ON CONVOLUTION FOR GENERAL NOVEL FRACTIONAL WAVELET TRANSFORM

ASHISH PATHAK*, PRABHAT YADAV** AND M M DIXIT**
*DEPARTMENT OF MATHEMATICS & STATISTICS
DR. HARISINGH GOUR CENTRAL UNIVERSITY
SAGAR-470003, INDIA.
** DEPARTMENT OF MATHEMATICS
NERIST, NIRJULI-791-109, INDIA

Abstract. Using Pathak and Pathak techniques, the basic function \( D^\alpha(u, v, w) \) associated with general novel fractional wavelet transform (GNFrWT) is defined and its properties are investigated. By using basic function \( D^\alpha(u, v, w) \) translation and convolution associated with GNFrWT are defined and certain existence theorems are proved for basic function and associated convolution.

1. Introduction

The general novel fractional wavelet transform (GNFrWT) of a function \( h(t) \) with respect to wavelet \( \phi \) can be defined as [1],

\[
(W^\alpha_\phi h)(a, b) = \int_{\mathbb{R}} h(t) \overline{\phi^a_{\alpha,a,b}(t)} \, dt,
\]

(1.1)

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*E-mail: pathak_maths@yahoo.com.
where
\[ \phi_{a,b}^{\alpha}(t) = e^{(-i/2)(t^2-b^2)\cot\theta}|a|^{-\rho} \phi \left( \frac{t-b}{a} \right); \quad a \in \mathbb{R}_+, \ b \in \mathbb{R} \text{ and } \rho \geq 0. \] (1.2)

The inversion formula for (1.1) with respect to the \( e^{(-i/2)(t^2-b^2)\cot\theta} \phi_{a,b}(t) \) can be given as
\[ h(t) = \frac{1}{c_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\phi^\alpha h)(a,b) \phi_{a,b}^{\alpha}(t)|a|^{2\rho-3} db da, \] (1.3)

where
\[ O < C_\phi = \int_{\mathbb{R}} \frac{\hat{\phi}(\omega)^2}{|\omega|} d\omega < \infty. \] (1.4)

If \( \alpha = 1 \), the GNFrWT correlate with the wavelet transform (WT). As [1], the GNFrWT can be expressed as in terms of the fractional fourier transform \( H^\alpha(v) \) of the signal \( h(t) \).
\[ (W_\phi^\alpha h)(a,b) = \int_{\mathbb{R}} 2\pi a^{-\rho+1} H^\alpha(v) \hat{\phi}(av cosec\theta) K_{-\alpha}(v,b) dv \] (1.5)

where \( \hat{\phi}(av cosec\theta) \) indicates the fourier transform of \( \phi(t) \). Also, the GNFrWT can be rewritten as [1]
\[ (W_\phi^\alpha h)(a,b) = e^{(i/2)b^2\cot\theta} \int_{\mathbb{R}} h(t)e^{(i/2)t^2\cot\theta} \phi_{a,b}(t) dt. \] (1.6)

**Theorem 1.1. (Parseval formula).** If wavelet \( \phi \in L^2(\mathbb{R}) \) and \( (W_\phi^\alpha h)(a,b) \) is the GNFrWT of \( h \in L^2(\mathbb{R}) \), then for any function \( h, g \in L^2(\mathbb{R}) \),
\[ \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\phi^\alpha h)(a,b) \overline{(W_\phi^\alpha g)(a,b)} |a|^{2\rho-3} db da = C_\phi(h,g). \] (1.7)

where
\[ O < C_\phi = \int_{\mathbb{R}} \frac{\hat{\phi}(\omega)^2}{|\omega|} d\omega < \infty. \]
Proof 1. By using (1.5), we have

\[ W_\alpha^\phi(h)(a,b) = \int_{\mathbb{R}} \sqrt{2\pi} a^{\rho+1} H_\alpha(h)(u) \hat{\phi}(\text{aucosec}(\theta)) K_{-\alpha}(u,b) du \] (1.8)

\[ W_\alpha^\phi(g)(a,b) = \int_{\mathbb{R}} \sqrt{2\pi} a^{\rho+1} H_\alpha(g)(v) \hat{\phi}(\text{aucosec}(\theta)) K_{-\alpha}(v,b) dv \] (1.9)

from (1.8) and (1.9), we get

\[ \int_{\mathbb{R}} \int_{\mathbb{R}} W_\alpha^\phi(h)(a,b) W_\alpha^\phi(g)(a,b) |a|^{2\rho-3}dbda = <h, g > C_\phi. \] (1.10)

