A ŠVARC-MILNOR LEMMA FOR MONOIDS ACTING BY ISOMETRIC EMBEDDINGS

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Abstract. We continue our programme of extending key techniques from geometric group theory to semigroup theory, by studying monoids acting by isometric embeddings on spaces equipped with asymmetric, partially-defined distance functions. The canonical example of such an action is a cancellative monoid acting by translation on its Cayley graph. Our main result is an extension of the Švarc-Milnor Lemma to this setting.

1. Introduction

Over the past few decades, combinatorial group theory has been increasingly influenced by its connections with geometry. The resulting subject of geometric group theory (see for example \cite{1, 2, 4}) is based on two principles. The first is that the use of diagrams associated to a group can be a valuable aid to combinatorial reasoning with generators and relations. The second is that a group can be understood by studying how it acts, in a suitably controlled way, upon a metric space. Arguably the greatest power for understanding groups, though, comes from a synthesis of these ideas, in which diagrams (and, by extension, groups themselves) are endowed with a metric structure, and the groups made to act upon the resulting spaces.

It is very natural to ask whether these geometric methods are particular to groups, or if they have any potential for wider application. Finitely generated semigroups and monoids are probably the most obvious candidates for such a generalization, and indeed a number of authors have employed geometric ideas in semigroup theory. However, while some success has been enjoyed with “diagrammatic” techniques (see for example \cite{6, 8, 9}), the same cannot yet be said for truly geometric methods, and there is as yet no coherent subject of geometric semigroup theory studying semigroups of isometries on metric spaces. In fact, we contend that there are two fundamental obstructions to the existence of such a subject.

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The first concerns the way in which monoids act: a monoid acting faithfully by bijections is necessarily group-embeddable, and monoids in general are not group-embeddable. Hence, to study more general monoids through their actions it is necessary to consider actions by more general functions than just permutations. In the case of actions on spaces with distance functions, this means that it is not sufficient to consider actions by isometries. The second fundamental obstruction concerns the nature of the spaces acted upon. In the group case, the essential philosophy (made precise in the Švarc-Milnor Lemma [7, 10]) is that a group acting in a suitably controlled way upon any metric space must resemble (more precisely, be quasi-isometric to) the space, and so properties of the group can be read off from properties of the space and vice versa. In a monoid or semigroup, by contrast, distance is neither symmetric (since there are no inverses) nor everywhere defined (since there may be ideals). Hence, there is no hope that a general monoid will “resemble” a metric space. Instead, the development of a true semigroup-theoretic analogue of geometric group theory will require the study of spaces which have the flexibility to resemble monoids, that is, spaces with asymmetric, partially-defined distance functions.

This paper forms part of an ongoing programme of research, in which we seek to transfer techniques of geometric group theory to semigroup theory, by replacing metrics with what we call semimetrics\(^3\). In [5] we initiated this study, by introducing a natural notion of quasi-isometry for these spaces, and proving an analogue of the Švarc-Milnor lemma for groups acting on them by isometries. We also presented some applications in semigroup theory, arising from the action of a Schützenberger group on the corresponding Schützenberger graph. Here, we begin to extend this theory to cover actions of monoids on semimetric spaces. Specifically, we consider monoids acting by isometric embeddings on semimetric spaces. While still not general enough to permit the study of monoids in absolute generality, this setting does encompass many more monoids than just groups, including most notably all cancellative monoids. In future research we shall explore the extent to which these methods can be extended yet further, to consider yet more general maps and hence yet larger classes of monoids; however, initial indications are that such results will become markedly more technical once cancellativity conditions are dropped.

In addition to this introduction, this paper comprises three sections. In Section 2 we briefly recall some definitions and foundational results from [5]. We then consider actions of monoids by isometric embeddings on semimetric spaces, identifying some key “well-behavedness” conditions. Section 3 considers monoids acting on their own Cayley graphs (viewed as semimetric spaces), and the extent to which these actions satisfy the conditions of the previous section. Finally, Section 4 proves an analogue of the Švarc-Milnor

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\(^3\)Such functions are ubiquitous in applied mathematics and arise with increasing frequency also in pure mathematics, but terminology for them is not standardized across the different areas in which they appear. Terms used include premetric, pseudo-metric, quasi-metric, quasi-semi-metric and extended quasi-metric, but some of these are also used for other generalizations of metrics.
lemma and gives some corollaries and example applications, including a complete description of the quasi-isometry types of free products of free monoids and finite groups.

