DIFFERENTIAL CALCULUS ON QUANTUM COMPLEX
GRASSMANN MANIFOLDS II: CLASSIFICATION

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Abstract. For differential calculi over certain right coideal subalgebras of quantum groups the notion of quantum tangent space is introduced. In generalization of a result by Woronowicz a one to one correspondence between quantum tangent spaces and covariant first order differential calculi is established. This result is used to classify differential calculi over quantum Grassmann manifolds $O_q(Gr(r,N))$. It turns out that up to special cases in low dimensions there exists exactly one such calculus of classical dimension $2r(N−r)$.

In the framework of quantum groups and quantum spaces there appear many examples of $q$-deformed coordinate algebras which allow well behaved covariant first order differential calculi (FODC) in the sense of Woronowicz [Wor89], [CP94], [KS97]. On the other hand there exists no general construction of a deformation of classical Kähler differentials in this setting. Therefore the task of classification of all covariant FODC over quantum spaces naturally arises and has been settled for many examples [Pod92], [AS94], [SS95b], [SS95a], [BS98], [HS98], [Maj98], [Wel98], [Her].

There are several techniques to perform classification. In [PW89] W. Pusz and S.L. Woronowicz consider calculi over quantum vector spaces generated as left modules by the differentials of the generators. A general ansatz is made and coefficients are determined by covariance. This approach has been applied to other quantum spaces [Pod92], [AS94], [Wel98], yet the calculational effort of this method becomes very large for more involved examples.

For any Hopf algebra $A$ there exists a one to one correspondence between differential calculi and certain right (or left) ideals of $A^\pm = \ker(\varepsilon) \subset A$ [Wor89]. Right ideals have been used in [SS95b], [SS95a], [BS98] to classify bicovariant differential calculi over quantum groups. In [Her], U. Hermisson has generalized this method to certain right (or left) coideal subalgebras $B \subset A$ and classified all 2-dimensional covariant FODC over Podleś’ quantum sphere.

A reformulation of the method of right ideals involves the notion of quantum tangent space. In the case of a Hopf algebra $A$ the quantum tangent space $T_\Gamma$ of a covariant FODC $\Gamma$ is a subspace of the dual Hopf algebra $A^*$ which determines the calculus $\Gamma$ uniquely. Quantum tangent spaces have been used for classification in [HS98], [Maj98]. The advantage of the quantum tangent space approach in the case of coordinate algebras of quantum groups $A = O_q(G)$ stems from the fact that $A^*$ possesses a well understood subalgebra $U$ which is closely related to the $q$-deformed universal enveloping algebra $U = U_q(g)$. The main strategy is to reduce

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the classification problem of covariant FODC to a classification problem of quantum tangent spaces in $\mathcal{U}$.

In this paper the quantum tangent space method is generalized to quantum spaces. More precisely we consider right coideal subalgebras $B$ of a quantum group $A = O_q(G)$ obtained as right $K$ invariants for certain left coideal subalgebras $K$ of $U$. In this situation a quantum tangent space is a subspace of the dual coalgebra $B^\circ$. Now the strategy for classification is the following. One has to find a suitable sufficiently small subcoalgebra $\mathcal{U} \subset B^\circ$. Using representation theory of $B$ one has to show that any quantum tangent space lies in $\mathcal{U}$. Finally quantum tangent spaces in $\mathcal{U}$ have to be classified. The crucial point of this strategy is the right choice of $\mathcal{U}$.

Here this program is performed for quantum Grassmann manifolds $O_q(\text{Gr}(r,N))$ [NDS97], [DS99], [Kol01] in the so called quantum subgroup case. All covariant FODC over $O_q(\text{Gr}(r,N))$ of dimension up to $2r(N-r)$ are classified. It turns out that up to special cases in low dimensions there exists exactly one covariant FODC which has the same dimension as its classical counterpart. This calculus has been constructed and investigated in [Kol01].

The ordering of this paper is as follows. In Section 1 the notion of quantum tangent space for a certain class of quantum spaces is introduced. A one to one correspondence between covariant FODC and quantum tangent spaces is established. This result strongly relies on the identification of covariant FODC with certain left ideals of $B^+$ given in [Heck].

Quantum Grassmann manifolds $O_q(\text{Gr}(r,N))$ are recalled in Section 2.

In Section 3 the structure of $\mathcal{U} = U/K^+U$ in the case $B = O_q(\text{Gr}(r,N))$ is investigated. It is shown that there exists a nondegenerate pairing between $B/(B^+)^{k+1}$ and the set $U_k \subset \mathcal{U}$ of elements of degree $k$ with respect to the coradical filtration of $\mathcal{U}$.

Section 4 is devoted to the representation theory of $O_q(\text{Gr}(r,N))$. Technical lemmata are obtained by explicit calculations using generators and relations of $O_q(\text{Gr}(r,N))$. For this reason some of the relations are collected in Appendix A.

Combining these results it is shown in Section 5 that any element of a finite dimensional quantum tangent space vanishes on $(O_q(\text{Gr}(r,N))^+)^k$ for some $k$ and therefore belongs to $\mathcal{U}$. Finally all quantum tangent spaces in $\mathcal{U}$ of dimension up to $2r(N-r)$ are determined.

If not stated otherwise all notations and conventions coincide with those introduced in [KS97].

1. QUANTUM TANGENT SPACE

Let $U$ denote a Hopf algebra with bijective antipode and $K \subset U$ a left coideal subalgebra, i.e. $\Delta_K : K \to U \otimes K$. Consider a tensor category $\mathcal{C}$ of finite dimensional left $U$-modules. Let $\mathcal{A} := U_0^\circ$ denote the dual Hopf algebra generated by the matrix coefficients of all $U$-modules in $\mathcal{C}$. Assume that $\mathcal{A}$ separates the elements of $U$. Define a right coideal subalgebra $B \subset \mathcal{A}$ by

$$B := \{ b \in \mathcal{A} \mid \langle k, b_{(1)}b_{(2)} \rangle = 0 \text{ for all } k \in K^+\},$$

where $K^+ = \{ k - \varepsilon(k) \mid k \in K \}$.

Assume $K$ to be $\mathcal{C}$-semisimple, i.e. the restriction of any $U$-module in $\mathcal{C}$ to the subalgebra $K \subset U$ is isomorphic to the direct sum of irreducible $K$-modules. By [MS99] Theorem 2.2 (2) this implies that $\mathcal{A}$ is a faithfully
flat $B$ module. Let $\tilde{A}$ denote the left $A$-module coalgebra $\tilde{A} := A/AB^+$, where $B^+ = \{ b \in B \mid \varepsilon(b) = 0 \}$. Further for any right $B$-module $M$ set $\tilde{M} := M/MB^+$, and for any right $A$-comodule $\Delta : N \to N \otimes \tilde{A}$ define $\Delta : N \to N \otimes \tilde{A}$ to be the compositor of $\Delta$ with the canonical projection.

Lemma 1.1. The pairing $K \times A/AB^+ \to \mathbb{C}$ is nondegenerate.

Proof. It is shown in the proof of [MS99] Theorem 2.2 (1), (2) that $A/AB^+$ is equal to the image of $A$ under the restriction map $U^0 \to K^0$.

Let $\mathcal{L} \subset B^+$ be a subspace of finite codimension. Then the orthogonal complement $\mathcal{L}$ in the dual vector space $\mathcal{B}'$ of $\mathcal{B}$ is defined by

$$T_\mathcal{L} := \{ f \in \mathcal{B}' \mid f(x) = 0 \text{ for all } x \in \mathcal{L} \}.$$  

Obviously, $\varepsilon \in T_\mathcal{L}$. If $\mathcal{L}$ is the left ideal corresponding to a unique right-covariant first order differential calculus $\Gamma$ over $\mathcal{B}$, see [Her], then $T_\mathcal{L}^+ = \{ t \in T_\mathcal{L} \mid t(1) = 0 \}$ is called the quantum tangent space of $\Gamma$.

On the other hand, for any finite dimensional subspace $T \subset B'$, $\varepsilon \in T$, define

$$L_T := \{ b \in B \mid f(b) = 0 \text{ for all } f \in T \}.$$  

Then $L_T \subset B^+$.

Proposition 1.1. Let $\mathcal{L}$ and $T_\mathcal{L}$ be given as above, then

1. $(B\mathcal{L} \subset \mathcal{L} \text{ and } T_\mathcal{L} \subset B^0) \Rightarrow \Delta T_\mathcal{L} \subset B^0 \otimes T_\mathcal{L},$
2. $\tilde{A}\mathcal{L} \subset \mathcal{L} \otimes \tilde{A} \Rightarrow T_\mathcal{L} K \subset T_\mathcal{L},$
3. $(B\mathcal{L} \subset \mathcal{L} \text{ and } \tilde{A}\mathcal{L} \subset \mathcal{L} \otimes \tilde{A}) \Rightarrow T_\mathcal{L} \subset B^0,$
4. $\dim \mathcal{L} T_\mathcal{L} = \dim \mathcal{B}^+ / \mathcal{L} + 1,$
5. $L_{T_\mathcal{L}} = \mathcal{L}.$

Proof. 1. Let $f \in T_\mathcal{L} \subset B^0$ and consider $\Delta f = \sum_i f_i \otimes g_i$, where $f_i$ are linearly independent in $B^0$. By assumption $0 = f(bx) = \sum_i f_i(b)g_i(x)$ for all $b \in B, x \in \mathcal{L}$. Moreover, there exist $b_j \in B$ such that $f_i(b_j) = \delta_{ij}$ for all $i, j$. Thus $g_i(x) = 0$ for all $x \in \mathcal{L}$.

