AN INTEGER OPTIMIZATION PROBLEM FOR NON-HAMILTONIAN PERIODIC FLOWS

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ABSTRACT. Let $\mathcal{C}$ be the class of compact $2n$-dimensional symplectic manifolds $(M, \omega)$ for which the first or $(n-1)$ Chern class vanish. We point out an integer optimization problem to find a lower bound $B(n)$ on the number of equilibrium points of non-Hamiltonian symplectic periodic flows on manifolds $(M, \omega) \in \mathcal{C}$. As a consequence, we confirm in dimensions $2n \in \{8, 10, 12, 14, 18, 20, 22\}$ a conjecture for unitary manifolds made by Kosniowski in 1979 for the subclass $\mathcal{C}$.

1. Introduction

We point out how to pose a classical geometric problem concerning the size of fixed point sets of $S^1$-actions on $2n$-dimensional manifolds, as an integer programming problem. This classical problem goes back to Frankel (1959) and Kosniowski (1979). By studying the integer programming problem we obtain some information on the classical problem for low values of $n$, under certain topological assumptions.

1.1. A geometric problem. A symplectic form on a smooth manifold $M$ is a closed, non-degenerate two-form $\omega \in \Omega^2(M)$. For simplicity, throughout we assume that $M$ is connected. Let $S^1 \subset \mathbb{C}$ be the multiplicative group of unit complex numbers. We say that an $S^1$-action on a symplectic manifold $(M, \omega)$ is symplectic if it preserves $\omega$. Let $\mathcal{X}_M$ be the vector field on $M$ induced by the $S^1$ action. An $S^1$-action is Hamiltonian if the 1-form

$$\iota_{\mathcal{X}_M}\omega := \omega(\mathcal{X}_M, \cdot)$$

is exact, that is, there exists a smooth map $\mu : M \to \mathbb{R}$ such that $-d\mu = \iota_{\mathcal{X}_M}\omega$. The map $\mu$ is called a momentum map. In the terminology of dynamical systems, we refer to symplectic $S^1$-actions as symplectic periodic flows. The fixed points of the action correspond to the equilibrium points of the flow.

A classical question in symplectic geometry is whether symplectic $S^1$-actions on compact symplectic manifolds which possess nonempty discrete fixed point set are Hamiltonian. It is unknown whether there is a non-Hamiltonian symplectic $S^1$ action on a compact symplectic manifold $M$ for which the fixed set $M^{S^1}$ is discrete but not empty. If the point set is not required to be discrete, there are non-Hamiltonian symplectic $S^1$-actions
with no fixed points; for instance, given an even-dimensional torus, consider the natural action of any circle subgroup. In [Mc88, Proposition 1 and Section 3], McDuff constructs a non-Hamiltonian symplectic $S^1$-action on a six-dimensional compact symplectic manifold, whose fixed sets are tori. If the fixed point set is discrete and nonempty, the problem has a history of partial answers, but the solution is unknown in general. The following question is closely related to this problem.

**Question 1.** Given a compact symplectic $2n$-dimensional manifold $(M, \omega)$ with a symplectic $S^1$-action and nonempty discrete fixed point set $M^{S^1}$, what are lower bounds $B(n)$ on the cardinality of $M^{S^1}$ depending on the $n$?

Question 1 goes back to Frankel (1959), who gave a sharp bound for compact Kähler manifolds. A conjecture in this direction (Conjecture 1) was made for the larger class of unitary manifolds by Kosniowski (1979), which remains open in general, as far as we know. The work of McDuff in 1988 (Theorem 2.2) gives a sharp answer for compact symplectic 4-manifolds. We review these results in Section 2.

1.2. **Summary of results.** In Section 3 we prove a result (Theorem A), the proof of which uses [GoSa12, Theorem 1.2] to give a non-explicit formula for $B(n)$, provided that the first or $(n - 1)$ Chern class of $M$ vanish. Theorem A gives rise to an integer optimization problem (Problem 3.8, see also Problem 3.9) to find $B(n)$. The problem may be solved by hand for several values of $n$. In this way we confirm a conjecture of Kosniowski of 1979 in dimensions $8 \leq 2n \leq 22$, $2n \neq 16$ for the class of symplectic manifolds with vanishing first or $(n - 1)$ Chern class.

