Subnormal weighted shifts on directed trees and composition operators in $L^2$ spaces with non-densely defined powers

Piotr Budzyński, Piotr Dymek, Zenon Jan Jabłoński, and Jan Stochel

Abstract. It is shown that for every positive integer $n$ there exists a subnormal weighted shift on a directed tree (with or without root) whose $n$th power is densely defined while its $(n+1)$th power is not. As a consequence, for every positive integer $n$ there exists a non-symmetric subnormal composition operator $C$ in an $L^2$ space over a $\sigma$-finite measure space such that $C^n$ is densely defined and $C^{n+1}$ is not.

1. Introduction

The question of when powers of a closed densely defined linear operator are densely defined has attracted considerable attention. In 1940 Naimark gave a surprising example of a closed symmetric operator whose square has trivial domain (see [16]; see also [10] for a different construction). More than four decades later, Schmüdgen discovered another pathological behaviour of domains of powers of symmetric operators (cf. [17]). It is well-known that symmetric operators are subnormal (cf. [1, Theorem 1 in Appendix I.2]). Hence, closed subnormal operators may have non-densely defined powers. In turn, quasinormal operators, which are subnormal as well (see [4] and [20]), have all powers densely defined (cf. [20]). In the present paper we discuss the above question in the context of subnormal weighted shifts on directed trees and subnormal composition operators in $L^2$ spaces (over $\sigma$-finite measure spaces).

As recently shown (cf. [15, Proposition 3.1]), formally normal, and consequently symmetric, weighted shifts on directed trees are automatically bounded and normal (in general, formally normal operators are not subnormal, cf. [11]). The same applies to symmetric composition operators in $L^2$ spaces (cf. [8, Proposition B.1]). Formally normal composition operators in $L^2$ spaces, which may be unbounded (see [8, Appendix C]), are still normal (cf. [7, Theorem 9.4]). As a consequence, all powers of such operators are densely defined (see e.g., [18, Corollary 5.28]).

The above discussion suggests the question of whether for every positive integer $n$ there exists a subnormal weighted shift on a directed tree whose $n$th power is densley defined while its $(n+1)$th power is not. As a consequence, for every positive integer $n$ there exists a non-symmetric subnormal composition operator $C$ in an $L^2$ space over a $\sigma$-finite measure space such that $C^n$ is densely defined and $C^{n+1}$ is not.

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a closed densely defined operator \( A \) in a complex Hilbert space \( \mathcal{H} \) is denoted by \( \mathcal{D}(A) \) (all operators considered in this paper are linear). Set \( \mathcal{D}(A) = \bigcap_{n=0}^{\infty} \mathcal{D}(A^n) \). Recall that a closed densely defined operator \( A \) in \( \mathcal{H} \) is said to be \( \mathcal{D}(A) \)-normal if \( AA^* = A^*A \) (see \cite{3, 18, 23} for more on this class of operators). We say that a densely defined operator \( A \) in \( \mathcal{H} \) is \( \mathcal{D}(A) \)-subnormal if there exists a complex Hilbert space \( \mathcal{K} \) and a normal operator \( N \) in \( \mathcal{K} \) such that \( \mathcal{H} \subseteq \mathcal{K} \) (isometric embedding) and \( Ah = Nh \) for all \( h \in \mathcal{D}(S) \). We refer the reader to \cite{12} and \cite{19, 20, 21, 22} for the foundations of the theory of bounded and unbounded subnormal operators, respectively.

