Energy conservation via a combination of velocity and its gradient in the Navier-Stokes system

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Abstract

In the spirit of recent work [15], it is shown that $v \in L^{\frac{2q}{p}}(0, T; L^{\frac{2q}{q-1}}(\mathbb{T}^d))$ and $\nabla v \in L^p(0, T; L^q(\mathbb{T}^d))$ imply the energy equality in homogeneous incompressible Navier-Stokes equations and together with bounded density with positive lower bound yields the energy conservation in the general compressible Navier-Stokes equations. This unifies the known energy conservation criteria via the velocity and its gradient in incompressible Navier-Stokes equations. This also helps us to extend the conditions via the velocity or gradient of the velocity for energy equality from the incompressible fluid to compressible flow and improves the recent results due to Nguyen-Nguyen-Tang [15, Nonlinearity 32 (2019)] and Liang [12, Proc. Roy. Soc. Edinburgh Sect. A (2020)].

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1 Introduction

The compressible Navier-Stokes equations in $\Omega \times [0, T)$ read

\[
\begin{aligned}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) - \text{div} (\nu(\rho) \mathcal{D} v) - \nabla (\mu(\rho) \text{div} v) + \nabla \Pi(\rho) &= 0, \\
(\rho, v)|_{t=0} &= (\rho_0, v_0),
\end{aligned}
\]

where $\rho$ stands for the density of the flow, $v$ represents the flow velocity field and $\Pi(\rho)$ is the scalar pressure; $\mathcal{D} v = \frac{1}{2}(\nabla v \otimes \nabla v^T)$ is the strain tensor; $\nu(\rho)$ and $\mu(\rho)$ are the viscosity coefficients. The classical isentropic compressible Navier-Stokes equations (1.2) and compressible Navier-Stokes equations with degenerate viscosity (1.3) are special cases of system (1.1) as follows:

\[
\begin{aligned}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla \Pi(\rho) - 2\mu \Delta v - \lambda \nabla \text{div} v &= 0,
\end{aligned}
\]

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\[
\begin{aligned}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) - \text{div} (\nu(\rho)Dv) + \nabla \Pi(\rho) &= 0.
\end{aligned}
\] (1.3)

In this paper, we consider the periodic problem, which means \( \Omega = T^3 \). The general pressure \( \Pi(\rho) \in C^1(0, \infty) \) and the viscosity coefficients \( \nu, \mu : (0, \infty) \to [0, \infty) \) are continuous functions of density.

Formally, the energy conservation in (1.1) is given by, for all \( t \in [0, T) \),
\[
\int_{\Omega} \left( \frac{1}{2} \rho |v|^2 + P(\rho) \right) dx + \int_0^T \int_{\Omega} \left( \nu(\rho) |Dv|^2 + \mu(\rho) |\text{div} v|^2 \right) dx dt
= \int_{\Omega} \left( \frac{1}{2} \rho_0 |v_0|^2 + P(\rho_0) \right) dx,
\] (1.4)

where \( P(\rho) = \rho \int_1^\rho \frac{\Pi(z)}{z^2} dz \).

In the spirit of well-known Shinbrot’s conditions in [19], Yu [23] showed that a weak solution \((\rho, v)\) of (1.3) or (1.2) satisfies
\[
\sqrt{\rho} v \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\rho} \nabla v \in L^2(0, T; L^2(\Omega)),
\]
\[
0 \leq \rho \leq c < \infty, \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))
\]
\[
v \in L^p(0, T; L^q(\Omega)), \quad \frac{1}{p} + \frac{1}{q} \leq \frac{5}{12} \quad \text{and} \quad q \geq 6,
\] (1.5)

then the energy conservation (1.4) holds.

Subsequently, there has been extensive studies involving energy equality of weak solutions of the compressible flow (see e.g. [1, 6, 10, 12, 15, 17]). Before we present these recent progress, we turn our attention to Shinbrot’s conditions in [19] for energy conservation of Leray-Hopf weak solutions in the homogeneous incompressible Navier-Stokes equations. Shinbrot extended Lions’ condition in [14] \( v \in L^4(0, T; (L^4(\Omega))) \) for energy conservation to
\[
v \in L^p(0, T; L^q(\Omega)) \quad \text{with} \quad \frac{2}{p} + \frac{2}{q} = 1, q \geq 4.
\] (1.6)

It is worth remarking that condition (1.5) can not yield the full regularity of the weak solutions. The energy conservation condition (1.6) can be replaced by
\[
v \in L^p(0, T; L^q(\Omega)) \quad \text{with} \quad \frac{1}{p} + \frac{3}{q} = 1, 3 < q < 4;
\] (1.7)

which was stated by Taniuchi in [21] and whose proof was given by Beirao da Veiga-Yang in [3]. Very recently, Berselli-Chiodaroli [4] and Zhang [24] obtained energy equality via the following condition,
\[
\nabla v \in L^p(0, T; L^q(\Omega)), \quad \frac{1}{p} + \frac{3}{q} = 2, \quad \frac{2}{2} < q < \frac{9}{5} \quad \text{or} \quad \frac{1}{p} + \frac{6}{5q} = 1, \frac{9}{5} \leq q.
\] (1.8)

We turn our attentions back to the compressible Navier-Stokes equations. For equations (1.2), Chen-Liang-Wang-Xu [7] obtained the energy balance in a bounded domain with physical boundaries under the following condition
\[
\sqrt{\rho} v \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\rho} \nabla v \in L^2(0, T; L^2(\Omega)),
\]
\[
0 \leq \rho \leq c < \infty, \quad \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))
\]
\[
v \in L^p(0, T; L^q(\Omega)), p \geq 4, q \geq 6.
\] (1.9)
For the weak solutions to the general compressible models (1.1), Nguye-Nguye-Tang [15] established the energy conservation criterion below

\[
0 < c_1 \leq \rho \leq c_2 < \infty, \quad v \in L^\infty(0, T; L^2(\mathbb{T}^3)), \quad \nabla v \in L^2(0, T; L^2(\mathbb{T}^3)),
\]

\[
\sup_{t \in (0, T)} \sup_{|h| < \varepsilon} |h|^{-\frac{2}{3}} \|\rho(\cdot + h, t) - \rho(\cdot, t)\|_{L^2(\mathbb{T}^3)} < \infty,
\]

\[
v \in L^p(0, T; L^q(\mathbb{T}^3)) \quad \text{with} \quad \begin{cases} \frac{2}{p} + \frac{3}{q} = 1, q \geq 4, \\ \frac{1}{p} + \frac{3}{q} = 1, 3 < q < 4. \end{cases}
\]  

