THE SHARP LOWER BOUND OF THE FIRST DIRICHLET EIGENVALUE FOR GEODESIC BALLS

HAIBIN WANG, GUOYI XU, JIE ZHOU

ABSTRACT. On complete noncompact Riemannian manifolds with non-negative Ricci curvature, Li-Schoen proved the uniform Poincaré inequality for any geodesic ball. In this note, we obtain the sharp lower bound of the first Dirichlet eigenvalue of such geodesic balls, which implies the sharp Poincaré inequality for geodesic balls.

Mathematics Subject Classification: 58J50.

1. Introduction

In 1984, Li and Schoen [LS84] proved the uniform Poincaré inequality for geodesic balls of complete noncompact Riemannian manifold with $Rc \geq 0$, their method is the integration estimate. Their Poincaré inequality is not sharp. In this note, we prove the sharp Poincaré inequality by obtaining the the sharp lower bound of the first Dirichlet eigenvalue of geodesic balls as follows.

Theorem 1.1. Suppose that $M^n$ is a complete non-compact Riemannian manifold with $Rc \geq 0$, for any $p \in M^n$, we have $\lambda_1(B_1(p)) > \frac{\pi^2}{16}$, where $B_1(p) \subseteq M^n$ is the unit geodesic ball centered at $p$. Furthermore, this lower bound is sharp.

Remark 1.2. The above lower bound is sharp because we find a sequence of Riemannian manifolds $(M^n_i, g_i)$ satisfying the above assumption and $p_i \in M^n_i$, such that $\lim_{i \to \infty} \lambda_1(B_1(p_i)) = \frac{\pi^2}{16}$ (see Figure 2).

The above sharp lower bound holds only for geodesic balls on complete noncompact Riemannian manifolds, because we know the following example from [CF78]. Consider $M_i = S^n\left(\frac{1+2^{-i}}{\pi}\right) \subseteq \mathbb{R}^{n+1}$, which is the boundary of the ball with radius $\frac{1+2^{-i}}{\pi}$ in $\mathbb{R}^{n+1}$; then $\lim_{i \to \infty} \lambda_1(B_1(p_i)) = 0$, where $p_i$ is the north pole of $M_i$ and $B_1(p_i)$ is the unit geodesic ball centered at $p_i$ (see Figure 1).

One well-known tool to get the sharp estimate of eigenvalues on manifolds is, the gradient estimate of eigenfunction introduced by Li-Yau [LY80]. By establishing the sharp gradient estimate of Neumann eigenfunction, Zhong-Yang [ZY84] proved that $\mu_2(M^n) \geq \frac{\pi^2}{4}$ for compact Riemannian manifold $M^n$ with $Rc \geq 0$.

Date: February 8, 2022.

The second author was partially supported by NSFC 11771230, NSFC 12026409 and Beijing Natural Science Foundation Z190003.
and $\text{diam}(M^n) = 2$, where $\mu_2(M^n)$ is the second Neumann eigenvalue of $M^n$ and $\mu_1(M^n) = 0$. Furthermore, Hang-Wang [HW07] showed the equality holds if and only if $M^n$ is isometric to $\mathbb{S}^1$. Later, Yang [Yan99] applied the gradient estimate method to obtain the lower bound of the first Dirichlet eigenvalue for domain with mean convex boundary.

For any domain $\Omega \subseteq (M^n, g)$ with $Rc \geq 0$ and maximal volume growth (i.e. $\text{AVR}(M^n) = \lim_{r \to \infty} \frac{\Vol(B_r(p))}{\omega_n r^n} > 0$, where $\omega_n$ is the volume of unit ball in $\mathbb{R}^n$); using Brendle’s sharp isoperimetric inequality (see [Bre]), recently Kristály [Kri] proved the sharp lower bound of the first Dirichlet eigenvalue as follows:

$$\lambda_1(\Omega) \geq j_v^2 \left( \frac{\omega_n \text{AVR}(g)}{\omega_n r^n} \right)^{\frac{2}{n}} \Vol(\Omega)^{-\frac{2}{n}},$$

where $j_v$ is the first positive root of the Bessel function $J_v$ of the first kind with degree $v \in \mathbb{R}$. Especially, the equality holds if and only if $(M^n, g)$ is $\mathbb{R}^n$.