2. Assume $k \in K, f \in T_\mathcal{L}$ and $x \in \mathcal{L}$. Since the pairing of $K$ and $\tilde{A}$ is well-defined, we get $(fk)(x) = f(x_0)k(x_{11}) = 0$ as $f(\mathcal{L}) = 0$.

3. and 5. This follows from the fact that the codimension of $\mathcal{L}$ in $\mathcal{B}$ is finite.

3. By assumption and Theorem 2 of [Her], and exchanging left and right, $\mathcal{L}$ corresponds to a right-covariant FODC $\Gamma$ over $\mathcal{B}$. Choose a basis $dx_1, \ldots, dx_k$ of $\Gamma^0 = \Gamma / \Gamma B^+$ and define functionals $\chi^i, i = 1, \ldots, k$, in $B'$ by

$$\overline{db} = dx_i \chi^i(b).$$  

Here $\overline{db}, b \in B$, denotes the element of $\Gamma^0$ represented by $db$. Since $\Gamma = \text{Lin}\{ da \mid a \in \mathcal{B} \}$, we get $\Gamma^0 = \text{Lin}(\overline{db} \mid b \in \mathcal{B})$. Moreover, $\chi^i(1) = 0$ for all $i$. Therefore the functionals $\{ \chi^1, \ldots, \chi^k, \varepsilon \}$ are linearly independent.

Let $y_i \in \mathcal{B}$ such that $\overline{dy_i} = dx_i$. Then the left $\mathcal{B}$-module structure of $\Gamma$ induces a finite dimensional representation $\rho$ of $\mathcal{B}$ by

$$\overline{db} dy_i = dx_j \rho^i_j(b).$$  

By the Leibniz rule

$$\chi^i(ab) = \chi^i(a)\varepsilon(b) + \rho^i_j(a)\chi^j(b).$$
which implies $\chi^i \in B^\circ$. On the other hand in terms of $\Gamma$

\begin{equation}
(1.7) \quad \mathcal{L} = \{ a_i^+ \varepsilon(b_i) | da_i b_i = 0 \}.
\end{equation}

Thus $\chi^i(x) = 0$ for all $x \in \mathcal{L}$, i.e. $\chi^i \in T_{\mathcal{L}}$. Now, dim$_C T_{\mathcal{L}} = \dim_C B^+/\mathcal{L} + 1 = \dim_C \Gamma/\Gamma B^+ + 1$ by 4. Therefore $\{\chi^1, \ldots, \chi^k, \varepsilon\}$ is a sufficiently large set to span all of $T_{\mathcal{L}}$. 

**Proposition 1.2.** Let $T$ and $\mathcal{L}_T$ be given as above. Then

1. $(T \subset B^\circ$ and $\Delta T \subset B^\circ \otimes T) \Rightarrow BL_T \subset \mathcal{L}_T,$
2. $TK \subset T \Rightarrow \overline{\Delta}L_T \subset \mathcal{L}_T \otimes \overline{\Delta},$
3. $\dim_C T = \dim_C B^+/\mathcal{L}_T + 1,$
4. $T_{\mathcal{L}_T} = T.$

**Proof.** 1. Let $x \in \mathcal{L}_T, b \in B$ and $f \in T$. Then $f(bx) = f(1)(b)f(2)(x) = 0$ by assumption.

2. Since $B$ is a right $A$ comodule, it is also a right $\overline{\Delta}$ comodule with coaction $\overline{\Delta}$. Let $x \in \mathcal{L}_T$ and $\overline{\Delta}x = \sum x_i \otimes y_i \in B \otimes \overline{\Delta}$. Assume $y_i$ to be linearly independent in $\overline{\Delta}$. Then by Lemma 1.1 there exist $k_j \in K$ such that $k_j(y_i) = \delta_{ij}$. Now for any $t \in T$ one has $t(x_j) = (tk_j)(x) = 0$ by assumption.

3. The subspace of $B$ where all elements of $T$ vanish has codimension $\dim_C T$.

4. This claim follows for instance from $T \subset T_{\mathcal{L}_T}$ and $\dim_C T = \dim_C T_{\mathcal{L}_T}$.

**Corollary 1.2.** Under the above assumptions there is a one to one correspondence between $n$-dimensional covariant FODC over $B$ and $(n + 1)$-dimensional subsets $T^e \subset B^\circ$ such that

\begin{equation}
(1.8) \quad \varepsilon \in T^e, \quad \Delta T^e \subset B^\circ \otimes T^e, \quad T^e K \subset T^e.
\end{equation}

2. Quantum Grassmann manifolds

In the remaining sections of this paper all considerations will be restricted to the following example of quantum complex Grassmann manifolds. Let $q \in \mathbb{R}$ be transcendental. This assumption is needed only in the proofs of Lemma 2.3 and of the main Theorem 5.4 where duality between $\overline{U}_+$ and $\mathcal{O}_q(\text{Mat}(r,N-r))$ is used. All other arguments only use $q \in \mathbb{R} \setminus \{-1,0,1\}$. Consider the $q$-deformed universal enveloping algebra $U := U_q(\mathfrak{sl}_N)$. The subalgebra $K$ generated by

\begin{equation}
(2.1) \quad \{ E_i, F_i, K_j | i \neq r, j = 1, \ldots, N - 1 \}
\end{equation}

is a coalgebra and hence a left coideal. The category $\mathcal{C}$ of type one representations of $U_q(\mathfrak{sl}_N)$ is a tensor category. The matrix coefficients of $\mathcal{C}$ generate $\mathcal{O}_q(\text{SL}(N))$ and $K$ is C-semisimple. Moreover, since $q$ is not a root of unity, the pairing between $\mathcal{O}_q(\text{SL}(N))$ and $U_q(\mathfrak{sl}_N)$ is nondegenerate. The subalgebra $B$ defined by (1.1) is just the $q$-deformed coordinate algebra $\mathcal{O}_q(\text{Gr}(r,N))$ of the Grassmann manifold of $r$-dimensional subspaces in $\mathbb{C}^N$ [NDS97, DS99] in the quantum subgroup case, i.e. the case when $K$ is a Hopf subalgebra of $U$.
It has been shown in [Kol01] that $O_q(\text{Gr}(r, N))$ can be written in terms of generators $z_{ij}, i, j = 1, \ldots, N$ which satisfy the relations
\begin{align}
q^{2b-2i}z_{ij}z_{ik}R_{kij}R_{cd} &= q^{2l-2i}z_{al}z_{jk}R_{id}R_{kl}
\sum_{i=1}^{N} z_{ii} &= \frac{1-q^{-2s}}{q-q^{-1}}
q^{2N+1}\sum_{n=1}^{N} q^{-2n}z_{in}z_{nk} &= z_{ik},
\end{align}
where $s = N-r$. The explicit form of some of the relations (2.2) – (2.4) is given in Appendix A and will be used in Section 4.

The right coaction of $O_q(\text{SL}(N))$ on the generators $z_{ij}$ is given by
\begin{equation}
\Delta z_{ij} = z_{kl} \otimes u^k_i S(u^l_j)
\end{equation}
where $u^i_j, i, j = 1, \ldots, N$ denote the matrix elements of the fundamental corepresentation of $O_q(\text{SL}(N))$. One obtains an induced left action of $U_q(\mathfrak{s}\mathfrak{l}_N)$ which in the conventions of [KS97], Sect. 8.4.1 reads as
\begin{align}
E_k \triangleright z_{ij} &= \delta_{ik} q^\delta_{kj} - \delta_{j,k+1}q z_{i,j-1}
F_k \triangleright z_{ij} &= \delta_{ik+1} z_{i,j-1} - \delta_{j,k} q^{\delta_{kj}-\delta_{j,k+1}} z_{i,j+1}
K_k \triangleright z_{ij} &= q^{\delta_{kj} - \delta_{j,k+1}} z_{i,j}.
\end{align}
The embedding $i : O_q(\text{Gr}(r, N)) \rightarrow O_q(\text{SL}(N))$ is given by
\[
i(z_{ij}) = \sum_{i=r+1}^{N} q^{-2N-1+2k} u^k_i S(u^l_j),
\]
thus in particular
\begin{equation}
\varepsilon(z_{ij}) = \begin{cases} q^{-2N-1+2i} & \text{if } i = j > r, \\ 0 & \text{else.} \end{cases}
\end{equation}

3. The coalgebra $U/K^+U$

Recall from [CK90] that there exists a filtration of $U$ such that the associated graded algebra is $q$-commutative, i.e. given by generators $E_\alpha, F_\beta, K_\pm$ and relations $t_1t_2 = q^{n_1}t_2t_1$ for all $t_1, t_2 \in \{E_\alpha, F_\beta, K^\pm\}$ and some $n = n(t_1, t_2) \in \mathbb{Z}$. Here $\alpha, \beta \in \Phi^+$ denote the positive roots of $\mathfrak{s}\mathfrak{l}_N$. Fix a reduced decomposition of the longest element $w$ of the Weyl group.

