Detecting entanglement in arbitrary two-mode Gaussian state: a Stokes-like operator based approach

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Detection of entanglement in quantum states is one of the most important problems in quantum information processing. However, it is one of the most challenging tasks to find a universal scheme which is also desired to be optimal to detect entanglement for all states of a specific class—as always preferred by experimentalists. Although, the topic is well studied at least in case of lower dimensional compound systems, e.g., two-qubit systems, but in the case of continuous variable systems, this remains as an open problem. Even in the case of two-mode Gaussian states, the problem is not fully solved. In our work, we have tried to address this issue. At first, a limited number of Hermitian operators is given to test the necessary and sufficient criterion on the covariance matrix of separable two-mode Gaussian states. Thereafter, we present an interferometric scheme to test the same separability criterion in which the measurements are being done via Stokes-like operators. In such case, we consider only single-copy measurements on a two-mode Gaussian state at a time and the scheme amounts to the full state tomography. Although this latter approach is a linear optics based one, nevertheless it is not an economic scheme. Resource-wise a more economical scheme than the full state tomography is obtained if we consider measurements on two copies of the state at a time. However, optimality of the scheme is not yet known.

I. INTRODUCTION

Quantum entanglement is the fundamental aspect which separates quantum mechanics from its classical counterpart. Entanglement is used as resources in various quantum computation and information theory protocols [1]. Hence, detection of entanglement is important in this area. By definition a separable state of two subsystems \( \rho_{12} \) can be written as

\[
\rho_{12} = \sum_i p_i \rho_1(i) \otimes \rho_2(i) . \tag{1}
\]

It is a convex combination of product states of two different subsystems 1 and 2. The above formula represents situation in which with probability \( p_i \) one of the system is in the state \( \rho_1(i) \) and the other is in \( \rho_2(j) \). The states \( \rho_1(i) \), \( \rho_2(i) \), and \( \rho_{12} \) are defined on the Hilbert spaces \( \mathcal{H}_1 \), \( \mathcal{H}_2 \), and \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), respectively. If a given state cannot be written in the form given by (1), then the state is entangled.

In general, given two arbitrary Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), declaring whether an arbitrary quantum state \( \rho_{12} \) acting on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is separable or entangled is a difficult problem. In fact this problem is NP-hard even for any Hilbert spaces of finite composite dimensions greater than six [2]. However, entanglement is used as resource in almost all quantum information protocols. Hence, it is important to find out methods of detection of entanglement, even for specific classes of states only.

Since the beginning of quantum information science, continuous variable systems have also been exploited as substitute for finite dimensional systems and possibly as more powerful tool as well, as there is a plethora of such research coming primarily from quantum optics, both from theory and experiment. Most of these continuous variable works centred around Gaussian states, as they are easy to prepare, manipulate, and measure [3]. Many of the protocols in finite dimensions have analogous protocols for continuous variables cases as well. To name a few, entanglement is used in continuous variables quantum teleportation [4–8] and continuous variables quantum key distribution [9].
In fact, squeezed states, a specific type of Gaussian states, which are under consideration in this paper, have been used by Cerf et al. \cite{10} for secret encoding. This has further been developed by Grosshans et al. \cite{11, 12}. Details of applications of continuous variables, in particular Gaussian states, can be seen in the excellent survey article by Weedbrook et al. \cite{3}.

Given any entangled state $\rho_{12}$ of a bi-partite system $1 + 2$, one can, in principle, find out a hermitian operator $\hat{W}_\rho$ (called as entanglement witness) acting on the Hilbert space of the bi-partite system such that $\text{Tr}(\hat{W}_\rho \rho_{12}) < 0$ but $\text{Tr}(\hat{W}_\rho \sigma_{12}) \geq 0$ for all separable states $\sigma_{12}$ of the system. Although $\hat{W}_\rho$ detects entanglement in $\rho_{12}$, it cannot detect entanglement in all the states of the system, and hence, it looses the ‘universality’ property.

On the other hand, the well-known Peres-Horodecki necessary-sufficient criterion for separability of two-qubit states $\rho_{12}$ \cite{2, 13} gives rise to the existence of a Hermitian operator $\hat{W}_u$ (say) – acting on four copies of the system Hilbert space – such that $\text{Tr}(\rho_{12}^4 \hat{W}_u) = \det(\rho_{12}^2)$ \cite{14}. As the Peres-Horodecki separability criterion for any two-qubit state $\rho_{12}$ is equivalent to the condition $\det(\rho_{12}^2) \geq 0$ for separability of $\rho_{12}$, therefore, the signature of $\text{Tr}(\rho_{12}^4 \hat{W}_u)$ will provide the desired universal witnessing of entanglement in two-qubit states. Using the notion of weak values, a resource-wise less demanding universal entanglement witnessing scheme (using two copies of the state at a time) for two qubits was proposed in Ref. \cite{15}, although it eventually amounts to state tomography. In Ref. \cite{16}, based on the work of Ref. \cite{14}, a measurement-device-independent (MDI) universal entanglement witnessing was given for two qubits. Moreover, for a two qudit system, an MDI universal scheme for witnessing the NPT-PPT divide was designed in Ref. \cite{16} for all finite values of $d$. Also, it was conjectured in Ref. \cite{16} that a universal entanglement witnessing scheme for bipartite systems with dimension higher than six can not exist.

In the case of continuous variables, the structure of states is naturally more complicated than in case of discrete systems. One such class of states, namely the Gaussian states, are of particular use, as they are relatively easy to prepare and handle. For the Gaussian state, the central role is played by the covariance matrix of the state. In a seminal paper \cite{17} by Simon, the first result on the detection of entanglement in two-mode Gaussian state was presented where a continuous variable version of the Peres-Horodecki PPT criterion was obtained. It had been shown that the criterion is necessary and sufficient to test separability of two-mode Gaussian states. Further generalizations of the criterion are given in Refs. \cite{18, 19}, where the criterion has been extended for higher number of modes under certain symmetry conditions. Tempted by the aforesaid necessary-sufficient condition of Ref. \cite{17}, one might expect that a universal entanglement witnessing scheme for (bosonic) two-mode Gaussian states should exist, like in the case of two qubits. Unfortunately, given any two-mode Gaussian state $\rho_{12}$, the signature of $\det(\rho_{12}^2)$ (or, something similar to it) does not capture the necessary-sufficient condition for separability of two-mode Gaussian states, mentioned in Ref. \cite{17}. Although this necessary-sufficient condition can be cast in terms of conditions on the parameters of the covariance matrix of the two-mode Gaussian state, nevertheless, re-casting these latter conditions in terms of statistics of measurements of observables on one or more copies of the two-mode state is a non-trivial task.

There are more general works in this direction which illustrate necessary and sufficient criterion for testing separability of any Gaussian state \cite{20, 21}, where semi-definite programming can be used for detection of entanglement. However, such methods can be used for computational purposes only, and cannot be used for actual experiment, where the state itself remains unknown. An equivalent form of the separability criterion was presented in Refs. \cite{24, 25}, where the separability criterion was derived in terms of the symplectic eigenvalues of the covariance matrix of a unknown two-mode Gaussian state. An experimental friendly approach to estimate entanglement in two-mode Gaussian state was provided in Ref. \cite{22}. However, these methods require apriori knowledge of the covariance matrix, and as such cannot be directly used for detection of entanglement in an unknown state. Similarly, in Ref. \cite{24}, an experimentally feasible scheme to test the separability criterion was proposed for partially known multimode Gaussian states. Precisely, in this case, the expectation values of quadrature observables are known along with the apriori knowledge whether or not the covariance matrix of the Gaussian state is symmetric. In Ref. \cite{26}, a scheme to test a modified version of separability criterion was discussed. However, this works for two-mode Gaussian states with a special form of covariance matrix. In Ref. \cite{27}, the authors considered the question of universal detection entanglement in a bi-partite CV system (with one or more modes in possession of each part) in which the relevant set of entanglement witness operators are being generated via semi-definite programming (SDP). These witness operators can be realized by acting on single copy of the state at a time and using random measurements involving homodyne detections, polarizing beam splitters, and polarization rotators, a scheme which was introduced in Ref. \cite{28} to find out the covariance matrix of any two-mode Gaussian state (described by 14 real parameters) using five different homodyne detections (five different field modes) in total. As this latter work actually amounts to state tomography, therefore, the scheme of Ref. \cite{27} also amounts to state tomography, even though, one may need less (or, more) number of measurements (in comparison with the full state tomography) for certain class of states. This is a signature of choosing the option for random measurements.

Given this background, in the present work, our aim is to detect entanglement in an arbitrary two-mode Gaussian state universally with fixed set of measurement settings. For testing entanglement in two-qubit systems, there are
efficient formulae. Hence, the question is whether any such formula can also be adopted for two-mode Gaussian states. In this paper, the question has been answered affirmatively by using an equivalent method of detection of entanglement in terms of determinants of block matrices as given in Eq. (8) (see Section II for details). In this regard, two schemes of entanglement detection have been given in which only five particular measurements are sufficient. This is followed by implementation of the universal scheme by performing Stokes-like measurements [35, 36] on a single copy of the state at a time. Interestingly, Stokes operators are efficient tools to describe the polarization degree of freedom of states of light and phase properties of the light fields [35]. In addition, as the set of observables can be measured with a pair of detectors at the two outputs of an optical device, e.g., beam splitter, the scheme can be executed in a standard experimental protocol. However, one of the drawbacks of such a scheme is that, if the photons are detected at the outputs of a measurement setup consisting of only passive linear optical devices, e.g., beam splitters and phase shifters with a fixed set of settings, then the scheme leads to the full state tomography. Thus, the scheme in Section III can be considered as an experimental realization of the scheme discussed in Ref. [28] to perform the full state tomography by homodyne detection. Interestingly, the full state tomography can be avoided by performing a special set of measurements with SWAP operators and active nonlinear devices, e.g., optical parametric amplifiers (OPAs) [38] on two copies of the state at a time.

The paper is organized as follows. In Section II, an introductory discussion on Gaussian state and its entanglement detection have been given. This is followed by a scheme to test separability criterion (8) with only five measurements. In Appendix A, a similar scheme with the same five measurements has also been described. In Section III, we discuss a method to test the separability criterion (8) by implementing Stokes-like measurements. However, as mentioned earlier, the scheme amounts to the full state tomography. In Section IV, we present another scheme to test the separability criterion (8), in which one can skip the full state tomography by considering measurements on two copies of the unknown Gaussian state at a time. Finally, in Section V, we discuss the main results and future directions and talk about a few open problems.

