A PREY-PREDATOR MODEL WITH A FREE BOUNDARY
AND SIGN-CHANGING COEFFICIENT IN
TIME-PERIODIC ENVIRONMENT

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Abstract. This paper is concerned with a prey-predator model with sign-chang-
ing intrinsic growth rate in heterogeneous time-periodic environment,
where the prey species lives in the whole space but the predator species lives in
a region enclosed by a free boundary. It is shown that the results for the case
of the non-periodic environment remain true in time-periodic environment. In
fact, we first establish a similar spreading-vanishing dichotomy, which implies
that if the predator species could spread successfully, then the two species will
coexist, and this is certainly for the situation that the predation is relatively
weak. Furthermore, some criteria are also obtained for spreading and vanishing.
At last, some rough estimates of the asymptotic spreading speed are given if
spreading occurs.

1. Introduction. A variety of mathematical models obtaining from nature are
based on the famous Lotka-Volterra predator-prey system. One of its modified
versions has the following form:

\[
\begin{aligned}
\frac{du}{dt} - d_1 u_{xx} &= u(a - eu + cv), \quad (t, x) \in (0, \infty) \times \Omega, \\
\frac{dv}{dt} - d_2 v_{xx} &= v(b - fv - du), \quad (t, x) \in (0, \infty) \times \Omega, \\
B[u] &= B[v] = 0, \quad (t, x) \in (0, \infty) \times \partial \Omega,
\end{aligned}
\]

where \( B[u] = \alpha u + \beta \frac{\partial u}{\partial n} \), \( \alpha \) and \( \beta \) are non-negative constants and satisfy \( \alpha + \beta > 0 \),
\( n \) is the outward unit normal vector of the boundary \( \partial \Omega \). \( d_1, d_2, a, b, c, d, e \) and
\( f \) are positive constants. In particular, when \( \beta = 0 \), the predator invades into
the prey’s living habitat. It is shown that whether the predator can invade success-
fully and coexist with the prey or not depends on the sign of the principal eigenvalue
of the corresponding eigenvalue problem (4.96) – (4.98) in [2] no matter how small or
large the predator’s initial density of population is, the readers can refer to detailed
argument of section 4.5 in [2]. Here, we emphasize that the sign of the principal
eigenvalue depends on the parameters in system. However, the introduction of
several bird species from Europe to North America in the 1900s was successful only
after several attempts by changing the initial data. This reveals that the species will
not invade unconditionally. Just like the problem raised by Lin [21], it is necessary
to consider the impact of the free boundary on the dynamics of the species.

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lem, spreading and vanishing.
In the natural world, the intrinsic growth rates of the predator and prey are not always positive constants as a result of the inhomogeneous distribution of the species. Meanwhile, due to seasonal variation and the cycle of day and night, their birth and death rates will present certain periodic characteristic. Particularly, when the environment becomes unfavourable to the growth of the species, the intrinsic growth rates may be negative in some periods and some areas. Hence, considering a predator-prey model with a free boundary and sign-changing coefficients in time-periodic environment has the greater theoretical and realistic significance. By the analysis above, we can use the corresponding periodic functions $f(t, x)$ to replace all constant coefficients $a, b, c, d, e$ and $f$ respectively, where the intrinsic growth rates $a(t, x)$ and $b(t, x)$ can change sign.

In the real world, the following phenomenon happens frequently: one kind of insect pests entering the favorable environment breeds wildly and expands its territory to a large area (almost all over the whole space). In order to control the pests, the most environment-friendly and effective strategy is to introduce one kind of natural enemies in this region. At the beginning, we put some natural enemy in some bounded area (initial habitat).

In general, the predator has a tendency to emigrate from the boundary to obtain its new habitat for improving its living environment, while the prey occupies the whole space, so it is reasonable to assume that the free boundary is caused only by the predator and the movement speed of the spreading front is proportional to the gradient of the population density of the predator at the front. In order to simplify the mathematics, in this paper we assume that the species can only invade further into the environment from the right end of the initial region and the left boundary is fixed at $x = 0$. Moreover, we take $c(t, x), f(t, x), e(t, x)$ and $d(t, x)$ to be positive constants, and by the suitable rescaling we may think that $c(t, x) = f(t, x) = 1$.

According to the above argument, we shall use the following free boundary problem with no-flux condition on the fixed boundary to describe the above phenomenon,

\[
\begin{align*}
    u_t - d_1 u_{xx} &= u(a(t, x) - u + cv), \quad t > 0, \quad 0 < x < h(t), \\
    u(t, x) &= 0, \quad t > 0, \quad x \geq h(t), \\
    v_t - d_2 v_{xx} &= v(b(t, x) - v - du), \quad t > 0, \quad 0 < x < \infty, \\
    u_x(t, 0) &= v_x(t, 0) = 0, \quad t > 0, \\
    h'(t) &= -\mu u_x(t, h(t)), \quad t > 0, \\
    h(0) &= h_0, \quad u(0, x) = u_0(x), \quad x \in [0, h_0], \\
    v(0, x) &= v_0(x), \quad x \in [0, \infty),
\end{align*}
\]

where $d_1, d_2, \mu, h_0$ are given positive constants, $x = h(t)$ stands for the free boundary to be determined, which grows at a speed proportional to the gradient of the population density of the predator at the front: $h'(t) = -\mu u_x(t, h(t))$. The ecological background and derivation of this free boundary condition can refer to [1]. We mainly want to realize the dynamics of predator, prey and the free boundary.

Throughout this paper, we assume that the functions $a(t, x)$ and $b(t, x)$ satisfy the following condition:

**($H$)** Functions $a(t, x)$ and $b(t, x) \in (C^{2,1} \cap L^\infty)([0, \infty) \times [0, \infty))$ for some $\nu \in (0, 1)$, and are $T$-periodic in time $t$ for some $T > 0$ and positive somewhere in $[0, T] \times [0, \infty)$; and the initial functions $u_0(x)$ and $v_0(x)$ satisfy:
We remark that there are a series of researches associated with (1) recently. When the functions $a, b$ are positive constants, Wang and Zhao [34] studied problem (1) for double free boundaries. In 2014, Wang [25] investigated some free boundary problems for that the predator and prey share the common region, which was extended by Wang and Zhang [30]. For the higher dimensions and radially symmetric case, problem (1) was studied by Zhao and Wang [36] for the heterogeneous environment. In [36], the authors showed that many results of [25, 34] continue to hold in this more general and ecologically realistic situation. In [34, 25, 30, 36], the complete description about spreading-vanishing dichotomy was given, namely, as time $t \to \infty$, the species either successfully establishes itself in the new environment (called spreading), or fails to establish and dies out eventually (called vanishing). Moreover, criteria for spreading and vanishing, long time behaviour of $(u, v)$ and asymptotic spreading speed of the spreading front when spreading happens had also been obtained. For the time-periodic case, Chen et al. [4] and Wang [32] studied the diffusive competition model in time-periodic environment. A large number of related works but for the diffusive Lotka-Volterra type competition model have been studied intensively, and we refer the readers to [8, 14, 33, 15, 35, 23, 24] and some of the references cited therein.

In the absence of $v$, problem (1) reduces to the following diffusive logistic model:

\[
\begin{aligned}
&u_t - d_1 u_{xx} = u(a(t, x) - u), \quad t > 0, \quad 0 < x < h(t), \\
&u_x(t, 0) = 0, \quad u(t, h(t)) = 0, \quad t > 0, \\
&h'(t) = -\mu u_x(t, h(t)), \quad t > 0, \\
&h(0) = h_0, \quad u(0, x) = u_0(x), \quad x \in [0, h_0],
\end{aligned}
\]  

which has been studied by Wang [25]. In the special case that the function $a$ has positive lower and upper bounds, problem (3) was studied by Du et al. [6], in which the authors discussed the higher dimension and radially symmetric case. Chen et al. [3] also have considered a diffusive logistic problem with a free boundary in time-periodic environment, including favorable habitat and unfavorable habitat.

If the function $a$ in problem (3) is independent of time $t$ and changes sign, this problem was considered by Lei et al. [20], Wang [26] and Zhou and Xiao [37]. In particular, when $a$ is a positive constant, Du and Lin [7] investigated problem (3) for the first time, which has been extended to the higher dimensional and radially symmetric case by Du and Guo [5].

Besides, Gu et al. [11] studied initially the long time behavior of solutions of a diffusion-advection logistic model with double free boundaries in one dimensional space when the influence of advection is small. Further, Gu et al. [12] considered the rightward and leftward asymptotic spreading speeds when spreading happens. Later, Gu et al. [13] investigated the long time behavior of solutions of Fisher-KPP equation with advection and free boundaries. For the general nonlinear advection-diffusion equations, Kaneko and Matsuzawa [17] discussed the spreading speeds of the fronts and sharp asymptotic profiles of solutions in free boundary problems in detail.

The rest of this paper is organized as follows. The global existence, uniqueness and estimates of solution are given in Section 2. To establish the criteria for spreading and vanishing, in Section 3 we provide some basic results. Section 4 is devoted
to the long time behavior of \((u,v)\) and we get a spreading-vanishing dichotomy. The criteria for spreading and vanishing will be given in Section 5. In Section 6, we give the rough estimation of asymptotic spreading speed. We will give some discussions in Section 7.

2. Existence and uniqueness. In this section, we will give some fundamental results on solutions of problem (1) under \((H)\). We firstly prove the following local existence and uniqueness result by the contradiction mapping theorem.