2. Semimetric Spaces and Monoid Actions

We denote by $\mathbb{R}^\infty$ the set $\mathbb{R}_{\geq 0}\cup\{\infty\}$ of non-negative real numbers with $\infty$ adjoined, equipped with the obvious total order, addition and multiplication (leaving $0\cdot\infty$ undefined). A semimetric on a set $X$ is a function $d : X \times X \to \mathbb{R}^\infty$ satisfying:

(i) $d(x, y) = 0$ if and only if $x = y$; and
(ii) $d(x, z) \leq d(x, y) + d(y, z)$;

for all $x, y, z \in X$. A semimetric space is a set equipped with a semimetric. A point $x_0 \in X$ is called a basepoint for the space $X$ if $d(x_0, y) \neq \infty$ for all $y \in X$. The space is called strongly connected if every point is a basepoint, that is, if the distance function never takes the value $\infty$.

We define the distance between subsets of $X$ by

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$$

for all $A, B \subseteq X$.

A path of length $n \in \mathbb{R}$ from $x$ to $y$ is a map $p : [0, n] \to X$ such that $p(0) = x$, $p(n) = y$ and $d(p(a), p(b)) \leq b - a$ for all $0 \leq a \leq b \leq n$. If $d(x, y) \neq \infty$ then a geodesic from $x$ to $y$ is a path of length $d(x, y)$ from $x$ to $y$. The semimetric space $X$ is called geodesic if for all $x, y \in X$ with $d(x, y) \neq \infty$ there exists at least one geodesic from $x$ to $y$.

Let $x_0 \in X$ and let $r$ be a non-negative real number. The out-ball of radius $r$ based at $x_0$ is

$$\overline{B}_r(x_0) = \{y \in X \mid d(x_0, y) \leq r\}.$$ 

Dually, the in-ball of radius $r$ based at $x_0$ is defined by

$$\overline{B}_r(x_0) = \{y \in X \mid d(y, x_0) \leq r\},$$

and the strong ball of radius $r$ based at $x_0$ is

$$B_r(x_0) = \overline{B}_r(x_0) \cap \overline{B}_r(x_0).$$

For $1 \leq \mu < \infty$, a subset $X'$ of $X$ is called $\mu$-quasi-dense if every point in $X$ is contained in the strong ball of radius $\mu$ around some point in $X'$.

Let $f : X \to Y$ be a map between semimetric spaces. Then $f$ is an isometric embedding if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$; a surjective (and hence bijective) isometric embedding is an isometry. More generally, let $1 \leq \lambda < \infty$, $1 \leq \mu < \infty$ and $0 < \epsilon < \infty$ be constants. The map $f$ is called a $(\lambda, \epsilon)$-quasi-isometric embedding, and $X$ embeds quasi-isometrically in $Y$, if

$$\frac{1}{\lambda} d(x, y) - \epsilon \leq d(f(x), f(y)) \leq \lambda d(x, y) + \epsilon$$

for all $x, y \in X$. If $f : X \to Y$ is a $(\lambda, \epsilon)$-quasi-isometric embedding and its image is $\mu$-quasi-dense, then $f$ is called a $(\lambda, \epsilon, \mu)$-quasi-isometry, and the spaces $X$ and $Y$ are quasi-isometric. Quasi-isometry is an equivalence relation on the class of semimetric spaces [5 Proposition 1]. A semimetric space is called quasi-metric if it is quasi-isometric to a metric space, or
equivalently [5, Proposition 2] if there are constants \( \lambda, \mu < \infty \) such that \( d(x, y) \leq \lambda d(y, x) + \mu \) for all points \( x \) and \( y \).