II. TEST OF SEPARABILITY AND ENTANGLEMENT MONOTONE

In our present work, we are interested in detection of entanglement in an unknown two-mode Gaussian state. Since the state itself is unknown, we don’t have any information about its covariance matrix, and hence one can not use Simon’s criterion [17] directly. Moreover, it also requires us to perform tomography on the state to estimate all necessary parameters. It is well known fact that a Gaussian state \( \rho_{12} \) can be completely characterized by the first and second moments. Precisely, the Wigner function of any two-mode Gaussian state \( \rho_{12} \) reads as

\[
W_{\rho_{12}}(\vec{\xi}) = \frac{1}{4\pi^2 \sqrt{\det \Gamma_{\rho_{12}}}} \exp \left\{ -\frac{1}{2} \left( \vec{\xi} - \langle \hat{R} \rangle \right)^T \Gamma_{\rho_{12}}^{-1} \left( \vec{\xi} - \langle \hat{R} \rangle \right) \right\}
\]

where \( \vec{\xi} = (q_1, p_1, q_2, p_2)^T \) is a vector of phase-space observables satisfying commutation relation \( [\hat{R}_k, \hat{R}_l] = iJ_{k,l} \), where \( J \) is a matrix defined as \( J = \bigoplus_{i=1}^4 \omega \) for \( \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The first moments and covariance matrix of the state \( \rho_{12} \) are defined as \( d_j = \langle \hat{R}_j \rangle \), and

\[
\Gamma_{\rho_{12}} = ((\Gamma_{\rho_{12}})_{k,l})_{k,l=1}^4 = \left( \frac{1}{2} \langle \hat{R}_k \hat{R}_l \rangle + \langle \hat{R}_l \hat{R}_k \rangle - 2 \langle \hat{R}_k \rangle \langle \hat{R}_l \rangle \right)_{k,l=1}^4
\]

receptively, where \( \langle \hat{O} \rangle = \text{Tr}(\rho_{12} \hat{O}) \).

For a two-mode Gaussian state \( \rho_{12} \), the following condition must hold (because of positive semi-definiteness of \( \rho_{12} \))

\[
\Gamma_{\rho_{12}} + \frac{i}{2} J \succeq 0.
\]

Here \( \Gamma_{\rho_{12}} \) can be taken as

\[
\begin{pmatrix}
A & C \\
C^T & B
\end{pmatrix},
\]

where \( A, B, C \) are matrices of dimension \( 2 \times 2 \). Precisely,

\[
A = \begin{pmatrix}
\langle q_1^2 \rangle - \langle \hat{q}_1 \rangle^2 & \frac{1}{2} \langle \hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1 \rangle - \langle \hat{q}_1 \rangle \langle \hat{p}_1 \rangle \\
\frac{1}{2} \langle \hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1 \rangle - \langle \hat{q}_1 \rangle \langle \hat{p}_1 \rangle & \langle \hat{p}_1^2 \rangle - \langle \hat{p}_1 \rangle^2
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\langle q_2^2 \rangle - \langle \hat{q}_2 \rangle^2 & \frac{1}{2} \langle \hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2 \rangle - \langle \hat{q}_2 \rangle \langle \hat{p}_2 \rangle \\
\frac{1}{2} \langle \hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2 \rangle - \langle \hat{q}_2 \rangle \langle \hat{p}_2 \rangle & \langle \hat{p}_2^2 \rangle - \langle \hat{p}_2 \rangle^2
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
\langle q_1 \otimes q_2 \rangle - \langle \hat{q}_1 \rangle \langle \hat{q}_2 \rangle & \langle q_1 \otimes \hat{p}_2 - \hat{q}_1 \hat{p}_2 \rangle - \langle \hat{q}_1 \rangle \langle \hat{p}_2 \rangle \\
\langle \hat{p}_1 \otimes q_2 \rangle - \langle \hat{p}_1 \rangle \langle \hat{q}_2 \rangle & \langle \hat{p}_1 \otimes \hat{p}_2 - \hat{p}_1 \hat{p}_2 \rangle - \langle \hat{p}_1 \rangle \langle \hat{p}_2 \rangle
\end{pmatrix}.
\]
It is worth to mention that, by local symplectic operations on each single modes, one can set the first moments of the state to zero, i.e., \( \langle \hat{q}_1 \rangle = \langle \hat{q}_2 \rangle = \langle \hat{p}_1 \rangle = \langle \hat{p}_2 \rangle = 0 \), which, of course requires apriori information regarding the values of these first moments of the state.

To detect entanglement in case of a two-mode Gaussian state, we employ an equivalent version \cite{24, 25} of Simon’s separability criterion \cite{17}, which is direct generalization of the Peres-Horodecki separability criterion \cite{2, 13} of partial transpose to continuous variable systems. Under partial transpose, Wigner distribution of the two-mode Gaussian state undergoes mirror reflection, which leads to the transformation in \( \Gamma_{\rho_{12}} : \Gamma_{\rho_{12}} \rightarrow \tilde{\Gamma}_{\rho_{12}} = \Lambda \Gamma_{\rho_{12}} \Lambda \), where \( \Lambda \) is phase space mirror reflection. Using \eqref{3}, Simon’s criterion here reads

\[
\tilde{\Gamma}_{\rho_{12}} + \frac{i}{2} J \geq 0. 
\]

Interestingly, one can show that, the two-mode Gaussian state is separable, iff the minimum symplectic eigenvalue \( \xi_{\min} \) of the covariance matrix \( \tilde{\Gamma}_{\rho_{12}} \) of the partially transposed state is greater or equal to 1/2 \cite{24, 25}. One can obtain an analytical formula for \( \xi_{\min} \) by solving a bi-quadratic equation \cite{23, 24}:

\[
\xi^4 - (\det A + \det B - 2 \det C)\xi^2 + \det \Gamma_{\rho_{12}} = 0.
\]

Symplectic eigenvalues of \( \tilde{\Gamma}_{\rho_{12}} \) are two positive roots of \eqref{6}, and the minimum symplectic eigenvalue \( \xi_{\min} \) \cite{24, 25} reads:

\[
\xi_{\min} = \sqrt{\frac{D - \sqrt{D^2 - 4 \det \Gamma_{\rho_{12}}}}{2}},
\]

where \( D = \det A + \det B - 2 \det C \). Inserting \( \xi_{\min} \) in separability criterion \( \xi_{\min} \geq 1/2 \), we obtain the following necessary-sufficient criterion for separability of a two-mode Gaussian state \( \rho_{12} \):

\[
D - 4 \det \Gamma_{\rho_{12}} \leq \frac{1}{4}.
\]

In addition, there exist entanglement monotones \cite{18, 23, 30}, which measure entanglement in a two-mode Gaussian state \( \rho_{12} \). For example, one can calculate the logarithmic negativity \( E_2 \) to quantify entanglement in \( \rho_{12} \). The logarithmic negativity \( E_2 \) is defined as \( E_2 = - \log \xi_{\min} \). In this context, it is worth to mention that in Ref. \cite{24} a scheme has been provided to measure squeezing and detect entanglement in multimode Gaussian states with phase-insensitive devices without homodyning. Interestingly, the scheme doesn’t provide full information of the two-mode Gaussian state. Precisely, the estimation of the determinants of \( \det A, \det B, \det C, \) and \( \det \Gamma_{\rho_{12}} \) do not lead to the full state tomography. However, in Ref. \cite{24}, the information about the expectation values of quadratures is necessary to choose the measurement setups to estimate the determinants of the block matrices. Precisely, in case of no apriori knowledge about the expectation values of quadratures one does not seem to have the potentiality to decide if the measurements for entanglement detection should be performed on a single copy or two copies of the two-mode Gaussian state. In addition, the method to determine \( \det C \), requires a symmetric form of the block matrix \( C \). If the matrix is not in a symmetric form, then a proper unitary transformation is needed to symmetrize the matrix \( C \). Hence, apriori knowledge on the symmetry of matrix \( C \) is also necessary. Therefore, the method described in Ref. \cite{24} does not address the issue of universal entanglement detection. Similarly, in Ref. \cite{26}, the authors proposed a scheme to test and quantify entanglement in an arbitrary two-mode Gaussian state with minimal requirements via local measurements and a classical communication channel. To this end, the authors developed a method to test a variant of separability criterion \eqref{8} and estimate entanglement of formation to quantify entanglement of the Gaussian state. However, the method was discussed for a specific form of the covariance matrix associated to a two-mode Gaussian state. Note that, a covariance matrix of an arbitrary two-mode Gaussian state can be transformed to this specific form by local symplectic transformations, but, to get such form one must have knowledge on the elements of the covariance matrix before the transformation is being applied upon. Apart from that, the number of measurements to execute such a scheme is more than the number of measurements required to perform the full state tomography. In the present work, our main aim is to provide schemes to test the separability criterion \eqref{8} without the requirement of any apriori knowledge of the unknown two-mode Gaussian state. For measurements on a single copy of the state at a time, the scheme leads to the full state tomography, and the number of measurements is same as that required for the full state tomography using projective measurement. However, for a special set of measurements on two copies of the state at a time, one doesn’t need to find each elements of the covariance matrix of an arbitrary two-mode Gaussian state. In other words, this latter scheme is not identical to the full state tomography.

We first discuss schemes to obtain the information regarding the quadrature mean values as well as the covariance matrix of a two-mode Gaussian state with a limited number of measurements. From discussions in previous section, one can see that the covariance matrix of a two-mode Gaussian state can be computed by averaging the phase space observables, e.g., \( q_1, q_1^2, q_2, q_2^2, \hat{p}_1, \hat{p}_1^2, \hat{p}_2, \hat{p}_2^2, q_1 \otimes q_2, q_1 \otimes \hat{p}_2, \hat{p}_1 \otimes q_2, \hat{p}_1 \otimes \hat{p}_2, \frac{1}{2} (q_1 \hat{p}_1 + \hat{p}_1 q_1), \) and \( \frac{1}{2} (q_2 \hat{p}_2 + \hat{p}_2 q_2) \) over
many copies of the state. For a pair of commuting operators, the corresponding observables can be measured jointly. If we have a set of observables some of which are pair-wise co-measurable, then we can group them in such a way that the entire set can be measured with a choice from the limited number of measurements. We systematically describe a scheme to re-construct the covariance matrix of a two-mode Gaussian state with a limited number of measurements.

