BESTVINA–BRADY GROUPS WITH POLYNOMIAL DEHN FUNCTIONS

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Abstract
Let $\Gamma$ be a finite simplicial graph that contains no induced $K_4$ subgraphs. We show that with some assumptions, the Dehn function of the associated Bestvina–Brady group is polynomial.

1. Introduction

Dehn functions are one of the quasi-isometry invariants of finitely presented groups, and they have been studied by many people. One of the reasons people study Dehn functions is that they provide the solvability of the word problem for finitely presented groups. That is, a finitely presented group has a solvable word problem if and only if its Dehn function is recursive. Besides the solvability of the word problem, Dehn functions can also detect certain structures in groups. For example, a group is hyperbolic if and only if it has a linear Dehn function [13]. We refer to [7] for more facts about Dehn functions. In this paper, we study Dehn functions of Bestvina–Brady groups, which are subgroups of right-angled Artin groups.

Given a finite simplicial graph $\Gamma$, the associated right-angled Artin group $A_\Gamma$ is generated by the vertices of $\Gamma$; the relators are commutators: two generators $u$, $v$ commute if they are adjacent vertices of $\Gamma$. Right-angled Artin groups have become an important object that people study in geometric group theory; see [10] for a general survey. They are known to be CAT(0) groups; both categories of groups have at most quadratic Dehn functions. But subgroups can have larger Dehn functions. Brady and Forester [3] gave examples of CAT(0) groups that contain finitely presented subgroups whose Dehn functions are of the form $n^\rho$, for a dense set of $\rho \in [2, \infty)$. Brady and Soroko [5] proved that for each positive integer $\rho$, there is a right-angled Artin group that contains a finitely presented subgroup whose Dehn function is $n^\rho$.

For a right-angled Artin group $A_\Gamma$, the associated Bestvina–Brady group $H_\Gamma$ is defined to be the kernel of the homomorphism $A_\Gamma \to \mathbb{Z}$, where the homomorphism sends all the generators of $A_\Gamma$ to 1. Bestvina and Brady [2] introduced these groups to provide examples of groups which satisfy the finiteness property $FP_n$ but not $FP_{n+1}$. For instance, let $\Gamma$ be a cycle $C_4$ on four vertices, then $A_\Gamma$ is $F_2 \times F_2$, and the group $H_\Gamma = \text{ker}(F_2 \times F_2 \to \mathbb{Z})$ is finitely generated ($FP_1$) but not finitely presented ($FP_2$) [2, Example 6.3]. Moreover, Bestvina–Brady groups are either counterexamples to the Eilenberg–Banea Conjecture or counterexamples to the Whitehead Conjecture [2, Theorem 8.7].

Dison proved that the Dehn functions of Bestvina–Brady groups are at most quartic [12]. There are Bestvina–Brady groups whose Dehn functions are linear, quadratic, cubic, or quartic [4, Part I]. It is natural to ask whether all the Bestvina–Brady groups have polynomial Dehn functions. Our main theorem is a partial answer to this question, and it also provides a way to identify the Dehn function of a given Bestvina–Brady groups by its defining graph.
Main Theorem. Let $\Gamma$ be a finite simplicial graph such that the flag complex $\Delta_\Gamma$ on $\Gamma$ is a 2-dimensional triangulated disk whose boundary is a square. If the maximal dimension of simplices in the interior of $\Delta_\Gamma$ is $d$ then the Dehn function $\delta_{H_\Gamma}(n)$ of the Bestvina–Brady group $H_\Gamma$ is $n^d$.

When $d = 0$, we can eliminate the square boundary condition; see Theorem 3.1. In this case, the Bestvina–Brady group $H_\Gamma$ has a graph of groups decomposition where the edge groups are $\mathbb{Z}$ and the vertex groups are right-angled Artin groups. We remark that some cases of $d = 0$ in the main theorem can be recovered by Carter and Forester [9]; see Example 4.3. In [4], Brady gave an example of a Bestvina–Brady group that has a cubic Dehn function. The defining graph in Brady’s example satisfies the assumptions of our main theorem when $d = 1$. In fact, we give a family of graphs such that the associated Bestvina–Brady groups have cubic Dehn functions; see Remark 4.5. Since there is a universal quartic upper bound, we only need to establish the upper bound for $d = 1$. To achieve the cubic upper bound, we use the corridor schemes [3]; see Section 4.2 for a detail proof. In [1], the authors introduced the height-pushing map to obtain the lower bound on the higher Dehn functions of orthoplex groups. Their method can be adapted to our proof to obtain the lower bound for the cases $d = 1, 2$ of the main theorem. We want to point out that their theorem ([1], Theorem 5.1) recovers Dison’s quartic upper bound in [12].

In our main theorem, we require the flag complex $\Delta_\Gamma$ to be 2-dimensional. This is equivalent to saying that the graph $\Gamma$ does not have induced $K_4$ subgraphs. When $\Gamma$ contains induced $K_4$ graphs and some specific subgraphs, we give a lower bound on $\delta_{H_\Gamma}$:

Proposition. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is simply-connected. If $\Gamma$ contains an induced subgraph $\Gamma'$ such that $\Delta_{\Gamma'}$ is a 2-dimensional triangulated subdisk of $\Delta_\Gamma$ that has square boundary and $\dim_I(\Delta_{\Gamma'}) = d$ for $d \in \{0, 1, 2\}$ then $n^{d+2} \preceq \delta_{H_\Gamma}(n)$.

This paper is organized as follows. Section 2 provides some necessary background. Section 3 contains the proof of the main theorem for the case $d = 0$. Section 4 is devoted to the proof of the main theorem for the cases $d = 1, 2$. In Section 5, we establish the lower bound on $\delta_{H_\Gamma}$ when $\Gamma$ has induced $K_4$ subgraphs.

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2. PRELIMINARIES

2.1. Dehn functions. Let $G$ be a group with a finite presentation $\mathcal{P} = \langle S | R \rangle$. Let $w$ be a word that represents the identity of $G$, denoted by $w \equiv_G 1$. The area of $w$, denoted by $\text{Area}(w)$, is defined as follows:

$$\text{Area}(w) = \min \left\{ N \left| w = \prod_{i=1}^N x_i r_i^{\pm 1} x_i^{-1}, x_i \in F(S), r_i \in R \right. \right\},$$

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where $F(S)$ is the free group generated by $S$. The Dehn function $\delta_G : \mathbb{N} \to \mathbb{N}$ of a group $G$ over the presentation $P = \langle S | R \rangle$ is defined by

$$
\delta_P(n) = \max \left\{ \text{Area}_G(w) \left| w \equiv_G 1, |w| \leq n \right. \right\},
$$

where $|w|$ denotes the length of the word $w$.

**Definition 2.1.** Let $f, g : [0, \infty) \to [0, \infty)$ be two functions. We say that $f$ is bounded above by $g$, denoted by $f \preceq g$, if there is a number $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all $n > 0$. We say that $f$ and $g$ are $\simeq$-equivalent, or simply equivalent, denoted by $f \simeq g$, if $f \preceq g$ and $g \preceq f$.

If $P_1$ and $P_2$ are finite presentations of a group $G$, then $\delta_{P_1}$ is equivalent to $\delta_{P_2}$; we refer to [4] for a proof of this fact. We denote the $\simeq$-equivalent class of $G$ by $\delta_G$, and call it the Dehn function of $G$. We say that the Dehn function $\delta_G$ is linear, quadratic, cubic or quartic if for all $n \in \mathbb{N}$, $\delta_G(n) \simeq n$, $\delta_G(n) \simeq n^2$, $\delta_G(n) \simeq n^3$ or $\delta_G(n) \simeq n^4$, respectively.

### 2.2. Right-angled Artin groups, Bestvina–Brady groups, and the Dicks–Leary presentation.

Let $\Gamma$ be a finite simplicial graph and $V(\Gamma)$ the set of vertices of $\Gamma$. The right-angled Artin group $A_\Gamma$ associated to $\Gamma$ has the following presentation:

$$A_\Gamma = \left\langle V(\Gamma) \left| [v_i, v_j] \text{ whenever } v_i \text{ and } v_j \text{ are connected by an edge of } \Gamma \right. \right\rangle.$$

When $\Gamma$ is a complete graph $K_n$ on $n$ vertices, $A_\Gamma = \mathbb{Z}^n$; when $\Gamma$ is a set of $n$ distinct points, $A_\Gamma = F_n$, the free group of rank $n$.

For each finite simplicial graph $\Gamma$, its associated right-angled Artin group $A_\Gamma$ is the fundamental group of a cubical complex $X_\Gamma$, called the Salvetti complex. It is well-known that the Salvetti complex $X_\Gamma$ is compact and non-positively curved, and its universal cover is a CAT(0) cube complex. Moreover, right-angled Artin groups are CAT(0) groups; thus, they have at most quadratic Dehn functions. We refer to [10] for more details of these facts.

Given a finite simplicial graph $\Gamma$, we define a group homomorphism $\phi : A_\Gamma \to \mathbb{Z}$ by sending all the generators of $A_\Gamma$ to 1. The kernel of this homomorphism is called the Bestvina–Brady group defined by $\Gamma$, and is denoted by $H_\Gamma$.

