ON THE HOMOTOPY CLASSIFICATION OF SPACES BY THE
FIXED LOOP SPACE HOMOLOGY

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Abstract. Let \( R \subseteq \mathbb{Q} \) be a subring of the rationals and let \( p \) be the least prime (if none, \( p = \infty \)) which is not invertible in \( R \). For an \( R \)-local \( r \)-connected CW-complex \( X \) of dimension \( \leq \min(r+2p-3,rp-1), r \geq 1 \), a complete homotopy invariant is constructed in terms of the loop space homology \( H_*(\Omega X) \). This allows us to classify all such \( R \)-local spaces up to homotopy with a fixed loop space homology.

1. Introduction

The Pontrjagin (Hopf) algebra \( H = H_*(\Omega X) \), where \( \Omega X \) is the (based) loop space on the \( X \), is not a complete homotopy invariant of a given space \( X \) even in the rational category. For instance, the 2-dimensional complex projective space \( \mathbb{C}P^2 \) and the direct product \( K(\mathbb{Q},2) \times S^5 \) of the rational Eilenberg-MacLane space and the 5-dimensional sphere have isomorphic loop space homologies as Hopf algebras, but distinct rational homotopy types [11]. So the problem is how to introduce an additional structure on the loop space homology which would be a complete homotopy invariant of a space and then to classify all homotopy types with a given Pontrjagin algebra. In the rational homotopy theory such a classification is obtained in [9].

Here we continue the classification beyond the rational spaces. An algebraic formalism for this is a modification of the methods developed in [1], [3], [5],[10], [9] and is done by the following steps:

1. To give the homotopy classification of differential Hopf algebras up to homotopy by a fixed homology Hopf algebra;
2. To establish a connection with the theory of Hopf algebras up to homotopy [1], [2];
3. To give the homotopy classification of spaces with fixed Pontrjagin (Hopf) algebra of their loop spaces, in particular, in the rational homotopy theory.

In this way, motivated by Berikashvili’s work [3], where a perturbation theory on the additive level was developed and phrased in terms of what he called the functor \( D \), and by Halperin-Stasheff’s one [5], we have constructed a multiplicative variant of the functor \( D \) in [9]. However, the classification problem beyond rational spaces requires to perturb a coproduct simultaneously with a differential on a multiplicative resolution for a given Hopf algebra \( H \). This is conceptually a new fact leading to the set \( D_H \), the set of Hopf perturbations, which just provides an additional structure on \( H \) discussed above.

In particular, for each space \( X \) with \( H = H_*(\Omega X; R) \) and satisfying the hypotheses of Theorem 6.1 there is the element \( d[X] \in D_H \) determining its \( R \)-local homotopy type. It should be emphasized that the use of multiplicative resolutions for Hopf algebras with not necessarily coassociative and cocommutative coproduct,
we basically exploit here, has a sense in the rational case too, since it in fact avoids the Milnor-Moore theorem, and, hence, an information about the homotopy groups of spaces under consideration.

It is a pleasure to dedicate this article to 70’s birthday of Professor Nodar Berikashvili.

2. Preliminaries

Let $R$ be a commutative ring with 1. A differential graded algebra (dga) is a graded $R$-module $C = \{C_i\}, i \in \mathbb{Z}$, with an associative multiplication $\omega: C_i \otimes C_j \to C_{i+j}$ and a homomorphism $d: C_i \to C_{i+1}$ such that

$$d^2 = 0, \quad d(xy) = d(x)y + (-1)^{|x|}xd(y),$$

where $xy \in C_{i+j}$ is the element $\omega(x \otimes y), x \in C_i, y \in C_j, |x| = i$. We assume that $\omega(x \otimes y)$ is a dga $C$ contains a unit $1 \in C_0$. A dga $C$ is called commutative (cdga) if $\omega = \omega T$, where $T(x \otimes y) = (-1)^{|x||y|}y \otimes x$. A non-negatively graded dga $C$ is called connected if $C_0 = R$. A derivation of degree $i$ on a dga $C$ is a homomorphism $\theta: C_n \to C_{n+i}$ such that $\theta(xy) = \theta(x)y + (-1)^{|x|}x\theta(y)$ or $\theta \omega = \omega(\theta \otimes 1 + 1 \otimes \theta)$, where the signs appear in the definition of $f \otimes g$ for the graded maps $f$ and $g$ according to the rule

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|}f(x) \otimes g(y).$$

The set of all derivations on $C$ is a sub-dg space of the dga of all homomorphisms $\text{Hom}(C, C)$ and is denoted by $\text{Der} C$.

