Global existence and non-existence of stochastic parabolic equations

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February 21, 2019

Abstract

This paper is concerned with the blowup phenomenon of stochastic parabolic equations both on bounded domain and in the whole space. We introduce a new method to study the blowup phenomenon on bounded domain. Comparing with the existing results, we delete the assumption that the solutions to stochastic heat equations are non-negative. Then the blowup phenomenon in the whole space is obtained by using the properties of heat kernel. We obtain that the solutions will blow up in finite time for nontrivial initial data.

Keywords: Itô’s formula; Blowup; Stochastic heat equation; Impact of noise.
AMS subject classifications (2010): 35K20, 60H15, 60H40.

1 Introduction

For deterministic partial differential equations, finite time blowup phenomenon has been studied by many authors, see the book [13]. There are two cases to study this problem. One is bounded domain and the other is whole space. On the bounded domain, the $L^p$-norm of solutions ($p > 1$) will blow up in finite time. The methods used for bounded domain include: Kaplan’s first eigenvalue method, concavity method and comparison method, see Chapter 5 of [13]. The main result is the following: under the assumptions that the initial data is suitable large and that the nonlinear term $f(u)$ satisfies $f(u) \geq u^{1+\alpha}$ with $\alpha > 0$, the solution of $u_t - \Delta u = f(u)$ with Dirichlet boundary condition will blow up in finite time.

For the whole space, the following ”Fujita Phenomenon” has been attraction in the literature. Consider the following Cauchy problem

\[
\begin{cases}
  u_t = \Delta u + u^p, & x \in \mathbb{R}^d, \quad t > 0, \quad p > 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^d.
\end{cases}
\]

(1.1)

It has been proved that:

(i) if $0 < p < 1$, then every nonnegative solution is global, but not necessarily unique;
(ii) if $1 < p \leq 1 + \frac{2}{d}$, then any nontrivial, nonnegative solution blows up in finite time;
(iii) if $p > 1 + \frac{2}{d}$, then $u_0 \in \mathcal{U}$ implies that $u(t, x, u_0)$ exists globally;
(iv) if $p > 1 + \frac{2}{d}$, then $u_0 \in \mathcal{U}_\infty$ implies that $u(t, x, u_0)$ blows up in finite time,
where \( \mathcal{U} \) and \( \mathcal{U}_\infty \) are defined as follows

\[
\mathcal{U} = \left\{ v(x) | v(x) \in BC(\mathbb{R}^d, \mathbb{R}_+), v(x) \leq e^{-k|x|^2}, \ k > 0, \delta = \delta(k) > 0 \right\},
\]

\[
\mathcal{U}_\infty = \left\{ v(x) | v(x) \in BC(\mathbb{R}^d, \mathbb{R}_+), v(x) \geq ce^{-k|x|^2}, \ k > 0, \ c \geq 1 \right\}.
\]

Here \( BC = \{ \text{bounded and uniformly continuous functions} \} \), see Fujita [11, 12] and Hayakawa [14].

It is easy to see that for the whole space, there are four types of behaviors for problem (1.1), namely, (1) global existence unconditionally but uniqueness fails in certain solutions, (2) global existence with restricted initial data, (3) blowing up unconditionally, and (4) blowing up with restricted initial data. The occurrence of these behaviors depends on the combination effect of the nonlinearity represented by the parameter \( p \), the size of the initial datum \( u_0(x) \), represented by the choice of \( \mathcal{U} \) or \( \mathcal{U}_\infty \), and the dimension of the space.

Now, we recall some known results of stochastic partial differential equations (SPDEs). In this paper, we only focus on the stochastic parabolic equations. It is known that the existence and uniqueness of global solutions to SPDEs can be established under appropriate conditions ([2, 7, 16, 17, 30]). For the finite time blowup phenomenon of stochastic parabolic equations, we first consider the case on bounded domain. Consider the following equation

\[
\begin{cases}
    du = (\Delta u + f(u))dt + \sigma(u)dW_t, & t > 0, \quad x \in D, \\
    u(x, 0) = u_0(x) \geq 0, & x \in D, \\
    u(x, t) = 0, & t > 0, \quad x \in \partial D.
\end{cases}
\]

Da Prato-Zabczyk [26] considered the existence of global solutions of (1.2) with additive noise (\( \sigma \) is constant). Manthey-Zausinger [20] considered (1.2), where \( \sigma \) satisfied the global Lipschitz condition. Dozzi and López-Mimbela [8] studied equation (1.2) with \( \sigma(u) = u \) and proved that if \( f(u) \geq u^{1+\alpha} \) (\( \alpha > 0 \)) and initial data is large enough, the solution will blow up in finite time, and that if \( f(u) \leq u^{1+\beta} \) (\( \beta \) is a certain positive constant) and the initial data is small enough, the solution will exist globally, also see [23]. A natural question arises: If \( \sigma \) does not satisfy the global Lipschitz condition, what can we say about the solution? Will it blow up in finite time or exist globally? Chow [3, 4] answered part of this question. Lv-Duan [18] described the competition between the nonlinear term and noise term for equation (1.2). Bao-Yuan [1] and Li et al. [14] obtained the existence of local solutions of (1.2) with jump process and Lévy process, respectively.

For blowup phenomenon of stochastic functional parabolic equations, see [31, 19] for details. In a somewhat different case, Mueller [21] and, later, Mueller-Sowers [22] investigated the problem of a noise-induced explosion for a special case of equation (1.2), where \( f(u) \equiv 0, \sigma(u) = u^\gamma \) with \( \gamma > 0 \) and \( W(x, t) \) is a space-time white noise. It was shown that the solution will explode in finite time with positive probability for some \( \gamma > 3/2 \).

We remark that the method used to prove the finite time blowup on bounded domain is the stochastic Kaplan’s first eigenvalue method. In order to make sure the inner product \( (u, \phi) \) is positive, the authors firstly proved the solutions of (1.2) keep positive under some assumptions, see [1, 3, 4, 15, 18]. We find that under some special case the positivity of solution can be deleted. What’s more, in present paper, we will give a new method (stochastic concavity method) to prove the solutions blow up in finite time. The advantage of this method is that we need not the positivity of solution.

For the whole space, Foondun et al. [10] considered the finite time blowup phenomenon for the Cauchy problem of stochastic parabolic equations. Comparing with the deterministic parabolic equations, they only obtained the result similar to type (4). In this paper, we establish the similar results to types (1) and (3). The method used here is comparison principle and the properties of
heat kernel. We obtain some different phenomenon with or without noise. Moreover, many types of noise are considered.

Comparing with the results of deterministic partial differential equations, there are a lot of work to do and we will study this issue in our further paper.

This paper is arranged as follows. In Sections 2 and 3, we will consider the global existence and non-existence of stochastic parabolic equations on bounded domain and in the whole space, respectively. This paper ends with a short discussion in Section 4.

Throughout this paper, we write $C$ as a general positive constant and $C_i$, $i = 1, 2, \cdots$ as a concrete positive constant.

2 Bounded domain

In this section, we first recall some known results on bounded domain, and then give some non-trivial generalizations. Consider the following SPDE

\[
\begin{cases}
  du = (\Delta u + f(u, x, t))dt + \sigma(u, \nabla u, x, t)dW_t, & t > 0, \ x \in D, \\
  u(x, 0) = u_0(x), & x \in D, \\
  u(x, t) = 0, & t > 0, \ x \in \partial D,
\end{cases}
\]  

(2.1)

where $\sigma$ is a given function, and $W(x, t)$ is a Wiener random field defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t$. The Wiener random field has mean $\mathbb{E}W(x, t) = 0$ and its covariance function $q(x, y)$ is defined by

\[
\mathbb{E}W(x, t)W(y, s) = (t \wedge s)q(x, y), \quad x, y \in \mathbb{R}^n,
\]

where $(t \wedge s) = \min\{t, s\}$ for $0 \leq t, s \leq T$. The existence of strong solutions of (2.1) has been studied by many authors [2, 26]. To consider positive solutions, they start with the unique solution $u \in C(\bar{D} \times [0, T]) \cap L^2((0, T); H^2)$ for equation (2.1). Chow [3, 4] considered the finite time blowup problem of (2.1). They used the positivity of solution to prove the finite time blowup. Under the following conditions

(P1) There exists a constant $\delta \geq 0$ such that

\[
\frac{1}{2} q(x, x) \sigma^2(r, \xi, x, t) - \sum_{i,j=1}^{n} a_{ij}\xi_i \xi_j \leq \delta r^2
\]

for all $r \in \mathbb{R}, x \in \bar{D}, \xi \in \mathbb{R}^n$ and $t \in [0, T]$;

(P2) The function $f(r, x, t)$ is continuous on $\mathbb{R} \times \bar{D} \times [0, T]$ and such that $f(r, x, t) \geq 0$ for $r \leq 0$ and $x \in D, t \in [0, T]$; and

(P3) The initial datum $u_0(x)$ on $\bar{D}$ is positive and continuous,

**Proposition 2.1** [3, Theorem 3.3] Suppose that the conditions (P1),(P2) and (P3) hold true. Then the solution of the initial-boundary problem for the parabolic Itô’s equation (2.1) remains positive, i.e., $u(x, t) \geq 0$, a.s. for almost every $x \in D$ and for all $t \in [0, T]$.

