CONFORMAL KILLING FORMS ON NEARLY KÄHLER MANIFOLDS

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Abstract. We study conformal Killing forms on compact 6-dimensional nearly Kähler manifolds. Our main result concerns forms of degree 3. Here we give a classification showing that all conformal Killing 3-forms are linear combinations of $d\omega$ and its Hodge dual $\ast d\omega$, where $\omega$ is the fundamental 2-form of the nearly Kähler structure. The proof is based on a fundamental integrability condition for conformal Killing forms. We have partial results in the case of conformal Killing 2-forms. In particular we show the non-existence of $J$-anti-invariant Killing 2-forms.

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1. Introduction

Conformal Killing forms are a natural generalization of conformal vector fields on Riemannian manifolds. They are defined as sections in the kernel of a certain conformally invariant first order differential operator. Conformal Killing forms which are in addition co-closed are called Killing forms. Equivalently Killing forms are characterized by a totally skew-symmetric covariant derivative, thus generalizing Killing vector fields in degree one. Killing forms were intensively studied in physics since they define first integrals of the equations of motion, i.e. functions which are constant along geodesics.

There are not too many known examples of compact Riemannian manifolds admitting non-parallel conformal Killing forms and even less for Killing forms. On the standard sphere the space of conformal Killing forms is of maximal dimension. It coincides with the sum of two eigenspaces of the Laplace operator on forms corresponding to minimal eigenvalues. Otherwise examples of Killing forms are usually related to special geometric situations, e.g. Killing forms exist on Sasakian, nearly Kähler or nearly parallel $G_2$ manifolds.

It is known that there are no non-parallel Killing forms on compact manifolds of special holonomy, e.g. for symmetric spaces, Kähler manifolds, or manifolds of holonomy $G_2$ or $Spin_7$ (cf. [2], [13], [15]). Much less is known about conformal Killing forms on such manifolds or on manifolds with a weakened holonomy condition, i.e. with some special structure group reduction. There is a classification of conformal Killing forms on compact Kähler manifolds.
In particular, in degree 2 they turn out to be $J$-invariant and related to so-called Hamiltonian 2-forms, e.g. studied in [1]. In the more general case there are only partial and unpublished results, e.g. in [14] it could be shown that any conformal Killing form on a compact Sasaki-Einstein manifold has to be one of the standard examples.

In the present article we study conformal Killing forms on compact 6-dimensional nearly Kähler manifolds. These are by definition almost Hermitian manifolds $(M, g, J)$ where the fundamental 2-form $\omega$ is a Killing 2-form, i.e. has totally skew-symmetric covariant derivative. In dimension 6 nearly Kähler manifolds are particularly interesting, e.g. they are Einstein and the metric cone has holonomy $G_2$. Moreover, with $\omega$ also $d\omega$ is a conformal Killing form and, since the Hodge $*$-operator preserves the space of conformal Killing forms, also $*\omega$ and $*d\omega$ are conformal Killing forms. Our main result states that there are no other examples in degree 3, i.e. we have

**Theorem 1.1.** Let $(M^6, g, J)$ be a compact 6-dimensional strict nearly Kähler manifold. Then any conformal Killing 3-form on $M$ is a linear combination of $d\omega$ and $*d\omega$.

It is interesting to note that the proof of this theorem is based on a fundamental integrability condition for conformal Killing forms. This condition, which is related to the fact that conformal Killing forms are components of a parallel section in the so-called prolongation bundle, is known for a long time but so far stayed with almost no application.

In degree 2 the situation seems to be more complicated. Here it is not possible to apply the integrability condition and also other methods useful in the case of degree 3 can not be applied. However, it is still possible to show the non-existence of Killing 2-forms of special type, e.g. we show in Propositions 5.1 and 5.2 that on a compact 6-dimensional strict nearly Kähler manifold there are no $J$-anti-invariant Killing or $*$-Killing forms of degree 2.

In a recent work I. Dotti and C. Herrera study Killing 2-forms on homogeneous spaces $G/K$ invariant under the group $G$. In particular they can show that there are no other Killing 2-forms on the nearly Kähler flag manifold $SU(3)/T^2$ (cf. [4]). Given this result it seems to be a natural to conjecture that compact strict nearly Kähler manifold in dimension 6 do not admit Killing 2-forms non-proportional to the fundamental 2-form $\omega$.

