SHARP ILL-POSEDNESS FOR THE MAXWELL-DIRAC EQUATIONS IN ONE SPACE DIMENSION

SIGMUND SELBERG AND ACHENEF TESFAHUN

ABSTRACT. The Maxwell-Dirac equations in one space dimension are proved to be well posed in the charge class, that is, with $L^2$ data for the spinor. We also prove that this result is sharp, in the sense that well-posedness fails for spinor data in $H^s$ with $s < 0$, as well as in $L^p$ with $1 \leq p < 2$. More precisely, we give an explicit example of such data for which no local solution can exist. Our proof of well-posedness applies to a class of systems which includes also the Dirac-Klein-Gordon system, but it does not require any null structure in the system.

1. INTRODUCTION

We consider the Maxwell-Dirac equations on the Minkowski space-time $\mathbb{R}^{1+1}$,

\begin{align}
(-i\gamma^\mu \partial_\mu + M)\psi &= A_\mu \gamma^\mu \psi, \\
\Box A_\mu &= -\overline{\psi} \gamma_\mu \psi,
\end{align}

with initial conditions at time $t = 0$,

$$
\psi(0, x) = \psi_0(x), \quad A_\mu(0, x) = a_\mu(x), \quad \partial_\mu A_\mu(0, x) = b_\mu(x).
$$

The unknowns are the Dirac spinor field $\psi : \mathbb{R}^{1+1} \to \mathbb{C}^2$, regarded as a column vector, and the electromagnetic potential components $A_\mu : \mathbb{R}^{1+1} \to \mathbb{R}$, $\mu = 0, 1$. Here $\Box = \partial_\mu \partial_\mu$ is the d’Alembertian, $M \in \mathbb{R}$ is a mass constant, and $\overline{\psi} = \psi^* \gamma^0$ with $\psi^*$ the complex conjugate transpose. The equations are written in covariant form on $\mathbb{R}^{1+1}$ with coordinates $x^\mu$ and metric $(g^{\mu\nu}) = \text{diag}(1, -1)$, where $x^0 = t$ is time and $x^1 = x$ is spatial position, and we write $\partial_\mu = \partial / \partial x^\mu$, so that $\partial_0 = \partial_t$, $\partial_1 = \partial_x$ and $\Box = \partial_t^2 - \partial_x^2$. The $2 \times 2$ Dirac matrices $\gamma^\mu$ should satisfy

$$
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}I \quad (g^{00} = 1, g^{11} = -1, g^{01} = g^{10} = 0)
$$

and

$$
(\gamma^0)^* = \gamma^0, \quad (\gamma^1)^* = -\gamma^1.
$$

We choose the representation

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

The Maxwell-Dirac system describes the motion of an electron interacting with its self-induced electromagnetic field, and it is the fundamental PDE system in relativistic quantum electrodynamics.
A key fact about this system is that it enjoys a U(1) gauge freedom, and the particular form (1) appears when the Lorenz gauge condition \( \partial_\mu A_\mu = 0 \) is chosen, that is,
\[
\partial_t A_0 = \partial_x A_1.
\]
(3)
Since the latter reduces to a constraint on the initial data, we do not include it in (1). A second, less obvious constraint on the data, arising from (1b) and (3) combined, is the Gauss law
\[
\partial_x E = |\psi|^2,
\]
where
\[
E := \partial_x A_0 - \partial_t A_1
\]
is the electric field. If the constraints (3) and (4) are satisfied by the data at time \( t = 0 \), then they will also be satisfied at all later times, for a sufficiently regular solution of (1).

Another key feature of the Maxwell-Dirac system is the conservation of charge,
\[
\int_\mathbb{R} |\psi(x,t)|^2 \, dx = \int_\mathbb{R} |\psi(x,0)|^2 \, dx,
\]
(5)
for sufficiently regular solutions. For this reason, a solution for which the map \( t \mapsto \psi(t, \cdot) \) is continuous into \( L^2(\mathbb{R}) \) and satisfies (5), will be referred to as a charge class solution.

The final key property that we want to mention, is that in the massless case \( M = 0 \), the system (1) is invariant under the rescaling
\[
\psi(t, x) \rightarrow \lambda^{3/2} \psi(\lambda t, \lambda x), \quad A_\mu(t, x) \rightarrow \lambda A_\mu(\lambda t, \lambda x) \quad (\lambda > 0).
\]
By the usual heuristics, this provides some information about possible obstructions to well-posedness in a given data space \( X_0 \). Specifically, if we send \( \lambda \) to zero, then the existence time of the rescaled solution goes to infinity, and this is only reasonable if the \( X_0 \) norm of the rescaled data tends to zero, or at least stays bounded. A data space \( X_0 \) is called subcritical, critical or supercritical according to whether the norm of the rescaled data tends to zero, remains constant or tends to infinity, respectively, as \( \lambda \) tends to zero. In a supercritical data space \( X_0 \) one does not expect well-posedness to hold.

