THE FIBERED ISOMORPHISM CONJECTURE FOR COMPLEX MANIFOLDS

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Abstract. In this paper we show that the Fibered Isomorphism Conjecture of Farrell and Jones corresponding to the stable topological pseudoisotopy functor is true for the fundamental groups of a class of complex manifolds. A consequence of this result is that the Whitehead group, reduced projective class groups and the negative $K$-groups of the fundamental groups of these manifolds vanish whenever the fundamental group is torsion free. We also prove the same results for a class of real manifolds including a large class of 3-manifolds which has a finite sheeted cover fibering over the circle.

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0. Introduction

In this paper we consider proving the Farrell-Jones Fibered Isomorphism Conjecture (FIC) corresponding to the stable topological pseudoisotopy functor for the fundamental groups of complex manifolds. This conjecture is already proved for several classes of groups including fundamental groups of closed nonpositively curved Riemannian manifolds ([6]), for cocompact discrete subgroup of virtually connected Lie groups ([6]) and for the class of virtually strongly poly-free groups ([9]). A rich class of complex manifolds are smooth complex projective algebraic varieties. Some results are known on the structure of the fundamental groups of a large class of complex surfaces. We make use of these informations for the proofs of our results. The method of the proofs also generalizes to consider some real manifolds and to prove the FIC for the fundamental groups.

The main input to the proofs of Theorem 1.3 and 1.5 comes from theorem 7.1 of Farrell and Linnell in [7] where they proved that if the Fibered Isomorphism Conjecture is true for all the groups in a directed system of groups then it is true for the direct limit also. The proof of our Main Lemma uses this result to show that the FIC is true for a large class of mapping torus of the fundamental group of a closed orientable real 2-manifold and also for a certain class of mapping tori of infinitely generated free groups. The key idea to prove the Main Lemma was that except for two closed surfaces the covering space corresponding to the commutator subgroup of the fundamental group of all other surfaces have one topological end. Though the spaces involved in the theorems have finitely presented fundamental groups, during the proof we encounter some infinitely generated groups and these are the places where we use the Main Lemma crucially.

Throughout the paper whenever we encounter a 3-manifold which fiber over the circle with fiber a surface of genus $\geq 2$ we make the assumption that the monodromy diffeomorphism is special (see Definition in Section 1). In [27] we have proved that we can remove this assumption of being ‘special’ provided the FIC is true for $A$-groups (see [27] for definition). Roughly speaking a torsion free $A$-group is a discrete group which is isomorphic to the fundamental group of a complete nonpositively curved Riemannian manifold whose metric is $A$-regular. In a recent paper ([16]) L.E. Jones proved that the assembly map in the statement of the FIC induces a surjective homomorphism on the homotopy group level for any torsion free $A$-group. He also stated some conjectured theorems ([[16], conjectured theorems 6.7 and 6.9]) which imply that the ‘torsion free’ assumption can be dropped from the statement of the result. On the other hand the FIC states that the above homomorphism should be an isomorphism.
In Section 1 using Lefschetz hyperplane section theorem we deduce that if the FIC is true for the fundamental group of smooth complex algebraic surfaces then it is true for the fundamental group of any smooth complex algebraic variety. Also we state our results in this section. Section 2 recalls the Farrell and Jones Fibered Isomorphism Conjecture and states the known results we need. In Section 3 we make a brief trip to classification of complex surfaces and their topological properties. Sections 4 and 5 contain the proofs of the results stated in Section 1 and we prove the Main Lemma in Section 6. In Section 7 we prove the FIC for a class of virtually fibered 3-manifolds. Section 8 contains examples of special diffeomorphisms.

1. Reduction to surface case and statements of results

Let $X$ be a smooth complex projective algebraic variety of dimension $n$. By definition $X \subset \mathbb{CP}^m$ for some $m$ and is a complex submanifold of $\mathbb{CP}^m$. It is well-known that not all complex manifold is a complex submanifold of some complex projective space; otherwise it will become algebraic. In Section 3 we will mention such examples in the case of surfaces.

There is a natural collection of complex submanifolds of $X$ arising from taking intersection of $X$ with hyperplanes $H$ in $\mathbb{CP}^m$. For a general hyperplane $H$ the intersection of $H$ with $X$ is a connected complex submanifold of $X$ ([[2], chapter I, corollary 20.3]). Let $H_0$ be such a hyperplane. Then the two manifolds $X$ and $X \cap H_0$ shares similar homological and homotopical properties up to a certain degree. This is the content of the Lefschetz hyperplane section theorem ([[2], chapter I, theorem 20.4]).

**Lefschetz hyperplane section theorem.** For $n \geq 2$ the inclusion map $X \cap H_0 \subset X$ induces the following isomorphisms.

$$H_i(X \cap H_0, \mathbb{Z}) \to H_i(X, \mathbb{Z})$$

$$\pi_i(X \cap H_0, \mathbb{Z}) \to \pi_i(X, \mathbb{Z})$$

for $0 \leq i \leq n - 2$.

Let $G_n$ be the class of fundamental groups of all smooth complex projective algebraic varieties of dimension $n$. Then successively applying Lefschetz hyperplane section theorem we get the following Lemma.

**Lemma 1.1.** $\bigcup_{n=2}^{\infty} G_n \subset G_2$.

Thus from Lemma 1.1 we see that if we want to prove the FIC for the fundamental group of smooth projective algebraic varieties then it is enough to consider the smooth projective algebraic surfaces.
The rest of this section contains the statements of the results. Before that we recall some standard definition from algebraic geometry. By a complex surface we mean a closed complex 2-manifold. By an algebraic complex surface we mean it is a complex surface and is defined by finitely many homogeneous polynomials in $n+1$ variable in the complex projective space $\mathbb{CP}^n$. By a curve we will mean a complex projective algebraic variety of dimension 1. $\kappa(X)$ denotes the Kodaira dimension of $X$. For definition of $\kappa(X)$ see [[2], p. 23] or [[10], definition 1.6]. When $X$ is a complex surface $\kappa(X) \in \{-\infty, 0, 1, 2\}$.

**Theorem 1.2.** Let $X$ be a complex surface of one of the following types.

1. $X$ is algebraic and $\kappa(X) = -\infty$
2. $X$ is a Hopf surface
3. $\kappa(X) = 0$
4. $X$ is an Inoue surface
5. $X$ is an elliptic surface

Then the Fibered Isomorphism Conjecture is true for $\pi_1(X)$.

The theorem below deals with some more complex manifolds. To state the theorem we need to make some definition. At first recall that it follows from a result of Hillman [[12], theorem 7] that if a complex surface fibers over the circle then the fiber is a Seifert fibered space (see Theorem 4.1). Hence if $X$ is such a surface then $\pi_1(X) \cong \pi_1(S) \rtimes \mathbb{Z}$ where $S$ is a Seifert fibered space. Assume that the monodromy diffeomorphism is a fiber preserving diffeomorphism of the Seifert fibered space $S$. If $\pi_1(S)$ is infinite then there is an infinite cyclic normal subgroup of $\pi_1(S)$ with quotient $\pi_1^{orb}(B)$ where $B$ is the base orbifold of $S$ ([[11], chapter 12]). Again if $\pi_1^{orb}(B)$ is infinite then one can find a finite index characteristic subgroup $K$ of $\pi_1^{orb}(B)$ which is isomorphic to a closed surface group (see Section 4). Note that in this situation the monodromy diffeomorphism of the fiber bundle $X \to S^1$ induces an automorphism of $\pi_1^{orb}(B)$. Since $K$ is characteristic we have an exact sequence

$$1 \to K \to \pi_1^{orb}(B) \rtimes \mathbb{Z} \to (\pi_1^{orb}(B)/K) \rtimes \mathbb{Z} \to 1$$

Let $l$ be an element of $(\pi_1^{orb}(B)/G) \rtimes \mathbb{Z}$ which generates an infinite cyclic normal subgroup of finite index. Since $K$ is a closed surface group the action of $l$ (by conjugation by a lift of $l$) on $K$ is induced, up to conjugation, by a diffeomorphism $f_l$ of a closed surface $F$ so that $\pi_1(F)$ is isomorphic to $K$. Let us call $f_l$ a base diffeomorphism associated to the infinite cyclic normal subgroup generated by $l$.

**Theorem 1.3.** Let $X$ be a complex surface which is the total space of a fiber bundle over the circle $S^1$. Under the above notations when $F$ is not the 2-torus assume
that there is a base diffeomorphism \( f_1 \) which is special (see Definition below). Then the FIC is true for \( \pi_1(X) \).

In Section 4 we will also prove that the FIC is true for a class of complex surfaces of Kodaira dimension 2. A large class of examples of surfaces of Kodaira dimension 2 are ramified 2-sheeted covering of \( \mathbb{CP}^2 \) ramified along a curve. We will prove the FIC for a class of such surfaces and will give an example to show that, given the methods available, this is the best possible result we can prove.

Complex surfaces with Kodaira dimension 1 are elliptic surfaces (Theorem 3.3.1). There is a notion of elliptic surfaces in the smooth category called \( C^\infty \)-elliptic surface. These are smooth real 4-manifold which are locally modelled on complex elliptic surfaces. We recall the definition of \( C^\infty \)-elliptic surface in Section 3. The fundamental groups of these 4-manifolds have close properties with that of complex elliptic surfaces. We prove in Corollary 1.4 that the FIC is true for a class of these manifolds also.

**Corollary 1.4.** The FIC is true for the fundamental group of a \( C^\infty \)-elliptic surface \( X \) if \( X \) has no singular fiber and with cyclic monodromy.

There is another natural collection of smooth 4-manifolds which are fiber bundles over real 2-manifolds with real 2-manifolds as fiber. In Theorem 1.5 below we prove that the FIC is true for the fundamental group of a large class of manifolds from this collection. A large class of complex surfaces belong to this collection where both the fiber and base are 2-manifolds of genus \( \geq 2 \). In this particular case the fiber bundle projection is called Kodaira fibration (see [[2], chapter V, section 14] for details). To state our next results we need the following definition.

