Abstract. Snake modules are a family of modules of quantum affine algebras which were introduced by Mukhin and Young. The aim of this paper is to prove that the Hernandez–Leclerc conjecture is true for snake modules of types $A_n$ and $B_n$. We prove that prime snake modules are real. We introduce $S$-systems consisting of equations satisfied by the $q$-characters of prime snake modules in types $A_n$ and $B_n$. Moreover, we show that every equation in the $S$-system of type $A_n$ (respectively, $B_n$) corresponds to a mutation in some cluster algebra $A$ (respectively, $A'$) and every prime snake module of type $A_n$ (respectively, $B_n$) corresponds to some cluster variable in $A$ (respectively, $A'$). In particular, this proves that the Hernandez–Leclerc conjecture is true for all snake modules of types $A_n$ and $B_n$.

Key words: cluster algebras; quantum affine algebras; snake modules; $S$-systems; $q$-characters; Frenkel–Mukhin algorithm

2010 Mathematics Subject Classification: 13F60; 17B37

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the field of complex numbers and $U_q\widehat{\mathfrak{g}}$ the corresponding quantum affine algebra. Snake modules were introduced by Mukhin and Young in [MY12a] and [MY12b]. They are modules of quantum affine algebras. The family of snake modules contains all minimal affinizations which were introduced by Chari in [C95].

A simple $U_q\widehat{\mathfrak{g}}$-module $M$ is called real if $M \otimes M$ is simple, see [Le03]. A simple $U_q\widehat{\mathfrak{g}}$-module $M$ is called prime if either $M$ is trivial or if there does not exist non-trivial $U_q\widehat{\mathfrak{g}}$-modules $M_1, M_2$ with $M = M_1 \otimes M_2$, see [CP97].

Chari and Pressley classified all prime $U_q\widehat{\mathfrak{sl}_2}$-modules in [CP91]. Some prime $U_q\widehat{\mathfrak{g}}$-modules including minimal affinizations were classified in [CMY13] by considering certain homological properties. In [MY12b], Mukhin and Young classified all prime snake modules of types $A_n$ and $B_n$ and proved that snake modules of types $A_n$ and $B_n$ can be uniquely (up to permutation) decomposed into a tensor of prime snake modules. We show that all prime snake modules of types $A_n$ and $B_n$ are real (Theorem 3.4).

The theory of cluster algebras were introduced by Fomin and Zelevinsky in [FZ02]. It has many applications to mathematics and physics.

Let $\mathcal{C}$ be the category of all finite-dimensional $U_q\widehat{\mathfrak{g}}$-modules. In [HL10], Hernandez and Leclerc introduced a full subcategory $\mathcal{C}_\ell$ ($\ell \in \mathbb{Z}_{\geq 0}$) of $\mathcal{C}$. Let $I$ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$ and let $I = I_0 \sqcup I_1$ be a partition of $I$ such that every edge connects a vertex of $I_0$ with a vertex of $I_1$. For $i \in I$, let $\xi_i = 0$ if $i \in I_0$ and $\xi_i = 1$ if $i \in I_1$. Every object $V$ in $\mathcal{C}_\ell$ satisfies: for every composition factor $S$ of $V$ and every $i \in I$, the roots of the Drinfeld polynomial $\pi_{i,S}(u)$ belong to $\{q^{-2k-\xi_i} | 0 \leq k \leq \ell\}$. 

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In \cite{HL10}, Hernandez and Leclerc introduced the concept of monoidal categorifications of cluster algebras. They proposed the following conjecture, see \cite{HL10, Le10, HL13}.

**Conjecture 1.1** (\cite{HL10}, Conjecture 13.2; \cite{HL13}, Conjecture 5.2; \cite{Le10}, Conjecture 9.1). The Grothendieck ring of $C_\ell$ has a cluster algebra structure. The simple $U_q\widehat{g}$-modules, which are prime and real, are cluster variables in some cluster algebra.

In the case of types $A_n$ and $D_4$, $\ell = 1$, Conjecture \cite{HL11} was proved in \cite{HL10}. In the case of types $ADE$, $\ell = 1$, Conjecture \cite{HL11} was proved in \cite{Nak11}. The work of \cite{Nak11} was generalized to all acyclic quivers by Kimura and Qin \cite{KQ14} and Lee \cite{Lee13}. In the case of type $A_3$, $\ell = 2$, Conjecture \cite{HL11} was proved in \cite{YMLZ15}. It was proved in \cite{HL13} that Conjecture \cite{HL11} is true for Kirillov–Reshetikhin modules in all types. Qin proved some cases of the Conjecture \cite{HL11} in \cite{QL14}. It is shown that Conjecture \cite{HL11} is true for all minimal affinizations of types $G_2$, $A_n$ and $B_n$, \cite{OL14, ZDL15}.

In this paper, we prove that Hernandez–Leclerc conjecture is true for all snake modules of types $A_n$ and $B_n$. More precisely, we prove that every prime snake module is a cluster variable in some cluster algebra introduced in \cite{HL13}. To this aim, we introduce two systems of equations consisting of equations satisfied by the $q$-characters of prime snake modules of types $A_n$ and $B_n$. We call these systems the $S$-systems of types $A_n$ and $B_n$ respectively. The equations in the $S$-systems of types $A_n$ and $B_n$ are of the form

$$[S_1][S_2] = [S_3][S_4] + [S_5][S_6],$$

(1.1)

where $S_i$ ($i \in \{1, 2, \ldots, 6\}$) is a prime snake module and $[S_i]$ is the equivalence class of $S_i$ in the Grothendieck ring of $C$. Moreover, $S_3 \otimes S_4$ and $S_5 \otimes S_6$ are simple (Theorem 4.3). By Equation (1.1), $S_1 \otimes S_2$ is not simple. Therefore, some tensor products of prime snake modules are simple and some tensor products of prime snake modules are not simple.

Let $\mathcal{A}$ (respectively, $\mathcal{A}'$) be the cluster algebra for the quantum affine algebra of type $A_n$ (respectively, $B_n$) introduced in \cite{HL13}. We show that the equations in the $S$-system of type $A_n$ (respectively, $B_n$) correspond to mutations in $\mathcal{A}$ (respectively, $\mathcal{A}'$) and prime snake modules of type $A_n$ (respectively, $B_n$) correspond to some cluster variables in $\mathcal{A}$ (respectively, $\mathcal{A}'$). In particular, this proves that the Hernandez–Leclerc conjecture is true for all snake modules of types $A_n$ and $B_n$.

The procedure of proving that prime snake modules of type $A_n$ (respectively, $B_n$) correspond to some cluster variables in $\mathcal{A}$ (respectively, $\mathcal{A}'$) is as follows. For a prime snake module $L(S)$ with highest $l$-weight monomial $S$, we define a set (Section 5.2)

$$\mathcal{FS}(S) = \{M_1, M_2, \ldots, M_q\},$$

where every $M_i$ is the highest weight monomial of a minimal affinization or a certain simple $U_q\widehat{g}$-module. We construct a mutation sequence $\text{Seq}_1, \text{Seq}_2, \ldots, \text{Seq}_q$ for $L(S)$ (Section 5.7), where $\text{Seq}_q$ is the mutation sequence for the simple $U_q\widehat{g}$-module $L(M_i)$ with highest $l$-weight monomial $M_i$. Therefore, prime snake modules of type $A_n$ (respectively, $B_n$) correspond to some cluster variables in $\mathcal{A}$ (respectively, $\mathcal{A}'$).

When $M_i$ is the highest weight monomial of a minimal affinization, the mutation sequence $\text{Seq}_i$ is similar to the mutation sequence for a minimal affinization in \cite{ZDL15}. We use the idea of the sequence for a minimal affinization in \cite{ZDL15}. The mutation sequences for minimal affinizations introduced in this paper is more convenient since the sequences produce all minimal
affinizations of type $A_n$ (respectively, $B_n$) in the same cluster algebra (in \cite{ZDL15}, half of the minimal affinizations of type $A_n$ (respectively, $B_n$) are in a cluster algebra $\mathcal{A}$ and the other half of the minimal affinizations of type $A_n$ (respectively, $B_n$) are in $\mathcal{F}$ which is dual to $\mathcal{A}$).

The paper is organized as follows. In Section 2 we give some background information about cluster algebras and finite-dimensional representations of quantum affine algebras. In Section 3 we recall the definition of snake modules and path description of $q$-characters for snake modules of types $A_n$ and $B_n$. Moreover, we show that all prime snake modules of types $A_n$ and $B_n$ are real (Theorem 3.4). In Section 4 we describe the $S$-systems of types $A_n$ and $B_n$. In Section 5 we show that the Hernandez–Leclerc conjecture is true for all snake modules of types $A_n$ and $B_n$. In Section 6 we give some examples of mutation sequences for some snake modules. In Sections 7, 8 and 9, we prove Theorem 3.4, Theorem 4.1 and Theorem 4.3 respectively.

2. Preliminaries

2.1. Cluster algebras. Cluster algebras were invented by Fomin and Zelevinsky in \cite{FZ02}. Let $Q$ be the rational field and $\mathcal{F} = \mathbb{Q}(x_1, x_2, \ldots, x_n)$ the field of rational functions. A seed in $\mathcal{F}$ is a pair $\Sigma = (y, Q)$, where $y = (y_1, y_2, \ldots, y_n)$ is a free generating set of $\mathcal{F}$, and $Q$ is a quiver with vertices labeled by $1, 2, \ldots, n$. Assume that $Q$ has neither loops nor 2-cycles. For $k = 1, 2, \ldots, n$, one defines a mutation $\mu_k$ by $\mu_k(y, Q) = (y', Q')$. Here $y' = (y'_1, \ldots, y'_n)$, $y'_i = y_i$, for $i \neq k$, and

$$y'_k = \frac{\prod_{i \to k} y_i + \prod_{k \to j} y_j}{y_k}, \tag{2.1}$$

where the first (respectively, second) product in the right-hand side is over all arrows of $Q$ with target (respectively, source) $k$, and $Q'$ is obtained from $Q$ by

(i) adding a new arrow $i \to j$ for every existing pair of arrow $i \to k$ and $k \to j$;

(ii) reversing the orientation of every arrow with target or source equal to $k$;

(iii) erasing every pair of opposite arrows possible created by (i).

The mutation class $C(\Sigma)$ is the set of all seeds obtained from $\Sigma$ by a finite sequence of mutation $\mu_k$. If $\Sigma' = ((y'_1, y'_2, \ldots, y'_n), Q')$ is a seed in $C(\Sigma)$, then the subset $\{y'_1, y'_2, \ldots, y'_n\}$ is called a cluster, and its elements are called cluster variables. The cluster algebra $A_{\Sigma}$ is the subring of $\mathcal{F}$ generated by all cluster variables. Cluster monomials are monomials in the cluster variables supported on a single cluster.

In this paper, the initial seed in the cluster algebra we use is of the form $\Sigma = (y, Q)$, where $y$ is an infinite set and $Q$ is an infinite quiver.

Definition 2.1 (\cite{GG14, Definition 3.1}). Let $Q$ be a quiver without loops or 2-cycles and with a countably infinite number of vertices labeled by all integers $i \in \mathbb{Z}$. Furthermore, for each vertex $i$ of $Q$ let the number of arrows incident with $i$ be finite. Let $y = \{y_i \mid i \in \mathbb{Z}\}$. An infinite initial seed is the pair $(y, Q)$. By finite sequences of mutation at vertices of $Q$ and simultaneous mutation of the set $y$ using the exchange relation (2.1), one obtains a family of infinite seeds. The sets of variables in these seeds are called the infinite clusters and their elements are called the cluster variables. The cluster algebra of infinite rank of type $Q$ is the subalgebra of $\mathbb{Q}(y)$ generated by the cluster variables.

Two quivers $Q_1$ and $Q_2$ related by a sequence of mutations are called mutation equivalent, and we write $Q_1 \sim Q_2$. 


2.2. Quantum affine algebras. Let $\mathfrak{g}$ be a simple Lie algebra and $I = \{1, \ldots, n\}$ the indices of the Dynkin diagram of $\mathfrak{g}$ (we use the same labeling of the vertices of the Dynkin diagram of $\mathfrak{g}$ as the one used in [Car05]). Let $C = (C_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, where $C_{ij} = \frac{2(a_i,a_j)}{(a_i,a_i)}$. There is a matrix $D = \text{diag}(d_i \mid i \in I)$ with entries in $\mathbb{Z}_{>0}$ such that $B = DC = (b_{ij})_{i,j \in I}$ is symmetric. We have $D = \text{diag}(d_i \mid i \in I)$, where $d_i = 1$, $i \in I$, for type $A_n$ and $d_i = 2$, $i = 1, \ldots, n - 1$, $d_n = 1$, for type $B_n$. Let $t = \max\{d_i \mid i \in I\}$. Then $t = 1$ for type $A_n$ and $t = 2$ for type $B_n$.

Let $q_i = q^{d_i}$, $i \in I$. Let $Q$ (respectively, $Q^+$) and $P$ (respectively, $P^+$) denote the $\mathbb{Z}$-span (respectively, $\mathbb{Z}_{>0}$-span) of the simple roots and fundamental weights respectively. Let $\leq$ be the partial order on $P$ in which $\lambda \leq \lambda'$ if and only if $\lambda' - \lambda \in Q^+$.

Quantum groups were introduced independently by Jimbo [Jim 85] and Drinfeld [Dri87]. Quantum affine algebras form a family of infinite-dimensional quantum groups. Let $\widehat{\mathfrak{g}}$ denote the untwisted affine algebra corresponding to $\mathfrak{g}$ as the one used in [Car05]). Let $\mathfrak{g}$ be the free abelian multiplicative group of monomials in infinitely many formal variables $x_{i,n}^\pm (i \in I, n \in \mathbb{Z})$, $h_{i,n} (i \in I, n \in \mathbb{Z} \setminus \{0\})$ and central elements $c^\pm 1/2$, subject to certain relations.

The algebra $U_q\mathfrak{g}$ is isomorphic to a subalgebra of $U_q\widehat{\mathfrak{g}}$. Therefore, $U_q\widehat{\mathfrak{g}}$-modules restrict to $U_q\mathfrak{g}$-modules.

