No Perfect Cuboid

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Abstract
A rectangular parallelepiped is called a cuboid (standing box). It is
called perfect if its edges, face diagonals and body diagonal all have integer
length. Euler gave an example where only the body diagonal failed to be
an integer (Euler brick). Are there perfect cuboids? We prove that there
is no perfect cuboid.

1 Introduction
Cuboids have been studied extensively. It suffices to look at rational cuboids [1,
2]. Rational cuboids are characterized by seven positive rational numbers (three
different edges, three different face diagonals and the body diagonal). Examples
are known where all but one of the seven quantities are rational. Our approach
uses the concept of a rational leaning box. This is a parallelepiped with two
different rectangular faces and a face that is a parallelogram. Rational leaning
boxes are characterized by nine positive rational numbers (three different edges,
two different face diagonals belonging to the rectangular faces, two different face
diagonals belonging to the face parallelogram and two different body diagonals).
If the parallelogram face becomes a rectangle, then we have a standing box.
Computer aided discoveries have shown the existence of perfect leaning boxes
[3, 4, 5]. We found a two-parameter family of solutions for rational leaning
boxes analytically. The two diagonals of the face parallelogram can never be
equal. Thus there is no standing rational box in this family. Finally we use
an equivalent description of leaning boxes to show that in general there is no
perfect cuboid.
In the appendices we use generic symbols which do not necessarily coincide with
the ones used in the main text.

2 The equations for the perfect leaning box
All the following nine quantities are positive integers.
x,y,z, denote the three different edges. The face rectangle (x,y) has diagonal
a, the face rectangle (x,z) has diagonal b and the face parallelogram (y,z) has
diagonals c₁,c₂. The two different body diagonals are denoted by d₁,d₂.
These quantities satisfy the equations

\[ x^2 + y^2 = a^2 \]  \hspace{1cm} (1)
\[ x^2 + z^2 = b^2 \]  \hspace{1cm} (2)
\[ x^2 + c_1^2 = d_1^2 \]  \hspace{1cm} (3)
\[ x^2 + c_2^2 = d_2^2 \]  \hspace{1cm} (4)
\[ 2y^2 + 2z^2 = c_1^2 + c_2^2 \]  \hspace{1cm} (5)

The last equation represents a perfect parallelogram [7, 8]

3 Parameterization for the rational leaning box

We look for solutions of the equations (1), (2), (3), (4), (5) in rational positive numbers.

We now scale these equations as follows

\[ u_1 = \frac{y}{x}, \quad u_2 = \frac{z}{x}, \quad u_3 = \frac{c_1}{x}, \quad u_4 = \frac{c_2}{x} \]  \hspace{1cm} (6)
\[ v_1 = \frac{a}{x}, \quad v_2 = \frac{b}{x}, \quad v_3 = \frac{d_1}{x}, \quad v_4 = \frac{d_2}{x} \]  \hspace{1cm} (7)

Then \( u_k \) and \( v_k, k=1,2,3,4 \), are positive rational numbers.

The scaled equations are

\[ 1 + u_1^2 = v_1^2 \]  \hspace{1cm} (8)
\[ 1 + u_2^2 = v_2^2 \]  \hspace{1cm} (9)
\[ 1 + u_3^2 = v_3^2 \]  \hspace{1cm} (10)
\[ 1 + u_4^2 = v_4^2 \]  \hspace{1cm} (11)
\[ 2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \]  \hspace{1cm} (12)

The last equation represents a rational parallelogram (Appendix C)

The scaled equations can be parameterized by the four Heron angles \( \psi_k \) and their generators \( s_k, k=1,2,3,4 \) (Appendix A) as follows
\[
\begin{align*}
    u_k &= \cot \psi_k = \frac{1 - s_k^2}{2s_k} = \frac{1}{2} \left( \frac{1}{s_k} - s_k \right) \\
    v_k &= \frac{1}{\sin \psi_k} = \frac{1 + s_k^2}{2s_k} = \frac{1}{2} \left( \frac{1}{s_k} + s_k \right) \\
    v_k - u_k &= s_k, v_k + u_k = \frac{1}{s_k}, 0 < s_k < 1
\end{align*}
\]

## 4 The three parallelograms

Besides the face parallelogram there are two interior parallelograms

\[
\begin{align*}
    I. & \quad 2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \\
    II. & \quad 2u_1^2 + 2v_2^2 = v_3^2 + v_4^2 \\
    III. & \quad 2v_1^2 + 2v_2^2 = v_3^2 + v_4^2
\end{align*}
\]

From Appendix (D.32) we have the representation

\[
\begin{align*}
    I. & \quad 2u_1 = u_3 \omega_+ (\alpha) + u_4 \omega_- (\alpha) \\
    & \quad 2u_2 = -u_3 \omega_- (\alpha) + u_4 \omega_+ (\alpha) \\
    II. & \quad 2u_1 = v_3 \omega_+ (\alpha_1) + v_4 \omega_- (\alpha_1) \\
    & \quad 2v_2 = -v_3 \omega_- (\alpha_1) + v_4 \omega_+ (\alpha_1) \\
    III. & \quad 2v_1 = v_3 \omega_+ (\alpha_2) + v_4 \omega_- (\alpha_2) \\
    & \quad 2u_2 = -v_3 \omega_- (\alpha_2) + v_4 \omega_+ (\alpha_2)
\end{align*}
\]

with \( m \) as the generator of \( \alpha \)

\[
m = \frac{2u_2 + u_3 - u_4}{2u_1 + u_3 + u_4}
\]

and with \( m_1 \) as the generator of \( \alpha_1 \)

\[
m_1 = \frac{2v_2 + v_3 - v_4}{2u_1 + v_3 + v_4}
\]

and with \( m_2 \) as the generator of \( \alpha_2 \)

\[
m_2 = \frac{2v_2 + v_3 - v_4}{2v_1 + v_3 + v_4}
\]
\(\alpha, \alpha_1, \alpha_2\) are Heron angles in the first quadrant.

In terms of generators \(s_3, s_4\), with

\[
Q = s_3s_4
\]  

we get from Appendix E the representation

I.  
\[4Qu_1 = s_4M(\alpha) + s_3H(\alpha)\]  
\[4Qu_2 = -s_4K(\alpha) + s_3N(\alpha)\]  

II.  
\[4Qu_1 = s_4N(\alpha_1) + s_3K(\alpha_1)\]  
\[4Qu_2 = -s_4H(\alpha_1) + s_3M(\alpha_1)\]  

III.  
\[4Qu_1 = s_4N(\alpha_2) + s_3K(\alpha_2)\]  
\[4Qu_2 = -s_4H(\alpha_2) + s_3M(\alpha_2)\]  

Comparing Equations (29-32) and using (15) we find

\[
0 = s_4[M(\alpha) - N(\alpha_1)] + s_3[H(\alpha) - K(\alpha_1)]
\]  

\[
8Qu_1 = s_4[M(\alpha) + N(\alpha_1)] + s_3[H(\alpha) + K(\alpha_1)]
\]  

\[
4Qu_2 = s_4[K(\alpha) - H(\alpha_1)] - s_3[N(\alpha) - M(\alpha_1)]
\]  

\[
4Q_{s_2} = -s_4[K(\alpha) + H(\alpha_1)] + s_3[N(\alpha) + M(\alpha)]
\]

With

\[
\alpha + \alpha_1 = 2\sigma_1, \quad \alpha - \alpha_1 = 2\delta_1
\]

\[
\omega_+ (\sigma_1) = \sqrt{2} \cos \psi, \quad \omega_- (\sigma_1) = \sqrt{2} \sin \psi
\]

\[
\sqrt{2} \cos \delta_1 = \omega_+ (\alpha + \psi), \quad \sqrt{2} \sin \delta_1 = -\omega_- (\alpha + \psi)
\]

and Appendix E (Lemma 7) Equations (35-38) become
\begin{align*}
0 &= s_3 \cos \psi H(\alpha + \psi) - s_4 \sin \psi K(\alpha + \psi) \quad (39) \\
4Q u_1 &= s_4 \cos \psi M(\alpha + \psi) + s_3 \sin \psi N(\alpha + \psi) \quad (40) \\
2Q s_2 &= s_4 \cos \psi K(\alpha + \psi) + s_3 \sin \psi H(\alpha + \psi) \quad (41) \\
2Q s_2 &= -s_4 \sin \psi M(\alpha + \psi) + s_3 \cos \psi N(\alpha + \psi) \quad (42)
\end{align*}

or in matrix form

\begin{align*}
\begin{pmatrix}
s_4 \sin \psi & -s_3 \cos \psi \\
s_4 \cos \psi & s_3 \sin \psi
\end{pmatrix}
\begin{pmatrix}
K(\alpha + \psi) \\
H(\alpha + \psi)
\end{pmatrix}
= 2Q
\begin{pmatrix}
0 \\
s_2
\end{pmatrix} \quad (43)
\end{align*}

\begin{align*}
\begin{pmatrix}
s_4 \cos \psi & s_3 \sin \psi \\
-s_4 \sin \psi & s_3 \cos \psi
\end{pmatrix}
\begin{pmatrix}
M(\alpha + \psi) \\
N(\alpha + \psi)
\end{pmatrix}
= 2Q
\begin{pmatrix}
\frac{2u_1}{s_2} \\
s_2
\end{pmatrix} \quad (44)
\end{align*}

