D-Walls and Junctions in Supersymmetric Gluodynamics

in the Large $N$ Limit

Suggest the Existence of Heavy Hadrons

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Abstract

A number of arguments exists that the “minimal” BPS wall width in large $N$ supersymmetric gluodynamics vanishes as $1/N$. There is a certain tension between this assertion and the fact that the mesons coupled to $\lambda\lambda$ have masses $\mathcal{O}(N^0)$. To reconcile these facts we argue that there should exist additional soliton-like states with masses scaling as $N$. The BPS walls must be “made” predominantly of these heavy states which are coupled to $\lambda\lambda$ stronger than the conventional mesons. The tension of the BPS wall junction scales as $N^2$, which serves as an additional argument in favor of the $1/N$ scaling of the wall width. The heavy states can be thought of as solitons of the corresponding closed string theory. They are related to certain fivebranes in the M-theory construction. We study the issue of the wall width in toy models which capture some features of supersymmetric gluodynamics. We speculate that the special hadrons with mass scaling as $N$ should also exist in the large $N$ limit of non-supersymmetric gluodynamics.
1 Introduction

Supersymmetric gluodynamics is the simplest non-Abelian supersymmetric (SUSY) gauge theory which turns out hardest to handle. The only global symmetry in this model (besides SUSY itself) is $\mathbb{Z}_{2N}$ discrete symmetry which is spontaneously broken by the gluino condensate down to $\mathbb{Z}_2$ (the gauge group here is assumed to be $\text{SU}(N)$). The lack of a continuous moduli space of vacua with a rich set of global symmetries prevents one from studying the theory in the spirit of Seiberg [1].

Possible existence of the BPS domain walls in SUSY gluodynamics [2, 3, 4] adds a significant, albeit indirect, information to a rather fragmentary knowledge of the strong coupling dynamics of this theory. What is the most surprising property of the BPS walls?

It was argued in [5] that the width $l$ of the “minimal” wall (i.e. the wall connecting the neighboring vacua) must scale as $1/N$ in the large $N$ limit. By the width we mean the transverse dimension saturating the wall tension. At first sight it is natural to expect this dimension to be of order 1 rather than $1/N$. Indeed, the order parameter $\lambda\lambda$ is a source of particles with mass $O(1)$, the “gluinoballs.” This means that at large separations from the wall, $\Delta z \gg \Lambda^{-1}$, the wall tails should fall off as $\exp(-\Lambda\Delta z)$, implying that the wall width is $O(1)$. (Here and below the scale parameter is denoted by $\Lambda$.)

The conclusion that $l \sim 1/N$ was further supported in Ref. [5] by constructing the BPS walls in a certain model for SUSY gluodynamics. One may wonder to which extent this construction is model-independent. We first summarize evidence in favor of $l \sim 1/N$. A new argument comes from a rather unexpected side. It was pointed out recently that several walls can join together and form a wall junction configuration preserving $1/4$ of the original SUSY [6, 7] (see Fig. 1). The models considered in the literature previously are those of the Wess-Zumino type. We will assume that the BPS junctions with $N$ minimal walls exist in $\text{SU}(N)$ supersymmetric gluodynamics. Then, the $N$ dependence of the junction tension comes out natural if $l$ is assumed to be $O(1/N)$; on the contrary, it is very hard to get the proper junction tension under any other scaling law for $l$.

Given that the BPS wall width $l \sim 1/N$ one can raise a legitimate question how this can possibly be explained in terms of the physical excitations of the theory. We will argue that there should exist certain distinguished (soliton-like) excitations in the theory with mass $m \sim O(N)$. These excitations are coupled to $\lambda\lambda$ stronger that the conventional mesons, so that the wall is “made” predominantly of these heavy states. Some arguments in favor of the existence of such states can be found within the D-brane construction of SUSY gluodynamics due to Witten [8], which also leads to the conclusion, that $l \sim 1/N$. We believe that this conclusion is general – as long as the minimal wall is BPS saturated, its width has to be $O(N^{-1})$, which entails, in turn, that matter of which it is (predominantly) built has mass $O(N)$. Various simple toy models illustrate this assertion.

If one introduces the gluino mass, supersymmetry is broken. At small masses
the breaking is small – we are pretty close to the supersymmetric picture. Once the
gluino mass becomes larger than Λ, the gluinos decouple, and we find ourselves in
non-supersymmetric Yang-Mills theory. One can try to extrapolate in the gluino
mass, assuming that there is no phase transition. Speculating along these lines,
we have to conclude that the $M \sim N$ hadrons of a special nature must exist in
non-supersymmetric gluodynamics too. This conclusion can also be supported by
the D-brane construction in non-supersymmetric Yang-Mills theory where domain
walls can be continuously extrapolated to a certain wrapped D-brane.

The paper is organized as follows. In Sec. 2 we briefly summarize what is known
about the minimal BPS wall width in SUSY gluodynamics. The $N$ counting for the
BPS junction is discussed. In Sec. 3 we give a field-theoretic argument in favor of
$M \propto N$ (here $M$ is the mass of the quantum of which the wall is “built”). In Sec. 4
we identify possible (soliton-like) candidates for the “building blocks” of the minimal
walls within the D-brane construction. In Sec. 5 we introduce a (SUSY violating)
gluino mass, and perform the extrapolation to large masses. This gives us a hint of
the existence of the $M \sim N$ hadrons in the large $N$ limit of non-supersymmetric
 gluodynamics. We ameliorate these arguments by an evidence coming from the
D-brane construction. Finally, Sec. 6 is devoted to toy models. The BPS wall
junctions are studied in this section as well.

2 Why the minimal wall width must scale as $1/N$

There are at least three different arguments supporting this conclusion. First, as
is well-known, a natural behavior of the volume energy density inside the wall is
\[ \varepsilon \sim N^2 \] (since there are \( N^2 \) degrees of freedom in the theory). Then, the fact that the BPS wall tension is \( \sim N \) \[ \varepsilon \] implies that \( l \sim 1/N \) \[ \varepsilon \]. This conclusion is supported by the solutions found in \[ \varepsilon \] and \[ \varepsilon \].

The second argument is based on the D-brane construction \[ \varepsilon \], which also implies that \( l \sim 1/N \) (see Sec. 4 for further details).

Here we will focus on a new argument based on the wall junction \[ \varepsilon \]. We will assume that the BPS saturated junction of \( N \) minimal walls exists in supersymmetric gluodynamics. Then, as was shown in Ref. \[ \varepsilon \], the junction tension \( T_{\text{Junction}} \) is

\[
T_{\text{Junction}} \propto \oint a_k \, dx^k \propto N^2. \tag{1}
\]

Here \( a_\mu \) is the axial current of gluinos (which scales as \( N^2 \)), and the integration contour runs over the large circle in the plane perpendicular to the wall junction. Let us try to understand geometrically how this behavior could occur.

Let us examine the “spokes” and the “hub” in Fig. 1a. If the width of the spokes is \( 1/N \), and there are \( N \) of them, then the diameter of the hub is \( \mathcal{O}(N^0) \). The area of the hub is \( \mathcal{O}(N^0) \) too. Given that the volume energy density scales as \( N^2 \), we naturally arrive at \( T_{\text{Junction}} \propto N^2 \).

At the same time, if the width of the spokes is \( \mathcal{O}(N^0) \), as was naively believed before \[ \varepsilon \], the diameter of the hub is \( \mathcal{O}(N) \) while the area of the hub is \( \mathcal{O}(N^2) \). To explain Eq. (1) one must assume that the volume energy density inside the hub differs from the vacuum energy density by \( \mathcal{O}(N^0) \), which is extremely counterintuitive.

Although, the arguments listed above are qualitative and present only circumstantial evidence, taken together they seem compelling. We will accept that the BPS wall width \( l \sim 1/N \) as a starting point. The conclusion of the existence of heavy soliton-like hadrons with mass scaling as \( \Lambda N \) will ensue \[ \varepsilon \].

