A co-free construction for elementary doctrines

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Abstract

We provide a co-free construction which adds elementary structure to a primary doctrine. We show that the construction preserves comprehensions and all the logical operations which are in the starting doctrine, in the sense that it maps a first order many-sorted theory into the same theory formulated with equality. As a corollary it forces an implicational doctrine to have an extensional entailment.

Introduction

This paper deals with the notion of internal equality in doctrines. Doctrines were introduced by Lawvere ([7], [6] and [8]) and we postpone in section 1 their formal definition. For the purpose of this introduction it is enough to think of doctrines as those presheaves such that, given a theory \( T \) over a many-sorted relational language \( L \), one looks at objects and morphisms of the domain category as types and terms of \( L \) respectively, while a well formed formula in \( T \) of type \( A \) is an element in the fiber over \( A \). Lawvere made extensive use of the language of adjoints and Jacobs [3] described equality between terms of a given type as a formula in the fiber over the product of that type with itself, satisfying the following rule of inference

\[ \Gamma, x: X | \phi \vdash \psi[x/y] \]
\[ \Gamma, x: X, y: X | \phi \land x =_X y \vdash \psi \]

where the double line indicates that one of the two sequents holds exactly when the other holds. A doctrine is a first order theory with equality if it possesses a formula \( =_X \), for every sort \( X \), which satisfies the previous rule.

A way to introduce higher order quantification is to consider a new type \( \Omega \) in the underlying signature and thinks of terms of type \( \Omega \) as propositions. From the categorical viewpoint this generates a correspondence between terms of type \( \Omega \) and formulas, and therefore it makes sense to investigate how the notion of internal equality \( =_\Omega \) is related to logical equivalence. A link is in the following rule, taken from [1] and [3]

\[ \Gamma \vdash \xi \land \phi \vdash \psi \quad \Gamma \vdash \xi \land \psi \vdash \phi \]
\[ \Gamma \vdash \xi \vdash \phi =_\Omega \psi \]

where it is implicit that if \( \phi \) and \( \psi \) are formulas over the context \( \Gamma \), then \( \phi =_\Omega \psi \) is still a formula over \( \Gamma \). We say that a doctrine is a higher order many-sorted
theory with extentional entailment if there is an object Ω in the base category and a formula =_Ω in the fiber over Ω × Ω which satisfies both the previous rules. In the present paper we provide a co-free construction that, starting from any doctrine \( P \), produces a new doctrine \( P_D \) with equality. That is to say that for every object \( X \) in the domain category of \( P_D \) there exists a well formed formula in \( P_D(X \times X) \) which satisfies the first one of the previous rules. We show also that if the starting doctrine \( P \) is an higher order implicational theory, the resulting doctrine \( P_D \) will have an internal equality over Ω satisfying both the previous rules; in other words: =_Ω and logical equivalence comes to coincides.

In section 1 we give the definitions of doctrines and some relevant examples. In section 2 we introduce the construction of Maietti and Rosolini of the category of quotients and the doctrine of descent data which is the base of the co-free construction we are going to provide in 3. In the last section we show which properties are preserved by the construction and some applications.

1 Doctrines

We recall those structures which we will be concerned with in the paper, see [10] and [11].

Definition 1.1. A primary doctrine is a functor \( P : C^{op} \rightarrow ISL \), where ISL is the subcategory of Posets consisting of inf-semilattices and homomorphisms and \( C \) is a category with binary products.

For the rest of the paper we will write \( f^* \) instead of \( P(f) \), to indicate the action of the functor \( P \) on a morphism \( f \) of \( C \). We shall refer to \( f^* \) as the reindexing functor along \( f \). Left and right adjoints to reindexing functor \( f^* \) will be \( \exists_f \) and \( \forall_f \) respectively. We say that a doctrine has finite joins if every fiber has finite joins. Analogously we say that a doctrine is implicational if every fiber has relative pseudo complements which commute with reindexing. For every pair of element \( x \) and \( y \) we will denote their meet by \( x \land y \), by \( x \lor y \) their join and by \( x \Rightarrow y \) their relative pseudo complements. Top and bottom elements will be \( \top \) and \( \bot \) respectively. Joins are said to be distributive if for every \( x, y \) and \( z \) it holds that \( x \land (y \lor z) = (x \land y) \lor (x \land z) \).

Definition 1.2. A primary doctrine \( P \) is said to be elementary if for every \( A \) in \( C \) there exists an object \( \delta_A \) in \( P(A \times A) \) such that for every \( X \) in \( C \)

i) the assignment \( \pi_1^*(\alpha) \land \delta_A \) determines a left adjoint to \( \Delta_A^* \)

ii) the assignment \( \langle \pi_1, \pi_2 \rangle^*(\alpha) \land \langle \pi_2, \pi_3 \rangle^*(\delta_A) \) determines a left adjoint to \( (id_X \times \Delta_A)^* \)

Primary doctrines are the objects of the 2-category \( PD \) in which
the 1-arrows are pairs $(F, f): P \to R$

\[
\begin{array}{c}
\text{C}^{\text{op}} \\
\uparrow P \\
\downarrow F \\
\text{D}^{\text{op}}
\end{array}
\begin{array}{c}
\text{ISL} \\
\downarrow f \\
\downarrow R
\end{array}
\]

where the functor $F$ preserves products and $f$ is a natural transformation from the functor $P: \text{C}^{\text{op}} \to \text{ISL}$ to the functor $R \circ F: \text{D}^{\text{op}} \to \text{ISL}$

the 2-arrows are those natural transformations $\nu$

\[
\begin{array}{c}
\text{C}^{\text{op}} \\
\uparrow P \\
\downarrow F \\
\text{D}^{\text{op}}
\end{array}
\begin{array}{c}
\text{ISL} \\
\downarrow G \\
\downarrow f
\end{array}
\]

such that, for every object $A$ in $\text{C}$ and every $\alpha$ in $P(A)$, it holds that $\nu^*_A(f_A(\alpha)) \leq g_A(\alpha)$.

We call ED the 2-subcategory of PD, in which the object are elementary doctrines and the 1-arrows are those 1-arrows in PD such that

$$f_{A \times A}(\delta_A) = (F\pi_1, F\pi_2)^*\delta_{F A}$$

for every 1-arrows $(F, f)$ and for every object $A$ in $\text{C}$.

