Counting the number of Killing vectors in a 3D spacetime

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Abstract

We devise an algorithm which allows one to count the number of Killing vectors for a Lorentzian manifold of dimension 3. Our algorithm relies on the principal traces of powers of the Ricci tensor and branches intricately according to the values of differential invariants arising from the compatibility conditions of the Killing equation. As illustrating examples, we classify the Lifshitz and pp-wave spacetimes into a hierarchy based on their level of symmetry. A complete classification of spacetimes admitting 4 Killing vectors is also presented.

Keywords: Killing vector, isometry group, differential invariant, Cartan scalar, Cartan–Karlhede algorithm, vanishing scalar invariant space

1. Introduction and summary of our work

Klein’s Erlangen programme [1] provided an attempt to make a connection between geometry and symmetry. In modern parlance, this programme tried to classify the geometry according to the invariant properties under a certain (finite) group action. Whilst this paradigm has turned out to be too narrow to encompass the Riemannian geometry, it nevertheless has significant implications and influences to Riemannian geometry, as well as for theoretical physics. The advent of general relativity stimulated Klein to investigate the role of groups from relativistic standpoint. He had regarded special relativity as the theory of invariants of Minkowski spacetime under the Lorentz group action and also contributed to the formulation of the conservation laws in general relativity, see [2] for a review.

An isometry group acting on a Lorentzian manifold \((M, g_{\mu\nu})\) is of fundamental importance in understanding the geometrical and physical properties of a spacetime. For instance, the isometry group allows one to set the privileged local coordinate system of \(M\) and also gives
rise to first integrals of geodesic flow that are linear in momenta. In particular, the globally conserved quantities associated with isometries—such as the mass and the angular momentum—play a significant role in the analysis of black holes. It is then natural to ask whether there exists a set of invariants in Klein’s sense which is associated with the isometry group.

In this paper, we focus on the local existence of infinitesimal isometries of \((M, g_{ab})\). The problem of finding all of them is equivalent to that of finding all independent Killing vectors (KVs). To give an exhaustive list of KVs for a given Riemannian/Lorentzian manifold has a long history in geometry, which dates back at least to Darboux [3]. For dimension \(n = 2\), he found a set of invariants for determining the existence and the number of KVs as shown in figure 1. See also [4–8] for pertinent results.

For determining the number of KVs in arbitrary dimension, Cartan’s equivalence method or the Cartan–Karlhede algorithm [9, 10] is known as an effective technique (see also section 9.2 of [11]). Though it was originally developed to solve alias the equivalence problem, that is the problem of deciding whether given two (semi-)Riemannian manifolds are locally isometric, this amounts to determining the dimension of their local isometry group, as well as the structure constants of the group. Their machinery is explained as follows. Let \((M, g_{ab})\) be a (semi-)Riemannian manifold of dimension \(n\) and \(C^p\) be the set of Cartan scalars of order \(p\). Note that Cartan scalars of order \(p\) are defined as the frame components of the Riemann–Christoffel tensor and its first \(p\) covariant derivatives. As with Newman–Penrose scalars, Cartan scalars are functions on the tangent frame bundle \(F(M)\) but not on \(M\). Cartan showed that the local geometry of \(M\) is completely determined by \(C^{p_0}\) with \(p_0 \leq \frac{1}{2}n(n + 1)\), where \(p_0\) is the smallest natural number such that the elements of \(C^{p_0 + 1}\) are functionally dependent on those in \(C^{p_0}\). Subsequently, Karlhede demonstrated that in dimension \(n = 4\) at most 7 differentiations suffice, whereat 3156 functionally independent scalars are required. Given the set \(C^{p_0}\) consisting of \(q\) functionally independent scalars, the manifold \(M\) admits \(\frac{1}{2}(n(n + 1) - q)\) independent KVs. Unfortunately, the actual computations required to perform the Cartan–Karlhede algorithm remain formidable, even though it indeed ensures that the problem of finding all KVs is computable.

For a Riemannian space of dimension \(n = 3\), an important progress has been made in this problem rather recently in [12]. The scheme exploits the compatibility condition of the Killing equation (see equation (1.1) below), and aims exclusively at determining the dimension of the isometry group. This provides a more efficient and effective algorithm to count the number of KVs, allowing us to circumvent enormous amount of computational efforts. We discuss in this paper its extension to a Lorentzian manifold \((M, g_{ab})\) of dimension 3. An essential ingredient which operates the mechanism is as follows: Recall that any vector \(K^a\) is a KV on \((M, g_{ab})\) if and only if the Killing equation is satisfied

\[ \mathcal{L}_K g_{ab} = 2\nabla_{(a}K_{b)} = 0. \] (1.1)
Here $\mathcal{L}_K$ is the Lie derivative along $K^a$, $\nabla_a$ denotes the Levi-Civita connection and indices are raised and lowered with $g_{ab}$ and its inverse. The round brackets denote symmetrisation over the enclosed indices. As the compatibility condition of equation (1.1), one finds the curvature collineation

$$\mathcal{L}_K R_{abc} = 0 ,$$

(1.2)

where $R_{abc}^d$ is the Riemann–Christoffel tensor defined by $2\nabla_a \nabla_b V_c = R_{abc}^d V_d$. Here the square brackets over indices is used for skew-symmetrisation. Any solution to the Killing equation (1.1) automatically solves the equation (1.2), but in general the converse is not true.

In dimension 3, the following condition is an immediate corollary of (1.2):

$$\mathcal{L}_K R = 0 , \quad \mathcal{L}_K S^{(2)} = 0 , \quad \mathcal{L}_K S^{(3)} = 0 ,$$

(1.3)

where $R_{ab} \equiv R_{abc}^d g^{db}$ is the Ricci tensor, $R \equiv R_{aa} = g_{ab} R_{ab}$ is the scalar curvature, and $S^{(2)} \equiv S^a_b S^b_a, S^{(3)} \equiv S^a_b S^b_c S^c_a$ are the principal traces of powers of the traceless Ricci tensor $S_{ab} \equiv R_{ab} - (R/3) g_{ab}$. Thus, any solution to equation (1.1) must satisfy the following matrix equation\(^4\)

$$R_a K^a = 0 , \quad R_a \equiv \begin{bmatrix} \nabla_a R \\ \nabla_a S^{(2)} \\ \nabla_a S^{(3)} \end{bmatrix} .$$

(1.4)

We shall refer to the $3 \times 3$ matrix $R_a$ as the first obstruction matrix. This equation implies that any KV must be in $\ker R_a$ and hence $\det R_a = 0$ for $\ker R_a \neq \emptyset$, where the determinant of $R_a$ is given by

$$\det R_a = dR \wedge dS^{(2)} \wedge dS^{(3)} .$$

(1.5)

The first obstruction matrix $R_a$ is classified by the dimension of its kernel,

$$d \equiv \dim \ker R_a = 3 - \rank R_a ,$$

(1.6)

which can be determined according to the minors of $R_a$:

$$dR \wedge dS^{(2)}, \quad dS^{(2)} \wedge dS^{(3)}, \quad dS^{(3)} \wedge dR ,$$

(1.7a)

$$dR , \quad dS^{(2)}, \quad dS^{(3)} .$$

(1.7b)

It follows that $(M,g_{ab})$ enjoys a local isometry group of dimension $d_{iso} \leq \frac{1}{2}d(d+1)$ with an isotropy subgroup of dimension $d_{sub} \leq \frac{1}{2}d(d-1)$, acting on orbits of dimension $d_{orb} \leq d$.

In any of these cases, a general solution of equation (1.4) can be written in the form

$$K^a = \sum_{\alpha} \omega_\alpha t_\alpha^a , \quad (\alpha = 1, \ldots, d)$$

(1.8)

where $\{t_\alpha^a\}$ are linearly independent vectors that span $\ker R_a$ and the coefficients $\{\omega_\alpha\}$ are left arbitrary. In what follows, we refer to the case in which equation (1.8) holds true as class $d$.

Substituting the form (1.8) into equation (1.1), we obtain a PDE system of the form

$$\nabla_b \omega = \Omega_\beta \omega , \quad \omega \equiv [\omega_\alpha \varpi_\beta]^T , \quad (\beta = 1, \ldots, m \equiv \frac{1}{2}d(d-1))$$

(1.9)

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\(^4\)The terms $S^{(m)}$ ($m \geq 4$) fail to contribute to the obstruction matrix, since the Cayley–Hamilton theorem allows one to express them as lower matrix powers.
where \( \{\varpi_d\} \) are the 1-jet variables and the connection \( \Omega_a \) is expressed in terms of the Ricci rotation coefficients and their ratio. Since equation (1.9) is the first-order system, its compatibility gives rise to algebraic constraints on \( \varpi \),

\[
(\nabla_{[a}\Omega_{b]} - \Omega_{[a}\partial_{b]\Omega_{]}) \, \varpi = 0.
\] (1.10)

Equivalently, equation (1.10) can be written in component form of curvature of the bundle

\[
R_{cls,d} \varpi = 0.
\] (1.11)

We henceforth call \( R_{cls,d} \) as the second obstruction matrix of class \( d \). Since the matrix equation (1.4) is a necessary condition for equation (1.1), the second obstruction matrix yields obstructions to the existence of KVs as differential invariants. A noteworthy asset of this method is that the obstruction is measured by a purely algebraic fashion.

Let us now outline our strategy to be carried out. In each class \( d \), we solve equation (1.10) and then update the form (1.8). When this achieves a decrease in the number \( d \) in equation (1.8) or in the number \( m \) in equation (1.9), which are initially given by \( d = \dim \ker R_a \) and \( m = \frac{1}{2}d(d - 1) \) respectively, we write out equations (1.9)–(1.11) with the updated form, that is the form (1.8) with \( \varpi \) being the solution of equation (1.10). Once again, we solve equation (1.10) and re-update the form (1.8). We iterate this procedure until the latest compatibility is met trivially, or until the number \( d \) vanishes as a consequence thereof. This procedure is amendable to a follow-up study. This is a prime advantage of our formulation over the treatment of Cartan–Karlhede.

In this paper, we classify the number of local isometry group for a Lorentzian manifold of dimension 3, by presenting the explicit forms of the second obstruction matrix \( R_{cls,d} \) in all classes. This survey is essentially based on the procedure developed in [12], but differs from it in that: In Lorentzian signature, there appear null KVs and the Ricci tensor is not always diagonalisable. It is this aspect that prohibits the direct application of the previous analysis of Riemannian case [12] and requires the separate study, complicating attempts to pin it down discursively. The strategy employed here is similar in spirit to the Erlangen programme, since the symmetry is classified in terms of differential invariants. On top of the intrinsic interest in 3 spacetime dimensions, the method developed here can be applicable also for the 3 dimensional induced metrics as well as for quotient metrics. For \( n \geq 4 \) dimensions, a considerable number of loose ends are left over and the study of counting KVs remains open. See e.g. [14] for the analysis giving rise to an upper bound of KVs. We hope to address the issue for \( n \geq 4 \) in the future.

Our main results in this paper can be summarised as follows:

**Theorem.** Let \((M, g_{\alpha\beta})\) be a 3-dimensional Lorentzian manifold. The number of linearly independent Killing vectors is counted by an algorithm described in figure 2. It includes sub-algorithms given in figures 3–7.

It is noteworthy that the algorithm shown in figure 2 has the nest structures: The sub-algorithm for the class 2 contains that for the class 1 as sub-sub-algorithms; Similarly, the sub-algorithm for the class 3 contains not just that for the class 1 but also that for the class 2 as sub-sub-algorithms. As we will see in sections 3 and 4, such structures serve to avoid unnecessary repetition and to simplify the whole algorithm.

The rest of this paper is organised as follows. In sections 2–4, we analyse the Killing equation in classes 1 to 3 in sequence. The corresponding obstruction matrices and sub-algorithm are given explicitly. We enlighten readers about the power of our algorithm with some instructive examples in section 5. We first inspect the Lifshitz spacetime admitting a single parameter...
the value of which controls the number of local isometry. After this simple exercise, a special attention is payed to the plane-fronted wave with parallel rays (pp-wave), which admits a covariantly constant null vector and vanishing scalar invariants. This metric epitomises the Lorentzian signature and is specified by a single function. We provide the complete classification of local isometry which turns out to be controlled by the profile of this function. We close this paper with some comments in section 6. Technical formulae are summarised in appendix A. An exhaustive classification of spacetimes admitting 4 KVs is given in appendix B. This also serves as an insightful guide to confirm the vindication of the present paper.

Remark that we shall use the same symbol for different sections and subsections recurrently in order to minimise the number of symbols and to lighten the notation. We caution the readers not to be confused by this abuse of notation.

2. Analysis of class 1

Let us begin our analysis with the class 1, in which any KV can be written as

\[ K^u \propto u^u, \quad (2.1) \]

where \( u^u \) is a vector that annihilates \( R_a \). The annihilator \( u^u \) must be specified beforehand, but the results in this section does not depend on the explicit form of \( u^u \).

For \( \dim \ker R_a = 1 \), there is at least one non-vanishing 2-form in equation (1.7a), which allows us to take \( u^u \) to be the Hodge dual of it. For instance, \( u^u \equiv \epsilon^{abc} \nabla_b R \nabla_c S(2) \) for \( dR \wedge dS^{(2)} \neq 0 \), where \( \epsilon^{abc} \) is the Levi-Civita tensor.

Our discussion has two offshoots according to whether \( u^u \) is timelike or spacelike (section 2.1), whilst \( u^u \) is null (section 2.2). Section 2.3 gives short summary of this section.