To see what this heuristic tells us in the case of the Maxwell-Dirac system, let us start with the \( L^2 \) based Sobolev spaces \( H^s(\mathbb{R}) \). For data
\[
(\psi_0, a_\mu, b_\mu) \in X_0 := H^s(\mathbb{R}) \times H^r(\mathbb{R}) \times H^{r-1}(\mathbb{R}),
\]
(6)
the critical regularity is seen to be \( s = -1 \) and \( r = -1/2 \) (for the homogeneous spaces), so based on scaling alone, one does not expect well-posedness if \( s < -1 \) or \( r < -1/2 \) (supercritical scaling). In fact, we shall see that there are far stronger restrictions on well-posedness than this, excluding the range \( s < 0 \). But before we get to this, let us mention some earlier results on well-posedness and ill-posedness of the Maxwell-Dirac system in one space dimension.

Chadam [2] proved local well-posedness of (1) in the space (6) with \( s = r = 1 \), and moreover using the conservation of charge he showed that the solution extends globally in time. Okamoto [3] proved local well-posedness for \( s > 0, \ r > 1/2, \ s \leq r \leq \min(s+1, 2s+1/2) \) and \( (s, r) \neq (1/2, 3/2) \), thus barely failing to reach the point \( (s, r) = (0, 1/2) \). Moreover, he proved that for \( s > 0 \), the data-to-solution map fails to be \( C^2 \) if \( r \) is outside the range specified above. In the massless case \( M = 0 \), Okamoto also proved that the data-to-solution map fails to be continuous.
at the point \((s, r) = (0, 1/2)\). This last result shows that, if one wants to prove well-posedness for \(s = 0\) (or below), the data for the electromagnetic potential \(A_\mu\) cannot be taken in the Sobolev spaces. A result in this direction was obtained by Hub \([3]\) in the massless case \(M = 0\): Using the interesting fact that the system can then be explicitly integrated, he proved global existence of \((1)\) in the case \(s = 0\) with \(a_\mu, b_\mu \in BC(\mathbb{R})\), where \(BC(\mathbb{R})\) denotes the space of bounded and continuous functions. This is however not a well-posedness result, since \(\partial_t A_\mu\) does not persist in the space \(BC(\mathbb{R})\). Global existence and uniqueness of weak solutions for \(s = 0\) with data \((a_\mu, b_\mu) \in L^\infty(\mathbb{R}) \times L^1(\mathbb{R})\) was obtained by You and Zhang \([11]\), without the restriction to zero mass. But this is also not a well-posedness result, since continuity of the solution map is not proved, and it is also not proved that \(\partial_t A_\mu\) persists in \(L^1(\mathbb{R})\).

Thus, no proper well-posedness result for the Cauchy problem \((1), (2)\) has been obtained previously in the charge class, that is for \(\psi_0 \in L^2(\mathbb{R})\) (see, however, Remark \([3]\) below). Here we prove such a result, with data for the potential \(A_\mu\) taken in the following space.

**Definition 1.** Let \(Y = Y(\mathbb{R})\) be the space with norm \(\|f\|_Y = \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^1(\mathbb{R})}\).

Thus, \(Y\) is the space of absolutely continuous functions \(f: \mathbb{R} \rightarrow \mathbb{C}\) with bounded variation (cf. Corollary 3.33 in \([3]\)), and \(Y_{\text{loc}}\) is the space of locally absolutely continuous functions.

Our first main result is then the following.

**Theorem 1.** The Cauchy problem \((1), (2)\) is globally well posed for initial data 
\[
(\psi_0, a_\mu, b_\mu) \in X_0 := L^2(\mathbb{R}) \times Y(\mathbb{R}) \times L^1(\mathbb{R}).
\]
That is, for any \(T > 0\), the problem has a unique solution \((\psi, A_\mu)\) on \((-T, T) \times \mathbb{R}\), satisfying
\[
(\psi, A_\mu, \partial_t A_\mu) \in C([-T, T]; X_0).
\]
Moreover, the data-to-solution map is continuous from \(X_0\) to \(C([-T, T]; X_0)\), and higher regularity persists. In particular, the solution is a limit in \(C([-T, T]; X_0)\) of smooth solutions.

**Remark 1.** The above data space has a subcritical scaling. In fact the scaling is the same as for the homogeneous version of \((1)\) with \((s, r) = (0, 1/2)\).

**Remark 2.** By persistence of higher regularity we mean that if, for some \(N \in \mathbb{N}\), we have \(\partial^j_t (\psi_0, a_\mu, b_\mu) \in X_0\) for \(j \in \{0, \ldots, N\}\), then it follows that \(\partial^j_t \partial^k_x (\psi, A_\mu, \partial_t A_\mu) \in C([-T, T]; X_0)\) for \(j, k \in \{0, \ldots, N\}\) with \(j + k \leq N\).

