Abstract

In this paper we generalize and put in a new light part of “Fourier analysis on Number fields and Hecke’s zeta function” [14] by Tate. We express the relative Euler characteristic using purely adelic language. By using certain natural normalization of Haar measure on adeles we obtain the relative Riemann-Roch theorem. In particular we show that using our relative normalization of the Haar measure on adeles we can obtain the relative Riemann-Roch theorem from the adelic Poisson summation formulae. In addition, using our methods we define the relative 'size of cohomology' numbers, i.e. extract the $h^0$ and $h^1$ part of the relative Euler characteristic. Our theory not only covers both absolute and relative cases, but also the case of an arithmetic curve and a nonsingular, projective curve over a finite field.

1 Introduction

Tate’s Fourier analysis on Number Fields and Hecke’s zeta function, preceded by work of Iwasawa ([6],[7]), showed that adelic language can be extremely useful. Even though [14] only considers number fields, to both the function field of an algebraic curve over a finite field and a number field one can associate its ring of adeles which is locally compact, Pontryagin self-dual abelian group. It is the restricted direct product of all the completions of the field. In his PhD thesis [14], John Tate used Fourier analysis on adelic rings of number fields to prove analytic continuation and functional equation for Dedekind zeta functions. Another important result appearing in that work is the Riemann-Roch theorem obtained as a direct consequence of the Poisson summation formula for the ring of adeles. One can see that this method allows us to obtain uniformly both the case of a (completed) spectrum of a number field and a proper algebraic curve over a finite field ([13, Chapter 7.2], [15, Proposition.
It is fascinating as in the geometric case the Euler characteristic is just an alternating sum of the dimensions of the cohomology groups viewed as vector spaces over the field of definition whereas in the arithmetic case it is a covolume of an appropriate lattice. In spite of that, harmonic analysis on rings of adeles miraculously provides us with the point of view from which both the geometric and the arithmetic case look the same. The analogy between number fields and function fields is one of the most important aspects of modern number theory. In particular, it triggered the development of geometric methods in number theory. As geometric counterparts of some of the most fundamental open problems in number theory are solved (see [9],[16]), deepening the understanding of the geometric-arithmetic analogy is of utmost importance.

This work revisits the global(adelic) Fourier analysis on global fields. We write the Euler characteristic for a given divisor as a single integral of the associated eigenfunction of the adelic Fourier transform. With this approach it is easy to see that the Riemann-Roch theorem holds. Serre’s duality can be obtained via Fourier transform correspondences. We extend the method to the relative case, i.e. morphisms between curves like in [10, Section V.1] (except that we also cover its geometric counterpart). It is also compatible with functorial approach that gives the Grothendieck-Riemann-Roch theorem for one-dimensional global fields (see for example [12, Chapter III] for the number field case). Surprisingly, this can be achieved by using a natural renormalisation of the Haar measure on the ring of adeles. We show that the Poisson summation formulae gives the relative Riemann-Roch theorem if applied to adeles with our new relative Haar measure.

In particular, we can write the relative Euler characteristics using just a single integral

\[ \chi_{K/L}(D_\alpha) = \log \int_{K} f_\alpha \, d\mu_{K/L}^{K/L} \]

where \( D_\alpha \) is a divisor on the given curve associated with an idele (a unit in the ring of adeles) \( \alpha \). \( f_\alpha \) is an eigenfunction of the global Fourier transform associated to \( \alpha \) and the integral is taken over the ring of adeles \( K \) and \( \mu_{K/L}^{K/L} \) is the ‘relative’ measure on \( K \) associated to the finite and separable field extension \( K/L \). The measure \( \mu_{K/L}^{K/L} \) is an appropriate “relative” normalisation of the Haar measure on \( K \)

\[ \mu_{K/L}^{K/L} \left( \prod_v O_{K_v} \times D_\infty \right) = N(\mathfrak{d}_{K/L})^{-1/2} N(\mathfrak{d}_{K/Q})^{1/2} \]

i.e. the measure of the fundamental domain of \( K \) with respect to the action of the field \( K \) is equal to the square root of the ideal norm \( N \) of the relative discriminant \( d_{K/L} \) in \( O_K \). Here \( D_\infty \) is the fundamental domain of \( K \otimes_\mathbb{Q} \mathbb{R} \) with respect to \( O_K \) if \( K \) is a number field and is trivial otherwise. In addition, we are able to directly define the ‘relative size of cohomology’ i.e. relative \( h^0 \) and \( h^1 \) using the Fourier analysis insights.

The paper is organized in the following way: first we recall the theory of local and global discriminant which will later play a role in the normalization of the Haar measures. We recall briefly the theory of adeles for one dimensional global fields.

In the following section we define the size of cohomology spaces as certain adelic integrals of eigenfunctions of adelic Fourier transform on significant adelic spaces. We see that the Riemann-Roch theorem holds and deduce Serre’s duality using harmonic analysis on number fields. All these definitions are compatible with the previous related work (e.g. [15],[1],[11])

In the section 4,5 and 6 we present the relative analogue of the theory i.e. for finite separable extensions of global fields which correspond to morphisms of curves. We show that in our setting we obtain the arithmetic Riemann-Roch theorem as stated in [10, Theorem V.1.2]. In section 6 we define relative size of cohomology, i.e. the \( h^0 \) and \( h^1 \) in the relative case, and show that Serre’s duality holds for such objects. In the appendix we quickly outline harmonic analysis on local non-archimedean fields that is used multiple times to complete calculations presented in this paper and is less classical than harmonic analysis on archimedean local fields.

These developments are not only inspired by mentioned Tate’s thesis. A person familiar with A.Boriso’s work on arithmetic cohomology theory introduced in [1] can easily notice similarities between his and our constructions. Even though we operate with eigenfunctions of the global[adelic] Fourier transform instead of explicit cohomology spaces, one can see that those two approaches are compatible.

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2 Background

2.1 Preliminaries

In this paper we uniformly treat all one-dimensional global fields $K$ i.e. number fields and the fields of functions of projective geometrically integral algebraic curves over a finite field. In particular, $K$ is a finite extension of the field of rational numbers $\mathbb{Q}$ or $K$ is a finite separable extension of $\mathbb{Q} = \mathbb{F}_p(t)$, where $t$ is transcendental over $\mathbb{F}_p$ and $\mathbb{F}_p$ is a finite field with $p$ elements. We will refer to number fields as arithmetic curves. The results in this paper are uniform for both arithmetic and algebraic curves. We denote by $O_K$ the ring of integers of $K$, which is the integral closure of $\mathbb{Z}$, $Z = \mathbb{F}_p[t]$ for the archimedean and the geometric case respectively. We denote by $X_K$ the "geometric object" associated to the global field $K$: if $K$ is a number field, then $X_K$ is the completion of the set of prime ideals of $O_K$ Spec $O_K$ by the places at infinity i.e. the set of all $\mathbb{C}$-embeddings of $K$ up to conjugacy; if $K$ is a function field, then $X_K$ is the corresponding algebraic curve. When $K$ is a number field, we denote by $\sigma : K \to \mathbb{C}$ a generic field embedding, and we identify conjugate embeddings of $K$. In this way $\sigma$ can be thought as an archimedean place $v$ of $K$ and moreover we fix the constant

$$e_v := \begin{cases} 1 & \text{if } v \text{ is real} \\ 2 & \text{if } v \text{ is complex} \end{cases}$$

which is the degree of $K_v$ over the field of real numbers $\mathbb{R}$. Let $K/L$ be a finite extension of global fields. We have the notion of a relative different $\mathcal{D}_{K/L}$ and a relative discriminant $\mathfrak{d}_{K/L}$. Recall that $\mathcal{D}_{K/L}$ is an ideal of $O_K$ divisible exactly by the ramified ideals of the extension, and $\mathfrak{d}_{K/L}$ is an ideal of $O_L$ divisible exactly by the ramified ideals. We denote by

$$d_{K/L} := N(\mathfrak{d}_{K/L})$$

where $N$ is the ideal norm function inside $O_L$. We will frequently write $d_K$ instead of $d_{K/L}$ if $L = \mathbb{Q}$.

$$d_K = \begin{cases} N(\mathfrak{d}_K) & K \text{ number field}, \\ q_K^{2g_K-2} & K \text{ function field}. \end{cases}$$

where $q_K$ is the cardinality of the field of constants of $K$ and $g_K$ is the genus of the curve $X_K$.