Let $\phi$ be the eigenfunction with respect to the first eigenvalue $\lambda_1$ on the bounded domain, i.e.,

\[
\begin{cases}
  -\Delta \phi = \lambda_1 \phi, & \text{in } D, \\
  \phi = 0, & \text{on } \partial D.
\end{cases}
\]
And we normalize it in such a way that
\[ \phi(x) \geq 0, \quad \int_D \phi(x) dx = 1. \]

In paper [4], Chow assumed that the following conditions hold

(N1) There exist a continuous function \( F(r) \) and a constant \( r_1 > 0 \) such that \( F \) is positive, convex and strictly increasing for \( r \geq r_1 \) and satisfies
\[ f(r, x, t) \geq F(r) \]
for \( r \geq r_1, x \in \bar{D}, t \in [0, \infty); \)
(N2) There exists a constant \( M_1 > r_1 \) such that \( F(r) > \lambda_1 r \) for \( r \geq M_1; \)
(N3) The positive initial datum satisfies the condition
\[ (\phi, u_0) = \int_D u_0(x)\phi(x) dx > M_1; \]

(N4) The following condition holds
\[ \int_{M_1}^{\infty} \frac{dr}{F(r) - \lambda_1 r} < \infty. \]

Alternatively, he imposes the following conditions \( S \) on the noise term:

(S1) The correlation function \( q(x, y) \) is continuous and positive for \( x, y \in \bar{D} \) such that
\[ \int_D \int_D q(x, y)v(x)v(y) dx dy \geq q_1 \int_D v^2(x) dx \]
for any positive \( v \in H \) and for some \( q_1 > 0; \)
(S2) There exist a positive constant \( r_2, \) continuous functions \( \sigma_0(r) \) and \( G(r) \) such that they are both positive, convex and strictly increasing for \( r \geq r_2 \) and satisfy
\[ \sigma(r, x, t) \geq \sigma_0(r) \quad \text{and} \quad \sigma_0^2(r) \geq 2G(r^2) \]
for \( x \in \bar{D}, t \in [0, \infty); \)
(S3) There exists a constant \( M_2 > r_2 \) such that \( q_1 G(r) > \lambda_1 r \) for \( r \geq M_2; \)
(S4) The positive initial datum satisfies the condition
\[ (\phi, u_0) = \int_D u_0(x)\phi(x) dx > M_2; \]

(S5) The following integral is convergent so that
\[ \int_{M_2}^{\infty} \frac{dr}{q_1 G(r) - \lambda_1 r} < \infty. \]

**Proposition 2.2** [4] Theorem 3.1 Suppose the initial-boundary value problem (2.1) has a unique local solution and the conditions (P1)-(P3) are satisfied, where \( \sigma \) does not depend on \( \nabla u. \)

In addition, we assume that either the conditions (N1)-(N4) or the alternative conditions (S1)-(S5) given above hold true. Then, for a real number \( p > 0, \) there exists a constant \( T_p > 0 \) such that
\[ \lim_{t \to T_p-} \mathbb{E}\|u\|_p = \lim_{t \to T_p-} \mathbb{E} \left( \int_D |u(x, t)|^p dx \right)^{\frac{1}{p}} = \infty, \]
where \( p \geq 1 \) under conditions \( N, \) while \( p \geq 2 \) under conditions \( S. \)
The positivity of solutions is needed for the case that the nonlinear term induces the finite time blowup. But for a special case, we can prove the positivity of solutions can be deleted. Now, we consider the following SPDEs

\[
\begin{cases}
    du = \Delta u dt + \sigma(u, x, t) dW(x, t), & t > 0, \ x \in D, \\
    u(x, 0) = u_0(x), & x \in D, \\
    u(x, t) = 0, & t > 0, \ x \in \partial D,
\end{cases}
\]

where \( W(x, t) \) is time-space white noise and \( D \subset \mathbb{R} \) is an interval in \( \mathbb{R} \).

**Theorem 2.1** Assume that the initial-boundary problem (2.2) has a unique local solution. Assume further that \( C_1 |u|^\gamma \leq |\sigma(u, x, t)| \leq C_2 |u|^\gamma \) with \( C_1 > 0 \) and \( \gamma_1 \geq \gamma > 1 \), \( u_0 \geq 0 \) and

\[
\left( \int_D u_0(x) \phi(x) dx \right)^{2(\gamma - 1)} \geq \frac{\lambda_1}{q_1 C_1^2}.
\]

Then there exist constants \( T^* > 0 \) and \( p \geq 2 \gamma_1 \) such that

\[
\lim_{t \to T^*} \mathbb{E} \|u_t\|_{L^p}^p = \lim_{t \to T^*} \mathbb{E} \int_D |u(x, t)|^p dx = \infty.
\]

**Proof.** We will prove the theorem by contradiction. Suppose finite time blowup is false. Then there exist a global positive solution \( u \) and \( p \geq 2 \gamma_1 \) such that for any \( T > 0 \)

\[
\sup_{0 \leq t \leq T} \mathbb{E} \|u(\cdot, t)\|_{L^p}^p < \infty,
\]

which implies that

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left| \int_D u(x, t) \phi(x) dx \right|^2 \leq \|\phi\|_{L^p(D)}^2 \sup_{0 \leq t \leq T} \mathbb{E} \|u(\cdot, t)\|_{L^p}^p < \infty,
\]

where \( 1/p + 1/q = 1 \), \( \phi \) is defined as below Proposition 2.1 and satisfies \( \int_D \phi(x) dx = 1 \). Define

\[
\hat{u}(t) := \int_D u(x, t) \phi(x) dx.
\]

By applying Itô’s formula to \( \hat{u}^2(t) \), we get

\[
\hat{u}^2(t) = (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s) ds + 2 \int_0^t \int_D \hat{u}(s) \sigma(u, x, t) \phi(x) dW(x, s) dx + \int_0^t \int_D \sigma^2(u, x, s) \phi^2(x) dx ds,
\]

We note that the stochastic term is usually a local martingale. Thus we need use the technique of stopping time. Let

\[
\tau_n = \inf \{t \geq 0 : \int_0^t \int_D \sigma^2(u, x, s) \phi^2(x) dx ds \geq n\}.
\]

Let \( \eta(t \land \tau_n) = \mathbb{E} \hat{u}^2(t \land \tau_n) \). By taking an expectation over \( \mathbb{E} \), we obtain

\[
\eta(t \land \tau_n) = (u_0, \phi)^2 - 2\lambda_1 \int_0^{t \land \tau_n} \eta(s) ds + \int_0^{t \land \tau_n} \mathbb{E} \int_D \sigma^2(u, x, s) \phi^2(x) dx ds.
\]

Noting that

\[
\eta(t \land \tau_n) \leq (u_0, \phi)^2 + \int_0^t \mathbb{E} \int_D \sigma^2(u, x, s) \phi^2(x) dx ds,
\]
and letting \( n \to \infty \), we have

\[
\eta(t) = (u_0, \phi)^2 - 2\lambda_1 \int_0^t \eta(s)ds + \int_0^t \mathbb{E} \int_D \sigma^2(u, x, s)\phi^2(x)dxds.
\]

