Upscaling Singular Sources in Weighted Sobolev Spaces by Sub-Grid Corrections

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Abstract

In this paper, we develop a numerical multiscale method to solve elliptic boundary value problems with heterogeneous diffusion coefficients and with singular source terms. When the diffusion coefficient is heterogeneous, this adds to the computational costs, and this difficulty is compounded by a singular source term. For singular source terms, the solution does not belong to the Sobolev space $H^1$, but to the space $W^{1,p}$ for some $p < 2$. Hence, the problem may be reformulated in a distance-weighted Sobolev space. Using this formulation, we develop a method to upscale the multiscale coefficient near the singular sources by incorporating corrections into the coarse-grid. Using a sub-grid correction method, we correct the basis functions in a distance-weighted Sobolev space and show that these corrections can be truncated to design a computationally efficient scheme with optimal convergence rates. Due to the nature of the formulation in weighted spaces, the variational form must be posed on the cross product of complementary spaces. Thus, two such sub-grid corrections must be computed, one for each multiscale space of the cross product. A key ingredient of this method is the use of quasi-interpolation operators to construct the fine scale spaces. Therefore, we develop a weighted projective quasi-interpolation that can be used for a general class of Muckenhoupt weight functions. We verify the optimal convergence of the method in some numerical examples with singular point sources and line fractures, and with oscillatory and heterogeneous diffusion coefficients.

Keywords: localization, multiscale methods, singular data, weighted Sobolev spaces

1 Introduction

Computing flow in heterogeneous porous media is a difficult problem due to the high-contrast in material properties as well as the large disparate scales of the permeability or hydraulic conductivity. To simplify the calculation, an upscaled or effective model is preferred so that many models and scenarios may be tested. The computational upscaling, or numerical homogenization, of complex porous media has a large literature in various areas of applications in petroleum, environmental, and materials engineering. One key aspect, particularly in subsurface modeling, is the upscaling of material properties in the neighborhood of singular sources, i.e. near wells or fractured injection/production sites. The upscaling of numerical simulations near the singular wellbore source in petroleum engineering has its roots in the work of Peaceman [44]. Here, special care must be taken in upscaling near the well as it is modeled by a singular Dirac source at the production site. There are various procedures for upsampling near wells in subsurface modeling, cf. [15, 19] for a general survey. In addition to point sources, complex fracture networks of linear or planar type sources are also considered and often need to be upscaled for fast efficient simulation [22, 24, 30].

The simulation of the fine-grid (non-upscaled) problem also poses unique challenges in this setting. Given the standard regularity, a $H^{-1}$ source, material properties that are $L^\infty$-elliptic, and a sufficiently regular domain, the solution of the elliptic partial differential equation is in the Hilbert space $H^1$. However, from

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2 Problem Setting and Background

In this section we will introduce the problem setting and some notation for the relevant distance-weighted Sobolev spaces. We introduce the idea of a Muckenhoupt weight, which yields a class of weighted spaces that have a valid Poincaré inequality. For a certain subclass of distance-weighted exponents we have a trace inequality from the singular source to the interior of the domain. This fact, along with a useful $L^2$-type decomposition will give us well-posedness, as well as a-priori bounds.

2.1 Elliptic Problems with Singular Sources

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and connected domain for $d = 2, 3$, with Lipschitz boundary. We seek to solve the following heterogeneous Laplace equation with Dirichlet boundary condition for $u$

$$\begin{align*}
-\text{div}(A \nabla u) &= f \delta_\Lambda \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}$$

where $\delta_\Lambda$ is the Dirac mass on $\Lambda$. Here $\Lambda$ is a sufficiently smooth closed submanifold of $\Omega$, such that dim($\Lambda$) = $\ell < d$, for $\ell = 0, 1, 2$. For simplicity we suppose, $\ell = 0$ is a point-source $x_0$, $\ell = 1$ corresponds to a piecewise line fracture, and for $\ell = 2$ this corresponds to a planar-type fracture.

2.2 Weighted Sobolev Spaces

To facilitate the solution of (1) we need additional notation. We define the following class of weighted Sobolev spaces for a positive weight $d^{2\beta}_\Lambda$. For $x \in \mathbb{R}^d$, let $dx$ be the Lebesgue measure on $\mathbb{R}^d$, and $ds$ on $\mathbb{R}^\ell$. We will use the notation $A \lesssim B$ if there exists a $C > 0$ such that $A < CB$, where $C$ is independent of the mesh, but may depend on other parameters such as $\beta, d, \ell, \Omega, \gamma_1, \gamma_2$, etc. For an open set $\omega \subset \mathbb{R}^d$, we define $L^2_{\beta}(\omega)$ to be all measurable functions $u$ on $\omega$, such that

$$\|u\|_{L^2_{\beta}(\omega)} = \left(\int_\omega u^2 d^{2\beta}_\Lambda dx\right)^{1/2} < \infty,$$

for $\beta \in (-\frac{d-\ell}{2}, \frac{d-\ell}{2})$, so that the weight is of Muckenhoupt class $A_2(\mathbb{R})$, cf. [3]. Define $H^1_{\beta}(\omega)$ similarly, all measurable functions $u$ on $\omega$, such that

$$\|u\|_{H^1_{\beta}(\omega)} := \left(\|u\|_{L^2_{\beta}(\omega)}^2 + \|\nabla u\|_{L^2_{\beta}(\omega)}^2\right)^{1/2} < \infty,$$

and we denote the space incorporating the vanishing boundary condition as

$$H^1_{0,\beta}(\omega) = \{u \in H^1_{\beta}(\omega) : u = 0 \text{ on } \partial \omega\}.$$

Integrating (1) by parts we obtain the following weak form. We seek a solution $u \in H^1_{0,\beta}(\Omega)$, so that

$$a(u, \psi) = l(\psi) \text{ for all } \psi \in H^1_{0,-\beta}(\Omega),$$

where

$$a(u, \psi) = \int_\Omega \text{div}(A \nabla u) \psi \, dx - \int_\Omega f \delta_\Lambda \psi \, dx + \int_{\partial \Omega} u \frac{\partial \psi}{\partial n} \, ds,$$

and

$$l(\psi) = \int_\Omega \psi \, dx.$$
where \(a(\cdot, \cdot) : H^1_{0,\beta}(\Omega) \times H^1_{0,-\beta}(\Omega) \to \mathbb{R}\) is the bilinear form

\[
a(u, \psi) = \int_\Omega A \nabla u \nabla \psi \, dx,
\]

and we suppose, with more generality than the source term in (1), that \(l \in (H^3_{0,-\beta}(\Omega))'\).

A key property of the distance weight \(d^{2\beta}_\Lambda\) is that it belongs to the Muckenhoupt class \(A_2(\mathbb{R}^d)\), [23, 25, 40, 43]. For a general weight, \(w \in L^1_{\text{loc}}(\mathbb{R}^d)\), we say that \(w \in A_p(\mathbb{R}^d)\) if there exists a \(C_{p,w} > 0\) such that

\[
\sup_{B} \left( \frac{1}{|B|} \int_B w \, dx \right) \left( \frac{1}{|B|} \int_B w^{\frac{1}{p'}} \, dx \right)^{p-1} = C_{p,w} < \infty,
\]

for all balls \(B \subset \mathbb{R}^d\). We will denote the Muckenhoupt weight constant for \(w\) as \(C_{p,w}\).

**Proposition 2.1** Suppose that \(\beta \in (-\frac{d-\ell}{2}, \frac{d-\ell}{2})\), then the weight \(d^{2\beta}_\Lambda \in A_2(\mathbb{R}^d)\), where \(\dim(\Lambda) = \ell\). More explicitly, we have for balls \(B\) in \(\mathbb{R}^d\) that

\[
\sup_{B} \left( \frac{1}{|B|} \int_B d^{2\beta}_\Lambda \, dx \right) \left( \frac{1}{|B|} \int_B d^{-2\beta}_\Lambda \, dx \right) = C_{2,\beta} < \infty.
\]

**Proof** The case of a point-source, \(\ell = 0\) and \(d = 2, 3\), can be found in [33]. The case of a linear fracture, \(\ell = 1\) and \(d = 3\) in [18, 17]. The general case can be found in [25] and references therein. □

The key inequality that holds existence, uniqueness, and the general analysis together is the weighted Poincaré inequality. The weighted Poincaré inequality for Muckenhoupt weights is well studied in nonlinear potential theory of degenerate problems [21, 35, 26] and references therein.

**Lemma 2.2 (Distance-Weighted Poincaré Inequality)** Let \(\omega \subset \Omega\), be a bounded, star-shaped domain (with respect to the ball \(B\)) and \(\text{diam}(\omega) \approx H\). Suppose that \(\beta \in (-\frac{d-\ell}{2}, \frac{d-\ell}{2})\), if \(w \in H^1_\beta(\omega)\) then we have

\[
\|w - \langle w \rangle_\omega\|_{L^2_\beta(\omega)} \lesssim H \|\nabla w\|_{L^2_\beta(\omega)},
\]

where the constants are independent of \(H\) and \(\langle w \rangle_\omega = \frac{1}{|\omega|} \int_\omega w \, dx\).

