Relativistic Quantum Newton’s Law
and Photon Trajectories

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Abstract
Using the relativistic quantum Hamilton-Jacobi equation within the framework of the equivalence postulate, and grounding oneself on both relativistic and quantum Lagrangians, we construct a Lagrangian of relativistic quantum system in one dimension and derive a third order equation of motion representing a first integral of the relativistic quantum Newton’s law. Then, we investigate the free particle case and establish the photon’s trajectories.

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1 Introduction

Deriving quantum mechanics from an equivalence postulate, Faraggi and Matone showed that the Schrödinger wave function must have the form

\[ \phi(x) = \left( \frac{\partial S_0}{\partial x} \right)^{-\frac{1}{2}} \left[ \alpha \exp \left( \frac{i}{\hbar} S_0 \right) + \beta \exp \left( -\frac{i}{\hbar} S_0 \right) \right], \]

where \( \alpha \) and \( \beta \) are complex constants and \( S_0 \) a real function representing the quantum reduced action. They established that the conjugate momentum given by

\[ P = \frac{\partial S_0}{\partial x} \]

never vanishes for bound and unbound states making possible a dynamical approach of the quantum motion of particles. This conjugate momentum is always real even in classically forbidden regions. They showed also, within the framework of differential geometry, that the quantum stationary Hamilton-Jacobi equation (QSHJE) which leads to the Schrödinger equation is

\[ \frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 + V(x) - E = \frac{\hbar^2}{4m_0} \left[ \frac{3}{2} \left( \frac{\partial S_0}{\partial x} \right)^{-2} \left( \frac{\partial^2 S_0}{\partial x^2} \right)^2 - \left( \frac{\partial S_0}{\partial x} \right)^{-1} \left( \frac{\partial^3 S_0}{\partial x^3} \right) \right], \]

where \( V(x) \) is the potential and \( E \) the energy. The solution of Eq. (3) investigated by Floyd and Faraggi and Matone is given in Ref. [9] as

\[ S_0 = \hbar \arctan \left( \frac{\theta}{\phi} + b \right), \]

where \( a \) and \( b \) are real constants. \( \theta \) and \( \phi \) are two real independent solutions of the Schrödinger equation. Taking advantage on these results, Bouda and Djama have recently introduced a quantum Lagrangian

\[ L(x, \dot{x}, \mu, \nu) = \frac{1}{2} m \dot{x}^2 f(x, \mu, \nu) - V(x), \]

from which they derived the quantum law of motion. They stated that the conjugate momentum of the non relativistic and spin-less particle is written as

\[ \frac{\partial S_0}{\partial x} = 2 \frac{(E - V)}{\dot{x}}. \]

From this last equation, they derived the first integral of the quantum Newton’s law (FIQNL).

\[ (E - V)^4 - \frac{m \dot{x}^2}{2} (E - V)^3 + \frac{\hbar^2}{8} \left[ \frac{3}{2} \left( \frac{\dot{x}^2}{\dot{x}} \right)^2 - \frac{\dot{x}}{\dot{x}} \right] (E - V)^2 \]

\[ - \frac{\hbar^2}{8} \left[ \dot{x} \frac{d^2 V}{dx^2} + \ddot{x} \frac{dV}{dx} \right] (E - V) - \frac{3\hbar^2}{16} \left[ \dot{x} \frac{dV}{dx} \right]^2 = 0, \]

which goes at the classical limit \( (\hbar \to 0) \) to the classical conservation equation

\[ \frac{m \dot{x}^2}{2} + V(x) = E. \]
Bouda and Djama have also plotted some trajectories of the particle for several potentials \[10\].

The construction of the Lagrangian (5) and the establishment of Eqs. (6) and (7) are important steps to build a deterministic theory which restores the existence of trajectories \[9, 10\]. Nevertheless, such a formalism cannot approach both relativistic velocities cases, and more than one dimension motions.