2.3. Finite-dimensional $U_q\widehat{\mathfrak{g}}$-modules and their $q$-characters. We recall some known results on finite-dimensional $U_q\widehat{\mathfrak{g}}$-modules and their $q$-characters, [CP94], [CP95a], [FR98], [MY12].

Let $P$ be the free abelian multiplicative group of monomials in infinitely many formal variables $(Y_{i,a})_{i \in I, a \in \mathbb{C}^\times}$. Then $\mathbb{Z}P = \mathbb{Z}[Y_{i,a}^\pm]_{i \in I, a \in \mathbb{C}^\times}$. For each $j \in I$, a monomial $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, where $u_{i,a}$ are some integers, is said to be $j$-dominant (respectively, $j$-anti-dominant) if and only if $u_{j,a} \geq 0$ (respectively, $u_{j,a} \leq 0$) for all $a \in \mathbb{C}^\times$. A monomial is called dominant (respectively, anti-dominant) if and only if it is $j$-dominant (respectively, $j$-anti-dominant) for all $j \in I$.

Every finite-dimensional simple $U_q\widehat{\mathfrak{g}}$-module is parametrized by a dominant monomial in $P^+$, [CP94], [CP95a]. That is, for a dominant monomial $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{u_{i,a}}$, there is a corresponding simple $U_q\widehat{\mathfrak{g}}$-module $L(m)$.

The $q$-character of a $U_q\widehat{\mathfrak{g}}$-module $V$ is given by

$$\chi_q(V) = \sum_{m \in \mathbb{Z}P} \dim(V_m) m \in \mathbb{Z}P,$$

where $V_m$ is the $l$-weight space with $l$-weight $m$, see [FR98]. We use $\mathcal{M}(V)$ to denote the set of all monomials in $\chi_q(V)$ for a finite-dimensional $U_q\widehat{\mathfrak{g}}$-module $V$. Let $P^+ \subset P$ denote the set of all dominant monomials. For $m_+ \in P^+$, we use $\chi_q(L(m_+))$ to denote $\chi_q(L(m_+))$. We also write $m \in \chi_q(m_+)$. Let $\mathcal{M}(m_+) \in \mathcal{M}(L(m_+))$. The following lemma is well-known.

Lemma 2.2. Let $m_1, m_2$ be two monomials. Then $L(m_1 m_2)$ is a sub-quotient of $L(m_1) \otimes L(m_2)$. In particular, $\mathcal{M}(L(m_1 m_2)) \subseteq \mathcal{M}(L(m_1)) \mathcal{M}(L(m_2))$. □

A finite-dimensional $U_q\widehat{\mathfrak{g}}$-module $V$ is said to be special if and only if $\mathcal{M}(V)$ contains exactly one dominant monomial. It is anti-special if and only if $\mathcal{M}(V)$ contains exactly one anti-dominant monomial. It is thin if and only if no $l$-weight space of $V$ has dimension greater than 1. Clearly, if a module is special or anti-special, then it is simple.
The elements $A_i,a \in \mathcal{P}, i \in I, a \in \mathbb{C}^\times$, are defined by
\[
A_i,a = Y_i,\hat{a} Y_i,\hat{aq}^{-1} \left( \prod_{j:C_{ji}=-1} Y_{j,a}^{-1} \right) \left( \prod_{j:C_{ji}=-2} Y_{j,aq} Y_{j,aq^{-1}}^{-1} \right) \left( \prod_{j:C_{ji}=-3} Y_{j,aq^2} Y_{j,aq^{-2}} Y_{j,aq}^{-1} \right),
\]
see [FR98]. Let $Q$ be the subgroup of $\mathcal{P}$ generated by $A_i,a, i \in I, a \in \mathbb{C}^\times$. Let $Q^\pm$ be the monoids generated by $A_i^\pm, i \in I, a \in \mathbb{C}^\times$. There is a partial order $\leq$ on $\mathcal{P}$ in which
\[
m \leq m' \text{ if and only if } m'm^{-1} \in Q^+.
\]
For all $m_+ \in \mathcal{P}^+, \mathcal{M}(L(m_+)) \subset m_+ Q^-$, see [FM01].

The concept of right negative was introduced in Section 6 of [FM01].

**Definition 2.3.** A monomial $m$ is called right negative if for all $a \in \mathbb{C}^\times$, for $L = \max\{ l \in \mathbb{Z} | u_{i,\hat{a}l}(m) \neq 0 \text{ for some } i \in I \}$ we have $u_{i,\hat{a}l}(m) \leq 0$ for $j \in I$.

For $i \in I, a \in \mathbb{C}^\times$, $A_i^{-1}$ is right-negative. A product of right-negative monomials is right-negative. If $m$ is right-negative and $m' \leq m$, then $m'$ is right-negative, see [FM01], [Her06].

### 2.4. $q$-Characters of $U_q\hat{sl}_2$-modules and the Frenkel–Mukhin algorithm.

We recall the results of the $q$-characters of $U_q\hat{sl}_2$-modules which are well-understood, see [CP91], [FR98].

Let $W_k^{(a)}$ be the irreducible representation $U_q\hat{sl}_2$ with highest weight monomial
\[
X_k^{(a)} = \prod_{i=0}^{k-1} Y_{aq^{2i-1}},
\]
where $Y_a = Y_1,a$. Then the $q$-character of $W_k^{(a)}$ is given by
\[
\chi_q(W_k^{(a)}) = X_k^{(a)} \sum_{i=0}^{k-1} \prod_{j=0}^{i-1} A_{aq^{2j-2}},
\]
where $A_a = Y_{aq^{-1}} Y_{aq}$.

For $a \in \mathbb{C}^\times, k \in \mathbb{Z}_{\geq 1}$, the set $\Sigma_k^{(a)} = \{ aq^{2i-1} | i=0,\ldots,k-1 \}$ is called a string. Two strings $\Sigma_k^{(a)}$ and $\Sigma_k^{(a')}$ are said to be in general position if the union $\Sigma_k^{(a)} \cup \Sigma_k^{(a')}$ is not a string or $\Sigma_k^{(a)} \subset \Sigma_k^{(a')}$ or $\Sigma_k^{(a')} \subset \Sigma_k^{(a)}$.

Denote by $L(m_+)$ the irreducible $U_q\hat{sl}_2$-module with highest weight monomial $m_+$. Let $m_+ \neq 1$ and $m_+ \in \mathbb{Z}[Y_a,a \in \mathbb{C}^\times$ be a dominant monomial. Then $m_+$ can be uniquely (up to permutation) written in the form
\[
m_+ = \prod_{i=1}^{s} \left( \prod_{b \in \Sigma_{k_i}^{(a_i)}} Y_b \right),
\]
where $s$ is an integer, $\Sigma_{k_i}^{(a_i)}, i = 1,\ldots,s$, are strings which are pairwise in general position and
\[
L(m_+) = \bigotimes_{i=1}^{s} W_{k_i}^{(a_i)}, \quad \chi_q(L(m_+)) = \prod_{i=1}^{s} \chi_q(W_{k_i}^{(a_i)}).
\]
For \( j \in I \), let
\[
\beta_j : \mathbb{Z}[Y_{i,a}^\pm]_{i \in I, a \in \mathbb{C}^\times} \to \mathbb{Z}[Y_{i,a}^\pm]_{a \in \mathbb{C}^\times}
\]
be the ring homomorphism such that for all \( a \in \mathbb{C}^\times \), \( \beta_j(Y_{k,a}) = 1 \) for \( k \neq j \) and \( \beta_j(Y_{j,a}) = Y_a \).

Let \( V \) be a \( U_q\hat{\mathfrak{g}} \)-module. Then \( \beta_i(\chi_q(V)) \), \( i \in I \), is the \( q \)-character of \( V \) considered as a \( U_q\hat{\mathfrak{g}} \)-module.

The Frenkel–Mukhin algorithm was introduced to compute the \( q \)-characters of \( U_q\hat{\mathfrak{g}} \)-modules in Section 5 of [FM01]. The algorithm is based on the \( q \)-characters of \( U_q\hat{s\mathfrak{l}_2} \)-modules. In Theorem 5.9 of [FM01], it is shown that the Frenkel–Mukhin algorithm works for modules which are special.

In some cases, the Frenkel–Mukhin algorithm does not return all terms in the \( q \)-character of a module. There are some counterexamples given in [NN11]. However, the Frenkel–Mukhin algorithm produces the correct \( q \)-characters of modules in many cases. In particular, if a module \( L(m_+) \) is special, then the Frenkel–Mukhin algorithm applied to \( m_+ \), see [FM01], produces the correct \( q \)-character \( \chi_q(L(m_+)) \).

We will need the following proposition from [HL10].

**Proposition 2.4** ([Her05], Proposition 3.1; [HL10], Proposition 5.9). Let \( V \) be a \( U_q\hat{\mathfrak{g}} \)-module and fix \( i \in I \). Then there is a unique decomposition of \( \chi_q(V) \) as a finite sum
\[
\chi_q(V) = \sum_{m \in \mathcal{P}_{i,+}} \lambda_m \varphi_i(m),
\]
and the \( \lambda_m \) are non-negative integers.

Here \( \varphi_i(m) \) \( (m \in \mathcal{P}_{i,+}) \) is a polynomial defined as follows, see Section 5.2.1 of [HL10]. Let \( m \in \mathcal{P}_{i,+} \) be an \( i \)-dominant monomial. Let \( \overline{m} \) be the monomial obtained from \( m \) by replacing \( Y_{j,a} \) with \( Y_a \) if \( j = i \) and by 1 if \( j \neq i \). Then the \( q \)-character \( \chi_q(L(\overline{m})) \) of the \( U_q\hat{s\mathfrak{l}_2} \)-module \( L(\overline{m}) \) is given by (2.4), (2.5). Write \( \chi_q(L(\overline{m})) = \overline{m}(1 + \sum_p \overline{M}_p) \), where the \( \overline{M}_p \) are monomials in the variables \( A_a^{-1} \) \( (a \in \mathbb{C}^\times) \). Then one sets \( \varphi_i(m) := m(1 + \sum_p M_p) \) where each \( M_p \) is obtained from the corresponding \( \overline{M}_p \) by replacing each variable \( A_a^{-1} \) by \( A_a^{-1} \).

The following corollary follows from Proposition 2.4 see [HL10].

**Corollary 2.5** ([HL10]). Let \( m \in \mathcal{P}_+ \) and \( mM \) a monomial of \( \chi_q(L(m)) \), where \( M \) is a monomial in the \( A_{i,a}^{-1} \) \( (j \in I) \). If \( M \) contains no variable \( A_{i,a} \), then \( mM \in \mathcal{P}_{i,+} \) and \( \varphi_i(mM) \) is contained in \( \chi_q(L(m)) \). In particular, \( \varphi_i(m) \) is contained in \( \chi_q(L(m)) \).

3. Snake modules of types \( A_n \) and \( B_n \)

In this section, we recall the definition of snake modules which were introduced by Mukhin and Young in [MY12a], [MY12b]. In the following, we assume that \( \mathfrak{g} \) is of type \( A_n \) or \( B_n \).

3.1. Snake positions and minimal snake positions. We recall the definitions of snake positions and minimal snake positions introduced in Section 4 of [MY12a] and Section 3 of
A subset $X \subset I \times \mathbb{Z}$ and an injective map $\iota : X \to \mathbb{Z} \times \mathbb{Z}$ are defined as follows.

**Type $A_n$**: Let $X := \{(i, k) \in I \times \mathbb{Z} : i-k \equiv 0 \pmod{2}\}$ and $\iota(i, k) = (i, k)$.

**Type $B_n$**: Let $X := \{(n, 2k) : k \in \mathbb{Z}\} \sqcup \{(i, k) \in I \times \mathbb{Z} : i < n \text{ and } k \equiv 1 \pmod{2}\}$ and

$$
\iota(i, k) = \begin{cases} 
(2i, k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 1 \pmod{4}, \\
(4n - 2 - 2i, k), & \text{if } i < n \text{ and } 2n + k - 2i \equiv 3 \pmod{4}, \\
(2n - 1, k), & \text{if } i = n.
\end{cases}
$$

Let $A, B$ be two sets. We define a map $\text{pr}_1 : A \times B \to A$ given by $\text{pr}_1(a, b) = a$.

Let $(i, k) \in X$. A point $(i', k') \in X$ is said to be in *snake position* with respect to $(i, k)$ if and only if

**Type $A_n$**: $k' - k \geq |i' - i| + 2$ and $k' - k \equiv |i' - i| \pmod{2}$.

**Type $B_n$**:

$$
\begin{align*}
&i = i' = n : k' - k \geq 2 \text{ and } k' - k \equiv 2 \pmod{4}, \\
&i \neq i' = n \text{ or } i' \neq i = n : k' - k \geq 2|i' - i| + 3 \text{ and } k' - k \equiv 2|i' - i| - 1 \pmod{4}, \\
&i < n \text{ and } i' < n : k' - k \geq 2|i' - i| + 4 \text{ and } k' - k \equiv 2|i' - i| \pmod{4}.
\end{align*}
$$

The point $(i', k')$ is in *minimal* snake position to $(i, k)$ if and only if $k' - k$ is equal to the given lower bound.

**Remark 3.1.** The above condition for type $A_n$ is slightly different from the condition for type $A_n$ in Section 4.2 of [MY12a] and Section 3.2 of [MY12b].

### 3.2. Prime snake positions

Let $(i, k) \in X$. A point $(i', k') \in X$ is said to be in *prime snake position* with respect to $(i, k)$ if and only if

**Type $A_n$**: $\min\{2n + 2 - i - i', i + i'\} \geq k' - k \geq |i' - i| + 2$ and $k' - k \equiv |i' - i| \pmod{2}$.

**Type $B_n$**: $\begin{align*}
&i = i' = n : 4n - 2 \geq k' - k \geq 2 \text{ and } k' - k \equiv 2 \pmod{4}, \\
&i \neq i' = n \text{ or } i' \neq i = n : 2i' + 2i - 1 \geq k' - k \geq 2|i' - i| + 3 \text{ and } k' - k \equiv 2|i' - i| - 1 \pmod{4}, \\
&i < n \text{ and } i' < n : 2i' + 2i - 1 \geq k' - k \geq 2|i' - i| + 4 \text{ and } k' - k \equiv 2|i' - i| \pmod{4}.
\end{align*}$

**Remark 3.2.** The above condition for type $A_n$ is slightly different from the condition for type $A_n$ in Section 3.3 of [MY12a].

