Equivalence groupoid for (1+2)-dimensional linear Schrödinger equations with complex potentials

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Abstract. We describe admissible point transformations in the class of (1+2)-dimensional linear Schrödinger equations with complex potentials. We prove that any point transformation connecting two equations from this class is the composition of a linear superposition transformation of the corresponding initial equation and an equivalence transformation of the class. This shows that the class under study is semi-normalized.

1. Introduction

In the theory of group classification of differential equations, the equivalence groupoid of a class of differential equations is the set of admissible transformations of this class together with the composition operation of admissible transformations [1]. More specifically, an admissible (point) transformation of the class consists of a pair of similar equations (an initial and a target ones) from the class and a point transformation between these equations. The composition of two admissible transformations is well defined whenever the target equation of the first admissible transformation coincides with the initial equation of the second admissible transformation. For each equation from the class, there exists the admissible identity transformation. Each admissible transformation is invertible. The associativity of the composition of admissible transformations directly follows from those of point transformations.

The study of admissible transformations was initiated by Kingston and Sophocleous [2], who called an unformalized version of such transformations form-preserving [3]. Form-preserving transformations under the name allowed transformations also arose in the course of symmetry analysis of variable-coefficient Korteweg–de Vries equations [4] and variable-coefficient (1+1)-dimensional cubic Schrödinger equations [5]. Later on, formalizing the framework of admissible transformations was initiated [6] and the range of applicability of admissible transformations was extended to various classes of differential equations being important for applications, including nonlinear Schrödinger equations [7], variable-coefficient reaction–diffusion equations [8, 9, 10], eddy vorticity flux parameterizations of the inviscid barotropic vorticity equation [11] and nonlinear wave equations from the theory of elasticity [1].

In order to rigorously pose the problem under consideration, we need to precisely define the notions of classes of differential equations and their equivalence groupoids. Other notions and definitions related to classes of differential equations can be found, e.g., in [1, 7]. Consider a system of differential equations \( \mathcal{L}_\theta: L(x, u(p), \theta^{(q)}(x, u(p))) = 0 \), parameterized by the tuple of arbitrary elements \( \theta(x, u(p)) = (\theta^1(x, u(p)), \ldots, \theta^k(x, u(p))) \), where \( x = (x_1, \ldots, x_n) \) is the tuple of independent variables and \( u(p) \) is the set of the dependent variables \( u = (u^1, \ldots, u^m) \) together with all derivatives of \( u \) with respect to \( x \) up to order \( p \). The symbol \( \theta^{(q)} \) stands for the set of
derivatives of \( \theta \) of order not greater than \( q \) with respect to the variables \( x \) and \( u_{(p)} \). The tuple of arbitrary elements \( \theta \) runs through the set \( S \) of solutions of an auxiliary system of differential equations \( S(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0 \) and differential inequalities, like \( \Sigma(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) \neq 0 \), in which both \( x \) and \( u_{(p)} \) play the role of independent variables and \( S \) and \( \Sigma \) are tuples of smooth functions depending on \( x \), \( u_{(p)} \) and \( \theta_{(q)} \). Then the set \( \{ L_{\theta} \mid \theta \in S \} := L\mid S \) is called a class of differential equations defined by the parameterized form \( L_{\theta} \) and the set \( S \) run by the arbitrary elements \( \theta \). Denote by \( T(\theta, \tilde{\theta}) \) with \( \theta, \tilde{\theta} \in S \) the set of point transformations in the space of the variables \( (x, u) \) that map the system \( L_{\theta} \) to the system \( L_{\tilde{\theta}} \). A triple constituted by two arbitrary elements \( \theta, \tilde{\theta} \in S \) with \( T(\theta, \tilde{\theta}) \neq \emptyset \) and a point transformation \( \varphi \in T(\theta, \tilde{\theta}) \) is called an admissible transformation of the class \( L\mid S \). The set of admissible transformations of the class \( L\mid S \) equipped by the natural composition of such transformations as an algebraic operation is called the equivalence groupoid of this class.

The notion of equivalence group originated from the work of Ovsiiannikov [12] and became a powerful tool which the group classification of differential equations relies on. Thus, the use of equivalence groups constitutes a basis for group classification. Here and in what follows, we are concerned with usual equivalence groups. This notion can be generalized in several ways [6, 7, 13]. The (usual) equivalence (pseudo)group \( G\sim = G\sim(L\mid S) \) of the class \( L\mid S \) consists of point transformations in the space of independent and dependent variables and arbitrary elements each of which is projectable to the space of \( (x, u_{(p)}(\theta)) \) for any \( 0 \leq p \leq p \), is consistent with the contact structure on the space of \( (x, u_{(p)}(\theta)) \), and maps every system from the class \( L\mid S \) to another system from the same class. Its elements are called equivalence transformations [7].

