Automatic Differentiation via Effects and Handlers
An Implementation in Frank

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Abstract
Automatic differentiation (AD) is an important family of algorithms which enables derivative based optimization. We show that AD can be simply implemented with effects and handlers by doing so in the Frank language. By considering how our implementation behaves in Frank’s operational semantics, we show how our code performs the dynamic creation of programs during evaluation.

CCS Concepts: • Theory of computation → Operational semantics; Control primitives; • Mathematics of computing → Automatic differentiation.

Keywords: automatic differentiation, algebraic effects, effect handlers, differentiable programming

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1 Introduction
Machine learning, artificial intelligence, scientific modelling, information analysis, and other data heavy fields have driven the demand for tools which enable derivative based optimization. The family of algorithms known as automatic differentiation (AD) is the foundation of the tools which achieve this. The family can be coarsely divided into forward mode and reverse mode. Multiple modes exist because their asymptotics depend on different features of the differentiated programs. Forward mode AD was introduced in 1964 by Wengert [14], and reverse mode AD was created by Speelpenning in his 1980 thesis [12].

It is not surprising that, given its history, AD has been implemented in many different ways. Many popular tools such as ADIFOR [1], ADIC [2], and Tapenade [6, 9] work via source transformation. These transformations take place on languages such as C and FORTRAN, and thus all of the aforementioned tools work externally from the program being written. We shall show here that the recent Frank language [3, 8] and its operational semantics, which leverages effects and handlers, can be informally seen as dynamically performing partial evaluation and program manipulation.

2 Background

2.1 Automatic Differentiation
We are most interested in showing the structure of AD algorithms, so we shall only give a short intuition for AD. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be smooth functions (i.e. infinitely differentiable at all points). The chain rule states that \( (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \). AD algorithms use this compositional property to incrementally calculate the derivative of an entire program one basic operation at a time during evaluation. We refer the reader to the textbook of Griewank and Walther [4] for general knowledge and to Hascoët and Araya-Polo [5] for checkpointed reverse mode, our most interesting example.

2.2 Effects and Handlers
Effects and handlers are a structured method of including side-effects into programs. Algebraic effects were introduced in 2001 by Plotkin and Pretnar [10] handlers for them in 2009 by Plotkin and Pretnar [11]. Effects and handlers can be viewed as an extension of the common feature of catchable exceptions. Catching an exception terminates the program delimited by the exception handling code, but effect handlers can resume the handled code and pass a value to it. Effects and handlers can implement many common side effects such as state, exceptions, non-determinism, logging, and input-output.

2.3 Frank
We will be using the Frank language to implement AD. Frank’s typing and operational semantics are inspired by call-by-push-value [7], meaning there is a distinction between values and computations. We note Frank has a fixed left-to-right evaluation order. Frank combines the concepts of functions and handlers by unifying them into what Frank refers to as operators, which act by application. However, we shall usually say handler for operators which handle effects and functions otherwise. We shall also simplify certain aspects for ease of exposition, see Convent et al. [3] for a tutorial and details.

Let us consider a simple example of a handler for a program which uses state of type \( \mathbb{S} \).
We will cover the implementation of four different handlers. We first explain the type of \(\text{state}\). The handler state takes two arguments, one of type \(s\) and one of type \(x\). In order for \(\text{state}\) to be used, the context in which it is called must support the ability \([\text{Console}]\) which is a snoc-list containing exactly one instance \([\text{Console}]\) of the interface \([\text{Console}]\) (the ability \([\text{Console}, \text{Console}]\) contains two distinct instances of the same interface). The ability \([\text{Console}]\) means we can use the command \(\text{print}\) defined by the interface \([\text{Console}]\). In the term \(\text{state}\ s\ x\), the value produced by \(s\) can only be computed using commands from the instances in the ability \([\text{Console}]\). On the other hand, the value produced by \(x\) can use commands from \([\text{Console}, \text{State}\ s]\). The value \(x\) has access to \(\text{state}\) commands because the adjustment \(<\text{State}\ s>\) extends the ambient ability \([\text{Console}]\). We note that the adjustment \(<\text{State}\ s>\) guarantees that \(\text{state}\) handles all commands of the State interface (\(\text{get}\) and \(\text{set}\)). The full type of \(\text{state}\) includes braces, showing that \(\text{state}\) is a suspended computation. Frank automatically inserts these if they are absent.