(1.10)

Though the part \(\frac{1}{p} + \frac{3}{q} = 1, 3 < q < 4\) was not mentioned in [15], it is a direct consequence of \(v \in L^4(0, T; L^4(\mathbb{T}^3))\) (see [3] and the corresponding proof in Theorem 1.1). Conservation of energy in the incompressible Euler equations is closely related to Onsager’s conjecture [18]. Onsager’s conjecture is concerned with the critical regularity of weak solutions in Euler system to guarantee persistence of energy. In this direction, recent progress can be found in [2, 3, 8, 9, 11, 16, 17]. As pointed in [15], the generalized Onsager’s conjecture for fluid equations addresses the question how much regularities are needed for a weak solution to conserve energy. Note that, blow-up criteria only via the density for strong solutions to the 3-D compressible Navier-Stokes equations yields strong solutions. The strong solutions are expected to meet energy conservation. Based on this, we try to remove (1.10) to obtain the persistence of energy. In addition, it is worth pointing out that Liang [12] recently derived the energy conservation for isentropic Navier-Stokes equations under the following condition

\[
0 < c_1 \leq \rho \leq c_2 < \infty, \quad v \in L^\infty(0, T; L^2(\mathbb{T}^3)), \quad \nabla v \in L^2(0, T; L^2(\mathbb{T}^3)),
\]

\[
\nabla v \in L^p \left(0, T; L^s(\mathbb{T}^3)\right) \quad \text{with} \quad \begin{cases} \frac{1}{p} + \frac{3}{s} < 2, & 3 < s < \frac{9}{5}, \\ \frac{5}{p} + \frac{6}{s} < 5, & \frac{9}{5} \leq s \leq 3, \\ \frac{1}{p} + \frac{2}{s + 2} < 1, & 3 < s < \infty. \end{cases}
\]  

(1.11)

Inspired by these works, the second objective of this paper is to show (1.10) and (1.7) or (1.8) guarantee the energy equality in system (1.1). Before stating the main results, we introduce the definition of the weak solutions.

**Definition 1.1.** A pair \((\rho, v)\) is called a weak solution to (1.1) with initial data \((\rho_0, v_0)\) if \((\rho, v)\) satisfy

(i) \(P(\rho), \rho|v|^2 \in L^\infty(0, T; L^1(\Omega)), \quad \nabla v \in L^2(0, T; L^2(\Omega)),\)

(ii) \(\rho(\cdot, t) \rightharpoonup \rho_0 \quad \text{in} \ D'(\Omega) \quad \text{as} \quad t \rightarrow 0, \quad \text{i.e.}\)

\[
\lim_{t \rightarrow 0} \int_{\Omega} \rho(x, t) \varphi(x) dx = \int_{\Omega} \rho_0(x) \varphi(x) dx,
\]  

(1.13)

for every test function \(\varphi \in C_0^\infty(\Omega).\)
(iii) \((\rho v)(\cdot, t) \to \rho_0 v_0\) in \(D'(\Omega)\) as \(t \to 0\) i.e.

\[
\lim_{t \to 0} \int_{\Omega} (\rho v)(x, t) \psi(x) dx = \int_{\Omega} (\rho_0 v_0)(x) \psi(x) dx,
\]

for every test vector field \(\psi \in C^\infty_0(\Omega)^d\).

(iv) the energy inequality holds

\[
\int_{\Omega} \left[ \frac{1}{2} \rho |v|^2 + P(\rho) \right] dx + \int_0^T \int_{\Omega} \left[ \nu(\rho) |\nabla v|^2 + \mu(\rho) |\text{div} \, v|^2 \right] dx dt 
\leq \int_{\Omega} \left( \frac{1}{2} \rho_0 |v_0|^2 + P(\rho_0) \right) dx,
\]

where \(P(\rho) = \rho \int_1^\rho \Pi(z) dz\).

We formulate our first result as follows

**Theorem 1.1.** The energy equality (1.4) of weak solutions \((\rho, v)\) to the Navier-Stokes equation (1.1) is valid if one of the following two conditions is satisfied

1. \(0 < c_1 \leq \rho \leq c_2 < \infty\), \(v \in L^\infty(0, T; L^2(\mathbb{T}^3))\), \(\nabla v \in L^2(0, T; L^2(\mathbb{T}^3))\)

   \[
v \in L^p(0, T; L^q(\mathbb{T}^3)) \quad \text{with} \quad \begin{cases}
   \frac{2}{p} + \frac{2}{q} = 1, q \geq 4, \\
   \frac{1}{p} + \frac{3}{q} = 1, 3 < q < 4;
\end{cases}
\]

2. \(0 < c_1 \leq \rho \leq c_2 < \infty\), \(v \in L^\infty(0, T; L^2(\mathbb{T}^3))\) and \(\nabla v \in L^2(0, T; L^2(\mathbb{T}^3))\)

   \[
   \nabla v \in L^p(0, T; L^q(\mathbb{T}^3)) \quad \text{with} \quad \begin{cases}
   \frac{1}{p} + \frac{3}{q} = 2, \frac{3}{2} < q < \frac{9}{5}, \\
   \frac{1}{p} + \frac{6}{5q} = 1, \frac{9}{5} \leq q.
\end{cases}
\]

**Remark 1.1.** Compared with result (1.10), conditions (1.16) in Theorem 1.1 only required the bounded density with positive lower bound. Result (1.16) is an improvement of (1.5) without vacuum.