The aim of this paper is to generalize the dynamical formalism that we have recalled above \[8\] into the one dimensional relativistic velocities cases. In this purpose let us recall the finding of Faraggi, Matone and Bertoldi concerning the relativistic quantum systems. They stated that the relativistic quantum wave function is given by Eq. (1), where $S_0$ defines the relativistic quantum reduced action, and wrote the relativistic quantum stationary Hamilton-Jacobi equation (RQSHJ) as \[3, 4\]

\[
\frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 - \frac{\hbar^2}{4m_0} \left[ \frac{3}{2} \left( \frac{\partial S_0}{\partial x} \right)^2 - \left( \frac{\partial^2 S_0}{\partial x^2} \right)^2 \right] - \left( \frac{\partial S_0}{\partial x} \right)^{-1} \left( \frac{\partial^3 S_0}{\partial x^3} \right) + \frac{1}{2m_0c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 ,
\]

where $V$ is the potential, $E$ is the total energy of the particle of mass equal to $m_0$ at rest and $c$ is the light velocity in vacuum. The solution of Eq. (8) can be expressed by Eq. (4), where $\theta_1$ and $\theta_2$ represent now two real independent solutions of the Klein-Gordon equation

\[
-c^2 \hbar^2 \frac{\partial^2 \phi}{\partial x^2} + \left[ m_0^2 c^4 - (E - V)^2 \right] \phi(x) = 0 .
\]

Taking advantage on these results, we will introduce in the following sections a relativistic quantum formalism with which we can study the dynamics of high energy particles. In Sec. 2, we present the relativistic quantum formalism with which we establish, in Sec. 3, the expression of the conjugate momentum. Still in Sec. 3, we derive the relativistic quantum law of motion. Finally, we investigate in Sec. 4 the free particle case and photon’s trajectories.

### 2 Construction of the Relativistic Quantum Lagrangian

Before introducing the relativistic quantum Lagrangian, let us recall the Lagrangian formalism of special relativity. The relativistic law of motion can be derived from a Lagrangian of the form \[11\]

\[
L = -m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} - V(x) ,
\]

where $c$ is the light velocity in vacuum, $V(x)$ is the potential and $m_0$ the mass of the particle at rest. The action of the system is given by

\[
S(x,t) = \int L \ dt ,
\]

Using expression (10) of the Lagrangian in the Lagrange equation derived from the least action principle, we get to

\[
\frac{m_0 \ddot{x}}{(1 - \dot{x}^2/c^2)^{3/2}} + \frac{dV}{dx} = 0 ,
\]
which represents the relativistic Newton’s law. Integrating Eq. (12), we find

\[ \frac{m_0 c^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} + V(x) = E, \]

(14)

where \( E \) define the total energy of the particle including the energy at rest \( m_0 c^2 \).

Eq. (14) represents the relativistic conservation equation. We note that in this case the light velocity in vacuum \( c \) appears as an upper limit of the particle velocity since \( \dot{x} \) and \( c \) take only real values.

Now consider a relativistic quantum system. As we have noticed for the quantum systems \[ \text{[9]}, \] the relativistic quantum reduced action \( S_0 \) expressed by Eq. (4) contains two constants more than the usual constant \( E \) appearing in the expression of the classical reduced action. This suggests that the relativistic quantum law of motion is a fourth order differential equation. Then, as in the quantum case \[ \text{[9]}, \] we introduce in the expression of the Lagrangian a function \( f \) of \( x \) depending on a set \( \Gamma \) of constants playing the role of hidden parameters.

As the relativistic quantum Lagrangian must goes at the classical limit (\( \hbar \to 0 \)) to the relativistic one, we postulate the following form for the Lagrangian

\[ L = -m_0 c^2 \sqrt{1 - \frac{\dot{x}^2}{c^2}} f(x, \Gamma) - V(x), \]

(15)
in which the function \( f(x, \Gamma) \) satisfies

\[ \lim_{\hbar \to 0} f(x, \Gamma) = 1. \]

(16)

The Hamiltonian corresponding to the Lagrangian (14) can be expressed by

\[ H = P\dot{x} - L, \]

(17)
since \( L \) depends only on the variables \( x, \dot{x} \) and the set \( \Gamma \) of constants, while the conjugate momentum is given by the relation

\[ P = \frac{\partial L}{\partial \dot{x}} = \frac{m_0 \dot{x} f(x, \Gamma)}{\sqrt{1 - \frac{\dot{x}^2}{c^2}} f(x, \Gamma)}. \]

