Analytical Solution of Ordinary Fractional Differential Equations by Modified Homotopy Perturbation Method and Laplace Transform

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Abstract. In this paper, a modified Laplace transform homotopy perturbation method (MLT-HPM) is used to find the approximate solution of ordinary fractional differential equations (OFDEs). The proposed method is developed to improve the accuracy of the approximate solutions provided by the LT-HPM method. The appropriate initial approximation will be chosen, furthermore, the residual error will be cancelled at several points of the interest interval (RECP). A couple of OFDEs are considered in order to demonstrate the effectiveness of the suggested method. Comparisons are made with the results obtained from LT-HPM, continuous analytical method (CAM), modified homotopy analysis method (MHAM), and modified optimal homotopy analysis method (MOHAM). The resulting solutions require only a first order approximation of MLT-HPM, compared to LT-HPM, which requires more iterations for the same examples studied.

1. Introduction

Fractional differential equations (FDEs) have been studied by many authors in recent years because of their extensive applications in applied mathematics, physics, engineering, economics such as control theory [1], oil industries [2], and other fields. Further information on those applications can be found in [3–7].

There are no exact analytical solutions for most FDEs because most of them reflect real-world problems; therefore, it is difficult to model these problems and find an exact solution for them. Thus, we often find approximate or numerical solutions for these equations.

Recently, several studies focused on the numerical and analytical solutions of FDEs, and some numerical and analytic methods have been developed, such as homotopy perturbation method [8–10], homotopy analysis method [11], Laplace homotopy perturbation method [12], optimal homotopy analysis methods [13], variational iteration method [14], Laplace homotopy analysis method [15], adomian decomposition method [16], and Bessel collocation method [17].

The homotopy perturbation method (HPM) was first introduced by He in 1999 [18] and was developed by him in 2003 [19] and 2006 [20], and the HPM was also studied by many authors to treat nonlinear equations established in various fields [21, 22]. The HPM has some advantages that attracted the attention of the authors, such as; it can
be used to many nonlinear problems, and the HPM availabilities a simple way to ensure the convergence of the solution. Moreover it can be merged with several mathematical methods such as transform methods, numerical methods etc. But in some applications when the series solution is used for the HPM method, it creates a disadvantage which will limit the effectiveness of the method. For instance, we need to suitably select an initial approximation; otherwise infinite iterations are required that will repeat calculations, and calculate unnecessary terms. Therefore, many authors had avoided this disadvantage by integrating it with other methods like the Laplace transform.

In particular, we are interested to solve the Pagley-Torvik equation which is a type of FDE that describes motion of a thin rigid plate immersed in a Newtonian fluid, see [23]. Some numerical and analytical methods have been applied to solve this type of equation such as cubic spline [24], hybridizable discontinuous Galerkin method [25], and Adomian decomposition method [26].

In this paper, we apply the Laplace transform (LT) with homotopy perturbation method (MHPM) to obtain approximate analytical solutions for OFDEs whereby the advantage of this technique is its ability to combine two effective methods; LT and HPM, and we have modified it to obtain approximate analytical solutions for nonlinear OFDEs. This modification was first used in 2017 [27] for solving nonlinear ordinary functional differential equations; now, we will apply the same modification in solving ordinary fractional differential equations, and then we compare the obtained results with those obtained by the LT-HPM.

2. Basic Definitions

In this section, we will provide some of the essential definitions which will be used in this study.

**Definition 1** A real function \( f(r) \), \( r > 0 \), is said to be in the space \( C_\alpha \), \( \alpha \in \mathbb{R} \), if there exists a real number \( p > \alpha \), such that \( f(r) = r^p h_1(r) \), where \( h_1(r) \in C(0,1) \), and it is said to be in space \( C_m^\alpha \) if and only if \( f^{(m)} \in C_\alpha \), \( m \in \mathbb{N} \).

**Definition 2** The Caputo fractional derivatives of order \( \alpha > 0 \) is defined as:

\[
D_\alpha^a f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi)d\xi,
\]

for \( m-1 < \alpha \leq m \), \( m \in \mathbb{N} \), \( t > 0 \), \( f(t) \in C_m^\alpha \).

