Non-Hermitian Floquet topological phases: Exceptional points, coalescent edge modes, and the skin effect

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Periodically driven non-Hermitian systems can exhibit rich topological band structure and non-Hermitian skin effect, without analogs in their static or Hermitian counterparts. In this work we investigate the exceptional band-touching points in the Floquet quasi-energy bands, the topological characterization of such exceptions points and the Floquet non-Hermitian skin effect (FNHSE). Specifically, we exploit the simplicity of periodically quenched two-band systems in one dimension or two dimensions to analytically obtain the Floquet effective Hamiltonian as well as locations of the many exceptional points possessed by the Floquet bulk bands. Two different types of topological winding numbers are used to characterize the topological features. Bulk-edge correspondence (BBC) is naturally found to break down due to FNHSE, which can be drastically different among different bulk states. Remarkably, given the simple nature of our model systems, recovering the BBC is doable in practice only for certain parameter regime where a low-order truncation of the characteristic polynomial (which determines the Floquet band structure) becomes feasible. Furthermore, irrespective of which parameter regime we work with, we find a number of intriguing aspects of Floquet topological zero modes and π modes. For example, under the open boundary condition zero edge modes and π edge modes can individually coalesce and localize at two different boundaries. These anomalous edge states can also switch their accumulation boundaries when certain system parameter is tuned. These results indicate that non-Hermitian Floquet topological phases, though more challenging to understand than their Hermitian counterparts, can be extremely rich in the presence of FNHSE.

I. INTRODUCTION

Periodic driving, when applied to spatially periodic systems, can yield rich topological phases of matter, now often termed as Floquet topological matter. Edge states of Floquet topological matter are pinned at special quasienergy values and the associated anomalous edge states may not obey the usual bulk boundary correspondence (BBC). The classification of Floquet topological phases has been established and corresponding topological states have been observed in cold atom, photonic, phononic and acoustic systems. Effects of interaction and disorder on Floquet topological phases have also been explored theoretically. In addition to the generation of novel topological phases, periodic time modulation as a control protocol is also of great interest to the realization of nonreciprocal propagation and the design of nonreciprocal devices for photonic and acoustic applications.

In a realistic experimental setup, a quantum system is likely to interact with its environment. To account for this, certain non-Hermitian terms can be introduced in our theoretical modelling. Over recent years, non-Hermitian systems have gained a great deal of attention, especially due to rapid progresses in experimental implementations of non-Hermiticity. Such experiments include examples from photonic, acoustics, vacancy centers in solids, and cold atoms, where non-Hermiticity were introduced through judiciously incorporating gain and loss. Not surprisingly then, these available experimental setups therefore make concrete realizations of non-Hermitian lattice model possible, such as a non-Hermitian version of the topological Su-Schrieffer-Heeger model. Non-Hermitian topological phases often exhibit significantly different physics from their parent Hermitian counterparts. A well-known example is the emergence of exceptional point (EP), where a spectral degeneracy is accompanied by a coalescence of the corresponding eigenstates. Another remarkable feature is the stark difference between the band spectra under periodic boundary condition and that under open boundary condition (hereafter PBC and OBC, respectively). In particular, the bulk eigenstates under OBC are generally localized near boundaries, a phenomenon termed the non-Hermitian skin effect (NHSE). It is now widely accepted that the presence of NHSE often signifies a breakdown of the bulk-boundary correspondence (BBC), a concept already experimentally verified.

Given the above two fruitful topics, namely, Floquet topological phases and non-Hermitian topological phases, we are motivated to examine how non-Hermiticity impacts on Floquet topological phases, especially the interplay of NHSE and periodic driving. Specifically, we ask the following questions: (i) What is the main effect of non-Hermiticity on Floquet-Bloch band structure? (ii) What will be the interesting features of a Floquet system with NHSE? (iii) Do there exist EPs in the quasi-energy spectrum of a periodically driven system? If yes, what is the topology of the EPs? (iv) What is the destiny of BBC in non-Hermitian Floquet systems? In this work, we attempt to answer these questions by focusing on two concrete non-Hermitian two-band systems subject to periodic quenching. One main reason to choose such systems is that the Floquet effective Hamiltonian can be easily obtained.
A number of findings are in order. (i) A periodically quenched system can exhibit the skin effect (dubbed as Floquet non-Hermitian skin effect (FNHSE)) even though the quenched Hamiltonian in each step does not have NHSE. (ii) The introduction of a non-Hermitian term can not only split spectral degenerate point (DP) of the parent Hermitian Floquet system into two EPs but also induce many other EPs. Such EPs are unique insofar as they each carry half-integer topological winding numbers. These results were totally absent in a previously studied non-Hermitian Floquet system that does not possess FNHSE. (iii) The presence of FNHSE breaks the BBC. However, within certain parameter regime, the existence of two different types of Floquet edge modes can still be predicted exactly by introducing the generalized Brillouin zone (GBZ) in two time-symmetric frames. Moreover, FNHSE can either push the edge modes to one side of the system or separate two different types of Floquet edge modes at two opposite boundaries. It is hoped that these results can motivate further studies of both fundamental aspects and potential applications of non-Hermitian Floquet topological phases.

The rest of this paper is organized as follows: In Sec. II we outline a general treatment of two-step quenched non-Hermitian Floquet systems and introduce two types of winding numbers for topological characterization of EPs. With these preparations, in Sec. III we study a two-dimensional (2D) model, with an emphasis placed on topological characterization of the EPs in the PBC spectrum. To better digest FNHSE, we treat a one-dimensional (1D) model in Sec. IV. There we examine the possibility of recovering the BBC with certain parameter regime. Sec. V concludes this paper. Some nonessential details of our calculation are placed in Appendix.

