Abstract

We present a Lagrangian-Hamiltonian formalism of a moving dielectric sphere interacting with radiation fields. By including the interaction up to the first order in the speed of the sphere, we derive the Hamiltonian and perform quantization of both the field and the mechanical motion of the sphere. In particular, we show how independent degrees of freedom can be consistently identified under the generalized radiation gauge via instantaneous mode projection. Our Hamiltonian indicates the form of coupling due to velocity-dependent interactions beyond adiabatic approximation. In addition, the Hamiltonian predicts that a geometrical quantum phase can be gained by the sphere moving in a light field.

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I. INTRODUCTION

The coupling between an optically levitated sphere and cavity fields has been a subject of research interests recently [1-4]. Such a system corresponds to a basic configuration in cavity optomechanics which explores quantum phenomena arising from the interaction between mechanical and optical degrees of freedom [5]. In particular, since mechanical systems such as a dielectric sphere or moving mirror have masses much greater than that of an atom, cavity optomechanics could open a door for testing quantum theory in macroscopic scale [4, 6], as well as probing Planck-scale physics [7]. Recently, progress in experiments has been made in demonstrating cooling and trapping of a dielectric sphere by optical fields [8].

In this paper, we present a Lagrangian and Hamiltonian formalism of the interaction of a moving dielectric sphere in quantized radiation fields. Previously, Chang et al. [1] and Romero-Isart et al. [2] have studied the (adiabatic) interaction Hamiltonian of the system to zeroth order in $v$, where $v$ is the speed of the sphere. Within this accuracy, the interaction takes the usual form $-\frac{1}{2} \int \mathbf{P} \cdot \mathbf{E} \, d^3x$ (with the polarization $\mathbf{P}$ proportional to the electric field $\mathbf{E}$) as if the dielectric were stationary. While such a treatment is suitable for a trapped sphere at low temperatures, it is interesting to ask for a general theory that takes into account interactions due to the motional states of the sphere. In particular, it is known that a moving dielectric possesses a velocity dependent polarization and magnetization [8]: $\mathbf{P} = (\epsilon - 1) (\mathbf{E} + v \times \mathbf{B})$ (to first order in $v$), $\mathbf{M} = -v \times \mathbf{P}$, and these motion-induced quantities could modify the dynamics of the sphere and the field. In order to account for such effects nonrelativistically, one has to keep the interaction part of the Lagrangian at least up to first order in $v$, and this is our task in this paper – to set up a consistent Hamiltonian leading to the quantization of the full system.

We point out that the moving sphere is a higher dimensional system due to the translational and rotational motion in three dimensions, and it requires a treatment different from that in one-dimensional optomechanical systems [9]. For example, an optical field can affect the rotational motion of a dielectric via the electromagnetic torque exerting to it [10, 12], and this has been studied in a sequence of experiments [13-17]. Indeed, by including the rotation degrees of freedom of the sphere in the Lagrangian, we will show that there is a kind of rotational optomechanical coupling caused by the velocity-dependent interactions. There
is also some theoretical subtlety in identifying independent degrees of freedom subjected to
the generalized transverse gauge. Such a gauge is commonly used for field quantization in
the presence of a stationary dielectric medium \[18\text{-}20\]. However, for a moving dielectric,
the generalized transverse gauge somehow mixes the mechanical degrees of freedom with the
field. In this paper, we resolve the problem by using instantaneous normal-mode projection,
and the quantization scheme is compatible with the generalized transverse gauge.

II. LAGRANGIAN

The system under investigation is depicted in Fig. 1 in which a rigid dielectric sphere of
mass \(m\) and radius \(R\) moves in an electromagnetic field. The sphere is free to rotate about
any axis through its center of mass (c.m.) at \(q\). In our study, we assume a nondispersive and
nonabsorptive linear dielectric. When the sphere is stationary, its dielectric permittivity is
given by

\[
\epsilon(r, q) = \begin{cases} n^2, & |r - q| \leq R \\ 1, & \text{otherwise.} \end{cases}
\]

where the convention \(\epsilon_0 = \mu_0 = 1\) and \(c = 1\) is used for convenience. In addition, the sphere
is assumed nonmagnetic in its rest frame.

\[L = \frac{1}{2} m q^2 + \frac{1}{2} I \omega^2 + \int d^3 r \mathcal{L}(r), \quad (2)\]

where \(I\) and \(\omega\) are the moment of inertia and angular velocity of the sphere, respectively.
The orientation of the sphere is specified by three Euler angles \(\alpha, \beta, \gamma\) \[21\], and hence \(\omega\]
can be explicitly expressed as \( \mathbf{\omega} = (\dot{\gamma} \sin \beta \cos \alpha - \dot{\beta} \sin \alpha) \hat{e}_x + (\dot{\gamma} \sin \beta \sin \alpha + \dot{\beta} \cos \alpha) \hat{e}_y + (\dot{\alpha} + \dot{\gamma} \cos \beta) \hat{e}_z \), where \( \hat{e}_l \) \((l = x, y, z)\) are the basis vectors. \( \mathcal{L}(\mathbf{r}) \) is the Lagrangian density of the field after eliminating the electronic degrees of freedom of the dielectric. To obtain the form of \( \mathcal{L} \), we go to an inertial frame \( S'(\mathbf{r}) \) in which the dielectric element at \( \mathbf{r} \) is instantaneously at rest. Assuming the acceleration of the dielectric does not change its macroscopic properties, the field Lagrangian density at \( \mathbf{r} \) in \( S'(\mathbf{r}) \) is given by the familiar form,