By studying Theorem A we obtain some properties related to the fixed points of the $S^1$-action, namely, we give necessary conditions on the number of negative weights at the fixed points (e.g. Theorem B). As a consequence, when $n = 2m + 1$ and $m \notin \{6k(k + 1) + 1 \mid k \in \mathbb{Z}_{\geq 0}\}$, we show that $B(n) \geq 4$. The estimates are in some cases higher than known estimates: for instance, $B(5) = 24$ which improves the known estimate of 3. However, we have not solved the integer optimization problem we introduce to give a formula for $B(n)$ for general $n \in \mathbb{N}$. We suspect that it would in some cases be amenable to techniques of linear programming, and if so, it could possibly lead to the solution of the conjecture in more cases.

2. **Preliminaries**

We review some results which are relevant to the answers given to Question 1 in this article, including some which we will need in the proofs.

2.1. **Origins.** At least from the point of view of equivariant symplectic geometry of torus actions, the interest in Question 1 has been to a large extent motivated by a result by T. Frankel on $S^1$-actions on compact Kähler manifolds.
Theorem 2.1 (Frankel '59). A symplectic Kähler $S^1$-action on a compact Kähler manifold is Hamiltonian if and only if it has fixed points.

The following appeared in McDuff [Mc88, Proposition 2].

Theorem 2.2 (McDuff, '88). A symplectic $S^1$-action on a 4-dimensional compact and connected symplectic manifold with at least one fixed point is Hamiltonian.

On the other hand, the following is a classical fact.

Proposition 2.3. A Hamiltonian $S^1$-action on a compact symplectic $2n$-manifold has at least $n + 1$ fixed points.

Proposition 2.3 follows from the fact the fixed points of the $S^1$-action are in correspondence with the critical points of the the momentum map $\mu$ of the action, and $\mu$ is a perfect Morse function which satisfies the Morse inequalities; we provide a detailed proof in Section 5. In view of Proposition 2.3, Theorem 2.1 gives the following.

Corollary 2.4 (Frankel '59). A symplectic Kähler $S^1$-action on a compact and connected Kähler $2n$-manifold has at least $n + 1$ fixed points.

We recall that a manifold $M$ is called unitary if the stable tangent bundle is endowed with a complex structure. Thus, every almost complex manifold, and hence every symplectic manifold, is unitary. For a real number $r$, let $\lfloor r \rfloor$ denotes its integer part. The following can be found in [Ko79].

Conjecture 1 (Kosniowski '79). Let $M$ be a $2n$-dimensional unitary $S^1$-manifold with isolated fixed points. If $M$ does not bound equivariantly, then the number of fixed points is at least $f(n)$, where $f(n)$ is a linear function, expected to be equal to $\lfloor n/2 \rfloor + 1$.

2.2. Some recent contributions. Let $S^1$ act on a compact symplectic manifold $(M, \omega)$ with momentum map $\mu: M \to \mathbb{R}$. Because the set of compatible almost complex structures $J: TM \to TM$ is contractible, there is a well-defined total Chern class of the tangent bundle $TM$, which we denote by $c^M = \sum_{j=0}^{n} c_j^M$. For every fixed point $p$ there is a well-defined multiset of integers, namely the weights of the $S^1$ action on $TM|_p$. Let $c_1(M)|_p$ be the first equivariant Chern class of $TM$ at $p \in M^{S^1}$, which one can naturally identify with an integer $c_1(M)(p)$: the sum of the weights at $p$. The map

$$ c_1(M): M^{S^1} \to \mathbb{Z}, \quad p \mapsto c_1(M)(p) \in \mathbb{Z}, $$

is called the Chern class map of $M$. Using the Atiyah-Bott and Berline-Vergne localization formula in equivariant cohomology and properties of Poincaré polynomials and Morse functions, Pelayo and Tolman ([PT11]) proved the following result.
Theorem 2.5 ([PT11]). Let $S^1$ act symplectically on a compact symplectic $2n$-dimensional manifold $M$. If $c_1(M): M^{S^1} \to \mathbb{Z}$ is somewhere injective\footnote{Let $f: X \to Y$ be a map between sets. We recall that $f$ is somewhere injective if there is $y \in Y$ such that $f^{-1}({\{y\}})$ is the singleton.} then the $S^1$-action has at least $n + 1$ fixed points.