Our idea is applying the maximum principle on the suitable function related to eigenfunction, and we obtain the $C^0$ estimate of Dirichlet eigenfunction directly without involving the gradient. This argument is partly inspired by the modulus continuity estimate of eigenfunctions in [Ni13], and it also has close relationship with the original parabolic method of [AC11] (also see [AC13]). As we know, the modulus continuity estimate of eigenfunctions can not yield the $C^0$ estimate of eigenfunction in our case, and is not enough to yield $\lambda_1(B_1(p)) > \frac{\pi^2}{16}$. This strict sharp inequality is novel, comparing the facts that the sharp lower bound of $\mu_2(M^n)$ can be realized by $M^n = \mathbb{S}^1$ in [ZY84] and the sharp lower bound of $\lambda_1(\Omega)$ can be realized by $\Omega = B_1^2 \subseteq \mathbb{R}^n$ in [Kri].

2. THE SHARP LOWER BOUND OF THE FIRST DIRICHLET EIGENVALUE

On complete non-compact Riemannian manifold $M^n$, let $\gamma$ be a ray starting at $p$, define the Busemann function with respect to $\gamma$ as $b_\gamma(x) = \lim_{t \to \infty} [t - d(x, \gamma(t))].$ When the context is clear, we also use $b$ instead of $b_\gamma$ for simplicity.
Proposition 2.1. Let $M^n$ be a complete noncompact Riemannian manifold with $Rc \geq 0$, for any ray $\gamma \subseteq M^n$ and region $\Omega \subseteq b_\gamma^{-1}[a, a + D]$ where $a \in \mathbb{R}, D > 0$, we have $\lambda_1(\Omega) \geq \frac{\pi^2}{4D^2}$.

Remark 2.2. By establishing the $C^0$-estimate of the first Dirichlet eigenfunction, we get the sharp estimate of $\lambda_1$. One novel thing about our $C^0$-estimate (2.1) is that there is no boundary assumption, although we use the noncompact property of the whole manifold $M^n$.

Proof: For simplicity, we use $b$ instead of $b_\gamma$ in the rest of the argument. Assume $\Delta u = -\lambda_1 u$ for some constant $\lambda_1 > 0$ with $u \in C_0^\infty(\Omega), u(x) \geq 0, \max_{x \in \Omega} u(x) = 1$. Let $\alpha = a + D$, firstly we show that

$$\sin^{-1} u \leq \sqrt{\lambda_1}(\alpha - b(x)). \tag{2.1}$$

To prove (2.1), we only need to show for any $\delta \in (0, 1)$ such that

$$\sin^{-1}((1 - \delta)u) \leq \sqrt{\lambda_1}(\alpha - b(x)). \tag{2.2}$$

Then (2.1) follows from letting $\delta \to 0$ in (2.2).

By contradiction. If (2.2) does not hold for some $\delta \in (0, 1)$, then there is some $\epsilon > 0$ such that for $\phi(x) = \sin^{-1}((1-\delta)u(x)) - \sqrt{\lambda_1} + \epsilon(\alpha - b(x))$, we have $\max_{x \in \Omega} \phi(x) > 0$. Note $\max_{x \in \Omega} \phi(x) = -\sqrt{\lambda_1} + \epsilon(\alpha - b(x)) \leq 0$, so there is $x_0 \in \Omega$ such that $\phi(x_0) = \max_{x \in \Omega} \phi(x) > 0$.

Now we have $\nabla \phi(x_0) = 0$, which implies $\frac{(1 - \delta)|\nabla u|}{\sqrt{1 - ((1 - \delta)u)^2}} = \sqrt{\lambda_1} + \epsilon$. Also note $\Delta b \geq 0$ by $Rc \geq 0$ and the Laplace Comparison Theorem. Now we have

$$0 \geq \Delta \phi(x_0) = \frac{(1 - \delta)u}{\sqrt{1 - ((1 - \delta)u)^2}}(-\lambda_1 + \frac{(1 - \delta)^2|\nabla u|^2}{1 - ((1 - \delta)u)^2}) + \sqrt{\lambda_1} + \epsilon \Delta b$$

$$\geq \frac{\epsilon(1 - \delta)u}{\sqrt{1 - ((1 - \delta)u)^2}} > 0.$$

This is the contradiction, then (2.2) is proved.

From $b(x) \geq a$, we have $0 \leq \alpha - b(x) \leq D$ for any $x \in \Omega$. Assume $u(x_1) = 1$ for some $x_1 \in \Omega$, we have $\lambda_1 \geq \frac{\sin^{-1} u(x_1)}{\alpha - b(x_1)} \geq \frac{\pi^2}{4D^2}$. \hfill $\Box$

Note $B_\gamma(p) \subseteq b_\gamma^{-1}[-1, 1]$ for some ray starting from $p$, we have the following sharp lower bound of the first Dirichlet eigenvalue.

