Brane cosmology with variable tension

José Antonio Belinchón

Dpt. Matemáticas. Universidad de Atacama. Av. Copayapu 485, Copiapó, Chile

Sami Dib

Max Planck Institute for Astronomy, Königstuhl 17, 69117, Heidelberg, Germany

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We study brane-world models and demonstrate that such models do not admit self-similar solutions through the matter collineation approach. By introducing the hypothesis of variable brane tension, $\lambda$, we outline the new effective field equation (EFE) in the most simple case (symmetric embedding) under the assumption that the fundamental constants in 5D are constants. In this case, we find the exact form that each physical quantity may take in order that the EFE become invariant under scale transformations. By taking into account such assumptions, we find that in 4D, the gravitational constant $\kappa^2 \sim \lambda$ while the cosmological constant $\Lambda \sim \lambda^2$ are always decreasing. These results are quite general and valid for any homogeneous self-similar metric. Nevertheless, the study of the EFE under scale symmetries suggests that $\rho \sim \lambda$ (as a functional relationship). This allows to get a growing $\kappa^2$ but, in this case, the fundamental constants in 5D must vary as well. We outline a toy model allowing such a possibility.

I. INTRODUCTION

Recent cosmological observations of Supernovae Type Ia (SNIa) at high redshift show evidence of an accelerated expansion of the Universe at late times (see for instance [1]-[3]). This is also supported by the observations of anisotropies in the CMB as well as in baryon acoustic oscillations (BAO).

Several ideas have been proposed in order to explain such acceleration. An interesting paradigm is to consider extra dimensions of space-time which could be the source of the current accelerated expansion. For instance, the Dvali-Gabadadze-Porrati model [4, 5] generates a natural accelerated expansion with a geometrical threshold associated to a five-dimensional space-time. Another example are the models proposed by Randall and Sundrum [6, 7] (hereafter RS models). They imagined our four-dimensional (4D) space-time could be a three-brane embedded in a 5-dimensional (5D) space-time (the bulk). According to the brane-world scenario, the physical fields (electromagnetic, Yang-Mills etc.) observed in our 4D Universe are confined to the three-brane. Only gravity can freely propagate both on the brane and in the bulk, with the gravitational self-couplings not significantly modified. Even with the fifth dimension uncompactified, standard 4D gravity can be reproduced on the brane in a certain limit [8]. At very high energies, in the presence of large 5D curvatures, significant deviations from the standard Einstein theory could occur in brane-world models, due to the nonstandard model 5D fields, and possible asymmetric embeddings of the brane into the bulk. Nevertheless, RS models drive a late time acceleration only by adding a dark energy field [9]. In order to alleviate this discrepancy between the theoretical predictions and the observations, it has been suggested that the brane tension could be variable. In this way, in [10] the authors investigate a brane model with variable brane tension as a function of redshift called chrono-brane. They showed that the model is consistent with the observational data and that it can drive the Universe to an accelerated phase at late times.

Brane-worlds with non-constant tension have been proposed by several authors (see for instance [11]-[15]). This new degree of freedom allows for evolving gravitational and cosmological constants, the latter being a natural candidate for dark energy. For cosmological branes, a variable brane tension leads to several important consequences that have been studied for example in ([15] and [16]-[18]). But this is not the only way for introducing the hypothesis of evolving constants in 4D. Higher-dimensional theories imply that some constants, such as the gravitational constant and the strength of the gauge-couplings, are not fundamental constants. Instead, they are related to the sizes of the extra-dimensional space, which are moduli fields in the four-dimensional effective theory [19]. In another approach, Ponce de Leon [20, 22] study the possibility that the variation of the fundamental constants, in the framework of brane-world models, are provoked by the fifth dimension described by a scalar function varying with time. Other works contemplate the variation of the fundamental constant, as well as other constants such as the speed of light [23]-[28].

As we have discussed above, the brane-world models with variable tension fit better with the observations than those with constant tension. Therefore our goal is to study brane-world cosmological models from different points of view. In the first place we determine that the models with constant tension cannot admit self-similar solutions. Secondly, we study whether the models with variable tension admit such kind of solutions. If the field equations are invariant under scale transformations then study how must vary each physical quantity, putting special emphasis in
determining how the gravitational and the cosmological “constants” vary. Thirdly, we explore other frameworks that allow the fundamental constants to vary in 5D.

The present paper is organized as follows. In Section II, we introduce the basic effective field equations (EFE) of the brane-world cosmological models. Then, we determine if the outlined EFE are invariant under scaling transformations. In this way, we show that the model does not admit homothetic solutions. The basic idea behind this geometrical procedure is a generalization of the Dimensional Analysis (DA). In Section III, we introduce the hypothesis of a variable brane tension. We start by formulating the new EFE in the most basic case and under the assumption that the fundamental constants behave as true constants in 5D. The hypothesis of a time varying brane tension implies that the gravitational and the cosmological constants must vary with time. Then, we determine the exact form that each physical quantity may take in order for the new EFE to admit exact self-similar solutions through the matter collineation approach, that is, we study if the new EFE are invariant under scaling transformations. This simple approach allows us to find exact solutions for a wide variety of homogeneous cosmological models since the results are quite general and valid for any self-similar metric. As examples, we study two cases by finding exact solutions to these particular models. In section IV, in a phenomenological way, we outline the EFE of a toy model that allows the fundamental constants in 5D to vary. In this way, we are able to generate new cosmological scenarios. In Section V, we discuss our findings and conclude.