**Definition 1.3.** A primary doctrine is called **universal** if for every projection arrows $\pi$ in $\text{C}$ the functor $\pi^*$ has a right adjoint $\forall\pi$ satisfying Beck-Chevalley condition: given a pullback diagram of the kind

\[
\begin{array}{ccc}
X \times Y' & \xrightarrow{\pi'} & Y' \\
\downarrow \text{id} \times f & & \downarrow f \\
X \times Y & \xrightarrow{\pi} & Y
\end{array}
\]

it holds that $\forall\pi' \circ (\text{id} \times f)^* = f^* \circ \forall\pi$

A primary doctrine is **existential** if the reindexing functors along a projection have a left adjoint satisfying Beck-Chevalley and Frobenius reciprocity: $\exists\pi(\alpha \land \pi^*\beta) = \exists\pi(\alpha) \land \beta$, for $\alpha$ in $P(X \times Y)$ and $\beta$ in $P(Y)$.

**Remark 1.4.** Recall from [6, 11] that in an elementary existential doctrine $P$ for every morphism $f: A \to B$ in the base category there exists a functor $\exists_f: P(B) \to P(A)$ such that $\exists f^*$. Indeed if $\pi_A$ and $\pi_B$ are the projections from $A \times B$ to $A$ and $B$ respectively, for $\alpha$ in $P(A)$

$$\exists_f(\alpha) = \exists_{\pi_B}((\text{id}_B \times f)^*\delta_B \land \pi_A^*\alpha)$$

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Such a generalized quantification satisfies Frobenius Reciprocity. For \( \beta \) in \( P(B) \), we have that \((id_B \times f)^* \delta_B \land f^* \beta = (id_B \times f)^* \delta_B \land \pi_B^* \beta \). Therefore

\[
\exists_{\pi_B} ((id_B \times f)^* \delta_B \land \pi_A^* \alpha \land f^* \beta) =
\exists_{\pi_B} ((id_B \times f)^* \delta_B \land \pi_A^* \alpha \land \pi_B^* \beta) =
\exists_{\pi_B} ((id_B \times f)^* \delta_B \land \pi_A^* \alpha) \land \beta
\]

And for a pullback square such as that in \( [1,2] \) the Beck-Chevalley condition holds: \( \exists_{(id_X \times f)} \pi^* = \pi^* \exists f \).

**Definition 1.5.** A primary doctrine is said to have a weak power objects if for every \( A \) in \( C \) there exists an object \( \pi A \) in \( C \) and an element \( \in A \) in \( P(A \times \pi A) \) such that, for every object \( B \) in \( C \) and element \( \phi \) in \( P(A \times B) \) there exists a morphism \( \{\phi\}: B \rightarrow \pi A \) such that \( \phi = (id_A \times \{\phi\})^* \in A \).

**Remark 1.6.** In the case the base category \( C \) has a terminal object \( 1 \): the first item in the definition \( [1,2] \) is redundant, since it becomes a particular instance of the second; when the doctrine has weak power objects, for every object \( A \) in \( C \) each element \( \phi \) in \( P(A) \) determines a term of type \( \pi 1 \) via the following isomorphism:

\[
\begin{array}{c}
1 \times 1 \xrightarrow{id_1 \times (\delta^*)} 1 \\
\downarrow \quad \downarrow \phi \\
1 \times A \xrightarrow{f} A
\end{array}
\]

we will denote with \( \epsilon_1 \) the element \((i^{-1})^* \in 1; \) in the case \( C \) has all comprehensions, defined to be those morphisms \([\phi]: X \rightarrow A \) which are terminal with respect to the property that \( \top_X \leq [\phi]^*(\phi) \), for every \( \phi \) in \( A \) (see \( [10] \)), then \( [\phi] \) is weakly classified by \( \{\phi\} \), where the true arrow is \( [\epsilon_1]: 1 \rightarrow 1 \).

There are several examples of doctrines, we list a few.

**Example 1.7. (Syntactic)** Given a theory \( T \) in a first order language \( L \), the base category \( \forall \) has lists of distinct variables \( \vec{x} = (x_1, x_2, \ldots x_n) \) as objects and lists of substitutions \( [\vec{f}/\vec{y}]: \vec{x} \rightarrow \vec{y} \) as morphisms. Composition is given by simultaneous substitution. For an object \( \vec{x} \) in \( \forall \), the fiber over \( \vec{x} \) consists of equivalence classes of well-formed formulae of \( L \) with no more free variables than \( x_1, x_2, \ldots x_n \), with respect to reciprocal entailment of \( T \), see \( [10] \).

**Example 1.8. (Subobjects)** Suppose \( C \) a small category with binary products and pullbacks. Consider the functor that assigns for every object \( A \) in \( C \) the collection \( \text{Sub}(A) \) of subobjects with codomain \( A \), ordered by factorization. The top element is \((\text{the equivalence class of}) \) the identity arrow. \( A \) represents \( \alpha \land \beta \) is any pullback of \( \alpha \) along \( \beta \). Given a morphism \( f \) in \( C \), \( f^* \alpha \) is the class of any pullback of \( \alpha \) along \( f \). If \( C \) is regular, the doctrine has left adjoints of all reindexing functors. It is elementary with \( \delta_A = \Delta_A: A \rightarrow A \times A \).

\( \text{Sub}: C^{op} \rightarrow \text{ISL} \) has full comprehensions. An element \( \alpha: X \rightarrow A \) in \( \text{Sub}(A) \) has itself as its own comprehension. Consider the following diagrams

\[
\begin{align*}
X \xrightarrow{\tau_X} X & \quad Y \xrightarrow{k} P \xrightarrow{p} X \\
X \xrightarrow{\alpha} A & \quad Y \xrightarrow{f^* \alpha} A
\end{align*}
\]

\[
\begin{align*}
X \xrightarrow{h} Q \xrightarrow{q} X \\
X \xrightarrow{\alpha^* \beta} A
\end{align*}
\]
where $P$ is a pullback of $\alpha$ along $f$ and $Q$ the pullback of $\beta$ along $\alpha$. The left one is a pullback and says that $\top_X \leq \alpha^* \alpha$. The second proves that if $\top_Y \leq f^* \alpha \ (\leq k)$, then $f$ factorizes through $\alpha$. Third pullback shows that if $\top_X \simeq \alpha^* \alpha \leq \alpha^* \beta \ (\leq h)$, then $\alpha \leq \beta \ (\leq q \circ h)$.

A particular case is when $C$ is a small, full subcategory of $\text{Set}$ closed under binary products and subsets and the functor $\text{Sub}: C^{op} \rightarrow ISL$ coincides with the powerset functor.

