Knot Floer homology detects genus-one fibred knots

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July 6, 2019

Abstract

Ozsváth and Szabó conjectured that knot Floer homology detects fibred knots. We propose a strategy to approach this conjecture based on Gabai’s theory of sutured manifold decomposition and contact topology. We implement this strategy for genus-one knots, obtaining as a corollary that if rational surgery on a knot $K$ gives the Poincaré homology sphere $\Sigma(2,3,5)$, then $K$ is the left-handed trefoil knot.

1 Introduction

Knot Floer homology was introduced independently by Ozsváth and Szabó in [14] and by Rasmussen in [22]. For any knot $K$ in $S^3$ and any integer $d$ the knot Floer homology group $\widehat{HF}(K,d)$ is a finitely generated graded Abelian group.

Knot Floer homology can be seen as a categorification of the Alexander polynomial in the sense that

$$\sum \chi(\widehat{HF}(K,d))T^i = \Delta_K(T)$$

where $\Delta_K(T)$ denotes the symmetrised Alexander polynomial. However the groups $\widehat{HF}(K,d)$, and in particular the top non trivial group, contain more information than just the Alexander polynomial, as the following results show.
Theorem 1.1. (Theorem 1.2) Let $g(K)$ denote the genus of $K$. Then

$$g(K) = \max \{ d \in \mathbb{Z} : \hat{HF}(K, d) \neq 0 \}.$$

Theorem 1.2. (Theorem 1.1) Let $g$ be the genus of $K$. If $K$ is a fibred knot, then $\hat{HF}(K, g) = \mathbb{Z}$.

Ozsváth and Szabó formulated the following conjecture, whose evidence is supported by the computation of knot Floer homology for a large number of knots.

Conjecture 1.3. (Ozsváth–Szabó) If $K$ is a knot in $S^3$ with genus $g$ and $\hat{HF}(K, g) = \mathbb{Z}$ then $K$ is a fibred knot.

In this article we propose a strategy to attack Conjecture 1.3 and we implement it in the case of genus-one knots. More precisely, we will prove the following result.

Theorem 1.4. Let $K$ be an oriented genus-one knot in $S^3$. Then $K$ is fibred if and only if $\hat{HF}(K, 1) = \mathbb{Z}$

Our strategy to prove Theorem 1.4 is to deduce information about the top knot Floer homology group of non-fibred knots from topological properties of their complement via sutured manifold decomposition and contact structures in a way that is reminiscent of the proof of Theorem 1.1.

It is well known that the only fibred knots of genus one are the trefoil knots and the figure-eight knot, therefore Theorem 1.4 together with the computation of knot Floer homology for such knots, implies the following

Corollary 1.5. Knot Floer homology detects the trefoil knots and the figure-eight knot.

The following conjecture was formulated by Kirby in a remark after Problem 3.6(D) of his problem list, and by Zhang in [23].

Conjecture 1.6. (Conjecture $\hat{I}$) If $K$ is a knot in $S^3$ such that there exists a rational number $r$ for which the 3–manifold obtained by $r$–surgery on $K$ is homeomorphic to the Poincaré homology sphere $\Sigma(2,3,5)$, then $K$ is the left-handed trefoil knot.
Conjecture $I$ was proved for some knots by Zhang in [23], and a major step toward its complete proof was made by Ozsváth and Szabó in [18], where they proved that a counterexample to Conjecture $I$ must have the same knot Floer Homology groups as the left-handed trefoil knot. Corollary 1.5 provides the missing step to prove it in full generality.

**Corollary 1.5.** Conjecture $I$ holds.

Corollary 1.5 has also been used by Ozsváth and Szabó to prove that the trefoil knot and the figure-eight knot are determined by their Dehn surgeries [12].

Acknowledgements

We warmly thank Steve Boyer, Ko Honda, Joseph Maher, and Stefan Tillmann for many inspiring conversations.

2 Overview of Heegaard Floer theory

Heegaard Floer theory is a family of invariants introduced by Ozsváth and Szabó in the last few years for the most common objects in low-dimensional topology. In this section we will give a brief overview of the results in Heegaard Floer theory we will need in the following, with no pretension of completeness. The details can be found in Ozsváth and Szabó's papers [16] [15] [20] [14] [19] [13].

2.1 Heegaard Floer homology

Let $Y$ be a closed, connected, oriented 3–manifold. For any Spin$^c$–structure $t$ on $Y$ Ozsváth and Szabó [16] defined an Abelian group $HF^+(Y, t)$ which is an isomorphism invariant of the pair $(Y, t)$. When $c_1(t)$ is not a torsion element in $H^2(Y)$ the group $HF^+(Y, t)$ is finitely generated; see [15 Theorem 5.2 and Theorem 5.11]. If there is a distinguished surface $\Sigma$ in $Y$ which is clear from the context, we use the shortened notation

$$HF^+(Y, d) = \bigoplus_{t \in \text{Spin}^c(Y)} HF^+(Y, t).$$

$$\langle c_1(t), [\Sigma] \rangle = 2d$$
This notation makes sense because $HF^+(Y, t) \neq 0$ only for finitely many Spin$^c$–structures. Heegaard Floer homology is symmetric; in fact there is a natural isomorphism $HF^+(Y, d) \cong HF^+(Y, -d)$ for any 3–manifolds $Y$ and any integer $d$: see [15, Theorem 2.4]. There is an adjunction inequality relating $HF^+$ to the minimal genus of embedded surfaces which can be stated as follows.

**Theorem 2.1.** [15, Theorem 1.6] If $\Sigma$ has genus $g$, then $HF^+(Y, d) = 0$ for all $d > g$.

Any connected oriented cobordism $X$ from $Y_1$ to $Y_2$ induces a homomorphism

$$F_X : HF^+(Y_1, t_1) \to HF^+(Y_2, t_2)$$

which splits as a sum of homomorphisms indexed by the Spin$^c$–structures on $X$ extending $t_1$ and $t_2$. When $X$ is obtained by a single 2–handle addition $F_X$ fit into a surgery exact triangle as follows.

**Theorem 2.2.** ([15, Theorem 9.12]) Let $(K, \lambda)$ be an oriented framed knot in $Y$, and let $\mu$ a meridian of $K$. Denote by $Y_{\lambda}(K)$ the manifold obtained from $Y$ by surgery on $K$ with framing $\lambda$, and by $X$ the cobordism induced by the surgery. If we choose a surface $\Sigma \subset Y \setminus K$ to partition the Spin$^c$–structures, then the following triangle

$$
\begin{array}{ccc}
HF^+(Y, d) & \xrightarrow{F_X} & HF^+(Y_{\lambda}(K), d) \\
& \nearrow & \downarrow \\
& & HF^+(Y_{\lambda+\mu}(K), d)
\end{array}
$$

is exact for any $d \in \mathbb{Z}$.