2. Weighted composition operators

Assume that \((X, \mathcal{A}, \nu)\) is a \( \sigma \)-finite measure space, \( w: X \to \mathbb{C} \) is an \( \mathcal{A} \)-measurable function and \( \phi: X \to X \) is an \( \mathcal{A} \)-measurable mapping. Define the \( \sigma \)-finite measure \( \nu_w: \mathcal{A} \to \mathbb{R}_+ \) by \( \nu_w(\Delta) = \int_\Delta |w|^2 \, d\nu \) for \( \Delta \in \mathcal{A} \). Let \( \nu_w \circ \phi^{-1}: \mathcal{A} \to \mathbb{R}_+ \) be the measure given by \( \nu_w \circ \phi^{-1}(\Delta) = \nu_w(\phi^{-1}(\Delta)) \) for \( \Delta \in \mathcal{A} \). Assume that \( \nu_w \circ \phi^{-1} \) is absolutely continuous with respect to \( \nu \). By the Radon-Nikodym theorem (cf. \cite[Theorem 2.2.1]{2}), there exists a unique (up to a.e. \([\nu]\) equivalence) \( \mathcal{A} \)-measurable function \( h = h_{\phi,w}: X \to \mathbb{R}_+ \) such that
\[
\nu_w \circ \phi^{-1}(\Delta) = \int_\Delta h \, d\nu, \quad \Delta \in \mathcal{A}.
\]
Then the operator \( C = C_{\phi,w} \) in \( L^2(\nu) \) given by
\[
\mathcal{D}(C) = \{ f \in L^2(\nu): w \cdot (f \circ \phi) \in L^2(\nu) \},
\]
\[
Cf = w \cdot (f \circ \phi), \quad f \in \mathcal{D}(C),
\]
(2.1)
is well-defined (cf. \cite[Proposition 6]{9}). Call \( C \) a weighted composition operator. By \cite[Proposition 9]{9}, \( C \) is densely defined if and only if \( h < \infty \) a.e. \([\nu]\); moreover, if this is the case, then \( \nu_w |_{\phi^{-1}(\mathcal{A})} \) is \( \sigma \)-finite and, by the Radon-Nikodym theorem,

\[1\text{ i.e., a system } \{\mu_v\}_{v \in V} \text{ of Borel probability measures on } \mathbb{R}_+ \text{ which satisfies (3.1).} \]
for every $\mathcal{A}$-measurable function $f: X \to [0, \infty]$ there exists a unique (up to a.e. $\nu_w$ equivalence) $\phi^{-1}(\mathcal{A})$-measurable function $E(f) = E_{\phi, w}(f): X \to [0, \infty]$ such that
\[
\int_{\phi^{-1}(\Delta)} f \, d\nu_w = \int_{\phi^{-1}(\Delta)} E(f) \, d\nu_w, \quad \Delta \in \mathcal{A}.
\]

We call $E(f)$ the conditional expectation of $f$ with respect to $\phi^{-1}(\mathcal{A})$ (see [9] for more information). A mapping $P: X \times \mathcal{B}([0, \infty]) \to [0, 1]$ is called an $\mathcal{A}$-measurable family of probability measures if the set-function $P(x, \cdot)$ is a probability measure for every $x \in X$ and the function $P(\cdot, \sigma)$ is $\mathcal{A}$-measurable for every $\sigma \in \mathcal{B}([0, \infty])$.

The following criterion (read: a sufficient condition) for subnormality of unbounded weighted composition operators is extracted from [9, Theorem 27].

**Theorem 2.1.** If $C$ is densely defined, $h > 0$ a.e. $\nu_w$ and there exists an $\mathcal{A}$-measurable family of probability measures $P: X \times \mathcal{B}([0, \infty]) \to [0, 1]$ such that
\[
E(P(\cdot, \sigma))(x) = \int_{\sigma} tP(\phi(x), dt) \quad \text{for } \nu_w\text{-a.e. } x \in X, \quad \sigma \in \mathcal{B}([0, \infty]), \quad (CC)
\]
them $C$ is subnormal.

Regarding Theorem 2.1, recall that if $C$ is subnormal, then $h > 0$ a.e. $\nu_w$ (cf. [9, Corollary 12]).

3. Weighted shifts on directed trees

Let $\mathcal{T} = (V, E)$ be a directed tree ($V$ and $E$ stand for the sets of vertices and edges of $\mathcal{T}$, respectively). Set $\text{Chi}(u) = \{v \in V: (u, v) \in E\}$ for $u \in V$. Denote by $\text{par}$ the partial function from $V$ to $V$ which assigns to each vertex $u \in V$ its parent $\text{par}(u)$ (i.e. a unique $v \in V$ such that $(v, u) \in E$). A vertex $u \in V$ is called a root of $\mathcal{T}$ if $u$ has no parent. A root is unique (provided it exists); we denote it by $\text{root}$. Set $V^0 = V \setminus \{\text{root}\}$ if $\mathcal{T}$ has a root and $V^0 = V$ otherwise. We say that $u \in V$ is a branching vertex of $V$, and write $u \in V_{<0}$, if $\text{Chi}(u)$ consists of at least two vertices. We refer the reader to [13] for all facts about directed trees needed in this paper.