Using the assumptions \( \inf_{x,y\in D} q(x, y) \geq q_1 > 0 \) and \( \sigma^2(u, x, s) \geq C_1|u|^{2\gamma} \) with \( \gamma > 1 \) and Jensen’s inequality, we have

\[
\eta(t) \geq \eta(0) - 2\lambda_1 \int_0^t \eta(s)ds + 2q_1C_1^2 \int_0^t \eta^2(s)ds,
\]

or, in the differential form,

\[
\begin{cases}
\frac{d\eta(t)}{dt} = -2\lambda_1\eta(t) + 2q_1C_1^2\eta(t) \\
\eta(0) = \eta_0.
\end{cases}
\]

Noting that

\[
\eta(0) = \left( \int_D u_0(x)\phi(x)dx \right)^2 \geq \left( \frac{\lambda_1}{q_1C_1^2} \right) \frac{1}{\gamma-1},
\]

we have \( \eta'(0) \geq 0 \). This implies that \( \eta(t) > 0 \). An integration of the differential equation gives that

\[
T \leq \int_{\eta_0}^{\eta(T)} \frac{dr}{C_1^2q_1r^{\gamma} - \lambda_1r} \leq \int_{\eta_0}^{\infty} \frac{dr}{C_1^2q_1r^{\gamma} - \lambda_1r} < \infty,
\]

which implies \( \eta(t) \) must blow up at a time \( T^* \leq \int_{\eta_0}^{\infty} \frac{dr}{C_1^2q_1r^{\gamma} - \lambda_1r} \). Hence this is a contradiction. This completes the proof. \( \square \)

The advantage of Theorem 2.1 is that the positivity of the solution is not needed. And in above Theorem, we assume that the initial-boundary problem (2.2) has a unique local solution. In fact, if \( \sigma \) satisfies the local Lipschitz condition, one can follow the method of [30] to obtain the existence and uniqueness of local solution, also see [24]. In [24, 30], the authors established the existence and uniqueness of energy solution, where the solutions belong to \( H_0^1(D) \) for any fixed time almost surely. Noting that \( H^{\frac{2}{p}}(D) \hookrightarrow L^{\infty}(D) \) for \( D \subset \mathbb{R} \), our assumptions are valid.

If we only consider the case \( \sigma \) does not depend on \( \xi \), that is, \( \sigma := \sigma(u, x, t) \). Then it follows the assumption (P1) that \( \sigma(0, x, t) = 0 \), which implies that for additive noise, the solutions maybe not keep positive. Hence the first eigenvalue method will fail. Next, we introduce another method. For simplicity, we consider the following SPDEs

\[
\begin{cases}
du = [\nabla u + |u|^{p-1}u]dt + \sigma(x, t)dB_t, & t > 0, \ x \in D, \\
u(x, 0) = u_0(x), & x \in D, \\
u(x, t) = 0, & t > 0, \ x \in \partial D,
\end{cases}
\]

(2.4)

where \( B_t \) is an one-dimensional Brownian motion. If the initial data belongs to \( H^1(D) \), Debussche et al. [6] proved the solution of (2.4) belongs to \( H_0^3(D) \) during the lifespan.

**Theorem 2.2** Suppose that \( p > 1 \) and \( u_0 \) satisfies

\[
-\frac{1}{2} \int_D |\nabla u_0(x)|^2dx + \frac{1}{p+1} \int_D |u_0(x)|^{p+1}dx - \frac{1}{2} \int_0^{\infty} \mathbb{E} \int_D |\nabla \sigma(x, t)|^2dxdt > 0,
\]

then the solution of (2.4) must blow up in finite time in sense of mean square.
Proof. We will prove the theorem by contradiction. First we suppose there exist a global solution $u$ such that

$$\sup_{t \in [0,T]} E \int_D u^2 dx < \infty$$

for any $T > 0$. Similar to the proof of Theorem 2.1 by using Itô formula, we have

$$E \int_D u^2 - \int_D u_0^2 = -2E \int_0^t \int_D |\nabla u|^2 + 2E \int_0^t \int_D |u|^{p+1} + E \int_0^t \int_D |\sigma(x,s)|^2.$$

Denote

$$v(t) = E \int_D u^2, \; h(t) = E \int_D (-2|\nabla u|^2 + 2|u|^{p+1} + |\sigma(x,t)|^2),$$

then we have

$$v(t) - v(0) = \int_0^t h(s) ds.$$

Let

$$I(t) = \int_0^t v(s) ds + A, \; A \text{ is a positive constant},$$

then we have $I'(t) = v(t)$, $I''(t) = h(t)$. Set

$$J(t) = E \int_D \left( -\frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} |u|^{p+1} \right).$$

Itô formula implies that

$$\frac{1}{2} E \int_D |\nabla u|^2 - \frac{1}{2} E \int_D |\nabla u_0|^2 = - \int_0^t E \int_D \Delta u (\Delta u + |u|^{p-1}u) + \frac{1}{2} \int_0^t E \int_D |\nabla \sigma(x,t)|^2,$$

and

$$\frac{1}{p+1} E \int_D |u|^{p+1} - \frac{1}{p+1} E \int_D |u_0|^{p+1} = \int_0^t E \int_D |u|^{p-1}u (\Delta u + |u|^{p-1}u) + \frac{p}{2} \int_0^t E \int_D |u|^{p-1} \sigma^2(x,t).$$

Therefore, we have

$$J(t) = J(0) + \int_0^t E \int_D (\Delta u + |u|^{p-1}u)^2 - \frac{1}{2} \int_0^t E \int_D |\nabla \sigma(x,s)|^2 + \frac{p}{2} \int_0^t E \int_D |u|^{p-1} \sigma^2(x,s).$$

By comparing $I''(t)$ and $J(t)$, we have, for $1 < \delta < \frac{p+1}{2}$,

$$I''(t) = h(t) \geq 4(1 + \delta) J(t).$$

Clearly,

$$I'(t) = v(t) = v(0) + \int_0^t h(s) ds,$$

$$= v(0) + \int_0^t E \int_D |\sigma(x,t)|^2 + \int_0^t E \int_D (-2|\nabla u|^2 + 2|u|^{p+1}) dx ds$$

$$= v(0) + \int_0^t E \int_D |\sigma(x,t)|^2 + \int_0^t E \int_D (2u \Delta u + 2|u|^{p+1}) dx ds.$$
It follows that, for any $\varepsilon > 0$,
\[
I'(t)^2 \leq 4(1 + \varepsilon) \left[ \int_0^t E \int_D (\Delta u + |u|^{p-1}u)^2 \, dx \, ds \right] \left[ \int_0^t E \int_D u^2 \, dx \, ds \right] + \frac{1}{1 + \varepsilon} \left[ v(0) + \int_0^t E \int_D |\sigma(x,t)|^2 \, dx \, ds \right]^2.
\]
Combining the above estimates, we obtain
\[
I''(t)I(t) - (1 + \alpha)I'(t)^2 \geq 4(1 + \delta) \left[ J(0) + \int_0^t E \int_D (\Delta u + |u|^{p-1}u)^2 - \frac{1}{2} \int_0^t E \int_D |\nabla \sigma(x,s)|^2 \right.
+ \frac{P}{2} \int_0^t E \int_D |u|^{p-1} \sigma^2(x,s) \cdot (\int_0^t E \int_D u^2 \, dx \, ds + A) \bigg] - 4(1 + \alpha)(1 + \varepsilon) \left[ \int_0^t E \int_D (\Delta u + |u|^{p-1}u)^2 \, dx \, ds \right] \left[ \int_0^t E \int_D u^2 \, dx \, ds \right] - \frac{(1 + \alpha)}{1 + \varepsilon} \left[ v(0) + \int_0^t E \int_D |\sigma(x,t)|^2 \, dx \, ds \right]^2.
\]
Now we choose $\varepsilon$ and $\alpha$ small enough such that
\[
1 + \delta > (1 + \alpha)(1 + \varepsilon).
\]
By assumption,
\[
J(0) - \frac{1}{2} \int_0^t E \int_D |\nabla \sigma(x,s)|^2 > 0.
\]
We can choose $A$ large enough such that
\[
I''(t)I(t) - (1 + \alpha)I'(t)^2 > 0,
\]
which implies that
\[
\frac{d}{dt} \left( \frac{I'(t)}{I^{1+\alpha}(t)} \right) > 0.
\]
Then we have
\[
\frac{I'(t)}{I^{1+\alpha}(t)} > \frac{I'(0)}{I^{1+\alpha}(0)} \quad \text{for} \quad t > 0.
\]
It follows that $I(t)$ cannot remain finite for all $t$. This is a contradiction. The proof is complete. $\square$

**Remark 2.1** The advantage of concavity method is that we did not use the positivity of solutions. Meanwhile, the disadvantage of Theorem 2.2 is that we only deal with the additive noise. For multiplicative noise, when we deal with the term $E \int_D |\nabla u|^2$, by using Itô formula, we will have the term $-\frac{1}{2} \int_0^t E \int_D |\nabla \sigma(u)|^2$, and we cannot control this term.

