AN INTRODUCTION TO NONCOMMUTATIVE DEFORMATIONS OF MODULES

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Abstract. Let \( k \) be an algebraically closed (commutative) field, let \( A \) be an associative \( k \)-algebra, and let \( M = \{M_1, \ldots, M_p\} \) be a finite family of left \( A \)-modules. We study the simultaneous formal deformations of this family, described by the noncommutative deformation functor \( \text{Def}_M : \mathfrak{a}_p \to \text{Sets} \) introduced in Laudal \[8\]. In particular, we prove that this deformation functor has a pro-representing hull, and describe how to calculate this hull using the cohomology groups \( \text{Ext}^n_A(M_i, M_j) \) and their matric Massey products.

Introduction

In this paper, I shall give an elementary introduction to the noncommutative deformation theory for modules, due to Laudal. This theory, which generalizes the classical deformation theory for modules, was introduced by Laudal in \[8\]. Earlier versions of this material appeared in the preprints Laudal \[3, 4, 5, 6, 7\].

This noncommutative deformation theory has several applications: In the paper Laudal \[8\], Laudal used it to construct algebras with a prescribed set of simple modules, and also to study the moduli space of iterated extensions of modules. In the preprint Laudal \[7\], he also showed that this theory is a useful tool in the study of algebras, and in establishing a noncommutative algebraic geometry.

These applications are an important part of the motivation for the noncommutative deformation theory. But we shall not go into the details of these applications in this elementary introduction. Instead, we refer to the papers and preprints of Laudal mentioned above for applications and further developments of the theory.

Throughout this paper, we shall fix the following notations: Let \( k \) be an algebraically closed (commutative) field, let \( A \) be an associative \( k \)-algebra, and let \( M = \{M_1, \ldots, M_p\} \) be a finite family of left \( A \)-modules. Notice that this notation differs from Laudal’s: While Laudal considers families of right modules in all his papers, I consider families of left modules. Of course, the difference is only in the appearance — the resulting theories are obviously equivalent.

We shall present a noncommutative deformation functor \( \text{Def}_M : \mathfrak{a}_p \to \text{Sets} \), which describes the simultaneous formal deformations of the family \( M \) of left \( A \)-modules. Furthermore, we shall prove that this deformation functor has a pro-representable hull \((H, \xi)\) when the family \( M \) satisfy a certain finiteness condition. We shall also describe a method for finding the pro-representable hull explicitly.

In section \[11\] we describe the category \( \mathfrak{a}_p \). It is a full sub-category of the category \( A_p \) of \( p \)-pointed \( k \)-algebras. The objects of \( A_p \), are the \( k \)-algebras \( R \) equipped with \( k \)-algebra homomorphisms \( k^p \to R \to k^p \), such that the composition \( k^p \to k^p \) is the identity. For any such object, \( R = (R_{ij}) \) is a \( k \)-algebra of \( p \times p \) matrices. The radical of this object is the ideal \( I(R) = \ker(R \to k^p) \subseteq R \). The category \( \mathfrak{a}_p \) is the

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full sub-category of \( A_p \) consisting of objects such that \( R \) is Artinian and complete in the \( I(R) \)-adic topology.

In section 2, we describe the noncommutative deformation functor associated to the family \( M \) of left \( A \)-modules,

\[
\text{Def}_M : a_p \to \text{Sets}
\]

It is constructed in the following way: Let \( R \) be an object of \( a_p \), and consider the vector space \( M_R = (M_i \otimes_k R_{ij}) \), equipped with the natural right \( R \)-module structure induced by the multiplication in \( R \). A deformation of \( M \) to \( R \) consists of the following data:

- A left \( A \)-module structure on \( M_R \) making \( M_R \) a left \( A \otimes_k R_{op} \)-module,
- Isomorphisms \( \eta_i : M_R \otimes_R k_i \to M_i \) of left \( A \)-modules for \( 1 \leq i \leq p \).

The set of equivalence classes of such deformations is denoted \( \text{Def}_M(R) \), and this defines the covariant functor \( \text{Def}_M \). Notice that the fact that \( M_R \cong (M_i \otimes_k R_{ij}) \) as right \( R \)-modules replaces the flatness condition in classical deformation theory. If \( p = 1 \) and \( R \) is commutative, the above condition is of course equivalent to the flatness condition, so the noncommutative deformation functor generalizes the classical one.

In section 3, we look at noncommutative deformations from the point of view of resolutions. Let \( R \) be any object of \( a_p \). An \( M \)-free module over \( R \) is a left \( A \otimes_k R_{op} \)-module \( F \) of the form \( F = (L_i \otimes R_{ij}) \), where \( L_1, \ldots, L_p \) are free left \( A \)-modules. \( M \)-free complexes and \( M \)-free resolutions are defined similarly. Let us fix a free resolution of \( M_i \) the form

\[
0 \leftarrow M_i \leftarrow L_{0,i} \leftarrow \cdots \leftarrow L_{m,i} \leftarrow \cdots
\]

for \( 1 \leq i \leq p \). We prove that there is a bijective correspondence between deformations of \( M \) to \( R \) and complexes of \( M \)-free modules over \( R \) of the form

\[
(L_{0,i} \otimes_k R_{ij}) \leftarrow \cdots \leftarrow (L_{m,i} \otimes_k R_{ij}) \leftarrow \cdots
\]

In fact, each such complex of \( M \)-free modules is an \( M \)-free resolution of the corresponding deformation \( M_R \) of \( M \) to \( R \).

In section 4, we recall some general facts about pointed functors and their representability. In section 5, we consider the special case of the noncommutative deformation functor \( \text{Def}_M \). From this point in the text, we assume that the family \( M \) satisfy the finiteness condition

\[
\text{(FC)} \quad \dim_k \text{Ext}_A^n(M_i, M_j) \text{ is finite for } 1 \leq i, j \leq p, \quad n = 1, 2.
\]

When this condition holds, we define \( T^1, T^2 \) to be the formal matrix rings (in the sense of section 1) given by the families of \( k \)-vector spaces \( V_{ij} = \text{Ext}_A^n(M_i, M_j) \) for \( n = 1, 2 \). Assuming condition \( \text{(FC)} \), we show the following theorem of Laudal, which generalizes the corresponding theorem for the classical deformation functor:

**Theorem 0.1.** There exists an obstruction morphism \( o : T^2 \to T^1 \), such that \( H = T^1 \otimes T^2 k^p \) is a pro-representable hull for the noncommutative deformation functor \( \text{Def}_M : a_p \to \text{Sets} \).

In the rest of the paper, we show how to construct the hull \( H \) explicitly, which can be accomplished by using matric Massey products. In section 6, we introduce the immediately defined matric Massey products. In section 7, we define the matric Massey products in general, and show that the hull \( H \) of the noncommutative deformation functor \( \text{Def}_M \) is determined by the vector spaces \( \text{Ext}_A^n(M_i, M_j) \) for
n = 1, 2 and 1 ≤ i, j ≤ p and their matric Massey products. We also describe a general method for calculating the hull H in concrete terms.

In appendix A we describe the Yoneda and Hochschild representations of the cohomology groups Ext^n_A(M_i, M_j). In this paper, we have chosen to express the matric Massey products using the Yoneda representation and M-free resolutions. It is also possible to express the matric Massey products using the Hochschild representation, see for instance Laudal. 

1. Categories of pointed algebras

Let p be a fixed natural number, and consider the ring k^p. This ring has a natural k-algebra structure given by the map α ↦ (α, ..., α) for α ∈ k. Let pr_i : k^p → k^p be the i’th projection, and consider the ideal k_i = pr_i(k^p) ⊆ k^p as a k^p-module for 1 ≤ i ≤ p. Clearly, k^p is an Artinian k-algebra and {k_1, ..., k_p} is the full set of isomorphism classes of simple k^p-modules, each of them of dimension 1 over k. This simple example will serve as a model for the p-pointed algebras that we shall consider in this section.

A p-pointed k-algebra is a triple (R, f, g), where R is an associative ring and f : k^p → R, g : R → k^p are ring homomorphisms such that g ∘ f = id. A morphism u : (R, f, g) → (R’, f’, g’) of p-pointed k-algebras is a ring homomorphism u : R → R’ such that the natural diagrams commute (that is, such that u ∘ f = f’ and g’ ∘ u = g). We shall denote the category of p-pointed k-algebras by A_p. Notice that if (R, f, g) is an object of A_p, then f is injective and g is surjective, and we shall identify k^p with its image in R. We often write R for the object (R, f, g) to simplify notation.

Let (R, f, g) be an object in A_p. We define the radical of R to be I(R) = ker(g), which is an ideal in R. Furthermore, we denote by J(R) the Jacobson radical of R

J(R) = \{x ∈ R : xM = 0 for all simple left R-modules M\},

which is also an ideal in R. We shall write I, J for the radicals I(R), J(R) when there is no danger of confusion. Notice that the Jacobson radical J depends only on the ring R, while the radical I depends on the structural morphism g as well.

For all objects R in A_p, we have an inclusion J(R) ⊆ I(R): We have J(k^p) = 0 since k^p is semi-simple, and g(J(R)) ⊆ J(k^p) = 0 since g : R → k^p is a surjection. In general, we know that R and R/J(R) have the same simple left modules. So if we consider k_i as a left R-module via the morphism g : R → k^p for 1 ≤ i ≤ p, we see that \{k_1, ..., k_p\} is contained in the set of isomorphism classes of simple left R-modules, and the equality J(R) = I(R) holds if and only if \{k_1, ..., k_p\} is the full set of isomorphism classes of simple left R-modules. Equivalently, the equality I(R) = J(R) holds if and only if there are exactly p isomorphism classes of simple left R-modules.

It is therefore clear that the equality I(R) = J(R) does not hold in general: It is easy to find examples where R has ‘too many’ simple modules. For instance, consider R = k[x]/(x − x^2) with the natural k-algebra structure f : k → R and let g : R → k be given by x ↦ 0. Then R is an object of A_1, but J(R) ≠ I(R) because R has two non-isomorphic simple left R-modules (given by x ↦ 0 and x ↦ 1).

Let e_i be the idempotent \{(0, 0, ..., 1, ..., 0) \in k^p\} for 1 ≤ i ≤ p. Notice that e_i e_j = 0 if i ≠ j, and that e_1 + ... + e_p = 1. For any object R in A_p, we identify \{e_1, ..., e_p\} with idempotents in R via the inclusion k^p ↠ R. Denote by R_{ij} the k-linear sub-space e_iR e_j ⊆ R. We immediately see, using the properties of the idempotents, that the following relations hold for 1 ≤ i, j, l, m ≤ p:

1. R_{ij}R_{lm} \subseteq \delta_{ij}R_{lm},
2. R_{ij} ∩ R_{lm} = 0 if (i, j) ≠ (l, m),
In particular, we have that to call an object \( R \) in \( A_p \), a matrix ring, and to write it \( R = (R_{ij}) \). Notice that \( R_i \) is an associative ring (with identity \( e_i \)), and that \( R_{ij} \) is a (unitary) \( R_i \)-\( R_j \) bimodule for \( 1 \leq i, j \leq p \). For any ideal \( K \subseteq R \), we see that \( e_i K e_j = K \cap R_{ij} \), and we shall denote this \( k \)-linear subspace \( K_{ij} \) for \( 1 \leq i, j \leq p \). Since \( K = \oplus K_{ij} \), we write \( K = (K_{ij}) \).

Let \( R \) be an object of \( A_p \), so \( R = (R_{ij}) \) is a matrix ring in the above sense. The following standard result gives useful information on when \( R \) is an Artinian or Noetherian ring:

**Proposition 1.1.** Let \( R = (R_{ij}) \) be an object in \( A_p \). Then \( R \) is Noetherian (Artinian) if and only if the following conditions hold:

i) \( R_{ii} \) is Noetherian (Artinian) for \( 1 \leq i \leq p \),

ii) \( R_{ij} \) is a Noetherian (Artinian) left \( R_{ii} \)-module and a Noetherian (Artinian) right \( R_{jj} \)-module for \( 1 \leq i \neq j \leq p \).

We recall that a finitely generated, associative \( k \)-algebra is not necessarily Noetherian. That is, Hilbert’s basis theorem does not hold for associative rings. For a counter-example, let \( R = k\{x_1, \ldots, x_n\} \) be the free associative \( k \)-algebra on \( n \) generators. It is well-known that \( R \) is Noetherian only if \( n = 1 \). However, we know from the Hopkins-Levitzki theorem that an associative Artinian ring is Noetherian.

A \( k \)-algebra \( R \) of finite dimension as vector space over \( k \) is Artinian. This is clear, since every one-sided ideal is a vector space over \( k \) of finite dimension. We have a converse statement under the following conditions:

**Lemma 1.2.** Let \( R \) be an object of \( A_p \). If \( R \) is Artinian and \( I(R) \) is nilpotent, then \( R \) has finite dimension as a vector space over \( k \).

**Proof.** We write \( I = I(R) \). Since \( R \) is Artinian and therefore Noetherian, \( I^m \) is finitely generated as a left \( R \)-module for all \( m \). Consequently, \( I^m / I^{m+1} \) is a finitely generated \( R/I \)-module for all \( m \), and hence has finite \( k \)-dimension. But \( I^n = 0 \) for some \( n \), so \( I^m \) has finite \( k \)-dimension for all \( m \geq 0 \). In particular, \( R \) has finite dimension as a vector space over \( k \). \(\square\)

We define the category \( A_p \) to be the full sub-category of \( A_p \) consisting of objects \( R \) in \( A_p \) such that \( R \) is Artinian and \( I(R) = J(R) \). The condition \( I(R) = J(R) \) might equivalently be replaced by the condition that \( I(R) \) is a nilpotent ideal, since the Jacobson radical is the largest nilpotent ideal in an Artinian ring. So by lemma 1.2, all objects \( R \) in \( A_p \) have finite \( k \)-dimension. Since \( R \) is Artinian, the condition that \( I(R) \) is nilpotent is also equivalent to \( \cap I(R)^n = 0 \). Finally, there is a geometric interpretation of the condition \( I(R) = J(R) \): By the comment earlier in this section, \( I(R) = J(R) \) if and only if \( \{ k_1, \ldots, k_p \} \) is the full set of isomorphism classes of simple left \( R \)-modules (or equivalently, that the number of such isomorphism classes is exactly \( p \)).

**Lemma 1.3.** Let \( R \) be an associative ring. Then there exists morphisms \( f : k^p \to R \) and \( g : R \to k^p \) making \( (R, f, g) \) an object of \( A_p \) if and only if \( R \) is an Artinian \( k \)-algebra with exactly \( p \) isomorphism classes of simple left \( R \)-modules, each of them of dimension 1 over \( k \).

**Proof.** One implication follows from the comments above. For the other, assume that \( R \) is Artinian with the prescribed isomorphism classes of simple left \( R \)-modules. This defines a morphism \( g : R \to k^p \). Clearly, \( I = \ker(g) = J(R) \) by the comments
above. So it is enough to lift the idempotents \( \{e_1, \ldots, e_p\} \) of \( k^n \) to idempotents \( \{r_1, \ldots, r_p\} \) in \( R \) such that \( r_1 + \cdots + r_p = 1 \) and \( r_ir_j = 0 \) when \( i \neq j \). But \( \hat{R} \) is Artinian and therefore \( I = \text{idempotents} \) is nilpotent, so this is clearly possible.

Let \( R \) be an object in \( \mathbf{A}_p \) with radical \( I = I(R) \). Then the \( I \)-adic filtration defines a topology on \( R \) compatible with the ring operations, and we shall always consider \( R \) a topological ring in this way. We say that the topology on \( R \) is Hausdorff (or separated) if and only if \( \cap I^n = 0 \).

For all objects \( R \) in \( \mathbf{A}_p \), there is an \( I \)-adic completion \( \hat{R} \) of \( R \) and a canonical morphism \( R \to \hat{R} \) in \( \mathbf{A}_p \). The \( I \)-adic completion \( \hat{R} \) is defined by the projective limit

\[
\hat{R} = \lim_{\leftarrow} R/I^n,
\]

and the morphism \( R \to \hat{R} \) is the natural one induced by this projective limit. Notice that the kernel of this morphism is \( \cap I^n \). We say that \( R \) is complete (or separated complete) if the natural morphism \( R \to \hat{R} \) is an isomorphism in \( \mathbf{A}_p \). In particular, this implies that the morphism is injective, so \( R \) is Hausdorff (or separated). This gives a new characterization of the category \( \mathbf{a}_p \):

**Lemma 1.4.** The category \( \mathbf{a}_p \) is the full sub-category of \( \mathbf{A}_p \) consisting of objects such that \( R \) is Artinian and \( I \)-adic complete.

We define the pro-category \( \hat{\mathbf{a}}_p \) of \( \mathbf{a}_p \) to be the full sub-category of \( \mathbf{A}_p \) consisting of objects such that \( R \) is complete and \( R/I(R)^n \) belongs to \( \mathbf{a}_p \) for all \( n \geq 1 \). It is clear that we have an inclusion of (full) sub-categories \( \mathbf{a}_p \subseteq \hat{\mathbf{a}}_p \).

Let \( R \) be an object in \( \hat{\mathbf{a}}_p \) with radical \( I = I(R) \). To fix notation, we write \( \text{gr}_n(R) = R^n/I^n+1 \) for \( n \geq 0 \) (with \( I^0 = R \)). We also write \( \text{gr} R = \bigoplus \text{gr}_n(R) \), this is the graded ring associated to the \( I \)-adic filtration of \( R \). The **tangent space** of \( R \) is defined to be the \( k \)-linear space dual to \( \text{gr}_1(R) \),

\[
t_R = \text{Hom}_k(I/I^2, k) = (I/I^2)^*,
\]

which is clearly of finite dimension over \( k \). In particular, we have \( (t_R)^* \cong I/I^2 \).

