IN Variant Measures for FIlippov Systems

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Abstract. We are interested in Filippov systems which preserve a probability measure on a compact manifold. Using the formalism coming from the theory of differential inclusions, we define a measure to be invariant for a Filippov system as the natural analogous definition of invariant measure for flows. Our first main result states that if a differential inclusion admits an invariant probability measure, this measure does not see the trajectories where there is a break of uniqueness. Our second main result provides a necessary and sufficient condition in order to exist an invariant probability measure preserved by a Filippov system. Our third main result concerns Filippov systems which preserve a probability measure equivalent to the volume measure. As a corollary the volume preserving Filippov systems are the refractive ones. Then, in light of our previous results, we analyze the existence of invariant measures for many examples.

1. Introduction

Filippov systems belong to a class of dynamical systems which are very useful to model many physical systems. The understanding of their chaotic behavior is an active area of interest in dynamical systems (one may see [3, 4, 6], and the references therein). A better comprehension of the chaoticity of a dynamical systems can be achieved through the ergodic theory point of view. Ergodic theory deals with dynamical systems admitting an invariant measure, hence on ergodic theory one may talk about statistical properties of the dynamics. In this work we try to understand the invariant measures of Filippov systems, which is the very first step in order to use ergodic theory to understand these systems.

In general, the Filippov solution of a discontinuous differential system passing through a point is not unique. This implies that their solutions, in general, do not enjoy the flow properties. This adds an extra difficulty when studying invariant measures.

For some prototypical situations we may intuit the existence or not of invariant probability measures. For instance, consider the following Filippov
systems defined on a open set \( U \subset \mathbb{R}^2 \):

\[
Z_1(x, y) = \begin{cases} 
(1, -1), & \text{if } y > 0, \\
(1, 1), & \text{if } y < 0,
\end{cases}
\]

\[
Z_2(x, y) = \begin{cases} 
(1, -1), & \text{if } y > 0, \\
(-1, 1), & \text{if } y < 0.
\end{cases}
\]

We know that every open set \( V \subset U \) flowing through the trajectories of \( Z_1 \) eventually collapses on \( \Sigma = \{ y = 0 \} \). This phenomenon prevents the existence of any invariant probability measure (see Figure 1). Indeed, assume that \( Z_1 \) admits an invariant measure \( \mu \). We know that there exists \( t_0 > 0 \) such that \( Z_{t_0}(J_1) = Z_{t_0}(J_2) = I \). Therefore \( \mu(J_1) = \mu(I) = \mu(J_2) \). However \( J_1 \cup J_2 \subset J = Z_{-t_0}(I) \). Hence \( \mu(I) = \mu(J) \) and \( \mu(J_1) + \mu(J_2) \leq \mu(J) \) which leads to a contradiction.

On the other hand every open set \( V \subset U \) flowing through the trajectories of \( Z_2 \) keeps its area unchanged. Hence the Lebesgue measure is invariant for \( Z_2 \).

The Filippov convention, stated in [5], for solutions of discontinuous differential systems takes advantage of a well developed theory of differential inclusions. In the present paper we shall use this theory to study their invariant measures. We prove that if a differential inclusion admits an invariant probability measure, it should somehow not be able to see the points where there is a break of uniqueness (Theorem A). The same happens to Filippov systems, in this case we are able to provide a necessary and sufficient condition for the existence of invariant probability measures (Theorem B). We also study the special case of volume preserving systems (Theorem C). In particular, we prove that these systems are those ones known in the literature as refractive systems (Corollary A). On Section §4 we present several examples of Filippov systems defined on the compact manifolds \( \mathbb{T}^2 \) (torus) and \( \mathbb{K}^2 \) (Klein
bottle). The existence of invariant probability measures for these examples are analyzed in light of the previous results.

1.1. Differential Inclusion. In what follows we briefly introduce the concept of differential inclusions. For more details on this subject we recommend the books [2, 7]. Let \( U \) be an open subset set of \( \mathbb{R}^n \) and \( F: U \to \mathbb{R}^n \) be a set-valued function, that is, for each \( x \in U \), \( F(x) \subset \mathbb{R}^n \). A function \( \phi: (-T, T) \to U \) is said to be a solution of the differential inclusion

\[
\dot{x} \in F(x)
\]

if \( \phi \) is an absolutely continuous function satisfying (1) almost everywhere. Usually, given \( x \in U \), \( S_F(x) \) denotes the set of all maximal solutions \( \phi(t) \) of (1) satisfying \( \phi(0) = p \). Let \( AC(\mathbb{R}^n) \) denote the set of all absolutely continuous function \( \phi: (-T, T) \to \mathbb{R}^n, T \in \mathbb{R} \), and

\[
S(F) = \bigcup_{x \in U} S_F(x) \subset AC(\mathbb{R}^n).
\]

Note that \( S_F: U \to AC(\mathbb{R}^n) \) is a set-valued map. In order to get some useful properties on \( S_F \) some hypothesis on \( F \) must be assumed:

(i) \( F(x) \subset \mathbb{R}^n \) is a closed convex set for every \( x \in U \).

(ii) \( F \) is Lipschitzian that is, there exists \( L > 0 \) such that \( F(x_1) \subset F(x_2) + L|x_1 - x_2|B_1(0) \) for every \( x_1, x_2 \in U \), where \( B_1(0) = \{ y \in \mathbb{R}^n : \|y\| \leq 1 \} \).