A differential graded coalgebra is a graded $R$-module $C = \{C_i\}$ with an associative comultiplication $\Delta: C \to C \otimes C$ and differential $\partial: C_i \to C_{i-1}$ such that $\Delta \partial = (\partial \otimes 1 + 1 \otimes \partial)\Delta$. A coalgebra $C$ is assumed to have counit $\epsilon: C \to R$, $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta = 1$. $C$ is said to be cocommutative if $\Delta = T\Delta$. A homomorphism $\theta$ on $C$ is called a coderivation if $\Delta \theta = (\theta \otimes 1 + 1 \otimes \theta)\Delta$. The set of all coderivations on $C$ is a sub-dg space of the dg module of all homomorphisms $\text{Hom}(C, C)$ and is denoted by $\text{Coder} C$.

A connected differential graded Hopf algebra is a connected dga $C$ together with a coalgebra structure such that $\Delta: C \to C \otimes C$ is a map of dga’s.

A homomorphism on $C$ is a Hopf derivation if it is a derivation and coderivation at the same time.

A differential graded Lie algebra is a $R$-module $L = \{L_i\}$ with a multiplication $\omega: L \otimes L \to L$ and a differential $\partial: L_i \to L_{i-1}$ such that $\omega = -\omega T, \omega(1 \otimes \omega) = \omega(\omega \otimes 1) + \omega(1 \otimes \omega)(T \otimes 1)$ and $\partial \omega = \omega(\partial \otimes 1 + 1 \otimes \partial)$.

The set of primitive elements $PC$ of a dg Hopf algebra $C$, i.e.

$$PC = \{x \in C | \Delta x = x \otimes 1 + 1 \otimes x\},$$

is a dg Lie algebra via the Lie bracket.

A multiplicative resolution, $(R, H_m, d)$, of a graded algebra $H_m$ is a bigraded tensor algebra $TV$ generated by a free bigraded $R$-module $V = \bigoplus_{j,m} V_{j,m}$, $j \leq 0, V_{j,m} \subset R_i H_m$, $m \in \mathbb{Z}$, where $d$ is of bidegree (-1,0), together with a map of bigraded algebras $(RH, d)$ inducing an isomorphism $H(RH, d) \xrightarrow{\approx} H$.

A map of dga’s $f: A \to B$ is a weak equivalence or a (co)homology isomorphism, if it induces an isomorphism on (co)homology.

Two (c)dga’s $A$ and $B$ are weak homotopically equivalent, if there is a (c)dga $C$ with weak equivalences $A \leftarrow C \to B$.

Two maps of dga’s $f, g: A \to B$ are ($g, f$)– derivation homotopic, if there exists a map $s: A \to B$ of degree 1 such that $sd_A + d_B s = f - g$ and $sw_A = w_B (f \otimes s + s \otimes g)$.

A Hopf algebra up to homotopy (Hah) is a dga with coproduct being compatible with the product and coassociative and cocommutative up to derivation homotopy.
A Hopf resolution of $H$ is a Hah $(RH,d,\Delta)$ with a coproduct $\Delta : R_qH \to \oplus_{i+j=q} R_iH \otimes R_jH$, $q \geq 0$, and a weak equivalence of dga’s
\[ \rho : (RH,d,\Delta) \to H \]
which preserves coproducts.

3. Multi algebras

A dga $(A,\cdot,d)$ is multi algebra if it is bigraded $A_n = \oplus_{i+j=n} A_{i,j}$, $i,j \in \mathbb{Z}$, and $d = d_0 + d_1 + \cdots + d_n + \cdots$, $d_n : A_{p,q} \to A_{p-n,q+n-1}$ [7]. A multi algebra $A$ is homological one if it is free as $R$-module, $d_0 = 0$ and $H_i(A,1) = 0, i > 0$. In this case, the sum $d_2 + \cdots + d_n + \cdots$ is called as a perturbation of the differential $d_1$. A multi algebra is free if it is a tensor algebra over a free bigraded $R$-module.

