Imploding Scalar Fields.

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Abstract

Static spherically symmetric uncoupled scalar space-times have no event horizon and a divergent Kretschmann singularity at the origin of the coordinates. The singularity is always present so that non-static solutions have been sought to see if the singularities can develop from an initially singular free space-time. In flat space-time the Klein-Gordon equation $\Box \varphi = 0$ has the non-static spherically symmetric solution $\varphi = \sigma(v)/r$, where $\sigma(v)$ is a once differentiable function of the null coordinate $v$. In particular the function $\sigma(v)$ can be taken to be initially zero and then grow, thus producing a singularity in the scalar field. A similar situation occurs when the scalar field is coupled to gravity via Einstein’s equations; the solution also develops a divergent Kretschmann invariant singularity, but it has no overall energy. To overcome this Bekenstein’s theorems are applied to give two corresponding conformally coupled solutions. One of these has positive ADM mass and has the properties: i) it develops a Kretschmann invariant singularity, ii) it has no event horizon, iii) it has a well-defined source, iv) it has well-defined junction condition to Minkowski space-time, v) it is asymptotically flat with positive overall energy. This paper presents this solution and several other non-static scalar solutions. The properties of these solutions which are studied are limited to the following three: i) whether the solution can be joined to Minkowski space-time, ii) whether the solution is asymptotically flat, iii) and if so what the solutions’ Bondi and ADM masses are.

1 Introduction.

Singularities appear in many physical theories. A singularity can be defined as a domain where the description provided by the physical theory breaks down. A prime example is the infinite electromagnetic potential of a point particle in Maxwell’s theory. A common approach to a theory which has singularities is to produce another theory governed by more general differential equations and then investigate whether the singularities still occur. For example in electromagnetic theory Born and Infeld investigated a generalization of Maxwell’s Lagrangian to see if the infinite potential was still present. In gravitational theory the situation is more complex: for a point particle the Kretschmann invariant $R_{abcd}R^{abcd}$ diverges at the position of the particle, also on occasion the particle is surrounded by an event horizon. The event horizon is not singular in the sense defined above because a description of its effects can be made: but the effects are so bizarre that
along with singularities they can be called pathological. Vacuum general relativity frequently has both pathologies an example being the Schwarzschild solution for a point particle. It is sometimes argued that the existence of both pathologies is palatable because the event horizon makes the divergent Kretschmann invariant invisible at infinity: but the physical description provided is still incomplete because it does not describe what happens at the divergent Kretschmann invariant. Vacuum general relativity does not furnish a good description of many astrophysical phenomena such as gravitational collapse because the pertinent space-time can contain many fields and fluids with non-vanishing stress. Astronomical observations purporting to be of "black holes" in fact do not directly observe event horizons, the models which describe the situation merely use a steeper potential than that of Newtonian theory, the Newtonian limit of most relativistic theories produces such a potential.

To find exact solutions of gravitational field equations to fit a particular physical requirement is notoriously difficult. An example is the two-body problem: since the inception of general relativity the solution for two point particles acting only through gravity has been sought. Another example is the Yukawa problem: in the absence of gravitation a massive scalar field has a shorter range than the corresponding massless case, Yukawa’s discovery of this lead to the postulation of nuclear forces. How gravitation alters the shape of the potential is unknown, and this would be of experimental interest as the Yukawa potential can be accurately measured in accelerators; also it is unknown whether the mass of the interacting scalar field is the same as the ADM mass. Scalar fields coupled to gravity produce unusual potentials, complicated by the fact that the luminosity radial coordinate is often of the form $R = r \exp(\varphi)$, where the metric is explicitly expressed in terms of $r$. Perhaps the simplest modification of vacuum general relativity is to choose a stress with an uncoupled scalar field. When this is done the situation is mitigated; for the static case there is no event horizon. The problem with static space-times is that the Kretschmann invariant is always present, the space-time does not develop so as to produce it. Non-static spherically symmetric scalar solutions have been found, one of which is asymptotically flat, Roberts (1986)\cite{1}, (1989)\cite{2}. This solution has unusual energetics; there is no overall energy, the positive energy of the scalar field and the negative energy of the gravitational field cancel out: as the space-time develops energy is just exchanged between them. To overcome this Bekenstein’s theorems are applied to uncoupled scalar field solutions to give two corresponding conformal scalar solutions. One of these is asymptotically
flat and has overall positive ADM energy.

In general relativity scalar fields can implode (and explode) like the example in the abstract; the situation here is more pathological than in flat space because not only is it possible to produce a singularity in the scalar field but in addition there is a co-locational singularity of the gravitational field as indicated by the divergence of the Kretschmann curvature invariant $R_{abcd} R^{abcd}$. Einstein-scalar space-times are sometimes not covered by theorems concerned with the general global and singular structure of space-time. These often assume the space-time stress tensor is restricted to: vacuum, or electromagnetic stress, or obey energy inequalities. An example of this is the formal definition of asymptotic flat space-times which assume that there is only an electromagnetic and gravitational field present. Numerical studies show that the rate of decay of scalar fields is between these two Roberts(1986)[1], furthermore many spherical symmetric perfect fluid stresses do not have (in the sense of taking a radial coordinate $r \to \infty$) asymptotically flat space-times, Roberts (1998)[3]. Here a scalar solution is taken to be asymptotically flat if it reduces to Minkowski space-time as $r \to \infty$. The rate of decay of fields seems to be: most fluids and some conformal scalars are not asymptotically flat, then gravitation $>$ uncoupled scalars $>$ uncoupled vectors $>$ interacting fields.

In section 2 some Robertson-Walker scalar field solutions are given. The examples chosen have scale factors which can be expressed in terms of straightforward functions. Ordinary scalar stresses obey the energy conditions but conformal scalar stresses sometimes do not. Violation of the energy conditions allow the possibility of a singular freee Robertson-Walker space-time, this can happen for conformal scalar fields Bekenstein and Meisels (1980)[4], and when cosmological constant is present, Murphy (1973)[5]. Here the junction conditions of Robertson-Walker space-time are studied, and possible applications mentioned. In section 3 Penny’s (1976)[6] solution is presented, this can implode, but is not asymptotically flat. In section 4 the solution previously found by the author, Roberts (1986)[1] and (1989)[2], and also its two conformal scalar extensions as found by Bekenstein’s theorems, are presented. These solutions are explicitly dependent on a radial coordinate so that they are more similar to the example in the abstract than the examples in sections 2 and 3. The solution and one of its Bekenstein extensions are asymptotically flat, and also have well defined junction conditions, contrary to what has been stated by Szabodos (1990)[7]. The Bondi and ADM masses of the solutions are calculated. The first two appendices derive the Clarke and Dray (1987)[8] junction conditions subject to spherical
symmetry. In all the specific cases looked at here if a solution has a continuous metric (and hence first fundamental form) across a junction then its surface stress (which depends on the derivatives of the metric) vanishes and the junction is well defined. The third appendix derives a the general expression for the ADM mass subject to spherical symmetry.

