Gravitational waves in the presence of a dynamical four-form

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Abstract

Propagation of weak gravitational waves, when a dynamical four-form is around, has been investigated. Exact, self-consistent solutions corresponding to plane, monochromatic gravitational waves have been studied.
INTRODUCTION

There are strong observational evidence in support of an accelerating universe. A small but positive cosmological constant or some dark energy could be responsible for the accelerated expansion [1,2 and references therein]. Is there a geometrical origin to the dark energy? In special relativity, $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ are both invariant under proper Lorentz transformations. Space-time geometry represented by $g_{\mu\nu}$ (that becomes $\eta_{\mu\nu}$ in a local inertial frame) is dynamical in general relativity (GR). What about $\epsilon_{\mu\nu\rho\sigma}$ in GR? Taking this as a cue, we extend GR by considering a dynamical four-form $\tilde{w}$ [3],

$$\tilde{w} = \frac{1}{4!} w_{\mu\nu\rho\sigma} \tilde{dx}^\mu \wedge \tilde{dx}^\nu \wedge \tilde{dx}^\rho \wedge \tilde{dx}^\sigma$$

so that,

$$w_{\mu\nu\rho\sigma}(x^\alpha) = \phi(x^\alpha) \epsilon_{\mu\nu\rho\sigma} \quad (1b)$$

with $\phi(x^\alpha)$ being a scalar-density of weight +1, that transforms as $\phi \rightarrow \phi/J$, under a general coordinate transformation.

Dynamical scalar-densities have been considered previously in the context of non-metric volume-forms [4,5 and references therein]. Our approach however is different, stressing on its association with Chern-Simons extension and dark energy [3,6]. We study the action $S$,

$$S = -\frac{m_{Pl}^2}{16\pi} \int R\sqrt{-g}d^4x + \int L\sqrt{-g}d^4x + \frac{A}{4!} \int \phi \, w^{\mu\nu\alpha\beta} \omega_{\mu\nu\rho\sigma} \phi \, d^4x + B \int \phi(x)d^4x + S_{GCS} \quad (2)$$

with a Chern-Simons part $S_{GCS}$, inspired by Jackiw and Pi’s paper [6],

$$S_{GCS} = H \int w^{\mu\nu\alpha\beta} \left[ \Gamma^\sigma_{\nu\tau} \partial_\alpha \Gamma^\tau_{\beta\sigma} + \frac{2}{3} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\alpha\eta} \Gamma^\eta_{\beta\rho} \right] \phi_{\mu} \, d^4x \quad (3)$$

where $L$ is the Lagrangian density for matter fields while $A$ and $B$ are real valued parameters with dimensions (mass)$^2$ and (mass)$^4$, respectively. $H$ is a dimensionless real constant in eq(3).

Equations of motion that ensue from extremizing $S$ are given by,

$$\psi_{;\alpha}^{\cdot \alpha} = \frac{1}{2} \left[ g_{\mu\nu} \psi_{;\mu} \psi_{;\nu} \psi - \frac{B}{A} \psi \, w^{\mu\nu\alpha\beta} R^\tau_{\sigma\alpha\beta} R^\sigma_{\tau\mu\nu} \right] \quad (4)$$

and,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{m_{Pl}^2} [T_{\mu\nu} + \Theta_{\mu\nu} + C_{\mu\nu}] \quad (5)$$
where the scalar field $\psi$ is related to $\phi$ simply as $\psi \equiv \frac{\phi}{\sqrt{-g}}$ while,

$$\Theta_{\mu\nu} = 2A \left[ \frac{\psi_{,\mu} \psi_{,\nu}}{\psi} - \frac{g_{\mu\nu}}{2} \left( g_{\alpha\beta} \frac{\psi_{,\alpha} \psi_{,\beta}}{\psi} + \frac{B}{A} \right) \right]$$

(6)

and $C^{\mu\nu}$ is the standard Cotton tensor [6],

$$C^{\mu\nu} \equiv 2 \left[ (\ln \psi)_{,\alpha;\beta} \left( * R_{\beta\alpha\mu\nu} + * R_{\beta\mu\alpha\nu} \right) - (\ln \psi)_{,\alpha} \left( \epsilon_{\alpha\mu\sigma\tau} R_{\sigma\tau}^{\nu} + \epsilon_{\alpha\mu\nu\tau\sigma} R_{\sigma\tau}^{\nu} \right) \right]$$

(7)

with,

$$* RR \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\sigma\alpha\beta} R_{\sigma\mu\nu} \quad * R_{\mu\nu}^{\rho\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\rho\alpha\beta} .$$

For a smooth FRW universe, the Chern-Simons extension has no effect since the Cotton tensor vanishes identically. But inhomogeneity and anisotropy resulting from gravitational collapse of cold dark matter would result in a non-vanishing $* RR$ on smaller length scales, which would act as a source of the dynamical four-form due to eq.(4). This can lead to a late time accelerated expansion of the universe if $A$ and $B$ have suitable values [3]. If $B < 0$, one can interpret $\pm \sqrt{\psi}$ to be a quintessence field with a mass $\mu = \sqrt{-B} / 4A$. A Chern-Simons extension to electrodynamics follows naturally from the dynamical four-form. Its effect on magnetohydrodynamics equations in the context of Cowling’s theorem and pulsars has been examined [7].

