Finding temporal patterns using algebraic fingerprints

Suhas Thejaswi∗ Aristides Gionis†

Abstract

In this paper we study a family of pattern-detection problems in vertex-colored temporal graphs. In particular, given a vertex-colored temporal graph and a multi-set of colors as a query, we search for temporal paths in the graph that contain the colors specified in the query. These types of problems have several interesting applications, for example, recommending tours for tourists, or searching for abnormal behavior in a network of financial transactions.

For the family of pattern-detection problems we define, we establish complexity results and design an algebraic-algorithmic framework based on constrained multilinear sieving. We demonstrate that our solution can scale to massive graphs with up to hundred million edges, despite the problems being NP-hard. Our implementation, which is publicly available, exhibits practical edge-linear scalability and highly optimized. For example, in a real-world graph dataset with more than six million edges and a multi-set query with ten colors, we can extract an optimal solution in less than eight minutes on a haswell desktop with four cores.

1 Introduction

Pattern mining in graphs has become increasingly popular due to applications in analyzing and understanding structural properties of data originating from information networks, social networks, transportation networks, and many more. Searching for patterns in graphs is a fundamental graph-mining task that has applications in computational biology and analysis of metabolic networks [40], discovery of controversial discussions in social media [15], and understanding the connectivity of the brain [30], among others. At the same time, real-world data are inherently complex. To accurately represent the heterogeneous and dynamic nature of real-world graphs, we need to enrich the basic graph model with additional features. Thus, researchers have considered labeled graphs [53], or heterogeneous graphs [43], where vertices and/or edges are associated with additional information represented with labels, and temporal graphs [28], where edges are associated with timestamps that indicate when interactions between pairs of vertices took place.

In this paper we study a family of pattern-detection problems in graphs that are both labeled and temporal. In particular, we consider graphs in which each vertex is associated with one (or more) labels, to which we refer as colors, and each edge is associated with a timestamp. We then consider a motif query, which is a multi-set of colors. The problem we consider is to decide whether there exists a temporal path whose vertices contain exactly the colors specified in the motif query. A temporal path in a temporal graph refers to a path in which the timestamps of consecutive edges are strictly increasing. If such a path exists, we also want to find it and return it as output.

∗Department of Computer Science, Aalto University, Finland.
†Department of Computer Science, KTH Royal Institute of Technology, Sweden, and Department of Computer Science, Aalto University, Finland.
The family of problems we consider have several interesting applications. One application is in the domain of tour recommendations [17], for travelers or tourists in a city. In this case, vertices correspond to locations. The colors associated with each location represent different activities that can be enjoyed in that particular location. For example, activity types may include items such as museums, archaeological sites, restaurants, etc. Edges correspond to transportation links between different locations, and each transportation link is associated with a timestamp indicating departure time and duration. Furthermore, for each location we may have information about the amount of time recommended to spent in that location, e.g., minimum amount of time required to finish a meal or appreciate a museum. Finally, the multi-set of colors specified in the motif query represents the multi-set of activities that a user is interested in enjoying. In the tour-recommendation problem we would like to find a temporal path, from a starting location to a destination, which satisfies temporal constraints (e.g., feasible transportation links, visit times, and total duration) as well as the activity requirements of the user, i.e., what kind of places they want to visit.

Another application is in the domain of analyzing networks of financial transactions. Here, the vertices represent financial entities, the vertex colors represent features of the entities, and the temporal edges represent financial transactions between entities, annotated with the time of the transaction, amount, and possibly other features. An analyst may be interested in finding long chains of transactions among entities that have certain characteristics, for example, searching for money-laundering activities may require querying for paths that involve public figures, companies with certain types of contracts, and banks in off-shore locations.

In this paper we study the following problem variants:

- **k-TempPath**: find a temporal path whose length is at least $k - 1$;
- **PathMotif**: find a temporal path whose vertices contain the set of colors specified by a motif query;
- **RainbowPath**: find a temporal path of length at least $k - 1$, whose vertices have distinct colors.

All the problems we consider are NP-hard; thus, there is no known efficient algorithm to find exact solutions. In such cases most algorithmic solutions resort to approximation schemes. In this paper we present an (exact) algebraic approach based on constrained multilinear sieving for pattern detection in temporal graphs.

The algorithms based on constrained multilinear detection offer the theoretically best-known results for a set of combinatorial problems including $k$-path [5], Hamiltonian path [6], graph motifs [8] and many more. The implementations based on multilinear sieving are known to saturate the empirical arithmetic and memory bandwidth on modern CPU and GPU micro-architectures. Furthermore, these implementations can scale to large graphs as well as large query sizes [9, 31].

Even though these algebraic techniques have been studied extensively in the algorithms community, they are not been applied to data-mining problems. To the best of our knowledge this is the first paper that applies these approaches for data mining and exploratory graph analysis. Furthermore, this is the first work that applies these techniques for pattern detection in temporal graphs.

Our key contributions are as follows:

- We introduce a set of pattern-detection problems that originate in the vertex-colored and temporal graphs. For the problems we define we present NP-hardness results, while showing that they are fixed-parameter tractable [16], meaning that, if we restrict the size of motif query the problems are solvable in polynomial time in the size of host graph.
• We present a general algebraic-algorithmic framework based on constrained multilinear sieving. Our solution exhibits edge-linear scalability. The applications of the algorithmic approach described in this work is not limited to temporal paths, but rather it can be extended to study information cascades, temporal arborescences and temporal subgraphs.

• We engineer an implementation of the algebraic algorithm and demonstrate that the implementation can scale for graphs with up to hundred million edges.

• Open-source release: our implementation and datasets are released as open source [47].

2 Related work

Pattern detection and pattern counting are fundamental problems in data mining. In the context of paths and trees, pattern matching problems have been extensively studied in non-temporal graphs both in theory [5, 8, 21, 24, 39] as well as applications [4, 13, 29, 45]. For many restricted variants of path problems Kowalik and Lauri presented complexity results and deterministic algorithms with probably optimal runtime bounds [39]. Most of these problems are known to be fixed-parameter tractable and the best known randomized algorithms for a subset of path and subgraph pattern detection problems is due to Björklund et al. [5, 8]. Color coding can be used to approximately count the patterns in $O^*(2^k)$ time, however, these algorithms require $O^*(2^k)$ memory [1]. A practical implementation of color coding using adaptive sampling and succinct encoding was demonstrated by Bressan et al. [11] for pattern counting problem. However, the techniques based on color coding is mostly used to detect and count patterns in graphs with no vertex labels.

Algebraic algorithms based on multilinear and constrained multilinear sieving are due to the pioneering work of Koutis [33, 34, 35], Williams [50], Koutis and Williams [36, 37]. The approach has been extended to various combinatorial problems using a multivariate variant of the sieve by Björklund et al. [5]. Dell et al. [19] used the decision oracles introduced by Björklund et al. to approximately count the motifs. A practical implementation of multilinear sieving and its scalability to large graphs has been demonstrated by Björklund et al. [9]. Furthermore, its parallelizability to vector-parallel architectures and scalability to large multi-set sizes was shown by Kaski et al. [31].

In the recent years there has been a lot of progress with respect to mining temporal graphs. The most relevant work includes methods for efficient computation of network measures, such as centrality, connectivity, density, motifs, etc. [18, 27, 28, 32, 41], as well as mining frequent subgraphs in temporal networks [44, 46, 49]. Path problems in temporal graphs are well studied [23, 51]. Many variants of the path problems are known to be solvable in polynomial-time [51, 52]. Surprisingly, a simple variant to check the existence of a temporal path with waiting time constraints was shown to be NP-complete by Casteigts et al. [12], more strongly, they proved that the problem is W[1]-hard. A known variant of the temporal-path problem is finding top-$k$ shortest paths, which not only asks us to find a shortest path, but also the next $k - 1$ shortest paths — which may be longer than the shortest path [26]. Here by shortest path we mean that the total elapsed time of the temporal path is minimized. Note that the top-$k$ shortest path is different from the $k$-TEMPATH problem studied in our work.