**Lemma 2.1.** For any given \((u_0, v_0)\) satisfying (2), any \(\nu \in (0,1)\) and \(p > \frac{3}{1-\nu}\), there is a \(\tau > 0\) such that problem (1) admits a unique solution

\[
(u, v, h) \in W^{1,2}_p(D^h_\tau) \times W^{1,2}_p(\Delta^\infty_\tau) \times C^{1+\frac{\nu}{2}}([0, \tau]),
\]

where \(D^h_\tau := \{(t, x) \in \mathbb{R}^2 : t \in [0, \tau], x \in [0, h(t)]\}\) and \(\Delta^\infty_\tau := \{(t, x) \in \mathbb{R}^2 : t \in [0, \tau], x \in [0, \infty)\}\). Moreover,

\[
\|u\|_{W^{1,2}_p(D^h_\tau)} + \|v\|_{W^{1,2}_p(\Delta^\infty_\tau)} + \|h\|_{C^{1+\frac{\nu}{2}}([0, \tau])} \leq C,
\]

where positive constants \(C\) and \(\tau\) depend only on \(a, b, c, d, h_0, \mu, \|u_0\|_{W^2_p}, \|v_0\|_{W^2_p}, \nu\) and \(p\).

**Proof.** **Step 1.** Transformation of the problem (1).

Let

\[
y = \frac{x}{h(t)}, \quad w(t, y) = u(t, h(t)y), \quad z(t, y) = v(t, h(t)y),
\]

\[
\alpha(t, y) = a(t, h(t)y), \quad \beta(t, y) = b(t, h(t)y),
\]

then (1) is equivalent to

\[
\begin{align*}
&w_t - \frac{d_1}{h(t)} w_{yy} - \frac{h'(t)}{h(t)} yw_y = w(\alpha(t, x) - w + cz), \quad 0 < t \leq \tau, \; 0 < y < 1, \\
&w_y(t, 0) = w(t, 1) = 0, \quad 0 \leq t \leq \tau, \\
&w(0, y) = u_0(h_0y) := w_0(y), \quad 0 \leq y \leq 1, \\
&h'(t) = -\frac{\mu}{n+1} w_y(t, 1), \quad 0 \leq t \leq \tau, \\
&h(0) = h_0,
\end{align*}
\]

and

\[
\begin{align*}
&z_t - \frac{d_2}{\alpha(t)} z_{yy} - \frac{h'(t)}{h(t)} yz_y = z(\beta(t, x) - z - dw), \quad 0 < t \leq \tau, \; 0 < y < \infty, \\
&z_y(t, 0) = 0, \quad 0 \leq t \leq \tau, \\
&z(0, y) = v_0(h_0y) := z_0(y), \quad 0 \leq y < \infty.
\end{align*}
\]

**Step 2.** Existence of the solution \((w, z, h)\) to (5) and (6).

For \(0 < \tau \ll 1\), we set

\[
Z_\tau = \{z \in C(\Delta^\infty_\tau) : z(0, y) = z_0(y), \; \|z - z_0\|_{L^\infty(\Delta^\infty_\tau)} \leq 1\}.
\]

Then \(Z_\tau\) is a bounded and closed convex set of \(C(\Delta^\infty_\tau)\), and for any given \(z \in Z_\tau\), we have

\[
\|z\|_{L^\infty(\Delta^\infty_\tau)} \leq \|z\|_{L^\infty(\Delta^\infty_\tau)} \leq 1 + \|z_0\|_{L^\infty([0, \infty))} \leq 1 + \|v_0\|_{L^\infty}.
\]

For the given \(z \in Z_\tau\), similarly to the argument of \([7]\) Theorem 2.1, by using the contraction mapping theorem we can prove that, when \(0 < \tau \ll 1\), the problem (5)
has a unique solution \((w(t,y), h(t))\), which continuously depends on \(z\) and satisfies

\[
\begin{align*}
\theta'(0) &= -\mu \theta_0 - w_y(0,1) = -\mu \theta_0 - w_0(h_0), \\
\theta' &\geq 0, \quad h(t) \leq h_0 + 1 \text{ in } [0, \tau]; \quad w(t,y) > 0 \text{ in } [0, \tau] \times [0, 1), \\
\|w\|_{W^{1,2}_p(\Delta T)} + \|w\|_{C^{1+\nu, 1+\nu}(\Delta T)} + \|h\|_{C^{1+\nu, 1+\nu}(\Delta T)} &\leq C_1,
\end{align*}
\]

where \(C_1\) depends only on \(\alpha, c, d, \mu, h_0, \nu, p, \|u_0\|_{W^2_p} \) and \(1 + \|v_0\|_{L^\infty}\).

Moreover, from the above discussion, we see that \((w(z), h(z))\) solves (1), and \((u, v, h)\) has a unique solution \((1)\), which is defined for \(t \in \mathbb{R}\). Therefore, if we take \(0 < \tau < 1\), then \((w(z), h(z))\) solves (1), and \((u, v, h)\) has at least one fixed point \(h \in C^{1+\nu, 1+\nu}(\Delta T)\), \(w \in W^{1,2}_p(\Delta T) \cap C^{1+\nu, 1+\nu}(\Delta T)\), \(z \in W^{1,2}_p(\Delta^\infty) \cap C^{1+\nu, 1+\nu}(\Delta^\infty)\).

Define a map

\[
\mathcal{F} : Z_\tau \to C(\Delta^\infty), \quad \mathcal{F}(z) = \pi.
\]

As mentioned above, we see that \(\mathcal{F}\) is continuous in \(Z_\tau\), and \(z \in Z_\tau\) is a fixed point of \(\mathcal{F}\) if and only if \((w, z, h)\) solves the problem \((5)\) and \((6)\). Estimation (9) indicates that \(\mathcal{F}\) is compact.

Recall the fact \(\pi(0, y) = \zeta_0(y)\). Using the mean value theorem and (9), we have

\[
\|\pi - \zeta_0\|_{L^\infty(\Delta T)} \leq \|\pi\|_{C^{1+\nu, 1+\nu}(\Delta T)} \tau^{\frac{1}{2}} \leq C_2 \tau^{\frac{1}{2}}.
\]

Therefore, if we take \(0 < \tau < 1\), then \(\mathcal{F}\) maps \(Z_\tau\) into itself. Hence, \(\mathcal{F}\) has at least one fixed point \(z \in Z_\tau\), namely, \((5)\) and \((6)\) has at least one solution \((w, z, h)\).

Moreover, from the above discussion, we see that \((w, z, h)\) satisfies

\[
\begin{align*}
h &\in C^{1+\nu, 1+\nu}(\Delta T), \quad h'(t) \geq 0 \text{ in } [0, \tau], \\
w &\in W^{1,2}_p(\Delta T) \cap C^{1+\nu, 1+\nu}(\Delta T), \quad z \in W^{1,2}_p(\Delta^\infty) \cap C^{1+\nu, 1+\nu}(\Delta^\infty).
\end{align*}
\]

**Step 3.** Existence and uniqueness of the solution \((w, z, h)\) to (1).

Define

\[
\begin{align*}
u(t, x) &= w(t, \frac{x}{h(t)}), \\
u(t, x) &= z(t, \frac{x}{h(t)}),
\end{align*}
\]

then \((u, v, h)\) satisfies \((1)\), and \((u, v)\) satisfies

\[
\begin{align*}
u &\in W^{1,2}_p(D^\mu h) \cap C^{1+\nu, 1+\nu}(D^\mu h), \quad v \in W^{1,2}_p(\Delta^\infty) \cap C^{1+\nu, 1+\nu}(\Delta^\infty).
\end{align*}
\]

It is easy to see that \((1)\) holds.

In the following, we prove the uniqueness. Let \((u_i, v_i, h_i)\), with \(i = 1, 2\), be two solutions of \((1)\), which are defined for \(t \in [0, \tau]\) and \(0 < \tau < 1\). We can think of that

\[
\begin{align*}
h_0 &\leq h_i(t) \leq h_0 + 1 \text{ in } [0, \tau], \quad i = 1, 2, \\
\|u_i(t, x) - u_0(x)\|_{L^\infty(D^\mu h_i)} &\leq 1, \quad \|v_i(t, x) - v_0(x)\|_{L^\infty(\Delta^\infty)} \leq 1, \quad i = 1, 2.
\end{align*}
\]
Taking $k = h_0 + 1$. For each $t \in [0, \tau]$, define $u_i(t, \cdot) = 0$ in $[h_i(t), \infty)$. We have $v_{i0} \in L^\infty(\Delta_\tau^\infty)$. Let

$$w_i(t, y) = u_i(t, h_i(t) y), \quad z_i(t, y) = v_i(t, h_i(t) y), \quad 0 \leq t \leq \tau, \quad 0 \leq y \leq 1.$$ 

Then $(w_i, h_i)$ solves $[3]$ with $z = z_i$ and satisfies $[7]$. Setting $W = w_1 - w_2$ and $H = h_1 - h_2$, we have

$$\begin{cases}
W_t - \frac{h_i}{h_1} W_{yy} - \frac{h_i}{h_1} y W_y - W(\alpha(t, x) - w_1 - w_2 + c z_1) \\
\quad = d_1(\frac{h_i}{h_1} - \frac{h_i}{h_2}) w_{2yy} + (\frac{h_i}{h_1} - \frac{h_i}{h_2}) y w_{2y} + c w_2 (z_1 - z_2), \\
W_y(t, 0) = W(t, 1) = 0, \\
W(0, y) = 0,
\end{cases} \quad 0 \leq t \leq \tau, \quad 0 < y < 1,$

and

$$\begin{cases}
H'(t) = -\mu_{\frac{1}{h_1(t)}} W_y(t, 1) - \mu_{\frac{1}{h_1(t)}} w_{2y}(t, 1), \\
H(0) = 0.
\end{cases} \quad 0 \leq t \leq \tau, \quad 0 \leq y \leq 1,$$

(10)

Remembering the facts $\|w_2\|_{W^{1,2}(\Delta_\tau^1)} \leq C_1$, $h_0 \leq h_i(t) \leq h_0 + 1$, $|h_i'(t)| \leq C_1$ and $\|z_i\|_{L^\infty(\Delta_\tau^1)} \leq 1 + \|v_0\|_{L^\infty}$, $i = 1, 2$. We can apply the standard $L^p$ theory to (10) and then use the Sobolev's imbedding theorem to derive

$$\|W\|_{C^{\frac{1}{1+p}, 1+p}(\Delta_\tau^1)} \leq C_4(\|z_1 - z_2\|_{L^\infty(\Delta_\tau^1)} + \|H\|_{C^1([0, \tau])}). \quad (12)$$