II. FIELD EQUATIONS IN THE BRANE-WORLD MODELS

In a five dimensional space-time (called the bulk), it is possible to confine matter fields in a four dimensional hypersurface (called 3-brane). The effective Einstein equations on the bulk are \[29, 30, 31\] and \[8\]:

\[ G_{ab}^{5} = \kappa_5^2 T_{ab}^{5} - \Lambda_{5} g_{ab}^{5}, \]  

(1)

where \( \Lambda_{5} \) is the 5D cosmological constant which is assumed to be negative and \( T_{ab}^{5} \) is the total energy-momentum tensor on the brane (vacuum and matter) defined by

\[ T_{ab}^{5} = \delta (\chi) (T_{ab}^{m} - \lambda g_{ab}), \]  

(2)

where \( T_{ab}^{m} \) represents ordinary matter and \( \lambda \) is the vacuum energy density interpreted as the tension of the brane, considered as “constant”, with the induced metric \( g_{ab} \). The Dirac delta function indicates the fact that the matter is confined in the hypersurface \( \chi = 0 \).

The basic equations on the brane are obtained by projections onto the brane with Gauss-Codazzi equations, Israel junction condition and assuming a \( \mathbb{Z}_2 \) symmetry of the bulk. Then the effective Einstein equations (4D) on the brane are:

\[ G_{ij} = \kappa^2 T_{ij}^{m} + \kappa_5^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij}, \]  

(3)

where \( \kappa^2 \) is the 4D gravitational coupling “constant” given by \( 6\kappa^2 = \kappa_5^4 \lambda \). For attracting gravity, the brane tension \( \lambda \) should be positive, \( \lambda > 0 \). There are several constraints which have been obtained for the brane tension (see for instance \[32, 33\]). The cosmological constant \( \Lambda \), is defined by \( \Lambda = \frac{\kappa_5^4}{\kappa_5^2} (\Lambda_{5} + \kappa_5^4 \lambda^2 / 6) / 2 \).

The above effective field equations \[33\] have two new correction terms. The first correction term is the quadratic term \( S_{ij} \), arising from the extrinsic curvature term in the projected Einstein tensor, and is given by

\[ S_{ij} = \frac{1}{12} T_{\alpha}^{\alpha} T_{ij} - \frac{1}{4} T_{\alpha\beta} T^{\alpha\beta}_{ij} + \frac{1}{24} g_{ij} \left[ 3 T_{\alpha\beta} T^{\alpha\beta} - T_{\alpha}^{\alpha} T_{\beta}^{\beta} \right]. \]  

(4)

The second correction term, \( \mathcal{E}_{ij} \), is the projection of the 5D Weyl tensor \( C_{abcd} \) onto the brane, and is defined as

\[ \mathcal{E}_{ij} = C_{ijkl} n^l n^d, \]  

(5)

and encompasses nonlocal effects from the free gravitational field in the bulk. This term vanishes if the bulk spacetime is purely anti-de Sitter. The only general known property of this nonlocal term is that it is traceless. \( n^d \) is the unit normal to the hypersurface \( \chi = 0 \).

The Einstein equation in the bulk, \( \nabla^{\nu} T_{\nu\beta}^{5} = 0 \), also implies the conservation of the energy momentum tensor of the matter on the brane, that is, \( \nabla^{j} (\kappa^2 T_{ij}^{m}) = 0 \). In this way, we have a new constraint on \( S_{ij} \) and \( \mathcal{E}_{ij} \) as

\[ \kappa_5^4 \nabla^{j} (S_{ij}) = \nabla^{j} (\mathcal{E}_{ij}) \]  

(6)
The symmetry properties of $\mathcal{E}_{ij}$ (the bulk Weyl tensor) imply that, in general, we can decompose it irreducibly with respect to the 4-velocity $u^i$, in the following way \cite{8,31}:

$$\mathcal{E}_{ij} = -\frac{6}{\kappa^2 \lambda} \left[ \mathcal{U} \left( u_i u_j + \frac{1}{3} h_{ij} \right) + \mathcal{P}_{ij} + Q_i u_j + Q_j u_i \right],$$

(7)

where the constant, $-6/(\kappa^2 \lambda)$, is introduced ad hoc for dimensional considerations. $\mathcal{U}$ is a scalar quantity, called dark energy density, $\mathcal{P}_{ij}$ is a spatial symmetric and trace-free tensor and $Q_i$ is a spatial vector. $h_{ij}$ is the projection tensor orthogonal to $u^i$ on the brane. By using the constraint equation (6), it is possible to derive the evolution equations for $\mathcal{U}$ and $Q_i$ but not for $\mathcal{P}_{ij}$, so the system of equations on the brane is not in general closed. In this work, for simplicity, we only consider a flat FLRW metric, that is, homogeneous and isotropic, so $Q_i = \mathcal{P}_{ij} = 0$, and therefore, the only non-zero contribution from the 5D Weyl tensor from the bulk is given by the scalar quantity $\mathcal{U}$. We would like to point out that other authors (see for instance \cite{14}) use other definitions for $\mathcal{E}_{ij}$ such as

$$\mathcal{E}_{ij} = -\kappa^2 \mathcal{U} \left( u_i u_j + \frac{1}{3} h_{ij} \right),$$

(8)

the unique difference are the coupling constants, but they change the nature of $\mathcal{U}$ and its physical dimensions and therefore the cosmological implications as we will show below.