\[
\mathcal{L}' = \frac{1}{2} \left( \varepsilon |\mathbf{E}'|^2 - |\mathbf{B}'|^2 \right),
\]

(3)

where \( \mathbf{E}' \) and \( \mathbf{B}' \) are the electric and magnetic fields in \( S'(\mathbf{r}) \), respectively. As the Lagrangian density is Lorentz invariant, \( \mathcal{L} \) can be readily obtained from the Lorentz transformation of the fields from \( S'(\mathbf{r}) \) to the laboratory frame \( S \).

In this paper we confine ourselves to a nonrelativistic motion of the sphere, so that the velocity \( \mathbf{v}(\mathbf{r}) = \dot{\mathbf{q}} + \mathbf{\omega} \times (\mathbf{r} - \mathbf{q}) \) of the dielectric element at any point \( \mathbf{r} \) satisfies \( v \equiv |\mathbf{v}(\mathbf{r})| \ll c \). By keeping terms up to first order of \( v \) in \( \mathcal{L} \), the Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} m \dot{\mathbf{q}}^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} \int d^3r \left( \varepsilon |\mathbf{E}|^2 - |\mathbf{B}|^2 \right) - \dot{\mathbf{q}} \cdot \mathbf{\Lambda} - \mathbf{\omega} \cdot \mathbf{\Gamma},
\]

(4)

where

\[
\mathbf{\Lambda} = \int d^3r (\epsilon - 1) (\mathbf{E} \times \mathbf{B}),
\]

\[
\mathbf{\Gamma} = \int d^3r (\epsilon - 1)(\mathbf{r} - \mathbf{q}) \times (\mathbf{E} \times \mathbf{B})
\]

are defined. The Lagrangian is a generalization to that of Barton and Eberlein and Salamone, which consider a one-dimensional configuration and focus only on the uniform c.m. motion of a dielectric slab. Here we will take both \( \mathbf{q} \) and \( \mathbf{\omega} \) as dynamical degrees of freedom which interact with the field through the \( -\dot{\mathbf{q}} \cdot \mathbf{\Lambda} - \mathbf{\omega} \cdot \mathbf{\Gamma} \) term. These velocity dependent interaction terms are not considered in previous studies. Interestingly, \( \mathbf{\Lambda} \) and \( \mathbf{\Gamma} \) are proportional to the linear momentum and angular momentum of the field inside the sphere.

The Lagrangian should be expressed in terms of the scalar potential \( V \) and vector potential \( \mathbf{A} \) via the substitution: \( \mathbf{E} = -\partial_t \mathbf{A} - \nabla V \) and \( \mathbf{B} = \nabla \times \mathbf{A} \). In this paper we shall fix the potentials by using the generalized radiation gauge:

\[
\nabla \cdot [\varepsilon(\mathbf{r}, \mathbf{q}) \mathbf{A}] = 0.
\]

(7)
Such a gauge condition has been employed for performing field quantization in the presence of stationary dielectric media \[18–20\]. It has the advantage that if the dielectric is at rest, the scalar potential $V$ is exactly zero. For a moving dielectric, it can be shown that $V$ under condition (7) contributes an interaction term of order $v^2$ in the Lagrangian, and hence the effects of $V$ can be consistently neglected in our nonrelativistic Lagrangian which keeps interaction terms up to the first order in $v$.

It is important to note that the generalized gauge (7) acts as a constraining equation of the vector potential $A$ for a given c.m. position of the sphere $q$. Therefore $A$ and $q$ cannot be treated as independent degrees of freedom under Eq. (7). To overcome the difficulty, we identify the independent degrees of freedom via the mode expansion of $A$:

$$A(r, t) = \sum_k Q_k(t)u_k[r, q(t)],$$

where $\{u_k(r, q)\}$ is a set of normal-mode functions of the field when the sphere is sitting at rest at the c.m. position $q$. Specifically, the mode function $u_k$ with the mode frequency $\omega_k^2(q)$ is defined by

$$\nabla \times (\nabla \times u_k) - \epsilon(r, q)\omega_k^2(q)u_k = 0,$$

subjected to suitable boundary conditions and $\nabla \cdot [\epsilon(r, q)u_k(r, q)] = 0$ from the gauge condition (7). In addition, these modes are orthonormal according to,

$$\int d^3r \, \epsilon(r, q)u_k(r, q) \cdot u_j(r, q) = \delta_{kj}.$$  

Note that we have considered $u_k$ as real functions for convenience. For a moving sphere, $q$ is a function of time, therefore $u_k$ can be interpreted as an instantaneous normal-mode of the field.