Now we estimate $\|z_1 - z_2\|_{L^\infty(\Delta_\tau^1)}$. For any fixed $(t, y) \in \Delta_\tau^1$, we have

$$|z_1(t, y) - z_2(t, y)| \leq |v_1(t, h_1(t) y) - v_1(t, h_2(t) y)| + |v_1(t, h_2(t) y) - v_2(t, h_2(t) y)| \leq \|v_1\|_{L^\infty(\Delta_\tau^1)} \|H\|_{C^1([0, \tau])} + \|v_1 - v_2\|_{C(\Delta_\tau^1)},$$

which implies,

$$\|z_1 - z_2\|_{L^\infty(\Delta_\tau^1)} \leq \|v_1 - v_2\|_{C(\Delta_\tau^1)} + \|v_1\|_{L^\infty(\Delta_\tau^1)} \|H\|_{C^1([0, \tau])}.$$

Combining this with (12), we get

$$\|W\|_{C^{\frac{1}{1+p}, 1+p}(\Delta_\tau^1)} \leq C_5(\|v_1 - v_2\|_{C(\Delta_\tau^1)} + \|H\|_{C^1([0, \tau])}). \quad (13)$$

Therefore, by use of (11),

$$\|H'\|_{C^p} \leq \mu_{\frac{1}{h_1(t)}} W_y \|C^{\frac{1}{p}, 0}(\Delta_\tau^1) \leq \mu_{\frac{1}{h_1(t)}} (h_1^{-1} - h_2^{-1}) w_{2y} \|C^{\frac{1}{p}, 0}(\Delta_\tau^1) \leq C_6(\|v_1 - v_2\|_{C(\Delta_\tau^1)} + \|H\|_{C^1([0, \tau])}). \quad (14)$$

Recall $W(0, y) = 0$, $H(0) = H'(0) = 0$. Taking advantage of the mean value theorem and (13) and (14), it follows that

$$\|W\|_{C(\Delta_\tau^1)} \leq \tau^{\frac{1}{p}} \|W\|_{C^{\frac{1}{p}, 0}(\Delta_\tau^1)} \leq C_5 \tau^{\frac{1}{p}} (\|v_1 - v_2\|_{C(\Delta_\tau^1)} + \|H\|_{C^1([0, \tau])}),$$

$$\|H\|_{C^1([0, \tau])} \leq 2 \tau \|H'\|_{C^1} \leq 2 C_6 \tau^{\frac{1}{p}} (\|v_1 - v_2\|_{C(\Delta_\tau^1)} + \|H\|_{C^1([0, \tau])}).$$

Make the zero extension of $w_i(t, \cdot)$ to $[1, \frac{1}{h_i(t)}]$ for each $t \in [0, \tau]$. The above estimates lead to

$$\|W\|_{C(\Delta_\tau^{1/h_0})} + \|H\|_{C([0, \tau])} \leq C_7 \tau^{\frac{1}{p}} \|v_1 - v_2\|_{C(\Delta_\tau^1)}$$

(15)

provided $0 < \tau \ll 1$. 

Thus, when $0 < \tau \ll 1$, we have that

$$\varepsilon \approx 0,$$
Since \( w(t, y) > 0 \) for \( 0 < t < \tau_0 \) and \( 0 < y < 1 \), by the Hopf boundary lemma, we have \( u_y(t, 1) < 0 \) for \( 0 < t < \tau_0 \). This fact combined with the relation \( u_x = \frac{1}{M} w_y \) allows us to derive \( u_x(t, h(t)) < 0 \), and so \( h'(t) = -\mu u_x(t, h(t)) > 0 \) for \( 0 < t < \tau_0 \).

It remains to show that \( h'(t) \leq M_3 \) for all \( t \in (0, \tau_0) \) with some \( M_3 \) independent of \( \tau_0 \). To this aim, we define

\[
\Omega_M := \left\{ (t, x) : 0 < t < \tau_0, \ h(t) - \frac{1}{M} < x < h(t) \right\}
\]

and construct an auxiliary function

\[
\overline{\eta}(t, x) = M_1[2M(h(t) - x) - M^2(h(t) - x)^2].
\]

We will choose \( M \) so that \( \overline{\eta}(t, x) \geq u(t, x) \) holds over \( \Omega_M \).

Direct calculations show that, for \( (t, x) \in \Omega_M \),

\[
\overline{\eta}_t = 2M_1Mh'(t)(1 - M(h(t) - x)) \geq 0, \quad -\overline{\eta}_{xx} = 2M_1M^2,
\]

\[
u(a(t, x) - u + cv) \leq M_1(||a||_\infty + cM_2).
\]

It follows that

\[
\overline{\eta}_t - d_1 \overline{\eta}_{xx} \geq 2d_1M_1M^2 \geq M_1(||a||_\infty + cM_2) \geq u(a(t, x) - u + cv) = u_t - d_1u_{xx} \text{ in } \Omega_M
\]

if \( M^2 \geq \frac{||a||_\infty + cM_2}{2d_1} \). It is obvious that

\[
\overline{\eta}(t, h(t) - \frac{1}{M}) = M_1 \geq u(t, h(t) - \frac{1}{M}), \quad \overline{\eta}(t, h(t)) = 0 = u(t, h(t)).
\]

On the other hand, for \( x \in [h_0 - \frac{1}{M}, h_0] \),

\[
\overline{\eta}_0(x) = -\int_x^{h_0} u_y'(y) dy \leq (h_0 - x)||u_y'||_{C([0, h_0])},
\]

\[
\overline{\eta}(0, x) = M_1[2M(h_0 - x) - M^2(h_0 - x)^2] \geq M_1M(h_0 - x).
\]

Therefore,

\[
\overline{\eta}(0, x) \geq u_0(x) \text{ for } x \in [h_0 - \frac{1}{M}, h_0] \text{ if } M_1M \geq ||u_y'||_{C([0, h_0])}.
\]

Let

\[
M = \max \left\{ \frac{1}{h_0}, \sqrt{\frac{||a||_\infty + cM_2}{2d_1}}, \frac{||u_y'||_{C([0, h_0])}}{M_1} \right\}.
\]

Applying the maximum principle to \( \overline{\eta} - u \) over \( \Omega_M \) gives that \( \overline{\eta}(t, x) \geq u(t, x) \) for \( (t, x) \in \Omega_M \), which implies that

\[
-2M_1M = \overline{\eta}_x(t, h(t)) \leq u_x(t, h(t)), \quad h'(t) = -\mu u_x(t, h(t)) \leq 2\mu M_1M := M_3
\]

for \( t \in [0, \tau_0) \).

\[\square\]

**Theorem 2.3.** The solution of problem [1] exists and is unique for all \( t \in (0, \infty) \).

**Proof.** Since the priori estimates have been established in Lemma 2.2, the proof can be done by a similar process in [2, Theorem 2.4]. Hence, we omit the proof here. \[\square\]

According to the above lemmas, we have the following result.
Theorem 2.4. The problem \([1]\) has a unique global solution \((u, v, h)\) in time and
\[ u \in C^{1+\frac{1}{2}+\nu}(D_\infty), \quad v \in C^{1+\frac{1}{2}+\nu}((0, \infty) \times [0, \infty)), \quad h \in C^{1+\frac{1}{2}+\nu}((0, \infty)), \]
where \(D_\infty = \{(t, x) : t > 0, \ x \in [0, h(t)]\}. \) Furthermore, there exist positive constants \(M = M(\|a, b, u_0, v_0\|_\infty)\) and \(C = C(\mu, \|a, b, u_0, v_0\|_\infty),\) such that
\[ 0 < u(t, x) \leq M, \quad 0 < h'(t) \leq \mu M, \ \forall \ (t, x) \in (0, \infty) \times (0, h(t)), \]
\[ 0 < v(t, x) \leq M, \ \forall \ (t, x) \in (0, \infty) \times (0, \infty), \]
and
\[ \begin{cases} \|u(t, \cdot\|_{C^1([0,h(t)])}, \|v(t, \cdot\|_{C^1([0,\infty))}) \leq C, & \forall \ t \geq 1, \\ \|h'(\cdot\|_{C^\frac{1}{2}([n+1,n+3])} \leq C, & \forall n \geq 0. \end{cases} \]  (20)

Proof. The estimate of \([19]\) can be proved by the similar method to that of \([28, \text{Theorem 2.1}]\). The proof of \((20)\) can be done by a similar argument of \([29, \text{Theorem 2.2}]\). The details are omitted here. 

3. Preliminaries. Assume that \((u, v, h)\) is the unique solution of \([1]\) obtained in Section 2. We need the following comparison principle for later applications.

Theorem 3.1. Let \(\tau > 0, \ \bar{h} \in C^1([0, \tau])\) and \(\bar{h}(t) > 0 \in [0, \tau]. \) Let \(\bar{v} \in C(\bar{D}) \cap C^{1,2}(\bar{Q})\) with \(O = \{(t, x) : 0 < t \leq \tau, \ 0 < x < \bar{h}(t)\}\) and \(\bar{v} \in C(\bar{Q}) \cap C^{1,2}(\bar{Q})\) with \(Q = \{(t, x) : 0 < t \leq \tau, \ 0 < x < \infty\}. \) Assume that \((\bar{v}, \bar{v}, \bar{h})\) satisfies
\[ \begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} \geq \bar{v}(a(t, x) - \bar{v} + c\bar{v}), & 0 < t \leq \tau, \ 0 < x < \bar{h}(t), \\ \bar{v}_t - d_2 \bar{v}_{xx} \geq \bar{v}(b(t, x) - \bar{v}), & 0 < t \leq \tau, \ 0 < x < \infty, \\ \bar{v}_x(t, 0) = 0, \ \bar{v}_x(t, 0) = 0, & 0 \leq t \leq \tau, \\ \bar{h}(t) = \bar{h}(t) = 0, & 0 \leq t \leq \tau, \\ \bar{h}(t) \geq - \mu \bar{v}_x(t, \bar{h}(t)), & 0 \leq t \leq \tau. \end{cases} \]  (21)

If \(\bar{h}(0) \geq h_0\) and \(u_0(x) \leq \bar{v}(0, x) \in [0, h_0], \ v_0(x) \leq \bar{v}(0, x) \in [0, \infty), \) then the solution \((u, v, h)\) of \([1]\) satisfies
\[ h(t) \leq \bar{h}(t) \text{ in } [0, \tau], \ u(t, x) \leq \bar{v}(t, x) \text{ in } \bar{D} \text{ and } v(t, x) \leq \bar{v}(t, x) \text{ in } \bar{Q}. \]

Proof. The proof is similar to that of \([8, \text{Lemma 2.6}]\), we omit the details here. 

