Delayed blow-up of nonlinear time fractional stochastic differential equations

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Abstract

In this paper, we study the nonlinear Caputo time fractional stochastic partial differential equations driven by Brownian motion of the form

\[ D_t^\beta u = \left[ -(-\Delta)^s u + v\Delta u + F(u) \right] dt + \sqrt{C_d} \sum_{k,i} \theta_{k,i} \sigma_{k,i} \nabla u(t) dW_{t}^{k,i}, \]

where \( s \geq 1, \frac{1}{2} < \beta < 1 \). The existence and uniqueness of the solutions are proved by using Galerkin approximations and priori estimates. We make hypotheses on the nonlinear term \( F \) and state that noise can postpone the life span of solutions to a stochastic nonlinear system when \( F \) satisfies these hypotheses. The hypotheses are verified to hold in the time fractional Keller-Segel and time fractional Fisher-KPP equations in 3D case.

Keywords: Caputo fractional integral and derivation, Blow-up solution, Time fractional Keller-Segel equation, Time fractional Fisher-KPP equation, Nonlinear fractional stochastic partial differential equation

1. Introduction

In this paper, we consider the blow-up problem for the nonlinear time fractional stochastic partial differential equation

\[
\begin{aligned}
D_t^\beta u &= \left[ -(-\Delta)^s u + v\Delta u + F(u) \right] dt + \sqrt{C_d} \sum_{k,i} \theta_{k,i} \sigma_{k,i} \nabla u(t) dW_{t}^{k,i}, \\
\quad & \quad u(x,0) = u_0(x), \quad u_0 \in L^2(\mathbb{T}^d),
\end{aligned}
\]

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on the tours $T^d = \mathbb{R}^d / \mathbb{Z}^d$, where $D_t^\beta$ is the left sided Caputo fractional derivative of order $\beta \in \left(\frac{1}{2}, 1\right)$, $d \geq 2$, and $s \geq 1$ is fixed, $F$ is a nonlinear term that satisfies specific assumptions. The meaning of the parameters in the equation is detailed in Section 3.

The phenomenon that the solutions of ordinary or partial differential equations diverge in finite time is known as the blow up of solutions [1], which exists in various fields, such as chemotaxis in biology [2], curvature flow in geometry [3], and fluid mechanics [4], etc. Compared with integer-order models, scholars have stated that fractional-order models can preserve the genetic and memory properties of functions in practical problems, and the physical meaning of parameters in fractional models is also more explicit. Consequently, motivated by practical applications in various fields such as statistical mechanics, electrical engineering, physics, and control theory, fractional models have been significantly developed in the past decades.

Recently, there are numerous research results for the solution of the blow-up phenomena, the conditions for the blow-up, the blow-up moment, the blow-up rate and the set of blow-up solutions have been studied widely. The sufficient conditions for the blow-up solutions of semi-discrete partial differential equations are given in [5, 6]. Later on, they proposed an upper bound for the blow-up time of solutions of fully discrete partial differential equations [7]. The relationship between the blow-up set of semi-discrete equations and the continuous equations was discussed in [8], moreover, they pointed out that the blow-up set of semi-discrete equations converges to the blast set of continuous equations when the spatial grid parameters are sufficiently small.

In the case $F = -u (1 - u)$, the Eqs. (1.1) is known as the time fractional Fisher-KPP reaction-diffusion equation, which has been studied by Alsaeedi and al. in [9]

$$\begin{align*}
D_t^\beta u &= -(-\Delta)^s u - u (1 - u), x \in \Omega, t > 0, \\
u (x, t) &= 0, x \in \mathbb{R}^N \setminus \Omega, t > 0, \\
u (x, 0) &= u_0 (x), x \in \Omega,
\end{align*}
$$

with $s \in (0, 1]$. They showed the initial conditions that make the system experience the blow-up in a finite time analyzed the asymptotic behavior of bounded solutions.

Li and Li in [10] considered the blow-up and global existence of the solution to a semilinear time-space Caputo–Hadamard fractional diffusion equation with fractional Laplacian

$$\begin{align*}
CHD_t^\beta u (x, t) + (-\Delta)^s u (x, t) &= |u (x, t)|^{p-1} u (x, t), x \in \mathbb{R}^d, t > a > 0, \\
u (x, a) &= u_a (x), x \in \mathbb{R}^d,
\end{align*}
$$

where $\beta \in (0, 1)$, $s \in (0, 1)$, $p > 1$, $d \in \mathbb{N}$. They studied the local existence and uniqueness of the mild solution by applying contraction mapping principle, and shown that the mild solution is a weak solution.
Zhang and Sun in [11] studied

\[
\begin{align*}
\begin{cases}
D^\beta_t u(x, t) - \Delta u(x, t) &= |u(x, t)|^{p-1} u(x, t), \; x \in \mathbb{R}^N, \; t > a > 0 \\
|u|_{t=0} = u_0, \; u_0 \in C_0(\mathbb{R}^N),
\end{cases}
\end{align*}
\]

(1.4)

where \(0 < \beta < 1, p > 1\), i.e., taking \(s = 1, F = |u|^{p-1} u\) in Eqs.(1.1). They proved the following results:

(i) For any \(u_0 \in C_0(\mathbb{R}^N)\), \(u_0 \geq 0, \; u_0 \not\equiv 0\), if \(1 < p < 1 + \frac{2}{N}\), then the solution of (1.4) blows up in finite time.

(ii) For any \(u_0 \in C_0(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)\), \(q_c = \frac{N(p-1)}{2}\), if \(p \geq 1 + \frac{2}{N}\) and \(\|u_0\|_{L^{q_c}(\mathbb{R}^N)}\) is sufficiently small then (1.4) has a global solution.

Scholars believe that the explosion of the solution may be affected by some deterministic perturbations or stochastic perturbations, and examples of deterministic perturbations affecting the blow-up phenomenon can be found in [12]. In practical problems, complex systems often have a large amount of noise perturbations, so stochastic models have attracted the interest of many scholars. The properties of these systems, such as well-posedness, invariant measure, stability and invariant manifold, have been discussed.(see ref [13–17]). In contrast to deterministic fractional differential equations, which have a wealth of results, there are relatively few theories for fractional stochastic differential equations in the Caputo sense, and most of the existing studies focus on discussing the well-posedness and asymptotics of the solutions.

Doan and al. in [18] established a result on the global existence and uniqueness of solutions for Caputo fractional stochastic differential equation of order \(\beta \in \left(\frac{1}{2}, 1\right)\) of the following form

\[
\begin{align*}
\begin{cases}
D^\beta_t u_t = f(t, u_t) + g(t, u_t) \frac{dW_t}{dt}, \quad t \geq 0, \\
\quad u_0 = u(0), \; u_0 \in L^2(\Omega, H),
\end{cases}
\end{align*}
\]

(1.5)

where \(f, g : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d\) are measurable and \((W_t)_{t \in [0, \infty)}\) is a standard scalar Brownian motion on an underlying complete filtered probability space \((\Omega, F, F := \{F_t\}_{t \in [0, \infty)}, \mathbb{P})\). They showed that the asymptotic distance between two distinct solutions is greater than \(t^{-\frac{1}{2\alpha}} - \varepsilon\) as \(t \to \infty\) for any \(\varepsilon > 0\). As a consequence, the mean square Lyapunov exponent of an arbitrary non-trivial solution of a bounded linear Caputo fractional stochastic differential equation is always non-negative. Later on, Wang and al. studied the Eqs.(1.5) of order \(\beta \in \left(\frac{1}{2}, 1\right)\) in [19][41]. They established the well-posedness for the Eqs.(1.5), and obtained the global existence and uniqueness of solution under some conditions consistent with the integral order stochastic differential equations. Finally, they discussed the continuity of solutions on the fractional order of those equations.

In [20], Zhang and al. considered a nonlinear variable-order fractional SDE

\[
\begin{align*}
\begin{cases}
\frac{du}{dt} = -\lambda(t) D_+^\alpha(t) (t) u + f(t, u) \; dt + \sigma(t, u) \; dW, \quad t \in [0, T], \\
u(0) = u_0,
\end{cases}
\end{align*}
\]

(1.6)
where $D_t^{\alpha(t)}$ denotes the variable-order fractional derivative in the Riemann-Liouville sense, and it holds

$$D_t^{\alpha(t)} g := \left. \frac{1}{\Gamma(1 - \alpha(t))} \frac{d}{d\xi} \int_0^\xi g(s)(\xi - s)^{-\alpha(t)} ds \right|_{\xi=t} = D_t^{\alpha(t)} g \frac{g(0) t^{-\alpha(t)}}{\Gamma(1 - \alpha(t))}.$$  

They shown the well-posedness of a variable-order fractional stochastic differential equation driven by a multiplicative white noise, which models random phenomena with memory effects, and the regularity of its solutions.

For a class of nonlinear time fractional stochastic partial differential equations

$$D_t^\beta u_t (x) = -v (-\Delta)^{s/2} u_t (x) + I_t^{1-\beta} \left[ \lambda \sigma (u) \dot{W} (t, x) \right],$$  

(1.7)

where $v > 0, \beta \in (0, 1), s \in (0, 2], d < \min \{2, \beta^{-1}\}, I_t^{1-\beta}$ is the Riesz fractional integral operator, $\dot{W} (t, x)$ is a space-time white noise, and $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. In [21], Mijena and Erkan studied the Eq.(1.7) in a $d + 1$ dimensions for $\lambda = 1$, and proved the following conclusions:

(i) absolute moments of the solutions of this equation grows exponentially;

(ii) the distances to the origin of the farthest high peaks of those moments grow exactly linearly with time.

Later on, as a further extension of the results in [21], Foondun and al. studied the asymptotic behavior of the solution with respect to time and the parameter $\lambda$ in [22], and given a precise rate with respect to the parameter $\lambda$.

Motivated by the above results, in this paper we consider the nonlinear time fractional stochastic partial differential system (1.1), and use Galerkin approximations and priori estimates to prove the existence and uniqueness of the solutions. Firstly, we consider the deterministic nonlinear fractional partial differential equation of the form

$$D_t^\beta u = \left[ - (-\Delta)^s u + F (u) \right] dt,$$  

(1.8)

with $s \geq 1$, and establish the assumptions for the nonlinear $F$. Then we claim that the noise can effectively postpone the explosion of the solution to problem (1.1), with the nonlinear $F$ satisfying the following conditions:

(H1) There exists $a_1 \geq 0$ and $\gamma_1 \in (0, s)$ such that $F : H^{s-\gamma_1} \to H^{-s}$ is a continuous mapping and holds

$$\|F(u)\|_{H^{-s}} \lesssim (1 + \|u\|_{L^2}^{\alpha_1}) (1 + \|u\|_{H^s});$$

(H2) There exists $a_2 \geq 0$ and $\gamma_2 \in (0, 2)$ such that

$$|\langle F(u), u \rangle| \lesssim (1 + \|u\|_{L^2}^{\alpha_2}) (1 + \|u\|_{H^s}^{\gamma_2});$$

(H3) There exists $a_3 \geq 0, \gamma_3 \in (0, 2)$ and $\eta \geq 0$ such that $a_3 + \gamma_3 \geq 2, \gamma_3 + \eta \leq 2$, and

$$|\langle u - v, F(u) - F(v) \rangle| \lesssim \|u - v\|_{L^2}^{\alpha_3} \|u - v\|_{H^s}^{\gamma_3} (1 + \|u\|_{H^s}^\eta + \|v\|_{H^s}^\eta);$$

4
(H4) There exists a set $\mathcal{M} \subset L^2(T^d)$, which is composed of bounded, closed and convex function, it holds that for any $u_0 \in \mathcal{M}$ and $T > 0$, we can find $v > 0$ big enough such that the Eqs. (1.9)

$$\begin{cases}
D^\alpha_t u = -(-\Delta)^\alpha u + v\Delta u + F(u), \\
 u|_{t=0} = u_0,
\end{cases}$$

(1.9)

has a global solution $u \in L^2(0, T; \mathcal{H}_a) \cap C([0, T]; L^2)$, it holds

$$\sup_{u_0 \in \mathcal{M}} \sup_{t \in [0, T]} \|u(t; u_0, v)\|_{L^2} < \infty,$$

(1.10)

where $u(\cdot; u_0, v)$ stands for the unique solution for (1.9) with initial data $u_0$.

The paper is organized as follows. In Section 2, we will first state some definitions relative to (1.1). In Section 3, we will introduce our model, hypotheses and main results. In Section 4, we prove that the assumptions in Section 2 hold for two examples. In Section 5, we will present the proofs for our main results based on some lemmas.

### 2. Preliminaries and notations

In this section, let us recall some necessary definitions of fractional order operators, present some symbols and auxiliary lemmas used later.

