SOME RESULTS ON THE RATIONAL BERNSTEIN MARKOV PROPERTY IN THE COMPLEX PLANE

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Abstract. The Bernstein Markov Property, shortly BMP, is an asymptotic quantitative assumption on the growth of uniform norms of polynomials or rational functions on a compact set with respect to $L^2_\mu$-norms, where $\mu$ is a positive finite measure.

We consider two variants of BMP for rational functions with restricted poles and compare them with the polynomial BMP finding out some sufficient conditions for the latter to imply the former. Moreover, we recover a sufficient mass-density condition for a measure to satisfy the rational BMP on its support.

1. Introduction

Let $K \subset \mathbb{C}$ be a polynomial determining compact set, that is if a polynomial vanishes on $K$ then it is the zero polynomial. In such a case $\|p\|_K := \max_{z \in K} |p(z)|$ is a norm on the space $\mathcal{P}^k$ of polynomials of degree not greater than $k$ for any $k \in \mathbb{N}$.

Let us pick a positive locally finite Borel measure $\mu$. When $\| \cdot \|_{L^2_\mu(K)}$ is a norm on $\mathcal{P}^k$ we can compare it with the uniform norm on $K$. In fact, since $\mathcal{P}^k$ is a finite dimensional normed vector space, there exist positive constants $c_1, c_2$ depending only on $(K, \mu, k)$ such that

$$c_1 \|p\|_{L^2_\mu} \leq \|p\|_K \leq c_2 \|p\|_{L^2_\mu} \quad \forall p \in \mathcal{P}^k.$$  

Notice that there exists such a $c_1$ because the measure $\mu$ is locally finite while $c_2$ is finite precisely when $\mu$ induces a norm.

The Bernstein Markov property is a quantitative asymptotic growth assumption on $c_2$ as $k \to \infty$. Namely, the couple $(K, \mu)$ is said to enjoy the Bernstein Markov property if for any sequence $\{p_k\} : p_k \in \mathcal{P}^k$ we have

$$(\text{BMP}) \quad \limsup_k \left( \frac{\|p_k\|_K}{\|p_k\|_{L^2_\mu}} \right)^{1/k} \leq 1.$$  

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The Bernstein Markov property can be equivalently defined in several complex variables and/or for weighted polynomials, i.e., functions of the type $p w^{\deg(p)}$ where $w$ is an admissible weight as in [17], see Definition 2.1.

A first motivation to its study comes from approximation theory. If $(K, \mu)$ have the Bernstein Markov property then, given any holomorphic function $f$, the error of best polynomial approximation $p_k$ of degree not greater than $k$ to $f$ and the error of the approximation $q_k$ given by projection in $L^2_\mu$ on the subspace $\mathcal{P}_k$ are asymptotically the same in the sense that for any $0 < r < 1$ and $f \in \mathcal{C}(K)$

$$\limsup_k \|f - p_k\|_K^{1/k} \leq r \text{ if and only if } \limsup_k \|f - q_k\|_{L^2_\mu}^{1/k} \leq r.$$  

Consequently, one has a $L^2$ version of the Bernstein-Walsh Lemma [20] for Bernstein Markov measures relating the rate of best $L^2$ approximation of a function to its maximum radius of holomorphic extension; [12, Prop. 12.2]. The several complex variables version of the Bernstein Walsh Lemma is usually referred as the Bernstein Walsh Siciak Theorem, see for instance [12, Th. 9.7].

Moreover, the Bernstein Markov property has been studied (see for instance [2, 1, 6, 4, 14]) in relation to (pluri-)potential theory, the study of plurisubharmonic functions in several complex variables. It turns out that such a property is fundamental both to recover the Siciak Zaharyuta extremal plurisubharmonic function and the (pluripotential) equilibrium measure (see [12]) by $L^2$ methods.

Lastly, Bernstein Markov measures play a central role in a recent theory of Large Deviation for random arrays and common zeroes of random polynomials; see for instance [7, 9] and references therein.

In the present paper we investigate two slightly modified versions of (BMP). To do that we define the following classes of sequences of rational functions

$$\mathcal{R}(P) := \bigcup_{k \in \mathbb{N}} \{ p_k/q_k : p_k, q_k \in \mathcal{P}_k, Z(q_k) \subseteq P \}$$  
and

$$\mathcal{Q}(P) := \bigcup_{k \in \mathbb{N}} \{ p_k/q_k : p_k, q_k \in \mathcal{P}_k, \deg q_k = k, Z(q_k) \subseteq P \},$$

where $P \subset \mathbb{C}$ is any compact set that from now on we suppose to be not intersecting $K$ and $Z(p) := \{ z \in \mathbb{C} : p(z) = 0 \}$.

Throughout the paper we use the symbol $M^+(K)$ to denote the cone of positive Borel finite measures $\mu$ such that $\text{supp} \mu \subseteq K$, adding a subscript $1$ for probability measures.

**Definition 1.1** (Rational Bernstein Markov Property). Let $K, P \subset \mathbb{C}$ be compact disjoint sets and $\mu \in M^+(K)$.

(i) (Rational Bernstein Markov Property.) If

$$\limsup_k \left( \frac{\|r_k\|_K}{\|r_k\|_{L^2_\mu}} \right)^{1/k} \leq 1 \quad \forall \{r_k\} \in \mathcal{R}(P),$$

Then we say that $(\mu, K, P)$ has the Rational Bernstein Markov Property.
then $(K, \mu, P)$ is said to enjoy the rational Bernstein Markov Property.

(ii) (sub-diagonal Rational Bernstein Markov Property.) If

$$
(Q\text{-BMP}) \quad \limsup_k \left( \frac{\|r_k\|_{K}}{\|r_k\|_{L_2^\mu}} \right)^{1/k} \leq 1 \quad \forall \{r_k\} \in Q(P),
$$

then $(K, \mu, P)$ is said to enjoy the rational sub-diagonal Bernstein Markov Property.

A motivation to study such properties is mainly given by the discretization of a quite general class of vector energy problems performed in [8]. Bloom, Levenberg and Wielonsky introduce a probability $\text{Prob}(\cdot)$ on the space of sequences of arrays of points $\{z^{(1)}, \ldots, z^{(m)}\}$, where $z^{(l)} = \{z_0^{(l)}, \ldots, z_k^{(l)}\} \in K_i^{k+1}$, on a vector of compact sets $\{K_1, \ldots, K_m\}$ in the complex plane based on a vector of probability measures $\mu_i \in M_+^1(K_i)$ such that $(K_l, \mu_l, \cup_{l \neq i} K_l)$ has the rational Bernstein Markov property. In [8] the authors actually deal with strong rational Bernstein Markov measures, which is a variant of RBMP where weighted rational function are considered instead of standard ones, however their paper can be read in the un-weighted setting picking (in their notation) $Q = 0$. Then they prove a Large Deviation Principle (LDP) for measures canonically associated to arrays of points randomly generated according to $\text{Prob}$. Also, they show that the validity of the LDP is not affected by the particular choice of $\{\mu_1, \mu_2, \ldots, \mu_m\}$ that are only required to form a vector of rational Bernstein Markov measures.

Measures having the rational Bernstein Markov property are worth to be studied also from the approximation theory point of view. In fact, for such measures it turns out that the radius of maximum meromorphic extension with exactly $m$ poles of a function $f \in C(K)$ is related to the asymptotic of its $L_2^\mu$ approximation numbers

$$
\min_{\deg p \leq k, \deg q = m} \|f - p/q\|_{L_2^\mu}^{1/k}.
$$

The reader is referred to Section 3.1 for a precise statement.

The paper is organized as follows.

In Section 2 we compare Definition 1.1 to the polynomial Bernstein Markov property. We address the following question. Are there sufficient additional conditions on $(K, \mu, P)$ for the polynomial BMP to imply the R-BMP or the Q-BMP? A positive answer to both instances of such a question is given in Theorem 2.3, by means of an equivalent formulation of the problem suggested in Propositions 2.1 and 2.2.

In Section 3 we give two sufficient conditions for the Rational Bernstein Markov Property. Namely, we show in Theorem 3.2 that, in the case of $P$ not intersecting the polynomial hull $\hat{K}$ of $K$ (see (9)), if the measure $\mu$ is thick on the compact set $K$ in the sense of the mass-density condition (30), then $(K, \mu, P)$ have the rational
Bernstein Markov property. Such mass density condition is given in terms of logarithmic capacity and goes back to [18] where it have been first formulated in the polynomial case.

To relate convergence of logarithmic capacities and Green functions we prove an equivalence result (Theorem 3.1) in the spirit of [5].

In the case $\hat{K} \cap P \neq \emptyset$ we show in Proposition 3.1 that it is possible to build a suitable conformal mapping $f$ such that the images $E$ of $K$ and $Q$ of $P$ under $f$ are in the relative position of the hypothesis of Theorem 3.2. Thus, we derive (Theorem 3.3) a sufficient mass density condition for $K$ and $P$ in the general case.

Finally we present, as an application, a meromorphic $L^2_\mu$ version of the Bernstein Walsh Lemma.

2. POLYNOMIAL VERSUS RATIONAL BMP

Let us illustrate some significantly different situations which can occur by providing some easy examples where we are able to perform explicit computations.

We recall that, given an orthonormal basis $\{q_j\}_{j=1,2,...}$ of a separable Hilbert space $H$ (endowed with its induced norm $\|\cdot\|_H$) of continuous functions on a given compact set, the Bergman Function $B_k(z)$ of the subspace $H_k := \langle q_1, q_2, \ldots, q_k \rangle$ is

$$B_k(z) := \sum_{j=1}^k |q_j(z)|^2.$$ 

It follows by its definition and by Parseval Identity that for any function $f \in H_k$ one has $|f(z)| \leq \sqrt{B_k(z)}\|f\|_H$, while the function $f(z) := \sum_{j=1}^k q_j(z_0)q_j(z)$ achieves the equality at the point $z_0$, thus

$$B_k(z) = \max_{f \in H_k \setminus \{0\}} \left( \frac{|f(z)|}{\|f\|_H} \right)^2.$$ 

Example 1.

(a) Let $\mu$ be the arc length measure on the boundary $\partial \mathbb{D}$ of the unit disk. Let $K = \partial \mathbb{D}$ and $P = \{0\}$.

