Non-commutative resolutions of quotient singularities

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Resolution of singularities

Let $X$ be a (singular) algebraic variety. Then $\pi : Y \to X$ is a resolution of $X$ if $Y$ is smooth, $\pi$ is proper and birational.
Resolution of singularities

Let \( X \) be a (singular) algebraic variety. Then \( \pi : Y \to X \) is a \textbf{resolution} of \( X \) if \( Y \) is smooth, \( \pi \) is proper and birational.

A resolution is \textbf{crepant} if \( X \) is Gorenstein and \( \pi^*\omega_X = \omega_Y \).
Bondal-Orlov conjecture

Conjecture
All crepant resolutions are derived equivalent (relative to $X$).

(Bondal, Orlov)
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True if dim $X = 3$.

(Bridgeland 2002)
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In the proof

$$D^b(\text{coh} Y_1) \cong D^b(\text{coh} \mathcal{A}) \cong D^b(\text{coh} Y_2),$$

$\mathcal{A}$ a sheaf of non-commutative algebras.
Non-commutative crepant resolution of singularities

Definition

\[ k = \bar{k}, \ \text{char} \ k = 0 \]
\[ S \text{ a Noetherian normal Gorenstein domain} \]

A **non-commutative crepant resolution** (NCCR) of \( S \) is an \( S \)-algebra \( \Lambda \) satisfying

- \( \text{gldim} \ \Lambda < \infty \),
- \( \Lambda = \text{End}_S(M) \) for a finitely generated reflexive \( S \)-module \( M \),
- \( \Lambda \) is maximal Cohen-Macaulay \( S \)-module.

*(Van den Bergh 2004)*
CCR vs NCCR

$Y$ Noetherian normal Gorenstein variety, $\dim Y = \dim S$

$f : Y \to \text{Spec } S$ a projective birational map

If $Y$ is derived equivalent to a ring $\Lambda$, then the following are equivalent:

• $f$ is crepant ($f^* \omega_S = \omega_Y$),
• $\Lambda \in \text{CM } S$.

In this case $\Lambda \sim \text{End } S(M)$ for some $M \in \text{ref } S$.

If $Y$ is derived equivalent to a ring $\Lambda$, then the following are equivalent:

• $f : Y \to \text{Spec } S$ is a crepant resolution of $\text{Spec } S$,
• $\Lambda$ is an NCCR of $S$.

(Iyama, Wemyss 2014)
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Finite groups

$G$ finite group, $W$ finite dimensional $G$-representation
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$S = (SW)^G$
NCCR
Finite groups

$G$ finite group, $W$ finite dimensional $G$-representation
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G finite group, W finite dimensional G-representation
\[ S = (SW)^G, \quad U = \bigoplus_{\chi \in \hat{G}} V(\chi) \]
\[ M(V) = (V \otimes SW)^G \] module of covariants
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\( \text{mod}(G, SW) \) category of finitely generated \( G \)-equivariant \( SW \)-modules
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\[ \Rightarrow \text{mod}(G, SW) \cong \text{mod} \text{End}_{G,SW}(U \otimes SW) \cong \text{mod} M(\text{End}(U)) \]
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$\Rightarrow \text{gldim} M(\text{End}(U)) < \infty$
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$G \subseteq SL(W) \Rightarrow S$ Gorenstein, $\text{End}_S(M(U))$ NCCR of $S$

(Auslander 1962)
Quotient singularities
Reuctive groups

• determinantal varieties
  \( S_{h,n} \) – variety of \( h \times h \) matrices of rank \( \leq n \)
Quotient singularities
Reductive groups

- determinantal varieties

- Pfaffian varieties
  \( S^-_{h,2n} \) – variety of skew-symmetric \( h \times h \) matrices of rank \( \leq 2n \)
Quotient singularities
Reductive groups

- determinantal varieties

- Pfaffian varieties

- determinantal varieties of symmetric matrices
  \( S^+_{h,n} \) – variety of symmetric \( h \times h \) matrices of rank \( \leq n \)
Quotient singularities
Reductive groups

• determinantal varieties

• Pfaffian varieties

• determinantal varieties of symmetric matrices

• affine toric varieties
Quotient singularities
Reductive groups

- determinantal varieties
- Pfaffian varieties
- determinantal varieties of symmetric matrices
- affine toric varieties
- (commutative) trace rings
NCCR
Finite $\rightarrow$ Reductive

- $\text{mod}(G, SW)$ does not have a projective generator
- Modules of covariants are often not Cohen-Macaulay
NCCR
Finite $\rightarrow$ Reductive

- $\text{mod}(G, SW)$ does not have a projective generator
  //via complexes relate different projectives
- modules of covariants are often not Cohen-Macaulay
  //use results on Cohen-Macaulayness of modules of covariants by Stanley for the torus case, and by Van den Bergh for the general group case
Non-commutative resolution of singularities

Definition

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NCRs appear in Kuznetsov’s Homological Projective Duality program.
Weak NCR
Reductive groups

$G$ reductive group, $W$ finite dimensional $G$-representation
Weak NCR
Reductive groups

$G$ reductive group, $W$ finite dimensional $G$-representation

There exists a finite-dimensional $G$-representation $U$ such that $\text{gldim } M(\text{End } U) < \infty$. 
$X = \text{Spec } SW$, $X^s = \{x \in X \mid x \text{ has a closed orbit and trivial stabilizer}\}$

$\mathcal{W}$ is **generic** if $\text{codim}(X - X^s) \geq 2$.

