The universal deformation of the Witt ring scheme

C. Deninger and Y.-T. Oh

Abstract. We determine the universal deformation over reduced base rings of the Witt ring scheme enhanced by a Frobenius lift and Verschiebung. It agrees with a $q$-deformation introduced earlier by the second author; we also give a simpler description for this. In the appendix we discuss a Witt vector theory for ind-rings which may be of independent interest.

Bibliography: 7 titles.

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§ 1. Introduction

Witt vectors play an important role in several branches of mathematics. Combinatorial considerations led to the study of certain $q$-deformations over spec $\mathbb{Z}[q]$ of the big Witt vector scheme $W$ over spec $\mathbb{Z}$, see [6]. In this article we consider another and simpler $q$-deformation of $W$ to a non-unital ring scheme $W^{(q)}$. The main result, Theorem 7 asserts that $W^{(q)}$ enhanced by a Frobenius lift, Verschiebung and the choice of a coordinate for the first component is the universal deformation over reduced bases of $W$ with the corresponding structures. The proof is based on an algebraic result, Proposition 2. It follows that the triple $(W, \text{Frobenius lift}, \text{Verschiebung})$ has no deformations within unital ring schemes and a one-parameter deformation in the non-unital category with parameter ‘space’ $\mathbb{G}_a/\mathbb{G}_m$ over $\mathbb{Z}[q]$, see Corollary 8. The $q$-deformation of [6] turns out to be isomorphic to $W^{(q)}$.

Over non-reduced base rings $R$ the deformation theory of the triple $(W, \text{Frobenius lift}, \text{Verschiebung})$ is richer in general. The remark at the end of § 5 explains how to determine it from a knowledge of the one-dimensional polynomial ring laws over $R$. Our theory also yields a simple proof of the decomposition theorem $W_{S,T} = W_S \circ W_T$ for coprime divisor stable sets $S$ and $T$ due to Auer, see [1]. In § 5 we also discuss the deformations of $W$ for integer values of $q$ studied in [5] and [7].

Throughout the paper we work with Witt vectors for divisor stable sets. Since in general Frobenius and Verschiebung maps are not endomorphisms of these sets, we have to work with projective systems of rings indexed by divisor stable subsets. As our main technical tool we prove a Cartier-Dieudonné lemma for them. A direct proof is given in § 3. A more conceptual proof is also possible using the theory

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of Witt vectors for certain inductive systems of rings which is developed in the appendix. This theory is a very natural generalization of Witt vector theory for individual rings and may be of independent interest.

§ 2. $q$-deformed Witt vectors

Convention: in this article all rings and algebras will be commutative and associative but not always unital. As usual ‘non-unital’ means ‘not necessarily unital’.

A non-empty subset $S$ of the natural numbers $\mathbb{N} = \{1, 2, \ldots \}$ is called divisor stable if $n \in S$ and $d \mid n$ imply that $d \in S$. In particular $1 \in S$. For $n \in S$ the sets $S/n = \{ \nu \in S \mid \nu n \in S \} \subset S$ and $S(n) = \{ \nu \in S \mid n \nmid \nu \} \subset S$ are again divisor stable. We assume that the reader is familiar with the rings $S/n$ of additive groups schemes over $W$ since $W$ is a (unital) ring scheme over $Z$ whose underlying scheme is $A^S = \text{spec} Z[S]$. Here $Z[S] = Z[t_n \mid n \in S]$ is the free commutative unital algebra on the set $S$ or, in other words, the polynomial algebra in the indeterminates $t_n$ for $n \in S$. The values of co-addition and co-multiplication

$$\Delta_+, \Delta_+: Z[S] \longrightarrow Z[S] \otimes Z[S] = Z[x_n; y_m \mid n, m \in S]$$

on the generators $n \in S$ are the Witt polynomials $\Sigma_n$ and $\Pi_n$ defining addition and multiplication in $W_S(A) = A^S$, that is,

$$\Sigma_n = \Delta_+(t_n), \quad \Pi_n = \Delta_+(t_n) \quad \text{in} \quad Z[x_\nu; y_\mu \mid \nu|n, \mu|m].$$

Here $x_\nu = t_\nu \otimes 1$ and $y_\mu = 1 \otimes t_\mu$. For every divisor stable subset $S'$ of $S$ there is a natural projection morphism $\pi: W_S \rightarrow W_{S'}$ of ring schemes. There is a morphism $s: W_{S'} \rightarrow W_S$ of schemes such that $\pi \circ s = \text{id}$. Equivalently $\pi: W_S(A) \rightarrow W_{S'}(A)$ is surjective for all rings $A$. For $n \in S$ the Verschiebung $V_n: W_{S/n} \rightarrow W_S$ is an additive morphism which fits into an exact sequence

$$0 \longrightarrow W_{S/n} \xrightarrow{V_n} W_S \xrightarrow{\pi} W_{S(n)} \longrightarrow 0.$$

Since $W_{\{1\}} = \mathbb{G}_a$ additively, it follows that for finite $S$ the underlying additive group scheme of $W_S$ is an iterated extension of $\mathbb{G}_a$. In particular, $W_S \otimes \mathbb{Q}$ is a commutative unipotent group scheme over $\text{spec} \mathbb{Q}$ and hence the logarithm provides an isomorphism

$$\log: W_S \otimes \mathbb{Q} \sim\rightarrow T_0 W_S \otimes \mathbb{Q}$$

of additive group schemes over $\text{spec} \mathbb{Q}$ (see [3], Ch. IV, § 2, n° 4, Proposition 4.1(iii)). Here $T_0 W_S$ is the tangent group scheme to $W_S$ along the zero section. It is canonically identified with $A^S$ with componentwise addition. Consider the morphism $S: A^S \rightarrow A^S$ which maps $(x_d)_{d \in S}$ to $(dx_d)_{d \in S}$ on $A$-valued points. Then the additive isomorphism

$$\log_S = S \circ \log: W_S \otimes \mathbb{Q} \sim\rightarrow T_0 W_S \otimes \mathbb{Q}$$

is induced by a unique additive morphism

$$\log_S: W_S \rightarrow T_0 W_S \cong A^S.$$
This is the so-called ghost map. Explicitly, we have
\[
\log_S((a_d)_{d \in S}) = \left( \sum_{d|n} da_d^{n/d} \right)_{n \in S}
\]
on \(A\)-valued points. The preceding assertions also hold for arbitrary \(S\) as we can see by taking the projective limit over the finite divisor stable subsets \(S_0\) of \(S\).

In the standard approach to Witt vector theory, the ghost map is used to define the ring structure. A new approach giving the ring structure directly was introduced in [2]. The morphism \(\log_S\) is a morphism of (unital) ring schemes if the ring scheme structure on \(T_0W_S \cong A^S\) is defined componentwise and \(A^1\) carries the standard ring structure. With the identifications \(\Gamma(W_S, \mathcal{O}) = \mathbb{Z}[x_d \mid d \in S]\) and \(\Gamma(T_0W_S, \mathcal{O}) = Z[u_d \mid d \in S]\) the induced ring homomorphism
\[
\log^*: \Gamma(T_0W_S, \mathcal{O}) \longrightarrow \Gamma(W_S, \mathcal{O})
\]
is determined by the formulae
\[
\log^*(u_n) = \sum_{d|n} dx_d^{n/d} \quad \text{for } n \in S.
\]
These are also the components of the formal logarithm with Jacobian \(S\) for the formal \(|S|\)-dimensional group law determined by the polynomials \(\Sigma_n\) for \(n \in S\).

**Remark.** The map \(S: A^S \rightarrow A^S\) on the product ring \(A^S\) is additive. Viewing the elements of \(A^S\) as formal power series in \(x\) over \(A\) with exponents in \(S\) we have \(S = x \frac{d}{dx}\). In this context recall that if additively \(W(A)\) is viewed as the formal multiplicative group of power series over \(A\) the ghost map is given by \(x \frac{d}{dx} \circ \log\).

We now recall the Frobenius and Verschiebung morphisms. For \(n \in S\), the Frobenius morphism \(F_n : W_S \rightarrow W_{S/n}\) is a morphism of ring schemes. The following relations hold in an obvious sense:
\[
V_n \circ V_m = V_{nm} : W_{S/nm} \rightarrow W_S, \quad F_n \circ F_m = F_{nm} : W_S \rightarrow W_{S/nm}, \quad \text{if } nm \in S;
\]
\[
F_n \circ V_n = n \text{id} : W_{S/n} \rightarrow W_{S/n} \quad \text{for } n \in S;
\]
\[
F_n \circ V_m = V_m \circ F_n : W_{S/m} \rightarrow W_{S/n} \quad \text{for } (n, m) = 1 \text{ and } nm \in S.
\]
For a prime number \(p \in S\), the Frobenius morphism \(F_p\) reduces mod \(p\) to the \(p\)th power map: the diagram
\[
\begin{array}{ccc}
W_S(A) & \xrightarrow{F_p} & W_{S/p}(A) \\
\downarrow & & \downarrow \\
W_S(A)/p & \xrightarrow{\cdot^p} & W_S(A)/p \xrightarrow{\pi} W_{S/p}(A)/p
\end{array}
\]
commutes for all rings \(A\). Here the vertical maps are the reduction maps mod \(p\). Of course the bottom line could also be replaced by
\[
W_S(A)/p \xrightarrow{\pi} W_{S/p}(A)/p \xrightarrow{\cdot^p} W_{S/p}(A)/p.
\]
Finally consider the functor $P$ on rings defined by $P(A) = (A, \cdot)$. It is represented by the monoid scheme $P = \text{spec} \mathbb{Z}[t]$ with co-multiplication $\Delta_\bullet$ and co-unit $\varepsilon_1$ determined by $\Delta_\bullet(t) = t \otimes t$ and $\varepsilon_1(t) = 1$. The Teichmüller map is the multiplicative morphism

$$\omega: P \longrightarrow W_S$$

which on $A$-valued points sends $a \in A = P(A)$ to $(a \delta_{d,1})_{d \in S}$ in $A^S = W_S(A)$. Here $\delta_{\nu,\mu} = 1$ if $\nu = \mu$ and $\delta_{\nu,\mu} = 0$ if $\nu \neq \mu$. For every commutative unital ring $R$, the whole situation can be base changed to $\text{spec} R$ and we then speak of Witt vector schemes over $R$ and so on.

We now describe two deformations of Witt vector theory which will later turn out to be isomorphic and universal in a suitable sense. The first, $W^{(q)}$, is obtained by a simple modification of the usual Witt vector functor. The second, $\overline{W}^{1-q}$, was introduced in [6] using a $q$-deformed ghost map.