We shall briefly explain some aspects of Frank’s operational semantics before we go into more detail during AD examples. Consider the example top-level use of \(\text{state}\) where semicolon is sequencing and postfix \(!\) is nullary function application.

\[2 + (\text{state} ! (\text{put} (\text{get!} + \text{get!}); \text{get!}))\]

The ability \([\text{Console}]\) is permitted at the top-level as Frank’s implementation will handle it. As the program executes, the underlined get is encountered and a continuation of the program delimited by \(\text{state}\) is captured, namely the operator \((r \rightarrow (\text{put} (r + \text{get!}); \text{get!})))\), and bound to \(k\) in the body of the second line of \(\text{state}\)’s definition. Once the execution of \((\text{put} (\text{get!} + \text{get!}); \text{get!}))\) finishes, the first line of \(\text{state}\)’s definition is matched and \(\text{state}\) exits.

### 3 Algorithm Implementations

We will cover the implementation of four different handlers in Frank:

- \(\text{evaluate}\) : the most basic handler which dispatches to built-in arithmetic operations;
- \(\text{diff}\) : an implementation of forward mode AD;
- \(\text{reverse}\) : an implementation of reverse mode AD which makes use of the built-in mutable state interface; and
- \(\text{reverse}\) : an implementation of checkpointed reverse mode AD which extends reverse.

Each of the handlers handle the interface \(\text{Smooth}\), which conceptually corresponds to smooth functions on the real numbers. We only include constants, negation, addition, and multiplication for simplicity, but any number of other smooth functions could be included. Additionally, Frank currently does not support floats, so we use integers instead, however with language support floats could be used.

\[
\begin{align*}
\text{data Nullary} &= \text{constE Int} \\
\text{data Unary} &= \text{negateE} \\
\text{data Binary} &= \text{plusE} | \text{timesE} \\

\text{interface Smooth X} &= \text{eval} \ 	ext{nullaryE} X \\
| \text{ap1 : Unary} \rightarrow X \rightarrow X \\
| \text{ap2 : Binary} \rightarrow X \rightarrow X \rightarrow X
\end{align*}
\]

The above definition says the \(\text{Smooth}\) interface is parameterized by \(x\) and has three effectful commands. The command \(\text{ap}\) is the \(n\)-ary application of a smooth function. Note that the nullary functions are constants. For ease of use, we define the following helper functions.

\[
\begin{align*}
\text{c} : \text{Int} \rightarrow [\text{Smooth}\ X] X \\
\text{c} i &= \text{ap}\ (\text{constE}\ i) \\
\text{n} : X \rightarrow [\text{Smooth}\ X] X \\
n x &= \text{ap}\ \text{negateE}\ x \\
p : X \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
p x y &= \text{ap}\ \text{plusE}\ x\ y \\
t : X \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
t x y &= \text{ap}\ \text{timesE}\ x\ y
\end{align*}
\]

The operational semantics of Frank allows us to treat the above helper functions as if they were commands themselves, which we will do throughout. We will also define helper functions for the dispatching of \(n\)-ary functions and their derivatives to make the similarity between different AD modes more apparent.

\[
\begin{align*}
\text{op0 : Nullary} \rightarrow [\text{Smooth}\ X] X \\
\text{op1 : Unary} \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
\text{op2 : Binary} \rightarrow X \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
\text{op0 (constE}\ i) &= c i \\
\text{op1 negateE}\ x &= n x \\
\text{op2 plusE}\ x y &= p x y \\
\text{op2 timesE}\ x y &= t x y \\
\text{der1 : Unary} \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
\text{der2L : Binary} \rightarrow X \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
\text{der2R : Binary} \rightarrow X \rightarrow X \rightarrow [\text{Smooth}\ X] X \\
\text{der1 negateE}\ x &= n (c 1) \\
\text{der2L plusE}\ x y &= c 1 \\
\text{der2L timesE}\ x y &= c 1 \\
\text{der2R plusE}\ x y &= c 1 \\
\text{der2R timesE}\ x y &= c 1 \\
\text{der1} \text{negateE}\ x &= n (c 1) \\
\text{der2L plusE}\ x y &= c 1 \\
\text{der2L timesE}\ x y &= c 1 \\
\text{der2R plusE}\ x y &= c 1 \\
\text{der2R timesE}\ x y &= c 1 \\
\text{der1} \text{negateE}\ x &= n (c 1)
\end{align*}
\]