The inverse equations are

\begin{align*}
\begin{pmatrix}
K(\alpha + \psi) \\
H(\alpha + \psi)
\end{pmatrix}
= \begin{pmatrix}
s_3 \sin \psi & s_3 \cos \psi \\
-s_4 \cos \psi & s_4 \sin \psi
\end{pmatrix}
\begin{pmatrix}
0 \\
2s_2
\end{pmatrix} \quad (45)
\end{align*}

\begin{align*}
\begin{pmatrix}
M(\alpha + \psi) \\
N(\alpha + \psi)
\end{pmatrix}
= \begin{pmatrix}
s_3 \cos \psi & -s_3 \sin \psi \\
\frac{2u_1}{s_2} & s_4 \cos \psi
\end{pmatrix}
\begin{pmatrix}
4u_1 \\
\frac{2}{s_2}
\end{pmatrix} \quad (46)
\end{align*}

Explicitly

\begin{align*}
K(\alpha + \psi) &= 2s_2 s_3 \cos \psi \quad (47) \\
H(\alpha + \psi) &= 2s_2 s_4 \sin \psi \quad (48) \\
M(\alpha + \psi) &= 4u_1 s_3 \cos \psi - 2 \frac{s_3}{s_2} s_3 \sin \psi \quad (49) \\
N(\alpha + \psi) &= 4u_1 s_4 \sin \psi + 2 \frac{s_4}{s_2} \cos \psi \quad (50)
\end{align*}

Comparing equations 29,30,33,34 and using (15) we find
\[0 = s_3[N(\alpha) - M(\alpha_2)] - s_4[K(\alpha) - H(\alpha_2)] \]  
\[8Qu_2 = s_3[N(\alpha) + M(\alpha_2)] - s_4[K(\alpha) + H(\alpha_2)] \]  
\[4Qs_1 = s_4[N(\alpha_2) - M(\alpha)] + s_3[K(\alpha_2) - H(\alpha)] \]  
\[4Q \frac{1}{s_1} = s_4[N(\alpha_2) + M(\alpha)] + s_3[K(\alpha_2) + H(\alpha)] \]

With

\[\alpha + \alpha_2 = 2\sigma_2, \quad \alpha - \alpha_2 = 2\delta_2\]

\[\omega_+(\sigma_2) = \sqrt{2} \cos\phi, \quad \omega_-(\sigma_2) = \sqrt{2} \sin\phi\]

\[\sqrt{2} \cos\delta_2 = \omega_+(\alpha + \phi), \quad \sqrt{2} \sin\delta_2 = -\omega_-(\alpha + \phi)\]

and Appendix E (Lemma 7), equations (51-54) become

\[0 = -s_4 \cos\phi K(\alpha + \phi) - s_3 \sin\phi H(\alpha + \phi) \]  
\[4Qu_2 = -s_4 \sin\phi M(\alpha + \phi) + s_3 \cos\phi N(\alpha + \phi) \]  
\[2Qs_1 = s_4 \sin\phi K(\alpha + \phi) - s_3 \cos\phi H(\alpha + \phi) \]  
\[2Q \frac{1}{s_1} = s_4 \cos\phi M(\alpha + \phi) + s_3 \sin\phi N(\alpha + \phi) \]

or in matrix form

\[
\begin{pmatrix}
-s_4 \cos\phi & -s_3 \sin\phi \\
 s_4 \sin\phi & -s_3 \cos\phi
\end{pmatrix}
\begin{pmatrix}
K(\alpha + \phi) \\
H(\alpha + \phi)
\end{pmatrix}
=
\begin{pmatrix}
0 \\
2Qs_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\sin\phi & s_3 \cos\phi \\
 s_4 \cos\phi & s_3 \sin\phi
\end{pmatrix}
\begin{pmatrix}
M(\alpha + \phi) \\
N(\alpha + \phi)
\end{pmatrix}
=
\begin{pmatrix}
4Qu_2 \\
2Q \frac{1}{s_1}
\end{pmatrix}
\]

The inverse equations are
\[
\begin{pmatrix}
K(\alpha + \phi) \\
H(\alpha + \phi)
\end{pmatrix} = 
\begin{pmatrix}
-s_3 \cos \phi & s_3 \sin \phi \\
-s_4 \sin \phi & -s_4 \cos \phi
\end{pmatrix}
\begin{pmatrix}
0 \\
2s_1
\end{pmatrix} \quad (61)
\]
\[
\begin{pmatrix}
M(\alpha + \phi) \\
N(\alpha + \phi)
\end{pmatrix} = 
\begin{pmatrix}
-s_3 \sin \phi & s_3 \cos \phi \\
s_4 \cos \phi & s_4 \sin \phi
\end{pmatrix}
\begin{pmatrix}
4u_2 \\
\frac{1}{2} s_1
\end{pmatrix} \quad (62)
\]

Explicitly

\[
K(\alpha + \phi) = 2s_1 s_3 \sin \phi \quad (63)
\]
\[
H(\alpha + \phi) = -2s_1 s_4 \cos \phi \quad (64)
\]
\[
M(\alpha + \phi) = -4u_2 s_3 \sin \phi + 2 \frac{s_3}{s_1} \cos \phi \quad (65)
\]
\[
N(\alpha + \phi) = 4u_2 s_4 \cos \phi + 2 \frac{s_4}{s_1} \sin \phi \quad (66)
\]

Finally, comparing equations (31-34), we get from equations (37,38)

\[
4Qu_2 = -s_4 K(\alpha) + s_3 N(\alpha) \quad (67)
\]
\[
4Qv_2 = -s_4 H(\alpha) + s_3 M(\alpha) \quad (68)
\]

and from equations (53,54)

\[
4Qu_1 = s_4 M(\alpha) + s_3 H(\alpha) \quad (69)
\]
\[
4Qv_1 = s_4 N(\alpha) + s_3 K(\alpha) \quad (70)
\]

Thus we are left with the equations

\[
4Qu_2 = -s_4 K(\alpha) + s_3 N(\alpha)
\]
\[
4Qu_1 = s_4 M(\alpha) + s_3 H(\alpha)
\]
\[
4Qv_2 = -s_4 H(\alpha) + s_3 M(\alpha)
\]
\[
4Qv_1 = s_4 N(\alpha) + s_3 K(\alpha)
\]
or

\[-s_4 K(\alpha) + s_3 N(\alpha) = -s_4 H(\alpha_2) + s_3 M(\alpha_2)\]
\[s_4 M(\alpha) + s_3 H(\alpha) = s_4 N(\alpha_1) + s_3 K(\alpha_1)\]

But these are precisely equations (51,35). Thus there are only the equations (47-50) and (63-66).

Remark 1. Equation 50 follows from the equations (47-49) and the identity (E.22)

Proof.

\[N(\alpha + \psi) = \frac{4Q + H(\alpha + \psi)M(\alpha + \psi)}{K(\alpha + \psi)}\]
\[= \frac{4Q + 4Q[2u_1 s_2 \sin \psi \cos \psi - \sin^2 \psi]}{2s_2 s_3 \cos \psi}\]
\[= \frac{4Q \cos \psi[2u_1 s_2 \sin \psi + \cos \psi]}{2s_2 s_3 \cos \psi}\]
\[= 2u_1 s_4 \sin \psi + 2 \frac{s_4}{s_2} \cos \psi\]

Remark 2. Equation 66 follows from equations (63-65) and the identity (E.22)

Proof.

\[N(\alpha + \phi) = \frac{4Q + H(\alpha + \phi)M(\alpha + \phi)}{K(\alpha + \phi)}\]
\[= \frac{4Q - 2s_1 s_4 \cos \phi[-4u_2 s_3 \sin \phi + 2 \frac{s_4}{s_1} \cos \phi]}{2s_1 s_3 \sin \phi}\]
\[= 4u_2 s_4 \cos \phi + 2 \frac{s_4}{s_1} \sin \phi\]
Remark 3. Equations (47,48,63,64) result in the relation

\[ s_1 s_2 = \tan (\phi - \psi) \]  

(71)

Proof.

\[ K(\alpha + \psi)H(\alpha + \phi) - K(\alpha + \phi)H(\alpha + \psi) = -2 s_3 s_2 \sin \psi + 2 s_1 s_3 \sin \phi + 2 s_2 s_4 \sin \phi \]

\[ = -4 Q s_1 s_2 \cos (\phi - \psi) \]

From the identity (E.24) we get

\[ K(\alpha + \psi)H(\alpha + \phi) - K(\alpha + \phi)H(\alpha + \psi) = 4 Q \sin (\psi - \phi) \]

and then

\[ s_1 s_2 \cos (\phi - \psi) = \sin (\phi - \psi) \]

Remark 4. Equations (47,63) together with equation 71 result in equation 65

Proof.

let \( \phi - \psi = \delta, \psi = \phi - \delta \)

Then

\[ K(\alpha + \psi) = K(\alpha + \phi - \delta) = K(\alpha + \phi) \cos \delta + M(\alpha + \phi) \sin \delta \]

and

\[ 2 s_3 s_4 \cos (\phi - \delta) = \cos \delta K(\alpha + \phi) \sin \delta \]

\[ 2 s_4 s_2 [\cos \phi + \sin \phi \tan \delta] = 2 s_1 s_3 \sin \phi + \tan \delta M(\alpha + \phi) \]

\[ 2 s_2 s_3 \cos \phi + 2 s_1 s_3 \sin \phi \delta = 2 s_1 s_3 \sin \phi + s_1 s_2 M(\alpha + \phi) \]

\[ 2 s_2 s_3 \cos \phi - s_1 s_2 M(\alpha + \phi) = 2 s_1 s_3 \sin \phi [1 - s_2^2] \]

\[ 2 \frac{s_3}{s_1} \cos \phi - M(\alpha + \phi) = 4 s_3 \sin \phi u_2 \]
The equations for the rational leaning box

