SHARP BOUNDS IN TERMS OF THE POWER OF THE
CONTRA-HARMONIC MEAN FOR NEUMAN-SÁNDOR MEAN

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ABSTRACT. In the paper, the authors obtain sharp bounds in terms of the
power of the contra-harmonic mean for Neuman-Sándor mean.

1. INTRODUCTION

For positive numbers \(a, b > 0\) with \(a \neq b\), the second Seiffert mean \(T(a, b)\), the
root-mean-square \(S(a, b)\), Neuman-Sándor mean \(M(a, b)\), and the contra-harmonic
mean \(C(a, b)\) are respectively defined in [9, 13] by

\[
T(a, b) = \frac{a - b}{2 \arctan[(a - b)/(a + b)]}, \quad S(a, b) = \left(\frac{a^2 + b^2}{2}\right)^rac{1}{2},
\]

\[
M(a, b) = \frac{a - b}{2 \arcsinh[(a - b)/(a + b)]}, \quad C(a, b) = \frac{a^2 + b^2}{a + b}.
\]

It is well known [7, 8, 10] that the inequalities

\[
M(a, b) < T(a, b) < S(a, b) < C(a, b)
\]

hold for all \(a, b > 0\) with \(a \neq b\).

In [2, 3], the inequalities

\[
S(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < T(a, b) < S(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)
\]

and

\[
C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < T(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)
\]

were proved to be valid for \(\frac{1}{2} < \alpha, \beta, \lambda, \mu < 1\) and for all \(a, b > 0\) with \(a \neq b\) if and
only if

\[
\alpha \leq \frac{1}{2} \left(1 + \sqrt{\frac{16}{\pi^2} - 1}\right), \quad \beta \geq \frac{3 + \sqrt{6}}{6},
\]

\[
\lambda \leq \frac{1}{2} \left(1 + \sqrt{\frac{4}{\pi^2} - 1}\right), \quad \mu \geq \frac{3 + \sqrt{3}}{6},
\]

respectively. In [12], the double inequality

\[
S(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < M(a, b) < S(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)
\]
was proved to be valid for $\frac{1}{2} < \alpha, \beta < 1$ and for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1}{2} \left\{ 1 + \frac{1}{\sqrt{\ln(1 + \sqrt{2})}} - 1 \right\} \quad \text{and} \quad \beta \geq \frac{3 + \sqrt{3}}{6}. \quad (1.7)$$

For more information on this topic, please refer to recently published papers [4, 5, 6, 11] and references cited therein.

For $t \in \left(\frac{1}{2}, 1\right)$ and $p \geq \frac{1}{2}$, let

$$Q_{t, p}(a, b) = C^p(ta + (1 - t)b, tb + (1 - t)a)A^{1-p}(a, b), \quad (1.8)$$

where $A(a, b) = \frac{a + b}{2}$ is the classical arithmetic mean of $a$ and $b$. Then, by definitions in (1.1) and (1.2), it is easy to see that

$$Q_{t, 1/2}(a, b) = S(ta + (1 - t)b, tb + (1 - t)a),$$
$$Q_{t, 1}(a, b) = C(ta + (1 - t)b, tb + (1 - t)a),$$

and $Q_{t, p}(a, b)$ is strictly increasing with respect to $t \in \left(\frac{1}{2}, 1\right)$.

Motivating by results mentioned above, we naturally ask a question: What are the greatest value $t_1 = t_1(p)$ and the least value $t_2 = t_2(p)$ in $\left(\frac{1}{2}, 1\right)$ such that the double inequality

$$Q_{t_1, p}(a, b) < M(a, b) < Q_{t_2, p}(a, b) \quad (1.9)$$

holds for all $a, b > 0$ with $a \neq b$ and for all $p \geq \frac{1}{2}$?

The aim of this paper is to answer this question. The solution to this question may be stated as the following Theorem 1.1.