**Remark 3.** So far, we did not take into account the data constraints \((3)\) and \((4)\). Typically, these constraints are not compatible with the choice of data space for \(\partial_t A_\mu\). Indeed, Okamoto \([8]\) observed that in Chadam’s result \([2]\), the electric field \(E = \partial_x A_0 - \partial_t A_1\) would initially belong to \(L^2(\mathbb{R})\), but this is not compatible with \((1)\), which implies that \(E(0, x) = c + \int_0^x \psi_0(x)^2 \, dx\) is an increasing function in \(x\). A similar incompatibility occurs in our Theorem \([11]\) since \(E\) would belong to \(L^1(\mathbb{R})\) initially. However, these incompatibilities are easily resolved by using the finite speed of propagation and localising.
Remark 4. Another way of resolving the incompatibility issue discussed in the previous remark, is to use the constraints (3) and (4) directly in the statement of the Cauchy problem. Then in (2) one has the constraints

\[ b_0 = \frac{d}{dx}a_1, \quad b_1 = \frac{d}{dx}a_0 - E_0, \]

where the initial value \( E_0 \) of the electric field is required to satisfy the Gauss law (4). Then the initial data are \( (\psi_0, a_0, a_1, E_0) \). Global well-posedness of (1) with such data was proved by the first author in [10] with \( \psi_0 \in L^2(\mathbb{R}) \) and \( a_0, a_1, E_0 \in BC(\mathbb{R}) \).

Our next main result is that Theorem 1 is sharp. For this we take \( \psi_0(x) = \chi_{[-1,1]}(x) \frac{1}{|x|^{1/2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) (7) where \( \chi_{[-1,1]} \) is the characteristic function of the interval \([-1,1]\). Then \( \psi_0 \in L^p(\mathbb{R}) \) for \( 1 \leq p < 2 \), so by the dual of the Sobolev embedding \( H^s(\mathbb{R}) \subset L^q(\mathbb{R}) \) for \( 2 \leq q < \infty \) and \( r = 1/2 - 1/q \), it follows that also

\[ \psi_0 \in H^s(\mathbb{R}) \] for \( s < 0 \).

But clearly \( \psi_0 \) fails to belong to \( L^2(\mathbb{R}) \).

Theorem 2. The Cauchy problem (1), (2) is ill posed in

\[ (\psi_0, a_\mu, b_\mu) \in X_0 := H^s(\mathbb{R}) \times D_0 \times D_1 \] for \( s < 0 \),

and in

\[ (\psi_0, a_\mu, b_\mu) \in X_0 := L^p(\mathbb{R}) \times D_0 \times D_1 \] for \( 1 \leq p < 2 \),

regardless of the choice of spaces \( D_0, D_1 \subset D'(\mathbb{R}) \). In fact, with \( \psi_0 \) as in (7) and with \( a_\mu = b_\mu = 0 \) for \( \mu = 0, 1 \), the problem has no local solution near the origin in \( \mathbb{R}^{1+1} \) which is a distributional limit of charge solutions.

In the next two sections we give the proofs of well-posedness and ill-posedness, respectively. In fact, our proof of well-posedness applies to a fairly general class of systems which includes not only the Maxwell-Dirac system (MD) but also the Dirac-Klein-Gordon system (DKG) as special cases.

2. Global well-posedness in the charge class of generic systems of MD/DKG type

Here we prove Theorem 1. In fact, we prove it for a more general system of the form

\[ (-i\gamma^\mu \partial_\mu + M)\psi = \sum_{j=1}^N V_j \gamma^0 B_j \psi, \] (8a)

\[ (\Box + m^2)V_j = \psi^* C_j \psi, \] (8b)

with initial conditions

\[ \psi(0, x) = \psi_0(x), \quad V_j(0, x) = v_j(x), \quad \partial_t V_j(0, x) = w_j(x) \] (9)

and unknowns \( \psi: \mathbb{R}^{1+1} \to \mathbb{C}^2 \) and \( V = (V_1, \ldots, V_N): \mathbb{R}^{1+1} \to \mathbb{R}^N \). Here \( N \in \mathbb{N}, m, M \in \mathbb{R} \) are constants, and the \( B_j \) and \( C_j \) are constant \( 2 \times 2 \) hermitian matrices. The assumption \( C_j^* = C_j \) guarantees that \( V_j \) stays real valued given that its data
are real valued. From (8a) and $j^* j = j j$ it then follows that $j^\mu := \psi^* \gamma^0 \gamma^\mu \psi$ satisfies $\partial_\mu j^\mu = 0$, hence the conservation of charge holds.

