A flexural crack model for damage detection in reinforced concrete structures

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Abstract. The use of changes in vibration data for damage detection of reinforced concrete structures faces many challenges that obstruct its transition from a research topic to field applications. Among these is the lack of appropriate damage models that can be deployed in the damage detection methods. In this paper, a model of a simply supported reinforced concrete beam with multiple cracks is developed to examine its use for damage detection and structural health monitoring. The cracks are simulated by a model that accounts for crack formation, propagation and closure. The beam model is studied under different dynamic excitations, including sine sweep and single excitation frequency, for various damage levels. The changes in resonant frequency with increasing loads are examined along with the nonlinear vibration characteristics. The model demonstrates that the resonant frequency reduces by about 10% at the application of 30% of the ultimate load and then drops gradually by about 25% at 70% of the ultimate load. The model also illustrates some nonlinearity in the dynamic response of damaged beams. The appearance of super-harmonics shows that the nonlinearity is higher when the damage level is about 35% and then decreases with increasing damage. The restoring force-displacement relationship predicted the reduction in the overall stiffness of the damaged beam. The model quantitatively predicts the experimental vibration behaviour of damaged RC beams and also shows the damage dependency of nonlinear vibration behaviour.

1. Introduction
The application of vibration-based techniques for damage detection and structural health monitoring (SHM) has been a popular subject of investigation for many years [1-3]. This is because the vibration data can easily be acquired and provide global information on the structural condition. However, the successful implementation of these techniques in the field of reinforced concrete (RC) structures is extremely limited, principally because of the absence of suitable damage models. For RC structures there are several various damage mechanisms, some of which introduce significant nonlinearity [4, 5]. This nonlinearity has been attributed to the nature of crack open to crack closed transition during the vibration cycle about equilibrium position [6]. In this particular case representing damage by a localised reduction in stiffness (e.g. open crack models) or even breathing crack models, where the closure effect is considered, is inadequate.

Breathing crack models treat the stiffness behaviour in a bilinear fashion which produces responses that do not reflect the inherent nonlinear behaviour of damaged RC structures. Researchers have therefore augmented these models by including the transition as the crack opens and closes in modelling the vibration response of cracked RC beams. Some researchers have suggested modelling the response of cracked beams as a sequence of four stages, namely; crack in tension, transition
interval as crack closes, crack in compression, and transition interval as crack opens (e.g. [7]). Others have related changes in stiffness to displacement of the vibrating beam and developed a hyperbolic tangent function to introduce the transition intervals [5]. However, although these nonlinear crack models have replicated the nonlinear phenomenon, they have not reproduced the experimental behaviour quantitatively. This is believed to be due to the lack of clear understanding of both the crack formation process and the vibration behaviour about the residual deformation.

This paper presents a model of a simply supported RC beam with distributed flexural cracks to study its vibration behaviour for damage detection and SHM. The model consists of a number of beam elements, some of which describe the cracked regions and others represent the undamaged parts, figure 1. The residual flexural rigidity of each cracked region is obtained from a crack model that considers crack initiation, growth and closure in a RC beam section.

To investigate its application in SHM and damage detection, the model is used to predict the response to different dynamic excitation types for various damage levels. Resonant frequencies with increasing damage are extracted and changes are compared with previous experimental results. The nonlinear vibration characteristics, such as appearance of super-harmonics and restoring force-displacement relationship, are also examined.

![Figure 1. Simply supported RC beam (a) dimensions and reinforcement details (b) model with distributed cracks.](image)

2. Crack model

The fictitious crack (FC) approach is used in the crack modelling process, where cracks are described using the stress-crack width relationship. The crack forms when the tensile strength of concrete ($f_{ct}$) is reached, and the effect of the crack spreads over a zone of width ($hc$), known as the equivalent elastic zone. The width of this zone is a vital parameter of the model as it defines the stiffness of the cracked section. It is taken as half the overall depth of the section [8].