#### 2.1. Preliminaries

**Definition 2.1.** [23] The Fourier transform $\hat{f}$ of distribution $f$ on $\Omega$ is defined as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\Omega} f(x) e^{-i\xi \cdot x} dx.$$ 

**Definition 2.2.** [23] Sobolev spaces $H^a(T^d), a \in \mathbb{R}$, is given by

$$H^a(T^d) = \left\{ f = \sum_k f_k e_k |f_{-k} = \hat{f}_k, \sum_k \left(1 + |k|^2\right)^{a} |f_k|^2 < \infty \right\},$$

where $f \in L^2(T^d, \mathbb{C}), \{e_k\}_{k \in \mathbb{Z}^d}$ given by $e_k = e^{ik \cdot x}$ is a complete orthonormal system. For any $f$ on $\Omega$, $a \in \mathbb{R}$, set $\Lambda = (-\Delta)^{\alpha}$, the form of Sobolev space is defined as

$$\|f\|_{H^a} = \|\Lambda^a f\|_{L^2} = \left(\int_{\Omega} |\xi|^{2a} |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

(2.1)

**Definition 2.3.** [23] $L^p(\Omega)$ denotes the $p$th-power integral space, and its form is defined as

$$\|f\|_{L^p} = \left(\int_{\Omega} |f(x)|^p dx\right)^{\frac{1}{p}}, \quad \|f\|_{L^\infty} = \text{ess sup}_{x \in \Omega} |f(x)|.$$

For $1 \leq p \leq \infty, a \in \mathbb{R}$, the space $H^{\alpha,p}(\Omega)$ is a subspace of $L^p(\Omega)$.
Definition 2.4. [24] A function \( u(t) \) \((t > 0)\) is said to be in the space \( C_{\kappa,k} \subseteq \mathbb{R} \) if there exists a real number \( k \) \((k > \kappa)\), such that \( u(t) = t^k u_1(t) \), where \( u_1(t) \in C(0,\infty) \), and it is said to be in the space \( C^m_{\kappa} \) if and only if \( f^{(m)} \) \( \in C_{\kappa} \), \( m \in \mathbb{N} \).

Definition 2.5. [24] The left sided Riemann-Liouville fractional integral operator \( I_t^\beta \) of order \( \beta > 0 \), of a function \( u \in C_{\kappa}, \kappa \geq -1 \) is defined as

\[
I_t^\beta u(t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} u(\tau) \, d\tau, & n-1 < \beta < n, t > 0, \\
\frac{t^\beta}{\Gamma(n+\beta)} u(t), & \beta = n \in \mathbb{N}.
\end{cases}
\]

where \( \Gamma(\beta) = \int_0^\infty e^{-t}t^{\beta-1} \, dt \) is a Gamma function.

Definition 2.6. [24] The left sided Riemann-Liouville fractional derivative operator \( D_t^\beta \) of order \( 0 < \beta < 1 \), of a function \( u \in C \), is defined as

\[
D_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t (t-\tau)^{-\beta} u(\tau) \, d\tau.
\]

Definition 2.7. [24] The left sided Caputo fractional derivative operator \( D_t^\beta \) of order \( \beta > 0 \), of a function \( u \in C_{m-1} \), is defined as

\[
D_t^\beta u(t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} \frac{d^n u(\tau)}{d\tau^n} \, d\tau, & n-1 < \beta < n, t > 0, \\
\frac{t^\beta}{\Gamma(n+\beta)} \frac{d^n u(t)}{d\tau^n}, & \beta = n \in \mathbb{N}.
\end{cases}
\]

Definition 2.8. [24] The left sided Caputo fractional derivative operator \( D_t^\beta \) of order \( \beta > 0 \), of a function \( u \in C_{m-1} \), is defined as

\[
D_t^\beta u(t) = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} \frac{d^n u(\tau)}{d\tau^n} \, d\tau, & n-1 < \beta < n, t > 0, \\
\frac{t^\beta}{\Gamma(n+\beta)} \frac{d^n u(t)}{d\tau^n}, & \beta = n \in \mathbb{N}.
\end{cases}
\]

Definition 2.9. [24] The Caputo time-fractional derivative operator \( D_t^\beta \) of order \( \beta > 0 \), is defined as

\[
D_t^\beta u(x,t) = \frac{\partial^n u(x,t)}{\partial t^n} = \begin{cases} 
\frac{1}{\Gamma(n-\beta)} \int_0^t (t-\tau)^{n-\beta-1} \frac{\partial^n u(\tau)}{\partial \tau^n} \, d\tau, & n-1 < \beta < n, t > 0, \\
\beta = n \in \mathbb{N}.
\end{cases}
\]

Definition 2.10. [24] The one- and two-parameter Mittag-Leffler function is defined as

\[
\begin{align*}
E_{\beta} (z^\beta) &= \sum_{k=0}^{\infty} \frac{z^{\beta k}}{\Gamma(\beta k+1)}, \quad \beta > 0 \\
E_{\beta,\gamma} (z^\beta) &= \sum_{k=0}^{\infty} \frac{z^{\beta k}}{\Gamma(\beta k+\gamma)}, \quad \beta, \gamma > 0
\end{align*}
\]

Definition 2.11. [24] For given \( f \in L^2 (\mathbb{T}^d; \mathbb{R}^d) \), \( f \) is divergence free in sense of if

\[
\langle f, \nabla g \rangle = 0 \quad \forall g \in C^\infty (\mathbb{T}^d).
\]
Definition 2.12. [22] The orthogonal projection $\Pi$ is given by

$$\Pi : f = \sum_{k \in \mathbb{Z}^d} f_k e_k \mapsto \Pi f = \sum_{k \in \mathbb{Z}^d} P_k f_k e_k,$$

where $P_k \in \mathbb{R}^d \times \mathbb{R}^d$ is the d-dimensional projection on $k^\perp$, $P_k = I - \frac{k}{|k|} \otimes \frac{k}{|k|} (k \neq 0)$. Moreover, $\Pi_N$ is given by

$$f = \sum_{k \in \mathbb{Z}^d} f_k e_k \mapsto \Pi_N f = \sum_{k \in \mathbb{Z}^d} f_k e_k,$$

where $\Pi_N : C^\infty (\mathbb{T}^d)' \to C^\infty (\mathbb{T}^d)$.

Definition 2.13. [23] Let \{B^{k,i} : k \in \mathbb{Z}^d_0, i = 1, \ldots, d-1\} be a family of independent standard real Brownian motions; then the complex Brownian motions can be defined as

$$W^{k,i} = \begin{cases} B^{k,i} + iB^{-k,i}, k \in \mathbb{Z}^d_0; \\ B^{-k,i} - iB^{k,i}, k \in \mathbb{Z}^d. \end{cases}$$

Definition 2.14. [24] Let $\mu \in \mathcal{M}(\Omega)_+$ be a finite measure on whose support is exactly that is, $\Omega$ is the smallest closed set such that $\mu(\mathbb{R}^d/\Omega) = 0$. Let $\#\mu$ be the pushforward measure on $\mathbb{R}$ of $\#\mu$ with respect to (w.r.t.) the mapping $f : \Omega \to \mathbb{R}$. That is

$$\#\mu = \mu(f^{-1}(C)), \quad \forall C \in \mathcal{B}(\mathbb{R}).$$

In particular, its moments $\#\mu = (\#\mu_k)_{k \in \mathbb{N}}$, read

$$\#\mu_k = \int z^k d\#\mu(z) = \int f(x)^k d\#\mu(x), k = 0, 1, \ldots.$$

Definition 2.15. [25] We say that a random time $\tau$ is a blow-up time (or explosion time) of the solution $u(t,x)$ to (1.1), if the following two conditions are fulfilled:

(i) For any $t < \tau$, $\sup_{x \in \Omega} |u(t,x)| < \infty$ a.s.;
(ii) If $\tau < \infty$, then $\lim_{t \to \tau} \sup_{x \in \Omega} |u(t,x)| = \infty$.

Lemma 2.1. [26] For $-1 < a < \beta + 1$, $E_{\beta,a}(t)$ is bounded on $(-\infty, 0]$, and

$$\lim_{t \to -\infty} t E_{\beta,a}(-t) = \frac{1}{\Gamma(a - \beta)}.$$

Lemma 2.2. [27] Let $u(t)$ be a continuous function on $(-\infty, 0]$ and satisfies

$$D^\beta u(t) \leq -\lambda u(t),$$

where $\beta \in (0,1)$, $\lambda$ is a constant. Then

$$u(t) \leq u(0) E_\alpha(-\lambda t^\beta), \quad t \geq 0.$$
Lemma 2.3. Suppose that $q > 1$, $p \in [q, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Suppose that $\Lambda^\sigma f \in L^q$, then $f \in L^p$ and there is a constant $C \geq 0$ such that

$$\|f\|_{L^p} \leq C \|\Lambda^\sigma f\|_{L^q}.$$ 

Lemma 2.4. (Fractional comparison principle) Let $u(0) = v(0)$, $u(t)$ and $v(t)$ satisfies

$$D_t^\sigma u(t) \geq D_t^\sigma v(t).$$

Lemma 2.5. For $u(t) \geq 0$,

$$D_t^\sigma u(t) + c_1 u(t) \leq c_2(t) \quad (2.7)$$

for almost all $t \in [0, T]$, where $c_1 > 0$ and the function $c_2(t)$ is non-negative and integrable for $t \in [0, T]$. Then

$$u(t) \leq u(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_2(s) \, ds. \quad (2.8)$$

Lemma 2.6. For any function $v(t)$ absolutely continuous on $[0, T]$, one has the inequality

$$v(t) D_t^\alpha v(t) \geq \frac{1}{2} D_t^\alpha v^2(t), \quad 0 < \alpha < 1. \quad (2.9)$$

Lemma 2.7. Let $s \in (0, 1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n, p, s, \Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}(\Omega)} \quad (2.10)$$

for any $q \in [p, p^*)$, where $p^* = p^*(N, s) = \frac{Np}{N-sp}$ is the so-called fractional critical exponent; i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p, p^*)$. If, in addition, $\Omega$ is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1, p^*)$.

Lemma 2.8. Let a nonnegative absolutely continuous function $u(t)$ satisfy the inequality

$$D_t^\alpha u(t) \leq c_1 u(t) + c_2(t), \quad 0 < \alpha \leq 1, \quad (2.11)$$

for almost all $t$ in $[0, T]$, where $c_1 > 0$ and $c_2(t)$ is an integrable nonnegative function on $[0, T]$. Then

$$u(t) \leq u(0) E_\alpha(c_1 t^\alpha) + \Gamma(\alpha) E_{\alpha,\alpha}(c_1 t^\alpha) I_\alpha^* c_2(t) \quad (2.12)$$

where $E_\alpha(z) = \sum_{n=0}^\infty z^n \Gamma(\alpha n + 1)$ and $E_{\alpha,\mu}(z) = \sum_{n=0}^\infty z^n \Gamma(\alpha n + \mu)$ are the Mittag-Leffler functions.
Lemma 2.9. \cite{kolmogorov_continuity_modification_theorem}(Kolmogorov continuity modification theorem) Suppose that 
\((\Omega, \mathcal{F}, P, (X_t)_{t \in \mathbb{R}_0^+})\) is a stochastic process with state space \(\mathbb{R}^d\). If there are \(\alpha, \beta, c > 0\) such that 
\[ E(|X_t - X_s|^{\alpha}) \leq c|t - s|^{1+\beta}, \quad s, t \in \mathbb{R}_{\geq 0}, \]  
then the stochastic process has a continuous modification that itself satisfies.

Lemma 2.10. \cite{vitali_convergence_theorem} For \(1 \leq p < \infty\), there exist constants such that the following holds: for every \(N \in \mathbb{N}\) and every martingale \((X^*_k)_{k=0}^N\), we have 
\[ E[X_N^{p/2}] \leq a_p E[(X^*_N)^p], \quad E[(X^*_N)^p] \leq b_p E[X_N^{p/2}], \]  
(2.14)

Lemma 2.11. \cite{skorokhod_representation_theorem} Let \(a(t)\) be a continuous on \([0, T)\) \((0 < T \leq \infty)\), \(l(t)\) is nonnegative and locally integrable on \([0, T)\), and suppose \(u(t)\) be a continuous nonnegative function on \([0, T)\) with 
\[ u(t) \leq a(t) + \int_0^t l(s)u(s)ds, t \in [0, T). \]  
(2.15)

Then 
\[ u(t) \leq a(t) + \int_0^t l(s)a(s) exp\left(\int_s^t l(\tau) d\tau\right) ds, t \in [0, T). \]  
(2.16)

If \(a(t)\) is a nonnegative non-decreasing on \([0, T)\), the inequality (2.16) is reduced to 
\[ u(t) \leq a(t) exp\left(\int_0^t l(s) ds\right) \]  
(2.17)

If \(a(t) \equiv 0\), then we can get \(u(t) \equiv 0\) on \([0, T)\).

Lemma 2.12. \cite{vitali_convergence_theorem}(Vitali convergence theorem) Assume that a sequence of Lebesgue integrable functions \(f_k : I \to \mathbb{R}, n \in \mathbb{N}\), is given such that \(f_k\) converge to \(f\) in measure. If the set \(\{f_k; k \in \mathbb{N}\}\) is uniformly absolutely continuous then the function \(f\) is Lebesgue integrable and 
\[ \lim_{k \to \infty} \int_I f_k = \int_I f. \]  
(2.18)

Lemma 2.13. \cite{skorokhod_representation_theorem}(Skorokhod representation theorem) Suppose \(P_n, n = 1, 2, \cdots\) and \(P\) are probability measures on \(S\) (endowed with its Borel algebra) such that \(P_n \Rightarrow P\). Then there is a probability space \((\Omega, \mathcal{F}, P)\) on which are defined \(S\)-valued random variables \(X_n, n = 1, 2, \cdots\) and \(X\) with distributions \(P_n\) and \(P\), respectively, such that \(\lim_{n \to \infty} X_n = X\) a.s..
Property 2.1. If $0 < \alpha < 1$, $u \in AC^{1} [0, T]$ or $u \in C^{1} [0, T]$, then the equality
\[ I_{t}^{\alpha} \left( D_{t}^{\alpha} u \right)(t) = u(t) - u(0), \]
and
\[ D_{t}^{\alpha} \left( I_{t}^{\alpha} u \right)(t) = u(t), \]
hold almost everywhere on $[0, T]$. In addition
\[ D_{t}^{1-\beta} \int_{0}^{t} D_{\tau}^{\beta} u(\tau) \, d\tau = \left( I_{t}^{\beta} \frac{d}{dt} I_{t}^{\beta} I_{t}^{1-\beta} \frac{d}{dt} u \right)(t) = u(t) - u(0). \] (2.19)

2.2. Notations

In what follows, $m \lesssim n$ refers to $m \leq Kn$, where $K$ is a constant greater than 0, and its value will change under different circumstances. In the following discussion, the dependence of the constants on parameters will be clearly written only when necessary.