**Theorem 1.1.** Let $t_1, t_2 \in \left(\frac{1}{2}, 1\right)$ and $p \in \left[\frac{1}{2}, \infty\right)$. Then the double inequality (1.9) holds for all $a, b > 0$ with $a \neq b$ if and only if

$$t_1 \leq \frac{1}{2} \left[ 1 + \sqrt{\left(\frac{1}{t^*}\right)^{1/p} - 1} \right] \quad \text{and} \quad t_2 \geq \frac{1}{2} \left( 1 + \frac{1}{\sqrt{6p}} \right), \quad (1.10)$$

where

$$t^* = \ln(1 + \sqrt{2}) = 0.88\ldots \quad (1.11)$$

**Remark 1.1.** When $p = \frac{1}{2}$ in Theorem 1.1, the double inequality (1.9) becomes (1.6).

**Remark 1.2.** If taking $p = 1$ in Theorem 1.1, we can conclude that the double inequality

$$C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < M(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \quad (1.12)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\frac{1}{2} < \lambda \leq \frac{1}{2} \left[ 1 + \sqrt{\frac{1}{\ln(1 + \sqrt{2})} - 1} \right] \quad \text{and} \quad 1 > \mu \geq \frac{1}{2} \left( 1 + \frac{\sqrt{6}}{6} \right). \quad (1.13)$$
2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

**Lemma 2.1** ([1, Theorem 1.25]). For \(-\infty < a < b < \infty\), let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable on \((a, b)\). If \(g'(x) \neq 0\) and \(\frac{f(x)}{g(x)}\) is strictly increasing (or strictly decreasing, respectively) on \((a, b)\), so are the functions

\[
\begin{align*}
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\end{align*}
\]

**Lemma 2.2.** The function

\[
h(x) = \frac{(1 + x^2) \arcsinh x}{x}
\]

is strictly increasing and convex on \((0, \infty)\).

*Proof.* This follows from the following arguments:

\[
h'(x) = \frac{x\sqrt{1 + x^2} - \arcsinh x + x^2 \arcsinh x}{x^2} \triangleq h_1(x),
\]

\[
h_1'(x) = x\left(\frac{3x}{\sqrt{1 + x^2}} + 2 \arcsinh x\right) \triangleq xh_2(x),
\]

\[
h_2'(x) = \frac{5 + 2x^2}{(1 + x^2)^{3/2}} > 0
\]
on \((0, \infty)\) and

\[
\lim_{x \to 0^+} h_1(x) = \lim_{x \to 0^+} h_2(x) = 0. \quad \square
\]

**Lemma 2.3.** For \(u \in [0, 1]\) and \(p \geq \frac{1}{2}\), let

\[
f_{u,p}(x) = p \ln(1 + ux^2) - \ln x + \ln \arcsinh x
\]
on \((0, 1)\). Then the function \(f_{u,p}(x)\) is positive if and only if \(6pu \geq 1\) and it is negative if and only if \(1 + u \leq \left(\frac{1}{t^*}\right)^{1/p}\), where \(t^*\) is defined by (1.11).

*Proof.* It is ready that

\[
\lim_{x \to 0^+} f_{u,p}(x) = 0
\]
and

\[
\lim_{x \to 1^-} f_{u,p}(x) = p \ln(1 + u) + \ln(t^*).
\]

An easy computation yields

\[
f_{u,p}'(x) = \frac{2pu}{1 + ux^2} + \frac{1}{\sqrt{1 + x^2} \arcsinh x} - \frac{1}{x}
\]

\[
= u[(2p - 1)x^2 \sqrt{1 + x^2} \arcsinh x + x^3] - \left[\sqrt{1 + x^2} \arcsinh x - x\right] \\
\frac{x(1 + ux^2) \sqrt{1 + x^2} \arcsinh x}{(2p - 1)x^2 \sqrt{1 + x^2} \arcsinh x + x^3} \left[u - g_1(x)\right]
\]

where

\[
g_1(x) = \arcsinh x - \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad g_2(x) = (2p - 1)x^2 \arcsinh x + \frac{x^3}{\sqrt{1 + x^2}}.
\]
Furthermore, we have
\begin{equation}
(2.7) \quad g_1(0) = g_2(0) = 0
\end{equation}
and
\begin{equation}
(2.8) \quad \frac{g_1'(x)}{g_2'(x)} = \frac{1}{2(2p-1)\sqrt{1+x^2} h(x) + (2p+1)x^2 + 2p + 2}
\end{equation}
where \(h(x)\) is defined by (2.2). From Lemma 2.2, it follows that the quotient \(\frac{g_1'(x)}{g_2'(x)}\) is strictly decreasing on \((0, 1)\). Accordingly, from Lemma 2.1 and (2.7), it is deduced that the ratio \(\frac{g_1(x)}{g_2(x)}\) is strictly decreasing on \((0, 1)\).

