Renormalization group approach to soft gluon resummation

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Abstract

We present a simple proof of the all-order exponentiation of soft logarithmic corrections to hard processes in perturbative QCD. Our argument is based on proving that all large logs in the soft limit can be expressed in terms of a single dimensionful variable, and then using the renormalization group to resum them. Beyond the next-to-leading log level, our result is somewhat less predictive than previous all-order resummation formulae, but it does not rely on non-standard factorization, and it is thus possibly more general. We use our result to settle issues of convergence of the resummed series, we discuss scheme dependence at the resummed level, and we provide explicit resummed expressions in various factorization schemes.
1 The structure of soft resummation

The resummation of large logarithms related to soft radiation near the boundary of phase space is a classic subject, which stems from very early results in quantum electrodynamics [1], and is of great phenomenological and theoretical interest for QCD at colliders [2]. The exponentiation of soft logs related to gluon emission has been proven in QCD to leading order [3], and to next-to-leading order (NLO) through an explicit analysis of the relevant Feynman graphs, either using eikonal [4] or factorization [5] techniques, which lead to equivalent [6] resummation formulae. These resummation formulae were subsequently used [7] to obtain resummed results for physical cross sections.

More recently [8], it was shown that a large class of resummation formulae can be derived in a unified way and to all orders by assuming the validity of a suitable factorization [9], and then applying the renormalization group to it. The price to pay for this generality is that one has to rely on a rather nontrivial two-scale factorization, which goes beyond the standard factorization of collinear large logs. Also, the resummed result obtained in this way looks quite formal: in particular, its equivalence with resummation formulae of the form of Refs. [4, 5] is only established [8] to next-to-leading order.

In this paper, we present a simple argument for the all-order resummation of soft logs. Our argument is based on the following steps. First, we establish to all orders the relation between the resummation of large logs of the kinematic variable $1-x$, and large logs of its Mellin conjugate variable $N$, up to power-suppressed terms in the soft limit $x \to 1$, i.e. $N \to \infty$. This relation generalizes to all orders the relation between the next-to-leading resummations of Refs. [4] and [5], which can be respectively viewed as a resummation of $\ln(1-x)$ contributions to the splitting function, and a resummation of $\ln \frac{1}{N}$ contributions to the anomalous dimension.

Next, we prove, that the large logarithms of $(1-x)$ all stem from logs of a dimensionful variable $Q^2(1-x)^a$, where $a$ is an integer which characterizes the physical process. This origin of this result is essentially kinematic. It is related to the structure of the phase space for tree-level $n$-particle emission, which turns out not to be modified by loop corrections. Finally, we resum the logs of this latter dimensionful variable using standard renormalization-group techniques. This leads to a resummed result which generalizes to all orders the next-to-leading log resummation of Refs. [4, 5] in a way which is somewhat less predictive than the all-order result of Ref. [8]. Indeed, we also conclude, like Ref. [8], that logs of $N$ exponentiate to all orders. However, using the resummation formula of Ref. [8] the $N^{k-1}$LO resummed result is determined by a fixed-order $N^k$LO calculation, whereas using our resummation the $N^{k-1}$LO resummed result is determined by a fixed-order $N^{\frac{k(k+1)}{2}}$-1LO computation. On the other hand, our result does not require two-scale factorization, and it is thus easier to derive, and possibly more general.

We then exploit our relation between the resummation of logs of $N$ and $1-x$ to study the relation between various resummed results. First, given that the resummed result can be equivalently viewed as a $\ln(1-x)$ resummation of the splitting function or a $\ln \frac{1}{N}$ resummation of the anomalous dimension, we discuss the possibility of further casting it as a $\ln(1-x)$ resummation of the physical cross section. We show that, whereas this is formally possible, the ensuing result is meaningless since it is the sum of a factorially divergent series. We trace the origin of this divergence, and we show that it is related to subleading logs, and thus in particular unrelated
to power corrections. We then turn to the derivation of explicit expressions for the resummed cross section in a variety of factorization schemes, and discuss the wider freedom in choice of factorization which is available at the resummed level.

2 Next-to-leading log resummation

In this section, we review the well-known resummation formulae of Refs. [4, 5], and recast them in a form which is best suited to our subsequent derivation and generalization.

We consider the total cross section for a partonic hard process in the vicinity of the kinematic boundary for the production of the final state. We assume that the kinematics of the process is specified by a hard scale $Q^2$, and a dimensionless variable $x$, defined in such a way that $0 \leq x \leq 1$, and $x = 1$ is the kinematic boundary, i.e. the limit in which there is no phase space for the emission of extra particles. Kinematically, the processes we consider fall into two broad categories, according to whether the large scale which makes the process perturbative is the virtuality of one of the incoming particles, as in deep-inelastic scattering (DIS), or the invariant mass of the final state, as in Drell-Yan (DY). We will not consider processes, such as jet or prompt-photon production, in which the hard scale is provided by a transverse energy. As is well known, the radiation of extra soft partons close to the kinematic boundary enhances the total cross section by contributions proportional to powers of $\ln(1-x)$. These logs are generated by the integration over the phase space of the (on-shell) emitted partons, and therefore, since this phase space has an azimuthal symmetry, we expect an extra emission in the final state to be accompanied by at most two powers of the large log.

Consider specifically the Mellin moments of the cross section

$$\sigma(N, Q^2) \equiv \int_0^1 dx x^{N-1} \sigma(x, Q^2)$$

and assume that for DIS-like processes they can be written in factorized form as

$$\sigma(N, Q^2) = C \left( N, Q^2/\mu^2, \alpha_s(\mu^2) \right) F(N, \mu^2)$$

$$= C \left( N, 1, \alpha_s(Q^2) \right) F(N, Q^2),$$

where $F(N, \mu^2)$ are Mellin moments of a parton density $F(x, \mu^2)$, $C(N, Q^2/\mu^2, \alpha_s(\mu^2))$ is a hard perturbative coefficient, and the last equation gives the standard renormalization-group improved form of the factorized cross section. In the case of Drell-Yan-like processes, there are two partons in the initial state, and thus Eq. (2.2) should be modified accordingly. In general, $F$ in Eq. (2.2) is an array of parton distributions (quarks and gluons), but the ensuing complication of the formalism is trivial. Furthermore, to next-to-leading order, as we will see explicitly in the sequel, there is no quark-gluon mixing, so $F$ can be effectively taken to be a single parton distribution. The validity of a factorization of the form Eq. (2.2) is the only assumption we will make on the physical processes to which our resummation applies, and it amounts to the assumption that the hard coefficient $C(N, Q^2/\mu^2, \alpha_s(\mu^2))$ can be multiplicatively renormalized. We will discuss this in more detail in Sect. 4.2.
Because resummation takes the form of an exponentiation, it is convenient to classify large logs in terms of the log derivative of the cross section $\sigma$, i.e., the so-called physical anomalous dimension $[10]$ defined as

$$Q^2 \frac{\partial \sigma(N, Q^2)}{\partial Q^2} = \gamma(N, \alpha_s(Q^2)) \sigma(N, Q^2).$$

(2.3)

When addressing issues of operator mixing, it is convenient to choose a basis of physical cross sections (or physical observables) equal in number to the independent parton distributions, so the physical anomalous dimension is a matrix. The physical anomalous dimension $\gamma$ Eq. (2.3) is independent of factorization scale, and it is related to the standard anomalous dimension $\gamma^{AP}$, defined by

$$\mu^2 \frac{\partial F(N, \mu^2)}{\partial \mu^2} = \gamma^{AP}(N, \alpha_s(\mu^2)) F(N, \mu^2),$$

(2.4)

according to

$$\gamma(N, \alpha_s(Q^2)) = \frac{\partial \ln C(N, Q^2/\mu^2, \alpha_s(\mu^2))}{\partial \ln Q^2} = \gamma^{AP}(N, \alpha_s(Q^2)) + \frac{\partial \ln C(N, 1, \alpha_s(Q^2))}{\partial \ln Q^2}.$$  

(2.5)

It follows that the physical and standard anomalous dimensions coincide at leading order, but differ beyond leading order. In terms of the physical anomalous dimension, the cross section can be written as

$$\sigma(N, Q^2) = K(N; Q_0^2, Q^2) \sigma(N, Q_0^2) = \exp \left[ E(N; Q_0^2, Q^2) \right] \sigma(N, Q_0^2),$$

where

$$E(N; Q_0^2, Q^2) = \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma(N, \alpha_s(k^2))$$

$$= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{AP}(N, \alpha_s(k^2)) + \ln C(N, 1, \alpha_s(Q^2)) - \ln C(N, 1, \alpha_s(Q_0^2)).$$

(2.7)

The structure of leading logs is then found by noting that upon Mellin transformation Eq. (2.1) the large $x$ region is mapped onto the large $N$ region, so logs of $1 - x$ become logs of $N$ and conversely. Explicitly

$$\int_0^1 dx \frac{x^{N-1}}{1-x} \ln^p(1-x) = \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \ln^p(1-x)$$

$$= \frac{1}{p+1} \ln^{p+1} \frac{1}{N} - \gamma_E \ln^p \frac{1}{N} + O \left( \frac{\ln^{p-1} 1}{N} \right),$$

(2.8)

where $\gamma_E$ is the Euler gamma, i.e., $\gamma_E = -\psi(1)$. The resummed next-to-leading order result of Refs. [4, 5] has the form

$$E^{\text{res}}(N; Q_0^2, Q^2) = a \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a} \frac{dk^2}{k^2} \hat{g}(\alpha_s(k^2)) + O(N^0)$$

(2.9)
anomalous dimension. Similarly, it is apparent that $\hat{g}$ to the two-loop anomalous dimension.

where

$$\hat{g}(\alpha_s) = \hat{g}_1 \alpha_s + \hat{g}_2 \alpha_s^2,$$

and $a = 1$ for processes with deep-inelastic kinematics, and $a = 2$ for Drell-Yan kinematics. This result implies that the physical anomalous dimension is given by

$$\gamma(N, \alpha_s(k^2)) = a \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \left[ \hat{g}_1 \alpha_s(k^2(1-x)^a) + \hat{g}_2 \alpha_s^2(k^2(1-x)^a) \right] + O\left(\alpha_s^{k+2}(k^2) \ln^k N \right) + O(N^0) \tag{2.11}$$

where $O(N^0)$ denotes non-logarithmic terms.

This means that, up to terms which are finite as $N \to \infty$, the physical anomalous dimension can be expanded as a power series in $\alpha_s$ at fixed $\alpha_s \ln N$. The leading log contribution to the physical anomalous dimension is the sum of terms to all orders in $\alpha_s$ where the powers of $\alpha_s$ and $\ln 1/N$ coincide; the next-to-leading order terms have one more power of $\alpha_s$, and so on. Because the QCD beta function starts at order $\alpha_s^2$, the order $\alpha_s^k$ contribution to $\gamma$ Eq. (2.11) leads to a contribution of order $\alpha_s^k(Q^2) \ln^{k+1} N$ to $E$ Eq. (2.7). In particular, the $O(\alpha_s)$ contribution corresponds to a series of double-log contributions to the cross section (in $\ln N$), in agreement with the simple power counting which leads to expect at most two extra powers of $\ln N$ when the cross section is computed at an extra order in $\alpha_s$. Notice, however, that the power counting for the physical anomalous dimension Eq. (2.11) is in fact more restrictive than what allowed on the basis of the fact that the cross section can contain at most double logs, in that it is free of contributions of order $\alpha_s^{k+1} \ln^{2k} N$ with $k > 1$. Indeed, Eq. (2.11) implies that the anomalous dimension contains at most single logs, i.e., each extra power of $\alpha_s$ is accompanied at most by an extra power of $\ln N$.

The resummed result Eq. (2.11) implies that all the leading-log coefficients are determined in terms of the single parameter $\hat{g}_1$, and the leading and next-to-leading ones by $\hat{g}_1$ and $\hat{g}_2$: at leading log we have

$$\gamma(N, \alpha_s(Q^2)) = a \hat{g}_1 \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \alpha_s(Q^2) \sum_{j=0}^{\infty} (-\beta_0)^j \alpha_s^j(Q^2) \ln^j(1-x)^a, \tag{2.12}$$

where we have expanded the beta function as

$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu^2) \equiv \beta(\alpha_s) = -\beta_0 \alpha_s^2(\mu^2) - \beta_1 \alpha_s^3(\mu^2) + O(\alpha_s^4). \tag{2.13}$$

The term with $j = 0$ in the series Eq. (2.12) is a contribution to the leading-order anomalous dimension: hence, $\hat{g}_1$ is just the coefficient of the $\ln(1/N)$ contribution to the usual one-loop anomalous dimension. Similarly, it is apparent that $\hat{g}_2$ is the coefficient of the $\ln N$ contribution to the two-loop anomalous dimension.

If Eqs. (2.9, 2.10) hold beyond next-to-leading order, as claimed in Ref. [8], i.e., if $\hat{g}(\alpha_s)$ is a power series in $\alpha_s$ with numerical coefficients $\hat{g}_k$, then $\hat{g}_k$ is just the coefficient of the $\ln N$
contribution to the $k$-loop physical anomalous dimension: the next $k$-to-leading log resummation is determined by knowledge of fixed-order $(k + 1)$-loop result. In Sect. 4 we will prove a somewhat less predictive form of the resummation, where the next $k$-to-leading log resummation is determined in general only by knowledge of a higher fixed-order computation.

Beyond leading order the standard anomalous dimension differs from the physical one, so $\hat{g}_k$ with $k > 1$ receives a contribution both from the standard anomalous dimension and from the coefficient function. It is thus natural to rewrite the resummation formula Eq. (2.9) in analogy to the unresummed separation Eq. (2.7) of contributions to $E$ from the anomalous dimension and coefficient function, by separating off the contribution which originates from the anomalous dimension $\gamma_{AP}$ Eq. (2.4):

$$E^{\text{res}}(N; Q_0^2, Q^2) = a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[ \int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s(Q^2(1-x)^a)) - B(\alpha_s(Q_0^2(1-x)^a)) \right]$$  \hspace{1cm} (2.14)

where $A(\alpha_s) = A_1 \alpha_s + A_2 \alpha_s^2$ is defined as the sum of one and two-loop coefficients of $\ln(1/N)$ in the standard anomalous dimension, and

$$B(\alpha_s) = B_1 \alpha_s $$  \hspace{1cm} (2.15)

makes up for the difference between physical and standard anomalous dimensions:

$$\hat{g}(\alpha_s) = A(\alpha_s) + \frac{\partial B(\alpha_s(k^2))}{\partial \ln k^2}. $$  \hspace{1cm} (2.16)

We can then rewrite the resummed cross section

$$\sigma^{\text{res}}(N, Q^2) = \exp \left[ E^{\text{res}}(N; Q_0^2, Q^2) \right] \sigma^{\text{res}}(N, Q_0^2) $$  \hspace{1cm} (2.17)

in factorized form according to Eq. (2.2) by collecting all $Q^2$-dependent contributions to the resummation Eq. (2.14) into a resummmed perturbative coefficient $C^{\text{res}}$

$$\sigma^{\text{res}}(N, Q^2) = C^{\text{res}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) F(N, \mu^2) $$  \hspace{1cm} (2.18)

$$C^{\text{res}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) = \exp \left\{ a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[ \int_{\mu^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + B(\alpha_s(Q^2(1-x)^a)) \right] \right\} $$  \hspace{1cm} (2.19)

The precise definition of the parton distribution $F$ and the factorization scale $\mu^2$ will depend on the choice of factorization scheme: according to the choice of scheme, the resummed terms will be either part of the hard coefficient $C^{\text{res}}$, or of the evolution of the structure function $F$. A detailed discussion of factorization scheme choices will be given in Sect. 4.