II. TIME-PERIODIC QUENCHING AND TOPOLOGICAL CHARACTERIZATION UNDER PBC

A. Floquet bands

Periodic driving such as periodic quenching is capable of producing topological phases with counter-intuitively large topological invariants. One qualitative explanation is that periodic driving can effectively induce long-range hopping. With this physical picture in mind, we anticipate that the interplay of non-Hermiticity and periodic driving can be highly nontrivial. For simplicity we focus on periodic quenching applied to physically realistic non-Hermitian tight-binding lattice models with only nearest-neighbor (NN) hoppings.

Consider then a periodically driving protocol with overall driving period $T = T_1 + T_2$, such that $H = H_1$ for duration $T_1$ and $H = H_2$ for duration $T_2$. We further confine our discussions to simple two-band dimensionless (with $\hbar = 1$) Hamiltonians that only contain Pauli matrices but are enough to produce rich topological phases. That is, we assume

$$H_i (i=1,2) = \sum_k \mathbf{B}_i (k) \cdot \sigma | k \rangle \langle k |,$$

where $k \in (-\pi, \pi]$ is the quasimomentum vector, and $\mathbf{B}_i (k) = \mathbf{h}_i (k) + ig_i (k)$ with $\mathbf{h}_i (k)$ and $g_i (k)$ being a real vector function, with their components attached to different Pauli matrices. The non-Hermiticity of $H_i$ stems from the non-Hermitian term $ig_i (k)$, which can represent either on-site complex potential or non-reciprocal hopping along a lattice. Regarding possible experimental setups, ultracold atomic gas in optical lattices, photonic crystals, and coupled resonators provide versatile platforms to realize such non-Hermitian systems, with tunability in the system parameters. For example, the non-Hermiticity may be realized by asymmetric scattering between a clockwise and a counterclockwise propagating mode within each resonator, or by introducing atom loss (to effectively induce some non-reciprocal hopping) in the cold-atom context. Given the realizability of FNHSE, we treat a one-dimensional (1D) model in Sec. IV. There we examine the possibility of recovering the BBC with certain parameter regime. Sec. V concludes this paper. Some nonessential details of our calculation are placed in Appendix.

Our general consideration starts from the following single-period Floquet operator:

$$\bar{U}_T = \sum_k U (k) | k \rangle \langle k |,$$

with

$$U (k) = e^{-iT_2 \mathbf{B}_2 (k) \cdot \sigma} e^{-iT_1 \mathbf{B}_1 (k) \cdot \sigma},$$

which defines an effective Floquet Hamiltonian $H_{\text{eff}} (k) = i \ln U (k) / T$. The corresponding Floquet eigenstates of $\bar{U} (k)$ are obtained from

$$U (k) | \varphi \rangle = e^{-i\varepsilon (k) T} | \varphi \rangle,$$

where $\varepsilon (k)$ are the quasienergies and $| \varphi \rangle$ are the corresponding Floquet eigenstates. Due to the non-Hermiticity of $H_i$, the Floquet operator is not unitary and therefore $\varepsilon (k)$ is usually not real. Nevertheless, the real parts of $\varepsilon (k)$ can still be deemed as a phase defined up to a multiple of $2\pi / T$, generally chosen to lie in the first quasienergy Brillouin zone $(-\pi / T, \pi / T]$.

The closed $SU(2)$ algebra here provides an explicit form of $U (k) = \exp [-i\varepsilon (k) \tilde{n} (k) \cdot \sigma]$, where $\tilde{n} = n (k) / \| n (k) \|$ is in general a complex unit vector with

$$n (k) = \sin (\| \mathbf{B}_2 (k) \| T_2) \cos (\| \mathbf{B}_1 (k) \| T_1) \tilde{B}_2 (k)$$

$$+ \cos (\| \mathbf{B}_2 (k) \| T_2) \sin (\| \mathbf{B}_1 (k) \| T_1) \tilde{B}_1 (k)$$

$$+ [\sin (\| \mathbf{B}_1 (k) \| T_1) \sin (\| \mathbf{B}_2 (k) \| T_2)$$

$$\tilde{B}_2 (k) \times \tilde{B}_1 (k)].$$
The expression of the quasienergies $\varepsilon(k)$ can be given as
\[
\cos[\varepsilon(k)] = \cos(||B_1(k)|| T_1) \cos(||B_2(k)|| T_2) \\
- [\sin(||B_1(k)|| T_1) \sin(||B_2(k)|| T_2) \times \hat{B}_1(k) \cdot \hat{B}_2(k)]
\]
with $||B_1(k)|| = \sqrt{B_{i,x}^2(k) + B_{i,y}^2(k) + B_{i,z}^2(k)}$ and $\hat{B}_1(k) = B_1(k) / ||B_1(k)||$ are also the complex unit vectors. From the expression of $\varepsilon(k)$ given above, it can be seen that there may exist two quasienergy gaps at $\varepsilon(k_c) = 0$ and $\varepsilon(k_c) = \pi$ in the complex band structure. The band touching points are located at $\varepsilon(k_c) = 0$ and $\varepsilon(k_c) = \pi$, equivalent to two criteria $\cos[\varepsilon(k_c)] = 1$ or $\cos[\varepsilon(k_c)] = -1$.

### B. Exceptional points

In Hermitian two-band Floquet systems, the closing of two bands occurs at either $\varepsilon(k_c) = 0$ or $\varepsilon(k_c) = \pi$. In the presence of non-Hermiticity here, i.e., $g_i(k) \neq 0$, the Floquet band closing points are in fact exceptional points where two Floquet eigenstates coalesce. This can be seen from the behavior of Floquet operator $U(k) = \cos[\varepsilon(k)] - i \sin[\varepsilon(k)] \hat{n}(k) \cdot \sigma$ at the band touching points $\varepsilon(k_c) = 0$ or $\varepsilon(k_c) = \pi$, which yields the identity $\cos[\varepsilon(k_c)] = \pm 1$. However, the presence of non-Hermitian term $g_i(k)$ does not need $\hat{n}(k_c) = 0$ but leads to $\sin[\varepsilon(k_c)] = [\hat{n}(k_c)] = 0$. Hence the Floquet operator at the band closing points reduces to
\[
U(k_c) = \pm 1 - i \hat{n}(k_c) \cdot \sigma,
\]
with $||\hat{n}(k_c)|| = 0$. It turns out that for our model systems below, $U(k_c)$ obtained above is a Jordan block form accompanied by the coalescence of two eigenstates $|\varphi\rangle$.

