Kinetic transport is needed to reliably extract shear viscosity from pA and AA data.

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The azimuthal anisotropies \( v_n \) of particle spectra measured in proton-nucleus (pA) and nucleus-nucleus (AA) collisions play a key role in constraining QCD matter properties like the shear viscosity over entropy density ratio \( \eta/s \). Here, we provide a calculation of \( v_2 \) in kinetic transport theory and compare it to calculations in viscous fluid dynamics with the same initial geometry and the same matter properties. Both descriptions agree if the transverse system size is large enough and the viscous fluid dynamics is initialized from the full kinetic transport calculation at a time larger than the microscopic isotropization timescale. We find that kinetic transport in the pre-hydrodynamic stage of viscous fluid dynamic calculations cannot be replaced by free-streaming without introducing a fine-tuning of the switching time and corresponding systematic uncertainties. We argue that the need of realistic kinetic transport for a reliable calculation of azimuthal anisotropies in pA and AA collisions may provide a tool for phenomenologically constraining the isotropization time which is a characteristic and as-yet-unknown length scale in the matter produced in pA and AA collisions.

Ultra-relativistic nucleus-nucleus (AA) and proton-nucleus (pA) collisions display remarkably large signatures of collectivity, in particular in the hadronic transverse momentum spectra and their azimuthal asymmetries [1–6]. To infer the properties of the ultra-dense and strongly expanding QCD matter in the collision region from these data, a dynamical modelling of collectivity is indispensable. From comparing fluid dynamic models to data, one generally infers matter properties consistent with a perfect fluid that exhibits minimal dissipation (minimal shear viscosity over entropy ratio, \( \eta/s \)) [7–8]. In marked contrast, transport models can account for the same signals of collectivity with material properties that allow for significant mean free path, thus exhibiting non-minimal dissipation [9–16].

Many microscopic models with boost-invariant longitudinal dynamics satisfy hydrodynamic constitutive equations in situations significantly out of equilibrium, an observation dubbed “hydrodynamization without thermalization” [17–23]. Ultra-relativistic pA and AA collisions realize such out-of-equilibrium scenarios since they are initiated with a highly anisotropic momentum distribution. Here, we ask as a function of system size which dynamical description of collectivity is required to extract for such longitudinally boost-invariant systems a quantitatively reliable estimate of \( \eta/s \) from the azimuthal asymmetries \( v_n \) measured in pA and AA collisions. Is a full transport calculation needed or is the study of the fluid dynamic limit sufficient to extract a reliable value of \( \eta/s \) from \( v_n \)?

We address this question in a two-pronged approach. First, we calculate azimuthal asymmetries from a simple but generic transport model in which the effects of collisions are accounted for by a single finite isotropization time. Second, following the current phenomenological practice, we free-stream the same initial conditions up to a time \( \tau_s \) at which we switch to viscous fluid dynamics, or we replace the pre-hydrodynamic stage up to \( \tau_s \) by full kinetic transport. The comparison of these approaches will reveal marked differences whose origin we clarify.

Our study focusses on azimuthal asymmetries \( v_n \) of the transverse energy \( dE_\perp \) that are trivially obtained from those of measured particle spectra \( dN \), but that do not depend on a hadronization prescription,

\[
\frac{dE_\perp}{dn_\perp d\phi} = \int dp_\perp^2 p_\perp dN = dE_\perp 2\pi dn_\perp \left( 1 + 2 \sum_{n=1}^{\infty} v_n \cos(n \phi) \right).
\]

To determine \( dE_\perp \), we evolve the energy-momentum tensor \( T^{\mu\nu} \) of the system to late times, either with transport...
theory, or with viscous fluid dynamics.

**Kinetic transport:** The energy momentum tensor $T^\mu_\nu = \int d^4x \int d\phi \, \epsilon^{\mu\nu} F$ can be written in terms of the first momentum moment $F(\vec{x}, \Omega, \tau) = \int d^4p e^{i\vec{p}\vec{x}} p f$ of the distribution function $f$, where we use normalized momenta $\nu_i \equiv p_i/p$ with $p^\mu \nu^\mu \equiv 0$ and $\nu^0 = 1$. For massless boost-invariant kinetic transport in the slice of central spatial rapidity $\eta/s$, the evolution equation for $F$ reads [16]

$$\partial_\tau F + \vec{v}_\perp \cdot \partial_{\vec{x}_\perp} F - \frac{v_\perp}{\tau} (1-v_\perp^2) \partial_{\vec{v}_\perp} F + \frac{4v_\perp^2}{\tau} F = -C[F].$$

(2)