A multialgebra morphism $f : A \to B$ is a dga map which preserves the column filtration, so that $f$ has the form $f = f_0 + \cdots + f_i + \cdots$, $f_i : A_{p,q} \to B_{p-i,q+i}$. A homotopy between two such morphisms is a derivation homotopy which raises the column filtration by 1.

The useful property of a free multi algebra is presented by the following

**Proposition 3.1.** If $f : A \to B$ is a weak equivalence of dga’s, then

(i) For a free multi algebra $C$, there is a bijection on the sets of homotopy classes of dga maps
\[ f_* : [C,A] \cong [C,B], \]

(ii) If $A$ (or $B$) is a homological multi algebra, then, in addition to (i), $[C,A]$ or $[C,B]$ means the set of homotopy classes of multi algebra morphisms.

**Proof.** The proof goes by induction on column grading using the standard Adams-Hilton argument (see Theorem 3.4 in [8] and Theorem 2.4 in [7]).

**Lemma 3.1.** Let $g' : A_{i,*} \to B_{i,*}$ be a multi algebra morphism preserving column grading where $A$ is free. Let $f : A \to B$ be any multi algebra morphism with $f_0 \simeq g'$. Then there is a multi algebra morphism $g : A \to B$ such that $g_0 = g'$ and $g \simeq f$.

**Proof.** Let $s_0$ be a derivation homotopy between $f_0$ and $g_0$. By the standard way we define a multi algebra morphism $g$ with $g \simeq_s f$ where $s|_V = s_0|_V$. By the given data $g$ is uniquely determined ($V$ denotes multiplicative generators of $A$; cf. Lemma 3.3.7 in [8], Lemma 2.3 in [1]). Since $s_0$ raises the column filtration by 1, it is not hard to see that $g_0 = g'$.

4. The set $D_H$ for differential Hopf algebras

Let $\text{HAH}$ denote the category of Hopf algebras up to homotopy (Hah’s) over a fixed commutative ring $R$ with 1 as in [1]. Let $H$ be a graded Hopf algebra (gha) and let
\[ \rho : (RH,d) \to H \]
be its multiplicative (algebra) resolution. By the standard way (using the Adams-Hilton theorem mentioned above) one can introduce on $RH$ a coproduct $\Delta : R_qH \to \oplus_{i+j=q} R_iH \otimes R_jH$, $q \geq 0$, preserving the resolution degree such that
\[ \rho : (RH,d,\Delta) \to H \]
is a morphism of $\text{HAH}$. So that we get a Hopf resolution of $H$ which we fix henceforth.

Recall that a perturbation of $(RH,d)$ is a derivation $h$ on $RH$ of total degree -1 such that $h : R_qH \to \oplus_{i \leq q-2} R_iH$ and $d_h^2 = 0$, $d_h = d + h$. We will refer to an $R$-linear map
\[ \nu : R_qH \to \oplus_{i+j} R_iH \otimes R_jH \]

of total degree 0 as a *perturbation* of \((RH, \Delta)\) if \(i + j \leq q - 1\) and \(\Delta_\nu = \Delta + \nu\) is an algebra map, that is, it belongs to the submodule

\[
\text{Copr} \ RH \subset \text{Hom}^0(RH, RH \otimes RH).
\]

We will refer to a pair \((h, \nu)\) as a *perturbation* of the triple \((RH, d, \Delta)\) if \(h\) and \(\nu\) are perturbations of \((RH, d)\) and \((RH, \Delta)\) respectively and \((RH, h, \nu, \Delta)\) is an object of \(\text{HAH}\).

In the sequel we assume for simplicity that a graded Hopf algebra \(H\) is \(R\)-torsion free.

Let denote differentials in \(\text{Der} RH\) and in \(\text{Hom}(RH, RH)\) by the same symbol \(\nabla\).

