FRACTIONAL SKELLAM PROCESS OF ORDER k

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ABSTRACT. We introduce and study a fractional version of the Skellam process of order k by time-changing it with an independent inverse stable subordinator. We call it the fractional Skellam process of order k (FSPoK). An integral representation for its one-dimensional distributions and their governing system of fractional differential equations are obtained. We derive the probability generating function, mean, variance and covariance of the FSPoK which are utilized to establish its long-range dependence property. Later, we considered two time-changed versions of the FSPoK. These are obtained by time-changing the FSPoK by an independent Lévy subordinator and its inverse. Some distributional properties and particular cases are discussed for these time-changed processes.

1. Introduction

Let \( \{N(t)\}_{t \geq 0} \) be a Poisson process with intensity \( k\lambda \) and \( \{X_i\}_{i \geq 1} \) be a sequence of independent and identically distributed (iid) discrete uniform random variables with support \( S = \{1, 2, \ldots, k\} \). Consider the following compound Poisson process:

\[
N^k(t) = \sum_{i=1}^{N(t)} X_i,
\]

where \( \{N(t)\}_{t \geq 0} \) is independent of \( \{X_i\}_{i \geq 1} \). The process \( \{N^k(t)\}_{t \geq 0} \) is known as the Poisson process of order \( k \) (PPoK) (see Kostadinova and Minkova (2013)). For \( k = 1 \), the PPoK reduces to the Poisson process \( \{N(t)\}_{t \geq 0} \) with intensity \( \lambda \). In ruin theory, the PPoK is used to model the number of claims where these arrive in groups of size \( k \).

Recently, Gupta et al. (2020) introduced and studied a Lévy process \( \{S^k(t)\}_{t \geq 0} \) by considering the difference of two PPoK, that is,

\[
S^k(t) = N^k_1(t) - N^k_2(t),
\]

where \( \{N^k_1(t)\}_{t \geq 0} \) and \( \{N^k_2(t)\}_{t \geq 0} \) are independent PPoK with intensities \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), respectively. It is called the Skellam process of order \( k \) (SPoK) whose state probabilities \( p^k(n, t) = \Pr\{S^k(t) = n\}, n \in \mathbb{Z} \) satisfy the following system of differential equations:

\[
\frac{d}{dt}p^k(n, t) = -k(\lambda_1 + \lambda_2)p^k(n, t) + \lambda_1 \sum_{j=1}^{k} p^k(n-j, t) + \lambda_2 \sum_{j=1}^{k} p^k(n+j, t),
\]

with initial conditions \( p^k(0, 0) = 1 \) and \( p^k(n, 0) = 0, n \neq 0 \). For \( k = 1 \), the SPoK reduces to an integer-valued Lévy process, namely, the Skellam process \( \{S(t)\}_{t \geq 0} \) (see Barndorff-Nielsen et al. (2012)). For each \( t \geq 0 \), the random variable \( S(t) \) has the Skellam distribution (see Skellam (1946)). It has several real life applications, for example, it is used as a sensor...
noise model for cameras (see Hwang et al. (2007)) and in modeling the score differences in a game of two competing teams (see Karlis and Ntzoufras (2009)).

In the past two decades, the time-changed point processes attracted the interest of several researchers due to their potential applications in different fields such as finance, hydrology, econometrics, etc. The Poisson process time-changed by a stable subordinator and by the inverse of a stable subordinator leads to the space fractional Poisson process (see Orsingher and Polito (2012)) and the time fractional Poisson process (see Meerschaert et al. (2011)), respectively. For other time-changed version of the Poisson process, we refer the reader to Beghin (2012), Orsingher and Toaldo (2015), Aletti et al. (2018), etc., and the references therein.

A subordinator \( \{D_f(t)\}_{t \geq 0} \) is a one-dimensional Lévy process whose Laplace transform is given by (see Applebaum (2009), Section 1.3.2)

\[
\mathbb{E} \left( e^{-sD_f(t)} \right) = e^{-tf(s)},
\]

where the function

\[
f(s) = bs + \int_0^\infty \left( 1 - e^{-sx} \right) \mu(dx), \quad s > 0,
\]

is called the Bernstein function. Here, \( b \geq 0 \) is the drift coefficient and \( \mu \) is a non-negative Lévy measure that satisfies \( \mu([0, \infty)) = \infty \) and \( \int_0^\infty (x \wedge 1) \mu(dx) < \infty \) where \( s \wedge t = \min\{s, t\} \). The sample paths of a subordinator \( \{D_f(t)\}_{t \geq 0} \) are non-decreasing and \( D_f(0) = 0 \) almost surely (a.s.). The first hitting time of \( \{D_f(t)\}_{t \geq 0} \) is called the inverse subordinator. It is defined as

\[
H_f(t) := \inf\{r \geq 0 : D_f(r) > t\}, \quad t \geq 0.
\]

A driftless subordinator, i.e., \( b = 0 \) with \( f(s) = s^\alpha \), \( 0 < \alpha < 1 \) is known as the stable subordinator. Its first hitting time \( \{Y_\alpha(t)\}_{t \geq 0} \) is called the inverse stable subordinator.

Recently, Gupta et al. (2020) introduced a time-changed SPoK by time-changing it with an independent subordinator. They obtained its probability mass function (pmf), mean, variance and covariance. Here, we introduce and study a time-changed version of the SPoK by time-changing it with an independent inverse stable subordinator. It is defined as

\[
S_\alpha^k(t) = \begin{cases} 
S^k(Y_\alpha(t)), & 0 < \alpha < 1, \\
S^k(t), & \alpha = 1.
\end{cases}
\]

We call the process \( \{S_\alpha^k(t)\}_{t \geq 0} \) as the fractional Skellam process of order \( k \) (FSPoK). For \( k = 1 \), the FSPoK reduces to the fractional Skellam process (FSP), for more details on FSP and its application to high frequency financial data set we refer the reader to Kerss et al. (2014). For \( \alpha = k = 1 \), the FSPoK reduces to the classical Skellam process.

We obtain an integral representation for the one-dimensional distributions of FSPoK. Also, the governing system of fractional differential equations for its state probabilities is obtained. The probability generating function (pgf), mean, variance and covariance of the FSPoK are derived and its long-range dependence (LRD) property is established. Later, we explore its time-changed versions. We consider the FSPoK time-changed by an independent Lévy subordinator and its inverse. We compute the mean, variance and covariance for these time-changed FSPoK. For the first time-changed version we establish the law of iterated logarithm and the LRD property under suitable restrictions on the Lévy subordinator. Some particular cases of these time-changed versions are also considered by taking specific subordinators such as the gamma subordinator, tempered stable subordinator and inverse
Gaussian subordinator. Also, we obtain the associated system of governing differential equations for these particular cases. The results obtained in this paper generalize and complement the results of Kerss et al. (2014) and Gupta et al. (2020).

2. Fractional Skellam process of order $k$

In this section, we introduce and study a stochastic process, namely, the fractional Skellam process of order $k$ (FSPoK) that is obtained by time-changing the SPoK by an independent inverse stable subordinator. Let \( \{Y_\alpha(t)\}_{t \geq 0}, \ 0 < \alpha < 1, \) be an inverse stable subordinator which is independent of SPoK \( \{S^k(t)\}_{t \geq 0} \). The FSPoK \( \{S^k_\alpha(t)\}_{t \geq 0} \) is defined as

\[
S^k_\alpha(t) = \begin{cases} 
S^k(Y_\alpha(t)), & 0 < \alpha < 1, \\
S^k(t), & \alpha = 1.
\end{cases}
\]

(2.1)

For $k = 1$, the process defined in (2.1) reduces to the fractional Skellam process (FSP) which is introduced and studied by Kerss et al. (2014). Moreover, for $\alpha = k = 1$ the FSPoK reduces to the classical Skellam process.