For the collision kernel $C[F]$, we use the isotropization-time approximation (ITA)

$$-C[F] = -\gamma \epsilon^{1/4}(x) [-v_\mu u^\mu (F-F_{\text{iso}})],$$

(3)

where $\epsilon$ is the local energy density and $F_{\text{iso}}(\vec{x}, \Omega, \tau) = \frac{\epsilon(\vec{x}, \tau)}{-\nu_\mu u^\mu}$. The ITA is closely related to the relaxation time approximation. We emphasize, however, that for observables constructed from $T^\mu_\nu$, it is not necessary to specify the momentum-dependence of $C[F]$. Eq. [3] is solely based on the mild assumption that any system evolves towards an isotropic distribution and that this can be characterized for $p$-integrated quantities by a single isotropization time $t_{\text{mfp}} \sim (\gamma \epsilon^{1/4})^{-1}$, set by the only model parameter $\gamma$. The ITA has been used extensively in the hydrodynamical limit and its transport coefficients are known: $\tau_\eta = (\gamma \epsilon^{1/4})^{-1}$ and kinetic shear viscosity $\eta_\tau = \frac{\eta}{\gamma \epsilon^{1/4}}$. While our calculation does not require it, we may further specify the relation between energy density and temperature $\epsilon = a T^4$. This allows us to extract a value for specific viscosity $\eta/s$ for a given $\gamma$ and thereby make a connection to fluid dynamic calculations that conventionally use $\eta/s$ as an input parameter. Motivated by lattice results, we choose in the following $a \approx 13$. This input fixes the relation between specific shear viscosity and our model parameter: $\eta/s \approx 0.11/\gamma$. ITA is found to reproduce the $T^\mu_\nu$ evolution of the QCD weak coupling effective kinetic theory [23] within $\sim 15\%$ [24]. However, the following does not assume that the collision kernel is dominated by perturbative physics.

In pA and AA collisions, azimuthal asymmetries $v_n$ in the final momentum distributions arise from azimuthal eccentricities $\delta_n$ in the initial spatial distribution. To choose a longitudinally boost-invariant initial condition that shares pertinent phenomenologically relevant features, we assume maximal initial anisotropy in momentum space $(x \delta(v_\perp))$ and an azimuthally isotropic Gaussian density profile supplemented by eccentricities $\delta_n$. Focusing for simplicity on the second harmonic, we write

$$F(\vec{x}, \phi, \tau_0) = 2\epsilon_0 \delta(v_\perp) \exp \left[ -\frac{r^2}{R^2} \right] \left( 1 + \delta_2 \frac{r^2}{R^2} \cos 2\phi \right).$$

(4)

The normalization of [4] corresponds to an initial central energy density $\epsilon(\tau_0, r = 0) = \epsilon_0$. We take $\tau_0 \to 0$ keeping $\epsilon_0 \tau_0$ fixed. Then, evolving this initial condition [4] with the kinetic theory [2], dimensionless observables can depend only on $\hat{\gamma} = R^{1/4}(\epsilon \tau_0)^{1/4}$, which measures the transverse system size $R$ in units of mean free path at the time $\tau = R$ at which collectivity is built up, $\hat{\gamma} \approx R \gamma (\epsilon \tau = R, \tau = 0) = R / t_{\text{mfp}}(\tau = R)$. From previous studies of this kinetic theory to first order in $\hat{\gamma}$, i.e., for small system sizes, we know already that all linear and non-linear structures observed in the azimuthal anisotropies $v_2$ arise, and that $v_2/\delta_2 = -0.2125$ [16].

The first novel result reported here is that we have extended the analysis of the kinetic theory [2] from first order in $\hat{\gamma}$ to all orders in $\hat{\gamma}$ and thus from peripheral AA collisions to collisions of any centrality. We do so by discretizing the transport equation [2] in comoving coordinates that leave the distribution $F$ of free streaming particles unchanged as function of time and evolve it in time numerically

$$\partial_\tau \hat{F}_n(\vec{x}, \phi, \tau) = -\frac{e^{-i(n\phi - \theta)}}{\Delta^4} C_n [\Delta^4 e^{i(n\phi - \theta)} \hat{F}_n],$$

(5)

with

$$\vec{x} = \vec{x} - \frac{\vec{v}_\perp}{\sqrt{1 - c^2}} (\tau_0 - \tau \Delta), \quad v_\perp = \frac{\tau_0}{\tau \Delta} v_\perp.$$  

(6)