Let \( u : R \to S \) be a morphism in \( \hat{\mathbf{a}}_p \). As usual, we consider \( R \) and \( S \) with the \( I \)-adic filtrations, where \( I \) is \( I(R) \) and \( I(S) \) respectively. Since \( u \) preserves these filtrations, it induces a morphism of graded rings \( \text{gr}(u) : \text{gr} R \to \text{gr} S \). This morphism is homogeneous of degree 0, so \( u \) also induces morphisms of \( k \)-vector spaces \( \text{gr}_n(u) : \text{gr}_n(R) \to \text{gr}_n(S) \) for all \( n \geq 0 \). In particular, we have a morphism of \( k \)-vector spaces \( \text{gr}_1(u) : \text{gr}_1(R) \to \text{gr}_1(S) \), and a dual morphism \( t_u : t_S \to t_R \).

**Proposition 1.5.** Let \( u : R \to S \) be a morphism in \( \hat{\mathbf{a}}_p \). Then \( u \) is a surjection if and only if \( \text{gr}_1(u) \) is a surjection. Furthermore, \( u \) is injective if \( \text{gr}(u) \) is injective.

**Proof.** If \( u \) is surjective, then clearly \( \text{gr}_1(u) \) is also surjective. To prove the other implication, let us consider the map \( \text{gr}(u) : \text{gr}(R) \to \text{gr}(S) \). Since \( \text{gr} S \) is generated by the elements in \( \text{gr}_1 S \) as an algebra, it follows that if \( \text{gr}_1(u) \) is surjective, then \( \text{gr}(u) \) is also surjective. From Bourbaki [1], chapter III, §2, no. 8, corollary 1 and 2, we have that \( u \) is surjective (injective) if \( \text{gr}(u) \) is surjective (injective), and the result follows.

Let \( n \) be any natural number. We define the **category** \( \mathbf{a}_p(n) \) to be the full sub-category of \( \mathbf{a}_p \) consisting of objects \( R \) in \( \mathbf{a}_p \) such that \( I(R)^n = 0 \). Notice that \( \mathbf{a}_p(n) \subseteq \mathbf{a}_p(n+1) \) for all \( n \geq 1 \). Furthermore, each object \( R \) in \( \mathbf{a}_p(n) \) belongs to a sub-category \( \mathbf{a}_p(n) \) for some integer \( n \).

Let \( u : R \to S \) be a morphism in \( \mathbf{a}_p \), and denote by \( K = \text{ker}(u) \) the kernel of \( u \). We say that \( u \) is a **small morphism** if we have \( I(R) \cdot K = K \cdot I(R) = 0 \). We prove the following important fact about small surjections:
Lemma 1.6. Let \( u : R \to S \) be a surjection in \( \mathfrak{a}_p \). Then \( u \) can be factored into a finite number of small surjections.

Proof. Let \( I = I(R) \), then \( I^n K = 0 \) for some \( n \geq 0 \). Consider the surjection \( u_q : R/I^n K \to R/I^{n-1} K \) for \( 1 \leq q \leq n \). Clearly \( I(R/I^n K) \ker(u_q) = 0 \) for all \( q \). Moreover, \( u_1 \circ \cdots \circ u_n = u \) when \( u_1 : R/K \to R/K \) is considered as a morphism onto \( S \cong R/K \). It is therefore enough to prove the lemma for a surjection \( u : R \to S \) with \( IK = 0 \). In this situation, \( K^l = 0 \) for some \( n \geq 0 \). Now consider the surjection \( v_q : R/KI^n \to R/KI^{n-1} \) for \( 1 \leq q \leq n \). Clearly, \( v_q \) is a small surjection for all \( q \). Moreover, \( u = v_1 \circ \cdots \circ v_n \) when \( v_1 : R/KI \to R/K \) is considered as a morphism onto \( S \cong R/K \). It follows that \( u \) can be factorized in a finite number of small surjections in \( \mathfrak{a}_p \).

We conclude this section with an important family of examples: Let \( V_{ij} \) be a finite dimensional \( k \)-vector space for \( 1 \leq i, j \leq p \), with \( \dim_k V_{ij} = d_{ij} \). Let furthermore \( \{r_{ij}(l) : 1 \leq l \leq d_{ij}\} \) be a basis of \( V_{ij} \) for \( 1 \leq i, j \leq p \) (or simply \( \{r_{ij}\} \) if \( d_{ij} = 1 \)). We define the free matrix ring \( R = R(\{V_{ij}\}) \) defined by the vector spaces \( V_{ij} \) in the following way: We say that a monomial in \( R \) of type \((i, j)\) and degree \( n \) is an expression of the form

\[
r_{i0i1}(l_1)r_{i1i2}(l_2)\cdots r_{i_{n-1}in}(l_n)
\]

with \( i_0 = i, i_n = j \). To these, we add the monomials \( e_i \) for \( 1 \leq i \leq p \), which we consider to be of type \((i, i)\) and degree 0. We define \( R \) to be the \( k \)-linear space generated by all monomials in \( R \), with the obvious multiplication: If \( M \) is a monomial of type \((i, j)\), and \( M' \) is a monomial of type \((l, m)\), then \( MM' = 0 \) if \( j \neq l \), and \( MM' \) is the monomial obtained by juxtapositioning \( M \) and \( M' \) (possibly after having erased unnecessary \( e_i \)'s) if \( j = l \). We see that \((R, f, g)\) is an object of the category \( \mathfrak{A}_p \), where \( f, g \) are the obvious maps \( k^p \to R \to k^p \). In fact, \( R_{ij} \) is the \( k \)-linear subspace generated by monomials in \( R \) of type \((i, j)\), and the ideal \( I = I(R) \) is the \( k \)-linear subspace generated by all monomials of positive degree.

We denote by \( \hat{R} = \hat{R}(\{V_{ij}\}) \) the completion of \( R = R(\{V_{ij}\}) \), and call this the formal matrix ring defined by the vector spaces \( V_{ij} \). Explicitly, every element in \( \hat{R}_{ij} \) is an infinite \( k \)-linear sum of monomials in \( R \) of type \((i, j)\). Let \( I = I(R) \), then we have that \( \hat{R}_n = R/I^n \cong \hat{R}/I(\hat{R})^n \) belongs to \( \mathfrak{a}_p \) for \( n \geq 1 \). Clearly, \( R_n \) has finite dimension as \( k \)-vector space, so \( R_n \) is Artinian, and \( I(R_n) = I/R^n \), so the radical is nilpotent. Since \( \hat{R} \) clearly is complete, it follows that \( \hat{R} \) belongs to \( \hat{\mathfrak{a}}_p \).

Notice that neither the free matrix ring \( R \) nor the formal matrix ring \( \hat{R} \) is Noetherian in general. For a counter-example, it is enough to consider the case when \( p = 1 \) and \( d_{11} = 2 \), or the case when \( p = 2 \) and \( d_{11} = d_{12} = d_{21} = 1, d_{22} = 0 \). In the first case, \( R \cong k \{x, y\} \), which we know is not Noetherian. In the second case, we have that \( \hat{R}_{11} = k \{r_{11}, r_{12}r_{21}\} \cong k \{x, y\} \), which again is not Noetherian. So by proposition \( \boxed{14} \) \( R \) is not Noetherian in this case either. A similar argument shows that \( \hat{R} \) is not Noetherian in any of the two cases.

2. Noncommutative deformations of modules

We recall that \( k \) is an algebraically closed (commutative) field, \( A \) is an associative \( k \)-algebra, and \( \mathbf{M} = \{M_1, \ldots, M_p\} \) is a finite family of left \( A \)-modules. In this section, we shall define the noncommutative deformation functor

\[
\text{Def}_\mathbf{M} : \mathfrak{a}_p \to \text{Sets}
\]

describing the simultaneous formal deformations of the family \( \mathbf{M} \).

Let \( R \) be an object of \( \mathfrak{a}_p \). A lifting of the family \( \mathbf{M} \) of left \( A \)-modules to \( R \) is a left \( A \otimes_k R^{\text{op}} \)-module \( M_R \), together with isomorphisms \( \eta_i : M_R \otimes_R k_i \to M_i \) of
left $A$-modules for $1 \leq i \leq p$, such that $M_R \cong (M_i \otimes_k R_{i})$ as right $R$-modules. We remark that a left $A \otimes_k R^\text{op}$-module is the same as an $A-R$ bimodule such that the left and right $k$-vector space structures coincide. Furthermore, the notation $(M_i \otimes_k R_{i})$ refers to the $k$-vector space 

$$(M_i \otimes_k R_{i}) = \bigoplus_{i,j} (M_i \otimes_k R_{ij})$$

with the natural right $R$-module structure coming from the multiplication in $R$. The condition that $M_R \cong (M_i \otimes_k R_{i})$ as right $R$-modules generalizes the flatness condition in commutative deformation theory.

Let $M'_R, M''_R$ be two liftings of $M$ to $R$. We say that these two liftings are equivalent if there exists an isomorphism $\tau : M'_R \rightarrow M''_R$ of left $A \otimes_k R^\text{op}$-modules such that the natural diagrams commute (that is, such that $\eta''_i \circ (\tau \otimes k_i) = \eta'_i$ for $1 \leq i \leq p$). We let $\text{Def}_M(R)$ denote the set of equivalence classes of liftings of $M$ to $R$, and we refer to these equivalence classes as deformations of $M$ to $R$. We shall often denote a deformation represented by $(M_R, \eta_i)$ by $M_R$ to simplify notation.

Let $u : R \rightarrow S$ be a morphism in $\mathfrak{a}_p$, and let $M_R$ be a lifting of $M$ to $R$, representing an element in $\text{Def}_M(R)$. We define $M_S = M_R \otimes_R S$, which has a natural structure as a left $A \otimes_k S^\text{op}$-module. Since $u$ is a morphism in $\mathfrak{a}_p$, we have natural isomorphisms of left $A$-modules 

$$(M_R \otimes_R S) \otimes_S k_i \cong M_R \otimes_R k_i,$$

inducing isomorphisms of left $A$-modules $\rho_i : M_S \otimes_S k_i \rightarrow M_i$ via $\eta_i$ for $1 \leq i \leq p$. A straight-forward calculation shows that $M_S$ together with the isomorphisms $\rho_i$ for $1 \leq i \leq p$ constitutes a lifting of $M$ to $S$, and furthermore that the equivalence class of this lifting is independent upon the representative of the equivalence class of $M_R$. Hence, we obtain a map $\text{Def}_M(u) : \text{Def}_M(R) \rightarrow \text{Def}_M(S)$, and we see that $\text{Def}_M : \mathfrak{a}_p \rightarrow \text{Sets}$ is a covariant functor.

Let $R = (R_{i})$ be an object in $\mathfrak{a}_p$. We shall describe how one, in principle, could attempt to calculate $\text{Def}_M(R)$ explicitly: We may assume that every element of $\text{Def}_M(R)$ is represented by a lifting $M_R$, such that $M_R = (M_i \otimes_k R_{i})$ considered as a right $R$-module. In order to describe this lifting completely, it is enough to describe the left action of $A$ on $M_R$. Furthermore, it is enough to describe this action on elements of the form $m_i \otimes e_i$ with $m_i \in M_i$, since we have 

$$a(m_i \otimes r_{ij}) = (a(m_i \otimes e_i))r_{ij}$$

for all $a \in A$, $m_i \in M_i$, $r_{ij} \in R_{ij}$. For a fixed $a \in A$, $m_i \in M_i$, assume that 

$$a(m_i \otimes e_i) = \sum (m'_j \otimes r'_{ji})$$

with $m'_j \in M_j$, $r'_{ji} \in R_{ji}$. Then multiplication by $e_i$ on the right gives the equality 

$$a(m_i \otimes e_i) = \sum_j (m'_{ij} \otimes r'_{ji}),$$

and the isomorphism $\eta_i$ gives a further restriction on the left action of $A$, expressed by the formula 

$$(1) \quad a(m_i \otimes e_i) = (am_i) \otimes e_i + \sum_j m'_{ij} \otimes r'_{ji},$$

where $a \in A$, $m_i \in M_i$, $m'_{ij} \in M_j$, $r'_{ji} \in I(R)_{ji}$. Consequently, the set $\text{Def}_M(R)$ consists of all possible choices of left $A$-actions on elements of the form $m_i \otimes e_i$, fulfilling condition (1) and the associativity condition, up to equivalence.

Let $R$ be any object in $\mathfrak{a}_p$. Then the formula $a(m_i \otimes e_i) = (am_i) \otimes e_i$ for $a \in A$, $m_i \in M_i$ defines a left $A$-module structure on $(M_i \otimes R_{ij})$ compatible with the right $R$-module structure. Hence, there exists a trivial lifting $M_R$ to $R$ for all objects $R$ in $\mathfrak{a}_p$, and $\text{Def}_M(R)$ is non-empty. Notice that in the case $R = k^p$, we
have $I = I(R) = 0$, so this trivial lifting is the only one possible. Consequently, we have $\text{Def}_M(k^p) = \{\ast\}$, where $\ast$ denotes the equivalence class of the trivial lifting.

Let $\nu : R \to S$ be a morphism in $\mathbf{a}_p$, and let $M_S \in \text{Def}_M(S)$ be a given deformation. We say that a deformation $M_R \in \text{Def}_M(R)$ is a lifting of $M_S$ if and only if $M_R \cong M_S$ when $\nu(R) = S$. Given any object $R$ in $\mathbf{a}_p$ and a deformation $M_R \in \text{Def}_M(R)$, we see that $M_R$ is a lifting of the trivial deformation $\ast$ in $\text{Def}_M(k^p)$ in the above sense via the structural morphism $g : R \to k^p$. Hence, our notation is consistent.

For another example, consider the test algebras $R(\alpha, \beta)$ for $1 \leq \alpha, \beta \leq p$, constructed in the following way: Let $R$ be the free matrix algebra defined by the $k$-vector spaces $V_{ij}$ with dimensions $d_{ij} = 1$ and $d_{ij} = 0$ when $(i, j) \neq (\alpha, \beta)$. We define $R(\alpha, \beta) = R/I(R)^2$, which is an object in $\mathbf{a}_p(2)$ by construction. We know that any lifting of $M$ to $R(\alpha, \beta)$ is defined by a left $A$-action

$$a(m_{ij} \otimes e_{ij}) = (am_{ij}) \otimes e_{ij} + \psi(a)(m_{ij} \otimes e_{ij})$$

for all $a \in A$, $m_{ij} \in M_{ij}$, where $\psi : A \times M_{ij} \to M_{ij}$ is a $k$-bilinear map and $\varepsilon_{ij}$ is the class of $e_{ij}$. Clearly, we must have $\varepsilon_{ij} = (am_{ij}) \otimes e_{ij}$ for all $a \in A$, $m_{ij} \in M_{ij}$, when $i \neq j$. Moreover, $\psi$ defines an associative $A$-module structure if and only if $\psi \in \text{Der}_k(A, \text{Hom}_k(M_{ij}, M_{ij}))$. In this case, we shall denote the corresponding lifting by $M(\psi) \in \text{Def}_M(R(\alpha, \beta))$. Given two derivations $\psi, \psi'$, we see that $M(\psi)$ and $M(\psi')$ are equivalent liftings if and only if there is a $\phi \in \text{Hom}_k(M_{ij}, M_{ij})$ such that

$$\psi - \psi'(a)(m_{ij}) = a\phi(m_{ij}) - \phi(a)m_{ij}$$

for all $a \in A$, $m_{ij} \in M_{ij}$.

**Lemma 2.1.** There is a bijective correspondence $\text{Def}_M(R(\alpha, \beta)) \cong \text{Ext}_A^1(M_{ij}, M_{ij})$ for $1 \leq \alpha, \beta \leq p$.

**Proof.** From the definition of Hochschild cohomology (see appendix A), we see that $\psi : A \to M_{ij}$ induces a bijective correspondence between $\text{HH}^1(A, \text{Hom}_k(M_{ij}, M_{ij}))$ and $\text{Def}_M(R(\alpha, \beta))$. Moreover, $\text{HH}^1(A, \text{Hom}_k(M_{ij}, M_{ij})) \cong \text{Ext}_A^1(M_{ij}, M_{ij})$ by proposition A.4. \qed

3. M-free resolutions and noncommutative deformations

We recall that $k$ is an algebraically closed (commutative) field, $A$ is an associative $k$-algebra, and $M = \{M_1, \ldots, M_p\}$ is a finite family of left $A$-modules. In this section, we shall define M-free resolutions and relate them to noncommutative deformations of modules. In particular, we shall show that M-free resolutions are useful computational tools in order to study the deformation functor $\text{Def}_M$.

Let $R$ be any object of $\mathbf{a}_p$. An *M-free module over* $R$ is a left $A \otimes_k R^{op}$-module $F$ of the form

$$F = (L_i \otimes_k R_{ij}),$$

where $L_1, \ldots, L_p$ are free left $A$-modules, and the left $A$-module structure on $F$ is the trivial one. In other words, $F$ is the trivial lifting of a family $\{L_1, \ldots, L_p\}$ of free left $A$-modules to $R$.

