AdS–Flows and Weyl Gravity

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Abstract

An analogy is noted between the RG flow equations in 4–dimensional gauge theory, as derived from the AdS/CFT correspondence, and the RG flow equations in 4–dimensional field theory coupled to a particular limit of Weyl supergravity. This suggests a possible theory of dynamical 3-branes with fluctuating 4–dimensional conformal factor. The argument involves a map from flows in 4-dimensional gauge theories to flows in a class of 2-dimensional sigma models.
1. Introduction

According to recent conjectures, $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang Mills theory with 't Hooft coupling

$$\lambda = g_{YM}^2 N,$$

as well as various other conformally invariant 4$d$ gauge theories have strong coupling descriptions in terms of type IIB superstring theory on $AdS_5$ times various Einstein manifolds $\mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H}$. String loop corrections are proportional to $1/N^2$, while $\alpha'$-corrections are proportional to $1/\sqrt{\lambda}$. In particular, the strong-coupling limit $\lambda \to \infty$ can be investigated in the supergravity approximation.

This dual description has been used to study RG flows in 4$d$ gauge theory in the large-$N$, large-$\lambda$ limit, based on the interpretation of the radial coordinate of $AdS_5$ as the scale of the 4$d$ theory $\mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H}$. The type IIB supergravity equations of motion then become RG flow equations.

Here it is first demonstrated how - at least in the vicinity of fixed points - these 4$d$ flows can be related to flows in 2$d$ sigma models with 5$d$ target space and Ramond-Ramond backgrounds. Fixed points are mapped to fixed points, and c-functions, beta functions and phase diagrams of the 4$d$ and 2$d$ theories are related to each other. This puts the previous supergravity results into a form in which they might be extendable to all orders in $\alpha'$.

It is then noted that the 4$d$ flows look very much like flows in 4$d$ field theories coupled to “conformally self-dual 4$d$ gravity” $\mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \times \mathbb{H}$, where the dynamics of the conformal factor originates from the 4$d$ conformal anomaly. It is speculated that this suggests that the world-brane theory of $N$ D-branes is really a theory that contains this version of 4$d$ supergravity; and that this suggests a theory of dynamical 3-branes that might be analogous to the Polyakov formulation of dynamical strings.

2. General setup

We consider type IIB string theory in an $AdS_5 \times E^5$ background. $E^5$ is an Einstein manifold, so its Ricci tensor is $R_{mn} = \Lambda g_{mn}$ with some $\Lambda$ and $m, n \in \{5, 6, 7, 8, 9\}$. Let us parametrize
the Einstein manifold by coordinates $\theta^m$ and denote by $x_\parallel$ the coordinates $x^\mu$, $\mu \in \{0, 1, 2, 3\}$, of the 4d space parallel to the boundary of $AdS_5$. The radial coordinate of $AdS_5$ will be called $\phi$. The $AdS_5 \times E^5$ metric is then

$$ds^2 = d\phi^2 + e^{2\alpha(\phi)} dx_\parallel^2 + L^2 \hat{g}_{mn} d\theta^m d\theta^n,$$

(2.1)

where $\hat{g}$ is defined such that the volume of $E^5$ as measured in the metric $\hat{g}$ is one; by a proper choice of coordinates, $\hat{g}$ can also locally be made to have unit determinant:

$$\text{Vol } \hat{E}^5 = 1, \quad \det \hat{g} = 1.$$

$L$ is a parameter that is related to $\Lambda$ by

$$\Lambda = \frac{\lambda}{L^2}, \quad \text{where} \quad \hat{R}_{mn} = \lambda \hat{g}_{mn}.$$

With these definitions, we find that $E^5$ and $AdS_5$ have the same curvature scalar up to a sign if $5\Lambda = 16\dot{\alpha}^2$ ("dot" means "$d/d\phi$"), i.e.

$$\alpha(\phi) = -\frac{q}{4} \frac{\phi}{L} \quad \text{with} \quad q \equiv \sqrt{5\lambda}.$$

(2.2)

The sign is such that the boundary of $AdS_5$ is at $\phi \to -\infty$. $\hat{g}$, $L \equiv e^\beta$ and the dilaton $\Phi$ are independent of $\phi$:

$$\hat{g} = \hat{g}_0 = \text{const.} \quad (2.3)$$

$$\beta = \beta_0 = \text{const.} \quad (2.4)$$

$$\Phi = \Phi_0 = \text{const.} \quad (2.5)$$

All other fields are zero, except that there are also $N$ units of electric and magnetic Ramond-Ramond 5-form flux:

$$\oint_{E^5} F^{(5)} = \oint_{E^5} *F^{(5)} \sim N,$$

such that

$$- g^{00} g^{11} g^{22} g^{33} g^{44} F_{01234} F_{01234} = g^{55} g^{66} g^{77} g^{88} g^{99} F_{56789} F_{56789} \sim \frac{N^2}{L^{10}}.$$

Now, the target space fields of type IIB superstring theory appear as coupling constants both in the 4d Yang-Mills theory and in the 2d sigma model that lives on the string
world-sheet. This is just the general statement (which does not depend on having an AdS–
background) that type IIB string fields can be coupled both to D3–branes and to fundamental
1–branes. In the 4d Lagrangean the string fields appear as follows:

$$\mathcal{L}_{YM} \sim e^{-\Phi} \text{tr } F^2$$  \hspace{1cm} (2.6)

$$+ \chi \text{ tr } F \wedge F$$ \hspace{1cm} (2.7)

$$+ \bar{g}_{ab} \text{ tr } \partial \phi^a \partial \phi^b$$ \hspace{1cm} (2.8)

$$+ A^{(4)}_{abcd} \text{ tr } \partial \phi^a \wedge \partial \phi^b \wedge \partial \phi^c \wedge \partial \phi^d + \ldots$$ \hspace{1cm} (2.9)

Here, $\chi$ and $A^{(4)}$ are the Ramond-Ramond 0-form and 4-form, and $\phi^a$ are the scalar fields
of the SYM theory, where $a, b \in \{1, \ldots, 6\}$. We have omitted the two-form gauge fields, but
they can be included. So we have the well-known relations

$$e^{\Phi} = g_{YM}^2$$ \hspace{1cm} (2.10)

$$\chi = \theta - \text{angle} ,$$ \hspace{1cm} (2.11)

etc. Furthermore, $\bar{g}$ is directly related to $\hat{g}$. E.g., in the case $E^5 = S^5$, $\bar{g} = \delta$ and, using
spherical coordinates $(r, \theta^m)$ in $\phi^a$–space, we have

$$\bar{g}_{ab} \, d\phi^a \, d\phi^b \sim d r^2 + r^2 \hat{g}_{mn}(\theta) d\theta^m d\theta^n .$$