2.1. Non-null case

When \( u^u \) is timelike or spacelike, we can normalise an annihilator of \( R_a \) to unity by setting

\[ e^a \equiv \frac{u^a}{\sqrt{1 - g_{bc} u^b u^c}}, \quad g_{abc} e^a e^b = 1, \quad (2.2) \]
where $\iota \equiv \text{sgn}(g_{ab}e^a u^b)$. We also introduce the tensor
\[ h_{ab}(\epsilon) \equiv g_{ab} - \iota e_a e_b , \] (2.3)
that is endowed with a projection property and an orthogonality
\[ h^a_c h^c_b = h^a_b , \quad h_{ab} e^b = 0 . \] (2.4)

In this case any KV takes the form
\[ K^a = \omega e^a , \] (2.5)
where $\omega$ is an unknown scalar. By using the projection tensor (2.3) and the form (2.5), one can boil down each component of equation (1.1) to
\[ 0 = e^a e^b \nabla (a K_b) = \iota \xi \omega , \] (2.6a)
\[ 0 = e^a h^b_c \nabla (a K_b) = \frac{1}{2} (\xi \nabla_c \omega - e_c \xi \omega - \iota \Omega^c \omega) , \] (2.6b)
\[ 0 = h^a_c h^b_d \nabla (a K_b) = \omega \kappa_{cd} , \] (2.6c)
where
\[ \Omega^a (\epsilon) \equiv - \iota e^b \nabla_b e_a , \quad \kappa_{ab} (\epsilon) \equiv h^a_c h^b_d \nabla (e_c e_d) . \] (2.7)

It follows that the Killing equation (1.1) amounts to
\[ \kappa_{ab} = 0 , \quad \nabla_{a} \omega = \Omega^a \omega . \] (2.8)

It is noted that the condition (2.6a) follows from the second equation in (2.8). The compatibility condition of the latter equation reads $\nabla_{[a} \Omega_{b]} = 0$.

As a result, the necessary and sufficient conditions for the local solvability of equation (1.1) are aggregated into an algebraic equation
\[ (R_{\text{ch.1}})_{ab} = 0 , \quad (R_{\text{ch.1}})_{ab} \equiv \left[ \begin{array}{c} \kappa_{ab} \\ \nabla_{[a} \Omega_{b]} \end{array} \right] , \] (2.9)
yielding tests for $e^a$. If the equation (2.9) is satisfied, there are no extra obstructions for the existence of the Killing vector. This means that precisely one KV exists. On the other hand, the failure of (2.9) means that there exist no KVs.

### 2.2. Null case

In this case, we directly write the KV as
\[ K^a = \omega u^a , \] (2.10)
where $\omega$ is an unknown scalar, keeping the same notation in section 2.1. We also define the projection tensor
\[ q_{ab}(u, v) \equiv g_{ab} - u_a v_b - v_a u_b , \] (2.11)
where $v^a$ is a vector field satisfying $g_{ab} v^a v^b = 0$ and $g_{ab} v^a u^b = 1^5$. The tensor (2.11) is projective and orthogonal.

\[ ^5\text{Remark that the two conditions do not determine } v^a \text{ uniquely. We need to make a particular choice of } v^a \text{ in } M \text{ (a section of a frame bundle } F(M)). \text{ In spite of this ambiguity, the final outcomes are insensitive to the choice of } v^a. \]
\[ q^a c q^b = q^c b, \quad q^a b u^b = 0, \quad q^a b v^b = 0. \] (2.12)

With the help of equations (2.10) and (2.11), the components of equation (1.1) can be written as follows:

\[ 0 = v^a b \nabla_a (a K_b) = \omega a v^a b \nabla_a u_b + \xi \omega, \] (2.13a)

\[ 0 = u^a b \nabla_b (a K_b) = \left( \frac{1}{2} \omega a u^a b \nabla_a u_b - \xi \omega \right), \] (2.13b)

\[ 0 = u^a c \nabla_a (a K_b) = \left( \frac{1}{2} \omega a u^a c \nabla_a u_c + \frac{1}{2} \nabla_b \omega \right), \] (2.13d)

\[ 0 = q^a c q^b d \nabla_a (a K_b) = \omega a q_c d. \] (2.13e)

where we remark that the uu-component is satisfied automatically and the ‘shear term’ in (2.13e) identically vanishes since \( q_{ab} \) admits only a single nonvanishing component. Here we have introduced

\[ \kappa(u, v) \equiv q^{ab} \nabla_a u_b, \quad \theta_a (u, v) \equiv u^b \nabla_b u_a - (u^b v^c \nabla_b u_c) u_a. \] (2.14)

From above equations, it follows that the satisfaction of the Killing equation is tantamount to

\[ \kappa = 0, \quad \theta_a = 0, \quad \nabla \omega = \Omega_a \omega, \] (2.15)

where

\[ \Omega_a (u, v) \equiv -2 v^b \nabla_b (u_b) + u_a v^b \nabla_b u_c. \] (2.16)

The compatibility condition of the third equation in (2.15) reads \( \nabla [\Omega a \omega] = 0. \)

As a result, the necessary and sufficient conditions for the local solvability of equation (1.1) are aggregated into an algebraic equation

\[ (R_{\text{cls.1}})_{ab} = 0, \quad (R_{\text{cls.1}})_{ab} \equiv \begin{bmatrix} \kappa q_{ab} \\ u_a \theta_b \\ \nabla_a [\Omega b] \end{bmatrix}, \] (2.17)

yielding tests for \( u^a \) and \( v^a \). If the equation (2.17) is satisfied, there are no extra conditions to be satisfied. Hence, one null KV exists.

### 2.3. Short summary of class 1

We summarise the results here in figure 3.

It deserves to emphasise that the sub-algorithm in figure 3 is applicable also for some cases in class 2 and 3 as explained in section 1: It might be seemingly appreciated that the prescription in class 2 and 3 is not be adaptive to class 1, since KVs in either case are expressed as a linear combination of two (or three) annihilators of \( R_a \) with \( \dim \ker R_a > 1 \). Nevertheless, an essential terminus a quo for the argument in class 1 is the assumption \( K^a \propto u^a \) in (2.1), rather than \( \dim \ker R_a = 1 \). Indeed, the situation we shall encounter in class 2 and 3 is that the KVs in several branches are proportional to an annihilator of \( R_a \), whereas \( \dim \ker R_a > 1 \).
Since $K^u \propto u^a$ conforms to the applicability of class 1, no harm is caused in pretending that the recipe in this section is reusable also for such branches. The recyclability of the analysis significantly reduces the total amount of calculations.

3. Analysis of class 2

Our focus in this section is centred on the class 2, in which any KV can be written as

$$K^u = \sum_{\alpha=1}^{2} \omega_{\alpha} u^a_{\alpha},$$

where $\{u^a_{\alpha}\}$ are vectors that annihilates $R_a$. The annihilators must be specified beforehand, but the results in this section do not depend on their explicit form.

For $\dim \ker R_a = 2$, there is only one linearly independent 1-form in equation (1.7b), say $u^a_{6}$. It turns out that any vector orthogonal to $u^a_{6} = g^{ab} u^b_{6}$ is an annihilator of $R_a$. In particular, $u^a_{6}$ itself is the annihilator if it is null, for which a special handling is required.

We are proceeding along two cases where section 3.1 treats the case where the two annihilators are both non-null, whilst section 2.2 discusses either of them is null. We do not try to examine the case in which two annihilators are null and non-parallel, since we can bring this case to the non-null case by a suitable change of basis. Section 3.3 gives short summary of this section.

3.1. Non-null case

In this case, it is assumed that $\{e^a_{i}, i = 1, 2, 3\}$ forms an orthonormal basis of $T(M),$

$$g^{ab} = \epsilon e^a_{i} e^b_{i} + \epsilon e^a_{2} e^b_{2} - \epsilon e^a_{3} e^b_{3},$$

where $\epsilon \equiv \text{sgn}(g_{ab} e^a_{i} e^b_{i})$, and two vectors $\{e^a_{2}, e^a_{3}\}$ are two annihilators of $R_a$. Remark that $e^a_{2}$ is spacelike, whereas $e^a_{3}$ is either spacelike for $\epsilon = -1$ or timelike for $\epsilon = +1$.

For $\dim \ker R_a = 2$, the basis $\{e^a_{i}\}$ is taken as follows: One can choose $e^a_{1}$ in such a way that it is proportional to $u^a$, i.e.

$$e^a_{1} \equiv \frac{u^a}{\sqrt{t u^b g_{bc} t^c}}.$$
where \( \kappa \equiv \text{sgn}(g_{ab}e^a e^b) \). So \( e^2_u \) can be taken as a vector such that \( g_{ab}e^a e^b = 0 \) and \( g_{ab}e^a_2 e^b_2 = 1 \), in terms of which one can specify \( e^3_u \) to be \( e^3_u \equiv e^{abc} e_{1b} e_{2c} \).

Given these orthogonal frame \( \{ e^i_u \} \), any KV can be written in the form

\[
K^a = \omega_2 e^2_u + \omega_3 e^3_u ,
\]

where scalars \( \{ \omega_2, \omega_3 \} \) are yet indeterminate.

Instead of writing down the components of equation (1.1) with equation (3.4) by the coordinate basis, it is much more convenient to work with the connection components. For this purpose, let us introduce the Ricci rotation coefficients as

\[
e^i_u \nabla_b e^i_u = \begin{bmatrix} 0 & \kappa_1 & -\eta_1 \\ -\kappa_1 & 0 & -\tau_1 \\ -\eta_1 & -\tau_1 & 0 \end{bmatrix} e^i_u , \quad (3.5a)
\]

\[
e^2_u \nabla_b e^3_u = \begin{bmatrix} 0 & -\kappa_2 & -\tau_2 \\ \kappa_2 & 0 & -\eta_2 \\ -\tau_2 & -\eta_2 & 0 \end{bmatrix} e^3_u , \quad (3.5b)
\]

\[
e^3_u \nabla_b e^1_u = \begin{bmatrix} 0 & \tau_3 & \kappa_3 \\ -\tau_3 & 0 & \eta_3 \\ \kappa_3 & \eta_3 & 0 \end{bmatrix} e^1_u , \quad (3.5c)
\]

where \( \kappa_i, \eta_i, \) and \( \tau_i \) are respectively the geodesic curvature, normal curvature and relative torsion of an integral curve of \( e^i_u \). Their derivatives entail relations amongst each other, which are collected in appendix A.1.

By using equations (3.2)–(3.5), the 11-part of equation (1.1) can be formally boiled down to

\[
\kappa_1 \omega_2 + \eta_1 \omega_3 = 0 ,
\]

yielding tests for \( e^1_1 \). This implies that the analysis branches off, depending on whether \( e^1_1 \) satisfies the geodesic equation \( e^1_u \nabla_b e^1_u = 0 \).

3.1.1. Branch where \( e^1_1 \) is not a geodesic tangent \( \kappa_1 \eta_1 \neq 0 \). From equation (3.6), \( \omega_2 \) and \( \omega_3 \) are related to each other, \( \omega_3 = - (\kappa_1 / \eta_1) \omega_2 \), or \( \omega_2 = - (\eta_1 / \kappa_1) \omega_3 \). This allows us to rewrite equation (3.4) as

\[
K^a = \omega_2 \left( e^2_u - \frac{\kappa_1 e^3_u}{\eta_1} \right) , \quad \text{or} \quad K^a = \omega_3 \left( e^3_u - \frac{\eta_1 e^2_u}{\kappa_1} \right) ,
\]

which matches the assumption (2.1) of the class 1. It turns out that the annihilator of \( R_a \) is specified as \( e^2_u - (\kappa_1 / \eta_1) e^3_u \) or \( e^3_u - (\eta_1 / \kappa_1) e^2_u \). As explained in section 2.3, the sub-algorithm described in figure 3 can be immediately testable.

3.1.2. Branch where \( e^1_1 \) is a geodesic tangent \( \kappa_1 = \eta_1 = 0 \). In this branch, the 11-part of the Killing equation (3.6) is satisfied automatically. The remaining parts read

\[
\mathcal{L}_1 \omega_2 = - \kappa_2 \omega_2 + (\tau_1 + \tau_2) \omega_3 ,
\]

\[8\] Similar to \( \nu^a \) in section 2.2 the two conditions do not determine \( e^2_u \) uniquely. We need to make a particular choice of \( e^2_u \) in \( M \). Again, the results here do not depend on the choice of \( e^2_u \).
where $\xi$ denotes the Lie derivative along $\xi^a$ and equation (3.8d) is the defining equation of the 1-jet variable $\omega$. Since the PDE system (3.8) is not closed with respect to unknown scalars $\{\omega_2, \omega_3, \omega\}$, we need additional relations between them. Such relations come from several parts of the identities $\nabla_{[a} \nabla_{b]} \omega_2 = \nabla_{[a} \nabla_{b]} \omega_3 = 0$ (see equation (A.2a)), leading to

$$2(\tau_3 - \tau_2) \omega = - \left( \ell \xi_2 (\kappa_2 + \iota \kappa_3) + \tau \xi_3 (\tau_3 - \tau_2) - 2i \eta_2(\tau_3 - \tau_2) \right) \omega_2$$

and

$$2(\tau_3 + \tau_2) \omega = \left( \ell \xi_2 (\kappa_2 + \iota \kappa_3) - \iota \xi_3 (\tau_3 - \tau_2) \right) \omega_2$$

This implies that $\omega$ can be expressed in terms of $\omega_2$ and $\omega_3$ except for $\kappa_2 + \iota \kappa_3 = \tau_3 = \tau_2 = 0$. Depending on the nonzeroness of the coefficients $\{\kappa_2 + \iota \kappa_3, \tau_2, \tau_3, \tau_2 - \tau_3\}$, the analysis further falls into four sub-branches.