Now let \( M \) be a monoid acting by isometric embeddings on a semimetric space \( X \). We say that the action is \textit{cobounded} if there is a strong ball \( B \) of finite radius such that \((mB)_{m \in M} \) covers \( X \). We say that the action is \textit{outward proper} if for every out-ball \( B \) of finite radius the set \( \{m \in M \mid d(B, mB) = 0\} \) is finite. Note that our definitions of outward proper and cobounded coincide with the usual notions of proper and cobounded in the special case of a group action on a metric space. Also, notice that coboundedness of the action is sufficient to ensure (if we do not assume this \textit{a priori}) that \( M \) is acting as a monoid, that is, that the identity element of \( M \) acts as the identity function on \( X \).

In addition to the preceding conditions, which are relatively straightforward generalisations of conditions imposed in the group case, we will also need to impose a fundamentally semigroup-theoretic restriction to ensure that the right ideal structure of the monoid is reflected in the space. If \( x_0 \in X \) is a basepoint, we say that the action is \textit{idealistic at} \( x_0 \) if

\[
d(mx_0, nx_0) < \infty \Rightarrow nM \subseteq mM
\]

for all \( m, n \in M \). We say that the action is \textit{idealistic} if it is idealistic at some basepoint. Notice that in the special case that \( M \) is a group, \( mM \subseteq nM \) for all \( m, n \in M \), so the action is idealistic exactly if the space has a basepoint.

3. \textbf{Cayley Graphs of Monoids and Cancellative Monoids}

Let \( M \) be a monoid generated by a finite set \( S \). Then we may define a semimetric on \( M \) by setting \( d_S(x, y) \) to be the shortest length of a word \( w \) over the generating set \( S \) such that \( xw = y \) in \( M \), or \( \infty \) if there is no such word. With this notion of distance, the natural action of \( M \) on itself by left translation is an action by \textit{contractions} (maps which do not increase distance). The action is by isometric embeddings exactly if \( M \) is left cancellative.

The semimetric space \( M \) is clearly not (unless \( M \) is trivial) geodesic. We extend \( M \) to a geodesic semimetric space \( \Gamma_S(M) \) by “stitching in”, for each element \( m \in M \) and generator \( s \in S \), a copy of the open interval \((0, 1)\) between \( m \) and \( ms \), which we view as an \textit{edge} from \( m \) to \( mx \). The semimetric is defined so that point \( \mu \) in this interval is at distance \( \mu \) from \( m \) in both directions, but at distance \( 1 - \mu \) from \( mx \) in only one direction.

Formally, our new semimetric space, which we denote \( \Gamma_S(M) \) and call the \textit{(continuous) Cayley graph of \( M \) with respect to \( S \)}, has point set \( M \cup (M \times S \times (0, 1)) \) and distance function defined by

- \( d(m, n) = d_S(m, n) \) for all \( m, n \in M \);
- \( d((m, (n, y, \nu)) = d(m, n) + \nu \) for all \( m, n \in M \), \( y \in S \), \( \nu \in (0, 1) \);
- \( d((m, x, \mu), (n, x, \nu)) = \min(\mu + d(m, n), (1 - \mu) + d(mx, n)) \) for all \( m, n \in M \), \( x \in S \), \( \mu \in (0, 1) \);
- \( d((m, x, \mu), (m, x, \nu)) = |\mu - \nu| \) for all \( m \in M \), \( x \in S \), \( \mu, \nu \in (0, 1) \);
- \( d((m, x, \mu), (n, y, \nu)) = d((m, x, \mu), n) + \nu \) for all \( m, n \in M \), \( x, y \in S \), \( \mu, \nu \in (0, 1) \) such that \( m \neq n \) or \( x \neq y \).
Note that this semimetric is slightly different from that used for Cayley graphs in [5]; the distinction is immaterial in the group case, but very important when considering monoids, where the definition used here is necessary to allow the following elementary result.

