to be in NP.

3.4 Crossing Minimal $x$-Monotone Drawings

Note that in a simple $x$-monotone drawing of $K_n$, the crossings appear only between edges whose endpoints induce a 4-tuple of one of the forms $++++$, $------$, $+-+-$, $----$, $++--$, $+-+$. Analogously as for the rectilinear drawings of $K_n$, we may...
call these 4-tuples convex. Then, for a simple x-monotone drawing \( D \) of \( K_n \) the crossing number of \( D \) equals the number of convex 4-tuples. A similar notion of convexity for general \( k \)-tuples was used by Peters and Szekeres [41].

This description of crossings is convenient for computer calculations. Using it, we have obtained a complete list of optimal \( x \)-monotone drawings of \( K_n \) for \( n \leq 10 \). To enumerate “essentially different” drawings we used the following approach.

Let \( D \) be an \( x \)-monotone drawing of \( K_n \) which induces a signature function \( \sigma \). We can assume that the vertices are points placed on the same horizontal line (the \( x \)-axis). The following operations on \( D \) and \( \sigma \) produce a signature function \( \sigma' \) of a simple monotone drawing \( D' \) that is homeomorphic to \( D \) on the sphere, by a homeomorphism that does not necessarily preserve the labels of vertices. In some cases we just describe the transformation of the drawing; the new signature function \( \sigma' \) can be then computed in a straightforward way.

(a) **Vertical reflection**: setting \( \sigma'(i, j, k) = \overline{\sigma(i, j, k)} \) for every \( (i, j, k) \in T_n \).

(b) **Horizontal reflection**: setting \( \sigma'(i, j, k) = \sigma(n+1-k, n+1-j, n+1-i) \) for every \( (i, j, k) \in T_n \).

(c) **Shifting** \( v_1 \): if every edge incident to \( v_1 \) lies completely above or completely below the \( x \)-axis, that is, \( \sigma(1, i, k) = \sigma(1, j, k) \) for every \( k \in \{3, \ldots, n\} \) and \( 1 < i, j < k \), then we can move \( v_1 \) to the position of \( v_n \) and move every \( v_{i+1} \) to the position of \( v_i \), for every \( 1 \leq i \leq n-1 \).

(d) **Switching consecutive points**: let \( j \in \{n-1\} \). If there is a \( \xi \in \{-, +\} \) such that \( \sigma(j, j+1, k) = \xi \) for every \( j+1 < k \leq n \) and \( \sigma(i, j, j+1) = \overline{\xi} \) for every \( 1 \leq i < j \), then we can switch the positions of \( v_j \) and \( v_{j+1} \). After the switch, we have \( \sigma'(j, j+1, k) = \overline{\xi} \) for every \( j+1 < k \leq n \) and \( \sigma'(i, j, j+1) = \xi \) for every \( 1 \leq i < j \).

(e) **Redrawing the edge** \( v_1v_n \): in every crossing minimal \( x \)-monotone drawing, the edge \( v_1v_n \) crosses no other edge, since we can always redraw this edge along the top or the bottom part of the boundary of the outer face. The signature function \( \sigma \) thus satisfies \( \sigma(1, i, n) = \xi \) for some \( \xi \in \{+, -\} \) and for every \( i, 1 < i < n \). We may thus simultaneously change all the signatures \( \sigma(1, i, n) \).

We say that two \( x \)-monotone drawings \( D \) and \( D' \) are **switching equivalent** if there is a sequence of operations (a)–(e) such that, when applied to \( D \), we obtain a drawing which has the same signature function as \( D' \). We have found representatives of all switching equivalence classes of crossing minimal \( x \)-monotone drawings of \( K_n \), for \( n \leq 10 \). Their numbers are given in Table 1.

Ábrego et al. [1] proved that for every even \( n \), there is a unique crossing minimal 2-page book drawing of \( K_n \), up to a homeomorphism of the sphere. We have found crossing minimal \( x \)-monotone drawings of \( K_8 \) and \( K_{10} \) that are not homeomorphic to 2-page book drawings. There are exactly two such drawings of \( K_8 \); see Fig. 14. We do not have a construction of such drawings of \( K_n \) for arbitrarily large \( n \).

### 4 Weakly Semisimple and Shellable Drawings

Our proof of Theorem 1.1 for semisimple monotone drawings, as well as the earlier proof by Ábrego et al. [2, Theorem 1.1], do not use all properties of monotone draw-
Table 1  Numbers of switching equivalence classes of crossing minimal \(x\)-monotone drawings of \(K_n\) for \(n \leq 10\)

| Number of vertices | 5   | 6   | 7   | 8   | 9   | 10  |
|--------------------|-----|-----|-----|-----|-----|-----|
| Number of drawings | 1   | 1   | 5   | 3   | 510 | 38  |

Fig. 14  Left a crossing minimal \(x\)-monotone drawing of \(K_8\) homeomorphic to the cylindrical drawing. Right a crossing minimal \(x\)-monotone drawing of \(K_8\) that is not homeomorphic to a 2-page book drawing and neither to the cylindrical drawing

...ings. Both rely only on the fact that the vertices of the drawing can be ordered as \(v_1, v_2, \ldots, v_n\) so that for every pair \(i, j\) with \(1 \leq i < j \leq n\), the vertices \(v_i\) and \(v_j\) are on the outer face of the drawing induced by the interval of vertices \(v_i, v_{i+1}, \ldots, v_j\).

Pedro Ramos [35] introduced the term shellable drawings for these drawings of \(K_n\). Ábrego et al. [3] later observed that a still more general condition, \(s\)-shellability for some \(s \geq n/2\), is sufficient, since the depth of the recursion in the proof is only \(n/2\). A drawing of a complete graph with a vertex set \(V\) is called \(s\)-shellable if there is a subset of vertices \(v_1, v_2, \ldots, v_s \in V\) such that for every pair \(i, j\) with \(1 \leq i < j \leq s\), the vertices \(v_i\) and \(v_j\) are on the outer face of the drawing induced by \(V \setminus \{v_1, v_2, \ldots, v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s\}\). In our version of this definition, we require \(v_1\) and \(v_s\) to be incident with the outer face; this is slightly more restrictive compared to the original definition in [3]. Informally speaking, \(s\)-shellable drawings consist of two parts: the first part is a shellable drawing of \(K_s\), the second part is an arbitrary drawing of the remaining vertices and edges that does not block the shelling of the first part. If \(s \geq 3\), this means, in particular, that all vertices from the second part “see” the vertices in the first part in the same cyclic order. The class of \(s\)-shellable drawings includes, for example, all drawings with a crossing-free cycle of length \(s\), with at least one edge of the cycle incident with the outer face [3]. Note that the notions shellable and \(n\)-shellable coincide for drawings of \(K_n\).

Following this notation, we call the sequence \(v_1, v_2, \ldots, v_n\) from the definition of a shellable drawing of \(K_n\) a shelling sequence of the drawing, which is similar to the term \(s\)-shelling introduced by Ábrego et al. [3].