Theorem 1.2. (Inversion Formula) If wavelet \( \phi \in L^2(\mathbb{R}) \) and \( W_\alpha^\phi(h)(a,b) \) is the GNFrWT of \( h \in L^2(\mathbb{R}) \), then the reconstruction of \( h \) is given by

\[ h(u) = \frac{1}{C_\phi} \int_{\mathbb{R}} \int_{\mathbb{R}} (W_\alpha^\phi(h)(a,b) \phi_{a,b}(t) |a|^{2\rho-3}dbda \] (1.11)

Proof 2. By using (1.10) for \( h = g \), then we can find (1.11).

This paper is arranged in the following manner: - In the next section, we define basic function \( D_\alpha(u,v,w) \), translation and associated convolution for GNFrWT. In the last third section, we obtained and established the certain existence theorem and convolution theorem, by using Pathak and Pathak techniques [6]

2. Basic function, translation and associated convolution for GNFrWT

Now, by using Pathak and Pathak techniques [6], we define the basic function \( D_\alpha(u,v,w) \) translation \( \tau^\alpha_u \) and associated convolution \( \#^\alpha \) operators for GNFrWT.

The basic function \( D_\alpha(u,v,w) \) for (1.1) is define as

\[ W_\phi[D_\alpha(u,v,w)](a,b) = \int_{\mathbb{R}} D_\alpha(u,v,w) \phi_{a,b}(t) dt = \psi_{a,b}(w) \chi_{a,b}(v), \] (2.1)
where $\psi^\alpha, \phi^\alpha$ and $\chi^\alpha$ are three fractional wavelets satisfying certain conditions (1.2).

Now, by using (1.3) we get,

$$D^\alpha(u, v, w) = C^{-1}_\phi \int_R \int_R \overline{\psi_{a,b}(w)} \overline{\chi_{a,b}(v)} \phi_{a,b}(u) |a|^{2\rho-3}dad.$$  \hfill (2.2)

The translation $\tau^\alpha_u$ is defined as \[6\]

$$(\tau^\alpha_u h)(v) = h^*(u, v) = \int_R D^\alpha(u, v, w)h(w)dw$$

$$= C^{-1}_\phi \int_R \int_R \int_R \int_R \overline{\psi_{a,b}(w)} \overline{\chi_{a,b}(v)} \phi_{a,b}(u) h(w) |a|^{2\rho-3}dadbw.$$  \hfill (2.3)

The associated convolution is defined as

$$(h\#^\alpha g)(u) = \int_R h^*(u, v)g(v)dv$$

$$= \int_R \int_R D^\alpha(u, v, w)h(w)g(v)dvdw$$

$$= C^{-1}_\phi \int_R \int_R \int_R \int_R \overline{\psi_{a,b}(w)} \overline{\chi_{a,b}(v)} \phi_{a,b}(u)h(w)g(v) |a|^{2\rho-3}dadbdv.$$  

Example 2.1. Basic function $D^\alpha(u, v, w)$ for general novel fractional morlet wavelet transform

Let $\psi(t) = \chi(t) = \phi(t) = e^{iw_o t - \frac{1}{2}t^2}$ be a morlet wavelet $[2]$. Then the general novel fractional morlet wavelet transform is given by $\phi^{\alpha}_{a,b}(t) = |a|^{-\rho} e^{iw_o(t-b) - \frac{1}{2}(t-b)^2}$. 
Now, from (2.2) 

\begin{align*}
D^\alpha(u, v, w) &= C^{-1}_\phi \int_\mathbb{R} \int_\mathbb{R} e^{(i/2)(w^2 + v^2 - u^2 - b^2) \cot \theta} \psi_{a,b}(w) \chi_{a,b}(v) \phi_{a,b}(u) \, |a|^{2\rho - 3} \, da \, db.
\end{align*}

\begin{align*}
&= C^{-1}_\phi e^{(i/2)(w^2 + v^2 - u^2) \cot \theta} \int_\mathbb{R} \int_\mathbb{R} e^{(-i/2)b^2 \cot \theta} e^{-iw_\alpha(a-b)} e^{iw_\alpha(a-b)} \, |a|^{\rho - 3} \, da \, db
\end{align*}

