A DISCRETE MODEL OF $S^1$-HOMOTOPY THEORY

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Abstract. We construct a discrete model of the homotopy theory of $S^1$-spaces. We define a category $\mathcal{P}$ with objects composed of a simplicial set and a cyclic set along with suitable compatibility data. $\mathcal{P}$ inherits a model structure from the model structures on the categories of simplicial sets and cyclic sets. We then show that there is a Quillen equivalence between $\mathcal{P}$ and the model category of $S^1$-spaces in which weak equivalences and fibrations are maps inducing weak equivalences and fibrations on passage to all fixed point sets.

1. Introduction

Simplicial techniques are often unavailable in the context of equivariant homotopy theory. When $G$ is not a discrete group, simplicial $G$-sets do not provide a model for the homotopy theory of $G$-spaces. The lack of an adequate replacement for simplicial sets is a substantial inconvenience. Cyclic sets [3] provide a useful discrete model of a portion of $S^1$-homotopy theory. Specifically, Spalinski [15] (following Dwyer, Hopkins, and Kan [4]) constructs a model structure on cyclic sets which is Quillen equivalent to the model structure on $S^1$-spaces in which weak equivalences and fibrations are detected on passage to fixed point subspaces for finite groups. However, since the $S^1$-fixed points of the geometric realization of a cyclic set must be discrete ([7], [15]), it is unreasonable to expect a model structure on cyclic sets which will capture all of $S^1$-homotopy theory.

Restating this observation, the category of cyclic sets encodes all of $S^1$-homotopy theory except for the information detected by the $S^1$-fixed points. A fundamental insight of Elmendorf [5] is that the homotopy theory of $G$-spaces is equivalent to the homotopy theory of appropriate diagrams of fixed-point information. See also Mandell and Scull [12] for a comprehensive modern discussion of this. This suggests that a natural avenue of attack is to consider a category consisting of a cyclic set appropriately coupled (via compatibility data) with a simplicial set to represent the information at the $S^1$ fixed points. Let $X$ be an $S^1$-space, and consider the following diagram:

\[
\begin{array}{ccc}
X \times S^1 \times E\mathcal{F} & \longrightarrow & X \times E\mathcal{F} \\
\downarrow & & \downarrow \\
X^{S^1} & & \allowbreak \\
\end{array}
\]

(1.1)

Here $E\mathcal{F}$ is the classifying space for the family of finite subgroups of $S^1$, the horizontal map is the inclusion and the vertical map is the projection. The associated pushout is weakly equivalent to $X$. This picture provides the inspiration for our construction. We think of the cyclic set as akin to $X \times E\mathcal{F}$, the simplicial set as $X^{S^1}$, and the compatibility data as the gluing along $X^{S^1} \times E\mathcal{F}$.

Given a simplicial set $A$ and a cyclic set $B$ we will describe the required compatibility in terms of a map $\nabla A \rightarrow B$, where $\nabla A$ is a homotopical cyclic approximation of $A$. Specifically, we construct a functor $\nabla : S \rightarrow S^c$ which has the property that there is a natural map $|\nabla A|_c \rightarrow |A|_c$ which is a weak equivalence upon passage to all fixed point sets for finite subgroups of $S^1$. The category $\mathcal{P}$ of compatible pairs is an instance of a more general construction.
Definition 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. The category $\mathcal{C}_F \mathcal{D}$ has

1. Objects specified by triples $(A, B, FA \to B)$ where $A$ is an object of $\mathcal{C}$ and $B$ is an object of $\mathcal{D}$.
2. Morphisms specified by maps $f_1 : A \to A'$ and $f_2 : B \to B'$ such that the following diagram commutes:

\[
\begin{array}{ccc}
FA & \longrightarrow & B \\
Ff_1 & \downarrow & \downarrow f_2 \\
FA' & \longrightarrow & B'.
\end{array}
\]

Remark 1.4. This is an example of a comma category [10].

Definition 1.5. The category $P$ is the comma category $S \leftarrow S_c$.

When there are model structures on $\mathcal{C}$ and $\mathcal{D}$, there is an induced model structure on $\mathcal{C}_F \mathcal{D}$ for suitable functors $F$.

Definition 1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is Reedy admissible if $F$ preserves colimits (e.g. $F$ is a left adjoint) and $F$ has the property that given a morphism $(A, B, FA \to B) \to (A', B', FA' \to B')$ in $\mathcal{C}_F \mathcal{D}$ such that $A \to A'$ is a trivial cofibration in $\mathcal{C}$ and $FA' \cup FA B \to B'$ is a trivial cofibration in $\mathcal{D}$ then $B \to B'$ is a weak equivalence in $\mathcal{D}$ (e.g. $F$ is a left Quillen functor).

Theorem 1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $F : \mathcal{C} \to \mathcal{D}$ be a Reedy admissible functor. Then $\mathcal{C}_F \mathcal{D}$ admits a model structure. A map $(A, B, FA \to B) \to (A', B', FA' \to B')$ is

1. a weak equivalence if $A \to A'$ is a weak equivalence in $\mathcal{C}$ and $B \to B'$ is a weak equivalence in $\mathcal{D}$,
2. a fibration if $A \to A'$ is a fibration in $\mathcal{C}$ and $B \to B'$ is a fibration in $\mathcal{D}$,
3. a cofibration if $A \to A'$ is a cofibration in $\mathcal{C}$ and $FA' \cup FA B \to B'$ is a cofibration in $\mathcal{D}$.

We use this theorem to obtain the model structure on $P$.

Lemma 1.8. The functor $\nabla : S \to S^c$ is Reedy admissible.

Corollary 1.9. There is a model structure on $\mathcal{P}$ inherited from the model structures on $S$ and $S^c$.

There is an adjunction specified by functors $L : \mathcal{P} \to \text{Top}^{S^1}$ and $R : \text{Top}^{S^1} \to \mathcal{P}$. We will provide more details about these functors later.