**Proposition 3.1.** The inclusion map from $M$ to $\Gamma_S(M)$ is an isometric embedding and a quasi-isometry.

*Proof.* That the map preserves distances, and hence is an isometric embedding and a quasi-isometric embedding, is immediate from the definition of the semimetric in $\Gamma_S(M)$. Moreover, it is also immediate that any point $(m, x, \mu)$ on an edge lies in the strong ball of radius 1 around $m$, so that $M \subseteq \Gamma_S(M)$ is quasi-dense and the map is a quasi-isometry. □

Note that while the semimetric spaces $M$ and $\Gamma_S(M)$ depend upon the choice of finite generating set $S$, their quasi-isometry type does not [5, Proposition 4], and hence is an isomorphism invariant of $M$. Thus, provided $M$ admits a finite generating set, it makes sense to speak of a semimetric space being quasi-isometric to the abstract monoid $M$.

For any monoid $M$, we can extend the left translation action of $M$ on itself to an action on $\Gamma_S(M)$, by defining $p(m, x, \mu) = (pm, x, \mu)$ for all $p, m \in M$, $x \in S$ and $\mu \in (0, 1)$. Recall (from for example [9]) that a monoid $M$ has finite (right) geometric type (also called bounded indegree) if for every $b, c \in M$ there are only finitely many elements $a$ satisfying $ab = c$. Note in particular that this condition is satisfied by right cancellative monoids.

**Proposition 3.2.** Let $M$ be a left cancellative monoid generated by a finite set $S$. Then $M$ acts on $\Gamma_S(M)$ by isometric embeddings, and the action is cobounded and idealistic at the identity of $M$. If $M$ has finite geometric type then the action is outward proper.

*Proof.* That the action is by isometric embeddings follows easily from the definitions and the fact that $M$ is left cancellative.

Let $e$ denote the identity of $M$. Then the strong ball of radius 1 around $e$ contains $e$ and all points of the form $(e, x, \mu)$ with $x \in S$ and $\mu \in (0, 1)$. Clearly every point in $\Gamma_S(M)$ is a translate of such a point by an element of $M$, so the action is cobounded.

If $m, n \in M$ are such that $d(me, ne) = d_S(m, n) < \infty$ then there is a word $w$ over the generating set $S$ such that $n = mw$. But now $n \in mM$ so that $nM \subseteq mM$. Thus, the action is idealistic at $e$.

Now suppose $M$ has finite geometric type. Since $e$ is a basepoint, every out-ball of finite radius is contained in an out-ball of finite radius around $e$, so it will suffice to assume $C$ is an out-ball of finite radius around $e$, and show that the set $\{m \in M \mid d(C, mC = 0)\}$ is finite. Let $B$ be any out-ball around $e$ of strictly larger radius than $C$. Then if $m$ is such that $d(C, mC) = 0$ it is easily seen that $B \cap mB \neq \emptyset$, so it will suffice to show that

$$Q = \{m \in M \mid B \cap mB \neq \emptyset\}$$

is finite. Suppose $m \in Q$. Then we may choose $b_m, c_m \in B$ such that $mb_m = c_m$. If either $b_m$ or $c_m$ is not in $M$ then it follows from the definition of the action that we have $b_m = (b', x, \mu)$ and $c_m = (c', x, \mu)$ for some $x \in S$,
\( \mu \in (0, 1) \) and \( b', c' \in M \) with \( mb' = c' \). Moreover, since all paths from \( e \) to \( b_m, c_m \) must lead through \( b' \) and \( c' \), we have \( b', c' \in B \). Thus, replacing \( b_m, c_m \) with \( b', c' \) if necessary, we may assume without loss of generality that \( b_m, c_m \in B \cap M \). Now by the finite geometric type assumption, there are only finitely many elements \( x \in M \) satisfying \( xb_m = c_m \). Hence, the map

\[
Q \to (B \cap M) \times (B \cap M), \quad m \mapsto (b_m, c_m)
\]

is finite-to-one. But since \( B \) has finite radius and \( M \) is finitely generated, it is easily seen that \( B \cap M \) is finite, and so \( Q \) is finite. \( \square \)

One case of the above proposition deserves particular note.

**Corollary 3.3.** A cancellative monoid acts outward properly, coboundedly and idealistically by isometric embeddings on its Cayley graph.