With the availability of social media data in the recent years there has been growing interest to study pattern mining problems in temporal graphs. Paranjape et al. [46] presented efficient algo-
rithms for counting small temporal patterns. Liu et al. [44] presented complexity results and approximation methods for counting patterns in temporal graphs. However, they mainly study temporal graphs with no vertex-labels (colors). Kovanen et al. [38] studied a general variant of the temporal subgraph problem in temporal graphs with vertex labels. Aslay et al. [2] presented methodologies for counting frequent patterns with vertex and edge labels in streaming graphs. However, most of these approaches are limited to small pattern sizes (up to 3 vertices).

To the best of our knowledge, there is no existing work related to detecting and extracting temporal patterns with vertex labels. The problems considered in this paper are closely related to variants of classical problems such as orienteering problem, TSP and Hamiltonian path [48, 22]. A motivating application for the problems can be traced to the context of tour recommendations [17, 25].

3 Method overview

Our method relies on the algebraic-fingerprinting technique [36, 50]. As this technique is not well known in the data-mining community, we provide a bird's eye view. The approach is described in more detail in Section 6.

In a nutshell, the problem is to decide the existence of a pattern, or a structure in the data. The idea is to encode the pattern-discovery problem as a polynomial over a set of variables. The variables represent entities of the problem instance (e.g., vertices or edges), and their values represent possible solutions (e.g., whether a vertex belongs to a path). The challenge is to find a polynomial encoding that has the property that a solution to our problem exists if and only if the polynomial evaluates to a non-zero term. We can then verify the existence of a solution, using polynomial identity testing, in particular, by evaluating random substitutions of variables: if one of them does not evaluate to zero, then the polynomial is not identically zero. Thus, the method can give false negatives, but the error probability can be brought arbitrarily close to zero.

It should also be noted that an explicit representation of the polynomial can be exponentially large. However, we do not need to represent the polynomial explicitly, since we only need to be able to evaluate the variable substitutions very fast.

This paper is organized as follows. In the next section we will introduce the terminology. In Section 5 we introduce the path problems in temporal graphs and in Section 6 we present an algebraic algorithm to solve the temporal-path problems. In Sections 7, 8, and 9 we discuss implementation, experimental setup and experimental evaluation, respectively. Finally, we conclude and present directions for future work in Section 10.

4 Terminology

In this section we introduce the basic terminology used in the paper.

A graph \( G \) is a tuple \((V, E)\) where \( V \) is a set of vertices and \( E \) is a set of unordered pairs of vertices called edges. We denote the number of vertices \(|V| = n\) and the number of edges \(|E| = m\). Vertices \( u \) and \( v \) are adjacent if there exists an edge \((u, v) \in E\). The set of vertices adjacent to vertex \( u \) is denoted by \( N(u) \). A walk between any two vertices is an alternating sequence of vertices and edges.
number of edges in the walk. A path vertices \( u \) in timestamp temporal edge \( u \) \( u \) vertices and temporal edges use the former definition of a temporal graph. j graph at time instance \( G \) A temporal graph can also be defined as j step \( G \). Given a temporal graph \( k \) exists a path of length at least \( k \). Björklund et al. time in the size of the host graph \( 5.2 \)-path problem in temporal graphs (\( -T \)-path problem in static graphs (\( -P \)-path problem in static graphs). More precisely, if we \( 5.1 \)-path problem in temporal graphs (\( -T \)-path problem to \( 22, ND29 \). Fortunately, the problem is fixed-parameter tractable. More precisely, if we fix the length of the path then the problem can be solved in polynomial time in the size of the host graph [5]. The best known fixed-parameter tractable algorithm is due to Björklund et al. [5] and has complexity \( O^*(1.66^k) \).

5 Path problems in temporal graphs

In this section we will introduce a set of path problems in temporal graphs. An exact algorithm based on multilinear sieving is presented in the next section.

Let us begin our discussion with the \( k \)-path problem for static graphs before continuing to path problems in temporal graphs.

5.1 \( k \)-path problem in static graphs (\( k \)-PATH)

Given a graph \( G = (V,E) \) and an integer \( k \leq n \) the \( k \)-PATH problem asks to decide whether there exists a path of length at least \( k - 1 \) in \( G \).

The \( k \)-PATH problem is \( \text{NP} \)-complete [22, ND29]. Fortunately, the problem is fixed-parameter tractable. More precisely, if we fix the length of the path then the problem can be solved in polynomial time in the size of the host graph [5]. The best known fixed-parameter tractable algorithm is due to Björklund et al. [5] and has complexity \( O^*(1.66^k) \).

5.2 \( k \)-path problem in temporal graphs (\( k \)-TEMP\( \text{PATH} \))

Given a temporal graph \( G^\tau = (V,E^\tau) \) and an integer \( k \leq n \) the \( k \)-\( \text{TEMP\text{PATH}} \) problem asks to decide whether there exists a temporal path of length at least \( k - 1 \) in \( G^\tau \).

For the \( \text{NP} \)-hardness, we reduce the \( k \)-PATH problem to \( k \)-TEMP\( \text{PATH} \) problem.

Lemma 5.1. Problem \( k \)-TEMP\( \text{PATH} \) is \( \text{NP} \)-complete.

\(^2\text{For convenience we represent }\{1,2,\ldots,k\} \text{ as } [k].\)
Problem straightforward. M agrees with lem asks to decide whether there exists a temporal path \( P \). Given a vertex-colored temporal graph \( G \), we know that all edges in \( P \) are present in \( G \). We construct a path \( G \) as shown in Figure 1. So we replace it by \( e_i \) for every edge \( e_i = (u_i, u_{i+1}) \) in \( P \) (by construction such an edge always exist in \( G \)). So \( P^\tau = e_1^\tau u_1 e_2^\tau u_2 \ldots e_{k-1}^\tau u_{k-1} \) is a temporal path of length \( k-1 \) in \( G \).

Inversely, assume that there exists a temporal path \( P^\tau = u_1 e_1^\tau u_2 \ldots e_{k-1}^\tau u_{k-1} \) of length \( k \) in \( G \). We construct a path \( P = u_1 e_1 u_2 \ldots e_{k-1} u_{k-1} \) by replacing \( e_i^\tau = (u_i, u_{i+1}, i) \) by \( e_i = (u_i, u_{i+1}) \) (such an edge always exist in \( G \) by construction). So \( P \) is a path of length \( k-1 \) in \( G \).

\[ \square \]

5.3 Path motif problem in temporal graphs (PATHMOTIF)

Given a vertex-colored temporal graph \( G^\tau = (V, E^\tau) \) and multi-set \( M \) of colors the PATHMOTIF problem asks to decide whether there exists a temporal path \( P^\tau \) in \( G^\tau \) such that the vertex colors of \( P^\tau \) agrees with \( M \).

The PATHMOTIF problem is NP-complete and a reduction from \( k\)-TEMPPATH to PATHMOTIF is straightforward.

**Lemma 5.2.** Problem PATHMOTIF is NP-complete.
Figure 3: An example of RAINBOWPath problem in temporal graphs.

Proof. Given an instance of a PATHMOTIF problem and a solution we can easily verify if the solution is a valid temporal path and the vertex colors of the solution agrees with that of the multi-set in polynomial time. So PATHMOTIF problem is in \textbf{NP}.