The flag complex $\Delta_\Gamma$ on a finite simplicial graph $\Gamma$ is a simplicial complex such that each complete subgraph $K_n$ of $\Gamma$ spans an $(n - 1)$-simplex in $\Delta_\Gamma$. When $\Delta_\Gamma$ is connected, $H_\Gamma$ is finitely generated; when $\Delta_\Gamma$ is simply-connected, $H_\Gamma$ is finitely presented; see [2] for the proof of these facts. When $H_\Gamma$ is finitely presented, we can write down its Dicks–Leary presentation [11]:

**Theorem 2.2.** ([11], Corollary 3) Let $\Gamma$ be a finite simplicial oriented graph. Suppose that $\Delta_\Gamma$ is simply-connected. Then the Bestvina–Brady group $H_\Gamma$ has the following finite presentation:

$$H_\Gamma = \left\langle E(\Gamma) \left| ef = g = fe \text{ whenever } e, f, g \text{ form an oriented triangle} \right. \right\rangle,$$

where $E(\Gamma)$ is the set of oriented edges of $\Gamma$, and the oriented triangle is shown in Figure 2.4.
Figure 2.1. Realator of the Dicks–Leary presentation.

In fact, the generating set in the above theorem can be reduced further:

**Corollary 2.3.** ([14], Corollary 2.3) If the flag complex on a finite simplicial graph is simply-connected, then \( H_\Gamma \) has a presentation \( H_\Gamma = F/R \), where \( F \) is the free group generated by the edges in a maximal tree of \( \Gamma \), and \( R \) is a finitely generated normal subgroup of the commutator group \([F,F]\).

While Dehn functions of right-angled Artin groups are at most quadratic, Dison [12] proved that Dehn functions of Bestvina–Brady groups are bounded above by quartic functions.

**Theorem 2.4.** ([12]) Dehn functions of Bestvina–Brady groups are at most quartic.

### 2.3. Interior dimensions.

Let \( D \) be a triangulated disk. An interior \( i \)-simplex of \( D \) is an \( i \)-simplex whose faces do not intersect \( \partial D \). We also call an interior 0-simplex an interior vertex, an interior 1-simplex an interior edge, and an interior 2-simplex an interior triangle. We say that \( D \) has interior dimension \( d \), denoted by \( \dim_I(D) = d \), if the maximal dimension of interior simplices of \( D \) is \( d \). If \( D \) contains no interior simplices, then we define \( \dim_I(D) = 0 \).

**Example 2.5.** The flag complexes on the following graphs are triangulated disks with square boundaries, and they have interior dimensions 0, 1, and 2, respectively:

![Diagram](image)

Figure 2.2. Disks of interior dimensions 0, 1, 2, respectively, and with square boundaries.

### 3. Disks with interior dimension 0

In this section, we prove the following theorem.

**Theorem 3.1.** Let \( \Gamma \) be a finite simplicial graph. If \( \Delta_\Gamma \) is a 2-dimensional triangulated disk satisfying \( \dim_I(\Delta_\Gamma) = 0 \), then \( \delta_{H_\Gamma}(n) \simeq n^2 \).

For such a graph \( \Gamma \), we will see later that the associated Bestvina–Brady group \( H_\Gamma \) has a graph of group decomposition, where the edges groups are infinite cyclic groups; and the
vertex groups are Bestvina–Brady groups on some induced subgraphs of $\Gamma$, namely, fans and wheels. Moreover, each of the vertex groups is isomorphic to a non-hyperbolic right-angled Artin group.

Recall that the join of two graphs $\Gamma_1$ and $\Gamma_2$, denoted by $\Gamma_1 \ast \Gamma_2$, is the graph obtained by taking the disjoint union of $\Gamma_1$ and $\Gamma_2$ together with all the edges that connect the vertices of $\Gamma_1$ and the vertices of $\Gamma_2$.

**Definition 3.2.** A fan $F_{n+1}$ is the join of a vertex and a path $P_n$. A wheel $W_{n+1}$ is the join of a vertex and a cycle $C_n$.

**Remark 3.3.** The flag complex on $W_4$ is a tetrahedron, which is not 2-dimensional. Throughout this paper, unless otherwise stated, all the wheels have at least five vertices, that is, $W_n$ for $n \geq 5$. Note that a triangle is also a fan $F_3$.

Fans and wheels have a special structure: they can be decomposed as a join of a vertex and graph. When a finite simplicial graph $\Gamma$ decomposes as a join $\Gamma = \{v\} \ast \Gamma'$, the Dicks–Leary presentation gives $H_\Gamma \cong A_{\Gamma'}$ (see Example 2.5 in [14]). That is, $H_\Gamma$ is a right-angled Artin group. Thus, $\delta_{H_\Gamma}$ is at most quadratic.

**Proposition 3.4.** Suppose a finite simplicial graph decomposes as a join $\Gamma = \{v\} \ast \Gamma'$. Then $H_\Gamma \cong A_{\Gamma'}$ and $\delta_{H_\Gamma}$ is at most quadratic. Moreover, if $\Gamma'$ contains an edge, then $A_{\Gamma'}$ is non-hyperbolic and $\delta_{H_\Gamma}$ is quadratic.

**Proof.** Since $\Gamma = \{v\} \ast \Gamma'$, we have $A_{\Gamma} = \mathbb{Z} \times A_{\Gamma'}$. We claim that $H_{\Gamma} \cong A_{\Gamma'}$. Label the edges that have $v$ as the common endpoint by $e_1, \ldots, e_k$; label the other end points of $e_1, \ldots, e_k$ by $v_1, \ldots, v_k$. Since $e_1, \ldots, e_k$ form a maximal tree of $\Gamma$, they form a generating set of $H_\Gamma$; see Corollary 2.3. Meanwhile, $v_1, \ldots, v_k$ form a generating set of $A_{\Gamma'}$. Define a map $\psi : H_{\Gamma} \to A_{\Gamma'}$ by sending $e_i$ to $v_i$ for $i = 1, \ldots, k$. This is a bijection between the generating sets of $H_{\Gamma}$ and $A_{\Gamma'}$. We now argue that $\psi$ preserves relators. Note that the relators of $H_{\Gamma}$ and $A_{\Gamma'}$ are commutators. The generators $e_i$ and $e_j$ commute when they are two edges of the same triangle, that is, when their end points $v_i$ and $v_j$ are connected by an edge. Thus, $v_i, v_j$ commute whenever $e_i, e_j$ commute; the converse is true. Hence, $\psi$ is an isomorphism.

Since $H_{\Gamma} \cong A_{\Gamma'}$ and $\delta_{A_{\Gamma'}}$ is at most quadratic, $\delta_{H_{\Gamma}}$ is at most quadratic. If $\Gamma'$ contains an edge, then $H_{\Gamma} \cong A_{\Gamma'}$ contains $\mathbb{Z} \times \mathbb{Z}$ as a subgroup. Therefore, $H_{\Gamma'}$ cannot be hyperbolic and $\delta_{H_{\Gamma}}$ has to be quadratic. $\square$

We want to point out that there are Bestvina–Brady groups that are not isomorphic to any right-angled Artin groups; see [14]. The following corollary is an immediate consequence of Proposition 3.4.

**Corollary 3.5.** If $\Gamma$ is a fan or wheel, then $\delta_{H_{\Gamma}}$ is quadratic.
Lemma 3.6. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is a 2-dimensional triangulated disk with $\dim_I(\Delta_\Gamma) = 0$. Then $\Gamma$ can be represented as a tree $T$: each vertex of $T$ represents a fan or a wheel; two vertices $v, w$ of $T$ are adjacent if the intersection of the graphs that are represented by $v$ and $w$ is an edge.

Proof. We observe that for any interior vertex of $\Delta_\Gamma$, the induced subgraph on the interior vertex, together with its adjacent vertices, is a wheel. Also, note that if $\Delta_\Gamma$ has two interior vertices, then they are not connected. Otherwise, the edge that connects the two interior vertices would be an interior edge of $\Delta_\Gamma$, which contradicts our assumption. Thus, there are three types of edges of $\Delta_\Gamma$: (1) edges on $\partial \Delta_\Gamma$, (2) edges that connect interior vertices and vertices on $\partial \Delta_\Gamma$, and (3) edges that intersect $\partial \Delta_\Gamma$ at two vertices. Cutting along all the edges of type (3), we obtain connected components of $\Delta_\Gamma$ whose 1-skeletons are wheels and fans (triangles). For each connected component, we assign a vertex to it; two vertices are connected if the corresponding connected components intersect in $\Delta_\Gamma$. Thus, we have the desire decomposition of $\Gamma$ with an underlying graph $T$.

Now we show that $T$ is a tree. Suppose $T$ is not a tree, then $T$ contains a circle $C_n$ of length $n$. The flag complex on the subgraph of $\Gamma$ whose decomposition corresponds to $C_n$ would be a triangulated annulus. Thus, $\Delta_\Gamma$ would not be a triangulated disk, and this contradicts the assumption that $\Delta_\Gamma$ is a triangulated disk. Hence, $T$ is a tree.