**Sub-branch where $\tau_2 = \tau_3 = \kappa_2 + \iota \kappa_3 = 0$.** In this sub-branch, the 1-jet variable $\omega$ cannot be expressed in terms of $\omega_2$ and $\omega_3$. The differential equations for $\omega$ come from the remaining parts of the identities $\nabla_{[a} \nabla_{b]} \omega_2 = \nabla_{[a} \nabla_{b]} \omega_3 = 0$. By combining this and equations (3.8), we obtain a PDE system

$$\nabla_{[a} \omega = \Omega_{[a} \omega, \quad \omega = [\omega_2 \omega_3 \omega]^T,$$  

where

$$\Omega_{[a} = i e_{[a} \left[ \begin{array}{ccc} -\kappa_2 & \tau_1 & 0 \\ \iota \tau_1 & -\kappa_2 & 0 \\ i \xi_2 \tau_1 + \eta_3 \tau_1 & \eta_2 \tau_1 - \xi \kappa_2 & 0 \\ \ell \xi_2 \eta_2 & 0 & 0 \\ 0 & 0 & 1 \\ \ell \xi_2 \eta_2 + i \eta_3 \tau_2 + i \xi \kappa_3 & 0 & 0 \\ -\eta_3 & -\iota \tau_3 & 0 \\ -\eta_2 & 0 & 0 \\ -i \eta_2 \eta_3 - \eta_2 \eta_2 & -i \eta_3 \eta_3 & \eta_3 \end{array} \right].$$

The compatibility condition, $(\nabla_{[a} \Omega_{b]} - \Omega_{[a} \Omega_{b]}^b) \omega = 0$, for equation (3.10) reads

$$R_{[a,b]}^c \omega = 0,$$

where

$$R_{[a,b]}^c = \left[ \begin{array}{ccc} \ell \xi_2 \kappa_2 & \xi_2 \kappa_2 & 0 \\ \ell \xi_2 \lambda_2 & \xi_2 \lambda_2 & 0 \\ \ell \xi_2 \lambda_2 & \xi_2 \lambda_2 & 0 \\ \end{array} \right],$$

and
where $\lambda_2 \equiv R_{ab} e^a e^b_2$. Remark that the first line corresponds to equation (3.9a), and some remaining components are derivable from its derivative.

In this sub-branch, the rank of $R_{cl,2}$ controls the number of KVs: If $\text{rank}R_{cl,2} = 0$, three KVs exist; if $\text{rank}R_{cl,2} = 2$, there is no KV; otherwise $\text{rank}R_{cl,2} = 1$, $\omega_2$ and $\omega_3$ are related to each other. This implies that the KVs in this branch are proportional to an annihilator of $R_c$. As explained in section 2.3, the sub-algorithm described in figure 3 can be immediately testable.

**Other sub-branches.** Except for the case of $\tau_2 = \tau_3 = 0$, $\kappa_2 + i\kappa_3 = 0$, equations (3.9) allow us to write the 1-jet variable $\psi$ in terms of the variables $\{\omega_2, \omega_3\}$. In these sub-branches, equation (3.8) is closed with respect to $\{\omega_2, \omega_3\}$ and then one needs to consider separately the compatibility of PDE systems of the form

$$\nabla_a \omega = \Omega^i_a \omega, \quad \omega = [\omega_2, \omega_3]^T.$$

Remark that explicit forms of the connection $\Omega^i_a$ and the obstruction matrix $R^i_{cl,2}$ depend on the nonzeroness of the coefficients $\{\kappa_2 + i\kappa_3, \tau_2, \tau_3\}$. We thus number $R^i_{cl,2}$ as follows:

\((\#1)\quad \text{For the case of } \tau_2 = \tau_3 = 0, \kappa_2 + i\kappa_3 \neq 0,\)

$$\psi = \eta_3 \omega_3,$$  \hspace{1cm} (3.13a)

$$\Omega^i_a = i e_2 \left[ -\kappa_2 \frac{\partial}{\partial \tau_1} \right] + e_{2a} \left[ 0 \quad \eta_2 \right] - i e_{3a} \left[ -\eta_2 \quad 0 \right],$$  \hspace{1cm} (3.13b)

$$R^i_{cl,2} \equiv \left[ \begin{array}{ccc} \ell_2 \kappa_2 & \ell_1 \kappa_2 \\ \ell_2 \kappa_3 & \ell_1 \kappa_3 \\ \ell_2 \eta_2 & \ell_3 \eta_2 \\ \ell_2 \eta_3 & \ell_3 \eta_3 \\ \ell_2 \tau_1 & \ell_3 \tau_1 \end{array} \right].$$  \hspace{1cm} (3.13c)

\((\#2)\quad \text{For the case of } \tau_2 = \tau_3 \neq 0,\)

$$\psi = \left( \frac{\ell_2 \kappa_3}{4 \tau_2} \right) \omega_2 + \left( \frac{\ell_1 \kappa_3}{4 \tau_2} + \eta_3 \right) \omega_3,$$  \hspace{1cm} (3.14a)

$$\Omega^i_a = i e_2 \left[ -\kappa_2 \frac{\partial}{\partial \tau_1} \right] + e_{2a} \left[ 0 \quad \eta_2 \right] - i e_{3a} \left[ -\eta_2 \quad 0 \right],$$  \hspace{1cm} (3.14b)

$$R^i_{cl,2} \equiv \left[ \begin{array}{ccc} \ell_2 \kappa_3 & \ell_3 \kappa_3 \\ \ell_2 \left( \frac{\kappa_2^2}{4 \tau_2^2} - 4i \tau_2 \right) & \frac{1}{2} (\ell_2 \kappa_3) (\ell_1 \tau_2) \quad \ell_3 \left( \frac{\kappa_2^2}{4 \tau_2^2} - 4i \tau_2 \right) \\ \ell_2 \psi_3 + \frac{\psi_3 \kappa_3}{4 \tau_2} & \frac{1}{2} (\ell_2 \kappa_3) (\ell_1 \tau_2) \quad \ell_3 \psi_3 + \frac{\psi_3 \kappa_3}{4 \tau_2} \end{array} \right].$$  \hspace{1cm} (3.14c)

where

$$\lambda_2 \equiv R_{ab} e^a e^b_2, \quad \lambda_3 \equiv R_{ab} e^a e^b_3,$$

$$\kappa_2 \equiv \kappa_2 + i\kappa_3, \quad \kappa_3 \equiv \kappa_2 - i\kappa_3.$$  \hspace{1cm} (3.14d)
\[ \psi_\alpha \equiv \frac{i\mathcal{L}_a \kappa_\delta}{\tau_2} + (-1)^{\alpha-1}4\eta_\alpha . \quad (\alpha = 2, 3) \]  

(3.14f)

For the case of \( \tau_2 \neq \tau_3 \),

\[ \mathcal{W} = -\left( \frac{\kappa_2 \tau_3 - 2\tau_2 \eta_2}{2\tau_8} \right) \omega_2 - \left( \frac{\kappa_3 \tau_2 + \kappa_2 \tau_3}{2\tau_8} \right) \omega_3, \]  

(3.15a)

\[ \Omega^a = \frac{i}{\tau_3} \begin{pmatrix} -\frac{\kappa_2 \tau_3 - 2\tau_2 \eta_2}{2\tau_8} \tau_1 - \frac{\kappa_2 \tau_3 + \kappa_3 \tau_2}{2\tau_8} \tau_3 \frac{1}{\tau_3} \end{pmatrix} \left[ \tau_2 - \frac{\kappa_2 \tau_3 + \kappa_3 \tau_2}{\tau_3} \right] + e_2 a \frac{\eta_2 - \frac{\kappa_2 \tau_3 + \kappa_3 \tau_2}{\tau_3}}{2\tau_3} \omega_2 - \frac{\kappa_3 \tau_2 + \kappa_2 \tau_3}{2\tau_8} \omega_3, \]  

(3.15b)

where \( R_{ij} \equiv R_{ab} e^a b \) and

\[ \tau_\sigma \equiv \tau_3 + \tau_2, \quad \tau_3 \equiv \tau_3 - \tau_2, \]  

(3.15d)

\[ \sigma_2 \equiv \frac{\kappa_2 \tau_3 + \kappa_3 \tau_2}{\tau_3}, \quad \sigma_3 \equiv \frac{\kappa_3 \tau_2 + \kappa_2 \tau_3}{\tau_3}, \]  

(3.15e)

\[ \Upsilon \equiv R_{22} - i\left( R_{11} + R_{33} \right) - \frac{2}{3} \left( \kappa_3^2 - \kappa_2^2 \right). \]  

(3.15f)

In a way parallel to that of \( R^a \), the rank of \( R^a\#^1 \), \( R^a\#^2 \), \( R^a\#^3 \) controls the number of KVs. For instance, in the case of \( \tau_2 = \tau_3 = 0, \kappa_2 + i\kappa_3 \neq 0 \), rank \( R^a\#^1 \) = 0 implies that two KVs exist. If rank \( R^a\#^2 \) = 2, we have no KVs. Otherwise, rank \( R^a\#^2 \) = 1 and then the KVs in this branch are proportional to an annihilator of \( R_a \). As explained in section 2.3, the sub-algorithm described in figure 3 can be immediately testable.

3.2. Null case

In this case, we suppose that \( \{ u^a, v^a, e^a \} \) spans a double-null basis of \( T(M) \),

\[ g^{ab} = u^a u^b + v^a v^b + e^a e^b, \]  

(3.16)

where \( \{ u^a, v^a \} \) are null vectors such that \( g_{ab} u^a v^b = 1 \); \( e^a \) is a spacelike unit vector orthogonal to \( u^a \) and \( v^a \); \{ \( u^a, e^a \) \} are two annihilators of \( R_a \).

When \( \dim \ker R_a = 2 \) and the 1-form in equation (1.7b) is null, a null vector \( u^a \) can be taken as its contravariant counterpart. By constructing another null vector \( v^a \) satisfying \( g_{ab} u^a v^b = 1 \), a spacelike unit vector \( e^a \) can also taken as \( e^a \equiv e^{abc} u^b v^c \).

Given the above assumptions, any KV can be written in the form

\[ K^a = \omega_\mu u^\mu + \omega_\nu v^\nu + e^a, \]  

(3.17)

where \( \{ \omega_\mu, \omega_\nu \} \) are unknown scalars to be determined.
In order to write the components of equation (1.1) with equation (3.17) explicitly, we introduce the Ricci rotation coefficients as

\[
\begin{align*}
\eta^a \nabla_b [u^a] & = \left[ \begin{array}{ccc}
\kappa_u & 0 & \eta_u \\
0 & -\kappa_u & \tau_u \\
-\tau_u & -\eta_u & 0
\end{array} \right] \left[ \begin{array}{c} u^a \\
v^a \\
e^a
\end{array} \right], \\
\eta^a \nabla_b [v^a] & = \left[ \begin{array}{ccc}
-\kappa_v & 0 & \tau_v \\
0 & \kappa_v & \eta_v \\
-\eta_v & -\tau_v & 0
\end{array} \right] \left[ \begin{array}{c} u^a \\
v^a \\
e^a
\end{array} \right], \\
\eta^a \nabla_b [e^a] & = \left[ \begin{array}{ccc}
\tau_e & 0 & -\kappa_e \\
0 & -\tau_e & -\eta_e \\
\eta_e & \kappa_e & 0
\end{array} \right] \left[ \begin{array}{c} u^a \\
v^a \\
e^a
\end{array} \right],
\end{align*}
\]

where \( \kappa_u, \eta_u, \) and \( \tau_u \) are respectively the geodesic, normal curvature and relative torsion of an integral curve of \( u^a \). The same geometric interpretation is bestowed with quantities for \( v^a \) and \( e^a \). Their derivatives entail relations amongst each other, which are collected in appendix A.2.

By using equations (3.17)–(3.18), the \( uu \)-component of equation (1.1) reads

\[
\eta_u \omega_e = 0. \tag{3.19}
\]

Depending on whether \( u^a \) satisfies the geodesic equation \( u^b \nabla_b u^a = \kappa_u u^a \), the analysis branches off.

3.2.1. Branch where \( u^a \) is not a geodesic tangent. In this branch, \( \eta_u \neq 0 \). Hence equation (3.19) implies \( \omega_e = 0 \) and then equation (3.17) takes the form

\[
K^a = \omega_a u^a, \tag{3.20}
\]

which conforms with the assumption (2.1) of the class 1. As explained in section 2.3, the subalgorithm described in figure 3 can be immediately testable.

3.2.2. Branch where \( u^a \) is a geodesic tangent \( \eta_u = 0 \). In this branch, the remaining parts of equation (1.1) lead to

\[
\begin{align*}
\mathcal{L}_u \omega_u & = -\kappa_u \omega_u + (\tau_u + \tau_v) \omega_e, \\
\mathcal{L}_u \omega_v & = -\kappa_v \omega_e, \\
\mathcal{L}_v \omega_u & = \kappa_v \omega_u + \eta_v \omega_e, \\
\mathcal{L}_v \omega_e & = \varpi, \\
\mathcal{L}_e \omega_u & = -(\tau_v + \tau_e) \omega_u - \eta_e \omega_e - \varpi, \\
\mathcal{L}_e \omega_e & = \kappa_e \omega_u, 
\end{align*}
\]

where equation (3.21d) defines the 1-jet variable \( \varpi \). As is the case in section 3.1, the PDE system (3.21) is not closed with respect to unknown scalars \( \{ \omega_u, \omega_v, \varpi \} \). The supplementary equations follow from several parts of the identities \( \nabla[a \nabla_b] \omega_u = \nabla[a \nabla_b] \omega_v = 0 \), yielding
\[ 2 \tau_v \varpi = \left( \xi_u \kappa_v + \xi_v \kappa_u + 2 \kappa_u \kappa_v + (\tau_u - \tau_v)(\tau_v + \tau_u) \right) \omega_u \]
\[- \left( \xi_v (\tau_u + \tau_v) - \xi_u \eta_v - \eta_v (2 \kappa_u - \kappa_v) - \eta_v (\tau_u - \tau_v) \right) \omega_v, \]
\[ \kappa_v \varpi = \left( \xi_u \tau_v + \kappa_v (\tau_u - \tau_v) \right) \omega_u + \left( \xi_v \tau_v - \kappa_v \eta_v \right) \omega_v. \]
\[(3.22a) \quad (3.22b)\]

This implies that the 1-jet variable \( \varpi \) is written in terms of \( \omega_u \) and \( \omega_v \) except for \( \tau_v = \kappa_v = 0 \). Depending on the vanishing of the coefficients \( \{ \tau_v, \kappa_v \} \), the analysis falls into three sub-branches.

