Qubit stabilizer states are complex projective 3-designs

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(Dated: October 12, 2015)

A complex projective $t$-design is a configuration of vectors which is “evenly distributed” on a sphere in the sense that sampling uniformly from it reproduces the moments of Haar measure up to order $2t$. We show that the set of all $n$-qubit stabilizer states forms a complex projective 3-design in dimension $2^n$. Stabilizer states had previously only been known to constitute 2-designs. The main technical ingredient is a general recursion formula for the so-called frame potential of stabilizer states. To establish it, we need to compute the number of stabilizer states with pre-described inner product with respect to a reference state. This, in turn, reduces to a counting problem in discrete symplectic vector spaces for which we find a simple formula. We sketch applications in quantum information and signal analysis.

I. INTRODUCTION AND MAIN RESULTS

A. Introduction

In its simplest incarnation, a $D$-dimensional complex projective $t$-design is a set of unit-length vectors in $\mathbb{C}^D$ that is evenly distributed on the sphere in the sense that sampling uniformly from this set reproduces the moments of Haar measure up to order $2t$ [1–5] (see Definition 1 below for a precise definition). In a variety of contexts such a design structure is important:

In numerical integration, designs are known as cubatures. It follows from the definition that the average of a homogeneous polynomial $p$ of order $2t$ over the complex unit sphere equals $p$’s average over the design. If the design has small order, this realization can be made the basis for fast numerical procedures that compute integrals of smooth functions over high-dimensional spheres.

In quantum information theory, designs are a widely-employed tool for derandomizing probabilistic constructions. Recall that the probabilistic method [6] is a powerful proof technique originally designed to tackle problems in combinatorics. At its core is the observation that the existence of certain extremal combinatorial structures often can be be proved by showing that a suitably chosen random construction would produce an example with high probability. In quantum information, randomized construction often rely on randomly chosen Hilbert space vectors [7]. While this method has brought about spectacular successes (such as the celebrated proof of strict sub-additivity of entanglement formation [8]), it suffers e.g. from the problem that generic Haar-random states of large quantum systems are unphysical: they cannot be prepared from separable inputs using a polynomial number of operations [9]. Designs, in contrast, can be chosen to consist solely of highly-structured and efficiently preparable vectors, while retaining “generic” properties in a precise sense. Thus considerable efforts have been expended at designing complex projective designs (and their unitary cousins) [3, 10–13].

Lastly, randomized constructions in Hilbert spaces have completely classical applications, e.g. in signal analysis. Take for instance the highly active field of compressed sensing and related topics [14]: There, one is interested in reconstructing objects that possess some non-trivial structure (e.g. sparsity, or low rank) from a small number of linear measurements. Strong recovery guarantees can be proven for randomly constructed measurement vectors. Once more, this raises the problem of finding sets of structured and well-understood measurements that sufficiently resemble the properties of generic random vectors. The use of designs for this purpose has been proposed in [15–17].

Despite this wealth of applications and non-constructive existence proofs [18], explicit constructions for complex designs remain rare. There are various infinite families of complex projective 2-designs (e.g. maximal sets of mutually unbiased bases [19, 20], stabilizer states, or symmetric informationally complete POVMs [2]); sporadic solutions for higher orders [11, 21, 22]; and approximate constructions involving random circuits [13]. To the best of our knowledge, an infinite set of explicit complex projective 3-designs has not been identified before.

Here, we show that the set of all stabilizer states in dimension $2^n$ forms a complex projective 3-design for all $n \in \mathbb{N}$. Recall that the stabilizer formalism is a ubiquitous tool in quantum information theory [9, 23]. Stabilizer states (and, slightly more general, stabilizer codes) are joint eigenvectors of generalized Pauli matrices. Constituting the main realization of quantum error correcting codes [23], they can be efficiently prepared [24] and described in terms of polynomially many parameters [9]. Yet they exhibit non-trivial properties like multi-partite entanglement [25]. Stabilizer states were instrumental in the development of measurement-based quantum computation [26, 27]. In several precise ways, they can be seen as the discrete analogue of Gaussian states [28]. Beyond quantum information, stabilizer states have proved to
be versatile enough to provide powerful models for one of the most influential recent development in theoretical condensed matter physics: the study of topological order [29, 30].

Our main result thus identifies yet another aspect according to which stabilizer states capture properties of generic state vectors.

### B. Designs and frame potential

In order to state our results more precisely, we need to give a formal definition of complex projective designs and introduce the related notion of frame potential. Following [4, 31, 32], we define

**Definition 1.** Fix a dimension $D$ and let $\mu$ be a probability measure on the unit sphere in $\mathbb{C}^D$. The measure $\mu$ is a complex projective $t$-design if, for any order-$t$ polynomial $p$, we have

$$
\mathbb{E}_{x,y \sim \mu} \left[ p \left( |\langle x,y \rangle|^2 \right) \right] = \int_{x,y} p \left( |\langle x,y \rangle|^2 \right) \mathrm{d}x \mathrm{d}y ,
$$

where the right-hand-side integration is with respect to the uniform (Haar) measure on the sphere.

In other words, sampling according to $\mu$ should give the same expectation values as sampling according to the uniform measure for any random variable that is a polynomial in $|\langle x,y \rangle|^2$ of order at most $t$. From now on, we will only be concerned with the case where $\mu$ is the uniform measure on a finite set of unit vectors.