In the following result we derive the system of governing differential equations that is satisfied by the state probabilities $p^k_\alpha(n,t) = \Pr\{S^k_\alpha(t) = n\}$ of FSPoK.

**Proposition 2.1.** The state probabilities $p^k_\alpha(n,t), \ n \in \mathbb{Z}$ of FSPoK solves the following system of fractional differential equations:

\[
\partial^\alpha_t p^k_\alpha(n,t) = -k(\lambda_1 + \lambda_2)p^k_\alpha(n,t) + \lambda_1 \sum_{j=1}^{k} p^k_\alpha(n-j,t) + \lambda_2 \sum_{j=1}^{k} p^k_\alpha(n+j,t),
\]

with the initial conditions $p^k_\alpha(0,0) = 1$ and $p^k_\alpha(n,0) = 0$, $n \neq 0$. Here, $\partial^\alpha_t$ denotes the Caputo fractional derivative defined as (see Kilbas et al. (2006))

\[
\partial^\alpha_t f(t) := \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) \, ds, & 0 < \alpha < 1, \\
f'(t), & \alpha = 1.
\end{cases}
\]

**Proof.** From (2.1), we have

\[
p^k_\alpha(n,t) = \int_0^\infty p^k_\alpha(n,u)h_\alpha(u,t) \, du,
\]

(2.3)

where $h_\alpha(u,t)$ is the pdf of \( \{Y_\alpha(t)\}_{t \geq 0} \). For $\gamma \geq 0$, the Riemann-Liouville (R-L) derivative is defined as (see Kilbas et al. (2006))

\[
D^\gamma_t f(t) := \begin{cases} 
\frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-m}} \, ds, & m-1 < \gamma < m, \\
\frac{d^m}{dt^m} f(t), & \gamma = m,
\end{cases}
\]

(2.4)

where $m$ is a positive integer. Using the following result (see Meerschaert and Straka (2013))

\[
D^\gamma_t h_\alpha(u,t) = -\frac{\partial}{\partial u} h_\alpha(u,t),
\]
we get
\[ D_t^\alpha p_k(n, t) = - \int_0^\infty p^k(n, u) \frac{\partial}{\partial u} h_\alpha(u, t) \, du \]
\[ = p^k(n, 0) h_\alpha(0+, t) + \int_0^\infty h_\alpha(u, t) \frac{d}{du} p^k(n, u) \, du \]
\[ = p^k(n, 0) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + \int_0^\infty h_\alpha(u, t) \frac{d}{du} p^k(n, u) \, du, \tag{2.5} \]
where in the last step we used \( h_\alpha(0+, t) = t^{-\alpha}/\Gamma(1 - \alpha) \) (see Meerschaert and Straka 2013). The following relation holds:
\[ \partial_t^\alpha p_\alpha(n, t) = D_t^\alpha p_\alpha(n, t) - p_\alpha(n, 0) \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}. \tag{2.6} \]

From (2.3), we have \( p_\alpha(n, 0) = p^k(n, 0) \) as \( h_\alpha(u, 0) = \delta_0(u) \). Substituting (1.2) in (2.5) and then using it in (2.6), we get
\[ \partial_t^\alpha p_\alpha(n, t) = \int_0^\infty h_\alpha(u, t) \frac{d}{du} p^k(n, u) \, du \]
\[ = \int_0^\infty \left( -k(\lambda_1 + \lambda_2)p^k(n, u) + \lambda_1 \sum_{j=1}^k p^k(n-j, u) + \lambda_2 \sum_{j=1}^k p^k(n+j, u) \right) h_\alpha(u, t) \, du \]
\[ = -k(\lambda_1 + \lambda_2)p_\alpha(n, t) + \lambda_1 \sum_{j=1}^k p_\alpha(n-j, t) + \lambda_2 \sum_{j=1}^k p_\alpha(n+j, t). \]

This completes the proof. □

On substituting \( k = 1 \) in (2.2), we get the system of governing differential equations for the state probabilities \( p_\alpha(n, t) = \text{Pr}\{S_\alpha(t) = n\}, \ n \in \mathbb{Z} \) of FSP (see Kerss et al. (2014), Eq. (3.5)) as follows:
\[ \partial_t^\alpha p_\alpha(n, t) = -(\lambda_1 + \lambda_2)p_\alpha(n, t) + \lambda_1 p_\alpha(n-1, t) + \lambda_2 p_\alpha(n+1, t), \]
with \( p_\alpha(0, 0) = 1 \) and \( p_\alpha(n, 0) = 0, \ n \neq 0. \)

To state the next result we need two special functions, namely, the Wright function \( M_\alpha(\cdot), \ 0 < \alpha < 1 \) and the modified Bessel function of first kind \( I_n(\cdot) \). These are defined as follows (see Mainardi (2010), Abramowitz and Stegun (1972)):
\[ M_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m! \Gamma(1 - m\alpha - \alpha)} \]
and
\[ I_n(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+n}}{m!(m+n)!} \]

Theorem 2.1. For \( n \in \mathbb{Z}, \) the state probability \( p_\alpha^k(n, t) \) of FSPoK is given by
\[ p_\alpha^k(n, t) = \frac{1}{t^{\alpha}} \left( \frac{\lambda_1}{\lambda_2} \right)^{n/2} \int_0^\infty e^{-ku(\lambda_1+\lambda_2)} I_{|n|} \left( 2uk\sqrt{\lambda_1\lambda_2} \right) M_\alpha \left( \frac{u}{t^{\alpha}} \right) du. \]
Proof. For \( n \in \mathbb{Z} \), the state probability \( p^k(n,t) \) of SPoK is given by (see Gupta et al. (2020), Eq. (38)): \[
p^k(n,t) = e^{-kt(\lambda_1+\lambda_2)} \left( \frac{\lambda_1}{\lambda_2} \right)^{n/2} I_{|n|} \left( 2tk\sqrt{\lambda_1\lambda_2} \right).
\] (2.7)

The following result holds (see Meerschaert et al. (2015), Section 3):
\[
h_\alpha(u,t) = \frac{1}{t^\alpha} M_\alpha \left( \frac{u}{t^\alpha} \right).
\] (2.8)

The result follows on substituting (2.7) and (2.8) in (2.3). □

The three-parameter Mittag-Leffler function is defined as (see Kilbas et al. (2006), p. 45)
\[
E_{\alpha,\beta}^\gamma(x) := \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+k)x^k}{k!\Gamma(k\alpha+\beta)}, \quad x \in \mathbb{R},
\] (2.9)
where \( \alpha > 0, \beta > 0 \) and \( \gamma > 0 \). For \( \gamma = 1, \) it reduces to two-parameter Mittag-Leffler function. For \( \gamma = \beta = 1, \) it further reduces to the Mittag-Leffler function.

In the next result we obtain the pgf of FSPoK.