By a weighted shift on $\mathcal{T}$ with weights $\lambda = \{\lambda_v\}_{v \in V^0} \subseteq \mathbb{C}$ we mean the operator $S_\lambda$ in $\ell^2(V)$ defined by
\[
D(S_\lambda) = \{f \in \ell^2(V): \Lambda_\mathcal{T} f \in \ell^2(V)\},
\]
\[
S_\lambda f = \Lambda_\mathcal{T} f, \quad f \in D(S_\lambda),
\]
where $\Lambda_\mathcal{T}$ is the mapping defined on functions $f: V \to \mathbb{C}$ via
\[
(\Lambda_\mathcal{T} f)(v) = \begin{cases} 
\lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^0, \\
0 & \text{if } v = \text{root}.
\end{cases}
\]
(As usual, $\ell^2(V)$ is the Hilbert space of square summable complex functions on $V$ with standard inner product.) For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one-point set $\{u\}$. Then $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$.

The following useful lemma is an extension of part (iv) of [14, Theorem 3.2.2].

**Lemma 3.1.** Let $S_\lambda$ be a weighted shift on a directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^0}$ and let $n \in \mathbb{Z}_+$. Then $S_\lambda^n$ is densely defined if and only if $e_u \in D(S_\lambda^n)$ for every $u \in V_{<0}$.
As opposed to this covers the case of classical weighted shifts and their adjoints. 

of Borel probability measures on \( \mathbb{R} = (T, V, E) \), the underlying \( \ell^2 \)-tree is rootless and leafless, which is required in \([14, \text{Proposition 3.1.10}]\). In particular, this covers the case of classical weighted shifts and their adjoints.

Now we give a criterion for subnormality of weighted shifts on directed trees. As opposed to [5, Theorem 5.1.1], we do not assume the density of \( \mathcal{C} \)-vectors in the underlying \( \ell^2 \)-space. Moreover, we do not assume that the underlying directed tree is rootless and leafless, which is required in [8, Theorem 47], and that weights are nonzero. The only restriction we impose is that the directed tree is countably infinite. This is always satisfied if the weighted shift in question is densely defined and has nonzero weights (cf. \([13, \text{Proposition 3.1.10}]\)).

Theorem 3.2. Let \( S_\lambda \) be a weighted shift on a countably infinite directed tree \( \mathcal{T} = (V, E) \) with weights \( \lambda = \{\lambda_e\}_{e \in V} \). Suppose there exist a system \( \{\mu_v\}_{v \in V} \) of Borel probability measures on \( \mathbb{R}_+ \) and a system \( \{\sigma_v\}_{v \in V} \) of nonnegative real numbers such that

\[
\mu_v(\sigma) = \sum_{e \in \text{Chi}(u)} |\lambda_e|^2 \int_{\sigma} \frac{1}{t} \mu_v(\text{d}t) + \varepsilon_v \delta_0(\sigma), \quad \sigma \in \mathcal{B}(\mathbb{R}_+), \quad u \in V. \tag{3.1}
\]

Then the following two assertions hold:

(i) if \( S_\lambda \) is densely defined, then \( S_\lambda \) is subnormal,

(ii) if \( n \in \mathbb{N} \), then \( S^n_\lambda \) is densely defined if and only if \( \int_0^\infty s^n \text{d}\mu_u(s) < \infty \) for all \( u \in V_\subset \).

Proof. (i) Assume that \( S_\lambda \) is densely defined. Set \( X = V \) and \( \mathcal{A} = 2^V \). Let \( \nu: \mathcal{A} \to \mathbb{R}_+ \) be the counting measure on \( X \) (\( \nu \) is \( \sigma \)-finite because \( V \) is countable).

Define the weight function \( w: X \to \mathbb{C} \) and the mapping \( \phi: X \to X \) by

\[
w(x) = \begin{cases} 
\lambda_x & \text{if } x \in V^\circ \\
0 & \text{if } x = \text{root}
\end{cases} \quad \text{and} \quad \phi(x) = \begin{cases} 
\text{par}(x) & \text{if } x \in V^\circ \\
\text{root} & \text{if } x = \text{root}
\end{cases}.
\]

Clearly, the measure \( \nu_w \circ \phi^{-1} \) is absolutely continuous with respect to \( \nu \) and

\[
h(x) = \nu_w(\phi^{-1}(\{x\})) = \nu_w(\text{Chi}(x)) = \sum_{y \in \text{Chi}(x)} |\lambda_y|^2, \quad x \in X. \tag{3.2}
\]