Given an instance of a \textit{k-TEMPPath} problem on a temporal graph $G^\tau = (V, E^\tau)$ we reduce it to PATHMOTIF problem. We construct a vertex colored temporal graph $G^\tau_c = (V_c, E^\tau_c)$ such that the vertex set $V_c = V$, the edge set $E^\tau_c = E^\tau$, the color mapping $c : V \rightarrow 1$ and multi-set $M = \{1^k\}$. More precisely, all the vertices in $V_c$ have the same color and the multi-set $M$ has color 1 for $k$ times.

We claim that there exists a \textit{k-TEMPPath} in $G^\tau$ if and only if there exists a PATHMOTIF in $G^\tau_c$. So, PATHMOTIF is at least as hard as \textit{k-TEMPPath}. From Lemma 5.1 we know that \textit{k-TEMPPath} is \textbf{NP}-hard. So, PATHMOTIF is also \textbf{NP}-hard.

Let $P$ be a temporal path of length $k - 1$ in $G^\tau$. We choose $P_c = P$ as a PATHMOTIF of length $k$ in $G^\tau_c$ since all vertices have same color and the multi-set $M$ agrees with the colors of vertices in $P_c$. Any PATHMOTIF of length $k - 1$ in $G^\tau_c$ is also a temporal path in $G$. \hfill \Box

5.4 Rainbow path problem in temporal graphs (RAINBOWPath)

Given a temporal graph $G^\tau = (V, E^\tau)$, an integer $k \leq n$, and a coloring function $c : V \rightarrow [k]$, the RAINBOWPath problem asks us to decide whether there exists a temporal path $P^\tau$ of length $k - 1$ such that all vertex colors of $P^\tau$ are different.

The RAINBOWPath problem is a special case of the PATHMOTIF problem, where all the colors in the multi-set $M$ are different, that is $M = [k]$. It is easy to see that the RAINBOWPath problem in static graphs can be reduced to the RAINBOWPath problem in temporal graphs by replacing each static edge with $k - 1$ temporal edges.\footnote{Given a static graph $G = (V, E)$ and a coloring function $c : V \rightarrow [k]$, the RAINBOWPath problem in static graphs asks us to find a path $P$ of length $k - 1$ such that all vertex colors of $P$ are different. The RAINBOWPath problem in static graphs is known to be \textbf{NP}-complete [20].} So, the RAINBOWPath problem is \textbf{NP}-complete. We skip the proof as the construction is similar to that of Lemma 5.1.

Lemma 5.3. Problem RAINBOWPath is \textbf{NP}-complete.

6 Algebraic algorithm for temporal paths

We now present an algorithm for the \textit{k-TEMPPath} and PATHMOTIF problems. Our algorithm relies on a polynomial encoding of temporal walks and the algebraic fingerprinting technique [8, 33, 36, 50]. The algorithm is presented in three steps:
(i) we present a dynamic-programming recursion to generate polynomial encoding of temporal walks;
(ii) we present an algebraic algorithm to detect the existence of an multilinear monomial in the polynomial generated using the recursion in (i) — furthermore, we prove that existence of a multilinear monomial implies existence of a temporal path; and
(iii) finally, we extend the approach to detect temporal paths with additional color constraints using constrained multilinear detection.

Let us begin our discussion with the concept of polynomial encoding of temporal walks.

Let \( P \) be a multivariate polynomial such that every monomial \( M \) is of the form

\[
M = x_1^{d_1} x_2^{d_2} \cdots x_q^{d_q} y_1^{f_1} y_2^{f_2} \cdots y_r^{f_r}.
\]

A monomial is multilinear if \( d_i \in \{0, 1\} \) for all \( i \in [q] \), and \( f_j \in \{0, 1\} \) for all \( j \in [r] \). A monomial is \( x\)-multilinear if \( d_i \in \{0, 1\} \) for all \( i \in [q] \) — in other words, we do not take into account the degrees of the \( y \)-variables. The degree of a monomial \( M \) is the sum of the degrees of all its variables. However, for a \( x \)-multilinear monomial the degree is the sum of degrees of \( x \)-variables.

### 6.1 Monomial encoding of a temporal walk

Let \( W^\tau = v_1 \ell_1 v_2 \cdots e_{k-1} v_k \) be a temporal walk in a temporal graph \( G^\tau = (V, E^\tau) \). Let \( \{x_{v_1}, \ldots, x_{v_k}\} \) be a set of variables representing vertices in \( V \) and \( \{y_{uv, i} : (u, v, i) \in E^\tau, \ell \in [k]\} \) be a set of variables such that \( y_{uv, i} \) correspond to an edge \( (u, v, i) \in E^\tau \) that appears at position \( \ell \) in \( W^\tau \). A monomial encoding of \( W^\tau \) is represented as

\[
x_{v_1} y_{v_1 v_2, i_1} x_{v_2} y_{v_1 v_2, i_2} \cdots y_{v_{k-1} v_k, k-1, i_{k-1}} x_{v_k},
\]

where \( i_1, \ldots, i_{k-1} \) denote the timestamps on the edges \( e_1, \ldots, e_{k-1} \), respectively.

It can be shown that the above encoding of \( W^\tau \) is \( x \)-multilinear if and only if \( W^\tau \) is a temporal path.

**Lemma 6.1.** The monomial encoding of a temporal walk \( W^\tau \) is \( x \)-multilinear (and multilinear) if and only if the temporal walk is a temporal path.

**Proof.** Let us assume that the temporal walk \( W^\tau \) is a temporal path, that is, each vertex in the walk appears exactly once so the \( x \) terms in the encoding are unique; furthermore, the \( y \) terms are unique since we will not visit any vertex more than once. So the monomial encoding of a temporal path is multilinear. Let us assume that the temporal walk is not a temporal path, that is, we have at least one vertex repetition. So there exists at least one \( x \)-variable in the encoding with degree at least two. So the encoding is not multilinear. \( \Box \)

### 6.2 Generating polynomial for temporal walks

In this section, we present a dynamic-programming recursion to generate temporal walks.

Let \( P_{u, i, \ell} \) denote the encoding of all walks of length \( \ell - 1 \) ending at vertex \( u \) at latest time \( i \in [t] \).

As an example illustrated in Figure 4, let \( v_1 \) be a vertex such that \( N_i(v_1) = \{v_2, v_3, v_4\} \). Let \( P_{v_2, \ell, i-1}, P_{v_3, \ell, i-1} \) and \( P_{v_4, \ell-1, i-1} \) represent the polynomial encoding of walks ending at vertices...
Figure 4: An illustration of the polynomial encoding of temporal walks.

\( v_2, v_3 \) and \( v_4 \), respectively, such that all walks have length \( \ell - 2 \) and end at latest time \( i - 1 \). Let \( P_{v_1, \ell, i-1} \) denote polynomial encoding of all walks of length \( \ell - 1 \), ending at \( v_1 \) at latest time \( i - 1 \).

The polynomial encoding to represent walks of length \( \ell - 1 \) ending at \( v_1 \) and at latest time \( i \) can be written as:

\[
P_{v_1, \ell, i} = x_{v_1} y_{v_2 v_1, \ell, i} P_{v_2, \ell-1, i-1} + x_{v_1} y_{v_3 v_1, \ell, i} P_{v_3, \ell-1, i-1} + x_{v_1} y_{v_4 v_1, \ell, i} P_{v_4, \ell-1, i-1} + P_{v_1, \ell, i-1}.
\]

Intuitively, the above equation represents that we can reach vertex \( v_1 \) at time step \( i \) if we have already reached any of its neighbors in \( N_i(v_1) \) by latest timestamp \( i - 1 \). Furthermore, the term \( P_{v_1, \ell, i-1} \) is included so that if we have reached \( v_1 \) at latest time \( i - 1 \) we can choose to stay at \( v_1 \) for timestamp \( i \).