Let $A$ be a unital algebra over the polynomial ring $\mathbb{Z}[q]$. We denote the ring with underlying additive group $(A, +)$ and twisted multiplication $x \ast y = qxy$ by $A^{(q)}$. It is unital if and only if $A$ is a $\mathbb{Z}[q, q^{-1}]$-algebra. There is a homomorphism of rings $A^{(q)} \rightarrow A$ which is an isomorphism if and only if $q$ is invertible in $A$. Setting $W^{(q)}_S(A) = W_S(A^{(q)})$ we obtain a commutative non-unital ring scheme $W^{(q)}_S$ over $\text{spec} \mathbb{Z}[q]$ whose underlying scheme is $A^S$. The base change of $W^{(q)}_S$ to $\text{spec} \mathbb{Z}[q, q^{-1}]$ is a unital ring scheme which is isomorphic to $W_S \otimes \mathbb{Z}[q, q^{-1}]$. By the Yoneda lemma the Frobenius and Verschiebung maps for $n \in S$

$$F_n: W_S(A^{(q)}) \longrightarrow W_{S/n}(A^{(q)}) \quad \text{and} \quad V_n: W_{S/n}(A^{(q)}) \longrightarrow W_S(A^{(q)})$$

come from morphisms $F_n: W^{(q)}_S \rightarrow W^{(q)}_{S/n}$ and $V_n: W^{(q)}_{S/n} \rightarrow W^{(q)}_S$. They have the same properties as those recalled above for the usual Witt vector schemes. In particular, the commutative group schemes $W^{(q)}_S$ are unipotent over $\text{spec} \mathbb{Q}[q]$ and hence there is the log-isomorphism

$$\log: W^{(q)}_S \otimes \mathbb{Q} \sim \rightarrow T_0 W^{(q)}_S \otimes \mathbb{Q}$$

of additive group schemes over $\text{spec} \mathbb{Q}[q]$. Here $T_0 W^{(q)}_S$ is the tangent group scheme to $W^{(q)}_S$ over $\text{spec} \mathbb{Z}[q]$ along the zero section. As before $T_0 W^{(q)}_S \cong A^S$ canonically and we have the morphism $\underline{S}$ defined as before. The additive isomorphism

$$\log_s = \underline{S} \circ \log: W^{(q)}_S \otimes \mathbb{Q} \sim \rightarrow T_0 W^{(q)}_S \otimes \mathbb{Q}$$

is induced by a unique additive morphism over $\mathbb{Z}[q]$

$$\log_s: W^{(q)}_S \longrightarrow T_0 W^{(q)}_S \cong A^S.$$

On $A$-valued points the induced (ghost-)map

$$\log_s: A^S \longrightarrow (A^{(q)})^S \quad \text{(2)}$$

is given by the formula

$$\log_s((a_d)_{d \in S}) = \left( \sum_{d | n} d q^{n-1} a_d^{n/d} \right)_{n \in S}.$$
Note here that $a^{*n} = q^{n-1}a^n$ for $a \in A$ and $n \geq 1$ by the definition of $q$-twisted multiplication in $A(q)$. The map $\log_\mathcal{S}$ in (2) is a ring homomorphism if on the right we view $(A(q))_\mathcal{S}$ as a ring under componentwise addition and multiplication. On the left we take the ring structure on the set $A^S$ which comes from the identification $W^q_S(A) = A^S$ as sets. Via the identifications $\Gamma(W^q_S, \mathcal{O}) = \mathbb{Z}[x_d \mid d \in S]$ and $\Gamma(T_0W^q_S, \mathcal{O}) = \mathbb{Z}[u_d \mid d \in S]$ the $\mathbb{Z}[q]$-algebra homomorphism

$$\log_\mathcal{S}^*: \Gamma(T_0W^q_S, \mathcal{O}) \longrightarrow \Gamma(W^q_S, \mathcal{O})$$

is given by the formulae

$$\log_\mathcal{S}^*(u_n) = \sum_{d \mid n} dq^{n-1}x_d^{d/n} = q^{-1}\sum_{d \mid n} d(qx_d)^{n/d}.$$

It follows that the universal polynomials for addition and multiplication and the Frobenius and Verschiebung morphisms are obtained from the usual ones by multiplying the variables by $q$ and dividing the resulting polynomial by $q$. For $S = \{1, p\}$ for example, setting $a = (a_1, a_p)$, $b = (b_1, b_p)$ we have:

$$\Sigma_1(a, b) = a_1 + b_1,$$
$$\Sigma_p(a, b) = a_p + b_p - q^{p-1}\sum_{\nu=1}^{p-1} p^{-1} \left(\begin{array}{c}p \\ \nu \end{array}\right) a_\nu^p b_{1 - \nu},$$
$$\Pi_1(a, b) = qa_1 b_1,$$
$$\Pi_p(a, b) = qpa_p b_p + q^p (a_1^p b_p + a_p b_1^p).$$

Consider the functor $P(q)$ on $\mathbb{Z}[q]$-algebras defined by $P(q)(A) = (A(q), *) = (A, *)$ where $x * y = qxy$ as above. It is represented by the semigroup scheme $P(q) = \text{spec } \mathbb{Z}[q][t]$ over $\text{spec } \mathbb{Z}[q]$ with co-multiplication $\Delta_\bullet$ determined by $\Delta_\bullet(t) = qt \otimes t$. The base change to $\mathbb{Z}[q, q^{-1}]$ is a monoid scheme with co-unit $\varepsilon_1(t) = q^{-1}$. There is a unique morphism $\omega(q) : P(q) \rightarrow W^q_S$ of multiplicative semigroup schemes which becomes the ordinary Teichmüller map

$$(A(q), \bullet) \longrightarrow W_S(A(q)), \quad \omega(q)(a) = (a\delta_{d, 1})_{d \in S}$$

on $\mathbb{Z}[q]$-algebras $A$. Base changed to $\mathbb{Z}[q, q^{-1}]$ the map $\omega(q)$ becomes a morphism of monoid schemes.

Remark. In [5] and [7] certain $q$-deformations of $W$ over $\text{spec } \mathbb{Z}$ were studied for integer values of $q$. The ghost map is the same as above but the induced multiplication on the Witt vectors is different since the ghost side is viewed as the ring $A^S$ and not as $(A(q))^S$. The additive structure is the same though.

A ring scheme $\overline{W}^{q}(q)$ over $\mathbb{Z}[q]$ was introduced for every polynomial $g(q)$ in $\mathbb{Z}[q]$ in [6]. The construction can be generalized to every divisor stable subset $S$ of $\mathbb{N}$ using the same arguments. The most relevant case for us is $\overline{W}^{1-q}_S$. It is defined as follows: there is a unique functorial ring structure on $A^S$ for all $\mathbb{Z}[q]$-algebras $A$
such that the following ‘ghost’ map is a homomorphism of non-unital rings
\[ \mathcal{G}_S : A^S \to (A^q)^S, \quad \mathcal{G}_S((a_n)_{n \in S}) = \left( \sum_{d|n} dq^{-1}(1 - (1 - q)^{n/d})a_n^{n/d} \right)_{n \in S}. \]

This was proved in [6], §3, using [6], Lemma 3.3.

It follows from Propositions 6.1 and 6.2 of [6] that there are unique Frobenius and Verschiebung morphisms \( F_n : W^{1-q}_S \to W^{1-q}_{S/n} \) and \( V_n : W^{1-q}_{S/n} \to W^{1-q}_S \) which correspond via the ghost maps to the maps
\[ F_n((a_\nu)_{\nu \in S}) = (a_{n\nu})_{\nu \in S/n} \quad \text{and} \quad V_n((a_\nu)_{\nu \in S}) = (n\delta_{n\nu}a_{\nu/n})_{\nu \in S}. \]

They satisfy the same relations as the usual Witt vectors and in particular \( F_p \) reduces mod \( p \) to the \( p \)th power morphism. The modified log-morphism \( \log_S = S \circ \log \) exists over \( \mathbb{Z}[q] \), that is, \( \log_S : W^{1-q}_S \to T_0 W^{1-q}_S \) and on \( A \)-valued points it equals \( \mathcal{G}_S \). Writing
\[ \Gamma(W^{1-q}_S, \mathcal{O}) = \mathbb{Z}[q][x_d \mid d \in S] \quad \text{and} \quad \Gamma(T_0 W^{1-q}_S, \mathcal{O}) = \mathbb{Z}[q][u_d \mid d \in S], \]
we have
\[ \log^*_S(u_n) = \sum_{d|n} dq^{-1}(1 - (1 - q)^{n/d})x_d^{n/d} \quad \text{for} \quad n \in S. \]

The universal polynomials describing \( W^{1-q}_S \) are more complicated than those for \( W^{1-q}_S \). For \( S = \{1, p\} \), for example, they are the following:
\[ \Sigma_1(a, b) = a_1 + b_1, \]
\[ \Sigma_p(a, b) = a_p + b_p - h(q) \sum_{\nu=1}^{p-1} \left( \begin{array}{c} p \\ \nu \end{array} \right) a_1^{\nu} b_1^{p-\nu}, \]
\[ \Pi_1(a, b) = qa_1b_1, \]
\[ \Pi_p(a, b) = qpa_pb_p + qh(q)(a_1^p b_p + b_1^p a_p) + qh(q)r(q)a_1^p b_1^p. \]

Here we have set
\[ h(q) = q^{-1}(1 - (1 - q)^p) = 1 + (1 - q) + \cdots + (1 - q)^{p-1} \]
and
\[ r(q) = p^{-1}(h(q) - q^{p-1}) = p^{-1}q^{-1}(1 - q^p - (1 - q)p) \in \mathbb{Z}[q]. \]

For the integrality of \( r(q) \) note that
\[ 1 - q^p - (1 - q)^p \equiv 1 - q^p - (1 - q)^p \mod p \equiv 0 \mod p \]
and
\[ 1 - q^p - (1 - q)^p \equiv 0 \mod q. \]

For any polynomial \( g(q) \in \mathbb{Z}[q] \) consider the ring homomorphism \( \alpha : \mathbb{Z}[q] \to \mathbb{Z}[q] \) with \( \alpha(q) = 1 - g(q) \) and set
\[ \overline{W}^{g(q)}_S = \overline{W}_S^{1-q} \otimes_{\mathbb{Z}[q], \alpha} \mathbb{Z}[q]. \]
In [6] the non-unital ring schemes $\overline{W}_S^{(q)}$ and their underlying group schemes were investigated in some detail for $S = \mathbb{N}$. It is possible to prove directly that $W_S^{(q)}$ and $\overline{W}_S^{1-q}$, together with their extra structures, are isomorphic. However this also follows without effort from the universality property of $W_S^{(q)}$ (see Corollary 8 and the example which follows it).