**Lemma 3.1.** Let $g_i, i, j = 1, 2, \ldots$ denote the generators $E_\alpha, F_\beta, K_j$ of $U$ with respect to $w$ in an arbitrary order. Then the elements
\[
\prod_{i} g_i^{n_i}
\]
are $n_i \in \mathbb{N}_0$ if $g_i = E_\alpha, F_\alpha$ and $n_i \in \mathbb{Z}$ if $g_i = K_j$ form a vector space basis of $U$.

**Proof.** This follows from the $q$-commutativity [CK90], Prop 1.7 d. \qed
Write the set $\Phi^+$ of positive roots as a disjoint union
\[ \Phi^+ = \Phi^+_+ \cup \Phi^+_-, \]
where $\Phi^+_+ = \{ \alpha_{ij} = \sum_{k=i}^j \alpha_k | i \leq r \leq j \}$ and $\alpha_k$ denote the simple roots.

**Proposition 3.1.** Let $\beta_i$ and $\beta'_i$ denote the elements of $\Phi^+_-$ in arbitrary fixed orders. Then the elements
\[ \prod_i (F_{\beta_i})^{n_i} \prod_j (E_{\beta'_j})^{m_j} \]
normalsize \[ n_i, m_j \in \mathbb{N}_0 \] form a vector space basis of $U/K^+U$.

**Proof.** Let $\gamma_i$ and $\gamma'_i$ denote the elements of $\Phi^+_-$ in arbitrary fixed orders. By Lemma 3.1 the elements
\[ (K^{i_1} \ldots K^{i_{N-1}}) \prod_i (F_{\gamma_i})^{r_i} \prod_j (E_{\gamma'_j})^{s_j} \prod_k (F_{\delta_k})^{n_k} \prod_l (E_{\gamma'_l})^{m_l} \]
\[ i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0 \] form a vector space basis of $U$. Thus it suffices to show that the elements
\[ (K^{i_1} \ldots K^{i_{N-1}} - 1) \prod_i (F_{\beta_i})^{n_i} \prod_j (E_{\beta'_j})^{m_j} \]
\[ (K^{i_1} \ldots K^{i_{N-1}} \prod_i (F_{\gamma_i})^{r_i} \prod_j (E_{\gamma'_j})^{s_j} \prod_k (F_{\delta_k})^{n_k} \prod_l (E_{\gamma'_l})^{m_l} \]
\[ i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0, \sum r_i + \sum s_j \geq 1, \] form a vector space basis of $K^+U$. Indeed the expressions \[ \text{(3.2)} \] and \[ \text{(3.3)} \] form a set of linearly independent elements of $K^+U$. Any element of $K^+U$ can be written as a sum of expressions of the form
\[ G(K^{i_1} \ldots K^{i_{N-1}} \prod_i (F_{\gamma_i})^{r_i} \prod_j (E_{\gamma'_j})^{s_j} \prod_k (F_{\delta_k})^{n_k} \prod_l (E_{\gamma'_l})^{m_l} \]
where $G \in \{ K_i - 1, F_j, E_j | j \neq r \}$ and $i_k \in \mathbb{Z}, r_i, s_j, n_k, m_l \in \mathbb{N}_0$. If $G = E_j$ then $G$ commutes with $K^{i_1} \ldots K^{i_{N-1}}$. If $\prod_i (F_{\gamma_i})^{r_i} \neq F_j$ then by reordering $E_j \prod_i (F_{\gamma_i})^{r_i}$ according to the above basis \[ \text{(3.4)} \] one obtains monomials of the form
\[ K^{i_1} \ldots K^{i_{N-1}} \prod_i (F_{\gamma_i})^{r_i} E_j^\delta \]
where $\delta = 1$ or $\sum r_i^\delta \neq 0$. As the elements $E_{\gamma_i^\delta}$ generate a subalgebra with basis $\prod_j (E_{\gamma_j})$ and $E_j$ is an element of this subalgebra the expression \[ \text{(3.4)} \] for $G = E_j$ can indeed be written as a linear combination of elements of the form \[ \text{(3.3)} \]. If on the other hand $\prod_i (F_{\gamma_i})^{r_i} = F_j$ then the relation
\[ E_j F_j - F_j E_j = \frac{K_j - K_j^{-1}}{q - q^{-1}} \]
implies the claim. The cases $G = K_i - 1$ and $G = F_j$ are dealt with in a similar way.

Let $\overline{U}$ denote $U/K^+U$. By Corollary 5.3.5 in [Mon93] $\overline{U}$ is pointed. Recall that the coradical $U_0$ of $U_q(\mathfrak{sl}_N)$ is the subalgebra generated by the elements $K_i, i = 1, \ldots, N - 1, \overline{\text{Mon93}}$, Lemma 5.5.5. By [Swe99] p. 182, Ex. 4 the coradical of $\overline{U}$ is contained in $\pi(U_0)$ where $\pi : U_q(\mathfrak{sl}_N) \to \overline{U}$ denotes the canonical projection. Thus $\overline{U}$ is connected, i. e. the coradical of $\overline{U}$ is equal to $\mathbb{C} \cdot 1$. 


For any coalgebra $C$ let $P(C) = \{ x \in C \mid \Delta x = 1 \otimes x + x \otimes 1 \}$ denote the vector space of primitive elements of $C$.

**Lemma 3.2.** $P(\mathcal{U})K \subset P(\mathcal{U})$.

**Proof.** Since $K$ is a coalgebra, for $p \in P(\mathcal{U})$, $k \in K$ we get
\[
\Delta(pk) = pk_{(1)} \otimes k_{(2)} + k_{(1)} \otimes pk_{(2)} = pk_{(1)} \otimes \epsilon(k_{(2)})1 + \epsilon(k_{(1)})1 \otimes pk_{(2)} = pk \otimes 1 + 1 \otimes pk.
\]
\[
\square
\]

Set $\mathcal{U}_- = \text{Lin}(\prod_i F_{\beta_i}^{m_i}), \mathcal{U}_+ = \text{Lin}(\prod_i E_{\beta_i'}^{n_j})$, where the products are taken over all $i$ such that $\beta_i, \beta_i' \in \Phi_r^+$. As $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) is the image of the Hopf subalgebra $U_q(b^\perp)$ (resp. $U_q(b^-)$) under the canonical projection $\pi$, the subspace $\mathcal{U}_+ \subset \mathcal{U}$ (resp. $\mathcal{U}_- \subset \mathcal{U}$) is a subcoalgebra.

In what follows several $U$-module coalgebra filtrations of $\mathcal{U}$ and $U_q(b_{\pm})$-module coalgebra filtrations of $\mathcal{U}_{\pm}$ will prove quite useful. Let $\mathcal{F}_1$ denote the filtration of $\mathcal{U}$ defined by
\[
\deg_1 \left( \prod_i (F_{\beta_i})^{n_i} \prod_j (E_{\beta_j'})^{m_j} \right) = \sum_i n_i + \sum_j m_j.
\]
The corresponding filtration of $U$ is defined by
\[
\deg_1(E_i) = \delta_{i,s} = \deg_1(F_i), \quad \deg_1(K_i) = 0.
\]
This induces a filtration on $\mathcal{U}_{\pm}$ which will also be denoted by $\mathcal{F}_1$. Further let $\mathcal{F}_2$ denote the filtration of $\mathcal{U}$ defined by
\[
\deg_2(E_i) = \deg_2(F_i) = 1.
\]
The corresponding filtration of $U$ is defined by
\[
\deg_2(E_i) = 1 = \deg_2(F_i), \quad \deg_2(K_i) = 0.
\]

Finally there exists a $\mathbb{Z}^{N-1}$ grading of $\mathcal{U}$ and $\mathcal{U}_{\pm}$ induced by the standard $\mathbb{Z}^{N-1}$ grading of $U$.

