Solution Of nth-Order Ordinary Differential Equations Using Lie Group

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Abstract. In the recent work, methods of solution nth-order linear and nonlinear ODE's of Lie group was introduced and the calculations of Lie point symmetries with higher order for ODEs were also achieved.

1. Introduction
Symmetry of DE mapped any solution to another one of DE. Symmetry based on methods to attack (in particular, non-linear) deterministic differential equations were introduced long ago. Generally the symmetries of a mathematical object are to be its invertible morphisms to itself. Symmetries of DEs are examined by [1],[2]. Lie symmetries of DEs may be used to construct exact solutions for the equations. The author in [4] realize a group and abelian group of symmetry, while authors in [3] define group of transformation (1-parameter Lie group of point transformation (1-P.L.G.O.P.T)), invariant of (function, Lie group of transformation, family of surface) also he established canonical coordinate and specified example group of scaling. Furthermore, he established the first fundamental theorem of Lie and construct an example of this theorem. Where the authors in [9] realize an invariant surface. Now in this work we insert the employment symmetry method to solve higher order of ODE. Moreover we employed this method to solve two cases of linear and nonlinear ODE's and give applications for that.

2. Preliminaries
In this section, some fundamental and necessary concepts were inserted in theory of symmetries which are needed later in studying Lie point of DE's and driving appropriate applications for the method.

Definition (2.1),[3],[5],[6]: Lie group transformations of (c)

\[ x^* = W(x;c) \quad \text{...(2.1)} \]

By employing Taylor expansion about neighborhood of \( c=0 \) and law of composition
\[ x^* = x + \left[ \frac{\partial W}{\partial (x; \varepsilon)} \right]_{\varepsilon=0} \varepsilon + \left[ \frac{\partial^2 W}{\partial \varepsilon^2} (x; \varepsilon) \right]_{\varepsilon=0} \frac{\varepsilon^2}{2!} + \ldots \]

\[ x^* = x + \varepsilon \xi(x) \quad \ldots (2.2) \]

(2.3)

Where

\[ \xi(x) = \frac{\partial W}{\partial (x; \varepsilon)} \bigg|_{\varepsilon=0} \quad \ldots (2.4) \]

The transformation \((T)\) of equation (2.3) is said to be the infinitesimal transformation \((\text{I.T})\) of (2.1), where component \(\xi(x) = (\xi_1(x), \xi_2(x), \ldots, \xi_n(x))\) are said to be the infinitesimal vector field of group (2.1).

Definition (2.2),[7],[4]: The infinitesimal operator may be sometimes called infinitesimal generator, group operator, Lie operator, group generator, admitted operator, group admitted, symmetry operator, infinitesimal symmetry, invariant group or Lie symmetry (2.1):

\[ X = X(x) = \xi(x).\nabla = \sum_{i=1}^{M} \xi_i(x) \frac{\partial}{\partial x_i} \quad \ldots (2.5) \]

\[ \nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_1} \right) \quad \ldots (2.6) \]

Where, \(\nabla\) is the gradient operator.

Theorem (2.1),[3],[2]: The (1-P.L.G.O.P.T) of (2.1) equal to:

\[ x^* = e^{\varepsilon X} x = \sum_{k=0}^{\infty} \frac{\varepsilon^k X^k x}{k!} \quad \ldots (2.7) \]

Where \(X^k = X X^{k-1}, K = 1, 2, \ldots \) and \(X^0 = I\)

2.1. Invariance of a Lie Group

In this section, an introducing and studying the invariance principle because since Lie group analysis based on it.

Definition (2.3),[8]: A mark \(x \in \mathbb{R}^N\) is called invariant mark if it stay same by every \((\text{L.G.O.P.T})\), i.e. \(x^* = x\), for all \(\varepsilon\).