### 2.2 The Ozsváth–Szabó contact invariant

A contact structure $\xi$ on a 3–manifold $Y$ determines a Spin$^c$–structure $t_\xi$ on $Y$ such that $c_1(t_\xi) = c_1(\xi)$. To any contact manifold $(Y, \xi)$ we can associate an element $c^+(\xi) \in HF^+(-Y, t_\xi)/\pm 1$ which is an isotopy invariant of $\xi$, see [19]. In the following we will always abuse the notation and consider $c^+(\xi)$ as an element of $HF^+(-Y, t_\xi)$, although it is, strictly speaking, defined only
up to sign. This abuse does not lead to mistakes as long as we do not use
the additive structure on $HF^+(-Y, t_\xi)$.

The proof of the following lemma is contained in the proof of [13, Corol-
larly 1.2]; see also [10, Theorem 2.1] for a similar result in the setting of
monopole Floer homology.

**Lemma 2.3.** Let $Y$ be a closed, connected oriented 3–manifold with $b_1(Y) = 1$, and let $\xi$ be a weakly symplectically fillable contact structure on $Y$ such that $c_1(\xi)$ is non trivial in $H^2(Y; \mathbb{R})$. Then $c^+(\xi)$ is a primitive element of $HF^+(-Y, t_\xi)/ \pm 1$.

Given a contact manifold $(Y, \xi)$ and a Legendrian knot $K \subset Y$ there is an
operation called contact $(+1)$–surgery which produces a new contact mani-
fold $(Y', \xi')$; see [2] and [3]. The Ozsváth–Szabó contact invariant behaves
well with respect to contact $(+1)$–surgeries.

**Lemma 2.4.** ([11, Theorem 2.3]; see also [19]) Suppose $(Y', \xi')$ is obtained
from $(Y, \xi)$ by a contact $(+1)$–surgery. Let $-X$ be the cobordism induced by
the surgery with opposite orientation. Then

$$F^+_{-X}(c^+(\xi)) = c^+(\xi').$$

### 2.3 Knot Floer homology

Knot Floer homology is a family of finitely dimensional graded Abelian
groups $\widehat{HF}_K(K, d)$ indexed by $d \in \mathbb{Z}$ attached to any oriented knot $K$ in
$S^3$; see Ozsváth and Szabó [14]. Denote the 0–surgery on $K$ by $Y_K$. The
knot Floer homology of $K$ is related to the Heegaard Floer homology of $Y_K$
by the following

**Proposition 2.5.** ([14, Corollary 4.5] and [13, Corollary 1.2]). Let $K$
be a knot of genus $g > 1$. Then

$$\widehat{HF}_K(K, g) = HF^+(Y_K, g - 1).$$

Another property of knot Floer homology we will need later is a Künneh-
like formula for connected sums.

**Proposition 2.6.** ([14, Corollary 7.2]) Let $K_1$ and $K_2$ be knots in $S^3$, and
denote by $K_1 \# K_2$ their connected sum. If $\widehat{HF}_K(K_1, d)$ is a free Abelian
group for every $d$ (or if we work with coefficients in a field), then

$$\widehat{HF}_K(K_1 \# K_2, d) = \bigoplus_{d_1 + d_2 = d} \widehat{HF}_K(K_1, d_1) \otimes \widehat{HF}_K(K_1, d_1).$$
The definition of Knot Floer homology can be extended to links. To any link $L$ in $S^3$ with $|L|$ components, Ozsváth and Szabó associate a nullhomologous knot $\kappa(L)$ in $\#^{|L|-1}(S^2 \times S^1)$. For a link with two components the construction is the following: choose points $p$ and $q$ on different components of $L$, then replace two balls centred at $p$ and $q$ with a 1–handle $S^2 \times [0, 1]$ and define $\kappa(L)$ as the banded connected sum of the two components of $L$ performed with a standardly embedded band in the 1–handle. If $L$ has more connected components this operation produces a link with one component less, so we repeat it until we obtain a knot. For the details see [14, Section 2.1].

**Definition 2.7.** ([14, Definition 3.3]) If $L$ is a link in $S^3$ we define

$$\widehat{HF}_K(L, d) = \widehat{HF}_K(\kappa(L), d).$$

### 3 Taut foliations and Heegaard Floer homology

#### 3.1 Controlled perturbation of taut foliations

Eliashberg and Thurston in [5] introduced a new technique to construct symplectically fillable contact structures by perturbing taut foliations. In this section we show how to control the perturbation in the neighbourhood of some closed curves. We need to introduce some terminology about confoliations, following Eliashberg and Thurston [5].

**Definition 3.1.** A confoliation on an oriented 3–manifold is a tangent plane field $\eta$ defined by a 1–form $\alpha$ such that $\alpha \wedge d\alpha \geq 0$.

Given a confoliation $\eta$ on $M$ we define its contact part $H(\eta)$ as

$$H(\eta) = \{ x \in M : \alpha \wedge d\alpha(x) > 0 \}.$$ 

**Lemma 3.2.** Let $\Sigma$ be a compact leaf with trivial germinal holonomy in a taut smooth foliation $F$ on a 3–manifold $M$, and let $\gamma$ be a non-separating closed curve in $\Sigma$. Then we can modify $F$ in a neighbourhood of $\gamma$ so that we obtain a new taut smooth foliation with non trivial linear holonomy along $\gamma$.

**Proof.** The holonomy of $\Sigma$ determines the germ of $F$ along $\Sigma$ (see [1, Theorem 3.1.6]), therefore $\Sigma$ has a neighbourhood $N = \Sigma \times [-1, 1]$ such that $F|_N$ is the product foliation. Pick $\gamma' \subset \Sigma = \Sigma \times \{0\}$ such that it intersects $\gamma$ in a
unique point, and call $\Sigma' = \Sigma \setminus \gamma'$. The boundary of $\Sigma'$ has two components $\gamma'_+ \setminus \gamma_-$.