**Proposition 2.2.** The pgf \( G^k_\alpha(\theta,t) = \mathbb{E} \left( \theta^{S^k_\alpha(t)} \right), 0 < \theta < 1, \) of FSPoK is given by
\[
G^k_\alpha(\theta,t) = E_{\alpha,1} \left( - \left( k(\lambda_1+\lambda_2) - \lambda_1 \sum_{j=1}^{k} \theta^j - \lambda_2 \sum_{j=1}^{k} \theta^{-j} \right) t^\alpha \right).
\]

**Proof.** The pgf of SPoK is given by (see Gupta et al. (2020), Eq. (39))
\[
G^k(\theta,t) = \exp \left( -t \left( k(\lambda_1+\lambda_2) - \lambda_1 \sum_{j=1}^{k} \theta^j - \lambda_2 \sum_{j=1}^{k} \theta^{-j} \right) \right),
\]
Using the above result, we get
\[
G^k_\alpha(\theta,t) = \int_{0}^{\infty} G^k(\theta,u)h_\alpha(u,t) \, du
\]
\[
= \int_{0}^{\infty} \exp \left( -u \left( k(\lambda_1+\lambda_2) - \lambda_1 \sum_{j=1}^{k} \theta^j - \lambda_2 \sum_{j=1}^{k} \theta^{-j} \right) \right) h_\alpha(u,t) \, du
\]
\[
= E_{\alpha,1} \left( - \left( k(\lambda_1+\lambda_2) - \lambda_1 \sum_{j=1}^{k} \theta^j - \lambda_2 \sum_{j=1}^{k} \theta^{-j} \right) t^\alpha \right).
\]

This completes the proof. □

Using the fact that the Mittag-Leffler function is an eigenfunction of the Caputo fractional derivative, it follows that
\[
\partial_t^\alpha G^k_\alpha(\theta,t) = - \left( k(\lambda_1+\lambda_2) - \lambda_1 \sum_{j=1}^{k} \theta^j - \lambda_2 \sum_{j=1}^{k} \theta^{-j} \right) G^k_\alpha(\theta,t), \quad G^k_\alpha(\theta,0) = 1.
\]

**Proposition 2.3.** The one-dimensional distributions of FSPoK \( \{S^k_\alpha(t)\}_{t \geq 0} \) are not infinitely divisible.
Proof. From (1.1) and (2.1), we get

\[ S_\alpha^k(t) = N_1^k(Y_\alpha(t)) - N_2^k(Y_\alpha(t)) \]
\[ = \frac{d}{N_1^k(t^\alpha Y_\alpha(1)) - N_2^k(t^\alpha Y_\alpha(1))}, \]

where \( \frac{d}{\ } \) means equal in distribution. Here, we have used the self-similarity property of \( \{Y_\alpha(t)\}_{t \geq 0} \). The following result holds for PPoK (see Sengar et al. (2020), Eq. (9)):

\[ \frac{N^k(t)}{t} \to \frac{k(k+1)}{2} \lambda, \quad \text{in probability as } t \to \infty. \]  

(2.10)

Thus, \( N^k(t)/t \to k(k+1)\lambda/2 \), in distribution as \( t \to \infty \). Therefore,

\[ \lim_{t \to \infty} \frac{S^k_\alpha(t)}{t^\alpha} = \lim_{t \to \infty} \frac{N^k_1(t^\alpha Y_\alpha(1)) - N^k_2(t^\alpha Y_\alpha(1))}{t^\alpha} \]
\[ = Y_\alpha(1) \left( \lim_{t \to \infty} \frac{N^k_1(t^\alpha Y_\alpha(1))}{t^\alpha Y_\alpha(1)} - \lim_{t \to \infty} \frac{N^k_2(t^\alpha Y_\alpha(1))}{t^\alpha Y_\alpha(1)} \right) \]
\[ = \frac{k(k+1)}{2}(\lambda_1 - \lambda_2)Y_\alpha(1), \quad \text{in distribution.} \]

It is known that \( Y_\alpha(1) \) is not infinitely divisible (see Vellaisamy and Kumar (2018)). Let \( S^k_\alpha(t) \) be infinitely divisible. Then, it follows that \( S^k_\alpha(t)/t^\alpha \) is infinitely divisible (see Steutel and van Harn (2004), Proposition 2.1). As the limit of a sequence of infinitely divisible random variables is infinitely divisible (see Steutel and van Harn (2004), Proposition 2.2), it follows that \( Y_\alpha(1) \) is infinitely divisible. This leads to a contradiction. \( \square \)

Let us assume that

\[ r_1 = k(k+1)(\lambda_1 - \lambda_2)/2 \quad \text{and} \quad r_2 = k(k+1)(2k+1)(\lambda_1 + \lambda_2)/6. \]  

(2.11)

The mean, variance and covariance function of \( \{S^k(t)\}_{t \geq 0} \) are given by (see Gupta et al. (2020))

\[ \mathbb{E}(S^k(t)) = r_1t, \quad \text{Var}(S^k(t)) = r_2t \quad \text{and} \quad \text{Cov}(S^k(s), S^k(t)) = r_2s, \quad 0 < s \leq t. \]  

(2.12)

The mean, variance and covariance of FSPoK are

\[ \mathbb{E}(S^\alpha_\alpha(t)) = r_1 \mathbb{E}(Y_\alpha(t)), \]  

(2.13)

\[ \text{Var}(S^\alpha_\alpha(t)) = r_2 \mathbb{E}(Y_\alpha(t)) + r_1^2 \text{Var}(Y_\alpha(t)), \]  

(2.14)

\[ \text{Cov}(S^\alpha_\alpha(s), S^\alpha_\alpha(t)) = r_2 \mathbb{E}(Y_\alpha(s)) + r_1^2 \text{Cov}(Y_\alpha(s), Y_\alpha(t)), \]  

(2.15)

which are obtained by using Theorem 2.1 of Leonenko et al. (2014).

Remark 2.1. On substituting \( k = 1 \) in (2.13)-(2.15), we get the mean, variance and covariance of FSP (see Kerss et al. (2014), Remark 3.2).

Next, we show that the FSPoK possesses the LRD property. The following definition will be used (see D’Ovidio and Nane (2014), Maheshwari and Vellaisamy (2016)):

Definition 2.1. Let \( s > 0 \) be fixed and \( \{X(t)\}_{t \geq 0} \) be a stochastic process such that

\[ \lim_{t \to \infty} \frac{\text{Cov}(X(s), X(t))}{\sqrt{\text{Var}(X(s))} \sqrt{\text{Var}(X(t))} t^{-\gamma}} = c(s). \]

If \( \gamma \in (0, 1) \) the process \( \{X(t)\}_{t \geq 0} \) has the LRD property, and if \( \gamma \in (1, 2) \) it has the SRD property.
**Theorem 2.2.** The FSPoK has the LRD property.

*Proof.* For a fixed $s > 0$, we have

$$
\lim_{t \to \infty} \frac{\text{Cov} \left( S^k_\alpha(s), S^k_\alpha(t) \right)}{\sqrt{\text{Var} \left( S^k_\alpha(t) \right) \sqrt{\text{Var} \left( S^k_\alpha(t)^{t-\alpha} \right)}}} = c(s),
$$

(2.16)

where

$$
c(s) = \left( \frac{1}{\Gamma(2\alpha)} - \frac{1}{\alpha(\Gamma(\alpha))^2} \right)^{-1} \left( \frac{\alpha r_2}{\Gamma(1+\alpha)r_1^2} + \frac{\alpha s^\alpha}{\Gamma(1+2\alpha)} \right).$$

The result in (2.16) follows by using a result on p. 10 of Leonenko *et al.* (2014), and the mean and variance of SPoK which are given in (2.12). Thus, the FSPoK exhibits the LRD property as $0 < \alpha < 1$. □

**Remark 2.2.** For a fixed $h > 0$, the increment of FSPoK is defined as

$$
Z^k_\alpha(t) = S^k_\alpha(t + h) - S^k_\alpha(t), \quad t \geq 0.
$$

It can be shown that the increment process $\{Z^k_\alpha(t)\}_{t \geq 0}$ exhibits the SRD property. The proof follows similar lines to that of Theorem 1 of Maheshwari and Vellaisamy (2016), and thus it is omitted.