Similarly, we have the following result.

Theorem 3.2. Let \(\tau > 0, \ \bar{h} \in C^1([0, \tau])\) and \(\bar{h}(t) > 0 \in [0, \tau]. \) Let \(\bar{u} \in C(\bar{D}) \cap C^{1,2}(\bar{D})\) with \(D = \{(t, x) : 0 < t \leq \tau, \ 0 < x < \bar{h}(t)\}. \) Assume that \((\bar{u}, \bar{h})\) satisfies
\[ \begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} \leq \bar{u}(a(t, x) - \bar{u}), & 0 < t \leq \tau, \ 0 < x < \bar{h}(t), \\ \bar{u}_x(t, 0) = \bar{u}(t, \bar{h}(t)) = 0, & 0 \leq t \leq \tau, \\ \bar{h}(t) - \mu \bar{u}_x(t, \bar{h}(t)), & 0 \leq t \leq \tau. \end{cases} \]

If \(\bar{h}(0) \leq h_0\) and \(u_0(x) \geq \bar{u}(0, x) \in [0, h_0], \) then the solution \((u, v, h)\) of \([1]\) satisfies
\[ h(t) \geq \bar{h}(t) \text{ in } [0, \tau], \ u(t, x) \geq \bar{u}(t, x) \text{ for } (t, r) \in \bar{D}. \]

Next, we will state some known results about the principal eigenvalue, which play an important role in later sections.
For any given $l > 0$, let $\lambda_1(l, d, c(t, x))$ be the principal eigenvalue of the following $T$-periodic eigenvalue problem
\[
\begin{aligned}
\phi_t - d\phi_{xx} - c(t, x)\phi &= \lambda \phi, & 0 \leq t \leq T, & 0 < x < l, \\
\phi_x(t, 0) &= 0, & 0 \leq t \leq T, \\
\phi(0, x) &= \phi(T, x), & 0 \leq x \leq l.
\end{aligned}
\] (22)

**Proposition 1.** ([28, Proposition 3.1]) The principal eigenvalue $\lambda_1(l, d, c(t, x))$ is continuous and strictly decreasing in $c$ and $l$. Moreover,
\[
\lim_{l \to 0^+} \lambda_1(l, d, c(t, x)) = +\infty, \quad \lim_{d \to \infty} \lambda_1(l, d, c(t, x)) = +\infty.
\]

We introduce some sets. For any given $d > 0$, we define
\[
\Sigma_d := \{ l > 0 : \lambda_1(l, d, c) = 0 \}.
\]
For any given $l > 0$, we denote
\[
\Sigma_l^- := \{ d > 0 : \lambda_1(l, d, c) \leq 0 \} \quad \text{and} \quad \Sigma_l^+ := \{ d > 0 : \lambda_1(l, d, c) > 0 \}.
\]

In what follows, we present some properties of $\lambda_1(l, d, c)$.

**Proposition 2.** ([28, Proposition 3.2]) Assume that the function $c(t, x)$ satisfies

(A1) there exists $c > 0$, $-2 < \rho \leq 0$, $k > 1$ and $x_n$, satisfying $x_n \to \infty$ as $n \to \infty$, such that $c(t, x) \geq cx^\rho$ in $[0, T] \times [x_n, kx_n]$.

Then $\lambda_1(\infty, d, c(t, x)) < 0$, and so $\Sigma_d \neq \emptyset$ for any $d > 0$.

**Proposition 3.** ([28, Proposition 3.3]) Assume that the function $c(t, x)$ satisfies

(A2) there exists $\tilde{x} > 0$ such that $\int_0^T c(t, \tilde{x})dt > 0$.

Then for any $l > \tilde{x}$, the set $\Sigma_l^- \neq \emptyset$. Moreover, there exists $d_0 > 0$ such that $(0, d_0] \subset \Sigma_l^-$.

4. **Spreading-vanishing dichotomy.** Lemma 2.2 implies that $x = b(t)$ is monotonically increasing. Thus, there exists $h_\infty \in (0, +\infty)$ such that $\lim_{t \to \infty} h(t) = h_\infty$.

Before giving the main results, we first give a definition of the spreading and vanishing for the species $u$:

(i) The species $u$ spreads if
\[
h_\infty = \infty \quad \text{and} \quad \lim_{t \to \infty} \| u(t, \cdot) \|_{C([0, h(t)])} > 0;
\]

(ii) The species $u$ vanishes if
\[
h_\infty < \infty \quad \text{and} \quad \lim_{t \to \infty} \| u(t, \cdot) \|_{C([0, h(t)])} = 0.
\]

Then we state the following condition:

(Cr) The functions $a(t, x)$, $b(t, x) \in C_r(T)$ for some $-2 < r \leq 0$. That is, there exist positive $T$-periodic functions $a_\infty(t)$, $b_\infty(t)$, $a^\infty(t)$ and $b^\infty(t) \in C^{r/2}([0, T])$, such that
\[
\begin{aligned}
a_\infty(t) &\leq \liminf_{x \to \infty} \frac{a(t,x)}{x^r}, \quad \limsup_{x \to \infty} \frac{a(t,x)}{x^r} \leq a^\infty(t), \\
b_\infty(t) &\leq \liminf_{x \to \infty} \frac{b(t,x)}{x^r}, \quad \limsup_{x \to \infty} \frac{b(t,x)}{x^r} \leq b^\infty(t)
\end{aligned}
\]
uniformly in $[0, T]$.

We can see that if the function $a \in C_r(T)$ for some $-2 < r \leq 0$, then $a$ satisfies the condition (A1).
When the condition (Cr) holds, we define
\[ a_\infty = \min_{[0,T]} a_\infty(t), \quad \overline{a}_\infty = \max_{[0,T]} a_\infty(t), \quad b_\infty = \min_{[0,T]} b_\infty(t), \quad \overline{b}_\infty = \max_{[0,T]} b_\infty(t). \]

**Theorem 4.1.** (Vanishing) Assume that the condition (Cr) holds. If \( h_\infty < \infty \), then
\[
\lim_{t \to \infty} \| u(t, \cdot) \|_{C([0,h(t)])} = 0, \tag{24}
\]
\[
\lim_{n \to \infty} v(t + nT, x) = V(t, x) \text{ uniformly in } [0, T] \times [0, L] \tag{25}
\]
for any \( L > 0 \), where \( V \) is the unique positive solution of the following \( T \)-periodic boundary value problem
\[
\begin{cases}
    V_t - dV_{xx} = V(b(t, x) - V), & 0 \leq t \leq T, 0 < x < \infty, \\
    V_x(t, 0) = 0, & 0 \leq t \leq T, \\
    V(0, x) = V(T, x), & 0 \leq x < \infty.
\end{cases} \tag{26}
\]

This shows that if the invasive species \( u \) cannot spread successfully, it will die out in the long run.

**Proof.** Applying Theorem 2.4 and [25] Proposition 3.1, we have that
\[
\lim_{t \to \infty} \| u(t, \cdot) \|_{C([0,h(t)])} = 0.
\]

We divide the proof of (25) into two parts, and show that, respectively,
\[
\lim_{n \to \infty} \inf v(t + nT, x) \geq V(t, x) \text{ uniformly in } [0, T] \times [0, L], \tag{27}
\]
\[
\lim_{n \to \infty} \sup v(t + nT, x) \leq V(t, x) \text{ uniformly in } [0, T] \times [0, L]. \tag{28}
\]

Arguing as Step 2 in the proof of [22] Theorem 3.2, we can get (27). Next, we prove (28).

For any given \( l > 0 \), let \( w(t, x) \) be the unique positive solution of
\[
\begin{cases}
    w_t - d_2w_{xx} = w(b(t, x) - w), & t > 0, \ 0 < x < l, \\
    w_x(t, 0) = 0, \ w(t, l) = w(t, l), & t > 0, \\
    w(0, x) = v_0(x), & 0 \leq x \leq l.
\end{cases} \tag{29}
\]

By the comparison principle, \( v(t, x) \leq w(t, x) \) for all \( (t, x) \in (0, \infty) \times [0, l] \). Using the argument of [25] Lemma 3.2, we can prove that
\[
\lim_{n \to \infty} w(t + nT, x) = V_l(t, x) \text{ uniformly in } [0, T] \times [0, l],
\]
where \( V_l(t, x) \) is the unique positive solution of the following initial-boundary value problem
\[
\begin{cases}
    V_t - d_2V_{xx} = V(b(t, x) - V), & 0 \leq t \leq T, \ 0 < x < l, \\
    V_x(t, 0) = 0, \ V(t, l) = v(t, l), & 0 \leq t \leq T, \\
    V(0, x) = V(T, x), & 0 \leq x \leq l,
\end{cases} \tag{30}
\]
and
\[
\lim_{l \to \infty} V_l(t, x) = V(t, x) \text{ uniformly in } [0, T] \times [0, L] \text{ for any } L > 0.
\]
Thus, we get \( \limsup_{n \to \infty} v(t + nT, x) \leq V(t, x) \) uniformly in \( [0, T] \times [0, L] \) for any \( L > 0 \).

By contrast, if the invasive species \( u \) can spread successfully, it can survive in the long run.