Denote by \( N_F \) the set of points of \( x \in U \) such that \( \#S_F(x) > 1 \), that is there exist at least two solutions \( \phi_1, \phi_2 \in S_F(x) \) such that \( \phi_1 \neq \phi_2 \). In this case we are able to find \( t_0 \neq 0 \) for which \( \phi_1(t_0) \neq \phi_2(t_0) \). In other words \( N_F \) constitutes the set of points in \( U \) for which the uniqueness of solution is lost. For a point \( x \in N_F \) we denote its saturation by

\[
\text{Sat}(N_F) = \bigcup_{x \in N_F} \text{Sat}(x), \quad \text{where} \quad \text{Sat}(x) = \bigcup_{\phi \in S_F(x)} \{ \phi(t) : t \in I_{x,\phi} \},
\]

where \( I_{x,\phi} \) is the domain of \( \phi \).

1.2. Filippov systems. Let \( M \) be a compact Riemannian manifold and let \( N \subset M \) be a codimension 1 compact submanifold. Denote by \( C_i, i = 1, 2, \ldots, k \), the connected components of \( M \setminus N \). Let \( X_i: M \to TM, \) for \( i = 1, 2, \ldots, k \), be vector fields on \( M \), i.e. \( X_i(p) \in T_pM \). Consider a piecewise smooth vector field on \( M \) given by

\[
Z(p) = X_i(p) \text{ if } p \in C_i, \text{ for } i = 1, 2, \ldots, k.
\]

Since \( N \) is a codimension 1 compact submanifold of \( M \), we can find, for each \( p \in N \), a neighborhood \( D \subset M \) of \( p \) and a function \( h: D \to \mathbb{R} \), having 0 as a regular value, such that \( \Sigma = N \cap D = h^{-1}(0) \). Moreover, the neighborhood
$D$ can be taken sufficiently small in order that $D \setminus \Sigma$ is composed by two disjoint region $\Sigma^+$ and $\Sigma^-$ such that $F^+ = Z|_{\Sigma^+}$ and $F^- = Z|_{\Sigma^-}$ are smooth vector fields. Accordingly, the piecewise smooth vector field (2) may be locally described as follows:

\begin{align*}
Z(p) = (F^+, F^-)_h = \begin{cases} 
F^+(p), & \text{if } h(p) > 0, \\
F^-(p), & \text{if } h(p) < 0,
\end{cases} \quad \text{for } p \in D.
\end{align*}

In [5], Filippov stated that the local trajectories of system (3) is a solution of a differential inclusion $\dot{p} \in F_Z(p)$, where $F_Z$ is the following set-valued function:

\begin{align*}
F_Z(p) = \frac{F^+(p) + F^-(p)}{2} + \text{sign}(h(p)) \frac{F^+(p) - F^-(p)}{2},
\end{align*}

and

\begin{align*}
\text{sign}(u) = \begin{cases} 
-1 & \text{if } u < 0, \\
[-1, 1] & \text{if } u = 0, \\
1 & \text{if } u > 0.
\end{cases}
\end{align*}

We point out that for the case of Filippov systems [3], the solutions of the differential inclusion $\dot{p} \in F_Z(p)$ have an easy geometrical interpretation. We shall briefly discuss it in the beginning of Section §3. For sake of simplicity we denote by $S_F(p)$ and $N_F$ the sets $S_{F_Z}(p)$ and $N_{F_Z}$, respectively.

1.3. **Measure preserving.** Throughout this paper we shall only work with Borel measures, that is the ones which $\sigma$-algebra associated is the Borel $\sigma$-algebra. Let $X_t$ denote the flow of a smooth vector field $X : M \to TM$ and $\mu$ a measure on $M$. We say that a flow $X_t$ preserves a measure $\mu$ if: for any subset Borel set $A \subset M$, $\mu(X_t(A)) = \mu(A)$, $\forall t \in \mathbb{R}$. Nevertheless, when one consider differential inclusions and, in particular, Filippov systems, we have seen that for a given initial condition $p_0 \in M$ it may exist several solutions starting at $p_0$. Consequently, the previous definition of flow and measure preserving fails. In order to overcome this difficulty, considering the analogous definition of measure preserving for flow, we say that the differential inclusions (1) preserves a measure $\mu$ if

\begin{align*}
\mu(S_F(A)(t)) = \mu(A),
\end{align*}

for any Borel subset $A \subset M$, where

\begin{align*}
S_F(A)(t) = \bigcup_{x \in A} S_F(x)(t).
\end{align*}

For Filippov systems we denote

\begin{align*}
Z_t(p) = S_F(p)(t) = \{\phi(t) : \phi \in S_F(p)\}.
\end{align*}
Hence, from (5), we say that the Filippov system (3) preserves a measure \( \mu \) if
\[
\mu(Z_t(A)) = \mu(A),
\]
for any Borel subset \( A \subset M \).

Due to the nonuniqueness of solutions this concept may be very restrictive
for differential inclusion in general, indeed one may find different approaches
to work with a measure preserving differential inclusions (e.g. \[ 1 \] and the
references therein). However, as we shall see, the definition of measure pre-
serving \( \mathbb{F} \) provides interesting results for Filippov systems. In this work we
are able to clearly see how the nonuniqueness of solution becomes an issue for
the existence of invariant measures.

