1. Introduction

As one of the basic relationships in ecological relationships, competition has been extensively studied. For competition models governed by differential equations of Lotka-Volterra type, we refer to [1, 2] and the references therein. Dispersal of species between patches is very common in nature. For example, many zooplankton species move downward into the darkness to reduce the predation risk by fish, while at night time, they move upward to consume the phytoplankton [3]. It is of great importance to examine how the dispersal affects local and global dynamics of the resulting metapopulations. To this purpose, many mathematical models incorporating the dispersal of species over patches have been proposed and studied. For instance, the movement of a single species has been considered in [4–9], and models allowing the dispersal of two species have appeared in [10–13]. See also [14–16] for studies on models with competition and predator-prey interactions in patchy environments.

We note that, in the above-mentioned work, dispersal is often assumed to be instantaneous. However, in reality, it always takes time for species dispersing from one patch to another. As a result, dispersal delays do exist and should be incorporated into modeling the dispersal process. With this in mind, recently, Zhang et al. considered a two-patch predator-prey model with delayed dispersal of prey only and showed that such dispersal delay exhibits a stabilizing role on the stability of the coexistence equilibrium [17]. Later, Mai et al. generalized the model considered in [17] into the case with an arbitrary number of patches and showed that the dispersal delay can indeed induce stability switches [18]. Sun et al. considered a two-patch predator-prey model with two dispersal delays and showed that the dispersal delays can destabilize and stabilize the coexistence equilibrium [19].

Motivated by above work, in this paper, we consider a two-species competition model and assume both species move randomly between two identical patches. Our objective is to explore the effects of delays on the stability and instability of the symmetric coexistence equilibrium.
short compared to the lifespan of each species, and no mortality will be included in the dispersal. Therefore, our two-patch competition model with delayed dispersal of two competing species are described by the following system:

\[
\begin{align*}
\frac{dX_i(t)}{dt} &= r_i X_i(t) \left[ 1 - \frac{X_i(t)}{K_1} - b_{i2} Y_i(t) \right] + D_1 \left( X_j(t - \eta_1) - X_i(t) \right), \\
\frac{dY_i(t)}{dt} &= r_2 Y_i(t) \left[ 1 - \frac{Y_i(t)}{K_2} - b_{21} X_i(t) \right] + D_2 \left( Y_j(t - \eta_2) - Y_i(t) \right),
\end{align*}
\]

(1)

where \(i, j \in \{1, 2\} \) and \( i \neq j \). \( X_i \) and \( Y_i \) denote the densities of two competing species \( X \) and \( Y \) in patch \( i \) \((i = 1, 2)\), respectively. \( r_i \) and \( r_2 \) are the intrinsic growth rates of species \( X \) and \( Y \), respectively. \( K_1 \) and \( K_2 \) are the carrying capacities of species \( X \) and \( Y \), respectively. \( b_{i2} \) and \( b_{21} \) measure the competition strengths of \( Y \) on \( X \) and \( X \) on \( Y \). \( D_1 \) and \( D_2 \) are the dispersal rates of species \( X \) and \( Y \). \( \eta_1 \) and \( \eta_2 \) denote the dispersal periods of species \( X \) and \( Y \) from one patch to the other, respectively. All parameters are assumed to be positive.

By rescaling \( x_i = X_i/K_i \), \( y_i = Y_i/K_i \), \( s = r_i t \), \( \tau_i = r_1 \eta_i \), \( \rho = r_2/r_1 \), \( a = b_{21}(K_2/K_1) \), \( b = b_{i2}(K_2/K_1) \), and \( D_j = r_1 d_j \), we obtain the dimensionless system (1)

\[
\begin{align*}
\frac{dx_i(s)}{ds} &= x_i(s) \left[ 1 - x_i(s) - ay_i(s) \right] \\
&\quad + d_1 \left( x_j(s - \tau_1) - x_i(s) \right), \\
\frac{dy_i(s)}{ds} &= \rho y_i(s) \left[ 1 - y_i(s) - bx_i(s) \right] \\
&\quad + d_2 \left( y_j(s - \tau_2) - y_i(s) \right),
\end{align*}
\]

(2)

where \(i, j \in \{1, 2\} \) and \( i \neq j \).

For the case with no dispersal (i.e., \( d_j = 0 \)), the dynamics of the competition model in a single patch is given below (See [2]).