Here, $\Delta = 1 - c^2 + (\tau_0 / \tau)^2 c^2$ and $F_n = e^{i(n\phi - \theta)} \Delta^4 \hat{F}_n$ and $C_n$ correspond to the $n$th harmonic of the distribution function and the appropriately linearized collision kernel for $F_n$. Fig. 1 shows how the full kinetic theory result for $v_2/\delta_2$ approaches the first order result for small $\hat{\gamma}$, and how it levels off in the multiple scattering regime (large $\hat{\gamma}$) to the value obtained from perfect fluid dynamics. Fig. 1 also provides explicit proof that one-hit dynamics is a reasonable approximation for $\hat{\gamma} \lesssim 1$, thus strengthening an earlier conclusion [16] based on parametric estimates. Before discussing the results of this transport theory in more detail, we evolve now the same initial condition [4] with

**Viscous fluid dynamics:** We parallel the set-up of massless transport theory by considering a conformally symmetric system with $\epsilon = 3p$. The tensor decomposition $T^{\mu\nu} = \epsilon (u^{\mu} u^{\nu} + \frac{1}{3} \delta^{\mu\nu}) + \Pi^{\mu\nu}$ defines the local rest frame $u^{\mu}$, energy density $\epsilon$ and shear viscous tensor $\Pi^{\mu\nu}$. To set the initial values of these fluid dynamic fields at the switching time $\tau_s$, we match this tensor decomposition at $\tau_s$ to the energy-momentum tensor calculated from the distribution [4] free-streamed up to $\tau_s$. Alternatively, we consider for the pre-hydrodynamic evolution up to $\tau_s$ instead of free-streaming the full kinetic theory, with $\gamma$ setting the kinetic viscosity in fluid dynamics by the kinetic theory relation $\gamma = 5 \frac{\Pi}{\Pi^{\mu\nu}} = (\epsilon \tau_0)^{-1}$. From time $\tau_s$ onwards, these fluid dynamic fields are then evolved with the first order Israel-Stewart viscous fluid dynamic
In Fig. 2, we compare at the 1st order hydrodynamical constitutive equation (green) for different values of transverse system size $\hat{\tau}/R$.

$$D\varepsilon + (\epsilon + p)\nabla u^{\mu} + \Pi_{\mu\nu}\Delta^{\alpha\beta}\nabla_{\alpha}u^{\nu} = 0,$$  \hspace{1cm} (7)

$$\epsilon + p)D\varepsilon^{\alpha} + \Delta^{\alpha\beta}\nabla_{\beta}p + \Delta^{\alpha}_{\nu}\nabla_{\mu}\Pi^{\mu\nu} = 0,$$  \hspace{1cm} (8)

$$\tau_{\pi}(D\Pi^{\mu\nu} + \frac{3}{5}\Pi^{\alpha\nu}\nabla_{\alpha}u^{\mu}) = - (\Pi^{\mu\nu} + 2\eta\sigma^{\mu\nu}).$$  \hspace{1cm} (9)

Here, $\Delta^{\mu\nu} = u^{\rho}u^{\nu} + g^{\mu\nu}$ is the projector on the subspace orthogonal to the flow field, $\nabla_{\mu}$ is the covariant derivative and $D = u^{\mu}\nabla_{\mu}$ is the comoving time derivative. Eqs. (7) and (8) result from energy and momentum conservation $\nabla_{\mu}T^{\mu\nu} = 0$, respectively. Eq. (9) ensures for a conformal system that within the shear relaxation time $\tau_{\pi}$, the shear viscous tensor relaxes to its Navier-Stokes value $-2\eta\sigma^{\mu\nu}$, where $\eta$ is the shear viscosity and $\sigma^{\mu\nu} = \frac{3}{5}(\Delta^{\mu\nu}\nabla_{\alpha}u^{\nu} + \Delta^{\alpha\beta}\nabla_{\alpha}u^{\mu}) - \frac{3}{5}\Delta^{\mu\nu}\nabla_{\alpha}u^{\alpha}$.

In practice, we follow [26, 27] and linearize eqs. (7)-(9) with respect to small eccentricity perturbations on top of an azimuthally symmetric background, $\varepsilon = \varepsilon_{BG} + \delta\varepsilon$, $u^{\mu} = u^{\mu}_{BG} + \delta u^{\mu}$, $\Pi^{\mu\nu} = \Pi^{\mu\nu}_{BG} + \delta\Pi^{\mu\nu}$. After harmonic decomposition, this leads to a coupled set of evolution equations for 10 $\tau$- and $r$-dependent fluid field components, namely four background field components and six components of second harmonic perturbations. The initial conditions at switching time $\tau_{s}$ are then evolved with a routine adapted from [26]. We calculate from the evolved fluid dynamic fields the zeroth and second order harmonics of the component $T^{\mu\nu}(\tau, r)$ of the energy momentum tensor, and we determine $\tau_{2}$ from the ratio of the $r$-integrals of these components. The values for $\tau_{2}$ shown here are the $\tau \to \infty$ limit of this procedure. Because of conformal symmetry, the elliptic momentum asymmetry extracted from viscous fluid dynamics can be shown to depend only on two parameters,

$$\tau_{2} = \tau_{2}(\hat{\gamma}, \tau_{s}/R).$$  \hspace{1cm} (10)