**Definition 3** The Laplace transform \( \mathcal{L}[f(t)] \) of the Caputo derivative is defined as [28]:

\[
\mathcal{L}[D_\alpha^a f(t)] = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \quad m-1 < \alpha \leq m.
\]

3. Laplace Transform Homotopy Perturbation Method (LT-HPM)

To illustrate how (LH-HPM) works, consider a general nonlinear fractional differential equation in the form:

\[
D_\alpha^a y(t) = f(t) - Ly(t) - Ny(t),
\]

where \( m-1 < \alpha \leq m \), \( m \in \mathbb{N} \), \( t > 0 \), with the following boundary conditions,

\[
\beta(y, \frac{dy}{dt}).
\]
where $L$ is a linear operator, while $N$ is a nonlinear operator, $f(t)$ is a known analytical function and $D^\alpha_t$ denotes the fractional derivative in the Caputo sense. The solution $y(t)$ is assumed to be a known function, $\beta$ is a boundary operator.

At first, we will construct a homotopy as:

$$ (1 - p) [D^\alpha_t y(t) - D^\alpha_t y_0(t)] + p [D^\alpha_t y_0(t) - f(t) + Ly(t) + Ny(t)] = 0, \quad (5) $$

or

$$ D^\alpha_t y(t) = D^\alpha_t y_0(t) + p [-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t)], \quad p \in [0, 1], \quad (6) $$

where $p$ is a homotopy parameter, and $p \in [0, 1]$, $y_0$ is the first approximation for the solution of (3) that satisfies the boundary condition.

Now, we will apply the Laplace transform to both sides of equation (6), we obtain:

$$ \mathcal{L} [D^\alpha_t y(t)] = \mathcal{L} [D^\alpha_t y_0(t) + p(-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t))]. \quad (7) $$

Applying the formula for the Laplace transform, we get:

$$ s^\alpha \mathcal{L}[y(t)] - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) = \mathcal{L} [D^\alpha_t y_0(t) + p(-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t))], \quad (8) $$

where $m - 1 < \alpha \leq m$, or

$$ \mathcal{L}[y(t)] = \frac{1}{s^\alpha} \left[ \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \right] + \frac{1}{s^\alpha} \mathcal{L} [D^\alpha_t y_0(t) + p(-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t))]. \quad (9) $$

Using the inverse Laplace transform on both sides of (9), we get:

$$ y(t) = \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \left( \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \right) + \frac{1}{s^\alpha} \mathcal{L}[D^\alpha_t y_0(t) + p(-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t))] \right]. \quad (10) $$

We can write the solution of (10) as a power series of $p$ as follows:

$$ y(t) = \sum_{n=0}^{\infty} p^n y_n. \quad (11) $$

Next, substituting (11) into (10), we obtain:

$$ \sum_{n=0}^{\infty} p^n y_n = \mathcal{L}^{-1} \left[ \frac{1}{s^\alpha} \left( \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \right) + \frac{1}{s^\alpha} \mathcal{L}[D^\alpha_t y_0(t) + p(-D^\alpha_t y_0(t) + f(t) - Ly(t) - Ny(t))] \right]. \quad (12) $$
Equating the terms with the identical powers of $p$, we have:

$$p^0 : y_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) + \frac{1}{s^\alpha} \mathcal{L}[D_+^\alpha y_0(t)]\right],$$

(13)

$$p^1 : y_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left(-D_+^\alpha y_0(t) + f(t) - L(y_0) - N(y_0)\right)\right],$$

$$p^2 : y_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left(-L(y_0, y_1) - N(y_0, y_1)\right)\right],$$

$$p^3 : y_3(t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\mathcal{L}\left(-L(y_0, y_1, y_2) - N(y_0, y_1, y_2)\right)\right],$$

...

Assuming that the initial approximation has the form $y(0) = \alpha_0$, $y'(0) = \alpha_1$, ..., $y^{(m-1)} = \alpha_{m-1}$; therefore, the exact solution may be obtained as follows:

$$y = \lim_{p \to 1} y = y_0 + y_1 + y_2 + ...$$

(14)

4. Modified Laplace Transform Homotopy Perturbation Method (MLT-HPM)

To figure out the basic idea of this method, we follow the same standard steps as LT-HPM, but with a slight difference; we replace $D_+^\alpha y_0(t)$ with the trial function $z(t)$ that provides some unknown parameters $A$, $B$, $C$, ... to be determined. We will determine this parameter by a system of algebraic equations, and the solution satisfies boundary conditions and by RECP.

The advantage of this replacement is the employment of the freedom by this parameter, and provides us a high accuracy by using only one iteration, unlike the LT-HPM, which will require more iterations, as we will see in the ensuing examples.