The resummed results Eq. (2.9), or Eqs. (2.18,2.19), can be explicitly cast in the form of an exponentiation of logs of $1/N$. To this purpose, we note that Eq. (2.8) implies that, up to
next-to-leading log level,
\[
\int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \ln^p(1 - x) = \frac{1}{p+1} \left( \ln \frac{1}{N} - \gamma_E \right)^{p+1} \\
= - \int_0^{1-\frac{1}{N}} \frac{dx}{1 - x} \left[ 1 - \gamma_E \frac{d}{d\ln(1 - x)} \right] \ln^p(1 - x).
\] (2.20)

The derivative term in the last expression is the first subleading correction to the leading-log term. Note that, whereas a next-to-leading log term can always be written as a derivative of the leading-log result, it is nontrivial that the coefficient of this derivative in Eq. (2.20) does not depend on the power \( p \) of \( \ln(1 - x) \). This fact can be exploited to rewrite the resummed result Eq. (2.19) as

\[
\ln C^{\text{res}}(N, Q^2/\mu^2, \alpha_s(\mu^2)) = -a \int_0^{1-\frac{1}{N}} \frac{dx}{1 - x} \int_{\mu^2}^{Q^2(1-x)^\alpha} \frac{dk^2}{k^2} A(\alpha_s(k^2)) + \tilde{B}(\alpha_s(Q^2(1-x)^\alpha)) \right] \\
= - \int_1^{N^a} \frac{dn}{n} \left[ \int_{\mu^2}^{Q^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) + \tilde{B}(\alpha_s(Q^2/n)) \right],
\] (2.21)

where in the second step we have performed the change of variable \( n = (1 - x)^{-\alpha} \), and to this order \( \tilde{B}(\alpha_s) = B(\alpha_s) - a\gamma_E A_1 \alpha_s \). In Eq. (2.21) the exponentiation is directly expressed in terms of logs of \( 1/N \).

Similarly, we can express the resummed physical anomalous dimension Eq. (2.3) directly in terms of logs of \( 1/N \) by using Eq. (2.20) to perform the Mellin transform. We get

\[
E^{\text{res}}(N; Q_0^2, Q^2) = a \int_0^{1-\frac{1}{N}} \frac{dx}{1 - x} \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} g(\alpha_s(k^2(1-x)^\alpha)) \\
= \int_1^{N^a} \frac{dn}{n} \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} g(\alpha_s(k^2/n)),
\] (2.22)

where

\[
g(\alpha_s(\mu^2)) = - \left( 1 - a\gamma_E \frac{d}{d\ln \mu^2} \right) \hat{g}(\alpha_s(\mu^2)).
\] (2.24)

The physical anomalous dimension Eq. (2.3) which corresponds to the resummed result Eq. (2.23) is particularly simple: we get

\[
\gamma(N, \alpha_s(Q^2)) = \int_1^{N^a} \frac{dn}{n} \left[ g_1 \alpha_s(Q^2/n) + g_2 \alpha_s^2(Q^2/n) \right] \\
= \frac{g_1}{\beta_0} \ln \alpha_s(Q^2/N^a) + \frac{1}{\beta_0} \left( g_2 - \frac{\beta_1}{\beta_0} g_1 \right) \left[ \alpha_s(Q^2/N^a) - \alpha_s(Q^2) \right],
\] (2.25)

where the coefficients of the two expansions Eq. (2.11) and Eq. (2.23) up to next-to-leading order are related by Eq. (2.24):

\[
g_1 = -\hat{g}_1; \quad g_2 = - (\hat{g}_2 - a\gamma_E \beta_0 \hat{g}_1).
\] (2.27)
The leading- and next-to-leading-log coefficients in the expansion of Eq. (2.11) are given by

\[
\gamma_1 = -\frac{g_1}{\beta_0} \ln \left[ 1 + \beta_0 \alpha_s(Q^2) \ln \frac{1}{N^a} \right],
\]

\[
\gamma_2 = -\left( g_2 - \frac{\beta_1}{\beta_0} g_1 \right) \frac{\alpha_s(Q^2) \ln \frac{1}{N^a}}{1 + \beta_0 \alpha_s(Q^2) \ln \frac{1}{N^a}} - \frac{g_1}{\beta_0} \frac{\beta_1}{\beta_0} g_1 \frac{1 + \beta_0 \alpha_s(Q^2) \ln \frac{1}{N^a}}{1 + \beta_0 \alpha_s(Q^2) \ln \frac{1}{N^a}},
\]

where we have used the beta function Eq. (2.13).

Summarizing, the resummation of Refs. \[4, 5\] can be compactly expressed in the two equivalent forms Eq. (2.11) or Eq. (2.25) of the resummed physical anomalous dimension. The former shows that the resummed anomalous dimension is the Mellin transform of a resummed leading-, next-to-leading-, . . . , \(\ln(1-x)\) splitting function, which in turn is a power series in \(\alpha_s(Q^2(1-x))^n\) with numerical coefficients. The latter shows that this Mellin transform can be just expressed as a series in \(\alpha_s(Q^2/N^a)\), with numerical coefficients. Knowledge of the \(k\)-th coefficient of either expansion determines the next\(k-1\)-to-leading resummation. This coefficient can be read off the coefficient of the \(\ln N\) term in a fixed \(k\)-th order computation of the physical anomalous dimension. Up to next-to-leading order, the two forms of the resummation are related by Eq. (2.24).

Our aim here is to derive an all-order generalization of these resummation formulae. A preliminary step is thus to obtain an all-order generalization of the relation between the two available forms of the next-to-leading log resummation. This generalization will be derived in the next section.

### 3 \(N\) space and \(x\) space beyond next-to-leading order

We wish to find an all-order generalization of the relation between leading \(\ln N\) and leading \(\ln(1-x)\) resummation discussed in the previous section. The leading \(\ln N\) resummation is expressed in terms of

\[
L_p \equiv \ln^p \frac{1}{N} = -p \int_0^1 \frac{dx}{1-x} \ln^{p-1}(1-x),
\]

while the leading \(\ln(1-x)\) resummation is expressed in terms of

\[
I_p \equiv \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \ln^p(1-x).
\]

In order to relate the two resummations we must therefore generalize to all orders the next-to-leading order relation Eq. (2.20), i.e., compute the Mellin transform Eq. (3.2) to all-order logarithmic accuracy, up to power corrections.

To this purpose, we notice that all integrals \(I_p\) can be obtained from a generating function \(G(\eta)\):

\[
I_p = \frac{d^p}{d\eta^p} G(\eta) \bigg|_{\eta=0},
\]

\[
G(\eta) = \int_0^1 dx \left( x^{N-1} - 1 \right) (1-x)^{\eta - 1}.
\]
The integral can be determined at large $N$ using the Stirling formula:

$$G(\eta) = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N+\eta)} - \frac{1}{\eta} = \frac{1}{\eta} \left[ \frac{\Gamma(1+\eta)}{N^\eta} - 1 \right] + O\left(\frac{1}{N}\right). \quad (3.4)$$

On the other hand, the generating function for $L_p$ is just $N^{-\eta}$:

$$L_p = \frac{d^p}{d\eta^p} N^{-\eta} \bigg|_{\eta=0}. \quad (3.5)$$

Hence, Eq. (3.4) can be viewed as a relation between the generating functions for $I_p$ and $L_p$. In particular, Taylor-expanding $\Gamma(1+\eta)$ in the expression Eq. (3.4) of $G(\eta)$ leads to leading, next-to-leading, ... $\ln N$ relations:

$$G(\eta) = \frac{1}{\eta} \left[ \frac{1}{N^\eta} \sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!} \eta^k - 1 \right] = -\sum_{k=0}^\infty \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{d\ln^k \frac{1}{N}} \int_0^{1-\frac{1}{N}} dx (1-x)^{\eta-1}. \quad (3.6)$$

For the individual $I_p$ Eq. (3.3) we get

$$I_p = -\sum_{k=0}^{p+1} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{d\ln^k \frac{1}{N}} \int_0^{1-\frac{1}{N}} dx \ln^p(1-x) + O\left(\frac{1}{N}\right). \quad (3.7)$$

Because

$$\frac{d^k}{d\ln^k \frac{1}{N}} \int_0^{1-\frac{1}{N}} dx \ln^p(1-x) = \int_0^{1-\frac{1}{N}} dx \frac{d^k}{d\ln^k (1-x)} \ln^p(1-x), \quad (3.8)$$

Eq. (3.7) can be rewritten as

$$I_p = -\sum_{k=0}^p \frac{\Gamma^{(k)}(1)}{k!} \int_0^{1-\frac{1}{N}} dx \frac{d^k}{1-x \ln^k(1-x)} \ln^p(1-x) + \Gamma^{(p+1)}(1) + O\left(\frac{1}{N}\right). \quad (3.9)$$

This is the desired generalization of Eq. (2.20) to all orders. Notice that the coefficient in the expansion remain $p$-independent to all orders. Performing the derivatives explicitly, we get

$$I_p = \frac{1}{p+1} \sum_{k=0}^{p+1} \binom{p+1}{k} \frac{\Gamma^{(k)}(1)}{\Gamma(1+\eta)} \left( \ln \frac{1}{N} \right)^{p+1-k} + O\left(\frac{1}{N}\right). \quad (3.10)$$

The inverse result, expressing $L_p$ in terms of $I_p$, can be analogously found by inverting the relation between generating functions:

$$N^{-\eta} = \frac{\eta G(\eta) + 1}{\Gamma(1+\eta)}. \quad (3.11)$$
Proceeding as above, we get
\[
\ln^n \frac{1}{N} = \sum_{k=1}^{n} \frac{\Delta^{(k-1)}(1)}{(k-1)!} \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \frac{d^k}{d \ln^k(1 - x)} \ln^n(1 - x) + \Delta^{(n)}(1) + O\left(\frac{1}{N}\right),
\]
where \(\Delta^{(k)}(\eta)\) is the \(k\)-th derivative of
\[
\Delta(\eta) \equiv \frac{1}{\Gamma(\eta)}.
\]
Evaluating the integrals explicitly we get
\[
\ln^n \frac{1}{N} = \sum_{i=1}^{n} \left( \frac{n}{i} \right) \Delta^{(n-i)}(1) I_{i-1} + \Delta^{(n)}(1) + O\left(\frac{1}{N}\right).
\]
Because of the \(p\)-independence of the coefficients of the expansion Eq. (3.9), we can determine explicitly the Mellin transform of a generic function
\[
\hat{g}(\ln(1 - x)) = \sum_{p=0}^{\infty} \hat{g}_p \ln^p(1 - x)
\]
of \(\ln(1 - x)\), up to power corrections:
\[
\int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \hat{g}(\ln(1 - x)) = \sum_{p=0}^{\infty} \hat{g}_p \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \ln^p(1 - x)
\]
\[
= - \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \int_0^{1 - \frac{1}{N}} dx \frac{d^k}{d \ln^k(1 - x)} \hat{g}(\ln(1 - x)) + O(N^0)
\]
\[
= \int_0^{1 - \frac{1}{N}} dx \frac{d}{1 - x} g(\ln(1 - x)) + O(N^0),
\]
where
\[
g(\ln(1 - x)) \equiv - \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{d \ln^k(1 - x)} \hat{g}(\ln(1 - x)),
\]
and the last (constant) term in Eq. (3.9) has been dropped, so Eq. (3.19) is only correct up to non-logarithmic terms, denoted by \(O(N^0)\).

The inverse relation can be analogously derived. Namely, we can cast the integral of any function
\[
g(\ln(1 - x)) = \sum_{p=0}^{\infty} g_p \ln^p(1 - x)
\]
as a Mellin transform, up to non-logarithmic terms:
\[
\int_0^{1 - \frac{1}{N}} dx \frac{d}{1 - x} g(\ln(1 - x)) = - \sum_{k=0}^{\infty} \frac{\Delta^{(k)}(1)}{k!} \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \frac{d^k}{d \ln^k(1 - x)} g(\ln(1 - x)) + O(N^0)
\]
\[
= \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \hat{g}(\ln(1 - x)) + O(N^0)
\]
where in the first step we have used Eq. (3.12), and now
\[
\hat{g}(\ln(1 - x)) \equiv -\sum_{k=0}^{\infty} \frac{\Delta^{(k)}(1)}{k!} \frac{d^k}{d\ln^k(1 - x)} g(\ln(1 - x)).
\] (3.20)

Equations (3.14-3.20) immediately provide us with the sought-for all-order generalization of the relation between the two forms of the resummed anomalous dimension Eq. (2.11) and Eq. (2.25), or of the resummation factor, Eqs. (2.9) and (2.23). Specifically,
\[
\int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \hat{g}[\alpha_s(k^2(1 - x)^a)] = \int_1^{N^0} \frac{dn}{n} g[\alpha_s(k^2/n)]
\] (3.21)

with
\[
g(\alpha_s(\mu^2)) = -\sum_{p=0}^{\infty} \frac{\Gamma^{(p)}(1) a^p}{p!} d^{p} \hat{g}(\alpha_s(\mu^2)).
\] (3.22)

By choosing a factorization scheme, we can then obtain from the resummation of the physical anomalous dimension a resummation of the coefficient function e.g. of the form of Eq. (2.19), with the functions $A$ and $B$ computed to the corresponding order.

For future applications, it is interesting to observe that, through similar manipulations, it is also possible to derive a relation between a generic function of $\ln \frac{1}{N}$, and a function of $\ln(1 - x)$. Indeed, given a function
\[
h\left(\ln \frac{1}{N}\right) = \sum_{p=0}^{\infty} h_p \ln^p \frac{1}{N}
\] (3.23)

Eq. (3.12) implies that
\[
h\left(\ln \frac{1}{N}\right) = \sum_{k=1}^{\infty} \frac{\Delta^{(k-1)}(1)}{(k-1)!} \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \frac{d^k}{d\ln^k(1 - x)} h(\ln(1 - x)) + O(N^0).\] (3.24)

The right-hand side of Eq. (3.24) can be viewed as the Mellin transform of a function (more properly a distribution) $\hat{h}(\ln(1 - x))$:
\[
h\left(\ln \frac{1}{N}\right) = \int_0^1 dx \frac{x^{N-1}}{1 - x} \hat{h}(\ln(1 - x));
\]
\[
\hat{h}(\ln(1 - x)) = \sum_{k=1}^{\infty} \frac{\Delta^{(k-1)}(1)}{(k-1)!} \left[ \frac{1}{1 - x} \frac{d^k}{d\ln^k(1 - x)} f(\ln(1 - x)) \right]_+ + O(N^0).\] (3.25)

### 4 All-order resummation

This section contains the main result of our paper, namely the proof of an all-order generalization of the next-to-leading order resumptions discussed in Sect. 2. This proof exploits the relation between $\ln(1 - x)$ resummation and $\ln N$ resummation established in Sect. 3, and proceeds in two steps. First, (in Sect. 4.1) we determine the $N$ dependence of the regularized coefficient function in the large-$N$ limit. We show that the kinematic structure of the $k$-particle phase
space essentially determines the form of the $N$-dependence in the large $N$ limit, and that loop integrations do not modify this result. Then (in Sect. [1.2]) we prove that, given this form of the $N$-dependence of the regularized cross section, renormalization group invariance fixes the all-order dependence of the physical anomalous dimension Eq. (2.3) in such a way that an infinite class of leading, next-to-leading, . . . resummations are found in terms of corresponding fixed order results. In particular, in Sect. [1.3] we show that the next-to-leading-log resummation Eq. (2.23) can be obtained by exploiting available fixed-order results.

4.1 Kinematic structure of soft logs

We consider the perturbative expansion of the bare coefficient function in powers of the bare coupling constant $\alpha_0$,

$$C^{(0)}(x, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(x, Q^2, \epsilon)$$

and its Mellin transform

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(N, Q^2, \epsilon)$$

in dimensional regularization with $d = 4 - 2\epsilon$ space-time dimensions. By coefficient function here we mean a vector whose elements are the parton-level cross section for the partonic subprocesses that contribute to the given process: the formalism and the results of this section remain true even in the presence of parton mixing.