We will prove the following result, which contains Theorem 1 as a special case.

**Theorem 3.** If $m = 0$, the Cauchy problem (8), (9) is globally well posed for initial data

$$(\psi_0, v, w) \in X_0 := L^2(\mathbb{R}; C^2) \times Y(\mathbb{R}; \mathbb{R}^N) \times L^1(\mathbb{R}; \mathbb{R}^N).$$

In general (that is, not assuming $m = 0$), the same result holds for data

$$(\psi_0, v, w) \in X_{0, \text{loc}} := L^2_{\text{loc}}(\mathbb{R}; C^2) \times Y_{\text{loc}}(\mathbb{R}; \mathbb{R}^N) \times L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^N).$$

**Remark 5.** For $m = 0$, the second statement in the theorem is a consequence of the first statement and finite speed of propagation.

**Remark 6.** Since we apply a contraction argument, we get well-posedness in the strong sense, including existence, uniqueness, and smooth dependence on the data. Moreover, higher regularity persists, so smooth initial data give a smooth solution.

**Remark 7.** By invariance of the system (8) under the reflection $(t, x, M, B_j) \rightarrow (-t, -x, -M, -B_j)$, it suffices to prove Theorem 3 for positive times.

**Remark 8.** The system (8) includes as special cases not only the Maxwell-Dirac system (1) but also the Dirac-Klein-Gordon system (DKG)

$$(-i \gamma^\mu \partial_\mu + M) \psi = \phi \psi,$$

$$(\Box + m^2) \phi = \overline{\psi} \psi,$$

for which Bourgain [1] proved global well-posedness in the charge class, improving the earlier $H^1$-result of Chadam [2]. The proof of Bourgain relies crucially on a null structure in the DKG system, whereas our proof of Theorem 3 does not require any such structure (of course, the two results are not quite identical, since the choice of data spaces for $\phi$ and $\partial_t \phi$ differs). On the other hand, the null structure in DKG is certainly necessary if one wants to go below the charge, and in fact it is possible to go down to $\psi_0 \in H^s$ for $s > -1/2$, but not further; see [5, 7, 6].

The remainder of this section is devoted to the proof of Theorem 3. For convenience we rewrite the system in terms of the Dirac matrices $\alpha = \gamma^0 \gamma^1$ and $\beta = \gamma^0$:

$$(-i \partial_t - i \alpha \partial_x + M \beta) \psi = \sum V_j B_j \psi,$$

$$(\partial_t^2 - \partial_x^2 + m^2) V_j = \psi^* C_j \psi.$$  

(10a)  

(10b)

2.1. **Preliminaries.** In preparation for the proof we recall some pertinent facts.

2.1.1. **Estimates for the Klein-Gordon and wave equations.** For

$$(\partial_t^2 - \partial_x^2 + m^2) u = F, \quad (u, \partial_t u)|_{t=0} = (f, g),$$
we recall the solution formula (see [9] Section 4.1.3])

$$u(t, x) = \frac{f(x + t) + f(x - t)}{2} - m^2 t J_1 \left( \frac{m \sqrt{t^2 - (x - y)^2}}{m \sqrt{t^2 - (x - y)^2}} \right) f(y) dy$$

$$+ \frac{1}{2} \int_{x-t}^{x+t} J_0 \left( \frac{m \sqrt{t^2 - (x - y)^2}}{m \sqrt{t^2 - (x - y)^2}} \right) g(y) dy$$

$$+ \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} J_0 \left( \frac{m \sqrt{(t-s)^2 - (x - y)^2}}{m \sqrt{(t-s)^2 - (x - y)^2}} \right) F(s, y) dy ds,$$

where $J_0(x) = \sum_{n=0}^{\infty} (-1)^n \left( \frac{x}{2} \right)^{2n}$ and $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n(n+1)!} \left( \frac{x}{2} \right)^{2n+1}$ are the Bessel functions of the first kind. It is well known that $J_0(x)$ and $J_1(x)$ are $O(1)$ (in fact they are $O(1/\sqrt{x})$) as $x \to \infty$, hence $J_0(x), x^{-1}J_1(x) \leq C < \infty$ for all $x \geq 0$. Thus, for $t > 0$,

$$\|u(t)\|_{L^\infty} \leq C(1 + m^2 t^2) \|f\|_{L^\infty} + C \|g\|_{L^1} + C \int_0^t \|F(s)\|_{L^1} ds. \quad (11)$$

For $m = 0$ one recovers D’Alembert’s formula for the wave equation,

$$u(t, x) = \frac{f(x + t) + f(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy + \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+t-s} F(s, y) dy ds,$$

and (11) holds with $C = 1$. Moreover, differentiating one obtains

$$\|\partial_x u(t)\|_{L^1}, \|\partial_t u(t)\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1} + \int_0^t \|F(s)\|_{L^1} ds$$

for $t > 0$, hence also

$$\|u(t)\|_Y + \|\partial_x u(t)\|_{L^1} \leq 3 \left( \|f\|_Y + \|g\|_{L^1} + \int_0^t \|F(s)\|_{L^1} ds \right). \quad (12)$$