1.4. Main results. Now we state our main results. Theorem A and B are
proved in Section \( \S 2 \) and Theorem C is proved in Section \( \S 3 \).

Theorem A. Suppose that \( F : U \rightarrow \mathbb{R}^n \) is a Lipschitzian set-valued map
such that \( F(x) \subset \mathbb{R}^n \) is a closed convex set for every \( x \in U \). If the differential
inclusion \( \mathbb{F} \) admits an invariant probability measure \( \mu \), then there exists an
open set \( A \subset U \) such that \( \text{Sat}(N_F) \subset A \) and \( \mu(A) = 0 \).

Regarding Filippov system, the saturation of the sets \( \Sigma^s \) and \( \Sigma^c \) through
the trajectories of \( Z \) is contained in \( \text{Sat}(N_Z) \). Theorem A applied to Filippov
systems, which preserve a probability measure \( \mu \), implies directly that there
exists an open set \( A \subset M \) such that \( \text{Sat}(N_Z) \subset A \) and \( \mu(A) = 0 \). Particularly,
in this case \( \mu(\text{Sat}(\Sigma^s \cup \Sigma^c)) = 0 \).

Our second main result provides a necessary and sufficient condition in order
to exist an invariant probability measure preserved by a Filippov system.

Theorem B. A Filippov systems \( Z \) defined on a compact manifold admits an
invariant probability measure if, and only if, there is a compact set \( K \subset M \) such
that the trajectories of the Filippov systems restricted to \( K \), \( Z_t|_K \), determines
a invariant flow.

Our third main result provides a necessary and sufficient condition in order
to a Filippov system to preserve a volume measure.

Theorem C. Let \( f : M \rightarrow (0, \infty) \) be a piecewise constant function defined as
being \( \alpha^\pm \) if \( h(x) \geq 0 \). The Filippov system \( Z = (F^+, F^-)_h \) preserves \( \nu = f \cdot \lambda \)
if and only if the vector fields \( F^\pm \) preserve the measures \( \nu^\pm = \alpha^\pm \cdot \lambda \) on \( \Sigma^\pm \)
and \( \alpha^+ F^+ h(p) = \alpha^- F^- h(p) \) for every \( p \in \Sigma \).

In section \( \S 3 \) we provide two main consequences of Theorem C. The first
consequence (Corollary A) shows that the Filippov systems preserving Lebesgue
measure are the refractive ones which preserve Lebesgue measure in the
regions of continuity. The second consequence (Corollary B) gives a necessary
condition for a tangency-tangency point of a planar Filippov system to be a
center point.
In section §4 we study several examples of Filippov systems and their invariant measures. The first example (see subsection §4.1) deals with constant piecewise vector fields defined on the torus \( T^2 \) and on the Klein bottle \( \mathbb{K}^2 \). We provide conditions (Proposition 1) in order for these systems to preserve absolutely continuous measures. The second example (see subsection §4.2) is a Filippov system defined on \( T^2 \) with no invariant probability measures and such that \( \text{Sat}(\mathcal{N}_Z) = T^2 \). The third example (see subsection §4.3) is a Filippov system defined on \( T^2 \) preserving an absolutely continuous probability measure, for which \( \text{Sat}(\mathcal{N}_Z) \) is an open set strictly contained in \( T^2 \). The fourth and last example (see subsection §4.4) is a Filippov system defined on \( T^2 \) with no invariant probability measures, for which \( \text{Sat}(\mathcal{N}_Z) \) is a closed set strictly contained in \( T^2 \).

2. INVARIANT PROBABILITY MEASURES FOR DIFFERENTIAL INCLUSIONS AND FILIPPOV SYSTEMS

We start this section introducing a preliminary result (see Theorem 4.12 from [7], page 109) that will be needed to prove Theorems A and B. Firstly a set-valued map \( F : U \to Y \) (\( Y \) topological space) is called upper semi-continuous at \( x_0 \in X \) if for any open subset \( W \) of \( Y \) containing \( F(x_0) \) there exists a neighborhood \( V \subset U \) of \( x_0 \) such that \( F(V) \subset W \).

**Theorem 1.** Assume that \( \mathcal{F} : U \to \mathbb{R}^n \) is a Lipschitzian set-valued map, such that \( \mathcal{F}(x) \subset \mathbb{R}^n \) is a closed convex set for every \( x \in U \). Then the set valued map \( S_{\mathcal{F}} : U \to AC(\mathbb{R}^n) \) is also Lipschitzian, in particular it is upper semi-continuous. Moreover, given \( \phi_0 \in S_{\mathcal{F}}(x_0) \) there exists a continuous function \( \Phi : U \to S(\mathcal{F}) \) satisfying \( \Phi(x) \in S_{\mathcal{F}}(x) \) and \( \Phi(x_0) = \phi_0 \).

**Remark 1.** The set-valued map \( \mathcal{F}_Z : D \to \mathbb{R}^n \), defined in [4], is Lipschitzian and, for each \( x \in U \), \( \mathcal{F}(x) \subset \mathbb{R}^n \) is a closed convex set.

In the remainder of this section we have the proofs of Theorems A and B.

2.1. Proof of Theorem A. Assuming the existence of an invariant probability measure \( \mu \) we shall prove that, for each \( x_0 \in \text{Sat}(\mathcal{N}_F) \), there exists a small neighborhood \( V_{x_0} \subset U \) such that \( \mu(V_{x_0}) = 0 \).