The field equations considered are Einstein’s equations with scalar fields as source. Specifically the ordinary scalar-Einstein equations are

\[ R_{ab} = 2\varphi_a \varphi_b, \]  

where the coupling constant is taken to be contained in the scalar field \( \varphi \). On occasion a null radiation field is also taken to be present with

\[ R_{ab} = \Phi^2 k_a k_b, \]  

where \( k_a \) is a null vector and \( \Phi \) is a function of \( x^a \). Conformal scalar solutions can be obtained from ordinary scalar solutions by using Bekenstein’s (1973) theorems. To derive these let barred quantities denote these quantities for an ordinary scalar solution, i.e. \( \bar{R}_{ab} = 2\bar{\varphi}_a \bar{\varphi}_b \). Then under a conformal transformation

\[ \bar{g}_{ab} = \Omega g_{ab}; \]  

the connection is transformed

\[ \bar{\Gamma}^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma} = \Omega^{-1}(\delta^{\alpha}_{\beta}\Omega_{\gamma} + \delta^{\alpha}_{\gamma}\Omega_{\beta} - g_{\beta\gamma}\Omega^{\alpha}); \]  

and the Ricci tensor is transformed

\[ \bar{R}^{c}_{ab} - R_{ab} = -2\Omega(\Omega^{-1})_{;ab} + \Omega^{-2}(\Omega^2)_{;c}g_{ab}, \]  

where the covariant derivatives ";'" are taken in the unbarred system. Now take

\[ \Omega = \sqrt{\pm(1 - 2\xi^2\varphi^2)} = \begin{cases} \text{sech} & \xi \varphi \\ \text{cosech} & \end{cases} \]  

\[ \xi \varphi = \sqrt{1 \mp \Omega^2} = \begin{cases} \text{tanh} & \xi \varphi \\ \text{coth} & \end{cases} \]  

(1.6)

where \( \xi \) is a constant and \( \varphi \) is a function, then substituting into the Ricci tensor obeys the equations for a conformal scalar field

\[ (1/\xi^2 - \psi^2)R_{ab} = 4\psi_a \psi_b - 2\psi_{;ab} - (\psi \psi^c)_{;c}g_{ab}. \]  

(1.7)
Thus given an ordinary scalar field solution $1.1$ Bekenstein’s theorems give two conformal scalar solutions $1.6$. Bekenstein refers to the upper sign conformal solution as type A and the lower sign conformal solution as type B, as no confusion with blood types should occur the ordinary scalar solution is here called type O. In the conventions used here, the coupling constant is taken to be in the scalar field and thus $\xi^2 = \frac{1}{3}$. The conformal scalar solutions are traceless and also obey $\psi^a_a = 0$. For type A the theorem generalizes for additional stress present, in the case of a null radiation field this must transform as $\Phi^2 \rightarrow \Omega^{-2}\Phi^2$, c.f.Bekenstein (1973) [9] equation 15; this generalization does not work for type B. Type B solutions are anticipated to have unusual global properties, for example as $\varphi \rightarrow 0$: $\Omega^{-1} \rightarrow 0$, $\psi \rightarrow \infty$; and also as $\varphi \rightarrow \infty$: $\Omega^{-1} \rightarrow \infty$, $\psi \rightarrow 1$; but applying Bekenstein’s theorems does not a priori produce a maximally extended space-time so that exact solutions have to be specified before precise pronouncements on there global properties can be made. Examples of static spherically symmetric conformal scalar fields with unusual properties can be found in Agnese and LaCamera (1985) [10].

Having presented some of the properties of scalar fields we can now come back the questions: general relativity and other gravitational theories are primarily macroscopic theories which couple to stresses that have macroscopic effect, such electromagnetism and perfect fluids are natural choices - why choose scalar fields? First it only takes an infinitesimal scalar field to change the global nature of a space-time. For example Wyman’s solution, which is the general static spherically symmetric O-scalar-Einstein solution, is a two parameter solution $(M, \sigma)$ with $M$ the Schwarzschild mass and $\sigma$ the scalar charge, an infinitesimal is sufficient for there to be no event horizon present. Thus microscopic fields, such as those of particle physics, can have a global effect on space-time. Secondly, for $\varphi$ time-like an O-scalar field is a particular example of a perfect fluid. Perfect fluids which are well-behaved and permeate the whole space-time can usually be shown to have no horizons, Roberts (1998) [3]. The fluid conservation equations often allow the fluid index $\omega$ to be equated with the fluid vector and hence the metric. Typically this results in equations such as the lapse $N = 1/\omega$; thus a well-behaved fluid index can often imply a well-behaved metric. In appendix D an attempt is made to extend to fields the techniques that lead to this result. Thirdly in microscopic physics hypothetical particles, the Higgs scalars, are used to introduce ”mass” terms while preserving gauge invariance. Although other mechanisms have been proposed, for example by using fermion composites, or using fluids Roberts (1989) [4], (1996) [12], the resulting Lagrangians have
terms similar to scalar fields. Fourthly many quantum and unified theories have gravitational actions with terms of higher order. The quadratic action can be split into two independent parts, the traceless part being the Bach tensor and the other part the Pauli tensor. Using a conformal factor solutions to these equations can be found by Barrow and Cotsakis' (1988) method. The Bach tensor has several similarities to conformal scalar fields and there might be a theorem connecting their solutions. Vacuum solutions can generate O-scalar solutions by Buchdahl’s trick (the analogous theorem for vector gauge theory is called the Julia-Zee correspondence) see for example Roberts (1986); O-scalar solutions can generate A and B scalar solutions by Bekenstein’s theorems; and perhaps A and B scalar solutions can generate Bach-Einstein solutions. Fifthly O-scalar solutions obey the energy conditions. The energy conditions are inequalities designed to judge whether macroscopic fluids have reasonable energetics: they break down when considering the interacting fields necessary for particle physics, see for example Hawking and Ellis (1973) p. 95.

Apart from the above five reasons for investigating scalar fields they can be viewed as merely a scalar function defined on a region of space, and such a requirement seems to be fairly ubiquitous in physics. Systematic discussion of them is hindered because there is no recognized way of classifying them. Some indications of their properties are given by their:

- **Coupling classification**, call scalar fields coupled only to Einstein’s equations type O: conformal scalar fields coupled to Einstein’s equations type C, scalar fields with mass self-interaction type Y, renormalisable scalar fields with fourth order self-interaction type l, inflationary scalar fields with potential \( V(\phi) \) type V, scalar fields that can be represented as fluids type F, symmetry breaking scalar fields coupled to vector fields type H - and so on.