**Sub-branch where** \( \tau_v = \kappa_v = 0 \). In this sub-branch, the 1-jet variable \( \varpi \) cannot be expressed in terms of \( \omega_u \) and \( \omega_v \). The differential equations for \( \varpi \) come from the remaining parts of the identities \( \nabla_y \nabla_b \omega_y = \nabla_y \nabla_b |\omega_v| = 0 \). By combining these with equations (3.21), we obtain a PDE system of the form
\[ \nabla_y \omega = \Omega_y \omega, \quad \omega \equiv [\omega_u \omega_v \varpi]^T, \]
where
\[ \Omega_y \equiv \begin{bmatrix} \kappa_v & \eta_v & 0 \\ 0 & 0 & 1 \\ \xi_v \tau_u - \xi_u \eta_v - \eta_v \tau_u & -\xi_v \eta_v - \xi_u (\eta_v + \kappa_v) + \eta_v (\tau_u - 2 \tau_v) & \kappa_v \\ \end{bmatrix} \]
\[ + v_a \begin{bmatrix} -\kappa_u & \tau_u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_v \end{bmatrix} + e_a \begin{bmatrix} -\tau_v & -\eta_v & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_v \end{bmatrix}. \]
\[(3.24)\]

The compatibility condition, \( (\nabla_y \Omega_y) - \Omega_y (\nabla_y \omega) = 0 \), for equation (3.23) reads
\[ R_{\text{cl.2}} \omega = 0, \quad R_{\text{cl.2}} \equiv \begin{bmatrix} R_{uv} & R_{uv} & 0 \\ \xi_u \tau_v + \tau_\nu \tau_v & -\xi_u \tau_v - 2 \tau_v \tau_v & -\tau_v \eta_v & -R_{uv} \end{bmatrix}, \]
where \( R_{uv} \equiv R_{ab} \nu^a \nu^b, R_{uv} \equiv R_{ab} \nu^a \nu^b \) and \( R_{uv} \equiv R_{ab} \nu^a \nu^b \). The first line corresponds to equation (3.22a).

In this sub-branch, the rank of \( R_{\text{cl.2}} \) influences the number of KVs in the same way as that presented in section 3.1.2.

**Other sub-branches.** Except for \( \tau_v = \kappa_v = 0 \), equations (3.22) allow us to express the 1-jet variable \( \varpi \) in terms of \( \{ \omega_u, \omega_v \} \). In these sub-branches, equation (3.21) is closed with respect to \( \{ \omega_u, \omega_v \} \) and then the compatibility of PDE systems is of the form
\[ \nabla_y \omega = \Omega_y \omega, \quad \omega \equiv [\omega_u \omega_v \tau_v]^T. \]
The upshot is as follows:

\( \neq 1 \) For the case of \( \tau_v = 0, \kappa_v \neq 0 \),
\[ \varpi = \tau_v \omega_u - \eta_v \omega_v, \]
\[ \Omega_y = \begin{bmatrix} \kappa_v & \eta_v & 0 \\ \tau_u & -\eta_v & 0 \\ \end{bmatrix} \]
\[ + v_a \begin{bmatrix} -\kappa_u & \tau_u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_v \end{bmatrix} + e_a \begin{bmatrix} -\tau_u & -\kappa_v \tau_v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau_v \end{bmatrix}. \]
\[(3.26a) \quad (3.26b)\]
\[ R_{\text{ch}_2}^{\#1} = \begin{bmatrix} R_{av} - \frac{1}{2} R_{ee} & R_{ve} + \eta_v \kappa_v \\ \xi_u \tau_u & \xi_v \tau_v \end{bmatrix} , \]  

(3.26c)

where \( R_{ee} \equiv R_{ab} e^a e^b \).

(\#2) For the case of \( \tau_v \neq 0 \),

\[
\omega = \left( \xi_u \kappa_v + \xi_v \kappa_u + \eta_v \kappa_v + \kappa_u \kappa_v + (\tau_u - \tau_v)(\tau_v + \tau_v) \right) \frac{\omega_u}{2\tau_v} 
- \left( \xi_v (\tau_u + \tau_v) - \xi_u \eta_v - \eta_v \kappa_v + \kappa_u \kappa_v + \kappa_v \kappa_v + \kappa_v \kappa_v \right) \frac{\omega_v}{2\tau_v} ,
\]  

(3.27a)

\[
\Omega_u = \left[ a_u \left[ \frac{\xi_u \kappa_v + \xi_v \kappa_u + \eta_v \kappa_v + \kappa_u \kappa_v + (\tau_u - \tau_v)(\tau_v + \tau_v)}{2\tau_v} \right] + \eta_u \left[ \begin{array}{c} -\kappa_u \\ \tau_u + \tau_v \end{array} \right] \right] 
+ \left[ e_u \left[ \frac{\xi_u \kappa_v + \xi_v \kappa_u + \eta_v \kappa_v + \kappa_u \kappa_v + \kappa_v \kappa_v + \kappa_v \kappa_v}{2\tau_v} \right] \right] .
\]  

(3.27b)

and \( R_{\text{ch}_2}^{\#2} \omega = 0 \) can be written as

\[ 0 = \left( \xi_u \kappa_v - \kappa_u \kappa_v \right) \omega_u + \left( \xi_v \kappa_v + (\tau_v - \tau_v) \kappa_v \right) \omega_v , \]  

(3.27c)

\[ 0 = \left( 2\xi_u \tau_v + (\psi_1 + 2\tau_u) \kappa_v \right) \omega_u + \left( 2\xi_v \tau_v + (\psi_2 - 2\eta_v) \kappa_v \right) \omega_v , \]  

(3.27d)

\[ 0 = \left( \xi_u \psi_1 \right) \omega_u + \left( \xi_u \psi_2 + (\kappa_u - \kappa_v) \psi_2 + (\tau_u + \tau_v) \psi_2 + 2\tau_u \tau_v - R_{ve} \right) \omega_v , \]  

(3.27e)

\[ 0 = \left( \xi_v (\psi_1 - 2\tau_v) - \frac{\psi_1}{2} (\psi - 2\tau_v) - 2\eta_v \kappa_v \right) \omega_u 
+ \left( \xi_v \psi_2 - \frac{\psi_2}{2} (\psi + 2\kappa_v) + (\psi_1 + 2\tau_v) \eta_v - 2R_{uv} \right) \omega_v , \]  

(3.27f)

\[ 0 = \left( \xi_v (\psi_1 + \frac{\psi_1^2}{2} + \kappa_v \psi_2 + R_{ve} \right) \omega_u + \left( \xi_v \psi_2 + \frac{\psi_1}{2} (\psi_2 - 2\eta_v + \tau_v \psi_2 - 2\eta_v \tau_v \right) \omega_v , \]  

(3.27g)

where

\[ \psi_1 \equiv \frac{1}{\tau_v} \left( R_{av} - \frac{1}{2} R_{ee} \right) - \tau_v , \quad \psi_2 \equiv \frac{1}{\tau_v} \left( R_{av} + \eta_v \kappa_v + \xi_v \right) . \]  

(3.27h)

In these sub-branches, the rank of \( R_{\text{ch}_2}^{\#1} \) and \( R_{\text{ch}_2}^{\#2} \) governs the number of KVs in the same way as that presented in section 3.1.2.
3.3. Short summary of class 2

We summarise the results obtained in this section in figures 4–6. It should be emphasised that the sub-algorithm in figures 4–6 can be again applicable for some cases in class 3 as explained in section 1: At the outset, KVs in class 3 is expressed as a linear combination of the three annihilators of $R_a$. In some branches, the KVs eventuates into the form of a linear combination of two (or less) annihilators of $R_a$, while keeping the property $\dim \ker R_a = 3$. Since $K^a = \sum \omega_\alpha e^\alpha_a$ accords with the prerequisite of class 2 (3.1), the results in this section is adapted to such branches as well.

4. Analysis of class 3

In this section, we address the case of class 3, for which all criteria in equations (1.7) are vanishing. This implies $R_a$ is a zero matrix, whence any vector can be an annihilator of $R_a$. Since the first obstruction matrix $R_a$ has been intentionally designed to ensure that all the eigenvalues of the traceless Ricci operator $S^b_b$ are constants if $\text{rank} R_a$ is zero, it is thereby reasonable to resort to the Jordan basis of $S^b_b$. Perhaps the other choices for the basis of $T(M)$ fail to lessen the burden of computations, despite the fact that Jordan basis inevitably demands us to solve the eigenvalue problem. Thus, our proposed formulation is based on the Jordan decomposition of the matrix $S^a_a$, which is nothing but the classification of the Segre type of $S^a_a$. It can be found in [16] that the Segre classification is carried out by an examination of the minimal polynomial of $S^a_a$ as shown in figure 7. See also [17] for the Segre classification of symmetric tensors in Lorentzian geometry.

Let us pause here to declare the Segre notation [11]. The eigenvalue equation $S^a_a V^b_b = \lambda V^a_a$ determines the orders of elementary divisors which belong to the several eigenvalues. A characteristic feature in Lorentzian geometry is that the elementary divisors can be non-simple and the eigenvalues can be complex. The Segre notation stands for the orders of elementary divisors with the round brackets specifying which eigenvalues coincide. If two eigenvalues are complex conjugates, they are denoted by $z$ and $\bar{z}$.

With these notations in mind, we are proceeding along four cases: We discuss the Segre type [1, 11] in section 4.1, the Segre type [21] in section 4.2, the Segre type [3] in section 4.3 and the Segre type $[zz1]$ in section 4.4.

4.1. Type [1, 11] and its degeneracies

In this case, we have the following Jordan chains:

$$S^a_a e^b_b = \lambda_\alpha e^\alpha_a, \quad (\alpha = 1, 2, 3)$$  \hspace{1cm} (4.1)

with $\sum_\alpha \lambda_\alpha = 0$. Here $e^\alpha_a$ is an eigenvector of $S^a_a$ belong to the eigenvalue $\lambda_\alpha$. In this subsection, it is assumed that $\{e^\alpha_a\}$ are normalised and $e^1_1$ is timelike, so that $\{e^\alpha_a\}$ form an orthonormal basis of $T(M)$.

$$g^{ab} = -e^1_1 e^1_1 + e^2_2 e^2_2 + e^3_3 e^3_3.$$  \hspace{1cm} (4.2)

Then, any KV can be written as

$$K^a = \sum_{\alpha=1}^3 \omega_\alpha e^\alpha_a,$$  \hspace{1cm} (4.3)

where scalars $\{\omega_\alpha\}$ are yet indeterminate.
It is an elementary computation to write down the the first compatibility condition, \( \mathcal{L}_K S_{ab} = 0 \), of equation (1.1), giving

\[
0 = (\lambda_1 - \lambda_2) \left( \varpi_2 + \kappa_2 \omega_2 - (\tau_1 + \tau_3) \omega_3 \right), \tag{4.4a}
\]

\[
0 = (\lambda_2 - \lambda_3) \left( \varpi_3 - (\tau_1 - \tau_2) \omega_1 + \eta_3 \omega_3 \right), \tag{4.4b}
\]

\[
0 = (\lambda_3 - \lambda_1) \left( \varpi_1 - \eta_1 \omega_1 - (\tau_2 - \tau_3) \omega_3 \right), \tag{4.4c}
\]

where the Ricci rotation coefficients are defined by equations (3.5), and the 1-jet variables \( \{ \varpi_\alpha \} \) are respectively defined as

\[
\varpi_1 \equiv \xi_1 \omega_1, \quad \varpi_2 \equiv \xi_1 \omega_2, \quad \varpi_3 \equiv \xi_2 \omega_3. \tag{4.5}
\]

The eigenvalues \( \lambda_\alpha \) are constrained by the second Bianchi identity \( \nabla_a R^b_b - (1/2) \nabla_b R = 0 \) as

\[
0 = (\lambda_1 - \lambda_2) \kappa_2 - (\lambda_3 - \lambda_1) \kappa_3, \tag{4.6a}
\]

\[
0 = (\lambda_1 - \lambda_2) \kappa_1 + (\lambda_2 - \lambda_3) \eta_3, \tag{4.6b}
\]

\[
0 = (\lambda_2 - \lambda_3) \eta_2 + (\lambda_3 - \lambda_1) \eta_1. \tag{4.6c}
\]

The compatibility conditions (4.4) are fulfilled trivially if the Segre type is \([1, (11)]\), \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \), thereby yielding the result shown in figure 7. In the remaining parts of this subsection, we investigate the Segre types \([1, (11)]\), \([(1, 1)]\) and \([1, 11]\) separately.