**Definition.** Let \( F \) be a closed orientable surface and \( f \) is an orientation preserving diffeomorphism of \( F \). Let \( \tilde{F} \) be the covering of \( F \) corresponding to the commutator subgroup of \( \pi_1(F) \) and let \( \tilde{f} \) be the lift of \( f \) to \( \tilde{F} \to \tilde{F} \). Let \( p : M_{\tilde{f}} \to M_f \) be the covering projection from the mapping torus of \( \tilde{f} \) to that of \( f \).

We say \( f \) is a special diffeomorphism if one of the following holds.

1. the mapping torus of \( f \) supports a nonpositively curved Riemannian metric.
2. some power of \( f \) is isotopic to identity.
3. the fundamental group of any component of \( p^{-1}(S) \) is not free, where \( S \) varies over all Seifert fibered pieces in the Jaco-Shalen and Johannson (JSJ) decomposition of the mapping torus \( M(f) \).

We now recall JSJ-decomposition of a 3-manifold briefly. A 3-manifold is called irreducible if any embedded 2-sphere in the manifold bounds an embedded 3-disc.
A compact orientable irreducible 3-manifold \( M \) is called \textit{Haken} if there is a compact orientable surface \( S \) embedded in \( M \) such that \( \pi_1(S) \) is infinite and the inclusion map \( S \to M \) induces an injective homomorphism \( \pi_1(S) \to \pi_1(M) \). The JSJ-decomposition states that any Haken 3-manifold admits a decomposition along finitely many mutually nonparallel tori embedded in \( M \) so that the decomposed pieces are either Seifert fibered or simple (see [14], [15]). Thurston proved that these simple pieces admits complete hyperbolic metric in the interior.

For examples of special diffeomorphisms, recall that the mapping tori of pseudo-Anosov diffeomorphisms are hyperbolic ([22], [28]). Also a large class of examples of special diffeomorphisms satisfying condition (3) above is given in Section 8.

\textbf{Theorem 1.5.} Let \( X \) be an orientable real 4-dimensional manifold which is the total space of a fiber bundle over an orientable real 2-dimensional manifold with fiber an orientable real 2-dimensional manifold. Assume all the monodromy diffeomorphisms of the fiber are special. Then the FIC is true for \( \pi_1(X) \).

In fact we prove that the FIC is true for any extension of a (real) surface group by a (real) surface group under a similar hypothesis.

From the proof of the above theorem the following more general corollary is easily deduced.

\textbf{Corollary 1.6.} Let \( M^{n+2} \to N^n \) be a fiber bundle projection of real manifolds with fiber \( F \). Let \( \pi_1(N) \) be torsion free, \( \pi_2(N) = 1 \) and the FIC is true for \( \pi_1(N) \). If \( M \) is not compact then assume in addition that the fiber of the fiber bundle projection \( M^{n+2} \to N^n \) is either of finite topological type or \( F \) has infinitely generated fundamental group and \( F \) is the covering of a compact surface \( F' \) corresponding to the commutator subgroup of \( \pi_1(F') \) and any monodromy diffeomorphism of the fiber \( F \) is a lift of a special diffeomorphism of \( F' \). Also assume the monodromy diffeomorphisms of the fiber \( F \) are special when \( F \) is closed and of genus \( \geq 2 \). Then the FIC is true for \( \pi_1(M) \).

The main ingredient behind the proof of the above results is the following proposition.

\textbf{Proposition 1.7.} Let \( N \) be a closed orientable 3-manifold and there is a finite sheeted cover \( M \) of \( N \) which fibers over the circle. Assume that the monodromy diffeomorphism of the fiber bundle \( M \to S^1 \) is special when the fiber is a surface of genus \( \geq 2 \). Then the FIC is true for \( \pi_1(N) \).

An important corollary to the above results is the following.
Corollary 1.8. Let $G$ be a torsion free subgroup of $\pi_1(X)$ where $X$ is a space appearing in the Theorems 1.2, 1.3 and 1.5, Corollaries 1.4 and 1.6 and Proposition 1.7. Then $Wh(G) = K_0(\mathbb{Z}G) = K_{-i}(\mathbb{Z}G) = 0$ for all $i \geq 1$.

We end this section with the following remark.

Remark 1.9. We have already mentioned in the introduction that throughout this section the assumption ‘special’ for the monodromy diffeomorphism of a fibered 3-manifold can be dropped provided the FIC is true for $A$-groups. Here we remark that in fact we do not need to assume this strong result. In [27] we show that we only need to assume that the FIC is true for the fundamental groups of compact irreducible 3-manifolds whose boundary components are all incompressible and are surfaces of genus $\geq 2$ (we called these groups as $B$-groups). Also we showed that a $B$-group is an $A$-group. We also proved (in [27]) that the FIC is true for a large class of $B$-groups.

2. Farrell-Jones fibered isomorphism conjecture

In this section we recall the Fibered Isomorphism Conjecture of Farrell and Jones made in [6].

Let $\mathcal{S}$ denotes one of the three functors from the category of topological spaces to the category of spectra: (a) the stable topological pseudoisotopy functor $\mathcal{P}()$; (b) the algebraic $K$-theory functor $\mathcal{K}()$; (c) and the $L$-theory functor $\mathcal{L}^{-\infty}()$.

Let $\mathcal{M}$ be the category of continuous surjective maps. The objects of $\mathcal{M}$ are continuous surjective maps $p : E \to B$ between topological spaces $E$ and $B$. And a morphism between two maps $p : E_1 \to B_1$ and $q : E_2 \to B_2$ is a pair of continuous maps $f : E_1 \to E_2$, $g : B_1 \to B_2$ such that the following diagram commutes.

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p} & & \downarrow{q} \\
B_1 & \xrightarrow{g} & B_2
\end{array}
\]

There is a functor defined by Quinn [25] from $\mathcal{M}$ to the category of $\Omega$-spectra which associates to the map $p : E \to B$ the spectrum $\mathbb{H}(B, S(p))$ with the property that $\mathbb{H}(B, S(p)) = S(E)$ when $B$ is a single point. For an explanation of $\mathbb{H}(B, S(p))$ see [[6], section 1.4]. Also the map $\mathbb{H}(B, S(p)) \to S(E)$ induced by the morphism: id: $E \to E$; $B \to \ast$ in the category $\mathcal{M}$ is called the Quinn assembly map.

Let $\Gamma$ be a discrete group and $\mathcal{E}$ be a $\Gamma$ space which is universal for the class of all virtually cyclic subgroups of $\Gamma$ and denote $\mathcal{E}/\Gamma$ by $\mathcal{B}$. For definition of universal space see [[6], appendix]. Let $X$ be a space on which $\Gamma$ acts freely and properly.
discontinuously and \( p : X \times \Gamma \mathcal{E} \to \mathcal{E}/\Gamma = \mathcal{B} \) be the map induced by the projection onto the second factor of \( X \times \mathcal{E} \).

The Fibered Isomorphism Conjecture states that the map

\[
\mathbb{H}(\mathcal{B}, \mathcal{S}(p)) \to \mathcal{S}(X \times \Gamma \mathcal{E}) = \mathcal{S}(X/\Gamma)
\]

is a (weak) equivalence of spectra. The equality is induced from the map \( X \times \Gamma \mathcal{E} \to X/\Gamma \) and using the fact that \( \mathcal{S} \) is homotopy invariant.

Let \( Y \) be a connected \( CW \)-complex and \( \Gamma = \pi_1(Y) \). Let \( X \) be the universal cover \( \tilde{Y} \) of \( Y \) and the action of \( \Gamma \) on \( X \) is the action by group of covering transformation. If we take an aspherical \( CW \)-complex \( Y' \) with \( \Gamma = \pi_1(Y') \) and \( X \) is the universal cover \( \tilde{Y}' \) of \( Y' \) then by [6], corollary 2.2.1 if the FIC is true for the space \( \tilde{Y}' \) then it is true for \( \tilde{Y} \) also. Thus whenever we say that the FIC is true for a discrete group \( \Gamma \) or for the fundamental group \( \pi_1(X) \) of a space \( X \) we would mean it is true for the Eilenberg-MacLane space \( K(\Gamma, 1) \) or \( K(\pi_1(X), 1) \) and the functor \( \mathcal{S}() \).

Throughout this paper we consider only the stable topological pseudoisotopy functor; that is the case when \( \mathcal{S}() = \mathcal{P}() \). And by the FIC we mean the FIC for \( \mathcal{P}() \).

Now we recall few known results we need about the FIC.

**Lemma A.** ([6], theorem A.8) If the FIC is true for a discrete group \( \Gamma \) then it is true for any subgroup of \( \Gamma \).

**Lemma B.** Let \( \Gamma \) be an extension of the fundamental group \( \pi_1(M) \) of a closed nonpositively curved Riemannian manifold or a compact orientable surface (may be with nonempty boundary) \( M \) by a finite group \( G \) then the FIC is true for \( \Gamma \). Moreover the FIC is true for the wreath product \( \Gamma \wr G \).

**Lemma C.** ([6], proposition 2.2) Let \( f : G \to H \) be a surjective homomorphism. Assume that the FIC is true for \( H \) and for \( f^{-1}(C) \) for all virtually cyclic subgroup \( C \) of \( H \) (including \( C = 1 \)). Then the FIC is true for \( G \).

We will use Lemma A, Lemma C and [[9], algebraic lemma] throughout the paper, sometimes even without referring to it.

**Lemma D.** Let \( S \) be a closed 2-dimensional orbifold. Then the FIC is true for \( \pi_1^{orb}(S) \).

Before we give the proofs of Lemma B and D we recall some group theoretic definition. Let \( G \) and \( H \) be two groups. Assume \( G \) is finite. Then \( H \wr G \) denotes the wreath product with respect to the regular action of \( G \) on \( G \). Recall that actually \( H \wr G \cong H^G \rtimes G \) where \( H^G \) is product of \( |G| \) copies of \( H \) indexed by the
elements of $G$ and the action of $G$ on the product is induced by the regular action of $G$ on $G$.