\begin{align*}
&= C^{-1}_\phi e^{(i/2)(w^2 + v^2 - u^2) \cot \theta} \int_\mathbb{R} \int_\mathbb{R} e^{(-i/2)b^2 \cot \theta} e^{(i/2)b^2} \, |t|^{\rho} \, dt \, db
\end{align*}

\begin{align*}
&= 2C^{-1}_\phi e^{(i/2)(w^2 + v^2 - u^2) \cot \theta} \int_\mathbb{R} \int_\mathbb{R} e^{(-i/2)b^2 \cot \theta} \cos \left[w_\alpha(t)(b + u - w - v)\right] \, \Gamma(1 + \rho/2) \frac{1}{2} 
\end{align*}

\begin{align*}
&= \frac{1}{2} \Gamma(1 + \rho/2) \frac{1}{\sqrt{|w - b|^2 + (v - b)^2 + (u - b)^2}} \left[ -w_\alpha^2(b + u - w - v)^2 \right] \frac{1}{2} \, db \, \rho > 0,
\end{align*}

by [4, p.15(14)] where \( \text{\(1 F_1(a; b; u)\)} \) is confluent hypergeometric function.

**Example 2.2. Basic function** \( D^\alpha(u, v, w) \) for general novel fractional mexican hat wavelet transform

The corresponding Mexican-Hat wavelet is \( \psi(t) = \chi(t) = \phi(t) = (1 - t^2) e^{-t^2} \) . Then the general novel fractional mexican hat wavelet transform is given by \( \phi_{b,a}^\alpha(t) = |a|^{-\rho} \left( 1 - \frac{(t-b)^2}{a^2} \right) e^{-\frac{(t-b)^2}{2a^2}} \).

Now by using (2.2), we have

\[
D^\alpha(u, v, w) = C^{-1}_\phi e^{i/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(i/2)b^2 \cot \theta} \left(1 - \frac{(w - b)^2}{a^2}\right) \\
\times \left(1 - \frac{(v - b)^2}{a^2}\right) \left(1 - \frac{(u - b)^2}{a^2}\right) \\
\times e^{-(1/2a^2)((w-b)^2+(v-b)^2+(u-b)^2)}d\theta dbda.
\]

\[
= C^{-1}_\phi e^{i/2} \int_{\mathbb{R}} e^{-(i/2)b^2 \cot \theta} e^{-t^2 L/2} t^{\rho+1} \\
\times (t^4 - Nt^2 + M) dt db.
\]

\[
= C^{-1}_\phi e^{i/2} \int_{\mathbb{R}} e^{-(i/2)b^2 \cot \theta} db (-1/2)^{\rho+1} \\
\times \left(\Gamma((\rho+6)/2)(L/2)^{-(\rho+6)/2} - \Gamma((\rho+4)/2)N(L/2)^{-(\rho+4)/2}
\right) \\
+ \Gamma((\rho+2)/2) M(L/2)^{-\frac{\rho+2}{2}}
\]

by [H, p.313(13)], where \( \rho > 0 \), \( L = [(w - b)^2 + (v - b)^2 + (u - b)^2] \),

\( M = [(u - b)^2(v - b)^2 + (u - b)^2(w - b)^2 + (v - b)^2(w - b)^2] \) and

\( N = [L + (u - b)^2(v - b)^2(w - b)^2] \).

In the following section we have obtained boundedness result for the basic function \( D^\alpha(u, v, w) \) and then establish existence theorem for the general novel fractional wavelet convolution and prove \( W^\alpha_\phi(h#^\alpha g) = (W^\alpha_\psi h)(W^\alpha_\chi g) \).

3. Existence Theorems

First we obtain boundedness results for the basic function \( D^\alpha(u, v, w) \).
Theorem 3.1. Let \((1 + |z|^\rho)\phi(z) \in L^p(\mathbb{R}), \chi \in L^q(\mathbb{R}), \frac{1}{p} + \frac{1}{q} = 1\) and \((1 + |z|^\rho)\psi(z) \in L^1(\mathbb{R}), \rho \geq 0\). Then

\[
|D^\alpha(u, v, w)| \leq 2^{\rho + \frac{1}{q}} C_\phi^{-1} |v - w|^{-\frac{1}{q}} |u - w|^{-\frac{1}{p}} \|\chi\|_q \|(1 + |z|^\rho)\phi(z)\|_p
\]

\times \|\chi\|_q \|(1 + |z|^\rho)\psi(z)\|_1,

(3.1)

where \(O < C_\phi = \int_{\mathbb{R}} \frac{|\hat{\phi}(\omega)|^2}{|\omega|} d\omega < \infty\).