Ábrego et al. [3] also considered the class of \(x\)-bounded drawings, which form a subclass of shellable drawings and generalize \(x\)-monotone drawings. A drawing of a graph is \(x\)-bounded if no two vertices share the same \(x\)-coordinate and every interior point of every edge \(uv\) lies in the interior of the strip bounded by two vertical lines passing through the vertices \(u\) and \(v\). Fulek et al. [19] showed that every \(x\)-bounded drawing \(D\) can be transformed into an \(x\)-monotone drawing \(D'\), while keeping the
rotation system and the parity of the number of crossings of every pair of edges fixed. This implies, in particular, that $\text{ocr}(D) = \text{ocr}(D')$. Also $D'$ is weakly semisimple if and only if $D$ is weakly semisimple. Therefore, the lower bound from Theorem 1.1 extends to all weakly semisimple $x$-bounded drawings of $K_n$.

It is not a priori clear that shellable drawings are essentially different from monotone or $x$-bounded drawings, since the conditions for shellability and $x$-boundedness are very similar at first sight. In Sect. 4.2 we show that simple shellable drawings are indeed more general than simple monotone drawings, but the difference is rather subtle. By a somewhat detailed analysis, which we do not include here, it can be shown that every simple shellable drawing of $K_n$ can be decomposed into three monotone drawings, in a very specific way.

Apart from following the proof of Theorem 1.1, we may obtain a lower bound on the crossing number of shellable drawings of $K_n$ by the following straightforward reduction to the monotone crossing number of $K_n$, using the combinatorial characterization of $x$-monotone drawings.

**Proposition 4.1** Let $D$ be a semisimple shellable drawing of $K_n$. There is a semisimple $x$-monotone drawing $D'$ of $K_n$ with $\text{ocr}(D') = \text{ocr}(D)$.

We note that the drawing $D'$ obtained in Proposition 4.1 does not necessarily preserve the parity of the number of crossings between a given pair of edges. Moreover, it is also possible that for a simple shellable drawing $D$, we obtain a monotone drawing $D'$ where some pair of edges cross more than once; see Fig. 15.

Let $v_1, v_2, \ldots, v_n$ be the vertices of a semisimple drawing $D$ of $K_n$. The order type of $D$ is the function $\sigma : \binom{[n]}{3} \to \{+,-\}$ defined in the following way: for $1 \leq i < j < k \leq n$, $\sigma(i,j,k) = +$ if the triangle $v_iv_jv_k$ is drawn counter-clockwise and $\sigma(i,j,k) = -$ if the triangle $v_iv_jv_k$ is drawn clockwise. This generalizes the definition of the signature function for semisimple monotone drawings. As in the previous section, we use the shortcut $\sigma(i,j,k,l)$ for the string of four signs $\sigma(i,j,k), \sigma(i,j,l), \sigma(i,k,l), \sigma(j,k,l)$.

**Proof** Let $v_1, v_2, \ldots, v_n$ be a shelling sequence of $D$. Let $\sigma$ be the order type of $D$. We show that $\sigma$ satisfies the assumptions of Theorem 3.1, and therefore can be realized by a semisimple monotone drawing. Let $v_i, v_j, v_k, v_l$ be a 4-tuple of vertices with

![Fig. 15](https://example.com/fig15.png) A simple shellable drawing (left) and the corresponding semisimple monotone drawing (right). Note that the left drawing is both shellable and monotone; however, its shelling sequence $1, 2, 3, 4, 5$ is not its monotone sequence.
1 ≤ i < j < k < l ≤ n. Then the drawing of $K_4$ induced by $v_i, v_j, v_k, v_l$ has $v_i$ and $v_l$ on its outer face. To verify the assumptions of Theorem 3.1, it is sufficient to show that none of the cases $\sigma(i, j, k, l) = +--\xi, \sigma(i, j, k, l) = --+\xi, \sigma(i, j, k, l) = \xi++$ or $\sigma(i, j, k, l) = \xi--$, with $\xi \in \{+, -, \}$, occurs. Suppose the contrary. Due to symmetry, we may suppose that $\sigma(i, j, k, l) = +--\xi$. This means that reading the linear counter-clockwise order of the edges incident with $v_j$ starting from the outer face, we encounter the edge $v_i v_j$ before the edge $v_i v_k$, $v_j v_k$ before $v_j v_l$, and $v_l v_j$ before $v_i v_l$; a contradiction.

Let $D'$ be a semisimple monotone drawing realizing $\sigma$. Every 4-tuple of vertices in $D$ induces a drawing of $K_4$ with at most one pair of edges crossing oddly. This is clear if $D$ is simple; for semisimple drawings this is proved in the claim in the proof of Lemma 2.2. Call a 4-tuple of vertices in $D$ or $D'$ odd if it induces exactly one pair of edges crossing oddly and even otherwise. To finish the proof, it remains to show that odd (even) 4-tuples of vertices in $D$ correspond to odd (even, respectively) 4-tuples in $D'$.

Odd (also convex) 4-tuples in $D'$ are of one of the forms $++++$, $-----$, $+---$, $-+++$, $-+-+$, $-+++$. Even 4-tuples in $D'$ are of one of the forms $++++$, $-+++$, $-+-+$, $+---$.

Let $v_i, v_j, v_k, v_l$, with $i < j < k < l$, be a 4-tuple of vertices in $D$, inducing a drawing $H$ of $K_4$. By deforming the plane, we may assume that $v_i = (0, 0), v_l = (1, 0)$, and that the vertices $v_j, v_k$ and the interiors of all six edges of $H$ lie in the interior of the strip between the vertical lines passing through $v_i$ and $v_l$. Note, however, that $H$ is not necessarily deformable to an $x$-bounded drawing with $v_j$ to the left of $v_k$: see Fig. 17, left.

Due to symmetry, we may assume that $\sigma(i, j, l) = +\cdot$. That is, the vertex $v_j$ and the interiors of the edges $v_i v_j$ and $v_j v_l$ lie below the edge $v_i v_l$. Now if $\sigma(i, k, l) = --\cdot$, then the vertex $v_k$ and the interiors of the edges $v_i v_k$ and $v_l v_j$ lie above the edge $v_i v_l$. See Fig. 16a. Thus, the edges $v_i v_l$ and $v_j v_k$ are forced to cross an odd number of times, and no other pair of edges in $H$ cross. Also, the triangle $v_i v_j v_k$ is drawn counter-clockwise and the triangle $v_j v_k v_l$ clockwise, so we have $\sigma(i, j, k, l) = +--\cdot$. Therefore, the 4-tuple $v_i, v_j, v_k, v_l$ is odd in both drawings $D$ and $D'$.