Although an M-free module over $R$ is not free considered as a left $A \otimes_k R^{op}$-module, it behaves as a free module when interpreted as a module of matrices in the correct way:

**Lemma 3.1.** Let $u : R \to S$ be a surjection in $\mathbf{a}_p$, and consider a left $A \otimes_k R^{op}$-module $M_R = (M_i \otimes_k R_{ij})$ and a left $A \otimes_k S^{op}$-module $M_S = (M_i \otimes_k S_{ij})$ such that the natural map $v : M_R \to M_S$ induced by $u$ is left $A$-linear. If $F^S$ is any M-free module over $S$ given by the free left $A$-modules $L_1, \ldots, L_p$ and $f_S : F^S \to M_S$ is any
Lemma 3.2. Let $M \leq v$ where $F$ commutative, where $M$-free resolution $(F, d)$ of $M$-free resolution of $M$. An $M$-free resolution of $M$ is an exact sequence of left $A \otimes_k R^{op}$-linear maps

$$0 \leftarrow M_R \leftarrow F^R \leftarrow F^1 \leftarrow \cdots \leftarrow F^m \leftarrow \cdots$$

where $F^m$ is an M-free module over $R$ for $m \geq 0$. So we have $F^m = (L_{m,i} \otimes_k R_{ij})$ where $L_{m,i}$ are free left $A$-modules for $1 \leq i \leq p$, $m \geq 0$. We shall denote the differentials by $u^m_i : F^R_{m+1} \rightarrow F^R_m$ for $m \geq 0$.

We fix a $k$-linear basis $\{ r_{ij}(l) : 1 \leq l \leq \dim_k R_{ij} \}$ of $R_{ij}$ for $1 \leq i, j \leq p$ such that $e_i$ is contained in the basis of $R_{ii}$ for $1 \leq i \leq p$. Consider the differential $d^R_m$ in the M-free resolution of $M$ above. Clearly, we can write $d^R_m$ uniquely in the form

$$d^R_m = \sum_{i, j, l} \alpha(r_{ij}(l))_m \otimes r_{ij}(l)$$

for all $m \geq 0$, where $\alpha(r_{ij}(l))_m : L_{m+1, i, j} \rightarrow L_{m, i}$ is a homomorphism of left $A$-modules for $1 \leq i, j \leq p$, $1 \leq l \leq \dim_k R_{ij}$. In particular, the M-free resolution of $M_R$ defines a family of 1-cochains $\alpha(r_{ij}(l)) \in \text{Hom}^1(L_{s, j}, L_{s, i})$, indexed by a $k$-linear basis for $R$.

From now on, we fix a free resolution $(L_{s, i}, d_{s, i})$ of $M_i$ considered as left $A$-module for $1 \leq i \leq p$. These free resolutions correspond to an M-free resolution $(F, d)$ of the trivial deformation $(M, k)$ considered as left $A$-module for $1 \leq i \leq p$. In fact, the M-free resolution $(F, d)$ is given by $F_m = (L_{m, i} \otimes_k (k^p)_{ij})$ and $d_m = \sum d_{m, i} \otimes e_i$ for $m \geq 0$. We have therefore fixed an M-free resolution $(F, d)$ of the trivial lifting of $M$ to $k^p$.

Let $R$ be any object of $a_p$. We say that a complex $(F^R_\ast, d^R_\ast)$ of M-free modules $F^R_m = (L_{m, i} \otimes_k R_{ij})$ over $R$ is a lifting of the complex $(F, d)$ if the following diagram commutes

where $u_m : F^R_m \rightarrow F^R_m$ are the natural maps induced by $R \rightarrow k^p$.

Lemma 3.2. Let $R$ be any object of $a_p$, and let $(F^R_\ast, d^R_\ast)$ be a lifting of the complex $(F_\ast, d_\ast)$. Then we have:

1. $H^m(F^R_\ast, d^R_\ast) = 0$ for all $m \geq 0$,
2. $H^0(F^R_\ast, d^R_\ast)$ is a lifting of the family $M$ to $R$. 
Proof. Clearly, the lemma holds for $R = k^p$. We shall consider a small surjection $u : R \to S$ in $a_p$ and liftings of complexes $(F^U_*, d^U_*)$ of $(F_*, d_*)$ to $U$ for $U = R, S$ such that the following diagram commutes:

$$
\begin{array}{cccccccc}
F^R_0 & d^R_0 & F^R_1 & d^R_1 & F^R_2 & \cdots \\
v_0 & v_1 & v_2 & \\
F^S_0 & d^S_0 & F^S_1 & d^S_1 & F^S_2 & \cdots \\
\end{array}
$$

In this situation, we shall prove that if the conclusion of the lemma holds for $S$, it holds for $R$ as well. This is clearly enough to prove the lemma.

Let $K = \ker(u)$, then we clearly have $\ker(v_m) = (F^R_{m,i} \otimes_k K_{ij})$ with the trivial left $A$-action for all $m \geq 0$. We denote this kernel by $F^R_m$, then $(F^K_*, d^K_*)$ is a complex of left $A \otimes_k R^{op}$-modules, where $d^K_*$ is the restriction of $d^R_0$. Moreover, it is clear that $v_m$ is surjective for $m \geq 0$. Define $M_U = H^0(F^U_*, d^U_*)$ for $U = R, S$, let $v : M_R \to M_S$ be the induced map, and denote the kernel by $M_K = \ker(v)$. Then clearly $v$ is surjective, and we have the following commutative diagram of complexes:

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 \\
& M_K & F^K_0 & F^K_1 & F^K_2 & \cdots \\
& i & i_0 & i_1 & i_2 & \\
& 0 & M_R & F^R_0 & F^R_1 & F^R_2 & \cdots \\
& v & v_0 & v_1 & v_2 & \\
& 0 & M_S & F^S_0 & F^S_1 & F^S_2 & \cdots \\
\end{array}
$$

Clearly all columns are exact, so the diagram gives a short exact sequence of complexes. By assumption, the bottom row is exact and $M_S = (M_i \otimes_k S_{ij})$ is a lifting of $M$ to $S$. Let us first show that $H^m(F^R_*, d^R_*) = 0$ for $m \geq 1$: This follows since the complex is a lifting of $(F_*, d_*)$ and because $I(R)K = 0$ (since $u : R \to S$ is small). The long exact sequence of cohomologies of the complexes above now implies that $H^m(F^R_*, d^R_*) = 0$ for all $m \geq 1$ and that we have a short exact sequence

$$
0 \to H^0(F^K_*, d^K_*) \to M_R \to M_S = (M_i \otimes_k S_{ij}) \to 0,
$$

of left $A$-modules, so in particular $M_K \cong H^0(F^K_*, d^K_*)$. But since $I(R)K = 0$, it follows that $H^0(F^K_*, d^K_*) \cong (H^0(L_{s,i}, d_{s,i}) \otimes_k K_{ij}) = (M_i \otimes_k K_{ij})$ with the trivial left $A$-module structure. It follows that $M_R \cong (M_i \otimes_k R_{ij})$ considered as a $k$-vector space, and therefore $M_R$ is a lifting of $M$ to $R$.

Lemma 3.3. Let $R$ be any object of $a_p$, and let $M_R$ be a lifting of $M$ to $R$. Then there exists an $M$-free resolution of $M_R$ which lifts the complex $(F_*, d_*)$ to $R$.

Proof. Clearly, the lemma holds for $R = k^p$. We shall consider a small surjection $u : R \to S$ in $a_p$, deformations $M_U \in \text{Def}_M(U)$ for $U = R, S$ such that $M_R$ lifts $M_S$ to $R$, and an $M$-free resolution $(F^S_*, d^S_*)$ of $M_S$ which lifts the complex $(F_*, d_*)$ to $S$. In this situation, we shall prove that there exists an $M$-free resolution $(F^R_*, d^R_*)$
of $M_R$ compatible with the M-free resolution of $M_S$. This is clearly enough to prove the lemma.

Let $F^R_m = (L_{m,i} \otimes_k R_{ij})$ for all $m \geq 0$. Moreover, we write $F^K_m = (L_{m,i} \otimes_k K_{ij})$ for all $m \geq 0$, where $K = \ker(u)$. To complete the proof, we have to find the differentials $d^K_m$ for $m \geq 0$ and the augmentation map $\rho_R$: By lemma 3.1, we can find a homomorphism $\rho_R: F^R_0 \to M_R$ lifting $\rho_S$. Denote by $\rho_K: F^K_0 \to M_K$ its restriction, where $M_K = \ker(M_R \to M_S)$. Since $u$ is small, $\rho^K$ is surjective, and this implies that the induced map $\ker(\rho^K) \to \ker(\rho^S)$ is surjective. By lemma 3.1, we can find a homomorphism $d^K_0: F^K_1 \to F^K_0$ lifting $d^K_0$ such that $d^K_0 = 0$. Let $F^K_0$ be the restriction of $d^K_0$, then clearly $\ker(\rho^K) = \text{Im}(d^K_0)$ since $u$ is small. An easy induction argument shows that we can construct a complex $(F^K_n, d^K_n)$ in such a way that the restriction $(F^K_n, d^K_n)$ is a resolution of $M_K$. By the proof of lemma 3.2 it follows that $H^m(F^R_n, d^R_1) = 0$ for $m \geq 1$ and that there is an exact sequence

$$0 \to M_R \to H^0(F^R_1, d^R_1) \to M_S \to 0.$$ 

This implies that $M_R = H^0(F^R_1, d^R_1)$, and $(F^R_n, d^R_n)$ is the required M-free resolution of $M_R$ compatible with the given M-free resolution of $M_S$. \hfill $\square$

**Proposition 3.4.** Let $u: R \to S$ be a surjection in $a_p$ and consider a deformation $M_S \in \text{Def}_M(S)$ and any M-free resolution $(F^S_n, d^S_n)$ of $M_S$ which lifts the complex $(F^S_n, d^S_n)$ to $S$. There is a bijective correspondence between the set of liftings

$$\{ M_R \in \text{Def}_M(R) : \text{Def}_M(u)(M_R) = M_S \}$$

and the set of M-free complexes $(F^R_n, d^R_n)$ which lift the resolution $(F^S_n, d^S_n)$ to $R$, up to equivalence.

**Proof.** For a small surjection, this follows from lemma 3.2 and lemma 3.3. But any surjection in $a_p$ is a composition of small surjections. \hfill $\square$

Let $R$ be any object in $a_p$. In section 2 we described how to, in principle, calculate $\text{Def}_M(R)$ by considering the possible left $A$-module structures on the right $R$-module $(M_i \otimes_k R_{ij})$. The M-free resolutions give us another way of viewing deformations in $\text{Def}_M(R)$: By proposition 3.4, we can view $\text{Def}_M(R)$ as the set of liftings of the complex $(F^S_n, d^S_n)$ to $R$, up to equivalence. Using equation 2, each lifting of complexes corresponds to a family of 1-cochains $\alpha(r_{ij}(l)) \in \text{Hom}^1(L_{e_j}, L_{e_i})$, parametrized by a $k$-basis for $R$. We leave it as an exercise for the reader to use this approach to calculate $\text{Def}_M(R)$ in the case $R = R_{\alpha, \beta}$ — this will give a new proof of lemma 2.1 via the Yoneda representation of $\text{Ext}^1_A(M_\beta, M_\alpha)$.

## 4. Pro-representing hulls of pointed functors

We say that a covariant functor $F: a_p \to \text{Sets}$ is pointed if $F(k^p) = \{ * \}$. In this section, we shall consider pointed functors defined on the category $a_p$, and study their representability. Of course, the motivation for this is the fact that $\text{Def}_M$ is such a pointed functor.

Let $R$ be any object of $\hat{a}_p$, and consider the functor $h_R: a_p \to \text{Sets}$ given by $h_R(S) = \text{Mor}(R, S)$ for all objects $S$ in $a_p$. The notation $\text{Mor}(R, S)$ denotes the set of morphisms from $R$ to $S$ in the pro-category $\hat{a}_p$. Then $h_R$ is clearly a pointed functor defined on $a_p$.

We say that a pointed functor $F: a_p \to \text{Sets}$ is representable if $F$ is isomorphic to $h_R$ for some object $R$ in $a_p$, and pro-representable if $F$ is isomorphic to $h_R$ for some object $R$ in $\hat{a}_p$. However, it is well-known that deformation functors seldom are representable or even pro-representable. So a weaker notion is required, and we
shall define the notion of a pro-representing hull of a pointed functor on \( \mathfrak{a}_p \). We start by introducing some notation:

Any pointed functor \( F : \mathfrak{a}_p \to \text{Sets} \) has an extension to a functor \( \tilde{F} : \tilde{\mathfrak{a}}_p \to \text{Sets} \) defined on the pro-category \( \tilde{\mathfrak{a}}_p \). This extension is defined by the formula

\[
\tilde{F}(R) = \lim_{\leftarrow} F(R/I^n)
\]

for any object \( R \) in \( \tilde{\mathfrak{a}}_p \) with \( I = I(R) \). Clearly, any pointed functor \( F : \mathfrak{a}_p \to \text{Sets} \) also has a restriction to the sub-category \( \mathfrak{a}_p(n) \subseteq \mathfrak{a}_p \) for all \( n \geq 1 \). We shall denote this restriction by \( F_n : \mathfrak{a}_p(n) \to \text{Sets} \).

**Lemma 4.1.** Let \( R \) be an object in \( \tilde{\mathfrak{a}}_p \), and let \( F : \mathfrak{a}_p \to \text{Sets} \) be a pointed functor. Then there is a natural isomorphism of sets \( \alpha \colon \tilde{F}(R) \to \text{Mor}(h_R, F) \).

**Proof.** Let \( \xi \in \tilde{F}(R) \), then \( \xi = (\xi_n) \) with \( \xi_n \in F(R/I^n) \) for all \( n \geq 1 \). For any object \( S \) in \( \mathfrak{a}_p \), we construct a map of sets \( \alpha(\xi)_S : \text{Mor}(R, S) \to F(S) \): Let \( u : R \to S \) be a morphism in \( \tilde{\mathfrak{a}}_p \), then \( u(I(R)) \subseteq I(S) \), and \( I(S) \) is nilpotent since \( S \) is in \( \mathfrak{a}_p \), so there exists \( n \geq 1 \) such that \( u \) factorizes through \( u_n : R/I(R)^n \to S \). We define \( \alpha(\xi)_S(u) = F(u_n)(\xi_n) \), and a straight-forward calculation shows that this expression is independent upon the choice of \( n \), and gives rise to a natural transformation of functors. Conversely, let \( \phi : h_R \to F \) be a natural transformation of functors on \( \mathfrak{a}_p \). Then we define \( \xi_n \in F(R/I(R)^n) \) to be \( \xi_n = \phi_{R/I(R)^n}(R \to R/I(R)^n) \), where \( R \to R/I(R)^n \) is the natural morphism. Again, a straight-forward calculation shows that \( \xi = (\xi_n) \) defines an element in \( \tilde{F}(R) \), and that this map of sets defines an inverse to \( \alpha \).

There is also a version of lemma 4.1 for the category \( \mathfrak{a}_p(n) \): For an object \( R \) in \( \mathfrak{a}_p(n) \), and a pointed functor \( F : \mathfrak{a}_p(n) \to \text{Sets} \), there is a natural isomorphism of sets \( \alpha_n : F(R) \to \text{Mor}(h_R, F) \). The construction of this isomorphism is similar to the construction in lemma 4.1.

We recall that a morphism \( \phi : F \to G \) of pointed functors \( F, G : \mathfrak{a}_p \to \text{Sets} \) is smooth if the following condition holds: For all surjective morphisms \( u : R \to S \) in \( \mathfrak{a}_p \), the natural map of sets

\[
F(R) \to F(S) \times_{G(S)} G(R),
\]

given by \( x \mapsto (F(u)(x), \phi_R(x)) \) for all \( x \in F(R) \), is a surjection. Clearly, it is enough to check this for small surjections in \( \mathfrak{a}_p \). Also notice that any morphism \( \phi : F \to G \) of functors naturally extends to a morphism \( \tilde{\phi} : \tilde{F} \to \tilde{G} \) of functors on \( \tilde{\mathfrak{a}}_p \), and if \( \phi \) is a smooth morphism, then \( \tilde{\phi}_R : \tilde{F}(R) \to \tilde{G}(R) \) is surjective for all objects \( R \) in \( \tilde{\mathfrak{a}}_p \).

Similarly, we say that a morphism \( \phi : F \to G \) of functors \( F, G : \mathfrak{a}_p(n) \to \text{Sets} \) on \( \mathfrak{a}_p(n) \) is smooth if the map of sets is surjective for all surjective morphisms \( u : R \to S \) in \( \mathfrak{a}_p(n) \). Clearly, a morphism \( \phi : F \to G \) of functors on \( \mathfrak{a}_p \) is smooth if and only if the restriction \( \phi_n : F_n \to G_n \) is smooth for all \( n \geq 1 \).

Let \( F \) be a pointed functor on \( \mathfrak{a}_p \). A **pro-couple** for \( F \) is a pair \((R, \xi)\), where \( R \) is an object in \( \tilde{\mathfrak{a}}_p \) and \( \xi \in \tilde{F}(R) \). A morphism \( u : (R, \xi) \to (R', \xi') \) of pro-couples is a morphism \( u : R \to R' \) in \( \tilde{\mathfrak{a}}_p \) such that \( \tilde{F}(u)(\xi) = \xi' \). If \((R, \xi)\) is a pro-couple for \( F \) such that \( R \) is also an object of \( \mathfrak{a}_p \), then it is called a **couple** for \( F \).

We say that a pro-couple \((R, \xi)\) **pro-represents** \( F \) if \( \alpha(\xi)_R : h_R \to F \) is an isomorphism of functors on \( \mathfrak{a}_p \). If \((R, \xi)\) pro-represents \( F \) and \((R, \xi)\) is also a couple for \( F \), then we say that \((R, \xi)\) **represents** \( F \). It is clear that if the couple \((R, \xi)\) represents \( F \), then \((R, \xi)\) is unique up to a unique isomorphism of couples.

Similarly, let \( F \) be a pointed functor on \( \mathfrak{a}_p(n) \). A couple for \( F \) is a pair \((R, \xi)\), where \( R \) is an object of \( \mathfrak{a}_p(n) \) and \( \xi \in F(R) \). We say that the couple \((R, \xi)\) represents \( F \) if and only if \( \alpha_n(\xi) \) is an isomorphism of functors defined on \( \mathfrak{a}_p(n) \). It is clear
that if this is the case, the couple \((R, \xi)\) is unique up to a unique isomorphism of couples.

