Induced packings of cycles

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Abstract

Two cycles of a graph are mutually induced if there is no edge between them in the graph. Given a graph $G$ and an integer $r$, the problem \textsc{Induced-Cycles} asks whether $G$ contains a packing of $r$ pairwise mutually induced cycles. A reduction from \textsc{Disjoint-Cycles} shows that this problem has no polynomial kernel when parameterized by $r$, unless \text{NP} \subseteq \text{coNP}/\text{poly}, according to the results in [Bodlaender, Thomassé, Yeo, 2012]. In this paper, we show that the problem \textsc{Induced-Cycles} parameterized by the $r$ and the maximum degree $\Delta$ has an $O(\Delta^2)$-kernel for $r = 2$ and an $O(r\Delta^2 \log(r\Delta))$-kernel for $r > 2$. As a consequence, the problem \textsc{Disjoint-Cycles} also has a polynomial kernel for the same parameters.

1 Introduction

Several problems in Graph Theory can be expressed in terms of containment problems, where given two graphs, one have to decide whether the structure of one of them can be found in the other. For instance, a consequence of the Graph Minor Theorem [17] of Robertson and Seymour is that as soon as the class of \textsc{Yes}-instances of a decision problem is minor closed, then this problem can be translated into a finite number of minor containment testing problems.

Containment problems have been extensively studied (cf. [1, 2, 4, 7, 8, 10–12, 15] for instance) and beside their theoretical interest they are known to have practical applications, for instance in VLSI design [13] and in Computer Graphics [3]. However, most attention have been brought so far to the minor relation, whereas several natural problems related to other containment relations still remain unsolved. Before we go further into details, let us introduce some basic definitions.

We say that a graph $G$ is contractible to a graph $H$ if a graph isomorphic to $H$ can be obtained by contracting some edges of $G$. The graph $G$ contains $H$ as a \textit{minor} (resp. \textit{induced minor}) if it has a subgraph (resp. induced subgraph) which is contractible to $G$. For each of these two containment relations, the problem of...
deciding whether $H$ is contained in $G$ has been proven in [14] to be NP-complete, even when $G$ and $H$ are restricted to be very simple graphs. Therefore we consider the variant of these problems where the graph $H$ is not a part of the input, that we respectively denote by $H$-MINOR and $H$-INDUCED-MINOR.

A consequence of the results of [16] is that the $H$-MINOR problem can be solved in cubic time. However, the same techniques, in particular the Linkage Theorem, cannot directly be used to deal with induced substructures. In 1995, Fellows et al. [7] constructed a graph on 68 vertices for which the problem $H$-INDUCED-MINOR is NP-complete. This initiated a quest to detect for which restrictions on the input graph $G$ the problem $H$-INDUCED-MINOR becomes tractable. To cite a few results, this problem can be solved in polynomial time when $G$ is planar [7], chordal [4], AT-free [10], or belongs to a non-trivial minor-closed class [18]. Another angle of attack is to look at which instances of $H$ make this problem easier. For instance, the detection of “star-like” induced minors can be performed in polynomial time [8].

Two cycles of a graph are mutually induced if there is no edge between them in the graph. The motivation of this paper is a question of [6] asking whether we can easily check whether a graph contains two mutually induced cycles (equivalently, two triangles as induced minor). This question was also raised by the authors of [9] who proved that the problem of detection of graphs not containing two odd cycles is NP-complete. Given a graph $G$ and an integer $r$, the problem INDUCED-CYCLES asks whether $G$ contains $r$ pairwise mutually induced cycles. Our contribution is the following.

**Theorem 1.** The problem of detecting two mutually induced cycles in a graph has $O(\Delta^2)$-kernel when parameterized by the maximum degree $\Delta$.

Remark that this corresponds to the case $r = 2$ of the problem INDUCED-CYCLES. General remarks about graphs not containing two mutually induced cycles and the proof of Theorem 1 will be presented in Section 2. Following the same ideas, we generalize Theorem 1 to an induced cycle packing of any size.

**Theorem 2.** The problem INDUCED-CYCLES has a $O(r \Delta^2 \log(r \Delta))$-kernel when parameterized by the maximum degree $\Delta$ and $r$.