Note that, $\hat{q}_1^2, \hat{q}_2^2,$ and $\hat{q}_1 \otimes \hat{q}_2$ are pair-wise measurable. Also, $\hat{p}_1^2, \hat{p}_2^2,$ and $\hat{p}_1 \otimes \hat{p}_2$ can be measured simultaneously. Another pair of observables $\frac{1}{2}(\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1)$, and $\frac{1}{2}(\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2)$ can be co-measured. However, $\hat{q}_1 \otimes \hat{p}_2,$ and $\hat{p}_1 \otimes \hat{q}_2$ need to be measured separately. Thus, the repeated measurements of the five groups of observables over many copies of the two-mode Gaussian state will yield complete knowledge of the covariance matrix.

Let’s consider that $5N$ $(N \gg 1)$ copies of a two-mode Gaussian state $\rho_{12}$ are shared between two parties, say Alice and Bob, where each of them possesses one subsystem with one mode. Each of the five groups of observables is to be measured on $N$ copies of $\rho_{12}$. The scheme goes as follows: a) Alice measures quadrature observable $\hat{q}_1$ on mode 1 of $1$st $N$ copies of the shared state $\rho_{12}$. Also, Bob measures $\hat{q}_2$ on mode 2 of the same $N$ copies of $\rho_{12}$. b) Next, Alice chooses to measure quadrature observable $\hat{p}_1$ on mode 1 of $2$nd $N$ copies of the shared state $\rho_{12}$ and Bob measures $\hat{p}_2$ on mode 2 of the same $N$ copies of $\rho_{12}$. c) Then, Alice measures $\frac{1}{2}(\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1)$ on mode 1 of $3$rd $N$ copies of the shared state $\rho_{12}$. Bob chooses to measure $\frac{1}{2}(\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2)$ on mode 2 of the same $N$ copies of $\rho_{12}$. d) Thereafter, Alice and Bob measure quadrature observables $\hat{q}_1$ on mode 1 and $\hat{p}_2$ on mode 2 of $4$th $N$ copies of $\rho_{12}$, respectively. e) At the end, quadrature observable $\hat{p}_1$ is measured by Alice on mode 1 of the last $N$ copies of $\rho_{12}$, whereas Bob measures quadrature observable $\hat{q}_2$ on mode 2 of the last $N$ copies of $\rho_{12}$. Thus, altogether five observables: $\hat{A} = \hat{q}_1 \otimes \hat{q}_2, \hat{B} = \hat{p}_1 \otimes \hat{p}_2, \hat{C} = \frac{1}{2}(\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1) \otimes \frac{1}{2}(\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2), \hat{D} = \hat{q}_1 \otimes \hat{p}_2,$ and $\hat{E} = \hat{p}_1 \otimes \hat{q}_2$ are measured separately on many copies of $\rho_{12}$. Now, we are going show that, measurements of the five observables are enough to compute all elements of the covariance matrix of the two-mode Gaussian state.

Measurement outcome of $\hat{A}$ will be of the form $q_1 q_2$, where $q_1, q_2 \in \mathbb{R}$. Here, a value $q_1$ will be obtained whenever Alice measures $\hat{q}_1$ on mode 1 of the shared state $\rho_{12}$. Similarly, measurement of $\hat{q}_2$ by Bob on mode 2 of $\rho_{12}$ will yield a value $q_2$. Also, assume that, the pair of values $(q_1, q_2)$ of quadrature observables $(\hat{q}_1, \hat{q}_2)$ (where $\hat{A} = \hat{q}_1 \otimes \hat{q}_2$) occurs with probability $P(q_1, q_2)$. Then, one can compute $P(q_1)$ and $P(q_2)$ as marginals of $P(q_1, q_2)$. Thus, by measuring $\hat{A} = \hat{q}_1 \otimes \hat{q}_2$ together with finding the individual values $q_1$ and $q_2$ one can compute respective probabilities $P(q_1)$, and $P(q_2)$ of occurrence of these values. With the results, one can obtain the values $\hat{q}_1^2$ and $\hat{q}_2^2$ of respective observables $\hat{q}_1^2$, and $\hat{q}_2^2$ together with the associated probabilities $P(q_1)$ and $P(q_2)$. Thus measurement of $\hat{A} = \hat{q}_1 \otimes \hat{q}_2$ yields triplet of values $(\hat{q}_1^2, \hat{q}_2^2, q_1 q_2)$ of observable-triplet $(\hat{q}_1^2, \hat{q}_2^2, \hat{q}_1 \otimes \hat{q}_2)$. The expectation values of the observable-triplet are given as follows:

$$\langle \hat{q}_1^2 \rangle = \text{Tr}_{12}(\hat{q}_1^2 \rho_{12})$$

$$= \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \ q_1^2 P(q_1 = q_1, q_2 = q_2 | \rho_{12})$$

$$= \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \ q_1^2 P(\hat{A} = q_1 q_2 | \rho_{12}), \quad (9)$$

$$\langle \hat{q}_2^2 \rangle = \text{Tr}_{12}(\hat{q}_2^2 \rho_{12})$$

$$= \int_{-\infty}^{+\infty} dq_2 \int_{-\infty}^{+\infty} dq_1 \ q_2^2 P(\hat{A} = q_1 q_2 | \rho_{12}), \quad (10)$$

and

$$\langle \hat{q}_1 \otimes \hat{q}_2 \rangle = \text{Tr}_{12}(\hat{q}_1 \otimes \hat{q}_2 \rho_{12})$$

$$= \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \ q_1 q_2 P(\hat{A} = q_1 q_2 | \rho_{12}). \quad (11)$$

Similarly, $\langle \hat{p}_1^2 \rangle, \langle \hat{p}_2^2 \rangle,$ and $\langle \hat{p}_1 \otimes \hat{p}_2 \rangle$ can be obtained by measuring $\hat{B} = \hat{p}_1 \otimes \hat{p}_2$ on $\rho_{12}$. Again, measurement of $\hat{C} = \frac{1}{2}(\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1) \otimes \frac{1}{2}(\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2)$ leads to expectation values $\langle \frac{1}{2}(\hat{q}_1 \hat{p}_1 + \hat{p}_1 \hat{q}_1) \rangle,$ and $\langle \frac{1}{2}(\hat{q}_2 \hat{p}_2 + \hat{p}_2 \hat{q}_2) \rangle$. Finally, $\langle \hat{q}_1 \otimes \hat{p}_2 \rangle$ and $\langle \hat{p}_1 \otimes \hat{q}_2 \rangle$ can be computed by separately measuring $\hat{D} = \hat{q}_1 \otimes \hat{p}_2$ and $\hat{E} = \hat{p}_1 \otimes \hat{q}_2$, respectively.

As mentioned earlier, although, the first moments of the state or expectation values of quadrature observables are not relevant in detection and quantification of entanglement in two-mode Gaussian state, the scheme discussed here can be used to obtain $\langle \hat{q}_1 \rangle, \langle \hat{q}_2 \rangle, \langle \hat{p}_1 \rangle,$ and $\langle \hat{p}_2 \rangle$ using the same set of measurements. Precisely, measuring $\hat{A}$ and $\hat{B}$ or $\hat{D}$ and $\hat{E}$ one can estimate $\langle \hat{q}_1 \rangle, \langle \hat{q}_2 \rangle, \langle \hat{p}_1 \rangle,$ and $\langle \hat{p}_2 \rangle$. For example, expectation value of quadrature observable $\langle \hat{q}_1 \rangle$ is
FIG. 1: a) A schematic diagram of the measurement setup to estimate all elements of a covariance matrix associated to a single mode of a Gaussian state. k-th mode of an unknown Gaussian state and a reference state which are represented by annihilation operators \( \hat{a}_k \), and \( \hat{a}_r \), respectively, interfere at a 50-50 beam-splitter (B.S). A phase shifter is introduced at the reference mode. The outputs of the beam-splitter are represented by annihilation operators \( \hat{a}_1 \), and \( \hat{a}_2 \). The detectors generate photocurrents \( I_1 \) and \( I_2 \) which are proportional to the intensities at the output modes of B.S. b) A schematic diagram of the measurement setup to obtain all elements of a covariance matrix corresponding to a two-mode Gaussian state. Two modes of \( \rho_{12} \) represented by annihilation operators \( \hat{a}_1 \), and \( \hat{a}_2 \), interfere at a 50-50 beam-splitter 1 (B.S1). The outputs of B.S1 are made to interfere at two separate 50-50 beam-splitters, e.g., beam-splitter 2 (B.S2) and beam-splitter 3 (B.S3) along with the two reference states represented by annihilation operators \( \hat{a}_c \), and \( \hat{a}_d \). Phase shifters are introduced at the reference modes. The outputs of the measurement setup are represented by annihilation operators \( \hat{a}_3 \), \( \hat{a}_4 \), \( \hat{a}_5 \), and \( \hat{a}_6 \). Two pairs of photocurrents \( (I_3, I_4) \), and \( (I_5, I_6) \) are proportional to the intensities of the output modes of B.S2 and B.S3, respectively.

Given as follows:

\[
\langle \hat{q}_1 \rangle = \text{Tr}_{12}(\hat{q}_1 \rho_{12})
\]

\[
= \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \, q_1 P(\hat{q}_1 = q_1, \hat{q}_2 = q_2 | \rho_{12})
\]

\[
= \int_{-\infty}^{+\infty} dq_1 \int_{-\infty}^{+\infty} dq_2 \, q_1 P(\hat{A} = q_1 q_2 | \rho_{12}).
\]

(12)

Here, measurement outcomes of \( \hat{A} = \hat{q}_1 \otimes \hat{q}_2 \), and quadrature observables \( \hat{q}_1, \hat{q}_1 \) must be known along with the probability of occurrence of values \( \hat{q}_1 = q_1, \hat{q}_2 = q_2 \), or \( \hat{A} = q_1 q_2 \), i.e., \( P(\hat{q}_1 = q_1, \hat{q}_2 = q_2 | \rho_{12}) \). Applying the similar method one obtains the first moments of the state. Thus, interestingly, the set of five measurements is sufficient to reconstruct the covariance matrix of a two-mode Gaussian state. Note that the aforesaid method provides the mean values \( \langle \hat{q}_1 \rangle, \langle \hat{p}_1 \rangle, \langle \hat{q}_2 \rangle, \) and \( \langle \hat{p}_2 \rangle \) as well as the covariance matrix for any two-mode state, and not necessarily only for two-mode Gaussian states. As a result, this method gives rise to tomography of any two-mode Gaussian states using measurements on the individual modes.