Theorem (2.2)[8]: A point \(x \in \mathbb{R}^N\) is said to be invariant point of a \((\text{L.G.O.P.T})\) (2.1) with (2.5) iff
\[ \xi_r(x) = 0; \forall r = 1, 2, \ldots, M \]  
\hspace{7cm} (2.8)

**Remark (2.1), [10]:**
1-Any \((1-P.L.G.O.P.T)\) of plane has exactly one independent invariant.
2-Each \((1-P.L.G.O.P.T)\) of \(M(x)\) has exactly \(N-1\) functionally independent invariants for \(M\). Then the characteristic equation is follow

\[
X M(x) = \sum_{i=1}^{N} \xi_i(x) \frac{\partial M}{\partial x_i} = 0
\]

\[
\frac{dx^1}{\xi_1(x)} = \frac{dx^2}{\xi_2(x)} = \cdots = \frac{dx^N}{\xi_N(x)}, \quad dM = 0
\]

\hspace{7cm} (2.9)

We get \(J_1(x), J_2(x), \ldots, J_{N-1}(x)\) where
\(J_1(x) = c_1, J_2(x) = c_2, \ldots, J_{N-1}(x) = c_{N-1}, c_1, c_2, \ldots, c_{N-1}\) are constants
3-Any set of \(N-1\) functionally independent invariants is called a basis of invariants for \(M\), and \(M(x) = \phi(J_1(x), J_2(x), \ldots, J_{N-1}(x))\), is an invariant function.

**Theorem (2.3) [9], [3]:** \(M(x)\) is called Invariance Criterion under (2.1) iff

\[ X M(x) \equiv 0 \]  
\hspace{7cm} (2.11)

\[ M(x) = 0 \]  
\hspace{7cm} (2.12)

2.2. Prolongation Formula, [10]:

The derivatives \(y', y'', \ldots\) under performance of point transformations:
\[ x^* = M(x, y; \epsilon) \]  
\hspace{7cm} (2.13)

\[ y^* = N(x, y; \epsilon) \]  
\hspace{7cm} (2.14)

We write by employing the total differentiation of these transformation as:
\[
D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots
\]

\hspace{7cm} (2.15)

Whereas the determination the 1,2-derivatives can be put as:
\[
y'' = \frac{dy''}{dx} = \frac{DN}{DM} = \frac{N_x + y' N_y}{M_x + y'M_y} \equiv P(x, y, y', \epsilon)
\]

\hspace{7cm} (2.16)
\[ y^{*}\prime \prime = \frac{dy^{*}\prime}{dx^{*}} = \frac{DP}{DN} = \frac{P_{x} + y^{\prime} P_{y} + y^{\prime\prime} P_{y\prime}\prime}{N_{x} + y^{\prime} N_{y}} \]  \hspace{1cm} \text{(2.17)}

Now, group P of point transformations (2.13), (2.14) and besides the transformation (2.15) one can get the first prolongation denoted by \( P^{(1)} \), and the space of three variables \( (x, y, y\prime) \). Moreover, by the transformation (2.17) we find the group \( P^{(2)} \) in the space \( (x, y, y\prime, y\prime\prime) \), also the higher prolongations, \( P^{(1)}, P^{(2)} \) were recognized in:

\[ P^{(1)} = \zeta_{1} \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \zeta_{1}, \zeta_{i} = D(\eta) - y^{\prime} D(\zeta) \]  \hspace{1cm} \text{(2.18)}

\[ P^{(2)} = P^{(1)} + \zeta_{2} \frac{\partial}{\partial y_{\prime}} = D(\zeta_{2}) - y^{\prime\prime} D(\zeta) \]  \hspace{1cm} \text{(2.19)}

We said of (2.18) and (2.19) are first and 1, 2- prolongation

\[ \frac{dx^{*}}{d} = \zeta(x^{*}, y^{*}), \quad x^{*}|_{x=0} = x \]  \hspace{1cm} \text{(2.20)}

\[ \frac{dy^{*}}{d} = \eta(x^{*}, y^{*}), \quad y^{*}|_{x=0} = y \]

