Time-Dependent Variational Approach to the Non-Abelian Pure Gauge Theory

Its Application to Evaluation of the Shear Viscosity of Quantum Gluonic Matter

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The time-dependent variational approach to the pure Yang-Mills gauge theory, especially a color $su(3)$ gauge theory, is formulated in the functional Schrödinger picture with a Gaussian wave functional approximation. The equations of motion for the quantum gauge fields are formulated in the Liouville-von Neumann form. This variational approach is applied in order to derive the transport coefficients, such as the shear viscosity, for the pure gluonic matter by using the linear response theory. As a result, the contribution to the shear viscosity from the quantum gluons is zero up to the lowest order of the coupling $g$ in the quantum gluonic matter.

§1. Introduction

One of recent interests for the quark and gluon physics, governed by the quantum chromodynamics (QCD), is to investigate properties of the quark-gluon plasma (QGP) and/or the quark-gluon matter. In the recent progress of the Relativistic Heavy Ion Collider (RHIC) experiments, it is said that the QGP may be not free gas but the strongly interacting quark-gluon matter.¹ The matter composed of quarks and gluons seems to reveal the properties of the liquid, not gas, like the perfect liquid. This conjecture is derived by comparing the obtained experimental data with the phenomenological analysis by using the hydrodynamical simulation with rather small shear viscosity, which leads to the near perfect liquid. The small shear viscosity is also near the lower bound which is conjectured in the AdS/CFT correspondence.²

Many works to understand properties of the gluonic matter were performed recently,³–⁵ while the transport coefficients, especially the shear viscosity for pure gluonic matter, were evaluated up to the lowest order of the QCD coupling constant $g$ in the early study.⁶ Namely, the shear viscosity $\eta_C$ for the gluonic matter at temperature $T$ can be expressed as⁵,⁶

$$\eta_C = d_f T^3 / g^4 \log(1/g^2),$$

up to the lowest order of $g$. Here, $d_f$ is numerically determined constant. For the quark matter, the shear viscosity is also evaluated, for example, in the Nambu-Jona-
Lasinio (NJL) model \(^7\) by using the linear response theory \(^8, 9\) in which a rather small shear viscosity is derived \(^10\). The small shear viscosity leads to the short mean free path in general. Thus, it may be shown that the constituents, namely quarks and gluons, of the matter under consideration, are strongly correlated.

However, as was shown by Ref. \(^11\), the anomalous contribution to the shear viscosity in the turbulent plasma fields gives the small shear viscosity even in the weak coupling QCD in which the quarks and gluons are weakly correlated. Namely, the shear viscosity \(\eta\) can be expressed by including the anomalous contribution \(\eta_A\) as \(^11\)

\[
\eta^{-1} = \eta_C^{-1} + \eta_A^{-1}.
\]  

(1.2)

If the anomalous viscosity \(\eta_A\) is small, the total shear viscosity \(\eta\) becomes to small even if the usual shear viscosity \(\eta_C\) is large for the small coupling constant \(g\) in Eq. (1.1). Thus, the small shear viscosity does not always lead to the strong coupling QCD, namely strongly correlated quark-gluon matter.

In this paper, thus, we consider the pure gluonic matter as the weak coupling system. we are concentrated our interest to calculating the shear viscosity for pure quantum gluonic matter without quarks by using the linear response theory. In this paper, the quantum gluon means the quantum fluctuation part around the mean field which leads to the Eq. (1.1). Thus, the shear viscosity of the quantum gluonic matter gives the contribution of the next and higher order of \(g\) in comparison with Eq. (1.1). One of purposes in this paper is to investigate the shear viscosity under small QCD coupling \(g\) for the quantum gluonic matter. It is important to investigate the contribution to the shear viscosity from the quantum gluonic fields. The reason is as follows: If there is a contribution to the shear viscosity of the order of \(g^0\), the finite value of the shear viscosity is remained even if the coupling \(g\) is small.

To deal with the quantum gluons and to investigate the dynamics of the quantum gluons, the time-dependent variational method with the Gaussian functional as a trial wave functional in the functional Schrödinger picture may gives a useful tool \(^12\). The reason why is that the mean fields and the quantum fluctuations around them can be treated on an equal footing and the higher order contributions for \(g\) are automatically included because certain kinds of the Feynman diagrams are taken into account in this variational approach. In this variational approach, the equations of motion for the mean fields and the fluctuation modes around them are derived in a self-consistent manner. Especially, the equations of motion for the quantum gluon fields are formulated in a form of the Liouville-von Neumann equations.

Another merit to use the time-dependent variational method for the pure gluonic matter is that the expectation values for various field operators and their products can easily be calculated because the state or the wave functional is prepared in the process of the variational calculations. When the transport coefficients such as the shear viscosity are calculated by using the linear response theory, the expectation values or thermal averages for the various operators such as the energy-momentum tensor operator are necessary. Thus, the variational approach may be suitable and give a practical method to calculate the transport coefficients.

This paper is organized as follows. In the next section, the time-dependent
variational approach to the pure Yang-Mills theory, especially the $su(3)$ gauge theory as the QCD, is formulated in the Hamiltonian formalism. In §3, the time-dependent variational equations for the quantum gluon fields are reformulated in a form of the Liouville-von Neumann equation for the reduced density matrix of the quantum fluctuation fields at zero and the finite temperatures. In §4, the shear viscosity is evaluated in our framework for pure gluonic matter by using the linear response theory from the viewpoint of weakly coupled QCD or weakly correlated pure gluonic matter. The last section is devoted to a summary and concluding remarks.

§2. Time-dependent variational approach to QCD

In this section, we give a variational method for the pure Yang-Mills gauge theory with color $su(3)$ symmetry in the functional Schrödinger picture with a Gaussian approximation, which is developed in Ref.12), in a slightly different manner. The trial state is constructed, paying an attention to the canonicity condition$^{13), 14}$ in our time-dependent variational approach. As a result, the equations of motion for variational functions are obtained as canonical equations of motion in classical mechanics.

2.1. Hamiltonian formalism of pure gauge theory

In this subsection, we summarize the Hamiltonian formalism of the pure Yang-Mills gauge theory for the sake of the definiteness of notations.

Let us start with the following Lagrangian density for the pure gauge theory with the color $su(N)$ symmetry:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a,$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c,$$  \hspace{1cm} (2.1)

where $A_\mu^a$ represents the gauge field and the Greek indices such as $\mu$, $\nu$ etc. and the Roman indices such as $a$, $b$ etc. mean the Lorentz and the color indices, respectively. The repeated indices are summed up. Later, we use another Roman indices such as $i$, $j$, $\cdots$, which means the space components of the Lorentz indices, that is 1, 2 and 3. Here, $g$ represents the coupling constant and $f_{abc}$ is the structure constant for the color $su(N)$:

$$[ T_a, T_b ] = i f_{abc} T_c,$$  \hspace{1cm} (2.2)

where $\{T_a\}$ is the $su(N)$ generators. In the adjoint representation, the $su(N)$ generator can be expressed as $(T_a)_c^b = -if_{abc}$.