### 4.1.1 Branch where the Segre type is \([1, (11)]\)

In this branch, two eigenvalues in the space-like direction coincide, i.e. \( \lambda_2 = \lambda_3 = -(1/2) \lambda_1 \). Then, it immediately follows from equations (4.4) and (4.6) that

\[
\kappa_1 = 0, \quad \eta_1 = 0, \quad \kappa_3 = - \kappa_2, \tag{4.7a}
\]

\[
\varpi_1 = (\tau_2 - \tau_3) \omega_2, \quad \varpi_2 = - \kappa_2 \omega_2 + (\tau_1 + \tau_3) \omega_2. \tag{4.7b}
\]

Given these conditions (4.7), the Killing equation (1.1) and the identities \( \nabla_\mu \nabla_\nu \omega_1 = \nabla_\mu \nabla_\nu \omega_2 = \nabla_\mu \nabla_\nu \omega_3 = 0 \) produce a PDE system of the form

\[
\nabla_\mu \omega = \Omega_\mu \omega, \quad \omega \equiv [\varpi_1 \omega_2 \varpi_3]^T, \tag{4.8}
\]

where

\[
\Omega_\mu = - e_{1a} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -\kappa_2 & \tau_1 + \tau_3 & 0 \\
0 & -\tau_1 - \tau_2 & \kappa_2 & 0 \\
-2\kappa_2 \tau_1 & \kappa_2 \eta_2 - \xi_1 \eta_2 & (\tau_1 + \tau_3) \eta_2 & 0
\end{bmatrix}
+ e_{2a} \begin{bmatrix}
0 & 0 & -\tau_2 + \tau_3 & 0 \\
\kappa_2 & 0 & \eta_2 & 0 \\
0 & 0 & 0 & 1 \\
-\kappa_2 \eta_2 - \xi_2 \tau_2 & -\xi_2 \eta_2 & \eta_3 - (\tau_1 - \tau_2)(\tau_2 - \tau_3) - \xi_2 \eta_3 - \xi_3 \eta_2 & 0
\end{bmatrix}
+ e_{3a} \begin{bmatrix}
0 & \tau_2 - \tau_3 & 0 & 0 \\
-\tau_2 - \tau_3 & -\eta_2 & -\eta_3 & 1 \\
-\kappa_2 & \eta_3 & 0 & 0 \\
2\kappa_2 \eta_3 - \xi_2 \eta_2 - \eta_3^2 + (\tau_1 - \tau_2)(\tau_2 - \tau_3) + \xi_2 \eta_3 & \eta_2 \eta_3 & \eta_2
\end{bmatrix}. \tag{4.9}
\]
The compatibility condition for equation (4.8) leads to
\[
(\tau_2 + \tau_3) \varpi_3 = - \tau_2 (\tau_2 + \tau_3) \omega_1 + \left( \xi_2 \tau_2 - \eta_2 (\tau_2 + \tau_3) + 2 \eta_2 \kappa_2 \right) \omega_2 - \left( \xi_2 \tau_3 - 2 \eta_2 \kappa_2 \right) \omega_3 ,
\]
Equation (4.8) is closed with respect to \( \kappa_2 \) and \( \omega_2 \).
\[
4 \kappa_2 \varpi_3 = - 4 \kappa_2 \tau_2 \omega_1 - \left( \xi_2 (\tau_2 + \tau_3) \right) \omega_2 - \left( \xi_3 (\tau_2 + \tau_3) + 4 \eta_2 \kappa_2 \right) \omega_3 .
\]
Contrary to the conclusion in Sub-section A.3a, this implies that \( \varpi_3 \) can be expressed in terms of \( \omega_1, \omega_2, \omega_3 \) except for \( \kappa_2 = \tau_2 + \tau_3 = 0 \). Depending on the nonzeroness of the coefficients \( \kappa_2, \tau_2 + \tau_3 \), the analysis falls into three sub-branches.

**Sub-branch where** \( \kappa_2 = \tau_2 + \tau_3 = 0 \). In this sub-branch, the conditions (4.10) gives
\[
(\xi_2 \tau_2) \omega_2 + (\xi_3 \tau_3) \omega_3 = 0,
\]
whilst the relations in equations (A.3) imply that \( \xi_2 \tau_2 = \xi_3 \tau_2 = 0 \). We conclude that four KVs exist in this sub-branch, since the compatibility is trivially fulfilled.

**Other sub-branches.** Except for \( \kappa_2 = \tau_2 + \tau_3 = 0 \), the 1-jet variable \( \varpi_3 \) is not an independent variable, but is expressed in terms of \( \omega_1, \omega_2, \omega_3 \) by virtue of equations (4.10). In these sub-branches, equation (4.8) is closed with respect to \( \omega_1, \omega_2, \omega_3 \) and then the compatibility of PDE systems is of the form
\[
\nabla_{\alpha} \omega = \Omega_{\alpha} \omega , \quad \omega \equiv [\omega_1 \omega_2 \omega_3]^T
\]
The results for two cases (#1) and (#2) are presented as follows:

**(#1) For the case of** \( \tau_2 + \tau_3 = 0, \kappa_2 \neq 0 \),

\[
\varpi_3 = - \tau_2 \omega_1 - \tau_3 \omega_3 ,
\]
\[
\Omega_{\alpha} = - \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ - \kappa_2 & - \tau_2 & 0 \\ \tau_2 & \kappa_2 & 0 \end{array} \right] + \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & 0 & - 2 \tau_2 \\ \kappa_2 & 0 & \eta_2 \\ - \tau_2 & 0 & - \eta_3 \end{array} \right] + \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & 2 \tau_2 & 0 \\ - \kappa_2 & - \eta_3 & 0 \end{array} \right] ,
\]
\[
R_{\mu \nu \rho}^{\#1} = \left[ \begin{array}{ccc} 0 & \xi_2 \tau_2 & \xi_3 \tau_2 \\ \xi_1 \eta_2 & \xi_2 \eta_2 & \xi_3 \eta_2 \\ \xi_1 \eta_3 & \xi_2 \eta_3 & \xi_3 \eta_3 \end{array} \right].
\]

**(#2) For the case of** \( \tau_2 + \tau_3 \neq 0 \),
\[
\varpi_3 = \left( \frac{\tau_2 - \tau_3}{2} \right) \omega_1 + \left( \frac{\xi_2 \kappa_2}{\tau_\sigma} \right) \omega_2 + \left( \frac{\xi_3 \kappa_2}{\tau_\sigma} - \eta_3 \right) \omega_3 ,
\]
\[
\Omega_{\alpha} = - \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & 0 & 0 \\ - \tau_2 & - \kappa_2 & \tau_1 + \frac{\tau_2 + \tau_3}{2} \\ - \tau_2 - \frac{\tau_2 + \tau_3}{2} & \kappa_2 & 0 \end{array} \right] + \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & 0 & \tau_3 \\ \kappa_2 & 0 & \eta_2 \\ \frac{\tau_2 + \tau_3}{2} & \xi_2 \eta_2 & \xi_3 \eta_2 \end{array} \right] \\
+ \epsilon_{\alpha\mu} \left[ \begin{array}{ccc} 0 & - \tau_3 & 0 \\ - \kappa_2 & - \eta_3 & - \xi_2 \kappa_2 \\ - \tau_2 + \frac{\tau_2 + \tau_3}{2} & \eta_2 & 0 \end{array} \right] ,
\]
\[
R_{\mu \nu \rho}^{\#2} = \left[ \begin{array}{ccc} 0 & \xi_2 \tau_2 & \xi_3 \tau_2 \\ \xi_1 \eta_2 & \xi_2 \eta_2 & \xi_3 \eta_2 \\ \xi_1 \eta_3 & \xi_2 \eta_3 & \xi_3 \eta_3 \end{array} \right].
\]
\[
R_{[1, (11)]}^2 = \begin{bmatrix}
0 & \xi_2 \tau_3 & \xi_3 \tau_3 \\
\xi_2 & \Pi_1 & \Pi_2 \\
\xi_3 & \Pi_3 & \Xi_1 \\
\Xi_2 & \Xi_3 
\end{bmatrix},
\]

where
\[
\tau_\delta \equiv \tau_3 - \tau_2, \\
\tau_\sigma \equiv \tau_3 + \tau_2, \\
\Pi_\alpha \equiv \xi_\alpha \eta_2 + \frac{\xi_\alpha \xi_2 \kappa_2}{\tau_\sigma} - \left(\xi_2 (\tau_\delta + 3 \tau_\sigma) - 4 \eta_2 \kappa_2\right) \frac{\xi_\alpha \kappa_2}{2 \tau_\sigma^2}, \\
\Xi_\alpha \equiv \xi_\alpha \eta_3 - \frac{\xi_\alpha \xi_3 \kappa_2}{\tau_\sigma} - \left(\xi_3 (\tau_\delta - 3 \tau_\sigma) - 4 \eta_3 \kappa_2\right) \frac{\xi_\alpha \kappa_2}{2 \tau_\sigma^2}.
\]

In these sub-branches, the rank of \(R_{[1, (11)]}^1\) and \(R_{[1, (11)]}^2\) is linked to the number of KVs as follows:

If \(\text{rank} R_{[1, (11)]}^1 = 0\), three KVs exist; If \(\text{rank} R_{[1, (11)]}^1 = 3\), there is no KV; If \(\text{rank} R_{[1, (11)]}^1 = 2\), the sub-algorithm described in figure 3 can be testable since the KVs in this case are proportional to an annihilator of \(R_a\); Otherwise \(\text{rank} R_{[1, (11)]}^1 = 1\), the sub-algorithm described in figures 4–6 can be testable since the KVs in this case can be written as a linear combination of two annihilators of \(R_a\). The argument for \(R_{[1, (11)]}^2\) parallels with above.

4.1.2. Branch where the Segre type is \([1, 1, 1]\). In this branch, we have \(\lambda_1 = \lambda_3 = -\frac{1}{2} \lambda_2\). Then, it immediately follows from equations (4.4) and (4.6) that

\[
\kappa_2 = 0, \quad \eta_2 = 0, \quad \kappa_1 = \eta_3, \\
\omega_2 = (\tau_1 + \tau_3) \omega_3, \quad \omega_3 = (\tau_1 - \tau_2) \omega_1 - \eta_3 \omega_3.
\]

Given these conditions (4.14), the Killing equation (1.1) and the identities \(\nabla_a \nabla_b \omega_1 = \nabla_a \nabla_b \omega_2 = \nabla_a \nabla_b \omega_3 = 0\) produce a PDE system of the form

\[
\nabla_a \omega = \Omega_a \omega, \quad \omega \equiv [\omega_1 \omega_2 \omega_3 \omega_3] \quad \text{T},
\]

where

\[
\Omega_a \equiv -\epsilon_{1a} \begin{bmatrix}
0 & -\kappa_1 & -\eta_1 & 0 \\
0 & 0 & \tau_1 + \tau_3 & 0 \\
-\eta_1 & -\tau_1 + \tau_3 & -\kappa_3 & 1 \\
-\eta_1 \kappa_3 & -\xi_1 \kappa_3 - 2 \eta_1 \kappa_1 & -\eta_1^2 + (\tau_1 + \tau_3)^2 - \xi_2 \eta_1 & \kappa_1 \\
\kappa_1 & 0 & -\tau_2 + \tau_3 & 0 \\
0 & 0 & 0 & 0 \\
\tau_1 - \tau_2 & 0 & -\kappa_1 & 0 \\
(\tau_1 - \tau_2) \kappa_3 & -2 \kappa_1 \kappa_2 & \xi_2 \kappa_3 - \kappa_1 \kappa_3 & 0
\end{bmatrix} + \epsilon_{2a} \begin{bmatrix}
0 & 0 & 0 & 1 \\
-\tau_1 - \tau_3 & 0 & 0 & 0 \\
\kappa_3 & -\kappa_1 & 0 & 0 \\
\eta_1^2 - (\tau_1 + \tau_3)(\tau_2 - \tau_3) + \xi_1 \kappa_3 + \xi_3 \eta_1 & \kappa_1 \kappa_3 - \xi_3 \tau_3 & \xi_3 \kappa_3 & 0
\end{bmatrix}.
\]
The compatibility condition for equation (4.15) leads to
\[(\tau_1 - \tau_3) \omega_1 = \left(2 \kappa_1 \kappa_3 - \ell_3 \tau_1 \right) \omega_1 - \tau_3 (\tau_1 - \tau_3) \omega_2 - \left(2 \eta_1 \kappa_1 - \kappa_3 (\tau_1 - \tau_3) - \ell_1 \tau_3 \right) \omega_3 . \tag{4.17a}\]
\[4 \kappa_1 \omega_1 = \left(4 \eta_1 \kappa_1 + \ell_1 (\tau_1 - \tau_3) \right) \omega_1 - 4 \kappa_1 \tau_3 \omega_2 + \left(\ell_3 (\tau_1 - \tau_3) \right) \omega_3 \tag{4.17b}.\]
This implies that \(\omega_1\) can be expressed in terms of \(\{\omega_1, \omega_2, \omega_3\}\) except for \(\kappa_1 = \tau_1 - \tau_3 = 0\). Depending on whether the coefficients \(\{\kappa_1, \tau_1 - \tau_3\}\) of \(\omega_0\) are vanishing, the analysis falls into three sub-branches.

**Sub-branch where** \(\kappa_1 = \tau_1 - \tau_3 = 0\). In this sub-branch, the conditions (4.17) gives
\[(\ell_1 \tau_1) \omega_1 + (\ell_3 \tau_1) \omega_3 = 0 \tag{4.18},\]
whilst the relations in equations (A.3a) imply that \(\ell_1 \tau_1 = \ell_3 \tau_1 = 0\). We conclude that four KVs exist in this sub-branch, as the compatibility conditions are trivially met.

