Ornate Necklaces and the Homology of the Genus One
Mapping Class group

James Conant

Abstract

According to seminal work of Kontsevich, the unstable homology of the mapping class

group of a surface can be computed via the homology of a certain lie algebra. In a recent paper,

S. Morita analyzed the abelianization of this lie algebra, thereby constructing a series of candi-

dates for unstable classes in the homology of the mapping class group. In the current paper,

we show that these cycles are all nontrivial, representing homology classes in $H_k(M^k_1; \mathbb{Q})_{\mathfrak{g}_k}$ for

all $k \geq 5$ satisfying $k \equiv 1 \mod 4$. Here $M^k_1$ is the mapping class group of a genus one surface

with $k$ punctures.

1 Introduction

Although the stable cohomology of the mapping class group has been completely computed [6],

the unstable cohomology remains an interesting target of exploration. A well-known theorem

of Kontsevich [5] (see also [1]) relates this unstable cohomology with the homology of a certain

symplectic lie algebra of derivations, $\mathfrak{a}_\infty$. To produce nontrivial elements in the cohomology of

some lie algebra $\mathfrak{a}$, it can be profitable to find an abelian quotient, $\mathfrak{b}$, of $\mathfrak{a}$, which yields a map

$$\bigwedge \mathfrak{b} = H^*(\mathfrak{b}) \rightarrow H^*(\mathfrak{a}).$$

This produces many cocycles in the image which become candidate cohomology classes. In [8],

Morita uses this technique to construct a sequence of cycles for the homology of $\text{Out}(F_n)$, which

Vogtmann and I analyzed in [2], showing that the first two are nontrivial.

The positive degree part of the lie algebra $\mathfrak{a}_\infty$ is the direct limit of lie algebras $\mathfrak{a}^+_n$. In this case,

Morita [7] calculates the weight 2 part of the abelianization of $\mathfrak{a}^+_n$, $H_1(\mathfrak{a}^+_n)_2$, from which he pro-

duces a sequence of cycles in the homology of the mapping class group. It is the purpose of this

paper to show that these are all nonzero, and in fact represent nontrivial classes in

$$H_k(M^k_1; \mathbb{Q})_{\mathfrak{g}_k}, \quad k \geq 5, \quad k \equiv 1 \mod 4$$

where $M^m_g$ represents the genus $g$ mapping class group with $m$ punctures, and $\mathfrak{g}_m$ is the symmetric
group permuting the punctures.
The homology of the genus one mapping class group has been studied previously by Getzler [4]. In fact, he explicitly computes the Euler characteristic of both $\mathcal{M}_1^m$ and $\mathcal{M}_1^m/\mathcal{S}_m$. The generating function for the latter is

$$\sum_{m=1}^{\infty} \chi(\mathcal{M}_1^m)x^m = \frac{(x + x^2 + x^3)(1 - x^4 - 2x^8 - x^{12} + x^{16})}{(1 - x^8)(1 - x^{12})}$$

$$= x + x^2 + x^3 - x^5 - x^6 - x^7 - x^9 - x^{10} - x^{11} - x^{14} - x^{15} - x^{17} - x^{18} - x^{19} - 3x^{21} - 3x^{22} - 3x^{23} - x^{25} - x^{26} - x^{27} - 3x^{29} - 3x^{30} - 3x^{31} - 3x^{33} - 3x^{34} - 3x^{35} - 3x^{37} - 3x^{38} - 3x^{39} - 3x^{41} - 3x^{42} - 3x^{43} - 5x^{45} - 5x^{46} - 5x^{47} - 3x^{49} - 3x^{50} \cdots$$

Thus the homology is growing, although it may only be growing very slowly. Thus these classes constructed from $H_1(\mathcal{M}_1^m)$ only form a small part of the homology. The genus one mapping class group remains an interesting object of investigation!