### 3.1 Evaluation

The most basic handler we will consider is the \(\text{evaluate}\) handler. It only handles \(\text{Smooth}\ X\) where \(X\) is instantiated to \(\text{Int}\).

\[
\begin{align*}
\text{evaluate : <Smooth}\ Int\ X \rightarrow X \\
\text{evaluate x} &= x \\
\text{evaluate <ap0 (constE}\ i) \rightarrow k\> &= \text{evaluate}\ (k\ i) \\
\text{evaluate <ap1 negateE}\ x \rightarrow k\> &= \text{evaluate}\ (k\ (-x)) \\
\text{evaluate <ap2 plusE}\ x\ y \rightarrow k\> &= \text{evaluate}\ (k\ (x\ +\ y)) \\
\text{evaluate <ap2 timesE}\ x\ y \rightarrow k\> &= \text{evaluate}\ (k\ (x\ \ast\ y))
\end{align*}
\]
In the case of \texttt{constE 1}, its integer parameter \( i \) is returned. Each other case of \texttt{evaluate} takes the integer arguments passed to the command and performs the corresponding integer operation.

The \texttt{evaluate} handler will always be our top-level handler, and it is the only way to remove all \texttt{Smooth} interfaces. We shall evaluate an example program where \texttt{evaluate} is the top-level handler to illustrate how Frank executes. We will be paying special attention to how delimited continuations are captured. We will use underlining to show what term is currently at the focus of evaluation.

Our initial program is below, and represents the term \( 1 + x^3 + \cdots \). We shall evaluate an example program similar to our previous one. The program will represent the same mathematical expression, but with \( x = 2 \) and \( y = 4 \), which equals \(-7\).

\[
\text{evaluate} \ (p \ (c \ 1) \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

The current focus of evaluation is the command \( c \ 1 \).

\[
\text{evaluate} \ (p \ (c \ 1) \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

The argument \( t \) is in normal form (fully evaluated). Therefore, we can handle the command \( c \ 1 \). The handling process begins by capturing the proper delimited continuation by incrementally freezing the stack of evaluation frames. We represent freezing by highlighting and boldface.

\[
\text{evaluate} \ (p \ (c \ 1) \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

We have now reached a handler, \texttt{evaluate}, for the command in focus. The frozen command (highlighted) is the captured delimited continuation. The \texttt{apN} case of \texttt{evaluate} is then matched to the command \( c \ 1 \), where \( k \) is bound to the continuation with \( c \ 1 \) removed and \( i \) is bound to \( 1 \). The bound variables \( k \) and \( i \) are then substituted into the corresponding body of \texttt{evaluate}.

\[
\text{evaluate} \ ((x \rightarrow \ (p \ x (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4)))))) \ 1)
\]

The next step applies the continuation to \( 1 \).

\[
\text{evaluate} \ (p \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

The focus of evaluation now moves to \( t \ 2 \ 2 \), and a new delimited continuation is dynamically captured.

\[
\text{evaluate} \ (p \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

\[
\text{evaluate} \ (p \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

\[
\text{evaluate} \ (p \ (p \ (t \ (t \ 2 \ 2) \ 2) \ (n \ (t \ 4 \ 4))))
\]

We have now again reached the \texttt{evaluate} handler, and this time match the \texttt{ap2} case, resulting in the following.

\[
\text{evaluate} \ ((x \rightarrow \ (p \ (1 \ (p \ (t \ x \ 2) \ (n \ (t \ 4 \ 4)))))) \ (2 \ \times \ 2))
\]

\[
\text{evaluate} \ ((x \rightarrow \ (p \ (1 \ (p \ (t \ x \ 2) \ (n \ (t \ 4 \ 4)))))) \ 4)
\]

\[
\text{evaluate} \ (p \ (p \ (t \ 4 \ 2) \ (n \ (t \ 4 \ 4))))
\]

Evaluation will continue as such until the final answer of \(-7\) is calculated.