Proof of Lemma $B$ and $D$. Let us start with the hypothesis of Lemma $D$. If $\pi_1^{orb}(S)$ is finite then there is nothing to prove because the FIC is true for any finite group. So assume it is infinite. If the orbifold fundamental group is infinite then there is a finite index subgroup $H$ of $\pi_1^{orb}(S)$ such that $H$ is isomorphic to the fundamental group of a closed surface. Taking intersection of all conjugates of $H$ in $\pi_1^{orb}(S)$ we get a finite index normal subgroup $H_1 < H$ of $\pi_1^{orb}(S)$. Clearly $H_1$ is again the fundamental group of a closed surface, say $\tilde{S}$. Thus we have an exact sequence

$$1 \to \pi_1(\tilde{S}) \to \pi_1^{orb}(S) \to G \to 1.$$ 

Here $G$ is a finite group.

Let $\Gamma = \pi_1^{orb}(S)$ and $M = \tilde{S}$. We have reached the hypothesis of Lemma $B$ in the case when $M$ is closed.

By [[9], algebraic lemma] or [[4], theorem 2.6A] we have an embedding of $\Gamma$ in the wreath product $\pi_1(M) \wr G$, where the wreath product is taken using the regular action of the group $G$ on $G$. Let $U = M \times \cdots \times M$ be the $|G|$-fold product of $M$. Then $U$ is a closed nonpositively curved Riemannian manifold. By [[9], fact 3.1] it follows that the FIC is true for $\pi_1(U) \rtimes G \simeq (\pi_1(M))^G \rtimes G \simeq \pi_1(M) \wr G$. Lemma A now proves that the FIC is true for $\Gamma$.

If $M$ is a compact surface with nonempty boundary then $\pi_1(M) < \pi_1(N)$ where $N$ is a closed nonpositively curved surface. Hence $\Gamma < \pi_1(M) \wr G < \pi_1(N) \wr G$. Using Lemma A we complete the proof. \qed

Finally we recall the following important case when the FIC is true.

**Theorem E.** ([6], theorem 2.1) Let $\Gamma$ be a cocompact discrete subgroup of a virtually connected Lie group. Then the FIC is true for $\Gamma$.

We will also use the following consequence of [[6], proposition 2.4] frequently. Recall that a poly-$Z$ group is a group having a normal series whose normal quotients are infinite cyclic. And a group is virtually poly-$Z$ if it has a finite index subgroup which is poly-$Z$.

**Theorem F.** ([6]) The FIC is true for any virtually poly-$Z$ group.

3. Classification of complex surfaces and related topological results

In this section we recall the classification of complex surfaces and their topological properties which we need in the next section for the proof of Theorems 1.2 and Corollary 1.4. The references for this material we follow are [10] and [2].
By a complex surface we mean a compact complex manifold of complex dimension 2. There are two classes of complex manifolds; the algebraic and the non-algebraic ones. Throughout the paper by algebraic surface we will mean (unless otherwise stated) a complex surface which is a complex submanifold of some complex projective space \( \mathbb{CP}^n \) and also this condition is equivalent to saying that it can be obtained as a set of zeros of finitely many homogeneous polynomial in some complex projective space.

Let \( X \) be a complex surface. For a point \( x \in X \) the blow up of \( X \) at \( x \) is the surface \( X \# \overline{\mathbb{CP}^2} \) where the connected sum is taken around a ball at \( x \). Here \( \overline{\mathbb{CP}^2} \) is \( \mathbb{CP}^2 \) with the opposite orientation. By blowing up at a point \( x \) we introduce a curve \( C \) which is isomorphic to \( \mathbb{CP}^1 \) in the space replacing the point \( x \). \( C \) has self-intersection number \(-1\). This is also called a \(-1\) curve. Conversely if there is a \(-1\) curve \( C \) in a complex surface \( X' \) then there is another complex surface \( X \) and a point on it so that \( X' \) is obtained from \( X \) by blowing up \( X \) at the point \( x \). This is called blowing down the curve \( C \). In this discussion our main interest is that by blowing down or blowing up in a complex surface we do not change the fundamental group.

For the definition of Kodaira dimension \( \kappa(X) \) of a complex surface \( X \) we refer the reader to \([10], \text{definition 1.6}\). \( \kappa(X) \) can assume only 4 values; \(-\infty, 0, 1 \) and 2. There is a classification of complex surface in terms of this dimension and we recall the known topological properties of surfaces of different Kodaira dimension.

We recall these results without proof and give the references where the proof can be found.

### 3.1 The case \( \kappa(X) = -\infty \).

A complex surface \( X \) is called ruled ([10], definition 1.9]) if there is a holomorphic map \( f : X' \to S \), where \( S \) is a complex 1-manifold and all the fibers of \( f \) are isomorphic to \( \mathbb{CP}^1 \) and \( X \) is a blow up of \( X' \). The map \( f : X' \to S \) is called a ruling of \( X' \). It is well known that given a ruling \( f : X \to S \) on a complex surface there is a rank 2 complex vector bundle on \( S \) whose associated \( \mathbb{CP}^1 \) bundle is isomorphic to \( X \) (see \([10], \text{chapter I, section 1.2.1}\)). Thus we see that \( f \) is in fact a fiber bundle projection. Hence for any ruled surface \( X \), \( \pi_1(X) \) is isomorphic to \( \pi_1(S) \) for some curve \( S \).

**Proposition 3.1.1.** ([10], chapter I, theorem 1.10]) Let \( X \) be a complex algebraic surface with \( \kappa(X) = -\infty \) then \( X \) is either \( \mathbb{CP}^2 \) or is ruled.

There are nonalgebraic surfaces with \( \kappa(X) = -\infty \). All of them has first Betti number equal to one and with infinite fundamental group. A classification of these surfaces are not yet known. A class of examples of such surfaces are Hopf surfaces.
These are complex surfaces which has universal cover biholomorphic to $\mathbb{C}^2 - (0,0)$. Also it is known ([10], chapter I, proposition 7.15) that any Hopf surface has a finite sheeted cover homotopy equivalent to $S^1 \times S^3$. Hence we have the following Proposition.

**Proposition 3.1.2.** Any Hopf surface has virtually infinite cyclic fundamental group.

Another class of example of nonalgebraic surfaces with $\kappa(X) = -\infty$ are Inoue surfaces ([2], chapter V, section 19). These surfaces are by construction fiber bundle over $S^1$ with fiber a 3-manifold which is a principal $S^1$-bundle over the 2-torus $S^1 \times S^1$.

**Proposition 3.1.3.** [23] The fundamental group of an Inoue surface is isomorphic to $G \times \mathbb{Z}$ where $G$ is the fundamental group of a principal $S^1$ bundle over the 2-torus.

**Corollary 3.1.4.** The fundamental group of an Inoue surface is poly-$\mathbb{Z}$.

### 3.2 The case $\kappa(X) = 0$

The following two propositions give the required topological properties we need of surfaces with $\kappa(X) = 0$.

**Proposition 3.2.1.** ([10], chapter I, theorem 2.6]) Let $X$ be an algebraic surface and $\kappa(X) = 0$. Then $\pi_1(X) \cong \mathbb{Z}^4, \{1\}, \mathbb{Z}/2\mathbb{Z}$, or $\pi_1(X_{\text{min}})$. Here $X_{\text{min}}$ is a hyperelliptic surface. In the last case we have an exact sequence

$$1 \to \Lambda \to \pi_1(X) \to G \to 1$$

where $G$ is a finite group and $\Lambda$ is a lattice in $\mathbb{C}^2$.

By definition a hyperelliptic surface is a finite quotient of a complex torus ([10], chapter I, section 1.1.4])

**Proposition 3.2.2.** ([10], chapter I, section 1.1.4]) If $X$ is a nonalgebraic surface and $\kappa(X) = 0$. Then $\pi_1(X)$ lies in an exact sequence.

$$1 \to \mathbb{Z} \times \pi_1(N) \to \pi_1(X) \to G \to 1$$

where $N$ is a 3-dimensional nilmanifold and $G$ is a finite group.

### 3.3 The case $\kappa(X) = 1$

An elliptic structure or elliptic fibration ([10], chapter I, section 1.14]) on a complex surface $X$ is a holomorphic map $\pi : X \to C$, where $C$ is a curve such that for a general point $t \in C$, $\pi^{-1}(t)$ is a curve of genus 1. An elliptic surface is a surface with an elliptic structure. Note that given an elliptic fibration the base curve $C$ inherits an orbifold structure.
Theorem 3.3.1. ([10], chapter I, theorem 1.15) If $X$ is a complex surface with $\kappa(X) = 1$ then $X$ is an elliptic surface.

3.4 The case $\kappa(X) = 2$.

Surfaces with $\kappa(X) = 2$ are called surface of general type. There is no complete classification of these surfaces. As examples of these surfaces we recall some of them from [2], p. 189. (a) complete intersections of sufficiently high degree; (b) products of curves of genus $\geq 2$; (c) Kodaira fibration; (d) quotients of symmetric domains; and (e) practically all ramified double covering of $\mathbb{C}P^2$. In the last case the surface can be a singular algebraic surface.

3.5 On elliptic surfaces.

We have already defined elliptic surfaces. In this subsection we recall the structure of the fundamental group of an elliptic surface. Also we recall the definition of $C^\infty$-elliptic surface and describe their fundamental groups. We always follow the terminology of [10].

Proposition 3.5.1. ([10], chapter II, theorem 2.3 and proposition 2.1) Let $\pi : X \to C$ be an elliptic fibration. If the Euler number of $X$ is positive then $\pi$ induces an isomorphism $\pi_1(X) \to \pi_1^{orb}(C)$.

Proposition 3.5.2. ([10], chapter II, lemma 7.3 and proposition 7.4) Let $\pi : X \to C$ be an elliptic surface with Euler number 0. Then we have

1. if $\pi_1^{orb}(C)$ is infinite then there is an exact sequence
   $$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(X) \to \pi_1^{orb}(C) \to 1.$$

2. if $\pi_1^{orb}(C)$ is finite and the orbifold $C$ is good then there is an exact sequence
   $$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(X) \to \pi_1^{orb}(C) \to 1.$$

3. if $C$ is bad then $\pi_1(X)$ is abelian and is isomorphic to either $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$.