**Theorem 1.** Consider a single patch competition model governed by

\[
\begin{align*}
\frac{dx(s)}{ds} &= x(s) \left[ 1 - x(s) - ay(s) \right], \\
\frac{dy(s)}{ds} &= \rho y(s) \left[ 1 - y(s) - bx(s) \right].
\end{align*}
\]

(3)

The following conclusions hold:

1. There are three trivial equilibria: \((0, 0)\), \((1, 0)\), and \((0, 1)\).
2. The coexistence equilibrium \((x^*, y^*)\) exists if and only if \( a > 1, b > 1 \) or \( a < 1, b < 1 \) and \( x^* = (1-a)/(1-ab) \) and \( y^* = (1-b)/(1-ab) \).
3. If \( a > 1 \) and \( b > 1 \), then the two boundary equilibria \((1, 0)\) and \((0, 1)\) are both stable.
4. The coexistence equilibrium \((x^*, y^*)\) is stable if \( a < 1 \) and \( b < 1 \) and \( a + b > 1 \).
5. The boundary equilibrium \((1, 0)\) is stable if \( b > 1 \) and \( b < 1 \).
6. The boundary equilibrium \((0, 1)\) is stable if \( a > 1 \) and \( a < 1 \).

3. Stability Analysis of the Symmetric Equilibria

Since the two patches under consideration are assumed to be identical, system (2) admits four symmetric equilibria as follows:

\[
\begin{align*}
&\text{Extinction of both species in each patch; } E_0 = (0, 0, 0, 0), \\
&\text{Extinction of species } Y \text{ and persistence of species } X \text{ in each patch; } E_1 = (1, 0, 1, 0), \\
&\text{Extinction of species } X \text{ and persistence of species } Y \text{ in each patch; } E_2 = (0, 1, 0, 1), \\
&\text{Coexistence of both species in each patch; } E^* = (x^*, y^*, x^*, y^*).\end{align*}
\]

Linearizing system (2) at a symmetric equilibrium \((x, y, x, y)\), we obtain the associated characteristic equation given by \( \det \Delta = 0 \) with

\[
\Delta = \begin{bmatrix} I_1 & J_2 \\ J_2 & I_1 \end{bmatrix},
\]

(4)

where \( I_1 = \begin{bmatrix} 1 - ax - \lambda & -\rho y \\ -b\rho y & 1 - bx - \lambda \end{bmatrix} \) and \( J_2 = \begin{bmatrix} d_1 e^{-\lambda \tau_1} & 0 \\ 0 & d_2 e^{-\lambda \tau_2} \end{bmatrix} \).

The symmetric equilibrium \((x, y, x, y)\) is locally asymptotically stable if all characteristic roots have negative real parts [20]. Next we will study the stability of all 4 symmetric equilibria by analyzing the associated characteristic equation \( \det \Delta = 0 \).

3.1. The Trivial Equilibrium \( E_0 \). Note that \( \det \Delta = \det(I_1 + J_2) \cdot \det(I_1 - J_2) \). Thus, at the trivial equilibrium \( E_0 \), the characteristic equations consist of the following 4 transcendental equations:

\[
\begin{align*}
\lambda - 1 + d_1 + d_1 e^{-\lambda \tau_1} &= 0, \\
\lambda - 1 + d_1 - d_1 e^{-\lambda \tau_1} &= 0, \\
\lambda - \rho + d_2 + d_2 e^{-\lambda \tau_2} &= 0, \\
\lambda - \rho + d_2 - d_2 e^{-\lambda \tau_2} &= 0.
\end{align*}
\]

(5) (6) (7) (8)

If \( \tau_1 = \tau_2 = 0 \), then the characteristic roots are \( 1, 1 - 2d_1, \rho, \rho - 2d_2 \). Clearly, there exist at least two positive characteristic roots \( 1 \) and \( \rho \). Thus \( E_0 \) is unstable.

According to the results of [21–23], the change of stability at the equilibrium can happen only if characteristic roots appear on or cross the imaginary axis as \( \tau \) increases. We assume that \( d_1 \neq 1/2 \) and \( d_2 \neq \rho/2 \) to ensure that zero is not a root of (5) and (7), respectively.