By generalizing the above idea, a generating function can be written as follows. We set

\[
P_{u_1, i} = x_u,
\]

for each \( u \in V \) and \( i \in [\ell] \), and

\[
P_{u, \ell, i} = x_u \sum_{v \in N_i(u)} y_{uv, \ell, i} P_{v, \ell-1, i-1} + P_{u, \ell, i-1}.
\] (1)

for each \( u \in V \), \( \ell \in [k] \), and \( i \in [\ell] \).

Furthermore, let us form the polynomial \( P_{\ell, i} = \sum_{u \in V} P_{u, \ell, i} \) for each \( \ell \in [k] \) and \( i \in [\ell] \). More precisely, \( P_{\ell, i} \) denotes the polynomial encoding of all walks of length \( \ell - 1 \) ending at latest timestamp \( i \). Now the problem of detecting a \( k \)-TEMPPATH is equivalent to finding a \( x \)-multilinear monomial in \( P_{k,f} \). From the construction of the generating function (1) it is clear that the \( y \) variables are always distinct, and thus, detecting a \( x \)-multilinear monomial is equivalent to detecting a multilinear monomial.

**Lemma 6.2.** The polynomial encoding \( P_{u, \ell, i} \) in Equation (1) contains a \( x \)-multilinear monomial of degree \( \ell \) if and only if there exists a temporal path of length \( \ell - 1 \) ending at vertex \( u \) at latest time \( i \).

**Proof.** Let us assume that there exists a temporal path of length \( \ell - 1 \) that ends at vertex \( u \) at latest time \( i \). From Lemma 6.1 there exists a multilinear monomial in \( P_{u, \ell, i} \). Since the length of the path is \( \ell - 1 \) there are \( \ell \) unique terms of \( x \) and \( \ell - 1 \) terms of \( y \). So the \( x \)-degree of the monomial is \( \ell \).

Let us assume that there exists a \( x \)-multilinear monomial term of degree \( \ell \) in \( P_{u, \ell, i} \). From Lemma 6.1 we know that such a monomial represents a temporal path and the length of path is
The complexity of the algorithm is \( O((2^k) k(nt + m)) \) time and \( \Theta(nt) \) space.

**Proof.** The algorithm decides the existence of a multilinear monomial term in the polynomial \( P_{k,t} \). However, computing \( P_{k,t} \) requires \( \Theta(n + m) \) additions and multiplications in the field \( GF(2^b) \), where \( m \) is the number of edges at time instance \( i \in [t] \). Furthermore, computing the polynomial for all \( i \in [t] \) requires \( \Theta(nt + m) \) additions and multiplications. Finally, we need to iterate the computations over \( \ell \in [k] \), so we have \( \Theta(k(nt + m)) \) additions and multiplications. The overall complexity of the algorithm is \( \Theta(2^k k(nt + m) A(2^b) + (2^b)) \). Here \( A(2^b) \) and \( M(2^b) \) is the complexity of addition and multiplication in the field \( GF(2^b) \), respectively. Asymptotically it is known that \( M(2^b) = \Theta(b \log b) \) and \( A(2^b) = \Theta(b \log b) \) [42]. So the overall complexity of the algorithm is \( \Theta(2^k k(nt + m)b \log b) \).

In order to compute \( P_{k,t} \) we need to remember the computations of \( P_{u,t,i−1} \) and \( P_{k−1,i−1} \) for all \( v \in V \setminus u \) and for all \( i \in [t] \). However, we can reuse space since computing the \( \ell \)-th polynomial encoding only depends on the \((\ell−1)\)-th polynomial encoding. For each vertex \( v \in V \) we need a field variable and we need to compute this for all \( i \in [t] \) (this is also required for even staying at the vertex without moving). So the memory requirement is \( \Theta(nt) \).

**Lemma 6.5.** \( k\text{-TEMPPath} \) is fixed-parameter tractable.
Proof. The runtime of the algorithm is of the form $O((f(k))g(n))$, where $f(k) = 2^k k$, which is a exponential, and $g(n) = (nt + m)$, which is polynomial. If we fix the size of $k$ then the algorithm can decide the existence of a $k$-TEMPPATH in polynomial time. So the $k$-TEMPPATH problem is fixed-parameter tractable.

\[ \]
6.6 Obtaining an optimal solution

In this section we describe a procedure to obtain an optimal solution. By optimal we mean that the maximum timestamp of the edges in the temporal path is minimized. For simplicity, we refer to our algorithm for the decision version as decision oracle.

To find the minimum (optimal) timestamp $t' \in [t]$, we make at most $O(\log t)$ queries to the decision oracle using binary search on range $[t]$. More precisely, we construct a polynomial encoding of all walks of length $k - 1$ which end at latest time $t'$ and query the oracle for the existence of a path. So we need at most $\log t$ queries to the decision oracle to find the minimum timestamp $t' \in [t]$ for which there exists a match. So the complexity of deciding the existence of an optimal solution is $O(2^k k(nt + m) \log t)$.

6.7 Extracting a solution

In the previous sections we described an algebraic solution for the decision version of the PATHMOTIF problem. In many cases we need to extract a solution, if such a path exists. Our solution extraction process works in two steps:

(i) extract a $k$-vertex temporal subgraph which contains a PATHMOTIF matching the multi-set query;
(ii) extract a temporal path in the subgraph from (i) using temporal DFS.

Extract subgraph. We use the decision oracle as a subroutine to find a solution in at most $O(n)$ queries as follows: for each vertex $v \in V$ we remove the vertex $v$ and the edges incident to it and query the oracle. If there is a solution, then we continue to next vertex; otherwise we put back $v$ and the edges incident to it, and continue to next vertex. In this way, we can obtain a subgraph with $k$ vertices in at most $n - k$ queries to the oracle. However, the number of queries to the decision oracle can be reduced to $O(k \log n)$ queries in expectation by recursively dividing the graph in to two halves [6];

Extract temporal path. We pick an arbitrary start vertex in the subgraph obtained from step (i) and find a temporal path connecting all the $k$ vertices using temporal DFS, if such a path do not exist then continue to next vertex. Even though the worst case complexity is $O(k!)$, in practice this approach works very fast. However, extracting a solution can be done using $O(k)$ queries to the decision oracle using vertex-localized sieving. We will leave vertex-localization variant of sieving for our future work.

Finally, the overall complexity of extracting an optimal solution is $O(2^k k(nt + m)(k \log n + \log t))$.

6.8 Path motif problem with delays

In a real-world transport network a transition between any two locations would involve a transition time and a minimum delay time at a location before continuing the journey, for example a museum. In this section, we introduce a problem setting with transition and delay times, and present generating polynomials to solve the problems.

For a temporal graph $G^\tau = (V, E^\tau)$, an edge $e \in E^\tau$ is a quadruple $(u, v, i, \Delta)$ where $u, v \in V$, $i \in \mathbb{Z}_{\geq 0}$ is a time instance and $\Delta \in \mathbb{Z}_{\geq 0}$ is transition time from $u$ to $v$. Additionally, each vertex has a delay time $\delta : V \rightarrow \mathbb{Z}_{\geq 0}$.