Definition 3.5.3. ([10], chapter II, definition 1.1) A $C^\infty$-elliptic surface is a smooth map $\pi : X \to C$ from a closed smooth oriented 4-manifold $X$ to a smooth oriented real 2-manifold $C$ such that for each point $p \in C$ there is an open disk $p \in \Delta \subset C$ and a complex elliptic surface $S \to \Delta$ and a smooth orientation preserving diffeomorphism $\pi^{-1}(\Delta) \to S$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\pi^{-1}(\Delta) & \overset{\simeq}{\longrightarrow} & S \\
\downarrow & & \downarrow \\
\Delta & = & \Delta
\end{array}
$$
It follows that for a $C^\infty$-elliptic surface $X$ the general fiber of the map $\pi : X \to C$ is a 2-torus. We say $X$ has no singular fiber if all the fibers of $\pi : X \to C$ are smooth submanifold. Note that $C$ inherits an orbifold structure and when the Euler number of $X$ is zero the monodromy action of $\pi_1^{\text{orb}}(C, p)$ on $H_1(f)$, where $f$ is the fiber of $\pi$ over a general base point $p$, factors through $\pi_1(C)$ and its image is a finite cyclic subgroup of $SL(H_1(f))$. In such a situation we say that the elliptic surface $X$ has cyclic monodromy (see last paragraph in [[10], p. 195]).

Now we are ready to state our next proposition whose proof is easily deduced from the proof of Proposition 3.5.2.

**Proposition 3.5.4.** Let $X$ be a $C^\infty$-elliptic surface without any singular fiber and has cyclic monodromy. Then the conclusions of Proposition 3.5.2 are true.

### 4. Proofs

**Proof of Theorem 1.2.** If $X$ is algebraic and $\kappa(X) = -\infty$ then by Proposition 3.1.1 either $\pi_1(X)$ is trivial or isomorphic to the fundamental group of a curve. Hence the FIC is true in this case (Lemma B).

By Proposition 3.1.2 any Hopf surface has virtually infinite cyclic fundamental group and hence the FIC is true for $\pi_1(X)$ by [[7], lemma 2.7].

When $X$ is algebraic and Kodaira dimension 0 then by Proposition 3.2.1 $\pi_1(X)$ is virtually abelian and hence the FIC is true by [[7], lemma 2.7]. If it is nonalgebraic and $\kappa(X) = 0$ then by Proposition 3.2.2 $\pi_1(X)$ can be embedded as a cocompact discrete subgroup of a virtually connected Lie group using [[9], algebraic lemma]. In short we have

$$\pi_1(X) < (\mathbb{Z} \times \pi_1(N)) \wr G < \text{Iso}(\mathbb{R} \times N)^G).$$

Hence the FIC is true for $\pi_1(X)$ by Theorem E.

Inoue surfaces has poly-$\mathbb{Z}$ fundamental group by Corollary 3.1.4. Hence by Theorem F the FIC is true for the fundamental group of an Inoue surface.

Now we come to the case of elliptic surfaces. In the case when the Euler number of the elliptic fibration is positive then by Proposition 3.5.1 the fundamental group is isomorphic to the fundamental group of a 2-dimensional real orbifold and hence the FIC is true by Lemma D.

Next assume that the elliptic fibration has Euler number 0. Then we had three cases as in Proposition 3.5.2. By Lemma D we know that $\pi_1^{\text{orb}}(C)$ satisfies the FIC. Assume $\pi_1^{\text{orb}}(C)$ is infinite. Let $E$ be a virtually cyclic subgroup of $\pi_1^{\text{orb}}(C)$. If $E$ is finite then $p^{-1}(E)$ is virtually abelian and hence the FIC is true for $p^{-1}(E)$ ([[7], lemma 2.7]). Here $p$ is the surjective homomorphism $p : \pi_1(X) \to \pi_1^{\text{orb}}(C)$. 

Now let $Z$ be an infinite cyclic normal subgroup of $E$ of finite index. Then the filtration $0 < Z < Z \oplus Z < p^{-1}(Z) < p^{-1}(E)$ of $p^{-1}(E)$ gives a virtually poly-$Z$ group structure on $p^{-1}(E)$ and hence the FIC is true for $p^{-1}(E)$. Thus the FIC is true for $\pi_1(X)$.

If $\pi_1^{orb}(C)$ is finite then from the second exact sequence in Proposition 3.5.2 it follows that $\pi_1(X)$ is virtually abelian. Hence the FIC is true for $\pi_1(X)$ by [[7], lemma 2.7]. In the last case again we apply [[7], lemma 2.7].

Thus we have proved the Theorem 1.2. □

Proof of Corollary 1.4. The proof is same as the proof of Theorem 1.2 in case when $X$ is an elliptic fibration. □

Proof of Theorem 1.3. Recall that we have a fiber bundle projection from the 4-dimensional real manifold $X$ over the circle and also $X$ has a complex structure. The following theorem shows that the fiber of this fiber bundle projection is of a particular type. The theorem is a consequence of [[12], theorem 7] and the discussion following it.

Theorem 4.1. Let $X$ be a complex surface which is the total space of a fiber bundle over $S^1$. Then the fiber is diffeomorphic to a Seifert fibered space.

Let $N$ be the fiber and $f$ be the monodromy diffeomorphism.

It is well known (see [[13], theorem VI.17] and [29], [30]) that apart from the following few cases (A to E) (see [[13], section VI.16]) the diffeomorphism $f$ is isotopic to a fiber preserving diffeomorphism, say $f$ again, that is $f$ sends the fiber circle to fiber circle of $N$.

A. The Lens spaces including $S^2 \times S^1$ and $S^3$.

B. Seifert fibered spaces with base $S^2$ and three exceptional fiber of index corresponding to the triple $(2, 2, \alpha)$ where $\alpha > 1$. These manifolds are called prism-manifolds and has finite fundamental group.

C. A class of torus bundles over the circle.

D. The solid torus.

E. A twisted $I$-bundle over the Klein bottle.

At first let us assume that $f$ is fiber preserving. We have two cases.

$\pi_1(N)$ is infinite. In this case as $N$ is a Seifert fibered space there is an exact sequence.

$$1 \to Z \to \pi_1(N) \to \pi_1^{orb}(B) \to 1$$

where $Z$ is infinite cyclic and is generated by a regular fiber and $\pi_1^{orb}(B)$ is the orbifold fundamental group of the base orbifold $B$ of $N$. 
As $f$ is fiber preserving and $Z$ is generated by a regular fiber the induced action of $f$ on $\pi_1(N)$ leaves $Z$ invariant and hence we get the following.

$$1 \to Z \to \pi_1(N) \rtimes \langle t \rangle \to \pi_1^{orb}(B) \rtimes \langle t \rangle \to 1.$$ 

Which reduces to:

$$1 \to Z \to \pi_1(X) \to \pi_1^{orb}(B) \rtimes \langle t \rangle \to 1.$$ 

Here, up to conjugation, the actions of $t$ on the various groups are induced by $f$.

We will apply Lemma C to this exact sequence. If $\pi_1^{orb}(B)$ is finite then $\pi_1(X)$ is virtually poly-$\mathbb{Z}$ and hence the FIC is true for this group by Theorem F. So we assume $\pi_1^{orb}(B)$ is infinite. In this case there is a finite index subgroup $H$ of $\pi_1^{orb}(B)$ which is the fundamental group of a closed surface.

**Claim.** There is a characteristic closed surface subgroup of $\pi_1^{orb}(B)$ of finite index.

**Proof of claim.** Let $H_1$ be the intersection of all conjugates of $H$ in $\pi_1^{orb}(B)$. Then $H_1 < H$ is a finite index normal subgroup of $\pi_1^{orb}(B)$. Also this implies that $H_1$ is again a closed surface group. Now let $G = \pi_1^{orb}(B)/H_1$ be the finite quotient group. Since $\pi_1^{orb}(B)$ is finitely presented there are only finitely many homomorphism from $\pi_1^{orb}(B)$ to $G$, say $f_1, \ldots, f_n$. Consider $K = \cap_i \ker f_i$. Then $K$ is a finite index characteristic subgroup of $\pi_1^{orb}(B)$. Also $K < H_1$ and hence $K$ is also a closed surface group.

**Remark 4.2.** We will use the procedure of finding a finite index characteristic subgroup as in the proof of the above claim in the remaining part of the paper without repeating the argument again.

Since $K$ is a characteristic subgroup we have an exact sequence.

$$1 \to K \to \pi_1^{orb}(B) \rtimes \langle t \rangle \to (\pi_1^{orb}(B)/K) \rtimes \langle t \rangle \to 1.$$ 

Note that $(\pi_1^{orb}(B)/K) \rtimes \langle t \rangle$ is a virtually cyclic group. Let $C$ be an infinite cyclic normal subgroup of $(\pi_1^{orb}(B)/K) \rtimes \langle t \rangle$ of finite index. Then we get the following.

$$\pi_1^{orb}(B) \rtimes \langle t \rangle < (K \rtimes C) \rtimes H.$$ 

Where $H \simeq ((\pi_1^{orb}(B)/K) \rtimes \langle t \rangle)/C$ is a finite group. Since $K$ is the fundamental group of a closed surface $K \rtimes C$ is the fundamental group of a closed 3-manifold which fibers over $S^1$. If $K$ is the fundamental group of a torus then $(K \rtimes C) \rtimes H$ is virtually poly-$\mathbb{Z}$ and hence the FIC is true. Otherwise, by hypothesis there is a base diffeomorphism (the monodromy diffeomorphism) which is special and
therefore we can apply Proposition 1.7 to deduce that the FIC is true for \((K \rtimes C) \wr H\). Consequently, the same is true for \(\pi_1^{orb}(B) \rtimes \langle t \rangle\). To complete the proof we need to check that \(p^{-1}(C)\) satisfies the FIC for any virtually cyclic subgroup \(C\) of \(\pi_1^{orb}(B) \rtimes \langle t \rangle\). Here \(p\) denotes the homomorphism \(\pi_1(X) \to \pi_1^{orb}(B) \rtimes \langle t \rangle\). Obviously \(p^{-1}(C)\) is virtually poly-Z and hence the FIC is true for \(p^{-1}(C)\). This completes the proof in this case.