The unknowns are

\[ s_1, s_2, s_3, s_4; \quad 0 < s_k < 1 \]  \hspace{1cm} (72)
\[ u_k = \frac{1 - s_k^2}{2s_k}, \quad Q = s_3s_4 \]  \hspace{1cm} (73)

The parameters are \( \alpha \) Heron angle with generator \( m \), \( \psi \) Euler angle and \( u_1 \) \hspace{1cm} (74)

Using Appendix E the equations (47-50) result in the relations

\[ K(\alpha) = 2s_3[s_2 \cos^2 \psi + \sin \psi\{2u_1 \cos \psi - \frac{1}{s_2} \sin \psi\}] \]  \hspace{1cm} (75)
\[ N(\alpha) = 2s_4[-s_2 \sin^2 \psi + \cos \psi\{2u_1 \sin \psi + \frac{1}{s_2} \cos \psi\}] \]  \hspace{1cm} (76)
\[ H(\alpha) = 2s_4[s_2 \sin \psi \cos \psi + \sin \psi\{2u_1 \sin \psi + \frac{1}{s_2} \cos \psi\}] \]  \hspace{1cm} (77)
\[ M(\alpha) = 2s_3[-s_2 \sin \psi \cos \psi + \cos \psi\{2u_1 \cos \psi - \frac{1}{s_2} \sin \psi\}] \]  \hspace{1cm} (78)

These relations reproduce equations (29-30)

\[ s_3H(\alpha) + s_4M(\alpha) = 4Qu_1 \]  \hspace{1cm} (79)
\[ s_3N(\alpha) - s_4K(\alpha) = 4Qu_2 \]  \hspace{1cm} (80)

With

\[ \lambda = \tan \psi \]  \hspace{1cm} (81)
the independent equations for the rational leaning box become

$$\omega_-(\alpha) - \lambda \omega_+(\alpha) = s_2 s_3 + \lambda s_2 s_4$$
$$Q[\omega_+(\alpha) + \lambda \omega_-(\alpha)] = s_2 s_3 - \lambda s_2 s_4$$
$$2u_1[\omega_-(\alpha) - \lambda \omega_+(\alpha)] + s_4 - \lambda s_3 = s_2[\omega_+(\alpha) + \lambda \omega_-(\alpha)]$$

From equations (82-84) with

$$\lambda = 0$$

the equations for the rational leaning box thus become

$$s_2 s_3 = \omega_-(\alpha)$$
$$s_2 s_3 = Q\omega_+(\alpha)$$
$$s_2 \omega_+(\alpha) = 2u_1 \omega_-(\alpha) + s_4$$

**Theorem 1.** For \( \lambda = 0 \) the solutions of the equations for the rational leaning box are given by the two rational parameters \( s_1, m \), where

$$0 < s_1 < 1, \quad u_1 = \frac{1 - s_1^2}{2s_1},$$

and \( \alpha \) has the generator \( m \),

$$0 < \alpha < \frac{\pi}{4}, \quad 0 < m < \sqrt{2} - 1$$

as follows
\[ s_2 = 2u_1 \cot (2\alpha) \quad (91) \]
\[ s_3 = \frac{\omega_-(\alpha)}{s_2} \quad (92) \]
\[ s_4 = \frac{s_2}{\omega_+(\alpha)} \quad (93) \]

We also have to respect the inequality (D.13)

**Proof.** Equation 92 follows from equation 86. Equation 87 reads

\[ s_2 = s_4 \omega_+(\alpha) \]

and gives equation 93.

From equation 88 we get

\[ s_2 \omega_+^2(\alpha) = 2u_1 \omega_-(\alpha) \omega_+(\alpha) + s_4 \omega_+(\alpha) \]

or

\[ s_2 [\omega_+^2(\alpha) - 1] = 2u_1 \omega_-(\alpha) \omega_+(\alpha) \]

resulting in

\[ s_2 \sin(2\alpha) = 2u_1 \cos(2\alpha) \]

Equation 90 assures that \( \omega_-(\alpha) > 0 \)

\( \square \)

**Theorem 2.** For \( \lambda = 0 \) the cuboid limit \( s_4 = s_3 \) is impossible. Thus there is no perfect cuboid in this family.
Proof. From equation (92,93) we find

\[ s_2^2s_3 = s_4 \cos(2\alpha) \]  

(94)

Since \(2\alpha\) is a Heron angle, \(\cos(2\alpha)\) can not be the square of a rational number, (Appendix A, Lemma 1)

Remark 5. For \(\psi = 0\), we get from page 4 that

\[ \omega_+(\sigma_1) = \sqrt{2}, \quad \omega_-(\sigma_1) = 0 \]

or

\[ \cos \sigma_1 = \frac{1}{\sqrt{2}}, \quad \sin \sigma_1 = \frac{1}{\sqrt{2}} \]

meaning that \(\sigma_1 = \frac{\pi}{4}\)

and thus \(\alpha + \alpha_1 = 2\sigma_1 = \frac{\pi}{2}\)

\[ \square \]
Example 1.

\[ s_1 = \frac{1}{2}, \quad m = \frac{1}{3} \]

\[ \cos \alpha = \frac{4}{5}, \quad \sin \alpha = \frac{3}{5}, \quad \cos (2\alpha) = \frac{7}{25}, \quad \sin (2\alpha) = \frac{24}{25}, \quad \omega_+(\alpha) = \frac{7}{5}, \quad \omega_-(\alpha) = \frac{1}{5} \]

Then

\[ s_2 = \frac{7}{16}, \quad s_3 = \frac{16}{35}, \quad s_4 = \frac{5}{16} \]

and

\[ u_1 = \frac{840}{1120}, \quad u_2 = \frac{1035}{1120}, \quad u_3 = \frac{969}{1120}, \quad u_4 = \frac{1617}{1120} \]
\[ v_1 = \frac{1400}{1120}, \quad v_2 = \frac{1525}{1120}, \quad v_3 = \frac{1481}{1120}, \quad v_4 = \frac{1967}{1120} \]

The leaning box is then given by

\[ x = 1120, \quad y = 840, \quad z = 1035 \]
\[ a = 1400, \quad b = 1525 \]
\[ c_1 = 969, \quad c_2 = 1617 \]
\[ d_1 = 1481, \quad d_2 = 1967 \]
Example 2.

\[ s_1 = \frac{12}{25}, \quad m = \frac{1}{3} \]

\[ \cos (2\alpha) = \frac{7}{25}, \quad \sin (2\alpha) = \frac{24}{25}, \quad \omega_+ (\alpha) = \frac{7}{5}, \quad \omega_- (\alpha) = \frac{1}{5} \]

Then

\[ s_2 = \frac{3367}{7200}, \quad s_3 = \frac{1440}{3367}, \quad s_4 = \frac{2405}{7200} \]

and

\[ u_1 = \frac{38868648}{48484800}, \quad u_2 = \frac{40503311}{48484800}, \quad u_3 = \frac{46315445}{48484800}, \quad u_4 = \frac{64478365}{48484800} \]
\[ v_1 = \frac{62141352}{48484800}, \quad v_2 = \frac{63176689}{48484800}, \quad v_3 = \frac{67051445}{48484800}, \quad v_4 = \frac{80673635}{48484800} \]

The leaning box is then given by

\[ x = 48484800, \quad y = 38868648, \quad z = 40503311 \]
\[ a = 62141352, \quad b = 63176689 \]
\[ c_1 = 46315445, \quad c_2 = 64478365 \]
\[ d_1 = 67051445, \quad d_2 = 80673635 \]
6 Symmetry and the general equations

We have solved the general equations (82-84) for

\[ \lambda = \tan \psi = 0 \]

For a given \( s_1 \), we found \( s_2, s_3, s_4 \) in terms of \( u_1 \) and \( \alpha \) (91-93). The two parallelograms I, II in (16-17) are described by the two Heron angles \( \alpha \) and \( \alpha_1 \) and according to the observation in Appendix D are also described by the two Heron angles \( \beta \) and \( \beta_1 \). Their generators are given by

\[
m(\alpha) = \frac{2u_2 + u_3 - u_4}{2u_1 + u_3 + u_4}, \quad m(\alpha_1) = \frac{2v_2 + v_3 - v_4}{2u_1 + v_3 + v_4}
\]

(95)

\[
m(\beta) = \frac{2u_2 - u_3 + u_4}{2u_1 + u_3 + u_4}, \quad m(\beta_1) = \frac{2v_2 - v_3 + v_4}{2u_1 + v_3 + v_4}
\]

(96)

Now the equation (16) is invariant under the interchange of \( u_3 \) and \( u_4 \), and the equation (17) is invariant under the interchange of \( v_3 \) and \( v_4 \). According to (13-14) this means that the equations (16-17) are invariant under the interchange of \( s_3 \) with \( s_4 \).