Let us take a sequence $\{r_k\} = \left\{ \frac{p_k}{z^k} \right\}$ in $\mathcal{R}(P)$ where $\deg p_k = l_k \leq k$, then we have

$$\|r_k\|_K = \left\| \frac{p_k}{z^k} \right\|_K = \|p_k\|_K \leq$$

$$\|B_k^{\mu}(z)\|^{1/2}\|p_k\|_{L^2(\mu)} = \|B_k^{\mu}(z)\|^{1/2}\|r_k\|_{L^2(\mu)}.$$ 

Here we indicated by $B_k^{\mu}(z)$ the Bergman function of the space $\left( \mathbb{D}^k, \langle \cdot, \cdot \rangle_{L^2_\mu} \right)$. 
For this choice of $\mu$ the orthonormal polynomials $q_k(z,\mu)$ are simply the normalized monomials $\left\{\frac{z^k}{\sqrt{2^k}}\right\}$, thus we have

$$
(3) \quad (\max_k B_k^0)^{1/2k} = \left(\max_k \frac{\sum_{j=0}^k |z|^j}{2 \pi}\right)^{1/2k} = \left(\frac{k + 1}{2 \pi}\right)^{1/2k}.
$$

It follows by (2) and (3) that $(K,\mu,P)$ has the rational Bernstein Markov Property. The same computation shows that actually any $\nu$ such that $(K,\nu)$ has the Bernstein Markov Property is such that $(K,\nu,P)$ has the rational Bernstein Markov Property.

(b) On the other hand, the same measure $\mu$ does not enjoy the sub-diagonal rational Bernstein Markov Property in the triple $(K,\mu,P)$ with $K = \{1/2 \leq |z| \leq 1\}$ and $P = \{0\}$ as the sequence of functions $\{1/z^k\}$ clearly shows: $\|z^{-k}\|_K = 2^k$, $\|z^{-k}\|_{L^2_0} = 1$. A fortiori the rational Bernstein Markov Property is not satisfied by $(K,\mu,P)$.

(c) On the contrary, the arc length measure on the inner boundary of $A$ has the sub-diagonal rational Bernstein Markov Property but neither the regular rational Bernstein Markov Property nor the polynomial one, as is shown by the sequence $\{z^k\}$. Notice that

$$
\left(\int_{\partial A} |z|^2 ds\right)^{1/2} = \sqrt{\pi} 2^{-k}, \quad \|z^k\|_K = 1, \quad \left(\frac{\|z^k\|_K}{\|z^k\|_{L^2_0}}\right)^{1/k} = 2^{-1/2k} \to 2 \neq 1.
$$

(d) Lastly, the measure $d\mu := d\mu_1 + d\mu_2 := 1/2 d\sigma|_{\partial D} + 1/2 d\sigma|_{1/2\partial D}$ (here $d\sigma$ denotes the standard length measure) has the rational Bernstein Markov property for the same sets $(K,P)$ as in the previous case (c).

In order to show that, we pick any sequence of polynomials $\{p_k\}$ of degree not greater than $k$, we consider the orthonormal basis for $\mu_1$ and $\mu_2$ and using (1) we get

$$
\left\|\frac{p_k}{z^k}\right\|_{L^2_0} \geq \|p_k\|_{L^2_0} + 2^k |p_k|_{L^2_0} \geq (B_k^0(z_1))^{-1/2} |p_k(z_1)|_{L^2_0} + 2^k (B_k^0(z_2))^{-1/2} |p_k(z_2)|_{L^2_0} = \left(\frac{2 \pi}{\sum_{j=0}^k |z_j|^2}\right)^{1/2} |p_k(z_1)|_{L^2_0} + 2^k \left(\frac{2 \pi}{\sum_{j=0}^k |z_j|^2}\right)^{1/2} |p_k(z_2)|_{L^2_0}.
$$

Now we pick $z_1 \in \partial D$ and $z_2 \in 1/2\partial D$ maximizing $|p_k|$ and we get

$$
\left\|\frac{p_k}{z^k}\right\|_{L^2_0} \geq \sqrt{\frac{2 \pi}{k + 1}} \|p_k\|_{L^2_0} + 2^k \sqrt{\frac{3 \pi}{4^{k+1} - 1}} \|p_k\|_{L^2_0} \geq \sqrt{\frac{3 \pi}{4^{k+1} - 1}} \left(\|p_k\|_{L^2_0} + 2^k \|p_k\|_{1/2D}\right) = \sqrt{\frac{3 \pi}{4^{k+1} - 1}} \left\|\frac{p_k}{z^k}\right\|_K.
$$
It follows that, denoting \( p_k/\zeta_k \) by \( r_k \), we have
\[
\limsup_k \left( \frac{\|r_k\|_A}{\|r_k\|_{L_2^\mu}} \right)^{1/k} \leq \lim_k \left( \frac{4k+1 - 1}{3\pi} \right)^{1/(2k)} = 1,
\]
hence \((A,\mu,\{0\})\) has the rational Bernstein Markov property.

The relation between these three properties is a little subtle: the examples above show that different aspects come in play as the geometry of \( K, A \) and \( P \) and the classes \( R(P), Q(P) \), it will be clear later that the measure theoretic and potential theoretic features are important as well.

We relate the Q-BMP and R-BMP to the weighted Bernstein Markov property with respect to a specific class of weights in Proposition 2.1 and 2.2; to do that we first recall the definition of weighted Bernstein Markov Property.

**Definition 2.1 (Weighted Bernstein Markov Property).** Let \( K \subset \mathbb{C} \) be a closed set and \( w : K \to [0, +\infty) \) be an uppersemicontinuous function, let \( \mu \in \mathcal{M}_+^+(K) \), then the triple \([K, \mu, w]\) is said to have the weighted Bernstein Markov property if for any sequence of polynomials \( p_k \in \mathcal{P} \) we have
\[
(WBMP) \quad \limsup_k \left( \frac{\|p_k w_k\|_K}{\|p_k w_k\|_{L_2^\mu}} \right)^{1/k} \leq 1.
\]

In what follows we deal with weak* convergence of measures. We recall that, given a metric space \( X \) and a Borel measure \( \mu \) on \( X \), the sequence of measures \((\mu_i)\) on \( X \) is said to weak* converge to \( \mu \) if for any bounded continuous function \( f \) we have
\[
\lim_i \left| \int_X f \, d\mu - \int_X f \, d\mu_i \right| = 0; \text{ in such a case we write } \mu_i \rightharpoonup^* \mu.
\]
Also, we recall that the space of Borel probability measures \( \mathcal{M}_1^+(X) \) is weak* sequentially compact, that is for any sequence there exists a weak* converging subsequence.

If \( X \) is a compact space, then \( \mathcal{C}(X) \) is a separable Banach space. It turns out that the space of Borel measures is isometrically isomorphic to the dual space \( \mathcal{C}^*(X) \) and the topology of weak* convergence is generated by the family of semi-norms \( \{p_f : f \in \mathcal{F}\} \) where \( p_f(\mu) := \|f \|_{L_1(X)} \) and \( \mathcal{F} \) is any countable dense subset of \( \mathcal{C}(X) \).

Using these facts it is not difficult to prove the following statement that we will use in the proof of the next proposition.

**Proposition.** Let \( P \) be a compact set in \( \mathbb{C} \) and \( \sigma \) a Borel measure supported on it having total mass equal to 1. There exists a sequence of arrays \( \{(\zeta_1^{(k)}, \ldots, \zeta_k^{(k)})\} \) of points of \( P \) such that we get
\[
\sigma_k := \frac{1}{k} \sum_{j=1}^k \delta_{\zeta_j^{(k)}} \rightharpoonup^* \sigma.
\]

For any compact set \( P \) we introduce the following notation
\[
\mathcal{W}(P) := \{e^{U_\sigma} : \sigma \in \mathcal{M}^+(P)\}, \quad \overline{\mathcal{W}}(P) := \mathcal{W}(P) \cup \{1\},
\]
Proposition 2.1. Let \( K \subset \mathbb{C} \) be a non polar compact set, \( \mu \in M^+(K) \) and \( P \) any compact set disjoint by \( K \). Then the following are equivalent

(i) \( \forall w \in \mathcal{W}(P) \) the triple \([K, \mu, w] \) has the weighted Bernstein Markov Property.

(ii) \((K, \mu, P)\) has the sub-diagonal rational Bernstein Markov Property.

Proof of (i) implies (ii). Let us pick a sequence \( \{r_k\} = \{p_k/q_k\} \) in \( \mathcal{Q}(P) \), where \( q_k := \prod_{j=1}^{k}(z - z_j) \), and let us set \( \sigma_k := \frac{1}{k} \sum_{j=1}^{k} \delta_{z_j} \). Then we can notice that

\[
U_{\sigma_k} = \int \log \frac{1}{|z - \zeta|} d\sigma_k(\zeta) = \frac{1}{k} \sum_{j=1}^{k} \log \frac{1}{|z - z_j|} = -\frac{1}{k} \log |q_k|.
\]

Thus, setting \( U_k := U_{\sigma_k} \), we have

\[
a_k := \left( \frac{\|r_k\|_{\mathbb{C}^2}}{\|r_k\|_{L^2_{\mu}}} \right)^{1/k} = \left( \frac{\|p_k e^{(kU_k)}\|_{\mathbb{C}^2}}{\|p_k e^{(kU_k)}\|_{L^2_{\mu}}} \right)^{1/k}.
\]

Now we pick any maximizing subsequence \( j \mapsto k_j \) for \( a_k \), that is \( \limsup_k a_k = \lim_j a_{k_j} \). Let us pick any weak* limit \( \sigma \in M^+_1(P) \) and a subsequence \( l \mapsto j_l \) such that \( \tilde{\sigma}_l := \sigma_{k_{j_l}} \rightarrow^{*} \sigma \). Moreover \( \lim_l b_l := \lim_l a_{k_{j_l}} = \limsup_k a_k \).

Let us notice that \( U := U_{\sigma} \) and all \( U_l := U_{\tilde{\sigma}_l} \) are harmonic functions on \( \mathbb{C} \setminus P \), moreover, due to [17], Th. 6.9 1.6, \( \{U_l\} \) converges quasi everywhere to \( U \). Notice that \( U_{\tilde{\sigma}_l} := -E^* \tilde{\sigma}_l \), where \( E(z) := \log |z| \) is a locally absolutely continuous function on \( \mathbb{C} \setminus \{0\} \), hence weak convergence of measures supported on \( P \) implies local uniform convergence of potentials on \( \mathbb{C} \setminus P \).