**Remark 2.2** The effect of noise on the blowup problem can be described as the followings:

(i) for an additive noise, without help of the nonlinear term, the solutions will not blow up in finite time; but if the solutions blow up in finite time without noise, the additive noise can make the finite time blowup hard to happen. In other words, the assumption on initial data will be stronger if we add the additive noise.

(ii) for multiplicative noise, without the help of nonlinear term, the solutions blow up in finite time under some assumptions on initial data.
Look back at Proposition 2.2 and Theorems 2.1 and 2.2, we find the finite time blowup appear in the $L^p$-norm of the solutions, $p > 1$. Maybe we will ask what about the case $0 < p < 1$. The following result answer this equation. Consider the following stochastic parabolic equations

$$
\begin{aligned}
    &\left\{ \begin{array}{ll}
        du = [\Delta u + f(u)]dt + \sigma(u)dW(x,t), & t > 0, \ x \in D, \\
        u(x,0) = u_0(x), & x \in D, \\
        u(x,t) = 0, & t > 0, \ x \in \partial D.
    \end{array} \right.
\end{aligned}
$$

(2.5)

**Theorem 2.3** Assume $f(r) \geq 0$ for $r \leq 0$. Then we have:

(i) Assume further that $f(r) \geq C_0r^p$, $q(x,y) \leq q_0$ for $x, y \in D$ and $\sigma^2(u) \leq C_1u^2$. If the initial data satisfies

$$
\left( \int_D u_0(x)\phi(x)dx \right)^{p-1} > \frac{\lambda}{C_0^p}, \quad \dot{\lambda} = \epsilon \lambda_1 + \frac{\epsilon}{2}(1 - \epsilon)q_0C_1^2.
$$

then the solution $u(x,t)$ of (2.5) will blow up in finite time in $L^1$-norm and $\epsilon$-order moment, where $0 < \epsilon < 1$ and $p > 1$, i.e., there exists a positive $T > 0$ such that

$$
\mathbb{E}\|u(\cdot, t)\|_{L^1(D)} \to \infty, \quad \text{as} \quad t \to T;
$$

(ii) Assume further that $f(r) \leq C_0r^p$, $q(x,y) \geq q_1$ for $x, y \in D$ and $\frac{1}{C_1}u^m \leq \sigma^2(u) \leq C_2u^m$. Then, if $m > p > 1$, $(m - p)(2m - 1) > mp$ and the initial data are bounded, then the solution $u(x,t)$ of (2.5) will exist globally in the following sense: $\mathbb{E}[\|u(\cdot, t)\|_1^r] \to \infty$ for any $t > 0$.

**Proof.** (i) It follows from Proposition 2.4 that (2.5) has a unique positive solution. Similar to the proof of Theorem 2.1 we will prove the theorem by contradiction. Suppose the claim is false. Then there exists a global positive solution $u$ such that for any $T > 0$

$$
\sup_{0 \leq t \leq T} \mathbb{E}\|u(\cdot, t)\|_{L^1(D)}^r < \infty,
$$

which implies that

$$
\sup_{0 \leq t \leq T} \mathbb{E}\left( \int_D u(x,t)\phi(x)dx \right)^{\epsilon} \leq \|\phi\|_{L^\infty(D)} \mathbb{E}\sup_{0 \leq t \leq T}\|u(\cdot, t)\|_{L^1(D)} < \infty.
$$

Set $\hat{u} = (u, \phi)$. Itô formula gives that

$$
\begin{aligned}
    \hat{u}^\epsilon(t) &= (u_0, \phi)^\epsilon - \epsilon \lambda_1 \int_0^t \hat{u}^\epsilon(s)ds + \epsilon \int_0^t \hat{u}^\epsilon(s)^{\epsilon-1} \int_D f(u)\phi dx ds \\
    &\quad + \epsilon \int_0^t \int_D \hat{u}^\epsilon(s)^{-1}\sigma(u)\phi(x)dW(x,s)dx \\
    &\quad + \frac{\epsilon(\epsilon - 1)}{2} \int_0^t \int_D \int_D q(x,y)\sigma(u)\phi(x)\sigma(u)\phi(y)dxdyds
\end{aligned}
$$

(2.6)

Let $\eta(t) = \mathbb{E}\hat{u}^\epsilon(t)$. Similar to the proof of Theorem 2.1 by taking an expectation over (2.6), we obtain

$$
\begin{aligned}
    \eta(t) &= (u_0, \phi)^\epsilon - \epsilon \lambda_1 \int_0^t \eta(s)ds + \epsilon \int_0^t \mathbb{E}\hat{u}^\epsilon(s)^{\epsilon-1} \int_D f(u)\phi dx ds \\
    &\quad + \epsilon \int_0^t \mathbb{E}\hat{u}^\epsilon(s)^{-2} \int_D \int_D q(x,y)\sigma(u)\phi(x)\sigma(u)\phi(y)dxdyds
\end{aligned}
$$
Using the assumptions $\inf_{x,y \in D} q(x, y) \leq q_0$ and $\sigma^2(u) \leq C_1 |u|^2$ and Jensen’s inequality, we have

$$\eta(t) \geq \eta(0) - \varepsilon \lambda_1 \int_0^t \eta(s) ds + C_0 \varepsilon \int_0^t \frac{\eta^{\frac{p+1}{p}}(s)}{\lambda r} ds - \frac{\varepsilon}{2} (1 - \varepsilon) q_0 C_1^2 \int_0^t \eta(s) ds,$$

or, in the differential form,

$$\frac{d\eta(t)}{dt} = -\lambda \eta(t) + C_0 \varepsilon \eta^{\frac{p+1}{p}}(t) \quad \eta(0) = \eta_0.$$

Noting that $\eta'(0) > 0$. This implies that $\eta(t) > 0$. An integration of the differential equation gives that

$$T \leq \int_0^{\eta(T)} \frac{dr}{C_0 \varepsilon \eta^{\frac{p+1}{p}}(r)} \leq \int_0^\infty \frac{dr}{C_0 \varepsilon \eta^{\frac{p+1}{p}}(r)} < \infty,$$

which implies $\eta(t)$ must blow up at a time $T^* \leq \int_0^\infty \frac{dr}{C_0 \varepsilon \eta^{\frac{p+1}{p}}(r)}$. Hence this is a contradiction. Thus we obtain the desired result.