Let \(F\) be a functor on \(a_p\), and let \((R, \xi)\) be a pro-couple for \(F\). For all \(n \geq 1\), let \((R_n, \xi_n)\) be given by \(R_n = R/I(R)^n\) and \(\xi_n = F(u_n)(\xi)\), where \(u_n : R \to R_n\) is the natural surjection. Then \((R_n, \xi_n)\) is a couple for the restriction \(F_n : a_p(n) \to \text{Sets}\) of \(F\) for all \(n \geq 1\). Notice that \(\alpha_n(\xi)\) is the restriction of the morphism \(\alpha(\xi)\) to \(a_p(n)\) for all \(n \geq 1\). Consequently, \((R, \xi)\) pro-represents \(F\) if and only if \((R_n, \xi_n)\) represents \(F_n\) for all \(n \geq 1\). In particular, it follows that if \((R, \xi)\) pro-represents \(F\), then \((R, \xi)\) is unique up to a unique isomorphism of pro-couples.

Let \(F : a_p \to \text{Sets}\) be a pointed functor on \(a_p\). A pro-representing hull of \(F\) is a pro-couple \((R, \xi)\) of \(F\) such that the following conditions hold:

1. \(\alpha(\xi) : h_R \to F\) is a smooth morphism of functors on \(a_p\).
2. \(\alpha_2(\xi_2) : h_{R_2} \to F_2\) is an isomorphism of functors on \(a_p(2)\).

To simplify notation, we sometimes call the pro-representing hull \((R, \xi)\) a hull of \(F\).

**Proposition 4.2.** Let \(F : a_p \to \text{Sets}\) be a pointed functor on \(a_p\), and assume that \((R, \xi), (R', \xi')\) are pro-representing hulls of \(F\). Then there exists an isomorphism of pro-couples \(u : (R, \xi) \to (R', \xi')\).

**Proof.** Let \(\phi = \alpha(\xi), \phi' = \alpha(\xi')\). Since \(\phi, \phi'\) are smooth morphisms, we have that \(\phi_{R_t} = \phi'_{R_t}\) are surjective. So we can find morphisms \(u : (R, \xi) \to (R', \xi')\) and \(v : (R', \xi') \to (R, \xi)\) of pro-couples of \(F\). The restriction to \(a_p(2)\) gives us morphisms \(u_2 : (R_2, \xi_2) \to (R'_2, \xi'_2)\) and \(v_2 : (R'_2, \xi'_2) \to (R_2, \xi_2)\). But both \((R_2, \xi_2)\) and \((R'_2, \xi'_2)\) represent \(F_2\), so \(u_2\) and \(v_2\) are inverses. In particular, \(gr_1(u_2)\) and \(gr_1(v_2)\) are inverses, and \((v \circ u_2) = v_2 \circ u_2 = id\). From the proof of proposition 1.5, we see that \(gr(v \circ u)\) is surjective. This means that \(gr_n(v \circ u)\) is a surjective endomorphism of a finite dimensional \(k\)-vector space for all \(n \geq 1\), so \(gr(v \circ u)\) is an isomorphism. By proposition 1.5, \(v \circ u\) is an isomorphism as well, and the same holds for \(u \circ v\) by a symmetric argument. It follows that \(u\) and \(v\) are isomorphisms. \(\square\)

So if there exists a pro-representing hull of a pointed functor \(F\), we know that it is unique, and we shall denote it by \((H, \xi)\). Notice that \((H, \xi)\) is only unique up to *non-canonical* isomorphism. By abuse of language, we shall sometimes omit \(\xi\) from the notation, and say that \(H\) is the hull of \(F\).

**5. Hulls of Noncommutative Deformation Functors**

We recall that \(k\) is an algebraically closed (commutative) field, \(A\) is an associative \(k\)-algebra, and \(M = \{M_1, \ldots, M_p\}\) is a finite family of left \(A\)-modules. In this section, we prove that if the family \(M\) satisfy the finiteness condition \([FC]\), then there exists a hull \(H = H(M)\) of the noncommutative deformation functor \(\text{Def}_M\). The proof follows Laudal \([L]\), and the essential point is the following obstruction calculus:

**Proposition 5.1.** Let \(u : R \to S\) be a small surjective morphism in \(a_p\), with kernel \(K = \ker(u)\), and let \(M_S \in \text{Def}_M(S)\) be a deformation. Then there exists a canonical obstruction

\[
o(u, M_S) \in (\text{Ext}_A^2(M_j, M_i) \otimes_k K_{ij}),
\]

such that \(o(u, M_S) = 0\) if and only if there exists a deformation \(M_R \in \text{Def}_M(R)\) lifting \(M_S\). If this is the case, the set of deformations in \(\text{Def}_M(R)\) lifting \(M_S\) is a torsor under the \(k\)-vector space \((\text{Ext}_A^1(M_j, M_i) \otimes_k K_{ij})\).

**Proof.** We recall from section 2 that up to equivalence, we may assume that \(M_S\) has the following form: \(M_S = (M_j \otimes_k S_{ij})\) with right \(S\)-module structure given by the multiplication in \(S\), and with left \(A\)-module structure given by \(k\)-linear
Therefore, the set of deformations $M$ for all $n$ spaces $HH$ is a left $A$-module structure on $M_S$. We let $Q^\prime = (\text{Hom}_k(M_j, M_i \otimes_k R_{ij}))$ and remark that this is an associative $k$-algebra in a natural way: We compose the $k$-linear morphisms in $Q^\prime$ by using the multiplication in $R$.

For $a, b \in A$, consider the expression $L(ab) - L(a)L(b) \in Q^\prime$. By the associativity of the left $A$-module structure on $M_S$, we see that $L(ab) - L(a)L(b) \in Q$, where $Q = (\text{Hom}_k(M_j, M_i \otimes_k K_{ij})) \subseteq Q^\prime$. Furthermore, we notice that $Q \subseteq Q^\prime$ is an ideal, and $Q$ has a natural structure as an $A$-$A$ bimodule via $L$, since $K^2 = 0$. We define $\psi \in \text{Hom}_k(A \otimes_k A, Q)$ to be given by $\psi(a, b) = L(ab) - L(a)L(b)$ for all $a, b \in A$. A straight-forward calculation shows that $\psi$ is a 2-cocycle in $HC^2(A, Q)$, so $\psi$ gives rise to an element $o(u, M_S) \in HH^2(A, Q)$ — see appendix A for the definition of the Hochschild complex and its cohomology. Since $K^2 = 0$, it follows that if $L^\prime$ is another $k$-linear lifting of the left $A$-module structure on $M_S$, then the $A$-$A$ bimodule structures of $Q$ given by $L$ and $L^\prime$ coincide. Therefore, $HH^2(A, Q)$ is independent upon the choice of $L$, and a straight-forward calculation shows that the same holds for the obstruction $o(u, M_S)$.

We remark that there exists a deformation $M_R \in \text{Def}_M(R)$ lifting $M_S$ if and only if there exists some $k$-linear lifting $L^\prime: A \to Q^\prime$ of the left $A$-module structure of $M_S$ such that $L(ab) = L(ab)L^\prime(b)$ for all $a, b \in A$. Let $\tau = L^\prime - L$, then $\tau: A \to Q$ is a $k$-linear map, and a straight-forward calculation shows that $L(ab) = L(ab)L^\prime(b)$ if and only if the relation

$$L(ab) - L(a)L(b) = L(\tau(b)b) - \tau(ab) + \tau(a)L(b) + \tau(a)\tau(b)$$

holds. Since $K^2 = 0$, the last term vanishes. The fact that the above relation holds for all $a, b \in A$ is therefore equivalent to the fact that $o(u, M_S) = 0$ in $HH^2(A, Q)$. So we have established that there exists a canonical obstruction $o(u, M_S)$ in $HH^2(A, Q)$ such that $o(u, M_S) = 0$ if and only if there is a lifting of $M_S$ to $R$.

Assume that $L: A \to Q^\prime$ is such that $L(ab) = L(a)L(b)$ for all $a, b \in A$, that is, such that it defines a deformation $M_R$ lying over $M_S$. For any other $k$-linear lifting $L^\prime: A \to Q^\prime$ of the left $A$-module structure on $M_S$, we may consider the difference $\tau = L^\prime - L: A \to Q$. A straight-forward calculation shows that $\tau$ is a 1-cocycle in $HC^1(A, Q)$ if and only if $L(ab) = L(ab)L^\prime(b)$ for all $a, b \in A$, that is, if and only if $L^\prime$ defines a left $A$-module structure on $M_R$. Furthermore, we have that $L$ and $L^\prime$ give rise to equivalent deformations if and only if $\tau$ is a 1-coboundary: It is clear that any equivalence between the left $A$-module structures of $M_R = (M_i \otimes_k R_{ij})$ given by $L$ and $L^\prime$ has the form $id + \psi$, where $\psi \in Q$. Furthermore, the map $id + \psi: M_R \to M_R$ (with the left $A$-module structure from $L^\prime$ and $L$ respectively) is a left $A$-module homomorphism if and only if $L(\tau)id + \psi) = (id + \psi)L^\prime(\tau)$ holds for all $a \in A$, and this last condition is equivalent with the fact that $\tau = d(\psi)$, so that $\tau$ is a 1-coboundary. If $\tau$ is a 1-boundary in $HC^1(A, Q)$, it is also clear that $id + \psi$ defines an equivalence between the two deformations given by $L$ and $L^\prime$. Therefore, the set of deformations $M_R$ lying over $M_S$ is a torsor under the $k$-vector space $HH^1(A, Q)$.

To end the proof, we have to show that there are isomorphisms of $k$-vector spaces $HH^n(A, Q) \cong (\text{Ext}^n_A(M_j, M_i)) \otimes_k K_{ij}$ for $n = 1, 2$: Since $L(a)$ is a lifting to $M_R$ of the left multiplication of $a$ on $M_S$ (satisfying equation 1), $L(a)$ satisfies equation 2 as well. That is, we have $L(a)_{ji}(m_i) - \delta_{ij}(am_i) \otimes e_i \in M_j \otimes_k I_{ji}$ for
all \( a \in A, m_i \in M_i, 1 \leq i, j \leq p \). Since \( K^2 = 0 \), this means that the \( A-A \) bimodule structure of \( Q \) defined via \( L \) coincides with the following natural one: Since \( M_i, M_j \otimes_k K_{ji} \) are left \( A \)-modules, we have that \( Q_{ij} = \text{Hom}_k(M_j, M_i \otimes_k K_{ij}) \) and \( Q = \oplus Q_{ij} \) has natural \( A-A \) bimodule structures. Clearly, we have

\[
\text{HH}^n(A, Q) \cong \oplus_{i,j} \text{HH}^n(A, Q_{ij}) = (\text{HH}^n(A, Q_{ij})).
\]

By appendix A, proposition A.3, we have that \( \text{HH}^n(A, Q_{ij}) \cong \text{Ext}_A^n(M_j, M_i \otimes_k K_{ij}) \) for \( n \geq 0 \). Moreover, \( \text{Ext}_A^n(M_j, M_i \otimes_k K_{ij}) \cong \text{Ext}_A^n(M_j, M_i) \otimes_k K_{ij} \) since \( K_{ij} \) is a \( k \)-vector space of finite dimension. This completes the proof of the proposition. □

We remark that it is easy to find an alternative proof of proposition 5.1 using resolutions and the Yoneda representation of \( \text{Ext}_A^n(M_i, M_j) \). This is straight-forward, but makes essential use of proposition A.3.

Also notice that the obstruction calculus is functorial in the following sense: Let \( u : R \to S \) and \( u' : R' \to S' \) be two small surjections in \( \mathfrak{a}_p \), and write \( K = \ker(u) \) and \( K' = \ker(u') \). Assume that \( v : R \to R' \) and \( w : S \to S' \) are morphisms such that \( u' \circ v = w \circ u \). Then \( v(K) \subseteq K' \), and the map \( v \) induces a \( k \)-linear map of obstruction spaces

\[
(\text{Ext}_A^2(M_j, M_i) \otimes_k K_{ij}) \to (\text{Ext}_A^2(M_j, M_i) \otimes_k K'_{ij}).
\]

If \( M_S \) is a deformation of \( M \) to \( S \) and \( M_{S'} = \text{Def}_M(w)(M_S) \) is the corresponding deformation to \( S' \), then this map of obstruction spaces maps \( o(u, M_S) \) to \( o(u', M_{S'}) \). This follows from the proof of proposition 5.1.

Let us start the construction of the pro-representing hull \((H, \xi) \) of \( \text{Def}_M \), using the obstruction calculus for \( \text{Def}_M \) given above. From now on, we shall assume that the family \( M \) satisfy the finiteness condition

\[
(\text{FC}) \quad \dim_k \text{Ext}_A^n(M_i, M_j) \text{ is finite for } 1 \leq i, j \leq p, n = 1, 2.
\]

We fix the following notation: Let \( \{x_{ij}(l) : 1 \leq l \leq d_{ij} \} \) be a basis for \( \text{Ext}_A^1(M_j, M_i)^* \) and \( \{y_{ij}(l) : 1 \leq l \leq r_{ij} \} \) be a basis for \( \text{Ext}_A^2(M_j, M_i)^* \) for \( 1 \leq i, j \leq p \) with \( d_{ij} = \dim_k \text{Ext}_A^1(M_j, M_i) \) and \( r_{ij} = \dim_k \text{Ext}_A^2(M_j, M_i) \). Moreover, we consider the formal matrix rings in \( \mathfrak{a}_p \) corresponding to these vector spaces, and denote them by \( T^1 = \hat{R}(\{\text{Ext}_A^1(M_j, M_i)^*\}) \) and \( T^2 = \hat{R}(\{\text{Ext}_A^2(M_j, M_i)^*\}) \).

First, let us show that \( \text{Def}_M \) restricted to \( \mathfrak{a}_p(2) \) is representable: We define \( H_2 \) to be the object \( H_2 = T_2^1 = T^1(I(T^1)^2) \) in \( \mathfrak{a}_p(2) \). For all objects \( R \) in \( \mathfrak{a}_p(2) \), we get \( \text{Mor}(H_2, R) \cong (\text{Hom}_k(\text{Ext}_A^1(M_j, M_i)^*, I(R))) \cong (\text{Ext}_A^1(M_j, M_i) \otimes_k I(R_{ij})) \), and \( \text{Def}_M(R) \cong (\text{Ext}_A^1(M_j, M_i) \otimes_k I(R_{ij})) \) by proposition 5.1 applied to the small surjection \( R \to k^p \). The isomorphisms we obtain in this way are compatible, so they induce an isomorphism \( \phi_2 : h_{H_2} \to \text{Def}_M \) of functors on \( \mathfrak{a}_p(2) \). From the version of lemma 5.1 for the category \( \mathfrak{a}_p(2) \), we see that there is a unique deformation \( \xi_2 \in \text{Def}_M(H_2) \) such that \( a_2(\xi_2) = \phi_2 \). By definition, \( (H_2, \xi_2) \) represents the deformation functor \( \text{Def}_M \) restricted to \( \mathfrak{a}_p(2) \).

Let us also give an explicit description of the deformation \( \xi_2 \): We have \( H_2 = T_2^1 \), so let us denote by \( c_{ij}(l) \) the image of \( x_{ij}(l) \) in \( H_2 \) for \( 1 \leq i, j \leq p, 1 \leq l \leq d_{ij} \). In this notation, \( \xi_2 \) is represented by the right \( H_2 \)-module \( (M_i \otimes_k (H_2)_{ij}) \), with left \( A \)-module structure defined by

\[
a(m_j \otimes c_{ij}) = am_j \otimes e_j + \sum_{i,l} \psi_{ij}(a)(m_j) \otimes c_{ij}(l)
\]

for all \( a \in A, m_j \in M_j, 1 \leq j \leq p \), where \( \psi_{ij} \in \text{Der}_k(A, \text{Hom}_k(M_j, M_i)) \) is a representative of \( x_{ij}(l)^* \in \text{Ext}_A^1(M_j, M_i) \) via Hochschild cohomology.
There is also an alternative description of $\xi_2$ using $M$-free resolutions and the Yoneda representation of $\text{Ext}^1_A(M_i, M_j)$: Let $\alpha(\epsilon_{ij}(l)) \in \text{Hom}^1(L_{s_j}, L_{s_i})$ be a 1-cocycle representing $x_{ij}(l)^* \in \text{Ext}^1_A(M_j, M_i)$ for $1 \leq i, j \leq p$, $1 \leq l \leq d_{ij}$. Then by construction, the formula

$$d_{H^2} = \sum_i d_{m,i} \otimes \epsilon_i + \sum_{i,j,l} \alpha(\epsilon_{ij}(l)) \otimes \epsilon_{ij}(l)$$

defines a differential which lifts the complex $(F_*, d_*)$ to $H_2$. By proposition 5.6, the lifted complex is in fact an $M$-free resolution of some deformation of $M$ to $H_2$, and this deformation is $\xi_2 \in \text{Def}_M(H_2)$.

**Theorem 5.2.** Assume that $\text{dim}_k \text{Ext}^n_A(M_i, M_j)$ is finite for $1 \leq i, j \leq p$, $n = 1, 2$. Then there exists a morphism $\alpha : T^2 \to T^1$ in $\mathfrak{a}_p$ such that $H(M) = T^1 \otimes T^2 k^p$ is a pro-representing hull for $\text{Def}_M$.