Note that the decomposition in Proposition 3.6 is not unique. Since for a fan $F_n+1$, there are $p(n)$ such decompositions, where $p(n)$ is the partition function.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a finite simplicial graph. If $\Gamma_1 \cap \Gamma_2$ is a single edge, then the Bestvina–Brady group $H_\Gamma$ splits over $\mathbb{Z}$ as $H_{\Gamma_1} \ast_\mathbb{Z} H_{\Gamma_2}$ by the Dicks–Leary presentation (see Theorem 2.2). It is not hard to see that Lemma 3.6 implies that $H_\Gamma$ splits over $\mathbb{Z}$, and the vertex groups are CAT(0) groups. We summarize these statements in the following proposition.

Proposition 3.7. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is a triangulated disk with $\dim_I(\Delta_\Gamma) = 0$. Then $H_\Gamma$ has a graph of group decomposition, where the underlying graph is the tree $T$ in Lemma 3.6; the edge groups are $\mathbb{Z}$, and the vertex groups are Bestvina–Brady groups defined by the graphs that are represented by the vertex of $T$ in the Lemma 3.6. Moreover, the vertex groups are right-angled Artin groups, then hence, CAT(0) groups.

To prove our main theorem in this section, we need the following proposition in [8]:

Proposition 3.8. ([8], Chapter II.11, 11.17 Proposition) If each of the groups $G_1$ and $G_2$ is the fundamental group of a non-positively curved compact metric space, then so is $G_1 \ast_\mathbb{Z} G_2$. In particular, if $G_1$ and $G_2$ are CAT(0) groups, so is $G_1 \ast_\mathbb{Z} G_2$.

Proof of Theorem 3.7. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is a 2-dimensional triangulated disk with $\dim_I(\Delta_\Gamma) = 0$. By Proposition 3.7 $H_\Gamma$ has a graph of group decomposition, where the edge groups are $\mathbb{Z}$, and the vertex groups are CAT(0). Repeating Proposition 3.8 gives us that $H_\Gamma$ is a non-hyperbolic CAT(0) group. Hence, $\delta_{H_\Gamma}$ is quadratic.

4. Disks with square boundaries

The goal of this section is to prove the following:

Theorem 4.1. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is a 2-dimensional triangulated disk whose boundary is a square. If $\dim_I(\Delta_\Gamma) = d$ for $d \in \{0, 1, 2\}$, then $\delta_{H_\Gamma}(n) \cong n^{d+2}$.  

When \( d = 0 \), the Theorem 4.1 is a corollary of Theorem 3.1. For \( d = 1 \), we need to establish both the lower bound and the upper bound for \( \delta_{\Gamma} \). For \( d = 2 \), since there is a universal quartic upper bound given by Dison \([12]\), we only need to establish the lower bound for \( \delta_{\Gamma} \).

Certain special cases of Theorem 4.1 for \( d = 0 \) can be recovered by a result of Carter and Forester \([9]\):

**Lemma 4.2.** ([9], Corollary 4.3) If a finite simplicial graph \( \Gamma \) is a join of three graphs \( \Gamma = \Gamma_1 \ast \Gamma_2 \ast \Gamma_3 \), then \( \delta_{\Gamma} \) is quadratic.

**Example 4.3.** Let \( \Gamma \) be \( \mathcal{W}_5 \). Label the vertices as follows:

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d} \\
\text{e}
\end{array}
\]

Let \( \Gamma_1 = \{c\} \), \( \Gamma_2 = \{b, d\} \), and \( \Gamma_3 = \{a, e\} \), then \( \Gamma = \Gamma_1 \ast \Gamma_2 \ast \Gamma_3 \). Therefore, \( \delta_{\Gamma} \) is quadratic by Lemma 4.2.

In subsection 4.1 and subsection 4.2 we establish the lower bound and the upper bound for \( \delta_{\Gamma} \), respectively. Before we proceed with the proof, we prove the following lemma which will be used later on.

**Lemma 4.4.** Let \( \Gamma \) be a finite simplicial graph such that \( \Delta_{\Gamma} \) is a 2-dimensional triangulated disk with square boundary. If \( \dim_1(\Delta_{\Gamma}) = 1 \), then \( \Gamma \) is the suspension of a path of length at least 3.

**Figure 4.1.** The suspension of a path of length at least 3.

*Proof.* Denote \( \Gamma' \) to be the graph whose edge set consists of all the interior edges of \( \Delta_{\Gamma} \). Label the four vertices on \( \partial \Delta_{\Gamma} \) by \( a, b, c, d \) as shown in Figure 4.2. Since \( \Delta_{\Gamma} \) is a triangulated disk, each edge of \( \Gamma' \) together with these two vertices on \( \partial \Delta_{\Gamma} \) form two adjacent triangles. These two vertices on \( \partial \Delta_{\Gamma} \) are not adjacent. Otherwise, there would be a \( K_4 \) subgraph contained in \( \Gamma \); see figure 4.2.
Figure 4.2. An interior edge together with two vertices on $\partial \Delta \Gamma$ form two adjacent triangles. These two vertices on $\partial \Delta \Gamma$ are not adjacent, see the picture on the left. Otherwise, $\Gamma$ would contain a $K_4$, see the picture on the right.

We make two claims on $\Gamma$. The first claim is: $\Gamma'$ has no vertices whose valency is greater than 3. Suppose $\Gamma'$ has a vertex whose valency is greater than 3, that is, $\Gamma'$ contains a tripod. Since each edge of the tripod together with two non-adjacent vertices on $\partial \Delta \Gamma$, say $b, d$, form two adjacent triangles, there are two triangles in $\Delta \Gamma$ share two consecutive edges; see Figure 4.3. Thus, $\Gamma$ is not a simplicial graph, we get a contradiction. Hence, no vertex of $\Gamma'$ has valency greater than 3. This proves the first claim.

Figure 4.3. The picture shows the simplest case: $\Gamma'$ is a tripod, drew in thick lines. There are two edges connecting the central vertex of the tripod and the vertex $d$.

The second claim is that $\Gamma'$ is a connected path. Suppose $\Gamma'$ is not connected, say $\Gamma'$ has $k$ connected components $\Gamma'_1, \cdots, \Gamma'_k$. Since none of vertices of $\Gamma'$ has valency greater than 3, each of $\Gamma'_1, \cdots, \Gamma'_k$ is a connected path. Also, all the pairs of adjacent vertices of $\Gamma'$ are connected either to vertices $a, c$ or vertices $b, d$ simultaneously, say vertices $b, d$: 
In Figure 4.4, label from the left-most connected component to the right-most component of $\Gamma'$ by $\Gamma'_1, \cdots, \Gamma'_k$, and denote $\Gamma_i$ to be the join of $\Gamma'_i$ and $\{b, d\}$. Since $\Delta_\Gamma$ is a triangulated disk, there must be an edge connecting $b$ and $d$ between $\Gamma_i$ and $\Gamma_{i+1}$, $i = 1, \cdots k - 1$:

In Figure 4.5 we see that connecting $b, d$ by an edge creates $K_4$ subgraphs in $\Gamma$. This contradicts our assumption. Thus, the graph $\Gamma'$ is a connected path.

Now, we have shown that $\Gamma'$ is a connected path, and all pairs of adjacent vertices of $\Gamma'$ are connected either to vertices $a, c$ or $b, d$ simultaneously. Again, suppose $b, d$ are connected to all vertices of $\Gamma'$. The two end vertices of $\Gamma'$ have no choices but connected to $a, c$, respectively. Hence, $\Gamma$ is a suspension of $\Gamma'$. This proves the lemma. \qed
Remark 4.5. In [4], Brady gave an example of a cubic Dehn function $\delta_{H_\Gamma}$, where $\Gamma$ is the suspension of a path of length 3. Theorem 4.1 plus Lemma 4.4 gives a family of graphs $\Gamma$ satisfying $\delta_{H_\Gamma}(n) \simeq n^3$. Our family of graphs includes Brady’s example.

4.1. Lower bound. In [1], the authors introduced height-pushing maps to give a lower bound on the higher dimensional Dehn functions of orthoplex groups. The same technique can be used here to obtain the desired lower bound. We give necessary background here; see [1] for more details.

Let $\Gamma$ be a finite simplicial graph. We use the notations $X_\Gamma$, $\tilde{X}_\Gamma$, $Z_\Gamma$, $h$ from subsection 2.2. We consider $\tilde{X}_\Gamma$ as a metric space by putting $l^1$-metric on it. Denote $B_r(x)$ and $S_r(x)$ to be the ball and the sphere in $\tilde{X}_\Gamma$ centered at $x$ with radius $r$.

A subspace $F \subseteq \tilde{X}_\Gamma$ is called a $k$-flat if it is isometric to the Euclidean space $\mathbb{E}^k$.

Theorem 4.6. ([1], Theorem 4.2) There is an $H_\Gamma$-equivariant retraction, called the height-pushing map

$$P : \tilde{X}_\Gamma \setminus \cup_{v \in Z_\Gamma} B_{1/4}(r) \to Z_\Gamma$$

such that when $P$ is restricted to $h^{-1}([-t,t])$, $P$ is a $(ct + c)$-Lipschitz map, where $c$ is a constant which only depends on $\Gamma$.