2.1.2. Energy inequality for the Dirac equation. Consider the Dirac equation

$$(-i\partial_t - i\alpha \partial_x + M\beta)\psi = F, \quad \psi|_{t=0} = f.$$

Applying $i\partial_t - i\alpha \partial_x + M\beta$ to both sides and using $\alpha\beta + \beta\alpha = 0$, $\alpha^2 = I$ and $\beta^2 = I$, one obtains

$$(\partial_t^2 - \partial_x^2 + M^2)\psi = G, \quad (\psi, \partial_t\psi)|_{t=0} = (f, g),$$

where $G = (i\partial_t - i\alpha \partial_x + M\beta)F$ and $g = iF(0) - \alpha F' - iM\beta f$. Thus, the Klein-Gordon solution formula from the previous subsection applies, so assuming for the moment that $f$ and $F$ are smooth and compactly supported, it follows that $\psi$ is smooth and that $\psi(t)$ is compactly supported for each $t$. Now premultiply the Dirac equation by $i\psi^*$, take real parts, and use $\alpha^* = \alpha$ and $\beta^* = \beta$, to get $\partial_t \rho + \partial_x j = 2 \text{Re}(i\psi^* F)$, where $\rho = \psi^* \psi$ and $j = \psi^* \alpha \psi$. Integration in $x$ gives

$$\frac{d}{dt} \int \psi^* \psi \, dx = 2 \text{Re} \int i\psi^* F \, dx \leq 2 \|\psi(t)\|_{L^2} \|F(t)\|_{L^2},$$

implying the energy inequality,

$$\|\psi(t)\|_{L^2} \leq \|f\|_{L^2} + \int_0^t \|F(s)\|_{L^2} \, ds. \quad (13)$$
By a density argument, the smoothness and support assumptions on \( f \) and \( F \) can now be removed, so that the inequality is valid for any \( f \in L^2 \) and \( F \in L^1_t L^2_x \), in which case \( \psi \in C(\mathbb{R}; L^2) \).

2.2. **Proof of Theorem 3**  
Solving for the potentials, we first prove local well-posedness for the non-linear and non-local Dirac equation thus obtained, with a time of existence depending on the \( X_0 \) norm of the data \((\psi_0, v, w)\). To obtain local well-posedness of the full system (10) we then show that \((v, \partial_t v)\) persists in \( Y \times L^1 \) (or its local version if \( m \neq 0 \)). Moreover the \( Y \times L^1 \) norm is a priori bounded on any finite time interval, and this together with the conservation of charge implies that the local result extends globally.

2.2.1. **Step 1: Local well-posedness for a non-linear and non-local Dirac equation.**  
Fix the data \((\psi_0, v, w) \in X_0\). Solving for the \( V_j \) in (11), we obtain

\[
(-i\partial_t - i\alpha \partial_x + M\beta)\psi = \sum \mathfrak{V}_j[\psi] B_j \psi, \quad \psi(0, x) = \psi_0(x),
\]

where the operators \( \psi \to \mathfrak{V}_j[\psi] \) are given by

\[
\mathfrak{V}_j[\psi](t, x) = \frac{v_j(x + t) + v_j(x - t)}{2} - \frac{m^2 t}{2} \int_{x-t}^{x+t} J_1 \frac{m \sqrt{t^2 - (x-y)^2}}{m \sqrt{t^2 - (x-y)^2}} v_j(y) dy
\]

\[
+ \frac{1}{2} \int_{x-t}^{x+t} J_0 \frac{m \sqrt{t^2 - (x-y)^2}}{m \sqrt{t^2 - (x-y)^2}} w_j(y) dy
\]

\[
+ \frac{1}{2} \int_0^{t} \int_{x-(s-t)}^{x+(s-t)} J_0 \left( m \sqrt{(t-s)^2 - (x-y)^2} \right) (\psi^* C_j \psi)(s, y) dy ds.
\]