### 4. Švarc-Milnor Lemma for Isometric Embeddings

The aim of this section is to establish the following theorem, which is an extension of the Švarc-Milnor Lemma \([7, 10]\) to the setting of monoids acting by isometric embeddings on semimetric spaces. The proof is adapted from our proof of a corresponding result for groups acting by isometries on semimetric spaces \([5]\), which in turn is based on the standard proof for groups acting on metric spaces (see for example \([4]\)). However, extra steps are required to handle the extra complications arising from the right ideal structure of the monoid and the directedness of the space acted upon.

**Theorem 4.1 (Švarc-Milnor Lemma for Isometric Embeddings).** Let \( M \) be a monoid acting idealistically, outward properly and coboundedly by isometric embeddings on a geodesic semimetric space \( X \). Then \( M \) is finitely generated and quasi-isometric to \( X \).

**Proof.** Let \( x_0 \in X \) be a basepoint such that the action is idealistic at \( x_0 \). Since the action is cobounded, we may choose a strong ball \( D \) of finite radius such that \( (mD)_{m \in M} \) covers \( X \). In particular, there exists \( g \in M \) and \( x'_0 \in D \) with \( x_0 = gx'_0 \). Let \( p \in X \) be arbitrary. Since the action is by isometric embeddings and \( x_0 \) is a basepoint, we have

\[
d(x'_0, p) = d(gx'_0, gp) = d(x_0, gp) < \infty.
\]

We conclude that \( x'_0 \) is a basepoint. Since \( x_0 \) and \( x'_0 \) are both basepoints, and \( D \) is a strong ball of finite radius containing \( x'_0 \) whose translates cover \( X \), it follows easily that there is a strong ball \( B \supseteq D \) of finite radius based at \( x_0 \) whose translates cover \( X \). Let \( R \) be the radius of \( B \).

Now let

\[
S = \{ m \in M \mid d(B, mB) = 0 \}.
\]

Since the strong-ball \( B \) is contained in an out-ball of the same radius, and \( M \) is acting outward property, the set \( S \) is finite. Clearly \( e \in S \), where \( e \) denotes the identity element of \( M \).

Let \( C = \overline{B}_{5R}(x_0) \), noting that \( B \subseteq C \) and define

\[
Q = \{ mB \mid d(B, mB) \neq 0 \text{ and } d(C, mB) = 0 \}.
\]
Note that $Q$ is finite, since it is contained in $\{mB \mid d(C, mC) = 0\}$, which is finite since the action is outward proper. Hence, we may choose a positive real number $r$ such that $r < R$ and $r < d(B, mB)$ for every $mB \in Q$.

**Claim.** For all $h \in M$ if $d(B, hB) < r$ then $d(B, hB) = 0$.

**Proof of Claim.** Suppose, seeking a contradiction, that $d(B, hB) < r$ but $d(B, hB) \neq 0$. Since $d(B, hB) < r$ there exist $u \in B$ and $v \in hB$ with $d(u, v) < r$. Since $u \in B$ we have $d(x_0, u) \leq R$. Therefore

$$d(x_0, v) \leq d(x_0, u) + d(u, v) < R + r \leq 2R < 5R$$

so that $v \in C = \overline{B}_{5R}(x_0)$. Thus $v \in C \cap hB$ so $d(C, hB) = 0$ and $hB \in Q$. But by the choice of $r$ it now follows that $r < d(B, hB)$, giving the required contradiction and completing the proof of the claim. \( \square \)

**Claim.** For all $m, n \in M$, if $d(mB, nB) < r$ then $n = mu$ for some $u \in S$.

**Proof of Claim.** Since $d(mB, nB) < r < \infty$, we may choose $a, b \in B$ such that $d(ma, nb) < r$. Now since $B$ is a strong ball of radius $R$, and the action is distance-preserving, we have

$$d(mx_0, nx_0) \leq d(mx_0, ma) + d(ma, nb) + d(nb, nx_0)$$

$$= d(x_0, a) + d(ma, nb) + d(b, x_0)$$

$$\leq R + r + R$$

$$< \infty.$$

Since the action is idealistic at $x_0$, this means that there exists $u \in M$ with $mu = n$.