In Appendix A, we present another scheme to estimate the elements of the covariance matrix with the same set of five measurements. It will be interesting to find a set of measurable quantities to experimentally realize the set of five measurements, e.g., using homodyne measurements. In the next section, we discuss simple interferometric setups to re-construct the covariance matrix with a different set of measurements.

III. MEASUREMENTS (EXPERIMENTALLY FEASIBLE)

As discussed before, to test the separability criterion (8), one needs to compute determinants of the block matrices \( A, B, \) and \( C \) of the covariance matrix of a two-mode Gaussian state \( \rho_{12} \) together with \( \text{det} \Gamma_{\rho_{12}} \). One of the methods
is to estimate all the elements of the $\Gamma_{\rho_{23}}$ and calculate the determinants. Here, we present such a scheme which can be realized in linear optical setups. Note that, as the two-mode Gaussian state is unknown, we don’t assume any properties of the Gaussian state to perform the measurements. Hence, a general method is adopted such that it works for an arbitrary two-mode Gaussian state.

Interestingly, in Ref. [31], a similar approach was implemented for a single-mode Gaussian state. To this end, the authors proposed a scheme in which a reference Gaussian state is made to interfere with the unknown single-mode Gaussian state on a beam-splitter and Stokes-like measurements are performed by measuring intensity difference at the two output modes of the beam-splitter. The experimental data along with the known values of means, variances and expectation value of a symmetric function of a pair of quadrature observables related to the reference mode are sufficient to estimate the means, variances and expectation value of the symmetric function of the pair of quadrature observables associated to the unknown single-mode Gaussian state. We consider the same method to estimate all the elements of matrices $A$ and $B$. To obtain the elements of matrix $C$, we extend the scheme by modifying the experimental setup with additional beam-splitters and phase-shifters. The details of the scheme with Stokes-like measurements are given below.

Let’s consider $\hat{a}_{k}$ be the annihilation operator (see Figure 1(a)) associated to k-th mode of a single-mode Gaussian state and an annihilation operator $\hat{a}_{r}$ represents a reference mode $r$. A phase-shifter is placed at the input mode of the 50-50 beam-splitter corresponding to the reference state to introduce phase shifts between two interfering modes. After relevant unitary transformations by a combination of a phase shifter and a beam splitter, annihilation operators $\hat{a}_{1}$ and $\hat{a}_{2}$ of the output modes of the beam-splitter are given as follows: $\hat{a}_{1} = (\hat{a}_{k} - \hat{a}_{r} e^{i\phi})/\sqrt{2}$, and $\hat{a}_{2} = (\hat{a}_{k} + \hat{a}_{r} e^{i\phi})/\sqrt{2}$. Also, quadrature observables $\hat{q}_{k}$, and $\hat{p}_{k}$ associated to k-th mode are defined as $(\hat{a}_{k} + \hat{a}_{k}^\dagger)/\sqrt{2}$, and $i(\hat{a}_{k}^\dagger - \hat{a}_{k})/\sqrt{2}$, respectively. Throughout the manuscript we follow the similar convention. Note that, usually Stokes operators $S_{0}, S_{1}, S_{2}$, and $S_{3}$ are associated to two orthogonal polarization modes of a single beam [35]. However, Stokes-like measurements can be realized on two output modes of a beam-splitter where the interfering input beams are generated from two different sources. To estimate the elements of matrices $A$, and $B$, which are associated to single modes, $S_{1}(\phi), S_{2}^{2}(\phi)$ are measured for different phase shifts $\phi$.

Considering photon number difference at the two outputs, we obtain

$$\hat{S}_{1}(\phi) = \hat{I}_{2} - \hat{I}_{1} = \hat{a}_{2}^\dagger \hat{a}_{2} - \hat{a}_{1}^\dagger \hat{a}_{1} = \hat{q}_{r}^{\phi} + \hat{p}_{r}^{\phi}$$

Note that, the expectation values of quadrature observables associated to reference state are given (Here we consider a displaced squeezed thermal state as a reference state. Please see Appendix B for the details). Thus, one can estimate $\langle \hat{q}_{k} \rangle$ and $\langle \hat{p}_{k} \rangle$ for experimentally obtained expectation values of $\hat{S}_{1}(\phi)$, and given values of $(\langle \hat{q}_{r}^{\phi} \rangle, \langle \hat{p}_{r}^{\phi} \rangle)$ by solving two linear equations obtained from (14) for two different phase shifts, e.g., $\phi = 0, \pi/2$.

Thereafter, estimating $\langle \hat{q}_{r}^{2} \rangle$ and $\langle \hat{p}_{r}^{2} \rangle$ we can compute variances of quadrature observables. To this end, we calculate expectation values of the square of the difference of photon counts at the two output modes of the beam-splitter for two different values of $\phi$. Precisely, the expectation values read

$$\langle \hat{S}_{1}^{2}(\phi = 0) \rangle = \langle (\hat{I}_{2} - \hat{I}_{1})^{2} \rangle_{\phi = 0} = \langle \hat{q}_{k}^{2} \rangle + \langle \hat{p}_{k}^{2} \rangle - \langle \hat{q}_{k} \hat{p}_{k} \rangle - \langle \hat{q}_{k} \hat{p}_{k} \rangle = \langle \hat{q}_{r}^{2} \rangle + \langle \hat{p}_{r}^{2} \rangle + \langle \hat{q}_{k} \hat{p}_{k} \rangle + \langle \hat{q}_{k} \hat{p}_{k} \rangle,$$

and

$$\langle \hat{S}_{1}^{2}(\phi = \pi/2) \rangle = \langle (\hat{I}_{2} - \hat{I}_{1})^{2} \rangle_{\phi = \pi/2} = \langle \hat{q}_{k}^{2} \rangle + \langle \hat{p}_{k}^{2} \rangle - \langle \hat{q}_{k} \hat{p}_{k} \rangle - \langle \hat{q}_{k} \hat{p}_{k} \rangle = \langle \hat{q}_{r}^{2} \rangle + \langle \hat{p}_{r}^{2} \rangle + \langle \hat{q}_{k} \hat{p}_{k} \rangle + \langle \hat{q}_{k} \hat{p}_{k} \rangle.$$
where we use $[q_k, p_k] = [q_r, p_r] = i \mathbb{1}$. Thereafter, inserting the values of $\langle q_k^2 \rangle$, $\langle p_k^2 \rangle$, $\langle q_r^2 \rangle$, and $\langle q_r p_r + p_r q_r \rangle$ in (16), we can find out $\langle q_k p_k + p_k q_k \rangle$.

In Figure 1(b), we present a schematic diagram of Stokes-like measurements. It is shown below that three 50-50 beam-splitters and two phase shifters $(\phi_1, \phi_2)$ are sufficient to estimate all the elements of $C$. Here $\hat{a}_1$ and $\hat{a}_2$ are annihilation operators associated to the two modes of the Gaussian state $\rho_{12}$. Similarly, $\hat{a}_c$ and $\hat{a}_d$ are annihilation operators of the two modes of a pair of single-mode reference states. Here, $\hat{a}_1, \hat{a}_2, \hat{a}_c$, and $\hat{a}_d$ are the annihilation operators of input modes of the measurement setup, whereas $\hat{a}_3, \hat{a}_4, \hat{a}_5$, and $\hat{a}_6$ are the annihilation operators associated to output modes. Phase shifters at the input modes of the reference states can be changed by an observer. Here, the elements of the matrix $C$ are computed by measuring $\hat{S}_1(\phi_1), \hat{S}_1(\phi_2), \hat{S}_2^1(\phi_1), \hat{S}_2^1(\phi_2)$, and $\hat{S}_3(\phi_1, \phi_2)$. The intensity difference at the two output modes of beam-splitter 2 is computed as follows

$$\hat{S}_1(\phi_1) = \hat{I}_4 - \hat{I}_3 = \hat{a}_4^\dagger \hat{a}_4 - \hat{a}_3^\dagger \hat{a}_3 = \frac{1}{\sqrt{2}}((\hat{q}_1 - \hat{q}_2)^2 + (\hat{p}_1 - \hat{p}_2)^2).$$

(18)

Similarly, the intensity difference at the two output modes of beam-splitter 3 reads

$$\hat{S}_1(\phi_2) = \hat{I}_6 - \hat{I}_5 = \hat{a}_6^\dagger \hat{a}_6 - \hat{a}_5^\dagger \hat{a}_5 = \frac{1}{\sqrt{2}}((\hat{q}_1 + \hat{q}_2)^2 + (\hat{p}_1 + \hat{p}_2)^2).$$

(19)

Another type of Stokes-like measurement associated to joint measurement on the two output modes of beam-splitters 2 and 3 is given as

$$\hat{S}_2(\phi_1, \phi_2) = i(\hat{a}_6^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_6) = \frac{1}{\sqrt{2}}((\hat{q}_1 - \hat{q}_2)\hat{p}_1^c - (\hat{q}_1 + \hat{q}_2)\hat{p}_2^c - (\hat{p}_1 - \hat{p}_2)\hat{q}_d^c - (\hat{p}_1 + \hat{p}_2)\hat{q}_d^c)$$

$$+ \frac{1}{2}((\hat{q}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{q}_2) + \frac{1}{2}(\hat{q}_c \otimes \hat{q}_d \sin(\phi_1 - \phi_2))$$

$$+ \hat{q}_c \otimes \hat{p}_d \cos(\phi_1 - \phi_2) - \hat{q}_d \otimes \hat{p}_c \cos(\phi_1 - \phi_2)).$$

(20)

Here, anti-coincidence of the detected photons is to be detected by using photon number resolving detectors at the two output modes.

