We have calculated the $q\bar{q}$ potential over a wide range of lattice separations on a $48^3 \cdot 56$ lattice at $\beta = 2.85$. We are able to investigate both long-range and (by correcting for the effects of lack of rotational invariance) short-range potentials. From the former we estimate the string tension and from the latter we are able to investigate asymptotic scaling. Fitting to this enables us to give an estimate for the $\Lambda$-parameter of pure-gauge SU(2).

1. Introduction

We discuss recent results for the potential between (hypothetical) static quarks calculated on the 64-node Meiko Computing Surface at Edinburgh University. Our lattice size, $48^3 \cdot 56$, is to our knowledge the largest so far used in a lattice gauge theory simulation. Thus we are in a good position to examine the approach to the continuum limit and comment on the remaining effects of finite lattice spacing $a$. A fuller write-up is available as a preprint [1], so after briefly mentioning our data and techniques we proceed directly to a discussion of the results.

2. Monte Carlo data

We have chosen $\beta = 2.85$ based on previous experience to make our lattice physically as large as necessary. It is known that the potential is less sensitive to finite-volume effects than, for example, the glueball spectrum but we do not wish to rely on this. We also have results for glueballs which will not be discussed here. We have performed a total of over $10^{12}$ link updates on this lattice.

Our technique for extracting the potential $V_{q\bar{q}}$ is standard. The effective potential in lattice units is extracted from $R \times T$ Wilson loops as $V_{\text{eff}}(R, T) = \ln(W(R, T)/W(R, T - 1))$; as $T$ becomes large, $V_{\text{eff}}(R, T)$ tends to the ground state potential $V_{q\bar{q}}(R)$ which is what we need. To improve our overlap onto this state we have used two popular ‘fuzzing’ schemes [2,3]; the ‘wave functions’ generated have been used as the basis for a variational analysis. The resulting overlap approaches unity in many cases.

We are faced as always with the problem of deciding how to extract $V_{q\bar{q}}(R)$ from $V_{\text{eff}}(R, T)$. We find that the results at $T = 5$ agree within statistical errors with those at $T = 4$ and use these as our main results. We have also performed an extrapolation to $T = \infty$ by assuming that the signal is contaminated only by one excited state. This assumption gives a lower bound on the potential. In this case we are less in control of statistical errors; furthermore, we expect this procedure to be less effective for larger $R$ where the first excited state is closer to the ground state. Thus we use the extrapolated potentials mainly as a way of gauging our systematic errors.

3. Fits to data

We have performed two separate fits for large and for small separations $R$. In fact, we have used different data analysis techniques in the two cases: the large-separation analysis required larger statistics but we reduced the total amount of data produced by blocking the lattice by a factor of two in each spatial direction. In the other case we needed the full lattice volume but lower statistics for accurate results.
3.1. Large $R$

Here our major object is to investigate the string tension $K$. We fit to the form

$$V(R) = C - \frac{E}{R} + KR$$

for $R \geq 6a$. In fact, the Coulomb term is modified to include corrections (from first order perturbation theory) for lack of rotational invariance, but the effect in this case is negligible. We are not interested in the constant $C$ since the ground state energy is in large measure an artefact of the cut-off. Our results for both the $T = 5$ and extrapolated potentials are shown in table 1.

| Table 1 | Result of fits for $R \geq 6a$ |
|---------|--------------------------------|
| $V(R,5)$ | $0.247(7)$ | $0.00401(8)$ | $15.2/7$ |
| $V_\infty(R)$ | $0.247(14)$ | $0.00363(17)$ | $8.2/7$ |

As already mentioned, we use $V_\infty$ to establish a systematic error. Hence the value we use for the string tension is from $V(R,5)$: $Ka^2 = 0.00401(8)(38)$.

3.2. Small $R$

Here our fit includes all separations $R > a$. Furthermore, we also include off-axis potentials — that is, those with separation vectors such as $(1,1,0)$, $(1,1,1)$, and so forth. This makes some means of controlling lattice artefacts imperative. Our fit in this case has the form

$$V(R) = C - \frac{A}{R} + \frac{B}{R^2} + KR - Af \delta G(R),$$

in which $K$ is fixed from the preceding fit; $A$, $B$ and $f$ are the free parameters.

There are two new elements. In this range of separations we are hoping to be able to see the coupling run; the term in $B$ is a simple-minded way of correcting the Coulomb term to account for this over a small range of $R$.

Both $A/R$ and $B/R^2$ terms describe effects which are smooth and monotonic in $R$. By contrast, the final term describes lattice artefacts: $\delta G(R)$ is the one-gluon correction to the Coulomb term, so that if $f$ were unity the two terms involving $A$ would combine to give the first-order-corrected Coulomb term which was used in the previous fit. By allowing $f$ to vary we account for the following observation: that the dominant effect of higher-order corrections is to rescale the first-order term.

| Table 2 | Result of fits for $R > a$ (with $K$ fixed) |
|---------|--------------------------------|
| $A$ | $0.238(7)$ | $0.63(3)$ | $0.050(6)$ | $9.7/14$ |

The $\chi^2$ of the resulting fit (table 2) justifies our use of this ansatz to correct for lack of rotational invariance. However, it is difficult to estimate the systematic error one should attach to this. We have arbitrarily assigned 10% of the correction which has been applied.

4. Scaling

The running coupling is conventionally defined in terms of $V_{\bar{q}q}$ as

$$\alpha_{\bar{q}q} \left( \frac{R_1 + R_2}{2} \right) = \frac{4}{3} R_1 R_2 \frac{V_{\bar{q}}(R_1) - V_{\bar{q}}(R_2)}{R_1 - R_2}, \quad (3)$$

in which we use the potentials $V_{\bar{q}}(R)$ with the lattice-artefact correction applied. This is a convenient quantity to use in investigating scaling as it is dimensionless.