The result that will be proven in this section, and then renormalization-group improved in the next section, is that in the large-$N$ limit, $C_n^{(0)}(N, Q^2, \epsilon)$ has the following structure:

$$C_n^{(0)}(N, Q^2, \epsilon) = \sum_{k=0}^{n} C_{nk}^{(0)}(\epsilon) (Q^2)^{-nk} \left( \frac{Q^2}{N^a} \right)^{-ke} + O(1/N),$$

where $O(1/N)$ denotes all terms which vanish as $N \to \infty$ in the limit $\epsilon \to 0$. We will also see that the coefficients $C_{nk}^{(0)}$ have a pole of order $2n$ in $\epsilon$, related to infrared singularities. Even though such poles cancel in the coefficient function, which is free of infrared singularities [11], the interference of the poles with the powers of $N^{-\epsilon}$ is responsible for the presence of powers of $\ln N$ in the renormalized four-dimensional cross section. Therefore, soft logs may be viewed as being due to an incomplete cancellation between real and virtual contributions to $C_n^{(0)}$ in the soft limit.

Equation (4.3) will be established by proving that

$$C_n^{(0)}(x, Q^2, \epsilon) = (Q^2)^{-ne} C_{n0}^{(0)}(\epsilon) \delta(1-x) + \sum_{k=1}^{n} C_{nk}^{(0)}(\epsilon) (1-x)^{-1-ak\epsilon} + O[(1-x)^0],$$

where $O[(1-x)^0]$ denote terms which are not divergent as $x \to 1$ in the $\epsilon \to 0$ limit. Eq. (4.4) implies Eq. (4.3) because

$$\int_0^1 dx x^{N-1} (1-x)^{-1-ak\epsilon} = \frac{\Gamma(N)\Gamma(-ak\epsilon)}{\Gamma(N-ak\epsilon)} = \Gamma(-ak\epsilon) N^{ak\epsilon} + O(1/N),$$

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while the Mellin transform of any function of $x$ which is not divergent as $x \to 1$ (e.g. any non-negative power of $1 - x$) vanishes as $N \to \infty$.

The content of Eq. (4.3) is that, in the large $N$ limit, the dependence of the regularized cross section on $N$ only goes through integer powers of the dimensionful variable $(Q^2/N^a)^{-\epsilon}$. Because in $d$ dimensions the coupling constant has the dimensions of $Q^{2\epsilon}$, the dependence on this dimensionful variable is related through the renormalization group to the running of the coupling. As we shall show in Sect. 4.2, this is sufficient to establish the desired all-order resummation. In the remainder of this subsection we will show that Eq. (4.4) essentially follows from the kinematics of soft emission. The reader who is not interested in technicalities can skip directly to Sect. 4.2 where only the result Eq. (4.3) will be used.

We consider deep-inelastic scattering and Drell-Yan production as representative cases of the two classes of processes discussed in Sect. 2. In the case of deep-inelastic scattering, the relevant parton subprocesses are

\begin{align}
\gamma^*(q) + Q(p) & \rightarrow Q(p') + X, \\
\gamma^*(q) + G(p) & \rightarrow Q(p') + X,
\end{align}

where $Q$ is a quark or an antiquark, $G$ a gluon, and $X$ is any collection of quarks and gluons. In the case of Drell-Yan, the relevant subprocesses are

\begin{align}
Q(p) + Q(p') & \rightarrow \gamma^*(q) + X, \\
Q(p) + G(p') & \rightarrow \gamma^*(q) + X, \\
G(p) + G(p') & \rightarrow \gamma^*(q) + X.
\end{align}

We are interested in the most singular contributions to the cross section as $x \to 1$, and specifically in the way these terms depend on $1 - x$. To this purpose, we consider the dependence on $1 - x$ of the phase space and of the amplitude for the generic processes Eq. (4.6-4.10). The structure of the amplitude will in turn be discussed by considering first tree-level processes, and then processes with loops. We will first carry out the proof for deep-inelastic scattering, and then for Drell-Yan.

As discussed in the Appendix, the phase space for a generic process with incoming momentum $P$ and $n$ bodies in the final state with outgoing momenta $k_1, \ldots, k_n$ can be expressed in terms of two-body phase space integrals by using recursively the identity Eq. (A.5). For DIS-like processes, all outgoing particles are massless partons. Indicating with $p$ and $p'$ the incoming and outgoing quark momenta and with $q$ the $\gamma^*$ momentum we have

\begin{equation}
\phi_{n+1}(p + q; k_1, \ldots, k_n, p') = \int dM_n^2 \frac{d}{2\pi} d\phi_2(p + q; k_n, P_n) d\phi_n(P_n, k_1, \ldots, k_{n-1}, p'),
\end{equation}

where

\begin{equation}
s = (p + q)^2 = \frac{Q^2(1 - x)}{x}.
\end{equation}

Applying Eq. (4.11) recursively we get

\begin{align}
\phi_{n+1}(P_{n+1}; k_1, \ldots, k_n, P) &= \int_0^s dM_n^2 \frac{d}{2\pi} d\phi_2(p + q; k_n, P_n) \int_0^{M_n^2} dM_{n-1}^2 \frac{d}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1}) \\
& \cdots \int_0^{M_3^2} dM_2^2 \frac{d}{2\pi} d\phi_2(P_3; k_2, P_2) d\phi_2(P_2; k_1, P_1),
\end{align}

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where we have defined $P_{n+1} \equiv p + q$, so $M_{n+1}^2 = s$, and $P_1 \equiv p'$, so $P_1^2 = 0$.

The dependence of the phase space on $1 - x$ can now be traced by performing the change of variables

$$z_i = \frac{M_i^2}{M_{i+1}^2}; \quad M_i^2 = s z_n \ldots z_i; \quad i = 2, \ldots, n.$$  \hspace{1cm} (4.14)

All $z_i$ range between 0 and 1, and

$$dM_n^2 \ldots dM_2^2 = s^{n-1} z_n^{-2} z_{n-1}^{-3} \ldots z_3 d z_n \ldots d z_2.$$  \hspace{1cm} (4.15)

The two-body phase space Eq. (4.9) is then

$$d\phi_2(P_{i+1}; k_i, P_i) = N(\epsilon) s^{-\epsilon} (z_n z_{n-1} \ldots z_{i+1})^{-\epsilon} (1 - z_i)^{1-2\epsilon} d\Omega_i,$$  \hspace{1cm} (4.16)

where $d\Omega_i$ is the angular integral in the center-of-mass frame of the $(k_i, P_i)$ system, and $N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}$.

Hence, the $n$-body phase space Eq. (4.13) can be rewritten as

$$d\phi_{n+1}(p + q; k_1, \ldots, k_n, p') = 2\pi \left[ \frac{N(\epsilon)}{2\pi} \right]^{n} s^{n-1-n\epsilon} d\Omega_n \ldots d\Omega_1$$

$$\int_0^1 d z_n z_n^{-2} (1 - z_n)^{1-2\epsilon} \ldots \int_0^1 d z_2 z_2^{-\epsilon} (1 - z_2)^{1-2\epsilon}.$$  \hspace{1cm} (4.17)

The dependence of the phase space on $1 - x$ comes entirely from the prefactor of $s^{n-1-n\epsilon}$: the dependence on $x$ and $Q^2$ has been entirely removed from the integration range. Now, the amplitude whose square modulus is integrated with the phase space Eq. (4.17) is in general a function $A_{n+1}(Q^2, s; z_2, \ldots, z_n; \Omega_1, \ldots, \Omega_n)$: it depends on $3(n + 3) - 10 + 1 = 3n$ independent variables for a process with two initial-state and $n + 1$ final-state particles, in which one of the masses (the virtuality $Q^2$) is not fixed. In the $x \to 1$ limit, $s \to 0$ and the dominant contribution is given by terms which are most singular as $s$ vanishes. Because of the cancellation of infrared singularities $[11], \quad |A_{n+1}|^2 \sim s^{-n+O(\epsilon)}$: a stronger singularity would lead to powerlike infrared divergences. Less singular terms instead correspond to contributions whose Mellin transform vanishes in the large-$N$ limit.

Hence only terms in the square amplitude which behave as $s^{-n+O(\epsilon)}$ contribute in the $x \to 1$ limit. In $d$ dimensions, these terms pick up an $s^{-n\epsilon}$ prefactor from the phase space Eq. (4.17). In order to establish the result Eq. (4.4), we must study the dependence of these terms on $s^\epsilon$. The case of a tree-level amplitude, i.e., purely real soft emission is in fact straightforward. Indeed, in this case, the amplitude manifestly does not contain any power of $s^\epsilon$, so the result Eq. (4.4) follows immediately, with $a = 1$. Notice that in this case, since the factors of $s^{-\epsilon}$ come from the phase space only, the sum over $k$ in Eq. (4.4) reduces to the single term $k = n$: the power of $[Q^2(1 - x)]^{-\epsilon}$ coincides with the number of $z$ integrations, i.e., with the number of emitted partons.

To understand the physical meaning of this result, note that in the center-of-mass frame of the $k_i, P_i$ system the energy of the $i$-th of the $n$ emitted partons is (see Appendix A)

$$\omega_i = k_i^0 = \frac{M_{i+1}}{2} \left( 1 - \frac{M_i^2}{M_{i+1}^2} \right).$$  \hspace{1cm} (4.18)
Now, concentrate on the emission of the $n$-th parton, whose energy is $k_n^0 = \frac{\sqrt{s}}{2}(1 - z_n) = \sqrt{s}\left(1 - \frac{M^2}{s}\right)$. The singularity arises if the squared amplitude behaves as $1/(k_n^0)^2$. This behaviour is seen to be present e.g. for gluon emission in the eikonal approximation. In such case we get

$$|A_{n+1}|^2 d\phi_{n+1} \sim \int_0^{k_n^0_{\max}} \frac{dk_n^0}{k_n^0 \sqrt{1 + 2\epsilon}} d\phi_n = \frac{s^{\epsilon}}{4} \int_0^1 \frac{dz_n}{(1 - z_n)^{1 + 2\epsilon}} d\phi_n$$

(4.19)

$$= -\frac{1}{2\epsilon} \left[\frac{Q^2(1 - x)}{4x}\right]^{-\epsilon} d\phi_n$$

(4.20)

where the maximum value of $k_n^0$ is attained when $M_n^2 = 0$ and is

$$k_n^0_{\max} = \frac{\sqrt{s}}{2} = \sqrt{\frac{Q^2(1 - x)}{4x}}.$$  

(4.21)

The $x$ dependence of the result is thus simply determined kinematically in terms of the upper limit of the logarithmic $k_n^0$ integration. Eq. (4.17) then shows that, thanks to the structure of the $n$-body phase space, this remains true in the general case of $n$-parton emission.

We will now study how this result is modified by the inclusion of loops. To this purpose, we notice that a generic amplitude with loops can be viewed as a tree-level amplitude formed with proper vertices. Powers of $s^\epsilon$ can only arise from loop integrations in the proper vertices. We thus consider only purely scalar loop integrals, since numerators of fermion or vector propagators and vertex factors cannot induce any dependence on $s^\epsilon$.

Let us therefore consider an arbitrary proper diagram $G$ in a massless scalar theory with $E$ external lines, $I$ internal lines and $V$ vertices. It can be shown [12] that, denoting with $P$ the set of $E$ external momenta and $P_E$ the set of independent invariants the corresponding amplitude $	ilde{A}_G(P_E)$ has the form

$$\tilde{A}_G(P_E) = K (2\pi)^d \delta^{(d)}(\sum \text{P}) A_G(P_E)$$

(4.22)

$$A_G(P_E) = \frac{\epsilon^{I - L(d - 1)}}{(4\pi)^{dL/2}} \frac{\delta(1 - \sum_{l=1}^I \beta_l)}{\Gamma(I - dL/2)} \frac{\Gamma(I - dL/2)}{I} \frac{\delta(1 - \sum_{l=1}^I \beta_l)}{[P_G(\beta)]^{d(I + 1)/2 - I}[D_G(\beta, P_E)]^{I - dL/2}},$$

In Eq. (4.23), $\beta_l$ are the usual Feynman parameters, $P_G(\beta)$ is a homogeneous polynomial of degree $L$ in the $\beta_l$, $D_G(\beta, P_E)$ is a homogeneous polynomial of degree $L + 1$ in the $\beta_l$ with coefficients which are linear functions of the scalar products of the set $P_E$, i.e. with dimensions of $(mass)^2$, and $K$ collects all overall factors, such as couplings and symmetry factors.

The amplitude $\tilde{A}_G(P_E)$ Eq. (4.23) depends on $s$ only through $D_G(\beta, P_E)$, which, in turn, is linear in $s$. We can then determine in general the dependence of $A_G(P_E)$ by considering two possible cases. The first possibility is that $D_G(\beta, P_E)$ is independent of all invariants except $s$, i.e. $D_G(\beta, P_E) = sd_G(\beta)$. In such case, $A_G(P_E)$ depends on $s$ as

$$A_G(P_E) = \left(\frac{1}{s}\right)^{1 - 2L + L\epsilon} a_G,$$

(4.23)

where $a_G$ is a numerical constant, obtained performing the Feynman parameter integrals. The second possibility is that $D_G(\beta, P_E)$ depends on some of the other invariants. In such case,
$A_G(P_E)$ is manifestly an analytic function of $s$ at $s = 0$, and thus it can be expanded in Taylor series around $s = 0$, with coefficients which depend on the other invariants. In the former case, Eq. (1.23) implies that the $s$ dependence induced by loop integrations in the square amplitude is given by integer powers of $s^{-\epsilon}$. In the latter case, the $s$ dependence induced by loop integrations in the square amplitude is given by integer positive powers of $s$.

Therefore, we conclude that Eq. (4.4) holds in general even when loop integrations are present. Unlike in the case of tree-level diagrams, however, the overall power of $s^{-\epsilon}$ is no longer determined by the phase space only, and thus no longer equal to the number of emitted partons, from which it can differ by an integer amount. Specifically, each loop integration can carry at most a factor of $s^{-\epsilon}$. Hence, at order $\alpha_0^n$ the highest power of $s^{-\epsilon}$ is $n$, given that either an extra real emission or an extra loop can give at most an extra factor $s^{-\epsilon}$, but otherwise contributions proportional to all integer powers of $s^{-\epsilon}$ up to $n$ are allowed. Because in $d$ dimensions the bare coupling constant $\alpha_0$ has mass dimensions $2\epsilon$, the dependence on $Q^2$, which is the only other dimensionful variable, is fixed by dimensional analysis, and Eq. (4.4) immediately follows.

Furthermore, it is clear (compare Eq. (1.20)) that each logarithmic $z_i$ integration produces a $1/\epsilon$ pole. Since each angular integration gives a $1/\epsilon$ pole from the collinear region, the highest soft singularity due to real emission in the coefficient $C^{(0)}_{nk}/\Gamma(-ak\epsilon)$ in Eq. (4.4) is an $\epsilon$ pole of order $2k - 1$. Hence, $C^{(0)}_n$ has a soft singularity $1/\epsilon^{2n}$, from the purely real emission term $C^{(0)}_{nk}$ in the sum Eq. (1.3). Because infrared singularities must cancel $\Gamma(1)$ in $C^{(0)}_n$, the sum of all other terms $C^{(0)}_{nk}$ in the sum also has an order $2n$ pole in $\epsilon$.