We have now seen how the \texttt{evaluate} handler interprets \texttt{Smooth} commands with the built-in arithmetic operations. Even though \texttt{evaluate} is simple, it allows us to write our other handlers in a polymorphic fashion independent of \texttt{Int}.

### 3.2 Forward mode

Our next handler is the \texttt{diff} handler, which implements forward mode AD via a method known as dual numbers. A dual number is a pair of real numbers where the second number represents the derivative of the first. The \texttt{diff} handler handles commands with dual number arguments. The mathematical justification of AD is not our focus, and thus we shall just focus on the patterns of computation present without proving their correctness. We define the \texttt{Dual} datatype and \texttt{diff} below.

\[
\text{data} \ 	exttt{Dual} \ X = \ dual \ X \ X
\]

\[
v : \text{Dual} \ X \rightarrow X
\]

\[
v \ (\text{dual} \ X \ _) = x
\]

\[
dv : \text{Dual} \ X \rightarrow X
\]

\[
dv \ (\text{dual} \_ \ dx) = dx
\]

\[
\text{diff} : \langle \text{Smooth} \ (\text{Dual} \ X) \rangle \ Y \rightarrow \langle \text{Smooth} \ X \ Y
\]

\[
diff \ x = x
\]

\[
diff \ \langle \text{opN} \ n \rightarrow k \rangle =
\]

\[
\text{let} \ r = \text{dual} \ \langle \text{opN} \ n \rangle \ (\langle c \ 0 \rangle \ in \ diff \ (k \ r) )
\]

\[
diff \ \langle \text{apN} \ u \ \langle \text{opN} \ x \ dx \rangle \rightarrow k \rangle =
\]

\[
\text{let} \ r = \text{dual} \ \langle \text{opN} \ u \ x \rangle \ (\text{t} \ (\text{derK} \ 1 \ u \ x) \ dx) \ in \ diff \ (k \ r)
\]

\[
diff \ \langle \text{ap2} \ b \ \langle \text{opN} \ x \ dx \rangle \ (\text{dual} \ y \ dy) \rightarrow k \rangle =
\]

\[
\text{let} \ r = \text{dual} \ \langle \text{opN} \ b \ x \ y \rangle \ (\text{t} \ (\text{derK} \ 2 \ b \ x \ y) \ dx) \ (\text{t} \ (\text{derK} \ 2 \ b \ x \ y) \ dy) \ in \ diff \ (k \ r)
\]

Notice the similarities between each of the \texttt{apN} cases. The command being handled by \texttt{diff} is evaluated with \texttt{opN} in the first component of \texttt{Dual}, and a calculation involving derivatives creates the second component.

We will evaluate an example program similar to our previous one. The program will represent the same mathematical term \( 1 + x^3 + \cdots \). We shall be calculating the derivative with respect to \( x \) at this point, which is \( 12 \). This is achieved by setting \( x \) to \texttt{dual 2 \ 1} and \( y \) to \texttt{dual 4 \ 0}, where \( x \) has its second component set to \( 1 \) to treat it as the differentiated variable and \( y \) has its second component set to \( 0 \) to treat it as a constant.

\[
\text{evaluate} \ ( \text{diff} \ ( \ p \ (c \ 1) \ (p \ (t \ (t \ (dual \ 2 \ 1) \ (dual \ 2 \ 1)) \ (dual \ 2 \ 1)) \ (n \ (t \ (dual \ 4 \ 0) \ (dual \ 4 \ 0))))
\]

Evaluation begins as before, with the \( c \ 1 \) command being in focus and a delimited continuation being captured.

\[
\text{evaluate} \ ( \text{diff} \ ( \ p \ (c \ 1) \ (p \ (t \ (t \ (dual \ 2 \ 1) \ (dual \ 2 \ 1)) \ (dual \ 2 \ 1)) \ (n \ (t \ (dual \ 4 \ 0) \ (dual \ 4 \ 0))))
\]