### 3. The FSPoK time-changed by a Lévy subordinator

In this section, we consider a time-changed version of the FSPoK. We call it the time-changed fractional Skellam process of order $k$ (TCFSPoK) and denote it by $\{Z^f_\alpha(t)\}_{t \geq 0}$, $0 < \alpha \leq 1$. It is defined as the FSPoK time-changed by an independent Lévy subordinator $\{D_f(t)\}_{t \geq 0}$ with $\mathbb{E} \left( D^f_j(t) \right) < \infty$ for all $r > 0$. Thus,

$$
Z^f_\alpha(t) := S^k_\alpha(D_f(t)), \quad t \geq 0,
$$

(3.1)

where $\{S^k_\alpha(t)\}_{t \geq 0}$ is independent of $\{D_f(t)\}_{t \geq 0}$.

For $\alpha = 1$, the TCFSPoK reduces to a time-changed version of the SPoK (see Gupta *et al.* (2020)), that is,

$$
Z^f(t) := S^k(D_f(t)), \quad t \geq 0.
$$

Its pmf $p^f(n, t) = \text{Pr}\{Z^f(t) = n\}$ is given by (see Gupta *et al.* (2020), Eq. (47))

$$
p^f(n, t) = \sum_{x=\max(0,-n)}^{\infty} \frac{(k\lambda_1)^n x^k (k\lambda_2)^x}{(n + x)!(n + x)!} \mathbb{E} \left( e^{-k(\lambda_1 + \lambda_2)D_f(t)} D^{2n+x}_f(t) \right), \quad n \in \mathbb{Z}.
$$

The mean and covariance of inverse stable subordinator are given by (see Leonenko *et al.* (2014), Eq. (8) and Eq. (10))

$$
\mathbb{E}(Y_\alpha(t)) = \frac{t^\alpha}{\Gamma(\alpha + 1)}
$$

(3.2)

and

$$
\text{Cov}(Y_\alpha(s), Y_\alpha(t)) = \frac{1}{\Gamma^2(\alpha + 1)} \left( \alpha s^{2\alpha} B(\alpha, \alpha + 1) + F(\alpha; s, t) \right), \quad 0 < s \leq t.
$$

(3.3)

where $F(\alpha; s, t) = \alpha t^{2\alpha} B(\alpha, \alpha + 1; s/t) - (ts)^\alpha$. Here, $B(\alpha, \alpha + 1)$ and $B(\alpha, \alpha + 1; s/t)$ denote the beta function and the incomplete beta function, respectively.

Let $l_1 = r_1/\Gamma(\alpha + 1)$, $l_2 = r_2/\Gamma(\alpha + 1)$ and $d = l_1^2 a B(\alpha, \alpha + 1)$ where $r_1$ and $r_2$ are given by (2.11).
The mean of TCFSPoK is obtained as follows:
\[
\mathbb{E}(Z^{f}_{\alpha}(t)) = \mathbb{E}(\mathbb{E}(S^{k}_{\alpha}(D_{f}(t))|D_{f}(t))) = l_{1}\mathbb{E}(D^{\rho}_{f}(t)). \tag{3.4}
\]

On substituting (3.2) and (3.3) in (2.15), we get
\[
\mathbb{E}(S^{k}_{\alpha}(s)S^{k}_{\alpha}(t)) = l_{2}s^{\alpha} + ds^{2\alpha} + \alpha l_{1}^{2}(t^{2\alpha}B(\alpha, \alpha + 1; s/t)).
\]

Thus,
\[
\mathbb{E}(Z^{f}_{\alpha}(s)Z^{f}_{\alpha}(t)) = \mathbb{E}(\mathbb{E}(S^{k}_{\alpha}(D_{f}(s))S^{k}_{\alpha}(D_{f}(t))|D_{f}(s), D_{f}(t))) = l_{2}\mathbb{E}(D^{\rho}_{f}(s)) + d\mathbb{E}(D^{2\alpha}_{f}(s)) + \alpha l_{1}^{2}\mathbb{E}(D^{2\alpha}_{f}(t)B(\alpha, \alpha + 1; D_{f}(s)/D_{f}(t))).
\]

Hence, the covariance of TCFSPoK is given by
\[
\text{Cov}(Z^{f}_{\alpha}(s), Z^{f}_{\alpha}(t)) = l_{2}\mathbb{E}(D^{\rho}_{f}(s)) + d\mathbb{E}(D^{2\alpha}_{f}(s)) - l_{1}^{2}\mathbb{E}(D^{\alpha}_{f}(s))\mathbb{E}(D^{\rho}_{f}(t)) + \alpha l_{1}^{2}\mathbb{E}(D^{2\alpha}_{f}(t)B(\alpha, \alpha + 1; D_{f}(s)/D_{f}(t))). \tag{3.5}
\]

Also, its variance is obtained by substituting \(s = t\) in (3.5) which is given by
\[
\text{Var}(Z^{f}_{\alpha}(t)) = \mathbb{E}(D^{\rho}_{f}(t)) \left(l_{2} - l_{1}^{2}\mathbb{E}(D^{\alpha}_{f}(t))\right) + 2d\mathbb{E}(D^{2\alpha}_{f}(t)). \tag{3.6}
\]

**Remark 3.1.** On substituting \(k = 1\) and taking \(\lambda_{2} \to 0\) in (3.4)–(3.6), we get the mean, covariance and variance of a time-changed version of the fractional Poisson process (see Maheshwari and Vellaisamy (2019), Theorem 3.2).

Next we show that the TCFSPoK exhibits the LRD property under certain restrictions on the Lévy subordinator. For positive functions \(f\) and \(g\), the notation \(f \sim g\) stands for \(f(t)/g(t) \to 1\) as \(t \to \infty\).

**Theorem 3.1.** Let \(\{D_{f}(t)\}_{t \geq 0}\) be a Lévy subordinator with \(f(\cdot)\) being its associated Bernstein function defined in (1.3) such that \(\mathbb{E}(D_{f}^{\rho}(t)) < \infty\) for all \(r > 0\). The TCFSPoK \(\{Z^{f}_{\alpha}(t)\}_{t \geq 0}\), \(0 < \alpha < 1\) exhibits the LRD property if
\[
\mathbb{E}(D_{f}^{\rho}(t)) \sim k_{i}t^{\rho}, \quad i = 1, 2, \tag{3.7}
\]
for some \(0 < \rho < 1\), and positive constants \(k_{1}\) and \(k_{2}\) such that \(k_{2} \geq k_{1}^{2}\).

**Proof.** Let \(0 < s < t\). In (3.5), we use the following asymptotic result for large \(t\) (see Maheshwari and Vellaisamy (2019), Theorem 3.3)
\[
\alpha\mathbb{E}(D^{2\alpha}_{f}(t)B(\alpha, \alpha + 1; D_{f}(s)/D_{f}(t))) \sim \mathbb{E}(D^{\alpha}_{f}(s))\mathbb{E}(D^{\rho}_{f}(t - s)),
\]





to obtain
\[
\text{Cov}(Z^{f}_{\alpha}(s), Z^{f}_{\alpha}(t)) \sim l_{2}\mathbb{E}(D^{\rho}_{f}(s)) + d\mathbb{E}(D^{2\alpha}_{f}(s)) - l_{1}^{2}\mathbb{E}(D^{\alpha}_{f}(s))\mathbb{E}(D^{\rho}_{f}(t)) + \alpha l_{1}^{2}\mathbb{E}(D^{2\alpha}_{f}(t)B(\alpha, \alpha + 1; D_{f}(s)/D_{f}(t)))
\]
\[
\sim l_{2}\mathbb{E}(D^{\rho}_{f}(s)) + d\mathbb{E}(D^{2\alpha}_{f}(s)) - l_{1}^{2}\mathbb{E}(D^{\alpha}_{f}(s))k_{1}(t^{\rho} - (t - s)^{\rho})
\]
\[
\sim l_{2}\mathbb{E}(D^{\rho}_{f}(s)) + d\mathbb{E}(D^{2\alpha}_{f}(s)) - l_{1}^{2}\mathbb{E}(D^{\alpha}_{f}(s))k_{1}st^{\rho - 1},
\]