$\alpha(\phi)$, on the other hand, is not a coupling constant of the Yang-Mills theory. It will be
regarded as an auxiliary parameter below and eliminated by its equations of motion. $L$ is
also not an independent Yang-Mills coupling constant, and will be regarded as an auxiliary
parameter, too – its equation of motion relates it, at fixed points, to the ‘t Hooft coupling:

$$L^4 \sim g_{YM}^2 N .$$

As mentioned, the same type IIB string fields also appear as coupling constants in the
2d Lagrangean of the world-sheet sigma model, but of course differently:

$$\mathcal{L}_\sigma \sim R^{(2)} \Phi$$ \hspace{1cm} (2.12)

$$+ (\partial \phi)^2$$ \hspace{1cm} (2.13)

$$+ L^2 \hat{g}_{mn} \partial \theta^m \partial \theta^n$$ \hspace{1cm} (2.14)
\[ + e^{2\alpha} \partial x^\mu \partial x^\mu \quad (2.15) \]

\[ + \text{RR-backgrounds} \quad (2.16) \]

Now, this sigma model contains more coupling constants than the Yang-Mills theory: it contains the auxiliary parameters \( \alpha(\phi) \) and \( L \), and it also allows in principle for a non-flat 4\( d \) metric \( g_{\mu\nu} \). It is therefore natural to instead consider a “reduced” sigma model that accounts only for the YM coupling constants - namely the sigma–model with only 5\( d \) internal target space, parametrized by \( \theta^m \), rather than the full 10\( d \) target space, and without the \( L^2 \) factor in the embedding space metric:

\[ \hat{\mathcal{L}}_\alpha \sim R^{(2)} \Phi + \hat{g}_{mn} \partial \theta^m \partial \theta^n + \text{RR-backgrounds on } E^5. \]

Then fixed points of the 4\( d \) gauge theory (with only \( \alpha \), but not the dilaton and the metric of the Einstein manifold changing as a function of \( \phi \)) correspond to fixed points of this 2\( d \) sigma model. We want to suggest how furthermore, at least in supergravity approximation, 4\( d \) \( c \)-functions are mapped to 2\( d \) \( c \)-functions, based on [10], and how the whole flow in the 4\( d \) gauge theory is mapped onto a flow in the 2\( d \) sigma model for the internal compact space.

This relation between the RG flow in this sigma model with 5\( d \) target space and the flow in 4\( d \) Yang–Mills theory will be based on the interpretation of \( \phi \) in terms of the scale of the Yang–Mills theory [13, 16]: for the AdS metric in (2.1,2.2), we can absorb overall scale transformations on the brane

\[ x_{||} \rightarrow x_{||} e^\tau \]

in shifts

\[ \phi \rightarrow \phi + \frac{4}{q} L \tau. \]

In this sense, \( \phi \) is “RG time” (the UV end \( \tau = -\infty \) is at the AdS boundary at \( \phi = -\infty \)), and the \( \phi \)-dependence of the string fields \( \hat{g}, \Phi, \chi, A^{(4)}, \ldots \) describes how the coupling constants “run” under scale transformations. RG trajectories in the gauge theory are thus viewed as time-dependent classical solutions \{\( \hat{g}(\phi), \Phi(\phi), \chi(\phi), A^{(4)}(\phi), \ldots \)\} of string theory [14]. 

\(^1\)This is quite analogous to the situation in the case of the RG flow in 2\( d \) field theory coupled to gravity, where RG trajectories are also classical solutions of string theory [17]. A difference is that in that case \( \phi \) represents the 2\( d \) scale, while here it represents the 4\( d \) scale.
If $\hat{g}, \Phi, \chi, A^{(4)}$ (and, as a consequence, $L$ and $\dot{\alpha}(\phi)$) are constant, we have a scale invariant theory or RG fixed point – both in the 4d and the 2d field theories. We will study the flow in the vicinity of such fixed points:

\[
\hat{g}_{mn} = \hat{g}^0_{mn} + \delta \hat{g}_{mn}(\phi, \theta) \tag{2.17}
\]
\[
\Phi = \Phi_0 + \delta \Phi(\phi, \theta) \tag{2.18}
\]
\[
\chi = \delta \chi(\phi, \theta), \tag{2.19}
\]
and so on. This is the most general ansatz consistent with Poincaré invariance on the brane (we also make the local gauge choice $g_{\phi \mu} = g_{\phi m} = 0$). We collectively denote (the modes of) the string field variations – i.e. the coupling constants of both the 4d and 2d field theory – by $\vec{\lambda}$:

\[
\vec{\lambda}(\phi) \equiv \{ \delta \hat{g}_{mn}, \delta \Phi, \delta \chi, \ldots \} .
\]
and assume that all the coupling constants $\vec{\lambda}$ are small.

As the other target space fields vary, $L = e^{2\beta}$ and $\alpha$ will also adjust:

\[
\beta = \beta_0 + \delta \beta , \quad \alpha = -\frac{q}{4} \frac{\phi}{L} + \delta \alpha .
\]

Then the scale $e^\tau(\phi)$ is (by holography)

\[
(e^\tau)'(\phi) = e^{-\alpha(\phi)} .
\]
The auxiliary parameters $\alpha, \beta$ will be eliminated below by solving their equations of motion in the vicinity of fixed points.

