COUNTING ALL REGULAR TETRAHEDRA IN \( \{0, 1, ..., n\}^3 \)

EUGEN J. IONASCU

Abstract. In this note we describe a procedure of calculating the number all regular tetrahedra that have coordinates in the set \( \{0, 1, ..., n\} \). We develop a few results that may help in finding good estimates for this sequence which is twice A103158 in the Online Encyclopedia of Integer Sequences [13].

1. INTRODUCTION

The story of regular tetrahedra having vertices of integer coordinates starts with the parametrization of some equilateral triangles in \( \mathbb{Z}^3 \) that begun in [9]. There was an additional hypothesis that did not cover all the generality in the result obtained in [9] but it was removed successfully in [2]. A few other related results appeared in [10] and [11]. In this note we are interested in the following problem.

How many regular tetrahedra, \( T(n) \), can be found if the coordinates of its vertices must be in the set \( \{0, 1, ..., n\} \)? We observe that \( A103158 = \frac{1}{2} T(n) \), see [13]. This sequence starts as in the next tables:

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|--------|---|---|---|---|---|---|---|---|---|----|----|
| A103158 | 1 | 9 | 36 | 104 | 257 | 549 | 1058 | 1896 | 3199 | 5154 | 7926 |
| \( n \) | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| A103158 | 11768 | 16967 | 23859 | 32846 | 44378 | 58977 | 77215 |

Using our method which is going to be described later we extended this sequence for all \( n \leq 100 \), and one can go far enough with this if time allows and powerful computer is used.

The rest of the terms are included at the end of the paper. Our approach begins with looking first at the faces of a regular tetrahedron, which must be equilateral triangles. It turns out that every equilateral triangle in \( \mathbb{Z}^3 \) after a translation by a vector with integer coordinates can be assumed to have the origin as one of its vertices. Then one can show that the other triangle’s vertices are contained in a lattice of points of the form

Date: December 5th, 2009.

Key words and phrases. diophantine equations, integers.
\( \mathcal{P}_{a,b,c} := \{ (\alpha, \beta, \gamma) \in \mathbb{Z}^3 | \ a\alpha + b\beta + c\gamma = 0, \ a^2 + b^2 + c^2 = 3d^2, \ a, b, c, d \in \mathbb{Z} \}. \)

Figure 1: The lattice \( \mathcal{P}_{a,b,c} \)

In general, the vertices of the equilateral triangles that dwell in \( \mathcal{P}_{a,b,c} \) form a strict sub-lattice of \( \mathcal{P}_{a,b,c} \) which is generated by only two vectors, \( \overrightarrow{\zeta} \) and \( \overrightarrow{\eta} \) (see Figure 1). These two vectors are described by the Theorem 1.1 proved in [2].

**Theorem 1.1.** Let \( a, b, c, d \) be odd integers such that \( a^2 + b^2 + c^2 = 3d^2 \) and \( \gcd(a, b, c) = 1 \). Then for every \( m, n \in \mathbb{Z} \) (not both zero) the triangle \( OPQ \), determined by

\[
\begin{align*}
\overrightarrow{OP} &= m \overrightarrow{\zeta} - n \overrightarrow{\eta}, \quad \overrightarrow{OQ} = n \overrightarrow{\zeta} - (n - m) \overrightarrow{\eta}, \quad \text{with} \quad \overrightarrow{\zeta} = (\zeta_1, \zeta_2, \zeta_3), \quad \overrightarrow{\eta} = (\eta_1, \eta_2, \eta_3), \\
(3) \quad &\begin{cases} 
\zeta_1 = -\frac{rac + dbs}{q}, \\
\zeta_2 = \frac{das - bcr}{q}, \\
\zeta_3 = r,
\end{cases} \quad \begin{cases} 
\eta_1 = -\frac{db(s - 3r) + ac(r + s)}{2q}, \\
\eta_2 = \frac{da(s - 3r) - bc(r + s)}{2q}, \\
\eta_3 = \frac{r + s}{2},
\end{cases}
\end{align*}
\]

where \( q = a^2 + b^2 \) and \( (r, s) \) is a suitable solution of \( 2q = s^2 + 3r^2 \) that makes all the numbers in (3) integers, forms an equilateral triangle in \( \mathbb{Z}^3 \) contained in the lattice (1) and having sides-lengths equal to \( d\sqrt{2(m^2 - mn + n^2)} \).
COUNTING ALL REGULAR TETRAHEDRA IN $\{0, 1, \ldots, n\}^3$

Conversely, there exists a choice of the integers $r$ and $s$ such that given an arbitrary equilateral triangle in $\mathbb{R}^3$ whose vertices, one at the origin and the other two in the lattice $[1]$, then there also exist integers $m$ and $n$ such that the two vertices not at the origin are given by $[2]$ and $[3]$.

The Diophantine equation

$$a^2 + b^2 + c^2 = 3d^2$$

has non-trivial solutions for every $d$ odd. As a curiosity, for $d = 2009$ one obtains 294 solutions satisfying also $0 < a \leq b \leq c$ and $\gcd(a, b, c) = 1$. We will refer to such a solution of (4) as a positive ordered primitive solution. For $d = 2008$, all these solutions satisfy even a stronger condition: $a < b < c$. Determining the exact number of solutions for (4) is certainly important if one wishes to find the number (or just an estimate) of equilateral triangles or the number of tetrahedra with vertices in $\{0, 1, 2, \ldots, n\}^3$. The number of solutions for (4), coincidentally, taken into account all permutations and changes of signs is given in a 1999 paper of Hirschhorn and Seller $[8]$:

$$\Lambda(d) := 8d \prod_{p|d, p \text{ prime}} \left( 1 - \frac{\left( \frac{-3}{p} \right)}{p} \right),$$

where $\left( \frac{-3}{p} \right)$ is the Legendre symbol.

We remind the reader that, if $p$ is prime then

Theorem 1.2. [Cooper-Hirschhorn] Given an odd number $d$, the number of primitive solutions of (4) taking into account all changing of signs and permutations, is equal to

$$\Lambda(d) := 8d \prod_{p|d, p \text{ prime}} \left( 1 - \frac{\left( \frac{-3}{p} \right)}{p} \right),$$

where $\left( \frac{-3}{p} \right)$ is the Legendre symbol.
(7) \[ \left\{ \begin{array}{ll}
0 & \text{if } p = 3 \\
1 & \text{if } p \equiv 1 \text{ or } 7 \pmod{12} \\
-1 & \text{if } p \equiv 5 \text{ or } 11 \pmod{12}
\end{array} \right. \]

We observe that the same type of prime partition are used into different calculations in both formulae (5) and (6). We have mentioned that the number of positive ordered primitive representations for \( d = 2009 \) was 294. This is exactly the number given by (6) modulo the number of permutations and changes of signs: indeed, \( 2009 = (41)(7^2) \), \( \left\langle -3 \frac{7}{7} \right\rangle = 1 \), \( \left\langle -3 \frac{41}{41} \right\rangle = -1 \) and \( \frac{\Lambda(2009)}{48} = \frac{8(41)(7^2)}{48}(1 - \frac{1}{7})(1 + \frac{1}{41}) = 7(42) = 294 \). This happens because there are no repeating values for \( a, b, \) and \( c \) in any of the positive ordered primitive solutions of (4). We will see later how the correct number or positive ordered primitive representations can be obtained in general.

For \( k \in \mathbb{N} \), we let \( \Omega := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m^2 - mn + n^2 = k^2 \} \). In [9] we showed that every regular tetrahedron with integer coordinates must have side length \( \lambda \sqrt{2} \), \( \lambda \in \mathbb{N} \), and in [11] we have found the following characterization of the regular tetrahedrons with integer coordinates.