(ii) Define

$$\tau_n = \inf \{ t > 0, \ (u, \phi)^\varepsilon > n \}.$$

Set $\hat{u} = (u, \phi)$. By using Itô formula, for $t \leq \tau_n$, we have

$$\begin{align*}
\hat{u}(t) &= (u_0, \phi)^\varepsilon - \varepsilon \lambda_1 \int_0^t \hat{u}(s) ds + \varepsilon \int_0^t \hat{u}(s)^{\varepsilon-1} \int_D f(u, \phi) dx ds \\
&\quad + \varepsilon \int_0^t \int_D \hat{u}(s)^{\varepsilon-1} \sigma(u, \phi) dW(x, s) dx \\
&\quad + \frac{\varepsilon (\varepsilon - 1)}{2} \int_0^t \int_D q(x, y) \sigma(u, \phi) \sigma(u, \phi) dxdy ds.
\end{align*} \tag{2.7}$$

Let $\eta(t) = E \hat{u}(t)$. By taking an expectation over $\mathbb{P}$, we obtain

$$\begin{align*}
\eta(t) &= (u_0, \phi)^\varepsilon - \varepsilon \lambda_1 \int_0^t \eta(s) ds + \varepsilon \int_0^t E \hat{u}(s)^{\varepsilon-1} \int_D f(u, \phi) dx ds \\
&\quad + \frac{\varepsilon (\varepsilon - 1)}{2} \int_0^t E \hat{u}(s)^{\varepsilon-2} \int_D q(x, y) \sigma(u, \phi) \sigma(u, \phi) dxdy ds \\
&\quad \leq \eta(0) - \varepsilon \lambda_1 \int_0^t \eta(s) ds + C_0 \varepsilon \int_0^t E \hat{u}(s)^{\varepsilon-1} \int_D |u|^p dxdy ds \\
&\quad + \frac{\varepsilon (\varepsilon - 1)}{2} \int_0^t E \hat{u}(s)^{\varepsilon-2} \int_D q(x, y) \sigma(u, \phi) \sigma(u, \phi) dxdy ds. \tag{2.8}
\end{align*}$$

Hölder inequality and $\varepsilon$-Young inequality yield that

$$C_0 \varepsilon \hat{u}(s)^{\varepsilon-1} \int_D |u|^p dx \leq C_0 \varepsilon \hat{u}(s)^{\varepsilon-1} \left( \int_D |u|^m dx \right)^{\frac{p}{m}} \leq C_0 \varepsilon \hat{u}(s)^{\varepsilon-1} \left( \int_D |u|^m dx \right)^{\frac{p}{m}} + C \hat{u}(s)^{\frac{2m}{2m-p-1}} (2p-2m-1).$$
Submitting the above inequality into (2.8), and using the assumptions on \( \sigma \), we have

\[
\eta(t) \leq \eta(0) - \epsilon \lambda_1 \int_0^t \eta(s)ds + C \int_0^t u(s) \frac{2m}{2m-p}(2p-\rho-1+\epsilon)ds
- \int_0^t eq_1(1-\epsilon) \frac{2C_1}{u(s)\epsilon^2} \left( \int_D |u|^m \phi dx \right)^2 ds
\leq \eta(0) - \epsilon \lambda_1 \int_0^t \eta(s)ds + C \int_0^t u(s) \frac{2m}{2m-p}(2p-\rho-1+\epsilon)ds
- \int_0^t eq_1(1-\epsilon) \frac{2C_1}{u(s)\epsilon^2} u(s)^{2m+\epsilon-2} ds.
\] (2.9)

The assumption \((m-p)(2m-1) > mp\) gives
\[
\epsilon < \frac{2m}{2m-p}(2p-\rho-1+\epsilon) < 2m + \epsilon - 2.
\]

Noting that for any \( r < m < n \) and \( u > 0 \), we have
\[
u^m = u^\beta u^{m-\beta} \leq \epsilon u^n + C(\epsilon) u^r, \quad \beta = \frac{r(n-m)}{n-r}.
\] (2.10)

So we can use (2.10) to deal with the second last term of right hand side of (2.9). Eventually, we get for \( t \leq \tau_n \)

\[
\eta(t) \leq \eta(0) + C \int_0^t \eta(s)ds.
\]

We remark the constant \( C \) does not depend on \( t \). The Gronwall’s lemma implies that

\[
\eta(t) \leq C + Ce^{Ct}, \quad t \leq \tau_n.
\]

Letting \( n \to \infty \), the above inequality implies that \( \mathbb{P}\{\tau_\infty < \infty\} = 0 \). The proof is complete. \( \Box \)

3 Whole space

In this section, we consider stochastic parabolic equations in whole space. Our aim is to establish the global existence and non-existence under some assumptions. We first recall the results of Foondun et al. \[10\], where the authors considered the following equation

\[
\partial_t u(x) = \mathcal{L}u(x) + \sigma(u(x)) \dot{F}(x,t) \quad t > 0, \ x \in \mathbb{R}^d.
\] (3.1)

Here \( \mathcal{L} \) denotes the fractional Laplacian, the generator of an \( \alpha \)-stable process and \( \dot{F} \) is the random forcing term which they took to be white in time and possibly colored in space. They obtained the following results.

**Proposition 3.1** \[10\] *Theorems 1.2,1.5,1.6,1.8,1.9]*

(i) Noise white both in time and space, i.e.,

\[
\mathbb{E}[\dot{F}(x,t)\dot{F}(y,s)] = \delta_0(t-s)\delta_0(x-y).
\]

Assume that there exists a \( \gamma > 0 \) such that

\[
\sigma(x) \geq |x|^{1+\gamma} \quad \text{for all} \ x \in \mathbb{R}^d,
\]
and that there is a positive constant $\kappa$ such that $\inf_{x \in \mathbb{R}^d} := \kappa$. Then there exists a $t_0 > 0$ such that for all $x \in \mathbb{R}^d$, the solution $u_t(x)$ of (3.1) blows up in finite time, i.e.,
\[
\mathbb{E}|u_t(x)|^2 = \infty \quad \text{whenever} \quad t \geq t_0. \tag{3.2}
\]
Furthermore, the initial condition can be weaken as the following,
\[
\int_{B(0,1)} u_0(x)dx := K_{u_0} > 0, \tag{3.3}
\]
where $B(0,1)$ is the ball centred in the point 0 and radius 1. The solution $u_t(x)$ of (3.1) also blows up in finite time whenever $K_{u_0} \geq K$, where $K$ is some positive constant.

(ii) Noise white in time and correlated in space, i.e.,
\[
\mathbb{E}[\dot{F}(x,t)\dot{F}(y,s)] = \delta_0(t-s)f(x,y).
\]
Assume that for fixed $R > 0$, there exists some positive number $K_f$ such that
\[
\inf_{x,y \in B(0,R)} f(x,y) \geq K_f. \tag{3.4}
\]
Then, for fixed $t_0 > 0$ there exists a positive unumber $\kappa_0$ such that for all $\kappa \geq \kappa_0$ and $x \in \mathbb{R}^d$ we have (3.2) holds.

In particularly, suppose that the correlation function $f$ is given by
\[
f(x,y) = \frac{1}{|x-y|^\beta} \quad \text{with} \quad \beta < \alpha \wedge d.
\]
Then for $\kappa > 0$ there exists a $t_0 > 0$ such that (3.2) holds.

Furthermore, under the assumptions (3.3) and (3.4), there exists a $t_0 > 0$ such that for all $x \in \mathbb{R}^d$ (3.3) holds.

In the above proposition, Foondun et al. [10] only considered the finite time blowup phenomenon driven by noise. Our aim in this paper is to find the effect of noise, including additive noise and multiplicative noise. And we are also very interested in the type (3) as introduction said.

We first consider the global existence of the following stochastic parabolic equations
\[
\begin{cases}
du = (\Delta u + f(u, x, t))dt + \sigma(u, x, t)dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\
u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^d,
\end{cases} \tag{3.5}
\]
where $B_t$ is one-dimensional Brownian motion. A mild solution to (3.3) in sense of Walsh [31] is any $u$ which is adapted to the filtration generated by the white noise and satisfies the following evolution equation
\[
u(x, t) = \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)f(u, y, s)dyds
+ \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)\sigma(u, y, s)dydB_s,
\]
where $K(t, x)$ denotes the heat kernel of Laplacian operator, i.e.,
\[
K(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right)
\]
satisfies
\[
\left(\frac{\partial}{\partial t} - \Delta\right)K(t, x) = 0 \quad \text{for} \quad (x, t) \neq (0, 0).
\]
We get the following results.
Theorem 3.1 Suppose that there exist positive constants $C_0$, $0 < p < 1$ such that

$$|h(u, x, t)| \leq C_0|u|^p, \quad h = f \text{ or } g.$$  

Then the solutions of (3.5) with bounded continuous initial data $u_0$ exist globally in any $r$-order moment, $r \geq 1$.