**Lemma 3.3.** The primitive elements in $\mathcal{U}_+$ and $\mathcal{U}_-$ are given by
\[
P(\mathcal{U}_+) = \text{Lin}\{E_{\beta_i'} \mid \beta_i' \in \Phi_r^+\}, \quad P(\mathcal{U}_-) = \text{Lin}\{F_{\beta_i} \mid \beta_i \in \Phi_r^+\}.
\]

**Proof.** The $U$-module coalgebra $\mathcal{U}_{\pm}$ is graded with respect to $\mathcal{F}_1$. Let $(\mathcal{U}_{\pm})_k$ denote the elements of degree $k$. As $(\mathcal{U}_{\pm})_0 = \mathbb{C}1$ all elements of $(\mathcal{U}_{\pm})_1$ have to be primitive. It remains to show that there are no primitive elements in the graded components $(\mathcal{U}_{\pm})_k$ for $k > 1$. The proof is carried out for the case $(\mathcal{U}_+)_k$.

Let $V(0)_+$ denote the right highest weight $U$-module given by one generator $v$ and relations
\[
vF_i = 0, \quad vK_i = v, \quad vE_j = 0 \quad \text{for all } i, j = 1, \ldots, N-1, j \neq r.
\]
The module $V(0)_+$ can be endowed with a right $U$-module coalgebra structure by (cf. [SV98], [SSV99])
\[
\Delta v = v \otimes v.
\]
Note that the coalgebras $\overline{U}_+$ and $V(0)_+$ are both isomorphic to
\[ U_q(b_+)|/(K_i - 1)U_q(b_+), E_jU_q(b_+)| j \neq r \].

It has been shown in [SSV99] that the $U$-module coalgebra $V(0)_+$ is the graded dual of the $U$-module algebra of $q$-deformed functions on $r \times (N-r)$ matrices $\mathcal{O}_q(\text{Mat}(r,N-r))$. Recall that the homogeneous component $(V(0)_+)_k$ of degree $k$ elements is spanned by all monomials $\prod_i (E_{\beta^i})^{n_i}$ with $\sum_i n_i = k$, while the degree of monomials in $\mathcal{O}_q(\text{Mat}(r,N-r))$ is given by the number of factors. Thus Lemma 3.2 implies that $P(\overline{U})$ is a graded vector space. Assume $x \in P(\overline{U}_+)$ to be a homogeneous primitive element with deg$_1(x) > 1$. As any monomial $u \in \mathcal{O}_q(\text{Mat}(r,N-r))$ of degree $k > 1$ can be written as a product $u = u_1u_2$ with deg$(u_1,2) < k$

\[ x(u) = x(u_1) \varepsilon(u_2) + \varepsilon(u_1)x(u_2) = 0 \]
as $x(u_1) = x(u_2) = 0$ and therefore $x = 0$. \hfill $\square$

**Lemma 3.4.** $P(\overline{U}) = \text{Lin}\{F_{\beta}, E_{\beta^i} | \beta_i, \beta_i^i \in \Phi^+_1\}$.

**Proof.** By Lemma 3.3 and Lemma 3.3 $P(\overline{U}_+ \oplus \overline{U}_-) = P(\overline{U}_+) \oplus P(\overline{U}_-)$. Suppose that $u = \sum \lambda_{n_1 \ldots n_m} \prod_i (F_{\beta_i})^{n_i} \prod_i (E_{\beta^i})^{n_i} \in P(\overline{U})$, $M = r(N-r)$ and $u \notin \overline{U}_+ \oplus \overline{U}_-$. By Lemma 3.2 one may assume that $u$ is homogeneous with respect to the $\mathbb{Z}^{N-1}$-grading. Let $S_u \subset \mathbb{N}_0^M$ denote the subset defined by

\[ S_u := \{(m_1, \ldots, m_M) | \exists (n_1, \ldots, n_M) \text{ such that } \lambda_{n_1 \ldots n_m}m_1 \ldots m_M \neq 0\}. \]

Choose a multiindex $(k'_1, \ldots, k'_M) \in S_u$ such that $\prod_i (E_{\beta^i})^{k'_i}$ is maximal among the $\prod_i (E_{\beta^i})^{l_i}$, $(l_1, \ldots, l_M) \in S_u$ with respect to the filtration $F_2$. By assumption $\prod_i (E_{\beta^i})^{k'_i} \neq 1$. Pick $(k_1, \ldots, k_M)$ such that $\lambda_{k_1 \ldots k_M}k'_1 \ldots k'_M \neq 0$. Write $\Delta u \in \overline{U} \otimes \overline{U}$ with respect to the basis given in Proposition 3.3 in the first tensor factor. The second tensor factor corresponding to $\prod_i (E_{\beta^i})^{k_i}$ is given by

\[ \sum_{(m_1, \ldots, m_M)} \lambda_{k_1 \ldots k_M}m_1 \ldots m_M \prod_i (E_{\beta^i})^{m_i} \neq 0 \]
as $u$ is homogeneous with respect to the $\mathbb{Z}^{N-1}$-grading. But this means that $u$ cannot be primitive. \hfill $\square$

**Lemma 3.5.** For any $x \in P(\overline{U}) \setminus \{0\}$ the functional $\langle \cdot, x \rangle : B \to \mathbb{C}$ is nonzero.

**Proof.** By Lemma 3.2 and Lemma 3.4 the primitive elements of $\overline{U}$ form a direct sum of two non isomorphic irreducible right $K$ modules with highest weight vectors $E_r$ and $F_{a_1, N-1}$. Therefore one can find an element $k \in K$ such that $xk = E_r$ or $xk = F_{a_1, N-1}$. Restrict to the first case. The relation

\[ \langle b, E_r \rangle = \langle b, xk \rangle = \langle k \triangleright b, x \rangle \]
implies that it suffices to find an element $b \in B$ such that $\langle b, E_r \rangle \neq 0$. This is achieved by choosing $b = z_{r, r+1}$ as follows from (2.6) and (2.9). \hfill $\square$

**Lemma 3.6.** The pairing $\langle \cdot, \cdot \rangle : B \otimes \overline{U} \to \mathbb{C}$ is nondegenerate.
Proof. Recall that the elements of $K^+U$ vanish on $B$. Hence the pairing $\langle \cdot, \cdot \rangle : B \otimes \mathcal{U} \to \mathbb{C}$ is just the restriction of the pairing between $A$ and $U$ to $B$ in the first component. Since the pairing between $A$ and $U$ is nondegenerate, $\mathcal{U}$ separates the elements of $B$. On the other hand, let $\mathcal{I}$ denote the subspace

$$\mathcal{I} := \{ f \in \mathcal{U} | \langle b_{(1)}, f \rangle b_{(2)} = 0 \text{ for all } b \in B \}$$

of $\mathcal{U}$. Clearly, $\mathcal{I}$ is a right $U$ submodule of $\mathcal{U}$. Moreover, $\mathcal{I}$ is the kernel of the coalgebra map $\mathcal{U} \to B^*$ and hence a coideal. Suppose that $\mathcal{I} \neq \{0\}$ and let $f \in \mathcal{I}$ be of minimal degree $k$ with respect to the coradical filtration. Since $f \in \ker \varepsilon$ and

$$\Delta(f) - 1 \otimes f - f \otimes 1 \in \mathcal{U}_{k-1} \otimes \mathcal{U}_{k-1} \cap (\mathcal{I} \otimes \mathcal{U} + \mathcal{U} \otimes \mathcal{I}),$$

by the minimality of $k$ we conclude that $f$ has to be primitive. This is a contradiction to Lemma 3.3.

We conclude this section with an auxiliary lemma which gives an upper bound for the dimension of $(B^+)^k/(B^+)^{k+1}$.

**Lemma 3.7.** $\dim(B^+)^k/(B^+)^{k+1} \leq \left(\frac{2r(N - r) + k - 1}{k}\right)$.

**Proof.** First note that (2.4) and (2.9) imply that $z_{ij}^+ \in (B^+)^2$ if $i, j \leq r$ or $i, j > r$. Take for instance $i = j = N$ then

$$z_{NN}^+ = q^{2N+1} \sum_{i=1}^{N} q^{-2i} z_{Ni} z_{iN} - \varepsilon(z_{NN}) = \sum_{i \leq N} q^{2N+1-2i} z_{Ni} z_{iN} + q(z_{NN}^+)^2 + 2z_{NN}^+. $$

Consider now $x \in (B^+)^k/(B^+)^{k+1}$. Then it follows that $x$ can be written as a linear combination of monomials of degree $k$ in the generators $z_{ij} \in M$,

$$M := \{z_{mn} | m \leq r, n > r \text{ or } m > r, n \leq r\}. $$

The Equations (2.2) and (2.4) yield linear relations between these monomials. Define an $\mathbb{N}_0^{N-1}$-valued vector space filtration $F$ of $(B^+)^k/(B^+)^{k+1}$ by

$$\deg \left( \prod_{m=1}^{k} z_{im,jm} \right) = \sum_{m=1}^{k} e_{|im-jm|}, \quad z_{im,jm} \in M$$

where $\mathbb{N}_0^{N-1}$ is ordered lexicographically ($e_1 > e_2 > \cdots > e_{N-1}$). To prove the lemma it suffices to show that for each pair $z_{kl}, z_{ij} \in M$ there exists $c_{ijkl} \in \mathbb{C} \setminus \{0\}$ such that

$$z_{ij} z_{kl} - c_{ijkl} z_{kl} z_{ij} \in (B^+)^3$$

up to terms of lower degree with respect to the filtration $F$. All occurring cases are checked by direct computation. As an example we consider the case $i < l < j = k$.