Where

\[ \zeta_{1} = D(\eta) - y^{\prime} D(\zeta) = \eta_{x} + (\eta_{y} - \zeta_{x}) y^{\prime} - y^{\prime\prime} \zeta_{y} \]

\[ \zeta_{2} = D(\zeta_{1}) - y^{\prime\prime} D(\zeta) = \eta_{xx} + (2\eta_{xy} - \zeta_{xx}) y^{\prime} + (\eta_{yy} - 2\zeta_{xy} - 3y^{\prime\prime} \zeta_{y}) y^{\prime\prime} \]  \hspace{1cm} \text{(2.21)}

\[ \zeta_{3} = \eta_{xxx} + (3\eta_{xxy} - \zeta_{xxx}) y^{\prime} + 3(\eta_{xyy} - \zeta_{xxy}) y^{\prime\prime} + (\eta_{yxy} - 3\zeta_{xyy}) y^{\prime\prime\prime} - \zeta_{yy} y^{\prime\prime\prime} + 3(\eta_{y} - \zeta_{y}) y^{\prime} - 2\zeta_{yy} y^{\prime\prime} - 3\zeta_{y} y^{\prime\prime} + (\eta_{y} - 3\zeta_{x} - 4\zeta_{y} y^{\prime}) y^{\prime\prime\prime} \]

2.3. Prolongation Formulas in Multidimension [11]:

This following with to multidimensional situation (with independent \( x^{i}, i=1,...,n \), and dependent variables \( u^{a}, a=1,...,m \)

Now, the reformulating the prolongation of (2.18), (2.19) reduce to:

\[ X = \xi^{i} \frac{\partial}{\partial x^{i}} + \eta^{a} \frac{\partial}{\partial u^{a}} \]  \hspace{1cm} \text{(2.22)}

With consideration of the the form
\( X^{(1)} = X + \zeta_{i}^{a} \frac{\partial}{\partial u_{i}^{a}} \)
\( X^{(2)} = X^{(1)} + \zeta_{ij}^{a} \frac{\partial}{\partial u_{ij}^{a}} \)

Where
\[
\zeta_{i}^{a} = D_{j}(\eta^{a}) - u_{j}^{a} D_{i}(\zeta^{j}) \\
\zeta_{ij}^{a} = D_{j}(\zeta^{a}_{i}) - u_{ik}^{a} D_{j}(\zeta^{k})
\]

And
\[
D_{i} = \frac{\partial}{\partial x^{i}} + u_{j}^{a} \frac{\partial}{\partial u_{j}^{a}} + u_{ij}^{a} \frac{\partial}{\partial u_{ij}^{a}} + \ldots 
\]

\( (2.23) \)

\[
(2.24) \)

2.4. Extended Prolongation of Lie Group and Infinitesimals:
the study of convenient extended prolongation for (1-P.L.G.O.P.T) and admitted by a kth-order with one dependent variable y and one independent variable x.

Theorem (2.4),[3]:
The (L.G.O.T) (2.13),(2.14) extends to its kth extension , k≥2 , in a form (1- P.L.G.O.P.T). It cuts on (x,y,y_{1},…,y_{k})-space:
\[
X^{*} = M(x, y; \epsilon) \\
y^{*} = N(x, y; \epsilon) \\
y^{*} = N_{i}(x, y, y' ; \epsilon)
\]

\( \ldots (2.26) \)

Where \( N_{1} = N_{1}(x, y, y';\epsilon) \) is defined by (2.26), \( N_{k-1} = N_{k-1}(x, y, y';\epsilon) \)