Next, we will replace $D_+^\alpha y_0(t)$ with the trial function $z(t)$ in (10), to obtain:

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) + \frac{1}{s^\alpha} \mathcal{L}[z(t) + p[-z(t) + f(t) - Ly(t) - Ny(t)]\right]$$

(15)

where $z(t) = Bt + C$.

We can write the solution of (15) as a power series of $p$ as follows:

$$y(t) = \sum_{n=0}^{\infty} p^n y_n.$$  

(16)

Next, substituting (16) into (15), we obtain:

$$\sum_{n=0}^{\infty} p^n y_n = \mathcal{L}^{-1}\left[\frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) + \frac{1}{s^\alpha} \mathcal{L}[z(t)]

+ p[-z(t) + f(t) - L \sum_{n=0}^{\infty} p^n y_n - N \sum_{n=0}^{\infty} p^n y_n]\right].$$

(17)
First, we will solve (20) via the LT-HPM, by substituting in (12), and choosing the following form:

\[ p^0 : y_0(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^\alpha} \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0) \right] + \frac{1}{s^{\alpha}} \mathcal{L}[z(t)], \]

(18)

\[ p^1 : y_1(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^\alpha} [ \mathcal{L}(-z(t))(t) + f(t) - L(y_0) - N(y_0)] \right], \]

\[ p^2 : y_2(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^\alpha} [ \mathcal{L}(-L(y_0, y_1) - N(y_0, y_1)) \right], \]

\[ p^3 : y_3(t) = \mathcal{L}^{-1}\left[ \frac{1}{s^\alpha} [ \mathcal{L}(-L(y_0, y_1, y_2) - N(y_0, y_1, y_2)) \right], \]

...\n
Assuming that the initial approximation has the form \( y(0) = \alpha_0, y'(0) = \alpha_1, ..., y^{(m-1)} = \alpha_{m-1} \); therefore, the exact solution may be obtained as follows:

\[ y = \lim_{p \to 1} y = y_0 + y_1 + y_3 + ... \]

(19)

5. Numerical Examples

In this section, we will show the efficiency of MLT-HPM for solving fractional differential equations with boundary conditions, through some examples, and the obtained solutions will be compared to the exact ones and to those obtained by the LT-HPM, CAM, MHAM, and MO-HAM.

Example 1 Consider the following Bagley-Torvik BVP:

\[ D^\frac{3}{2} y(t) = -y''(t) - y(t) + t^2 + t + \frac{4}{\sqrt{\pi}} \sqrt{t} + 3, \]

(20)

\[ 1 < \alpha \leq 2, 0 \leq t \leq 1, \]

(21)

subject to the boundary conditions,

\[ y(0) = 1, y(1) = 3. \]

(22)

The exact solution is \( y(t) = t^2 + t + 1 \).

LT-HPM

First, we will solve (20) via the LT-HPM, by substituting in (12), and choosing \( y_0(t) = 1 + At \), we obtain:

\[ \sum_{n=0}^{\infty} p^n y_n = 1 + At + \mathcal{L}^{-1}\left[ \frac{p}{s^{\alpha}} \mathcal{L}\left( -\sum_{n=0}^{\infty} p^n y''_n - \sum_{n=0}^{\infty} p^n y_n + t^2 + t + \frac{4}{\sqrt{\pi}} \sqrt{t} + 3 \right) \right], \]

(23)

where we defined \( A = y'(0) \) and employed the condition \( y(0) = 1 \). Now equating the terms with identical powers of \( p \) and solving the Laplace transforms, we can obtain a series of equations of the following form:

\[ p^0 : y_0(t) = 1 + At, \]

(24)

\[ p^1 : y_1(t) = \frac{8}{3} \frac{t^\frac{1}{2}}{\sqrt{\pi}} + \frac{32}{105} \frac{t^\frac{3}{2}}{\sqrt{\pi}} + t^2 - \frac{8}{15} \frac{(-1 + A)t^\frac{5}{2}}{\sqrt{\pi}}, \]

\[ p^2 : y_2(t) = \frac{4}{3} \frac{t^\frac{1}{2}}{\sqrt{\pi}} - 2t - \frac{2}{3} t^3 - \frac{1}{60} t^5 + \frac{8}{15} \frac{t^\frac{5}{2}}{\sqrt{\pi}} + \frac{1}{24} (-1 + A)t^4 + \frac{1}{2} (1 + A)t^2. \]