The conjugate momentum, $\pi^{a\mu}$, for the field $A_\mu^a$ is defined as

$$\pi^{a\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu^a} = F_{a\mu}.$$  \hspace{1cm} (2.3)

Here, $\dot{A} = \partial A/\partial t$. We introduce the vector notation such as $A_a = (A_1^a, A_2^a, A_3^a)$. Then, the conjugate momentum with a space component can be expressed as

$$\pi^a = -(F_{10}^a, F_{20}^a, F_{30}^a).$$
\[ -\dot{A}^a - \nabla A_0^a + gf_{abc} A^b A_0^c = E^a . \] (2.4)

Here, we define the color electric field \( E^a \). As is similar to the color electric fields, we define the color magnetic field as

\[ B^a = -(F_{23}^a, F_{31}^a, F_{12}^a) \]
\[ = \nabla \times A^a - \frac{1}{2} gf_{abc} A^b \times A^c . \] (2.5)

Thus, we define the Hamiltonian density \( \mathcal{H}_0 \) as

\[ \mathcal{H}_0 = \pi^{a \mu} \dot{A}^a_{\mu} - \mathcal{L} = \frac{1}{2} [(E^a)^2 + (B^a)^2] + \pi^a \cdot (\nabla A_0^a - gf_{abc} A_0^b A_0^c) . \] (2.6)

As is well known, the gauge theory leads to the constrained system. Namely, the conjugate momentum \( \pi_0^a \) is identical to zero, so it is necessary to impose a constraint condition and the consistency condition for the time evolution as

\[ \pi_0^a = F_{a00} = 0 , \quad \dot{\pi}_0^a = 0 . \] (2.7)

This fact leads to the Dirac theory of constrained system. In terms of the analytic mechanics, the constrained condition is written as

\[ \dot{\pi}_0^a = \{ \pi_0^a , \int d^3x \mathcal{H} \} \big|_P = D \cdot \pi^a = 0 , \]
\[ D \cdot \pi^a = \nabla \cdot \pi^a - gf_{abc} A^b \cdot \pi^c \]
\[ = \nabla \cdot E^a + ig \cdot if_{abc} A^b \cdot E^c , \] (2.8)

where \( \{ , \} \big|_P \) represents the Poisson bracket. Thus, the Hamiltonian is written as

\[ \int d^3x \mathcal{H}_0 = \int d^3x \frac{1}{2} [(E^a)^2 + (B^a)^2] + \int d^3x (\pi^a \cdot \nabla A_0^a - gf_{abc} \pi^a \cdot A_0^b A_0^c) \]
\[ = \int d^3x \frac{1}{2} [(E^a)^2 + (B^a)^2] , \] (2.9)

where we used the integrated by part in the second term and the constrained condition (2.7). Thus, hereafter, we use the Hamiltonian density as

\[ \mathcal{H}_0 = \frac{1}{2} [(E^a)^2 + (B^a)^2] . \] (2.10)

For the later convenience, we introduce the following variable, \( \mathcal{G} \):

\[ \mathcal{G} = G^a T^a \]
\[ = (\nabla \cdot E^a + ig \cdot if_{abc} A^b \cdot E^c) T^a \]
\[ = \nabla \cdot E + ig [ A^i , E^i] , \] (2.11)

where \( A^i = A^i_a T^a \) and so on, and \( G^a = D \cdot \pi^a \). Thus, it is understood that \( \mathcal{G} \) is nothing but the infinitesimal generator of the gauge transformation.
2.2. Variational approach to pure gauge theory in quantum field theory

In this subsection, we formulate the time-dependent variational method for the pure Yang-Mills gauge theory by using the functional Schrödinger picture\(^{15,16}\) within the Gaussian approximation. We formulate our variational method in the canonical form by the help of the canonical variable or canonicity conditions.\(^{13,14}\)

The time-dependent variational principle is formulated as

\[
\delta \int dt \langle \Phi | i \frac{\partial}{\partial t} - \int d^3 x H(\Phi) = 0 ,
\]

where \(H\) means the Hamiltonian density under consideration. In the functional Schrödinger picture, the commutation relation \([A_i^a(x), E_j^b(y)] = i\delta_{ij}\delta_{ab}\delta^3(x - y)\) leads to

\[
E_i^a(x)|\Phi\rangle = -i \frac{\delta}{\delta A_i^a(x)}|\Phi\rangle .
\]

It is restricted ourselves that the trial state \(|\Phi\rangle\) or the trial wave functional \(\Phi(A^a) = \langle A^a|\Phi\rangle\) has the following Gaussian form as

\[
\Phi(A^a) = \mathcal{N}^{-1} \exp(i\langle E|A - \bar{A}\rangle) \exp\left(-\langle A - \bar{A}\rangle \frac{1}{4G} - i\langle \Sigma|A - \bar{A}\rangle\right) .
\]

Here, we used abbreviated notations such as

\[
\langle E|A\rangle = \int d^3 x E_i^a(x, t) \cdot A_i^a(x) ,
\]

\[
\langle A| \frac{1}{4G} A\rangle = \int \int d^3 x d^3 y A_i^a(x) \frac{1}{4G} G^{-1ab}_{ij}(x, y, t) A_j^b(y) .
\]

Here, \(\bar{A}_i^a(x, t), E_i^a(x, t), G_{ij}^{ab}(x, y, t)\) and \(\Sigma_{ij}^{ab}(x, y, t)\) are the variational functions which are determined by the time-dependent variational principle. The reason why the form (2.14) is adopted is that the canonicity conditions for (\(\bar{A}_i^a, E_i^a\)) and (\(G_{ij}^{ab}, \Sigma_{ij}^{ab}\)) are automatically satisfied:

\[
\langle \Phi| \frac{\delta}{\delta \bar{A}_i^a}|\Phi\rangle = E_i^a , \quad \langle \Phi| \frac{\delta}{\delta E_i^a}|\Phi\rangle = 0 ,
\]

\[
\langle \Phi| \frac{\delta}{\delta G_{ij}^{ab}}|\Phi\rangle = 0 , \quad \langle \Phi| \frac{\delta}{\delta \Sigma_{ij}^{ab}}|\Phi\rangle = -G_{ij}^{ab} .
\]

Thus, our time-dependent variational method is formulated as a canonical form.

In the functional Schrödinger picture, the expectation values are easily calculated such as follows:

\[
\langle \Phi| A_i^a(x)|\Phi\rangle = \bar{A}_i^a(x, t) ,
\]

\[
\langle \Phi| E_i^a(x)|\Phi\rangle = E_i^a(x, t) ,
\]

\[
\langle \Phi| A_i^a(x) A_j^b(y)|\Phi\rangle = \bar{A}_i^a(x, t) \bar{A}_j^b(y, t) + G_{ij}^{ab}(x, y, t) ,
\]

\[
\langle \Phi| E_i^a(x) E_j^b(y)|\Phi\rangle = E_i^a(x, t) E_j^b(y, t) + \frac{1}{4} G^{-1ab}_{ij}(x, y, t) + 4(\Sigma G \Sigma)_{ij}^{ab}(x, y, t) ,
\]

\[
\langle \Phi| A^a(x) \cdot E^b(x)|\Phi\rangle = \bar{A}^a(x, t) \cdot \bar{E}^b(x, t) + 2(\Sigma G)_{a}^{ab}(x, x, t) .
\]
Thus, it is understood that \( \mathbf{A}_i^a \) represent the classical fields of gauge fields and the diagonal component of \( G_{ij}^{ab} \), that is, \( G_{i}^{aa} \), where indices \( i \) and \( a \) are no sum, is a quantum fluctuations around the classical field \( \mathbf{A}_i^a \). Thus, in this functional Schrödinger picture, the two-point function \( G_{ij}^{ab}(x, y, t) \) plays a role of the gauge-particle propagator.