For every point $x \in \gamma'$ denote by $x_\pm$ the points in $\gamma'_\pm$ which correspond to $x$. Choose a diffeomorphism $f: [-1, 1] \to [-1, 1]$ such that

1. $f(x) = x$ if $x \in [-1, -1+\epsilon] \cup [1-\epsilon, 1]$ for some small $\epsilon$;
2. $f(0) = 0$;
3. $f'(0) \neq 1$,

then re-glue $\gamma_- \times [-1, 1]$ to $\gamma_+ \times [-1, 1]$ identifying $(x_-, t)$ to $(x_+, f(t))$. \qed

**Lemma 3.3.** Let $(M, F)$ be a foliated manifold, and let $\gamma$ be a curve with non-trivial linear holonomy contained in a leaf $\Sigma$. Then $F$ can be approximated in the $C^0$-topology by confoliations such that $\gamma$ is contained in their contact parts and is a Legendrian curve with twisted number zero with respect to the framing induced by $\Sigma$.

*Proof.* We apply [5, Proposition 2.6.1] to make $F$ a contact structure in a neighbourhood of $\gamma$, then the proof of the lemma is a check on the explicitly given contact form. \qed

For any subset $A \subset M$ we define its *saturation* $\hat{A}$ as the set of all points in $M$ which can be connected to $A$ by a curve tangent to $\eta$.

**Definition 3.4.** A confoliation $\eta$ on $M$ is called *transitive* if $\hat{H}(\eta) = M$.

**Lemma 3.5.** Let $(M, F)$ be a smooth taut foliated manifold, let $\Sigma$ be a compact leaf of $F$ with trivial germinal holonomy, and let $\gamma \subset \Sigma$ be a closed non-separating curve. Then $F$ can be approximated in the $C^0$-topology by contact structures such that $\gamma$ is a Legendrian curve with twisting number zero with respect to $\Sigma$.

*Proof.* First we apply Lemma 3.2 to create non-trivial linear holonomy along $\gamma$, so that we can apply Lemma 3.3 to make $F$ a contact structure in a neighbourhood of $\gamma$, and $\gamma$ becomes a Legendrian curve with twisting number zero.

The approximation of a confoliation $F$ by contact structures is done in two steps. First $F$ is $C^0$-approximated by a transitive confoliation $\tilde{F}$, then $\tilde{F}$ is $C^1$-approximated by a contact structure $\xi$. The first step is done by
perturbing the foliation in arbitrarily small neighbourhoods of curves contained in $M \setminus \overline{H(F)}$, then we can assume that a neighbourhood $V$ of $\gamma$ is not touched in the first step.

Since $\widetilde{F}$ and $\xi$ are $C^1$–close, they are defined by $C^1$–close 1–forms $\alpha$ and $\beta$. Let $h: M \to [0, 1]$ be a smooth function supported in $V$ such that $h \equiv 1$ in a smaller neighbourhood of $\gamma$, then the 1–form $\beta + h(\alpha - \beta)$ coincides with $\alpha$ near $\gamma$ and with $\beta$ outside $V$. It defines a contact structure which is $C^0$–close to $F$ because $\alpha$ and $\beta$ are $C^1$–close and the contact condition open in the $C^1$–topology.

Proposition 3.6. ([5, Corollary 3.2.5] and [4, Corollary 1.4]) Let $F$ be a taut foliation on $M$. Then any contact structure $\xi$ which is sufficiently close to $F$ as a plane field in the $C^0$ topology is weakly symplectically fillable.

3.2 An estimate on the rank of $HF^+(Y)$ coming from taut foliations

Let $\eta$ be a field of tangent planes in the 3–manifold $Y$, and let $S$ be an embedded compact surface with non empty boundary such that $\partial S$ is tangent to $\eta$.

Definition 3.7. Let $v$ be the positively oriented unit tangent vector field of $\partial S$. We define the relative Euler class $e(\eta, S)$ of $\eta$ on $S$ as the obstruction to extending $v$ to a nowhere vanishing section of $\eta|_S$.

Properly speaking $e(\eta, S)$ is an element of $H^2(S, \partial S)$, but we can identify it with an integer number via the isomorphism $H^2(S, \partial S) = \mathbb{Z}$. If $\eta$ is the field of the tangent planes of the leaves of a foliation $F$ we write $e(F, S)$ for $e(\eta, S)$. If $\eta$ is a contact structure $e(\eta, S)$ is the sum of the rotation numbers of the components of $\partial S$ computed with respect to $S$.

Theorem 3.8. Let $Y$ be a 3–manifold with $H_2(Y) \cong \mathbb{Z}$, and let $\Sigma$ be a genus minimising closed surface representing a generator of $H_2(Y)$. Call $\Sigma_+$ and $\Sigma_-$ the two components of $\partial(Y \setminus \Sigma)$. Suppose that $\Sigma$ has genus $g(\Sigma) > 1$ and that $Y$ admits two smooth taut foliations $F_1$ and $F_2$ such that $\Sigma$ is a compact leaf for both, and the holonomy of $\Sigma$ has the same Taylor series as the identity. If there exists a properly embedded surface $S \subset Y \setminus \Sigma$ with boundary $\partial S = \alpha_+ \cup \alpha_-$ such that

1. $\alpha_+ \subset \Sigma_+$ and $\alpha_- \subset \Sigma_-$ are non separating curves, and
Figure 1: A somehow misleading picture of \( S \subset Y \setminus \Sigma \)

2. \( e(F_1, S) \neq e(F_1, S) \)

(see Figure 7), then \( \text{rk} \, HF^+(Y, g-1) > 1 \).

Requiring that the holonomy of \( \Sigma \) has the same Taylor series as the identity is not as strong a restriction as it seems, because foliations constructed by sutured manifold theory have this property; see the induction hypothesis in the proof of [6, Theorem 5.1].

The strategy of the proof is to view \( F_1 \) and \( F_2 \) as taut foliations on \(-Y\) and to approximate them by contact structures \( \xi_1 \) and \( \xi_2 \) on \(-Y\) so that \( c(\xi_i) \in HF^+(Y, g-1) \), then to construct a new 3–manifold \( Y_\phi \) together with a 4–dimensional cobordism \( W \) from \(-Y\) to \(-Y_\phi\) so that \( F^+_W(c^+(\xi_1)) \) and \( F^+_W(c^+(\xi_2)) \) are linearly independent in \( HF^+(Y_\phi) \). This implies that \( c^+(\xi_1) \) and \( c^+(\xi_2) \) are linearly independent in \( HF^+(Y, g-1) \). The entire subsection is devoted to the proof of Theorem 3.8.

We choose a diffeomorphism \( \phi: \Sigma_+ \to \Sigma_- \) such that \( \phi(\alpha_+) = \alpha_- \), and we form the new 3–manifold \( Y_\phi \) by cutting \( Y \) along \( \Sigma \) and re-gluing \( \Sigma_+ \) to \( \Sigma_- \) after acting by \( \phi \). The diffeomorphism \( \phi \) exists because \( \alpha_+ \) and \( \alpha_- \) are non separating.