(18)

At the classical limit, we see clearly by using Eq. (15) that the relativistic quantum momentum tends to the relativistic one expressed as

\[ P = \frac{m_0 \dot{x}}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}}. \]

Replacing Eqs. (17) and (14) in Eq. (16), one obtains

\[ H = \sqrt{m_0^2 c^4 + \frac{P^2 c^2}{f}} + V(x), \]

(19)

For the stationary cases the Hamiltonian \( H \) corresponds to the total energy \( E \) of the particle. Then, we can write Eq. (19) as (after taking in account Eq. (18))

\[ E = \frac{m_0 c^2}{\sqrt{1 - (\frac{\dot{x}^2}{c^2}) f(x, \Gamma)}} + V(x), \]

(20)
This last relation can be deduced from the Lagrangian expressed by (15). Indeed, the least action principle leads to

\[
\frac{m_0\dot{x}^2 f}{\sqrt{1 - (\dot{x}^2/c^2)}} + \frac{m_0 \dot{x}}{2} \frac{df/dt}{\sqrt{1 - (\dot{x}^2/c^2)}} + m_0 \dot{x} f \frac{d}{dt} \left( \frac{1}{\sqrt{1 - (\dot{x}^2/c^2)}} \right) + \frac{dV}{dx} = 0 ,
\]

(21)

where we have used the fact that \( \dot{x} (\partial f/\partial x) = df/dt \). After integrating Eq. (21), one finds Eq. (20).

Now, let us establish the explicit form of \( f \). From (20), we can write

\[
\frac{1}{2} \frac{m_0 \dot{x}^2 f}{1 - (\dot{x}^2/c^2)} f + \frac{1}{2m_0 c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 .
\]

(22)

Using Eq. (18) in Eq. (22), we obtain

\[
\frac{P^2}{2m_0 f(x, \Gamma)} + \frac{1}{2m_0 c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 ,
\]

(23)

from which and taking into account Eq. (2), we deduce

\[
f(x, \Gamma) = \frac{c^2 (\partial S_0/\partial x)^2}{m_0^2 c^4 - (E - V)^2} .
\]

(24)

If we substitute \( S_0 \) with its expression (4) in Eq. (24), we notice that \( f \) depends on \( x, a, b \) and \( E \). Then, we identify the set \( \Gamma \) to the integration constants \( a, b \) and \( E \). Remark that at the classical limit \( (\hbar \to 0) \), we have from the RQSHJE (Eq. (9))

\[
(\partial S_0/\partial x)^2 = \frac{1}{c^2} \left[ (E - V)^2 - m_0^2 c^4 \right] ,
\]

and then, we obtain in Eq. (24)

\[
\lim_{\hbar \to 0} f(x, \Gamma) = 1 ,
\]

which corresponds to Eq. (16). From Eq. (24), we note that the function \( f(x, a, b) \) is real positive when \( E - V > m_0 c^2 \) (classically allowed regions) and real negative when \( E - V < m_0 c^2 \) (classically forbidden regions). We note also from Eq. (20) that \( c \) does not appear as an upper limit since the square root contains the function \( f \) depending on \( x \), while for the classically forbidden regions there is no limit of the particle velocity since \( f \) is negative.

Now, as it is the case for the quantum Lagrangian \[9\], let us demonstrate that the use of expressions (15) for the Lagrangian and (19) for the Hamiltonian is justified by appealing to a coordinate transformation \( x \to \dot{x} \) after which, the RQSHJE takes the form

\[
\frac{1}{2m_0} \left( \frac{\partial S_0(\dot{x})}{\partial \dot{x}} \right)^2 + \frac{1}{2m_0 c^2} \left[ m_0^2 c^4 - (E - \dot{V}(\dot{x}))^2 \right] = 0 \quad (25)
\]

The coordinate \( \dot{x} \) defined by

\[
\dot{x} = \int^x \frac{c (\partial S_0/\partial x)}{\sqrt{(E - V)^2 - m_0^2 c^4}} \, dx ,
\]

(26)
and which we will call “Relativistic quantum coordinate” is analogue to the quantum coordinate introduced by Faraggi and Matone in Ref. [3, 12].