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### 2. Nearly Kähler manifolds

Let $(M, g, J)$ be an almost Hermitian manifold, i.e. the Riemannian manifold $(M, g)$ admits an almost complex structure $J$ compatible with $g$. In this situation the fundamental 2-form is defined as $\omega(X, Y) = g(JX, Y)$ for tangent vectors $X, Y$. Using a local orthonormal basis $\{e_i\}$ it can be written as $\omega = \frac{1}{2} \sum e_i \wedge Je_i$. 

Definition 2.1 (cf. [5]). A nearly Kähler manifold is an almost Hermitian manifold \((M, g, J)\) such that the fundamental 2-form \(\omega\) is a Killing 2-form, i.e. satisfies \(X_j \nabla_X \omega = 0\) for every tangent vector \(X\). A nearly Kähler manifold is called strict if \(\nabla_X J \neq 0\) for every \(X \neq 0\).

In this article we are mainly interested in strict 6-dimensional nearly Kähler manifolds. These manifolds are automatically Einstein with positive scalar curvature. As usual we will normalize the scalar curvature to \(\text{scal} = 30\). The 3-form \(\Psi^+ := d \omega\) is of type \((3,0)+(0,3)\) and the real part of a complex volume form \(\Psi := \Psi^+ + i \Psi^-,\) where \(\Psi^-\) is the Hodge dual of \(\Psi^+\). The differential forms \(\omega\) and \(\Psi^+\) define a structure group reduction from \(O(6)\) to \(SU(3)\). Conversely, 6-dimensional nearly Kähler manifolds can be described as Riemannian manifolds \((M^6, g)\) with a \(SU(3)\)-structure \((\omega, \Psi^+)\) satisfying the equations \(d \omega = 3 \Psi^+\) and \(d \Psi^- = -2 \omega \wedge \omega\). More details on \(SU(3)\)-structures and 6-dimensional nearly Kähler manifolds can be found in [9].

In dimension 6 there are only a few examples known. The homogeneous ones, classified in [5], are the standard sphere \(S^6\) and \(S^3 \times S^3, SU(3)/T^2\) and \(CP^3\) with their nearly Kähler metrics. Only quite recently new examples were found. These are metrics of cohomogeneity one on \(S^6\) and on \(S^3 \times S^3\) (cf. [5]).

In the rest of this section we collect some results on the curvature of nearly Kähler manifolds, which are needed later. Let \((M^n, g, J)\) be a nearly Kähler manifold. The Riemannian curvature \(R\) can be considered as a map \(R : \Lambda^2 T M \rightarrow \Lambda^2 T M\), the so-called curvature operator. It is defined by the equation \(g(R(X \wedge Y), Z \wedge V) = R(X, Y, Z, V)\) for any tangent vectors \(X, Y, Z, V\). Note that with this convention \(R\) acts as \(-i \Gamma_{XY}\) on the standard sphere \(S^n\).

For further use we have to recall the definition of the curvature endomorphism \(q(R)\). Let \(P = P_{SO(n)}\) be the frame bundle and \(EM\) any vector bundle associated to \(P\) via a \(SO(n)\)-representation \(\rho : SO(n) \rightarrow \text{Aut}(E)\), where \(E\) is some real or complex vector space. Then \(q(R) \in \text{End}(EM)\) is defined as

\begin{equation}
q(R) := \frac{1}{2} (e_i \wedge e_j)_* \circ R(e_i \wedge e_j)_*,
\end{equation}