From (11) we see that for any \( T > 0 \),

\[
\| \mathfrak{V}[\psi] \|_{C_T L^\infty} \leq C(1 + m^2 T^2) \| \psi \|_{L^\infty} + C \| w \|_{L^1} + C T \| \psi \|^2_{C_T L^2},
\]

\[
\| \mathfrak{V}[\psi] - \mathfrak{V}[\psi'] \|_{C_T L^\infty} \leq C T \left( \| \psi \|_{C_T L^2} + \| \psi' \|_{C_T L^2} \right) \| \psi - \psi' \|_{C_T L^2},
\]

where \( C_T L^p = C([0, T]; L^p(\mathbb{R})) \) with norm \( \| u \|_{C_T L^p} = \sup_{t \in [0, T]} \| u(t) \|_{L^p} \). From these estimates and the energy inequality (13), we now see that for a pair of equations in iterative form,

\[
(-i\partial_t - i\alpha \partial_x + M\beta)\psi = \sum \mathfrak{V}_j[\psi'] B_j \psi', \quad \psi(0, x) = \psi_0(x),
\]

\[
(-i\partial_t - i\alpha \partial_x + M\beta)\Psi = \sum \mathfrak{V}_j[\Psi'] B_j \Psi', \quad \Psi(0, x) = \psi_0(x),
\]

where \( \psi', \Psi' \in C_T L^2 \) (the previous iterates) are given, we get the estimates:

\[
\| \psi \|_{C_T L^2} \leq \| \psi_0 \|_{L^2} + T \| \mathfrak{V}[\psi'] \|_{C_T L^\infty} \| \psi' \|_{C_T L^2}
\]

\[
\leq \| \psi_0 \|_{L^2} + C T (1 + m^2 T^2) \| (v, w) \|_{L^\infty \times L^1} \| \psi' \|_{C_T L^2} + C T^2 \| \psi' \|^3_{C_T L^2}
\]

and

\[
\| \psi - \Psi \|_{C_T L^2} \leq CT \| \mathfrak{V}[\psi'] - \mathfrak{V}[\Psi'] \|_{C_T L^\infty} \| \psi' \|_{C_T L^2}
\]

\[
+ CT \| \mathfrak{V}[\Psi'] \|_{C_T L^\infty} \| \psi - \Psi' \|_{C_T L^2}
\]

\[
\leq CT^2 (\| \psi' \|_{C_T L^2} + \| \Psi' \|_{C_T L^2})^2 \| \psi' - \Psi' \|_{C_T L^2}
\]

\[
+ CT (1 + m^2 T^2) \| (v, w) \|_{L^\infty \times L^1} \| \psi' - \Psi' \|_{C_T L^2},
\]

where \( C \) changes from line to line and depends also on the matrices \( B_j \). It now follows by a standard iteration argument that we have local well-posedness for
and for any $R > 0$ we have a time of existence $T = T(R) > 0$ for data with $\|\psi_0\|_{L^2} + \|v\|_{L^\infty} + \|w\|_{L^2} \leq R$. Moreover, conservation of charge holds, since this is true for smooth solutions with compactly supported data, and the solutions we obtain are limits in $C_T L^2$ of such solutions.

2.2.2. Step 2: Persistence of $m$ non-existence. More precisely, approximating with data by finite speed of propagation we may assume that the data $(\psi, v, w) \in X_0$ are compactly supported, say in the interval $[-a, a]$. Then $(\psi, V, \partial_t V)(t)$ is supported in $[-a - t, a + t]$ for $t > 0$. Temporarily writing the equation (12) for $V_j$ as

$$(\partial_t^2 - \partial_x^2)V_j = -m^2 V_j + \psi^* C_j \psi,$$

we apply (12) and obtain

$$\|V(t)\|_Y + \|\partial_t V(t)\|_{L^1} = O(1 + t) + C m^2 t (a + t) \|V\|_{L^\infty ([0, t] \times \mathbb{R})}.$$ 

To control the last term we note that (11) implies $\|V(t)\|_{L^\infty} = O(1 + t^2)$, hence $\|V(t)\|_Y + \|\partial_t V(t)\|_{L^1} = O(1 + t^4)$. This concludes the proof of Theorem 3.

3. Ill-posedness of Maxwell-Dirac below charge

In terms of the components of $\psi = (u, v)^T$ and setting $A_+ = A_0 + A_1$ and $A_- = A_0 - A_1$, the system (11) becomes

$$(\partial_t + \partial_x) u = -iM v + iA_+ u, \quad (15a)$$

$$(\partial_t - \partial_x) v = -iM u + iA_- v, \quad (15b)$$

$$(\partial_t^2 - \partial_x^2) A_+ = -2|v|^2, \quad (15c)$$

$$(\partial_t^2 - \partial_x^2) A_- = -2|u|^2. \quad (15d)$$

We take initial data

$$u(0, x) = f(x), \quad v(0, x) = g(x), \quad A_\pm (0, x) = \partial_t A_\pm (0, x) = 0. \quad (16)$$

Then with

$$f(x) = g(x) = \chi_{[-1, 1]}(x) \frac{1}{|x|^{1/2}}, \quad (17)$$

which belongs to $L^p(\mathbb{R})$, $1 \leq p < 2$, and to $H^s(\mathbb{R})$, $s < 0$, we show ill-posedness by non-existence. More precisely, approximating with data

$$f_\varepsilon(x) = g_\varepsilon(x) = \chi_{[-1, 1]}(x) \frac{1}{(\varepsilon + |x|)^{1/2}} \quad \text{for } \varepsilon > 0, \quad (18)$$

and denoting by $(u_\varepsilon, v_\varepsilon, A_+, \varepsilon, A_-, \varepsilon)$ the corresponding charge solution (which exists globally by Theorem 1), we show that $A_+ \varepsilon$ fails to have a limit in the sense of distributions as $\varepsilon \to 0$, in the region $t > |x|$.