By setting

\[ \hat{S}_0(\hat{x}) = S_0(x), \hat{V}(\hat{x}) = V(x), \]

Eq. (25) takes the form

\[ \frac{1}{2m_0} \left( \frac{\partial S_0}{\partial x} \right)^2 \left( \frac{\partial x}{\partial \hat{x}} \right)^2 + \frac{1}{2m_0c^2} \left[ m_0^2c^4 - (E - V(x))^2 \right] = 0. \]

Eq. (28) is equivalent to Eq. (20) since, as we can deduce from Eqs. (24) and (26)

\[ f(x, a, b, E) = \left( \frac{\partial x}{\partial \hat{x}} \right)^2. \]

Then, we note that a classical formulation with respect to the relativistic quantum coordinate \( \hat{x} \) is strictly equivalent to both Lagrangian and Hamiltonian formulations (Eqs. (15) and (19)).

3 The Relativistic Quantum Law of Motion

Now, taking Eq. (24) into Eq. (20), we get to the following expression of the conjugate momentum

\[ \frac{\partial S_0}{\partial x} = \frac{E - V(x)}{\dot{x}} - \frac{m_0^2c^4}{(E - V(x))\dot{x}}, \]

where we have eliminated one of the roots for \( \partial S_0/\partial x \) since Eq. (18) indicates that \( \dot{x} \) and \( P = \partial S_0/\partial x \) have the same sign in classically allowed regions and are opposites in forbidden ones. Note that it is also possible to derive Eq. (30) by using a relativistic quantum version of the Jacobi’s theorem, exactly in the same way as it is done in Ref. [9].

Let us now consider both relativistic and classical limits in Eq. (30). For the relativistic limit \( (c \rightarrow \infty) \), the kinetic energy \( T \) given by

\[ T = E - V - m_0c^2, \]

satisfies the relation

\[ T << m_0c^2. \]

Using Eq. (31) in Eq. (31), we have

\[ \dot{x} \frac{\partial S_0}{\partial x} = T + m_0c^2 - \frac{m_0^2c^4}{T + m_0c^2}, \]

which reduces at the relativistic limit to

\[ \frac{\partial S_0}{\partial x} = \frac{2T}{\dot{x}}, \]

after having used the fact that \( (1 + T/m_0c^2)^{-1} \approx (1 - T/m_0c^2) \). Eq. (34) corresponds to the quantum expression of the conjugate momentum given in Ref. [6] as

\[ \frac{\partial S_0}{\partial x} = 2 \left( \frac{E' - V}{\dot{x}} \right). \]
where $E' - V$ represents the kinetic energy for the non-relativistic cases. For the classical limit ($\hbar \to 0$), we have $f \to 1$, then Eq. (30) reduces to

$$\dot{x} \frac{\partial S_0}{\partial x} = \frac{m_0 c^2}{\sqrt{1 - \dot{x}^2/c^2}} - m_0 c^2 \sqrt{1 - \dot{x}^2/c^2},$$

from which we deduce

$$\frac{\partial S_0}{\partial x} = \frac{m_0 \dot{x}}{\sqrt{1 - \dot{x}^2/c^2}},$$

representing the expression of the conjugate momentum for a relativistic system. Thus, at the classical limit ($\hbar \to 0$) the relativistic quantum motion reduces to the relativistic one, and, at the relativistic limit ($c \to \infty$), it reduces to the quantum one. Then, we can consider expression (30) as a generalization of the conjugate momentum to the relativistic quantum systems.