### 3.3. Snakes and snake modules

A finite sequence $(i_t, k_t)$, $1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 0}$, of points in $X$ is called a *snake* if and only if for all $2 \leq t \leq T$, $(i_t, k_t)$ is in snake position with respect to $(i_{t-1}, k_{t-1})$, [MY12a, MY12b]. It is called a *minimal* (respectively, *prime*) snake if and only if successive points are in minimal (respectively, prime) snake position, [MY12a, MY12b].

The simple module $L(m)$ is called a *snake module* (respectively, a minimal snake module) if and only if $m = \prod_{t=1}^{T} Y_{i_t, k_t}$ for some snake $(i_t, k_t)_{1 \leq t \leq T}$ (respectively, for some minimal snake $(i_t, k_t)_{1 \leq t \leq T}$, [MY12a, MY12b].

**Theorem 3.3 ([MY12b, Proposition 3.1]).** A snake module is prime if and only if its snake is prime. Every snake module can be uniquely (up to permutation) decomposed into a tensor of prime snake modules.

We have the following theorem.
Theorem 3.4. Prime snake modules are real.

Theorem 3.4 will be proved in Section 7.

In this paper, when we write the highest $l$-weight monomial $m$ of a snake module $L(m)$ explicitly, we write $m = Y_{i_1,k_1}Y_{i_2,k_2} \cdots Y_{i_T,k_T}$ such that $k_t, 1 \leq t \leq T$, are in increasing order.

3.4. Path description of $q$-characters for snake modules of types $A_n$ and $B_n$. We will review the path description of $q$-characters for snake modules of types $A_n$ and $B_n$, see Section 5 of [MY12a] and Section 6 of [MY12b] for further details.

A path is a finite sequence of points in the plane $\mathbb{R}^2$. We write $(j, \ell) \in p$ if $(j, \ell)$ is a point of the path $p$.

The following is the case of type $A_n$. For all $(i, k) \in \mathcal{X}$, let

$$\mathcal{P}_{i,k} = \{(0, y_0), (1, y_1), \ldots, (n, y_{n+1}) : y_0 = i + k, y_{n+1} = n + 1 - i + k, \text{ and } y_{i+1} - y_i \in \{1, -1\}, 0 \leq i \leq n\}.$$ 

The sets $C^\pm_p$ of upper and lower corners of a path $p = ((r, y_r))_{0 \leq r \leq n+1} \in \mathcal{P}_{i,k}$ are defined as follows:

$$C^+_p = \{(r, y_r) \in p : r \in I, y_{r-1} = y_r + 1 = y_{r+1}\},$$

$$C^-_p = \{(r, y_r) \in p : r \in I, y_{r-1} = y_r - 1 = y_{r+1}\}.$$ 

The following is the case of type $B_n$. Fix an $\varepsilon$, $0 < \varepsilon < 1/2$, $\mathcal{P}_{n,\ell}$ for all $\ell \in 2\mathbb{Z}$ are defined as follows.

For all $\ell \equiv 3 \text{ mod } 4$,

$$\mathcal{P}_{n,\ell} = \{((0, y_0), (2, y_1), \ldots, (2n - 4, y_{n-2}), (2n - 2, y_{n-1}), (2n - 1, y_n)) : y_0 = \ell, y_{n+1} - y_i \in \{2, -2\}, 0 \leq i \leq n - 2, \text{ and } y_{n+1} - y_i \in \{1 + \varepsilon, -1 - \varepsilon\}\}.$$ 

For all $\ell \equiv 1 \text{ mod } 4$,

$$\mathcal{P}_{n,\ell} = \{((4n - 2, y_0), (4n - 4, y_1), \ldots, (2n + 2, y_{n-2}), (2n, y_{n-1}), (2n - 1, y_n)) : y_0 = \ell + 2n - 1, y_{n+1} - y_i \in \{2, -2\}, 0 \leq i \leq n - 2, \text{ and } y_{n+1} - y_i \in \{1 + \varepsilon, -1 - \varepsilon\}\}.$$ 

For all $(i, k) \in \mathcal{X}$, $i < n$, $\mathcal{P}_{i,k}$ are defined as follows:

$$\mathcal{P}_{i,k} = \{(a_0, a_1, \ldots, a_n, \overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) : (a_0, a_1, \ldots, a_n) \in \mathcal{P}_{n,k - (2n - 2i - 1)}, (\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_n) \in \mathcal{P}_{n,k + (2n - 2i - 1)}, \text{ and } a_n - \overline{a}_n = (0, y) \text{ where } y > 0\}.$$ 

The sets of upper and lower corners $C^\pm_p$ of a path $p = ((j, \ell_j))_{0 \leq r \leq |p|-1} \in \mathcal{P}_{i,k}$, where $|p|$ is the number of points in the path $p$, are defined as follows:

$$C^+_p = \ell^{-1}\{(j, \ell) \in p : j_e \notin \{0, 2n - 1, 4n - 2\}, \ell_{r-1} > \ell_r, \ell_{r+1} > \ell_r \} \cup \{(n, \ell) \in \mathcal{X} : (2n - 1, \ell - \varepsilon) \notin p \text{ and } (2n - 1, \ell + \varepsilon) \notin p\},$$

$$C^-_p = \ell^{-1}\{(j, \ell) \in p : j_e \notin \{0, 2n - 1, 4n - 2\}, \ell_{r-1} < \ell_r, \ell_{r+1} < \ell_r \} \cup \{(n, \ell) \in \mathcal{X} : (2n - 1, \ell - \varepsilon) \notin p \text{ and } (2n - 1, \ell + \varepsilon) \notin p\}.$$
A map $m$ sending paths to monomials is defined by

$$m : \bigcup_{(i,k) \in \mathcal{X}} \mathcal{P}_{i,k} \rightarrow \mathbb{Z}[Y^\pm_{j,\ell} | (j,\ell) \in \mathcal{X}].$$

$$p \mapsto m(p) = \prod_{(j,\ell) \in C^+_p} Y_{j,\ell} \prod_{(j,\ell) \in C^-_p} Y_{j,\ell}^{-1}. \quad (3.1)$$

We identify a path $p$ with the monomial $m(p)$. Let $p, p'$ be paths. It is said that $p$ is strictly above $p'$ or $p'$ is strictly below $p$ if

$$(x, y) \in p \text{ and } (x, z) \in p' \implies y < z.$$ 

It is said that a $T$-tuple of paths $(p_1, \ldots, p_T)$ is non-overlapping if $p_s$ is strictly above $p_t$ for all $s < t$. For any snake $(i_t, k_t) \in \mathcal{X}$, $1 \leq t \leq T$, $T \in \mathbb{Z}_{\geq 1}$, $\mathcal{P}(i_t, k_t)_{1 \leq t \leq T}$ is defined by

$$\mathcal{P}(i_t, k_t)_{1 \leq t \leq T} = \{ (p_1, \ldots, p_T) : p_t \in \mathcal{P}_{i_t,k_t}, 1 \leq t \leq T, (p_1, \ldots, p_T) \text{ is non-overlapping} \}.$$ 

**Theorem 3.5** ([MY12a], Theorem 6.1; [MY12b], Theorem 6.5). Let $(i_t, k_t) \in \mathcal{X}$, $1 \leq \ell \leq T$, be a snake of length $T \in \mathbb{Z}_{\geq 1}$. Then

$$\chi_q(L(\prod_{\ell=1}^{T} Y_{i_t,k_t})) = \sum_{(p_1, \ldots, p_T) \in \mathcal{P}(i_t,k_t)_{1 \leq t \leq T}} \prod_{\ell=1}^{T} m(p_t). \quad (3.2)$$

The module $L(\prod_{\ell=1}^{T} Y_{i_t,k_t})$ is thin, special and anti-special.

By Theorem 3.5, the $q$-characters of snake modules of types $A_n$ and $B_n$ with length $T$ are given by a set of $T$-tuples of non-overlapping paths. The paths in each $T$-tuple are non-overlapping. This property is called the non-overlapping property.

We also need the following notations in this paper. For all $(i, k) \in \mathcal{X}$, let $p^+_{i,k}$ be the highest path which is the unique path in $\mathcal{P}_{i,k}$ with no lower corners and $p^-_{i,k}$ the lowest path which is the unique path in $\mathcal{P}_{i,k}$ with no upper corners.

## 4. $S$-systems of types $A_n$ and $B_n$

In this section, we introduce a closed system of equations which contains all prime snake modules of type $A_n$ (respectively, $B_n$) and only contains prime snake modules of type $A_n$ (respectively, $B_n$).

### 4.1. Another notation of snake modules

In order to introduce the $S$-systems, we need to use another notation of snake modules. We fix an $a \in \mathbb{C}^\times$ and denote $i_s = Y_{i_s,aq^s}$, where $i \in I$, $s \in \mathbb{Z}$.

Every snake module of type $A_n$ is a module with highest $l$-weight monomial of the form

$$S^{(l)}_{i_1, \ldots, i_m}(\ell_1, j_1, \ell_2, j_2, \ldots, \ell_{m-1}, j_{m-1}, \ell_m, j_m) := \prod_{j=1}^{m} \left( \prod_{r=0}^{k_j-1} (i_j)^{\ell_1+2r+\sum_{i=1}^{r} n_i} \right), \quad (4.1)$$

where $t \in \mathbb{Z}$, $i_j \in I$, $k_j \geq 0$, $1 \leq j \leq m$, $j_t \in \mathbb{Z}_{\geq 0}$, $1 \leq \ell \leq m - 1$, and

$$n_\ell = 2k_\ell + |i_{\ell+1} - i_\ell| + 2j_\ell. \quad (4.2)$$
In type $B_n$, for $i,j \in I$, we define $\varepsilon_{i,j} = -\delta_{in} - \delta_{jn}$, where $\delta_{ij}$ is the Kronecker delta. Every snake module of type $B_n$ is a module with highest $l$-weight monomial of the form

$$S^{(t)}_{k_1^{(i_1,j_1)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}} := \prod_{j=1}^{m} \left( \prod_{r=0}^{k_j-1} (i_j)^{t+2d_{ij}r+\sum_{s=1}^r j_s} \right),$$

(4.3)

where $t \in \mathbb{Z}$, $i,j \in I$, $k_j \geq 0$, $1 \leq j \leq m$, and

$$n_\ell = 2d_{ij}k_\ell + 2|i_\ell+1 - i_\ell| + 4 - 2d_{ij} + 4j_\ell + \varepsilon_{i_\ell,i_{\ell+1}}.$$  

(4.4)

Let $S$ be a dominant monomial. We also use $S$ to denote $L(S)$. For example, we use $S_{k_1^{(i_1,j_1)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}}$ to denote the irreducible finite-dimensional $U_q\hat{g}$-module with highest $l$-weight monomial $S_{k_1^{(i_1,j_1)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}}$.

For simplicity, if $j_\ell = 0$ for some $\ell$, $1 \leq \ell \leq m - 1$, then we use

$$S^{(t)}_{k_1^{(i_1,j_1)}, \ldots, k_\ell^{(i_\ell,j_\ell)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}}$$

to denote $S^{(t)}_{k_1^{(i_1,j_1)}, \ldots, k_\ell^{(i_\ell,j_\ell)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}}$. In this notation, $S^{(t)}_{k_1^{(i_1,j_1)}, \ldots, k_\ell^{(i_\ell,j_\ell)}, \ldots, k_m^{(i_{m-1},j_{m-1}), k_m^{(i_m)}}}$ is a minimal snake module.

Let $\mathcal{S}$ be the set of all snake modules and $\mathfrak{T}$ the set of all snakes. We define a map

$$\varphi : \mathcal{S} \rightarrow \mathfrak{T}$$
$$S \mapsto \text{the snake of } S.$$  

(4.5)

It is easy to see that the map $\varphi : \mathcal{S} \rightarrow \mathfrak{T}$ is a bijection.

4.2. Neighboring points. The concept of neighboring points was introduced in Section 3 of [MY12b]. Let $(i,k) \in \mathcal{X}$ and $(i',k') \in \mathcal{X}$ such that $(i',k')$ is in prime snake position with respect to $(i,k)$. The neighboring points to the pair $(i,k)$, $(i',k')$ are two finite sequences $X_{i,k}^{i',k'}$ and $Y_{i,k}^{i',k'}$ of points in $\mathcal{X}$ defined as follows.
In type $A_n$, let
\[
X_{i,k}^{i',k'} = \begin{cases} 
\emptyset, & k+i > k' - i', \\
\left\{ \left( \frac{1}{2}(i+k+i' - k'), \frac{1}{2}(i+k - i' + k') \right) \right\}, & k+i = k' - i'. 
\end{cases}
\]
\[
Y_{i,k}^{i',k'} = \begin{cases} 
\emptyset, & k+i > k' - i', \\
\left\{ \left( \frac{1}{2}(i'+k' + i - k), \frac{1}{2}(i' + k' - i + k) \right) \right\}, & k+n+1 - i > k' - n - 1 + i', \\
\left\{ \left( \frac{1}{2}(i'+k' + i - k), \frac{1}{2}(i' + k' - i + k) \right) \right\}, & k+n+1 - i = k' - n - 1 + i'. 
\end{cases}
\]

In type $B_n$, let
\[
\begin{align*}
(X_{i,k}^{i',k'}, Y_{i,k}^{i',k'}) &= \left\{ \begin{array}{ll}
\emptyset, & \text{if } i < n, 2n+k - 2i \equiv 1 \pmod{4}, \text{ or } i = n, k \equiv 0 \pmod{4}, \\
\left\{ \left( \frac{1}{2}(2i+k+2i'-k'), \frac{1}{2}(2i+k-2i'+k') \right) \right\}, & \text{if } i < n, i' < n, k'-k = 2i + 2i', \\
\left\{ \left( \frac{1}{2}(2i+k+2n-1-k'), \frac{1}{2}(2i+k-2n+1+k') \right) \right\}, & \text{if } i < n, i' < n, k'-k < 2i + 2i', \\
\left\{ \left( \frac{1}{2}(2i+k+2n-1-k'), \frac{1}{2}(2i+k-2n+1+k') \right) \right\}, & \text{if } i < n, i' = n, k'-k = 2i + 2n - 1, \\
\left\{ \left( \frac{1}{2}(2n-1+k+2i'-k'), \frac{1}{2}(2n-1+k-2i'+k') \right) \right\}, & \text{if } i < n, i' < n, k'-k < 2n + 2i' - 1, \\
\left\{ \left( \frac{1}{2}(4n-2+k+k') \right) \right\}, & \text{if } i = n, i' = n.
\end{array} \right.
\end{align*}
\]

4.3. $S$-systems of types $A_n$ and $B_n$. In types $A_n$ and $B_n$, every prime snake module can be written as
\[
S^{(t)}_{k_1^{(1,j_1)}, k_2^{(1,j_2)}, \ldots, k_{m-1}^{(1,j_{m-1})}, k_m^{(1,j_m)}},
\]
where $m \geq 1$, $j_\ell \geq 0$, $1 \leq \ell \leq m - 1$, if $j_{\ell} = 0$, then $i_\ell \neq i_{\ell+1}$, $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}$.