Knowing the equivalence groupoid of a class of differential equations simplifies the process of classifying Lie symmetry extensions within this class. In particular, this gives an easy way of finding the corresponding equivalence group. The presence of a nice relation between the equivalence groupoid and the equivalence group makes the entire procedure of symmetry classification less cumbersome and more harmonious and allows for presenting a final classification list in a compact explicit form. Moreover, properties of the equivalence groupoid influences the choice of methods for group classification.

The class \( L\mid S \) is normalized (in the usual sense) if its equivalence groupoid is generated by its (usual) equivalence group and is said to be semi-normalized if its equivalence groupoid is generated by the transformations from the equivalence group simultaneously with transformations from symmetry groups of initial or transformed systems [7].

The algebraic method is the best choice for group classification of normalized or specifically semi-normalized classes of differential equations. A number of classes of differential equations that are important for applications are normalized or specifically semi-normalized or can be partitioned into normalized classes and hence were classified within the framework of the algebraic method. See, e.g., [1, 7, 11, 14, 15, 16].

The equivalence groupoid of the class of linear Schrödinger equations with complex potentials has been computed in [16] in the (1+1)-dimensional case. Then properties of the groupoid (more precisely, semi-normalization of the class) was taken as a base for carrying out the group classification of these equations by the algebraic method similarly to [15], where Lie symmetries of \( n \)th order \( (n \geq 2) \) of linear ordinary differential equations were classified using semi-normalization of the class of such equations. The final classification result, which is a complete list of inequivalent potentials corresponding to equations with nontrivial Lie symmetries, may be used in quantum theory, quantum field theory, optics and other branches of physics, cf. [17, 18, 19, 20].

The aim of this paper is to study the equivalence groupoid, the equivalence group and normalization properties of the class \( \mathcal{F} \) of (1+2)-dimensional linear Schrödinger equations with complex potentials,

\[
ti\psi_t + \psi_{xx} + V(t, x)\psi = 0. \tag{1}
\]

Here and in what follows \( t \), \( x_1 \) and \( x_2 \) are the real independent variables, \( x = (x_1, x_2) \), \( \psi \) is the complex dependent variable and \( V \) is an arbitrary smooth complex-valued potential depending
on \( t \) and \( x \). Subscripts of functions denote differentiation with respect to the corresponding variables. In particular, \( \psi_t = \partial \psi / \partial t \) and \( \psi_{ab} = \partial^2 \psi / \partial x_a \partial x_b \). The indices \( a, b, c, \) and \( d \) run from 1 to 2, and we use summation convention over repeated indices.

The structure of this paper is organized as follows: In Section 2 we compute the equivalence groupoid of the class \( \mathcal{F} \). It appears that the transformational parts of admissible transformations of \( \mathcal{F} \) are uniformly parameterized by the corresponding initial values of \( V \). The expression for target values of the arbitrary element \( V \) in terms of its initial values and transformation parameters is also found. Then the equivalence group of the class \( \mathcal{F} \) and normalization properties of this class are described in Section 3. The last section includes a short summary and suggests a direction for the future work.

2. Equivalence groupoid

We find the equivalence groupoid \( \mathcal{G}^\sim \) of the class \( \mathcal{F} \) by using the direct method. We seek for all invertible point transformations of the form

\[
\tilde{t} = T(t, x, \psi, \psi^*), \quad \tilde{x}_a = X^a(t, x, \psi, \psi^*), \quad \tilde{\psi} = \Psi(t, x, \psi, \psi^*), \quad \tilde{\psi}^* = \Psi^*(t, x, \psi, \psi^*)
\]

with \( J = \partial(T, X^a, \Psi, \Psi^*) / \partial(t, x, \psi, \psi^*) \neq 0 \) that map a fixed equation from the class \( \mathcal{F} \) to another equation of the same class,

\[
i \tilde{\psi} + \tilde{\psi}_{xx} x_a x_a + \tilde{V}(\tilde{t}, \tilde{x}) \tilde{\psi} = 0.
\]