**Remark 1.2.** We extend the energy conservation criteria (1.6)-(1.8) from incompressible Navier-Stokes equations to general compressible Navier-Stokes equations.

**Remark 1.3.** In contrast with (1.11), the generalization in (1.17) is two fold: first, to improve the corresponding results in (1.17), second, to consider the more general equations.

It should be pointed out that Theorem 1.1 is a direct consequence of the following theorem.

**Theorem 1.2.** For any dimension \(d \geq 2\), the energy equality (1.4) of weak solutions \((\rho, v)\) to the Navier-Stokes equation (1.1) is valid provided

\[
\begin{cases}
0 < c_1 \leq \rho \leq c_2 < \infty, v \in L^\infty(0, T; L^2(\mathbb{T}^d)), \nabla v \in L^2(0, T; L^2(\mathbb{T}^d)), \\
v \in L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(\mathbb{T}^d)), \\
\nabla v \in L^p(0, T; L^q(\mathbb{T}^d)).
\end{cases}
\]
Remark 1.4. At first glance, energy conservation criteria (1.18) are based on a combination of velocity and its gradient and are more complicated than (1.16) and (1.17), however, (1.18) together with natural energy $v \in L^\infty(0,T; L^2(\mathbb{T}^3)), \nabla v \in L^2(0,T; L^2(\mathbb{T}^3))$ leads to (1.16) and (1.17). For the details, see the proof of Theorem 1.1.

Remark 1.5. From Theorem 1.2 $v \in L^{2p-1}(0,T; L^{2q-2}(\mathbb{T}^3))$ and $\nabla v \in L^p(0,T; L^q(\mathbb{T}^3))$ means the energy equality in the classical homogenous incompressible Navier-Stokes equations. As mentioned in latter remark, this result covers the (1.6)- (1.8), hence, roughly speaking, this unifies the known energy conservation criteria via the velocity and its gradient in incompressible Navier-Stokes equations.

Remark 1.6. We follow the path of [15] to prove Theorem 1.2. The new ingredient is the application of the following inequality

$$\left\| \nabla \left( \frac{(\rho v)^\xi}{\rho^{\xi}} \right) \right\|_{L^p(\mathbb{T}^d)} \leq C \left\| \nabla v \right\|_{L^p(\mathbb{T}^d)}. \quad (1.19)$$

This helps us to pass the limit of pressure term only with the positive bounded density, which removes the additional condition of the density (1.10) in [15]. For the proof of (1.19), we refer the readers to Lemma 2.3 (see also [12, page 7]).

Remark 1.7. One can consider Theorem 1.1 and Theorem 1.2 on smooth bounded domain. Combining the framework for bounded domain in [15] and the proof here, one need to deal with the boundary terms caused by integrating by parts. Fortunately, these additional terms are the lower order terms.

Remark 1.8. In dimension $d = 2$, the Gagliardo-Nirenberg inequality guarantees that

$$\left\| v \right\|_{L^4(0,T; L^4(\mathbb{T}^2))} \leq C \left\| v \right\|_{L^\infty(0,T; L^2(\mathbb{T}^2))}^{\frac{1}{2}} \left\| \nabla v \right\|_{L^2(0,T; L^2(\mathbb{T}^2))}^{\frac{1}{2}} \leq C.$$ 

Therefore, according to Theorem 1.2, the bounded density with positive lower bound and natural energy yield the energy conservation of the weak solutions.

Finally, as [15], one can establish the results parallel to Theorem 1.1 and 1.2 for the non-homogenous incompressible Navier-Stokes equations below

$$\begin{cases}
\rho_t + \text{div} (\rho v) = 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) - \text{div} (\rho \mathcal{D} v) + \nabla \Pi = 0, \\
\text{div} v = 0, \\
(\rho, v)|_{t=0} = (\rho_0, u_0),
\end{cases} \quad (1.20)$$

we leave this to the interested readers.

The remainder of this paper is organized as follows. Section 2 is devoted the auxiliary lemmas involving mollifier and the key inequality (1.19). In section 3, we first present the proof of Theorem 1.2. Then, based on Theorem 1.2 we complete the proof of Theorem 1.1.

2 Notations and some auxiliary lemmas

First, we introduce some notations used in this paper. For $p \in [1, \infty]$, the notation $L^p(0,T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in
Moreover, if \( p,q < \) and, if \((1.5)\) Suppose that Lemma 2.1.

**Proof.**

where \( \Omega \) is the bounded smooth functions on \( \Omega \). For simplicity, we denote by

\[
\int_0^t \int_{\mathbb{R}^d} f(x,\tau)dx\tau = \int_0^t \int f \quad \text{and} \quad \|f\|_{L^p(0,T;X)} = \|f\|_{L^p(X)}.
\]

Let \( \eta_\varepsilon : \mathbb{R}^d \to \mathbb{R} \) be a standard mollifier i.e. \( \eta(x) = C_0 e^{-\frac{1}{1-|x|^2}} \) for \( |x| < 1 \) and \( \eta(x) = 0 \) for \( |x| \geq 1 \), where \( C_0 \) is a constant such that \( \int_{\mathbb{R}^d} \eta(x)dx = 1 \). For \( \varepsilon > 0 \), we define the rescaled mollifier \( \eta_\varepsilon(x) = \frac{\varepsilon}{\varepsilon}\eta(\frac{x}{\varepsilon}) \). For any function \( f \in L^1_{loc}(\Omega) \), its mollified version is defined as

\[
f_\varepsilon(x) = (f \ast \eta_\varepsilon)(x) = \int_{\mathbb{R}^d} f(x-y)\eta_\varepsilon(y)dy, \quad x \in \Omega_\varepsilon,
\]

where \( \Omega_\varepsilon = \{x \in \Omega : d(x,\partial\Omega) > \varepsilon\} \).

We first recall the results involving the mollifier established in [13].