3. 4d flows from 2d flows

The relation we propose between the 4d and 2d flows is derived in appendix A in the classical supergravity approximation following the method used in [18]. This approximation is valid in the limit of large $N$ and large ‘t Hooft coupling $g_{YM}^2 N$. The result is the following. Write the perturbed gauge theory Lagrangean as

\[
\mathcal{L}_4 = \mathcal{L}^0_4 + \vec{\lambda} \cdot \vec{O}_4 ,
\]
where $\mathcal{L}_2^0$ corresponds to the 4d CFT, whose central charge we call $c_4$. Write the perturbed action of the 2d sigma model with 5d target space parametrized by $\theta^m$, and target space metric $\hat{g}$ (and not $g$!), as:

$$\mathcal{L}_2 = \mathcal{L}_2^0 + \vec{\lambda} \cdot \vec{O}_2,$$

where $\mathcal{L}_2^0$ corresponds to the 2d CFT with central charge $c_2$. Call the beta functions in the 2d sigma model $\vec{\beta}^{(5)}$, so the 2d flow obeys

$$\dot{\lambda}_i = \beta_i^{(5)} = \Delta_i^j \lambda_j + c_i^{jk} \lambda_j \lambda_k + \ldots,$$  \hspace{1cm} (3.1)

where $\Delta_i^j$ are the dimensions of the 2d field theory, $c_i^{jk}$ are the OPE coefficients, and more generally, the beta functions are gradients of the 2d $c$-function $c_2(\vec{\lambda})$. Then the flow of the same coupling constants in the 4d gauge theory obeys in the vicinity of fixed points, i.e. to linear order\footnote{I.e. the conformal anomaly coefficients are $c_4 = c = a$.} in $\lambda$, $\dot{\lambda}$: (but possibly to all orders in $\alpha'$)

$$q_2^2 \left( \ddot{\lambda} + 4 \dot{\lambda} \right) = \vec{\beta}^{(5)},$$

with $q_2^2 = \frac{5 - c_2}{3} + \text{h.o.}$ \hspace{1cm} (3.2)

The statement is that on the RHS of (3.2) we have the same beta functions (3.1) as in the 2d sigma model, only the LHS changes in going from the 2d sigma model to the 4d gauge theory. This implies that fixed points are mapped to fixed points, and the structure of fixed points and flows between them in 4d gauge theory seems to be at least as rich as in 2d field theory. Moreover, there might be a natural way in which the $c$-function of 2d field theory induces a $c$-function of 4d gauge theory, as discussed in point 4 below.

In the following we will occasionally redefine $\tau \rightarrow \tilde{\tau} = \frac{4}{q} \tau$. Then the equation becomes

$$\ddot{\lambda} + q \dot{\lambda} = \vec{\beta}^{(5)}, \quad \text{with} \quad q^2 = \frac{5 - c_2}{3}.$$  \hspace{1cm} (3.4)

To compare with some of the literature \cite{3}, \cite{11}, note that the same equations also apply to gauged supergravity. In this case the coupling constants $\vec{\lambda}$ correspond to the 42 scalar fields $\Phi^J$ that represent deformations of the metric, B-fields, $\chi$ and $\Phi$, and the $c$-function

\footnote{there are nonuniversal corrections of order $\dot{\lambda}^2$ to the LHS; also, at next order, the RHS of eqn. (3.3) contains a term proportional to $\dot{\lambda}^2$ (see (5.20)).}
\(c_2(\bar{\lambda}) - 5\) is represented by the potential \(v(\Phi) = \frac{1}{\bar{\tau}^2} V(\Phi)\). As in \([3, 10]\), in the supergravity approximation it can be seen from the appendix (cf. eqn. (5.12)) that it is consistent to set the dilaton \(\Phi\) constant. So the Yang–Mills coupling constant \(g_{YM} = e^{\frac{\Phi}{2}}\) does not run: it remains large and thus it is consistent to use the supergravity approximation.

In the case of supersymmetric flows describing BPS-saturated kink solitons of gauged supergravity \([10]\), the second-order flow equations become standard first-order flow equations (the Bogomol’nyi equations). Moreover, the beta functions become independent of \(g_{YM}\) and can be interpolated from strong coupling to the perturbative regime.

Even without supersymmetry, it is possible to define modified beta functions

\[
\beta^{(4)}_i = \Delta^{(4)}_j \lambda^j + c^{(4)}_{ijk} \lambda^j \lambda^k ,
\]

such that the 4d flow becomes first-order: \(\dot{\lambda}^i = \beta^{(4)}_i\). Plugging the first-order equations into the second order equations (3.2) yields the modified beta function coefficients order by order (see below and \([19]\)).

The above statements are well-known in the supergravity approximation. One reason for rewriting them in the form of a relation between 4d gauge theory and 2d field theory (rather than supergravity) is to motivate the comments in the next section. Another reason is that now it makes sense to ask whether relation (3.2,3.3) might hold, in the vicinity of fixed points, beyond the supergravity approximation. Before commenting on this, however, let us summarize what the relation is good for:

1. **Dimensions:** Keeping only the linear part of the 2d beta functions \(\beta_i = \Delta_i^{(2)} \lambda_i\), the dimensions \(\Delta_i^{(4)}\) of the 4d operators obey

\[
\Delta^{(4)}(\Delta^{(4)} + 4) = \frac{16}{q^2} \Delta_i^{(2)} .
\]

We have to drop one of the two “dressings”. In supergravity approximation, where

\[
q^2 = \frac{1}{3}(c_2 - 5) = \hat{R} = 5\lambda ,
\]

this relation takes on a form that is well-known \([3, 4]\).

2. **OPE coefficients:** the quadratic beta function coefficients \(c^{(2)}_{ijk}\) are universal if \(\Delta_i^{(2)}\) is small (of order \(\lambda\)). To relate the OPE coefficients in two and four dimensions, we
assume that $\Delta_i^{(2)} = 0$ and make the ansatz that the 4d flow $\dot{\lambda}$ in (3.2) also obeys a first-order differential equation, similarly as in [19]:

$$\dot{\lambda} = c_{ijk}^{(4)} \lambda^i \lambda^j + ...$$

Plugging this into (3.2) and expanding yields:

$$c_{ijk}^{(4)} = \frac{4}{q^2} c_{ijk}^{(2)} .$$

3. **Phase diagrams:** since fixed points are mapped to fixed points and relevant, marginal and irrelevant directions of the flow near fixed points are again mapped onto relevant, marginal and irrelevant directions, the phase diagrams of the 4d and 2d theories should also agree.

4. **$c$-functions:** The 2d $c$-function $c_2(\bar{\lambda})$ always decreases along the 2d flow, as the beta functions are its gradients. But (3.2) describes the damped motion of a particle in the potential $c_2(\bar{\lambda})$, so at least near the fixed point $c_2(\bar{\lambda})$ also decreases along the 4d flow. In this sense, the $c$-function of 2d field theory might also give rise to a $c$-function of 4d gauge theory. Away from the fixed point it seems hard to make a clear statement.

