An infinite, two-parameter family of polynomials with factorization similar to $X^m - 1$

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ABSTRACT

For a suitable irreducible base polynomial $f(x) \in \mathbb{Z}[x]$ of degree $k$, a family of polynomials $F_m(x)$ depending on $f(x)$ is constructed with the properties:

(i) there is exactly one irreducible factor $U_{d,f}(x)$ for $F_m(x)$ for each divisor $d$ of $m$;

(ii) $\deg(U_{d,f}(x)) = \varphi(d) \cdot \deg(f)$ generalizing the factorization of $x^m - 1$ into cyclotomic polynomials;

(iii) when the base polynomial $f(x) = x - 1$ this $F_m(x)$ coincides with $x^m - 1$.

As an application, irreducible polynomials of degree 12, 24, 24 are constructed having Galois groups of order matching their degrees and isomorphic to $S_3 \oplus C_2$, $S_3 \oplus C_2 \oplus C_2$ and $S_3 \oplus C_4$, respectively.

1. Introduction

Cyclotomic polynomials occupy a central role in algebraic number theory: any single root of them generates a Galois extension of the field of rational numbers with abelian Galois group. And conversely the celebrated result of Kronecker and Weber states that any abelian extension of $\mathbb{Q}$ is contained in such extensions. The product of these polynomials $U_{d,f}(x)$ over $d$ chosen as divisors of a pre-determined $m$ gives the irreducible factorization of $x^m - 1$. In this article, we arrive at one generalization of such a factorization, which leads us to new irreducible polynomials that may be regarded as some kind of relativized cyclotomic polynomials (relative to a given fixed irreducible polynomial).

There have been many attempts to find generalizations earlier. One by K. Nageswara Rao [8] arises by generalizing the notion of the set of divisors of a number and using the corresponding Möbius inversion.

C. Kimberling [4] has generalized the (univariate) cyclotomic polynomial to a polynomial in three variables having integer coefficients which he denotes by $C_n(x, y, z)$. He shows that $C_n(x, 1, 0)$ is the usual cyclotomic polynomial. While the roots of classical cyclotomic polynomials are related to the values of the trigonometric function at $2\pi k/n$, his generalization has roots related to the values of hyperbolic sine functions.

In Kwon, Lee and Lee [5], the authors deal with a natural choice of primitive elements $\beta$ for subfields of cyclotomic fields, and the minimal polynomials of $\beta$ for $\mathbb{Q}(\beta) \subset \mathbb{Q}(\zeta)$ are their way of generalizing the cyclotomic polynomial.

Möbius inversion formula and inclusion-exclusion principle provide a combinatorial approach to defining the cyclotomic polynomial using $x^k - 1$ for various $k$. This has led to many...
generalizations named Unitary Cyclotomic polynomials and more. Further details on this may be found in the works of Moree [7] and Bachman [1].

In this article, our approach is to relativize the concept which has spawned new kinds of polynomials different from all the above. We also show that the roots of our polynomials generate certain Galois extensions with prescribed Galois groups.

2. Preliminaries

Conventions: We take the field of rational numbers, \( \mathbb{Q} \) as our base field. All the algebraic numbers may be taken as algebraic integers with no loss of generality. When we talk of degree or conjugates of an algebraic number it is all with respect to \( \mathbb{Q} \), unless stated otherwise.

We use the standard notation \( \Phi_m(x) \) for the classical \( m \)-th cyclotomic polynomial whose degree is given by the Euler totient function \( \varphi(m) \). We write \( \text{Gal}(f(x)) \) for the Galois group of a polynomial \( f(x) \) (which is tacitly assumed to be over \( \mathbb{Q} \)).

First we define the notion of uniform degree which is very handy for our purposes.

**Uniform Degree:** Let \( \alpha \) be an algebraic number. We say \( \alpha \) is of uniform degree, if for every integer \( m \neq 0 \), the degree of \( \alpha^m \) as an algebraic number is the same as the degree of \( \alpha \).