Now using again the fact that the action is distance preserving, we have

$$d(B, uB) = \inf \{d(y, uz) : y, z \in B\}$$

$$= \inf \{d(my, muz) : y, z \in B\}$$

$$= d(mB, muB)$$

$$= d(mB, nb)$$

$$< r.$$

Hence, by the previous claim, we have $d(B, uB) = 0$, which by definition gives $u \in S$, completing the proof of the claim. \( \square \)

Now choose a positive real number $l < r$. We claim that $S$ generates $M$ and that for all $m \in M$

$$d_S(e, m) \leq \frac{1}{l}d(x_0, mx_0) + 1$$

where $e$ is the identity element of the monoid $M$. To see this, let $m \in M$ be arbitrary. Since $x_0$ is a basepoint, $d(x_0, mx_0)$ is finite. Since the semimetric space $X$ is geodesic, there is geodesic from $x_0$ to $mx_0$, that is, a map $p : [0, d(x_0, mx_0)] \to X$ such that $p(0) = x_0$, $p(d(x_0, mx_0)) = mx_0$ and $d(p(a), p(b)) \leq b - a$ for all $0 \leq a \leq b \leq d(x_0, mx_0)$. Let $k$ be the integer part of $\frac{1}{l}d(x_0, mx_0)$, and for $1 \leq i \leq k$ define $x_i = p(il)$. Then for $0 \leq i < k$ we have

$$d(x_i, x_{i+1}) = d(p(il), p((i + 1)l)) \leq (i + 1)l - il = l.$$
If we set \( x_{k+1} = m x_0 \) then recalling the definition of \( k \) we also have
\[
d(x_k, x_{k+1}) = d(p(kl), p(d(x_0, m x_0))) \leq d(x_0, m x_0) - kl < l.
\]
Since the translates of \( B \) cover the space \( X \), we may choose \( m_0, \ldots, m_{k+1} \in M \) such that each \( x_i \in m_i B \). Clearly, we may assume \( m_0 = e \) and \( m_{k+1} = m \).

Now for \( 0 \leq i \leq k \)
\[
d(m_i B, m_{i+1} B) \leq d(x_i, x_{i+1}) \leq l < r,
\]
and so by the above claim there exists \( s_i \in S \) with \( m_{i+1} = m_i s_i \). But then
\[
m = m_{k+1} = m_k s_k = m_{k-1} s_{k-1} s_k = \ldots = m_0 s_0 s_1 \ldots s_k = s_0 s_1 \ldots s_k \in \langle S \rangle
\]
since \( m_0 = e \).

We have written an arbitrary element \( m \in M \) as a product of elements of \( S \), which proves the claim that \( S \) generates the monoid \( M \). Moreover, we have written \( m \) as a product of \( k + 1 \) generators from \( S \), and \( k \) was defined to be the integer part of \( \frac{1}{l} d(x_0, m x_0) \), so
\[
d_S(e, m) \leq k + 1 \leq \frac{1}{l} d(x_0, m x_0) + 1 \tag{1}
\]