First assume that \( x_0 \in \mathcal{N}_F \). Then there exists \( \phi_1, \phi_2 \in S_F \) and \( \overline{t} \neq 0 \) such that \( y_1 = \phi_1(\overline{t}) \neq \phi_2(\overline{t}) = y_2 \). Applying Theorem 1 we get the existence of continuous functions \( \Phi_1, \Phi_2 : U \to S(\mathcal{F}) \) such that \( \Phi_i(x) \in S_F(x) \) and \( \Phi_i(x_0) = \phi_i \) for \( i \in \{1, 2\} \). Therefore we can find a small neighborhood \( V_{x_0} \subset U \) of \( x_0 \) such that \( \Phi_1(V_{x_0})(\overline{t}) \cap \Phi_2(V_{x_0})(\overline{t}) = \emptyset \). Denote \( V_{i} = \Phi_i(V_{x_0})(\overline{t}) \). Since \( V_{i} \subset S_F(V_{x_0})(\overline{t}) \) we have that

\[
\mu(V_1) + \mu(V_2) \leq \mu(S_F(V_{x_0})(\overline{t})) = \mu(V_{x_0}).
\]
Nevertheless $V_{x_0} \subset S_F(V_i)(-\overline{t})$ for $i = 1, 2$. Indeed, let $x \in V_{x_0}$, so $v_i = \Phi_i(x)(\overline{t}) \in V_i$ and $\psi_i(t) = \Phi_i(x)(t + \overline{t}) \in S_F(v_i)$, which implies that $x = \psi_i(-\overline{t}) \in S_F(v_i)(-\overline{t})$. Hence

\begin{equation}
\mu(V_{x_0}) \leq \mu(S_F(V_i)(-\overline{t})) = \mu(V_i), \quad \text{for } i \in \{1, 2\}.
\end{equation}

From (6) and (7) we conclude that $\mu(V_{x_0}) = 0$.

Now assume that $x_0 \in \text{Sat}(N_F) \setminus N_F$. In this case, there exist $y_0 \in N_F$, $\phi_0 \in S_F(y_0)$, and $t_0 \neq 0$ such that $\phi_0(t_0) = x_0$. Since $x_0 \notin N_F$ we have that $S_F(x_0)(t) = \{\psi_0(t) = \phi_0(t + t_0)\}$. From the first part of the proof, there exists a neighborhood $V_{y_0} \subset U$ of $y_0$ such that $\mu(V_{y_0}) = 0$. Since $\psi_0(-t_0) = y_0$ there exists a small neighborhood $V \subset AC(\mathbb{R}^n)$ of $S_F(x_0)(t) = \{\psi_0(t)\}$ such that $\phi(t_0) \in V_{y_0}$ for every $\phi \in V$. Now, since $S_F$ is upper semi-continuous at $x_0$ there exists a neighborhood $V_{x_0} \subset U$ of $x_0$ such that $S_F(V_{x_0}) \subset V$, which implies that $S_F(V_{x_0})(-t_0) \subset V_{y_0}$. Hence

$$0 \leq \mu(V_{x_0}) = \mu(S_F(-t_0)) \leq \mu(V_{y_0}) = 0.$$

Finally, take $A$ as being the following open subset of $U$:

$$A = \bigcup \{V_x : x \in \text{Sat}(N_F)\}.$$

We conclude the proof of Theorem [A] by noticing that $\mu(A) = 0$. Indeed, otherwise we would be able to find a density point $x \in A$, which is an absurd because $x \in V_{x_0}$ for some $x_0 \in \text{Sat}(N_F)$ and $\mu(V_{x_0}) = 0$.

### 2.2. Proof of Theorem [B]

If $Z$ admits an invariant probability measure then, from Theorem [A] there exists a open set $A \subset M$ such that $\text{Sat}(N_Z) \subset A$ and $\mu(\text{Sat}(N_Z)) = 0$. Now take $C = M \setminus A$. Notice that $C$ is a closed set in a compact $M$, consequently also compact, and $Z_t|_C$ is a flow because $C$ is far from $\text{Sat}(N_Z)$. We claim that there exists an invariant compact set $K \subset C$. Indeed, since $\mu(M) = 1$ consider $K = \text{supp}(\mu) \neq \emptyset$, which is compact because the support of $\mu$ is closed. Clearly $K \subset C$. We claim that $K$ in invariant through $Z_t$. Indeed for $x_0 \in K$, $Z_t(x_0) = \{y_0\}$ is a unitary set. Since $Z_t$ is upper semi-continuous we have that for any neighborhood $W \subset U$ of $y_0$ there exists a neighborhood $V \subset U$ of $x_0$, $\mu(V) > 0$, such that $Z_t(V) \subset W$. Therefore $\mu(W) \geq \mu(Z_t(V)) = \mu(V) > 0$, which implies that $y_0 \in K$.

To prove the converse implication, just notice that every flow defined on a compact invariant set $K$ admits an invariant probability measure $\nu$. So for a Borel set $B \subset M$ define $\mu(B) = \nu(B \cap K)$. So $\mu$ is an invariant probability measure for $Z$. This concludes the proof of Theorem [B].