By the finite speed of propagation we may remove the characteristic function $\chi_{[-1, 1]}(x)$ in the above data. Indeed, this does not affect the solution in the region $|x| + |t| \leq 1$, and it suffices to prove the non-convergence in this region.
We first prove the massless case, \( M = 0 \). Then the system can be explicitly integrated, as observed in [4]. The general case will then be handled by comparing the massive solution with the massless one.

3.1. The massless case. Taking \( M = 0 \), the system (15), (16) is easily integrated. First, integrating (15a) and (15b) along characteristics gives

\[
\begin{align*}
u(t, x) &= f(x - t) e^{i\phi_+(t, x)}, \\
v(t, x) &= g(x + t) e^{i\phi_-(t, x)},
\end{align*}
\]

where

\[
\begin{align*}
\phi_+(t, x) &= \int_0^t A_+(\sigma, x - t + \sigma) \, d\sigma, \\
\phi_-(t, x) &= \int_0^t A_-(\sigma, x + t - \sigma) \, d\sigma
\end{align*}
\]

are real valued. Then, since \( |u(t, x)|^2 = |f(x - t)|^2 \) and \( |v(t, x)|^2 = |g(x + t)|^2 \), we can integrate (15c) and (15d) to get

\[
\begin{align*}
A_+(t, x) &= -\int_0^t \int_{x-(t-s)}^{x+t-s} |g(y + s)|^2 \, dy \, ds, \\
A_-(t, x) &= -\int_0^t \int_{x-(t-s)}^{x+t-s} |f(y - s)|^2 \, dy \, ds.
\end{align*}
\]

These formal computations are valid for well-posed solutions, in particular for the charge solutions \( (u_\varepsilon, v_\varepsilon, A_+\varepsilon, A_-\varepsilon) \) with data as in (13) (with the characteristic function removed, as remarked above), and one can now easily compute the complete solution. For our purposes, however, the following lower bound suffices: In the region \( t > |x| \),

\[
-A_-\varepsilon(t, x) = \int_0^t \int_{x-(t-s)}^{x+t-s} \frac{1}{\varepsilon + |y - s|} \, dy \, ds
\]

\[
\geq \int_0^t \int_{s}^{x+t-s} \frac{1}{\varepsilon + y - s} \, dy \, ds
\]

\[
= \frac{x + t}{2} (-\log \varepsilon + \frac{1}{2} (\varepsilon + x + t) \log(\varepsilon + x + t) - 1) - \frac{1}{2} \varepsilon (\log \varepsilon - 1).
\]

Now fix a non-negative test function \( \theta \in C_\infty^\infty(\mathbb{R}^2) \) supported in the region \( t > |x| \). Then it follows that

\[
-\int A_-\varepsilon(t, x) \theta(t, x) \, dt \, dx \geq (-\log \varepsilon) \int t + x \ \theta(t, x) \, dt \, dx + R_\varepsilon,
\]

where \( R_\varepsilon \) converges, by the dominated convergence theorem, to

\[
R = \int \left( \frac{1}{2} (x + t) |\log(x + t) - 1| \right) \theta(t, x) \, dt \, dx,
\]

as \( \varepsilon \to 0 \). We conclude that \( A_-\varepsilon \) cannot converge in the sense of distributions on the region \( t > |x| \), and this proves Theorem 2 in the case \( M = 0 \).
3.2. The massive case. In the case $M \in \mathbb{R}$, $M \neq 0$, it suffices to show the lower bound, uniformly in $\varepsilon > 0$,
\[ |u_\varepsilon(t,x)|^2 \geq \frac{1}{2} |f_\varepsilon(x-t)|^2 \quad \text{for } 0 < t \ll 1 \text{ and } t < x < 1 - t, \]
(19)
since then for $|x| < t \ll 1$ we obtain
\[
-A_{-,\varepsilon}(t,x) = \int_0^t \int_{x-(t-s)}^{x+t-s} |u_\varepsilon(s,y)|^2 \, dy \, ds
\geq \int_0^{x+t-s} |u_\varepsilon(s,y)|^2 \, dy \, ds
\geq \frac{1}{2} \int_0^{x+t-s} \frac{1}{\varepsilon + y - s} \, dy \, ds,
\]
hence the argument from the case $M = 0$ goes through and proves Theorem 2.