It should be noted here that the trial state (2.14) does not have the gauge symmetry, that is \( \mathcal{G}[\Phi] \neq 0 \). Thus, we impose the gauge invariance by introducing the Lagrange multiplier. From (2.8), the constraint \( \mathbf{D} \cdot \pi^a = 0 \) is recast into another form \( \mathcal{G}^a = 0 \) from (2.11), where \( \mathcal{G}^a \) is the generator of the gauge transformation. Thus, we introduce the effective Hamiltonian density \( \mathcal{H} \) by considering the gauge invariance in the space of the trial states as\(^{12}\)

\[
\mathcal{H} = \mathcal{H}_0 - \omega^a(x) \mathcal{G}^a(x), \tag{2.18}
\]

where \( \omega^a(x) \) represents a Lagrange multiplier, which insures the constraint \( \mathcal{G}^a(x) = 0 \). Thus, we use the above Hamiltonian density in order to determine the time dependences of the variational functions \( \mathbf{A}_i^a(x, t) \), \( \mathbf{E}_i^a(x, t) \), \( G_{ij}^{ab}(x, y, t) \) and \( \Sigma_{ij}^{ab}(x, y, t) \).

The expectation value of the Hamiltonian can be expressed as following simple form:

\[
\langle H \rangle = \langle \Psi | \int d^3x [\mathcal{H}_0 - \omega^a(x) \mathcal{G}^a(x)] | \Psi \rangle
\]

\[
= \langle \mathcal{H}_0 \rangle - \int d^3x \omega^a(x) \langle \mathcal{G}^a(x) \rangle,
\]

\[
\langle \mathcal{H}_0 \rangle = \int d^3x \left( \frac{1}{2} \mathbf{B}^a(x) \cdot \mathbf{E}^a(x) + \frac{1}{2} \mathbf{E}^a(x) \cdot \mathbf{E}^a(x) + \frac{1}{8} \text{Tr}(x|G^{-1}|x) \right.
\]

\[
+ 2 \text{Tr}(x|\Sigma G \Sigma|x) + \frac{1}{2} \text{Tr}(x|K G|x) + \frac{g^2}{8} \left( \text{Tr}[S^i T^a(x|G|x)] \right)^2
\]

\[
+ \frac{g^2}{4} \text{Tr} \left[ S^i T^a(x|G|x) S^i T^a(x|G|x) \right],
\]

\[
\langle \mathcal{G}^a(x) \rangle = \nabla \cdot \mathbf{E}^a(x) - ig \text{Tr}(x|T^a[S, G]|x) + ig \cdot if_{abc} \mathbf{A}^b \cdot \mathbf{E}^c, \tag{2.19}
\]

where we define

\[
\mathbf{B}^a_i = \epsilon_{ijk} \partial_j \mathbf{A}^a_k - \frac{1}{2} g f_{abc} \epsilon_{ijk} \mathbf{A}^b_j \mathbf{A}^c_k,
\]

\[
(S^i)_{jk} = i \epsilon_{ijk}, \quad (T^a)_{bc} = -i f_{abc},
\]

\[
K = (-i \mathbf{S} \cdot \mathbf{D})^2 - g \mathbf{S} \cdot \mathbf{B},
\]

\[
\mathbf{D} = \nabla - ig \mathbf{A}, \quad \mathbf{A} = \mathbf{A}^a_i T^a, \quad \mathbf{B} = \mathbf{B}^a T^a. \tag{2.20}
\]

Here, we can use the abbreviated notation such as \( \langle x|G|y \rangle = G_{ij}^{ab}(x, y, t) \). In the above representation, \( \mathbf{S} \) represent the spin 1 matrices whose spatial component with \( i \) is \( S^i \).

2.3. Variational equations and their solutions in the time-independent case

The equations of motion for the variational functions are derived from the time-dependent variational principle in Eq.(2.12) with the Hamiltonian density (2.18).
The results are summarized in the form of canonical equations of motion as

\[
\begin{align*}
\hat{A}^a(x, t) &= \frac{\delta\langle H \rangle}{\delta \dot{E}^a(x, t)}, \\
\hat{E}^a(x, t) &= -\frac{\delta\langle H \rangle}{\delta \hat{A}^a(x, t)}, \\
\hat{G}^a_{ij}(x, y, t) &= \frac{\delta\langle H \rangle}{\delta \Sigma^a_{ij}(x, y, t)}, \\
\hat{\Sigma}^a_{ij}(x, y, t) &= -\frac{\delta\langle H \rangle}{\delta \hat{G}^a_{ij}(x, y, t)},
\end{align*}
\]  

(2.21a)

where \( \langle H \rangle \) is given in Eq. (2.19).

In the time-independent case, the above equations of motion in Eq. (2.21a) allow the following solutions within the lowest order of \( g \) as

\[
\hat{A}^a(x) = 0, \quad \hat{E}^a(x) = -\nabla \omega^a(x).
\]  

(2.22)

In the above solutions, the mean field or the classical field \( \hat{A}^a(x) \) is identical to zero, namely, under this situation, only the quantum field is dealt with in this matter system. Further, in the lowest order of \( g \), the time-independent equations (2.21b) present the following solutions under \( \hat{A}^a = 0 \):

\[
\begin{align*}
G^a_{ij}(x, y) &= \int \frac{d^3k}{(2\pi)^3} e^{ik(x-y)} G^a_{ij}(k) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3} e^{ik(x-y)} G_k \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right), \\
G^a_{ij}(k) &= \delta^{ab} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) G_k, \quad G_k = \frac{1}{2|k|}, \\
\Sigma^a_{ij}(x, y) &= \int \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{i k' x} e^{-i k y} \langle k' a | \Sigma_{ij} | b k \rangle, \\
\langle k' a | \Sigma_{ij} | b k \rangle &= \frac{1}{2} \left( \delta_{ii} - \frac{k_i k_j}{k^2} \right) \left( \delta_{jj} - \frac{k_i k_j}{k^2} \right) g \omega^c(q) \delta^{(3)}(k' - k - q) \\
&\quad \times f^{abc} \frac{G_k - G_{k'}}{G_k + G_{k'}}, \\
\omega^a(q) &= \int d^3x \omega^a(x) e^{-i k x}.
\end{align*}
\]  

(2.23a)

(2.23b)

(2.23c)

Thus, \( G^a_{ij}(k) \) has only the transverse component. This feature is plausible for the gauge-particle propagation.