**Remark 1.** Since \( b(t, x) \in C_r(T) \), it follows from [23] Theorem 4.2 that \( V(t, x) \in C_r(T) \).
Theorem 4.2. (Spreading) Assume that (Cr) holds and
\[ b_\infty - d(\bar{a}_\infty + c\bar{b}_\infty) > 0. \]  \hfill (31)
If \( h_\infty = \infty \), then we have
\[ U(t, x) \leq \liminf_{n \to \infty} u(t + nT, x), \quad \limsup_{n \to \infty} u(t + nT, x) \leq \bar{U}(t, x), \]  \hfill (32)
\[ V(t, x) \leq \liminf_{n \to \infty} v(t + nT, x), \quad \limsup_{n \to \infty} v(t + nT, x) \leq \bar{V}(t, x) \]  \hfill (33)
uniformly in \([0, T] \times [0, L]\) for any \( L > 0 \), where \( \bar{U}, \bar{V}, \bar{U} \) and \( \bar{V} \) will be given in the proof.

Remark 2. When the relationship between the predator and prey is relatively weak, namely, \( c \) and \( d \) is small enough, it is easy to see that (31) holds.

Proof. Step 1. The construction of \( \bar{U}, \bar{V}, \bar{U} \) and \( \bar{V} \).

Since \( b \in C_r(T) \), it follows from [28, Theorem 4.2] that the problem
\[
\begin{align*}
V_t - d_2V_{xx} &= V(b(t, x) - V), \quad 0 \leq t \leq T, \quad 0 < x < \infty, \\
V_x(t, 0) &= 0, \quad 0 \leq t \leq T, \\
V(0, x) &= V(T, x), \quad 0 \leq x < \infty
\end{align*}
\]  \hfill (34)
admits a unique positive solution \( \bar{V}(t, x) \in C_r(T) \), and
\[
\bar{b}_\infty = \liminf_{x \to \infty} \frac{\bar{V}(t, x)}{x^r}, \quad \limsup_{x \to \infty} \frac{\bar{V}(t, x)}{x^r} \leq \bar{b}_\infty
\]  \hfill (35)
uniformly in \([0, T]\).

It follows that \( a + c\bar{V} \in C_r(T) \) since \( a \in C_r(T) \). Applying [28, Theorem 4.2] again, the problem
\[
\begin{align*}
U_t - d_1U_{xx} &= U(a(t, x) + c\bar{V}(t, x) - U), \quad 0 \leq t \leq T, \quad 0 < x < \infty, \\
U_x(t, 0) &= 0, \quad 0 \leq t \leq T, \\
U(0, x) &= U(T, x), \quad 0 \leq x < \infty
\end{align*}
\]  \hfill (36)
has a unique positive solution \( \bar{U}(t, x) \in C_r(T) \), and
\[
\bar{a}_\infty + c\bar{b}_\infty = \liminf_{x \to \infty} \frac{\bar{U}(t, x)}{x^r}, \quad \limsup_{x \to \infty} \frac{\bar{U}(t, x)}{x^r} \leq \bar{a}_\infty + c\bar{b}_\infty
\]  \hfill (37)
uniformly in \([0, T]\).

It follows that \( b - d\bar{U} \in C_r(T) \) since \( \bar{b}_\infty - d(\bar{a}_\infty + c\bar{b}_\infty) > 0 \). Applying [28, Theorem 4.2] again, the problem
\[
\begin{align*}
V_t - d_2V_{xx} &= V(b(t, x) - d\bar{U}(t, x) - V), \quad 0 \leq t \leq T, \quad 0 < x < \infty, \\
V_x(t, 0) &= 0, \quad 0 \leq t \leq T, \\
V(0, x) &= V(T, x), \quad 0 \leq x < \infty
\end{align*}
\]  \hfill (38)
has a unique positive solution \( \bar{V}(t, x) \in C_r(T) \), and
\[
\bar{b}_\infty - d(\bar{a}_\infty + c\bar{b}_\infty) \leq \liminf_{x \to \infty} \frac{\bar{V}(t, x)}{x^r}, \quad \limsup_{x \to \infty} \frac{\bar{V}(t, x)}{x^r} \leq \bar{b}_\infty - d(\bar{a}_\infty + c\bar{b}_\infty)
\]  \hfill (39)
uniformly in \([0, T]\).
It follows that \(a + c\bar{V} \in C_r(T)\). Applying [28, Theorem 4.2] again, the problem

\[
\begin{aligned}
U_t - d_1U_{xx} &= U(a(t, x) + c\bar{V}(t, x) - U), \quad 0 \leq t \leq T, \quad 0 < x < \infty, \\
U_x(t, 0) &= 0, \quad 0 \leq t \leq T, \\
U(0, x) &= U(T, x), \quad 0 \leq x < \infty
\end{aligned}
\]

(37)

has a unique positive solution \(U(t, x) \in C_r(T)\).

In addition, by [32, Proposition 2.3] we have \(\bar{V} \leq \underline{V}\) and \(\underline{U} \leq \bar{U}\) in \([0, T] \times [0, \infty)\), and hence in \([0, \infty) \times [0, \infty)\) as they are \(T\)-periodic functions in time \(t\).

**Step 2.** Arguing as the step 1 in the proof of [32, Theorem 3.2], we can get the second inequality of (4.11).

For any given \(L > 0\), let \(w(t, x)\) be the unique positive solution of

\[
\begin{aligned}
w_t - d_2w_{xx} &= w(b(t, x) - w), \quad t > 0, \quad 0 < x < \infty, \\
w_x(t, 0) &= 0, \quad t > 0, \\
w(0, x) &= v_0(x), \quad 0 \leq x < \infty.
\end{aligned}
\]

(38)

It follows from [32, Proposition 2.2] that

\[
\lim_{n \to \infty} w(t + nT, x) = \bar{V}(t, x) \text{ uniformly in } [0, T] \times [0, L],
\]

On the other hand, by the comparison principle, we have \(v(t, x) \leq w(t, x)\) for all \((t, x) \in (0, \infty) \times [0, L]\). Thus, we get \(\limsup v(t + nT, x) \leq \bar{V}(t, x)\) uniformly in \([0, T] \times [0, L]\) for any \(L > 0\).

**Step 3.** For any \(\varepsilon > 0\), denote \(a_\varepsilon(t, x) = a(t, x) + c[\bar{V}(t, x) + \varepsilon(1 + x)^r]\). It follows from the condition (Cr) that

\[
a_\infty + c(b_\infty + \varepsilon) \leq \liminf_{x \to \infty} \frac{a_\varepsilon(t, x)}{x^r}, \quad \limsup_{x \to \infty} \frac{a_\varepsilon(t, x)}{x^r} \leq a_\infty + c(b_\infty + \varepsilon)
\]

uniformly in \([0, T]\). Hence \(a_\varepsilon(t, x) \in C_r(T)\). For such fixed \(\varepsilon\), there exists \(l^*\) such that \(\lambda_1(l, d_1, a_\varepsilon) < 0\) for all \(l > l^*\).

For any fixed \(\varepsilon > 0\) and \(l > l^*\), capitalize on the second limit of [33] and \(h_\infty = \infty\), there exists \(\tau \gg 1\) such that

\[
h(t) > l, \quad v(t, x) \leq \bar{V}(t, x) + \varepsilon(1 + x)^r, \quad \forall \ (t, x) \in [\tau, \infty) \times [0, l].
\]

Consider the following auxiliary \(T\)-periodic boundary value problem

\[
\begin{aligned}
Z_t - d_1Z_{xx} &= Z(a_\varepsilon(t, x) - Z), \quad 0 \leq t \leq T, \quad 0 < x < l, \\
Z_x(t, 0) &= 0, \quad Z(t, l) = u(t, l), \quad 0 \leq t \leq T, \\
Z(0, x) &= Z(T, x), \quad 0 \leq x \leq l.
\end{aligned}
\]

(39)

Since \(\lambda_1(l, d_1, a_\varepsilon) < 0\), it is well known (see, for example, [2, Corollary 3.4]) that the above problem admits a unique positive solution, denoted by \(Z^\tau_0(t, x)\). Let \(U^\tau\) be the unique positive solution of the following initial-boundary value problem

\[
\begin{aligned}
U_t - d_1U_{xx} &= U(a_\varepsilon(t, x) - U), \quad t > \tau, \quad 0 < x < l, \\
U_x(t, 0) &= 0, \quad U(t, l) = u(t, l), \quad t \geq \tau, \\
U(\tau, x) &=MZ^\tau_0(\tau, x), \quad 0 \leq x \leq l
\end{aligned}
\]

(40)
where $M > 1$ is sufficiently large such that $MZ^l(t,x) > u(\tau, x)$ in $[0, l]$. Obviously, the function $\chi(t,x) := MZ^l(t,x)$ satisfies
\[
\begin{cases}
\chi_t - d_1\chi_{xx} > \chi(a_x(t,x) - \chi), & t > \tau, \quad 0 < x < l, \\
\chi_x(t,0) = 0, & \chi(t,l) > u(t,l), \quad t \geq \tau, \\
\chi(\tau,x) = MZ^l(\tau,x), & 0 \leq x \leq l.
\end{cases}
\tag{41}
\]

By the comparison principle, we have
\[
u(t,x) \leq U^l(t,x) \leq \chi(t,x), \quad \forall (t,x) \in [\tau, \infty) \times [0, l].
\]

Using the argument of [28, Lemma 3.2], we can prove that
\[
\lim_{n \to \infty} U^l(t+nT, x) = Z^l(t,x) \text{ in } C^{1,2}([0,T] \times [0,l]),
\]
and we have known that
\[
\lim_{t \to \infty} Z^l(t,x) = Z^\infty(t,x) \text{ in } C^{1,2}([0,T] \times [0,L]) \text{ for any } L > 0,
\]
where $Z^\infty$ is the unique positive solution of $T$-periodic boundary value problem
\[
\begin{cases}
Z_t - d_1Z_{xx} = \chi(a_x(t,x) - Z), & 0 \leq t \leq T, \quad 0 < x < \infty, \\
Z_x(t,0) = 0, & 0 \leq t \leq T, \\
Z(0,x) = Z(T,x), & 0 \leq x < \infty.
\end{cases}
\tag{42}
\]

The existence and uniqueness of $Z^\infty$ is guaranteed by [28, Theorem 4.2]. It follows that
\[
\limsup_{n \to \infty} u(t + nT, x) \leq Z^\infty(t,x) \text{ uniformly for } (t,x) \in [0,T] \times [0,L].
\]

Note that $a_x(t,x) \to a(t,x) + c\nabla(t,x)$ as $\varepsilon \to 0$, by the continuous dependence of solution with respect to parameter, we have that
\[
\lim_{\varepsilon \to 0} Z^\infty(t,x) = \overline{U}(t,x) \text{ uniformly in } [0,T] \times [0,L].
\]

Thus,
\[
\limsup_{n \to \infty} u(t + nT, x) \leq \overline{U}(t,x) \text{ uniformly in } [0,T] \times [0,L].
\]

In the same way, we can show that
\[
\underline{V}(t,x) \leq \liminf_{n \to \infty} v(t + nT, x) \text{ uniformly in } [0,T] \times [0,L]
\]

and
\[
\underline{U}(t,x) \leq \liminf_{n \to \infty} u(t + nT, x) \text{ uniformly in } [0,T] \times [0,L]
\]

for any $L > 0$. 