By substituting solution (24) into (11) and calculating the limit when \( p \to 1 \), we get:

\[
y(t) = 1 + At + t^2 + \frac{4t^\frac{3}{2}}{\sqrt{\pi}} + \frac{32}{105} \frac{t^\frac{5}{2}}{\sqrt{\pi}} + \frac{1}{2} (1 + A)t^2 - \frac{8}{15} \frac{(-1 + A)t^\frac{7}{2}}{\sqrt{\pi}} - 2t
\]

\[
- \frac{2}{3} t^3 - \frac{1}{60} t^5 + \frac{8}{15} \frac{t^\frac{7}{2}}{\sqrt{\pi}} + \frac{1}{24} (-1 + A)t^4.
\]

In order to calculate the value of \( A \), we will require that (25) satisfies the boundary condition \( y(1) = 3 \), hence we obtain the following value:

\[
A = 0.1567547961.
\]

Substituting (26) into (25), we get:

\[
y(t) = -1.843245204t + 1 + 4 \frac{t^\frac{3}{2}}{\sqrt{\pi}} + \frac{32}{105} \frac{t^\frac{5}{2}}{\sqrt{\pi}} + 1.578377398t^2
\]

\[
+ 0.253734189t^\frac{5}{2} - \frac{2}{3} t^3 - \frac{1}{60} t^5 - 0.03513521683t^4 + \frac{8}{15} \frac{t^\frac{7}{2}}{\sqrt{\pi}}.
\]

**MLT-HPM**

In order to obtain an approximate analytical solution by using MLT-HPM, we substitute (20) into (17), we obtain:

\[
\sum_{n=0}^{\infty} p^n y_n = 1 + At + 4 \frac{t^\frac{3}{2}}{15} (2Bt + 5C) \sqrt{\pi} + \mathcal{L}^{-1} \left[ \frac{p}{s^2} \mathcal{L}((-Bt - C - 1)) \right]
\]

\[
- \sum_{n=0}^{\infty} p^n y_n' - \sum_{n=0}^{\infty} p^n y_n + t^2 + t + \frac{4}{\sqrt{\pi}} \sqrt{t + 3} \right] ,
\]

where we defined \( A = y'(0) \) and employed the condition \( y(0) = 1 \).

Now equating the terms with identical powers of \( p \) and solving the Laplace transforms, we can obtain a series of equations of the following form:

\[
p^0 : y_0(t) = 1 + At + \frac{4}{15} \frac{t^\frac{3}{2}(2Bt + 5C)}{\sqrt{\pi}},
\]

\[
p^1 : y_1(t) = -\frac{1}{24} Bt^4 + \frac{32}{105} \frac{t^\frac{5}{2}}{\sqrt{\pi}} + \frac{8}{15} \frac{(1 + C - A)t^\frac{7}{2}}{\sqrt{\pi}}
\]

\[
- \frac{1}{6} C t(t^2 + 6) + \frac{4}{3} \frac{(2 + B)t^\frac{3}{2}}{\sqrt{\pi}} - \frac{1}{2} (-2 + B)t^2.
\]

By substituting solution (29) into (16) and calculating the limit when \( p \to 1 \), we get:

\[
y(t) = 1 + At + \frac{4}{15} \frac{t^\frac{3}{2}(2Bt + 5C)}{\sqrt{\pi}} - \frac{1}{24} Bt^4 + \frac{32}{105} \frac{t^\frac{5}{2}}{\sqrt{\pi}} + \frac{8}{15} \frac{(1 + C - A)t^\frac{7}{2}}{\sqrt{\pi}}
\]

\[
- \frac{1}{6} C t(t^2 + 6) + \frac{4}{3} \frac{(2 + B)t^\frac{3}{2}}{\sqrt{\pi}} - \frac{1}{2} (-2 + B)t^2.
\]
In order to calculate the values of A, B and C, three equations must be formed: to find the first equation, we will require that (30) satisfies the boundary condition \( y(1) = 3 \); to find the other equations we used residual error cancelation in some point of the interval \([0, 1]\), let us say \( t = 0.2 \) and \( t = 0.21 \). Subsequently we obtain an equation system for A, B, and C. Following the above procedure, we obtain the following values:

\[
A = 2.858710268, \quad B = -5.452594808, \quad C = 1.646852110.
\]