**Example:** For any square-free integer \( d \), the algebraic number \( 1 + \sqrt{d} \) is of uniform degree 2.

Besides the above example there are many more. The following result assures us such numbers are aplenty in any number field.

**Lemma 1.** For any algebraic number \( \alpha \), say of degree \( d \), there is an integer \( k \in \mathbb{Z} \) such that \( \alpha + k \) has uniform degree \( d \).

**Proof.** Let \( \alpha = \alpha_1, \alpha_2, ..., \alpha_d \) be all the conjugates of \( \alpha \).

Suppose that \( \alpha^m \) is “defective” in the sense that it has degree less than \( d \) for some \( m \geq 1 \). This means, \( \alpha_i^m = \alpha_j^m \) for some \( 1 \leq i, j \leq d \). The lemma claims that the above defect can be cured by adding a suitable integer \( k \) to \( \alpha \). We will show that almost all integers \( k \) will do. If a specific \( k \) is unable to cure this defect it follows that for suitable roots of unity \( \zeta_{ij}, \eta_{ij} \) we have simultaneously

\[
\alpha_i = \zeta_{ij} \alpha_j \quad (1)
\]

\[
(\alpha_i + k) = \eta_{ij}(\alpha_j + k) \quad (2)
\]

Substituting (1) in (2) we get

\[
\zeta_{ij} \alpha_j + k = \eta_{ij}(\alpha_j + k)
\]

Rewriting the above, for this specific \( k \), we get

\[
(\zeta_{ij} - \eta_{ij}) \alpha_j = (\eta_{ij} - 1) k \quad (3)
\]

We claim that Equation (3) will be true only for a finite number of \( k \)'s thereby proving the lemma. Suppose not. Then we will have infinitely many choices of \( k \) which will make the RHS of (3) grow unboundedly with such \( k \). However, we can easily see that the LHS will be bounded with the following argument: For all the choices of \( k \) in (3), LHS will change only in the choice of the roots of unity and the choice among the \( d \) conjugates of \( \alpha \). So applying triangle inequality to the LHS we get,

\[
| (\zeta_{ij} - \eta_{ij}) \alpha_j | \leq 2 \max \{ |\alpha_1|, |\alpha_2|, ..., |\alpha_d| \},
\]

which is an absolute bound independent of \( k \) depending only on \( \alpha \).

The following elementary fact about linear disjointness plays a very significant role in our article.
Lemma 2 (see Morandi [6], p. 184, Example 20.6). Suppose that $K$ and $L$ are finite extensions of a field $F$ with $K$ a Galois extension over $F$. Under this hypothesis, $K$ and $L$ are linearly disjoint over $F$ if and only if $K \cap L = F$.

The following lemma which must be well known to experts is provided with a proof for the reason that it is short, and also it will make our main theorem widely applicable.

Lemma 3. There exists number fields $F$ that are linearly disjoint over $\mathbb{Q}$ with every cyclotomic extension.

Proof. We first claim that any number field $F$ which is a Galois extension of $\mathbb{Q}$ having some non-abelian simple group $G$ as Galois group will do. If such an $F$ is not linearly disjoint with a cyclotomic extension $L$ then, by Lemma 2 it follows that $L \cap F = K$ will be a number field of degree $> 1$. Being contained in a cyclotomic extension, $K$ will be a Galois extension of $\mathbb{Q}$. As it is also a subfield of $F$, by fundamental Galois correspondence, this $K$ must be a fixed field of a normal subgroup of $G$. The simplicity of $G$ rules this out providing the contradiction.

To finish the argument we need to show that such an extension $F$ with a simple non-abelian Galois group exists. Schur (see e.g. [2]) has shown that for some generalized Laguerre polynomials $L_{\alpha}^{(\alpha)}(x)$, $\alpha = 1$ their splitting fields have the alternating group $A_{\alpha}$ as Galois groups. Also, the specific polynomial $x^2 - 154x + 99$ has been shown to have the simple group $PSL(2, F_7)$ as the Galois group by Erbach, Fisher and McKay in [3].