Definition 1.10. The functor $L : \mathcal{P} \to \text{Top}^{S^1}$ takes a triple $(A, B, \nabla A \to B)$, to the pushout in the diagram:

\[
\begin{array}{ccc}
|\nabla A|_c & \longrightarrow & |B|_c \\
\downarrow & & \downarrow \\
|A|_s & \longrightarrow & X.
\end{array}
\]
The functor $R : \text{Top}^{S^1} \to \mathcal{P}$ takes $X$ to the triple $(S(X^{S^1}), S_c(X), \nabla S(X^{S^1}) \to S_c(X))$. The map $\xi : \nabla S(X^{S^1}) \to S_c(X)$ is the adjoint of the composite
\begin{equation}
|\nabla S(X^{S^1})|_c \to |S(X^{S^1})|_s \to X^{S^1} \to X.
\end{equation}

Recall that there is a model structure on $\text{Top}^{S^1}$ given by defining a map $f : X \to Y$ to be a weak equivalence if $f^H : X^H \to Y^H$ is a weak equivalence for all $H \subset S^1$, a fibration if $f^H : X^H \to Y^H$ is a fibration for all $H \subset S^1$, and a cofibration if it has the left-lifting property with respect to acyclic fibrations [15].

Here is the main theorem of the paper.

**Theorem 1.2.** The functors $L$ and $R$ specify a Quillen equivalence between $\mathcal{P}$ with the model structure given by Theorem 1.1 and $\text{Top}^{S^1}$ with the model structure described above.

The problem of obtaining a discrete model for $S^1$-spaces was raised by Voevodsky in a 2002 e-mail to May [13].

The rest of the paper is organized as follows:

1. A brief review of simplicial and cyclic sets.
2. A review of the model structures on $S^1$ spaces.
3. Definition of $\nabla$ and demonstration that it is Reedy admissible.
4. The adjunction between $\mathcal{P}$ and $\text{Top}^{S^1}$.
5. Proof of Theorem 1.2.
6. Proof of Theorem 1.1.
7. Appendix: calculations from Spalinski’s thesis [14].

## 2. A REVIEW OF CYCLIC SETS

We give a very succinct review of cyclic sets. Good references for readers unfamiliar with the category are [15, 4, 7, 3]. A cyclic set can be regarded as a simplicial set with extra data, namely an action of $\Lambda_c$ on the $n$-simplices which is compatible with the face and degeneracy operators. Define the cyclic category $\Lambda^{op}$ to have the same objects as the category $\Delta^{op}$ and the same generating morphisms along with an extra degeneracy $s_{n+1} : [n] \to [n+1]$ and the “cyclic relations” $(d_0 s_{n+1})^{n+1} = \text{id}$. Define $t_n = d_0 s_{n+1} : [n] \to [n]$.

Every morphism in $\Lambda^{op}$ can be written as a composite $\phi = STD$ of a composite $S$ of degeneracy operators, a power $T$ of $t_n$ for some $n$, and a composite $D$ of boundary operators [15]. For further discussion of the properties of $\Lambda^{op}$ (e.g. $\Lambda^{op}$ is self dual) see [3] or [6].

Cyclic sets are contravariant functors from the category $\Lambda^{op}$ to sets. The category of cyclic sets will be denoted $\mathbf{S}^c$. As in the theory of simplicial sets, the represented cyclic sets $\Lambda[n] = \text{hom}_\Lambda(-, n)$ play an important role. The geometric realization of the underlying simplicial set of a cyclic set admits a natural $S^1$-action. The geometric realization, regarded as a functor from cyclic sets to $S^1$-spaces, will be denoted $| - |_c$. In particular, $|\Lambda[n]|_c \cong S^1 \times |\Delta[n]|$, with $S^1$ acting on the product by rotation on the first coordinate, where $\Delta[n]$ is the represented simplicial set with trivial action. By manipulation of coends one obtains $|X|_c = X \otimes_{\Lambda^{op}} |\Lambda|_c$. The adjoint to the realization is the “cyclic singular functor” $S_c$ defined to have $n$-simplices $\text{hom}_{\mathbf{S}^c}(|\Lambda[n]|_c, X)$ [4]. Here the cyclic structure is obtained by regarding $|\Lambda[n]|$ as homeomorphic to $S^1 \times |\Delta[n]|$, where the action of $t_n$ permutes the coordinates of a point in $|\Delta[n]|$ and rotates $S^1$ by $e^{2\pi i/(n+1)}$.

Now consider the subgroup $\mathbb{Z}/(r) \subset S^1$. Given a cyclic set, we can apply the subdivision functor $Sd_r$ to the underlying simplicial set [1]. This has a natural simplicial action of $\mathbb{Z}/(r)$ induced from the cyclic structure, and so we can define a composite functor $\Phi_r$ which takes a cyclic set $X$ to the simplicial set $(Sd_r X)^{\mathbb{Z}/(r)}$. There is a homeomorphism $|\Phi_r(X)|_s \cong |X|^{\mathbb{Z}/(r)}$. By Freyd’s adjoint functor theorem, $\Phi_r$ has an adjoint $\Psi_r$. It is useful to describe $\Psi_r$ more concretely and so we reproduce calculations of Spalinski [14] in the appendix.

The functors $\Phi_r$ are used to prove the following result.
Lemma 2.1 (Spalinski [15]). The counit of the adjunction between $|−|_c$ and $S_c(−)$ induces weak equivalences on passage to fixed point spaces for finite subgroups of $S^1$.

3. A review of model structures on simplicial sets, cyclic sets, and $\text{Top}^{S^1}$

We briefly review the model structures on $\text{S}$, $\text{S}^c$, and $\text{Top}^{S^1}$.

Theorem 3.1. There is a model structure on simplicial sets in which a map is

(1) a fibration if it is a Kan fibration,
(2) a weak equivalence if the induced map on passage to geometric realization is an equivalence,
(3) a cofibration if it is an injection.