Prior to Theorem 2.5, Tolman and Weitsman had proven a remarkable result for semifree actions.

Theorem 2.6 ([TW00]). Let $S^1$ act symplectically and semifreely on a compact symplectic $2n$-dimensional manifold $M$ with isolated fixed points. Then the $S^1$-action has at least $n + 1$ fixed points.

The following result complements Theorems 2.5 and 2.6 in low dimensions.

Theorem 2.7 ([PT11]). Let $S^1$ act symplectically on a compact symplectic manifold $M$. Suppose that the fixed point set $M^{S^1}$ is nonempty. Then there are at least two fixed points. If $\dim M \geq 8$, then there are at least three fixed points. Moreover, if $c_1(M): M^{S^1} \to \mathbb{Z}$ is not identically zero and $\dim M \geq 6$, then there are at least four fixed points.

The literature on non-necessarily Hamiltonian $S^1$-symplectic actions is extensive, see [TW00, Pe01, Li07, CHS10, Li07, LP12, Go05, Go06, LL11, LL10] and the references therein for related results. Non-necessarily Hamiltonian $(S^1)^k$-actions on manifolds of dimension $2k$ where studied in [DP07, Pe10].

3. An integer programming problem

3.1. Tools. In order to state and prove our results we need the following definition and the two previous results which follow it.

Definition 3.1 Let $(M, \omega)$ be a compact symplectic manifold on which a circle $S^1$ acts symplectically with nonempty discrete fixed point set $M^{S^1}$. For $i \in \mathbb{Z}$ let

$$N_i := \left| \{ p \in M^{S^1} \mid \lambda_p = i \} \right|,$$

where $\lambda_p$ is the number of negative weights at $p$ for all $p \in M^{S^1}$.

The following was proven in Pelayo-Tolman [PT11].

Theorem 3.2 ([PT11]). Let $S^1$ act symplectically on a compact symplectic $2n$-manifold with isolated fixed points. Then

$$N_i = N_{n-i} \quad \forall \ i \in \mathbb{Z}.$$

The following corresponds to [GoSa12, Theorem 1.2].
Theorem 3.3 ([GoSa12]). Let \((M, J)\) be an almost complex compact and connected manifold equipped with an \(S^1\)-action which preserves the almost complex structure \(J\) and has a discrete fixed point set. For every \(p = 0, \ldots, n\), let \(N_p\) be the number of fixed points with exactly \(p\) negative weights. Then

\[
\int_M c_1^M c_{n-1}^M = \sum_{p=0}^{n} N_p \left( 6p(p - 1) + \frac{5n - 3n^2}{2} \right),
\]

where \(c_1^M\) and \(c_{n-1}^M\) are respectively the first and \((n-1)\) Chern classes of \(M\).

Because of Theorem 3.3, the main equation for the purposes of this article is:

\[(3.1) \int_M c_1^M c_{n-1}^M = 0.\]

3.2. Discussion on assumption (3.1). We start with a remark.

Remark 3.4 The following hold:

(i) if \(c_1^M = 0\) or \(c_{n-1}^M = 0\) then (3.1) holds;
(ii) if a symplectic \(S^1\)-action is Hamiltonian then \(c_1^M \neq 0\) (see for example [T10, Lemma 3.8]).

Condition (3.1) is the assumption under which the results of this paper are proven. Next we show that in dimension 6 (3.1) is equivalent to requiring the \(S^1\)-action to be non-Hamiltonian. First we recall a few definitions and known facts.

By [Au91] (see also [Fe01, Proof of Theorem 1, paragraph 1]):

\[(3.2) \text{if the action is non-Hamiltonian then } N_0 = N_n = 0.\]

Fact (3.2) was used by Feldman in [Fe01] to characterize the Todd genus of a compact and connected symplectic manifold endowed with a symplectic \(S^1\)-action and discrete fixed point set. We recall that, given a compact almost complex manifold \((M, J)\), the Todd genus \(\text{Todd}(M)\) is the genus associated to the power series

\[
\frac{x}{1 - e^{-x}}.
\]

The following result is due to Feldman.