We can exploit this uniform convergence as follows. For any \( \varepsilon > 0 \) there exists \( l_{\varepsilon} \) such that for any \( l > l_{\varepsilon} \) we have

\[
(1 - \varepsilon)U \leq U_l \leq (1 + \varepsilon)U \quad \text{uniformly on } K.
\]

Now we denote \( k_{j_l} \) by \( \tilde{k}_l \) and \( p_{k_{j_l}} \) by \( \tilde{p}_l \). It follows by (6) that for \( l \) large enough

\[
\frac{\|\tilde{p}_l e^{(\tilde{k}_l U_l)}\|_{\mathbb{C}^2}}{\|\tilde{p}_l e^{(\tilde{k}_l U_l)}\|_{L^2_{\mu}}} \leq \frac{\|\tilde{p}_l e^{(1+\varepsilon)U_l}\|_{\mathbb{C}^2}}{\|\tilde{p}_l e^{(1+\varepsilon)U_l}\|_{L^2_{\mu}}} \leq e^{(2k_\varepsilon \|U_l\|_{\mathbb{C}^2})} \frac{\|\tilde{p}_l e^{(1-\varepsilon)U_l}\|_{\mathbb{C}^2}}{\|\tilde{p}_l e^{(1-\varepsilon)U_l}\|_{L^2_{\mu}}}.
\]
Hence, exploiting \( w_k := e^{(1-\varepsilon)U} \in \mathcal{W}(P) \) and \( \mu \) having the WBMP for such a weight, we have

\[
\limsup_k \alpha_k = \lim_l \left( \frac{\| \bar{p}_l e^{kU} \|_K}{\| \bar{p}_l e^{kU} \|_{L^2_\mu}} \right)^{1/k} \leq e^{-2\varepsilon\|U\|_K} \lim_l \left( \frac{\| \bar{p}_l e^{k(1-\varepsilon)U} \|_K}{\| \bar{p}_l e^{k(1-\varepsilon)U} \|_{L^2_\mu}} \right)^{1/k} \leq e^{-2\varepsilon\|U\|_K} \lim_l \left( \frac{\| \bar{p}_l e^{k\varepsilon U} \|_K}{\| \bar{p}_l e^{k\varepsilon U} \|_{L^2_\mu}} \right)^{1/k} \leq e^{2\varepsilon\|U\|_K} \rightarrow 1 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

\( \square \)

**Proof of (ii) implies (i).** Suppose by contradiction that there exists \( \sigma \in \mathcal{W}(P) \) such that \([K, \mu, \exp U^\sigma]\) does not have the weighted Bernstein Markov Property; without lost of generality we can suppose \( \sigma \) to be a probability measure.

We pick \( \{z^{(k)}_1, \ldots, z^{(k)}_j\}_{k=1, \ldots} \) and \( \sigma_k = \frac{1}{k} \sum_{j=1}^k \delta_{z^{(i)}_j} \) as in (4).

Let us set \( w = \exp U^\sigma \), \( w_k = \exp U^\sigma_k \). We can perform the same reasoning as above, using the absolute continuity of the log kernel away from 0, to get \( U^\sigma_k \rightarrow U^\sigma \) uniformly on \( K \). Thus for any \( \varepsilon > 0 \) we have \( (1-\varepsilon)U^\sigma_k \leq U^\sigma \leq (1+\varepsilon)U^\sigma_k \) uniformly on \( K \) for \( k \) large enough. That is

\[
(7) \quad w_k^{1-\varepsilon} \leq w \leq w_k^{1+\varepsilon} \quad \text{uniformly on} \ K \quad \text{for} \ k \quad \text{large enough}.
\]

Notice that given any sequence \( \{p_k\} \) such that \( p_k \in \mathcal{P}^k \) we have

\[
\{r_k\} := \{p_k w_k^{1-\varepsilon}\} = \left\{ \frac{P_k}{\prod_{j=1}^k (z - z_j)} \right\} \in \mathcal{Q}(P).
\]

Since we assumed that \([K, \mu, w]\) doesn’t have the WBMP we can pick \( p_k \) such that, using (7),

\[
1 < \limsup_k \left( \frac{\|p_k w_k^\varepsilon\|_K}{\|p_k w_k^\varepsilon\|_{L^2_\mu}} \right)^{1/k} \leq \limsup_k \left( \frac{\|p_k w_k^{(1+\varepsilon)}\|_K}{\|p_k w_k^{(1+\varepsilon)}\|_{L^2_\mu}} \right)^{1/k} \leq \limsup_k \left( \frac{\|p_k w_k^\varepsilon\|_K}{\|p_k w_k^\varepsilon\|_{L^2_\mu}} \right)^{1/k} \leq \|w_k\|_K^{2\varepsilon} \left( \frac{\|p_k\|_K}{\|p_k\|_{L^2_\mu}} \right)^{1/k} \leq \|w\|_K^{2\varepsilon} \rightarrow 1.
\]

\( \square \)

We can prove the following variant of the previous proposition by some minor modifications of the proof.

**Proposition 2.2.** Let \( K \subset \mathbb{R} \) be a non polar compact set, \( \mu \in \mathcal{M}^+(K) \) and \( P \) any compact set disjoint by \( K \). Then the following are equivalent

(i) \( \forall w \in \overline{\mathcal{W}}(P) \) the triple \([K, \mu, w]\) has the weighted Bernstein Markov Property.

(ii) \( (K, \mu, P) \) has the rational Bernstein Markov Property.
Proof of (i) implies (ii). We pick an extremal sequence in $\mathcal{R}(P)$ (i.e., for $a_k$ as in (5)) $r_k := \frac{p_k}{q_m}$, where $\deg p_k = l_k$ and $\deg q_m = m_k$. We can assume that $0 \leq m_k \leq l_k \leq k$ since the case $m_k > l_k$ (after relabelling the indexes) is exactly as in the previous proof.

We notice that

$$r_k = p_k e^{(m_k U^{\sigma_m})} = p_k e^{\left(l_k U^{\hat{\sigma}_{mk}}\right)} = r_k e^{(l_k U^{\sigma_k})},$$

where the sequence of measures $\{\hat{\sigma}_k\} := \left\{\frac{m_k}{l_k} \sigma_{mk}\right\}$ is bounded in total variation thus has some weak*-limit, which is a possibly zero measure. Here is the main difference between this case and Proposition 2.1 where each weak*-limit has the same positive mass. We take any converging (to any limit, say $\hat{\sigma}$) subsequence and relabel indexes to avoid a more complicated notation.

Notice that $U^{\hat{\sigma}_k}$ converges to $U^{\hat{\sigma}}$ uniformly on $K$ as in the previous proof, hence for any $\varepsilon > 0$ we can pick $k_\varepsilon$ such that for any $k > k_\varepsilon$ we have

$$U^{\hat{\sigma}_k} - \varepsilon \leq U^{\hat{\sigma}} \leq U^{\hat{\sigma}_k} + \varepsilon.$$

Therefore we have

$$r_k e^{-l_k \varepsilon} = p_k e^{(l_k U^{\sigma_k})} e^{(-l_k \varepsilon)} \leq p_k e^{(l_k U^{\hat{\sigma}})} \leq r_k e^{l_k \varepsilon}.$$  \hfill (8)

The result follows by the same lines as in proof of Proposition 2.1 using the weighted Bernstein Markov property of $[K, \mu, \nu]$ where each weak* limit has the same positive mass. We take any converging (to any limit, say $\hat{\sigma}$) subsequence and relabel indexes to avoid a more complicated notation.

Proof of (ii) implies (i). Pick $\sigma \in \overline{W}(P)$. If $\sigma$ is not the zero measure this follows by Proposition 2.1 and by the fact that $\mathcal{R}$-BMP is stronger than $Q$-BMP. If $\sigma = 0$ we notice that $\mathcal{R}$-BMP is stronger than the usual BMP. \hfill $\square$

Remark 2.2. The combination of the two previous propositions proves in particular that if $(K, \mu, P)$ has the $Q$-BMP and $(K, \mu)$ has the BMP, it follows that $(K, \mu, P)$ has the rational Bernstein Markov property.

On the other hand if $(K, \mu, P)$ has the $Q$-BMP but not the $R$-BMP, it follows that $(K, \mu)$ doesn’t satisfy the BMP.

According to Proposition 2.2 our original question boils down to whether the BMP implies the WBMP for any weight in the class $\overline{W}(P)$. In the next theorem we give two possible sufficient conditions for that, corresponding to two different situations that are rather extremal in a sense. The reader is invited to compare them with situation of Example 1(a) and 1(b).

We denote by $S_K$ the Shilov boundary of $K$ with respect to the uniform algebra of functions that are uniform limits on $K$ of entire functions (or equivalently polynomials), while the standard notation for the polynomial hull of a compact set $K$.
is
\[(9)\quad \hat{K} := \{z \in \mathbb{C} : |p(z)| \leq \|p\|_K, \forall p \in \mathcal{P}\},\]
where \(\mathcal{P} := \cup_{k \in \mathbb{N}} \mathcal{P}_k\).

**Theorem 2.3.** Let \(K \subset \mathbb{C}\) be a compact non polar set and \(\mu \in \mathcal{M}^+(K)\) be such that \(\text{supp} \mu = K\) and \((K, \mu)\) has the Bernstein Markov Property. Pick any compact set \(P \subset \mathbb{C}\) disjoint by \(K\), suppose that one of the following occurs.

**Case a.** \(S_K = K\).

**Case b.** \(K = \hat{K}\).

Then the couple \([K, \mu, w]\) has the weighted Bernstein Markov Property w.r.t. any weight \(w \in \mathcal{W}(P)\) and thus \((K, \mu, P)\) has the rational Bernstein Markov Property.

**Proof.** Let us pick \(\sigma \in \mathcal{M}^+(P)\) and set \(w = \exp U^\sigma\), also we pick a sequence \(\{p_k\}\), where \(p_k \in \mathcal{P}_k\). We show that in both cases \([K, \mu, w]\) has the weighted Bernstein Markov Property w.r.t. any weight \(w \in \mathcal{W}(P)\), the rest following by Proposition 2.2.

**Case a.** We first recall (see [18, Lemma 3.2.4 pg. 70]) that the set \(\{|g| : g \in \mathcal{P}\}\) is dense in the cone of positive continuous functions on \(S_K\), which \(w\) belongs to.

For any \(\varepsilon > 0\) we can pick \(g_\varepsilon \in \mathcal{C}_{\mu_r}\) such that
\[(10)\quad (1 - \varepsilon)|g_\varepsilon| \leq w \leq (1 + \varepsilon)|g_\varepsilon|\]
Notice that \(|g_\varepsilon|^k = |g_\varepsilon|^k = |\tau_{\varepsilon, k}|\), where \(\tau_{\varepsilon, k} \in \mathcal{C}_{\mu_r}\).

