THE SATO–TATE DISTRIBUTION IN FAMILIES OF
ELLIPTIC CURVES WITH A RATIONAL PARAMETER
OF BOUNDED HEIGHT

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Abstract. We obtain new results concerning the Sato–Tate conjecture on the distribution of Frobenius angles over parametric families of elliptic curves with a rational parameter of bounded height.

1. Introduction

1.1. Background and motivation. Let \( f(Z), g(Z) \in \mathbb{Z}[Z] \) be polynomials satisfying

\[
\Delta(Z) \neq 0 \quad \text{and} \quad j(Z) \notin \mathbb{Q},
\]

where

\[
\Delta(Z) = -16(4f(Z)^3 + 27g(Z)^2) \quad \text{and} \quad j(Z) = \frac{-1728(4f(Z))^3}{\Delta(Z)}.
\]

Then we consider the elliptic curve

\[
E(Z) : \quad Y^2 = X^3 + f(Z)X + g(Z)
\]

over the function field \( \mathbb{Q}(Z) \). Thus, \( \Delta(Z) \) and \( j(Z) \) are the discriminant and \( j \)-invariant of this elliptic curve, respectively; see [29] for a general background on elliptic curves.

In what follows, we also refer to [29] for the definition of the conductor \( N_E \) of an elliptic curve \( E \) as well as for the notions of CM curves and non-CM curves.

The properties of the specialisations \( E(t) \) modulo consecutive primes \( p \leq x \) for a growing parameter \( x \) and for the parameter \( t \) that runs through some interesting sets \( \mathcal{T} \) have recently being investigated quite intensively, see [10, 23, 27] and also Section 1.2. These sets \( \mathcal{T} \) can be of integer or rational numbers of limited size, and sometimes also of certain arithmetic structure; for example \( \mathcal{T} \) can be a set of primes in a given interval \([1, T]\), see [8].

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Throughout the paper, for an elliptic curve $E$ over $\mathbb{Q}$ and a prime $p \nmid N_E$ we denote by $E_p$ the reduction of $E$ modulo $p$, which is an elliptic curve defined over the finite field $\mathbb{F}_p$ of $p$ elements. Furthermore, we use $E_p(\mathbb{F}_p)$ to denote the group of $\mathbb{F}_p$-rational points on $E_p$. In particular, $a_p(E) = p + 1 - \#E_p(\mathbb{F}_p)$ is the Frobenius trace.

From the Hasse bound (see [29]): $|a_p(E)| \leq 2\sqrt{p}$, we can define the Frobenius angle $\psi_p(E) \in [0, \pi]$ by the equation

$$\cos \psi_p(E) = \frac{a_p(E)}{2\sqrt{p}}.$$  

Then, the Sato–Tate conjecture predicts that the angles $\psi_p(E)$ are distributed in $[0, \pi]$ with the Sato–Tate density

$$\mu_{ST}(\alpha, \beta) = \frac{2}{\pi} \int_\alpha^\beta \sin^2 \theta \, d\theta = \frac{2}{\pi} \int_{\cos \beta}^{\cos \alpha} (1 - z^2)^{1/2} \, dz,$$

where $[\alpha, \beta] \subseteq [0, \pi]$.

The Sato–Tate conjecture has been settled only quite recently in the series of works of Barnet-Lamb, Geraghty, Harris and Taylor [6], Clozel, Harris and Taylor [9], Harris, Shepherd-Barron and Taylor [18], and Taylor [30]. In particular, given a non-CM elliptic curve $E$ over $\mathbb{Q}$ of conductor $N_E$, for the number $\pi_E(\alpha, \beta; x)$ of primes $p \leq x$ with $p \nmid N_E$ for which $\psi_p(E) \in [\alpha, \beta] \subseteq [0, \pi]$, we have

$$\pi_E(\alpha, \beta; x) \sim \mu_{ST}(\alpha, \beta) \cdot \frac{x}{\log x}$$

as $x \to \infty$.

However, the above asymptotic formula is lack of an explicit error term. So, it makes sense to study $\pi_E(\alpha, \beta; x)$ on average over some natural families of elliptic curves. In this paper, we continue this line of research and in particular introduce new families of curves with a rational parameter.

1.2. Previous results. As one of the possible relaxation of the still open Lang–Trotter conjecture, see [20], Fouvry and Murty [16] have introduced the study of the reductions $E_p$ for $p \leq x$ on average over a family of elliptic curves $E$. More precisely, in [16] the frequency of vanishing $a_p(E_{u,v}) = 0$ is investigated for the family of curves

$$E_{u,v} : Y^2 = X^3 + uX + v,$$

with the integer parameters $(u, v) \in [-U, U] \times [-V, V]$. This has been extended to arbitrary values $a_p(E_{u,v}) = a$ by David and Pappalardi [14] and more recently by Baier [2], see also [3].
The approach of [2, 3, 14, 16] also applies to the Sato–Tate conjecture on average for the family (1.5), see [4], provided that $U$ and $V$ are reasonably large compared to $p$. Banks and Shparlinski [5] have shown that using a different approach, based on bounds of multiplicative character sums and the large sieve inequality (instead of the exponential sum technique employed in [16]), one can study “thinner” families, that is, establish the Sato–Tate conjecture on average for the curves (1.5) for smaller values of $U$ and $V$.