Theorem 3.2. For any family $\mathcal{F}$ of subgroups of $S^1$, there is a model structure on $\text{Top}^{S^1}$ in which a map $f : X \to Y$ is

(1) a fibration if the induced maps $f^H : X^H \to Y^H$ are fibrations for all $H \in \mathcal{F}$,
(2) a weak equivalence if the induced maps $f^H : X^H \to Y^H$ are weak equivalences for all $H \in \mathcal{F}$,
(3) a cofibration if it has the left-lifting property with respect to trivial fibrations.

In particular, this holds when $\mathcal{F}$ is the family of all subgroups of $S^1$ and when $\mathcal{F}$ is the family of all finite subgroups of $S^1$.

Theorem 3.3 (Spalinski [15]). There is a model structure on cyclic sets in which a map is

(1) a fibration if $\Phi_r(f)$ is a fibration of simplicial sets for all $r \geq 1$,
(2) a weak equivalence if $\Phi_r(f)$ is a weak equivalence of simplicial sets for all $r \geq 1$,
(3) a cofibration if it has the left-lifting property with respect to trivial fibrations.

Remark 3.1. Spalinski [15] also shows that cofibrations can be characterized as retracts of transfinite composites of pushouts of coproducts of maps $\Psi_r(\delta(k)) \to \Phi_r(k)$.

The homotopy theory of cyclic sets is the same as the homotopy theory of $\text{Top}^{S^1}$ with respect to the family of finite subgroups of $S^1$.

Theorem 3.4 (Spalinski [15]). The cyclic realization functor and the cyclic singular functor induce a Quillen equivalence between $\text{Top}^{S^1}$ with the model structure in which $\mathcal{F}$ is the family of finite subgroups of $S^1$ and the category of cyclic sets with the model structure described above.

4. The functor $\nabla$

We shall construct a functor $\nabla : \text{S} \to \text{S}^c$ such that $|X|^H \simeq |\nabla X|^H$ for all finite $H \subset S^1$. One's first guess is that $\nabla$ ought to be the left adjoint to the forgetful functor which assigns to a cyclic set its underlying simplicial set (Kan extension). However, this is the free cyclic set associated to the underlying simplicial set [2], and does not have the properties we need.

Another obvious guess is to define $\nabla X = S_c(|X|)$. By Lemma 2.1, we know the counit provides a map $|S_c(|X|)|_c \to |X|_c$ which is an equivalence on passage to all finite subgroups. Unfortunately, as a composite of a left adjoint and a right adjoint, this functor has rather unpleasant properties. For instance, it preserves neither colimits nor limits.

We want a functor from simplicial sets to cyclic sets which is a left adjoint and so preserves colimits. All such functors arise from cosimplicial cyclic sets. In fact, there is an equivalence between the category of cosimplicial objects in $\mathcal{C}$ and adjunctions from simplicial sets to $\mathcal{C}$ for categories $\mathcal{C}$ with all small colimits [9, 3.1.5].

Definition 4.1. Set $\nabla_n = S_c(|\Delta[n]|)$. Then $\nabla_*$ is a cosimplicial cyclic set and so we can define a functor $\nabla : \text{S} \to \text{S}^c$ by letting $\nabla X = X \otimes_{\Delta^\text{op}} \nabla_*$. The functor $\nabla$ has the right adjoint $A : \text{S}^c \to \text{S}$ specified by $A(Y)_n = \text{hom}_{\text{S}^c}(\nabla_n, Y)$. 
We will repeatedly use the following result, which we quote from [11].

**Lemma 4.2.** The functor $(-)^G$ on based $G$-spaces preserves pushouts of diagrams one leg of which is a closed inclusion.

**Lemma 4.3.** There is a natural map $ζ : |∇|c \rightarrow |X|_s$ which induces weak equivalences on passage to fixed point subspaces for all finite subgroups of $S^1$.

**Proof.** By construction, the counit map $γ_n : |∇|_c \rightarrow |Δ[n]|_s$ induces weak equivalences on passage to all fixed point subspaces for finite subgroups of $S^1$. Define $ζ$ to be the following map:

$$|∇X|_c = ((X \otimes_{Δop} ∇) \otimes_{Δop} |Δ|_s) = X \otimes_{Δop} |∇|_c \rightarrow X \otimes_{Δop} |Δ|_s = |X|_s.$$  

Both the domain and the codomain can be regarded as a succession of pushouts with one leg a cofibration. Therefore the fixed-point functor commutes with each of these coends by Lemma 4.2 and so $ζ$ induces weak equivalences on passage to fixed subspaces.

**Remark 4.4.** The essential aspect of the $∇_n$ is that they come equipped with maps from $|∇|_c$ to $|Δ[n]|_s$ which induce weak equivalences on passage to fixed point subspaces for all finite subgroups of $S^1$. Any other cosimplicial cyclic set which had this property would suffice for our purposes. One might prefer a functorial approximation of $∇_1$. Alternatively, as the singular construction we give is rather bloated, we expect that other explicit models of $∇_n$ may well be preferable for specific applications.

To use Theorem 1.1 to show that there is a model structure on $P$, we must verify that $∇$ is Reedy admissible. By construction, $∇$ is a left adjoint and so preserves colimits.

**Lemma 4.5.** Given a map $(A, B, ∇A \rightarrow B) \rightarrow (A', B', ∇A' \rightarrow B')$ in $P$ such that $A \rightarrow A'$ is a trivial cofibration and $∇A' \cup_{ΓA} B \rightarrow B'$ is a trivial cofibration, the map $B \rightarrow B'$ is a weak equivalence. Therefore $∇$ is Reedy admissible.