2.5. Extended Infinitesimals transformations of Lie group of 1-Dependent Variable and 1- Independent Variable.[3]:
Consider the (1-P.L.G.O.T)
\[
x^{*} = M(x, y; \epsilon) = x + \epsilon \zeta(x, y) + O(\epsilon^{2}) \quad \ldots (2.27) \\
y^{*} = N(x, y; \epsilon) = y + \epsilon \eta(x, y) + O(\epsilon^{2}) \quad \ldots (2.28)
\]

has \( \zeta(x) = (\zeta(x, y), \eta(x, y)) \)

with vector field
Now, kth extending (2.27),(2.28) leads:

\[ x^* = M(x,y;\epsilon) = x + \epsilon \xi(x,y) + O(\epsilon^2) \]
\[ y^* = N(x,y;\epsilon) = y + \epsilon \eta(x,y) + O(\epsilon^2) \]
\[ y^{*'} = N_1(x,y,y';\epsilon) = y' + \epsilon \eta(1)(x,y,y') + O(\epsilon^2) \]
\[ \vdots \]
\[ y^{*k} = N_k(x,y,y',\ldots,y_k;\epsilon) = y_k + \epsilon \eta(k)(x,y,y',\ldots,y_k) + O(\epsilon^2) \]

The above expression has infinitesimal of the form

\[ (\xi(x,y), \eta(x,y), \eta(1)(x,y,y'), \ldots, \eta(k)(x,y,y',\ldots,y_k)) \]

Then the vector field of (2.29) the above expression:

\[ X^k = \zeta(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} + \eta(1)(x,y,y') \frac{\partial}{\partial y'} + \ldots \]
\[ + \eta(1)(x,y,y',\ldots,y^{k-1}) \frac{\partial}{\partial y^k} + \epsilon \frac{\partial}{\partial \epsilon} \]

Where \( k=1,2,\ldots \). Explicit formula for the expanded infinitesimals \( \eta^k \) is held.

**Theorem (2.5),[3]:** Consider

\[ \eta^k(x,y,y',\ldots,y^{k-1}) = D^k \eta(x,y), k = 1,2,\ldots \]

where \( \eta^0(x,y) \)

**3. Lie group of higher order for ODE's:**

Consider nth-order ODE given as:

\[ V(x,y,y',y'',\ldots,y^{(n)}) = 0 \]

Which can takes the following cases

1- **Case I:** equation (3.1) is non-linear.

2- **Case II:** The following equation (linear).

\[ m_n(x)y^{(n)} + m_{n-1}(x)y^{(n-1)} + \ldots + m_1(x)y' + m_0(x)y = Q(x) \]

Where \( m_0,m_1,\ldots,m_n \) are constant with \( m_n \neq 0 \), we solved (3.1), (3.2) in 2-cases:

**Algorithm (1):** we will insert this procedure involved by following steps to calculate the Lie point symmetry of higher order ODEs for case I:

Step 1: Write all terms of equation (3.1) on the left hand side.
Step 2: Write the generator of symmetry with unknown \( \xi \) and \( \eta \) of the form:

\[
X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}
\]

Step 3: Write n-prolongation of the symmetry generator \( X \) in style:

\[
X^{[n]} = X + \zeta^{(1)} \frac{\partial}{\partial y'} + \zeta^{(2)} \frac{\partial}{\partial y''} + \ldots + \zeta^{(n)} \frac{\partial}{\partial y^{(n)}}
\]

Step 4: Apply the prolonged generator \( X^{[n]} \) on \( V \) of (3.1) we locate the following:

\[
X^{[n]} \left( V(x, y, y', y'', \ldots, y^{(n)}) \right) \bigg|_{y=0} = 0
\]

Step 5: By using expansion of \( \zeta^{[1]}, \zeta^{[2]}, \zeta^{[3]} \) from (2.21) and \( \zeta^{(4)}, \zeta^{(5)}, \ldots, \zeta^{(n)} \) found by using maple package.

Step 6: Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in determined system of linear homogenous ODE's in the terms of \( \xi \) and \( \eta \).

Step 7: Find the general solution.

Algorithm (2): We calculate the Lie point symmetry of higher order ODEs for case II are given in the following steps:

Step 1: Write all terms of equation (3.2) on the left hand side.