If $\sigma(i, k, l) = +\cdot$, then the vertex $v_k$ and the interiors of the edges $v_i v_k$ and $v_l v_l$ lie below the edge $v_i v_l$. We have four cases according to the signs $\sigma(i, j, k)$ and $\sigma(j, k, l)$, which determine the vertical order of the edges near $v_j$ and $v_l$, respectively, but do not determine completely which edges cross oddly. This is true even when the drawing $H$ is simple; see Fig. 17. If $\sigma(i, j, k, l) = +++++$ or $\sigma(i, j, k, l) = -++--\cdot$, then either the edges $v_i v_k$ and $v_j v_l$ cross oddly, or the edges $v_i v_j$ and $v_k v_l$ cross oddly, and some other pair of edges may cross evenly; see Fig. 16 b, c. In both cases, the 4-tuple $v_i, v_j, v_k, v_l$ is odd in both drawings $D$ and $D'$. If $\sigma(i, j, k, l) = -++++$ or $\sigma(i, j, k, l) = +++--\cdot$, then no two edges cross; see Fig. 16 d, e. In these last two cases, the 4-tuple $v_i, v_j, v_k, v_l$ is even in both drawings $D$ and $D'$.

Proposition 4.1 can be generalized to weakly semisimple shettltube drawings, but the equality of the odd crossing numbers has to be replaced by inequality, since there are weakly semisimple shettltube drawings of $K_4$ with odd crossing number 2; see Fig. 18, left.
Fig. 16 Examples of semisimple shellable drawings of $K_4$

Fig. 17 Two drawings of $K_4$ with the same order type

Fig. 18 Left a weakly semisimple shellable drawing of $K_4$ with two pairs of edges crossing oddly. Right a weakly semisimple drawing of $K_4$ with three pairs of edges crossing oddly

For general weakly semisimple drawings, the triangles are not necessarily simple closed curves. Nevertheless, we may still define the orientation of a triangle when every two of its edges cross evenly. Let $uvw$ be a triangle in a weakly semisimple drawing $D$ of $K_n$. Orient the closed curve $\gamma$ representing the triangle $uvw$ so that it passes through the vertices $u$, $v$, $w$ in this cyclic order. Then for each point $p$ on $\gamma$ that is not a crossing, a sufficiently small neighborhood of $p$ is divided by $\gamma$ into
the right neighborhood and the left neighborhood of \( p \), consistently with the chosen orientation of \( \gamma \).

Let \( x \) be a point in the complement of \( \gamma \) in the plane. The winding number of \( \gamma \) around \( x \) is, informally speaking, the number of counter-clockwise turns of \( \gamma \) around \( x \). More formally, if \( \gamma \) is parametrized by continuous polar coordinates \((r(t), \varphi(t)) : [0, 1] \rightarrow (0, \infty) \times \mathbb{R}\), with center at \( x \), then the winding number of \( \gamma \) around \( x \) is \( \frac{\varphi(1) - \varphi(0)}{2\pi} \). We use only the parity of the winding number, which is independent of the chosen orientation of \( \gamma \).

We say that the triangle \( uvw \), represented by the curve \( \gamma \), is oriented counter-clockwise if for some point \( x \) in the right neighborhood of \( u \), the winding number of \( \gamma \) around \( x \) is even. Similarly, the triangle \( uvw \) is oriented clockwise if the winding number of \( \gamma \) around \( x \) is odd. Due to the fact that every two edges of \( uvw \) cross an even number of times, the definition does not change if we choose \( x \) in the right neighborhood of \( v \) or \( w \). We may thus generalize the notion of the order type to every weakly semisimple drawing of \( K_n \) with vertices labeled \( v_1, v_2, \ldots, v_n \).

**Proposition 4.2** Let \( D \) be a weakly semisimple shellable drawing of \( K_n \). There is a semisimple \( x \)-monotone drawing \( D' \) of \( K_n \) with \( \text{ocr}(D') \leq \text{ocr}(D) \).

**Proof** We proceed in the same way as in the proof of Proposition 4.1. Let \( v_1, v_2, \ldots, v_n \) be a shelling sequence of \( D \) and let \( \sigma \) be the order type of \( D \). The fact that \( \sigma \) satisfies the assumptions of Theorem 3.1 can be proved exactly in the same way as in the proof of Proposition 4.1. Let \( D' \) be a semisimple monotone drawing with signature function \( \sigma \).

To prove the inequality, it is sufficient to show that every 4-tuple of vertices in \( D \) that induces a \( K_4 \) subgraph with odd crossing number 0, corresponds to a 4-tuple with no crossing in \( D' \). For that, we only need to show that the 4-tuple in \( D \) is of the type ++++, −++++, −−−− or +−−−. All other 4-tuples in \( D \) induce subgraphs with odd crossing number 1 or 2, which is at least as large as the odd crossing number of any \( K_4 \) subgraph in \( D' \).

Let \( v_i, v_j, v_k, v_l \), with \( i < j < k < l \), be vertices in \( D \) inducing a subgraph \( H \) with all pairs of edges crossing evenly. We will show that there is a planar drawing \( H'' \) of the complete graph with vertices \( v_i, v_j, v_k, v_l \), with \( v_i \) and \( v_l \) on its outer face, such that the orientation of each triangle in \( H'' \) is the same as in \( H \). This will finish the proof, since such a drawing \( H'' \) is homeomorphic to one of the drawings in Fig. 16 d, e.

The drawing \( H \) satisfies the assumptions of the weak Hanani–Tutte theorem [39]. The weak Hanani–Tutte theorem says that for every drawing \( D \) of a graph \( G \) in the plane where every two edges cross an even number of times, there is a planar drawing \( D' \) of \( G \) which has the same rotation system as \( D \) (that is, the cyclic orders of the edges around each vertex are preserved). The shortest proof of the weak Hanani–Tutte theorem, based on a more general version for arbitrary surfaces [34], was given by Fulek et al. [18, Lemma 3].

We may assume that \( v_i \) is the unique point in \( H \) with smallest \( x \)-coordinate and that \( v_l \) is the unique point in \( H \) with largest \( x \)-coordinate. We extend the drawing \( H \) to a drawing \( K \) by adding a vertex \( y \) placed below \( H \), a vertex \( z \) placed above \( H \), and adding four edges \( v_i y, y v_l, v_l z, z v_i \), drawn as monotone curves and forming a simple...
cycle \(v_i y v_l z\). The cycle \(v_i y v_l z\) forms the boundary of the outer face of \(K\). By the weak Hanani–Tutte theorem, there is a planar drawing \(K'\) having the same rotation system as \(K\). In particular, the cycle \(v_i y v_l z\) bounds a face \(F\) in \(K'\). Without loss of generality, we may assume that \(F\) is the outer face of \(K'\). By the weak Hanani–Tutte theorem, there is a planar drawing \(K'\) having the same rotation system as \(K\). In particular, the cycle \(v_i y v_l z\) forms the boundary of the outer face of \(K\).

Let \(H'\) be the drawing obtained from \(K'\) by removing the vertices \(y, z\) and their adjacent edges. Clearly, the drawings \(H'\) and \(H\) have the same rotation system, \(H'\) has no crossings, and \(v_i\) and \(v_l\) are on the boundary of the outer face of \(H'\). The orientation of triangles \(v_i v_j v_k\), \(v_i v_j v_l\) and \(v_i v_k v_l\) is determined by the rotation at \(v_i\), and the orientation of the triangle \(v_j v_k v_l\) is determined by the rotation at \(v_l\). It follows that \(H\) and \(H'\) have the same order type, and the proof is finished.