**Lemma 2.1.** ([13]) Suppose that \( f \in L^p(0,T;L^q(\mathbb{T}^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\|\nabla f_\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}, \tag{2.1}
\]

and, if \( p,q < \infty \)

\[
\limsup_{\varepsilon \to 0} \varepsilon \|\nabla f_\varepsilon\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0.
\]

Moreover, if \( 0 < c_1 \leq g \leq c_2 < \infty \), then there holds, for any \( \varepsilon > 0 \),

\[
\left\|\frac{\nabla f_\varepsilon}{g}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1}\|f\|_{L^p(0,T;L^q(\mathbb{T}^d))}, \tag{2.2}
\]

and if \( p,q < \infty \)

\[
\limsup_{\varepsilon \to 0} \varepsilon \left\|\frac{\nabla f_\varepsilon}{g}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{2.3}
\]

The next lemma with \( p = q, p_1 = q_1, p_2 = q_2 \) was proved in [13]. We generalize it by extending the integral norms with different exponents in space and time.

**Lemma 2.2.** Let \( 1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume \( f \in L^{p_1}(0,T;W^{1,q_1}(\mathbb{T}^d)) \) and \( g \in L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\left\|(fg)_\varepsilon - \varepsilon^{\frac{1}{p_1}}\varepsilon^{\frac{1}{p_2}}g\frac{1}{q_1}\frac{1}{q_2}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} \leq C\varepsilon^{\frac{1}{p_1}}\varepsilon^{\frac{1}{p_2}}\varepsilon^{\frac{1}{q_1}}\varepsilon^{\frac{1}{q_2}}\|f\|_{L^{p_1}(0,T;W^{1,q_1}(\mathbb{T}^d))}\|g\|_{L^{p_2}(0,T;L^{q_2}(\mathbb{T}^d))}. \tag{2.4}
\]

Moreover, if \( q_1, q_2 < \infty \) then

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-\frac{1}{p}}\left\|(fg)_\varepsilon - \frac{1}{q_1}\frac{1}{q_2}g\frac{1}{q_1}\frac{1}{q_2}\right\|_{L^p(0,T;L^q(\mathbb{T}^d))} = 0. \tag{2.5}
\]

**Proof.** Thanks to the fact observed in [9] and the ideas in [13], we know that

\[
(fg)_\varepsilon - \frac{1}{q_1}\frac{1}{q_2}g\frac{1}{q_1}\frac{1}{q_2} = R_\varepsilon - (f_\varepsilon - f)(g_\varepsilon - g), \tag{2.6}
\]

\[
(\text{f(x,y,z,t)}) = \int_a^b \int_0^1 \int_0^1 \int_0^1 f(x,y,z,t)dzdydt,
\]

\[
(\text{g(x,y,z,t)}) = \int_0^b \int_0^1 \int_0^1 \int_a^b g(x,y,z,t)dzdydt.
\]
where
\[ R^\varepsilon(x, s) := \int_{\mathbb{R}^d} (f(y, s) - f(x, s)) (g(y, s) - g(x, s)) \eta_{\varepsilon}(x - y) dy. \]

Triangle inequality yields that
\[ \| (f \varepsilon)^* - f^\varepsilon g^\varepsilon \|_{L^q(\mathbb{T}^d)} \leq \| R^\varepsilon \|_{L^q(\mathbb{T}^d)} + \| (f - f^\varepsilon)(g - g^\varepsilon) \|_{L^q(\mathbb{T}^d)}. \] (2.7)

By means of Hölder’s inequality and direct computation, we see that
\[
| R^\varepsilon | \leq \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^d} | f(y) - f(x) | | g(y) - g(x) | dy
\]
\[
\leq C \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} | f(y) - f(x) |^{q_1} dy \right)^{\frac{1}{q_1}} \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} | g(y) - g(x) |^{q_2} dy \right)^{\frac{1}{q_2}}
\]
\[
\leq C \varepsilon \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} \int_0^1 | \nabla f(x + (y - x)s) |^{q_1} ds dy \right)^{\frac{1}{q_1}} \left( \int_{B(0, 1)} | g(x + \omega x) |^{q_2} d\omega + | g(x) |^{q_2} \right)^{\frac{1}{q_2}}
\]
\[
\leq C \varepsilon \left( \int_{\mathbb{R}^d} | \nabla f(x - z) |^{q_1} \int_0^1 \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) | ds dz \right)^{\frac{1}{q_1}} \left( \int_{\mathbb{R}^d} | g(x - z) |^{q_2} \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) | dz + | g(x) |^{q_2} \right)^{\frac{1}{q_2}}
\]
\[
\leq C \left( | \nabla f |^{q_1} * J^\varepsilon \right)^{\frac{1}{q_1}} \left( | g |^{q_2} * J_{1\varepsilon} + | g(x) |^{q_2} \right)^{\frac{1}{q_2}} ,
\] (2.8)

where \( J^\varepsilon = \int_0^1 \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) | ds \geq 0 \), \( J_{1\varepsilon} = \int_0^1 \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) | ds = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) | dz = | B(0, 1) | \).

In view of the Young’s inequality, we conclude that
\[
\| R^\varepsilon \|_{L^q} \leq C \varepsilon \left( | \nabla f |^{q_1} * J^\varepsilon \right)^{\frac{1}{q_1}} \left( | g |^{q_2} * J_{1\varepsilon} + | g(x) |^{q_2} \right)^{\frac{1}{q_2}} \|_{L^q} \]
\[
\leq C \varepsilon \left( \| \nabla f \|_{L^{q_1}} \| g \|_{L^{q_2}} \right)^{\frac{1}{q_1}} \left( \| g \|_{L^{q_2}} + \| g \|_{L^{q_2}} \right)^{\frac{1}{q_2}} \]
\[
\leq C \varepsilon \| \nabla f \|_{L^{q_1}} \| g \|_{L^{q_2}} .
\] (2.9)