By projecting $A$ onto instantaneous normal-modes, the gauge condition (7) is automatically satisfied and we can treat $Q_k$ and $q$ as generalized coordinates, i.e., independent degrees of freedom. The Lagrangian (4) becomes

$$L = \frac{1}{2}mq^2 + \frac{1}{2}I\omega^2 + \frac{1}{2} \sum_k \left[ \dot{Q}_k^2 - \omega_k^2(q)Q_k^2 \right] - \dot{q} \cdot \sum_{kj} \eta_{kj} \dot{Q}_k Q_j - \omega \cdot \sum_{kj} g_{kj} \dot{Q}_k Q_j,$$  

where $\eta_{kj}$ and $g_{kj}$ are given by

$$\eta_{kj} = \frac{1}{2} \epsilon \int d^3r \, u_k(r, q) \cdot u_j(r, q)$$

and

$$g_{kj} = \frac{1}{2} \int d^3r \, \epsilon \nabla \cdot [\epsilon u_k(r, q) u_j(r, q)].$$
with corrections on the order $O(v^2)$. Here the $\eta_{kj}$ and $g_{kj}$ are coefficients depending on $q$:

\[
\eta_{kj}(q) = -\int d^3r [\epsilon \sum_{l=x,y,z} (u_k \cdot \hat{e}_l) \nabla_q (u_j \cdot \hat{e}_l) + (\epsilon - 1) u_k \times (\nabla \times u_j)],
\]

\[
g_{kj}(q) = -\int d^3r (\epsilon - 1) (r - q) \times [u_k \times (\nabla \times u_j)].
\]

We remark that validity of $L$ in Eqs. (4) and (11) requires that interaction terms involving $v^2$ and higher orders are negligible. Those $v^2$ terms omitted in the Lagrangian are corrections to translation and rotational kinetic energies such as $\delta m_{ij} \dot{q}_i \dot{q}_j$ and $\delta I_{ij} \dot{\omega}_i \dot{\omega}_j$, where $\delta m_{ij}$ and $\delta I_{ij}$ are respectively the mass tensor and rotational inertial tensor due to the field energy stored inside the sphere. Because $\delta m_{ij}$ is generally time dependent, it contributes a velocity dependent force $\delta \dot{m}_{ij} \dot{q}_j$. Similarly, there is an angular velocity dependent torque $\delta \dot{I}_{ij} \omega_j$ coming from the rate of change of $\delta I_{ij}$. If the sphere’s motion is consistently described by $L$ in Eq. (11) to the first order in $v$, then the force correction $\delta \dot{m}_{ij} \dot{q}_j$ and torque correction $\delta \dot{I}_{ij} \omega_j$ should be small compared with that generated by the $-\dot{q} \cdot \Lambda - \omega \cdot \Gamma$ term. Such a condition depends on the physical configuration of the system. In the single mode situations with nonzero $\Lambda$ and $\Gamma$ (see Sec. IV), one can observe that $\delta m_{ij}$ and $\delta I_{ij}$ oscillate rapidly at the field frequency, hence their effects can be averaged out and become negligible in the spirit of the rotating wave approximation.

### III. HAMILTONIAN AND QUANTIZATION

The Hamiltonian defined from $L$ of the system is given by

\[
H \equiv p \cdot \dot{q} + \sum_{\zeta=\alpha,\beta,\gamma} \pi_\zeta \dot{\zeta} + \sum_k P_k \dot{Q}_k - L,
\]

where $p$, $P_k$, and $\pi_\zeta$ are canonical momenta conjugate to $q$, $Q_k$, and the Euler angle $\zeta$ respectively. However, since it is rather inconvenient to handle Euler angles, we introduce a canonical angular momentum defined by $J = (\partial L/\partial \omega)$, which is related to $\pi_\zeta$ via the relation: $J \cdot \omega = \dot{\alpha} \pi_\alpha + \dot{\beta} \pi_\beta + \dot{\gamma} \pi_\gamma$. From the Lagrangian, $p$, $J$, and $P_k$ are given by

\[
p = m \dot{q} - \sum_{kj} \eta_{kj} \dot{Q}_k Q_j,
\]

\[
J = I \omega - \sum_{kj} g_{kj} \dot{Q}_k Q_j,
\]

\[
P_k = \dot{Q}_k - \sum_j (q \cdot \eta_{kj} + \omega \cdot g_{kj}) Q_j.
\]
Note that \( p \) and \( J \) differs from the kinetic momentum \( m \dot{q} \) and angular momentum \( I \omega \) for nonzero fields, respectively.