Moreover, making use of L'Hôpital's rule leads to
\begin{equation}
(2.9) \quad \lim_{x \to 0} \frac{g_1(x)}{g_2(x)} = \frac{1}{6p}
\end{equation}
and
\begin{equation}
(2.10) \quad \lim_{x \to 1} \frac{g_1(x)}{g_2(x)} = \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1}.
\end{equation}

When \(u \geq \frac{1}{6p}\), combining (2.6) and (2.9) with the monotonicity of \(\frac{a_1(x)}{g_1(x)}\) shows that the function \(f_{u,p}(x)\) is strictly increasing on \((0, 1)\). Therefore, the positivity of \(f_{u,p}(x)\) on \((0, 1)\) follows from (2.4) and the increasingly monotonicity of \(f_{u,p}(x)\).

When \(u \leq \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1}\), combining (2.6) and (2.10) with the monotonicity of \(\frac{a_1(x)}{g_2(x)}\) reveals that the function \(f_{u,p}(x)\) is strictly decreasing on \((0, 1)\). Hence, the negativity of \(f_{u,p}(x)\) on \((0, 1)\) follows from (2.4) and the decreasingly monotonicity of \(f_{u,p}(x)\).

When \(\frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1} < u < \frac{1}{6p}\), from (2.6), (2.9), (2.10), and the monotonicity of the ratio \(\frac{a_1(x)}{g_2(x)}\), we conclude that there exists a number \(x_0 \in (0, 1)\) such that \(f_{u,p}(x)\) is strictly decreasing in \((0, x_0)\) and strictly increasing in \((x_0, 1)\). Denote the limit in (2.5) by \(h_p(u)\). Then, from the above arguments, it follows that
\begin{equation}
(2.11) \quad h_p \left( \frac{1}{6p} \right) = p \ln \left( 1 + \frac{1}{6p} \right) + \ln(t^*) > 0
\end{equation}
and
\begin{equation}
(2.12) \quad h_p \left( \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1} \right) = p \ln \left[ 1 + \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1} \right] + \ln(t^*) < 0.
\end{equation}

Since \(h_p(u)\) is strictly increasing for \(u > -1\), so it is also in \(\left[ \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1}, \frac{1}{6p} \right]\). Thus, the inequalities in (2.11) and (2.12) imply that the function \(h_p(u)\) has a unique zero point \(u_0 = \left( \frac{1}{t^*} \right)^{1/p} - 1 \in \left( \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1}, \frac{1}{6p} \right)\) such that \(h_p(u) < 0\) for \(u \in \left[ \frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1}, u_0 \right)\) and \(h_p(u) > 0\) for \(u \in \left( u_0, \frac{1}{6p} \right]\). As a result, combining (2.4) and (2.5) with the piecewise monotonicity of \(f_{u,p}(x)\) reveals that \(f_{u,p}(x) < 0\) for all \(x \in (0, 1)\) if and only if \(\frac{\sqrt{2}t^* - 1}{\sqrt{2} (2p-1)t^* + 1} < u < u_0\). The proof of Lemma 2.3 is complete.
3. Proof of Theorem 1.1

Now we are in a position to prove our Theorem 1.1.

Since both \( Q_{t,p}(a, b) \) and \( M(a, b) \) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \( a > b \). Let \( x = \frac{a - b}{a + b} \in (0, 1) \). From (1.2) and (1.8), we obtain

\[
\ln \frac{Q_{t,p}(a, b)}{T(a, b)} = \ln \frac{Q_{t,p}(a, b)}{A(a, b)} - \ln \frac{T(a, b)}{A(a, b)}
= p \ln \left[ 1 + (1 - 2t)^2 x^2 \right] - \ln x + \ln \text{arcsinh } x.
\]

Thus, Theorem 1.1 follows from Lemma 2.3.

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