§3. A variant of the Cartier-Dieudonné lemma

In this section we prove two technical results which are the basis for the deformation theory of the Witt vector scheme in §4. For a divisor stable subset $T$ of $\mathbb{N}$ we write $S \preceq T$ to signify that $S$ is a divisor stable subset of $T$. Let $A = (A_S)_{S \preceq T}$ be a projective system of rings on $T$, that is, a contravariant functor from the ordered set $\{S \preceq T\}$ viewed as a category to the category of (commutative) unital or non-unital rings. For $S_1 \preceq S_2 \preceq T$ we denote the transition maps simply by $\pi: A_{S_2} \to A_{S_1}$. We say that $A$ is equipped with commuting Frobenius lifts if for all prime numbers $p \in S \preceq T$ there are ring homomorphisms

$$F_p: A_S \to A_{S/p}$$

with the following properties:

1) for all $a \in A_S$ we have the congruence

$$F_p(a) \equiv \pi(a)^p \mod p A_{S/p};$$

2) the $F_p$ are natural in the sense that for $p \in S_1 \preceq S_2 \preceq T$ the diagram

$$\begin{array}{ccc}
A_{S_2} & \xrightarrow{F_p} & A_{S_2/p} \\
\pi \downarrow & & \pi \downarrow \\
A_{S_1} & \xrightarrow{F_p} & A_{S_1/p}
\end{array}$$

commutes;

3) for prime numbers $l$ with $pl \in S \preceq T$, the diagram

$$\begin{array}{ccc}
A_S & \xrightarrow{F_p} & A_{S/p} \\
F_l \downarrow & & F_l \downarrow \\
A_{S/l} & \xrightarrow{F_p} & A_{S/pl}
\end{array}$$

commutes.

For each $n \in S$ we define $F_n: A_S \to A_{S/n}$ as the composition $F_n = F_{p_1^{\nu_1}} \circ \cdots \circ F_{p_r^{\nu_r}}$ where $n = p_1^{\nu_1} \cdots p_r^{\nu_r}$ is the prime decomposition of $n$. By 2) this is well defined and the properties 2) and 3) of being natural and commutation then hold without the assumption that $p$ and $l$ are prime numbers. Morphisms of projective systems of rings on $T$ with commuting Frobenius lifts are defined in the obvious way and we obtain a category $\mathcal{RF}_T$ both in the unital and in the non-unital cases. Note that
for a ring $A$, Witt vector theory gives us an object $W(A) := (W_S(A))_{S \leq T}$ in $\mathcal{RF}_T$.

Generalizing a well-known fact from the theory of Witt vector rings we show that $\sim$ has the following universal property.

**Proposition 1.** Assume that for $A$ in $\mathcal{RF}_T$ the ring $A = A\{1\}$ has no $T$-torsion. Then there is a unique morphism $\alpha = (\alpha_S)_{S \leq T} : A \to W(A)$ in $\mathcal{RF}_T$ with $\alpha_S = \text{id}_A$ for $S = \{1\}$. The morphism $\alpha$ is functorial in $A$. It is given explicitly as follows. The composition of $\alpha_S : A_S \to W_S(A)$ with the (injective) ghost map $\log_S : W_S(A) \to A^S$ is given by the formula

$$(\log_S \circ \alpha_S)(a) = (\pi F_n(a))_{n \in S} \quad \text{for } a \in A_S.$$ 

Here $\pi$ denotes the map $\pi : A_{S/n} \to A\{1\} = A$.

**Proof.** We have to show that there is a unique family of ring homomorphisms $\alpha_S : A_S \to W_S(A)$ for $S \leq T$ which commute with the projections $\pi$ and the Frobenius maps. Since $A$ has no $T$-torsion by assumption, the ghost maps

$$\log_S : W_S(A) \to A^S$$

give isomorphisms onto their images $X_S(A) := \log_S(W_S(A))$. On the ghost side the projection $\pi : X_{S_2}(A) \to X_{S_1}(A)$ for $S_1 \leq S_2$ is induced by the projection $A^{S_2} \to A^{S_1}$. The Frobenius map

$$F_n : X_S(A) \to X_{S/n}(A) \quad \text{for } n \in S \leq T$$

is the restriction of the map

$$F_n : A^S \to A^{S/n}, \quad (a_\nu)_{\nu \in S} \mapsto (a_d)_{d \in S/n}.$$ 

Note that the pro-system $(A^S)_{S \leq T}$ satisfies all the requirements for being an object of $\mathcal{RF}_T$ except for the congruence between $F_p$ and the $p$th power modulo $p$.

The uniqueness assertion follows from the next claim.

**Claim.** There is a unique family of ring homomorphisms

$$\beta_S : A_S \to A^S \quad \text{for } S \leq T, \quad \text{where } \beta_{\{1\}} = \text{id},$$

commuting with the corresponding projections $\pi$ and Frobenius maps $F_p$. It is given by the formula

$$\beta_S(a) = (\pi F_n(a))_{n \in S} \quad \text{for } a \in A_S,$$

where $\pi : A_{S/n} \to A\{1\} = A$. (3)

Uniqueness and formula (3) follow from the formula $\pi F_n(b) = b_n$ for $b \in A^S$. It is straightforward to check that (3) does indeed define the desired family of maps.

For the proof of Proposition 1 it remains to show that $\beta_S(A_S) \subset X_S(A)$ for all $S \leq T$. Thus for every $a \in A_S$ there have to be (uniquely determined) elements $a_d \in A$ for $d \in S$ with

$$\pi F_n(a) = \sum_{d \mid n} da_d^{n/d} \quad \text{in } A \quad \text{for every } n \in S.$$ 

(4)

To establish this we prove the following stronger statement by induction on $n \in S$. 

Claim. Given $n \in S$ and $b \in A_S$, there exist elements $b_d \in A_{S/d}$ for all $d \mid n$ such that
\[ F_n(b) = \sum_{d \mid n} d \pi(b_d)^{n/d} \quad \text{in } A_{S/n}. \tag{5} \]

Here $\pi(b_d)$ is the projection of $b_d$ along $\pi: A_{S/d} \to A_{S/n}$.

For $n = 1$ we take $b_1 = b$. Assume that the claim holds for proper divisors of $n$. Then, for any prime divisor $p$ of $n$ we know that
\[ F_{n/p}(b) = \sum_{d \mid (n/p)} d \pi(b_d)^{n/(pd)} \quad \text{in } A_{S/(n/p)} \]
for elements $b_d \in A_{S/d}$ and the corresponding $\pi: A_{S/d} \to A_{S/(n/p)}$. Applying $F_p$ we get
\[ F_n(b) = \sum_{d \mid (n/p)} d \pi F_p(b_d)^{n/(pd)} \quad \text{in } A_{S/n}, \]
where the $\pi$’s are projections $A_{S/(dp)} \to A_{S/n}$. By assumption
\[ F_p(b_d) \equiv \pi(b_d)^p \mod p A_{S/(dp)} \]
and therefore
\[ F_p(b_d)^{n/(pd)} \equiv \pi(b_d)^{n/d} \mod p v_p(n/d) A_{S/(dp)}. \]

Here we have used the fact that $\alpha \equiv \beta \mod p$ implies that $\alpha^{p^i} \equiv \beta^{p^i} \mod p^{i+1}$ and hence $\alpha^k \equiv \beta^k \mod p^{v_p(k)+1}$ for $k \geq 1$. It follows that
\[ F_n(b) \equiv \sum_{d \mid (n/p)} d \pi(b_d)^{n/d} \mod p v_p(n) A_{S/n} \equiv \sum_{d \mid n, d \neq n} d \pi(b_d)^{n/d} \mod p v_p(n) A_{S/n}. \]

Here and in what follows the notation $d \mid n$ means that $d \mid n$ and $d \neq n$.

For the last step, note that if $d \mid n$ and $d \nmid (n/p)$ then $v_p(d) = v_p(n)$. Since $p \mid n$ was arbitrary we conclude that
\[ F_n(b) \equiv \sum_{d \mid n, d \neq n} d \pi(b_d)^{n/d} \mod n A_{S/n}. \]

Hence an element $b_n \in A_{S/n}$ can be found so that (5) holds. The explicit formula for $\log_S \circ \alpha_S$ shows that $\alpha = (\alpha_S)$ depends functorially on $\sim_A$.

Proposition 1 is proved.

Remark. In the appendix we sketch a theory of Witt vector rings for ind-rings which elucidates the somewhat ad hoc proof of Proposition 1.

In general the morphism $\alpha$ in Proposition 1 will not be an isomorphism. For this to hold more structure is required. We call a projective system $A = (A_S)_{S \preceq T}$ continuous if for all $S \preceq T$ we have $A_S = \lim_{\rightarrow} A_{S_0}$ where $S_0$ runs over the finite divisor stable subsets of $S$. Note that for finite $T$ continuity is automatic.
Consider the category $\mathcal{RFV}_T$ whose objects are continuous projective systems $A = (A_S)_{S \preceq T}$ of rings on $T$ with commuting Frobenius lifts $F_p$ together with Verschiebung maps for all prime numbers $p \in S \preceq T$

$$V_p: A_{S/p} \longrightarrow A_S$$

with the following properties in addition to 1), 2) and 3) above:
4) the $V_p$ are additive homomorphisms which are natural in the sense that for $p \in S_1 \preceq S_2 \preceq T$ the diagram

$$\begin{array}{ccc}
A_{S_2/p} & \xrightarrow{V_p} & A_{S_2} \\
\pi \downarrow & & \downarrow \pi \\
A_{S_1/p} & \xrightarrow{V_p} & A_{S_1}
\end{array}$$

commutes;
5) for prime numbers $l$ with $pl \in S \preceq T$ the diagram

$$\begin{array}{ccc}
A_{S/pl} & \xrightarrow{V_p} & A_{S/l} \\
V_l \downarrow & & \downarrow V_l \\
A_{S/p} & \xrightarrow{V_p} & A_S
\end{array}$$

commutes;
6) the composition $A_{S/p} \xrightarrow{V_p} A_S \xrightarrow{F_p} A_{S/p}$ is $p$-multiplication, that is, $F_p \circ V_p = p$ for $p \in S \preceq T$. For any prime $l$ with $l \neq p$ and $pl \in S \preceq T$ the diagram

$$\begin{array}{ccc}
A_{S/p} & \xrightarrow{V_p} & A_S \\
F_l \downarrow & & \downarrow F_l \\
A_{S/pl} & \xrightarrow{V_p} & A_{S/l}
\end{array}$$

commutes;
7) there are exact sequences of additive groups

$$0 \longrightarrow A_{S/p} \xrightarrow{V_p} A_S \xrightarrow{\pi} A_{S(p)} \longrightarrow 0,$$

where $S(p) = \{d \in S \mid p \nmid d\}$.

Morphisms between objects in $\mathcal{RFV}_T$ are defined in the obvious way. For a ring $A$, Witt vector theory gives an object $W(A) = (W_S(A))_{S \preceq T}$ in $\mathcal{RFV}_T$.