Theorem 1.3. Every tetrahedron whose side lengths are \( \lambda \sqrt{2} \), \( \lambda \in \mathbb{N} \), which has a vertex at the origin, can be obtained by taking as one of its faces an equilateral triangle having the origin as a vertex and the other two vertices given by (2) and (3) with \( a, b, c \) and \( d \) odd integers satisfying (4) with \( d \) a divisor of \( \lambda \), and then completing it with the fourth vertex \( R \) with coordinates

\[
\left( \begin{array}{c}
\frac{(2\zeta_1 - \eta_1)m}{\pm 2ak} \\
\frac{-(\zeta_1 + \eta_1)n}{3} \\
\frac{-2bk}{3}
\end{array} \right), \left( \begin{array}{c}
\frac{(2\zeta_2 - \eta_2)m}{\pm 2bk} \\
\frac{-(\zeta_2 + \eta_2)n}{3} \\
\frac{-2ck}{3}
\end{array} \right), \left( \begin{array}{c}
\frac{(2\zeta_3 - \eta_3)m}{\pm 2ck} \\
\frac{-(\zeta_3 + \eta_3)n}{3} \\
\frac{+2ak}{3}
\end{array} \right), \text{ for some } (m, n) \in \Omega(k), k := \frac{\lambda}{d}.
\]

Conversely, if we let \( a, b, c \) and \( d \) be a primitive solution of (4), let \( k \in \mathbb{N} \) and \( (m, n) \in \Omega(k) \), then the coordinates of the point \( R \) in (8), which completes the equilateral triangle \( OPQ \) given as in (2) and (3), are

(a) all integers, if \( k \equiv 0 \pmod{3} \) regardless of the choice of signs or

(b) integers, precisely for only one choice of the signs if \( k \not\equiv 0 \pmod{3} \).

The following graph (Figure 2) is constructed on the positive ordered primitive solutions of (4), with edges defined by:

two vertices, say \([a_1, b_1, c_1, d_1]\) and \([a_2, b_2, c_2, d_2]\), are connected, if and only if
\[ a_1 a_2' \pm a_2 b_2' \pm c_1 c_2' \pm d_1 d_2 = 0 \]

for some choice of the signs and permutation \((a_2', b_2', c_2')\) of \((a_2, b_2, c_2)\).

Equation (9) insures basically that the planes \(P_{a_1, b_1, c_1}\) and \(P_{a_2', b_2', c_2'}\) associated to two faces make a dihedral angle of \(\text{arccos}(1/3) \approx 70.52878^\circ\). In fact, this equality characterizes the existence of a regular tetrahedron having integer coordinates with one of its faces in the plane \(P_{a_1, b_1, c_1}\) and another contained in the plane \(P_{a_2', b_2', c_2'}\). For instance, \([(1, 1, 5), 3]\) is connected to \([(1, 5, 11), 7]\) since \(1(11) + (1)5 + 5(1) - 3(7) = 0\). An example of a regular tetrahedron which has a face in \(P_{-5, -1, 1}\) and one face in \(P_{-1, -5, 11}\) is given by the vertices: \([19, 23, 0], [0, 12, 20], [27, 0, 17], \text{and} [24, 27, 29]\).

![Figure 2: The graph \(RT, d \leq 19\).](image)

A few questions related to this graph appear naturally at this point. Is it connected? Is there a different characterization of the existence of an edge between two vertices in terms of only \(d_1\) and \(d_2\)? We do not have an answer to the second question. This graph seems to have a fractal structure.
Each edge in this graph, determined by \([a_1, b_1, c_1, d_1]\) and \([a_2, b_2, c_2, d_2]\), gives rise to a minimal tetrahedra (the side lengths are at most \(\max\{d_1, d_2\} \sqrt{2}\)) which is determined up to the set of isometric transformations that are generated by the symmetries of the cube \(C(m)\) where \(m\) is the size of the smallest “cube” \(\{0, 1, \cdots, m\}^3\) containing the tetrahedron or a translation of it.

2. Some preliminaries

We would like to have a good estimate of the primitive solutions of (4) which satisfy in addition \(0 < a \leq b \leq c\). Let us observe that we cannot have \(a = b = c\) unless \(d = 1\). So, the counting in (6) via (7) would give what we want if we can count the number of positive primitive solutions of the following equation in terms of \(d\):

\[
2a^2 + c^2 = 3d^2.
\]

A similar description to the Pythagorean triples, which gives the nature of the solutions of (10), is stated next.

**Theorem 2.1.** For every positive integers \(l\) and \(k\) such that, \(\gcd(k, l) = 1\) and \(k\) is odd, then \(a, c\) and \(d\) given by

\[
(11) \quad d = 2l^2 + k^2 \quad \text{and} \quad \begin{cases} a = \lfloor 2l^2 + 2kl - k^2 \rfloor, & c = \lfloor k^2 + 4kl - 2l^2 \rfloor, \text{ if } k \not\equiv l \pmod{3} \\ a = \lfloor 2l^2 - 2kl - k^2 \rfloor, & c = \lfloor k^2 - 4kl - 2l^2 \rfloor, \text{ if } k \equiv -l \pmod{3} \end{cases}
\]

constitute a positive primitive solution for (10).

Conversely, with the exception of the trivial solution \(a = c = d = 1\), every positive primitive solution for (11) appears in the way described above for some \(l\) and \(k\).

**Proof.** First, one can check that (11) satisfy (10) for every \(l\) and \(k\). As a result it follows that \(a, c\) and \(d\) are positive integers. Let \(p\) be a prime dividing \(a, c\) and \(d\). Then \(p\) must divide \(\pm a - d = 2k(\pm l - k)\) and so \(p\) is equal to 2, \(p\) divides \(k\) or it divides \(\pm l - k\). If \(p = 2\) then, \(p\) must divide \(k\) but this contradicts the assumption that \(k\) is odd.

In case \(p\) is not equal to 2 and it divides \(k\), we see \(p\) must divide \(l^2 = (d - k^2)/2\). Since we assumed \(\gcd(l, k) = 1\) it remains that \(p\) must divide \(\pm l - k\). By our assumptions on \(k\) and \(l\), \(p\) cannot be equal to 3. Then \(p\) divides \(\pm a + (\pm l - k)^2 = 3l^2\). Because \(p \neq 3\) then \(p\) must divide \(l^2\) and so \(p\) should divide \(l\) and then \(k\). This contradiction shows that \(a, c\) and \(d\) cannot have prime common factors. So, we have a primitive solution in (11).

For the converse, let us assume that \(a, c\) and \(d\) is a positive primitive solution of (10), which is different of the trivial one. We denote by \(u = \frac{a}{d}\) and \(v = \frac{c}{d}\). Then the point of rational coordinates
\((u, v)\) (different of \((1, 1)\)) is on the ellipse \(\frac{x^2}{3^2/2} + \frac{y^2}{3} = 1\) (Figure 3) in the first quadrant. This ellipse contains the following four points with integer coordinates: \((1, 1), (-1, 1), (-1, -1)\) and \((1, 1)\). This gives the lines \(y + 1 = t_1(x + 1), y + 1 = t_2(x - 1), y - 1 = t_3(x + 1),\) and \(y - 1 = t_4(x - 1),\) passing through \((u, v)\) and one of the points mentioned above. Hence, the slopes \(t_1, t_2, t_3,\) and \(t_4\), are rational numbers. This gives expressions for the point \((u, v)\) in terms of \(t_i\) \((i = 1, ..., 4)\). Let us assume that \(t_i = \frac{k_i}{l_i}\) with \(k_i, l_i \in \mathbb{Z}\), written in the reduced form. Then we must have

\[
\begin{align*}
u &= \frac{|2 \pm 2t_i - t_i^2|}{2 + t_i^2} = \frac{|2l_i^2 \pm 2k_i l_i - k_i^2|}{2l_i^2 + k_i^2}, \quad v &= \frac{|t_i^2 \pm 4t_i - 2|}{2 + t_i^2} = \frac{|k_i^2 \pm 4k_i l_i - 2l_i^2|}{2l_i^2 + k_i^2},
\end{align*}
\]