After performing measurements on many copies of the state $\rho_{12} \otimes \rho_c \otimes \rho_d$ one can estimate the expectation values of $\hat{S}_1^2(\phi_1 = 0), \hat{S}_1^2(\phi_2 = 0), \hat{S}_2^2(\phi_2 = \frac{\pi}{2}), \hat{S}_1(\phi_1 = 0) \otimes \hat{S}_1(\phi_2 = 0)$, and $\hat{S}_3(\phi_1 = 0, \phi_2 = 0)$. After, simplification we can write

$$\langle \hat{S}_1^2(\phi_1 = 0) \rangle = \langle (\hat{I}_4 - \hat{I}_3)^2 \rangle = \langle \hat{a}_4^\dagger \hat{a}_4 - \hat{a}_3^\dagger \hat{a}_3 \rangle$$

$$= \frac{1}{2}(\langle (\hat{q}_1 - \hat{q}_2)^2 \rangle + \langle (\hat{q}_1 - \hat{q}_2)(\hat{p}_1 + \hat{p}_2) \rangle + \langle (\hat{p}_1 - \hat{p}_2)(\hat{q}_1 - \hat{q}_2) \rangle + \langle (\hat{p}_1 - \hat{p}_2)^2 \rangle)$$

$$= \frac{1}{2}(\langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle - \langle \hat{q}_1 \hat{q}_2 \rangle - \langle \hat{p}_1 \hat{p}_2 \rangle + \langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle)$$

$$+ \langle (\hat{p}_1 \hat{q}_1 - \hat{q}_1 \hat{p}_1 - \hat{q}_1 \hat{q}_2 + \hat{p}_1 \hat{p}_2 \rangle)$$

$$+ \langle (\hat{p}_1 \hat{p}_2 - 2(\hat{p}_1 \hat{q}_1 + \hat{q}_1 \hat{p}_1) \rangle)$$

(21)

$$\langle \hat{S}_1^2(\phi_2 = 0) \rangle = \langle (\hat{I}_6 - \hat{I}_5)^2 \rangle = \langle \hat{a}_6^\dagger \hat{a}_6 - \hat{a}_5^\dagger \hat{a}_5 \rangle$$

$$= \frac{1}{2}(\langle (\hat{q}_1 - \hat{q}_2)^2 \rangle + \langle (\hat{q}_1 + \hat{q}_2)(\hat{p}_1 + \hat{p}_2) \rangle + \langle (\hat{p}_1 - \hat{p}_2)(\hat{q}_1 - \hat{q}_2) \rangle + \langle (\hat{p}_1 + \hat{p}_2)^2 \rangle)$$

$$= \frac{1}{2}(\langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle + \langle \hat{q}_1 \hat{q}_2 \rangle - \langle \hat{p}_1 \hat{p}_2 \rangle + \langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle)$$

$$+ \langle (\hat{p}_1 \hat{q}_1 - \hat{q}_1 \hat{p}_1 - \hat{q}_1 \hat{q}_2 + \hat{p}_1 \hat{p}_2 \rangle)$$

$$+ \langle (\hat{p}_1 \hat{p}_2 - 2(\hat{p}_1 \hat{q}_1 + \hat{q}_1 \hat{p}_1) \rangle)$$

(22)

$$\langle \hat{S}_2^2(\phi_2 = \frac{\pi}{2}) \rangle = \frac{1}{2}(\langle (\hat{q}_1 + \hat{q}_2)(\hat{q}_d - \hat{p}_d) + (\hat{p}_1 + \hat{p}_2)(\hat{q}_d + \hat{p}_d) \rangle)^2$$

$$= \frac{1}{4}(\langle \hat{q}_1^2 + 2(\hat{q}_1 \hat{q}_2 + \hat{q}_2 \hat{q}_1) \rangle - \langle \hat{q}_d \hat{q}_d \rangle - \langle \hat{p}_d \hat{p}_d \rangle + \langle \hat{q}_d \hat{p}_d \rangle)$$

$$+ \langle (\hat{p}_1 \hat{q}_1 - \hat{q}_1 \hat{p}_1 - \hat{q}_1 \hat{q}_2 + \hat{p}_1 \hat{p}_2 \rangle)$$

$$+ \langle (\hat{p}_1 \hat{p}_2 - 2(\hat{p}_1 \hat{q}_1 + \hat{q}_1 \hat{p}_1) \rangle)$$

(23)
\[ \langle \hat{S}_1(\phi_1 = 0) \rangle = \frac{1}{2} ((\langle \hat{q}_1^2 \rangle - \langle \hat{q}_2^2 \rangle)\langle \hat{q}_c \rangle\langle \hat{q}_d \rangle + (\langle \hat{q}_1^2 \rangle - \langle \hat{q}_2^2 \rangle)\langle \hat{p}_c \rangle\langle \hat{p}_d \rangle + (\langle \hat{q}_1 \hat{p}_1 \rangle - \langle \hat{q}_2 \hat{p}_2 \rangle)\langle \hat{q}_c \rangle\langle \hat{q}_d \rangle + (\langle \hat{p}_1 \hat{q}_1 \rangle - \langle \hat{p}_2 \hat{q}_2 \rangle)\langle \hat{p}_c \rangle\langle \hat{q}_d \rangle + (\langle \hat{q}_1 \hat{p}_2 \rangle - \langle \hat{p}_1 \hat{q}_2 \rangle)\langle \hat{q}_c \rangle + (\langle \hat{p}_1 \hat{q}_2 \rangle - \langle \hat{p}_2 \hat{q}_1 \rangle)\langle \hat{q}_c \rangle \rangle, \] 

(24)

and

\[ \langle \hat{S}_2(\phi_1 = 0, \phi_2 = 0) \rangle = \frac{1}{2\sqrt{2}} ((\langle \hat{q}_1 \rangle - \langle \hat{q}_2 \rangle)\langle \hat{p}_d \rangle + (\langle \hat{q}_1 \rangle + \langle \hat{q}_2 \rangle)\langle \hat{p}_c \rangle - (\langle \hat{p}_1 \rangle - \langle \hat{p}_2 \rangle)\langle \hat{q}_d \rangle - (\langle \hat{p}_1 \rangle + \langle \hat{p}_2 \rangle)\langle \hat{q}_c \rangle) \]

+ \frac{1}{2} (\langle \hat{q}_1 \rangle\langle \hat{p}_d \rangle - \langle \hat{p}_1 \rangle\langle \hat{q}_2 \rangle) + \frac{1}{2} (\langle \hat{q}_d \rangle\langle \hat{p}_c \rangle - \langle \hat{p}_d \rangle\langle \hat{q}_c \rangle). \]

(25)

The expectation values of the quadrature observables and square of the quadrature observables for reference modes can be computed directly (see Appendix B for details). For the single mode states of the two-mode Gaussian state we have already discussed the method to estimate \( \hat{q}_k, \hat{p}_k \), \( \langle \hat{q}_k^2 \rangle, \langle \hat{p}_k^2 \rangle \), \( \langle \hat{q}_k \hat{p}_k + \hat{p}_k \hat{q}_k \rangle \). Using commutation relation \( [q_k, p_k] = i\hbar \text{I} \), and for the unbiased reference state \( \langle \hat{q}_r \langle \hat{p}_r \rangle = 1/2 \), and \( \langle \hat{p}_r \rangle = -i/2 \), we can simplify Eqs. (21), and (22). As one can estimate \( \langle \hat{S}_1^2(\phi_1 = 0) \rangle \) and \( \langle \hat{S}_2^2(\phi_2 = 0) \rangle \) from the experimental data, the two unknown quantities \( \langle \hat{q}_1 \rangle, \langle \hat{q}_2 \rangle \) and \( \langle \hat{p}_1 \rangle, \langle \hat{p}_2 \rangle \) can be computed by solving the two linear equations (21), and (22). Similarly, for biased reference state, one can obtain \( \langle \hat{q}_r \rangle, \langle \hat{p}_r \rangle = 1/2 \), and \( \langle \hat{p}_r \rangle = -i/2 \) for certain choices of parameters (for details, see the explanation after Eq. (B6)).

Similarly, following such a method one can estimate \( \langle \hat{q}_1 \rangle\langle \hat{p}_2 \rangle \) and \( \langle \hat{p}_1 \rangle\langle \hat{q}_2 \rangle \) by solving the two linear equations (23), and (24), for certain choices of the pair of reference states. The expectation values \( \langle \hat{q}_1 \rangle\langle \hat{p}_2 \rangle \) and \( \langle \hat{p}_1 \rangle\langle \hat{q}_2 \rangle \) can be obtained by computing expectation values of \( \hat{S}_1^2(\phi = \frac{\pi}{2}) \) and \( \hat{S}_1(\phi_1 = 0) \rangle \otimes \hat{S}_1(\phi_2 = 0) \rangle \), along with the previously estimated expectation values of the set of quadrature observables. In the case of reference states for which the first moments are zero, i.e., \( \langle \hat{q}_c \rangle = 0 \) and/or \( \langle \hat{p}_c \rangle = 0 \), the same quantities can be estimated after solving Eqs. (23), and (25). In such a case, additionally, the expectation value of \( \hat{S}_3(\phi_1 = 0, \phi_2 = 0) \rangle \) must be obtained from the experimental data.

Thus, for a given reference state, the elements of the covariance matrix associated to each of the single modes of the two-mode Gaussian state can be estimated for a single measurement setup with three phase shifts. In addition, considering the measurement setup in Figure 1(b), for \( \phi_1 = 0 \) and \( \phi_2 = 0, \pi/2 \), the elements of the matrix \( C \rangle \) are computed. Once we know the elements of the covariance matrix, we can test (8) and quantify entanglement in the two-mode Gaussian state by computing \( \text{det}A, \text{det}B, \text{det}C, \text{and det} \Gamma_{\rho_{12}} \). Note that, the choice of phase shifts associated to Stokes-like measurements is not unique. It depends on the parameters of the reference state. However, as we don’t require any apriori knowledge of the first moments and the second moments of the unknown two-mode Gaussian state, the scheme discussed here is universal. On the other hand, following this scheme, to test the separability criterion (8) one needs to estimate all elements of the covariance matrix as well as the first moments of the two-mode Gaussian state. Thus, the proposed scheme leads to the full state tomography. This may be due to the fact that the measurements are performed on a single copy of the two-mode Gaussian state at a time. Similar results were obtained in the case of finite dimensional systems. According to the Refs. [32, 33], for finite dimensional systems, any universal entanglement detection scheme for a given bi-partite system which uses single copy of the state at a time amounts to the full state tomography. However, for continuous variable systems, e.g., two-mode Gaussian states, we are not aware of such result. In the next section (Sec. IV), we provide a scheme to estimate \( \text{det}A, \text{det}B, \text{det}C, \text{and det} \Gamma_{\rho_{12}} \) without performing the full state tomography. However, measurements on two copies of the two-mode Gaussian state are necessary. It is evident from the above discussion (related to Figure 1) that the aforesaid method of universal entanglement detection in two-mode Gaussian states is achieved here via LOCC only. On the other hand, measurement in an entangled basis is used in Sec. IV while dealing with two copies of the two-mode Gaussian states.