It is not hard to see that $\mu$ fulfills (1) for all polynomials of order $t$ or less, if equality holds for the specific case of $p(z) = z^t$. The resulting value is the $t$-th order frame potential [33]

$$
F_t(\mu) := \mathbb{E}_{x,y \sim \mu} \left[ |\langle x,y \rangle|^2 \right].
$$

It is known that the Haar integral on the r.h.s. of (1) minimizes the frame potential over the set of all measures $\mu$ and that, in fact, its value is given by

$$
F_t(\mu) \geq W_t(D) := \left( \frac{D + t - 1}{t} \right)^{-1} .
$$

This relation is known as Welch bound [34] or Sidelnikov inequality [35]. In summary, we have:

**Theorem 1 ([4, 31–33]).** Fix a dimension $D$ and let $\mu$ be a probability measure on the unit sphere in $\mathbb{C}^D$. The measure $\mu$ is a complex projective $t$-design if and only if its frame potential meets the Welch bound

$$
F_t(\mu) = W_t(D).
$$

### C. Main results

At the heart of this work is an explicit characterization of the frame potential assumed by the uniform distribution over stabilizer states in prime power dimensions $D = d^n$. We denote the set of stabilizer states on $(\mathbb{C}^d)^{\otimes n} \simeq \mathbb{C}^D$ by $\text{Stabs}(d,n)$. The unitary symmetry group of the set of stabilizer states is the Clifford group (for a precise definition, see Section II C). All results are then implied by the following recursion formula over the dimension’s exponent $n = \log_d(D)$.

**Theorem 2 (Main Theorem).** Let $d$ be a prime number and let $t \in \mathbb{N}_+$. Then for all dimensions $D = d^n$, the frame potential $F_t(\text{Stabs}(d,n))$ of stabilizer states in $\mathbb{C}^D$ is determined by the following recursion formula over $n$:

1. $F_t(\text{Stabs}(d,1)) = \frac{d^{2t-1} + 1}{(d+1)d}$,
2. $F_t(\text{Stabs}(d,n+1)) = \frac{d^{n-t-2} + 1}{d(d^{n+1} + 1)}.$

Comparing this explicit characterization of the frame potential to the Sidelnikov inequality (3) allows us to draw the following conclusions:

**Corollary 1.** Let $d^n$ be a prime-power dimension. Then the following statements are true

1. $\text{Stabs}(d,n)$ forms a complex projective 2-design.
2. $\text{Stabs}(d,n)$ constitutes a complex projective 3-design if and only if $d = 2$.
3. The set $\text{Stabs}(d,n)$ does not constitute a complex projective 4-design.
4. The Clifford group does not act irreducibly on $\text{Sym}^4(\mathbb{C}^D) \subset (\mathbb{C}^D)^{\otimes 4}$. In particular, it is not a unitary 4-design.

As indicated before, the first fact was already widely known [11, 19, 20]. The other results, however, are new to the best of our knowledge. We reemphasize that these assertions follow immediately form the Main Theorem, which may be of independent interest.

### D. Applications and Outlook

Here, we sketch relations of the result to problems from signal analysis and quantum physics. Elaborating on these connections will be the focus of future work.

In low-rank recovery [14, 36–38], a low-rank matrix $X$ is to be reconstructed from few linear measurements of the form $y_i = \text{tr}(X a_i)$. In the phase retrieval problem [15, 39, 40] one aims to recover a complex vector $x \in \mathbb{C}^D$ from the absolute value of a small number of measurements $y_i = |\langle x, a_i \rangle|$ that are ignorant towards
phase information. This task can be reduced to a particular instance of rank-one matrix recovery by rewriting the measurements as [41, 42]

\[ y_i^2 = \text{tr} \left( |x\rangle \langle x| |a_i\rangle \langle a_i| \right), \]

eq by setting \( X = |x\rangle \langle x| \) and \( A_i = |a_i\rangle \langle a_i| \). For both problems, strong recovery guarantees for randomly constructed measurements are known. Oftentimes these rely on generic (e.g. Gaussian) measurement ensembles and employing complex projective designs to partially derandomize these results has been proposed in both contexts [15, 16, 43].

Regarding both low rank matrix recovery and phase retrieval, it is known that sampling measurement vectors independently from a 2-design does not do the job [15], while 4-designs already have an essentially optimal performance [43, 44]. However, the remaining intermediate case for \( t = 3 \) is not yet fully understood. Numerical studies conducted by Drave and Rauhut [45] indicate that random stabilizer-state measurements perform surprisingly well at that task. The combinatorial properties of prime power stabilizer states – e.g. Theorem 2 – may help to clarify this situation. We believe this to be a potentially very insightful open problem.

Finally, we want to point out that one nice structural property of stabilizer states is that they come in bases, i.e. the set of all stabilizer states is a union of different orthonormal bases (see e.g. Theorem 3 below). This allows for a considerably more structured random measurement protocol: Select one such basis at random and iteratively measure the trace inner product of an unknown low rank matrix with all projectors onto the individual basis vectors. After having acquired \( D \) data points that way, choose a new stabilizer basis at random and repeat. We refer to [46] for a detailed description of such a protocol. It should be clear that it has immediate applications to quantum state tomography. In the above paper, non-trivial recovery statements have been announced for \( t \)-designs that admit such a basis structure and have strength \( t \geq 3 \). Again, stabilizer states obey these criteria and have been used for the numerical experiments conducted there. However the announced recovery statement suffers from a non-optimal sampling rate for 3-designs and the rich combinatorial structure of stabilizer bases might help to amend that situation.

### E. Relation to previous work and history

After completion of this work (first announced at the QIP 2013 conference [47]), we became aware of the fact that a close analogue of our main result follows from a statement proved in the field of algebraic combinatorics [48] in 1999. The object of study there is a real version of stabilizer states in \( \mathbb{R}^{2^n} \), as well as their symmetries, which are given by a real version of the Clifford group. The key result is that under the action of the real Clifford group, the space \( \text{Sym}^3(\mathbb{R}^{2^n}) \) decomposes into irreps in exactly the same way as it does under the action of the full orthogonal group \( O(2^n) \) [48, 49]. This implies [50, 51] that any orbit of the real Clifford group gives rise to a set that reproduces moments of Haar measure up to order 6 (the established – if confusing – terminology is to refer to such sets as spherical 6-designs [1], while the complex-valued analogue would be called a complex projective 3-design [2]).