where \(\{e_i\}, i = 1, \ldots, n\), is a local orthonormal frame. Here and henceforth, we use Einsteins summation convention on repeated subscripts. We also identify \(TM\) with \(T^* M\) using the metric. The 2-form \(X \wedge Y \in \Lambda^2 T \cong \mathfrak{so}(n)\) acts via the differential \(\rho_*\) of the representation \(\rho\), we write \((X \wedge Y)_* = \rho_*(X \wedge Y)\). In particular we have for any tangent vectors \(X, Y, Z\) the standard action of \(\mathfrak{so}(n)\) written as \((X \wedge Y)_* Z = g(X, Z) Y - g(Y, Z) X\). Moreover for any section \(\varphi \in \Gamma(EM)\) we have \(R(X \wedge Y)_* \varphi = R_{XY} \varphi\). It is easy to check that \(q(R)\) acts as the Ricci endomorphism on tangent vectors. We remark that \(q(R)\) may be defined in this way for any curvature tensor \(R\), e.g. for the curvature tensor \(\bar{R}\) of the canonical hermitian connection \(\nabla\). Recall that the canonical connection \(\nabla\) is defined by \(\nabla_X Y := \nabla_X Y - \frac{1}{2} J(\nabla_X J) Y\) for any vector fields \(X, Y\). This is a \(U_m\)-connection, i.e. \(\nabla g = 0\) and \(\nabla J = 0\).

It is well-known that the Riemannian curvature tensor \(R\) of a nearly Kähler manifold can be written as \(R_{XY} = -(X \wedge Y)_* + R^C_{XY},\) where \(R^C_{XY}\) is a curvature tensor of Calabi-Yau...
type (cf. [8], p. 253). In other words, the curvature operator $R^{CY} = R + \text{id} : \Lambda^2 T \rightarrow \Lambda^2 T$ takes values only in $\Lambda^{1,1}_0 T \cong \mathfrak{su}(n)$. Equivalently we have for every tangent vectors $X, Y$ the equations

\begin{equation}
R^{CY}_{XY} \circ J = J \circ R^{CY}_{XY} \quad \text{and} \quad \text{tr} (R^{CY}_{XY} \circ J) = 0.
\end{equation}

Note, that the second equation can be written as $R^{CY}(\omega) = 0$ and also as $R(\omega) = -\omega$. The first equation in \eqref{c1} can be rewritten in several ways, giving compatibility equations for the Riemannian curvature $R$ and the almost complex structure $J$. In particular we have

**Proposition 2.2.** Let $(M^n, g, J)$ be a nearly Kähler manifold. Then for any tangent vectors $X, Y, Z$ the following equations hold

\begin{enumerate}[(i)]
  \item $R_{XY}JZ = JR_{XY}Z + J(X \wedge Y)_* Z - (X \wedge Y)_* JZ$
  \item $R_{JX}JYZ = R_{XY}Z + (X \wedge Y)_* Z - (JX \wedge JY)_* Z$
  \item $R_{XJY}Z = -R_{JXY}Z - (JX \wedge JY)_* Z$
\end{enumerate}

**Proof.** The first equation is exactly \eqref{c1} written in terms of $R$. The second equation follows from the first by the following calculation

\[ g(R_{JX}JYZ, V) = g(R_{ZW}JX, JY) = g(R_{ZW}X, Y) + g(J(Z \wedge V)_* X - (Z \wedge V)_* JX, JY) \]

\[ = g(R_{XY}Z, V) + g((Z \wedge V)_* X, Y) - g((Z \wedge V)_* JX, JY) \]

\[ = g(R_{XY}Z, V) + g(Z \wedge V, X \wedge Y) - g(Z \wedge V, JX \wedge JY) \]

\[ = g(R_{XY}Z, V) + g((X \wedge Y)_* Z, V) - g((JX \wedge JY)_* Z, V). \]

The third equation follows from the second by replacing $X$ with $JX$. \hfill \Box

As a consequence of these curvature equations we have several equations on the space of $p$-forms, which we will later apply to conformal Killing forms on nearly Kähler manifolds. For any $p$-form $\sigma$ and any tangent vector $X$ we define the curvature expressions

\begin{equation}
R^+(X)\sigma := e_i \wedge R_{Xe_i}\sigma \quad \text{and} \quad R^-(X)\sigma := e_i \hook R_{Xe_i}\sigma,
\end{equation}

where $\{e_i\}$ is a local orthonormal frame. Recall that the almost complex structure $J$ extends to a map on $p$-forms. It is defined as $J_\ast \sigma = J e_i \wedge e_i \hook \sigma = \omega_\ast \sigma$. In particular, we note that $R(\omega) = -\omega$ can be written as $R_{e_i}J_{e_i}\sigma = -2 J_\ast \sigma$. Moreover, we recall that $(J_\ast)^2$ acts as $-(p-q)^2 \text{id}$ on forms of type $(p, q) + (q, p)$ (cf. [7], p. 60). For later use we note that $J_\ast$ is injective on the space of 3-forms. Indeed, the space of 3-forms on an almost Hermitian manifold decomposes into the sum of spaces of forms of types $(3, 0) + (0, 3)$ and $(1, 2) + (2, 1)$.