More generally, a good 4d $c$-function $c_4(\bar{\lambda})$ might be some function of the 2d $c$-function $c_2(\bar{\lambda})$. To see precisely what function, note that it has been argued in [10] within the supergravity approximation that $(\dot{\alpha})^{-3}$ (\dot{\alpha} is defined in section 2) is proportional to to the 4d conformal anomaly at fixed points and might be taken as a 4d $c$-function. Since $\dot{\alpha}^2 \propto (\bar{c} - 5)$ at fixed points (see appendix A), it is tempting to conjecture a possible generalization of this statement to higher orders in $\alpha'$: that the 4d $c$–function follows from the 2d $c$-function by

$$[c_4(\bar{\lambda}) - k] \sim -[c_2(\bar{\lambda}) - 5]^{-\frac{3}{2}} ,$$

where we have allowed for some constant $k$. The relative minus sign is needed to make $c_4$ decrease, rather than increase along the flow.

Could the relations (3.2,3.3) be, in the vicinity of fixed points, exact in $\alpha'$? One possibility to address this question would be to include $\alpha'$-corrections to the string effective action and check whether they are of higher order in $|\bar{\lambda}|$, i.e. vanish in the vicinity of fixed points.
Another possibility is suggested by comparing with an analogous situation that arises in quite a different context: suppose we perturb a 2d CFT:

\[ \mathcal{L} = \mathcal{L} + \lambda^i O_i \]

Then the coupling constants will typically flow under scale transformations:

\[ \dot{\lambda} = \beta . \]

How is this flow modified if the 2d field theory is coupled to gravity? The answer is (see the appendix of [20] for a review):

\[ \frac{\alpha^2}{4} (\tilde{\lambda} + Q \tilde{\lambda}) = \beta \quad \text{with} \quad Q^2 = \frac{D_{\text{crit}} - c}{3} \]

and some constant \( \alpha \). Now in 2d gravity we can prove that these equations are exact in \( \alpha' \) in the vicinity of the fixed point (i.e. to order \( |\tilde{\lambda}| \)), and hold not just in the supergravity approximation. This works as follows [21]: The CFT coupled to gravity is described by a CFT with one more field \( \phi \), the Liouville mode. At the fixed point, the two CFT’s decouple:

\[ S_{\text{CFT}} + \int d^2 \xi \left( \partial^2 \phi + Q \hat{R} \phi \right). \]

Using OPE’s, we know the exact central charge \( c \) and dimension of the operator \( e^{\gamma \phi} \) in the second CFT:

\[ c = 1 + 3Q^2 \quad \text{,} \quad \text{dim}(e^{\gamma \phi}) \propto \gamma (\gamma + Q). \]

Conformal invariance of the perturbed theory then implies the two flow equations above [17], defining \( \tau = \frac{4}{Q} \phi \). In the case of 4d SYM, it is of course much harder to check whether the flow equations are exact in \( \alpha' \) because of the Ramond-Ramond backgrounds.\(^4\) It might be possible, though, to turn the Ramond–Ramond backgrounds in the \( \phi - x^\mu \) sector (which is the analog of the above \( \phi \) sector) into Neveu-Schwarz backgrounds by duality transformations. We must leave this subject to future work.

\(^4\)See [22] for attempts to deal with Ramond-Ramond backgrounds. One should first fully understand the simpler case of the 2d conformal field theories that are dual to string theory on \( AdS_3 \) times some compact manifolds, where there are no RR backgrounds; see, e.g., [23] in this context.
4. Dynamical 4d conformal factor?

Let us now come to a key point of this paper. Second–order flow equations like (3.2) seem to be typical for RG flows in theories with a fluctuating scale, as in the case of 2d gravity just mentioned. In fact, the equation (3.4),

\[
\ddot{\lambda} + q \dot{\lambda} \propto \beta, \quad q^2 \propto \text{conformal anomaly}
\]

looks precisely like the flow equation in 4d field theory coupled to a particularly interesting limit of Weyl supergravity: “conformally self–dual gravity”. By “conformally self–dual gravity” we mean 4d gravity with action

\[
\rho \int d^4x \sqrt{g} W^2 \quad \text{in the limit} \quad \rho \to \infty,
\]

where \( W^2 \) is the Weyl^2 term. The cosmological constant, and the coefficients of the \( R \) and \( R^2 \) terms are assumed to be zero. This theory is not quite renormalizable, because there is also a conformal anomaly term; but in the limit \( \rho \to \infty \) the anomaly term can nicely be dealt with, as will be discussed below.

However, one point must be addressed before going any further. There is a well-known problem common to all fourth-order derivative actions, like the Weyl^2 action: we can rewrite them in terms of new fields with two derivatives only, but some of them will have the wrong sign in the kinetic term. With Minkowskian signature, this leads to perturbative non-unitarity. For this reason, we can only consider Euclidean signature in this section.

Below it will be explained why this limit of Weyl gravity leads to flow equations of the type (4.1). But before, let us give another independent argument that indicates that conformally self–dual gravity might indeed arise as part of the world–volume theory.

Kaluza-Klein reduction

Why might (4.2) arise as part of the world–brane theory? Let us start with the dual formulation of the D3–world–brane theory in terms of 5-dimensional gauged supergravity in an \( AdS_5 \) background. Suppose we further “Kaluza-Klein reduce” this 5d gravity theory along

\footnote{For another (different) discussion of the role of Weyl gravity in the context of the AdS/CFT correspondence, see [24].}
$z = \exp \left\{ \frac{\phi}{L} \right\}$ onto the 4d boundary of $AdS_5$ (similarly as in [25]). By this we mean that all fields except for the “warp factor” in the line element $\frac{1}{z^2} dx^2$ are assumed to be independent of the 5th coordinate $z$, and $z$ is integrated over. (In standard Kaluza-Klein reduction there would be no warp factor and $z$ would be some circular coordinate.) Now, as in standard Kaluza-Klein reduction, the 5d Hilbert-Einstein action induces a 4d Hilbert-Einstein action. Its coefficient (the inverse 4d Newton constant) diverges (we neglect factors of $L$):

$$\frac{1}{\kappa_5^2} \int_\epsilon dz \sqrt{g^{(5)}} R^{(5)} \rightarrow \frac{1}{\kappa_4^2} \sqrt{g^{(4)}} R^{(4)} \quad \text{with} \quad \frac{1}{\kappa_4^2} \sim \frac{1}{\kappa_5^2} \int_\epsilon^\infty dz \frac{1}{z^3} \sim \frac{1}{e^2},$$