Step 2: Write the vector filed of the form:

\[
X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}
\]

Step 3: We need find n-prolongation in style:

\[
X^{[n]} = X + \zeta^{(1)} \frac{\partial}{\partial y'} + \zeta^{(2)} \frac{\partial}{\partial y''} + \ldots + \zeta^{(n)} \frac{\partial}{\partial y^{(n)}}
\]

Step 4: Write \( X^{[n]} (y^{(n)} - \frac{1}{m_n} Q(x) - m_{n-1} y^{(n-1)} - \ldots - m_1 y' - m_0 y) \bigg|_{y^{(n)} = \frac{1}{m_n} Q(x) - m_{n-1} y^{(n-1)} - \ldots - m_1 y' - m_0 y} = 0 \)

Step 5: By using expansion of \( \zeta^{[1]}, \zeta^{[2]}, \zeta^{[3]} \) from (2.21) and \( \zeta^{(4)}, \zeta^{(5)}, \ldots, \zeta^{(n)} \) found by using maple package.

Step 6: Now replacing \( y^{(n)} \) by \( \frac{1}{m_n} Q(x) - m_{n-1} y^{(n-1)} - \ldots - m_1 y' - m_0 y \)

Step 7: Find the general solution.

4. Applications

The followings are some examples of higher order of ODEs solved by using above algorithm (1).

Example (1): Consider the 10th-order of ODE in:

\[
y^{(10)} = \frac{1}{x^3} y' y^2 \quad \ldots (4.1)
\]
Now, calculate the Lie point symmetry of (4.1) with respect vector filed:

\[ X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad \text{...(4.2)} \]

considering 10th- prolongation in a style:

\[ X^{[10]} = X + \zeta^{(1)} \frac{\partial}{\partial y} + \zeta^{(2)} \frac{\partial}{\partial y''} + \ldots + \zeta^{(10)} \frac{\partial}{\partial y^{(10)}} \quad \text{...(4.3)} \]

\[ \frac{3}{x^2} y' y^2 \zeta - \frac{2}{x^2} y' y \eta - \frac{1}{x^3} y^2 \zeta^{(1)} + \zeta^{(10)} = 0 \quad \text{...(4.4)} \]

Now, by using expansion \( \zeta^{(1)}, \zeta^{(2)}, \text{and } \zeta^{(3)} \) of (2.21), where \( \zeta^{(4)}, \ldots, \zeta^{(10)} \) found by maple package, and replacing \( y^{(10)} \) by \( \frac{1}{x^2} y' y^2 \)

we get the general solution

\[ \zeta = c_1 x \quad \text{...(4.5)} \]

\[ \eta = -3c_1 y \quad \text{...(4.6)} \]

Then (4.1) has Lie point symmetry

\[ X = x \frac{\partial}{\partial x} - 3 y \frac{\partial}{\partial y} \quad \text{...(4.7)} \]

Example(2): The 5th–order of ODE for (case II) is:

\[ y^{(5)} = y^{-(F+1)} \quad F \text{ is constant} \quad \text{...(4.8)} \]

The vector field

\[ X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad \text{...(4.9)} \]

we must calculate:

\[ X^{[5]} \left( y^{(5)} - y^{-(k+1)} \right) \bigg|_{y^{(5)} = y^{-(k+1)}} = 0 \quad \text{...(4.10)} \]

The five extension compute In a manner:

\[ X^{[5]} = X + \zeta^{(1)} \frac{\partial}{\partial y} + \zeta^{(2)} \frac{\partial}{\partial y''} + \zeta^{(3)} \frac{\partial}{\partial y^{(3)}} + \zeta^{(4)} \frac{\partial}{\partial y^{(4)}} + \zeta^{(5)} \frac{\partial}{\partial y^{(5)}} \quad \text{...(4.11)} \]

Where

\[ \eta^{(i)}(x, y, y', \ldots, y^{(i)}) = \frac{d}{dx} \zeta^{(i-1)} - y^{(i)} \frac{d}{dx} \zeta, i = 1, 2, 3, 4, 5 \quad \text{...(4.12)} \]
With \( \eta^{(i)} = \frac{d^{i}y}{dx^{i}} \), \( \eta^{(0)} = \zeta^{(0)} = \eta(x, y) \)

d \frac{d}{dx} \) is the total differentiation realize:

\[
\frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''} + \ldots
\]