### 3. Invariant volume measure for Filippov systems

Consider the Filippov system [2] defined on the compact Riemannian manifold $M$. Note that $M$ can also be seen as a measurable space, where the sigma
algebra $\mathcal{B}$ is the Borel sigma algebra that is, the one generated by the open sets of $M$. Recall that a probability measure on $M$ is a map $\mu : \mathcal{B} \to [0, 1]$ such that $\mu(M) = 1$, $\mu(\bigcup_{i \in \mathbb{N}} A_i) = \sum_i \mu(A_i)$ if the sets $A_i$ are disjoint, and $\mu(A) \leq \mu(B)$ if $A \subset B$. Furthermore, since $M$ is a Riemannian manifold we fix throughout the paper its metric, that is for each $p \in M$ we associate an inner product $\langle \cdot, \cdot \rangle_p$ on $T_pM$. When the context is clear we shall denote $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_p$.

For the case of Filippov systems [2] the solutions of the associated differential inclusion [4] (and the sets $Z_t(p_0)$) are well described and fairly known in the literature (see [5]). In order to state these conventions we distinguish some regions on $\Sigma$. The points on $\Sigma$ where both vector fields $F^+$ and $F^-$ simultaneously point outward or inward from $\Sigma$ define, respectively, the escaping $\Sigma^e$ and sliding $\Sigma^s$ regions, and the complement of its closure in $\Sigma$ defines the crossing region $\Sigma^c$. The complement of the union of those regions $\Sigma^t$ constitute the tangency points between $F^+$ or $F^-$ with $\Sigma$. Denoting $F^\pm h(p) = \langle \nabla h(p), F^\pm(p) \rangle_p$, we have

$$
\begin{align*}
\Sigma^c &= \{ p \in \Sigma : F^+ h(p) \cdot F^- h(p) > 0 \}, \\
\Sigma^s &= \{ p \in \Sigma : F^+ h(p) < 0, F^- h(p) > 0 \}, \\
\Sigma^e &= \{ p \in \Sigma : F^+ h(p) > 0, F^- h(p) < 0 \}, \\
\Sigma^t &= \{ p \in \Sigma : F^+ h(p) F^- h(p) = 0 \}.
\end{align*}
$$

For $p \in \Sigma^c$ the solutions either side of the discontinuity $\Sigma$, reaching $p$, can be joined continuously, forming a solution that crosses $\Sigma^c \subset N$. Alternatively, for $p \in \Sigma^{s,e} = \Sigma^s \cup \Sigma^e \subset N$ the solutions either side of the discontinuity $\Sigma$, reaching $p$, point both toward or outward $\Sigma$. For these points the solutions either side of the discontinuity $\Sigma$ can be joined continuously to solutions that slide on $\Sigma^{s,e}$ following the sliding vector field:

$$
Z^s(p) = \frac{F^- h(p) F^+(p) - F^+ h(p) F^-(p)}{F^- h(p) - F^+ h(p)}, \text{ for } p \in \Sigma.
$$

In what follows we prove our first main result regarding volume preserving Filippov systems.

3.1. Proof of Theorem [C]. First of all, a necessary condition for $Z$ to preserve $\nu$ is that $\Sigma^s \cup \Sigma^e = \emptyset$. Indeed, if $\Sigma^s \neq \emptyset$ (resp. $\Sigma^e \neq \emptyset$) we may find sets $A \subset M$, with positive measure, such that the forward flow (resp. backward flow) of $Z$ collapses $A$ into a set $\tilde{A} \subset \Sigma^s$ (resp. $\tilde{A} \subset \Sigma^e$), but since $\Sigma^s$ (resp. $\Sigma^e$) is a codimension one manifold it has zero volume measure, hence $\tilde{A}$ has zero volume measure. Another important point is that the saturation of $\Sigma^t$ through the orbits of $Z$ has zero volume measure. So we are not worried with this set.
Now, to say that the vector fields $F^\pm(x)$ and $Z(x)$ preserve, respectively, the measures $\nu^\pm$ and $\nu$ is equivalent to say that the vector fields $G^\pm(x) = \alpha^\pm F^\pm(x)$ and $G(x) = f(x)Z(x)$ preserve the Lebesgue measure $\lambda$.

Since $\Sigma = h^{-1}(0)$, with 0 a regular value of $h$, the following map

\[ \eta : x \in U \mapsto \frac{\nabla h(x)}{||\nabla h(x)||} \in T_xM \]

is well defined on some neighborhood $U$ of $\Sigma$. Notice that $\eta$ is a unit vector field on $U$ which is normal to the codimension one manifold $\Sigma$.

Let $\sigma$ be a small disk inside $\Sigma$. The flux $V^\pm(\sigma)$ of the vector fields $G^\pm$ through $\sigma$, that is the total amount of flow of $G^\pm$ passing through $\sigma$, is measured by the surface integral

\[ V^\pm(\sigma) = \int_\sigma \langle G^\pm, \eta \rangle \, d\Sigma = \int_\sigma \frac{\langle G^\pm, \nabla h \rangle}{||\nabla h||} \, d\Sigma = \int_\sigma G^\pm h ||\nabla h|| \, d\Sigma. \]

where $d\Sigma$ denotes the volume form of $\Sigma$. Since the vector fields $G^\pm$ preserve volume measure, the vector field $G$ will preserve volume measure if and only if $V^+(\sigma) = V^-(\sigma)$ for every small $\sigma \subset \Sigma$. Hence we conclude that $Z$ preserves $\nu$ if and only if $\alpha^+ F^+ h(p) = \alpha^- F^- h(p)$ for every $p \in \Sigma$.