The whole argument can be reproduced for Drell-Yan-like processes with minor modifications which account for the different kinematics. Specifically, in the case of Drell-Yan we have massless partons in the initial state with momenta $p, p'$, while the Drell-Yan pair in the final state has momentum $Q$, so $x = Q^2/s$, with $s = (p + p')^2$. The recursive expression for the $n$-body phase space Eq. (A.3) now becomes

$$d\phi_{n+1}(p + p'; Q, k_1, \ldots, k_n) = \int_{Q^2}^{s} \frac{dM^2_n}{2\pi} d\phi_2(p + p'; k_n, P_n) d\phi_n(P_n; Q, k_1, \ldots, k_{n-1}),$$

(4.24)

leading to

$$d\phi_{n+1}(P_{n+1}; Q, k_1, \ldots, k_n) = \int_{Q^2}^{s} \frac{dM^2_n}{2\pi} d\phi_2(p + p'; k_n, P_n) \int_{Q^2}^{M^2_n} \frac{dM^2_{n-1}}{2\pi} d\phi_2(P_{n-1}; k_{n-1}, P_{n-1}) \cdots \int_{Q^2}^{M^2_2} \frac{dM^2_1}{2\pi} d\phi_2(P_2; k_2, P_2) d\phi_2(P_2; k_1, P_1),$$

(4.25)

where now $P_{n+1} \equiv p + p'$, so $M^2_{n+1} = s$, and $P_1 \equiv Q$.

The change of variables which separates off the dependence on $1 - x$ is now

$$z_i = \frac{M^2_i - Q^2}{M^2_{i-1} - Q^2}; \quad M^2_i - Q^2 = (s - Q^2)z_n \ldots z_i; \quad i = 2, \ldots, n$$

(4.26)

so all $z_i$ still range between 0 and 1 and

$$dM^2_n \ldots dM^2_2 = (s - Q^2)^{n-1} z_n^{-1} z_{n-2}^{-1} \ldots z_3^{-1} dz_n \ldots dz_2.$$  

(4.27)
The two-body phase space Eq. (A.9) is now

\[ d\phi_2(P_{i+1}; k_i, P_i) = N(\epsilon) (M_{i+1}^2)^{-1+\epsilon} \left[ (M_{i+1}^2 - Q^2) - (M_i^2 - Q^2) \right]^{1-2\epsilon} d\Omega_i \]

\[ = N(\epsilon) (Q^2)^{-1+\epsilon} (s - Q^2)^{1-2\epsilon} (z_n \ldots z_{i+1})^{1-2\epsilon} (1 - z_i)^{1-2\epsilon} d\Omega_i, \quad (4.28) \]

where in the last step we have replaced \((M_{i+1}^2)^{-1+\epsilon}\) by \((Q^2)^{-1+\epsilon}\) in the \(x \to 1\) limit (compare Eq. (4.26)).

We finally get

\[ d\phi_{n+1}(p + q; k_1, \ldots, k_n, p') = 2\pi \left[ \frac{N(\epsilon)}{2\pi} \right]^n (Q^2)^{-n(1-\epsilon)} (s - Q^2)^{2n-1-2n\epsilon} d\Omega_n \ldots d\Omega_1 \]

\[ \int_0^1 dz_n z_n^{(n-2)+(n-1)(1-2\epsilon)} (1 - z_n)^{1-2\epsilon} \ldots \int_0^1 dz_2 z_2^{1-2\epsilon} (1 - z_2)^{1-2\epsilon}. \quad (4.29) \]

The dependence on \(1 - x\) is now entirely contained in the prefactor

\[ (Q^2)^{-n(1-\epsilon)} (s - Q^2)^{2n-1-2n\epsilon} \frac{x^{1-2n+2n\epsilon}}{Q^2(1 - x)^2} [Q^2(1 - x)^2]^{n-n\epsilon}. \quad (4.30) \]

Again, this proves the result Eq. (4.3) for real (tree-level) emission, since the corresponding amplitude is free of factors of \((s - Q^2)^{-1-\epsilon}\). Note however that now \(a = 2\). The physical interpretation is the same as in the deep-inelastic case, except that the upper limit of the logarithmic \(k_i^0\) integration Eq. (4.19) for the \(i\)-th parton is now attained when \(M_i^2 = Q^2\) and is

\[ k_{i,\text{max}}^0 = \frac{\sqrt{s}}{2} \left( 1 - \frac{Q^2}{s} \right) = \sqrt{\frac{Q^2(1 - x)^2}{4x}}. \quad (4.31) \]

Finally, loops can be included as in the case of deep-inelastic processes, with the possible cut at \(x = 1\) now being given by a factor of \((s - Q^2)^{-L+L\epsilon}\).

We conclude that Eq. (4.4) holds for both deep-inelastic and Drell-Yan-like processes, with \(a = 1\) in the former case and \(a = 2\) in the latter. This difference is merely a reflection of the underlying kinematics, Eqs. (4.21) and (4.31), but in both cases the soft logs (\(i.e.\) powers of \((1 - x)^{-\epsilon}\)) can be traced to the fact that the phase space variables \(M_i^2\) can all be related to a single dimensionful variable \(s\) (DIS) or \(s - Q^2\) (DY) through dimensionless variables \(z_i\) which range from 0 to 1, according to Eqs. (4.14) and (4.26).

It is interesting to ask which are the kinematic configurations which contribute to the \(x \to 1\) limit: in fact, they turn out not to be the same in deep-inelastic and Drell-Yan. Indeed, consider a contribution to the cross section that involves \(n\) massless partons with momenta \(k_1, \ldots, k_n\) in the final state. We have

\[ p + p' = q + k_1 + \ldots + k_n. \quad (4.32) \]

Squaring both sides of Eq. (4.32) we get

\[ s(1 - x) = \sum_{i,j=1}^n k_i \cdot k_j + 2 \sum_{i=1}^n q \cdot k_i. \quad (4.33) \]
The quantities
\[ q \cdot k_i = k_i^0 \left( q_0 - \sqrt{q_0^2 - Q^2 \cos \theta_i} \right) \tag{4.34} \]
are positive semi-definite, and vanish only for \( k_i^0 = 0 \). Hence, for Drell-Yan processes, when \( x \) approaches 1, all \( k_i^0 \)'s must go to zero. i.e. all emitted partons must be soft.

On the other hand, in the case of DIS we have
\[ p + q = k_1 + \ldots + k_n \tag{4.35} \]
(where \( k_1 = p' \)), which gives
\[ \frac{Q^2(1 - x)}{x} = \sum_{i,j=1}^{n} k_i \cdot k_j = \sum_{i,j=1}^{n} k_i^0 k_j^0 (1 - \cos \theta_{ij}), \tag{4.36} \]
where \( \theta_{ij} \) is the angle between \( \vec{k}_i \) and \( \vec{k}_j \). Now take the limit \( x \to 1 \). The most general solution to Eq. (4.36) is not \( k_i^0 = 0 \) for all \( i \), but rather
\[ k_i^0 = 0 \quad \text{for} \quad 1 \leq i \leq \bar{n} \]
\[ \theta_{ij} = 0; k_i^0, k_j^0 \text{ arbitrary} \quad \text{for} \quad \bar{n} \leq i, j \leq n. \tag{4.37} \]
In other words, for DIS the kinematical configurations that correspond to \( x = 1 \) include not only the case when all partons in the final state have small energies, but also the cases when a subset of them are collinear to each other.

### 4.2 Renormalization group improvement

We now study the restrictions that renormalization–group invariance imposes on the cross section. As already mentioned, our only assumption is the standard factorization Eq. (2.2), namely, that \( C(N, Q^2 / \mu^2, \alpha_s(\mu^2)) \) can be multiplicatively renormalized. This means that all divergences can be removed from the bare coefficient \( C^{(0)}(N, Q^2, \alpha_0, \epsilon) \) Eq. (4.2) by defining a renormalized coupling \( \alpha_s(\mu^2) \) according to the implicit equation
\[ \alpha_0(\mu^2, \alpha_s(\mu^2), \epsilon) = \mu^{2\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{4.38} \]
and a renormalized coefficient
\[ C \left( N, Q^2 / \mu^2, \alpha_s(\mu^2), \epsilon \right) = Z^{(C)}(N, \alpha_s(\mu^2), \epsilon) C^{(0)} \left( N, Q^2, \alpha_0, \epsilon \right), \tag{4.39} \]
where \( \mu \) is the renormalization scale and \( Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \) and \( Z^{(C)}(N, \alpha_s(\mu^2), \epsilon) \) are computable in perturbation theory and have multiple poles at \( \epsilon = 0 \). The renormalized coefficient function \( C \left( N, Q^2 / \mu^2, \alpha_s(\mu^2), \epsilon \right) \) is finite at \( \epsilon = 0 \), and, because the renormalized coupling \( \alpha_s \) is dimensionless, it can only depend on \( Q^2 \) through \( Q^2 / \mu^2 \).

In a mass-independent renormalization (and factorization) scheme such as \( \overline{\text{MS}} \), both \( Z^{(\alpha_s)} \) and \( Z^{(C)} \) are series in \( \alpha_s \) with \( Q^2 \)–independent coefficients. Therefore, the physical anomalous dimension is
\[ \gamma \equiv Q^2 \frac{d}{dQ^2} \ln C \left( N, Q^2 / \mu^2, \alpha_s, \epsilon \right) = -\epsilon \alpha_0 \frac{\partial}{\partial \alpha_0} \ln C^{(0)} \left( N, Q^2, \alpha_0, \epsilon \right), \tag{4.40} \]
where we have exploited the fact that, for dimensional reasons, $C^{(0)}$ can only depend on $Q^2$ through the combination $\alpha_0 Q^{-2\epsilon}$. Equation (4.40) implies that in $d$ dimensions the physical anomalous dimension $\gamma$ viewed as a function of $\alpha_0$ admits an expansion in powers of $\alpha_0 Q^{-2\epsilon}$, while as a function of $\alpha_s(\mu^2)$ it admits an expansion in powers of $(Q^2/\mu^2)^{-\epsilon}$.

In the large $N$ limit, the dependence of the bare coefficient function on $N$ is given by Eq. (4.3). But Eq. (4.40) implies that $\gamma$ has the same property, namely

$$\gamma\left(N, Q^2, \alpha_0, \epsilon\right) = \sum_{i=1}^\infty \alpha_0^i \sum_{j=0}^i \gamma_{ij}(\epsilon) (Q^2)^{-i-j} \left(\frac{Q^2}{N^a}\right)^{-j\epsilon} + O\left(\frac{1}{N}\right)$$

$$= \sum_{i=1}^\infty \sum_{j=0}^i \gamma_{ij}(\epsilon) \left[(Q^2)^{-\epsilon} \alpha_0\right]^{i-j} \left[\left(\frac{Q^2}{N^a}\right)^{-\epsilon} \alpha_0\right]^j + O\left(\frac{1}{N}\right). \tag{4.41}$$

Hence, in the large $N$ limit the $d$-dimensional physical anomalous dimension is a power series in $Q^{-2\epsilon}\alpha_0$ and $(Q^2/N^a)^{-\epsilon}\alpha_0$, with $N$–independent coefficients.

The renormalized expression of the physical anomalous dimension is found expressing in Eq. (4.41) the bare coupling in terms of the renormalized one by means of Eq. (4.38): this yields an expression of the physical anomalous dimension which is finite in the limit $\epsilon \to 0$. Now, the function

$$\bar{\alpha}_0(Q^2/\mu^2, \alpha_s(\mu^2), \epsilon) = Q^{-2\epsilon} \alpha_0(\mu^2, \alpha_s(\mu^2), \epsilon) = \left(\frac{Q^2}{\mu^2}\right)^{-\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{4.42}$$

is manifestly renormalization-group invariant:

$$\mu^2 \frac{d\bar{\alpha}_0}{d\mu^2} = 0. \tag{4.43}$$

It follows that

$$\bar{\alpha}_0(Q^2/\mu^2, \alpha_s(\mu^2), \epsilon) = \bar{\alpha}_0(1, \alpha_s(Q^2), \epsilon) = \alpha_s(Q^2) Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon). \tag{4.44}$$

Explicitly, noting that the $d$ dimensional beta function is given by

$$\mu^2 \frac{d}{d\mu^2} \alpha_s(\mu^2) \equiv \beta^{(d)}(\alpha_s(\mu^2), \epsilon) = -\epsilon \alpha_s(\mu^2) + \beta(\alpha_s(\mu^2))$$

$$= -\epsilon \alpha_s(\mu^2) - \beta_0 \alpha_s^2(\mu^2) + O(\alpha_s^3), \tag{4.45}$$

1Note that the standard anomalous dimension $\gamma^{AP}$ Eq. (2.3) is instead given by

$$\gamma^{AP}(N, \alpha_s(\mu^2)) = -\frac{\partial \ln Z^{(C)}(N, \alpha_s(\mu^2), \epsilon)}{\partial \ln \mu^2},$$

due to the fact that the bare coefficient $C^{(0)}$ is independent of $\mu^2$. It follows that $\gamma^{AP} = \gamma$ if and only if $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) = 1$, i.e. if the four–dimensional beta function vanishes.
in terms of the four-dimensional beta function of Eq. (2.13), and exploiting the \( \mu \)-independence of \( \alpha_0 \), Eq. (1.38) gives

\[
Z^{(\alpha_s)}(\alpha_s, \epsilon) = \exp \int_0^\alpha \frac{d\alpha'}{\alpha'} \frac{\beta(\alpha')}{\epsilon \alpha' - \beta(\alpha')} = \left( 1 + \frac{\beta_0 \alpha_s}{\epsilon} \right)^{-1} + O\left( \frac{\alpha_s^{k+1}}{\epsilon^k} \right).
\] (4.46)

The renormalized anomalous dimension is found by rewriting Eq. (4.41) as

\[
\gamma \left( N, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon \right) = \sum_{i=1}^\infty \sum_{j=0}^i \gamma_{ij}(\epsilon) \left[ \tilde{\alpha}_0(1, \alpha_s(Q^2), \epsilon) \right]^{i-j} \left[ \alpha_0(1, \alpha_s(Q^2/N^a), \epsilon) \right]^j,
\] (4.47)

and then re-expanding in powers of the renormalized coupling:

\[
\gamma \left( N, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon \right) = \sum_{m=1}^\infty \sum_{n=0}^m \gamma_{mn}^R(\epsilon) \alpha_s^{m-n}(Q^2) \alpha_s^n(Q^2/N^a),
\] (4.48)

The physical anomalous dimension is thus seen to be a power series in \( \alpha_s(Q^2) \) and \( \alpha_s(Q^2/N^a) \), with coefficients \( \gamma_{mn}^R \) determined by \( \gamma_{ij}(\epsilon) \) and the renormalization constant \( Z^{(\alpha_s)} \). If \( \alpha_s(Q^2/N^a) \) is re-expanded in terms of \( \alpha_s(Q^2) \), the physical anomalous dimension is seen to be of order \( \alpha_s(Q^2) \) (and thus vanish when \( \alpha_s(Q^2) \to 0 \)), as it ought to.