**Example 1.9. (Triposes)** We refer to the definition given by Pitts in [11].

Given a category $C$ with binary products, a tripos is a primary doctrine $P: C^{op} \rightarrow ISL$ such that: (i) for every object $A \in C$, $P(A)$ is a Heyting Algebra (ii) for every arrow $f$ in $C$, $f^*$ is an homomorphism of Heyting algebras (iii) for every projection arrow $\pi$ in $C$ the functor $\pi^*$ has left and right adjoints satisfying the Beck-Chevalley conditions (iv) $P$ has weak power objects (v) for every object $A$ in $C$ there exists an element $\delta_A$ in $P(A \times A)$ such that, for all $\alpha$ in $P(A \times A)$, $\top_A \leq \Delta_A^*(\alpha)$ if and only if $\delta_A \leq \alpha$.

All triposes are universal doctrine with weak power objects. They are elementary, since the assignment $\exists_{\Delta_X}(\alpha) = \pi_1^*(\alpha) \land \delta_X$ provides a left adjoint to $\Delta_X^*$, in fact

$$
\begin{align*}
\alpha &\leq \Delta_X^*(\beta) \\
\top_X \leq \alpha &\Rightarrow \top_X \leq \Delta_X^*(\beta) \\
\top_X &\leq \top_X \leq \Delta_X^*(\beta) \\
\delta_X \leq \pi_1^*(\alpha) &\Rightarrow \beta \\
\delta_X \land \pi_1^*(\alpha) &\leq \beta
\end{align*}
$$

and the assignment $\exists_\alpha(\alpha) = (\pi_1, \pi_2)^*(\alpha) \land (\pi_2, \pi_3)^*(\delta_A)$ determines a left adjoint to the reindexing of $e: = id_X \times \Delta_A: X \times A \rightarrow X \times A \times A$

$$
\begin{align*}
\alpha \leq e^*(\beta) \\
\top_X \times A &\leq \alpha \Rightarrow e^*(\beta) \\
\top_A &\leq \top_A \leq \alpha \Rightarrow e^*(\beta) \\
\top_A &\leq \top_A \leq \Delta_A^*(\alpha) \Rightarrow e^*(\beta) \\
\delta_A &\leq \delta_A \leq \Delta_A^*(\alpha) \Rightarrow e^*(\beta) \\
\delta_A &\leq \delta_A \leq \Delta_A^*(\alpha) \Rightarrow e^*(\beta) \\
\alpha &\leq e^*(\beta)
\end{align*}
$$

Similarly it can be proved that Frobenius reciprocity is verified (see also [13], pag 60). Two important examples of triposes are $H(-)$, for a complete Heyting algebra $H$, and $P(N)(-)$, for a partial combinatory algebra over a set $N$. In each case $C$ is $\text{Set}$, the category of sets and functions. There is no need for a tripos to have comprehensions. But this is the case for localic triposes $H(-)$ and realizability triposes $P(N)(-)$.

Take a set $X$ and an object $\phi: X \rightarrow H$: a comprehension of $\phi$ is given by the inclusion $[\phi]: \{ x \in X \mid \top \leq \phi(x) \} \hookrightarrow X$. The same holds for realizability triposes, for which $[\phi]: \{ x \in X \mid N \subseteq \phi(x) \} \hookrightarrow X$.

These comprehensions can not be full. Take $[\phi]: A \hookrightarrow X$ and consider the
function $\psi: X \rightarrow \mathbb{H}$ defined by $\psi(x) = \top$ if $x \in A$ and $\bot$ otherwise. For this function certainly holds $[\phi]^*(\psi) = \top$, but it is not the case that $\phi \preceq \psi$.

**Example 1.10. (Topologies)** Consider the category $\text{TOP}$ of topological spaces and continuous functions. For every topological space $X$, $\mathcal{O}(X)$ is the collection of its open sets, and then it possesses finite meets and arbitrary joins. Take the functor $\mathcal{O}: \text{TOP}^{\text{op}} \rightarrow \text{ISL}$ determined by the following assignment

$$(X, \mathcal{O}(X)) \mapsto (Y, \mathcal{O}(Y))$$

Even though each fiber is an Heyting algebra, and therefore it has pseudo relative complements (see [12], page 51), $\mathcal{O}$ is not implicational as a doctrine: given a generic continuous function $f$, we have that pseudo relative complements need not commute with reindexing (see [3], page 39). $\mathcal{O}$ is existential, since every projection functor has a left adjoint (see [12], page 58) satisfying Beck-Chevalley condition and Frobenius reciprocity (recall that projections are open functions). $\mathcal{O}$ has full comprehensions. Given a set $X$, for any open set $S$ in $\mathcal{O}(X)$, define its comprehension to be the inclusion function $\lfloor S \rfloor : (X, \mathcal{O}(X)) \rightarrow (X, \mathcal{O}(X))$, where $O_S(X)$ is the topology induced by $S$. These comprehensions are also full.

Suppose $Q$ in $\mathcal{O}(X)$ such that $\lfloor S \rfloor^{-1}(Q) = S$, this means $\{x \in S \mid x \in Q\} = S \cap Q = S$, so $S \subseteq Q$.

The doctrine has weak power objects. We call $\Sigma$ the Sierpinski space consisting of two points 0 and 1 and a third non trivial open set $\{1\}$. If a topological space $T$ is locally compact, then there exists in $\text{TOP}$ the function space $\Sigma^T$ (see [2] and [5]). $\Sigma$ extends the subobjects classifier from $\text{Set}$ to $\text{TOP}$ in the sense that for every $\phi$, open set of $T$, the characteristic function of the inclusion $\lfloor \phi \rfloor$ is the unique arrow making the following a pullback

$$
\begin{array}{ccc}
X & \rightarrow & 1 \\
\downarrow & & \downarrow \top \\
T & \xrightarrow{\chi_{\phi}} & \Sigma
\end{array}
$$

for which it holds that $\chi_{\phi}^{-1}(\{1\}) = \phi$. Now for every topological space $A$ consider any construction that produces a larger locally compact space $\tilde{A}$ such that the inclusion morphism $i_A: A \hookrightarrow \tilde{A}$ is continuous and open, e.g. Alexandroff compactifications, see [3]: the following lemma holds: if $f: A \times B \rightarrow \Sigma$ is continuous, then the extension $\tilde{f}: \tilde{A} \times \tilde{B} \rightarrow \Sigma$ is continuous, where $\tilde{f}(a, b) = f(a, b)$ if $a \in A$, then $f(a, b) = 0$.