By (2.6)

$$z_{ij} z_{jl} = q^{-1} z_{jl} z_{ij} + q^{-1} q z_{jj} z_{il} - q^{-1} q \sum_{m<j} q^{2j-2m} z_{im} z_{ml}$$

$$= q^{-1} z_{jl} z_{ij} + q^{-1} q z_{jj}^+ z_{il} + q^{-1} q \sum_{m>j} q^{2j-2m} z_{im} z_{ml} + q^{-1} q z_{ij} z_{jl}$$

which implies

$$z_{ij} z_{jl} = q z_{jl} z_{ij} + q q z_{jj}^+ z_{il} + q q \sum_{m>j} q^{2j-2m} z_{im} z_{ml}. $$
As the last term of the right hand side is of lower degree with respect to \( F \) and \( z_{ij}, z_{il} \in (B^+)^2 \) the obtained relation is of the desired form (3.7).

Let \( \overline{U}_k \) denote the elements of degree \( k \) in \( \overline{U} \) with respect to the filtration \( F_1 \).

**Corollary 3.8.** The pairing \( \langle \cdot, \cdot \rangle : B/(B^+)^{k+1} \otimes \overline{U}_k \to \mathbb{C} \) is nondegenerate.

**Proof.** By Lemma 3.7 and Proposition 3.1

\[
\dim B/(B^+)^{k+1} \leq \sum_{l=0}^{k} \left( 2r(N - r) + l - 1 \right) = \dim(\overline{U}_k).
\]

On the other hand \( \overline{U}_k(B^+)^{k+1} = 0 \). Using Lemma 3.6 one obtains that \( B/(B^+)^{k+1} \) separates \( \overline{U}_k \) and hence \( \dim(\overline{U}_k) = \dim(\overline{U}) = \dim(B/(B^+)^{k+1}) \).

**Corollary 3.9.** The coradical filtration of \( \overline{U} \) coincides with \( F_1 \).

**Proof.** By Proposition 11.0.5 in [Swe69] and Lemma 3.6 one has \( C_k = \{ f \in \overline{U} \mid \langle (B^+)^{k+1}, f \rangle = 0 \} \) where \( C_k \) denotes the elements of \( \overline{U} \) of degree \( k \) with respect to the coradical filtration. By Corollary 3.8 the right hand side coincides with \( \overline{U}_k \).

4. Graded representations of \( \mathcal{O}_q(\text{Gr}(r, N)) \)

Let \( V_0 \) denote the subalgebra of \( B = \mathcal{O}_q(\text{Gr}(r, N)) \) generated by the little generators \( z_{ii}, i = 1, \ldots, N \), and let \( V_+ \) (resp. \( V_- \)) denote the subalgebra of \( B^+ \) generated by \( z_{ij}, i < j \) (resp. \( i > j \)).

**Lemma 4.1.** The map \( V_- \otimes V_0 \otimes V_+ \to B \) given by multiplication is onto.

**Proof.** Note first that the left hand side of the expressions in \( A.2, A.3, A.4 \) consist of all possible products of generators \( z_{db}z_{ca} \) with \( z_{db} \in V_\alpha, z_{ca} \in V_\beta \) and \( (\alpha, \beta) \in \{ (+, 0), (0, -), (+, -) \} \). Let now \( W_k \) denote the vector space generated by all \( k \)-fold products of the elements \( z_{ij}, i, j = 1, \ldots, N \). It suffices to show that any element of \( W_k \) can be written as a linear combination of monomials

\[
\text{(4.1)}
\]

in standard form where \( v_{\alpha,j} \) denote generators of \( V_\alpha \) for all \( j = 1, \ldots, k_\alpha, \alpha \in \{ -, 0, + \} \) and \( k = k_- + k_0 + k_+ \). This is achieved by induction over \( k \) and the relation

\[
W_{k+1} = \sum_{c,\alpha} W_k z_{ca}.
\]

Assume first \( w \in W_k \) and \( c = a \). By \( A.2 \) one obtains

\[
wz_{ca} \in W_k V_+
\]

if \( w \in \sum_{i<j} W_{k-1} z_{ij} \).

On the other hand, if \( c > a, c - a = l \) and \( w \in W_k \) is a monomial \( (4.1) \) then \( A.3, A.4 \) imply

\[
wz_{ca} \in W_k V_0 + W_k V_+ + \sum_{j<i<l+j} W_k z_{ij}.
\]

Induction over \( l \) yields the result. \( \square \)
Let $T$ denote the quantum tangent space of a covariant first order differential calculus over $B$, and let $\{t_i | i = 1, \ldots, d\}$ be a basis of $T^e = T \oplus \mathbb{C}$. By Corollary 1.2, one has $\Delta T^e \subset B^e \otimes T^e$. The elements $a_{ij} \in B^e$ defined by

$$\Delta t_i = a_{ij} \otimes t_j$$

generate a finite dimensional subcoalgebra $T' \subset B^e$. Thus they can be considered as matrix coefficients of a finite dimensional representation of $B$.

**Lemma 4.2.** Let $K' \subset U_q(\mathfrak{g})$ denote the commutative subalgebra generated by the elements $K_i, i = 1, \ldots, N - 1$. Then $T'K' \subset T'$.

**Proof.** By Corollary 1.2 one has $T^e K \subset T^e$, further by (2.1) $K_j \in K$ for all $j$. Therefore one can define $\lambda_{ij}^l \in \mathbb{C}$ by $t_i K_j = \lambda_{ij}^l t_l$. As $K_j$ is invertible in $U_q(\mathfrak{g})$ the matrix $\lambda_{ij}^l$ is invertible for fixed $j$. Then

$$\Delta(t_i K_j) = a_{ik} K_j \otimes t_k K_j = a_{ik} K_j \otimes \lambda_{ij}^l t_l = \lambda_{ik}^l a_{ik} K_j \otimes t_l.$$ 

On the other hand

$$\Delta(t_i K_j) = \Delta(\lambda_{ij}^m t_m) = \lambda_{ij}^m a_{ml} \otimes t_l.$$ 

Comparing coefficients of the linear independent elements $t_l$ one obtains

$$\lambda_{ij}^l a_{ik} K_j = \lambda_{ij}^m a_{ml}$$

for all $l, i, j$. The invertibility of $\lambda_{ij}^l$ implies $a_{ik} K_j \in T'$.

Let $P_+$ (resp. $P_-$) denote the set of monomials in the elements $z_{ij}, i < j$ (resp. $i > j$).

**Corollary 4.3.** The matrix coefficients $a_{ij}$ vanish on all but finitely many elements of $P_+$ and $P_-$.

**Proof.** Note that for any $x \in B$ and for any $K_k \in K'$

$$(a_{ij} K_k)(x) = a_{ij}(K_k \triangleright x).$$

Assume that $a_{ij}(x) \neq 0$ for infinitely many elements $x \in P_+$. Among the elements $x$ are common eigenvectors of the $K_k \triangleright$ with infinitely many different eigenvalues. Therefore $\text{dim}(a_{ij} K') = \infty$ in contradiction to Lemma 1.2 and $\text{dim} T' < \infty$. $\square$

In particular the generators $z_{ij}, i \neq j$, act as nilpotent operators on the representation determined by the matrix coefficients $a_{ij}$. Such representations are also annihilated by certain powers of $V_0^+$. 

**Lemma 4.4.** Let $W$ denote a finite dimensional representation of $B$ such that the generators $z_{ij}, i \neq j$, act nilpotently. Then the commutative nonunital subring $V_0^+ \subset B^+$ also acts nilpotently on $W$.

**Proof.** The proof is performed in several steps.

**Step 1a:** Consider the following ordering on the generators $z_{ij}, i < j$ of $V_+$:

$$z_{ij} < z_{kl} \iff (i < k) \text{ or } (i = k \text{ and } j < l).$$

Let $v$ denote a common eigenvector of all $z_{ii}$, i. e.

$$z_{ii} v = \lambda_n v \quad \forall n = 1 \ldots N$$

such that

$$(4.2) \quad z_{ij} v = 0 \quad \forall z_{ij} < z_{kl}.$$
Then \( w_0 = z_{kl}v \) satisfies

1. \( z_{nn}w = \mu_n w \) where

\[
\mu_n = \begin{cases} 
q^{-2} \lambda_k & \text{if } n = k, \\
\lambda_i + (1 - q^{-2}) \lambda_k & \text{if } n = l, \\
\lambda_n & \text{else.}
\end{cases}
\]  

(4.3)

2. \( z_{ij}w = 0 \) \( \forall z_{ij} < z_{kl} \).