Theorem 3.5 ([Fe01]). The Todd genus associated to a compact and connected symplectic manifold with a symplectic \(S^1\)-action and discrete fixed point set is either one, in which case the action is Hamiltonian, or zero, in which case the action is not Hamiltonian.

We are ready to prove the following.
Proposition 3.6. Suppose that $S^1$ acts symplectically on a compact and connected $6$-dimensional manifold $M$ with nonempty discrete fixed point set. Then the $S^1$-action is non-Hamiltonian if and only if (3.1) holds.

Proof. It is sufficient to observe that when $\dim(M) = 6$, $$\text{Todd}(M) = \int_M \frac{c_1^M c_2^M}{24}.$$ The conclusion follows from Theorem 3.5. \qed

Question 3.7 Under which conditions does the claim in Proposition 3.6 hold if $2n \geq 8$?

3.3. Integer programming problem. In this section we present an approach for finding the minimal number of fixed points on a compact and connected symplectic manifold satisfying (3.1) endowed with a symplectic, but non-Hamiltonian, $S^1$-action with discrete fixed point set. In virtue of Theorem 2.2 we will henceforth assume that

$$2n = \dim(M) \geq 6.$$ Let $m \in \mathbb{Z}_{\geq 0}$ and let

(3.3) $F_1(N_1, \ldots, N_m) := N_m + 2 \sum_{k=1}^{m-1} N_{m-k}$;

(3.4) $F_2(N_1, \ldots, N_m) := 2 \sum_{k=1}^{m} N_k$;

$G_1(N_1, \ldots, N_m) := -mN_m + 2 \sum_{k=1}^{m-1} (6k^2 - m)N_{m-k}$;

$G_2(N_1, \ldots, N_m) := \sum_{k=0}^{m-1} (6k(k+1) - m + 1)N_{m-k}$.

For $i \in \{1, 2\}$ let

(3.5) $\mathcal{Z}_i := \left\{ N := (N_1, \ldots, N_m) \in (\mathbb{Z}_{\geq 0})^m \mid F_i(N) > 0, \ G_i(N) = 0 \right\}$.

We are ready to state our main result.

Theorem A. Let $(M, \omega)$ be a $2n$-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian $S^1$-action with nonempty, discrete fixed point set and such that (3.1) holds. Let $F_1, F_2$ be respectively
given in (3.3), (3.4), and let $Z_1, Z_2$ be given in (3.5). Then the number of fixed points of the $S^1$-action is greater than or equal to:

$$B(n) := \begin{cases} 
\min_{Z_1} F_1 & \text{if } n = 2m; \\
\min_{Z_2} F_2 & \text{if } n = 2m + 1.
\end{cases}$$

**Proof.** By Theorem 3.2 we have that $N_i = N_{n-i}$ for every $i \in \mathbb{Z}$. Since the $S^1$-action is symplectic but not Hamiltonian, by (3.2) we have $N_0 = N_n = 0$. Thus, since the total number of fixed points is

$$\sum_{k=1}^{n-1} N_k,$$

it follows that $F_1(N_1, \ldots, N_m)$ (resp. $F_2(N_1, \ldots, N_m)$) counts the total number of fixed points when $n = 2m$ (resp. $n = 2m + 1$), and, since the fixed point set is nonempty, we have $F_1 > 0$ (resp. $F_2 > 0$).

Moreover the constraint $G_1 = 0$ (resp. $G_2 = 0$) comes from combining (3.1) with Theorem 3.3. Let $g(p, n)$ be

$$6p(p-1) + \frac{5n - 3n^2}{2}.$$ 

If $n = 2m$, by Theorem 3.2 and (3.2), a computation shows that

$$\sum_{p=0}^{n} N_p g(p, n) = -m N_m + \sum_{k=1}^{m-1} \left( g(m-k, n) + g(m+k, n) \right) N_{m-k}$$

$$= -m N_m + 2 \sum_{k=1}^{m-1} (6k^2 - m) N_{m-k}$$

$$= G_1(N_1, \ldots, N_m).$$

Analogously, if $n = 2m + 1$ we have that

$$\sum_{p=0}^{n} N_p g(p, n) = \sum_{k=0}^{m-1} N_{m-k} \left( g(m-k, n) + g(m+k+1, n) \right)$$

$$= 2 \sum_{k=0}^{m-1} \left( 6k(k+1) - m + 1 \right) N_{m-k}$$

$$= 2 G_2(N_1, \ldots, N_m).$$

If $n = 2m$ (resp. $n = 2m + 1$), by Theorem 3.3 the constraint $G_1 = 0$ (resp. $G_2 = 0$) is equivalent to (3.1). □