Now, we will derive the relativistic quantum equation of motion using expression (30) of the conjugate momentum. Then, if one computes the derivatives

$$\frac{\partial^2 S_0}{\partial x^2} = -\frac{1}{\dot{x}} \frac{dV}{dx} \left[1 + \frac{m_0 c^4}{(E - V)^2}\right] + \frac{\dot{x}}{\dot{x}^3} \frac{1}{E - V} \left[m_0^2 c^4 - (E - V)^2\right],$$

$$\frac{\partial^3 S_0}{\partial x^3} = -\frac{1}{\dot{x}^2} \frac{d^2 V}{dx^2} \left[1 + \frac{m_0 c^4}{(E - V)^2}\right] + \left(3 \frac{\ddot{x}^2}{\dot{x}^5} - \frac{\dddot{x}}{\dot{x}^4}\right) \frac{(E - V)^2 - m_0^2 c^4}{(E - V)^2} + \frac{2}{\dot{x}^3} \frac{dV}{dx} \left[\frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4}\right]^2 \frac{\dddot{x}}{E - V} \frac{dV}{dx}^2 \frac{\dddot{x}}{E - V} = 0,$$

one can deduce the following expression of the Schwarzian derivative of $S_0$ with respect to $x$.

$$\{S_0, x\} = \left(\frac{\dot{x}}{\dot{x}^3} - \frac{3}{2} \frac{\dddot{x}}{\dot{x}^4}\right) + \left(\frac{\dddot{x}}{\dot{x}^3} \frac{dV}{dx} + \frac{\ddot{x}^2}{\dot{x}^2} \frac{d^2 V}{dx^2}\right) \frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} + \frac{3}{2} \frac{1}{(E - V)^2} \left(\frac{dV}{dx}\right)^2 \frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} \left[\frac{(E - V)^2 + m_0^2 c^4}{(E - V)^2 - m_0^2 c^4}\right]^2 + \frac{2}{(E - V)^2} \left(\frac{dV}{dx}\right)^2 \frac{m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} = 0.$$

Replacing Eqs. (30) and (40) in Eq. (9), we get to

$$[(E - V)^2 - m_0^2 c^4]^2 + \frac{\dot{x}^2}{c^2}(E - V)^2 [(E - V)^2 - m_0^2 c^4] + \frac{\hbar^2}{2} \left[3 \left(\frac{\dot{x}}{\dot{x}^3}\right)^2 - \frac{\dddot{x}}{\dot{x}}\right].$$

$$\frac{(E - V)^2 - \frac{\hbar^2}{2} \left(\frac{\dot{x}^2}{c^2} \frac{d^2 V}{dx^2}\right) \left[(E - V)^2 + m_0^2 c^4\right]}{(E - V)^2 - m_0^2 c^4} = \frac{\hbar^2}{2} \left(\frac{dV}{dx}\right)^2 \frac{m_0^2 c^4}{(E - V)^2 - m_0^2 c^4} = 0.$$
derivatives of the potential $V$ with respect to $x$. Then, the solution $x(t, E, a, b, c)$ of (41) contains four integration constants which can be determined by the initial conditions. It is clear that if we set $\hbar = 0$, Eq. (41) reduces to Eq. (14) representing the relativistic conservation equation. Note also that, after taking the relativistic limit ($c \to \infty$) in Eq. (41), one gets to Eq. (7) representing the FIQNL.

Remark that the FIRQNL can be derived by using the solution (4) of the RQSHJE as the same way presented in Ref. [9] to derive the FIQNL. This method leads straightforwardly to Eq. (41).

4 The Free Particle Case and Photon’s trajectories

We will study now the motion of a free particle with $m_0$ rest mass. For this case the expressions (30) of the conjugate momentum and (41) of the FIRQNL reduce respectively to

$$\frac{\partial S_0}{\partial x} = \frac{E}{\dot{x}} - \frac{m_0^2c^4}{xE},$$

and

$$(E^2 - m_0^2c^4)^2 \frac{\dot{x}^2}{c^2} E^2 (E^2 - m_0^2c^4) + \frac{\hbar^2}{2} \left( \frac{3}{2} \left( \frac{\dot{x}}{x} \right)^2 - \frac{\dot{x}}{x} \right) E^2 = 0. \quad (43)$$