By assumption this implies the existence of a common eigenvector \( v_+ \) of all \( z_{ii} \), \( i = 1, \ldots, N \) such that \( V_+ v_+ = 0 \). Such a common eigenvector will be called a maximal eigenvector.

**Proof of Step 1a:** The value of \( \mu_n \) can be computed by means of the list in the Appendix A. Consider for example the case \( n = l \). Then by the fourth relation in A.2 one has

\[
z_{ll}z_{kl}v = z_{kl}z_{ll}v + (1 - q^{-2}) \sum_{i<l} q^{2l-2i} z_{ki}z_{il}v
\]

By assumption and by means of A.6 the last term is simplified to

\[-q^2 \sum_{k<i<l} q^{2l-2i} z_{kk}z_{kl}v.
\]

Combining this with the result in the case \( n = k \) one gets

\[
z_{ll}w = \lambda_l w + (1 - q^{-2}) \lambda_k w.
\]

The second property follows at once from the second and third relation of A.3

**Step 1b:** In analogy to Step 1a consider the following ordering on the generators \( z_{ij}, i > j \) of \( V_- \):

\[
z_{ij} < z_{kl} \iff \text{ } (i > k) \text{ or } (i = k \text{ and } j > l).
\]

Let \( v \) denote a common eigenvector of all \( z_{ii} \), i.e.

\[
z_{nn}v = \lambda_n v \quad \forall n = 1 \ldots N
\]

such that

\[
z_{ij}v = 0 \quad \forall z_{ij} < z_{kl}.
\]

Then \( w_0 = z_{kl}v \) satisfies

1. \( z_{nn}w = \mu_n w \) where

\[
\mu_n = \begin{cases} 
q^2 \lambda_k + (1 - q^2)q^{2k-2N-1} & \text{if } n = k, \\
\lambda_i + (1 - q^2) \lambda_k + (q^2 - 1)q^{2k-2N-1} & \text{if } n = l, \\
\lambda_n & \text{else.}
\end{cases}
\]  

(4.5)

2. \( z_{ij}w = 0 \) \( \forall z_{ij} < z_{kl} \).

As for the case of \( V_+ \) this implies the existence of a common eigenvector \( v_- \) of all \( z_{ii} \), \( i = 1, \ldots, N \) such that \( V_- v_- = 0 \). Such a common eigenvector will be called a minimal eigenvector.
Proof of Step 1b: As above the value of $\mu_n$ can be computed by means of the list in Appendix A. Consider again the case $n = l$. Then by the second relation in A.3 and the projector property (2.4) one has

\[
z_{ll} z_{kl} v = q^2 z_{kl} z_{ll} v + (q^2 - 1) \left[ q^{2l-2N-1} z_{kl} - \sum_{i \geq l} q^{2l-2i} z_{ki} z_{il} \right] v
\]

\[
= q^2 z_{kl} z_{ll} v + (q^2 - 1) q^{2l-2N-1} z_{kl} v - (q^2 - 1) z_{kl} z_{ll} v - q \sum_{i > l} q^{2l-2i} z_{ki} z_{il} v.
\]

By assumption and by means of A.9.1 one obtains

\[
\sum_{i=l+1}^{k-1} q^{2l-2i} z_{ki} z_{il} v = (1 - q^{2(l-k+1)})(q^{-2} z_{kk} w - q^{2k-2N-3} \hat{w}).
\]

Combination with the result in the case $n = k$ yields the desired expression.

Step 2a: Let $v_+$ denote a maximal eigenvector. Then there exists a subset $\{i_1 < i_2 < \cdots < i_s\} \subset \{1, \ldots, N\}$ such that for all $k$ the eigenvalue $\lambda_k$ of $z_{kk}$ is given by

\[
\lambda_k = \begin{cases} q^{2l-2s-1} & \text{if } k = i_l \text{ for some } l \\ 0 & \text{else.} \end{cases}
\]

Proof of Step 2a: It follows from (2.4) and A.6.1 that

\[
z_{kk} v_+ = q^{-1} z_{kk} w - q^{2k-2N-3} \hat{w}.
\]

Thus $\lambda_k = 0$ or $q \lambda_k + q \sum_{j > k} \lambda_j = 1$. Therefore there exists a subset $\{i_1 < i_2 < \cdots < i_n\} \subset \{1, \ldots, N\}$ such that

\[
\lambda_k = \begin{cases} q^{2l-2n-1} & \text{if } k = i_l \text{ for some } l \\ 0 & \text{else.} \end{cases}
\]

The relation (2.3) implies $\sum_{k=1}^N \lambda_k = (1 - q^{-2s})/(q - q^{-1})$ which leads to $n = s$ as $q$ is not a root of unity.

Step 2b: Let $v_-$ denote a minimal eigenvector. Then there exists a subset $\{i_1 < i_2 < \cdots < i_t\} \subset \{1, \ldots, N\}$ such that for all $k$ the eigenvalue $\lambda_k$ of $z_{kk}$ is given by $\lambda_k = q^{2k-2N-1} - \lambda'_k$ where

\[
\lambda'_k = \begin{cases} q^{2l-2N-1} & \text{if } k = i_l \text{ for some } l \\ 0 & \text{else.} \end{cases}
\]
Proof of Step 2b: It follows from (2.4) and A.7.2 that
\[ z_{kk}v_- = \sum_{j \geq k} q^{2N+1-2j} z_{kj} z_{jk} v_- \]
\[ = q^{2N+1-2k} \lambda_k^2 v_- + \hat{q} \sum_{j < k} q^{2N+1-2j} \left[ -q^{-1} \lambda_j \lambda_k + q^{2k-2j-1} \lambda_j^2 \right. \]
\[ + \left. \hat{q} \sum_{i < j} \lambda_i \lambda_k - q^{2k-2j} \hat{q} \sum_{i < j} \lambda_i \lambda_j \right] v_- . \]
This implies
\[ q^2 \lambda_k - \lambda_{k+1} = q^{2N-2k-1} (q^2 \lambda_k - \lambda_{k+1}) \left( \lambda_{k+1} + \lambda_k - \hat{q} \sum_{j < k} \lambda_j \right) . \]

Now the result follows in analogy to the proof of Step 2a.

Step 3: Using Steps 1 and 2 we will now prove the claim of the Lemma. Let \( v_- \) denote a minimal eigenvector with eigenvalues \( \lambda_i \) and pick \( k \) minimal such that \( \lambda_k \neq 0 \). Then by Step 2b we have \( k \leq r + 1 \) and \( \lambda_k = q^{2k-2N-1} \). Multiplication with suitable elements of \( V_+ \) transforms \( v_- \) into a maximal eigenvector \( v_+ \). By (4.3)
\[ z_{kk}v_+ = q^{2k-2l-2N-1} v_+ \]
where \( l \geq 0 \). Further by Step 2a
\[ z_{kk}v_+ = q^{-2m+1} v_+ \]
where \( m \leq s \). Comparison of the exponents yields \( k = r + 1 \) as \( q \) is not a root of unity. This implies in particular that
\[ z_{kk}v_- = \begin{cases} 
q^{2k-2N-1} v_- & \text{if } k \geq r + 1 \\
0 & \text{else.}
\end{cases} \]

Let now \( v \) be an arbitrary common eigenvector with eigenvalues \( \nu_k \) of \( z_{kk} \), \( k = 1, \ldots, N \). Multiplication with suitable elements of \( V_- \) transforms \( v \) into a minimal vector \( v_- \) with eigenvalues \( \lambda_k \). By the above considerations \( \lambda_k \) is given by (4.6). Note that the tuple \( (\lambda_1, \ldots, \lambda_N) \) is invariant under the invertible transformation (4.5). This implies \( \nu_k = \lambda_k \) and therefore all common eigenvectors correspond to the same set of eigenvalues independently of the representation. Hence for any common eigenvector \( v \) of \( V_0 \) one obtains \( (z_{ii} - \varepsilon(z_{ii})) v = 0 \).

5. Classification

Lemma 5.1. Let \( I \subset O_q(\text{Gr}(r,N)) \) denote the ideal generated by
\[ \{ z_{kl} \mid k \leq r \text{ or } l \leq r \} . \]
Then \( I = B^+ \).

Proof. By Proposition 2.3 in [Kol01] \( I \) is equal to the kernel of the projection
\[ \pi : B = O_q(\text{Gr}(r,N)) \to \mathbb{C} \]
induced by the surjective Hopf algebra homomorphism
\[ O_q(SU(N)) \to O_q(SU(N-r)) \]
\[ u^i_j \mapsto \begin{cases} 
  \epsilon(u^i_j) & \text{if } i \leq r \text{ or } j \leq r, \\
  u^{i-r}_{j-r} & \text{else.}
\end{cases} \]

Thus \( \mathcal{I} \subset \mathcal{B}^+ \) has codimension 1 in \( \mathcal{B} \) and therefore \( \mathcal{I} = \mathcal{B}^+ \).