So it only remains to prove (19). To this end, observe that (15a) and (15b) imply
\[
\begin{align*}
(\partial_t + \partial_x)|u|^2 &= -2M \operatorname{Im}(u\overline{v}), \\
(\partial_t - \partial_x)|v|^2 &= 2M \operatorname{Im}(u\overline{v}),
\end{align*}
\]
which integrates to
\[
|u(t,x)|^2 = |f(x-t)|^2 - 2M \int_0^t \operatorname{Im}(u\overline{v})(\sigma,x-t+\sigma) \, d\sigma,
\]
(20)
\[
|v(t,x)|^2 = |g(x+t)|^2 + 2M \int_0^t \operatorname{Im}(u\overline{v})(\sigma,x+t-\sigma) \, d\sigma.
\]
(21)

Now fix $\varepsilon > 0$ and define, for $\rho > 0$,
\[
B_{\rho}(t) = \sup_{t+\rho \leq x \leq 1-t} \left( |u_\varepsilon(t,x)|^2 + |v_\varepsilon(t,x)|^2 \right).
\]
Note that this quantity is finite, since the solution is smooth in the region $x > t > 0$.

Applying (20) and (21) we then find
\[
B_{\rho}(t) \leq \frac{2}{\varepsilon + \rho} + 2|M| \int_0^t B_{\rho}(\sigma) \, d\sigma
\]
so by Grönwall’s inequality,
\[
B_{\rho}(t) \leq \frac{2}{\varepsilon + \rho} e^{2|M|t} \leq \frac{4}{\varepsilon + \rho}
\]
if $0 < t < (2|M|)^{-1}$, which we assume from now on.

Applying (20) again we now conclude that, for $t + \rho \leq x \leq 1 - t$,
\[
|u_\varepsilon(t,x)|^2 \geq \frac{1}{\varepsilon + x - t} - |M| \int_0^t B_{\rho}(\sigma) \, d\sigma
\geq \frac{1}{\varepsilon + x - t} - |M|t \frac{4}{\varepsilon + \rho}.
\]
Choosing $\rho = x - t$ we obtain
\[
|u_\varepsilon(t,x)|^2 \geq \frac{1/2}{\varepsilon + x - t} \quad \text{for } 0 < t < \frac{1}{8|M|} \text{ and } t < x < 1 - t,
\]
proving (19).
ILL-POSEDNESS FOR 1D MAXWELL-DIRAC

REFERENCES

1. N. Bournaveas, *A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension*, J. Funct. Anal. 173 (2000), no. 1, 203–213. MR 1760283 (2001c:35128)

2. John M. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension*, J. Functional Analysis 13 (1973), 173–184. MR 0368640 (51 #4881)

3. Gerald B. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication. MR 1681462

4. Hyungjin Huh, *Global charge solutions of Maxwell-Dirac equations in \( \mathbb{R}^{1+1} \)*, J. Phys. A 43 (2010), no. 44, 445206, 7. MR 2733825

5. Shuji Machihara, Kenji Nakanishi, and Kotaro Tsugawa, *Well-posedness for nonlinear Dirac equations in one dimension*, Kyoto J. Math. 50 (2010), no. 2, 403–451. MR 2666663

6. Shuji Machihara and Mamoru Okamoto, *Sharp ill-posedness of the Dirac-Klein-Gordon system in one space dimension*, arXiv:1808.07642.

7. ____, *Remarks on ill-posedness for the Dirac-Klein-Gordon system*, Dyn. Partial Differ. Equ. 13 (2016), no. 3, 179–190. MR 3522179

8. Mamoru Okamoto, *Well-posedness and ill-posedness of the Cauchy problem for the Maxwell-Dirac system in \( 1 + 1 \) space time dimensions*, Adv. Differential Equations 18 (2013), no. 1-2, 179–199. MR 3052714

9. Andrei D. Polyanin, *Handbook of linear partial differential equations for engineers and scientists*, Chapman & Hall/CRC, Boca Raton, FL, 2002. MR 1935578 (2003i:35001)

10. Sigmund Selberg, *Global existence in the critical space for the Thirring and Gross-Neveu models coupled with the electromagnetic field*, Discrete Contin. Dyn. Syst. 38 (2018), no. 5, 2555–2569. MR 3800049

11. Aiguo You and Yongqian Zhang, *Global solution to Maxwell-Dirac equations in \( 1 + 1 \) dimensions*, Nonlinear Anal. 98 (2014), 226–236. MR 3158454

Department of Mathematics, University of Bergen, PO Box 7803, 5020 Bergen, Norway

E-mail address: Sigmund.Selberg@uib.no
E-mail address: Achenef.Temesgen@uib.no