It seems there is no name for an algebraic number which generates a Galois extension of $\mathbb{Q}$. Nor does the minimal polynomial for such a number have any special name.

As we will be constructing such polynomials later, we honor them formally here with a name:

**Galois polynomial**: An irreducible polynomial $f(x) \in \mathbb{Q}[x]$ with the property that for any root $\alpha$ of $f$, the field $\mathbb{Q}(\alpha)$ is also a splitting field of $f(x)$ over $\mathbb{Q}$ is called a *Galois polynomial*.

Evidently, cyclotomic polynomials are Galois polynomials.

### 3. Relative cyclotomic polynomials

We state our main theorem under milder assumptions, though one can show lots of examples where stronger hypothesis (and consequently stronger conclusions) are valid.

**Main Theorem**: Let $\alpha$ be an algebraic integer of degree $k$ and $f(x)$ its minimal polynomial over $\mathbb{Q}$. Choose an integer $m > 2$ such that

(i) The $m$-th cyclotomic field $\mathbb{Q}(\zeta)$ and $\mathbb{Q}(\alpha)$ are linearly disjoint over $\mathbb{Q}$. (here $\zeta = \exp(2\pi i/m)$).

(ii) the degree of $x^m$ is the same as the degree of $\alpha$.

Denoting by $f_m(x)$ the minimal polynomial of $\alpha^m$ over $\mathbb{Q}$, define $F_m(x) := f_m(x^m)$. With this notation, $F_m(x)$ factorizes in $\mathbb{Q}[x]$ with exactly one irreducible factor, denoted $\Phi_{d,f}(x)$, with degree $= \deg f \varphi(d)$ for each divisor $d$ of $m$. That is,

$$F_m(x) = f_m(x^m) = \prod_{d|m} \Phi_{d,f}(x) \quad \text{where} \quad \deg(\Phi_{d,f}) = \deg(f) \cdot \varphi(d).$$

We call these polynomials $\Phi_{d,f}(x)$ as relative $d$-th cyclotomic polynomials with respect to $f(x)$.

Proof. Let $\alpha_1 = \alpha, \alpha_2, ..., \alpha_k$ be the conjugates of $\alpha$. By definition, $F_m(x)$ has degree $km$. The $k$ conjugates of $x^m$, namely $\alpha_1^m, \alpha_2^m, ..., \alpha_k^m$ are among the $km$ roots of $F_m(x)$. We first claim that the roots of $F_m(x)$ are all distinct. In fact for any irreducible polynomial $h(x)$ in characteristic zero $h(x^m)$ has distinct roots (because $h$ has).
Let $S$ denote the set of all those $km$ roots of $F_m$. Clearly
\[ S = \{ \zeta_d^j \mid i = 1, 2, \ldots, k, \text{ and } j = 1, 2, \ldots, m \}. \]

Our plan is to partition $S$ into disjoint subsets $S_d$, one for each divisor $d$ of $m$, with $S_d$ a set of cardinality $k\varphi(d)$ as the roots of an irreducible polynomial $\Phi_{d,f}(x)$.

With this in mind, for any divisor $d$ of $m$ we define
\[ S_d = \{ \zeta_d^j \mid i = 1, 2, \ldots, k \text{ and } \gcd(j, d) = 1 \text{ with } j < d \}. \]

Then $S = \bigcup_{d|m} S_d$. We can see that for any two distinct divisors $d, d'$ of $m$ we have $S_d \cap S_{d'} = \emptyset$ and $|S_d| = k\varphi(d)$.

We denote by $\zeta_d$ a primitive $d$-th root of unity, except for the case $d = m$, where we use simply $\zeta$.

Now we proceed to show that elements of $S_d$ are the roots of the minimal polynomial of $\zeta_s d$.