The solutions of \( \dot{q}, \omega \) and \( \dot{Q}_k \) in terms of the canonical momenta are complicated because of the coupled equations (15)–(17). However, since field-dependent terms that are quadratic in \( v \) have been discarded the Lagrangian, we have to maintain the consistency by dropping terms of same order in eliminating \( \dot{q}, \omega \) and \( \dot{Q}_k \) for the Hamiltonian. Specifically, we shall write

\[
\begin{align*}
\dot{q} &= p + \sum_{kj} \eta_{kj} \dot{Q}_k Q_j 
\approx p + \sum_{kj} \eta_{kj} P_k Q_j, \quad (18) \\
\dot{\omega} &= J + \sum_{kj} g_{kj} \dot{Q}_k Q_j 
\approx J + \sum_{kj} g_{kj} P_k Q_j. \quad (19)
\end{align*}
\]

By substituting Eqs. (18) and (19) into Eq. (14) and again neglecting field corrected inertia contributions, we obtain an explicit expression of the Hamiltonian (14),

\[
H = \frac{(p + \Lambda')^2}{2m} + \frac{(J + \Gamma')^2}{2I} + \frac{1}{2} \sum_k \left[ P_k^2 + \omega_k^2(q) Q_k^2 \right], \quad (20)
\]

where \( \Lambda' = \sum_{kj} \eta_{kj} P_k Q_j \) and \( \Gamma' = \sum_{kj} g_{kj} P_k Q_j \) are defined.

It is interesting that this Hamiltonian takes a form similar to the minimal-coupling Hamiltonian in electrodynamics, with \( \Lambda' \) and \( \Gamma' \) somehow playing the role of vector potential in the kinetic energy term. In addition, the Hamiltonian indicates that there is a rotational coupling of the sphere described by the second term in (20). It is useful to rewrite \( \Gamma' \) as

\[
\Gamma' = -\int d^3r \left( \frac{\epsilon - 1}{\epsilon} \right) [(r - q) \times (\Pi \times B)], \quad (21)
\]

where \( B = \sum_k Q_k (\nabla \times u_k) \) and \( \Pi = \epsilon \sum_k P_k u_k \) are the magnetic field and field canonical momentum density, respectively. The form of \( \Gamma' \) is very similar to the field angular momentum stored in the dielectric, apart from a proportionality constant. Therefore, approximately speaking, the second term of Hamiltonian (20) represents an angular momentum coupling, i.e., the interaction corresponds to an exchange of angular momenta between the field and the sphere. Furthermore, since \( \Gamma'(q) \) depends on \( q \), there is also a coupling between the mechanical rotation and c.m. motion of the sphere, mediated by the fields.

With the classical Hamiltonian (20), the canonical quantization of the system is readily achieved by postulating the dynamical variables into operators with the commutation relations:

\[
[q_{\mu}, p_{\nu}] = i\hbar \delta_{\mu \nu}, \quad [J_{\mu}, J_{\nu}] = i\hbar \epsilon_{\mu \nu \kappa} J_{\kappa}, \quad [K_{\mu}, K_{\nu}] = -i\hbar \epsilon_{\mu \nu \kappa} K_{\kappa}, \quad \text{and} \quad [Q_k, P_j] = i\hbar \delta_{kj},
\]
where the Greek subscripts refer to the three axes in rectangular coordinates, and $K_{\mu}$ are the body-axis components of $\mathbf{J}$ [21]. In this way the quantum Hamiltonian of the system takes the same expression as Eq. (20), but with $P_k Q_j$ symmetrized by $(P_k Q_j + Q_j P_k)/2$ in $\Lambda'$ and $\Gamma'$.

In order to represent photon states of the system, we introduce the $\mathbf{q}$-dependent annihilation and creation operators for each cavity field mode:

$$a_k(\mathbf{q}) = \sqrt{\frac{1}{2\hbar\omega_k(\mathbf{q})} \left[ \omega_k(\mathbf{q})Q_k + iP_k \right]},$$

$$a_k^\dagger(\mathbf{q}) = \sqrt{\frac{1}{2\hbar\omega_k(\mathbf{q})} \left[ \omega_k(\mathbf{q})Q_k - iP_k \right]},$$

which satisfy the commutation relation $[a_k(\mathbf{q}), a_j^\dagger(\mathbf{q})] = \delta_{kj}$. Since $a_k(\mathbf{q})$ depends on $\mathbf{q}$, for each position of the dielectric we have a set of Fock states associated with that position. These states can be labeled as $|\{n_k\}, \mathbf{q}, \xi\rangle$, where $\{n_k\} = \{n_1, n_2, n_3, \ldots\}$ denotes the occupation number of each photon mode, and $\xi = (j, m, k)$ denotes the eigenbasis vectors of $\mathbf{J}$ (see [21]):

$$\mathbf{J}^2|\{n_k\}, \mathbf{q}, \xi\rangle = \hbar^2 j(j+1)|\{n_k\}, \mathbf{q}, \xi\rangle,$$

$$J_z|\{n_k\}, \mathbf{q}, \xi\rangle = \hbar m|\{n_k\}, \mathbf{q}, \xi\rangle,$$