Proof 3. From (2.2), we have

\[
|D^\alpha(u, v, w)| = |C_\phi^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} |a|^{-\rho} e^{(-i/2)(u^2 - b^2)\cot \theta} \psi \left( \frac{w - b}{a} \right) |a|^{-\rho} e^{(-i/2)(u^2 - b^2)\cot \theta} \phi \left( \frac{u - b}{a} \right) |a|^{2\rho - 3} dbda|
\]

\[
= C_\phi^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi \left( \frac{w - b}{a} \right) |a|^{-\rho} e^{(-i/2)(u^2 - b^2)\cot \theta} \phi \left( \frac{u - b}{a} \right) |a|^{2\rho - 3} dbda
\]

(3.2)

By using Theorem 2.1 [6] we get the required result.

Theorem 3.2. (i) Let \(\psi \in L^1(\mathbb{R}), \phi \in L^p(\mathbb{R}), \chi \in L^q(\mathbb{R}), p, q > 1, 0 < \rho < 1\) and \(\frac{1}{p} + \frac{1}{q} = 1 + \rho\). Then

\[
\int_{\mathbb{R}} |D^\alpha(u, v, w)| dw \leq C_\phi^{-1} C(p, \rho) |u - v|^{-\rho} \|\psi\|_1 \|\phi\|_p \|\chi\|_q,
\]

(3.3)

where \(C(p, \rho)\) is a constant.

(ii) Let \(\psi \in L^1(\mathbb{R}), (1 + |y|^\rho^{-1})\phi(y) \in L^1(\mathbb{R}), (1 + |y|^\rho^{-1})\chi(v) \in L^1(\mathbb{R}), \) and \(\rho \geq 1\). Then

\[
\int_{\mathbb{R}} |D^\alpha(u, v, w)| dw \leq C_\phi^{-1} 2^{\rho - 1} |u - v|^{-\rho} \left[ \|\phi(x)x^{-1}\|_1 \|\chi\|_1 + \|\chi(y)y^\rho^{-1}\|_1 \|\phi\|_1 \right]
\]

\times \|\psi\|_1.

(3.4)
Proof 4. From (3.2) and using Theorem 2.2 the required results follows.

Theorem 3.3. (i) Let $\psi \in L^p(\mathbb{R}), \phi \in L^1(\mathbb{R}), \chi \in L^q(\mathbb{R}), p, q > 1, 0 < \rho < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \rho$. Then
\[
\int_{\mathbb{R}} |D^\alpha (u, v, w)| \, du \leq C^{-1}_\phi C(p, \rho) |v - w|^{-\rho} \|\psi\|_p \|\phi\|_1 \|\chi\|_q . \tag{3.5}
\]

(ii) Let $\phi \in L^1(\mathbb{R}), (1 + |x|^{\rho-1}) \chi(x) \in L^1(\mathbb{R}), (1 + |x|^\rho) \psi(x) \in L^1(\mathbb{R})$, and $\rho \geq 1$. Then
\[
\int_{\mathbb{R}} |D^\alpha (u, v, w)| \, du \leq C^{-1}_\phi 2^{\rho-1} |v - w|^{-\rho} \left[ \|\psi(x) x^{\rho-1}\|_1 \|\chi\|_1 + \|\chi(x) x^{\rho-1}\|_1 \|\psi\|_1 \right] \\
\times \|\phi\|_1 . \tag{3.6}
\]

The proof is similar to that Theorem 3.2.

Theorem 3.4. (i) Let $\psi \in L^q(\mathbb{R}), \phi \in L^p(\mathbb{R}), \chi \in L^1(\mathbb{R}), p, q > 1, 0 < \rho < 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \rho$. Then
\[
\int_{\mathbb{R}} |D^\alpha (u, v, w)| \, du \leq C^{-1}_\phi C(p, \rho) |u - w|^{-\rho} \|\psi\|_q \|\phi\|_p \|\chi\|_1 . \tag{3.7}
\]

(ii) Let $\chi \in L^1(\mathbb{R}), (1 + |x|^\rho) \phi(x) \in L^1(\mathbb{R}), (1 + |x|^{\rho-1}) \psi(x) \in L^1(\mathbb{R})$, and $\rho \geq 1$. Then
\[
\int_{\mathbb{R}} |D^\alpha (u, v, w)| \, dv \leq C^{-1}_\phi 2^{\rho-1} |u - w|^{-\rho} \left[ \|\psi(x) x^{\rho-1}\|_1 \|\phi\|_1 + \|\phi(x) x^{\rho-1}\|_1 \|\psi\|_1 \right] \\
\times \|\chi\|_1 . \tag{3.8}
\]

The proof is similar to that Theorem 3.2.