In the next Lemma, we establish the lower bound for $\delta_{H_\Gamma}$ in Theorem 4.1.

Lemma 4.7. Let $\Gamma$ be a finite simplicial graph such that $\Delta_\Gamma$ is a 2-dimensional triangulated disk with square boundary. If $\dim_I(\Delta_\Gamma) = d$, then $\delta_{H_\Gamma}(n) \gtrsim n^{d+2}$.

We give the idea of the proof of Lemma 4.7. Recall that Bestvina–Brady groups $H_\Gamma$ act geometrically on the zero level set $Z_\Gamma = h^{-1}(0)$. So the filling function of $Z_\Gamma$ is equivalent to $\delta_{H_\Gamma}$. We construct a loop in $Z_\Gamma$ whose length is $\simeq r$, and it bounds a disk in the ambient space $\tilde{X}_\Gamma$. The height-pushing map pushes the disk into $Z_\Gamma$ to fill the loop, and this filling is at least $r^{d+2}$.

Proof of Lemma 4.7. The case $d = 0$ follows from Theorem 3.1. We prove the case when $d = 1$; the case $d = 2$ is similar and appears in [1]. When $d = 1$, recall Lemma 4.4 that $\Gamma$ is the suspension of a path of length at least 3. Label the four vertices on the boundary of $\Delta_\Gamma$ as follows:

![Figure 4.6](image_url) Figure 4.6. The square boundary of $\Delta_\Gamma$ and an interior 1-simplex $\sigma$.
Define bi-infinite geodesic rays $\gamma_i$, $i = 0, 1$, in the 1-skeleton of $\tilde{X}_\Gamma$:

$\gamma_i(0) = e, \quad \gamma_i|_{\mathbb{R}^+} = a_ib_ia_i\cdots, \quad \gamma_i|_{\mathbb{R}^-} = b_ia_i\cdots$.

Define a map $F : \mathbb{Z} \times \mathbb{Z} \to A_\Gamma$ by

$F(x) = \gamma_0(x_0)\gamma_1(x_1), \quad x = (x_0, x_1) \in \mathbb{Z} \times \mathbb{Z}$.

The image of $F$ consists of elements in the non-abelian group $\langle a_0, a_1, b_0, b_1 \rangle$. Let $\overline{F}$ be the non-standard 2-dimensional flat in $\tilde{X}_\Gamma$ such that the set of its vertices is the image of $F$; see figure 4.7. Since $h(x) = h(F(x_0, x_1)) = |x_0| + |x_1|$, $x = (x_0, x_1) \in \mathbb{Z} \times \mathbb{Z}$, the flat $\overline{F}$ is at non-zero height and has a unique vertex $F(0, 0) = \gamma_0(0)\gamma_1(0) = e$ at height 0. We think of $\overline{F}$ as the boundary of a reversed infinite square pyramid with the apex $F(0, 0)$. For each $r > 0$, the intersection $\overline{F} \cap h^{-1}(r)$ is homeomorphic to $S^1$, which is a loop at height $r$. Note that this loop $\overline{F} \cap h^{-1}(r)$ bounds a 2-disk $\overline{F} \cap h^{-1}([0, r])$ in $\overline{F}$.

\[ F(0, 0) = e \]

Figure 4.7. The non-standard 2-dimensional flat $\overline{F}$. The parallelogram drew in the thick lines is the intersection $\overline{F} \cap h^{-1}(r)$.

Using the group action $A_\Gamma$ on $\tilde{X}_\Gamma$, we translate the 2-disk $\overline{F} \cap h^{-1}([0, r])$ to $(a_0)^{-r}\overline{F} \cap h^{-1}([-r, 0])$. We denote the new 2-disk by $D_r$ and its interior by $\tilde{D}_r$. The loop $\overline{F} \cap h^{-1}(r)$ at height $r$ is translated to a loop $S_r$ in $Z_\Gamma$ that is also homeomorphic to $S^1$:

$S_r := [(a_0)^{-r}\overline{F}] \cap h^{-1}(0) = [(a_0)^{-r}\overline{F}] \cap Z_\Gamma$.

Notice that after performing this translation, the unique vertex $(a_0)^{-r}F(0, 0)$ is at height $-r$. Now we have that $S_r$ is a loop in $Z_\Gamma$ and it bounds a 2-disk $D_r$. The next step is to use the height-pushing map $P$ to push this filling $D_r$ of $S_r$ in $\tilde{X}_\Gamma$ to fill $S_r$ in $Z_\Gamma$.

In order to use the height-pushing map, we need to do some surgery on $D_r$. At each vertex $v \in \tilde{D}_r$, replace $B_{1/4}(v)$ by a copy of $\Delta_\Gamma$. We now show that after applying the height-pushing map to the surgical $\tilde{D}_r$, the scaled copies of 1-simplices in the interior of
\[ \Delta \] at each vertex of the surgical \( \tilde{D} \) do not intersect much in \( Z \). We first consider two different 1-simplices \( \sigma_1 \) and \( \sigma_2 \) in the interior of \( \Delta \) at a vertex \( v \) in \( \tilde{D} \). Since \( \sigma_1 \) and \( \sigma_2 \) either are disjoint or intersect at a 0-simplex, the images \( P(\sigma_1) \) and \( P(\sigma_2) \) intersect at most at a 0-simplex. We next consider 1-simplices in the interior of \( \Delta \) at different vertices \( v_1, v_2 \) in \( \tilde{D} \). Let \( \sigma = [x, y] \) be an 1-simplex in the interior of \( \Delta \); see Figure 4.6. Let \( v'_1 \) and \( v'_2 \) be vertices of the scaled copies \( P(\sigma) \) in \( Z \), based at vertices \( v_1 \) and \( v_2 \) in \( \tilde{D} \), respectively. Then we have

\[
v'_1 = (a_0)^{-r}v_1 x^{s_1} y^{r_1} \quad \text{and} \quad v'_2 = (a_0)^{-r}v_2 x^{s_2} y^{r_2}
\]

for some \( s_1, t_1, s_2, t_2 \in \mathbb{Z}_{\geq 0} \). If \( v'_1 = v'_2 \), that is, the intersection of scaled copies \( P(\sigma) \) based at \( v_1 \) and \( v_2 \) is non-empty, then we have \( v_1 = v_2 \) since \( \{a_0, a_1, b_0, b_1\} \cap \langle x, y \rangle = \{0\} \). Thus, no vertices of different scaled copies of \( \sigma \) are the same. That is, scaled copies of \( \sigma \) in \( Z \) are disjoint. Each interior 1-simplex in \( \Delta \) is an intersection of two 2-simplices in \( \Delta \), so the image of the surgical \( \tilde{D} \) under \( P \) has the area \( 2 \cdot | \{\text{number of 1-simplices}\} | \). Since we only need a lower bound, we don’t need to count all the 1-simplices. For each vertex in \( \tilde{D} \), we take one 1-simplex \( \sigma \) in \( \Delta \) based at that vertex. As we have seen that the scaled copies \( P(\sigma) \) don’t cancel out. So the filling of \( S_r \) in \( Z \) will be at least the sum of the length of these scaled copies \( P(\sigma) \). For each interior 1-simplex \( \sigma \) in \( \Delta \) based at a vertex \( v \) of \( \tilde{D} \), it follows by Theorem 4.6 that the length of the scaled copied \( P(\sigma) \) in \( Z \) grows linearly in terms of \( r \), that is, the length of \( P(\sigma) \) depends on how far is \( \sigma \) pushed into \( Z \) by the height-pushing map. Thus, the further the \( \sigma \) is from \( Z \), the longer the length of \( P(\sigma) \). At height \(-r \), there is only one vertex in \( \tilde{D} \), namely, the apex \( (a_0)^{-r}F(0, 0) \). The image of the 1-simplex \( \sigma \) based at the apex under \( P \) is \( cr + c \), by Theorem 4.6. At each height \( i \) \( = -(r - 1), \cdots, -1 \), there are \( 4i \) vertices and the image of 1-simplex \( \sigma \) under \( P \) has length \( c(r - i) + c \). We have

\[
\text{Area}(S_r) \geq (cr + c) + \sum_{i=1}^{r-1} 4i[c(r - i) + c] \simeq r^3.
\]

Thus, the filling function of \( Z \) is at least cubic. Hence, \( \delta_{H_r} \) is at least cubic. This proves the lemma.

4.2. Upper bound. As we mentioned at the beginning of this section, we only need to establish the cubic upper bound for \( d = 1 \) in Theorem 4.1. The proof relies on analyzing the van Kampen diagram of an arbitrary word \( w \in H_r \) that represents the identity. The tool that we will be using is corridor schemes. Here we only give the definition; more details can be found in [3]. Let \( \Gamma \) be a finite simplicial graph with labelings on the edges such that \( \Delta \) is a 2-dimensional triangulated disk. A corridor scheme for \( \Gamma \) is a collection \( \sigma \) of labeled edges of \( \Gamma \) such that every triangle of \( \Delta \) has either zero or two edges in \( \sigma \). Given a van Kampen diagram \( \Delta \), a \( \sigma \)-corridor is a corridor in \( \Delta \) that consists of triangles; each triangle has exactly two edges in \( \sigma \), and every pair of adjacent triangles intersect at an edge in \( \sigma \). If \( \alpha \) and \( \beta \) are two disjoint corridor schemes, then \( \alpha \)-corridors and \( \beta \)-corridors never cross each other. If \( \Gamma \) is oriented, then each triangle in a diagram \( \Delta \) is orientated.