**Corollary 2.3.** Let $(M^6, g, J)$ be a 6-dimensional nearly Kähler manifold. Then for any tangent vector $X$ and any $p$-form $\sigma$ the following equations hold

\begin{enumerate}[(i)]
  \item $Je_i \hook R_{Xe_i}\sigma = R^-(JX)\sigma - J_\ast (X \hook \sigma) + 2X \wedge \omega \hook \sigma + (5-p)JX \hook \sigma$
  \item $Je_i \hook R^+(e_i)\sigma + Je_i \wedge R^-(e_i)\sigma = -6 J_\ast \sigma$
\end{enumerate}
Conformal Killing forms

Proof. We use the third equation of Proposition 2.2 and the definition of $J$ to obtain

$$Je_i \mathcal{R}_{X, e_i} \sigma = -e_i \mathcal{R}_{X, Je_i} \sigma$$

$$= R^-(JX) \sigma + e_i \mathcal{R}(e_i \wedge JX \mathcal{R} - JX \wedge e_i) \sigma$$

$$+ e_i \mathcal{R}(Je_i \wedge X \mathcal{R} - X \wedge Je_i) \sigma$$

$$= R^-(JX) \sigma + 6JX \mathcal{R} \sigma - (p - 1) JX \mathcal{R} \sigma - JX \mathcal{R} \sigma$$

$$- J(X \mathcal{R} \sigma) - JX \mathcal{R} \sigma + X \wedge 2\omega \mathcal{R} \sigma$$

$$= R^-(JX) \sigma - J(X \mathcal{R} \sigma) + 2X \wedge \omega \mathcal{R} \sigma + (5 - p) JX \mathcal{R} \sigma$$

This proves the first equation of Corollary 2.3. Rewriting the first summand of the second equation we find

$$Je_i \mathcal{R}^+(e_i) \sigma = Je_i \mathcal{R}(e_j \wedge Re_i e_j) \sigma = g(Je_i, e_j) Re_i e_j \sigma + e_j \wedge Je_i \mathcal{R} e_i e_j \sigma .$$

Using the first equation of Corollary 2.3 with $X = e_j$ and the equation $Re_i, e_i = -2J$ we obtain

$$Je_i \mathcal{R}^+(e_i) \sigma = Re_i e_i \sigma + e_j \mathcal{R}(e_j \mathcal{R} - J(e_j) \sigma - J(e_j) \sigma + 2e_j \wedge \omega \mathcal{R} + (5 - p) Je_j \mathcal{R} \sigma)$$

$$= -2J \sigma - Je_j \mathcal{R}(e_j) \sigma - e_j \wedge J(e_j) \sigma - (5 - p) J e_j \wedge e_j \mathcal{R} \sigma$$

$$= -Je_j \wedge R^-(e_j) \sigma - (7 - p) J \sigma - e_j \wedge J(e_j) \mathcal{R} \sigma$$

It remains to determine the sum $e_j \wedge J(e_j) \mathcal{R} \sigma = e_j \wedge Je_i \mathcal{R} e_i e_j \mathcal{R} \sigma$. Here we compute

$$e_j \wedge J(e_j) \mathcal{R} \sigma = -Je_i \wedge e_j \wedge e_i \mathcal{R} e_j \sigma = Je_i \wedge e_i \mathcal{R} e_j \wedge e_j \mathcal{R} \sigma - Je_i \wedge e_i \mathcal{R} \sigma$$

$$= (p - 1) J \sigma$$

Substituting this result into the last expression for $Je_i \mathcal{R}^+(e_i) \sigma$ we obtain the second equation of Corollary 2.3. □

3. Conformal Killing forms

In this section we will recall the definition, examples and important properties of conformal Killing forms. More details can be found in [12].