$$K_z|\{n_k\}, \mathbf{q}, \xi\rangle = \hbar k|\{n_k\}, \mathbf{q}, \xi\rangle,$$

where $K_z$ is the $z$ component of $\mathbf{J}$ in the body coordinates. Here $|\{n_k\}, \mathbf{q}, \xi\rangle$ is a simultaneous eigenstate of the photon-number operator $a_k^\dagger(\mathbf{q}) a_k(\mathbf{q})$ and the position operator $\mathbf{q}$ i.e.,

$$a_k^\dagger(\mathbf{q}) a_k(\mathbf{q})|\{n_k\}, \mathbf{q}, \xi\rangle = n_k |\{n_k\}, \mathbf{q}, \xi\rangle,$$

$$\mathbf{q}|\{n_k\}, \mathbf{q}, \xi\rangle = \mathbf{q}|\{n_k\}, \mathbf{q}, \xi\rangle.$$  

Such a set of eigenstates is orthonormal and complete, so that any quantum state of the whole system $|\Psi\rangle$ can be expanded in the basis of these eigenstates, i.e.,

$$|\Psi\rangle = \sum_{\xi, \{n_k\}} \int d^3 \mathbf{q} C(\{n_k\}, \mathbf{q}, \xi)|\{n_k\}, \mathbf{q}, \xi\rangle,$$

where $C(\{n_k\}, \mathbf{q}, \xi)$ is the probability amplitude.

With the help of the $\mathbf{q}$-dependent annihilation and creation operators, the Hamiltonian Eq. (20) becomes

$$H = \left( \frac{\mathbf{p} + \Lambda'}{2m} \right)^2 + \left( \frac{\mathbf{J} + \Gamma'}{2I} \right)^2 + \sum_k \hbar \omega_k(\mathbf{q}) \left( a_k^\dagger a_k + \frac{1}{2} \right),$$
where we have used a shorthand $a_k = a_k(q)$ for convenience, and

$$\Lambda'(q) = -\frac{i\hbar}{2} \sum_{k,j} \eta_{kj}(q) \sqrt{\frac{\omega_k(q)}{\omega_j(q)}} \left( a_k a_j - a_k^\dagger a_j^\dagger + a_k^\dagger a_j - a_j^\dagger a_k \right),$$  \hspace{1cm} (31)$$

$$\Gamma'(q) = -\frac{i\hbar}{2} \sum_{k,j} g_{kj}(q) \sqrt{\frac{\omega_k(q)}{\omega_j(q)}} \left( a_k a_j - a_k^\dagger a_j^\dagger + a_k^\dagger a_j - a_j^\dagger a_k \right).$$  \hspace{1cm} (32)$$

Note that $\Lambda'$ and $\Gamma'$ contain photon-number nonconserving terms $a_k^\dagger a_j^\dagger$ which are responsible for photon generation in the dynamical Casimir effect [24], but this is a subject beyond the scope of this paper. For fields at optical frequencies, the $a_k^\dagger a_j^\dagger$ terms are fast oscillating in the interaction picture, and so in the spirit of rotating wave approximation, only the photon-number conserving terms $a_k^\dagger a_j$ will be kept in $\Lambda'$ and $\Gamma'$. These terms describe the scattering of photons between different modes due to the motion of the sphere.

**IV. SINGLE-MODE SITUATIONS**

In this section, we discuss the Hamiltonian under the single-mode approximation. This applies to situations when the field is dominantly contributed by a single mode $k$, and the scattering of photons from the $k$ mode to other modes is negligible within a coherent interaction time. From Eqs. (31) and (32), it follows that under the single-mode consideration, $\Lambda'$ and $\Gamma'$ only contain photon-number nonconserving terms $a_k^\dagger a_j^\dagger$ and $a_k^\dagger a_j^\dagger$, and vanish under the rotating wave approximation. Hence the corresponding Hamiltonian reads

$$H \approx \frac{p^2}{2m} + \frac{J^2}{2I} + \hbar \omega_k(q) a_k^\dagger a_k,$$  \hspace{1cm} (33)$$

in which the rotational motion of the sphere is decoupled from the field, and the optomechanical coupling appears only through the position dependent mode frequency $\omega(q)$. Such a form of the Hamiltonian has been considered in Refs. [1, 2].