By Theorem 4.1 and Theorem 4.2, we immediately have that the following spreading-vanishing dichotomy holds under the conditions (Cr) and [31].

**Corollary 1.** Let $(u,v,h)$ be the solution of the free boundary problem (1). Assume that the conditions (Cr) and [31] hold. Then, the following alternative holds:

(i) Spreading: $h_\infty = \infty$ and $\liminf_{t \to \infty} \|u(t,\cdot)\|_{C([0, h(t)])} > 0$;

(ii) Vanishing: $h_\infty < \infty$ and $\lim_{t \to \infty} \|u(t,\cdot)\|_{C([0, h(t)])} = 0$. 

5. Criteria for spreading and vanishing. Since $a(t,x)$ and $V(t,x)$ satisfy the condition (Cr), then $a+cv$ satisfies the condition (A1). Making use of Proposition 2 it yields that $\Sigma_{d_1} = \{ l > 0 : \lambda_1(l,d_1,a+cv) = 0 \} \neq \emptyset$. By the monotonicity of $\lambda_1(l,d_1,a+cv)$ in $l$, we see that $\Sigma_{d_1}$ contains at most one element. We write it as $h^*$, namely, $\lambda_1(h^*,d_1,a+cv) = 0$.

Now we give a necessary condition of vanishing.

**Lemma 5.1.** Assume that $\Sigma_{d_1} \neq \emptyset$. If $h_\infty < \infty$, then $h_\infty \leq h^*$. Hence, $h_0 \geq h^*$ implies $h_\infty = \infty$ for all $\mu > 0$.

**Proof.** We assume $h_\infty > h^*$ to get a contradiction. If $h_\infty > h^*$, then $\lambda_1(h_\infty,d_1,a+cv) < 0$. By the continuity of $\lambda_1(h_\infty,d_1,a+cv)$, there exists $\varepsilon > 0$ such that $\lambda_1(h_\infty,d_1,a+cv-\varepsilon) < 0$. In view of (25) and the fact that $V(t,x)$ is $T$-periodic in $t$, there exists $\tau > 1$ such that $\lambda_1(h(\tau),d_1,a+cv-\varepsilon) < 0$ and $v(t,x) \geq V(t,x) - \frac{\varepsilon}{\varepsilon}$ for all $(t,x) \in (\tau, \infty) \times [0,h(\tau)]$.

Let $w(t,x)$ be the unique solution of

\[
\begin{cases}
  w_t - d_1w_{xx} = w(a(t,x) + cv(t,x) - \varepsilon - w), & t > \tau, \ 0 < x < h(\tau), \\
  w_x(t,0) = 0, \ w(t,h(\tau)) = 0, & t > \tau, \\
  w(\tau,x) = u(\tau,x), & 0 \leq x \leq h(\tau).
\end{cases}
\]

Then $u \geq w$ in $[\tau, \infty) \times [0,h(\tau)]$ by the comparison principle.

Noting that $\lambda_1(h(\tau),d_1,a+cv-\varepsilon) < 0$, it follows from [16, Theorem 28.1] that

\[
\lim_{n \to \infty} w(t+nT,x) = Z(t,x) \text{ uniformly in } [0,T] \times [0,h(\tau)],
\]

where $Z(t,x)$ is the unique positive solution of the following $T$-periodic boundary value problem

\[
\begin{cases}
  Z_t - d_1Z_{xx} = Z(a(t,x) + cv(t,x) - \varepsilon - Z), & 0 \leq t \leq T, \ 0 < x < h(\tau), \\
  Z_x(t,0) = 0, \ Z(t,h(\tau)) = 0, & 0 \leq t \leq T, \\
  Z(0,x) = Z(T,x), & 0 \leq x \leq h(\tau).
\end{cases}
\]

Since $u \geq w$ in $[\tau, \infty) \times [0,h(\tau)]$, we immediately deduce that

\[
\liminf_{n \to \infty} u(t+nT,x) \geq Z(t,x), \ \forall (t,x) \in [0,T] \times [0,h(\tau)].
\]

This is in contradiction to Theorem 3.1. \hfill \Box

In the following, for the parameter $h_0$ satisfying $h_0 < h^*$ and $(u_0,v_0)$ fixed, we discuss the effect of the coefficient $\mu$ on the spreading and vanishing for the species $u$.

**Lemma 5.2.** Assume that $\Sigma_{d_1} \neq \emptyset$. If $h_0 < h^*$, then there exists $\mu^0 > 0$ depending on $u_0$ such that $h_\infty = \infty$ for $\mu \geq \mu^0$.

**Proof.** Since $\Sigma_{d_1} \neq \emptyset$, there exists $T^* > 0$ such that $\lambda_1(\sqrt{T^*},d_1,a+cv) < 0$.

Inspired by the argument of [9, proposition 5.3], let $\lambda$ be the eigenvalue of

\[
\begin{cases}
  -d_1\varphi'' - \frac{1}{\varphi'} = \lambda\varphi, & 0 < x < 1, \\
  \varphi'(0) = \varphi(1) = 0,
\end{cases}
\]

then the corresponding eigenfunction $\varphi > 0$, $\varphi' \leq 0$ in $[0,1)$, and $\|\varphi\|_{L^\infty([0,1))} = 1$. 

Define
\[ h(t) = \sqrt{t + \delta}, \quad 0 \leq t \leq T^*, \]
\[ u(t, x) = \begin{cases} \frac{m}{(t + \delta)^3} \varphi\left(\frac{x}{\sqrt{t + \delta}}\right), & 0 \leq t \leq T^*, \quad 0 \leq x \leq h(t), \\ 0, & 0 \leq t \leq T^*, \quad h(t) < x < \infty, \end{cases} \]
where \( \delta, m \) and \( k \) are positive constants to be chosen later.

Choosing \( 0 < \delta \leq \min\{1, h_0^2\} \) and \( k > \lambda + (T^* + 1)(\|a\|_{L^\infty} + \|u\|_{L^\infty}) \), selecting \( m \) to be sufficiently small such that \( u(0, x) = \frac{m}{\sigma^3} \varphi\left(\frac{x}{\delta}\right) < u_0(x) \), and \( \mu \geq \frac{(T^* + 1)^k}{2m\sigma^3(1)} : = \mu^0 \), we obtain
\[
u_t - d_1 \nu_x - \nu(u(t, x) - u) \leq - \frac{m}{(t + \delta)^3} \left[ k\varphi + \frac{x}{2\sqrt{t + \delta}} \varphi' + d_1 \varphi'' - (t + \delta)\varphi\|a\|_{L^\infty} - (t + \delta)\varphi\|\nabla u\|_{L^\infty} \right] \leq - \frac{m}{(t + \delta)^3} \left[ d_1 \varphi'' + \frac{1}{2} \varphi' + k\varphi - (T^* + 1)\varphi(\|a\|_{L^\infty} + \|u\|_{L^\infty}) \right] \leq - \frac{m}{(t + \delta)^3} (d_1 \varphi'' + \frac{1}{2} \varphi' + \lambda \varphi) = 0,
\]
for \( (t, x) \in (0, T^*) \times (0, h(t)) \), and
\[ h'(t) + \mu \nu_x(t, h(t)) = \frac{1}{2\sqrt{t + \delta}} + \frac{\mu m}{(t + \delta)^{k+\frac{3}{2}}} \varphi'(1) \leq 0 \text{ for } 0 < t \leq T^* \]
Moreover,
\[ \nu_x(t, 0) = (t, h(t)) = 0 \text{ for } 0 < t \leq T^*, \quad h(0) = \sqrt{\delta} \leq h_0. \]
Then we can use Theorem 3.2 to conclude that \( h(t) \geq h(t) \) in \( [0, T^*] \). Specially, we obtain
\[ h(T^*) \geq h(T^*) = \sqrt{T^* + \delta} > \sqrt{T^*}. \]
Since \( \lambda_1(\sqrt{T^*}, d_1, a + cV) < 0 \), we deduce that
\[ h(T^*) > \sqrt{T^*} > h^*, \]
which implies that \( h_\infty > h^* \). It follows from Lemma 5.1 that \( h_\infty = \infty. \)

Similarly to the argument of Lemma 5.2, we can get the following corollary.

**Corollary 2.** For fixed \( d_1 \) and \( \mu \), assume that \( \Sigma_{d_1} \neq \emptyset \). Let \( u_0(x) = \sigma \psi(x) \) for some positive constant \( \sigma \), where \( \psi(x) \) satisfies
\[ \psi(x) \in C^2([0, h_0]), \quad \psi(x) > 0 \text{ in } (0, h_0) \text{ and } \psi_x(0) = \psi(h_0) = 0. \] (45)
If \( h_0 < h^* \), then there exists \( \sigma^0 \) such that \( h_\infty = \infty \) for \( \sigma \geq \sigma^0 \).