The proof of Theorem 2 will be presented in Section 3. Note that the problem INDUCED-CYCLES is the induced version of the problem DISJOINT-CYCLES, which asks if a graph has $r$ vertex-disjoint cycles. A motivation for our choice of both $\Delta$ and $r$ as parameters of INDUCED-CYCLES is that when parameterized only by $r$, this problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly. Indeed, let us consider the following reduction from DISJOINT-CYCLES to INDUCED-CYCLES: for every instance $(G, k)$ of DISJOINT-CYCLES, we construct an instance $(G', k)$ of INDUCED-CYCLES where $G'$ has been obtained from $G$ by subdividing once every edge. Clearly, $|V(G')| = O(|V(G)|^2)$ and $G'$ contains $k$ pairwise mutually induced cycles iff $G$ has $k$ vertex-disjoint cycles. Therefore, the existence of a polynomial kernel for INDUCED-CYCLES parameterized by $r$ would imply that DISJOINT-CYCLES also have a polynomial kernel. This would, in turn, imply that NP $\subseteq$ coNP/poly according to the following result.
Proposition 1 ([5]). The problem DISJOINT-CYCLES parameterized by the number of cycles has no polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.

As an immediate consequence of the reduction above and our main results, we get a kernel for the problem DISJOINT-CYCLES parameterized by $r$ and the maximum degree $\Delta$.

Corollary 1. The problem DISJOINT-CYCLES has a $O(r\Delta^2\log(r\Delta))$-kernel when parameterized by the maximum degree $\Delta$ and $r$.

Further research. In this paper, we investigate the problem of detecting graphs with a packing of $r$ mutually induced cycles, and we prove that it has a polynomial kernel when parameterized by $r$ and the maximum degree of the graph. This also implies the existence of a kernel of the same size for the non-induced version of the problem for the same parameters. In particular, we answer positively the question of [6] for graphs of bounded degree. Moreover, the size of this kernel is the best (both in terms of $r$ and $\Delta$) that we can obtain from the techniques that we used, up to a logarithmic factor. This work also gives some insight on the parameterized complexity of the problem $H$-INDUCED-MINOR, for the case where $H$ is a union of disjoint triangles. Besides, we prove bounds of independent interest on the number of vertices that of a reduced graph can have without containing a packing of induced cycles. However, whether the case $r = 2$ of the problem INDUCED-CYCLES can be solved in polynomial time on any graph is still an open question. Another question of interest is whether our kernel can be made quadratic.

Basic definitions. In this paper, $V(G)$ (resp. $E(G)$) denotes the set of vertices (resp. edges) of a graph $G$. Two vertices are said to be adjacent if they share an edge. For every vertex $v \in V(G)$, we denote by $N_G(v)$ the set of neighbors of $v$, i.e. vertices that are adjacent to $v$. The degree of a vertex is the size of its neighborhood: $\deg_G(v) = |N_G(v)|$. We drop the subscript from these notations when it is obvious from the context. The maximum degree $\Delta(G)$ of $G$ is the maximum degree over the vertices of $G$. We define the distance between two vertices $u, v$ of a graph $G$ as the number of edges of a shortest path between them if they live in the same connected component, $\infty$ otherwise. The girth $\text{girth}(G)$ of $G$ is the length of a smallest cycle in $G$. Every two subgraphs $G_1, G_2$ of $G$ are said to be adjacent if $G$ has an edge, one endpoint of which is in $V(G_1)$ and the other is in $V(G_2)$. For every $n \in \mathbb{N}^*$, we denote by $K_n$ the complete graph on $n$ vertices. For every $a, b \in \mathbb{N}$ with $a \leq b$, $[a, b]$ stands for the interval $\{i \in \mathbb{N}, a \leq i \leq b\}$. All along this paper, logarithms are binary.

2 Detecting two induced cycles

In this section, we will describe a polynomial kernel for the problem 2-INDUCED-CYCLES, that asks whether a graph contains two mutually induced cycles, parameterized by the maximum degree. We first describe reduction rules whose aim is to
simplify graphs without changing the maximum number of mutually induced cycles they contain. We then state general remarks about reduced graphs not containing two mutually induced cycles, which will be used in proving the theorem.

**Reduced graph.** Let us call an edge redundant if both endpoints are of degree 2 and the edge does not belong to a triangle. A graph is said to be reduced if it contains no redundant edge, nor a vertex of degree zero or one. In order to obtain a reduced graph from any graph, we consider the two following reduction rules:

(R1) If the graph $G$ has a redundant edge, contract it.