**Proof** By Proposition 2.1, we have \(d^{2\beta}_\Lambda \in A_2(\mathbb{R}^d)\), and so by the general Muckenhoupt weighted Poincaré inequality [43, Corollary 3.2], we easily obtain the result. □

**Remark** As noted in [9], the above inequality may be extended to a connected union of star-shaped domains where the average can be taken over a subdomain. This can be proven in a similar way to [22, Corollary 4.4]. We will refer to both of these results simply as the weighted Poincaré inequality when there is no ambiguity. Further, for completeness, we note a similar Friedrich’s type inequality also holds for \(w \in H^1_{0,\beta}(\omega)\),

\[
\|w\|_{L^2_\beta(\omega)} \lesssim H \|\nabla w\|_{L^2_\beta(\omega)}.
\]

We have the following decomposition of \(L^2_\beta(\Omega; \mathbb{R}^d)\) that is critical for existence and uniqueness of solutions to (2) in weighted spaces.

**Lemma 2.3 (Decomposition of \(L^2_\beta(\Omega; \mathbb{R}^d)\))** Let \(\beta \in (-\frac{d-\ell}{2}, \frac{d-\ell}{2})\), for \(\tau \in L^2_\beta(\Omega; \mathbb{R}^d)\) there exist a pair \((\sigma, z) \in L^2_\beta(\Omega; \mathbb{R}^d) \times H^1_{0,\beta}(\Omega)\) such that

\[
\tau = \nabla z + \sigma, \quad \langle A \sigma, \nabla w \rangle_\Omega = 0, \forall w \in H^1_{0,-\beta}(\Omega),
\]

\[
\|\nabla z\|_{L^2_\beta(\Omega)} \lesssim \|\tau\|_{L^2_\beta(\Omega)}, \quad \|\sigma\|_{L^2_\beta(\Omega)} \lesssim \|\tau\|_{L^2_\beta(\Omega)}.
\]
Proof Since we have the weighted Poincaré-Friedrich’s inequalities from Lemma 2.2 for \( \beta \in \left(-\frac{d-\ell}{2}, \frac{d-\ell}{2}\right) \), we see that an immediate generalization of [17, Lemma 2.1] is possible, and the same abstract proof holds. □

From this Lemma, as in [3, 17] we establish the well-posedness of the abstract problem (2).

**Theorem 2.4** Let \( \beta \in \left(-\frac{d-\ell}{2}, \frac{d-\ell}{2}\right) \), then the abstract problem (2) is well-posed, and we have the following stability bound

\[
\|u\|_{H_0^1(\Omega)} \lesssim \|f\|_{H_{\alpha}^1(\Omega)}.
\]  

**Proof** This is an immediate corollary of Lemma 2.3, cf. [17, Corollary 2.2]. □

The above theorem is for more general source terms than we will consider in this work. We will focus on singular source terms and so must consider a smaller class of function spaces, and values here, which we will denote as \( \alpha \). To this end, we introduce the natural trace space related to Dirac measures \( \delta_\Lambda \).

**Lemma 2.5 (Distance-Weighted Trace Inequality)** Suppose \( \dim(\Lambda) = \ell, \dim(\Omega) = d, \) and \( 0 \leq \ell \leq d - 1 \), and that \( \alpha \) is so that

\[
\frac{d-\ell}{2} - 1 < \alpha < \frac{d-\ell}{2}.
\]  

Then, there exists a bounded continuous trace operator \( \text{tr}_{\Lambda}(\cdot) : L^2(\Lambda) \to H^1_{-\alpha}(\Omega) \). We have the following bound

\[
\|v\|_{L^2(\Lambda)} \lesssim \|v\|_{H_{\alpha}^1(\Omega)},
\]  

where the hidden constant depends on \( \alpha \) and \( \Lambda \).

**Proof** The case of a point-source, \( \ell = 0 \) and \( d = 2, 3 \), can be found in [8]. The case of a linear fracture, \( \ell = 1 \) and \( d = 3 \) in [17, 18]. For a general discussion on trace spaces of distance-weighted spaces we refer to [41], where one can see the general bounds; in particular for the case of planar type fractures \( \ell = 2 \) and \( d = 3 \), as well as the case of a linear fracture \( \ell = 1 \) in \( d = 2 \) dimensional space. □

Thus, we have the following well-posedness for singular source terms.

**Corollary 2.6** Suppose \( \dim(\Lambda) = \ell, \dim(\Omega) = d, \) and \( 0 \leq \ell \leq d - 1 \). Let \( \alpha \in \left(\frac{d-\ell}{2} - 1, \frac{d-\ell}{2}\right) \), then with Dirac measure data, \( f\delta_\Lambda \), for \( f \in L^2(\Lambda) \), is well-posed, and we have the following stability bound

\[
\|u\|_{H^1_{\alpha}(\Omega)} \lesssim \|f\|_{L^2(\Lambda)}.
\]  

**Proof** This is an immediate corollary of Lemma 2.3 and the trace Lemma 2.5. The arguments can be see in more detail in [17, Remark 1]. □

3 Quasi-Interpolation in Distance-Weighted Sobolev Spaces

The multiscale method utilized in this paper, as well as previous works [7, 9, 10, 38], relies on the construction of a projective quasi-interpolation operator. Here we construct a quasi-interpolation operator for distance-weighted Sobolev spaces using weighted local \( L^2 \) projections onto simplices in a similar vein to the authors in [5]. Much of this presentation will follow that of [9], where the authors handled a specific type of weight for fractional Laplacians. We introduce the discretization and classical nodal basis. We then state the local stability and approximability properties of these operators.
3.1 Coarse Grid Finite Elements

Here we follow much of the notation in [9, 37]. We suppose that we have a coarse quasi-uniform, shape-
regular discretization $T_H$ of the domain $\Omega$ with characteristic mesh size $H$. In this work, we will not consider
errors from the fine-grid $h$. We denote the nodes of the mesh $\mathcal{N}$. The interior nodes of $\Omega$ (not including
vanishing Dirichlet condition) we denote as $\mathcal{N}_{\text{int}}$, and the Dirichlet nodes as $\mathcal{N}_{\text{dir}}$. We will write $\mathcal{N}(\omega)$ for
nodes restricted to $\omega$, similarly for interior, or Dirichlet nodes. We suppose further that there is a $T \in T_H$
such that $\Lambda \subset T$. Note that if the source intersected a small number of triangles, taking the intersected
patch would also be sufficient here.

Let the classical conforming $P_1$ finite element space over $T_H$ be given by $S_H$, and let $V_H = S_H \cap H^1_{0,\beta}(\Omega)$.
Utilizing the notation in [42], we denote $v \in \mathcal{N}$ as nodal values. The $P_1$ nodal basis function $\lambda_v$, for a node $v \in \mathcal{N}$, is written as

$$
\lambda_v(v) = 1 \text{ and } \lambda_v(w) = 0, \; v \neq w \in \mathcal{N}.
$$

This is a basis for $V_H$. We define the patch around $v$ as

$$
\omega_v = \bigcup_{T \ni v} T,
$$

for $T \in T_H$. We define for any patch $\omega_v$ the extension patch

$$
\omega_v = \omega_{v,0} = \text{supp}(\lambda_v) \cap \Omega, \quad \omega_v,k = \text{int}(\{T \in T_H | T \cap \omega_{v,k-1} \neq \emptyset\} \cap \Omega),
$$

for $k \in \mathbb{N}_+$. We will denote $V_H|\omega$ to be the coarse grid space restricted to some domain $\omega$.

We further suppose that for these patches $\frac{|B|}{|\omega_v,k|} \lesssim 1$, for some ball $B$ containing $\omega_{v,k}$. Thus, for
$\beta \in \left(-\frac{d-k}{2}, \frac{d-k}{2}\right)$, we have the bound

$$
\left(\frac{1}{|\omega_{v,k}|} \int_{\omega_{v,k}} d_\Lambda^{2\beta} dx\right) \left(\frac{1}{|\omega_{v,k}|} \int_{\omega_{v,k}} d_\Lambda^{-2\beta} dx\right) \lesssim \left(\frac{|B|}{|\omega_{v,k}|} \frac{1}{|B|} \int_B d_\Lambda^{2\beta} dx\right) \left(\frac{1}{|\omega_{v,k}|} \frac{1}{|B|} \int_B d_\Lambda^{-2\beta} dx\right)
$$

$$
\lesssim \left(\frac{|B|}{|\omega_{v,k}|}\right)^2 \left(\frac{1}{|B|} \int_B d_\Lambda^{2\beta} dx\right) \left(\frac{1}{|B|} \int_B d_\Lambda^{-2\beta} dx\right) \lesssim \left(\frac{|B|}{|\omega_{v,k}|}\right)^2 C_{2,\beta} \lesssim C_{2,\beta}.
$$

(12)

Where we utilized the bound (3) and Proposition 2.1 hence we can apply the Muckenhoupt weight bounds to
our patches.

3.2 Quasi-Interpolation Operator

In a related setting, the authors in [42, 43] construct a quasi-interpolation based on a higher order Clément
type of operator. In this section, we will construct a slightly different quasi-interpolation that is also projec-
tive. This projective quasi-interpolation satisfies the requisite stability and approximability properties. This
is a modification of the operator of [5] and was utilized in perforated domains in [10] and in [9] for fractional
Laplacians.