Hereafter in case of a complex value \( \beta \) we use the notation

\[
\tilde{\beta} = \beta \quad \text{if} \quad T_t > 0 \quad \text{and} \quad \tilde{\beta} = \beta^* \quad \text{if} \quad T_t < 0.
\]

**Lemma 1.** Any point transformation \( \mathcal{T} \) connecting two equations from the class \( \mathcal{F} \) satisfies the conditions

\[
T_x = T_\psi = T_{\psi^*} = 0, \quad X^a_\psi = X^a_{\psi^*} = 0,
\]

\[
\Psi_\psi = 0 \quad \text{if} \quad T_t < 0 \quad \text{and} \quad \Psi_{\psi^*} = 0 \quad \text{if} \quad T_t > 0.
\]

**Proof.** The proof is similar to that given in [7].

**Theorem 1.** The equivalence groupoid \( \mathcal{G}^\sim \) of the class \( \mathcal{F} \) consists of the triples of the form \((V, V, \mathcal{T})\), where \( \mathcal{T} \) is a point transformation in the space of variables, whose components are

\[
\begin{align*}
\tilde{t} &= T, \quad \tilde{x}_a = |T_t|^{1/2} O^{ab} x_b + X^a, \\
\tilde{\psi} &= \exp \left( i \frac{T_t}{8 |T_t|^2} x_a x_a + i \frac{c'}{2 |T_t|^2} O^{ba} x_a + \Lambda + i \Sigma \right) (\tilde{\psi} + \tilde{\Omega}),
\end{align*}
\]

and the transformed potential \( \tilde{V} \) is expressed via \( V \) as

\[
\tilde{V} = \frac{\tilde{V}}{|T_t|} + \frac{2T_{tt}T_t - 3 T_{tt}^2}{16 c' T_t^2} x_a x_a + \frac{c'}{2 |T_t|^2} \left( \frac{X^a_{\psi^*}}{T_t} \right) O^{ba} x_a + \frac{\Sigma_t - i \Lambda_t}{T_t} - \frac{X^a_{\psi^*} X^a_{\psi} + i T_{tt}}{4 T_t^2}.
\]

Here \( T, X^a, \Lambda \) and \( \Sigma \) are arbitrary smooth real-valued functions of \( t \) with \( T_t \neq 0 \), \( c' = \text{sgn} T_t \), \( \Omega = \Omega(t, x) \) is an arbitrary solution of the initial equation and \( O = (O^{ab}) \) is an arbitrary constant 2 × 2 orthogonal matrix.

**Proof.** Let a point transformation \( \mathcal{T} \) connect two equations from the class \( \mathcal{F} \). Lemma 1 implies that \( T = T(t) \) with \( T_t \neq 0 \), \( X^a = X^a(t, x) \) and \( \Psi_{\psi} \neq 0 \). Applying the chain rule for total derivatives with respect to \( t \) and \( x \) to the equality \( \tilde{\psi}(\tilde{t}, \tilde{x}) = \Psi(t, x, \psi) \), we derive

\[
D_t \tilde{\psi}(\tilde{t}, \tilde{x}) = \tilde{\psi}_t T_t + \tilde{\psi}_{xx} X^b_t = D_t \Psi, \quad D_a \tilde{\psi}(\tilde{t}, \tilde{x}) = \tilde{\psi}_{xa} X^a_b = D_a \Psi,
\]

\[
D_0 D_a \tilde{\psi}(\tilde{t}, \tilde{x}) = \tilde{\psi}_{xa, x_a} X^b_{x_b} + \tilde{\psi}_{xa} X^a_{ab} = D_0 D_a \Psi.
\]
where $D_t$ and $D_a$ are the total derivative operators with respect to $t$ and $x_a$, respectively. The above equations are equivalent to

$$
\begin{align*}
\hat{\psi}_t &= \frac{1}{T_t} \left( D_t \Psi - Y^a_b X^b_t D_a \Psi \right), \\
\hat{\psi}_{x_a x_b} &= Y^a_c Y^b_d (D_b D_a \Psi - Y^d_c X^d_{ab} D_d \Psi),
\end{align*}
$$

(6)

$$
\hat{\psi}_{x_a, x_b, x_c} = Y^a_v Y^b_d (D_b D_a \Psi - Y^d_v X^d_{ab} D_d \Psi),
$$