**Other sub-branches.** Except when \(\kappa_1 = \tau_1 - \tau_3 = 0\), equations (4.17) allow us to write the 1-jet variable \(\omega_1\) in terms of \(\{\omega_1, \omega_2, \omega_3\}\). In these sub-branches, equation (4.15) is closed with respect to \(\{\omega_1, \omega_2, \omega_3\}\) and then the compatibility of PDE systems takes the form

\[\nabla_\omega \omega = \Omega_\omega \omega \tag{4.19}, \quad \omega \equiv [\omega_1 \omega_2 \omega_3]^T.\]

The individual results are described as follows:

(\#1) For the case of \(\tau_1 = \tau_3, \kappa_1 \neq 0\),
\[\omega_1 = \eta_1 \omega_1 - \tau_1 \omega_2 \tag{4.19a} ,\]
\[\Omega_\omega = -\epsilon_{1a} \begin{bmatrix} 0 & -\kappa_1 & -\eta_1 \\ 0 & 0 & 2 \tau_1 \\ -\tau_1 & -\kappa_3 \end{bmatrix} + \epsilon_{2a} \begin{bmatrix} \kappa_1 & 0 & \tau_1 \\ 0 & 0 & 0 \\ \tau_1 & 0 & -\kappa_1 \end{bmatrix} + \epsilon_{3a} \begin{bmatrix} -2 \tau_1 & 0 & 0 \\ \kappa_3 & \kappa_1 & 0 \end{bmatrix} \tag{4.19b} ,\]
\[R^{\#1}_{[1,1,1]} = \begin{bmatrix} \ell_1 \tau_1 & 0 & \ell_3 \tau_1 \\ \ell_1 \eta_1 & \ell_2 \eta_1 & \ell_3 \eta_1 \\ \ell_1 \kappa_3 & \ell_2 \kappa_3 & \ell_3 \kappa_3 \end{bmatrix} \tag{4.19c} ,\]

(\#2) For the case of \(\tau_1 \neq \tau_3\),
\[\omega_1 = \left(\eta_1 - \frac{\ell_1 \kappa_1}{\tau_6} \right) \omega_1 - \left(\frac{\tau_6 + \tau_\sigma}{2} \right) \omega_2 - \left(\frac{\ell_3 \kappa_1}{\tau_6} \right) \omega_3 \tag{4.20a} ,\]
\[\Omega_\omega = -\epsilon_{1a} \begin{bmatrix} 0 & -\kappa_1 & -\eta_1 \\ 0 & 0 & \tau_\sigma \\ -\frac{\ell_1 \kappa_1}{\tau_6} & -\frac{\tau_6 - \tau_\sigma}{2} & -\frac{\ell_1 \kappa_1}{\tau_3} \end{bmatrix} + \epsilon_{2a} \begin{bmatrix} \kappa_1 & 0 & \frac{\tau_6 + \tau_\sigma}{2} - \tau_3 \\ 0 & 0 & 0 \\ -\frac{\tau_6 - \tau_\sigma}{2} & \tau_3 & 0 \end{bmatrix} + \epsilon_{3a} \begin{bmatrix} -\tau_\sigma & 0 & 0 \\ \kappa_3 & \kappa_1 & 0 \end{bmatrix} \tag{4.20b}.\]
\[
R_{\frac{\#2}{\left(1,1\right)11}}^{(2)} = \begin{bmatrix}
\xi_1 \tau_\sigma & 0 & \xi_3 \tau_\sigma \\
\Pi_1 & \Pi_2 & \Pi_3 \\
\xi_1 & \xi_2 & \xi_3
\end{bmatrix},
\]  
(4.20c)

where

\[
\tau_\sigma \equiv \tau_3 + \tau_1, \\
\tau_\delta \equiv \tau_3 - \tau_1, \\
\Pi_\alpha \equiv \xi_\alpha \eta_1 - \frac{\xi_\alpha \xi_1 \kappa_1}{\tau_\delta} + \left(\xi_1 (\tau_\sigma + 3 \tau_3) - 4 \eta_1 \kappa_1\right) \frac{\xi_\alpha \kappa_1}{2 \tau_\delta}, \\
\xi_\alpha \equiv \xi_\alpha \kappa_3 + \frac{\xi_\alpha \xi_3 \kappa_1}{\tau_\delta} + \left(\xi_3 (\tau_\sigma - 3 \tau_3) - 4 \eta_3 \kappa_3\right) \frac{\xi_\alpha \kappa_1}{2 \tau_\delta}.
\]  
(4.20d, e, f, g)

In these sub-branches, the rank of \(R_{\frac{\#1}{\left(1,1\right)11}}\) and \(R_{\frac{\#2}{\left(1,1\right)11}}\) controls the number of KVs in the same way as that presented in section 4.1.1.

4.1.3. Branch where the Segre type is \([1, 11]\). In this branch, the eigenvalues \(\{\lambda_\alpha\}\) differ from each other. Then, it immediately follows from equations (4.4) and (4.6) that

\[
\kappa_3 = \delta \lambda_1 \kappa_2, \quad \kappa_1 = - \delta \lambda_2 \eta_3, \quad \eta_2 = - \delta \lambda_3 \eta_1,
\]  
(4.21a)

\[
\varpi_1 = \eta_1 \omega_1 + (\tau_2 - \tau_3) \omega_2, \quad \varpi_2 = - \kappa_2 \omega_3 + (\tau_1 + \tau_3) \omega_3, \quad \varpi_3 = (\tau_1 - \tau_2) \omega_1 - \eta_3 \omega_3,
\]  
(4.21b)

where \(\delta \lambda_1 \equiv (\lambda_1 - \lambda_3)/(\lambda_3 - \lambda_1)\), \(\delta \lambda_2 \equiv (\lambda_2 - \lambda_3)/(\lambda_2 - \lambda_1)\), and \(\delta \lambda_3 \equiv (\lambda_3 - \lambda_1)/(\lambda_2 - \lambda_3)\).

Given these conditions (4.21), the Killing equation (1.1) produces a PDE system of the form

\[
\nabla_\alpha \omega = \Omega_\alpha \omega, \quad \omega \equiv [\omega_1 \omega_2 \omega_3]^T,
\]  
(4.22)

where

\[
\Omega_\alpha \equiv - e_\alpha \begin{bmatrix}
0 & \delta \lambda_2 \eta_3 & -\eta_1 \\
0 & -\kappa_2 & \tau_1 + \tau_3 \\
0 & -\tau_1 + \tau_2 & -\delta \lambda_1 \kappa_2
\end{bmatrix} + e_\omega \begin{bmatrix}
-\delta \lambda_2 \eta_3 & 0 & -\tau_2 + \tau_3 \\
\kappa_2 & 0 & -\delta \lambda_1 \eta_3 \\
\tau_1 - \tau_2 & 0 & -\eta_3
\end{bmatrix} + e_\omega \begin{bmatrix}
-\eta_1 & \tau_2 - \tau_3 & 0 \\
-\tau_1 & \delta \lambda_1 \kappa_2 & \eta_3 \\
\delta \lambda_1 \kappa_2 & \eta_3 & 0
\end{bmatrix}.
\]  
(4.23)

The compatibility condition for equation (4.22) leads to

\[
R_{\left(1,1\right)11} = \begin{bmatrix}
\xi_1 \kappa_1 & \xi_2 \kappa_1 & \xi_3 \kappa_1 \\
\xi_1 \kappa_2 & \xi_2 \kappa_2 & \xi_3 \kappa_2 \\
\xi_1 \eta_1 & \xi_2 \eta_1 & \xi_3 \eta_1 \\
\xi_1 \tau_1 & \xi_2 \tau_1 & \xi_3 \tau_1 \\
\xi_1 \tau_2 & \xi_2 \tau_2 & \xi_3 \tau_2 \\
\xi_1 \tau_3 & \xi_2 \tau_3 & \xi_3 \tau_3
\end{bmatrix}.
\]  
(4.24)

In this sub-branch, the rank of \(R_{\left(1,1\right)11}\) governs the number of KVs in the same way as that presented in section 4.1.1.

4.1.4. Short summary of class 3 type \([1, 11]\). Let us visually abridge the results obtained in this subsection in figures 8–10.
4.2. Type [21] and its degeneracy

In this case, we have the following Jordan chains:

\[ S^\alpha_{a} f^b_1 = \lambda_1 f^b_1, \quad (4.25a) \]

\[ S^\alpha_{a} f^b_2 = \lambda_1 f^b_2 + f^b_1, \quad (4.25b) \]

\[ S^\alpha_{a} f^b_3 = \lambda_3 f^b_3, \quad (4.25c) \]

with \( 2\lambda_1 + \lambda_3 = 0 \). Here \( f^b_i \) is a generalised eigenvector of \( S^\alpha_{a} \). It can be shown that the eigenvectors \( f^b_1, f^b_2, f^b_3 \) are respectively null and spacelike. The causal nature of \( f^b_2 \) is indeterminate.

In this branch, three eigenvalues of \( S^\alpha_{a} \) are

\[ \text{indeterminate}. \]

Taking the double-null basis \( \{ u^a, v^a, e^a \} \) of \( T(M) \) as

\[ g^{ab} = u^a u^b + v^a v^b + e^a e^b, \quad (4.26) \]

it turns out useful to choose \( u^a \equiv f^b_1 \) and \( e^a \equiv f^b_3 \), with \( v^a \) being a null vector such that

\[ g_{ab} u^a v^b = 1 \quad \text{and} \quad g_{ab} e^a v^b = 0. \]

Then, any KV can be written as

\[ K^a = \omega_u u^a + \omega_v v^a + \omega_e e^a, \quad (4.27) \]

where scalars \( \{ \omega_u, \omega_v, \omega_e \} \) are yet indeterminate.

The first compatibility condition, \( \mathcal{L}_b S_{ab} = 0 \), of equation (1.1) yields

\[ \lambda \omega_x = -\lambda(\tau_u - \tau_v)\omega_v - \lambda\kappa_e \omega_e, \quad (4.28a) \]

\[ 3\lambda \omega_u - S_{uv} \omega_v = -3\lambda(\tau_u + \tau_v)\omega_v - \left(3\eta \lambda - (\tau_u - \tau_v)S_{uv}\right)\omega_v + \kappa_v S_{uv} \omega_v, \quad (4.28b) \]

\[ 2S_{uv} \omega_u = -\left(\mathcal{E}_u S_{uv} + 2\kappa_u S_{uv}\right)\omega_u - (\mathcal{E}_v S_{uv})\omega_v - \left(\mathcal{E}_u S_{uv} - 2(\tau_u - \tau_v)S_{uv}\right)\omega_v, \quad (4.28c) \]

where \( \lambda \equiv \lambda_1 = -(1/2)\lambda_3 \) and \( S_{uv} \equiv S_{ab} v^a v^b \). The Ricci rotation coefficients are defined by equations (3.18), and the 1-jet variables \( \{ \omega_u, \omega_v, \omega_e \} \) are respectively defined as

\[ \omega_u \equiv \mathcal{E}_u \omega_u, \quad \omega_v \equiv \mathcal{E}_v \omega_v, \quad \omega_e \equiv \mathcal{E}_e \omega_e. \quad (4.29) \]

The second Bianchi identity \( \nabla_a R^a_{\beta} - (1/2)\nabla_b R = 0 \) puts the following constraints

\[ 0 = \lambda\kappa_e, \quad (4.30a) \]

\[ 0 = \mathcal{E}_u S_{uv} - 3\eta \lambda + (2\kappa_u - \kappa_v)S_{uv}, \quad (4.30b) \]

\[ 0 = \eta \kappa_v S_{uv} + 3\lambda(\tau_u + \tau_v). \quad (4.30c) \]

Suppose \( S_{uv} = 0 \). Then equations (4.25) and (4.26) imply that the basis \( \{ u^a, v^a, e^a \} \) satisfies

\[ S^\alpha_{a} u^b = \lambda u^b, \quad S^\alpha_{a} v^b = \lambda v^b, \quad S^\alpha_{a} e^b = -2\lambda e^b, \]

which contradicts the assumption that the Segre type of \( S^\alpha_{a} \) is [21]. It therefore follows that \( S_{uv} \neq 0 \). In the remaining parts of this subsection, we investigate the Segre types [(21)] and [21] separately.

4.2.1. Branch where the Segre type is [(21)]. In this branch, three eigenvalues of \( S^\alpha_{a} \) are coincident and then \( \lambda = \lambda_1 = \lambda_3 = 0 \) follows from traceless property. Then, equations (4.28) and (4.30) are combined to give

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\[ \bar{\omega}_v = - \left( \frac{\kappa_v}{2} \right) \omega_u - \left( L_v \varphi \right) \omega_v - \left( L_v \varphi - (\tau_v - \tau_e) \right) \omega_e, \quad (4.31a) \]
\[ \bar{\omega}_e = - \left( \tau_u - \tau_v \right) \omega_v - \kappa_v \omega_e, \quad (4.31b) \]

where \( \varphi \equiv (1/2) \log S_v \). Given these conditions (4.31), the Killing equation (1.1) and the identities \( \nabla_u \nabla_j \omega_u = \nabla_j \nabla_v \omega_v = \nabla_u \nabla_v \omega_e = 0 \) produce a PDE system of the form
\[ \nabla_u \omega = \Omega_u \omega, \quad \omega \equiv [\omega_u \omega_v \omega_e \bar{\omega}_u]^T, \quad (4.32) \]

where
\[
\Omega_u = \kappa_u \begin{bmatrix}
\kappa_v & 0 & \eta_v & 0 \\
-\frac{\kappa_v}{2} & -L_v \varphi & \tau_v - \tau_e - L_v \varphi & 0 \\
-\tau_v - \tau_e & -\eta_v & -\eta_v & -1 \\
\frac{\kappa_v (\kappa_v + \kappa_e)}{2} - \kappa_v (\tau_v - \tau_e) + \kappa_v \kappa_e & \eta_v (\tau_v - \tau_e - (\tau_v + \tau_e) (L_v \varphi - \kappa_v) & \eta_v + \eta_v (\tau_v - \tau_e - 2 \tau_e) - \kappa_v \eta_v + \eta_v (L_v \varphi - \kappa_v) & \eta_v (L_v \varphi - \kappa_v) + \eta_v \eta_v - \kappa_v (\tau_v - \tau_e) \kappa_v \\
0 & \kappa_v & 0 & 0 \\
0 & -\tau_v - \tau_e & 0 & 0 \\
\frac{\kappa_v (\kappa_v - 2 \kappa_e)}{2} + \tau_v^2 - \frac{\tau_v \tau_e}{2} - \kappa_v \eta_v - \kappa_v (\tau_v + \tau_e) & \eta_v (\tau_v + \tau_e - 2 \tau_e) - \kappa_v \eta_v - (L_v \varphi - \kappa_v) + \eta_v (L_v \varphi - \kappa_v) & \eta_v (L_v \varphi - \kappa_v) \eta_v - \kappa_v (\tau_v - \tau_e) \kappa_v \eta_v & \eta_v (L_v \varphi - \kappa_v) \eta_v - \kappa_v (\tau_v - \tau_e) \kappa_v \\
\end{bmatrix}
\]
(4.33)

Several parts of the compatibility condition for equation (4.32) lead to
\[ \sigma \bar{\omega}_u = - \left( e^{\varphi} L_v \varphi + \tau_v + \tau_e \right) \sigma \omega_u - \left( e^{\varphi} L_v \varphi + \eta_v \sigma \right) \omega_v - \left( e^{\varphi} L_v \varphi \right) \omega_e, \quad (4.34a) \]
\[ \kappa_v \bar{\omega}_u = - \kappa_v \left( \tau_v + \tau_e \right) \omega_u - \left( L_v \varphi + \eta_v \kappa_v \right) \omega_v - \left( L_v \tau_v \right) \omega_e, \quad (4.34b) \]

where
\[ \sigma \equiv L_v \varphi + \tau_v + \tau_e, \quad \Sigma \equiv L_v (e^{-\varphi}) + \kappa_v e^{-\varphi}. \quad (4.34c) \]

This implies that \( \bar{\omega}_u \) can be expressed in terms of \( \{ \omega_u, \omega_v, \omega_e \} \) except when \( \sigma = \kappa_v = 0 \). Depending on the nonzeroness of the coefficients \( \{ \sigma, \kappa_v \} \), the analysis falls into three sub-branches.