If for any \(p_k \in \mathcal{P}_k\) we set \(\hat{p}_k := \tau_{\varepsilon, k} p_k \in \mathcal{C}_{\mu_r}\), then we have
\[
\begin{align*}
\|p_k w^k\|_K &\leq (1 + \varepsilon)^k \|\tau_{\varepsilon, k} p_k\|_K = \|\hat{p}_k\|_K, \\
\|p_k w^k\|_{L^2_\mu} &\geq (1 - \varepsilon)^k \|\tau_{\varepsilon, k} p_k\|_{L^2_\mu} = \|\hat{p}_k\|_{L^2_\mu},
\end{align*}
\]
and thus
\[(11)\quad \left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}}\right)^{1/k} \leq 1 + \varepsilon \left[\frac{\|\hat{p}_k\|_K}{\|\hat{p}_k\|_{L^2_\mu}}\right]^{1/(m_r + 1)} \leq 1.
\]
Using the polynomial BMP of \((K, \mu)\) and the arbitrariness of \(\varepsilon > 0\) we can conclude that \(\lim \sup \left(\frac{\|p_k w^k\|_K}{\|p_k w^k\|_{L^2_\mu}}\right)^{1/k} \leq 1\).

**Case b.** Suppose first that \(K\) is connected, then it follows that there exists an open neighbourhood \(D\) of \(K\) which is a simply connected domain and \(P \cap D = \emptyset\). We recall that any harmonic function on a simply connected domain is the real part of an holomorphic one. Hence, being \(U^\sigma\) harmonic on \(D\), we can pick \(f\) holomorphic on \(D\) such that
\[(12)\quad w = \exp U^\sigma = \exp \text{Re } f = |\exp f|\]
Since \( g := \exp f \) is an holomorphic function on \( D \), by Runge Theorem, we can uniformly approximate it by polynomials \( g_\varepsilon \) on \( K \). Now we can conclude the proof by the same argument (11) and (12) of the Case a above.

If otherwise \( K \) is not known to be connected, we apply the following version of the Hilbert Lemniscate Theorem [11, Th. 16.5.6], given any open neighbourhood \( U \) of \( K \) not intersecting \( P \) we can pick a polynomial \( s \in \mathcal{P} \) such that \( |s(z)| > \|s\|_K \) for any \( z \in \mathbb{C} \setminus U \).

It follows that, picking a suitable positive \( \delta \), the set \( E := \{ |s| \leq \|s\| + \delta \} \) is a closed neighbourhood of \( K \) not intersecting \( P \).

Notice that the set \( E \) has at most \( \deg s \) connected components \( E_j \) and by definition it is polynomially convex. Moreover the Maximum Modulus Theorem implies that each \( D_j := \text{int} E_j \) is simply connected or the disjoint union of a finite number of simply connected domains that we do not relabel.

For any \( j = 1, 2, \ldots, \deg s \) we set \( w_j := w|_{D_j} \), so we can find holomorphic functions \( f_j \) and \( g_j \) on \( D_j \), continuous up to its boundary, such that \( w_j = |\exp f_j| = |g_j| \).

Now notice that being \( D \) the disjoint union of the sets \( D_j \)'s the function \( g_j(z) = g_j(z) \) if \( z \in D_j \) is holomorphic on \( D \) and continuous on \( E \), we can apply the Mergelyan Theorem to find for any \( \varepsilon > 0 \) a polynomial \( g_\varepsilon \) such that

\[
(1 - \varepsilon)|g_\varepsilon(z)| \leq w(z) \leq (1 + \varepsilon)|g_\varepsilon(z)| \quad \forall z \in E \supseteq K.
\]

We are back to the Case a and the proof can be concluded by the same lines. \( \square \)

3. **A sufficient Mass-Density Condition for the RBMP**

In the previous section we shown two instances where the rational and the polynomial Bernstein Markov property are essentially the same. In the case that \( K \neq S_K \) and \( K \neq \hat{K} \) we cannot derive the RBMP from a polynomial property, but still there exists plenty of measures satisfying such a property, see for instance Example 1 (d). Thus the aim of the present section is to work out a sufficient condition for the RBMP by different tools, we do it in terms of potential and measure theoretic properties of the considered measure.

In the case of \( K = \text{supp} \mu \) being a regular set for the Dirichlet problem, the Bernstein Markov Property for \( (K, \mu) \) is equivalent (cfr. [4, Th. 3.4]) to \( \mu \in \text{Reg} \).

A positive Borel measure is in the class \( \text{Reg} \) or has regular \( n \)-th root asymptotic behaviour if for any sequence of polynomials \( \{p_k\} \) one has

\[
(13) \quad \lim_{k} \sup \left( \frac{|p_k(z)|}{\|p_k\|_{L^2(E)}} \right)^{1/\deg p_k} \leq 1 \quad \text{for} \ z \in E \subseteq K, \ \text{cap}(E) = \text{cap}(K).
\]

However, the definition can be given in terms of other equivalent conditions, see [18 Th. 3.1.1, Def. 3.1.2].
Moreover in [18, Th. 4.2.3] it has been proven that any Borel compactly supported finite measure having regular support \( K \subset \mathbb{C} \) and enjoining a mass density condition \( (\Lambda^*\text{-criterion [18, pag. 132])} \) is in the class \( \text{Reg} \), consequently \( (K, \mu) \) has the BMP. In order to fulfil such \( \Lambda^* \) condition a measure needs to be thick in a measure-theoretic sense on a subset of its support which has full logarithmic capacity (see equation (15) below for the rigorous statement).

Notice that, even if this \( \Lambda^* \) criterion is not known to be necessary for the BMP, in [18] authors show that the criterion has a kind of sharpness property and no counterexamples to the conjecture of \( \Lambda^* \) being necessary for the BMP are known. Moreover, this mass density sufficient condition has been extended (here the logarithmic capacity has been substituted by the relative Monge-Ampere capacity with respect to a ball containing the set \( K \)) to the case of several complex variables by Bloom and Levenberg [5]. Here we present the extension to the rational functions case.

More precisely, we provide two similar sufficient conditions for the RBMP: Theorem 3.2 applies when \( P \) is a subset of the unbounded component of the complement of the set \( K \) while Theorem 3.3 in a more general case.

We recall that given a proper sub-domain \( D \) of the one point compactification \( \mathbb{C}_\infty \) of \( \mathbb{C} \) the Green function of \( D \) is the unique function \( G_D : D \times D \rightarrow ]-\infty, \infty[ \) such that \( G_D(\cdot, \zeta) \) is harmonic in \( D \setminus \{\zeta\} \) and bounded out from any neighbourhood of \( \zeta \), \( G_D(\cdot, \zeta) \) has a logarithmic pole at \( \zeta \) and \( \lim_{z \rightarrow z_0} G_D(z, w) = 0 \) for all \( z_0 \in \partial D \setminus N \) where \( N \) is a polar set (i.e., locally is the \([-\infty]\) level set of a sub-harmonic function).

Let \( K \subset \mathbb{C} \) be any compact set, then we can consider the standard splitting in connected components

\[
\mathbb{C} \setminus K := \Omega_K \cup \left( \bigcup_{j \in I} \Omega_j \right),
\]

where \( \Omega_j \)'s are open bounded, while \( \Omega_K \) is the only unbounded connected component of \( \mathbb{C} \setminus K \).

To simplify the notation from now on we denote by \( g_K(z, \zeta) \) the Green function \( G_{\Omega_K}(z, \zeta) \) of the unbounded domain \( \Omega_K \).

We will make repeated use of this classical result (see for instance [15])

\[
g_K(z, a) = g_{\eta_a(K)}(\eta_a(z), \infty),
\]

where \( \eta_a(z) = \frac{1}{z-a} \).

The main tool in this section is to relate the convergence of logarithmic capacities of the subsets \( K_j \) of a given compact regular non polar set \( K \) to the uniform convergence of the Green functions \( g_{K_j}(z, a) \) to \( g_K(z, a) \) with poles \( a \) in a given disjoint compact set \( P \). We need a one variable version (see Th. 3.1 below) of [5][Th. 1.2] adapted to our setting of moving poles.
Here we deal with logarithmic capacity $\text{cap}(\cdot)$ of compact subset of the complex plane
\begin{equation}
\text{cap}(K) := \sup_{\mu \in M^+_1(K)} \exp(-I[\mu]),
\end{equation}
where we denote by
\begin{equation}
I[\mu] := \int U^\mu d\mu = \int \int \log \frac{1}{|z - \zeta|} d\mu(z)d\mu(\zeta)
\end{equation}
the logarithmic energy of the measure $\mu$.

The existence of a minimizers for $I[\cdot]$ holds true provided $K$ is a non polar set \[17\] Part I (e.g. $\text{cap}(K) > 0$) while the uniqueness follows by the strict convexity of $I[\cdot]$. The unique minimizer is named \textit{equilibrium measure} or \textit{extremal measure} and denoted by $\mu_K$. It is a fundamental result that, for non-polar $K$,
\begin{equation}
\mu_K = \Delta g_K(z, \infty).
\end{equation}
Here the Laplacian has to be intended in the sense of distributions and has been normalized to get a probability measure.

There exists another characterization of the Green function that allow also a generalization to several complex variables. Namely one considers the Lelong class $\mathcal{L}(\mathbb{C})$ of all subharmonic functions on the complex plane having a logarithmic pole at $\infty$, then introduce the \textit{extremal subharmonic function}
\begin{equation}
V_K(z) := \sup\{u \in \mathcal{L}(\mathbb{C}), u|_K \leq 0\}.
\end{equation}
The upper envelope defining $V_K$ has been proved to be equal to the logarithm of the Siciak function
$$\Phi_K := \sup\{|p(z)|^{1/\deg(p)}, p \in \mathcal{P}(\mathbb{C}), \|p\|_K \leq 1\}.$$ By these definitions it follows the Bernstein Walsh Inequality
\begin{equation}
|p(z)| \leq \|p\|_K \exp(\deg(p)V_K(z)), \ p \in \mathcal{P}(\mathbb{C}).
\end{equation}
Moreover, it turns out that the uppersemicontinuous regularization
$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta)$$
coincides with $g_k(z, \infty)$ for all non polar compact $K$. For this reason we refer to $V_K$ as the un-regularized Green function.

Lastly, we recall that a compact set $K$ is said to be \textit{regular} if $g_K(\cdot, \infty)$ (or equivalently $V_K^*$) is continuous on $K$ and hence on $\mathbb{C}$.