The technique of [5] has been used in several other problems, see [11, 15, 25, 26]. In particular, the Sato–Tate conjecture has been established on average for several other families of curves. For example, Shparlinski [26] has studied the family of elliptic curves $Y^2 = X^3 + f(u)X + g(v)$ with integers $|u| \leq U$, $|v| \leq V$, where $f, g \in \mathbb{Z}[Z]$.

Sha and Shparlinski [23] have established the Sato–Tate conjecture on average for the families of curves $Y^2 = X^3 + f(u+v)X + g(u+v)$, where $u, v$ both run through some subsets of $\{1, 2, \ldots, T\}$, or both run over the set

$$\mathcal{F}(T) = \{u/v \in \mathbb{Q} : \gcd(u, v) = 1, 1 \leq u, v \leq T\}.$$  

For the size of $\mathcal{F}(T)$, it is well known that

$$\# \mathcal{F}(T) \sim \frac{6}{\pi^2} T^2,$$

as $T \to \infty$, see [17, Theorem 331]. We recall that the set $\mathcal{F}(T) \cap [0, 1]$ is the well-known set of Farey fractions. We note that all the related results of [23] hold without any changes if one replaces the set $\mathcal{F}(T)$ with $\mathcal{F}(T) \cap [0, 1]$.

In addition, for the family of curves (1.2), Cojocaru and Hall [12] have given an upper bound on the frequency of the event $a_p(E(t)) = a$ for a fixed integer $a$, when the parameter $t$ runs through the set $\mathcal{F}(T)$. This bound has been improved by Cojocaru and Shparlinski [13] and then further improved by Sha and Shparlinski [23].

Most recently, de la Bretèche, Sha, Shparlinski and Voloch [8] have established the Sato–Tate conjecture on average for the polynomial family (1.2) of elliptic curves when the variable $Z$ is specialised to a parameter $t$ from sets of prescribed multiplicative structure, such as prime numbers, and geometric progressions. Particularly, the Sato–Tate conjecture on average is true for the families of curves $Y^2 = X^3 + f(uv)X + g(uv)$, where $u, v$ both run through some subsets of $\{1, 2, \ldots, T\}$.

1.3. General notation. As usual the expressions $A = O(B)$ and $A \ll B$ (sometimes we will write this also as $B \gg A$) are both equivalent to
the inequality $|A| \leq cB$ with some absolute constant $c > 0$, $A = o(B)$ means that $A/B \to 0$ and $A \sim B$ means that $A/B \to 1$. We also write $A \asymp B$ if $A \ll B \ll A$.

Throughout the paper the implied constants may, where obvious, depend on the polynomials $f$ and $g$ in (1.2) and the real positive parameter $\varepsilon$, and are absolute otherwise.

Furthermore, the letter $p$ always denotes a prime number. We always assume that the elements of $\mathbb{F}_p$ are represented by the set $\{0, \ldots, p-1\}$ and thus we switch freely between the equations in $\mathbb{F}_p$ and congruences modulo $p$.

As usual, we use $\pi(x)$ to denote the number of primes $p \leq x$.

For a subset $S$ in the real plane, we denote by $N(S) = \#(S \cap \mathbb{Z}^2)$ the number of integral lattice points in $S$.

2. Main Results

In this paper, we establish the Sato–Tate conjecture on average for some families of elliptic curves with a rational parameter.

Recall that for any $t \in \mathbb{Q}$ with $\Delta(t) \neq 0$, we use $\pi_{E(t)}(\alpha, \beta; x)$ to denote the number of primes $p \leq x$ with $p \nmid N_{E(t)}$ (or equivalently, $\Delta(t) \neq 0 \pmod{p}$, see Section 3.1) and $\psi_p(E(t)) \in [\alpha, \beta]$.

We start with a general result. Let

\begin{align*}
I(A, T) &= [A + 1, A + T], \\
J(B, T) &= [B + 1, B + T],
\end{align*}

be two intervals of the form (2.1).

**Theorem 2.1.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1). Let $I(A, T)$ and $J(B, T)$ be two intervals of the form (2.1). Let $W \subseteq I(A, T) \times J(B, T)$ be an arbitrary convex subset. Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have

\[
\frac{1}{\pi(x)N(W)} \sum_{(u, v) \in W \cap \mathbb{Z}^2_{\Delta(u/v) \neq 0}} \pi_{E(u/v)}(\alpha, \beta; x) - \mu_{ST}(\alpha, \beta) \ll \frac{T \log x}{x} + \frac{T}{N(W)} + \frac{T^{1/4}x^{1/2+o(1)}}{N(W)^{1/2}}.
\]

In particular, we have:

**Corollary 2.2.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1). Let $W \subseteq I(A, T) \times J(B, T)$ be an arbitrary convex subset such that

\[
N(W) \gg T^\eta
\]
for some real $\eta > 3/2$. Assume that for sufficiently small $\varepsilon > 0$, 
\[ x^{2/(2\eta - 1) + \varepsilon} \leq T \leq x^{1 - \varepsilon}. \]

Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have 
\[
\frac{1}{\pi(x)\mathcal{N}(\mathcal{W})} \sum_{(u,v) \in \mathcal{W} \cap \mathbb{Z}^2 \setminus \Delta(u/v) \neq 0} \pi_{E(u/v)}(\alpha, \beta; x) = \mu_{\text{ST}}(\alpha, \beta) + O \left( x^{-\varepsilon/2 + o(1)} \right).
\]

Note that the fact $\mathcal{N}(\mathcal{W}) \ll T^2$ and the condition (2.2) imply that we also have $\eta \leq 2$ in Corollary 2.2.