**Proof.** Since $B \rightarrow B'$ is the composite

$$(4.6) \quad B \rightarrow ∇A' \cup_{ΓA} B \rightarrow B'$$

and $∇A' \cup_{ΓA} B \rightarrow B'$ is a weak equivalence by hypothesis, it suffices to show that $B \rightarrow ∇A' \cup_{ΓA} B$ is a weak equivalence. This is equivalent to showing that $|B|_c^H \rightarrow |∇A' \cup_{ΓA} B|_c^H$ is a weak equivalence of spaces for all finite $H \subset S^1$. Since geometric realization is a colimit, $|∇A' \cup_{ΓA} B|_c^H$ is isomorphic to $(|∇A'|_c \cup |ΓA|_c^H |B|_c^H$. Since $A \rightarrow A'$ is a cofibration of simplicial sets and hence an inclusion, $|ΓA|_c \rightarrow |ΓA'|_c$ is a closed inclusion. Therefore by Lemma 4.2 the fixed-point functor commutes with the pushout, and so $|(∇A'|_c \cup |ΓA|_c^H |B|_c^H$ is equivalent to $|∇A'|_c^H \cup |ΓA|_c^H |B|_c^H$. Since $|ΓA|_c^H \rightarrow |∇A'|_c^H$ is a trivial cofibration when $A \rightarrow A'$ is a trivial cofibration, $|B|_c^H \rightarrow |∇A'|_c^H \cup |ΓA|_c^H |B|_c^H$ is the pushout of a trivial cofibration and so is itself a trivial cofibration.

**Corollary 4.7.** There is a model structure on $P$ in which a map is

1. a weak equivalence if $A \rightarrow A'$ is a weak equivalence of simplicial sets and $B \rightarrow B'$ is a weak equivalence of cyclic sets,
2. a fibration if $A \rightarrow A'$ is a fibration of simplicial sets and $B \rightarrow B'$ is a fibration of cyclic sets,
3. a cofibration if $A \rightarrow A'$ is a cofibration of simplicial sets and $∇A' \cup_{ΓA} B \rightarrow B'$ is a cofibration of cyclic sets.

**Proof.** This follows immediately from Lemma 4.5 and Theorem 1.1.
5. The adjunction between $\mathcal{P}$ and $\text{Top}^{S^1}$

There are natural functors from $\mathcal{P}$ to $\text{Top}^{S^1}$ and from $\text{Top}^{S^1}$ to $\mathcal{P}$ defined as follows.

**Lemma 5.1.** Given a morphism in $\mathcal{P}$ from $(A, B, \nabla A \to B)$ to $(A', B', \nabla A' \to B')$, the induced diagram

\[
\begin{array}{ccc}
|\nabla A|_c & \xrightarrow{\zeta} & |A|_s \\
\downarrow & & \downarrow \\
|\nabla A'|_c & \xrightarrow{\zeta} & |A'|_s
\end{array}
\]  

is commutative.

**Proof.** Rewriting the diagram as follows

\[
\begin{array}{ccc}
A \otimes_{\Delta_{op}} |\nabla| & \longrightarrow & A \otimes_{\Delta_{op}} |\Delta| \\
\downarrow & & \downarrow \\
A' \otimes_{\Delta_{op}} |\nabla| & \longrightarrow & A' \otimes_{\Delta_{op}} |\Delta|
\end{array}
\]

makes the commutativity apparent. \qed

**Definition 5.4.** The functor $L: \mathcal{P} \to \text{Top}^{S^1}$ takes a triple $(A, B, \nabla A \to B)$, to the pushout in the diagram:

\[
\begin{array}{ccc}
|\nabla A|_c & \longrightarrow & |B|_c \\
\zeta \downarrow & & \downarrow \\
|A|_s & \longrightarrow & X.
\end{array}
\]

A morphism $(A, B, \nabla A \to B) \to (A', B', \nabla A' \to B')$ induces a commutative diagram:

\[
\begin{array}{ccc}
|A|_s & \longleftarrow & |\nabla A|_c & \longrightarrow & |B|_c \\
\downarrow & & \downarrow & & \downarrow \\
|A'|_s & \longleftarrow & |\nabla A'|_c & \longrightarrow & |B'|_c.
\end{array}
\]

The lefthand square commutes by the preceding lemma and the righthand square commutes because of the definition of a morphism. Therefore there is an induced map of pushouts, which specifies the action of $L$ on morphisms.

**Lemma 5.7.** A morphism $X \to Y$ in $\text{Top}^{S^1}$ induces a commutative diagram:

\[
\begin{array}{ccc}
\nabla S(X^{S^1}) & \xrightarrow{\xi} & S_c(X) \\
\downarrow & & \downarrow \\
\nabla S(Y^{S^1}) & \xrightarrow{\xi} & S_c(Y).
\end{array}
\]

**Proof.** This diagram commutes if and only if the adjoint diagram

\[
\begin{array}{ccc}
|\nabla S(X^{S^1})|_c & \longrightarrow & X \\
\downarrow & & \downarrow \\
|\nabla S(Y^{S^1})|_c & \longrightarrow & Y
\end{array}
\]
commutes. The latter diagram can be written as the composite:

\[
\begin{array}{c}
|\nabla S(X^S^1)|_c \longrightarrow |S(X^S^1)|_s \longrightarrow X^S^1 \longrightarrow X \\
\downarrow \quad \downarrow \quad \downarrow \\
|\nabla S(Y^S^1)|_c \longrightarrow |S(Y^S^1)|_s \longrightarrow Y^S^1 \longrightarrow Y.
\end{array}
\]

(5.10)

Here the lefthand square commutes by Lemma 5.1. The middle square commutes by the naturality of the counit. The righthand square commutes trivially. Therefore the original diagram commutes. \(\square\)

**Definition 5.11.** The functor \(R : \text{Top}^{S^1} \to P\) takes \(X\) to the triple \((S(X^{S^1}), S_c(X), \nabla S(X^{S^1}) \to S_c(X))\). The map \(\nabla S(X^{S^1}) \to S_c(X)\) is the adjoint of the composite:

\[
|\nabla S(X^S^1)|_c \rightarrow |S(X^S^1)|_s \rightarrow X^{S^1} \rightarrow X
\]

(5.12)

A map \(X \to Y\) in \(\text{Top}^{S^1}\) induces maps \(S(X^{S^1}) \rightarrow S(Y^{S^1})\) and \(S_c(X) \to S_c(Y)\) by functoriality. By the preceding lemma, these maps fit into a commutative diagram:

\[
\begin{array}{c}
\nabla S(X^{S^1}) \longrightarrow S_c(X) \\
\downarrow \\
\nabla S(Y^{S^1}) \longrightarrow S_c(Y).
\end{array}
\]

(5.13)

We think of \(L\) as a realization functor and \(R\) as a singular functor.