2.4. Thouless-Valatin correction

In the Hamiltonian density (2.18), the constrained term \( \omega^a \hat{G}^a \) is introduced. This treatment is resemble to that of the nuclear rotation. In the nuclear many-body theory, the collective rotational motion of the axially symmetric deformed nuclei for \( z \)-axis is described in the same way used in this section. If the nuclear rotation occurs in the perpendicular to the \( x \) axis, the state \( |\Psi(t)\rangle = e^{-i Et} |\Psi\rangle \) is replaced as

\[
|\Psi_\omega(t)\rangle = e^{-i \omega t J_x} e^{-i E_\omega t} |\Psi\rangle,
\]  

(2.24)

where the angular momentum operator \( J_x \) and the angular velocity \( \omega \) are introduced. Then, the Schrödinger equation, \( i \partial_t |\Psi_\omega(t)\rangle = \hat{H} |\Psi_\omega(t)\rangle \) is recast into

\[
(\hat{H} - \omega \hat{J}_x) |\Psi\rangle = E_\omega |\Psi\rangle,
\]  

(2.25)
where $\hat{H}$ is the original nuclear Hamiltonian. It is known that we have to get rid of the effect of nuclear rotation from the total energy as

$$E = \langle \Psi | \hat{H} | \Psi \rangle - \frac{\langle \Psi | \hat{J}_x^2 | \Psi \rangle}{2I} = \langle \Psi | \hat{H} | \Psi \rangle - \Delta E_{TV}, \quad (2.26)$$

where $I$ is the moment of inertia and is defined as

$$I = \lim_{\omega \to 0} \frac{\langle \Phi_\omega | \hat{J}_x | \Phi_\omega \rangle}{\omega}. \quad (2.27)$$

This energy correction, $\Delta E_{TV}$, in Eq.\( (2.26) \) is well known as the Thouless-Valatin correction.\(^{19}\)

Thus, in the approach to the pure Yang-Mills theory, it is first pointed out that the same correction term is necessary in Ref.\(^{12}\). For the pure Yang-Mills theory in the treatment of the variational method, the Thouless-Valatin correction term can be expressed as

$$\Delta E_{TV} = \int \int d^3x d^3y \langle \Phi | G^a(x) G^b(y) | \Phi \rangle \langle ax | \frac{1}{2I} | by \rangle, \quad (2.28)$$

where the moment of inertia for the gauge rotation, $\mathcal{I}^{ab}(x, y) = \langle ax | \mathcal{I} | by \rangle$, is defined as

$$\mathcal{I}^{ab}(x, y) = \lim_{\omega^b(y) \to 0} \frac{\langle \Phi | G^a(x) | \Phi \rangle}{\omega^b(y)}. \quad (2.29)$$

It is first shown that, in Ref.\(^{12}\), the above Thouless-Valatin correction term and the contribution of the moment of inertia for the gauge rotation play essential roles in order to reproduce the one-loop running coupling constant. The validity of our time-dependent variational approach owes the fact that the one-loop running coupling constant is exactly reproduced in the lowest order approximation of $g$ under $\mathcal{T}^a = 0$ developed in Ref.\(^{12}\).

§3. Time-dependent variational equations for quantum gauge fields

In this section, we present the equations of motion for the quantum fluctuations around the classical field configurations $\mathcal{T}^a$ and $E^a$, namely, $G^{ab}_{ij}$ and $\Sigma^{ab}_{ij}$ for quantum gauge fields, in a slightly different forms from Eq.\( (2.21b) \). We can formulate the equations of motion for quantum gauge fields as the Liouville-von Neumann equation.

3.1. Liouville-von Neumann equation for quantum gauge fields

First, the reduced density matrix $\mathcal{M}$ is introduced as is similar to the Hartree-Bogoliubov theory for many-body physics in the boson systems. We define the reduced density matrix\(^{20),21}\) for the quantum gauge fields as

$$\mathcal{M}^{ab}_{ij}(x, y, t) = \begin{pmatrix}
-i \langle \hat{A}_i^a(x, t) \hat{E}_j^b(y, t) \rangle - \frac{1}{2} & \langle \hat{A}_i^a(x, t) \hat{A}_j^b(y, t) \rangle \\
\langle E_i^a(x, t) \hat{E}_j^b(y, t) \rangle & i \langle E_i^a(x, t) \hat{A}_j^b(y, t) \rangle - \frac{1}{2}
\end{pmatrix}$$
where the symbol \( \langle \cdots \rangle \) represents the expectation values for the state \( |\Phi(t)\rangle \) in Eq. (2.14) as is shown in Eq. (2.17). In the later, the expectation values are replaced into the thermal averages. Here, \( \hat{A}_i^a \) means the quantum fluctuations around the classical configuration \( \overline{A}_i^a \). Thus, this reduced density matrix \( \mathcal{M} \) can be regarded as the one consist by the quantum gauge fields.

By the help of the Heisenberg equations of motion in the Heisenberg picture, the time evolution of the reduced density matrix is easily derived. As the result, we can obtain the following Liouville-von Neumann type equation of motion for the reduced density matrix composed of the quantum gauge fields as

\[
i \mathcal{M}_{ij}^{ab}(x, y, t) = \left[ \mathcal{H}, \mathcal{M} \right]_{ij}^{ab}(x, y, t),
\]

\[
\hat{\mathcal{H}}_{ij}^{ab}(x, y, t) = \left( \lambda_{ij}^{ab}(x) \delta_{ij} \delta_{ab} - \frac{\delta_{ij} \lambda_{ab}(x)}{\Gamma_{ij}^{ab}(x, t)} \right) \delta^3(x - y),
\]

\[
\Gamma_{ij}^{ab}(x, t) = K_{ij}^{ab} + g^2 \langle S_i^c(x)|G|S_j^c \rangle \delta_{ij} + g^2 \langle S_i^c(x)|T^c \rangle \delta_{ij} \text{Tr}[S_i^c(x)|G|x]
\]

\[
\lambda_{ij}^{ab}(x) = -ig\omega^p(x) f_{pab}\delta_{ij} = g(\omega^p(x) T^p)_{ij}^{ab},
\]

where \( K_{ij}^{ab}, S_i^c \) and \( T^c \) have been defined in Eq. (2.20). Here, \( \mathcal{H} \) is the Hamiltonian matrix which governs the time evolution of the reduced density matrix.

At this stage, it is important to indicate that the square of the reduced density matrix \( \mathcal{M} \) is easily obtained from the second line of Eq. (3.1) as

\[
\mathcal{M}^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.
\]

Thus, the eigenvalues of the reduced density matrix are \( \pm 1/2 \) as is similar to the case of the linear sigma model.\(^{20,21} \)

The eigenvector for the eigenvalue 1/2 can be expressed as

\[
\langle x | 1/2_n ai \rangle = \begin{pmatrix} u_{n_i}^a(x, t) \\ v_{n_i}^a(x, t) \end{pmatrix},
\]

where \( n \) represents a certain quantum number. Then the following eigenvalue equation should be satisfied:

\[
\int d^3y \mathcal{M}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{n_i}^b(y, t) \\ v_{n_j}^a(y, t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_{n_i}^a(y, t) \\ v_{n_j}^a(y, t) \end{pmatrix}.
\]

From the above eigenvalue equation, we can derive the following equation:

\[
\int d^3y \mathcal{M}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{n_j}^b(y, t) \\ -v_{n_i}^a(y, t) \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} u_{n_i}^a(y, t) \\ -v_{n_j}^a(y, t) \end{pmatrix}.
\]
Thus, we conclude that the $i(u^*_{n_i} - v^*_{n_i})$ is the eigenvector for the reduced density matrix $M$ with the eigenvalue $-1/2$, which is expressed as $\langle x \mid -1/2_n ai \rangle$. Further, we can derive the following:

$$
\int d^3y M^{ab}_{ij}(x, y, t) \left( \frac{v^a_{n_1}(y, t)}{u^a_{n_1}(y, t)} \right) = \frac{1}{2} \left( \frac{v^a_{n_1}(y, t)}{u^a_{n_1}(y, t)} \right).
$$

(3.7)

Thus, we can introduce another vector as

$$
\langle x \mid 1/2^a_n ai \rangle = \left( \frac{v^a_{n_1}(x, t)}{u^a_{n_1}(x, t)} \right).
$$

(3.8)

Then, we can express $\langle 1/2^a_n ai \mid x \rangle = (v^*_{n_1}(x, t), u^*_{n_1}(x, t))$.