However, we emphasize that the single-mode Hamiltonian (33) is based on the real mode function $u_k$ in the derivation. The situation is different if complex mode functions are involved, for example, in a ring cavity which supports traveling wave modes. In the Appendix, we show how the Hamiltonian (30) can be modified to incorporate complex modes, in which a complex mode function is formed by a linear combination of real modes of the same frequency. In particular, when photons mainly occupy a complex mode $f$, by the single mode...
approximation the Hamiltonian becomes

\[ H = \frac{(p + \lambda b^\dagger b)^2}{2m} + \frac{(J + \gamma b^\dagger b)^2}{2I} + \hbar \omega(q) \left( b^\dagger b + \frac{1}{2} \right), \]  

(34)

where \( b \) and \( \omega(q) \) are the annihilation operator and mode frequency associated with the \( f \) mode, respectively, and \( \lambda(q) \) and \( \gamma(q) \) are coupling strengths determined by the mode function,

\[ \lambda(q) = -\hbar \text{Im} \int d^3r \left( \epsilon \sum_{l=x,y,z} (f^* \cdot \mathbf{\hat{e}}_l) \nabla_q (f \cdot \mathbf{\hat{e}}_l) + (\epsilon - 1) f^* \times (\nabla \times f) \right), \]  

(35)

\[ \gamma(q) = -\hbar \text{Im} \int d^3r (\epsilon - 1) (\mathbf{r} - q) \times [f^* \times (\nabla \times f)]. \]  

(36)

We see that the velocity dependent coupling reappears in the single complex mode. Physically, \( \lambda(q) \) and \( \gamma(q) \) can roughly be understood as a measure of the field momentum and angular momentum stored in the sphere, contributed by a \( f \) mode photon. A complex mode, such as that of a traveling wave, can carry net field momentum (angular momentum). On the other hand, a real mode can be decomposed as a linear combination of complex modes, whose contributions of momenta (angular momenta) cancel each other. A familiar example is the standing wave mode function \( \sin(\mathbf{k} \cdot \mathbf{r}) \), which is a superposition of two traveling wave modes \( e^{i\mathbf{k} \cdot \mathbf{r}} \) and \( e^{-i\mathbf{k} \cdot \mathbf{r}} \). This explains why the velocity dependent coupling only appears in the complex mode situation.

To estimate the magnitude of \( \lambda(q) \) and \( \gamma(q) \), we consider a physical situation where a dielectric sphere of subwavelength size is placed in a ring cavity, and the field excitation is dominantly contributed by a Gaussian beam of an optical tweezer. The configuration of ring cavity supports traveling wave modes, so that \( \lambda(q) \) and \( \gamma(q) \) can be nonvanishing. Under the paraxial approximation, the field mode function is given by,

\[ f(x, y, z) = \left[ u(x, y, z) \mathbf{\hat{e}}_x + \frac{i}{k} \frac{\partial u}{\partial x} \mathbf{\hat{e}}_z \right] e^{ikz} \sqrt{L_c}, \]  

(37)

\[ u(x, y, z) = \sqrt{\frac{2}{\pi w(z)^2}} \exp \left[ -\frac{x^2 + y^2}{w(z)^2} \left( 1 - i \frac{z}{z_R} \right) \right] \exp \left[ -i \tan^{-1} \left( \frac{z}{z_R} \right) \right], \]  

(38)

where \( L_c \) is the effective length of the cavity. The beam is linearly polarized in \( \mathbf{\hat{e}}_x \), propagating along its wavevector \( \mathbf{k} = k \mathbf{\hat{e}}_z \), with a beam radius \( w(z) = \sqrt{2(z^2 + z_R^2)/kz_R} \) (focal plane at \( z = 0 \)), and \( z_R \) is the Rayleigh range. For a subwavelength sphere satisfying \( kR \ll 1 \) and \( R/z_R \ll 1 \), the Gaussian beam should be a good approximation to the normal-mode, and
the \( q \) dependence of the mode function should be weak such that the contribution of the term involving \( \nabla_q \) in Eq. (35) is negligible, i.e.,

\[
\lambda(q) \approx -\hbar \text{Im} \int d^3r (\epsilon - 1) f^* \times (\nabla \times f). \tag{39}
\]

Therefore \( \lambda \) and \( \gamma \) can be determined by substituting Eqs. (37) and (38) into Eqs. (36) and (39). In particular, near the beam focus where their magnitudes are largest, we find

\[
\lambda = -\frac{4\hbar(n^2-1)}{3} \left( \frac{R}{z_R} \right)^3 (kz_R)^2 \hat{e}_z, \tag{40}
\]

\[
\gamma = -\frac{4\hbar(n^2-1)z_R}{15} \left( \frac{R}{z_R} \right)^5 (kz_R)^2 \left( 1 + 2kz_R \right) \left( -\frac{q_y}{z_R} \hat{e}_x + \frac{q_x}{z_R} \hat{e}_y \right). \tag{41}
\]

Numerical calculations were performed based on the parameters in Ref. [2], where \( n = 1.45 \), \( R = 100 \text{ nm} \), \( L_c = 4 \text{ mm} \), \( \lambda = 2\pi/k = 1064 \text{ nm} \) and \( z_R = 0.53 \mu m \) for the optical tweezer at a power \( P = 15 \text{ mW} \) (corresponds to an average photon number \( \langle b^\dagger b \rangle \approx 10^6 \)). We find that if the sphere moves under thermal fluctuations at room temperature, the optical phase shift \( \hbar^{-1} (\dot{q} \cdot \lambda + \omega \cdot \gamma) \Delta t \) accumulated within the coherent time scale \( \Delta t \approx 0.1 \text{ ms} \) would be on the order \( 5.1 \times 10^{-5} \text{ rad} \), which is very small. Furthermore, under the strong coherent field of the optical tweezer, we find that the magnitude of the nonadiabatic force \( F = \dot{q} \times (\nabla \times \lambda) - \nabla (\omega \cdot \gamma) \) is negligible compared with the restoring force of the optical tweezer.