The following lemma will be used later in verifying the transversality condition.
Lemma 2. Suppose, at certain \( \tau \), the characteristic equation
\[
\lambda + p + q e^{-\lambda \tau} = 0
\] (9)
admits a pair of purely imaginary roots \( \pm i \omega \) \((\omega > 0)\). Then
\[ (d \text{Re}(\lambda)/d\tau)_{\lambda=i\omega} > 0. \]

Proof. For (9), it is easy to get \( d\lambda/d\tau = q\lambda(e^{\lambda \tau} - q\tau) \). Thus
\[
\text{sign} \left( \frac{d \text{Re} \lambda}{d\tau} \right)_{\lambda=i\omega} = \text{sign} \left( \frac{\omega^2}{(\cos \omega \tau - q\tau)^2 + \sin^2 \omega \tau} \right) = \text{sign} \left( \omega^2 \right). \] (10)
Consequently, \((d \text{Re}(\lambda)/d\tau)_{\lambda=i\omega} > 0.\) The proof is complete. \( \square \)

Next we look for a pair of purely imaginary roots of characteristic equations (5)-(8) by setting \( \lambda = i \omega \) with \( \omega > 0 \). Substituting \( \lambda = i \omega \) into (5)-(6), we obtain
\[
\omega_01 = \sqrt{d_1^2 - (1 - d_1)^2}. \] (11)
Thus if \( 0 < d_1 < 1/2 \), then (5)-(6) admit no purely imaginary roots; while if \( d_1 > 1/2 \), there is a pair of purely imaginary roots \( \pm i \omega_{01} \) for (5) and also for (6). Further we have \((d \text{Re}(\lambda)/d\tau_1)_{\lambda=i\omega_{01}} > 0\) by Lemma 2. This implies that if \( d_1 > 1/2 \), the characteristic roots of (5) and (6) cross the imaginary axis through \( \pm i \omega_{01} \) at \( \tau = \tau_1 \) from left to right and the number of characteristic roots with positive real parts is increased by 2, as \( \tau_1 \) crosses the critical value.

Similarly, for (7) and (8), we can show that there are no imaginary roots if \( 0 < d_2 < \rho/2 \); while if \( d_2 > \rho/2 \), then there exists a pair of purely imaginary roots \( \pm i \omega_{02} \) with \((d \text{Re}(\lambda)/d\tau_2)_{\lambda=i\omega_{02}} > 0\). Note that \( E_0 \) is unstable for \( \tau_1 = \tau_2 = 0 \). Thus the above analysis shows that the trivial equilibrium \( E_0 \) remains unstable for \( \tau_1 > 0 \) and \( \tau_2 > 0 \) and hence we have the following result on the stability of the trivial equilibrium \( E_0 \).

Theorem 3. Consider system (2). The trivial equilibrium \( E_0 \) is unstable for \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \).

3.2. The Boundary Equilibrium \( E_1 \). At the boundary equilibrium \( E_1 \), the characteristic equations read as
\[
\lambda + 1 + d_1 + d_1 e^{-\lambda \tau_1} = 0 \] (12)
\[
\lambda + 1 + d_1 - d_1 e^{-\lambda \tau_1} = 0 \] (13)
\[
\lambda - \rho (1 - b) + d_2 + d_2 e^{-\lambda \tau_2} = 0 \] (14)
\[
\lambda - \rho (1 - b) + d_2 - d_2 e^{-\lambda \tau_2} = 0 \] (15)

When \( \tau_i = 0, (i = 1, 2) \), the characteristic roots of equations (12)-(15) are \(-1, -1 - 2d_1, \rho(1-b), \rho(1-b)/2 - 2d_2 \). Here we have two cases to consider: Case (1): \( b > 1 \), all roots are negative, and the equilibrium \( E_1 \) is locally asymptotically stable; and Case (2): \( 0 < b < 1 \), there are at least one positive root, and the equilibrium \( E_1 \) is unstable.

We also assume that \( d_2 \neq \rho(1-b)/2 \) to ensure that 0 is not a characteristic root of (14). Substituting \( \lambda = i \omega \) with \( \omega > 0 \) into (12)-(15), we find that if (12) or (13) admits purely imaginary roots \( \pm i \omega_{11} \), then \( \omega_{11} = \sqrt{d_1^2 - (1 + d_1)^2} \); and if (14) or (15) admits purely imaginary roots \( \pm i \omega_{12} \), then \( \omega_{12} = \sqrt{d_2^2 - (d_2 - \rho(1-b))^2} \). However, \( \omega_{11} \) does not exist since \( d_1 > 0 \). If \( b > 1 \), then \( \omega_{12} \) can never exist; while if \( 0 < b < 1 \) and \( 0 < d_2 < \rho(1-b)/2 \), then \( \omega_{12} \) can never exist either; only if \( 0 < b < 1 \) and \( d_1 > \rho(1-b)/2 \), there are purely imaginary roots for (14) and (15). However, it follows from Lemma 2 that \((d \text{Re}(\lambda)/d\tau_1)_{\lambda=i\omega_{12}} > 0.\) Consequently, the boundary equilibrium \( E_1 \) remains unstable for all \( \tau_1 > 0 \) and \( \tau_2 > 0 \).

Summarizing the above analysis, we have the following result.