(R2) If the graph $G$ contains a vertex $v$ of degree 0 or 1, delete it.

It is easy to see that these reduction rules do not change the property of containing two mutually induced cycles and also that every graph can be reduced by a finite number of applications of (R1) and (R2). Algorithm 1 is a linear time implementation of the reduction.

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**Algorithm 1:** Reduction algorithm.

**Input:** a graph $G$

**Output:** a reduced graph

$V := V(G)$

$S := \emptyset$

**while** $S \neq V$ **do**

pick $v \in V\setminus S$

**if** $\deg(v) \geq 3$ **then**

$S := S \cup \{v\}$

**else if** $\deg(v) = 2$ **then**

**if** for some $u \in N(v)$ we have $u \in S$ and $\deg(u) = 2$ and $N(u) \cap N(v) = \emptyset$

**then**

contract edge $\{u, v\}$ and keep the resulting vertex in $V\setminus S$

**else**

$S := S \cup \{v\}$

**else if** $\deg(v) = 1$ **then**

**if** the only $u \in N(v)$ is such that $u \in S$ and $\deg(u) \leq 3$

**then**

contract edge $\{u, v\}$ and keep the resulting vertex in $V\setminus S$

**else**

$V = V \setminus \{v\}$

**else** delete $v$ (which in this case is isolated)

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**Lemma 1.** Algorithm 1 runs in linear time and outputs a reduced graph.

**Proof.** Every step in the while loop is performed in constant time. It is easy to check that each iteration of the while loop decreases the quantity $2|V| - |S|$ by 1 or 2, and since $2|V| - |S| \geq |V| \geq 0$, the algorithm will perform the loop at most $2|V(G)|$ times. This means that the algorithm is linear. Also notice, that the set $S$ does not contain any vertex of degree 1 nor any redundant edges. Hence the resulting graph is reduced as required. \[\square\]
Remark 1. Reducing a graph does not increase its maximum degree.

Lemma 2. If $T$ is a tree with $l \geq 2$ leaves and no redundant edge, then $|V(T)| \leq 4l - 5$.

Proof. Let $T$ be a tree as in the statement of the lemma and let $T'$ be the tree obtained from $T$ by dissolving every vertex of degree 2. We denote by $l \geq 2$ the number of leaves in $T'$ (which remains the same as in $T$) and by $t$ the number of internal vertices of $T'$. Since $T'$ is a tree, we have:

$$|E(T')| = |V(T')| - 1 = l + t - 1.$$  

This together with handshaking lemma and observation that every internal vertex has degree at least 3 imply:

$$2(l + t - 1) = 2|E(T')| = \sum_{v \in V(T')} \deg(v) \geq l + 3t.$$  

Subtracting $2t$ from both sides we get the following bound on the number of vertices of $T'$:

$$2l - 2 \geq l + t = |V(T')|.$$  

Now, remark that since $T$ does not contain a redundant edge, it has at most $|E(T')|$ vertices of degree two, and hence

$$|V(T)| \leq |V(T')| + |E(T')| = 2|V(T')| - 1 \leq 4l - 5.$$  

\[\square\]

Corollary 2. If $F$ is a forest with $l \geq 2$ leaves or isolated vertices and without redundant edges, then $|V(T)| \leq 4l - 5$.

Proof. First remark that if $F$ has no connected component of size at least 3, then we have $|V(F)| = l \leq 4l - 5$ (holds for all $l \geq 2$). On the other hand, if $F$ has a connected component of size at least 3, we can add edges between internal vertices of different connected components in order to obtain a forest with the same vertex set and the same number of isolated vertices and leaves and containing exactly one tree $T$ on at least 3 vertices. If $l_1$ is the number of leaves in $T$, then by Lemma 2 we have $|V(T)| \leq 4l_1 - 5$. The rest of the forest (consisting of components of size 1 and 2) contains $l - l_1$ vertices. Hence $|V(F)| \leq 4l_1 - 5 + l - l_1 \leq 4l - 5$.  

\[\square\]

Lemma 3. If $G$ is a connected reduced graph not containing two mutually induced cycles, then $|V(G)| < 4\text{girth}(G)\Delta(G)^2$.