The findings of [48] are formulated in the language of algebraic invariant theory. While the present authors were trying to relate them to the results we had established in the context of quantum information, we became aware of yet another development. Huangjun Zhu [52] independently derived a very simple and elegant proof showing that the complex Clifford group in dimensions \( d = 2^n \) actually forms a unitary 3-design [10, 11]. This means that the irreducible representation spaces of the action of the Clifford group on \( (\mathbb{C}^{2^n})^\otimes 3 \) coincide with those of the full unitary group \( U(d) \). In particular, the Clifford group acts irreducibly on \( \text{Sym}^3(\mathbb{C}^d) \) which, in turn, implies that that any orbit of the group constitutes a complex projective 3-design. The work of Zhu thus fully implies our main result. What is more, the proof is simpler.

The appeal of the question treated here was underscored even more, when we learned a few days prior to submission of this paper to the arxiv e-print server, that yet another researcher – Zak Webb – had independently obtained results related to the ones of Zhu [53].

In comparison to these works, our proof methods are completely different: We rely on counting structures in discrete symplectic vector spaces in order to compute the angle set between stabilizer states, whereas [48] is based on algebraic invariant theory and [52] on character theory. As a corollary, we derive an expression for the number of stabilizer states with prescribed inner product to a reference state. This finding might be of independent interest. Also, we show that the set of stabilizer states fails to be a 4-design in dimensions \( 2^n \) and that stabilizer states in dimensions other than powers of two do not even constitute a 3-design. The simultaneously submitted papers seem to have left this possibility open.

### II. PROOF OF THE MAIN STATEMENT

#### A. Outline

We already mentioned in the introduction that there is a geometric approach to stabilizer states building on
the theory of discrete symplectic vector spaces. This phase space formalism will be introduced in Section II B. We formally define stabilizer states and explain how to compute inner products in this language in Section II C. We then move on to briefly introducing Grassmannians and some core concepts of discrete symplectic geometry. These tools will be used to establish Theorem 2 in Section III.

## B. Phase Space Formalism

We start by considering a $d$-dimensional Hilbert space $\mathcal{H}$, equipped with a basis $\{|q\rangle | q \in Q\}$, where the configuration space $Q$ is given by $Q := \{0, \ldots, d-1\} \subset Z$ with arithmetics modulo $d$. Following [54, 55], we define two phase factors $\tau := e^{\pi i (d^2+1)/d} = (-1)^d e^{\pi i /d}$ and $\omega := \tau^2 = e^{2\pi i/d}$. For $q, p \in Q$, we introduce the shift and boost operators defined by the relations

\[
\text{shift: } \hat{\epsilon}(q)|x\rangle = |x+q\rangle, \quad \text{boost: } \hat{\epsilon}(p)|x\rangle = \omega^{px}|x\rangle \quad (6)
\]

for all $x \in Q$.

For $p, q \in Q$, the corresponding Weyl operator (or generalized Pauli operator) is defined as

\[
w(p,q) = \tau^{-pq}\hat{\epsilon}(p)\hat{\epsilon}(q). \quad (7)
\]

Again following [54, 55], we adopt the convention that any arithmetic expression in the exponent of $\tau$ is not understood to be modulo $d$, but rather as taking place in the integers. This makes a difference for even dimensions (see below). One could argue that it would be slightly cleaner to syntactically distinguish the modular shifts with elements $\tau$ and $\omega$ understood to be modulo $d$. This makes a difference for even dimensions, however, $\tau$ has order $2d$. This somewhat complicates the theory of stabilizer states in the even-$d$ case - c.f. Section II C.

The preceding definitions have been made with a single $d$-dimensional system in mind. We now extend our formalism to $n$ such systems. The corresponding configuration space is $Q = Z_d^n$ with elements $q = (q_1, \ldots, q_n)$ and $q_i \in Z_d$. The associated phase space will be denoted by $V := Q \times Q \simeq Z_d^{2n}$ (dim $V = 2n$). It carries a symplectic form given by the natural multi-dimensional analogue of (8):

\[
[u, v] := u^T J v, \quad J = \begin{pmatrix} 0_{n \times n} & \mathbb{1}_{n \times n} \\ -\mathbb{1}_{n \times n} & 0_{n \times n} \end{pmatrix}.
\]

With elements $(p, q) \in V$, we associate Weyl operators

\[
w(p,q) = w(p_1, \ldots, p_n, q_1, \ldots, q_n) = w(p_1, q_1) \otimes \ldots \otimes w(p_n, q_n)
\]

acting on the tensor product space $(C^d)^\otimes n$. With these definitions, the composition and commutation relations (9, 10) remain valid for $n > 1$.

We conclude this section with two formulas that will be important in what follows and can both be verified immediately. First, the Weyl operators are trace-less, with the exception of the trivial one:

\[
\text{tr} \ w(v) = d^n \delta_{v,0}. \quad (11)
\]

Second, for any vector $v \in V$ and any subspace $W \subseteq V$ one has

\[
\sum_{w \in W} \omega^{[v,w]} = \begin{cases} |W| & \text{if } [v, w] = 0 \ \forall w \in W, \\ 0 & \text{else.} \end{cases} \quad (12)
\]

## C. Stabilizer States

Here, we will cast the established theory [9, 23] of stabilizer states into the language of symplectic geometry required for our proof. For previous similar expositions, see [28, 56].

\footnote{This is connected to the fact that stabilizer states are the natural discrete analogue of Gaussian states of bosonic systems, where the symplectic structure is well-appreciated. For a concise introduction of this point of view, see [28].}
Note that Equation (10) implies that two Weyl operators \( w(u) \) and \( w(v) \) commute if and only if \([u,v] = 0\). Now consider the image of an entire subspace \( M \subseteq V \) under the Weyl representation. We define

\[
w(M) = \{w(m) : m \in M\}
\]

and observe that \( w(M) \) consists of mutually commuting operators if and only if \([m,m'] = 0\) holds for all \( m,m' \in M \). Spaces having this property are called isotropic. Assume now that \( M \) is isotropic.