Therefore the EFE on the brane are defined by:

$$G_{ij} = \kappa^2 T^m_{ij} + \kappa^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij},$$

(9)

$$\nabla^j T^m_{ij} = 0,$$

(10)

$$\kappa^4 \nabla^j \left( S_{ij} \right) = \nabla^j \left( \mathcal{E}_{ij} \right).$$

(11)

We assume a flat FLRW metric given by

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2),$$

(12)

and a perfect fluid whose energy-momentum tensor on the brane is defined by

$$T^m_{ij} = (\rho + p) u_i u_j + pg_{ij},$$

(13)

where $\rho$ is the energy density and $p$ is the isotropic pressure, with the equation of state $p = \gamma \rho$, with $\gamma$ being a constant. Under all these assumptions, the effective gravitational field equation EFE \cite{9,11}, with $\mathcal{E}_{ij}$ defined by Eq. (7), become

$$3H^2 = \kappa^2 \rho + \frac{\kappa^2}{2 \lambda} \rho^2 + \frac{6}{\lambda \kappa^2} \mathcal{U} + \Lambda,$$

(14)

$$2 \dot{H} + 3H^2 = -\kappa^2 p - \frac{\kappa^2}{2 \lambda} (2 \rho p + \rho^2) - \frac{2}{\lambda \kappa^2} \mathcal{U} + \Lambda,$$

(15)

$$\dot{\rho} + 3 (\rho + p) H = 0, \quad \rho = \rho_0 a^{-3(\gamma+1)},$$

(16)

$$\dot{\mathcal{U}} + 4H \mathcal{U} = 0, \quad \mathcal{U} = \mathcal{U}_0 a^{-4},$$

(17)

where $\dot{\cdot} = d/dt, \ H = \dot{a}/a$. Eq. (17) is obtained from Eq. (6), since for a perfect fluid, the left hand side of Eq. (6) vanishes due to the energy-momentum conservation of the fluid. Hence, in this case the non-local energy density obeys the radiation like energy conservation (17). However, unlike radiation, the non-local energy density may be negative.

Nevertheless, if we assume Eq. (8) then the EFE \cite{9,11}, become

$$3H^2 = \kappa^2 \rho + \frac{\kappa^2}{2 \lambda} \rho^2 + \kappa^2 \mathcal{U} + \Lambda,$$

(18)

$$2 \dot{H} + 3H^2 = -\kappa^2 p - \frac{\kappa^2}{2 \lambda} (2 \rho p + \rho^2) - \frac{\kappa^2}{3} \mathcal{U} + \Lambda,$$

(19)

$$\dot{\rho} + 3 (\rho + p) H = 0, \quad \rho = \rho_0 a^{-3(\gamma+1)},$$

(20)

$$\dot{\mathcal{U}} + 4H \mathcal{U} = 0, \quad \mathcal{U} = \mathcal{U}_0 a^{-4}.$$  

(21)
There is no difference between Eqs. (14-17) and Eqs. (18-21) while the “constants” \((\kappa^2, \lambda, \Lambda)\) behave as true “constants”.

Once the EFE are defined, we would like to study if they are invariant under a conformal motion, that is, if they admit homothetic or power-law solutions. In particular, we will study if the EFE are invariant under scale transformations. In fact, this is a generalization of Dimensional Analysis (DA) (see for instance [34-37]). To this purpose, we apply the matter collineations method, that is, we calculate the Lie derivative of the effective energy momentum tensor along a homothetic vector field \((\ref{39})\). We would like to point out that there are no self-similar or power-law solutions for the FE \((\ref{14-17})\), since the same physical quantity, \(\rho\), has different order of magnitude.

By taking into account the homothetic vector field for the metric \((\ref{12})\), that is \((\ref{39})\), we aim to impose that the right hand side of Eq. (27) has divergence zero, that is

\[
\nabla \cdot (T_{ab}^{m}) = 0
\]

where obviously there is no matter conservation \((\nabla \cdot (T_{ab}^{m}) \neq 0)\), and it is possible to give a thermodynamical interpretation of this new model as done, for example, by Wong et al in \((\ref{41})\). In the case of variable brane tension, the projected 4D EFE on the brane have a similar form to the general case, therefore they are

\[
\nabla \cdot (T_{ab}^{m}) = \nabla \cdot \lambda,
\]

where obviously there is no matter conservation \((\nabla \cdot (T_{ab}^{m}) \neq 0)\), and it is possible to give a thermodynamical interpretation of this new model as done, for example, by Wong et al in \((\ref{41})\). In the case of variable brane tension, the projected 4D EFE on the brane have a similar form to the general case, therefore they are

\[
G_{ij} = \kappa^2 T_{ij}^{m} + \kappa_S^4 S_{ij} - \Lambda g_{ij},
\]

where

\[
\kappa^2 = \kappa_S^4 \frac{6}{\lambda}, \quad \Lambda = \frac{1}{2} (\kappa_S^2 \Lambda_5 + \kappa^2 \lambda),
\]
so the assumption of variable brane tension imply that the 4D gravitational and cosmological “constant” must vary. Note that from Eq. (30) we may assume that \( \kappa^2 \) must vary as \( \lambda \) in order to keep \( \kappa_5^2 \) constant and therefore \( \Lambda \) varies as \( \lambda^2 \). However, the evolution and conservation equations change in this framework. In order to deduce the new equations, we impose the Bianchi identity to the right hand side of Eq. (29), that is