For any $p, q \in [1, \infty)$, the norm of space $L^{p} (0, T; L^{q} (T^{d}))$ is denoted as $\| \cdot \|_{L^{p} L^{q}}$, the other symbols such as $\| \cdot \|_{L^{2} H^{s}}$ and $\| \cdot \|_{W^{1, 2} H^{s}}$ have the same meaning. The notation $\langle \cdot, \cdot \rangle$ represents the inner product in $L^{2} (T^{d})$ or the duality on $H^{s} (T^{d}) \times H^{-s} (T^{d})$.

3. Models and main result

In this section, we consider the hypotheses of nonlinear term $F$ in the deterministic nonlinear time fractional partial differential equation as below:
\[ D_{t}^{\beta} u = \left( -(-\Delta)^{s} u + F(u) \right) dt, \] (3.1)
where $s \geq 1$ is fixed, $\frac{1}{2} < \beta < 1$. The we introduce a nonlinear fractional stochastic partial differential equations, which is driven by driven by a Brownian motion of the form
\[ D_{t}^{\beta} u = \left( -(-\Delta)^{s} u + v\Delta u + F(u) \right) dt + \sqrt{C_{d} v} \sum_{k,i} \theta_{k,i} \sigma_{k,i} \nabla u(t) \, dW_{k,i}^{\theta}, \] (3.2)
where $\Delta$ is the periodic Laplacian operator, $s \geq 1$ is fixed, $\frac{1}{2} < \beta < 1$, $C_{d} = d/d - 1$, see [5,3] for details, and $\nu > 0$ represents noise intensity. $\sum_{k,i} T^{\beta}$ is equivalent to $\sum_{k,i}$. Set $\theta = \{\theta_{k}\} \in \ell^{2} (Z_{d}^{0})$, $\ell^{2}$ stands for the sum of squares sequence space. In what follows, we mainly discuss those $\theta$ with only finitely many non-zero components, and always assume that the parameter $\theta$ satisfies $\|\theta\|_{\ell^{2}} = 1$ and the components of its have symmetry property, namely for all $|k| = |l|$ ($k, l \in Z_{d}^{0}$), it holds $\theta_{k} = \theta_{l}$. Recall that $Z_{d}^{0} = Z^{d} \setminus \{0\}$ is the nonzero lattice points, and holds
\[ Z_{d}^{0} = Z_{d}^{+} \cup Z_{d}^{-}, \quad Z_{d}^{+} = -Z_{d}^{-}, \quad Z_{d}^{+} \cap Z_{d}^{-} = \emptyset. \]
For any $k \in \mathbb{Z}_+^d$, take a set of orthonormal basis $\{b_{k,1}, \ldots, b_{k,d-1}\}$ of $k^\perp = \{y \in \mathbb{R}^d : y \cdot k = 0\}$, note that the choice of orthonormal basis is not unique. Let the family $\{\sigma_{k,i} : k \in \mathbb{Z}_0^d, i = 1, \ldots, d-1\}$ as the periodic divergence free smooth vector fields. For any $k \in \mathbb{Z}_+^d$, set $b_{k,i} = -b_{k,i}$, hence it can be defined as
\[
\sigma_{k,i}(x) = b_{k,i}e^{2\pi ik \cdot x}, \quad x \in \mathbb{T}^d, \quad k \in \mathbb{Z}_0^d, \quad i = 1, 2, \ldots, d-1,
\]
where $i$ represents the imaginary unit.

Define the family $\{W_t^{k,i} : k \in \mathbb{Z}_0^d, i = 1, \ldots, d-1\}$ as the complex Brownian motions on the filtered probability space $\mathbb{Q}F, \mathbb{P}$, according to definition, it holds $W_t^{k,i} = W_t^{-k,i}$ and for all $k, m \in \mathbb{Z}_0^d, i, j \in \{1, \ldots, d-1\}$, $W_t^{k,i}$ and $W_t^{m,j}$ are independent, and the following is satisfied:
\[
[W_t^{k,i}, W_t^{m,j}]_t = 2t\delta_{k,-m}\delta_{i,j}, \quad (3.3)
\]

Take a sequence $\{\theta^N\}_{N \geq 1} \subset \ell^2$, which satisfies
\[
\|\theta^N\|_{\ell^2} = 1 \quad (\forall N \geq 1), \quad \lim_{N \to \infty} \|\theta^N\|_\infty = 0, \quad (3.4)
\]

**Hypothesis 3.1.** Assume that the nonlinear $F$ satisfies

- (H1) There exists $a_1 \geq 0$ and $\gamma_1 \in (0, s)$ such that $F : H^{s-\gamma_1} \to H^{-s}$ is a continuous mapping and holds
  \[
  \|F(u)\|_{H^{-s}} \lesssim (1 + \|u\|_{L^2}^{\alpha_3}) \left(1 + \|u\|_{H^s}\right);
  \]

- (H2) There exists $a_2 \geq 0$ and $\gamma_2 \in (0, 2)$ such that
  \[
  |\langle F(u), u \rangle| \lesssim (1 + \|u\|_{L^2}^{\alpha_4}) \left(1 + \|u\|_{H^s}^{\gamma_2}\right);
  \]

- (H3) There exists $a_3 \geq 0$, $\gamma_3 \in (0, 2)$ and $\eta \geq 0$ such that $a_3 + \gamma_3 \geq 2$, $\gamma_3 + \eta \leq 2$, and
  \[
  |\langle u - v, F(u) - F(v) \rangle| \lesssim \|u - v\|_{L^2}^{\alpha_5} \left(\|u - v\|_{H^s}^{\gamma_4}\right) \left(1 + \|u\|_{H^s}^\eta + \|v\|_{H^s}^\eta\right);
  \]

- (H4) There exists a set $\mathcal{M} \subset L^2(\mathbb{T}^d)$, which is composed of bounded, closed and convex function, it holds that for any $u_0 \in \mathcal{M}$ and $T > 0$, we can find $v > 0$ big enough such that the equation
  \[
  \left\{ \begin{array}{l}
  D_t^\beta u = -(-\Delta)^s u + v \Delta u + F(u), \\
  u|_{t=0} = u_0,
  \end{array} \right. \quad (3.5)
  \]
  has a global solution $u \in L^2(0, T; H^s) \cap C([0, T]; L^2)$, and it holds
  \[
  \sup_{u_0 \in \mathcal{M}} \sup_{t \in [0, T]} \|u(t; u_0, v)\|_{L^2} < \infty, \quad (3.6)
  \]

where $u(\cdot; u_0, v)$ stands for the unique solution for (3.5) with initial data $u_0$.  

Remark 3.1. For condition (H3), it can be extended to more general case:

(H3') There exists $N \in \mathbb{N}$ and non-negative $a_k^\alpha, \gamma_k^\alpha, \eta_k, \eta_k$, $k = 1, \ldots, N$ such that for all $k$ holds $a_k^\alpha + \gamma_k^\alpha \geq 2$, $\gamma_3^\alpha + \eta_3 \leq 2$, and

$$|\langle u - v, F(u) - F(v) \rangle| \lesssim \sum_{j=1}^{N} \|u - v\|_{L^2}^a \|u - v\|_{H^\alpha}^{\gamma_j^\alpha} (1 + \|u\|_{H^\alpha}^{\eta_j}) \left(1 + \|u\|_{L^2}^{\eta_3} + \|v\|_{L^2}^{\eta_3}\right).$$

Remark 3.2. By combining hypothesis (H2), Lemma 4.2 and Young inequality, we can conclude that any solutions of (3.5) satisfy

$$D_t^\beta \|u\|_{L^2}^2 = \int_{\mathbb{R}^d} D_t^\beta u^2 dx \leq \int_{\mathbb{R}^d} 2 u D_t^\beta u dx$$

where $C_1, C_2 > 0, \alpha' = 2\alpha/ (2 - \gamma_2) > 0$. By using inequality (see (3.2))

$$\|u\|_{H^\alpha}^2 \leq 2^{\alpha - 1} \left(\|u\|_{L^2}^2 + \|(-\Delta)^{\alpha/2} u\|_{L^2}^2\right)$$

and Poincaré inequality

$$\left\|f - \int_{\mathbb{T}^d} f(x) dx\right\|_{L^2}^2 \leq (2\pi)^{-2} \left\|\nabla f\right\|_{L^2}^2,$$

combining (3.7) and (3.8) we can obtain that

$$D_t^\beta \|u\|_{L^2}^2 \leq -2 (4\pi^2 v - 1) \|u\|_{L^2}^2 + C_2 \left(1 + \|u\|_{L^2}^\alpha\right) = -\chi_v \|u\|_{L^2}^2 + C_2 \left(1 + \|u\|_{L^2}^\alpha\right),$$

where $\chi_v := 2 (4\pi^2 v - 1)$, $\chi_v$ denotes that the value of $\chi$ depends on $v$, taking $v$ big enough is equivalent to $\chi_v$ big enough.

The main aim of the paper is to consider the effect of noise on the lifetime of the solutions to nonlinear fractional stochastic partial differential equations, we introduce the two main conclusions of the paper.

For given $u_0 \in L^2$, denote the random maximal time of existence of solutions $u(t; u_0, v, \theta)$ to nonlinear fractional stochastic differential equation (3.2) by $\eta = \eta(u_0, v, \theta)$, which has trajectories in $C ([0, \eta); L^2 (\mathbb{T}^d))$.

Theorem 3.1. Assume $F$ satisfies (H1)-(H3), $M \subset L^2 (\mathbb{T}^d)$ satisfies (H4), then for any $T \in (0, \infty)$, $v = v(T) \in (0, \infty)$, for every $\varepsilon > 0$, $u_0 \in \mathcal{M}$, there exists $\theta \in \ell^2$ such that

$$P (\eta(u_0, v, \theta) \geq T) > 1 - \varepsilon$$

In others words, the solution to (3.2) with initial data $u_0 \in \mathcal{M}$ does not experience blow-up within time $T$. 

12
From Theorem 3.1., it can be deduced that the following result holds.

**Theorem 3.2.** Assume hypotheses (H1)-(H4) are true, moreover, $F$ satisfies

(i) For $u_0 \in \mathcal{M}$, The $L^2$ norm of the solution $u(t; u_0, v)$ to Eqs. (3.5) decreases exponentially;

(ii) Under small initial conditions, the Eqs. (3.2) has a unique global solution. Then, we claim that there exists $\theta \in \ell^2$, for all $u_0 \in \mathcal{M}$, $t > 0$, the solution $u(t; u_0, v, \theta)$ to (3.2) exists with a probability no less than $1 - \varepsilon$.

4. Examples

In this section, we verify that Hypothesis 3.1 holds for fractional Keller-Segel and fractional Fisher-KPP equations when $d = 3$.

4.1. Fractional Keller-Segel Equation

Keller and Segel proposed a partial differential equation for modeling the chemotaxis behavior in cellular systems, which is known as the Keller-Segel model \[43\]. The simplified form is shown as

\[
\begin{align*}
\partial_t \rho &= \Delta \rho - \chi \nabla \cdot (\rho \nabla c) \quad \text{in } \Omega, \\
\partial_t c &= \gamma \Delta c + \beta \rho - \mu c, \\
\rho(0, \cdot) &= \rho_0 \geq 0, \quad c(0, \cdot) = c_0 \geq 0, \\
\partial_x \rho(t, \cdot) &= \partial_x c(t, \cdot) = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where $\chi, \gamma, \mu, \beta$ are positive constants.

Jäger and Luckhaus \[44\] studied the dynamic behavior of the Keller-Segel approximation system and stated that, for small $\chi$, there exists a unique smooth global solution $\rho$ if the initial data are smooth, and in the high-dimensional case, there exists solution $\rho$ that explode in finite time for large $\chi$.

Winkler \[45\] considered the Neumann initial-boundary value problem for the higher-dimensional parabolic Keller-Segel system and proved that for any prescribed $m = \int_{\Omega} \rho_0 > 0$, there exists radially symmetric positive initial data $(\rho_0, c_0)$ can make the corresponding solution blow up in finite time. Moreover, by providing an essentially explicit blow-up criterion shown that within the space of all radial functions, the set of such blow-up enforcing initial data indeed is large in an appropriate sense.

The fractional Keller-Segel model is an extension of the Keller-Segel system for the case that the motion of the cell cannot be described by random walk. Next, we verify that the fractional Keller-Segel system (4.1) satisfies the Hypothesis 3.1.

\[
\begin{align*}
\mathcal{D}_t^\beta \rho &= \Delta \rho - \chi \nabla \cdot (\rho \nabla c), \\
-\Delta c &= \rho - \rho_0, \\
\rho_0 &= \int_{\Omega} \rho(x) \, dx.
\end{align*}
\]  

where $\chi > 0$ is a fixed parameter indicating sensitivity, $\rho$ represents the bacterial population density and $c$ represents the density of a chemoattractant. $\Omega \in \mathbb{R}^d$ is bounded and satisfies Neumann boundary condition. In order to simplify the proof, set parameter $\chi = 1$ (the proof of $\chi \neq 1$ is similar to $\chi = 1$).
Remark 4.1. In the following discussion, \( w_{T^3} \) means \( \int_{T^3} u(x,t) \, dx \). If \( \omega \) is the solution to equation (4.1), then it holds \( \omega_{T^3}(t) = \omega_{T^3}(0) =: \mu \).