2.2 Local fields

Let $K$ be a global field and let $| \cdot |_1$ and $| \cdot |_2$ be absolute value functions on $K$.

**Definition 2.1.** We say that $| \cdot |_1$ and $| \cdot |_2$ are equivalent if they define the same topology on $K$.

It is a known fact that $| \cdot |_1$ and $| \cdot |_2$ are equivalent if and only if there exists a real number $c > 0$ such that $|a|_1 = |a|^c_2$ for all $a \in K$.

**Definition 2.2.** We will refer to an equivalence class of absolute value functions on $K$ as a place $v$ of $K$.

The completion of $K$ with respect to a norm is the same within its equivalence class $v$. We will denote it as $K_v$. $K_v$ is a locally compact additive group with the topology induced by the norm function. We will refer to it as a local field. As a result of being locally compact, there exists a Haar measure $\mu_{K_v}$ on $K_v$ which is unique up to a positive constant. When $v$ is non-archimedean one puts

- $O_v := \{ a \in K_v : |a|_v \leq 1 \}$,
- $\mathfrak{p}_v := \{ a \in K_v : |a|_v < 1 \}$.

The ring $O_v$ is a complete discrete valuation ring and $\mathfrak{p}_v$ is its maximal ideal. We will denote by $\pi_v \in \mathfrak{p}_v \cap O_v$ the uniformizing parameter, by $k(v) = O_v/\mathfrak{p}_v$ the residue field of $O_v$ and by $\nu_v : K_v \to \mathbb{Z}$ the associated valuation function.

We chose a representative of the place $v$ that we will refer to as the normalised absolute value $| \cdot |_v : K_v \to \mathbb{R}_{\geq 0}$ on $K_v$ as

$$|a|_v = \frac{\mu_{K_v}(aV)}{\mu_{K_v}(V)},$$

where $V$ is any $\mu_{K_v}$-measurable set. It is easy to see that $| \cdot |_v$ does not depend on the choice of the Haar measure and the measurable set. One can check that

- $|a|_v$ is the standard absolute value when $K_v = \mathbb{R}$
- $|a|_v$ is the square of the standard absolute value when $K_v = \mathbb{C}$
- $|a|_v = (\#k(v))^{-\nu_v(a)}$. 

3
If \( v \) is real archimedean or non-archimedean then \(|\cdot|_v\) is indeed an absolute value function on \( K_v \). When \( v \) is a complex archimedean place then \(|\cdot|_v\) does not satisfy the triangle inequality. As a consequence, \(|\cdot|_v\) does not belong to the equivalence class of norms defining the place \( v \). Nevertheless, \(|\cdot|_v\) still gives us the desired topology on \( K_v \) i.e. the topology given by any representative of \( v \). This is due to the fact that we identify complex embeddings that are conjugate to each other, but still need to count the absolute value functions associated to each of them.

Let \( K/L \) be a finite extension of global fields. We say that a place \( v \) of \( K \) lies above a place \( w \) of \( L \) (or equivalently that \( w \) lies under \( v \)) when \( K_v \) is a finite extension of \( L_w \). In this case we often write \( v|w \). The relation between \(|\cdot|_v\) on \( K_v \) and \(|\cdot|_w\) on given by the following formula

\[
|a|_v = |N_{K_v/L_w}(a)|_w
\]

(see for example [12, Prop III.1.2.(v)]). In particular the normalised absolute value functions we chose satisfy the product formula, i.e. for \( a \in K \) one has

\[
\prod_v |a|_v = 1.
\]

(see for example [12, Prop III.1.3]).

For a non-archimedean place \( w \) of \( L \) lying under \( v \) we fix the Haar measure on \( K_v \) in the following way

\[
\mu_{K_v|L_w}(O_{K_v}) := |\delta_{K_v/L_w}|^{-1/2} := N(\delta_{K_v/L_w})^{-1/2}
\]

where \( \delta_{K_v/L_w} \) is the local discriminant for \( K_v/L_w \) (see [12, Chapter III §2]). \( L_w \) is the local field of \( L \) at the place \( w \) and \( N \) is the integral norm function inside \( O_L \). When \( v \) is archimedean we set the standard Lebesgue measure if \( K_v = \mathbb{R} \) and twice the standard Lebesgue measure if \( K_v = \mathbb{C} \). We define the Fourier transform \( \hat{f} \) by

\[
\hat{f}(\eta) = \int f(a) \chi_v(a \cdot \eta) \, \mathrm{d}\mu_{K_v/L_w}(a)
\]

for any \( f \) for which this integral is defined. In this paper we will only need to deal with eigenfunctions of the Fourier transform, i.e. functions of the form

- \( \text{char}_{v|\mathbb{Q}_v}, m \in \mathbb{Z} \) for \( v \) non-archimedean,
- \( \exp(-e_v \pi |a|^{-2} a \alpha^{-\alpha}) \), \( \alpha \in K_v^\times \) \( e_v = [K_v : \mathbb{R}] \) for \( v \) archimedean.

Each \( K_v \) is non-canonically Pontryagin self-dual for any \( v \). To see this, one can write the following map

\[
\begin{array}{ccc}
K_v & \xrightarrow{\psi} & \hat{K}_v \\
\downarrow & & \downarrow \\
\chi_v & \xmapsto{} & x \mapsto \chi_v(a \cdot x)
\end{array}
\]

where \( \chi_v \) is a non-trivial character of \( K_v \). One can check that this correspondence is a topological isomorphism (see for example [14, 2.2]). To choose \( \chi_v \), isomorphism we are going to choose a so-called standard character for each \( v \) (check A for details).

On the Pontryagin dual \( \hat{K}_v \), isomorphic to \( K_v \), we choose the measure \( \hat{\mu}_{K_v/L_w} \) dual to \( \mu_{K_v/L_w} \) i.e. such that for each local Fourier transform the following inversion formula holds (see [14, Theorem 2.2.2]):

\[
f(\xi) = \hat{f}(-\xi) .
\]

For \( K_v = Q_p \), where \( p|v \) and \( Q_p \) is the localisation of \( Q \) at \( p \), the dual measure \( \hat{\mu}_{K_v/L_w} \) is equal to \( \mu_{K_v/L_w} \) under the identification \( \hat{K}_v \simeq K_v \). For all the remaining case the Haar measure on the dual space \( \hat{K}_v \) is different from the one on \( K_v \).

The relative discriminant \( \delta_{K_v/L_w} \) with respect to the extension \( K_v/L_w \), can be seen as an ideal of \( O_L \).