The outline of the paper is as follows. In the first section we review the definition of ribbon graph homology, which is a useful way to compute the cohomology of the mapping class group. We then introduce cocycles, $\Theta_k$, on this complex, and show they are nontrivial classes by constructing explicit cycles, $Z_k$, with which they pair nontrivially. In the last section we explain why these cocycles correspond to the ones constructed by Morita.

We finish the introduction with a couple of questions. In the main construction, the cocycles $\Theta_k$ are only nonzero for $k \in \{5, 9, 13, \ldots \}$, whereas the cycles are nonzero whenever $k \geq 3$ is odd.

**Question:** Do the cycles $Z_k$ represent nontrivial elements of $H^k(\mathcal{M}_1^m; \mathbb{Q})_{\mathcal{S}_k}$ for all odd $k \geq 3$?

Also, these classes $\Theta_k$ only form a small part of the homology, but arise in a very elegant way via the lie algebra theory. Hence it is natural to wonder what is special about them within the homology of the mapping class group.

**Question (Morita):** Can one identify these (co)cycles in the context of Getzler’s [4] computation using algebraic geometry and number theory?

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### 2 The ribbon graph complex

We review the definition of a well-known graphical chain complex computing the homology of the mapping class group. This is the ribbon graph complex.

**Definition 1**

1. A ribbon graph is a finite connected graph with vertices of valency at least 3, with the additional structure that each vertex has a specified cyclic order of all incoming half-edges. By convention, when
These graphs are drawn, they inherit these cyclic orders from the orientation of the plane of the paper. Every ribbon graph can be thickened canonically into an oriented surface.

2. An orientation of a graph, X, can be defined in many equivalent ways. It is an equivalence class of decorations on a graph, and in the connected case, there are exactly two orientations. For our purposes, an orientation will be determined by an ordering of the vertices and a direction for all the edges. Two such assignments of vertex order and edge direction are equivalent iff they differ by an even number of vertex swaps and edge reversals.

3. If \( m > 0 \), let \( rG^m_g \) be the rational vector space spanned by oriented ribbon graphs which thicken to a surface of genus \( g \) with \( m \) punctures, modulo the relations that \( (X, -\text{or}) = -(X, \text{or}) \), where or is an orientation of the graph \( X \).

4. Let \( rG = \bigoplus_{g \geq 0, m > 0} rG^m_g \).

5. The vector space \( rG^m_g \) is graded by number of vertices. Let the degree \( k \) subspace be denoted by \( rG^m_g[k] \).

6. The boundary operator \( d : rG^m_g[k] \rightarrow rG^m_g[k-1] \) is defined by letting \( d(X) \) be the sum of oriented graphs obtained by contracting each non-loop edge of \( X \). Note that the two vertices which are joined by the edge contraction inherit a canonical cyclic order. The orientation is determined by reordering the vertices so that the contracted edge will go from vertex 1 to vertex 2. Then in the contracted graph, the newly created vertex is ordered first, and the other vertices are shifted down by one from their previous numerical label.

The following result is well-known, and follows from work of Penner [9]. See also [1, Theorem 4].

**Theorem 1** There is an isomorphism

\[
H^{4g+2m-4-k}(\mathcal{M}^m_g, \mathbb{Q})_{\Sigma_m} \cong H_k(rG^m_g).
\]

It will be convenient to consider a quasi-isomorphic quotient complex \( \overline{rG} \).

**Definition 2**

1. A ribbon graph \( X \) is said to have a cut vertex, if in the thickening of \( X \), there is a properly embedded arc which meets \( X \) only at the vertex and disconnects the thickened surface into two pieces.

2. Let \( C \) be the subcomplex spanned by graphs with cut vertices, and let \( \overline{rG} = rG/C \).

**Proposition 1** The natural map \( rG \rightarrow \overline{rG} \) induces a quasi-isomorphism.

**Proof:** This follows from remarks at the end of Section 4.2 of [1]. If one considers the associative operad to be spanned by planar rooted binary trees as we did in that paper, then a separating edge
inside a tree at a vertex makes the vertex a cut vertex. If there is a separating edge of the graph outside of a vertex, then either of its two endpoints will correspond to cut vertices of the ribbon graph.