We now give a natural class of subsets that meet the condition (2.2) in Corollary 2.2. Namely, by the Pick’s theorem (see [7, Theorem 2.8]) this holds for any convex simple polygon with vertices on the integral lattice $\mathbb{Z}^2$ whose area is not less than $T^\eta$ up to a constant.

Assume that we further have 
\[
\#\{(u, v) \in \mathcal{W} \cap \mathbb{Z}^2 : \gcd(u, v) = 1\} \asymp \mathcal{N}(\mathcal{W}).
\]

Then, one can similarly establish the Sato-Tate conjecture on average for $\pi_{E(u/v)}(\alpha, \beta; x)$, where $(u,v) \in \mathcal{W} \cap \mathbb{Z}^2$ with $\gcd(u, v) = 1$, as in Corollary 2.2.

If we choose $\mathcal{W} = \mathcal{I}(A, T) \times \mathcal{J}(B, T)$, then we can take $\eta = 2$ and $x^{2/3 + \varepsilon} \leq T \leq x^{1 - \varepsilon}$ in Corollary 2.2. In the following we want to relax this condition on $T$ in the case when $A, B \leq T$.

We now define the sets:
\[
\mathcal{Z}(A, B, T) = (\mathcal{I}(A, T) \times \mathcal{J}(B, T)) \cap \mathbb{Z}^2,
\]

and
\[
\mathcal{Z}^*(A, B, T) = \{(u,v) \in \mathcal{Z}(A, B, T) : \gcd(u, v) = 1\}.
\]

**Theorem 2.3.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1). Let $A, B \leq T$ and let $\mathcal{Y}$ be one of the sets $\mathcal{Z}(A, B, T)$ or $\mathcal{Z}^*(A, B, T)$. Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have 
\[
\frac{1}{\pi(x)\#\mathcal{Y}} \sum_{(u,v) \in \mathcal{Y}} \pi_{E(u/v)}(\alpha, \beta; x) - \mu_{\text{ST}}(\alpha, \beta)
\ll x^{-1/4} + T^{-1/2 + o(1)} x^{1/4}.
\]

We remark that $\#\mathcal{Z}(A, B, T) = (T + 1)^2$ and 
\[
\#\mathcal{Z}^*(A, B, T) \asymp T^2.
\]
If $T \geq x^{1/2+\varepsilon}$ with some positive $\varepsilon \leq 1/2$, then the error term in Theorem 2.3 becomes $O(x^{-\varepsilon/2+o(1)})$. Note that the set $F(T)$ defined in (1.6) is exactly the following set

$$\{u/v : (u, v) \in \mathbb{Z}^*(0, 0, T)\}.$$ 

Thus, we have:

**Corollary 2.4.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), and for some sufficiently small $\varepsilon > 0$ we have $T \geq x^{1/2+\varepsilon}$.

Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have

$$\frac{1}{\pi(x)\#F(T)} \sum_{s \in F(T), \Delta(s) \neq 0} \pi_{E(s)}(\alpha, \beta; x) = \mu_{ST}(\alpha, \beta) + O(x^{-\varepsilon/2+o(1)}) .$$

One can also consider the Sato-Tate conjecture on average with the product $uv$ for $u, v \in \mathcal{Z}(A, B, T)$.

Here, we present the following result for the family of elliptic curves parameterized by products $rs$ with $r, s$ from arbitrary subsets of $F(T)$.

**Theorem 2.5.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1). Then, for any subsets $R, S \subseteq F(T)$, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$ we have

$$\frac{1}{\pi(x)\#R\#S} \sum_{r \in R, s \in S, \Delta(rs) \neq 0} \pi_{E(rs)}(\alpha, \beta; x) = \mu_{ST}(\alpha, \beta) - \mu_{ST}(\alpha, \beta)$$

$$\ll T^4x^{-1/4} + T^3x^{1/4}\log x \frac{\#R\#S}{\#F(T)} .$$

Thus, for $R = S = F(T)$, recalling (1.7), we derive the following multiplicative analogue of [23, Theorem 6]. In turn, we also note that [23, Theorem 6] can be extended to sum sets of arbitrary sets $\mathcal{R}, \mathcal{S} \subseteq F(T)$.

**Corollary 2.6.** Suppose that the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), and for some sufficiently small $\varepsilon > 0$ we have $T \geq x^{1/4+\varepsilon}$.

Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have

$$\frac{1}{\pi(x)(\#F(T))^2} \sum_{r, s \in F(T), \Delta(rs) \neq 0} \pi_{E(rs)}(\alpha, \beta; x) = \mu_{ST}(\alpha, \beta) + O(x^{-\varepsilon}\log x) .$$
3. Preliminaries

3.1. Primes of good reduction. We start with the observation that the condition (1.1) (over any field $\mathbb{K}$ of characteristic $p > 3$) implies that $\Delta(Z) \in \mathbb{K}[Z]$ is not a constant polynomial. Indeed, if $\Delta(Z) = c \neq 0$ for some $c \in \mathbb{K}$, then $f(Z)$ and $g(Z)$ have no common roots. Since $j(Z)$ is not constant, both $f$ and $g$ are also not constant. Now, considering the derivative $\Delta(Z)' = 0$, we easily see that $f$ and $g$ must have common roots, which leads to a contradiction.

For $t \in \mathbb{Q}$, let $N(t)$ denote the conductor of the specialisation of $E(Z)$ at $Z = t$. We always consider rational numbers in the form of irreducible fraction.

Note that for $t \in \mathbb{Q}$, the discriminant $\Delta(t)$ may be a rational number. However, we know that the elliptic curve $E(t)$ has good reduction at prime $p$ if and only if $p$ does not divide both the numerator and denominator of $\Delta(t)$; see [29, Chapter VII, Proposition 5.1(a)]. So, we can say that for any prime $p$, $p \nmid N(t)$ (that is, $E(t)$ has good reduction at $p$) if and only if $\Delta(t) \not\equiv 0 \pmod{p}$ (certainly, it first requires that $p$ does not divide the denominator of $\Delta(t)$).

3.2. Preparations for distribution of angles. Given an angle $\vartheta \in [0, \pi]$ and an integer $n \geq 1$, we define the function

$$
\text{sym}_n(\vartheta) = \frac{\sin((n+1)\vartheta)}{\sin \vartheta}.
$$

(3.1)

Note that for any $n \geq 2$, we have

$$
\text{sym}_n(\vartheta) = \frac{\sin n\vartheta}{\sin \vartheta} \cos \vartheta + \cos n\vartheta = \text{sym}_{n-1}(\vartheta) \cos \vartheta + \cos n\vartheta,
$$

which implies (via a simple inductive argument) that

$$
\text{sym}_n(\vartheta) \ll n.
$$

(3.2)

The following result is based on the ideas of Niederreiter [22], and has been used implicitly in a number of works (see, for example, [23]). It is also explicitly given in [8, Corollary 3.2].

**Lemma 3.1.** Given $m$ arbitrary angles $\psi_1, \ldots, \psi_m \in [0, \pi]$ (not necessarily distinct), assume that for every integer $n \geq 1$ we have

$$
\left| \sum_{i=1}^{m} \text{sym}_n(\psi_i) \right| \leq n\sigma
$$

for some real $\sigma \geq 2$. Then, uniformly over $[\alpha, \beta] \subseteq [0, \pi]$, we have

$$
\# \{ \psi_i \in [\alpha, \beta] : 1 \leq i \leq m \} = \mu_{\text{ST}}(\alpha, \beta)m + O \left( \sqrt{m\sigma} \right).
$$
3.3. Exponential sums with ratios. For an integer $m$, we denote
\[ e_m(z) = \exp(2\pi iz/m). \]
The following result is essentially given in [28, Lemma 7].

**Lemma 3.2.** Let $T < p$ for a prime $p$ and let $\mathcal{I}(A,T)$ and $\mathcal{J}(B,T)$ be two intervals of the form (2.1). Let $W \subseteq \mathcal{I}(A,T) \times \mathcal{J}(B,T)$ be an arbitrary convex subset. Then
\[
\max_{a \in \mathbb{F}_p^*} \left| \sum_{(u,v) \in W \cap \mathbb{Z}^2} e_p(au/v) \right| \leq T^{1/2}p^{1/2+o(1)}.
\]

The following bound that holds for any prime $p$ and integer $T \geq 2$,

\[
(3.3) \quad \max_{a \in \mathbb{F}_p^*} \left| \sum_{u, v \in \mathcal{F}(T)} e_p(au/v) \right| \leq T(Tp)^{o(1)} + T^2/p,
\]
is actually a corrected form of [23, Lemma 15] where also the term $T^2/p$ has to be added. Indeed in the proof of [23, Lemma 15], we omitted the case $\gcd(v, p) \neq 1$, which makes contributions at most $T(T/p + 1)$, and so one should also add $T^8/p^4$ to the error term in [23, Lemma 16]. Fortunately, this change does not affect the result in [23, Theorem 5], whose proof relies on [23, Lemma 16].

Now, we need to generalize the bound (3.3). We first note that the proof in [24, Lemma 3] (the condition $L_x < m$ there can be deleted) actually gives the following estimate.