Recalling the fact we only have to establish the cubic upper bound for \( d = 1 \) in Theorem 3.1 and Lemma 1.4, the graph \( \Gamma \) in the rest of this subsection will be the suspension of a path of length at least 3 with orientations and labels on the edges given by Figure 4.8.
The Dicks–Leary presentation (see Theorem 2.2) for $H_\Gamma$ is $\langle A, B, X | R \rangle$, where

$$A = \{a_1, \ldots, a_{k+1} \}$$

$$B = \{b_1, \ldots, b_{k+1} \}$$

$$X = \{x_1, \ldots, x_k \}$$

$$R = \{a_i x_i = a_{i+1}, b_i x_i = b_{i+1} = x_i b_i, \ i = 1, \ldots, k \}$$

Let $w$ be a freely reduced word of length at most $n$ that represents the identity in $H_\Gamma$ and $\Delta$ a minimal van Kampen diagram for $w$. Choose two disjoint corridor schemes $\alpha = \{a_1, \ldots, a_{k+1} \}$ and $\beta = \{b_1, \ldots, b_{k+1} \}$ for $\Gamma$. Note that $\alpha$-corridors and $\beta$-corridors never cross each other. For each corridor in $\Delta$, fix a vertex $p$ of the corridor that is on $\partial \Delta$. When reading along the boundary of a corridor starting from $p$, we get a boundary word $u'v''u''v'$ or its cyclic permutation, where $u', v''$ are letters in $\alpha$ or $\beta$; $v', v''$ are words in the free group $F(x_1, \ldots, x_k)$.

Remark 4.8. Since $\Gamma$ is an oriented simplicial graph, all the corridors and triangles of a diagram $\Delta$ are oriented. For the pictures in the rest of this subsection, we omit the arrows on some of the edges for simplicity. Since each edge is labeled with a letter, and consecutive edges form a path, we don’t distinguish letters and edges, and words and paths.

Definition 4.9. Let $C_i, 1 \leq i \leq h$, be a corridor in $\Delta$ with the boundary word $u'_i v''_i u''_i v'_i$ (or its cyclic permutation) where $u'_i, v''_i$ are letters in $\alpha$ or $\beta$; $v'_i, v''_i$ are words in the free group $F(x_1, \ldots, x_k)$; see Figure 4.9. If $u'_1, \ldots, u'_h$ and $u''_1, \ldots, u''_h$ are two sets of consecutive letters on $\partial \Delta$ and $v''_i = v''_{i+1}$ for $i = 1, \ldots, h - 1$, then the subdiagram $T$ of $\Delta$ obtained by gluing corridors $C_1, \ldots, C_h$ along the words $v''_1 = v''_2, \ldots, v''_{h-1} = v''_h$ is called a stack. The shorter word of the words $v''_1$ and $v''_h$ is called the top of $T$; the longer word of the words $v'_1$ and $v'_h$ is
called the bottom of $T$. The words $u'_1 \cdots u'_h$ and $u''_1 \cdots u''_h$ are called the legs of $T$, and the number $h$ is called the height of $T$.

Roughly speaking, a stack is a pile of corridors where one corridor sits on top of another. The shape of a stack looks like an isosceles trapezoid, whose legs are parts of $\partial \Delta$. The height of a stack is the length of its legs. The bases of a stack, namely, the top and the bottom, are not parts of $\partial \Delta$. The top of a stack is the shorter base, and the bottom of a stack is the longer base.

![Figure 4.10. A simple example of a stack.](image)

**Definition 4.10.** Let $C$ be either a single $\alpha$-corridor or $\beta$-corridor with boundary word $u'v''u'v'$ (or its cyclic permutation), as shown in Figure 4.9. We say that a vertex $q$ on the boundary of $C$ labeled by the word $v'$ (respectively $v''$) is a $j$-vertex, or $q$ has type $j$, if there are $j+1$ edges connecting $q$ to $j+1$ distinct consecutive vertices on the part of boundary of $C$ labeled by the word $v''$ (respectively $v'$).

![Figure 4.11. Picture of a $j$-vertex. The number $j$ around the vertex indicates that the vertex is a $j$-vertex.](image)

We say that a pair of adjacent vertices $q_1$ and $q_2$ on the boundary of $C$ labeled by the word $v'$ (respectively $v''$) generate a vertex $q_3$ in $v''$ (respectively $v'$) if there is a triangle shown as follows:

![Figure 4.12. A pair of adjacent vertices $p_1$ and $p_2$ generates $w$.](image)
**Lemma 4.11.** Let $\Gamma$ be the graph as shown in Figure 4.8. Let $w \in H_\Gamma$ be a freely reduced word that represents the identity. Let $T$ be a stack in a minimal van Kampen diagram $\Delta$ of $w$ such that the top of $T$ has length $l$ and the height of $T$ is $h$. Then the area of $T$ satisfies

$$\text{Area}(T) \leq C(lh^2 + h^3),$$

where $C$ is a positive constant that does not depend on $l$ and $h$.

The proof of Lemma 4.11 relies on a series of lemmas that analyze the boundaries of corridors of a stack. All the van Kampen diagrams are assumed to be minimal. This implies that for a fixed $k \geq 3$ in Lemma 4.11, each vertex on the boundary of a corridor is of type at most $k$. We label the boundaries boundary words of corridors of a stack $T$ as shown in Figure 4.13:

![Figure 4.13. Boundary words of corridors in a stack $T$.](image)

Let’s look at a single corridor $C_i$ as shown in Figure 4.14:

![Figure 4.14. A single corridor with boundary words.](image)

Each edge $x_m$ of $t_i$, $1 \leq m \leq k$, is part of the boundary of a triangle in the interior of $C_i$: 

![Figure 4.15.](image)
Depending on $C_i$ is an $\alpha$-corridor or $\beta$-corridor, the orientation of $x_m$, and the orientation of $u'_i, u''_i$ (they have the same orientation), we have the following eight possibilities:

![Figures 4.16 and 4.17](image)

**Figure 4.16.** The first row are the cases where the arrows on $u'_i$ and $u''_i$ are pointing from $t_i$ to $t_{i+1}$; the second row are the cases where the arrows on $u'_i$ and $u''_i$ are pointing from $t_{i+1}$ to $t_i$. The left two columns are the cases where $C_i$ is an $\alpha$-corridor; the right two columns are the cases where $C_i$ is a $\beta$-corridor.

Note that each of the boundaries of the triangles in Figure 4.16 is a relator of $H_T$.

**Lemma 4.12.** Let $C_i$ be either an $\alpha$-corridor or $\beta$-corridor in a stack $T$ as shown in Figure 4.13 and Figure 4.14.

(1) The area of $C_i$ is $|t_i| + |t_{i+1}|$.

(2) Suppose $|t_i| < |t_{i+1}|$, then

$$|t_{i+1}| = \sum_j j \cdot |\{j\text{-vertices on } t_i\}|.$$

**Proof.**

(1) Since each edge on $t_i$ and $t_{i+1}$ is part of a unique triangle in $C_i$, the result follows.

(2) Since each $j$-vertex on $t_i$ contributes $j$ edges on $t_{i+1}$, the statement follows immediately.

**Lemma 4.13.** Let $C_i$ be either an $\alpha$-corridor or $\beta$-corridor in a stack $T$ as shown in Figure 4.13 and Figure 4.14. There are eight combinations of edges on $t_i$ (respectively $t_{i+1}$) that will create 1-vertices on the $t_i$ (respectively $t_{i+1}$) for a given $m$, $2 \leq m \leq k - 1$, as shown in Figure 4.17.

![Figures 4.17](image)

**Figure 4.17.** Possible combinations of edges that create 1-vertices.

**Proof.** We prove the case when $C_i$ is an $\alpha$-corridor; the case for a single $\beta$-corridor is similar. To see the statement, we only need to know how to fill $C_i$. Take the left-most picture in the first row of Figure 4.17. Suppose the $x_m$'s are edges on either $t_i$ or $t_{i+1}$. By Figure 4.16 we have two choice to complete the triangles based on the $x_m$'s:
Thus, the vertex is a 1-vertex. The arguments for other combinations are similar. □

**Lemma 4.14.** Let $C_i$ and $C_{i+1}$ be two consecutive corridors in a stack $T$. Suppose that the arrows on the edges $u'_i, u''_i, u'_{i+1}, u''_{i+1}$ have the same orientation, as shown in Figure 4.19. Assume that there are no $0$-vertices on $t_i$. Then

1. All the vertices on $t_{i+1}$ are 1-vertices, 2-vertices, or 3-vertices, except possibly the two vertices at the ends of $t_{i+1}$.
2. All the vertices on $t_{i+2}$ are either 1-vertices or 2-vertices, except possibly the two vertices at the ends of $t_{i+1}$.
3. We have $|t_i| \leq |t_{i+1}| \leq |t_{i+2}|$.