\[
\nabla^j (\kappa^2 T_{ij}^m + \kappa_5^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij}) = 0,
\]

with the constraint \( \nabla^j T_{ij}^m = \nabla^j \lambda \). Therefore we may consider

\[
\nabla^j \mathcal{E}_{ij} = \nabla^j (\kappa^2 T_{ij}^m) + \kappa_5^4 \nabla^j S_{ij} - \nabla^j (\Lambda g_{ij}),
\]

where \( \mathcal{E}_{ij} \) also depends on \( \kappa^2 \) and \( \lambda \). From Eq. (30) we get:

\[
\nabla^j \kappa^2 = \kappa^2 \frac{\nabla^j \lambda}{\lambda}, \quad \nabla^j \Lambda = \kappa^2 \nabla^j \lambda.
\]

Simplifying Eq. (32) with (33 and 28) we get

\[
\nabla^j \mathcal{E}_{ij} = \nabla^j(\kappa^2 T_{ij}^m) + 6 \kappa^2 \lambda \nabla^j S_{ij},
\]

and therefore the EFE become

\[
G_{ij} = \kappa^2 T_{ij}^m + \kappa_5^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij},
\]

\[
\nabla^j T_{ij}^m = \nabla^j \lambda,
\]

\[
\nabla^j \mathcal{E}_{ij} = \nabla^j(\kappa^2 T_{ij}^m) + 6 \kappa^2 \lambda \nabla^j S_{ij}.
\]

For a perfect fluid and the flat FLRW metric with \( \mathcal{E}_{ij} \) defined by (7), the EFE (35-37) yield

\[
3H^2 = \kappa^2 \rho + \kappa^2 \frac{\rho^2}{\lambda} + 6 \frac{\kappa^2}{\lambda^2} \mathcal{U} + \Lambda,
\]

\[
2\dot{H} + 3H^2 = -\kappa^2 p - \kappa^2 \frac{(2\rho p + \rho^2)}{\lambda} - \frac{2}{\lambda^2} \mathcal{U} + \Lambda,
\]

\[
\dot{\rho} + 3 (\rho + p) H = -\dot{\lambda},
\]

\[
\dot{\mathcal{U}} + 4H \mathcal{U} - 2U \frac{\dot{\lambda}}{\lambda} = -\frac{5}{6} \rho \kappa^4 \dot{\lambda}.
\]

Note that if \( \mathcal{U} = 0 \), or \( \rho = 0 \), then \( \dot{\lambda} = 0 \), that is, the geometry of the bulk must have a non vanishing Weyl tensor or the matter model of the brane is not the vacuum.

If \( \mathcal{E}_{ij} \) is defined by (8) then the EFE (35-37) become

\[
3H^2 = \kappa^2 \rho + \kappa^2 \frac{\rho^2}{\lambda} + \kappa^2 \mathcal{U} + \Lambda,
\]

\[
2\dot{H} + 3H^2 = -\kappa^2 p - \kappa^2 \frac{(2\rho p + \rho^2)}{\lambda} - \frac{1}{3} \kappa^2 \mathcal{U} + \Lambda,
\]

\[
\dot{\rho} + 3 (\rho + p) H = -\dot{\lambda},
\]

\[
\dot{\mathcal{U}} + 4H \mathcal{U} + 2U \frac{\dot{\lambda}}{\lambda} = -5 \frac{\rho}{\lambda} \kappa^4 \dot{\lambda}.
\]

noting that the main difference with respect to Eqs. (38-41) is given by Eq. (45).
A. Matter collineation approach

As above, we study if these new EFE may admit homothetic solutions. By taking into account the matter collineation approach we can determine the behavior of each physical quantity. Hence

\[ \mathcal{L}_H (\Lambda g_{ij}) = 0 \quad \iff \quad \frac{\dot{\Lambda}}{\Lambda} = -\frac{\dot{\lambda}}{\lambda} = -\frac{2}{t}, \quad \iff \quad \Lambda \approx \frac{\kappa^2 \lambda}{\Lambda_0 t^{-2}}, \quad (46) \]

\[ \mathcal{L}_H (\kappa^2 T_{ij}^m) = 0 \quad \iff \quad \frac{\dot{\kappa}^2}{\kappa^2} + \frac{\dot{\rho}}{\rho} = -\frac{2}{t}, \quad \iff \quad \kappa^2 \rho \approx t^{-2}, \quad (47) \]

\[ \mathcal{L}_H (\kappa^4 S_{ij}) = 0 \quad \iff \quad \frac{\dot{\rho}}{\rho} = -\frac{1}{t^2}, \quad \iff \quad \rho \approx t^{-1}, \quad (48) \]

\[ \mathcal{L}_H (\mathcal{E}_{ij}) = 0 \quad \iff \quad \frac{\dot{U}}{U} - \frac{\kappa^2}{\kappa^2} = -\frac{2}{t^2}, \quad \iff \quad U \approx \frac{\kappa \lambda}{t^2}. \quad (49) \]

where in Eq. (49) we are considering its definition given by Eq. (7).