where we have cut off $AdS_5$ at some small distance $z = \epsilon$ from the boundary. This divergence is powerlike and not universal, in the sense that it depends on precisely how we cut off $AdS_5$ at its boundary. There is however a universal $\log \epsilon$ divergence of the form [26]

$$\rho \int d^4x \sqrt{g} (W^2 + G) , \quad \text{with} \quad \rho = c \log \epsilon \rightarrow \infty ,$$

where the number $c$ is proportional to the 4d conformal anomaly and $G$ is the Gauss–Bonnet density, whose integral is a topological invariant (the Euler character). So if we assume that we can pick a “renormalization scheme” (i.e. a way to regularize the divergences at the boundary) in which the non-universal Hilbert-Einstein (and cosmological) term vanishes, then what remains is – up to a topological invariant – indeed the “conformally self-dual gravity” action.

Precisely the bosonic sector of this supergravity theory in the limit $\rho \rightarrow \infty$ has previously been studied by Fradkin and Tseytlin [12] and as a theory closely analogous to 2d gravity by Antoniadis and Mottola [13] and by the author [14]. It is a renormalizable theory whose nice properties are summarized in appendix B; the main points will be recalled below.

Is this version of 4d gravity dynamical here, or does it correspond to a fixed background? There is a standard argument that says that there is no dynamical 4d gravity on the $AdS -$ boundary (see e.g. [5]). It asserts that the norm of those states of the 5d theory that would correspond to 4d gravitons diverges, so these states are not normalizable. However, this argument may not apply here, because this divergence of the norm is – like the inverse Newton constant – not universal. The divergence may also vanish in a scheme in which the the inverse Newton constant vanishes, leaving us with a finite norm state.
Let us therefore try out the hypothesis that dynamical conformally self–dual $4d$ gravity is indeed present on the $3$–brane, and show that this could reproduce at least qualitatively the form of the flow equation (4.1).

Conformally self-dual gravity

The following is briefly reviewed in appendix B: up to topological invariants the $4d$ gravity action to be studied is actually [14]

$$
\rho \int d^4x \sqrt{g} W_+^2, \quad \rho \to \infty .
$$

$W_+$ is the self-dual part of the Weyl tensor. The limit $\rho \to \infty$ restricts the metric to be “conformally self-dual”: $W_+ = 0$ (see appendix). The path integral over metrics then reduces, in analogy with $2d$ gravity, to a path integral over the conformal factor $\phi$ and an integral over the moduli space of conformally self-dual metrics.

In $2d$ gravity, the dynamics for the conformal factor arises from the nonlocal conformal anomaly term in the effective action. In conformal gauge the anomaly term becomes local; it consists of a kinetic term for $\phi$ plus a background charge term (neglecting the Liouville potential)

$$
\phi \Box \phi - Q R \phi \quad \text{with} \quad Q^2 = \frac{c - 25}{3} .
$$

(4.3)

$Q$ is determined to make the total conformal anomaly vanish. In our $4d$ theory, everything is almost completely analogous [27, 13] (see appendix B). The nonlocal anomaly term becomes a local term for $\phi$, if evaluated for conformally self-dual metrics (which are the only metrics that survive the limit $\rho \to \infty$). The relevant part of the induced Lagrangean for $\phi$ is

$$
\phi \Box^2 \phi - \frac{1}{2} Q G \phi \quad \text{with} \quad Q^2 = \frac{\tilde{b} + 1538}{90}
$$

with Gauss-Bonnet density $G$ and $\tilde{b} = -360b$ where the $4d$ conformal anomaly coefficients are called $a, b$: essentially,

$$
<T_\mu^\mu> = \frac{1}{16\pi^2} (aW^2 + bG)
$$
(assuming $g$ is an Einstein metric). Some coefficients $a$ (which plays a minor role here) and $b$ are given in the appendix.

Due to the $\Box^2$ kinetic term for $\phi$, its propagator is logarithmic and therefore the exponential $e^{2\alpha\phi}$ has definite scaling dimension $^{[13, 14]}$

$$\text{dim}(e^{\alpha\phi}) = -\alpha(\alpha + Q).$$

As in $2d$, scaling operators $\Phi_i$ of the matter theory with dimension $\Delta_i$ are gravitationally dressed: $\Phi_i \rightarrow e^{\gamma_i \phi} \Phi_i$. Scale invariance determines the dressed scaling dimension $\gamma_i$:

$$\Delta_i - \gamma_i(\gamma_i + Q) = 4$$

Now, perturbing the CFT by operators $\lambda^i e^{\gamma_i \phi} \Phi_i$, defining $\lambda^i(\phi) = \lambda^i e^{\gamma_i \phi}$, multiplying the above equation by $\lambda^i(\phi)$, replacing $\gamma_i \rightarrow \partial_\phi$ and identifying $\beta_i = (\Delta_i - 4)\lambda_i + O(\lambda^2)$ yields the following flow equations:

$$\ddot{\lambda} + Q \dot{\lambda} \propto \beta, \quad Q^2 \propto \text{conformal anomaly}. \quad (4.4)$$

These are RG flow equations in view of the interpretation of $\phi$ as the $4d$ conformal factor – so overall $4d$ scale transformations correspond to constant shifts of $\phi$.

These flow equations indeed have the same form as (4.1). It is tempting to interpret this as additional evidence that dynamical 3–branes are described in terms of conformally self–dual gravity coupled to matter – analogous perhaps to the description of dynamical one–branes (strings) in terms of $2d$ gravity coupled to matter.$^6$

It remains to compare the coefficients $q$ in (4.1) and $Q$ in (4.4), as well as the beta functions. This is left for future work. It is presently not even clear to the author whether one should expect quantitative agreement, since the calculation leading to (4.1) is done completely within classical string theory. A fascinating possibility would be that studying the flow on dynamical 3-branes might yield nonperturbative information about string theory, instead of reproducing the classical result.