Then define the set \(M_H\) and the group \(G_H\) as

\[
M_H = \{(h, \nu), h \in \text{Der}^1 RH, \nu \in \text{Hom}^0(RH, RH \otimes RH) | \nabla(h) = -hh, \quad h = h_{2,-1} + h_{3,-2} + \cdots, \quad h_{r+1,-r} \in \text{Der}^{r+1,-r} RH, \nu = \nu_{1,-1} + \nu_{2,-2} + \cdots, \quad \nu_{r,-r} \in \text{Hom}^{r,-r}(RH, RH \otimes RH), \quad \Delta_\nu \in \text{Copr} RH, \quad d_h \in \text{Coder}(RH, \Delta_\nu)\},
\]

\[
G_H = \{(p, s), p \in \text{Aut} RH, s \in \text{Hom}^1(RH, RH \otimes RH) | p = 1 + p_{1,-1} + p_{2,-2} + \cdots, \quad p_{r,-r} \in \text{Hom}^{r,-r}(RH, RH)\}.
\]

The group structure on \(G_H\) is defined by \((p, s)(p', s') = (pp', (p' \otimes p)sp' + s').\)

Then we define the action \(M_H \times G_H \to M_H\) by

\[
(h, \nu) \ast (p, s) = (\tilde{h}, \tilde{\nu}),
\]

in which

\[
\tilde{h} = p^{-1}hp + p^{-1}\nabla(p)
\]

\[
\tilde{\nu} = (p \otimes p)spp^{-1} + sd_h + (d_h \otimes 1 + 1 \otimes d_h)s.
\]

Note that by the definition of \(\tilde{\nu}\) the chain homotopy \(s\) becomes a derivation homotopy in a standard way (cf. Lemma 3.3.7 in [8]).

5. Homotopy classification of differential Hopf algebras

Here we state and prove our main theorem about the homotopy classification of Hopf algebras up to homotopy with fixed homology algebra. We have the following

**Theorem 5.1.** Let \(H\) be a graded Hopf algebra and let \(p : (RH, d, \Delta) \to H\) be its Hopf resolution. Let \((A, d, \psi)\) be an object of the category \(\text{HAH}\) with \(i_A : H \approx H(A, d).\) Then

Existence. There exists a triple \((h, \nu, k)\), where a pair \((h, \nu)\) is a perturbation of \((RH, d, \Delta)\) and

\[
k : (RH, h, \nu, \Delta) \to (A, d, \psi)
\]

is a morphism of \(\text{HAH}\) inducing an isomorphism in homology such that \(k|_{R_0H}\) induces the composition \(i_A \circ \rho|_{R_0H}\).

Uniqueness. If there exits another triple \((\tilde{h}, \tilde{\nu}, \tilde{k})\) satisfying the above conditions, then there is an isomorphism of \(\text{HAH}\)

\[
p : (RH, h, \nu, \Delta) \to (RH, \tilde{h}, \tilde{\nu})
\]

such that \(p\) has the form \(p = 1 + p',\) where \(p'\) lowers the resolution degree at least 1, and \(k\) is homotopic to \(\tilde{k} \circ p.\)

**Proof.** Existence. First we define a perturbation \(h\) and a dga map (weak equivalence) \(k : (RH, h) \to (A, d)\) by induction similarly to [5], [9]. Since \(R_0H\) is a free algebra we can define a dga map \(k_0 : R_0H \to (A, d)\) inducing on homology \(i_A \circ \rho|_{R_0H}\). Then there is \(k_1 : V_{1,*} \to A_{*+1}\) with \(k_0 \circ d_R = d_A \circ k_1,\) where \(V_{*,*}\) denotes
bigraded generators of $R_sH_*$. Extend the restriction of $k_0+k_1$ to $V_{(1),*} = V_{0,*} + V_{1,*}$ to obtain a dga map

$$k_{(1)} : (R_{(1)}H_*, d_R) \to (A, d).$$

Suppose we have constructed a pair $(h_{(n)}, k_{(n)})$, $h_{(n)} = h_2 + \cdots + h_n$ is a derivation on $R_nH_*$, $k_{(n)}|_{V_{(n),*}} = (k_0 + \cdots + k_n)|_{V_{(n),*}}$, and $k_{(n)} : R_nH \to A$ is multiplicative, such that