To prove the lemma it suffices to note that there are no open sets in $\Sigma$ containing the point 0 other than the top element, then $\tilde{f}^{-1}(\{1\}) = f^{-1}(\{1\})$ and the inclusion function is open. Note that $f = \tilde{f} \circ (i_A \times \text{id})$. Now consider the diagram

$$
\begin{array}{ccc}
A \times \Sigma & \xrightarrow{i_A \times \text{id}} & \tilde{A} \times \Sigma \\
\downarrow & & \downarrow \chi_{\tilde{A}} \\
A \times B & \xrightarrow{i_A \times \text{id}} & \tilde{A} \times B
\end{array}
$$
define \( \in_A := (ev_A \circ (i_A \times id))^{-1}(\{1\}) \) and for every open set \( \phi \) in \( A \times B \) define \( \{\phi\} := \chi_{\phi^{-1}} \) the exponential transpose of the extension of \( \chi_{\phi^{-1}} \). \( (id_A \times \chi_{\phi^{-1}})^{-1}(\in_A) = \chi_{\phi^{-1}}^{-1}(\{1\}) = \phi. \)

The doctrine fails to be elementary. Given a topological space \( X \), we have that \( \delta_X \) should be the smallest open set \( U \) of \( X \times X \) such that \( X \subseteq \Delta^{-1}(U) \).

In other words \( \delta_X = (\bigcap_{X \subseteq \Delta^{-1}(U)} U)^o \)

if \( X \) is the interval \([0, 1]\) with the euclidean topology, then \( \delta_X \) would be empty.

### 2 Quotients and descents

Recall a construction presented in [9, 10], which is based on the notion of equivalence relation in a doctrine.

**Definition 2.1.** Given a primary doctrine \( P: \mathbb{C}^{op} \to \text{ISL} \) and an object \( A \) of \( \mathbb{C} \), an element \( \rho \) in \( P(A \times A) \) is said to be an equivalence relation on \( A \) if

- reflexivity: \( \top_A \leq \Delta_A^*(\rho) \)
- symmetry: \( \rho \leq (\pi_1, \pi_2)^*(\rho) \)
- transitivity: \( (\pi_1, \pi_2)^*(\rho) \wedge (\pi_2, \pi_3)^*(\rho) \leq (\pi_1, \pi_3)^*(\rho) \)

Note that if the doctrine \( P \) is also elementary, then \( \delta_A \) is an equivalence relation on \( A \) for every object \( A \) in \( \mathbb{C} \).

In [9, 10] the authors consider a certain category \( Q_P \), when \( P: \mathbb{C}^{op} \to \text{ISL} \) is elementary. In the category \( Q_P \)

**objects** are pairs \((A, \rho)\) such that \( \rho \) is an equivalence relation on \( A \)

**morphisms** \( f: (A, \rho) \to (B, \sigma) \) are arrows \( f: A \to B \) in \( \mathbb{C} \) such that \( \rho \leq (f \times f)^* \sigma \)

and composition is given as in \( \mathbb{C} \).

A first remark is that the construction gives a category in the more general case of \( P \) primary. The category \( Q_P \) has binary products: given \((A, \rho)\) and \((B, \sigma)\) in \( Q_P \), \((A, \rho) \times (B, \sigma) := (A \times B, \rho \boxtimes \sigma)\), where \( \rho \boxtimes \sigma \) is \((\pi_1, \pi_3)^* \rho \wedge (\pi_2, \pi_4)^* \sigma \). Moreover if \( \mathbb{C} \) has a terminal object, \( Q_P \) has a terminal object.

There is an obvious forgetful functor \( \mathbb{U}: Q_P \to \mathbb{C} \), and a functor \( \triangledown: \mathbb{C} \to Q_P \), determined by the following assignments:

\[
\begin{array}{cccc}
(A, \rho) & \quad A & \quad A & \quad (A, \delta_A) \\
\mathbb{U} \quad f & \quad \mapsto & \quad f & \quad \triangledown \quad f & \quad \mapsto & \quad f \\
(B, \sigma) & \quad B & \quad B & \quad (B, \delta_B)
\end{array}
\]
∇ is clearly a functor since, for every morphism $f$ in $C$, $\delta_A \leq (f \times f)^* \delta_B$.

**Lemma 2.2.** Given an elementary doctrine $P: \mathcal{C}^{op} \to \text{ISL}$, the functor $\nabla$ is left adjoint to $U$.

**Proof.** For every object $(B, \sigma)$ in $Q_P$, the map $\varepsilon_B := id_B: (B, \delta_B) \to (B, \sigma)$ is the $B$-component of a natural transformation. This is the counit of the adjunction, since for every object $A$ in $C$ and every arrow $f: (A, \delta_A) \to (B, \sigma)$ in $Q_P$ the diagram commutes

$$
\begin{array}{ccc}
(B, \delta_B) & \overset{id_B}{\longrightarrow} & (B, \sigma) \\
\downarrow^f & & \downarrow^f \\
(A, \delta_A) & \overset{f}{\longrightarrow} & (B, \sigma)
\end{array}
$$

and $f$ is the unique such arrow. \qed

**Definition 2.3.** Given a primary doctrine $P: \mathcal{C}^{op} \to \text{ISL}$ and an equivalence relation $\rho$ on an object $A$ of $C$, the poset of descent data $\text{Des}_\rho$ is the sub-order of $P(A)$ made by those $\alpha$ such that

$$
\pi_1^*(\alpha) \land \rho \leq \pi_2^*(\alpha)
$$

The order $\text{Des}_\rho$ is closed under meets and it has trivially $\top_A$, then $\text{Des}_\rho$ is an inf-semilattice.

The following proposition generalizes to primary doctrines a similar result given for elementary doctrine in [9, 10].

**Proposition 2.4.** Given a primary doctrine $P: \mathcal{C}^{op} \to \text{ISL}$, the assignment

$$
\begin{array}{ccc}
(A, \rho) & \overset{\text{Des}_\rho}{\longmapsto} & (B, \sigma) \\
\downarrow^f & & \downarrow^f \\
(B, \sigma) & \overset{\text{Des}_\sigma}{\longmapsto} & (B, \sigma)
\end{array}
$$

determines a primary doctrine $P_D: \mathcal{Q}_P^{op} \to \text{ISL}$.

**Proof.** It suffices to note that, for every $\beta$ in $\text{Des}_\sigma$, $f^* \beta$ is in $\text{Des}_\rho$, that can be proved by taking the descent condition on $\beta$, applying to both sides $(f \times f)^* \beta$ and use the fact that $\rho \leq (f \times f)^* \sigma$. \qed

**3 A co-free construction**

There is an obvious forgetful functor $U: \mathcal{E} \longrightarrow \mathcal{P}$, which maps every elementary doctrine to itself. We shall show that the construction in [2] extends to a 2-right adjoint to it.