### C. Floquet non-Hermitian skin effect

Now we turn to another well-known non-Hermitian effect, i.e., NHSE. Due to NHSE, there is a marked difference between band structure under OBC and that under PBC. Generally, OBC skin modes can be extrapolated from the PBC eigenmodes upon inserting a complex flux to the system with increasing amplitude. During that process, the PBC spectra, which are generically closed loops in the complex plane as a function of quasi-momentum, collapse into open lines or arcs that precisely represent the edge modes under OBC. It is hence curious to examine NHSE in periodically driven systems. Interestingly, to our knowledge, except for a quantum walk experimental study, little is known about Floquet NHSE (FNHSE). Following [81, 83, 89], it is now understood that if for each $k$, there always exists $k'$ such that $\varepsilon(k) = \varepsilon(k')$, then the FNHSE spectrum does not possess any loop structure in the first place and hence FNHSE cannot present under OBC. On the other hand, if such a relation does not hold, then the spectrum may form some loop structure in the complex quasi-energy plane. As a result, FNHSE can be manifested drastically upon introducing OBC.

### D. Winding numbers

In this subsection, we will define two different types of winding numbers to capture the topological property of the Floquet spectrum under PBC. As seen below, the first type of winding numbers reflects the vorticity of the quasi-energy spectrum and the second type of winding numbers characterize the BBC instead. Together they offer complementary topological characterization of the model systems we shall introduce later.

Consider first two specific time windows from $t = T_i/2$ to $t = 3T_i/2 + T_2$ and $t = T_2/2$ to $t = 3T_2/2 + T_1$. The second time frame represents a shift of first time frame. The resulting respective Floquet operators in such two symmetric time frames are given by
\[
U_i(k) = n_0 + i n_i(k) \cdot \sigma, \quad i = 1, 2
\]
where
\[
n_0 = \cos[\varepsilon(k)], \quad n_i(k) = \cos(||B_1(k)|| T_1) \sin(||B_2(k)|| T_2) \hat{B}_i(k) \cdot \hat{B}_j(k) + \sin(||B_1(k)|| T_1) \cos(||B_2(k)|| T_2) \hat{B}_i(k) \cdot \hat{B}_j(k) + \sin(||B_1(k)|| T_1) \hat{B}_j(k) - (\hat{B}_i(k) \cdot \hat{B}_j(k)) \hat{B}_i(k),
\]


with \( i = 1, j = 2 \) or \( i = 2, j = 1 \). To digest the results here, we consider a periodically quenched system with sublattice symmetry, i.e., \( SH_jS^{-1} = -H_i \), where \( S \) represents certain chiral-symmetry (unitary) operators. Because there is no cross product of any two Pauli matrices appearing the expressions above, it can be easily seen that the Floquet operator still possesses the same chiral symmetry, namely, \( SU_j(k)\, S^{-1} = U^{-1}_i(k) \). As such, the effective Floquet Hamiltonian \( H^{\text{eff}}_j(k) = i\ln U_j(k)/T \) with \( j = 1, 2 \) also contains only two Pauli matrices such that the complex-quasi energy spectrum is symmetric with respect to zero.

As an example, we assume that \( H^{\text{eff}}_j(k) \) only includes two Pauli matrices and commutes with \( \sigma_y \). That is,

\[
H^{\text{eff}}_j(k) = n_{j,x}(k)\, \sigma_x + n_{j,z}(k)\, \sigma_z,
\]

with \( (j = 1, 2) \). In the Hermitian case, one then uses the winding of the vector \([n_{j,x}(k)], n_{j,z}(k)\] as a function of \( k \) in the \( xz \)-plane to define winding numbers, and this vector would be ill-defined at any DP because \([n_{j,x}(k)], n_{j,z}(k)\) = 0 for a DP. However, at EPs in non-Hermitian systems, as illuminated above, the \([n_{j,x}(k)], n_{j,z}(k)\) is generally nonzero. So we need to resort an alternative definition of winding numbers. This is made possible by the observation that at any EP at zero or \( \pi \) quasi-energy, the expectation value of \( H^{\text{eff}}_j(k) \) on the associated quasi-energy eigenstate must be zero. As a result, the corresponding expectation values of \( \sigma_x \) and \( \sigma_z \) must be zero at such EPs. This facilitates us to define a vector field

\[
D_j(k, s) = \left( \langle \sigma_x \rangle_j, \langle \sigma_z \rangle_j \right),
\]

where \( s \) denotes other system parameters. This vector field becomes ill-defined at an EP, indicating that EPs represent topological defects under this new definition of winding numbers for non-Hermitian systems. \(^{[17,22]}\)

In a more formal language, the topological characterization of the effective Floquet Hamiltonian of \( H^{\text{eff}}_j(k) \) can be captured by

\[
w_{1,j} = \frac{1}{2\pi i} \int_C \Xi^{-1}_{j,S} \, d\Xi_{j,S}, \quad j = 1, 2
\]

where \( \Xi_{j,S} = \langle \sigma_x \rangle_j + i \langle \sigma_z \rangle_j \) and \( C \) denotes a closed loop in the \( \langle \sigma_x \rangle_j - \langle \sigma_z \rangle_j \) vector field space. This closed loop can be specified after we define an arbitrary closed loop in the momentum space first and then accordingly compute the expectation values \( \langle \sigma_x \rangle_j \) and \( \langle \sigma_z \rangle_j \) on the associated Floquet eigenstates as a function of \( k \). As elaborated below, choosing different closed loops in the momentum space will result in different closed loops in the vector field space and hence different winding numbers. The sign of \( w_{1,j} \) also indicates the winding direction of vector field \( D_j(k, s) \). Since Floquet operators associated with the two different time-symmetry time frames are connected by a unitary transformation, which does not change the location of EPs, the two Floquet effective Hamiltonians in both symmetric time frames should yield equivalent topological winding numbers. Here we want to stress that EPs play the same role as DPs in the parent Hermitian Floquet system. The value of the winding number depends on whether the loop in the momentum space encircle the EP. In the following concrete examples, the different topological phases can be identified with the aid of \( w_{1,j} \), especially in the 1D example. In this sense, the EP can be deemed as the topological phase transition points.