**Proof.** We prove the lemma for the case when $C_i, C_{i+1}$ are consecutive $\alpha$-corridors, and the arrows on $u'_i, u''_i, u'_{i+1}, u''_{i+1}$ are pointing away from $t_i$. Other cases are similar.

1. We claim that every pair of adjacent vertices on $t_i$ generates either a 1-vertex or 3-vertex on $t_{i+1}$. Let two adjacent vertices on $t_i$ be connected by an edge $x_m$. These two adjacent vertices generate a vertex on $t_{i+1}$; Figure 4.20 lists all possible combinations of edges that meet at the vertex on $t_{i+1}$ that is generated by a pair of adjacent vertices on $t_i$. 

![Figure 4.18](image)

![Figure 4.19](image)

![Figure 4.20](image)
The 1-vertices in Figure 4.20 are recognized by Lemma 4.13. The following picture shows that the vertex in Figure 4.20 is a 3-vertex:

![Diagram showing a 3-vertex](image)

**Figure 4.21.**

There are four more cases obtained by flipping the pictures in Figure 4.20 such that the arrow on the edge $x_m$ of $t_i$ has the opposite orientation. This proves the claim.

Next, we show that the vertices on $t_{i+1}$ that are not generated by pairs of adjacent vertices on $t_i$ are 2-vertices, except the two vertices at the end of $t_{i+1}$. We claim that each $j$-vertex on $t_i$ creates $j - 1$ vertices on $t_{i+1}$ that are all 2-vertices. For any $j$-vertex on $t_i$ (respectively $t_{i+1}$), by definition, there are $j + 1$ edges connecting the $j$-vertex to $j + 1$ consecutive distinct vertices on $t_{i+1}$ (respectively $t_i$). That is, there are $j$ distinct triangles based on $j$ consecutive edges on $t_{i+1}$ (respectively $t_i$) that has the $j$-vertex as the common vertex; see Figure 4.22.

![Diagram showing that each j-vertex creates j-1 2-vertices](image)

**Figure 4.22.** Each $j$-vertex on $t_i$ (respectively $t_{i+1}$) creates $j - 1$ vertices on $t_{i+1}$ (respectively $t_i$) that are all 2-vertices.

Let $x_{r-1}, x_r$ be two consecutive edges in $x_m, \ldots, x_{m+j-1}$, and let the vertex $v_r$ be the intersection of $x_{r-1}$ and $x_r$. Figure 4.23 shows that $v_r$ is a 2-vertex. This proves the claim.
We have shown that all the vertices on $t_{i+1}$ are 1-vertices, 2-vertices, or 3-vertices. Each of the vertices on $t_{i+1}$ creates numbers of 0, 1, or 2 vertices on $t_{i+2}$, respectively, and they are all 2-vertices. Other vertices on $t_{i+2}$ are generated by pairs of adjacent vertices on $t_{i+1}$. We now show that these vertices on $t_{i+2}$ are 1-vertices. We prove the claim by showing all the possible combinations of types of vertices on $t_{i+1}$. We have six cases: pairs of adjacent 1-vertices, adjacent 2-vertices, and adjacent 3-vertices; pairs of adjacent 1-vertex and 2-vertex; pairs of adjacent 1-vertex and 3-vertex; and pairs of adjacent 2-vertex and 3-vertex. In the following pictures, all the 1-vertices are recognized by Lemma 4.13.

Figure 4.24 shows that a pair of adjacent 1-vertices on $t_{i+1}$ generates a 1-vertex on $t_{i+2}$.

Next, Figure 4.25 shows that a pair of adjacent 1-vertex and 2-vertex on $t_{i+1}$ generates a 1-vertex on $t_{i+2}$. Recall that Figure 4.20 and Figure 4.23 are the only situations of 1-vertex and 2-vertex on $t_{i+1}$.
Figure 4.25. A pair of adjacent 1-vertex and 2-vertex on \( t_{i+1} \) generates a 1-vertex on \( t_{i+2} \).

For a pair of adjacent 2-vertices on \( t_{i+1} \), we have one combination up to orientation:

Figure 4.26. A pair of adjacent 2-vertices on \( t_{i+1} \) generates a 1-vertex on \( t_{i+2} \).

For a pair of adjacent 1-vertex and 3-vertex on \( t_{i+1} \), we have:

Figure 4.27. A pair of adjacent 1-vertex and 3-vertex on \( t_{i+1} \) generates a 1-vertex on \( t_{i+2} \).

For a pair of adjacent 2-vertex and 3-vertex on \( t_{i+1} \), since the number of combinations of edges on \( t_{i+1} \) that will create 2-vertices and 3-vertices are limited, we have fewer cases here:

Figure 4.28. A pair of adjacent 2-vertex and 3-vertex on \( t_{i+1} \) generates a 1-vertex on \( t_{i+2} \).
Finally, for a pair of adjacent 3-vertices, we have only one possibility up to orientation:

![Figure 4.29](image)

**Figure 4.29.** A pair of adjacent 3-vertex on \( t_{i+1} \) generates a 1-vertex on \( t_{i+2} \).

(3) Since we assume that there are no 0-vertices on \( t_i \), the number of vertices on \( t_{i+1} \) is no less than the number of vertices on \( t_i \); the number of vertices on \( t_{i+2} \) is also no less than the number of vertices on \( t_{i+1} \). Thus, we have

\[
|t_i| = (\text{number of vertices on } t_i) - 1 \\
\leq (\text{number of vertices on } t_{i+1}) - 1 \\
= |t_{i+1}|
\]

and

\[
|t_{i+1}| = (\text{number of vertices on } t_{i+1}) - 1 \\
\leq (\text{number of vertices on } t_{i+2}) - 1 \\
= |t_{i+2}|.
\]

Hence, \(|t_i| \leq |t_{i+1}| \leq |t_{i+2}|.|\)

**Lemma 4.15.** Let \( C_i \) and \( C_{i+1} \) be two consecutive corridors in a stack \( T \) as in the Lemma 4.14. Let the arrows on the edges labeled by \( u_i', u_i'' \) and \( u_{i+1}', u_{i+1}'' \) have different orientations. There are two cases, as shown in Figure 4.30. Assume that there are no 0-vertices on \( t_i \). Then \(|t_{i+2}'| \leq |t_{i+2}|\), where \( t_{i+2} \) is in Figure 4.19.

![Figure 4.30](image)

**Figure 4.30.** Edges \( u_i', u_i'' \) and \( u_{i+1}', u_{i+1}'' \) have different orientations.

**Proof.** We prove the case when \( u_i', u_i'', u_{i+1}', u_{i+1}'' \) are pointing toward \( t_{i+1} \). The other case is similar. We claim that in this case, \( C_i \) and \( C_{i+1} \) cannot be both \( \alpha \)-corridors or \( \beta \)-corridors.
Suppose $C_i$ and $C_{i+1}$ are both $\alpha$-corridors (respectively $\beta$-corridors), then there are two triangles that share a unique common edge on $t_{i+1}$:

![Figure 4.31](image1)

**Figure 4.31.** Two triangles from different $\alpha$-corridors with opposite orientations.

The situation as shown in Figure 4.31 contradicts the fact that $T$ is part of a minimal van Kampen diagram since we can obtain a smaller van Kampen diagram by canceling these two triangles. This proves the claim.

Now suppose $C_i$ is an $\alpha$-corridor and $C_{i+1}$ is a $\beta$-corridor. For each $j$-vertex on $t_i$, by definition, it connects to $j + 1$ distinct vertices on $t_{i+1}$:

![Figure 4.32](image2)

**Figure 4.32.** Every pair of adjacent vertices of the distinct $j + 1$ vertices on $t_{i+1}$ generates the same vertex on $t'_{i+2}$.

From Figure 4.32 we see that every pair of adjacent vertices of these $j + 1$ vertices on $t_{i+1}$ generates the same vertex on $t'_{i+2}$. That is, each $j$-vertex on $t_i$ creates $j - 1$ vertices on $t_{i+1}$ and they are 0-vertices. Since there are no 0-vertices on $t_{i+1}$ by Lemma 4.14, the number of vertices on $t'_{i+2}$ is less than the numbers of vertices on $t_{i+2}$ in Lemma 4.14. Thus, we have

$$|t'_{i+2}| = (\text{number of vertices on } t'_{i+2}) - 1 \leq (\text{number of vertices on } t_{i+2}) - 1 = |t_{i+2}|.$$

This completes the proof. □

Now we prove Lemma 4.11.