Thus, by \([13, \text{Proposition 3.1.3}]\), \( h(x) < \infty \) for every \( x \in X \). We claim that \( h > 0 \) a.e. \( [\nu_w] \). This is the same as to show that if \( x \in V^\circ \) and \( \nu_w(\text{Chi}(x)) = 0 \), then \( \lambda_x = 0 \). Thus, if \( x \in V^\circ \) and \( \nu_w(\text{Chi}(x)) = 0 \), then applying (3.1) to \( u = x \), we deduce that \( \mu_x = \delta_0 \); in turn, applying (3.1) to \( u = \text{par}(x) \) with \( \sigma = \{0\} \), we get \( \lambda_x = 0 \), which proves our claim.

\[\text{We adopt the conventions that } 0 \cdot \infty = \infty \cdot 0 = 0, \frac{1}{0} = \infty \text{ and } \sum_{v \in \varnothing} \xi_v = 0.\]
Note that $X = \bigsqcup_{x \in X} \phi^{-1}(\{x\})$ (the disjoint union). Hence, the conditional expectation $E(f)$ of a function $f : X \to \mathbb{R}_+$ with respect to $\phi^{-1}(\mathcal{A})$ is given by

$$E(f)(z) = \frac{\int_{\text{Chi}(x)} f \, \nu_w}{h(x)} , \quad z \in \phi^{-1}(\{x\}), \quad x \in X_+, \quad (3.3)$$

where $X_+ := \{x \in X : \nu_w(\text{Chi}(x)) > 0\}$ (see also (3.2)); on the remaining part of $X$ we can put $E(f) = 0$.

Substituting $\sigma = \{0\}$ into (3.1), we see that $\mu_y(\{0\}) = 0$ for every $y \in V^\circ$ such that $\lambda_y \neq 0$. Thus, using the standard measure-theoretic argument and (3.1), we deduce that

$$\int t \, d \mu_x(t) = \sum_{y \in \text{Chi}(x)} |\lambda_y|^2 \mu_y(\sigma), \quad \sigma \in \mathcal{B}(\mathbb{R}_+) , \quad x \in X. \quad (3.4)$$

Set $P(x, \sigma) = \mu_x(\sigma)$ for $x \in X$ and $\sigma \in \mathcal{B}(\mathbb{R}_+)$. It follows from (3.3) and (3.4) that $P : X \times \mathcal{B}(\mathbb{R}_+) \to [0, 1]$ is a ($\mathcal{A}$-measurable) family of probability measures which fulfils the following equality

$$E(P(\cdot, \sigma))(z) = \frac{\int_{\sigma} t P(\phi(z), d t)}{h(\phi(z))} , \quad z \in \phi^{-1}(\{x\}), \quad x \in X_+. \quad (3.5)$$

This implies that $P$ satisfies (CC). Hence, by Theorem 2.1, the weighted composition operator $C$ (see (2.1)) is subnormal. Since $S_{\lambda} = C$, assertion (i) is proved.

(ii) It is easily seen that if $\mu$ is a finite positive Borel measure on $\mathbb{R}_+$ and $\int_{\mathbb{R}_+} s^n \, d \mu(s) < \infty$ for some $n \in \mathbb{N}$, then $\int_{\mathbb{R}_+} s^k \, d \mu(s) < \infty$ for every $k \in \mathbb{N}$ such that $k \leq n$. This fact combined with Lemma 3.1 and [5, Lemmata 2.3.1(i) and 4.2.2(i)] implies assertion (ii).

**Remark 3.3.** Assume that $S_{\lambda}$ is a densely defined weighted shift on a countably infinite directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$. A careful inspection of the proof of Theorem 3.2 reveals that if $\{\mu_x\}_{x \in X}$ (with $X = V$) is a system of Borel probability measures on $\mathbb{R}_+$ which satisfies (3.1), then $h > 0$ a.e. $\nu_w$, the family $P$ defined by $P(x, \cdot) = \mu_x$ for $x \in X$ satisfies (CC) and $\mu_x = \delta_0$ for every $x \in X \setminus X_+$. We claim that if $h > 0$ a.e. $\nu_w$ and $P : X \times \mathcal{B}(\mathbb{R}_+) \to [0, 1]$ is any family of probability measures which satisfies (CC), then the system $\{\tilde{\mu}_x\}_{x \in X}$ of probability measures defined by