Note how the continuation captured is delimited by \texttt{diff} and not \texttt{evaluate}. This behavior is due to the effect typing system of Frank. There are two different instances of the \texttt{Smooth} interface available to the portion of the program being handled. By default, the innermost handler provides the
instance being used by extending the ambient ability with an adaptor. As we shall see later, Frank provides constructs allowing us to select handlers other than the innermost. The top case of diff is matched by c 1 with the following result.

```plaintext
evaluate (
    let r = dual \(\text{op0 } \text{constE} \ 1\) \(\text{c 0}\) in 
    diff ( 
        (x -> \(\text{p x (p (t (\text{dual 2 1} (\text{dual 2 1})) (\text{dual 2 1})) (n (t (\text{dual 4 0} (\text{dual 4 0}))))})\) r) 
    )
)

evaluate ( 
    let r = dual \(\text{c 1}\) \(\text{c 0}\) in 
    diff ( 
        (x -> \(\text{p x (p (t (\text{dual 2 1} (\text{dual 2 1})) (\text{dual 2 1})) (n (t (\text{dual 4 0} (\text{dual 4 0}))))})\) r) 
    )
)
```

We now have two c commands which will be be handled by evaluate, producing dual 1 0 for r's value. After handling, r will be substituted and the continuation applied.

```plaintext
evaluate (diff ( 
    p (dual 1 0) \(\text{p (t (\text{dual 2 1} (\text{dual 2 1})) (\text{dual 2 1})) (n (t (\text{dual 4 0} (\text{dual 4 0}))))}\) r)
))
```

Evaluation will then continue in a similar manner for all remaining commands. Each command will first be handled by diff, and the commands in the body of each diff case handled by evaluate, eventually producing dual -7 12.

We will now focus on Frank's ability to dynamically determine which handler handles a command. First, we define two auxiliary functions.

```plaintext
lift : X -> [\text{Smooth} X, \text{Smooth} (\text{Dual} X)] (\text{Dual} X)
lift x = dual x [<\text{Smooth}> (c 0)]

d : ((\text{Dual} X) -> [\text{Smooth} X, \text{Smooth} (\text{Dual} X)] (\text{Dual} X)) 
  -> X -> [\text{Smooth} X] X
d f x = dv (diff (f (dual x (<\text{Smooth}> (c 1)))))
```

The adaptor <Smooth> in lift causes the command c 0 to be associated with Smooth X and not the rightmost instance Smooth (Dual X). The d function returns the derivative of a unary function and lift will enable us to nest d. Note that as in lift, <Smooth> in d causes the command c 1 to be associated with Smooth X. Consider the expression \(d_{\alpha \beta}(x; x')^{d_{\alpha \beta}(\alpha y+y)}{|_{y=1}}x=1\) (which equals 1). The corresponding program requires lift.

```plaintext
evaluate (d (x -> t x (d (y -> p (lift x) y) (c 1)))) (c 1))
```

We evaluate until the delimited continuation is captured.

```plaintext
evaluate (d (x -> t x (d (y -> p (lift x) y) (c 1)))) (c 1)) 1)
```

```plaintext
evaluate (dv (diff ( 
    (x -> t x (d (y -> p (lift x) y) (c 1)))) 
    (dual 1 (<\text{Smooth}> (c 1))))
))
```

```plaintext
evaluate (dv (diff ( 
    (x -> t x (d (y -> p (lift x) y) (c 1)))) 
    (dual 1 (<\text{Smooth}> (c 1))))
))
```

The continuation for c 1 is delimited by evaluate due to <Smooth>.

We conclude by noting that Frank will reject the nested program if lift is not present.

### 3.3 Reverse mode

The evaluate and diff handlers manipulate programs by capturing delimited continuations, but only in quite simple ways. They each eventually compute a value based on the command being handled and then continue with the original program with the computed value substituted in. The reverse handler will be different, and will build up a secondary program during the evaluation of the initial program.