**Proof of Lemma 4.11.** Let $T$ be a stack with labelings as shown in Figure 4.13. Recall that $t_0, \ldots, t_h$ are words in $F(x_1, \ldots, x_k)$ and $u'_i, u''_i$ are both letters either in the corridor scheme
\( \alpha = \{a_1, \cdots, a_{k+1}\} \) or \( \beta = \{b_1, \cdots, b_{k+1}\} \). Since the area of a stack \( T \) is the sum of the areas of the corridors that contained in \( T \), by Lemma 4.12 we have:

\[
\text{Area}(T) = |t_0| + 2|t_1| + \cdots + 2|t_{h-1}| + |t_h| \leq 2 (|t_0| + \cdots + |t_h|).
\]

Thus, to have the largest possible area of \( T \), we need the longest possible length of the word \( t_i \) for each \( i = 1, \cdots, h \). In order to have the longest possible length of \( t_i \), we need a few assumptions. From Lemma 4.12 we know that the length of \( t_i \) depends on the types of vertices on \( t_{i-1} \). Note that for a fix \( k \), each vertex of \( T \) is at most a \( k \)-vertex. We assume that the vertices on \( t_0 \) and the two vertices at the ends of \( t_i \) are \( k \)-vertices. This assumption gives the longest possible length of the word \( t_i \). By Lemma 4.13 we assume that the arrows on \( u'_i, u''_i \) are pointing away from \( t_0 \). We compute \( |t_i| \) as follows.

For \( |t_1| \), since \( |t_0| = l \) and all the vertices on \( t_0 \) are \( k \)-vertex, we have \( |t_1| = k(l+1) \). On \( t_1 \), there are two \( k \)-vertices at the two ends. Lemma 4.14 tells us that every pair of adjacent vertices on \( t_0 \) generates either a 1-vertex or a 3-vertex on \( t_1 \), but we may assume that they are all 3-vertices so that we can get the largest possible \( |t_2| \). So the number of 3-vertices on \( t_1 \) is \( l \); other vertices on \( t_1 \) are 2-vertices and there are \( (k-1)(l+1) \) of them. Knowing the types of vertices on \( t_1 \) gives the length of \( t_2 \):

\[
|t_2| = 2 \cdot k + l \cdot 3 + (k-1)(l+1) \cdot 2 = 2kl + l + 4k - 2.
\]

For \( i \geq 2 \), every pair of adjacent vertices on \( t_{i-1} \) generates a 1-vertex on \( t_i \) by Lemma 4.14 so the number of 1-vertices on \( t_i \) is \( |t_{i-1}| \). Every 2-vertex on \( t_{i-1} \) creates a 2-vertex on \( t_i \) and the two \( k \)-vertices at the ends of \( t_{i-1} \) creates \( (k-1)2 \)-vertices on \( t_i \). So the number of 2-vertices on \( t_i \) is the number of 2-vertices on \( t_{i-1} \) plus \( 2(k-1) \):

\[
(kl + l + k - 1) + (i - 1) \cdot 2(k-1) = kl + l + (2i - 1)k - 2i + 1.
\]

Having the information of the vertices on \( t_i \) we get

\[
|t_{i+1}| = |t_{i-1}| \cdot 1 + [kl + l + (2i - 1)k - 2i + 1] \cdot 2 + 2 \cdot k
\]

\[
= |t_{i-1}| + 2kl + 2l + 4ik - 4i + 2
\]

for \( i = 2, \cdots, h - 1 \). Let

\[
d(i) = |t_{i+1}| - |t_{i-1}| = 2kl + 2l + 4ik - 4i + 2, \quad i = 2, \cdots, h - 1,
\]

then \( \{d(i)\} \) is an arithmetic sequence whose difference is \( 4k - 4 \) and

\[
d(i + 2) - d(i) = 8k - 8.
\]

When \( i \) is even, we have

\[
|t_{i+1}| = |t_1| + d(2) + d(4) + \cdots + d(i)
\]

\[
= |t_1| + \left[ \frac{i}{2} \cdot \frac{d(2) + d(i)}{2} \right]
\]

\[
= |t_1| + \frac{i}{4} \left[ 2d(2) + \left( \frac{i}{2} - 1 \right) (8k - 8) \right]
\]

\[
= |t_1| + \frac{i}{2} d(2) + i(i - 2)(k - 1).
\]

23
When $i$ is odd, we have

\[
|t_{i+1}| = |t_2| + d(3) + d(5) + \cdots + d(i) \\
= |t_2| + \left[ \frac{i-1}{2} \cdot \frac{d(3) + d(i)}{2} \right] \\
= |t_2| + \frac{(i-1)}{4} \left[ 2d(3) + \left( \frac{i-3}{2} \right) (8k-8) \right] \\
= |t_2| + \left( \frac{i-1}{2} \right) d(3) + (i-1)(i-3)(k-1).
\]

When $h$ is odd, we have

\[
\text{Area}(T) \leq 2 \sum_{i=1}^{h} |t_i| = 2 \left( |t_0| + |t_1| + |t_2| + \sum_{i=2, \text{i is even}}^{h-1} |t_{i+1}| + \sum_{i=3, \text{i is odd}}^{h-2} |t_{i+1}| \right).
\]

Furthermore,

\[
\sum_{i=2, \text{i is even}}^{h-1} |t_{i+1}| = \sum_{i=2, \text{i is even}}^{h-1} \left[ |t_1| + \frac{i}{2} \left( \frac{d(3)}{2} + i(i-2)(k-1) \right) \right] \\
= \sum_{i=2, \text{i is even}}^{h-1} |t_1| + \left[ \frac{d(3)}{2} - 2(k-1) \right] \sum_{i=2, \text{i is even}}^{h-1} i + (k-1) \sum_{i=2, \text{i is even}}^{h-1} i^2 \\
= \left( \frac{h-1}{2} \right) |t_1| + \left[ \frac{d(3)}{2} - 2(k-1) \right] \left( \frac{h^2 - 1}{4} \right) + (k-1) \frac{h(h^2 - 1)}{24} \\
\leq h|t_1| + [d(2) - 2(k-1)] h^2 + (k-1)h^3 \\
= hk(l+1) + (2kl + 2l + 6k - 4)h^2 + (k-1)h^3 \\
\leq hk(l+1) + (2kl + 2l + 6k)h^2 + kh^3 \\
\leq 2kh + 10kh^2 + kh^3 \\
\leq 12kh^2 + kh^3.
\]
Thus, $h$ permutation of the following word, and we also denoted by $\partial T$
where each of the words $A$, $U$

Let $\alpha$-corridors and the corridors form stacks whose heights are greater than 1. Recall

Denote the stacks by $T$, $U$ = \ldots, $U$, $A$

$T_{i-1} = \sum_{i=3}^{h-2} \left| t_{i+1} \right| = \sum_{i=3}^{h-2} \left[ t_{2} + \left( i - \frac{1}{2} \right) d(3) + (i - 1)(i - 3)(k - 1) \right]

= \frac{h - 3}{2} \left| t_{2} \right| + \frac{d(3)}{2} \sum_{i=3}^{h-2} (i - 1) + (k - 1) \sum_{i=3}^{h-2} (i^2 - 4i + 3)

= \frac{h - 3}{2} \left| t_{2} \right| + \frac{d(3)}{8} (h - 3) (h - 1)

+ (k - 1) \left[ \frac{1}{24} (h - 3) (h - 2) (h - 1) - (h - 3) (h + 1) + \frac{3}{2} (h - 3) \right]

\leq h \left| t_{2} \right| + d(3) h^2 + k h^3 + k h^2 + 2h

= h(2kl + l + 4k - 2) + (2kl + 2l + 12k - 10)h^2 + k h^3 + k h^2 + 2h

= (2kl + l + 4k)h + (2kl + 2l + 13k - 10)h^2 + k h^3

\leq 7klh^2 + 17klh^2 + k h^3

= 24klh^2 + k h^3.

Thus,

\text{Area}(T) \leq 2 \left[ \left| t_{0} \right| + \left| t_{1} \right| + \left| t_{2} \right| + (12klh^2 + k h^3) + (24klh^2 + k h^3) \right]

= 2 \left[ l + k(l + 1) + 2kl + l + 4k - 2 + 36klh^2 + 2k h^3 \right]

\leq (6kl + 4l + 10k) + 72klh^2 + 4k h^3

\leq 92k(lh^2 + h^3).

When $h$ is even, the computation is similar. Hence, $\text{Area}(T) \leq C(h^3 + lh^2)$, where $C$ is a positive constant which does not depend on $l$ and $h$.

\textbf{Lemma 4.16}. Let $\Gamma$ be the graph as shown in Figure 4.8. Then $\delta_{H_r}(n) \leq n^3$.

\textbf{Proof}. Let $w$ be a freely reduced word of length at most $n$ that represents the identity in $H_r$. Let $\Delta$ be a minimal van Kampen diagram for $w$. Choose corridor schemes $\alpha = \{a_1, \ldots, a_{k+1}\}$ and $\beta = \{b_1, \ldots, b_{k+1}\}$ for $\Gamma$. The van Kampen diagram $\Delta$ is cut up by $\alpha$-corridors and $\beta$-corridors and some of the corridors form stacks whose heights are greater than 1. Recall that a single corridor is a stack of height 1. Thus, the diagram $\Delta$ is cut up by stacks.