Theorem 2.3. Let $M^n$ be a complete noncompact Riemannian manifold with $Rc \geq 0$, for any region $\Omega \subseteq M^n$ with $\text{diam}(\Omega) = D < \infty$, we have $\lambda_1(\Omega) \geq \frac{\pi^2}{4D^2}$. Furthermore $\lambda_1(B_1(p)) > \frac{\pi^2}{16}$ and this inequality is sharp.

Proof: Step (1). For $p \in \Omega$, take a ray $\gamma$ starting from $p$, let the Busemann function with respect to $\gamma$ be $b(x)$. From $\text{diam}(\Omega) = D$ and $|\nabla b| = 1$, we have
\[ \lambda_1(B_1(p_i)) = \frac{\pi^2}{16} + \delta_i \to \frac{\pi^2}{16} \]
\[ r_i = 2 - 2\pi \epsilon_i - \epsilon_i \to 2 \]

**Figure 2.** Collapsing to the sharp lower bound

\[ \max_{x \in \Omega} b(x) - \min_{x \in \Omega} b(x) \leq D, \text{ hence } \Omega \subseteq b^{-1}[a, a + D] \text{ for some } a. \text{ Then } \lambda_1(\Omega) \geq \frac{\pi^2}{4D^2} \]

Assume \( \Delta u = -\lambda_1 u \) for some constant \( \lambda_1 > 0 \) with \( u \in C_0^\infty(B_1(p)), u(x) \geq 0, \max_{x \in B_1(p)} u(x) = 1. \) To prove \( \lambda_1 > \frac{\pi^2}{16}, \) from the above, we only need to show that \( \lambda_1 \neq \frac{\pi^2}{16}. \) If \( \lambda_1 = \frac{\pi^2}{16}, \) assume \( u(x_1) = 1 \) for some \( x_1 \in B_1(p). \) Then from (2.1), we have

\[ \frac{\pi}{4}(1 - b(x_1)) \geq \sin^{-1} u(x_1) = \frac{\pi}{2}. \]

This implies \( b(x_1) = -1. \) On the other hand, from the definition of \( b(x), \) we know \( b(x_1) \geq -d(x_1, p) > -1, \) it is the contradiction.

**Step (2).** Next we show that \( \lambda_1(B_1(p)) > \frac{\pi^2}{16} \) is sharp. Consider \((\mathbb{R}^n, g_i)\) with 

\[ g_i = dr^2 + f_i^2(r)dS^{n-1}, \text{ where } \epsilon_i = 2^{-i} \text{ and } \]

\[ f_i(r) = \begin{cases} 
  \epsilon_i - \epsilon_i \epsilon_i^{-1} e_i r^{-i-1} & 0 \leq r < \epsilon_i, \\
  \epsilon_i & r \geq \epsilon_i.
\end{cases} \]

Then it is easy to check that \( Rc(g_i) \geq 0. \) Assume the origin point of \((\mathbb{R}^n, g_i)\) is \( q_i. \)

Take \( p_i = \exp_{q_i}((1 - \epsilon_i)\theta_0) \) for some \( \theta_0 \in S^{n-1}, \) we claim

\[ B_{2 - 2\pi \epsilon_i}(q_i) \subset B_1(p_i). \]
For any \( x \in B_{2-2\pi\epsilon_1}(q_1) \), we know \( x = \exp_{q_1}(r_1, \theta) \) for some \( r_1 \leq 2 - 2\pi\epsilon_1 - \epsilon_1 \) and \( \theta \in S^1 \). Take \( x_1 = \exp_{q_1}(1 - \epsilon_1)\theta \), by \( f_1(r) \leq \epsilon_1 \), we know
\[
d(x, p_i) \leq d(x, x_1) + d(x_1, p_i) \leq 1 - 2\pi\epsilon_1 + \pi\epsilon_1 < 1.
\]
This implies \( B_{2-2\pi\epsilon_1}(q_1) \subset B_1(p_i) \). So, by the monotonicity of Dirichlet eigenvalue with respect to the domain, we know \( \lambda_1(B_1(p_i)) \leq \lambda_1(B_{2-2\pi\epsilon_1}(q_1)) \).