By [12, Chapter III.2, (2.11)] we have the following relation between local and global discriminants:

\[
\delta_{K/L} = \prod_{w|v} \prod_{v|w} \delta_{K_w/L_w}
\]

where \( v \) runs through places of \( K \) and \( w \) runs through places of \( L \). One can deduce that

\[
d_K = \prod_v d_{K_v}
\]

and

\[
d_{K/L} = \prod_{w|v} \prod_{v|w} d_{K_w/L_w} = d_K/d_L^{d_{K/L}} ,
\]

(2)
where \( d_{K/L} := N(\mathfrak{d}_{K/L}) \), \( d_{K_v/L_v} := N(\mathfrak{d}_{K_v/L_v}) \) is the ideal norm inside \( O_L \) of the relative discriminant \( \mathfrak{d}_{K/L} \), \( \mathfrak{d}_{K_v/L_v} \) accordingly (see for example [8]).

When \( v \) is archimedean we put \( d_{K_v/L_v} = 1 \), i.e. we always see the archimedean places as unramified. In the geometric case we can always choose the transcendental element \( t \) of \( L \) such that the extension \( K/L \) is unramified for each infinite place, i.e. the local discriminant \( \mathfrak{d}_{K_v/L_v} \) is trivial for \( v \mid w \) infinite (see [8][Lemma]). We need to work with this choice of \( t \) so that the equation 2 holds (see [8][Theorem]).

### 2.3 Adeles

We define the ring of adeles of \( K \) as

\[
\mathbb{A}_K = \{(a_v)_v \in \prod_v K_v \mid a_v \in O_v \text{ for all but finitely many } v\},
\]

with componentwise addition and multiplication. The adeles form a topological ring endowed with the restricted direct product topology (see [14, Section 3]). It is important to keep in mind that it is different from the topology induced by the product topology on \( \prod_v K_v \). Adele ring is a special case of restricted direct product of topological abelian groups with respect to their compact subgroups.

In particular the conditions imposed on the direct product make the additive group of adeles locally compact. The Haar measure \( \mu_{\mathbb{A}_K} \) on \( \mathbb{A}_K \) is just a restricted direct product of local Haar measures ([14, Section 3.3]). In particular an adelic function

\[
f = \prod_v f_v
\]
satisfying certain convergence conditions can be integrated in the following way

\[
\int f d\mu_{\mathbb{A}_K} = \prod_v \int f_v.
\]

Given the collection of standard characters \( \{\chi_v\} \) we define the global standard character

\[
\chi = \prod_v \chi_v
\]
on \( \mathbb{A}_K \). We can check that

\[
\mathbb{A}_K \xrightarrow{\cup} \mathbb{A}_K \xrightarrow{\cup} \mathbb{A}_K \quad a \xrightarrow{x} \chi(a \cdot x)
\]

where \( a = (a_v) \in \mathbb{A}_K, x = (x_v) \in \mathbb{A}_K \), is a topological isomorphism. Therefore \( \mathbb{A}_K \) is (non-canonically) Pontryagin self dual. We define the Fourier transform

\[
\hat{f}(\eta) = \int f(a)\chi(\eta \cdot a) d\mu(a)
\]
on \( \mathbb{A}_K \) for \( f = \prod_v f_v \), for which the integral exists. If the Fourier inversion formula holds for the chosen normalizations of the Haar measures on the local fields, then in holds for the induced normalisation of the Haar measure on the Ring of adeles (see [14, Section 3]).

It can be shown that:

- With the diagonal embedding \( K \) can be seen as a discrete subgroup of \( \mathbb{A}_K \).
- The quotient \( \mathbb{A}_K/K \) is compact.

Check [14][Corollary 4.1.1] for details.

The invertible adelic elements

\[
\mathbb{A}_K^* = \{(a_v)_v \in \prod_v K_v^* \mid a_v \in O_v^* \text{ for all but finitely many } v\}
\]

are called ideles. We have induced an absolute value function \( \mathbb{A}_K^* \): if \( \alpha = (a_v) \), then we put \( |\alpha| = \prod_v |a_v|_v \). We have a surjective group homomorphism

\[
\mathbb{A}_K^* \xrightarrow{\cup} \text{Div}(X_K) \xrightarrow{\cup} D_\alpha := \sum_v n_v(\alpha_v)[v].
\]

In the number field case by \( \text{Div}(X) \) we mean the group of Arakelov divisors of \( X \). We will denote by \( \kappa_{K/L} \in \mathbb{A}_K^* \) an idele that maps to the canonical divisor of \( K \), i.e. the divisor associated to \( \mathfrak{d}_{K/L} \). We will write \( \kappa_K \) instead of \( \kappa_{K/Q} \). Notice that

\[
|\kappa_{K/L}| = d_{K/L}.
\]
2.4 Integration on subspaces and dual spaces

In this section we will discuss the theory of integration on the spaces \( \mathcal{A}_K, K, \mathcal{A}_K/K \), their dual spaces in the sense of Pontryagin, and relations between them.

We have the following identifications

\[
\mathcal{A}_K \cong \mathcal{A}_K, \quad \mathcal{A}_K/K \cong K, \quad \hat{K} \cong \mathcal{A}_K/K,
\]

i.e. \( \mathcal{A}_K \) is Pontryagin self-dual, and \( \mathcal{A}_K/K \) and \( K \) are Pontryagin dual to each other. For a fixed Haar measure on a locally compact group \( G \) we have a uniquely determined dual measure on \( \hat{G} \), i.e. a measure such that the Fourier inversion formula holds. Notice however even though \( \mathcal{A}_K \) is self-dual, a Haar measure on \( \mathcal{A}_K \) does not need to be self-dual. Since Haar measure is unique up to a constant, there exists exactly one which is equal to its dual measure.

We call an adelic function periodic if for all \( w \in K, u \in \mathcal{A}_K \) \( f(u + w) = f(u) \). A periodic function \( f \) on \( \mathcal{A}_K \) naturally induces a function on \( \mathcal{A}_K/K \). Given a function \( f \) on \( \mathcal{A}_K \) we call the induced function

\[
\hat{f}(\bar{u}) = \sum_{w \in K} f(\bar{u} + w)
\]
on \( \mathcal{A}_K/K \) the periodification of \( f \).

Having chosen a normalisation \( \mu_{\mathcal{A}_K} \) of the Haar measure on \( \mathcal{A}_K \), we still need to fix the normalisation \( \mu_K, \mu_{\mathcal{A}_K/K} \) of Haar measures on \( K \) and \( \mathcal{A}_K/K \) respectively, in a way that the three measures are compatible, i.e. that the Weil’s formula

\[
\int_{\mathcal{A}_K} f(u) d\mu_{\mathcal{A}_K} = \int_{\mathcal{A}_K/K} \left( \sum_{w \in K} f(w + \bar{u}) d\mu_{\mathcal{A}_K/K} \right) = \int_{K} \int_{\mathcal{A}_K/K} f(w + \bar{u}) d\mu_K d\mu_{\mathcal{A}_K/K} \tag{3}
\]

holds (see for example [4, Chapter 5 §11.1]). Compatibility amounts to the fact that \( \mu_{\mathcal{A}_K/K}(\hat{K}/K) \cdot \mu_K(\{u\}) = \mu_{\mathcal{A}_K/K}(\{u\}) \cdot \mu_K(\{u\}) \), where \( D \) is the fundamental region of \( \mathcal{A}_K \) with respect to the action of \( K \). As a consequence of the Weil’s formula the choice of normalisation of any two measures \( \mu_{\mathcal{A}_K/K}, \mu_K, \mu_{\mathcal{A}_K/K} \) determines the third. The Haar measure normalisation on \( \mathcal{A}_K/K \) will be always uniquely determined by the choice of normalisation of the Haar measure of \( \mathcal{A}_K \) by the means of the following proposition:

**Proposition 2.3.** For our choice of a normalisations of \( \mu_K, \mu_{\mathcal{A}_K} \) and a continuous periodic function \( f \) we have

\[
\int_{\mathcal{A}_K/K} f(u) d\mu_{\mathcal{A}_K/K} = \int_{\mathcal{A}_K/K} f(u) d\mu_{\mathcal{A}_K/K} \]

Proof. It is a generalisation of [14, Lemma 4.2.1] and the proof is the same, i.e. one sees that

\[
I(f) = \int_{\mathcal{A}_K/K} f(u) d\mu_{\mathcal{A}_K/K}
\]
is a functional on \( \mathcal{A}_K/K \) satisfying the properties of the Haar integral. \( \square \)

As a consequence of the Weil’s formula and 2.3 the measure on \( K \) will be always chosen as the counting measure (atomic measure with the coefficient 1). This rule does not apply to the Pontryagin dual of \( \mathcal{A}_K/K \)

\[
\mathcal{A}_K/K \cong K.
\]

Even though it can be identified with \( K \), we are forced to consider the measure \( \widetilde{\mu}_{\mathcal{A}_K/K} \) dual to \( \mu_{\mathcal{A}_K/K} \) chosen on \( \mathcal{A}_K/K \) for the inversion formula to hold. Analogously, the normalisation of the measure on

\[
\hat{K} \cong \mathcal{A}_K/K
\]
is the probabilistic measure, i.e. the measure dual to the counting measure we fixed on \( K \). In particular, it can be different from the normalisation we choose on \( \mathcal{A}_K/K \) by the means of 2.3.

**Remark 2.4.** If the measure we fix on \( \mathcal{A}_K \) is the self-dual one, then the situation simplifies, i.e. the measures on \( K \) and \( \mathcal{A}_K/K \) agree, and the measures on \( \mathcal{A}_K/K \) and \( \hat{K} \) agree.

**Example 2.5.** Let \( K \) be a number field. For any non-archimedean place \( v \) of \( K \) consider the relative measure \( \mu_{K,v} \), i.e. the Haar measure such that

\[
\mu_{K,v}(O_K) = d_{K,v}/q_v.
\]
For an archimedean \( v \) we take \( \mu_K \) to be the Lebesgue measure if \( v \) is real and twice the Lebesgue measure if \( v \) is complex. One can check that the measure

\[
\mu_{\mathbb{A}_K} = \prod_v \mu_{K_v}
\]

is the self-dual Haar measure on \( \mathbb{A}_K \). Denote by \( \mu_K \) and \( \mu_{\mathbb{A}_K/K} \) the counting measure and the measure on \( \mathbb{A}_K/K \) induced by the Weil’s formula (or 2.3) accordingly. It is easy to check that \( \mu_{\mathbb{A}_K/K} \) is the probabilistic measure.

### 3 Adelic representation of the size of cohomology and the Euler characteristics

Cohomology groups are not defined for arithmetic curves as our object is no longer a scheme after adding the part at infinity. Nevertheless, Riemann-Roch formulae requires only the “size of the cohomology”. In the geometric case this is the dimension of the groups of cohomology.

In this section we will work with a global field \( K \) with the self-dual measure from 2.5. We consider the following adelic function associated to an idele \( \alpha \)

\[
f_\alpha = \prod_v f_{\alpha_v}(u_v) : \mathbb{A}_K \to \mathbb{R}
\]

where

\[
f_{\alpha_v}(u_v) = \begin{cases} 
\text{char}_{O_v}(\alpha_v^{-1}u_v) & v \text{ non-archimedean} \\
\exp(-e_v \pi |\alpha_v^{-1}u_v|^{2/e_v}) & v \text{ archimedean}
\end{cases}
\]

where \( \text{char}_{O_v} \) is the characteristic function of \( O_v \subset K_v \) for non-archimedean \( v \), and \( e_v = [K_v : \mathbb{R}] \) for archimedean \( v \). This class of functions is the only one we need for our theory. One can check ([14, Lemma 3.2.3]) that the functions in this class are integrable with respect to our measure. As mentioned before \( f_{\alpha_v} \) are eigenfunctions of the local Fourier transforms.

We define a function

\[
\bar{f}_\alpha : \mathbb{A}_K/K \to \mathbb{R}
\]

where

\[
\bar{f}_\alpha(u + K) = \left( \int_K f_\alpha(w)dw \right)^{-1} \left( \int_{\mathbb{A}_K/K} \bar{f}_\alpha(u + w)dw \right)
\]

It is easy to see that \( \bar{f}_\alpha \) is well defined on the quotient \( \mathbb{A}_K/K \). In particular \( \bar{f}_\alpha \) is just the periodification \( \tilde{f}_\alpha \) of \( f_\alpha \) normalised so that \( \bar{f}_\alpha(\bar{0}) = 1 \).

We define

\[
h^0(D_\alpha) := \log \int_K f_\alpha(w)d\mu_K(w) \quad (4)
\]

\[
h^1(D_\alpha) := \log \left( \int_{\mathbb{A}_K/K} \bar{f}_\alpha(\bar{u})d\mu_{\mathbb{A}_K/K}(\bar{u}) \right)^{-1} \quad (5)
\]

Recall that \( \mu_K \) and \( \mu_{\mathbb{A}_K/K} \) are the counting and the probabilistic measure correspondingly. It makes sense to define

\[
\chi(D_\alpha) := \log \int_{\mathbb{A}_K} f_\alpha d\mu_{\mathbb{A}_K}.
\]

We chose the Haar measures in the way that the Weil’s equation 3 holds. In other words we have the following relation:

\[
\int_{\mathbb{A}_K} f_\alpha(u)d\mu_{\mathbb{A}_K}(u) = \int_{\mathbb{A}_K/K} \left( \int_K f_\alpha(\bar{u} + w)d\mu_K(w) \right) d\mu_{\mathbb{A}_K/K}(\bar{u}),
\]

and therefore one sees that

\[
\int_{\mathbb{A}_K} f_\alpha = \int_K f_\alpha \cdot \int_{\mathbb{A}_K/K} \bar{f}_\alpha.
\]

It is equivalent to

\[
\chi(D_\alpha) = h^0(D_\alpha) - h^1(D_\alpha)
\]
and justifies the definition of $\chi$. It is easy to see that

$$\chi_K(D_\alpha) = \log |\alpha| - \frac{1}{2} \log |\kappa_K|.$$ 

Indeed, the value of the integral for any $D_\alpha$ is uniquely determined by the adelic norm $|\alpha|$ (see [17, Chapter 1]) and the integral

$$\int_{\mathcal{A}_K} f_\alpha \mu_{\mathcal{A}_K}.$$ 

Note that the adelic norm is a product of local norms (see [14, Lemma 4.1.2] ) $|\alpha|_v$. Hence, one sees that

$$\int_{\mathcal{A}_K} f_\alpha \mu_{\mathcal{A}_K} = |\alpha| \int_{\mathcal{A}_K} f_\alpha \mu_{\mathcal{A}_K}.$$ 

To conclude one just needs

$$\log \int_{\mathcal{A}_K} f_\alpha d\mu_{\mathcal{A}_K} = \chi_K(D_1) = \log \prod_v \int_{K_v} \text{char}_{O_{K_v}}(u_v) d\mu_{K_v} = \log \left( \prod_v \prod_{w|v} d_{K_v/w}^{1/2} \right) = -\frac{1}{2} \log d_K$$