Figure 3: The ellipse \(\frac{x^2}{3^2/2} + \frac{y^2}{3} = 1\)

and so, these equalities give

\[
(12) \quad \frac{a}{d} = \frac{|2l_i^2 \pm 2k_i l_i - k_i^2|}{2l_i^2 + k_i^2}, \quad \text{and} \quad \frac{c}{d} = \frac{|k_i^2 \pm 4k_i l_i - 2l_i^2|}{2l_i^2 + k_i^2}, \quad i = 1, ..., 4.
\]

We claim that the function \(t_i \to 2l_i^2 + k_i^2\) \((i = 1, ..., 4)\) is injective. If for some \(2l_i^2 + k_i^2 = 2l_j^2 + k_j^2\) \((i \neq j)\), that would imply that the corresponding numerators in \((12)\) are equal. This gives enough information to conclude a contradiction. There are \(\binom{4}{2} = 6\) possibilities here but we are going to include the details only in the case \(i = 1\) and \(j = 2\). The rest of the cases can be done in a similar fashion. For this situation we have, \(2l_1^2 + 2k_1 l_1 - k_1^2 = k_2^2 + 2k_2 l_2 - 2l_2^2\) and \(k_1^2 + 4k_1 l_1 - 2l_1^2 = k_2^2 - 4k_2 l_2 - 2l_2^2\). The first equality implies

\[
2k_1 l_1 = k_1^2 + k_2^2 + 2k_2 l_2 - 2l_1^2 - 2l_2^2 = 2k_1^2 + 2k_2 l_2 - 4l_2^2
\]
which substituted into the second equality gives

$$6k_1^2 = 2k_2^2 - 8k_2l_2 + 8l_2^2 \iff 3k_1^2 = (k_2 - 2l_2)^2.$$  

Because $\sqrt{3}$ is irrational, the last equality is impossible for $k_1$, $k_2$, $l_2$ integers and $k_1$ nonzero. For the other cases one will get a contradiction based on the facts that $\sqrt{\frac{3}{2}}$ and $\sqrt{2}$ are irrational numbers.

A similar argument to the one in the first part of the proof shows that the fractions in the right-hand side of the equalities of (12) can be simplified only by a factor of 2, 3 or 6. Having four distinct possibilities in (12) for the denominators, exactly one of the fractions (simultaneously in the first and second equalities) must be in reduced form. This one will give the wanted representation. ■

Similar to Fermat’s theorem about the representation of primes as a sum of two squares and the number of such representations one can show the next result.

**Theorem 2.2.** (Fermat [4]) An odd prime $p$ can be written as $2x^2 + y^2$ with $x, y \in \mathbb{Z}$ if and only if $p \equiv 1$ or $3 \pmod{8}$. If $d = 2^k \prod p_i^{\alpha_i} \prod q_j^{\beta_j}$ is the prime factorization of $d$, with $q_j$ primes as before and $p_i$ the rest of them, then the number of representations $d = 2x^2 + y^2$ with $x, y \in \mathbb{Z}$ is either zero if not all $\alpha_i$ are even and otherwise given by

$$\lfloor \frac{1}{2} \prod (\beta_i + 1) \rfloor.$$  

The number of positive primitive representations $d = 2x^2 + y^2$ for $d$ odd, i.e. $x, y \in \mathbb{N}$ and $gcd(x, y) = 1$, is equal to

$$\Gamma_2(d) = \begin{cases} 0 & \text{if } d \text{ is divisible by a prime factor of the form } 8s+5 \text{ or } 8s+7, \ s \geq 0, \\ 2^{k-1} & \text{where } k \text{ is the number of distinct prime factors of } d \\ & \text{of } d \text{ of the form } 8s+1, \text{ or } 8s+3 (s \geq 0). \end{cases}$$  

Putting the two results together (Theorem 2.2 and Theorem 1.2) we obtain the following proposition:

**Proposition 2.3.** For every $d$ odd, the number of representations of (4) which satisfy $0 < a \leq b \leq c$ and $gcd(a, b, c) = 1$ is equal to

$$\pi_\epsilon(d) = \frac{\Lambda(d) + 24\Gamma_2(3d^2)}{48}.$$  

A regular tetrahedron whose vertices are integers is said to be irreducible if it cannot be obtained by an integer dilation and a translation from a smaller one also with integer coordinates. An important question at this point about irreducible tetrahedra is included next.
COUNTING ALL REGULAR TETRAHEDRA IN \{0, 1, ..., n\}³

Does every irreducible tetrahedron with integer coordinates have a face with a normal vector \((a, b, c)\) satisfying \(a^2 + b^2 + c^2 = 3d^2\), such that \(d\) gives the side lengths \(ℓ\) of the tetrahedron by the formula \(ℓ = d\sqrt{2}\)? In other words, is there a face for which \(k = 1\) in the Theorem 1.3?

Unfortunately the answer to this question is no. The following points together with the origin, \([-6677, -2672, 1445]\), \([-5940, 4143, -1167]\), \([-3837, 2595, 5688]\) form a regular tetrahedron of side-lengths equal to \(5187\sqrt{2}\) and the highest \(d\) for the faces is 1729. We observe that 3, 7, 13 and 19 are the first three distinct primes numbers of the form \(u^2 + 3v^2\), \(u, v \in \mathbb{Z}\).

3. The Code

The program is written in Maple code and it is based on the Theorem 1.3. The main idea is to create a list of irreducible regular tetrahedra that can be used to generate all the others in \{0, 1, 2, ..., n\}³ by certain transformations generating a partition for the set of all the tetrahedra. Each such irreducible tetrahedron is constructed out of the equation of one face using Theorem 1.3.

One important problem that appears here is to make sure this list contains distinct elements, elements which may appear theoretically in this list from four different constructions, one for each face. It turns out that there is a simple way of making sure that this doesn’t happen.

Let us observe that if \(gcd(m, n) = d > 1\) then all the coordinates of the vertices of the initial face are multiple of \(d\) and by the formula (8), so are the coordinates of the fourth point of the tetrahedron. This is because the numbers \(k\) in (8) satisfy a Diophantine equation of the form \(k^2 = m^2 - mn + n^2\). We go one step further, if \(k^2\) is of the form

\[ m^2 - mn + n^2 = \left(\frac{m + n}{2}\right)^2 + 3\left(\frac{m - n}{2}\right)^2, \]

then one can see that for \(k\) even, the \(gcd(m, n) \geq 2\). Also, if \(k\) is odd then both \(m\) and \(n\) must be odd and so we have a representation of \(k^2\) as \(u^2 + 3v^2\), \(u, v \in \mathbb{Z}\). If \(k\) is divisible by 3 then it is easy to see that 3 divides \(u\) an \(v\), and this attracts \(gcd(m, n) \geq 3\). Hence we are going to look only for those odd values \(k \leq n\), which are not multiples of 3, in the Theorem 1.3. This means that only one choice of signs in (8) is useful. A similar fact to Theorem 2.2 takes place.

**Theorem 3.1. (Fermat [4])** A prime \(p\) can be written as \(x^2 + 3y^2\) with \(x, y \in \mathbb{Z}\) if and only if \(p = 3\) or \(p \equiv 1 \mod 3\). If \(d = \prod \pi_i^{\alpha_i} \prod q_j^{\beta_j}\) with \(q_j\) primes as before and \(p_i\) the rest of them, then the number of representations \(d = x^2 + 3y^2\) with \(x, y \in \mathbb{Z}\) is either zero if not all \(\alpha_i\) are even and otherwise given by
The number of positive primitive representations \( d = x^2 + 3y^2 \) for \( d \) odd, i.e. \( x, y \in \mathbb{N} \) and \( \gcd(x, y) = 1 \), is equal to

\[
\frac{1}{2} \prod (\beta_i + 1).
\]

(17)

As a result of these facts we first calculate all \( k \leq n \) such that \( k^2 = m^2 - mn + n^2 \) has a solution with \( \gcd(m, n) = 1 \).