**Lemma 3.3.** Let $U, V, W$ be arbitrary positive integers with $V < W$. Assume that for each integer $v$ we are given two integers $L_v, U_v$ with $0 \leq L_v < U_v \leq U$. Then for any integer $a \not\equiv 0 \pmod m$, we have
\[
\left| \sum_{v=V}^{W} \sum_{u=L_v+1}^{U_v} e_m(au/v) \right| \leq (U + W)(Wm)^{o(1)}.
\]

We recall the definitions (2.3) and (2.4) of the sets $\mathcal{Z}(A,B,T)$ and $\mathcal{Z}^*(A,B,T)$.

Applying similar arguments as in the proof of [23, Lemma 15] and using Lemma 3.3 (taking $m = p$), we derive:
Lemma 3.4. Let $A, B \leq T$ and let $\mathcal{I}(A,T)$ and $\mathcal{J}(B,T)$ be two intervals of the form (2.1). Then for any prime $p$, we have

$$\max_{a \in \mathbb{F}_p^*} \left| \sum_{(u,v) \in \mathbb{Z}(A,B,T)} e_p(au/v) \right| \leq T(Tp)^o(1) + T^2/p,$$

and

$$\max_{a \in \mathbb{F}_p^*} \left| \sum_{(u,v) \in \mathbb{Z}^*(A,B,T)} e_p(au/v) \right| \leq T(Tp)^o(1) + T^2/p.$$ 

3.4. Bounds on some single sums. Michel [21, Proposition 1.1] gives the following bound for the sum of the function $\text{sym}_n(\vartheta)$, given by (3.1) twisted by additive characters, we refer to [19] for a background on characters.

Lemma 3.5. If the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), then for any prime $p$, we have

$$\sum_{w \in \mathbb{F}_p^*} \text{sym}_n(\psi_p(E(w))) e_p(mw) \ll np^{1/2},$$

uniformly over all integers $m \geq 0$ and $n \geq 1$.

We also need the following analogue of Lemma 3.5, which is given in [8, Lemma 3.4].

Lemma 3.6. If the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), then for any prime $p$ and any multiplicative character $\chi$ of $\mathbb{F}_p$, we have

$$\sum_{w \in \mathbb{F}_p^*} \text{sym}_n(\psi_p(E(w))) \chi(w) \ll np^{1/2},$$

uniformly over all integer $n \geq 1$.

3.5. Frobenius angles over ratios. In this section, we estimate several sums of the function $\text{sym}_n(\vartheta)$ when $\vartheta$ runs through Frobenius angles over ratios. We start with a general result.

Lemma 3.7. Let $T < p$ for a prime $p$ and let $\mathcal{I}(A,T)$ and $\mathcal{J}(B,T)$ be two intervals of the form (2.1). Let $\mathcal{W} \subseteq \mathcal{I}(A,T) \times \mathcal{J}(B,T)$ be an arbitrary convex subset. Then

$$\sum_{(u,v) \in \mathcal{W} \cap \mathbb{Z}^2 \atop \Delta(u/v) \neq 0 \pmod{p}} \text{sym}_n(\psi_p(E(u/v))) \ll nT^{1/2}p^{1+o(1)},$$

uniformly over all integers $n \geq 1$. 

Proof. Using the orthogonality of the exponential function, we write

\[
\sum_{(u,v) \in W \cap \mathbb{Z}^2} \mathbb{S}_n(\psi_p(E(u/v))) \\Delta(u/v) \not\equiv 0 \pmod{p}
\]

\[
= \sum_{w \in \mathbb{F}_p} \mathbb{S}_n(\psi_p(E(w))) \sum_{(u,v) \in W \cap \mathbb{Z}^2} \frac{1}{p} \sum_{m=0}^{p-1} e_p(m(w - u/v)) + O(nT),
\]

where the term \(O(nT)\) comes from the case \(\gcd(v,p) \neq 1\) by using (3.2). Note that since \(T < p\), at most one \(v\) is divisible by \(p\). Now, changing the order of summation we obtain:

\[
\sum_{(u,v) \in W \cap \mathbb{Z}^2} \mathbb{S}_n(\psi_p(E(u/v))) \\Delta(u/v) \not\equiv 0 \pmod{p}
\]

\[
= \frac{1}{p} \sum_{m=0}^{p-1} \sum_{w \in \mathbb{F}_p} \mathbb{S}_n(\psi_p(E(w))) e_p(mw) \sum_{\Delta(w) \neq 0} e_p(-mu/v) + O(nT).
\]

Combining Lemma 3.5 with Lemma 3.2, we have

\[
\sum_{(u,v) \in W \cap \mathbb{Z}^2} \mathbb{S}_n(\psi_p(E(u/v))) \\Delta(u/v) \not\equiv 0 \pmod{p}
\]

\[
\ll np^{-1/2} \sum_{m=0}^{p-1} \left| \sum_{(u,v) \in W \cap \mathbb{Z}^2 \atop \gcd(v,p) = 1} e_p(-mu/v) \right| + nT
\]

\[
\ll np^{-1/2} (\mathcal{N}(W) + T^{1/2}p^{3/2+o(1)}) + nT
\]

\[
\ll np^{-1/2} T^2 + nT^{1/2}p^{1+o(1)} + nT
\]

\[
\ll nT^{1/2}p^{1+o(1)},
\]

which concludes the proof. \(\square\)