Theorem 3.5. Let $\phi \in L^1(\mathbb{R}), \psi \in L^p(\mathbb{R}), \chi \in L^q(\mathbb{R}), p, q > 1, 0 < \rho < 1$, $\frac{1}{p} + \frac{1}{q} = \rho + 1, h \in L^r(\mathbb{R})$ and $g \in L^{r'}(\mathbb{R}), r, r' > 1, \frac{1}{r} + \frac{1}{r'} + \rho = 2$. Then
\[ ||(h^{#\alpha}g)||_1 \leq C^{-1}_\phi C(\rho, p, r) |||\phi||_1 |||\psi||_p |||\chi||_q |||g||_{r'} |||h||_r \]

where \( C_\phi \) is given by (1.4) and \( C(\rho, p, r) \) is a constant.

**Proof 5.** we have

\[
\int_{\mathbb{R}} |(h^{#\alpha}g)(u)| du \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |h^\ast(u, v)| |g(v)| dv \right) du = \int_{\mathbb{R}} |g(v)| dv \int_{\mathbb{R}} |h^\ast(u, v)| du \leq \int_{\mathbb{R}} |g(v)| dv \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |D^\alpha(u, v, w)| |h(w)| dw \right) du
\]

by using above Theorem 3.3, we get

\[
\int_{\mathbb{R}} |(h^{#\alpha}g)(u)| du \leq C^{-1}_\phi C(p, \rho) |||\phi||_1 |||\chi||_q |||\psi||_p \int_{\mathbb{R}} |g(v)| dv \int_{\mathbb{R}} |h(w)| |v-w|^{-\rho} dw
\]

Therefore, by using Theorem 2.5 [6], we get the required result.

**Theorem 3.6.** Let \( \phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \chi \in L^q(\mathbb{R}), \psi \in L^p(\mathbb{R}), p, q > 1, 0 < \rho < 1, \frac{1}{p} + \frac{1}{q} = \rho + 1 \). Assume further that \( h \in L^r(\mathbb{R}), g \in L^{r'}(\mathbb{R}), r, r' > 1 \) and \( \frac{1}{r} + \frac{1}{r'} + \rho = \frac{2}{\rho} \). Then

\[
W_\phi^\alpha(h^{#\alpha}g)(a, b) = (W_\psi^\alpha h)(a, b)(W_\chi^\alpha g)(a, b)
\]

**Proof 6.** By using above Theorem 3.5, \( (h^{#\alpha}g) \in L^1(\mathbb{R}) \). As given basic wavelet \( \phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), W_\phi^\alpha(h^{#\alpha}g)(a, b) \) exist.
From (2.1), we have

\[
W_{\phi}^{\alpha}(h^{\#^\alpha}g)(a, b) = \int_{\mathbb{R}} (h^{\#^\alpha}g(u)) \phi^\alpha \left(\frac{u-b}{a}\right) |a|^{-\rho} du \\
= \int_{\mathbb{R}} \phi^\alpha \left(\frac{u-b}{a}\right) du \int_{\mathbb{R}} \int_{\mathbb{R}} D^\alpha(u, v, w) h(w) g(v) |a|^{-\rho} dwdv \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} h(w) g(v) dwdv \int_{\mathbb{R}} D^\alpha(u, v, w) \phi^\alpha \left(\frac{u-b}{a}\right) |a|^{-\rho} du \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} h(w) g(v) dwdv \int_{\mathbb{R}} D^\alpha(u, v, w) \frac{\psi_{a,b}^{\alpha}(w)}{\chi_{a,b}^{\alpha}(v)} \\
= \int_{\mathbb{R}} h(w) \psi_{a,b}^{\alpha}(w) dw \int_{\mathbb{R}} g(v) \chi_{a,b}^{\alpha}(v) dv \\
= (W_{\psi}^{\alpha} h)(a, b) (W_{\chi}^{\alpha} g)(a, b)
\]

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