**Proposition 5.14.** The functors

\[
L : P \rightleftarrows \text{Top}^{S^1} : R
\]

form an adjoint pair.

*Proof.* Given a map \(|A|_s \cup_{\nabla A}|B|_c \rightarrow X\), we must show that there is a unique corresponding map \((A, B, \nabla A \to B) \rightarrow (S(X^{S^1}), S_c(X), \nabla S(X^{S^1}) \to S_c(X))\). We clearly get unique maps \(A \to S(X)\) and \(B \to S_c(X)\) as adjoints to the maps \(|A|_s \to X\) and \(|B|_c \to X\) induced by the map from the pushout. It suffices to verify that the compatibility imposed by the pushout square is equivalent to the compatibility condition for a morphism in \(P\).

So consider the square induced by our adjoint maps:

\[
\nabla A \quad \longrightarrow \quad B \\
\downarrow \\
\nabla S(X^{S^1}) \quad \longrightarrow \quad S_c(X).
\]

(*)

We must show that it commutes. Now, the map \(|A|_s \cup_{\nabla A}|B|_c \rightarrow X\) provides us with a commuting square:

\[
\begin{array}{c}
|\nabla A|_c \longrightarrow |B|_c \\
\downarrow \\
|A|_s \longrightarrow X.
\end{array}
\]

(5.16)

Such squares are in bijective correspondence with commuting squares:

\[
\begin{array}{c}
\nabla A \quad \longrightarrow \quad B \\
\downarrow \\
S_c(|A|_s) \quad \longrightarrow \quad S_c(X).
\end{array}
\]

(**)
The two composites $\nabla A \rightarrow B \rightarrow S_c(X)$ are the same. Therefore to verify the correspondence of the compatibility conditions it suffices to show that the maps

$$(5.17) \quad (\ast) \quad \nabla A \rightarrow \nabla S(X^{S^1}) \rightarrow S_c(X) \quad (\ast\ast) \quad \nabla A \rightarrow S_c(|A|_s) \rightarrow S_c(X)$$

are identical. We do this by explicitly chasing elements around these two paths. Start with the map $g: |A|_s \rightarrow X$. Regarding $|A|_s$ as the coend $A \otimes_{\Delta^p} |\Delta|$, we view $g$ as taking $(a, \delta)$ to $g(a, \delta)$ and its adjoint as taking $a$ to the map $(\delta \rightarrow g(a, \delta))$.

So let’s unwind the two maps. The map

$$(\ast\ast) \quad \nabla A \rightarrow S_c(|A|_s) \rightarrow S_c(X)$$

is the composite

$$(5.18) \quad \nabla A \rightarrow S_c(|A|_s) \rightarrow S_c(|A|_s) \rightarrow S_c(X).$$

The first constituent map is adjoint to the map $|\nabla A|_c \rightarrow |A|_s$ which we defined as

$$(5.19) \quad A \otimes_{\Delta^p} |\nabla|_c \rightarrow A \otimes_{\Delta^p} |\Delta|_s$$

via the map $\gamma: |\nabla|_c \rightarrow |\Delta|_s$. In order to calculate the adjoint map, we write the first coend as

$$(5.20) \quad (A \otimes_{\Delta^p} \nabla) \otimes_{\Delta^p} |A|_c \rightarrow A \otimes_{\Delta^p} |\Delta|_s$$

where the map takes $((a, \nu), \lambda)$ to $(a, \gamma(\nu, \lambda))$. Then the adjoint is the map

$$(5.21) \quad \lambda \rightarrow ((a, \nu) \rightarrow (a, \gamma(\nu, \lambda))).$$

Next, we have the map $S_c(|A|_s) \rightarrow S_c(X)$ which is obtained by applying $S_c$ to the map $g: |A|_s \rightarrow X$. That is, the induced map takes the map $\lambda \rightarrow (a, \delta)$ to the map $\lambda \rightarrow g(a, \delta)$. Finally, the composite is

$$(\ast\ast) \quad (a, \nu) \rightarrow (\lambda \rightarrow g(a, \gamma(\nu, \lambda)))$$

On the other hand, we can decompose the map

$$(\ast) \quad \nabla A \rightarrow \nabla S(X^{S^1}) \rightarrow S_c(X)$$

as the composite

$$(5.22) \quad \nabla A \rightarrow \nabla S(X^{S^1}) \rightarrow \nabla S(X^{S^1}) \rightarrow S_c(X).$$

The first constituent map is obtained by applying $\nabla$ to the map $A \rightarrow S(X^{S^1})$ adjoint to $g$. Explicitly, this is

$$(5.23) \quad (a, \nu) \rightarrow ((a \rightarrow (\delta \rightarrow g(a, \delta))), \nu).$$

The second map is the adjoint to the map $|\nabla S(X^{S^1})|_c \rightarrow X$, which decomposes as the composite

$$(5.24) \quad |\nabla S(X^{S^1})|_c \rightarrow |S(X^{S^1})|_s \rightarrow X^{S^1} \rightarrow X$$

that takes $(h, \nu, \lambda)$ to $(h, \gamma(\nu, \lambda))$ and then to $h(\gamma(\nu, \lambda))$. The adjoint can be written as:

$$(5.25) \quad (h, \nu) \rightarrow (\lambda \rightarrow h(\gamma(\nu, \lambda))).$$

Composing, we have

$$(\ast) \quad (a, \nu) \rightarrow (\lambda \rightarrow g(a, \gamma(\nu, \lambda))).$$

$\square$
6. Proof of Theorem 1.2

The functors $L$ and $R$ are compatible with our model structures.