- **Generational classification**: exact scalar field solutions can be generated from exact solutions to simpler differential equations. For a given configuration usually the generated solution is not the most general one.

- **Stress classification**: stress tensors can be classified by the Segre or Plebanski methods.

- **Energy classification**: for a space-time with Lorenz signature (-,+,+,+) rather than positive definite signature (+,+,+,+), vectors can be time-like, null, or space-like. The existence of time-like vectors allows measures of the energy to be defined. For a given space-time \( \psi_a \) and \( \psi_b T^b_a \) are vectors that can be time-like, null, or space-like; the energy conditions can be investigated and on occasion the overall energy measured.

For conformal scalar fields the energy conditions can be complicated; hand calculations of them for the specific solutions presented here are too
long to be practicable. Three observations are now stated which give a rough indication of what energetics to expect. The first observation is that the general stress for conformal scalar fields contains terms of undetermined sign and this remains the case even if the conformal scalar stress has been obtained by using Bekenstein’s theorems; this can been seen from \[1.6\] and \[1.7\].

The second observation is for an O-scalar solution it is possible to consider whether the vector \( \varphi_a \) is space-like, time-like, or null: this just depends on the Ricci scalar because \( g^{ab} = \frac{1}{2} \bar{R} \); now using Bekenstein’s theorems both for type A and B there is the equation \( g^{ab} \varphi_a \varphi_b = \frac{1}{2} \bar{R} \). Thus there is no change in the causal direction of the scalar field. The third observation is achieved by direct inspection of the scalar fields. Neglect that the equation for the stress of a conformal scalar field differs from that of the ordinary case and also that Bekenstein’s theorems introduce a conformal factor into the metric, then the energy conditions will just depend on the relative size of the scalar fields involved. The type O scalar field is a negative real scalar quantity, Bekenstein’s theorems give that the type A scalar is the tanh of this and that the type B is the coth of this, thus the scalar fields take real values such that \( 0 > A > O > B > -\infty \). Now the type O scalar field on occasion (an example being that of section IV) contains the same quantity of energy, but of the opposite sign as the gravitational field; the above inequalities suggest that the type A solution would have total positive energy because the scalar field is not so negative, and also that the type B solution would have total negative energy. This is found to be the case for the type A solution described in the conclusion. The above suggests that it is a reasonable guess that type A solutions have well defined energy conditions and that type B do not: this is what would be expected from the known exact solutions where it would account for type A have mundane properties whereas those of type B solutions are bizarre.

## 2 Robertson-Walker Scalar Solutions.

The Robertson-Walker line element can be put in the form

\[
ds^2 = -N(t)^2 dt^2 + R(t)^2 d\Sigma_{3,k}^2,
\]

where

\[
d\Sigma_{3,k}^2 = d\chi^2 + f(\chi)^2 (d\theta^2 + \sin^2(\theta) d\varphi^2),
\]
and

\[ f(\chi) = \sin(\chi), \chi, \sinh(\chi), \]

for \( k = +1, 0, -1 \) respectively. Taking \( A = R^2, B = R^2f^2, C = N^2, \) and \( \chi = r \) gives the line element in the spherically symmetric form [B.1]. \( N \) is called the lapse and \( R \) the scale factor. \( N \) can be absorbed into the line element, the choice \( N = 1 \) gives the Robertson-Walker line element in proper time. For the choice \( N = R \) Robertson-Walker space-time is conformal to the Einstein static universe and by convention the time coordinate is denoted by \( \eta \). For \( N = 1 \) the scale factor \( R \) can be expanded as a Taylor series around a fixed time \( t = t_0 \) thus

\[ R = R_0[1 + H_0(t - t_0) - \frac{1}{2}q_0H_0^2(t - t_0)^2 + O(t - t_0)^3], \]  

(2.2)

where

\[ H \equiv \frac{\dot{R}}{R}, \]

(2.3)

is called the Hubble parameter and

\[ q \equiv \frac{\ddot{R}.R}{\dot{R}^2}, \]

(2.4)

is called the deceleration parameter, the subscript ”0” indicates that the parameter is evaluated at \( t = t_0 \), and \( \dot{R} = \partial R \).

The equation of state

\[ p = (\gamma - 1), \]

(2.5)

produces equations equivalent to those of an ordinary scalar field in the particular case of \( \gamma = 2 \) (see appendix D). Einstein’s equations have been solved by Vajk(1968)\[15\] for the metric 2.1 and equation of state 2.5, specializing to the \( \gamma = 2 \) ordinary scalar field case gives

\[ k = 0, \quad \Xi = \alpha \eta^{1/2}, \quad \varphi = \frac{\sqrt{3}}{2}ln \eta, \]

\[ k = +1, \quad \Xi = \alpha(sin \eta \cos \eta)^{1/2}, \quad \varphi = \frac{\sqrt{3}}{2}ln \tan \eta, \]

\[ k = -1, \quad \Xi = \alpha(sinh \eta \cosh \eta)^{1/2}, \quad \varphi = \frac{\sqrt{3}}{2}ln \tanh \eta, \]

(2.6)

where

\[ \alpha = 2R_0\sqrt{\frac{H_0^2R_0^2}{c^2} + k}, \]
c is the speed of light and Ξ is equal to both the scale factor and the lapse, i.e. Ξ = N = R. The $k = 0$ solution is one of the few solutions known to have an exact form for the world function, Roberts (1993) [16].

Applying Bekenstein’s theorems

$$k = 0, \quad \Upsilon = \frac{\alpha}{2}(\eta \pm 1), \quad \frac{1}{\sqrt{3}}\psi = \frac{\eta \mp 1}{\eta \pm 1},$$

$$k = +1, \quad \Upsilon = \frac{\alpha}{2}(\sin \eta \pm \cos \eta), \quad \frac{1}{\sqrt{3}}\psi = \frac{\sin \eta \mp \cos \eta}{\sin \eta \pm \cos \eta},$$

$$k = -1, \quad \Upsilon = \frac{\alpha}{2}exp(\pm \eta), \quad \frac{1}{\sqrt{3}}\psi = -exp(\mp \eta),$$

where $\Upsilon = \Xi \Omega^{-1}$. In the $k = 0$ case the $\pm 1$ can be absorbed into the line element by defining $\eta' = \eta \pm 1$, giving a single solution. Transferring the $k = +1$ solution to proper time by defining $t = (\alpha/2)(-\cos et \pm \sin \eta)$ gives

$$ds^2 = -dt^2 + \left(\frac{\alpha^2}{2} - t^2\right)d\Sigma^2_{3,+1}, \quad \frac{1}{\sqrt{3}}\psi = \left(\frac{\alpha^2}{2t^2} - 1\right),$$

showing that there is only one $k = +1$ metric. In the $k = -1$ case define $t = (\alpha/2)exp(\pm \eta)$ to give

$$ds^2 = -dt^2 + t^2d\Sigma^2_{3,-1}, \quad \frac{1}{\sqrt{3}}\psi = -\frac{\alpha^2}{4t^2},$$

this is just the Milne universe, further discussed at equation 2.16; the field $\psi$ is a ghost field that does not contribute to the stress. Conformal scalar stresses are traceless, this can be used to reduce the number of equations, in particular the Einstein-conformal scalar equations with Robertson-Walker metric can be quickly reduced to one equation

$$R_{,\eta \eta} = -kR,$$

this gives solutions more general than those of 2.8, however they are particular instances of the conformal scalar and incoherent radiation solutions of Bekenstein (1974) [9] #6. Bekenstein’s theorems can then be used in reverse to give generalizations of 2.7.