**PROPAGATION OF GRAVITATIONAL WAVES**

In this section, we study the effect of the four-form on gravitational waves (GWs) by assuming that, except for a weak GW and $\psi$, there is no other matter present. Adopting a TT-gauge for the plane GW travelling along the x-axis,

$$h_{\oplus} \equiv h_{22}(t, x) = -h_{33}(t, x) \quad h_{\otimes} \equiv h_{23}(t, x) = h_{32}(t, x)$$

we express $S$ of eq.(2), using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and retaining only upto quadratic terms in $h_{\oplus}$ and $h_{\otimes}$ for the gravitational part of the action [6,8],

$$S \approx -\frac{m_{Pl}^2}{8\pi} \int \left[ h_{\oplus} (\ddot{h}_{\oplus} - h_{\oplus}^{,\nu} ; h_{\otimes} - h_{\otimes}^{,\nu} ; h_{\otimes}^{,\nu} - h_{\otimes}^{,\nu} \right] d^4 x + S_{GCS} + S_\phi$$

(8)

with,

$$S_{GCS} = -H \int \eta^{,\lambda} \left( \frac{\partial^2 \ln \psi}{\partial x \partial \tau} (h_{\oplus} h_{\oplus,0\lambda} - h_{\otimes} h_{\otimes,0\lambda}) + \frac{\partial^2 \ln \psi}{\partial t \partial \tau} (h_{\otimes} h_{\otimes,1\lambda} - h_{\oplus} h_{\oplus,1\lambda}) \right) d^4 x$$
\[-H \int \frac{\partial \ln \psi}{\partial x} \left( h_{\oplus}(\dot{h}_{\oplus,0} - h''_{\oplus,0}) - h_{\oplus}(\ddot{h}_{\oplus,0} - h''_{\oplus,0}) \right) \, d^4 x \]

\[+ H \int \frac{\partial \ln \psi}{\partial t} \left( h_{\oplus}(\dot{h}_{\oplus,1} - h''_{\oplus,1}) - h_{\oplus}(\ddot{h}_{\oplus,1} - h''_{\oplus,1}) \right) \, d^4 x \]

and,

\[S_\phi = \int \left[ \frac{A}{\psi} \eta^{\mu\nu} \psi_{\mu} \psi_{\nu} + B \psi - \frac{A}{\psi} \left( h_{\oplus}(\dot{\psi},2,2 - \psi,3,3) + 2h_{\oplus,2,2,3,3} \right) \right] \, d^4 x \]

Equations of motion that follow from variation of \( S \) (eq.(8)) with respect to \( h_{\oplus}, h_{\odot} \) and \( \psi \) are given by,

\[\dot{h}_{\oplus} - h''_{\oplus} = - \frac{8\pi}{m_{P}^2} \left[ H \left\{ \frac{\partial^2 \chi}{\partial t \partial x} (\dot{h}_{\oplus} + h''_{\oplus}) - h_{\oplus,01} \left( \frac{\partial^2 \chi}{\partial t^2} + \frac{\partial^2 \chi}{\partial x^2} \right) + \frac{\partial \chi}{\partial x} (\dot{h}_{\oplus,0} - h''_{\oplus,0}) - \frac{\partial \chi}{\partial t} (\dot{h}_{\oplus,1} - h''_{\oplus,1}) \right\} + + A \exp \chi \left\{ \left( \frac{\partial \chi}{\partial y} \right)^2 - \left( \frac{\partial \chi}{\partial z} \right)^2 \right\} \right] \]

\[\dot{h}_{\odot} - h''_{\odot} = \frac{8\pi}{m_{P}^2} \left[ H \left\{ \frac{\partial^2 \chi}{\partial t \partial x} (\dot{h}_{\odot} + h''_{\odot}) - h_{\odot,01} \left( \frac{\partial^2 \chi}{\partial t^2} + \frac{\partial^2 \chi}{\partial x^2} \right) + \frac{\partial \chi}{\partial x} (\dot{h}_{\odot,0} - h''_{\odot,0}) - \frac{\partial \chi}{\partial t} (\dot{h}_{\odot,1} - h''_{\odot,1}) \right\} + + 2A \exp \chi \left( \frac{\partial \chi}{\partial y} \frac{\partial \chi}{\partial z} \right) \right] \]

and

\[\partial^\mu \partial^\nu \chi + \frac{1}{2} \eta^{\mu\nu} \frac{\partial \chi}{\partial x^\mu} \frac{\partial \chi}{\partial x^\nu} - \frac{B}{4A} = h_{\oplus} \left[ \frac{\partial^2 \chi}{\partial y^2} \frac{\partial \chi}{\partial z^2} + \frac{1}{2} \left( \frac{\partial \chi}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial \chi}{\partial z} \right)^2 \right] + 2h_{\odot} \left[ \frac{\partial^2 \chi}{\partial y \partial z} + \frac{1}{2} \frac{\partial \chi}{\partial y} \frac{\partial \chi}{\partial z} \right] \]

where \( \chi \equiv \ln \psi \). The derivatives with respect to time \( t \) and x-coordinate are denoted by dot and prime, respectively.