3.2. Spectral decomposition of reduced density matrix

In the previous subsection, it is learned that the reduced density matrix $M$ has the eigenvalues $\pm 1/2$. In this subsection, the eigenvalue equation is summarized taking into account the extension to the finite temperature systems. Further, by using the eigenstates for $M$, the reduced density matrix is expressed in the form of the spectral decomposition.

Considering the extension to finite temperature systems, the eigenvalue equations are described with the abstract representation as

$$
M^{ab}_{ij} |\sigma f_n, b, j \rangle = \sigma f_n |\sigma f_n, a, i \rangle, \quad M^{ab}_{ij} |\sigma f_n^\dagger, b, j \rangle = \sigma f_n |\sigma f_n^\dagger, a, i \rangle,
$$

(3.9)

where $f = 1/2$ and $\sigma = \pm$ at zero temperature developed in the previous subsection. From the second equation in (3.9), we obtain

$$
\langle \sigma f_n^\dagger, b, j | M^{ba}_{ij} = \sigma f_n \langle \sigma f_n^\dagger, a, i \rangle.
$$

(3.10)

By using Eqs. (3.9) and (3.10), we can easily derive the following orthogonal relations as

$$
\sum_{a, i} \langle \sigma f_n^\dagger, a, i | \sigma f_n, a, i \rangle = \delta_{\sigma \sigma'} \delta_{n n'}, \quad (3.11)
$$

where the normalization condition of $|\sigma f_n, a, i \rangle$ is taken into account. By using the completeness relation $\int d^3x |x\rangle \langle x| = 1$ and the expression such as (3.4) with $\sigma = \sigma' = 1$ or $\sigma = 1$ and $\sigma' = -1$, the above condition (3.11) is rewritten as

$$
\sum_{a, i} \int d^3x (v^a_{n_1}(x)u^a_{n_1}(x) + u^*_{n_1}(x)v^a_{n_1}(x)) = \delta_{nn'}, \quad (3.12)
$$

$$
\sum_{a, i} \int d^3x (u^a_{n_1}(x)v^a_{n_1}(x) - v^*_{n_1}(x)u^a_{n_1}(x)) = 0, \quad (3.12)
$$

$$
\sum_{a, i} \int d^3x (u^a_{n_1}(x)v^a_{n_1}(x) - v^*_{n_1}(x)u^a_{n_1}(x)) = 0. \quad (3.12)
$$
Thus, the reduced density matrix $\mathcal{M}$ itself can be expressed in terms of the eigenstates of $\mathcal{M}$ as follows:

$$
\mathcal{M}_{ij}^{ab}(x, y, t) = \sum_{n(\sigma > 0)} f_n \left[ \begin{pmatrix} u_{ni}^a(y, t) \\ v_{ni}^a(y, t) \end{pmatrix} (v_{nj}^b(y) , u_{nj}^b(y)) \\
+ \begin{pmatrix} u_{ni}^a(y, t) \\ -v_{ni}^a(y, t) \end{pmatrix} (-v_{nj}^b(y) , u_{nj}^b(y)) \right].
$$

(3.13)

We can easily verify that the above $\mathcal{M}$ satisfy the eigenvalue equations (3.5) and (3.6) with $f_n$ instead of $1/2$.

Next, let us determine the eigenvalue $f_n$ in the system at finite temperature. In the time-independent case, from Eq.(3.2a), it is seen that the reduced density matrix and the Hamiltonian matrix commute each other. Thus, there exist the simultaneous eigenstates for $\mathcal{M}$ and $\tilde{\mathcal{H}}$ whose eigenvalues are $\sigma f_n$ and $\sigma E_n$, respectively. Namely,

$$
\int d^3 y \tilde{\mathcal{H}}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{ni}^b(y, t) \\ v_{ni}^b(y, t) \end{pmatrix} = E_n \begin{pmatrix} u_{ni}^a(y, t) \\ v_{ni}^a(y, t) \end{pmatrix},
$$

$$
\int d^3 y \tilde{\mathcal{H}}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{nj}^b(y, t) \\ -v_{nj}^b(y, t) \end{pmatrix} = -E_n \begin{pmatrix} u_{nj}^a(y, t) \\ -v_{nj}^a(y, t) \end{pmatrix}.
$$

(3.14)

Thus, we can derive $\text{Tr}(\tilde{\mathcal{H}} \mathcal{M}) = \sum_m E_m$. Here, the Helmholtz free energy is defined as

$$
F = \langle \tilde{\mathcal{H}} \rangle - TS,
$$

$$
\langle \tilde{\mathcal{H}} \rangle = \text{Tr}(\tilde{\mathcal{H}} \mathcal{M}) = \sum_m E_m,
$$

$$
S = \sum_{m\sigma} [(1 + n_{m\sigma}) \ln(1 + n_{m\sigma}) - n_{m\sigma} \ln n_{m\sigma}] ,
$$

(3.15)

where $T$ is temperature. Thus, the minimization condition is imposed:

$$
\delta F = \text{Tr}(\tilde{\mathcal{H}} \delta \mathcal{M}) - T \delta S = 0.
$$

(3.16)

Here, we assume that $\sigma f_n$ depends on $n_{m\sigma}$ linearly. Under this assumption, $\delta \mathcal{M}/\delta n_{m\sigma} = \delta(u, v)(v^*, u^*)$ for $\sigma = +$ and $\delta \mathcal{M}/\delta n_{m\sigma} = \delta(u^*, -v^*)(-v, u)$ for $\sigma = -$ are obtained respectively. Thus, we obtain

$$
\frac{\delta F}{\delta n_{m\sigma}} = E_m - T \ln \frac{1 + n_{m\sigma}}{n_{m\sigma}} = 0,
$$

i.e.,

$$
n_{m\sigma} = \frac{1}{e^{E_m/T} - 1},
$$

(3.17)

where $\sigma = \pm$. Thus, we omit the suffix $\sigma$ in $n_{m\sigma}$. Of course, the eigenvalue $f$ is reduced to $1/2$ when $T \to 0$. Thus, finally, we obtain

$$
f_m = n_m + \frac{1}{2}, \quad n_m = \frac{1}{e^{E_m/T} - 1}.
$$

(3.18)
From the (1,2)-component of the reduced density matrix in Eq. (3.1), the two-point function $G$ can be decomposed by each spectral function as

$$G_{ij}^{ab}(x, y, t) = \sum_{n} f_{n} \left[ u_{ni}^{a}(x) u_{nj}^{b}(y) + u_{ni}^{a}(x) u_{nj}^{b}(y) \right].$$  \hspace{1cm} (3.19)

Hereafter, we take the quantum number $n$ as momentum $k$. Then, we obtain

$$G_{ij}^{ab}(x, y, t) = \int \frac{d^{3}k}{(2\pi)^{3}} f_{k} \left[ u_{ki}^{a}(x) u_{kj}^{b}(y) + u_{ki}^{a}(x) u_{kj}^{b}(y) \right].$$  \hspace{1cm} (3.20)

At zero temperature, we have already derived the expression of $G$ in Eq. (2.23). In the above expression in (3.20), $f_{k} = 1/2$ and $uu^{*}$ is also expressed. By using the knowledge of the zero temperature case, the above expression in (3.20) can be recast into

$$G_{ij}^{ab}(x, y, t) = \delta_{ij} \int \frac{d^{3}k}{(2\pi)^{3}} G_{k} \left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right) e^{ik \cdot (x-y)},$$

$$G_{k} = (2n_{k} + 1) \frac{1}{2|k|}. \hspace{1cm} (3.21)$$

From the eigenvalue equation (3.14), the energy eigenvalue for the quantum gauge fields is easily obtained. For simplicity, we neglect the gauge rotating term, namely, we put $\omega_{i}(x) = 0$. Then from (3.14), we obtain

$$E_{k} \delta_{ij} \delta_{ab} = \Gamma_{ij}^{1/2} \frac{k_{i}k_{j}}{k^{2}},$$

$$\Gamma_{ij}^{1/2}(k) = \delta_{ij} |k| \left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right). \hspace{1cm} (3.22)$$

up to the order of $g$.