One could also reproduce the algebraic argument from [3], where the only change needed is to slightly modify the proof of Lemma 2.1 to account for the ribbon structure.

\[ \square \]

3 Morita’s unstable classes

In this section, for every \( k \geq 5 \) satisfying \( k \equiv 1 \mod 4 \), we construct a cocycle \( \Theta_k \), on the ribbon graph complex, and show that it represents a nontrivial class in

\[ H^k(\mathcal{R}_1^k) \cong H_k(\mathcal{M}_1^k, \mathbb{Q}_{\Theta}). \]

In the next section we will review Morita’s construction and show that the classes \( \Theta_k \) indeed coincide with the ones he constructed.

**Definition 3** For any integer \( k \geq 1 \), let \( X_k \in \mathcal{R}_1^k \) be the graph pictured in Figure 1.

**Proposition 2** The ribbon graphs \( X_k, k \geq 1 \), are nonzero if and only if \( k \equiv 1 \mod 4 \) and \( k \neq 1 \).

**Proof:**

First I would like to argue that when \( k > 1 \), the symmetry group of \( X_k \) as an unoriented ribbon graph is the dihedral group \( D_{2k} \). There is clearly a cyclic symmetry which preserves the ribbon structure. Reflection in a line is achieved by rotation of \( X_k \) by \( \pi \) through an axis, as in the picture on the right. This clearly preserves the ribbon structure. Thus the dihedral group forms a subgroup of the symmetry group. Conversely any symmetry of the graph induces a symmetry of the big loop, and this gives a map from the symmetry group onto the dihedral group. We claim that this is a monomorphism. To see this, suppose
that the induced element of the dihedral group is the identity. Then all vertices are fixed, and the
only thing that could change is the order in which the ends of each small loop attach. However,
these cannot change because that would change the cyclic order at the vertex, and therefore not
preserve the ribbon structure. Hence the original symmetry was the identity.

In the case that $k$ is even, the generator of the cyclic symmetry group is orientation-reversing.
Hence $X_k = -X_k$ and so $X_k = 0$. In the case that $k \equiv 3 \mod 4$, the rotation by $\pi$ around an
axis, as pictured above, is orientation-reversing, and so $X_k = 0$. To see this, note that the edge
directions are all reversed, while the vertices are changed by an odd permutation. Since there are
an even number of edges, the result follows. In the case that $X_k \equiv 1 \mod 4$ and $k \neq 1$, it is easy to
check that both of these types of symmetries are orientation preserving, and since they generate
the entire symmetry group, the graph is nonzero.

Finally, when $k = 1$, there is an orientation-reversing symmetry which exchanges the two loops.
Visualize the single vertex with the edge of one loop emanating from the top and feeding into the
bottom. The other loop emanates from the right and feeds into the left. If one rotates the picture
by $-\pi/4$, the direction of the vertical edge gets switched.

Definition 4 If $k \equiv 1 \mod 4$ and $k \neq 1$, then let $\Theta_k: rG^k_1 \to \mathbb{Q}$ be the characteristic function for $X_k$.

Lemma 1

1. $\Theta_k$ is a cocycle.

2. $\Theta_k$ induces a cocycle $\overline{\Theta}_k$ on $\overline{rG}^k_1$.

Proof: Let $Y$ be a ribbon graph. We wish to show that $\Theta_k(dY) = 0$. If $X_k$ appears as a summand
in $dY$, then $\pm Y$ must be of the form

But then $dY$ can be written

Thus $\Theta_k(dY) = 0$ as desired.
We get a well-defined induced cocycle $\Theta_k$ because $\Theta_k$ vanishes on all graphs with cut vertices. □

We now come to the main theorem:

**Theorem 2** For all $k \geq 5$ where $k \equiv 1 \mod 4$, $\Theta_k$ represents a nontrivial ribbon graph cohomology class, and therefore represents a nontrivial homology class in

$$H_k(M_k^i; \mathbb{Q})_{\Theta_k}.$$

To see this, we will find a cycle $Z_k \in \overline{\mathcal{G}}_k^1$ such that $\Theta_k(Z_k) \neq 0$.