An attempt to generalize the approach in Proposition 4.1 to general non-shellable drawings fails, for the following reason. If \(v_1, v_2, \ldots, v_n\) is a chosen ordering of the vertices which is not a shelling sequence, we can have a 4-tuple \(v_i, v_j, v_k, v_l\), with \(i < j < k < l\), inducing a planar drawing of \(K_4\) such that \(v_i\) or \(v_l\) is the only vertex not incident with the outer face. These 4-tuples are of type \(+−−+, \quad +−−+, \quad −+−−\), or \(−−−+\). In monotone drawings, such 4-tuples are not semi-simple and, moreover, have monotone odd crossing number 2. On the other hand, this is the only obstacle in generalizing Proposition 4.1 to all simple drawings. Indeed, it is easy to see that all simple drawings of \(K_4\) with one crossing and arbitrary ordering of the vertices are of type \(++++\), \(++−−\), \(−+−−\), or \(−−−−\), and thus correspond to a simple monotone drawing of \(K_4\) with one crossing. In fact, this is still true also for semisimple drawings, by the claim in the proof of Lemma 2.2.

We may thus generalize Proposition 4.1 and consequently Theorem 1.1 to every drawing of \(K_n\) such that there is an ordering \(v_1, v_2, \ldots, v_n\) of its vertices such that for every 4-tuple \(v_i, v_j, v_k, v_l\), with \(i < j < k < l\), inducing a planar drawing \(H\) of \(K_4\), the vertices \(v_i\) and \(v_l\) are on the outer face of \(H\). We call such a drawing weakly shellable. Trivially, every drawing of \(K_n\) with \(\binom{n}{4}\) crossings is weakly shellable, with arbitrary ordering of its vertices.

**Corollary 4.3** Let \(D\) be a semisimple weakly shellable drawing of \(K_n\). There is a semisimple \(x\)-monotone drawing \(D'\) of \(K_n\) with \(ocr(D') = ocr(D)\).

**Corollary 4.4** Let \(D\) be a semisimple weakly shellable drawing of \(K_n\). Then \(ocr(D) \geq Z(n)\).

We note that there are simple drawings of complete graphs that are not weakly shellable. For example, the drawing \(F_6\) of \(K_6\) in Fig. 23, left, has the property that every vertex is the central vertex of a planar drawing of \(K_4\) induced by some 4-tuple of vertices. Moreover, by taking two disjoint copies \(F_6\) and adding all remaining 36 edges, we obtain a simple drawing of \(K_{12}\) which will not become weakly shellable even if we change its outer face by an arbitrary sequence of edge flips.

By removing the central vertex in \(F_6\) we obtain a weakly shellable simple drawing of \(K_5\) that is not shellable. This shows that weakly shellable drawings are more general than shellable drawings.
4.1 Local Characterization of Shellable Drawings

The definition of a shellable drawing of a complete graph involves testing a quadratic number of subgraphs. It is easy to see that only linearly many of the subgraphs are sufficient.

**Observation 4.5** A sequence of vertices \(v_1, v_2, \ldots, v_n\) is a shelling sequence of a drawing of a complete graph if and only if for every \(i \in [n]\), the vertex \(v_i\) is on the outer face of the two subgraphs induced by the subsets of vertices \(\{v_1, v_2, \ldots, v_i\}\) and \(\{v_i, v_{i+1}, \ldots, v_n\}\).

In a similar spirit as in Theorem 3.1, we may obtain a local characterization of shellable drawings, by testing only the subgraphs with four vertices. Like in Theorem 3.1, we need to assume a fixed ordering of the vertices, as there are arbitrarily large minimal non-shellable (and non-monotone) drawings of complete graphs—for example, “flowers” generalizing the drawing \(F_6\) in Fig. 23, left. Unlike in the case of monotone drawings, the order type does not necessarily determine a unique shellable drawing; see Fig. 17.

**Theorem 4.6** Let \(D\) be a simple drawing of \(K_n\). A sequence \(v_1, v_2, \ldots, v_n\) of the vertices is a shelling sequence of \(D\) if and only if every 4-tuple \(v_i, v_j, v_k, v_l\), with \(i < j < k < l\), induces a drawing of \(K_4\) having \(v_i\) and \(v_l\) on its outer face.

To show Theorem 4.6, we use the following generalization of Carathéodory’s theorem.

**Lemma 4.7** (Carathéodory’s theorem for simple complete topological graphs). Let \(D\) be a simple drawing of \(K_n\) and let \(x\) be a point in the interior of a bounded face of \(D\). Then there is a triangle \(uvw\) in \(D\) containing \(x\) in its interior. Moreover, there is a set of at most \(n - 2\) triangles covering all bounded faces of \(D\) and such that every edge of \(D\) is in at most two of these triangles.

We use only the first part of the lemma. The stronger conclusions are included since they follow easily from the proof and might be interesting on their own.

**Proof** We proceed by induction on the number of vertices. For \(n \leq 2\) the assumptions are vacuous and for \(n = 3\) the statement is obvious. Now let \(n \geq 4\) and suppose that the lemma has been proved for drawings with at most \(n - 1\) vertices. Let \(v_1, v_2, \ldots, v_n\) be the vertices of \(D\). Let \(D_{n-1}\) be the drawing of the complete subgraph induced by \(v_1, v_2, \ldots, v_{n-1}\). Let \(C\) be the simple curve forming the boundary of the outer face of \(D_{n-1}\). By induction, all bounded faces of \(D_{n-1}\) are covered by a set \(\mathcal{T}_{n-1}\) of at most \(n - 3\) triangles so that no edge is contained in more than two triangles from \(\mathcal{T}_{n-1}\). We assume (and prove) an even stronger induction statement: if two triangles from \(\mathcal{T}_{n-1}\) share an edge \(e\), then they do not cover the same face incident with \(e\). That is, the two triangles are “attached” to \(e\) from the opposite sides of \(e\).

By adding \(v_n\) with its incident edges to \(D_{n-1}\), the outer face of \(D_{n-1}\) is partitioned into the outer face of \(D_n\) and several bounded faces. We show that all these new bounded faces can be covered by a single triangle. We distinguish two cases.
(a) The vertex $v_n$ is in the outer face of $D_{n-1}$. First we observe that no edge $v_i v_n$ has more than one crossing with $C$. See Fig. 19a. Suppose the contrary and let $x_1$ and $x_2$ be two crossings of $v_n v_i$ with $C$ closest to $v_n$. Then the portion of $v_n v_i$ between $x_1$ and $x_2$ separates the drawing $D_{n-1}$ into two parts, each of them containing at least one vertex. In particular, the part that does not contain $v_i$ contains some other vertex $v_j$. The edge $v_i v_j$ has to lie in the closed region bounded by $C$, thus it is forced to cross the edge $v_i v_n$; a contradiction.