Theorem 4. For the boundary equilibrium \( E_1 \) of system (2), we have the following conclusions:
(i) If \( b > 1 \), then \( E_1 \) is locally asymptotically stable for \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \).
(ii) If \( 0 < b < 1 \), then \( E_1 \) is unstable for \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \).

3.3. The Boundary Equilibrium \( E_2 \). The stability of the boundary equilibrium \( E_2 \) can be dealt with similarly as that of \( E_1 \) and we have the following result.

Theorem 5. For the boundary equilibrium \( E_2 \) of system (2), we have the following conclusions:
(I) If \( a > 1 \), then \( E_2 \) is locally asymptotically stable for \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \).
(II) If \( 0 < a < 1 \), then \( E_2 \) is unstable for \( \tau_1 \geq 0 \) and \( \tau_2 \geq 0 \).

3.4. The Symmetric Coexistence Equilibrium \( E^* \). In this subsection, we assume that \( a > 1, b > 1 \) or \( a < 1, b < 1 \) to ensure the existence of the symmetric coexistence equilibrium \( E^* \). At the coexistence equilibrium \( E^* \), the associated characteristic equations reduce to
\[
\lambda^2 - \lambda (A + B - d_1 - d_2) - \lambda \left( d_1 e^{-\lambda \tau_1} + d_2 e^{-\lambda \tau_2} \right)
+ (Bd_1 - d_1 d_2) e^{-\lambda \tau_1} + (Ad_2 - d_1 d_2) e^{-\lambda \tau_2}
+ (1 - a) AB + d_1 d_2 - Bd_1 - Ad_2
+ d_1 d_2 e^{-\lambda (\tau_1 + \tau_2)} = 0,
\]
\[
\lambda^2 - \lambda (A + B - d_1 - d_2) + \lambda \left( d_1 e^{-\lambda \tau_1} + d_2 e^{-\lambda \tau_2} \right)
- (Bd_1 - d_1 d_2) e^{-\lambda \tau_1} - (Ad_2 - d_1 d_2) e^{-\lambda \tau_2}
+ (1 - a) AB + d_1 d_2 - Bd_1 - Ad_2
+ d_1 d_2 e^{-\lambda (\tau_1 + \tau_2)} = 0,
\] (16)
where \( A = (a - 1)/(1 - ab) \) and \( B = (b - 1)/(1 - ab) \). If \( a > 1 \), \( b > 1 \), or \( a < 1, b < 1 \), then \( A < 0 \) and \( B < 0 \).
We will mainly use numerical simulations to explore if there has been some excellent work (see, for example, [24, 25]). We have shown that the resulting two-patch competition model admits 4 symmetric equilibria: the trivial equilibrium $E_0$, two boundary equilibria $E_i$ and $E_j$, and the symmetric coexistence equilibrium $E^*$. For $E_0$, $E_i$, and $E_j$, we have analytically proven that the dispersal delays do not affect their stability and instability. For the symmetric coexistence equilibrium $E^*$, though we were not able to provide analytical analysis due to the complexity arisen from two delays, we have numerically demonstrated that the dispersal delays are also harmless in the sense that they do not affect the stability and instability of the symmetric coexistence equilibrium.

5. Summary

In this paper, we have incorporated dispersal of both species between two patches into a two-species competition model. We have shown that the resulting two-patch competition model admits 4 symmetric equilibria: the trivial equilibrium $E_0$, two boundary equilibria $E_i$ and $E_j$, and the symmetric coexistence equilibrium $E^*$. For $E_0$, $E_i$, and $E_j$, we have analytically proven that the dispersal delays do not affect their stability and instability. For the symmetric coexistence equilibrium $E^*$, though we were not able to provide analytical analysis due to the complexity arisen from two delays, we have numerically demonstrated that the dispersal delays are also harmless in the sense that they do not affect the stability and instability of the symmetric coexistence equilibrium.
Figure 2: Numerical solutions of system (2). The symmetric coexistence equilibrium is unstable with parameters $a = 2$, $b = 4$, and $\rho = 2$. Initial condition is set as $(x_1(0), y_1(0), x_2(0), y_2(0)) = (0.06, 0.44, 0.06, 0.44)$.

Figure 3: Numerical solutions of system (2). The symmetric coexistence equilibrium is unstable with parameters $a = 2$, $b = 4$, and $\rho = 2$. Initial condition is set as $(x_1(0), y_1(0), x_2(0), y_2(0)) = (0.22, 0.42, 0.22, 0.42)$. 
This conclusion differs from that of predator-prey models discussed in [9, 17–19], where the dispersal delays can induce stability switches.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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