**Results and conclusions:** In Fig. 2 we compare at the center $r = 0$ of the collision, where transverse velocity is absent, the transport theory to the first order viscous constitutive equation $P_{L}^{hydro} = \frac{\varepsilon}{3} [1 - \frac{16}{3} \eta \frac{\sigma}{\varepsilon}] \frac{1}{1 - \partial_{\tau}u_{\tau}}]$. With increasing system size and evolution time, fluid dynamic expectations are seen to coincide better and better with transport results. We observe that the kinetic theory (2)-(4) shows hydrodynamization—that is, approximate overlap of green and red curves in Fig. 2—for $\tau \gtrsim R/\hat{\gamma}$. Consistent with many recent studies [21–23], this takes place prior to thermalization, $P_{L} \sim \epsilon/3$.

As seen in Fig. 1, the value $\tau_{2}/\delta_{2} = -0.51$ which results from a calculation within ideal fluid dynamics with initial conditions of (4) with $\tau_{s} \to 0$ is still substantially larger than the kinetic theory value at $\hat{\gamma} = 6$, and in order to study how the ideal fluid dynamics is reached we turn to viscous fluid dynamics. In this viscous case, we still can take the $\tau_{s} \to 0$ limit, but the viscous fluid dynamic corrections diverge at early times when the dynamics is close to free-streaming. The sign change of $P_{L}^{hydro}$ in Fig. 2 marks the point at which the first viscous correction to $P_{L}$ becomes of order $O(1)$ indicating catastrophic breakdown of fluid dynamics. It is therefore no surprise that $\tau_{2}$ differs from kinetic theory in this limit (green dash-dotted line in Fig. 1). The fluid dynamic description must be supplemented with some pre-hydrodynamic evolution between $0 < \tau < \tau_{s}$. We next ask as a function of system size $R$, switching time $\tau_{s}$, and different choices of pre-hydrodynamic evolution up to $\tau_{s}$, how well viscous fluid dynamics can approximate the late time behaviour of kinetic theory for the calculation of $\tau_{2}/\delta_{2}$.

We first ask whether a full kinetic theory description of the pre-hydrodynamic evolution can be avoided completely by initializing viscous fluid dynamics at switching time $\tau_{s}$ from free-streamed distributions. This simplified procedure is in current phenomenological use [28]. It may be justified qualitatively on the grounds that both free-streaming and kinetic transport smoothen gradients in initial distributions and that any difference between free-streaming and transport will emerge only gradually at times $\tau \sim l_{mfp} \sim R/\hat{\gamma}$. Similarly, as one observes from Fig. 2, fluid dynamics starts to give a good description of the $T^{\mu\nu}$-evolution at the same timescale. Therefore, one may attempt to fine-tune the switching time to $\tau_{s} \sim l_{mfp}$ to minimize the systematic error made both in the free streaming and in the fluid dynamic stages of the evolution. To obtain a quantitative assessment about how well this line of reasoning works, we have initialized fluid dynamics from free-streamed distributions at time $\tau_{s}$, see
erratum of first arXiv-version (v1): In v1 of this work, we had argued that in viscous fluid dynamics, the \( \eta/s \)-dependence of \( v_2/\delta_2 \) is a function of \( \frac{\eta}{\pi l_{\mathrm{mfp}}^2} \) with \( T_0 \) the temperature at initialization time \( \tau_0 \). The evaluation of eqs. (7)-(9) with which this claim was supported did not include terms arising from expanding kinetic viscosity \( \eta_{\mathrm{kin}} = \frac{\eta}{2T} \) and relaxation time \( \tau_s = 5\tau_{\mathrm{kin}} \) to first order in fluid perturbations. After correction of this error, we find a significant \( R \)-dependence of \( v_2/\delta_2 \) in solving eqs. (7)-(9). In this revised manuscript, all kinetic theory results are unchanged, all viscous fluid dynamic results are corrected, and the corresponding conclusions about viscous fluid dynamics are changed substantially.
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