In order to determine $x(t, E, a, b, c)$, we can solve Eq. (43) with appealing to the solutions of the Klein-Gordon equation for a vanishing potential. Indeed, in this purpose let us introduce the new variables

$$U = \frac{1}{c} \sqrt{E^2 - m_0^2c^4} ; q = \frac{c}{E} \sqrt{E^2 - m_0^2c^4} t,$$

with which Eq. (43) takes the form

$$\frac{1}{2m_0} \left( \frac{dU}{dq} \right)^2 - \frac{\hbar^2}{4m_0} \left[ \frac{3}{2} \left( \frac{dU}{dq} \right)^2 - \left( \frac{dU}{dq} \right)^{-1} \left( \frac{d^3U}{dq^3} \right) \right] + \frac{1}{2m_0} [m_0^2c^4 - E^2] = 0, \quad (45)$$

Note that $U$ and $q$ have the dimensions of, respectively, action and distance. Eq. (45) has as solution

$$U = \hbar \arctan \left( \frac{\psi_1}{\psi_2} + b \right) + U_0, \quad (46)$$

$a$ and $b$ being real constants, $\psi_1$ and $\psi_2$ are two solutions of the Klein-Gordon equation with vanishing potential

$$- c^2 \hbar^2 \frac{\partial^2 \psi}{\partial x^2} + [m_0^2c^4 - E^2] \psi(x) = 0. \quad (47)$$

If we choose the two solutions of Eq. (47) as $\psi_1 = \sin \left( \sqrt{E^2 - m_0^2c^4} \frac{q}{\hbar c} \right)$ and $\psi_2 = \cos \left( \sqrt{E^2 - m_0^2c^4} \frac{q}{\hbar c} \right)$, Eq. (46) will be written as

$$x(t) = \frac{\hbar c}{\sqrt{E^2 - m_0^2c^4}} \arctan \left[ a \tan \left( \frac{E^2 - m_0^2c^4}{\hbar E} t \right) + b \right] + x_0. \quad (48)$$
This equation represents the time equation of trajectories. As we have mentioned above, \( x(t) \) contains four constants since the fundamental equation of motion is a fourth order differential equation. Because the arctangent function is defined on the interval \( ] - \frac{\pi}{2}, \frac{\pi}{2} [ \), Eq. (48) shows that the particle is contained between

\[
- \frac{\hbar c}{\sqrt{E^2 - m_0^2 c^4}} \frac{\pi}{2} + x_0
\]

and

\[
\frac{\hbar c}{\sqrt{E^2 - m_0^2 c^4}} \frac{\pi}{2} + x_0.
\]

This is not possible since the particle is free. Then, it is necessary to readjust the additive integration constant \( x_0 \) after every interval of time in which the tangent function goes from \(-\infty\) to \(+\infty\) in such a way to guarantee the continuity of \( x(t) \).

In this purpose, expression (48) must be rewritten as

\[
x(t) = \frac{\hbar c}{\sqrt{E^2 - m_0^2 c^4}} \arctan \left[ a \tan \left( \frac{E^2 - m_0^2 c^4}{\hbar E} t \right) + b \right] + \frac{\pi \hbar c}{\sqrt{E^2 - m_0^2 c^4}} n + x_0.
\]

(49)

with

\[
t \in \left[ \frac{\pi \hbar E}{E^2 - m_0^2 c^4} \left( n - \frac{1}{2} \right), \frac{\pi \hbar E}{E^2 - m_0^2 c^4} \left( n + \frac{1}{2} \right) \right]
\]

for every integer number. The purely relativistic trajectory is obtained for \( a = 1 \) and \( b = 0 \). Indeed, for these values, Eq. (49) reduces to the relativistic relation

\[
x(t) = c \sqrt{E^2 - m_0^2 c^4} t + x_0.
\]

(50)

Eq. (49) indicates that all the trajectories defined by the set \((a, b)\) pass through some nodes exactly as we have seen for quantum trajectories \([10]\). These nodes correspond to the times

\[
t_n = \frac{\pi \hbar E}{E^2 - m_0^2 c^4} \left( n + \frac{1}{2} \right).
\]

(51)

The distance between two adjacent nodes is on time axis

\[
\Delta t_n = t_{n+1} - t_n = \frac{\pi \hbar E}{E^2 - m_0^2 c^4}.
\]

(52)

and space axis

\[
\Delta x_n = x_{n+1} - x_n = \frac{\pi \hbar c}{\sqrt{E^2 - m_0^2 c^4}}.
\]

(53)

These distances are both proportional to \( \hbar \) meaning that at the classical limit \( \hbar \to 0 \) the nodes becomes infinitely near, and then, all quantum trajectories tend to the purely relativistic one. As it is explained in Ref. \([10]\), this is the reason why in problems for which \( \hbar \) can be disregarded, relativistic quantum trajectories reduces to the purely relativistic one.