Furthermore, one has
\[
| (f^\varepsilon - f)(g^\varepsilon - g) |
\]
\[
\leq \int | (f(y) - f(x)) | \eta_{\varepsilon}(x - y) dy \int | (g(y) - g(x)) | \eta_{\varepsilon}(x - y) dy
\]
\[
\leq C \varepsilon \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} \int_0^1 | \nabla f(x + (y - x)s) | ds dy \right)^{\frac{1}{q_1}} \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} | g(y) - g(x) | dy \right)^{\frac{1}{q_2}}
\]
\[
\leq C \varepsilon \left( \int_{B(x, \varepsilon)} \int_0^1 | \nabla f(x + (y - x)s) |^{q_1} ds dy \right)^{\frac{1}{q_1}} \left( \frac{1}{\varepsilon^d} \int_{B(x, \varepsilon)} | g(y) - g(x) |^{q_2} dy \right)^{\frac{1}{q_2}} .
\] (2.10)

Along the same lines of derivation of (2.8) and (2.9), we arrive at
\[
\| (f^\varepsilon - f)(g^\varepsilon - g) \|_{L^q} \leq C \varepsilon \| \nabla f \|_{L^{q_1}} \| g \|_{L^{q_2}} .
\] (2.11)
In combination with (2.6), (2.9) and (2.11) and using the Hölder’s inequality, we can deduce the result (2.4).

Furthermore, if \( q_1, q_2 < \infty \), let \( \{f_n\}, \{g_n\} \in C_0^\infty (\mathbb{T}^d) \) with \( f_n \to f, \ g_n \to g \) strongly in \( W^{1,q_1} \) and \( L^{q_2} \), respectively. Thus, by the density arguments, we find that

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^q} \leq C \|((fg - g_n)^\varepsilon + (fg_n)^\varepsilon - f^\varepsilon (g - g_n)^\varepsilon)\|_{L^q} \\
\leq C \|((fg - g_n)^\varepsilon - f^\varepsilon (g - g_n)^\varepsilon)\|_{L^q} + \|((fg_n)^\varepsilon - f^\varepsilon g_n^\varepsilon)\|_{L^q}
\]

(2.12)

which means

\[
\varepsilon^{-1} \|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(L^q)} \leq C (\|\nabla f\|_{L^p} \|g - g_n\|_{L^{q_2}} + \varepsilon \|\nabla g_n\|_{L^{q_2}} \|\nabla g_n\|_{L^{q_2}}),
\]

(2.13)

hence, as \( n \to \infty \) and \( \varepsilon \to 0 \), we infer that

\[
\varepsilon^{-1} \|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(L^q)} \leq C \left( \int_0^T \|\nabla f\|_{L^{p_1}} \|g - g_n\|_{L^{q_2}} + \varepsilon \|\nabla g_n\|_{L^{q_2}} \|\nabla g_n\|_{L^{q_2}} \right)^{\frac{1}{p}} + C \varepsilon \left( \int_0^T \|\nabla f\|_{L^p} \|\nabla g_n\|_{L^{q_2}} \|\nabla g_n\|_{L^{q_2}} \right)^{\frac{1}{p}}
\]

(2.14)

Then, we have completed the proof of Lemma 2.2. \( \square \)

The next lemma is the critical point to remove \( (1.10)_2 \).

**Lemma 2.3.** Assume that \( 0 < p \leq \rho(x,t) \leq \overline{p} < \infty \) and \( v \in W^{1,p}(\mathbb{T}^d) \) with \( 1 \leq p \leq \infty \). Then

\[
\left\| \partial \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^p(\mathbb{T}^d)} \leq C \|\nabla v\|_{L^p(\mathbb{T}^d)}.
\]

(2.15)

**Proof.** By direct computation, one has

\[
\partial \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) = \frac{\partial (\rho v)^\varepsilon}{\rho^\varepsilon} - \frac{(\rho v)^\varepsilon - \rho^\varepsilon v}{(\rho^\varepsilon)^2} \partial \rho^\varepsilon = I_1 + I_2.
\]

(2.16)

Let \( B(x,\varepsilon) = \{y \in \Omega : |x - y| < \varepsilon \} \), then Using the Hölder’s inequality, we have

\[
|I_1| \leq C |\int \rho(y) (v(y) - v(x)) \nabla x \eta_y(x-y)dy|
\]

\[
\leq C \|\rho\|_{L^\infty} \int_{\mathbb{R}^d} |v(y) - v(x)| \frac{1}{\varepsilon^d} \nabla \eta_y \frac{x-y}{\varepsilon} dy
\]

(2.17)

\[
\leq C \left( \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} |v(y) - v(x)|^p \frac{1}{\varepsilon^p} dy \right)^{\frac{1}{p}}.
\]
Using the mean value theorem, one has
\[ \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} |v(y) - v(x)|^p \frac{dy}{\varepsilon^p} \leq C \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \int_0^1 |\nabla v(x + (y - x)s)|^p \frac{|y - x|^p}{\varepsilon^p} ds \, dy \]
\[ \leq C \int_0^1 \int_{B(x,\varepsilon)} |\nabla v(x + s\varepsilon)|^p d\omega ds \]
\[ \leq C \int_{\mathbb{R}^d} |\nabla v(x - z)|^p \int_0^1 1_{B(0,\varepsilon)}(z) \frac{1}{\varepsilon^d} ds \, dz \]
\[ = |\nabla v|^p * J_\varepsilon(z), \tag{2.18} \]

where \( J_\varepsilon(z) = \int_0^1 1_{B(0,\varepsilon)}(z) \frac{1}{\varepsilon^d} ds \, dz \geq 0 \) and it’s easy to check that \( \int_{\mathbb{R}^d} J_\varepsilon dz = \text{meas}(B(0,1)) \).