The null junction conditions are studied by defining

$$v = \eta + r.$$
Robertson-Walker space-time in the conformal time coordinate $\eta$ takes the single null form with $X = S = Y f^{-2} = R(v - r)^2$. The $\theta$ and $\varphi$ components of the surface stress vanish identically, the $v$ and $r$ components are given by

$$\tau_{ab} = -f^{-2}R^{-2}[(R^2f^2)']_an_a n_b,$$  \hspace{1cm} (2.12)

which do not vanish at a junction with Minkowski space-time.

The non-null junctions are studied by calculating the second fundamental form as in Appendix B. The second fundamental form across the space-like surface normal to $B.3$ is

$$K_{rr} = \frac{R\dot{R}}{N},$$

$$K_{\theta\theta} = \sin^{-2}(\theta)K_{\varphi\varphi} = -\frac{f^2R\dot{R}}{N},$$

which gives no junctions to Minkowski space-time. The surface normal to the radial space-like vector $B.5$ has second fundamental form

$$K_{tt} = 0,$$

$$K_{\theta\theta} = \sin^{-2}(\theta)K_{\varphi\varphi} = ff' R,$$

which gives no junctions to Minkowski space-time. The radial space-like vector is not well suited to Robertson-Walker geometries, choosing the isotropic space-like vector $B.7$ gives second fundamental form

$$K_{tt} = K_{tr} = 0,$$

$$K_{r\theta} = \frac{K_{r\varphi}}{\sin \theta} = \frac{K_{\theta\varphi}}{\sin \theta} = -\frac{1}{2}fK_{rr} = \frac{-f'R}{3\sqrt{3}},$$

$$K_{\theta\theta} = \frac{K_{r\theta}}{\sin \theta} = \frac{f}{3\sqrt{3}}\{-\cot \theta + 2f'R\}.$$

Again there are no junctions to Minkowski space-time.

Consider Minkowski space-time in the form with $k = 0$ and $N = R = 1, f = r$, and apply the coordinate transformation

$$t = \bar{t}\cosh \chi, \quad r = \bar{t}\sinh \chi.$$  \hspace{1cm} (2.16)

\footnote{footnote added 1999: junctions between Schwarzschild space-time and the pressure-free Friedman universe are discussed in Stephani,H. General Relativity, An introduction to the theory of the gravitational field, Cambridge University Press (1982) §27.3}
This transformation gives the Milne universe which has $N = 1$, $R = t$, and $f = \sinh r$. The Milne universe is flat and is identical to Minkowski space-time except that there is a point removed at the origin $t = 0$. At first sight it might be expected that the Milne universe could be joined to Minkowski space-time across the surface chosen here. The reason that this does not happen is that the space and time coordinates have been "mixed" by so that if there was a well defined junction it would be across a different surface from those chosen here. A general treatment of redefinitions of space and time coordinates in Robertson-Walker space-time can be found in Infeld and Schild (1945) [17].

The junction conditions of Robertson-Walker space-time have two further applications. The first is the production of a spherical Minkowski cavity which has implications for Mach’s principle, see for example Weinberg (1972) [18] p.474. A point inside the cavity is an inertial frame if it does not rotate with respect to the reference frame of the rest of the Universe, which is taken to be given by the Robertson-Walker space-time surrounding the cavity. A different approach to Mach’s principle is discussed in Roberts (1985) [19]. The second is to the cell universe models. The surface normal to the vectors chosen here do not allow junctions between Robertson-Walker space-time and Schwarzschild space-time, thus for the Schwarzschild cell universe of Lindquist and Wheeler (1957) [20] to work a different vector has to be chosen or different physical assumptions made.

3 Penny’s Solution

Penny’s solution (1976) [6], and the related solutions of Gurses (1977) [21] and Ray (1977) [22] are conformally flat. The conformal factor generating technique used to find these solutions is also used to study solutions of higher order gravity theories, Barrow and Cotsakis (1988) [13]. Here attention is restricted to Penny’s solution where the conformal factor and the ordinary scalar field are given by

$$\Xi = k_a x^a + a, \quad k_{a,b} = 0, \quad \varphi = \sqrt{3} l n \Xi,$$

(3.1)

respectively. Defining

$$K_a = -\Xi k_a,$$

(3.2)

$K_a$ is a Killing vector which is null iff $k_a$ is null. The conformal factor can be expressed as

$$\Xi^2 = a + bt + cx + dy + ez,$$

(3.3)
where \(a,b,c,d\) are constants. There is no asymptotically flat solution. For 
\(a = c = d = e = 0,\ b = 2R_0^2H_0\), this is the \(k = 0\) solution.

Using Bekenstein’s theorems, conformal scalar solutions are

\[
\Upsilon = \Omega^{-1} \Xi, \\
2\Omega = 2 \begin{cases} \cosh \varphi = \Xi \pm \Xi^{-1}, \\ \sinh \varphi = \Xi \mp \Xi^{-1} \end{cases}, \quad (3.4)
\]

\[
\xi\psi = \begin{cases} \tanh \varphi = \frac{\Xi^2 + 1}{\Xi^2 - 1}, \\ \coth \varphi = \frac{\Xi^2 - 1}{\Xi^2 + 1} \end{cases},
\]

giving

\[
\Upsilon = \frac{2\Xi^2}{(\Xi^2 \pm 1)}.
\]

Defining

\[
K_a = \Upsilon^2 k_a, \quad (3.5)
\]

again \(K_a\) is a Killing vector which is null iff \(k_a\) is null.

### 4 The solution previously found by the Author

The solution found in Roberts (1986) and (1989) is

\[
ds^2 = -(1 + 2\alpha_v)dv^2 + 2dvdr + r(r - 2\alpha)d\Sigma_2^2, \quad (4.1)
\]

where \(d\Sigma_2^2 = d\theta^2 + \sin^2\theta\ d\varphi\), and \(\alpha\) is a twice differentiable function of \(v\).