Next we turn to the second definition of winding numbers \( w_{11,j} \), analogous to some known treatments in Hermitian cases. That is,

\[
w_{11,j} = \frac{1}{2\pi i} \int_C \frac{dL^{-1}_{j,S}(k)}{dk} L_{j,S}(k),
\]

where \( L_{j,S}(k) = n_{j,x}(k) + in_{j,z}(k) \), with \( n_{j,x}(k) \) and \( n_{j,z}(k) \) still describing the effective Floquet Hamiltonian [see Eq. (12)]. At first glance this definition seems to advocate complex winding numbers. However, due to the assumed chiral symmetry of the system, it can be shown that the winding numbers thus defined are actually real. \(^{[12]}\) Note also that \( w_{11,j} \) represents and only represents the winding behavior of \( L_{j,S}(k) \) around the origin of the momentum space (this is thus different from our choice of a rather arbitrary closed loop in the momentum space when examining the first type of winding numbers). Following previous extensive results in Hermitian Floquet systems, winding numbers \( w_{11,j} \) in two time-symmetric frames are expected to predict the number of Floquet edge states, thereby reflecting BBC. Specifically, using \( w_{11,1} \) and \( w_{11,2} \), one can obtain \( W_0 \) and \( W_\pi \) as

\[
W_0 = \frac{w_{11,1} + w_{11,2}}{2}, \quad W_\pi = \frac{w_{11,1} - w_{11,2}}{2}.
\]

FIG. 1: Schematic illustration of a 2D quenched system. Each purple dot represents a unit cell with two internal degrees of freedom. The system Hamiltonian is \( H_1 \) for duration \( T_1 \) for which the couplings between the adjacent unit cells are only along the square-side directions. The system Hamiltonian is then quenched to \( H_2 \) for duration \( T_2 \), where only intercell coupling along the diagonal direction exists.
III. MODEL: 2D NON-HERMITIAN FLOQUET SYSTEM

With all the preparations above, we are now ready to develop more physical insights by working on a concrete system.

Let us consider a 2D non-Hermitian Floquet system, the two Hamiltonians that the system is periodically quenched between are assumed to be

\[ H_1 = \sum_{k_x,k_y} B_{1,x} (k_x, k_y) \sigma_1 |k_x, k_y\rangle \langle k_x, k_y|, \]
\[ H_2 = \sum_{k_x,k_y} B_{2,z} (k_x, k_y) \sigma_2 |k_x, k_y\rangle \langle k_x, k_y|, \]

where

\[ B_{1,x} (k_x, k_y) = 2t_1 (\cos k_x + \cos k_y) + t_2, \]
\[ B_{2,z} (k_x, k_y) = 4t_3 \sin k_x \sin k_y + i\gamma. \]

with \( t_1 \) and \( t_3 \) being the intercell couplings along the square-side and diagonal directions. \( t_2 \) describes the intracell coupling and \( \gamma \) represents the balance gain and loss in each unitcell. Obviously, the Floquet operators of two symmetric time frames respect the chiral symmetry, i.e., \( \sigma_y U_i (k_x, k_y) \sigma_y^{-1} = U_i^{-1} (k_x, k_y) \). In Fig. 1 we sketch two alternating lattice configurations of the system. In addition to the rather familiar optical lattice and waveguide array platforms, such 2D non-Hermitian Floquet system could be also realized with circuits and nonreciprocal electrical elements or in discrete-time non-unitary quantum-walk systems. Note also that in the high-frequency limit \( T \to 0 \) (with \( T_1 = T_2 \)), the Floquet effective Hamiltonian \( H_{\text{eff}} \) is simply given by \((H_1 + H_2)/2\), which in the coordinate space represents a tight-binding bilayer square lattice. In that limit the EPs always occur at either \( k_x = 0, \pi \) or \( k_y = 0, \pi \). Thus, it is interesting to see how periodic quenching in general may change the locations of the EPs. Note in passing that, according to Eq. (9), the quasi-energy spectrum of our 2D model is determined by

\[ \cos \varepsilon (k_x, k_y) = \cos [2t_1 (\cos k_x + \cos k_y) + t_2] \times \cos (4t_3 \sin k_x \sin k_y + i\gamma), \]

where we set lattice constant, and driving period all equal to 1. For either \( H_1 \) or \( H_2 \) alone, one can see \( H_1 (k) = -H_1 (-k) \) for all \( k \). Thus, \( H_1 \) or \( H_2 \) alone as a static system does not possess NHSE. That also indicates that in the high frequency limit, the Floquet effective Hamiltonian becomes \((H_1 + H_2)/2\) (with \( T_1 = T_2 \)) and does not possess NHSE either. In general, by periodically quenching between these two Hamiltonians, we may induce some loop structure in the spectrum of the resulting Floquet effective Hamiltonian and hence FNHSE. This will be elaborated in the next section using a reduced model.