$$\tilde{\mu}_x = \begin{cases} P(x, \cdot) & \text{if } x \in X_+, \\ \delta_0 & \text{otherwise,} \end{cases}$$

satisfies (3.1) with $\{\tilde{\mu}_x\}_{x \in X}$ in place of $\{\mu_x\}_{x \in X}$. Indeed, (CC) implies (3.5). Hence, by (3.3), equality in (3.4) holds for every $x \in X_+$ with $\mu_z = P(z, \cdot)$ for $z \in X$. This implies via the standard measure-theoretic argument that equality in (3.1) holds for every $u \in X_+$. Since $h > 0$ a.e. $\nu_w$, we deduce that equality in (3.1) holds for every $u \in X_+$ with $\{\mu_x\}_{x \in X}$ in place of $\{\mu_x\}_{x \in X}$. Clearly, this is also the case for $u \in X \setminus X_+$. Thus, our claim is proved.

4. Trees with one branching vertex

Theorem 3.2 will be applied in the case of weighted shifts on leafless directed trees with one branching vertex. First, we recall the models of such trees (see
For \( \eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\} \) with \( \eta \geq 2 \), we define the directed tree \( \mathcal{T}_{\eta, \kappa} = (V_{\eta, \kappa}, E_{\eta, \kappa}) \) as follows (the symbol “\( \sqcup \)” denotes disjoint union of sets)

\[
V_{\eta, \kappa} = \{-k: k \in J_\kappa\} \sqcup \{0\} \sqcup \{(i, j): i \in J_\eta, j \in \mathbb{N}\},
\]

\[
E_{\eta, \kappa} = E_\kappa \sqcup \{(0, (i, 1)): i \in J_\eta\} \sqcup \{((i, j), (i, j + 1)): i \in J_\eta, j \in \mathbb{N}\},
\]

where \( J_n = \{k \in \mathbb{N}: k \leq n\} \) for \( n \in \mathbb{Z}_+ \cup \{\infty\} \). Clearly, \( \mathcal{T}_{\eta, \kappa} \) is leafless and 0 is its only branching vertex. From now on, we write \( \lambda_{i,j} \) instead of the more formal expression \( \lambda_{(i,j)} \) whenever \((i, j) \in V_{\eta, \kappa}\).

**Theorem 4.1.** Let \( \eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\} \) be such that \( \eta \geq 2 \) and let \( S_\lambda \) be a weighted shift on a directed tree \( \mathcal{T}_{\eta, \kappa} \) with nonzero weights \( \lambda = \{\lambda_v\}_{v \in V_{\eta, \kappa}} \). Suppose that there exists a sequence \( \{\mu_i\}_{i=1}^\eta \) of Borel probability measures on \( \mathbb{R}_+ \) such that

\[
\int_0^\infty s^n d\mu_i(s) = \left| \prod_{j=2}^{n+1} \lambda_{i,j} \right|^2, \quad n \in \mathbb{N}, \ i \in J_\eta,
\]

and that one of the following three disjunctive conditions is satisfied:

(i) \( \kappa = 0 \) and

\[
\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} d\mu_i(s) \leq 1,
\]

(ii) \( 0 < \kappa < \infty \) and

\[
\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s} d\mu_i(s) = 1,
\]

\[
\left| \prod_{j=0}^{l-1} \lambda_{-j} \right|^2 \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{l+1}} d\mu_i(s) = 1, \quad l \in J_{\kappa-1},
\]

(iii) \( \kappa = \infty \) and equalities (4.2) and (4.3) are valid.

Then the following two assertions hold:

(a) if \( S_\lambda \) is densely defined, then \( S_\lambda \) is subnormal,
(b) if \( n \in \mathbb{N} \), then \( S^n_\lambda \) is densely defined if and only if
\[
\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_{0}^{\infty} s^{n-1} \, d\mu_i(s) < \infty.
\] (4.5)

**Proof.** As in the proof of [6, Theorem 4.1], we define the system \( \{\mu_v\}_{v \in V_{\eta,\kappa}} \) of Borel probability measures on \( \mathbb{R}_+ \) and verify that \( \{\mu_v\}_{v \in V_{\eta,\kappa}} \) satisfies (3.1). Hence, assertion (a) is a direct consequence of Theorem 3.2(i).