Reverse mode AD works by creating a mutable cell for each value which accumulates contributions to its derivative. The method of accumulation is a generalized version of the backpropagation algorithm made prominent by machine learning. We define the datatype Prop for backpropagation where Ref X is a reference to a mutable cell containing a value of type X. The reverse handler handles commands containing Prop's.

```plaintext
data Prop X = prop X (Ref X)
```

```plaintext
fwd : Prop X -> X
fwd (prop x _) = x

deriv : Prop X -> Ref X
deriv (prop _ r) = r

reverse : <Smooth (Prop X)> Unit -> [RefState, Smooth X] Unit
reverse x = x
reverse <ap0 n -> k> = 
    let r = prop (op0 n) (new (c 0)) in 
    reverse (k r)
reverse <ap1 u (prop x dx) -> k> = 
    let r = prop (op1 u x) (new (c 0)) in 
    reverse (k r);
    write dx (p (read dx) (t (der1 u x) (read (deriv r))))
reverse <ap2 b (prop x dx) (prop y dy) -> k> = 
    let r = prop (op2 b x y) (new (c 0)) in 
    reverse (k r);
    write dx (p (read dx) (t (der2 b x y) (read (deriv r))))
    write dy (p (read dy) (t (der2R b x y) (read (deriv r))))
```

The reverse handler makes use of the same op and der functions as diff, but is different from evaluate and diff in two important ways. Firstly, the type of reverse shows that it requires access to the RefState interface of mutable state (a built-in effect of Frank that can be handled by the language implementation). Secondly, the body of the ap1 and ap2 cases contains code after the use of the captured delimited continuation k. We shall see these writes to memory will form the secondary program which actually accumulates derivatives.

To properly calculate derivatives with reverse, we require a small helper function which starts the process of backpropagation, which we call grad for gradient.

```plaintext
grad : ((Prop X) 
    -> [RefState, Smooth X, Smooth (Prop X)] (Prop X)) 
  -> X -> [RefState, Smooth X] X
```
grad f x =
  let z = prop x (new (c 0)) in
  reverse (write (deriv (f z)) (<Smooth> (c 1)));
  read (deriv z)

We evaluate the same term as before.

   evaluate (grad ((x ->
     let y = c 4 in p (c 1) (p (t (t x x) x) (n (t y y))))
   ) 2))

evaluate (grad ((x ->
  let z = prop 2 (new (c 0)) in
  reverse (write (deriv ((x ->
    let y = c 4 in p (c 1) (p (t (t x x) x) (n (t y y))))
  ) z)) (<Smooth> (c 1));
  read (deriv z))

The term new (c 0) is handled first by evaluate for c 0 (returning 0), and the command new 0 is handled by the Frank implementation and returns a new reference <z> whose cell contains 0. The result is then substituted for z.

evaluate (reverse (write (deriv (prop 2 <z>))
  let y = c 4 in p (c 1) (p (t (t x x) x) (n (t y y))))
  ) (<Smooth> (c 1));
  read (deriv prop 2 <z>))

Next, the anonymous function is applied to prop 2 <z>.

evaluate (reverse (write (deriv (prop 2 <z>))
  let y = c 4 in p (c 1) (p (t (t x x) x) (n (t y y))))
  ) (<Smooth> (c 1));
  read (deriv prop 2 <z>))

The command c 4 is handled by the ap0 case of reverse, which as before creates a new reference <r1>, and thus y is substituted by prop 4 <r1>. The command c 1 will create <r2>.

evaluate (reverse (write (deriv (prop 1 <r2>)
  p (c 1) (p (t (t (prop 2 <r2>) (prop 2 <r2>))
  (prop 2 <r2>))
  (n (t y y))))
  ) (<Smooth> (c 1));
  read (deriv prop 2 <r2>))

We have now reached the first interesting command, which matches the ap2 case of reverse. The captured delimited continuation is now explicitly highlighted.

evaluate (reverse (write (deriv (prop 1 <r2>)
  p (t (t (prop 2 <r2>) (prop 2 <r2>))
  (prop 2 <r2>))
  (n (t (prop 4 <r1>) (prop 4 <r1>)))))
  ) (<Smooth> (c 1));
  read (deriv prop 2 <r2>))

The result of reverse handling the command produces a new reference <r3>.