It follows that for every edge $v_n v_i$, either the relative interior of $v_n v_i$ lies outside $C$ and $v_i$ lies on $C$, or $v_n v_i$ crosses $C$ in exactly one point, $x_i$, and the portion of $v_n v_i$ between $x_i$ and $v_i$ lies in the closed region bounded by $C$. In all cases, only the initial
portion of the edge \( v_n v_i \) lies in the outer face of \( D_{n-1} \). Consequently, only two edges incident with \( v_n \) are incident with the outer face of \( D_n \).

Let \( v_n v_k \) and \( v_n v_l \) be the two edges incident with \( v_n \) and with the outer face of \( D_n \). Since the relative interior of the edge \( v_k v_l \) lies inside \( C \), the triangle \( \triangle v_n v_k v_l \) covers all bounded faces of \( D_n \) lying outside \( C \). If no triangle from \( \mathcal{T}_{n-1} \) has the edge \( v_k v_l \), or if exactly one such triangle, \( \triangle v_m v_k v_l \), exists but has the opposite orientation from \( \triangle v_n v_k v_l \), we let \( \mathcal{T}_n = \mathcal{T}_{n-1} \cup \{ v_n v_k v_l \} \). If some triangle \( \triangle v_m v_k v_l \) from \( \mathcal{T}_{n-1} \) has the edge \( v_k v_l \) and has the same orientation as \( \triangle v_n v_k v_l \), then \( v_m \) cannot lie outside \( v_n v_k v_l \), as then the edge \( v_n v_m \) would be incident with the outer face. Hence \( v_m \) is inside \( v_n v_k v_l \). The orientation of the triangle \( \triangle v_m v_k v_l \) then implies that the whole triangle \( \triangle v_m v_k v_l \) is covered by \( v_n v_k v_l \), and so we let \( \mathcal{T}_n = (\mathcal{T}_{n-1} \setminus \{ v_m v_k v_l \}) \cup \{ v_n v_k v_l \} \).

(b) The vertex \( v_n \) is in the interior of some bounded face of \( D_{n-1} \). By a similar argument as in part (a), every edge \( v_n v_i \) has at most two crossings with \( C \). See Fig. 19 b. If no edge incident with \( v_n \) is incident with the outer face of \( D_n \), then \( C \) is the boundary of the outer face of \( D_n \) and thus we let \( \mathcal{T}_n = \mathcal{T}_{n-1} \). If two edges \( v_n v_j \) and \( v_n v_j \) cross \( C \), they separate the closed region bounded by \( C \) into two parts. The vertices \( v_i \) and \( v_j \) must be in the same part, otherwise the edge \( v_i v_j \) would cross \( v_n v_i \) or \( v_n v_j \), which is forbidden.

It follows that at most two edges incident with \( v_n \), \( v_n v_k \) and, possibly, \( v_n v_l \), are incident with the outer face of \( D_n \). All other edges \( v_n v_i \) that cross \( C \) do so in a “nested fashion” in the interval bounded by the crossings of \( v_n v_k \) with \( C \), or in the interval bounded by the crossings of \( v_n v_l \) with \( C \); see Fig. 19b. Hence, if \( v_n v_k \) and \( v_n v_l \) are incident with the outer face, then the triangle \( \triangle v_n v_k v_l \) covers all bounded faces of \( D_n \) that lie outside \( C \).

If there is no triangle \( \triangle v_m v_k v_l \) in \( \mathcal{T}_{n-1} \) with the same orientation as \( \triangle v_n v_k v_l \), we let \( \mathcal{T}_n = \mathcal{T}_{n-1} \cup \{ v_n v_k v_l \} \). If there is a triangle \( \triangle v_m v_k v_l \) in \( \mathcal{T}_{n-1} \) with the same orientation as \( \triangle v_n v_k v_l \), then \( v_m \) has to be inside \( \triangle v_n v_k v_l \). For if \( v_m \) was outside \( \triangle v_n v_k v_l \) in the region bounded by \( v_n v_k, v_n v_l \) and \( C \), then one of the edges \( \triangle v_m v_k \) or \( \triangle v_m v_l \) would be forced to cross an adjacent edge of \( C \). Similarly, if \( v_m \) was in the other region outside \( v_n v_k v_l \) and inside (or on) \( C \), then the edge \( \triangle v_m v_n \) would be forced to cross an adjacent edge or it would separate \( v_n v_k \) or \( v_n v_l \) from the outer face. Like in case (a), if \( v_m \) is inside \( v_n v_k v_l \), then the orientation of \( \triangle v_m v_k v_l \) implies that \( \triangle v_m v_k v_l \) is covered by \( v_n v_k v_l \). We let \( \mathcal{T}_n = (\mathcal{T}_{n-1} \setminus \{ v_m v_k v_l \}) \cup \{ v_n v_k v_l \} \).

We are left with the case when \( v_n v_k \) is the only edge incident with \( v_n \) and with the outer face. Let \( x_1 \) and \( x_2 \) be the crossings of \( v_n v_k \) with \( C \), so that \( x_1 \) is between \( v_n \) and \( x_2 \). Without loss of generality, assume that the portion of the edge \( v_n v_k \) starting at \( x_1 \) and ending at \( x_2 \) is oriented counter-clockwise on the boundary of the outer face. Let \( v_n v_l \) be the edge following \( v_n v_k \) clockwise in the rotation at \( v_n \).

If \( v_n v_l \) does not cross \( C \), then the triangle \( \triangle v_n v_k v_l \) covers all bounded faces of \( D_n \) outside \( C \). Similarly as in the previous case, we argue that if there is a triangle \( \triangle v_m v_k v_l \) in \( \mathcal{T}_{n-1} \) with the same orientation as \( \triangle v_n v_k v_l \), then \( v_m \) is inside \( \triangle v_n v_k v_l \) and so \( \triangle v_m v_k v_l \) is covered by \( v_n v_k v_l \), otherwise the edge \( v_n v_m \) would have to cross some adjacent edge. Here we use the fact that no edge leaves \( v_n \) outside the triangle \( \triangle v_n v_k v_l \). Again, we let \( \mathcal{T}_n = \mathcal{T}_{n-1} \setminus \{ v_m v_k v_l \} \cup \{ v_n v_k v_l \} \) or \( \mathcal{T}_n = \mathcal{T}_{n-1} \cup \{ v_n v_k v_l \} \), according to the existence of the triangle \( \triangle v_m v_k v_l \) covered by \( v_n v_k v_l \).

Finally, suppose that \( v_n v_l \) crosses \( C \). By induction, there is a triangle \( \triangle v_m v_i v_j \in \mathcal{T}_{n-1} \) containing \( v_n \) in its interior. Hence, each of the edges \( v_m v_k \) and \( v_n v_l \) crosses at least one edge of \( v_m v_i v_j \). If \( v_n v_k \) and \( v_n v_l \) cross the same edge, say, \( v_m v_i \), then the edge
Lemma 4.8 Let $D$ be a simple drawing of $K_n$. A sequence of vertices $v_1, v_2, \ldots, v_n$ is an $x$-monotone sequence of $D$ if and only if it is a shelling sequence of $D$ and the induced by the 4-tuple $v_i v_j v_k v_l$ is symmetric).

