THE NON-TEMPERED $\theta_{10}$ ARTHUR PARAMETER AND GROSS-PRASAD CONJECTURES

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Abstract. We provide a construction of local and automorphic non-tempered Arthur packets $A_\Psi$ of the group $SO(3,2)$ and its inner form $SO(4,1)$ associated with Arthur’s parameter

$$\Psi : \mathcal{L}_F \times SL_2(\mathbb{C}) \to O_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to Sp_4(\mathbb{C})$$

and prove a multiplicity formula.

We further study the restriction of the representations in $A_\Psi$ to the subgroup $SO(3,1)$. In particular, we discover that the local Gross-Prasad conjecture, formulated for generic $L$-packets, does not generalize naively to a non-generic $A$-packet. We also study the non-vanishing of the automorphic $SO(3,1)$-period on the group $SO(4,1) \times SO(3,1)$ and $SO(3,2) \times SO(3,1)$ for the representations above.

The main tool is the local and global theta correspondence for unitary quaternionic similitude dual pairs.

1. Introduction

Let $F$ be a global field and let $\mathcal{L}_F$ be its conjectural Langlands group. Recall that for every place $v$ of $F$ there is an embedding $i_v : W'_F \to \mathcal{L}_F$, where $W'_F$ is the Weil-Deligne group of the local field $F_v$. In this paper we consider a non-tempered Arthur parameter $\Psi : \mathcal{L}_F \times SL_2(\mathbb{C}) \to O_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to Sp_4(\mathbb{C})$ such that the image of a unipotent element of $SL_2(\mathbb{C})$ belongs to the orbit generated by the short root element. Such a parameter is called a parameter of $\theta_{10}$ type. The global parameter $\Psi$ gives rise to a family of local non-tempered parameters

$$\Psi_v = \Psi \circ i_v : W'_F \times SL_2(\mathbb{C}) \to Sp_4(\mathbb{C}).$$

Let $(V,q^\pm)$ be a non-degenerate quadratic space over a non-archimedean field $F_v$ of dimension 5, discriminant 1 and the normalized Hasse invariant $\pm 1$. The groups $SO(V,q^\pm)$ are pure inner forms of each other and share the same dual group $Sp_4(\mathbb{C})$. According to Arthur’s conjecture there exist finite sets $A_{\Psi_v}$ of admissible unitary representations of $SO(V,q^\pm)(F_v)$ corresponding to the parameter $\Psi_v$, called local $A$-packets. The construction of Arthur packets is a deep question which is far from being solved. For many small rank cases the packets are constructed by the theta correspondence method.

1.1. Construction of $A$-packets. In this paper we construct the local $A$-packets of $\theta_{10}$ type for the split group $SO(V,q^+)$ and its inner form $SO(V,q^-)$ using the similitude theta correspondence for quaternionic unitary dual pairs. The quaternionic unitary dual pairs are relevant to the problem since the groups $SO(V,q^\pm)(F_v)$ are isomorphic to the group of projective similitude automorphisms of the two-dimensional Hermitian space over $D$, where $D$ runs over the set of all quaternion algebras over $F_v$. When $F_v = \mathbb{R}$ the same similitude theta correspondence allows to define the Arthur packets for the groups $SO(3,2)$ and $SO(4,1)$, but not for the anisotropic form $SO(5)$.

We justify our construction using global methods. Let $(V,q)$ be a non-degenerate quadratic space of dimension 5 over a number field $F$ that is not anisotropic over any real place. Taking tensor products of the representations of $SO(V,q)(F_v)$ in the local $A$-packets one can form a set of nearly equivalent
representations of $SO(V, q)(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $F$. This set is called the global $A$-packet. For any representation in the global $A$-packet Arthur predicts a formula for its multiplicity in the discrete automorphic $L^2$ spectrum of $SO(V, q)$. We construct an automorphic realization of certain representations in the global $A$-packets and prove this multiplicity formula.

As another justification we show that the constructed cuspidal representations form a nearly equivalence class of cuspidal representations. The proof of the latter statement involves an $L$-function argument.

The cuspidal automorphic representations of $\theta_{10}$ type of the split group $Sp_4 \simeq Spin_5$ were considered by Piatetskii-Shapiro and Howe [HPS] using the theta correspondence for the dual pair $(O_2, Sp_4)$ as the first counterexamples to the Ramanujan conjecture. Later, Soudry in [S] used the similitude global theta correspondence to construct non-tempered CAP representations of $GSp_4$ and investigated their properties. In [Ya2] Yasuda has constructed the local Arthur packets of $\theta_{10}$ type of the group $Sp_4$ and its inner form using the theta correspondence for quaternionic dual pairs. We modify his results for the similitude theta correspondence. Note, that for the similitude group the multiplicity one property holds for the cuspidal representations in the packets of type $\theta_{10}$, when the representations of $Sp_4$ constructed by Yasuda can have high multiplicity in the discrete spectrum.

1.2. The restriction problem. Our second goal is to investigate the restriction problem over a local non-archimedean field $F_v$. Assume for now that $(V, q)$ is an arbitrary non-degenerate quadratic space and $(U, q_U)$ is its non-degenerate subspace of codimension one.

The main object of the restriction problem is to compute, for all irreducible representations $\Pi$ of $SO(V, q)$ and $\pi$ of $SO(U, q_U)$, the dimension of

$$\text{Hom}_{SO(U)}(\Pi, \pi).$$

The recent multiplicity one result in [AGSR] with a refinement in [W] shows that the dimension of this space is at most one.

About twenty years ago Gross and Prasad in [GP1] and [GP2] have formulated a conjecture according to which, given two generic local Langlands parameters

$$\Phi_1 : W'_F \to L(SO(V, q))(\mathbb{C}), \quad \Phi_2 : W'_F \to L(SO(U, q_U))(\mathbb{C}),$$

there exists a unique quadratic space $(V', q')$ with a non-degenerate subspace $(U', q'|_{U'})$ of codimension one such that

$$\dim V' = \dim V, \quad \text{disc}(V', q') = \text{disc}(V, q), \quad \dim U' = \dim U, \quad \text{disc}(U', q'|_{U'}) = \text{disc}(U, q_U),$$

and unique representations $\Pi$ of $SO(V')$ and $\pi$ of $SO(U')$ in the local Langlands packets associated with the parameters $\Phi_1$ and $\Phi_2$ respectively such that

$$\text{Hom}_{SO(U')}(\Pi, \pi) \neq 0.$$ 

Equivalently,

$$\sum_{V' \subset U'} \sum_{\Pi, \pi} \dim \text{Hom}_{SO(U')}(\Pi, \pi) = 1. \tag{1.1}$$

Here $SO(V') \supset SO(U')$ runs over all relevant pure inner forms of $SO(V) \supset SO(U)$ and $\Pi$ and $\pi$ run over representations in the Langlands packets associated with $\Phi_1, \Phi_2$ respectively.

The conjecture has first been proven in several low rank cases (see [P1], [P2]) and later proven in its full generality by Moeglin and Waldspurger in a series of papers (see [MW]) assuming some natural properties of the generic Langlands packets. It has also recently been revised and generalized by Gan, Gross and Prasad in [GPP] for all the classical groups.

It is natural to ask what happens if the parameters are not generic. Having global applications in mind it is natural to replace the Langlands parameters $\Phi_j$ and the associated Langlands packets...
by the Arthur’s parameters $\Psi_j$ and the Arthur’s packets $A_{\Psi_j}$ associated with them. By Shahidi’s conjecture every tempered Arthur parameter is generic Langlands parameter. Thus we concentrate on the non-tempered Arthur’s parameters.

At the moment there is no satisfactory generalization of the Gross-Prasad conjecture for the non-tempered Arthur packets. However, examining the small rank cases for which the construction of the Arthur packets is known we quickly see that the picture turns out to be quite different. In particular, the sum (1.1) is not always positive. In the cases it is positive, we call the pair of parameters $(\Psi_1, \Psi_2)$ locally admissible. Furthermore, for locally admissible pairs the sum $(1.1)$ can be greater than one.

Let us elaborate on the picture for the case $\dim V = 5$ and $\text{disc}(q) = 1$. In this case $U$ is a non-degenerate subspace of $V$ of dimension 4 and $\text{disc}(q|_U) = d$. The discriminant algebra $K$ of $q|_U$ is defined by

$$K = \left\{ \frac{\sqrt{d}}{F \times F} \ | \ d \notin (F^\times)^2 \right\}.$$  

By abuse of notations we shall write $\text{disc}(q|_U) = K$.

The non-tempered parameters $\Psi_1$ are partitioned into families according to the orbit of the image of $SL_2(C)$, or, by the Jacobson-Morozov theorem, according to a non-trivial unipotent orbit of $Sp_4(C)$. There exist three families of non-tempered Arthur parameters corresponding to the three non-trivial unipotent orbits of $L^1SO(V) \simeq Sp_4(C)$: the regular one, the one generated by a long root and the one generated by a short root of $Sp_4(C)$.

When $\Psi_1$ is associated with the regular orbit, the packet is a singleton and consists of a one-dimensional representation. Thus, $(\Psi_1, \Psi_2)$ is admissible only for non-tempered $\Psi_2$ corresponding to the regular orbit of $L^1(SO(U)) = SO_4(C)$ and the restriction question is trivial.

The parameters $\Psi_1$ associated with the long root orbit are called parameters of Saito-Kurokawa type. Arthur’s packets for all the inner forms were constructed in [G] and the restriction problem has been considered in [GG]. In particular, the condition for a pair $(\Psi_1, \Psi_2)$ to be admissible was determined. Note that $\Psi_2$ must be tempered. The sum $(1.1)$ can equal 1 or 2. However, for a fixed space $V'$ we have

$$\sum_{\Pi, \pi} \dim \text{Hom}_{SO(U')}(\Pi, \pi) \leq 1.$$  

Here the representations $\Pi$ of $SO(V')$ and $\pi$ of $SO(U')$ run over the representations in the packets associated with the parameters $\Psi_1$ and $\Psi_2$ respectively.

Finally, for $\Psi_1$ associated with the short root orbit, i.e., of $\theta_{10}$ type, we solve the restriction problem in this paper. Assuming $\text{disc}(q|_U)$ is a field we determine the parameters $\Psi_2$ such that $(\Psi_1, \Psi_2)$ is locally admissible. Similar to Saito-Kurokawa case, the parameter $\Psi_2$ must be tempered. We compute the value of $(1.1)$: it can be either 2 or 4. Furthermore, even for a fixed form $V'$ the sum (1.2) can be bigger than one. This phenomenon has not occurred before. The local restriction theorem appears in Section 8.

For the case $\text{disc}(q|_U)$ is a split quadratic algebra we obtain the restriction of representations in $A_{\Psi_1}$ to the split group $SO(2, 2)$, but not to its anisotropic inner form $SO(4)$.

**Remark 1.3.** The difficulty that prevents us from computing the restriction to the anisotropic group is of technical nature. Our main tool is the see-saw duality for a pair of similitude quaternionic unique dual pairs. However there is a difficulty to define the theta correspondence for such similitude dual pairs when neither one of the quaternionic Hermitian spaces admits a polarization as a sum of two isotropic subspaces. Hence, the construction will not work whenever the group $SO(U)$ is anisotropic. The same difficulty prevents one to construct representations in the Arthur packet of the anisotropic group $SO(5)$ over the field of real numbers.
Including the anisotropic case would bring a lot of new notations and discussions which we feel do not belong here. We shall treat the remaining case elsewhere.

The local restriction problem has a global counterpart. For any \((V', U')\) as above, define a functional \(P_{V', U'}\) on the space of automorphic forms \(A(SO(V')) \otimes \mathcal{A}_{cusp}(SO(U'))\) by

\[
P_{V', U'}(F, f) = \int_{SO(U')(F) \backslash SO(U')(\mathbb{A})} F(h) f(h) \, dh.
\]

Given an automorphic representation \(\Pi\) of \(SO(V')(\mathbb{A})\) and a cuspidal automorphic representation \(\pi\) of \(SO(U')(\mathbb{A})\), we investigate the non-vanishing of \(P_{V', U'}\) on \(\Pi \boxtimes \pi\). Two parameters \(\Psi_1\) and \(\Psi_2\) such that \(P_{V', U'}\) is non-trivial on some representation \(\Pi \boxtimes \pi\) of \(SO(V') \times SO(U')\) from the global packet \(A_{\Psi_1} \times A_{\Psi_2}\) are called globally admissible.

Let \(\Psi_1\) be a parameter of the type \(\theta_{10}\), and \(E\) be a quadratic field extension naturally associated with it. Let \(\Psi_2\) be a tempered parameter of \(SO(U, q|_{U})\).

Our main global theorem states:

**Theorem 1.4.** Let \(\Pi\) be an automorphic representation of \(SO(V')(\mathbb{A})\) in \(A_{\Psi_1}\), and let \(\pi\) be a cuspidal representation of \(SO(U')(\mathbb{A})\). The following statements are equivalent.

1. The period \(P_{V', U'}\) does not vanish on the \(\Pi \boxtimes \pi\).
2. \(\text{Hom}_{SO(U')(\mathbb{A})}(\Pi, \pi) \neq 0\), and \(E \neq \text{disc}(q|_{U'})\).

This agrees with a version of the refined Ichino-Ikeda conjecture. For tempered cuspidal representations \(\Pi\) of \(SO(V')(\mathbb{A})\) and \(\pi\) of \(SO(U')(\mathbb{A})\), Ichino and Ikeda in [II] have conjectured that the period \(P_{V', U'}\) does not vanish on \(\Pi \boxtimes \pi\) if and only if

\[
L(\Pi \boxtimes \pi, 1/2) \neq 0, \quad \frac{L(\Pi \boxtimes \pi, 1)}{L(\Pi \boxtimes \pi, Ad, 1)} \neq 0.
\]

We show that the second statement of Theorem 1.4 is equivalent to (1.5).

**Remark 1.6.** For the same reasons as before we investigate the non-vanishing of \(P_{V', U'}\) assuming

- \((V', q')\) is not anisotropic over any real place.
- \((U', q'|_{U'})\) is not anisotropic for any local place.

The paper is organized as follows: after explaining in Section 2 some generalities about (skew)-Hermitian spaces over division algebras, we introduce in the Sections 3 and 4 the group of similitude automorphisms of these spaces.

In Section 5 we recall the notion of Howe duality for similitude unitary quaternionic groups. This is our main tool for constructing the \(A\)-packets. In Section 6 we determine the theta correspondence as explicitly as possible. The local \(A\)-packets are defined in the Section 7. The restriction problem is solved in the Section 8 using the see-saw duality. The rest of the paper addresses global questions. In section 9 we define the global Arthur packet and compute the multiplicity predicted by Arthur’s multiplicity formula. The automorphic realization of the global Arthur packets is obtained in Sections 10 and 11 using the global theta correspondence. The Arthur’s multiplicity formula is established in the Section 12. Section 13 is devoted to the Rankin-Selberg integral representation of an \(L\)-function of degree 5 of a representation of \(SO(V')\). When the group \(SO(V')\) is split, this integral representation has been constructed by Piatetskii-Shapiro and Rallis. This \(L\)-function is used in Section 14 to show that the cuspidal representations that we have constructed constitute a full nearly equivalence class, so that our construction is exhaustive. The global restriction problem is solved using the global see-saw duality in the Sections 15–17. The main global theorem is (1.4). Finally, in Section 18 we show the compatibility of the obtained results with the Ichino-Ikeda conjecture.
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2. **Hermitian and skew-Hermitian spaces**

Let $F$ be a local non-archimedean field. Let $D$ be a (possibly split) quaternion algebra over $F$. Denote by $\sigma$ the main involution on $D$ and by $\Nm_{D/F}$ the reduced norm. To any right (left) $D$-module $M$ we associate a left (right) $D$ module $\overline{M}$ by

\[
\overline{M} = \{ \bar{m} : m \in M \}, \quad \overline{m_1 + m_2} = \overline{m_1} + \overline{m_2}, \quad d \cdot \overline{m} = \overline{m \cdot \sigma(d)} \quad (\overline{m} \cdot d = \sigma(d) \cdot \overline{m}).
\]

We define a right skew-Hermitian space $(V, s)$ to be a right free $D$-module together with a map $s : V \times V \to D$ such that

\[
s(d_1 v_1, v_2 d_2) = d_1 s(v_1, v_2) d_2, \quad s(v_1, v_1) = -\sigma(s(v_1, v_2)).
\]

All the non-degenerate skew-Hermitian free modules of rank $n$ over $D$ are classified up to isometry by the discriminant in $F^\times / (F^\times)^2$, or equivalently by quadratic algebras over $F$.

We also define a left Hermitian space $(W, h)$ to be a left free $D$-module $W$ together with the map $h : W \times \overline{W} \to D$ such that

\[
h(d_1 w_1, \overline{w_2} d_2) = d_1 h(w_1, \overline{w_2}) d_2, \quad h(w_1, \overline{w_1}) = h(w_1, \overline{w_2}).
\]

For any even $n$ there exists a unique, up to isometry, non-degenerate left Hermitian space of rank $n$ over $D$.

2.1. **Morita equivalence.** Let $D$ be a split algebra. Equivalently, $D = \End_F(M)$ for a two-dimensional space $M$ over $F$, viewed as a right $D$ module. The space $M \otimes_D \overline{M}$ is one dimensional over $F$. Fix an isomorphism $\varphi_M : M \otimes_D \overline{M} \to F$. The choice of $\varphi_M$ fixes an isomorphism $\overline{M} \simeq M^\ast$. In particular, there is an isomorphism $\overline{M} \otimes_F M \simeq \End_F(M) = D$.

For any left Hermitian space $(W_D, h_D)$ over $D$ there corresponds a symplectic space $(W, h)$ over $F$ defined by

\[
W = M \otimes_D W_D, \quad h(m_1 \otimes w_1, m_2 \otimes w_2) = \varphi_M(m_1 h_D(w_1, \overline{w_2}) \otimes \overline{m_2}).
\]

Similarly, for any right skew-Hermitian space $(V_D, s_D)$ there corresponds a quadratic space $(V, s)$ over $F$ defined by

\[
V = V_D \otimes_D \overline{M}, \quad s(v_1 \otimes \overline{m_1}, v_2 \otimes \overline{m_2}) = \varphi_M(m_1 s_D(v_1, v_2) \otimes \overline{m_2}).
\]

Obviously,

\[
\dim_F V = 2 \dim_D V_D, \quad \dim_F W = 2 \dim_D W_D, \quad \disc(V) = \disc(V_D).
\]

Note that the isomorphism $\varphi_M$ is not canonical. For an element $a \in F^\times \setminus \Nm_{E/F}(E^\times)$ the isomorphism $\varphi_M$ and $a \varphi_M$ give rise to two quadratic spaces $(V, s^\pm)$ having the same discriminant but different normalized Hasse invariants. Hasse invariant $h(V, q)$ for a quadratic space is normalized so that it is constant in any Witt tower.