If \( d \) is odd, then the \( w(M) \) not only commute, but actually form a group \( w(u)w(v) = w(u+v) \). That’s because in (9), the phase factor depends on \([u,v]\) modulo \( d \), which is zero by assumption for \( u,v \in M \). For even dimensions, however, \([u,v]\) might equal \( d \) and in this case, the product \( w(u)w(v) = -w(u+v) \) does not lie in \( w(M) \) (in other words, \( v \mapsto w(v) \) is only a projective representation of the additive group of \( M \)). This would create problems in our analysis below. Fortunately, it turns out that one can choose phases \( c(v) \in \{\pm 1\} \) such that \( v \mapsto c(v)w(v) \) does become a true representation of \( M \). We will now describe this construction.

To this end, choose a basis \( B = \{u_1, \ldots, u_{\dim M}\} \) of \( M \). For a given element \( m \in M \), let \( m = \sum i m_i u_i \) be the expansion of \( m \) with respect to this basis. Define the (basis-dependent) Weyl operators to be:

\[
w_B(m) := \prod_{i=1}^{\dim M} w(u_i)^{m_i}.
\]

Using the fact that the \( w(u_i) \) commute, one then obtains for \( m, m' \in M \)

\[
w_B(m)w_B(m') = \prod_{i=1}^{\dim M} w(u_i)^{m_i} \prod_{i=1}^{\dim M} w(u_i)^{m'_i} = \prod_{i=1}^{\dim M} w(u_i)^{m_i+m'_i} = w_B(m+m').
\]

This is the desired representation of \( M \).

Stabilizer states turn out to be related to maximal isotropic spaces \( M \). We call a subspace \( M \subseteq V \) Lagrangian (LAG) – or maximally isotropic – if every vector \( v \in V \) that commutes with all elements of \( M \) is already contained in \( M \). This is precisely the case if

\[
M = \{v \in V : [v,m] = 0 \quad \forall m \in M\} =: M^\perp,
\]

where \( M^\perp \) denotes the symplectic complement of \( M \). A basic result of symplectic geometry (e.g. Satz 9.11 in [57]) states that this condition is fulfilled if and only if \( \dim M = \frac{d}{2}\dim V = n \), or equivalently \( |M| = d^n \).

We are now ready to state the relation between Lagrangian subspaces and state vectors in Hilbert space:

**Theorem 3** (Stabilizer States). Let \( M \subseteq V \) be a Lagrangian subspace, let \( B \) be a basis of \( M \). Then the following assertions are valid:

1. Up to a global phase, every \( v \in M \) singles out one unit vector \(|M,v\rangle \in \mathcal{H} \) – called a stabilizer state that fulfills the eigenvalue equations

\[
\omega^{[v,m]}w_B(m)|M,v\rangle = |M,v\rangle \quad \forall m \in M.
\]

2. Two elements \( u,v \in M \) define the same stabilizer state if and only if they belong to the same affine space \([v]_M := \{v+m, m \in M\} \mod M \). If this is not the case, the resulting stabilizer states are orthogonal, i.e. \( \langle M,u|v,M\rangle = 0 \).

3. \( V \) can be decomposed into a union of \( d^n = \dim(\mathcal{H}) \) different affine spaces modulo \( M \). Via (14), this union defines an orthonormal basis of stabilizer states associated with \( M \).

This statement implies that each stabilizer state is uniquely characterized by a Lagrangian subspace \( M \subseteq V \) and one particular affine space \([v]_M \mod M \). In the remainder of this article it will be convenient to represent each such affine space by a representative \( \zeta \in [v]_M \in V \) contained in it. We have opted to denote such representatives of cosets \( \zeta, r \in V \) by greek letters to notionally underline their origin.

**Proof of Theorem 3.** Define

\[
\rho_{M,v} := d^{-n} \sum_{m \in M} \omega^{[v,m]}w_B(m)
\]

and compute

\[
\rho_{M,v}^2 = d^{-2n} \sum_{m,m' \in M} \omega^{[v,m]}\omega^{[v,m']}w_B(m)w_B(m')
\]

\[
= d^{-2n} \sum_{m,m' \in M} \omega^{[v,m+m']}w_B(m+m')
\]

\[
= d^{-n} \sum_{m \in M} \omega^{[v,m]}w_B(m) = \rho_{M,v}
\]

as well as

\[
\text{tr } \rho_{M,v} := d^{-n} \sum_{m \in M} \omega^{[v,m]} \text{tr } w_B(m)
\]

\[
= d^{-n} \text{tr } w_B(0) = 1
\]

where we have employed (11). The first relation implies that \( \rho_{M,v} \) is a projection and the second one that is has rank one. One can check by direct calculation that

\[
\omega^{[v,m]}w_B(m)\rho_{M,v} = \rho_{M,v}
\]

holds for every \( m \in M \). Consequently, the so that the any vector from the range of \( \rho_{M,v} \) fulfills all eigenvalue equations. However, since \( \rho_{M,v} \) has rank one, its range corresponds to a single vector that we can associate with \(|M,v\rangle \in \mathcal{H} \) up to a global phase. This proves the first claim up to uniqueness which we are going to establish later on.
For the second claim, fix \( u, v \in V \) and observe
\[
\text{tr} \left( \rho_{M, u} \rho_{M, v} \right) = d^{-2n} \sum_{m, m' \in M} \omega^{[u, m]} \omega^{[v, m']} \text{tr} \left( w_B(m + m') \right)
\]
\[
= d^{-2n} \sum_{m, m' \in M} \omega^{[u, v]} \omega^{[v, m']} d^n \delta_{m + m', 0}
\]
\[
= d^{-n} \sum_{m \in M} \omega^{[u - v, m]}
\]
\[
= d^{-n} \begin{cases} 
|M| & \text{if } |u - v, m| = 0 \forall m \in M, \\
0 & \text{else},
\end{cases}
\]
where we have used (12). But because \( M \) is maximally isotropic, \( |u - v, m| = 0 \forall m \in M \) implies \( u - v \in M \). Thus, there is one \( \rho_{M, u} \) for each affine space \( u + M \subseteq V \), and two distinct affine spaces give rise to orthogonal states which is just the second claim.