The following integer programming problem, which is motivated by the conjecture of Kosniowski (Conjecture 1), arises from Theorem A.
Problem 3.8. Let \( i \in \{1, 2\} \). Let \( F_1, F_2 \) be respectively given in (3.3), (3.4), and let \( Z_1, Z_2 \) be given in (3.5). Find conditions on \( n \in \mathbb{Z}, n \geq 3, \) such that

\[
\min_{Z_i} F_i \geq \lceil n/2 \rceil + 1
\]

holds.

The table in Figure 3.3 provides a solution to Problem 3.8 when \( 2n \in \{8, 10, 12, 14, 18, 20, 22\} \). We have not solved it for \( 2n \geq 26 \).

Similarly, motivated by Frankel’s Theorem (Theorem 4.3), we propose the following sharper version of Problem 3.8.

Problem 3.9. Let \( i \in \{1, 2\} \). Let \( F_1, F_2 \) be respectively given in (3.3), (3.4), and let \( Z_1, Z_2 \) be given in (3.5). Find conditions on \( n \in \mathbb{Z}, n \geq 3, \) such that

\[
\min_{Z_i} F_i \geq n + 1
\]

holds.

Example 3.10 Using Theorem A we can compute \( B(n) \) for some values of \( n \). The table in Figure 3.3 gives \( B(n) \) for \( n \leq 12 \). For the sake of clarity we compute \( B(n) \) when \( n = 4 \) and \( 5 \), the other cases are analogous.

- \( 2n = 8 \). We have to minimize \( F_1(N_1, N_2) = N_2 + 2N_1 \) subject to the conditions \( G_1(N_1, N_2) = -2N_2 + 8N_1 = 0, F_1 > 0 \) and \( N_1, N_2 \in \mathbb{Z}_{\geq 0} \).
  
  One immediately sees that the values of \( N_1 \) and \( N_2 \) which minimize \( F_1 \) are respectively 1 and 4, yielding \( B(4) = 6 \).

- \( 2n = 10 \). In this case we have to minimize \( F_2(N_1, N_2) = 2(N_1 + N_2) \) subject to the conditions \( G_2(N_1, N_2) = -N_2 + 11N_1 = 0, F_2 > 0 \) and \( N_1, N_2 \in \mathbb{Z}_{\geq 0} \).
  
  A computation shows that the values of \( N_1 \) and \( N_2 \) which minimize \( F_2 \) are respectively \( N_1 = 1 \) and \( N_2 = 11 \), yielding \( B(5) = 24 \).

Remark 3.11 The proof of Theorem 3.3 in [GoSa12] makes use of equivariant \( K \)-theory. Theorem 3.3 is used to prove Theorem A. Theorem A leads to estimates for \( B(n) \) which, in some cases, improve previous estimates (see Figure 3.3). On the other hand, [PT11] uses equivariant cohomology to estimate \( B(n) \) (under different assumptions), eg. see Theorem 2.5.

We have not solved the optimization problem arising from Theorem A to estimate \( B(n), n \in \mathbb{N} \) (see also Problems 3.8, 3.9). However, we suspect that it could be amenable to techniques from linear programming. If this is the case, Theorem A could lead to the solution of the Kosniowski’s conjecture (Conjecture 1) for more values of \( n \in \mathbb{N} \), provided (3.1) holds.
| dim $M = 2n$ | $n+1$ | $[n/2]+1$ | minimal $|M^{S^1}|$ if $\int_M c_1^M c_{n-1}^M = 0$ |
|-------------|-------|----------|------------------------|
| 8           | 5     | 3        | 6                      |
| 10          | 6     | 3        | 24                     |
| 12          | 7     | 4        | 4                      |
| 14          | 8     | 4        | 12                     |
| **16**      | **9** | **5**    | **3**                  |
| 18          | 10    | 5        | 8                      |
| 20          | 11    | 6        | 12                     |
| 22          | 12    | 6        | 6                      |
| **24**      | **13**| **7**    | **2**                  |
| :           | :     | :        |                        |
| $2n$        | $n+1$ | $\mathcal{B}(n)$ |
| :           | :     | :        |                        |