**Proposition 2.** Assume that for $A$ in $\mathcal{RFV}_T$ the ring $A = A_{\{1\}}$ has no $T$-torsion. Then the morphism $\alpha: A \rightarrow W(A)$ in $\mathcal{RFV}_T$ from Proposition 1 defines an isomorphism in $\mathcal{RFV}_T$. In particular, it is an isomorphism in $\mathcal{RF}_T$. 
Proof. The main point is that for any prime \( p \in S \preceq T \) the diagram

\[
\begin{array}{ccc}
A_{S/p} & \xrightarrow{V_p} & A_S \\
\downarrow{\alpha_{S/p}} & & \downarrow{\alpha_S} \\
W_{S/p}(A) & \xrightarrow{V_p} & W_S(A)
\end{array}
\]

commutes. Since \( A \) has no \( T \)-torsion, the ghost map \( \log_S \) on \( W_S(A) \) is injective and hence it suffices to show that we have

\[
\log_S \circ V_p = \log_S \circ \alpha_{S/p} \quad \text{on} \quad A_{S/p}.
\]

For \( a \in A_{S/p} \), using the explicit description of \( \log_S \circ \alpha \) in Proposition 1, setting \( \delta_{p|n} = 1 \) for \( p \mid n \) and \( \delta_{p|n} = 0 \) for \( p \nmid n \), we find:

\[
(\log_S \circ \alpha_{S/p} \circ V_p)(a) = (\log_S \circ \alpha_S)(V_p(a)) = (\pi F_n V_p(a))_{n \in S} = (\delta_{p|n} \pi F_n V_p(a))_{n \in S}.
\]

Namely, by 6) and 7) above, for \( p \nmid n \) we have

\[
\pi F_n V_p = \pi V_p F_n = 0.
\]

Using 6) again we therefore obtain

\[
(\log_S \circ \alpha_{S/p} \circ V_p)(a) = (\delta_{p|n} \pi F_n V_p(a))_{n \in S} = p(\delta_{p|n} \pi F_n V_p(a))_{n \in S}
= V_p((\pi F_n(a))_{n \in S/p}) = (V_p \circ \log_{S/p} \circ \alpha_{S/p})(a)
= (\log_S \circ V_p \circ \alpha_{S/p})(a)
\]

Thus we have shown (7) and hence (6). Combining (6) and property 7) we get a commutative diagram with exact lines for all \( p \in S \preceq T \)

\[
\begin{array}{ccc}
0 & \xrightarrow{\alpha_{S/p}} & A_{S/p} \\
V_p & & \downarrow{\alpha_S} \\
W_{S/p}(A) & \xrightarrow{V_p} & W_S(A)
\end{array}
\]

Since \( \alpha_{\{1\}} = \text{id} \), induction with respect to \(|S|\) now shows that \( \alpha_S \) is an isomorphism for all finite \( S \). The general case follows by the continuity of \( A \) and \( W \).

Proposition 2 is proved.

§ 4. Universality of Witt vector schemes

As before, fix a divisor stable subset \( T \subset \mathbb{N} \). Consider a projective system \( M = (M_S)_{S \preceq T} \) of affine commutative ring schemes \( M_S = \text{spec} \ B_S \) over a base ring \( R \). As before, the transition maps are denoted by \( \pi \). We say that \( M \) is equipped with commuting Frobenius lifts if for all prime numbers \( p \in S \preceq T \) there are morphisms of ring schemes over \( R \)

\[
F_p: M_S \rightarrow M_{S/p},
\]
such that for all \( R \)-algebras \( C \), the projective system \( M(C) = (M_S(C))_{S \leq T} \) together with the maps \( F_p(C) \) is an object of \( \mathcal{R} \mathcal{F}_T \). In particular, the relations \( \pi \circ F_p = F_p \circ \pi \) and \( F_p \circ F_l = F_l \circ F_p \) hold in the obvious sense, see properties 2) and 3) in §3. The ensuing category is denoted by \( \mathcal{R} \mathcal{F}_T/R \). We can view its objects as the ‘representable functors’ from the category of \( R \)-algebras to \( \mathcal{R} \mathcal{F}_T \). We define another category \( \mathcal{R} \mathcal{F} \mathcal{V}_T/R \) as follows. The objects are continuous projective systems \( M \) in \( \mathcal{R} \mathcal{F}_T/R \) equipped with morphisms of the underlying additive group schemes \( V_p: M_{S/p} \to M_S \) for all prime numbers \( p \in S \leq T \) such that on \( R \)-algebras \( C \) we get objects of \( \mathcal{R} \mathcal{F}_T \). We can view its objects as the \( \pi \)-representable functors’ from the category of \( R \)-algebras to \( \mathcal{R} \mathcal{F}_T \).

We can view the objects of \( \mathcal{R} \mathcal{F} \mathcal{V}_T/R \) as the representable functors from the category of \( R \)-algebras to \( \mathcal{R} \mathcal{F} \mathcal{V}_T \).

Given any affine ring scheme \( M = \text{spec} B \) over \( R \), the functor

\[
W_S^M = W_S \circ M: \{R\text{-algebras}\} \to \{\text{rings}\}
\]

is an affine ring scheme over \( R \) with underlying scheme \( M_S = \text{spec} B^\otimes S \). By the Yoneda lemma the Frobenius and Verschiebung maps on Witt vectors for all prime numbers \( p \in S \)

\[
F_p: W_S(M(C)) \to W_{S/p}(M(C)) \quad \text{and} \quad V_p: W_{S/p}(M(C)) \to W_S(M(C))
\]

for the rings \( M(C) \) where \( C \) runs over \( R \)-algebras come from morphisms of ring schemes

\[
F_p: W_S^M \to W_{S/p}^M \quad \text{and} \quad V_p: W_{S/p}^M \to W_S^M.
\]

By the usual theory of Witt vectors we see that equipped with the \( F_p \)'s (and \( V_p \)'s) the projective system of ring schemes \( W^M := (W_S^M)_{S \leq T} \) becomes an object in \( \mathcal{R} \mathcal{F}_T/R \) (in \( \mathcal{R} \mathcal{F} \mathcal{V}_T/R \), respectively).

We need a technical condition in the following definition.

**Definition.** An object \( M \) in \( \mathcal{R} \mathcal{F}_T/R \) or \( \mathcal{R} \mathcal{F} \mathcal{V}_T/R \) has no Hopf \( T \)-torsion if for \( M = M_{(1)} \) the abelian groups \( M(R) \), \( M(B_S) \) and \( M(B_S \otimes_R B_S) \) have no \( T \)-torsion for all \( S \leq T \) where \( M_S = \text{spec} B_S \).
Theorem 3. a) Assume that $M$ in $\mathcal{RF}_{T/R}$ has no Hopf $T$-torsion. Then there is a unique morphism

$$\alpha = (\alpha_S)_{S \in T}: \sim M \to W^M$$

in $\mathcal{RF}_{T/R}$ with $\alpha_S = \text{id}$ for $S = \{1\}$.

b) Assume that $M$ in $\mathcal{RF}_T$ has no Hopf $T$-torsion. Then the above unique morphism $\alpha: \sim M \to W^M$ in $\mathcal{RF}_{T/R}$ defines an isomorphism in $\mathcal{RF}_T$ (and hence in $\mathcal{RF}_{T/R}$).

Proof. For the class $\mathcal{C}$ of $R$-algebras $C$ with $M(C)$ $T$-torsion free, Propositions 1 and 2 ensure the existence of unique morphisms in $\mathcal{RF}_T$ (respectively isomorphisms in $\mathcal{RF}_T$)

$$\alpha(C) = (\alpha(C)_S)_{S \in T}: \sim \sim \sim M(C) \to W^M(C) = W(M(C)).$$

Moreover, they are functorial in $C \in \mathcal{C}$. For a) we only need to show that the functorial ring homomorphisms $\alpha(C)_S: M_S(C) \to W^M_S(C)$ come from uniquely determined morphisms of ring schemes $\alpha_S: M_S \to W^M_S$ which commute with the maps $\pi$, $F_p$ and $V_p$. Since $M$ has no Hopf $T$-torsion, the existence of $\alpha_S$ follows from the following version of the Yoneda lemma (see Lemma 4) applied to $F = M_S$, $G = W^M_S$ and the class $\mathcal{C}$ above. Another application of (the first part of) the lemma shows that the $\alpha_S$'s commute with $\pi$, $F_p$ and $V_p$.

For b) we need to show in addition that the morphisms $\alpha_S: M_S \to W^M_S$ are isomorphisms. Again this follows from Lemma 4, this time applied to the inverses of the isomorphisms $\alpha(C)$ above. To do this we use the fact that $W^M$ has no Hopf $T$-torsion, which follows from the corresponding assumption on $M$ because as schemes $W^M_S \cong M^S$ (by definition) and $M_S \cong M^S$ (since the sequences (8) are scheme-theoretically split).

Theorem 3 is proved.

Lemma 4. Let $F = \text{spec } A$ and $G = \text{spec } B$ be two affine ring schemes over $R$. Assume that for a class $\mathcal{C}$ of $R$-algebras there are functorial ring homomorphisms for all $C \in \mathcal{C}$,

$$\alpha(C): F(C) \longrightarrow G(C).$$

If $A \in \mathcal{C}$, then there is a unique morphism of $R$-schemes $\alpha: F \to G$ which induces $\alpha(C)$ for all $C \in \mathcal{C}$. If in addition $R \in \mathcal{C}$ and $A \otimes_R A \in \mathcal{C}$, then $\alpha$ is a morphism of ring schemes.

Proof. The morphism $\text{spec } \alpha^\sharp: F \to G$ induced by

$$\alpha^\sharp = \alpha(A)(\text{id}_A) \in \text{Hom}_{R\text{-alg}}(B, A),$$

induces the given maps $\alpha(C)$, since for $\psi \in F(C) = \text{Hom}_{R\text{-alg}}(A, C)$ we have

$$\alpha(C)(\psi) = \psi \circ \alpha^\sharp = (\text{spec } \alpha^\sharp)(\psi).$$

On the other hand, this equation applied to $C = A$, $\psi = \text{id}$ implies that $\alpha^\sharp = \alpha(A)(\text{id})$ is unique, hence $\alpha := \text{spec } \alpha^\sharp$ is unique as well. Since $\alpha(C)$ is an additive
(multiplicative) map for $C \in \mathcal{C}$ and since $\alpha(C) = (\alpha^\sharp)^*\alpha(C) = (\alpha^\sharp \otimes \alpha^\sharp)^*$, we have

$$(\alpha^\sharp)^* \circ \Delta^* = \Delta^* \circ (\alpha^\sharp \otimes \alpha^\sharp)^*$$

on $F(C) \times F(C) = \text{Hom}_{R, \text{alg}}(A \otimes A, C)$ where $\Delta$ is the co-addition (co-multiplication) on $A$ and on $B$, respectively. If $A \otimes A \in \mathcal{C}$, we can apply this to $C = A \otimes A$ and id $\in \text{Hom}(A \otimes A, A \otimes A)$ and obtain

$$\Delta \circ \alpha^\sharp = (\alpha^\sharp \otimes \alpha^\sharp) \circ \Delta.$$ 

If $R \in \mathcal{C}$, then $\alpha(R) = (\alpha^\sharp)^*$ and from $\alpha(R)(0) = 0$ we get $(\alpha^\sharp)^*(\varepsilon_A) = \varepsilon_B$, that is, $\varepsilon_B = \varepsilon_A \circ \alpha^\sharp$ for the co-zeroes (or co-units) $\varepsilon_A$ and $\varepsilon_B$ of $A$ and $B$.