The following lemma is a strengthening of a similar result in [9].

**Lemma 3.1.** Given a primary doctrine $P: \mathcal{C}^{op} \to \text{ISL}$, the doctrine $P_D: \mathcal{Q}_P^{op} \to \text{ISL}$ built as in [2.4] is elementary. 8
Proof. Consider \((A, \rho)\) in \(Q_P\). Note that \(\rho\) is an element of \(\text{Des}_P\), since
\[
\pi^*_1 \rho \land (\rho \boxtimes \rho) = \langle \pi_1, \pi_2 \rangle^* \rho \land \langle \pi_1, \pi_3 \rangle^* \rho \land \langle \pi_2, \pi_4 \rangle^* \rho
\]
and by transitivity of \(\rho\)
\[
\pi^*_1 \rho \land \rho \boxtimes \rho \leq \langle \pi_3, \pi_4 \rangle^* \rho = \pi^*_2 \rho
\]
Let \(\delta_{(A, \rho)}\) be \(\rho\) and define \(\exists_{\Delta A} \alpha := \pi^*_1 \alpha \land \rho\). We want to prove that, for every \(\alpha\) in \(\text{Des}_P\) and \(\beta\) in \(\text{Des}_{P_\rho}\), \(\exists_{\Delta A} \alpha \leq \beta\) if and only if \(\alpha \leq \Delta^*_A \beta\) Suppose \(\exists_{\Delta A} \alpha \leq \beta\), which means \(\pi^*_1 (\alpha) \land \rho \leq \beta\), and apply \(\Delta^*_A\) to both sides, to obtain \(\alpha \land \Delta^*_A \rho \leq \Delta^*_A \beta\). So \(\alpha \leq \Delta^*_A \beta\), by reflexivity of \(\rho\). Assume now \(\alpha \leq \Delta^*_A \beta\), the descent condition for \(\beta\) gives:
\[
\langle \pi_1, \pi_2 \rangle^* \beta \land \langle \pi_1, \pi_3 \rangle^* \rho \land \langle \pi_2, \pi_4 \rangle^* \rho \leq \langle \pi_3, \pi_4 \rangle^* \beta
\]
By reindexing along \(\Delta_A \times id_A \times id_A\) and \(\Delta_A \times id_A\) one obtains
\[
\pi^*_1 \Delta^*_A \beta \land \rho \leq \beta
\]
by reflexivity of \(\rho\)
\[
\alpha \leq \Delta^*_A \beta \\
\pi^*_1 \alpha \land \rho \leq \pi^*_1 \Delta^*_A \beta \land \rho \\
\pi^*_1 \Delta^*_A \beta \land \rho \leq \beta \\
\exists_{\Delta A} \alpha \leq \beta
\]
To verify the conditions ii) of 1.2, consider an object \((X, \tau)\) and let \(e := \text{id}_X \times \Delta_A\) be a morphism in \(Q_P\). The proof that if \(\exists_{\Delta}(\alpha) \leq \beta\), then \(\alpha \leq e^*(\beta)\) is similar to that in example 1.10 (where \(\rho\) is \(\delta_A\)). The proof of the converse, is essentially as before where:
\[
\langle \pi_1, \pi_2, \pi_3, \pi_4 \rangle^* \beta \land \langle \pi_1, \pi_4 \rangle^* \tau \land \langle \pi_2, \pi_5 \rangle^* \rho \land \langle \pi_3, \pi_6 \rangle^* \rho \leq \langle \pi_4, \pi_5, \pi_6 \rangle^* \beta
\]
and reindexing along the following composition
\[
\begin{array}{ccc}
X \times A \times A & \rightarrow & X \times A \times X \times A \times A \\
\downarrow \text{id}_X \times \Delta_A \times \text{id}_A & & \downarrow \text{id}_X \times \Delta_A \times \text{id}_X \times \text{id}_A \times \text{id}_A \\
X \times A \times A & \rightarrow & X \times A \times X \times A \times A \\
\downarrow \Delta_A \times \text{id}_A \times \text{id}_A \times \text{id}_A & & \downarrow \text{id}_X \times \text{id}_A \times \text{id}_A \times \text{id}_A \\
X \times X \times A \times A & \rightarrow & X \times X \times A \times A \\
\end{array}
\]

Given a 1-morphism in \(PD\), \((F, f): P \rightarrow R\), consider the functor \(F_D\) defined by the following assignment
\[
\begin{array}{ccc}
(A, \rho) & \rightarrow & (FA, \langle \pi_1, \pi_2 \rangle^* f_A \times A (\rho)) \\
q & \rightarrow & F q \\
(B, \sigma) & \rightarrow & (FB, \langle \pi_1, \pi_2 \rangle^* f_B \times B (\sigma))
\end{array}
\]
and the \(Q_P\)-indexed family of arrow \(f_D\) whose \((A, \rho)\)-component is the restriction of \(f_A: P(A) \rightarrow R(F A)\) to \(\text{Des}_P\).
Lemma 3.2. Given a 1-morphism in $\mathbf{PD}$, $(F, f): P \to R$ the pair $(F_\mathcal{D}, f_\mathcal{D}): P_\mathcal{D} \to R_\mathcal{D}$ determines a 1-morphism in $\mathbf{ED}$.

Proof. First note that $(\pi_1, \pi_2)^\ast f_{\mathcal{A}}(\rho)$ is an equivalence relation since $\rho$ is and $f$ is natural. $Fq$ is a morphism in $\mathcal{Q}_\mathcal{P}$, since $(\pi_1, \pi_2)^\ast f_{\mathcal{A}}(\rho) \leq (Fq \times Fq)(\pi_1, \pi_2)^\ast f_{\mathcal{B}}(\sigma) = (\pi_1, \pi_2)^\ast F(q \times q)^\ast f_{\mathcal{B}}(\sigma) = (\pi_1, \pi_2)^\ast f_{\mathcal{A}}(q \times q^\ast \sigma)$, for naturality of $f$. It is left to show that the images of the restriction is $\mathcal{D}e_{\mathcal{S}\mathcal{A}}(\pi_1, \pi_2)^\ast f_{\mathcal{A}}(\rho)$, but this is true since, for $\alpha$ in $\mathcal{D}e_{\mathcal{S}\mathcal{A}}$, $\pi_1^\ast \alpha \land \pi_2^\ast \alpha$, then apply $f_{\mathcal{A}}$ to both sides and, recalling that $f_{\mathcal{A}} \circ \pi_1^\ast = \pi_1^\ast \circ f_{\mathcal{A}}$ for naturality of $f$, one has $\pi_1^\ast f_{\mathcal{A}} \land f_{\mathcal{A}}(\rho) \leq \pi_2^\ast f_{\mathcal{A}}$. Now it suffices to reinforce both sides along $(\pi_1, \pi_2)$. The last step is to show that $f_\mathcal{D}$ preserves the elementary structure, i.e. $f_{\mathcal{D}(A, \rho) \times \mathcal{A}(A, \rho)}(\delta_{\mathcal{A}(A, \rho)}) = (F_\mathcal{D} \pi_1, \pi_2)^\ast f_{\mathcal{A}}(\rho)$, which reduces to the following equality $f_{\mathcal{A}}(\rho) = (F_\pi_1, F_\pi_2)^\ast f_{\mathcal{A}}(\rho)$, where $(\pi_1, \pi_2) \circ (F_\pi_1, F_\pi_2) = id_f_{\mathcal{A}(A, \rho)}$.