From (p.4) and (81) we find

\[ \alpha + \alpha_1 = 2\sigma_1 \]

\[ \omega_+(\sigma_1) = \sqrt{2} \cos \psi, \quad \omega_-(\sigma_1) = \sqrt{2} \sin \psi, \quad \lambda = \tan \psi = \frac{\omega_-(\sigma_1)}{\omega_+(\sigma_1)} \]

let \( k \) be the generator of \( \alpha + \alpha_1 \)

\[ k = \frac{m(\alpha) + m(\alpha_1)}{1 - m(\alpha)m(\alpha_1)} \]

(97)
Then
\[
\sin(2\sigma_1) = \frac{2k}{1+k^2}, \quad \cos(2\sigma_1) = \frac{1-k^2}{1+k^2}
\]
and
\[
\lambda = \tan\psi = \frac{\omega_-(\sigma_1)}{\omega_+(\sigma_1)} = \frac{\omega^2_-(\sigma_1)}{\omega_+(\sigma_1)\omega_-(\sigma_1)} = \frac{1 - \sin(2\sigma_1)}{\cos(2\sigma_1)}
\]
resulting in
\[
\lambda = \frac{1-k}{1+k}
\]

Let $\tilde{k}$ be the generator of $\beta + \beta_1$

\[
\tilde{k} = \frac{m(\beta) + m(\beta_1)}{1 - m(\beta)m(\beta_1)}
\]
and thus
\[
\tilde{\lambda} = \frac{1-\tilde{k}}{1+k}
\]

Therefore the parameters $u_1, \beta, \tilde{\lambda}$ also satisfy the general equations, however with the interchange of $s_3$ with $s_4$. 

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7 An equivalent set of equations for the rational leaning box

For the three parallelograms we use

\[ \begin{align*}
I. & \quad 2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \\
II. & \quad 2u_1^2 + 2v_2^2 = v_3^2 + v_4^2 \\
III. & \quad v_2^2 - u_2^2 = 1, v_3^2 - u_3^2 = 1, v_4^2 - u_4^2 = 1
\end{align*} \]

where \( u_k = \cot \psi_k, \quad v_k = \frac{1}{\sin \psi_k}, \) \( \psi_k \) Heron angles with generators \( s_k \in (0, 1), k=1,2,3,4. \)

We give \( s_1 \) resulting in giving \( u_1 \) and \( v_1. \)

According to Appendix D we have the following parameterization

I. parameters \( u_1, \alpha, \beta; \alpha, \beta \in (0, \frac{\pi}{2}) \) (103)
\[ \alpha + \beta = 2\sigma, \alpha - \beta = 2\delta; \alpha = \sigma + \delta, \beta = \sigma - \delta \] (104)

II. parameters \( u_1, \alpha_1, \beta_1; \alpha_1, \beta_1 \in (0, \frac{\pi}{2}) \) (105)
\[ \alpha_1 + \beta_1 = 2\sigma_1, \alpha_1 - \beta_1 = 2\delta_1; \alpha_1 = \sigma_1 + \delta_1, \beta_1 = \sigma_1 - \delta_1 \] (106)

and thus the representation

\[ \begin{align*}
2u_2 &= u_1 \tan \sigma, & 2u_3 &= u_1 \frac{\omega_+ (\delta)}{\cos \sigma}, & 2u_4 &= u_1 \frac{\omega_- (\delta)}{\cos \sigma} \\
2v_2 &= u_1 \tan \sigma_1, & 2v_3 &= u_1 \frac{\omega_+ (\delta_1)}{\cos \sigma_1}, & 2v_4 &= u_1 \frac{\omega_- (\delta_1)}{\cos \sigma_1}
\end{align*} \] (107)

Conversely

\[ \begin{align*}
\tan \sigma &= \frac{u_2}{u_1}, & \tan \delta &= \frac{u_3 - u_4}{u_3 + u_4} \\
\tan \sigma_1 &= \frac{v_2}{u_1}, & \tan \delta_1 &= \frac{v_3 - v_4}{v_3 + v_4}
\end{align*} \] (109)

Observe that \( \psi_1 \) is a Heron angle and \( \sigma, \delta, \sigma_1, \delta_1 \) are Euler angles. We rename \( \psi_1 = \psi \)

We now have the conditions
\[ u_1^2 \left[ \tan^2 \sigma_1 - \tan^2 \sigma \right] = u_1^2 \left[ \frac{1}{\cos^2 \sigma_1} - \frac{1}{\cos^2 \sigma} \right] = 1 \]  \hspace{1cm} (111)

\[ u_1^2 \left[ \frac{\omega_+^2(\delta)}{\cos^2 \sigma_1} - \frac{\omega_+^2(\delta)}{\cos^2 \sigma} \right] = u_1^2 \left[ \frac{1 + \sin (2\delta_1)}{\cos^2 \sigma_1} - \frac{1 + \sin (2\delta)}{\cos^2 \sigma} \right] = 1 \]  \hspace{1cm} (112)

\[ u_1^2 \left[ \frac{\omega_-^2(\delta)}{\cos^2 \sigma_1} - \frac{\omega_-^2(\delta)}{\cos^2 \sigma} \right] = u_1^2 \left[ \frac{1 - \sin (2\delta_1)}{\cos^2 \sigma_1} - \frac{1 - \sin (2\delta)}{\cos^2 \sigma} \right] = 1 \]  \hspace{1cm} (113)

or using eq.(111), the conditions reduce to

\[ \tan^2 \sigma_1 - \tan^2 \sigma = \tan^2 \psi \]  \hspace{1cm} (114)

\[ \frac{\sin(2\delta)}{\cos^2 \sigma} = \frac{\sin(2\delta_1)}{\cos^2 \sigma_1} \]  \hspace{1cm} (115)

Using the generators, we introduce the abbreviations

\[ M = m(\alpha + \beta) = m(2\sigma) = \tan \sigma > 0 \]  \hspace{1cm} (116)

\[ M_1 = m(\alpha_1 + \beta_1) = m(2\sigma_1) = \tan \sigma_1 > 0 \]  \hspace{1cm} (117)

\[ N = m(\alpha - \beta) = m(2\delta) = \tan \delta \]  \hspace{1cm} (118)

\[ N_1 = m(\alpha_1 - \beta_1) = m(2\delta_1) = \tan \delta_1 \]  \hspace{1cm} (119)

Then

\[ \tan \alpha = \tan (\sigma + \delta) = \frac{\tan \sigma + \tan \delta}{1 - \tan \sigma \tan \delta} = \frac{M + N}{1 - MN} \]  \hspace{1cm} (120)

\[ \tan \alpha_1 = \tan (\sigma_1 + \delta_1) = \frac{\tan \sigma_1 + \tan \delta_1}{1 - \tan \sigma_1 \tan \delta_1} = \frac{M_1 + N_1}{1 - M_1 N_1} \]  \hspace{1cm} (121)

resulting in

\[ N = \frac{\tan \alpha - M}{1 + M \tan \alpha} \]  \hspace{1cm} (122)

\[ N_1 = \frac{\tan \alpha_1 - M_1}{1 + M_1 \tan \alpha_1} \]  \hspace{1cm} (123)

The conditions (114) (115) then become

\[ M_1^2 - M^2 = \tan^2 \psi \]  \hspace{1cm} (124)

\[ \frac{2N}{1 + N^2 (1 + M^2)} = \frac{2N_1}{1 + N_1^2 (1 + M_1^2)} \]  \hspace{1cm} (125)

Or, using eqs (122), (123) we find
\[(1 - M^2)\sin(2\alpha) - 2M\cos(2\alpha) = (1 - M_1^2)\sin(2\alpha_1) - 2M_1\cos(2\alpha_1)\]  \hspace{1cm} (126)

Now

\[
\cos \delta \cos \sigma = \frac{1}{2} [\cos(\delta - \sigma) + \cos(\delta + \sigma)] = \frac{1}{2} [\cos \beta + \cos \alpha] \quad (127)
\]

\[
\sin \delta \cos \sigma = \frac{1}{2} [\sin(\delta - \sigma) + \sin(\delta + \sigma)] = \frac{1}{2} [\sin \alpha - \sin \beta] \quad (128)
\]

\[
\frac{1}{\cos^2 \sigma} = 1 + \tan^2 \sigma = 1 + M^2 \quad (129)
\]

gives the following representation

\[
\frac{\omega_+(\delta)}{\cos \sigma} = \frac{\cos \delta + \sin \delta}{\cos^2 \sigma} \cos \sigma = \frac{1}{2} (1 + M^2) [\omega_+(\alpha) + \omega_-(\beta)] \quad (130)
\]

\[
\frac{\omega_-(\delta)}{\cos \sigma} = \frac{\cos \delta - \sin \delta}{\cos^2 \sigma} \cos \sigma = \frac{1}{2} (1 + M^2) [\omega_-(\alpha) + \omega_+(\beta)] \quad (131)
\]

From

\[
\cos \beta = \cos(\alpha + \beta - \alpha) = \cos(\alpha + \beta) \cos \alpha + \sin(\alpha + \beta) \sin \alpha \quad (132)
\]

\[
\cos \beta = \frac{1 - M^2}{1 + M^2} \cos \alpha + \frac{2M}{1 + M^2} \sin \alpha \quad (133)
\]

\[
\sin \beta = \sin(\alpha + \beta - \alpha) = \sin(\alpha + \beta) \cos \alpha - \cos(\alpha + \beta) \sin \alpha \quad (134)
\]

\[
\sin \beta = \frac{2M}{1 + M^2} \cos \alpha - \frac{1 - M^2}{1 + M^2} \sin \alpha \quad (135)
\]

we find

\[
\omega_+(\beta) = \frac{1 - M^2}{1 + M^2} \omega_-(\alpha) + \frac{2M}{1 + M^2} \omega_+(\alpha) \quad (136)
\]

\[
\omega_-(\beta) = \frac{1 - M^2}{1 + M^2} \omega_+(\alpha) - \frac{2M}{1 + M^2} \omega_-(\alpha) \quad (137)
\]

and then

\[
\omega_+(\alpha) + \omega_-(\beta) = \frac{2}{1 + M^2} [\omega_+(\alpha) - M \omega_-(\alpha)] \quad (138)
\]

\[
\omega_-(\alpha) + \omega_+(\beta) = \frac{2}{1 + M^2} [\omega_-(\alpha) + M \omega_+(\alpha)] \quad (139)
\]