Note that, in case of standard homodyne detection (for example in Refs. [27, 28]) the intensity of signal state must be less than the intensity of the reference state. Thus, it is may not be a feasible task to look for a reference state which has a higher intensity than the intensity of a signal state of undefined photon numbers, e.g., macroscopic states [31]. In such a case, standard homodyne detection is difficult to perform. In Ref. [31], the elements of covariance matrix of a single mode macroscopic Gaussian state are estimated by performing Stokes-like measurements with a low-intensity reference state. In such a scheme, a strong coherent reference state can be replaced with a low-intensity displaced squeezed thermal reference state. In short, Stokes-like measurements generalize homodyne detection. The similar method we follow here to estimate elements of the covariance matrix of a two-mode Gaussian state. Also, it is worth mentioning that when the signal state is a macroscopic state then it is impractical to use photon number resolving detectors for photon counting. Thus, intensity detectors which estimates only moments of the photon numbers are useful for such scenario. However, the method forbids to reconstruct the whole density matrix, i.e., the full state tomography. Note that, this discussion is important while testing the scheme in an experiment. However, this is beyond the scope of the present work.
IV. ESTIMATION OF DETERMINANTS WITHOUT THE FULL STATE TOMOGRAPHY

In this section we discuss a different scheme of testing separability criterion (8), without doing a tomography of the state. Thus, the certification of entanglement can be recast as a task to compute determinants of block matrices $A$, $B$, $C$, and the covariance matrix $\Gamma_{\rho_{12}}$ with less resources than the resources required for the full state tomography.

From the definition of the Wigner function (2) of the two-mode Gaussian state one can check that

$$
(4\pi)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dp_{2} dq_{2} dp_{1} dq_{1} \ W_{\rho_{12}}^{2}(\hat{q}_{1}, \hat{p}_{1}, \hat{q}_{2}, \hat{p}_{2}) = \frac{1}{\sqrt{\det \Gamma_{\rho_{12}}}}
$$

Interestingly, the left-hand side of the equality is defined as $\text{Tr} \rho_{12}^{2}$. Similarly, we have $\text{Tr} \rho_{2}^{2} = 1/\sqrt{\det B}$, where $\rho_{1}$ and $\rho_{2}$ can be obtained by tracing out mode 2 and mode 1 of the two-mode Gaussian state $\rho_{12}$, respectively. Therefore, experimentally realizable schemes to estimate $\text{Tr} \rho_{12}^{2}$, $\text{Tr} \rho_{2}^{2}$, and $\text{Tr} \rho_{12}^{2}$ will yield det $A$, det $B$, and det $\Gamma_{\rho_{12}}$, respectively.

To this end we consider the SWAP operator $\hat{S} : \hat{S}(|n\rangle_{1} \otimes |m\rangle_{2}) = |n\rangle_{1} \otimes |m\rangle_{2}$, where $|k\rangle_{j}$ is the $k$-th Fock state of mode $j$. It can now be checked that: $\text{Tr} [\hat{S}(\rho_{1} \otimes \rho_{1})] = \text{Tr} \rho_{12}^{2}$, $\text{Tr} [\hat{S}(\rho_{2} \otimes \rho_{2})] = \text{Tr} \rho_{2}^{2}$. Such a SWAP operator acting on the two-mode system together is known to be self-adjoint, although physical realization of measurement of the SWAP operator on the two-mode system may turn out to be quite difficult (see Ref. [34]). Similar argument may also be provided for the physical realization of measurement of the $(2+2)$-modes SWAP operator $\hat{S}'$. Thus $\text{Tr} \rho_{12}^{2}$ can be obtained from the relation $\text{Tr} [\hat{S}'(\rho_{12} \otimes \rho_{12})] = \text{Tr} \rho_{2}^{2}$, where $\hat{S}'$ is the SWAP operator acting on 2 + 2 modes i.e., $\hat{S}'(|n\rangle_{1} \otimes |m\rangle_{1} \otimes |n\rangle_{2} \otimes |m\rangle_{2}) = (|n\rangle_{1} \otimes |m\rangle_{1} \otimes |n\rangle_{2} \otimes |m\rangle_{2})$ for all $n, m, r, s = 0, \ldots$. As $|n\rangle_{1} \otimes |n\rangle_{2}$ for $n = 0, 1, \ldots$ and $1/\sqrt{2}(|n\rangle_{1} \otimes |n+k\rangle_{1} \otimes |n\rangle_{2} \otimes |n+k\rangle_{2})$ for $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$ are the eigen states of the SWAP operator corresponding to the eigenvalue $+1$ while $(1/\sqrt{2})(|n\rangle_{1} \otimes |n+k\rangle_{1}-|n\rangle_{1} \otimes |n+k\rangle_{2})$ for $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$ are the eigenstates of the operator corresponding to the eigenvalue $-1$, therefore measurement of the SWAP operator $\hat{S}'$ would correspond to the projective measurement in the basis $\{|n\rangle_{1} \otimes |n\rangle_{2} : n = 0, 1, \ldots \} \cup \{(1/\sqrt{2})(|n\rangle_{1} \otimes |n+k\rangle_{1} \otimes |n\rangle_{2} \otimes |n+k\rangle_{2}) : n = 0, 1, \ldots, k = 1, 2, \ldots \}$. On the other hand, measurement of the SWAP operator $\hat{S}'$ would amount to measurement in the basis $\{|n\rangle_{1} \otimes |m\rangle_{1} \otimes (|n\rangle_{2} \otimes |m\rangle_{2}) : n, m = 0, 1, \ldots, k, l = 1, 2, \ldots \} \cup \{(1/\sqrt{2})(|n\rangle_{1} \otimes |m\rangle_{1} \otimes (|n+k\rangle_{2} \otimes |m+l\rangle_{2}) + (|n+k\rangle_{1} \otimes |m+l\rangle_{1} \otimes (|n\rangle_{2} \otimes |m\rangle_{2})) : n, m = 0, 1, \ldots, k, l = 1, 2, \ldots \}$ - a global measurement on all the four modes $1', 2', 2''$ together. Note that such a measurement is practically impossible to perform as it may seem to require photon number resolving detectors with infinite resolution, although recent experimental work [37] does provide implementation of $(1+1)$-mode SWAP operator as a unitary operator on two motional states in a system of trapped $^{177}Yb^{+}$ ions. In case there is some restriction on the average photon numbers of the input two-mode Gaussian states, one can, in principle, perform the aforesaid projective measurement with restricted photon number resolving detectors. We see that the measurement of $\hat{S}$ on the two copies of the single-mode reduced density matrix $\rho_{1}$, two copies of the single-mode reduced density matrix $\rho_{2}$, and also measurement of $\hat{S}'$ on two copies of the state $\rho_{12}$ provide us the values of the three local symplectic invariants det $A$, det $B$, and det $\Gamma_{\rho_{12}}$. Thus, we are left with the computation of det $C$. Three different methods of obtaining elements of matrix $C$ are given below.

**Method 1**: On single copy of the unknown two-mode Gaussian state Alice (in possession of mode 1) performs at random measurements of one of the two observables $\hat{q}_{1}$, $\hat{p}_{1}$. Also, on that same copy of the two-mode Gaussian state, Bob (in possession of mode 2) performs at random measurements of one of the two observables $\hat{q}_{2}$, $\hat{p}_{2}$. They will then communicate classically regarding the choice of their measurements. By this method, Alice and Bob together can find out the matrix $C$, and thereby another local symplectic invariant, det $C$.

**Method 2**: If we assume that the covariance matrix is of Simon type; i.e., $A = \lambda I_{2}$, $B = \mu I_{2}$, and $C = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$, then the computation can be simplified. ($I_{2}$ denotes the $2 \times 2$ identity matrix). Note that det $A$, det $B$, and det $C$ are invariant under local symplectic transformations. Hence by the method described above, we may calculate $\lambda$ and $\mu$ which are positive square roots of det $A$ and det $B$ respectively. To calculate $|\det C|$ we use the fact that due to the special structure of Simon’s form det $\Gamma_{\rho_{12}} = \det(\lambda \mu I_{2}-\lambda \mu I_{2})$. Simple algebra shows that det $\Gamma_{\rho_{12}} = (\lambda \mu)^{2} = (\lambda \mu)(s+t)+st$. Similarly we may apply a Gaussian rotation matrix of the form $\begin{bmatrix} \cos \theta I_{2} & \sin \theta I_{2} \\ -\sin \theta I_{2} & \cos \theta I_{2} \end{bmatrix}$. For the value $\theta = \frac{\pi}{4}$ we observe that the marginal covariance matrix with respect to mode 1 takes the form $(\lambda+\mu)I_{2}-(C+C')$. Determinant of this can be calculated by the previous method which will take the form $det((\lambda+\mu)I_{2}-(C+C')) = (\lambda+\mu)^{2}-2(\lambda+\mu)(s+t)+4st$. Since the value of $\lambda$ and $\mu$ are known, and the determinants of the left hand sides can be estimated, $s$ and $t$ can be calculated by solving the two equations. As a result the det $C$ can also be calculated.
and thereby, applying the corresponding phase-space displacements to bring down the aforesaid mean values to zero.

First finding out the mean values of the pump beam, respectively, then by definition, (see Refs. [38, 39]) for strength, which depends on the intensity of the pump beam as well as the nonlinearity of the OPA crystal and phase

\[ \hat{a}_1^{(1)}, \hat{a}_1^{(2)} \text{ and } \hat{a}_2^{(1)}, \hat{a}_2^{(2)} \]

two pairs of annihilation operators associated to the two pairs of input modes of the OPAs on Alice’s and Bob’s side, respectively. Note that, the correlated modes are \( \hat{a}_1^{(j)}, \hat{a}_2^{(j)} \). Pump1 and Pump2 are sources of pump beams incident on the OPA on Alice’s and Bob’s side, respectively. The output modes of the OPA on Alice’s and Bob’s side are represented by annihilation operators \( \hat{A}_k \) and \( \hat{B}_k \), respectively. Here \( j \in \{1, 2\} \) and \( k \in \{3, 4\} \).

**Method 3**: Here, our aim is to estimate the elements of the matrix \( C \) by performing Stokes-like measurements. To implement such a scheme, Alice and Bob are given two copies of a two-mode Gaussian state and each of them possesses one OPA [38]. In addition, we assume that the first moments of the Gaussian state are zero, i.e., \( \langle \hat{q}_i \rangle = \langle \hat{p}_i \rangle = 0 \) for \( i = 1, 2 \). This, one can take without loss of generality, as, otherwise by phase-space displacement one can bring the two-mode Gaussian state in that form. The amount of the aforesaid phase-space displacements can be obtained by first finding out the mean values \( \langle \hat{q}_1 \rangle, \langle \hat{p}_1 \rangle, \langle \hat{q}_2 \rangle, \langle \hat{p}_2 \rangle \) (following, as for example, the method described in Figure 1(a)), and thereby, applying the corresponding phase-space displacements to bring down the aforesaid mean values to zero.