Moreover, there are isomorphism of $D$-modules

\[
V \otimes_F M = V_D \otimes_D \overline{M} \otimes_F M \simeq V_D, \quad \overline{M} \otimes_F W = \overline{M} \otimes_F M \otimes_D W_D \simeq W_D.
\]
2.2. Main players. Let us fix the Hermitian and skew-Hermitian spaces that will be considered in the paper. First we fix notations for the algebras.

- Let $D$ be a quaternion algebra over $F$ with the main involution $\sigma$. Define $h(D) = 1$ if $D$ split and $h(D) = -1$ otherwise.
- Let $E$ be a quadratic algebra over $F$ contained in $D$. The involution $\sigma$ restricted to $E$ defines a non-trivial Galois action.
- Let $K$ be a quadratic algebra over $F$.
- Denote $L = E \otimes_F K$. It is a quadratic algebra over $K$.
- Denote $D_K = D \otimes_F K$. It is a quaternion algebra over $K$. The main involution is still denoted by $\sigma$ and $\text{Nm}_{D_K/K}$ denotes the reduced norm.

1. Let $(V_D, s_D)$ be the one-dimensional right skew-Hermitian space over $D$ of discriminant $E$.
2. Let $(V_{D_K} = V_D \otimes_F K, s_{D_K})$ be the one-dimensional right skew-Hermitian space over $D_K$, where
   $$ s_{D_K}(v_1 \otimes k_1, v_2 \otimes k_2) = s_D(v_1, v_2)k_1k_2. $$
   It has discriminant $L = E \otimes_F K$.
3. Let $(W_{D_K}, h_{D_K})$ be the one-dimensional left Hermitian space over $D_K$.
4. Let $(W_D = R_{K/F}W_{D_K}, h_D)$ be the two-dimensional right Hermitian space over $D$ obtained from $W_{D_K}$ by a restriction of scalars. The form $h_D$ is defined by
   $$ h_D(w_1, \overline{w_2}) = \text{tr}_{D_K/D}h_{D_K}(w_1, \overline{w_2}). $$

Assume that the algebra $D_K$ splits.

1. Denote by $(V_K, s_K^2)$ the two-dimensional quadratic spaces over $K$ of discriminant $L$, Morita equivalent to $(V_{D_K}, s_{D_K})$.
2. Denote by $(W_K, h_K)$ the two-dimensional symplectic space over $K$, Morita equivalent to $(W_{D_K}, h_{D_K})$.

Assume that the algebra $D$ splits.

1. Denote by $(V_F, s_F^2)$ the two-dimensional quadratic spaces over $F$ of discriminant $E$, Morita equivalent to $(V_D, s_D)$.
2. Denote by $(W_F, h_F)$ the four-dimensional symplectic space over $F$, Morita equivalent to $(W_D, h_D)$.

Note the obvious relations:

$$ V_K = V_F \otimes_F K, \quad W_F = R_{K/F}W_K. $$

2.3. Symplectic forms on tensor products. The space $V_D \otimes_D W_D$ admits a symplectic form defined by

$$ \langle v \otimes w, v' \otimes w' \rangle = \text{tr}_{D/F} s_D(v, v')\sigma(h_D(w, \overline{w'})). $$

Similarly, the space $V_F \otimes_F W_F$ admits a symplectic form defined by

$$ \langle v \otimes w, v' \otimes w' \rangle = s(v, v')h(w, w'). $$

Finally, the space $V_K \otimes_K W_K$ is a symplectic space over $K$. Composing the symplectic form with $\text{tr}_{K/F}$ we obtain the symplectic form over $F$.

**Lemma 2.1.**

1. Suppose that $D$ splits. There is a natural isomorphism of symplectic $F$ spaces
   $$ V_F \otimes_F W_F \simeq V_D \otimes_D W_D. $$
2. Suppose that $D_K$ splits. There is a natural isomorphisms of symplectic $F$ spaces
   $$ V_K \otimes_K W_K \simeq V_D \otimes_D W_D. $$
Proof. Since $D$ splits, $D = \text{End}_F(M)$. To prove (1) note that
\[ V_F \otimes W_F = V_D \otimes_D M \otimes_F M \otimes_D W_D \simeq V_D \otimes_D W_D. \]
To prove (2) note that
\[ V_K \otimes_K W_K \simeq V_{D_K} \otimes_{D_K} W_{D_K} \]
and so
\[ V_K \otimes_K W_K \simeq V_D \otimes_D D_K \otimes D_{D_K} W_D \simeq V_D \otimes_D W_D. \]
Clearly, all the natural isomorphisms above are isomorphisms of symplectic spaces.

2.4. Compatibility of polarizations. If $D_K$ splits there exists a two-dimensional space $M_K$ such that $D_K = \text{End}_K(M_K)$. Given a polarization of $W_K$ as a sum of two isotropic spaces
\[ W_K = X_K \oplus Y_K, \]
the compatible polarization of $W_D$ is defined by
\[ W_D = X_D \oplus Y_D, \quad X_D = \overline{M_K} \otimes_K X_K, \quad Y_D = \overline{M_K} \otimes_K Y_K. \]
Similarly if $D$ splits, the polarizations
\[ W_F = X_F \oplus Y_F, \quad W_K = X_K \oplus Y_K \]
are called compatible if
\[ X_F = R_{K/F} X_K, \quad Y_F = R_{K/F} Y_K. \]

Lemma 2.2. (1) For compatible polarizations there is a natural isomorphism of $F$-spaces
\[ V_F \otimes_F X_F \simeq V_K \otimes_K X_K. \]
(2) Suppose that $D_K$ splits. For compatible polarizations there is a natural isomorphism of $F$-spaces
\[ V_D \otimes_D X_D \simeq V_K \otimes_K X_K. \]

Proof. Part one is trivial. To prove Part (2) simply note that
\[ V_D \otimes_D X_D \simeq V_D \otimes_D D_K \otimes D_{D_K} M_K \otimes_K X_K \simeq V_{D_K} \otimes_{D_K} M_K \otimes_K X_K \simeq V_K \otimes_K X_K. \]
The forms $h_D, h, h_K$ define natural isomorphisms
\[ X_D \simeq Y_D, \quad X_F \simeq Y_F, \quad X_K \simeq Y_K. \]

3. Hermitian Unitary Groups and their representations

Many groups will be used in the course of the paper. Let us introduce some preliminary notations:
- For any algebraic group $M$ denote by $M^c$ its connected component.
- If $M$ is a subgroup of a group of similitude of a Hermitian/skew-Hermitian space, denote by $M^1$ the subgroup of elements of $M$ whose similitude is $1$.
- For a group $M$ over a field $F$ and an extension $E$ over $F$ we denote by $R_{E/F} M$ the $F$-group obtained from $M$ by restriction of scalars.
- Let $M$ be an algebraic group over a local field $F$. A representation $\pi$ of $M(F)$ (or just of $M$ if there is no confusion) is a smooth representation if $F$ is non-archimedean and smooth Frechet representation of moderate growth if $F$ is archimedean.
- Let $N(F)$ be a normal subgroup of $M(F)$ and let $\pi$ be a representation of $N(F)$. For $m \in M(F)$. Denote by $\pi^m$ the conjugate representation, i.e., $\pi^m(n) = \pi(mnm^{-1})$. 

3.1. The group $G_D$. Let $M_2$ be the $F$-group of $2 \times 2$ matrices. The $F$-group $G_D$ is defined by

$$G_D = \{ g \in M_2(D) : \exists \lambda(g) \in \mathbb{G}_m : gJ_0(g)^t = \lambda(g)^{-1}J \},$$

where $J = (0 \ 1 \ 1 \ 0) \in M_2$. The group $G_D$ acts on the algebraic Hermitian vector space $(W_D, h_D)$ by $g.w = wg^{-1}$ and it is isomorphic to the group of the similitude automorphisms with the similitude character $\lambda$. That is,

$$G_D \simeq \{ g \in \text{Aut}(W_D) : \forall w_1, w_2 \in W_D \quad h_D(g.w_1, g.w_2) = \lambda(g)h_D(w_1, w_2) \}.$$ 

We denote by $Z_D$ the center of $G_D$. Obviously $Z_D \simeq \mathbb{G}_m$.

3.1.1. The parabolic subgroup $P_D$. We fix a polarization $W_D = X_D \oplus Y_D$ and define $P_D$ to be the subgroup of $G_D$ preserving the subspace $Y_D$. It is a maximal parabolic subgroup. If $D$ does not split then $P_D$ is unique parabolic subgroup of $G_D$.

One has a Levi decomposition $P_D = M_D \cdot U_D$, where $M_D \simeq GL(X_D) \times \mathbb{G}_m$. We realize $M_D$ in $G_D$ as

$$M_D = \left\{ m(a, t) = \begin{pmatrix} ta & 0 \\ 0 & (\bar{a}^\ast)^{-1} \end{pmatrix}, \quad a \in GL(X_D), t \in \mathbb{G}_m \right\},$$

where $\bar{a}^\ast \in GL(Y_D)$ is characterized by

$$h_D(x, \overline{\bar{a}^\ast(y)}) = h_D(a(x), y).$$

In particular $\lambda(m(a, t)) = t^{-1}$.

The unipotent radical $U_D$ can be identified with

$$\{ S \in \text{Hom}(X_D, Y_D), S = -S^\ast \} \simeq \{ \text{skew-Hermitian forms on } X_D \},$$

where $S^\ast \in \text{Hom}(X_D, Y_D) = \text{Hom}(X_D, X_D^\ast)$ is characterized by

$$h_D(y_1, S(y_2)) = h_D(S^\ast(y_1), y_2).$$

For any such $S$, the element $u(S) \in U_D$ acts trivially on $Y_D$ and for $x \in X_D$ one has $u(S)(x) = x + S(x)$.

3.1.2. The action of $M_D$ on $U_D$. The group $M_D$ acts on $U_D$ by conjugation

$$m(a, t).u(S) = u(\bar{a}^\ast Sa).$$

Note that two Hermitian forms lie in the same orbit if and only if they have the same discriminants in $F^\times/(F^\times)^2$. Hence, these orbits are classified by the quadratic algebras inside $D$.

3.1.3. The characters of $U_D$. Fix a non-trivial additive character $\psi$ of $F$. The group of unitary characters of $U_D$ can be identified with $\text{Hom}(U_D, \mathbb{G}_a)$ via composition with $\psi$. The group $M_D$ acts on $\text{Hom}(U_D, \mathbb{G}_a)$ by the dual action to the one discussed above.

**Proposition 3.1.**

1. The set $\text{Hom}(U_D, \mathbb{G}_a)$ is naturally identified with the set of skew-Hermitian forms on $Y_D$.

2. $M_D$-orbits on $\text{Hom}(U_D, \mathbb{G}_a)$ are parameterized by quadratic algebras over $F$.

3. For any skew-Hermitian form $T$ on $Y_D$ one has $\text{Stab}_{M_D}(\Psi_T) \simeq GU(Y_D, T)$.

**Proof.** (1). Indeed, given two maps $T \in \text{Hom}(Y_D, X_D)$ and $S \in \text{Hom}(X_D, Y_D)$ such that $T = -T^\ast, S = -S^\ast$ there is a non-degenerate pairing

$$(T, S) = \text{tr}_{D/F} TS.$$

In particular, for $T \neq 0$ we define a non-trivial character

$$\Psi_T(u(S)) = \psi(\text{tr}_{D/F} TS).$$

(2) follows from (1).
3.1.4. Induced representations. Any irreducible representation of $M_D$ has the form $\tau \boxtimes \chi$ where $\tau$ is an irreducible representation of $GL(X_D)$ and $\chi$ is a character of $\mathbb{G}_m$. We denote the induced representation associated with $\tau \boxtimes \chi$ by

$$I_{P_D}(\tau \boxtimes \chi) = \text{Ind}_{P_D}^{G_F} \tau \boxtimes \chi.$$ 

If $\tau \boxtimes \chi$ is a product of a tempered representation and a positive character the representation $I_{P_D}(\tau \boxtimes \chi)$ has a unique irreducible quotient denoted by $J_{P_D}(\tau \boxtimes \chi)$. Note that $I_{P_D}(\tau \boxtimes \chi) \simeq (\chi \circ \lambda^{-1}) \otimes I_{P_D}(\tau \boxtimes 1)$.

3.2. The group $G_F$. The group $G_F = GSp_4$, acting on $W_F$ by $g.w = w g^{-1}$, is the group of similitude automorphisms of the symplectic space $(W_F, h_F)$. The group $G_D$ is an inner form of $G_F$ and is isomorphic to it if and only if $D$ splits. We realize $G_F$ as a group of matrices

$$\{g \in GL(W_F) : g J_4 g^t = \lambda(g)^{-1} J_4\}$$

$$J_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

The group $G_D$ is an inner form of $G_F$ and is isomorphic to it if and only if $D$ splits.

3.2.1. Parabolic subgroups of $G_F$. Fix a polarization $W_F = X_F \oplus Y_F$. The maximal Siegel parabolic $P_F$ consists of the elements stabilizing the space $Y_F$. One has a Levi decomposition $P_F = M_F \cdot U_F$ with $M_F \simeq GL_2 \times \mathbb{G}_m$.

The other maximal parabolic subgroup of $G_F$ is the Heisenberg parabolic subgroup. It is denoted by $Q_F = L_F \cdot V_F$ or just by $Q$. The Levi subgroup $L_F \simeq GL_2 \times \mathbb{G}_m$ is embedded in $G_F$ as

$$L_F = \left\{ l(t,g) = \begin{pmatrix} t & & \\ g & & \\ t^{-1} \text{det}(g) & & \end{pmatrix} \mid g \in GL_2, t \in \mathbb{G}_m \right\}.$$ 

We also fix a Borel subgroup $B_F = Q_F \cap P_F$ with a Levi decomposition $B_F = T_F N_F$. The split torus $T_F$ is realized inside $G_F$ as

$$T_F = \{t(a,b,s) = \text{diag}(a, b, b^{-1} s, a^{-1} s)\}.$$ 

The similitude factor is given by $\lambda(t(a,b,s)) = s^{-1}$.

3.2.2. Induced representations. Any irreducible representation of $L_F$ has the form $\chi \boxtimes \tau$ where $\tau$ is an irreducible representation of $GL_2$ and $\chi$ is a character of $\mathbb{G}_m$. If $\chi \boxtimes \tau$ is a product of a tempered representation and a positive character then the representation $\text{Ind}_{Q_F}^{G_F} (\chi \boxtimes \tau)$ has a unique irreducible quotient denoted by $J_Q(\chi \boxtimes \tau)$.

In the course of determining the theta correspondence in Section 6 we shall need the decomposition of the representation $I_Q(\mu) := \text{Ind}_{Q_F}^{G_F} \mu^{-1} \boxtimes \mu \circ \text{det}$.

**Lemma 3.2.** ([K] Theorem 4.2)

1. For $\eta \neq 1, | \cdot |^{\mp 2}$ the representation $I_{Q_D}(\mu)$ is irreducible.
2. $I_Q(1) = J_Q(1 | \cdot | \boxtimes | \text{det}^{-1/2} \text{Ind}_{GL_2}^{G_F} 1) \oplus J_F(| \text{det}^{1/2} \text{St} | \cdot |^{-1/2}).$
3. The length of $I_Q(1 | \cdot |^{-2})$ is two. It has $| \cdot | \circ \lambda$ as a unique quotient.

Here $\text{St}$ denotes the Steinberg representation of $GL_2$. 

For a character $\chi(t(a,b,s)) = \chi_1(a)\chi_2(b)\chi(s)$ denote the associated induced representation by $I_B(\chi_1, \chi_2; \chi)$.

### 3.3. The groups $G_{D_K}, G_K$ and $G_{D_K}^0, G_K^0$.

The group $G_{D_K}$ over $K$ is the group of similitude automorphisms of the one-dimensional Hermitian space $W_{D_K}$ acting by $g.w = wg^{-1}$ with the similitude character $\lambda_K$, i.e.,

$$G_{D_K} = \{g \in \text{Aut}_{D_K}(W_{D_K}) : \forall w_1, w_2 \in W_{D_K} \quad h_{D_K}(g.w_1, g.w_2) = \lambda_{D_K}(g) h_{D_K}(w_1, w_2)\}.$$

Since the space $W_{D_K}$ is one dimensional over $D_K$ one has

$$G_{D_K}(K) \simeq D_K^\times, \quad \lambda_{D_K}(g) = (\text{Nm}_{D_K/K}(g))^{-1}.$$

The group $G_K$ over $K$ is the group of similitude automorphisms of the two-dimensional Hermitian space $W_K$ acting by $g.w = wg^{-1}$ with the similitude character $\lambda_K$, i.e.,

$$G_K = \{g \in \text{Aut}_K(W_K) : \forall w_1, w_2 \in W_{D_K} \quad h_K(g.w_1, g.w_2) = \lambda_K(g) h_K(w_1, w_2)\}.$$

Since $h_K$ is a symplectic form it follows that

$$G_K \simeq GL_2, \quad \lambda_K = (\text{det})^{-1}.$$

Fixing a polarization $W_K = X_K \oplus Y_K$ fixes a Borel subgroup $B_K$ that preserves $Y_K$. There is a Levi decomposition $B_K = T_K \cdot N_K$.

The group $G_{D_K}$ is an inner form of $G_K$ and is isomorphic to $G_K$ if and only if the algebra $D_K$ splits over $K$.

The similitude factor $\lambda_K$ induces the map of $F$-groups:

$$R_{K/F} \lambda_{D_K} : R_{K/F} G_{D_K} \to R_{K/F} G_m, \quad R_{K/F} \lambda_K : R_{K/F} G_K \to R_{K/F} G_m$$

and there is a canonical embedding $G_m \hookrightarrow R_{K/F} G_m$.

We define the following algebraic groups over $F$.

$$G_{D_K}^0 = \{g \in R_{K/F} G_{D_K} : R_{K/F} \lambda_{D_K}(g) \in G_m\}.$$

$$G_K^0 = \{g \in R_{K/F} G_K : R_{K/F} \lambda_K(g) \in G_m\}.$$

In particular

$$G_{D_K}^0(F) = \{g \in G_{D_K}(F) : \lambda_{D_K}(g) \in F^\times\}.$$

There are natural inclusions of $F$-groups

$$G_{D_K}^0 \hookrightarrow G_{D_K}, \quad G_K^0 \hookrightarrow G_F.$$

Note that under this embedding $B_K^0 \hookrightarrow P_F$.

The group $G_m$ acts on $Res_{K/F} V_{D_K}$ and $Res_{K/F} V_K$ by scalar multiplication. Denote its image in $G_K^0$ and $G_{D_K}^0$ by $Z_K^0$ and $Z_{D_K}^0$ respectively. The groups $Z_K^0$ and $Z_{D_K}^0$ are contained in the center of $G_K^0$ and $G_{D_K}^0$ respectively.

#### 3.3.1. Dihedral representations.

Let $\eta_L$ be a character of $L^\times$. Denote by $\pi(\eta_L)$ the dihedral representation of $G_K$ associated to $\eta_L$. If $L$ is a field and $\eta_L \neq \eta_L^\sigma$ then $\pi(\eta_L)$ is supercuspidal.
3.4. The unitary quaternionic groups as special orthogonal groups. The groups \( G_D, G^0_{D_K} \) play main role in this paper because of the following accidental isomorphisms

**Proposition 3.3.** Let \((V, q)\) be a 5-dimensional quadratic spaces over \(F\) with discriminant 1. Let \((U, q_K)\) be a non-degenerate quadratic subspace of \(V\) of codimension 1 and discriminant algebra \(K\).