3 Suggestive Arguments from Field-Theoretic Consideration

Supersymmetric gluodynamics is a strongly coupled theory. This means that we have no reliable tools for addressing the problem we are interested in at the quantitative level. However, we can consider the issue at the qualitative level if, instead of SUSY gluodynamics, we will deal with a simpler theory belonging to the same universality class. The most natural choice seems to be SQCD with \( N_f = N - 1 \). If the matter mass term is small, \( m \to 0 \), this is a weakly coupled theory (in the Higgs phase). It explicitly exhibits the \( \mathbb{Z}_N \) structure of the vacuum state. The minimal BPS walls can be treated quasiclassically. We will show that their width shrinks with \( N \) as \( 1/N \), while the mass of the quantum they are built of grows linearly with \( N \).

\[ ^1 \text{More accurately, one should speak of the difference between the volume energy density inside the wall and in the vacuum. The latter vanishes, however, in supersymmetric theories.} \]
At the same time, since all matter fields are in the fundamental representation, it seems reasonable to assume that there is no phase transition that would separate the Higgs phase at small $m$ from the strongly coupled phase at large $m$. At large $m$ the matter fields decouple, and we return to SUSY gluodynamics. It is natural to think that the interpolation is smooth in this transition. The field with mass $\propto N$ comprising the wall at small coupling goes into a gluon/gluino “soliton” with mass $M \sim NA$ in SUSY gluodynamics.

In more detail, the construction is as follows. One introduces $N_f$ chiral superfields $Q_f$ ($f = 1, 2, ..., N_f$) which are fundamentals of SU($N$) and $N_f$ chiral superfields $\tilde{Q}^g$ ($g = 1, 2, ..., N_f$) which are antifundamentals. Here $N_f = N - 1$. We then introduce the tree level superpotential

$$W_{\text{tree}} = m \sum_{f=1}^{N_f} M_f^f,$$  \hspace{1cm} (2)

where

$$M_f^g \equiv Q_f \tilde{Q}^g$$ \hspace{1cm} (3)

are the moduli, and the mass term $m$ is assumed to be very small and diagonal. Then the theory is fully Higgsed \[12\]. A superpotential is generated on the moduli space (via instantons) once the heavy fields are integrated out,

$$W_{\text{inst}} = \frac{\tilde{\Lambda}^{2N+1}}{\det M},$$ \hspace{1cm} (4)

where $\tilde{\Lambda}$ is a scale parameter of SQCD (it will drop out in what follows).

If the mass term is chosen as in Eq. (2), the vacuum expectation values of the moduli are diagonal too,

$$\langle M_f^g \rangle = v^2 \delta_{fg}.$$ \hspace{1cm} (5)

There are $N$ solutions for $v^2$,

$$v^2 = |v^2| \exp \left( \frac{2\pi i k}{N} \right), \hspace{1cm} k = 1, 2, ..., N.$$

The absolute value of $v^2$ is very large in the limit of small $m$. This explains why the vector bosons and their superpartners are heavy and can be integrated out. Equation (3) implies that although there are $N_f^2$ light moduli fields, the domain walls occur only for the field $\sum_{f=1}^{N_f} M_f^f$. All other $N_f^2 - 1$ moduli fields, say $M_1^1$ or $M_1^2 - M_2^2$, etc., oscillate near zero; they do not experience the wall-type transition.

Next, we combine Eqs. (2) and (4), find the stationary points of the superpotential to determine the vacua, and then find the masses of all $N_f^2$ fields on the moduli space. The mass of the quantum for which $\sum_{f=1}^{N_f} M_f^f$ is the interpolating field is $Nm$; all other mass eigenvalues do not contain the $N$ factor.
The minimal BPS walls in this model were considered in Ref. [13]. Although the $N$ dependence of the wall width was not explicitly discussed, one can infer from these works that the wall is built of the field $\sum_{f=1}^{N_f} M_f$, and its width is $(mN)^{-1}$. Although other $N_f^2 - 1$ “genuinely light” moduli do couple to the field $\sum_{f=1}^{N_f} M_f$ through Eq. (4), they are not excited on the wall. This explains why there is no wall broadening. The same phenomenon will be discussed in details in Sec. 6.

To avoid confusion, a remark is in order here concerning the relation between the argument above and the results of Ref. [13]. Smilga and collaborators did find a phase transition in the matter mass parameter $m$. This is explained by the fact that they were forced to deal with the Taylor-Veneziano-Yankielowicz model on the large $m$ side, rather than with genuine SQCD. In the description of walls this model is not adequate (in fact, it is inadequate both in the small and in the large $m$ domains). If $m$ is small, and one retains only moduli, the corresponding effective theory is an exact low-energy expansion, in the Wilsonian sense. As such, it describes the dynamics of the moduli in full. Thus, when $m$ is small, (more exactly, $Nm$ has to be small) and the wall profile is broad, it is fully legitimate to use the effective theory of the moduli to exhaustively describe the walls. As $Nm$ approaches $\Lambda$, the description of the walls in the effective theory of moduli becomes wrong. At this point we must roll over from the theory of moduli to full SQCD. Then, there will be no phase transition in $m$. Thus, the observation of this phase transition in Ref. [13] is the artifact of the approximation used.

4 Suggestive Arguments from D-branes

The aim of this section is to use the D-brane construction of the BPS domain wall [8] to study the origin of the states with the mass of order $O(N)$. These states should be responsible for the $1/N$ scaling of the domain wall width.

In the large $N$ limit SUSY gluodynamics is expected to be described by a certain non-critical closed string theory (let us call it tentatively the “closed QCD string”). This theory is hard to formulate precisely. However, it is believed to lie in the universality class of the theory obtained via D-brane construction [8]. Excitations of the closed QCD string should give rise to the colorless bound state spectrum of SUSY gluodynamics. These are gluino-gluino, gluino-gluon and pure gluonic bound states (including higher radial excitations). All these states have masses of order $O(1)$. Moreover, they couple to each other with couplings suppressed by powers of $1/N$. This exhausts the perturbative part of the closed QCD string theory. In addition, there should exist a nonperturbative sector of the closed QCD string theory. In analogy with type IIA, B strings, one might expect to find point-like or extended objects in the nonperturbative sector [8]. These objects have little to do with mesons and glueballs – they represent independent degrees of freedom of the theory.

2 In the given context the reference to point-like objects means that their size is $1/N$. 

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As a particular consequence of this fact, the BPS domain walls in SUSY gluodynamics manifest themselves as certain D-branes, on which the open QCD strings can end [8]. Notice, that (naively) the open strings are not expected in SUSY Yang-Mills (SYM) theory, since in this theory there are no fundamentals on which the open strings could end. Nevertheless, the open strings appear at nonperturbative level, precisely as it happens in type IIA, B critical closed string theories. As was mentioned, these open strings terminate on the D-branes, i.e., on the BPS domain walls of the underlying SYM theory [8] (see Fig. 2). For these reasons, in what follows let us call these BPS walls D-walls. It seems clear that one could not find a D-wall in the theory where only mesons with masses \( O(1) \) are included.

The effective meson theory for the large \( N \) gluodynamics is believed to be the theory of a closed QCD string (as opposed to the open string theory of conventional QCD with quarks). What kind of new nonperturbative point-like objects exist in the closed string theory? A point-like soliton of the closed string theory with the mass of order \( O(N) \) reminds a zero-brane. Unfortunately, the question whether the QCD string has zero-branes is hard to study in the fundamental theory (since no consistent string theory is known for this). What we could study instead is the theory which is in the same universality class, and, which can be realized in a particular D-brane construction [8]. We will argue below that the brane construction calls for the inclusion of the states with mass \( O(N) \) in order to be able to describe the BPS domain walls.