**Figure 3.1.** Minimal number of fixed points of a symplectic $S^1$-action with nonempty discrete fixed point set on a compact and connected $2n$-dimensional manifold, under assumption (3.1). The $[n/2]+1$ bound appears in Conjecture 1. The $n+1$ bound is motivated by Theorem 4.3. Values in boldface are those for which $\mathcal{B}(n)$ is smaller than the value predicted in Conjecture 1.

4. Applications of Theorem A

The following result and its corollary are consequences of Theorem A. They provide some necessary conditions on the number of negative weights at the fixed points of the action.

**Theorem B.** Let $(M, \omega)$ be a $2n$-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian $S^1$-action. Suppose that the fixed point set of the action is nonempty and discrete, and that (3.1) holds. Then the following hold:

- If $n = 2m$ and $m \notin \{6k^2 \mid k \in \mathbb{Z}\}$, then:

  \[
  N_m + \sum_{k=1}^{\ell} N_{m-k} > 0, \quad \text{and} \quad \sum_{k=\ell+1}^{m-1} N_{m-k} > 0
  \]

  where

  $$\ell = \lfloor \sqrt{m/6} \rfloor.$$  

- If $n = 2m+1$ and $m \notin \{6k(k+1) + 1 \mid k \in \mathbb{Z}_{\geq 0}\}$, then:

  \[
  \sum_{k=0}^{\ell} N_{m-k} > 0, \quad \text{and} \quad \sum_{k=\ell}^{m-1} N_{m-k} > 0
  \]
where \( \ell = \left[ -3 + \frac{\sqrt{6m + 3}}{6} \right] \).

**Proof.** We may write the condition \( G_1 = 0 \) as

\[
2 \sum_{k=\lfloor \sqrt{m/6} \rfloor + 1}^{m-1} (6k^2 - m)N_{m-k} = mN_m + 2 \sum_{k=1}^{\lfloor \sqrt{m/6} \rfloor} (m - 6k^2)N_{m-k}.
\]

Then the coefficients of the \( N_i \)'s in the sums above are strictly positive in the range over which they are added. Since

\[
\sum_{i=1}^{n-1} N_i > 0,
\]

formula (4.1) follows.

The proof of (4.2) is similar to the proof of (4.1) in view of the fact that we may write the condition \( G_2 = 0 \) as

\[
\sum_{k=\lfloor \sqrt{m/6} \rfloor + 1}^{m-1} \left( m - 1 - 6k(k + 1) \right)N_{m-k} = \sum_{k=\lfloor \sqrt{m/6} \rfloor + 1}^{m-1} \left( 6k(k + 1) - m + 1 \right)N_{m-k},
\]

and each of the coefficients multiplying the \( N_i \)'s is positive. \( \square \)

From Theorem B we obtain the following (which complements Theorem 2.7).

**Corollary 4.1.** Let \((M, \omega)\) be a 2n-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian \( S^1 \)-action. Suppose that the fixed point set of the action is nonempty and discrete, and that (3.1) holds. Then the following hold:

1. Let \( n = 2m \) and \( m \notin \{6k^2 \mid k \in \mathbb{Z}\} \). Then:
   a. if \( N_m > 0 \) or there exists a \( \lfloor \sqrt{m/6} \rfloor + 1 \leq k \leq m - 1 \) such that \( N_{m-k} \neq 0 \) then the minimal number of fixed points is 3;
   b. if there exists a \( 1 \leq k \leq \lfloor \sqrt{m/6} \rfloor \) such that \( N_{m-k} \neq 0 \) then the minimal number of fixed points is 4;
   c. if \( m < 6 \) then \( N_m > 0 \).

2. Let \( n = 2m + 1 \) and \( m \notin \{6k(k + 1) + 1 \mid k \in \mathbb{Z}_{\geq 0}\} \). Then the minimal number of fixed points is 4.