It is well known that the mapping class group of a closed surface of genus \( g > 1 \) is generated, as a monoid, by positive Dehn twists; see [4, Footnote 3]. In order to construct the 4–dimensional cobordism \( W \) from \( Y \) to \( Y_\phi \) we
decompose $\phi$ as a product $\phi = \prod \tau_{c_1} \ldots \tau_{c_k}$ where $\tau_{c_i}$ is a positive Dehn twist around a curve $c_i \subset \Sigma$. Then we identify a tubular neighbourhood $N$ of $\Sigma$ with $\Sigma \times [-1, 1]$, we choose distinct points $t_1, \ldots, t_k$ in $(-1, 1)$ and see $c_i$ as a curve in $\Sigma \times \{t_i\}$. The surface $\Sigma \times \{t_i\}$ induces a framing on $c_i$, and $Y_{\phi}$ is obtained by $(−1)$–surgery on the link $C = c_1 \cup \ldots \cup c_k \subset Y$, where the surgery coefficient of $c_i$ is computed with respect to the framing induced by $\Sigma \times \{t_i\}$. Equivalently, $−Y_{\phi}$ is obtained by $(+1)$–surgery on the same link $C$ seen as a link in $−Y$. We denote by $W$ the smooth 4–dimensional cobordism obtained by adding 2–handles to $−Y$ along the curves $c_i$ with framing $+1$, and by $−W$ the same cobordism with opposite orientation, so that $−W$ is obtained by adding 2–handles to $Y$ along the curves $c_i$ with framing $−1$.

**Lemma 3.9.** The map

$$F^+_{−W}: HF^+(Y, g − 1) \to HF^+(Y_{φ}, g − 1)$$

induced by the cobordism $−W$ is an isomorphism.

**Proof.** The map $F^+_{−W}$ is a composition of maps induced by elementary cobordisms obtained from a single 2–handle addition. We apply the surgery exact triangle (Theorem 2.2) and the adjunction inequality (Theorem 2.1) as in [17, Lemma 5.4] to prove that each elementary cobordism induces an isomorphism in Heegaard Floer homology. □

We can assume that the tubular neighbourhood $N$ of $\Sigma$ is foliated as a product in both foliations. In fact there is a diffeomorphism $Y \cong (Y \setminus \Sigma) \cup (\Sigma \times [-1, 1])$ such that $\Sigma_+$ is identified with $\Sigma \times \{-1\}$, and $\Sigma_-$ is identified with $\Sigma \times \{1\}$, then we can extend the foliations $F_1 | Y \setminus \Sigma$ to foliations on $(Y \setminus \Sigma) \cup (\Sigma \times [-1, 1])$ which are product foliations on $\Sigma \times [-1, 1]$. We call the resulting foliated manifolds $(Y, F_1)$ and $(Y, F_2)$ again. This operation does not destroy the smoothness of $F_1$ and $F_2$ because their holonomies along $\Sigma$ have the same Taylor series as the identity.

We see $S$ as a surface in $Y \setminus N$, so that $\alpha_+$ is identified to a curve in $\Sigma \times \{-1\}$, and $\alpha_-$ is identified to a curve in $\Sigma \times \{1\}$. By Lemma 3.5 we can control the perturbations of $F_1$ and $F_2$ so that $\alpha_+, \alpha_-$, and $c_i$ for all $i$ become Legendrian curves with twisting number zero for both $\xi_1$ and $\xi_2$, where the twisting number is computed with respect to the framing induced by $\Sigma$. This implies that we can construct contact structures $\xi'_1$ and $\xi'_2$ on $−Y_{φ}$ by $(+1)$–contact surgery on $\xi_1$ and $\xi_2$. 

10
By hypothesis $b_1(Y) = 1$, and $c_1(\xi_1)$ and $c_1(\xi_2)$ are non torsion because

$$\langle c_1(\xi_i), [\Sigma] \rangle = \langle c_1(F_i), [\Sigma] \rangle = \chi(\Sigma) < 0,$$

therefore Lemma 2.3 applies and gives $c^+(\xi_i) \neq 0$ for $i = 1, 2$. Moreover $c^+(\xi'_1) = F^+_W(c^+(\xi_1))$ and $c^+(\xi'_1) = F^+_W(c^+(\xi_1))$ by Lemma 2.4 because $\xi'_i$ is obtained form $\xi_i$ by a sequence of contact $(+1)$–surgeries, therefore from Lemma 3.9 it follows $c^+(\xi'_i) \neq 0$ for $i = 1, 2$.

**Lemma 3.10.** Let $N_\phi$ be the subset of $Y_\phi$ diffeomorphic to $\Sigma \times [-1, 1]$ obtained from $N \subset Y$ by performing $(-1)$–surgery on $C = c_1 \cup \ldots \cup c_n$, and call $S'$ the annulus in $N_\phi$ bounded by the curves $\alpha_+$ and $\alpha_-$. If we define $S = S \cup S'$, then

$$\langle c_1(\xi'_1), [S'] \rangle \neq \langle c_1(\xi'_2), [S'] \rangle.$$

**Proof.** Because $\alpha_+$ and $\alpha_-$ are Legendrian curves with twisting number 0 with respect to both $\xi'_1$ and $\xi'_2$, and $\xi'_1$ and $\xi'_2$ are both tight, from the Thurston–Bennequin inequality we obtain

$$e(S', \xi'_1) = e(S', \xi'_2) = 0.$$

In the complements of $N$ and $N_\phi$ we have $\xi_i|_{(Y \setminus N)} = \xi'_i|_{(Y_\phi \setminus N_\phi)}$, therefore $e(S, \xi_i) = e(S, \xi_i)$. Since $\xi_1$ and $\xi_2$ are $C^0$–close to $F_1$ and $F_2$ and $\partial S = \alpha_+ \cup \alpha_-$ is tangent to both $\xi_i$ and $F_i$, we have $e(S, \xi_i) = e(S, F_i)$ for $i = 1, 2$. From the additivity property of the relative Euler class we obtain

$$\langle c_1(\xi'_i), [S] \rangle = e(S, \xi'_i) + e(S', \xi'_i) = e(S, F_i),$$

therefore $\langle c_1(\xi'_i), [S] \rangle \neq \langle c_1(\xi'_i), [S] \rangle$. $\square$

Lemma 3.10 implies that the the Spin$^c$–structures $s_{\xi'_1}$ and $s_{\xi'_2}$ induced by $\xi'_1$ and $\xi'_2$ are not isomorphic, therefore $c^+(\xi'_1)$ and $c^+(\xi'_2)$ are linearly independent because $c^+(\xi'_1) \in HF^+(Y, s_{\xi'_1})$ and $c^+(\xi'_2) \in HF^+(Y, s_{\xi'_2})$. This implies that $c^+(\xi_1)$ and $c^+(\xi_2)$ are linearly independent too, therefore it proves Theorem 3.8