Therefore, we may conclude that each physical quantity must behave as

\[ a \sim t^{a_1}, \quad \rho \sim t^{-\gamma}, \quad \kappa^2 \sim t^{-1}, \quad \Lambda \sim t^{-2}, \quad \lambda \sim t^{-1}, \quad U \sim t^{-4}, \quad (50) \]

in order to obtain homothetic or power law solutions. At this point, we would like to emphasize one important theoretical issue. If we set \( \Lambda \sim t^{-2} \), that is, \( \Lambda = \Lambda_0 t^{-2} \), where \( \Lambda_0 \) is a dimensional constant, this implies that from the definition of \( \Lambda, 2\Lambda = (\kappa^2 \Lambda_5 + \kappa^2 \lambda) \), that \( \Lambda_5 \) must be zero, \( \Lambda_5 = 0 \), otherwise we have to write \( \Lambda = K + \Lambda_0 t^{-2} \), \( K = \text{const.} \), which is a kinematic self-similar solution instead of self-similar one (see for instance [37]). So for the self-similar solutions we are assuming that \( \Lambda_5 = 0 \).

As can be observed, we have obtained \( \kappa^2 \approx \lambda \), without any assumption. This result, which is not new in the literature, has been assumed (or imposed) by several authors (see for instance [11]-[12]-[13] and [28]). Note furthermore that, \( \kappa^2 \sim \lambda \sim t^{-1} \), so the gravitational “constant” is always decreasing as the brane tension. Therefore, in the framework of the homothetic solutions, the model always predicts a decreasing gravitational constant.

These relationships keep the 5D constants as true constants. At this point we have only obtained the order of magnitude for each quantity. Now, we have to find the exact values of the numerical constants \( a_1, \rho_0, \kappa_0^2, \) etc, in order to find an exact solution, if it is possible, since we have six unknowns and only four equations. These results [50] are dimensionally consistent. By inserting them into the system of equations (38-41), we may observe that the system collapses to an algebraic one, and whose solution is:

\[ a_1 = \frac{\lambda_0 + \rho_0}{3\rho_0 (\gamma + 1)}, \quad \kappa_0^2 = \frac{2\lambda_0 \left( -3\gamma + 1 \rho_0^2 + (1 - 3\gamma) \lambda_0 \rho_0 + 2\lambda_0^2 \right)}{3\rho_0 \left( \gamma + 1 \right)^2 \left( -3\gamma + 1 \rho_0^2 + (11 - 3\gamma) \lambda_0 \rho_0 + 2\lambda_0^2 \right)}, \]

\[ \Lambda_0 = \frac{\lambda_0 \left[ -5(3\gamma + 1) \rho_0^2 + 2(2 - 9\gamma) \lambda_0 \rho_0^3 + (11 - 3\gamma) \lambda_0^2 \rho_0 + 2\lambda_0^2 \right]}{3\rho_0 \left( \gamma + 1 \right)^2 \left( -3\gamma + 1 \rho_0^2 + (11 - 3\gamma) \lambda_0 \rho_0 + 2\lambda_0^2 \right)}, \]

\[ U_0 = \frac{5}{9\rho_0^2} \frac{\gamma (\gamma + 1)^3}{\left( -3\gamma + 1 \rho_0^2 + (11 - 3\gamma) \lambda_0 \rho_0 + 2\lambda_0^2 \right)} \lambda_0^3. \quad (51) \]

Since each numerical constant (with dimensions) \( a_1, \rho_0, \kappa_0^2, \) etc, depends on three parameters \( (\lambda_0, \rho_0, \gamma) \) then, it is necessary to study them numerically. For example, we have fixed, \( \lambda_0 = 2 > 0 \), and plotted each parameter in the plane \( (\rho_0, \gamma) \) showing that if \( \gamma \in (-1, 1) \), and \( \rho_0 \in (0, 1] \), then:

\[ a_1 > 0, \quad \kappa_0^2 > 0, \quad \Lambda_0 > 0, \quad U_0 > 0, \quad (52) \]

which, at least, is physically reasonable. We can even find regions within the subset \( \gamma \in (-1, 1) \), and \( \rho_0 \in (0, 1] \) where \( a_1 > 1 \), so the solution is accelerative. For values of \( \gamma < -1 \) some of this numerical constants take negative values. Hence, we may conclude that \( \gamma \) can only take values in the interval \( (-1, 1] \). This approach has one advantage which is related to the possibility of determining the integration constants that appear in the theoretical models. Usually, these constants are determined from the initial conditions, which are not or poorly known. The present approach allows an independent determination of the free parameters of the different cosmological models, leading to the possibility of the direct confrontation of the theoretical results with observations.

To solve the system of equations (42-45), we need to determine the behavior of \( U \), since we are taking into account the definition of \( \mathcal{E}_{ij} \) given by Eq. (5). If we consider this definition, then we get:

\[ \mathcal{L}_H (\mathcal{E}_{ij}) = 0 \quad \iff \quad \frac{\dot{U}}{U} = \frac{\kappa^2}{\kappa^2} = -\frac{2}{t}, \quad \iff \quad \kappa^2 U \approx t^{-2}. \quad (53) \]
which implies that $U \sim t^{-1}$. Hence, in this case the quantities behave as:

$$a \sim t^{a_1}, \quad \rho \sim t^{-1}, \quad \kappa^2 \sim t^{-1}, \quad \Lambda \sim t^{-2}, \quad \lambda \sim t^{-1}, \quad U \sim t^{-1}.$$ (54)

The main and unique difference between (50) and (54) is

$$U \sim t^{-1}, \quad U \sim t^{-1}. \quad \text{(55)}$$

This is an important issue, since they are different physical quantities, with different dimensional equations and therefore they bring us to obtain different cosmological implications.