Lemma 4 is proved.

We end this section with an application of Theorem 3. Two divisor stable subsets $T_1$ and $T_2$ of $\mathbb{N}$ are coprime if and only if $T_1 \cap T_2 = \{1\}$. In this case the set $T_1 \cdot T_2 = \{nm \mid n \in T_1, m \in T_2\}$ is again divisor stable. Lenstra conjectured and Auer proved in [1] that there are natural isomorphisms of functors on unital rings

$$W_{T_1 \cdot T_2} \sim \rightarrow W_{T_1} \circ W_{T_2}.$$ 

We will explain how this follows from Theorem 3. For $S \leq T_1$ set

$$M_S = W_{S \cdot T_2}.$$ 

For $n \in S$ we have $(S \cdot T_2)/n = (S/n) \cdot T_2$. Hence the usual Witt vector Frobenius and Verschiebung morphisms give morphisms

$$F_n : M_S \rightarrow M_{S/n} \quad \text{and} \quad V_n : M_{S/n} \rightarrow M_S.$$ 

Using these we obtain an object $M = (M_S)_{S \leq T_1}$ of $\mathcal{R}_F \mathcal{V}_{T_1/\mathbb{Z}}$ with $M := M_{\{1\}} = W_{T_2}$. The coordinate rings $B_S$ of all $M_S$ are polynomial algebras over $\mathbb{Z}$ and so are $B_S \otimes \mathbb{Z} B_S$. The ring $W_{T_2}(A)$ is a subring of $A^{T_2}$ via the ghost map if $A$ has no $T_2$-torsion. Therefore, if $A$ has no $\mathbb{Z}$-torsion it follows that $M$ has no Hopf $T_1$-torsion. Hence we can apply Theorem 3 b) and obtain a uniquely determined isomorphism

$$\alpha : M = (W_{S \cdot T_2})_{S \leq T_1} \rightarrow W^M = (W_S \circ W_{T_2})_{S \leq T_1}$$

in $\mathcal{R}_F \mathcal{V}_{T_1/\mathbb{Z}}$ with $\alpha_S = \text{id}$ for $S = \{1\}$. In particular, we get a natural isomorphism

$$\alpha_T_1 : W_{T_1 \cdot T_2} \sim \rightarrow W_{T_1} \circ W_{T_2}.$$ 

The Witt vector functor $W_S$ transforms short exact sequences

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of rings into short exact sequences of rings

$$0 \rightarrow W_S(I) \rightarrow W_S(A) \rightarrow W_S(B) \rightarrow 0.$$
Hence the isomorphism (10) is also valid on non-unital rings (they are the kernels of surjections of unital rings). Applying (10) to $A^q$ for $\mathbb{Z}[q]$-algebras $A$ we obtain an isomorphism of ring schemes over $\text{spec} \mathbb{Z}[q]$

$$W_{T_1,T_2}^{(q)} \sim W_{T_1} \circ W_{T_2}^{(q)}.$$  

Note that for $T_2 = \{1\}$ this holds by definition.

§ 5. Deformations of Witt vector schemes

In this section we prove that the $q$-deformation $W^{(q)}$ introduced in § 2 is universal over reduced bases and show that $W^{(q)}$ is isomorphic to $\overline{W}^{1-q}$.

For an $R$-algebra $C$ and an element $r \in R$ we denote the ring with underlying additive group $(C, +)$ and twisted multiplication $x * y = rxy$ by $C^{(r)}$. The ring $C^{(r)}$ is unital if and only if $r \in R^\times$. In this case $1_{C^{(r)}} = r^{-1}$ is the unity. Let $G^{(r)}_R$ be the ring scheme over $R$ defined by $G^{(r)}_R(C) = C^{(r)}$ for all $R$-algebras $C$. It is represented by the polynomial algebra $R[t]$ together with

$$\Delta_+(t) = t \otimes 1 + 1 \otimes t \quad \text{and} \quad \Delta_-(t) = rt \otimes t.$$  

Moreover, $\varepsilon_0(t) = 0$ and $G^{(r)}_R$ is unital if and only if $r \in R^\times$. In this case the co-unit $\varepsilon_1$ is given by $\varepsilon_1(t) = r^{-1}$.

In particular, we have a non-unital ring scheme $G^{(q)}_R$ over $\mathbb{Z}[q]$ and a unital ring scheme $G^{(q)}_R$ over $\mathbb{Z}[q, q^{-1}]$. An element $r \in R$ corresponds to a ring homomorphism $\mathbb{Z}[q] \to R$ and we have

$$G^{(r)}_R = G^{(q)} \otimes_{\mathbb{Z}[q]} R$$  

as non-unital ring schemes over $R$. Similarly, a unit $r \in R^\times$ determines a ring homomorphism $\mathbb{Z}[q, q^{-1}] \to R$ and

$$G^{(r)}_R = G^{(q)} \otimes_{\mathbb{Z}[q, q^{-1}]} R$$  

as unital ring schemes. For any two units $r, r' \in R^\times$ the ring schemes $G^{(r)}_R$ and $G^{(r')}_R$ are isomorphic in the category of unital ring schemes. Now let $r, r' \in R$ be arbitrary. In the category of non-unital ring schemes $G^{(r)}_R$ and $G^{(r')}_R$ are isomorphic if and only if $r' = ru$ for some $u \in R^\times$. There is a natural bijection

$$\{u \in R^\times \mid r = r'u\} \sim \text{Iso}(G^{(r)}_R, G^{(r')}_R)$$  

(12) and, in particular, an isomorphism of groups

$$\{u \in R^\times \mid r = ru\} \sim \text{Aut}(G^{(r)}_R).$$

Here Iso and Aut are taken in the category of non-unital ring schemes over $R$. If $r \in R$ is not a zero divisor then $\text{Aut}(G^{(r)}_R) = \{\text{id}\}$.

We denote by $(G^{(r)}_R, t)$ the pair consisting of $G^{(r)}_R$ and the chosen local coordinate $t$ above. Consider two ring schemes $M_1$ and $M_2$ over $R$ whose underlying schemes are isomorphic to $\mathbb{A}^1$ and which are enhanced by coordinates $z_1$ and $z_2$ at zero. We say that $(M_1, z_1)$ and $(M_2, z_2)$ are isomorphic if there is an isomorphism (necessarily unique) of ring schemes $\varphi : M_1 \to M_2$ with $z_1 = z_2 \circ \varphi$. Clearly $(G^{(r)}_R, t) \cong (G^{(r')}_R, t)$ if and only if $r = r'$. 
Proposition 5. Let $R$ be a reduced ring and let $M$ be a ring scheme over $R$ with underlying scheme isomorphic to $\mathbb{A}^1$, and coordinate $z$ at zero. Then there exists a unique element $r \in R$ such that $(M, z) \cong (G^{(r)}_R, t)$. If $M$ is unital and the category is that of unital ring schemes, then $r \in R^\times$.

Proof. We have $M = \text{spec } R[z]$. The ring scheme structure of $M$ corresponds to co-addition $\Delta_+$, co-multiplication $\Delta_*$, co-zero $\varepsilon_0$ and in the unital case a co-unit $\varepsilon_1$. By the choice of $z$ we have $\varepsilon_0(z) = 0$. Set $F(X, Y) = \Delta_+(z)$ and $G(X, Y) = \Delta_*(z)$ where we set $X = z \otimes 1$ and $Y = 1 \otimes z$. Then $F, G$ determine a one-dimensional polynomial ring law over $R$. Now the assertion follows from the following lemma.

Lemma 6. Let $R$ be a reduced ring and let $F, G \in R[X, Y]$ be a one-dimensional polynomial ring law over $R$. Then there is a unique element $r \in R$ such that

$$F(X, Y) = X + Y \quad \text{and} \quad G(X, Y) = rXY.$$ 

Proof. Let $n \geq 1$ be the highest power of $x$ in $F(x, y)$. Inspecting the formula

$$F(x, F(y, z)) = F(F(x, y), z)$$

shows that $n = n^2$ since $R$ has no nilpotent elements except 0. Hence $n = 1$ and similarly for $y$. Thus

$$F(x, y) = x + y + cxy \quad \text{for some } c \in R.$$ 

By assumption there is a polynomial $I(x) = -x + \text{deg} \geq 2$ such that $F(x, I(x)) = 0$. Since $R$ is reduced, it follows that $c = 0$. Hence $F(x, y) = x + y$. The associativity of $G$ implies in the same way as for $F$ that $G(x, y) = rxy$ for some $r \in R$.

Lemma 6 and Proposition 5 are proved.

Since we are interested in deformations of Witt vector schemes it is natural to introduce the full subcategory $\mathcal{A}\mathcal{F}\mathcal{V}_{T/R}$ of $\mathcal{A}\mathcal{R}\mathcal{V}_{T/R}$ whose objects $M$ satisfy $M_{\{1\}} \cong \mathbb{A}^1$ as schemes. We rigidify $\mathcal{A}\mathcal{F}\mathcal{V}_{T/R}$ by considering the category $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/R}$ of pairs $(M, z)$ where $z$ is a coordinate at zero of $M_{\{1\}}$. A morphism $\varphi: (\sim M, z) \to (\sim M', z')$ is a morphism $\varphi: M \to M'$ in $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/R}$ such that $z = z' \circ \varphi_{\{1\}}$. In particular, $\varphi_{\{1\}}: M_{\{1\}} \to M'_{\{1\}}$ is an isomorphism.

For $S \ll T$ recall the non-unital ring scheme $W^{(q)}_S$ over $\mathbb{Z}[q]$ from § 2. We have $W^{(q)}_S = W^{G(q)}_S$ and $W^{(q)}_{\{1\}} = \mathbb{A}^1$ is enhanced by the fixed coordinate $t$. Thus we may view $W^{(q)}_S \cong W^{G(q)}_S$ as objects of $\mathcal{A}\mathcal{F}\mathcal{V}_{T/\mathbb{Z}[q]}$ and $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/\mathbb{Z}[q]}$. Similar definitions over the ring $\mathbb{Z}[q, q^{-1}]$ hold in the unital case.

Theorem 7. Let $R$ be a reduced ring without $T$-torsion and $M$ an object in $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/R}$. In the non-unital case there is a unique homomorphism $\mathbb{Z}[q] \to R$ such that there exists a morphism in $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/R}$

$$\alpha: \sim M \longrightarrow \sim W^{(q)}_\sim \otimes_{\mathbb{Z}[q]} R.$$ 

The morphism $\alpha$ is uniquely determined and it is an isomorphism. In the unital case the corresponding assertions hold with $\mathbb{Z}[q]$ replaced by $\mathbb{Z}[q, q^{-1}]$. 
Proof. Since the sequence (8) is scheme-theoretically split and since $M = M_{(1)}$ is isomorphic to $\mathbb{A}^1$, it follows inductively that $M_S \cong \mathbb{A}^S$ as schemes if $S$ is finite. The continuity of $M$ implies that $M_S \cong \mathbb{A}^S$ for all $S \leq T$. Writing $M_S = \text{spec} B_S$, the algebra $B_S$ is therefore a polynomial algebra over $R$. Since $R$ has no $T$-torsion by assumption and $M \cong \mathbb{A}^1$ we see that $M$ has no Hopf $T$-torsion. Now the corollary follows from Theorem 3 and Proposition 5.