Consider two parallel BPS walls at a certain distance from each other. One can stretch between them one, two, and so on, open QCD strings. At the point of each string-wall junction one observes a lump of energy which corresponds to an object similar to a quark in the fundamental representation. This lump of energy is localized on the wall. When the number of the QCD strings stretched between the walls becomes equal to \( N \), the lumps can fuse forming an object which is color singlet, has mass \( m \sim N \) and reminds a baryon. Since it is color singlet, there are no reasons why it should be localized on the wall – it seems likely that it will propagate in the bulk.

Thus, we come to a conclusion that there might exist some new states in SYM theory which are neither mesons nor glueballs; nevertheless, they couple to the BPS domain wall.

In the field-theoretic language one may recall that the baryons in conventional QCD emerge as the Skyrme solitons in the effective meson theory [14]. Their masses scale as \( O(N) \) in the large \( N \) limit, although the masses of the original mesons are \( O(1) \) [15].

To reiterate the argument at a slightly more quantitative level we first briefly review the construction of Ref. [8]. Following [16, 17, 8] we start with type IIA string theory. The brane setup is as follows. There is one Neveu-Schwarz (NS) fivebrane which spans the worldvolume \((x_0, x_1, x_2, x_3, x_4, x_5)\) and which lives at the point \(x_6 = x_7 = x_8 = x_9 = 0\). Another fivebrane (called NS') with the worldvolume \((x_0, x_1, x_2, x_3, x_7, x_8)\) is separated from the NS fivebrane at some distance \(S_0\) along
There are $N$ coincident D4-branes suspended between the fivebranes (see Fig. 3). The worldvolume of these branes is in the $(x_0, x_1, x_2, x_3, x_6)$ part of space-time. One defines $v = x_4 + ix_5$, $w = x_7 + ix_8$. This configuration is argued to describe $\mathcal{N} = 1$ SYM theory without chiral multiplets at low energies. In the infrared limit this field theory lives in the $(x_0, x_1, x_3)$ subspace. The gauge coupling of this model is related to the string coupling constant, $g_s$, as follows: $g_{\text{YM}}^2 \sim g_s l_s / S_0$, where $l_s$ is the type IIA string length. The model is studied by elevating the type IIA construction to M-theory. Thus, one takes $g_s$ to be large so that the eleventh dimension opens up in the form of a circle $S^1$ of radius $R$.

In this case, the D4-branes discussed above are just the M-theory fivebranes wrapped on the circle $S^1$. Thus, all the branes in Fig. 3 are the M-theory fivebranes. All these fivebranes can be described as $\mathbb{R}^4 \times \Sigma$, where $\mathbb{R}^4$ stands for space of SYM theory and $\Sigma$ is a complex Riemann surface in a space with the following complex variables: $v$, $w$ and $t \equiv \exp(-s) = \exp(-R^{-1}(x_6 + ix_{10}))$. From the D-brane construction

$$R = g_s l_s .$$

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\footnote{Below, we will see that in the particular case at hand the coupling which effectively defines the transition to M-theory is $Ng_s$ instead of $g_s$. This is related to a membrane which is wrapped $N$ times around the compact eleventh dimension.}
described above one finds that the curve $\Sigma$ is defined by the equations
\[ w = \zeta v^{-1} \]
\[ w^N = \zeta^N t^{-1}, \]  \hspace{1cm} (7)
where $\zeta$ is some complex constant. The discrete $\mathbb{Z}_N$ transformations of SUSY gluodynamics are realized as
\[ w \rightarrow \exp \left( \frac{i2\pi}{N} \right) w, \quad t \rightarrow t, \quad v \rightarrow v. \]
The QCD strings discussed above are identified in this picture with the boundary of the M-theory membrane [8] which intersects the corresponding fivebrane [18]. The M-theory membrane lives in $\mathbb{R}^4 \times Y \times \mathbb{R}$, where $Y$ is a complex three-fold which contains $\Sigma$. One can consider a membrane which is a product of a onebrane in $\mathbb{R}^4$ and a onebrane in $Y$. This membrane would look as a string to the four-dimensional observer living in $\mathbb{R}^4$. Using this construction one identifies both the type IIA and QCD strings (more precisely the strings which are in the same universality class). The type IIA string, from the M-theory standpoint, would correspond to a membrane wrapped on the eleventh-dimensional circle $S^1$. This gives just conventional type IIA string in the double dimensional reduction of M-theory [19]. The tension of this string in terms of the M-theory variables is
\[ T_{IIA} \sim \frac{R}{l_M^3}, \]  \hspace{1cm} (8)
where, $l_M$ denotes the fundamental length of M-theory (inverse of the eleven-dimensional Planck scale).

Using the relations between the M-theory parameters and type IIA parameters, $R = g_s l_s$ and $l_M^3 = g_s l_s^3$, one finds that $T_{\text{IIA}} \sim 1/l_s^2$, as is expected for the fundamental type IIA string.

Let us now turn to the QCD strings; their existence is more subtle to see. Consider an open curve $C$ in $Y$ parametrized by some parameter $0 \leq \sigma \leq 1$. One chooses $C$ in such a way that the endpoints of the curve are in $\Sigma$ defined by Eq. (7). Picking one particular $C$ we determine a string in space-time. If there are $N$ different curves $C_k \ (k = 0, ..., N - 1)$, they define $N$ strings. $C_k$’s can be parametrized as follows:

$$v \propto \exp(i2\pi \sigma/N)\exp(i2\pi k/N)$$

$$w = \zeta v^{-1}.$$  \hspace{1cm} (9)

As a result, in this construction the QCD string tension is

$$T_{\text{QCD}} \sim \frac{|\zeta|^{1/2}}{N l_M^3}.$$  \hspace{1cm} (10)

Since $\sum_{k=0}^{N-1} \exp(i2\pi k/N) = 0$, the boundaries of $C_k$ add up to zero. Thus, $N$ QCD strings can join to create a closed loop in the complex $v$ plane which, subsequently, can be contracted with no obstruction, to a point. Thus, the QCD strings determined above can annihilate in groups of $N$. [8]

Let us now turn to the issue of how the BPS domain walls of SYM theory can be seen in the D-brane construction. If the domain wall interpolates between $x_3 \to -\infty$ to $x_3 \to +\infty$, it can be described by a M-theory fivebrane which interpolates between the fivebranes used to set the vacuum states at two ends. Thus, one can construct a domain wall as a fivebrane with the worldvolume $\mathbb{R}^3 \times S$, where $\mathbb{R}^3$ is just the wall worldvolume $x_0, x_1, x_2$, and $S$ is a certain three-surface embedded in the M-theory space-time. When $x_3$ approaches $-\infty$, $S$ looks as $\mathbb{R} \times \Sigma$, with $\Sigma$ being defined by $w = \zeta v^{-1}, \ v^n = t$. When $x_3 \to +\infty, \ S = \mathbb{R} \times \Sigma'$, where $\Sigma'$ is defined by $w = \exp(i2\pi/N)\zeta v^{-1}, \ v^n = t$. Thus, the brane interpolates between the neighboring chirally asymmetric vacua of the model. The tension of this wall can be also calculated [8], namely,

$$T_D \sim \frac{R|\zeta|}{l_M^3}.$$  \hspace{1cm} (11)

The requirement for the wall defined by $\mathbb{R}^3 \times S$ to be a BPS state is equivalent to the condition that $S$ is a supersymmetric three-cycle [20]. Given these walls, it can be shown explicitly that the open QCD string which interpolates from $v = v_0$ to $v = v_0 \exp(i2\pi/N)$ can actually terminate on the domain wall [8]. As a consequence, there should exist some degrees of freedom on the wall, the open string endpoints, which transform in the fundamental representation of the gauge group. These states...
are not present in the Lagrangian, but appear at the nonperturbative level in the theory. Since QCD strings can annihilate in groups of $N$, this indicates that $N$ copies of fundamental representation can form a singlet “baryon” \[8\]. The mass of this state will scale as $N$.