Now, let us investigate the case where the free particle is a photon which have a vanishing mass at rest \( (m_0 = 0) \). Thus, for the photon, the conjugate momentum is

\[
\frac{\partial S_0}{\partial x} = \frac{E}{\dot{x}}.
\]

(54)
and the FIRQNL is

$$E^2 - \dot{\dot{x}}^2 - \frac{\hbar^2}{2} \left[ \frac{3}{2} \left( \frac{\dot{x}}{E} \right)^2 - \frac{\dot{x}}{E} \right] = 0.$$  \hspace{1cm} (55)

From Eqs. (54) and (55), we can establish the time equation of photon’s trajectories

$$x(t) = \frac{\hbar c}{E} \arctan \left[ a \tan \left( \frac{E}{\hbar} t \right) + b \right] + \frac{\pi \hbar c}{E} n + x_0.$$  \hspace{1cm} (56)

with

$$t \in \left[ \frac{\pi \hbar}{E} (n - \frac{1}{2}), \frac{\pi \hbar}{E} (n + \frac{1}{2}) \right].$$

We can check easily that the purely relativistic trajectory can be obtained from (56) by setting $a = 1$ and $b = 0$, we then find

$$x(t) = ct + x_0.$$  \hspace{1cm} (57)

As the free massive particle case, we note that all these trajectories even the purely relativistic one $(a = 1, b = 0)$ pass through nodes corresponding to the times

$$t_n = \frac{\pi \hbar}{E} \left( n + \frac{1}{2} \right),$$  \hspace{1cm} (58)

for which the positions

$$x_n = \frac{\pi \hbar c}{E} \left( n + \frac{1}{2} \right),$$  \hspace{1cm} (59)

do not depend on $a$ and $b$. The distance between two adjacent nodes on time axis

$$\Delta t_n = t_{n+1} - t_n = \frac{\pi \hbar}{E},$$  \hspace{1cm} (60)

and space axis

$$\Delta x_n = x_{n+1} - x_n = \frac{\pi \hbar c}{E},$$  \hspace{1cm} (61)

are both proportional to $\hbar$, meaning that at the classical limit ($\hbar \to 0$), the nodes become infinitely near, and all relativistic quantum trajectories of the photon tend to the purely relativistic one. In order to check this fact, we refer the reader to Ref. [10].

An important remark should be made concerning the fact that the particle’s velocity is less than $c$ in some regions and greater than $c$ in other regions. This fact contradict the noticing, in special relativity that the light velocity in vacuum $c$ is the upper limit of the velocities of any massive particle. However, we note from Eqs. (49), (52), (53), (56), (60) and (61) that the free particle and the photon cover the distance between two nodes in such a way that the average velocity between two nodes is

$$v_{mean_e} = \frac{\Delta x_n}{\Delta t_n} = \frac{c \sqrt{E^2 - m_0^2 c^4}}{E},$$  \hspace{1cm} (62)

for the electron, and

$$v_{mean_{ph}} = \frac{\Delta x_n}{\Delta t_n} = c,$$  \hspace{1cm} (63)
for the photon. The velocity given by Eq. (62) is the purely relativistic one (see Eq. (50)), it is less than $c$, while the velocity given by Eq. (63) is the light velocity in vacuum $c$. These results can be interpreted as follows: In subquantic scales, the light velocity does not appear as an upper limit, otherwise, the particle may have a velocity greater than $c$ between two nodes. But, in the average and between two or more than two nodes, $c$ appears as an upper limit, since for all possible trajectories the distance between two nodes (Eqs. (53) and (61)) is covered at constant times (Eqs. (52) and (60)) making that the average velocity is less than (Eq. (62)) or equal (Eq. (63)) to $c$.