The two modes of the \( j \)-th copy of the Gaussian state on Alice’s and Bob’s side are represented by the annihilation operators \( \hat{a}_1^{(j)} \) and \( \hat{a}_2^{(j)} \), respectively (see Figure 2). Note that, the correlated modes are denoted by the annihilation operators \( \hat{a}_1^{(j)}, \hat{a}_2^{(j)} \). Input modes of the each of the two OPAs are represented by the annihilation operators \( \hat{a}_1^{(1)}, \hat{a}_1^{(2)} \). After the interference separately at the two OPAs, the annihilation operators of the outputs on Alice’s and Bob’s side are \( \hat{A}_k \) and \( \hat{B}_k \), respectively. If \( \mu_1 = \cosh g_1 \) and \( \nu_1 = e^{i \Phi_1} \sinh g_1 \), where \( g_1 \) and \( \Phi_1 \) are parametrical strength, which depends on the intensity of the pump beam as well as the nonlinearity of the OPA crystal and phase of the pump beam, respectively, then by definition, (see Refs. [38, 39]) for \( l \in \{1, 2\} \), we can write

\[
A_3 = \mu_1 a_1^{(2)} + \nu_1 a_1^{(1)} \quad A_4 = \mu_1 a_1^{(1)} + \nu_1 a_1^{(2)} \quad \\
B_3 = \mu_2 a_2^{(2)} + \nu_2 a_2^{(1)} \quad B_4 = \mu_2 a_2^{(1)} + \nu_2 a_2^{(2)} ,
\]

where, on Alice’s side and Bob’s side, the OPA transformations are parametrized by \( (\mu_1 = \cosh g_1, \nu_1 = e^{i \Phi_1} \sinh g_1) \), and \( (\mu_2 = \cosh g_2, \nu_2 = e^{i \Phi_2} \sinh g_2) \), respectively.

To compute the elements of the matrix \( C \), we need to estimate expectation values of a set of Stokes-like measurements, e.g., \( \langle \hat{A}_1^\dagger \hat{B}_3 - \hat{B}_1^\dagger \hat{A}_3 \rangle, \langle \hat{A}_1^\dagger \hat{B}_3 + \hat{B}_1^\dagger \hat{A}_3 \rangle, \langle \hat{A}_1^\dagger \hat{B}_4 - \hat{B}_1^\dagger \hat{A}_4 \rangle, \) and \( \langle \hat{A}_1^\dagger \hat{B}_4 + \hat{B}_1^\dagger \hat{A}_4 \rangle \). Using (27) and replacing the annihilation and the creation operators with quadrature observers, a pair of expectation values, which requires measurements jointly on two different modes, is given as follows:

\[
\langle \hat{A}_1^\dagger \hat{B}_3 - \hat{B}_1^\dagger \hat{A}_3 \rangle = M_1(\langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle) + N_1(\langle \hat{q}_1 \hat{p}_2 \rangle - \langle \hat{p}_1 \hat{q}_2 \rangle),
\]

\[
\langle \hat{A}_1^\dagger \hat{B}_3 + \hat{B}_1^\dagger \hat{A}_3 \rangle = M_2(\langle \hat{q}_1 \hat{q}_2 \rangle + \langle \hat{p}_1 \hat{p}_2 \rangle) + N_2(\langle \hat{q}_1 \hat{p}_2 \rangle - \langle \hat{p}_1 \hat{q}_2 \rangle),
\]

where \( M_1, N_1, M_2, \) and \( N_2 \) [41] are constants. For another pair of Stokes-like measurements we obtain

\[
\langle \hat{A}_1^\dagger \hat{B}_4 - \hat{B}_1^\dagger \hat{A}_4 \rangle = M_1'(\langle \hat{q}_1 \hat{q}_2 \rangle - \langle \hat{p}_1 \hat{p}_2 \rangle) + N_1'(\langle \hat{q}_1 \hat{p}_2 \rangle + \langle \hat{p}_1 \hat{q}_2 \rangle),
\]

FIG. 2: A schematic diagram of the measurement setup to estimate all elements of the matrix \( C \) associated to a two-mode Gaussian state. Two copies of the state are shared between Alice and Bob, \( (\hat{a}_1^{(1)}, \hat{a}_1^{(2)}) \) and \( (\hat{a}_2^{(1)}, \hat{a}_2^{(2)}) \) are two pairs of annihilation operators associated to the two pairs of input modes of the OPAs on Alice’s and Bob’s side, respectively.
where $M_1', N_1', M_2'$, and $N_2'$ [42] are constants. Here we consider the fact that $\langle \hat{q}_1^{(j)} \rangle = \langle \hat{p}_1^{(j)} \rangle = 0$. Also, $\langle \hat{q}_1^{(j)} \otimes \hat{q}_2^{(j)} \pm \hat{p}_1^{(j)} \otimes \hat{p}_2^{(j)} \rangle = \langle \hat{q}_1 \otimes \hat{q}_2 \pm \hat{p}_1 \otimes \hat{p}_2 \rangle = \langle \hat{q}_1 \otimes \hat{q}_2 \pm \hat{p}_1 \otimes \hat{p}_2 \rangle$. Similarly, we use $\langle \hat{q}_1^{(j)} \otimes \hat{p}_2^{(j)} \pm \hat{p}_1^{(j)} \otimes \hat{q}_2^{(j)} \rangle = \langle \hat{q}_1 \otimes \hat{p}_2 \pm \hat{p}_1 \otimes \hat{q}_2 \rangle$. This is so because $\langle \hat{q}_1^{(j)} \otimes \hat{q}_2^{(j)} \rangle, \langle \hat{q}_1^{(j)} \otimes \hat{p}_2^{(j)} \rangle, \langle \hat{q}_2^{(j)} \otimes \hat{p}_1^{(j)} \rangle$, and $\langle \hat{p}_1^{(j)} \otimes \hat{p}_2^{(j)} \rangle$ are independent, where $j \in \{1, 2\}$. Note that, $\langle \hat{q}_m^{(s)} \otimes \hat{p}_n^{(t)} \rangle = \langle \hat{q}_m^{(s)} \rangle \langle \hat{p}_n^{(t)} \rangle = 0$, where $s \neq t$. As expectation values of the Stokes-like measurements can be estimated from experiments, all the elements of matrix $C$ can be obtained by solving Eqs. (28), (29), (30), and (31). It is worth mentioning that the elements of the $C$ matrix can also be found by computing expectation values of generators of SU(1, 1) group [40].

Note that, the total number of measurements to obtain $\det A, \det B,$ and $\det \Gamma_{p12}$ by SWAP operations is three. In addition, the required numbers of measurements for computing $\det C$ are four (random measurements of quadrature observables), one (for special forms of the matrices $A, B, C,$ one can perform SWAP operation $S'$), and four (for given first moments of the quadratures of the state, four Stokes-like measurements are required here) for Method 1, Method 2, and Method 3, respectively. Although, for Methods 1, and 3, the $\det C$ can be computed from the elements of matrix $C$, but det $B$, and det $\Gamma_{p12}$ are found without obtaining the elements of the matrices. Precisely, we don’t need to estimate $\langle \hat{q}_m^2 \rangle, \langle \hat{p}_m^2 \rangle$, and $\frac{1}{2} \langle \hat{q}_m \hat{p}_m + \hat{p}_m \hat{q}_m \rangle$. We have thereby, described a scheme to test the separability criterion (8) for two-mode Gaussian state without the full state tomography. Thus, resource-wise, the scheme is more economical than the full state tomography.

V. DISCUSSION

In this paper we present altogether four schemes for detection of entanglement in an unknown two-mode Gaussian state. The first three schemes are described with measurements on a single copy of the state at a time. An identical set of five measurements is required to execute the schemes demonstrated in Section II and in Appendix A. However, it will be interesting to find a set of experimentally measurable quantities for such five measurements which will be resource-wise more economical than a set of measurements for the full state tomography although single copy usage of the state does not seem to be better than state tomography in terms of resource requirements. In Section III, we provide an experimental friendly scheme to test the separability criterion (8). The elements of the covariance matrix of the unknown Gaussian state are estimated by measuring intensity at the outputs of the interferometric setups (see Figure 1). We conjecture that as in such cases the manipulations are made on single copy of the state, the scheme leads to the full state tomography. Finally, in Section IV, we discuss a scheme by considering measurements on two copies of the state at a time. Interestingly, one can exploit the structure of SWAP operator and estimate the determinant (taking two copies of the state at a time) of matrices $A, B$. In addition, considering OPA transformations on two copies of the two-mode Gaussian state, we compute $\det C$. As it does not require knowledge of each and every parameter of the state, the scheme does not amount to the full state tomography. It is worth mentioning that the schemes described in present work, also give rise to the estimation of the measure of entanglement in the two-mode Gaussian state. For example, one can quantify the entanglement of a two-mode Gaussian state in terms of the logarithmic negativity (see the discussion below Eq. (8)). Note that, the advantage of the scheme presented in Section III over other schemes with measurements on two copies of the state at a time lies in the fact that the former can be more robust against errors than the later if the source which prepares the unknown two-mode Gaussian state is erroneous. In addition, in Ref. [31], it was shown that the robustness of the estimation of covariance matrix of the single mode Gaussian state does not depend on the choice of biased or unbiased displaced squeezed thermal state (reference state) for about $10^4 - 10^5$ measurements. We expect the same for the two-mode Gaussian state. However, we need to investigate the robustness of the scheme by considering imperfect detectors, and mode-matching error between signal states and reference states interfering on beam-splitters. We also need to look for realistic measurements rather than the measurement of the $(1+1)$-modes as well as $(2+2)$-modes SWAP operators $\hat{S}$, and $\hat{S}'$, respectively while dealing with two copies of the two-mode Gaussian states together.

In future, we would like to extend the scheme in Section III for multimode systems together with finding out the optimal universal entanglement witnessing scheme for any given multi-mode Gaussian states. Also, a measurement-device independent scheme to detect entanglement in Gaussian states will be presented in a forthcoming paper.