By Lemma 4.7, there is a triangle $v_j v_k v_l$ with $1 \leq j < k < l < i$ containing $v_i$ in its interior. In particular, $v_i$ is not incident with the outer face of the drawing of $K_4$ induced by the 4-tuple $v_j, v_k, v_l, v_i$.

4.2 Shellable Drawings and Monotone Drawings

Here we show that shellable drawings form a more general class than monotone drawings. We also show how monotone drawings may be characterized as a special case of shellable drawings.

Two drawings $D_1, D_2$ of a graph $G = (V, E)$ are weakly isomorphic if for every two edges $e, f \in E$, $e$ and $f$ cross in $D_1$ if and only if they cross in $D_2$. Let $D$ be a simple drawing of $K_n$ with vertex set $\{v_1, v_2, \ldots, v_n\}$. We say that a sequence of vertices $v_1, v_2, \ldots, v_n$ is an $x$-monotone sequence of $D$ if $v_1$ and $v_n$ are incident with the outer face of $D$ and $D$ is weakly isomorphic to a simple monotone drawing where $v_i = (i, 0)$ for every $i \in [n]$.

We have the following characterization of $x$-monotone sequences in terms of shelling sequences.

Lemma 4.8 Let $D$ be a simple drawing of $K_n$. A sequence of vertices $v_1, v_2, \ldots, v_n$ is an $x$-monotone sequence of $D$ if and only if it is a shelling sequence of $D$ and the path $v_1 v_2 \ldots v_n$ does not cross itself.

Proof The “only if” part is obvious. Let $v_1, v_2, \ldots, v_n$ be a shelling sequence such that the path $v_1 v_2 \ldots v_n$ does not cross itself. We claim that for every $v_i, v_j, v_k, v_l$ with $1 \leq i < j < k < l \leq n$, the path $v_i v_j v_k v_l$ does not cross itself. Let $H$ be the drawing of $K_4$ induced by the vertices $v_i, v_j, v_k, v_l$.

Suppose for contrary that the path $v_i v_j v_k v_l$ in $H$ crosses itself. That is, the edges $v_i v_j$ and $v_k v_l$ cross. Let $i, j, k, l$ be such a 4-tuple with the pair $(l - i, j - i)$ lexicographically smallest. The drawing $H$ is homeomorphic to one of the drawings in Fig. 20. Since $v_i$ is on the outer face of the complete subgraph $D_{l,i}$ with vertices $v_i, v_{i+1}, \ldots, v_l$, there is an unbounded curve $\gamma_l$ starting at $v_l$ going to infinity and avoiding all edges of $D_{l,i}$. Similarly, there is an unbounded curve $\gamma_k$ starting at $v_k$ going to infinity and avoiding all edges of the complete graph induced by $v_i, v_{i+1}, \ldots, v_k$. In particular, $\gamma_l$ and $\gamma_k$ do not cross the path $P_{i,j} = v_i v_{i+1} \ldots v_j$, the curve $\gamma_l$ lies completely in the outer face of $H$, and $\gamma_k$ lies completely outside the triangle $v_i v_j v_k$.

By the minimality of $l - i$, the edge $v_k v_l$ crosses no edge $v_{i+a} v_{i+a+1}$ with $1 \leq a \leq j - i - 1$. By the minimality of $j - i$, the edge $v_i v_{i+1}$ does not cross $v_k v_l$, unless
\[ j = i + 1. \] Since the double-infinite curve formed by \( \gamma_k, v_kv_l \) and \( \gamma_l \) separates \( v_i \) from \( v_j \), it must cross the path \( P_{i,j} \). This implies that \( j = i + 1 \).

Similarly, there are unbounded curves \( \gamma_i \) and \( \gamma_j \) starting at \( v_i \) and \( v_j \), respectively, that do not cross the path \( P_{k,l} = v_kv_{k+1} \ldots v_l \), the curve \( \gamma_i \) lies completely in the outer face of \( H \), and \( \gamma_j \) lies completely outside the triangle \( v_jv_kv_l \). Since \( j = i + 1 \) and by the assumption, the edge \( v_iv_j \) does not cross the path \( P_{k,l} \) either. The double-infinite curve formed by \( \gamma_i, v_iv_j \) and \( \gamma_j \) thus separates \( v_k \) from \( v_l \) but does not cross \( P_{k,l} \); this is a contradiction.

Since the path \( v_iv_jv_kv_l \) does not cross itself, the order type of \( H \) determines the drawing up to an isotopy. Indeed, the drawings in Fig. 17 represent, up to relabeling, the only two isotopy classes of simple shellable drawings of \( K_4 \) that have the same order type. There are two possible shelling sequences common for both drawings. For the shelling sequence 1, 2, 3, and 4, the corresponding path is noncrossing only in the right drawing. For the shelling sequence 1, 3, 2, 4, the corresponding path is noncrossing only in the left drawing.

Let \( \sigma \) be the order type of \( D \). By the proof of Proposition 4.1, there is a semisimple monotone drawing \( D' \) with signature function \( \sigma \) such that two edges cross oddly in \( D \) if and only if they cross oddly in \( D' \).

It remains to show that \( D' \) is simple. By Theorem 3.1, it is sufficient to show that there is no 5-tuple \( (a, b, c, d, e) \) with \( a < b < c < d < e \) such that \( \sigma(a, b, e) = \sigma(a, d, e) = \sigma(b, c, d) = \sigma(a, c, e) = \xi \), where \( \xi \in \{+,-\} \). Suppose for contrary that there is such a 5-tuple. By symmetry, we may assume that \( \xi = + \). The vertices \( v_a, v_b, v_c, v_d, v_e \) induce a shellable drawing \( K \) of \( K_5 \) in \( D \). We may deform the plane by an isotopy so that \( v_a = (0, 0), v_e = (1, 0) \), and so that all edges of \( K \) are drawn between the vertical lines going through \( v_a \) and \( v_e \). From \( \sigma(a, b, e) = \sigma(a, c, e) = - \) we have \( \sigma(a, b, c, e) = +++- \). Similarly, from \( \sigma(a, c, e) = - \) and \( \sigma(a, d, e) = + \) we have \( \sigma(a, c, d, e) = ----+ \). This further implies that \( \sigma(a, b, c, d) = +---- \). In particular, the edges \( v_av_c \) and \( v_bv_d \) cross. The signatures also imply that \( v_b \) and \( v_d \) are below the edge \( v_av_c \) and \( v_c \) is above the edge \( v_av_e \). For a simple drawing this means that the edge \( v_bv_d \) is below \( v_av_c \) and the relative interior of the edge \( v_av_c \) is above \( v_av_e \), therefore the edges \( v_av_c \) and \( v_bv_d \) cannot cross; a contradiction. \( \Box \)

It is easy to see that some drawings of \( K_n \) have more shelling sequences than \( x \)-monotone sequences. For example, for the convex geometric drawing of \( K_n \), all \( n! \) permutations of vertices are shelling sequences, whereas at most \( n \cdot 2^{n-2} \) permutations of vertices, inducing a noncrossing Hamiltonian path, are \( x \)-monotone sequences.
To show that shellable drawings are indeed more general than monotone drawings, we provide an example of a shellable drawing that has no $x$-monotone sequence.