Denote the stacks by $T_1, \ldots, T_m$ that cut up $\Delta$; denote the two legs of $T_j$ by $U_j'$ and $U_j''$, and the height of $T_j$ by $\left| U_j' \right| = \left| U_j'' \right| = h_j$, $j = 1, \ldots, m$. Note that $U_j'$ and $U_j''$ are words on $\partial \Delta$ that consist of letters in the corridor schemes $\alpha$ and $\beta$. The boundary word $w$ is a cyclic permutation of the following word, and we also denoted by $w$:

\[ w = A_1 B_1 A_2 B_2 \cdots A_r B_s, \]

where each of the words $A_1, \ldots, A_r$ consists of one or more words from $U_1', \ldots, U_m'$ and $U_1'', \ldots, U_m''$; each of the words $B_1, \ldots, B_s$ is a word in the free group $F(x_1, \ldots, x_k)$. The length of each of the words $A_1, \ldots, A_r$ is either the height of a single stack or the sum of
the heights of multiple stacks. Each of the words \( B_1, \ldots, B_s \) is a base of a stack or part of a base of a stack. Denote the length of the words \( B_1, \ldots, B_s \) by \( l_1, \ldots, l_s \). Let

\[
h = |A_1| + \cdots + |A_r| = 2(h_1 + \cdots + h_m) = 2 \sum_{i=1}^{m} h_i
\]

and

\[
l = |B_1| + \cdots + |B_s| = \sum_{i=1}^{s} l_i.
\]

Consider a stack \( T' \) whose top is the word \( B_1 \cdots B_s \) of length \( l \) and whose legs are \( U'_1 \cdots U'_m \) and \( U''_1 \cdots U''_m \) and the height \( h = |U'_1 \cdots U'_m| = |U''_1 \cdots U''_m| \), assuming that all the arrows on \( U'_i, U''_i \) are pointing away from the top. We claim that the area of \( \Delta \) is less than the area of the stack \( T' \).

![Figure 4.33](image-url)

**Figure 4.33.** The stack \( T' \) with labels on the top and the legs.

There are three possible cases that the van Kampen diagram \( \Delta \) could be. The first case is that every stack has a base that is part of \( \partial \Delta \), as shown in Figure [4.34]. The second case is that every stack has a base that is either on \( \partial \Delta \), or part of it is on \( \partial \Delta \), as shown in Figure [4.35]. The third case is that there is a stack whose bases are not part of \( \partial \Delta \), as shown in Figure [4.36].
Figure 4.34. Case 1: Every stack $T_i$ has a base $B_i$ that is part of $\partial \Delta$.

Figure 4.35. Case 2: At least one of the stacks such that part of its bases is part of $\partial \Delta$. 

27
Case 1. Recall that each of $A_i$ is a leg of one or more stacks; each $B_i$ is part of a base of a stack or a base of a stack. By assumption, each stack $T_i$ has a base that is on $\partial\Delta$, say $B_i$ is a base of the stack $T_i$. If $B_i$ is the top of the stack $T_i$, then the top of the stack $T_i'$ in Figure 4.37 is longer than $B_i$. Thus, Area($T_i$) \leq Area($T_i'$). If $B_i$ is the bottom of the stack $T_i$, then consider a stack whose heights are $U_i, V_i$ and whose top is $B_i$. The area of this stack is obviously greater than the area of $T_i$, but less than the area of $T_i'$ in Figure 4.37. Thus, for each stack $T_i, j = 1, \ldots, m$, there is a stack $T_j'$ in $T'$ satisfying Area($T_j$) \leq Area($T_j'$). These substacks $T_1', \ldots, T_m'$ are disjoint inside $T'$ because their legs are disjoint. Each substack $T_j'$ is at different height inside $T'$, as shown in Figure 4.37.
We have

\[
\text{Area}(w) = \text{Area}(\Delta) \\
= \text{Area}(T_1) + \cdots + \text{Area}(T_m) \\
\leq \text{Area}(T'_1) + \cdots + \text{Area}(T'_m) \\
\leq \text{Area}(T') \\
\leq C(h^3 + lh^2) \\
\leq 2Cn^3
\]

for some positive constant \(C\), which does not depend on \(|w| = n\). The second last inequality follows by Lemma 4.11, and the last inequality holds since \(l \leq n\) and \(h \leq n\). Thus, we prove the claim for the first case.

**Case 2.** Suppose that the van Kampen diagram \(\Delta\) has some stacks whose bases are not all on \(\partial \Delta\), as shown in Figure 4.35. Divide those stacks into smaller stacks as following:

In Figure 4.38, each of the stacks \(T_i\) has at least one base \(\overline{B}_i\) that is on \(\partial \Delta\). Thus, Case 2 follows by Case 1.

**Case 3.** Suppose that there is a stack in \(\Delta\) whose bases are not on \(\partial \Delta\), as shown in Figure 4.36. Divide \(\Delta\) and rearrange the stacks as follows:
Again, all the stacks in Figure 4.39 have a base that is on $\partial \Delta$. Thus, Case 3 follows by Case 1.

This completes the proof of the lemma. $\square$

5. Graphs with induced $K_4$ subgraphs

In the previous sections, we considered finite simplicial graphs $\Gamma$ that do not contain $K_4$ as induced subgraphs. In this section, we consider finite simplicial graphs that do contain $K_4$ as induced subgraphs. The main change is that the flag complexes on finite simplicial graphs with $K_4$ induced subgraphs are not 2-dimensional.

Unfortunately, we have not been able to characterize the Dehn function $\delta_{H_{\Gamma}}$ when $\Delta_{\Gamma}$ is not 2-dimensional. Instead, we obtain a lower bound for $\delta_{H_{\Gamma}}$ when $\Gamma$ contains induced subgraphs as shown in Figure 2.5.

**Definition 5.1.** We say that a subgroup $H$ is a retract of a group $G$ if there is a homomorphism $r : G \to H$, such that $r : H \to H$ is the identity. We call the homomorphism $r$ a retraction.

A standard fact about group retract is that if $H$ is a retract of a finitely presented group $G$, then $H$ is also finitely presented. The following lemma says that group retractions do not increase Dehn functions.

**Lemma 5.2.** ([6], Lemma 2.2) If $H$ is a retract of a finitely presented group $G$, then $\delta_H \preceq \delta_G$.

**Proposition 5.3.** Let $\Gamma$ be a finite simplicial graph. If $\Gamma'$ is a connected induced subgraph of $\Gamma$, then $H_{\Gamma'}$ is a retract of $H_{\Gamma}$.

**Proof.** Since $\Gamma'$ is an induced subgraph of $\Gamma$, $A_{\Gamma'}$ is a retract of $A_{\Gamma}$. Let $r : A_{\Gamma} \to A_{\Gamma'}$ be a retraction; define $r' = r|_{H_{\Gamma}} : H_{\Gamma} \to H_{\Gamma'}$. Since $\Gamma$ and $\Gamma'$ are connected, $H_{\Gamma}$ and $H_{\Gamma'}$ are finitely generated and their generating sets are sets of directed edges of $\Gamma$ and $\Gamma'$, respectively. It suffices to show that $r'$ is the identity on a generating set of $H_{\Gamma'}$. Let $e$ be a oriented edge
of $\Gamma'$ with initial vertex $v$ and terminal vertex $w$. The generator $e$ of $H_{\Gamma'}$ can be expressed in terms of the generators of $A_{\Gamma'}$, $e = vw^{-1}$. Since $r : A_{\Gamma} \to A_{\Gamma'}$ is a retraction, we have

$$r'(e) = r(vw^{-1}) = vw^{-1} = e.$$  

This shows that $r' : H_{\Gamma} \to H_{\Gamma'}$ is a retraction.

We establish the lower as promised.

**Proposition 5.4.** Let $\Gamma$ be a finite simplicial graph such that $\Delta_{\Gamma}$ is simply-connected. If $\Gamma$ contains an induced subgraph $\Gamma'$ such that $\Delta_{\Gamma'}$ is a 2-dimensional triangulated subdisk of $\Delta_{\Gamma}$ that has square boundary and $\dim_I(\Delta_{\Gamma'}) = d$, $d \in \{0, 1, 2\}$, then $n^{d+2} \leq \delta_{H_{\Gamma}}(n)$.

**Proof.** By Theorem 4.1 we have $\delta_{H_{\Gamma'}}(n) \simeq n^{d+2}$. Since $\Gamma'$ is an induced subgraph of $\Gamma$, it follows from Proposition 5.3 and Lemma 5.2 that $H_{\Gamma'}$ is a retract of $H_{\Gamma}$ and $n^{d+2} \simeq \delta_{H_{\Gamma'}}(n) \leq \delta_{H_{\Gamma}}(n)$. \qed

We remark that Proposition 5.4 can be used to prove Theorem 4.1 for the case $d = 1$. We close this section with an example.

**Example 5.5.** Let $\Gamma$ be the graph as shown in the following figure:

![Graph Γ](image)

*Figure 5.1. The graph $\Gamma$.*

The flag complex $\Delta_{\Gamma}$ is not 2-dimensional since $\Gamma$ contains induced $K_4$ subgraphs. Observe that $\Gamma$ contains an induced subgraph $\Gamma'$ that is the suspension of a path of length 3 in the middle of $\Gamma$. Hence, by Theorem 4.1 and Proposition 5.3, we have $n^3 \simeq \delta_{H_{\Gamma'}}(n) \leq \delta_{H_{\Gamma}}(n)$.

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31
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