**Lemma 6.1.** Let $\mathcal{P}$ have the model structure described in Corollary 4.7 and $\text{Top}^{S^1}$ have the model structure generated by the family of all subgroups of $S^1$. Then the adjoint functors $L$ and $R$ form a Quillen adjunction.

**Proof.** It suffices to show that $R$ preserves fibrations and trivial fibrations. If $X \rightarrow Y$ is a fibration or a trivial fibration, then $S(X) \rightarrow S(Y)$ and $S_c(X) \rightarrow S_c(Y)$ are as well since $S_c(-)$ and $S(-)$ are themselves right Quillen adjoints.

**Remark 6.2.** In fact, $R$ preserves weak equivalences since both $S(-)$ and $S_c(-)$ preserve weak equivalences.

Now, one potential problem with this model for $\text{Top}^{S^1}$ is that while cyclic sets don’t capture “useful” data at the $S^1$ fixed points, they do have some information there which might corrupt the data encoded in the simplicial set. In fact, it isn’t in general the case that the counit $|S(X^{S^1})| \cup |\nabla S(X^{S^1})| \rightarrow |S_c(X)|$ is a weak equivalence of $S^1$-spaces. However, the following lemmas show that this map is an equivalence once we pass to cofibrant approximations. Observe that $(A, B, \nabla A \rightarrow B)$ cofibrant implies that $A$ is cofibrant and $\nabla A \rightarrow B$ is a cofibration.

**Lemma 6.3.** If $X \rightarrow Y$ is cofibration of cyclic sets, then the induced map $|X|_{c}^{S^1} \rightarrow |Y|_{c}^{S^1}$ is a homeomorphism.

**Proof.** As noted previously, a cofibration of cyclic sets is a retract of a relative cell complex with respect to the family $\Psi_s(\partial \Delta[k]) \rightarrow \Psi_s(\Delta[k])$. A retract of a homeomorphism is a homeomorphism. Thus, it will suffice to observe that the domains and codomains of these generating cofibrations have no $S^1$-fixed points, as the fixed-point functor commutes with these pushouts after passage to cyclic realization by Lemma 4.2. But this is true by an explicit calculation [14] which we reproduce in the appendix.

**Corollary 6.4.** If $X$ is a cofibrant cyclic set, then $|X|_{c}^{S^1}$ is empty.

Lemma 6.3 enables us to show that for cofibrant objects in $\mathcal{P}$ the gluing behaves properly.

**Lemma 6.5.** Let $(A, B, \nabla A \rightarrow B)$ be a cofibrant object in $\mathcal{P}$ and define $Z = |A|_s \cup |\nabla A|_c \rightarrow |B|_c$. Then $Z_{c}^{S^1} \simeq |A|_s$ and for all finite $H \subset S^1$, $Z^H \simeq |B|^H$.

**Proof.** Observe that by Lemma 4.2, passage to fixed points commutes with the pushout since $\nabla A \rightarrow B$ is a cofibration. First consider the $S^1$-fixed points. We have a map $|A|_s \rightarrow Z_{c}^{S^1}$ induced by the pushout. Since $\nabla A \rightarrow B$ is a cofibration, Lemma 6.3 tells us that $|\nabla A|_{c}^{S^1} \simeq |B|_{c}^{S^1}$. But this immediately implies that $|(A|_s \cup |\nabla A|_c \rightarrow |B|_c)^{S^1} \simeq |A|_s$. Now consider a finite subgroup $H \subset S^1$. Since $|\nabla A|_c^H \simeq |A|_s^H$ and $|\nabla A|_c^H \rightarrow |B|_c^H$ is a cofibration, $|B|_c^H \rightarrow Z^H$ is the pushout along a cofibration of a weak equivalence. Therefore $|B|_c^H \rightarrow Z^H$ is a weak equivalence since $\nabla$ is proper.

**Theorem 1.2.** The functors $L$ and $R$ specify a Quillen equivalence between $\mathcal{P}$ with the model structure given by Theorem 1.1 and $\text{Top}^{S^1}$ with the model structure in which $\mathcal{F}$ is the family of all subgroups of $S^1$.

**Proof.** We must show that given a cofibrant object $(A, B, \nabla A \rightarrow B)$ in $\mathcal{P}$ and a fibrant $S^1$-space $X$, a map $(A, B, \nabla A \rightarrow B) \rightarrow RX$ is a weak equivalence if and only if the adjoint $L(A, B, \nabla A \rightarrow B) \rightarrow X$ is a weak equivalence. Writing out the functors, we need to show that

$$\begin{align*}
(A, B, \nabla A \rightarrow B) & \rightarrow (S(X^{S^1}), S_c(X), \nabla S(X^{S^1}) \rightarrow S_c(X))
\end{align*}$$

is a weak equivalence if and only if

$$\begin{align*}
|A|_s \cup |\nabla A|_c \rightarrow X
\end{align*}$$
is a weak equivalence.

So assume that \(|A|_s \cup |\nabla A|_c \mid |B|_c \to X\) is a weak equivalence. This implies that the induced map

\begin{equation}
(6.8) \quad ([A|_s \cup |\nabla A|_c \mid |B|_c)^{S^1} \to X^{S^1}
\end{equation}

is a weak equivalence. Furthermore, by Lemma 6.5 the map

\begin{equation}
(6.9) \quad |A|_s \to ([A|_s \cup |\nabla A|_c \mid |B|_c)^{S^1}
\end{equation}

is a weak equivalence, and so the composition is a weak equivalence. This implies that the adjoint \(A \to S(X^{S^1})\) is a weak equivalence of simplicial sets. Similarly, for any finite \(H \subset S^1\) the assumption implies that the induced map

\begin{equation}
(6.10) \quad ([A|_s \cup |\nabla A|_c \mid |B|_c)_H \to X^H
\end{equation}

is a weak equivalence and Lemma 6.5 tells us that

\begin{equation}
(6.11) \quad |B|_c \to ([A|_s \cup |\nabla A|_c \mid |B|_c)_H
\end{equation}

is a weak equivalence. Therefore the composite is a weak equivalence, and this implies that the adjoint \(B \to S_c(X)\) is a weak equivalence of cyclic sets.