From now on we use the following notation, given any compact set $K$ and a positive $\varepsilon$ we set
$$K^\varepsilon := \{z : d(z, K) \leq \varepsilon\},$$
where $d(z, K) := \min_{\zeta \in K} |z - \zeta|$ is the standard euclidean distance.
Finally we notice that

\[(20)\]

\[\gamma\]

Proof of pole when \(a\)

We proceed along the following steps:

1. \(\lim \cap (K_j) = \cap(K)\).
2. \(\lim g_{K_j}(z, a) = g_K(z, a) \text{ loc. unif. for } z \in D, \text{ unif. for } a \in P\).

Proof. By Hilbert Lemniscate Theorem for any \(\varepsilon < d(K, P)\) we can pick a polynomial \(q\) such that

\[\tilde{K} \subset D := \{|q| < ||q||K\} \subset \hat{K}^\varepsilon, \hat{K}^\varepsilon \cap P = \emptyset.\]

Let \(D\) be fixed in such a way.

We introduce a more concise notation for the Green functions involved in the proof: we denote by \(g(z, a)\) the Green function with pole at \(a\) for the set \(\Omega_K\), we omit the pole when \(a = \infty\), we add a subscript \(j\) if \(K\) is replaced by \(K_j\) and a superscript \(b\) if \(K\) or \(K_j\) are replaced by \(\eta_b(K)\) or \(\eta_b(K_j)\), where \(\eta_b(z) := 1/(z - b)\). In symbols

\[g(z) := g_{K(z, \infty)}, \quad g_j(z, a) := g_{K_j(z, a)},\]

\[g_{j}\)(z) := g_{K_j(z, \infty)}, \quad g_{b}(z, a) := g_{\eta_{b}K(z, a)},\]

\[g(z, a) := g_{K}(z, a), \quad g_{j}(z, a) := g_{\eta_{b}K_j(z, a)}.\]

Moreover we set \(E_j := \eta_{a_j}(K_j)\) and \(E := \eta_{\hat{a}}(K)\).

Proof of \((\mathbb{S}) \Rightarrow (\mathbb{S}').\) In order to prove the local uniform convergence of \(g_j(\cdot, a)\) to \(g(\cdot, a)\), uniformly with respect to \(a \in P\), we pick any converging sequence \(P \ni a_j \to \hat{a}\), we set \(\hat{D} := \eta_{\hat{a}}(D)\) and we prove

\[(20)\]

\[g_{a_j} \to \tilde{g} \text{ loc. unif. in } \hat{D}.
\]

Finally we notice that \(g_j(\cdot, a_j) = g_{a_j} \circ \eta_{a_j}^{-1} \to g_{\hat{a}} \circ \eta_{\hat{a}}^{-1} = g(\cdot, \hat{a}) \text{ loc. unif. in } D\) hence the result follows.

We proceed along the following steps:

\[(S1)\]

\[\lim \cap (E_j) = \cap(E)\]

\[(S2)\]

\[\mu_{E_j} \rightharpoonup \mu_E\]

\[(S3)\]

\[\lim g_{E_j}(z, \infty) = g_E(z, \infty) \text{ loc. unif. in } \mathbb{C}\]

Here we used the standard notation (see \((17)\)) \(\mu_E\) for the equilibrium measure of the compact non-polar set \(E\).

To prove \((S1)\) we use [15, Th. 5.3.1] applied to the set of maps \(\varphi_j := \eta_{a_j} \circ \eta_{\hat{a}}^{-1}\) and \(\psi_j := \varphi_j^{-1}\) together with the assumption \((\mathbb{S})\). Each map is bi-holomorphic on a
Notice that without loss of generality we can assume

\[ \text{let} \]

The proof of (S2) is by the Direct Method of Calculus of Variation. More explicitly,

Finally, (S1) follows by combining (23) and (24) and this last statement.

Since \( \text{cap}(E) \leq \text{Lip}_E(f) \text{cap}(E) \), where \( \text{Lip}_E(f) := \inf \{ L : |f(x) - f(y)| < L|x - y| \forall x, y \in E \} \) for any Lipschitz mapping \( f : E \to \mathbb{C} \); [15] Th. 5.3.1. Therefore, due to (21) and (22), we have the following upper bounds.

\[
\text{cap}(E_j) = \text{cap}(\varphi_j(\eta_0(K_j))) \leq L_j \text{cap}(\eta_0(K_j)),
\]

\[
\text{cap}(\eta_0(K_j)) = \text{cap}(\eta_0 \circ \eta_{a_j}^{-1}(E_j)) = \text{cap}(\psi_j(E_j)) \leq L_j \text{cap}(E_j).
\]

Thus, using \( \lim_{j} L_j = 1 \), we have

\[ \liminf_j \text{cap}(E_j) \geq \liminf_j \frac{1}{L_j} \text{cap}(\eta_0(K_j)) = \liminf_j \text{cap}(\eta_0(K_j)), \]

\[ \limsup_j \text{cap}(E_j) \leq \limsup_j L_j \text{cap}(\eta_0(K_j)) \leq \limsup_j \text{cap}(\eta_0(K_j)). \]

Since \( \eta_0 \) is a local diffeomorphism near \( K \) we have \( \text{Lip}_E(\eta_0^{-1}) = (\text{Lip}_K(\eta_0))^{-1} \) and we can conclude (again by [15] Th. 5.3.1) that

\[ \lim_j \text{cap}(\eta_0(K_j)) = \text{cap}(E). \]

Finally, (S1) follows by combining (23) and (24) and this last statement.

The proof of (S2) is by the Direct Method of Calculus of Variation. More explicitly, let \( \mu_j := \mu_{E_j} \) be the sequence of equilibrium measures, i.e., the minimizers of \( I[\cdot] \) as defined in [16] among the classes \( \mu \in \mathcal{M}_1(E_j) \). From (S1) it follows that \( \liminf_j I[\mu_j] = I[\mu_E] \). Therefore, if \( \mu \) is any weak* closure point of the sequence, by lower semi-continuity of \( I \), we get \( I[\mu] \leq I[\mu_E] \).

Notice that without loss of generality we can assume \( K_j \), and thus \( E_j \), to be not polar, since \( \text{cap}(K_j) > 0 \) for \( j \) large enough.

If \( \text{supp} \mu \subseteq E \), by the strict convexity of the energy functional, we have that \( \mu = \mu_E \) and the whole sequence is converging to \( \mu_E \); see [17] Part I, Th. 1.3]. Then we are left to prove \( \text{supp} \mu \subseteq E \), this follows by the uniform convergence of \( \eta_{a_j} \) to \( \eta_0 \) and by properties of weak* convergence of measures.

To this aim, we suppose by contradiction \( \text{supp} \mu \cap (\mathbb{C} \setminus E) \neq \emptyset \). It follows that there exists a Borel set \( B \subseteq \mathbb{C} \setminus E \) with \( \mu(B) > 0 \). Since \( \mu \) is Borel we can find a closed set \( C \subseteq B \) still having positive measure. Being \( \mathbb{C} \) a metric space and we can find an open neighbourhood \( A \) of \( C \) disjoint by \( E \) with \( \mu(A) > 0 \).
Due to the Porte-manteau Theorem we have
\[ 0 < \mu(A) \leq \liminf_j \mu_j(A). \]
Therefore \( C \subseteq A \subset E_{j_m} \) for an increasing subsequence \( j_m \).

By the uniform convergence \( \eta_{a,j_m} \to \eta_a \) it follows that \( C \subseteq A \subseteq E \), a contradiction since we assumed \( C \cap E = \emptyset \).

Let us prove (S3).

First, we recall (see for instance [17, pg. 53]) that for any compact set \( M \subset C \) we have \( g_M(z, \infty) = \text{cap}(M) - U^{\mu_M}(z) \). Hence it follows that
\[ g_{E_j}(\zeta, \infty) = \text{cap}(E_j) - U^{\mu_j}(\zeta). \]

Due to (S2) and by the Principle of Descent [17, I.6, Th. 6.8] for any \( \zeta \in C \) we have
\[ \limsup_j -U^{\mu_j}(\zeta) \leq -U^{\mu_E}(\zeta). \]

It follows by (S1), (25) and (26) that
\[ \limsup_j g_{E_j}(\zeta, \infty) \leq g_E(\zeta, \infty), \forall \zeta \in C. \]

The sequence of subharmonic functions \( \{g_{E_j}(\zeta, \infty)\} \) is locally uniformly bounded above and non negative, therefore we can apply the Hartog’s Lemma. For each \( \epsilon > 0 \) there exists \( j(\epsilon) \in \mathbb{N} \) such that
\[ \|g_{E_j}(\zeta, \infty)\|_E \leq \|g_E(\zeta, \infty)\|_E + \epsilon = \epsilon. \]

Here the last equality is due to the regularity of \( K \) and thus of \( E \) (e.g. \( g_E(\zeta, \infty) \equiv 0 \forall \zeta \in E \)). Therefore we have
\[ g_{E_j}(\zeta, \infty) - \epsilon \leq g_E(\zeta, \infty), \forall \zeta \in E. \]

By the extremal property of the Green function (see [18] and lines below) and the upper bound (27) it follows that
\[ g_{E_j}(\zeta, \infty) - \epsilon \leq g_E(\zeta, \infty), \forall \zeta \in C, j \geq j(\epsilon). \]

Since \( g_E(\cdot, \infty) \) is continuous (hence uniformly continuous on a compact neighbourhood \( M \) of \( E \) containing all \( E_j \)) for any \( \epsilon > 0 \) we can pick \( \delta > 0 \) such that \( g_E(\zeta, \infty) \leq \epsilon \) for any \( \zeta \in E^\delta \).

Let us set \( j'(\epsilon) := \min\{j : E_j \subseteq E^\delta \forall j \geq j'\} \), notice that \( j'(\epsilon) \in \mathbb{N} \) for any (sufficiently small) \( \epsilon > 0 \) since
\[ E_j \subset \eta_{a,j}(K) \subseteq L_j \eta_{\hat{a}}(K) = L_j E \subseteq E^{(L_j-1)\|z\|_E}, \]
where \( L_j \) is defined in equations (21) (22) and \( L_j \to 1 \).