On the other hand, for \( r_i = 2 - 2\pi\epsilon_1 - \epsilon_1 \), by taking \( \tau_i = \frac{\epsilon_1}{2r_1} \) and \( \varphi_i(x) = \cos \tau_i d(x, q_i) \in W_0^{1,2}(B_1(q_i)) \). We know
\[
\lambda_1(B_{2-2\pi\epsilon_1}(q_1)) \leq \int_{B_{r_i}(q_i)} |\nabla \varphi_i|^2 d\mu_{g_i} \leq \frac{\tau_i^2}{r_i^2} \int_0^{r_i} \cos^2(s) f_i^{m-1}(\frac{s}{r_i}) ds \leq \frac{\epsilon_i^{m-1}}{r_i^2} \int_0^{r_i} \sin^2(s) f_i^{m-1}(\frac{s}{r_i}) ds.
\]
Thus \( \lim_{i \to \infty} \lambda_1(B_1(p_i)) \leq \lim_{i \to \infty} \int_0^{r_i} \frac{\sin^2(s)}{\cos^2(s)} ds \leq \frac{\pi^2}{16} \). \( \square \)

Similar to Sakai’s conjecture for Zhong-Yang’s sharp eigenvalue estimate (see [Sak05]) and Hang-Wang’s rigidity result [HW07], we may ask whether the unit geodesic ball in complete noncompact manifolds with nonnegative Ricci curvature is close to an interval \([0, 2]\) in the Gromov-Hausdorff sense if the first eigenvalue is nearly \( \frac{\pi^2}{16} \). From (2.1), we in fact have \( \lim_{\lambda_1(B_1(p)) \to \frac{\pi^2}{16}} \text{diam}(B_1(p)) = 2 \). One natural question is whether the width of \( B_1(p) \) is nearly 0 when \( \lambda_1(B_1(p)) \) is nearly \( \frac{\pi^2}{16} \).

**Acknowledgments**

We thank one anonymous referee for the helpful comment on the earlier version of this note.

**References**

[AC11] Ben Andrews and Julie Clutterbuck, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. 24 (2011), no. 3, 899–916, DOI 10.1090/S0894-0347-2011-00699-1.

[AC13] ________, *Sharp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue*, Anal. PDE 6 (2013), no. 5, 1013–1024, DOI 10.2140/apde.2013.6.1013.

[Bre] Simon Brendle, *Sobolev inequalities in manifolds with nonnegative curvature*, arXiv:2009.13717v4 [math.DG].

[CF78] I. Chavel and E. A. Feldman, *Spectra of domains in compact manifolds*, J. Functional Analysis 30 (1978), no. 2, 198–222, DOI 10.1016/0022-1236(78)90070-8.

[HW07] Fengbo Hang and Xiaodong Wang, *A remark on Zhong-Yang’s eigenvalue estimate*, Int. Math. Res. Not. IMRN 18 (2007), Art. ID rnm064, 9, DOI 10.1093/imrn/rnm064.

[Kri] Alexandru Kristály, *Explicit sharp constants in Sobolev inequalities on Riemannian manifolds with nonnegative Ricci curvature*, arXiv:2012.11862v2 [math.DG].

[LS84] Peter Li and Richard Schoen, *L^p and mean value properties of subharmonic functions on Riemannian manifolds*, Acta Math 153 (1984), no. 3–4, 279–301.

[LY80] Peter Li and Shing Tung Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), 1980, pp. 205–239.
[Ni13] Lei Ni, Estimates on the modulus of expansion for vector fields solving nonlinear equations, J. Math. Pures Appl. (9) 99 (2013), no. 1, 1–16, DOI 10.1016/j.matpur.2012.05.009.

[Sak05] Takashi Sakai, Curvature—up through the twentieth century, and into the future? [translation of Sūgaku 54 (2002), no. 3, 292–307; MR1929898], 2005, pp. 165–187. Sugaku Expositions.

[Yan99] DaGang Yang, Lower bound estimates of the first eigenvalue for compact manifolds with positive Ricci curvature, Pacific J. Math. 190 (1999), no. 2, 383–398, DOI 10.2140/pjm.1999.190.383.

[ZY84] Jia Qing Zhong and Hong Cang Yang, On the estimate of the first eigenvalue of a compact Riemannian manifold, Sci. Sinica Ser. A 27 (1984), no. 12, 1265–1273.

Haibin Wang, Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China
Email address: wanghb20@mails.tsinghua.edu.cn

Guoyi Xu, Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China
Email address: guoyixu@tsinghua.edu.cn

Jie Zhou, Department of Mathematical Sciences, Tsinghua University; Academy for Multidisciplinary Studies, Capital Normal University, Beijing, P. R. China
Email address: zhoujie2014@mails.ucas.ac.cn