1. 
   \[ G_D / Z_D \simeq SO(V, q), \quad h(V, q) = h(D). \]

2. 
   \[ G^0_{D_K} / Z^0_{D_K} \simeq SO(U, q_K), \quad h(U, q_K) = h(D_K). \]

In particular the groups \( G_D / Z_D \) and \( G^0_{D_K} / Z^0_{D_K} \) over a local non-archimedean field vary over all pure inner forms of \(SO(V)\) and \(SO(U)\) as \(D\) varies over the set of quaternion algebras. Note that \(SO(U, q_K)\) is anisotropic if and only if the algebra \(D_K\) splits.

**Proof.** We shall construct the isomorphisms explicitly. One has \(D = \text{Span}_F\{i, j, k\}\), where

\[ i^2 = \alpha, j^2 = \beta, k = ij = -ji, \quad \alpha, \beta \in F^\times. \]

It is known that \(h(D) = (\alpha, \beta)_F\), where \((\cdot, \cdot)_F\) denotes the Hilbert symbol.

Consider the space

\[ M_2(D) \supset X = \{ x \in M_2(D) : Jx^tJ^{-1} = \sigma(x) \}. \]

The space \(X\) has dimension 6 over \(F\) and admits the symmetric form \(q(x, y) = \text{tr}_{D/F}(\text{tr}(xy))\). The group \(G_D\) acts on \(X\) by conjugation. In particular \(Z_D\) acts trivially. This action preserves the form \(q\) and fixes the identity matrix. Denote by \(V\) the orthogonal complement to the identity matrix. More precisely,

\[ V = \{ \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right) : b, c \in F, a \in D, \text{tr}_{D/F}(a) = 0 \} \]

We obtain an isomorphism between \(G_D / Z_D\) and \(SO(V, q)\). Considering the orthogonal basis of \(V\)

\[ \left\{ \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \left( \begin{array}{cc} j & 0 \\ 0 & -j \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \right\} \]

it is easy to see that \(\text{disc}(V, q) = 1\) and the normalized Hasse invariant \(h(V, q)\) equals \(h(D)\). This proves (1).

For an quadratic algebra \(K = F[\sqrt{d}]\) consider a vector \(v_K = (\frac{1}{d} 0) \in V\). In particular \(q(v_K, v_K) = d\). The stabilizer of \(v_K\) in \(G_D\) isomorphic to \(G^0_{D_K}\), i.e.,

\[ \{ g \in G_D : gv_Kg^{-1} = v_K \} \simeq G^0_{D_K}. \]

Denote by \(U\) the orthogonal complement to the one-dimensional space \(\langle v_K \rangle\) and denote by \(q_K\) the restriction of the form \(q\) to \(U\). Then, \(G^0_{D_K}\) acts irreducibly on \(U\) and the subgroup \(Z^0_{D_K}\) acts trivially. This defines an isomorphism

\[ G^0_{D_K} / Z^0_{D_K} \simeq SO(U, q_K). \]

Considering the orthogonal basis of \(U\)

\[ \left\{ \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right), \left( \begin{array}{cc} j & 0 \\ 0 & -j \end{array} \right), \left( \begin{array}{cc} k & 0 \\ 0 & -k \end{array} \right), \left( \begin{array}{cc} 0 & \text{disc}(q_K) \frac{1}{d} \\ \frac{1}{d} & 0 \end{array} \right) \right\} \]

it is easy to see that \(\text{disc}(U, q_K) = K\) and \(h(U, q_K) = h(D_K)\). This proves (2). \(\square\)

Thus, instead of considering restrictions of representations of \(SO(V')\) to \(SO(U')\) we consider the equivalent question of restriction of representations of \(G_D\) with trivial central character to \(G^0_{D_K} / Z^0_{D_K}\).
Remark 3.4. When $F = \mathbb{R}$ the non-degenerate Hermitian forms over $D$ are classified by their signature and we have

$$G_D/Z_D \simeq \begin{cases} 
SO(3,2) & h(D) = 1, \quad \text{sig}(W_D) = (1,1) \\
SO(4,1) & h(D) = -1, \quad \text{sig}(W_D) = (1,1) \\
SO(5,0) & h(D) = -1, \quad \text{sig}(W_D) = (2,0) 
\end{cases}.$$ 

Note that the space $W_D$ with the signature $(2,0)$ is not hyperbolic.

4. The skew-Hermitian unitary groups

We shall now define and discuss some auxiliary groups that will be used to construct the packets of representations for the groups $G_D$ and $G^0_{Dk}$ and to determine the restrictions.

4.1. The group $H_D$, $H_F$ and their representations. Define $H_D$ to be the group of similitude automorphisms of $(V_D,s_D)$. It is a disconnected algebraic group. Denote its connected component by $H^c_D$. Its properties are described below.

Proposition 4.1. (1) One has $H^c_D = R_{E/F}\mathbb{G}_m$ and the group $H_D$ fits into the exact sequence

$$1 \to H^c_D \to H_D \to \mu_2 \to 1.$$ 

This sequence splits if and only if $D$ splits.

(2) The sequence of $F$-points is also exact

$$1 \to E^\times \to H_D(F) \to \mu_2(F) \to 1.$$ 

In particular, both connected components of $H_D$ have points over $F$.

(3) \[ \lambda(H_D(F)) = \begin{cases} 
\text{Nm}(E^\times) & D \text{ is split} \\
F^\times & D \text{ is not split} 
\end{cases} \]

where $\lambda$ is the similitude character.

(4) \[ H^1_D(F) \simeq \begin{cases} 
O(V_F)(F) & D \text{ is split} \\
E^1 & D \text{ is not split} 
\end{cases} \]

In particular, the non-identity connected component of the disconnected algebraic group $H^1_D$ does not have $F$-points for the non-split $D$.

It is easy to describe the irreducible representations of $H_D(F)$. By abuse of notations we shall write $H_D$ for $H_D(F)$ when there is no ambiguity.

Proposition 4.2. Let $\eta$ be a character of $H^c_D \simeq E^\times$.

(1) Any irreducible representation of $H_D$ is a direct summand of some two-dimensional representation $\text{Ind}_{H^c_D}^{H^1_D} \eta$.

(2) If $\eta \neq \eta^\sigma$ then $\tau^+_\eta = \text{Ind}_{H^c_D}^{H^1_D} \eta$ is irreducible.

(3) If $\eta = \eta^\sigma$ then $\text{Ind}_{H^c_D}^{H^1_D} \eta$ is a sum of two one-dimensional representations. We denote them by $\tau^\pm_\eta$.

(4) For any two characters $\eta$ and $\eta'$

$$\dim \text{Hom}_{H^c_D}(\tau^\pm_\eta, \eta') = \begin{cases} 
1 & \eta' \in \{\eta, \eta^\sigma\} \\
0 & \text{otherwise} 
\end{cases}.$$
4.1.1. **Labeling.** We wish to distinguish between the two representations $\tau^+_\eta$ when $\eta = \eta^\sigma$. If $D$ splits we choose labeling such that

$$\tau^+_\eta|_{H^0_\eta(F)} = 1, \quad \tau^-_\eta|_{H^0_\eta(F)} = \text{sgn}.$$

If $D$ does not split then labeling is equivalent to a choice of $\eta_F$ such that $\eta = \eta_F \circ \text{Nm}$. There exist two such characters: $\eta^+_F$ where $\eta^+_F = \chi_{E/F} \eta^+_E$. For any choice of $\eta^+_F$ define

$$\tau^+_\eta = \eta^+_F \circ \lambda.$$

Clearly, replacing $\eta^+_F$ by $\eta^-_F$ changes the labeling.

4.2. **The group $H_F$ and its representations.** Define $H_F$ to be a group of similitude automorphisms of $(V_F, s_F)$. One has $\lambda(H_F) = \text{Nm}_{E/F}(E^\times)$.

If $D$ splits then $H_F \simeq H_D$. In particular any irreducible representation of $H_F$ is a constituent of $\text{Ind}^{H_F}_{H_E} \eta$.

If $E$ is a split quadratic algebra over $F$ and $\sigma \in \text{Aut}_F(E)$ is a non trivial automorphism we fix an isomorphism

$$H^c_F \simeq \mathbb{G}_m \times \mathbb{G}_m, \quad \sigma(x, y) = (x^{-1} y, y).$$

Any character of $\sigma$ of $H^c_F$ can be written in the form $\mu \boxtimes \eta_F$, where $\mu$ and $\eta_F$ are characters of $\mathbb{G}_m$.

**Lemma 4.3.**

1. $(\mu \boxtimes \eta_F)^\sigma = \mu^{-1} \boxtimes \mu \eta_F$. In particular, $\eta$ is Galois invariant if and only if $\mu = 1$.
2. $\tau^{+}_{\mu \boxtimes \eta_F} = (\eta_F \circ \lambda) \otimes \tau^{+}_{\mu \boxtimes 1}$. We shall write $\tau^{+}_{\mu}$ for $\tau^{+}_{\mu \boxtimes 1}$.
3. $\tau^{\pm}_{\mu} = (\mu \circ \lambda) \otimes \tau^{\pm}_{\mu^{-1}}$.

4.3. **The groups $H_{D_K}, H_K$ and $H^0_{D_K}, H^0_K$.** Define $H_{D_K}$ and $H_K$ to be the group of similitude automorphisms of $(V_{D_K}, s_{D_K})$ and of $(V_K, s_K)$ respectively.

The similitude characters are denoted by $\lambda_{D_K}$ and $\lambda_K$. Also denote

$$H^0_{D_K}(F) = \{ h \in H_{D_K}(K) : \lambda_{D_K}(h) \in F^\times \}.$$

$$H^0_K(F) = \{ h \in H_K(K) : \lambda_K(h) \in F^\times \}.$$

Note the following properties:

1. There are natural embeddings

$$H_D \hookrightarrow H^0_{D_K}, \quad H_F \hookrightarrow H^0_K.$$

2. $\lambda_K(H_K(F)) = \text{Nm}_{L/K} L^\times, \quad \lambda_K(H^0_K(F)) = \text{Nm}_{L/K} L^\times \cap F^\times = F^\times$.

3. If $D_K$ splits over $K$ then

$$H_K \simeq H_{D_K}, \quad H^0_K \simeq H^0_{D_K}.$$

5. ** Theta correspondences for similitude dual pairs**

5.1. **Howe duality.** Let us recall the notion of the abstract Howe duality after Roberts, [R]. Let $A$ and $B$ be reductive groups over a local field of characteristic zero and let $\rho$ be a smooth representation of $A \times B$. For any smooth irreducible representation $\tau$ of $A$ (resp. $B$) the maximal $\tau$ isotypic component of $\rho$ has the form $\tau \boxtimes \Theta(\tau)$ for some smooth representation $\Theta(\tau)$ of $B$ (resp. of $A$), possibly zero.

We say that Howe duality holds for the triplet $(\rho, A, B)$ if for any irreducible representation $\tau$ of $A$ (resp. $B$) such that $\Theta(\tau) \neq 0$ the maximal semisimple quotient $\theta(\tau)$ of $\Theta(\tau)$ is irreducible. The representations $\Theta(\tau)$ and $\theta(\tau)$ are called big theta lift and small theta lift of $\tau$ respectively. Theta correspondence is another name for Howe duality.
Let \((H^1, G^1)\) be a classical dual pair of isometry groups acting on the spaces \(V\) and \(W\) respectively and let \((H, G)\) be the corresponding similitude groups with the similitude character \(\lambda\). Define
\[
H^+ = \{ h \in H : \lambda(h) \in \lambda(G) \}, \quad G^+ = \{ g \in G : \lambda(g) \in \lambda(H) \}
\]
and
\[
R = \{(h, g) \in H \times G : \lambda(h) = \lambda(g)\} \subseteq H^+ \times G^+.
\]

Let \(\omega_\psi\) be a Weil representation of \(Sp(V \otimes W)\) associated to a non-trivial additive character \(\psi\). Any splitting \(s : H^1 \times G^1 \to \overline{Sp}(V \otimes W)\) defines the representation \(\omega_{\psi,s} = \omega_\psi \circ s\) of \(H^1 \times G^1\).

The Weil representations \(\omega_{\psi,s}\) has been suitably extended from \(H^1 \times G^1\) to \(R\) for symplectic-orthogonal pairs in \([R]\) and for the quaternionic unitary dual pairs where one of the spaces is hyperbolic even-dimensional in \([GT]\).

**Proposition 5.1.** ([R, GT])

1. Howe duality for the triplet \((\omega_{\psi,s}, G^1, H^1)\) implies Howe duality for the triplet \((\Omega^+ = \text{ind}_{R}^{G^+ \times H^+} \omega_{\psi,s}, G^+, H^+)\).

2. If \(\tau\) is an irreducible representation of \(H^+\) such that \(\tau|_{H^1} = \oplus k\tau_i\), then \(\Theta(\tau)|_{G^1} = \oplus k \cdot \Theta(\tau_i)\) and if \(\Theta(\tau_i) = \Theta(\tau_i)\) for some \(i\) then \(\Theta(\tau) = \Theta(\tau)\).

**Remark 5.2.** Proving his results, Roberts has made the assumption that the restriction of irreducible representations from \(G^+\) to \(G^1\) is multiplicity free. This assumption holds for symplectic-orthogonal dual pairs ([AP]), but does not necessarily hold for quaternion dual pairs. The proof in \([GT]\), Prop. 3.3 that works for all classical dual pairs does not require this assumption.

Although \([GT]\) deals with smooth representations over non-archimedean fields, the proof of Prop. 3.3. applies in the case \(F\) is archimedean as well. Indeed, it uses only Frobenius reciprocity and the technique of restriction a representation to a subgroup of finite index.

Another general result was proven for isometry theta correspondence in \([MVW]\) and generalized for a similitude theta correspondence in \([GT]\).

**Theorem 5.3.** Let \(F\) be a non-archimedean field. For any supercuspidal representation \(\tau\) of \(H^+\), its theta lift \(\Theta(\tau)\) is either zero or irreducible. In particular \(\Theta(\tau) = \Theta(\tau)\).

### 5.2. Orthogonal-symplectic dual pairs

Howe duality does not hold for a general similitude orthogonal-symplectic triplet \((\Omega = \text{ind}_{R}^{G^+ \times H} \omega_{\psi,s}, H, G)\). However it does hold in the case of interest in this paper. We have the following proposition

**Proposition 5.4.** Let \((H^1 = SO(V, q), G^1 = Sp(W))\) be an orthogonal-symplectic dual pair where \((V, q)\) be an even-dimensional quadratic space and \(\dim V \leq \dim W\). Then Howe duality for the triplet \((\omega_{\psi,q}, H^1, G^1)\) implies Howe duality for the triplet \((\Omega = \text{ind}_{G^+ \times H}^{G^+ \times H} \Omega^+, H, G)\).

**Proof.** Note that \(H^+ = H\). If disc\((V, q)\) is a split algebra then \(G^+ = G\) and there is nothing to prove. Assume now that disc\((V, q) = E\) is a field. In this case \(G^+\) is a subgroup of \(G\) of index 2.

It has been shown in \([R]\), Prop. 1.2. that the following statements are equivalent:

1. Howe duality for \((\Omega, H, G)\) holds.
2. for any irreducible representation \(\pi\) of \(G^1\) and any \(g \in G\) at most one of the representations \(\pi, \pi^g\) has a non-zero theta lift to \(H^1\).

Let us show that the statement (2) above holds. Otherwise there exists an irreducible representation \(\pi\) of \(G^1\) and an element \(g \in G\) with \(\lambda(g) = a \notin \text{Nm}_{E/F}(E^*)\) such that
\[
\theta_{\psi,q}(\pi) \neq 0, \quad \theta_{\psi,q}(\pi^g) \neq 0
\]
Hence \( \theta_{\psi,aq}(\pi^q) \) is non-zero representation of \( SO(V,aq) \). The spaces \( (V,aq) \) and \( (V,q) \) belong to the different Witt towers. It follows from the conservation principal recently proven in [SZ] for all local fields that

\[
\dim(V,q) + \dim(V,aq) \geq 2 \dim W + 2.
\]

This contradicts the condition \( \dim(V) \leq \dim(W) \). \( \square \)

Let us get back to the notations of Sections 2–4. The group \( (H^1_F,G^1_F) \) is an orthogonal-symplectic dual pair inside \( Sp(V_F \otimes_F W_F) \). Given a polarization \( W_F = X_F \oplus Y_F \) there is a canonical splitting served by [Ku]

\[ i_{s_F} : H^1_F \times G^1_F \hookrightarrow \widetilde{Sp}(V_F \otimes_F W_F). \]

The pullback of the Weil representation \( \omega_\psi \) of \( \widetilde{Sp}(V_F \otimes_F W_F) \) defines a representation \( \omega_{s_F,\psi} \) of \( H^1_F \times G^1_F \).

We shall realize the Weil representation \( \omega_\psi \) of \( H^1_F \times G^1_F \) on the space of the Schwarz functions \( S(V_F \otimes X_F) \). The group \( H^1_F \times M^1_F \) acts naturally on the space \( S(V_F \otimes X_F) \). The action of \( H^1_F \times M^1_F \) is given by the usual formulas:

\[
\begin{aligned}
\omega_{s_F,\psi}(h)\phi(x) &= \phi(h^{-1}x) & h \in H^1_F, \\
\omega_{s_F,\psi}(u(S))\phi(x) &= \psi(\langle x, u(S)x \rangle)\phi(x) & S \in \text{Hom}(X_F,Y_F), S = -S^* . \\
\omega_{s_F,\psi}(m(a))\phi(x) &= \lambda_{E_F}(\det(a))|\det(a)|\phi(axa) & a \in GL(X_F)
\end{aligned}
\]

The representation \( \omega_{s_F,\psi} \) can be extended to the group

\[ R_F = \{ (h,g) \in H^1_F \times G^1_F : \lambda(h) = \lambda(g) \} \]

by defining

\[
\omega_{s_F,\psi}(h,g)\phi(x) = |\lambda(h)|^{-\frac{\dim V_F \cdot \dim W_F}{\dim V_F}} \omega_{s_F,\psi}(1,g')\phi(h^{-1}x) = |\lambda(h)|^{-1} \omega_{s_F,\psi}(1,g')\phi(h^{-1}x),
\]

where

\[ g' = \begin{pmatrix} \lambda(h) & 0 \\ 0 & 1 \end{pmatrix} g \in G^1_F. \]

**Remark 5.5.** The definition above agrees with the definition of Weil representation in [GTak]. Indeed, the similitude factor of a matrix \( g \in GSp_4 \) in [GTak] is defined to be \( \det(g)^{1/2} \), while here \( \lambda(g) = \det(g)^{-1/2} \). The groups \( R_D \) and the group \( R \) in [GTak] are the same subgroups of \( GO(V_F) \times GSp(W_F) \) and the representations are identical.