We consider the following cases.
Encoding only with delay:

\[ P_{u,\ell,i} = x_u \sum_{(u,v,i,\Delta)\in E} y_{uv,i}P_{v,\ell-1,i-\delta(v)} + P_{u,\ell,i-1}. \]

Encoding only with transition time:

\[ P_{u,\ell,i+\Delta} = x_u \sum_{(u,v,i,\Delta)\in E} y_{uv,i}P_{v,\ell-1,i-1} + P_{u,\ell,i-1}. \]

Encoding with transition time and delay:

\[ P_{u,\ell,i+\Delta} = x_u \sum_{(u,v,i,\Delta)\in E} y_{uv,i}P_{v,\ell-1,i-\delta(v)} + P_{u,\ell,i-1}. \]

From Lemmas 6.2 and 6.6 it follows that existence of a multilinear monomial in the polynomial generated above would imply the existence of a PATHMOTIF.

7 Implementation

We use the design of Björklund et al. [9] as a starting point for our implementation, in particular we make use of fast finite-field arithmetic implementation.

Intuition. An high-level intuition of the approach is as follows: we assign each variable in a monomial a value in the field \( GF(2^b) \). The multiplication between any two field variables is defined as an XOR operation. Likewise, if we multiply two variables with the same value they cancel out each other and the resulting monomial has a zero value. So even though the generating polynomial has monomials which are not \( x \)-multilinear, the contributions from such monomials will cancel out during the evaluation. It is to be noted that the actual implementation is not identical to this description, but the high-level idea is similar. For the implementation details of the finite-field arithmetic, we refer the reader to the work of Björklund et al. [9, § 3.3].

A natural challenge in implementation engineering is to saturate the arithmetic bandwidth of the hardware, while simultaneously keeping the memory pipeline busy. Modern computing architectures have high memory bandwidth, however, the increase in bandwidth comes at the cost of latency. More precisely, after the processor issues a memory fetch instruction it takes many clock cycles to fetch the data from the main memory and make it available on the registers. Often the memory latency is orders of magnitude greater than the latency of arithmetic operations. Now, the challenge is to keep the processor busy with enough arithmetic instructions for computation meanwhile the memory pipeline is busy fetching the data for subsequent computations. The memory interface can be effectively utilized using coalesced memory accesses, by arranging the memory layout such that the data used for consecutive computation is available in consecutive memory addresses. The arithmetic bandwidth can be saturated by enabling parallel executions of same arithmetic operations, enabled using vector extensions. More precisely, if we are executing same arithmetic operation on different operands, then we can group the operands using vector extensions to execute arithmetic operations in parallel, thereby increasing the arithmetic throughput. The combination of memory coalescence and vector extensions are often used to speedup the computation in the algorithm-engineering community [3].
Our engineering effort boils down to implementing the generating function $1$ and evaluating
the recurrence at $2^k$ random points. Specifically, we introduce a domain variable $x_v$ for each $v \in V$
and a support variable $y_{uv,\ell,i}$ for each $\ell \in [k]$ and $(u, v, i) \in E^\tau$. In total, there are $O(n)$ domain
variables and $O(mk)$ support variables. The values of variables $x_v$ are computed using Equation $4$
and the values of variables $y_{uv,\ell,i}$ are assigned uniformly at random using a pseudorandom number
generator, on the fly without storing in memory. Observe that the variables $x_v$ and $y_{uv,\ell,i}$ are used
exactly once during the computation of $P_{k,t}$. Recall that, in theory, our algorithm has a false negative
probability of $2^{k-1}2^b$, however, in practice the false negative probability depend on the quality of
the random number generator.

Our implementation of the recurrence in Equation $1$ loops over three variables: the outermost
loop is over $[k]$, second loop over $[\tau]$ and final loop over $V$. So we compute the value of $P_{u,\ell,i}$ for
all $u \in V$ with $\ell$ and $i$ fixed. Precisely, for each iteration of inner most loop we compute the value
of $P_{u,\ell,i}$. In order to reduce the memory access latency we arrange our memory layout as $k \times \tau \times n$;
furthermore, we employ hardware pre-fetching [9, § 3.6] to saturate the memory bandwidth. The
inner most loop on $V$ is independent of each other, likewise we employ, OpenMP API using the ‘omp
parallel for’ construct with default scheduling over the vertices in $V$ to achieve thread-level
parallelism. Our current experimental results still lacks a detailed comparison of benefits of these
optimization techniques, and it is left as a future work.

If we observe carefully, the dynamic-programming recursion $1$, computing $P_{u,\ell,i}$ is independent
for each vertex. More precisely, computing $P_{u,\ell,i}$ is independent for each $u \in V$, provided that we fix
$\ell$ and $i$ fixed. So the algorithm is thread-parallelizable up to $n$ threads. Furthermore, we need to evaluate
at $2^k$ random points, that is, we need $2^k$ random substitutions of the variables in $z$ to evaluate
$P_{k,t}$ and these substitutions are independent of each other. More precisely, vector parallelization is
achieved on $2^k$ random points of evaluation, since we will perform the same arithmetic operation
on different set of data. So the algorithm is vector-parallelizable up to $2^k n$.

Our implementation is written in C programming language with OpenMP constructs to achieve
thread-level parallelism. The source code is compiled using -march=native and -O5 optimization
flags to enable architecture-specific optimization and instruction set extensions. The running time is
measured using OpenMP time interface omp_get_wtime and memory usage is tracked using wrapper
functions around standard C memory allocation subroutines malloc and free. We support both
directed and undirected graphs. Furthermore, we support extracting an optimal solution, for which
we rely on the self-reducibility of the interval oracles, based on the work of Björklund et al. [7].
However, our current implementation still lacks the functionality of listing all the solutions in the
input instance, which we leave for future work.

Our software is available as open source [47].

8 Experimental setup

In this section we discuss our experimental setup.

8.1 Baseline

For the problems considered in this paper we are not aware of any known baselines to compare.
Thus, we implemented two naive baselines:
(i) an exhaustive-search algorithm using temporal DFS, and
(ii) a brute-force algorithm based on random walks.

**Exhaustive-search.** In this technique, we pick a vertex \( v \in V \) and perform temporal DFS starting from \( v \) by restricting the depth of the DFS to \( k \). Every time we reach depth \( k \), we check if the vertices in the path satisfies the multi-set colors. If yes, then we update the solution to our optimal solution only if the maximum timestamp in the temporal path is less than the current solution, otherwise we will continue. Finally, we repeat the process for each vertex.

The runtime of exhaustive-search algorithm is bounded by \( \mathcal{O}(nd^k) \), where \( d \) is the maximum degree of the graph. The runtime analysis is as follows: as a worst case let us assume that each vertex has degree \( d \). While performing temporal DFS each vertex has one incoming edge and \( d-1 \) outgoing edges. So the temporal DFS tree of depth \( k \) has at most \( \mathcal{O}(d^k) \) vertices to visit. Furthermore we perform temporal DFS starting from each vertex in the graph making the overall complexity of the algorithm \( \mathcal{O}(nd^k) \). Observe carefully that a temporal DFS on each vertex is independent and it can be parallelized up to \( n \)-threads. So we use thread-level parallelism to speedup the computation. Note that the runtime bound of exhaustive search is loose, in practice the algorithm performs much faster than the theoretical bounds. A more tighter bound can be obtained with an assumption on degree and timestamp distribution, for example, \( d \)-regular or power-law distribution for graph degrees.

**Random-walk.** In this approach we pick a vertex \( v \) uniformly at random and perform a random temporal walk by restricting the length to \( k-1 \). We check if the random walk is a path and the vertex colors of the walk matches the multi-set. If yes then we update the solution to optimal solution provided that the maximum timestamp in the current solution is less than the optimal solution, otherwise we continue to next random walk. The runtime of the algorithm is bounded by the number of random walk iterations. This approach is also thread parallelizable since each random walk is independent.

It is to be noted that, our random-walk implementation failed to report an optimal solution even for small graph instances with \( m = 10^4 \) and \( k = 5 \) even after hundred million random iterations. For this reason, we experiment only with the exhaustive-search baseline.