**Proof.** For simplicity, let us write $I$ for the ideal $I = I(T^1)$, and for all $n \geq 1$, let us write $T_n^1$ for the quotient $T_n^1 = T^1/I^n$, and $t_n : T_{n+1}^1 \to T_n^1$ for the natural morphism. From the paragraphs preceding this theorem, we know that $(H_2, \xi_2)$ represents $\text{Def}_M$ restricted to $\mathfrak{a}_p(2)$. Let $o_2 : T^2 \to T_2^1$ be the trivial morphism given by $o_2(I(T^2)) = 0$ and let $a_2 \subseteq I^2$, then $H_2 \cong T^1/a_2 \cong T^1_2 \otimes T^2 k^p$. Using $o_2$ and $\xi_2$ as a starting point, we shall construct $o_n$ and $\xi_n$ for $n \geq 3$ by an inductive process. So let $n \geq 2$, and assume that the morphism $o_n : T^2 \to T_n^1$ and the deformation $\xi_n \in \text{Def}_M(H_n)$ is given, with $H_n = T_n^1 \otimes T^2 k^p$. We shall also assume that $t_{n-1} \circ o_n = o_{n-1}$ and that $\xi_n$ is a lifting of $\xi_{n-1}$.

Let us now construct the morphism $o_{n+1} : T^2 \to T_{n+1}^1$. We let $a_n'$ be the ideal in $T_1^n$ generated by $o_n(I(T^2)^2))$. Then $a_n' = a_n/I^n$ for an ideal $a_n \subseteq T^1$ with $I^n \subseteq a_n$, and $H_n \cong T^1/a_n$. Let $b_n = Ia_n + a_n I$, then we obtain the following commutative diagram:

$\begin{array}{ccc}
T^2 & \xrightarrow{o_n} & T^1/b_n \\
\downarrow T_{n+1} & & \downarrow T_n/I_n \\
T_n^1 & \to & H_n = T^1/a_n,
\end{array}$

Observe that $T^1/b_n \to T^1/a_n$ is a small surjection. So by proposition 5.4 there is an obstruction $o_{n+1}' = o(T^1/b_n \to H_n, \xi_n)$ for lifting $\xi_n$ to $T^1/b_n$, and we have

$$o_{n+1}' \in (\text{Ext}^2_A(M_j, M_i) \otimes_k (a_n/b_n)_{ij}) \cong (\text{Hom}_k(\text{gr}_1(T^2), (a_n/b_n)_{ij})).$$

Consequently, we obtain a morphism $o_{n+1}' : T^2 \to T^1/b_n$. Let $a''_{n+1}$ be the ideal in $T^1/b_n$ generated by $o_{n+1}'(I(T^2))$. Then $a''_{n+1} = a_{n+1}/b_n$ for an ideal $a_{n+1} \subseteq T^1$ with $b_n \subseteq a_{n+1} \subseteq a_n$. We define $H_{n+1} = T^1/a_{n+1}$ and obtain the following commutative diagram:

$\begin{array}{ccc}
T^2 & \xrightarrow{o_{n+1}} & T_{n+1}^1 \\
\downarrow T^1 & & \downarrow T^1/b_n \\
T_n^1 & \to & H_{n+1} = T^1/a_{n+1}
\end{array}$

By the choice of $a_{n+1}$, the obstruction for lifting $\xi_n$ to $H_{n+1}$ is zero. We can therefore find a lifting $\xi_{n+1} \in \text{Def}_M(H_{n+1})$ of $\xi_n$ to $H_{n+1}$.

The next step of the construction is to find a morphism $o_{n+1} : T^2 \to T^1_{n+1}$ which commutes with $o_{n+1}'$ and $o_n$: We know that $t_{n-1} \circ o_n = o_{n-1}$, which means that $a_{n-1} = I^{n-1} + a_n$. For simplicity, let us write $O(K) = (\text{Hom}_k(\text{gr}_1(T^2)_{ij}, K_{ij}))$ for
any ideal $K \subseteq T^1$. Consider the following commutative diagram of $k$-vector spaces, in which the columns are exact:

\[
\begin{array}{ccc}
O(b_n/I^{n+1}) & \xrightarrow{j_n} & O(b_{n-1}/I^{n}) \\
\downarrow & & \downarrow \\
O(a_n/I^{n+1}) & \xrightarrow{k_n} & O(a_{n-1}/I^{n}) \\
\downarrow r_{n+1} & & \downarrow r_n \\
O(a_n/b_n) & \xrightarrow{l_n} & O(a_{n-1}/b_{n-1}) \\
\end{array}
\]

We may consider consider $o_n$ as an element in $O(a_{n-1}/I^{n})$, since $a_n \subseteq a_{n-1}$. On the other hand, $o_{n+1}' \in O(a_n/b_n)$. Let $o_n' = r_n(o_n)$, then the natural map $T^1/b_n \to T^1/b_{n-1}$ maps the obstruction $o_{n+1}'$ to the obstruction $o_n'$ by the second remark following proposition 5.1. This implies that $o_{n+1}'$ commutes with $o_n'$, so $l_n(o_{n+1}') = o_n' = r_n(o_n)$. But we have $o_n(I(T^2)) \subseteq a_n$, so we can find an element $\xi_{n+1} \in O(a_n/I^{n+1})$ such that $k_n(\xi_{n+1}) = o_n$. Since $a_{n-1} = a_n + I^{n-1}$, $j_n$ is surjective. Elementary diagram chasing using the snake lemma implies that we can find $o_{n+1} \in O(a_n/I^{n+1})$ such that $r_{n+1}(o_{n+1}) = o_{n+1}'$ and $k_n(o_{n+1}) = o_n$. It follows that the obstruction $o_{n+1}$ defines a morphism $o_{n+1}: T^2 \to T^1$ compatible with $o_n$ such that $T^1_{n+1} \otimes T^2 k^n \cong H_{n+1}$.

By induction, it follows that we can find a morphism $o_n: T^2 \to T^1$ and a deformation $\xi_n \in \text{Def}_M(H_n)$, with $H_n = T^1_n \otimes T^2 k^n$, for all $n \geq 1$. From the construction, we see that $l_n o_n = o_{n-1}$ for all $n \geq 2$, so we obtain a morphism $o: T^2 \to T^1$ by the universal property of the projective limit. Moreover, the induced morphisms $h_n: H_{n+1} \to H_n$ are such that $\xi_{n+1} \in \text{Def}_M(H_{n+1})$ is a lifting of $\xi_n \in \text{Def}_M(H_n)$ to $H_{n+1}$. Notice that $I(H_n)^n = 0$ and that $H_{n+1}/I(H_{n+1})^{n+1} = H_{n+1}$ for all $n \geq 2$. It follows that $H/I(H)^n = H_n$ for all $n \geq 1$, so $H$ is an object of the pro-category $\mathbf{a}_p$. Let $\xi = (\xi_n)$, then clearly $\xi \in \text{Def}_M(H)$, so $(H, \xi)$ is a pro-couple for $\text{Def}_M$. It remains to show that $(H, \xi)$ is a pro-representable hull for $\text{Def}_M$.

It is clearly enough to show that $(H, \xi_n)$ is a pro-representing hull for $\text{Def}_M$ restricted to $\mathbf{a}_p(n)$ for all $n \geq 3$. So let $\phi_n = a_n(\xi_n)$ be the morphism of functors on $\mathbf{a}_p(n)$ corresponding to $\xi_n$. We shall prove that $\phi_n$ is a smooth morphism. So let $u: R \to S$ be a small surjection in $\mathbf{a}_p(n)$, and assume that $M_R \in \text{Def}_M(R)$ and $v \in \text{Mor}(H_n, S)$ are given such that $\text{Def}_M(u)(M_R) = \text{Def}_M(v)(\xi_n) = M_S$. Let us consider the following commutative diagram:
Let \( v' : T^1 \to R \) be any morphism making the diagram commutative. Then \( v'(a_n) \subseteq K \), where \( K = \ker(u) \), so \( v'(b_n) = 0 \). But the induced map \( T^1/b_n \to R \) maps the obstruction \( o_{n+1}^{nu} \to o(u,M_S) \), and we know that \( o(u,M_S) = 0 \). So we have \( v'(a_{n+1}) = 0 \), and \( v' \) induces a morphism \( v' : H_{n+1} \to R \) making the diagram commutative. Since \( v'(I(H_{n+1})^\nu) = 0 \), we may consider \( v' \) a map from \( H_{n+1}/I(H_{n+1})^\nu \cong H_n \). So we have constructed a map \( v' \in \text{Mor}(H_n,R) \) such that \( u \circ v' = v \). Let \( M_R' = \text{Def}_M(v'(\xi_n)) \), then \( M_R' \) is a lifting of \( M_S \) to \( R \). By proposition 5.1, the difference between \( M_R \) and \( M_R' \) is given by an element

\[
d \in (\text{Ext}_A^1(M_j,M_i) \otimes_k K_{ij}) = (\text{Hom}_k(\text{gr}_1(T^1)_{ij},K_{ij})).
\]

Let \( v'' : T^1 \to R \) be the morphism given by \( v''(x_{ij}(l)) = v'(x_{ij}(l)) + d(x_{ij}(l)) \) for \( 1 \leq i,j \leq p, 1 \leq l \leq d_{ij} \). Since \( a_{n+1} \subseteq I(T^1)^2 \), we have

\[
v''(a_{n+1}) \subseteq v'(a_{n+1}) + I(R)K + KI(R) + K^2.
\]

But \( u \) is small, so \( v''(a_{n+1}) = 0 \) and \( v'' \) induces a morphism \( v'' : H_n \to R \). Clearly, \( u \circ v'' = u \circ v' = v \), and \( \text{Def}_M(v'')(\xi_n) = M_R \) by construction. It follows that \( \phi_n \) is smooth for all \( n \geq 3 \).

We remark that the conclusion of the theorem still holds if we relax the finiteness condition \( \text{FC} \). If we only assume that

\[
\dim_k \text{Ext}_A^1(M_i,M_j) \text{ finite for } 1 \leq i,j \leq p,
\]
then the object \( T^2 \) is in \( A_p \), but not necessarily in \( A_p \). However, the rest of the proof is still valid as stated, so the finiteness condition on \( \text{Ext}_A^1(M_i,M_j) \) is clearly not essential.

In general, it is possible to generalize theorem 5.2 to the case when \( \text{Ext}_A^1(M_i,M_j) \) has countable dimension as a vector space over \( k \) for \( 1 \leq i,j \leq p, n = 1,2 \), see Laudal [2]. However, we shall always assume \( \text{FC} \) in this paper.

Assume that \( M \) satisfy \( \text{FC} \). If \( \text{Ext}_A^1(M_i,M_j) = 0 \) for \( 1 \leq i,j \leq p \), we say that the deformation functor \( \text{Def}_M \) is \emph{unobstructed}. For instance, \( \text{Def}_M \) is unobstructed for any finite family \( M \) of left \( A \)-modules satisfying \( \text{FC} \) if \( A \) is left hereditary (that is, the left global homological dimension of \( A \) is at most 1). If \( \text{Def}_M \) is unobstructed, \( H = T^1 \) is the hull of \( \text{Def}_M \).

In general, \( \text{Def}_M \) can be obstructed, and there is no simple formula for the hull \( H \) of \( \text{Def}_M \) if this is the case. However, there exists an algorithm for calculating the hull \( H \) using matrix Massey products. In the next sections, we shall introduce the matrix Massey products and explain how the hull can be calculated when \( M \) satisfy \( \text{FC} \).

6. IMMEDIATELY DEFINED MATRIC MASSEY PRODUCTS

We recall that \( k \) is an algebraically closed (commutative) field, \( A \) is an associative \( k \)-algebra, and \( M = \{ M_1, \ldots, M_p \} \) is a finite family of left \( A \)-modules. From now on, we also assume that the family \( M \) satisfy the finiteness condition \( \text{FC} \). In this section, we shall define the immediately defined matric Massey products and their defining system, and show how to calculate these products using matrices.

Let us fix a monomial \( X \in I(T^1) \) of type \( (i,j) \) and degree \( n \geq 2 \). Then we can write \( X \) uniquely in the form

\[
X = x_{i_0i_1}(l_1) x_{i_1i_2}(l_2) \ldots x_{i_{n-1}i_n}(l_n),
\]

where \( (i_0,i_n) = (i,j) \). Let \( X' \) be another monomial in \( T^1 \). We shall say that \( X' \) \emph{divides} \( X \) if there exist monomials \( X(l),X(r) \in T^1 \) such that \( X = X(l)X'X(r) \), and write \( X' | X \) if this is the case.
Consider the set of monomials \( \{ X' \in I(T^1) : X' \not\parallel X \} \), and denote by \( J(X) \) the ideal in \( T^1 \) generated by these monomials. We define \( R(X) = T^1/J(X) \) and \( S(X) = R(X)/(X) = T^1/(J(X), X) \). Then the natural map
\[
\pi(X) : R(X) \to S(X)
\]
is a small surjection in \( a_p \), and it has a 1-dimensional kernel which is generated by the monomial \( X \). We write \( I(X) = I(S(X)) \) and \( S(X)_n = S(S(X))/I(X)^n \) for all \( n \geq 1 \).

Let us consider the set \( B(X) = \{(i, j, l) : 1 \leq i, j \leq p, 1 \leq l \leq d_{ij}, x_{ij}(l) \mid X \} \), and denote by \( v_{ij}(l) \) the image of \( x_{ij}(l) \) in \( S(X)_2 \) for all \( (i, j, l) \in B(X) \). Then the set
\[
\{ v_{ij}(l) : (i, j, l) \in B(X) \}
\]
is a natural \( k \)-basis for \( I(X)/I(X)^2 \).

Assume that a morphism \( \phi(X) : H \to S(X) \) is given, and denote the composition of \( \phi(X) \) with the natural morphism \( S(X) \to S(X)_2 \) by \( \phi(X)_2 : H \to S(X)_2 \). This morphism can be written uniquely in the form
\[
\phi(X)_2 = \sum_{(i,j,l) \in B(X)} \alpha_{ij}(l) \otimes v_{ij}(l),
\]
where \( \alpha_{ij}(l) \in \text{Ext}^2_A(M_j, M_i) \) for all \( (i, j, l) \in B(X) \).

Conversely, consider a family \( \{ \alpha_{ij}(l) \in \text{Ext}^2_A(M_j, M_i) : (i, j, l) \in B(X) \} \) of extensions indexed by \( B(X) \), corresponding to a morphism \( \phi(X)_2 : H \to S(X)_2 \) given by \( \phi(X)_2 = \sum \alpha_{ij}(l) \otimes v_{ij}(l) \). If there exists a lifting of \( \phi(X)_2 \) to a morphism \( \phi(X) : H \to S(X) \), we say that the matric Massey product
\[
\langle \alpha ; X \rangle = \langle \alpha_{0,i_1}(l_1), \alpha_{i_1,i_2}(l_2), \ldots, \alpha_{i_{n-1},i_n}(l_n) \rangle
\]
is defined, and that \( \phi(X) \) is a defining system for this matric Massey product. If this is the case, we denote the deformation induced by the defining system \( \phi(X) \) by \( M_X \in \text{Def}_M(S(X)) \), and by proposition \( 64 \) the obstruction for lifting \( M_X \) to \( R(X) \) is an element
\[
o(\pi(X), M_X) \in (\text{Ext}^2_A(M_j, M_i) \otimes_k K(X))_{ij} \cong \text{Ext}^2_A(M_j, M_i),
\]
where \( K(X) = \ker(\pi(X)) \cong kX \). In general, this element depends upon the deformation \( M_X \), and therefore on the defining system \( \phi(X) \). We define the value of the matric Massey product to be
\[
\langle \alpha ; X \rangle = \langle \alpha_{0,i_1}(l_1), \alpha_{i_1,i_2}(l_2), \ldots, \alpha_{i_{n-1},i_n}(l_n) \rangle = o(\pi(X), M_X).
\]
Consequently, the value of the matric Massey product \( \langle \alpha ; X \rangle \) will in general depend upon the chosen defining system.

Let us fix the monomial \( X \). Then the matric Massey product \( \alpha \mapsto \langle \alpha ; X \rangle \) is a not everywhere defined \( k \)-linear map
\[
\text{Ext}_A^1(M_{i_1}, M_{i_0}) \otimes_k \cdots \otimes_k \text{Ext}_A^1(M_{i_{n-1}}, M_{i_n}) \to \text{Ext}_A^2(M_{i_n}, M_{i_0}).
\]
In fact, this map is defined for \( \alpha \) if and only if the morphism \( \phi(X)_2 : H \to S(X)_2 \) corresponding to \( \alpha \) can be lifted to a morphism \( \phi(X) : H \to S(X) \). Moreover, even when this map is defined for \( \alpha \), it is not necessarily uniquely defined: In general, its value \( \langle \alpha ; X \rangle \) depends upon the chosen lifting \( \phi(X) \), the defining system. The matric Massey products \( \langle \alpha ; X \rangle \) defined above are called the immediately defined matric Massey products.

We remark that if \( X \) is a monomial of degree \( n = 2 \), then the situation is much simpler: We have \( S(X) = S(X)_2 \), so the matric Massey product \( \langle \alpha ; X \rangle \) is uniquely defined for any family of extensions \( \{ \alpha_{ij}(l) : (i, j, l) \in B(X) \} \). In fact, the matric Massey product is just the usual cup product in this case.
Let us fix a monomial $X \in I(T^1)$ of degree $n \geq 2$. Then there exists a natural family of extensions indexed by $B(X)$ given by $\alpha_{ij}(l) = x_{ij}(l)^*$,

$$\{x_{ij}(l)^* \in \text{Ext}_A^1(M_j, M_i) : (i, j, l) \in B(X)\}.$$

The matric Massey products of these extensions are the ones that we shall use for the construction of the hull $H$ of $\text{Def}_M$ in the next section. We therefore introduce the notation

$$\langle \alpha : X \rangle = \langle \alpha_{i_0 i_1}(l_1)^*, \alpha_{i_1 i_2}(l_2)^*, \ldots, \alpha_{i_{n-1} i_n}(l_n)^* \rangle$$

for their immediately defined matric Massey products.