Finally, note that there are \( |V / M| = d^n = \dim \mathcal{H} \) such affine spaces, which proves that one obtains an orthonormal basis in this way. Moreover, this establishes the uniqueness part of the first statement and implies, justifying that \( |M, v\rangle \) is well-defined up to a global phase.

In the remainder of this section, we will show how to choose consistent bases for two, possibly intersecting, Lagrangian spaces \( M, N \) and use these results to come up with formulas for the inner product between two arbitrary stabilizer states.

**Lemma 1** (Compatible bases). Let \( M, N \subseteq V \) be two Lagrangian subspaces. Then there exists bases \( B_M \) of \( M \) and \( B_N \) of \( N \) such that \( w_{B_M}(m) = w_{B_N}(m) \) for any \( m \in M \cap N \). What is more, for \( m \in M \) and \( n \in N \), it holds that
\[
\text{tr} \left( w_{B_M}(m) w_{B_N}(-n) \right) = d^n \delta_{m, n}.
\]

**Proof.** Choose a basis \( \{ u_1, \ldots, u_{\dim M} \} \) of \( M \cap N \). By elementary linear algebra, it can be extended to a basis \( B_M \) of \( M \) and to a basis \( B_N \) of \( N \). The first claim follows immediately from (13). For the second claim, note that for from (9), we have that \( w_{B_M}(m) w_{B_N}(-n) = \pm w(m - n) \). Thus, by (11), the trace in (15) vanishes unless \( m = -n \). In that case, however, \( m, n \in K \) and thus, by construction of the bases, \( w_{B_M}(m) = w_{B_K}(m) \) and \( w_{B_N}(-n) = w_{B_K}(-n) \). Thus
\[
w_{B_M}(m) w_{B_N}(-n) = w_{B_K}(m - n) = w_B(0) = w(0).
\]
The claim then follows from (11). \( \square \)

We conclude this subsection with an important observation: The overlap of different stabilizer states is fully characterized by the geometric intersection of their underlying Lagrangian subspaces.

**Lemma 2** (Overlap of stabilizer states). Let \( |M, \zeta\rangle, |N, \iota\rangle \in \mathcal{H} \) be two stabilizer states characterized by Lagrangian subspaces \( M, N \subseteq V \) (as well as corresponding bases \( B_M \) and \( B_N \) if \( d \) is even) and representatives \( \zeta, \iota \in V \) of cosets \( [\zeta]_M \in V / M \) and \( [\iota]_N \in V / N \), respectively. Then, setting \( K = M \cap N \), their inner product is given by
\[
|\langle M, \zeta | N, \iota \rangle|^2 = \begin{cases} 
\left( \frac{d^{|K|}}{d^{n}} \right)^2 & \text{if } |\zeta, m| = |\iota, m| \forall m \in K, \\
0 & \text{else}.
\end{cases}
\]

**Proof.** The claim follows from direct computation. According to Lemma 1 we can pick bases \( B_K \) of \( K := M \cap N, B_M \) of \( M \) and \( B_N \) of \( N \) that are compatible with each other. With respect to these bases we can write
\[
|\langle M, \zeta | N, \iota \rangle|^2 = \text{tr} \left( |M, \zeta\rangle \langle N, \iota| \right) = d^{-2n} \sum_{m \in M, m' \in N} \omega^{[\zeta, m]} \omega^{-[\iota, m']} \text{tr} \left( w_{B_M}(m) w_{B_N}(-m') \right)
\]
\[
= d^{-n} \sum_{m \in M \cap N} \omega^{[\zeta - \iota, m]}
\]
\[
= d^{-n} \begin{cases} 
|M \cap N| & \text{if } |\zeta - \iota, m| = 0 \forall m \in M \cap N, \\
0 & \text{else},
\end{cases}
\]
where the last equation follows from formula (12). \( \square \)

### D. Grassmannian subspaces and discrete symplectic geometry

Let \( Q \) be a \( n \)-dimensional vector space over the finite field \( \mathbb{Z}_d \). The Grassmannian \( G(d, n, k) \) is the set of \( k \)-dimensional subspaces of \( V \). A standard result – e.g formula (9.2.2) in [58] – says that the size of \( G \) is given by the Gaussian binomial coefficient:
\[
|G(d, n, k)| = \binom{n}{k}_d := \begin{cases} 
\prod_{i=0}^{k-1} d^{n-i}_i & \text{if } k \leq n, \\
0 & \text{else}.
\end{cases}
\]

This is the analogue of the familiar binomial coefficient for the finite field \( \mathbb{Z}_d \). As such it exhibits similar properties, such as \( \binom{n}{k}_d = \binom{n}{n-k}_d \) (symmetry), \( \binom{n}{k}_d = \binom{k}{n}_d = \binom{k}{k}_d = 1 \) (trivial coefficients) and Pascal’s identity
\[
\binom{n}{k}_d = d^k \binom{n-1}{k}_d + d^{n-1} \binom{n-1}{k-1}_d.
\]

For further reading and proofs of these identities we refer to Chapter 9 in [58] and move on to introducing some core concepts of symplectic geometry:

Let \( V \) be a \( 2n \)-dimensional symplectic vector space over the finite field \( \mathbb{Z}_d \). A polarization \((M, N)\) of \( V \) is
the choice of two Lagrangian subspaces $M, N$ which are transverse in the sense that their direct sum spans the entire space, i.e. $M \oplus N = V$. For a fixed Lagrangian $M$ we define the set

$$\mathcal{T}(M) = \{ N \mid N \text{ Lagrangian; } (M, N) \text{ is a polarization of } V \}$$

of all Lagrangian subspaces transverse to $M$. The set $\mathcal{T}(M)$ appears in various contexts. For instance it labels all graph states (in a sense explained below) in quantum information theory [25].