$^6$For possibly related suggestions about the existence of such an analogy see [29].
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Appendix A: Supergravity approximation

This appendix is based on ref. [18], generalizing the procedure used there to the case at hand, which includes a Ramond-Ramond background. We start with the superstring effective action [30], setting \( \alpha' = 2 \) and keeping only the metric, the dilaton and the RR-5-form for simplicity:

\[
S^{(10)} = \int d^D x \sqrt{G} \, e^{-2\Phi} \left\{ \frac{1}{3} [10 - C^{(D)}(\vec{x})] - \frac{1}{2 \cdot 5!} e^{2\Phi} F_{mnopq} F^{mnopq} \right\},
\]

where \( D \) is the dimension of embedding space and the function

\[
C^{(D)}(\vec{x}) = D - 3\left[ R - 4(\nabla \phi)^2 + 4\Box \phi \right]
\]

becomes \( \vec{x} \)-independent and equal to the central charge when the sigma model represents a conformal theory [31]. \( C \) is defined in terms of the variation of \( S \) with respect to the constant mode of the dilaton, and therefore does not explicitly involve the RR background \( F^{(5)} \). For the metric we make the ansatz of section 2:

\[
ds^2 = d\phi^2 + e^{2\alpha(\phi)} dy_{II}^2 + L^2 \hat{g}_{mn} d\theta^m d\theta^n,
\]

with

\[
\det \hat{g} = 1, \quad L^2 \equiv e^{2\beta}.
\]

The RR 5-form field strength is

\[
\frac{1}{5!} F^2 = N^2 e^{-10\beta}.
\]

The string equations of motion are the requirements that the beta functions of the 2d sigma model with 10d target space are zero. The beta functions for the metric and the dilaton are,
respectively (with $A, B \in \{0, \ldots, 9\}$):

$$0 = \beta_{AB} = 2 \{ R_{AB} + 2 \nabla_A \nabla_B \Phi - \frac{5}{2 \cdot 5!} e^{2\Phi} F_{A\ldots B} \}.$$  

$$0 = \frac{1}{3} (10 - C) = R - 4(\nabla \Phi)^2 + 4 \Box \Phi.$$  

We now make a $1+4+5$ split of the 10 coordinates into $(\phi, x^\mu, \theta^m)$, assume that all fields are independent of $x^\mu$, and find to leading order:

$$\beta^{(10)}_{mn} = \beta^{(5)}_{mn} - (g''_{mn} - \varphi' g'n_m - \varphi' g'n_m g')$$  

$$\frac{1}{3} [10 - C^{(10)}] = \frac{1}{3} [5 - C^{(5)}] + \varphi'' - (\varphi')^2,$$

and so on, where primes denote derivatives with respect to $\phi$.

$$\varphi = 2\Phi - \log \sqrt{g} = 2\Phi - 4\alpha - 5\beta$$

is the shifted dilaton, and $\beta^{(5)}$ are the beta functions of the 2d sigma model with 5d target space metric $g = L^2 \hat{g}$.

In this way, we learn from the $\beta_{\phi\phi}, \beta_{\mu\nu}, \beta_{mn}$ and $\Phi$-equations, respectively:

$$- \varphi'' + 4(\alpha')^2 + 5(\beta')^2 + \frac{1}{4} (\hat{g}')^2 = \frac{5}{8} N^2 e^{2\Phi - 10\beta}$$  

$$\alpha'' - \varphi' \alpha' = \frac{5}{8} N^2 e^{2\Phi - 10\beta}$$  

$$\beta'' - \varphi' \beta' = \frac{1}{5} R - \frac{1}{2} N^2 e^{2\Phi - 10\beta}$$  

$$\hat{g}''_{mn} - \varphi' g'_{mn} - \hat{g}'^{kl} \hat{g}''_{mk} \hat{g}''_{nl} = 2 e^{-2\beta} (\hat{R}_{mn} - \frac{1}{5} \hat{g}_{mn} \hat{R})$$  

$$- \varphi'' + (\varphi')^2 = \frac{1}{3} (5 - C^{(5)}) = R$$  

$$\Phi'' - \varphi' \Phi' = 0.$$

A combination of the first and the second-last equations is first order in derivatives, constraining initial values. The last equation follows from the other ones. The $\beta_{ij}$ equation has been split into its trace and its trace-free part. $\hat{g}, \Phi$ correspond to coupling constants $\vec{\lambda}$, as defined in section 2, while the other equations (which are not independent) fix $\alpha$ and $\beta$.

E.g., $\alpha$ and $\beta$ are easily fixed at fixed points, where

$$\beta' = \hat{g}' = \Phi' = \varphi'' = 0.$$
Denoting \(\kappa = e^\Phi\), \(Q = -\varphi'\), so that \(\alpha = \frac{Q}{\kappa}\phi\) at the fixed point, we read off (using \(R = \hat{R}/L^2\))

\[
\hat{R} = \frac{5N^2\kappa^2}{2L^8} = 16(\alpha')^2 L^2 = \text{constant}.
\]

Defining \(\lambda\) as in section 2 by

\[
\hat{R}_{mn} = \lambda \hat{g}_{mn} \to \hat{R} = 5\lambda,
\]

we have

\[
L^8 = \frac{\kappa^2N^2}{2\lambda}, \quad Q^2 = \frac{5\lambda}{L^2}.
\]

In order to expand in the vicinity of fixed points, it is useful to first average \([\ref{4.1}],[\ref{4.2}]\), i.e. to split the shifted dilaton \(\varphi(\vec{\theta}, \phi)\) into an \(\vec{\theta}\)-dependent part \(\tilde{\varphi}(\vec{\theta}, \phi)\) and an \(\vec{\theta}\)-independent part \(\varphi_0(\phi)\) as follows:

\[
\varphi_0(\phi) \equiv -\log\left[\int d^5x \ e^{-\varphi(\vec{\theta}, \phi)}\right], \quad \tilde{\varphi}(\vec{\theta}, \phi) \equiv \varphi(\vec{\theta}, \phi) - \varphi_0(\phi).
\] (5.13)

Let us define the space average of a function \(f(\vec{\theta})\) by

\[
<f(\vec{\theta})> \equiv \frac{\int d^5\theta \ f(\vec{\theta}) \ e^{-\varphi(\vec{\theta})}}{\int d^5\theta \ e^{-\varphi(\theta)}}.
\] (5.14)

Then define

\[
Q(\phi) \equiv -\varphi'(\phi) = -<\varphi'(\vec{\theta}, \phi)>.
\] (5.15)

Integrating (5.6) weighted by \(e^{-\varphi}\) yields

\[
Q' + Q^2 = \frac{1}{3}(5 - \bar{c}),
\] (5.16)

where (5.14) has been used and we have defined the function

\[
\bar{c}(\phi) = \bar{c}(G, B, \varphi) = <C(5)(\vec{\theta}, \phi) >.
\] (5.17)

\(\bar{c}\) can be considered as a generalisation of the ‘c–function’ \([\ref{3.2}]\) of 2d field theory.