Proof. Let $C$ be a cycle in $G$ of length $g = \text{girth}(G)$. For $i \in \{1, 2\}$, we denote by $N_i$ the set of vertices at distance $i$ from $C$. We also set $R = V(G) \setminus (V(C) \cup N_1)$ and $\Delta = \Delta(G)$. Remark that if $R$ contains a cycle, then it is not adjacent to $C$ and hence $G$ has two mutually induced cycles. Therefore $R$ induces a forest in $G$. Also notice
that every leaf or isolated vertex of \(G[R]\) belongs to \(N_2\), otherwise it would have degree one in \(G\), which would contradict the fact that \(G\) is reduced. Besides, if we have two adjacent vertices of degree two, one of them must belong to \(N_2\), otherwise it contradicts the fact that \(G\) is reduced. To avoid this situation we construct graph \(R^+\) from \(G[R]\) by adding a neighbor of degree one to each vertex of \(N_2\) of degree two. Now \(R^+\) is a forest which has at most \(|N_2|\) leaves or isolated vertices and has no redundant edge: by Corollary 2 we have \(|R^+| \leq 4|N_2| - 5\).

The cycle \(C\) has \(g\) vertices each of degree at most \(\Delta\) and with two neighbors in \(C\), therefore \(|N_1| \leq g(\Delta - 2)\) and by a similar argument we obtain \(|N_2| \leq g(\Delta - 2)(\Delta - 1)\).

We are now able to give a bound on the size of \(G\):

\[
|V(G)| = |V(C)| + |N_1| + |R| \\
\leq |V(C)| + |N_1| + |V(R^+)| \\
\leq g + g(\Delta - 2) + 4(g(\Delta - 2)(\Delta - 1) - 5 \\
\leq 4g\Delta^2 - 11g\Delta + 7g - 5 \\
< 4g\Delta^2.
\]

\(\square\)

Now we show that reduced graphs without two mutually induced induced cycles have small girth.

**Lemma 4.** If \(G\) is a connected reduced graph not containing two mutually induced cycles, then \(\text{girth}(G) \leq 10\).

**Proof.** Let us assume by contradiction that \(G\) has girth \(g > 10\). We use the same notation for \(C, N_1, N_2, R\) as in Lemma 3. First of all, note that every vertex \(v \in N_1\) has a unique neighbor in \(C\). Indeed, if \(v\) had at least two neighbors \(u\) and \(u'\) in \(C\) then together with the shortest of the two paths from \(u\) to \(u'\) along \(C\) it would create a cycle of length at most \(\left\lceil \frac{g}{2} \right\rceil + 1 < g\), a contradiction. Similarly, we obtain that \(N_1\) is an independent set, that each vertex in \(N_1\) has a unique neighbor in \(N_2\) and that \(N_2\) is an independent set (otherwise we would have a cycle of length at most \(\left\lceil \frac{g}{2} \right\rceil + 4 < g\) in \(G\)).

Now, as in Lemma 3 \(R\) induces a forest, all of which leaves and isolated vertices lie in \(N_2\). But since \(N_2\) is independent and each \(v \in N_2\) has a unique neighbor in \(N_1\) we deduce that \(G[R]\) contains only components of size at least 3. Moreover, if we pick any leaf \(v \in V_2\) with its only neighbor \(u \in R \setminus V_2\), we have that \(\deg(u) \geq 3\) as otherwise \(\deg(u) = 1\) or \(\{u, v\}\) redundant would contradict the fact that \(G\) is reduced.

Having learned the structure of the graph, we are ready to derive a contradiction on the size of the girth as follows. Pick an arbitrary component in \(G[R]\) and a path \(P\) of maximal length in it. Let \(v\) and \(t\) be the two endpoints of the path. Since \(v\) is a leaf, by the previous paragraph its unique neighbor \(u\) has degree at least 3, so let \(v'\) be a neighbor of \(u\) not belonging to \(P\). It is easy to see that distance between \(t\) and \(v'\) is \(|P|\), so by maximality of the path, we deduce that \(v'\) is a leaf. Now, take \(z, z' \in N_1\), \(y, y' \in V(C)\) such that \(uvzy\) and \(uv'z'y'\) are two paths from \(u\) to the
cycle. These paths together with the shorter path from $y$ to $y'$ along $C$ form a cycle of length $5 + \left\lfloor \frac{g}{2} \right\rfloor < g$. Contradiction proves the result. \qed