Let \( \psi_K = \psi \circ tr_{K/F} \) be a non-trivial additive character of \( K \). Similarly to above \( H^1_K \times G^1_K \) is a dual pair in \( \widetilde{Sp}(V_K \otimes W_K) \)

We denote by \( \omega_{s_K,\psi_K} \) the realization of the Weil representation of \( H^1_K \times G^1_K \) on the space \( S(V_K \otimes Y_K) \) such that

\[
\begin{aligned}
\omega_{s_K,\psi_K}(h,1)\phi(x) &= \phi(h^{-1}x) & h \in H^1_K, \\
\omega_{s_K,\psi_K}(1,u(S))\phi(y) &= \psi_K(\langle x, u(S)x \rangle)\phi(x) & S \in \text{Hom}(X_K,Y_K), S = -S^*. \\
\omega_{s_K,\psi_K}(1,m(a))\phi(y) &= \chi_{L/F}(a)|a_K|\phi(axa) & a \in GL(X_K)
\end{aligned}
\]

Similar to the previous cases, we extend \( \omega_{s_K,\psi_K} \) to the group

\[ R_K = \{ (h,g) \in H^1_K \times G^1_K : \lambda_K(h) = \lambda_K(g) \} \]

by

\[
\omega_{s_K,\psi_K}(h,g) = |\lambda(h)|^{\frac{\dim V_K \cdot \dim W_K}{\dim V_K}} \omega_{s_K,\psi_K}(1,g')\phi(h^{-1}x) = |\lambda(h)|^{-1/2} \omega_{s_K,\psi}(1,g')\phi(h^{-1}x),
\]

where

\[ g' = \begin{pmatrix} \lambda(h) & 0 \\ 0 & 1 \end{pmatrix} g \in G^1_K. \]
The representation $\omega_{s,K}^0$ will denote the restriction of $\omega_{s,K}^0$ to $R_K^0 = R_K \cap (H_K \times G_K^0)$.

5.3. **The induced representations** $\Omega_E$, $\Omega_L$. The compactly induced representations

$$\text{ind}_{R_F}^{H_F \times G_F} \omega_{s,F,\psi}, \quad \text{ind}_{R_K}^{H_K \times G_K} \omega_{s,K,\psi}\]$$

no longer depend on the character $\psi$ nor on the Hasse invariants of $s_F, s_K$ but only on the discriminant of the form and hence will be denoted by $\Omega_E$ and $\Omega_L$ respectively. Indeed, for any $a \in F^\times$ we denote by $\psi_a$ the additive character defined by $\psi_a(x) = \psi(ax)$. Let $g_a$ be any element of $G_F$ whose similitude factor is $a$. From the formulas above it follows easily that

$$\omega_{s,F,\psi_a} \simeq \omega_{a,s,F,\psi} \simeq \omega_{s,F,\psi}^{g_a}.$$ 

Hence the induction of $\omega_{s,F,\psi}$ to $H_F \times G_F$ does not depend on $a$. Similar argument applies for $\omega_{s,K,\psi}$.

We also define $\Omega_L^0 = \text{ind}_{R_K}^{H_K^0 \times G_K^0} \omega_{s,K,\psi,K}^0$.

**Proposition 5.6.** Howe duality holds for the triplets $(\Omega_E, H_F, G_F)$, $(\Omega_L, H_K, G_K)$ and $(\Omega_L^0, H_K^0, G_K^0)$.

**Proof.** Howe duality for the isometry triplet $(\omega_{s,F,\psi}, H_F^1, G_F^1)$ is proved in [Ya2]. Since $\dim V_F \leq \dim W_F$ Howe duality for the triplet $(\Omega_E, H_F, G_F)$ follows from Proposition 5.3.

Howe duality for the triplet $(\omega_{s,K,\psi,K}, H_K^1, G_K^1)$ is well known. The convenient reference is [Ca] for non-archimedean field. When $F$ is archimedean, Howe duality holds by the general result of Howe in [H].

Since $\dim V_K = \dim W_K$, Howe duality for the triplet $(\Omega_E, H_F, G_F)$ follows from Proposition 5.3.

Finally, $\lambda(H_K^0(F)) = \text{Nm}_{L/K}(L^\times) \cap F^\times = F^\times$ and $\lambda(G_K^0(F)) = F^\times$. Hence $(H_K^0)^+ = H_K^0$, $(G_K^0)^+ = G_K^0$ and Howe duality for the triplet $(\Omega_L^0, H_K^0, G_K^0)$ also holds.

5.4. **Quaternionic unitary dual pair.** The pair $H_D^1 \times G_D^1$ constitute a commuting pair inside $Sp(V_D \otimes W_D)$. This is a dual pair if and only if $h(D) = 1$.

Assume that $W_D$ admits a polarization $X_D \oplus Y_D$. This is always the case when $F$ is non-archimedean or if $F = \mathbb{C}$. If $F = \mathbb{R}$ the space $W_D$ must have signature $(1, 1)$.

The group $H_D \times GL(X_D)$ acts naturally on the space of Schwarz functions $S(V_D \otimes X_D)$.

We shall denote by $\omega_{s,D,\psi}$ the realization of the Weil representation of $H_D^1 \times G_D^1$ on the space $S(V_D \otimes X_D)$ such that

$$\begin{align*}
\omega_{s,D,\psi}(h)(x) &= \phi(h^{-1}y), \\
\omega_{s,D,\psi}(m)(x) &= \chi_{E/F}(\text{Nm}_{D/F}(a)) \text{Nm}_{D/F}(a) \phi(xa), \\
\omega_{s,D,\psi}(u)(x) &= \psi((x, u(S)x)) \phi(x).
\end{align*}$$

This definition agrees with the one given in [Ya2]. We extend $\omega_{s,D,\psi}$ to $R_D = \{(h, g) \in H_D \times G_D : \lambda(h) = \lambda(g)\}$ by setting

$$\omega_{s,D,\psi}(h, g)(x) = |\lambda(h)|^{-\frac{\dim_D - \dim_D}{2}} \omega_{s,D,\psi}(1, g')(h^{-1}x) = |\lambda(h)|^{-1} \omega_{s,D,\psi}(1, g') \phi(h^{-1}x)$$

where

$$g' = \begin{pmatrix} \lambda(h) & 0 \\ 0 & 1 \end{pmatrix} g \in G_D^1.$$ 

Note that the center of $R_D$ acts trivially. The representation $\text{ind}_{R_D}^{H_D \times G_D} \omega_{s,D}$ does not depend on $\psi$ and $s_D$ but only on its discriminant $E$ and hence will be denoted by $\Omega_D^E$.

The lemma below follows easily from the explicit Weil representation formulas and from Lemma 2.2.
Lemma 5.7. If $D$ splits then for compatible polarizations of $W_D$ and $W_F$ the canonical isomorphism $S(V_D \otimes X_D) \to S(V_F \otimes X_F)$ defines an isomorphism of representations $\omega_{s_D, \psi} \simeq \omega_{s_F, \psi}$.

Corollary 5.8. If $D$ splits then

$$H_D \simeq H_F, \quad G_D \simeq G_F, \quad \Omega^D_E \simeq \Omega_E$$

and Howe duality for the triplet $(\Omega^D_E, H_D, G_D)$ is equivalent to Howe duality for the triplet $(\Omega_E, H_F, G_F)$.

Proposition 5.9. Howe duality holds for $(\Omega^D_E, H_D, G_D)$.

Proof. In view of the last corollary it remains to consider the case where $b(D) = -1$. In this case $\lambda(H_D^1(F)) = F^\times$ and hence $G_D^+ = G_D$. The Howe duality for $(\Omega^D_E, H_D, G_D)$ follows now from Howe duality for $(\omega_{s_D, s_D}, H_D^1, G_D^1)$ that is proved in [Ya2]. \hfill \Box

6. Explicit Theta Correspondence

For any representation $\tau$ of $H_D$ we denote by $\Theta^D_E(\tau)$ and $\theta^D_E(\tau)$ the representations of $G_D$ that are the big and small theta lifts of $\tau$ respectively. Similarly, for an irreducible representation $\tau$ of $G_K$ (resp. $G^K_0$) we denote by $\Theta_L(\tau)$ and $\theta_L(\tau)$ (resp. $\Theta^L(\tau)$ and $\theta^L(\tau)$) the representations of $H_K$ (resp. $H^K_0$) which are the big and small theta lifts of $\tau$.

In this section we shall give more details on the theta correspondence $\theta^D_E$, $\theta_L$ and $\theta^0_L$ when $F$ is non-archimedean local field.

6.1. Explicit theta correspondence $\theta^D_E$. The theta lift $\theta^D_E$ will be used to define the Arthur packet on $G_D$. Hence it is desirable to know this theta lift as explicitly as possible.

To study the restriction problem we will apply see-saw duality technique which makes use of big theta lift $\Theta^D_E$ rather than small theta lift $\theta^D_E$. Thus it is important to determine $\Theta^D_E(\tau)$ as well.

First we prove the following reduction lemma.

Lemma 6.1. (1) Let $D$ be a non-split algebra and $\eta = \eta^E_F \circ Nm_{E/F}$. Then

$$\Theta^D_E(\tau^\pm_\eta) = (\eta^E_F \circ \lambda^{-1}) \otimes \Theta^D_E(1).$$

(2) Let $E = F \times F$ and $\eta = \mu \boxtimes \eta_F$. Then

$$\Theta^D_E(\tau^\pm_\eta) = (\eta^F \circ \lambda^{-1}) \otimes \Theta^D_E(\tau^\pm_\mu)$$

and the same relations hold for $\theta^D_E$.

Proof. Both statements follow immediately from the fact that for any character $\chi$ of $F^\times$

$$\tau \boxtimes \Theta(\tau) \simeq ((\chi \circ \lambda) \otimes \tau) \boxtimes ((\chi \circ \lambda^{-1}) \otimes \Theta^D_E(\tau))$$

as $R_D$ modules. \hfill \Box

Theorem 6.2. Let $E$ be a field extension of $F$.

(1) For any irreducible representation $\tau$ of $H_D$, the representation $\Theta^D_E(\tau)$ is a non-zero irreducible representation of $G_D$. In particular $\Theta^D_E(\tau) = \theta^D_E(\tau)$.

(2) Let $D$ be a non-split algebra.

(a) If $\eta \neq \eta^F$ then $\Theta^D_E(\tau^+_\eta)$ is supercuspidal representation.

(b) If $\eta = \eta^F = \eta^E_F \circ Nm$ then

$$\Theta^D_E(\tau^+_\eta) = (\eta^E_F \circ \lambda^{-1}) \otimes J_{P_D}(\chi_{E/F} \cdot \cdot |^{-1/2} \circ \text{Nm}_{D/F}).$$

(3) Let $D$ be a split algebra.

(a) If $\eta \neq \eta^F$ then $\Theta^D_E(\tau^+_\eta) = J_{Q}(\chi_{E/F} \cdot \cdot | \pi(\eta) |^{-1/2})$, where $\pi(\eta)$ is the dihedral supercuspidal representation of $GL_2$ associated to the character $\eta$.
Lemma 6.4. For $\eta = \eta^\sigma$ then $\Theta^D_E(\tau^+_\eta) = \text{the unique irreducible quotient of } I_B(\chi_{E/F} \cdot \lambda^{-1/2})$ and $\Theta^D_E(\tau^-_\eta)$ is supercuspidal.

Here the representation $\pi(\eta)$ denotes the dihedral supercuspidal representation of $GL_2$ with respect to a character $\eta$.

Proof. If $E$ is a field then the group $H_D$ is compact modulo its center. Hence, the irreducibility of $\Theta^D_E(\tau)$ follows from Theorem 5.5. Yasuda in [Ya2] has shown that $\theta_D(\tau)$ is not zero for any representation $\tau$ of $H_D$. Therefore, the non-vanishing follows from Prop. 5.1 Part (2).

Let us prove the second part. If $\eta \neq \eta^\sigma$ then the restriction of $\tau^+_\eta$ to $H_D$ is a sum of two non-trivial representations. Yasuda has shown that the lift of a non-trivial representations to $G_D$ is non-trivial and supercuspidal. Hence, by Prop. 5.1 Part (2) the representation $\Theta^D_E(\tau^+_\eta)$ is also supercuspidal.

If $\eta = \eta^\sigma = \eta_F \circ \text{Nm}_{D/F}$ then, by the reduction lemma above, it is enough to determine the theta lift of the trivial representation. We construct a map

$$T \in \text{Hom}_{H_D \times M_D}((\Omega^D_E)_{U_D}, 1 \boxtimes ((\chi_{E/F} \cdot \lambda) \circ \text{Nm}_{D/F}) \boxtimes | \cdot |^{-1/2})$$

by

$$T(f) = \int_{F^\times} f(1, m(1, t))(0)|t|^{-1/2} d^\times t.$$ 

The equivariance properties are easily checked.

Hence by Frobenius reciprocity the theta lift $\Theta^D_E(1)$ is a subrepresentation of $\text{Ind}^{G_D}_{P_D} \delta_{P_D}^{-1/2}(\chi_{E/F} \cdot \nu \circ \text{Nm}_{D/F}) \boxtimes | \cdot |^{-1/2}$

and hence it is a Langlands quotient of $\text{Ind}^{G_D}_{P_D}(\chi_{E/F} \cdot | \cdot |^{1/2} \circ \text{Nm}_{D/F} \boxtimes 1$, i.e.,

$$J_{P_D}(\chi_{E/F} \cdot | \cdot |^{1/2} \circ \text{Nm}_{D/F} \boxtimes 1).$$

The third part is proved in [GI] Prop. A.8. The small discrepancy in the notations is resolved by remark 5.5. □

Next we consider the case where $E$ and hence $D$ are split algebras. Recall that any representation of $H_D$ has the form $\tau^\pm_\eta$, where $\eta = \mu \boxtimes \eta_F$.

Theorem 6.3. Let both $D$ and $E$ be split algebras.

1. If $\mu \neq 1$, $| \cdot |^{\pm 2}$ then

$$\Theta^D_E(\tau^+_\eta) = \Theta^D_E(\tau^-_\eta) = (\eta_F \circ \lambda^{-1}) \circ I_Q(\mu).$$

2. If $\mu = | \cdot |^{\pm 2}$ then

$$\Theta^D_E(\tau^+_\eta) = (\eta_F \cdot | \cdot |^{\pm 1}) \circ \lambda^{-1}.$$

3. If $\mu = 1$ then $\Theta^D_E(\tau^+_\eta)$ are both irreducible and

$$\Theta^D_E(\tau^+_\eta) = (\eta_F \circ \lambda^{-1}) \circ I_Q(\det | \cdot |^{-1/2} \circ \text{Ind}^{GL_2}_{B} 1), \quad \Theta^D_E(\tau^-_\eta) = (\eta_F \circ \lambda^{-1}) \circ J_P(\det | \cdot |^{1/2} \circ \text{St}) \boxtimes | \cdot |^{-1/2}).$$

Lemma 6.4 reduces the proof of this theorem to the $\eta_F = 1$ case.

Proof. The key step in the proof is the following lemma.

Lemma 6.4. For $\mu \neq | \cdot |^{2}$ there is an injective map of $G_D$ modules

$$\text{Hom}_{H_D}(\Omega^D_E, \text{Ind}^{H_D}_{M_D} \mu \boxtimes 1) \rightarrow I_Q(\mu)^*.$$
Proof. By Frobenius reciprocity
\[
\text{Hom}_{H_D}(\Omega^D_E, \text{Ind}^H_{H_D} \mu \boxtimes 1) = \text{Hom}_{H_D}(\Omega^D_E, \mu \boxtimes 1).
\]

The restriction of \(\Omega^D_E\) to \(H^1_D \times G_D\) is in fact the Jacquet module \(\Omega^D_E\) with respect to the maximal parabolic subgroup \(H^1_D\) in \(H_D\). The result of Gan and Takeda in [GTak] implies the following filtration of \(H^1_D \times G_D\) modules
\[
0 \to J \to \Omega^D_E \to \Omega^D_E/J \to 0.
\]

Here \(J = \text{ind}^{H^1_D \times G_D}_{H^1_D \times Q_D} S(F^x \times F^x)\) where the action of \(H^1_D \times Q_D\) on \(S(F^x \times F^x)\) is given by
\[
(h(a,r), l(t,g) \phi)(x,y) = |r \det(g)^{-1}| \phi(xa \det(g)b^{-1}, yr \det(g)).
\]

The quotient \(\Omega/J\) is isomorphic to \(S(F^x)\) and the action of \(H^1_D \times G_D\) is given by
\[
(h(a,r), g) \phi(x) = |r|^{-1} |a|^2 \phi(xr \det(g)).
\]

In particular, if \(\mu \neq | \cdot |^2\) one has \(\text{Hom}_{H^1_D}(\Omega/J, \mu \boxtimes 1) = 0\) and hence by Lemma 9.4 of [GG1] there is an isomorphism of \(G_D\) modules
\[
\text{Hom}_{H^1_D}(\Omega^D_E, \mu \boxtimes 1) = \text{Hom}_{H^1_D}(J, \mu \boxtimes 1) = (\text{Ind}^D_{G_D} V)^*,
\]

where \(V^* = \text{Hom}_{H^1_D}(S(F^x \times F^x), \mu \boxtimes 1)\).

A straightforward computation shows that \(V\) is a one-dimensional space on which \(Q\) acts by \(\mu^{-1} \boxtimes \mu \circ \det\).

To derive the theorem from the last lemma assume first that \(\mu \neq 1\) so that \(\tau_\mu = \text{Ind}^H_{H_D} \mu \boxtimes 1\) is irreducible. By the lemma, there exists a surjective map \(I_Q(\mu) \to \Theta^D_E(\tau_\mu \boxtimes 1)\).

By Lemma 3.2 we conclude that \(\Theta^D_E(\tau_\mu \boxtimes 1) = \Theta^D_E(\tau_\mu \boxtimes 1) = I_Q(\mu)\) for \(\mu \neq | \cdot |^\pm 2\). It also follows that \(\theta^D_{\text{E}}(\tau_{| \cdot |^2 \boxtimes 1}) = | \cdot | \circ \lambda\). By Lemma 3.3, \(\tau_{| \cdot |^2 \boxtimes 1} \simeq (| \cdot |^2 \circ \lambda) \otimes \tau_{| \cdot |^2 \boxtimes 1}\). Lemma 6.1 implies now that
\[
\theta^D_{\text{E}}(\tau_{| \cdot |^2 \boxtimes 1}) = (| \cdot |^2 \circ \lambda^{-1}) \otimes \theta^D_{\text{E}}(\tau_{| \cdot |^2 \boxtimes 1}) = | \cdot | \circ \lambda^{-1}.
\]

Assume now that \(\mu = 1\) so that \(\text{Ind}^H_{H_D} \mu \boxtimes 1 = \tau^+_1 \oplus \tau^-_1\). By the lemma above there is a surjective map \(I_Q(1) \to \Theta^D_E(\tau^+_1) \oplus \Theta^D_E(\tau^-_1)\) and hence using Lemma 3.2 both
\[
\{\Theta^D_E(\tau^+_1), \Theta^D_E(\tau^-_1)\} = \{J_Q(| \cdot | \boxtimes |(\det^{-1/2} \otimes \text{Ind}^{GL_2}_{B_2} 1)), J_P(|(\det^{-1/2} \otimes St) | \boxtimes | \cdot |^{-1/2}|)\}
\]

are irreducible.