**Definition 5** Let $T_i$ be the sum of all isomorphism classes of planar binary rooted trees with $i$ leaves. Each such tree has a canonical orientation defined by directing the edges away from the root, and numbering the internal vertices from left to right.

The first few $T_i$ are pictured in Figure 2.

**Figure 2**: The first few $T_i$.

**Definition 6** An ornate necklace is a sum of ribbon graphs of the form pictured in Figure 3. It is oriented so that the edges of the large loop are directed counterclockwise as indicated, and the edges in the $T_{i_j}$ are directed away from the root. The vertices are numbered so that the $T_{i_j}$ vertices lie before the $T_{i_{j+1}}$ vertices. The root of $T_{i_j}$ is numbered before its other vertices, which are ordered left to right as in the definition of $T_{i_j}$.

Let the ornate necklace of Figure 3 be denoted $[i_1, \ldots, i_n]$.

**Examples:**

1. $[1, \ldots, 1] = X_k$, if there are $k$ ‘1’s.
2. $[3, 2] = $
Definition 7 The notation \((i_1, i_2)\) represents the local picture:

and can be inserted into the above bracket notation, as in \([(i_1, i_2), i_3, \ldots, i_n]\), with evident meaning.

Lemma 2 In \(d[i_1, \ldots, i_n]\) the only edge contractions which contribute are

1. The root edges of each \(T_{i_j}\).
2. The edges of the big loop.

Proof: The two types of edges not mentioned are the interior edges of the \(T_{i_j}\) and the edges emanating from the tops of the \(T_{i_j}\).

To see that contracting the interior edges of the \(T_{i_j}\) cancel, note that such an edge contraction will create a 4-valent vertex which can be expanded in two different ways, and thus appears twice when contracting the \(T_{i_j}\) interior edges. Moreover, the signs are opposite, as indicated in the picture on the right.

For edges emanating from the top of the \(T_{i_j}\), we show that the resulting ribbon graphs have cut vertices, and are therefore 0 in \(\mathcal{R}\). Such an edge, \(e\), emanates from a trivalent vertex \(v\). When \(e\) is contracted, the part of \(T_{i_j}\) also emanating from \(v\) will form a cut component at the root vertex, as indicated in the following picture.
The grey strips represent sets of parallel edges, whereas the boxes represent trees. In the above picture, we have drawn $e$ emanating from the right of $v$, but it also evidently works if $e$ emanates from the left of $v$. \hfill \Box

**Definition 8** Let $|i_1, \ldots, i_n|$ be the number of cyclic symmetries possessed by $[i_1, \ldots, i_n]$. For example $|1,1,1,1,1| = 5$, $|1,2,1,2,1,2| = 3$ and $|1,1,2,1| = 1$.

Now we are ready to define the cycle $Z_k$.

**Definition 9** Let
\[
Z_k = \sum (-1)^n \frac{|i_1, \ldots, i_n|}{|i_1, \ldots, i_n|},
\]
where the sum is over all isomorphism classes of ornate necklaces where $i_1 + \cdots + i_n = k$.

This is well-defined, in the sense that the terms $[i_1, \ldots, i_n]$ have a canonical orientation invariant under cyclic symmetries. (This follows because $k$ is odd.)

For example,
\[
Z_5 = -\frac{1}{5} [1,1,1,1,1] + [2,1,1,1] - [1,2,2] - [1,1,3] + [2,3] + [1,4] - [5].
\]

Clearly, when $k \equiv 1 \mod 4$ and $k \neq 1$, we have $\Theta_k(Z_k) = -1/k$. So it suffices to show $d(Z_k) = 0$.