Substituting (31) into (30), we get:

\[
y(t) = 1 + 2.858710268t + \frac{4}{15}t^\frac{4}{5}(-10.90518962t + 8.234260550) + .2271914503t^4
\]
\[
+ \frac{32}{105} \frac{t^7}{\sqrt{\pi}} - 0.06374835515t^\frac{5}{2} - 2.744753517t(t^2 + 6) - 2.597224036t^\frac{3}{2}
\]
\[
+ 3.726297404t^2.
\]

**Example 2** Consider the following fractional differential equation,

\[
D^\frac{5}{2}y(t) + y''''(t) + y(t)^2 - t^4 = 0,
\]

\[
2 < \alpha \leq 3, \quad 0 \leq t \leq 1,
\]

subject to,

\[
y(0) = 0, \quad y(1) = 1, \quad y''(0) = 2,
\]

with the exact solution \( y(t) = t^2 \).

**LT-HPM**

Applying the LT-HPM to find a solution for (33), and chose \( y_0(t) = t^2 + At \), the second-order approximation solution is given by:

\[
y(t) = t^2 + At - \frac{64}{10395} \frac{A(12t + 11A)t^\frac{9}{2}}{\sqrt{\pi}} + \frac{1}{10} At^5 - \frac{2147483648}{99613755115875} \frac{A^4t^{\frac{23}{2}}}{\pi^{\frac{1}{2}}}
\]
\[
+ \frac{1}{65745078376477500} (-1099511627776t^{\frac{27}{2}} + 5478756531373125t^4 \pi^{\frac{3}{2}}).A^2
\]
\[
+ \frac{1024}{45045} \frac{t^{13}}{\sqrt{\pi}} - \frac{68719476736}{1826252177124375} \frac{A^3t^{\frac{25}{2}}}{\pi^{\frac{3}{2}}}
\]

In order to calculate the value of A, we will require that (35) satisfies the boundary condition \( y(1) = 1 \), where we obtain the following value:

\[
A = -0.01212515598.
\]

By substituting (36) into (35), we get:

\[
y(t) = -0.01212515598t + t^2 + 0.4211801336 \times 10^{-4} t^\frac{7}{2}(12t - .1333767158)
\]
\[
- 0.1212515598 \times 10^{-2} t^6 - 8.368258381 \times 10^{-14} t^\frac{25}{2} + 1.204637256 \times 10^{-11} t^\frac{31}{2}
\]
\[
+ \frac{1024}{45045} \frac{t^{13}}{\sqrt{\pi}} - 4.415566586 \times 10^{-10} t^\frac{25}{2} + 0.2200232691 \times 10^{-5} t^4(\pi^{\frac{3}{2}}).
\]
Applying the MLT-HPM to find a solution for (33), the first-order approximation solution is given by:

\[ y(t) = t^2 + At + \frac{8}{105} \frac{t^5 (2Bt + 7C)}{\sqrt{\pi}} - \frac{256}{3465} \frac{At^{10}}{\sqrt{\pi}} - \frac{64}{945} \frac{A^2 t^7}{\sqrt{\pi}} - \frac{4194304}{22915517625} \frac{B^2 t^{10}}{\pi^{\frac{17}{2}}} - \frac{131072}{30405375} \frac{C^2 t^{15}}{\pi^{\frac{17}{2}}} - \frac{1}{1120} (2AB + 7C)t^7 \]

\[ + \frac{1}{1929728000} \frac{C(-33554432Bt^{12} - 187612425At^6\pi^2 + 2940537600t^7\pi - 9648639000t^2\pi^2)}{\pi^2} \]

\[ + \frac{1}{26880} \frac{B(-33t^8\sqrt{\pi} + 14336t^5\pi - 4480t^3\sqrt{\pi})}{\sqrt{\pi}}. \]

In order to calculate the values of \( A, B \) and \( C \) we will require that (38) satisfies the boundary condition \( y(1) = 1 \), and we used residual error cancelation in some points of the interval \([0, 1]\), let us say \( t = 0.2 \) and \( t = 0.21 \), after that we obtain an equation system for \( A, B, \) and \( C \). Following the above procedure, we obtain the following values:

\[ A = B = C = 0. \]  

(39)

Substituting (39) into (38), we get:

\[ y(t) = t^2. \]  

(40)

### 6. Discussion

In this work, MLT-HPM was used in order to obtain accurate approximate solutions for ordinary fractional differential equations with boundary conditions.

Table 1 illustrates the comparison of absolute error between the approximate solution obtained by the MLT-HPM and to those obtained by the LT-HPM at 10 points in the interval \([0, 1]\) of example 1. It is clear that the absolute error in our method is much lower at each point of the ten points; it is also shown that the given solution obtained by MLT-HPM is very close to the exact solution which leads to the solution to converge, demonstrating the reliability of the method.