On the other hand, the representation \(\tau^+_1\) restricted to \(H^1_D\) is the trivial representation whose lift \(\Theta_{s,o}(1)\) to \(G^1_D\) is determined by [Yal]. It equals \(\overline{\text{Ind}}_{B_2}\) \((\text{Ind}^{GL_2}_{B_2} 1, 1, 4)\). Hence by 5.3, Part (2)
\[
\Theta^D_E(\tau^+_1) = J_Q(| \cdot | \boxtimes |(\det^{-1/2} \otimes \text{Ind}^{GL_2}_{B_2} 1)|), \quad \Theta^D_E(\tau^-_1) = J_P(|(\det^{-1/2} \otimes St) \boxtimes | \cdot |^{-1/2}|).
\]

Note that in the only case where \(\theta^D_{\text{E}}(\tau) \neq \Theta^D_{\text{E}}(\tau)\) the representation \(\tau\) is not unitary. In fact it is easy to prove that \(\Theta^D_{\text{E}}(\tau)\) is not one dimensional. For \(\mu = | \cdot |^\pm 2\) it follows from the proof above that \(\Theta^D_{\text{E}}(\tau_\mu \boxtimes 1) = I_Q(\mu)\).

Finally we can write explicitly the theta lift of unramified representations.

**Proposition 6.5.** Let \(D\) be a split algebra and let \(\eta\) be an unramified character. Then, \(\theta^D_{\text{E}}(\tau^+_q)\) is unramified representation and

- (1) If \(E\) is a field and \(\eta = \eta_{P} \circ \text{Nm}_{E/F}\) then \(\theta^D_{\text{E}}(\tau^+_q)\) is the unique irreducible quotient of \(I_B(\chi_{E/F} | \cdot |, \chi_{E/F} \cdot \eta_{F} | \cdot |^{-1/2}).\)
(2) If $E = F \times F$, and $\eta = \mu \otimes \eta_F$ then $\theta_E^D(\tau^\perp_\eta)$ is the unique irreducible quotient of $I_B([ \cdot | \cdot, \mu; \eta_F | \cdot ]^{1/2})$.

Part one is proved in [GI], A.8, while the second part follows from Theorem 6.3.

6.2. Twisted Jacquet modules of $G_D$. For any irreducible representation $\Pi$ of $G_D$ denote its wave front by

$$\hat{F}(\Pi) = \{ E \subset D : \exists T : \text{disc}(T) = E \text{ and } \Pi_{U_D, \Psi} \neq 0 \}.$$ 

Proposition 6.6. Let $\tau$ be an irreducible representation of $H_D$.

1. $\hat{F}(\theta_E^D(\tau)) = \{ E \}.$

2. If $\text{disc}(T) = E$ then $\theta_E^D(\tau)_{U_D, \Psi}$ equals $\tau^\vee \otimes \chi_{D,-1}$ where $\ker \chi_{D,-1} = H_D^t$ whenever either $D$ is split and $-1 \notin \text{Nm}_{E/F}(E^s)$ or $D$ is non-split and $-1 \in \text{Nm}_{E/F}(E^s)$. Otherwise $\chi_{D,-1}$ is a trivial character of $H_D$.

Proof. Fix non-zero vectors $e \in X_D, e^* \in Y_D$ such that $h_D(e, e^*) = 1$. Note that $V_D \otimes D \simeq V_D \otimes e$.

In the course of the proof define for any $t \in \mathbb{G}_m$ by $t = t\text{Id}_{X_D} + \text{Id}_{Y_D} \in G_D$.

There is an isomorphism of $H_D \times P_D$ modules

$$B : \Omega_E^D \to S((V_D \otimes e) \oplus F^s)$$

given by

$$B(\phi)(v \otimes e, y) = \phi(1, \tilde{y})(v \otimes e).$$

By the formulas of the Weil representations

$$(\omega_{s_D, \psi}(u(S)) - \Psi_T(u(S))) B(\phi)(v \otimes e, y) =$$

$$\psi(y \text{tr}_{D/F} \sigma(s_D(S, \tau)) h_D(e, S(e)) - \text{tr}_{D/F}(TS)) B(\phi)(v \otimes e, y) =$$

$$\psi(\text{tr}_{D/F}(y \sigma(s_D(S, \tau)) h_D(e, S(e)))) B(\phi)(v \otimes e, y).$$

The expression above vanishes for all the skew-Hermitian forms $S$ if and only if $B(\phi)$ is supported on the set

$$A_{s_D, T} = \{(v \otimes e, y) : y s_D(S, \tau) = T(e^*, \tau^*) \}.$$ 

Hence, the restriction of functions from $S(V_D \otimes X_D)$ to $A_{s_D, T}$ defines an isomorphism of $H_D \times GU(Y_D, T)$ modules

$$(\Omega_E^D)_{U_D, \Psi_T} \simeq S(A_{s_D, T}).$$

If $\text{disc}(s_D) \neq \text{disc}(T)$ then $A_{s_D, T} = \emptyset$ and hence $(\Omega_E^D)_{U_D, \Psi_T} = 0$. If $\text{disc}(s_D) = \text{disc}(T)$, the skew-Hermitian left $D$-modules $(Y_D, T)$ and $(\overline{V_D}, s_D)$ are equivalent. Hence, there exists an element $v_0 \in V_D$ such that $s_D(\overline{v_D}, v_0) = T(e^*, \overline{\tau})$.

There is a natural bijection of the sets $GU(V_D, s_D) \simeq H_D$ and $A_{s_D, T}$ via

$$h \mapsto (hv_0, \lambda(h^{-1})).$$

Using this isomorphism we identify $(\Omega_E^D)_{U_D, \Psi_T}$ with $S(H_D)$. By the formulas of the Weil representation, the action of $H_D \times GU(Y_D, T)$ on $S(H_D)$ is

$$\omega_{s_D, \psi}(h_1, h_2) \phi(\alpha) = \chi_{E/F} \circ \text{Nm}_{D/F}(h_2) | \lambda(h_1^{-1})| \cdot |\lambda(h_2)| \phi(h_1^{-1} ah_2).$$

The character $\chi_{E/F} \circ \text{Nm}_{D/F}$ on $H_D$ equals to $\chi_{D,-1}$. Hence, for any $\tau$ of $H_D$ the isotypic component of $\tau$ in $(\Omega_E^D)_{U_D, \Psi_T}$ is $\tau \otimes (\tau^\vee \otimes \chi_{D,-1})$. The proposition now follows. $\square$
6.3. Explicit theta correspondence $\theta_L$. This correspondence is well-known. Consider a character $\eta_L : L^\times \to \mathbb{C}$. If $L$ is a split algebra over $K$ fix an isomorphism $L^\times \simeq K^\times \times K^\times$ such that $\sigma(x,y) = (x^{-1}y, y)$. Then, the character $\eta_L$ has the form $\mu_K \boxtimes \eta_K$ so that

$$\eta_L(x,y) = \mu_K(x)\eta_K(y).$$

Obviously, $\eta_L$ is Galois invariant if and only if $\mu_K = 1$.

**Proposition 6.7.** Let $\pi$ be an irreducible representation of $G_K$. Then $\Theta_L(\pi) = \theta_L(\pi)$. More precisely,

1. Let $L$ be a field. Then $\Theta_L(\pi) = 0$ unless $\pi = \pi(\eta_L)$ for some $\eta_L : L^\times \to \mathbb{C}^\times$ and $\Theta_L(\pi(\eta_L)) = \tau_L^{+\eta_L}$.
2. Let $L = K \times K$. Then

$$\theta_L(\pi) = \begin{cases} 
\tau_L^{+\eta_L} \boxtimes \eta_K & \pi = \text{Ind}_{E_K}^{G_K} \mu_K \boxtimes \eta_K, \mu_K \neq \cdot |^{-1} \\
\tau_L^{-1} \eta_K & \pi = \eta_K \circ \lambda_K^{-1} \\
0 & \text{otherwise}
\end{cases}.$$ 

6.4. Relation between $\Theta_L$ and $\Theta_L^0$. The relation between $\Theta_L$ and $\Theta_L^0$ is given in the following proposition whose proof is identical to the proof of Prop. 5.1 part (2).

**Proposition 6.8.**

1. Let $\pi$ be an irreducible representation of $G_K^0$. Then $\Theta_L^0(\pi)$ is an irreducible representation of $H_K^0$.
2. Let $\pi$ be an irreducible representation of $G_K$ such that $\pi|_{G_K^0} = \oplus \pi_i$, sum of irreducible representations. Then $\Theta_L(\pi)|_{H_K^0} = \oplus \Theta_L^0(\pi_i)$.

7. Local parameters and local packets

In this section we describe the structure of the non-tempered Arthur packets on $SO(V)$ and define them using the theta correspondence described above.

7.1. Local parameters and local packets on $G_D$. Let $F$ be a local field and let $W_F$ be the Weil-Deligne group of $F$.

**Definition 7.1.** The local Arthur parameter of $\theta_{10}$ type is a map

$$\Psi : W_F^L \times SL_2(\mathbb{C}) \to Sp_4(\mathbb{C}),$$

where the image of $W_F^L$ is bounded and the image of a unipotent element of $SL_2(\mathbb{C})$ is conjugated to a short root unipotent element of $Sp_4(\mathbb{C})$.

The centralizer of the image of $SL_2(\mathbb{C})$ in $Sp_4(\mathbb{C})$ is the group $O_2(\mathbb{C})$. Hence, by restriction, the parameter $\Psi$ gives rise to a Langlands parameter $\Phi : W_F \to O_2(\mathbb{C})$. Obviously $\Phi$ determines $\Psi$.

The parameter $\Phi : W_F \to O_2(\mathbb{C})$ determines a quadratic algebra $E$ over $F$ as follows. If the image of $\Phi$ is contained in $SO_2(\mathbb{C})$ then $E = F \times F$. Otherwise there exists a quadratic field extension $E$ such that $\Phi(W_E) \subset SO_2(\mathbb{C})$. Denote by $\sigma$ the non-trivial automorphism in $\text{Aut}(E/F)$. By class field theory $\Phi$ determines the character $\eta : E^\times \to \mathbb{C}^\times$. To stress this dependence we shall write $\Psi_{E,\eta}, \Phi_{E,\eta}$ for $\Psi$ and $\Phi$ as above. Note that the parameters $\Psi_{E,\eta}$ and $\Psi_{E,\eta'}$ are conjugate and the same is true for $\Phi_{E,\eta}, \Phi_{E,\eta'}$.

The dual Langlands group of $Z_D \setminus G_D$, where $D$ runs over quaternionic algebras, is $Sp_4(\mathbb{C})$. Hence, by Arthur’s conjecture the parameter $\Psi_{E,\eta}$ gives rise to a set $A_{E,\eta}^{D}$ of unitary admissible representations of $G_D$. Similarly, the parameter $\Phi_{E,\eta}$ gives rise to a set $L_{E,\eta}^{D}$ of unitary admissible representations of $H_D$. The unions

$$L_{E,\eta} = \bigcup_{D \supset E} L_{E,\eta}^{D}, \quad A_{E,\eta} = \bigcup_{D \supset E} A_{E,\eta}^{D}$$
must stay in bijection with \( \hat{S}_{E,\eta} \), the set of characters of \( S_{E,\eta} \), where the local component group \( S_{E,\eta} \) of both \( \Psi_{E,\eta} \) and \( \Phi_{E,\eta} \) is given by

\[
S_{E,\eta} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \eta = \eta^\sigma \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise} \end{cases}.
\]

The \( L \)-packet \( L^D_{E,\eta} \) of representations of \( H_D \) is defined to consist of the constituents of \( \text{Ind}_{H_D}^{G_D} \eta \). More precisely,

\[
L^D_{E,\eta} = \{ \{ \tau_+^\eta, \tau_-^\eta \} : \eta = \eta^\sigma \} \quad \text{if} \quad \eta = \eta^\sigma.
\]

The authenticity of this construction will be evident from global considerations.

The structure of the packets \( L^D_{E,\eta} \) and \( A^D_{E,\eta} \) is identical. Thus, it is natural to construct the Arthur packet \( A^D_{E,\eta} \) of representations of \( G_D \) using the theta-correspondence.

\[
A^D_{E,\eta} = \{ \Pi^+_\eta = \theta^D_E(\tau^\eta) \}.
\]

Note that the labeling of \( L^D_{E,\eta} \) and hence of \( A^D_{E,\eta} \) is not canonical when \( D \) does not split and \( \eta = \eta^\sigma \).

Let us describe the bijection \( r \) between the sets \( A_{E,\eta} \) (resp. \( L_{E,\eta} \)) and \( \hat{S}_{E,\eta} \).

- If \( S_{E,\eta} = \mathbb{Z}/2\mathbb{Z} \) then
  \[
  r(\Pi^+_\eta) = r(\tau^+_\eta) = \begin{cases} 1 & D \text{ is split} \\ \text{sgn} & D \text{ is not split} \end{cases}.
  \]

- If \( S_{E,\eta} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) then
  \[
  r(\Pi^+_\eta) = r(\tau^+_\eta) = \begin{cases} 1 \otimes 1 & D \text{ is split} \\ \text{sgn} \otimes \text{sgn} & D \text{ is not split} \end{cases}.
  \]

7.2. The local packet of \( SO(U) \simeq G_K^0/F^\times \). The packet of representations of \( G_K^0(F) \) is defined as the set of constituents of a single irreducible representation \( \pi \) of \( G_K^0(K)/F^\times \) after restriction to \( G_K^0(F)/F^\times \). We denote the corresponding parameter by \( \Psi_\pi \).

8. See-saw duality and the restriction theorem

In this section assume that the algebra \( D_K \) splits.

8.1. See-saw duality. There is a natural embedding \( i_G : G_K^0 \simeq G_D^0 \hookrightarrow G_D \). The following proposition is straightforward.

**Proposition 8.1.** Let \( R^0_{D,K} = \{ (h,g) \in H_D \times G_K^0 : \lambda(h) = \lambda_K(g) \} \). The group \( R^0_{D,K} \) can be identified with the subgroup of \( R^0_K \) and of \( R_D \) via the imbeddings

\[
i_H \times \text{id} : R^0_{D,K} \hookrightarrow R^0_K, \quad \text{id} \times i_G : R^0_{D,K} \hookrightarrow R_D.
\]

For compatible polarizations, the natural isomorphism \( S(V_K \otimes X_K) \to S(V_D \otimes X_D) \) defines an isomorphism

\[
\omega^0_{s,K,\psi_K} |_{R^0_{D,K}} \simeq \omega_{s,D,\psi} |_{R^0_{D,K}}.
\]

Using this we prove the see-saw duality theorem.

**Theorem 8.2.** Let \( \tau \) and \( \pi \) be two irreducible representations of \( H_D \) and \( G_K \) respectively. Then

\[
\text{Hom}_{G_K^0}(\Theta^D_E(\tau), \pi) = \text{Hom}_{H_D}(\Theta^D_E(\pi), \tau).
\]
Proof. Assume that after the restriction to \( G_K^0 \) the representation \( \pi \) decomposes as \( \pi = \oplus \pi_i \). By definition one has
\[
\text{Hom}_{H_D}(\Theta_L^0(\pi_i), \tau) = \text{Hom}_{H_D}(\tau^\vee, \text{Hom}_{G_K^0}(\Omega^0_L, \pi_i)) = \text{Hom}_{H_D \times G_K}(\Omega^0_L, \tau \boxtimes \pi_i).
\]
Now, \( \Omega^0_L|_{H_D \times G_K} = \text{ind}_{\Omega^0_D \times G_K}^{H_D \times G_K} \omega_{s_K, \psi_K} \) and \( \Omega^D|_{H_D \times G_K} = \text{ind}_{\Omega^D_D \times G_K}^{H_D \times G_K} \omega_{s_D, \psi} \). Hence, by Frobenius reciprocity and by Proposition 8.1 the above equals
\[
\text{Hom}_{R_D \times G_K}(\omega_{s_K, \psi_K}, \tau \boxtimes \pi_i) = \text{Hom}_{R_D \times G_K}(\omega_{s_D, \psi}, \tau \boxtimes \pi_i) = \text{Hom}_{H_D \times G_K}(\Omega^D_L, \tau \boxtimes \pi_i) = \text{Hom}_{G_K}(\Theta^D_L(\tau), \pi_i).
\]
Using Prop. 8.3 we obtain
\[
\text{Hom}_{H_D}(\Theta_L(\pi), \tau) = \oplus_i \text{Hom}_{H_D}(\Theta^0_L(\pi_i), \tau) = \oplus_i \text{Hom}_{G_K}(\Theta^D_L(\tau), \pi_i) = \text{Hom}_{G_K}(\Theta^D_L(\tau), \pi).
\]
\( \square \)

8.2. The main restriction theorem.

**Definition 8.3.** We say that a character \( \eta_L \) of \( L^\times \) matches a character \( \eta \) of \( E^\times \) if there exists \( s \in \text{Aut}(L/K) \) such that \( s(\eta_L)|_{E^\times} = \eta \).

**Theorem 8.4.** Let \( \Pi^+_L \) and \( \pi \) be irreducible representations of \( G_D \) and \( G_K \) respectively.

1. \( \text{Hom}_{G_K}(\Pi^+_L, \pi) = 0 \) unless \( \pi = \pi(\eta_L) \) for some \( \eta_L : L^\times \to \mathbb{C} \) such that \( \eta_L \) matches \( \eta \).
2. Assume that the condition in the first part holds.
   (a) If \( \eta \neq \eta^\sigma \) then \( \dim \text{Hom}_{G_K}(\Pi^+_L, \pi) = 1 \).
   (b) If \( \eta = \eta^\sigma \) and \( \eta_L \neq \eta^\sigma_L \) then
   \[
   \dim \text{Hom}_{G_K}(\Pi^+_L, \pi) = 1, \quad \dim \text{Hom}_{G_K}(\Pi^-_L, \pi) = 1.
   \]
   (c) Assume that \( \eta = \eta^\sigma \), \( \eta_L = \eta^\sigma_L \). In particular \( \eta = \eta^\sigma_F \circ \text{Nm}_{E/F} \) and \( \eta_L = \eta_K \circ \text{Nm}_{L/K} \).
   Then \( \eta_K|_{E^\times} = \eta^\sigma_F \) or \( \eta^\sigma_F \).
   If \( D \) splits then
   \[
   \dim \text{Hom}_{G_K}(\Pi^+_L, \pi) = 1, \quad \dim \text{Hom}_{G_K}(\Pi^-_L, \pi) = 0.
   \]
   If \( D \) does not split then the labeling is not canonical and depends on the choice \( \eta^\sigma_F \). More precisely, if \( \eta_K|_{E^\times} = \eta^\sigma_F \) then
   \[
   \dim \text{Hom}_{G_K}(\Pi^+_L, \pi) = 1, \quad \dim \text{Hom}_{G_K}(\Pi^-_L, \pi) = 0.
   \]

Proof. First assume that \( \Theta^D_L(\tau) = \Theta^D_L(\tau) \). Then
\[
\text{Hom}_{G_K}(\Pi^+_L, \pi) = \text{Hom}_{G_K}(\Theta^D_L(\tau^\sigma_L), \pi) = \text{Hom}_{H_D}(\tau^\sigma_L, \tau^\sigma_L)
\]
when \( \pi = \pi(\eta_L) \) and zero otherwise. It is easy to describe the latter space using Frobenius reciprocity.