Could the properties of these states be understood within SYM theory or in the brane construction given above? In field theory this would be a difficult task, since it requires solution at strong coupling. One might hope that the problem is easier in the brane construction. However, as we will argue below, the corresponding effective string coupling is large in the regime we deal with. Thus, the problem is equally complicated. The QCD string tension is given in Eq. (10). On the other hand, $T_{\text{QCD}} \sim \Lambda^2$. Using these two expressions one finds

$$|\zeta| \sim N^2 \Lambda^4 l_s^6.$$  \hfill (12)

Moreover, the domain wall tension (11) must be proportional to $N\Lambda^3$ [2]. Combining this assertion with (11) and (12), we obtain

$$R \sim \frac{1}{N\Lambda}.$$  \hfill (13)

Finally, using the relation (6) we get

$$Ng_s \sim \frac{1}{l_s \Lambda}.$$  \hfill (14)

First of all, we see that the string coupling scales as $g_s \sim 1/N$ in the large $N$ limit. Moreover, since $l_s^{-1} \gg \Lambda$, the combination $Ng_s \gg 1$. This is the effective coupling in the large $N$ limit. It is turns out to be large. Therefore, quantitative studies are not feasible \[4\] (for more detailed discussions of these and related issues see Ref. [21]). Nevertheless, we can draw certain qualitative conclusions.

The result most important for our purposes can be summarized as follows. The QCD strings, as well as type IIA strings, are described by the M-theory membranes. A fivebrane describes the domain wall of SYM theory. The domain wall is a BPS state. This is guaranteed by the fact that the three-surface $S$ in the domain wall construction is a supersymmetric three-cycle. As we mentioned above, the presence of this state in the theory is intrinsically related to the possibility of interpolation between distinct vacuum states of the theory. Therefore, if one is to describe the D-walls using some “order parameter effective Lagrangian approach” in the large $N$ limit, the heavy states should necessarily be included in consideration (despite

\[4\] Note that there is another reason why the brane construction cannot be used for quantitative studies. In addition to the D0-branes of type IIA, which decouple from SYM theory, there are Kaluza-Klein states of the membrane worldvolume fields which are wrapped $N$ times on $S^1$. The masses of these states scale as $1/(NR) \sim \Lambda$; they do not decouple from the SYM degrees of freedom \[8\]. That is why the model obtained in the D-brane construction is not SYM theory per se, but rather its close relative \[8\].

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the fact that they are very heavy and, naively, should decouple). In other words, all conventional mesons are perturbative excitations of the closed QCD string. If one writes down an effective Lagrangian including these fields only, one would not be able to describe the BPS domain walls. To get such walls we should necessarily include a field (or fields) with mass of order $\mathcal{O}(N)$. Such a model Lagrangian was introduced in [22], and used in [3] to study the BPS domain walls.

In Sec. 6 we discuss simple toy models engineered for the purpose of studying the problem of interaction of the D-walls and wall junctions with the light hadronic states. Before we proceed to these models, however, we will make a remark regarding non-supersymmetric Yang-Mills theory.

## 5 Large $N$ (Non-Supersymmetric) Gluodynamics

If supersymmetry is broken, the degeneracy of the vacuum states inherent to SUSY gluodynamics is gone. Gone with it are the (perfectly stable) walls. Recently it was argued, however [23, 24], that in the large $N$ limit there exist infinitely many quasistable “vacua” [the decay rate of the false vacua was argued [24] to be suppressed as $\exp(-\text{const} N^4)$]; the walls interpolating between them are quasistable. Their life time is exponentially large, $\tau \sim \exp(\text{const} N^4)$. One may ask how the width of such walls scales with $N$.

The arguments of Ref. [23] were based on the D-brane picture; similar arguments were presented in Sec. 4. In Ref. [24] a field-theoretic approach was exploited. One starts from supersymmetric gluodynamics, adds the gluino mass to break SUSY, and then continuously interpolates from the limit of small masses where reliable estimates are possible to the limit of large masses when gluinos decouple, and one finds oneself in non-supersymmetric gluodynamics. If no phase transition happens en route, then the qualitative dependences obtained in the small mass limit persist in non-supersymmetric gluodynamics.

Adopting the same approach and making the same assumption of no phase transition in the problem at hand, we arrive at the conclusion that in the large $N$ limit of non-supersymmetric gluodynamics the width of the (quasistable) domain walls scales as $l \sim (N\Lambda)^{-1}$. As in the supersymmetric case, this implies that the dominant degrees of freedom comprising the wall have masses $M \sim N\Lambda$. In other words, in the large $N$ limit there should exist some distinguished hadrons with mass scaling as $N$ in pure Yang-Mills theory.

One can supplement the field-theoretic argument above by D-brane considerations which point in the same direction: the wall width in (non-SUSY) gluodynamics scales as $1/N$. Let us briefly review this construction. One starts with type IIA string theory on $\mathbb{R}^4 \times S^1 \times \mathbb{R}^5$. To obtain the low-energy gauge theory one puts $N$ D4-branes on top of each other. The D4-brane worldvolume is taken to be $\mathbb{R}^4 \times S^1$. The boundary conditions for fermions on $S^1$ are chosen to be antiperiodic. As a

\[ \text{The reasoning for introducing a new field in [22] was based on symmetry arguments.} \]
result, SUSY is broken on the worldvolume and the low-energy theory is nothing but non-supersymmetric Yang-Mills theory with the gauge group $U(N)$.

As follows from the results of Refs. [25, 26, 27, 28], the large $N$ behavior of the $SU(N)$ part of this theory can be described by string theory on a certain background $X$. The topology of $X$ is $\mathbb{R}^4 \times D \times S^4$, where $D$ is a two-dimensional disc. The metric of $X$ can be found explicitly [28]. We just emphasize here the feature which is crucial for our discussion. The metric on $X$ depends on a certain parameter, let us call it $\eta$. A (non-supersymmetric) gauge theory is expected to emerge [28] in the limit $\eta \to 0$. On the other hand, if $\eta \gg 1$, one expects supergravity to be a good description of the string theory on $X$. Thus, in order to use supergravity results for studying the gauge theory, one has to assume that there is no phase transition in $\eta$. Given this assumption, it is possible to identify an object in string theory which corresponds to a domain wall separating a given pair of distinct vacua. This is a D6-brane which is wrapped on the $S^4$ factor of $X$ [23]. As a result, one can easily establish the large $N$ scaling of the domain wall tension. The D6-brane tension is proportional to $\sim 1/g_s l_s^7$. The string coupling $g_s$ scales as $1/N$. Hence, the domain wall tension should scale as $\sim N$. On the other hand, the volume energy density generically scales as $N^2$. In order to make the surface energy (tension) of the domain wall scale as $N$ it is necessary to accept that its width is proportional to $1/N$.

These arguments, combined with the field-theoretic arguments based on extrapolation from the supersymmetric limit, reinforce each other and indicate that there is no phase transition in the gluino mass. Indeed, in the D-brane consideration one assumes that there is no phase transition in the parameter $\eta$ which, in general, has nothing to do with the gluino mass.

We conclude that the scaling law for the non-supersymmetric walls is the same as in the supersymmetric case, $l \sim 1/N$. The consequence which immediately follows is the existence of the “solitonic glueballs” which “build” the wall and have mass of order $N$.

The question as to the nature of these hadrons remains open. As far as we know, the wall-based consideration presented above presents the first hint that such special hadrons may exist; so far they have never been discussed in the literature.

