EPR and Linear GUP

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It was found by Einstein, Podolsky, and Rosen that the non-simultaneous reality of position and momentum implies that quantum mechanics cannot be considered a complete theory of reality. We show that the linear generalized uncertainty principle (LGUP) implies a simultaneous reality of position and momentum as a possible state in quantum mechanics. This possible state may ameliorate the EPR argument.

The Einstein–Podolsky–Rosen (EPR) argument \[1\] proved through logical analysis that the description of physical reality provided by quantum mechanics is incomplete due to non-simultaneous reality of incompatible quantities (in particular position and momentum). This can be expressed by the non-commutative relation between position and momentum as follows

\[ [x_i, p_j] = i \delta_{ij} \hbar \]  

(1)

The position and momentum cannot have real eigenvalue for the same eigenstate. The EPR argument was extended by Bohm \[2\] into deriving paradox with Einstein’s theory of relativity. Simply if spin 0 particle decays into two particles, one particle of them will be measured with Spin 1/2 and simultaneously the other particle will be measured as Spin -1/2. This may imply simultaneous communication between the two particles that sets paradox with Einstein’s theory of relativity. Furthermore, The EPR was studied in details by Bell \[3\] where he defined Bell’s inequalities that give information about the correlations two entangled particles. It was found that Bell’s inequalities are violated by experiments \[1, 5\]. The violation of Bell’s inequalities has been interpreted as following: Either physical reality or locality is wrong. If quantum mechanics describes physical reality, it seems to be “non-local”.

It is clear that EPR sets a dilemma between locality and completeness. A possible way to solve this dilemma is to seek the possible physical state in which incompatible quantities (position and momentum) can have a simultaneous reality. Why do we seek this physical state? Because the physical reality in this state would be considered complete, and at the same time it could explain the instantaneous message between the two entangled particles according to the interpretation of the violation of Bell’s inequalities.

In order to find a physical state in which incompatible quantities (position and momentum) have simultaneous reality, we investigate the modified theories of the uncertainty principle that is collectively known as the generalized uncertainty principle (GUP). Various approaches to quantum gravity such as string theory, loop quantum gravity, and quantum geometry suggest a generalized form of the uncertainty principle (GUP) that implies the existence of a minimum measurable length. Several forms of the GUP that include non-relativistic and relativistic forms have been proposed in \[6–14\]. Phenomenological and experimental implications of the GUP have been investigated in low and high energy regimes. These include atomic systems \[12, 16\], quantum optical systems \[17\], gravitational bar detectors \[18\], gravitational decoherence \[19\], composite particles \[20\], astrophysical systems \[21\], condensed matter systems \[22\], and macroscopic harmonic oscillators \[24\]. Reviews of the GUP, its phenomenology, and its experimental implications can be found in \[24, 25\]. The GUP motivated by string theory takes the following form:

\[ \Delta x \Delta p \geq \frac{\hbar}{2} (1 + \beta \Delta p^2) \]

(2)

where \( \beta = \beta_0 \ell_p^2 / \hbar^2 \), \( \beta_0 \) is a dimensionless constant, and \( \ell_p = 1.6162 \times 10^{-35} \text{m} \) is the Planck length. In 1926, Dirac realized that the commutator between any two variable in quantum mechanics is isomorphic to the Poisson bracket \[26\]. The isomorphic Poisson bracket to the quadratic GUP commutator is given by \[27\]

\[ \{x_i, p_j\} = [\delta_{ij} + \beta \delta_{ij} p^2 + 2 \beta p_i p_j] \]

(3)

where \( x_i \) and \( p_j \) are now c-numbers. If we look at the quadratic GUP model in Eq. \[3\], we see that the Poisson bracket can equal to zero if we allow the right hand side of Eq. \[3\] to be zero as follows

\[ \delta_{ij} + \beta \delta_{ij} p^2 + 2 \beta p_i p_j = 0 \]

(4)

For simplicity, let us consider one dimensional case

\[ 1 + 3 \beta p^2 = 0 \]

(5)

which has an imaginary solution as follows

\[ p = \frac{i}{\sqrt{3} \beta} \]

(6)

Based on the isomorphism between Poisson bracket and quantum commutator, the solution in Eq. \[6\] set a
Let us look at another model that is known as Linear GUP which was motivated by doubly special relativity (DSR) that was proposed by Magueijo and Smolin [28]. DSR suggests the existence of an invariant length/energy scale in addition to the invariance of the speed of light. The corresponding uncertainty principle of doubly special relativity was introduced in [22] and was further investigated in [13, 30]. Moreover, when linear GUP was studied with Schrödinger equation, Klein-Gordon equation and Dirac equation, a discrete picture of space is obtained along with discrete picture of energy from the same wavefunction solutions [13, 30].

The discreteness results have been obtained in weak gravity cases [31] and strong gravity case [32]. The linear GUP reads

\[ [x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) \right] \]  

where \( \alpha = \alpha_0 l_p / \hbar \), and \( \alpha_0 \) is a dimensionless constant.

The corresponding Poisson bracket of linear GUP takes the form

\[ \{x_i, p_j\} = \left[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) \right] \]

where \( x_i \) and \( p_j \) are c-numbers in Eq. [3]. Let us see a case at which Poisson bracket vanishes. This can be achieved if we allow for the right hand side of Eq. [8] to be equal to zero

\[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) = 0 \]

Multiplying both sides of Eq. [9] by \( \delta_{ij} \), we get

\[ 3 - \alpha \left( 3p + \frac{p_j^2}{p} \right) = 0 \]

where \( p^2 = p_i p_j \). Eq. [10] can be rearranged as follows

\[ 3 = 4\alpha p \]

This lead to a solution in 3 dimensional case as

\[ p = \frac{3}{4\alpha} \]

This solution is the maximal energy that corresponds to a minimal length in the linear GUP model. This solution represents the simultaneous reality between \( x \) and \( p \) through the isomorphism or correspondence between the quantum commutator and Poisson bracket. This indicates that the position and momentum will commute with each other at the minimal length scale provided by linear GUP. This implies that incompatible quantities (position and momentum) would have simultaneous reality at the minimal length scale. This simultaneous reality induces a complete picture of quantum mechanics according to the logical analysis presented in EPR analysis [1]. On the other hand, this simultaneous reality may be used to explain the instantaneous and non-local correlations when measuring the spin of two entangled particles as we explained above which was set by violation of Bell’s inequalities. Possible conceptual connection between minimal length implied by Linear GUP and spin concept was studied in [34].

In the last decade, GUP has been used to explain several phenomenological and experimental points in low and high energy systems such as atomic systems [15, 16], quantum optical systems [17], gravitational bar detectors [18], gravitational decoherence [19], composite particles [20], astrophysical systems [21], condensed matter systems [22], and macroscopic harmonic oscillators [23]. In this letter, we show that linear GUP could explain the quantum entanglement and non-local correlations of quantum mechanics.

To conclude, linear GUP is found to generate a physical state of simultaneous reality between position (minimal length) and momentum (maximum energy). This state may contain a loophole to introduce a complete picture of quantum mechanics and the EPR argument still holds.

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