**Proposition 3** $d(Z_k) = 0$.

**Proof:**
By the previous lemma, \( d(Z_k) \) has two types of terms. If an edge in the large loop is contracted, two adjacent \( T_{ij} \)'s will be joined:

\[
[\ldots, i_j, i_{j+1}, \ldots] \rightarrow [\ldots, (i_j, i_{j+1}), \ldots].
\]

Note that we can apply a cyclic symmetry to ensure that it of this form, and we don’t need to join the last tree to the first tree. This simplifies sign considerations and notation. If you contract a root edge, a tree \( T_{ij} \) will become

\[
\pm \sum_{a+b=i_j} (a, b).
\]

Thus, to show \( d(Z_k) = 0 \), it suffices to show that each isomorphism class \( \alpha = [(i_1, i_2), i_3, \ldots, i_n] \) appears twice with opposite sign. Well, \( \alpha \) arises from two terms \( A = [i_1, i_2, i_3, \ldots, i_n] \) and \( B = [i_1 + i_2, i_3, \ldots, i_n] \). Notice that \( \alpha \) appears once in \((-1)^n \frac{dA}{|A|}\) and in \((-1)^{n-1} \frac{dB}{|B|}\). Thus it suffices to note that it appears with the same sign in \( dA \) as it does in \( dB \). This is illustrated in the picture in the above right. Note that \( m \) is the same in both cases: the number of vertices of \( T_{ij} \) plus one. \( \square \)

4 Comparison to Morita’s definition

We start by defining the lie algebra \( a_n \) for any positive integer \( n \).

**Definition 10** Let \( V_n \) be the symplectic vector space which is the rational homology of a genus \( n \) surface. Explicitly, let \( V_n \) be the \( 2n \) dimensional vector space with basis \( p_1, \ldots, p_n, q_1, \ldots, q_n \) and symplectic form \( \omega \) defined so that \( \omega(p_i, q_i) = 1 = -\omega(q_i, p_i) \) and is trivial on all other pairs of basis vectors.

One way to define \( a_n \) is given in the following way [7 Prop. 2].

**Definition 11** Let

\[
(a_n)_i = (V_n^\otimes i+2)^{\mathbb{Z}_{i+2}},
\]

where \( \mathbb{Z}_{i+2} \) acts by cyclic permutation, and let \( a_n = \bigoplus_{i=1}^\infty (a_n)_i \). This defines \( a_n \) as a vector space. The bracket \( [\cdot, \cdot] : (a_n)_i \otimes (a_n)_j \rightarrow (a_n)_{i+j} \) is induced by the contraction \( C_{1, i+3} : V_n^\otimes i+2 \otimes V_n^\otimes j+2 \rightarrow V_n^\otimes i+j+2 \). (Recall that the contraction \( C_{i,j} : V^\otimes l \rightarrow V^\otimes l-2 \) is defined by \( C_{i,j}(v_1 \otimes \cdots \otimes v_k) = \omega(v_i, v_j)v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_k \).)

We note that \( a_n \) is isomorphic to the lie algebra \( LO_n \) associated to the associative operad, defined in [11 Section 2.4.1]. The difference is that in [11], we considered the space of coinvariants \((V_n^\otimes i+2)^{\mathbb{Z}_{i+2}}\) instead of the isomorphic space of invariants.
Definition 12

1. Let $a_n^+ = \bigoplus_{i=2}^{\infty} (a_n)_i$.
2. Let $a_\infty^+ = \lim \{ a_1^+ \subset a_2^+ \subset \cdots \}$.

The cohomology $H^*(a_\infty^+)$ has a Hopf algebra structure, as described in [1, Prop. 7], so that we can consider the primitive elements, denoted with the prefix "P". We also have natural maps $PH^k(a_{n+1}^+)^{sp} \to PH^k(a_\infty^+)^{sp}$, allowing us to consider the limit $PH^k(a_\infty^+)^{sp}$.