3.1 Serre’s duality

In this section we will prove Serre’s duality in the framework of our theory. We know that the Pontryagin dual of $K$ is $\mathcal{A}_K/K$. We will start by justifying that

$$\widehat{f_\alpha} = \left( \int_{\mathcal{A}_K} f_\alpha d\mu_{\mathcal{A}_K} \right)^{-1} \cdot \widehat{f_\alpha} = f_{\alpha^{-1}|_K} \cdot \widehat{f_\alpha}(0)$$ (6)

where $\widehat{f_\alpha}$ is the Fourier transform of $f_\alpha$ with respect to the measure and the character (determined by the standard character of $\mathcal{A}_K$) of $\mathcal{A}_K/K$ and $\widehat{f_\alpha}$ is the Fourier transform of $f_\alpha$ with respect to the measure and the standard character of $\mathcal{A}_K$. Notice that the equation 6 is equivalent to saying that the periodification of $f_\alpha$, i.e. the natural function induced on $\mathcal{A}_K/K$ by $f_\alpha$, and $\widehat{f_\alpha}$ have the same Fourier transform, i.e:

$$\widehat{f_\alpha} \cdot \int_{\mathcal{A}_K} f_\alpha d\mu_{\mathcal{A}_K} = \widehat{f_\alpha}|_K,$$ (7)

(check [14][Lemma 4.2.3] for details). We also use the fact that

$$\widehat{f_\alpha} = f_{\alpha^{-1}} \cdot \widehat{f_\alpha}(0) \cdot \int_{\mathcal{A}_K} f_\alpha d\mu_{\mathcal{A}_K}$$

(see A.4 in A for details). Directly from 6 one can deduce that

$$\left( \widehat{f_\alpha}(0) \right)^{-1} \cdot \widehat{f_\alpha}$$

is the inverse Fourier transform of $f_{\alpha^{-1}|_K}$ with respect to the measure and the character on $K$ coming from the identification with the space dual to $\mathcal{A}_K/K$, i.e.

$$\left( \widehat{f_\alpha}(0) \right)^{-1} \cdot f_\alpha = \widehat{f_{\alpha^{-1}|_K}}.$$ 

where $\widehat{f_{\alpha^{-1}|_K}}$ is taken with respect to the measure dual to $\mu_{\mathcal{A}_K/K}$ which is equal to $\mu_K$ in this special case. Evaluating this relation at zero one can deduce

$$1 = \widehat{f_\alpha}(0) = \int_{\mathcal{A}_K} f_{\alpha^{-1}} d\mu_{\mathcal{A}_K} \cdot \widehat{f_\alpha}(0) = \int_{\mathcal{A}_K} f_{\alpha^{-1}} d\mu_{\mathcal{A}_K} \cdot \widehat{f_\alpha}(0).$$ (8)

Indeed, we have

$$\int_{\mathcal{A}_K} f_{\alpha^{-1}} d\mu_{\mathcal{A}_K} = \widehat{f_{\alpha^{-1}}}(0).$$

One easily sees that 8 and

$$\int_{\mathcal{A}_K} \widehat{f_\alpha} d\mu_{\mathcal{A}_K} = \widehat{f_\alpha}(0),$$

leads us to the desired relation

$$\hat{h}^0(D_{\alpha^{-1}}) = h^1(D_\alpha).$$ (9)

Relation 9 is nothing else, but Serre’s duality.
Remark 3.1. Notice that our theory is compatible with the theory introduced in [1]. Even though we don’t explicitly point out the cohomology spaces, one can see that the functions we use for obtaining the ”size of cohomology” encodes the date of a ghost space of the first kind (see [1, Lemma 2.2, Definition 5.1]). Indeed, the non-archimedean part of the function \( f_\alpha \) are characteristic functions that encode the fractional ideal
\[
I_{D_\alpha} = \prod_v D_v^{-n_v},
\]
whereas the archimedean part of \( f_\alpha \) corresponds to the function \( u : I_{D_\alpha} \to \mathbb{R} \). Notice, (see [1, Section 2]) that the function \( u \) satisfies all but one (idempotency) of the properties characterising characteristic functions. From this point of view local components of \( f_\alpha \) are somewhat analogous. At the same time the local factors of \( f_\alpha \) are all eigenfunctions of the local Fourier transform. To give broader perspective to our approach it is worth stressing out that we have an exact sequence
\[
K \to \mathcal{A}_K \to \mathcal{A}_K/K,
\]
to which we associate the functions \( f_\alpha \) (restricted to \( K \)), \( \hat{f}_\alpha \), \( f_\alpha \) for an idele \( \alpha \) associated to a divisor \( D_\alpha \). We can compare it to the sequence
\[
I_{D_\alpha} \to I_{D_\alpha} \otimes \mathbb{R} \to (I_{D_\alpha} \otimes \mathbb{R})/I_{D_\alpha}
\]
of ghost spaces of the first kind (see [1, Section 5]). One can see that the integral of our functions over the adelic spaces in the first sequence are equal to the dimensions of the ghost spaces in the second sequence. In particular both approaches give the same Euler characteristic as defined in [10].

Remark 3.2. Due to Serre’s duality being a consequence of Fourier duality we can see that \( h^0(D_\alpha) \) and \( h^1(D_{\alpha-1}) \) are bond by Heisenberg indeterminacy principle. Indeed, we have the Heisenberg inequality, with in case of the Gaussian distribution becomes an equality. See [1, Page 9] for a related comment about ghost spaces.

4 Relative Euler characteristic

Fix a finite extension of global fields \( K/L \) and let \( v \) be a place lying above \( w \). We fix a normalization of the Haar measure \( \mu_{K_v/L_w} \) on a local field \( K_v \) with respect to a local field for which the measure of the ring of integers \( O_{K_v} \) is the square root of the relative discriminant of \( K_v/L_w \), i.e:
\[
\int_{K_v} \chi_{O_{K_v}}(u_v) d\mu_{K_v/L_w}(u_v) = d_{K_v/L_w}^{-1/2}
\]
where \( d_{K_v/L_w} = N(d_{K_v/L_w}) \) is the norm of the relative discriminant. We will call it the local relative measure. By taking an integral over \( K_v/L_w \) we mean an integral with respect to the relative measure. Given the local relative measures for each place we define a global relative measure \( \mu_{K/L} \) as the product of the local measures(see [14] 3.3 for a detailed description for the procedure). Note that the relative discriminant is equal to the local ring of integers almost everywhere and as a consequence
\[
\int_{K_v} \chi_{O_{K_v}}(u_v) d\mu_{K_v/L_w}(u_v) = 1 \quad \text{for almost all} \; v,
\]
which is a necessary condition for defining a global measure. Then we define the relative Euler characteristic
\[
\chi_{K/L}(D_\alpha) = \log \int_{\mathcal{A}_K} f_\alpha(u) d\mu_{K/L}(u).
\]
One has
\[
\chi_{K/L}(D_1) = \log \prod_w \int_{K_v/L_w} \chi_{O_{K_v}}(u_v) d\mu_{K_v/L_w} = \log \left( \prod_v d_{K_v/L_w}^{-1/2} \right) = -\frac{1}{2} \log d_{K/L}
\]
By 2 we get
\[
\log d_{K/L} = \log d_L - [K : L] \log d_K.
\]
Similarly to the absolute case the value of the integral for any \( D_\alpha \) is uniquely determined by the adelic norm \( |\alpha| = \prod_v |\alpha_v|_v \) and the integral
\[
\int_{\mathcal{A}_K} f_1 \mu_{K/L} = d_{L/K}^{-1/2}.
\]
Therefore, one has
\[ \chi_{K/L}(D_1) = \log |\alpha| - \frac{1}{2} \log |\kappa_{K/L}| \]
where \( \kappa_{K/L} \) is the idele associated to the divisor of the canonical divisor i.e. a divisor associated to relative discriminant \( \Delta_{K/L} \). Note that this definition agrees with the relative Euler characteristic defined in [10] V §1. Indeed

\[ \chi_{K/L}(D_1) = \log |\alpha| - \frac{1}{2} \log |\kappa_{K/L}| = \log |\alpha| - \frac{1}{2} \log |\kappa_L| + \frac{1}{2} |K : L| \log |\kappa_L| = \chi_K(D_1) - |K : L| \chi_L(D_1, L), \]

where by \( D_{1,L} \) we mean the zero divisor on Spec \( L \) i.e. the divisor associated to the trivial idele.