The stress is given by a scalar field and a null radiation field

\[
\varphi = \frac{1}{2} \ln \left(1 - 2\frac{2\alpha}{r}\right), \quad \Phi^2 = \frac{2\alpha\alpha_{,vv}}{r(r - 2\alpha)}. \quad (4.2)
\]

Defining the luminosity distance \(R^2 = r(r - 2\alpha)\), and taking the positive sign of the square root the solution becomes

\[
ds^2 = \left(-1 + \frac{2\alpha\alpha_v}{\lambda}\right)dv^2 + \frac{2R}{\lambda}dRdv + R^2 d\Sigma_2^2, \quad (4.3)
\]

\[
\varphi = \frac{1}{2} \ln \left(\frac{\lambda - \alpha}{\lambda + \alpha}\right), \quad \Phi^2 = \frac{2\alpha\alpha_v}{R^2},
\]

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where \( \lambda^2 = \alpha^2 + R^2 \). The Bondi mass \( M(v) \) is half the coefficient of the \( R^{-1} \) term of \( g_{vv} \), expanding gives

\[
M(v) = \alpha \alpha_v. \tag{4.4}
\]

For the null radiation field to vanish \( \alpha_{vv} = 0 \) or \( \alpha = \sigma v + \beta \), where \( \sigma \) and \( \beta \) are constants. It is straightforward to show that \( \beta \) can be absorbed into the line element leaving \( \alpha = \sigma v \); this can be substituted into (4.3) for a form of the metric using the luminosity radial coordinate, alternatively it can be substituted into (4.1) giving

\[
ds^2 = -(1 + 2 \sigma) dv^2 + 2 dv dr + r(r - 2 \sigma v)d\Sigma^2_2, \tag{4.5}
\]

\[
\varphi = \frac{1}{2} \ln \left( 1 - \frac{2 \sigma v}{r} \right).
\]

Defining \( v' = \sqrt{1 + 2 \sigma} \) and \( r' = r/\sqrt{1 + 2 \sigma} \) gives

\[
ds^2 = -dv'^2 + 2 dv' dr' + r'((1 + 2 \sigma)r' - 2 \sigma v')d\Sigma^2_2, \tag{4.6}
\]

\[
\varphi = \frac{1}{2} \ln \left( 1 - \frac{2 \sigma}{1 + 2 \sigma} \frac{v'}{r'} \right),
\]

then defining \( t' = v' - r' \) gives

\[
ds^2 = -dt'^2 + dr'^2 + r'((1 + 2 \sigma)t' - 2 \sigma t')d\Sigma^2_2,
\]

\[
\varphi = \ln(R/r). \tag{4.7}
\]

also defining \( t = t'/1 + 2 \sigma \) the solution can be put in the form

\[
ds^2 = -(1 + 2 \sigma) dt^2 + \frac{dr^2}{(1 + 2 \sigma)} + r \left( \frac{r}{1 + 2 \sigma} - 2 \sigma t \right)d\Sigma^2_2,
\]

\[
\varphi = \frac{1}{2} \ln \left( \frac{1}{(1 + 2 \sigma)} - \frac{2 \sigma t}{r} \right). \tag{4.8}
\]

and from this form using (4.3) the ADM mass is seen to vanish identically.

Using Bekenstein’s theorems to find conformal scalar solutions (4.3) gives

\[
R = \tilde{R} \Omega^{-1} = \tilde{R} \left\{ \begin{array}{ll}
\cosh & (\lambda - \alpha)^{\xi} \pm (\lambda + \alpha)^{\xi} \\
\sinh & (\lambda - \alpha)^{\xi} \mp (\lambda + \alpha)^{\xi}
\end{array} \right\},
\]

\[
\xi \varphi = \frac{1}{2} \tilde{R}^{1-\xi} [(\lambda - \alpha)^{\xi} \pm (\lambda + \alpha)^{\xi}],
\]

\[
\xi \varphi = (\lambda - \alpha)^{\xi} \mp (\lambda + \alpha)^{\xi}, \tag{4.9}
\]

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where $\lambda^2 = \bar{R}^2 + \alpha^2$ and $\bar{R}$ denotes the luminosity coordinate for the O-scalar solution, and $R$ denotes it for the conformal scalar solution. For the type A solution general can be retained if the null radiation field is transformed, but for the type B solution $\alpha$ must equal $\sigma v$. Inspecting 4.9 gives limiting behaviour of the conformal scalar solution in terms of the luminosity coordinate for the ordinary solution,

\[
\begin{align*}
\bar{R} &\uparrow \infty, \text{ TypeA: } R \uparrow \infty, \psi \uparrow 0, \text{ TypeB: } R \downarrow -\xi, \psi \downarrow -\infty, \\
\bar{R} &\downarrow 0, \text{ TypeA: } R \downarrow 0, \psi \downarrow -1, \text{ TypeB: } R \downarrow -0, \psi \uparrow -1.
\end{align*}
\]

The type B solution does not have an asymptotically flat region, so that attention is restricted to the type A solution. Expanding 4.9 for large $R$ gives

\[
R = \bar{R} \left( 1 + \frac{\xi^2 \alpha^2}{2R^2} + O \left( \frac{\alpha}{R} \right)^3 \right),
\]

(4.11)

Solving this quadratic

\[
\bar{R} = \frac{R}{2} \left( 1 \pm \sqrt{1 - \frac{2\xi^2\alpha^2}{R^2}} \right),
\]

(4.12)

the upper sign is taken so that $R = \bar{R}$ when $\xi = 0$. Differentiating

\[
d\bar{R} = \frac{1}{2} \left( 1 + \frac{2}{\sqrt{1 - 2\xi^2\alpha^2/R^2}} \right) - \frac{2\xi^2\alpha\alpha_v dv}{R\sqrt{1 - 2\xi^2\alpha^2/R^2}},
\]

(4.13)

inserting into the line element and using 4.3, 4.12, 4.13 and that $\lambda^2 = \alpha^2 + \bar{R}^2$ gives

\[
g_{vv} = -1 + \frac{2(1 - \xi^2)}{R}\alpha\alpha_v + O(R^{-2}).
\]

(4.14)

Thus the Bondi mass is given by

\[
M(v) = (1 - \xi^2)\alpha\alpha_v,
\]

(4.15)

which is two thirds of 4.4. To calculate the ADM mass note that the conformal factor can be used

\[
A' = \bar{A}' + \lim_{r \to \infty} \left[ -\frac{1}{2r} \left( \frac{1}{\Omega} \right)' \right],
\]

(4.16)
where \( \bar{B}, \bar{A} \) are the values of these quantities in the O-scalar solution, using the metric in the form \( 4.8 \) and noting that \( A \) vanishes at \( r \to \infty \).

\[
A' = \lim_{r \to \infty} -\frac{\bar{B}}{2} \sinh(2\xi \varphi) \cdot \varphi' = \lim_{r \to \infty} -\frac{\xi \varphi}{2} \sinh(2\xi \varphi) \cdot t. 
\]  \hspace{1cm} (4.17)

Expanding \( \sinh(2\xi \varphi) \) for \( r \to \infty \) gives

\[
A'_t = \begin{cases} 
\xi^2 \sigma^2 t, & \text{for } -1 < 2\sigma < 1, \\
2\xi^2 \sigma^{-2 - \xi + 1} t, & \text{for } 2\sigma \geq 1,
\end{cases} 
\]  \hspace{1cm} (4.18)

for \( 2\sigma \geq -1 \) the signature of the metric is not retained.