The Floquet EPs can now be determined by the fol-
Following equations

$$4t_3 \sin k_x \sin k_y = n\pi,$$

$$2t_1 (\cos k_x + \cos k_y) + t_2 = m\pi \pm \arccos \left(\frac{1}{\cosh \gamma}\right),$$

where $m$, $n$ are integers of the same parity (opposite parity) if the EPs locates at $\varepsilon (k_x, k_y) = 0$ [w (k_x, k_y) = $\pi$]. These specific solutions give many EPs, as shown in Fig. 2 (for simplicity, we set $T_1 = T_2 = 1$ throughout this paper). Regarding how the presence of the non-Hermitian term affects the parent Hermitian Floquet system, two important observations are in order. First, the $i\gamma$ term can induce the splitting of a single DP into two EPs. Each pair of the EPs denoted by either red star or blue circle stemming from a black DP as determined by $2t_1 (\cos k_x + \cos k_y) + t_2 = m\pi$ and $4t_3 \sin k_x \sin k_y = n\pi$. This is plotted in Fig. 2 where different colors are also used to represent quasi-energy values at zero or $\pi$. In principle, each EP inherits half of the topological winding number of the DP associated with the parent Hermitian Floquet system. These EPs cannot be annihilated unless two EPs with same quasi-energy collide with each other and therefore the presence of EP is topological. Second, as compared with the Hermitian case, the $i\gamma$ term can generate new intersections between the two closed curves described by Eq. (21) and Eq. (21), thus creating brand-new second-type EPs unrelated to the DPs of the parent Hermitian system. This second type of EPs are denoted by diamonds in Fig. 2. It is seen that the second-type of EPS always emerge in pairs.

To further characterize the vorticity of the EPs in the quasi-energy spectrum, we next plot the planar vector field $\mathbf{D}_1 (k)$ in Fig. 3. The vorticity of the EPs can be quantified by the winding number $w_{1,j}$. To that end, we first choose a closed loop in the momentum space, then we obtain a closed loop in the vector field space. Similar to the Refs. 89, $w_{1,j}$ will be a half integer if the momentum-space loop encircles an EP of the first type), and it must be zero if the loop does not encircle any EP of the first type. It is also found that each pair of the second type of EPs, if both enclosed by the chosen momentum-space
closed trajectory. Hence the winding number around the origin, but the momentum-space loop used to define the vector field are the same with those in 2(b). It is seen that when the loop surrounds one EP, the vector field black lines are for results when a closed loop encircles an EP, two EPs with the same quasi-energy, two EPs with different quasi-energies and four type-II EPs. The system parameters are the same with those in 2(b). It is seen that when the loop surrounds one EP, the vector field $D_1(k)$ does wind around the origin, but the momentum-space loop used to define the trajectory must continue to wind twice in order to form a closed trajectory. Hence the winding number $w_{1,1}$ has a fractional value $\pm 1/2$. When the momentum-space loop encircles two EPs with same (opposite) charge, the obtained winding number will be 1 (0). Note that the points of black curve are double degenerate and hence the winding number is 0.

To further illustrate our observations above, we now look into the trajectory of $\langle \sigma_x \rangle_1$, $\langle \sigma_z \rangle_1$ as $k$ encircles the EPs found in Fig. 4. There it is seen the trajectory is half of a closed curve, thus indicating that their winding number is one half. By contrast, when $k$ encircles both EPs with the same topological charge, the resulting trajectory forms a closed curve and therefore the corresponding winding number is 1.

IV. FLOQUET NON-HERMITIAN SKIN EFFECT AND TOPOLOGICAL CHARACTERIZATION

A. EPs in a 1D non-Hermitian Floquet system

As become clear from explicit calculations presented in Appendix, it is highly non-trivial to examine FNHSE in Floquet systems, even for 1D situations. Recognizing this challenge, in this section we start with a 1D non-Hermitian Floquet system periodically quenched between the following two Hamiltonians,

$$H_1 = \sum_k B_{1,x}(k) \sigma_x \langle k \rangle \langle k \rangle,$$

$$H_2 = \sum_k B_{2,z}(k) \sigma_z \langle k \rangle \langle k \rangle,$$

where

$$B_{1,x}(k) = w + v \cos k,$$

$$B_{2,z}(k) = v \sin k + i\gamma,$$

with $w$ and $v$ are the intracell and intercell couplings. $i\gamma$ denotes the staggered on-site imaginary potential. Although the driven Hamiltonian in each period is simple, the Floquet system can still induce many non-trivial physical properties. This 1D Floquet Hamiltonian can be also deemed as the equivalent Hamiltonian describing the subspace of the 2D Floquet system in Eqs. [16]-[17] associated with one particular quasi-momentum along the $y$ direction. In this sense, examining 1D Floquet systems not only provides some insights into low-dimensional FNHSE but also captures important aspects of the edge states of certain 2D non-Hermitian Floquet systems. The locations of EPs of this 1D model are found to be determined from

$$v \sin k = n\pi,$$

$$w + v \cos k = m\pi \pm \arccos \left(\frac{1}{\cosh \gamma}\right).$$

Again, $m$, $n$ are integers of the same parity (opposite parity) if the gap closes at $\varepsilon(k) = 0$ [$\varepsilon(k) = \pi$]. In Fig. 5 we plot the obtained EPs with different quasi-energies in the $w - k$ space. We choose the $w - k$ space to plot because, now given only one quasi-momentum variable, we are forced to use a second system parameter to form a 2D parameter space so as to define winding numbers etc. For example, the winding numbers $w_{1,j}$ now defined in the $w - k$ space also capture the topological features of the EPs. As a specific example, we may examine $w_{1,j}$ on a circle around the EPs, defined as $k = r_1 \sin \theta + k_c$ and $w = r_2 \cos \theta + w_c$ with $\theta$ varying from 0 to $2\pi$, $r_1$ the radius of the ellipse and $(k_c, w_c)$ the coordinate of the EP in this parameter space. According to the Ref. [23], the winding number can be therefore represented as

$$\sum_i w_{1,\alpha}^i = v_{L,\alpha} - v_{R,\alpha},$$

FIG. 4: Trajectory of $\langle \sigma_x \rangle_1$, $\langle \sigma_z \rangle_1$ obtained along a momentum-space closed loop. Green, purple, orange, and black lines are for results when a closed loop encircles an EP, two EPs with the same quasi-energy, two EPs with different quasi-energies and four type-II EPs. The system parameters are the same with those in 2(b).
FIG. 5: Many EPs found in our 1D non-Hermitian Floquet system. Each of the EPs carries half-integer topological defect, as manifested by the planar vector field $D_i(k,w)$. The system parameters are $v = 1.8$, $\gamma = 0.5$. Note that there is no type-II EPs for 1D model here. All the EPs stem from the splitting of DPs of the parent Hermitian Floquet system.