To conclude, we point out that the absolute Euler \( \chi \) characteristic is a special case of the relative Euler characteristic \( \chi_{K/L} \); i.e. it is the relative Euler characteristic taken for \( L = Q \).

Remark 4.1. The relative Euler characteristic in the number fields case is frequently referred to as the Minkowski Euler characteristics. Since we work simultaneously with number fields and function fields we decided to refer to it simply as Euler characteristic.

5 Riemann-Roch for one-dimensional global fields

As the consequence of our considerations we can easily get

\[ \chi_K(D_1) - \chi_K(D_0) = \log |\alpha| \]

which is the Riemann-Roch theorem as stated in [10, Chapter V, Theorem 1.2] and [12, Chapter III, Proposition 3.4]. We also have

\[ \chi_{K/L}(D_0) - \chi_{K/L}(D_1) = \log |\alpha| \]

which can be seen as relative Riemann-Roch theorem. It agrees with ([10, Chapter V, Theorem 1.2]) except that here we state it uniformly for all one-dimensional global fields. It is also worth noticing that the formula also agrees with what we would obtain in [14] using the relative normalisation of the Haar measure. To see this, one has to realize that the Poisson summation formula for in the relative case \( L \neq Q \) contains a constant which is not present (or equal to 1) in the Poisson summation formula for a self-dual Haar measure for the extension \( K/Q \). Indeed, in [14] several steps are used to obtain the Poisson summation formula. One can repeat them in our more general setting, i.e. allowing any relative Haar measure normalisation. We normalized our measures on \( h_K, K \) and \( h_K/K \) is such a way that the Riemann-Roch statement has the following form

\[
\frac{1}{|\alpha| \cdot \text{vol}(D)} \sum_{u \in K} \hat{f}(u \cdot \alpha^{-1}) = \sum_{u \in K} f(\alpha \cdot u),
\]

(10)

where \( f = f_1 = \prod_v \text{char}_{O_v} \times \prod_{v \in \infty} \exp(-e_v \pi |u_v|^{2/e_v}) \) and \( \text{vol}(D) \) is the volume of the fundamental domain of \( h_K \) with respect to the action of \( K \) (see [14][4.2]). The equation rewrites in the following way

\[
\frac{|\kappa_{K/L}|^{-1/2}}{|\alpha| \cdot \text{vol}(D)} \sum_{u \in K} f(\kappa_K \cdot u \cdot \alpha^{-1}) = \sum_{u \in K} f(\alpha \cdot u).
\]

(11)

We have that \( \text{vol}(D) = |\kappa_L|^{1/2[L:K]} \) and \( |\kappa_{K/L}| = |\kappa_L|^{-[L:K]} \cdot |\kappa_K| \) therefore (as in [10, Chapter V, §1]) it simplifies to the classical (absolute) Riemann-Roch formula

\[
\frac{|\kappa_K|^{-1/2}}{|\alpha|} \sum_{u \in K} f(\kappa_K \cdot u \cdot \alpha^{-1}) = \sum_{u \in K} f(\alpha \cdot u).
\]

(12)

as in [14, Theorem 4.2.1]. Nevertheless, comparing the ingredients of the equation 12

\[
e^{\chi_K(D_1)} = e^{1-g} \quad e^{h^1(D_{0}-1)} = e^{h^1(D_{\kappa-1}u)} \quad e^{h^0(D_{\kappa-1})}
\]

\[
e^{\chi_K(D_0)} = e^{1-g} = \sum_{u \in K} f(\alpha \cdot u)
\]

corresponding to the [absolute] Riemann-Roch theorem, and the equation 11
one can indeed see that the equation 11 can be seen as the relative Riemann-Roch theorem (see for example [10, Theorem V.1.2]). In particular, according to 11 the relative Euler characteristic is equal to
\[
\chi_{K/L}(D_{\alpha-1}) = \log \sum_{u \in K} f(\alpha \cdot u) - \log \frac{1}{\text{vol}(D)} \sum_{u \in K} \kappa_{K}^{-1} \cdot u \cdot \alpha^{-1},
\]
which agrees with our definition after applying Serre’s duality 9. Later in 6.1 we will discuss the relative Serre’s duality which is more natural to use here. The $1/\text{vol}(D)$ coefficient should be seen as the scaling factor of the atomic measure on $K$ identified with $\hat{\kappa}_{K}/\hat{K}$ as discussed in 2.4.

Remark 5.1. In [14] one takes $K = Q$. The relative measure for this extension is the unique self dual measure on $\kappa_{L}$. As a different normalisation of the Haar measure, the relative measures for extensions $L/K$ where $K \neq Q$ are not self-dual. Therefore the Poisson summation formula holds with a non-trivial constant. It is exactly this constant that produces the correction term in the relative Euler characteristic, as shown in the above calculation.

Remark 5.2. This formula agrees with [1, Theorem 5.3] and [15, Proposition 1] for $L = Q$. It also agrees with the one given by the Grothendieck-Riemann-Roch theorem (see [12, Chapter III §7, §8] for the arithmetic case. We have the same formulas for the geometric case). Therefore we have the full right to call our Euler characteristic and Riemann-Roch theorem relative.

Note that the Euler characteristics can be written in the following form
\[
\chi_{K/L}(D_{\alpha-1}) = \log \int_{\kappa_{L}} f_{\alpha} d\mu_{\kappa_{L}}^{K/Q} \left( \int_{\kappa_{L}} f_{1} d\mu_{L}^{L/Q} \left( \int_{\kappa_{L}} f_{L,1} d\mu_{\kappa_{L}}^{K/Q} \right)^{-1} \right),
\]
where $L \subset K$ is a finite extension of one-dimensional global fields and $f_{L,1}$ is a function on $\kappa_{K}$ such that
\[
(f_{L,1})_{v} = \begin{cases} 
\text{char}_{O_{w}}(u_{w}) & w \text{ non-archimedean} \\
\exp(-\gamma_{v}u_{w}^{2}) & w \text{ archimedean}
\end{cases}
\]
where $v, w$ ranges over places of $K$ and $L$ respectively and $O_{w}$ is the ring of integers of the local field $L_{w}$ of $L$.