6 Modeling the $Z_N$ Vacua in the Large $N$ Limit

As we discussed above, the vacua in SUSY gluodynamics are defined by the vacuum expectation value (VEV) of the order parameter $\lambda \lambda$. The effective Lagrangian for this order parameter contains, generally speaking, an infinite number of massive fields. It is not known at present how to truncate self-consistently this Lagrangian. However, in the large $N$ limit certain simplifications are expected to happen. Namely, the couplings of all physical states in the effective Lagrangian are expected to be suppressed by powers of $1/N$. Thus, it seems that in the large $N$ limit one could concentrate on the part of the effective Lagrangian which includes
\(\lambda\lambda\) (and its SUSY partners) only.

In terms of the closed QCD string theory, this would correspond to a certain truncated perturbative approximation to the string spectrum. However, as was discussed in the previous sections, this is not enough for the description of the D-walls of the model. As we established above, one should also include in the effective Lagrangian certain states (with mass of the order \(O(N)\)) which have something to do with the \(\mathbb{Z}_N\) structure of the ground state. Let us study how this feature can actually be realized. Below we will work in the large \(N\) limit, to the leading order in \(1/N\). In fact, one appropriate toy model has been already discussed in Sec. 3. Here we dwell on some other toy models with the appropriate \(\mathbb{Z}_N\) structure.

### 6.1 A model ascending to the Veneziano-Yankielowicz Lagrangian

The order parameter effective Lagrangian can be written in terms of the chiral superfield \(^{29}\)

\[
S \equiv \langle \text{Tr}(W_\alpha W^\alpha) \rangle = \langle \text{Tr}(\lambda^\alpha \lambda_\alpha) \rangle + \ldots \equiv \langle \lambda \lambda \rangle + \ldots ,
\]

where \(S\) is regarded as a classical superfield and the matrix elements above are defined in the presence of an appropriate coordinate-dependent background (super)source. The effective superpotential reproducing all the anomalies of the model is given by \(^{29}\)

\[
W_{VY} = NS \left[ \ln \left( \frac{N \Lambda^3}{S} \right) + 1 \right].
\]

As far as dynamics of a single superfield \(S\) is concerned, the superpotential (15) is (locally) exact. The corresponding scalar potential describes the spontaneous chiral symmetry breaking, with a nonzero gluino condensate \(^{29}\).

In terms of the closed QCD string theory, the expression (15) includes only a single superfield of the closed QCD string perturbative spectrum. From this perspective it becomes clear that (15) should not be adequate for description of the D-walls, since these require some nonperturbative stringy input as well.

From the field-theoretical standpoint this is reflected in the fact that the superpotential (15) does not respect the \(\mathbb{Z}_N\) discrete symmetry and should be modified \(^{30}\). For the purpose of studying the vacuum structure per se, the modification by a constant integer-valued Lagrange multiplier is good enough \(^{30}\). However, to describe the domain walls, a “smoother” modification is required \(^{31}\). This conclusion is supported by extensive analysis \(^{13, 32}\) of the domain walls emerging in the Kovner-Shifman Lagrangian per se.

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\(^{6}\)It also supports, for nonsingular Kähler potentials, a chirally symmetric vacuum (the so-called Kovner-Shifman vacuum), with the vanishing gluino condensate \(^{30}\).
One possibility to deal with this problem is to introduce an additional superfield \( X \) which would restore the \( \mathbb{Z}_N \) invariance of the model \([22]\). From the physical point of view, as we elucidated above, this would correspond to some effective parametrization of the \( \mathbb{Z}_N \) structure and the properties of the D-walls. In general, it might be more relevant to introduce more than a single superfield, let us say \( X_1, X_2, \ldots \), to parametrize the strong coupling dynamics of SYM theory. A new superpotential with multiple fields will be discussed in the next section. In the present section we concentrate on the case of a single superfield \( X \) in order to elucidate how the construction works. The \( \mathbb{Z}_N \) symmetric superpotential with \( S \) and \( X \) fields can be regarded as the Veneziano-Yankielowicz superpotential \([13]\) in which the scale parameter \( \Lambda^3 \) is promoted to some \( X \) dependent chiral superfield, \( \Lambda^3 N \rightarrow \Lambda^3 F(X) \) \([22]\). Then, dynamics of the \( X \) field determines the phase of the gluino condensate. The \( \mathbb{Z}_N \) symmetric superpotential can be written as follows \([22]\):

\[
W_{\mathbb{Z}_N} = NS \left[ \ln \left( \frac{\Lambda^3 F(X)}{S} \right) + 1 \right]. \tag{16}
\]

Here the function \( F(X) \) can be presented as:

\[
F(X) = X \exp \left[ \frac{1}{N} \sum_n c_n \left( \frac{X}{N} \right)^n \right]. \tag{17}
\]

The following properties are required:

\[
F'(X_k^{\text{vac}}) = 0, \quad F(X_k^{\text{vac}}) = X_k^{\text{vac}} \equiv N \exp(2\pi ik/N). \tag{18}
\]

It is not difficult to see that the superpotential (16) respects the \( \mathbb{Z}_N \) discrete symmetry (with the transformations \( S \rightarrow S \exp(2\pi i l/N), \ X \rightarrow X \exp(2\pi i l/N) \)). The corresponding vacua are given by

\[
S^{\text{vac}} = S_k = \Lambda^3 X_k^{\text{vac}}, \quad X^{\text{vac}} = N \exp(2\pi ik/N), \quad k = 1, 2, \ldots, N. \tag{19}
\]

Thus, the superpotential (16) can be used to describe the BPS domain walls interpolating between two distinct chirally asymmetric vacua at large \( N \). Moreover, one can argue that, for the purpose of description of the nearest-neighbor wall transition, the superpotential (16) can be simplified further \([4]\). Indeed, the following relation holds for the vacuum state labeled by the phase \( k \):

\[
S|_k = \Lambda^3 F(X)|_k.
\]

This relation is nothing but the definition of the gluino condensate with the phase set by the vacuum value of the \( X \) superfield.

It is useful to introduce the chiral superfield \( S/F(X) \). In all vacua this superfield takes the same value equal to \( \Lambda^3 \). Let us now concentrate on interpolation between

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For simplicity we use notations of \([4]\) instead of those of \([22]\).
a pair of the nearest-neighbor vacua. In this case the relative change in the gluino bilinear is of order $1/N$. Hence, the chiral superfield $S/F(X)$ can only deviate from its vacuum value by a quantity of order $1/N$. Thus, we introduce the parametrization

$$\frac{S}{F(X)} = \Lambda^3 \left( 1 - \frac{\Sigma}{N} \right),$$

with $\Sigma$ being a new chiral superfield. Substituting Eq. (20) in the superpotential (16), one finds

$$W_{Z_N} = N\Lambda^3 F(X) \left[ 1 - \frac{\Sigma^2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right) \right].$$

The superfield $\Sigma$ enters this superpotential in the subleading order in $1/N$. Thus, it is the superfield $X$ which should describe the domain walls between the adjacent vacua in the large $N$ limit. Neglecting higher-order terms, the superpotential then is given by

$$W_{Z_N} = N\Lambda^3 F(X).$$

In addition, one should keep in mind that on a solution the $S$ superfield is related to the $X$ superfield as follows:

$$S = \Lambda^3 F(X) \left( 1 + \mathcal{O}(1/N^2) \right).$$

The superpotential (22) can be further reduced to the Landau-Ginzburg superpotential [5]

$$F(X) = X - \frac{N}{N+1} \left( \frac{X}{N} \right)^{N+1}.$$

For this expression, the BPS domain wall solution, with the width $l \sim 1/N$, can be explicitly found [9, 5]. Indeed, if one introduces a new variable $\Phi \equiv X/N$, the corresponding BPS equation in the large $N$ limit takes the form

$$\partial_z \Phi^* = i \left( 1 - \Phi^N \right).$$

This equation can be solved in the large $N$ limit. Introducing the notation

$$\Phi \equiv \left( 1 - \frac{\sigma}{N} \right) e^{i\tau/N},$$

where $\sigma$ and $\tau$ are two real functions of $z$ with the boundary conditions $\sigma(\pm\infty) = 0$ and $\tau(-\infty) = 0$, $\tau(+\infty) = -2\pi$, one finds the solution [3],

$$\cos(\tau) = (1 - \sigma)\exp(\sigma),$$

$$\int_{\sigma(0)}^{\sigma(z)} dt \left[ \exp(-2t) - (1 - t)^2 \right]^{-\frac{1}{2}} = -N \left| z \right|.$$

The subleading corrections in this expression are suppressed as $1/N^2$ since the $\Sigma$ field vanishes on the solution as $1/N$ [3].
The width of the wall is of order $1/N$. In this case, the presence of the D-wall in the theory is guaranteed by the presence of the field $X$ (which has mass $\mathcal{O}(N)$) in the superpotential (22). In reality, however, there might exist some light states with masses of order $\mathcal{O}(1)$, which would couple to $X$ with couplings suppressed by the $1/N$ factors. The $S$ particle (or $\Sigma$) is the example of such a state. We would like to study the impact of this state on the energy density and the width of the wall.