The surface stress \( A.6 \) must vanish at any null junction; this implies \( [Y'] \) must vanish if \( \tau_{rr}, \tau_{ph}, \tau_{rv}, \text{and } \tau_{rr} \) are to vanish, and \( [(X/S)'/2S] \) must vanish if \( \tau_{gg} \) and \( \tau_{ph} \) are to vanish. For the type O solution take the line element \( 4.3 \), \( [Y'] \) vanishes as this line element is expressed in terms of the luminosity radial coordinate already and

\[
\frac{1}{2S} \left( \frac{X}{S} \right)' = \frac{\alpha}{2R^\xi} (\alpha - 2\lambda \alpha_v). 
\]  \hspace{1cm} (4.19)

Now the metric can be chosen to continuously join to Minkowski space-time by taking \( \alpha \) to continuously increase from (or decrease to) \( \alpha = 0 \); \( 4.19 \) shows that there is no surface stress at the join \( \alpha = 0 \) so that the field equations are obeyed throughout the space-time. For the type A and B solutions \( 4.9 \) gives

\[
Y = \frac{1}{4} R^2 - 2 \xi \{ (\lambda - \alpha)^\xi \pm (\lambda + \alpha)^\xi \}, 
\]  \hspace{1cm} (4.20)

\[
Y' = \frac{1}{2} R^2 - 2 \xi \{ (\lambda - \alpha)^\xi \pm (\lambda + \alpha)^\xi \} 
\]

\[
\times \left\{ (1 - \xi)((\lambda - \alpha)^\xi \pm (\lambda + \alpha)^\xi) + \frac{\xi}{\alpha} R^2 ((\lambda - \alpha)^{\xi-1} \pm (\lambda + \alpha)^{\xi-1}) \right\}, 
\]

\[
\frac{1}{2S} \left( \frac{X}{S} \right)' = \frac{4\alpha(\alpha - \lambda \alpha_v)}{R^{3 - 2\xi} \{ (\lambda - \alpha)^\xi \pm (\lambda + \alpha)^\xi \}}. 
\]

For the type B solution the metric is not continuous at \( \alpha = 0 \) as would be anticipated from the general remarks in the introduction. The type A solution again has a metric which can be chosen to continuously join to Minkowski space-time by taking \( \alpha \) to behave as before.
At any junction across a time-like surface the limits of the second fundamental form either side of the junction must coincide. For the type O solution, dropping the primes on the metric gives the second fundamental form

\[ K_{\theta\theta} = \sin^{-2}\theta K_{\varphi\varphi} = r - \sigma t, \]

which gives a junction where the field equations are defined with Minkowski space-time at \( t=0 \). For type A and B conformal solutions the extension of the metric has second fundamental form

\[ K_{tt} = -\xi \sigma t (r(r - 2\sigma t))^{-\xi/2-1} \cdot \{(r - 2\sigma t)^{\xi} \mp r^{\xi}\}, \]

Again there is a junction at \( t = 0 \).

5 Conclusion

Solutions to the Einstein-scalar equations which can represent an imploding scalar field were presented. Bekenstein’s theorems were used to generate the corresponding Einstein-conformal scalar solutions. The Robertson-Walker solutions presented here are examples of solutions previously found by Vajk and Bekenstein; they are not asymptotically flat and cannot be joined to Minkowski space-time by the methods used here. Penny’s solution also can represent an imploding scalar field but it is not asymptotically flat, and only when it reduces to a Robertson-Walker metric is it spherically symmetric. The solution previously found by the author, and its Bekenstein type A extension, are asymptotically flat and have well defined junctions with Minkowski space-time, and therefore can represent a scalar field imploding from nothing, thus generalizing the example in the abstract. This solution has Bondi mass \( \alpha_{\text{A},v} \), and zero ADM mass, the zero ADM mass is because the energy of the gravitational field is negative and equals the positive energy of the scalar field. The type A extension has Bondi mass \( (1 - \xi^2)\alpha_{\text{A},v} \), and ADM mass \( \xi^2 \sigma^2 t, (|2\sigma| < 1) \). Assuming that the null radiation field vanishes, so that there is only the conformal scalar field present the Type A solution has \( \alpha = \sigma v \), therefore
TypeO: \[ M(v) = \sigma^2 v, \quad A^t = 0, \]

TypeA: \[ M(v) = \frac{2}{3} \sigma^2 v, \quad A^t = \frac{1}{3} \sigma^2 t. \]

The type A solution might violate the energy conditions, but subject to this proviso it is possible to start with Minkowski space-time and join the type A solution at \( t = 0 \) generating non-zero ADM mass. This only goes to show that you can get something (as measured by ADM mass) from nothing.

A Appendix A: Junction Conditions Across A Null Surface

In single null coordinates a spherically symmetric line element can be written as
\[ ds^2 = -X\, dv^2 + 2S\, dv dr + Y(d\theta^2 + \sin^2 \theta d\phi^2). \quad (A.1) \]

A suitable null tetrad is
\[ l^a = (S, 0, 0, 0), \quad n^a = (X/2S, -1, 0, 0), \quad m^a = (0, 0, 1, isin\phi) \sqrt{Y/2}. \quad (A.2) \]

The projection tensor is defined by
\[ q^{ab} = g^{ab} + 2l^a n^b, \]
and has non-vanishing components
\[ q_{\theta\theta} = q_{\phi\phi} = 1. \]

The internal second fundamental form \( \chi^{ab} \) involves covariant derivatives of \( n^a \) which can be calculated using the Christoffel symbols in Roberts (1989) [3], and it has non-vanishing components
\[ \chi_{\theta\theta} = \sin^{-2}\theta \chi_{\phi\phi} = -(XY'/2S + Y')/2S. \quad (A.3) \]