We found the following solution for the system (48, 50)

$$a_1 = \frac{2U_0 + 5\rho_0}{4U_0}, \quad \kappa_0^2 = \frac{3 \left[ 75 (\gamma + 1)^2 \rho_0^2 + 4 (3\gamma + 1) U_0 (5 + U_0) \right]}{2U_0 \left[ 45 (\gamma + 1)^2 \rho_0^2 + 6 (3\gamma + 13) (\gamma + 1) U_0 \rho_0 + 8 (3\gamma + 1) U_0^2 \right]},$$

$$\Lambda_0 = \frac{3\rho_0 \left[ 1125 (\gamma + 1)^2 \rho_0^2 + 450 (3\gamma + 5) (\gamma + 1) U_0 \rho_0^2 + 20 (90\gamma + 27\gamma^2 + 47) U_0^2 \rho_0 + 8 (3\gamma + 11) (3\gamma + 1) U_0^3 \right]}{16U_0^3 \left[ 45 (\gamma + 1)^2 \rho_0^2 + 6 (3\gamma + 13) (\gamma + 1) U_0 \rho_0 + 8 (3\gamma + 1) U_0^2 \right]}, \quad \text{(56)}$$

$$\lambda_0 = \frac{15 (\gamma + 1) \rho_0^2 + 2 (3\gamma + 1) U_0 \rho_0}{4U_0},$$

where from $\lambda_0$ we may obtain $U_0 = U_0 (\rho_0, \lambda_0, \gamma)$

$$U_0 = \frac{15\rho_0 (\gamma + 1)}{2 \left[ - (3\gamma + 1) \rho_0 + 2\lambda_0 \right]}, \quad \text{(57)}$$

and to replace this result in the other numerical constants given in (56) in order to compare them with (51). The performed numerical analysis indicates that in the subset of the plane $(\rho_0, \gamma)$ given by $(\rho_0 \times \gamma)$ with $\rho_0 \in (0, 1]$ and $\gamma \in (-1, 1],$

$$a_1 > 0, \quad \kappa_0^2 > 0, \quad \Lambda_0 > 0, \quad \lambda_0 > 0,$$ (58)

that is, this solution has similar behavior as the above one given by (51). The main difference is in the behavior of $a_1$. If $U_0$ is large enough in the region $((0, 1] \times (-1, 1]),$ then $a_1 \rightarrow 1/2,$ and therefore the solution does not accelerate.

IV. ANOTHER APPROACH

We may now reason in a different way. If we consider $\kappa_0^4 = 6\kappa^2/\lambda,$ and redo the calculations, in particular the Eq. (48), we get again

$$L_H (\Lambda g_{ij}) = 0 \iff \dot{\Lambda} + 2\Lambda = 0, \iff \Lambda = \Lambda_0 t^{-2},$$ (59)

$$L_H (\kappa^2 T_{ij}^{\text{m}}) = 0 \iff \frac{\kappa^2}{\rho} - \frac{\dot{\rho}}{\rho} = -\frac{2}{t}, \iff \kappa^2 \rho \approx t^{-2},$$ (60)

$$L_H \left( \frac{\kappa^2}{\lambda} S_{ij} \right) = 0 \iff \frac{\kappa^2}{\lambda} \frac{\dot{\lambda}}{\lambda} + 2\frac{\dot{\rho}}{\rho} = -\frac{2}{t}, \iff \frac{\kappa^2}{\lambda} \rho^2 \approx t^{-2};$$ (61)

$$L_H (E_{ij}) = 0 \iff \frac{\dot{U}}{U} - \frac{\kappa^2}{\lambda} \frac{\dot{\lambda}}{\lambda} = -\frac{2}{t}, \iff \frac{U}{\kappa^2 \lambda} \approx t^{-2},$$ (62)

with $E_{ij}$ given by Eq. (7). From (61), we get $\kappa^2 \rho^2 / \lambda \approx t^{-2},$ while from (60) we have obtained $\kappa^2 \rho \approx t^{-2},$ and therefore we get a new relationship

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\rho}}{\rho} \iff \lambda \approx \rho,$$ (63)

that is, $\lambda$ and $\rho$ behave in the same way, $\lambda \approx \rho.$ From the definition (2) we observe that these two quantities have the same dimensional equation, $[\lambda] = [\rho],$ and therefore, from this point of view, they must behave in the same way.
If we take into account all these considerations, we may check that all the EFE are dimensionally consistent (with \([c] = 1\)), for example, the Friedman equation

\[
[H^2] = T^{-2} = [\kappa^2 \rho] = \frac{\kappa^2}{\lambda^2} [\rho^2] = \left[ \frac{\mathcal{U}}{\kappa^2} \right] = [\Lambda].
\]  

(64)

Now, from (62) we obtain \(\mathcal{U} \sim t^{-4}\), since \(\lambda \approx \rho\) and \(\kappa^2 \rho \approx t^{-2}\) and hence \(\kappa^2 \lambda \approx t^{-2}\). Therefore we have the following relationships

\[
a \sim t^{a_1}, \quad \rho \sim t^r, \quad \kappa^2 \sim t^g, \quad \Lambda \sim t^{-2}, \quad \lambda \sim t^r, \quad \mathcal{U} \sim t^{-4},
\]