Definition 13

1. Let $G$ be the reduced associative graph complex, as described in [1]. (Recall the adjective “reduced” means all vertices have valency at least 3.) Note that $rG \subset \overline{\mathcal{O}G}$ is the subspace of connected graphs.
2. For $t \geq 0$, let $\mathcal{O}G_{2t}$ be the subcomplex of graphs such that the number of edges minus the number of vertices is equal to $t$. This is the weight $2t$ part of the complex, which induces the weight $2t$ part of the homology or cohomology, also indicated with a subscript. Note that $\mathcal{O}G_{2t}$ is finite dimensional.
3. Let
   \[ \psi_n: \bigwedge a_n^+ \to \overline{\mathcal{O}G} \]
   be defined as in [1, Section 2.5.2], using the identification of $\mathcal{L}O_n$ with $a_n$, and let $\psi_\infty$ be the limit map.
4. For every $t \geq 0$, let the induced map on the weight $2t$ part be denoted $(\psi_\infty)_{2t}: \mathcal{O}G_{2t} \to (\bigwedge a_\infty^+).$ (The fact that the image lies in the weight $2t$ part is easily checked.)

Theorem 3 For every $t \geq 0$, the dual map $(\psi_\infty)_{2t}^*$ induces an isomorphism $PH^k(\mathcal{O}G)_{2t} \to PH^k(a_\infty^+)_{2t}^{sp}$.

Proof: This follows from [1, Corollary 5], with some modification. A significant difference is that we are considering the lie algebra $a_\infty^+$ and not the lie algebra $a_\infty$ as in [1]. This has the effect of eliminating bivalent vertices from $\mathcal{O}G$, allowing us to consider $\overline{\mathcal{O}G}$ instead, but since [1, Proposition 8] is no longer true one must consider the space of sp-invariants $PH^k(a_\infty^+)^{sp}$ instead of simply $PH^k(a_\infty^+)$. Finally, to get the exact statement above, one restricts to the weight $2t$ part and takes the dual. \qed

Since we have

$$PH^k(\overline{\mathcal{O}G}) \cong H^k(P\overline{\mathcal{O}G}) \cong H^k(rG),$$

Theorem 3 implies the following suitably modified result of Kontsevich [5].

Theorem 4 We have

$$PH^k(a_\infty^+)_{2t}^{sp} \cong \bigoplus_{2g-2+m = t, m \geq 0} H_{2t-k}(M_{g, m}^m; \mathbb{Q})_{\mathcal{G}w}.$$
Morita precisely determines $H_1(a_n^+)\otimes \mathbb{Q}$ as follows \cite[Theorem 6]{7}.

**Definition 14** Let $b_n = \Lambda^2 H / \mathbb{Q}(\omega_0)$, where $\omega_0$ is the symplectic element. Let $b_n$ be given an abelian lie algebra structure.

**Proposition 4** (Morita \cite[Thm. 6]{7}) There is an isomorphism of $\mathfrak{sp}(2n)$ modules, $H_1(a_n^+)\otimes \mathbb{Q} \cong b_n$, which is induced by the map $h_1 \otimes h_2 \otimes h_3 \otimes h_4 \mapsto \omega(h_1, h_3)h_2 \wedge h_4$.

**Definition 15**

1. Let $\pi_n : a_n^+ \to b_n$ be the corresponding map of lie algebras.

2. There are maps $b_n \to b_{n+1}$ defined via the maps $H_1(a_n^+)\otimes \mathbb{Q} \to H_1(a_{n+1}^+)\otimes \mathbb{Q}$. Let $b_\infty$ be the direct limit, with limit map $\pi_\infty : a_\infty^+ \to b_\infty$. The cohomology of $b_\infty$ also forms a Hopf algebra.