**Theorem 4.9** The drawing in Fig. 21 is a simple shellable drawing of $K_9$ which is not weakly isomorphic to a simple monotone drawing.

**Proof** Clearly, the sequence $1, 4, f, e, x, a, b, 2, 5$ is a shelling sequence of the drawing $S_9$ in Fig. 21. Suppose that $\mu$ is an $x$-monotone sequence of $S_9$. We write $v \prec w$ for vertices $v, w$ if $v$ precedes $w$ in $\mu$. By symmetry, we may assume that $1 \prec 5$. The subgraphs induced by 4-tuples $\{1, 2, 4, 5\}$, $\{1, 2, a, b\}$ and $\{4, 5, e, f\}$ have unique $x$-monotone sequences, up to reversal. In particular, we have $1 \prec 2 \prec 4 \prec 5$, which in turn implies that $1 \prec a \prec b \prec 2 \prec 4 \prec f \prec e \prec 5$. To uncover the vertex $a$, it is not sufficient to remove the vertex $1$, we have to remove at least one more vertex. Since all vertices except for $x$ are preceded by $a$ in $\mu$, we have $x \prec a$. Similarly, to uncover the vertex $e$, it is not sufficient to remove the vertex $5$, and the only available vertex is $x$. Therefore, $e \prec x$. These conditions cannot be fulfilled, thus $S_9$ has no $x$-monotone sequence. $\square$

### 4.3 Crossing Number and $k$-Edges in Weakly Semisimple Drawings

Here we show a generalization of Lemma 2.2 to weakly semisimple drawings, which may be used to generalize Theorem 1.1 and the result of Ábrego *et al.* [3] to weakly semisimple $s$-shellable drawings with $s \geq n/2$. As in Proposition 4.2, the equality has to be replaced by an inequality. Since the orientation of triangles and hence the order type can be still defined in weakly semisimple drawings (see the definition before Proposition 4.2), the notions of $k$-edges, $\leq k$-edges, $\leq \leq k$-edges and separations generalize to weakly semisimple drawings as well.
Lemma 4.10 For every weakly semisimple drawing \( D \) of \( K_n \) we have

\[
\text{ocr}(D) \geq 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D) - \frac{1}{2} \left( \frac{n}{2} \right) \left[ \frac{n-2}{2} \right] - \frac{1}{2} (1 + (-1)^n) E_{\leq \lfloor n/2 \rfloor - 2}(D).
\]

Proof The lemma follows in the same way as Lemma 2.2 or Lemma 2.1, after proving that every weakly semisimple drawing \( D \) of \( K_4 \) satisfies the inequality \( \text{ocr}(D) + E_1(D) \geq 3 \). The equality is not always attained as there are weakly semisimple drawings of \( K_4 \) with odd crossing number 3 and with six separations; see Fig. 18, right.

Let \( D \) be a weakly semisimple drawing of \( K_4 \). The separation graph of \( D \) is the subgraph of \( D \) formed by the 1-edges in \( D \). The separation graph depends only on the order type of \( D \). Every order type can be obtained from each other by changing the orientation of some triangles. By changing the orientation of a triangle \( uvw \), the edges \( uv, uw, vw \) change from 0-edges to 1-edges and vice versa. It follows that the degree of each vertex in the separation graph either remains the same or changes by 2. Since in the planar drawing of \( K_4 \) the separation graph is isomorphic to \( K_{1,3} \), it follows that the separation graph of \( D \) has all vertices of odd degree. That is, it is isomorphic to \( K_2 + K_2, K_{1,3}, \) or \( K_4 \). In particular, \( E_1(D) \geq 2 \).

Every edge flip (see the definition in the proof of Lemma 2.2) in a drawing of \( K_4 \) changes the orientation of two adjacent triangles. The separation graph is thus transformed by taking the symmetric difference with a cycle \( C_4 \). Clearly, if the separation graph is isomorphic to \( K_{1,3} \), then its symmetric difference with arbitrarily positioned \( C_4 \) is isomorphic to \( K_{1,3} \) as well. We may transform \( D \) by a sequence of edge flips into a drawing \( D'' \) which has at least one vertex \( v \) on the outer face. Let \( w_1, w_2, w_3 \) be the other three vertices of \( D'' \), so that the initial portions of the edges \( vw_1 \) and \( vw_3 \) are incident with the outer face of \( D'' \) and the rotation at \( v \) is \( w_1, w_2, w_3 \). See Fig. 22, left.

We extend the drawing \( D'' \) by adding one auxiliary vertex \( x \) close to \( w_1 \) and edges \( vx \) and \( xw_1 \), so that \( x \) follows immediately after \( v \) in the rotation at \( w_1 \), the rotation...
at \( v \) is \( x, w_1, w_2, w_3 \), the triangle \( vxw_1 \) is oriented clockwise and the path \( vxw_1 \) is drawn close to the edge \( vv_1 \). We denote this new drawing as \( K \).

Since every two edges cross evenly in \( D'' \), the same is true for the drawing \( K \) and thus we may apply the weak Hanani–Tutte theorem to \( K \). We obtain a planar drawing \( K' \) with the same rotation system as \( K \). We may assume that \( vw_1w_3 \) forms a boundary of the outer face of \( K' \). See Fig. 22, right. Let \( D' \) be the subgraph of \( K' \) obtained after removing \( x \) and its adjacent edges. The orientations of all three triangles incident to \( v \) are the same in \( D'' \) and in \( D' \), since \( v \) is on the outer face in both drawings and the rotation at \( v \) is the same in \( K \) and in \( K' \).

It remains to compare the orientation of the triangle \( w_1w_2w_3 \) in \( D'' \) and \( D' \). Let \( \gamma (\gamma') \) be the closed curve formed by the edges of the triangle \( w_1w_2w_3 \) in \( K (K', \) respectively). Since the curve \( vx \) crosses every edge of \( D'' \) an even number of times, the winding number of \( \gamma \) around \( x \) has the same parity as the winding number of \( \gamma \) around \( v \). Since \( v \) is in the outer face of \( D'' \), both winding numbers are even. Since \( x \) is outside \( \gamma' \) in \( K' \), the winding number of \( \gamma' \) around \( x \) is even as well. Together with the fact that in both drawings \( K \) and \( K' \), the rotation at \( w_1 \) is the same, this implies that the triangle \( w_1w_2w_3 \) is oriented counter-clockwise in both drawings. Therefore, \( D'' \) and \( D' \) have the same order type.

Combining Lemma 4.10 with the proof by Ábrego et al. [3], we obtain the following generalization.