**Corollary 3.** If $G$ is a reduced graph not containing two mutually induced cycles, then $|V(G)| < 40 \Delta(G)^2$.

**Proof.** Since every component of reduced graph contains a cycle, $G$ contains at most one connected component. Combining Lemmas 3 and 4 we obtain that $|V(G)| < 40 \Delta(G)^2$. \qed

**Remark 2** of Theorem 7. Consider the following procedure. Given a graph $G$, we apply Algorithm 1 and obtain a graph $G'$. If $|V(G')| \geq 40 \Delta(G)^2$, then we output the graph $K_3 + K_3$, otherwise we output $G'$. The call to the reduction algorithm runs in linear time, as explained in Lemma 1. Moreover, remark that either the procedure outputs the reduced input graph, or $K_3 + K_3$ in which case, the input graph is known to contain two mutually induced cycles, by Lemma 3. According to Remark 1, the maximum degree of a reduced graph is never more that the one of the original graph. Therefore the output instance is equivalent to the input with regard to the problem \textsc{2-Induced-Cycles}. At last, the output graph has size quadratic in the maximum degree of the input graph. Consequently, this procedure is a kernelization for the problem \textsc{2-Induced-Cycles} parameterized by maximum degree. This proves the existence of $O(\Delta^2)$ kernel for this problem. \qed

## 3 Dealing with more cycles

In this part, we investigate the question of deciding whether a graph contains $r$ pairwise mutually induced cycles, for some fixed $r \in \mathbb{N}$. Using the ideas of the above section, we show that the problem \textsc{Induced-Cycles} parameterized by the maximum degree $\Delta$ and the number of cycles $r$ admits a $O(r \Delta^2 \log(r \Delta))$-kernel.

**Definition 1.** For positive integers $r \geq 2, d$ we denote by $h_r(d)$ the least integer such that every connected reduced graph $G$ of degree at most $d$ and with more than $h_r(d)$ vertices has $r$ mutually induced cycles. When such a number does not exists, we set $h_r(d) = \infty$.

We showed in the previous section that $h_2(d) \leq 40d^2$. In this section we will show for every $r \geq 2, d \geq 1$ we have $h_r(d) = O(rd^2 \log(rd^2))$.

**Remark 2.** If $G$ is a reduced graph with at least $h_r(\Delta(G) + 1)$ vertices, then $G$ contains $r$ mutually induced cycles. This remark is trivial when $G$ is connected, by definition of $h_r$. Let us now assume that $G$ is not connected, and has $k > 1$ connected components $D_1, \ldots, D_k$. Remark that since $G$ is reduced, every connected component has at least three vertices. For every $i \in [1, k]$, let us denote by $v_i, u_i$ two arbitrarily chosen distinct vertices of $D_i$, and let $G'$ be the graph obtained by adding to $G$ the edge $\{v_i, u_{i+1}\}$ for every $i \in [1, k-1]$. Notice that $\Delta(G') \leq \Delta(G) + 1$. Moreover, $G'$
Proof. Let a shortest path from \( v \) follows.\( w \)

For every \( i \in \mathbb{N} \) holds

Let us consider for every \( i \in \mathbb{N} \) the graph \( G_i \), constructed from the disjoint union of a complete binary tree \( T \) of height \( 2i \) and a clique on vertices \( v_1, \ldots, v_{2i} \) by partitioning the \( 2^{2i} \) leaves of \( T \) into \( 2^i \) subsets \( L_1, \ldots, L_{2^i} \), each of size \( 2^i \), and adding all possible edges between \( v_i \) and the vertices of \( L_i \), for every \( i \in \mathbb{N} \).

Clearly, this graph is reduced: it has no vertex of degree less than two and every vertex of degree two, which is a leaf of \( T \), is adjacent to a vertex of degree three (its parent in \( T \)) and to a vertex of high degree (in the clique). Remark that the size of this graph is the sum of the sizes of the tree and the clique; it has therefore \( 2^{2i+1} + 2^i - 1 \) vertices. Besides, every vertex of the tree has maximum degree three in \( G \), therefore the maximum degree is reached in the clique where every vertex is adjacent to the \( 2^i - 1 \) other vertices of the clique and to \( 2^i \) leaves of the tree, thus \( \Delta(G) = 2^{i+1} - 1 \). Every cycle in this graph must use at least one vertex of the clique, therefore for every \( i \in \mathbb{N} \), \( G_i \) does not contain two mutually induced cycles. We get \( h_2(x) > x^2/2 \) by remarking that for every \( i \in \mathbb{N} \), \( |V(G_i)| \geq \Delta(G_i)^2/2 \).