3. Suppose $k \in \{5, 9, 13, \ldots\}$. Let $\xi_k^n : \Lambda^k(b_n) \to \mathbb{Q}$ be the map induced by the product of contractions:
   
   $$C_{2,3}C_{4,5} \cdots C_{2k-1,2k}C_{2k,1} : V_n^{\otimes 2k} \to \mathbb{Q},$$
   
   where $\Lambda^2 V_n$ is regarded as a subspace of $V_n^{\otimes 2}$ and $\Lambda^k(\Lambda^2 V_n)$ is then regarded as a subspace of $V_n^{\otimes 2k}$.

   Let $\xi_k : \Lambda^k(b_\infty) \to \mathbb{Q}$ be the limit map. One must check that $\xi_k^n$ is well-defined, which amounts to showing that $\xi_k^n$ vanishes on $\omega_0 \wedge x_2 \wedge \cdots \wedge x_k$, which is easy to check provided $k \neq 1$.

**Proposition 5** (Morita \cite[Prop. 10]{7}) We have that

$$PH^k(b_\infty)^{\mathfrak{sp}} = \begin{cases} \mathbb{Q} & \text{if } k = 5, 9, 13, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Moreover, for $k = 5, 9, 13, \ldots$, the map $\xi_k : P \Lambda^k b_\infty \to \mathbb{Q}$ is a generator.

**Sketch of proof:** One calculates $P \Lambda^k(b_n)^{\mathfrak{sp}} = \begin{cases} \mathbb{Q} & \text{if } k = 5, 9, 13, \ldots \\ 0 & \text{otherwise} \end{cases}$ by classical invariant theory.

The single class, $e_n^k$, in degree $k$ comes from a $k$-gon and is defined as

$$e_n^k = \sum_{i=1}^n [p_i \wedge q_i] \wedge \cdots \wedge [p_i \wedge q_i],$$

where $[p_i \wedge q_i]$ is the element in $b_n$ represented by $p_i \wedge q_i \in \Lambda^2 V_n$.

To see that $\xi_k$ is a nonzero invariant, we show that each $\xi_{k,n}$ has this property. It is straightforward to show that $\xi_{k,n}$ is killed by the $\mathfrak{sp}$ action, and it is nonzero, for example, since $\xi_{k,n}(e_n^k) = -2n$. \hfill $\blacksquare$

**Definition 16** Morita’s classes are defined from the weight $2k$ classes $\xi_k$, as follows. Note that the map $\pi_\infty$ induces $(\pi_\infty)^*_{2k} : PH^k(b_\infty)^{\mathfrak{sp}} \to PH^k(a_\infty^+)^{\mathfrak{sp}}$. The cocycles constructed by Morita are then $(\pi_\infty)^*_{2k}(\xi_k)$. 

11
We are now ready to relate Morita’s cocycles with the cocycles $\Theta_k$ defined earlier in the paper.

**Proposition 6** We have $(\pi_\infty)_{2k}^*(\xi_k) = \pm (\psi_\infty)_{2k}^*(\Theta_k)$.

**Proof:** Let $x \in (\wedge^k a_\infty^+)_{2k}$, be a wedge of $k$ symplecto-spiders of weight $2k$, using the terminology of [1]. Then
\[ \langle (\psi_\infty)_{2k}^*(\Theta_k), x \rangle = \langle (\pi_\infty)_{2k}^*(\xi_k), x \rangle \]
if and only if
\[ \langle \Theta_k, \psi_\infty(x) \rangle = \langle \xi_k, \pi_\infty(x) \rangle. \]
Now $\psi_\infty(x)$ glues the labeled edges of $x$ together, multiplying by the contraction of the coefficients, and $\Theta_k$ will only be nonzero if the gluing matches opposite edges on each of the degree 2 symplecto-spiders, and then joins the symplecto-spiders up in a $k$-cycle. On the other hand $\pi_\infty$ by definition contracts opposite edges in each symplecto-spider, and then $\xi_k$ contracts the remaining edges into a $k$-cycle in all possible ways. \(\square\)

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