For the purpose of our counting argument, we need to compute the size of $\mathcal{T}(M) \in V$.

**Proposition 1.** Let $V$ be a $2n$-dimensional symplectic space over $\mathbb{Z}_d$ and let $M$ be an arbitrary Lagrangian subspace. Then, the cardinality of $\mathcal{T}(M)$ amounts to

$$|\mathcal{T}(M)| = d^{2n(n+1)}.$$ 

Proof. Fix $M$ and note that a subset $N \subset V$ has to be both Lagrangian and transverse to $M$ in order to lie in $\mathcal{T}(M)$. These conditions can be made more explicit if we choose a basis $b_1, \ldots, b_{2n}$ of $V$ which obeys

$$M = \text{span} \{ b_1, \ldots, b_n \} \quad \text{and} \quad [b_i, b_j] = \delta_{ij} = \delta_{i+j}.$$ 

where $\oplus$ denotes addition modulo $2n$. Such a basis allows us to fully characterize any subspace $N$ by a $n \times 2n$-generator matrix $G_N$ with column vectors $a_1, \ldots, a_n$ obeying $\text{span} \{ a_1, \ldots, a_n \} = N$. Moreover, it will be instructive to partition each generator matrix into two $n \times n$ blocks $A$ and $B$, i.e. $G_N = \begin{pmatrix} A & B \end{pmatrix}$. Due to our choice of basis the generator matrix $G_M$ of $M$ is particularly simple, namely $G_M = \begin{pmatrix} I_{n \times n} & 0_{n \times n} \end{pmatrix}^T$. Transversality can be restated in terms of these generator matrices: $M \oplus N = V$ if and only if the $2n \times 2n$-matrix $(G_M G_N)$ has full rank. Due to the particular form of $G_M$ this is however equivalent to demanding $\text{rank}(B) = n$. Thus we can convert $G_N$ into the equivalent generator matrix $\tilde{G}_N = \begin{pmatrix} A^T & I_{n \times n} \end{pmatrix}$ (and generators $\tilde{a}_1, \ldots, \tilde{a}_n$ as above) by applying a Gauss-Jordan elimination in the columns of $G_N$.

The generator matrix $\tilde{G}_N$ characterizes a Lagrangian subspace if and only if $[\tilde{a}_i, \tilde{a}_j] = 0$ holds for all $i, j = 1, \ldots, n$. These requirements can be summarized in a single matrix equality, namely $\tilde{G}_N^T \tilde{G}_N$ must identically vanish. Inserting the particular form of $\tilde{G}_N$ and carrying out the math reveals that this is equivalent to demanding that $A^T - A$ must be the zero matrix. Hence, a subspace $N$ is a polarization of $M$ if and only if its generator matrix (with respect to the basis chosen above) is Gauss-Jordan equivalent to $G_N = \begin{pmatrix} A & I_{n \times n} \end{pmatrix}^T$, where $A$ is a symmetric $n \times n$-matrix over $\mathbb{Z}_d$. Therefore there is a one-to-one correspondence between polarizations $N$ of $M$ and symmetric $n \times n$-matrices over $\mathbb{Z}_d$. The dimensionality of the latter is $\frac{1}{2}n(n+1)$ which completes the proof.

The one-to-one correspondence between polarizations of $M$ and symmetric matrices in this proof gives additional meaning to the set $\mathcal{T}(M)$. Recall that a stabilizer state $|N, \zeta\rangle$ is a graph state if $N$ possesses a generator matrix of the form $(A \ I_{n \times n})^T$, where $A$ is a symmetric $n \times n$-matrix. Hence, $\mathcal{T}(M)$ is the set of all Lagrangian subspaces $N$ which lead to graph states.

The name graph state pays tribute to the fact that $A$ can be interpreted as the adjacency matrix of a (possibly weighted) graph. Graph states possess a rich structure and many properties of $|N, \zeta\rangle$ can be deduced from the corresponding graph alone. However, here we content ourselves with pointing out the analogy between graph states and $\mathcal{T}(M)$. For further reading we defer the reader to [25].

Let us now turn to subspaces of the symplectic vector space $V$. It is clear that a proper subspace $W \subset V$ is itself a vector space, however in general it fails to be symplectic. This is due to the fact that the standard symplectic inner product (8) of $V$ becomes degenerate if we restrict it to $W$. Therefore important tools – such as Proposition 1 – cannot be directly applied to the proper subspace $W$. However, this problem can be (partly) circumvented by applying a linear symplectic reduction. For $W \subset V$ we define the quotient

$$\hat{W} = W / (W \perp \cap W).$$

This space carries the non-degenerate symplectic form

$$[[v], [w]]_{\hat{W}} := [v, w]_V$$

which is easily seen not to depend on the representatives for $[v]$ and $[w]$. Consequently, the space $\hat{W}$ endowed with $[,]_{\hat{W}}$ is a symplectic vector space. We will need such a reduction in the proof of Theorem 4. 

### III. PROOF OF THE MAIN THEOREM

In this section we show our main result – Theorem 2 – which provides an explicit recursion fully characterizing the frame potential $\mathcal{F}(\text{Stabs}(d, n))$ of stabilizer states in prime power dimensions $D = d^n$. We denote the set of all stabilizer states by $\text{Stabs}(d, n) = \{ x_{1, \ldots, n} \}_{S(d, n)} \subset \mathbb{C}^D$, where $S(d, n) := |\text{Stabs}(d, n)|$ is just the cardinality of that set. Recall that in our framework each stabilizer state $x_i \in \mathbb{C}^D$ is specified by a Lagrangian subspace $M$ in $V = \mathbb{Z}_d^2$ and a representative $\zeta \in V$ of the coset $|\zeta\rangle_M \in \mathbb{C}^M$.

The Clifford invariance [28] of stabilizer states allows us to calculate any frame potential $\mathcal{F}(\text{Stabs}(d, n))$ by counting intersections of Lagrangian subspaces. This is the content of the following result that considerably simplifies the expression for frame potentials.