It follows by this choice that
\[ \|g_{E_j}(\zeta, \infty)\|_{E_j} \leq \epsilon, \forall j \geq j'(\epsilon). \]
Therefore, again by the extremal property of \( g_{E_j}(\zeta, \infty) \), we have
\[
(29) \quad g_{E_j}(\zeta, \infty) - \varepsilon \leq g_{E_j}(\zeta, \infty), \quad \forall \zeta \in \mathbb{C}, \ j \geq j'(\varepsilon).
\]

Now simply observe that (29) and (28) imply
\[
g_{E_j}(\zeta, \infty) - \varepsilon \leq g_{E_j}(\zeta, \infty) \leq g_{E_j}(\zeta, \infty) + \varepsilon, \quad \forall \zeta \geq \max\{j'(\varepsilon), j'(\varepsilon)\}.
\]

Therefore, again by the extremal property of \( g_{E_j}(\zeta, \infty) \), we have
\[
g_{E_j}(\zeta, \infty) - \varepsilon \leq g_{E_j}(\zeta, \infty) \leq g_{E_j}(\zeta, \infty) + \varepsilon.
\]

Therefore we can conclude that \( \liminf_{j} \cdot \) converges locally uniformly to \( g_{E}(\cdot, \infty) \).

To conclude the proof of (ii) \( \Rightarrow \) (ii) let us pick any compact subset \( L \) of \( D \).
\[
\|g^\alpha_j - g^\beta\|_L = \|g_{E_j}(\eta_{a_j}(z), \infty) - g_{E}(\eta_{a}(z), \infty)\|_L \leq \|g_{E_j}(\eta_{a_j}(z), \infty) - g_{E}(\eta_{a_j}(z), \infty)\|_L + \|g_{E}(\eta_{a_j}(z), \infty) - g_{E}(\eta_{a}(z), \infty)\|_L \rightarrow 0
\]

Here we used the continuity of \( g_{E}(z, \infty) \) and the local uniform convergence of \( \eta_{a_j} \) to \( \eta_0 \). By the arbitrariness of the sequence of poles \( \{a_j\} \) follows.

Proof of (ii) \( \Rightarrow \) (i). Fix any pole \( a \in P \), by the continuity of \( \eta_{a_j} \) and \( \{a_j\} \) we have \( g_j \rightarrow g \) locally uniformly in \( D \).

Due to the local uniform convergence we have the weak* convergence of distributional Laplacian \( \mu_j = \Delta g^\alpha_j \rightarrow^* \Delta g^\alpha = \mu_E \).

By Lower Envelope Theorem [17] 1.6 Th. 6.9 we have
\[
\liminf_j U^{\mu_E} = \liminf_j -g^\alpha_j + \text{cap}(E_j) =_{\text{q.e.}} U^{\mu_E} = -g^\alpha + \text{cap}(E).
\]

Therefore we can conclude that \( \liminf_j \text{cap}(E_j) = \text{cap}(E) \), being by definition \( \limsup_j \text{cap}(E_j) \leq \text{cap}(E) \) we have
\[
\lim_j \text{cap}(E_j) = \text{cap}(E).
\]

Now we can perform the same reasoning as in proving (ST) above working with maps \( \varphi_j \) and \( \psi_j \) showing that the same holds true for \( K_j \) and \( K \).

\[\square\]

The previous theorem is the main tool in proving that, for measures having regular compact support, the classical sufficient mass density condition in [17] or \( \Lambda^* \) condition [18] implies a rational Bernstein Markov Property, provided \( P \subset \Omega_K \).

It is worth to notice that, due to Theorem [2.3] and [18] Th. 4.2.3, if \( K = S_K \) or \( K = \hat{K} \), then the same \( \Lambda^* \) condition implies the rational Bernstein Markov property even if \( P \subset \Omega_K \) does not hold.

\textbf{Theorem 3.2.} Let \( K \subset \mathbb{C} \) be a compact regular set and \( P \subset \Omega_K \) be compact. Let \( \mu \in \mathcal{M}^+(K) \), \( \text{supp} \mu = K \) and suppose that there exists \( t > 0 \) such that
\[
(30) \quad \lim_{r \to 0^+} \text{cap}\{z \in K : \mu(B(z, r)) \geq r^t\} = \text{cap}(K).
\]

Then \((K, \mu, P)\) has the rational Bernstein Markov Property.
Proof: The proof of Theorem 3.2 goes along the same lines of [18], except for the lack of the Bernstein Walsh Inequality (19) which is not available for rational functions.

In place of it we use the following variant due to Blatt [3, eqn. 2.2] which holds for any rational function.

\[
\frac{p_k(\zeta)}{q_k(\zeta)} = \frac{c_k \prod_{j=0}^{m_k} (\zeta - z_j)}{\prod_{j=0}^n (\zeta - a_j)}
\]

(31)

\[
|r_k(\zeta)| \leq \|r_k\|_K \exp \left( \sum_{j=1}^n g_{K_j}(\zeta, a_j) + (m_k - n_k)g_{K}(\zeta, \infty) \right) \forall \zeta \notin \{a_1, \ldots, a_n\}.
\]

(32)

Thus in particular we have

\[
|r_k(\zeta)| \leq \|r_k\|_{K_j} \exp \left( n_k \max_{a \in P} g_{K_j}(\zeta, a) + (m_k - n_k)g_{K}(\zeta, \infty) \right) \forall \zeta \in \mathbb{C} \setminus P.
\]

(33)

Notice that, for any sequence \(K_j \subset K\) such that \(\cap K_j \rightarrow \cap K\), from Theorem 3.1 it follows that

\[
g_{K_j}(\zeta, a) \rightarrow g_{K}(\zeta, a) \text{ locally uniformly in } \mathbb{C} \setminus P.
\]

Moreover, it is well known that under the same condition we have

\[
g_{K_j}(\zeta, \infty) \rightarrow g_{K}(\zeta, \infty) \text{ locally uniformly in } \mathbb{C}.
\]

Pick any \(r_k \in \mathcal{R}(P)\). By the regularity of \(K\) and the compactness of \(P\) for any \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
g_{K}(\zeta, a) \leq \varepsilon \forall \zeta : \text{dist}(\zeta, K) \leq \delta, \forall a \in P
\]

\[
g_{K}(\zeta, \infty) \leq \varepsilon \forall \zeta : \text{dist}(\zeta, K) \leq \delta.
\]

Let us pick \(\varepsilon > 0\), it follows by (32) that there exists \(\delta > 0\) such that \(\forall \zeta : \text{dist}(\zeta, K) \leq \delta\) we have

\[
|r_k(\zeta)| \leq \|r_k\|_K \exp \left( n_k \max_{a \in P} g_{K_j}(\zeta, a) + (m_k - n_k)g_{K}(\zeta, \infty) \right) \leq e^{\|r_k\|_K} \varepsilon.
\]

(34)

By Theorem 3.1 (possibly shrinking \(\delta\)) we have, for any \(A \subset K\), with \(\text{cap}(A) > \text{cap}(K) - \delta\) and locally uniformly in \(\mathbb{C} \setminus P\),

\[
\max_{w \in P} g_A(\zeta, w) \leq \max_{w \in P} g_{K}(\zeta, w) + \varepsilon,
\]

(35)

\[
g_A(\zeta, \infty) \leq g_{K}(\zeta, \infty) + \varepsilon.
\]

(36)

Using (34) and (36) we have

\[
|r_k(\zeta)| \leq e^{2\|r_k\|_K} \varepsilon, \forall \zeta \in K^\delta, \forall A \subset K\text{ with } \text{cap}(A) > \text{cap}(K) - \delta.
\]

(37)

Let \(\zeta_0 \in A\) be such that \(\|r_k\|_A = |r_k(\zeta_0)|\), we show that a lower bound for \(|r_k|\) holds in a ball centred at \(\zeta_0\). By the Cauchy Inequality we have \( |r_k(\zeta)| < \frac{\|r_k(\zeta_0)\|}{s} \leq \frac{e^{2\|r_k\|_K}}{s} \).
for any $|ζ - ζ₀| < s$, $s < δ$. Taking $s = δ/2$ we can integrate such an estimates as follows $∀z ∈ B(ζ₀, δ/2)$

$$
∥r_k∥_A = |r_k(ζ₀)| = \left| r_k(z) + \int_{|ζ|<\delta} r'_k(ζ)dζ \right| ≤ |r_k(z)| + |z - ζ₀|\frac{e^{(2ζ_k)||r_k||_A}}{δ/2}.
$$

It follows by the above estimate that

$$
\min_{z ∈ B(ζ₀, e^{(2ζ_k)})} |r_k(z)| ≥ \frac{∥r_k∥_A}{2} ∀A ⊂ K \text{ with } \text{cap}(A) > \text{cap}(K) - δ.
$$

Now we provide a lower bound for $L^2_δ$ norms of $r_k$ by integrating the last inequality on a (possibly smaller) ball and picking $A ⊂ K$ according to the mass density condition (43).

Precisely, set $ρ_k := e^{(-3δk)}$, by the hypothesis we can pick $t > 0$ and $A_k ⊂ K$ with $\text{cap}(A_k) > \text{cap}(K) - δ$ such that $μ(B_k) := μ(B(η, ρ_k)) ≥ ρ_k^t ∀η ∈ A_k$. We pick $k ≥ \bar{k}$ such that $ρ_k < \frac{δ}{4}$, thus using (38) we get

$$
∥r_k∥^2_{L^2_δ} ≥ \int_{B_k} |r_k|^2 dμ ≥ \min_{z ∈ B_k} |r_k(z)|^2 μ(B_k) ≥ \frac{∥r_k∥^2_A}{4} ρ_k^t ≥ \frac{e^{(-3δk)}}{4}∥r_k∥^2_A ≥ \frac{e^{(-4+3δk)}}{4}∥r_k∥^2_K.
$$

It follows that $\left(\frac{∥r_k∥_{L^2_δ}}{∥r_k∥_A}\right)^{1/k} ≤ 41^k e^{(4+3δk)}$, by arbitrariness of $ε > 0$ we can conclude that

$$
\limsup_k \left(\frac{∥r_k∥_{L^2_δ}}{∥r_k∥_A}\right)^{1/k} ≤ 1
$$

If we remove the hypothesis $P ⊂ Ω_k$, then Theorem 3.1 is no more applicable. We go around such a difficulty by a suitable conformal mapping $f$ of a neighbourhood of $K ∪ P$ given by the Proposition 3.1 below.

We recall the definitions of Fekete points and transfinite diameter for the reader convenience. Given any compact set $K$ in the complex plane for any positive integer $k$ a set of Fekete points of order $k$ is an array $z_k = \{z_0, \ldots, z_k\} ∈ K^k$ that maximizes the product of distances of its points among all such arrays, that is

$$
V_k(z_k) := \prod_{1≤i≤j≤k} |z_i - z_j| = \max_{ζ ∈ K^k} \prod_{1≤i≤j≤k} |ζ_i - ζ_j|.
$$

Notice that such maximizing array does not need to be unique.