**4 Applications of Theorem 3.8**

**4.1 sutured manifolds**

In order to apply Theorem 3.8 we need a way to construct taut foliations in 3–manifolds. This is provided by Gabai’s sutured manifold theory.
Definition 4.1. ([6] Definition 2.6) A \textit{sutured manifold} \((M, \gamma)\) is a compact oriented 3–manifold \(M\) together with a set \(\gamma \subset \partial M\) of pairwise disjoint annuli \(A(\gamma)\) and tori \(T(\gamma)\). Each component of \(A(\gamma)\) is a tubular neighbourhood of an oriented simple closed curve called \textit{suture}. Finally every component of \(\partial M \setminus \gamma\) is oriented, and the orientations must be coherent with the orientations of the sutures.

We define \(R_+(\gamma)\) the subset of \(R(\gamma) = \partial M \setminus \gamma\) where the orientation agrees with the orientation induced by \(M\) on \(\partial M\), and \(R_-(\gamma)\) the subset of \(\partial M \setminus \gamma\) where the two orientations disagree. We define also \(R(\gamma) = R_+ (\gamma) \cup R_- (\gamma)\).

Definition 4.2. ([6] Definition 2.10) A sutured manifold \((M, \gamma)\) is tight if \(R(\gamma)\) minimises the Thurston norm in \(H_2 (M, \gamma)\).

We will give the following definition only in the simpler case when no component of \(\gamma\) is a torus, because this is the case we are interested in.

Definition 4.3. ([6] Definition 3.1 and [8] Correction 0.3) Let \((M, \gamma)\) be a sutured manifold with \(T(\gamma) = \emptyset\), and \(S\) a properly embedded oriented surface in \(M\) such that

1. no component of \(S\) is a disc with boundary in \(R(\gamma)\)
2. no component of \(\partial S\) bounds a disc in \(R(\gamma)\)
3. for every component \(\lambda\) of \(S \cap \gamma\) one of the following holds:
   a. \(\gamma\) is a non-separating properly embedded arc in \(\gamma\), or
   b. \(\lambda\) is a simple closed curve isotopic to a suture in \(A(\gamma)\).

The \(S\) defines a \textit{sutured manifold decomposition}

\[(M, \gamma) \xrightarrow{S} (M', \gamma')\]

where \(M' = M \setminus S\) and

\[
\gamma' = (\gamma \cap M') \cup \nu(S_+ \cap R_-(\gamma)) \cup \nu(S_- \cap R_+(\gamma)),
\]

\[
R_+ (\gamma') = ((R_+ (\gamma) \cap M') \cup S_+) \setminus \text{int}(\gamma'),
\]

\[
R_- (\gamma') = ((R_- (\gamma) \cap M') \cup S_-) \setminus \text{int}(\gamma'),
\]

where \(S_+\) and \(S_-\) are the portions of \(\partial M'\) corresponding to \(S\) where the normal vector to \(S\) points respectively out of or into \(\partial M'\).
A taut sutured manifold decomposition is a sutured manifold decomposition
\((M, \gamma) \xrightarrow{S} (M', \gamma')\) such that both \((M, \gamma)\) and \((M', \gamma')\) are taut sutured manifolds.

**Definition 4.4.** (\cite{6} Definition 4.1) A **sutured manifold hierarchy** is a sequence of decompositions

\[(M_0, \gamma_0) \xrightarrow{S_1} (M_1, \gamma_1) \xrightarrow{\cdots} (M_n, \gamma_n)\]

where \((M_n, \gamma_n) = (R \times [0, 1], \partial R \times [0, 1])\) for some surface with boundary \(R\).

The main results in sutured manifold theory are that for any taut sutured manifold \((M, \gamma)\) there is a sutured manifold hierarchy starting form \((M, \gamma)\) \cite{6} Theorem 4.2], and that we can construct a taut foliation on \((M, \gamma)\) such that \(R(\gamma)\) is union of leaves from a sutured manifold hierarchy starting form \((M, \gamma)\) \cite{6} Theorem 5.1]. Thus sutured manifold theory translates the problem about the existence of taut foliations into a finite set of combinatorial data. The particular result we will use in our applications is the following.

**Theorem 4.5.** Let \(M\) be a closed, connected, irreducible, orientable 3–manifold, and let \(\Sigma\) be a genus minimising connected surface representing a non-trivial class in \(H_2(M; \mathbb{Q})\). Denote by \((M_1, \gamma_1)\) the taut sutured manifold where \(M_1 = M \setminus \Sigma \) and \(\gamma_1 = \emptyset\). If \(g(\Sigma) > 1\) and there is a properly embedded surface \(S\) in \(M_1\) yielding a taut sutured manifold decomposition, then \(M\) admits a smooth taut foliation \(\mathcal{F}\) such that:

1. \(\Sigma\) is a closed leaf,

2. if \(f\) is a representative of the germ of the holonomy map around a closed curve \(\delta \subset \Sigma\), then

   \[
   \frac{d^nf}{dt^n}(0) = \begin{cases} 1, & i = 1, \\ 0, & i > 1, \end{cases}
   \]

3. \(e(\mathcal{F}, S) = \chi(S)\).

**Proof.** By \cite{6} Theorem 4.2\] \(M\) admits a taut sutured manifold hierarchy

\[(M_1, \gamma_1) \xrightarrow{S_1} (M_2, \gamma_2) \xrightarrow{\cdots} (M_n, \gamma_n),\]

then take the foliation \(\mathcal{F}_1\) constructed from that hierarchy using the construction in \cite{6} Theorem 5.1]. In particular \(\mathcal{F}_1\) is smooth because \(g(\Sigma) > 1\).
and \( \partial M_1 \) is union of leaves. We obtain \( \mathcal{F} \) by gluing the two components of \( \partial M_1 \) together.