Let us now consider the case \( r > 2 \). For every \( i \in \mathbb{N} \), let \( G_{i,r} \) be the disjoint union of \( r - 1 \) copies of the graph \( G_i \). By the remarks above, none of these graphs contain \( r \) mutually induced cycles. By Remark 2, \( G_{i,r} \) must therefore have less than \( h_r(\Delta(G_i') + 1) \) vertices for every \( i \in \mathbb{N} \). As \( |V(G_{i,r})| = (r - 1) |V(G_i)| \geq (r - 1) \Delta(G_i)^2/2 \), we finally get that for every \( r > 2 \) and every \( x \in \mathbb{N} \), \( h_r(x+1) > (r - 1)x^2/2 \). This concludes the proof.

Lemma 6. If \( G \) is a reduced graph of girth \( g \geq 6 \) then \( |V(G)| \geq g \times 2^{|g/8|-1} \).

Proof. Let \( C \) be a cycle of length \( g = \text{girth}(G) \geq 6 \) in \( G \) and let \( h = \lceil \frac{g}{4} \rceil \). For every \( i \in [0, h] \), let \( N_i \) be the set of vertices at distance \( i \) from \( C \).

Remark 3. For every \( i \in [1, h - 1] \), any vertex \( u \in N_i \) has exactly one neighbor in \( N_{i-1} \). In fact, let us assume that \( u \) has two neighbors \( v \neq w \) in \( N_{i-1} \) and let \( P \) be a shortest path from \( v \) to \( w \) not using \( u \). Remark that \( v \) and \( w \) are not adjacent, otherwise \( \{u, v, w\} \) would induce a triangle, whereas \( \text{girth}(G) \geq 6 \). Since both \( v \) and \( w \) are at distance \( i - 1 \leq \lceil \frac{g}{4} \rceil - 2 \) from \( C \), we can upper-bound the length of \( P \) as follows.

\[
|V(P)| \leq 2 \left\lfloor \frac{g}{4} \right\rfloor - 4 + \left\lfloor \frac{g}{2} \right\rfloor \leq g - 4
\]
Together with $u$, $P$ forms a cycle of length at most $g - 3$, which is impossible since $G$ has girth $g$. Therefore $v = w$. For a similar reason, $N_i$ is an independent set for every $i \in [1, h - 1]$.

A consequence of the previous observation is that for every $i \in [1, h - 1]$, $|N_i| \leq |N_{i+1}|$.

**Remark 4.** For every $i \in [i, h - 3]$, $|N_{i+2}| \geq 2|N_i|$. Indeed, since $G$ is reduced, it has no vertex of degree one neither an edge whose ends are of degree two. Therefore, every $v \in N_i$ either has exactly one neighbor $u$ in $N_{i+1}$ and $u$ has at least two neighbors in $N_{i+2}$, or $v$ has at least two neighbors in $N_{i+1}$, each having at least one neighbor in $N_{i+2}$. Thus, the subset of vertices of $N_{i+2}$ that are at distance 2 from $v$ is of size at least two. By Remark 3, these subsets are disjoint for different choices of the vertex $v \in N_i$, so we get $|N_{i+2}| \geq 2|N_i|$.

As a consequence of Remark 4 for every $i \in [1, h - 1]$, we have $|N_i| \geq |N_1| \cdot 2^{((i-1)/2)}$. Since $C$ has no two adjacent vertices both of degree two, $|N_1| \geq \left\lceil \frac{9}{4} \right\rceil$, hence for every $i \in [1, h - 1]$, $|N_i| \geq \left\lceil \frac{9}{4} \right\rceil \cdot 2^{((i-1)/2)} \geq \frac{9}{4} \cdot 2^{((i-1)/2)}$. We are now able to give a lower bound on the number of vertices of $G$:

$$|V(G)| \geq \sum_{i=0}^{h-1} |N_i|$$

$$\geq g + \sum_{i=1}^{h-1} \frac{g}{2} \cdot 2^{((i-1)/2)}$$

$$\geq g + g \sum_{i=0}^{\lfloor \frac{h}{2} \rfloor - 1} 2^i$$

$$\geq g \cdot 2^{\lfloor h/2 \rfloor - 1}$$

$$\geq g \cdot 2^{\lfloor g/8 \rfloor - 1}$$

(because $h = \lfloor g/4 \rfloor$).