(7)

where $Y^a_b X^b_c = \delta^a_c$ and $\delta^a_c$ is the Kronecker delta. In fact, the vector-function $(Y^1, Y^2)$ is the inverse of the vector-function $(X^1, X^2)$ with respect to $x$ and $Y^a_c = \partial Y^a / \partial x_c$. We substitute the values of $\hat{\psi}_t$ and $\hat{\psi}_{x_a x_b}$ defined in (6) and (7) into the equation (3) and take into account the expression for $\hat{\psi}_t$, $\hat{\psi}_t = i \varepsilon (\hat{\psi}_{aa} + \hat{V} \hat{\psi})$, where $\varepsilon' = \text{sgn} T_t$. As a result, we derive the equation

$$
\begin{align*}
&i \frac{1}{T_t} \left( \Psi_t + \Psi_{\psi} (i \varepsilon \hat{\psi}_{aa} + i \varepsilon' \hat{V} \hat{\psi}) - Y^a_b (\Psi_a + \Psi_{\psi} \hat{\psi}_a) X^b_t \right) + Y^a_b Y^b_c \left( \Psi_{ab} + \Psi_{\psi b} \hat{\psi}_b + \Psi_{\psi a} \hat{\psi}_a \right) \\
&+ Y^a_c Y^b_c \left( \Psi_{\psi b} \hat{\psi}_b \hat{\psi}_a + \Psi_{\psi a} \hat{\psi}_{ab} - Y^d_c (\Psi_d + \Psi_{\psi d} \hat{\psi}_d) X^c_{ab} \right) + \hat{V} \Psi = 0.
\end{align*}
$$

Then splitting this equation with respect to various derivatives of $\hat{\psi}$ and additionally arranging leads to the system

$$
\begin{align*}
Y^a_b Y^b_c = 0, \quad a \neq b, & \quad Y^a_c Y^c_a = \frac{1}{|T_t|}, \quad \Psi_{\psi \psi} = 0, \quad (8) \\
2 \frac{1}{|T_t|} \Psi_{ab} + i \frac{1}{T_t} Y^a_b \Psi_{\psi} X^b_t - \frac{1}{|T_t|} Y^a_c Y^c_{ab} = 0, & \quad (9) \\
i \frac{1}{T_t} \Psi_t - \frac{1}{|T_t|} \hat{V} \Psi_{\psi} - i \frac{1}{T_t} Y^a_b \Psi_{a} X^b_t + \frac{1}{|T_t|} \Psi_{aa} - Y^d_c \Psi_{d} X^c_{aa} + \hat{V} \Psi = 0. & \quad (10)
\end{align*}
$$

We first show that $X^c_{ab} = 0$ for all $a, b$ and $c$. The first two equations in (8) together with the condition $Y^a_b X^b_t = \delta^a_b$ imply that $X^b_t = |T_t| Y^a_b$. Therefore, $X^c_a X^e_b = |T_t| \delta^c_e$, i.e. $X^c_a = |T_t|^{1/2} O^{ca}$, where $O = (O^{ca})$ is a $2 \times 2$ orthogonal matrix-function of $t$ and $x$. Suppose that $O$ is a special orthogonal matrix, i.e. det $O = 1$. Then the matrix $(X^c_a)$ can be written as

$$
(X^c_a) = |T_t|^{1/2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
$$

(11)

for a smooth function $\theta = \theta(t, x)$. The representation (11) implies that $X^1_1 = X^2_2 = |T_t|^{1/2} \cos \theta$ and $X^1_2 = -X^2_1 = -|T_t|^{1/2} \sin \theta$, which means that transformation components $X^1$ and $X^1$ satisfy the Cauchy–Riemann system

$$
X^1_1 = X^2_2, \quad X^1_2 = -X^2_1.
$$

Hence both the components $X^c_a$ are solution of the Laplace equation, i.e. $X^c_{ab} = 0$. Applying the Laplace operator $\partial_{ab}$ to both sides of the equation $X^c_a X^e_b = |T_t|$, where there is no summation with respect to $a$, we derive $X^c_a X^e_b = 0$, i.e. $X^c_{ab} = 0$ for all $a, b$ and $c$. The same result is obtained when det $O = -1$. Thus, $X^c_a$ is a linear function in $x$, which implies that the matrix $O$ may depend only on $t$. From the previous consideration, the equation (9) becomes

$$
\Psi_{\psi} = \frac{i}{2T_t} X^b_a X^b_t \Psi_{\psi}.
$$

(12)