Remark. For a fixed $T$ it follows that $(\sim W(q), t)$ over $\text{spec} \mathbb{Z}[q]$ (resp. $\text{spec} \mathbb{Z}[q, q^{-1}]$) is the universal deformation over reduced bases of $(\sim W, t)$ in $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T}$.

We say that a ring homomorphism $\varphi: \mathbb{Z}[q] \to R$ is determined up to a unit in $R$ if $\varphi(q)$ is determined up to a unit in $R$. Forgetting the coordinate of $M$ we get the following.

Corollary 8. Let $R$ be a reduced ring without $T$-torsion and $M$ an object of the category $\mathcal{A}\mathcal{F}\mathcal{V}_{T/R}$.

a) In the non-unital case there is a homomorphism $\varphi: \mathbb{Z}[q] \to R$ which is determined up to a unit in $R$ such that there is an isomorphism in $\mathcal{A}\mathcal{F}\mathcal{V}_{T/R}$

$$\alpha: \sim W(q) \otimes_{\mathbb{Z}[q], \varphi} R.$$ 

If $\varphi(q)$ is not a zero divisor in $R$ then $\alpha$ is uniquely determined. In the general case the set of $\alpha$’s is parametrized by the stabilizer of $\varphi(q)$ under the action of $R^\times$ on $R$.

b) In the unital case there is a unique isomorphism in $\mathcal{A}\mathcal{F}\mathcal{V}_{T/R}$

$$\alpha: \sim W \longrightarrow \sim W \otimes_{\mathbb{Z}} R.$$ 

Thus $W$ has no non-trivial deformations over reduced bases in $\mathcal{A}\mathcal{F}\mathcal{V}_{T}$.

Example. Recall the second family $\mathcal{W}_S^{g(q)}$ of $q$-deformed Witt vector schemes over $\text{spec} \mathbb{Z}[q]$ from §2. Together with their Frobenius and Verschiebung structures and the first coordinate they make

$$\mathcal{W}_S^{g(q)} := (\mathcal{W}_S^{g(q)})_{S \leq T}$$

an object in $\mathcal{A}\mathcal{F}\mathcal{V}^+_{T/\mathbb{Z}[q]}$. We have $\mathcal{W}_S^{g(q)}(\{1\}) = \mathcal{G}(1-g(q))$ by construction and this identification respects the preferred coordinates at zero of both sides.

It follows from Corollary 8 that there is a unique isomorphism

$$\mathcal{W}_S^{g(q)} \sim \mathcal{W}_S^{(1-g(q))} \, \text{ in } \mathcal{A}\mathcal{F}\mathcal{V}^+_{T/\mathbb{Z}[q]}.$$ 

Moreover, this is the only isomorphism of $\sim \mathcal{W}_S^{g(q)}$ with $\sim \mathcal{W}_S^{(r)}$ for some $r \in \mathbb{Z}[q]$ in this category. Forgetting the coordinate at zero and working in the category $\mathcal{A}\mathcal{F}\mathcal{V}_{T/\mathbb{Z}[q]}$, Corollary 8 implies that there are exactly two values of $r \in \mathbb{Z}[q]$ for which an isomorphism (automatically unique) $\mathcal{W}_S^{g(q)} \sim \mathcal{W}_S^{(r)}$ exists, namely $r = 1 - g(q)$ and $r = g(q) - 1$. Note here that $\mathbb{Z}[q]$ is an integral domain with $\mathbb{Z}[q]^\times = \{\pm 1\}$. In particular, it follows that $\mathcal{W}_S^{g(q)} \cong \mathcal{W}_S^{2g(q)}$ as observed in [6], Proposition 3.9.
Example. In [5] Lenart introduced modified Witt rings $W^r(A)$ for every integer $r \in \mathbb{Z}$. In [7] truncated versions $W^r_S(A)$ of these rings were studied. They are obtained as follows. Define $W^r_S(A) = A^S$ as sets and consider the ghost map

$$\Phi^r_S: W^r_S(A) \rightarrow A^S,$$

given by the formula

$$\Phi^r_S((a_d)_{d \in S}) = \left( \sum_{d|n} d^{n/d-1} a_d^{n/d} \right)_{n \in S}.$$

It is the same ghost map as for $W^{(q)}_S(A)$ in formula (2) but on the ghost side $A^S$ is taken componentwise with the usual ring structure (instead of $(A^{(q)})^S$ as in (2)). Using Fermat’s little theorem it was shown in [5] and [7] that for any $r \in \mathbb{Z}$ there is a unique, possibly non-unital ring structure on $W^r_S(A)$ which is functorial in $A$ such that $\Phi^r_S$ becomes a ring homomorphism. Moreover, the usual Frobenius and Verschiebung operators for $n \in S$ on the ghost side

$$F_n((a_\nu)_{\nu \in S}) = (a_{n\nu})_{\nu \in S/n} \quad \text{and} \quad V_n((a_\nu)_{\nu \in S}) = (n\delta_{n|\nu}a_{\nu/n})_{\nu \in S}$$

induce corresponding morphisms on the Lenart-Witt side. It is immediate that all conditions for $W^r := (W^r_S)_{S \leq T}$ to be an object of $\mathcal{AFV}^+_T/\mathbb{Z}$ are satisfied except possibly for the property that $F_p$ should reduce mod $p$ to the $p$th power map for all primes $p \in S \leq T$. It can be shown with some effort that this condition is satisfied if and only if no prime divisor of $r$ is contained in $T$. Since $W^{r}_{\{1\}}(A) = A$ as rings for any $r \in \mathbb{Z}$, it follows from Theorem 7 that $W^r$ is uniquely isomorphic to $W^r_{\{1\}}$ if no prime divisor of $r$ is in $T$. Thus in this case there are natural isomorphisms of rings

$$W^r_S(A) \sim W_S(A).$$

This fact is a special case of [7], Theorem 14. If $T$ contains a prime divisor of $r$ then our theory does not apply to $W^r$. Let us illustrate the preceding discussion with the case $T = \{1, p\}$ where the calculations are easy.

The ghost maps are $\Phi^r_{\{1\}} = \text{id}$ and

$$\Phi^r_{\{1, p\}}(a_1, a_p) = (a_1, pa_p + r^{p-1}a_1^p).$$

Hence $W^r_{\{1\}}(A) = A$ as rings and on $W^r_{\{1, p\}}(A)$ addition and multiplication are given as follows:

$$(a_1, a_p) + (b_1, b_p) = (a_1 + b_1, a_p + b_p - r^{p-1}\sum_{\nu=1}^{p-1} \frac{1}{p}(\nu)^{p}\nu^{-\nu} a_1^{\nu}b_1^{p-\nu})$$

and

$$(a_1, a_p) \cdot (b_1, b_p) = \left( a_1b_1, pa_p b_p + r^{p-1}(a_p b_1^p + a_1^p b_p) + \frac{1}{p} r^{p-1}(r^{p-1} - 1)a_1^p b_1^p \right).$$
The Frobenius morphism

\[ F_p : W_{\{1,p\}}^r(A) \longrightarrow W_{\{1\}}^r(A) = A \]

is given by the formula

\[ F_p(a_1, a_p) = pa_p + r^{p-1}a_1^p. \]

With the projection

\[ \pi : W_{\{1,p\}}^r(A) \longrightarrow W_{\{1\}}^r(A) = A, \quad a = (a_1, a_p) \longmapsto a_1, \]

we therefore have

\[ F_p(a) \equiv r^{p-1} \pi(a)^p \mod pA. \]

Thus the relation

\[ F_p(a) \equiv \pi(a)^p \mod pA \]

for all rings \( A \) is equivalent to \( r^{p-1} \equiv 1 \mod p \), that is, to the assertion that \( p \in T = \{1, p\} \) is not a prime divisor of \( r \). In this case the map

\[ \alpha : W_{\{1,p\}}^r(A) \rightarrow W_{\{1\}}^r(A) \]

with

\[ \alpha(a_1, a_p) = \left( a_1, a_p + r^{p-1} - 1 \frac{a_1^p}{p} \right) \]

is an isomorphism of rings.

**Remark.** Over non-reduced bases there are more one-dimensional polynomial ring laws than those given in Lemma 6. Over \( R = \mathbb{Z}[\varepsilon]/(\varepsilon^2) \) for example the polynomials \( F(X, Y) = X + Y + \varepsilon XY \) and \( G(X, Y) = r\varepsilon XY \) for \( r \in R \) define a different class. Once the one-dimensional polynomial ring laws over a possibly non-reduced \( T \)-torsion free base ring \( R \) are known the deformation theory of \((W, \text{Frobenius}, \text{Verschiebung})\) over \( R \) can be determined using Lemma 4. The proofs of Theorem 7 and Corollary 8 just have to be generalized accordingly.

**Appendix: Witt vectors of inductive systems of rings**

We sketch a natural generalization of the theory of Witt vectors to ind-rings. The somewhat technical motivation is given at the end of the section.

Let \( S \subset \mathbb{N} \) be a divisor stable subset and \( \mathcal{A} = (A_n)_{n \in S} \) an inductive system of unital or non-unital commutative rings. This means that for \( n \in S \) and \( d \mid n \) there are ring homomorphisms

\[ \pi_{d,n} : A_d \rightarrow A_n, \]

with \( \pi_{n,n} = \text{id} \) and \( \pi_{d_1,n} = \pi_{d,n} \circ \pi_{d_1,d} \) if \( d_1 \mid d \).

Consider the set

\[ WS(\mathcal{A}) = \prod_{n \in S} A_n \]

and the ghost map

\[ \mathcal{G}_S : WS(\mathcal{A}) \rightarrow \prod_{n \in S} A_n \]
The universal deformation of the Witt ring scheme

\[ \mathcal{G}_{S,n}((a_\nu)_\nu) = \sum_{d|n} d\pi_{d,n}(a_d)^{n/d} \quad \text{in } A_n. \]

**Proposition 9.** Assume that \( A_n \) has no \( n \)-torsion for every \( n \in S \). Then \( \mathcal{G}_S : W_S(A) \to \prod_{n \in S} A_n \) is injective.