\(\pi_1(N)\) is finite. As \(\pi_1(N)\) is finite \(\pi_1(X) \simeq \pi_1(N) \rtimes \langle t \rangle\) has a normal subgroup of finite index isomorphic to \(\pi_1(N) \times \mathbb{Z}\). Hence \(\pi_1(X)\) is virtually infinite cyclic. The theorem now follows.

Now we come to the situation when \(f\) is not fiber preserving. We have mentioned before there are the following possibilities: A to E.

**Case A.** \(\pi_1(X)\) is isomorphic to one of the following.

1. fundamental group of a closed flat 2-manifold.
2. infinite cyclic.
3. has a finite normal subgroup with infinite cyclic quotient. And hence is virtually infinite cyclic.

We have already mentioned that the FIC is true in cases (1) to (3).

**Case B.** This case goes to (3) in Case A.

**Case C.** In this case \(\pi_1(X)\) is poly-Z and hence the FIC is true by Theorem F.

As \(X\) is closed Case D and Case E do not occur.

This completes the proof of the Theorem. □

**Remark 4.3.** The crucial fact we used in the proof is that the fiber \(N\) is a Seifert fibered space. We would like to point out here that except for only two examples all orientable Seifert fibered space appear as fiber of fiber bundle projection \(M^4 \to S^1\) where \(M^4\) is a complex surface (see [12], theorem 7.)

**Proof of Corollary 1.8.** If \(G\) is torsion free and the FIC is true for \(G\) then using Lemma C to the projection \(G \times \mathbb{Z}^n \to G\) and noting that the FIC is true for free abelian groups we get that the FIC is true for \(G \times \mathbb{Z}^n\) also. Now we can apply [[9], theorem D] to see that \(Wh(G \times \mathbb{Z}^n) = 0\) for all \(n \geq 0\). It now follows from Bass’s contracted functor argument ([[1], §7, chapter XII]) that \(\tilde{K}_0(\mathbb{Z}G) = K_{-i}(\mathbb{Z}G) = 0\) for all \(i \geq 1\). □

In this paragraph we will say few words about the case of surfaces of general type. Recall the examples (a) to (e) from 3.4 of Section 3. The examples in (a) are simply connected ([[2], chapter V, proposition 2.1]). Examples (b) and (c) are already considered in Theorem 1.5. The fundamental groups of compact quotients of symmetric domains are discrete cocompact subgroups of the Lie groups \(SU(2,1)\)
or $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ (see [[2], chapter V, section 20]). In case (e) we have complex surfaces $X$ with a surjective morphism $p : X \to \mathbb{CP}^2$ such that outside a curve $C$ in $\mathbb{CP}^2$ the morphism is a 2-sheeted covering; that is $p$ is a ramified covering with ramification locus $C$. It is well known that for any curve $C' \subset X$ the inclusion map induces a surjective homomorphism $\pi_1(X - C') \to \pi_1(X)$. Now combining the two results from [[21], corollary(Zariski Conjecture) and corollary 2.8] we get that if $C \subset \mathbb{CP}^2$ is either a smooth curve or a singular curve with only nodal singularities then $\pi_1(\mathbb{CP}^2 - C)$ is abelian. Hence $X - p^{-1}(C)$ is also abelian. On the other hand we have a surjective homomorphism $\pi_1(X - p^{-1}(C)) \to \pi_1(X)$. Thus we have proved the following.

**Proposition 4.4.** Let $X$ be an algebraic surface (possibly singular). Then $\pi_1(X)$ is abelian whenever $X$ is a 2-sheeted ramified covering of $\mathbb{CP}^2$ with ramification locus is either a smooth or a nodal curve.

Summarizing the above discussions we note that for these surfaces of general type the fundamental groups are equal to 1, groups already contained in Theorem 1.5, discrete cocompact subgroup of Lie group or abelian. We have already shown that the FIC is true in these situations.

**Remark 4.5.** Recall from [[2], chapter I, section 17] that the algebraic surface in the above proposition is singular whenever the branch locus is singular.

We give an example below to show that, given the method of proof we have to prove the FIC for groups, among the examples of ramified coverings of $\mathbb{CP}^2$ the above class of examples are best possible for which we could prove the FIC.

**Examples 4.6.** ([[21], example 6.7]) Let $f$ and $g$ be two homogeneous polynomials in three variables of degree 2 and 3 respectively. Consider the curve $C$ defined by $f^3 - g^2 = 0$ in $\mathbb{CP}^2$. $C$ is smooth outside the points where $f = 0$ and $g = 0$. And the singular points are ordinary cusp. Then $\pi_1(\mathbb{CP}^2 - C) \simeq \mathbb{Z}_2 \ast \mathbb{Z}_3$. Now if $X$ is a complex surface which is a 2-sheeted ramified covering of $\mathbb{CP}^2$ with ramification locus $C$ then $\pi_1(X)$ is the image of an index 2 subgroup $H$ of $\mathbb{Z}_2 \ast \mathbb{Z}_3$. On the other hand $H$ contains the commutator subgroup of $\mathbb{Z}_2 \ast \mathbb{Z}_3$ which is a nonabelian free group. Hence $\pi_1(X)$ can be a highly complicated group.

5. The FIC for some real manifolds

We need the following crucial Lemma to prove Theorem 1.5. Apart from being the crucial ingredient throughout the paper the Main Lemma is of independent interest. The proof of this Lemma is given in Section 6. The definition of special diffeomorphism is given in the Introduction.
Main Lemma. Let $M^3$ be a closed 3-dimensional manifold which is the total space of a fiber bundle projection $M^3 \to \mathbb{S}^1$ with orientable fiber. When the fiber is a surface of genus $\geq 2$ assume that the fiber bundle has special monodromy diffeomorphism. Then the FIC is true for $\pi_1(M)$.

Proof of Theorem 1.5. We prove a general version of Theorem 1.5. Let $F$ and $B$ be two closed orientable 2-dimensional real manifold. Let $G$ be a group which fits in an exact sequence.

$$1 \to \pi_1(F) \to G \to \pi_1(B) \to 1.$$ 

Also by assumption the action of any element of $\pi_1(B)$ on $\pi_1(F)$ is induced by a special diffeomorphism of $F$.

Then we prove that the FIC is true for $G$. Note that $\pi_1(X)$ also fits in such an exact sequence. Without loss of generality we can assume that both the surfaces $F$ and $B$ are of genus $\geq 1$. Then as $B$ supports a nonpositively curved Riemannian metric, $\pi_1(B)$ satisfies the FIC. Let $C$ be a virtually cyclic subgroup of $\pi_1(B)$. As $\pi_1(B)$ is torsion free, $C$ is infinite cyclic. We have an exact sequence $1 \to \pi_1(F) \to p^{-1}(C) \to C \to 1$. We have $p^{-1}(C) \simeq \pi_1(F) \rtimes \langle t \rangle$ where the image of $t$ in $C$ generates $C$. It is well known that as $F$ is closed, up to conjugation the action of $t$ on $\pi_1(F)$ is induced by a diffeomorphism of $F$. Hence $p^{-1}(C) \simeq \pi_1(N)$ where $N$ is a 3-manifold which is a fiber bundle over the circle with fiber diffeomorphic to $F$. By hypothesis this diffeomorphism is special. By the Main Lemma the FIC is true for $p^{-1}(C)$. We apply Lemma C to complete the proof. □

Proof of Corollary 1.6. The proof is the same as the proof of Theorem 1.5 when $M$ is compact. The only exception will be if the fiber has nonempty boundary. If the fiber is simply connected then there is nothing to prove. Otherwise by theorems 3.2 and 3.3 from [18] the interior of the mapping torus $M$ of a fiber supports a complete nonpositively curved Riemannian metric so that near the boundary the metric is a product metric. Hence the double of $M$ will support a nonpositively curved metric. Also, the mapping torus $M$ will have incompressible boundary and hence $\pi_1(M) < \pi_1(DM)$ where $DM$ denotes the double of $M$. By Lemma B we complete the proof in this case.

Next we assume $M$ is noncompact and of finite topological type. Consider the surjective homomorphism $p : \pi_1(M) \to \pi_1(N)$ with kernel $\pi_1(F)$. If $C$ is an infinite cyclic subgroup of $\pi_1(N)$ then $p^{-1}(C) \simeq \pi_1(F) \rtimes \mathbb{Z} \simeq \pi_1(P)$ where $P$ is the mapping torus of a (monodromy) diffeomorphism of $F$. Note that $\pi_1(F)$ is a finitely generated free group. Now we use Lemma 6.1 and the previous argument in case of compact fiber with nonempty boundary to complete the proof of the
Corollary in this case. If the fiber has infinitely generated fundamental group then 
\( \pi_1(P) \) is a subgroup of a 3-manifold fibering over the circle with special monodromy
diffeomorphism and hence the FIC is true for \( \pi_1(P) \) by the Main Lemma and Lemma A. □

6. Proof of the Main Lemma

Proof of the Main Lemma. We start the proof without assuming the monodromy is special. The following exact sequence is obtained from the long exact homotopy
sequence of the fibration \( M \to \mathbb{S}^1 \).

\[
1 \to \pi_1(F) \to \pi_1(M) \to \pi_1(\mathbb{S}^1) \to 1
\]

where \( F \) is the fiber of the fiber bundle projection. Let \([A, A]\) denotes the commutator subgroup of the group \( A \). Then we have

\[
1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(F) \to H_1(F, \mathbb{Z}) \to 1.
\]

Let \( t \) be a generator of \( \pi_1(\mathbb{S}^1) \). Since \([\pi_1(F), \pi_1(F)]\) is a characteristic subgroup of \( \pi_1(F) \) the action (induced by the monodromy) of \( t \) on \( \pi_1(F) \) leaves \([\pi_1(F), \pi_1(F)]\) invariant. Thus we have another exact sequence

\[
1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(F) \rtimes \langle t \rangle \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1.
\]

Which reduces to the sequence

\[
1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1.
\]

We would like to apply Lemma C to this exact sequence.