We again recall the definitions (2.3) and (2.4) of the sets \(Z(A,B,T)\) and \(Z^*(A,B,T)\). The following results gives an improvement upon the estimate in Lemma 3.7 when \(W = I(A,T) \times J(B,T)\). Note that here we do not need the condition \(p > T\) any more. The proof is fully
analogous to that of Lemma 3.7, except that instead of Lemma 3.2 one has to apply Lemma 3.4:

**Lemma 3.8.** Let $A, B \leq T$, and let $\mathcal{Y}$ be one of the sets $\mathcal{Z}(A, B, T)$ or $\mathcal{Z}^*(A, B, T)$. If the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), then for any prime $p$, uniformly over all integers $n \geq 1$ we have

$$\sum_{(u,v) \in Y \atop \Delta(u/v) \not\equiv 0 \pmod{p}} \text{sym}_n(\psi_p(E(u/v))) \ll nT^2 p^{-1/2} + nT^{1+o(1)} p^{1/2+o(1)}.$$ 

Similarly, we get the following over the product set $\mathcal{F}(T) \times \mathcal{F}(T)$.

**Lemma 3.9.** If the polynomials $f(Z), g(Z) \in \mathbb{Z}[Z]$ satisfy (1.1), then for any prime $p$, integer $T \geq 2$ and sets $\mathcal{R}, \mathcal{S} \subseteq \mathcal{F}(T)$, we have

$$\sum_{r \in \mathcal{R}, s \in \mathcal{S} \atop \Delta(rs) \not\equiv 0 \pmod{p}} \text{sym}_n(\psi_p(E(rs))) \ll n \left(T^4 p^{-1/2} + T^2 p^{1/2} (\log p)^2\right),$$

uniformly over all integers $n \geq 1$.

**Proof.** We fix a primitive multiplicative character $\chi$ of $\mathbb{F}_p$. Using the orthogonality of the character function, we write

$$\sum_{r \in \mathcal{R}, s \in \mathcal{S} \atop \Delta(rs) \not\equiv 0 \pmod{p}} \text{sym}_n(\psi_p(E(rs))) = \Omega_1 + \Omega_2,$$

where

$$\Omega_1 = \sum_{r \in \mathcal{R}, s \in \mathcal{S} \atop \Delta(rs) \not\equiv 0 \pmod{p}} \text{sym}_n(\psi_p(E(rs))),$$

$$\Omega_2 = \sum_{w \in \mathbb{F}_p^* \atop \Delta(w) \not\equiv 0} \text{sym}_n(\psi_p(E(w))),$$

and

$$\sum_{u_1/v_1 \in \mathcal{R}, \gcd(u_1 v_1, p) = 1} \sum_{u_2/v_2 \in \mathcal{S}, \gcd(u_2 v_2, p) = 1} \frac{1}{p-1} \sum_{m=1}^{p-1} \chi^m(wv_1 v_2/(u_1 u_2)),$$

where $p \mid t$ means that $p$ divides the denominator or numerator of a rational number $t \neq 0$.

Using (3.2), we directly have

$$(3.4) \quad \Omega_1 \ll nT^3 (T/p + 1).$$
By the Cauchy inequality, we get
\[
\Omega_2 = \frac{1}{p-1} \sum_{m=1}^{p-1} \sum_{w \in \mathbb{F}_p^*} \text{sym}_m(\psi_p(E(w))) \chi^m(w) \sum_{u_1/v_1 \in \mathcal{R}} \chi^m(v_1/u_1) \sum_{u_2/v_2 \in \mathcal{S}} \chi^m(v_2/u_2).
\]

Changing the order of summation, we obtain:
\[
\Omega_2 \ll n^{p-1/2} \sum_{m=1}^{p-1} \sum_{u_1/v_1 \in \mathcal{R}} \chi^m(v_1/u_1) \left| \sum_{u_2/v_2 \in \mathcal{S}} \chi^m(v_2/u_2) \right|^2.
\]

Using Lemma 3.6, we have
\[
\Omega_2 \ll n^{p-1/2} \sum_{m=1}^{p-1} \sum_{u_1/v_1 \in \mathcal{R}} \chi^m(v_1/u_1) \left| \sum_{u_2/v_2 \in \mathcal{S}} \chi^m(v_2/u_2) \right|^2.
\]

By the Cauchy inequality, we get
\[
\Omega_2^2 \ll \frac{n^2}{p} \sum_{m=1}^{p-1} \chi^m(v_1/u_1) \left| \sum_{u_2/v_2 \in \mathcal{S}} \chi^m(v_2/u_2) \right|^2.
\]

Note that for any subset \( Q \subseteq \mathcal{F}(T) \), by the orthogonality of characters, we have
\[
\sum_{m=1}^{p-1} \left| \sum_{u/v \in Q} \chi^m(u/v) \right|^2 = \sum_{m=1}^{p-1} \sum_{u_1/v_1, u_2/v_2 \in Q} \chi^m(u_2v_1/(u_1v_2)) = \sum_{m=1}^{p-1} \sum_{u_1/v_1, u_2/v_2 \in Q} \chi^m(u_2v_1/(u_1v_2)) = (p-1)W,
\]

where \( W \) is number of solutions to the congruence
\[
u_1/v_1 \equiv u_2/v_2 \pmod{p}, \quad u_1/v_1, u_2/v_2 \in Q, \quad \gcd(u_1u_2v_1v_2, p) = 1.
\]