**Sub-branch where** \( \sigma = \kappa_v = 0 \). In this sub-branch, the 1-jet variable \( \bar{\omega}_u \) cannot be expressed in terms of \( \{ \omega_u, \omega_v, \omega_e \} \). The remaining parts of the compatibility condition for equation (4.32) read
\[
R_{(21)}[\omega] = 0, \quad R_{(21)}[\omega] = 0, \quad (4.35) \]

Remark that some remaining components are derivable from its derivative. In this sub-branch, the rank of \( R_{(21)} \) governs the number of KVs in the same way as that presented in section 4.1.1.

**Other sub-branches.** Except when \( \sigma = \kappa_v = 0 \), equations (4.34) allow us to write the 1-jet variable \( \bar{\omega}_u \) in terms of \( \{ \omega_u, \omega_v, \omega_e \} \). In these sub-branches, equation (4.32) reduces to a PDE system of the form
\[ \nabla u \omega = \Omega_u \omega , \quad \omega \equiv [\omega_u \omega_v \omega_e]^T. \]

Here, the compatibility condition of equation (4.36a) is considered collectively. The results are displayed as follows:

\((\#1)\) For the case of \(\sigma = 0, \kappa_e \neq 0\),
\[ \omega_u = - (\tau_v + \tau_e) \omega_u - \left( \frac{\xi_e \tau_v}{\kappa_e} + \eta_e \right) \omega_v - \left( \frac{\xi_e \tau_v}{\kappa_e} \right) \omega_e , \quad (4.36a) \]
\[ \Omega_u = u_u \left[ \begin{array}{ccc} \kappa_v & 0 & \eta_v \\ -\frac{\tau_v}{\kappa_v} - \xi_v \tau_v & \eta_v & -\xi_v \eta_v \\ 0 & \frac{\tau_v}{\kappa_v} - \eta_v & 0 \end{array} \right] + u_v \left[ \begin{array}{ccc} -\kappa_u + \frac{\eta_u}{\kappa_e} & \xi_u \phi - \kappa_v & \xi_u \phi + \tau_u + \tau_e \\ 0 & \kappa_u & 0 \\ 0 & -\tau_u + \tau_v & -\kappa_e \end{array} \right] \]
\[ + u_e \left[ \begin{array}{ccc} -\tau_v + \tau_e & -\eta_v - \xi_e \kappa_e & -\xi_e \kappa_e \\ 0 & \tau_v - \tau_e & 0 \\ \kappa_e & \eta_v & 0 \end{array} \right], \quad (4.36b) \]

\[ R^\#1_{([21])} = \left[ \begin{array}{ccc} \xi_v \tau_v & \xi_v \tau_v & \xi_v \tau_v \\ \xi_v \eta_v + \kappa_e \xi_v \phi & \xi_v \eta_v + \kappa_e \xi_v \phi & \xi_v \eta_v + \kappa_e \xi_v \phi \\ \xi_v \eta_v - \eta_e \xi_v \phi & \xi_v \eta_v - \eta_e \xi_v \phi & \xi_v \eta_v - \eta_e \xi_v \phi \end{array} \right]. \quad (4.36c) \]

\((\#2)\) For the case of \(\sigma \neq 0\),
\[ \omega_u = - \left( \frac{\epsilon^2 \xi_v \Sigma}{\sigma} + \tau_u + \tau_e \right) \omega_u - \left( \frac{\epsilon^2 \xi_v \Sigma}{\sigma} + \eta_v \right) \omega_v - \left( \frac{\epsilon^2 \xi_v \Sigma}{\sigma} \right) \omega_e , \quad (4.37a) \]
\[ \Omega_u = u_u \left[ \begin{array}{ccc} \kappa_v & 0 & \eta_v \\ -\frac{\tau_v}{\kappa_v} - \xi_v \tau_v & \eta_v & -\xi_v \eta_v \\ 0 & \frac{\tau_v}{\kappa_v} - \eta_v & 0 \end{array} \right] + u_v \left[ \begin{array}{ccc} -\kappa_u + \frac{\eta_u}{\kappa_e} & \xi_u \phi - \kappa_v & \xi_u \phi + \tau_u + \tau_e \\ 0 & \kappa_u & 0 \\ 0 & -\tau_u + \tau_v & -\kappa_e \end{array} \right] \]
\[ + u_e \left[ \begin{array}{ccc} -\tau_v + \tau_e & -\eta_v - \xi_e \kappa_e & -\xi_e \kappa_e \\ 0 & \tau_v - \tau_e & 0 \\ \kappa_e & \eta_v & 0 \end{array} \right], \quad (4.37b) \]

\[ R^\#2_{([21])} = \left[ \begin{array}{ccc} \xi_u \left( \sigma - \frac{5}{2} \tau_v \right) & \xi_v \left( \sigma - \frac{5}{2} \tau_v \right) & \xi_e \left( \sigma - \frac{5}{2} \tau_v \right) \\ \xi_u \tau_v & \xi_v \tau_v & \xi_e \tau_v \\ \xi_u \eta_v + \kappa_e \xi_u \phi & \xi_v \eta_v + \kappa_e \xi_v \phi & \xi_e \eta_v + \kappa_e \xi_e \phi \end{array} \right] \Phi_u \Phi_v \Phi_e , \quad (4.37c) \]

where
\[ \Phi_\alpha \equiv \xi_u \eta_v - 2 \eta_e \xi_v \phi + \frac{\epsilon^2 \xi_u \Sigma}{\sigma} \]
\[ - \frac{\epsilon^2 \left( \xi_u \phi \right)}{\sigma} \left( \xi_u \Sigma - \xi_v \sigma + \left( \xi_v \phi - \kappa_v - \eta_e \right) \sigma \right) \frac{\epsilon^2 \xi_u \Sigma}{\sigma^2} , \quad (4.37d) \]
\[ \Theta_\alpha \equiv \xi_u \eta_e - \eta_e \xi_u \phi - \frac{\epsilon^2 \xi_u \Sigma}{\sigma} + \left( \xi_u \Sigma + \xi_v \sigma + \left( \tau_u + \tau_v - \sigma \right) \sigma \right) \frac{\epsilon^2 \xi_u \Sigma}{\sigma^2} , \quad (4.37e) \]
In these sub-branches, the rank of \( R_{[(21)]}^1 \) and \( R_{[(21)]}^2 \) governs the number of KVs in the same way as that presented in section 4.1.1.

4.2.2. Branch where the Segre type is [21]. In this branch, it is assumed that \( \lambda = \lambda_1 = -(1/2)\lambda_3 \neq 0 \). Then, it immediately follows from equations (4.28) and (4.30) that

\[
\varpi_u = -(\tau_u + \tau_v)\omega_u - \eta_v \omega_v, \tag{4.38a}
\]

\[
\varpi_v = -\left(\xi_v \varphi + \kappa_u\right)\omega_u - \left(\xi_v \varphi\right)\omega_v - \left(\xi_v \varphi - \tau_v + \tau_u\right)\omega_e, \tag{4.38b}
\]

\[
\varpi_e = -(\tau_u - \tau_v)\omega_v, \tag{4.38c}
\]

and

\[
\kappa_v = 0, \quad \eta_u = -\left(\frac{3\lambda}{S_{vv}}\right)(\tau_u + \tau_v). \tag{4.38d}
\]

Given these conditions (4.38), the Killing equation (1.1) produces a PDE system of the form

\[
\nabla_a \omega = \Omega_a \omega, \quad \omega \equiv [\omega_u, \omega_v, \omega_e]^T, \tag{4.39}
\]

where

\[
\Omega_a \equiv u_a \begin{bmatrix}
\kappa_v & 0 & \eta_v \\
-\kappa_u - \xi_u \varphi & -\xi_v \varphi - \xi_v \varphi + \tau_v - \tau_e & 0 \\
\tau_u - \tau_v & 0 & -\eta_e \\
\frac{3\lambda}{S_{vv}}(\tau_u + \tau_v) - \tau_v & -\eta_e & 0 \\
0 & \eta_e & 0
\end{bmatrix} + v_u \begin{bmatrix}
\xi_u \varphi & \xi_v \varphi - \kappa_v & \xi_v \varphi + \tau_u + \tau_e \\
\kappa_u & 0 & \frac{-3\lambda}{S_{vv}}(\tau_u + \tau_v) \\
0 & -\tau_u + \tau_v & 0 \\
0 & \frac{3\lambda}{S_{vv}}(\tau_u + \tau_v) - \tau_v & \eta_e \\
0 & 0 & \eta_e
\end{bmatrix} + e_a \begin{bmatrix}
\xi_u \tau_u & \xi_v \tau_u & \xi_e \tau_u \\
\xi_u \tau_v & \xi_v \tau_v & \xi_e \tau_v \\
\xi_u \sigma & \xi_v \sigma & \xi_e \sigma \\
\xi_u \eta_v - 2\eta_v \xi_u \varphi & \xi_v \eta_v - 2\eta_v \xi_v \varphi & \xi_e \eta_v - 2\eta_v \xi_e \varphi \\
\xi_u \eta_e - \eta_e \xi_u \varphi & \xi_v \eta_e - \eta_e \xi_v \varphi & \xi_e \eta_e - \eta_e \xi_e \varphi
\end{bmatrix}, \tag{4.40}
\]

with \( \varphi \equiv (1/2) \log S_{vv} \). The compatibility condition for equation (4.39) leads to

\[
R_{[21]} = \begin{bmatrix}
\xi_u \tau_u & \xi_v \tau_u & \xi_e \tau_u \\
\xi_u \tau_v & \xi_v \tau_v & \xi_e \tau_v \\
\xi_u \sigma & \xi_v \sigma & \xi_e \sigma \\
\xi_u \eta_v - 2\eta_v \xi_u \varphi & \xi_v \eta_v - 2\eta_v \xi_v \varphi & \xi_e \eta_v - 2\eta_v \xi_e \varphi \\
\xi_u \eta_e - \eta_e \xi_u \varphi & \xi_v \eta_e - \eta_e \xi_v \varphi & \xi_e \eta_e - \eta_e \xi_e \varphi
\end{bmatrix}, \tag{4.41a}
\]

where

\[
\Sigma \equiv \xi_e (e^{-\varphi}) + \kappa_v e^{-\varphi}. \tag{4.41b}
\]

In this sub-branch, the rank of \( R_{[21]} \) governs the number of KVs in the same way as that presented in section 4.1.1.
4.2.3. Short summary of class 3 type [21]. We synopsise the results obtained in this subsection in figures 11 and 12.

4.3. Type [3]

In this case, we have the following Jordan chain:

\[ S^a_b f_1^b = \lambda f_1^a, \]  
\[ S^a_b f_2^b = \lambda f_2^a + f_1^a, \]  
\[ S^b_b f_3^a = \lambda f_3^a + f_1^a, \]

with \( \lambda = 0 \). Here \( f_1^a \) is an generalised eigenvector of \( S^a_b \). It can be shown that the vectors \( f_1^a, f_2^a \) are respectively null and spacelike, whereas the causal nature of \( f_3^a \) is free from restriction. In this subsection, it is assumed that \( \{ u^a, v^a, e^a \} \) forms a double-null basis of \( T(M) \),

\[ g^{ab} = u^a v^b + v^a u^b + e^a e^b, \]

where \( u^a \equiv f_1^a \); \( e^a \equiv f_3^a \); \( v^a \) is defined as a null vector such that \( g_{ab} u^a v^b = 1 \) and \( g_{ab} v^a e^b = 0 \). Then, any KV can be written as

\[ K^a = \omega_u u^a + \omega_v v^a + \omega_e e^a, \]

where scalars \( \{ \omega_u, \omega_v, \omega_e \} \) are yet indeterminate.