We now define the $d_\Lambda^{2\beta}$-weighted local $L^2$ projections, for $\beta \in \left(-\frac{d-k}{2}, \frac{d-k}{2}\right)$. For $v \in \mathcal{N}_{\text{int}}$, $P_v : L^2_\beta(\omega_v) \rightarrow V_H|\omega_v$ is a weighted projection in the sense that

$$
\int_{\omega_v} (P_v u) v_H d_\Lambda^{2\beta} dx = \int_{\omega_v} (v_H u) d_\Lambda^{2\beta} dx \text{ for all } v_H \in V_H|\omega_v.
$$

(13)

From this we define the quasi-interpolation operator $I^\beta_H : H^1_\beta(\Omega) \rightarrow V_H$ as

$$
I^\beta_H u(x) = \sum_{v \in \mathcal{N}_{\text{int}}} (P_v u)(v) \lambda_v(x).
$$

(14)

6
Note that this quasi-interpolation assumes zero Dirichlet boundary conditions as we sum over the interior nodes \( \mathcal{N}_{\text{int}} \) only. If we have non-trivial Dirichlet conditions, techniques to handle the boundary nodes will have to be employed as in [9, 47], or even additional boundary corrections have to be computed as in [27].

### 3.3 Local Stability and Approximability

The quasi-interpolation operator \( I^\beta_H \) defined by (14) satisfies the following stability and local approximation properties. The proof of this lemma is based on that presented in [9, 39], but is slightly simpler here since we do not have to treat non-zero boundary terms. Since the proof is only slightly different, we leave it for the Appendix A.

**Lemma 3.1** Let \( I^\beta_H \) be given by (14) and \( v \in \mathcal{N} \). Suppose that \( \beta \in \left( -\frac{d-\ell}{2}, \frac{d-\ell}{2} \right) \). The quasi-interpolation satisfies the following local stability estimates for all \( u \in H^1(\Omega) \),

\[
\left\| \nabla I^\beta_H u \right\|_{L^2(\omega_v)} \lesssim \left\| \nabla u \right\|_{L^2(\omega_v)},
\]

(15b)

Further, the quasi-interpolation satisfies the following local approximation properties

\[
\left\| u - I^\beta_H u \right\|_{L^2(\omega_v)} \lesssim H \left\| \nabla u \right\|_{L^2(\omega_v)},
\]

(16a)

\[
\left\| \nabla (u - I^\beta_H u) \right\|_{L^2(\omega_v)} \lesssim \left\| \nabla u \right\|_{L^2(\omega_v)}.
\]

(16b)

Moreover, the quasi-interpolation \( I^\beta_H \) is a projection.

**Proof** See Appendix A. □

### 4 Numerical Upscaling Method

We now will construct our multiscale approximation space to handle the oscillations created by the heterogeneities of the coefficient and the sub-grid singular source terms. The singular source terms are incorporated into the coarse-grid corrections. This splitting can be found in [29, 37] and references therein. We begin by constructing fine-scale spaces, that contain the small scale information, as well as singular source information via the distance weight.

#### 4.1 Construction of the Multiscale Space

We define the kernel quasi-interpolation operator for \( \beta \in \left( -\frac{d-\ell}{2}, \frac{d-\ell}{2} \right) \) to be the subspace

\[
V^\beta_f = \{ v \in H^1(\Omega) \mid I^\beta_H v = 0 \}.
\]

These spaces will capture the sub-grid scale singular features not resolved by \( V_H \). Note that by stability of \( I^\beta_H \), this is a closed subspace of \( H^1(\Omega) \), as this will be needed for the fine-scale decomposition of \( L^2(\Omega) \).

We define the corrector \( Q^\beta_{\Omega} : V_H \to V^\beta_f \) to be the projection operator such that for \( v_H \in V_H \) we compute \( Q^\beta_{\Omega}(v_H) \in V^\beta_f \) as

\[
\int_{\Omega} A \nabla Q^\beta_{\Omega}(v_H) \nabla w d^2 \Lambda = \int_{\Omega} A \nabla v_H \nabla w d^2 \Lambda, \quad \text{for all } w \in V^\beta_f.
\]

(17)
We use the correctors to define the multiscale space

$$V_{H}^{ms,\beta} = (V_H - Q_{\Omega}^{\beta}(V_H)) \subset H^{1}_{0,\beta}(\Omega). \quad (18)$$

This projection gives a $Ad^{2\beta}_{\Lambda}$-weighted orthogonal splitting

$$H^{1}_{0,\beta}(\Omega) = V_{H}^{ms,\beta} \oplus Ad^{\beta}_{\Lambda} V_{\beta}^{f},$$

so that for $u \in H^{1}_{0,\beta}(\Omega)$ and $u_{H}^{ms} \in V_{H}^{ms,\beta}$, we have $u - u_{H}^{ms} \in V_{\beta}^{f}$. Further, we note in the following Proposition 4.1 that these correctors are well posed, and thus the multiscale space exists.

**Proposition 4.1** The corrector problem (17) is well posed and $Q_{\Omega}^{\beta}$ satisfies the bound

$$\left\| \nabla Q_{\Omega}^{\beta}(v_H) \right\|_{L^{2}_{\Lambda}(\Omega)} \lesssim \left\| \nabla v_H \right\|_{L^{2}_{\Lambda}(\Omega)}. \quad (19)$$

**Proof** It is trivial to see that the variational form (17) is coercive and bounded on the closed subspace $V_{\beta}^{f} \subset H^{1}_{0,\beta}(\Omega)$, so the Lax-Milgram theorem holds. The difficulty is to obtain the bounds and to make sure that the right-hand side is well posed. In problem (17), take $w \in V_{\beta}^{f}$ to be $Q_{\Omega}^{\beta}(v_H)$, then we have

$$\left\| \nabla Q_{\Omega}^{\beta}(v_H) \right\|_{L^{2}_{\Lambda}(\Omega)}^{2} \lesssim \int_{\Omega} |A^{1/2} \nabla Q_{\Omega}^{\beta}(v_H)|^{2} d^{2\beta}_{\Lambda} \, dx = \int_{\Omega} A \nabla v_H \nabla Q_{\Omega}^{\beta}(v_H) d^{2\beta}_{\Lambda} \, dx$$

$$\lesssim \left( \int_{\Omega} |A^{1/2} \nabla v_H|^{2} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} \left( \int_{\Omega} |A^{1/2} \nabla Q_{\Omega}^{\beta}(v_H)|^{2} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} .$$

Here we used the coercivity and the boundedness of the bilinear form, and we obtain

$$\left\| \nabla Q_{\Omega}^{\beta}(v_H) \right\|_{L^{2}_{\Lambda}(\Omega)} \lesssim \left( \int_{\Omega} |\nabla v_H|^{2} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} \lesssim \left\| \nabla v_H \right\|_{L^{2}_{\Lambda}(\Omega)} < \infty .$$

This is due to the fact that $V_{H} \subset H^{1}_{0,\beta}(\Omega)$ for $\beta \in \left(-\frac{d-f}{2}, \frac{d-f}{2}\right)$, which can be seen from two cases. The trivial case is when $\beta \in (0, \frac{d-f}{2})$, and so $d^{2\beta}_{\Lambda} \|_{L^{\infty}} \lesssim C(\Omega) < \infty$, and thus

$$\left( \int_{\Omega} |\nabla v_H|^{2} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} \lesssim C(\Omega) \left( \int_{\Omega} |\nabla v_H|^{2} \, dx \right)^{1/2} < \infty .$$

Now we suppose $\beta \in (-\frac{d-f}{2}, 0)$, and denote with $T$ the triangle where $\nabla v_H$ obtains its maximum, then

$$\left( \int_{\Omega} |\nabla v_H|^{2} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} \lesssim \left\| \nabla v_H \right\|_{L^{\infty}(T)} \left( \int_{\Omega} d^{2\beta}_{\Lambda} \, dx \right)^{1/2} < \infty ,$$

since $d^{2\beta}_{\Lambda} \in L^{1}(\Omega)$, if $\beta \in (-\frac{d-f}{2}, 0)$, which can be seen from [3] Lemma 2.2].

Note here this $\beta$ interval is the most general, and only takes into account the values where the distance function is of Muckenhoupt class. For a singular source term, we need the restricted interval $\alpha \in \left(\frac{d-f}{2} - 1, \frac{d-f}{2}\right)$. The multiscale problem is defined on a cross product:

$$V_{H}^{ms,\alpha} \times V_{H}^{ms,-\alpha} \subset H^{1}_{0,\alpha}(\Omega) \times H^{1}_{0,-\alpha}(\Omega).$$

We refer to these modified coarse spaces, $V_{H}^{ms,\pm \alpha}$, as the “ideal” multiscale spaces. The multiscale Galerkin approximation $u_{H}^{ms} \in V_{H}^{ms,\alpha}$ to (1) satisfies

$$\int_{\Omega} A \nabla u_{H}^{ms} \nabla v \, dx = \int_{\Lambda} f v \, ds \ \text{for all} \ v \in V_{H}^{ms,-\alpha}. \quad (20)$$
Remark: Note that we must have this $\pm \alpha$ pairing due to the bilinear form acting on the cross product
\[ a(\cdot, \cdot) : V_H^{ms,\alpha} \times V_H^{ms,-\alpha} \subset H^1_{0,\alpha}(\Omega) \times H^1_{0,-\alpha}(\Omega) \to \mathbb{R}. \]

In addition, due to the requirements of the trace theorem for singular data, we must have
\[ v \in V_H^{ms,-\alpha} \subset H^1_{0,-\alpha}(\Omega) \]
so that error bounds may be obtained.