**Corollary 4.11** Let \( s \geq n/2 \) and let \( D \) be a weakly semisimple \( s \)-shellable drawing of \( K_n \). Then \( ocr(D) \geq Z(n) \).

### 5 Concluding Remarks

It would be interesting to see if techniques similar to those used in the proof of Theorem 1.1 can be used to prove Hill’s conjecture for general drawings of complete graphs. We note that the same approach does not generalize to all drawings. For example, a particular planar realization of the so-called cylindrical drawing [23,25] of \( K_{10} \), with crossing number \( Z(10) \), does not satisfy the lower bound on \( \leq 1 \)-edges from Theorem 2.6. See Fig. 23, right. Fig. 23, left, shows an even smaller example, but this drawing of \( K_6 \) is not crossing optimal. Analogous cylindrical drawings of \( K_{4k+6} \), for \( k \geq 2 \), violate the lower bound on \( \leq k \)-edges from Theorem 2.6.

Extrapolating the definitions of \( \leq k \)-edges and \( \leq k \)-edges, we define the number of \( \leq \leq k \)-edges, \( E_{\leq \leq k} (D) \), by the following identity.

\[
E_{\leq \leq k} (D) = \sum_{j=0}^{k} E_{\leq j} (D) = \sum_{i=0}^{k} \binom{k + 2 - i}{2} E_i (D).
\]

In our context, using \( \leq \leq k \)-edges seems to be even more natural than using \( \leq k \)-edges, since the formula from Lemma 2.1 can be rewritten in the following compact form:
Fig. 23 A general simple drawing of $K_6$ (left) and a cylindrical drawing of $K_{10}$ (right) where $E_0 = 5$ and $E_1 = 0$, hence $E_{\leq 1} = 10 < 12 \leq 3(1+3)$.

$$
\text{cr}(D) = 2E_{\leq \lfloor n/2 \rfloor - 2}(D) - \frac{1}{8}n(n-1)(n-3) \quad \text{for } n \text{ odd, and} \\
\text{cr}(D) = E_{\leq \lfloor n/2 \rfloor - 3}(D) + E_{\leq \lfloor n/2 \rfloor - 2}(D) - \frac{1}{8}n(n-1)(n-2) \quad \text{for } n \text{ even.}
$$

We conjecture that the following lower bound on $\leq \leq k$-edges is satisfied by all simple drawings of complete graphs.

**Conjecture 1** Let $n \geq 3$ and let $D$ be a simple drawing of $K_n$. Then for every $k$ satisfying $0 \leq k < n/2 - 1$, we have

$$
E_{\leq \leq k}(D) \geq 3\binom{k+4}{4}.
$$

Conjecture 1 is stronger than Hill’s conjecture. Theorem 2.6 implies Conjecture 1 for all simple $x$-monotone drawings. All our examples of simple drawings of complete graphs, including the cylindrical drawings, also satisfy Conjecture 1. We note that Conjecture 1 is trivially satisfied for $k = 0$, since every simple drawing of a complete graph with at least three vertices has at least three 0-edges—those incident with the outer face.

We have no counterexample even to the following conjecture, which further generalizes Conjecture 1 to arbitrary graphs.

**Conjecture 2** Let $k \geq 0$ and let $D$ be a simple drawing of a graph with at least $\binom{2k+3}{2}$ edges. Then

$$
E_{\leq \leq k}(D) \geq 3\binom{k+4}{4}.
$$
Note that in a drawing of a general graph with \( n \) vertices, a \( k \)-edge contained in \( t \) triangles is also a \((t - k)\)-edge, but not necessarily an \((n - 2 - k)\)-edge. Thus, for example, in every drawing of a triangle-free graph, every edge is a 0-edge. This suggests that it might be easier to prove Conjecture 2 for non-complete graphs. Also, Conjecture 2 or some still stronger variant might be susceptible to a proof by induction on the number of edges.

Further, it would be interesting to generalize Theorem 1.1 to arbitrary monotone drawings, where adjacent edges are also allowed to cross oddly. For such drawings, two notions of the crossing number are of interest. The monotone odd crossing number, \( \text{mon-ocr}(G) \), counting the minimum number of pairs of edges crossing an odd number of times, and the monotone independent odd crossing number, \( \text{mon-iocr}(G) \), or, \( \text{mon-ocr}_- (K_n) \), counting the number of pairs of nonadjacent edges crossing an odd number of times. By definition, for every graph \( G \) we have \( \text{mon-ocr}_- (G) \leq \text{mon-ocr}_+ (G) \leq \text{mon-ocr}_- (G) \).

5.1 Order Types and \( \lambda \)-Matrices

By Lemma 2.2, the crossing number of a semisimple drawing of \( K_n \) is determined by the number of \( k \)-edges for all \( k \). For a set of points \( p_1, p_2, \ldots, p_n \) in the plane, Goodman and Pollack [21] introduced the \( \lambda \)-matrix \( (\lambda(i, j)) \), where for every \( i \neq j \), \( \lambda(i, j) \) is the number of points to the left of the directed line \( p_i p_j \), and \( \lambda(i, i) = 0 \). They showed that the \( \lambda \)-matrix determines the order type of the point set. Aichholzer et al. [9] used \( \lambda \)-matrices to represent point sets for computing lower bounds on the rectilinear crossing number of complete graphs.

The \( \lambda \)-matrix may be defined for semisimple drawings of \( K_n \) with vertices \( v_1, v_2, \ldots, v_n \) in a similar way: for every \( i \neq j \), \( \lambda(i, j) \) is the number of triangles \( v_i v_j v_l \) oriented counter-clockwise. Clearly, \( v_i v_j \) is a \( k \)-edge if and only if \( \lambda(i, j) \in \{ k, n - 2 - k \} \). The order type of a drawing determines its \( \lambda \)-matrix, but not the drawing itself (see Fig. 17 or Fig. 15). Therefore, the \( \lambda \)-matrix does not determine the drawing either. However, a generalization of Goodman’s and Pollack’s result to semisimple drawings is true.

**Observation 5.1** The \( \lambda \)-matrix of a semisimple drawing of \( K_n \) determines its order type.

This is easily seen by induction over all subgraphs of \( K_n \): in every semisimple drawing of a graph with at least one edge, all edges incident with the outer face are 0-edges. In particular, there is an edge \( v_i v_j \) such that \( \lambda(i, j) = 0 \). Every 0-edge determines the orientation of all incident triangles. Therefore, we may remove such an edge, update the \( \lambda \)-matrix and use induction for the smaller graph.

The same observation is no longer true for weakly semisimple drawings: in the drawing in Fig. 18, right, every edge is a 1-edge. Therefore, its \( \lambda \)-matrix is identical with the \( \lambda \)-matrix of a mirror-symmetric drawing, but these two drawings have mutually inverse order types.
Since the crossing number of a semisimple drawing of a complete graph is determined by its \( \lambda \)-matrix, it might be interesting to investigate the properties of \( \lambda \)-matrices that can be realized by semisimple drawings of complete graphs.

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