Let $S_1$ be the prime snake module (4.6) and sgn($x$) the sign function. We define $S_1$ in Table 1 (respectively, Table 2) for type $A_n$ (respectively, $B_n$).
| Conditions | $\mathcal{S}_1$ |
|------------|----------------|
| $m = 1$    | $\mathcal{S}_1^{(1)}$ |
| $j_1 = 0$  | $\mathcal{S}_1^{(2)}$ if $i_2 \equiv 1 \pmod{2}$ |
| $j_1 = 0$  | $\mathcal{S}_1^{(2)}$ if $i_2 \equiv 0 \pmod{2}$ |

for $j_2 = 0$, $m \geq 3$, $(t_2 - t_1) (t_3 - t_2) > 0$, $j_2 \geq 0$, $3 \leq \ell \leq m - 1$

and $j_2 = 1$, $m \geq 3$, $(t_2 - t_1) (t_3 - t_2) > 0$, $j_2 \geq 0$, $3 \leq \ell \leq m - 1$

for $j_2 = 0$, $m \geq 3$, $(t_2 - t_1) (t_3 - t_2) \leq 0$, $j_2 \geq 0$, $3 \leq \ell \leq m - 1$

and $j_2 = 1$, $m \geq 3$, $(t_2 - t_1) (t_3 - t_2) \leq 0$, $j_2 \geq 0$, $3 \leq \ell \leq m - 1$

| Conditions | $\mathcal{S}_1$ |
|------------|----------------|
| $m = 1$    | $\mathcal{S}_1^{(2)}$ |
| $j_1 = 0$, $i_1 \neq n$, $i_2 = n$, $m = 2$, $k_2$ is odd | $\mathcal{S}_1^{(2)}$ |
| $j_1 = 0$, $i_1 \neq n$, $i_2 = n$, $m = 2$, $k_2$ is even | $\mathcal{S}_1^{(2)}$ |

| Conditions | $\mathcal{S}_1$ |
|------------|----------------|
| $m = 2$    | $\mathcal{S}_1^{(2)}$ |
| $j_1 = 0$, $i_1 = n$ | $\mathcal{S}_1^{(2)}$ |

| Conditions | $\mathcal{S}_1$ |
|------------|----------------|
| $m = 2$    | $\mathcal{S}_1^{(2)}$ |
| $j_1 = 0$, $i_2$ is odd, $m = 3$ | $\mathcal{S}_1^{(2)}$ |
| $j_1 = 0$, $i_2$ is even, $m = 3$ | $\mathcal{S}_1^{(2)}$ |

| Conditions | $\mathcal{S}_1$ |
|------------|----------------|
| $m = 2$    | $\mathcal{S}_1^{(2)}$ |
| $j_1 \geq 1$, $i_1 = n$ | $\mathcal{S}_1^{(2)}$ |

Table 1. Definition of $\mathcal{S}_1$ in type $A_n$.

Table 2. Definition of $\mathcal{S}_1$ in type $B_n$. 
Let $X_1 = \varphi(S_1)$ and $X_2 = \varphi(S_2)$, where $\varphi$ is defined in (15). We define
\begin{equation}
S_3 = L(Y_{i,t}) \prod_{(i,k) \in X_1} Y_{i,k}, \quad S_4 = L( \prod_{(i,k) \in X_2 \setminus \{i,t\}} Y_{i,k}). \tag{4.7}
\end{equation}

Let
\begin{align*}
X'_{i_1} &= \{ (i_1, t + 2d_{i_1}, j) : 1 \leq j \leq k_1 \} \subset X_1, \\
X_i &= \{ (i_1, t + 2d_{i_1}, j - 2d_i) : 1 \leq j \leq k_1 \} \subset X_2,
\end{align*}

and
\begin{align*}
X &= \prod_{(i,k) \in X_1} X_{i,k}^{i+k+2d_i}, \\
Y &= \prod_{(i,k) \in X_1} X_{i,k}^{i+k+2d_i}.
\end{align*}

If $m = 1$, let
\begin{equation}
S_5 = L( \prod_{(i,k) \in X} Y_{i,k}), \quad S_6 = L( \prod_{(i,k) \in Y} Y_{i,k}). \tag{4.8}
\end{equation}

If $m \geq 2$, we define $S_5, S_6$ as follows. In the case of type $A_n$, let
\begin{align*}
S_5 &= \begin{cases} 
L((\prod_{(i,k) \in X} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & i_1 \leq i_2, \\
L((\prod_{(i,k) \in X} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & i_1 > i_2,
\end{cases} \\
S_6 &= \begin{cases} 
L((\prod_{(i,k) \in Y} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & i_1 \leq i_2, \\
L((\prod_{(i,k) \in Y} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & i_1 > i_2.
\end{cases} \tag{4.9}
\end{align*}

In the case of type $B_n$, let
\begin{align*}
S_5 &= \begin{cases} 
L((\prod_{(i,k) \in X} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & \text{pr}_1((i_1, t)) < \text{pr}_1((i_2, t + n_1)), \text{ or } i_1 = i_2 \neq n, \text{ or } i_1 = i_2 = n, \text{ or } t + n_1 \equiv 2 \pmod{4}, \\
L((\prod_{(i,k) \in X} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & \text{pr}_1((i_1, t)) \geq \text{pr}_1((i_2, t + n_1)), \text{ or } i_1 = i_2 = n, \text{ or } t + n_1 \equiv 0 \pmod{4},
\end{cases} \\
S_6 &= \begin{cases} 
L((\prod_{(i,k) \in Y} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & \text{pr}_2((i_1, t)) = \text{pr}_2((i_2, t + n_1)), \text{ or } i_1 = i_2 \neq n, \text{ or } i_1 = i_2 = n, \text{ or } t + n_1 \equiv 2 \pmod{4}, \\
L((\prod_{(i,k) \in Y} Y_{i,k})(\prod_{(i,k) \in X_1 \setminus \{i_1\}} Y_{i,k})), & \text{pr}_2((i_1, t)) > \text{pr}_2((i_2, t + n_1)), \text{ or } i_1 = i_2 = n, \text{ or } t + n_1 \equiv 0 \pmod{4},
\end{cases} \tag{4.10}
\end{align*}

where $n_1$ is defined in (1.3) for type $B_n$, the map $\iota$ is defined in Section 3.1.

We have the following theorem.

**Theorem 4.1.** In type $A_n$ (respectively, $B_n$), let $S_2$ be the prime snake module (4.6). We have the following system of equations
\begin{equation}
[S_1][S_2] = [S_3][S_4] + [S_5][S_6], \tag{4.11}
\end{equation}

where $S_1$ is defined in Table 1 (respectively, Table 2), $S_3, S_4$ are defined in (4.7), $S_5, S_6$ are defined in (4.8), (4.9) (respectively, (4.10)).

We call the system of equations in Theorem 4.1 the $S$-system for type $A_n$ (respectively, $B_n$). In particular when $m = 1$, the system of equations in Theorem 4.1 is $T$-system for types $A_n$ and $B_n$, [Hero06], [KNS94]. The equations in the $S$-systems are different from the equations in the extended $T$-systems, [MY12]. Theorem 4.1 will be proved in Section 8.
Example 4.2. The following are some equations in the $S$-system for type $A_3$.

\[
[3_{-3}3_{-1}1][3_{-3}3_{-3}3_{-3}] = [3_{-3}3_{-3}3_{-3}][3_{-3}3] + [2_{-2}2_{-2}],
\]

\[
[3_{-3}3_{-1}][3_{-3}3_{-3}3_{-1}] = [3_{-3}3_{-3}3_{-3}][1_{-1}][2_{-1}][3_{-1}2_{-2}],
\]

\[
[3_{-3}3_{-1}3_{-3}3_{-2}] = [3_{-3}3_{-3}3_{-3}][2_{-2}3_{-2}] + [2_{-2}2_{-2}][2_{-2}2_{-2}3_{-2}].
\]

Moreover, we have the following theorem.

- Theorem 4.3. The modules in the summands on the right-hand side of each equation in Theorem 4.1 are simple.

Theorem 4.3 will be proved in Section 9.

4.4. The $s$-systems of types $A_n$ and $B_n$. Let $S$ be a $U_qG$-module. We use $\text{Res}(S)$ to denote the restriction of $S$ to $U_qG$. Let $\chi(M)$ be the character of a $U_qG$-module $M$. We have a system of equations

\[
\chi(\text{Res}(S_i))\chi(\text{Res}(S_2)) = \chi(\text{Res}(S_3))\chi(\text{Res}(S_4)) + \chi(\text{Res}(S_5))\chi(\text{Res}(S_6)),
\]

where $[S_1][S_2] = [S_3][S_4] + [S_5][S_6]$ are equations of the $S$-system for type $A_n$ (respectively, $B_n$). We call this system of equations the $s$-system of type $A_n$ (respectively, $B_n$).

5. Relation between $S$-systems and cluster algebras

In this section, we show that every equation in the $S$-system of type $A_n$ (respectively, $B_n$) corresponds to a mutation in some cluster algebra $\mathcal{A}$ (respectively, $\mathcal{A}'$) and every prime snake module of type $A_n$ (respectively, $B_n$) corresponds to some cluster variable in $\mathcal{A}$ (respectively, $\mathcal{A}'$). In particular, this proves that the Hernandez–Leclerc conjecture (Conjecture 1.1) is true for snake modules of types $A_n$ and $B_n$. 
5.1. Definition of cluster algebras $\mathcal{A}$ and $\mathcal{A}'$. We recall the definition of the cluster algebras introduced in [HL13]. Let $\tilde{V} = I \times \mathbb{Z}$ and let $\tilde{\Gamma}$ be a quiver with the vertex set $\tilde{V}$ whose arrows are given by $(i, r) \to (j, s)$ if and only if $b_{ij} \neq 0$ and $s = r + b_{ij}$, where $B = (b_{ij})_{i,j \in I} = DC$ is defined in Section 2.2.

It is shown that $\tilde{\Gamma}$ has two isomorphic components in [HL13]. Let $\Gamma$ be one of the components and $V$ its vertex set. Let $\psi$ be a function defined by $\psi(i, t) = (i, t + d_i)$ for $(i, t) \in V$. Let $W \subseteq I \times \mathbb{Z}$ be the image of $V$ under the map $\psi$ and let $G$ be the same quiver as $\Gamma$ but with vertices labeled by $W$. Let $W^{-} = W \cap (I \times \mathbb{Z}_{\leq 0})$ and let $Q$ be the full sub-quiver of $G$ with vertex set $W^{-}$.

Let $z^{-} = \{z_{i,t} : (i, t) \in W^{-}\}$ and let $\mathcal{A}$ be the cluster algebra defined by the initial seed $(z^{-}, Q)$. For convenience, we denote by $Q'$ and $\mathcal{A}'$ the quiver $\tilde{Q}$ and the cluster algebra $\mathcal{A}$ in the case of type $B_n$, respectively.

In the case of type $A_n$, let

$$s = \{s_{k(i)}^{(-2k+2)} | i \text{ is even, } k \in \mathbb{Z}_{\geq 1}\} \cup \{s_{k(i)}^{(-2k+1)} | i \text{ is odd, } k \in \mathbb{Z}_{\geq 1}\}.$$  \hspace{1cm} (5.1)

In the case of type $B_n$, let $s' = s_1 \cup s_2$, where

$$s_1 = \{s_{k(i)}^{(-2k-2)} | k \in \mathbb{Z}_{\geq 1}\},$$  \hspace{1cm} (5.2)

$$s_2 = \{s_{k(i)}^{(-4k+3)} , s_{k(i)}^{(-4k+1)} | i \in \{1, \ldots, n-1\}, k \in \mathbb{Z}_{\geq 1}\}. \hspace{1cm} (5.3)$$

Let $\mathcal{A}$ (respectively, $\mathcal{A}'$) be the cluster algebra defined by the initial seed $(s, Q)$ (respectively, $(s', Q')$). Here we identify $s$ (respectively, $s'$) with $z^{-}$ as follows. For $(i, t) \in W^{-}$, we identify $s_{k(i)}^{(l)}$ with $z_{i,t}$.

We say that $s_{k(i)}^{(l)}$ is at this vertex and we say that the label of this vertex is $(i, t)$. Let $Q$ (respectively, $Q'$) be a quiver which is mutation equivalent to $Q$ (respectively, $Q'$) in type $A_n$ (respectively, $B_n$). After we mutate at a vertex $v$ of $Q$ (respectively, $Q'$), the variable at $v$ is changed and the label of $v$ is changed.

In this paper, our mutation sequences satisfy this property: after we mutate a quiver using a mutation sequence, any two vertices in the current quiver we obtain have different labels. Suppose that the label of a vertex $v$ in $Q$ (respectively, $Q'$) is $(i, t)$. After we mutate at $v$, the label of $v$ becomes $(i, t - 2d_i)$. We use $(i, t)$ to denote the vertex with the label $(i, t)$.