To describe all the solutions of (16), for each \( k \) found by the previous procedure, there are usually at least eighteen solutions if signs and order are counted, but if we impose the conditions \( \gcd(m, n) = 1 \), \( 0 < m, n \) and \( 2m < n \), we slice these solutions by a factor of 18. Such a solution is going to be referred to
as a *primitive* solution and these primitive solution of (16) can be calculated with the following procedure.

```markdown
listofmn:=proc(k)
local a,b,i,x,m,n,nx,L;
x:=[isolve(k^2 = m^2 - mn + n^2)]; nx:=nops(x); L:={
for i from 1 to nx do
if lhs(x[i][1])=m then a:=rhs(x[i][1]); b:=rhs(x[i][2]);
else b:=rhs(x[i][1]); a:=rhs(x[i][2]); fi;
if gcd(a,b)=1 and a > 0 and b > 0 and 2a < b then L:=L union [a,b];fi;
od;L;
end:
```

For example, if k = 91 we get two primitive solutions, m = 1991, n = 9095, and m = 3401, n = 9440. It turns out that it is enough to know just the primitive solutions of (16) in order to find all integer solutions. For each k ≤ n, output of the procedure `kvalues`, we need to find the values of d as in Theorem 1.3, which are only restricted to two conditions: d must be an odd positive integer and dk ≤ n. The last restriction follows from the fact that the sides of the regular tetrahedron constructed from say d and k, with a primitive solution (m, n) of (16), must be equal to dk√2. One can see that the biggest regular tetrahedron inscribed in the cube [0, n]^3 has sides equal to n√2 (see Proposition 2.1 in [10]). The next procedure is then very simple.

```markdown
determined:=proc(n)
local i,x,m,L,j;
x:=kvalues(n);m:=nops(x);
for i from 1 to m do
j:=1;L[i]:=\{\};
```

![Figure 4: Largest tetrahedrons in a cube](image-url)
while \((2j-1)x[i]\leq n\) do
\[L[i]:=L[i] \cup \{2j-1\}; j:=j+1;\] od;
\[\text{od}; [\text{seq}(L[i], i=1..m)];\]
end:

Next, we need to find all primitive solutions \((a, b, c)\) of the equation \(a^2 + b^2 + c^2 = 3d^2\). The number of such solution is given by \((15)\).

\[
\text{abcsol}:=\text{proc}(d)\\
\text{local } i,j,k,m,u,x,y,\text{sol,cd};\\
\text{sol}:=\{\};\\
\text{for } i \text{ from } 1 \text{ to } d \text{ do }\\
u:=\text{isolve}(3d^2 - i^2 = x^2 + y^2)]; k:=\text{nops}(u);\\
\text{for } j \text{ from } 1 \text{ to } k \text{ do }\\
\text{if } \text{rhs}(u[j][1])>=i \text{ and } \text{rhs}(u[j][2])>=i \text{ then }\\
cd:=\text{gcd}(\text{gcd}(i, \text{rhs}(u[j][1])), \text{rhs}(u[j][2]));\\
\text{if } cd=1 \text{ then } \text{sol}:=\text{sol} \cup \{\text{sort}([i, \text{rhs}(u[j][1]), \text{rhs}(u[j][2])])\};fi;\\
fi;\\
\text{od}; \text{ od};\\
\text{convert(sol,list)};\\
\text{end:}
\]

For example, if \(d = 2009\) we get 294 solutions, as seen before, of which one of them is \(a = 1, b = 1159\) and \(c = 3281\). Next, we are going to use the construction of an equilateral triangle in the plane of equation \(ax + by + cz = 0\) with \(a^2 + b^2 + c^2 = 3d^2\) using the formulae \((9)\) with \(m\) and \(n\) given by the procedure \(\text{listofmn}\). The fourth point is then calculated using the formula \((8)\). We are using only two of the possible values of \(m\) and \(n\) \((m' = m, n' = n\) and \(m' = n, n' = n - m\), in order to obtain two equilateral triangles that share a side, \(\overrightarrow{OQ}\) since all other tetrahedra as in the figure below, can be obtained from these two by a simple translation, and as a result they will translate into the same minimal tetrahedron inside the first quadrant (see \([11]\)).
COUNTING ALL REGULAR TETRAHEDRA IN \{(0,1,...,n)^3\}

Figure 5: All six tetrahedrons with one face in $\mathcal{P}_{a,b,c}$

```
findpar:=proc(a,b,c,mm,nn)
local i,j,sol,mx,nx,r,s,my,ny,q,d,u,v,w,x,y,z,nu,nu,nu,v,nz, nw,om1,om2,l,uu,t,R1,R2,fc,k;
q := a^2 + b^2; k := sqrt(mm^2 - mm * nn + nn^2);
sol:=convert(isolve(2*q = x^2 + 3*y^2),list); ns:=nops(sol); d:=sqrt((a^2 + b^2 + c^2)/3);
if ef=0 then r:=rhs(sol[1][1]); s:=rhs(sol[1][2]);
   uu:=(s^2 + 3*r^2 - 2*q)^2; if uu > 0 then t:=s;s:=r;r:=t; fi;
   mx:=-((d*b*(3+r+s) + a*c*(r-s))/(2*q));nx:=-(r*a*c + d*b*s)/q;
   my:=(d*a*(3*r+s) - b*c*(r-s))/(2*q);ny:=-(r*b*c - d*a*s)/q;
   mz:=(r-s)/2;nu:=my;nx:=nu;ny:=nu;mw:=nz;nu:=nx-nz;nv:=ny-my;
   if mx=floor(mx) and nx=floor(nx) and my=floor(my) and ny=floor(ny) then
      u:=nu*m-nu*n;v:=mv*m-nv*n;w:=mw*m-nw*n;
      x:=mx*m-nx*n;y:=my*m-ny*n;z:=mz*m-nz*n;
      R1:=[(x+u+2*a*k)/3,(v+y+2*b*k)/3,(z+w+2*c*k)/3];
      R2:=[(x+u+2*a*k)/3,(v+y+2*b*k)/3,(z+w+2*c*k)/3];
      fc:=subs(m=mm,n=nn,R1[1]); if fc=floor(fc) then
         om1:=subs(m=mm,n=nn,[u,v,w],[x,y,z],R1[1]); else
         om1:=subs(m=mm,n=nn,[u,v, w],[x,y, z],R2);fi;
      fc:=subs(m=nn,n=nn-mm,R1[1]); if fc=floor(fc) then
         om2:=subs(m=nn,n=nn-mm,[u,v, w],[x,y, z],R1[1]); else
         om2:=subs(m=nn,n=nn-mm,[u,v, w],[x,y, z],R2);fi;
      ef:=1; fi;fi; od;
om1,om2; end:
```
Before we translate these two tetrahedrons we need a small sub-routine for subtraction of two vectors.

\[
\text{subtrv} := \text{proc}(U,V) \text{ local } W; \\
W[1] := U[1] - V[1]; W[2] := U[2] - V[2]; W[3] := U[3] - V[3]; [W[1],W[2],W[3]]; \text{ end:}
\]

The next procedure translates a tetrahedron which comes as output of \textit{findpar} into the positive octant of the space in such a way that for each component at least one of the tetrahedron’s vertex has a zero coordinate on that component. Let us observe that this operation is invariant to translations of the tetrahedron; this justifies the choice of looking at only two tetrahedrons out of six in the procedure \textit{findpar}.

\[
\text{tmttopq} := \text{proc}(T) \text{ local } i,a,b,c,v,O,TR; \\
a := \text{min}(T[1][1],T[2][1],T[3][1],0); \\
b := \text{min}(T[1][2],T[2][2],T[3][2],0); \\
c := \text{min}(T[1][3],T[2][3],T[3][3],0); O := [0,0,0]; v := [a,b,c]; \\
TR := \{\text{subtrv}(O,v),\text{subtrv}(T[1],v),\text{subtrv}(T[2],v),\text{subtrv}(T[3],v)\}; \text{ end:}
\]

Next, we calculate the size of smallest cube \(C_m := [0,m]^3\) which contains the tetrahedron resulted from the \textit{tmttopq}.