For any \( f \in L^2(T^3) \), define the operator

\[
\nabla^{-1} f := \nabla (-\Delta)^{-1} (f - f_{T^3})
\]

where \((-\Delta)^{-1} f \in H^2(T^3)\), and it holds \(-\nabla \cdot \nabla^{-1} f = f - f_{T^3} \). Then, we deduce that, for any \( \alpha \in \mathbb{R} \), the operator \( \nabla^{-1} \) can be extends to a continuous operator from \( H^\alpha(T^3) \) to \( H^{1+\alpha}(T^3) \).

Define \( u(t) := \omega(t) - \omega_{T^3}(t) = \omega(t) - \mu \), It is obvious that, \( \omega \) solves (4.1) is equivalent to \( u \) with zero mean satisfies

\[
D_\beta^t u = \Delta u - \nabla \cdot [(u + \mu) \nabla^{-1} u],
\]

the above formula can be transformed into

\[
D_\beta^t u = \Delta u - \nabla \cdot [u \nabla^{-1} u] + \mu u,
\]

by using the result \(-\nabla \cdot (\mu \nabla^{-1} u) = \mu u\).

Lemma 4.1. The nonlinear portion \( F(u) = -\nabla \cdot (u \nabla^{-1} u) \) in (4.2) satisfies the assumptions (H1) - (H3), and for any initial data \( u_0 \in L^2(T^3) \) with zero mean, the assumption (H4) holds.

Proof. It is clear that, the linear part \( \mu u \) of \( F(u) \) satisfies (H1) - (H3), hence, we mainly consider nonlinear portion \( H(u) = -\nabla \cdot (u \nabla^{-1} u) \).

From the relationship between the norm of Sobolev space and \( L^2 \) space, the inequality

\[
\| H(u) - H(v) \|_{H^{-1}} = \nabla \cdot \| (u \nabla^{-1} u - v \nabla^{-1} v) \|_{H^{-1}} \leq \| u \nabla^{-1} u - v \nabla^{-1} v \|_{L^2}
\]

holds. By using Sobolev embeddings we obtain

\[
H^{7/4}(T^3) \subset L^\infty(T^3), \quad H^{3/4}(T^3) \subset L^4(T^3),
\]

combining with Hölder's inequality, it holds

\[
\| u \nabla^{-1} u - v \nabla^{-1} v \|_{L^2} = \| u \nabla^{-1} u - v \nabla^{-1} u + v \nabla^{-1} u - v \nabla^{-1} v \|_{L^2} \\
\leq \| (u-v) \nabla^{-1} u \|_{L^2} + \| v \nabla^{-1} (u-v) \|_{L^2} \\
\leq \| u-v \|_{L^2} \| \nabla^{-1} u \|_{L^\infty} + \| v \|_{L^4} \| \nabla^{-1} (u-v) \|_{L^4} \\
\lesssim \| u-v \|_{L^2} (\| u \|_{H^{3/4}} + \| v \|_{H^{3/4}}),
\]

namely, \( \| H(u) - H(v) \|_{H^{-1}} \leq C \| u-v \|_{L^2} (\| u \|_{H^{3/4}} + \| v \|_{H^{3/4}}) \), where \( C \) is a constant independent of parameters. Hence, we claim that the mapping \( H : H^\frac{7}{4} \rightarrow H^{-1} \) is continuous, setting \( v = 0 \), then

\[
\| H(u) \|_{H^{-1}} \leq \| u \|_{L^2} \| u \|_{H^{3/4}} \leq \| u \|_{L^2} \| u \|_{H^1}.
\]
so (H1) is satisfied with $a_1 = 1, \gamma_1 = 1/4$.

Recall that
\[ \langle f, g \cdot \nabla f \rangle = -\frac{1}{2} \langle f^2, \nabla \cdot g \rangle, \]
using Sobolev embedding $H^{1/2} (T^3) \subset L^3 (T^3)$, we can conclude
\[ |\langle H(u), u \rangle| = |\langle u, \nabla^{-1} u \cdot \nabla u \rangle| = \frac{1}{2} \int_{T^3} u^3 (x) \, dx \leq \|u\|_{L^6}^3 \leq \|u\|_{H^{1/2}}^3 \leq \|u\|_{L^2}^{3/2} \|u\|_{H^{1/2}}^{3/2}, \]
so (H2) is satisfied with $a_2 = \gamma_2 = 3/2$.

From the above proof process, we can obtain
\[ \|H(u) - H(v)\|_{H^{-1}} \leq \|u - v\|_{H^{1/2}} \|u - v\|_{H^{1/4}} \leq \|u - v\|_{L^2} \|u - v\|_{H^1} \leq \|u - v\|_{H^{1/2}} \|u - v\|_{H^1}, \]
so (H3) is satisfied with $a_3 = \gamma_3 = \eta = 1$.

**Remark 4.2.** From Remark 3.2, any solution of (3.3) satisfies
\[ D_t^\beta \|u\|_{L^2}^2 \leq -\chi_v \|u\|_{L^2}^2 + C_2 \left(1 + \|u\|_{L^2}^{a'}\right), \]
let $f$ be the solution to fractional ordinary equation
\[ D_t^\beta f = -\chi_v f + C_2 \left(1 + f^{a'/2}\right), \]
with initial data $f_0 = \|u(0)\|_{L^2}^2$. By the fractional comparison principle, it holds $\|u(t)\|_{L^2} \leq f$. For any deterministic $R \geq 0$, it is possible to choose $\chi_v$ and a constant $M$ big enough, and $L$ is a constant with parameters $v, R$. The above equation has a global solution satisfying $f \leq L(v, R)$ with initial $f_0 \in [0, R]$. Then hypothesis (H4) holds for
\[ \mathcal{M} = \left\{ u \in L^2 (T^d) : \int_{T^d} u \, dx = 0, \|u\|_{L^2} \leq R \right\}. \]

It is clear to see that the below conclusion holds.

**Corollary 4.1.** For any $\mu \in \mathbb{R}$ and $R \geq 0$, let
\[ \mathcal{M}_{R, \mu} = \left\{ f \in L^2 (T^d) : \|f - f_{T^d}\|_{L^2} \leq R, f_{T^d} = \mu \right\}, \]
such that hypothesis (H4) holds.

**Proof.** If $\omega \in \mathcal{M}_{R, \mu}$ solves (4.1), then $u = \omega - \omega_{T^d}$ solves
\[ \begin{cases} D_t^\beta u = -(-\Delta)^\alpha u + F(u), \\ u|_{t=0} = u_0, \end{cases} \]
where $F(u) = -\nabla \cdot (u \nabla^{-1} u) + \mu u$, and $u_0$ with zero mean and satisfies $\|u_0\|_{L^2} \leq R$. Combing continuity of $F$ and Remark 4.2, we can deduce that hypothesis (H4) holds for choosing
\[ \mathcal{M}_{R, \mu} = \left\{ f \in L^2 (T^d) : \|f - f_{T^d}\|_{L^2} \leq R, f_{T^d} = \mu \right\}. \]
4.2. Fractional Fisher-KPP Equation

The reaction-diffusion equation controls the temporal evolution of the concentration or population density of species spreading, and its applications include the spatial and temporal spread of epidemics, the spatial spread of invasive species, etc. A well-known example of reaction-diffusion equation is the Fisher-KPP model, which is named after Fisher [46], Kolmogorov, and Petrovsky and Piskunov [47]. The standard reaction-representation of this equation [3] is

$$\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + ru(x,t)(1-u(x,t)), \quad D > 0, r > 0.$$  

As the research progressed, scholars found that the classical Fisher-KPP model has great limitations in modeling practical problems, so they combined the fractional order method with the reaction diffusion equation, and then focus more attention on the fractional order diffusion problem. Furthermore, they claim that the later method is more suitable for modeling sub-diffusion problems [48]. Regarding the blow up and global existence of the solution of the system, a large number of results have been presented.

Ahmad at al. [49] have studied a reaction diffusion

$$D^\beta_t u = \Delta u + u^2 - u$$  (4.4)

with a Caputo fractional derivative in time and various boundary conditions. With some conditions on the initial data, they demonstrate that the solutions may experience explosion in a finite time. However, for realistic initial conditions, solutions are global in time. Besides, they analyzed the asymptotic behavior of bounded solutions. Xu at al. [50] theoretically demonstrated the blow-up phenomenon and the conditions for its appearance. By fixing the other parameters in the model, they found that the lower the order, the earlier the blow-up comes.

Next, we discuss the fractional Fisher KPP model (4.4) satisfies the assumption (H1)-(H4) in the 3D case.

Lemma 4.2. The nonlinear portion $G(u) = u^2$ satisfies condition (H1)-(H4).

Proof. For any $\varphi \in H^1(T^3)$, by using Sobolev embedding $H^{1/2}(T^3) \subset L^3(T^3)$ and Hölder’s inequality, it holds

$$|\langle G(u) - G(v), \varphi \rangle| = |\langle u^2 - v^2, \varphi \rangle| \leq \|u + v\|_{L^3} \|u - v\|_{L^3} \|\varphi\|_{L^3}$$

$$\lesssim \|u + v\|_{H^{1/2}} \|u - v\|_{H^{1/2}} \|\varphi\|_{H^{1/2}}.$$

Then

$$\|G(u) - G(v)\|_{H^{-1}} \leq \|G(u) - G(v)\|_{H^{-1/2}} \lesssim (\|u\|_{H^{1/2}} + \|v\|_{H^{1/2}}) \|u - v\|_{H^{1/2}}$$

which implies the continuity of mapping $G : H^{1/2} \rightarrow H^{-1}$, so (H1) is satisfied with $a_1 = 1, \gamma_1 = 1/2$. 

16
Recall that interpolation inequality
\[
\|u\|_{L^{p_k}(\mathbb{R}^n)}^{p_k} \leq \|u\|_{L^{\frac{n}{n+2}}(\mathbb{R}^n)}^{\varsigma_1(p_k)} \|u\|_{L^{p_k-1}(\mathbb{R}^n)}^{(1-\varsigma_1)},
\]
(4.5)
and Sobolev inequality (see [51])
\[
\|u\|_{L^{p_k}(\mathbb{R}^n)}^{p_k} \leq \|u\|_{L^{\frac{n}{n+2}}(\mathbb{R}^n)}^{\varsigma_1(p_k)} \|u\|_{L^{p_k-1}(\mathbb{R}^n)}^{(1-\varsigma_1)},
\]
(4.6)
with \(\varsigma_1 = \left(\frac{1}{p_k-1} - \frac{1}{p_k}\right)/\left(\frac{1}{p_k-1} - \frac{n-2}{np_k}\right) \sim O(1), \quad 1 - \varsigma_1 \sim O(1), \quad p_k \to \infty.

By using Sobolev embedding and interpolation inequality (4.5), it holds
\[
\|(G(u),u)\| = \|(u^2,u)\| \leq \|u\|_{L^3}^3 < \|u\|_{H^{1/2}}^3 \lesssim \|u\|_{L^2}^{3/2} \|u\|_{H^1}^{3/2}
\]
so (H2) is satisfied with \(a_2 = \gamma_2 = 3/2\).

Take \(\varphi\) in the above formula as \(u - v\), we claim
\[
\|(G(u) - G(v),u-v)\| \leq \|G(u) - G(v)\|_{H^{-1/2}} \|u-v\|_{H^{1/2}} \lesssim \|u-v\|_{H^1/2} (\|u\|_{H^{1/2}} + \|v\|_{H^{1/2}}) \lesssim \|u-v\|_{H^1} \|u-v\|_{L^2} (\|u\|_{H^{1}} + \|v\|_{H^{1}})
\]
so (H3) is satisfied with \(a_3 = \gamma_3 = \eta = 1\).

The following proves there exists \(\mathcal{M} \subset L^2(\mathbb{T}^3)\), such that for nonlinear \(F(u) = u^2 - u\), hypothesis (H4) holds.