Finally, from eq.(107)
\[ u_2 = u_1 M \] (140)
\[ u_3 = u_1 [\omega_+(\alpha) - M \omega_- (\alpha)] \] (141)
\[ u_4 = u_1 [\omega_- (\alpha) + M \omega_+ (\alpha)] \] (142)

and from eq.(108)

\[ v_2 = u_1 M_1 \] (143)
\[ v_3 = u_1 [\omega_+(\alpha_1) - M_1 \omega_- (\alpha_1)] \] (144)
\[ v_4 = u_1 [\omega_- (\alpha_1) + M_1 \omega_+ (\alpha_1)] \] (145)

From the symmetry of interchanging \( \alpha \) and \( \beta \), which corresponds to the interchange of \( u_3 \) and \( u_4 \), we also have the representation

\[ u_3 = u_1 [\omega_- (\beta) + M \omega_+ (\beta)] \] (146)
\[ u_4 = u_1 [\omega_+ (\beta) - M \omega_- (\beta)] \] (147)

**Example**

For the special case of \( \alpha + \alpha_1 = \frac{\pi}{2} \), resulting in

\[
\cos(2\alpha_1) = -\cos(2\alpha), \sin(2\alpha_1) = \sin(2\alpha)
\]

and given the two Heron angles \( \psi, \alpha \); \( u_1 = \cot \psi \),

the conditions (124) (126) read

\[
M_1 - M = 2 \cot(2\alpha), M_1 + M = \frac{1}{2} \tan^2 \psi \tan(2\alpha)
\] or

\[
M = \frac{1}{4} [\tan^2 \psi \tan(2\alpha) - 4 \cot(2\alpha)]
\]
\[
M_1 = \frac{1}{4} [4 \cot(2\alpha) + \tan^2 \psi \tan(2\alpha)]
\]

Observe that this two-parameter family of rational leaning boxes has no cuboid limit, because in the cuboid limit \( N=0, N_1 = 0 \), implying

\[
M = \tan \alpha, M_1 = \tan \alpha_1 = \frac{1}{\tan \alpha}
\]

Then
\[ M_1^2 - M^2 = \tan^2 \psi \]

reads

\[ 1 - \tan^4 \alpha = \tan^2 \alpha \tan^2 \psi \]

which has no rational solutions [6].

We now split the eq.(126) into two parts

\[ (M^2 - 1) \sin(2\alpha) + 2M \cos(2\alpha) = 4D \tag{148} \]
\[ (M_1^2 - 1) \sin(2\alpha_1) + 2M_1 \cos(2\alpha_1) = 4D \tag{149} \]

According to the Appendix F, they have the following parameter representations, replacing \( \lambda \) by \( r \), respectively by \( r_1 \).

\[
\begin{align*}
M &= 2r + \tan \alpha \\
M_1 &= 2r_1 + \tan \alpha_1
\end{align*}
\]

\[
\begin{align*}
D &= r[1 + r \sin(2\alpha)] \tag{150} \\
D &= r_1[1 + r_1 \sin(2\alpha_1)] \tag{151}
\end{align*}
\]

Now, using [8], the equation

\[ r[1 + r \sin(2\alpha)] = r_1[1 + r_1 \sin(2\alpha_1)] \tag{152} \]

has the parameter representation

\[
\begin{align*}
r &= f \frac{1 + f \sin(2\alpha_1)}{1 - f^2 \sin(2\alpha) \sin(2\alpha_1)} \\
r_1 &= f \frac{1 + f \sin(2\alpha)}{1 - f^2 \sin(2\alpha) \sin(2\alpha_1)}
\end{align*}
\]

Conversely, for \( f \neq 0 \)

case (i)

\[ r_1 \neq r, f = \frac{r_1 - r}{r \sin(2\alpha) - r_1 \sin(2\alpha_1)} \tag{155} \]

case (ii)

\[ r_1 = r, f = \frac{r}{1 + r \sin(2\alpha)} \tag{156} \]
and $\sin(2\alpha_1) = \sin(2\alpha)$

or $\cos(\alpha_1 + \alpha) \sin(\alpha_1 - \alpha) = 0$

For the non-trivial case, $\alpha \neq \alpha_1$ we have $\alpha_1 + \alpha = \frac{\pi}{2}$, which is the example above.

This parameter representation can be verified directly.

8 Cuboid Limit

The cuboid limit is given by

$$\beta \to \alpha \quad \text{and} \quad \beta_1 \to \alpha_1$$  

(157)

From (118, 119) this means

$$N \to 0 \quad \text{and} \quad N_1 \to 0$$  

(158)

and from (122, 123) we get

$$M \to \tan \alpha \quad \text{and} \quad M_1 \to \tan \alpha_1$$  

(159)

or equivalently from (150, 151) that

$$r \to 0 \quad \text{and} \quad r_1 \to 0$$  

(160)

Looking at the equation (152) and letting $\frac{r_1}{r} = s$, we have two cases

(i) $s = 1$ implies $\sin(2\alpha) = \sin(2\alpha_1)$ or $\alpha + \alpha_1 = \frac{\pi}{2}$  

(161)

which, according to the example on page 21 has no cuboid limit. This is also the family found in section 5.

(ii) $s \neq 1$. Let

$$F(r, r_1) = \frac{r_1}{r} - \frac{1 + r \sin(2\alpha)}{1 + r_1 \sin(2\alpha_1)}$$  

(162)

For the cuboid limit

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\[ F(r, r_1) = s - 1 \neq 0, \quad (163) \]

which contradicts equation (152).

Thus there is also no cuboid limit.

In conclusion, there is no perfect cuboid.
9 Consequence

Corollary

Let \( \alpha, \alpha_1 \) be Euler angles and \( \psi \) be a Heron angle. Then in the equation

\[
\tan^2 \alpha_1 - \tan^2 \alpha = \tan^2 \psi \tag{164}
\]

not both \( \alpha \) and \( \alpha_1 \) can be Heron angles

Proof. In the cuboid limit only equation (124) is the surviving condition. If both \( \alpha \) and \( \alpha_1 \) were Heron angles we would have a perfect cuboid. This is a contradiction. \( \square \)

Example

Euler Cuboid (body diagonal not rational)

\[
\begin{align*}
\tan \alpha_1 &= \frac{125}{240}, & \cos \alpha_1 &= \frac{48}{\sqrt{2929}}, & \sin \alpha_1 &= \frac{25}{\sqrt{2929}} \\
\tan \alpha &= \frac{44}{240}, & \cos \alpha &= \frac{60}{61}, & \sin \alpha &= \frac{11}{61}, & m(\alpha) = \frac{1}{11} \\
\tan \psi &= \frac{17}{240}, & \cos \psi &= \frac{80}{89}, & \sin \psi &= \frac{39}{89}, & m(\psi) = \frac{3}{13}
\end{align*}
\]

Example

Face Cuboid (one face diagonal not rational)

\[
\begin{align*}
\tan \alpha_1 &= \frac{765}{520}, & \cos \alpha_1 &= \frac{104}{185}, & \sin \alpha_1 &= \frac{153}{185}, & m(\alpha_1) = \frac{9}{17} \\
\tan \alpha &= \frac{756}{520}, & \cos \alpha &= \frac{130}{\sqrt{52621}}, & \sin \alpha &= \frac{189}{\sqrt{52621}} \\
\tan \psi &= \frac{117}{520}, & \cos \psi &= \frac{40}{41}, & \sin \psi &= \frac{9}{41}, & m(\psi) = \frac{1}{9}
\end{align*}
\]
Appendix A

Generator of an angle
Heron angle, Euler angle

Definition 1
For an arbitrary angle $\alpha$, its generator is defined by

$$m(\alpha) = \frac{\sin \alpha}{1 + \cos \alpha} = \tan \left(\frac{\alpha}{2}\right) \quad (A.1)$$

Consequently

$$\cos \alpha = \frac{1 - m^2(\alpha)}{1 + m^2(\alpha)}, \quad \sin \alpha = \frac{2m(\alpha)}{1 + m^2(\alpha)} \quad (A.2)$$

We have the following properties

$$m(-\alpha) = -m(\alpha) \quad (A.3)$$

$$m^2(\alpha) = \frac{1 - \cos \alpha}{1 + \cos \alpha} \quad (A.4)$$

$$m(\alpha + \beta) = \frac{m(\alpha) + m(\beta)}{1 - m(\alpha)m(\beta)} \quad (A.5)$$

$$\frac{dm(\alpha)}{d\alpha} = \frac{1}{2} \frac{1}{\cos^2 \left(\frac{\alpha}{2}\right)} > 0 \quad (A.6)$$

Then $m(\alpha)$ is an increasing function of $\alpha$, with

$$m(0) = 0, \quad m\left(\frac{\pi}{2}\right) = 1$$

For $0 < \alpha < \frac{\pi}{2}$ the generator satisfies

$$0 < m(\alpha) < 1 \quad (A.7)$$
**Definition 2**

An angle \( \alpha \) is called a Heron angle if both \( \sin \alpha \) and \( \cos \alpha \) are rational.

The generator of a Heron angle is rational and vice versa. The sum and the difference of two Heron angles are Heron angles.