Let us now consider the motion in classically forbidden regions ($E < m_0c^2$). For this case, the solutions of Klein-Gordon equation are

$$\psi_1 = \exp\left(-\frac{\sqrt{m_0^2c^4 - E^2}}{\hbar c} x\right),$$

(64)

and

$$\psi_2 = \exp\left(+\frac{\sqrt{m_0^2c^4 - E^2}}{\hbar c} x\right).$$

(65)

Then, by integrating Eq. (42), we get

$$S_0 = \left(E - \frac{m_0^2c^4}{E}\right) t,$$

(66)

Replacing Eqs. (64) and (65) in Eq. (66), and taking in account expression (4) of $S_0$, we find

$$x(t) = \frac{\hbar c}{2\sqrt{m_0^2c^4 - E^2}} \ln \left| a \tan \left( \frac{E^2 - m_0^2c^4}{\hbar E} t + b \right) \right| + x_0.$$

(67)

This equation represents the relativistic quantum time equation for a particle moving in the classically forbidden regions. Its velocity is given by

$$\dot{x}(t) = -\frac{1}{2} \frac{a c}{E} \frac{1 + \tan^2\left(E^2 - m_0^2c^4/\hbar E\right)}{a \tan(E^2 - m_0^2c^4/\hbar E) + b},$$

(68)

from which we can see that at the times $-(2n+1)\pi\hbar E/4(E^2 - m_0^2c^4)$ the velocity becomes infinite. Let us recall that we have already faced in Ref. [10] infinite velocities for quantum problems. Nevertheless, in non-relativistic problems such velocities do not contradict any postulate, while for the relativistic problems the infinite velocities appears to contradict the fact that in special relativity the light velocity in vacuum $c$ is an upper limit.

To sustain our finding, let us recall the results of R. Y. Chiao’s experiments [13]. Chiao asserts that "Experiments have shown that individual photons penetrate an optical tunnel barrier with an effective group velocity considerably greater than the vacuum speed of light." The experiments were done with photon pairs emitted in slightly different directions so that one photon passed through the tunnel barrier, while the other photon passed through the vacuum. Chiao found that the photon’s transit time through the barrier was smaller than the twin photon’s transit time through an equal distance in vacuum. These results are in agreement with our approach, since it illustrate infinite (greater than $c$) velocities in classically forbidden regions (tunnel barrier).
Conclusions To conclude, we would like to stress that we exposed in this article an original approach of the relativistic quantum mechanics. It is a generalization of the one exposed in Ref. [9]. Thus, we have derived the relativistic quantum Newton’s law (41) from (30) which represents the relation between the conjugate momentum and the speed of the particle. These two equations are obtained in different contexts:

• a Lagrangian formulation;

• a Hamiltonian formulation, grounding oneself on the solution of the RQSHJE;

• a relativistic quantum version of Jacobi’s theorem

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REFERENCES

1. A. E. Faraggi and M. Matone, Phys. Lett. B 450, 34 (1999); hep-th/9705108.

2. A. E. Faraggi and M. Matone, Phys. Lett. B 437, 369 (1998); hep-th/9711028.

3. A. E. Faraggi and M. Matone, Int. J. Mod. Phys. A 15, 1869 (2000); hep-th/9809127.

4. G. Bertoldi, A. E. Faraggi and M. Matone, Class. Quant. Grav. 17, 3965 (2000); hep-ph/9909201.

5. E. R. Floyd, Phys. Rev. D 34, 3246 (1986).

6. E. R. Floyd, Found. Phys. Rev. 9, 489(1996); quant-ph/9707051.

7. E. R. Floyd, Phys. Lett. A 214, 259 (1996);

8. E. R. Floyd, quant-ph/0009070.

9. A. Bouda and T. Djama, Phys. Lett. A 285, 27 (2001); quant-ph/0103071.

10. A. Bouda and T. Djama, quant-ph/0108022.

11. Boudenot, Electromagntisme et gravitation relativiste.

12. A. E. Faraggi and M. Matone, Phys. Lett. A 249, 180 (1998); hep-ph/9801033.

13. R. Y. Chiao, quant-ph/9811019.