**Lemma 7.** For every positive integers $r \geq 2$, $d$, we have $h_r(d) \leq f_r(d) := 96d^2 r \log(32d^2r)$.

Equivalently, a reduced connected graph $G$ without $r \geq 2$ mutually induced cycles and with degree bounded by $d$ has less than $f_r(d)$ vertices.

**Proof.** Let $C$ be a cycle of length $g = \text{girth}(G)$ in $G$. We respectively denote by $N_1$ and $N_2$ the vertices at distance one and two from $C$. Also, we denote by $R$ (resp. $S$) the vertices at distance at least two (resp. at least three) from $C$. Clearly, $G[R]$ does not contain $r - 1$ mutually induced cycles. Notice that $R = S \cup N_2$ and that the set $S$ does not contain a vertex of degree less than two nor a redundant edge in $G[R]$. In what follows, we will reduce the graph $G[R]$ to the graph $R^+$. Since $R^+$ is reduced graph without $r - 1$ mutually induced cycles, it has bounded size, by induction. From this we will conclude the bound on $|R|$ and hence the bound on the $|V(G)|$. Now, we need to count the number of vertices lost in reduction procedure. To make the calculation easier, we consider the slightly modified reduction routine Algorithm 2.
**Input:** a graph $G$ and the sets $N_2$ and $R$

**Output:** the graph $R^+$

$N := N_2$

$S := R \setminus N_2$

**while** $N$ contains a vertex $v$ of degree one **do**

- let $u$ be the unique neighbor of $v$
  - **if** $\deg(u) = 2$ **then** contract $N(u) \cup u$ to a single vertex and keep it in $N$
    - **else** contract edge $\{u, v\}$ and keep the resulting vertex in $N$

**while** $N$ contains a vertex $v$ of degree two **do**

- let $u_1, u_2$ be the two neighbors of $v$
  - **if** $\deg(u_i) > 2$ for both $i = 1, 2$ **then** move vertex $v$ from $N$ to $S$
    - **else if** $\deg(u_i) = 2$, $N(u_i) = \{v, z\}$ and $\deg(z) = 2$
      - contract vertices $v, u_i, z$ into a single vertex and keep it in $N$
    - **else** contract $v$ with its neighbor(s) of degree 2 into single vertex and keep it in $S$

**while** $N$ contains an isolated vertex $v$ **do**

- delete $v$

**Algorithm 2: Reduction of $G[R]$.**

Let $d_1$ be the initial number of vertices of degree one in $G[R]$. As there are no vertices of degree one in $R \setminus N_2$ we have $d_1 \leq |N_2|$. It is not hard to see that after each step of the first while loop of Algorithm 2 the quantity $|N| + d_1$ decreases by at least one. Notice also, that after each step of the second or third while loop the quantity $|N|$ decreases by at least one. Hence, all in all, at most $|N| + d_1$ steps are performed in the reduction algorithm. Now, notice that each step reduces the number of vertices in $G[R]$ by at most two. Hence, at the end of the algorithm we will have a reduced graph $R^+$ such that $|G[R]| - |R^+| \leq 2(|N_2| + d) \leq 4|N_2|$

The graph $R^+$ is reduced and does not contain $r - 1$ cycles: by Remark 2 we have $|R^+| \leq h_{r-1}(d + 1)$. Putting these bounds together, we obtain an inequality:

$$|V(G)| = |C| + |N_1| + |R| \leq |C| + |N_1| + 4|N_2| + |R^+| \leq g + g(d - 2) + 4g(d - 2)(d - 1) + h_{r-1}(d + 1) \leq 4gd^2 + h_{r-1}(d + 1)$$

(1)