The matric Massey products are called *matric* because these products (and their defining systems) can be described completely in terms of linear algebra and matrices. We shall end this section by giving such a description.

Let $\{\alpha_{ij}(l) \in \text{Ext}_A^1(M_j, M_i) : (i, j, l) \in B(X)\}$ be a family of extensions indexed by $B(X)$, and consider the corresponding matric Massey product

$$(4) \quad \langle \alpha : X \rangle = \langle \alpha_{i_0 i_1}(l_1), \alpha_{i_1 i_2}(l_2), \ldots, \alpha_{i_{n-1} i_n}(l_n) \rangle.$$  

We assume that there exists a defining system $\phi(X) : H \to S(X)$ for this matric Massey product. Then $\phi(X)$ induces a deformation $M_X \in \text{Def}_M(S(X))$. We notice that the matric Massey product (4) only depends upon this deformation. By abuse of language, we shall therefore let the notion defining system refer to the deformation $M_X$ as well as the morphism $\phi(X) : H \to S(X)$ which induces $M_X$.

We know that any deformation $M_X \in \text{Def}_M(S(X))$ can be described by a complex which lifts $(F_*, d_*)$ to $S(X)$. Such a complex is given by differentials of the form

$$d^S_m(X) : (L_{m+i, i} \otimes_k S(X)_{ij}) \to (L_{m, i} \otimes_k S(X)_{ij}).$$

We write $u(X')$ for the image of $X'$ in $S(X)$ whenever $X'$ is a monomial in $T^1$, and define $\overline{B}(X) = \{X' \in I(T^1) : X'$ is a monomial such that $X' \mid X \} \cup \{e_1, \ldots, e_p\}$. Then the set

$$\{v(X') : X' \in \overline{B}(X)\}$$

is a natural $k$-basis for $S(X)$, and $\overline{B}(X)$ contains $\{x_{ij}(l) : (i, j, l) \in B(X)\}$ and $\{e_1, \ldots, e_p\}$ as subsets. Let us write $\overline{B}(X)_{ij} = \overline{B}(X) \cap S(X)_{ij}$ for $1 \leq i, j \leq p$.

With this notation, the above differentials have the form

$$d^S_m(X) = \sum_{1 \leq i \leq p} d_{m, i} \otimes e_i + \sum_{X' \in \overline{B}(X)} \alpha(X')_m \otimes v(X'),$$

where $\alpha(X') \in \text{Hom}_k^1(L_{s_j}, L_{t_i})$ is a 1-cochain whenever $X' \in \overline{B}(X'_{ij})$.

Let $d^S_m$ be arbitrary maps between M-free modules over $S(X)$ defined by a family of 1-cochains $\{\alpha(X') : X' \in \overline{B}(X)\}$ as above. These maps lift the complex $(F_*, d_*)$ if $\alpha(e_i) = d_{e_i}$ for $1 \leq i \leq p$. Moreover, these maps are differentials if and only if the following condition holds: For all monomials $Z \in \overline{B}(X)$ and for all integers $m \geq 0$, we have

$$(5) \quad \sum_{X', \alpha(X') \in \overline{B}(X)} \alpha(X')_m \circ \alpha(X'')_{m+1} = \sum_{X', \alpha(X') \in \overline{B}(X)} \alpha(X'')_{m+1} \alpha(X')_m = 0.$$

In the first sum, the symbol $\circ$ denotes composition of maps. We recall that each of the maps involved can be considered as right multiplication by a matrix. In the second summation, we identify the maps with such matrices, and re-write the composition of maps as multiplication of the corresponding matrices.

Assume that these conditions hold. Then the family $\{\alpha(X') : X' \in \overline{B}(X)\}$ of 1-cochains defines a lifting of complexes of $(L_*, d_*)$ to $S(X)$ given by the differentials $d^S(X)$ as above, and this lifting corresponds to a deformation $M_X \in \text{Def}_M(S(X))$. 

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**References**

[1] Eivind Eriksen, *On the Matric Massey Products*, *Algebraic Topology and Applications to Physics*, Lecture Notes in Mathematics, Vol. 2259, Springer, 2017.

[2] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, Vol. 271, Springer, 1972.

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**Further Reading**

For a deeper understanding of the matric Massey products and their applications, the reader is encouraged to consult the aforementioned references and related literature. The topic is rich and far-reaching, with connections to various areas of mathematics and theoretical physics.
The deformation $M_X$ is a defining system for the matric Massey product \((\mathcal{M})\) if and only if $\alpha(X')$ is a 1-cocycle which represents $\alpha_{ij}(l) \in \text{Ext}_A^1(M_j, M_i)$ whenever $X' = x_{ij}(l)$ for some $(i,j,l) \in B(X)$. In this case, we shall refer to the family of 1-cochains $\{\alpha(X') : X' \in \overline{B}(X)\}$ as a defining system for the matric Massey product \((\mathcal{M})\).

Finally, assume that the family of 1-cochains $\{\alpha(X') : X' \in \overline{B}(X)\}$ is a defining system of the matric Massey product \((\mathcal{M})\). Then the value of this matric Massey product is given by

\[
\langle \alpha ; X \rangle_m = \sum_{X' , X'' \in \overline{B}(X) \atop X' ; X'' = X} \alpha(X'')m+1 \alpha(X')m
\]

for all $m \geq 0$, where the multiplication denotes matrix multiplication of the corresponding matrices.

**Proposition 6.1.** Let $\{\alpha_{ij}(l) \in \text{Ext}_A^1(M_j, M_i) : (i,j,l) \in B(X)\}$ be a family of extensions. A defining system for the matric Massey product

\[
\langle \alpha ; X \rangle = (\alpha_{i_0 i_1} (l_1), \ldots, \alpha_{i_{n-1} i_n} (l_n))
\]

corresponds to a family $\{\alpha(X') \in \text{Hom}_A^1(L_{+j}, L_{+i}) : 1 \leq i, j \leq p, X' \in \overline{B}(X)_{ij}\}$ of 1-cochains satisfying the following conditions:

- $\alpha(e_j) = d_{n, j}$ for $1 \leq i \leq p$,
- $\alpha(X')$ is a 1-cocycle representing $\alpha_{ij}(l)$ whenever $X' = x_{ij}(l)$ for some $(i,j,l) \in B(X)$,
- For all $Z \in \overline{B}(X)$ and for all $m \geq 0$, we have
  \[
  \sum_{X' , X'' \in \overline{B}(X) \atop X' ; X'' = Z} \alpha(X'')m+1 \alpha(X')m = 0.
  \]

Moreover, given such a family of 1-cochains, the matric Massey product $\langle \alpha ; X \rangle$ is represented by the 2-cocycle given by

\[
\langle \alpha ; X \rangle_m = \sum_{X' , X'' \in \overline{B}(X) \atop X' ; X'' = X} \alpha(X'')m+1 \alpha(X')m
\]

for all $m \geq 0$.

Hence we have described the immediately defined matric Massey products and their defining systems in terms of linear algebra and matrices, as we set out to do. We remark that the description given in proposition 6.1 is extremely useful for doing concrete calculations with matric Massey products, and even for implementing such computations on computers. It also justifies the name matric.

7. Calculating Hulls Using Matric Massey Products

We recall that $k$ is an algebraically closed (commutative) field, $A$ is an associative $k$-algebra, and $\mathcal{M} = \{M_1, \ldots, M_p\}$ is a finite family of left $A$-modules. We also assume that the family $\mathcal{M}$ satisfy the finiteness condition \((\mathcal{FC})\). In this section, we show how to calculate the hull $H$ of the deformation functor $\text{Def}_{\mathcal{M}}$ using matric Massey products.

By theorem 5.2, there exists an obstruction morphism $o : T^2 \to T^1$ in $\mathfrak{a}_p$ such that $H = T^1 \otimes_{T^2} k^p$ is a hull for the deformation functor $\text{Def}_{\mathcal{M}}$. We shall write $I = I(T^1)$ and $f_{ij}(l) = o(y_{ij}(l))$ for $1 \leq i, j \leq p, 1 \leq l \leq r_{ij}$. Then $f_{ij}(l)$ is a
formal power series in $I^2_{ij}$ by construction. Let us define $a \subseteq T^1$ to be the ideal generated by $\{f_{ij}(l) : 1 \leq i, j \leq p, 1 \leq l \leq r_{ij}\}$. Then $a \subseteq I^2$, and we have

$$H = T^1 \hat{\otimes}_{T^2} k^a \cong T^1 / a.$$  

We shall use the matric Massey products from section 8 to calculate the coefficients of the power series $f_{ij}(l)$. Clearly, this is sufficient to determine the hull $H$.

Let us fix an integer $N \geq 2$ such that $a \subseteq I^N$. This is always possible, since $a \subseteq I^2$. So $f_{ij}(l) \in I^N$ for all $f_{ij}(l)$, and we can write $f_{ij}(l)$ in the form

$$f_{ij}(l) = \sum_{|X|=N} a_{ij}^l(X) \cdot X + \sum_{|X|>N} a_{ij}^l(X) \cdot X$$

for $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$, with $a_{ij}^l(X) \in k$ for all monomials $X \in I^N$. As usual, we use the notation $|X|$ to denote the degree of the monomial $X$.

Let $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$ and let $n \geq N$. Then we agree to write $f_{ij}(l)^n$ for the truncated power series

$$f_{ij}(l)^n = \sum_{|X|=N} a_{ij}^l(X) \cdot X.$$

Moreover, let $a_{n+1} = I^{n+1} + (f^n)$ for all $n \geq N$, where $(f^n) \subseteq T^1$ is the ideal generated by $\{f_{ij}(l)^n : 1 \leq i, j \leq p, 1 \leq l \leq r_{ij}\}$, and let $a_n = I^n$ for $2 \leq n \leq N$. We write $H_n = H/I(H^n)$ as usual, then $H_n = T^1/a_n$ for all $n \geq 2$, in accordance with the notation in the proof of theorem 5.2.

Recall that $H_2 = T_2$ and that $\xi_2 \in \text{Def}_M(H_2)$ denotes the universal deformation with the property that the couple $(H_2, \xi_2)$ represents $\text{Def}_M$ restricted to $a_p(2)$. We have assumed that $a \subseteq I^N$, and this means that there exists a lifting of $\xi_2$ to $H_N = T^1/a_N = T^1_K$. Let us proceed to find such a lifting $M_N \in \text{Def}_M(H_N)$ explicitly.

We choose to describe the deformation $M_N$ in terms of $M$-free resolutions. Let us define $\overline{B}(N-1)$ to be the set of all monomials in $T^4$ of degree at most $N-1$. Then $\{X : X \in \overline{B}(N-1)\}$ is a monomial basis of $H_N$, and any $M$-free resolution of $M_N$ can be described by a family $\{\alpha(X) : X \in \overline{B}(N-1)\}$ of 1-cochains satisfying the following conditions:

- $\alpha(e_i) = d_{si}$ for $1 \leq i \leq p$,
- $\alpha(x_{ij}(l))$ is a 1-cocycle representing $x_{ij}(l)^*$ for $1 \leq i, j \leq p$, $1 \leq l \leq d_{ij}$,
- For all $Z \in \overline{B}(N-1)$ and for all $m \geq 0$, we have

$$\sum_{X', X'' \in \overline{B}(N-1), X'X''=Z} \alpha(X''_{m+1}) \alpha(X')_m = 0.$$  

We know that a family of 1-cochains with the above properties exists, since we can find a lifting $M_N$ of $\xi_2$ to $H_N$ and this deformation must have some $M$-free resolution. So we choose one such family $\{\alpha(X) : X \in \overline{B}(N-1)\}$ and fix this choice. This means that we have fixed a deformation $M_N \in \text{Def}_M(H_N)$ with an $M$-free resolution given by the corresponding differentials. So $(H_N, M_N)$ is a pro-representing hull for $\text{Def}_M$ restricted to $a_p(N)$.

**Lemma 7.1.** Let $\pi : R \to S$ be any small surjection in $a_p$, let $\phi : H \to S$ be any morphism, and denote by $M_\phi \in \text{Def}_M(S)$ the deformation induced by $\phi$. Then we
can lift \( \phi \) to a morphism \( \overline{\phi} : T^1 \to R \) making the diagram

\[
\begin{array}{ccc}
T^1 & \xrightarrow{\overline{\phi}} & R \\
\downarrow & & \downarrow \\
H & \xrightarrow{\phi} & S
\end{array}
\]

commutative, and the obstruction \( o(\pi, M_{\phi}) \) for lifting \( M_{\phi} \) to \( R \) is given by

\[
o(\pi, M_{\phi}) = \sum_{i,j,l} y_{ij}(l)^* \otimes \overline{\phi}(f_{ij}(l)).
\]

**Proof.** By construction and functoriality, the obstruction \( o(\pi, M_{\phi}) \) is given as the restriction of the composition \( \overline{\phi} \circ o \) to the \( k \)-linear subspace \( (\text{Ext}^2(M_j, M_i)^*) \subseteq T^2 \). Since \( \{y_{ij}(l)\} \) is a \( k \)-linear basis for this subspace, we get the desired expression for the obstruction. \( \square \)

Let us define \( b_N \subseteq T^1 \) to be the ideal \( b_N = I_{a_N} + a_N I = I^{N+1} \), and consider the natural map \( r_N : R_N \to H_N \), where \( R_N = T^1/b_N = T^1_{N+1} \). By construction, \( r_N \) is a small surjection in \( a_p \), and the natural surjection \( \overline{\phi}_N : T^1 \to R_N \) makes the diagram

\[
\begin{array}{ccc}
T^2 & \xrightarrow{o} & T^1 \\
\downarrow & & \downarrow \\
H & \xrightarrow{\phi_N} & H_N
\end{array}
\]

commutative. Let \( B'(N) \) be the set of all monomials in \( T^1 \) of degree \( N \). Since \( \ker(r_N) = I^N/I^{N+1} \), we see that \( \{X : X \in B'(N)\} \) is a monomial basis for \( \ker(r_N) \). Moreover, let \( \overline{B}(N) = B'(N) \cup \overline{B(N - 1)} \). Then clearly \( \{X : X \in \overline{B}(N)\} \) is a monomial basis for \( R_N \).

Since \( r_N \) is a small surjection, there is an obstruction \( o(r_N, M_N) \) for lifting \( M_N \) to \( R_N \), and we see from lemma 7.1 that this obstruction can be expressed as

\[
o(r_N, M_N) = \sum_{i,j,l} y_{ij}(l)^* \otimes \overline{\phi}_N(f_{ij}(l))
\]

\[
= \sum_{i,j,l} y_{ij}(l)^* \otimes \overline{f}_{ij}(l)
\]

\[
= \sum_{i,j,l} \sum_{X \in B'(N)} y_{ij}(l)^* \otimes (a^l_{ij}(X) \cdot X),
\]

where \( \overline{f}_{ij}(l) \) and \( \overline{X} \) denote the images of \( f_{ij}(l) \) and \( X \) in \( R_N \).

We say that the family \( D(N) = \{\alpha(X) : X \in \overline{B(N - 1)}\} \) of 1-cochains is a defining system for the matric Massey products of order \( N \),

\[
\langle \alpha^*; X \rangle \text{ for } X \in B'(N).
\]

Let \( X \in B'(N) \) be any monomial of type \((i,j)\). We define the matric Massey product \( \langle \alpha^*; X \rangle \) to be the coefficient of \( \overline{X} \) in the obstruction \( o(r_N, M_N) \) above. Then we immediately see that this matric Massey product has value

\[
\langle \alpha^*; X \rangle = \sum_{l=1}^{r_{ij}} a^l_{ij}(X) \cdot y_{ij}(l)^*.
\]
In other words, the coefficient of $X$ in the power series $f_{ij}(l)$ is given by the matric Massey product $\langle x^*: X \rangle$ above as

$$a^l_{ij}(X) \equiv y_{ij}(l)(\langle x^*: X \rangle)$$

for $1 \leq l \leq r_{ij}$.

We notice that the matric Massey products of order $N$ defined above are immediately defined. In other words, they can be expressed in terms of the matric Massey products of section 6. In fact, the defining system $D(N)$ induces a defining system $\{ \alpha(X') : X' | X, X' \neq X \}$ in the sense of section 6 and the value of the corresponding matric Massey product $\langle x^*: X \rangle$ is exactly the coefficient of $X$ in the obstruction $o(r_N, M_N)$.

On the other hand, we can calculate the obstruction $o(r_N, M_N)$ using the defining system $D(N)$, and therefore also the coefficient of $X$ in this obstruction for each $X \in B'(N)$. A straight-forward calculation show that this coefficient is given by the 2-cocycle $y(X)$ defined by

$$y(X)_m = \sum_{X', X'' \in B(N-1)} \alpha(X'')_{m+1} \alpha(X')_m$$

for all $m \geq 0$. This means that the matric Massey product $\langle x^*: X \rangle$ is represented by $y(X)$, so we can easily calculate all matric Massey products of order $N$ using the defining system $D(N)$. This determines the truncated power series $f_{ij}(l)^N$, since we have

$$f_{ij}(l)^N = \sum_{X \in B'(N)} a^l_{ij}(X) \cdot X = \sum_{X \in B'(N)} y_{ij}(l)(\langle x^*: X \rangle) \cdot X$$

for $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$.