It turns out that, denoting by $δ_k(K) := \left(\max_{ζ ∈ K^k} V_k(ζ)\right)^{\frac{2}{k(k+1)}}$ the $k$-th diameter of $K$, we have

$$
\lim_k δ_k(K) =: δ(K) = \text{cap}(K),
$$

(39)
where $\delta(K)$ is the transfinite diameter of $K$ (existence of the limit being part of the statement). We refer the reader to [15, 17, 16] for further details.

**Proposition 3.1.** Let $K, P \subset \mathbb{C}$ be compact sets, where $K \cap \hat{P} = \emptyset$. Then there exist $w_1, w_2, \ldots, w_m \in \mathbb{C} \setminus (K \cup \hat{P})$ and $R_2 > R_1 > 0$ such that denoting by $f$ the function $z \mapsto \prod_{j=1}^{m} (z - w_j)$ we have

\[
K \subset \{ |f| < R_1 \},
\]

\[
P \subset \{ R_1 < |f| < R_2 \}.
\]

**Proof.** We first suppose $P$ to be not polar.

Moreover we show that we can suppose without lost of generality that

\[
(40) \quad \log \delta(P) < \min_{K} g_P(z, \infty).
\]

To do that, consider $0 < \lambda < \frac{1}{\delta(P)}$ and notice that

\[
\log(\lambda P) = \log \lambda \delta(P) < 0.
\]

On the other hand one has $g_{\lambda P}(z, \infty) = g_P(z \lambda, \infty)$, thus it follows that

\[
\min_{z \in K} g_P(z, \infty) = \min_{z \in \lambda K} g_{\lambda P}(z, \infty) > 0 > \log(\lambda P),
\]

where the first inequality is due to the assumption $K \cap \hat{P} = \emptyset$.

If we build $\tilde{f}$ as in the proposition for the sets $P' := \lambda P$ and $K' := \lambda K$, then $f := \tilde{f} \circ \lambda^{-1}$ enjoys the right properties for the original sets $P, K$. Hence in the following we can suppose (40) to hold.

Let us pick $0 < \rho < \hat{\rho} := \text{dist}(\hat{P}, K)/2$ and consider the set $\hat{P}^\rho$.

For the sake of an easier notation we denote by $g(z)$ and $g_\rho(z)$ the functions $g_P(z, \infty)$ and $g_{\rho P}(z, \infty)$.

For any $k \in \mathbb{N}$ let us pick any set $Z_k(\rho) := \{z_1^{(k)}, \ldots, z_k^{(k)}\}$ of Fekete points for $\hat{P}^\rho$, moreover we denote the polynomial $\prod_{j=1}^{k} (z - z_j^{(k)})$ by $q_k$. Notice that $Z_k(\rho) \subset (\partial \hat{P}^\rho)^k \subset (\mathbb{C} \setminus (K \cup P))^k$, hence $\{z_1^{(k)}, \ldots, z_k^{(k)}\}$ is an admissible tentative choice for $w_1, w_2, \ldots, w_k$.

Let us set

\[
a(\rho) := \min_{K} g_\rho, \quad a := \min_{\rho \in [0, \hat{\rho}]} a(\rho) = a(\hat{\rho}),
\]

\[
b := \max_{\rho \in [0, \hat{\rho}]} \max_{K} g_\rho = \max_{K} g.
\]

We recall that (see [17, III Th. 1.8])

\[
\lim_{k \to \infty} \frac{1}{\log^+ |q_k|} g_\rho = g_\rho, \quad \text{locally uniformly on } \mathbb{C} \setminus \hat{P}^\rho.
\]
Thus for any \( \epsilon > 0 \) we can choose \( m(\epsilon) \in \mathbb{N} \) such that
\[
\left\| \frac{1}{m} \log^+|q_m| - g \right\|_{B(\rho)} < \epsilon \quad \forall m \geq m(\epsilon),
\]
where \( B(\rho) := \{ z \in \mathbb{C} : a \leq g_\rho(z) \leq b \} \), notice that \( \hat{P}^\rho \cap B(\rho) = \emptyset \).

Then, taking \( \epsilon < a \) we have \( \forall m \geq m(\epsilon) \)
\[
K \subset \left\{ a(\rho) - \epsilon \leq \frac{1}{m} \log^+|q_m| \leq b + \epsilon \right\} = \left\{ e^{m(a(\rho) - \epsilon)} \leq |q_m| \leq e^{m(b + \epsilon)} \right\} =: A(\epsilon, \rho, m).
\]

On the other hand by the same reason\( \log \delta_m(P^\rho) < a(\rho) - \epsilon \).

In such a case the function \( f(z) := \frac{1}{q_m(z)} \) satisfies the properties of the proposition since
\[
\|f\|_K \leq e^{-(m(a(\rho) - \epsilon))} < \delta_m(P^\rho)^{-m} \leq \min_P |f|.
\]

To conclude, we are left to prove that we can choose admissible \( m, \rho > 0 \) and \( \epsilon > 0 \) such that \( (42) \) holds. To do that we recall that, since \( P = \cap_{l \in \mathbb{N}} P_{l/4} \), by \[15\] Th. 5.1.3 we have
\[
\delta(P) = \lim_{l} \delta(P_{l/4}) = \lim_{m} \lim_{l} \delta_m(P_{l/4}).
\]

On the other hand by the same reason \( g_{1/m} \) is uniformly converging by the Dini’s Lemma to \( g \) on a neighbourhood of \( K \) not intersecting \( P_{\rho} \).

Therefore, it follows by \( (41) \) and \( (40) \) that possibly shrinking \( \epsilon \) to get
\[
0 < \epsilon \leq m(\epsilon) - \log \delta(P) \quad \text{we have}
\]
\[
\lim_{l} \lim_{m} \log \delta_m(P_{l/4}) = \log \delta(P) < \min_{k} m(\epsilon) - \epsilon = \lim_{k} \min_{m} g_{1/m} - \epsilon.
\]

Hence (possibly taking \( \epsilon' < \epsilon \)) there exists an increasing subsequence \( k \rightarrow l_k \) with
\[
\lim_{m} \log \delta_m(P_{1/l_k}) < \lim_{m} \min_{k} g_{1/m} - \epsilon' \quad \text{for any} \ k \in \mathbb{N}.
\]

In the same way we can pick a subsequence \( k \rightarrow m_k \) such that \( \log \delta_m(P_{1/l_k}) < \min_{k} g_{1/m_k} - \epsilon'' \) for all \( k \in \mathbb{N} \). Taking \( k \) large enough to get \( m_k > m(\epsilon'') \) and setting \( m := m_k, \rho := 1/l_k \) suffices.

In the case of \( P \) being a polar subset of \( \mathbb{C} \) we observe that for any positive \( \rho \) the set \( \hat{P}^\rho \) is not polar since it contains at least one disk. Moreover notice that \( \lim m \delta_m(P_{1/m}) = \log \delta(P) = -\infty \) whereas the sequence of harmonic (on a fixed
suitable neighbourhood of \( K \) functions \( g_{1/m} \) is positive and increasing. Equation (41) is then satisfied for \( m \) large enough. The rest of the proof is identical. \( \square \)

We use the standard notation \( f_\ast \mu(A) := \int_{f^{-1}(A)} d\mu \) for any Borel set \( A \subset \mathbb{C} \).

If we set \( E := f(K) \), \( Q := f(P) \) we can see that \( E \cap Q = \emptyset \) thus \( E, Q \) are precisely in the same relative position as in the Theorem 3.1. Therefore we are now ready to state a sufficient condition for the rational Bernstein Markov property under more general hypothesis.

**Theorem 3.3 (Mass-Density Sufficient Condition).** Let \( K, P \subset \mathbb{C} \) be compact disjoint sets where \( K \) is non polar regular w.r.t. the Dirichlet problem and \( \hat{P} \cap K = \emptyset \). Let \( \mu \in M^+(K) \) be such that \( \text{supp} \mu = K \) and suppose that there exist \( t > 0 \) and \( f \) as in Proposition 3.1 such that the following holds

\[
\lim_{r \to 0^+} \cap \left\{ z \in f(K) : f_\ast \mu(B(z, r)) \geq r^t \right\} = \cap(f(K)).
\]

Then \( (K, \mu, P) \) has the rational Bernstein Markov Property.

**Proof.** By Theorem 3.2 it follows that the triple \( (E, f_\ast \mu, Q) \) has the rational Bernstein Markov Property.

To conclude the proof it is sufficient to notice that for any sequence \( \{r_k\} \) in \( \mathcal{R}(P) \), the sequence

\[
\tilde{r}_j := r_{j/m} \circ f \quad j = 1, 2, \ldots
\]

is an element of \( \mathcal{R}(Q) \). Moreover by the RBMP of \( (E, f_\ast \mu, Q) \) we can pick \( c_j > 0 \) such that \( \lim sup_j c_j^{1/j} \leq 1 \) and

\[
\|r_k\|_E = \|\tilde{r}_{mk}\|_{E} \leq c_{mk}\|\tilde{r}_{mk}\|_{L^2(f_\ast \mu)} \leq c_{mk}\|r_k\|_{L^2(\mu)}.
\]

Thus we have

\[
\left( \frac{\|r_k\|_E}{\|r_k\|_{L^2(\mu)}} \right)^{1/k} \leq \left( c_{mk}^{1/(mk)} \right)^m \to 1^m = 1.
\]

\( \square \)

**Example 2.** We go back to the case of the Example 1 (d) to show that the same conclusion follows by applying Theorem 3.3. Let us recall the notation. We consider the annulus \( A := \{ z : 1/2 \leq |z| \leq 1 \} \), set \( K := \partial A \), \( P := \{0\} \) and \( \mu := 1/2 ds_{|\partial A|} + 1/2 ds_{|\partial D|} \), where \( ds \) is the standard length measure.

We proceed as in Proposition 3.1 to build the map \( f \): we take \( \rho = 0.1 \) and for each \( m \in \mathbb{N} \) we pick a set of Fekete points for \( P^\rho = \{ |z| \leq 0.1 \} \).