5.2. Fundamental segments and distinguished factors.

Definition 5.1. Let $S$ be a prime snake module and $S$ its highest $l$-weight monomial. Then $S$ can be written as

$$S = S_{k_1(j_1, j_2)}^{(l_1, j_1, j_2, \ldots, j_{m-1}, j_m)} k_{m-1}^{(i_1, \ldots, i_{m-1}, i_m)}.$$  \hspace{1cm} (5.4)

where $m \geq 1$, $j_\ell \geq 0$, $1 \leq \ell \leq m - 1$, if $j_\ell = 0$, then $i_\ell \neq i_{\ell+1}$, $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}$.

Let

$$FS(S) = FS_1 \cup FS_2 \cup FS_3,$$  \hspace{1cm} (5.5)

where

$$FS_1 = \left\{ S_{k_m}^{(l_m)} : t_m = t + \sum_{j=1}^{m-1} n_j \right\},$$  \hspace{1cm} (5.6)
We call \( \mathcal{FS}(S) \) the set of fundamental segments of \( S \).

**Example 5.2.** In type \( A_5 \), we have

\[
\mathcal{FS}(2_{-124-85-55-340}) = \{4_0, 5_{-55-340}, 2_{-124-85-5} \}.
\]

In type \( A_4 \), we have

\[
\mathcal{FS}(2_{-163-133-112-82-43-1}) = \{3_{-1}, 2_{-43-1}, 2_{-82-4}, 3_{-133-112-8}, 2_{-163-13} \},
\]

\[
\mathcal{FS}(2_{-302-261-231-212-183-152-122-102-64-240}) = \{4_{-240}, 2_{-64-2}, 2_{-122-160-6, 3_{-152-12, 1_{-231-212-183-15, 2_{-261-23, 2_{-302-26} \}.}
\]

In type \( B_3 \), we have

\[
\mathcal{FS}(1_{-312-252-173-123-62-1}) = \{2_{-1}, 3_{-62-1}, 3_{-123-6, 2_{-173-12, 2_{-252-17}, 1_{-312-25} \},
\]

\[
\mathcal{FS}(2_{-432-352-312-253-183-83-3230}) = \{3_{-30}, 3_{-33-2}, 3_{-183-8}, 1_{-253-18, 2_{-352-312-25, 2_{-432-35} \}.}
\]

The following proposition is easy to prove.

**Proposition 5.3.** Let \( S \) be a prime snake module and \( S \) its highest \( l \)-weight monomial. Then \( S \) is uniquely determined by \( \mathcal{FS}(S) \).

**Definition 5.4.** Let \( S \) be a prime snake module and \( S \) its highest \( l \)-weight monomial. Let \( M \) be a monomial in \( \mathcal{FS}(S) \), the last factor of \( M \) is called the distinguished factor of \( M \).

**Example 5.5.** In type \( A_4 \), let \( S = 2_{-163-133-112-82-43-1} \). Then the set of distinguished factors of \( S \) is \( \{3_{-1}, 2_{-4}, 2_{-8, 3_{-13} \} \), see Figure 7.

In type \( B_3 \), let \( S = 1_{-312-252-173-123-62-1} \). Then the set of distinguished factors of \( S \) is \( \{2_{-1}, 3_{-6}, 3_{-12}, 2_{-17, 2_{-25} \} \), see Figure 8.
5.3. Distinguished sub-quivers. Let \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) be a quiver which is mutation equivalent to \( Q \) (respectively, \( Q' \)) and any two vertices in \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) have different labels. We define a subset \( \mathcal{Y} \) of the set of vertices in \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) as follows. In the case of type \( A_n \), let

\[
\mathcal{Y} = \{(i, k) \in \tilde{Q} : i - k \equiv 0 \pmod{2}\}.
\]

In the case of type \( B_n \), let

\[
\mathcal{Y} = \{(n, 2k) \in \tilde{Q}' : k \in \mathbb{Z}_{\leq 0}\} \cup \{(i, k) \in \tilde{Q}' : i < n \text{ and } k \equiv 1 \pmod{2}\}.
\]

For a quiver \( \mathcal{L} \), we use \( V(\mathcal{L}) \) to denote the set of its vertices. We define a distinguished sub-quiver \( \mathcal{L}_{i,t}^{\tilde{Q}} \) (respectively, \( \mathcal{L}_{i,t}^{\tilde{Q}'} \)) with respect to \( (i, t) \in V(\tilde{Q}) \) (respectively, \( V(\tilde{Q}') \)) in type \( A_n \) (respectively, \( B_n \)).

The map \( \iota \) is defined in Section 3.1. In the case of type \( A_n \), let

\[
V(\mathcal{L}_{i,t}^{\tilde{Q}}) = \{(i(j, y_j)) \in \mathcal{Y} : j \in I, y_i = t - 2, y_j = y_{j+1} + 1, 1 \leq j \leq i - 1, \text{ and } y_{j+1} = y_j + 1, i \leq j \leq n - 1\}.
\]

Figures 3, 4 illustrate distinguished sub-quivers \( \mathcal{L}_{4,0}^{Q} \), \( \mathcal{L}_{5,-3}^{Q} \) in the original quiver of type \( A_8 \) respectively.
In the case of type $B_n$,

- for all $i = n$, let

$$V(\mathcal{L}_{n,\ell}) = \{\iota(j, y_j) \in \mathcal{Y} : j \in I, \, y_n \in \{\ell - 4, \ell - 6\}, \, y_{n-1} = y_n + 3, \, y_j = y_{j+1} + 2, \, 1 \leq j \leq n - 2\};$$

- for all $i < n$, let

$$V(\mathcal{L}_{i,\ell}) = \{\iota(j, y_j) \in \mathcal{Y} : j \in I, \, y_i = \ell - 4, \, y_j = y_{j+1} + 2, \, 1 \leq j \leq i - 1, \, y_{j+1} = y_j + 2, \, i \leq j \leq n - 2, \, y_n = y_{n-1} + 1\} \cup \{\iota(j, y_j) \in \mathcal{Y} : j \in I, \, y_n = \ell + 2n - 2i - 7, \, y_{n-1} = y_n + 3, \, y_j = y_{j+1} + 2, \, 1 \leq j \leq n - 2\}.$$
Similarly, we define mutation sequences with respect to a quiver.

5.4. Mutation sequences with respect to a quiver. Let \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) be a quiver which is mutation equivalent to \( Q \) (respectively, \( Q' \)) and any two vertices in \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) have different labels. By saying that we mutate \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)), we mean that we mutate at the vertex of \( \tilde{Q} \) (respectively, \( \tilde{Q}' \)) which has the label \((i, t)\) in the \( i \)-th column and so on until the vertex at infinity in the \( i \)-th column.

For convenience, in the case of type \( B_n \), let

\[
V'(l(\tilde{Q}')) = \begin{cases} 
V(l(\tilde{Q}')) , & i \neq n, 2n - 2i + \ell \equiv 1 \ (\text{mod} \ 4), \text{or} \ i = n, \\
V(\tilde{Q}')(2n - 1, \ell + 2n - 2i - 7) , & i \neq n, 2n - 2i + \ell \equiv 3 \ (\text{mod} \ 4), \\
\end{cases}
\]

\[
V'(r(\tilde{Q}')) = \begin{cases} 
V(r(\tilde{Q}')) - \{(2n - 1, \ell + 2n - 2i - 7)\} , & i \neq n, 2n - 2i + \ell \equiv 1 \ (\text{mod} \ 4), \\
\end{cases}
\]

For type \( A_n \) (respectively, \( B_n \)), suppose that

\[
V(l(\tilde{Q}')) \ (\text{respectively}, \ V'(l(\tilde{Q}'))) = \{(j_1, t_1), (j_2, t_2), \ldots, (j_m, t_m)\},
\]

where \( j_1 < j_2 < \cdots < j_m < \text{pr}_1(\ell(i, \ell)) \),

and

\[
V(r(\tilde{Q}')) \ (\text{respectively}, \ V'(r(\tilde{Q}'))) = \{(j_1, t_1), (j_2, t_2), \ldots, (j_m, t_m)\},
\]

where \( j_1 > j_2 > \cdots > j_m > \text{pr}_1(\ell(i, \ell)) \).

In type \( A_n \) (respectively, \( B_n \)), we say that we mutate \( l(\tilde{Q}) \) (respectively, \( l(\tilde{Q}') \)), we mean that we mutate \( C_{j_1, t_1}, C_{j_2, t_2}, \ldots, C_{j_m, t_m} \). We say that we mutate \( r(\tilde{Q}) \) (respectively, \( r(\tilde{Q}') \)),
we mean that we mutate $C_{j_1,t_1}$, $C_{j_2,t_2}$, \ldots, $C_{j_m,t_m}$. We say that we mutate $\mathcal{Q}_{i,t}^\varrho$ (respectively, $\mathcal{Q}_{i,t'}^\varrho$), we mean that we mutate $l(\mathcal{Q}_{i,t}^\varrho)$ (respectively, $l(\mathcal{Q}_{i,t'}^\varrho)$), $r(\mathcal{Q}_{i,t}^\varrho)$ (respectively, $r(\mathcal{Q}_{i,t'}^\varrho)$), and then mutate $C_{i,t-2}$ (respectively, $C_{i,t-4}$).

5.5. Definitions of the maps $\tau_i$, $\tau_r$, $\tau$. Let $\tilde{Q}_1$ (respectively, $\tilde{Q}_1'$) be a quiver which is mutation equivalent to $\tilde{Q}$ (respectively, $\tilde{Q}'$).

Let $\mathcal{L} = \{\mathcal{Q}_{i,t}^\varrho : (i,t) \in \mathcal{Y}\}$ (respectively, $\{\mathcal{Q}_{i,t}^\varrho : (i,t) \in \mathcal{Y}\}$). We define three maps $\tau_i$, $\tau_r$, $\tau$ on $\mathcal{L}$ as follows. In the case of type $A_n$, let

$$\tau_i(\mathcal{Q}_{i,t}^\varrho) = \mathcal{Q}_{i-1,t-1}^\varrho,$$  

$$\tau_r(\mathcal{Q}_{i,t}^\varrho) = \mathcal{Q}_{i+1,t-1}^\varrho,$$  

$$\tau(\mathcal{Q}_{i,t}^\varrho) = \mathcal{Q}_{i,t-2}^\varrho,$$  

where the quivers $\tilde{Q}_1$'s in (5.8), (5.9), (5.10) are obtained from $\tilde{Q}$ by mutating $l(\mathcal{Q}_{i,t}^\varrho)$, $r(\mathcal{Q}_{i,t}^\varrho)$, $\mathcal{Q}_{i,t}^\varrho$ respectively.

In the case of type $B_n$, let

$$\tau_i(\mathcal{Q}_{i,t}^\varrho) = \begin{cases} 
\mathcal{Q}_{i-1,t-2}^\varrho, & i \leq n-1, \ 2n + t - 2i \equiv 1 \pmod{4}, \\
\mathcal{Q}_{i-1,t-3}^\varrho, & i = n, \ t \equiv 2 \pmod{4}, \\
\mathcal{Q}_{i-1,t-1}^\varrho, & i = n, \ t \equiv 0 \pmod{4}, \\
\mathcal{Q}_{i-2,t-1}^\varrho, & i = n, \ 2n + t - 2i \equiv 3 \pmod{4}, \\
\mathcal{Q}_{i-2,t-2}^\varrho, & i < n-1, \ 2n + t - 2i \equiv 3 \pmod{4}, \\
\end{cases}$$  

$$\tau_r(\mathcal{Q}_{i,t}^\varrho) = \begin{cases} 
\mathcal{Q}_{i+1,t-2}^\varrho, & i < n-1, \ 2n + t - 2i \equiv 1 \pmod{4}, \\
\mathcal{Q}_{i+1,t-3}^\varrho, & i = n, \ 2n + t - 2i \equiv 1 \pmod{4}, \\
\mathcal{Q}_{i+1,t-1}^\varrho, & i = n, \ t \equiv 2 \pmod{4}, \\
\mathcal{Q}_{i+1,t-3}^\varrho, & i = n, \ t \equiv 0 \pmod{4}, \\
\mathcal{Q}_{i+1,t-2}^\varrho, & i \leq n-1 \text{ and } 2n + t - 2i \equiv 3 \pmod{4}, \\
\end{cases}$$  

$$\tau(\mathcal{Q}_{i,t}^\varrho) = \mathcal{Q}_{i,t-4}^\varrho,$$  

where the quivers $\tilde{Q}_1'$s in (5.11), (5.12), (5.13) are obtained from $\tilde{Q}'$ by mutating $l(\mathcal{Q}_{i,t}^\varrho)$, $r(\mathcal{Q}_{i,t}^\varrho)$, $\mathcal{Q}_{i,t}^\varrho$ respectively.

We define $\tau_i^m(\mathcal{Q}_{i,t}^\varrho) = \mathcal{Q}_{i,t}^\varrho$, $\tau_r^m(\mathcal{Q}_{i,t}^\varrho) = \tau_l(\tau_r^{m-1}(\mathcal{Q}_{i,t}^\varrho))$. We use the following convention: if $m < 0$, then $\tau_i^m(\mathcal{Q}_{i,t}^\varrho) = \emptyset$. The quivers $\tau_r^m(\mathcal{Q}_{i,t}^\varrho)$, $\tau_r^m(\mathcal{Q}_{i,t}^\varrho)$, $\tau_i^m(\mathcal{Q}_{i,t}^\varrho)$, $\tau_r^m(\mathcal{Q}_{i,t}^\varrho)$, $\tau_i^m(\mathcal{Q}_{i,t}^\varrho)$ are defined similarly.
By definition, $\tau_i^m(\mathcal{L}_{i,t}^{\tilde{Q}})$ (respectively, $\tau_i^m(\mathcal{L}_{i,t}^{\tilde{Q}'}))$ is a sub-quiver of some quiver $\tilde{Q}_m$ (rep. $\tilde{Q}_m'$) which is mutation equivalent to $\tilde{Q}$ (respectively, $\tilde{Q}'$). For simplicity, we write $\mathcal{L}_{i,t'} = \tau_i^m(\mathcal{L}_{i,t})$ for $\mathcal{L}_{i,t'}^{\tilde{Q}_m} = \tau_i^m(\mathcal{L}_{i,t}^{\tilde{Q}})$ (respectively, $\mathcal{L}_{i,t'}^{\tilde{Q}'} = \tau_i^m(\mathcal{L}_{i,t}^{\tilde{Q}'}))$. Similarly, we write $\mathcal{L}_{i,t'} = \tau_r^m(\mathcal{L}_{i,t})$ for $\mathcal{L}_{i,t'}^{\tilde{Q}_m} = \tau_r^m(\mathcal{L}_{i,t}^{\tilde{Q}})$ (respectively, $\mathcal{L}_{i,t'}^{\tilde{Q}'} = \tau_r^m(\mathcal{L}_{i,t}^{\tilde{Q}'}))$, and write $\mathcal{L}_{i,t'} = \tau_0^m(\mathcal{L}_{i,t})$ for $\mathcal{L}_{i,t'}^{\tilde{Q}_m} = \tau_0^m(\mathcal{L}_{i,t}^{\tilde{Q}})$ (respectively, $\mathcal{L}_{i,t'}^{\tilde{Q}'} = \tau_0^m(\mathcal{L}_{i,t}^{\tilde{Q}'}))$.