Proof. By taking the second moment and using the Walsh isometry, we get for any $T > 0$

$$E|u(x, t)|^2 = \left(\int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy + \int_0^T \int_{\mathbb{R}^d} K(t, x - y)u(y, s)dyds \right.$$

$$\left. + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)\sigma(u, y, s)dydB_s\right)^2$$

$$\leq 4 \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy + 4C_0^2 \int_0^T \int_{\mathbb{R}^d} K(t, x - y)|E[u(y, s)]|^2dyds$$

$$+ 4C_0^2 \int_0^T \sup_{y, s} |E[u(y, s)]|^2 \int_0^T \int_{\mathbb{R}^d} K(t, x)dt \cdot dx$$

Then taking supremum for $t, x$ over $[0, T] \times \mathbb{R}^d$ (the right hand is independent of $t$ and $x$), we get

$$\sup_{t \in [0, T], x \in \mathbb{R}^d} E|u(x, t)|^2 \leq 4 \sup_{x \in \mathbb{R}^d} |u_0(x)|^2 + 8C_0^2 T \sup_{t \in [0, T], x \in \mathbb{R}^d} [E[u(y, s)]|^2]^p \int_0^T \int_{\mathbb{R}^d} K(t, x)dt \cdot dx$$

Notice that $0 < p < 1$, we have for any $T > 0$

$$\sup_{t \in [0, T], x \in \mathbb{R}^d} E|u(x, t)|^2 \leq C(T) < \infty,$$

which implies that $\mathbb{P}\{|u(x, t)| = \infty\} = 0$. The proof is complete. 

We remark that the heat kernel $K$ belongs to $L^1(\mathbb{R}^d)$ but not $L^2(\mathbb{R}^d)$. Hence this result does not hold for the noise white in both time and space. Meanwhile, if we assume the covariance function $q(x, y)$ is uniformly bounded, then the above result also hold for the noise white in time and correlated in space.

Next, we establish the result similar to the case of type (3). In order to do that, we will consider the following Cauchy problem

$$\begin{cases}
    du_t = \Delta u + \sigma(u, x, t)dW(x, t), \quad t > 0, & x \in \mathbb{R}^d, \\
    u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^d,
\end{cases}$$

where $W(t, x)$ is white noise both in time and space. In the rest of paper, we always assume that the initial data is nonnegative continuous function. A mild solution to (3.6) in the sense of Walsh is any $u$ which is adapted to the filtration generated by the white noise and satisfies the following evolution equation

$$u(x, t) = \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)\sigma(u, y, s)W(dy, ds),$$

where $K(t, x)$ denotes the heat kernel of Laplacian operator. We get the following results.
Theorem 3.2 Suppose \( d = 1 \) and \( \sigma^2(u, x, t) \geq C_0 u^{2m}, \) \( C_0 > 0, \) then for \( 1 < m \leq \frac{3}{2}, \) the solutions of (3.5) blows up in finite time for any nontrivial nonnegative initial data \( u_0. \) That is to say, there exists a positive constant \( T \) such that for all \( x \in \mathbb{R} \)

\[
\mathbb{E} u^2(x, t) = \infty \quad \text{for} \quad t \geq T.
\]

Proof. We assume that the solution remains finite for all finite \( t \) almost surely and want to derive a contradiction. By taking the second moment and using the Walsh isometry, we get

\[
\mathbb{E}|u(x, t)|^2 = \left( \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy \right)^2 + \int_0^t \int_{\mathbb{R}^d} K^2(t - s, x - y)\mathbb{E}\sigma^2(u, y, s)dyds
\]

\[=: I_1^2(x, t) + I_2(x, t).\]

We may assume without loss of generality that \( u_0(x) \geq C_1 > 0 \) for \( |x| < 1 \) by the assumption. A direct computation shows that

\[
I_1(x, t) \geq \frac{C_1}{(2\pi t)^{d/2}} \int_{B_1(0)} \exp \left( -\frac{|x|^2 + |y|^2}{2t} \right) dy
\]

\[
\geq \frac{C_1}{(2\pi t)^{d/2}} \exp \left( -\frac{|x|^2}{2t} \right) \int_{|y| \leq \frac{1}{\sqrt{t}}} \exp \left( -\frac{|y|^2}{2t} \right) dy
\]

\[
\geq \frac{C}{(2\pi t)^{d/2}} \exp \left( -\frac{|x|^2}{2t} \right)
\]

(3.7)

for \( t > 1 \) and \( C > 0. \)

It is easy to see that

\[
I_2(x, t) \geq C_0 \int_0^t \int_{\mathbb{R}^d} K^2(t - s, x - y)\mathbb{E}|u(y, s)|^{2m}dyds
\]

\[
\geq C_0 \int_0^t \int_{\mathbb{R}^d} K^2(t - s, x - y)[\mathbb{E}|u(y, s)|^{2m}]^{m}dyds.
\]

Denote \( v(x, t) = \mathbb{E}|u(x, t)|^2. \) Let

\[
G(t) = \int_{\mathbb{R}^d} K(t, x)v(x, t)dx.
\]

Then for \( t > 1,

\[
G(t) = \int_{\mathbb{R}^d} I_1^2(x, t)K(t, x)dx + \int_{\mathbb{R}^d} I_2(x, t)K(t, x)dx
\]

\[
\geq \frac{C_2}{t^d} + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(t, x)K^2(t - s, x - y)v^m(y, s)dydxds.
\]

(3.8)

It is clear that

\[
\int_{\mathbb{R}^d} K(t, x)K^2(t - s, x - y)dx
\]

\[
= \frac{1}{(2\pi t)^{d/2}[2\pi(t - s)]} \int_{\mathbb{R}^d} \exp \left( -\frac{|x|^2}{2t} - \frac{|x - y|^2}{t - s} \right) dx
\]

\[
= K(s, y) \frac{(2\pi s)^{d/2}}{(2\pi t)^{d/2}[2\pi(t - s)]} \int_{\mathbb{R}^d} \exp \left( \frac{|y|^2}{2s} - \frac{|x|^2}{2t} - \frac{|x - y|^2}{t - s} \right) dx.
\]
Since
\[
\frac{|y|^2}{2s} \frac{|x|^2}{2t} - \frac{|x-y|^2}{t-s} \geq \frac{|y|^2}{2s} \left( \frac{2|x-y|}{2t} + |y|^2 + 2|x-y||y| \right) - \frac{|x-y|^2}{t-s}
\]
\[
= \frac{1}{2t} \left( -2|x-y||y| + \frac{t-s}{s}|y|^2 \right) - \frac{|x-y|^2}{t-s}
\]
\[
\geq -\frac{s|x-y|^2}{2t(t-s)} - \frac{|x-y|^2}{2t} - \frac{|x-y|^2}{t-s}
\]
\[
\geq -\frac{2|x-y|^2}{t-s} \text{ for } 0 < s < t,
\]
we get for \(0 < s < t\)
\[
\int_{\mathbb{R}^d} \exp \left( \frac{|y|^2}{2s} - \frac{|x|^2}{2t} - \frac{|x-y|^2}{t-s} \right) dx \geq \int_{\mathbb{R}^d} \exp \left( \frac{-2|x-y|^2}{t-s} \right) dx = C_3(t-s)^{d/2}.
\]
Substituting the above estimate into (3.8) and applying Jensen’s inequality, we obtain
\[
G(t) \geq \frac{C_2}{t^d} + C_4 \int_0^t \frac{s^{d/2}}{t^d} \int_{\mathbb{R}^d} K(s,y)v^m(y,s)dydxds
\]
\[
\geq \frac{C_2}{t^d} + C_4 \int_0^t \frac{s^{d/2}}{t^d} G^m(s)ds
\]
We can rewrite the above inequality as
\[
t^dG(t) \geq C_2 + C_4 \int_0^t s^{d/2} G^m(s)ds =: g(t).
\](3.9)
Then for \(t > 1\), we have
\[
g(t) \geq C_2,
\]
\[
g'(t) \geq C_4 t^{d/2} G^m(t) \geq C_4 t^{d/2} \left( \frac{1}{t^d} g(t) \right)^m = C_4 t^{\frac{d}{2} - dm} g^m(t),
\]
which implies
\[
\frac{C_2^{1-m}}{m-1} \geq \frac{1}{m-1} g^{1-m}(t) \geq C_4 \int_t^T s^{\frac{d}{2} - dm} dx \text{ for } T > t \geq 1.
\]
If \(m \leq \frac{d+2}{2d}\), that is, \(\frac{d}{2} - dm + 1 \geq 0\), the right-hand side of the above inequality is unbounded as \(T \to \infty\), which gives a contradiction. Noting that we must let \(m > 1\) because we used the Jensen’s inequality, thus we get \(1 < m \leq \frac{d}{2}\) and \(d = 1\). And thus we complete the proof. \(\square\)

If the noise is just one-dimensional Brownian motion, the result will be different. For this, we consider the following stochastic
\[
\begin{cases}
du_t = \Delta u t + \sigma(u, x, t)dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\
u(x, 0) = u_0(x) \geq 0, & x \in \mathbb{R}^d,
\end{cases}
\](3.10)
where \(B_t\) is one-dimensional Brownian motion. A mild solution to (3.10) in sense of Walsh [31] is any \(u\) which is adapted to the filtration generated by the white noise and satisfies the following evolution equation
\[
u(x, t) = \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t-s, x - y)\sigma(u, y, s)dydB_s,
\]
where \(K(t, x)\) denotes the heat kernel of Laplacian operator.
**Theorem 3.3** Suppose \( d = 1 \) and \( \sigma^2(u, x, t) \geq C_0 u^2 \), \( C_0 > 0 \), then the solutions of (3.10) blows up in finite time for any nontrivial nonnegative initial data \( u_0 \).