3.2. Some consequences of Theorem C. Piecewise continuous systems of kind (2) satisfying $F^\pm(x)$ preserve volume measure, the vector field $G$ is normal to the codimension one manifold $\Sigma$.

Let $\sigma$ be a small disk inside $\Sigma$. The flux $V^\pm(\sigma)$ of the vector fields $G^\pm$ through $\sigma$, that is the total amount of flow of $G^\pm$ passing through $\sigma$, is measured by the surface integral

\[ V^\pm(\sigma) = \int_\sigma \langle G^\pm, \eta \rangle \, d\Sigma = \int_\sigma \frac{\langle G^\pm, \nabla h \rangle}{||\nabla h||} \, d\Sigma = \int_\sigma G^\pm h ||\nabla h|| \, d\Sigma. \]

where $d\Sigma$ denotes the volume form of $\Sigma$. Since the vector fields $G^\pm$ preserve volume measure, the vector field $G$ will preserve volume measure if and only if $V^+(\sigma) = V^-(\sigma)$ for every small $\sigma \subset \Sigma$. Hence we conclude that $Z$ preserves $\nu$ if and only if $\alpha^+ F^+ h(p) = \alpha^- F^- h(p)$ for every $p \in \Sigma$.

**Corollary A.** The Filippov system $Z = (F^+, F^-)_h$ preserves volume measure if and only if $F^\pm$ preserve volume measure in $\Sigma^\pm$ and $Z$ is a refractive system.

A point $p \in \Sigma$ is called a tangency of order $k$ for $F^\pm$ if $(F^\pm)^{k-1}h(p) = 0$ and $(F^\pm)^{k-1}h(p) \neq 0$. If $(F^\pm)^{k-1}h(p) \leq 0$ then it is called invisible, otherwise it is called visible. It is fairly known that if $F^\pm$ are planar vector fields and $p$ is a visible tangency of both vector fields $F^\pm$ of even order such that $F^+(p)F^-(p) < 0$ (see Figure 2) then a first return map is well defined on a small neighborhood of $p$ in $\Sigma$. In this case, as an application of Corollary A, the next result provides sufficiently conditions in order to assure that $p$ is a center point, that is there exists a small neighborhood $U$ of $p$ in $M$ such that all the orbits contained in $U \setminus \{0\}$ are closed.

**Corollary B.** Consider the Filippov vector field $Z = (F^+, F^-)_h$ and let $p \in \Sigma$ be a invisible tangency for both vector field. If $tr(dF^\pm(p)) = 0$ and $Z$ is refractive then $p$ is a center point.

**Proof.** From Corollary A $Z$ is a volume preserving Filippov system. The refractive condition also implies that $F^+(p)F^-(p) < 0$. If the first return map is not the identity then $p$ would be attractive or repulsive, which is an absurd.
Figure 2. Visible tangency point for both vector fields $F^\pm$ of even order such that $F^+(p)F^-(p) < 0$.

Therefore the first return map is the identity which implies that $p$ is a center point. □

4. Examples of piecewise smooth vector fields

This section is devoted to provide examples of piecewise smooth vector fields defined on the torus $T^2$ and on the Klein bottle $K^2$ for which the main results of the previous section may be applied. In subsection §3.1, we deal with piecewise constant vector fields on $T^2$ and on $K^2$. As a consequence of Theorem C, it is established conditions for these systems to admit an invariant volume measure. In subsection §3.2, it is provided an example of a piecewise constant vector field on $T^2$ such that $\mathcal{N}_Z = T^2$ and therefore, as a consequence of Theorem B, does not admit an invariant probability measure. In subsection §3.3 we provide an example of a piecewise smooth vector field on $T^2$ such that $T^2 \setminus \mathcal{N}_Z$ is a nonempty closed set and therefore, as a consequence of Theorem B, admit an invariant probability measure. In subsection §3.4 we provide an example of a piecewise smooth vector field on $T^2$ for which the hypotheses of Theorem B do not hold, namely when $T^2 \setminus \mathcal{N}_Z$ is a nonempty open set. We show that this vector field admits an invariant absolutely continuous probability measure. Furthermore, we also observe that this vector field can be perturbed in order to obtain a second example for which $T^2 \setminus \mathcal{N}_Z$ is still a nonempty open set, but with no invariant probability measure.

First of all consider the following piecewise smooth vector field defined on the square $S = [\alpha, \alpha + p] \times [\beta, \beta + q] \subset \mathbb{R}^2$:

\begin{equation}
Z(x, y) = \begin{cases}
X_i(x, y) & \text{if } x \in [h_i, h_{i+1}], \ for \ i = 1, 2, \ldots, n - 1, \\
X_n(x, y) & \text{if } x \in [0, h_1],
\end{cases}
\end{equation}

where each $X_i(x, y)$, $i = 1, 2, \ldots, n$, is a smooth vector field defined on $S$. Denote the sets of discontinuity by $\Sigma_i = [\alpha, \alpha + p] \times \{h_i\}$, for $i = 1, 2, \ldots, n$, with $\beta < h_1 < h_2 < \cdots < h_n = \beta + q$.