The smoothness of \( \mathcal{F} \) along \( \Sigma \) and part (2) come from the Induction Hypothesis (iii) in the proof of [6, Theorem 5.1]. Part (3) is a consequence of Case 2 of the construction of the foliation in the proof of [6, Theorem 5.1]; in fact \( \mathcal{F} \) has a leaf which coincides with \( S \) outside a small neighbourhood of \( \Sigma \), and spirals toward \( \Sigma \) inside that neighbourhood.

\[ \Box \]

**Remark 4.6.** A properly embedded surface \( S \) in \( M_1 = M \setminus S \) gives a taut sutured manifold decomposition

\[
(M_1, \gamma_1) \overset{S}{\sim} (M_2, \gamma_2)
\]

if for translates \( \Sigma'_+ \) and \( \Sigma'_- \) of the boundary components of \( \partial M \setminus \Sigma \) the surfaces \( S + \Sigma'_+ \) and \( S + \Sigma'_- \) obtained by cut-and-paste surgery (see Figure 2) are Thurston norm minimising in \( H_2(M_1, \partial S) \). In fact \( S + \Sigma'_+ \) and \( S + \Sigma'_- \) are isotopic to \( R_+ (\gamma_2) \) and to \( R_- (\gamma_2) \) respectively, and minimising the Thurston norm in \( M_1 \) clearly implies minimising the Thurston norm in the smaller manifold \( M_2 = M_1 \setminus S \).

### 4.2 Application to genus-one knots

Let \( K \) be a genus-one knot in \( S^3 \), and let \( Y_K \) be the the 3–manifold obtained as 0–surgery on \( K \). Let \( T \) be a minimal genus Seifert surface for \( K \) and let \( \widehat{T} \) be the torus in \( Y_K \) obtained by capping \( T \) off. Denote \( M_{\widehat{T}} = Y_K \setminus \widehat{T} \) and \( \partial M_{\widehat{T}} = \widehat{T}_+ \cup \widehat{T}_- \), where \( \widehat{T}_+ \) is given the orientation induced by the
orientation of $M_\hat{F}$ by the outward normal convention, and $\hat{T}_-$ is given the opposite one.

Let $\mu$ be a properly embedded curve in $M_\hat{F}$ joining $\hat{T}_+$ and $\hat{T}_-$ which closes to the core of the surgery torus in $Y_K$, then $M_\hat{F} \setminus \nu(\mu)$ is homeomorphic to $S^3 \setminus \nu(T)$. We can divide $\partial(M_\hat{F} \setminus \nu(\mu))$ in two pieces: $\partial_h(M_\hat{F} \setminus \nu(\mu)) = \partial M_\hat{T} \setminus \nu(\mu)$ called the horizontal boundary, and $\partial_v(M_\hat{T} \setminus \nu(\mu)) = \partial \nu(\mu) \setminus \partial M_\hat{T}$ called the vertical boundary.

Let $\alpha_+$ and $\beta_+$ be two simple closed curves in $\hat{T}_+$ which generate $H_1(\hat{T}_+)$ and intersect transversally in a unique point. Because the maps

$$(\iota_\pm)_*: H_1(\hat{T}_\pm, \mathbb{Z}) \rightarrow H_1(M_\hat{F}, \mathbb{Z})$$

induced by the inclusions are isomorphisms there are simple closed curves $\alpha_-$ and $\beta_-$ in $\hat{T}_-$ such that both $\alpha_+ \cup -\alpha_-$ and $\beta_+ \cup -\beta_-$ bound a surface in $M_\hat{F}$. We may assume also that $\alpha_-$ and $\beta_-$ intersect transversally in a unique point.

Denote by $\mathcal{S}_n^+(\alpha)$ the set of the surfaces which are bounded by $\alpha_+ \cup -\alpha_-$ and which intersect $\mu$ transversally in exactly $n$ positive points and in no negative points, and by $\mathcal{S}_n^-(\alpha)$ the set of the surfaces with the same property bounded by $-\alpha_+ \cup \alpha_-$. Let $\mathcal{S}_n^+(\beta)$ and $\mathcal{S}_n^-(\beta)$ be the same for the curves $\beta_+$ and $\beta_-$. Let $\kappa_n^+(\alpha)$ be the minimal genus of the surfaces in $\mathcal{S}_n^+(\alpha)$ and define $\kappa_n^-(\alpha)$, $\kappa_n^+(\beta)$, and $\kappa_n^-(\beta)$ in analogous ways.

**Lemma 4.7.** The sequences $\{\kappa_n^+(\alpha)\}$, $\{\kappa_n^-(\alpha)\}$, $\{\kappa_n^+(\beta)\}$, and $\{\kappa_n^-(\beta)\}$ are non increasing.

**Proof.** We prove the lemma only for $\{\kappa_n^+(\alpha)\}$ because the other cases are similar. Let $S_n^+$ be a surface in $\mathcal{S}_n^+(\alpha)$ with genus $g(S_n^+) = \kappa_n^+(\alpha)$, and call $S_{n+1}^+$ the surface in $\mathcal{S}_{n+1}^+(\alpha)$ constructed by cut-and-paste surgery between $S_n^+$ and $\hat{T}_+$. By definition $g(S_{n+1}^+) \geq \kappa_{n+1}^+(\alpha)$, and $g(S_{n+1}^+) = g(S_n^+) = \kappa_n^+(\alpha)$ because $\hat{T}_+$ is a torus. \hfill $\Box$

**Lemma 4.8.** If $K$ is not fibred, then for all $n \geq 0$ either $\kappa_n^+(\alpha) \neq 0$ and $\kappa_n^-(\alpha) \neq 0$, or $\kappa_n^+(\beta) \neq 0$ and $\kappa_n^-(\beta) \neq 0$.

**Proof.** Assume that there are annuli $A_\alpha \in \mathcal{S}_n^+(\alpha) \cup \mathcal{S}_n^-(\alpha)$ and $A_\beta \in \mathcal{S}_n^+(\beta) \cup \mathcal{S}_n^-(\beta)$. If we make $A_\alpha$ and $A_\beta$ transverse their intersection consists of one segment from $\alpha_+ \cap \beta_+$ to $\alpha_- \cap \beta_-$ and a number of homotopically trivial closed curves. By standard arguments in three-dimensional topology we can
isotope $A_\alpha$ and $A_\beta$ in order to get rid of the circles because $M_T$ is irreducible, therefore we can assume that $A_\alpha \cap A_\beta$ consists only of the segment. The boundary of $M_T \setminus (A_\alpha \cup A_\beta)$ is homeomorphic to $S^2$, therefore $M_T \setminus (A_\alpha \cup A_\beta)$ is homeomorphic to $(\hat{T}_+ \setminus (\alpha_+ \cup \beta_+)) \times [0,1] \cong D^3$ because $M_T$ is irreducible. This proves that $M_T$ is homeomorphic to $\hat{T}_+ \times [0,1]$. However, if $K$ is not fibred, then $Y_K$ is not fibred either by \[9 Corollary 8.19, \] therefore $M_T$ is not a product.