We therefore compare these results with those obtained previously on $32^4$ lattices at $\beta = 2.7$, plotting $\alpha_{\bar{q}q}$ against $R$ and rescaling only the horizontal axis. That the agreement is entirely comfortable, with no indication of scaling violation, will become clear when we discuss figure 1 which imposes the even stronger condition that the rescaling factor is $\sqrt{K}$.

For now, we show in table 3 various values for the ratios of the lattice spacings $a(\beta=2.7)/a(\beta=2.85)$. The first value is obtained by a full fit to the Monte Carlo results, including an estimate of the systematic error. The last is
Figure 1. The running coupling versus a length scale

Table 3
The ratio $a(2.7)/a(2.85)$ in various calculations

|                | UKQCD | $\beta_{E}$ | L"uscher scheme et al. | Asymptotic (two loops) |
|----------------|-------|-------------|------------------------|------------------------|
|                | 1.60(6) | 1.53        | 1.48                   | 1.46                   |

the naive expectation of 2-loop perturbation theory using the bare couplings. It is clear there is some discrepancy. The other two values require some comment.

It is useful to use instead of the bare coupling something more closely related to physical processes. One popular choice is the $\beta_{E}$-scheme $\mathbb{R}$, where the plaquette expectation value $\langle U_{pl} \rangle$ is used as the basis for calculating the coupling. Here we use the simplest variant where $\beta_{E} = c_2/(1 - \langle U_{pl} \rangle)$, in which $c_2$ is a perturbative coefficient. Using even this simple correction changes the $a$-ratio by about half of the discrepancy between Monte Carlo and naive perturbation theory, giving some confidence our results are in the right region.

The remaining estimate is taken from the fit in figure 3 of ref. $\mathbb{R}$. In this paper (see the talk by Martin L"uscher at this conference) the authors calculate the running coupling by a recursive finite-size-scaling technique. The value shown is in terms of the bare coupling with lattice sizes determined at fixed physical coupling. The result quoted, close to the two-loop value, is not central to ref. $\mathbb{R}$; it is indicative of the fact that the starting point for that paper is in small (and therefore perturbative) volumes.

We have also compared the new results with those on $24^4$ lattices at $\beta = 2.4$. Even here we find excellent agreement, showing scaling over a range of lattice spacings $a(\beta=2.4)/a(\beta=2.85) = 4.12(2)$, which is certainly impressive.

5. Running coupling

Equation $\mathbb{R}$ defines a running coupling in the $\Lambda_R$-scheme; $\Lambda_R = 1.048\Lambda_{MS}$ so these schemes are perturbatively close. We plot $\alpha_{\bar{q}q}$ against the
dimensionless $R \sqrt{K}$ which gives a physical length scale. This plot is shown in figure [1].

In the figure we show results for both $\beta = 2.85$ (\bigcirc) and $\beta = 2.7$ (\bigtriangleup). For clarity only the larger of the two error bars (statistical and systematic) is shown here. (Both sets appear in ref. [1].) The effect of running is clear and the curve appears to be universal.

The solid lines are plots of $\alpha_q$ from perturbation theory:

$$\alpha_q(R) = \frac{1}{4\pi} \left( b_0 \ln(\Lambda_R R)^{-2} + (b_1/b_0) \ln(\Lambda_R R)^{-2} \right)$$

with the usual perturbative coefficients. The upper line corresponds to $\alpha \Lambda_R = 0.044$ and the lower to $\alpha \Lambda_R = 0.038$; note that our results include increasing contributions from the linear potential as $R$ increases which will tend to make our results larger than the perturbative estimate. From these limits our estimate of the scale parameter is $\alpha \Lambda_R = 0.041(3)$. In terms of the string tension this is $\sqrt{K}/\Lambda_R = 1.54(15)$ or $\sqrt{K}/\Lambda_{MS} = 1.62(14)$; using the (in this case entirely unjustifiable) value $\sqrt{K} = 0.44$ GeV we get $\Lambda_{MS} = 272(24)$ MeV. (In these units our shortest-distance point, at $R = \sqrt{2a}$, would correspond to a momentum scale $q = 1/R \approx 4.4$ GeV.)

The remaining points with horizontal error bars are converted from ref. [6]. We have used the value of the string tension calculated here to set the scale, however the error in the string tension is not included. (This is consistent with the other points in the figure.) The region of interest is at the lower end; the error bars just fail to overlap our lines for $\Lambda_R$, so the conclusions are not entirely clear. If our estimate of systematic errors is to be believed there appears to be some discrepancy.

6. Concluding remarks

It has recently become clear [7] that attempts to compare Monte Carlo results from the lattice with perturbation theory using the bare coupling are doomed. Here we have presented confirmatory evidence which further suggests that when one uses more physical quantities, the usual Wilson theory for SU(2) without fermions is able to produce results entirely in accord with those in the continuum. Our bare coupling is not able to give us agreement with perturbation theory, but if we fit to the $\Lambda$-parameter ourselves we see a running coupling entirely in agreement with the asymptotic two-loop result. In addition we see scaling of $V_{q\bar{q}}$ results between lattices differing by a factor of four in lattice spacing.

In ref. [8] these new results are displayed in the $\beta_E$ scheme referred to above and it is clearly seen that the approach to asymptotic behaviour is much smoother when the bare coupling is not used, enabling an extrapolated value of the ratio $\sqrt{K}/\Lambda_{MS}$ to be extracted. Their value is $1.79(12)$, slightly higher than our unextrapolated number which is $1.62(14)$.

Overall, we believe that pure-gauge lattice SU(2) is already able to give results which accurately reflect continuum dynamics.

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