Our baselines are implemented in C programming language, furthermore, optimized for architecture-specific instruction set and parallelized to achieve thread-level parallelism.

### 8.2 Hardware

To evaluate our algorithm implementations we experiment with two hardware configurations.

**Workstation:** A Fujitsu Esprimo E920 with 1×3.2 GHz Intel Core i5-4570 CPU, 4 cores, 16 GB memory, Ubuntu, and gcc v 5.4.0.

**Computenode:** A Dell PowerEdge C4130 with 2×2.5 GHz Intel Xeon 2680 V3 CPU, 24 cores, 12 cores/CPU, 128 GB memory, Red Hat, and gcc v 6.3.0.

Our executions make use of all cores, advanced vector extensions (AVX-2) and PCMULQDQ instruction set for fast finite field arithmetic. All the experiments are executed on the workstation configuration with an only exception for experiments with scaling to large graphs, which are executed on the computenode.

---

4 More precisely, the number of vertices in the temporal DFS tree is \( \frac{(d-1)^{k-1}+d^{-3}}{d-2} \).
8.3 Input graphs

We evaluate our methods using both synthetic and real-world graphs.

**Synthetic graphs.** We use two types of synthetic graphs: (a) random \(d\)-regular graphs; (b) power-law graphs. The regular graphs are generated using the configuration model [10] § 2.4. The configuration model for power-law graphs is as follows: given non-negative integers \(D, n, w\) and \(\alpha < 0\); we generate a \(n\)-vertex graph such these properties roughly hold: (i) the sum of vertex degrees is \(Dn\); (ii) the distribution of degrees is supported at \(w\) distinct values with geometric spacing; and (iii) the frequency of vertices with degree \(d\) is proportional to \(d^\alpha\). The edge timestamps are assigned uniformly at random in range \([t]\). Both directed and undirected graphs are generated using the same configuration model, however, for directed graphs the orientation is preserved. We ensure that the graph generator produces identical graph instances in all the hardware configurations.

**Real-world graphs.** We use the road transport networks from the cities of Helsinki and Madrid. The description of the dataset is as follows: each row in the dataset is a temporal edge between two locations, i.e., an unique identifier describing origin and destination, starting time and duration of travel; origin and destination locations. We pre-process the dataset to generate a graph by renaming the location identifiers in range starting from 1 to the maximum number of locations available in the dataset. We assign an unique identifier for each discrete starting time beginning with 1 and incrementing the identifier by one iteratively for each next available starting time, in this way we are avoiding the timestamps for which there are no temporal edges, thereby reducing the maximum timestamp value. If the time values in dataset are unix timestamps, then we approximate the value to the closest (floor) second. The vertex colors are assigned uniformly at random in range \(\{1, \ldots, 30\}\) and the multi-set colors are chosen uniformly at random in range \(\{1, \ldots, 30\}\). All datasets used in the experiments are made public [47].

9 Experimental evaluation

We will now describe our results and key findings. Recall that decision time is the time required to decide the existence of one solution, while extraction time is the time required to extract such a solution. As discussed previously, extracting a solution requires multiple calls to the decision oracle. Our baseline and scalability experiments are performed on RAINBOWPath problem, remember that, in RAINBOWPath problem every vertex matches with a multi-set color. Likewise, no trivial preprocessing step can be employed to reduce the graph size.

9.1 Baseline

Our first set of experiments compares the extraction time to obtain an optimal solution using our algebraic algorithm and the exhaustive-search baseline. In Table [1] we report extraction times of algebraic algorithm and baseline to extract an optimal solution for: (i) five independent \(d\)-regular random graphs with \(n = 10^2, \ldots, 10^5\) and fixed values of \(d = 20, t = 100, k = 5\); (ii) five independent power-law graphs with \(n = 10^2, \ldots, 10^5, D = 20, w = 100, k = 5, \alpha = -0.5\); and (iii) \(\alpha = -1.0\). Vertex colors are assigned randomly in the range \([k]\) and the multi-set is \([k]\). Each graph instance has at least ten target instances agreeing multi-set colors with different timestamps chosen uniformly at random. For the baseline we report the minimum time of five independent runs, however, for the algebraic algorithm we report the maximum. Speedup is the ratio of baseline and
Table 1: Comparison of extraction time for baseline and algebraic algorithms.

| No. of edges ($m$) | Regular | Powlaw $d^{-0.5}$ | Powlaw $d^{-1.0}$ |
|-------------------|---------|-------------------|-------------------|
|                   | Baseline | Algebraic | Speedup | Baseline | Algebraic | Speedup | Baseline | Algebraic | Speedup |
| 1 040             | 0.05 s   | 0.04 s       | 1.2     | 0.05 s   | 0.04 s   | 1.2     | 0.05 s   | 0.04 s   | 1.2     |
| 10 040            | 0.48 s   | 0.12 s       | 4.1     | 1.03 s   | 0.11 s   | 9.4     | 10.82 s  | 0.10 s   | 103.6   |
| 100 040           | 5.62 s   | 1.06 s       | 5.3     | 30.38 s  | 1.07 s   | 28.5    | 20 430.16 s | 0.92 s | 22 306.1 |
| 1 000 040         | 74.01 s  | 12.02 s      | 6.2     | 808.24 s | 11.18 s  | 72.3    | –        | 10.03 s  | –       |

Figure 5: Comparison of extraction time of baseline and algebraic algorithms. We report the decision time as a function of the number of edges for $d$-regular random graph (left), power-law graph with $d = -0.5$ (center), and power-law graph with $d = -1.0$ (right).

Surprisingly, the baseline can compete with the algebraic algorithm in the case of $d$-regular random graphs, however, the runtimes have high variance across different graph topologies. On the other hand, the algebraic algorithm is very stable. For the power-law graphs with $m = 10^5$ edges and multi-set size $k = 5$, the algebraic algorithm is at least twenty thousand times faster than the baseline. Our exhaustive-search implementation failed to report a solution in small graphs $m = 10^3$ with large multi-set size $k = 10$.

In Figure 5, we report the extraction time for baseline and algebraic algorithms to obtain an optimal solution for five independent $d$-regular random graphs with $n = 10^2, \ldots, 10^5$ and fixed values of $d = 20$, $t = 100$, $k = 5$ (left); five independent power-law graphs with $n = 10^2, \ldots, 10^5$, $D = 20$, $w = 100$, $t = 100$, $k = 5$, $\alpha = -0.5$ (center); and $\alpha = -1.0$ (right). The vertex colors are assigned uniformly at random in range $[k]$ and the multi-set is $[k]$. Each graph instance has at least ten target instances satisfying the multi-set colors with different timestamps chosen uniformly at random. The experiments are executed on workstation.

The runtimes of both baseline and algebraic algorithms are consistent with very little variance across independent graph inputs of same graph topology. However, the exhaustive-search baseline has high variance in runtime with change in graph topology. For a power-law of size $m = 10^5$ with $\alpha = -1.0$, the runtime of exhaustive-search is at least three thousand times greater than a $d$-regular graph of same size.
Figure 6: Scalability results. Runtime as a function of the number of edges (left); multi-set size (center-left); number of timestamps (center-right); and degree (right).

9.2 Scalability

Our second set of experiments study scalability with respect to: (i) number of edges; (ii) multi-set size; (iii) number of timestamps; and (iv) vertex degree.