The complement

\[
\tilde{\alpha} = \frac{\pi}{2} - \alpha
\]  

of a Heron angle \( \alpha \) is a Heron angle.

**Definition 3**

An angle \( \alpha \) is called an Euler angle if \( \tan \alpha \) is rational.

If \( \alpha \) is an Euler angle then \( 2\alpha \) is a Heron angle. A Heron angle is an Euler angle.

**Lemma 1**

Let \( \alpha \) be a Heron angle, \( 0 \leq \alpha \leq \pi \).

Then the equation

\[
\sin \alpha = \lambda^2
\]  

where \( \lambda \) is a rational number, \( 0 \leq \lambda \leq 1 \), has only the trivial solutions \( \alpha = 0 \), \( \alpha = \frac{\pi}{2} \), \( \alpha = \pi \)

**Proof.** Let \( \lambda = \frac{a}{b} \), \( 0 < a < b \), \( a \) and \( b \) integers.

Then \( \sin \alpha \) is rational and

\[
\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \lambda^4 = \frac{b^4 - a^4}{b^4}
\]  

But according to Euler [6], \( b^4 - a^4 \) can not be the square of an integer. Thus \( \cos \alpha \) is not rational, i.e. \( \alpha \) is not a Heron angle, except for the trivial cases. □
Lemma 2
Let $\alpha$ be a Heron angle, $0 \leq \alpha \leq \frac{\pi}{2}$.

Then the equation

$$\tan \alpha = \lambda^2 \quad (A.11)$$

where $\lambda$ is a rational number, $\lambda \geq 0$, has only the trivial solution $\alpha = 0$.

Proof. Let $\lambda = \frac{a}{b}$, $a > 0$, $b > 0$, $a$ and $b$ integers.

Then $\tan \alpha$ is rational and

$$\frac{1}{\cos^2 \alpha} = 1 + \tan^2 \alpha = 1 + \lambda^4 = \frac{b^4 + a^4}{b^4} \quad (A.12)$$

But according to Euler [6], $b^4 + a^4$ is not the square of an integer. Thus $\cos \alpha$ is not rational, except for the trivial case.

Thus in any case, for a Heron angle $\alpha$, $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\cot \alpha$ can not be the square of a rational number, except for the trivial cases. \qed

Corollary 1
The elliptic curve

$$y^2 = x(1 - x^2) \quad (A.13)$$

has only the trivial rational points $(x,y)$, namely $(-1,0), (0,0), (1,0)$.

Proof. Let $\alpha$ be a Heron angle and

$$\lambda^2 = \sin 2\alpha$$

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Then $\lambda$ can not be rational, except for the trivial cases. Let $x$ be the generator of $\alpha$

$$
\lambda^2 = 2 \sin \alpha \cos \alpha = 2 \frac{2x}{1 + x^2} \frac{1 - x^2}{1 + x^2}
$$

$$
\lambda^2 = \left[ \frac{2}{1 + x^2} \right]^2 x(1 - x^2)
$$

Then

$$
y^2 = x(1 - x^2)
$$

has only the trivial rational points.

\[\square\]

**Corollary 2**

The elliptic curve

$$
y^2 = 2x(1 - x^2)
$$

has only the trivial rational points $(x,y)$, namely $(-1,0),(0,0),(1,0)$

**Proof.** Let $\alpha$ be a Heron angle with generator $x$, and

$$
\lambda^2 = \tan \alpha
$$

Then $\lambda$ can not be rational, except for the trivial case.

Now

$$
\lambda^2 = \frac{2x}{1 - x^2} = \left[ \frac{1}{1 - x^2} \right]^2 2x(1 - x^2)
$$
Then

\[ y^2 = 2x(1 - x^2) \]

has only the trivial rational points. \qed
Appendix B

Rotations

A rotation in two dimensions is given by the matrix

\[
R(\alpha) = \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\]  

(B.1)

These form an Abelian group.

Group multiplication  \( R(\alpha)R(\beta) = R(\alpha + \beta) \)  
(B.2)

Identity  \( R(0) \)  
(B.3)

Inverse  \( R(\alpha)^{-1} = R(-\alpha) \)  
(B.4)

For two two-dimensional vectors, related by a rotation, we have

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = R(\alpha) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\]  
(B.5)

They have the same length, i.e.

\[
x_1^2 + y_1^2 = x_2^2 + y_2^2
\]  
(B.6)

Conversely we get the rotation angle \( \alpha \) through

\[
\cos \alpha = \frac{1}{x_2^2 + y_2^2} [x_1 x_2 + y_1 y_2]
\]  
(B.7)

\[
\sin \alpha = \frac{1}{x_2^2 + y_2^2} [y_1 x_2 - x_1 y_2]
\]  
(B.8)
Appendix C

The $\omega$-functions

Definition

For an angle $\alpha$ we introduce the $\omega$-functions by

$$\omega_+(\alpha) = \cos \alpha + \sin \alpha, \quad \omega_-(\alpha) = \cos \alpha - \sin \alpha$$  \hfill (C.1)

We then have the following properties

$$\omega_+(-\alpha) = \omega_-(\alpha)$$  \hfill (C.2)
$$\omega_-(\alpha) = \omega_+(\alpha)$$  \hfill (C.3)
$$\omega_2^+ + \omega_2^- = 2$$  \hfill (C.4)
$$\omega_+(\alpha)\omega_-(\alpha) = \cos(2\alpha)$$  \hfill (C.5)
$$\omega_2^+ + \omega_2^- = 1 + \sin(2\alpha)$$  \hfill (C.6)
$$\omega_2^+ + \omega_2^- = 1 - \sin(2\alpha)$$  \hfill (C.7)

For two angles $\alpha$ and $\beta$ we introduce

$$\alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta$$  \hfill (C.8)
$$\alpha = \sigma + \delta, \quad \beta = \sigma - \delta$$  \hfill (C.9)

We then find

$$\omega_+(\alpha)\omega_+(\beta) = \cos(2\delta) + \sin(2\sigma)$$  \hfill (C.10)
$$\omega_-(\alpha)\omega_-(\beta) = \cos(2\delta) - \sin(2\sigma)$$  \hfill (C.11)
$$\omega_+(\alpha)\omega_-(\beta) = \cos(2\sigma) + \sin(2\delta)$$  \hfill (C.12)

We also have the relation

$$R(\beta)\begin{pmatrix} \omega_+(\alpha + \beta) \\ \omega_-(\alpha + \beta) \end{pmatrix} = \begin{pmatrix} \omega_+(\alpha) \\ \omega_-(\alpha) \end{pmatrix}$$ \hfill (C.13)

or explicitly

$$\omega_+(\alpha + \beta) = \cos \beta \omega_+(\alpha) + \sin \beta \omega_-(\alpha)$$  \hfill (C.14)
$$\omega_+(\alpha + \beta) = \cos \alpha \omega_+(\beta) + \sin \alpha \omega_-(\beta)$$  \hfill (C.15)
$$\omega_-(\alpha + \beta) = -\sin \beta \omega_+(\alpha) + \cos \beta \omega_-(\alpha)$$  \hfill (C.16)
$$\omega_-(\alpha + \beta) = -\sin \alpha \omega_+(\beta) + \cos \alpha \omega_-(\beta)$$  \hfill (C.17)
This gives the following relations

\[ \omega_+(\alpha) + \omega_+(\beta) = 2 \cos \delta \omega_+(\sigma) \quad \text{(C.18)} \]
\[ \omega_+(\alpha) - \omega_+(\beta) = 2 \sin \delta \omega_-(\sigma) \quad \text{(C.19)} \]
\[ \omega_-(\alpha) + \omega_-(\beta) = 2 \cos \delta \omega_-(\sigma) \quad \text{(C.20)} \]
\[ \omega_-(\alpha) - \omega_-(\beta) = -2 \sin \delta \omega_+(\sigma) \quad \text{(C.21)} \]
\[ \omega_+(\alpha) + \omega_-(\beta) = 2 \cos \sigma \omega_+(\delta) \quad \text{(C.22)} \]
\[ \omega_+(\alpha) - \omega_-(\beta) = 2 \sin \sigma \omega_-(\delta) \quad \text{(C.23)} \]
\[ \omega_-(\alpha) + \omega_+(\beta) = 2 \cos \sigma \omega_-(\delta) \quad \text{(C.24)} \]
\[ \omega_-(\alpha) - \omega_+(\beta) = -2 \sin \sigma \omega_+(\delta) \quad \text{(C.25)} \]

and finally

\[
\begin{pmatrix}
\omega_-(\alpha) \\
\omega_+(\alpha)
\end{pmatrix} = R(2\alpha)
\begin{pmatrix}
\omega_+(\alpha) \\
\omega_-(\alpha)
\end{pmatrix}
\]

\text{(C.26)}
Appendix D

The Rational Parallelogram

A parallelogram with its sides \(u_1, u_2\) and diagonals \(u_3, u_4\) being positive rational numbers is called a rational parallelogram. It is governed by the parallelogram equation

\[
2u_1^2 + 2u_2^2 = u_3^2 + u_4^2 \tag{D.1}
\]

In [7, 8] we found a bijective parameter representation for all rational parallelograms. It is given by the rational scaling parameter

\[
u > 0 \tag{D.2}
\]

and two rational parameters \(m, n\)

\[
0 < m < 1, \ 0 < n < 1 \tag{D.3}
\]