**Lemma 5.3.** Assume that \( \Sigma_{d_1} \neq \emptyset \). If \( h_0 < h^* \), then there exists \( \mu_0 > 0 \) depending on \( u_0 \) such that \( h_\infty \leq h^* \) for all \( \mu \leq \mu_0 \).

**Proof.** This proof can be done by a similar process in [31] Lemma 3.2 with minor modification.

Since \( h_0 < h^* \), we have that \( \lambda_1(h_0, d_1, a + cV) > 0 \).
Let \( z(t) \) be the solution of
\[
\begin{cases} z' = z(\|b\|_{L^\infty} - z), & t > 0, \\ z(0) = \|v_0\|_{L^\infty}. \quad (46) \end{cases}
\]
Then \( v(t, x) \leq z(t) \).
Consider the following auxiliary free boundary problem

\[
\begin{cases}
  w_t - d_1 w_{xx} = w(a(t, x) - w + cz(t)), & t > 0, \ 0 < x < s(t), \\
  w_x(t, 0) = 0, \ w(t, s(t)) = 0, & t > 0, \\
  s'(t) = -\mu w_x(t, s(t)), & t > 0, \\
  s(0) = h_0, \ w(0, x) = u_0(x), & 0 \leq x \leq h_0. 
\end{cases}
\]  

(47)

**Step 1.** We shall prove that there exists \( \mu_0 > 0 \) such that

\[ s_\infty < \infty, \ \forall \ 0 < \mu \leq \mu_0. \]

Define

\[
\begin{align*}
  f(t) &= Me^{\int_0^t (c + cz(s)) ds}, \ t > 0; \\
  \eta(t) &= (1 + \delta)^2 + \frac{\delta^2}{0\int_0^t f(s) ds} \int_0^t f(s) ds \frac{1}{2}, \ t > 0; \\
  \xi(t) &= \int_0^t \eta^{-2}(\tau) d\tau, \ t \geq 0; \\
  \overline{w}(t, x) &= f(t) \phi(\xi(t), y), \ y = \frac{x}{\eta(t)}, \ t > 0, \ 0 \leq x \leq h_0 \eta(t).
\end{align*}
\]

For any given \( 0 < \varepsilon \ll 1 \), since \( a(t, x) \) is uniformly continuous in \([0, T] \times [0, 3h_0]\) and \( T \)-periodic in \( t \), there exists \( 0 < \delta_0(\varepsilon) \ll 1 \) such that, for all \( 0 < \delta < \delta_0(\varepsilon) \),

\[
|a(\xi, y) - \eta^2(\xi) a(t, x)| \leq \varepsilon \text{ for } \forall (t, x) \in [0, \infty) \times [0, h_0 \eta(t)].
\]

Direct calculation yields

\[
\begin{align*}
  \overline{w}_t - d_1 \overline{w}_{xx} - \overline{w}(a(t, x) - \overline{w} + cz(t)) \\
  &= f' \phi + f \eta^{-2} (\phi \phi_x - d_1 \phi_{yy}) - f \phi_x \eta^{-2} \eta' - f \phi [a(t, x) - f \phi + cz(t)] f \phi_x \eta^{-2} \eta' \\
  &\geq f' \phi + f \eta^{-2} (\phi - d_1 \phi_{yy}) - f \phi [a(t, x) + cz(t)] \\
  &= f' \phi + f \eta^{-2} [a(\xi, y) + cV + \lambda_1 (h_0; d_1, a + cV)] - f \phi [a(t, x) + cz(t)] \\
  &= f \phi [f' \frac{f}{f} + \frac{a(\xi, y) - a(t, x)}{\eta^2} + (cV + \lambda_1) \eta^{-2} - cz(t)] \\
  &\geq f \phi [\varepsilon + cz(t) - \varepsilon + \lambda_1 \eta^{-2} - cz(t)] \\
  &\geq 0,
\end{align*}
\]

and

\[ \overline{w}_x(t, 0) = 0, \ \overline{w}(t, h_0 \eta(t)) > 0. \]

We now choose \( M > 0 \) sufficiently large such that

\[ \overline{w}(0, x) = M \phi(0, \frac{x}{1 + \delta}) \geq u_0(x), \ \forall x \in [0, h_0]. \]

The direct calculation yields

\[
\begin{align*}
  h_0 \eta'(t) &= \frac{h_0 \delta^2 f(t)}{2 \eta(t) \int_0^\infty f(s) ds} \text{ and } -\mu \overline{w}_x(t, h_0 \eta(t)) = \frac{\mu f(t) \phi_y(\xi, h_0)}{\eta(t)}.
\end{align*}
\]

Therefore, if we take

\[
\mu_0 = \frac{h_0 \delta^2}{2 |\phi_y(\xi, h_0)| \int_0^\infty f(s) ds},
\]
then for any $0 < \mu \leq \mu_0$, there holds $h_0\eta'(t) \geq -\mu \overline{\eta}_x(t, h_0\eta(t))$. We can apply the comparison principle to conclude that $s(t) \leq h_0\eta(t)$. Therefore,

$$s_\infty \leq \lim_{t \to \infty} h_0\eta(t) \leq h_0(1 + 2\delta) < \infty.$$  

**Step 2.** Noting that $v(t, x) \leq z(t)$, we see that $(u, h)$ satisfies

$$\begin{cases}
u t - d_1 u_{xx} \leq u(a(t, x) - u + cz(t)), & t > 0, \ 0 < x < h(t), \\
u x(t, 0) = 0, \ u(t, h(t)) = 0, & t > 0, \\
u h'(t) = -\mu u_x(t, h(t)), & t > 0, \\
u h(0) = h_0, \ u(0, x) = u_0(x), & 0 \leq x \leq h_0.
\end{cases} \tag{48}$$

Applying the comparison principle, it is derived that

$$h(t) \leq s(t), \ u(t, x) \leq w(t, x) \text{ for } \forall (t, x) \in (0, \infty) \times [0, h(t)].$$

Hence, $h_\infty \leq s_\infty < \infty$ for all $0 < \mu \leq \mu_0$. 

Similarly to the argument of Lemma 5.3, we can get the following corollary.

**Corollary 3.** For fixed $d_1$ and $\mu$, assume that $\Sigma_{d_1} \neq \emptyset$. Let $u_0(x) = \sigma \psi(x)$ for some positive constant $\sigma$, where $\psi(x)$ satisfies (15). If $h_0 < h^*$, then there exists $\sigma_0$ such that $h_\infty \leq h^*$ for $\sigma \leq \sigma_0$.

Now we fix $d_1$, and consider $h_0$ and $\mu$ as varying parameters to depict the sharp criteria for spreading and vanishing.

**Theorem 5.4.** Assume that (Cr) holds. For fixed $d_1$, we have

(i) If $h_0 \geq h^*$, then $h_\infty = \infty$ for all $\mu > 0$.
(ii) If $h_0 < h^*$, then there exist $\mu^* > 0$ such that $h_\infty = \infty$ for $\mu > \mu^*$, and $h_\infty \leq h^*$ for $0 < \mu \leq \mu^*$ or $\mu = \mu^*$.

**Proof.** The proof is similar to that of [23, Theorem 5.2]. We give the details below for completeness.

Since (Cr) holds, $\Sigma_{d_1} \neq \emptyset$. The conclusion (i) follows from Lemma 5.1. Next, we prove the conclusion (ii).

Define $\Sigma^* := \{\mu > 0 : h_\infty \leq h^*\}$. By Lemma 5.3, we have $\Sigma^* \supset (0, \mu_0]$. It follows from Lemma 5.2 and Spreading-Vanishing dichotomy that $\Sigma^* \cap [\mu_0, \infty) = \emptyset$. Therefore, $\mu^* := \sup \Sigma^* \in [\mu_0, \mu^0]$. By this definition and Lemma 5.1, we find that $h_\infty = \infty$ when $\mu > \mu^*$.

We claim that $\mu^* \in \Sigma^*$. Otherwise $h_\infty = \infty$ for $\mu = \mu^*$. Hence, we can find $T > 0$ such that $h(T) > h^*$. To stress the dependence of the solution $(u, v, h)$ in $[\boldsymbol{1}]$ on $\mu$, we write $(u_\mu, v_\mu, h_\mu)$ instead of $(u, v, h)$. So we have $h_\mu^*(T) > h^*$. By the continuous dependence of $(u_\mu, v_\mu, h_\mu)$ on $\mu$, we can find $\epsilon > 0$ small so that $h_\mu(T) > h^*$ for $\mu \in [\mu^* - \epsilon, \mu^* + \epsilon]$. It follows that for all such $\mu$,

$$\lim_{t \to \infty} h_\mu(t) > h_\mu^*(T) > h^*.$$  

This implies that $[\mu^* - \epsilon, \mu^* + \epsilon] \cap \Sigma^* = \emptyset$, and $\sup \Sigma^* \leq \mu^* - \epsilon$, contradicting to the definition of $\mu^*$. This proves our claim.

Define $\Sigma_* := \{\nu > 0 : \nu \geq \mu_0 \text{ such that } h_\infty \leq h^* \text{ for all } 0 < \mu < \nu\}$, then $\mu_* := \sup \Sigma_* \leq \mu^*$ and $(0, \mu_*) \subset \Sigma_*$. Similarly to the above, we can prove that $\mu_* \in \Sigma_*$. The proof is completed. 

\[\square\]
Remark 3. We should emphasize that \( \mu^* \) may not be equal to \( \mu_* \) in Theorem 5.4, which is a little different from results in competition models (see, for example, [8]). This mainly results from the lack of monotonicity of solution to (1) in \( \mu \). The monotonicity can not be deduced from the comparison principle (Theorem 3.1 and 3.2).

Now we fix \( h_0 \), and regard \( d_1 \) and \( \mu \) as the variable parameters to describe the criteria for spreading and vanishing.

**Theorem 5.5.** For fixed \( h_0 \), we have

(i) Assume that \( h_0 > \tilde{h} \), where \( \tilde{h} \) is given in (A2). If \( d_1 \in \Sigma_{h_0}^- \), then \( h_{\infty} = \infty \) for all \( \mu > 0 \).