Differentiating this equation for $a = 1$ with respect to $x_2$ and for $a = 2$ with respect to $x_1$, subtracting the first result from the second and taking into account the coincidence of mixed derivatives, we find the compatibility condition $X^b_1 X^b_2 = X^b_2 X^b_1$. The representation (11) together with the compatibility condition results in $\theta_t = 0$. Therefore, the matrix $O$ is a constant orthogonal matrix.
The general solution of the third equation in (8) is \( \Psi = \Psi^1 \hat{\psi} + \Psi^0 \), where \( \Psi^0 \) and \( \Psi^1 \) are smooth complex-valued functions of \( t \) and \( x \), and \( \Psi^1 = \Psi \hat{\psi} \neq 0 \). The integration of the equation (12) gives the expression of \( \Psi^1 \),

\[
\Psi^1 = \exp \left( \frac{i}{8|T_1|} x_a x_a + \frac{i}{2|T_1|^1/2} O^{b\alpha} x_a + \Lambda + i\Sigma \right),
\]

where \( \Lambda \) and \( \Sigma \) are arbitrary smooth real-valued function of \( t \) arising in the course of the integration. Finally, we consider the equation (10), which reduces, under the derived conditions, to

\[
 \frac{i}{T_1} \Psi_t - \frac{1}{|T_1|} \hat{V} \Psi \hat{\psi} - \frac{i}{T_1} X^b_a X^b_1 \Psi_a + \frac{1}{|T_1|} \Psi^{1a} + \hat{V} \Psi = 0.
\]

Splitting with respect to \( \hat{\psi} \) in view of the representation for \( \Psi \) and rearranging, we obtain

\[
 \hat{V} = \frac{\hat{V}}{|T_1|} - \frac{i}{T_1 \Psi_t} \left( \Psi^1 X^b_a X^b_1 \Psi^1 - \frac{1}{|T_1|} \Psi^{1a} \right),
\]

\[
 i\epsilon \Psi^0_t - \frac{i}{T_1} X^b_a X^b_1 \Psi^0_a + \Psi^{0a} + |T_1| \hat{V} \Psi^0 = 0.
\]

Let us introduce the function \( \Omega = \hat{\Psi}^0/\Psi^1 \), i.e. \( \Psi^0 = \Psi^1 \hat{\Omega} \). The equation (15) is equivalent to the initial linear Schrödinger equation in terms of \( \Omega \). After the substitution of \( \Psi^1 \) by its expression from (13) into (14) and then additionally collecting coefficients of \( x \), we derive the final expression for \( \hat{V} \).

\[
 \frac{i}{T_1} \Psi_t - \frac{1}{|T_1|} \hat{V} \Psi \hat{\psi} - \frac{i}{T_1} X^b_a X^b_1 \Psi_a + \frac{1}{|T_1|} \Psi^{1a} + \hat{V} \Psi = 0.
\]

3. Equivalence group and normalization properties

There are several ways of computing the equivalence group for a given class of differential equations. Some of them are the direct or the infinitesimal method. At the same time, it is unnecessary to do the computation of the equivalence group \( G^- \) of the class \( \mathcal{F} \) using one of the above alternatives since we can derive the equivalence group \( G^- \) of the class \( \mathcal{F} \) from the knowledge of its equivalence groupoid \( G^- \).

**Corollary 1.** The (usual) equivalence group \( G^- \) of the class \( \mathcal{F} \) consists of point transformations in the space of independent and dependent variables and arbitrary element that are of the form (5) with \( \Omega = 0 \).

**Proof.** Let \( \mathcal{T} \) be a point transformation connecting two equations from the class \( \mathcal{F} \). Then \( \mathcal{T} \) is necessarily of the form (5a)–(5b), and potentials of the equations are related by (5c). Any transformation from the group \( G^- \) generates a family of admissible transformations of the class \( \mathcal{F} \) and hence it has the form (5). On the other hand, from the definition of equivalence group, the group \( G^- \) contains only point transformations whose components corresponding to variables do not depend on the arbitrary element \( V \). This condition is satisfied if and only if \( \Omega = 0 \) is a common solution for all equations from the class \( \mathcal{F} \). The only common solution is \( \Omega = 0 \).