While the velocity dependent effects on the subwavelength sphere appears to be quite weak under the single mode field, we should point out that even under adiabatic motion of the sphere, there can be an appreciable effect of geometrical phase due to \( \lambda \). Let us suppose that the wave packet of the sphere travels from \( q_i \) to \( q_f \) along a path \( C \), under a strong coherent field (e.g., of the optical tweezer). In this case, we may take the field state as classical by replacing \( b^\dagger b \) by its expectation value \( \langle b^\dagger b \rangle \). The minimal coupling due to \( \lambda \) then affects the wave function of the sphere \( \psi(q) \) by attaching to it a path-dependent, quantum mechanical (geometrical) phase, apart from an overall dynamical phase:

\[
\psi(q_i) \to e^{-i\Theta} \psi(q_f), \quad \Theta = \hbar^{-1} \int_C dq \cdot \lambda(q) \langle b^\dagger b \rangle. \tag{42}
\]

In particular, if the sphere travels along the beam axis of the optical tweezer,

\[
\Theta = -\frac{2}{3}(n^2-1) \frac{k^2R^3}{L_c} \langle b^\dagger b \rangle \left[ \frac{(q_z/z_R)}{1 + (q_z/z_R)^2} + \tan^{-1} \left( \frac{q_z}{z_R} \right) \right]_{q_z}^{q_f}. \tag{43}
\]
Hence under the numerical parameters used above, the sphere accumulates a phase of $\Theta \approx -5.4\pi$ by traveling one optical wavelength across the beam focus (i.e. from $z = -\lambda/2$ to $z = \lambda/2$) (see Fig. 2). Note that since $\lambda$ is proportional to $|f|^2$, the Gouy phase factor (i.e., $\exp[-i \tan^{-1}(z/z_R)]$) and the optical phase factor $e^{ikz}$ do not contribute to the calculation of $\Theta$.

We point out that in setups which make use of real modes [4], we have $\Theta = 0$, but for other configurations involving complex modes, the geometrical phase could be nonzero. We also remark that for a subwavelength sphere, $\Theta$ is proportional to the time averaged Poynting vector integrated along the path $C$, which is analogous to the case of an induced dipole [25]. However, if the sphere size is appreciable relative to the field wavelength, the change of mode structure as the sphere moves should be properly taken into account. In our approach, this can be readily achieved by including the contribution of the first term involving $\nabla q$ in Eq. (35).

![FIG. 2: Magnitude of geometrical phase $|\Theta|$ accumulated as the dielectric sphere travels from $q_z = -\lambda/2$ to $q_z = \lambda/2$ along the beam axis. The beam focus is at $q_z = 0$. We follow the numerical parameters as in Ref. [2], where $n = 1.45$, $R = 100$ nm, $L_c = 4$ mm, $\lambda = 1064$ nm, and $z_R = 0.53$ $\mu$m for the optical tweezer at a power $P = 15$ mW (hence $\langle b^\dagger b \rangle \approx 10^6$).]
V. CONCLUSION

To conclude, we have presented a nonrelativistic Lagrangian-Hamiltonian formalism of a moving dielectric sphere interacting with radiation fields. We see that in this three-dimensional system, the sphere’s c.m. degree of freedom \( q \) is no longer independent with the vector potential \( A \) under the generalized radiation gauge, and this poses an interesting subtlety in the theory. We have resolved this issue by making use of the instantaneous normal-mode projection to consistently identify the independent degrees of freedom subject to the gauge, enabling canonical quantization of the system in the usual manner.

By including the sphere-field interaction up to first order in \( v \), our Hamiltonian (20) and (30) should capture velocity-dependent optomechanical processes that are not described under the adiabatic approximation. For example, coupling between the field (normal) modes can result from both the translational and rotational motion of the sphere through the interaction described by \( \Lambda' \) and \( \Gamma' \), respectively. In addition, these two mechanical degrees of freedom become coupled in the presence of radiation fields, due to the coupling characterized by \( \Gamma'(q) \). Such motion-induced coupling can become significant in nonadiabatic regimes, especially when the oscillation (or rotation) frequency of the dielectric sphere is close to the frequency spacing between two field modes, in which case transitions between the two modes can be resonantly enhanced.