(ii) For any fixed \( d_1 \in \Sigma_{h_0}^+ \), there exist \( \mu_0 > 0 \) such that \( h_{\infty} = \infty \) for all \( \mu \leq \mu_0 \).

If, in addition, \( \Sigma_{d_1} \neq \emptyset \) for such \( d_1 \), then there exist \( \mu^* \geq \mu_* > 0 \), such that \( h_{\infty} = \infty \) for \( \mu > \mu^* \), and \( h_{\infty} < \infty \) for \( 0 < \mu \leq \mu_* \) or \( \mu = \mu^* \).

6. **Spreading speed.** For the case of spreading successfully, we estimate roughly the asymptotic spreading speed of the free boundary \( h(t) \) in this section.

**Theorem 6.1.** Under the condition (Cr) with \( r = 0 \), we further assume that \( \mu_0 = 0 \) holds. When the spreading occurs, i.e., \( h_{\infty} = \infty \), we have

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \frac{1}{T} \int_0^T k_0(\mu, a^\infty + cM, 1)(t) dt,
\]

\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq \frac{1}{T} \int_0^T k_0(\mu, a^\infty, 1)(t) dt,
\]

where \( k_0(\mu, \cdot, \cdot)(t) \) is given in [28, Proposition 5.1].

**Proof.** Let \((w, g)\) and \((z, p)\) be solutions of the free boundary problems

\[
\begin{align*}
&w_t - d_1 w_{xx} = w(a(t, x) + cM - u), \quad t > 0, \quad 0 < x < g(t), \\
&w_x(t, 0) = w(t, g(t)) = 0, \quad t > 0, \\
&g'(t) = -\mu w_x(t, g(t)), \quad t > 0, \\
&g(0) = g_0 \geq h_0, \quad w(0, x) = w_0(x) \geq u_0(x), \quad x \in [0, g_0],
\end{align*}
\]

and

\[
\begin{align*}
&z_t - d_1 z_{xx} = z(a(t, x) - u), \quad t > 0, \quad 0 < x < p(t), \\
&z_x(t, 0) = z(t, p(t)) = 0, \quad t > 0, \\
&p'(t) = -\mu z_x(t, p(t)), \quad t > 0, \\
&p(0) = p_0 \leq h_0, \quad z(0, x) = z_0(x) \leq u_0(x), \quad x \in [0, p_0],
\end{align*}
\]

respectively, where \( M \) is given in Theorem 2.1. Assume that \( g_{\infty} = \infty \) and \( p_{\infty} = \infty \).

It follows from the result of [28] that

\[
\limsup_{t \to \infty} \frac{g(t)}{t} \leq \frac{1}{T} \int_0^T k_0(\mu, a^\infty + cM, 1)(t) dt,
\]

\[
\liminf_{t \to \infty} \frac{p(t)}{t} \geq \frac{1}{T} \int_0^T k_0(\mu, a^\infty, 1)(t) dt.
\]
For the solution \((u, v, h)\) of the problem \((1)\), by the comparison principle, we have that
\[ p(t) \leq h(t) \leq g(t). \]
Hence,
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \frac{1}{T} \int_0^T k_0(\mu, a^\infty + cM, 1)(t) dt,
\]
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq \frac{1}{T} \int_0^T k_0(\mu, a^\infty, 1)(t) dt.
\]

7. **Discussion.** If the initial region \((0, h_0)\) satisfies
\[
\lambda_1(h_0, d_1, a + cV) \leq 0 \quad (d_1 \text{ is fixed})
\]

or the diffusion rate \(d_1\) satisfies
\[
\lambda_1(h_0, d_1, a + cV) \leq 0 \quad (h_0 \text{ is fixed}),
\]
then the predator species will spread successfully regardless of the expansion capacity \(\mu\) and initial population size \(u_0(x)\). Moreover, if, in addition, \((31)\) holds, then they will coexist friendly in the environment. Otherwise, we can select the predator with the larger expansion capacity \(\mu\) or enlarge its initial population size \(u_0(x)\) such that the predator will spread throughout the whole space to control the size of the prey.

Depicting asymptotic behavior accurately relies on the alternate iterative form constructed by the upper and lower solution. So we only describe the long time behavior roughly in Theorem 4.2. In this theorem, we add an assumption \(b_\infty - d(a^\infty + cV) > 0\), which indicates the relationship between the predator and prey is relatively weak, namely, the coefficient \(c\) and \(d\) are small enough. If \((31)\) does not hold true, then the existence of the positive solution to problem \((36)\) will not be guaranteed. This needs us to find other appropriate methods or new techniques.

Relatively speaking, the asymptotic spreading speed for a two-species system with free boundaries has been studied weakly so far. Apart from \((11)\), the related researches almost only give the rough estimates, see \([14, 25, 8, 24]\) for example. Due to the method limited, we only estimate roughly the asymptotic spreading speed in this paper.

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**REFERENCES**

[1] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, 7 (2012), 583–603.
[2] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, John Wiley and Sons Ltd, 2003.
[3] Q. Chen, F. Li and F. Wang, A diffusive logistic problem with a free boundary in time-periodic environment: Favorable habitat or unfavorable habitat, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 13–35.
[4] Q. Chen, F. Li and F. Wang, The diffusive competition problem with a free boundary in heterogeneous time-periodic environment, *J. Math. Anal. Appl.*, 433 (2016), 1594–1613.
[5] Y. Du and Z. Guo, Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary, *J. Differential Equations*, 250 (2011), 436–4366.
[6] Y. Du, Z. Guo and R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, *J. Funct. Anal.*, 265 (2013), 2089–2142.
[7] Y. Du and Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, 42 (2010), 377–405.
[8] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. Ser. B*, 19 (2014), 3105–3132.
1. Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries. J. Eur. Math. Soc., 17 (2015), 2673–2724.
2. Y. Du, M. Wang and M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary. J. Math. Pures Appl., 107 (2017), 253–287.
3. H. Gu, Z. Lin and B. Lou, Long time behavior of solutions of a diffusion-advection logistic model with free boundaries. Appl. Math. Lett., 37 (2014), 49–53.
4. H. Gu, Z. Lin and B. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries. Proc. Amer. Math. Soc., 143 (2015), 1109–1117.
5. H. Gu, B. Lou and M. Zhou, Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries. J. Funct. Anal., 269 (2015), 1714–1768.
6. J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system. J. Dynam. Differential Equations, 24 (2012), 873–895.
7. J. S. Guo and C. H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries. Nonlinearity, 28 (2015), 1–27.
8. P. Hess, Periodic-Parabolic Boundary Value Problems and Positivity. Pitman Res. Notes Math., vol.247. Longman Sci. Tech., Harlow, 1991.
9. Y. Kaneko and H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection-diffusion equations. J. Math. Anal. Appl., 428 (2015), 43–76.
10. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’ceva, Linear and Quasilinear Equations of Parabolic Type. Amer. Math. Soc., Providence, RI, 1968.
11. C. Lei, Z. Lin and H. Wang, The free boundary problem describing information diffusion in online social networks. J. Differential Equations, 254 (2013), 1326–1341.
12. C. Lei, Z. Lin and Q. Zhang, The spreading front of invasive species in favorable habitat or unfavorable habitat. J. Differential Equations, 257 (2014), 145–166.
13. Z. Lin, A free boundary problem for a predator-prey model. Nonlinearity, 20 (2007), 1883–1892.
14. R. Peng and X. Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession. Discrete Contin. Dyn. Syst. Ser. A, 33 (2013), 2007–2031.
15. J. Wang, The selection for dispersal: A diffusive competition model with a free boundary. Z. Angew. Math. Phys., 66 (2015), 2143–2160.
16. J. Wang and L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment. J. Math. Anal. Appl., 423 (2015), 377–398.
17. M. Wang, On some free boundary problems of the prey-predator model. J. Differential Equations, 256 (2014), 3365–3394.
18. M. Wang, The diffusive logistic equation with a free boundary and sign-changing coefficient. J. Differential Equations, 258 (2015), 1252–1266.
19. M. Wang, Spreading and vanishing in the diffusive prey-predator model with a free boundary. Commun. Nonlinear Sci. Numer. Simul., 23 (2015), 311–327.
20. M. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment. J. Funct. Anal., 270 (2016), 483–508.
21. M. Wang, Dynamics for a diffusive prey-predator model with different free boundaries, preprint. arXiv:1511.06479v2
22. M. Wang and Y. Zhang, Two kinds of free boundary problems for the diffusive predator-prey model. Nonlinear Anal. Real World Appl., 24 (2015), 73–82.
23. M. Wang, W. Sheng and Y. Zhang, Spreading and vanishing in a diffusive prey-predator model with variable intrinsic growth rate and free boundary. J. Math. Anal. Appl., 441 (2016), 309–329.
24. M. Wang and Y. Zhang, The time-periodic diffusive competition models with a free boundary and sign-changing growth rates. Z. Angew. Math. Phys., 67 (2016), Art. 132, 24 pp.
25. M. Wang and J. Zhao, Free boundary problems for a Lotka-Volterra competition system. J. Dynam. Differential Equations, 26 (2014), 655–672.
26. M. Wang and J. Zhao, A free boundary problem for a predator-prey model with double free boundaries. J. Dynam. Differential Equations, 26 (2014), 655–672.
27. M. Wang and J. Zhao, A free boundary problem for a predator-prey model with double free boundaries. J. Dynam. Differential Equations, 26 (2014), 655–672.
28. M. Wang and J. Zhao, A free boundary problem for a predator-prey model with double free boundaries. J. Dynam. Differential Equations, 26 (2014), 655–672.
29. M. Wang and J. Zhao, A free boundary problem for a predator-prey model with double free boundaries. J. Dynam. Differential Equations, 26 (2014), 655–672.
30. M. Wang and J. Zhao, A free boundary problem for a predator-prey model with higher dimension and heterogeneous environment. Nonlinear Anal. Real World Appl., 16 (2014), 250–263.
P. Zhou and D. Xiao, The diffusive logistic model with a free boundary in heterogeneous environment, *J. Differential Equations*, **256** (2014), 1927–1954.

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