Finally, it should be noted that the gauge-particle has only transverse component as is realized in Eq. (2.23) with factor $\left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right)$. Since the reduced density matrix $\mathcal{M}$ includes $G$ and $\Sigma$ composed by the quantum gauge field, $\mathcal{M}$ also contains the factor $\left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right)$. Thus, we define the following projection operator $\hat{P}$:

$$\frac{\langle k' | \hat{P}_{ij} | k \rangle}{\langle k' | k \rangle} = \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}}. \hspace{1cm} (3.23)$$

Then, $\hat{P}$ certainly has a property of the projection operator, namely, $\hat{P}^{2} = \hat{P}$. As is easily shown, the following relations are satisfied:

$$G = \hat{P} G \hat{P}, \quad \Sigma = \hat{P} \Sigma \hat{P},$$

$$\mathcal{M} = \hat{P} \mathcal{M} \hat{P} = \hat{P} \mathcal{M} = \mathcal{M} \hat{P}. \hspace{1cm} (3.24)$$

Since there always exists the projection factor to the transverse component, $\left( \delta_{ij} - \frac{k_{i}k_{j}}{k^{2}} \right)$, the inverse of the two-point function $G$ should be regarded as the two point function which satisfies the following relation:

$$G^{-1}G = \hat{P}. \hspace{1cm} (3.25)$$
§4. Transport coefficients of quantum gluonic matter

Hereafter, we deal with the color $su(3)$ pure gauge theory, namely the QCD without quarks. In this section, we present the expression of the transport coefficients in our variational approach by using the Kubo formula\(^8,9\) for the quantum gluonic matter with the color $su(3)$ symmetry.

4.1. Kubo formula based on the variational approach

By taking into account an external source field $\hat{A}$ and its conjugate force $F(t)$, the Hamiltonian $\hat{H}$ is modified from $\hat{H}_0$ to

$$\hat{H}(t) = \hat{H}_0 - \hat{A}F(t). \quad (4.1)$$

If the external force $F(t)$ is adopted as $F(t) = Fe^{-it}$, then, an observable $\hat{B}$ at time $t$, $\langle \hat{B} \rangle_t$, can be expressed in the linear response theory as

$$\langle \hat{B} \rangle_t = \langle \hat{B} \rangle_{eq} + \chi_{BA}(\omega)Fe^{-it}, \quad (4.2)$$

where $\langle \cdots \rangle_{eq}$ means the thermal average with respect to the equilibrium state. Here, $\chi_{BA}(\omega)$ is called the complex admittance and is defined as

$$\chi_{BA}(\omega) = \lim_{\epsilon \to 0^+} i\frac{\hbar}{\omega} \int_{0}^{\infty} dt \langle [\hat{A}, \hat{B}(t)] \rangle_{eq} e^{i\omega t - \epsilon t}, \quad (4.3)$$

where

$$\varphi_{BA}(t) = -i\frac{\hbar}{\omega} \langle [\hat{A}, \hat{B}(t)] \rangle_{eq}
= \int_{0}^{\beta} d\lambda \langle \hat{A}(-i\hbar\lambda)\hat{B}(t) \rangle_{eq}, \quad (4.4)$$

is a quantum response function. By using the integrated by part and an property of the equilibrium state, namely, $\langle \hat{A}\hat{B} \rangle_{eq} = -\langle \hat{A}\hat{B} \rangle_{eq}$, the complex admittance is recast into

$$\chi_{BA}(\omega) = \frac{1}{\hbar\omega} \lim_{\epsilon \to 0^+}\int_{0}^{\infty} dt e^{i\omega t - \epsilon t} \langle [\hat{B}(t), \hat{A}(0)] \rangle_{eq}
- \frac{1}{\hbar\omega} \lim_{\epsilon \to 0^+} \int_{0}^{\infty} dt e^{-\epsilon t} \langle [\hat{A}(t), \hat{B}(0)] \rangle_{eq}. \quad (4.5)$$

Following the general theory,\(^9\) the transport coefficients are obtained by adopting both operators $\hat{A}$ and $\hat{B}$ being currents $J(t)$ as

$$\chi_{BA}(\omega, k=0) = \frac{1}{\omega} \lim_{\epsilon \to 0^+} \int_{0}^{\infty} dt \int d^3r e^{i\omega t - \epsilon t} \langle [J(r, t), J(0, 0)] \rangle_{eq}
- \frac{1}{\omega} \lim_{\epsilon \to 0^+} \int_{0}^{\infty} dt \int d^3r e^{-\epsilon t} \langle [J(r, t), J(0, 0)] \rangle_{eq}, \quad (4.6)$$
where we return to the natural unit, $\hbar = 1$. Here, current $J(t)$ is defined as

$$J(t) = \int d^3r J(r, t)e^{-ik \cdot r}$$

(4.7)

for an application to the gluonic matter in mind.

4.2. Shear viscosity in the quantum gluonic matter

In the gluonic matter, the dependence of the coupling constant $g$ for the usual, not anomalous, shear viscosity is given as (1.1) in the lowest order of $g$. In this paper, since we deal with the quantum gluonic field described by the variables $G$ and $\Sigma$ with $A = 0$, so we can evaluate the usual shear viscosity $\eta_C$ with higher order of $g$ compared with those developed in the previous papers. Hereafter, we denote $\eta_C$ as $\eta$ simply because, as a final result in this paper, the small value of $\eta_C$ can be derived from the viewpoint of the weak coupling QCD. Thus, it is expected that $\eta$ is also small from Eq.(1.2) for the quantum gluonic matter.