The material on the crack formation path is assumed to be in one of three possible states. These are: (a) linear-elastic state before crack formation, (b) a fracture state where material is softening due to the cohesive forces in the fracture process zone, and (c) a state of zero stress when the crack width is beyond the critical crack width ($wc = 160 \, \mu m$). The assumptions and modelling steps that were followed in developing the crack model can be found in [9].

2.1. Stress-strain/crack width relationship for concrete

The constitutive relationship for concrete is assumed to be dependent on the level of damage and whether the section is being loaded or unloaded. When load is applied, concrete is modelled as a linear-elastic material in compression and also in tension up to $f_{ct}$. Afterwards, the crack forms and the stress decays with increasing elongation until it reaches zero at $wc$. The stress-crack width relationship is modelled adopting the relationship developed by Cornelissen et al. [10], equation (1).
where \( f_c \) is concrete stress in tension zone, \( v_c \) is elongation of concrete, \( v_{ct} \) is the elongation of concrete at the ultimate tensile strength, and \( C_1 \) and \( C_2 \) are empirical values equal 3 and 6.93 respectively.

The stress-strain relationship for compressive and linear-elastic tensile regions is combined with the stress-crack width relationship into one single stress-elongation \((f_c, v_c)\) relationship, figure 2. On the ascending branch, the stress in concrete varies linearly with the elongation as there is no crack, equation (2). On the descending branch, after crack initiation, the total elongation consists of the linear elastic elongation plus the crack width, equation (3).

\[
\begin{align*}
    f_c &= \frac{E_c v_c}{h_c} \\
    v_c &= \frac{f_c h_c}{E_c} + w
\end{align*}
\] (2)

in which \( E_c \) is the concrete modulus of elasticity, and \( w \) is the crack width.

The procedure followed in unloading is based on two main assumptions. First, points in the compression region and those in the linear-elastic zone of the tension region are unloaded linearly. Second, points in the descending branch of the envelope are unloaded using the constitutive model developed by Hordijk [11], modified by considering the total elongation rather than the crack width, equation (4). The unloading stress-elongation relationship for concrete is illustrated in figure 2.

\[
\begin{align*}
    f_{cu} &= f_{eu} + \left( \frac{f_{eu}}{3\left( \frac{v_{eu}}{v_{ct}} \right) + 0.4} \right) \left[ 0.014 \left( \ln \left( \frac{v_{eu}}{v_{ct}} \right) \right)^{-5} - 0.57 \left( 1 - \frac{v_{eu}}{v_{ct}} \right)^{1/2} \right]
\end{align*}
\] (4)

where \( f_{eu} \) is the unloading concrete stress, \( v_{eu} \) is the unloading concrete elongation, \( f_{eu} \) is the envelope unloading concrete stress, and \( v_{eu} \) is the envelope unloading concrete elongation.

\[\text{Figure 2. Loading/unloading stress-elongation relationship for concrete.}\]

### 2.2. Stress-strain relationship for steel reinforcement

The envelope stress-strain curve developed by Bai et al. [12], which ignores the Baushinger effect, is adopted for steel reinforcement constitutive relationship. For the loading scenario, the elastic region is followed by a yield plateau as described in equation (5).
where $f_s$ is the steel stress, $v_s$ is the steel elongation, $E_s$ is the steel modulus of elasticity, $f_y$ is the yield strength, and $v_y$ is the yield elongation.

For the unloading scenario, points in the elastic branch of the envelope are unloaded linearly, whereas those in the yield plateau are unloaded using the bilinear constitutive relationship, equation (6).

$$ f_{su} = E_s \left( v_{su} - v_{pu} \right) $$

where $f_{su}$ is the unloading steel stress, $v_{su}$ is the unloading steel elongation, and $v_{pu}$ is the residual elongation for inclined path obtained from equation (7).

$$ v_{pu} = \frac{f_{su} h_c}{E_s} + v_{seau} $$

where $f_{seau}$ and $v_{seau}$ are the envelope unloading steel stress and elongation respectively.