Extending the range of variables to the whole interval \([1, T]\), and using [23, Lemma 14] (in a slightly more precise form with \((\log p)^2\) instead of \(p^{o(1)}\) given in the proof of [23, Lemma 14]), which in turn is essentially
a version of a result of Ayyad, Cochrane and Zheng [1, Theorem 2], we obtain
\[ W \ll T^4/p + T^2(\log p)^2. \]
Hence
\[
\left| \sum_{m=1}^{p-1} \sum_{\substack{u/v \in \mathbb{Q} \\
gcd(uv,p) = 1}} \chi^m(v/u) \right|^2 \ll T^4 + T^2p(\log p)^2,
\]
and recalling (3.5) we derive
\[
(3.6) \quad \Omega_2 \ll nT^4p^{-1/2} + nT^2p^{1/2}(\log p)^2.
\]
Since
\[ T^4p^{-1/2} + T^2p^{1/2}(\log p)^2 \geq nT^3, \]
the bounds (3.4) and (3.6) imply the desired result. \(\Box\)

4. Proofs of Main Results

4.1. Proof of Theorem 2.1. For any prime \(p\), let
\[ \mathcal{W}_p = \{(u, v) \in \mathcal{W} \cap \mathbb{Z}^2 : \Delta(u/v) \not\equiv 0 \pmod{p}\}. \]
We denote by \(M_p(\alpha, \beta; \mathcal{W})\) the number of pairs \((u, v) \in \mathcal{W}_p\) such that \(\psi_p(E(u/v)) \in [\alpha, \beta]\).

It follows from Lemma 3.7 that for prime \(p > T\), we have
\[
\left| \sum_{(u,v)\in \mathcal{W}_p} \text{sym}_n(\psi_p(E(u/v))) \right| \ll nT^{1/2}p^{1+o(1)}.
\]
So, using Lemma 3.1, we have
\[
M_p(\alpha, \beta; \mathcal{W}) = \mu_{\text{ST}}(\alpha, \beta)\#\mathcal{W}_p + O \left( \sqrt{T^{1/2}p^{1+o(1)}\#\mathcal{W}_p} \right).
\]
Noticing for \(p > T\),
\[ \mathcal{N}(\mathcal{W}) - T - T\deg \Delta \leq \#\mathcal{W}_p \leq \mathcal{N}(\mathcal{W}) \]
(where the term \(-T\) comes from the exceptional case \(p \mid v\)), we obtain
\[
M_p(\alpha, \beta; \mathcal{W}) - \mu_{\text{ST}}(\alpha, \beta)\mathcal{N}(\mathcal{W}) \ll T + T^{1/4}p^{1/2+o(1)}\mathcal{N}(\mathcal{W})^{1/2}. \tag{4.1}
\]
For \(p \leq T\), we use the trivial bound
\[
M_p(\alpha, \beta; \mathcal{W}) - \mu_{\text{ST}}(\alpha, \beta)\mathcal{N}(\mathcal{W}) \ll \mathcal{N}(\mathcal{W}). \tag{4.2}
\]
Besides, it is easy to see that
\[
\sum_{(u,v) \in W \cap \mathbb{Z}^2 \Delta(u/v) \neq 0} \pi_{E(u/v)}(\alpha, \beta; x) = \sum_{p \leq x} \sum_{\substack{\Delta(u/v) \neq 0 \mod p \\psi_p(E(u/v)) \in [\alpha, \beta]}} 1
\]
\[
= \sum_{p \leq x} \sum_{\Delta(u/v) \neq 0 \mod p} \sum_{\psi_p(E(u/v)) \in [\alpha, \beta]} 1 = \sum_{p \leq x} M_p(\alpha, \beta; W).
\]

Thus, applying (4.1) and (4.2) we deduce that
\[
\sum_{(u,v) \in W \cap \mathbb{Z}^2 \Delta(u/v) \neq 0} \pi_{E(u/v)}(\alpha, \beta; x) - \sum_{p \leq x} \mu_{ST}(\alpha, \beta) \mathcal{N}(W)
\]
\[
= \sum_{p \leq x} (M_p(\alpha, \beta; W) - \mu_{ST}(\alpha, \beta) \mathcal{N}(W))
\]
\[
\ll T \mathcal{N}(W) + \sum_{T < p \leq x} (T + T^{1/4} p^{1/2+o(1)} \mathcal{N}(W)^{1/2})
\]
\[
\ll T \mathcal{N}(W) + T \pi(x) + T^{1/4} x^{1/2+o(1)} \mathcal{N}(W)^{1/2} \pi(x).
\]

Now, the desired result follows from dividing both sides by \(\pi(x) \mathcal{N}(W)\).

4.2. **Proof of Theorem 2.3.** Since the proofs of the two cases are similar, we only present a proof for one case.