6.1.1 The energy density and the wall width

In this subsection we study how the presence of the $S$ (or $\Sigma$) field affects the energy density and the width of the BPS wall. Let us make a step back and consider the superpotential (16) before the field $S$ (or $\Sigma$) is eliminated. The tension of the wall is the sum of the kinetic and potential energy contributions. The BPS equations guarantee that these two contributions are equal. Therefore, we can study only the potential energy. The expression for the energy density of the wall takes the form

$$E \propto \int dz \left[ g^{-1}_{XX^*} \left| \frac{\partial W_{ZN}}{\partial X} \right|^2 + \left( g^{-1}_{XS^*} \frac{\partial W_{ZN}}{\partial X} \frac{\partial W^*_{ZN}}{\partial S^*} + \text{h.c.} \right) + g^{-1}_{SS^*} \left| \frac{\partial W_{ZN}}{\partial S} \right|^2 \right]. \quad (27)$$

Generically, the Kähler potential is of order $N^2$. The fields $S$ and $X$ scale as $N$. Hence, the Kähler metric is of order $N^0$ (more precisely, it cannot be larger than $N^0$). On the other hand, the derivatives of the superpotential with respect to $X$ and $S$ have a distinct large $N$ behavior. Indeed,

$$\frac{\partial W_{ZN}}{\partial S} = N \ln \left( \frac{\Lambda^3 F(X)}{S} \right) \sim N^0, \quad (28)$$

and,

$$\frac{\partial W_{ZN}}{\partial X} = NS \frac{F'(X)}{F(X)} \sim N. \quad (29)$$

We observe a very special pattern here. The superpotential (16) scales as $N^2$, while the $S$ field scales as $N$. Nevertheless, the corresponding derivative scales as $N^0$. This is a consequence of a very specific dependence of the superpotential on the $S$ superfield.

From these results we find the contributions of each term in the energy (27). The first term scales as $\sim N^2$, the second and third terms scale as $N$ (at most), and the last term scales as $N^0$. All these terms are to be multiplied by the factor $1/N$. The latter arises due to the integration over the wall width which scales as $1/N$. Summarizing, the BPS wall tension scales as $N$, in accordance with [2]. The dominant contribution to the tension is due to the heavy $X$ superfield. The $S$ superfield can at most contribute to the tension at the level of $N^0$. This is negligible in the large $N$ limit. Therefore, in the leading order of the large $N$ expansion the
wall is made of the $X$ field. This is in full accord with the intuitive expectations, of course.

Having established this, let us analyze whether the interactions of the $X$ superfield with the $S$ (or $\Sigma$) field can cause the wall broadening. As was mentioned before, the mass of these states is of order $N^0$. If the walls were able to emit these states, then the tails of the wall would behave as $\exp(-\Lambda z)$. This would indicate that the wall has a finite width, of order $1/\Lambda$, due to the “cloud” of finite mass states emitted by the wall. However, as we will show below, the wall in our model cannot emit a finite number of light states in the large $N$ limit.

This assertion is related to the fact that the corresponding couplings are suppressed as $1/N$. Indeed, let us consider the interaction vertex of the $X$ superfield with the $\Sigma$ field (it is helpful to deal with $\Sigma$ rather than $S$). This vertex is defined as

$$ V_{X^* \Sigma} \propto \partial^2 \frac{\partial^2 V(X, X^*, \Sigma, \Sigma^*)}{\partial X \partial \Sigma} \bigg|_{\text{vac}}. $$

Here $V$ stands for the potential of the model. Using the superpotential (21) we find that the vertex is proportional to the off-diagonal element of the inverse Kähler metric

$$ V_{X^* \Sigma} \propto g^{-1}_{X^* \Sigma} \bigg|_{\text{vac}}. $$

For a generic form of the Kähler potential, the large $N$ scaling of this expression is not known. However, in Ref. [5] it was shown that the wall solution of the theory with the superpotential (16) exists if and only if

$$ g^{-1}_{X^* \Sigma} \bigg|_{\text{solution}} \propto \frac{1}{N}. $$

(32)

Since the wall interpolates between the pair of distinct vacua, this implies that

$$ g^{-1}_{X^* \Sigma} \bigg|_{\text{vac}} \propto 1/N. $$

Therefore, the coupling of the $\Sigma$ superfield to the wall is suppressed by the factor $1/N$. As a result, the light states cannot be emitted by the wall in the large $N$ limit. The broadening of the wall will not happen, and the wall width will scale as $1/N$.

### 6.1.2 Domain wall junctions

It has been found recently that in some models with the $Z_N$ symmetry the domain walls can form the so-called wall junctions which can be BPS-saturated (preserve 1/4 of SUSY) [6, 7] (see also [33]). Such a configuration is depicted in Fig. 1 where we see that $N$ domain walls join at some localized region of space. The intersection of these walls forms a tube.
Given that the model one deals with has domain walls, it is natural to look for the junction type solution as well. Let us assume that the \( x \) and \( y \) coordinates parametrize the plane of Fig. 1. Introduce a complex variable

\[
2\zeta \equiv x + iy.
\]

(Note the unconventional normalization.) In terms of this variable the corresponding BPS equations for the junction take the form \([4, 6, 7]\)

\[
\partial_\zeta \star X^* (\zeta) = \frac{\partial W_{Z_N}}{\partial X}.
\] (33)

Using the superpotential \((22), (24)\) we obtain the BPS equation for the junction,

\[
\partial_\zeta \star \Phi^* = 1 - \Phi^N.
\] (34)

Here, as above, we defined a new variable \( \Phi = X/N \). A possible constant phase in the right-hand side of the BPS equation is absorbed in \( \zeta^* \).

There are two possible solutions of the junction type in the large \( N \) limit. We will consider them in turn. First, observe that if \( |\Phi| < 1 \), the right-hand side of this equation equals to unity in the large \( N \) limit. Thus, the solution of the equation is

\[
\Phi = \zeta, \quad \text{for} \quad |\zeta| < 1.
\] (35)

If, on the other hand, \( |\Phi| = 1 \) then the solution of Eq. (34) is just a constant phase,

\[
\Phi = \exp(i\delta k), \quad k = 1, 2, \ldots, N, \quad \text{for} \quad |\Phi| = 1.
\] (36)

Finally, if \( |\Phi| > 1 \), the right-hand side of (34) tends to infinity in the large \( N \) limit. Therefore, the solution of this kind does not exist.