In order to calculate the shear viscosity, the energy-momentum tensor for the pure gluonic field is necessary. In the symmetric representation, the energy-momentum tensor $T^\mu_\nu$ is obtained as

$$T^\mu_\nu = F^\mu_{\rho a} F^\nu_{\rho a} - \frac{1}{4} \delta^\mu_\nu F^\alpha_{\rho a} F^\rho_{\alpha a} .$$

(4.8)

Then, $T^\mu_\nu$ can be expressed in terms of the color electric and color magnetic fields as

$$T_{00}(r) = \frac{1}{2}(E^a(r) \cdot E^a(r) + B^a(r) \cdot B^a(r)) ,$$

$$T_{0i}(r) = -\epsilon_{ijk} E^a_j(r) B^a_k(r) ,$$

$$T_{ij}(r) = -E^a_i(r) E^a_j(r) - B^a_i(r) B^a_j(r) + \frac{1}{2} \delta_{ij} (E^a(r) \cdot E^a(r) + B^a(r) \cdot B^a(r)) .$$

(4.9)

The shear viscosity $\eta(\omega)$ is obtained by taking the current $J$ as $T_{xy} = T_{12}$ in Eq.(4.6),

$$\eta(\omega) = \frac{i}{\omega} \left[ \Pi^R(\omega) - \Pi^R(0) \right] ,$$

$$\Pi^R(\omega) = -i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r e^{i\omega t - ct} \langle [ T_{12}(r, t) , T_{12}(0, 0) ] \rangle_{eq} .$$

(4.10)

By taking a limit $\omega \to 0$, the shear viscosity $\eta(0)$ has a simple form as

$$\eta(0) = -\frac{d}{d\omega} \text{Im} \Pi^R(\omega) \bigg|_{\omega \to +0} .$$

(4.11)

Here, the thermal average $\langle \cdots \rangle_{eq}$ can be replaced to the expectation value given in Eq.(3.21) for $G$ and (2.23b) for $\Sigma$ with $G^T_k$ instead of $G_k$ in our variational approach at finite temperature. Thus, it is necessary for calculating the shear viscosity to evaluate the thermal average $\langle [ T_{12}(r, t) , T_{12}(0, 0) ] \rangle$ in our framework.
Here, we can derive

\[ T_{12}(r, t) = -E_i^a(r, t)E_i^a(r, t) - B_i^a(r, t)B_i^a(r, t) \, . \]  

The operators \( E_i^a(r, t) \) and \( B_i^a(r, t) \) are obtained as a result of the time evolution governed by the Hamiltonian \( \int d^3x H_0 \) with \( [2.10] \):

\[ E_i^a(r, t) = e^{iH_0t}E_i^a(r)e^{-iH_0t} , \quad B_i^a(r, t) = e^{iH_0t}B_i^a(r)e^{-iH_0t} , \]  

\[ H_0 = \int d^3r \frac{1}{2} \left[ (E_i^a(r))^2 + (B_i^a(r))^2 \right] . \]

Here, we can derive

\[ [ H_0 , E_i^a(r) ] = -i\epsilon_{ijk}\partial_kB_j^a(r) - \frac{i}{2}g\epsilon_{ijk}\varepsilon^{abc}(B_j^b(r)A_k^c(r) + A_k^b(r)B_j^b(r)) \]

\[ = -i(S \cdot \mathbf{p})_{ik}B_k^a(r) + O(g) , \]

\[ [ H_0 , B_i^a(r) ] = i\epsilon_{ijk}\partial_kE_j^a(r) + \frac{i}{2}g\epsilon_{ijk}\varepsilon^{abc}(E_j^b(r)A_k^c(r) + A_k^b(r)E_j^b(r)) \]

\[ = i(S \cdot \mathbf{p})_{ik}E_k^a(r) + O(g) , \]

\[ \hat{\mathbf{p}} = -i \frac{\partial}{\partial r} , \]

where \( S \) is defined in \( [2.20] \). Thus, up to the lowest order of \( g \) in our quantum gluonic matter, we can derive the operators at time \( t \) as

\[ E_i^a(r, t) = \hat{C}_{ij}^{ab}(t)E_j^b(r) + \hat{D}_{ij}^{ab}(t)B_j^b(r) + O(g) , \]

\[ B_i^a(r, t) = \hat{C}_{ij}^{ab}(t)B_j^b(r) - \hat{D}_{ij}^{ab}(t)E_j^b(r) + O(g) , \]

\[ \hat{C}_{ij}^{ab}(t) = \delta^{ab}[\cos(t(S \cdot \hat{\mathbf{p}}))]_{ij} , \quad \hat{D}_{ij}^{ab}(t) = \delta^{ab}[\sin(t(S \cdot \hat{\mathbf{p}}))]_{ij} . \]

Thus, we can derive the energy-momentum tensor operator at time \( t \) in Eq.(4.12).

From \( [1.10] \), we need the \( \langle [ T_{12}(r, t), T_{12}(r, 0) ] \rangle \) to estimate the shear viscosity in quantum gluonic matter. After lengthy but straightforward calculation, we can derive the following form up to the lowest order of \( g \):

\[ \langle [ T_{12}(r, t) , T_{12}(r', 0) ] \rangle \]

\[ = M_{12ij}^{ab}(t) \cdot 2i\partial_t^r \delta(r - r') \cdot \left[ \epsilon_{ijl}\Xi_{12}^{ab}(r, r', t) + \epsilon_{j2l}\Xi_{11}^{ab}(r, r', t) + \epsilon_{ilj}\Xi_{2j}^{ab}(r', r, t) + \epsilon_{i2j}\Xi_{1j}^{ab}(r', r, t) \right] \]

\[ - \hat{N}_{12ij}^{ab}(t) \cdot i\partial_t^r \delta(r - r') \cdot \left[ \epsilon_{ijl}T_{12}^{ab}(r, r', t) + \epsilon_{j2l}T_{11}^{ab}(r, r', t) - \epsilon_{ilj}T_{2j}^{ab}(r', r, t) - \epsilon_{i2j}T_{1j}^{ab}(r', r, t) \right] \]

\[ + O(g^2) , \]  

\[ (4.17) \]
Thus, in Eq. (4.17), the first term vanishes, namely, 

$$\Xi_{ij}(t) = F_{ij}(t) + \hat{D}_{ij}(t) \ ,$$

$$N_{ij}^{ab}(t) = \hat{F}_{ij}^{ab}(t) - \hat{D}_{ij}^{ab}(t) \ ,$$

$$\Xi_{ij}^{ab}(r, r', t) = \epsilon_{imk} \partial_m^{(G \Sigma)_{kj}}(r, r', t) + \epsilon_{jmk} \partial_m^{(G \Sigma)_{ik}}(r, r', t) + O(g^2)$$

$$= - (S \cdot \hat{p})_{ik}(G \Sigma)_{kj}^{ab}(r, r', t) - (\Sigma G)_{ik}^{ab}(r, r', t) (S \cdot \hat{p})_{kj} + O(g^2)$$

$$= \frac{1}{2} \left[ \langle E^a(r) B_j^b(r') \rangle + \langle B_i^a(r) E_j^b(r') \rangle \right] + O(g^2) \ ,$$

$$\gamma_{ij}^{ab}(r, r', t) = - \frac{1}{4} (G^{-1})_{ij}^{ab}(r, r', t) + \epsilon_{ipm} \epsilon_{jkn} \partial_p^{(G \Sigma)} G_{mn}^{ab}(r, r', t) + O(g^2)$$

$$= - \frac{1}{4} (G^{-1})_{ij}^{ab}(r, r', t) + (S \cdot \hat{p})_{im} G_{mn}^{ab}(r, r', t) + O(g^2)$$

$$= \langle B_i^a(r) B_j^b(r') \rangle - \langle E^a(r) E_j^b(r') \rangle + O(g^2) \ ,$$

(4.18)

where we define $f(r, r', t) \hat{p} = -i \partial f(r, r', t) / \partial r$. Here, we used the fact that $\Sigma$ is of order of $g$ as is seen in Eq. (2.23).