**Proposition 4.1.** For deterministic \(k_0 < 1\) and \(\delta_0 \in [0, +\infty)\), it is possible to find \(v = v(k_0,\delta_0)\) large enough, such that for any \(u_0 \subset L^2(\mathbb{T}^3)\) satisfying
\[
\int_{\mathbb{T}^3} u_0(x) \, dx \leq k_0 < 1, \quad \left\| u_0 - \int_{\mathbb{T}^3} u_0(x) \, dx \right\|_{L^2} \leq \sqrt{\delta_0}
\]
(4.7)
then the equation
\[
\begin{cases}
D_t^\delta u = (1 + v) \Delta u + u^2, \\
u|_{t=0} = u_0.
\end{cases}
\]
(4.8)
admits a global solution \(u \in C([0, +\infty); L^2(\mathbb{T}^3))\). Furthermore, the solution \(u\) satisfies
\[
\int_{\mathbb{T}^3} u(t,x) \, dx \leq 1, \quad \left\| u(t,) - \int_{\mathbb{T}^3} u(t,x) \, dx \right\|_{L^2} \leq \sqrt{\delta_0}, \quad \|u(t)\|_{L^2} \leq 1 + \sqrt{\delta_0} \quad \forall t \geq 0.
\]
(4.9)

**Proof.** In order to simplify the proof, it is replaced \(v + 1\) by \(v\) in the following discussion. Integrating \(D_t^\delta u = v \Delta u + u^2 - u\) over \(\mathbb{T}^3\) yields
\[
D_t^\delta u_{\mathbb{T}^3} = v \Delta u_{\mathbb{T}^3} + \|u\|_{L^2}^2 u_{\mathbb{T}^3} = \|u\|_{L^2}^2 - u_{\mathbb{T}^3}^2 + u_{\mathbb{T}^3}^3 - u_{\mathbb{T}^3} = \|u - u_{\mathbb{T}^3}\|_{L^2}^2 + u_{\mathbb{T}^3}^3 - u_{\mathbb{T}^3}^2
\]
Hence, it holds
\[
D_t^\delta (u - u_{\mathbb{T}^3}) = v \Delta (u - u_{\mathbb{T}^3}) + \left(u^2 - \|u\|_{L^2}^2\right) - (u - u_{\mathbb{T}^3}).
\]

17
By using Lemma, we deduce
\[
D_t^\beta \| u - u_{T^3} \|_{L^2}^2 \leq -2v \| \nabla (u - u_{T^3}) \|_{L^2}^2 - 2 \| u - u_{T^3} \|_{L^2}^2 + 2 \int_{T^3} \left( u^2 - \| u \|_{L^2}^2 \right) (u - u_{T^3}) \, dx
\]
where
\[
\int_{T^3} u^2 (u - u_{T^3}) \, dx = \int_{T^3} (u - u_{T^3})^3 \, dx + 2 \int_{T^3} u u_{T^3} (u - u_{T^3}) \, dx - u_{T^3}^2 \int_{T^3} (u - u_{T^3}) \, dx
\]
\[
= \int_{T^3} (u - u_{T^3})^3 \, dx + 2u_{T^3} \int_{T^3} u (u - u_{T^3}) \, dx
\]
\[
= \int_{T^3} (u - u_{T^3})^3 \, dx + 2u_{T^3} \int_{T^3} (u - u_{T^3})^2 \, dx
\]
Applying Sobolev inequality (4.6) and interpolation inequality (4.5), we have
\[
\left| \int_{T^3} (u - u_{T^3})^3 \, dx \right| \lesssim \| u - u_{T^3} \|_{L^3}^3 \lesssim \| u - u_{T^3} \|_{H^{1/2}}^3
\]
\[
\lesssim \| u - u_{T^3} \|_{L^6}^{3/2} \| u - u_{T^3} \|_{L^6}^{3/2}
\]
\[
\lesssim K \| u - u_{T^3} \|_{L^2}^3 + \| \nabla u \|_{L^2}^3.
\]
Hence, the above inequality can be transformed into
\[
\int_{T^3} u^2 (u - u_{T^3}) \, dx \leq \| \nabla u \|_{L^2}^2 + K \| u - u_{T^3} \|_{L^2}^6 + 2u_{T^3} \| u - u_{T^3} \|_{L^2}^2.
\]
Combing with Poincaré inequality \((4.33)\), it holds
\[
D_t^\beta \| u - u_{T^3} \|_{L^2}^2 \leq -2v \| \nabla u \|_{L^2}^2 + 2 \int_{T^3} u^2 (u - u_{T^3}) \, dx
\]
\[
\leq -2 (v - 1) \| \nabla u \|_{L^2}^2 + K \| u - u_{T^3} \|_{L^2}^6 + 4u_{T^3} \| u - u_{T^3} \|_{L^2}^2
\]
\[
\leq (-C_v + 4u_{T^3}) \| u - u_{T^3} \|_{L^2}^6 + K \| u - u_{T^3} \|_{L^2}^2,
\]
for the choice \(C_v = 8\pi^2 (v - 1)\), the constant \(K\) in the above formula is positive and independent of the parameter \(v\).

Based on the above discussion, we can obtain the two equalities
\[
D_t^\beta u_{T^3} = \| u - u_{T^3} \|_{L^2}^2 + u_{T^3}^2 - u_{T^3},
\]
\[
D_t^\beta \| u - u_{T^3} \|_{L^2}^2 \leq \| u - u_{T^3} \|_{L^2}^2 + (C_v + 4u_{T^3}) \| u - u_{T^3} \|_{L^2}^6 + K \| u - u_{T^3} \|_{L^2}^6.
\]
Next, set \(f(t) = u_{T^3}(t), g(t) = \| u - u_{T^3} \|_{L^2}^2, \lambda = C_v\), we obtain a system of differential inequalities as the form
\[
\begin{cases}
D_t^\beta f = g + f^2 - f, \\
D_t^\beta g \leq (-\lambda + 4f) g + Kg^3, \\
f(0) \leq m_0 < 1, \\
g(0) \leq \delta_0.
\end{cases}
\quad (4.11)
\]
It is obvious that for all \(t \geq 0\), \(g(t) \geq 0\). Next, we prove that exists \(\lambda > 0\) such that the above system satisfies
\[
\sup_{t \geq 0} f(t) \leq 1, \quad \sup_{t \geq 0} g(t) \leq \delta_0.
\]
18
For \( f(0) \in [0, k_0] \), combing with the fact \( g(t) = \|u - u^{*}\|_{L^{2}}^{2} \geq 0 \). Consider \( \varepsilon \in (0, 1) \), such that \( k_0 + \varepsilon < 1 \). Define
\[
T_{\varepsilon} = \inf \{ t \geq 0 : (f(t), g(t)) \notin [0, k_0 + \varepsilon] \times [0, \delta_0 + \varepsilon] \}.
\]

Hence, for any \( t \in [0, T_{\varepsilon}] \), it holds
\[
\begin{align*}
D_{t}^{\beta} f(t) &= g(t) + f^2(t) - f(t), \\
D_{t}^{\beta} g(t) &\leq \left[ -\lambda + 4 + K (\delta_0 + 1)^2 \right] g(t),
\end{align*}
\]
Set \( \gamma = \lambda - 4 - K (\delta_0 + 1)^2 \), as an application of Lemma 2.5, we have \( g(t) \leq g(0) \leq \delta_0 \).

For any \( t \in [0, T_{\varepsilon}] \), \( 0 \leq f(t) \leq 1 \), therefore
\[
D_{t}^{\beta} f(t) + f(t) = g(t) + f^2(t) \leq \delta_0 + 2 \leq \delta_0 + 2 + e^{-\lambda t}.
\]
Applying the Lemma 2.5 again, we know that
\[
f(t) \leq f(0) + \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1} \left( \delta_0 + 2 + e^{-\lambda s} \right) ds,
\]
since the growth rate of exponential function is faster than that of power function, there exists a constant \( M \) and \( \lambda \) large enough, which make
\[
e^{-\lambda s} + 2 + \delta_0 \leq Me^{-\lambda s} \text{ and } M \int_{0}^{t} (t-s)^{\beta-1} e^{-\lambda s} ds \leq \varepsilon.
\]
Namely,
\[
f(t) \leq f(0) + \frac{M}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1} e^{-\lambda s} ds
\leq k_0 + \frac{M}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1} e^{-\lambda s} ds
\leq k_0 + \varepsilon < 1.
\]

For \( f(0) < 0 \), the choice of \( \varepsilon, \lambda \) are the same as above. Define
\[
\tau_{\varepsilon} = \inf \{ t \geq 0 : (x(t), y(t)) \notin (-\infty, 0) \times [0, \delta_0 + \varepsilon] \},
\]
for any \( t \leq \tau_{\varepsilon} \), observe that
\[
D_{t}^{\beta} g(t) \leq \left( -\lambda + 4f(t) \right) g(t) + K g(t)^3 \leq \left[ -\lambda + K (\delta_0 + 1)^2 \right] g(t).
\]
By selecting the appropriate \( \lambda \) such that \( -\lambda + K (\delta_0 + 1)^2 < 0 \). Similarly, it holds that
\[
g(t) \leq \delta_0 \quad \forall t \in [0, \tau_{\varepsilon}).
\]
Therefore, for all \( t \in [0, \tau_{\varepsilon}) \), we conclude that
\[
(f(t), g(t)) \in (-\infty, 0) \times [0, \delta_0].
\]
The above proof shows that, we can find \( \lambda \) large enough such that the conclusion holds.

It is easy to deduce the following corollary from Proposition 4.1.
Corollary 4.2. For any \( k_0 < 1, \delta_0 < \infty \), selects the set with the following form
\[
\mathcal{M}_{k_0, \delta_0} = \left\{ f \in L^2 \left( \mathbb{T}^3 \right) : f_{T^3} \leq k_0, \| f - f_{T^3} \|_{L^2} \leq \sqrt{\delta_0} \right\},
\] (4.12)
the hypothesis (H4) holds.

Proposition 4.2. For any \( u_0 \in L^2 \left( \mathbb{T}^3 \right) \), \( u_0 \) is nonnegative and satisfies
\[
\int_{\mathbb{T}^3} u_0(x) \, dx > 1,
\]
and not depend on \( v > 0 \), the solution to the equation
\[
D_t^\beta u = v \Delta u + u^2 - u
\] (4.13)
blows up in \( L^2 \left( \mathbb{T}^3 \right) \).

Proof. Integrating the Eq. (4.13) on torus \( \mathbb{T}^3 \), yields
\[
D_t^\beta u_{T^3} = v \Delta u_{T^3} + \int_{\mathbb{T}^3} u^2 \, dx - u_{T^3} \geq u_{T^3}^2 - u_{T^3},
\]
where \( \Delta u_{T^3} = \int_{\mathbb{T}^3} \Delta u(x, t) \, dx = 0 \), the second step holds due to the application of Jensen inequality. We establish a system of differential inequalities as the form
\[
\begin{cases}
D_t^\beta f(t) = f^2(t) - f(t), \\
f(0) = \int_{\mathbb{T}^3} u_0(x) \, dx > 1.
\end{cases}
\] (4.14)
Since \( f(0) > 1 \), from [52] we can obtain that the solution to Eqs. (4.14) blows up in a finite time \( T^* \). As an application of Lemma 2.4, it holds
\[
u_{T^3} \geq f(t),
\]
then we can deduce
\[
\lim_{t \to T^*} \| u(t) \|_{L^2} \geq \lim_{t \to T^*} f(t) = +\infty.
\]
Namely, the solution to Eq. (4.13) blows up in \( L^2 \left( \mathbb{T}^3 \right) \).

5. Proof of main conclusions

The purpose of this section is to prove Theorem 3.1 and Theorem 3.2 hold if the nonlinear \( F \) satisfies assumptions (H1) - (H4).

Due to the existence of nonlinear parts, fractional stochastic differential equations may have only local solutions for general initial value conditions. Therefore, we introduce a smooth non-increasing cut-off function \( L_R \), which has the form
\[
L_R(x) = \begin{cases} 
1, & x \in [0, R], \\
0, & x \in [R + 1, \infty),
\end{cases}
\] (5.1)
for sufficiently small $\gamma > 0$, denote $L_R (u)$ as $L_R (\|u\|_{H^{-\gamma}})$. Next, we discuss fractional stochastic differential equations with cut-off function $L_R$, the equation can be rewritten as

$$
\begin{cases}
D_t^\alpha u = \left[-(-\Delta)^\alpha u + v \Delta u + L_R (u) F (u)\right] dt + \sqrt{C_{dV}} \sum_{k,i} \theta_k \sigma_{k,i} \nabla u (t) dW^i_{k,t}, \\
\|u\|_{t=0} = u_0.
\end{cases}
$$ (5.2)

where $\alpha \geq 1, \frac{1}{2} < \beta < 1, C_d = d/d - 1,v > 0$, and set $\Lambda = (-\Delta)^{1/2}$.

Based on the assumptions of $\theta$ and condition $\|\theta\|_{\ell^2}^2 = 1$, we obtain that

$$
\sum_{k,i} \theta_k^2 \sigma_{k,i} (x) \otimes \bar{\sigma}_{k,i} (x) = \sum_{k,i} \theta_k^2 a_{k,i} \otimes a_{k,i} = \frac{d-1}{d} \|\theta\|_{\ell^2}^2 I_d = \frac{d-1}{d} I_d, \quad \forall x \in \mathbb{T}^d
$$ (5.3)

where $I_d$ is an identity matrix.

**Definition 5.1.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a given stochastic basis on which a family $\{W^{k,i}\}_{k,i}$ of complex Brownian motions satisfying (3.3) is defined, $u_0 \in L^2 (\mathbb{T}^d)$. A process $u$ with trajectories in $u \in C (\{0, T\}; L^2 (\mathbb{T}^d)) \cap L^2 (0, T; H^s (\mathbb{T}^d))$ is a solution to the equation (5.2) if it is $\mathcal{F}_t$-adapted and holds

$$
\langle u (t), \varphi \rangle = \langle u_0, \varphi \rangle - \frac{1}{\Gamma (2\beta)} \int_0^t (t-\tau)^{\beta-1} \langle \Lambda^\alpha u (\tau), \Lambda^\beta \varphi \rangle + v \langle \nabla u (\tau), \nabla \varphi \rangle d\tau + \frac{1}{\Gamma (\beta)} \int_0^t (t-\tau)^{\beta-1} L_R (u (\tau)) (F (u (\tau)), \varphi) d\tau - \sqrt{C_{dV}} \sum_{k,i} \theta_k \int_0^t (t-\tau)^{\beta-1} \langle u (\tau), \sigma_{k,i} \cdot \nabla \varphi \rangle dW^i_{k,t}
$$

with probability one, for any $\varphi \in H^s (\mathbb{T}^d), t \in [0, T]$.