**Proof.** Similar to the proof of Theorem 3.2, we assume that the solution remains finite for all finite \( t \) almost surely. By taking the second moment and using the Walsh isometry, we get

\[
\mathbb{E}|u(x, t)|^2 = \left( \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy \right)^2 + \int_0^t \left( \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}\sigma(u, y, s)dy \right)^2 ds
\]

\[=: u_1(x, t) + u_2(x, t).\]

We may assume without loss of generality that \( u_0(x) \geq C_1 > 0 \) for \( |x| < 1 \) by the assumption. The estimate (3.7) also holds, i.e.,

\[u_1(x, t) \geq \frac{C}{(2\pi t)^{d/2}} \exp \left( - \frac{|x|^2}{2t} \right)\]

for \( t > 1 \) and \( C > 0 \).

It is easy to see that, for \( m \geq 2 \),

\[u_2(x, t) \geq C_0 \int_0^t \left( \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}|u(y, x)|^m dy \right)^2 ds\]

\[\geq C_0 \int_0^t \left( \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}|u(y, x)|^{2m/2} dy \right)^2 ds\]

\[\geq C_0 \int_0^t \left( \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}|u(y, x)|^2 dy \right)^m ds.\]

Denote \( v(x, t) = \mathbb{E}|u(x, t)|^2 \). Let

\[G(t) = \int_{\mathbb{R}^d} K(t, x)v(x, t)dx.\]

Then for \( t > 1 \),

\[G(t) = \int_{\mathbb{R}^d} u_1^2(x, t)K(t, x)dx + \int_{\mathbb{R}^d} u_2(x, t)K(t, x)dx\]

\[\geq \frac{C_2}{t^{d/2}} + \int_0^t \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(t, x)K(t - s, x - y)v(y, s)dydx \right)^m ds. \quad (3.11)\]

It is clear that (see [13, Page 42])

\[\int_{\mathbb{R}^d} K(t, x)K(t - s, x - y)dx \geq C_3 K(s, y) \left( \frac{s}{t} \right)^{d/2}.\]

Substituting the above estimate into (3.11) and applying Jensen’s inequality, we obtain

\[G(t) \geq \frac{C_2}{t^{d/2}} + C_3 \int_0^t \left( \frac{s^{d/2}}{t^{d/2}} \right)^m G^m(s)ds\]

We can rewrite the above inequality as

\[t^{md/2}G(t) \geq C_2 t^{(m-2)d/2} + C_3 \int_0^t s^{dm/2}G^m(s)ds =: g(t). \quad (3.12)\]
Then for $t > 1$, we have
\[ g(t) \geq C_2 t^{(m-2)d/2}, \]
\[ g'(t) \geq C_3 t^{dm/2} G^m(t) \geq C_3 t^{d/2} \left( \frac{1}{t^{dm/2}} g(t) \right)^m = C_3 t^{(1-m)md/2} g^m(t), \]
which implies
\[ \frac{C_2^{1-m}}{m-1} t^{-d(m-1)(m-2)/2} \geq \frac{1}{m-1} g^{1-m}(t) \geq C_4 \int_t^T s^{(1-m)md/2} dx \text{ for } T > t \geq 1. \]

If $(m-1)md/2 \leq 1$, we will get a contradiction by letting $T \to \infty$. If $\frac{d(m-1)(m-2)}{2} > -1 + \frac{(m-1)md}{2}$, then we will get a contradiction by letting $T \to \infty$ and then taking $t \gg 1$. Noting that when $m = 2$, $d = 1$, we have $(m-1)md/2 = 1$ and $\frac{d(m-1)(m-2)}{2} > -1 + \frac{(m-1)md}{2}$ is equivalent to $m < 1 + \frac{1}{2}$. Since $m \geq 2$, we get a contradiction for the case that $m = 2$, $d = 1$. The proof is complete. □

**Remark 3.1** Comparing Theorem 3.2 with Proposition 3.1, the assumptions of Proposition 3.1 on initial data need the lower bound, but in Theorem 3.2 we did not.

Theorems 3.2 and 3.3 show that the time-space white noise and Brownian motion are different. But the method used here is not suitable to fractional Laplacian operator. Sugitani [29] established the Fujita index for Cauchy problem of fractional Laplacian operator. The main difficult is that we can not get the exact estimate of $\int_{\mathbb{R}^d} p^2(t, x)p(t-s, x-y)dx$, where $p(t, x)$ is the heat kernel of fractional Laplacian operator.

### 4 Discussion

An interesting issue of stochastic partial differential equations is to find the difference when we add the noise, i.e., the impact of noise. For stochastic partial differential equations, we want to know whether the solutions keep positive. In this section, we first consider the positivity of the solutions of stochastic parabolic equations in the whole space, and then consider the impact of noise.

In the followings, we will select a test function $\beta_\varepsilon(r)$. Define
\[ \beta_\varepsilon(r) = \int_r^\infty \rho_\varepsilon(s) ds, \quad \rho_\varepsilon(r) = \int_{r+\varepsilon}^\infty J_\varepsilon(s) ds, \quad r \in \mathbb{R}, \]
\[ J_\varepsilon(|x|) = \varepsilon^{-n} J \left( \frac{|x|}{\varepsilon} \right), \quad J(x) = \begin{cases} 
C \exp \left( \frac{1}{|x|^2 - 1} \right), & |x| < 1, \\
0, & |x| \geq 1. 
\end{cases} \]

Then by direct verification, we have the following result.

**Lemma 4.1** The above constructed functions $\rho_\varepsilon, \beta_\varepsilon$ are in $C^\infty(\mathbb{R})$ and have the following properties: $\rho_\varepsilon$ is a non-increasing function and
\[ \beta_\varepsilon'(r) = -\rho_\varepsilon(r) = \begin{cases} 
0, & r \geq 0, \\
-1, & r \leq -2\varepsilon. 
\end{cases} \]

Additionally, $\beta_\varepsilon$ is convex and
\[ \beta_\varepsilon(r) = \begin{cases} 
0, & r \geq 0, \\
-2\varepsilon - r + \varepsilon C, & r \leq -2\varepsilon, 
\end{cases} \]
where $\dot{C} = \int_{-2}^{0} \int_{t+1}^{1} J(s) dx dt < 2$. Furthermore,

$$0 \leq \beta''_\varepsilon(r) = J_\varepsilon(r + \varepsilon) \leq \varepsilon^{-d} C, \quad -2\varepsilon \leq r \leq 0,$$

which implies that

$$-2^d C \leq r^d \beta''(r) \leq 0 \quad \text{for} \quad -2\varepsilon \leq r \leq 0, \quad \text{and} \quad d \text{ is odd;}$$

$$0 \leq r^d \beta''(r) \leq 2^d C \quad \text{for} \quad -2\varepsilon \leq r \leq 0, \quad \text{and} \quad d \text{ is even.}$$

Now, we consider the following stochastic parabolic equations

$$\begin{aligned}
\begin{cases}
\frac{du}{dt} = (\Delta u + f(u, x, t)) dt + g(u, x, t) dW(x, t), & t > 0, \quad x \in \mathbb{R},
\end{cases}
\end{aligned}
$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

where $W(x, t)$ is time-space white noise.