Conformal Killing $p$-forms on a Riemannian manifold $(\mathcal{M}, g)$ are defined as differential forms $\sigma \in \Gamma(\Lambda^p TM)$ satisfying for any tangent vector $X$ the equation

$$\nabla_X \sigma = \frac{1}{p+1} X \mathcal{R} d \sigma - \frac{1}{n-p+1} X \wedge d^* \sigma .$$

Conformal Killing forms which in addition are co-closed are called Killing forms. Closed conformal Killing forms are also called $\ast$-Killing forms. Indeed the Hodge $\ast$-operator preserves the space of conformal Killing forms and maps Killing forms to $\ast$-Killing forms and vice versa.
Every parallel form is trivially a Killing form. For \( p = 1 \) Killing forms are dual to Killing vector fields. The standard sphere \((S^n, g_0)\) is the compact manifold with the maximal number of conformal Killing forms. Here every conformal Killing form is a linear combination of a Killing and a \(*\)-Killing form. The space of Killing forms on \( S^n \) coincides with the eigenspace of the Laplace operator on co-closed forms for the minimal eigenvalue. Other interesting examples are related to special geometric structures. For nearly Kähler manifolds the fundamental 2-form is by definition a Killing form. Similarly, the defining 3-form of a nearly parallel \( G_2 \)-structure is by definition a Killing form. If \( \eta \) is the 1-form dual to the Reeb vector field \( \xi \) defining a Sasakian structure then all the forms \( \eta \wedge d\eta^k \) for \( k = 0, \ldots, n \) are Killing forms.

These examples on the standard sphere, 6-dimensional nearly Kähler, nearly parallel \( G_2 \)- and on Sasakian manifolds are so-called special Killing forms. They are Killing forms \( \sigma \) satisfying the additional equation \( \nabla_X d\sigma = c X \wedge \sigma \) for some real constant \( c \) and every tangent vector \( X \). Special Killing forms on compact Riemannian manifolds \((M^n, g)\) were classified in [12]. They turn out to be in bijective correspondence to parallel forms on the metric cone over \( M \). In particular it follows that on 6-dimensional nearly Kähler manifolds \( d\omega \) is a closed conformal Killing form and that \( \Delta\omega = 12\omega \) (cf. [12], Prop. 4.2). As far as we know, the only examples of non-parallel, non-special Killing forms in degree larger than one are the fundamental 2-forms of strict nearly Kähler manifolds in dimension larger than 6 and the torsion forms of metric connections with skew-symmetric and parallel torsion, e.g. the torsion form on naturally reductive spaces.

In this article we will consider conformal Killing forms on compact manifolds. Here one has a rather useful additional characterization. First, we recall from [12] that conformal Killing \( p \)-forms \( \sigma \) on an arbitrary Riemannian manifold \((M^n, g)\) satisfy a second order Weitzenböck equation:

\[
q(R) \sigma = \frac{p}{p+1} d^* d \sigma + \frac{n-p}{n-p+1} d d^* \sigma ,
\]

where \( q(R) \in \text{End}(\Lambda^p T M) \) is the curvature term defined in [11].

Let \( \Delta = d^* d + dd^* \) denote the Hodge-Laplace operator on forms. Then it follows from [11] that \( \Delta \sigma = \frac{p+1}{p} q(R) \sigma \) holds for Killing \( p \)-forms \( \sigma \) and \( \Delta \sigma = \frac{n-p+1}{n-p} q(R) \sigma \) for \(*\)-Killing \( p \)-forms. A simple integration argument shows that these equations characterize Killing resp. \(*\)-Killing forms on compact manifolds, i.e. a \( p \)-form \( \sigma \) is a Killing form if and only if \( d^* \sigma = 0 \) and \( \Delta \sigma = \frac{p+1}{p} q(R) \sigma \). Similarly \( \sigma \) is a \(*\)-Killing form if and only if \( d\sigma = 0 \) and \( \Delta\sigma = \frac{n-p+1}{n-p} q(R) \sigma \). Moreover, we have a similar characterization for conformal Killing forms in middle dimension, i.e. a \( m \)-form \( \sigma \) on a \( 2m \)-dimensional compact manifold \( M \) is conformal Killing if and only if the equation \( \Delta\sigma = \frac{m+1}{m} q(R) \sigma \) is satisfied (cf. [12], Cor. 2.5).