We denote by $T^2$ the Torus given by the quotient $T^2 = S/\sim$, where

$$(x, y) \sim (z, w) \iff x - z \in p\mathbb{Z}, \ y - w \in q\mathbb{Z},$$

where $p$ and $q$ are positive integers.
which identifies \([\alpha, \alpha + p] \times \{b\}\) with \([\alpha, \alpha + p] \times \{b + q\}\) and \(\{a\} \times [\beta, \beta + q]\) with \(\{a + p\} \times [\beta, \beta + q]\), preserving the orientation. Accordingly, the vector field (9) can be seen as defined on \(T^2\). In this case the set of discontinuity \(\Sigma\) is given by the union of \(\Sigma_i\), for \(i = 1, 2, \ldots, n\). Clearly \(\Sigma_n = [\alpha, \alpha + p] \times \{\beta\} = [\alpha, \alpha + p] \times \{\beta + q\}\).

Analogously we denote by \(K^2\) the Klein bottle given by the quotient \(K^2 = S/\sim\), where now

\[(x, y) \sim (z, w) \Leftrightarrow x - z \in p\mathbb{Z}, y + w \in q\mathbb{Z},\]

which identifies \([\alpha, \alpha + p] \times \{b\}\) with \([\alpha, \alpha + p] \times \{b + q\}\) and \(\{a\} \times [\beta, \beta + q]\) with \(\{a + p\} \times [\beta, \beta + q]\), reversing the orientation in the last identification. The piecewise vector field (9) can be seen as defined on \(K^2\), but in this case an additional discontinuity is added, namely \(\Sigma_0 = \{a\} \times [\beta, \beta + q] = \{a + p\} \times [\beta, \beta + q]\). It is worthy to say that the Klein bottle could also be obtained by reversing the orientation of the first identification. In this case the set of discontinuity would coincide with the torus case.

4.1. Piecewise constant vector fields on \(T^2\) and on \(K^2\). Let \(a_i \in \mathbb{R}\) and \(b_i > 0\), for \(i = 1, 2, \ldots, n\). Consider the vector field (9) defined on \([0, 1]^2\), and assume that \(X_i(x, y) = (a_i, b_i)\), for \(i = 1, 2, \ldots, n\). The next result is obtained from Theorem \(\Box\):

**Proposition 1.** Let \(f : [0, 1]^2 \to \mathbb{R}\) be the following constant piecewise function:

\[
(10) \quad f(x, y) = \begin{cases} 
\alpha_i & \text{if } x \in [h_i, h_{i+1}], \text{ for } i = 1, 2, \ldots, n - 1, \\
\alpha_n & \text{if } x \in [0, h_1].
\end{cases}
\]

(a) The vector field (9) defined on \(T^2\) preserves the measure \(\nu = f \cdot \lambda\) if and only if, for each \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i = \alpha/b_i, \alpha \in \mathbb{R}\).

(b) The vector field (9) defined on \(K^2\) preserves the measure \(\nu = f \cdot \lambda\) if and only if, for each \(i \in \{1, 2, \ldots, n\}\), \(\alpha_i = \alpha/b_i, \alpha \in \mathbb{R}\), and \(a_i/b_i = a_{n-i+1}/b_{n-i+1}\)

**Remark 2.** Note that, from statement (a) of Proposition 1, when the piecewise vector field (9) is defined on \(T^2\), one can always find a piecewise constant function (10) such that (9) preserves the absolutely continuous measure \(\nu = f \cdot \lambda\). Nevertheless that is not the case when (9) is defined on \(K^2\). Indeed, from statement (b) of Proposition 1, some conditions on the parameters of (9) must be satisfied.

**Proof of Proposition 1.** We know that each vector field \(X_i\) preserves the measure \(\alpha_i \cdot \lambda\), for \(i = 1, 2, \ldots, n\). Applying Theorem \(\Box\) for each connected component \(\Sigma_i, i = 1, 2, \ldots, n\), of the discontinuity manifold \(\Sigma\) we get that \(Z\) preserves
the measure \( \nu = f \cdot \lambda \) if and only if

\[
\begin{cases}
0 = b_i \alpha_i - b_{i+1} \alpha_{i+1}, & \text{for } i = 1, 2, \ldots, n - 1, \\
0 = b_n \alpha_n - b_1 \alpha_1,
\end{cases}
\]

and

\[
\alpha_i \alpha_{i+1} > 0, \quad \text{for } i = 1, 2, \ldots, n - 1, \quad \text{and } \alpha_n \alpha_1 > 0.
\]

The last equality of system (11) is due to the identification \([\alpha, \alpha + p] \times \{b\} \sim [\alpha, \alpha + p] \times \{b + q\}\). Adding up the first \(n - 1\) equalities of (11) we get \(-b_n \alpha_n + b_1 \alpha_1 = 0\), which is equivalent to the last equality of (11). Therefore the system of linear equations (11) admits non-trivial solutions. Solving it we conclude that \((\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha(b_1^{-1}, b_2^{-1}, \ldots, b_n^{-1}), \alpha \in \mathbb{R}\). Since \(b_i > 0\) for \(i = 1, 2, \ldots, n\), the condition (12) holds. It concludes the proof of statement (a).

When the vector field (9) is defined on \(K^2\), system (11) is again a necessary condition for (9) to preserve \(\nu\), hence \((\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha(b_1^{-1}, b_2^{-1}, \ldots, b_n^{-1}), \alpha \in \mathbb{R}\). Moreover, applying Theorem C regarding the set of discontinuity \(\Sigma_0\), we see that \(\alpha_i a_i = \alpha_{n-i+1} a_{n-i+1}\), which implies \(a_i/b_i = a_{n-i+1}/b_{n-i+1}\), for \(i = 1, 2, \ldots, n\), is another necessary condition for (9) to preserve \(\nu\). It remains to prove that these conditions are sufficient.