**Proof.** (1) By (4.1), if \( N_m > 0 \) then there exists a \( \lfloor \sqrt{m/6} \rfloor + 1 \leq k \leq m - 1 \) such that \( N_{m-k} \neq 0 \); viceversa, if there exists a \( \lfloor \sqrt{m/6} \rfloor + 1 \leq k \leq m - 1 \)
such that \( N_{m-k} \neq 0 \) then either \( N_m \) or \( N_{m-h} > 0 \), where \( 1 \leq h \leq \lceil \sqrt{m/6} \rceil \), and (a) follows from Theorem 3.2.

If there exists a \( 1 \leq k \leq \lceil \sqrt{m/6} \rceil \) such that \( N_{m-k} \neq 0 \) then there exists \( \lceil \sqrt{m/6} \rceil + 1 \leq h \leq m-1 \) such that \( N_{m-h} \neq 0 \) which, by Theorem 3.2, implies (b).

The proof of (2) is analogous. \( \square \)

\textbf{Remark 4.2} Equation (3.6) may not hold, as it may be seen from Table 3.3 at \( n = 8 \). When \( \dim(M) = 6 \) the equation given by \( G_2 = 0 \) is an identity thus, by Theorem 3.2, \( B(3) = 2 \). More generally, it’s easy to see that when \( \dim(M) = 2n = 4m \) with \( m \in \{6k^2 \mid k \in \mathbb{Z}_{>0}\} \), or when \( \dim(M) = 2n = 4m + 2 \), with \( m \in \{6k(k+1) + 1 \mid k \in \mathbb{Z}_{\geq 0}\} \) then, by our procedure and Theorem 3.2, we get \( B(n) = 2 \).

The following is a consequence of Example 3.10.

\textbf{Corollary 4.3.} Let \((M, \omega)\) be a \( 2n \)-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian \( S^1 \) action with nonempty, discrete fixed point set and such that (3.1) holds. Suppose that \( 2n \in \{8, 10, 14, 20\} \). Then the \( S^1 \)-action has at least \( n+1 \) fixed points.

Under some assumptions, we can answer the following question.

\textbf{Question 4.4} ([PT11]) Suppose that \( n \) is an odd number. Is there a symplectic \( S^1 \)-action on a compact, connected symplectic \( 2n \)-manifold \((M, \omega)\) with exactly three fixed points, other than the standard actions on \( \mathbb{C}P^2 \)? \( \odot \)

Question 4.4 was settled by Jang recently.

\textbf{Theorem 4.5} ([Ja12]). Let \( S^1 \) act symplectically on a compact, connected symplectic manifold \((M, \omega)\). If there are exactly three fixed points, \( M \) is equivariantly symplectomorphic to \( \mathbb{C}P^2 \).

Corollary 4.1 (2) gives Theorem 4.5 with a simpler proof in the following cases.

\textbf{Corollary 4.6.} The answer to Question 4.4 is “No” whenever \( \dim(M) = 2m+1 \), (3.1) holds and \( m \notin \{6k(k+1) + 1 \mid k \in \mathbb{Z}_{>0}\} \).

\textbf{Theorem C.} Let \((M, \omega)\) be an 8-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian \( S^1 \)-action with nonempty, discrete fixed point set and such that \( c_1^M = 0 \). Then
\[
c_2^M \neq 0
\]
and
\[
\int_M (c_2^M)^2 \geq 2.
\]
Proof. The Todd genus of an 8-dimensional compact and connected symplectic manifold is given by

$$\text{Todd}(M) = \int_M \frac{-(c_1^M)^4 + 4(c_1^M)^2c_2^M + 3(c_2^M)^2 + c_1^M c_3^M - c_4^M}{720}. $$

Since by assumption $c_1^M = 0$ and the action is not Hamiltonian, by Theorem 3.5 we have

$$\int_M (c_2^M)^2 = \frac{1}{3} \int_M c_4^M. $$

By the Atiyah-Bott-Berline-Vergne Localization Theorem ([AtBo84, BV82]), it is straightforward to see that

$$\int_M c_4^M = \text{number of fixed points of the action}. $$

Since we are assuming $c_1^M = 0$, condition (3.1) is satisfied. Hence, by Example 3.10, the number of fixed points is greater or equal to 6, which, together with (4.3) gives the desired inequality, and hence $c_2^M \neq 0$. $\Box$

Remark 4.7 Theorem C does not immediately generalize to dimension 10. In this case the Todd polynomial is of the form $c_1^M C$, where $C$ is a combination of Chern classes of degree 8. Hence, if we assume $c_1^M = 0$, the Todd genus is zero. It would be interesting to understand under which conditions Theorem C generalizes to dimension 10 or higher.