The representation is given by

\[
\begin{align*}
u_1 &= (1 - mn)u \tag{D.4} \\
u_2 &= (m + n)u \tag{D.5} \\
u_3 &= (1 + mn - n + m)u \tag{D.6} \\
u_4 &= (1 + mn + n - m)u \tag{D.7}
\end{align*}
\]

Conversely

\[
\begin{align*}
m &= \frac{2u_2 + u_3 - u_4}{4u}, \quad n = \frac{2u_2 - u_3 + u_4}{4u} \tag{D.8}
\end{align*}
\]
Special cases:

Rectangle: \[ u_4 = u_3; \quad n = m \] (D.10)

Rhomboid: \[ u_2 = u_1; \quad n = \frac{1 - m}{1 + m} \] (D.11)

From (D.4) and (D.5) we find

\[(m + n)u_1 = (1 - mn)u_2\] (D.12)

and then

\[n = \frac{u_2 - mu_1}{u_1 + mu_2}, \quad 0 < n < 1\] (D.13)

\[m = \frac{u_2 - mu_1}{u_1 + mu_2}, \quad 0 < m < 1\] (D.14)

From (D.13) we find

\[1 - mn = \frac{u_1(1 + m^2)}{u_1 + mu_2}\] (D.15)

\[1 + mn = \frac{u_1(1 - m^2) + 2mu_2}{u_1 + mu_2}\] (D.16)

\[m - n = \frac{2mu_1 - u_2(1 - m^2)}{u_1 + mu_2}\] (D.17)

\[u = \frac{u_1 + mu_2}{1 + m^2}\] (D.18)

and then

\[u_3 = (u_2 + u_1)\frac{2m}{1 + m^2} - (u_2 - u_1)\frac{1 - m^2}{1 + m^2}\] (D.19)

\[u_4 = (u_2 + u_1)\frac{1 - m^2}{1 + m^2} + (u_2 - u_1)\frac{2m}{1 + m^2}\] (D.20)
This is a parameterization of a rational parallelogram by

\[ u_1 > 0, u_2 > 0, \text{ and } 0 < m < 1 \]

From (D.14) we find

\[
1 - mn = \frac{u_1 (1 + n^2)}{u_1 + nu_2} \quad (D.21)
\]
\[
1 + mn = \frac{u_1 (1 - n^2) + 2nu_2}{u_1 + nu_2} \quad (D.22)
\]
\[
m - n = \frac{-2nu_1 + u_2 (1 - n^2)}{u_1 + nu_2} \quad (D.23)
\]
\[
u = \frac{u_1 + nu_2}{1 + n^2} \quad (D.24)
\]

and then

\[
u_3 = (u_2 - u_1) \frac{2n}{1 + n^2} + (u_2 + u_1) \frac{1 - n^2}{1 + n^2} \quad (D.25)
\]
\[
u_4 = (u_2 + u_1) \frac{2n}{1 + n^2} - (u_2 - u_1) \frac{1 - n^2}{1 + n^2} \quad (D.26)
\]

This is a parameterization of a rational parallelogram by

\[ u_1 > 0, u_2 > 0, \text{ and } 0 < n < 1 \]

Now the two parameters m,n in (D.3) give rise to the Heron angles \( \alpha, \beta \) through
\begin{align*}
\cos \alpha &= \frac{1 - m^2}{1 + m^2}, \quad \sin \alpha = \frac{2m}{1 + m^2}, \quad 0 < \alpha < \frac{\pi}{2} \\
\cos \beta &= \frac{1 - n^2}{1 + n^2}, \quad \sin \beta = \frac{2n}{1 + n^2}, \quad 0 < \beta < \frac{\pi}{2}
\end{align*} (D.27)

And conversely

\begin{align*}
m &= \frac{\sin \alpha}{1 + \cos \alpha} = \tan\left(\frac{\alpha}{2}\right) \quad (D.29) \\
n &= \frac{\sin \beta}{1 + \cos \beta} = \tan\left(\frac{\beta}{2}\right) \quad (D.30)
\end{align*}

Using the \(\omega\)-functions we get in matrix notation the representation

\[
\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \omega_+ (\alpha) & -\omega_- (\alpha) \\ \omega_- (\alpha) & \omega_+ (\alpha) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (D.31)
\]

Inversely

\[
2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega_+ (\alpha) & \omega_- (\alpha) \\ -\omega_- (\alpha) & \omega_+ (\alpha) \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \quad (D.32)
\]

and conversely

\begin{align*}
\omega_+ (\alpha) &= \frac{1}{u_1^2 + u_2^2} [u_1 u_3 + u_2 u_4] \\
\omega_- (\alpha) &= \frac{1}{u_1^2 + u_2^2} [u_1 u_4 - u_2 u_3] \quad (D.33, D.34)
\end{align*}

Similarly we find the representation

\[
\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \omega_- (\beta) & \omega_+ (\beta) \\ \omega_+ (\beta) & -\omega_- (\beta) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (D.35)
\]
Inversely

\[
2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \omega_-(\beta) & \omega_+(\beta) \\ \omega_+(\beta) & -\omega_-(\beta) \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}
\]

(D.36)

and conversely

\[
\omega_+(\beta) = \frac{1}{u_1^2 + u_2^2}[u_1 u_4 + u_2 u_3]
\]

(D.37)

\[
\omega_-(\beta) = \frac{1}{u_1^2 + u_2^2}[u_1 u_3 - u_2 u_4]
\]

(D.38)

Finally we introduce the two Euler angles \(\sigma, \delta\) through

\[
\alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta
\]

(D.39)

\[
\alpha = \sigma + \delta, \quad \beta = \sigma - \delta
\]

(D.40)

From

\[
u_1 \omega_+(\alpha) - u_2 \omega_-(\alpha) = u_1 \omega_-(\beta) + u_2 \omega_+(\beta)
\]

(D.41)

and

\[
2u_3 = u_1 \omega_+(\alpha) - u_2 \omega_-(\alpha) + u_1 \omega_-(\beta) + u_2 \omega_+(\beta)
\]

(D.42)

\[
2u_4 = u_1 \omega_-(\alpha) + u_2 \omega_+(\alpha) + u_1 \omega_+(\beta) - u_2 \omega_-(\beta)
\]

(D.43)

we find, using (C.19), (C.22), (C.24), (C.25) the relations

\[
u_1 \sin \sigma = u_2 \cos \sigma
\]

(D.44)

\[
u_3 = [u_1 \cos \sigma + u_2 \cos \sigma] \omega_+(\delta)
\]

(D.45)

\[
u_4 = [u_1 \cos \sigma + u_2 \sin \sigma] \omega_-(\delta)
\]

(D.46)
resulting in the representation

\[ u_2 = u_1 \tan \sigma \quad \text{(D.47)} \]
\[ u_3 = u_1 \frac{\omega_+ (\delta)}{\cos \sigma} \quad \text{(D.48)} \]
\[ u_4 = u_1 \frac{\omega_- (\delta)}{\cos \sigma} \quad \text{(D.49)} \]

Conversely

\[ \tan \sigma = \frac{u_2}{u_1} \quad \text{(D.50)} \]
\[ \tan \delta = \frac{u_3 - u_4}{u_3 + u_4} \quad \text{(D.51)} \]

From (D.27), (D.28), (D.39), (D.40) we find the inequalities

\[ \tan \sigma > 0 \quad \text{(D.52)} \]
\[ -1 < \tan \delta < 1 \quad \text{(D.53)} \]

(D.47-49) is a parameterization of a rational parallelogram, given one side \( u_1 > 0 \), by two Euler angles.

Observe, that given \( u_1 \) and \( u_2 \) there are two representations of \( u_3 \) and \( u_4 \), (D.31) (D.35). One is through the Heron angle \( \alpha \), with generator \( m = m(\alpha) \) and the other through the Heron angle \( \beta \), with generator \( n = m(\beta) \). According to (D.8) and (D.9) they are however related through the interchange of \( u_3 \) with \( u_4 \).
Appendix E

Auxiliary Functions

For an angle $\alpha$ and a number $Q$ we introduce the functions

$$H(\alpha) = \omega_-(\alpha) - Q\omega_+(\alpha) \quad (E.1)$$
$$K(\alpha) = \omega_-(\alpha) + Q\omega_+(\alpha) \quad (E.2)$$
$$M(\alpha) = \omega_+(\alpha) - Q\omega_-(\alpha) \quad (E.3)$$
$$N(\alpha) = \omega_+(\alpha) + Q\omega_-(\alpha) \quad (E.4)$$

or in matrix form

$$\begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} = \begin{pmatrix} \omega_-(\alpha) & -\omega_+(\alpha) \\ \omega_+(\alpha) & \omega_-(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ Q \end{pmatrix} = R(\alpha) \begin{pmatrix} 1 - Q \\ 1 + Q \end{pmatrix} \quad (E.5)$$

$$\begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix} = \begin{pmatrix} \omega_-(\alpha) & \omega_+(\alpha) \\ \omega_+(\alpha) & -\omega_-(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ Q \end{pmatrix} = R(\alpha) \begin{pmatrix} 1 + Q \\ 1 - Q \end{pmatrix} \quad (E.6)$$

We now introduce the $T$-matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (E.7)$$

and find the following properties

$$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (E.8)$$

$$TR(\alpha)T = R(-\alpha) \quad (E.9)$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \quad (E.10)$$
\[ R(\alpha)TR(-\alpha) = \begin{pmatrix} -\sin(2\alpha) & \cos(2\alpha) \\ \cos(2\alpha) & \sin(2\alpha) \end{pmatrix} \quad (E.11) \]