Later we will study conformal Killing forms on 6-dimensional nearly Kähler manifolds, which are automatically Einstein. On Einstein manifolds more can be said about conformal Killing forms, e.g. it is known that in this situation for any conformal Killing 2-form \( \sigma \) the codifferential \( d^* \sigma \) is dual to a Killing vector field. However, here we are more interested in a
commutator rule for the differential and codifferential with the curvature endomorphism \( q(R) \). We have the following result. Let \((M^n, g)\) be an Einstein manifold. Then every conformal Killing \( p \)-form \( \sigma \) satisfies the two equations

\[
\begin{align*}
\text{(5) } d(\mathcal{R}(R) \sigma) &= \frac{\text{scal}}{n} d \sigma + \frac{n-1}{p+1} q(R) d \sigma \\
\text{and } d^*(\mathcal{R}(R) \sigma) &= \frac{\text{scal}}{n} d^* \sigma + \frac{n-p-1}{n-p+1} q(R) d^* \sigma .
\end{align*}
\]

The statements follow from easy local calculations (e.g. cf. [11], Prop. 4.4.12 and Cor. 7.1.2).

Finally we want to mention an important integrability condition for conformal Killing forms. Indeed, the conformal Killing equation is of finite type, i.e. there is a finite prolongation and conformal Killing forms are components of a parallel section in a larger bundle, with respect to a suitable connection. In particular, the curvature of this connection vanishes on conformal Killing forms and one component of the corresponding equation is the following integrability condition (cf. [12], Prop. 6.4). Let \( \sigma \) be a conformal Killing \( p \)-form then

\[
\begin{align*}
\text{(7) } R_{XY} \sigma &= \frac{1}{p(n-p)} (X \wedge Y)_* q(R) \sigma + \frac{1}{p} (Y \wedge R^+(X) \sigma - X \wedge R^+(Y) \sigma) \\
&\quad + \frac{1}{n-p} (Y \wedge R^-(X) \sigma - X \wedge R^-(Y) \sigma)
\end{align*}
\]

is satisfied for all tangent vectors \( X, Y \). The curvature expressions \( R^+(X) \) and \( R^-(X) \) were defined in (3). We will use this equation for conformal Killing 3-forms on 6-dimensional nearly Kähler manifolds.

**4. Conformal Killing 3-forms on nearly Kähler manifolds**

Let \((M^6, g, J)\) be a compact 6-dimensional strict nearly Kähler manifold with scalar curvature normalized to \( \text{scal} = 30 \) and let \( \sigma \) be a conformal Killing 3-form on \( M \). In this section we will give the proof of our main theorem, i.e. Theorem [11] and show that the 3-form \( \sigma \) has to be a linear combination of \( d\omega \) and \( *d\omega \).

In the first step we use the integrability condition (7) with \( X = e_i, Y = Je_i \), for an orthonormal frame \( \{e_i\} \). After summation we obtain

\[
\begin{align*}
\text{(8) } R_{ei}Je_i \sigma &= \frac{1}{9} (e_i \wedge Je_i)_* q(R) \sigma + \frac{2}{3} Je_i \wedge R^+(e_i) \sigma + \frac{2}{3} Je_i \wedge R^-(e_i) \sigma.
\end{align*}
\]

Since \( R(\omega) = -\omega \) for the fundamental 2-form \( \omega \), the left-hand side of this equation can be rewritten as \( R_{ei}Je_i \sigma = 2 R(\omega) \sigma = -2 \omega_* \sigma = -2 J \sigma \). The first summand on the right-hand side is equal to \( \frac{2}{3} J q(R) \sigma \). Moreover, substituting the second equation of Corollary [2.3] into (8), we see that (8) is equivalent to the equation \( J q(R) \sigma = 9 J \sigma \). But \( J \) is injective on 3-forms, thus we find \( q(R) \sigma = 9 \sigma \) for every conformal Killing 3-form \( \sigma \).