4.2 Truncated Multiscale Space

The solution of (17) requires the calculation of global correctors. However, it is now well established that in most diffusive regimes the correctors decay exponentially. To this end, we define the localized fine-scale space to be the fine-scale space extended by zero outside the patch, that is in the larger $\beta$ interval
\[ V^f_{\beta}(\omega_v,k) = \{ v \in V_H^f \mid \text{both zero on } \partial \Omega_\beta \}. \]

We let for some $v \in \mathcal{N}_{int}$ and $k \in \mathbb{N}$ the localized corrector operator $Q^\beta_{v,k} : V_H \to V^f_{\beta}(\omega_v,k)$, be defined such that given a $v_H \in V_H$
\[ \int_{\omega_v,k} A \nabla Q^\beta_{v,k}(v_H) \nabla w \, d^2 \Lambda \, dx = \int_{\omega_v,k} A \hat{\lambda}_v \nabla v_H \nabla w \, d^2 \Lambda \, dx, \quad \text{for all } w \in V^f_{\beta}(\omega_v,k), \quad (21) \]
where $\hat{\lambda}_v = \frac{\sum_{v' \in \mathcal{N}_{int}} \lambda_{v'}}{\sum_{v \in \mathcal{N}_{int}} \lambda_v}$ is augmented due to the zero Dirichlet condition. The collection \{ $\hat{\lambda}_v$ \}_{v \in \mathcal{N}_{int}} is a partition of unity [29]. We denote the global truncated corrector operator as
\[ Q^\beta_k(v_H) = \sum_{v \in \mathcal{N}_{int}} Q^\beta_{v,k}(v_H). \quad (22) \]

With this notation, we write the truncated multiscale space as
\[ V^{ms,\beta}_{H,k} = (V_H - Q^\beta_k(V_H)) \subset H^1_{0,\beta}(\Omega). \quad (23) \]

Then, the corresponding truncated multiscale approximation to (1) is: find $u^{ms}_{H,k} \in V^{ms,\beta}_{H,k}$ such that
\[ \int_{\Omega} A \nabla u^{ms}_{H,k} \nabla v \, dx = \int_{\Lambda} f v ds \quad \text{for all } v \in V^{ms,-\alpha}_{H,k}. \quad (24) \]
This more efficient scheme is utilized in the numerical experiments.

Note also that for sufficiently large $k$, we recover the full domain and obtain the ideal corrector with functions of global support, denoted $Q^\beta_{\Omega}$, from (17).

5 Error Analysis

In this section we present the error introduced by using (20) on the global domain to compute the solution to (1). Then, we show how localization effects the error when we use (24) on truncated domains. The key component of these error estimates is related to the trace spaces from Lemma 2.5 and the a-priori estimate from Corollary 2.6.
5.1 Weighted Trace Inequality

We begin first by a scaled weighted trace inequality.

**Lemma 5.1** Suppose that $\alpha \in \left( \frac{d-\ell}{2} - 1, \frac{d-\ell}{2} \right)$. Let $T \in \mathcal{T}_H$ such that $\Lambda \subset T$. Then, for $u \in H^{-\alpha}_\Lambda(\Omega)$, we have the following trace inequality

$$
\|u\|_{L^2(\Lambda)} \lesssim H^{\alpha - \frac{d-\ell}{2}} \|u\|_{L^2_{-\alpha}(T)} + H^{\alpha - \frac{d-\ell}{2} + 1} \|\nabla u\|_{L^2_{-\alpha}(T)}.
$$

**Proof** We proceed by using mapping arguments similar to [20, Lemma 7.2] and weighted-scaling arguments from [17, 18] for $\ell = 1$ and $d = 3$. The estimate for $\ell = 0$ and $d = 2, 3$ can be found in [3]. Using general scaling arguments, we generalize this to the case of $\ell = 2$ and $d = 3$, and $\ell = 1$ and $d = 2$.

We denote the the reference (unit size) element $\tilde{T}$ and similarly the reference sub-domain $\tilde{\Lambda}$. We let $A_T : \tilde{T} \to T$ be an affine mapping, and denote $\hat{u} = u \circ A_T$, $\hat{x} = A_T^{-1}(x)$ for $x \in T$. Clearly, $\Lambda = A_T^{-1}(\Lambda) \subset \tilde{T}$ and $\text{diam}(T) \approx H$. Note that from [17, Lemma 3.2] we have from shape regularity that $cH\hat{d}(\hat{x}) \leq d_\Lambda(A_T(\hat{x})) \leq CH\hat{d}(\hat{x})$, thus,

$$
\|u\|^2_{L^2_{-\alpha}(T)} = \int_T \hat{u}^2d\hat{\Lambda} \geq \frac{|T|}{|\tilde{T}|} H^{-2\alpha} \int_{\tilde{T}} \hat{u}^2(\hat{x})d\tilde{\Lambda}(\hat{x}) \gtrsim C H^{-2\alpha} |T| |\tilde{T}| \|\hat{u}\|^2_{L^2_{-\alpha}(\tilde{T})}.
$$

By using standard trace inequality arguments, the trace bound [8], and the above scaling (26), in the weighted norm we obtain

$$
\|u\|_{L^2(\Lambda)} = \left( \frac{|\Lambda|}{|\tilde{\Lambda}|} \right)^{\frac{1}{2}} \|\hat{u}\|_{L^2(\tilde{\Lambda})} \lesssim |\Lambda|^\frac{1}{2} \left( \|\hat{u}\|_{L^2_{-\alpha}(\tilde{T})} + \|\nabla \hat{u}\|_{L^2_{-\alpha}(\tilde{T})} \right)
\lesssim |\Lambda|^\frac{1}{2} |T|^{-\frac{1}{2}} H^\alpha \left( \|u\|_{L^2_{-\alpha}(T)} + \|\nabla A_T\| \|\nabla u\|_{L^2_{-\alpha}(T)} \right)
\lesssim H^\alpha H^{-\frac{1}{2}} H^\alpha \left( \|u\|_{L^2_{-\alpha}(T)} + H \|\nabla u\|_{L^2_{-\alpha}(T)} \right).
$$

Here we have used that $|\Lambda| \approx H^\ell$, for $\ell = 0, 1, 2$, where we take $|\Lambda| = |x_0| = 1$ for $\ell = 0$, and where $|\cdot|$ refers to the measure in the relevant dimension for $\ell = 1, 2$. Here we suppose a planar fracture has area $H^2$ and a line fracture has length $H$. □

Using local approximability of $I^\alpha_H(u)$, we have the following corollary.

**Corollary 5.2** Suppose the assumptions of Lemma 5.1 we then have

$$
\|u - I_H^{\alpha}(u)\|_{L^2(\Lambda)} \lesssim H^{\alpha - \frac{d-\ell}{2} + 1} \|\nabla u\|_{L^2_{-\alpha}(T)}.
$$

**Proof** This is an easy consequence of Lemma 5.1 and stability and approximability of $I_H^{\alpha}(u)$ from Lemma 5.1.

5.2 Error with Global Support

To obtain the error of the multiscale method with globally computed correctors [17], we must utilize the tools of existence and uniqueness for the cross-product space as in [17]. To this end, we have the following fine-scale decomposition of $L^2_\beta(\Omega; \mathbb{R}^d)$. This will then allow us to prove an error bound for the upscaling method.

**Theorem 5.3** (Fine-Scale Decomposition of $L^2_\beta(\Omega; \mathbb{R}^d)$) Let $\beta \in \left( -\frac{d-\ell}{2}, \frac{d-\ell}{2} \right)$. For each $\tau \in L^2_\beta(\Omega; \mathbb{R}^d)$, there is a unique pair $(\sigma, z) \in L^2_\beta(\Omega; \mathbb{R}^d) \times V^f_\beta$ so that

$$
\tau = \nabla z + \sigma, \quad \int_{\Omega} A\sigma \nabla w \, dx = 0, \quad \forall w \in V^f_\beta,
\|\nabla z\|_{L^2_\beta(\Omega)} \lesssim \|\tau\|_{L^2_\beta(\Omega)}, \quad \|\sigma\|_{L^2_\beta(\Omega)} \lesssim \|\tau\|_{L^2_\beta(\Omega)}.
$$
This can be written as the direct sum: \( L^2_\beta(\Omega; \mathbb{R}^d) = \nabla V^f_\beta \oplus (\nabla V^f_{-\beta})^{\perp} \).

**Proof** Here the proof is the same as [17, Lemma 2.1], with \( M_1 = L^2_\beta(\Omega; \mathbb{R}^d) \), \( M_2 = L^2_{-\beta}(\Omega; \mathbb{R}^d) \), \( X_1 = V^f_\beta \), and \( X_2 = V^f_{-\beta} \). This is primarily due to having appropriate Poincaré inequalities in this setting. \( \square \)

We have the following error for the approximation computed from \([20]\).