\[
\text{mscofmt} := \text{proc}(Q) \text{ local } a,b,c,T,m; T := \text{convert}(Q,\text{list}); \\
a := \text{max}(T[1][1],T[2][1],T[3][1],T[4][1]); \\
b := \text{max}(T[1][2],T[2][2],T[3][2],T[4][2]); \\
c := \text{max}(T[1][3],T[2][3],T[3][3],T[4][3]); m := \text{max}(a,b,c); \text{ end:}
\]

The tetrahedron obtained as a result of \textit{tmttopq} is then transformed within the cube found above through all the translations, rotations and symmetries of the cube. We denote this orbit of \(T\), by \(O(T)\).

\[
\text{orbit1} := \text{proc}(T) \text{ local } i,k,T1,a,b,c,x,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,T15,T16, \\
T17,T18,T19,T20,T21,T22,T23,T24,S,Q,d,a1,b1,c1; Q := \text{convert}(T,\text{list}); \\
d := \text{mscofmt}(T); T1 := T; T2 := \{\text{seq}([Q[k][2],Q[k][3],Q[k][1]],k=1..4)\}; \\
T3 := \{\text{seq}([Q[k][1],Q[k][3],Q[k][2]],k=1..4)\}; \\
T4 := \{\text{seq}([Q[k][1],Q[k][2],d-Q[k][3]],k=1..4)\}; \\
T5 := \{\text{seq}([Q[k][2],Q[k][3],d-Q[k][1]],k=1..4)\}; \\
T6 := \{\text{seq}([Q[k][1],Q[k][3],d-Q[k][2]],k=1..4)\}; \\
T7 := \{\text{seq}([Q[k][1],d-Q[k][2],Q[k][3]],k=1..4)\}; \\
T8 := \{\text{seq}([Q[k][2],d-Q[k][3],Q[k][1]],k=1..4)\}; \\
T9 := \{\text{seq}([Q[k][1],d-Q[k][3],Q[k][2]],k=1..4)\}; \\
T10 := \{\text{seq}([d-Q[k][1],Q[k][2],Q[k][3]],k=1..4)\};
\]
COUNTING ALL REGULAR TETRAHEDRA IN $\{0, 1, \ldots, n\}^3$

\[
T_{11} := \{\text{seq}([d-Q[k][2], Q[k][3], Q[k][1]], k=1..4)\};
\]
\[
T_{12} := \{\text{seq}([d-Q[k][1], Q[k][3], Q[k][2]], k=1..4)\};
\]
\[
T_{13} := \{\text{seq}([Q[k][1], d-Q[k][2], d-Q[k][3]], k=1..4)\};
\]
\[
T_{14} := \{\text{seq}([Q[k][1], d-Q[k][3], d-Q[k][2]], k=1..4)\};
\]
\[
T_{15} := \{\text{seq}([Q[k][1], d-Q[k][3], Q[k][2]], k=1..4)\};
\]
\[
T_{16} := \{\text{seq}([d-Q[k][1], d-Q[k][2], Q[k][3]], k=1..4)\};
\]
\[
T_{17} := \{\text{seq}([d-Q[k][2], d-Q[k][3], Q[k][1]], k=1..4)\};
\]
\[
T_{18} := \{\text{seq}([d-Q[k][1], d-Q[k][3], d-Q[k][2]], k=1..4)\};
\]
\[
T_{19} := \{\text{seq}([d-Q[k][1], Q[k][2], d-Q[k][3]], k=1..4)\};
\]
\[
T_{20} := \{\text{seq}([d-Q[k][2], Q[k][3], d-Q[k][1]], k=1..4)\};
\]
\[
T_{21} := \{\text{seq}([d-Q[k][1], d-Q[k][2], d-Q[k][3]], k=1..4)\};
\]
\[
T_{22} := \{\text{seq}([d-Q[k][2], d-Q[k][3], d-Q[k][1]], k=1..4)\};
\]
\[
T_{23} := \{\text{seq}([d-Q[k][1], d-Q[k][3], d-Q[k][2]], k=1..4)\};
\]
\[
T_{24} := \{\text{seq}([d-Q[k][1], Q[k][2], d-Q[k][3]], k=1..4)\};
\]
\[
S := \{T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}, T_{22}, T_{23}, T_{24}\};
\]

\text{orbit} := \text{proc}(T) \text{ local } S, Q, T1; \text{ Q := convert(T, list);} \\
\text{ T1 :=}\{\text{seq}([Q[k][3], Q[k][2], Q[k][1]], k=1..4), [Q[2][3], Q[2][2], Q[2][1]], \\
\{Q[3][3], Q[3][2], Q[3][1]], [Q[4][3], Q[4][2], Q[4][1]]\}; \text{ S := orbit1(T) union orbit1(T1); S; end:}

We recall from [10] a few variables that we are going to use in this calculation also. The theorem used there applies as well to this case because it is a pure set theoretic result. The meaning of those variables here is:

(i) $n$ - the dimension of the cube $C_n = [0, n]^3$,

(ii) $m$ - the maximum of all the coordinates in a tetrahedron $T$ computed by \texttt{tmtparam},

(iii) $\alpha(T)$ - the cardinality of $O(T)$ within $C_m$,

(iv) $\beta(T)$ - the cardinality of $O(T) \cap [O(T) + e_1]$,

(v) $\gamma(T)$ - the cardinality of $[O(T) + e_1] \cap [O(T) + e_2]$.

\textbf{Theorem 3.2.} (Theorem 2.2 in [10]) The number $f(T, n)$ of all tetrahedrons that can be obtained from $T$ within a cube $C_n$ by translations, rotations, or symmetries, is given by

\[
f(T, n) = (n + 1 - m)^3 \alpha - 3(n + 1 - m)^2(n - m)\beta + 3(n + 1 - m)(n - m)^2\gamma,
\]
for all $n \geq m$.

Hence, we need to calculate $\alpha$, $\beta$ and $\gamma$. 

addvec:=proc(U,V)
local W;
W[1]:=U[1]+V[1]; W[2]:=U[2]+V[2]; W[3]:=U[3]+V[3]; [W[1],W[2],W[3]];
end:

addvect:=proc(T,v)
local i,Q; Q:=
{ }
for i from 1 to 4 do
Q:=Q union addvec(T[i],v);
od; Q;
end:

transl:=proc(T)
local S,Q,i,j,k,a2,b2,c2,a,b,c,d;
Q:=convert(T,list);
a:=max(Q[1][1],Q[2][1],Q[3][1],Q[4][1]);
b:=max(Q[1][2],Q[2][2],Q[3][2],Q[4][2]);
c:=max(Q[1][3],Q[2][3],Q[3][3],Q[4][3]);
d:=max(a,b,c);
a2:=d-a;b2:=d-b;c2:=d-c;
S:=orbit(T);
for i from 0 to a2 do
for j from 0 to b2 do
for k from 0 to c2 do
S:=S union orbit(addvect(T,[i,j,k]));
end;
end;
end:

This last procedure gives the value of $\alpha$. Then $\beta$ is calculated by the following.

intersalongE1:=proc(T)
local S,m,i,S1,S2;
S2:=transl(T); S:=convert(S2,list); m:=nops(S); S1:=
{ }
for i from 1 to m do
S1:=S1 union addvec(S[i],[0,0,1]);
od; S2 intersect S1;
end:

Then the variable $\gamma$ is given by the procedure:

intersalongE2:=proc(T)
local S,m,i,S1,S2,S3,S4;
S2:=transl(T); S:=convert(S2,list); m:=nops(S); S1:=
{ }
for i from 1 to m do
S1:=S1 union addvec(S[i],[0,0,1]);
od; S3:=
{ };
for i from 1 to m do S3:=S3 union \{addvect(S[i],[0,1,0])\}; od; S1 intersect S3;
end:

The function in Theorem 3.2 is then implemented by

\[
f:=(n,d,alpha,beta,gamma)\rightarrow (n - d + 1)^3 \ast alpha - 3 \ast (n - d) \ast (-d + 1 + n)^2 \ast beta + 3 \ast gamma \ast (n - d + 1) \ast (n - d)^2:
\]

notetraincn:=proc(T,n)
local d,x,y,z,w;
d:=mscofmt(T);
x:=nops(transl(T));y:=nops(intersalongE1(T));w:=nops(intersalongE2(T));
if \( n \geq d \) then
z:=f(n,d,x,y,w);else z:=0;
fi;z;end:

Finally we put together a list of irreducible tetrahedrons whose orbits under the operations above are pairwise disjoint. The next four procedures are pretty simple and one can figure out what they do. They are used in the code of ExtendList.

distance:=proc(A,B) local C;
C:=subtrv(A,B); sqrt(C[1]^2 + C[2]^2 + C[3]^2);
end:

ccheckrt:=proc(T) local d1,d2,d3,d4,d5,d6,Q,D;
d1:=distance(T[4],T[1])/sqrt(2); d2:=distance(T[4],T[2])/sqrt(2);
d3:=distance(T[4],T[3])/sqrt(2); d4:=distance(T[1],T[2])/sqrt(2);
d5:=distance(T[1],T[3])/sqrt(2); d6:=distance(T[2],T[3])/sqrt(2);
min(d1,d2,d3,d4,d5,d6); end:
crossproductt:=proc(U,V) local x,i,j,k,d,i1,j1,k1,d1;
i:=U[2]*V[3]-U[3]*V[2];j:=U[3]*V[1]-U[1]*V[3];k:=U[1]*V[2]-V[1]*U[2];
d:=gcd(i,j);d1:=gcd(d,k);i1:=i/d1;j1:=j/d1;k1:=k/d1; sqrt((i1^2 + j1^2 + k1^2)/3);
end:
facheck:=proc(T) local N1,N2,N3,N4,U,V,W;
U:=subtrv(T[1],T[2]);V:=subtrv(T[1],T[3]);W:=subtrv(T[1],T[4]);
N1:=crossproductt(U,V);N2:=crossproductt(V,W);N3:=crossproductt(U,W);
N4:=crossproductt(subtrv(U,V),subtrv(U,W)); max(N1,N2,N3,N4); end:

ExtendList:=proc(n,L,mm,nn)
local i,sol,nsol,alpha,beta,gammmma,nel,mt,ttp1,ttp2,Or1,Or,t nel,NL,intcard,cio,normals,length,k;
nel:=nops(L);
k:=sqrt(mm^2 - mm * nn + nn^2);
sol:=abcsol(n);nsol:=nops(sol);
tnel:=nel;Or:={ }
for i from 1 to nsol do
    mt:=findpar(sol[i][1],sol[i][2],sol[i][3],mm,nn); ttp1:=tmttopq(mt[1]); ttp2:=tmttopq(mt[2]);
    normals:=facesnew(ttp1);length:=checkrt(ttp1);
    if k=1 or normals < length then
        cio:=evalb(ttp1 in Or);
        if cio=false then
            Or:=Or union transl(ttp1); Or1:=transl(ttp2);intcard:=nops(Or intersect Or1);
            if intcard > 0 then NL[tnel+1]:=[n,mscofmt(ttp1),ttp1,sol[i]];
            tnel:=tnel+1;else
                NL[tnel+1]:=[n,mscofmt(ttp1),ttp1,sol[i]];
                NL[tnel+2]:=[n,mscofmt(ttp2),ttp2,sol[i]];
                tnel:=tnel+2;
                fi;
            fi;
            fi;
        normals:=facesnew(ttp2);length:=checkrt(ttp2);
        if k=1 or normals < length then
            ttp2:=tmttopq(mt[2]);
            cio:=evalb(ttp2 in Or);
            if cio=false then
                Or:=Or union transl(ttp2); Or1:=transl(ttp1);intcard:=nops(Or intersect Or1);
                if intcard > 0 then NL[tnel+1]:=[n,mscofmt(ttp1),ttp1,sol[i]];
                tnel:=tnel+1;else
                    NL[tnel+1]:=[n,mscofmt(ttp1),ttp1,sol[i]];
                    NL[tnel+2]:=[n,mscofmt(ttp2),ttp2,sol[i]];
                    tnel:=tnel+2;
                    fi;
                fi;
            fi;
            fi;
        fi;
    fi;
    fi;
    od;
COUNTING ALL REGULAR TETRAHEDRA IN \( \{0, 1, ..., n\}^3 \)

\[
\{\text{seq}(L[i], i=1..\text{nel}), \text{seq}(NL[j], j=\text{nel}+1..\text{tnel})\};
\]
end;

Once this list is computed for every \( n \) we can add up all the contributions for each tetrahedron in the list. A few procedures are used in the code of the main calculation called *calculation*.

\[
multbyfactorv := \text{proc}(v, k) \\
\text{local } w; \\
w[1] := v[1] * k; w[2] := v[2] * k; w[3] := v[3] * k; \\
[w[1], w[2], w[3]]; \\
\]
end:

\[
multbyfactor := \text{proc}(T, k) \\
\text{local } i, NT, Q; NT := \{\}; \\
Q := \text{convert}(T, \text{list}); \\
\text{for } i \text{ from 1 to 4 do} \\
\quad NT := NT \cup \text{multbyfactorv}(Q[i], k); \\
\text{od}; NT; \\
\]
end:

\[
\text{addup} := \text{proc}(n, L) \\
\text{local } i, j, m, mm, nt, k, T, Q, alpha, beta, gamma, d; nt := 0; \\
m := \text{nops}(L); k := \text{floor}(\frac{n-1}{2}); i := 1; \\
\text{while } i \leq m \text{ do} \\
\quad \text{if } L[i][2] \leq n \text{ then} \\
\quad\quad mm := \text{floor}(n/L[i][1]); T := L[i][3]; j := 1; \\
\quad\quad \text{while } j \leq mm \text{ and } L[i][2] * j \leq n \text{ do} \\
\quad\quad\quad Q := \text{multbyfactor}(T, j); d := L[i][2] * j; \\
\quad\quad\quad alpha := \text{nops}(\text{transl}(Q)); beta := \text{nops}(\text{intersalongE1}(Q)); \\
\quad\quad\quad gamma := \text{nops}(\text{intersalongE2}(Q)); nt := nt + f(n, d, alpha, beta, gamma); \\
\quad\quad j := j + 1; \\
\quad\quad \text{od}; fi; \\
\quad i := i + 1; \\
\text{od}; \\
tn/2; \\
\]
end:

In the above procedure, *addup*, we get every irreducible from the list \( L \) together with all their appropriate dilations to compute their contribution in the cube \( [0, n]^3 \) using the formula given by
Theorem 3.2. The result is divided by two to match the sequence introduced in A103158. Finally, the list $L$ is calculated in terms of $n$ in two steps. First, construct the irreducible tetrahedrons using values of of $mm = 0$ and $nn = 1$ in \texttt{ExtendList} and then we take care of the other possible values of $mm$ and $nn$, making sure that we do not have duplicates by looking at the biggest $d$ that shows up from each face.