**Lemma 5.2.** Let \( k \in \mathbb{N} \) and \( f \in \mathcal{B}^o \) be a functional such that \( f(x) = 0 \) for all monomials

\[ x = x_{-1} \cdots x_{-k} x^+_0 \cdots x^+_{0,k_0} x^+_{+1} \cdots x^+_{+,k_+} \in V_\times \otimes V^+_0 \otimes V^+ \]

where \( k_- + k_0 + k_+ \geq k \) and \( x_{a,l} \) denote generators of \( V_\alpha \). Then \( f|_{(B^+)_{2k}} = 0 \).

**Proof.** Take \( y \in (B^+)_{2k} \). By Lemma 5.1 the element \( y \) can be written as a linear combination of monomials in the generators \( z_{ij} \) with at least 2\( k \) indices smaller than \( r + 1 \). If we write these monomials in standard form (5.1) using the relations (2.2) the number of such indices will not decrease. Hence the appearing monomials have at least \( k \) factors \( z_{ij} = z^+_{ij} \) with \( i \leq r \) or \( j \leq r \).

Recall that \( U = U/K+U \) can be considered as a subset of \( \mathcal{B}^o \).

**Lemma 5.3.** Let \( T \) denote the quantum tangent space of a covariant first order differential calculus over \( \mathcal{B} \). Then \( T \subset \overline{U} \).

**Proof.** By Corollary 4.2 \( T^c = T \oplus Cx \subset \mathcal{B}^o \) is a right \( K \)-module and a left \( \mathcal{B}^o \)-coideal. Consider the coalgebra \( T^c \) defined above Lemma 4.2. It is generated by functionals \( a_{ij} \in \mathcal{B}^o \) which are matrix coefficients of a finite dimensional representation of \( \mathcal{B} \). By Corollary 4.3 and Lemma 4.4 there exists \( k \in \mathbb{N} \) such that \( a_{ij}(x) = 0 \) for all monomials

\[ x = x_{-1} \cdots x_{-k} x^+_0 \cdots x^+_{0,k_0} x^+_{+1} \cdots x^+_{+,k_+} \in V_\times \otimes V^+_0 \otimes V^+ \]

where \( k_- + k_0 + k_+ \geq k \) and \( x_{a,l} \) denote generators of \( V_\alpha \). Therefore by Lemma 5.2 one has \( a_{ij}|_{(B^+)_{2k}} = 0 \). Corollary 5.8 implies \( a_{ij} \in \overline{U}_{2k-1} \). As \( T^c \subset T^c \) the claim of the lemma follows.

**Theorem 5.4.** Any covariant first order differential calculus \( \Gamma \) over \( O_q(Gr(r,N)) \) is uniquely determined by its quantum tangent space \( T(\Gamma) \). If \( \dim \Gamma \leq 2r(N-r) \) then \( T(\Gamma) \) belongs to the following list.

1. For any \( N, r \): \( T_0 = \{0\} \), \( \dim T(\Gamma) = 0 \)

\[ T_+ = \text{Lin}\{E_{\beta_i} \mid \beta_i \in \Phi_+^\times\}, \quad \Gamma(T_+) = r(N-r) \]

\[ T_- = \text{Lin}\{F_{\beta_i} \mid \beta_i \in \Phi_-^\times\}, \quad \Gamma(T_-) = r(N-r) \]

\[ T = T_- \oplus T_+, \quad \dim T(\Gamma) = 2r(N-r) \]

2. In addition

if \( N=2, r=1 \):

\[ T_{1,+} = \text{Lin}\{E_1, E^2_1\}, \quad T_{1,-} = \text{Lin}\{F_1, F^2_1\}, \]

if \( 4 \leq N \leq 7, 2 \leq r \leq N-2 \):

\[ T_{2,+} = T_+ \oplus V_+, \quad T_{2,-} = T_- \oplus V_- \]

where \( V_\pm \) denotes the \( K \)-invariant \( r(r-1)(N-r)(N-r-1)/4 \)-dimensional subspace of \( (U^\mp)_2 \).
Choose a multiindex \( \mathbf{k} \in \mathbb{Z}_u \) that the right \( K \)-module \( E_r K \subset P(\mathbb{U}_+) \) is \( r(N-r) \)-dimensional. By Lemma 3.3 \( \dim P(\mathbb{U}_+) = r(N-r) \) and therefore \( E_r K = P(\mathbb{U}_+) \). Now \( u \in T(\Gamma) \) and \( T(\Gamma) K \subset T(\Gamma) \) implies \( P(\mathbb{U}_+) \subset T(\Gamma) \).

Note that \( T(\Gamma) \) is \( \mathbb{Z}^{N-1} \)-graded as it is invariant under the action of all \( K_i, i = 1, \ldots, N - 1 \).

**Step 2:** Assume that there exist \( u_+ \in \mathbb{U}_+ \cap T(\Gamma) \) and \( u_- \in \mathbb{U}_- \cap T(\Gamma), u_\pm \neq 0 \) and \( \dim T(\Gamma) \leq 2r(N-r) \). Then \( T(\Gamma) = T(\mathbb{U}) \).

**Proof of Step 2:** By Step 1 \( P(\mathbb{U}_+) \subset T(\Gamma) \) and \( P(\mathbb{U}_-) \subset T(\Gamma) \). Since \( \dim P(\mathbb{U}_+) = r(N-r) \) the assertion follows.

**Step 3:** Suppose that \( u = \sum \lambda_{n_1 \ldots n_m} m_1 \ldots m_3 \prod (E_{\beta_i}^{k_i}) \prod (E_{\beta_i}^{n_i}) = 0 \) for all \( \beta_i \in \mathbb{U}_+ \cap T(\Gamma) \) and \( u \neq 0 \). Pick \( (k_1, \ldots, k_3) \) such that \( \lambda_{k_1 \ldots k_3} m_1 \ldots m_3 \neq 0 \). Write \( \Delta u = \mathbb{U}_- \cap T(\Gamma) \) with respect to the basis given in Proposition 3.2 in the first tensor factor. The second tensor factor corresponding to \( \prod (E_{\beta_i}^{k_i}) \prod (E_{\beta_i}^{n_i}) \neq 0 \) given by

\[
\sum_{(m_1 \ldots m_3)} \lambda_{k_1 \ldots k_3} m_1 \ldots m_3 \prod (E_{\beta_i}^{n_i}) \neq 0
\]

as \( u \) is homogeneous with respect to the \( \mathbb{Z}^{N-1} \)-grading. Therefore \( u_+ \in \mathbb{U}_+ \cap T(\Gamma) \neq \{0\} \). Similarly one obtains that \( \mathbb{U}_- \cap T(\Gamma) \neq \{0\} \). Now Step 2 and \( u \not\in \mathbb{U}_+ \cap \mathbb{U}_- \) imply the claim.

**Step 4:** By Steps 2 and 3 it remains to consider the cases where \( T(\Gamma) \subset \mathbb{U}_+ \) or \( T(\Gamma) \subset \mathbb{U}_- \). Consider the case \( T(\Gamma) \subset \mathbb{U}_+ \). Recall from the proof of Lemma 3.3 that the right \( K \)-module \( (\mathbb{U}_+)_k \) is dual to the left \( K \)-module \( \mathcal{O}_q(\text{Mat}(r, N-r))_k \) of homogeneous elements of degree \( k \) in \( \mathcal{O}_q(\text{Mat}(r, N-r)) \).

By Step 1 one has \( P(\mathbb{U}_+) \subset T(\Gamma) \). In what follows assume that there exists \( u \in T(\Gamma) \cap (\mathbb{U}_+)_k \) for some \( k \geq 2 \). Then the coproduct of \( u \) can be written as in (3.2). If no summands in \( (\mathbb{U}_+)_{l} \cap (\mathbb{U}_+)_{k-l} \) occur for some \( l \in \{1, \ldots, k-1\} \) then \( u(x) = 0 \) for all \( x \in \mathcal{O}_q(\text{Mat}(r, N-r))_k \). This is a contradiction to the duality between \( \mathbb{U}_+ \) and \( \mathcal{O}_q(\text{Mat}(r, N-r)) \). Thus for each \( l = 1, \ldots, k \) there exists a nonzero \( u_l \in T(\Gamma) \cap (\mathbb{U}_+)_{l} \).
If $r = 1$ or $r = N - 1$ then $\mathcal{O}_q(\text{Mat}(r, N - r))_2$ is an irreducible $K$-module of dimension $N(N - 1)/2$. Thus $\dim(\mathcal{U}_+^1) + \dim(\mathcal{U}_+^2) \leq 2(N - 1)$ if and only if $N = 2$. This proves the theorem if $r = 1$ or $r = N - 1$.