**Proof.** Assume that

\[ \mathcal{G}_S((a_\nu)) = \mathcal{G}_S((b_\nu)). \]

By definition we get \( a_1 = b_1 \). For \( n \in S, n \neq 1 \), assume that \( a_d = b_d \) in \( A_d \) has been shown for all \( d \nmid n \). The equation \( \mathcal{G}_{S,n}((a_\nu)) = \mathcal{G}_{S,n}((b_\nu)) \) gives

\[ n(a_n - b_n) = \sum_{d \nmid n} d(\pi_{d,n}(b_d)^{n/d} - \pi_{d,n}(a_d)^{n/d}) = 0; \]

and hence \( a_n = b_n \) since \( A_n \) has no \( n \)-torsion.

**Example.** If \( (x_n) = \mathcal{G}_S((a_\nu)) \) and \( p \in S \), then

\[ x_p = \pi_{1,p}(a_1)^p + pa_p \quad \text{and hence } x_p \equiv \pi_{1,p}(a_1)^p \mod pA_p. \]

The Witt polynomials \( \Sigma_n \) and \( \Pi_n \) for addition and multiplication depend only on the variables \( x_d \) with \( d \mid n \). Hence we can define addition and multiplication on \( W_S(A) \) by setting

\[ a \oplus b = c \quad \text{for } a, b \in W_S(A), \]

where

\[ c_n = \Sigma_n(\pi_{d,n}(a_d), \pi_{d,n}(b_d); d \mid n), \]

and similarly for multiplication. As in the usual case, this is the only ring structure on the set \( W_S(A) \) which is functorial in \( A \) and for which \( \mathcal{G}_S \) is a ring homomorphism if \( \prod_{n \in S} A_n \) is equipped with componentwise addition and multiplication. Similarly the polynomials defining the usual Frobenius morphisms \( F_n : W_S \to W_{S/n} \) define commuting ring homomorphisms

\[ F_n : W_S(A) \to W_{S/n}(F_n(A)), \]

where \( F_n(A) = (A_{n\nu})_{\nu \in S/n} \). There are also additive Verschiebung maps

\[ V_n : W_{S/n}(V_n(A)) \to W_S(A), \]

where \( V_n(A) = (A_{n\nu})_{\nu \in S/n} \) and \( V_n((a_\nu)_{\nu \in S/n}) = (\delta_{n|\mu} \pi_{\mu/n,\mu}(a_{\mu/n}))_{\mu \in S} \) with \( \delta_{n|\mu} = 1 \) if \( n \mid \mu \) and \( \delta_{n|\mu} = 0 \) if \( n \nmid \mu \). The projection

\[ \text{res} : W_S(A) = \prod_{n \in S} A_n \overset{\text{pr}_1}{\rightarrow} A_1 \]

is a surjective homomorphism of rings.
Example. A ring $A$ can be viewed as the inductive system $A_{\text{const}}$, where $A_n = A$ and $\pi_{d,n} = \text{id}$ for $d \mid n$, $n \in S$. Then $W_S(A_{\text{const}}) = W_S(A)$ together with Frobenius and Verschiebung maps. We can also form the trivial inductive system $A_{\text{triv}}$ of non-unital rings with $A_n = A$ and $\pi_{n,n} = \text{id}$, $\pi_{d,n} = 0$ for $d \mid n$. Let $A^{(n)} = A$ as an additive group but with ring structure $a \ast b = nab$. Then $W_S(A_{\text{triv}}) = \prod_{\nu \in S} A^{(\nu)}$, the product ring. The Frobenius map $F_n$ corresponds to the map

$$F_n : \prod_{\nu \in S} A^{(\nu)} = W_S(A_{\text{triv}}) \to W_S(F_n A_{\text{triv}}) = \prod_{\nu \in S} A^{(\nu)}$$

with

$$F_n((a_{\nu})_{\nu \in S}) = (na_{\nu n})_{\nu \in S/n}.$$ 

The ring $W_N(A_{\text{triv}})$ appeared previously as $W^0(A)$ in [5], Corollary 5.9,(1).

For a divisor stable subset $T \subset S$ and an inductive system $A = (A_\nu)_{\nu \in S}$ there is a natural surjective ring homomorphism

$$W_S(A) \underbrace{\text{proj}} \rightarrow W_T(\text{res}_S^T(A)), \quad (a_\nu)_{\nu \in S} \mapsto (a_\nu)_{\nu \in T},$$

where $\text{res}_S^T(A) = (A_\nu)_{\nu \in T}$.

**Proposition 10.** Consider an inductive system $A = (A_\nu)_{\nu \in S}$. Then the following diagram is commutative:

$$\begin{array}{ccc}
W_S(A) & \xrightarrow{F_p} & W_{S/p}(F_p(A)) \\
\downarrow & & \downarrow & \downarrow W_{S/p}(\pi) \\
W_S(A)/p & \xrightarrow{(\cdot)^p} & W_S(A)/p & \rightarrow W_{S/p}(\text{res}_S^{S/p}(A))/p
\end{array}$$

Here $\pi : \text{res}_S^{S/p}(A) \rightarrow F_p(A)$ is the map with $\nu$th component $\pi_{\nu,p} : A_\nu \rightarrow A_{\nu p}$ for $\nu \in S/p$.

**Proof.** Set $\Lambda = \mathbb{Z}[x_d \mid d \in S]$ and let $(f_\nu)_{\nu \in S/p}$ with $f_\nu \in \mathbb{Z}[x_d \mid d|p\nu] \subset \Lambda$ be the family of polynomials defining the morphism $F_p : W_S \rightarrow W_{S/p}$. Also let $(G_\mu)_{\mu \in S}$ with $G_\mu \in \mathbb{Z}[x_d \mid d|\mu] \subset \Lambda$ be the family of polynomials defining the $p$th power morphism $(\cdot)^p : W_S \rightarrow W_S$. It is known that for $\nu \in S/p$ the difference $f_\nu - G_\nu$ is divisible by $p$ in $\Lambda$. This is equivalent to the fact that diagram (1) commutes. Now for $(a_\mu)_{\mu \in S} \in W_S(A)$ the $\nu$th component for $\nu \in S/p$ of

$$F_p((a_\mu)_{\mu \in S}) \rightarrow W_{S/p}(\pi)(\pi_{S,S/p}(a_\mu)_{\mu \in S}))$$

is given by

$$f_\nu(\pi_{d,\nu p}(a_d); d | p\nu) - G_\nu(\pi_{d,\nu p}(a_d); d | p\nu).$$

This holds because the transition maps are ring homomorphisms and $\pi_{\nu,p} \circ \pi_{d,\nu} = \pi_{d,\nu p}$. The assertion follows.

There is a Dwork lemma for our Witt vector rings of inductive systems.
Dwork lemma. Let $A = (A_n)_{n \in S}$ be an inductive system of rings on a divisor stable subset $S$ of $\mathbb{N}$. Assume that for all $n \in S$ and primes $p | n$ ring homomorphisms are given (compatible with the transition maps)

$$\phi_p : A_{n/p} \rightarrow A_n$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
A_{n/p} & \xrightarrow{\phi_p} & A_n \\
\downarrow & & \downarrow \\
A_{n/p}/p & \xrightarrow{(\cdot)^p} & A_{n/p}/p \xrightarrow{\pi_{n/p,n}} A_n \\
\end{array}
$$

Thus, for $a \in A_{n/p}$, in $A_n$

$$\phi_p(a) \equiv \pi_{n/p,n}(a^p) \mod pA_n.$$

Then the image of the ghost map

$$G_S : W_S(A) \rightarrow \prod_{n \in S} A_n$$

is the following subring:

$$G_S(W_S(A)) = \{(x_n)_{n \in S} | \phi_p(x_{n/p}) \equiv x_n \mod p^{v_p(n)}A_n \text{ for } p | n, n \in S\}.$$ 

Proof. The argument in the proof of Lemma 1.1 in [4] can be easily adapted to our setting. Consider the case $S = \{1, p\}$, for example. If $(x_n) = G_S((a_\nu))$, then $x_1 = a_1$ and $x_p = \pi_{1,p}(x_1^p) + pa_p$. Hence $(x_1, x_p) \in G_S(W_S(A))$ if and only if $x_p \equiv \pi_{1,p}(x_1^p) \mod pA_p$ or equivalently, if and only if $x_p \equiv \phi_p(x_1) \mod pA_p$.

We need the following construction. Given an inductive system of rings $A = (A_n)_{n \in S}$, we obtain another such system $\tilde{W}(A)$ by setting

$$\tilde{W}(A)_n = W_{S/n}(F_n(A)) \text{ for } n \in S$$

and defining $\pi_{d,n}$ to be the composition

$$\pi_{d,n} : W_{S/d}(F_d(A)) \xrightarrow{proj} W_{S/n}(res_{S/d}(F_d(A))) \xrightarrow{W_{S/n}(\pi)} W_{S/n}(F_n(A)).$$

Here

$$\pi : res_{S/d}(F_d(A)) = (A_{\nu d})_{\nu \in S/n} \rightarrow (A_{\nu n})_{\nu \in S/n} = F_n(A)$$

is the map with $\nu$th component $\pi_{\nu d,\nu n} : A_{\nu d} \rightarrow A_{\nu n}$. It follows from Proposition 10 that the maps $F_p$ equip $\tilde{W}(A)$ with a commuting family of Frobenius lifts as in the Dwork lemma. The commutation property of the $F_p$ follows from the known commutation properties of the universal polynomials defining the Witt vector Frobenius morphisms. The surjective ring homomorphisms

$$\tilde{W}(A)_n = W_{S/n}(F_n(A)) \xrightarrow{res} F_n(A)_1 = A_n$$
are compatible with the transition maps \( \pi_{d,n} \) for \( d \mid n, \ n \in S \). Hence we obtain a map of inductive systems of rings

\[
\text{res}: W(A) \to A.
\]

The ghost maps

\[
\mathcal{G}_{S/n}: W_{S/n}(F_n(A)) \to \prod_{k \in S/n} A_{kn}
\]

combine into a morphism of inductive systems of rings

\[
\mathcal{G}: \tilde{W}(A) \to \left( \prod_{k \in S/n} A_{kn} \right)_{n \in S}
\]

such that \( \text{res} \circ \mathcal{G} = \text{res} \).

**Universal property of \( W \).** In the situation of the Dwork lemma, assume in addition that \( A_n \) has no \( n \)-torsion for each \( n \in S \). Moreover, suppose that \( \phi_p \) commutes with \( \phi_l \) for all primes \( p, l \in S \). This means that for every \( n \in S \) with \( p \mid n \) and \( l \mid n \) the following diagram commutes:

\[
\begin{array}{ccc}
A_{n/p} & \xrightarrow{\phi_p} & A_{n/l} \\
\phi_l & & \phi_l \\
A_{n/p} & \xrightarrow{\phi_p} & A_n
\end{array}
\]

Then there is a unique morphism \( \lambda: A \to \tilde{W}(A) \) of inductive systems of rings with the following properties:

a) \( \text{res} \circ \lambda = \text{id}_{\tilde{A}} \);

b) \( \lambda \) commutes with the Frobenius lifts on \( A \) and \( \tilde{W}(A) \), that is, the diagrams

\[
\begin{array}{ccc}
A_{n/p} & \xrightarrow{\lambda_{n/p}} & \tilde{W}(A)_{n/p} \\
\phi_p & & \phi_p \\
A_n & \xrightarrow{\lambda_n} & \tilde{W}(A)_n
\end{array}
\]

commute for all \( p \mid n, \ n \in S \).