**Lemma 3.** Let $D = d^n$ be a prime power. The $t$-th frame potential of the set of all stabilizer states in dimension $D$ is
Theorem 3 assures that any such subspaces and cosets instead. Such a reformulation allows us to employ Lemma 2 which implies

\[ |\langle N, \zeta | M, 0 \rangle|^{2l} = \begin{cases} d^{-li}|K|^l & \text{if } \zeta, m = 0 \forall m \in K, \\ 0 & \text{else,} \end{cases} \]

where \( K = M \cap N \) denotes the intersection. If this intersection is \( k \)-dimensional, \( |K| = d^k \) and consequently \( |\langle N, \zeta | M, 0 \rangle|^{2l} = d^{-l(n-k)} \), provided that \( \zeta, m = 0 \) for all elements \( m \in K \). This requirement for a non-vanishing overlap is met if and only if \( \zeta \in K^\perp \). The number of representatives \( \zeta \) which obey this property (and single out different stabilizer states) is given by the order of the quotient space \( |K^\perp / N| \). Since \( N \subseteq K^\perp \) (which follows from \( K \subseteq M \) and \( N^\perp = N \)), such a quotient space is well defined and its order amounts to

\[ |K^\perp / N| = d^{\dim(K^\perp / N)} = d^{2n-k-n} = d^{n-k}. \]

Consequently, for each pair of Lagrangians \( M, N \) with \( k \)-dimensional intersection, \( d^{n-k} \) out of a total of \( d^n \) stabilizer states specified by \( N \) give rise to a non-vanishing overlap \( |\langle N, \zeta | M, 0 \rangle|^{2l} = d^{-l(n-k)} \) with the fixed stabilizer state \( x_k = |M, 0 \rangle \). Inserting this insight into (23) reveals

\[
\mathcal{F}_I(\text{Stabs}(d, n)) = \frac{1}{S(d, n)} \sum_{n_{\text{LAG}}} \sum_{\zeta | N \in V / N} |\langle N, \zeta | M, 0 \rangle|^{2l},
\]

where we have replaced the summation over the different Lagrangian subspaces with an equivalent summation over the dimension \( k \) of the intersections \( M \cap N \).

Lemma 3 shows that we can compute the stabilizer frame potential \( \mathcal{F}_I(\text{Stabs}(d, n)) \) provided that the number \( \kappa_m(d, n, k) \) is known for any Lagrangian subspace \( M \) and any intersection space dimension \( k \in \{0, \ldots, n\} \). The following two statements characterize that number.

**Theorem 4.** Let \( V \) be a 2n-dimensional symplectic space over \( \mathbb{Z}_d \). Fix an arbitrary Lagrangian subspace \( M \) and a \( k \)-dimensional subspace \( K \) of \( M \). The number of Lagrangian subspaces \( N \) that obey \( M \cap N = K \) equals

\[
\mathcal{T}(d, n-k) = d^{\frac{k}{2}(n-k)(n-k+1)}. \]

The fact that each Lagrangian \( M \) admits \( |\mathcal{G}(d, n, k)| = \binom{n}{k} d \) different \( k \)-dimensional subspaces \( K \) (formula (17)) immediately yields the following corollary.

**Corollary 2** (Expression for \( \kappa_m(d, n, k) \)). Let \( V \) be a 2n-dimensional symplectic space over \( \mathbb{Z}_d \). For an arbitrary Lagrangian subspace \( M \subset V \) and \( k \in \{0, \ldots, n\} \), the number of Lagrangian subspaces \( N \) whose intersection with \( M \) is \( k \)-dimensional amounts to

\[
\kappa_m(d, n, k) = \binom{n}{k} d^{\frac{k}{2}(n-k)(n-k+1)}. \]

**Proof of Theorem 4.** We need to count in how many ways one can choose a Lagrangian space \( N \subset V \) that intersects \( M \) exactly in \( K \). Our strategy will be to relate the set of such extensions \( N \) of \( K \) to a set \( \mathcal{T} \) as in Proposition 1. To that end, set \( \mathcal{W} := K^\perp / K \). Note that \( K \subseteq K^\perp \) (because \( K \subseteq M \) and \( M \) is Lagrangian) implies

\[
\mathcal{W} = K^\perp / K = K^\perp / (K \cap K^\perp) = K^\perp / ((K^\perp)^\perp \cap K^\perp).
\]

Therefore \( \mathcal{W} \) is the linear symplectic reduction of \( K^\perp \) as defined in (19). The space \( \mathcal{W} \) endowed with the induced symplectic product \( [, ]_\mathcal{W} \) defined in (20) forms a symplectic vector space with dimension

\[
\dim \mathcal{W} = \dim K^\perp / K = 2n - k - k = 2(n-k).
\]

Note that any isotropic space \( N \) containing \( K \) is in particular contained in \( K^\perp \). The canonical projection \( N \mapsto N / K \) sets up a one-to-one correspondence between \( n \)-dimensional subspaces of \( K^\perp \) containing \( K \) and \( (n-k) \)-dimensional subspaces of \( \mathcal{W} \). We need two properties of this correspondence:
(i) $N/K \subset \tilde{W}$ is isotropic if and only if $N \subset V$ is. Proof: This follows immediately from (20).

(ii) $N/K \subset \tilde{W}$ is transverse to $M/K$ if and only if $M \cap N = K$. Proof: Basic linear algebra shows

$$(M + N)/K \simeq M/K + N/K.$$ 

For the left hand side:

$$\dim(M + N) = \dim(M) + \dim(N) - \dim(M \cap N) \leq 2n - k$$

with equality if and only if $M \cap N = K$. Hence $\dim(M + N)/K \leq 2(n - k)$ with the same condition for equality.

For the right hand side:

$$\dim(M/K) + \dim(N/K) \leq \dim M + \dim N - 2 \dim K = 2(n - k)$$

with equality if and only if the two spaces are transverse.