3. Beam model
The crack model is incorporated into a beam model in order to study the behaviour of a simply supported RC beam with multiple cracks. The cracks are assumed to form at equal spacing in between the shear links (i.e. 135mm), where the length of the cracked region is taken as $h_c$. The length of the undamaged parts ($l_2$) is considered as the clear spacing between the cracked regions (i.e. 30mm), except for the outer undamaged parts where the length ($l_1$) is 82.5mm.

For RC beams subject to flexure a crack forms when the tensile strength of concrete is reached. At the crack location, the contribution of concrete in transmitting the tensile force becomes negligible. At both sides of the crack, a discontinuity region forms in which the concrete is reactivated to carry a part of the tensile force by virtue of the bond stress. However, within this region the tensile strength of concrete cannot be reached. With increasing deformation, another crack forms just outside the discontinuity region of the adjacent crack. This stage, where regularly new cracks are formed, is called the crack-formation stage. In this stage cracks occur at almost regular spacing depending on the length of the discontinuity region which in turn relies on the bond and tensile stress of concrete [13].

The beam is subjected to static four-point bending under displacement control (figure 3) and the dynamic behaviour is scrutinised at various levels of damage by subjecting the beam to different sinusoidal excitations.

![Figure 3. Static four-point bending displacement control loading.](image-url)
loading step. Rayleigh damping is used in constructing the global damping matrix, where the damping of the cracked regions is assumed not to be influenced by the presence of cracks, equation (8).

\[ C = a_0M + a_1K \]  

(8)

where \( C \) is the damping matrix, \( M \) is the mass matrix, \( K \) is the stiffness matrix (constant for the undamaged parts and function of the interface rotation for the cracked regions), and \( a_0 \) and \( a_1 \) are the proportionality factors given by equation (9).

\[
\begin{bmatrix}
    a_0 \\
    a_1
\end{bmatrix} = \frac{2\zeta\omega_m\omega_n}{\omega_m + \omega_n} \begin{bmatrix}
    \omega_m & \omega_n \\
    \omega_n & 1
\end{bmatrix}
\]

(9)

in which \( \zeta \) is the damping ratio taken as 5%, \( \omega_m \) and \( \omega_n \) are considered as the 1st and 2nd un-damped natural frequencies of the system (302.87 rad/sec and 1211.47 rad/sec).

3.1. Results of static loading

When load is applied, the corresponding displacements and rotations at the interface of each beam element are calculated using a multivariate Newton-Raphson iterative solver. Then, the relative rotation of each cracked region is related to the elongation of the cracked section and thus the corresponding moment is found adopting the constitutive relationships of the crack model. When equilibrium of forces and compatibility are ensured, the next loading step is applied following the same procedure up to the required loading level.

Figure 4 illustrates the load-displacement and moment-rotation curves for various loading and unloading cases. It is clear that the relationship is linear up to the cracking load, where cracks start to form in the cracked regions so that their relative rotations are higher than the rotation that induces cracks. Thereafter, the relationship is not linear as the contribution of concrete under the neutral axis reduces and the tensile resistance of each cracked region is mainly provided by the steel reinforcement. The figure also depicts that when unloading from a point beyond the cracking load the unloading curve softens at the beginning of unloading and then starts to stiffen towards zero loading. The unloading curve further stiffens when unloading is continued to a negative value of about 5% of the ultimate load, representing crack closure. This observation is key to the behaviour of a crack under dynamic loading where it transitions between opening and closure during the vibration cycle.

**Figure 4.** (a) Loading/unloading load-midspan displacement curves (b) Loading/unloading moment-rotation curves for midspan cracked region.
Figure 5 shows load-rotation and moment rotation-curves for different cracked regions. It can be seen that all cracked regions behave linearly until cracks initiate in some of them and the relationship becomes nonlinear. In the remaining parts, where the relative rotation is less than the cracking rotation, the load-rotation curve is still linear. However, the moment-rotation curve of each cracked region follows the same stiffness relationship. This is clearly illustrated in figure 5(b).