$$d_Rh_n + h_nd_R + \sum_{i+j=n+1} h_i h_j = 0$$

on $R_{(n+1)}H_*$ and

$$k_{(n)}(d_R + h_{(n)}) = d_A k_{(n)}$$
on $R_{(n)}H_*$. Now consider $k_{(n)}(d_R + h_{(n)})|_{V_{n+1,*}} : V_{n+1,*} \to A_{*+n+1}$. Clearly, $d_A k_{(n)}(d_R + h_{(n)}) = 0$. Define a derivation

$$h_{n+1} : R_{(n+1)}H_* \to R_0H_{*+n}$$
with $p h_{n+1} = i_A^{-1}[k_{(n)}(d_R + h_{(n)})]$. Then extend $h_{n+1}$ on $R_{(1)}H_*$ as a derivation (denoted by the same symbol) by

$$d_Rh_{n+1} + h_{n+1}d_R + \sum_{i+j=n+2} h_i h_j = 0$$
on $R_{(n+2)}H_*$. Hence, there is $k_{n+1} : V_{n+1,*} \to A_{*+n+1}$ with

$$k_{(n)}(d_R + h_{(n+1)}) = d_A k_{n+1}$$on $V_{(n+1),*}$. Extend the restriction of $k_{(n)} + k_{n+1}$ to $V_{(n+1),*}$ to obtain a multiplicative map

$$k_{(n+1)} : R_{(n+1)}H \to A.$$Thus, the construction of the pair $(h_{(n+1)}, k_{(n+1)})$ completes the inductive step and, consequently, one obtains a pair $(h, k)$,

$$h = h_2 + \cdots + h_n + \cdots,$$k|_V = (k_0 + \cdots + k_n + \cdots)|_V.

To construct $\nu$ we consider the weak equivalence $k \otimes k : RH \otimes RH \to C \otimes C$ and the composition $RH \xrightarrow{k} C \xrightarrow{\psi} C \otimes C$. Then by Proposition 3.1 there is $f : RH \to RH \otimes RH$ with $(k \otimes k) \circ f \simeq \psi \circ k$. Obviously, we have that $f_0 \simeq \Delta$. Using Lemma 3.1 we find $g$ with $g \simeq f$ and $g_0 = \Delta$. Put $\nu = g - \Delta$. Then a triple $(h, \nu, k)$ is as desired.

Uniqueness. Using Proposition 3.1 and Lemma 3.1 we find a multi algebra morphism

$$p : (RH, d_h) \to (RH, d_{\bar{k}})$$with $\bar{k} \circ p \simeq k$ and $p_0 = \text{Id}$. Automatically we have that $(p \otimes p) \circ \Delta^e \simeq \Delta^e \circ p$. So that $p$ is a morphism in $\mathbf{HAH}$. \hfill $\Box$

This Theorem allows us to classify Hopf algebras up to weak homotopy equivalences similarly to [9]. Namely, let $\Omega H$ denote the set of weak homotopy types of Hah’s with homology isomorphic to $H$. We have that $\text{Aut } H$ canonically acts on the set $D_H$ and let $D_H / \text{Aut } H$ be the orbit set.

Then we obtain the following main classification theorem about Hopf algebras.

**Theorem 5.2.** There is a bijection on sets

$$\Omega_H \approx D_H / \text{Aut } H.$$

This page seems to be a continuation of a larger document discussing a specific mathematical topic, possibly in the field of algebra or topology, given the use of terms and concepts like Hopf algebras and derived algebraic geometry.
Proof. Let $A$ be from $\text{HAH}$ with $H(A) \approx H$. By Theorem 5.1 we assign to $A$ the class $d[A] \in D_H$ of a perturbation pair $(h, \nu)$. If $A$ is weak equivalent to $B$, then by using Proposition 3.1 and Theorem 5.1 we conclude that this class is the same one for $B$, too. So, we have a well defined map

$$\Omega_H \to D_H / \text{Aut} \, H.$$ 

On the other hand, there is an obvious map

$$\Omega_H \leftarrow D_H / \text{Aut} \, H$$

which corresponds to an element $d \in D_H / \text{Aut} \, H$ the class of a Hopf multialgebra $(RH, d_h, \Delta_h)$, where $(h, \nu)$ is a representative of the $d$. Clearly, these maps are converse to each other. \hfill \Box

Let $\text{HAH}_r^{\text{p}}$ denote the full subcategory of $\text{HAH}$ whose objects are $r - 1$-connected Hopf algebras up to homotopy having multiplicative generators in the range of dimensions $r$ through $n$, inclusive. Let $\Omega^{r,n}_H$ denote the set of homotopy types of Hah’s in $\text{HAH}_r^{\text{p}}$ with homology isomorphic to $H$.