Summarizing, the solution we just found looks as the one presented in Fig. 1a. There is a cylindrical core of a unit radius in the center of the solution where \( \Phi = \zeta \). There are domain wall lines joining the cylinder from the exterior of the core. The value of the field \( \Phi \) between these walls is a constant phase \( \exp(i\delta l) \), \( l = 1, 2, \ldots, N \), which takes different values for different sectors of Fig. 1a. Notice that the value of the \( X \) field in the center of the solution (35) is zero. If so, in accordance with (23), the value of the gluino condensate in this region vanishes too. Thus, the (hypothetical) chirally symmetric Kovner-Shifman phase \([30]\) is realized within this solution inside the string at the geometric axis of the wall junction solution. We should also point out that the solution given by (35) and (37) is defined up to terms which provide appropriate matching at the boundary circle and at the boundaries of the sectors. The width of the region where the matching between (35) and (37) should be performed vanishes in the large \( N \) limit and, therefore, these terms cannot be controlled in our approximation. Strictly speaking, the very fact of matching remains an assumption.
Let us now turn to the second solution. It is straightforward to check that the following expression is the solution of Eq. (34):

$$\Phi = \exp \left( \frac{i2\pi k}{N} \right) \text{ in the } k\text{-th sector, } k = 1, 2, ..., N.$$ (37)

This configuration describes the domain walls intersecting at the origin in the \((x, y)\) plane (see Fig. 1b). The intersection is just a straight line perpendicular to the \((x, y)\) plane. Different sectors between different walls are parametrized by the value of the phase of the solution (37). Below, at the end of this section, we will argue that the two solutions presented above have the same energy in the limit of infinite \(N\).

Before we turn to this discussion, however, let us calculate the tension of the tube which is formed at the intersection of the walls. This tension is defined by circulation of the axial current along the big circle (call it \(\Gamma\)) enclosing the tube \[10\]

$$T_{\text{Junction}} = \int_{\Gamma} a_l dx^l.$$ (38)

In the model at hand \(a_\mu = -iN^2(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*)/2\). Along the circle \(\Gamma\) the axial current is a total derivative, \(a_\mu = N^2 \partial_\mu \alpha\), where \(\alpha\) is the phase which changes along the circle \(\Gamma\). As a result, the expression for the tube tension is

$$T_{\text{Junction}} = 2\pi N^2.$$ (39)

Thus, the tension scales as \(N^2\) in the large \(N\) limit. This is consistent with our previous estimate \([1]\) and with the the expectation that there should be no BPS strings in SYM theory. The tension of the latter would scale \([9]\) as \(N\).

Let us now go back and show that the two junction solutions described above, although seemingly different, are both indeed BPS states in the large \(N\) limit. Both of these solutions preserve \(1/4\) of SUSY and satisfy the BPS equations. The only difference between them might be the energy of a configuration. However, simple counting based on (39) and the expression for the BPS domain wall tension \(T_{\text{DW}}\) \([2]\), shows that the energies of the two configurations enclosed by a circle of the radius \(R\) are equal to \(T_{\text{DW}} NR + T_{\text{Junction}}\). They differ from one another only in the subleading order of the \(1/N\) expansion. This may be interpreted as follows. Suppose we start with the configuration of Fig. 1b and consider the circle of a unit radius which encloses the center. As the number of walls tend to infinity, the angular separation between the walls inside the circle tends to zero. Thus, the interior of this circle will look pretty much as the core in Fig. 1a. Hence, we have the same energy density in the region enclosed by the circle. On the other hand, the angular separations between the walls cannot be set equal to zero outside of the circle, since these walls stretch out to the spatial infinity along the radial directions.

\[9\] In the D-brane construction the tension of a BPS D-string would scale as \(1/g_s \sim N\). Since QCD strings annihilate in groups of \(N\), they cannot possibly be BPS saturated objects. The theorem that in no non-Abelian theory BPS strings can exist at weak coupling is established in Ref. \([10]\).
6.2 Domain walls in a model with multiple fields

In the previous subsection we found that the width of the BPS wall is not affected by interactions with light gluinoballs as long as these interactions are suppressed as $1/N$. However, the story might be more complicated if there are open strings ending on the walls. In this case the wall is a sort of D-brane for a non-critical SYM string. If these strings do exist, their lowest excitations have nothing to do with the conventional bound states, glueballs and gluinoballs. Instead, they might appear as a result of the presence of D-walls. If so, the D-wall will be able to interact with states of the open strings; this interaction should not necessarily be suppressed by powers of $1/N$.

The existence of the open strings cannot be rigorously established in SUSY gluodynamics at present. Nevertheless, assuming that this phenomenon takes place, it is worth finding a toy model in which these features could be discussed. Here we present a prototype field-theoretic model which has certain properties described above.

The model is specified by allowing $F$ in Eqs. (16), (22) to be a function of $N + 1$ variables,

$$F = F(X_1, X_2, \ldots, X_{N+1}).$$

(40)

After the $S$ field is removed, as described in the previous section, the spectrum of the model consists of $N + 1$ states. One of these states is heavy, with mass of order $N$. The rest have finite masses. There is a BPS wall solution in the model. The wall is made of the heavy field. The light fields vanish on the solution. The width of the solution is of order $1/N$, just like in the previous section. What is different is that the interaction vertices of the wall with the light states are not suppressed by $1/N$. Nevertheless, the light degrees of freedom are not excited on the wall; there is no broadening.

6.2.1 The prototype superpotential

Here we study a $Z_N$ symmetric model which contains one heavy state and a number of light states. The model is written in terms of $N + 1$ fields denoted by $X_k$, $k = 1, 2, \ldots, N + 1$. The $Z_N$ symmetric superpotential of the model is (below we set $\Lambda = 1$)

$$W = X_1 + e^{i\delta} X_2 + e^{i2\delta} X_3 + \ldots + e^{iN\delta} X_{N+1} - N e^{-i\pi(N+1) - i\theta} \prod_{k=1}^{N+1} \frac{X_k}{N}.$$  

(41)

This superpotential replaces Eq. (22) after the $S$ field is eliminated and $F$ is allowed to be a function of $N + 1$ variables. Here $\delta \equiv 2\pi/N$, and $\theta$ denotes the theta angle. One could have easily absorbed the factors $\exp(i k \delta)$ in Eq. (11) in the definition of the fields $X_k$. We keep them for reasons which need not concern us here. The model (11) can be viewed as a simplified version of the model discussed in Sec. 3.
The corresponding Lagrangian is invariant under the simultaneous discrete \( \mathbb{Z}_N \) transformations of the fields \( X_k, \ k = 1, 2, \ldots, N+1 \)
\[
X_k \rightarrow X_k \exp(i\delta l), \quad l = 1, 2, \ldots, N.
\]
(42)
In addition, there is a symmetry which permutes different \( X \)’s among themselves. A simplest pattern of these transformations can be written as follows:
\[
X_j \rightarrow X_{j+1} \exp(i\delta), \quad j = 1, 2, \ldots, N, \quad X_{N+1} \rightarrow X_1.
\]
(43)
For simplicity let us chose the Kähler potential in the form
\[
K = \frac{1}{N} \left( X_1^* X_1 + X_2^* X_2 + \cdots + X_{N+1}^* X_{N+1} \right).
\]
(44)
The reason why we introduce the overall factor \( 1/N \) will become clear shortly (essentially, it is needed in order to make the Kähler potential scale as \( N^2 \) on the solution).

Let us first study the vacuum structure in this model. The vacua of the theory are determined by the equations
\[
\frac{\partial W}{\partial X_j} = \left[ e^{i\delta(j-1)} - \frac{1}{X_j} N e^{-i\pi(N+1)-i\theta} \prod_{k=1}^{N+1} \frac{X_k}{N} \right] = 0, \quad j = 1, 2, \ldots, N+1.
\]
(45)
This system of equations implies that
\[
X_1 = e^{i\delta} X_2 = e^{i2\delta} X_3 = \ldots = e^{iN\delta} X_{N+1} = N e^{-i\pi(N+1)-i\theta} \prod_{k=1}^{N+1} \frac{X_k}{N}.
\]
(46)
One finds as a solution
\[
\langle X_k \rangle = N \exp \left( -i\delta(k-1) - \frac{i\theta}{N} \right), \quad k = 1, 2, \ldots, N+1.
\]
(47)
All other solutions of Eq. (46) are related to the latter by the symmetry transformations. There are \( N \) different vacua in the theory described by the superpotential (11), in full accordance with the (spontaneously broken) \( \mathbb{Z}_N \).