**Remark 1.** In fact, the whole equivalence group \( G^- \) is generated by the continuous family of transformations of the form (5), where \( \Omega = 0 \), \( T_1 > 0 \) and \( \det O = 1 \), and two discrete transformations: the space reflection \( I_a \) for a fixed \( a \) \((\tilde{t} = t, \tilde{x}_a = -x_a, \tilde{x}_b = x_b, b \neq a, \tilde{\psi} = \psi, \tilde{V} = V)\) and the Wigner time reflection \( I_t \) \((\tilde{t} = -t, \tilde{x} = x, \tilde{\psi} = \psi^*, \tilde{V} = V^*)\).

Summing up, we state the following

**Corollary 2.** The class \( \mathcal{F} \) is semi-normalized. More precisely, for each admissible transformation \((V, V, \mathcal{T})\) in the class \( \mathcal{F} \), its translational part \( \mathcal{T} \) is the composition of a linear superposition transformation \( \mathcal{T}_1 \) of the initial equation with the potential \( V \) and the projection of an equivalence transformation \( \mathcal{T}_2 \) of the class \( \mathcal{F} \) to the space variables with \( \mathcal{T}_2 V = \tilde{V} \).
Proof. Consider two fixed similar equations from the class $\mathcal{F}$ with potentials $V$ and $\tilde{V}$ and let $\mathcal{T}$ be a point transformation connecting these equations. From Theorem 1, the transformation $\mathcal{T}$ is of the form $(5a)$–$(5b)$, and the potentials $V$ and $\tilde{V}$ are related by $(5c)$. We define two point transformations. The first transformation $\mathcal{T}^1$ is the point transformation in the variable space with the components $\dot{t} = t$, $\dot{x} = x$, $\dot{\psi} = \psi + \Omega$ with the same $\Omega$ as in $\mathcal{T}$. It does not change the potential $V$ and, therefore, is a point symmetry transformation of linear superposition for the initial equation. The second transformation $\mathcal{T}^2$ is the point transformation in the extended space of variables and the potential $V$ that is of the form $(5)$ with the same values of parameters as in $\mathcal{T}$ except $\Omega = 0$. Hence it is an equivalence transformation, which connects the equations with the potentials $V$ and $\tilde{V}$, $\tilde{V} = \mathcal{T}_2V$. As a result, the transformation $\mathcal{T}$ coincides with the composition of $\mathcal{T}^1$ and the projection $\mathcal{T}^2|_{(t,x,\psi)}$ of $\mathcal{T}^2$, $\mathcal{T} = \mathcal{T}^2|_{(t,x,\psi)} \circ \mathcal{T}^1$. \hfill \Box

4. Conclusion

The equivalence groupoid of the class $\mathcal{F}$ of $(1+2)$-dimensional linear Schrödinger equations with complex potentials, which has an interesting algebraic structure, is exhaustively described in Theorem 1 using the direct method. The method is standard for the study of point transformations between differential equations. At the same time, it is not quite algorithmic, especially when considering the entire set of admissible transformations of a class of differential equations. The versions of the direct method for finding point symmetry transformations of a single differential equation or equivalence transformations of a class of differential equations are much easier to be realized. Moreover, computations become trickier and more cumbersome if we increase the dimension of equations. New features also appear in the form of admissible transformations in comparison with the $(1+1)$-dimensional case. They are not exhausted by the formal extension of the set of space variables. Coupling of space variables leads, in particular, to involving rotations in the corresponding equivalence groupoid. In total, the above makes the entire consideration much more complicated than in $(1+1)$-dimensions.

Knowing the equivalence groupoid of the class $\mathcal{F}$, we construct the (usual) equivalence group of $\mathcal{F}$ in an easy way, roughly speaking, by singling out families of admissible transformations that are pointwise parameterized by the arbitrary element, which is the potential $V$ for the class $\mathcal{F}$, and having the same transformational part. We relate the equivalence groupoid, the equivalence group and normalization properties of the class $\mathcal{F}$, and this relation is similar to that for the $(1+1)$-dimensional counterpart of the class $\mathcal{F}$, which is studied in [16]. We show that, roughly speaking, any point transformation connecting two fixed equations from the class $\mathcal{F}$ is the composition of a linear superposition symmetry transformation of the initial equation and an equivalence transformation of this class. In other words, the class $\mathcal{F}$ is semi-normalized in a quite specific way, which guarantees, in view of our experience with the $(1+1)$-dimensional counterpart of the class $\mathcal{F}$ [16], the effective use of the algebraic method for the exhaustive group classification of the class $\mathcal{F}$. Therefore, the results of the present paper can be considered as a first step in our future work on this classification.

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