**Example 5.6.** Figures (a) and (b) illustrate the maps $\tau_i$, $\tau_r$, $\tau_0$ in the original quivers of types $A_8$, $B_4$ respectively.

**Figure 9.** In the original quiver of type $A_8$: (a) $\mathcal{L}_{4,-4} = \tau_i(\mathcal{L}_{5,-3})$; (b) $\mathcal{L}_{6,-4} = \tau_r(\mathcal{L}_{5,-3})$; (c) $\mathcal{L}_{5,-5} = \tau(\mathcal{L}_{5,-3})$.

**Figure 10.** In the original quiver of type $B_4$: (a) $\mathcal{L}_{3,-5} = \tau_i(\mathcal{L}_{4,-4})$; (b) $\mathcal{L}_{3,-7} = \tau_r(\mathcal{L}_{4,-4})$; (c) $\mathcal{L}_{4,-8} = \tau(\mathcal{L}_{4,-4})$.

5.6. **Mutation sequences of Kirillov–Reshetikhin modules.** In [HL13], Hernandez and Leclere defined a sequence of mutations for every Kirillov–Reshetikhin module whose highest weight monomial $m$ satisfies the property: $(i, t) \in W^-$ for every factor $Y_{i,t}$ in $m$. We recall the mutation sequences for Kirillov–Reshetikhin modules introduced in [HL13].
Let $\tilde{Q}$ (respectively, $\tilde{Q}'$) be a quiver which is mutation equivalent to $Q$ (respectively, $Q'$) defined in Section 5.1 and any two vertices in $\tilde{Q}$ (respectively, $\tilde{Q}'$) have different labels. By saying that we mutate $C_{i,t}$ of $\tilde{Q}$ (respectively, $\tilde{Q}'$), we mean that we mutate at the vertex of $\tilde{Q}$ (respectively, $\tilde{Q}'$) which has the label $(i,t)$ in the $i$-th column and so on until the vertex at infinity in the $i$-th column.

Let $\text{Seq}_j$, $m_1 \leq j \leq m_2$ be mutation sequences, we use

$$\left(\prod_{j=m_1}^{m_2} \text{Seq}_j\right)$$

and

$$\left(\text{Seq}_{m_1} \leq j \leq m_2 \text{ or } \prod_{j=m_1}^{m_2} \text{Seq}_j\right)$$

to denote mutation sequences

$$\text{Seq}_{m_2}, \text{Seq}_{m_2-1}, \ldots, \text{Seq}_{m_1} \text{ and } \text{Seq}_{m_1}, \text{Seq}_{m_1+1}, \ldots, \text{Seq}_{m_2}$$

respectively.

Consider the Kirillov–Reshetikhin module $S^{(\ell)}_{k(i)}$, $t \leq 0$, $k \in \mathbb{Z}_{\geq 1}$, $i \in I$. In the case of type $A_n$ ([HL13], Section 3), we mutate

$$j-1 \prod_{\ell=0}^{t-1} \left(\prod_{r=1}^{[\frac{j}{2}]} C_{2r-2\ell} \right) \left(\prod_{r=1}^{[\frac{j}{2}]} C_{2\ell-2\ell-1}\right)$$

starting from the quiver $Q$, where $j$ is defined by the formula

$$t = \begin{cases} 
-2k - 2j + 2, & i \in 2\mathbb{Z} \cap I, \\
-2k - 2j + 1, & i \in (2\mathbb{Z} + 1) \cap I.
\end{cases}$$

We use $Q_0$ to denote the current quiver. Then we obtain the Kirillov–Reshetikhin module $S^{(\ell)}_{k(i)}$, $t \leq 0$, $k \in \mathbb{Z}_{\geq 1}$, $i \in I$, at the vertex $(i,t)$ of $Q_0$.

In the case of type $B_n$ ([HL13], Section 3), let

$$KR(n, \ell) = C_{n,-4\ell} \left(\prod_{r=0}^{[\frac{j}{2}] - 1} C_{n-1-2r,-4\ell-1}\right) \left(\prod_{r=0}^{[\frac{j}{2}] - 2} C_{n-2-2r,-4\ell-2}\right) C_{n,-4\ell-2} \left(\prod_{r=0}^{[\frac{j}{2}] - 2} C_{n-2-2r,-4\ell-1}\right) \left(\prod_{r=0}^{[\frac{j}{2}] - 1} C_{n-1-2r,-4\ell-3}\right).$$

When $i \neq n$, $t = -4k - 2j + 3$, we mutate

$$\prod_{\ell=0}^{[\frac{j}{2}] - 1} KR(n, \ell)$$

starting from the quiver $Q'$. When $i = n$, $t = -2k - 2j + 4$, we mutate

$$\prod_{\ell=0}^{[\frac{j}{2}] - 2} KR(n, \ell)$$

if $j$ is odd, and mutate

$$\left(\prod_{\ell=0}^{[\frac{j}{2}] - 2} KR(n, \ell)\right) \left(\prod_{r=0}^{[\frac{j}{2}] - 1} C_{n,-2j+4} \prod_{r=0}^{[\frac{j}{2}] - 1} C_{n-1-2r,-2j+3}\right) \left(\prod_{r=0}^{[\frac{j}{2}] - 2} C_{n-2-2r,-2j+1}\right)$$

if $j$ is even, starting from the quiver $Q'$. We use $Q'_0$ to denote the current quiver. Then we obtain the Kirillov–Reshetikhin module $S^{(\ell)}_{k(i)}$, $t \leq 0$, $k \in \mathbb{Z}_{\geq 1}$, $i \in I$, at the vertex $(i,t)$ of $Q'_0$. 
5.7. Mutation sequences for snake modules of types $A_n$ and $B_n$. Let $S$ be a prime snake module and $S$ its highest $l$-weight monomial. Then $S$ can be written as

$$S = S_{k_1}^{(i_1,j_1)} \cdots k_m^{(i_{m-1},j_{m-1})},$$

where $m \geq 1$, $j_\ell \geq 0$, $1 \leq \ell \leq m - 1$, if $j_\ell = 0$, then $i_\ell \neq i_{\ell+1}$, $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}$.

By Definition 5.1, we have

$$FS(S) = FS_1(S) \cup FS_2(S) \cup FS_3(S),$$

where $FS_1 = \{M_1 = S_{k_m}^{(i_m)}\}$, $FS_2(S) \cup FS_3(S) = \{M_2, \ldots, M_q\}$. We reorder the elements in $FS_2(S) \cup FS_3(S)$ such that the distinguished factor $(l_p)_{s_p}$ of $M_p$ and the distinguished factor $(l_{p+1})_{s_{p+1}}$ of $M_{p+1}$ satisfy $s_p > s_{p+1}$, $2 \leq p \leq q - 1$.

Let $Q_1$, $Q_2$, ..., $Q_h$ be quivers in a mutation sequence. Let $(i_\ell, s_\ell) \in V(Q_\ell)$, $1 \leq \ell \leq h$. For simplicity, we write $L_{i_\ell, s_\ell}$ for $L_{i_\ell, s_\ell}$ and write $C_{i_\ell, s_\ell}$ for $C_{i_\ell, s_\ell}$.

Let $M_p \in FS(S)$, $1 \leq p \leq q$, and let $(l_p)_{s_p}$ be the distinguished factor of $M_p$. Using the mutation sequence defined in Section 5.6 starting from the initial quiver $Q$ in type $A_n$ (respectively, $Q'$ in type $B_n$) defined in Section 5.1 we can obtain a quiver $Q_0$ (respectively, $Q'_0$) and obtain the module $L(M_1) = L(S_{k_m}^{(i_1, j_1)})$ at the vertex $(i_1, t)$ of $Q_0$ (respectively, $Q'_0$).

In the following, we define mutation sequences $Seq_1$, $Seq_2$, $Seq_3$, ..., $Seq_q$ starting from the quiver $Q_0$ (respectively, $Q'_0$) of type $A_n$ (respectively, $B_n$) such that after we mutate $Seq_1$, $Seq_2$, ..., $Seq_q$, we obtain the snake module $S = L(S)$ at the vertex $(i_1, t)$.

The following is the case of type $A_n$.

1. Suppose that $M_p \in FS(S_2)$. Then there are some $\ell, r$ such that

$$M_p = S_{k_\ell}^{(i_\ell, j_\ell)} \cdots k_\ell^{(i_{\ell+1}, j_{\ell+1})}.$$

If the sequence $(i_u)_{\ell \leq u \leq \ell + r}$ is in decreasing order (respectively, in increasing order), then we mutate $(r(\tau_r^{(l_p)}(L_{i_p, s_p})))_{0 \leq h \leq n-\ell+1}$ (respectively, $(r(\tau_r^{(l_p)}(L_{i_p, s_p})))_{0 \leq h \leq n-\ell+1}$). If $r \geq 2$, we continue mutating

$$\ell + 1 \prod_{u=\ell+r-1}^{\ell+1} \left\{ \sum_{i=\ell+r-1}^{\ell+1} \frac{
abla}{k_i} \left( C_{i_\ell+1, s_\ell+1-2j, C_{i_\ell+2, s_\ell+2-2j, \ldots, C_{n, s_n-2j}} \right) \right\},$$

(respectively,

$$\ell + 1 \prod_{u=\ell+r-1}^{\ell+1} \left\{ \sum_{i=\ell+r-1}^{\ell+1} \frac{
abla}{k_i} \left( C_{i_\ell-1, s_\ell-1-2j, C_{i_\ell-2, s_\ell-2-2j, \ldots, C_{1, s_1-2j}} \right) \right\}),$$

where $i, s_i$ satisfy $L_{i, s_i} = \tau_r^{l_p-i}(L_{i_p, s_p})$, $i_\ell, i_{\ell+r} \leq i \leq n$ (respectively, $L_{i, s_i} = \tau_r^{l_p-i}(L_{i_p, s_p})$, $1 \leq i \leq i_\ell+1$).

2. Suppose that $M_p \in FS(S_3)$. Then there is some $\ell$ such that $M_p = S_{k_\ell}^{(i_\ell, j_\ell)} \cdots k_\ell^{(i_{\ell+1}, j_{\ell+1})}$.

- If $i_\ell \geq l_p$, then we mutate

$$(\tau_r^{(l_p)}(L_{i_p, s_p}))_{0 \leq h \leq i_\ell-l_p-1}, \quad (\tau_r^{(l_p)}(L_{i_p, s_p}))_{0 \leq h \leq j_\ell-1}.$$

- If $i_\ell < l_p$, then we mutate

$$(\tau_r^{(l_p)}(L_{i_p, s_p}))_{0 \leq h \leq \ell-p-1}, \quad (\tau_r^{(l_p)}(L_{i_p, s_p}))_{0 \leq h \leq \ell-1}.$$
• If \( i_\ell \leq l_p \), then we mutate
\[
(\tau_l^h(L_{l_p,s_p}))_{0 \leq h \leq l_p - i_\ell - 1}, \quad (\tau_r^h(L_{l_p,s_p}))_{0 \leq h \leq j_r - 1}.
\]

The following is the case of type \( B_n \).

(1) Suppose that \( M_p \in FS(S_2) \). Then there are some \( \ell, r \) such that
\[
M_p = S_{k_1^{(\ell)}}^{(l_p)} \cdots k_{\ell+1}^{(l_{\ell+1})} 1^{(\ell_{\ell+1})}.
\]
Suppose that the sequence \((i_u)_{\ell \leq u \leq \ell + r} \) is in increasing order. If \( l_p \neq n \), \( 2n + s_p - 2l_p \equiv 1 \) (mod 4) or \( l_p = n, s_p = 0 \) (mod 4) (respectively, \( l_p \neq n \), \( 2n + s_p - 2l_p \equiv 3 \) (mod 4) or \( l_p = n, s_p \equiv 2 \) (mod 4)), then we mutate
\[
(\tau_l^h(L_{l_p,s_p}))_{0 \leq h \leq l_p - 2} \quad \text{(respectively, } (\tau_r^h(L_{l_p,s_p}))_{0 \leq h \leq l_p - 2}).
\]
If \( r \geq 2 \), then we continue mutating
\[
\left\{ \sum_{i_u=\ell+1}^{\ell+\ell+1} k_i \prod_{j=1+\sum_{i_u=\ell+1}^{\ell+\ell+1} k_i} (C_{i_u-1,s_{i_u-1}-4j}, C_{i_u-2,s_{i_u-2}-4j}, \ldots, C_{1,s_1-4j}) \right\},
\]
where \( i, s_i \) satisfy \( L_{i_\ell,s_i} = \tau_l^{i-1}(L_{l_p,s_p}) \) for \( 1 \leq i \leq \ell + r \).

(2) Suppose that \( M_p \in FS(S_2) \). Then there are some \( \ell, r \) such that
\[
M_p = S_{k_1^{(\ell)}}^{(l_p)} \cdots k_{\ell+1}^{(l_{\ell+1})} 1^{(\ell_{\ell+1})}.
\]
Suppose that the sequence \((i_u)_{\ell \leq u \leq \ell + r} \) is in decreasing order.