FIG. 6: GBZs, denoted by $C_\beta$, obtained for two different time-symmetric frames (with different colors). Details are specified in the main text and in Appendix. (a) $w = 0.5$, $v = 0.8$, and $\gamma = 0.5$. (b) $w = 2$, $v = 0.6$, and $\gamma = 0.5$. For comparison, we also plot $C_k$ with blue dashed lines in the absence of the non-Hermitian term. It is seen that $|\beta|$ can be larger or less than unity.

where

$$w_{L(R),\alpha} = \frac{1}{2\pi i} \int_k dk \Xi^{-1}_{\alpha,S} d\Xi_{\alpha,S}/dk, \quad \alpha = 1, 2$$

denoting the winding numbers of different topological phases on two sides of the EP. $w^t_{i,\alpha}$ represents the winding number of the $i$-th EPs. In our 1D system shown in Fig. 5, only one EP is available for a given value of $w$. Therefore we can remove the summation in the Eq. (29). It is clear that the change of winding number $v_\alpha$ across the phase transition point equals to the summation of winding number $w^t_{i,\alpha}$ of each EPs. From this point of view, EP is also the topological phase transition point.

B. Skin effect and topological characterization

For the present 1D non-Hermitian Floquet system, the Floquet operators in two time-symmetric frames are given by

$$U_j(k) = n_0(k) + i n_j(k) \cdot \sigma, \quad j = 1, 2$$

where $j = 1, 2$, and the effective vector fields $n_j(k)$ are found to be

$$n_{1,x}(k) = \sin[B_{1,x}(k)] \cos[B_{2,z}(k)],$$
$$n_{1,z}(k) = \sin[B_{2,z}(k)],$$
$$n_{2,x}(k) = \sin[B_{1,x}(k)],$$
$$n_{2,z}(k) = \cos[B_{1,x}(k)] \sin[B_{2,z}(k)].$$

Apparently, if $w \neq m\pi$, then the $w$ value drops out from expressions $\cos[B_{1,z}(k)] = \cos[w + v \cos(k)] = \pm \cos[v \cos(k)]$ and $\sin[B_{1,z}(k)] = \sin[w + v \cos(k)] = \pm \sin[v \cos(k)]$ and the resulting Floquet operator reduces to a previously studied model that is known to have no FNHSE (this will be confirmed again below). In general situations, due to FNHSE, the Floquet eigenstates...
do not necessarily extend over the bulk. Instead, they can be localized at either end of the lattice. The bulk topological invariants [Eq. (15)] in terms of the Bloch wave vector can no longer predict exactly the number of edge modes under OBC. One main purpose of a careful topological characterization for our 1D model is to inspect the possibility of restoring BBC in the presence of FNHSE. This presents a challenge given the dramatic difference between the PBC spectrum and the OBC spectrum caused by FNHSE.

To attack this issue below we shall follow a generalized Bloch band theory by obtaining first the so-called generalized Brillouin Zone (GBZ) \[ \text{GBZ} \]. In short, we aim to recover the BBC by accounting for the non-Bloch-wave character of bulk states through GBZ and then re-calculate topological invariants based on GBZ. In the generalized Bloch band theory, one key step is to replace \( \exp (\pm i k) \) with \( \beta^{\pm 1} \), which can be determined by the characteristic polynomial \( \det [H_{\text{eff}}(\beta) - \varepsilon I] = 0 \) and other conditions. For non-Hermitian systems in general, \( |\beta| \) is not unity and the corresponding bulk states are localized at one boundary. Specifically, if we number the solutions to \( \det [H_{\text{eff}}(\beta) - \varepsilon I] = 0 \) as \( \beta_i (i = 1, ..., 2M) \) so as to satisfy \( |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_{M-1}| \), where \( 2M \) stands for the degree of algebraic equation for \( \beta \). The so-called GBZ, denoted \( C_\beta \), is obtained by all solutions satisfying the curious condition \( |\beta_M| = |\beta_{M+1}| \) (see Appendix for more details).

One can now appreciate that consideration of GBZ in Floquet systems can be highly technical. In static systems (or a Floquet system in the high-frequency limit), the effective Hamiltonian \( H_{\text{eff}}^j \) only contains some short-range couplings (e.g., nearest-neighbor hopping). Then the equation \( \det [H_{\text{eff}}^j(\beta) - \varepsilon I] = 0 \) involves only a finite-order polynomial of \( \beta \). However, even in our simple 1D Floquet system here, the periodic quenching effectively induces long-range hopping across the lattice. This fact can be also seen from the momentum-space function \( \sin[\cos(\cdot)] \) contained in Eqs. (32)-(35), which can be directly interpreted as a consequence of effectively long-range hopping across the lattice. This hence leads to a highly complicated characteristic equation \( \det [H_{\text{eff}}(\beta) - \varepsilon I] = 0 \). Indeed, for our 1D model here, \( \det [H_{\text{eff}}(\beta) - \varepsilon I] = 0 \) involves functions such as \( \sin[\pi + \pi \cos(\beta/2 + 1/2)] \), which is a polynomial of \( \beta \) to infinite order.

To provide part of the solutions, we now assume that the strength of \( v \) is less than 1 such that the involved polynomials about \( \beta \) can be effectively truncated up to

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**FIG. 7:** The restored BBC for our 1D non-Hermitian Floquet system depicted in the main text. (a) Upper panel: The OBC spectrum with \( v = 0.8 \), and \( \gamma = 0.5 \). (b) Lower panel: The topological bulk invariants calculated with the aid of GBZ. Red and blue points denote the winding numbers \( W_0 \) and \( W_\pi \), respectively. The obtained winding numbers can perfectly predict two types of Floquet edge modes.