Next, to estimate \( I_2 \), due to the Hölder’s inequality, one deduces
\[ |I_2| = |\int \rho(y) (v(y) - v(x)) \eta_\varepsilon(x - y) dy| \frac{\rho(y) \nabla \eta_\varepsilon(x - y) dy}{(\int \rho(y) \nabla \eta_\varepsilon(x - y) dy)^2} | \]
\[ \leq C \|\rho\|_{L^\infty} \int_{B(x,\varepsilon)} |v(y) - v(x)| \frac{1}{\varepsilon^d} dy \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^d} \nabla \eta_\varepsilon(x - y) dy | \frac{1}{\varepsilon^d} dy \]
\[ \leq C \left( \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \frac{|v(y) - v(x)|^p}{\varepsilon^p} dy \right)^{\frac{1}{p}}. \tag{2.19} \]

Therefore, by the same arguments as in \( \text{(2.18)} \), in combination with \( \text{(2.16)} - \text{(2.19)} \), we have
\[ |I_1| + |I_2| \leq C \left( |\nabla v|^p * J_\varepsilon(z) \right)^{\frac{1}{p}}. \tag{2.20} \]

Then from the Young’s inequality, we arrive at
\[ \left\| \partial_t (\rho v)^\varepsilon \right\|_{L^p(\mathbb{T}^d)} \leq C \left( |\nabla v|^p * J_\varepsilon(z) \right)^{\frac{1}{p}} \left\| \rho v \right\|_{L^p} \]
\[ \leq C \left\| \nabla v \right\|_{L^p} \left\| J_\varepsilon \right\|_{L^1}^{\frac{1}{p}} \leq C \left\| \nabla v \right\|_{L^p}. \tag{2.21} \]

Then we have completed the proof of lemma \( \text{(2.3)} \).

\[ \square \]

### 3 Proof of theorem \( \text{1.1} \) and \( \text{1.2} \)

In this section, we first present the proof of Theorem \( \text{1.2} \). Then, making the use of interpolation and the natural energy, we prove Theorem \( \text{1.1} \) by the results of Theorem \( \text{1.2} \).

**Proof of Theorem \( \text{1.2} \).** Multiplying \( \text{(1.1)}_2 \) by \( \left( \frac{\rho v}{\rho^\varepsilon} \right)^\varepsilon \), then integrating over \( (s,t] \times \mathbb{T}^d \) with \( 0 < s < t < T \), we have
\[ \frac{1}{\sqrt[T]{t-s}} \int_s^t \int_{\mathbb{T}^d} \left[ \partial_t (\rho v)^\varepsilon + \text{div} (\rho v \otimes v)^\varepsilon + \nabla \Pi(\rho)^\varepsilon - \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon - \nabla (\mu(\rho) \text{div} v)^\varepsilon \right] = 0. \tag{3.1} \]
We will rewrite every term of the last equality to pass the limit of $\varepsilon$. For the first term, a straightforward calculation and (3.1) yields that
\[
\int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \partial_t (\rho v)^\varepsilon = \int_s^t \int \frac{1}{2} \partial_t (\frac{|(\rho v)^\varepsilon|^2}{\rho^\varepsilon}) + \frac{1}{2} \partial_t \rho^\varepsilon \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2} = \int_s^t \int \frac{1}{2} \partial_t (\frac{|(\rho v)^\varepsilon|^2}{\rho^\varepsilon}) - \frac{1}{2} \text{div} (\rho v)^\varepsilon \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2},
\]
Integration by parts means that
\[
\int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \text{div} (\rho v \otimes v)^\varepsilon = -\int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) [(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon] - \int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) (\rho v)^\varepsilon \otimes v^\varepsilon.
\]
Making use of integration by parts once again, we infer that
\[
-\int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) (\rho v)^\varepsilon \otimes v^\varepsilon = \int_s^t \int \text{div} (\frac{v^\varepsilon}{\rho^\varepsilon}) |(\rho v)^\varepsilon|^2 + \frac{1}{2} \int_s^t \int \text{div} (\rho v)^\varepsilon \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2}
= \frac{1}{2} \int_s^t \int \text{div} (\rho^\varepsilon v^\varepsilon) \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2}
= \frac{1}{2} \int_s^t \int \left[ \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\rho v)^\varepsilon + \frac{1}{2} \int_s^t \int \text{div} (\rho v)^\varepsilon \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2}
\]
Inserting (3.4) into (3.3), we arrive at
\[
\int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \text{div} (\rho v \otimes v)^\varepsilon = -\int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) [(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon]
- \int_s^t \int \left[ \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\rho v)^\varepsilon + \frac{1}{2} \int_s^t \int \text{div} (\rho v)^\varepsilon \frac{|(\rho v)^\varepsilon|^2}{(\rho^\varepsilon)^2}.
\]
For the pressure term, by the integration by parts, one has
\[
\int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\Pi(\rho))^\varepsilon = \int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla [\Pi(\rho)]^\varepsilon + \int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla \Pi(\rho)^\varepsilon
= -\int_s^t \int \text{div} \left[ \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right] [\Pi(\rho)]^\varepsilon + \int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla \Pi(\rho)^\varepsilon.
\]
Using the mass equation (3.1), the second term on the right hand-side of (3.6) can be
rewritten as
\[
\int_{s}^{t} \int \frac{\rho v}{\rho^\varepsilon} \nabla \Pi(\rho^\varepsilon) = \int_{s}^{t} \int \frac{\rho v}{\rho^\varepsilon} \nabla \int_{s}^{\frac{z}{\rho^\varepsilon}} \frac{\Pi'(z)}{z^2} dz dx d\tau
\]
\[
= \int_{s}^{t} \int \frac{\partial \rho^\varepsilon}{\rho^\varepsilon} \left[ \frac{\Pi(\rho^\varepsilon)}{\rho^\varepsilon} + \int_{s}^{\frac{z}{\rho^\varepsilon}} \frac{\Pi'(z)}{z^2} dz \right] dx d\tau
\]
\[
= \int_{s}^{t} \int \partial \tau \rho^\varepsilon \frac{\Pi(\rho^\varepsilon)}{\rho^\varepsilon},
\]
where \( P(\rho^\varepsilon) = \rho^\varepsilon \int_{1}^{\rho^\varepsilon} \frac{\Pi(z)}{z^2} dz \).