6 Relative cohomology numbers

Having fixed a global field extension $K/L$ and the relative normalisation of the Haar measure $\mu_{\kappa_{K}}^{K/L}$ on the full ring of adèles $\kappa_{K}$ we fix measures $\mu_{K/L}^{K/L}$, $\mu_{\kappa_{K}/K}^{K/K}$, $\mu_{\kappa_{K}/L}^{K/L}$ and $\mu_{\hat{\kappa}_{K}/K}^{K/K}$ on $K$, $\kappa_{K}/K$, $\hat{K} \simeq \kappa_{K}/K$ and $\hat{\kappa}_{K}/\hat{K} \simeq K$ respectively following the convention established in 2.4, i.e. the measures on $\kappa_{K}$ and $K$ and $\kappa_{K}/K$ are chosen in a way that all the three measures are canonically related, i.e. satisfying the following Weil’s formula:
\[
\int_{\kappa_{K}} f(u) d\mu_{\kappa_{K}}^{K/L} = \int_{\kappa_{K}/K} \left( \int_{K} f(\bar{u} + w) d\mu_{K}^{L/K}(w) \right) d\mu_{\kappa_{K}/K}(\bar{u}).
\]

In addition the measure of the whole space $\kappa_{K}/K$ with respect to $\mu_{\kappa_{K}}^{K/L}$ is equal to the volume of the fundamental domain $D$ of $\kappa_{K}$ with respect to the action of $K$ like in refintd and each element of the discrete set $K$ with respect to $\mu_{\kappa_{K}}^{K/L}$ has weight 1.

Definition 6.1. We define the relative cohomology numbers in the following way
\[
h^{0}_{K/L}(D_{\alpha}) = -\log \int_{K} f_{\alpha}(w) d\mu_{\kappa_{K}/K}(w)
\]
\[
\hat{h}^1_{K/L}(D_a) = -\log \left( \int_{\mathbb{A}_K} \hat{f}_a(\bar{u}) d\mu^{K/L}_{\mathbb{A}_K}(\bar{u}) \right)^{-1}
\]

Notice that since the measures on \( K \) and \( \mathbb{A}_K/K \) are not dual to each other.

Recall that
\[
\text{vol}_{K/L}(D) = \sqrt{d_K} \prod_v \mu_{K_v/L_v}(O_{K_v}) = d_K^{1/2} \frac{d_L^{1/2[K:L]}}{d_K} = d_L^{1/2[K:L]}
\]
in the arithmetic case. One can also deduce that
\[
\text{vol}_{K/L}(D) = d_L^{1/2[K:L]}
\]
in the geometric case. Indeed, even though an analogous explicit computation of the volume of the fundamental domain is not classical, one can use the iterated Poisson summation formula to deduce that \( \text{vol}_{K/Q}(D) = 1 \) (notice that [14, the last remark of 4.2] also applies to the geometric case).

Therefore in both cases one has
\[
\text{vol}_{K/L}(D) = d_L^{1/2[K:L]}
\]
We can conclude that
\[
\hat{h}^1_{K/L}(D_a) = \text{vol}_{K/L}(D) h^1(D_a)
\]
and
\[
\hat{h}^0_{K/L}(D_a) = \hat{h}^0(D_a).
\]
In particular one can use our relative cohomology number to define the relative Euler characteristics as an alternating sum of the relative cohomology numbers obtaining the same result as in our direct single integral definition.

**Remark 6.2.** We tried to chose "the best" normalization of the Haar measure on \( K \) and \( \mathbb{A}_K/K \) i.e. such that both the Weil’s formula 3 and 2.3 hold. Nevertheless it might be that any pair related by Weil’s formula can be seen as ‘good’. The measures on \( \hat{K} \) and \( \mathbb{A}_K/K \) can always be chosen as dual to those on \( K \) and \( \mathbb{A}_K/K \) respectively.

### 6.1 Relative Serre’s duality

In the relative situation we still have a relation between \( \hat{h}^0_{K/L}(D_{\alpha^{-1}}) \) and \( \hat{h}^1_{K/L}(D_a) \) which can be seen as a relative Serre’s duality. Indeed, using the ‘absolute’ Serre’s duality \( \hat{h}^1(D_a) = \hat{h}^0(D_{\alpha^{-1}}) \) and knowing that
\[
\hat{h}^0_{K/L}(D_a) = \hat{h}^0(D_a) \quad \text{and} \quad \hat{h}^1_{K/L}(D_a) = d_L^{-1/2[K:L]} h^1(D_a)
\]
we can deduce the following relation
\[
\hat{h}^1_{K/L}(D_a) = \frac{1}{\text{vol}_{K/L}(D)} \hat{h}^0_{K/L}(D_{\alpha^{-1}}).
\]
This result can be also obtained by analogy with absolute Serre’s duality. Indeed, we can use [14][Lemma 4.2.3], which states that Fourier transform of an adelic function and its periodification are equal on \( K \). In our notation it writes as
\[
\hat{f}_a(\xi) \cdot \int_K f_a d\mu^{K/L}_K = \hat{f}_a|_K(\xi)
\]
(13)
since \( \hat{f}_a \) is a normalisation of the periodification of \( f_a \). Using the properties of the Fourier transform we can rewrite 13 as
\[
\hat{f}_a(\xi) \cdot \int_K f_a d\mu^{K/L}_K = f_{\alpha^{-1}}|_K(\xi) \cdot \int_K f_a d\mu^{K/L}_K,
\]
and further as
\[
\hat{f}_a(\xi) = f_{\alpha^{-1}}|_K(\xi) \cdot \hat{f}_a(0),
\]
(see A.4 in for details). Taking the Fourier transform of the equation and evaluating at 0 one obtains
\[
1 = \hat{f}_a(0) = \hat{f}_a(0) \cdot f_{\alpha^{-1}}|_K(0),
\]
where the Fourier transform of \( f_{\alpha^{-1}}|_K \) is taken with respect to the measure \( \hat{\mu}^{K/L}_{\mathbb{A}_K/K} \). Once we notice that
\[
(\hat{f}_a(0))^{-1} = e^{h^1(D_a)} \quad \text{and} \quad \frac{1}{\text{vol}_{K/L}(D)} \hat{f}_a|_K(0) = \int_K f_{\alpha^{-1}} d\mu^{K/L}_{\mathbb{A}_K/K}
\]
we can conclude the desired equation.

**Remark 6.3.** It might be helpful to go once again through the proof of Serre’s duality for the absolute measures in the section 3.1 as arguments there work also for the relative case and the presentation of the argument there is slightly different.

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7 Future perspectives

The generalization of the presented theory to the case of vector bundles is relatively straightforward and is going to appear in a follow-up paper. It is interesting to check if our methods can at least partially help to obtain Riemann-Roch theorem within the theory of adelic curves introduced in \[2\].

Unlike the vector bundle generalization, this question is not at all trivial.

As we mentioned before the relative Riemann-Roch theorem is used in the horizontal divisor part of the proof of the Faltings-Riemann-Roch theorem. More precisely it is a very important ingredient of the Arakelov adjunction formula which itself plays a crucial role in the horizontal part of the Faltings-Riemann-Roch theorem. It is an interesting question if one can extend the ideas presented in this paper to obtain a global (adelic) analogue of the formula. It is also natural to ask if the method generalises to higher dimensions. The generalization of adeles to any scheme is typically referred to as Beilinson adeles (\[5\]). It was extended to the case of arithmetic surfaces in \[3\], even though the authors believe that the construction might still need modification of its part at infinity.

Nevertheless, one can still hope that obtaining an adelic (in the Beilinson sense) interpretation of the Arakelov adjunction formula and Faltings Riemann Roch theorem is possible. It would be also interesting to check if our approach is compatible with \[11\].

Appendices

A Fourier analysis on local fields

In this appendix we briefly recall the basic results of Fourier analysis on [one-dimensional] local fields.