Assume without loss of generality that $\kappa^+_m(\alpha) \neq 0$ and $\kappa^-_m(\alpha) \neq 0$ for any $n \geq 0$. Lemma 4.7 implies that the sequences $\{\kappa^+_m(\alpha)\}$ and $\{\kappa^-_m(\alpha)\}$ are definitively constant. Fix from now on a positive integer $m$ such that $\kappa^+_m+i(\alpha) = \kappa^+_m(\alpha)$ and $\kappa^-_{m+i}(\alpha) = \kappa^-_m(\alpha)$ for all $i \geq 0$, and choose surfaces $S^+_m \in S^+_m(\alpha)$ and $S^-_m \in S^-_m(\alpha)$ such that $S^+_m$ has genus $\kappa^+_m(\alpha)$ and $S^-_m$ has genus $\kappa^-_m(\alpha)$.

**Lemma 4.9.** Let $K_0$ and $K$ be knots in $S^3$, and denote by $M$ the 3-manifold obtained by gluing $S^3 \setminus \nu(K_0)$ to $S^3 \setminus \nu(K)$ via an orientation-reversing diffeomorphism $f: \partial(S^3 \setminus \nu(K_0)) \to \partial(S^3 \setminus \nu(K))$ mapping the meridian of $K_0$ to the meridian of $K$ and the longitude of $K_0$ to the longitude of $K$. Then $M$ is diffeomorphic to the 3-manifold $Y_{K_0 \# K}$.

**Proof.** Glue a solid torus $S_1 = S^1 \times [-\epsilon, \epsilon] \times [0,1]$ to $S^3 \setminus \nu(K_0) \cup S^3 \setminus \nu(K)$ so that $S^1 \times [-\epsilon, \epsilon] \times \{0\}$ is glued to a neighbourhood of a meridian of $K_0$ and $S^1 \times [-\epsilon, \epsilon] \times \{1\}$ is glued to a neighbourhood of a meridian of $K$. The resulting manifold $M'$ is diffeomorphic to $S^3 \setminus \nu(K_0 \# K)$, so that if we glue a solid torus $S_2 = S^1 \times D^2$ to $M' = S^3 \setminus \nu(K_0 \# K)$ mapping the meridian of $S_2$ to the longitude of $K_0 \# K$ we obtain $Y_{K_0 \# K}$.

Now we look at the gluing in the inverse order. First we glue $S_1$ to $S_2$ so that $S^1 \times \{-\epsilon, \epsilon\} \times [0,1] \subset \partial S_1$ is glued to disjoint neighbourhoods of two parallel longitudes of $S_2$, therefore the resulting manifold is diffeomorphic to $T^2 \times [0,1]$. Then we glue $S_1 \cup S_2 = T^2 \times [0,1]$ to $S^3 \setminus \nu(K_0) \cup S^3 \setminus \nu(K)$ and we obtain $M$.

**Proof of Theorem 1.4.** In order to estimate the rank of $\widehat{HF}(K, 1)$ we cannot apply Theorem 3.8 directly; we have to increase the genus of $K$ artificially first. Let $K_0$ be any fibred knot with genus one, say the figure-eight knot. By \[7 Theorem 3\] $K$ is fibred if and only if $K \# K_0$ is fibred. Also, the Künneth formula for connected sums Proposition 2.6

\[ \widehat{HF}(K \# K_0, 2) \cong \widehat{HF}(K, 1) \otimes \widehat{HF}(K_0, 1) \cong \widehat{HF}(K, 1), \]
and by Proposition 2.5
\[ HF^+(Y_{K\#K_0}, 1) \cong \tilde{HF^+}(K\#K_0, 2). \]

Let \( T \) and \( T_0 \) be genus minimising Seifert surfaces for \( K \) and \( K_0 \) respectively, and call \( \tilde{T} \) and \( \tilde{T}_0 \) the corresponding capped-off surfaces in \( Y_K \) and \( Y_{K_0} \).

By Lemma 4.9 the 3–manifold \( Y_{K\#K_0} \) is diffeomorphic to the union of \( S^3 \setminus \nu(K) \) and \( S^3 \setminus \nu(K_0) \) along the boundary via an identification \( \partial(S^3 \setminus \nu(K)) \rightarrow \partial(S^3 \setminus \nu(K)) \) mapping meridian to meridian and longitude to longitude. In \( Y_{K\#K_0} \) the Seifert surfaces \( T \) and \( T_0 \) are glued together to give a closed surface \( \Sigma \) with \( g(\Sigma) = 2 \) which minimises the genus in its homology class. Call \( M_\Sigma = Y_{K\#K_0} \setminus \Sigma \).

If we denote by \( \mu_0 \) a segment in \( M_{\tilde{T}_0} \) which closes to the core of the surgery torus in \( Y_{K_0} \), we can see \( M_\Sigma \) as \((M_{\tilde{T}} \setminus \nu(\mu)) \cup (M_{\tilde{T}_0} \setminus \nu(\mu_0))\) glued together along their vertical boundary components.

From \( S^+_m \) and \( S^-_m \) we can construct surfaces \( \hat{S}^+ \) and \( \hat{S}^- \) in \( M_\Sigma \) by glueing a copy of \( T_0 \) to each one of the \( m \) components of \( S^+_m \cap \partial(M_{\tilde{T}} \setminus \nu(\mu)) \). From an abstract point of view \( \hat{S}^+ \) and \( \hat{S}^- \) are obtained by performing a connected sum with a copy of \( \tilde{T}_0 \) at each of the \( m \) intersection points between \( S^+_m \) or \( S^-_m \) and \( \mu \), therefore \( g(\hat{S}^\pm) = \kappa^{\pm}_m(\alpha) + m \).

Consider the taut sutured manifold \((M_\Sigma, \gamma)\) where \( \gamma = \emptyset \). We claim that
\[
(M_\Sigma, \gamma) \xrightarrow{\tilde{S}^+} (M_\Sigma \setminus \hat{S}^+, \gamma^+) \\
(M_\Sigma, \gamma) \xrightarrow{\tilde{S}^-} (M_\Sigma \setminus \hat{S}^-, \gamma^-)
\]
are taut sutured manifold decompositions. By Remark 4.9 this is equivalent to proving that the surfaces \( \hat{S}^+ + \Sigma_+, \hat{S}^- + \Sigma_-, \hat{S}^+ + \Sigma_-, \) and \( \hat{S}^- + \Sigma_- \) obtained by cut-and-paste surgery between \( S^\pm \) and \( \Sigma^\pm \) minimise the genus in their relative homology classes in \( H_2(M_\Sigma, \alpha_+ \cup \alpha_-) \).