(65)

such that \(g = -2 - r\). Note that these new results imply that, \(\kappa_5^2 \neq \text{const.}\) If \(r = -1\), then we may recover the solution (50) as a particular case. While, if we take into account Eq. (8), then the quantities must behave as

\[
a \sim t^{a_1}, \quad \rho \sim t^r, \quad \kappa^2 \sim t^{-2 - r}, \quad \Lambda \sim t^{-2}, \quad \lambda \sim t^r, \quad \mathcal{U} \sim t^r.
\]

(66)

Such solutions (65, 66) open the door to having a growing \(G\) if \(r < -2\), but we cannot use them to find solutions from Eqs. (49-51), since they have been deduced under the assumption that \(\kappa_5^2\) and \(\Lambda_5\) are constants. Therefore, we need to propose a more general framework in order to accommodate the possibility of having a \(\kappa_5^2 \neq \text{const.}\).

For example, if we fix the scale factor, \(a \sim \exp(a_1 t)\), then \(\kappa^2\) must be growing in order to keep \(\kappa^2 \rho \text{ const.}\) In the same way, \(\lambda\) must vary as \(\rho\) (that is, in a decreasing way) and if we use (7), then \(\mathcal{U} \sim \text{const.}\), while if we use Eq. (8) \(\mathcal{U} \sim \rho\), that is, it is decreasing. Note that \(\Lambda \approx \kappa^2 \lambda \approx \text{const}\), hence, it is possible to find solutions with \(\kappa^2\) growing and \(\lambda\) decreasing such that \(\Lambda \approx \text{const}\), so the hypothesis of a variable \(\lambda\) does not necessarily imply variation in all the 4D constants. These results may be formalized by employing the approach of the matter collineations, but instead of using a homothetic vector field, we need to use another VF, as for example, \(X = K \partial_t - x \partial_x - y \partial_y - z \partial_z \in \mathcal{X}(M), K \in \mathbb{R}\), since we are looking for other transformations that keep \(H \sim \text{const.}\).

These results suggest that we may consider \(\kappa_5^2\) and \(\Lambda_5\) as being variable as in the 4D model. If constants vary in 4D, why shouldn’t they vary in 5D? For example, the Kaluza-Klein models suggest that they must vary in nD (see for instance [12, 43 and 19]).

The approach is phenomenological and can be described in the following way. We consider the gravitational FE in 5D

\[
G_{5ab}^5 = \kappa_5^2 T_{5ab}^5 - \Lambda_5 g_{5ab}^5,
\]

(67)

and impose the condition \((44-46)\)

\[
\nabla^b (\kappa_5^2 T_{5ab}^5 - \Lambda_5 g_{5ab}) = 0
\]

(68)

where \(T_{5ab}^5\) is given by Eq. (2). Therefore, the above equation may written as:

\[
\nabla^b (\kappa_5^2 T_{5ab}^5) = \nabla^b (\kappa_5^2 \lambda g_{5ab}) + \nabla^b (\Lambda_5 g_{5ab}),
\]

(69)

hence

\[
\nabla^b (T_{5ab}^m) = \frac{1}{\kappa_5^2} \left[ \kappa_5^2 \nabla_a \lambda + \lambda \nabla_a \kappa_5^2 + \nabla_a \Lambda_5 - T_{5ab}^m \nabla^b \kappa_5^2 \right].
\]

(70)

We may now assume the condition \(\nabla^b (T_{5ab}^m) = 0\), in order to avoid matter creation processes (47), so simplifying Eq. (70) we get

\[
\nabla_a \Lambda_5 = (T_{5ab}^m + \lambda g_{5ab}) \nabla^b \kappa_5^2 - \kappa_5^2 \nabla_a \lambda,
\]

(71)

hence, in 4D the EFE yield

\[
G_{ij} = \kappa^2 T_{ij}^m + \kappa_5^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij},
\]

(72)

with

\[
\nabla^j \left( \kappa^2 T_{ij}^m + \kappa_5^4 S_{ij} - \mathcal{E}_{ij} - \Lambda g_{ij} \right) = 0,
\]

(73)

therefore

\[
\nabla^j \left( \mathcal{E}_{ij} \right) = T_{ij}^m \nabla^j \left( \kappa^2 \right) + \kappa^2 \nabla^j \left( T_{ij}^m \right) + S_{ij} \nabla^j \left( \kappa_5^4 \right) + \kappa_5^4 \nabla^j \left( S_{ij} \right) - \nabla^j \left( \Lambda g_{ij} \right),
\]

(74)
that can be simplified by assuming

$$\kappa^2 = \frac{\kappa^2}{6} \lambda, \quad \Lambda = \frac{1}{2} (\kappa^2 A_3 + \kappa^2 \lambda), \quad \nabla^b (T_{ab}^m) = 0. \quad (75)$$