We have the following consequence of Corollary 2.5 and Theorem A.

Theorem D. Let $(M, \omega)$ be an $2n$-dimensional compact and connected symplectic manifold with a symplectic but non-Hamiltonian $S^1$ action with nonempty, discrete fixed point set, such that (3.1) is satisfied. If the number of fixed points is in $[B(n), n]$ then the Chern class map is not somewhere injective.

5. Final remarks

Proposition 2.3 follows from the fact that the momentum map $\mu$ is a Morse-Bott function, whose set of critical points Crit($\mu$) is a submanifold of $M$, and coincides with the fixed point set of the action. Thus, if it is not zero dimensional, then there are infinitely many critical points of $\mu$ and the result is obvious. If it is zero dimensional, then $\mu$ is a perfect Morse function (i.e., the Morse inequalities are equalities) because of the following classical result: If $f$ is a Morse function on a compact and connected manifold whose critical points have only even indices, then it is a perfect Morse function (e.g., [Ni07, Corollary 2.19 on page 52]).
Let $m_k(\mu)$ be the number of critical points of $\mu$ of index $k$. The total number of critical points of $\mu$ is
\[
\sum_{k=0}^{2n} m_k(\mu) = \sum_{k=0}^{2n} b_k(M),
\]
where $b_k(M) := \dim \left( H^k(M, \mathbb{R}) \right)$ is the $k$th Betti number of $M$. The classes $[\omega^k]$ are nontrivial in $\mathbb{H}^{2k}(M, \mathbb{R})$ for $k = 0, \ldots, n$, so $b_{2k}(M) \geq 1$, and hence the number of critical points of $\mu$ is at least $n + 1$.

One can try to use Theorem 5.1 below to deduce a result analogous to Proposition 2.3 for circle valued momentum maps by replacing the Morse inequalities by the Novikov inequalities (see [Pa06, Chapter 11, Proposition 2.4], [Fa04, Theorem 2.4]), if all the critical points of $\mu$ are non-degenerate.

**Theorem 5.1** (McDuff, ‘88). Let the circle $S^1$ act symplectically on the compact connected symplectic manifold $(M, \sigma)$. Then either the action admits a standard momentum map or, if not, there exists a $S^1$-invariant symplectic form $\omega$ on $M$ that admits a circle valued momentum map $\mu : M \to S^1$. Moreover, $\mu$ is a Morse-Bott-Novikov function and each connected component of $M^{S^1} = \text{Crit}(\mu)$ has even index. If $\sigma$ is integral, then $\omega = \sigma$.

The number of critical points of the circle-valued momentum map $\mu$ in Theorem 5.1 is $\sum_{k=0}^{2n} m_k(\mu)$. This integer is estimated from below by
\[
\sum_{k=0}^{2n} \left( \hat{b}_k(M) + \hat{q}_k(M) + \hat{q}_{k-1}(M) \right),
\]
where $\hat{b}_k(M)$ is the rank of the $\mathbb{Z}((t))$-module $H_k(\widetilde{M}, \mathbb{Z}) \otimes \mathbb{Z}[t, t^{-1}] \mathbb{Z}(t)$, $\hat{q}_k(M)$ is the torsion number of this module, and $\widetilde{M}$ is the pull back by $\mu : M \to \mathbb{R}/\mathbb{Z}$ of the principal $\mathbb{Z}$-bundle $t \in \mathbb{R} \mapsto [t] \in \mathbb{R}/\mathbb{Z}$. Unfortunately, this lower bound can be zero. We refer to [PR12, Sections 3 and 4] for a detailed proof of Theorem 5.1 and [PR12, Remark 6] for further details.

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