Indeed, in case (a), if \( \eta_L \neq \eta^\sigma_L, \eta \neq \eta^\sigma \) one has
\[
\text{Hom}_{H_D}(\tau^\sigma_L, \tau^\sigma_L) = \text{Hom}_{H_D}(\eta_L, \eta) \oplus \text{Hom}_{H_D}(\eta_L, \eta^\sigma)
\]
that is one-dimensional if \( \eta_L \) matches \( \eta \) and zero otherwise.

In case (b), \( \eta_L \neq \eta^\sigma_L \) but \( \eta = \eta^\sigma \). One has
\[
\text{Hom}_{H_D}(\tau^\sigma_L, \tau^\sigma_L) = \text{Hom}_{H_D}(\eta_L, \eta)
\]
that is one-dimensional if \( \eta_L \) matches \( \eta \) and zero otherwise.

Finally assume that \( \eta_L = \eta^\sigma_L \) and that \( \eta = \eta^\sigma \). To prove (c) note that the condition \( \eta_L|_{E^\times} = \eta \) implies \( \eta_K \circ \text{Nm}_{L/K}|_{E^\times} = \eta^\sigma_F \circ \text{Nm}_{E/F}|_{E^\times} \) and hence the restriction of \( \eta_K \) to \( \text{Nm}_{E/F}(E^\times) \) coincides
with \( \eta_K^\pm \). In particular \( \eta_K = \eta_K^\pm \). Note that there are two possible choices for \( \eta_K \) but their restrictions to \( F^\times \) coincide.

If \( D \) splits then \( \tau_{nl}^+ \) restricted to \( H_K^1 \) is trivial and hence the restriction to the subgroup \( H_D \subset H_K^1 \) is also trivial. Thus, \( \tau_{nl}^+ \) restricted to \( H_D \) equals \( \tau_{nl}^+ \).

For non-split \( D \) we know already that \( \eta_K = \eta_K^\pm \). Assume that \( \eta_K = \eta_K^\pm \). Then \( \tau_{nl}^+ = \eta_K \circ \lambda_K \) whose restriction to \( H_D \) is \( \eta_K^\pm \circ \lambda = \tau_{nl}^+ \). Hence

\[
\dim \text{Hom}_{C^\times_K} (\Pi_\eta^+, \pi) = 1, \quad \dim \text{Hom}_{C^\times_K} (\Pi_\eta^-, \pi) = 0.
\]

The case \( \eta_K = \eta_K^\pm \) follows by the same argument.

Finally, if \( \Theta^D_E(\tau) \neq \Theta^D_E(\tau) \) then by Propositions 6.2 and 6.3 one has \( E = F \times F, \tau = \tau_{nl}^+ \otimes \eta_{np} \) and \( \Theta^D_E(\tau) = (\eta_{0}^F \cdot |T^1|) \circ \lambda \). In this case the restriction of \( \Theta^D_E(\tau) \) to \( G^0_K \) is \( (\eta_{0}^F \cdot |T^1|) \circ \lambda_K \). The latter representation is \( \pi(| \cdot |_{K}^{1/2} \otimes \eta_{FP}) \) and the character \( | \cdot |_{K}^{1/2} \otimes \eta_{FP} \) of \( K^\times \times K^\times \) matches the character \( | \cdot |_{K} \otimes K^\times \).

\[\Box\]

Corollary 8.5. Let \( \Psi_1 = \Psi_{E,\eta} \) and \( \Psi_2 = \Psi_{\pi} \). Then

\[
(8.6) \sum_{U \subset V} \sum_{\pi \in A_{E,\eta}} \dim \text{Hom}_{SO(\gamma)} (\Pi, \pi)
\]

vanishes unless \( \pi = \pi(\eta_L) \) for a character \( \eta_L \) matching \( \eta \).

If \( \pi = \pi(\eta_L) \) with \( \eta_L \) matching \( \eta \) then (8.6) equals

\[
\begin{cases} 4 & \eta = \eta', \eta_L \neq \eta_L^2 \\ 2 & \text{otherwise} \end{cases}
\]

9. Global Parameters and global packets

9.1. Global parameters. Let \( F \) be a number field. Denote by \( pl(F) \) the set of completions of \( F \). Let \( \mathcal{L}_E \) be the conjectural Langlands group associated with \( F \). A non-tempered global Arthur parameter of \( \theta_{10} \) type is a map

\[
\Psi : \mathcal{L}_E \times SL_2(\mathbb{C}) \to Sp_4(\mathbb{C}),
\]

where the image of the unipotent element of \( SL_2(\mathbb{C}) \) is conjugated to the short root unipotent element of \( Sp_4(\mathbb{C}) \) and the image of \( \mathcal{L}_E \) is bounded.

As before, it determines by restriction a global tempered parameter \( \Phi : \mathcal{L}_E \to O_2(\mathbb{C}) \). If \( \Phi(\mathcal{L}_E) \) is contained in \( SO_2(\mathbb{C}) \) then its centralizer is not finite modulo the center and hence the parameter is not expected to contribute to the discrete spectrum. Thus we assume that this is not the case. In particular, there exists a unique quadratic field extension \( E \) over \( F \), with the non-trivial automorphism \( \sigma \in Gal(E/F) \), such that \( \Phi(\mathcal{L}_E) \subset SO_2(\mathbb{C}) \). Moreover, by class field theory \( \Phi \) determines a unitary automorphic character

\[
\eta = \otimes_{w \in pl(E)} \eta_w : E^\times \backslash E \to \mathbb{C}^\times
\]

with a trivial restriction to \( I_E \). We shall write \( \Phi_{E,\eta} \) and \( \Psi_{E,\eta} \) for \( \Phi, \Psi \) respectively.

For any place \( v \) of \( F \) there is an inclusion \( W'_{F_v} = \mathcal{L}_{F_v} \hookrightarrow \mathcal{L}_E \). Thus, by composition, the parameters \( \Psi_{E,\eta}, \Phi_{E,\eta} \) give rise to a family of local parameters \( \Psi_{E,\eta_v}, \Phi_{E,\eta_v} \) where \( E_v = E \otimes_F F_v \) is a quadratic algebra over \( F_v \) and

\[
\eta_v = \begin{cases} \eta_w & v = w \\ \eta_{w_1} \otimes \eta_{w_2} & v = w_1 w_2 \end{cases}
\]
Definition 9.1. The automorphic characters \( \eta_1 \) and \( \eta_2 \) will be called globally equivalent if there exists \( s \in \text{Gal}(E/F) \) such that \( s(\eta_1) = \eta_2 \). The characters will be called almost everywhere equivalent if for almost all places \( v \) of \( F \) there exists \( s_v \in \text{Aut}(E_v/F_v) \) such that \( \eta_1 v = s_v(\eta_2 v) \).

Remark 9.2. The parameters \( \Psi_{E,\eta_1} \) and \( \Psi_{E,\eta_2} \) (resp. \( \Phi_{E,\eta_1} \) and \( \Phi_{E,\eta_2} \)) associated with the globally equivalent characters are conjugate.

In the next section the following lemma will be used

Lemma 9.3. Let \( \eta_1, \eta_2 : E^\times \setminus E \to \mathbb{C}^\times \) be two almost everywhere equivalent unitary characters. Then \( \eta_1, \eta_2 \) are globally equivalent.

Proof. Let \( S \) be a finite set of places outside of which the local equivalence holds. Denote by \( L^S(., s) \) the Euler product of the local L-functions over the places outside of \( S \) and by \( LS \) the finite product of the local L-functions over the places in \( S \).

Consider the L-function \( L(\eta_1 \eta_2^{-1}, s)L(\eta_1 \sigma(\eta_2)^{-1}, s) \). By assumption it equals

\[
\zeta_E^S(s)\zeta_E^{\sigma}(s)\zeta_E^{\eta_1}(s)\zeta_E^{\eta_2}(s)
\]

and hence has a pole at \( s = 1 \). Thus, one of the terms \( L(\eta_1 \eta_2^{-1}, s), L(\eta_1 \sigma(\eta_2)^{-1}, s) \) has a pole at \( s = 1 \). In particular, either \( \eta_1 = \eta_2 \) or \( \eta_1 = \sigma(\eta_2) \).

9.2. Global packets. Let \( D \) be a quaternion division algebra over \( F \) containing \( E \) and let \( \eta = \otimes_{v \in p(F)} \eta_v \) be an automorphic character of \( \mathbb{I}_E \). We shall define several subsets of \( p(F) \).

- Let \( S_E \) be the set of places \( v \) of \( F \) such that \( E_v = E \otimes_v F_v \) does not split over \( F_v \).
- Let \( S_D \) be the set of places \( v \) of \( F \) at which \( D_v \) does not split over \( F_v \). This set is always finite and of even cardinality. Clearly, \( S_D \subseteq S_E \).
- Let \( S_\eta \) be the set of places \( v \) of \( F \) such that \( \eta_v = \eta_v^\sigma \) for the non-trivial \( \sigma \) in \( \text{Aut}(E_v/F_v) \). In particular, for any \( D \) the local packets \( L_{E,v,\eta}^D \) and \( L_{E,v,\eta}^{D\sigma} \) constructed in Section 7.1 contain two elements for \( v \in S_\eta \) and are singletons otherwise. The set \( S_\eta \) is always infinite since any \( v \in S_E \) such that \( \eta_v \) is unramified belongs to it.
- Let \( S_{e-1} \) be the set of places \( v \) of \( F \) such that \( -1 \notin \text{Nm}_{E_v/F_v}(E_v^\times) \). This set is always finite and of even cardinality.

For the one-dimensional skew-Hermitian space \( (V_D, s_D) \) over \( D \) of discriminant \( E \) and for the two-dimensional hyperbolic Hermitian space \( (W_D, h_D) \) over \( D \) we denote

\[
H_D(\mathbb{A}) = \text{GU}(V_D, s_D)(\mathbb{A}), \quad G_D(\mathbb{A}) = \text{GU}(W_D, h_D)(\mathbb{A}).
\]

The parameters \( \Phi_{E,\eta} \) and \( \Psi_{E,\eta} \) give rise to the packets \( L_{E,\eta}^D \) and \( A_{E,\eta}^D \) of representations of \( H_D(\mathbb{A}) \) and \( G_D(\mathbb{A}) \) as follows.

For a collection of representations \( \epsilon = (\epsilon_v \in \hat{E}_{v,\eta_v}) \), where \( \epsilon_v \) is the trivial representation for almost all \( v \), define the representations \( \Pi_{\eta}^\epsilon \) of \( G_D(\mathbb{A}) \) and \( \tau_{\eta}^\epsilon \) of \( H_D(\mathbb{A}) \) by

\[
\Pi_{\eta}^\epsilon = \otimes \Pi_{\eta_v}^\epsilon, \quad \tau_{\eta}^\epsilon = \otimes \tau_{\eta_v}^\epsilon.
\]

Furthermore, for any finite set \( S \subseteq S_\eta \) consider the collection \( \epsilon_S = (\epsilon_v \in \hat{E}_{v,\eta_v}) \) such that

\[
\Pi_{\eta_v}^{\epsilon_v} = \begin{cases} \Pi_{\eta_v}^+ & v \notin S \\ \Pi_{\eta_v}^- & v \in S \end{cases}
\]

We shall denote by \( \tau_{\eta}^S \) and \( \Pi_{\eta}^S \) the representations of \( H_D(\mathbb{A}) \) and \( G_D(\mathbb{A}) \) respectively, corresponding to the collection \( \epsilon_S \).

The global Arthur packets for \( G_D \) and Langlands packets for \( H_D \) are defined to be

\[
A_{E,\eta}^D = \{ \Pi_{\eta}^S : S \subseteq S_\eta, |S| < \infty \}, \quad L_{E,\eta}^D = \{ \tau_{\eta}^S : S \subseteq S_\eta, |S| < \infty \}.
\]
9.3. Arthur multiplicity formula. The global component groups of the parameters \( \Psi_{E, \eta} \) \( \Phi_{E, \eta} \) are isomorphic and are equal to

\[
S_{E, \eta} = \pi_0(\text{Cent}_G(\text{Im} \, \Psi_{E, \eta})) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \eta = \eta^* \\
\mathbb{Z}/2\mathbb{Z} & \eta \neq \eta^*
\end{cases}
\]

and there is a natural inclusion \( i : S_{E, \eta} \rightarrow \Pi_v S_{E, \eta_v} \).

According to Arthur’s conjecture for a collection of representations \( \epsilon = (\epsilon_v \in \widehat{S_{E, \eta_v}}) \) where \( \epsilon_v \) is trivial for almost all \( v \), the multiplicities of the representations \( \Pi_{\eta_v} \) \( \tau_\eta^v \) in the discrete spectrum of \( G_D \) and \( H_D \) respectively are equal to

\[
m(\epsilon) = \frac{\sum_{s \in S_{E, \eta}} \Pi_v \epsilon_v(i(s))}{|S_{E, \eta}|}.
\]

Lemma 9.5. For an automorphic character \( \eta \) and a finite set \( S \subset S_\eta \) one has \( m(\epsilon^S) = m(\eta, S) \) where

\[
m(\eta, S) = \begin{cases} 
0 & \eta = \eta^*, |S| \text{ is odd} \\
1 & \text{otherwise}
\end{cases}
\]

Proof. If \( S_{E, \eta} = \mathbb{Z}/2\mathbb{Z} \) then \( \epsilon_S(-1) = (-1)^{|S|} = 1 \) since the set \( S_D \) has even cardinality.

If \( S_{E, \eta} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) then

\[
\epsilon_S(1, -1) = (-1)^{|S_D|}, \quad \epsilon_S(-1, 1) = (-1)^{|S|}, \quad \epsilon_S(-1, -1) = (-1)^{|S_D|}.
\]

Here \( \triangle \) denotes the symmetric difference of the sets.

Hence, for any finite \( S \subset S_\eta \), the RHS of (9.4) equals

\[
\begin{cases} 
0 & \eta = \eta^*, |S| \text{ is odd} \\
1 & \text{otherwise}
\end{cases}
\]

Recall that for \( v \in S_D \cap S_\eta \) the labeling of the representations in \( L_{E, \eta_v}^{D_v} \) (and hence in \( A_{E, \eta_v}^{D_v} \)) was not canonical. For the multiplicity formula to hold we should choose the labeling in a coherent way. If \( \eta \neq \eta^* \) then the labeling of \( L_{E, \eta_v}^{D_v} \) for \( v \in S_D \cap S_\eta \) can be chosen arbitrary. However, if \( \eta = \eta^* \) then \( \eta = \eta_F \circ \text{Nm}_{E/F} \) for an automorphic character \( \eta_F \). The character \( \eta_F \) is defined up to multiplication by \( \chi_{E/F} \). The choice of \( \eta_F \) fixes the labeling for every \( v \in S_D \). Replacing \( \eta_F \) by \( \chi_{E/F} \eta_F \) will change the labeling simultaneously at all the places in \( S_D \).

Our next task will be to prove that Arthur’s multiplicity conjecture holds for all the representations in \( L_{E, \eta} \) and \( A_{E, \eta} \).

10. Discrete spectrum of \( H_D \)

In this section we shall describe explicitly the decomposition of the space \( L^2(\mathbb{Z}_D(\mathbb{A}) H_D(F) \setminus H_D(\mathbb{A})) \). Since \( E \) is assumed to be a field, the space \( Z_D(\mathbb{A}) H_D(F) \setminus H_D(\mathbb{A})) \) is compact and hence the space of square integrable functions decomposes discretely.

Theorem 10.1.

\[
L^2(\mathbb{Z}_D(\mathbb{A}) H_D(F) \setminus H_D(\mathbb{A})) = \bigoplus_{\eta \sim \eta_F} V(\eta), \quad V(\eta) = \bigoplus_{S \subset S_\eta, |S| < \infty} m(\eta, S) \tau_\eta^S.
\]

Proof. First note that all the representations in the sum are non-isomorphic. Indeed \( \tau_\eta^S \simeq \tau_\eta^S \) if and only if for each place \( v \) of \( F \) there exists \( \sigma_v \in \text{Aut}(E_v/F_v) \) such that \( \eta_v = \eta_2^{\sigma_v} \). By Lemma 9.3 the characters \( \eta_1, \eta_2 \) are equivalent. Hence, also \( S_1 = S_2 \).
The representations $\tau^S_\eta$ with $m(\eta, S) = 1$ are realized in the discrete spectrum $L^2(Z_{H_D}(\mathbb{A})H_D(F)\backslash H_D(\mathbb{A}))$ by an Eisenstein series.

$$E_\eta : \text{Ind}_{H_D(\mathbb{A})}^{H_D(\mathbb{A})} \eta \to L^2(Z_{H_D}(\mathbb{A})H_D(F)\backslash H_D(\mathbb{A}))$$

defined by

$$(10.2) \quad E_\eta(\phi)(h) = \sum_{s \in H_D(\mathbb{F})\backslash H_D(\mathbb{A})} \phi(sh).$$

It follows immediately from the definition that

$$\text{Ker}(E_\eta) = \bigoplus_{(\eta, S) : m(\eta, S) = 0} \tau^S_\eta.$$ 

In particular, we have shown one inclusion. To establish the equality take $f$ such that $(f, E_\eta(\phi)) = 0$ for all $\eta, \phi \in \text{Ind}_{H_D(\mathbb{A})}^{H_D(\mathbb{A})} \eta$. One has

$$(f, E_\eta(\phi))_{H_D} = \int_{H_D(\mathbb{A})\backslash H_D(\mathbb{A})} \left( \int_{Z_{H_D}(\mathbb{A})H_D(\mathbb{F})\backslash H_D(\mathbb{A})} f(\rho h) \overline{\eta(\rho)} d\rho \right) \phi(h) dh.$$ 

If the integral vanishes for all $\phi$ then the inner integral vanishes for almost all $h$ and for all $\eta$. Hence $f = 0$. \qed

**Remark 10.3.** In particular, for a fixed equivalence class of $\eta$, the space $V(\eta)$ is a full nearly equivalence class of cuspidal representations of $H_D(\mathbb{A})$. This shows that the $L$-packets $L_{E, \eta}$ were defined correctly.

Since the global packets $L_{E, \eta}$ and $A_{E, \eta}$ have the same structure it is natural to attempt to construct the automorphic representations in $A_{E, \eta}$ using the automorphic representations of $L_{E, \eta}$. The global theta correspondence method provides such a construction.