In example 1, the MLT-HPM reached results having higher precision and wider range, requiring only one iteration unlike the LT-HPM method which requires more iterations for the same case study. This example was solved using the continuous analytical method (CAM) [29], where the initial approximation was unstable because it was based on the exact solution. Comparing the absolute error between the exact solution and LT-HPM, MLT-HPM and CAM respectively, using only one iteration we obtain absolute errors as shown in Table 1.

When we present the approximate solutions of both methods of Example 1 graphically and compare it to the exact solutions, we note that the approximate solutions obtained by MLT-HPM are very close to the exact solutions as shown in Figure 2, unlike the approximate solutions obtained by LT-HPM in Figure 1.

In Table 2, a comparison of absolute errors between the approximate solution obtained by the MLT-HPM and to those obtained by the LT-HPM at 10 points are presented for Example 2.

Figure 3 depict the graphical comparison of LT-HPM and the exact solution, while Figure 4 illustrates the absolute error between the exact solution and the approximation solution obtained.
Table 1: Comparison of absolute errors between LT-HPM and MLT-HPM on [0,1] for example 1

| t  | Exact solution | LT-HPM | MLT-HPM | CAM  |
|----|----------------|--------|---------|------|
| 0  | 1              | 0      | 0       | 1    |
| .1 | 1.11           | 0.2060378470 | 0.0000940943 | 1.107 |
| .2 | 1.24           | 0.3385214582 | 0.0017803082 | 1.216 |
| .3 | 1.39           | 0.4185379431 | 0.0013277952 | 1.309 |
| .4 | 1.56           | 0.4544899245 | 0.0000383856 | 1.368 |
| .5 | 1.75           | 0.4519495485 | 0.0010279003 | 1.375 |
| .6 | 1.96           | 0.4153034276 | 0.000900694  | 1.312 |
| .7 | 2.19           | 0.3483496321 | 0.000373300  | 1.161 |
| .8 | 2.44           | 0.2545835754 | 0.002068437  | .904 |
| .9 | 2.71           | 0.1373605034 | 0.002694330  | .523 |
| 1  | 3.00           | 0      | 0       | 0    |

by the LT-HPM. It is important to note that when we apply MLT-HPM on the same example, we get $y(x) = t^2$, which is the exact solution.
Our method of selecting the initial approximation is more stable than the methods used in [30], which has adopted the approach of selecting the exact solution on the first approximation. Moreover, the methods MHAM and MOHAM in [30] need two iterations to obtain the same result as MLT-HPM, which needs only one iteration.

Table 2: The exact and LT-HPM of Example 2

| t  | Exact solution | LT-HPM       | MLT-HPM       |
|----|----------------|--------------|--------------|
| 0  | 1              | 1            | 0            |
| .1 | 0.1            | 0.001212521021 | 0            |
| .2 | 0.4            | 0.002424964197 | 0            |
| .3 | 0.9            | 0.003634625061 | 0            |
| .4 | 0.16           | 0.004825757356 | 0            |
| .5 | 0.25           | 0.005947078975 | 0            |
| .6 | 0.36           | 0.006874399764 | 0            |
| .7 | 0.49           | 0.007356058843 | 0            |
| .8 | 0.64           | 0.006939144193 | 0            |
| .9 | 0.81           | 0.004874665531 | 0            |
| 1  | 1.00           | 1.0000000000  | 0            |

In fact, the trial function z(t) plays an important role in the behavior of the MLT-HPM; this additional freedom was used to obtain accurate approximate solutions. In addition, MLT–HPM is a method with Ability to increase the acceleration of convergence. For example, from the previous numerical examples, we see that MLT–HPM is more accurate than LT–HPM, even though the MLT–HPM is just a first order approximation while the LT–HPM is a second order approximate solution.

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7. Conclusion

In this paper, we used the MLT-HPM to find an approximate solution for ordinary fractional differential equations. Our study depends on the trial function, where we employed the first order approximation to obtain high accuracy of the solutions. It is shown that the introduction of the trial function into the MLT-HPM is the to provide, accelerated convergence of the approximate solution toward the exact solution. Finally, comparing the accuracy of the approximate solutions obtained by MLT-HPM with those obtained by LT-HPM, show us that this method can provide the exact solution by using only first iterations and improves the performance of the standard LT-HPM.

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