Let \(K\) be a global field and let \(v\) be a place of \(K\). When \(v\) is archimedean we choose the standard absolute value and the standard Lebesgue measure when \(K_v = \mathbb{R}\) and the square of the standard absolute value and twice the standard Lebesgue measure when \(K_v = \mathbb{C}\). When \(v\) is non-archimedean place of \(K\) we fix a uniformizing element \(\pi_v\). Then the completion \(K_v\) is a finite extension of the field \(K_0\) that can be either \(\mathbb{Q}_p\) or \(\mathbb{F}_p((t))\) for a prime number \(p\) and we fix \([K_v : \mathbb{Q}] = n\). The residue field of \(K_v\) is denoted by \(k(v)\), we have \([k(v) : \mathbb{F}_p]\) = \(d\le n\) and we fix the following absolute value on \(K_v\):

\[|x|_v := (\#k(v))^{-\nu_v(x)}\]

Let \(\mu_v\) be any Haar measure on \(K_v\); we recall that for any measurable set \(S \subseteq K_v\) and any \(x \in K_v\) it holds that:

\[\mu(xS) = |x|_v \mu_v(S)\]

We don’t specify the Haar measure normalization for the non-archimedean places as in what follows our arguments will be independent of this choice. We define the standard character

- For \(K_0 = \mathbb{Q}_p\), we fix the following map \(\Lambda : \mathbb{Q}_p \rightarrow \mathbb{R}\):

\[\Lambda(x) := \sum_{i=-k}^{-1} a_i \quad \text{if} \quad x = \sum_{i \geq -k} a_i p^i \quad \text{and} \quad a_i \in \{0, \ldots, p-1\}\]

and we put

\[\psi_0 : \mathbb{Q}_p \rightarrow \mathbb{S}^1 \quad \text{with} \quad x \mapsto \exp(-2\pi i \Lambda(x))\]

- For \(K_0 = \mathbb{F}_p((t))\)

\[\psi_0 : \mathbb{F}_p((t)) \rightarrow \mathbb{S}^1 \quad \text{with} \quad \sum_i a_it^i \mapsto \exp\left(\frac{2\pi i \tilde{a}_{-1}}{p}\right)\]

where \(\tilde{a}_{-1}\) is any lift to \(\mathbb{Z}\) of the coefficient \(a_{-1}\).

- For \(K_0 = \mathbb{R}\)

\[\psi_0 : \mathbb{R} \rightarrow \mathbb{S}^1 \quad \text{with} \quad x \mapsto \exp(-2\pi ix)\]

The standard character \(\chi_v : K_v \rightarrow \mathbb{S}^1\) is defined as \(\chi_v : \psi_0 \circ \text{Tr}_{K_0/K_v}\).
Lemma A.1. Let $m \in \mathbb{Z}$, then
\[ \int_{\pi_v^m O_v} \chi_v(x) dx = \begin{cases} \mu_v(\pi_v^m O_v) & \text{if } m \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{if } m \geq 0 \] and
\[ \int_{\pi_v^m O_v} \chi_v(x) dx = \begin{cases} (p^d)^{-m} \mu_v(O_v) & \text{if } m \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{otherwise} \]

Proof. Case $m \geq 0$. Remember that the trace map sends $O_v$ to the ring of integers of $K_0$, therefore then $\chi_v(x)$ restricted to $\pi_v^m O_v$ is identically 1. It means that the integral is
\[ \mu_v(\pi_v^m O_v) = (p^d)^{-m} \mu_v(O_v). \]

Case $m < 0$. There exists $y \in \pi_v^m O_v$ such that $\chi_v(y) \neq 1$, then after making the change of variables $x \mapsto x + y$ we get:
\[ \int_{\pi_v^m O_v} \chi_v(x) dx = \int_{\pi_v^m O_v} \chi_v(x + y) dx = \chi_v(y) \int_{\pi_v^m O_v} \chi_v(x) dx \]

Hence the only possibility is $\int_{\pi_v^m O_v} \chi_v(x) dx = 0$. \[ \square \]

Lemma A.2. Let $m \in \mathbb{Z}$, then
\[ \int_{K_v} \text{char}_{\pi_v^{-m} O_v}(x) \chi_v(-xy) dx = \begin{cases} \mu_v(\pi_v^{-m} O_v) & \text{if } y \in \pi_v^{-m} O_v \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{if } y \in \pi_v^{-m} O_v \]
and
\[ \int_{\pi_v^m O_v} \chi_v(x) dx = \begin{cases} (p^d)^{-m} \mu_v(O_v) & \text{if } y \in \pi_v^{-m} O_v \\ 0 & \text{otherwise} \end{cases} \]
\[ \text{otherwise} \]

Proof. If $y \in \pi_v^{-m} O_v$, then $xy \in O_v$ for $x \in \pi_v^m O_v$, hence the integral reduces to
\[ \int_{\pi_v^m O_v} \chi_v(x) dx = \mu_v(\pi_v^m O_v) = p^{-dm} \mu_v(O_v). \]

If $y \notin \pi_v^{-m} O_v$ then the integrand is not the constant 1 and with the same change of variables of the proof of Lemma (A.1) one shows that the integral is 0. \[ \square \]

Now we are able to write explicitly the Fourier transform
\[ \hat{f}(x) := \int_{K_v} f(y) \chi_v(-xy) dy \]
when $f = \text{char}_{\pi_v^m O_v}$:
\[ \text{char}_{\pi_v^m O_v}(x) = \int_{K_v} \text{char}_{\pi_v^{-m} O_v}(y) \chi_v(-xy) dy = \int_{\pi_v^m O_v} \chi_v(-xy) dy = \]
\[ \mu_v(\pi_v^{-m} O_v) \text{char}_{\pi_v^{-m} O_v}(x) = (p^d)^{-m} \mu_v(O_v) \text{char}_{\pi_v^{-m} O_v}(x) \]
where the last equality follows from Lemma A.2.

Remark A.3. Recall that when $v$ is archimedean then $f_{\alpha_v}(x) = \exp(-\varepsilon_v \pi |\alpha_v^{-1} x|^{2/\varepsilon_v})$, $\varepsilon_v = |K_v : \mathbb{R}|$ and
\[ f_{\alpha_v^{-1}}(x) = |\alpha_v| f_{\alpha_v^{-1}} = |\alpha_v| f_{\alpha_v}. \]

as metric corresponding to $\kappa$ is trivial i.e. $\kappa_v = 1$ for all archimedean $v$.

Proposition A.4. Let $\alpha = (\alpha_v)$ be an idele and let
\[ f_{\alpha} = \prod_v f_{\alpha_v} \]
be the eigenfunction of the Adelic Fourier transform associated to $\alpha$, then
\[ \hat{f}_{\alpha} = f_{\alpha(0)} \cdot f_{\alpha^{-1}}. \]

Proof. One has that
\[ \text{char}_{\pi_v^m O_v}(x) = |\alpha_v| f_{\alpha_v^{-1}} = |\alpha_v| f_{\alpha_v}. \]

Now it is enough to use A.2 and A.3. \[ \square \]

Remark A.5. Notice that
\[ \hat{f}_{\alpha} = |\kappa_{K/L}|^{-1/2} f_{\alpha^{-1}}. \]

In particular, since $f_{\alpha}(u) = f_{\alpha}(\alpha^{-1} \cdot u)$, one can write
\[ \hat{f}_{\alpha}(\alpha^{-1} u) = |\kappa_{K/L}|^{-1/2} f_{\alpha}(\kappa_{K/L} u). \]
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