**Theorem 5.4** Let \( \alpha \in (\frac{d-\ell}{2}, \frac{d-\ell}{2}) \). Suppose that \( u \in H^1_{0,\alpha}(\Omega) \) satisfies \([2]\) with source term \( f\delta_\Lambda \), \( f \in L^2(\Lambda) \), and that \( u_H^{ms} \in \tilde{V}^{ms,\alpha}_H \) satisfies \([20]\). Then, we have the following error estimate

\[
\| \nabla u - \nabla u_H^{ms} \|_{L^2(\Omega)} \lesssim H^{\alpha-\frac{d-\ell}{2}+1} \| f \|_{L^2(\Lambda)}. \tag{28}
\]

**Proof** From the orthogonal splitting of the spaces we have that \( u - u_H^{ms} = u^f \). Thus, by taking \( \tau = \nabla u^f d^2_{\alpha} \in L^2_{-\alpha}(\Omega; \mathbb{R}^d) \) and by Theorem 5.3, there exists a \( \sigma \in L^2_{-\alpha}(\Omega; \mathbb{R}^d) \) and a \( v^f \in V^{f}_{\alpha} \) so that \( \tau = \nabla v^f + \sigma \), where \( \int_\Omega A \nabla u^f \sigma \, dx = 0 \), and \( \| \nabla u^f \|_{L^2(\Omega)} \geq \| \tau \|_{L^2_{-\alpha}(\Omega)} \gtrsim \| \nabla v^f \|_{L^2_{-\alpha}(\Omega)} \). We have

\[
\int_\Omega A \nabla u^f \nabla v^f \, dx = \int_\Omega A \nabla u^f \tau \, dx - \int_\Omega A \nabla u^f \sigma \, dx \gtrsim \| \nabla u^f \|_{L^2(\Omega)}^2 \gtrsim \| \nabla u^f \|_{L^2(\Omega)} \| \nabla v^f \|_{L^2_{-\alpha}(\Omega)}. \]

Using the above together with \( I^{-\alpha}_H(v^f) = 0 \), and the trace estimate of Corollary 5.2 we have

\[
\| \nabla u^f \|_{L^2(\Omega)} \| \nabla v^f \|_{L^2_{-\alpha}(\Omega)} \lesssim \int_\Omega A \nabla u^f \nabla v^f \, dx = \int_\Lambda f(v^f - I^{-\alpha}_H(v^f)) \, ds \\
\lesssim \| f \|_{L^2(\Lambda)} \| v^f - I^{-\alpha}_H(v^f) \|_{L^2(\Lambda)} \lesssim H^{\alpha-\frac{d-\ell}{2}+1} \| \nabla v^f \|_{L^2_{-\alpha}(\Omega)} \| f \|_{L^2(\Lambda)}. \]

Dividing the last \( \| \nabla v^f \|_{L^2_{-\alpha}(\Omega)} \) term yields the result. \( \square \)

### 5.3 Error with Localization

In this section, we show the error due to truncation with respect to patch extensions. The standard result holds here, similarly to that in [18], and the references therein. The key lemma needed is the following estimate, the proof is standard and for completeness can be found in Appendix [13].

**Lemma 5.5** Let \( \beta \in \left(-\frac{d-\ell}{2}, \frac{d-\ell}{2}\right) \) and \( u_H \in V_H \subset H^1_{0,\beta}(\Omega) \), let \( Q^\beta_k \) be constructed from \([21]\) and \([22]\), and \( Q^\beta_\Omega \) defined to be the “ideal” corrector without truncation in \([17]\), then

\[
\left\| \nabla (Q^\beta_\Omega(u_H) - Q^\beta_k(u_H)) \right\|_{L^2_\beta(\Omega)} \lesssim k^{\frac{\ell}{2}} \| \nabla u_H \|_{L^2_\beta(\Omega)}, \tag{29}
\]

with \( \theta \in (0, 1) \).

**Proof** See Appendix [13]. \( \square \)

We then are able to derive an error bound with localized correctors.

**Theorem 5.6** Let \( \alpha \in \left(\frac{d-\ell}{2}, \frac{d-\ell}{2}\right) \). Suppose that \( u \in H^1_{0,\alpha}(\Omega) \) satisfies \([2]\) with source term \( f\delta_\Lambda \), \( f \in L^2(\Lambda) \), and that \( u_H^{ms} \in \tilde{V}^{ms,\alpha}_H \), with local correctors calculated from \([21]\), satisfies \([24]\). Then, we have the following error estimate

\[
\| \nabla u - \nabla u^{ms}_H \|_{L^2(\Omega)} \lesssim \left( H^{\alpha-\frac{d-\ell}{2}+1} + k^{\frac{\ell}{2}} \right) \| f \|_{L^2(\Lambda)}, \tag{30}
\]

with \( \theta \in (0, 1) \).
Figure 1: Convergence histories for the highly oscillatory example with point source (left), point sink together with point source (middle), and line source (right).

\textbf{Proof} We follow the proof given in [9]. We let $u_{H}^{m,s} = u_{H} - Q_{\Omega}^{0}(u_{H})$ be the ideal global multiscale solution satisfying (20), and $u_{H,k}^{m,s} = u_{H,k} - Q_{k}^{0}(u_{H,k})$ be the corresponding truncated solution to (24). Then, by Galerkin approximations being energy minimizers we have

$$\|\nabla u - \nabla(u_{H,k} - Q_{k}^{0}(u_{H,k}))\|_{L^2(\Omega)} \lesssim \|\nabla u - \nabla(u_{H} - Q_{\Omega}^{0}(u_{H}))\|_{L^2(\Omega)}.$$ 

Using this fact and Theorem 5.4 and Lemma 5.5 we have

$$\|\nabla u - \nabla u_{H,k}^{m,s}\|_{L^2(\Omega)} \lesssim \|\nabla u - \nabla(u_{H} - Q_{\Omega}^{0}(u_{H}) + Q_{\Omega}^{0}(u_{H}) - Q_{k}^{0}(u_{H}))\|_{L^2(\Omega)}$$

$$\lesssim \|\nabla u - \nabla u_{H}^{m,s}\|_{L^2(\Omega)} + \|\nabla(Q_{\Omega}^{0}(u_{H}) - Q_{k}^{0}(u_{H}))\|_{L^2(\Omega)}$$

$$\lesssim H^{a-\frac{d}{2}+1} \|f\|_{L^2(\Lambda)} + k^{\frac{d}{2}k}\|\nabla u_{H}\|_{L^2(\Omega)}.$$ 

In addition note that, by construction $u_{H} = I_{H}(u_{H}^{m,s})$. Thus, using local stability (16b) and a-priori bounds from (20), obtained via the trace inequality in Lemma 2.5 and Corollary 2.6 we have

$$\|\nabla u_{H}\|_{L^2(\Omega)} = \|\nabla u_{H}^{m,s}\|_{L^2(\Omega)} \lesssim \|\nabla u_{H}^{m,s}\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Lambda)}.$$ 

Thus, applying the above, we obtain our bound. \(\square\)

6 Numerical Examples

In this section we present numerical experiments for the unit square $\Omega = (0, 1)^2$ with three different singular source terms. First, a single singular source with $f = 1$ at (1/2, 1/2), then a singular sink and source with $f = -1$ at (3/4, 1/4) and $f = 1$ at (1/4, 3/4), finally a singular line fracture with $f = 1$ along $(3/2^3, 1/2) \times (5/2^4, 1/2)$. As indicated from the theory, for the point singular sources we consider $\alpha \in (0, 1)$, while for the line fracture we consider the range $\alpha \in (-1/2, 1/2)$. In the following we present the results for two different types of multiscale permeabilities $A$, the first one being a highly oscillatory periodic coefficient and the second one is constructed from the SPE10 benchmark data. For numerical efficiency we chose $k = 3$ layers for the localized corrector problems in all numerical experiments. Since for each problem below the solution $u$ is unknown, we compare the multiscale approximations $u_{H,3}^{m,s}$, for $H = 2^{-1}, \ldots, 2^{-5}$, to a reference approximation $u_h$ on a fine grid with $h = 2^{-9}$.

6.1 Highly Oscillatory Example

In this example we consider a highly oscillatory permeability

$$A(x) = 1 + \frac{1}{4} \left( \sin \left( \frac{\pi x_1}{2^a} \right) + \sin \left( \frac{\pi x_2}{2^b} \right) \right).$$
Figure 2: Permeability in the heterogeneous example with scaled data from the SPE10 data.

Figure 3: Approximations for the heterogeneous example with point source for \( \alpha = 0.5 \) (left), point sink together with point source for \( \alpha = 0.5 \) (middle), and line fracture for \( \alpha = 0 \) (right). Fine scale approximation with \( h = 2^{-9} \) (top) compared to the LOD approximation with \( H = 2^{-5} \) (bottom).

with values between 1/2 and 3/2. Note that none of the coarse meshes with mesh size \( H = 2^{-1}, \ldots, 2^{-5} \) resolves the oscillations. In Figure 4, we present the convergence results for the three different singular source terms. For the point singularity with a single source or two point singularities with a sink and a source, we observe \( O(H^\alpha) \) convergence of the error \( \|\nabla u_h - \nabla u_{H,3}^{ms}\|_{L^2(\Omega)} \) for \( \alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \). For the line fracture we observe the order of convergence \( O(H^{\alpha+1/2}) \) for \( \alpha = -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2} \). These results confirm the theoretical convergence rates of Theorem 5.6 and show that the convergence is independent of the highly oscillating coefficient.

### 6.2 Heterogeneous Example

In this example we choose a permeability \( \mathcal{A} \) without any (periodic) structure to demonstrate the generality of the method. We consider the permeability of Figure 2 with values between 1 and 11, taken from the SPE10 data, which has been rescaled with the function \( 1 + \log(1 + z) \) in order to reduce the high contrast of the data. Note that the theory here does not prevent issues from high contrast coefficients, and these ratios of material properties maybe tracked in the analysis. Still, in unreported numerical experiments we
observe convergence of the LOD method for singular sources and the original high contrast data, at the cost of slower or in some cases even faster convergence rates than predicted by the theory, which are arguably pre-asymptotic. Since high contrast is not in the focus of this paper, we reduced the contrast, in order to demonstrate the theoretical convergence rates for very coarse meshes.