4.3. Lowest order approximation for shear viscosity in quantum gluonic matter

In the lowest order of $g$, $M_{ij}^{ab}$ in Eq. (4.18) is proportional to $\delta^{ab}$. On the other hand, $\Xi_{ij}^{ab}$ is proportional to $f^{abc} \omega^c$ because both the variables $(G \Sigma)_{kj}^{ab}$ and $(G \Sigma)_{ik}^{ab}$ in $\Xi_{ij}^{ab}$ are proportional to $f^{abc} \omega^c$ which is derived from (2.23) with $G_k^T$ and (3.21). Thus, in Eq. (4.17), the first term vanishes, namely,

$$M_{ijkl}^{ab} \bar{m}_{mn} = 0 \quad \text{for any } i, j, k, l, m \text{ and } n \ .$$

(4.19)

Next, let us consider the second term in (4.17). At zero temperature, the equation of motion for $G$ can be derived from (2.19) or (2.21b) by $\delta \langle H \rangle / \delta G = 0$ which leads to

$$- \frac{1}{8} G^{-2} + \frac{1}{2} K = 0$$

(4.20)

in the lowest order of $g$. Thus, the solution of $G$ is written as

$$G = \frac{1}{2 \sqrt{K}} = \frac{1}{2 S \cdot \hat{p}} \ ,$$

(4.21)

where $K = (S \cdot \hat{p})^2$ with $\mathbf{A} = \mathbf{0}$ in (2.20). Using the above fact, from (4.18), the following is derived up to the lowest order of $g$:

$$\gamma_{ij}^{ab} = - \left[ \frac{1}{4} G^{-1} - (S \cdot \hat{p}) G (S \cdot \hat{p}) \right]_{ij}^{ab} = 0 \ .$$

(4.22)

As a result, the shear viscosity for the pure quantum gluonic matter is zero at zero temperature up to the order of $g^1$:

$$\eta(\omega) = 0 \quad (4.23)$$
From Eq.(4.10), the complex admittance for the shear viscosity \( \Pi \) line and we define \( g \). Here, up to the lowest order of \( N \) with \( G^T_k \) instead of \( G_k \). Thus, we obtain \( \gamma \) in the lowest order approximation as

\[
\gamma_{ij}^{ab}(r, r', t) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3} e^{ik(r-r')} \cdot \frac{2|k|}{\sinh\left(\frac{E_k}{T}\right)} \left( \delta_{ij} - \frac{k_ik_j}{|k|^2} \right)
\]

(4.24)

From Eq.(4.10), the complex admittance for the shear viscosity \( \eta(\omega) \) is written as

\[
\Pi^R(\omega) = -i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r \ e^{i\omega t - \epsilon t} \langle [T_{12}(r, t), T_{12}(0, 0)] \rangle
\]

\[
= i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r \ e^{i\omega t - \epsilon t} \tilde{N}_{12ij}(t) (i\partial_r \delta^3(r))
\times \left\{ \epsilon_{jil} \gamma_{12}^{ab}(r, 0, t) + \epsilon_{jil} \gamma_{j1}^{ab}(r, 0, t) - \epsilon_{il} \gamma_{j2}^{ab}(0, r, t) - \epsilon_{il} \gamma_{j1}^{ab}(0, r, t) \right\}
\]

(4.25)

where the integration by part has been carried out from the second line to the third line and we define \( \gamma_{t,ij}(r, r', t) \) as

\[
\gamma_{t,ij}(r, r', t) = \delta^{ab} \int \frac{d^3k}{(2\pi)^3} e^{ik(r-r')} \cdot \frac{k_i|k|}{\sinh\left(\frac{E_k}{T}\right)} \left( \delta_{ij} - \frac{k_ik_j}{|k|^2} \right).
\]

(4.26)

Here, up to the lowest order of \( g \) in this quantum gluonic matter, the operator \( \tilde{N}_{12ij}(t) \) is written in Eq.(3.13) with (3.16). Thus, we can expand \( \tilde{N}_{12ij}(t) \) as

\[
\tilde{N}_{12ij}(t) = \delta^{ab} \sum_{n=1} f(2n-1) e^{2n-1(S \cdot \hat{p})^2} \left[ \delta_{ij} t(S \cdot \hat{p})_{2j} - \delta_{2j} t(S \cdot \hat{p})_{1i} + O(\hat{p}^3) \right],
\]

(4.27)

for any \( i, j, k \) and \( m \) because the integrand is the total derivative with respect to \( r \). Thus, we can obtain the relation \( \Pi^R(\omega) = 0 \). Finally, from Eqs.(4.25) and (4.28), we conclude that the shear viscosity \( \eta(\omega) \) in Eq.(4.10) is as follows

\[
\eta(\omega) = 0
\]

(4.29)
up to the lowest order of \( g \) in the quantum gluonic matter. Here, the terms of the next order of \( \hat{C}_{ab}^{ij}(t) \) and \( \hat{D}_{ab}^{ij}(t) \) in \( \hat{N}_{12 ij}(t) \) include the coupling constant \( g \) without the spatial derivative. Thus, Eq. (4.28) is not satisfied in the order of \( g \). Therefore, the result (4.29) is valid up to the order of \( g^0 \) while (4.23) is valid up to the order of \( g^1 \) at zero temperature because the mechanism to vanish the value of shear viscosity is different.

§5. Summary and concluding remarks

In this paper, the time-dependent variational method for the pure Yang-Mills gauge theory is formulated in the functional Schrödinger picture with the Gaussian trial wave functional. In this variational method, the classical mean fields and the quantum fluctuations around them are treated self-consistently and both degrees of freedom are coupled each other. Further, the equations of motion for the quantum fluctuations around the mean fields were reformulated in a form of the Liouville-von Neumann equation for the reduced density matrix which was introduced in the Hartree-Bogoliubov approximation developed in the many-body problems for boson systems.

This variational method developed in this paper was applied to the pure quantum gluonic matter system in order to evaluate the shear viscosity, which is one of the transport coefficients of the gluonic matter in the system with the color \( su(3) \) symmetry, namely, the QCD without quarks. As a result, it was shown that there is no contribution of the quantum gluons to the shear viscosity in the pure gluonic matter up to the lowest order of the QCD coupling \( g \) at finite temperature. Namely, up to the order of \( g^0 \), the contribution of the quantum gluons to the shear viscosity is nothing. At zero temperature, adding to the order of \( g^0 \), there is no contribution up to the order of \( g \) due to the equations of motion. Thus, for small \( g \), namely, from the viewpoint of the weak coupling QCD,\(^{11} \) the shear viscosity in quantum gluonic matter may be small because the quantum gluons contribute to the shear viscosity from the order of \( g \) or higher at finite temperature.

Recently, Matsui and Matsuo give the transport equations for the Wigner distribution function and the anomalous distribution function\(^{22} \) which are coupled each other to determine the dynamics of the meson fields and the fluctuations around them in the linear sigma model, as is similar to our formalism. In order to compare the theoretical analysis with the experimental results, the information of the gluon distribution function may be necessary as was discussed in Ref.\(^{22} \) in the case of the linear sigma model. It is one of important further problems to investigate the gluon distribution function governed by the transport equation derived by this formalism such as the extended Boltzmann equations.\(^{22} \)

It may be interesting to investigate the higher order contribution to the shear viscosity because the approximation used in this paper corresponds to the Hartree-Bogoliubov like approximation. The random phase approximation (RPA) is missing in the treatment in this paper. It may be necessary to extend our treatment to including the RPA like modes as was developed for the linear sigma model.\(^{23} \) Further, it may be also interesting to investigate the behavior of other transport coefficients.
in this framework developed in this paper. They are future problems.

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