If the fiber is \( \mathbb{S}^2 \) or \( \mathbb{T}^2 \) then \( \pi_1(M) \) is poly-\( \mathbb{Z} \) and hence the FIC is true for \( \pi_1(M) \). So assume that the fiber has genus \( \geq 2 \).

Note that the group \( H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \) is poly-\( \mathbb{Z} \). Hence by Theorem F the FIC is true for \( H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \). Let \( C \) be a virtually cyclic subgroup of \( H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \). Let \( p \) denotes the surjective homomorphism \( \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \). We will show that the FIC is true for \( p^{-1}(C) \). Note that \( C \) is either trivial or infinite cyclic.

Case \( C = 1 \). In this case we have that \( p^{-1}(C) \) is a nonabelian free group and hence is the fundamental group of a surface. We need the following lemma to complete the proof of this case.
Lemma 6.1. Let $\Gamma$ be the fundamental group of a surface then the FIC is true for $\Gamma \wr G$ for any finite group $G$.

Proof. If $\Gamma$ is finitely generated then $\Gamma$ is the fundamental group of a compact surface and hence the lemma follows from Lemma B. In the infinitely generated case $\Gamma \wr G \simeq \lim_{i \to \infty} (\Gamma_i \wr G)$ where each $\Gamma_i$ is a finitely generated free group. Now recall that any finitely generated nonabelian free group is the fundamental group of a compact surface with nonempty boundary. By Lemma B, and [[7], theorem 7.1] the proof is complete. □

Case $C \neq 1$. We have $p^{-1}(C) = [\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle$ where $s$ is a generator of $C$. Let $\tilde{F}$ be the covering space of $F$ corresponding to the commutator subgroup $[\pi_1(F), \pi_1(F)]$.

Note that the monodromy diffeomorphism of $F$ lifts to a diffeomorphism of $\tilde{F}$ which in turn, up to conjugation, induces the action of $t$ on $[\pi_1(F), \pi_1(F)]$. Also, the induced action of $t$ on $H_1(F, \mathbb{Z})$ is given by $t(v) = \tilde{f} \circ v \circ \tilde{f}^{-1}$, where $\tilde{f} : \tilde{F} \to \tilde{F}$ is a lift of the monodromy diffeomorphism and $v \in H_1(F, \mathbb{Z})$. Here $H_1(F, \mathbb{Z})$ is identified with the group of covering transformation of the covering $\tilde{F} \to F$. From this observation it follows that, up to conjugation, the action of $s$ on $[\pi_1(F), \pi_1(F)]$ is induced by a diffeomorphism (say $f$) of $\tilde{F}$. Indeed, if $s = (s_1, t^k) \in H_1(F, \mathbb{Z}) \rtimes \langle t \rangle$ then $f = s_1 \circ \tilde{f}^k : \tilde{F} \to \tilde{F}$.

Let $g : F \to F$ be a diffeomorphism of $F$ so that $f$ and a lift of $g$ induce the same outer automorphism of $\pi_1(\tilde{F})$. Such a diffeomorphism exists. Namely, let $u$ be an element of $\pi_1(F) \rtimes \langle t \rangle$ which goes to $s$. Then the conjugation action by $u$ on $\pi_1(F) \rtimes \langle t \rangle$ leaves $\pi_1(F)$ invariant. Since $F$ is closed there is a diffeomorphism $g$ of $F$ which, up to conjugation, induces this action of $u$ on $\pi_1(F)$. Clearly $g$ has the required property. Let $M_g$ and $M_f$ be the mapping tori of $g$ and $f$ respectively. Then $\pi_1(M_f)$ is a subgroup of $\pi_1(M_g)$ and $\pi_1(M_f) \simeq [\pi_1(F), \pi_1(F)] \rtimes \langle s \rangle$.

Note that the topological type of the mapping torus $M_g$ is an invariant of the isotopy class of $g$. Below, whenever we say “$g$ is” we mean “an isotopy of $g$ is”. Also assume that $g$ is an orientation preserving diffeomorphism. The orientation reversing case can easily be tackled from the orientation preserving case and is left to the reader.

By Nielsen-Thurston classification of surface diffeomorphisms (see [3] or [[24], p. 175]) there are now three cases.

Case 1. $g$ is pseudo-Anosov. In this case $M_g$ supports a hyperbolic structure ([22], [28]) and hence the FIC is true for $\pi_1(M_g)$ by Lemma B. Thus the FIC is true for $\pi_1(M_f)$ also.

Case 2. $g$ is of finite order. In this case $M_g$ has a regular finite sheeted covering
diffeomorphic to $F \times S^1$. Hence $\pi_1(M_g)$ is a subgroup of $\pi_1(F \times S^1) \wr G_1$ where $G_1$ is a finite group. Since $F \times S^1$ supports a nonpositively curved Riemannian metric Lemma B applies again.

Case 3. $g$ is not pseudo-Anosov and of infinite order. By the above mentioned classification of surface diffeomorphism, in this case $g$ is reducible (see [[3], p. 75] for definition) and hence there are finitely many nontrivial elements $h_1, h_2, \cdots, h_n$ in $\pi_1(F)$ represented by pairwise disjoint mutually nonparallel simple closed curves (say, $C_1, C_2, \ldots, C_n$ respectively) on $F$ so that $g(\bigcup_i C_i) = \bigcup_i C_i$. Also there are pairwise disjoint tubular neighborhoods $N_i(C_i)$ of $C_i$ and submanifolds $F_j$ for $j = 1, 2, \cdots, l$ of $F$ satisfying the followings (see [[8], p. 219, Théorème 4.2]).

1. $F - \bigcup_{i=1}^n N_i(C_i) = \bigcup_{j=1}^l F_j$;
2. $g(F_j) = F_j$ for each $j = 1, 2, \cdots, l$;
3. $g|_{F_j}$ is isotopic either to a pseudo-Anosov diffeomorphism or to a finite order diffeomorphism.

Here some $F_j$ may have more than one connected component. Choose a positive integer $N$ so that $g^N(L) = L$ and $g^N|_L$ is isotopic either to a pseudo-Anosov diffeomorphism or to a finite order diffeomorphism, for each connected component $L$ of $F - \bigcup_{i=1}^n N(C_i)$. Hence the mapping torus of $g^N|_L$ is either a Seifert fibered space (in the case when $g^N|_L$ is isotopic to a finite order diffeomorphism) or supports a hyperbolic metric in the interior (pseudo-Anosov case) [see [28] or [22]]. In fact the mapping tori of $g^N|_L$ where $L$ varies over the connected components of $F - \bigcup_{i=1}^n N(C_i)$ give the JSJT decomposition of the mapping torus $M_{g^N}$ of $g^N$. If for some $L$, the mapping torus of $g^N|_L$ is hyperbolic then by [[18], theorem 3.2 and 3.3] $M_{g^N}$ supports a nonpositively curved Riemannian metric and since $M_{g^N}$ is a finite sheeted cover of $M_g$ we can apply Lemma B to conclude that the FIC is true for $\pi_1(M_g)$ and hence for $\pi_1(M_f)$ (Lemma A) also. Therefore for the rest of the proof we can assume that for each component $L$ of $F - \bigcup_{i=1}^n N(C_i)$ $g^N|_L$ is isotopic to a finite order diffeomorphism. From the above discussion it follows that the FIC is true for $\pi_1(M_f)$ when the monodromy is special and satisfies condition (1) or (2) in the definition.

From now onwards we assume that the monodromy diffeomorphism is special and satisfies condition (3) in the definition.

We will write $M_{f^N}$ as an increasing union of connected compact 3-manifolds with incompressible tori boundary components.

We need the following claim.

Claim. $M_f$ has one topological end.

Proof of claim. Recall that ([[5], p. 115]) by definition a group $G$ has $e(G)$ number
of ends if there is a regular covering projection \( \tilde{X} \to X \) where \( X \) is a finite complex, \( G \) is isomorphic to the group of covering transformation of \( \tilde{X} \to X \) and \( \tilde{X} \) has \( e(G) \) number of topological ends. Also this definition does not depend on the covering projection ([5], theorem 3).

As \( F \) has first Betti number \( \geq 2 \) the group \( H_1(F, \mathbb{Z}) \) is free abelian of rank greater than 1 and also \( H_1(F, \mathbb{Z}) \) is the group of covering transformations of \( \tilde{F} \to F \). Since \( F \) is compact, the manifold \( \tilde{F} \) has one topological end (see [5]). Figure 1 describes \( \tilde{F} \).

![Figure 1](image-url)

It now follows that \( M_f \) has one topological end. \( \square \)

The same proof shows that \( M_{fN} \) also has one topological end. This ensures that there exists a surjective proper smooth function \( \delta : M_{fN} \to \mathbb{R}^{\geq 0} \). Also since \( M_{fN} \) is connected and has one end we can always choose an \( r \in \mathbb{R}^{\geq 0} \) bigger than any given positive real number such that \( \delta^{-1}([0, r]) \) is connected.

We would like to construct submanifolds \( \tilde{N}_i^s \) of \( M_{fN} \) with the following properties.

1. each \( \tilde{N}_i^s \) is a compact connected submanifold of \( M_{fN} \) and has incompressible tori boundary.
2. \( \tilde{N}_i^s \subset \tilde{N}_{i+1}^s \) for \( i = 1, 2, \cdots \).
3. \( M_{fN} = \bigcup_{i=0}^{\infty} \tilde{N}_i^s \).

Consider the covering projection \( p : M_{fN} \to M_{gN} \). Recall that all the pieces in the JSJT decomposition of \( M_{gN} \) are Seifert fibered and are obtained by taking the mapping torus of \( g^N|_L \) for some component \( L \) of \( F - \bigcup_{i=1}^{n} N(C_i) \). Let \( P \) be such a Seifert fibered piece. We claim the following.