However, we cannot yet conclude that the four-dimensional physical anomalous dimension admits an expansion of the form of Eq. (4.48), because the coefficients \( \gamma_{mn}^R(\epsilon) \) are not necessarily finite as \( \epsilon \to 0 \). In order to understand this, it is convenient to separate off the \( N \)-independent terms in the renormalized anomalous dimension, \( \textit{i.e.} \) the terms with \( n = 0 \) in the internal sum in Eq. (4.48). Namely, we write

\[
\gamma \left( N, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon \right) = \gamma^{(l)} \left( \alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon \right) + \gamma^{(c)} \left( \alpha_s(Q^2), \epsilon \right),
\] (4.49)

where we have defined

\[
\gamma^{(l)} \left( \alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon \right) \equiv \sum_{i=1}^\infty \sum_{j=1}^i \gamma_{ij}(\epsilon) \left[ \tilde{\alpha}_0(1, \alpha_s(Q^2), \epsilon) \right]^{i-j} \left[ \alpha_0(1, \alpha_s(Q^2/N^a), \epsilon) \right]^j
\]

\[
= \sum_{m=0}^\infty \sum_{n=1}^m \gamma_{mn}^R(\epsilon) \alpha_s^m(Q^2) \alpha_s^n(Q^2/N^a),
\] (4.50)

and

\[
\gamma^{(c)} \left( \alpha_s(Q^2), \epsilon \right) \equiv \sum_{i=1}^\infty \gamma_{i0}(\epsilon) \left[ \tilde{\alpha}_0(1, \alpha_s(Q^2), \epsilon) \right]^i = \sum_{m=1}^\infty \gamma_{m0}^R(\epsilon) \alpha_s^m(Q^2).
\] (4.51)

Whereas \( \gamma \left( N, Q^2/\mu^2, \alpha_s, \epsilon \right) \) is finite in the limit \( \epsilon \to 0 \), where it coincides with the four-dimensional physical anomalous dimension of Eq. (2.3), \( \gamma^{(l)} \) and \( \gamma^{(c)} \) are not necessarily separately finite as \( \epsilon \to 0 \). However, Eq. (1.49) implies that \( \gamma^{(l)} \) and \( \gamma^{(c)} \) can be made finite by adding and subtracting to them a suitable counterterm \( Z^{(\gamma)}(\alpha_s(Q^2), \epsilon) \):

\[
\gamma^{(l)} \left( \alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon \right) = \gamma^{(l)} \left( \alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon \right) - Z^{(\gamma)} \left( \alpha_s(Q^2), \epsilon \right),
\] (4.52)

\[
\gamma^{(c)} \left( \alpha_s(Q^2), \epsilon \right) = \gamma^{(c)} \left( \alpha_s(Q^2), \epsilon \right) + Z^{(\gamma)} \left( \alpha_s(Q^2), \epsilon \right).
\] (4.53)
where the functions $\gamma^{(l)}$ and $\gamma^{(c)}$ have a finite $\epsilon \to 0$ limit. The counterterm cannot depend on $N$ because of the $N$-independence of $\hat{\gamma}^{(l)}$. A possible choice is therefore

$$Z^{(\gamma)}(\alpha_s(Q^2), \epsilon) \equiv \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon). \quad (4.54)$$

With this choice, $\gamma^{(l)}$ is clearly finite for $N = 1$, where it vanishes; because of the $N$-independence of the counterterm, $\gamma^{(l)}$ will then be finite for any $N$. Other forms of the counterterm can differ from this only in the choice of the finite part, i.e., can be obtained by adding to $Z^{(\gamma)}$ Eq. (4.54) a finite function of $\alpha_s(Q^2)$. Therefore, they correspond to the possibility of reshuffling finite $N$–independent terms between $\gamma^{(l)}$ and $\gamma^{(c)}$. The choice Eq. (4.54) is characterized by the fact that with this choice $\gamma^{(l)}$ is ‘purely logarithmic’, in that it vanishes at $N = 1$ where $\ln N = 0$. We thus get

$$\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^n), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon) - \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon), \quad (4.55)$$

$$\gamma^{(c)}(\alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon), \quad (4.56)$$

and the physical anomalous dimension is

$$\gamma(N, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^n), \epsilon) + \gamma^{(c)}(\alpha_s(Q^2), \epsilon) + O(1/N)$$

$$= \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^n), \epsilon) + O(N^0), \quad (4.57)$$

where now both $\gamma^{(l)}$ and $\gamma^{(c)}$ are finite as $\epsilon \to 0$, and $\gamma^{(l)}$ provides an expression of the resummed physical anomalous dimension in the large $N$ limit, up to non-logarithmic terms.

It is apparent from Eq. (4.50) and the definition Eq. (4.51) that $\gamma^{(c)}$ is a power series in $\alpha_s(Q^2)$ with finite coefficients. In order to understand the perturbative structure of $\gamma^{(l)}$ as well, we notice that

$$\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^n), \epsilon) = \int_1^{N^n} \frac{d\alpha_s(Q^n)}{n} g(\alpha_s(Q^2), \alpha_s(Q^2/n), \epsilon) \quad (4.58)$$

where

$$g(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) = -\mu^2 \frac{\partial}{\partial \mu^2} \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon)$$

$$= -\beta^{(d)}(\alpha_s(\mu^2), \epsilon) \frac{\partial}{\partial \alpha_s(\mu^2)} \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon), \quad (4.59)$$

where $\beta^{(d)}(\alpha_s)$ is the $d$-dimensional beta function of Eq. (4.43). It immediately follows from Eqs. (4.55, 4.59) that $g$ is a power series in $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$:

$$g(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} g_{mn}(\epsilon) \alpha_s^m(Q^2) \alpha_s^n(\mu^2). \quad (4.60)$$

The sum over powers of $\alpha_s(\mu^2)$ in Eq. (4.60) starts at $n = 1$ because the expansion of $\gamma^{(l)}$ contains at least one power of $\alpha_s(Q^2/N)$. The perturbative structure of $\gamma^{(l)}$ is thus dictated by
Eqs. (4.58,4.60). Because of Eq. (4.57), this provides us with the desired all-order generalization of the next-to-leading log resummation formula Eq. (2.23) discussed in Sect. 2.

Our final result for the four-dimensional resummed physical anomalous dimension is thus

\[ \gamma_{\text{res}}(N, \alpha_s(k^2)) = \int_1^N \frac{dn}{n} g(\alpha_s(k^2), \alpha_s(k^2/n)) + O(N^0) \]  

(4.61)

Exploiting the all-order relation between leading $\ln \frac{1}{N}$ and leading $\ln(1-x)$ resummation derived in Sect. 3, Eqs. (3.19,3.20), this result can be equivalently cast as the Mellin transform of a resummed splitting function $P_{\text{res}}(x, \alpha_s(k^2))$:

\[ \gamma_{\text{res}}(N, \alpha_s(k^2)) = \int_0^1 dx x^{N-1} P_{\text{res}}(x, \alpha_s(k^2)) + O(N^0) \]  

(4.63)

These resummations generalize to all orders the next-to-leading log results Eq. (2.23) and Eq. (2.11), respectively. The two forms Eq. (4.61) and (4.63) of the resummation are equivalent up to non-logarithmic terms, and can be used to compute the resummed evolution factor Eq. (2.6) $K_{\text{res}}(N; Q_0^2, Q^2)$:

\[ K(N; Q_0^2, Q^2) = \exp \int_{Q_0^2}^Q \frac{dk^2}{k^2} \gamma_{\text{res}}(N, \alpha_s(k^2)). \]  

(4.66)

Resummed expressions for the coefficient function and anomalous dimension in any factorization scheme can be obtained from Eq. (4.66), as we will discuss in Sect 6.

4.3 Next-to-leading resummation

Let us now study our all-order generalized resummation at the leading and next-to-leading level. This will also help us to understand the relation of our resummed result to the stronger one proven in Ref. [8]. The lowest-order term in Eq. (4.60), used in the expression of the resummed anomalous dimension Eqs. (4.57-4.58), leads to the leading log resummation:

\[ \gamma_{1}^{(l)} = -g_{01} \int_{\alpha_s(Q^2)}^{\alpha_s(Q^2/N^a)} \frac{d\alpha}{\beta_0^{(d)}(\alpha, \epsilon)} \alpha \]  

\[ = \frac{g_{01}}{\beta_0} \ln \frac{\beta_0 \alpha_s(Q^2/N^a) + \epsilon}{\beta_0 \alpha_s(Q^2) + \epsilon} + O(\alpha_s(Q^2/N)), \]  

(4.67)

which reduces to the leading-log result Eq. (2.28) as $\epsilon \rightarrow 0$.  

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Recalling that \( \hat{\gamma}^{(l)} \) Eq. (4.55) is independent of \( \alpha_s(Q^2) \) and is free of constant (i.e. \( \alpha_s \)-independent) terms, Equation (4.67) uniquely determines the leading log contribution to \( \hat{\gamma}^{(l)} \):

\[
\hat{\gamma}_1^{(l)} = \frac{g_{01}}{\beta_0} \ln \left( 1 + \frac{\beta_0 \alpha_s(Q^2/N^a)}{\epsilon} \right), \tag{4.68}
\]

This shows explicitly that \( \hat{\gamma}^{(l)} \) is an analytic function of \( \alpha_s(Q^2/N) \), but does not have a finite \( \epsilon \to 0 \) limit. By contrast, \( \gamma_1^{(l)} \) Eq. (4.67) does have a finite \( \epsilon \to 0 \) limit, but the limit is no longer an analytic function of both \( \alpha_s(Q^2/N) \) and \( \alpha_s(Q^2) \).

Subleading resummed terms are found by integrating according to Eq. (4.58) higher order contributions in the expansion Eq. (4.60), and consistently including higher-order contributions to the beta function. In general, the term \( g_{ij} \) will generate a next\(^{i+j-1}\)-to-leading order resummation (when the beta function is included at leading order). The NLO resummation of Eq. (2.23) is recovered if the only subleading contribution is the \( \hat{\gamma}^{(0)} \) term in the expansion Eq. (4.60), i.e., if \( g_{11} = 0 \).

As discussed in the introduction, an all-order generalization of the resummation Eq. (2.23) was derived in Ref. [8]. This resummation has the form of Eqs. (4.57-4.58), but with \( g \) a function of \( \alpha_s(\mu^2) \) only, i.e., with all \( g_{mn} = 0 \) when \( m > 0 \) in Eq. (4.60). In such case, knowledge of the first \( k \) coefficients \( g_{0n} \), \( 1 \leq n \leq k \) fully determines the next\(^{k-1}\)-to-leading log resummation. These coefficients, in turn, can be determined by performing a next\(^{k-1}\)-to-leading fixed-order computation. The more general resummation of Eq. (4.60) is less predictive. Indeed, if all coefficients \( g_{mn} \) are nonzero, knowledge of the \( k(k+1)/2 \) coefficients with \( m+n \leq k \) is necessary in order to determine the next\(^{k-1}\)-to-leading log resummation. In turn, this requires a next\(^{k-1}\)-to-leading fixed-order computation. So, while the leading-log resummation is determined by the \( O(\alpha_s) \) result, an \( O(\alpha_s^3) \) computation is required in order to determine both coefficients \( g_{20} \) and \( g_{11} \) which control the next-to-leading log resummation, and so forth.

The conditions under which the more restrictive result of Ref. [8] holds can be understood by comparing to our approach the derivation of that result. The approach of Ref. [8] is based on assuming the validity of a factorization formula which is more restrictive than the standard collinear factorization Eq. (2.22). This factorization was proven for a wide class of processes in Ref. [8], and it implies that the perturbative coefficient Eq. (2.2) in the large-\( N \) limit can be factored as:

\[
C(N, Q^2/\mu^2, \alpha_s(\mu^2)) = C^{(l)}(Q^2/(\mu^2 N^a), \alpha_s(\mu^2)) C^{(c)}(Q^2/\mu^2, \alpha_s(\mu^2)). \tag{4.69}
\]

or equivalently

\[
C^{(0)}(N, Q^2, \alpha_0) = C^{(0,l)}(Q^2/N^a, \alpha_0, \epsilon) C^{(0,c)}(Q^2, \alpha_0, \epsilon). \tag{4.70}
\]

Then, the physical anomalous dimension Eq. (4.40) manifestly becomes

\[
\gamma(N, Q^2/\mu^2, \alpha_s(\mu^2), \epsilon) = \gamma^{(l)}(Q^2/(N^a \mu^2), \alpha_s(\mu^2), \epsilon) + \gamma^{(c)}(Q^2/\mu^2, \alpha_s(\mu^2), \epsilon), \tag{4.71}
\]

i.e., \( \gamma^{(l)} \) depends on \( \alpha_s(Q^2/N^a) \) only. Proceeding as above, one then ends up with the resummation Eqs. (1.57-4.58), but with \( g = g(\alpha_s(\mu^2), \epsilon) \), or, equivalently, all \( g_{mn} = 0 \) when \( m > 0 \), as advertised.
By running this argument in reverse, the factorization Eq. (4.69) is thus seen to be a necessary and sufficient condition for the validity of the more restrictive factorization formula of Ref. [8], where $g = g(\alpha_s(\mu^2))$. This factorization is in general rather nontrivial. Indeed, recalling the expansion Eqs. (4.2, 4.3), Eq. (4.70) is seen to be satisfied if and only if in the large-$N$ limit the coefficients $C^{(0)}_n(\epsilon)$ can be written as

$$C^{(0)}_n(\epsilon) = \sum_{k=0}^{n} F_k(\epsilon) G_{n-k}(\epsilon) (Q^2)^{-n-k}\epsilon \left(\frac{Q^2}{N^a}\right)^{-k\epsilon}.$$  \hspace{1cm} (4.72)

Whereas an all-order proof of Eq. (4.72) is highly nontrivial, the structure of our more general, but weaker resummation Eqs. (4.57-4.58) implies that a fixed order proof of the factorization is in fact sufficient for the corresponding resummed result to hold.

Consider the simplest nontrivial case, namely, the next-to-leading log resummation: this is determined by knowledge of the coefficients $g_{10}$ and $g_{11}$. The stronger resummation holds when $g_{11} = 0$. But in order to check whether this is the case or not it is sufficient to verify that the desired factorization Eq. (4.72) holds to lowest nontrivial order, namely that

$$C^{(0)}_2(N, Q^2, \epsilon) = G_2(\epsilon)(Q^2)^{-2\epsilon} + F_1(\epsilon)G_1(\epsilon)(Q^2)^{-\epsilon} \left(\frac{Q^2}{N^a}\right)^{-\epsilon} + F_2(\epsilon) \left(\frac{Q^2}{N^a}\right)^{-2\epsilon} + O(1/N).$$  \hspace{1cm} (4.73)

Equivalently, we can compute the coefficients $g_{01}$, $g_{20}$ and $g_{11}$ which determine the next-to-leading log resummation by comparison with fixed-order results: in order to calculate three coefficients, we need a next-to-next-to leading fixed order computation. Indeed, expanding Eqs. (4.61) in powers of $\alpha_s(Q^2)$ to order $\alpha^3_s$ we get

$$\gamma(N, \alpha_s(Q^2)) = -g_{01}\alpha_s(Q^2) \ln \frac{1}{N} + \left[\frac{\beta_0g_{01}}{2} \ln^2 \frac{1}{N} - (g_{11} + g_{02}) \ln \frac{1}{N}\right] \alpha^2_s(Q^2)$$

$$- \frac{\beta_0^2g_{01}}{3} \ln^3 \frac{1}{N} - \left(\frac{\beta_0g_{11}}{2} + \beta_0g_{02} + \frac{\beta_1g_{01}}{2}\right) \ln^2 \frac{1}{N} + \left(\frac{\beta_1g_{02}}{\beta_0} - \frac{\beta_2g_{01}}{\beta_0^2}\right) \ln \frac{1}{N}\right] \alpha^3_s(Q^2)$$

$$+ O(\alpha^4_s) + O(N^0).$$

Hence, after having determined $g_{01}$ from the leading log result, the $O(\alpha^2_s)$, $O(\ln N)$ term only determines the combination $g_{11} + g_{02}$, but the $O(\alpha^3_s)$, $O(\ln^2 N)$ provides an independent linear combination.