Let \( \sigma \) be a conformal Killing 3-form on a 6-dimensional nearly Kähler manifold. Then \( \sigma \) is a conformal Killing form in middle dimension and we have \( \Delta \sigma = \frac{4}{3} q(R) \sigma = 12 \sigma \). Since the
Laplace operator \( \Delta = d d^* + d^* d \) commutes with \( d \) and \( d^* \), also \( \Delta d\sigma = 12 d\sigma \), \( \Delta d^* \sigma = 12 d^* \sigma \) and \( \Delta d d^* \sigma = 12 d d^* \sigma \) hold. Using (5) and (6) we obtain

\[
(9) \quad d(q(R)\sigma) = 5 d\sigma + \frac{1}{2} q(R) d\sigma \quad \text{and} \quad d^*(q(R)\sigma) = 5 d^* \sigma + \frac{1}{2} q(R) d^* \sigma.
\]

Since \( q(R)\sigma = 9 \sigma \) these equations imply

\[
q(R) d\sigma = 8 d\sigma \quad \text{and} \quad q(R) d^* \sigma = 8 d^* \sigma.
\]

and it follows that \( \Delta d^* \sigma = 12 d^* \sigma = \frac{3}{2} q(R) d^* \sigma \) and similarly that \( \Delta d\sigma = \frac{3}{2} q(R) d\sigma \). Hence, by the characterization of Killing and \( \ast \)-Killing forms on compact manifolds given above, we conclude that for every conformal Killing 3-form \( \sigma \) the form \( d^* \sigma \) is a Killing 2-form and \( d\sigma \) is a \( \ast \)-Killing 4-form.

Now, let \( \sigma \) be a Killing form, i.e. a co-closed conformal Killing 3-form, then \( d\sigma \) is a \( \ast \)-Killing 4-form and the \( \ast \)-Killing equation for \( d\sigma \) reads for any tangent vector \( X \) as

\[
\nabla_X d\sigma = -\frac{1}{3} X \wedge d^* d\sigma = -\frac{1}{3} X \wedge \Delta \sigma = -4 X \wedge \sigma
\]

and similarly \( \nabla_X d^* \sigma = 4 X \wedge \sigma \). Thus we see that every Killing 3-form \( \sigma \) already is a special Killing 3-form and similarly every \( \ast \)-Killing 3-form is the Hodge dual of a special Killing 3-form.

In general a conformal Killing 3-form \( \sigma \) needs not to be closed or co-closed. However, it follows, that the 3-form \( \sigma - \frac{1}{12} dd^* \sigma \) is co-closed and \( \sigma - \frac{1}{12} d^* d\sigma \) is closed. Indeed

\[
d^*(\sigma - \frac{1}{12} d d^* \sigma) = d^* \sigma - \frac{1}{12} d^* d d^* \sigma = d^* \sigma - \frac{1}{12} \Delta d^* \sigma = 0.
\]

A similar calculation shows that \( \sigma - \frac{1}{12} d^* d\sigma \) is closed. Using (5) once again, this time for the Killing 2-form \( d^* \sigma \), we obtain \( q(R) dd^* \sigma = 9 dd^* \sigma \). Hence, \( \Delta dd^* \sigma = 12 \sigma = \frac{3}{2} q(R) dd^* \sigma \) and we see as above that \( dd^* \sigma \) has to be a closed conformal Killing 3-form. But then \( \sigma - \frac{1}{12} dd^* \sigma \) is a conformal Killing form too and thus, since co-closed, it is in fact a Killing 3-form. We showed already that Killing 3-forms are automatically special. Thus we conclude that \( \sigma - \frac{1}{12} dd^* \sigma \) is a special Killing 3-form. In the same way we can show that \( \sigma - \frac{1}{12} d^* d\sigma \) is the Hodge dual of a special Killing 3-form.

Special Killing forms were classified in [12], Thm. 4.8. In particular, the only special Killing 3-forms on 6-dimensional nearly Kähler manifolds are constant multiples of \( *d\omega \). It follows that \( \sigma - \frac{1}{12} dd^* \sigma = \lambda *d\omega \) for some real constant \( \lambda \). Similarly we have that \( \sigma - \frac{1}{12} d^* d\sigma \) is the Hodge dual of a special Killing 3-form and thus \( \sigma - \frac{1}{12} d^* d\sigma = \mu d\omega \) for some real constant \( \mu \). Since \( \Delta \sigma = 12 \sigma \) we can write \( \sigma \) as

\[
\sigma = 2 \sigma - \frac{1}{12} \Delta \sigma = (\sigma - \frac{1}{12} d^* d\sigma) + (\sigma - \frac{1}{12} dd^* \sigma) = \lambda *d\omega + \mu d\omega
\]