**Corollary 5.1.** Let $H$ have a Hopf resolution which belongs to $\text{HAH}_r^{\text{p}}$. Then there is a bijection on the sets

$$\Omega^{r,n}_H \approx D_H / \text{Aut} \, H.$$

A natural question arises: when is a perturbation $\nu$ of $(RH, \Delta)$ zero in Theorem 5.1? It appears that the case Anick considers [1]-the category of $r$-mild Hopf algebras up to homotopy- is a special one of this question. More precisely, for an $r$-mild Hopf algebra $(A, d, \psi)$ (i.e. $A$ belongs to $\text{HAH}_{r}^{\text{p}-1}$, $p$ is the smallest non-invertible prime in $R \subseteq \mathbb{Q}$), there is a dg Lie algebra $L_A$ and an isomorphism $A \approx UL_A$ in $\text{HAH}$. The homology $H$ of $(A, d)$ has the same form $H \approx UL_H$. Now we can take a Hopf resolution $(RH, d, \Delta)$ of $H$ also having the form $RH = UL_R$ with the canonical Hopf algebra structure, where $d$ is a derivation and a coderivation at the same tame. Analogously to [5] one can find a perturbation $h$ of $(RH, d)$ and a weak equivalence

$$(RH, d_h, \Delta) \to (A, d, \psi).$$

In other words, when a special Hopf resolution for $H$ is taken, then any pair $(h, \nu)$ is equivalent to that of the form $(h', 0)$.

6. **Homotopy classification of spaces**

Let now $\text{CW}_r^{\text{m}}$ denote the category of $r$-connected CW-complexes of dimension $\leq m$ with trivial $(r - 1)$-skeleton and pointed CW-maps between them, and let $\text{CW}_r^{\text{m}}(R)$ be its $R$-local category where $m = \min(r + 2p - 3, rp - 1)$, $r \geq 1$, $p$ is the smallest non-invertible prime in $R \subseteq \mathbb{Q}$ (if none, $p = \infty$).

From [2],[6] we have that the homotopy category of $\text{CW}_r^{\text{m}}(R)$ is equivalent to the homotopy one of $\text{HAH}_r^{\text{p}-1}$.

Recall that (see, for example [4]) that a space $X$ is called $R$-coformal if $C_r(\Omega X; R)$ is weak equivalent to $H_r(C_r(\Omega X; R))$. We have the following main classification theorem about topological spaces.

**Theorem 6.1.** Let $\Omega^{r,m}_H(R)$ denote the set of the homotopy types of spaces $X$ from $\text{CW}_r^{\text{m}}(R)$ with an isomorphism $H \approx H_r(\Omega X)$ (assuming $H$ is $R$-torsion free). If there exists an element in $\Omega^{r,m}_H(R)$ corresponding to an $R$-coformal space from $\text{CW}_r^{\text{m}}$, then there is a bijection on the sets

$$\Omega^{r,m}_H(R) \approx D_H / \text{Aut} \, H.$$
Proof: We have that the Pontrjagin algebra \( H_*(\Omega Y; R) \) of an \( R \)-coformal space \( Y \) from \( CW_r^m \) has a Hopf resolution which belongs to \( HAHR^{m-1} \). Indeed, \( Y \) has a minimal Adams-Hilton model \( A_Y \) with a weak equivalence \( A_Y \rightarrow H(\Omega Y; R) \) (cf. \( [4] \)). On the other hand, one can easily construct a Hopf resolution \( RH(\Omega Y; R) \) for \( H_*(\Omega Y; R) \) which is a minimal model in the above sense. These two models do not need to be isomorphic, in general, but have isomorphic multiplicative generators \( [4] \). The generators of the Adams-Hilton model are concentrated in the range of dimensions \( r \) through \( m - 1 \), so that \( RH(\Omega Y; R) \) belongs to \( HAHR^{m-1} \). Then we have a bijection on the sets (in view of the above equivalence between the homotopy categories of \( CW_r^m(R) \) and \( HAHR^{m-1} \))

\[
\Omega_H^{r,m}(R) \approx \Omega_H^{r,m}.
\]