The expansion near fixed points follows as in \([\ref{1.8}]\). We collectively denote \(\delta \hat{g}, \delta \Phi, \ldots\) by \(\vec{\lambda}\) as in section 2, noting i.p. that \(g = L^2\hat{g}\). Next we expand the above string equations of
motion order by order in $\vec{\lambda}$. It is easy to see that to lowest order it suffices to set $L$ and $Q$ to their fixed point values, to ignore the terms of order $(\vec{\lambda}')^2$ in the flow equations and to set the beta functions for the sigma model with target space metric $\hat{g}$ equal to those for the sigma model with metric $g$. One then obtains

$$L^2(\vec{\lambda}'' + Q\vec{\lambda}') = \vec{\beta}^{(5)}$$
$$Q^2 = \frac{5 - \bar{c}}{3}.$$  \hfill (5.18)

The second equation is the equation of motion for the constant part of the shifted dilaton $\varphi$. To next order, $Q^2$ gets a new contribution [18]:

$$Q^2 = \frac{5 - \bar{c}(\vec{\lambda})}{3} + \frac{1}{4}(\vec{\lambda}')^2.$$  \hfill (5.19)

Finally, we define RG time near the fixed point (where $\alpha = \frac{Q}{4} \phi$) as in section 2:

$$\tau \equiv \frac{Q}{4} \phi \quad \text{with} \quad Q \equiv -\frac{q}{L}.$$  \hfill (5.20)

We denote derivatives with respect to $\tau$ by “·” , and pull out factors of $\frac{1}{L}$ using

$$\vec{\lambda}' = \frac{Q}{4} \vec{\lambda}' , \quad \frac{5 - \bar{c}}{3} = R = \frac{1}{L^2} \hat{R} = \frac{1}{L^2} \frac{5 - c_2}{3}.$$  \hfill (5.21)

c_2 is (a generalization of) the c-function for the sigma model with embedding space metric $\hat{g}$ (rather than $g = L^2 \hat{g}$). This yields to lowest order:

$$\frac{q^2}{26}(\dddot{\vec{\lambda}} + 4\dddot{\bar{\lambda}}) = \vec{\beta}^{(5)}$$
with $q^2 = \frac{5 - c_2}{3} + h.o.$  \hfill (5.22)

**Appendix B: Conformally self-dual gravity**

We consider four–dimensional Euclidean “gravity” with action

$$S = \int_M d^4x \sqrt{g}\{\lambda + \gamma R + \eta R^2 + \rho W^2\}$$
in the limit

$$\lambda = \gamma = \eta = 0 , \rho \to \infty.$$
$W$ is the Weyl tensor, the traceless part of the Riemann tensor:

$$W_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau} - \frac{1}{2}(g_{\mu\sigma}R_{\nu\tau} + g_{\mu\tau}R_{\nu\sigma} - g_{\mu\tau}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\tau}) + \frac{1}{6}(g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma})R.$$  

In the limit $\rho \to \infty$ this theory becomes free, as will become clear below; this limit corresponds to an UV fixed point, as shown by Fradkin and Tseytlin \cite{12}. What makes the theory in this limit nontrivial is the conformal anomaly term in the effective action, as we will review for the case of the bosonic theory.

There are two topological invariants in 4d, the Euler characteristic $\chi$ and the signature $\tau$ of a manifold

$$\tau = \frac{1}{48\pi^2} \int d^4x \sqrt{g} (W_2^2 - W_2^-) \quad \text{and} \quad \chi = \frac{1}{32\pi^2} \int d^4x \sqrt{g} G.$$  

with Gauss-Bonnet density

$$G = R^\mu\nu\sigma\tau R_{\mu\nu\sigma\tau} - 4R^\mu\nu R_{\mu\nu} + R^2$$

and (anti-) self-dual part of the Weyl tensor

$$W_{\pm\mu\nu\sigma\tau} \equiv \frac{1}{2}(W_{\mu\nu\sigma\tau} \pm \frac{1}{2}\epsilon_{\mu\nu}^{\ \alpha\beta}W_{\alpha\beta\sigma\tau}).$$

So after subtracting a topological invariant $\propto \tau$, we end up studying the action \cite{14}

$$\rho \int d^4x \sqrt{g} W_2^+ , \quad \rho \to \infty .$$

In the limit $\rho \to \infty$, all metrics are strongly suppressed in the path integral $\int \mathcal{D}g \ exp\{-S\}$ except for conformally self-dual metrics, i.e., metrics with $W_+ = 0$. $W_+$ has 5 independent components, so the condition $W_+ = 0$ kills 5 of the 10 components of the metric. This condition is conformally and diffeomorphism invariant. So the 5 remaining components of the metric must be the 4 diffeomorphisms and the 4d conformal factor. In addition, there is a moduli space of conformally self-dual metrics. This is a 4d analog of the moduli space of Riemann surfaces of 2d gravity. In the simplest case of $S^4$ topology, there is no moduli space: conformally self-dual metrics are conformally flat,

$$\hat{g} = \delta e^\phi$$

\footnote{Related work is \cite{28}.}
up to diffeomorphisms $\delta \xi_\mu$. (For K3, e.g., the moduli space is 57-dimensional.) We can now split up fluctuations around conformally self-dual metrics as

$$\delta g_{\mu\nu} = \hat{g}_{\mu\nu}\delta \phi + \nabla_\mu \delta \xi_\nu + \delta h_{\mu\nu}.$$ 