The physical anomalous dimension is in turn determined by the coefficient function $C$ Eq. (2.2) and the standard anomalous dimension $\gamma^{AP}$ Eq. (2.4) according to Eq. (2.3), which through order $\alpha^3_s$ gives

$$\gamma(N, \alpha_s(Q^2)) = \gamma^{AP}_1 \alpha_s(Q^2) + (\gamma^{AP}_2 - \beta_0 C_1) \alpha^2_s(Q^2) + (\gamma^{AP}_3 - 2\beta_0 C_2 + \beta_0 C_1^2 - \beta_1 C_1) \alpha^3_s(Q^2) + O(\alpha^4_s),$$  \hspace{1cm} (4.75)

where $\gamma^{AP}_i$ and $C_i$ are the order-$\alpha^i_s$ coefficients in the expansion of $\gamma^{AP}$ and $C$ respectively.

The coefficients $C$ are known up to $O(\alpha^3_s)$ [15] for Drell-Yan [13] and deep-inelastic [14] processes, while $\gamma^{AP}$ is known (in the MS scheme) up to $O(\alpha^2_s)$ [17], and has been very recently
determined to $O(\alpha_s^3)$ in the large $N$ limit \[17\]. However, knowledge of the $O(\alpha_s^2)$ anomalous dimension is in fact sufficient to determine $g_{11}$, thanks to the fact that in the $\overline{\text{MS}}$ scheme the anomalous dimension is linear in $\ln N$ to all orders \[16\], so its knowledge to $O(\alpha_s^2)$ is sufficient to establish the vanishing of $g_{11}$.

To see how this works, consider explicitly the case of the deep-inelastic structure function $F_2$. The coefficients in the $\overline{\text{MS}}$ scheme are

$$C_1^q = \frac{C_F}{4\pi} \left[ 4I_1 - 3I_0 - 4\zeta(2) - 9 \right] + O(1/N) \quad (4.76)$$

$$C_2^q = \frac{1}{8\pi^2} \left\{ C_F^2 \left[ 4I_3 - 9I_2 - \left( 16\zeta(2) + \frac{27}{2} \right) I_1 + \left( \frac{51}{4} + 18\zeta(2) - 4\zeta(3) \right) I_0 \right] 
+ C_A C_F \left[ -\frac{11}{3} I_2 + \left( \frac{367}{18} - 4\zeta(2) \right) I_1 + \left( -\frac{3155}{108} + \frac{22}{3} \zeta(2) + 20\zeta(3) \right) I_0 \right] 
+ n_f C_F \left[ \frac{2}{3} I_2 - \frac{29}{9} I_1 + \left( \frac{247}{54} - \frac{4}{3} \zeta(2) \right) I_0 \right] \right\} + O(N^0), \quad (4.77)$$

where $\zeta$ is the Riemann $\zeta$ function, $C_F$ and $C_A$ are the usual quadratic Casimir operators, and $I_p$ have been defined in Eq. (3.2) and, as proven in Sect. 3, contain the logarithmic $N$ dependence. Note that $O(N^0)$ terms have been included in Eq. (4.76) in view of the fact that $C_1^q$ appears in Eq. (4.75). The gluon coefficient to next-to-leading order is $O(1/N)$. The $\overline{\text{MS}}$ anomalous dimension in turn is

$$\gamma_{qq}^{AP} = \frac{C_F}{\pi} I_0 + O(N^0) \quad (4.78)$$

$$\gamma_{2qq}^{AP} = \frac{1}{4\pi^2} \left[ -\frac{10}{9} n_f C_F + \left( \frac{67}{9} - 2\zeta(2) \right) C_A C_F \right] I_0 + O(N^0), \quad (4.79)$$

while $\gamma_{qq}$ and $\gamma_{qq}$ are $O(1/N)$. The gluon anomalous dimension $\gamma_{qq}$ does contain soft logs, but its contribution to $F_2$ only goes through via interference with $C_g$, and it is thus power suppressed. Therefore, we see that through next-to-leading order there is no operator mixing, and only the quark sector contributes to the large $N$ limit. Substituting in Eq. (4.75), and solving for $g_{11}$ and $g_{02}$ we get, after tedious but straightforward algebra

$$g_{01} = -\frac{C_F}{\pi} \quad (4.80)$$

$$g_{02} = -\frac{1}{144\pi^2} \left[ (24\gamma_E + 58) C_F n_f + (12\pi^2 - 132\gamma_E - 367) C_A C_F \right] \quad (4.81)$$

$$g_{11} = 0. \quad (4.82)$$

In Drell-Yan there is likewise no mixing, and the check that $g_{11} = 0$, performed in Ref. \[18\], follows the same lines.

Hence, by combining available fixed-order results with our general resummation Eq. (4.61), we reproduce the next-to-leading log resummation Eq. (2.23). Beyond next-to-leading order the general result Eq. (4.61) holds, and will have to be used for the full matrix of two-by-two resummed physical anomalous dimensions.
5 Momentum-space resummation

We have seen that resummed results can be compactly expressed in terms of the physical anomalous dimension Eq. (1.61), or in terms of a resummed splitting function Eq. (1.64) obtained from it by inverse Mellin transformation, up to non-logarithmic terms. In the former case, the resummation is cast in Mellin $N$ space, and one resums large $\ln \frac{1}{k}$ (at large $N \to \infty$). In the latter case, the splitting function is defined in momentum space, i.e., as a function of the momentum fraction $x$ and one resums large $\ln(1-x)$ (at large $x \to 1$). In this section we address the question whether it is possible to construct a resummation of $\ln(1-x)$ directly at the level of the momentum-space cross section $\sigma(x, Q^2)$. Indeed, even when the resummation is performed at the level of splitting functions, the cross section is written as the inverse Mellin transform of the exponentiated result Eq. (2.11). It would seem more natural if the momentum-space cross section $\sigma(x, Q^2)$ could be written directly as the exponential of a suitable leading, next-to-leading,... $\ln(1-x)$ function.

A general way of performing the inverse Mellin transform of a leading-, next-to-leading,..., $\ln \frac{1}{N}$ function up to terms which are suppressed by powers of $1-x$ was given in Sect. 3 Eq. (3.23). Using this result, it is easy to rewrite the $N$-space resummation factor Eq. (1.60) as the Mellin transform of an $x$ space resummed expression:

\[
K^{\text{res}}(N; Q_0^2, Q^2) = \exp \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{(l)}(\alpha_s(k^2), \alpha_s(k^2/N^a)) + O(N^0) \tag{5.1}
\]

\[
= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^1 dx \frac{d^n}{d \ln^n(1-x)} \left( \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{(l)}(\alpha_s(k^2), \alpha_s(k^2(1-x)^a)) \right)^k + O(N^0),
\]

where $\gamma^{(l)}$ is the resummed physical anomalous dimension Eq. (1.58), and, even though the resummation factor $K^{\text{res}}$ is defined up to non-logarithmic terms, we have normalized it by the requirement that $K^{\text{res}} \to 1$ as $\alpha_s \to 0$. This immediately leads to the identification of the momentum-space resummation factor:

\[
K^{\text{res}}(x; Q_0^2, Q^2) = \delta(1-x) - \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=1}^{\infty} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \left[ \frac{d^n}{d x d \ln^n(1-x)} \left( \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{(l)}(\alpha_s(k^2), \alpha_s(k^2(1-x)^a)) \right)^k \right]_+. \tag{5.2}
\]

We can now derive a closed form of the sum of the exponential expansion in Eq. (5.2), using a representation of the $+$ distribution given in Ref. [7]:

\[
K^{\text{res}}(x; Q_0^2, Q^2) = - \lim_{\eta \to 0^+} \sum_{n=1}^{\infty} \frac{\Delta^{(n-1)}(1)}{(n-1)!} \left[ \frac{d^n}{d \ln^n(1-x)} \left( \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{(l)}(\alpha_s(k^2), \alpha_s(k^2(1-x)^a)) \right) \right]_+ \tag{5.3}
\]
where $\theta$ is the Heaviside function. The equality holds in the sense of distributions: it is proven by folding $K^{\text{res}}$ with a test function, integrating by parts, and noting that $\theta(-\eta) = 0$ and $\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2)) = 0$ because of Eq. (4.55). It is also convenient to define a momentum-space resummation exponent $E^{\text{res}}(x; Q_0^2, Q^2)$ through the implicit equation

$$K^{\text{res}}(x; Q_0^2, Q^2) = -\lim_{\eta \to 0^+} \frac{d}{dx} \left[ \theta(1 - x - \eta) \exp E^{\text{res}}(x; Q_0^2, Q^2) \right]. \quad (5.4)$$

Terms with $n = 1, 2, \ldots$ in Eqs. (5.2, 5.3) generate leading, next-to-leading, $\ldots$ $\ln(1 - x)$ contributions to the resummation factor. Hence, if $\gamma^{(l)}$ is computed at the leading log level Eq. (2.28), only the $n = 1$ terms should be retained,

$$K^{\text{LL}}(x; Q_0^2, Q^2) = -\lim_{\eta \to 0^+} \frac{d}{dx} \left[ \theta(1 - x - \eta) \exp \left( -\int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \frac{g_1}{\beta_0} \ln \left( 1 + \beta_0 \alpha_s(k^2) \ln(1 - x)^a \right) \right) \right]$$

$$= -\lim_{\eta \to 0^+} \frac{d}{dx} \left[ \theta(1 - x - \eta) \exp E^{\text{LL}}(x; Q_0^2, Q^2) \right], \quad (5.5)$$

where

$$E^{\text{LL}}(x; Q_0^2, Q^2) = -\frac{g_1}{\beta_0} \ln(1 - x)^a \left[ \frac{1 + \alpha_s \beta_0 \ln(1 - x)^a}{\alpha_s \beta_0 \ln(1 - x)^a} \right] \alpha_s(Q^2). \quad (5.6)$$

At next-to-leading and higher orders, the action of subsequent terms in the series of derivatives Eq. (5.3) should be accordingly truncated so that the resummation exponent $E^{\text{res}}$ is computed to the given order.

We would thus be led to conclude that Eq. (5.4), and its obvious subleading generalizations, provide us with the desired form of the leading-, next-to-leading-, $\ldots$, $\ln(1 - x)$ resummation factor, which can then be used to construct the corresponding resummed momentum-space cross sections. Surprisingly, however, it turns out that the formal expression Eq. (5.5) is actually ill-defined: the corresponding resummed cross section diverges. This difficulty was first pointed out in the fixed coupling limit in Ref. [7]. Also surprisingly, this divergence is a consequence of the leading- (next-to-leading, $\ldots$) $\ln(1 - x)$ truncation: it is absent if the Mellin transform of the leading- (next-to-leading, $\ldots$) $\ln \frac{x}{Q}$ result is performed up to power suppressed terms, $i.e.,$ by retaining all terms in the sum over $n$ Eq. (5.3).

To see this divergence, recall that the resummed evolution factor $K^{\text{res}}$ according to Eq. (2.6) satisfies

$$\sigma(z, Q^2) = \int_{z}^{1} \frac{dx}{x} K^{\text{res}}(x; Q_0^2, Q^2) \sigma \left( \frac{z}{x}, Q_0^2 \right) \equiv \int_{0}^{1} dx K^{\text{res}}(x; Q_0^2, Q^2) \tau(x) \quad (5.7)$$

where

$$\tau(x) = \frac{1}{x} \sigma \left( \frac{z}{x}, Q_0^2 \right) \theta(x - z). \quad (5.8)$$

Integrating by parts and noting that $\tau(0) = 0$, we get

$$\sigma(z, Q^2) = \int_{0}^{1} dx \frac{d\tau(x)}{dx} \exp E^{\text{res}}(x; Q_0^2, Q^2). \quad (5.9)$$
Specializing to the leading log resummation Eq. (5.5)
\[
\exp E^{LL}(x; Q_0^2, Q^2) = \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{G_0 \ln(1-x)} \exp \left[ G_1 \left( \alpha_s(Q^2) - \alpha_s(Q_0^2) \right) \ln^2(1-x)^a \right] \exp O(\alpha_s^2),
\]
where we have expanded the leading log exponent Eq. (5.9) in powers of \(\alpha_s\), and \(O(\alpha_s^2)\) denotes the next-to-next-to leading term in this expansion, which is in fact of order \((\alpha_s(Q^2) - \alpha_s(Q_0^2)) \ln^3(1-x)^a = \alpha_s(Q^2) \ln^3(1-x)^a [1 + O(\alpha_s(Q^2))]\). Explicitly, the coefficients of the first two orders of this expansion are \(G_0 = g_1/\beta_0\) and \(G_1 = -g_1/2\).

Now, as \(x \to 1\), the \(O(\alpha_s)\) contribution to \(E^{LL}\) diverges as \(\ln^2(1-x)\), so \(\exp E^{LL}\) diverges faster than any power of \(1-x\). However, \(\tau(x)\) is manifestly regular as \(x \to 1\), so the integral Eq. (5.9) does not exist. In fact, matching with the leading-order anomalous dimension shows that \(g_1 = -C_F/\pi < 0\), so the integral diverges for \(Q^2 < Q_0^2\); however, regardless of the sign of \(g_1\) the presence of this divergence means that the perturbative expansion of \(\sigma\) in powers of \(\alpha_s\) has vanishing radius of convergence.

This can also be seen explicitly by using the leading log \(K^{LL}\) Eq. (5.3) in Eq. (5.7):
\[
\sigma(z, Q^2) = \int_0^1 dx \frac{d\tau(x)}{dx} \left( \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)} \right)^{G_0 \ln(1-x)^a} \sum_{p=0}^{\infty} \frac{[G_1 \left( \alpha_s(Q^2) - \alpha_s(Q_0^2) \right)]^p}{p!} \ln^{2p}(1-x)^a \times \exp O(\alpha_s^2).
\]

The generic term in the sum over \(p\) is
\[
K_p \approx \left. \frac{d\tau(x)}{dx} \right|_{x=1} \frac{[a^2 G_1 \left( \alpha_s(Q^2) - \alpha_s(Q_0^2) \right)]^p}{p!} C_{2p}
\]
\[
C_{2p} \equiv \int_0^1 dx \ (1-x)^{a G_0 \ln \alpha_s/Q_0^2} \ln^{2p}(1-x) = (2p)! \left( \frac{1}{1 + a G_0 \ln \alpha_s/Q_0^2} \right)^{2p+1}
\]

where in we have Taylor expanded \(\tau(x)\) about \(x = 1\), and neglected terms which are suppressed by powers of \(1-x\), consistent with the fact that the resummed distribution \(K_{res}\) is determined in the same approximation. The factorial growth of \(C_p\) Eq. (5.12) implies that the sum over \(p\) in Eq. (5.11) diverges factorially.

Higher order contributions in the expansion of \(K^{LL}\) in powers of \(\alpha_s\) will make this divergence worse. Indeed, the \(O(\alpha_s^k)\) contribution is proportional to \(\ln^k(1-x)\), so the factorial divergence of the corresponding expansion is more severe. Furthermore, the expansion of \(K^{LL}\) amounts essentially to the expansion of \(\ln(1 + \alpha_s \beta_0 \ln(1-x)^a)\) in powers of \(\alpha_s\), which, since \(x < 1\) is a series of terms with the same sign, so these divergences add up.

In fact, the leading log exponent Eq. (5.4) blows up when \(\ln(1-x)^a = -\frac{1}{\alpha_s \beta_0}\). This divergence corresponds to the Landau pole in the coupling constant \(\alpha_s(k^2(1-x)^a)\) which appears
in the $x$–space resummation factor Eq. (5.2). Because of the Landau pole, the expansion of the 
resummed exponent $E$ in powers of $\alpha_s$ has a finite radius of convergence. However, as first 
clarified in Ref. [7], the factorial divergence Eq. (5.11) is entirely unrelated to the Landau pole. Indeed, 
we have seen that it is present even if we retain only one term in the expansion of the 
resummed exponent: rather, this divergence plagues the series expansion of the exponential, 
when integrated term by term.

Now, we exploit the power of our all-order relation between leading $\ln(1-x)$ and leading $\ln \frac{1}{N}$ 
series to show that this factorial divergence is just a byproduct of the truncation to leading (or 
next-to-leading, etc.) level of the full expression Eq. (5.2) of the $x$–space resummation factor.