### 11. The Global Theta Correspondence

Whenever $m(\eta, S) = 1$ we shall construct an automorphic realization of $\Pi^S_\eta$ using the global theta correspondence.

Recall the construction of the global theta correspondence for the dual pair $(G_D, H_D)$. Let $\psi : F\backslash \mathbb{A} \to \mathbb{C}^\times$ be a non-trivial additive character. The representation

$$\left( \omega_{s_D, \psi} = \otimes_v \omega_{s_D, \psi_v}, R_D(\mathbb{A}), S(V_D \otimes X_D)(\mathbb{A}) \right)$$

admits an automorphic realization

$$\theta \in \text{Hom}_{R_D(\mathbb{A})}(S(V_D \otimes X_D)(\mathbb{A}), \mathcal{A}(R_D(F)\backslash R_D(\mathbb{A})))$$

given by

$$\theta(\phi)(r) = \sum_{x \in (V_D \otimes X_D)(F)} \omega_{s_D, \psi}(r) \phi(x).$$
11.1. **The global theta correspondence from \( H_D \) to \( G_D \).** For a cuspidal representation \( \tau \) of \( H_D(\mathbb{A}) \) define a map

\[
\theta^D_E : \omega_{\psi,s_D} \boxtimes \psi \rightarrow \mathcal{A}(G_D(F) \backslash G_D(\mathbb{A}))
\]
as follows: For \( g \in G_D(\mathbb{A}) \) such that \( \lambda(g) \in \lambda(H_D(\mathbb{A})) \) take \( h \in H_D(\mathbb{A}) \) such that \( (h,g) \in R_D(\mathbb{A}) \) and define

\[
\theta^D_E(\phi \otimes f)(g) = \int_{H^1(F) \backslash H^1(\mathbb{A})} \phi(h_1 h, g) \overline{f(h_1 h)} dh_1.
\]

For \( \lambda(g) \notin \lambda(H_D) \) define \( \theta^D_E(\phi \otimes f)(g) \) to be zero.

We denote by \( \theta^D_E(\tau) \) the representation spanned by the functions

\[
\{ \theta^D_E(\phi \otimes f)(g), \phi \in \omega_{s_D,\psi}, f \in \tau \}.
\]

**Theorem 11.1.**

1. For any irreducible representation \( \tau^S_\eta \) of \( H_D(\mathbb{A}) \), the representation \( \theta^D_E(\tau^S_\eta) \) is irreducible, non-zero and contained in the discrete spectrum of \( G_D \). Moreover,

\[
\theta^D_E(\tau^S_\eta) \cong \otimes_v \mathbb{S} \theta^D_{E_v}(\tau^+_{\eta_v}) \otimes \otimes_v \mathbb{S} \theta^D_{E_v}(\tau^-_{\eta_v}).
\]

2. Let \( D \) be a split algebra. Then \( \theta^D_E(\tau^S_\eta) \) is cuspidal unless \( S = \emptyset \).

3. Let \( D \) be a non-split algebra. Then \( \theta^D_E(\tau^S_\eta) \) is cuspidal unless \( \eta = \eta^\sigma \) and \( S \subseteq S_D \).

**Proof.** The non-vanishing follows from the non-vanishing of the global theta lift for isometries, proven in [Ya2]. The irreducibility of the global lift follows from the irreducibility of the local lifts. Parts (2) and (3) follow from Lemma 1.3 of [S] and Lemma 3.1 of [Ya2]. If the conditions in Parts (2) and (3) do not hold then applying the square-integrability criterion it is shown in [Ya1] and [KRS] that the theta lifts are contained in the discrete spectrum. \( \square \)

In particular, for any \( \eta, S \) one has

\[
m(\Pi^S_\eta) \geq m(\eta, S).
\]

The other inequality will be proved in Section 12.

12. **Wave front and the multiplicity formula**

12.1. **The Fourier coefficient.** Fix a non-trivial automorphic additive character \( \psi : F \backslash \mathbb{A} \rightarrow \mathbb{C}^\times \). We identify the unitary characters of \( U_D(F) \backslash U_D(\mathbb{A}) \) with the space of skew-Hermitian forms on \( Y_D(F) \). Any skew-Hermitian form \( T \) on \( Y_D(F) \) gives rise to a form on \( Y_D(\mathbb{A}) \). Denote

\[
\Psi_T(u(S)) = \psi(\text{tr}_{D(\mathbb{A})/\mathbb{A}}(TS)).
\]

The \( M_D(F) \) orbits on the space of characters stay in bijection with the equivalence classes of locally isometric skew-Hermitian forms on \( Y_D(F) \) and hence are parameterized by the quadratic algebras \( E \) inside \( D \). The stabilizer in \( M_D(\mathbb{A}) \) of the character associated with the form \( T \) is isomorphic to \( GU(Y_D,T)(\mathbb{A}) \).

For any automorphic form \( \varphi \) on \( G_D \) define its Fourier coefficient with respect to the skew-Hermitian form \( T \) by

\[
F_T(\varphi) = \int_{U_D(F) \backslash U_D(\mathbb{A})} \varphi(u) \overline{\Psi_T(u)} du.
\]

For any automorphic representation \( \Pi \) of \( G_D(\mathbb{A}) \) the Fourier coefficient defines a map

\[
F_T \in \text{Hom}(\Pi_{U_D,\Psi_T}, \mathcal{A}(GU(Y_D,T)(F)) \backslash GU(Y_D,T)(\mathbb{A}))
\]

by

\[
F_T(\varphi)(h) = F_T(h\varphi).
\]
In particular, for $\Pi \simeq \Pi^S_\eta$ one has $F_T(\Pi) = 0$ unless $\text{disc} T = E$. For $\text{disc} T = E$ there is an isomorphism
\[ GU(Y_D, T) \simeq GU(V_D, s_D) \simeq H_D \]
and by Proposition 6.6 $F_T(\Pi)$ is a quotient of $(\tau^S_\eta)^V \otimes \chi_{D,-1}$, where
\[ (\chi_{D,-1})_v = \begin{cases} \text{sgn} & v \in S_D \triangle S_{-1} \\ 1 & \text{otherwise} \end{cases} \]

12.2. The wave front. For any irreducible automorphic representation $\Pi$ define the wave front set by
\[ \hat{F}(\Pi) = \{ E : \exists T : \text{disc}(T) = E, \varphi \in \Pi : F_T(\varphi) \neq 0 \} \]
If $\Pi = \otimes \Pi_v$ and $E \in \hat{F}(\Pi)$ then obviously $E_v \in \hat{F}(\Pi_v)$. We shall make use of the following important theorem by Jian-Shu Li, [Li].

**Theorem 12.1.** Let $\Pi$ be a cuspidal irreducible representation of $G_D(\mathbb{A})$. Then $\hat{F}(\Pi) \neq \emptyset$.

**Proposition 12.2.** Let $\Pi$ be an automorphic representation of $G_D(\mathbb{A})$ which is nearly equivalent to $\Pi^0_\eta$. Then, $\hat{F}(\Pi) = \{ E \}$.

**Proof.** First note that $\hat{F}(\Pi) \subseteq \{ E \}$. Indeed, assume that $E' \in \hat{F}(\Pi)$. For almost all places $v$ one has $\hat{F}(\Pi_v) = \hat{F}(\Pi^0_v) = \{ E_v \}$ by Proposition 6.6. Hence, the characters $\chi_{E/F}$ and $\chi_{E'/F}$ are nearly equivalent. By the strong multiplicity one theorem $E = E'$.

It remains to show that $\hat{F}(\Pi) \neq \emptyset$. For a cuspidal $\Pi$ this follows from Theorem 12.1. Assume now that $\Pi$ is not contained in the cuspidal spectrum of $G_D(\mathbb{A})$. Then, restricting $\Pi$ to $G_D^1(\mathbb{A})$ and restricting the functions in $\Pi$ to $G_D^1(F) \backslash G^1(\mathbb{A})$ we obtain a (possibly reducible) representation of $G_D^1(\mathbb{A})$ that is not contained in the cuspidal spectrum of $G_D^1(\mathbb{A})$. By [Ya1] every irreducible residual representation that is nearly equivalent to a constituent of $\Pi^0_\eta(G_D(\mathbb{A}))$ is isomorphic to $\theta^D_{s_D, \psi}(1)$ and hence by Lemma 4.12 in [Ya1] has some non-trivial Fourier coefficient. Thus, $\hat{F}(\Pi) \neq \emptyset$.

From this we can deduce Arthur’s multiplicity formula.

**Proposition 12.3.** For any $\eta$ and $S$ the multiplicity $m(\Pi^S_\eta)$ of $\Pi^S_\eta$ in the discrete spectrum of $G_D(\mathbb{A})$ equals $m(\eta, S)$.

**Proof.** Assume that $m(\eta, S) = 0$. Let us show that $m(\Pi^S_\eta) = 0$. Indeed, if $\Pi^S_\eta$ can be embedded into the discrete spectrum of $G_D$ then $\hat{F}(\Pi^S_\eta) = \{ E \}$. Hence, for a skew-Hermitian form $T$ with discriminant $E$, the space $F_T(\Pi^S_\eta)$ defines a non-zero irreducible automorphic representation of $H_D(\mathbb{A})$ isomorphic to $(\tau^S_\eta)^V \otimes \chi_{D,-1}$. Since by assumption $\eta = \eta^\sigma$, the above representation is
\[ \tau_{\eta^{-1}}^S \Delta(S_D \triangle S_{-1}) \]
In particular $\Delta(S_D \triangle S_{-1})$ has even cardinality and hence $S$ has even cardinality so $m(\eta, S) = 1$. This is a contradiction.

Assume now that $(\eta, S)$ is such that $m(\eta, S) = 1$. Let us show that the multiplicity of $\Pi^S_\eta$ in the discrete spectrum is 1. The multiplicity is at least one because one realization is given by the theta correspondence $\Pi^S_\eta = \theta^D_E(\tau^S_\eta)$. If there are two embeddings
\[ J_1, J_2 \in \text{Hom}_{G_D(\mathbb{A})}(\Pi^S_\eta, L^2_{disc}(Z_D(\mathbb{A}) G_D(F) \backslash G_D(\mathbb{A}))), \]
we know that $\hat{F}(J_1(\Pi^S_\eta)) = \{ E \}$. The image of the maps
\[ F_T \circ J_i \in \text{Hom}_{U_D(\mathbb{A})}(\Pi^S_\eta, L^2(H_D(F) \backslash H_D(\mathbb{A}))) \]
define an automorphic irreducible representation of \( H_D(\mathbb{A}) \) isomorphic to \((\tau_\eta^S)^\vee \otimes \chi_{D,-1} \).
By the multiplicity one property for the discrete spectrum of \( H_D \) there exists a constant \( c \in \mathbb{C}^\times \) such that \( F_I(J_1 - cJ_2)(\Pi^S_\eta) = 0 \). Hence, the automorphic representation \((J_1 - cJ_2)(\Pi^S_\eta)\) does not support any non-degenerate coefficients along \( U_D(\mathbb{A}) \) and therefore by Proposition 12.2 it is zero. In other words, \( J_1 \) and \( J_2 \) are proportional and hence the multiplicity of \( \Pi^S_\eta \) in the discrete spectrum is one. \( \square \)

13. L-function

Our next goal is to show that the constructed Arthur packets contain the full nearly equivalence class of cuspidal representations. Our approach exploits a Rankin-Selberg integral representation of an \( L \)-function of degree 5 of a cuspidal representation of \( G_D(\mathbb{A}) \). This is a generalization of the \( L \)-function studied in [PS-R].

13.1. Notations. Below \( F \) is a number field and \( \mathbb{A} \) is its ring of adeles. For any place \( v \) of \( F \), \( F_v \) denotes the \( v \)-adic completion of \( F \). If \( F_v \) is non-archimedean \( \mathcal{O}_v \) denotes the ring of integers of \( F_v \), \( \varpi_v \) denotes a uniformizer inside \( \mathcal{O}_v \) and \( q_v \) denotes the cardinality of the residue field. For any finite set of places \( S \) we denote \( \mathbb{A}_S = \prod_{v \in S} F_v \).

The group \( Sp_4(\mathbb{C}) \) which is the \( L \)-group of \( Z_D \backslash G_D \), admits a 5 dimensional irreducible complex representation \( \rho \), given by the accidental isomorphism \( PSp_4(\mathbb{C}) \simeq SO_5(\mathbb{C}) \) discussed above. Let \( \chi : E^\times \backslash \mathbb{I}_E \to \mathbb{C}^\times \) be an automorphic character and let \( \Pi = \otimes_v \Pi_v \) be an irreducible representation of \( G_D(\mathbb{A}) \). There exists a finite set of places \( \Omega \) which includes all the archimedean places and which contains \( S_D \) such that for \( v \notin \Omega \) the representation \( \Pi_v \) is unramified with a Satake parameter \( t_{\Pi_v} \), and the character \( \chi_v \) is unramified too. Define

\[
L^\Omega(\Pi, \chi, \rho, s) = \prod_{v \notin \Omega} \det(1 - \chi_v(\varpi_v)\rho(t_{\Pi_v})q_v^{-s})^{-1}.
\]

Let \( \Pi \) be an irreducible representation of \( G_D(\mathbb{A}) \) nearly equivalent to \( \Pi_\eta^0 \) for some automorphic character \( \eta \) of \( E^\times \backslash \mathbb{I}_E \). Hence \( \Pi \) and \( \Pi_\eta^0 \) share partial \( L \)-functions \( L^\Omega(\cdot, \chi_{E/F}, s) \) for a set \( \Omega \) large enough.

By Proposition 6.7

\[
L^\Omega(\Pi, \chi_{E/F}, \rho, s) = \zeta^\Omega(s-1)\zeta^\Omega(s)L^\Omega(\chi_{E/F}, s)^2\zeta^\Omega(s+1)
\]

and hence it has a simple pole at \( s = 2 \). We shall show that this property characterizes the cuspidal representations in the nearly equivalence class of \( \Pi_\eta^0 \).

13.2. Eisenstein series. Let \( K \) denote the maximal compact subgroup of \( G_D(\mathbb{A}) \). For any \( K \)-finite standard section \( f(\cdot, s) \) in the unitary induced representation \( Ind_{P_{D_-}}^{G_D} |Nm_D/F|^s \), consider the associated Eisenstein series for \( s \), whose real part is sufficiently large

\[
E(g, f, s) = \sum_{\gamma \in P_{D_-}(F) \backslash G_D(F)} f(\gamma g, s).
\]

The Eisenstein series admits a meromorphic continuation for the entire complex plane.

**Theorem 13.1.** For any standard section \( f(g, s) \), the Eisenstein series \( E(g, f, s) \) has at most a simple pole at \( s = 3/2 \). The pole is attained by the spherical section and the residue is the constant function.

**Proof.** For \( D \) split this is proved in Theorem 3.1 in [S], and for non-split \( D \) this is proved in [Ya1]. \( \square \)

Let us define the normalized Eisenstein series by

\[
E^*(g, f, s) = \zeta(2s-1)\zeta(s+1/2)E(g, f, s)
\]
13.3. Rankin-Selberg integral. Let $\Pi$ be an irreducible cuspidal representation of $G_D(\mathbb{A})$ such that $E \in \hat{F}(\Pi)$. For any $\varphi \in \Pi$, $\phi \in S(V_D \otimes X_D)$ and a section $f(\cdot, s)$ consider the integral

$$Z(\varphi, \phi, f, s) = \int_{G_D^{(1)}(F) \backslash G_D^{(1)}(\mathbb{A})} \varphi(g)\theta_D^D(\phi)(g)E^*(g, f, s - 1/2)\, dg.$$  

(13.2)

The function $\varphi$ is rapidly decreasing on $G_D^{(1)}(F) \backslash G_D^{(1)}(\mathbb{A})$. In particular the integral converges absolutely and hence defines a meromorphic function on $\mathbb{C}$.

Theorem 13.3. Let $\Omega$ be a finite set of places which includes all the archimedean places and the set $S_D$ such that outside of $\Omega$ the representation $\Pi_v$ and the field extension $E_v$ are unramified. Let $\varphi = \otimes_v \varphi_v$ and $f = \otimes_v f_v$ be factorizable data, that is spherical outside of the set $\Omega$. Then, for $\text{Re}(s)$ sufficiently large

$$Z(\varphi, \phi, f, s) = L^D(\Omega, \chi_{E/F}, \rho, s)d_\Omega(\varphi, \phi, f, s),$$

(13.4)

where

$$d_\Omega(\varphi, \phi, f) = \int_{U_D(\Lambda_\Omega) \backslash G_D^{(1)}(\Lambda_\Omega)} F_{s_D}(g\varphi)\omega_{s_D, \psi}(g)(r_0)f^*(g, s)\, dg$$

Moreover, for every $s_0 \in \mathbb{C}$ there exist $\varphi, f, \phi$ such that $d_\Omega(\varphi, \phi, f, s)$ defines a holomorphic non-zero function in the neighborhood of $s_0$.

The proof of this theorem will occupy the rest of this section. Let us list some immediate corollaries:

The zeta integral $Z(\varphi, \phi, f, s)$ has meromorphic continuation to the whole complex plane and hence the identity [13.4] can be used to define the meromorphic continuation of the partial $L$-function $L^D(\Pi, \chi_{E/F}, \rho, s)$.

For an irreducible cuspidal representation $\Pi$ of $G_D(\mathbb{A})$ define the representation $\theta_D^D(\Pi)$ of $H_D(\mathbb{A})$ spanned by the functions

$$\theta_D^D(\phi, \varphi)(h) = \int_{G_D^{(1)}(F) \backslash G_D^{(1)}(\mathbb{A})} \theta_D^D(\phi(h, g_1\phi)\varphi(g_1g))\, dg_1, \quad \lambda(g) = \lambda(h), \varphi \in \Pi, \phi \in \omega_{\psi, s_D}.$$  

Corollary 13.5. Let $\Pi$ be an irreducible cuspidal representation of $G_D(\mathbb{A})$ such that $E \in \hat{F}(\Pi)$ and the finite set $\Omega$ is as above. If $L^D(\Pi, \chi_{E/F}, \rho, s)$ has a pole at $s = 2$ then $\theta_D^D(\Pi) \neq 0$.

Proof. Let $\varphi, \phi, f$ be functions such that $d_\Omega(\varphi, \phi, f, s)$ is holomorphic, nonzero around $s = 2$. Hence $Z(\varphi, \phi, f, s)$ has a pole at $s = 2$ and the leading term of Laurent expansion of $Z(\varphi, \phi, f, s)$ at $s = 2$ is $\text{Res}_{s=2} E^*(g, f, s)\theta_D^D(\phi, \varphi)(1)$. □

If $D$ splits, the theorem was proven by [PS-R]. The proof in the case where $D$ does not split is almost identical and is sketched in the following three subsections.

13.4. Unfolding.

Proposition 13.6. For $\text{Re}(s)$ sufficiently large it holds

$$Z(\varphi, \phi, f, s) = \int_{U_D(\mathbb{A}) \backslash G_D^{(1)}(\mathbb{A})} F_{s_D}(g\varphi)f(g, s - 1/2)\omega_{\psi, s_D}(g)\phi(r_0)\, dg,$$

(13.7)

where $r_0$ is a fixed non-zero vector of $V_D \otimes X_D$.  