• If \( 2n + s_p - 2l_p \equiv 1 \) (mod 4), then we mutate \((\tau_r^h(L_{l_p,s_p}))_{0 \leq h \leq n - l_p - 1} \). If \( r \geq 2 \), then we continue mutating
\[
\left\{ \sum_{i_u=\ell+1}^{\ell+\ell+1} k_i \prod_{j=1+\sum_{i_u=\ell+1}^{\ell+\ell+1} k_i} (\tau(L_{n,s_n-4j+4}), C_{i_u+1,s_{i_u+1}-4j}, C_{i_u+2,s_{i_u+2}-4j}, \ldots, C_{n,s_n-4j}) \right\},
\]
where \( i, s_i \) satisfy \( L_{i_\ell,s_i} = \tau_r^{i-1}(L_{l_p,s_p}) \) for \( i_{\ell+r} \leq i \leq n \).

• If \( 2n + s_p - 2l_p \equiv 3 \) (mod 4), then we mutate \((\tau_l^h(L_{l_p,s_p}))_{0 \leq h \leq n - l_p - 1} \). If \( r \geq 2 \), then we continue mutating
\[
\left\{ \sum_{i_u=\ell+1}^{\ell+\ell+1} k_i \prod_{j=1+\sum_{i_u=\ell+1}^{\ell+\ell+1} k_i} (L_{n,s_n-4j+4}, C_{i_u+1,s_{i_u+1}-4j}, C_{i_u+2,s_{i_u+2}-4j}, \ldots, C_{n,s_n-4j}) \right\},
\]
where \( (i, s_i) \) such that \( L_{i_\ell,s_i} = \tau_l^{i-1}(L_{l_p,s_p}) \), \( i_{\ell+r} \leq i \leq n \).

(3) Suppose that \( M_p \in FS(S_3) \). Then there is some \( \ell \) such that \( M_p = S_{k_1^{(\ell)}}^{(l_p)} 1^{(\ell_{\ell+1})} \).

• If \( l_p \neq n \), \( 2n + s_p - 2l_p \equiv 1 \) (mod 4), \( i_\ell \geq l_p \) or \( l_p \neq n, 2n + s_p - 2l_p \equiv 3 \) (mod 4), \( i_\ell \leq l_p \) or \( l_p = n, s_p \equiv 2 \) (mod 4), then we mutate
\[
(\tau_l^h(L_{l_p,s_p}))_{0 \leq h \leq |i_\ell - l_p| - 1}, \quad (\tau_r^h(L_{l_p,s_p}))_{0 \leq h \leq j_r - 1}.
\]
In [ZDLL15], the mutation sequences for minimal affinizations which satisfy

\[ \iota \leq t_p \leq n, \quad s_p \equiv 0 \pmod{4}, \quad \iota \geq \ell_p \quad \text{or} \quad \ell_p = n, \quad s_p \equiv 0 \pmod{4}, \]

then we mutate

\[ (\tau^{\ell_p}(\mathcal{L}_{t_p,s_p}))_{0 \leq h \leq |t_p| - 1}, \quad (\tau^{h}(\mathcal{L}_{t_p,s_p}))_{0 \leq h \leq |t_p| - 1}. \]

**Remark 5.7.** Minimal affinizations are modules \( S^{(t)}_{k_1^{(i)}, k_2^{(i)}, \ldots, k_r^{(i)}} \) which satisfy \( i_1 < \cdots < i_r \) or \( i_1 > \cdots > i_r \). The mutation sequences above in the case (1) in type \( A_n \) (respectively, the cases (1), (2) in type \( B_n \)) are mutation sequences for minimal affinizations which satisfy \( i_1 < \cdots < i_r \) or \( i_1 > \cdots > i_r \). These mutation sequences are defined in the same cluster algebra. In [ZDLL13], the mutation sequences for minimal affinizations which satisfy \( i_1 < \cdots < i_r \) are defined in a cluster algebra \( \mathfrak{A} \) and the mutation sequences for minimal affinizations which satisfy \( i_1 > \cdots > i_r \) are defined in another cluster algebra \( \mathfrak{A}' \) which is dual to \( \mathfrak{A} \).

### 5.8. The equations in the \( S \)-system of type \( A_n \) (respectively, \( B_n \)) correspond to mutations in the cluster algebra \( \mathfrak{A} \) (respectively, \( \mathfrak{A}' \)).

In this section, we give the relation between prime snake modules and cluster variables.

Let \( S \) be the set of prime snake modules. Let

\[ S = \{ S_{k_1^{(i)}, k_2^{(i)}, \ldots, k_m^{(i)}} \colon \ell \geq 0, \quad 1 \leq \ell \leq m - 1, \quad k_1, \ldots, k_m \in \mathbb{Z}_{\geq 1}, \quad t \in \mathbb{Z} \}. \]

We define a map

\[ \psi : S \rightarrow S \]

\[ S_{k_1^{(i)}, k_2^{(i)}, \ldots, k_m^{(i)}} \mapsto S_{k_1^{(i)}, k_2^{(i)}, \ldots, k_m^{(i)}}. \] (5.14)

We apply the map \( \psi \) defined by (5.14) to the equations \([S_1][S_2] = [S_3][S_4] + [S_5][S_6]\) in the \( S \)-system for type \( A_n \) (respectively, \( B_n \)). Then we have a new system of equations:

\[ s_1 s_2 = s_3 s_4 + s_5 s_6, \] (5.15)

where \( s_i = \psi(S_i), \quad 1 \leq i \leq 6 \). For each equation in (5.15), we define \( s'_2 = s_2 \). Then we obtain a set of equations:

\[ s'_1 = s_2 = \frac{s_3 s_4 + s_5 s_6}{s_1} \] (5.16)

We find that the above set of equations is the set of equations of the mutations in Section 5.7. Therefore, we have the following theorem.

**Theorem 5.8.** The Hernandez–Leclerc conjecture (Conjecture 1.1) is true for snake modules of types \( A_n \) and \( B_n \).

### 6. Examples of mutation sequences for some snake modules

In this section, we give some examples of mutation sequences for some snake modules.

**Example 6.1.** In type \( A_5 \), let \( S = 2_{-1}2_{-4}5_{-5}5_{-3}4_{0} \). By Definition 5.7

\[ \mathcal{F}S(S) = \{ 4_{0}, \quad 5_{-5}4_{-3}4_{0}, \quad 2_{-1}2_{-4}5_{-5} \}. \]

The set of distinguished factors of \( S \) is \( \{ 4_{0}, \quad 5_{-5} \} \). The mutation sequence for \( S \) is

\[ \tau(\mathcal{L}_{4_{0}}), \quad \tau(\mathcal{L}_{5_{-5}}), \quad \tau(\mathcal{L}_{4_{0}}), \quad \tau(\mathcal{L}_{5_{-5}}), \quad \tau(\mathcal{L}_{4_{0}}), \quad \mathcal{L}_{5_{-5}}), \quad \mathcal{L}_{4_{0}}, \quad \mathcal{L}_{5_{-5}}, \quad C_{3_{-9}}, \quad C_{2_{-10}}, \quad C_{1_{-11}}. \]
We obtain the snake module $S = L(S)$ at the vertex which has the label $(2, -12)$, see Figure 7.

The initial quivers in this section are the initial quivers in [HL13]. The mutation sequences in this section are similar to the mutation sequences given in [HL13]. In [HL13], the mutation sequences produce Kirillov–Reshetikhin modules. In the following, the mutation sequences produce prime snake modules.
Example 6.2. In type $A_5$, let $S = 2_{-18} 4_{-4} 5_{-9} 4_{-6}$. By Definition 5.7,
\[
\mathcal{FS}(S) = \{4_{-6}, 5_{-11} 5_{-9} 4_{-6}, 2_{-18} 4_{-14} 5_{-11}\}.
\]
The set of distinguished factors of $S$ is $\{4_{-6}, 5_{-11}\}$. The mutation sequence for $S$ is
\[
C_{2,0}, C_{4,0}, C_{1,-1}, C_{3,-1}, C_{5,-1}, C_{2,-2}, C_{4,-2}, C_{1,-3}, C_{3,-3}, C_{5,-3}, C_{2,-4}, C_{4,-4}, C_{1,-5}, C_{3,-5}, C_{5,-5}, \tau(L_{4,-6}), \tau(L_{5,-11}), \tau(L_{5,-9} L_{5,-11}), L(L_{5,-11}), C_{3,-15}, C_{2,-16}, C_{1,-17}.
\]
We obtain the snake module $S = L(S)$ at the vertex which has the label $2_{-18}$.

Example 6.3. In type $A_4$, let $S = 3_{-25} 3_{-21} 2_{-16} 2_{-12} 3_{-9} 2_{-4} 1_{-1}$. \[
\mathcal{FS}(3_{-25} 3_{-21} 2_{-16} 2_{-12} 3_{-9} 2_{-4} 1_{-1}) = \{1_{-1}, 3_{-2} 6_{-4} 1_{-1}, 2_{-12} 3_{-9}, 2_{-16} 2_{-12}, 3_{-21} 2_{-16}, 3_{-25} 3_{-21}\}.$
The set of distinguished factors of $S$ is $\{1_{-1}, 3_{-9}, 2_{-12}, 2_{-16}, 3_{-21}\}$. The mutation sequence for $S$ is
\[ v(L_{1,-1}), v(\tau_r(L_{1,-1})), v(\tau_r^2(L_{1,-1})), C_{3,-5}, C_{4,-6}, C_{3,-7}, C_{4,-8}, l(L_{3,-9}), l(\tau(L_{3,-9})), L_{2,-12}, v(L_{2,-16}), \tau_r(L_{2,-16}), L_{3,-21}. \]

We obtain the snake module $S = L(S)$ at the vertex which has the label $(3,-25)$.

Example 6.4. In type $B_3$, let $S = 1_{-35}2_{-29}2_{-21}3_{-16}3_{-10}2_{-5}$.

$FS(1_{-35}2_{-29}2_{-21}3_{-16}3_{-10}2_{-5}) = \{2_{-5}, 3_{-10}2_{-5}, 3_{-16}3_{-10}, 2_{-21}3_{-16}, 2_{-29}2_{-21}, 1_{-35}2_{-29}\}$.

The set of distinguished factors of $S$ is $\{2_{-5}, 3_{-10}, 3_{-16}, 2_{-21}, 2_{-29}\}$. The mutation sequence for $S$ is
\[ C_{3,0}, C_{2,-1}, C_{1,-3}, C_{3,-2}, C_{1,-1}, C_{2,-3}, v(L_{2,-5}), L_{3,-10}, l(L_{3,-16}), l(\tau(L_{3,-16})), L_{2,-21}, l(L_{2,-29}). \]

We obtain the snake module $S = L(S)$ at the vertex which has the label $(1,-35)$.

Let $S = 2_{-43}2_{-35}2_{-31}1_{-25}3_{-18}3_{-3}8_{-23}0$. By Definition 7.1
\[ FS(S) = \{3_{-29}, 3_{-8}, 3_{-18}, 1_{-25}, 3_{-18}, 2_{-35}, 2_{-31}, 1_{-25}, 2_{-43}, 2_{-35}\}. \]

The set of distinguished factors of $S$ is $\{3_{0}, 3_{-2}, 3_{-8}, 3_{-18}, 1_{-25}, 2_{-35}\}$. The mutation sequence for $S$ is
\[ L_{3,-2}, L_{3,-8}, \tau(L_{3,-8}), v(L_{3,-18}), l(\tau_r(L_{3,-18})), l(L_{1,-25}), l(\tau(L_{1,-25})), L_{2,-35}. \]

We obtain the snake module $S = L(S)$ at the vertex which has the label $(2,-43)$.

7. Proofs of Theorem 3.4

In this section, we will prove Theorem 3.4.

7.1. Proof of Theorem 3.4. Let $S$ be a prime snake module and $S$ its highest $l$-weight monomial. Then $S$ can be written as
\[ S = S_{k_1}^{i_1} S_{k_2}^{i_2} \cdots S_{k_m}^{i_m}, \]
where $m \geq 1$, $j_0 \geq 0$, $1 \leq \ell \leq m - 1$, if $j_0 = 0$, then $i_\ell \neq i_{\ell+1}$, $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}$.

The theorem follows from the fact: $\chi_q(S)\chi_q(S)$ has only one dominant monomial $S^2$.

Let $L = \sum_{\ell=1}^{m} k_\ell$. Suppose that $m = \prod_{\ell=1}^{L} m(p_\ell)$ (respectively, $m' = \prod_{\ell=1}^{L} m(p'_\ell)$) is a tuple of non-overlapping paths and
\[ \sum_{\ell=1}^{m} k_\ell = 0, \quad d_r + \sum_{\ell=1}^{m} k_\ell = t + 2d_j r - 2d_j + \sum_{\ell=1}^{j-1} n_\ell, \]
where $1 \leq r \leq j$, $1 \leq j \leq m$, by convention $\sum_{\ell=1}^{0} n_\ell = 0$.

Suppose that $mm'$ is dominant. If $p_L \neq p_{L+} + d_L$, then $mm'$ is right-negative and not dominant. Therefore $p_L = p'_{L+}$. Similarly, we have $p'_L = p_{L+}$. By the non-overlapping property, we have $p'_j = p'_{j+} d'_j$; $p_j = p_{j+} d_j$ for all $1 + \sum_{\ell=1}^{m-1} k_\ell \leq j \leq L$. 
Suppose that \( p_{\sum_{\ell=1}^{m-1} k_\ell} \neq p_{\sum_{\ell=1}^{m-1} k_\ell}' \). Then \( m(p_{\sum_{\ell=1}^{m-1} k_\ell}) \) has some negative factor \( i_\ell \), where \((i, \ell) \in C_{\sum_{\ell=1}^{m-1} k_\ell} \). By Theorem 3.3, \( m \) has the negative factor \( i_\ell^{-1} \). Therefore, the negative factor \( i_\ell^{-1} \) is canceled by \( m' \). It follows that \( m' \neq S \) since \( i_\ell^{-1} \) is not in \( S \). But then \( mm' \) has one of the following factors:

- in type \( A_n \): \( 1^{-1}_{\ell+i-1}, 2^{-1}_{\ell+i-2}, \ldots, (i-2)^{-1}_{\ell+2}, (i-1)^{-1}_{\ell+1}, (i+1)^{-1}_{\ell}, (i+2)^{-1}_{\ell+1}, \ldots, (2n-1)^{-1}_{\ell} \)
- in type \( B_n \): \( i \neq n, 1^{-1}_{\ell+2i-3}, 2^{-1}_{\ell+2i-4}, \ldots, (i-2)^{-1}_{\ell+4}, (i-1)^{-1}_{\ell+2}, i = n, 1^{-1}_{\ell+2i-3}, 2^{-1}_{\ell+2i-5}, \ldots, (i-2)^{-1}_{\ell+3}, (i-1)^{-1}_{\ell+1} \).