In Figure 3 we display the fine scale FEM approximations together with the multiscale approximations on refinement level 5. We observe that the multiscale approximations resemble the fine scale features of the fine scale approximation very well. In Figure 4 we observe convergence of $O(H^\alpha)$ of the error $\|\nabla u_h - \nabla u_{ms, H,3}\|_{L^2(\Omega)}$ for the point singular source terms and $O(H^{\alpha+1/2})$ for the singular line fracture, which confirms Theorem 5.6 in the case of an unstructured permeability $A$ with moderate contrast.

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### A Quasi-Interpolation Stability

Here we present the proof of \eqref{eq:3.1} and \eqref{eq:3.2} from Lemma 3.1.

**Proof of Lemma 3.1** Suppose that $v' \in \mathcal{N}(\omega_v)$. If $v' \in \mathcal{N}_{int}(\omega_v)$, then noting that $\mathcal{P}_{v'} u$ is finite dimensional and using the following result from classical finite element inverse inequalities

$$\|\mathcal{P}_{v'} u\|_{L^r(\omega_v)} \lesssim |\omega_v|^{(s-\frac{1}{2})} \|\mathcal{P}_{v'} u\|_{L^s(\omega_v)}, \text{ for } 1 \leq s \leq r < \infty,$$

for $r = \infty, s = 1$, we obtain

$$\|\mathcal{P}_{v'} u\|_{L^\infty(\omega_v)} \lesssim |\omega_v|^{-1} \|\mathcal{P}_{v'} u\|_{L^1(\omega_v)} = |\omega_v|^{-1} \int_{\omega_v} |\mathcal{P}_{v'} u| (d_{A,2}^2)^{\frac{1}{2}} (d_{A,3}^2)^{-\frac{1}{2}} \, dx$$

$$\leq |\omega_v|^{-1} \|\mathcal{P}_{v'} u\|_{L^3(\omega_v)} \left( \int_{\omega_v} d_{A,2}^{-2} \, dx \right)^{\frac{1}{2}}.$$
Here we use the obvious notation \( \| \cdot \|_{L^p(\omega)} = \int_{\omega} (\cdot)^p \, dx \), with \( s \in [1, \infty) \). From (13), letting \( v_H = (P_{\omega}) u \),

\[
\| P_{\omega} u \|_{L^2(\omega)}^2 = \int_{\omega} |P_{\omega} u|^2 d\lambda = \int_{\omega} u P_{\omega} u d\lambda dx \\
\lesssim \| P_{\omega} u \|_{L^\infty(\omega')} \int_{\omega'} |u| (d\lambda)^{\frac{1}{2}} (d\lambda)^{\frac{1}{2}} dx \\
\lesssim \| P_{\omega} u \|_{L^\infty(\omega')} \| u \|_{L^2(\omega')} \left( \int_{\omega'} d\lambda dx \right)^{\frac{1}{2}}.
\]

Thus, manipulating the above identities

\[
\| P_{\omega} u \|_{L^\infty(\omega')} \lesssim |\omega|^{-2} \left( \int_{\omega'} d\lambda \frac{1}{2} \right) \left( \int_{\omega'} d\lambda \frac{1}{2} \right) \| u \|_{L^2(\omega')} \| P_{\omega} u \|_{L^\infty(\omega')}.
\]

Rearranging terms and by taking the larger patch \( \omega_{1,1} \supset \omega_{1} \), we have

\[
| P_{\omega} u(v') | \lesssim |\omega_{1,1}|^{-2} \left( \int_{\omega_{1,1}} d\lambda \frac{1}{2} \right) \left( \int_{\omega_{1,1}} d\lambda \frac{1}{2} \right) \| u \|_{L^2(\omega_{1,1})}.
\]

Finally, we note (again taking a larger domain \( \omega_{1,1} \) to \( \omega_{1} \)) that

\[
\| \lambda \|_{L^2(\omega)} \lesssim \left( \int_{\omega} d\lambda \frac{1}{2} \right)^{\frac{1}{2}}, \quad \text{and} \quad \| \nabla \lambda \|_{L^2(\omega)} \lesssim H^{-1} \left( \int_{\omega} d\lambda \frac{1}{2} \right)^{\frac{1}{2}}.
\]

For the quasi-interpolation \( I_H(\omega) \) we have

\[
I_H(\omega) = \sum_{v' \in N_{\omega}(\omega)} (P_{\omega}) u(v') \lambda_{\omega}, \quad \text{in} \ \omega.
\]

For \( L^2 \) stability, we note that from (31) and (32), we obtain

\[
\left\| I_H(\omega) \right\|_{L^2(\omega)} \leq \sum_{v' \in N_{\omega}(\omega)} \| (P_{\omega}) u(v') \|_{L^2(\omega)} \| \lambda \|_{L^2(\omega)} \\
\lesssim \left( \frac{|B|}{|\omega_{1,1}|} \right)^2 \frac{1}{|B|^2} \left( \int_B d\lambda \frac{1}{2} \right) \left( \int_B d\lambda \frac{1}{2} \right) \| u \|_{L^2(\omega_{1,1})} \\
\lesssim \left( \frac{|B|}{|\omega_{1,1}|} \right)^2 C_2 \| u \|_{L^2(\omega_{1,1})},
\]

where we used the Muckenhoupt weight condition from Proposition 2.1. We take \( B \) to be the ball containing the patch \( \omega_{1,1} \), and we suppose (by quasi-uniformity) that the ratio \( \left( \frac{|B|}{|\omega_{1,1}|} \right) \) is trivially bounded.

For the \( H^1 \) stability, first noting that \( \langle u \rangle_{\omega_{1,1}} = I_H(\langle u \rangle_{\omega_{1,1}}) \), we denote \( \bar{u} = u - \langle u \rangle_{\omega_{1,1}} \). Thus, from (31) and (32), and arguments used above, we obtain

\[
\| \nabla I_H(\omega) \|_{L^2(\omega)} = \| \nabla I_H(\bar{u}) \|_{L^2(\omega)} \lesssim \sum_{v' \in N_{\omega}(\omega)} \| (P_{\omega}) \bar{u}(v') \|_{L^2(\omega)} \| \nabla \lambda \|_{L^2(\omega)} \\
\lesssim H^{-1} |\omega_{1,1}|^{-2} \left( \int_{\omega_{1,1}} d\lambda \frac{1}{2} \right) \left( \int_{\omega_{1,1}} d\lambda \frac{1}{2} \right) \| \bar{u} \|_{L^2(\omega_{1,1})} \lesssim \left( \frac{|B|}{|\omega_{1,1}|} \right)^2 C_2 \| \nabla u \|_{L^2(\omega_{1,1})},
\]

(34)
where for the last inequality we used the weighted Poincaré inequality from Lemma 2.2. To prove local $L^2$ approximability, we note that for $\tilde{u} = u - \langle u \rangle_{\omega_{v,1}}$, using (33) and Lemma 2.2 we obtain

$$
\left\| u - T^\beta_H(u) \right\|_{L^2_v(\omega)} = \left\| \tilde{u} - T^\beta_H(\tilde{u}) \right\|_{L^2_v(\omega)} \leq \left\| \tilde{u} \right\|_{L^2_v(\omega)} + \left\| T^\beta_H(\tilde{u}) \right\|_{L^2_v(\omega)} \leq H \left\| \nabla u \right\|_{L^2_v(\omega)}.
$$

(35)

Thus, local approximability holds, and result (16b) trivially holds from $H^1$ stability. From arguments in [10], we deduce that $T^\beta_H$ is also a projection. □

B Truncation Estimates

Now we will prove and state the auxiliary lemmas used to prove the localized error estimate in Theorem 5.6. These proofs are entirely based on the works [9, 29, 37] and references therein. The proofs have been extended to weighted spaces in [9]. In that work the weight function was $y^\gamma$, for some $\gamma \in (-1, 1)$, we replace this weight with $d^\beta_2$ and will have a very similar proof. Here we present a version of these ideas and highlight any subtle differences.

We begin with defining some standard technical cutoff functions. For $v, v' \in N_{int}$ and $l, k \in \mathbb{N}$ and $m = 0, 1, \ldots$, with $k \geq l \geq 2$ we have

$$
\text{if } \omega_{v,m+1} \cap (\omega_{v,k} \setminus \omega_{v,l}) \neq \emptyset, \text{ then } \omega_{v,1} \subset (\omega_{v,k+m+1} \setminus \omega_{v,l-m-1}).
$$

(36)

We will use the cutoff functions defined in [29]. For $v \in N_{int}$ and $k > l \in \mathbb{N}$, let $\eta^{k,l}_v : \Omega \rightarrow [0, 1]$ be a continuous weakly differentiable function so that

$$
(\eta^{k,l}_v)|_{\omega_{v,k-l}} = 0,
$$

(37a)

$$
(\eta^{k,l}_v)|_{\Omega \setminus \omega_{v,k}} = 1,
$$

(37b)

$$
\forall T \in T_H, \left\| \nabla \eta^{k,l}_v \right\|_{L^\infty(T)} \leq C_{co} \frac{1}{lH},
$$

(37c)

where $C_{co}$ is only dependent on the shape regularity of the mesh $T_H$. We choose here the cutoff function as in [37] where we choose a function $\eta^{k,l}_v$ in the space of $P_1$ Lagrange finite elements over $T_H$ such that

$\eta^{k,l}_v(v') = 0$ for all $v' \in N_{int} \cap \omega_{v,k-l}$,

$\eta^{k,l}_v(v') = 1$ for all $v' \in N_{int} \cap (\Omega \setminus \omega_{v,k})$,

$\eta^{k,l}_v(v') = \frac{j}{l}$ for all $v' \in N_{int} \cap \omega_{v,k-j}, j = 0, 1, \ldots, l$.