The residual stiffness is determined by fitting a cubic polynomial to the moment-rotation curves about zero moments of each cracked region and the equivalent flexural rigidity is estimated using equation (10). The rotational stiffness and flexural rigidity of each cracked region are then related to the residual rotation in order to be used in the vibration analysis of the beam, figure 6.

\[ EI_{eq} = K_\theta h_c \]  

(10)

where \( EI_{eq} \) and \( K_\theta \) are the equivalent flexural rigidity and rotational stiffness of each cracked region respectively.

Figure 6. (a) Rotational stiffness-rotation curve during the vibration cycle for midspan cracked region (b) Flexural rigidity-rotation curve during the vibration cycle for midspan cracked region.
3.2. Results of dynamic loading

The dynamic behaviour is examined at different damage levels by subjecting the beam to sinusoidal excitations, at 1m from the left support, including sine sweep and single excitation frequency. This procedure is repeated for every loading step by first evaluating the flexural rigidity of each cracked region from the static loading using equation (10), and then extracting the vibration characteristics from the dynamic input.

The equation of motion, equation (11), is solved numerically using the built-in Matlab routine ode23t solver. This solver implements the trapezoidal rule using a “free” interpolant and it is suitable for moderately stiff problems.

\[
[M]\dddot{U}+[C]\ddot{U}+[K]U = F(t)
\]  

(11)

where \( \ddot{U}, \dot{U}, \) and \( U \) are the acceleration, velocity and displacement vectors respectively, and \( F(t) \) is the sinusoidal input force. The equation of motion is set up as in equation (12) in order to be solved numerically using the ode23t solver.

\[
\begin{bmatrix}
\dddot{u}_1 \\
\ddot{u}_2
\end{bmatrix} =
\begin{bmatrix}
I & 0 & -I & 0 & -I & 0 \end{bmatrix}
\begin{bmatrix}
\ddot{u}_1 \\
\dddot{u}_2
\end{bmatrix} +
\begin{bmatrix}
I & 0 \\
0 & [M]
\end{bmatrix}
\begin{bmatrix}
0 \\
F(t)
\end{bmatrix}
\]

(12)

The system is subjected first to sine sweep and the first resonant frequency is estimated for the undamaged beam. After applying the damaging loads the resonant frequency is predicted at various levels of damage. Figure 7 relates the reduction in the first resonant frequency with the damage level, which is taken as a percentage of the failure load of the beam (~45 kN). The model predicts that there is a weak reduction in the resonant frequency with increasing load, as it drops to around 90% at a damage load approximately equal to 30% of the failure load. Beyond this point, the resonant frequency continues to decrease more gradually until it has reduced by about 25% at 70% damaging load. These results have a similar trend of previous experimental data, which found that the resonant frequency varies in a tri-linear manner with increasing damage. It drops by around 10% at the application of 20-30% of the failure load and then changes gradually before it decreases by 25% after applying 70% of the failure load [14].

![Figure 7. Reduction in resonant frequency with damage level.](image-url)
However, changes in resonant frequency are deemed to be unsuitable for damage detection due to their susceptibility to environmental conditions and to the inherent nonlinear behaviour of cracked RC beams. Hence, nonlinear vibration characteristics are now examined by subjecting the system to a harmonic excitation and observing appearance of super-harmonics and changes in the relationship between the restoring force and modal displacement. The appearance of super-harmonics is a robust indication of nonlinearity in a system.

Figure 8 shows the dynamic response of the system to a single excitation frequency in the frequency domain. Figure 8(a) represents that the response of the undamaged beam is at the excitation frequency (30 Hz) as expected for a linear system. However, figures 8(b–d) show that super-harmonics of the 2nd and 3rd integers of the excitation frequency appear for the damaged beam. This is because the motion of the system for the damaged beam is not perfectly sinusoidal.