Consider the functor $(-)_\mathcal{D}: \mathbf{PD} \to \mathbf{ED}$

$$
\begin{array}{ccc}
P & \to & P_\mathcal{D} \\
(F, f) & \mapsto & (F_\mathcal{D}, f_\mathcal{D}) \\
R & \to & R_\mathcal{D}
\end{array}
$$

For every doctrine $P: \mathcal{C} \to \mathbf{ISL}$ in $\mathbf{PD}$ there is a 1-morphism $\varepsilon_P$ from $P_\mathcal{D}$ to $P$ given by the pair $(\mathbb{U}, i)$, where $\mathbb{U}: Q_P \to \mathcal{C}$ is the forgetful functor defined before 2.2, while the $A$-component of $i$ is the inclusion functor $\mathcal{D}e_{\mathcal{S}\mathcal{A}} \hookrightarrow P(A)$.

Proposition 3.3. The natural transformation $\varepsilon$ is the counit of an adjunction $\mathcal{U} \dashv (-)_\mathcal{D}$.

Proof. Note that $\mathcal{U}(P) = P$: given an elementary doctrine $P: \mathcal{C}^{\mathcal{op}} \to \mathbf{ISL}$, a morphism $(F, f): \mathcal{U}(P) \to R$ in $\mathbf{PD}$, consider the arrow $(F, f): P \to R_\mathcal{D}$ in $\mathbf{ED}$, determined by the following composition

$$
\begin{array}{ccc}
\mathcal{C}^{\mathcal{op}} & \xrightarrow{\mathcal{V}} & P \\
\downarrow & & \downarrow id_P \\
\mathcal{Q}_P^{\mathcal{op}} & \xrightarrow{F_P} & P_\mathcal{D} \\
\downarrow & & \downarrow f_\mathcal{D} \\
\mathcal{Q}_R^{\mathcal{op}} & \xrightarrow{f_R} & R_\mathcal{D} \\
\downarrow & & \downarrow R_\mathcal{D} \\
\mathbf{ISL} & \xrightarrow{F} & \mathbf{ISL}
\end{array}
$$

then $F = F_\mathcal{D} \circ \mathcal{V}$ and $f = f_\mathcal{D} \circ id_P$. Where the natural transformation $P \to P_\mathcal{D} \circ \mathcal{V}$ is the identity from the fact that $\mathcal{D}e_{\mathcal{S}\mathcal{A}} = P(A)$. What is left to prove is that $(F, f)$ is the unique arrow that makes the following diagram commutes
Commutativity: recall that, for an object \( A \) in \( \mathcal{C} \), \( U(\mathcal{F})(A) = U(F_D(\nabla(A))) \), then follow the assignments below

\[
A \mapsto (A, \delta_A) \mapsto (FA, \delta_F A) \mapsto FA
\]

moreover \((i \circ \mathcal{F})_A = i_A \circ f_{DA} \circ iD_{PA}\), then take \( \alpha \in P(A) \) and follow the assignments

\[
\alpha \mapsto f_A(\alpha) \mapsto i(f_A(\alpha)) = f_A(\alpha)
\]

Uniqueness is given by the fact that \((U, i)\) is mono, since \( U \) is the identity on objects and morphism and \( i \) is an inclusion functor.

\[\square\]

4 Applications

The co-free construction presented in the previous section preserves all the first order predicate structures which are in \( P \) in the sense of the following

**Proposition 4.1.** Given a primary doctrine \( P : \mathbb{C}^{op} \to \text{ISL} \) and the elementary doctrine \( P_D : \mathbb{Q}^{op}_P \to \text{ISL} \)

(i) if \( P \) has finite distributive joins, so has \( P_D \) and \( \varepsilon_P : P_D \to P \) preserves them

(ii) if \( P \) is implicational, so is \( P_D \) and \( \varepsilon_P \) preserves this

(iii) if \( P \) existential, so is \( P_D \) and \( \varepsilon_P \) preserves this

(iv) if \( P \) universal, so is \( P_D \) and \( \varepsilon_P \) preserves this

(v) if \( P \) has (full) comprehensions, so has \( P_D \) and \( \varepsilon_P \) preserves them

**Proof.** (i) Given \( \alpha \) and \( \beta \) in \( P_D(A, \rho) \), the join \( \alpha \vee \beta \) in \( P(A) \) is in \( \text{Des}_\rho \) by distributivity. (ii) Like before, given \( \alpha \) and \( \beta \) in \( P_D(A, \rho) \), take \( \alpha \Rightarrow \beta \) in \( P(A) \).