Conversely, assume that the adjoint

\begin{equation}
(6.12) \quad (A, B, \nabla A \to B) \to (S(X^{S^1}), S_c(X), \nabla S(X^{S^1}) \to S_c(X))
\end{equation}

is a weak equivalence. This implies that \(|A|_s \to X^{S^1}\) is a weak equivalence of simplicial sets and that \(|B|_c \to X\) is a weak equivalence of cyclic sets. The previous discussion and the “two out of three” property for weak equivalences now imply that \(|A|_s \cup |\nabla A|_c \mid |B|_c \to X\) is a weak equivalence. \(\square\)

7. Proof of Theorem 1.1

The proof that \(\mathcal{C}_F\mathcal{D}\) inherits a model structure from model structures on \(\mathcal{C}\) and \(\mathcal{D}\) when \(F\) is Reedy admissible uses the standard technique for lifting model structures to diagram categories indexed by Reedy categories [9], [8].

**Theorem 1.1.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be model categories and \(F : \mathcal{C} \to \mathcal{D}\) be a Reedy admissible functor. Then \(\mathcal{C}_F\mathcal{D}\) admits a model structure. A map \((A, B, FA \to B) \to (A', B', FA' \to B')\) is

1. a weak equivalences if \(A \to A'\) is a weak equivalence in \(\mathcal{C}\) and \(B \to B'\) is a weak equivalence in \(\mathcal{D}\),
2. a fibration if \(A \to A'\) is a fibration in \(\mathcal{C}\) and \(B \to B'\) is a fibration in \(\mathcal{D}\),
3. a cofibration if \(A \to A'\) is a cofibration in \(\mathcal{C}\) and \(FA' \cup_{FA} B \to B'\) is a cofibration in \(\mathcal{D}\).

**Proof.**

1. \(\mathcal{C}_F\mathcal{D}\) has all small limits and colimits since \(F\) preserves colimits and \(\mathcal{C}\) and \(\mathcal{D}\) have all small limits and colimits.
2. Weak equivalences satisfy the “two out of three” axiom since they do in \(\mathcal{C}\) and \(\mathcal{D}\).
3. It is clear that the weak equivalences and fibrations are closed under retracts, since they are defined levelwise. We need to verify that retracts of cofibrations are cofibrations. The commutative diagram:

\begin{equation}
(7.1) \quad \begin{array}{ccc}
FA & \rightarrow & B' \\
\downarrow & & \downarrow \\
FA' & \rightarrow & B' \\
\end{array} \quad \begin{array}{ccc}
FC & \rightarrow & D \\
\downarrow & & \downarrow \\
FC' & \rightarrow & D' \\
\end{array} \quad \begin{array}{ccc}
FA & \rightarrow & B \\
\downarrow & & \downarrow \\
FA' & \rightarrow & B' \\
\end{array}
\end{equation}

implies that \(FC' \cup_{FC} D \to D'\) is a retract of \(FA' \cup_{FA} B \to B'\). Since \(FA' \cup_{FA} B \to B'\) is a cofibration, we know from the model structure on \(\mathcal{D}\) that \(FC' \cup_{FC} D \to D'\) is itself a cofibration in \(\mathcal{D}\). Moreover, it is clear that \(C \to C'\) is a cofibration in \(\mathcal{C}\) because it is a retract of \(A \to A'\).
(4) Now we need to verify the factorization results. Assume we have a map \((A, B, FA \to B) \to (A', B', FA' \to B')\). We will construct a factorization of this map into a trivial cofibration and a fibration (the other case is analogous). Consider the following diagram:

\[
\begin{array}{ccc}
\text{FA} & \longrightarrow & B \\
\downarrow & & \downarrow \\
\text{FA'} & \longrightarrow & B'.
\end{array}
\]

We employ the standard latching space argument. Choose a factorization of \(A \to A'\) as \(A \to C \to A'\) where \(A \to C\) is a trivial cofibration in \(C\) and \(C \to A'\) is a fibration. This yields a factorization \(FA \to FC \to FA'\). So now we have the following diagram:

\[
\begin{array}{ccc}
\text{FA} & \longrightarrow & B \\
\downarrow & & \downarrow \\
\text{FC} & \longrightarrow & ? \\
\downarrow & & \downarrow \\
\text{FA'} & \longrightarrow & B'.
\end{array}
\]

To complete the diagram choose a factorization of \(FC \cup FA B \to B'\) as

\[
(7.4)
\]

where \(C \cup_A B \to C'\) is a trivial cofibration and \(C' \to B'\) is a fibration, and then put \(C'\) in for the \(?\).

By the assumption on \(F\), \(B \to C'\) is a weak equivalence. This yields the factorization

\[
(7.5)
\]

in which the first arrow is a trivial cofibration and the second a fibration.

(5) Finally, we must verify the lifting properties. Assume we have a trivial cofibration and a fibration (the other case is analogous). The lifting problem

\[
(7.6)
\]

splits into the following interlocked lifting problems:

\[
(7.7)
\]

First, take a lift \(X \to B\) in the lefthand diagram using the model structure on \(\mathcal{C}\). Now consider the diagram:

\[
(7.8)
\]
Here the map $FX \cup F_A A' \to B'$ is built using the map $FX \to FB$ obtained from the lift. Take a lift $X' \to B'$ in this diagram using the model structure in $\mathcal{D}$. Together, these two lifts provide the desired lifting.

Remark 7.9. There is a dual version of this result for categories with objects $(A, B, A \to GB)$ in which $G$ is a co-Reedy admissible functor. That is, $G$ preserves limits and satisfies an appropriate pullback condition.