It is clear that
\[
- \int_{s}^{t} \int \frac{\rho v}{\rho^\varepsilon} \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon = \int_{s}^{t} \int -\text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon v^\varepsilon - \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon},
\]
\[
- \int_{s}^{t} \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\mu(\rho) v)^\varepsilon = \int_{s}^{t} \int -\nabla (\mu(\rho) v)^\varepsilon v^\varepsilon - \nabla (\mu(\rho) v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon}.
\]

Substituting (3.2), (3.5)-(3.8) into (3.1), we see that
\[
\int_{s}^{t} \int \partial \tau \left( \frac{1}{2} \left| \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right|^2 + P(\rho^\varepsilon) \right) - \int_{s}^{t} \int \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon v^\varepsilon - \int_{s}^{t} \int \nabla (\mu(\rho) v)^\varepsilon v^\varepsilon
\]
\[
= \int_{s}^{t} \int \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon} + \int_{s}^{t} \int \nabla (\mu(\rho) v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon}
\]
\[
+ \int_{s}^{t} \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \left[ (\rho v \otimes v)^\varepsilon - (\rho v^\varepsilon \otimes v^\varepsilon) \right] + \int_{s}^{t} \int \left[ \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right).
\]

Next, we need to prove that the terms on the right hand-side of (3.9) tend to zero as \( \varepsilon \to 0 \).

It follows from Lemma 2.1 and Lemma 2.2 that
\[
\| \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \|_{L^2(L^2)} \leq C \varepsilon^{-1} \| \nu(\rho) \mathcal{D} v \|_{L^2(L^2)} \leq C \varepsilon^{-1} \| \nabla v \|_{L^2(L^2)},
\]
\[
\limsup_{\varepsilon \to 0} \| \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \|_{L^2(L^2)} = 0,
\]
\[
\| (\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon \|_{L^2(L^2)} \leq C \| \rho \|_{L^\infty(L^\infty)} \| v \|_{L^2(W^{1,2})}.
\]

We derive from this and the Hölder’s inequality that
\[
\left| \int_{s}^{t} \int \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon} \right|
\]
\[
\leq C \| \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \|_{L^2(L^2)} \| (\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon \|_{L^2(L^2)}
\]
\[
\leq C \varepsilon \| \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \|_{L^2(L^2)} \| \rho \|_{L^\infty(L^\infty)} \| v \|_{L^2(W^{1,2})}.
\]

As a consequence, we get
\[
\limsup_{\varepsilon \to 0} \left| \int_{s}^{t} \int \text{div} (\nu(\rho) \mathcal{D} v)^\varepsilon \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon} \right| = 0.
\]
Likewise, there holds
\[
\limsup_{\varepsilon \to 0} \int_s^t \int \nabla (\mu(\rho) \text{div} v) \frac{(\rho v)^\varepsilon - \rho^\varepsilon v^\varepsilon}{\rho^\varepsilon} = 0. \tag{3.12}
\]
By means of the triangle inequality, the Hölder inequality and Lemma 2.23, we obtain
\[
\int_s^t \int \text{div} \left[ \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right] \left[ (\Pi(\rho))^\varepsilon - \Pi(\rho^\varepsilon) \right] \leq \int_s^t \int \text{div} \left[ \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right] |(\Pi(\rho))^\varepsilon - \Pi(\rho)| + \int_s^t \int \text{div} \left[ \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right] |\Pi(\rho) - \Pi(\rho^\varepsilon)| 
\leq C \| \text{div} \left[ \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right] \|_{L^2 L^2} \left( \|(\Pi(\rho))^\varepsilon - \Pi(\rho)\|_{L^2 L^2} + \|\Pi(\rho) - \Pi(\rho^\varepsilon)\|_{L^2 L^2} \right) 
\leq C \| \nabla v \|_{L^2 L^2} \left( \|(\Pi(\rho))^\varepsilon - \Pi(\rho)\|_{L^2 L^2} + \sup \left\| \rho - \rho^\varepsilon \right\|_{L^2 L^2} \right),
\]
which implies that
\[
\limsup_{\varepsilon \to 0} \int_s^t \int \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \left( (\Pi(\rho))^\varepsilon - \Pi(\rho^\varepsilon) \right) = 0.
\]
At this stage, it is enough to show
\[
\limsup_{\varepsilon \to 0} \int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \left[ (\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon \right] + \int_s^t \int \left( \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right) \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} = 0, \tag{3.14}
\]
under the hypothesis
\[
v \in L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}}) \quad \text{and} \quad \nabla v \in L^p(L^q). \tag{3.15}
\]
Applying Lemma 2.22, we obtain that
\[
\|(\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon\|_{L^{p+1}(L^{q+1})} \leq C \varepsilon \|v\|_{L^p(W^{1,q})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \leq C \varepsilon \|v\|_{L^p(W^{1,q})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \leq C \varepsilon \|v\|_{L^p(W^{1,q})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})}, \tag{3.16}
\]
\[
\limsup_{\varepsilon \to 0} \|\nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right)\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} = 0,
\]
Using the Hölder’s inequality and Lemma 2.11, we find
\[
\left| \int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \left[ (\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon \right] \right| 
\leq C \left\| \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \left( \|v\|_{L^p(W^{1,q})} \right) \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} 
\leq C \varepsilon \left\| \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \leq C \varepsilon \left\| \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \right\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})} \|v\|_{L^p(W^{1,q})} \|\rho v\|_{L^{\frac{2p}{p-1}}(L^{\frac{2q}{q-1}})}, \tag{3.17}
\]
which in turn implies
\[
\limsup_{\varepsilon \to 0} \left| \int_s^t \int \nabla \left( \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right) \left[ (\rho v \otimes v)^\varepsilon - (\rho v)^\varepsilon \otimes v^\varepsilon \right] \right| = 0.
\]
We turn our attentions to the term $\int_t^T \int \left[ (\rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon) \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon}$. Since $\rho v \in L^{2p/(2q)}(L^{2q/(2q)})$, we derive from Lemma 2.4 that

$$\limsup_{\varepsilon \to 0} \varepsilon \left\| \nabla (\rho v)^\varepsilon \right\|_{L^{2p/(2q)}(L^{2q/(2q)})} = 0. \quad (3.18)$$