where we have used (3.7) in the penultimate step. Again by using (3.7) in (3.6), we get
\[
\text{Var}(Z^{f}_{\alpha}(t)) \sim l_{2}k_{1}t^{\rho} - l_{1}^{2}k_{1}^{2}t^{2\rho} + 2dk_{2}t^{2\rho}
\]
\[
\sim \left(2dk_{2} - k_{1}^{2}l_{1}^{2}\right)t^{2\rho} = \frac{r^{2}}{\alpha} \left(\frac{k_{2}}{\Gamma(2\alpha)} - \frac{k_{1}^{2}}{\alpha\Gamma^{2}(\alpha)}\right)t^{2\rho}.
\]
Thus, for large \( t \), we have
\[
\text{Corr} \left( Z^f_\alpha(t), Z^f_\alpha(s) \right) \sim \frac{l_2 \mathbb{E} \left( D^\gamma_\alpha(s) \right) + d \mathbb{E} \left( D^{2\alpha}_f(s) \right) - l_1^2 \mathbb{E} \left( D^\gamma_\alpha(s) \right) k_1 s \rho t^{\rho - 1}}{\sqrt{\text{Var} \left( Z^f_\alpha(s) \right) \sqrt{2dk_2 - k_1^2 l_1^2} t^{2\rho}}}
\]
\[
\sim c_1(s)t^{-\rho},
\]
where
\[
c_1(s) = \frac{l_2 \mathbb{E} \left( D^\gamma_\alpha(s) \right) + d \mathbb{E} \left( D^{2\alpha}_f(s) \right)}{\sqrt{\text{Var} \left( Z^f_\alpha(s) \right) \left(2dk_2 - k_1^2 l_1^2\right)}}.
\]
Hence, the TCFSPoK exhibits the LRD property as \( 0 < \rho < 1 \).

**Remark 3.2.** In a similar way it can be shown that \( \{Z^f(t)\}_{t \geq 0} \) exhibits the LRD property.

The following result will be used to prove the law of iterated logarithm (LIL) for TCFSPoK (see Bertoin (1996), Theorem 14, p. 92).

**Lemma 3.1.** Let \( \{D_f(t)\}_{t \geq 0} \) be a Lévy subordinator whose associated Bernstein function \( f \) is regularly varying at 0+ with index 0 < \( \gamma < 1 \), i.e., \( \lim_{x \to 0^+} f(\lambda x)/f(x) = \lambda^\gamma \), \( \lambda > 0 \). Also, let
\[
g(t) = \frac{\log \log t}{\phi(t^{-1} \log \log t)}, \quad t > e,
\]
where \( \phi \) is the inverse of \( f \). Then,
\[
\liminf_{t \to \infty} \frac{D_f(t)}{g(t)} = \gamma(1 - \gamma)^{(1-\gamma)/\gamma}, \quad \text{a.s.}
\]

**Theorem 3.2.** Let the Bernstein function \( f(\cdot) \) associated with Lévy subordinator \( \{D_f(t)\}_{t \geq 0} \) be regularly varying at 0+ with index 0 < \( \gamma < 1 \). Then,
\[
\liminf_{t \to \infty} \frac{Z^f_\alpha(t)}{(g(t))^{\alpha}} \overset{d}{=} \frac{k(k + 1)}{2} (\lambda_1 - \lambda_2) Y_\alpha(1) \gamma^\alpha (1 - \gamma)^{\alpha(1-\gamma)/\gamma},
\]
where \( g(t) \) is given in \((3.8)\).
where the last step follows from (3.9). □

**Remark 3.3.** On substituting $k = 1$ and taking $\lambda_2 \to 0$ in (3.10), we get the LIL for a time-changed version of the fractional Poisson process (see Maheshwari and Vellaisamy (2019), Theorem 3.5).

3.1. **Some special cases of the TCFSPoK.** In this subsection, we time-change the FSPoK and the SPoK by three specific Lévy subordinators, namely, the gamma subordinator, the tempered stable subordinator and the inverse Gaussian subordinator. We obtain the governing systems of differential equations for their one-dimensional distributions.

3.1.1. **FSPoK time-changed by gamma subordinator.** Let $\{Z(t)\}_{t \geq 0}$ be a gamma subordinator with the following probability density function (pdf):

$$g(x, t) = \frac{a^bt}{\Gamma(bt)}x^{bt-1}e^{-ax}, \ x > 0,$$

where $a > 0$ and $b > 0$. The Bernstein function $f_1(s)$ associated with $\{Z(t)\}_{t \geq 0}$ is given by $f_1(s) = b \log(1 + s/a)$, $s > 0$ (see Applebaum (2009), p. 55).

The FSPoK time-changed by an independent gamma subordinator is defined as

$$Z_{\alpha}^{f_1}(t) := S_{\alpha}^{k}(Z(t)), \ t \geq 0. \quad (3.11)$$

The following result will be used to obtain the governing system of differential equations for its pmf $p_{\alpha}^{f_1}(n, t) = \Pr\{Z_{\alpha}^{f_1}(t) = n\}, n \in \mathbb{Z}$.

**Lemma 3.2** (Vellaisamy and Maheshwari (2018)). For any $\gamma \geq 1$, the pdf $g(x, t)$ of gamma subordinator solves

$$D_t^\gamma g(x, t) = bD_t^{\gamma-1}(\log(ax) - \psi(bt)) g(x, t), \ x > 0,$$

$$g(x, 0) = 0.$$

Here, $\psi(x) := \Gamma'(x)/\Gamma(x)$ is the digamma function and $D_t^\gamma$ is the R-L fractional derivative defined in (2.4).

**Theorem 3.3.** Let $\gamma \geq 1$ and $\psi(x)$ be the digamma function. Then, the pmf $p_{\alpha}^{f_1}(n, t)$ solves the following equation:

$$D_t^\gamma p_{\alpha}^{f_1}(n, t) = bD_t^{\gamma-1}(\log(a) - \psi(bt)) p_{\alpha}^{f_1}(n, t) + b \int_0^\infty p_{\alpha}^{k}(n, x) \log(x) D_t^{\gamma-1}g(x, t) \, dx.$$

**Proof.** From (3.11), we have

$$p_{\alpha}^{f_1}(n, t) = \int_0^\infty p_{\alpha}^{k}(n, x)g(x, t) \, dx. \quad (3.12)$$

Taking the R-L fractional derivative in (3.12) and using Lemma 3.2 we get

$$D_t^\gamma p_{\alpha}^{f_1}(n, t) = \int_0^\infty p_{\alpha}^{k}(n, x)D_t^\gamma g(x, t) \, dx$$

$$= b \int_0^\infty p_{\alpha}^{k}(n, x)D_t^{\gamma-1}(\log(ax) - \psi(bt)) g(x, t) \, dx$$

$$= bD_t^{\gamma-1}\log(a) \int_0^\infty p_{\alpha}^{k}(n, x)g(x, t) \, dx + b \int_0^\infty p_{\alpha}^{k}(n, x) \log(x) D_t^{\gamma-1}g(x, t) \, dx$$

$$- bD_t^{\gamma-1}\psi(bt) \int_0^\infty p_{\alpha}^{k}(n, x)g(x, t) \, dx.$$
\[ = bD_t^{\alpha - 1}(\log(a) - \psi(bt)) p^{\alpha}_\alpha(n, t) + b \int_0^\infty p^{\alpha}_\alpha(n, x) \log(x) D_t^{\alpha - 1} g(x, t) \, dx. \]

This completes the proof.