**Remark 5.1.** For a given $L^2$-valued local martingale $M$, the quadratic variation process $[M]$ is the unique increasing process such that $\|M\|_{L^2} - [M]$ is a local martingale (see [33] for more details).

**Lemma 5.1.** Let $u$ be a solution to Eq. (5.2), then for any $\varphi \in H^\alpha$ it holds

$$
\langle u (t), \varphi \rangle - \langle u_0, \varphi \rangle = \langle G (t), \varphi \rangle + \langle M (t), \varphi \rangle
$$ (5.4)

where the process $G \in W^{1,2} (0, T; H^{-s})$, $M (t)$ is an $L^2$-valued continuous local martingale, and can be described as

$$
G (t) = \frac{1}{\Gamma (\beta)} \int_0^t (t-\tau)^{\beta-1} \left[-\Lambda^{2s} u (\tau) + v \Delta u (\tau) + L_R (u (\tau)) F (u (\tau))\right] d\tau,
$$ (5.5)

$$
M (t) = \frac{\sqrt{C_{dV}}}{\Gamma (\beta)} \int_0^t \sum_{k,i} \theta_k \sigma_{k,i} (t-\tau)^{\beta-1} \cdot \nabla u (\tau) dW^i_{k,t}.
$$ (5.6)

It is enough for us to find a constant $K = K (v, T)$, such that

$$
\|G\|_{W^{1,2} H^{-s}} \leq K \left(1 + \|u\|_{L^\infty L^2} (1 + \|u\|_{L^2 H^s})\right),
$$ (5.7)
and the quadratic variation of local martingale $M(t)$ satisfies

$$[M](t) = 2v \int_0^t (t - \tau)^{2\beta - 2} \|\nabla u(\tau)\|_{L^2}^2 \, d\tau. \quad (5.8)$$

**Proof.** Note that $\langle \Lambda^2 u, \varphi \rangle = \langle \Lambda^* u, \Lambda^* \varphi \rangle$, and $\Delta u \in H^{-2} \subset H^{-s}$, it follows that

$$\| -\Lambda^2 u(t) + v\Delta u(t)\|_{H^{-s}}^2 \lesssim (1 + v^2) \|u(t)\|_{H^s}^2,$$

combing with Cauchy-Schwarz inequality, we claim that

$$\left( \int_0^T (T - \tau)^{\beta - 1} \left\| -\Lambda^2 u(\tau) + v\Delta u(\tau) \right\|_{H^{-s}}^2 \, d\tau \right)^{\frac{1}{2}} \times \left( \int_0^T \left\| -\Lambda^2 u(\tau) + v\Delta u(\tau) \right\|_{H^{-s}}^4 \, d\tau \right)^{\frac{1}{4}} \lesssim (1 + v^2) \|u\|_{L^2}^2,$$

and by hypothesis (H1) it holds

$$\left( \int_0^T (T - \tau)^{\beta - 1} \left\| F(u(\tau)) \right\|_{H^{-s}}^2 \, d\tau \right)^{\frac{1}{2}} \lesssim \left( \int_0^T (T - \tau)^{\beta - 1} \left( 1 + \|u(\tau)\|_{H^s}^2 \right) \left( 1 + \|\nabla u(\tau)\|_{L^2}^{2\beta_1} \right) \, d\tau \right)^{\frac{1}{2}},$$

which implies $G \in W^{1,2}(0, T; H^{-s})$ and satisfies (5.6).

Next, we prove that $M$ is a $L^2$-value stochastic integral, assume that $E \left[ \int_0^T (T - \tau)^{2\beta - 2} \|u(\tau)\|_{H^s}^2 \, d\tau \right] < \infty$. By the condition (3.3) and (5.3), we claim

$$\sqrt{C_d v} \int_0^T (t - \tau)^{\beta - 1} \cdot \nabla u(\tau) \, dW_{t}^{k,i}(t) = 2C_d v \int_0^T \sum_{k,i} \theta_k \cdot \nabla u(\tau) \, dW_{t}^{k,i}(t) = 2C_d v \int_0^T \sum_{k,i} \theta_k \cdot \nabla u(\tau) \, dW_{t}^{k,i}(t) = 2C_d v \int_0^T \sum_{k,i} \theta_k \cdot \nabla u(\tau) \, dW_{t}^{k,i}(t) = 2C_d v \int_0^T \sum_{k,i} \theta_k \cdot \nabla u(\tau) \, dW_{t}^{k,i}(t)$$

which implies the formula (5.8) holds and

$$\mathbb{E} \left( \|M(t)\|_{L^2}^2 \right) = \mathbb{E} \left( [M](t) \right) = 2v \mathbb{E} \left[ \int_0^T (T - \tau)^{2\beta - 1} \|\nabla u(\tau)\|_{L^2}^2 \, d\tau \right] < \infty.$$

Using the integration by parts and the divergence free property of $\sigma_k$, we deduce the following:

$$(M(t), \varphi) = \sqrt{C_d v} \int_0^T \sum_{k,i} \theta_k \left( \sigma_{k,i}(t - \tau)^{\beta - 1} \cdot \nabla u(\tau), \varphi \right) \, dW_{t}^{k,i}(t) = -\sqrt{C_d v} \int_0^T \sum_{k,i} \theta_k (t - \tau)^{\beta - 1} (u(\tau), \sigma_{k,i} \cdot \nabla \varphi) \, dW_{t}^{k,i}(t)$$

Therefore, any solution of equation (5.2) can be decomposed into the form in (5.4). The proof is complete. \qed
Lemma 5.2. For any \( \alpha, \gamma, \delta > 0, \quad p < \infty \), define the space
\[
\mathcal{K} := C ([0, T]; L^2) \cap C^\gamma ([0, T]; H^{-\alpha}) \cap L^2 (0, T; H^\delta),
\]
\[
\mathcal{X} := L^p (0, T; L^2) \cap C ([0, T]; H^{-\delta}) \cap L^2 (0, T; H^{s-\delta}),
\]
then \( \mathcal{K} \to \mathcal{X} \) is a compact embedding. Moreover, the set
\[
\mathcal{X}_L := \left\{ f \in \mathcal{X} : \sup_{t \in [0, T]} \| f(t) \|_{L^2} + \| f \|_{L^2 H^s} \leq L \right\}
\]
is a closed subset of \( \mathcal{X} \).

Proof. Set a sequence \( \{ f_n \}_n \), which is bounded in \( \mathcal{K} \). It follows from the Ascoli–Arzela theorem [54] that, we can choose a subsequence \( \{ f_{n_k} \}_{n_k} \) of \( \{ f_n \}_n \) that converges to \( f \) in \( C ([0, T]; H^{-\alpha}) \), which indicates that \( \{ f_{n_k} \}_{n_k} \) uniform bound in \( C ([0, T]; H^{-\alpha}) \). Recall that
\[
\| f_{n_k} - f \|_{L^2} \leq \| f_{n_k} - f \|_{H^{-\alpha}},
\]
namely \( \{ f_{n_k} \}_{n_k} \) uniform bound in \( C ([0, T]; L^2) \). By using the inequality (5.9), it holds that for any \( \varepsilon > 0 \), \( f_{n_k} \to f \) in \( C ([0, T]; H^{-\delta}) \). Observe that
\[
\int_0^T \| f_{n_k} (t) - f(t) \|_{L^2} dt \leq \int_0^T \| f_{n_k} (t) - f(t) \|_{H^{-\delta}}^s \| f_{n_k} (t) - f(t) \|_{H^{-\delta}} dt,
\]
combining with the uniform bound in \( L^\infty (0, T; L^2) \), we can deduce that for general \( p < \infty \), \( f_{n_k} \to f \) in \( L^p (0, T; L^2) \). Applying the convergence of \( L^2 (0, T; L^2) \) and the uniform boundedness of \( L^2 (0, T; H^\delta) \), we claim that \( \{ f_{n_k} \}_{n_k} \) convergence in \( L^2 (0, T; H^{s-\delta}) \). Summarize the above, the embedding \( \mathcal{K} \to \mathcal{X} \) is compact.

Finally, recall the continuous property of norm and the definition of set \( \mathcal{X}_L \), it is obvious that \( \mathcal{X}_L \subset \mathcal{X} \), namely, \( \| \cdot \|_{L^\infty L^2} + \| \cdot \|_{L^2 H^s} \) has lower semicontinuity in the topology of \( \mathcal{X} \), which implies that \( \mathcal{X}_L \) is a closed subset of \( \mathcal{X} \). \( \square \)

Lemma 5.3. Assume that hypothesis (H1) is true. Then for any \( N < \infty, 1 \leq p < 2 \), the mapping \( g \mapsto F(g) \) is continuous from \( \mathcal{X}_L \) to \( L^p (0, T; H^s) \).

Proof. Recall the definition of \( \mathcal{X}_L \) (5.9) and the hypothesis (H1), it is easy to obtain that \( F(g) \in L^2 (0, T; H^{-s}) \), for any \( g \in \mathcal{X}_L \). From now on, we fix the parameter \( N < \infty \), and take a sequence \( \{ g_n \}_n \) from the space \( \mathcal{X}_L \), which converges to \( g \). Similarly to the proof of Lemma 6.2, it holds \( g_n \to g \) in \( L^2 (0, T; H^{s-\delta}) \). The continuity of \( F \) implies that \( F(g_n) \) converge to \( F(g) \) in measure in the space \( H^{-s} \). Take \( \delta < \gamma_1 \), by using the condition (H1), there exists a constant \( M \) such that
\[
\| F(g_n) \|_{L^2 H^{-s}}^2 = \int_0^T \| F(g_n(t)) \|_{H^{-s}}^2 dt \lesssim \int_0^T \left( 1 + \| g_n(t) \|_{L^2}^{2\alpha_1} \right) \left( 1 + \| g_n(t) \|_{H^s}^2 \right) dt \lesssim M,
\]
which is equivalent to the uniform boundedness of sequence \( \{ F ( g_n ) \} \) in \( L^2 ( 0, T ; H^{-s} ) \).

By applying the Vitali’s convergence theorem, see [39], we can obtain the convergence of \( \{ F ( g_n ) \} \) in \( L^p ( 0, T ; H^{-s} ) \), for any \( p < 2 \). From what has been discussed above, the map from \( X_L \) to \( L^p ( 0, T ; H^{-s} ) \) is continuous.

\[ \Box \]

**Remark 5.2.** Applying the interpolation inequality

\[ \| u \|_{L^2} \leq \| u \|_{H^{\delta}}^\frac{\delta}{H^{\delta}} \| u \|_{H^{-s}}^\frac{\delta}{H^{-s}} , \]

the inequality in (H2) can be rewritten as

\[ | \langle F ( u ) , u \rangle | \leq (1 + \| u \|_{H^s}^\gamma) \left( 1 + \| u \|_{H^{-s}}^{\gamma_2} \right)^\frac{\delta}{H^{\delta}} \left( 1 + \| u \|_{H^{-s}}^{\gamma_2} \right) \]

The inequality in \( H_2 \) can be rewritten as

\[ | \langle F ( u ) , u \rangle | \leq (1 + \| u \|_{H^s}^\gamma) \left( 1 + \| u \|_{H^{-s}}^{\gamma_2} \right)^\frac{\delta}{H^{\delta}} \left( 1 + \| u \|_{H^{-s}}^{\gamma_2} \right) . \]

By the above estimate, we can find \( \delta \) small enough such that assumption (H2) can be generalized to the more general case

\[ | \langle F ( u ) , u \rangle | \leq (1 + \| u \|_{H^s}^\gamma) \left( 1 + \| u \|_{H^{-s}}^{\gamma_2} \right) \]

(5.10)

where \( \tilde{\gamma}_2 > 0, \tilde{s}_2 < 2 \).

**Proposition 5.1.** For any \( \theta \in \ell^2, u_0 \in L^2 ( T^d ) \), the solution \( u \) to Eqs. (5.2) is unique, \( u \in C ([0, T] ; L^2 ( T^d ) ) \cap L^2 ( 0, T ; H^s ( T^d ) ) \), and satisfies

\[ \mathbb{P}-a.s., \sup_{t \in [0, T]} \| u ( t ) \|_{L^2}^2 + \frac{1}{\Gamma ( \beta )} \int_0^t ( t - \tau )^{\beta - 1} \| \Lambda^s u ( \tau ) \|_{L^2}^2 \, d\tau \leq K_1 \left( 1 + \| u ( 0 ) \|_{L^2}^2 \right) , \]

(5.11)

where \( K_1 = K_1 ( T, \delta, R ) \) is an unimportant constant. Moreover, for any \( p > 1, \gamma > p + 1, \kappa < \left( \frac{2\beta - 1}{2p} - 1 \right) \), there exists a constant \( K_2 \) such that the martingale \( M \) satisfies

\[ \mathbb{E} \left[ \left( \sup_{0 \leq \tau \leq T} \frac{\| M ( t ) - M ( \tau ) \|_{H^{-\gamma}}}{| t - \tau |^{\kappa}} \right)^{2p} \right] \leq K_2 \| \theta \|_{L^\infty}^{2p} \left( 1 + \| u_0 \|_{L^2}^2 \right)^p . \]

(5.12)

It is also important to note that \( K_2 \) does not depend on \( \| \theta \|_{L^2} \).