In addition, we conclude from Lemma 2.2 that

$$\left\| \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right\|_{L^p L^q} \leq C \varepsilon \left\| v \right\|_{L^p W^{1,q}} \left\| \rho \right\|_{L^\infty L^\infty}. \quad (3.19)$$

Using the Hölder’s inequality and (3.19), we find,

$$\left| \int_s^t \int \left[ \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\rho v)^\varepsilon \right| \leq C \left\| \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right\|_{L^p(L^q)} \left\| \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \right\|_{L^{2p/(2q)}(L^{2q/(2q)})} \left\| \nabla (\rho v)^\varepsilon \right\|_{L^{2p/(2q)}(L^{2q/(2q)})} \quad (3.20)$$

Together this with (3.18) yields that

$$\limsup_{\varepsilon \to 0} \left| \int_s^t \int \left[ \rho^\varepsilon v^\varepsilon - (\rho v)^\varepsilon \right] \frac{(\rho v)^\varepsilon}{\rho^\varepsilon} \nabla (\rho v)^\varepsilon \right| = 0.$$

Collecting all the above estimates, using the weak continuity of $\rho$ and $\rho v$, and the limits

$$\limsup_{\varepsilon \to 0} \frac{\left| \int_s^t \int \text{div} (\nu(\rho) Dv)^\varepsilon \cdot \nabla v^\varepsilon + \int_s^t \int \nu(\rho) |Dv|^2 \right|}{\left| \int_s^t \int \text{div} (\mu(\rho) Dv)^\varepsilon \cdot \nabla v^\varepsilon + \int_s^t \int \mu(\rho) |Dv|^2 \right|} = 0,$$

We complete the proof of Theorem 1.2.

We are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (i) The natural energy gives $v \in L^2(0,T;W^{1,2}(\mathbb{T}^3))$. Choosing $p = q = 2$ in (1.18), we immediately prove that the condition $v \in L^4(0,T;L^4(\mathbb{T}^3))$ yields energy equality. It is worth remarking that the rest proof in (1.16) can be reduced to this special case. Next, we first deal the cases (1.16) in Theorem 1.1 with $q \geq 4$ and $\frac{2}{p} + \frac{2}{q} = 1$. The Gagliardo-Nirenberg inequality guarantees that

$$\left\| v \right\|_{L^4(0,T;L^4(\mathbb{T}^3))} \leq C \left\| v \right\|_{L^{\infty}(0,T;L^2(\mathbb{T}^3))} \left\| v \right\|_{L^p(0,T;L^q(\mathbb{T}^3))} \leq C. \quad (3.21)$$

where we have used $q \geq 4$. From the the result just proved, we obtain energy equality via (1.16) with $q \geq 4$. We consider (1.16) with $3 < q < 4$ and $\frac{1}{2} + \frac{2}{q} = 1$. Using the Gagliardo-Nirenberg inequality again, we know that

$$\left\| v \right\|_{L^4(0,T;L^4(\mathbb{T}^3))} \leq C \left\| \nabla v \right\|_{L^2(0,T;L^2(\mathbb{T}^3))} \left\| v \right\|_{L^p(0,T;L^q(\mathbb{T}^3))} \leq C \left( \left\| \nabla v \right\|_{L^2(0,T;L^2(\mathbb{T}^3))} + \left\| v \right\|_{L^{\infty}(0,T;L^2(\mathbb{T}^3))} \right) \left\| v \right\|_{L^p(0,T;L^q(\mathbb{T}^3))} \leq C. \quad (3.22)$$
We finish the proof of (1.16).

Now, we focus on the proof of (1.17). Indeed, note that \( v \in L^p(0, T; W^{1,q}(T^3)) \), therefore, according to Theorem 1.1, it suffices to derive \( v \in L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(T^3)) \) from (1.17). For \( q \geq \frac{9}{5} \), by the Gagliardo-Nirenberg inequality, we get

\[
\|v\|_{L^{\frac{2p}{p-1}}(T^3)} \leq C \|v\|_{L^{\frac{5p-9}{3}}(T^3)}^{\frac{5p-9}{3}} \|\nabla v\|_{L^{\frac{9}{3q-6}}(T^3)}^{\frac{9}{3q-6}}.
\]

(3.23)

Thanks to \( \frac{1}{p} + \frac{6}{5q} = 1 \), we further infer that

\[
\|v\|_{L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(T^3))} \leq C \|v\|_{L^{\frac{5p-9}{3}}(0, T; L^{\frac{2q}{q-1}}(T^3))}^{\frac{5p-9}{3}} \|\nabla v\|_{L^{\frac{9}{3q-6}}(T^3)}^{\frac{9}{3q-6}} \leq C.
\]

(3.24)

In light of Theorem 1.2, we have proved (1.17) for \( q \geq \frac{9}{5} \).

Finally, for \( \frac{3}{2} < q < \frac{9}{5} \), it follows the Gagliardo-Nirenberg inequality that

\[
\|v\|_{L^{\frac{2p}{p-1}}(T^3)} \leq C \|v\|_{L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(T^3))}^{\frac{2p-9q}{6q-18}} \|\nabla v\|_{L^{\frac{2q}{q-1}}(T^3)}^{\frac{2q-9}{6q-18}}.
\]

(3.25)

Thanks to \( \frac{1}{p} + \frac{3}{q} = 2 \), we further have

\[
\|v\|_{L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(T^3))} \leq C \|v\|_{L^{\frac{2p}{p-1}}(0, T; L^{\frac{2q}{q-1}}(T^3))}^{\frac{2p-9q}{6q-18}} \|\nabla v\|_{L^{\frac{2q}{q-1}}(0, T; L^{\frac{2q}{q-1}}(T^3))}^{\frac{2q-9}{6q-18}} \leq C.
\]

(3.26)

We conclude the desired result from Theorem 1.2. The proof of this theorem is completed. \( \square \)

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