Now let \( m_1, m_2 \in M \) be arbitrary. If \( d_S(m_1, m_2) = \infty \) then by definition \( m_2 M \) is not contained in \( m_1 M \). Since the action is idealistic at \( x_0 \), this means that \( d(m_1 x_0, m_2 x_0) = \infty \). In particular, we have
\[
d_S(m_1, m_2) \leq \frac{1}{l} d(m_1 x_0, m_2 x_0) + 1.
\]
On the other hand, if \( d_S(m_1, m_2) < \infty \) then we can write \( m_2 = m_1 n \) for some \( n \in M \). Since \( M \) acts on \( X \) by isometric embeddings and on itself by contractions, applying equation \((1)\) we obtain
\[
d_S(m_1, m_2) = d_S(m_1, m_1 n) \leq d_S(e, n) \leq \frac{1}{l} d(x_0, n x_0) + 1 = \frac{1}{l} d(m_1 x_0, m_1 n x_0) + 1 = \frac{1}{l} d(m_1 x_0, m_2 x_0) + 1.
\]
Now let \( \lambda = \max\{d(x_0, s x_0) \mid s \in S\} \); since \( x_0 \) is a basepoint and \( S \) is finite, this maximum exists and is finite. We claim that for all \( m, n \in M \) we have
\[
d(m x_0, n x_0) \leq \lambda d_S(m, n).
\]
Indeed, when \( d_S(m, n) = \infty \) this is obviously true. Otherwise we can write \( n = m s_1 \ldots s_k \) where \( k = d_S(m, n) \) and \( s_1, \ldots, s_k \in S \). Now applying the triangle inequality and the fact that the action is by isometric embeddings
we obtain
\[
\begin{align*}
  d(mx_0, nx_0) & = d(mx_0, ms_1 \ldots s_k x_0) \\
  & = d(x_0, s_1 \ldots s_k x_0) \\
  & \leq d(x_0, s_1 x_0) + d(s_1 x_0, s_1 s_2 x_0) + \ldots + d(s_1 s_2 \ldots s_{k-1} x_0, s_1 s_2 \ldots s_k x_0) \\
  & = d(x_0, s_1 x_0) + d(x_0, s_2 x_0) + \ldots + d(x_0, s_k x_0) \\
  & \leq \lambda k = \lambda \delta_S(m, n).
\end{align*}
\]

Now consider the mapping \( f : M \to X \) defined by \( m \mapsto mx_0 \). It follows from the observations above that
\[
\delta_S(m_1, m_2) \leq \frac{1}{k} d(f(m_1), f(m_2)) + 1
\]
and also
\[
d(f(m_1), f(m_2)) \leq \lambda \delta_S(m_1, m_2)
\]
for all \( m_1, m_2 \in M \). Moreover, given \( x \in X \), since \((\alpha B)_{\alpha \in M} \) covers \( X \) we conclude that there exists \( h \in M \) with \( x \in hB \), and thus
\[
\max(d(f(h), x), d(x, f(h))) \leq R.
\]
Hence \( M \) and \( X \) are quasi-isometric. \( \square \)

Combining Theorem 4.1 with Proposition 3.2 yields the following characterisation of the property of finite generation for left cancellative monoids of finite geometric type (and so in particular for cancellative monoids).

**Corollary 4.2.** A left cancellative monoid of finite geometric type is finitely generated if and only if it acts outward properly, coboundedly, and idealistically by isometric embeddings on a geodesic semimetric space.

**Corollary 4.3.** Let \( M \) be a finitely generated monoid. Then \( M \) is a group if and only if \( M \) acts outward properly, coboundedly and idealistically by isometric embeddings on a quasi-metric space. If \( M \) is a group then every semimetric space on which it acts outward properly, coboundedly and idealistically by isometric embeddings is quasi-metric.

**Proof.** Suppose \( M \) acts properly, coboundedly and idealistically by isometric embeddings on a geodesic quasi-metric space \( X \). Then by Theorem 4.1, \( M \) is quasi-isometric to \( X \). By [5, Corollary 1], quasi-metricity is a quasi-isometry invariant, so it follows that \( M \) itself is quasi-metric. By [5, Proposition 8] this means that \( M \) is right simple, but a right simple monoid must be a group.

Conversely, if \( M \) is a group then it is certainly cancellative, so by Corollary 3.3 it admits an outward proper, cobounded, idealistic action by isometric embeddings (which must in fact be isometries) on a geodesic semimetric space. By Theorem 4.1, any space on which \( M \) so acts must be quasi-isometric to \( M \), and hence quasi-metric. \( \square \)

Recall that a subsemigroup \( S \) of a semigroup \( T \) is called **left unitary** if whenever \( s \in S \) and \( t \in T \) are such that \( st \in S \), we have also \( t \in S \). The following theorem, which provides a generalisation of the well-known fact
that a finitely generated group is quasi-isometric to each of its finite index subgroups, is an typical illustration of how Theorem 4.1 may be applied.