Then we have the following statements

**Lemma 1**

\[
\begin{align*}
2 &= H(\alpha)\omega_-(\alpha) + N(\alpha)\omega_+(\alpha) \\
2 &= K(\alpha)\omega_-(-\alpha) + M(\alpha)\omega_+(\alpha) \\
2Q &= N(\alpha)\omega_-(\alpha) - H(\alpha)\omega_+(\alpha) \\
2Q &= K(\alpha)\omega_+(\alpha) - M(\alpha)\omega_-(-\alpha)
\end{align*} \quad (E.12)-(E.15)
\]

**Lemma 2**

\[
\begin{align*}
\begin{pmatrix} H(\alpha + \beta) \\ N(\alpha + \beta) \end{pmatrix} &= R(\alpha) \begin{pmatrix} H(\beta) \\ N(\beta) \end{pmatrix} = R(\beta) \begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} \\
\begin{pmatrix} K(\alpha + \beta) \\ M(\alpha + \beta) \end{pmatrix} &= R(\alpha) \begin{pmatrix} K(\beta) \\ M(\beta) \end{pmatrix} = R(\beta) \begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix}
\end{align*} \quad (E.16)-(E.17)
\]

**Lemma 3**

With \( \bar{\alpha} = \frac{\pi}{2} - \alpha \) we have

\[
H(\bar{\alpha}) = -K(\alpha), \quad N(\bar{\alpha}) = M(\alpha) \quad (E.18)
\]

**Lemma 4**

\[
\begin{align*}
\begin{pmatrix} H(\alpha) \\ N(\alpha) \end{pmatrix} &= R(2\alpha) \begin{pmatrix} M(\alpha) \\ K(\alpha) \end{pmatrix} \\
\begin{pmatrix} K(\alpha) \\ M(\alpha) \end{pmatrix} &= R(2\alpha) \begin{pmatrix} N(\alpha) \\ H(\alpha) \end{pmatrix}
\end{align*} \quad (E.19)-(E.20)
\]
Lemma 5

1) \[ H^2(\alpha) + N^2(\alpha) = K^2(\alpha) + M^2(\alpha) = 2(1 + Q^2) \]  (E.21)
2) \[ K(\alpha)N(\alpha) - H(\alpha)M(\alpha) = 4Q \]  (E.22)
3) \[ K(\alpha)N(\alpha) + H(\alpha)M(\alpha) = 2(1 + Q^2)\cos(2\alpha) \]  (E.23)
4) \[ K(\alpha + \delta)H(\alpha + \beta) - K(\alpha + \beta)H(\alpha + \delta) = 4Q\sin(\delta - \beta) \]  (E.24)
5) \[ N(\alpha + \beta)M(\alpha + \delta) - N(\alpha + \delta)M(\alpha + \beta) = 4Q\sin(\delta - \beta) \]  (E.25)

Lemma 6

1) \[ K(\alpha) + H(\alpha) = 2\omega_-(\alpha) \]  (E.26)
2) \[ K(\alpha) - H(\alpha) = 2Q\omega_+(\alpha) \]  (E.27)
3) \[ N(\alpha) + M(\alpha) = 2\omega_+(\alpha) \]  (E.28)
4) \[ N(\alpha) - M(\alpha) = 2Q\omega_-(\alpha) \]  (E.29)
5) \[ M(\alpha) + H(\alpha) = 2\cos\alpha[1 - Q] \]  (E.30)
6) \[ M(\alpha) - H(\alpha) = 2\sin\alpha[1 + Q] \]  (E.31)
7) \[ N(\alpha) + K(\alpha) = 2\cos\alpha[1 + Q] \]  (E.32)
8) \[ N(\alpha) - K(\alpha) = 2\sin\alpha[1 - Q] \]  (E.33)

Now with

\[ \alpha + \beta = 2\sigma, \quad \alpha - \beta = 2\delta \]
\[ \alpha = \sigma + \delta, \quad \beta = \sigma - \delta \]

we introduce the angle \( \psi \) through

\[ \omega_+(\sigma) = \sqrt{2}\cos\psi, \quad \omega_-(\sigma) = \sqrt{2}\sin\psi \]  (E.34)

Then \( \psi \) is an Euler angle and for

\[ 0 < \alpha < \frac{\pi}{2}, \quad 0 < \beta < \frac{\pi}{2} \]
we find \[ -\frac{\pi}{4} < \psi < \frac{\pi}{4} \]
From $\sigma = \alpha - \delta$ and (C.13) we find

$$
\begin{pmatrix}
\omega_+(\sigma) \\
\omega_-(\sigma)
\end{pmatrix} = R(\delta)
\begin{pmatrix}
\omega_+(\alpha) \\
\omega_-(\alpha)
\end{pmatrix} = R(\psi)
\begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix}
$$

(E.35)

$$
\begin{pmatrix}
\omega_+(\alpha + \psi) \\
\omega_-(\alpha + \psi)
\end{pmatrix} = R(-\psi)
\begin{pmatrix}
\omega_+(\alpha) \\
\omega_-(\alpha)
\end{pmatrix} = R(-\delta)
\begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix}
$$

(E.36)

Explicitly

$$
\sqrt{2} \cos \delta = \omega_+(\alpha + \psi), \quad \sqrt{2} \sin \delta = -\omega_-(\alpha + \psi)
$$

(E.37)

**Lemma 7**

1) $M(\alpha) - N(\beta) = -2 \sin \psi K(\alpha + \psi)$ (E.38)
2) $M(\alpha) + N(\beta) = 2 \cos \psi M(\alpha + \psi)$ (E.39)
3) $N(\alpha) - M(\beta) = -2 \sin \psi H(\alpha + \psi)$ (E.40)
4) $N(\alpha) + M(\beta) = 2 \cos \psi N(\alpha + \psi)$ (E.41)
5) $K(\alpha) + H(\beta) = 2 \sin \psi M(\alpha + \psi)$ (E.42)
6) $K(\alpha) - H(\beta) = 2 \cos \psi K(\alpha + \psi)$ (E.43)
7) $H(\alpha) + K(\beta) = 2 \sin \psi N(\alpha + \psi)$ (E.44)
8) $H(\alpha) - K(\beta) = 2 \cos \psi H(\alpha + \psi)$ (E.45)

The easy proofs are left to the reader.
Appendix F

A Convenient Bijective Parameterization

For a given Heron angle $\alpha$, $0 < \alpha < \frac{\pi}{2}$, we look at the quadratic equation

$$(M^2 - 1) \sin(2\alpha) + 2M \cos(2\alpha) = 4D, M > 0 \quad (F.1)$$

or

$$M^2 \sin(2\alpha) + 2M \cos(2\alpha) - [\sin(2\alpha) + 4D] = 0 \quad (F.2)$$

Then the solutions are given by

$$M_+ = \frac{1}{\sin(2\alpha)}[-\cos(2\alpha) + \Delta] \quad (F.3)$$

$$M_- = \frac{1}{\sin(2\alpha)}[-\cos(2\alpha) - \Delta] \quad (F.4)$$

where

$$\Delta^2 = \cos^2(2\alpha) + \sin(2\alpha)[\sin(2\alpha) + 4D] \quad (F.5)$$

or

$$\Delta^2 + [\sin(2\alpha) - D]^2 = 1 + [\sin(2\alpha) + D]^2 \quad (F.6)$$

According to [8] this relation has the following bijective parameter representation with the parameters $a_1, a_2, \lambda$

$$\Delta = a_1 + \lambda a_2 \quad 1 = a_1 - \lambda a_2 \quad (F.7)$$

$$\sin(2\alpha) - D = a_2 - \lambda a_1 \quad \sin(2\alpha) + D = a_2 + \lambda a_1 \quad (F.8)$$

Conversely

$$2a_1 = 1 + \Delta \quad (F.9)$$

$$a_2 = \sin(2\alpha) \quad (F.10)$$

$$\lambda a_1 = D \quad (F.11)$$

From (F.5) and (F.9) we find that

$$a_1 \neq 0, \quad 1 + \lambda a_2 = a_1 \neq 0 \quad (F.12)$$

Then
\[ a_1 = 1 + \lambda a_2 = 1 + \lambda \sin(2\alpha) \quad \text{(F.13)} \]

\[ \Delta = 1 + 2\lambda a_2 = 1 + 2\lambda \sin(2\alpha) \quad \text{(F.14)} \]

and from (F.3), (F.4) the equation (F.1) is bijectively parameterized by the parameter \( \lambda \) as

\[ M_+ = 2\lambda + \tan \alpha \quad \text{(F.15)} \]

\[ M_- = -2\lambda - \cot \alpha \quad \text{(F.16)} \]

\[ D = \lambda [1 + \lambda \sin(2\alpha)] \quad \text{(F.17)} \]

Conversely

\[ 2\lambda = M_+ - \tan \alpha \quad \text{(F.18)} \]

**Remark 6.**

1) From (F.13) we find

\[ 1 + \lambda \sin(2\alpha) \neq 0 \quad \text{(F.19)} \]

2) For the limit

\[ \lambda \to 0 \quad \text{(F.20)} \]

we find from (F.15-17) that

\[ M_+ \to \tan \alpha \quad \text{(F.21)} \]

\[ M_- \to -\cot \alpha \quad \text{(F.22)} \]

\[ D \to 0 \quad \text{(F.23)} \]

Since the solution of (F.1) has to be positive, the only acceptable solution for the limit (F.20) is

\[ M = M_+ \quad \text{(F.24)} \]
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