**Theorem 4.1** Assume that (i) the function $f(r, x, t)$ is continuous on $\mathbb{R} \times \mathbb{R} \times [0, T]$; (ii) $f(r, x, t) \geq 0$ for $r \leq 0$, $x \in \mathbb{R}$ and $t \in [0, T]$; and (iii) $g^2(u, x, t) \leq k u^{2m}, \quad \text{where} \quad k > 0, \quad 2m > 1 \quad \text{and} \quad (-1)^{2^{m-1}} \in \mathbb{R}$. Then the solution of initial-boundary value problem (4.1) with nonnegative initial datum remains positive: $u(x, t) \geq 0$, a.s. for almost every $x \in \mathbb{R}$ and for all $t \in [0, T]$.

**Proof.** Define

$$\Phi(\varepsilon(u_t) = (1, \beta(\varepsilon(u_t))) = \int_\mathbb{R} \beta(\varepsilon(u(x, t))) dx.$$ 

By Itô's formula, we have

$$\begin{aligned}
\Phi(\varepsilon(u_t) &= \Phi(\varepsilon(u_0) + \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) \Delta u(x, s) dx ds \\
&+ \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) f(u(x, s), x, s) dx ds \\
&+ \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) g(u(x, s), x, s) dW(x, s) dx \\
&+ \frac{1}{2} \int_0^t \int_\mathbb{R} \beta''(u(x, s)) g^2(x, s, x, s) dx ds \\
&= \Phi(\varepsilon(u_0) + \int_0^t \int_\mathbb{R} \beta''_\varepsilon(u(x, s)) \left( \frac{1}{2} g^2(u(x, s), x, s) - |\nabla u|^2 \right) dx ds \\
&+ \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) f(u(x, s), x, s) dx ds \\
&+ \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) g(u(x, s), x, s) dW(x, s) dx.
\end{aligned}$$

Taking expectation over the above equality and using Lemma 4.1 we get

$$\begin{aligned}
\mathbb{E} \Phi(\varepsilon(u_t) &= \mathbb{E} \Phi(\varepsilon(u_0) + \mathbb{E} \int_0^t \int_\mathbb{R} \beta''_\varepsilon(u(x, s)) \\
&\times \left( \frac{1}{2} g^2(u(x, s), x, s) - |\nabla u|^2 \right) dx ds \\
&+ \mathbb{E} \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) f(u(x, s), x, s) dx ds \\
&\leq \mathbb{E} \Phi(\varepsilon(u_0) + \frac{k}{2} \int_0^t \int_\mathbb{R} \beta''(u(x, s)) u(x, s)^{2m} dx ds \\
&+ \mathbb{E} \int_0^t \int_\mathbb{R} \beta'_\varepsilon(u(x, s)) f(u(x, s), x, s) dx ds.
\end{aligned}$$
Here and after, we denote $\| \cdot \|_{L^1}$ by $\| \cdot \|_1$. Let $\eta(u) = u^-$ denote the negative part of $u$ for $u \in \mathbb{R}$. Then we have $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \Phi_\varepsilon(u_t) = \mathbb{E} \|\eta(u_t)\|_1$. It follows from Lemma 4.1 that

$$0 \geq u^{2m} \beta'_\varepsilon(u) \geq \begin{cases} 0, & u \geq 0 \text{ or } u \leq -2\varepsilon, \\ -2Cu^{2m-1}, & -2\varepsilon \leq u \leq 0, \text{ and } u^{2m-1} \geq 0, \end{cases}$$

or

$$0 \leq u^{2m} \beta''_\varepsilon(u) \leq \begin{cases} 0, & u \geq 0 \text{ or } u \leq -2\varepsilon, \\ -2Cu^{2m-1}, & -2\varepsilon \leq u \leq 0, \text{ and } u^{2m-1} \leq 0, \end{cases}$$

which implies that $\lim_{\varepsilon \rightarrow 0} u^{2m} \beta''_\varepsilon(u) = 0$ provided that $2m > 1$. By taking the limits termwise as $\varepsilon \rightarrow 0$ and using Lemma 4.1 we get

$$\mathbb{E} \|\eta(u_t)\|_1 \leq \mathbb{E} \|\eta(u_0)\|_1 - \mathbb{E} \int_0^t \int_{\mathbb{R}} \eta'(u(x,s)) f(u(x,s), x, s) dx ds$$

$$\leq 0,$$

which implies that $u^- = 0$ a.s. for a.e. $x \in D$, $\forall t \in [0, T]$. This completes the proof. □

If $W(x,t)$ is replaced by $B_t$ in (4.1), then Theorem 4.1 holds for any dimension. The reason why we only consider one dimension in Theorem 4.1 is that the Itô formula only holds for one-dimensional time-space white noise.

In order to find the impact of noise, we first recall a well-known result of deterministic parabolic equations. Consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u_t = \Delta u + u^p, & t > 0, \quad x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

**Proposition 4.1** (i) If $p > 1 + \frac{2}{d}$, then the solution of (4.2) is global in time, provided the initial datum satisfies, for some small $\varepsilon > 0$,

$$u_0(x) \leq \varepsilon K(1, x), \quad x \in \mathbb{R}^d.$$  

(ii) If $1 < p \leq 1 + \frac{2}{d}$, then all nontrivial solutions of (4.2) blow up in finite time.

Next we consider the stochastic parabolic equation

$$\begin{cases} du_t = [\Delta u + |u|^{p-1}u] dt + \sigma(u)dW(x,t), & t > 0, \quad x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^d. \end{cases} \tag{4.3}$$

It is well known that the mild solution of (4.3) can be written as

$$u(x,t) = \int_{\mathbb{R}^d} K(t, x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t-s, x-y)|u|^{p-1} u dy ds$$

$$+ \int_0^t \int_{\mathbb{R}^d} K(t-s, x-y)\sigma(u, y, s)W(dy, ds).$$

**Theorem 4.2** Assume all the assumptions of Theorem 4.1 hold. Then $1 < p \leq 1 + \frac{2}{d}$, then the expectation of all nontrivial solutions of (4.3) blow up in finite time. That is to say, there exists a positive constant $t_0 > 0$ such that $\mathbb{E} u(x,t) = \infty$, $t \geq t_0$ for all $x \in \mathbb{R}^d$. When $m > 1$, the mean square of solutions to (4.3) will blow up in finite time under the condition that the initial data is suitable large.
Proof. It follows from Theorem 4.1 that the solutions of (4.3) keep positive. Following the representation of mild solution, we have

\[ \mathbb{E}u(x, t) = \int_{\mathbb{R}^d} K(t, x - y)\mathbb{E}u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}|u|^p dyds, \]

which implies that

\[ \mathbb{E}u(x, t) \geq \int_{\mathbb{R}^d} K(t, x - y)\mathbb{E}u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(t - s, x - y)\mathbb{E}|u|^p dyds, \]

Denoting \( v(x, t) = \mathbb{E}u(x, t) \), we have that \( v(x, t) \) is a super-solution of (4.2). By the results of Proposition 4.1 and comparison principle, we obtain that there exists a positive constant \( t_0 > 0 \) such that \( \mathbb{E}u(x, t) = \infty, \quad t \geq t_0 \) for all \( x \in \mathbb{R}^d \). Meanwhile, noting that

\[ \mathbb{E}u(x, t) \leq (\mathbb{E}|u|^p(x, t))^{\frac{1}{p}}, \quad p > 1, \]

we have that \( \mathbb{E}u^p(x, t), \quad p > 1, \) will blow up in finite time.

When \( m > 1 \), we have

\[ \mathbb{E}|u(x, t)|^2 \geq \left( \int_{\mathbb{R}^d} K(t, x - y)u_0(y)dy \right)^2 + \int_0^t \int_{\mathbb{R}^d} K^2(t - s, x - y)\mathbb{E}\sigma^2(u, y, s)dyds =: w(x, t). \]

Foondun [10] proved the mean square of function \( w(x, t) \) will blow up in finite time under the condition that the initial data is suitable large. So the solution \( u \) will also blow up in finite time. The proof is complete. \( \Box \)

Acknowledgment The first author was supported in part by NSFC of China grants 11771123. The authors thanks Prof. Feng-yu Wang for discussing this manuscript.

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