\texttt{calculation:=proc(n) local i,j1,j2,k,k1,L,x,xx,y,B,ii,T;
  x:=kvalues(n);y:=determined(n); xx:=floor((n+1)/2);
i:=nops(x);L:=[ ];L:=ExtendList(1,L,0,1);
  for j from 1 to xx do
    L:=ExtendList(2*i+1,L,0,1);
  od; for j from 1 to i do
    k:=listofmn(x[j]);k1:=nops(k);
    for j2 from 1 to k1 do
      for j1 from 1 to nops(y[i]) do
        L:=ExtendList(y[j][j1],L,k[j2][1],k[j2][2]);
      od;
    od;
  od; for ii from 1 to n do B[ii]:=addup(ii,L); print([ii,B[ii]]); od;
  T:=[seq(B[ii],ii=1..n)]; T; end:
}

The result of the \texttt{calculation}(100) gives in less than a few hours of computations: [1, 1] [2, 9] [3, 36] [4, 104] [5, 257] [6, 549] [7, 1058] [8, 1896] [9, 3199] [10, 5145] [11, 7926] [12, 11768] [13, 16967] [14, 23859] [15, 32846] [16, 44378] [17, 58977] [18, 77215] [19, 99684] [20, 126994] [21, 159963] [22, 199443] [23, 246304] [24, 301702] [25, 366729] [26, 442587] [27, 530508] [28, 631820] [29, 748121] [30, 880941] [31, 1031930] [32, 1202984] [33, 1395927] [34, 1612655] [35, 1855676] [36, 2127122] [37, 2429577] [38, 2765531] [39, 3137480] [40, 3548434] [41, 4001071] [42, 4498685] [43, 5044606] [44, 5641892] [45, 6294195] [46, 7005191] [47, 7779872] [48, 8620242] [49, 9533105] [50, 10521999] [51, 11591474] [52, 12746562] [53, 13992107] [54, 15332971] [55, 16775590] [56, 18324372] [57, 19985523] [58, 21765013] [59, 23668266] [60, 25702480] [61, 27837699] [62, 30188259] [63, 32655348] [64, 35281418] [65, 38074085] [66, 41040945] [67, 44188592] [68, 47525856] [69, 51061295] [70, 54804647] [71, 58763604] [72, 62949850] [73, 67371219] [74, 72037311] [75, 76958126] [76, 82143618] [77, 87606245] [78, 93355379] [79, 99403446] [80, 10576278] [81, 11244333] [82, 11945658] [83, 12681497] [84, 13453274] [85, 14262185] [86, 15109369] [87, 15996413] [88, 16924526] [89, 17895409] [90, 189102295] [91, 199706864] [92, 210781424] [93, 222341631] [94, 23440252] [95, 246978962] [96, 260093046] [97, 273757925] [98, 287989943] [99, 302809940] [100, 318235290]
We observe a similar behavior with the sequence \( \frac{\ln(ET(n))}{\ln(n+1)} \), in [10]:

![Graph](image)

**Figure 6:** The graph \( \frac{\ln(T(n)/2)}{\ln(n+1)} \), \( 1 \leq n \leq 100 \).

**REFERENCES**

[1] N. C. Ankeny, Sums of Three Squares, Proceedings of AMS, vol. 8, No. 2, pp 316-319.
[2] R. Chandler and E. J. Ionascu, A characterization of all equilateral triangles in \( \mathbb{Z}^3 \), *Integers*, Art. A19 of Vol. 8, 2008.
[3] S. Cooper and M. Hirschhorn, On the number of primitive representations of integers as a sum of squares, *Ramanujan J.* (2007) 13, pp. 7-25.
[4] D. A. Cox, Primes of the Form \( x^2 + ny^2 \): Fermat, Class Field Theory, and Complex Multiplication, Wiley-Interscience, 1997.
[5] R. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, 2004.
[6] E. Grosswald, *Representations of integers as sums of squares*, Springer Verlag, New York, 1985.
[7] I. J. Schoenberg, *Regular Simplices and Quadratic Forms*, J. London Math. Soc. 12 (1937) 48-55.
[8] M. D. Hirschhorn and J. A. Sellers, On representations of numbers as a sum of three squares, *Discrete Mathematics*, 199 (1999), pp. 85-101.
[9] E. J. Ionascu, *A parametrization of equilateral triangles having integer coordinates*, Journal of Integer Sequences, Vol. 10, 09.6.7, (2007).
[10] E. J. Ionascu, *Counting all equilateral triangles in \( \{0, 1, 2, ..., n\}^3 \)*, Acta Math. Univ. Comenianae, vol. LXXVII, 1(2008), pp.129-140.
[11] E. J. Ionascu, *A characterization of regular tetrahedra in \( \mathbb{Z}^3 \)*, J. Number Theory, 129(2009), 1066-1074.
[12] K. Rosen, *Elementary Number Theory*, Fifth Edition, Addison Wesley, 2004.
[13] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, 2005, published electronically at [http://www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).

Current address: Department of Mathematics, Columbus State University, 4225 University Avenue, Columbus, GA 31907, Honorific Member of the Romanian Institute of Mathematics “Simion Stoilow”

E-mail address: ionascu_eugen@colstate.edu;
Finding the solutions $a,b,c$ of $a^2+b^2+c^2=3d^2$ ($0<=a<=b<=c$).

Only the solutions that satisfy $gcd(a,b,c)=1$ are in the output.

```plaintext
> abcsol:=proc(d)
    local i,j,k,m,u,x,y,sol,cd;sol:=[ ];
    for i from 1 to d do
        u:=[isolve(3*d^2-i^2=x^2+y^2)]; k:=nops(u);
        for j from 1 to k do
            if rhs(u[j][1])>=i and rhs(u[j][2])>=i then
                cd:=gcd(i,rhs(u[j][1]),rhs(u[j][2]));
                if cd=1 then sol:=sol union 
                {sort([[i,rhs(u[j][1]),rhs(u[j][2])]]);fi;
                fi;
            od;
        od;
    convert(sol,list);
end:
```

Finding the two minimal regular tetrahedra given $a,b,c$ such that $a^2+b^2+c^2=3d^2$

The fourth point is the origin. We have included only two since all the other are
going to be produced by cube transformations.

A few other programs check if the output is what is suppose to be.

```plaintext
> findpar:=proc(a,b,c)
    local i,j,sol,mx,nx,r,s,my,ny,q,d,u,v,w,x,y,z,mu,nv,mv,ef,ns,mz,nz
        ,mw,nw,om1,om2,l,uu,t,R1,R2,fc; q:=a^2+b^2;
    sol:=convert({isolve(2*q=x^2+3*y^2)},list); ns:=nops(sol); d:=sqrt((a^2+
        ^2+c^2)/3);
    ef:=0;
```
and $\mu=\text{floor}(\mu)$ and $\nu=\text{floor}(\nu)$ and $m=\text{floor}(m)$ and $v=\text{floor}(v)$

then

$u := \mu m - \nu n; v := \nu v - m w; w := m w - n v; n := m w - n v; n := m w - n v; n := m w - n v;$

$x := \mu n - x n; y := \nu n - y n; z := m n - x n; z := m n - x n; z := m n - x n;$

$R1 := [(x + u - 2 a) / 3, (v + y - 2 b) / 3, (z + w - 2 c) / 3];$

$R2 := [(x + u + 2 a) / 3, (v + y + 2 b) / 3, (z + w + 2 c) / 3];$

$fc := \text{subs}(m = 1, n = 0, R1[1]);$

if $fc = \text{floor}(fc)$ then

$om1 := \text{subs}(m = 1, n = 0, [[u, v, w], [x, y, z], R1]);$

else

$om1 := \text{subs}(m = 1, n = 0, [[u, v, w], [x, y, z], R2]);$

endif;

$fc := \text{subs}(m = 1, n = 1, R1[1]);$

if $fc = \text{floor}(fc)$ then

$om2 := \text{subs}(m = 1, n = 1, [[u, v, w], [x, y, z], R1]);$

else

$om2 := \text{subs}(m = 1, n = 1, [[u, v, w], [x, y, z], R2]);$

endif;

endif;

$= \text{subs}(m = 1, n = 1, [[u, v, w], [x, y, z], R1]);$

if $fc = \text{floor}(fc)$ then

$om2 := \text{subs}(m = 1, n = 1, [[u, v, w], [x, y, z], R2]);$

endif;

endif;

$= 1; fi; fi;$

end;

> substrv := proc(U,V)

local W;

W[1] := U[1] - V[1]; W[2] := U[2] - V[2]; W[3] := U[3] - V[3];

[ W[1], W[2], W[3] ];

end:

Translating the minimal tetrahedra to the positive quadrant.