**Claim.** each component of \( p^{-1}(P) \) is a Seifert fibered space.

**Proof of claim.** Recall that \( u \in \pi_1(F) \times \langle t \rangle \) was a lift of \( s = (s_1, t^k) \in H_1(F, \mathbb{Z}) \times \langle t \rangle \) and \( g \) induced the restriction to \( \pi_1(F) \) of the conjugation action by \( u \) on \( \pi_1(F) \times \langle t \rangle \). Let \( u = (u_1, t^k) \) and let \( (x, 1) \in \pi_1(F) \times \langle t \rangle \). Then \( u(x, 1)u^{-1} = (u_1t^k(x)u_1^{-1}, 1) \). That is, the action of \( u \) is composition of the action of \( t^k \) and a conjugation by an element of \( \pi_1(F) \). Since conjugation action by an element of \( \pi_1(F) \) is induced by
a diffeomorphism of $F$ isotopic to the identity ([20]), it follows that $g$ is isotopic to the $k$-th power of the monodromy diffeomorphism (say $h$). Hence $g^N$ is isotopic $h^k N$. Consequently $M_{h^k N}$ is diffeomorphic to $M_{g N}$.

Consider the following commutative diagram.

\[
    \begin{array}{ccc}
    M_{f N} & \xrightarrow{f_1} & M_{h^k N} \\
    p \downarrow & & q \downarrow \\
    M_{g N} & \xrightarrow{f_2} & M_{h^k N}
    \end{array}
\]

As $M_{g N} \to M_{h^k N}$ is a finite sheeted cover the JSJ decomposition of $M_{g N}$ can be obtained by taking the inverse image of the pieces of the JSJ decomposition of $M_{h^k N}$. So let $P$ be such a piece in $M_{g N}$ and let $S_P$ be a component of $p^{-1}(P)$ (as above) which goes to the component $S_{P'}$ of $q^{-1}(P')$ where $P' = f_2(P)$. Then we have a finite sheeted covering projection $S_P \to S_{P'}$ and by hypothesis $\pi_1(S_{P'})$ is not free. We check that $S_{P'}$ is a Seifert fibered space. Since $S_P \to S_{P'}$ is finite sheeted it will follow that $S_P$ is also Seifert fibered.

The claim now follows from the following Lemma.

**Lemma 6.2.** In the above notation $S_{P'}$ is a Seifert fibered space.

**Proof.** Recall the well-known theorem that if the fundamental group of a compact orientable irreducible 3-manifold contains an infinite cyclic central subgroup then the manifold admits a Seifert fibered structure. There is a generalization of this theorem for noncompact 3-manifolds (see [[19], theorem 1.1]) under certain conditions ([[19], definitions in the introduction]). Since $P'$ is a compact quotient of $S_{P'}$ these conditions are easily satisfied by $S_{P'}$. Hence we only have to show that there is an infinite cyclic central subgroup of $\pi_1(S_{P'})$. Since $P'$ is Seifert fibered we have the following short exact sequence.

\[ 1 \to C \to \pi_1(P') \to \pi_1^{orb}(B) \to 1 \]

where $C$ is infinite cyclic central (generated by a regular fiber) and $B$ is the base surface of the Seifert fibered space $P'$. If $C \cap p_* (\pi_1(S_{P'})) = (1)$ then since $B$ has nonempty boundary and $\pi_1(S_{P'})$ is torsion free it would follow that $\pi_1(S_{P'})$ is free. Which is a contradiction since $h$ is special and satisfies condition (3). □

This proves the claim. □

Thus we have found a JSJT type decomposition of $M_{f N}$ consisting of Seifert fibered spaces which are components of $p^{-1}(P)$ where $P$ varies over all Seifert
fibered pieces of $M_{g,N}$. Each component $S_{P}$ of $p^{-1}(P)$ has base surface (say $B_{S}$) with a discrete set (say, $D$) of orbifold points on $B_{S}$. Let $q : S_{P} \to B_{S}$ be the quotient map. Note that $B_{S}$ has a filtration by increasing sequence (under inclusion) of compact subsurfaces with incompressible circle boundary components so that each such circle boundary avoids $D$. By taking the inverse images of these subsurfaces under $q$ we can write $S_{P}$ as an increasing union of compact submanifolds $N_{i}^{S_{P}}$ with incompressible tori boundary components.

Now choose $r_{1} \in \mathbb{R}^{\geq 0}$ and $i_{1}^{1}, i_{2}^{1}, \ldots, i_{k_{1}}^{1}$ so that

$$\delta^{-1}([0, r_{1}]) \subset \bigcup_{j=1}^{j=k_{1}} N_{i_{j}^{1}}^{S_{P_{j}}},$$

where $P_{j}$ is some Seifert fibered piece of $M_{g,N}$ and $S_{P_{j}}$ denotes some component of $p^{-1}(P_{j})$. Since $M_{f,N}$ is connected and the family $\{N_{i}^{S_{P}}\}$ is a covering of $M_{f,N}$ we can choose $i_{1}^{l}, i_{2}^{l}, \ldots, i_{k_{l}}^{l}$ so that $\bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}}$ is connected. Let $N_{l}^{s} = \bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}}$. Similarly we can choose $r_{1} < r_{2} < \cdots < r_{m} < \cdots$ and $i_{1}^{l}, i_{2}^{l}, \ldots, i_{k_{l}}^{l}$, $l = 1, 2, \ldots$, so that $r_{m} \to \infty$ and

$$\bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}} \subset \delta^{-1}([0, r_{l}]) \subset \bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}}$$

and $\bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}}$ is connected. Write $N_{l}^{s} = \bigcup_{j=1}^{j=k_{l}} N_{i_{j}^{l}}^{S_{P_{j}}}$. It follows that $N_{l}^{s}$ satisfies the properties (2) and (3). To check (1) note that $N_{l}^{s}$ is a connected and compact 3-manifold. Also any two $N_{i_{j}^{l}}^{S_{P_{j}}}$ either do not intersect or intersect along incompressible tori boundary components or along some incompressible annuli on some incompressible tori boundary components. In any case it follows that $N_{l}^{s}$ has incompressible tori boundary. This proves (1).

Now we check that the FIC is true for $\pi_{1}(M_{f})$. By [[18], theorems 3.2 and 3.3] the interior of $N_{l}^{s}$ supports a complete nonpositively curved Riemannian metric so that near the boundary (tori) the metric is a product. Since the inclusion $N_{l}^{s} \subset N_{l}^{s} \cup_{\partial} N_{l}^{s}$ have the obvious retractions we get the following inclusion on fundamental groups: $\pi_{1}(N_{l}^{s}) < \pi_{1}(N_{l}^{s} \cup_{\partial} N_{l}^{s})$ where $N_{l}^{s} \cup_{\partial} N_{l}^{s}$ is the double of $N_{l}^{s}$. Also $N_{l}^{s} \cup_{\partial} N_{l}^{s}$ is a closed nonpositively curved manifold (by [18]). Hence the FIC is true for $\pi_{1}(N_{l}^{s}) \times G$ by Lemma A and B, where $G$ is a finite group. Also we have $\pi_{1}(M_{f,N}) \simeq \lim_{i \to \infty} \pi_{1}(N_{i}^{s})$ and hence we get that $\pi_{1}(M_{f}) < \pi_{1}(M_{f,N}) \times G < \lim_{i \to \infty} (\pi_{1}(N_{i}^{s}) \times G)$, where $G$ is the group of covering transformation of $M_{f,N} \to M_{f}$. And hence by [[7], theorem 7.1] the FIC is true for $\pi_{1}(M_{f}) \simeq [\pi_{1}(F), \pi_{1}(F)] \times \langle u \rangle$.

This completes the proof of the Main Lemma. \hfill \Box

Lastly we record, for later application, the following corollaries which are consequences of the proof of the Main Lemma.
Corollary 6.3. Let $F$ be a closed orientable surface of genus $> 1$ and $g : F \to F$ be an orientation preserving reducible and infinite order diffeomorphism. Also assume that all the pieces in the JSJ decomposition of $M_{g,N}$ (induced by $g$) for some large $N$ are Seifert fibered. Assume that $g$ is special. Let $f : \tilde{F} \to \tilde{F}$ be a lift of the diffeomorphism $g$ under the covering $\tilde{F} \to F$ corresponding to the commutator subgroup of $\pi_1(F)$. Then $\pi_1(M_f) \simeq \pi_1(\tilde{F}) \rtimes \langle t \rangle$ is a subgroup of $\lim_{i \to \infty} (\pi_1(N_i^s) \rtimes G)$. Where $M_f$ is the mapping torus of $f$, the action of $s$ on $\pi_1(\tilde{F})$ is induced by $f$, $N_i^s$ are increasing sequence (under inclusion) of compact connected irreducible 3-manifold with incompressible tori boundary components and $G$ is a finite group.

Corollary 6.4. In the hypothesis of the above corollary if $g$ is either pseudo-Anosov or is a finite order diffeomorphism or there is a hyperbolic piece in the JSJ decomposition of $M_{g,N}$ for some large $N$ then $\pi_1(M_f)$ is a subgroup of $\pi_1(N) \rtimes G$ where $N$ is a closed nonpositively curved Riemannian 3-manifold and $G$ is a finite group.

Remark 6.5. We will use the Main Lemma and the method of its' proof to deduce the FIC for a large class of 3-manifold groups in [27].

Remark 6.6. Here we remark that when the diffeomorphism $g$ belongs to Case 3 then $g$ is isotopic to the $k$-th power of the monodromy diffeomorphism of the fiber bundle projection $M \to S^1$.

7. Virtually fibered 3-manifold and the FIC

In this section we prove Proposition 1.7.

Proof of Proposition 1.7. Note that it is enough to prove that the FIC is true for $\pi_1(M) \rtimes G$ for any finite group $G$.

Let $F$ be the fiber of the fiber bundle projection $M \to S^1$. Passing to a finite cover we can make sure $F$ is orientable and the monodromy diffeomorphism of the fiber bundle projection $M \to S^1$ is orientation preserving. There are two cases to consider now.

Case A. Assume that the monodromy diffeomorphism belongs to the first two classes in the definition of special diffeomorphism. In this case by Corollary 6.4 and [26], lemma B] the proposition follows.

Case B. Assume that the monodromy diffeomorphism is special and condition (3) is satisfied.