(b) Fix \( n \in \mathbb{N} \). It follows from Theorem 3.2(ii) that \( S^n_\lambda \) is densely defined if and only if \( \int_{0}^{\infty} s^n \, d\mu_0(s) < \infty \). Using the explicit definition of \( \mu_0 \) and applying the standard measure-theoretic argument, we see that
\[
\int_{0}^{\infty} s^n \, d\mu_0(s) = \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_{0}^{\infty} s^{n-1} \, d\mu_i(s).
\]
This completes the proof of assertion (b) (the case of \( n = 1 \) can also be settled without using the definition of \( \mu_0 \) simply by applying Lemma 3.1 and [13, Proposition 3.1.3(iii)]).

Note that Theorem 4.1 remains true if its condition (ii) is replaced by the condition (iii) of [6, Theorem 4.1] (see also [6, Lemma 4.2] and its proof).

**Corollary 4.2.** Under the assumptions of Theorem 4.1, if \( n \in \mathbb{N} \), then the following two assertions are equivalent:

(i) \( S^n_\lambda \) is densely defined and \( S^{n+1}_\lambda \) is not,
(ii) the condition (4.5) holds and \( \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_{0}^{\infty} s^n \, d\mu_i(s) = \infty \).

5. The example

It follows from [5, Lemma 2.3.1(i)] that if \( S_\lambda \) is a weighted shift on \( \mathcal{T}_{\eta,\kappa} \) and \( \eta < \infty \), then \( \mathcal{D}_\infty(S_\lambda) \) is dense in \( \mathcal{L}^2(V_{\eta,\kappa}) \) (this means that Corollary 4.2 is interesting only if \( \eta = \infty \)). If \( \eta = \infty \), the situation is completely different. Using Theorem 4.1 and Corollary 4.2, we show that for every \( n \in \mathbb{N} \) and for every \( \kappa \in \mathbb{Z}_+ \cup \{\infty\} \), there exists a subnormal weighted shift \( S_\lambda \) on \( \mathcal{T}_{\infty,\kappa} \) such that \( S^n_\lambda \) is densely defined and \( S^{n+1}_\lambda \) is not. For this purpose, we adapt [13, Procedure 6.3.1] to the present context. In the original procedure, one starts with a sequence \( \{\mu_i\}_{i=1}^{\infty} \) of Borel probability measures on \( \mathbb{R}_+ \) (whose \( n \)-th moments are finite for every \( n \in \mathbb{Z} \) such that \( n \geq -(\kappa + 1) \)) and then constructs a system of nonzero weights \( \lambda = \{\lambda_i\}_{i \in V_{\infty,\kappa}} \) that satisfies the assumptions of Theorem 4.1 (in fact, using Lemma 5.2 below, we can also maintain the condition (4.5)). However, in general, it is not possible to maintain the condition (ii) of Corollary 4.2 even if \( \{\mu_i\}_{i=1}^{\infty} \) are measures with two-point supports (this question is not discussed here).

**Example 5.1.** Assume that \( \eta = \infty \). Consider the measures \( \mu_i = \delta_{q_i} \) with \( q_i \in (0, \infty) \) for \( i \in \mathbb{N} \). By [13, Notation 6.1.9 and Procedure 6.3.1], \( S_\lambda \in \mathcal{B}(\mathcal{L}^2(V_{\infty,\kappa})) \) if and only if \( \sup \{ q_i : i \in \mathbb{N} \} < \infty \). Hence, there is no loss of generality in assuming that \( \sup \{ q_i : i \in \mathbb{N} \} = \infty \). To cover all possible choices of \( \kappa \in \mathbb{Z}_+ \cup \{\infty\} \), we look for a system of nonzero weights \( \{\lambda_i\}_{i \in V_{\infty,\kappa}} \) which satisfies (4.1), (4.2), (4.3) with \( \kappa = \infty \), (4.5) and the equality \( \sum_{i=1}^{\infty} |\lambda_{i,1}|^2 \int_{0}^{\infty} s^n \, d\mu_i(s) = \infty \). Setting \( \lambda_{i,1} = \sqrt{\alpha_i q_i} \)
for \( i \in \mathbb{N} \), we reduce our problem to finding a sequence \( \{\alpha_i\}_{i=1}^{\infty} \subseteq (0, \infty) \) such that

\[
\sum_{i=1}^{\infty} \alpha_i q_i^l < \infty, \quad l \in \mathbb{Z} \text{ and } l \leq n, \tag{5.1}
\]

\[
\sum_{i=1}^{\infty} \alpha_i q_i^{n+1} = \infty. \tag{5.2}
\]