In this easy example \( m = 2 \) suffices to our aim, so we can choose \( w_1 = 0.1 \), \( w_2 = -0.1 \), \( f(z) = \frac{1}{(z-w_1)(z-w_2)} = \frac{1}{z^2 - 0.01} \).
We notice that \( f \) is a holomorphic map of a neighbourhood \( K^\delta \) of \( K \) and we can compute its Lipschitz constant as follows.

\[
L_\delta := \text{Lip}_{K^\delta}(f) = \max_{z \in K^\delta} \left| \frac{-2z}{(z^2 - 0.01)^2} \right|.
\]

For instance, taking \( \delta = 0.1 \) we get \( L_\delta = \frac{4(1 - 2\delta)}{1-4\delta} = 5.3 \).

For any \( z \in f(K) \) we use the following notation \( f^{-1}(z) = \{ \zeta_1, \zeta_2 \} \). Now we notice that for any \( z \in f(\partial \mathbb{D}) \) we have

\[
f_*\mu(B(z, r)) \geq \mu \left( B(\zeta_1, \frac{r}{L_\delta}) \cup B(\zeta_2, \frac{r}{L_\delta}) \right) = \frac{1}{2} \int_{B(\zeta_1, \frac{r}{L_\delta}) \cap \partial \mathbb{D}} ds + \frac{1}{2} \int_{B(\zeta_2, \frac{r}{L_\delta}) \cap \partial \mathbb{D}} ds =
\]

\[
\frac{1}{2} \int_{\zeta_1 - \arccos \left( 1 - \left( \frac{r}{L_\delta} \right)^2 \right)}^{\zeta_1 + \arccos \left( 1 - \left( \frac{r}{L_\delta} \right)^2 \right)} d\theta + \frac{1}{2} \int_{\zeta_2 - \arccos \left( 1 - \left( \frac{r}{L_\delta} \right)^2 \right)}^{\zeta_2 + \arccos \left( 1 - \left( \frac{r}{L_\delta} \right)^2 \right)} d\theta = \arccos \left( 1 - \left( \frac{r}{L_\delta} \right)^2 \right) \geq r',
\]

for any \( 0 < t < 1 \) and sufficiently small \( r \). If instead \( z \in f(\frac{1}{2} \partial \mathbb{D}) \), then the same computations show that \( f_*\mu(B(z, r)) \geq \arccos \left( 1 - \left( \frac{r}{2L_\delta} \right)^2 \right) \), hence it dominates \( r' \) for any \( 0 < t < 1 \) and sufficiently small \( r \).

This implies that (43) holds for \((K, \mu, f)\) and hence, due to Theorem 3.3 \((K, \mu, P)\) has the rational Bernstein Markov property.

Finally we notice that also \((A, \mu, P)\) has the rational Bernstein Markov property (as we observed in Example 1 (d)) since any rational function having poles on \( P \) achieves the maximum of its modulus on \( K \).

It is worth to notice that a measure \( \mu \) can satisfy (43) even if the mass of balls of radius \( r \) decays very fast (e.g. faster than any power of \( r \)) as \( r \to 0 \) at some points of the support of \( \mu \). This is the case of the following example.

**Example 3.** Let us consider the measure

\[
\frac{d\mu}{d\theta} := \exp \left( \frac{-1}{1 - \left( \frac{\theta}{\pi} \right)^2} \right), \quad -\pi \leq \theta \leq \pi
\]

defined on the unit circle \( \partial \mathbb{D} \) and pick as pole set \( P := \{ 0 \} \).

We use the map \( f(z) := \frac{1}{z - 0.1} \) to check (43), notice that \( f \) is a bi-holomorphism of \( \partial \mathbb{D} \) on its image, being \( L := 1/0.9 \) its Lipschitz constant on \( \partial \mathbb{D} \). It follows that for
any $r > 0$ we have $f_s \mu(B(\frac{1}{e^{\theta} - 0.1}, r)) \geq \mu(B(e^{i\theta}, r/L))$.

$$
\mu(B(e^{i\theta}, r/L)) = \int_{\theta - \arcsin(1 - \frac{r^2}{2L^2})}^{\theta + \arcsin(1 - \frac{r^2}{2L^2})} \exp\left(-\frac{\pi^2}{\pi^2 - u^2}\right) du
$$

$$
\geq \begin{cases}
2 \arcsin(1 - \frac{r^2}{2L^2}) \exp\left(-\frac{\pi^2}{\pi^2 - (\theta + \arcsin(1 - \frac{r^2}{2L^2}))^2}\right), & \theta > \arcsin(1 - \frac{r^2}{2L^2}) \\
2 \arcsin(1 - \frac{r^2}{2L^2}) \exp\left(-\frac{\pi^2}{\pi^2 - (\theta - \arcsin(1 - \frac{r^2}{2L^2}))^2}\right), & \theta < -\arcsin(1 - \frac{r^2}{2L^2}).
\end{cases}
$$

We use the simpler notation $s := \arcsin(1 - \frac{r^2}{2L^2})$, that is $r = 2L \sin(s/2)$. Notice that $s \to 0^+$ as $r \to 0^+$.

We are interested in computing $\lim_{s \to 0^+} \text{cap}(K_{s,t})$, where

$$
K_{s,t} := \left\{ \frac{1}{e^{i\theta} - 0.1}, -\pi \leq \theta \leq \pi, \mu(B(e^{i\theta}, 2 \sin(s/2))) > (2L \sin(s/2))^t \right\}.
$$

Since $f$ is a bi-holomorphism of a neighbourhood of $\partial \mathbb{D}$ we have

$$
\lim_{s \to 0^+} \text{cap}(K_{s,t}) \geq \lim_{s \to 0^+} \left( \text{cap} D_{s,t}(\text{Lip}(f^{-1}))^{-1} \right) = \lim_{s \to 0^+} \left( \text{cap} D_{s,t} \text{Lip}(f) \right),
$$

where $D_{s,t} := \left\{ e^{i\theta}, -\pi \leq \theta \leq \pi, \mu(B(e^{i\theta}, 2 \sin(s/2))) > (2L \sin(s/2))^t \right\}$.

In order to prove that (45) holds for $f_s \mu$ we are left to prove that there exists $t > 0$ such that

$$
\lim_{s \to 0^+} \text{cap}(D_{s,t}) = \text{cap}(\partial \mathbb{D})
$$

$$
\lim_{s \to 0^+} \text{Lip}(f)_{D_{s,t}} = \text{Lip}(f)_{\partial \mathbb{D}}
$$

To do that let us notice that, for $t > 1$,

$$
D_{s,t} =
\left\{ e^{i\theta} : \pi \leq \theta \leq s, 2s e^{\pi i - (\theta - s)^2} > L^t \left(2 \sin(\frac{s}{2})\right)^t \right\} \cup \left\{ e^{i\theta} : -\pi \leq \theta \leq -s, 2s e^{\pi i - (\theta + s)^2} > L^t \left(2 \sin(\frac{s}{2})\right)^t \right\} =
\left\{ e^{i\theta} : \pi \leq \theta \leq s, e^{\pi i - (\theta - s)^2} > L^t \left(\frac{\sin(s/2)}{s/2}\right)^t \right\} \cup \left\{ e^{i\theta} : -\pi \leq \theta \leq -s, e^{\pi i - (\theta + s)^2} > L^t \left(\frac{\sin(s/2)}{s/2}\right)^t \right\} \supseteq
$$

$$
D_{s,t} :=
\left\{ e^{i\theta} : \pi \leq \theta \leq s, \exp\left(-\frac{\pi^2}{\pi^2 - (\theta - s)^2}\right) > L^t \frac{s}{2} \right\} \cup \left\{ e^{i\theta} : -\pi \leq \theta \leq -s, \exp\left(-\frac{\pi^2}{\pi^2 - (\theta + s)^2}\right) > L^t \frac{s}{2} \right\} =
\left\{ e^{i\theta} : \max\left\{ -\pi, -s - \frac{1}{\log(L^t s^{-1}/2)} \right\} < \theta < \min\left\{ \pi, s + \pi \frac{1}{\log(L^t s^{-1}/2)} \right\} \right\}.
$$
We recall that the logarithmic capacity of an arc of circle of radius 1 and length \( \alpha \) is \( \sin(\alpha/4) \); see [15, pg. 135]. Therefore we have

\[
\text{cap}(\partial D) \geq \lim_{s \to 0^+} \text{cap}(D_{s,t}) \geq \lim_{s \to 0^+} \text{cap}(\tilde{D}_{s,t}) = \\
\lim_{s \to 0^+} \sin \left( \frac{1}{4} \left( 2\pi \sqrt{1 + \frac{1}{\log(L_s')} + 2s} \right) \right) = 1 = \text{cap}(\partial D),
\]

(47)

this proves (45).

To prove (46), simply observe that \( 0 \in D_{s,t} \) for any \( s, t > 0 \) and thus \( L \geq \text{Lip}_{D_{s,t}}(f) \geq |\partial_z f(0)| = L \). It follows by (44) that (43) holds and thus \((K, \mu, P)\) has the rational Bernstein Markov property.

3.1. **Application: a \( L^2 \) meromorphic Bernstein Walsh Lemma.** For a given compact set \( K \subset \mathbb{C} \) we denote by \( D_r \) the set \( \{ z \in \mathbb{C} : g_K(z, \infty) < -\log r \} \) and by \( \mathcal{M}_n(D_r) \) the class of meromorphic functions having precisely \( n \) poles (counted with their multiplicities) in \( D_r \). Let us denote by \( \mathcal{R}_{k,n} \) the class of rational functions having at most \( k \) zeroes and at most \( n \) poles (each of them counted with its multiplicity). It is a classical result of Walsh [19] that if \( f \in \mathcal{M}_n(D_r) \) is holomorphic on \( K \) and \( r_k \in \mathcal{R}_{k,n} \) is such that

\[
\limsup_k \|f - r_k\|_K^{1/k} \leq 1/r,
\]

(48)

then each \( r_k \) for \( k \) large enough has precisely \( n \) poles in \( D_r \) converging to the poles of \( f \) and the sequence \( r_k \) converges uniformly to \( f \) on each compact subset \( E \) of \( D_r \setminus \text{Poles}(f) \). This theorem has been sharpened including possible singularities on \( \partial D_r \) in [13]; see also [10].

If \((K, \mu, P)\) has the rational Bernstein Markov property for a compact set \( P \supset \text{Poles}(f) \), the condition (48) can be checked using the \( L^2_\mu \) norms in place of uniform ones. Notice that by the rational Bernstein Markov property if \( \limsup_k \|f - r_k\|_K^{1/k} \leq 1/r \), then \( \limsup_k \|f - r_k\|_L^{1/k} \leq 1/r \), thus the conclusions of the Walsh theorem hold.

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