We will now prove a lemma showing the quasi-invariance of the fine-scale functions under multiplication by cutoff functions in the distance-weighted Sobolev space.

Lemma B.1 Let $k > l \in \mathbb{N}$, $v \in N_{int}$, and $\beta \in (-\frac{d-1}{2}, \frac{d-1}{2})$. Suppose that $w \in V^f_\beta$, then we have the estimate

$$
\left\| \nabla T^\beta_H(\eta^{k,l}_v w) \right\|_{L^2_v(0)} \lesssim l^{-1} \left\| \nabla w \right\|_{L^2_v(\omega_{v,k+2} \setminus \omega_{v,k-1})}.
$$

Proof Fix $v$ and $k$, and denote the average as $\langle \eta^{k,l}_v \rangle_{\omega_{v,1}} = \frac{1}{|\omega_{v,1}|} \int_{\omega_{v,1}} \eta^{k,l}_v dx$. For an estimate on a single patch $\omega_{v_i}$, using the stability (15) and the fact that $T^\beta_H(w) = 0$, we have

$$
\left\| \nabla T^\beta_H(\eta^{k,l}_v w) \right\|_{L^2_v(\omega)} = \left\| \nabla T^\beta_H(\langle \eta^{k,l}_v \rangle_{\omega_{v,1}} w) \right\|_{L^2_v(\omega)} \lesssim \left\| \langle \eta^{k,l}_v \rangle_{\omega_{v,1}} \nabla w \right\|_{L^2_v(\omega_{v,1})} + \left\| \nabla \eta^{k,l}_v (w - T^\beta_H(w)) \right\|_{L^2_v(\omega_{v,1})}.
$$
Summing over all $v' \in N_{\text{int}}$, using the quasi-inclusion property \cite{36} yields

$$
\left\| \nabla I_H^{\beta}(\eta_{N,l}^{k,l}w) \right\|_{L_\beta^2(\Omega)}^2 \lesssim \sum_{\omega_{\nu,k} \subset \omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1}} \left\| (\eta_{N,l}^{k,l} - \langle \eta_{N,l}^{k,l} \rangle_{\omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1}}) \nabla w \right\|_{L_\beta^2(\omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1})}^2 + \sum_{\omega_{\nu,k} \subset \omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1}} \left\| \nabla I_H^{\beta}(\eta_{N,l}^{k,l}w - I_H^{\beta}(w)) \right\|_{L_\beta^2(\omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1})}^2.
$$

(38)

Here we used that $\nabla \eta_{N,l}^{k,l} \neq 0$ only in $\omega_{\nu,k} \setminus \omega_{\nu,k-1}$ and $\langle \eta_{N,l}^{k,l} \rangle_{\omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1}} \neq 0$ only if $\omega_{\nu,1}$ intersects $\omega_{\nu,k} \setminus \omega_{\nu,k-1}$.

We now denote $\mu_{N,l}^{k,l} = \eta_{N,l}^{k,l} - \langle \eta_{N,l}^{k,l} \rangle_{\omega_{\nu,k+1}}$, and let $T$ be a simplex in $\omega_{\nu,1}$ such that the supremum $\left\| \mu_{N,l}^{k,l} \right\|_{L^\infty(\omega_{\nu,1})}$ is obtained. On $T$, $\mu_{N,l}^{k,l}$ is an affine function, using the fact that $\eta_{N,l}^{k,l}$ is taken to be $P_1$, we use the following inverse estimate combined with the Muckenhoupt property Proposition \cite{2.1}. Note that by utilizing the following result from classical finite element inverse inequalities

$$
\left\| q \right\|_{L^r(T)} \lesssim |T|^{\frac{1}{2} - \frac{1}{r}} \left\| q \right\|_{L^{\frac{r}{r-1}}(T)}, \quad \text{for } 1 \leq s \leq r < \infty,
$$

for $r = \infty$, $s = 1$, we obtain

$$
\left\| q \right\|_{L^\infty(T)} \lesssim |T|^{-1} \left\| q \right\|_{L^1(T)} = |T|^{-1} \int_T |q| (|d_\alpha^{2\beta}|)^{\frac{1}{2}} (|d_\alpha^{2\beta}|)^{-\frac{1}{2}} \, dx \leq |T|^{-1} \left\| q \right\|_{L^\infty(T)} \left( \int_T d_\alpha^{-2\beta} \, dx \right)^{\frac{1}{2}}.
$$

So that

$$
\left\| \mu_{N,l}^{k,l} \right\|_{L^\infty(\omega_{\nu,1})} = \left\| \mu_{N,l}^{k,l} \right\|_{L^\infty(T)} \lesssim |T|^{-1} \left\| \mu_{N,l}^{k,l} \right\|_{L^\infty(T)} \left( \int_T d_\alpha^{-2\beta} \, dx \right)^{\frac{1}{2}}.
$$

Using the above estimate and taking the whole patch, we see that

$$
\left\| \eta_{N,l}^{k,l} - \langle \eta_{N,l}^{k,l} \rangle_{\omega_{\nu,1}} \right\|_{L^\infty(\omega_{\nu,1})} \lesssim \left| \omega_{\nu,1} \right|^{-1} \left( \int_{\omega_{\nu,1}} d_\alpha^{-2\beta} \, dx \right)^{\frac{1}{2}} \left\| \eta_{N,l}^{k,l} - \langle \eta_{N,l}^{k,l} \rangle_{\omega_{\nu,1}} \right\|_{L_\beta^2(\omega_{\nu,1})} \lesssim \left| \omega_{\nu,1} \right|^{-1} \left( \int_{\omega_{\nu,1}} d_\alpha^{-2\beta} \, dx \right)^{\frac{1}{2}} H \left\| \nabla \eta_{N,l}^{k,l} \right\|_{L^\infty(\omega_{\nu,1})} \lesssim \left( C_{2,\beta} \right) H \left\| \nabla \eta_{N,l}^{k,l} \right\|_{L^\infty(\omega_{\nu,1})},
$$

where we used the Muckenhoupt weight bound \cite{33}, as well as quasi-uniformity of the grid. Returning to \cite{38}, using the above relation on the first term and the approximation property \cite{16} on the second term, we obtain

$$
\left\| \nabla I_H^{\beta}(\eta_{N,l}^{k,l}w) \right\|_{L_\beta^2(\Omega)}^2 \lesssim H^2 \left\| \nabla \eta_{N,l}^{k,l} \right\|_{L^\infty(\Omega)}^2 \left\| \nabla w \right\|_{L_\beta^2(\omega_{\nu,k+1} \setminus \omega_{\nu,k-1-1})}^2 + H^2 \left\| \nabla \eta_{N,l}^{k,l} \right\|_{L^\infty(\Omega)}^2 \left\| \nabla w \right\|_{L_\beta^2(\omega_{\nu,k+2} \setminus \omega_{\nu,k-1-2})}^2
$$

Finally, we arrive at

$$
\left\| \nabla I_H^{\beta}(\eta_{N,l}^{k,l}w) \right\|_{L_\beta^2(\Omega)}^2 \lesssim I^{-2} \left\| \nabla w \right\|_{L_\beta^2(\omega_{\nu,k+2} \setminus \omega_{\nu,k-1-2})}^2
$$

where we used $\left\| \nabla \eta_{N,l}^{k,l} \right\|_{L^\infty(\Omega)}^2 \lesssim 1/(IH)^2$. \hfill $\Box$
For the distance-weighted Sobolev space, we have the following decay of the fine-scale space:

**Lemma B.2** Let \( \beta \in (\frac{-d+\epsilon}{2}, \frac{-d-\epsilon}{2}) \). Fix some \( v \in N_{int} \) and \( F \in (V^f_\beta)' \) the dual of \( V^f_\beta \) satisfying \( F(w) = 0 \) for all \( w \in V^f_\beta(\Omega, \omega_{v,1}) \). Let \( u \in V^f_\beta \) be the solution of

\[
\int_{\Omega} A \nabla u \nabla w^{2 \beta}_\Lambda \, dx = F(w) \quad \text{for all } w \in V^f_\beta.
\]

Then, there exists a constant \( \theta \in (0, 1) \) such that for \( k \in \mathbb{N} \) we have

\[
\| \nabla u \|_{L^2_\beta(\Omega, \omega_{v,k})} \lesssim \theta^k \| \nabla u \|_{L^2_\beta(\Omega)}.
\]

**Proof** Let \( \eta^{k,l}_v \) be the cut-off function as in the previous lemma for \( l < k - 2 \). Let \( \tilde{u} = \eta^{k,l}_v u - \mathcal{I}_H^\beta(\eta^{k,l}_v u) \in V^f_\beta(\Omega \setminus \omega_{v,k-1}) \), and note that from Lemma [B.1] we have

\[
\| \nabla (\eta^{k,l}_v u - \tilde{u}) \|_{L^2_\beta(\Omega)} = \| \nabla \mathcal{I}_H^\beta(\eta^{k,l}_v u) \|_{L^2_\beta(\Omega)} \lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,k-2} \setminus \omega_{v,k-l-2})}.
\]  

(40)