To see this is in \( P_D(A, \rho) \), recall that, since \( \rho \) is symmetric, the descent condition can be written as \( \pi_2^\alpha \wedge \rho = \pi_1^\alpha \wedge \rho \). One has that \( \pi_1^\alpha (\alpha \Rightarrow \beta) \wedge \rho \leq \pi_1^\beta (\alpha \Rightarrow \beta) \) if and only if \( \pi_1^\alpha (\alpha \Rightarrow \beta) \wedge \rho \leq \pi_2^\alpha \beta \) if and only if \( \pi_1^\alpha (\alpha \Rightarrow \beta) \wedge \rho \leq \pi_2^\beta \). (iii) For \( \alpha \) in \( P_D(A \times B, \rho \otimes \sigma) \), we have \( \pi_1^\beta \exists_{\pi_1} \omega \wedge \rho = \exists_{\pi_1, \pi_2} \langle \pi_1, \pi_3 \rangle^\alpha \wedge \rho \) by Beck-Chevalley. By Frobenius Reciprocity that is equal to \( \exists_{\pi_1, \pi_2} \langle \pi_1, \pi_3 \rangle^\alpha \wedge \langle \pi_1, \pi_2 \rangle^\rho \leq \exists_{\pi_1, \pi_2} \langle \pi_2, \pi_3 \rangle^\alpha = \pi_2^\exists_{\pi_1} \alpha \); (iv) we have that \( \pi_1^\exists_{\pi_1} (\alpha \wedge \rho \leq \pi_2^\exists_{\pi_1} (\alpha) \) if and only if \( \forall_{\pi_1, \pi_2} (\pi_1, \pi_3) \rangle^\alpha \wedge \rho \leq \forall_{\pi_1, \pi_2} \langle \pi_2, \pi_3 \rangle^\alpha \). Since \( \langle \pi_1, \pi_2 \rangle \Rightarrow \forall_{\pi_1, \pi_2} \) the inequality holds if and only if \( \langle \pi_1, \pi_3 \rangle^\alpha \wedge \langle \pi_1, \pi_2 \rangle^\rho \leq \langle \pi_2, \pi_3 \rangle^\alpha \) which is the descent condition for \( \alpha \). (v) Take an element \( \alpha \) in \( P_D(A, \rho) \), this is also in \( P(A) \), and consider its comprehension \( [\alpha] : X \to A \), this produces a comprehension morphism \( X, ([\alpha] \times \langle \alpha \rangle)^\rho \to (A, \rho) \) in \( \mathbb{Q}_P \).

Fullness directly derives from that in \( P \). In each case (i)-(iv) we shall show that \( P_D(A, \rho) = \text{Des}_\rho \subseteq P(A) \) is closed under the relevant constructions, thus obtaining immediately preservation by \( \varepsilon_P \).

\[\square\]

In Example 4.10 we presented a doctrine that fails to be implicational since, even though every fiber has pseudo relative complements, they do not distribute under reindexing. Moreover the doctrine is not universal: it has right adjoints along all the projections, but these do not satisfied Beck-Chevalley conditions. The next two propositions show that these two properties are gained with the co-free construction. The first is from 4.9 and the second is standard.
Proposition 4.2. If \( P: \mathcal{C}^{\text{op}} \to \text{ISL} \) is an elementary existential doctrine and every fiber has pseudo relative complements, then \( P \) is implicational.

Proof. Suppose \( f: A \to B \) is a morphism in \( \mathcal{C} \), by 1.4 there exists \( \exists f: P(A) \to P(B) \) satisfying Frobenius Reciprocity.

\[
(f^* \alpha \Rightarrow f^* \beta) \land f^* \alpha \leq f^* \beta \\
\exists f(f^* \alpha \Rightarrow f^* \beta) \land f^* \alpha \leq \beta \\
\exists f(f^* \alpha \Rightarrow f^* \beta) \land \alpha \leq \beta \\
f^* \alpha \Rightarrow f^* \beta \leq f^*(\alpha \Rightarrow \beta)
\]

To prove that \( f^*(\alpha \Rightarrow \beta) \leq f^* \alpha \Rightarrow f^* \beta \) it suffices to use the distributivity of reindexing functors on meets. \( \square \)

Proposition 4.3. If \( P: \mathcal{C}^{\text{op}} \to \text{ISL} \) is an existential elementary doctrine with right adjoints \( \forall \pi \) along every projection \( \pi \), then \( P \) is universal.

As a corollary of 4.1 and 4.2, we have that if \( P \) is an existential doctrine and every fiber has pseudo relative complements, then \( P_D \) is implicational. And, as a corollary of 4.1 and 4.3, if \( P \) is an existential doctrine with right adjoints along every projections, then \( P_D \) is universal. In particular the doctrine \( O_D \) is implicational and universal.

Power objects are not preserved, but it holds that

Proposition 4.4. If \( P: \mathcal{C}^{\text{op}} \to \text{ISL} \) is universal and implicational with weak power objects, then \( P_D \) has weak power objects.

Proof. A weak power object of \( (A, \rho) \) in \( Q_P \) is

\[
(\pi A, \forall_{(\pi_2, \pi_3)}((\pi_1, \pi_2)^* \in A \Leftrightarrow (\pi_1, \pi_3)^* \in A))
\]

where the membership predicate \( \in_{(A, \rho)} \) is

\[
\in_A \land \forall_{(\pi_1, \pi_3)}((\pi_1, \pi_2)^* \rho \Rightarrow (\pi_2, \pi_3)^* \in A)
\]

\( \square \)

It is worth to remark that power objects as defined in \( 4.3 \) are still weak, but they gain the property that, in \( Q_P \) if two morphisms \( \{\phi\} \) and \( \{\phi\}' \) classify the same element \( \phi \) in the fiber over \( (A, \rho) \times (B, \sigma) \), then it holds that \( \top_B \leq \langle \{\phi\}, \{\phi\}' \rangle^* \delta_{(A, \rho)} \). This lead to introduce internal extentionality. We said that for an object \( A \) in the base category of an elementary doctrine, \( \delta_A \) provides a notion of internal equality for terms of type \( A \). Certainly external equality implies internal, in the sense that given \( t_1, t_2: X \to A \), if it holds that \( t_1 = t_2 \) (i.e they are the same morphism in \( \mathcal{C} \)) then \( \top_X \leq \langle t_1, t_2 \rangle^* \delta_A \). The converse can be forced considering the category \( [\mathcal{C}] \), whose objects are the same as in \( \mathcal{C} \) and the morphism are equivalence classes of morphism of \( \mathcal{C} \) with respect to the relation: \( [t_1] = [t_2] \) if and only if \( \top_X \leq \langle t_1, t_2 \rangle^* \delta_A \). This construction is given directly in \( [9, 10] \), and named extensional collapse of \( \mathcal{C} \).
If $P$ is an elementary doctrine with power objects in which the base category $C$ has a terminal object $1$, then for every object $A$ in $C$, every element $\phi$ determines (at least) a term of type $\pi_1$, i.e. $\langle \{\phi\} \rangle : A \rightarrow \pi_1$ (see [1,6]). Hence we can use this correspondence to define a notion of internal equality for formulas

$$\phi \leftrightarrow \psi := \langle \{\phi\}, \{\psi\} \rangle^* \delta_{\pi_1}$$

which depends on a choice of the morphisms $\{\phi\}$ and $\{\psi\}$ and satisfies the following rule

$$\gamma \leq \phi \leftrightarrow \psi \quad \gamma \land \phi \leq \psi$$

by the fact that we have $\pi_1^* x \land \delta_{\pi_1} \leq \pi_2^* x$, for every $x$ in $P(\pi_1)$, then reindex both sides along $\langle \{\phi\}, \{\psi\} \rangle$ with $x = \epsilon_1$ to have $\phi \land (\phi \leftrightarrow \psi) \leq \psi$, which we use in the following tree

$$\gamma \land \phi \leq (\phi \leftrightarrow \psi) \land \phi$$

$$\gamma \land \phi \leq \psi$$

The converse of the previous rule does not holds in general. This motivates the following