Lemma 7.10. If $\mathcal{C}$ and $\mathcal{D}$ are left proper and $F$ is Reedy admissible, then $\mathcal{C}_F \mathcal{D}$ is left proper. If $\mathcal{C}$ and $\mathcal{D}$ are right proper and $F$ is a left Quillen functor, then $\mathcal{C}_F \mathcal{D}$ is right proper.

Proof. The first assertion follows since fibrations, weak equivalences, and pullbacks are defined levelwise. For the second assertion, we need that $(A, B, F A \to B)$ a cofibration implies that $B \to B'$ is a cofibration. If $F$ is a left Quillen functor, this follows from [8, 15.3.11]. □

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Appendix A. Calculations from Spalinski’s thesis

We reproduce several calculations which appeared in Spalinski’s thesis [14] but not in the paper based on the thesis [15].

A.1. Explicit calculation of $\Psi_r$. We need to calculate $\Psi_r(\Delta[k])$. Recall that $\Psi_r$ is the adjoint to $\Phi_r$, where $\Phi_r(X) = (Sd_r X)^{Z(\mathcal{C})}$. It is sufficient to find a cyclic set $A$ such that there is a natural equivalence

$$(A.1) \quad \text{hom}_{\mathcal{C}}(A, X) \to \text{hom}_{\mathcal{C}}(\Delta[k], \Phi_r(X)).$$

We know that there is an equivalence

$$(A.2) \quad \text{hom}_{\mathcal{C}}(\Delta[k], \Phi_r(X)) \to \Phi_r(X)_k$$

given by $f \mapsto f(\iota_k)$. So it will suffice to exhibit a cyclic set $A$ such that there is a natural equivalence

$$(A.3) \quad \text{hom}_{\mathcal{C}}(A, X) \to \Phi_r(X)_k.$$  

There is an action of $\mathbb{Z}/(n + 1)$ on $|\Lambda[n]|_c$ for $n \geq 1$. By the Yoneda lemma, each map $\Lambda[n] \to \Lambda[n]$ is of the form $\text{hom}_{\Lambda \to}(\phi, -)$ for some $\phi : [n] \to [n] \in \Lambda^{op}$. The map corresponding to $t_{n+1}$ has order $n + 1$. This provides the action of $\mathbb{Z}/(n + 1)$, and we refer to the generator of this action as $\alpha$. This action induces an action of $\mathbb{Z}/(n + 1)$ on $|\Lambda[n]|_c$.

Definition A.4. If $k$ divides $n + 1$, let $\Lambda[n \mid k]$ denote the orbit space of $\Lambda[n]$ with respect to the action of the subgroup of $\mathbb{Z}/(n + 1)$ generated by $\alpha^k$.

Proposition A.5. The map

$$(A.6) \quad \text{hom}_{\mathcal{C}}(\Lambda[r(k + 1) - 1 \mid k + 1], X) \to \Phi_r(X)_k$$

given by $f \mapsto f(\iota_{r(k+1)-1})$ is a bijection and so $\Psi_r(\Delta[n]) = \Lambda[r(n + 1) - 1 \mid n + 1]$.
Proof. First note that the image of $\gamma$ is actually contained in the above fixed point set:

$$f^{k+1}_{r(k+1)} \cdot \gamma(f) = f^{k+1}_{r(k+1)} \cdot f([r_{r(k+1)}]) = f([r_{r(k+1)}]) = f([r_{r(k+1)}]) = \gamma(f).$$

Next, observe that $\gamma$ is onto. Take $x \in X^H_{r(k+1)-1}$ and consider the map:

$$f : \Lambda[r(k+1) - 1] \to X, \quad t_{r(k+1)-1} \mapsto x.$$

Note that $\text{Im} f \subseteq X^H_{r(k+1)-1}$. Let $z = STD[t_{r(k+1)}] \in \Lambda[r(k+1) - 1]$. We need to show that $f(\alpha_{k+1}z) = f(z)$. We have:

$$f(\alpha_{k+1}z) = f(\alpha_{k+1}STD[t_{r(k+1)}]) = f(STD^{k+1}_{r(k+1)}[t_{r(k+1)}]) = STD^{k+1}_{r(k+1)}f([t_{r(k+1)}]) = STD^{k+1}_{r(k+1)}x = STDx = STDf([t_{r(k+1)}]) = f(z).$$

Hence $f$ factors as:

$$\Lambda[r(k+1) - 1] \to \Lambda[r(k+1) - 1]/\alpha^{k+1} \to X.$$  

Here the first map is the quotient map and the second map is $\bar{f}$. By construction $\gamma(\bar{f}) = x$. Finally, we need to check that $\gamma$ is injective. Suppose that $\gamma(f) = \gamma(g)$. Then $f([r_{r(k+1)}]) = g([r_{r(k+1)}])$. Since $[r_{r(k+1)}]$ generates $\Lambda[r(k+1) - 1]/\alpha^{k+1}$, $f = g$. □

A.2. Fixed points of $\Psi_t(\Delta[n])$. The explicit description of $\Psi_t(\Delta[n])$ makes it easy to calculate its $S^1$ fixed points.

**Proposition A.9.** For $k \mid (n + 1)$, $|\Lambda[n \mid k]|_{S^1}$ is empty.

**Proof.** Let $p \in (\Delta[n \times S^1])$. Then we have:

$$p = (x_0, x_1, \ldots, x_n, t), \quad x_i \geq 0, \quad \sum_{i=0}^{n} x_i = 1, \quad t \in S^1.$$  

Let $\sigma = (0, 1, \ldots, n)$, $\tau = \sigma^{-k}$, and $\gamma = e^{2\pi i / n + 1}$. The action of $\alpha^k$ on $(\Delta[n \times S^1])$ is given by

$$\alpha^k(x_0, x_1, \ldots, x_n, t) = (x_{\tau(0)}, x_{\tau(1)}, \ldots, x_{\tau(n)}, \gamma t).$$

Since $S^1$ is infinite and each orbit of $\alpha^k$ has only finitely many points,

$$((\Delta[n \times S^1]) / \alpha^k)^{S^1} = \emptyset.$$  

□

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