Figure 6(left) reports decision and extraction times for $d$-regular random graphs with $n = 10^2, \ldots , 10^5$ and fixed values of $d = 20$, $k = 8$, $t = 100$. Figure 6(center-left) shows decision time for $d$-regular random graphs with $k = 10, \ldots , 18$ and fixed values of $n = 10^3$, $d = 20$, $t = 100$. Vertex colors are assigned randomly in the range $[k]$ and the multi-set is $[k]$. We observe a linear scaling with increasing the number of edges and exponential scaling with increasing the multi-set size, as expected by the theory. The variance in decision time is very small for different inputs, however, it is higher for extraction time. The algorithm is able to decide the existence of a solution in less than two minutes for graphs up to one million edges with multi-set size $k = 8$ and extract a solution in less than sixteen minutes.

Next we study the effect of graph density on scalability. Figure 6(center-right) shows decision and extraction times for $d$-regular random graphs with $t = 10, \ldots , 100$ and fixed values of $n = 10^4$, $d = 20$, $k = 8$. Figure 6(right) shows decision and extraction times for $d$-regular random graphs with $d = 2, 20, 200, 2000$ and corresponding values of $n = 10^6, \ldots , 10^9$, with fixed $m = 10^6$ and $t = 100$. We observe that the algebraic algorithm performs better for dense graphs. A possible explanation is that for sparse graphs there is not enough work to keep both the arithmetic and memory pipeline busy, simultaneously.

All experiments are executed on the workstation configuration using all cores with undirected graphs. Additionally, we make sure that each input instance has at least ten solutions agreeing multi-set colors with different timestamps chosen uniformly at random. We also verified the correctness of our implementation with graph instances having an unique solution and no solution.

9.3 Scaling to large graphs

Next we study the scalability of the algebraic algorithm to graphs with up to hundred million edges. Figure 8 reports decision and extraction times for $d$-regular random graphs with $n = 10^3, \ldots , 10^6$, $d = 200$, $t = 100$ with $k = 5$ (left) and $k = 10$ (right). Vertex colors are assigned randomly in the range $[k]$ and the multi-set is $[k]$. The experiments are executed on the computenode configuration using all cores. In graphs with hundred million edges, the algebraic algorithm can extract an optimal solution in less than two minutes for $k = 5$ and less than two hours for $k = 10$.

Our next set of experiments study the scaling of the algorithm for graphs up to a billion edges provided the number of timestamps is small. Figure 8(left) shows a comparison of decision and
Figure 7: Scaling to large graphs. Runtime as a function of number of edges with $k = 5$ (left) and $k = 10$ (right).

Figure 8: Scaling up to a billion edges. We compare the runtime (left) and memory usage (right) for decision and extraction variants of the problem as a function of number of edges.

extraction time of algorithm and Figure 8 (right) reports memory usage of the algorithm for five independent $d$-regular random graphs with $n = 10^2, \ldots, 10^7$ and $d = 200$, $k = 5$, $t = 20$, fixed. It is to be noted that the number of timestamps is small. The vertex colors are assigned uniformly at random in range $[30]$ and multi-set is chosen uniformly at random. We ensure that each graph instance has at least ten target instances agreeing the multi-set colors. All experiments are performed on computenode configuration using all cores with undirected graphs. We employ a preprocessing step by removing the vertices whose colors do not match the multi-set colors.

We observe approximately factor ten difference in decision and extraction time for graphs with a small number of edges, and the difference in runtime decreases with the increase in graph size. The memory usage for deciding the existence of a solution and extracting a solution are approximately the same. On the haswell computenode configuration, we can extract a solution in a billion-edge graph in less than three minutes for small values of multi-set size $k = 5$ with at most sixty gigabytes of peak-memory usage. Remember that our current implementation uses $O(ntk)$ memory instead of $O(nt)$, which implies that there is a room for improvement in the implementation that can reduce the memory usage by factor $k$.

9.4 Experiments with real-world graphs

Finally, we report decision and extraction times for the algebraic algorithm on real-world data. The datasets used for these experiments are described in Section 8.3. Table 2 reports decision and extraction time (in seconds) for the experiments on real-world datasets. For each dataset we report the maximum time among the five independent executions by choosing multi-set colors at random.
Table 2: Experimental results on real-world graphs.

| Dataset               | $n$  | $m$     | $t$  | $k = 5$           | $k = 10$          |
|-----------------------|------|---------|------|------------------|-------------------|
|                       |      |         |      | Baseline | Algebraic | Baseline | Algebraic |
| Madrid tram           | 70   | 35 144  | 1265 | 1.37 s  | 0.24 s    | 1 337.98 s | 28.05 s   |
| Madrid train          | 91   | 43 677  | 1 181| 40.01 s | 0.25 s    | –        | 24.12 s   |
| Madrid bus            | 4 597| 2 254 993| 1 440| 6 337.89 s| 1.27 s    | –        | 278.91 s  |
| Madrid interurban bus | 7 543| 1 495 055| 1 440| 744.79 s| 1.30 s    | –        | 325.51 s  |
| Helsinki bus          | 7 959| 6 403 785| 1 440| –       | 1.67 s    | –        | 444.66 s  |

For multi-set size $k = 5$, the extraction time is at most two seconds. For larger multi-set size $k = 10$, the extraction time is at most eight minutes in all the datasets. The experiments are executed in the workstation setup using all cores. Additionally, we pre-process the graphs by removing vertices whose colors do not match with multi-set colors for both the baseline and the algebraic algorithm.

10 Conclusions and future work

In this paper we introduce several pattern-detection problems that arise in the context of mining large temporal graphs. We present complexity results and design algebraic algorithms for finding exact solutions, based on the constrained multilinear-sieving technique. Our implementation can scale to large graphs up to hundred million edges despite the problems being NP-hard. We present extensive experimental results that validate our scalability claims.

As future work we would like to consider a problem setting where we are searching for a temporal path with a specification of colors. However, the vertices need not be immediately adjacent but rather some uncertain distance away from each other. In such scenarios, the task can still be formulated as a PATHMOTIF problem with generalization to each vertex being colored with a set of colors instead of a single color. This enables us to have wild-card entry matches to accommodate for uncertainty. A possible direction for our future work is to design algorithms to accommodate wild-card entries in pattern-detection problems.

The application of our framework is not limited to temporal paths but rather can be extended to a wide range of pattern-detection problems where we search for information cascades, temporal arborescences, and temporal subgraphs.

11 Acknowledgements

This research was supported by the Academy of Finland project “Adaptive and Intelligent Data (AIDA)” (317085) and the EC H2020 RIA project “SoBigData++” (654024). We acknowledge the use of computational resources funded by the project “Science-IT” at Aalto University, Finland. Finally, we thank Juho Lauri for his valuable feedback, in particular his comments on the complexity of the RAINBOWPath problem.
References

[1] N. Alon, P. Dao, I. Hajirasouliha, F. Hormozdiari, and S. C. Sahinalp, Biomolecular network motif counting and discovery by color coding, Bioinformatics, 24 (2008), pp. 241–249.

[2] C. Aslay, A. Nasir, G. De Francisci Morales, and A. Gionis, Mining frequent patterns in evolving graphs, in CIKM, 2018, pp. 923–932.

[3] N. Bell and M. Garland, Efficient sparse matrix-vector multiplication on CUDA, Nvidia tech. rep., NVIDIA Corp., 2008.

[4] A. Benson, D. Gleich, and J. Leskovec, Higher-order organization of complex networks, Science, 353 (2016), pp. 163–166.

[5] A. Björklund, T. Husfeldt, P. Kaski, and M. Koivisto, Narrow sieves for parameterized paths and packings, JCSS, 87 (2017), pp. 119–139.

[6] A. Björklund, P. Kaski, and Ł. Kowalik, Determinant sums for undirected Hamiltonicity, SIAM J. Comput., 43 (2014), pp. 280–299.

[7] ———, Fast witness extraction using a decision oracle, in ESA, 2014, pp. 149–160.

[8] ———, Constrained multilinear detection and generalized graph motifs, Algorithmica, 74 (2016), pp. 947–967.