From this estimate and the properties of \( F \) we have

\[
\int_{\Omega \setminus \omega_{v,k-1}} A \nabla u \nabla \tilde{u} \, d^{2 \beta}_\Lambda \, dx = \int_{\Omega} A \nabla u \nabla \tilde{u} \, d^{2 \beta}_\Lambda \, dx = F(\tilde{u}) = 0.
\]  

(41)

We utilize a version of the Caccioppoli inequality [13] for the distance-weighted space, and the coercivity of the corrector problems [17] to obtain

\[
\| \nabla u \|_{L^2_\beta(\Omega, \omega_{v,k})}^2 \lesssim \int_{\Omega \setminus \omega_{v,k-1}} A \nabla u \nabla u \, d^{2 \beta}_\Lambda \, dx = \int_{\Omega \setminus \omega_{v,k-1}} A \nabla u \big( \nabla (\eta^{k,l}_v u) - \eta^{k,l}_v \nabla u \big) \, d^{2 \beta}_\Lambda \, dx.
\]

Using the fact that \( \mathcal{I}_H^\beta(u) = 0 \), estimate (40), and the relation (41), we have

\[
\| \nabla u \|_{L^2_\beta(\Omega, \omega_{v,k})}^2 \lesssim \int_{\Omega \setminus \omega_{v,k-1}} A \nabla u \nabla (\eta^{k,l}_v u - \tilde{u}) \, d^{2 \beta}_\Lambda \, dx - \int_{\Omega \setminus \omega_{v,k-1}} A \nabla u (u - \mathcal{I}_H^\beta(u)) \eta^{k,l}_v \, d^{2 \beta}_\Lambda \, dx
\]

\[
\lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,k-2})}^2 + (lH)^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,k-1})} \| u - \mathcal{I}_H^\beta(u) \|_{L^2_\beta(\omega_{v,k-1})}
\]

\[
\lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,k-1})}^2.
\]

On the last term we used the approximation property [16]. Successive applications of the above estimate leads to

\[
\| \nabla u \|_{L^2_\beta(\Omega, \omega_{v,k})}^2 \lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,k-2})}^2 \lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\omega_{v,1})}^2 \lesssim l^{-1} \| \nabla u \|_{L^2_\beta(\Omega)}^2.
\]

Finally, noting that

\[
\left\lfloor \frac{k-1}{l+2} \right\rfloor = \left\lfloor \frac{k-l-2}{l+2} \right\rfloor \geq \frac{k}{l+2} - 1,
\]

taking \( \theta = l^{-1} \frac{k+1}{k+2} \) yields the result. \( \square \)

We now are ready to restate our result on the error introduced from localization. This is merely Lemma 5.5 restated and proven. When \( k \) is sufficiently large so that the corrector problem is all of \( \Omega \), we denote \( Q^\beta_{v,k} = Q^\beta_{v,\Omega} \). Let \( u_H \in V_H \), let \( Q^\beta_k \) be constructed from [21], and \( Q^\beta_{v,\Omega} \) defined to be the “ideal” corrector without truncation, then

\[
\| \nabla (Q^\beta_{v,\Omega}(u_H) - Q^\beta_k(u_H)) \|_{L^2_\beta(\Omega)} \lesssim k^\frac{3}{2} \theta^k \| \nabla u_H \|_{L^2_\beta(\Omega)}.
\]  

(42)

Again we use techniques standard at this point in the view of [9], but presented for completeness.
Proof of Lemma 5.5] We denote \( v = Q_{\Omega}^\beta(u_H) - Q_{\beta}^\beta(u_H) \in V^f_\beta \), subsequently \( T_H^\beta(v) = 0 \). Taking the cut-off function \( \eta_v^{k,1} \) we have

\[
\|\nabla v\|_{L_\Lambda^2(\Omega)}^2 \lesssim \sum_{v \in \mathcal{N}_{int}} \int_\Omega A\nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H))\nabla(v(1-\eta_v^{k,1}))\,d\Lambda^2 \, dx \tag{43}
\]
\[
+ \sum_{v \in \mathcal{N}_{int}} \int_\Omega A\nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H))\nabla(v\eta_v^{k,1})\,d\Lambda^2 \, dx. \tag{44}
\]

Estimating the right hand side of (43) for each \( v \), we have, using the boundedness of \( A \)

\[
\int_\Omega A\nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H))\nabla(v(1-\eta_v^{k,1}))\,d\Lambda^2 \, dx \lesssim \left\| \nabla(v(1-\eta_v^{k,1})) \right\|_{L_\Lambda^2(\Omega)} \left( \left\| \nabla Q_{\nu,\Omega}^\beta(u_H) \right\|_{L_\Lambda^2(\Omega)} + \left\| \nabla Q_{\nu,\beta}^\beta(u_H) \right\|_{L_\Lambda^2(\Omega)} \right) \left( \left\| \nabla v \right\|_{L_\Lambda^2(\Omega)} + H^{-1} \left\| \nabla T_H^\beta(v) \right\|_{L_\Lambda^2(\Omega)} \right) \leq \left\| \nabla Q_{\nu,\Omega}^\beta(u_H) \right\|_{L_\Lambda^2(\Omega)} \left\| \nabla v \right\|_{L_\Lambda^2(\Omega)}.
\]

As in the proof of Lemma B.2 we denote \( \tilde{v} = \eta_v^{k,1}v - T_H^\beta(\eta_v^{k,1}v) \in V^f_\beta(\Omega \setminus \omega_{\nu,k-2}) \) and so \( \tilde{v} \) satisfies

\[
\int_\Omega A\nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H))\nabla(\tilde{v})\,d\Lambda^2 \, dx = 0.
\]

We have now the estimate for (44) for \( v \in \mathcal{N}_{int} \) using the above identity and (40)

\[
\int_\Omega A\nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H))\nabla(v\eta_v^{k,1} - \tilde{v})\,d\Lambda^2 \, dx \lesssim \left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)} \left\| \nabla v \right\|_{L_\Lambda^2(\omega_{\nu,k-2})}.
\]

Combing the estimates for (43) and (44) we obtain

\[
\left\| \nabla v \right\|_{L_\Lambda^2(\Omega)}^2 \lesssim \sum_{v \in \mathcal{N}_{int}} \left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)} \left\| \nabla v \right\|_{L_\Lambda^2(\omega_{\nu,k-2})}^2 \lesssim k^2 \left( \sum_{v \in \mathcal{N}_{int}} \left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)}^2 \right)^{1/2} \left\| \nabla v \right\|_{L_\Lambda^2(\Omega)},
\]

supposing that \( \# \{ v' \in \mathcal{N}_{int}[\omega_{\nu'} \subset \omega_{\nu,k+2} \} \leq k^d, \) as is guaranteed by quasi-uniformity of the coarse-grid.

For \( v \in \mathcal{N}_{int} \), we estimate \( \left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)} \) and we use the Galerkin orthogonality of the local problem, that is

\[
\left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)} \leq \inf_{q_v \in V^f(\omega_{\nu,k})} \left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - q_v) \right\|_{L_\Lambda^2(\Omega)}. \tag{46}
\]

Let \( q_v = (1 - \eta_v^{(k-1),1})Q_{\nu,\Omega}^\beta(u_H) - T_H^\beta((1 - \eta_v^{(k-1),1})Q_{\nu,\Omega}^\beta(u_H)) \in V^f(\omega_{\nu,k}) \), we have

\[
\left\| \nabla(Q_{\nu,\Omega}^\beta(u_H) - Q_{\nu,\beta}^\beta(u_H)) \right\|_{L_\Lambda^2(\Omega)} \leq \left\| \nabla((1 - \eta_v^{(k-1),1})Q_{\nu,\Omega}^\beta(u_H) + T_H^\beta((1 - \eta_v^{(k-1),1})Q_{\nu,\Omega}^\beta(u_H))) \right\|_{L_\Lambda^2(\Omega)} \lesssim \left\| \nabla Q_{\nu,\Omega}^\beta(u_H) \right\|_{L_\Lambda^2(\Omega \setminus \omega_{\nu,k-2})} + \left\| \nabla(T_H^\beta((1 - \eta_v^{(k-1),1})Q_{\nu,\Omega}^\beta(u_H))) \right\|_{L_\Lambda^2(\Omega)}.
\]
Using $I^β_H((1 - η^k_{v-1})Q^β_{v,Ω}(u_H)) = -I^β_H(η^k_{v-1}Q^β_{v,Ω}(u_H))$ and Lemma B.1 on the second term, and then Lemma B.2, we arrive at

$$\|\nabla (Q^β_{v,Ω}(u_H) - Q^β_{v,k}(u_H))\|^2_{L^2_β(Ω)} \lesssim \|\nabla Q^β_{v,Ω}(u_H)\|^2_{L^2_β(Ω\setminus(Ω\setminus k-4))} \lesssim θ^{2k-4} \|\nabla Q^β_{v,Ω}(u_H)\|^2_{L^2_β(Ω)}.$$

From the definition of $Q^β_{v,Ω}$ from [21] with global corrector patches, we get

$$\|\nabla (Q^β_{v,Ω}(u_H) - Q^β_{v,k}(u_H))\|^2_{L^2_β(Ω)} \lesssim θ^{2k}\|\nabla u_H\|^2_{L^2_β(Ω)},$$

where we used the bounds from Proposition 4.1 modified for $Q^β_{v,Ω}$ from [21] with localized right hand side, hence localized upper bounds. Thus, summing over all $v \in N_{int}$ and combining the above with (45) concludes the proof. □

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