Let $h_N : H_{N+1} \rightarrow H_N$ be the natural map. Then $\ker(h_N) = I^N/a_{N+1}$, so we can find a subset $B(N) \subseteq B'(N)$ of monomials in $T^1$ of degree $N$ such that $\{ X : X \in B(N) \}$ is a monomial basis for $\ker(h_N)$. Let $\overline{B}(N) = B(N) \cup B(N-1)$, then clearly $\{ X \in \overline{B}(N) \}$ is a monomial basis for $H_{N+1}$. So for each monomial $X \in T^1$ with $|X| \leq N$, we have a unique relation in $H_{N+1}$ of the form

$$X = \sum_{X' \in \overline{B}(N)} \beta(X, X') X',$$

with $\beta(X, X') \in k$ for all $X' \in \overline{B}(N)$. Since we have $o(h_N, M_N) = 0$, we deduce that

$$\sum_{|X|=N} \langle x^*: X \rangle \beta(X, X') = 0$$

for all $X' \in B(N)$. Notice that $\beta(X, X') = 0$ if the monomials $X$ and $X'$ do not have the same type. Therefore, it makes sense to consider the 1-cocycle

$$\sum_{|X|=N} \beta(X, X') y(X),$$

and by the relation above, this is a 1-coboundary. It follows that we can find a 1-cochain $\alpha(X')$ such that

$$d \alpha(X') = - \sum_{|X|=N} \beta(X, X') y(X),$$
and we fix such a choice. Consider the family \( \{ \alpha(X) : X \in \overline{B}(N) \} \). This defines an M-free complex over \( H_{N+1} \) if and only if we have

\[
\sum_{|X|=N} \beta(X, Z) \sum_{X', X'' \in \overline{B}(N)} \alpha(X'') \alpha(X') = 0
\]

for all \( Z \in \overline{B}(N) \). By the definition of \( \alpha(X') \) when \( X' \in B(N) \), this condition holds, and we denote by \( M_{N+1} \in \text{Def}_M(H_{N+1}) \) the deformation with the complex defined by \( \{ \alpha(X) : X \in \overline{B}(N) \} \) as M-free resolution. It is clear from the construction that \( M_{N+1} \) is a lifting of \( M_N \), so \( (H_{N+1}, M_{N+1}) \) is a pro-representing hull for \( \text{Def}_M \) restricted to \( a_p(N+1) \).

Let \( b_{N+1} \subseteq T^1 \) be the ideal \( b_{N+1} = Ia_{N+1} + a_{N+1}I = I^{N+2} + I(f^N) + (f^N)I \), and consider the natural map \( r_{N+1} : R_{N+1} \rightarrow H_{N+1} \), where \( R_{N+1} = T^1/b_{N+1} \).

By construction, \( r_{N+1} \) is a small surjection in \( a_p \), and it is clear that the natural morphism \( \bar{\phi}_{N+1} : T^1 \rightarrow R_{N+1} \) makes the diagram

\[
\begin{array}{ccc}
T^2 & \xrightarrow{\alpha} & T^1 \\
\downarrow & & \downarrow \\
H & \xrightarrow{\phi_{N+1}} & R_{N+1} \\
\downarrow & & \downarrow \\
H_{N+1} & \xrightarrow{\phi_N} & H_N \\
\end{array}
\]

commutative. We see that \( \ker(r_{N+1}) = a_{N+1}/b_{N+1} \), which we can re-write in the following way:

\[
\ker(r_{N+1}) = (I^{N+1} + (f^N))/(I^{N+2} + I(f^N) + (f^N)I) = (f^N)/(I(f^N) + (f^N)I) \oplus I^{N+1}/(I^{N+2} + I(f^N) + (f^N)I)
\]

Let us write \( c(N+1) = I^{N+1}/(I^{N+2} + I(f^N) + (f^N)I) \). Then \( c(N+1) \subseteq \ker(r_{N+1}) \) is an ideal, and we can clearly find a set \( B'(N+1) \) of monomials in \( T^1 \) of degree \( N+1 \) such that \( \{ \overline{X} : X \in B'(N+1) \} \) is a monomial basis for \( c_{N+1} \). Let us choose \( B'(N+1) \) such that for every \( X \in B'(N+1) \), there is a monomial \( X' \in B(N) \) such that \( X' \mid X \), this is clearly possible. We let \( \overline{B}(N+1) = B'(N+1) \cup \overline{B}(N) \), then

\[
\{ \overline{X} : X \in \overline{B}(N+1) \} \cup \{ f_{ij}(l)^N : 1 \leq i, j \leq p, 1 \leq l \leq r_{ij} \}
\]

is a basis for \( R_{N+1} \). So for each monomial \( X \in T^1 \) with \( |X| \leq N+1 \), we have a unique relation in \( R_{N+1} \) of the form

\[
\overline{X} = \sum_{X' \in \overline{B}(N+1)} \beta'(X, X') \overline{X'} + \sum_{i,j,l} \beta(X, i, j, l) f_{ij}(l)^N,
\]

with \( \beta'(X, X'), \beta(X, i, j, l) \in k \) for all \( X' \in \overline{B}(N+1), 1 \leq i, j \leq p, 1 \leq l \leq r_{ij} \).

Since \( r_{N+1} \) is a small surjection, there is an obstruction \( o(r_{N+1}, M_{N+1}) \) for lifting \( M_{N+1} \) to \( R_{N+1} \), and we see from lemma \[1\] that this obstruction can be expressed as

\[
o(r_{N+1}, M_{N+1}) = \sum_{i,j,l} y_{ij}(l)^* \otimes \overline{\phi}_{N+1}(f_{ij}(l))
= \sum_{i,j,l} y_{ij}(l)^* \otimes f_{ij}(l)
= \sum_{i,j,l} y_{ij}(l)^* \otimes (f_{ij}(l)^N) + \sum_{X \in B'(N+1)} a_{ij}(X) \cdot \overline{X},
\]

where \( a_{ij}(X) \) are the coefficients of the obstruction.
where $f_{ij}(l)$, $f_{ij}(l)^N$ and $X$ denote the images of $f_{ij}(l)$, $f_{ij}(l)^N$ and $X$ in $R_{N+1}$.

We say that the family $D(N+1) = \{\alpha(X) : X \in \overline{B}(N)\}$ is a defining system for the matric Massey products of order $N+1$,

$$\langle x^*; X \rangle \text{ for } X \in B'(N+1)$$

Let $X \in B'(N+1)$ be any monomial of type $(i, j)$. We define the matric Massey product $\langle x^*; X \rangle$ to be the coefficient of $X$ in the obstruction $o(r_{N+1}, M_{N+1})$ above. Then we immediately see that this matric Massey product has value

$$\langle x^*; X \rangle = \sum_{l=1}^{r_{ij}} a^l_{ij}(X) \cdot y_{ij}(l)^*.$$

In other words, the coefficient of $X$ in the power series $f_{ij}(l)$ is given by the matric Massey product $\langle x^*; X \rangle$ above as

$$a^l_{ij}(X) = y_{ij}(l)(\langle x^*; X \rangle)$$

for $1 \leq l \leq r_{ij}$.

On the other hand, we can calculate the obstruction $o(r_{N+1}, M_{N+1})$ using the defining system $D(N+1)$, and therefore also the coefficient of $X$ in this obstruction for each $X \in B'(N+1)$. A straightforward calculation show that this coefficient is given by the 2-cocycle $y(X)$ defined by

$$y(X)_m = \sum_{|Z| \leq N+1} \beta'(Z, X) \sum_{X', X'' \in \overline{B}(N)} \alpha(X'')_{m+1} \alpha(X')_m$$

for all $m \geq 0$. This means that the matric Massey product $\langle x^*; X \rangle$ is represented by $y(X)$, so we can easily calculate all matric Massey products of order $N+1$ using the defining system $D(N+1)$.

By the construction in the proof of theorem 16.2 we have that $H_{N+2}$ is the quotient of $R_{N+1}$ by the ideal generated by the obstruction $o(r_{N+1}, M_{N+1})$. On the other hand, we know that $H_{N+2} = T^1 / (I^{N+2} + f^{N+1})$. This implies that for all monomials $X \notin B'(N+1)$ of degree $N+1$, the coefficient $a^l_{ij}(X) = 0$ for $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$. In other words, the truncated power series $f_{ij}(l)^{N+1}$ is determined by the matric Massey products of order $N+1$ above, since we have

$$f_{ij}(l)^{N+1} = f_{ij}(l)^N + \sum_{X \in B'(N+1)} a^l_{ij}(X) \cdot X$$

$$= f_{ij}(l)^N + \sum_{X \in B'(N+1)} y_{ij}(l)(\langle x^*; X \rangle) \cdot X$$

for $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$.

Let $h_{N+1} : H_{N+2} \to H_{N+1}$ be the natural map, and consider its kernel. By definition, we have

$$\ker(h_{N+1}) = a_{N+1} / a_{N+2} = ((f^N) + I^{N+1}) / ((f^{N+1}) + f^{N+2}),$$

so we can clearly find a subset $B(N+1) \subseteq B'(N+1)$ of monomials of degree $N+1$ such that $\{X : X \in \overline{B}(N+1)\} \cup \{f_{ij}(l)^N : 1 \leq i, j \leq p, 1 \leq l \leq r_{ij}\}$ is a basis for $\ker(h_{N+1})$. Let $\overline{B}(N+1) = B(N+1) \cup \overline{B}(N)$, then clearly

$$\{X : X \in \overline{B}(N+1)\} \cup \{f_{ij}(l)^N : 1 \leq i, j \leq p, 1 \leq l \leq r_{ij}\}$$

is a monomial basis for $H_{N+2}$. So for each monomial $X \in T^1$ with $|X| \leq N+1$, we have a unique relation in $H_{N+2}$ of the form

$$X = \sum_{X' \in \overline{B}(N+1)} \beta(X, X') \overline{X'} + \sum_{i, j, l} \beta(X, i, j, l) f_{ij}(l)^N,$$
with $\beta(X, X'), \beta(X, i, j, l) \in k$ for all $X' \in \mathcal{B}(N + 1)$, $1 \leq i, j \leq p$, $1 \leq l \leq r_{ij}$. Since we have $a(h_{N+1}, M_{N+1}) = 0$, we deduce that
\[
\sum_{|X| \leq N+1} \langle x^n : X \rangle \beta(X, X') = 0
\]
for all $X' \in B(N + 1)$. Notice that $\beta(X, X') = 0$ if the monomials $X$ and $X'$ do not have the same type. Therefore, it makes sense to consider the 1-cocycle
\[
\sum_{|X| \leq N+1} \beta(X, X') y(X),
\]
and by the relation above, this is a 1-coboundary. It follows that we can find a 1-cocycle $\alpha(X')$ such that
\[
d \alpha(X') = - \sum_{|X| \leq N+1} \beta(X, X') y(X),
\]
and we fix such a choice. Consider the family $\{\alpha(X) : X \in \mathcal{B}(N+1)\}$. This defines an $M$-free complex over $H_{N+2}$ if and only if we have
\[
\sum_{|X| \leq N+1} \beta(X, Z) \sum_{X', X'' \in \mathcal{B}(N+1)} \alpha(X'') \alpha(X') = 0
\]
for all $Z \in \mathcal{B}(N+1)$. By the definition of $\alpha(X')$ when $X' \in B(N + 1)$, this condition holds, and we denote by $M_{N+2} \in \text{Def}_M(H_{N+2})$ the deformation with the complex defined by $\{\alpha(X) : X \in \mathcal{B}(N+1)\}$ as $M$-free resolution. It is clear from the construction that $M_{N+2}$ is a lifting of $M_{N+1}$, so $(H_{N+2}, M_{N+2})$ is a pro-representing hull for $\text{Def}_M$ restricted to $a_p(n + 2)$.

It is clear that we can continue in this way. For every $k \geq 1$, we can calculate the coefficients in the truncated power series $f_{ij}(l)^{N+k}$, and therefore find $H_{N+k+1}$. At the same time, we find the defining systems $\{\alpha(X) : X \in \mathcal{B}(N + k)\}$ necessary to calculate the matric Massey products of order $N + k + 1$, and these defining systems completely determine the deformation $M_{N+k+1}$. We have described how to do this in the case $k = 1$, and the general case is similar.

We conclude that the method that we have described above can be used to calculate the pro-representing hull $(H_n, M_n)$ for the deformation functor $\text{Def}_M$ restricted to $a_p(n)$ for any $n \geq N$. We can therefore, in principle, find the hull
\[
H = \lim_{\to} H_n
\]
of $\text{Def}_M$, and also the corresponding versal family defined over $H$,
\[
\xi = M = \lim_{\to} M_n.
\]
It follows that the pro-representing hull $(H, \xi)$ of the deformation functor $\text{Def}_M$ can be calculated using matric Massey products.

8. AN EXAMPLE

Let $k$ be an algebraically closed field of characteristic 0, and let $A = A_2(k)$ be the second Weyl algebra over $k$. We shall think of $A$ as the ring of differential operators in the plane defined over $k$ with coordinates $x$ and $y$. Thus, we can write $A = k[x, y] / (\partial x, \partial y)$, where $\partial x = \partial / \partial x$ and $\partial y = \partial / \partial y$. In other words, $A$ is the $k$-algebra generated by $x, y, \partial x, \partial y$ with relations $[\partial x, x] = [\partial y, y] = 1$. 

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Let us consider the family of left $A$-modules $\mathbf{M} = \{M_1, M_2, M_3, M_4\}$, where $M_i = A/I_i$ for $1 \leq i \leq 4$ and $I_i \subseteq A$ are left ideals given by

\[
I_1 = A(\partial x, \partial y) \quad I_2 = A(\partial x, y) \\
I_3 = A(x, \partial y) \quad I_4 = A(x, y)
\]

We immediately notice that the left $A$-modules in the family $\mathbf{M}$ have the following free resolutions:

\[
\begin{align*}
0 & \leftarrow M_1 \leftarrow A \leftarrow \frac{\partial x}{\partial y} A^2 \leftarrow \frac{\partial y}{\partial x} - \partial x A \leftarrow 0 \\
0 & \leftarrow M_2 \leftarrow A \leftarrow \frac{y}{x} A^2 \leftarrow \frac{y}{x} - \partial x A \leftarrow 0 \\
0 & \leftarrow M_3 \leftarrow A \leftarrow \frac{x}{\partial y} A^2 \leftarrow \frac{\partial y}{\partial x} - x A \leftarrow 0 \\
0 & \leftarrow M_4 \leftarrow A \leftarrow \frac{x}{y} A^2 \leftarrow \frac{y}{x} - x A \leftarrow 0
\end{align*}
\]

We consider the elements of the free $A$-modules $A^n$ as row vectors, and the maps in the free resolutions above as right multiplication of these row vectors by the given matrices. Notice that for $1 \leq i \leq 4$, the free $A$-module $L_{m,i}$ in the free resolution of $M_i$ does not depend upon $i$. We shall therefore write $L_m = L_{m,i}$ for all $m \geq 0$, $1 \leq i \leq 4$.

It is known that $\mathbf{M}$ is a family of simple holonomic left $A$-modules, so this family satisfies the finiteness condition $\text{FC}$. Therefore, there exists a pro-representing hull $(H, \xi)$ for the deformation functor $\text{Def}_A : \mathfrak{a}_4 \rightarrow \text{Sets}$ by theorem 5.2. We shall use the methods from section 7 to construct this hull explicitly.

Let us start by calculating $\text{Ext}^1_A(M_i, M_j)$ for $n = 1, 2$, $1 \leq i, j \leq 4$. We need both the dimensions and $k$-linear bases for these vector spaces, where each basis vector is represented by a cocycle in the corresponding Yoneda complex. The calculations are straight-forward, so we only state the results here:

\[
\dim_k \text{Ext}^1_A(M_i, M_j) = \begin{cases} 
1 & \text{if } i = 1 \text{ or } i = 4 \text{ and } j = 2 \text{ or } j = 3, \text{ or} \\
1 & \text{if } i = 2 \text{ or } i = 3 \text{ and } j = 1 \text{ or } j = 4, \\
0 & \text{otherwise}
\end{cases}
\]

\[
\dim_k \text{Ext}^2_A(M_i, M_j) = \begin{cases} 
1 & \text{if } (i, j) = (1, 4), (2, 3), (3, 2), (4, 1), \\
0 & \text{otherwise}
\end{cases}
\]

We denote the basis vectors of $\text{Ext}^1_A(M_j, M_i)$ by $x_{ij}^*$ since there is at most one for each pair of indices $(i, j)$. From the dimensions listed above, we see that we have the following basis vectors:

\[x_{12}^*, x_{13}^*, x_{21}^*, x_{24}^*, x_{23}^*, x_{31}^*, x_{34}^*, x_{42}^*, x_{43}^*\]

We choose a Yoneda representative for each vector $x_{ij}^*$, and we denote this representative by $\alpha(x_{ij})$. From the free resolutions above, we see that we can write each of these representatives in the form

\[\alpha(X) = \{\alpha(X)_0, \alpha(X)_1\},\]
where $\alpha(X)_0 : L_1 \to L_0$ is right multiplication by a matrix $(\alpha)^0$ with entries $a, b \in A$, and $\alpha(X)_1 : L_2 \to L_1$ is right multiplication by a matrix $(\alpha)^1$ with entries $c, d \in A$ for each monomial $X = x_{ij}$. We find the following representatives:

$$\alpha(x_{12}) = \alpha(x_{21}) = \alpha(x_{43}) = \{(\frac{q}{0}), (1, 0)\}
\alpha(x_{13}) = \alpha(x_{31}) = \alpha(x_{24}) = \{(\frac{0}{a}, 0, -1)\}$$

Similarly, we denote the basis vectors of $\text{Ext}^2_A(M_j, M_i)$ by $y^*_{ij}$ since there is at most one for each pair of indices $(i, j)$. From the dimensions listed above, we see that we have the following basis vectors:

$$y^*_{14}, y^*_{23}, y^*_{12}, y^*_{41}$$

We choose a Yoneda representative for each vector $y^*_{ij}$ in this list, and we denote this representative $\alpha(y_{ij})$. From the free resolutions above, we see that we can write each of these representatives in the form

$$\alpha(Y) = \{\alpha(Y)_0\},$$

where $\alpha(Y)_0 : L_2 \to L_0$ is given by right multiplication of an element $a \in A$ for each monomial $Y = y_{ij}$. We find the following representatives:

$$\alpha(y_{14}) = \alpha(y_{23}) = \alpha(x_{32}) = \alpha(x_{41}) = \{(1)\}$$

This completes the calculations of $\text{Ext}^n_A(M_j, M_i)$ for $n = 1, 2$ and $1 \leq i, j \leq 4$. We know that these calculations determine the hull at the tangent level, $(H_2, \xi_2)$.