A simplification occurs if we assume \( \psi \) to depend only on the x-coordinate and time, like the GW amplitude. With \( \chi = \chi(t, x) \), the last terms in eqs.(9) and (10) drop out. Introducing the complex circularly polarized GW amplitudes [8],

\[h_R = \frac{1}{\sqrt{2}}(h_{\oplus} + ih_{\odot}) \quad h_L = \frac{1}{\sqrt{2}}(h_{\oplus} - ih_{\odot})\]

one finds from eqs.(9) and (10) that,

\[\dot{h}_R - h''_R = - \frac{8\pi H i}{m_{P}^2} \left[ \chi \frac{\partial}{\partial x} (\dot{h}_R - h''_R) - \chi' \frac{\partial}{\partial t} (\dot{h}_R - h''_R) + \dot{h}_R(\dot{\chi} + \chi'') - \chi'(\dot{h}_R + h''_R) \right] \]

\[\dot{h}_L - h''_L = \frac{8\pi H i}{m_{P}^2} \left[ \chi \frac{\partial}{\partial x} (\dot{h}_L - h''_L) - \chi' \frac{\partial}{\partial t} (\dot{h}_L - h''_L) + \dot{h}_L(\dot{\chi} + \chi'') - \chi'(\dot{h}_L + h''_L) \right] \]
Since $\chi = \chi(t, x)$, the equation of motion for the four-form given by eq.(11) simplifies to,

$$\ddot{\chi} - \chi'' + \frac{1}{2}(\dot{\chi}^2 - \chi'') + 2\mu^2 = 0$$  \hspace{1cm} (14)

where $\mu \equiv \sqrt{-\frac{B}{4A}}$ acts as the mass of the four-form field [3]. Seeking exact solutions, involving monochromatic and circularly polarized GWs, we substitute,

$$h_R(t, x) = h \exp(i(\omega t - kx))$$

in eqs.(12), leading to a relation,

$$\omega^2 - k^2 = -\beta[k(\omega^2 - k^2)\dot{\chi} + \omega(\omega^2 - k^2)\chi' - i\omega k(\ddot{\chi} + \chi'') - i(\omega^2 + k^2)\dot{\chi}']$$  \hspace{1cm} (15)

where $\beta \equiv \frac{8\pi H}{m_P^2}$.

If $w = \pm k$, eq.(15) implies,

$$2\chi' \pm (\ddot{\chi} + \chi'') = 0$$  \hspace{1cm} (16)

The only self-consistent solution of eqs.(14) and (16) is,

$$\chi(t, x) = (a_0 - \frac{\mu^2}{a_0})t + (a_0 + \frac{\mu^2}{a_0})x$$  \hspace{1cm} (17)

where $a_0$ is an integration constant. But the above solution is unphysical, as it implies an exponentially growing $\psi$. On the other hand, when $w^2 > k^2$,

$$k\dot{\chi} + \omega\chi' + \frac{1}{\beta} = 0 \Rightarrow \chi'' = \frac{k^2}{w^2} \ddot{\chi}$$  \hspace{1cm} (18)

and therefore, $\chi = \chi(x - \frac{\omega}{k}t)$, leading to an exact solution for $a_1 \geq \lambda$,

$$\chi = \ln \left( \sqrt{\frac{a_1}{\lambda}} \cos \left( \pm \sqrt{\frac{\lambda}{2}} \left( t - \frac{k}{\omega}x \right) + a_2 \right) \right)$$  \hspace{1cm} (19)

where $\lambda \equiv \frac{2w^2\mu^2}{\omega^2 - k^2}$, $a_1$ and $a_2$ are integration constants. Hence, $\psi = \exp(\chi) \propto \cos \left( \pm \sqrt{\frac{\lambda}{2}} \left( t - \frac{k}{\omega}x \right) + a_2 \right)$, with $|\omega| > |k|$, is an acceptable solution. Indeed, it is possible to have plane and monochromatic GWs with phase velocity exceeding the speed of light. However, eq.(14) being a nonlinear differential equation, obtaining exact solutions corresponding to GWs with superposed wave modes is non-trivial.
SUMMARY

Gravitational waves propagating in the presence of a dynamical four-form can be decoupled using circularly polarized waveforms. In this paper, we have obtained exact, self-consistent solutions, and have shown that the phase velocity of a monochromatic gravitational wave can generically exceed the speed of light.

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