Now observe that when $g > 8(2 + \log(4d^2 + h_{r-1}(d + 1)))$, by Lemma 5 we get:

$$|V(G)| \geq g^{2^{g/8 - 2}} > g^{2^{\log(4d^2 + h_{r-1}(d + 1))}} \geq g(4d^2 + h_{r-1}(d + 1)) \geq 4gd^2 + h_{r-1}(d + 1) \geq |V(G)|$$

(using 1)
This contradiction leads to the conclusion that \( g \leq 8(2 + \log(4d^2 + h_{r-1}(d+1))) \) and putting this bound on the girth of \( G \) into \((\text{II})\) we get:

\[
V(G) \leq 32d^2(2 + \log(4d^2 + h_{r-1}(d+1))) + h_{r-1}(d+1)
\]

As this holds for every reduced \( G \) without \( r \) mutually induced cycles with degree bounded by \( d \) we obtain:

\[
h_r(d) \leq 32d^2 \log(16d^2 + 4h_{r-1}(d+1)) + h_{r-1}(d+1) \quad (2)
\]

To finish the proof, we will check by induction on \( r \) that \( f_r(d) = 96d^2 r \log(32d^2 r) \geq h_r(d) \). It is true for \( r = 2 \) by Corollary \((3)\) Suppose \( r > 2 \) and \( f_{r-1}(d) \geq h_{r-1}(d) \) and denote \( D = 32d^2 \) for convenience. Then we have the following.

\[
f_r(d) - h_{r-1}(d) \geq f_r(d) - f_{r-1}(d) \quad \text{(induction hypothesis)}
\]

\[
= 3Dr \log(Dr) - 3D(r-1) \log(D(r-1))
\]

\[
\geq D \log((Dr)^3)
\]

\[
\geq D \log(16d^2 + (32d^2 r)(16d^2 r)(32d^2 r))
\]

\[
\geq D \log(16d^2 + 96(4(d+1)^2 r) \log(32(d+1)^2 r)) \quad \text{(term by term, } r \geq 3\text{)}
\]

\[
= 32d^2 \log(16d^2 + 4f_r(d+1))
\]

\[
\geq 32d^2 \log(16d^2 + 4h_{r-1}(d+1)) \quad \text{(induction hypothesis)}
\]

Together with \((2)\) this imply: \( f_r(d) \geq h_{r-1}(d+1) + 32d^2 \log(16d^2 + 4h_{r-1}(d+1)) \geq h_r(d) \). Hence we are done. \( \Box \)

We are now able to prove our second Theorem.

**Proof of Theorem \((3)\).** We do not go much into detail since this proof is very similar to the proof of Theorem \((1)\). The idea is the following: if after being reduced, the graph of the instance \((G, r)\) has more than \( f_r(\Delta(G) + 1) \) vertices, by Remark \((2)\) and Lemma \((7)\) it contains a packing of \( r \) mutually induced cycles so we can replace this instance by the equivalent instance \((r \cdot K_3, r)\), where \( r \cdot K_3 \) is the disjoint union of \( r \) copies of \( K_3 \). Otherwise, the size of \( G \) is bounded by \( f_r(\Delta(G) + 1) \), a function of its maximum degree and \( r \) which are the parameters of this instance. Besides, the reduction algorithm runs in linear time (cf. Lemma \((1)\)) and does not increase the maximum degree. This proves the existence of a kernel of size \( f_r(\Delta + 1) = O(\Delta^2 r \log(\Delta r)) \) for the problem \textsc{Induced-Cycles} parameterized by maximum degree \( \Delta \) and \( r \). \( \Box \)

In this section, the technique we used in order to prove that \textsc{Induced-Cycles} has a polynomial kernel is show that a reduced graph with many vertices will always contain an induced packing of \( r \) cycles. For this, we introduced the family of functions \((h_r)_{r \in \mathbb{N}}\), where for every \( r \in \mathbb{N} \), the function \( h_r \) bounds the size of a reduced graph not having a packing of \( r \) induced cycles in function of its maximum degree. Lemma \((5)\) and Lemma \((7)\) give an estimate of the function \( h_r \): we proved that \( h_r(x) = \Omega(rx^2) \) and \( h_r(x) = O(rx^2 \log(rx)) \). Observe that our estimation of \( h_r \) is tight up to a logarithmic factor. Therefore this technique cannot not be used to prove more than a quadratic kernel.
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