evaluate (reverse (write (deriv (prop 1 <r2>)
  p (t (prop 4 <r3>) (prop 2 <r2>))
  (n (t (prop 4 <r1>) (prop 4 <r1>)))))
  ) (<Smooth> (c 1));
  write <r2> p (read <r2>)
  (t (der2L timesE 2 2) (read (deriv (prop 4 <r3>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 2 2) (read (deriv (prop 4 <r3>))));
  write <r3> p (read <r3>)
  (t (der2L plusE 1 -8) (read (deriv (prop -16 <r6>))));
  write <r5> p (read <r5>)
  (t (der2R plusE 16 16) (read (deriv (prop -16 <r6>))));
  write <r1> p (read <r1>)
  (t (der2L timesE 4 4) (read (deriv (prop 16 <r5>))));
  write <r1> p (read <r1>)
  (t (der2R timesE 4 4) (read (deriv (prop 16 <r5>))));
  write <r3> p (read <r3>)
  (t (der2L timesE 4 4) (read (deriv (prop 8 <r4>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 4 4) (read (deriv (prop 8 <r4>))));
  write <r2> p (read <r2>)
  (t (der2L timesE 4 4) (read (deriv (prop 8 <r4>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 4 4) (read (deriv (prop 8 <r4>))));

We see that the evaluation of the initial program has produced new expressions to be evaluated after the initial program finishes. The handling by reverse will eventually handle all commands meant for it, producing the following.

evaluate (reverse (write <r3> p (read <r3>)
  (t (der2L plusE 1 -8) (read (deriv (prop -7 <r8>))));
  write <r4> p (read <r4>)
  (t (der2R plusE 8 -16) (read (deriv (prop -8 <r7>))));
  write <r5> p (read <r5>)
  (t (der2L timesE 16 8) (read (deriv (prop 16 <r6>))));
  write <r1> p (read <r1>)
  (t (der2R timesE 4 4) (read (deriv (prop 16 <r5>))));
  write <r3> p (read <r3>)
  (t (der2L timesE 4 4) (read (deriv (prop 4 <r5>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 4 4) (read (deriv (prop 4 <r5>))));
  write <r2> p (read <r2>)
  (t (der2L timesE 4 4) (read (deriv (prop 4 <r5>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 4 4) (read (deriv (prop 4 <r5>))));
  write <r2> p (read <r2>)
  (t (der2L timesE 4 4) (read (deriv (prop 4 <r5>))));
  write <r2> p (read <r2>)
  (t (der2R timesE 4 4) (read (deriv (prop 4 <r5>))));

The above code is the secondary program created by reverse, which performs backpropagation. Furthermore, if a user wished to capture this secondary program, the definition of reverse could be changed to return a suspended computation. Thus, we could also partially evaluate the whole program (initial and backpropagation) by running only the initial program and capturing the backpropagation computation.

It could also be possible to use multi-stage programming by reifying the initial and secondary programs as a computation graph in the style of Wang et al. [13]. Their approach uses delimited continuations and combines normal execution with building an intermediate representation. As effects and handlers are essentially a structured use of delimited continuations, a similar story for Frank may be possible.

3.4 Checkpointed reverse mode

The final algorithm we shall cover is checkpointed reverse mode. Reverse mode has maximum memory residency proportional to the number of operations (as seen in the definition of reverse). Checkpointed reverse mode allows a tradeoff between space and time by recomputing checkpointed subprograms, once without allocating memory and an additional time with memory. However, any memory allocated in between these two runs can be safely deallocated, as it
corresponds to code after the checkpointed subprogram in the original program, thus reducing maximum memory residency.

To define our new handler, we introduce a 
Checkpoint effect which takes a suspended computation that will be run multiple times. We also define a simple evaluate like handler, evaluatet (see appendix for definition).

interface Checkpoint X =
  checkpoint :
    ([Checkpoint X , Smooth (Prop X)] Prop X) -> Prop X

Frank also contains a catch-all pattern match <> which matches values and commands not handled above it. We use this feature to extend reverse by delegating any Smooth commands received to reverse and only adding a case for checkpoint.

reversec : <Checkpoint X, Smooth (Prop X)> Unit
  -> [RefState, Smooth X] Unit
reversec x = x
reversec <checkpoint p -> k> =
  let s = new (c 0) in
  let res = RefState (evaluatet s ({<Smooth(s a b -> s b)> p!})) in
  let r = prop (fwd res) (new (c 0)) in
  reversec (k r);
reversec (write (deriv (<Smooth(s a b -> s b), RefState> p!)))
  (read (deriv r)))
reversec <> = reversec (<Smooth s a -> s> (reverse m!))