**FIG. 8:** Plots of the physical quantities of average position \( P \) (\( P_\pm \)), and fidelity \( F_\pm \) as functions of \( w \). The numerical simulation is performed for \( v = 0.8 \), and \( \gamma = 0.5 \). (a) Blue dots, red triangles, and black circles denote \( P^\pm \), \( P_+ \), and \( P_- \) respectively. (b) Red triangles and black circles represent fidelity \( F_+ \) and \( F_- \) respectively. It can be seen that FNHSE can push all the Floquet eigenstates to one side of lattice accompanied by the coalescence of the Floquet edge modes, which can be shown in panel (b). Note that the fidelity \( F_\pm \) quickly drops from 1 to 0 as the system parameter \( w \) approaches \( m\pi \). At these special points, the Floquet system is free of FNHSE and therefore the edge modes localize individually at both ends of the lattice.
β^2 and β^2 - 2 (for more details, see Appendix). This procedure also restricts ourselves to a parameter regime without the coexistence of many pairs of Floquet edge states. In Fig. 6, we plot the GBZ according to our calculations detailed in Appendix. It can be shown that |β| can be larger or smaller than 1. Furthermore, C_β can have cusps at which more than two solutions of β share the same absolute value.

To proceed, we replace the C_β with C_β in Eq. 14. We then calculate the generalized winding numbers W_0 and W_π in terms of the reduced vector u_β (β). Presented in Fig. 7 is the OBC spectrum of our non-Hermitian system and the corresponding generalized winding numbers. Excellent agreement is obtained and the BBC is perfectly recovered for small values of v. Note also that even when we only approximate the characteristic equation \text{Det} [H_{eff} (β) - \epsilon I] = 0 up to the second order, BBC for our 1D Floquet system can already be well recovered. One may perform similar calculations for based on a third-order expansion in terms of v, but that is already more accurate than necessary in terms of recovering the BBC here in the presence of FNHSE.

C. Anomalous 0 and π modes under OBC

In this subsection, we investigate the Floquet eigenstates under OBC. It can be envisioned that the Floquet eigenstates will be pushed to one lattice boundary due to FNHSE. However, because of the absence of FNHSE for w = mπ, we anticipate some interesting details as we tune w across these special points. In addition, for systems with larger and larger v, the emergence of long-range hopping can make the situation richer, a restoration of BBC will not be practical and this makes our investigation of the Floquet edge states more necessary.

Let us first focus on the case of with small v, namely v = 0.8 as an example. At w = mπ, the system is free from FNHSE. What we are curious about is how the Floquet eigenstates change when the system crosses such FNHSE-free points, and whether this transition is different for bulk and edge modes. To that end we define several quantities. We first define P and P± as follows:

\[
P = \frac{1}{N} \sum_n \frac{\langle \varphi_n | \tilde{x} | \varphi_n \rangle}{\langle \varphi_n | \varphi_n \rangle}, \quad P_\pm = \frac{1}{N_{\text{edge}}} \sum_j \frac{\langle \varphi_{\pm,j} | \tilde{x} | \varphi_{\pm,j} \rangle}{\langle \varphi_{\pm,j} | \varphi_{\pm,j} \rangle},
\]

where |φ_n⟩ represents the nth bulk eigenstates, and |φ_{±,j}⟩ represents the jth Floquet edge mode at quasi-energy 0 (for +) and π (for −), respectively. N (N_{\text{edge}}) is the number of of the bulk (edge) states, and \tilde{x} is the position operator. P therefore measures the average position of the bulk states, and P± measures the same for Floquet edge states at quasi-energy 0 and π. If all the bulk eigenstates are extended states or all the edge modes come in pair and are localized on both sides of the lattice, then the values of P and P± will be half of the lattice length. The other quantity describes the fidelity between two edge modes with the same quasi-energy:

\[
F_\pm (j, j') = |\langle \varphi_{\pm,j} | \varphi_{\pm,j'} \rangle|.
\]

If F_± (j, j') = 1, then the corresponding two edge modes coalesce. Below we call such coalescence edge modes as anomalous Floquet edge modes.

Figure 8 shows how FNHSE impacts on the Floquet eigenstates in terms of the quantities defined above. There it is seen that FNHSE pushes all Floquet eigenstates to one boundary. In particular, except for cases with w very close to special points of w = mπ, all Floquet edge modes coalesce and are localized on only one side of the lattice. The values of P and P± undergo dramatic changes as w crosses the skin-effect-free points. As such, the average location of both the bulk states and coalescent edge states switches from one side to the other side of the lattice. The preferred direction of FNHSE here can thus be tuned via the system parameter w.
Next we switch to cases with larger $v$, i.e., $v = 1.8$. This value is already beyond the regime where our theoretical treatment can restore BBC. Results are presented in Fig. [9] in terms of the populations on each lattice sites for individual eigenstates under OBC. It can now be seen that FNHSE pushes most of the Floquet eigenstates to both lattice boundaries instead of one boundary. Thus, FNHSE for larger values of $v$ is richer as its preferred direction changes from one bulk eigenstate to another. Furthermore, the two types of edge modes (0 and $\pi$) can now coexist under OBC spectrum, (see green rectangles of Fig. 10). Remarkably, we find that in the present parameter regime each of the coexisting edge modes coalesces individually and localizes at two different boundaries of the lattice. If we sweep the parameter $w$ across $m\pi$, then the Floquet zero modes and $\pi$ modes will also exchange their preferred boundaries. To our knowledge, such anomalous localization phenomena are not known previously and it would be stimulating to develop a theory to account for these findings. The separation of Floquet zero modes and $\pi$ modes by FNHSE may be also beneficial when considering dynamics control, information encoding and even braiding between Floquet zero modes and $\pi$ modes [95,96].