The next step is to find the the hull $H$ and the versal family $\xi$, and we shall employ the notations and methods of section 7 to accomplish this. Let $N = 2$, we know that this choice is always possible. As usual, we let $T^1$ be the formal matrix algebra generated by the monomials $x_{ij}$ in the above list, and let $I = I(T^1)$ be its radical. Furthermore, denote by $f_{ij} = o(y_{ij}) \in I^2_{ij}$ for $(i, j) = (1, 4), (2, 3), (3, 2), (4, 1)$, and by $f^1_{ij}$ the corresponding truncated power series for each $n \geq N$.

First, we have to find a defining system $\{o(X) : |X| < 2\}$ for the matric Massey products $\langle \alpha^*; X \rangle$ when $X$ is any monomial of degree 2 in $T^1$. This is easily done: The 1-cocycle $o(e_i)$ is the free resolution of $M_i$ for $1 \leq i \leq 4$, and the 1-cocycle $\alpha(X)$ was chosen above for each monomial $X = x_{ij}$ of degree 1.

Let us calculate the matric Massey products of order 2: Using the defining system given above, we find that the cocycles $y(X)$ representing the matric Massey products $\langle \alpha^*; X \rangle$ are given by

$$y(X)_0 = \begin{cases} 
-1 & \text{if } X = x_{12}x_{24}, x_{21}x_{13}, x_{34}x_{42}, x_{43}x_{31}, \\
1 & \text{if } X = x_{13}x_{34}, x_{24}x_{43}, x_{31}x_{12}, x_{42}x_{21}, \\
0 & \text{otherwise}
\end{cases}$$

for all monomials $X$ of degree 2 in $T^1$. This means that the corresponding matric Massey products are given by

$$\langle x_{12}, x_{24} \rangle = -y_{14} \quad \langle x_{13}, x_{34} \rangle = y_{14}
\langle x_{21}, x_{13} \rangle = -y_{23} \quad \langle x_{24}, x_{43} \rangle = y_{23}
\langle x_{31}, x_{12} \rangle = y_{32} \quad \langle x_{34}, x_{42} \rangle = -y_{32}
\langle x_{42}, x_{21} \rangle = y_{41} \quad \langle x_{43}, x_{31} \rangle = -y_{41},$$

$$\langle x_{i1}, x_{i2} \rangle = 0 \quad \langle x_{i1}, x_{i3} \rangle = 0 \quad \langle x_{i1}, x_{i4} \rangle = 0 \quad \langle x_{i2}, x_{i3} \rangle = 0 \quad \langle x_{i2}, x_{i4} \rangle = 0 \quad \langle x_{i3}, x_{i4} \rangle = 0$$
and all other matri Massey products of order 2 are zero. This translates to the
following truncated power series $f_{ij}^2$:

\[
\begin{align*}
  f_{14}^2 &= x_{13}x_{34} - x_{12}x_{24} \\
  f_{23}^2 &= x_{24}x_{43} - x_{21}x_{13} \\
  f_{32}^2 &= x_{31}x_{12} - x_{34}x_{42} \\
  f_{41}^2 &= x_{42}x_{21} - x_{43}x_{31}
\end{align*}
\]

By the general theory, we therefore have $H_3 = T^1/(f_{14}^2, f_{23}^2, f_{32}^2, f_{41}^2) + I^3$. We know
that we can find a lifting $\xi_3$ of $\xi_2$ to $H_3$, and that $(H_3, \xi_3)$ is a pro-representing hull of $\text{Def}_M$ restricted to $a_4(3)$.

In order to find $\xi_3$, we let $B(2) = \{X : |X| = 2\} \setminus \{x_{13}x_{34}, x_{24}x_{43}, x_{31}x_{12}, x_{42}x_{21}\}$. We also let $\overline{B}(2) = B(2) \cup \overline{B}(1)$, where $\overline{B}(1) = \{X : |X| \leq 1\}$. Then $\{X : X \in \overline{B}(2)\}$ is a monomial basis for $H_3$. We observe that if we choose $\alpha(X) = 0$ for all $X \in B(2)$, the family $\{\alpha(X) : X \in \overline{B}(2)\}$ defines an $M$-free complex over $H_3$. In other words, this family completely defines the deformation $\xi_3 \in \text{Def}_M(H_3)$ lifting $\xi_2$.

Clearly, we could continue in this way. But after the last computations, it is tempting to think that $f_{ij} = f_{ij}^2$ for $(i, j) = (1, 4), (2, 3), (3, 2), (4, 1)$. Let us check if this is the case: We put $T = T^1/(f_{14}^2, f_{23}^2, f_{32}^2, f_{41}^2)$, and choose a monomial basis $B$ of $T$ containing $\overline{B}(2)$. Furthermore, we let $\alpha(X)$ be as before when $X \in \overline{B}(2)$ and let $\alpha(X) = 0$ for all monomials $X \in B$ of degree at least 3. This choice corresponds to maps $d_0^T$, $d_1^T$ of $M$-free modules over $T$, and a computation shows that

\[
d_0^T \circ d_1^T = (1 \otimes (f_{14}^2 + f_{23}^2 + f_{32}^2 + f_{41}^2)) = 0.
\]

So the family $\{\alpha(X) : X \in B\}$ defines an $M$-free complex over $T$, and therefore a deformation $\xi \in \text{Def}_M(T)$ lifting $\xi_3$. This proves that $H = T$, or in other words, that $H = T^1/(x_{13}x_{34} - x_{12}x_{24}, x_{24}x_{43} - x_{21}x_{13}, x_{31}x_{12} - x_{34}x_{42}, x_{42}x_{21} - x_{43}x_{31})$ is a pro-representing hull of $\text{Def}_M$. In particular, $f_{ij} = f_{ij}^2$ for all $i, j$. Moreover, the family $\{\alpha(X) : X \in B\}$ defines the versal family $\xi \in \text{Def}_M(H)$.

**Appendix A. Yoneda and Hochschild Representations**

Let $k$ be an algebraically closed (commutative) field, let $A$ be an associative $k$-algebra, and let $M, N$ be left $A$-modules. In this appendix, we recall several different descriptions of the $k$-vector space $\text{Ext}_A^n(M, N)$ for $n \geq 0$. In particular, we show how to realize this cohomology group using the Yoneda and Hochschild complexes.

**A.1. The Yoneda representation.** Fix free resolutions $(L_*, d_*)$ of $M$ and $(L'_*, d'_*)$ of $N$. We shall write $d_i : L_{i+1} \to L_i$ and $d'_i : L'_{i+1} \to L'_i$ for the differentials, and denote the augmentation morphisms by $\rho : L_0 \to M$ and $\rho' : L'_0 \to N$.

For all integers $n \geq 0$, the cohomology group $\text{Ext}_A^n(M, N)$ is defined to be the $n$’th cohomology group of the complex $\text{Hom}_A(L_*, N)$,

\[\text{Ext}_A^n(M, N) = H^n(\text{Hom}_A(L_*, N)).\]

Notice that in general, this Abelian group does not have a left $A$-module structure, but only a left $C(A)$-module structure, where $C(A)$ is the centre of $A$. In particular, if $A$ is commutative, then $\text{Ext}_A^n(M, N)$ has the structure of an $A$-module, and if $A$ is a $k$-algebra, then $\text{Ext}_A^n(M, N)$ has the structure of a $k$-vector space.
We denote by $\text{Hom}^*(L_*, L'_*)$ the Yoneda complex given by the given free resolutions. This complex is defined in the following way: For each integer $n \geq 0$, let $\text{Hom}^n(L_*, L'_*)$ be the left $A$-module $\text{Hom}^n(L_*, L'_*) = \Pi_i \text{Hom}_A(L_{i+n}, L'_i)$. Moreover, let the differential $d^n : \text{Hom}^n(L_*, L'_*) \to \text{Hom}^{n+1}(L_*, L'_*)$ for $n \geq 0$ be the $A$-linear map given by the formula

$$d^n(\phi)_i = \phi_id_{n+i} + (-1)^{n+1}d'_i \phi_{i+1}$$

for all $i \geq 0$, where we write $\phi = (\phi_i)$ with $\phi_i \in \text{Hom}_A(L_{i+n}, L'_i)$ for all $i \geq 0$. It is easy to check that this map is a well-defined differential, so the Yoneda complex is a complex of Abelian groups. We shall write $H^n(\text{Hom}(L_*, L'_*))$ for the cohomology groups of the Yoneda complex. Since the differential $d = d^n$ is left $C(A)$-linear, these cohomology groups have a natural structure as left $C(A)$-modules.

**Lemma A.1.** For all integers $n \geq 0$, there is a canonical isomorphism of left $C(A)$-modules

$$H^n(\text{Hom}(L_*, L'_*)) \cong \text{Ext}^n_A(M, N).$$

**Proof.** There is a natural map $f_n : \text{Hom}^n(L_*, L'_*) \to \text{Hom}_A(L_n, N)$, given by $f(\phi) = \phi'\phi_0$, where $\phi = (\phi_i) \in \text{Hom}^n(L_*, L'_*)$. It is easy to see that these maps are compatible with the differentials, and a small calculation show that $f_n$ induces an isomorphism on cohomology $H^n(\text{Hom}(L_*, L'_*)) \to \text{Ext}^n_A(M, N)$ for all integers $n \geq 0$. \hfill \Box

**A.2. Definition of Hochschild cohomology.** Let $Q$ be an $A$-$A$ bimodule. We define the Hochschild complex of $A$ with values in $Q$ in the following way: Let $HC^n(A, Q) = \text{Hom}_k(\bigotimes^n A, Q)$ for all $n \geq 0$. So any $\psi \in HC^n(A, Q)$ corresponds to a $k$-bilinear map from $n$ copies of $A$ into $Q$, and we shall therefore write $\psi(a_1, \ldots, a_n)$ in place of $\psi(a_1 \otimes \cdots \otimes a_n)$ for $\psi \in \text{HC}^n(A, Q), a_1, \ldots, a_n \in A$. Moreover, let $d^n : \text{HC}^n(A, Q) \to \text{HC}^{n+1}(A, Q)$ for $n \geq 0$ be the $k$-linear map given by the formula

$$d^n(\psi)(a_0, \ldots, a_n) = a_0 \psi(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i \psi(a_0, \ldots, a_{i-1}a_i, \ldots, a_n)$$

$$+ (-1)^{n+1} \psi(a_0, \ldots, a_{n-1})a_n$$

for all $\psi \in \text{HC}^n(A, Q), a_0, \ldots, a_n \in A$.

**Lemma A.2.** $HC^*(A, Q)$ is a complex of $k$-vector spaces.

**Proof.** Let $\psi \in \text{HC}^n(A, Q)$. Then $\psi'' = d^n(\psi)$ is a sum of $n + 1$ summands, and we denote these by $\psi_0', \ldots, \psi_n'$, in the order they appear in formula (7). We let $\psi'' = d^{n+1} \psi = d^{n+1}d^n \psi$. Each $d^{n+1} \psi'$ for $0 \leq i \leq n$ is a sum of $n + 2$ summands, and we denote these by $\psi_{ij}'$ for $0 \leq j \leq n + 1$ in the order they appear in formula (7). A straight-forward calculation shows that we have $\psi_i'' + \psi_{ij}' = 0$ for all indices $i, j$ with $0 \leq j \leq n + 2, j \leq i \leq n + 1$. Since $\psi'' = \sum \psi_{ij}'$, it follows that $\psi'' = 0$ in $HC^{n+2}(A, Q)$. Consequently, $HC^*(A, Q)$ is a complex of $k$-vector spaces. \hfill \Box

We define the Hochschild cohomology of $A$ with values in $Q$ to be the cohomology of the Hochschild complex $HC^*(A, Q)$, so we have

$$HH^n(A, Q) = H^n(\text{HC}^*(A, Q)) = \ker(d^n)/\text{Im}(d^{n-1})$$

for all $n \geq 0$. In particular, the cohomology groups $HH^n(A, Q)$ have a natural structure as $k$-vector spaces.
Let $\psi \in HC^1(A, Q)$, then $\psi$ is a 1-cocycle if and only if $\psi(ab) = a\psi(b) + \psi(a)b$ for all $a, b \in A$. So we have ker$(d^1) = \text{Der}_k(A, Q)$. We say that a derivation $\psi \in \text{Der}_k(A, Q)$ is trivial if there is an element $q \in Q$ such that $\psi$ is of the form $\psi(a) = aq - qa$ for all $a \in A$. Clearly, the set of trivial derivations is the image Im$(d^1)$. So $\text{HH}^1(A, Q) \cong \text{Der}_k(A, Q)/T$ where $T$ is the trivial derivations of $A$ into $Q$.

A.3. The Hochschild representation. We remark that $Q = \text{Hom}_k(M, N)$ is an $A$-$A$ bimodule in a natural way: For any $a \in A$, let $L_a : M \to M$ denote left multiplication on $M$ by $a$, and $L'_a : N \to N$ left multiplication on $N$ by $a$. The bimodule structure is given by $a\phi = L'_a\phi$, $\phi a = \phi L_a$ for $a \in A$. $\phi \in \text{Hom}_k(M, N)$. We shall consider the Hochschild cohomology of $A$ with values in $Q = \text{Hom}_k(M, N)$.

By definition, we have that $\text{HH}^0(A, Q) = \text{Hom}_A(M, N)$ when $Q = \text{Hom}_k(M, N)$. So we have a natural isomorphism of $k$-vector spaces $\text{Ext}^0_A(M, N) \cong \text{HH}^0(A, Q)$. Notice that since $k \subseteq C(A)$, $\text{Ext}^n_A(M, N)$ has a natural $k$-vector space structure for all $n \geq 0$. It is possible to extend the above isomorphism to the higher cohomology groups:

**Proposition A.3.** For all integers $n \geq 0$, there is an isomorphism of $k$-vector spaces

$$\sigma_n : \text{Ext}^n_A(M, N) \to \text{HH}^n(A, \text{Hom}_k(M, N)).$$

**Proof.** From Weibel [9], lemma 9.1.9, there is an isomorphism of $k$-vector spaces between $\text{HH}^n(A, \text{Hom}_k(M, N))$ and $\text{Ext}^n_A(M, N)$ for $n \geq 0$. But since $k$ is a commutative field, there is a canonical isomorphism between $\text{Ext}^n_A(M, N)$ and $\text{Ext}^n_A(M, N)$, see theorem 8.7.10 in Weibel [9].

We shall give an explicit identification of $k$-vector spaces between $\text{Ext}^1_A(M, N)$ and $\text{HH}^1(A, \text{Hom}_k(M, N))$: Let $(L_\ast, d_\ast)$ be a free resolution of $M$, with augmentation morphism $\rho : L_0 \to M$, and let $\tau : M \to L_0$ be a $k$-linear section of $\rho$. For any 1-cocycle $\phi \in \text{Hom}_A(L_1, N)$, let $\psi = \psi(\phi) \in \text{Der}_k(A, \text{Hom}_k(M, N))$ be the following derivation: For any $a \in A$, $m \in M$, let $x = x(a, m) \in L_1$ be such that $d_0(x) = \tau(m) - \tau(am)$. Notice that such an $x$ exists, and is uniquely defined modulo the image Im$d_1$. We define $\psi$ by the equation $\psi(a)(m) = \phi(x)$ with $x = x(a, m)$. Since $\phi$ is a cocycle, $\psi$ is a well-defined homomorphism in $\text{Hom}_k(A, \text{Hom}_k(M, N))$, and a straight-forward calculation shows that $\psi$ is a derivation.

**Lemma A.4.** Assume that $\text{Ext}^1_A(M, N)$ is a finite dimensional $k$-vector space. Then the assignment $\phi \mapsto \psi(\phi)$ defined in the above paragraph induces an isomorphism $\sigma_1 : \text{Ext}^1_A(M, N) \to \text{HH}^1(A, \text{Hom}_k(M, N))$.

**Proof.** Assume that $\phi$ is a co-boundary, so $\phi = d^0(\phi')$, where $\phi' \in \text{Hom}_A(L_0, N)$. Then $\psi = d^0(\phi')$, where $\psi' = \phi'y \in \text{Hom}_k(M, N)$, so $\phi$ is a derivation. Consequently, the assignment induces a well-defined map of $k$-linear spaces. This map is furthermore injective: Assume that $\psi$ is a trivial derivation, so $\psi = d^0(\psi')$, where $\psi' \in \text{Hom}_k(M, N)$. Then, we can construct an $A$-linear map $\phi' \in \text{Hom}_A(L_0, N)$ in the following way: Choose a basis for $L_0$, and for each basis vector $y \in L_0$, choose $\psi(y) \in L_1$ such that $d_0(y') = -\psi'\rho(y)$. Then we define $\phi'(y) = \psi'(\rho(y) + \phi'(y))$ for each basis vector $y \in L_0$. We obtain a morphism $\phi' \in \text{Hom}_A(L_0, N)$ by $A$-linear extension, and $d^0(\phi') = \phi$, so $\phi$ is a co-boundary. To show that $\sigma_1$ is an isomorphism as well, it is enough to notice that $\dim \text{Ext}^1_A(M, N) = \dim_k \text{HH}^1(A, \text{Hom}_k(M, N))$ by proposition A.3 since $\text{Ext}^1_A(M, N)$ has finite $k$-dimension.

The identification $\sigma_n : \text{Ext}^n_A(M, N) \to \text{HH}^n(A, \text{Hom}_k(M, N))$ for $n \geq 2$ can be constructed in a similar way.
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