The external second fundamental form \( \psi^{ab} = l^c q^{d}_{a} q^{d}_{b} \) is
\[ \psi_{\theta\theta} = \sin^{-2}\theta \psi_{\phi\phi} = Y'/2, \quad (A.4) \]

and the normal fundamental form \( \eta^a = l^c q^{d}_{a} n^{d} \) vanishes. The surface gravity \( \omega = -l^c n^c n^d; c \) is
\[ \omega = -(X/S)'/2S. \quad (A.5) \]

The surface stress \( \tau_{ab} = -(Tr\psi)n_a n_b - 2[\eta (a)]_b - [\omega] q_{ab} \) is
\[ \tau_{ab} = -(Y') n_a n_b + [(X/S)'/2S] q_{ab}, \quad (A.6) \]

with \( n_a \) and \( q_{ab} \) given by \( [A.3] \) and where the bracket \( [\ ] \) is defined by
\[ [Q]_y = \lim_{x\to y^+} Q - \lim_{x\to y^-} Q, \quad (A.7) \]

for a point \( y \) in the surface.
Appendix B: Junction Conditions Across Space-like And Time-like Surfaces

The line element can be taken in the form

\[ ds^2 = -C \, dt^2 + A \, dr^2 + B(d\theta + \sin^2 \theta d\varphi^2). \]  (B.1)

For a non-null surface, the surface stress vanishes iff the second fundamental form obeys

\[ [K_{ab}] = 0, \]  (B.2)

where the bracket "[ ]" is defined by A.7. A suitable unit time-like vector field

\[ U_a = (\sqrt{C}, 0, 0, 0). \]  (B.3)

The projection tensor \( h^b_a = g^{b}_{a} + U_a U^b \), has three components \( h^r_r = h^\theta_\theta = h^\varphi_\varphi = 1 \) The second fundamental form is \( K_{ab} = U_c d h^c_a h^d_b \) it involves covariant derivatives of \( U_a \) which can be calculated using the Christoffel symbols in Roberts (1989)[2][3], the non-vanishing components are

\[ K_{rr} = -\dot{\theta}/(2\sqrt{C}), \quad K_{\theta\theta} = \sin^{-2} \theta K_{\varphi\varphi} = -\dot{B}/(2\sqrt{C}), \]  (B.4)

where \( \dot{A} = \partial_t A \). The radial unit space-like vector is

\[ U_a = (0, \sqrt{A}, 0, 0), \]  (B.5)

The space-like projection tensor \( h^b_a = g^{b}_{a} - U_a U^b \), has three components \( h^t_t = h^\theta_\theta = h^\varphi_\varphi = 1 \). The second fundamental form for the corresponding time-like surface is

\[ K_{tt} = -C'/(2\sqrt{A}), \quad K_{\theta\theta} = \sin^{-2} \theta K_{\varphi\varphi} = B'/(2\sqrt{A}), \]  (B.6)

where \( C' = \partial - rC \). Choosing the isotropic unit space-like vector

\[ U_a = \frac{1}{\sqrt{3}} (0, \sqrt{A}, \sqrt{B}, \sqrt{B} \sin \theta), \]  (B.7)

similarly the non-vanishing components of the second fundamental form are

\[ K_{tt} = -C'/(2\sqrt{3}A), \]  (B.8)

\[ K_{tr} = -2\sqrt{A/B} K_{t\theta} = \frac{-2 A}{\sin \theta B} K_{t\varphi} = \frac{-\dot{A}}{3\sqrt{3}A} + \frac{\dot{B}}{3B} \sqrt{A}3, \]
\[ K_{r\theta} = \frac{K_{r\varphi}}{\sin \theta} = \sqrt{AK_{\theta \varphi}} \sin \theta = -\frac{1}{2} \sqrt{\frac{B}{A}} K_{rr} = -\frac{B'}{6\sqrt{3}B}, \]
\[ K_{\theta \theta} = \frac{K_{r\theta}}{\sin \theta} = \frac{1}{3\sqrt{3A}} \{-B \cot \theta + B'\}. \]

The above three vectors \[B.3\], \[B.5\], and \[B.7\] are suitable for the majority of purposes; but for example, if there is "Mixing" between the space and time coordinates, like that of equation \[2.16\] then other vectors have to be used.

## Appendix C: The ADM Energy

The ADM energy for a spherically symmetric space-time is found by generalizing Weinberg’s (1972) derivation for the Schwarzschild solution. Define the rectilinear coordinates
\[ x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta, \quad (C.1) \]
the line element \[B.1\] becomes
\[ ds^2 = -C \, dt^2 + \left(\frac{A}{r^2} - \frac{B}{r^4}\right)(dx_i dx_i) + \left(\frac{B}{r^2}\right) dx_i dx_i. \quad (C.2) \]
Defining \[h_{ij} = g_{ij} - \eta_{ij}\], and \[n^{-i} = x^i/r\], \[i, j = 1, 2, 3\], gives
\[ h_{ij} = (A - B/r^2)n_i n_j + (B/r^2 - 1)\delta_{ij}. \quad (C.3) \]
The ADM mass is given by
\[ A_i^t = \oint dS^i \left(h_{i,j}^t - h_{j,i}^t\right). \quad (C.4) \]
using \[C.3\] this is
\[ A_i^t = -\frac{1}{2} r \left(\frac{-A}{r} + \left(\frac{B}{r^2}\right)' + \frac{B}{r^3}\right). \quad (C.5) \]
The remaining components of the ADM vector are given by
\[ 8\pi A_j = \lim_{r \to \infty} \oint dS^i (K_{ij} - \delta_{ij} K) \quad (C.6) \]
using the fundamental form \[B.3\] gives
\[ 8\pi A_j = \lim_{r \to \infty} \left(\frac{B}{B\sqrt{C}}, 0, 0\right) \quad (C.7) \]
and this vanishes in all the cases considered here.
D  A Field Index

A perfect fluid has Lagrangian \( L_d = p \) (the pressure), see for example Roberts (1996)\cite{12}, and Hamiltonian density \( H_d = \mu \) (the density). Taking metric variations of the Lagrangian produces the metric stress

\[
T_{ab} = (\mu + p)V_a V_b + p \delta_{ab}, \quad V_a V^a = -1, \tag{D.1}
\]

The absolute derivative is defined by

\[
\dot{X}_{abc...} = V_e X_{abc...;e}, \tag{D.2}
\]

The Bianchi identities give

\[
-V_a T^{ab} = \dot{\mu} + (\mu + p)\dot{\Theta}, \quad \dot{\Theta} = V^a, \tag{D.3}
\]

\[
h_{ab} T_{bc} = (\mu + p)a + h_{a} b. \tag{D.4}
\]