[9] A. Björklund, P. Kaski, Ł. Kowalik, and J. Lauri, Engineering motif search for large graphs, in ALENEX, 2015, pp. 104–118.

[10] B. Bollobás, Random Graphs, Cambridge UP, second ed., 2001.

[11] M. Bressan, S. Leucci, and A. Panconesi, Motivo: Fast motif counting via succinct color coding and adaptive sampling, PVLDB, 12 (2019), pp. 1651–1663.

[12] A. Casteigts, A. Himmel, H. Molter, and P. Zschoche, The computational complexity of finding temporal paths under waiting time constraints, CoRR, abs/1909.06437 (2019).

[13] F. Cicalese, T. Gagie, E. Giaquinta, E. S. Laber, Z. Lipták, R. Rizzi, and A. I. Tomescu, Indexes for jumbled pattern matching in strings, trees and graphs, in SPIRE, 2013, pp. 56–63.

[14] M. Coletto, K. Garimella, A. Gionis, and C. Lucchese, Automatic controversy detection in social media: A content-independent motif-based approach, Online Social Networks and Media, 3–4 (2017), pp. 22–31.

[15] M. Coletto, K. Garimella, A. Gionis, and C. Lucchese, A motif-based approach for identifying controversy, in Eleventh International AAAI Conference on Web and Social Media, 2017.

[16] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Parameterized algorithms, 2015.

[17] M. DeChoudhury, M. Feldman, S. Amer-Yahia, N. Golbandi, R. Lempel, and C. Yu, Automatic construction of travel itineraries using social breadcrumbs, in HT, 2010, pp. 35–44.
[18] R. Dechter, I. Meiri, and J. Pearl, *Temporal constraint networks*, Artificial intelligence, 49 (1991), pp. 61–95.

[19] H. Dell, J. Lapinskas, and K. Meeks, *Approximately counting and sampling small witnesses using a colourful decision oracle*, in SODA, 2020, pp. 2201–2211.

[20] E. Eiben, R. Ganian, and J. Lauri, *On the complexity of rainbow coloring problems*, Discrete Applied Mathematics, 246 (2018), pp. 38 – 48. The Combinatorics of Graphs and Strings.

[21] T. Gagie, D. Hermelin, G. M. Landau, and O. Weimann, *Binary jumbled pattern matching on trees and tree-like structures*, in ESA, 2013.

[22] M. R. Garey and D. S. Johnson, *Computers and intractability*, vol. 29, W. H. Freeman and Co., 2002.

[23] B. George, S. Kim, and S. Shekhar, *Spatio-temporal network databases and routing algorithms: A summary of results*, in International Symposium on Spatial and Temporal Databases, 2007, pp. 460–477.

[24] E. Giaquinta and S. Grabowski, *New algorithms for binary jumbled pattern matching*, IPL, 113 (2013), pp. 538–542.

[25] A. Gionis, T. Lappas, K. Pelechrinis, and E. Terzi, *Customized tour recommendations in urban areas*, WSDM, 2014, pp. 313–322.

[26] M. Gupta, C. C. Aggarwal, and J. Han, *Finding top-k shortest path distance changes in an evolutionary network*, in SSTD, 2011, pp. 130–148.

[27] P. Holme, *Modern temporal network theory: a colloquium*, European Physical Journal B, 88 (2015), p. 234.

[28] P. Holme and J. Saramäki, *Temporal networks*, Physics reports, 519 (2012), pp. 97–125.

[29] ———, *Temporal networks*, Physics reports, 519 (2012), pp. 97–125.

[30] C. J. Honey, R. Kötter, M. Breakspear, and O. Sporns, *Network structure of cerebral cortex shapes functional connectivity on multiple time scales*, PNAS, 104 (2007), pp. 10240–10245.

[31] P. Kaski, J. Lauri, and S. Thejaswi, *Engineering Motif Search for Large Motifs*, in SEA, 2018, pp. 1–19.

[32] V. Kostakos, *Temporal graphs*, Physica A: Statistical Mechanics and its Applications, 388 (2009), pp. 1007–1023.

[33] I. Koutis, *Faster algebraic algorithms for path and packing problems*, in ICALP, 2008.

[34] ———, *The power of group algebras for constrained multilinear monomial detection*, Dagstuhl meeting 10441, (2010).

[35] ———, *Constrained multilinear detection for faster functional motif discovery*, IPL, 112 (2012), pp. 889–892.
36 I. KOUTIS AND R. WILLIAMS, Limits and applications of group algebras for parameterized problems, in ICALP (1), 2009.

37 I. KOUTIS AND R. WILLIAMS, Algebraic fingerprints for faster algorithms, Comm. of the ACM, 59 (2016), pp. 98–105.

38 L. KOVANEN, M. KARSAI, K. KASKI, J. KERTÉSZ, AND J. SARAMÄKI, Temporal motifs in time-dependent networks, Journal of Statistical Mechanics: Theory and Experiment, 2011 (2011), p. P11005.

39 L. KOWALIK AND J. LAURI, On finding rainbow and colorful paths, TCS, 628 (2016), pp. 110 – 114.

40 V. LACROIX, C. G. FERNANDES, AND M.-F. SAGOT, Motif search in graphs: application to metabolic networks, IEEE Transactions on Computational Biology and Bioinformatics (TCBB), 3 (2006), pp. 360–368.

41 M. LATAPY, T. VIARD, AND C. MAGNIEN, Stream graphs and link streams for the modeling of interactions over time, Social Network Analysis and Mining, 8 (2018).

42 S.-J. LIN, T. Y. AL-NAFFOURI, Y. S. HAN, AND W.-H. CHUNG, Novel polynomial basis with fast Fourier transform and its application to Reed-Solomon erasure codes, ITIT, 62 (2016).

43 L. LIU, J. TANG, J. HAN, M. JIANG, AND S. YANG, Mining topic-level influence in heterogeneous networks, in CIKM, 2010, pp. 199–208.

44 P. LIU, A. BENSON, AND M. CHARIKAR, Sampling methods for counting temporal motifs, in WSDM, 2019, pp. 294–302.

45 R. MILO, S. SHEN-ORR, S. ITZKOVITZ, N. KASHTAN, D. CHKLOVSKI, AND U. ALON, Network motifs: Simple building blocks of complex networks, Science, 298 (2002), pp. 824–827.

46 A. PARANJAPE, A. BENSON, AND J. LESKOVEC, Motifs in temporal networks, WSDM, 2017, pp. 601–610.

47 S. THEJASWII AND A. GIONIS, 2019. https://github.com/suhastheju/temporal-patterns

48 P. VANSTEENWEGEN, W. SOUFFRIAU, AND D. V. OUDHEUSDEN, The orienteering problem: A survey, EJOR, 209 (2011), pp. 1 – 10.

49 B. WACKERSREUTHER, P. WACKERSREUTHER, A. OSWALD, C. BÖHM, AND K. BORGWARDT, Frequent subgraph discovery in dynamic networks, in MLG, 2010.

50 R. WILLIAMS, Finding paths of length k in O^*(2^k) time, IPL, 109 (2009).

51 H. WU, J. CHENG, S. HUANG, Y. KE, Y. LU, AND Y. XU, Path problems in temporal graphs, Proc. VLDB Endow., 7 (2014), pp. 721–732.

52 H. WU, J. CHENG, Y. KE, S. HUANG, Y. HUANG, AND H. WU, Efficient algorithms for temporal path computation, TKDE, 28 (2016), pp. 2927–2942.
[53] J. Yang, J. McAuley, and J. Leskovec, *Community detection in networks with node attributes*, in ICDM, 2013, pp. 1151–1156.