**Definition 4.5.** Given a primary doctrine $P: C^{op} \rightarrow ISL$ in which the base category has a terminal object $1$, a weak power object $\pi_1$ and an elementary structure $\delta_{\pi_1}$ in the poset over $\pi_1 \times \pi_1$, we say that $P$ has **extentional entailment** if, for every object $A$ in $C$ and every element $\phi, \psi$ and $\gamma$ in $P(A)$ the following rule

$$\gamma \land \psi \leq \phi \quad \gamma \land \phi \leq \psi$$

is satisfied.

As an immediate property we have that in an elementary doctrine $P: C^{op} \rightarrow ISL$ with extentional entailment, for every formulas $\phi$ and $\psi$, it holds that $\phi \leftrightarrow \psi$ if and only if $\top \leq \langle \{\phi\}, \{\psi\} \rangle^* \delta_{\pi_1}$, which means that every classifying morphism is unique in the extentional collapse of $C$.

**Remark 4.6.** There is a connection between extentional entailment and the presence of pseudo relative complements in every fibre of a doctrine. If a doctrine $P$ has extentional entailment, then for every object $A$ in $C$, $P(A)$ has pseudo relative complements: it suffices to define $\alpha \Rightarrow \beta := (\alpha \land \beta) \leftrightarrow \alpha$, in the spirit of logic of toposes (see [1]). If $P$ is elementary, then is also implicational by [4,2].

The converse need not to be true in the sense that, even if an elementary doctrine has pseudo relative complements over each fiber, we have that $\pi_1^* \epsilon_1 \Leftrightarrow \pi_2^* \epsilon_1$, may not be the left adjoint to $\Delta_{\pi_1}^*$, as we see, for instance, in example 4.8.

**Example 4.7.** *(Subobjects)* Let $C$ be a finitely complete small category. The doctrine $Sub: C^{op} \rightarrow ISL$ has extentional entailment if and only if $C$ has a subobjects classifier. Let $\Omega$ be the subobjects classifier of $C$. Then $\pi_1 = \Omega$. $\epsilon_1$ is the true arrow. To prove the converse, suppose $Sub$ to have extentional entailment. Define $\Omega$ to be $\pi_1$. The true arrow is $\epsilon_1: 1 \rightarrow \pi_1$. Every mono $\phi: X \rightarrow A$ is classified by $\{\phi\}$, since reindexing is given by pullbacks. $\{\phi\}$ is unique because of extentionality of entailment, which says that if $\phi = f^* \epsilon_1$ for
some $f$, then $\top \leq (\{\phi\}, f)^* \delta_\Omega$, where $\delta_\Omega$ is $\Delta_\Omega$. Under the same conditions, an immediate corollary is that $\mathcal{C}$ is an elementary topos if and only if $\text{Sub}$ has power objects. Suppose $\text{Sub}$ to have power objects. For each $A$ in $\mathcal{C}$, $\Omega^A$ is $\pi A$ and $\epsilon A$ is $\{\epsilon_A\}$. For every morphism $f: A \times B \rightarrow \Omega$, the transpose $\overline{f}$ is $\{f^* \epsilon_1\}$. Then $\{\epsilon_A\} \circ (id_A \times \{f^* \epsilon_1\}) = f$, since they both classify $f^* \epsilon_1$. The converse is proved in [3], page 336.

Example 4.8. (Tripes) In general a tripos need not have extentional entailment. Take the localic tripos. $1 = \{\ast\}$. $\delta_{\pi^1} : \mathbb{H}^1 \times \mathbb{H}^1 \rightarrow \mathbb{H}$ is given by the following assignment $(f, g) \mapsto \top$ if $f = g$, then $\bot$. But $f(*) \Leftrightarrow g(*)$ is not necessarily $\bot$ if $f \neq g$. Analogously for realizability tripes, see [3] page 331.

In 4.6 and 4.8 we showed that an implicational doctrine need not have an extentional entailment. The following proposition says that this holds once the elementary structure is co-freely added to a doctrine. In other words give a doctrine $P$, the canonical inequality $\delta = \pi^1 \epsilon_1 \Leftrightarrow \pi^2 \epsilon_1$ is an equality in $P_{\mathcal{D}}$.

Proposition 4.9. If $P : \mathcal{C}^{\text{op}} \rightarrow \text{ISL}$ is such that $\mathcal{C}$ has a terminal object 1 with a weak power object $\pi 1$ and pseudo relative complements in $P(\pi 1)$, then $P_{\mathcal{D}} : Q P \rightarrow \text{ISL}$ has extentional entailment.

Proof. Recalling that in $Q P$ the terminal object is $(1, \top \times 1)$, define $\pi(1, \top \times 1) := (\pi 1, \pi^1 \epsilon_1 \Leftrightarrow \pi^2 \epsilon_1)$. $\epsilon_1$ certainly belongs to the category of descent data, since $\pi^1 \epsilon_1 \wedge (\pi^1 \epsilon_1 \Leftrightarrow \pi^2 \epsilon_1) \leq \pi^2 \epsilon_1$. To prove that $P_{\mathcal{D}}$ has extentional entailment it is left to show that $\pi^1 \epsilon_1 \Leftrightarrow \pi^2 \epsilon_1$ is $\delta = \pi(1, \top \times 1)$, which is true by proposition 3.1.

As a final remark note that, under the hypothesis of proposition 4.4 the elementary doctrine $P_{\mathcal{D}}$ is a tripos, since it is possible to define finite joins and existential quantifications on the basis of implicational operations and (higher order) universal quantifications (See [?], [3] and [?]). Then a doctrine that differs from a tripos only by the lack of an elementary structure, thanks to propositions 4.1 and 4.3 comes to be a tripos and this tripos has extentional entailment by 4.9. On the other hand any tripos $P$, which is known to be an interpretation of higher order many-sorted non-extentional predicate logic (see [3] or [13]), generates a new tripos $P_{\mathcal{D}}$ which interpretes higher order many-sorted predicate logic with extentional entailment.

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