This finishes the proof of Theorem [12].
5. Killing 2-forms on nearly Kähler manifolds

In this last section we want to make a few remarks concerning conformal Killing 2-forms on nearly Kähler manifolds. Here we can not use an argument similar to the one for 3-forms. The coefficients of the last two summands in our integrability condition (8) are now different, so it is not possible to substitute the second equation of Corollary 2.3. Also, the extension of the almost complex structure \( J \) is not injective on the space of all 2-forms, it vanishes on 2-forms of type \((1,1)\), i.e. \( J \)-invariant 2-forms.

However, it is still possible to exclude the existence of \( J \)-anti-invariant 2-forms of a special type, e.g. of type \((2,0) + (0,2)\), i.e. anti-invariant 2-forms. Let \( \sigma \) be an anti-invariant 2-form on a compact nearly Kähler manifold \((M^6, g, J)\). Then there exists a vector field \( \xi \) with \( \sigma = \xi \downarrow \Psi^+ = \Psi^+_{\xi} \). First we compute the action of \( q(R) \) on anti-invariant 2-forms. Here we have \( q(R) \Psi^+ = 4 \Psi^+_{q(R)\xi} \) (cf. [8], p. 254 and Lemma 4.8). From Proposition 3.4 in [9] we then obtain \( q(R) \sigma = q(R) \sigma + 4 \sigma = 8 \sigma \).

Assume \( \sigma \) to be a Killing 2-form then we have \( \Delta \sigma = \frac{3}{2} q(R) \sigma = 12 \sigma \). But now we can use (9) for the 3-form \( d\sigma \) to get \( q(R)d\sigma = 9d\sigma \) and thus \( \Delta d\sigma = 12 \sigma = \frac{4}{3} q(R)d\sigma \). Hence, \( d\sigma \) is a closed conformal Killing 3-form and by Theorem it has to be a constant multiple of \( d\omega \), i.e. \( d\sigma = \lambda d\omega \) for some real number \( \lambda \). Finally we note that \( \Delta \sigma = d^*d\sigma = \lambda d^*d\omega = \lambda \Delta \omega = 12 \omega \). This would imply \( \omega = \sigma \), which is of course a contradiction. Summarizing we proved the following

**Proposition 5.1.** Let \((M^6, g, J)\) be a compact nearly Kähler manifold. Then \( M \) admits no non-trivial \( J \)-anti-invariant Killing 2-form.

Finally we consider the case of closed \( J \)-anti-invariant conformal Killing 2-forms, i.e. let \( \sigma := \Psi^+ \) be a \( \ast \)-Killing 2-form. Then by definition \( \nabla_X \sigma = -\frac{1}{3} X \wedge d^* \sigma \) and in particular \( \sigma \) is closed. The characterization of \( \ast \)-Killing 2-forms implies \( \Delta \sigma = \frac{5}{4} q(R) \sigma = 10 \sigma \). For the vector field \( V = d^* \sigma \) we have \( dV = d^*d\sigma = \Delta \sigma = 10 \sigma \). Hence, \( \nabla_X dV = 10 \nabla_X \sigma = -\frac{30}{3} X \wedge V \) and we obtain \( \Delta V = d^*dV = \frac{50}{3} V \). But with \( \Delta \sigma = 10 \sigma \) we also have \( \Delta V = 10 V \), which is possible only with \( V = 0 \). Because \( dV = 10 \sigma \) we conclude that also the conformal Killing 2-form \( \sigma \) has to vanish. Summarizing we just proved

**Proposition 5.2.** Let \((M^6, g, J)\) be a compact nearly Kähler manifold. Then \( M \) admits no non-trivial \( J \)-anti-invariant \( \ast \)-Killing 2-form.

As already mentioned in the introduction, it follows from the work of I. Dotti and C. Herrera in [4] that besides the fundamental 2-form there are no SU(3)-invariant Killing 2-forms on the nearly Kähler flag manifold \( SU(3)/T^2 \). It seems, to be a reasonable to conjecture that a Killing 2-forms on a compact 6-dimensional strict nearly Kähler manifold has to be a scalar multiple of the fundamental 2-form.
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