**Proof.** We use the Galerkin approximations and priori estimates to prove the conclusion. For any \( N \in \mathbb{N} \), set \( S_N \) be a finite dimensional subspace of \( S = L^2 ( T^d ) \) that is spanned by \( e^{2\pi i k \cdot x} \), and \( | k | \leq N \). Denote an orthogonal mapping \( \Pi_N : S \rightarrow S_N \). For the equation form

\[ D_t^\theta u_N ( t ) = \left[ -\Lambda^2 u_N ( t ) + v \Delta u_N ( t ) + L ( u_N ( t ) ) \Pi_N F ( u_N ( t ) ) \right] dt - \sqrt{C_0} \sum_{k,i} \theta_k \Pi_N ( \sigma_{k,i} \cdot \nabla u_N ( t ) ) dW_t^{k,i} \]

24
with initial data $u_N(0) = \Pi_N u_0$. It is obvious that the norms $\| \cdot \|_{L^2}$ are equivalent to $\| \cdot \|_{H^s}$ in space $S_N$. Recall the definition of cut-off function $L_R$ \[ (5.11) \], for sufficiently small $\delta$, $L_R(\|u_N\|_{H^{-\delta}})$ is continuous in $u_N$. Besides,

$$D_\beta^t \|u_N\|_{L^2}^2 \leq 2 \int_{T^+} u_N(t) D_\beta^t u_N(t) \, dx = -2 L_R(u_N(t)) \|u_N\|_{H^s}^2 + 2 L_R(u_N(t)) \langle F(u_N(t)), u_N(t) \rangle$$

Using the inequality $\|u_N(t)\|_{H^s} \leq 2^{s-1} \left( \|u_N(t)\|_{L^2}^2 + \|\Lambda^s u_N(t)\|_{L^2}^2 \right)$, substituting \[ (5.14) \] into Eq.\[ (5.13) \] yields

$$D_\beta^t \|u_N\|_{L^2}^2 \leq -\|\Lambda^s u_N(t)\|_{L^2}^2 + \|u_N(t)\|_{L^2}^2 + C'.$$

Therefore, there exists a constant $K_1$, such that

$$\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + \left. \frac{1}{\Gamma(\beta)} \int^t_0 (t - \tau)^{\beta-1} \|\Lambda^s u(\tau)\|_{L^2}^2 \, d\tau \right) \leq K_1 \left( 1 + \|u(0)\|_{L^2}^2 \right).$$

For any $\varphi \in C^\infty(T^2)$, $\langle M_N, \varphi \rangle$ is a real valued martingale with quadratic variation, $\{\sigma_{k,i}\}_{k,i}$ is an orthonormal system in $L^2(T^2)$, we have

$$[\langle M_N(t), \varphi \rangle(t) - \langle M_N(\tau), \varphi \rangle(\tau)] = \frac{C_{\text{cut}}}{\Gamma(\beta)} \sum_{k,i} \frac{\theta_k}{\Gamma(\beta)^2} \int^t_0 (t - r)^{\beta-1} \langle u_N(r), \sigma_{k,i} \cdot \Pi_N \nabla \varphi \rangle \, dW_r^{k,i}$$

Note that for any $p > 1, \gamma > 1 + p$, using the Jensen inequality and Lemma \[ 2.10 \].
we can claim that
\[
\mathbb{E} \left[ \| M_N(t) - M_N(\tau) \|_{H^{-\gamma}}^{2p} \right] \lesssim \sum_k \left( 1 + |k|^2 \right)^{-\gamma} \mathbb{E} \left[ \| (M_N(t) - M_N(\tau), e_k) \|_{L^2}^{2p} \right]
\]
\[
\lesssim \| \theta \|_{L^\infty}^{2p} \left( 1 + \| u_0 \|_{L^2}^2 \right)^{p} |t - \tau|^{p(2\beta - 1)} \sum_k \left( 1 + |k|^2 \right)^{-\gamma} |k|^{2p}
\]
\[
\lesssim \| \theta \|_{L^\infty}^{2p} \left( 1 + \| u_0 \|_{L^2}^2 \right)^{p} |t - \tau|^{p(2\beta - 1)}.
\]
where \( e_k(x) = e^{2\pi ik \cdot x} \) with the property \( \| \Pi_N \nabla e_k \|_{L^\infty} \lesssim |k| \). Since \( \gamma > 1 + p \), then \( \sum_k \left( 1 + |k|^2 \right)^{-\gamma} |k|^{2p} \) is convergent. Combining Lemma 2.9 and (5.15), it follows that
\[
\mathbb{E} \left[ \left( \sup_{0 \leq s < t \leq T} \frac{\| M(t) - M(s) \|_{H^{-\gamma}}}{|t - s|^\gamma} \right)^{2p} \right] \leq C_2 \| \theta \|_{L^\infty}^{2p} \left( 1 + \| u_0 \|_{L^2}^2 \right)^{p},
\]
since the parameters \( \kappa < \frac{(2\beta - 1)p - 1}{2p} \).

Next, we assume \( u_N = u_N(0) + G_N + M_N \), where
\[
D_t^p G_N = -\Lambda^2 u_N(t) + \nu \Delta u_N(t) + L_R(u_N(t)) F(u_N(t))
\]
From the above conclusions, it is clear that
\[
\| G_N \|_{C^{1/2} H^{-\gamma}} \leq \| G_N \|_{W^{1,2} H^{-\gamma}} \lesssim (1 + \| u_N \|_{L^2}^2) (1 + \| u_N \|_{L^2 H^{\gamma}}^2).
\]
In summary, combining with (5.11), there exists \( p > 1, \alpha, \gamma > 0 \) such that
\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \left( \| u_N \|_{L^2 H^\gamma} + \| u_N \|_{L^\infty L^2} + \| u_N \|_{C^{1/2} H^{-\alpha}} \right)^p \right] < \infty
\]
Denote \( \mu_N \) as the law of \( u_N \), according to Lemma 5.2 and Prokhorov’s theorem (see 5.2), we deduce \( \{ \mu_N \}_N \) is tight in \( \mathcal{X} \). Combing with 5.14, it is possible that exists sufficiently large \( L \), such that the laws \( \{ \mu_N \}_N \) are all supported and tight on \( \mathcal{X}_L \).

Set \( W = (W^k, i)_{k,i} = \left( W_t^k, i \right)_{0 \leq t \leq T} : k \in \mathbb{Z}^d, i = 1, \ldots, d - 1 \), denote the joint law of \( (u_N, W) \) by \( P_N \), the sequence \( \{ P_N \}_N \) is tight in \( \mathcal{X}_L \times \mathcal{L} = \mathcal{X}_L \times C \left( [0, T]; \mathbb{R}^2 \right) \). By Skorokhod’s representation theorem (see 40) it is known that there exists another probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{F}_t), \tilde{\mathbb{P}}) \) and a sequence \( \{ (\tilde{u}_N, \tilde{W}_N) \}_N \), which is distributed according to \( P_N \) and satisfies \( (\tilde{u}_N, \tilde{W}_N) \to (\tilde{u}, \tilde{W}), \tilde{\mathbb{P}} - a.s. \). Therefore \( \tilde{u}_N = u_N(0) + \tilde{G}_N + \tilde{M}_N \) solves the \( N \)-step SDE associated to \( \tilde{W}_N \) and satisfies the same prior estimates as \( u_N, M_N \). Note that the \( \tilde{\mathbb{P}} - a.s. \) convergence, it is obvious that \( \tilde{u} \) is the solution of (5.12) for taking Brownian motion as \( \tilde{W} \). By the similar steps as above, it can be seen that \( u_N \) and \( M_N \) also satisfy inequality (5.11) and 5.12.
Suppose that \( u_1 \) and \( u_2 \) are two solutions defined on the same probability space with the same initial \( u_0 \in L^2 \), corresponding to the same Brownian motion and satisfying (5.11). Thus for any \( \varphi \in H^s (\mathbb{T}^d) \), denote \( \tilde{u} = u_1 - u_2 = G + M \) as the difference between the two solutions, observe that

\[
\langle \tilde{u} \rangle, \varphi \rangle = -\frac{1}{4\beta^2} \int_0^1 (t - \tau) \beta^{-1} \langle \Lambda^\alpha \tilde{u} \rangle, \Lambda^\alpha \varphi \rangle d\tau + \frac{1}{4\beta^2} \int_0^1 (t - \tau) \beta^{-1} \langle \nabla \tilde{u} \rangle, \nabla \varphi \rangle d\tau
\]

\[
+ \frac{1}{4\beta^2} \int_0^1 (t - \tau) \beta^{-1} \langle L_R (u_1 (\tau)) F (u_1 (\tau)) - L_R (u_2 (\tau)) F (u_2 (t)) \rangle, \varphi \rangle d\tau
\]

\[
- \sqrt{\frac{\alpha \beta}{4\beta^2}} \sum_{k,i} \int_0^1 \theta_k (t - \tau) \beta^{-1} \langle \tilde{u} \rangle, \sigma_{k,i} \cdot \nabla \varphi \rangle dW_{t,i}.
\]

Similar to (5.13), it holds

\[
D_{t}^{\frac{\beta}{2}} \| \tilde{u} (t) \|_{L^2}^2 \leq -2 \| \Lambda^s \tilde{u} (t) \|_{L^2}^2 + 2 \langle L_R (u_1 (t)) F (u_1 (t)) - L_R (u_2 (t)) F (u_2 (t)) , \tilde{u} (t) \rangle
\]

(5.17)

Next, we estimate the second term of (5.17). Set

\[
2 | L_R (u_1 (t)) - L_R (u_2 (t)) | | \langle F (u_1 (t)) , \tilde{u} (t) \rangle | + L_R (u_2 (t)) | | \langle F (u_1 (t)) - F (u_2 (t)) , \tilde{u} (t) \rangle | =: I_1 + I_2
\]

Firstly, we consider \( I_1 \). By the definition of cut-off function \( L_R \) (5.1) and Lagrange mean value theorem, we obtain that

\[
| L_R (u_1 (t)) - L_R (u_2 (t)) | \leq \| L_R \|_{\infty} \| u_1 (t) \|_{H^{s-\gamma}} - \| u_1 (t) \|_{H^{s-\gamma}} \leq \| u_1 (t) - u_2 (t) \|_{H^{s-\gamma}} \leq \| \tilde{u} (t) \|_{L^2}
\]

Note that, under the assumption (H1), satisfying

\[
| \langle F (u_1 (t)) , \tilde{u} (t) \rangle | \leq (1 + \| u_1 (t) \|_{H^s}) (1 + \| u_1 (t) \|_{H^s}) \| \tilde{u} (t) \|_{H^s},
\]

the solutions \( u_1, u_2 \) both satisfy (5.11), which implies the above inequality can be simplified as below:

\[
| \langle F (u_1 (t)) , \tilde{u} (t) \rangle | \leq (1 + \| u_1 (t) \|_{H^s}) \| \tilde{u} (t) \|_{H^s}.
\]

Combining the above estimates, applying the Cauchy inequality and the inequality that (4.12)

\[
\| \tilde{u} (t) \|_{H^s}^2 \leq 2^{s-1} \left( \| \tilde{u} (t) \|_{L^2}^2 + \| \Lambda^s \tilde{u} (t) \|_{L^2}^2 \right),
\]

we can obtain

\[
I_1 = 2 | L_R (u_1 (t)) - L_R (u_2 (t)) | | \langle F (u_1 (t)) , \tilde{u} (t) \rangle |
\]

\[
\leq (1 + \| u_1 (t) \|_{H^s}) \| \tilde{u} (t) \|_{H^s} \| \tilde{u} (t) \|_{L^2}
\]

\[
\leq K \left( 1 + \| u_1 (t) \|_{H^s}^2 \right) \| \tilde{u} (t) \|_{L^2}^2 + \frac{1}{2} \| \tilde{u} (t) \|_{L^2}^2
\]

\[
\leq K \left( 1 + \| u_1 (t) \|_{H^s}^2 \right) \| \tilde{u} (t) \|_{L^2}^2 + \frac{1}{2} \| \Lambda^s \tilde{u} (t) \|_{L^2}^2.
\]

Next, we consider \( I_2 \). From assumption (H3) and the uniform bound (5.11), it holds

\[
I_2 = L_R (u_2 (t)) | | \langle (F (u_1 (t)) - F (u_2 (t)) , u_1 (t) - u_2 (t) \rangle |
\]

\[
\leq \| \tilde{u} (t) \|_{H^s}^{3s} \| \tilde{u} (t) \|_{L^2}^{3s} \left( 1 + \| u_1 (t) \|_{H^s}^{3s} \| u_2 (t) \|_{H^s}^{3s} \right)
\]

\[
\leq \frac{1}{2} \| \tilde{u} (t) \|_{H^s}^{3s} + K \| \tilde{u} (t) \|_{H^s}^{3s} \left( 1 + \| u_1 (t) \|_{H^s}^{3s} \| u_2 (t) \|_{H^s}^{3s} \right)\]
Since $\gamma_3 < 2$, Recall the requirement of parameters in assumption (H3), we obtain that $2\eta/\gamma_3 > 2$. Thus, it is clear that $u(t)$ satisfies the bound $\mathcal{H}(t)$, we state the above inequality can be rewritten as

$$I_2 \leq \frac{1}{2} \|\Lambda u(t)\|_{L^2}^2 + K \|u(t)\|_{H^s}^2 \left(1 + \|u_1(t)\|_{H^r}^2 + \|u_2(t)\|_{H^r}^2\right)$$

(5.19)

Following the results of (5.18) and (5.19), yields

$$D_t^\beta \|u(t)\|_{L^2}^2 \leq \left(1 + \|u_1(t)\|_{H^r}^2 + \|u_2(t)\|_{H^r}^2\right) \|u(t)\|_{L^2}^2,$$

Recall that $\|u(0)\|_{L^2} = \|u_1(0) - u_2(0)\|_{L^2} = 0$, as an application of Gronwall inequality, see Lemma 2.11 it holds $\|u(t)\|_{L^2} = 0$, which implies the uniqueness of the solution. The proof is complete.