It is also interesting to note that the amplitude and number of super-harmonics decrease with the level of damage indicating a reduction in nonlinearity of the system. Clear nonlinearity can be seen when the beam is subjected to about 35% of the ultimate load, as the amplitude of the 2nd harmonic is higher than that of 2nd harmonic of the other damage levels. These observations agree well with reported experimental work that found the level of nonlinearity increases with damaging load up to 30% and then decreases in a reverse trend [4].

Figure 8. Response to single excitation frequency.

The restoring force surface method is an efficient and reliable technique for systems in which nonlinearity plays a role in the dynamic structural behaviour. The restoring force of a system is only dependent on displacement and velocity, thus it can be represented by a surface over the phase plane, equation (13).
\[ f(u, \dot{u}) = F(t) - m\ddot{u} \]  \hspace{1cm} (13)

in which \( f(u, \dot{u}) \) is the internal restoring force and \( m\ddot{u} \) is the inertia force.

Each displacement and velocity value indicates a point in this phase plane whereas the restoring force gives the height of the restoring force surface. There are various methods of interpolating and plotting procedures of a continuous surface on a grid. One of these plotting procedures is a representation of a slice of the restoring force data on a plane of constant velocity or displacement [15]. In this study, the relationship between the restoring force and displacement is obtained by taking an intersection of the surface with the plane through zero velocity.

Figure 9 shows the relationship between the restoring force and the modal displacement at the point of excitation for an undamaged beam and a damaged beam. It is possible to note that there is a reduction in the overall stiffness of the damaged beam of about 10%. Another observation is that the restoring force curves do not overlap at any points apart from the origin. This is because the adopted crack model is a nonlinear model, where if it was a bilinear model the stiffness of the undamaged and damaged beam at crack closure (positive modal displacement) would be equal. This observation further suggests that cracks in RC beams do not fully close during the vibration cycle as the stiffness of the damaged beam is less than that of the undamaged beam for the closure state.

4. Conclusions
A simply supported RC beam model with multiple flexural cracks is developed in order to study its dynamic behaviour for damage detection and SHM applications. The model consists of a number of beam elements, some of which represent the undamaged parts and others describe the cracked regions. The flexural rigidity of each cracked region is estimated from a crack model based on a FC approach and accounts for crack formation, propagation and closure. The model predicts the response to static loading for various loading and unloading scenarios. Then, the dynamic behaviour is scrutinised at various levels of damage and the resonant frequency is extracted and related to the damaging load. It is found that the reduction in resonant frequency with increasing damage follows a similar trend to previous experimental observations.
The model also studies the nonlinear vibration behaviour of the cracked beam scenarios, such as appearance of super-harmonics and restoring force-displacement relationship. The results show that the undamaged beam responds at the excitation frequency while the damaged beam responds at the excitation frequency and also at the multiple integers of the excitation frequency. It is also observed that the level of nonlinearity decreases after 35% of the ultimate load as the amplitude of the super-harmonics reduces. A similar trend is also observed experimentally [4]. The restoring force method depicts the reduction in the overall stiffness of the damaged beam. It also suggests that breathing cracks models are inappropriate for RC structures as cracks do not close during the vibration cycle.

To conclude, the model developed in this paper has shown to be capable of quantitatively predicting the experimentally observed vibration behaviour of damaged RC beams. The model also illustrates that changes in nonlinear vibration behaviour are sensitive to the damage level. Therefore, if an appropriate means of parameterising the nonlinearity due to the presence of cracks is obtained, this will make changes in nonlinear vibration a potential damage detection tool for SHM. In future work, the crack model will be improved to include loading, unloading and reloading scenarios in order to examine the influence of repeated loading on the dynamic behaviour of RC beams.

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