For any prime \(p\), let
\[
Z_p = \{(u, v) \in Z(A, B, T) : \Delta(u/v) \equiv 0 \pmod{p}\}.
\]
We denote by \(M_p(\alpha, \beta; Z)\) the number of pairs \((u, v) \in Z_p\) such that \(\psi_p(E(u/v)) \in [\alpha, \beta]\).

It follows from Lemma 3.8 that
\[
\left| \sum_{(u,v) \in Z_p} \text{sym}_n(\psi_p(E(u/v))) \right| \ll nT^2 p^{-1/2} + nT^{1+o(1)} p^{1/2+o(1)}.
\]

So, using Lemma 3.1, we have
\[
M_p(\alpha, \beta; Z) = \mu_{ST}(\alpha, \beta) \# Z_p
\]
\[
+ O \left( \sqrt{(T^2 p^{-1/2} + T^{1+o(1)} p^{1/2+o(1)}) \# Z_p} \right).
\]

Noticing that
\[
\# Z(A, B, T) - T(T/p + 1) - T(T/p + 1) \deg \Delta \leq \# Z_p \leq \# Z(A, B, T)
\]
(where the term \(-T(T/p + 1)\) comes from the exceptional case \(p | v\)), we obtain

\[
M_p(\alpha, \beta; Z) - \mu_{\text{ST}}(\alpha, \beta) \# Z(A, B, T) \\
\ll T^2 p^{-1/4} + T^{3/2+o(1)} p^{1/4+o(1)}.
\]

In addition, as in the above it is easy to see that

\[
\sum_{(u, v) \in Z(A, B, T) \atop \Delta(u/v) \neq 0} \pi_{E(u/v)}(\alpha, \beta; x) = \sum_{p \leq x} M_p(\alpha, \beta; Z).
\]

Thus, applying (4.3) we deduce that

\[
\sum_{(u, v) \in Z(A, B, T) \atop \Delta(u/v) \neq 0} \pi_{E(u/v)}(\alpha, \beta; x) - \sum_{p \leq x} \mu_{\text{ST}}(\alpha, \beta) \# Z(A, B, T)
\ll \sum_{p \leq x} \left( T^2 p^{-1/4} + T^{3/2+o(1)} p^{1/4+o(1)} \right)
\ll T^2 x^{3/4} / \log x + T^{3/2+o(1)} x^{1/4+o(1)} \pi(x) \\
\ll T^2 x^{3/4} / \log x + T^{3/2+o(1)} x^{5/4+o(1)}.
\]

Then, the desired result follows easily from dividing both sides by \(\pi(x) \# Z(A, B, T)\) and the fact that \(\# Z(A, B, T) \asymp T^2\).

4.3. Proof of Theorem 2.5. Denote \(\mathcal{T} = \mathcal{R} \times \mathcal{S}\). For any prime \(p\), let

\[
\mathcal{T}_p = \{(r, s) \in \mathcal{T} : \Delta(rs) \equiv 0 \pmod{p}\}.
\]

We denote by \(M_p(\alpha, \beta; \mathcal{T})\) the number of pairs \((r, s) \in \mathcal{T}_p\) such that \(\psi_p(E(rs)) \in [\alpha, \beta]\).

It follows from Lemma 3.9 that

\[
\left| \sum_{(r, s) \in \mathcal{W}_p} \operatorname{sym}_n(\psi_p(E(rs))) \right| \ll n \left( T^4 p^{-1/2} + T^2 p^{1/2} (\log p)^2 \right).
\]

So, using Lemma 3.1, we have

\[
M_p(\alpha, \beta; \mathcal{T}) - \mu_{\text{ST}}(\alpha, \beta) \# \mathcal{T}_p \ll \sqrt{(T^4 p^{-1/2} + T^2 p^{1/2} (\log p)^2) \# \mathcal{T}_p}.
\]

Noticing that

\[
\# \mathcal{R} \# \mathcal{S} - 2T^3(T/p + 1) - T^3(T/p + 1) \deg \Delta \leq \# \mathcal{T}_p \leq \# \mathcal{R} \# \mathcal{S},
\]

we obtain

\[
(4.4) \quad M_p(\alpha, \beta; \mathcal{T}) - \mu_{\text{ST}}(\alpha, \beta) \# \mathcal{R} \# \mathcal{S} \ll T^4 p^{-1/4} + T^3 p^{1/4} \log p.
\]
Besides, as in the above we have
\[ \sum_{r \in R, s \in S, \Delta(rs) \neq 0} \pi_{E(rs)}(\alpha, \beta; x) = \sum_{p \leq x} M_p(\alpha, \beta; T). \]

Thus, applying (4.4) we deduce that
\[ \sum_{r \in R, s \in S, \Delta(rs) \neq 0} \pi_{E(rs)}(\alpha, \beta; x) - \mu_{ST}(\alpha, \beta) \# R \# S \pi(x) \ll \sum_{p \leq x} \left( T^4 p^{-1/4} + T^3 p^{1/4} \log p \right) \ll T^4 x^{3/4} / \log x + T^3 x^{5/4}, \]
and the desired result now follows.

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