Under the single mode approximation, we have shown that the velocity dependent effects are typically very weak for a subwavelength sphere, hence it may justify some of the approximations in Refs. [1, 2]. Nonetheless, even under such limiting considerations, we have also indicated an appreciable geometrical phase (in excess of \( \pi \)) acquired by the sphere wavepacket as it moves adiabatically under the single (complex) mode field. Moreover, we should emphasize that our theory does not require the single-mode adiabatic approximation. With the explicit form of interaction strengths between the sphere and various field modes in Eq. (30), our work here enables a further study on the quantum dynamics and applications in multimode nonadiabatic regimes.

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Appendix A: The complex mode Hamiltonian

The complex mode annihilation operators \( \{ b_k(q) \} \) are constructed as a linear combination of the (degenerate) real mode operators \( \{ a_k(q) \} \) by

\[
b_k(q) = \sum_j U_{kj} a_j(q),
\]

where the (complex) coefficients \( U_{kj} \) satisfy the unitary property \( \sum_j U_{kj} U_{kj}^* = \delta_{kk'} \), so that the commutation relation \([b_k(q), b_{k'}^\dagger(q)] = \delta_{kk'} \) is readily fulfilled. Furthermore, since \( b_k(q) \) only mixes degenerate real modes, we have \( U_{kj} = 0 \) except \( \omega_k(q) = \omega_j(q) \). The complex mode function associated with \( b_k(q) \) is given by

\[
f_k(r, q) = \sum_j U_{kj}^* u_j(r, q),
\]

so that in terms of the vector potential \( A(r) \) and the field canonical momentum density \( \Pi(r) \),

\[
b_k(q) = \sqrt{\frac{1}{2\hbar\omega_k(q)}} \int d^3r \left[ \varepsilon(r, q) \omega_k(q) A(r) + i\Pi(r) \right] \cdot f_k^\dagger(r, q).
\]

Using Eq. (A1) and the properties of \( U_{kj} \), we can rewrite the Hamiltonian (30) as

\[
H = \left( \frac{p + \Lambda'}{2m} \right)^2 + \left( \frac{J + \Gamma'}{2I} \right)^2 + \sum_k \hbar \omega_k(q) \left( b_k^\dagger b_k + \frac{1}{2} \right),
\]

where \( \Lambda' \) and \( \Gamma' \) reads

\[
\Lambda'(q) = -\frac{i\hbar}{2} \sum_{k,j} \sqrt{\frac{\omega_k(q)}{\omega_j(q)}} \left[ n_{kj}^{(1)}(q) b_k b_j + n_{kj}^{(2)}(q) b_k^\dagger b_j - \text{H.c.} \right],
\]

\[
\Gamma'(q) = -\frac{i\hbar}{2} \sum_{k,j} \sqrt{\frac{\omega_k(q)}{\omega_j(q)}} \left[ g_{kj}^{(1)}(q) b_k b_j + g_{kj}^{(2)}(q) b_k^\dagger b_j - \text{H.c.} \right],
\]

and the coefficients are given by

\[
n_{kj}^{(1)}(q) = -\int d^3r [\epsilon \sum_{l=x,y,z} (f_k \cdot \hat{e}_l) \nabla_q (f_j \cdot \hat{e}_l) + (\epsilon - 1) f_k \times (\nabla \times f_j)],
\]

\[
n_{kj}^{(2)}(q) = -\int d^3r [\epsilon \sum_{l=x,y,z} (f_k^\dagger \cdot \hat{e}_l) \nabla_q (f_j \cdot \hat{e}_l) + (\epsilon - 1) f_k^\dagger \times (\nabla \times f_j)],
\]

\[
g_{kj}^{(1)}(q) = -\int d^3r (\epsilon - 1) (r - q) \times [f_k \times (\nabla \times f_j)],
\]

\[
g_{kj}^{(2)}(q) = -\int d^3r (\epsilon - 1) (r - q) \times [f_k^\dagger \times (\nabla \times f_j)].
\]
It follows that when the field excitation is dominantly contributed by a single complex mode \( f_k \), Eqs. (A5) and (A6) can be reduced to, in the spirit of rotating wave approximation (where fast oscillating terms such as \( b_k^2 \) and \( b_k^{\dagger 2} \) are neglected),

\[
\Lambda'(q) = -\frac{i\hbar}{2} \left[ \eta^{(2)}_{kk}(q) - \eta^{(2)*}_{kk}(q) \right] b_k^\dagger b_k \equiv \lambda(q) b^\dagger b, \tag{A11}
\]

\[
\Gamma'(q) = -\frac{i\hbar}{2} \left[ g^{(2)}_{kk}(q) - g^{(2)*}_{kk}(q) \right] b_k^\dagger b_k \equiv \gamma(q) b^\dagger b, \tag{A12}
\]

where the explicit form of \( \lambda(q) \) and \( \gamma(q) \) are given in Eqs. (35) and (36), and \( b = b_k(q) \) and \( f = f_k \) are used for shorter notations. Hence the Hamiltonian (34) readily follows in the limit of single complex mode. As a remark, a single real mode can (effectively) be decomposed into sums of degenerate complex modes whose contributions to the photon-number-conserving terms in Eqs. (A5) and (A6) cancel out.

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