Let us comment on the role of the “theta angle”. When the \( \theta \) parameter changes
\[
\theta \rightarrow \theta + 2\pi,
\]
the vacuum values of different fields transform into one another,
\[
X_1^{\text{vac}} \rightarrow X_2^{\text{vac}} \rightarrow X_3^{\text{vac}} \rightarrow \ldots X_N^{\text{vac}} \rightarrow X_1^{\text{vac}}, \quad X_{N+1}^{\text{vac}} \rightarrow X_2^{\text{vac}}.
\]
(48)
Thus, the \( 2\pi \) shift of the theta parameter leads to a relabeling of the vacuum states, precisely as it should be in pure SYM theory (with massless gluinos and a nonzero
After the role of the theta term is elucidated, we will set \( \theta = 0 \) for simplicity.

Next, we turn to the task of finding the BPS domain walls for the superpotential \((\text{II})\). The BPS equations for the minimal wall, interpolating between the neighboring vacua \(m\) and \(m+1\) are

\[
\frac{1}{N} \partial_z X_j^* = e^{i\gamma} \frac{\partial W}{\partial X_j} = e^{i\gamma} \left[ e^{i(\delta-1)} - \frac{1}{X_j} N \prod_{k=1}^{N+1} \frac{X_k}{N} \right], \quad j = 1, 2, \ldots
\]

(49)

where

\[
\gamma = \text{Arg}\Delta W,
\]

and the following boundary conditions are implied

\[
X_j(-\infty) \to \langle X_j \rangle_m, \quad X_j(\infty) \to \langle X_j \rangle_{m+1}.
\]

In Eq. (49) and below it is assumed for simplicity that \(N\) is even.

It is easy to see that the Ansatz which goes through the system of equations (49) is

\[
X \equiv X_1 = X_2 e^{i\delta} = X_3 e^{i2\delta} = \ldots = X_{N+1} e^{iN\delta}.
\]

(50)

On this Ansatz one finds (introducing a new variable \(\Phi \equiv X/N\)) that the system reduces to a single equation,

\[
\partial_z \Phi^* = i \left(1 - \Phi^{N}\right),
\]

(51)

(in the large \(N\) limit the phase \(\gamma = \pi/2\) up to subleading terms).

This latter equation is identical to (25) and can be solved in the large \(N\) limit, as in Eq. (26). Therefore, expressions (50) and (26) define the BPS domain walls of the model \((\text{II})\). The width of the wall is of order \(1/N\), in accordance with the discussion in the previous sections.

It is interesting to understand what physical excitations “make” the wall. To answer this question we must determine the mass eigenstates of the model at hand. The fields \(X_k\) in the superpotential \((\text{II})\) are not diagonal. Indeed, the corresponding \(N+1\) by \(N+1\) mass matrix of fermions has the form

\[
M_{kj} = N \left. \frac{\partial^2 W}{\partial X_k \partial X_j} \right|_{\text{vac}} = \begin{cases} 0 & \text{if } k = j, \\ e^{i(k-j)\delta} & \text{if } k \neq j \end{cases}
\]

(52)

where the mass matrix is evaluated at the vacuum where \(\langle X_1/N \rangle = 1\); in all other vacua the results are essentially the same. Note that each entry in this matrix is \(O(1)\). Nevertheless, upon diagonalization we find one eigenstate – call it \(Y\) – with mass \(N\)

\[
Y = \frac{1}{\sqrt{N+1}} \left( X_1 + e^{i\delta} X_2 + e^{i2\delta} X_3 + \ldots + e^{iN\delta} X_{N+1} \right).
\]

(53)
Other $N$ states $Y_l$, $l = 1, 2, \ldots, N$ can be expressed in terms of $X$’s in a simple way, they have the same form as the diagonal generators of SU$(N)$, e.g.

$$Y_1 = \frac{1}{\sqrt{2}} \left(X_1 - e^{i\delta} X_2\right), \quad Y_2 = \frac{1}{\sqrt{6}} \left(X_1 + e^{i\delta} X_2 - 2e^{i2\delta} X_3\right),$$

(54)
and so on. They all have mass 1, i.e. are light.

### 6.2.2 Why the broadening of the wall does not take place

In this subsection we argue that although the wall interacts with the light states with unsuppressed couplings, nevertheless the broadening of the wall width does not take place.

It is important to note that all light states $Y_l$ have the vanishing VEV’s in the vacua of the theory. Moreover, these fields are identically zero on the domain wall solution defined by Eqs. (50) and (26). The only field which actually “makes” the BPS wall is the heavy state $\mathcal{Y}$. Moreover, the superpotential, being expressed in terms of $\mathcal{Y}$ and $Y_l$, contains no terms linear in $Y_l$.

Let us now briefly discuss interactions of the heavy field $\mathcal{Y}$ with the light fields $Y_l$. The expression for the interaction potential is rather cumbersome, so we merely summarize below its basic features. In the superpotential there are vertices with $N - 1$ heavy fields and two light fields, $N - 2$ heavy and three light, and so on. In the large $N$ limit the vertices have the structure

$$\left(\frac{\mathcal{Y}}{\sqrt{N}}\right)^{N-1} \frac{Y_l}{\sqrt{N}} \frac{Y_{l'}}{\sqrt{N}}.$$

The kinetic term is

$$\mathcal{K} = \frac{1}{N} \left(\mathcal{Y}^* \mathcal{Y} + \sum_{l=1}^{N} Y_l^* Y_l\right).$$

Rescaling the fields to cast the kinetic term in the canonic form, one finds that, say, the heavy-light-light vertex is not suppressed by powers of $1/N$. The production rate is proportional to the square of the amplitude multiplied by the inverse mass of the decaying state. Since the decaying state has the mass of order $N$, this rate is going to be suppressed as $1/N$.

$$\Gamma \propto \frac{\Lambda}{N}.$$

Likewise, one can find that the rates for the decays of the heavy state into arbitrary number of light states are suppressed by the corresponding powers of $1/N$. Correspondingly, the wall energy is completely saturated by the contribution of the heavy field $\mathcal{Y}$, the wall width $l \sim 1/N$, and there is no broadening.
7 Discussion and Conclusions

The vacuum structure of SUSY gluodynamics is rich and complicated. Essential ingredients in exploring this structure are the BPS domain walls and junctions. The theory is in the strong coupling regime. Therefore, quantitative studies of the fundamental model are not feasible. Instead, one is able to abstract certain qualitative features from exact results based on supersymmetry and from D-brane constructions. It is also possible to model essential properties of the vacuum in terms of effective Lagrangians. These model Lagrangians allow one to explore the BPS objects and to study the expected most salient features of the theory.

In particular, we argued that the minimal BPS wall in SUSY gluodynamics has width $(N \Lambda)^{-1}$ and is “made” of the field which is not present among conventional mesons of the model. The mass of the relevant state scales as $N \Lambda$.

The width of the BPS wall scales as $1/N$ despite the fact that the order parameter $\lambda \lambda$ interacts with the finite mass mesons. This property allows one to naturally interpret the tension of the BPS wall junction, which scales as $N^2$. The $1/N$ scaling law for the width implies the existence of a hadronic state with mass $\sim N \Lambda$ distinguished by its role in making the wall. We suggested speculative arguments hinting that such hadrons can persist in non-supersymmetric YM theory in the large $N$ limit. If so, there emerges a challenging task to understand the nature of this special glueball.

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