Let $\mathcal{W}$ be generic. There exists a finite-dimensional $G$-representation $U$ such that $\text{End}_{SW^G}(M(U))$ is a NCR of $SW^G$. 
$G$ connected, $T$ maximal torus in $G$, $B$ Borel subgroup
$X(T)^+$ dominant cone
$\bar{\rho} = \frac{1}{2}$(sum of positive roots)
$W$ $d$-dimensional $G$-representation
$\{\beta_i\}_{i=1}^d$ $T$-weights of $W$
$\Sigma = \{\sum_i a_i \beta_i \mid a_i \in (-1, 0]\}$
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$\mathcal{L} = X(T)^+ \cap (-\bar{\rho} + \Sigma)$
$U = \bigoplus_{\chi \in \mathcal{L}} V(\chi)$

Then $\text{gldim}(M(\text{End}U)) < \infty$. 
NCRs for affine toric varieties

$S$ finitely generated, normal, positive semigroup $\subset \mathbb{Z}^m$

$\text{End}_{k[S]} \left( k\left[\frac{1}{n}S\right]\right)$ is a NCR of $k[S]$ for all $n \gg 0$. 
NCRs

Example

- \( G = k^*, \ W = k^{\oplus 4} : (-3, -2, 1, 4) \)

\[ U = \bigoplus_{i=-4}^{4} V(i) \]

End_{SW^G}(M(U)) is a NCR of SW^G.
• $G = k^*^2$, $\mathcal{W} = k^\oplus^6$: $(1, 0), (1, 1), (0, 1), (−1, 0), (−1, −1), (0, −1)$

$U = V(1, 0) \oplus V(1, 1) \oplus V(0, 1) \oplus V(0, 0) \oplus V(−1, 0) \oplus V(−1, −1) \oplus V(0, −1)$

$\text{End}_{\mathcal{SW}^G}(M(U))$ is a NCR of $\mathcal{SW}^G$. 
Modules of covariants

CM criterion

\[ M(V(\chi)^*) \] is Cohen-Macaulay \( SW^G \)-module for \( \chi \in -2\bar{\rho} + \Sigma \).

(Stanley 1972, Van den Bergh 1991)
$W$ is quasi-symmetric if $\sum_{\beta_i \in \ell} \beta_i = 0$ for all lines $\ell \subseteq X(T)_\mathbb{R}$ through the origin.

If $W$ is generic, quasi-symmetric + technical conditions, then $SW^G$ has an NCCR.
NCCR

Example

- $G = k^*$, $W = k^* \oplus^4: (-3, -2, 1, 4)$

$U = \bigoplus_{i=-2}^{2} V(i)$

$\text{End}_{SW^G}(M(U))$ is an NCCR of $SW^G$. 
NCCR

Example

• \( G = k^*^2, \ W = k^{\oplus 6}: (1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1), (0, -1) \)

\( U = V(0, 0) \oplus V(1, 0) \oplus V(1, 1) \)

End\(_{SWG}(M(U))\) is an NCCR of \( SW^G \).
NCCR
Quotient singularities examples

- determinantal varieties
- Pfaffian varieties $S_{h,2n}^-$ if $h$ odd
- determinantal varieties of symmetric matrices $S_{h,n}^+$ if $h \equiv n + 1 \pmod{2}$, if $h \equiv n \pmod{2}$ twisted NCCRs
- (commutative) trace rings (twisted NCCRs)
NCCR

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- (commutative) trace rings (twisted NCCRs)
- Gorenstein affine toric varieties of dimension 3

(Broomhead 2012)
NCCR

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(Broomhead 2012)

• determinantal varieties

(Buchweitz, Leuschke, VdB, 2010)

• NCCRs for $S_{h,4}^- \setminus \{0\}$ for $h$ odd

(Kuznetsov 2008)

• NCRs for $S_{h,n}^-$, $S_{h,n}^+$

(Weyman, Zhao 2012)
NCCR
(Counter)examples

- $G = k^*^2$, $W = k^{\oplus 6}$: $(1, 0), (3, 3), (0, 1), (-3, 0), (-1, -1), (0, -3)$
  
  $SW^G$ does not have an NCCR given by modules of covariants.
NCCR
(Counter)examples

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$SW^G$ does not have an NCCR given by modules of covariants.

• $G = SL_2$, $W = V^{\oplus^4}$

$SW^G \cong k[x_1, \ldots, x_6]/(x_1^2 + \cdots + x_6^2)$
NCCR
(Counter)examples

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  $SW^G$ does not have an NCCR given by modules of covariants.

• $G = SL_2$, $W = V^{\oplus 4}$
  $SW^G \cong k[x_1, \ldots, x_6]/(x_1^2 + \cdots + x_6^2)$
  $SW^G$ does not have an NCCR.

  (Quarles 2005)
Non-commutative Bondal-Orlov conjecture

Conjecture
All NCCRs are derived equivalent.

(Van den Bergh; Iyama, Wemyss)
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Sufficient depth criterion. True if dim $S = 3$ for CM rings.

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Let $G = k^*$, $W = k^\oplus n$. Are all NCCRs for $SW^G$ derived equivalent?
Non-commutative Bondal-Orlov conjecture

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All NCCRs are derived equivalent.

(\text{Van den Bergh; Iyama, Wemyss})

Sufficient depth criterion. True if \text{dim } S = 3 \text{ for CM rings.}

(Iyama, Wemyss 2013)

Let \( G = k^* \), \( W = k^\oplus n \). Are all NCCRs for \( SW^G \) derived equivalent? (All modules of covariants giving an NCCR have the same number of indecomposable summands.)
Thank you.