...(4.13)

The determining equation of (4.10) is given as follows:

\[
(\zeta^{(5)} + (F + 1)\eta y^{-(F+2)}) \bigg|_{y^{(1)}=y^{-(F+3)}} = 0
\]

...(4.14)

by using maple package we get the following result

\[
\eta = 5c_1, y
\]

...(4.15)

\[
\zeta = (F + 2)c_1, x + c_2
\]

...(4.16)

Where \( c_1 \) and \( c_2 \) are constants, then equation (4.8) has 2-generators are:

\[
X_1 = \hat{\partial}x
\]

\[
X_2 = (F + 2)x \hat{\partial}x + 5y \hat{\partial}y
\]

...(4.17)

Example(3): Consider the 15th- order of ODE for (case II) is the form

\[
y^{(15)} = \frac{y'^2}{y'^3}
\]

...(4.18)

the Lie point symmetry of (4.18) with respect vector filed:

\[
X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}
\]

...(4.19)

we needed 15th- prolongation of (4.19):

\[
X^{[15]} = X + \zeta^{(1)} \frac{\partial}{\partial y'} + \zeta^{(2)} \frac{\partial}{\partial y''} + \zeta^{(3)} \frac{\partial}{\partial y'''} + \ldots + \zeta^{(15)} \frac{\partial}{\partial y^{(15)}}
\]

...(4.20)

Equation (4.19) generates a point symmetry of (4.18) if the determining equation

\[
\zeta^{(15)} - 2\zeta^{(1)} \frac{y'}{y'^3} + 3\eta \frac{y'^2}{y'^4} = 0
\]

...(4.21)

Where \( \zeta^{(1)} \), \( \zeta^{(2)} \), \( \zeta^{(3)} \) are given by (2.21) respectively and \( \zeta^{(4)}, \ldots \zeta^{(15)} \) found
by maple package, we get the general solution:
\[ \zeta = c_1 x + c_2 \]
\[ \eta = \frac{13}{2} c_1 y \]

5. Conclusion
The Lie group method is an efficient technique to solve higher order of ODEs and can be applied for solved linear and nonlinear for nth-order of ODE's

Reference
[1] Stephani H 1989 *Differential Equations: Their Solutions Using Symmetry* (Cambridge University Press, New York).
[2] Olver P J 1987 *Group Invariant Solutions of Differential Equations* Siam J. Appl. Math. Vol.(47) No.2 .PP. 263-277.
[3] Bluman G W and Kumei S 1989 *Symmetry and Differential Equations* (Springer-Verlag, New York, NY, USA).
[4] Cantwell B J 2002 *Introduction to Symmetry Analysis* (Cambridge University Press: Cambridge, UK).
[5] Dresner L 1999 *Application of Lie’s Theory of Ordinary and Partial Differential Equations* (USA, British Library).
[6] Stephani H 1989 *Differential Equations: Their Solutions Using Symmetries* (Cambridge University Press, Cambridge, UK).
[7] Bluman G W, Anco S C and Cheviakov A F 2009 *Applications of Symmetry Methods to Partial Differential Equations* (Springer, New York, NY, USA).
[8] Bluman G W and Anco S C 2002 *Symmetry and Integration Methods for Differential Equations* (Springer, New York, NY, USA).
[9] Bluman G W and Cole J D 1974 *Similarity Methods Differential Equations* (Springer-Verlag, New York, NY, USA).
[10] Ibragimov N H 1994 *CRC Handbook of Lie Group Analysis of Differential Equation* (Vol. I,
[11] Grigoriev Y N et al 2010 *symmetries of Integro-Differential Equations: With Application in Mechanics and Plasma Physics* (Lect. Notes Phys. 806, Springer, Dordrecht, DOI: 10.1007/978-90-481-3797-8).