Next we discuss two particular cases of the time-changed SPoK (see Gupta et al. (2020)).

3.1.2. SPoK time-changed by tempered stable subordinator. Let \( \{D_{\eta,\nu}(t)\}_{t \geq 0} \) denote the tempered stable subordinator (TSS) with stability index \( 0 < \nu < 1 \) and the tempering parameter \( \eta > 0 \). The Bernstein function \( f_2(s) \) associated with TSS is given by

\[
f_2(s) = (\eta + s)^\nu - \eta^\nu, \quad s > 0. \tag{3.13}\]

The SPoK time-changed by an independent TSS is defined as

\[
Z^{f_2}(t) := S_k(D_{\eta,\nu}(t)), \quad t \geq 0. \tag{3.14}\]

**Proposition 3.1.** The pmf \( p^{f_2}(n, t) = \Pr\{Z^{f_2}(t) = n\}, \ n \in \mathbb{Z}, \) is the solution of the following differential equation:

\[
\left( \eta^\nu - \frac{d}{dt} \right)^{1/\nu} p^{f_2}(n, t) = \eta p^{f_2}(n, t) + k(\lambda_1 + \lambda_2)p^{f_2}(n, t) - \lambda_1 \sum_{j=1}^k p^{f_2}(n-j, t) - \lambda_2 \sum_{j=1}^k p^{f_2}(n+j, t). \]

**Proof.** From (3.14), we have

\[
p^{f_2}(n, t) = \int_0^\infty p^k(n, x) h_{\eta,\nu}(x, t) \, dx.
\]

Here, \( h_{\eta,\nu}(x, t) \) denotes the pdf of TSS. It is known that (See Beghin (2015), Eq. (15))

\[
\frac{\partial}{\partial x} h_{\eta,\nu}(x, t) = -\eta h_{\eta,\nu}(x, t) + \left( \eta^\nu - \frac{\partial}{\partial t} \right)^{1/\nu} h_{\eta,\nu}(x, t),
\]

with initial conditions \( h_{\eta,\nu}(x, 0) = \delta_0(x) \) and \( h_{\eta,\nu}(0, t) = 0 \). Here, \( \delta_0(x) \) is the Dirac delta function. Using the following results: \( \lim_{x \to 0} h_{\eta,\nu}(x, t) = \lim_{x \to \infty} h_{\eta,\nu}(x, t) = 0 \), we get

\[
\left( \eta^\nu - \frac{d}{dt} \right)^{1/\nu} p^{f_2}(n, t) = \int_0^\infty p^k(n, x) \left( \frac{\partial}{\partial x} h_{\eta,\nu}(x, t) + \eta h_{\eta,\nu}(x, t) \right) \, dx
\]

\[= \eta p^{f_2}(n, t) - \int_0^\infty h_{\eta,\nu}(x, t) \frac{d}{dx} p^k(n, x) \, dx
\]

\[= \eta p^{f_2}(n, t) - \int_0^\infty \left( -k(\lambda_1 + \lambda_2)p^k(n, x) + \lambda_1 \sum_{j=1}^k p^k(n-j, x) + \lambda_2 \sum_{j=1}^k p^k(n+j, x) \right) h_{\eta,\nu}(x, t) \, dx, \quad \text{(using (1.2))}
\]

\[= \eta p^{f_2}(n, t) + k(\lambda_1 + \lambda_2)p^{f_2}(n, t)
\]

\[- \lambda_1 \sum_{j=1}^k p^{f_2}(n-j, t) - \lambda_2 \sum_{j=1}^k p^{f_2}(n+j, t).
\]

This completes the proof. \( \square \)
Proposition 3.2. The pmf \( p^{f_2}(n, t) \) solves
\[
\sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{1}{n^{1-k/m}} \frac{d^k}{dt^k} p^{f_2}(n, t) = k(\lambda_1 + \lambda_2) p^{f_2}(n, t) - \lambda_1 \sum_{j=1}^{k} p^{f_2}(n-j, t) - \lambda_2 \sum_{j=1}^{k} p^{f_2}(n+j, t).
\]

3.1.3. SPoK time-changed by inverse Gaussian subordinator. Let \( \{Y(t)\}_{t \geq 0} \) be an inverse Gaussian (IG) subordinator with the following pdf (see Applebaum (2009), Eq. (1.27))
\[
q(x, t) = (2\pi)^{-1/2} \delta t x^{-3/2} e^{\delta^2 t x^{-1} + \gamma^2 x}, \quad x > 0, \quad \delta > 0, \quad \gamma > 0
\]
and the associated Bernstein function
\[
f_3(s) = \delta \left( \sqrt{2s + \gamma^2} - \gamma \right), \quad s > 0.
\]
(3.15)
The SPoK time-changed by an independent IG subordinator is defined as
\[
Z^{f_3}(t) := S^k(Y(t)), \quad t \geq 0.
\]
(3.16)

Proposition 3.2. The pmf \( p^{f_3}(n, t) = \Pr\{Z^{f_3}(t) = n\}, n \in \mathbb{Z}, \) solves the following differential equation:
\[
\left(\frac{d^2}{dt^2} - 2\delta \gamma \frac{d}{dt}\right) p^{f_3}(n, t) = 2\delta^2 \left( k(\lambda_1 + \lambda_2) p^{f_3}(n, t) - \lambda_1 \sum_{j=1}^{k} p^{f_3}(n-j, t) - \lambda_2 \sum_{j=1}^{k} p^{f_3}(n+j, t) \right).
\]

Proof. From (3.16), we have
\[
p^{f_3}(n, t) = \int_{0}^{\infty} p^k(n, x) q(x, t) \, dx.
\]
(3.17)
On taking derivatives, we get
\[
\frac{d}{dt} p^{f_3}(n, t) = \int_{0}^{\infty} p^k(n, x) \frac{\partial}{\partial t} q(x, t) \, dx
\]
and
\[
\frac{d^2}{dt^2} p^{f_3}(n, t) = \int_{0}^{\infty} p^k(n, x) \frac{\partial^2}{\partial t^2} q(x, t) \, dx.
\]
The following results hold for \( q(x, t) \) (see Vellaisamy and Kumar (2018), Eq. (3.3))
\[
\frac{\partial^2}{\partial t^2} q(x, t) - 2\delta \gamma \frac{\partial}{\partial t} q(x, t) = 2\delta^2 \frac{\partial}{\partial x} q(x, t)
\]
(3.18)
and \( \lim_{x \to \infty} q(x, t) = \lim_{x \to 0} q(x, t) = 0. \) From (3.17), we get
\[
\left(\frac{d^2}{dt^2} - 2\delta \gamma \frac{d}{dt}\right) p^{f_3}(n, t) = 2\delta^2 \int_{0}^{\infty} p^k(n, x) \left( \frac{\partial^2}{\partial t^2} - 2\delta \gamma \frac{\partial}{\partial t} \right) q(x, t) \, dx
\]
\[
= 2\delta^2 \int_{0}^{\infty} p^k(n, x) \frac{\partial}{\partial x} q(x, t) \, dx, \quad \text{using (3.18)}
\]
\[
= -2\delta^2 \int_{0}^{\infty} q(x, t) \frac{d}{dx} p^k(n, x) \, dx
\]
\[
= -2\delta^2 \int_{0}^{\infty} \left( -k(\lambda_1 + \lambda_2) p^k(n, x) + \lambda_1 \sum_{j=1}^{k} p^k(n-j, x) \right)
\]
\[
= -2\delta^2 \int_{0}^{\infty} \left( -k(\lambda_1 + \lambda_2) p^k(n, x) + \lambda_1 \sum_{j=1}^{k} p^k(n-j, x) \right)
\]
\[
= -2\delta^2 \int_{0}^{\infty} \left( -k(\lambda_1 + \lambda_2) p^k(n, x) + \lambda_1 \sum_{j=1}^{k} p^k(n-j, x) \right)
\]
The proof follows along the similar lines to that of Theorem 2 of Gupta et al. □
This completes the proof.