We have the following exact sequence.

$$1 \to [\pi_1(F), \pi_1(F)] \to \pi_1(M) \to H_1(F, \mathbb{Z}) \rtimes \langle t \rangle \to 1.$$
If $F$ is the 2-sphere or the torus then $\pi_1(M) \wr G$ is virtually poly-$\mathbb{Z}$ and hence the FIC is true by Theorem F. So assume $F$ is not the 2-sphere or the torus. That is the genus of $F$ is $\geq 2$.

Taking wreath product with $G$ the above exact sequence gives the following.

$$1 \to ([\pi_1(F), \pi_1(F)])^G \to \pi_1(M) \wr G \to (H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G \to 1.$$ 

Note that $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$ is virtually poly-$\mathbb{Z}$ and hence the FIC is true for $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$.

By the following Lemma, [[7], theorem 7.1] and Lemma 6.1 it follows that the FIC is true for $([\pi_1(F), \pi_1(F)])^G$.

**Lemma 7.2.** Let $G_1$ and $G_2$ be two groups and assume the FIC is true for both $G_1$ and $G_2$ then the FIC is true for the product $G_1 \times G_2$.

**Proof.** Consider the projection $p_1 : G_1 \times G_2 \to G_1$. By [[26], lemma C] we need to check that the FIC is true for $p_1^{-1}(C)$ for any virtually cyclic subgroup $C$ of $G_1$. Note that $p_1^{-1}(C) = C \times G_2$. Now consider the projection $p_2 : C \times G_2 \to G_2$. Again we apply [[26], lemma C]. That is we need to show that the FIC is true for $p_2^{-1}(C')$ for any virtually cyclic subgroup $C'$ of $G_2$. But $p_2^{-1}(C') = C \times C'$ which is virtually poly-$\mathbb{Z}$ and hence the FIC is true for $p_2^{-1}(C')$. This completes the proof of the lemma. \(\square\)

Let $Z$ be a virtually cyclic subgroup of $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$. If $Z$ is finite then

$$p^{-1}(Z) < ([\pi_1(F), \pi_1(F)])^G \wr Z < ([\pi_1(F), \pi_1(F)]) \wr (G \times Z).$$

Here $p$ is the surjective homomorphism $\pi_1(M) \wr G \to (H_1(F, \mathbb{Z}) \rtimes \langle t \rangle) \wr G$. Now Lemma 6.1 applies on the right hand side group to show that the FIC is true for $p^{-1}(Z)$.

If $Z$ is infinite then let $Z_1$ be the intersection of $Z$ with the torsion free part $(H_1(F, \mathbb{Z}) \rtimes \langle t \rangle)^G$. Hence $Z_1 \simeq \langle u \rangle$ is an infinite cyclic normal subgroup of $Z$ of finite index. We get

$$p^{-1}(Z) < (p^{-1}(Z_1)) \wr Z/Z_1 \simeq (([\pi_1(F), \pi_1(F)])^G \rtimes \langle u \rangle) \wr Z/Z_1 \simeq ([\pi_1(F), \pi_1(F)] \times [\pi_1(F), \pi_1(F)] \times \cdots \times [\pi_1(F), \pi_1(F)]) \rtimes \langle u \rangle \wr Z/Z_1 = H(\text{say}).$$

In the above display there are $|G|$ number of factors of $[\pi_1(F), \pi_1(F)]$. Note that the action of $u$ on $([\pi_1(F), \pi_1(F)])^G$ is factorwise, that is, the $i$-th coordinate of $u$ acts on the $i$-th factor of $([\pi_1(F), \pi_1(F)])^G$. Let $u = (u_1, \cdots, u_{|G|})$. Recall that if $u_j \neq 1$ then the action of $u_j$ on $[\pi_1(F), \pi_1(F)]$ is induced by the lift of
a diffeomorphism (say $g_j$) of $F$. Without loss of generality we can assume that $g_1, \ldots, g_k$ are diffeomorphism of type as in Corollary 6.3 and $g_{k+1}, \ldots, g_l$ are of type as in Corollary 6.4 and the remaining $u_j$ are trivial element. Note that here it follows from Remark 6.6 that for $j = 1, 2, \ldots, k$ each $g_j$ has the same property as $g$. Applying Corollaries 6.3 and 6.4 we get

\[
H < \left( \lim_{i \to \infty} \left( \pi_1(N_i^{u_j}) \wr G_i \right) \times \cdots \times \left( \pi_1(N_i^{u_k}) \wr G_k \right) \right) \times \left( \pi_1(N_{k+1}) \wr G_{k+1} \right) \times \cdots \times
\]

\[
\left( \pi_1(N_l) \wr G_l \right) < \left( \lim_{i \to \infty} \left( \pi_1(N_i^{u_j}) \wr G_1 \right) \times \cdots \times \left( \pi_1(N_i^{u_k}) \wr G_k \right) \times \cdots \times \left( \pi_1(N_l) \wr G_l \right) \right)
\]

where $N_i^{u_j}$ are irreducible 3-manifolds with incompressible boundary as appeared in Corollary 6.3, $N_j$ are closed nonpositively curved Riemannian manifolds and $G_1, \ldots, G_l$ are finite groups. Let us denote the group inside the limit of the last expression by $K_i$. Then the last line becomes

\[
\lim_{i \to \infty} \left( K_i \wr (G_1 \times \cdots \times G_l) \times (\pi_1(F) \times \pi_1(F)) \right) < \left( \lim_{i \to \infty} \left( K_i^{G_1 \times \cdots \times G_l} \wr Z/Z_1 \right) \times \left( [\pi_1(F), \pi_1(F)] \right) \right)
\]

Let $M_i^{u_j}$ be the double of $N_i^{u_j}$. Then by [Le] $M_i^{u_j}$ is a closed nonpositively curved Riemannian manifold. Using the obvious retractions we find that $\pi_1(N_i^{u_j})$ is a subgroup of $\pi_1(M_i^{u_j})$. Hence $K_i$ is a subgroup of the fundamental group of a closed nonpositively curved Riemannian manifold, namely $(M_i^{u_1})^{G_1} \times \cdots \times (M_i^{u_k})^{G_k} \times N_{k+1}^{G_{k+1}} \times \cdots \times N_l^{G_l}$. And the last inclusion follows from the following easy to verify Lemma.

**Lemma 7.3.** Let $A$ and $B$ be two finite groups and $G$ be any group, then $(G \wr A) \wr B$ is a subgroup of $G^{A \times B} \wr (A \wr B)$

Now using [[7], theorem 7.1], Lemma 6.1, Lemma 7.2 and the following Lemma we complete the proof of the Proposition. □

**Lemma 7.4.** Let $A$ and $B$ be two groups and $G$ be a finite group then $(A \wr B) \wr G$ is a subgroup of $(A \wr G) \times (B \wr G)$.

**Proof.** The proof is easy and left to the reader. □
8. Examples

This section is devoted in giving examples of special diffeomorphisms.

**Lemma 8.1.** Let \( f : F \to F \) be a special diffeomorphism. Then for any positive integer \( n \), \( f^n \) is also a special diffeomorphism. Conversely, if \( f^n \) is special for some \( n \) then so is \( f \).

**Proof.** If \( f \) satisfies condition (2) in the definition of special diffeomorphism then obviously \( f^n \) is special and satisfies condition (2). Converse is also trivial. If \( f \) satisfies condition (1) then since \( M_{f^n} \) is a finite sheeted cover of \( M_f \) it follows \( M_{f^n} \) also satisfies condition (1). In fact the pull back metric does the job. The converse direction follows from \([17], \text{corollary 2.5}\).

So assume that \( f \) satisfies condition (3).

Consider the following commutative diagram.

\[
\begin{array}{ccc}
M_{f^n} & \xrightarrow{f_1} & M_f \\
\downarrow p & & \downarrow q \\
M_{f^n} & \xrightarrow{f_2} & M_f
\end{array}
\]

Note that Seifert fibered pieces of \( M_{f^n} \) are inverse images of Seifert fibered pieces of \( M_f \) under the map \( f_2 \). So let \( P \) be a Seifert fibered piece of \( M_{f^n} \) and \( f_2(P) = Q \). Let \( \tilde{P} \) be a component of \( p^{-1}(P) \) which goes to the component \( \tilde{Q} \) of \( q^{-1}(Q) \). Note that \( f_1|_{\tilde{P}} : \tilde{P} \to \tilde{Q} \) is a finite sheeted covering. By hypothesis \( \pi_1(\tilde{Q}) \) is not free. By Lemma 6.2 \( \tilde{Q} \) is Seifert fibered. Since \( \tilde{P} \) is a finite sheeted cover of \( \tilde{Q} \), \( \tilde{P} \) is also Seifert fibered and hence has an infinite cyclic normal subgroup generated by a regular fiber. Hence \( \pi_1(\tilde{P}) \) is not free. Hence \( f^n \) is also special.

Conversely if \( f^n \) is special for some \( n \) then in the above notation \( \pi_1(\tilde{P}) \) is not free, consequently \( \pi_1(\tilde{Q}) \) is also not free as \( \pi_1(\tilde{P}) \) is a subgroup of \( \pi_1(\tilde{Q}) \). \( \square \)

**Example 8.2.** Now we give examples of some special diffeomorphisms. We have already seen examples of special diffeomorphism which satisfies condition (1) or (2). Let \( f : F \to F \) be a diffeomorphism of a closed surface of genus \( \geq 2 \). Assume that \( f \) is reducible and in the JSJT decomposition of the mapping torus of \( f \) (as described in **Case 3** of the proof of the Main Lemma) there are only Seifert fibered pieces. Hence there are mutually disjoint simple closed curves \( C_1, \ldots, C_n \) on \( F \) which are not null homotopic and \( f(\bigcup C_i) = \bigcup C_i \). Assume that each \( C_i \) represents an element of the commutator subgroup of \( \pi_1(F) \). It is now easy to check that condition (3) in the definition of special diffeomorphism is satisfied. In fact one can
check that any component of $p^{-1}(S)$ contains a free abelian subgroup of rank 2 for every Seifert fibered piece $S$ of $M_f$.

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