Under all these assumptions and simplification, the resulting EFE in 4D become:

$$G_{ij} = \kappa^2 T_{ij}^m + \kappa^4 S_{ij} - E_{ij} - \Lambda g_{ij}, \quad (76)$$

$$\nabla^j T_{ij}^m = 0 \quad (77)$$

$$\nabla^j A_5 = (T_{im}^m + \lambda g_{ab}) \nabla^j (\kappa^2) - (\kappa^2) \nabla_i \lambda \quad (78)$$

$$\nabla^i (E_{ij}) = T_{ij}^m \nabla^j (\kappa^2) + S_{ij} \nabla^j \left( \frac{6 \kappa^2}{\lambda} \right) + \kappa^2 \nabla^j (S_{ij}) - \nabla^j (\Lambda g_{ij}), \quad (79)$$

which can be simplified further if we consider the special case $A_5 = 0$, hence

$$G_{ij} = \kappa^2 T_{ij}^m + \kappa^4 S_{ij} - E_{ij} - \Lambda g_{ij}, \quad (80)$$

$$\nabla^j T_{ij}^m = 0, \quad (81)$$

$$\kappa^2 \nabla_i \lambda = (T_{im}^m + \lambda g_{ab}) \nabla^j \kappa^2, \quad (82)$$

$$\nabla^i (E_{ij}) = T_{ij}^m \nabla^j (\kappa^2) + S_{ij} \nabla^j \left( \frac{6 \kappa^2}{\lambda} \right) + \frac{6 \kappa^2}{\lambda} \nabla^j (S_{ij}) - \nabla_i (\kappa^2 \lambda). \quad (83)$$

These equations are more general than $[35][37]$ and admit as solution, for example, $[55]$ or $[56]$ and avoid the case $\nabla^j T_{ij}^m \neq 0$, which is always difficult to justify from the observational point of view.

V. CONCLUSIONS

In the present paper, we have explored whether the brane-world cosmological models admit self-similar solutions. We have considered the most simple case (symmetric embedding) and a perfect fluid as matter source, and we have determined the behavior of the energy density $\rho$. We have found that this class of models do not admit self-similar solutions, since the EFE are not invariant under scale transformations. The result is quite general and valid for any self-similar metric.

Nevertheless, if we introduce the hypothesis of variable brane tension, we have shown that these new equations admit scaling symmetries and then they admit homothetic solutions. We have studied two cases, depending on the definition tensor $E_{ij}$, Eqs. $[54]$ or $[55]$. One drawback that is presented of this phenomenological model is that it depends crucially on having a non vanishing projected Weyl tensor. If $E_{ij} = 0$, then the brane tension must be constant, $\lambda = 0$, and therefore $G$ and $\Lambda$ are constant. If they really do vary, such “physical” property should not depend on the embedding.

From a cosmological, as well as a physical point of view, the relevance of the obtained results is twofold. Firstly, it gives the possibility of obtaining the functional forms of each physical quantity that admits cosmological power law solutions of the form $a(t) = t^{a_1}$, which are relevant for several phases of the evolution of the Universe. The deceleration parameter corresponding to this solution is given by $q = (1/a_1) - 1$, and it can describe accelerating cosmological models for $a_1 > 1$, decelerating evolution for $a_1 < 1$, and marginally accelerating solutions for $a_1 = 1$. Hence, one could obtain severe constraints for the allowed functional form of each physical quantity by requiring the presence in the theory of this power law form of the scale factor. Generally, for this form of the scale factor, the geometrical (Ricci scalar), and physical (energy density) parameters have a simple time dependence, being inversely proportional to the square of the time Eqs. $[54]$ and $[55]$. This indicates that the considered self-similar solutions do present a singular behavior at $t = 0$. The effective gravitational coupling has a power law dependence on time, and, it always decreases as $G_{eff} \propto t^{-1}$ while the cosmological constant behaves as $\Lambda \propto t^{-2}$, that is, they are “always” decreasing. The second implication of the self-similar approach is related to the possibility of determining the integration constants that appear in the theoretical models. Usually, these constants are determined from the initial conditions, which are not, or poorly known. The present approach allows an independent determination of the free parameters of the different cosmological models, leading to the possibility of the direct confrontation of the theoretical results with observations. We have studied two cases and both could be physically realistic Eqs. $[54]$ and $[55]$. The self-similar approach for the investigation of the gravitational theories represents a powerful approach that can give important information on the mathematical and physical structures of the particular models. In the present study we have performed such an
analysis for the brane-world cosmological models, with the results of our analysis indicating that simple power law cosmological models can be obtained in the framework of this modified gravity theory.

In the above approach, we have worked under the assumption that the fundamental constants of the theory in 5D must remain constants. The performed study of invariance of the EFE under scale transformations indicates that it could be better to consider \( \rho \sim \lambda \) as a functional relationship Eqs. (65 and 66). This result is more general than the one obtained in (50 or 54) and we recover them by fixing \( r = -1 \). In this way, we may obtain \( G_{\text{eff}} \propto t^{-2-r} \) while \( \Lambda \propto t^{-2} \) (is always valid). This new result allows the possibility of getting \( G \) as a growing, constant if \( r = -2 \) (in this limiting case, \( \Lambda \) vanishes) or as a decreasing. However, such working hypothesis could imply that the fundamental constants in 5D vary as well, and therefore the EFE (49-50) are not valid and new ones need to be derived. To this purpose, we have outlined a toy model which allows such a possibility (see Eqs. (76-79)) and which admits Eqs. (65 and 66) as solutions. In conclusion, we think that the most appropriate framework for considering the variation of the fundamental constants is the Jordan frame, where the “constants” are considered as scalar functions (see for instance [48-50]).

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