Hence, the corollary yields the desired bijection. \( \square \)

Note that to prove this theorem for any \( H_* \) or \( H_* \approx H_*(\Omega X) \), some \( X \) in \( CW_r^m(R) \), we must answer the following problems:

Problem 1. Under what conditions \( H_* \) has a Hopf resolution which belongs to \( HAHR^{m-1} \) and is realized as the Pontrjagin algebra for some space in \( CW_r^m(R) \)?

Problem 2. Is there an \( R \)-coformal space \( Y \) with \( H_*(\Omega Y) \approx H_* \approx H_*(\Omega X) \)?

Since \( D_H \) does not depend on resolutions used, from Theorem 6.1 we see that for calculating, for instance, rational homotopy types with a fixed Pontrjagin algebra \( H \approx H_*(\Omega X) \) it is enough to take an arbitrary Hopf resolution for \( H \) (which does not need a representation \( H = UL_H \)) and then to classify all perturbations \( (h, \nu) \) on it up to isomorphisms.

For a \( \mathbb{Q} \)-coformal space \( X \), such resolution \( RH \) lies in fact in the cobar construction \( \Omega A_X \), where \( A_X \) is any chain coalgebra model for \( X \) (commutative or not), for example, the dual of the Halperin-Stasheff filtered model of \( X \). Indeed, since \( \Omega A_X \) is a free algebra, one can choose an algebra map \( RH \rightarrow \Omega A_X \) to be monomorphism (therefore, \( RH \) can be identified with its image). Then the restriction of the canonical commutative coproduct to \( RH \) is homotopic (by the Adams-Hilton argument) to some coproduct on \( RH \) to obtain a Hopf resolution of \( H \).

Let us consider an example to show the existence of non-zero perturbations of coproducts under consideration.

Let

\[
X = K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times K(\mathbb{Q}, 5) \times K(\mathbb{Q}, 11),
\]

the product of Eilenberg-MacLane spaces. Then for \( H_* = H_*(\Omega X) \), we have

\[
(RH, d, \Delta) = (T(V), d, \Delta), \; V = \oplus V_{i,j},
\]

\[
x_0 \in V_{0,1}, \; y_0 \in V_{0,3}, \; z_0 \in V_{0,4}, \; w_0 \in V_{0,10},
\]

\[
x_1 \in V_{1,3}, \; y_1 \in V_{1,7},
\]

\[
x_2 \in V_{2,5}, \; y_2 \in V_{2,11}, \ldots
\]

\[
0 = dx_0 = dy_0 = dz_0 = dw_0,
\]

\[
dx_1 = x_0 x_0,
\]

\[
dy_1 = y_0 y_0,
\]

\[
\Delta x_0 = x_0 \otimes 1 + 1 \otimes x_0,
\]

\[
\Delta y_0 = y_0 \otimes 1 + 1 \otimes y_0,
\]

\[
\Delta z_0 = z_0 \otimes 1 + 1 \otimes z_0,
\]

\[
\Delta w_0 = w_0 \otimes 1 + 1 \otimes w_0 + y_0 z_0 \otimes y_0 + y_0 \otimes y_0 z_0, \ldots
\]

The possibility for a perturbation \( h \) of the differential \( d \) to be (homologically) non-zero is: \( h(x_2) = z_0 \) or \( h(w_2) = w_0 \), but the last case requires a perturbation of the coproduct defined by \( \nu(y_1) = y_0 z_0 \), so that a pair \( (h, \nu) \) is a non-zero perturbation.

\[
\nu(1) = y_0 z_0
\]

\[
\nu(0) = 0
\]
Then we obtain by using an obstruction theory that there are 4 rational homotopy types with the homology $H$.

**References**

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