To this purpose, let us again concentrate on the leading log expression, and separate off the 
first few terms in the expansion of the resummed exponent in powers of $\alpha_s$: retaining all terms 
in the sum over $n$ Eq. (5.2), we get

$$\sigma(z, Q^2) = \int_0^1 \frac{dz}{dz} \left[ \sum_{n=1}^{\infty} \frac{\Delta(n-1)}{(n-1)!} d\ln^{n-1}(1-x) \right]$$

$$\left[ \sum_{p=0}^{\infty} \frac{G_0 \ln \frac{\alpha_s(Q^2)}{(\alpha_s(Q_0^2))} \ln(1-x)^p}{p!} \right]$$

$$+ \left( \sum_{q=0}^{\infty} \frac{[G_1(\alpha_s(Q^2) - \alpha_s(Q_0^2))]^q}{q!} \right) \ln^{2q}(1-x)^a$$

(5.13)

$$\times \left( \sum_{r=0}^{\infty} \frac{G_2(\alpha_s^2(Q^2) - \alpha_s^2(Q_0^2))^r}{r!} \ln^{3r}(1-x)^a \right) \exp(O(\alpha_s^3))$$

$$= \int_0^1 \frac{dz}{dz} \left[ \exp(O(\alpha_s^3)) \left( \sum_{r=0}^{\infty} \frac{G_2(\alpha_s^2(Q^2) - \alpha_s^2(Q_0^2))^r}{r!} a^{3r} \frac{d^{3r}}{d\eta^{3r}} \right) \right.$$

$$\times \left( \sum_{q=0}^{\infty} \frac{[G_1(\alpha_s(Q^2) - \alpha_s(Q_0^2))]^q}{q!} \frac{d^q}{d\eta^q} \right) \right] \frac{(1-x)^\eta}{\Gamma(1+\eta)} \bigg|_{\eta=0},$$

where in the last step we have used the definition Eq. (3.13) of $\Delta(n)$, and the limit $\eta \to 0$ should 
be taken after performing all derivatives.

Now, it is easy to compute the inner “leading order” sum (index $p$): $\exp \frac{d}{d\eta}$ acts as a finite 
translation operator. We get

$$\sigma(z, Q^2) = \int_0^1 \frac{dz}{dz} \left[ \exp(O(\alpha_s^3)) \left( \sum_{r=0}^{\infty} \frac{G_2(\alpha_s^2(Q^2) - \alpha_s^2(Q_0^2))^r}{r!} a^{3r} \frac{d^{3r}}{d\eta^{3r}} \right) \right.$$

$$\times \left( \sum_{q=0}^{\infty} \frac{[G_1(\alpha_s(Q^2) - \alpha_s(Q_0^2))]^q}{q!} \frac{d^q}{d\eta^q} \right) \right] \frac{(1-x)^\eta+aG_0 \ln \frac{\alpha_s(Q^2)}{(\alpha_s(Q_0^2))}}{\Gamma(1+\eta+aG_0 \ln \frac{\alpha_s(Q^2)}{(\alpha_s(Q_0^2))})} \bigg|_{\eta=0}$$

(5.14)

$$\left( \sum_{r=0}^{\infty} \frac{G_2(\alpha_s^2(Q^2) - \alpha_s^2(Q_0^2))^r}{r!} a^{3r} \frac{d^{3r}}{d\eta^{3r}} \right) \right] \frac{(1-x)^\eta}{\Gamma(1+\eta+aG_0 \ln \frac{\alpha_s(Q^2)}{(\alpha_s(Q_0^2))})} \bigg|_{\eta=0}$$

(5.15)

We see therefore that, due to the inclusion of subleading corrections of the sum Eq. (5.2), the 
generic term in the sum over $q$ is now

$$K_q \approx \frac{d\tau(x)}{dx} \bigg|_{x=1} \frac{[a^2 G_1(\alpha_s(Q^2) - \alpha_s(Q_0^2))]^q}{q!} C_{2q}$$

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\[ C_{2q} = \frac{d^2 q}{d\eta^2} \frac{1}{\Gamma(2 + \eta + aG_0 \ln \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)})} \bigg|_{\eta=0}, \]  

(5.16)

where we have again Taylor expanded \( \tau(x) \) about \( x = 1 \).

Comparing this with the previous result Eq. (5.12) we see that the power in the denominator is replaced by a Gamma function. This is enough to tame the factorial divergence as we now show. We observe that

\[ \sum_{q=0}^{\infty} \frac{C_q}{q!^z} = \frac{1}{\Gamma(2 + z + A_0 \ln \frac{\alpha_s(Q^2)}{\alpha_s(Q_0^2)})}; \]

(5.17)

hence, \( C_q \) are the coefficients of the Taylor series expansion of an entire function, which, therefore, has infinite radius of convergence (note that if we had used the result Eq. (5.12) we would have found instead a finite radius of convergence). This in turn implies that

\[ \lim_{q \to \infty} \frac{C_{q+1}}{(q+1)!} \frac{q!}{C_q} = \lim_{q \to \infty} \frac{1}{q+1} \frac{C_{q+1}}{C_q} = 0, \]

(5.18)

and consequently

\[ \lim_{q \to \infty} \frac{C_{2q+1}}{(q+1)!} \frac{q!}{C_{2q}} = \lim_{q \to \infty} \frac{1}{q+1} \frac{C_{2q+1}}{C_{2q}} = 0. \]

(5.19)

This means that the coefficients \( C_{2q} \) no longer diverge factorially: in fact, they are also coefficients of the Taylor series expansion of an entire function, with an infinite convergence radius.

Proceeding in the same way, we can then show that the sum over \( r \) in Eq. (5.14), i.e., the expansion of the \( O(\alpha_s^2) \) contribution, is also convergent, and so forth. Therefore, we conclude that order by order in perturbation theory, the resummed result Eq. (5.2) leads to a convergent perturbative expansion, but if this is turned into a leading-, next-to-leading,\ldots \ln(1-x), a meaningless divergent expansion is found.

In summary, we have seen that the leading-, next-to-leading,\ldots \ln(1-x) expressions of the momentum-space cross section \( \sigma(x, Q^2) \) are all ill-defined. However, if the resummation is formulated at the leading-, next-to-leading,\ldots \ln \frac{1}{x} level, and the resummed \( \sigma(x, Q^2) \) is defined as the inverse Mellin of the ensuing result, up to power corrections, then finite results are obtained.

6 Factorization schemes and matching

The resummed results derived in Sect. 4 provide us with a prediction for the resummed scale dependence of the physical cross section, expressed by the physical anomalous dimension \( \gamma \). In practice, we would like to have resummed predictions for the coefficient function \( C \) Eq. (2.3) and standard anomalous dimension \( \gamma^{\text{AP}} \) Eq. (2.3). Because of Eq. (2.4), that relates \( \gamma \) to \( \gamma^{\text{AP}} \) and \( C \), all the relevant information is contained in the physical anomalous dimension, and the resummed anomalous dimension and coefficient function can be determined from it. In order to do this, we must suitably combine resummed and unresummed results, and choose a factorization scheme.

The simplest choice consists of choosing a physical factorization scheme, defined by

\[ \sigma(N, Q^2) = F(N, Q^2) \]

(6.1)
so that
\[ C(N, 1, \alpha_s(Q^2)) = 1; \quad \gamma^{\text{AP}}(N, \alpha_s(Q^2)) = \gamma(N, \alpha_s(Q^2)). \quad (6.2) \]

Since now the physical and standard anomalous dimension coincide, Eq. (1.61) gives us a resummed expression of this anomalous dimension, up to non-logarithmic terms. A full expression of the anomalous dimension, valid both at large and small \( N \), can be obtained by combining these resummed results with standard fixed-order ones, and subtracting the double counting. Specifically, a next-to-leading order and next-to-leading log expression of the physical anomalous dimension is given by

\[ \gamma(N, \alpha_s(k^2)) = \alpha_s(k^2) \gamma^{(0)}(N) + \alpha_s^2(k^2) \gamma^{(1)}(N) + \int_1^{N_a} \frac{dn}{n} \left[ g_{01} \alpha_s(k^2/n) + g_{02} \alpha_s^2(k^2/n) \right] \]

\[ - \left[ -g_{01} \alpha_s(k^2) \ln \frac{1}{N_a} + \alpha_s^2(k^2) \left( g_{01} \beta_0 \frac{\ln \frac{1}{N_a}}{2} - g_{02} \ln \frac{1}{N_a} \right) \right], \quad (6.3) \]

where \( \gamma^{(0)} \) is the standard leading-order anomalous dimension, \( \gamma^{(1)} \) is the two-loop physical anomalous dimension, obtained using Eq. (2.5) from the next-to-leading order anomalous dimension and one-loop coefficient function, and the term in square brackets is the double-counting subtraction of the logarithmic contributions to \( \gamma^{(0)} \) and \( \gamma^{(1)} \).

A different resummed expression can be obtained by using the expression Eq. (4.63) of the resummed physical anomalous dimension in terms of the resummed splitting function Eq. (4.61). In this case, Eq. (6.3) is replaced by

\[ \gamma(N, \alpha_s(k^2)) = \int_0^1 dx \, x^{-1} \left\{ \alpha_s(k^2) P^{(0)}(x) + \alpha_s^2(k^2) P^{(1)}(x) \right. \]

\[ + \left. \left[ \hat{g}_{01} \alpha_s(k^2(1-x)^a) + \hat{g}_{02} \alpha_s^2(k^2(1-x)^a) \right] \right\}_+, \]

\[ - \left[ \hat{g}_{01} \alpha_s(k^2) + \alpha_s^2(k^2) \left( \hat{g}_{02} - \hat{g}_{01} \beta_0 \ln(1-x)^a \right) \right] \right\}_+, \quad (6.4) \]

where \( P^{(i)}(x) \) are defined through

\[ \gamma^{(i)}(N, \alpha_s) = \int_0^1 dx \, x^{-1} P^{(i)}(x), \quad (6.5) \]

and the relation between \( g \) and \( \hat{g} \) was given in Eq. (4.63) (or in Eq. (2.27) to next-to-leading order). The two forms of the resummed anomalous dimension Eq. (6.3) and Eq. (6.4) coincide at order \( \alpha_s^2 \) and at the next-to-leading log level, but differ by next-to-next-to-leading terms, as well as by \( O(1/N) \) terms which are at least \( O(\alpha_s^3) \). Hence, their difference might be used to estimate the impact of the next-to-next-to-leading resummation in a factorization-scheme independent way.

Resummed coefficient function and anomalous dimension in a generic scheme can be obtained by generalizing the conventional factorization scheme change at the resummed level. A scheme change is performed by redefining the parton distribution by a finite function \( Z(N, \alpha_s) \), which
can be viewed as a redefinition of the finite part of the multiplicative renormalization \( Z^{(C)} \) of the hard coefficient, Eq. [4.39]

\[
F'(N, \mu^2) = Z(N, \alpha_s(\mu^2)) F(N, \mu^2).
\]  

(6.6)

Upon the scheme change, the anomalous dimension and coefficient function change by

\[
\ln C'(N, Q/\mu^2, \alpha_s(\mu^2)) = \ln C(N, Q/\mu^2, \alpha_s(\mu^2)) - \ln Z(N, \alpha_s(\mu^2))
\]  

(6.7)

\[
\gamma'^{\text{AP}}(N, \alpha_s(\mu^2)) = \gamma^{\text{AP}}(N, \alpha_s(\mu^2)) + \mu^2 \frac{d}{d\mu^2} \ln Z(N, \alpha_s(\mu^2)).
\]  

(6.8)

A conventional (unresummed) scheme change is performed by computing \( Z \) to fixed order in \( \alpha_s \):

\[
Z^u(N, \alpha_s(\mu^2)) = 1 + Z_1(N)\alpha_s(\mu^2) + Z_2(N)\alpha_s^2(\mu^2) + \ldots.
\]  

(6.9)

The term \( Z_k \) affects the next \( k \)-to-leading and higher order contributions to the anomalous dimension. At the resummed next \( k \)-to-leading log level, the physical anomalous dimension is computed including terms of \( O(\alpha_n^{n+k} \ln^k N) \) to all orders in \( \alpha_s \). Such terms are reshuffled between the coefficient function and the parton distribution by performing a scheme change Eq. (6.6) with

\[
Z^{\text{res}}(N, \alpha_s) = \exp \left[ \frac{1}{\alpha_s} z_0 \left( \alpha_s \ln \frac{1}{N} \right) + z_1 \left( \alpha_s \ln \frac{1}{N} \right) + \alpha_s z_2 \left( \alpha_s \ln \frac{1}{N} \right) + \ldots \right]
\]  

(6.10)

\[
z_k = \sum_{i=2}^{\infty} z_k^{(i)} \left( \alpha_s \ln \frac{1}{N} \right)^i,
\]  

(6.11)

where \( z_k \) affects the next \( k \)-to-leading log and higher contributions to the resummed anomalous dimension \( \gamma^{\text{AP}} \) and coefficient function \( C \).

Even though any resummed scheme change of the form Eq. (6.11) is in principle acceptable (i.e., it can be obtained by redefining the finite part of \( Z^{(C)} \) Eq. (1.39)), in practice it is convenient to consider in particular resummed scheme changes which preserve the structure Eq. (4.61) of the resummed anomalous dimension, i.e. such that after scheme change Eq. (6.8), the anomalous dimension can still be written in the form of Eq. (4.61), and the scheme change only affects the numerical values of the coefficients \( g_{ij} \). Such scheme changes are performed by choosing \( Z \) to be of the form

\[
Z^{\text{res}}(N, \alpha_s(\mu^2)) = \exp \int_1^{N^u} \frac{dn}{n} \int_{\Lambda^2} \frac{dk^2}{k^2} \left( z(\alpha_s(k^2), \alpha_s(k^2/n)) - z(\alpha_s(k^2), \alpha_s(k^2)) \right)
\]  

(6.12)

\[
z(\alpha_s(\mu^2), \alpha_s(\mu^2/n)) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} z_{ij} \alpha_s^i(\mu^2) \alpha_s^j(\mu^2/n); \quad z_{11} = 0,
\]  

(6.13)

where \( \Lambda \) is an arbitrary fixed scale, and the subtraction is necessary in order to ensure that \( \ln Z^{\text{res}} \) be free of linear \( \ln N \) terms, in agreement with Eq. (6.11).

Resummed anomalous dimension and coefficient function in any scheme can in general be obtained from the full physical scheme anomalous dimension, which combines resummed and
unresummed results (such as Eq. (6.3)) by performing the combination of a resummed (6.10) and unresummed (6.9) scheme change,

\[ Z(N, \alpha_s) = Z^u(N, \alpha_s)Z^{\text{res}}(N, \alpha_s). \]  

(6.14)

In practice, in the sequel we will take \( Z^{\text{res}} \) to be of the form Eq. (6.13). Then, \( Z^{\text{res}} \) contains higher order powers of \( \ln N \) but no linear terms in \( \ln N \) (nor, of course, non-logarithmic terms). Furthermore, we will take \( Z^u \) to contain at most a linear term in \( \ln N \), but not higher order powers. In such case, there is no overlap between a resummed next\(k\)-to-leading log scheme change \( Z^{\text{res}} \) and an unresummed next\(k\)-to-leading order scheme change \( Z^u \): the unresummed and resummed scheme changes affect different contributions in an expansion of the anomalous dimension in powers of \( \alpha_s \) and \( \ln N \).