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Appendix A: Measurement scheme II

In the second scheme, Alice and Bob divide the $5N(N \gg 1)$ copies of two-mode Gaussian state $\rho_{12}$ in two groups, e.g., a group with $4N$ copies of $\rho_{12}$ and another one with $N$ copies of $\rho_{12}$. Here, exchange of 1 bit of classical communication between Alice and Bob is required to distinguish between two groups. Let’s consider, at first Alice and Bob choose to perform measurements on $4N$ copies of $\rho_{12}$. Thereafter, Alice randomly measures $\hat{q}_{1}$ and $\hat{p}_{1}$ on mode 1, whereas Bob randomly measures $\hat{q}_{2}$ and $\hat{p}_{2}$ on mode 2 of $\rho_{12}$ from the group with $4N$ copies of the state. Note that, effectively, Alice and Bob measure $\hat{A} = \hat{q}_{1} \otimes \hat{q}_{2}$, $\hat{B} = \hat{p}_{1} \otimes \hat{p}_{2}$, $\hat{D} = \hat{q}_{1} \otimes \hat{p}_{2}$, and $\hat{E} = \hat{p}_{1} \otimes \hat{q}_{2}$ at random. Classical communication between Alice and Bob is necessary to know precisely if $\hat{q}_{1}$ is measured along with $\hat{q}_{2}$ or $\hat{p}_{2}$ and $\hat{p}_{1}$ is measured along with $\hat{q}_{2}$ or $\hat{p}_{2}$. In turn, this will lead to joint probabilities $P(\hat{q}_{1}, \hat{q}_{2})$, $P(\hat{q}_{1}, \hat{p}_{2})$, $P(\hat{p}_{1}, \hat{q}_{2})$, and $P(\hat{p}_{1}, \hat{p}_{2})$. On each of the remaining $N$ copies of $\rho_{12}$ Alice and Bob measure $\frac{1}{2}(\hat{q}_{1}\hat{p}_{1} + \hat{p}_{1}\hat{q}_{1})$, and $\frac{1}{2}(\hat{q}_{2}\hat{p}_{2} + \hat{p}_{2}\hat{q}_{2})$, respectively. This is equivalent to measure $\hat{C} = \frac{1}{2}(\hat{q}_{1}\hat{p}_{1} + \hat{p}_{1}\hat{q}_{1})$ and $\frac{1}{2}(\hat{q}_{2}\hat{p}_{2} + \hat{p}_{2}\hat{q}_{2})$. In this case, no classical communication between Alice and Bob is needed to compute joint probability $P\left(\frac{1}{2}(\hat{q}_{1}\hat{p}_{1} + \hat{p}_{1}\hat{q}_{1}), \frac{1}{2}(\hat{q}_{2}\hat{p}_{2} + \hat{p}_{2}\hat{q}_{2})\right)$.

Interestingly, Alice can estimate expectation values $\langle \hat{q}^{2}_{1} \rangle$, and $\langle \hat{p}^{2}_{1} \rangle$ by separately measuring $\hat{q}_{1}$, and $\hat{p}_{1}$ on many copies of mode 1 of $\rho_{12}$. Likewise, Bob obtains expectation values $\langle \hat{q}^{2}_{2} \rangle$, and $\langle \hat{p}^{2}_{2} \rangle$ by separately performing measurements of $\hat{q}_{2}$, and $\hat{p}_{2}$ on many copies of mode 2 of $\rho_{12}$. Note that, expectation values which are defining correlations between measurements on two modes e.g., $\langle \hat{q}_{1} \otimes \hat{q}_{2} \rangle$, $\langle \hat{p}_{1} \otimes \hat{p}_{2} \rangle$, $\langle \frac{1}{2}(\hat{q}_{1}\hat{p}_{1} + \hat{p}_{1}\hat{q}_{1}) \otimes \frac{1}{2}(\hat{q}_{2}\hat{p}_{2} + \hat{p}_{2}\hat{q}_{2}) \rangle$, $\langle \hat{q}_{1} \otimes \hat{p}_{2} \rangle$, and $\langle \hat{p}_{1} \otimes \hat{q}_{2} \rangle$ are computed by separately measuring $\hat{A} = \hat{q}_{1} \otimes \hat{q}_{2}$, $\hat{B} = \hat{p}_{1} \otimes \hat{p}_{2}$, $\hat{C} = \frac{1}{2}(\hat{q}_{1}\hat{p}_{1} + \hat{p}_{1}\hat{q}_{1}) \otimes \frac{1}{2}(\hat{q}_{2}\hat{p}_{2} + \hat{p}_{2}\hat{q}_{2})$, $\hat{D} = \hat{q}_{1} \otimes \hat{p}_{2}$, and $\hat{E} = \hat{p}_{1} \otimes \hat{q}_{2}$ on many copies of $\rho_{12}$. The first moments of the state are estimated in Section II. Thus, all the elements of the covariance matrix of a two-mode Gaussian state are estimated with less resources than the full state tomography.

Appendix B: Calculations of elements of the covariance matrix of a displaced squeezed thermal state

In our analysis, we have considered displaced squeezed thermal state as a reference state. The thermal state is represented by a density matrix of the following form:

$$\rho_{\tau} = \sum_{n=0}^{\infty} \frac{\bar{n}_{\tau}^{n}}{(\bar{n}_{\tau} + 1)^{n+1}} |n\rangle \langle n|,$$

where $\rho_{\tau}$ is expressed in the number state basis and $\bar{n}_{\tau}$ is the mean photon number of the thermal state. Displaced squeezed thermal state $\rho_{\tau}^{DS}$ is obtained after applying a unitary transformation $U_{\tau} = D(\alpha(\tau))S(\xi(\tau))$ on the state $\rho_{\tau}$, where $D(\alpha(\tau))$ is a displacement operator: $D(\alpha(\tau)) = \exp(\alpha_{\tau}^{\dagger}a - \alpha_{\tau}a^{\dagger})$, and $S(\xi(\tau))$ is a squeezing operator: $S(\xi(\tau)) = \exp(\frac{1}{2} (\xi^{2}a^{\dagger 2} - \xi^{*}a^{2}))$. Here $\alpha = \delta_{r}e^{i\beta_{r}}$, and $\xi = \theta_{r}e^{i\gamma_{r}}$. Note that, for an unbiased displaced squeezed thermal state $\beta_{r} = \gamma_{r} = 0$. A biased $\rho_{\tau}^{DS}$ is defined as follows:

$$\rho_{\tau}^{DS} = D(\alpha(\tau))S(\xi(\tau))\rho_{\tau}S(\xi(\tau))^{\dagger}D(\alpha(\tau))^{\dagger}.$$  

Thereafter, the expectation value of an observable $\hat{O}$ reads

$$\langle \hat{O} \rangle = \text{Tr} \left( \rho_{\tau}^{DS} \hat{O} \right) = \text{Tr} \left( D(\alpha(\tau))S(\xi(\tau))\rho_{\tau}S(\xi(\tau))^{\dagger}D(\alpha(\tau))^{\dagger} \hat{O} \right)$$

$$= \text{Tr} \left( \rho_{\tau}S(\xi(\tau))^{\dagger}D(\alpha(\tau))^{\dagger} \hat{O} D(\alpha(\tau))S(\xi(\tau)) \right),$$
where the last equality follows from invariance of trace under cyclic-permutation of operators. Next, in our derivation of expectation values of observables we consider the following set of transformations of annihilation and creation operators:

\begin{align}
\hat{a}_r &\rightarrow \hat{a}_r \cosh \theta_r - e^{-i \gamma_r} \hat{a}_r^\dagger \sinh \theta_r + \alpha_r \\
\hat{a}_r^\dagger &\rightarrow \hat{a}_r^\dagger \cosh \theta_r - e^{-i \gamma_r} \hat{a}_r \sinh \theta_r + \alpha_r^*.
\end{align}

(B4)

For displaced squeezed thermal state, the expectation values of quadrature observables and square of the quadrature observables reads

\begin{align}
\langle \hat{q}_r \rangle &= \sqrt{2} d_r \cos \beta_r, \\
\langle \hat{p}_r \rangle &= \sqrt{2} d_r \sin \beta_r \\
\langle \hat{q}_r^2 \rangle &= \left( \bar{n}_r + \frac{1}{2} \right) \left( e^{2 \theta_r} \sin^2 \gamma_r + e^{-2 \theta_r} \cos^2 \gamma_r \right) + 2 d_r^2 \cos^2 \beta_r, \\
\langle \hat{p}_r^2 \rangle &= \left( \bar{n}_r + \frac{1}{2} \right) \left( e^{2 \theta_r} \cos^2 \gamma_r + e^{-2 \theta_r} \sin^2 \gamma_r \right) + 2 d_r^2 \sin^2 \beta_r.
\end{align}

(B5)

In addition, we compute

\begin{align}
\langle \hat{q}_r \hat{p}_r \rangle &= d_r^2 \sin 2 \beta_r - \left( \bar{n}_r + \frac{1}{2} \right) \sinh 2 \theta_r \sin \gamma_r + \frac{i}{2} \\
\langle \hat{p}_r \hat{q}_r \rangle &= d_r^2 \sin 2 \beta_r - \left( \bar{n}_r + \frac{1}{2} \right) \sinh 2 \theta_r \sin \gamma_r - \frac{i}{2}.
\end{align}

(B6)

Note that, for unbiased displaced squeezed state \( \langle \hat{q}_r \hat{p}_r \rangle = - \langle \hat{p}_r \hat{q}_r \rangle = \frac{i}{2} \). Similarly, one can choose \( d_r, \bar{n}_r, \beta_r, \gamma_r \), and \( \theta_r \) in such a way that, we obtain \( \langle \hat{q}_r \hat{p}_r \rangle = - \langle \hat{p}_r \hat{q}_r \rangle = \frac{i}{2} \), for \( d_r^2 \sin 2 \beta_r = \left( \bar{n}_r + \frac{1}{2} \right) \sinh 2 \theta_r \sin \gamma_r \), where \( \beta_r, \gamma_r \), and \( \theta_r \) are non-zero. From (B5), one can calculate

\begin{align}
\langle \hat{q}_r^2 \rangle - \langle \hat{p}_r^2 \rangle &= 2 d_r^2 \cos 2 \beta_r - (2 \bar{n}_r + 1) \sinh 2 \theta_r \cos \gamma_r \\
\langle \hat{q}_r^2 \rangle + \langle \hat{p}_r^2 \rangle &= 2 d_r^2 + (2 \bar{n}_r + 1) \cosh 2 \theta_r.
\end{align}

(B7)