**Proof.** We first assume that \( \lambda \) satisfying the desired properties exists and determine its form. According to the Dwork lemma and Proposition 9, via the injective ghost map we have isomorphisms for all \( n \in S \)

\[
W(A)_n = W_{S/n}(F_n(A))
\]

\[
\cong \left\{ (x_k) \in \prod_{k \in S/n} A_{nk} \mid \phi_l(x_{k/l}) \equiv x_k \mod l^{v_l(k)} A_{nk} \text{ for } l \mid k, \ k \in S/n \right\},
\]

where \( l \) denotes prime numbers. In the proof which follows, we will view this isomorphism as an identification.
For $m = p_1^{\nu_1} \cdots p_r^{\nu_r}$ dividing $n \in S$ we set

$$\phi_m := \phi_{p_1}^{\nu_1} \circ \cdots \circ \phi_{p_r}^{\nu_r} : A_{n/m} \to A_n.$$  

By assumption, this is independent of the ordering of the primes $p_1, \ldots, p_r$ dividing $m$. For the $W$-Frobenius morphisms the corresponding formula

$$F_m = F_{p_1}^{\nu_1} \circ \cdots \circ F_{p_r}^{\nu_r} : W(A)_{n/m} \to W(A)_n$$

holds and we therefore have a commutative diagram for $m \mid n$, $n \in S$

$$
\begin{array}{ccc}
A_{n/m} & \xrightarrow{\lambda_{n/m}} & W(A)_{n/m} \\
\phi_m \downarrow & & \downarrow F_m \\
A_n & \xrightarrow{\lambda_n} & W(A)_n \\
\end{array}
$$

In the representation of $W(A)_{n/m}$ and $W(A)_n$ on the ghost side via the Dwork lemma, the map $F_m$ is given by $F_m((x_k)_{k \in S/(n/m)}) = (x_{km})_{k \in S/n}$. For $a \in A_{n/m}$ consider the relation

$$\lambda_n(\phi_m(a)) = F_m(\lambda_{n/m}(a)).$$

The property $\text{res} \circ \lambda = \text{id}$ implies that $\lambda_n(b)_1 = b$ for all $b \in A_n$. Hence

$$\phi_m(a) = \lambda_n(\phi_m(a))_1 = \lambda_{n/m}(a)_m.$$ 

Setting $\nu = n/m$, it follows that for all $\nu \in S$, $m \in S/\nu$ and $a \in A_{\nu}$, we have $\lambda_{\nu}(a)_m = \phi_m(a)$, that is, $\lambda_{\nu}(a) = (\phi_m(a))_{m \in S/\nu}$. Thus we have seen that a map $\lambda : A \to W(A)$ with properties a) and b) must have the form $\lambda = (\lambda_n)_{n \in S}$ where $\lambda_n : A_n \to W(A)_n = W_{S/n}(F_n A)$ is given by $\lambda_n(a) = (\phi_k(a))_{k \in S/n}$. On the other hand, setting $x_k = \phi_k(a)$ for $a \in A_n$ we have

$$\phi_l(x_{k/l}) = \phi_l(\phi_k(a)) = \phi_k(a) = x_k.$$ 

Hence $\lambda_n(a) = (\phi_k(a))_{k \in S/n}$ is indeed an element of $W(A)_n$ in the Dwork lemma description above. Clearly $\lambda_n$ so defined is a ring homomorphism with

$$\lambda_n(a)_1 = \phi_1(a) = a, \quad \text{that is, } \text{res} \circ \lambda = \text{id}.$$ 

The maps $\lambda_n$ are compatible with the transition maps of $A$ and $W(A)$ because the $\phi_p$ are compatible with transition maps by definition. Finally, we have

$$\lambda_n(\phi_p(a)) = (\phi_k(\phi_p(a)))_{k \in S/n} = (\phi_{kp}(a))_{k \in S/n} = F_p((\phi_k(a)))_{k \in S/(n/p)},$$

that is, $\lambda_n \circ \phi_p = F_p \circ \lambda_{n/p}$.

This completes the proof of the universal property of $W$.

**Corollary 11** (Universal property of $W$ over $S$). Fix a divisor stable set $S$ and consider inductive systems of rings $A = (A_n)_{n \in S}$ and $B = (B_n)_{n \in S}$ with the following properties:

i) $A_n$ and $B_n$ have no $n$-torsion for all $n \in S$;

ii) $A$ is equipped with commuting Frobenius lifts.
Then for any morphism $\alpha: A \to B$ of ind-rings there is a unique morphism $\beta: A \to W(B)$ of ind-sets commuting with the Frobenius maps such that the diagram

\[
\begin{array}{ccc}
W(B) & \xrightarrow{\beta} & B \\
\downarrow{\text{res}} & & \downarrow{\text{res}} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

commutes. The morphism $\beta$ is a morphism of ind-rings.

**Proof.** The existence follows from the Cartier-Dieudonné lemma by setting $\beta = W(\alpha) \circ \lambda$ and looking at the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & W(A) & \xrightarrow{W(\alpha)} & W(B) \\
\downarrow{\text{res}} & \xleftarrow{\text{res}} & \downarrow{\text{res}} & & \downarrow{\text{res}} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

For the uniqueness, assume that we are given another lift (of ind-sets) $\beta': A \to W(B)$ of $\alpha$ commuting with the Frobenius maps. Since the $B_n$ have no $n$-torsion the ghost map

\[W(B) \to \left( \prod_{k \in S/n} B_{kn} \right)_{n \in S}\]

has injective components.

Hence it suffices to show that a map $\gamma$ making the diagram

\[
\begin{array}{ccc}
 & & \left( \prod_{k \in S/n} B_{kn} \right)_{n \in S} \\
\gamma & \downarrow{\pi} & \downarrow{\pi} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

commutative is uniquely determined if it commutes with the Frobenius maps. Here $\pi = (\pi_n)_{n \in S}$, where $\pi_n: \prod_{k \in S/n} B_{kn} \to B_n$ maps $(x_k)_{k \in S/n}$ to $x_1$. Moreover, commutation with Frobenius maps means that all diagrams

\[
\begin{array}{ccc}
A_{n/p} & \xrightarrow{\gamma_{n/p}} & \prod_{k \in S/(n/p)} B_{kn/p} \\
\phi_p & \downarrow{\phi_p} & \downarrow{F_p} \\
A_n & \xrightarrow{\gamma_n} & \prod_{k \in S/n} B_{kn}
\end{array}
\]

for all prime numbers $p \mid n, n \in S$ are commutative. Here

\[F_p((x_k)_{k \in S/(n/p)}) = (x_{pk})_{k \in S/n}.\]
It follows that defining $\phi_m = \phi_{p_1}^{\nu_1} \circ \cdots \circ \phi_{p_r}^{\nu_r}$ for $m = p_1^{\nu_1} \cdots p_r^{\nu_r}$ as before, the following diagrams for $m | n$, $n \in S$ commute:

$$
\begin{array}{ccc}
A_{n/m} & \xrightarrow{\gamma_{n/m}} & \prod_{k \in S/(n/m)} B_{kn/m} \\
\phi_m & \downarrow & \downarrow F_m \\
A_n & \xrightarrow{\gamma_n} & \prod_{k \in S/n} B_{kn}
\end{array}
$$

Here $F_m((x_k)_{k \in S/(n/m)}) = (x_{mk})_{k \in S/n}$. The property $\pi_n \circ \gamma_n = \alpha_n$ implies that for $a \in A_{n/m}$ we have

$$
\alpha_n(\phi_m(a)) = \pi_n(\gamma_n(\phi_m(a))) = \pi_n(F_m(\gamma_{n/m}(a))) = \gamma_{n/m}(a)_m.
$$

Setting $\nu = n/m$ it follows that for all $\nu \in S$, $m \in S/\nu$ and $a \in A_{\nu}$ we have

$$
\gamma_{\nu}(a)_m = \alpha_{m\nu}(\phi_m(a)) \quad \text{in} \quad B_{m\nu}
$$
or, replacing the index $m$ by the letter $k$:

$$
\gamma_{\nu}(a) = (\alpha_{k\nu}(\phi_k(a)))_{k \in S/\nu} \quad \text{in} \quad \prod_{k \in S/\nu} B_{k\nu}.
$$

Corollary 11 is proved.

Remarks. a) Consider the map $\beta = (\beta_n) : A \to W(B)$, where

$$
\beta_n : A_n \to W(B)_n = W_{S/n}(F_n(B)).
$$

In the proof above we have seen that the composition of $\beta_n$ with the ghost map

$$
\mathcal{G} : W_{S/n}(F_n(B)) \to \prod_{k \in S/n} B_{kn}
$$
is given as follows: for $a \in A_n$ we have the formula:

$$
\mathcal{G}(\beta_n(a)) = (\alpha_{kn}(\phi_k(a)))_{k \in S/n}.
$$

b) For an object $\tilde{A} = (A_S)_{S \subseteq T}$ in $\mathcal{R}_T$ such that no $A_S$ has $T$-torsion, Corollary 11 leads to another proof of Proposition 1. Fix $S \triangleleft T$ and consider the inductive system of rings $\underline{A} = (A_{S/n})_{n \in S}$ with the evident transition maps. It is equipped with commuting Frobenius lifts. There is a canonical map of ind-rings $\underline{A} \to \underline{B} := A_{\text{const}}$ where $A = A_{\{1\}}$. From Corollary 11 we get a unique commutative diagram

$$
\begin{array}{ccc}
W(A) & \xrightarrow{\beta} & \underline{B} := A_{\text{const}} \\
\downarrow & \searrow \text{res} & \\
\underline{A} & \xrightarrow{\text{res}} & A_{\text{const}}
\end{array}
$$
where \( \beta \) commutes with the Frobenius maps. Passing to the \( n = 1 \) components, we obtain a commutative diagram

\[
\begin{array}{ccc}
WS(A) & \xrightarrow{\beta_S} & A \\
A_S & \xrightarrow{} & A
\end{array}
\]

where we have set \( \beta_S := \beta_1 \). Using the explicit description of \( \beta_1 \) in part a) of the remark one checks that \((\beta_S)_{S\in T}\) is a morphism in \( RF_T \).

c) If \( S/p \neq S \) for \( p \in S \), the Frobenius \( F_p \) is not an endomorphism of \( WS(A) \) and hence a Cartier-Dieudonné lemma cannot be formulated for the individual ring \( WS(A) \). However, as seen above, \( F_p \) defines a Frobenius lift on the ind-ring \( W(A) = (WS/n(A))_{n\in S} \). With a possible comonad structure of \( W \) in mind, it is therefore natural to extend Witt vector theory to ind-rings over a given truncation set.

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