Let us now consider explicitly the commonly used \( \overline{\text{MS}} \) scheme. This scheme is characterized by the fact that \( Z^{(C)} \) Eq. (4.39) is a series of pure poles in \( 1/\bar{\epsilon} \equiv 1/\epsilon - \gamma_E + \log(4\pi) \). It can then be shown [16, 19] that if a resummation of the form of Ref. [8] holds (i.e., of the form Eq. (4.61) with all \( g_{ij} = 0 \) when \( i > 0 \)), the \( \overline{\text{MS}} \) anomalous dimension \( \gamma_{\text{AP}}^{\overline{\text{MS}}} \) has, to all orders in \( \alpha_s \), at most a single \( \ln(1/N^a) \). In order to get from the physical to the \( \overline{\text{MS}} \) scheme one has to perform a suitable combination of the well–known unresummed scheme change which takes to \( \overline{\text{MS}} \) from the so–called DIS scheme [20], and a resummed scheme change. This latter resummed scheme change is in fact entirely determined by the requirement that the \( \overline{\text{MS}} \) anomalous dimension be linear in \( \ln(1/N^a) \).

To see this, assume that the resummed anomalous dimension in the physical scheme is given by

\[ \gamma_{\text{AP},\text{res}}^{\overline{\text{MS}}}(N, \alpha_s(\mu^2)) = \int_1^{N^a} \frac{dn}{n} \sum_{j=1}^{\infty} g_{0j} \alpha_s^j(\mu^2/n), \]  

(6.15)

and perform a resummed scheme change Eqs. (6.12-6.13), with \( z_{ij} = 0 \) for \( i > 0 \). After such a scheme change,

\[ \gamma_{\text{AP},\text{res}}^{\overline{\text{MS}}}(N, \alpha_s(\mu^2)) = \int_1^{N^a} \frac{dn}{n} \sum_{j=1}^{\infty} g_{0j} \alpha_s^j(\mu^2/n) + \int_1^{N^a} \frac{dn}{n} \sum_{j=1}^{\infty} z_{0j} \left[ \alpha_s^j(\mu^2/n) - \alpha_s^j(\mu^2) \right] \]  

\[ = \int_1^{N^a} \frac{dn}{n} \sum_{j=1}^{\infty} (g_{0j} + z_{0j}) \alpha_s^j(\mu^2/n) + \ln \frac{1}{N^a} \sum_{j=1}^{\infty} z_{0j} \alpha_s^j(\mu^2). \]  

(6.16)

Hence, imposing that \( \gamma_{\text{AP},\text{res}}^{\overline{\text{MS}}} \) be linear in \( \ln(1/N^a) \) fixes

\[ Z^{\text{res}} = -\sum_{j=1}^{\infty} g_{0j} \alpha_s^j(k^2/n). \]  

(6.17)

Performing the scheme change Eq. (6.17) one gets

\[ \gamma_{\text{AP},\text{res}}^{\overline{\text{MS}}}(N, \alpha_s(\mu^2)) = -\ln \frac{1}{N^a} \sum_{j=1}^{\infty} g_{0j} \alpha_s^j(\mu^2), \]  

(6.18)

\[ C_{\text{res}}^{\overline{\text{MS}}}(N, 1, \alpha_s(Q^2)) = \exp \sum_{j=1}^{\infty} g_{0j} \int_1^{N^a} \frac{dn}{n} \int_{Q^2}^{Q^2} \frac{dk^2}{k^2} \alpha_s^j(k^2/n). \]  

(6.19)
The resummed $\overline{\text{MS}}$ anomalous dimension and coefficient function can now be obtained by transforming those given in Eqs. (6.13,19) with an unresummed scheme change obtained by using $Z_{\overline{\text{MS}}}^u$ of Ref. [4], but omitting all contributions to $Z_{\overline{\text{MS}}}^u$ which are proportional to $\ln^m(1/N^a)$ with $m > 1$, so that the requirement on $Z_{\overline{\text{MS}}}^u$ discussed after Eq. (6.14) is satisfied.

In practice, however, it is easier to observe that the requirement that the $\overline{\text{MS}}$ anomalous dimension be free of higher order powers of $\ln N$ simply means that the $\overline{\text{MS}}$ resummed anomalous dimension coincides with the unresummed one. This further implies that the $\overline{\text{MS}}$ resummed coefficient function is found combining the resummed result Eq. (6.19) with the standard unresummed coefficient function, and subtracting double counting. At next-to-leading order, next-to-leading log we get

$$C_{\overline{\text{MS}}}(N, 1, \alpha_s(Q^2)) = \exp \left[ \int_1^N \frac{dn}{n} \int_{Q^2n}^{Q^2} \frac{dk^2}{k^2} \left( g_{01}\alpha_s(k^2/n) + g_{02}\alpha_s^2(k^2/n) \right) \right]$$

$$\times \left[ 1 + \alpha_s(Q^2) \left( C_1(N, 1) + \frac{g_{01}}{2} \ln^2 \frac{1}{N^a} \right) + O(\alpha_s^{k+2}\ln^k N) + O(\alpha_s^2) \right]$$

$$= \exp \left[ \int_1^N \frac{dn}{n} \int_{Q^2n}^{Q^2} \frac{dk^2}{k^2} \left( g_{01}\alpha_s(k^2/n) + g_{02}\alpha_s^2(k^2/n) \right) \right]$$

$$+ \alpha_s(Q^2) \left( C_1(N) + \frac{g_{01}}{2} \ln^2 \frac{1}{N^a} \right) + O(\alpha_s^{k+2}\ln^k N) + O(\alpha_s^2) \quad (6.20)$$

where $C_1(N)$ is the standard next-to-leading order contribution to the coefficient function. In the first line on the r.h.s. we have given the result which is obtained by performing the two scheme changes $Z_{\overline{\text{MS}}}^u Z_{\overline{\text{MS}}}^u$, which correspond respectively to the first and second factor in square brackets, and in the second line the result obtained by combining Eq. (6.19) with the unresummed result and explicitly performing the double-counting subtraction; of course the two results agree up to subleading corrections.

Resummed results in the $\overline{\text{MS}}$ scheme can be analogously obtained starting from the expression Eq. (6.63) of the resummed physical anomalous dimension in terms of a resummed splitting function. Proceeding in a similar way, with obvious modifications, we get

$$C_{\overline{\text{MS}}}(N, 1, \alpha_s(Q^2)) = \exp \left[ a \left( \int_0^1 dx \frac{x^{N-1} - 1}{1-x} \int_{Q^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} \left( \hat{g}_{01}\alpha_s(k^2) + \hat{g}_{02}\alpha_s^2(k^2) \right) \right) \right]$$

$$+ \alpha_s(Q^2) \int_0^1 x^{N-1} \left[ C_1(x) - \frac{\hat{g}_{01}}{2} \left( \frac{\ln(1-x)^a}{1-x} \right)_+ \right] + O(\alpha_s^{k+2}\ln^k N) + O(\alpha_s^2), \quad (6.21)$$

where now $\hat{g}_{00}$ are contributions to the coefficient of the $O(\ln(1-x)^a)$ term in the physical splitting function, and $C_1(x)$ is the $x$-space next-to-leading order contribution to the coefficient function. Note that the double-counting subtraction in Eq. (6.20) is not just the Mellin transform of the double counting subtraction of Eq. (6.21): this is only true at the leading log level. Again, the two forms Eq. (6.20) and (6.21) of the resummed $\overline{\text{MS}}$ coefficient function coincide at the next-to-leading order and next-to-leading log level, but differ by terms which are at least of $O(\alpha_s^2)$ and then either next-to-next-to-leading log or suppressed by powers of $N$.

The resummed result Eq. (6.21) can be finally brought in the form of Ref. [4] by recalling that $\hat{g}_{02}$ is determined in terms of the physical anomalous dimension, and separating off the
contributions which originate from the anomalous dimension and coefficient function, according to Eq. (2.10):

\[
C_{\text{MS}}(N, 1, \alpha_s(Q^2)) = \exp a \int_0^1 dx \frac{x^{N-1} - 1}{1 - x} \left[ \int_{Q^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} \left[ A_1 \alpha_s(k^2) + A_2 \alpha_s^2(k^2) \right] + B_1 \alpha_s(Q^2(1-x)^a) \right] + \alpha_s(Q^2)^{\frac{1}{2}} \frac{\ln(1-x)^a}{1-x} - \left( \frac{B_1}{1-x} \right) + O(\alpha_s^{k+2} \ln^k N) + O(\alpha_s^2).
\]

(6.22)

Note that the double counting subtraction is not the same as in Eq. (6.21), because the \(B\)-term in Eq. (6.22) is not free of \(O(\ln \frac{1}{N})\) terms.

The physical and \(\overline{\text{MS}}\) schemes which we discussed explicitly are extreme choices in the treatment of the resummed terms: in the former case, these are entirely included in the anomalous dimension, and in the latter case they are entirely included in the coefficient function. Of course, a variety of intermediate choices is possible. These resummed scheme choices can then be freely combined with the usual choice of unresummed factorization scheme. For instance, one can combine an \(\overline{\text{MS}}\)-like resummation (where the resummation is included in the coefficient function) with a physical-scheme anomalous dimension. In such case, the anomalous dimension is given, up to \(O(N^0)\) terms, by Eq. (6.18). Clearly, all such scheme choices can be obtained by a combination of the resummed and unresummed scheme changes Eqs. (6.9,6.11).

7 Summary

In this paper, we have discussed the resummation of large logarithmic contributions to perturbative coefficients that arise as a left-over of the cancellation of infrared singularities near the boundary of the phase space. Our approach is based on identifying the relevant dimensionful scale for the large logs, showing that it is determined essentially by kinematic considerations, and using the renormalization group to resum it.

The main result of this paper is a resummation formula which allows the computation of the physical anomalous dimension (i.e., the scale dependence of physical observables) to any logarithmic accuracy, starting from a fixed-order calculation. This result is less predictive than what one gets by extending the structure of familiar next-to-leading log resummation to high orders: specifically, we find that an order \(\frac{k(k+1)}{2}\) perturbative calculation is required for resummation to next\(^{k-1}\)-to-leading log accuracy. In particular, the standard next-to-leading log resummation turns out to be fully determined by the fixed next-to-leading order result only because of the vanishing of a coefficient which, in our approach, appears to be accidental.

The advantage of this approach, however, is that it does not rely on factorization properties of the physical cross section. Indeed, our proof does not require any detailed consideration of the individual Feynman diagrams which contribute in the soft limit, and thus in particular it does not require the classification of the different forms of radiation which contribute in the soft limit (soft radiation, ‘jet’ radiation collinear to incoming or outgoing partons etc.) which is the main complication in conventional next-to-leading [4, 3] or all-order [8] approaches to resummation.
An important feature of the present approach is that resummed results are provided directly for a physical observable, the physical anomalous dimension, rather than for partonic quantities. This makes it easier to study issues related to the choice of an appropriate resummation variable. In particular, we have shown that a leading (or next-to-leading, etc.) \( \ln(1-x) \) expression of the cross section is necessarily ill-defined, and traced the origin of this problem, first pointed out in Ref. [7], to subleading \( \ln(1-x) \) terms which appear in the inverse Mellin transform of a leading \( \ln N \) expression, which is instead well-defined. Also, the discussion of issues related to the choice of factorization scheme is considerably streamlined: our results are manifestly scheme-independent, and results within specific choices of scheme can be readily obtained as special cases.

Because of its generality, the approach to soft resummation presented in this paper lends itself naturally to a variety of extensions and generalizations. In particular, it should be easy to generalize it to other classes of soft resummation, such as jet or prompt-photon production, and, perhaps more interestingly, to the resummation of transverse momentum distributions, which can be unified [8] with the resummation discussed here by considering Mellin and Fourier transforms as special cases of a complex integral transform.

It is interesting to ask whether the formalism presented in this paper might also lead to further insight on resummation at small \( x \).

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A N-body phase space

The \( n \)-body phase space can be expressed in terms of the \( n-1 \)-body and the two-body phase spaces [21]. To see this, start with the standard definition of the phase space for a generic process with incoming momentum \( P \) and \( n \) bodies in the final state with outgoing momenta \( k_1, \ldots, k_n \) and masses \( k_i^2 = m_i^2 \):

\[
d\phi_n(P; k_1, \ldots, k_n) = \frac{d^{d-1}k_1}{(2\pi)^{d-1}2k_1^0} \cdots \frac{d^{d-1}k_n}{(2\pi)^{d-1}2k_n^0} (2\pi)^d \delta^{(d)}(P-k_1 - \ldots - k_n), \quad (A.1)
\]

The momentum-conservation delta function can be rewritten as

\[
\delta^{(d)}(P-k_1 - \ldots - k_n) = \int d^dP_n \delta^{(d)}(P-P_{n-1} - k_n) \delta^{(d)}(P_{n-1} - k_1 - \ldots - k_{n-1}). \quad (A.2)
\]

Now note that

\[
d^dP_{n-1} = (2\pi)^d \frac{d^{d-1}P_{n-1}}{(2\pi)^{d-1}2P_{n-1}^0} \frac{d(P_{n-1}^0)^2}{2\pi} = (2\pi)^d \frac{d^{d-1}P_{n-1}}{(2\pi)^{d-1}2P_{n-1}^0} \frac{dM_{n-1}^2}{2\pi}, \quad (A.3)
\]

where

\[
M_{n-1}^2 = (P_{n-1}^0)^2 - |\vec{P}_{n-1}|^2; \quad (m_1 + m_2 + \ldots + m_{n-1})^2 \le M_{n-1}^2 \le (\sqrt{P^2} - m_n)^2. \quad (A.4)
\]

We obtain immediately

\[
d\phi_n(P; k_1, \ldots, k_n) = \int_0^\infty \frac{dM_{n-1}^2}{(2\pi)} d\phi_2(P; P_{n-1}, k_n) d\phi_{n-1}(P_{n-1}; k_1, \ldots, k_{n-1}), \quad (A.5)
\]

which is the desired result. Using this result recursively, the \( n \) body phase space can be entirely expressed in terms of two-body phase space integrals.

For completeness, we also give the expression of the two-body phase space in the center-of-mass frame. We have

\[
d\phi_2(P_{i+1}; P_i, k_i) = \frac{d^{d-1}P_i}{(2\pi)^{d-1}2\vec{P}_i^0} \frac{d^{d-1}k_i}{(2\pi)^{d-1}2k_i^0} (2\pi)^d \delta^{(d)}(P_{i+1} - P_i - k_i)
\]

\[
= \frac{(2\pi)^{2d-2}}{4} \frac{d^{d-1}k_i}{P_i^0 k_i^0} \delta(P_{i+1}^0 - P_i^0 - k_i^0). \quad (A.6)
\]

In the center-of-mass frame, \( \vec{P}_{i+1} = 0 \), \( P_{i+1}^0 = M_{i+1} \) and \( |\vec{P}_i| = |k_i| = k_i^0 \), since \( k_i^2 = 0 \). We have therefore

\[
\delta(P_{i+1}^0 - P_i^0 - k_i^0) = \sqrt{(k_i^0)^2 + M_i^2} \frac{M_{i+1}^2}{2M_{i+1}} \delta \left( k_i^0 - M_{i+1}^2 - M_i^2 \right). \quad (A.7)
\]

Hence

\[
|\vec{k}_i| = k_i^0 = \frac{M_{i+1}^2 - M_i^2}{2M_{i+1}} = \frac{M_{i+1}}{2} \left( 1 - \frac{M_i^2}{M_{i+1}^2} \right), \quad (A.8)
\]

and

\[
d\phi_2(P_{i+1}; P_i, k_i) = \frac{(2\pi)^{2d-2}}{4} \frac{(k_i^0)^{1-2d}}{M_{i+1}} d\Omega_i \frac{1}{2(4\pi)^{2-2d}} M_{i+1}^{-2d} \left( 1 - \frac{M_i^2}{M_{i+1}^2} \right)^{1-2d} d\Omega_i. \quad (A.9)
\]
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