4.2. The saturation of \(N_Z\) is the whole \(T^2\). As a trivial example of a piecewise smooth system such that \(N_Z = T^2\), we may consider the following piecewise constant vector field defined on the torus \(T^2 = [0, 1]^2/\sim\):

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } y \geq 0,
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } y \geq 0,
\end{cases}
\]

Indeed, for a given \(p \subset M \setminus \Sigma\) its forward trajectory reach the sliding region, and its backward trajectory reach the escaping region.

4.3. The saturation of \(N_Z\) is open and strictly contained in \(T^2\). As an example of a piecewise smooth system such \(N_Z \neq T^2\) is open, we may consider
the vector field defined on the torus $[0, \pi] \times [-3/2, 3]/\sim$:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix}
(y - 5/2) \\
(y - 7/2) \\
\frac{1}{5} - \sin^2(x)
\end{pmatrix} & \text{if } 3/2 \leq y \leq 3, \\
\begin{pmatrix}
1 \\
(y - 2)(y - 1) \\
\frac{3}{5} + \sin^2(x)
\end{pmatrix} & \text{if } 0 < y < 3/2, \\
\begin{pmatrix}
1 \\
(y + 2)(y + 1) \\
\frac{3}{5} - \sin^2(x)
\end{pmatrix} & \text{if } -3/2 < y < 0.
\end{cases}
\]

Figure 3. Phase space of the piecewise smooth vector field (14) defined on the rectangle $[0, \pi] \times [-3/2, 3]$. The shaded region indicates the set $\mathcal{N}_Z$.

The set of discontinuity is given by $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = [0, \pi] \times \{0\}$ and $\Sigma_2 = [0, \pi] \times \{3/2\}$. Note that the vector field is continuous on the lines $[0, \pi] \times \{-3/2\}$ and $[0, \pi] \times \{-3/2\}$. The contact between the vector field and the discontinuous manifold $\Sigma$ occurs at the points $c_1 = (), c_2 = (), c_3 = ()$ and $c_4 = ()$. Moreover $\Sigma^{s,e} = \Sigma_1 \setminus \{c_1, c_2\}$ and $\Sigma^e = \Sigma_2 \setminus \{c_3, c_4\}$. We stress that the breaking of unicity occurs at the sliding and escaping sets and at the tangency.
\(c_1\), so \(n = \Sigma^s \cup \Sigma^u \cup \{c_1\}\). Furthermore, it is easy to see that \(\phi_1(t) = (t, 1)\) and \(\phi_2(t) = (t, -1)\) are limit cycles. After some simple computations we conclude that \(N_Z = \{(x, y) \in [0, \pi] \times [-3/2, 3]: -1 < y < 1\}\), which is the open region delimited by the limit cycles \(\phi_1\) and \(\phi_2\).

4.4. The saturation of \(N_Z\) is closed and strictly contained in \(T^2\). When \(N_Z \neq T^2\) is closed Theorem B does not apply. In this case we may find examples for which there exist invariant probability measures as well as examples for which there is not exist invariant probability measures.

In what follows we provide a piecewise smooth vector field defined on the torus \([-\pi/2, 3\pi/2] \times [-3\pi/2, 3\pi/2]\)/~ for which there exists an invariant probability measure. Moreover this probability measure is absolutely continuous:

\[
\begin{cases}
(\cos(x)(-\sqrt{3}\cos(y) + \sin(y)) & \text{if } y \leq 0, \\
(-\sin(x)(\cos(y) + \sqrt{3}\sin(y)) & \text{if } y \geq 0,
\end{cases}
\]

Indeed, let \(U = M \setminus N_Z\). Note that if \((x, y) \in U\) and \(y \leq 0\) then \(x \in [-\pi/2, 0) \cup (\pi, 3\pi/2]\). In this case \(\sin(x) > 0\). So taking \(h(x, y) = u, \alpha^+(x, y) = 1\) and \(\alpha^-(x, y) = \sin(x)\) define \(f : U \to \mathbb{R}\) as being \(\alpha^\pm\) if \(h(x, y) \geq 0\).

Figure 4. Phase space of the piecewise smooth vector field (13) defined on the rectangle \([-\pi/2, 3\pi/2] \times [-3\pi/2, 3\pi/2]\). The shaded region indicates the set \(N_Z\).

We note that the vector field (14) may be perturbed in order to get two limit cycles, each one tangent to \(\Sigma\) at two points. In this case \(N_Z \neq M\) is closed,
nevertheless there are no invariant probability measure. Indeed suppose this perturbed system has an invariant measure \( \mu \) such that \( \mu(U) \neq 0 \), for some \( U \subset M \setminus \mathcal{N}_Z \). In this case there exists a sequence of times \( (t_i) \) for which the sets \( \phi_{t_i}(U) \subset M \setminus \mathcal{N}_Z \) are disjoint, which implies that \( \mu(M) = \infty \).

Acknowledgements

D.D.N. was supported by a FAPESP grant 2016/11471-2 and R.V was supported by FAPESP grant 2016/22475-9.

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