By assumption $Q(\zeta)$ and $Q(\zeta_s d)$ are linearly disjoint over $Q$. Consequently, for any $d \mid m$, we have $Q(\zeta)$ and $Q(\zeta_s d)$ are also linearly disjoint. So, we have the following equality of field degrees:
\[ n \mid m \implies [Q(\zeta_s d) : Q] = [Q(\zeta) : Q] \times [Q(\zeta_s d) : Q] = k\varphi(d). \]

We know that $Q(\zeta_s d) \subseteq Q(\zeta_s d)$. So, $[Q(\zeta_s d) : Q] \leq [Q(\zeta) : Q] = k\varphi(d)$. If we show $\zeta_s d$ is an algebraic number of degree $k\varphi(d)$ we are done. This is equivalent to showing $\zeta_s d$ has relative degree $\varphi(d)$ over $Q(\zeta)$. Suppose, to the contrary, this relative degree $r < \varphi(d)$ over $Q(\zeta)$. Then for some $h(x) = \sum_{i=0}^{r} c_i x^i$ with $c_i \in Q[\zeta]$, we have $h(\zeta_s d) = 0$. That is,
\[ 0 = \sum_{i=0}^{r} c_i (\zeta_s d)^i = \sum_{i=0}^{r} (c_i \zeta)^i \zeta_s d = \sum_{i=0}^{r} b_i \zeta_s d \text{ for } b_i \in Q(\zeta). \]

This is a contradiction to the linear disjointness of $Q(\zeta)$ with $Q(\zeta_s d)$. So $\zeta_s d$ is an algebraic number of degree $\varphi(d)$ over $Q(\zeta)$. Its minimal polynomial over $Q$ is the polynomial denoted by $\Phi_{d,f}(x)$ in the statement of the theorem. This is irreducible and is of degree $k\varphi(d)$. \qed

**Remark.** If $Q(\zeta)$ and $Q(\zeta_s d)$ are not linearly disjoint then the factorization of $F_m(\zeta)$ was found to have more irreducible factors than the number of divisors of $m$.

**Corollary.** If, further, $Q(\zeta)$ over $Q$ is a Galois extension, then for each divisor $d$ of $m$, the above polynomial $\Phi_{d,f}(x)$ will be a Galois polynomial of degree $\varphi(d)\deg(f)$.

Now we would like to point out the scenarios where the hypothesis of our main theorem hold.

We will choose $\zeta$ to be of uniform degree, by adding an integer (provided by Lemma 1) so that condition (ii) of Main theorem will be true for all $m$. To find examples where condition (i) holds we can use examples of fields assured by Lemma 3.

But for cases where the splitting field of $f(x)$ intersects with some cyclotomic extension, one can still find infinitely many $m$ such that $m$-th cyclotomic field is linearly disjoint with it.

For example, if we choose $\zeta = 1 + \sqrt{2}$, the factorization fails (that is, $F_m(\zeta)$ does not imitate the factorization behavior of $x^m - 1$) whenever $m$ is a multiple of the discriminant (8 in this case). Otherwise the factorization will be along the classical lines.

**Remark.** The relative cyclotomic polynomials $\Phi_{d,f}(x)$ with respect to $f(x)$ (using the same notations as in the Main Theorem) may alternatively be expressed using Möbius inversion by
\[ \Phi_m(x) = \prod_{d|m} F_d(x)^{\mu(n/d)} = \prod_{d|m} f_d(x)^{\mu(n/d)} \]
4. Examples and applications

Using our main theorem, given a Galois polynomial we can indicate how to find more Galois polynomials of higher degrees. More specifically, starting with Galois polynomial \( f \) of degree \( k \) with Galois group \( G \) of order \( k \), we show how to find Galois polynomials with Galois group of the form \( G \oplus H \).

First requirement is \( H \) must be abelian, and it should be the Galois group of a suitable cyclotomic extension of \( \mathbb{Q} \).

All the calculations were done with the computer algebra system SAGE.

Example 1

We start with \( f = x^6 + 3x^5 - 2x^4 - 9x^3 + 5x + 1 \) a Galois polynomial with Galois group \( S_3 \) found in A Database for Number Fields in the website http://galoisdb.math.upb.de/home as the base polynomial.