4. THE FSPOK TIME-CHANGED BY INVERSE SUBORDINATOR

The first hitting time of subordinator \( \{D_f(t)\}_{t \geq 0} \) is called the inverse subordinator. It is defined as
\[
H_f(t) := \inf\{r \geq 0 : D_f(r) > t\}, \quad t \geq 0.
\]

It is known that \( \mathbb{E}\left(H'_f(t)\right) < \infty \) for all \( r > 0 \) (see Aletti et al. (2018), Section 2.1).

Let
\[
\tilde{Z}^f(\alpha)_t := \mathcal{G}^k(H_f(t)), \quad t \geq 0, \tag{4.1}
\]
be the FSPOK time-changed by an independent inverse subordinator.

For \( \alpha = 1 \), the process defined in (4.1) reduces to a time-changed version of the SPoK, that is,
\[
\tilde{Z}^f(t) := \mathcal{G}^k(H_f(t)), \quad t \geq 0.
\]
The pmf \( \tilde{p}^f(n, t) = \Pr\{\tilde{Z}^f(t) = n\} \) of \( \{\tilde{Z}^f(t)\}_{t \geq 0} \) is given by
\[
\tilde{p}^f(n, t) = \sum_{x=\max(0,-n)}^{\infty} \frac{(k \lambda_1)^{n+x}(k \lambda_2)^x}{(n+x)!x!} \mathbb{E}\left(e^{-k(\lambda_1 + \lambda_2) H_f(t)} H_{f}^{2n+x}(t)\right), \quad n \in \mathbb{Z}.
\]
The proof follows along the similar lines to that of Theorem 2 of Gupta et al. (2020).

The mean, variance and covariance of \( \{\tilde{Z}^f_{\alpha}(t)\}_{t \geq 0} \) are given as follows:

Let \( l_1 = r_1/\Gamma(\alpha + 1) \), \( l_2 = r_2/\Gamma(\alpha + 1) \) and \( d = l_1^2 \alpha B(\alpha, \alpha + 1) \) where \( B(\alpha, \alpha + 1) \) is the beta function, and \( r_1 \) and \( r_2 \) are given by (2.11). Then,
\[
(i) \quad \mathbb{E}\left(\tilde{Z}^f_{\alpha}(t)\right) = l_1 \mathbb{E}\left(H_{f}^{\alpha}(t)\right),
\]
\[
(ii) \quad \text{Var}\left(\tilde{Z}^f_{\alpha}(t)\right) = \mathbb{E}\left(H_{f}^{\alpha}(t)\right) (l_2 - l_1^2 \mathbb{E}\left(H_{f}^{\alpha}(t)\right)) + 2d \mathbb{E}\left(H_{f}^{2\alpha}(t)\right),
\]
\[
(iii) \quad \text{Cov}\left(\tilde{Z}^f_{\alpha}(s), \tilde{Z}^f_{\alpha}(t)\right) = l_2 \mathbb{E}\left(H_{f}^{\alpha}(s)\right) + d \mathbb{E}\left(H_{f}^{2\alpha}(s)\right) - l_1^2 \mathbb{E}\left(H_{f}^{\alpha}(s)\right) \mathbb{E}\left(H_{f}^{\alpha}(t)\right) + \alpha l_1^2 \mathbb{E}\left(H_{f}^{\alpha}(s)B(\alpha, \alpha + 1; H_{f}(s)/H_{f}(t))\right),
\]
where \( 0 < s \leq t \) and \( B(\alpha, \alpha + 1; H_{f}(s)/H_{f}(t)) \) is the incomplete beta function.

The proof of (i)-(iii) follows similar lines to the corresponding results of \( \{\tilde{Z}^f_{\alpha}(t)\}_{t \geq 0} \) given in the previous section.

Next we discuss two particular cases of the time-changed process \( \{\tilde{Z}^f(\alpha)\}_{t \geq 0} \).

4.1. SPoK time-changed by the inverse TSS. Let \( \{\mathcal{L}_{\eta, \nu}(t)\}_{t \geq 0} \) denote the inverse TSS which is defined as the first hitting time of TSS \( \{\mathcal{D}_{\eta, \nu}(t)\}_{t \geq 0}, \ 0 < \nu < 1, \ \eta > 0. \) That is,
\[
\mathcal{L}_{\eta, \nu}(t) = \inf\{s \geq 0 : \mathcal{D}_{\eta, \nu}(s) > t\}, \quad t \geq 0.
\]
The SPoK time-changed by an independent inverse TSS is defined as
\[
\tilde{Z}^{f_2}(t) := \mathcal{G}^k(\mathcal{L}_{\eta, \nu}(t)), \quad t \geq 0, \tag{4.2}
\]
where the associated Bernstein function \( f_2 \) is given in \((3.13)\).

**Proposition 4.1.** For \( n \in \mathbb{Z} \), the pmf \( \tilde{p}^{f_2}(n, t) = \Pr \{ \tilde{Z}^{f_2}(t) = n \} \) solves the following differential equation:

\[
\left( \eta + \frac{d}{dt} \right)^{\nu} \tilde{p}^{f_2}(n, t) = p^{k}(n, x)l_{\eta,\nu}(x, t)\Big|_{x=0} - k(\lambda_1 + \lambda_2)\tilde{p}^{f_2}(n, t) + \lambda_1 \sum_{j=1}^{k} \tilde{p}^{f_2}(n - j, t) + \lambda_2 \sum_{j=1}^{k} \tilde{p}^{f_2}(n + j, t) + \eta^{\nu}\tilde{p}^{f_2}(n, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)p^{k}(n, 0),
\]

where \( E_{1,1-\nu}^{1-\nu}(\cdot) \) is the three-parameter Mittag-Leffler function defined in \((2.9)\).

**Proof.** From \((4.2)\), we get

\[
\tilde{p}^{f_2}(n, t) = \int_{0}^{\infty} p^{k}(n, x)l_{\eta,\nu}(x, t) \, dx, \quad t \geq 0,
\]

\((4.3)\)

where \( p^{k}(n, x) \) and \( l_{\eta,\nu}(x, t) \) are the pmf of SPoK and the pdf of \( L_{\eta,\nu}(t) \), respectively.