**Proposition 5.2.** Set an sequence $\{u^S_N\}_{S \geq 1} \subset L^2 (\mathbb{T}^d)$, which is weakly converging to $u_0$ in $L^2 (\mathbb{T}^d)$, and consider a family of symmetric coefficients $\{\theta^S\}_{S \geq 1} \subset \ell^2$, satisfying

$$\|\theta^S\|_{\ell^2} = 1 \quad \forall S \in \mathbb{R}, \quad \lim_{S \to \infty} \|\theta^S\|_{\ell^\infty} = 0.$$  

(5.20)

Denote by $u^S$ the unique solution of (5.2) whose has initial data $u^N_0$ and is related to the parameter $\theta^S$, where $S \geq 1$. Then for every $\delta > 0, \rho \geq 2$, in the topology

$$\mathcal{X} = L^2 \left(0, T; H^s - \delta\right) \cap L^\rho \left(0, T; L^2\right) \cap C \left([0, T]; H^\delta\right),$$

$u^S$ converges to $u$ in probability, where $u$ is the unique solution of Eq. (5.21)

$$\left\{\begin{array}{l}
D_t^\beta u = -(-\Delta)^s u + \nu \Delta u + L_R(u) F(u), \\
u |_{t=0} = u_0.
\end{array}\right.$$  

(5.21)

**Proof** The elements of $L^2 (\mathbb{T}^d)$ are bounded, obviously, $\{u^N_0\}_{N \geq 1}$ is weakly convergent and bounded in $L^2 (\mathbb{T}^d)$. It is known from the boundedness of (5.11), there exists a deterministic constant $N$, such that

$$\sup_{N \in \mathbb{N}} \left(\sup_{t \in [0, T]} \|u^N(t)\|_{L^2}^2 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} \|\Lambda u(\tau)\|_{L^2}^2 d\tau\right) \leq N.$$  

(5.22)

Recalling Lemma 5.1, the solution $u^S$ can be decomposed as $u^S = u_0 + G^S + M^S$, where $G^S$ satisfies

$$\|G^S\|_{C_{1/2} H^{-\delta}} \leq \|G^S\|_{W_{1/2} H^{-\delta}}.$$

Therefore, synthesizing the conclusions from the above, we state that exists parameters $q > 1, \alpha, \gamma > 0$, satisfying

$$\sup_{t \in [0, T]} \mathbb{E} \left[\left(\|u^S\|_{L^2}^q + \|u^S\|_{L^\infty}^q + \|u^S\|_{C^\gamma H^{-\alpha}}^q\right)^q\right] < \infty.$$  

28
In the following claims, $\mu^S$ represents the law of $u^S$. By Lemma 5.9 the sequence $\{\mu^S\}_{S \geq 1}$ is tight in $\mathcal{X}$. Then we choose $L$, which is large enough, such that $\{\mu^S\}_{S \geq 1}$ is also tight $\mathcal{X}_L$. By the Prokhorov’s theorem (see [55]), there exists a subsequence $\{\mu^{S_k}\}_{S \geq 1}$ that $\mu^{S_k}$ weakly converges to a probability measure $\mu$ in $\mathcal{X}_L$. Next, we establish an operator $B^\varphi$. For any $\varphi \in C^\infty(\mathbb{T}^d)$, the definition of the mapping $B^\varphi: \mathcal{X}_L \to C([0, T]; \mathbb{R})$ is shown as follows:

$$(B^\varphi f)(t) := \langle f(t), \varphi \rangle - \langle u_0, \varphi \rangle - \frac{1}{\Gamma(d)} \int_0^t (t - \tau)^{d-1} L_R (f(\tau)) \langle F(f(\tau)), \varphi \rangle d\tau + \frac{1}{\Gamma(d)} \int_0^t (t - \tau)^{d-1} \langle \nabla^2 \varphi(\tau), d\tau - v \frac{1}{\Gamma(d)} \int_0^t (t - \tau)^{d-1} \langle f(\tau), \Delta \varphi(\tau) \rangle d\tau \nonumber$$

From the conclusion of Lemma 5.9, it follows that the mapping $B^\varphi$ is continuous on $\mathcal{X}_L$. To prove the conclusion, we introduce a pushforward measure $B^\varphi \# \mu$. Assume $B^\varphi \# \mu$ is the pushforward measure of $\mu$ under $B^\varphi$, denotes as $B^\varphi \# \mu = \mu(B^\varphi)^{-1}$. Constructing an operation

$$B^\varphi u^S = \langle u^S_0 - u_0, \varphi \rangle + \langle M^S, \varphi \rangle,$$

from that we can obtain $B^\varphi \# \mu^S$ is the law of $B^\varphi u^S$. Note that $u^S_0$ weakly converges to $u^S_0$ when $S \to \infty$, hence combing result 5.12. it holds

$$E \left[\sup_{t \in [0, T]} \left|\langle M^S(t), \varphi \rangle\right|^p\right] \lesssim \|\varphi\|_{H^s}^p E \left[\|M^S\|_{C^\gamma H^{-\alpha}}^p\right] \lesssim \|\theta^S\|_{L^\infty}^p \to 0.$$

We claim that for any fixed $\varphi \in C^\infty(\mathbb{T}^d)$, the support of $B^\varphi \# \mu$ is set $\{0\}$. Then, exact a countable set $\{\varphi_n\}_n$ from $C^\infty(\mathbb{T}^d)$, which dense in $H^s$, such that

$$\mu(\{f \in \mathcal{X}_L : B^\varphi f = 0, \forall n \in \mathbb{N}\}) = 0,$$

By using a density argument, it holds

$$\mu(\{f \in \mathcal{X}_L : f \text { solves } (5.21)\}) = 1,$$

which implies that $\mu = \delta_u$, where $u$ is the unique solution to equation (5.21), namely $\mu^S$ covers weakly to $\delta_u$, the conclusion is proved. Since this reasoning applies to any weakly convergent subsequence, we deduce that the whole sequence converges to $\mu = \delta_u$, and also converges in a probabilistic sense.

From now on, the proof of Theorem 3.1. and Theorem 3.2. will be given. **Proof of Theorem 3.1.** From now on, we fix the parameters $\varepsilon, T > 0$, it can be obtained from hypothesis (H4) that there exists $v, R > 0$, it holds

$$\sup_{u_0 \in \mathcal{M}} \sup_{t \in [0, T]} \|u(t; u_0, v)\|_{L^2} \leq R - 1, \quad (5.23)$$

for any $\theta \in \ell^2, \|\theta\|_{\ell^2} = 1$. In the following, we use $u^R(\cdot, u_0, v, \theta)$ to represent the solution of the equation (5.21) with initial data $u_0$. Similarly, denote the solution of the Eqs. (5.21) by $u^R(\cdot, u_0, v)$. Namely, for fixed $R$,

$$u^R(\cdot, u_0, v) = u(\cdot, u_0, v), \quad u^R\left(\cdot, u_0^j, v\right) = u\left(\cdot, u_0^j, v\right) \quad \forall j \in \mathbb{N}.$$
Proof by contradiction, we state that if the sequence \( \{\theta^S\} \subset \ell^2 \) satisfies Proposition \(5.2\), then

\[
\lim_{N \to \infty} \sup_{u_0 \in K} \mathbb{P} \left( \sup_{t \in [0,T]} \| u^R(t; u_0, v) - u(t; u_0, v) \|_{H^{-\gamma}} > 1 \right) = 0. \tag{5.24}
\]

Assuming that there exists \( \zeta > 0 \), \( \{S_j\}_{j} \subset \mathbb{N} \) and \( \{u^j_0\}_{j} \subset \mathcal{M} \), such that

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \left\| u^R(t; u^j_0, \theta^{S_j}, v) - u(t; u^j_0, v) \right\|_{H^{-\gamma}} > \zeta > 0 \right) > 0 \tag{5.25}
\]

Note that \( \mathcal{M} \) is a set of bounded, convex and closed functions, it is weakly tight in \( L^2(\mathbb{T}^d) \). Using the Eblein-Smulian theorem \( (50) \) and the reflexivity of the space \( L^p(1 < p < \infty) \) \( (57) \), it follows that there exists a subsequence \( \{u^{j^k}_0\}_{j^k} \) of \( \{u^j_0\}_{j} \) converging to \( u_0 \) in \( \mathcal{M} \), and \( u_0 \subset \mathcal{M} \). Following the Proposition \(5.2\) it is clear that, as \( k \to \infty \), \( u^R(\cdot; u^{j_k}_0, \theta^{S_j}, v) \) converges to \( u^R(\cdot; u^{j^k}_0, v) \) in \( C([0,T]; H^{-\gamma}) \) in probability. Recall that \( \{u^{j^k}_0\}_{j^k} \) weakly converges to \( u_0 \), so we state that \( u^R(\cdot; u^{j^k}_0, v) \) converges to \( u^R(\cdot; u_0, v) \) in \( C([0,T]; H^{-\gamma}) \) in probability, i.e.,

\[
\lim_{k \to \infty} \left[ u^R(t; u^{j^k}_0, \theta^{S_j}, v) - u(t; u^{j^k}_0, v) \right] = 0
\]

which contradicts with \( (5.25) \), then \( (5.24) \) holds.

Observe that

\[
\lim_{S \to \infty} \sup_{u_0 \in \mathcal{M}} \mathbb{P} \left( \sup_{t \in [0,T]} \left\| u^R(t; u_0, \theta^{S}, v) \right\|_{H^{-\gamma}} > R \right) \\
\leq \lim_{S \to \infty} \sup_{u_0 \in \mathcal{M}} \mathbb{P} \left( \sup_{t \in [0,T]} \left\| u^R(t; u_0, \theta^{S}, v) - u(t; u_0, v) \right\|_{H^{-\gamma}} > 1 \right) = 0
\]

which can be obtained from \( (5.23) \) and \( (5.24) \). Consequently, as \( u_0 \in \mathcal{M} \), it is easy to see that for fixed \( T \), there exists a constant \( S \) big enough, it holds

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \left\| u^R(t; u_0, \theta^{S}, v) \right\|_{H^{-\gamma}} \leq R \right) > 1 - \varepsilon,
\]

namely,

\[
\mathbb{P} \left( L_R^C \left( u^R(t; u_0, \theta^{S}, v) \right) = 1, \forall t \in [0,T] \right) > 1 - \varepsilon.
\]

For any \( t \in [0,T] \), Eq. \( (3.2) \) is equivalent to equation \( (5.2) \) while \( L_R \left( u^R(t; u_0, \theta^N, v) \right) = 1 \). Namely, the above result implies that the life span of solutions to \( (1.1) \) with initial data \( u_0 \) is greater than \( T \).
Proof of Theorem 3.2. For $T$ large enough, it follows from Theorem 3.1 that there exists $\theta \in \ell^2$, such that for all $u_0 \in \mathcal{M}$, the lifetime of the maximal solution of equation (3.2) is greater than $T$. Similar to the proof of Theorem 3.1, it is obvious that $u (\cdot; u_0, v, \theta)$ converges to $u (\cdot; u_0, v)$ in $L^2 (0, T; L^2)$.

Following condition (a), we state that for $t = t (\omega) \in [T - 1, T]$, it is possible to take $\| u (t; u_0, v, \theta) \|_{L^2}$ small enough, it also means that for any $\varepsilon > 0$, holding

$$\| u (t; u_0, v) \|_{L^2 (T-1, T; L^2)} \leq \varepsilon.$$ 

Hence, we have

$$\| u (t; u_0, v) \|_{L^2 (T-1, T; L^2)} \leq \| u (t; u_0, v, \theta) - u (t; u_0, v) \|_{L^2 (T-1, T; L^2)} + \| u (t; u_0, v) \|_{L^2 (T-1, T; L^2)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Redefining $u (t (\omega); u_0, v, \theta)$ as the initial data $u_0$, i.e., as the small initial data of the stochastic equation (3.2), the conclusion holds under the condition (b).

**Future work**

In this paper, we focus on the blow-up time of the fractional nonlinear stochastic partial differential equation, which has the form

$$\begin{cases}
D^\beta_t u = \left[ -(-\Delta)^s u + v u + F (u) \right] dt + \sqrt{C_d} v \sum_{k,i} \theta_k \sigma_{k,i} \nabla u (t) W^{k,i}_t, \\
u |_{t=0} = u_0.
\end{cases}$$

In the subsequent study, we will consider the explosion of the fractional nonlinear stochastic partial differential equation of the form

$$\begin{cases}
D^\beta_t u = \left[ -(-\Delta)^s u + v u + F (u) \right] dt + \sqrt{C_d} v \int_0^t \sum_{k,i} \theta_k \sigma_{k,i} \nabla u (\tau) D^\beta_{\tau} W^{k,i}_\tau, \\
u |_{t=0} = u_0,
\end{cases}$$
in $d \geq 2$ dimensions.

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**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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36