We illustrate this first for \( H = C_2 \oplus C_2 \), the Klein group which is the Galois group of the 8-th cyclotomic extension. As this extension is linearly disjoint with \( \mathbb{Q}(\alpha) \) (\( \alpha \) denoting a root of this \( f \)), we can appeal to our Main Theorem. So we look at \( x^8 \) which is again an algebraic number of degree 6 with minimal polynomial \( f_8(x) \). From the factorization of \( F_8(x) = f_8(x^8) \), we obtain new Galois polynomial, namely the unique relative cyclotomic polynomial factor \( \Phi_{8,f}(x) \):

\[
F_8(x) = (x^{48} - 1405x^{40} + 226310x^{32} - 3670777x^{24} + 1940230x^{16} - 201085x^8 + 1)
\]

\[
\Phi_{8,f}(x) = (x^{24} + 53x^{20} + 702x^{16} + 2553x^{12} + 2062x^8 + 453x^4 + 1)
\]

The polynomial \( \Phi_{8,f}(x) \) is of degree 24 = \( \deg f \cdot \varphi(8) \) and has Galois group \( \text{Gal}(f(x)) \oplus \text{Gal}(\Phi_8(x)) \) of order 24 which is isomorphic to \( S_3 \oplus C_2 \oplus C_2 \). This is a Galois polynomial.

Example 2

For the Galois polynomial \( g = x^6 + 3x^5 + 5x^4 + 5x^3 + 5x^2 + 3x + 1 \) with Galois group \( S_3 \), found in the same database we can compute and arrive at

\[
F_3(x) = x^{18} - 3x^{15} + 11x^{12} + 11x^9 - 3x^3 + 1
\]

which leads us to the following relative cyclotomic polynomial as its highest degree factor.

\[
\Phi_{3,g}(x) = x^{12} - 3x^{11} + 4x^{10} - 5x^9 + 5x^8 + 2x^7 - 7x^6 + 2x^5 + 5x^4 - 5x^3 + 4x^2 - 3x + 1
\]

The relative cyclotomic factor \( \Phi_{3,g}(x) \) is of degree 12 = \( \deg g \cdot \varphi(3) \) and has Galois group \( \text{Gal}(g(x)) \oplus \text{Gal}(\Phi_3(x)) \) of order 12 isomorphic to \( S_3 \oplus C_2 \). This is a Galois polynomial.

Example 3

Using the same \( g \) as in Example 2, but with \( m = 5 \) this time, we obtain

\[
F_5(x) = x^{30} - 17x^{25} + 95x^{20} - 135x^{15} + 95x^{10} - 17x^5 + 1
\]

\[
\Phi_{5,g}(x) = x^{24} - 3x^{23} + 4x^{22} - 2x^{21} - 4x^{20} - 3x^{19} + 27x^{18} - 45x^{17} + 37x^{16} + 8x^{15} - 11x^{14} - 45x^{13} + 73x^{12} - 45x^{11} - 11x^{10} + 8x^9 + 37x^8 - 45x^7 + 27x^6 - 3x^5 - 4x^4 - 2x^3 + 4x^2 - 3x + 1
\]

The relative cyclotomic factor \( \Phi_{5,g}(x) \) is a polynomial of degree 24 = \( \deg g \cdot \varphi(5) \) and has Galois group \( \text{Gal}(g(x)) \oplus \text{Gal}(\Phi_5(x)) \) of order 24 which is isomorphic to \( S_3 \oplus C_4 \).
5. Conclusion

When the linear disjointness hypothesis with the $m$-th cyclotomic field does not hold, it was found (using SAGE) that we get excess factors for $F_m(x)$. To elaborate, in all the examples computed for those failed cases we found that the number of irreducible factors of $F_m(x)$ was always more than the number of divisors of $m$. In fact, corresponding to the divisor $d = m$ of $m$, the factorization of $F_m(x)$ did not have an irreducible factor of degree $k \varphi(m)$; instead it had two irreducible factors of degree $k \varphi(m)/2$. We are yet to prove and this needs further investigation.

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