Indeed, if \( \{\alpha_i\}_{i=1}^{\infty} \) is such a sequence, then multiplying its terms by an appropriate positive constant, we may assume that \( \lambda \) and finally we set \( \lambda_{i,j} = \sqrt{\mu_i} \) for all \( i, j \in \mathbb{N} \) such that \( j \geq 2 \). The so constructed weights \( \{\lambda_{i,j}\}_{i,j} \) meets our requirements.

The following lemma turns out to be helpful when solving the reduced problem.

**Lemma 5.2.** If \( \{a_{i,j}\}_{i,j=1}^{\infty} \) is an infinite matrix with entries \( a_{i,j} \in \mathbb{R}_+ \), then there exists a sequence \( \{\alpha_i\}_{i=1}^{\infty} \subseteq (0, \infty) \) such that

\[
\sum_{i=1}^{\infty} \alpha_i a_{i,j} < \infty, \quad j \in \mathbb{N}. \]

**Proof.** First observe that for every \( i \in \mathbb{N} \), there exists \( \alpha_i \in (0, \infty) \) such that \( \alpha_i \sum_{k=1}^{\infty} a_{i,k} \leq 2^{-i} \). Hence, \( \sum_{i=1}^{\infty} \alpha_i a_{i,j} \leq 1 \) for every \( j \in \mathbb{N} \). \( \Box \)

Since \( \sup \{q_{i_k} : i \in \mathbb{N}\} = \infty \), there exists a subsequence \( \{q_{i_k}\}_{k=1}^{\infty} \) of the sequence \( \{q_i\}_{i=1}^{\infty} \) such that \( q_{i_k} \geq k \) for every \( k \in \mathbb{N} \). Set \( \Omega = \{i_k : k \in \mathbb{N}\} \). By Lemma 5.2, there exists \( \{\alpha_i\}_{i \in \mathbb{N}\setminus\Omega} \subseteq (0, \infty) \) such that

\[
\sum_{i \in \mathbb{N}\setminus\Omega} \alpha_i q_i^l < \infty, \quad l \in \mathbb{Z} \text{ and } l \leq n. \tag{5.3}
\]

Define the system \( \{\alpha_i\}_{i \in \Omega} \subseteq (0, \infty) \) by

\[\alpha_{i_k} = \frac{1}{k^2 q_{i_k}^l}, \quad k \in \mathbb{N}.\]

Since \( q_{i_k} \geq k \) for all \( k \in \mathbb{N} \), we get

\[
\sum_{i \in \Omega} \alpha_i q_i^l = \sum_{i \in \Omega} \alpha_{i_k} q_{i_k}^l = \sum_{k=1}^{\infty} \alpha_{i_k} \frac{1}{k^2 q_{i_k}^l} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty, \quad l \in \mathbb{Z} \text{ and } l \leq n, \tag{5.4}
\]

and

\[
\sum_{i \in \Omega} \alpha_i q_i^{n+1} = \sum_{i \in \Omega} \alpha_{i_k} q_{i_k}^{n+1} = \sum_{k=1}^{\infty} \frac{q_{i_k}}{k^2} \geq \sum_{k=1}^{\infty} \frac{1}{k} = \infty. \tag{5.5}
\]

Combining (5.3), (5.4) and (5.5), we get (5.1) and (5.2), which solves the reduced problem and consequently gives the required example.

**Remark 5.3.** It is worth mentioning that if \( \kappa = \infty \), then any weighted shift \( S_\lambda \) on \( \mathcal{S}_{\infty, \infty} \), with nonzero weights is unitarily equivalent to an injective composition operator in an \( L^2 \) space over a \( \sigma \)-finite measure space (cf. [14, Lemma 4.3.1]). This fact combined with Example 5.1 shows that for every \( n \in \mathbb{N} \), there exists a subnormal composition operator \( C \) in an \( L^2 \) space over a \( \sigma \)-finite measure space such that \( C^n \) is densely defined and \( C^{n+1} \) is not.
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