The Eisenhart-Synge fluid index is

\[
\omega = \ln I = \int \frac{dp}{\mu + p}, \tag{D.4}
\]

in the literature sometime \( I \) is called the index and sometimes \( \omega \). Assuming an equation of state

\[
p = f(\mu), \quad \mu = f^{-1}(p), \tag{D.5}
\]

the Bianchi identities can be expressed as

\[
\dot{\Theta} = \omega \frac{df^{-1}}{dp}, \quad \dot{V}_a = -h_{a} b \omega. \tag{D.6}
\]

For the equation of state\cite{23} the index is \( \omega = [(\gamma - 1)/\gamma]ln\mu \). An O-scalar field has this equation of state with \( \gamma = 2 \) and \( V_a = \varphi_a / \sqrt{-\varphi^2_c}, p = \mu = -\varphi^2_c \). Taking a time-like vector field, say

\[
V_a = (N, 0, 0, 0), \quad V^a = (-1/N, 0, 0, 0), \tag{D.7}
\]

it is possible to relate the behaviour of the lapse \( N \) to the fluid index \( \omega \). For many configurations they can be equated \( N = g(\omega) \), typically \( N = 1/\omega \). Thus if the fluid index is well-behaved throughout the space it is possible, without recourse to field equations, to discuss whether the metric is. Other choices of time-like vector-field can be made, for example an asymptotically flat space-time there is the normal vector to the three-sphere at infinity.
There is no direct analog of the preceding for fields with infinite degrees of freedom. This is because the stress for fields does not have an explicit dependence on vector fields. It is possible to introduce vector fields into an action and vary it to produce an extreme configuration between the metric, fields, and vector field, in the following manner. Let the fields be describable by a Lagrangian

$$I_l = \int \sqrt{-g} d^4x \mathcal{L}(\varphi, \varphi_a),$$  \hspace{1cm} (D.8)

Under infinitesimal coordinate variations this gives

$$\frac{\delta I}{\delta x^a} \int \sqrt{-g} d^4x \nabla \Theta_c^{ab},$$  \hspace{1cm} (D.9)

where $\Theta^{ab}_c$ is the canonical stress

$$\Theta^{ab}_c = \frac{\partial \mathcal{L}}{\partial \varphi_a} \partial^b \varphi - g^{ab} \mathcal{L}.$$

The Hamiltonian density is usually defined in terms of components $\mathcal{H}_d = \Theta^{0}_c$, more generally it can be defined as

$$\mathcal{H}_d = V^a V^b \Theta^{ab}_c,$$

where $V_a$ is a unit time-like vector-field. Variations of $I_l$ with respect to the metric produce the stress

$$T_{ab} = D_{ab} + \mathcal{L} g_{ab},$$

where $D_{ab}$ is given by $\delta I_d/\delta g_{ab}$. Applying the time-like vector-field and using $\mathcal{H}_d$ gives

$$\mathcal{H}_d + \mathcal{L}_d = V^a V^b D_{ab}.$$

Variations of actions corresponding to $\mathcal{H}_d$ and $\mathcal{L}_d$ give dynamical equations, this suggests considering a new action $I_n$ which is a sum of the Hamiltonian and Lagrangian actions

$$I_n = \int \sqrt{-g} d^4x (\mathcal{H}_d + \mathcal{L}_d) = \int \sqrt{-g} d^4x V^a V^b D_{ab},$$

which will give extremeal (maximally stable or unstable) configurations. Another possible way of producing further equations between the fields is to consider higher order variations. For example Bazanski (1977)[23],

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the second covariant of the point particle action gives the geodesic deviation equations.

A scalar-electrodynamic Lagrangian for a complex scalar field $\psi$ and a vector field $A_a$ is

$$
\mathcal{L} = -\frac{1}{4} F^2 + \frac{1}{2} \mu^2 A^2 - \lambda (A_a^2)^2 + J_a A^a - D_a \psi D^a \bar{\psi} - V(\psi^2) + 2 \beta \psi^2 R,
$$

$$
D_a \psi = \partial \psi + ie A_a, \quad \psi^2 = \bar{\psi} \psi, \quad (D.15)
$$

$$
D_a \bar{\psi} = \partial \bar{\psi} - ie \bar{\psi} A_a.
$$

Varying with respect to the metric gives

$$
D_{ab} = F_{ac} F_b^c - \mu^2 A_a A_b - 2 J_a A_b + 4 \lambda^2 A_{ab} A_c^c + 2 D_{(a} \psi dr_{b)} \bar{\psi} - 2 \beta (\psi^2)_{ab} - 2 \beta \psi^2 R. \quad (D.16)
$$

Setting up the new action $I_n$ and varying with respect to the fields gives

$$
\frac{\delta I_n}{\delta \psi} := -2 (V^a V^b D) b \psi a + 2 ie A_a V^a V^b D_b \bar{\psi} + 2 \beta \bar{\psi} [ (V^a V^b)_{ab} - R_{ab} V^a V^b ], \quad (D.17)
$$

$$
\frac{\delta I_n}{\delta A_a} := -2 (V^a V^b F_b^c + V^c V^b F_{ba}) c + 2 V_a ( -\mu^2 V^b A_b - J_b A^b ) + ie ( \psi D_a \bar{\psi} - \bar{\psi} D_a \psi ),
$$

It is also possible to vary with respect to $\dot{\psi}, \dot{A}_a, g_{ab}$, and $V_a$. Variations with respect to $V_a$ are best done using velocity potentials, see Roberts (1996)[10][12]. Varying with respect to $g_{ab}$ a new ”stress” tensor can be created. Applying the Bianchi identities to it the equations $D.18$ are not recovered, because the conservation equations as derived from $D.9$ are also changed. Choosing $D_{ab} = R_{ab}$ and varying with respect to $g_{ab}$ does not reproduce Raychaudhuri’s equations but instead $\Theta = R_{ab} V^a V^b + \Theta^2 - V_{a, b} + \frac{1}{2} \Box V^2$. For the Reissner-Nordstrom solution with vector-field $D.7$, and the null vector field $l_a$, $D.18$ becomes

$$
\psi_t = -\frac{4 \sqrt{2}}{r^3 N^3} e \left[ 1 + \frac{2m}{r} - \frac{3 e^2}{r} \right],
$$

$$
\psi_v = 4 \sqrt{2} \frac{e}{r^3}. \quad (D.18)
$$
For static O-scalar fields the field index vanishes everywhere, for non-static O-scalar fields using \( l_a \) gives

\[
\psi = -2 \left( \frac{2S'}{S} + \frac{Y'}{Y} \right) \phi'.
\]  

(D.19)

In the solution 4.3 this is

\[
\psi = \frac{4\sigma v}{Y^2} (r - \sigma v).
\]  

(D.20)

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