The following result will be used (see Kumar et al. (2019), Eq. (25)):

\[
\frac{\partial}{\partial x} l_{\eta,\nu}(x, t) = - \left( \eta + \frac{\partial}{\partial t} \right)^{\nu} l_{\eta,\nu}(x, t) + \eta^{\nu}l_{\eta,\nu}(x, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)\delta_{0}(x),
\]

where \( \delta_{0}(x) = l_{\eta,\nu}(x, 0) \). Also, we have \( \lim_{x \to \infty} l_{\eta,\nu}(x, t) = 0 \) (see Alrawashdeh et al. (2017), Lemma 4.6). Using the above results in \((4.3)\), we get

\[
\left( \eta + \frac{d}{dt} \right)^{\nu} \tilde{p}^{f_2}(n, t) = \int_{0}^{\infty} p^{k}(n, x) \left( - \frac{\partial}{\partial x} l_{\eta,\nu}(x, t) + \eta^{\nu}l_{\eta,\nu}(x, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)\delta_{0}(x) \right) \, dx
\]

\[
= p^{k}(n, x)l_{\eta,\nu}(x, t)\Big|_{x=0} + \int_{0}^{\infty} l_{\eta,\nu}(x, t) \frac{d}{dx} p^{k}(n, x) \, dx + \eta^{\nu}\tilde{p}^{f_2}(n, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)\int_{0}^{\infty} p^{k}(n, x)\delta_{0}(x) \, dx
\]

\[
= p^{k}(n, x)l_{\eta,\nu}(x, t)\Big|_{x=0} + \int_{0}^{\infty} \left( - k(\lambda_1 + \lambda_2)p^{k}(n, x) + \lambda_1 \sum_{j=1}^{k} p^{k}(n - j, x) + \lambda_2 \sum_{j=1}^{k} p^{k}(n + j, x) \right) l_{\eta,\nu}(x, t) \, dx + \eta^{\nu}\tilde{p}^{f_2}(n, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)p^{k}(n, 0)
\]

\[
= p^{k}(n, x)l_{\eta,\nu}(x, t)\Big|_{x=0} - k(\lambda_1 + \lambda_2)\tilde{p}^{f_2}(n, t) + \lambda_1 \sum_{j=1}^{k} \tilde{p}^{f_2}(n - j, t) + \lambda_2 \sum_{j=1}^{k} \tilde{p}^{f_2}(n + j, t) + \eta^{\nu}\tilde{p}^{f_2}(n, t) - t^{-\nu}E_{1,1-\nu}^{1-\nu}(-\eta t)p^{k}(n, 0).
\]

This gives the required result. \(\square\)

**4.2. SPoK time-changed by the first hitting time of IG subordinator.** Consider an IG subordinator whose associated Bernstein function \( f_3 \) is given in \((3.15)\). The first hitting time \( \{ H(t) \}_{t \geq 0} \) of the IG subordinator \( \{ Y(t) \}_{t \geq 0} \) is defined as

\[
H(t) := \inf\{ s \geq 0 : Y(s) > t \}, \quad t \geq 0.
\]
We define the following time-changed process:

\[
\tilde{Z}^{f_3}(t) := S^k(H(t)), \quad t \geq 0,
\]  

(4.4)

where the SPoK \( \{S^k(t)\}_{t \geq 0} \) is independent of \( \{H(t)\}_{t \geq 0} \).

**Proposition 4.2.** For \( n \in \mathbb{Z} \), the pmf \( \tilde{p}^{f_3}(n,t) = \Pr\{\tilde{Z}^{f_3}(t) = n\} \) solves the following differential equation:

\[
\delta \left( \gamma^2 + 2 \frac{d}{dt}\right) \frac{1}{2} \tilde{p}^{f_3}(n,t) = (\delta \gamma - k(\lambda_1 + \lambda_2)) \tilde{p}^{f_3}(n,t) + \lambda_1 \sum_{j=1}^{k} \tilde{p}^{f_3}(n-j,t) + \lambda_2 \sum_{j=1}^{k} \tilde{p}^{f_3}(n+j,t) - \delta \gamma \text{Erf} \left( \frac{\gamma \sqrt{t}}{\sqrt{2}} \right) p^k(n,0),
\]

where \( \text{Erf}(\cdot) \) is the error function.

**Proof.** From (4.4), we have

\[
\tilde{p}^{f_3}(n,t) = \int_0^\infty p^k(n,x)h(x,t) \, dx, \quad t \geq 0,
\]  

(4.5)

where \( h(x,t) \) is the pdf of \( H(t) \).

The following result will be used (see Wyłomańska et al. (2016), Eq. (2.22))

\[
\frac{\partial}{\partial x} h(x,t) = -\delta \left( \gamma^2 + 2 \frac{d}{dt}\right) \frac{1}{2} h(x,t) + \delta \gamma h(x,t) - \frac{\delta \sqrt{2e^{-\gamma^2 t/2}}}{\sqrt{\pi t}} \delta_0(x),
\]

with the initial condition \( h(x,0) = \delta_0(x) \). Using the above result in (4.5), we get

\[
\delta \left( \gamma^2 + 2 \frac{d}{dt}\right) \frac{1}{2} \tilde{p}^{f_3}(n,t) = \int_0^\infty p^k(n,x) \left( -\frac{\partial}{\partial x} h(x,t) + \delta \gamma h(x,t) - \frac{\delta \sqrt{2e^{-\gamma^2 t/2}}}{\sqrt{\pi t}} \delta_0(x) \right) \, dx
\]

\[
= p^k(n,0)h(0,t) + \int_0^\infty h(x,t) \frac{d}{dx} p^k(n,x) \, dx + \delta \gamma \tilde{p}^{f_3}(n,t)
\]

\[
- \frac{\delta \sqrt{2e^{-\gamma^2 t/2}}}{\sqrt{\pi t}} p^k(n,0)
\]

\[
= p^k(n,0)h(0,t) + \int_0^\infty \left( -k(\lambda_1 + \lambda_2)p^k(n,x) + \lambda_1 \sum_{j=1}^{k} p^k(n-j,x) + \lambda_2 \sum_{j=1}^{k} p^k(n+j,x) \right) h(x,t) \, dx + \delta \gamma \tilde{p}^{f_3}(n,t) - \frac{\delta \sqrt{2e^{-\gamma^2 t/2}}}{\sqrt{\pi t}} p^k(n,0)
\]

\[
= p^k(n,0)h(0,t) - k(\lambda_1 + \lambda_2)\tilde{p}^{f_3}(n,t) + \lambda_1 \sum_{j=1}^{k} \tilde{p}^{f_3}(n-j,t) + \lambda_2 \sum_{j=1}^{k} \tilde{p}^{f_3}(n+j,t) + \delta \gamma \tilde{p}^{f_3}(n,t) - \frac{\delta \sqrt{2e^{-\gamma^2 t/2}}}{\sqrt{\pi t}} p^k(n,0)
\]

\[
= (\delta \gamma - k(\lambda_1 + \lambda_2)) \tilde{p}^{f_3}(n,t) + \lambda_1 \sum_{j=1}^{k} \tilde{p}^{f_3}(n-j,t)
\]
\[ \lambda_2 \sum_{j=1}^{k} \bar{p}^f_j(n + j, t) - \delta \gamma \text{Erf} \left( \frac{\gamma \sqrt{t}}{\sqrt{2}} \right) p^k(n, 0). \]

The last step follows by using the following result (see Vellaisamy and Kumar (2018), Proposition 2.2):

\[ \lim_{x \to 0} h(x, t) = h(0, t) = \delta e^{-\gamma^2 t/2} \left( \sqrt{\frac{2}{\pi t}} - \gamma e^{\gamma^2 t/2} \text{Erf} \left( \frac{\gamma \sqrt{t}}{\sqrt{2}} \right) \right). \]

This completes the proof. \(\Box\)

5. Concluding remarks

In this paper, we introduced and studied a fractional version of the SPoK, namely, the FSPoK. It is obtained by time-changing the SPoK with an independent inverse stable subordinator. Its one-dimensional distribution, pgf, mean, variance and covariance are obtained. It is shown that the FSPoK exhibits the LRD property. Also, we considered two time-changed versions of the FSPoK where time-change is done using a Lévy subordinator and its inverse. Some distributional properties and particular cases are discussed for these time-changed processes. Kerss et al. (2014) introduced and studied a fractional version of the Skellam process, namely, the FSP. They showed its applications to high frequency financial data set. As the FSPoK is a generalized version of the FSP, we expect the FSPoK to have potential applications in finance and related fields.

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