Note how the checkpointed subprogram (the suspended computation p which is the argument of checkpoint) is called twice, once with evaluatet as the handler and once with reversec as the handler. Additionally, the last case will match every Smooth command, and then reinvokes the captured computation with a new reversec handler to handle the command.

Consider the following program where gradc is grad with reversec in the place of reverse.

evaluate (gradc ((x ->
  let y = c 2 in
  let z = checkpoint (p x y) in
  let a = checkpoint (let w = checkpoint (t x z) in p w y) in
  p a x
  ) (c 2)))

The first interesting evaluation step is after the underlined checkpoint has been handled.

evaluate (reversec (<Smooth(s a -> s)> (reverse (write (deriv (let z = prop 4 <r2> in
  let a = checkpoint (let w = checkpoint (t prop 2 <z> z in
  p w (prop 2 <r1>)) in
  p a (prop 2 <z>))
  ) (<Smooth> (c 1)))));
  reversec (write (deriv (<Smooth(s a b -> s b), RefState> (p (prop 2 <z>)) (prop 2 <r1>)))))
  (read (deriv (prop 4 <r2>))));
read (deriv (prop 2 <z>)))

Note how on the second line the reverse handler has been made the innermost delimiter of the remainder of the initial program, via the catch-all case of reversec. Additionally, note how the checkpointed code (underlined) is stored as a thunk to be run after the initial program in the second use of reversec. After the initial program has been evaluated away, we obtain the following.

evaluate (reversec (<Smooth(s a -> s)> (reverse (write <r1>)));
  write <r4> (p (prop 2 <r2>))
  (t (der2L plusE 10 2) (read (deriv (prop 12 <r4>)))))
  (read (deriv (prop 10 <r3>)));
write <r4> (p (prop 2 <r2>))
  (t (der2L plusE 10 2) (read (deriv (prop 12 <r4>)))))
  (read (deriv (prop 10 <r3>)));
reversec (write (deriv (<Smooth(s a b -> s b), RefState> (p (prop 2 <z>)) (prop 4 <r2>)) in
  p w (prop 2 <r1>)));
  (read (deriv (prop 10 <r3>)));
write <r4> (p (prop 2 <r2>))
  (t (der2L plusE 10 2) (read (deriv (prop 12 <r4>)))))
  (read (deriv (prop 4 <r2>))));
read (deriv (prop 2 <z>))

The remaining checkpoint command illustrates the recursive nature of reversec. It shows how even nested checkpointing in checkpointed code can be properly evaluated by delaying the program transformation happening via evaluation.

4 Conclusion

We have seen the implementation and evaluation of AD in Frank via Frank’s operation semantics and four handlers: evaluate, diff, reverse, and reversec. While evaluate and diff do effectively no program transformations, reverse and reversec build up ancillary programs via delimited continuations. The effects and handler style of Frank allowed us to compose and nest our defined handlers, which is especially apparent in the modular definition of reversec which delegates all Smooth commands to reverse. It may also be possible to integrate multi-stage programming by using the system of Wang et al.. In conclusion, we have illustrated in Frank that effects and handlers are a good match for AD, and that effects and handlers can be seen as a form of program manipulation.

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Appendix

The following is the definition of evaluatet use in reversec.

Note the similarities with evaluate.

```plaintext
evaluatet : Ref X /
  -> <Checkpoint X, Smooth (Prop X)> Y /
  -> [Smooth X] Y

evaluatet _ x = x

evaluatet s <checkpoint p -> k> =
  let res = evaluatet s (<Smooth (s a b -> s b)> p!) in
  evaluatet s (k (prop (fwd res) s))
```

```plaintext
evaluatet s <ap0 n -> k> =
  evaluatet s (k (prop (Smooth (op0 n)) s))

evaluatet s <ap1 u (prop x dx) -> k> =
  evaluatet s (k (prop (Smooth (op1 u x)) s))

evaluatet s <ap2 b (prop x dx) (prop y dy) -> k> =
  evaluatet s (k (prop (Smooth (op2 b x y)) s))
```