> tmttopq := proc(T)

local i, a, b, c, v, O, TR;

a := min(T[1][1], T[2][1], T[3][1], 0);

b := min(T[1][2], T[2][2], T[3][2], 0);

c := min(T[1][3], T[2][3], T[3][3], 0);

O := [0, 0, 0];

v := [a, b, c]; TR := [substrv(O, v), substrv(T[1], v), substrv(T[2], v), substrv(T[3], v)];

end:
Determining the number of tetrahedra in the orbit under the cube symmetries

\begin{verbatim}
> orbit1:=proc(T)
local i,k,T1,a,b,c,x,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,T15,T16,T17,T18,
    T19,T20,T21,T22,T23,T24,S,Q,d,a1,b1,c1:
Q:=convert(T,list):
d:=macrot(T):
T1:={};
T2:={seq([Q[k][2],Q[k][3],Q[k][1]],k=1..4)};
T3:={seq([Q[k][1],Q[k][3],Q[k][2]],k=1..4)};
T4:={seq([Q[k][1],Q[k][2],Q[k][3]],k=1..4)};
T5:={seq([Q[k][2],Q[k][3],Q[k][1]],k=1..4)};
T6:={seq([Q[k][1],Q[k][3],Q[k][2]],k=1..4)};
T7:={seq([Q[k][1],d-Q[k][2],Q[k][3]],k=1..4)};
T8:={seq([Q[k][2],d-Q[k][3],Q[k][1]],k=1..4)};
T9:={seq([Q[k][1],d-Q[k][3],Q[k][2]],k=1..4)};
T10:={seq([d-Q[k][1],Q[k][2],Q[k][3]],k=1..4)};
T11:={seq([d-Q[k][2],Q[k][3],Q[k][1]],k=1..4)};
T12:={seq([d-Q[k][1],Q[k][3],Q[k][2]],k=1..4)};
T13:={seq([Q[k][1],d-Q[k][2],d-Q[k][3]],k=1..4)};
T14:={seq([Q[k][2],d-Q[k][3],d-Q[k][1]],k=1..4)};
T15:={seq([Q[k][1],d-Q[k][2],d-Q[k][3]],k=1..4)};
T16:={seq([d-Q[k][1],d-Q[k][2],Q[k][3]],k=1..4)};
T17:={seq([d-Q[k][2],d-Q[k][3],Q[k][1]],k=1..4)};
T18:={seq([d-Q[k][1],d-Q[k][3],Q[k][2]],k=1..4)};
T19:={seq([d-Q[k][1],Q[k][2],d-Q[k][3]],k=1..4)};
T20:={seq([d-Q[k][2],Q[k][3],d-Q[k][1]],k=1..4)};
T21:={seq([d-Q[k][1],Q[k][3],d-Q[k][2]],k=1..4)};
T22:={seq([d-Q[k][1],d-Q[k][2],d-Q[k][3]],k=1..4)};
T23:={seq([d-Q[k][2],d-Q[k][3],d-Q[k][1]],k=1..4)};
T24:={seq([d-Q[k][1],d-Q[k][3],d-Q[k][2]],k=1..4)};
S:={T1,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,T15,T16,T17,T18,T19,T20,T21,T22,
    T23,T24};
S;
end:

> orbit:=proc(T)
local S,Q,T1:
Q:=convert(T,list):
T1:={Q[1][3],Q[1][2],Q[1][1],Q[2][3],Q[2][2],Q[2][1],Q[3][3],Q[3][2],Q[3][1],
    Q[4][3],Q[4][2],Q[4][1]};
S:=orbit1(T) union orbit1(T1);
\end{verbatim}
> addvec:=proc(T,v)
    local i, Q:=[];
    for i from 1 to 4 do
      Q:=Q union {addvec(T[i],v)};
    od;
    Q;
end:

> transl:=proc(T)
    local S, Q, i, j, k, a2, b2, c2, a, b, c, d;
    Q:=convert(T,list);
    a:=max(Q[1][1],Q[2][1],Q[3][1],Q[4][1]);
    b:=max(Q[1][2],Q[2][2],Q[3][2],Q[4][2]);
    c:=max(Q[1][3],Q[2][3],Q[3][3],Q[4][3]);
    d:=max(a,b,c);
    a2:=d-a; b2:=d-b; c2:=d-c;
    S:=orbit(T);
    for i from 0 to a2 do
      for j from 0 to b2 do
        for k from 0 to c2 do
          S:=S union orbit(addvec(T,[i,j,k]));
        od;
      od;
    S;
end:

Intersection of the extended orbit with its translation along \([0,0,1]\)
(calculation of beta).

> intersalongE1:=proc(T)
    local S, m, i, S1, S2;
    S2:=transl(T); S:=convert(S2,list); m:=nops(S); S1:={};
    for i from 1 to m do
      S1:=S1 union {addvec(S[i],[0,0,1])};
    od;
    S2 intersect S1;
S1:=S1 union {addvec(S[i],[0,0,1])};
end;
S3:=();
for i from 1 to m do
S3:=S3 union {addvec(S[i],[0,1,0])};
end;
S1 intersect S3;

The function that gives the number of tetrahedra generated in an arbitrary cube.

> f := (n,d,alpha,beta,gamma) -> (n-d+1)^3*alpha-3*(n-d)*(-d+1+n)^2*beta+3*
    gamma*(n-d+1)*(n-d)^2:

> notetraincn := proc(T,n)
    local d,x,y,z,w;
    d:=mstofmt(T);
    x:=nops(transl(T)); y:=nops(intersalongE1(T)); w:=nops(intersalongE2(T))
    if n>d then
        z:=f(n,d,x,y,w); else z:=0;
    fi;
    z;
end:

List of minimal tetrahedra and the main programs.

We need to keep for each minimal tetrahedra, the length of its sides (which is d\sqrt{2}), the dimension of the minimal cube containing it, the alpha, the beta, and the gamma numbers.

> ExtendList:=proc(n,L)
    local
    i,sol,nsol,alpha,beta,gamma,nel,mt,ttp1,ttp2,Or1,Or,tnel,NL,ntcard,c
    o;
    nel:=nops(L);
    sol:=abcsol(n); nsol:=nops(sol);
NL[tlen+2]:=[n, mscofmt(ttp2), ttp2, sol[i]];
    tlen:=tlen+2;
    fi;
    fi;
    cio:=evalb(ttp2 in Or);
    if cio=false then
        Or:=Or union transl(ttp2); Or1:=transl(ttp1); intcard:=nops(Or intersect Or1);
        if intcard>0 then NL[tlen+1]:=[n, mscofmt(ttp1), ttp1, sol[i]];
            tlen:=tlen+1; else
            NL[tlen+1]:=[n, mscofmt(ttp1), ttp1, sol[i]];
            NL[tlen+2]:=[n, mscofmt(ttp2), ttp2, sol[i]];
            tlen:=tlen+2;
        fi;
    fi;
    od;
    [seq(L[i], i=1..tlen), seq(NL[j], j=n+1..tlen)];
end:

> multbyfactorv:=proc(v,k)
    local w;
    w[1]:=v[1]*k; w[2]:=v[2]*k; w[3]:=v[3]*k;
    [w[1], w[2], w[3]];
end:

> multbyfactor:=proc(T,k)
    local i, NT, Q:=NT:=
    Q:=convert(T, list);
    for i from 1 to 4 do
        NT:=NT union {multbyfactorv(Q[i], k)};
    od;
    NT;
end:

> main:=proc(n,L)
    local i, j, m, mm, nt, k, T, Q, alpha, beta, gamma, d, nt:=0;
    n:=nops(L); k:=floor((n-1)/2);
    if 2*k+1=n then L:=ExtendList(n, L): fi;
    i:=1;
    while i<=n do
    end:

od;
nt/2;
end:

> calculation:=proc(n)
    local i,j,L;
j:=floor(n/2);
L:=[];L:=ExtendList(1,L);
for i from 1 to j+1 do
L:=ExtendList(2*i+1,L);
od;
main(n,L);
end:

Calculations of the list

> calculation(1); calculation(2); calculation(3); calculation(4); calculation(5);

     1
     9
    36
   104
   257

> calculation(30);

     880941

> calculation(31);

     1031930

> calculation(32);

     1202984

> calculation(33);

     1395927

> calculation(34);

     1619980