If $2 \leq r \leq N - 2$ then $\mathcal{O}_q(\text{Mat}(r, N - r))_2 = V_1 \oplus V_2$ is the direct sum of two irreducible $K$-modules of dimensions

$$\dim V_1 = r(r + 1)(N - r)(N - r + 1)/4, \quad \dim V_2 = r(r - 1)(N - r)(N - r - 1)/4.$$ Since $\dim V_1 > r(N - r)$ the component $V_1$ cannot be a subspace of $T(\Gamma)$. Moreover $\dim V_2 \leq r(N - r)$ if and only if $4 \leq N \leq 6$ or $N = 7, r = 2$. In the case $N = 6, r = 3$ one has $\dim V_2 = r(N - r)$. Let $V_+ \subset (\mathcal{U}_+^2)$ denote the $K$-module dual to $V_2$. Then if $u \in (\mathcal{U}_+^2)_k$, $k \geq 2$ and $\dim T(\Gamma) \leq 2r(N - r)$ one has $4 \leq N \leq 7$ and $T_{2, +} = (\mathcal{U}_+^1) \oplus V_+ \subset T(\Gamma)$. In particular if $N = 6, r = 3$ then $T(\Gamma) = T_{2, +}$.

It remains to consider the cases $r = 2$ and $r = N - 2$. Since $\mathcal{O}_q(\text{Mat}(r, N - r))_3 = V_{1, 2}^1 \oplus V_{1, 2}^2$ where $V_{1, 2}^1$ are the irreducible $K$-modules of dimensions

$$\dim V_{1, 2}^1 = 2(N - 2)(N - 1)/3, \quad \dim V_{1, 2}^2 = 2(N - 2)(N - 1)(N - 3)/3$$ one obtains

$$\dim V_{1, 2}^i + \dim(\mathcal{U}_+^1) + \dim V_+ > 4(N - 2).$$ Therefore $k \geq 3$ would imply $\dim T(\Gamma) > 2r(N - r)$ and hence $T(\Gamma) = T_{2, +}$. \hfill $\square$

In [Kol01] two differential calculi $\Gamma_+$ and $\Gamma_-$ of dimension $0 < \dim \Gamma_\pm \leq r(N - r)$ were constructed (Prop. 3.1 in [Kol01]). Theorem 5.4 implies in particular that these calculi coincide with $\Gamma(T_\pm)$ and $\Gamma(T_{-\pm})$. Therefore the differential calculus corresponding to the quantum tangent space $T = T_\pm \oplus T_{-\pm}$ is isomorphic to $\Gamma_q(\text{Gr}(r, N)) = \Gamma_+ \oplus \Gamma_-$. 

**Appendix A. Relations of $\mathcal{O}_q(\text{Gr}(r, N))$**

For notational convenience set $\hat{q} = q - q^{-1}$.

1. $V_0$ is commutative
   - $b = d < a = c$: $z_{aa} z_{dd} = z_{dd} z_{aa}$

2. $V_0 \otimes V_+$
   - $b = d < c < a$: $z_{bb} z_{ca} = z_{ca} z_{bb}$
   - $b = d = c < a$: $z_{bb} z_{ba} = q^{-2} z_{ba} z_{bb} - q^{-1} \hat{q} \sum_{i < b} q^{2b - 2i} z_{ba} z_{ia}$
   - $c < b = d < a$: $z_{bb} z_{ca} = z_{ca} z_{bb} - q \hat{q} z_{cb} z_{ba} + \hat{q}^2 \sum_{i < b} q^{2b - 2i} z_{cb} z_{ia}$
   - $c < b = d = a$: $z_{aa} z_{ca} = z_{ca} z_{aa} + q^{-1} \hat{q} \sum_{i < a} q^{2a - 2i} z_{ca} z_{ia}$
   - $c < a < b = d$: $z_{bb} z_{ca} = z_{ca} z_{bb}$

3. $V_0 \otimes V_-$
   - $b = d < a < c$: $z_{bb} z_{ca} = z_{ca} z_{bb}$
   - $b = d = a < c$: $z_{bb} z_{cb} = q^2 z_{cb} z_{bb} + q \hat{q} \sum_{i < b} q^{2b - 2i} z_{cb} z_{ib}$
   - $a < b = d < c$: $z_{bb} z_{ca} = z_{ca} z_{bb} + q \hat{q} z_{cb} z_{ba} + \hat{q}^2 \sum_{i < b} q^{2b - 2i} z_{cb} z_{ia}$
   - $a < b = d = c$: $z_{bb} z_{ba} = z_{ba} z_{bb} - q^{-1} \hat{q} \sum_{i < b} q^{2b - 2i} z_{bi} z_{ia}$
   - $a < c < b = d$: $z_{bb} z_{ca} = z_{ca} z_{bb}$

4. $V_+ \otimes V_-\vphantom{_0}$
   - $d < b < a < c$: $z_{db} z_{ca} = z_{ca} z_{db}$
   - $d < b = a < c$: $z_{db} z_{ca} = q z_{ca} z_{da}$
   - $d < a < b < c$: $z_{db} z_{ca} = z_{ca} z_{db} + \hat{q} z_{cb} z_{da}$
\[ d = a < b < c : \ z_{ab}z_{ca} = qz_{ca}z_{ab} + \hat{q}z_{cb}z_{da} + \hat{q}z_{db}z_{aa} + \hat{q}z_{dc}z_{db} + q^{2a-2i}z_{ci}z_{ib} \]

\[ a < d < b < c : \ z_{db}z_{ca} = z_{ca}z_{db} + \hat{q}z_{cb}z_{da} \]

\[ d < a < b < c : \ z_{db}z_{ba} = q^{-1}z_{ba}z_{db} + q^{-1}\hat{q}z_{bb}z_{da} - q^{-1}\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ d = a < b = c : \ z_{ab}z_{ba} = z_{ba}z_{ab} + q^{-1}\hat{q}z_{cb}z_{aa} + q^{-1}\hat{q}z_{db}z_{a} + q^{-1}q^{2a-2j}z_{bj}z_{jb} - q^{-1}\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ a < d < b < c : \ z_{db}z_{ba} = q^{-1}z_{ba}z_{db} + q^{-1}\hat{q}z_{bb}z_{da} - q^{-1}\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ d < a < c < b : \ z_{db}z_{ca} = z_{ca}z_{db} + \hat{q}z_{da}z_{cb} \]

\[ a < c < b : \ z_{cb}z_{ca} = qz_{ca}z_{ab} + \hat{q}z_{ca}z_{cb} + \hat{q}z_{db}z_{a} + q^{2a-2j}z_{ci}z_{ib} \]

\[ a < d < c < b : \ z_{db}z_{ca} = z_{ca}z_{db} + \hat{q}z_{da}z_{cb} \]

\[ a < c < d < b : \ z_{cb}z_{ca} = qz_{ca}z_{cb} \]

\[ a < c < d < b : \ z_{db}z_{ca} = z_{ca}z_{db} \]

5. \( V_+ \otimes V_+ \)

\[ d < b = c < a : \ z_{db}z_{ba} = q^{-1}z_{ba}z_{db} - q^{-1}\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} - q\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

By induction from 5.1

\[ d < b = c < a : \ z_{db}z_{ba} = q^{-1}z_{ba}z_{db} - q^{b-1}q^{2b-2j}z_{aj}z_{ja} - q\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

By induction from 4.7

\[ d = a < b = c : \ z_{ab}z_{ca} = z_{ba}z_{ab} + q^{-1}\hat{q}z_{ca}z_{db} - q^{2b-2j}z_{aj}z_{ja} - q\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ d = a < b < c : \ z_{ab}z_{ca} = z_{ba}z_{ab} + q^{-1}\hat{q}z_{ca}z_{db} - q^{2b-2j}z_{aj}z_{ja} - q\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ d = a < b < c : \ z_{ab}z_{ca} = z_{ba}z_{ab} + q^{-1}\hat{q}z_{ca}z_{db} - q^{2b-2j}z_{aj}z_{ja} - q\hat{q}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

V. \( V_+ \otimes V_+ \)

\[ d > b = c > a : \ z_{db}z_{ba} = q^{-1}z_{ba}z_{db} - q^{-1}\hat{q}z_{bb}z_{da} + q^{-1}z_{db}z_{a} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

\[ d = c > b > a : \ z_{db}z_{da} = qz_{db}z_{db} \]

\[ b < d > c > a : \ q^{b}z_{db}z_{ca} = q^{-1}z_{ba}z_{ab} - \hat{q}z_{ca}z_{db} + q^{-1}q^{2b-2j}z_{aj}z_{ja} - q^{-1}q^{2b-2j}z_{aj}z_{ja} \]

where \((a < b) = 1\) if \(a < b\) and \((a < b) = 0\) else.

6. By induction from 8.1

\[ d > b = c > a : \ z_{db}z_{ba} = qz_{ba}z_{db} - q^{-1}q^{2d-2N-1}z_{da} + \hat{q}z_{db}z_{a} - q^{-1}q^{2d-2j}z_{aj}z_{ja} \]

\[ d > b = c > a : \ z_{db}z_{ba} = qz_{ba}z_{db} - q^{-1}q^{2d-2N-1}z_{da} + \hat{q}z_{db}z_{a} - q^{-1}q^{2d-2j}z_{aj}z_{ja} \]

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