**Theorem 4.4.** Let $M$ be a left unitary submonoid of a finitely generated left cancellative monoid $N$ of finite geometric type, and suppose there is a finite set $P \subseteq N$ of right units such that $MP = N$. Then $M$ is finitely generated and quasi-isometric to $N$.

**Proof.** Consider the left translation actions of both $N$ and $M$ on $\Gamma_S(N)$. By Proposition 3.2 the action of $N$ is outward proper and by isometric embeddings, from which it follows immediately that the action of $M$ is outward proper and by isometric embeddings.

Since the action of $N$ is cobounded, there is a strong ball $B$ of finite radius whose translates by elements of $N$ cover $\Gamma_S(N)$. By a simple argument (or from the proof of Proposition 3.2) we may assume that this ball is centred at the identity $e$ of $N$. Since $P$ is finite and consists of right units, it is contained in a strong ball of finite radius around $e$ in $N$, and hence (since the embedding of $N$ into $\Gamma_S(N)$ is a quasi-isometry) also in $\Gamma_S(N)$. Consider now the set $PB$. Then for any $x \in P$ and $b \in B$ we have

$$d(e, xb) \leq d(e, x) + d(x, xb) = d(e, x) + d(e, b)$$

and similarly

$$d(xb, e) \leq d(xb, x) + d(x, e) = d(b, e) + d(x, e).$$

Since $P$ and $B$ are both contained in strong balls of finite radius around $e$, it follows that $PB$ is contained in a strong ball ($C$ say) of finite radius around $e$. Now for any point $y \in \Gamma_S(N)$ we have $y = nb$ for some $n \in N$ and $b \in B$. But $n = mx$ for some $m \in M$ and $x \in P$. Hence, $y = mxb \in mPB \subseteq mC$, so the translates of $C$ by elements of $M$ cover $\Gamma_S(N)$, and the action of $M$ is cobounded.

Finally, we claim that the action of $M$ is idealistic at $e$, considered as a point in $\Gamma_S(N)$. Indeed, suppose $m, n \in M$ are such that $d(me, ne) < \infty$ in $\Gamma_S(N)$. By Proposition 3.2 the action of $N$ is idealistic at $e$, so we have $nN \subseteq mN$. Then there is an $s \in N$ such that $ms = n$. Since $m \in M$, $ms = n \in M$ and $M$ is left unitary, we deduce that $s \in M$, whereupon $nM \subseteq mM$, as required.

Thus Theorem 4.1 applies, and tells us that $M$ is finitely generated and quasi-isometric to $\Gamma_S(N)$, which by Proposition 3.1 is quasi-isometric to $N$.

Theorem 4.4 allows us, for example, to completely describe the quasi-isometry types of free products of finitely generated free monoids and finite groups.

**Corollary 4.5.** Let $F$ be a finitely generated free monoid of rank $r$ and $G$ a finite group. Then the free product $F * G$ is quasi-isometric to a free monoid of rank $r|G|$.

**Proof.** It is well known (see for example [3, Lemma 1]) that the free product $F * G$ is cancellative. Let $\{f_1, \ldots, f_r\}$ be the free generating set for $F$. It
is easy to show that every element of \( F \ast G \) can be written uniquely as an alternating product of the form 

\[
  g_0 x_1 g_1 x_2 g_2 \cdots x_n g_n
\]

where \( n \geq 0 \), each \( g_i \in G \) and each \( x_i \in \{ f_1, \ldots, f_r \} \). Let \( M \subseteq F \ast G \) be the set of elements which admit decompositions as above with \( g_n \) the identity element of \( G \). It is readily seen that \( M \) is a left unitary submonoid of \( F \ast G \), and it also follows easily from the uniqueness of the above decompositions that \( M \) is freely generated by the elements of the form \( gf_i \) for \( g \in G \) and \( 1 \leq i \leq r \), and hence is free of rank \( r|G| \). Finally, \( G \) is a finite set of right units in \( F \ast G \), and \( MG = F \ast G \). Hence, by Theorem 4.4 we may conclude that \( F \ast G \) is quasi-isometric to \( M \), which is a free monoid of rank \( r|G| \). □

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