We recall that \( \hat{T}^+ \) and \( \Sigma_+ \) are oriented by the outward normal convention, while \( \hat{T}^- \) and \( \Sigma_- \) are oriented by the inward normal convention. For this reason \( \mu \cap \hat{T}^+ \) and \( \mu \cap \hat{T}^- \) consist both of one single positive point. We consider only \( \hat{S}^+ + \Sigma_+ \), the remaining cases being similar due to the above consideration.

Let \( \hat{S} \subset M_\Sigma \) be a genus minimising surface in the same relative homology class as \( \hat{S}^+ + \Sigma_+ \). We can see \( \hat{S} \) as the union of two (possibly disconnected) properly embedded surfaces with boundary \( S \subset M_{\tilde{T}} \setminus \nu(\mu) \) and...
$S_0 \subset M_{\tilde{F}_0} \setminus \nu(\mu_0)$, then $\chi(\tilde{S}) = \chi(S) + \chi(S_0)$. We can easily modify $\tilde{S}$ without increasing its genus so that it intersects $\partial_{\nu}(M_{\tilde{F}} \setminus \nu(\mu))$ and $\partial_{\nu}(M_{\tilde{F}_0} \setminus \nu(\mu_0))$ in homotopically non trivial curves. The number of connected components of $\partial S_0 = \partial S \cap \partial_{\nu}(M_{\tilde{F}} \setminus \nu(\mu))$ counted with sign is $m + 1$.

Since $\chi(S) + \chi(S_0) = \chi(\tilde{S}) = 2 - 2g(\tilde{S})$, and $\tilde{S}$ is genus minimising, both $S$ and $S_0$ maximise the Euler characteristic in their relative homology classes. $M_{\tilde{F}_0} \setminus \nu(\mu_0)$ is a product $T_0 \times [0, 1]$, therefore $\chi(S_0)$ is equal to the negative of the number of components of $\partial S_0$ counted with sign, i.e. $\chi(S_0) = -(m + 1)$. We can modify $S_0$ without changing $\chi(S_0)$ so that it consists of some boundary parallel annuli and $m + 1$ parallel copies of $T_0$, then we push the boundary parallel annuli into $M_{\tilde{F}} \setminus \nu(\mu)$, so that we have a new surface $S' \subset M_{\tilde{F}} \setminus \nu(\mu)$ whose intersection with $\partial_{\nu}(M_{\tilde{F}} \setminus \nu(\mu))$ consists of exactly $m + 1$ positively oriented non trivial closed curves. If we glue discs to these curves we obtain a surface $S_{m+1}^{+} \in S_{m+1}^{+}$ such that

$$g(\tilde{S}) = g(S_{m+1}^{+} \# (m + 1)\tilde{T}_0) = \kappa_{m+1}^{+} + m + 1.$$  

Since

$$g(\tilde{S}_+ + \Sigma_+) = g(\tilde{S}_+) + 1 = g(S_m^{+} \# m\tilde{T}_0) + 1 = g(S_m^{+}) + m + 1 = \kappa_{m}^{+} + m + 1$$

and $\kappa_{m+1}^{+} = \kappa_{m}$, we conclude that $g(\tilde{S}_+ + \Sigma_+) = g(\tilde{S})$, then $g(\tilde{S}_+ + \Sigma_+)$ minimises the genus in its relative homology class.

By Theorem 4.5, the taut sutured manifold decompositions

$$(M_{\Sigma}, \gamma = \emptyset) \xrightarrow{\tilde{S}_+^{+}} (M_{\Sigma} \setminus \tilde{S}_+, \gamma_+^{+})$$

$$(M_{\Sigma}, \gamma = \emptyset) \xrightarrow{\tilde{S}_-^{-}} (M_{\Sigma} \setminus \tilde{S}_-, \gamma_-^{-})$$

provide taut smooth foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ such that $\Sigma$ is a closed leaf for both so that, in particular,

$$\langle c_1(\mathcal{F}_+), [\Sigma] \rangle = \langle c_1(\mathcal{F}_-), [\Sigma] \rangle = \chi(\Sigma),$$

and moreover $e(\mathcal{F}_+, \tilde{S}_+^{+}) = \chi(\tilde{S}_+) + 1$ and $e(\mathcal{F}_-, \tilde{S}_-^{-}) = \chi(\tilde{S}_-)$. 

Take $R \in S_0^{\alpha}(\alpha)$, then $-R \in S_0^{\alpha}(\alpha)$, therefore $[\tilde{S}_+] = [R] + m[\Sigma]$ and $[\tilde{S}_-] = -[R] + m[\Sigma]$ as relative homology classes in $H_2(M_{\Sigma}, \alpha_+ \cup \alpha_-)$, therefore

$$e(\mathcal{F}_+, \tilde{S}_+) = e(\mathcal{F}_+, R) + m\chi(\Sigma) = e(\mathcal{F}_+, R) - 2m$$

18
and
\[ e(\mathcal{F}_-, \hat{S}_-) = e(\mathcal{F}_-, -R) + m\chi(\Sigma) = -e(\mathcal{F}_-, R) - 2m. \]

This implies
\[ e(\mathcal{F}_+, R) = \chi(\hat{S}_+) + 2m = \chi(S^+_m) = -2\kappa^+_m(\alpha) \quad (1) \]
and
\[ e(\mathcal{F}_-, R) = -\chi(\hat{S}_-) - 2m = -\chi(S^-_m) = 2\kappa^-_m(\alpha). \quad (2) \]

Recall that \( \chi(S^\pm_m) = -2\kappa^\pm_m(\alpha) \) because \( S^+_m \) and \( S^-_m \) have 2 boundary components each. Equations (1) and (2) imply that \( e(\mathcal{F}_+, R) \neq e(\mathcal{F}_-, R) \) because \( \kappa^\pm_m(\alpha) > 0 \), so we can apply Theorem 3.8.

**Proof of Corollary 1.5.** It is well known that the trefoil knots and the figure-eight knot are the only three fibred knots with genus one, therefore by Theorem 1.4 if \( \hat{HF}_K(K, 1) = \mathbb{Z} \) then \( K \) is either one of the trefoil knots or the figure-eight knot. These knots are alternating, therefore their knot Floer homology groups can be computed by using [21, Theorem 1.3], then the statement follows from the fact that these groups are distinct.

**Proof of Corollary 1.7.** The proof is immediate from Corollary 1.5 and from [18, Theorem 1.6] asserting that, if surgery on a knot \( K \) gives the Poincaré homology sphere, then the knot Floer homology of \( K \) is isomorphic to the knot Floer homology of the left-handed trefoil knot.

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