V. CONCLUSIONS

In this work, we have investigated some general aspects of periodically driven non-Hermitian lattice systems in 1D and 2D. For Floquet band structure under PBC, we focus on EPs and their topological characterization. For Floquet spectrum under OBC, we place our emphasis on FNHSE and the BBC. Though our explicit results are based on simple periodically quenched two-band models, we can still make a number of general conclusions, as listed below.

First, Non-Hermiticity in Floquet systems can induce many EPs. One type of EPs is inherited from the DPs of their parent Hermitian Floquet system with half-integer topological winding numbers. The second type of EPs, newly created by non-Hermiticity, always appear in pairs and have opposite half-integer winding numbers. The number and the locations of such EPs can be extensively controlled by tuning some system parameters.

Second, non-Hermitian Floquet systems in general have FNHSE, albeit with some special FNHSE-free points. This explains why in a previous study, FNHSE was not found. The preferred direction of FNHSE can be controlled if we tune relevant system parameters across such FNHSE-free points. This observation may lead to new opportunities for quantum control with the aid of non-Hermiticity.

Third, in the presence of FNHSE, BBC breaks down as expected. Recovering BBC can in principle be done in Floquet systems, in the same fashion as in static systems using, for example, the GBZ framework. However, even for our model systems whose bulk spectrum under PBC can be worked out analytically, restoration of BBC is only feasible in practice for certain parameter regimes where low-order truncation of the characteristic polynomial can be done with negligible error. As such, in general, predicting the Floquet edge states by use of the bulk properties may present a challenge.

Fourth, to motivate further studies, a few interesting aspects of non-Hermitian Floquet edge states deserve to be highlighted. In particular, Floquet zero and $\pi$ modes are found to be coalescent modes in general. Their existence can be well predicted from topological winding numbers of the bulk states if BBC can be restored. These two types of coalescent edge modes can also coexist (in our model system, it is already highly challenging to predict this coexistence using restored BBC). When they do coexist, they can be localized at different boundaries of...
the system. It is worth pointing out that such characteristics are very different from those in the work\textsuperscript{28} in which BBC holds and many non-coalescent edge modes can exist due to the absence of FNHSE.

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Appendix A: Generalized Brillouin Zone in two symmetric time frames

In this Appendix, we shall explain how we obtain the GBZ in two time-symmetric frames mentioned in the main text. Let us start from the Floquet operator in the first time frame, which can be written as

$$U_1(k) = n_0 + i n_1(k) \cdot \sigma,$$

where $n_{1x}(k) = \sin(w + v \cos k) \cos(v \sin k + i \gamma)$, and $n_{1z}(k) = \sin(v \sin k + i \gamma)$. The first step is to replace $e^{ik}$ with $\beta$ so that the effective magnetic field can be expressed as

$$n_{1x}(\beta) = \sin \left[ w + v (\beta + \beta^{-1}) / 2 \right] \times \cos \left[ -iv (\beta - \beta^{-1}) / 2 + i \gamma \right],$$

$$n_{1z}(\beta) = \sin \left[ -iv (\beta - \beta^{-1}) / 2 + i \gamma \right].$$

The corresponding characteristic equation of $n_1(k) \cdot \sigma$ can be written as

$$\varepsilon^2(\beta) = n_{1x}^2(\beta) + n_{1z}^2(\beta) \quad (A4)$$

We should choose the values of $\beta$ such that the energy levels become dense and asymptotically form continuum bands when the system size increases. However, the challenge here is that the polynomial with respect to $\beta$ is of infinite order. It is too cumbersome to arrive at the general solution to $\det[H_{\text{eff}}(\beta) - \varepsilon I] = 0$. To obtain an analytical result, we need to perform the Taylor expansion of $\varepsilon(\beta)$ up to finite order. This can be done by considering a small $v$ coupling. Then, we can approximate the characteristic equation up to the second order (as an example here. One may also try the same expansion up to the third order) with respect to $\beta$. Under this treatment, the approximate expression of $\varepsilon(\beta)$ can be given as

$$\varepsilon^2(\beta) = \sum_{j=-2}^{2} \beta^j X_j,$$

where

$$X_0 = \frac{v^2}{16} \left[ 1 + 2 \cos(2w) - 2 \cosh^2(2\gamma) \right],$$

$$X_1 = v \cos(2w) \cos(\gamma) \sinh(2\gamma),$$

$$X_{-1} = v \cos(2w) \cosh(\gamma) \sin(2\gamma),$$

$$X_{-2} = \frac{v^2}{16} \left[ 1 + 2 \cos(2w) - 2 \cosh^2(2\gamma) \right] - \cos(2\gamma) + 4 \sin(2w) \sin(2\gamma).$$

This is a quartic equation equation with respect to $\beta$ and therefore $M = 2$. Note that the degree of algebraic equation for $\beta$ depends on the order $n$ of our Taylor expansion here, i.e., $M = n$. Moreover, the trajectory of $\beta$ satisfying the continuum band can be determined by $|\beta_2| = |\beta_3|$. Suppose the two solutions $\beta$ and $\beta'$ have the same absolute values, then we have $\beta' = \beta e^{i\theta}$ where $\theta$ is a real number. Taking the difference between two characteristic equations $\varepsilon^2(\beta) = F(\beta)$ and $\varepsilon^2(\beta') = F(\beta')$, we have

$$0 = \sum_{j=-2}^{2} \beta^j X_j \left( 1 - e^{i\theta} \right). \quad (A10)$$

This equation allows us to obtain $\beta$ for a given value of $\theta \in [0, 2\pi)$. Then we obtain a set of values of $\beta$ that satisfies $|\beta| = |\beta'|$. Picking up $\beta$ from the constraint $|\beta_2| = |\beta_3|$, we finally arrive at the GBZ. One can follow the same method to obtain the GBZ associated with the Floquet operator for the second time frame.
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