This contradicts the assumption that \( mm' \) is dominant. Therefore, \( p_{\sum_{\ell=1}^{m-1} k_\ell} = p_{\sum_{\ell=1}^{m-1} k_\ell}' \).

By Theorem 3.5 we have \( p_j = p_{c_j,d_j}^+ \) for all \( 1 + \sum_{\ell=1}^{m-2} k_\ell \leq j \leq \sum_{\ell=1}^{m-1} k_\ell \). By the same argument, we have \( p_j = p_{c_j,d_j}^-, 1 \leq j \leq \sum_{\ell=1}^{m-1} k_\ell \). Therefore, \( m = S \).

By the same argument, we have \( m' = S \). Therefore, the only dominant monomial in \( \chi_q(S)\chi_q(S) \) is \( S^2 \).

8. Proof of Theorem 4.1

In this section, we will prove Theorem 4.1.

8.1. Classification of dominant monomials. First we classify all dominant monomials in each summand on the left- and right-hand sides of every equation in Theorem 4.1. We have the following lemma.

**Lemma 8.1.** Let \([S_1][S_2] = [S_3][S_4] + [S_5][S_6] \) be any equation in the \( S \)-system of type \( A_n \) (respectively, \( B_n \)) in Theorem 4.1. Let \( S_i \) be the highest \( l \)-weight monomial of \( S_i, i \in \{1, 2, \ldots, 6\} \). The dominant monomials in each summand on the left- and right-hand sides of \([S_1][S_2] = [S_3][S_4] + [S_5][S_6] \) are given in Table 3.

| Summands in the equations | \( M \) | Dominant monomials |
|----------------------------|--------|-------------------|
| \( \chi_q(S_1)\chi_q(S_2) \) | \( M=S_1S_2 \) | \( M \prod_{0 \leq j \leq r} A_{1+2d_1}^{-1} k_1 - \ldots - d_1, -1 \leq r \leq k_1-1 \) |
| \( \chi_q(S_3)\chi_q(S_4) \) | \( M=S_3S_4 \) | \( M \prod_{0 \leq j \leq r} A_{1+2d_1}^{-1} k_1 - \ldots - d_1, -1 \leq r \leq k_1-2 \) |
| \( \chi_q(S_5)\chi_q(S_6) \) | \( M=S_5S_6 \) | \( M \) |

**Table 3.** Dominant monomials in the \( S \)-systems of types \( A_n \) and \( B_n \).

8.2. Proof of Theorem 4.1. By Table 3, the dominant monomials of the \( q \)-characters of the left-hand side and of the right-hand side of every equation in Theorem 4.1 are the same. Therefore, Theorem 4.1 is true.

8.3. Proof of Lemma 8.1. We will prove the case of type \( A_n \) and the case of type \( B_n \) respectively.

**Proof of the case of type \( A_n \).** Let

\[
S_1 = \sum_{\ell}^{(t+2)} k_{1(b_1,b_2)} k_{2(b_2+1)} k_{3(b_3,b_4)} k_{4(b_4,b_5)} \ldots k_{m(b_m)}, \quad S_2 = \sum_{\ell}^{(t)} k_{1(b_1,b_2)} k_{2(b_2+1)} k_{3(b_3,b_4)} k_{4(b_4,b_5)} \ldots k_{m(b_m)},
\]
where \( m \geq 3, i_1 > i_2, i_3 > i_2, j \ell \geq 0, 3 \leq \ell \leq m - 1 \). The other cases are similar.

Let \( L = \sum_{\ell=1}^{m} k_\ell \). Suppose that \( m = \prod_{j=1}^{L} m(p_j) \) be a monomial in \( \chi_q(S_2) \), where \( (p_1, \ldots, p_L) \in \overrightarrow{F}(c_j, d_j)_{1 \leq j \leq L} \) is a tuple of non-overlapping paths and

\[
\begin{align*}
  c_{r+\sum_{\ell=1}^{j-1} k_\ell} &= i_j, \\
  d_{r+\sum_{\ell=1}^{j-1} k_\ell} &= t + 2r - 2 + \sum_{\ell=1}^{j-1} n_\ell,
\end{align*}
\]

where \( 1 \leq r \leq k_j, 1 \leq j \leq m \), by convention \( \sum_{\ell=1}^{0} k_\ell = 0 \).

Let \( m' = \prod_{j=1}^{L} m'(p'_j) \) be a monomial in \( \chi_q(S_1) \), where \( (p'_1, \ldots, p'_L) \in \overrightarrow{F}(c'_j, d'_j)_{1 \leq j \leq L} \) is a tuple of non-overlapping paths and

\[
\begin{align*}
  c'_1 = c'_2 = \cdots = c'_{k_1} &= i_1, \\
  c'_{k_1+1} = c'_{k_1+2} = \cdots = c'_{k_1+k_2} &= i_2 + 1, \\
  d'_1 &= t + 2, d'_2 = t + 4, \ldots, d'_{k_1} = t + 2k_1, \\
  d'_{k_1+1} &= t + n_1 + 1, d'_{k_1+2} = t + n_1 + 3, \ldots, d'_{k_1+k_2} = t + n_1 + 2k_2 - 1,
\end{align*}
\]

where \( 1 \leq r \leq k_j, 3 \leq j \leq m \).

We have \( c_j = c'_1, d_j = d'_j \), \( k_1 + k_2 + 1 \leq j \leq L \).

Suppose that \( mm' \) is dominant. By the same arguments as the arguments in the proof of Theorem 3.3, we have \( p_j = p^+_{c_j, d_j}, k_1 + 1 \leq j \leq L \) and \( p'_j = p^+_{c'_j, d'_j} \), \( 1 \leq j \leq L \).

If \( p_k = p^+_{c_k, d_k}, \) then \( p_j = p^+_{c_j, d_j} \) for all \( 1 \leq j \leq k_1 - 1 \). Therefore, \( mm' = S_1 S_2 \). If \( p_k = p^+_{c_k, d_k}, A_{i_1, t+2k_1-1} \), then \( p_j \in \{ p^+_{c_j, d_j}, p^+_{c_j, d_j} A^{-1}_{i_1, t+2j-1} \}, 1 \leq j \leq k_1 - 1 \). Therefore, \( mm' \) is one of the dominant monomials \( S_1 S_2 \prod_{j=0}^{k_1} A^{-1}_{i_1, t+2k_1-1} \), \( 0 \leq r \leq k_1 - 1 \). If \( p_k \notin \{ p^+_{c_k, d_k}, p^+_{c_k, d_k} A^{-1}_{i_1, t+2k_1-1} \} \), then by the same arguments as the arguments in the proof of Theorem 3.3, it follows that \( mm' \) is not dominant which contradicts our assumption. \( \square \)

**Proof of the case of type \( B_n \).** Let

\[
S_1 = S_{(k_1, k_1, k_1, k_{j_2}, k_{j_3}, \ldots, k_{j_m})}^{(j_1+2)}, \quad S_2 = S_{(k_1, k_1, k_{j_2}, k_{j_3}, \ldots, k_{j_m})}^{(j_1+1)},
\]

where \( j_1 \geq 1 \). The other cases are similar.

Let \( L = \sum_{\ell=1}^{m} k_\ell \). Suppose that \( m = \prod_{j=1}^{L} m(p_j) \) be a monomial in \( \chi_q(S_2) \), where \( (p_1, \ldots, p_L) \in \overrightarrow{F}(c_j, d_j)_{1 \leq j \leq L} \) is a tuple of non-overlapping paths and

\[
\begin{align*}
  c_{r+\sum_{\ell=1}^{j-1} k_\ell} &= i_j, \\
  d_{r+\sum_{\ell=1}^{j-1} k_\ell} &= t + 2d_{i_j} r - 2d_{i_j} + \sum_{\ell=1}^{j-1} n_\ell,
\end{align*}
\]

where \( 1 \leq r \leq k_j, 1 \leq j \leq m \), by convention \( \sum_{\ell=1}^{0} k_\ell = 0 \).

Let \( m' = \prod_{j=1}^{L} m'(p'_j) \) be a monomial in \( \chi_q(S_1) \), where \( (p'_1, \ldots, p'_L) \in \overrightarrow{F}(c'_j, d'_j)_{1 \leq j \leq L} \) is a tuple of non-overlapping paths and

\[
\begin{align*}
  c'_1 = c'_2 = \cdots = c'_{k_1} &= i_1, \\
  d'_1 &= t + 2, d'_2 = t + 4, \ldots, d'_{k_1} = t + 2k_1, d'_{k_1+1} = t + 2k_1 + 2,
\end{align*}
\]

where \( 1 \leq r \leq k_j, 2 \leq j \leq m \).
We have $c_j = c_{j+1}'$, $d_j = d_{j+1}'$, $k_1 + 1 \leq j \leq L$.

Suppose that $mm'$ is dominant. By the same arguments as the arguments in the proof of Theorem 3.4, we have $p_j = p_{c_j,d_j}^+$, $k_1 + 1 \leq j \leq L$ and $p_j' = p_{c_j',d_j}'$, $1 \leq j \leq L + 1$.

If $p_{k_1} = p_{c_{k_1},d_{k_1}}^+$, then $p_j = p_{c_j,d_j}^+$ for all $1 \leq j \leq k_1 - 1$. Therefore, $mm' = S_1S_2$. If $p_{k_1} = p_{c_{k_1},d_{k_1}}^+$, $A_{i_1,t+2k_1-1}^{-1}$, then $p_j \in \{p_{c_j,d_j}^+, p_{c_j,d_j}^+ A_{i_1,t+2j-1}^{-1}\}$, $1 \leq j \leq k_1 - 1$. Therefore, $mm'$ is one of the dominant monomials $S_1S_2 \prod_{j=0}^{r} A_{i_1,t+2k_1-2j-1}^{-1}$, $0 \leq r \leq k_1 - 1$. If $p_{k_1} \notin \{p_{c_{k_1},d_{k_1}}^+, p_{c_{k_1},d_{k_1}}^+ A_{i_1,t+2k_1-1}^{-1}\}$, then by the same arguments as the arguments in the proof of Theorem 3.4 it follows that $mm'$ is not dominant which contradicts our assumption. □

9. PROOF OF THEOREM 4.3

In this section, we will prove Theorem 4.3. By Lemma 8.1 we have the following corollary.

**Corollary 9.1.** The modules in the second summand on the right-hand side of every equation of the $S$-systems for types $A_n$ and $B_n$ are special. In particular, they are simple.

Therefore, in order to prove Theorem 4.3 we only need to prove that the modules in the first summand on every equation of the $S$-systems for types $A_n$ and $B_n$ are simple. We will prove that $\chi_q(S_3) \chi_q(S_4)$ is simple in the case of type $A_n$, where

$$S_3 = S_{(i_1),k_2(i_2+1),k_3(i_3+2),k_4(i_4+3),...,k_m(i_m)}^{(l)}, \quad S_4 = S_{(i_1),k_2(i_2),k_3(i_3+3),k_4(i_4+4),...,k_m(i_m)}^{(l+2)},$$

and $m \geq 3$, $i_1 > i_2$, $i_3 > i_4$, $j \leq 2$, $0 \leq l \leq m - 1$. The other cases are similar.

By Lemma 8.1, the dominant monomials of $\chi_q(S_3) \chi_q(S_4)$ are

$$M_r = S_3S_4 \prod_{j=0}^{r} A_{i_1,t+2k_1-2j-1}^{-1}, -1 \leq r \leq k_1 - 2,$$

where $S_3$ (respectively, $S_4$) is the highest $l$-weight monomial. We need to show that $\chi_q(M_r) \notin \chi_q(S_3) \chi_q(S_4)$ for $0 \leq r \leq k_1 - 2$. We will prove the case of $r = 0$, the other cases are similar.

Let $n_1 = S_3S_4 A_{i_1,t+2k_1-1}^{-2}$. By Corollary 2.25 the monomial $n_1 \in \chi_q(M_0)$. Suppose that $n_1 \in \chi_q(S_3) \chi_q(S_4)$. Then $n_1 = m_1 m_2$, where $n_1 \in \chi_q(S_3)$, $m_2 \in \chi_q(S_4)$. Since $n_1 = S_3S_4 A_{i_1,t+2k_1-1}^{-2}$, by the expressions $S_3$ and $S_4$ we must have

$$m_1 = S_3 A_{i_1,t+2k_1-1}^{-1}, \quad m_2 = S_4 A_{i_1,t+2k_1-1}^{-1}.$$

But by the Frenkel–Mukhin algorithm, $S_3 A_{i_1,t+2k_1-1}^{-1}$ is not in $\chi_q(S_3)$. This is a contradiction. Therefore, $n_1 \notin \chi_q(S_3) \chi_q(S_4)$ and hence $\chi_q(M_0) \notin \chi_q(S_3) \chi_q(S_4)$.

ACKNOWLEDGEMENT

J.-R. Li would like to express his gratitude to Professor Vyjayanthi Chari for helpful discussions about prime modules. This work was partially supported by the National Natural Science Foundation of China (no. 11371177, 11501267, 11401275), and the Fundamental Research Funds for the Central Universities of China (no. lzujbky-2015-78). J.-R. Li was also supported by ERC AdG Grant 247049 and the PBC Fellowship Program of Israel for Outstanding Post-Doctoral Researchers from China and India.
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