It follows that $M/K$ is a Lagrangian subspace of $\tilde{W}$ and there is a one-to-one correspondence between Lagrangian spaces $N$ intersecting $M$ in $K$ and Lagrangian subspaces of $\tilde{W}$ transverse to $M/K$. Employing Proposition 1 then yields the desired result. \(\square\)

Finally, we are going to require an explicit characterization of the number $S(d, n)$ of stabilizer states. We borrow it from [28, Corollary 21]:

**Proposition 2 (Number of stabilizer states).** For $\mathcal{H} = \left(C^d\right)^{\otimes n}$, the cardinality $S(d, n)$ of Stabs$(d, n) \subset \mathcal{H}$ amounts to

$$S(d, n) = |\text{Stabs}(d, n)| = d^n \prod_{j=1}^{n} \left(\begin{array}{c} d \end{array}\right) + 1$$

and thus obeys the recursion

$$S(d, n) \over S(d, n + 1) = \frac{1}{(d^n + 1)d}$$

(26)

Formula (25) combined with Corollary 2 allows us to write down the frame potential (Lemma 3) explicitly:

$$\mathcal{F}_l(\text{Stabs}(d, n)) = \frac{1}{S(d, n)} \sum_{k=0}^{n} \left(\begin{array}{c} n \end{array}\right) \left(\begin{array}{c} n+k \end{array}\right) d^\frac{1}{2}(n-k)(n-k+3-2t)$$

(27)

with $S(d, n)$ defined in (25). Note that this is a purely combinatorial expression that depends solely on $d$ and $n$. Analyzing its recursive dependence on $n$ allows us to establish the main result of this work – Theorem 2.

Proof of Theorem 2. Let us start with the base case (4) which is readily established. Indeed, setting $n = 1$ and evaluating formula (27) reveals that for any $d$ and $t$, $\mathcal{F}_l(\text{Stabs}(d, n))$ amounts to

$$\frac{1}{(d + 1)d} \left(\begin{array}{c} n \end{array}\right) d^\frac{1}{2}(n-k)(n-k+3-2t)$$

where we have used $\binom{n}{k} = \binom{n}{n-k} = 1$. Let us now move on to establishing the recursive behavior. Replacing $n$ by $(n + 1)$ in formula (27) and employing Pascal’s identity (18) as well as trivial coefficients for Gaussian binomials yields

$$\mathcal{F}_l(\text{Stabs}(d, n + 1)) = \frac{1}{S(d, n + 1)} \left(\begin{array}{c} n+1 \end{array}\right) d^\frac{1}{2}(n-k)(n-k+3-2t)$$

(28)

where we have incorporated the first and last terms in the first and second summation, respectively. Note that the second summation just corresponds to

$$\sum_{k=0}^{n} \left(\begin{array}{c} n+k \end{array}\right) d^\frac{1}{2}(n-k)(n-k+3-2t)$$

which in that very form also appears in (27). Importantly, a similar equivalence is true for the first sum appearing in (28). Taking a closer look at the overall exponent of $d$ in that summation re-
The two base values (31) and (33) coincide for \( d \leq 2 \). Otherwise, the former is strictly larger than the latter. Comparing the recursion factors yields

\[
\frac{\text{Eq. (34)}}{\text{Eq. (32)}} = \frac{(d^n + 2)(d^n + 1)}{(d^{n+1} + 2)(d^{n+1} - 1)}
\]

\[
= \frac{d^n + 3d^{n+2} + 2}{d^{2n} + (2/d + d)dn + 2} \leq 1
\]

with equality if and only if \( d = 1, 2 \). Consequently we have \( F_3(\text{Stabs}(d,n)) = W_3(d^n) \) for any \( n \in \mathbb{N}_+ \) if and only if \( d \leq 2 \).

Finally, let us move on the the 4-design case, where we have

\[
F_4(\text{Stabs}(d,1)) = \frac{1 + d^{-2}}{(d + 1)d'}
\]

\[
F_4(\text{Stabs}(d,n + 1)) = \frac{d^{n-2} + 1}{(d^n + 1)d'}
\]

and

\[
W_4(d) = \frac{24}{(d + 3)(d + 2)(d + 1)d}
\]

\[
W_4(d^{n+1}) = \frac{(d^n + 3)(d^n + 2)(d^n + 1)}{(d^{n+1} + 3)(d^{n+1} + 2)(d^{n+1} + 1)d}
\]

Comparing (37) to (40) reveals \( F_4(\text{Stabs}(d,1)) \geq W_4(d) \) with equality if and only if \( d = 1 \). An analogous relation holds for (38) and (41) which assures that \( F_4(\text{Stabs}(d,n)) \) and \( W_4(d^n) \) only ever coincide in the trivial case \( d = 1 \).

For the final claim of Corollary 1, note that the set of stabilizer states in prime-power dimensions form one orbit under the action of the Clifford group [28]. Also, any orbit of a unitary \( t \)-design is a complex projective \( t \)-design [10, 11]. Thus Claim 3 implies that the Clifford group is not a 4-design. Peter Turner has made us aware of the fact that the frame potential of group orbits only depends on the action of that group on the totally symmetric space \( \text{Sym}^3(\mathbb{C}^D) \). Following the reasoning of [11], a group acting irreducibly on that space has the property that any orbit constitutes a complex projective \( t \)-design. Thus, the stronger statement in Claim 4 is also implied by Claim 3.

Acknowledgements: The authors want to thank P. Turner for insightful discussions and H. Zhu, as well as Z. Webb for informing us of their impending work [52, 53].

The work of DG and RK is supported by the Excellence Initiative of the German Federal and State Governments (Grants ZUK 43 & 81), the ARO under contract W911NF-14-1-0098 (Quantum Characterization, Verification, and Validation), the DFG projects GRO 4334/1,2 (SPP1798 CoSIP), and the State Graduate Funding Program of Baden-Württemberg.
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