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Proof. Substituting the definition of the Eisenstein series for $\text{Re}(s)$ sufficiently large we obtain

$$Z(\varphi, f, s) = \int_{P^1_D(F) \backslash G^1_D(\mathbb{A})} \varphi(g) \sum_{x \in (V_D \otimes X_D)(F)} \omega_{s_D, \psi}(g) \phi(x) f(g, s - 1/2) dg =$$

$$\int_{U_D(F) \backslash G^1_D(\mathbb{A})} \varphi(g) \omega_{s_D, \psi}(g) \phi(r_0) f(g, s - 1/2) dg =$$

$$\int_{U_D(F) \backslash G^1_D(\mathbb{A})} F_{s_D}(g \varphi) \omega_{s_D, \psi}(g) \phi(r_0) f(g, s - 1/2) dg.$$

Since $D$ does not split there are only two orbits of the action of $M^1_D(F)$ on $V_D \otimes X_D(F)$: the zero orbit and the open orbit. The element $r_0$ is the representative of an open orbit. The contribution from the zero orbit vanishes because of cuspidality of $\Pi$. \hfill \Box

Remark 13.8. Note that collapsing the sum and the integration above will be justified if we show that the integral on the right hand side absolutely converges. We shall show it in the next subsection.

13.5. Unramified computation. For a general $\Pi$, the space $\text{Hom}_{U(F)}(\Pi_v, C_{\Psi_{s_D}})$ is not one-dimensional. Hence, the functional $F_{s_D}$ is not necessarily factorizable. However, the integral $\int_{G^1_D(F)}$ is factorizable due to the following striking proposition.

Proposition 13.9. Let $v \notin \Omega$ and let $v_0^0$ be a $K_v$-fixed vector of $\pi_v$. Let $f^*(g, s)$ be a spherical section normalized by $f^*(e, s) = \zeta(2s - 1)\zeta(s + 1/2)$. For any functional $L \in \text{Hom}_{U(F)}(\Pi_v, C_{\Psi_{s_D}})$ one has

$$\int_{U_D(F_v) \backslash G^1_D(F_v)} L(g v_0^0) \omega_{s_D, \psi}(g) \phi(r_0 \otimes \varepsilon) f^*(g, s - 1/2) dg = L(\Pi_v, \chi_{E_v/F_v}, \rho, s) L(v_0^0).$$

When $v \notin S_D$ one has $G^1_D \simeq Sp(4)$. Hence, the proposition is a special case of the main theorem in [PS-R] for $n = 2$. As a corollary we obtain the decomposition

$$Z(\varphi, \phi, f, s) = L^1(\Pi, \chi_{E/F}, \rho, s) d_{\Omega}(\varphi, \phi, f, s),$$

where

$$d_{\Omega}(\varphi, \phi, f, s) = \int_{U_D(\mathbb{A}) \backslash G^1_D(\mathbb{A})} F_{s_D}(g \varphi) \omega_{s_D}(g) (r_0 \otimes \varepsilon) f^*(g, s - 1/2) dg.$$

The integral on the right hand side converges for $\text{Re}(s)$ sufficiently large. Indeed, by Iwasawa decomposition its convergence is equivalent to the convergence of

$$J(\varphi, \phi, s) = \int_{M^1_D(\mathbb{A})} F_{s_D}(m \varphi) \phi(m r_0) |N_{D/F}(m)|^s \chi_{E/F}(N_{D/F}(m)) dm.$$
13.6. The ramified factor. It remains to show that for any \( s_0 \in \mathbb{C} \) there exists
\[
\varphi \in \Pi, \quad \phi \in S(V_D \otimes X_D)(F_\Omega), \quad f \in \text{Ind}_{G_D(\Lambda_\Omega)}^{G_D(\Lambda_\Omega)} |\text{Nm}_{D/F}(s-1/2)
\]
such that
\[
d_\Omega(\varphi, \phi, f) = \int_{\nu_D(\Lambda_\Omega) \backslash G_D(\Lambda_\Omega)} F_{s_D}(g \varphi) \omega_{s_D}(g)(r_0 \otimes \varepsilon)f^*(g, s - 1/2) dg
\]
is holomorphic and does not vanish in a neighborhood of \( s_0 \).

Define as above
\[
J(\varphi, \phi, s) = \int_{M^1_D(\Lambda_\Omega)} F_{s_D} (m \varphi) \phi(m r_0) |\text{Nm}_{D/F}(m)|^s \chi_{E/F}(\text{Nm}_{D/F}(m)) dm.
\]
Suppose \( \varphi \) is such that \( F_{s_D}(\varphi) \neq 0 \).

It is possible to find a Schwarz function \( \phi \) whose support is small enough to ensure the holomorphicity and non-vanishing of \( J(\varphi, \phi, s) \) around \( s = s_0 \). Then,
\[
d_\Omega(\varphi, f, \phi, s) = \int_{K_\Omega} J(k \varphi, k \phi, s) f(k) dk.
\]

Choose now a standard section \( f \) whose restriction to \( K \) has a support which is small enough to ensure the non-vanishing of \( d_\Omega(\varphi, f, \phi, s) \) around \( s = s_0 \).

14. The nearly equivalence classes

We shall use the results of the previous section to show that the constructed set of representations contains, together with every cuspidal representation, its full nearly equivalence class of cuspidal representations.

**Theorem 14.1.** Let \( \Pi \) be a cuspidal irreducible representation of \( G_D(\mathbb{A}) \) that is nearly equivalent to \( \Pi^0_\eta \). Then, there exists \( S \) such that \( \Pi = \theta^D_E(\tau^S_\eta) \).

**Proof.** We start with a cuspidal representation \( \Pi \) nearly equivalent to \( \Pi^0_\eta \). There is a finite set \( \Omega \) such that
\[
L^\Omega(\Pi, \chi_{E/F}, \rho, s) = L^\Omega(\Pi^0_\eta, \chi_{E/F}, \rho, s)
\]
and hence has a pole at \( s = 2 \). Moreover, by Proposition 12.2 one has \( \hat{F}(\Pi) = \{ E \} \). Hence, by Corollary 13.5 \( \theta^D_E(\Pi) \) is a non-zero irreducible representation of \( H_D(\mathbb{A}) \) which is nearly equivalent to \( \tau^S_\eta \). By Proposition 10.1 one has \( \theta^D_E(\Pi) = \tau^S_{S'} \) for some \( S \) with \( m(\eta, S) = 1 \). Hence \( \Pi \simeq \Pi^S_\eta \). From Proposition 12.3 it now follows that \( \Pi = \theta^D_E(\tau^S_\eta) \). \( \square \)

15. Global theta correspondence from \( G_K \) to \( H_K \)

15.1. The pairs \((G_K, H_K)\) and \((G^0_K, H^0_K)\). Denote by \( \mathbb{A}_K \) the ring of adeles of the field \( K \). The automorphic realization of the global Weil representation
\[
(\omega_{s_K}, R_K(\mathbb{A}_K), S(V_K \otimes X_K(\mathbb{A}_K))
\]
is given by
\[
\theta(\phi)(h, g) = \sum_{x \in V_K \otimes X_K(K)} \omega_{s_K, \psi_K}(h, g) \phi(x).
\]
For a cuspidal representation $\pi$ of $G_K(\mathbb{A})$ define its theta lift to be the space spanned by the functions of the form
\[ \theta_L(\phi \otimes \varphi)(h) = \int_{G^1_K(\mathbb{A}) \backslash G^1_K(\mathbb{A}_K)} \theta(\phi)(h, g_1 g) \varphi(g_1 g) \, dg, \]
where $\lambda(g) = \lambda(h), \phi \in \omega_{K,S_K}, \varphi \in \pi$.

This is a well-known lift. Below we summarize its properties.

**Proposition 15.1.** Let $\pi$ be an irreducible cuspidal representation of $G_K(\mathbb{A})$. The following statements are equivalent.

1. $\theta_L(\pi) \neq 0$.
2. $L(\pi, Ad \otimes \chi_{L/K}, s)$ has a pole at $s = 1$.
3. $\theta_L(\pi) = \tau^\theta_{\eta_L}$ for some character $\eta_L$.
4. $\pi = \pi(\eta_L)$ is a dihedral cuspidal representation for some character $\eta_L : L^X \backslash I_L \rightarrow \mathbb{C}^\times, \eta_L \neq s(\eta_L), s \in \text{Aut}(L/K)$.

**Remark 15.2.**

1. If $E = K$ then $V_K$ is a split quadratic space, $\chi_{L/K} = 1$. The $L$-function $L(\pi, Ad, s)$ is entire and hence $\theta_L(\pi) = 0$ for any cuspidal representation $\pi$ of $G_K(\mathbb{A})$.
2. The automorphic realization of the representations $\tau^\theta_{\eta_L}$ of $H_K(\mathbb{A}_K)$ is given as in [10,3] by
\[ E_{\eta_L} : \text{Ind}_{H_K^1}^{H_K} \eta_L \rightarrow L^2(Z_H(\mathbb{A})H_K(\mathbb{A}) \backslash H_K(\mathbb{A}_K)), \]
\[ E_{\eta_L}(\phi_K)(h) = \sum_{H_K^1(K) \backslash H_K(K)} \phi_K(\gamma h) = \sum_{s \in \text{Aut}(L/K)} \phi_K(sh). \]

Let $\pi$ be an irreducible cuspidal representation of $G_K(\mathbb{A}_K)$. Consider a space of automorphic functions on $G^0_K(\mathbb{A}_K)$ obtained by the restriction of functions in the space of $\pi$. This space decomposes as direct sum of nearly equivalent representations of $G^0_K(\mathbb{A}_K)$, that constitute an automorphic packet. Moreover, any automorphic packet on $G^0_K(\mathbb{A}_K)$ arises in this way.

16. Global See-Saw Identity

**Proposition 16.1.** Let $\tau$ and $\pi$ be two unitary cuspidal representations of $H_D(\mathbb{A})$ and $G_K(\mathbb{A}_K)$ respectively. Assume that the central character of $\tau$ is trivial and the central character of $\pi$ has trivial restriction to $\mathbb{I}_F$. Then, there is an equality of the Petersson inner products
\[ (g_D^0(\phi \otimes f_\tau), f_D)_{G^0_K} = (\theta_L(\phi \otimes f_\tau), f_D)_{H_D}. \]

**Proof.** In the course of this proof denote $G^0_K = G^0_K \cap G^1_D$. In particular
\[ G_{1,+}^0(F_v) = \left\{ g \in G_K(K_v) : \det(g) \in F_v \right\} \quad v \in S_D \]
\[ G_{0,+}^0(F_v) = \left\{ g \in G_K(K_v) : \det(g) \in \text{Nm}_{E/F}(E_v) \right\} \quad v \notin S_D. \]

Let
\[ C = (\mathbb{A}_K^\times)^2 \text{Nm}(E^\times) \backslash (\Pi_{v \in S_D} F_v) \text{Nm}_{E/F}(\mathbb{A}_E^S). \]
The similitude characters of the groups $G^0_{1,K}(\mathbb{A})$ and $H_D(\mathbb{A})$ induce isomorphisms
\[ Z^0_K(\mathbb{A})G^1_K(\mathbb{A})G^0_{1,+}(F) \backslash G^0_{1,+}(\mathbb{A}) \simeq C, \quad Z_D(\mathbb{A})H^1_D(\mathbb{A})H_D(F) \backslash H_D(\mathbb{A}) \simeq C. \]

Fix the splitting maps $\mathcal{C} \rightarrow G^0_{1,+}(\mathbb{A})$ and $\mathcal{C} \rightarrow H_D(\mathbb{A})$ and denote them by $c \mapsto g_c$ and $c \mapsto h_c$ respectively.
To obtain the global see-saw duality we write for $f_\tau, f_\pi \in \pi$

$$(\theta^D_D(\phi \otimes f_\tau), f_\pi)_{G_K^0} = \int_{Z_K(\mathbb{A})G_K(F) \backslash G_K(\mathbb{A})} \theta^D_D(\phi \otimes f_\tau)(g)f_\pi(g)dg = \int_{Z_K(\mathbb{A})G_K^0(F) \backslash G_K(\mathbb{A})} \theta^D_D(\phi \otimes f_\tau)(g)f_\pi(g)dg =$$

\[ \int_{Z_D(\mathbb{A})H_D(F) \backslash H_D(\mathbb{A})} \theta(\phi \otimes f_\pi)(h)f_\tau(h)dh = (\theta_L(\phi,f_\pi),f_\tau)_{H_D}. \]

17. The main global theorem

Let $K$ be a quadratic algebra and $D$ be a quaternion algebra such that $S_D \subset S_K$. Define the period integral

$$P_{D,K} : A(Z_{H_D}(\mathbb{A})G_D(F) \backslash G_D(\mathbb{A})) \otimes \mathcal{A}_{cusp}(Z_K(\mathbb{A})G_K(F) \backslash G_K(\mathbb{A})) \to \mathbb{C}$$

by

$$P_{D,K}(f,\varphi) = \int_{Z_K(\mathbb{A})G_K^0(F) \backslash G_K(\mathbb{A})} f(g)\varphi(g)dg.$$

The convergence of this period follows from the cuspidality of $\varphi$.

We investigate the non-vanishing of $P_{D,K}$ on the representation $\Pi \boxtimes \pi$ whenever $\Pi \in A^D_{E,\eta}$ and $\pi$ is a cuspidal representation of $G_K(\mathbb{A})$ whose central character has trivial restriction to $I_F$.

**Theorem 17.1.** Let $\Pi \in A^D_{E,\eta}$ be an automorphic representation and let $\pi$ be an irreducible cuspidal representations of $G_K(\mathbb{A})$.

1. If $K = E$ then $P_{D,K}$ vanishes on $\Pi \boxtimes \pi$.
2. If $K \neq E$ then $P_{D,K}$ vanishes on $\Pi \boxtimes \pi$ if and only if $\text{Hom}_{G_K(\mathbb{A})}(\Pi,\pi) = 0$.

**Proof.** (1) The first statement follows immediately from the see-saw identity and Remark 15.2.

(2) Assume $K \neq E$. If $P_{D,K} \neq 0$ then obviously $\text{Hom}_{G_K(\mathbb{A})}(\Pi^S_{\eta} \boxtimes \pi,\mathbb{C}) = 0$. Let us prove the other direction. If $\text{Hom}_{G_K(\mathbb{A})}(\Pi^S_{\eta} \boxtimes \pi,\mathbb{C}) \neq 0$ then for any finite place $v$ the representation $\pi_v$ is dihedral representation of $G_K(F_v)$ with respect to $L_v$ and some character. In particular, $\pi_v \simeq \pi_v \otimes \chi_{L_v/K_v}$ for any finite $v$. Equivalently, $\pi$ and $\pi \otimes \chi_{L/K}$ are nearly equivalent representations and hence by strong multiplicity one are isomorphic. So, $\pi = \pi(\eta_L)$ is a global dihedral representation of $G_K(\mathbb{A})$. By the main local theorem, $\eta_{L_v}$ matches $\eta_v$ for any finite $v \in pI(F)$. Hence by Lemma 9.3 $\eta_L$ matches $\eta$. Without loss of generality we may assume that $\eta|_E = \eta$.

By the see-saw identity, the non-vanishing of $P_{D,K}$ on $\Pi \boxtimes \pi$ is equivalent to the non-vanishing of the integral

$$\int_{Z_D(\mathbb{A})H_D(F) \backslash H_D(\mathbb{A})} E_{\eta_L}(\phi_K)(h)E_{\eta}(\phi)(h)dh,$$

for some pure tensor products vectors $\phi_K \in \tau^0_{\eta_L} \subset \text{Ind}_{H_K(\mathbb{A})}^{G_K(\mathbb{A})} \eta_L$ and $\phi \in \tau^S_{\eta} \subset \text{Ind}_{H_D(\mathbb{A})}^{G_K(\mathbb{A})} \eta$. One has

$$\int_{Z_D(\mathbb{A})H_D(F) \backslash H_D(\mathbb{A})} E_{\eta_L}(\phi_K)(h)E_{\eta}(\phi)(h)dh =$$
Let $\lambda$ be Arthur parameters of $SO(V) \times SO(U)$, and $\Phi_1 \times \Phi_2$ be associated Langlands parameters. Let $\Pi \boxtimes \pi$ be a cuspidal automorphic representation of $SO(V') \times SO(U')$ in the packet $\mathcal{A}_\pi \times \mathcal{A}_\pi$ such that $\text{Hom}_{SO(U')}(\Pi, \pi) \neq 0$. The non-vanishing of the period $P_{V',U'}$ on $\Pi \boxtimes \pi$ is equivalent to the non-vanishing of

$$L(\Phi_1 \times \Phi_2', s) \bigg|_{s=1/2} \quad \text{and} \quad L(\Phi_1, Ad, s + 1/2) \bigg|_{s=1/2}$$
The conjecture holds for Saito-Kurokawa packets as shown in [GG]. Let us show that it also holds for the packets of the type \( \theta_{10} \).

The L-parameter associated to \( \Psi_{E,\eta} \) equals \( \Phi_{E,\eta}|^{-1/2} \otimes \Phi_{E,\eta}|^{-1/2} \). The parameter \( \Psi_2 \) is associated to a representation \( \pi \) of \( G_K(\mathbb{A}) \) whose central character has trivial restriction to \( \mathbb{I}_E \).

Thus, \( (18.3) \) equals

\[
L(\Phi_{E,\eta} \times \Phi_{E,\eta}^\vee, s + 1/2) \bigl| \frac{L(\Phi_{E,\eta} \times \Phi_{E,\eta}^\vee, s - 1/2)}{L(\Phi_{E,\eta} \otimes \Phi_{E,\eta}, Ad, s + 1/2)} \bigr| \frac{L(\Phi_{E,\eta}, Ad, s + 3/2)}{L(\Phi_{E,\eta}, Ad, s - 1/2)} \bigl|_{s=1/2}.
\]

Assume that this expression does not vanish. The denominator has a simple pole at \( s = 1/2 \). Hence the numerator also must have a pole. The numerator is the product of two triple L-functions whose analytic behavior was studied in [I]. In particular, if \( L(\pi(\eta) \times \pi, s) \) has a pole at \( s = 1 \) then \( K \neq E \) and \( \pi = \pi(\eta_L) \) with \( \eta_L \) matches \( \eta \). In this case \( L(\pi(\eta) \times \pi, s) \) also has a pole at \( s = 0 \).

Conversely, assume \( P_{D,K}(\Pi \boxtimes \pi) \neq 0 \). Then \( K \neq E \) and \( \pi = \pi(\eta_L) \), where \( \eta_L \) matches \( \eta \). In this case

\[
L(\pi(\eta) \boxtimes \pi(\eta_L), s) = \zeta_E(s)L_E(\eta^{-1}\sigma(\eta), s)L_L(\eta_L^{-1}\sigma(\eta_L), s)
\]

and hence has a pole at \( s = 1 \) and at \( s = 0 \). Thus the \( (18.3) \) does not vanish.

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