The limit $\rho \to \infty$ can be regarded as the “classical limit” for the five other components $h_{\mu\nu}$ only, in the sense that the path integral over them becomes Gaussian and leaves us with a determinant

$$\det(O^\dagger O)^{-\frac{1}{2}}$$

where $O^\dagger$ is the linearized $W_+$-term, $O$ is its adjoint and $O^\dagger O$ is a 4th order, conformally invariant, linear differential operator [13]. What remains is a path integral over the conformal factor and an ordinary integral over the moduli space mentioned above. Changing variables from $g$ to $\phi$ and $\xi$ and integrating over $\xi$ contributes, as in 2d, another Jacobian

$$(\det L^\dagger L)^{\frac{1}{2}}.$$ 

The determinants $\det(O^\dagger O)$, $\det(L^\dagger L)$, as well as any conformally coupled matter fields contribute to the conformal anomaly, analogously to 2d. For conformally invariant differential operators $X$:

$$\det X_{\hat{g},\phi} = \det X_{\hat{g}} e^{-S_{\hat{g},\phi}}$$

where the induced action $S_i$ is obtained by integrating the trace anomaly of the stress tensor

$$-2 \frac{\delta S_i[\hat{g}, \phi]}{\delta \phi} = \sqrt{\hat{g}} < T^\mu_\mu >= \frac{1}{16\pi^2} \sqrt{\hat{g}}(a W^2 + b G).$$

(There is also a term $\frac{2}{3} a \Box R$, but it vanishes for Einstein manifolds.) Following [27, 13], the conformal anomaly can be integrated: the 4d analog of the (free part of the) 2d Liouville action is, essentially,

$$S_i[\hat{g}, \phi] = -\frac{1}{32\pi^2} \int d^4 x \sqrt{\hat{g}} \left\{ b (\phi \Box^2 \phi + \hat{G} \phi) + a \hat{W}^2 \phi \right\}.$$ 

Note that, after restricting to conformally self-dual metrics and going to conformal gauge, the originally nonlocal conformal anomaly term has become local. Some coefficients $a$ and $b$ are according to Fradkin and Tseytlin [33]: (there, $a = \frac{\beta}{2}, b = \beta_1 - \frac{\beta}{2}$):
| Field Type                                      | $a$ | $b$ |
|------------------------------------------------|-----|-----|
| conf. coupled real scalars $\phi$ ($\Delta \sim \Box - \frac{1}{6} R$): | 1   | 1   |
| spin $\frac{1}{2}$ (two-component) fermions $\chi$: | 3   | $\frac{11}{2}$ |
| massless gauge fields $A_\mu$:                   | 12  | 62  |
| Gravitino $\psi_\mu$:                           | $-298$ | $-548$ |
| Graviton $(\det O^1 O)^{-1/2}(\det L^1 L)^{1/2}$: | 796 | 1566 |
| Scalar $\varphi$ with kin. term $\Box^2$, min. coupled: | $-8$ | $-28$ |
| Fermion $\lambda$, superpartner of this $\varphi$: | $-1$ | $-\frac{27}{2}$ |

Similarly as in 2d \[21\], we now add the conformal factor $\phi$ as an independent new matter field \[13\] with standard measure and action

$$
\frac{1}{32\pi^2} \int d^4 x \sqrt{g} \left\{ (\phi \Box^2 \phi - \frac{1}{2} Q \hat{G} \phi - \frac{1}{2} P \hat{W}^2 \phi) \right\}
$$

($P$ will play a minor role here) to the theory and demand conformal invariance of the total theory. This means: i), conformal anomalies have to cancel; ii), scaling operators are dressed to have dimension 4, etc.

As for i), the free theory for $\phi$ has conformal anomaly

$$
a = a_{\Box^2} + \frac{1}{4} P Q, \quad b = b_{\Box^2} + \frac{1}{4} Q^2,
$$

where $a_{\Box^2}, b_{\Box^2}$ are read off from the table. Therefore cancellation of conformal anomalies fixes $Q, P$: $b + b_{O^1 O} + b_{L^1 L} + b_{\text{matter}} = 0, a + a_{O^1 O} + a_{L^1 L} + a_{\text{matter}} = 0$ implies

$$
Q^2 = \frac{1}{90} (N_0 + 11 N_{\frac{1}{2}} + 62 N_1 + 1538), \quad P Q = -\frac{1}{30} (N_0 + 6 N_{\frac{1}{2}} + 12 N_1 + 788).
$$

where $N_0, N_{\frac{1}{2}}, N_1$ are the number of conformally coupled scalars, spin $\frac{1}{2}$ fermions and massless gauge fields.

A key point, noted by Antoniadis and Mottola \[13\], is that due to the $\Box^2$ kinetic term for $\phi$, its propagator is logarithmic and therefore the exponential $e^{2a \phi}$ is a scaling operator, as in 2d (this is very unusual in 4d). Its dimension comes out to be

$$
\text{dim}(e^{a \phi}) = -\alpha(\alpha + Q)
$$
As in two dimensions, if \( \Phi_i \) is a scaling operator of the matter theory with conformal dimension \( \Delta_i \), the operator

\[
O_i \equiv \int d^4x \sqrt{g} e^{\gamma_i \phi} \Phi_i
\]

with \( \gamma_i \) determined by

\[
\Delta_i - \gamma_i (\gamma_i + Q) = 4
\]

is then a scale invariant operator that can be added to the action, at least infinitesimally.

We finally mention that there is also an analog of the \( c = 1 \) barrier of \( 2d \) gravity: instead of \( c \leq 1 \) we have \[13\]

\[+360 \ b \leq 98.\]

This is usually satisfied, as \( b \) is negative for conventional matter. One can also compute a scaling law e.g. for the fixed volume partition function \[14\]:

\[
Z(V) \sim V^{-3.67...}
\]

See part III of the author’s thesis for details; there, \( Q \equiv -\sqrt{-4B}, \phi \rightarrow -2\phi/Q, \gamma \rightarrow -\frac{Q}{2}\gamma. \)

Let us also quote the anomaly coefficients for some supersymmetry multiplets from \[33\]:

\[
\begin{array}{ccc}
\text{Multiplet} & \text{24} a & \text{-24} b \\
\text{N=1 supergravity multiplet:} & 102 & 72 \\
\text{N=1 scalar multiplet:} & 1 & \frac{1}{2} \\
\text{N=1 vector multiplet:} & 3 & \frac{9}{2} \\
\text{N=1 \( \Box^2 \) multiplet:} & -3 & -\frac{9}{2} \\
\text{N=2 supergravity multiplet:} & 52 & 41 \\
\text{N=2 hypermultiplet:} & 2 & 1 \\
\text{N=2 vector multiplet:} & 4 & 5 \\
\text{N=2 \( \Box^2 \) multiplet:} & 2 & -11 \\
\text{N=4 supergravity multiplet, min. coupled \( \phi \):} & -24 & -24 \\
\text{N=4 vector multiplet:} & 6 & 6
\end{array}
\]

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