Calculating the second Bianchi identity, \( \nabla_a R_{bc} - (1/2) \nabla_b R = 0 \), leads to

\[ \eta_u = 0, \quad \kappa_u = 2\kappa_e, \quad \xi_a S_{av} + 3\kappa_e S_{vv} = 2\tau_u + \tau_v - \tau_e, \]

where \( S_{vv} \equiv S_{ab} v^a v^b \) and the Ricci rotation coefficients are defined by equations (3.18). Note that \( S_{ab} v^a e^b = 1 \) by equations (4.42) and (4.43). Using this identity, the first compatibility of equation (1.1), \( \xi_a S_{ab} = 0 \), can be written in components

\[ \varpi_u = \frac{1}{2} (\tau_v - 3\tau_e - 3\kappa_e S_{vv}) \omega_u + \frac{1}{2} (\xi_a S_{av} - 2\eta_v) \omega_u + \frac{1}{2} (\xi_a S_{av}) \omega_e, \]
\[ \varpi_v = -2\kappa_e \omega_u + (\tau_v - \tau_e) \omega_e, \]
\[ \varpi_e = - (\tau_u - 3\tau_e) \omega_v - \kappa_e \omega_e, \]

where the 1-jet variables \( \{ \varpi_u, \varpi_v, \varpi_e \} \) are respectively defined as

\[ \varpi_u \equiv \xi_a \omega_u, \quad \varpi_v \equiv \xi_a \omega_v, \quad \varpi_e \equiv \xi_a \omega_e. \]

Then, the Killing equation (1.1) produces a PDE system of the form

\[ \nabla_a \omega = \Omega_a \omega, \quad \omega \equiv [\omega_u, \omega_v, \omega_e]^T, \]

where

\[ \Omega_a \equiv u_a \left[ \begin{array}{ccc} \kappa_v & 0 & -\eta_v \\ \frac{3\kappa_v - 3\kappa_e S_{vv}}{2} & \frac{\xi_a S_{av}}{2} - \frac{\xi_a S_{av}}{2} - \eta_v & \frac{\xi_a S_{av}}{2} \\ \frac{\tau_u - 3\tau_e - 3\kappa_e S_{vv}}{2} & \frac{\xi_a S_{av}}{2} & -\tau_u + \tau_e \end{array} \right] + v_a \left[ \begin{array}{ccc} 0 & -\kappa_v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\kappa_v + 2\kappa_e \end{array} \right]. \]
The compatibility condition for equation (4.48) leads to

\[
R_3 = \begin{pmatrix}
\xi_{u\kappa_e} & \xi_{v\kappa_e} & \xi_{e\kappa_e} \\
\xi_\kappa (\tau_e - 3\tau_v) & \xi_\kappa (\tau_e - 3\tau_v) & \xi_\kappa (\tau_e - 3\tau_v) \\
\xi_\kappa (\kappa_e S_{vv} + 2\tau_v) & \xi_\kappa (\kappa_e S_{vv} + 2\tau_v) & \xi_\kappa (\kappa_e S_{vv} + 2\tau_v) \\
\xi_\kappa (\tau_v + \frac{7}{2}\tau_e) & \xi_\kappa (\tau_v + \frac{7}{2}\tau_e) & \xi_\kappa (\tau_v + \frac{7}{2}\tau_e) \\
\Xi_u & \Xi_v & \Xi_e \\
\Theta_u & \Theta_v & \Theta_e
\end{pmatrix},
\]

(4.50a)

where

\[
\Xi_\alpha \equiv \xi_\alpha \eta_e + \xi_\alpha \xi_e S_{vv} \xi_4 + \left( \tau_v + \tau_e - 3\kappa_e S_{vv} \right) \frac{\xi_\alpha S_{vv}}{4},
\]

(4.50b)

\[
\Theta_\alpha \equiv \xi_\alpha \eta_v - \xi_\alpha \xi_v S_{vv} \xi_4 + \left( \xi_e S_{vv} + 2\kappa_v + 2\eta_e \right) \frac{\xi_\alpha S_{vv}}{4}.
\]

(4.50c)

In this sub-branch, the rank of \( R_3 \) governs the number of KVs in the way shown in figure 13.

4.4. Type \([zz1]\]

In this case, we have the following Jordan chains:

\[
S^a_{b+} j^b_+ = \lambda_+ j^b_+,
\]

(4.51a)

\[
S^a_{b-} j^b_- = \lambda_- j^b_-,
\]

(4.51b)

\[
S^a_{b} j^b = \lambda j^a,
\]

(4.51c)

where \( \lambda_\pm = \alpha \pm i\beta (\beta \neq 0) \) are complex eigenvalues corresponding to the complex eigenvectors \( j^b_\pm = x^a \pm iy^a \). It follows from the symmetric traceless property of \( S_{ab} \) that \( 2\alpha + \lambda = 0 \) and \( g_{ab} (x^a x^b + y^a y^b) = 0 \). On the other hand, the real/imaginary parts of equations (4.51) give

\[
S^a_{b} x^b = \alpha x^a - \beta y^a,
\]

(4.51d)

\[
S^a_{b} y^b = \beta x^a + \alpha y^a.
\]

(4.51e)

This implies that the real vectors \( \{x^a, y^a\} \) span a timelike surface. One can then fix \( x^a \) to be timelike and \( y^a \) to be spacelike without loss of generality. In this subsection, it is supposed that \( \{e^a_\alpha\} \) forms an orthonormal basis of \( T(M) \),

\[
g^{ab} = - e^a_1 e^b_1 + e^a_2 e^b_2 + e^a_3 e^b_3,
\]

(4.52)

where \( e^a_1 \propto x^a \), \( e^a_2 \propto y^a \) and \( e^a_3 \propto j^a \). Then, any KV can be written as

\[
K^a = \sum_{\alpha=1}^{3} \omega_\alpha e^a_\alpha,
\]

(4.53)

where scalars \( \{\omega_\alpha\} \) are yet indeterminate.

The first compatibility condition, \( \xi_\kappa S_{ab} = 0 \), of equation (1.1), gives rise to

\[
\omega_1 = \eta_1 \omega_1 + (\tau_2 - \tau_3) \omega_2,
\]

(4.54a)
where the Ricci rotation coefficients are defined by equations (3.5). The 1-jet variables \( \{ \varpi_u, \varpi_v, \varpi_e \} \) are respectively defined as
\[
\varpi_1 \equiv \mathcal{E}_3 \omega_1, \quad \varpi_2 \equiv \mathcal{E}_1 \omega_2, \quad \varpi_3 \equiv \mathcal{E}_2 \omega_3.
\] (4.55)

Then, the Killing equation (1.1) produces a PDE system of the form
\[
\nabla_a \omega = \Omega_a \omega, \quad \omega \equiv \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T,
\] (4.56)

where
\[
\Omega_a \equiv -e_{ia} \begin{bmatrix} 0 & -\kappa_1 & -\eta_1 \\ -\kappa_2 & 0 & \tau_1 - \tau_2 \\ \tau_1 - \tau_2 & 0 & -\eta_3 \end{bmatrix} + e_{ia} \begin{bmatrix} \kappa_1 & 0 & \tau_3 - \tau_2 \\ 0 & \kappa_2 & \eta_2 \\ \tau_1 - \tau_2 & 0 & -\eta_3 \end{bmatrix} + e_{ia} \begin{bmatrix} -\eta_1 & \tau_2 - \tau_3 & 0 \\ \eta_1 & 0 & \tau_2 - \tau_3 \\ \eta_1 & \eta_2 & 0 \end{bmatrix}.
\] (4.57)

The compatibility condition for equation (4.56) leads to
\[
R_{[z]1} = \begin{bmatrix} \mathcal{E}_1 \kappa_3 & \mathcal{E}_2 \kappa_3 & \mathcal{E}_3 \kappa_3 \\ \mathcal{E}_1 \eta_1 & \mathcal{E}_2 \eta_1 & \mathcal{E}_3 \eta_1 \\ \mathcal{E}_1 \eta_2 & \mathcal{E}_2 \eta_2 & \mathcal{E}_3 \eta_2 \\ \mathcal{E}_1 \eta_3 & \mathcal{E}_2 \eta_3 & \mathcal{E}_3 \eta_3 \\ \mathcal{E}_1 \tau_2 & \mathcal{E}_2 \tau_2 & \mathcal{E}_3 \tau_2 \\ \mathcal{E}_1 \tau_3 & \mathcal{E}_2 \tau_3 & \mathcal{E}_3 \tau_3 \end{bmatrix}. \] (4.59)

In this sub-branch, the rank of \( R_{[z]1} \) governs the number of KVs in the way shown in figure 14.

5. Application

In this section, a couple of examples is provided to illustrate how useful our prescription is. First, we shall consider the Lifshitz spacetime (see e.g. [18] and references therein) in section 5.1, whose metric has an arbitrary constant \( z \). Afterwards, we deal with a pp-wave spacetime in section 5.2, which serves as a typical example of vanishing scalar invariant spaces and is characterised by a single function \( h \). As we will see below, a complete classification of their local isometry groups depends respectively on the values of \( z \) and the profile of \( h \). This demonstrates the power of the present formulation.

5.1. The Lifshitz spacetime

In condensed matter systems, many phase transitions are governed by the fixed points admitting the anisotropic dynamical scaling. In light of holography, a great deal of attention has been recently focused on the gravity dual with this dynamical scaling, which is modelled by the Lifshitz metric [18]. The three-dimensional Lifshitz metric reads

\[
\varpi_2 = -\kappa_2 \omega_2 + (\tau_1 + \tau_3) \omega_3, \quad (4.54b)
\]
\[
\varpi_3 = (\tau_1 - \tau_2) \omega_1 - \eta_3 \omega_3, \quad (4.54c)
\]

where the Ricci rotation coefficients are defined by equations (3.5). Note that the second Bianchi identity imposes
\[
\eta_3 - 2\kappa_1 = \frac{3\alpha}{\beta} \kappa_3, \quad \kappa_3 + 2\kappa_2 = -\frac{3\alpha}{\beta} \eta_3, \quad \tau_1 + \tau_2 = \frac{3\alpha}{\beta} (\eta_1 - \eta_2). \] (4.58)

The compatibility condition for equation (4.56) leads to
\[
R_{[z]1} = \begin{bmatrix} \mathcal{E}_1 \kappa_3 & \mathcal{E}_2 \kappa_3 & \mathcal{E}_3 \kappa_3 \\ \mathcal{E}_1 \eta_1 & \mathcal{E}_2 \eta_1 & \mathcal{E}_3 \eta_1 \\ \mathcal{E}_1 \eta_2 & \mathcal{E}_2 \eta_2 & \mathcal{E}_3 \eta_2 \\ \mathcal{E}_1 \eta_3 & \mathcal{E}_2 \eta_3 & \mathcal{E}_3 \eta_3 \\ \mathcal{E}_1 \tau_2 & \mathcal{E}_2 \tau_2 & \mathcal{E}_3 \tau_2 \\ \mathcal{E}_1 \tau_3 & \mathcal{E}_2 \tau_3 & \mathcal{E}_3 \tau_3 \end{bmatrix}. \] (4.59)

In this sub-branch, the rank of \( R_{[z]1} \) governs the number of KVs in the way shown in figure 14.
where \( z \) is an arbitrary constant corresponding to the dynamical exponent and \( L \) is related to the curvature scale. For this metric, the scalar curvature \( R \) and the principal traces of of powers of the traceless Ricci operator \( S^\mu_\nu \) are all constants, which evaluate to

\[
R = -\frac{2(z^2 + z + 1)}{L^2}, \quad S^{(2)} = \frac{2(z - 1)(z^2 + z + 1)}{3L^2}, \quad S^{(3)} = \frac{(z - 1)^4(2z^2 + 5z + 2)}{9L^6}
\]

As the metric (5.1) belongs to the class 3, the criteria in figure 7 must be checked. After simple calculations, we find that \( S^\mu_\nu = 0 \) if \( z = 1 \). It then follows from the result in figure 7 that 6 KVs exist and the AdS metric is recovered. By solving the Killing equation (1.1), their explicit form can be read as

\[
\partial_t, \quad \partial_x, \quad x\partial_t + t\partial_x, \quad t\partial_t - r\partial_r + x\partial_x,
\]

\[
\frac{L^4 + r^2(t^2 + x^2)}{2r^2} \partial_t - rt \partial_t + tx \partial_x, \quad tx\partial_t - rx\partial_r - \frac{L^4 - r^2(t^2 + x^2)}{2r^2} \partial_x.
\]

If \( z = 0 \), the Segre type of \( S^\mu_\nu \) is \([1, (11)]\). Consequently, the number of KVs can be computed by the algorithm described in figure 8 and it equals 4. One sees that the metric culminates in \( \mathbb{R} \times H^2 \). Once again, solving equation (1.1) gives their explicit form

\[
\partial_t, \quad \partial_x, \quad r\partial_t - x\partial_x, \quad rx\partial_t + \frac{L^4 - r^2 x^2}{2r^2} \partial_x.
\]

If \( z \neq 0, 1 \), the Segre type of \( S^\mu_\nu \) is either \([1, 111]\) for \( z = -1 \) or \([1, 11]\) for \( z \neq -1 \). Then the number of KVs can be computed by the algorithm described in figure 9 or 10. In either case, there are 3 KVs in the form

\[
\partial_t, \quad \partial_x, \quad tz\partial_t - r\partial_r + x\partial_x.
\]

The last one captures the anisotropic scaling \( t \rightarrow \lambda^x t, r \rightarrow \lambda^{-1} r, x \rightarrow \lambda x \). This completes a classification of the metric (5.1) based on their level of symmetry, which is summarised in figure 15.

### 5.2. The pp-wave spacetime

For Lorentzian manifolds, a natural question to ask is whether our theorem works for the metric with vanishing scalar invariant (VSI) property. Here we show that it does.

A VSI spacetime is a Lorentzian manifold \( M \) in which scalar Weyl invariants of any order vanish identically, yet the Riemann–Christoffel tensor \( R_{abcd} \) is nonvanishing. Note that scalar Weyl invariants (or polynomial curvature invariants) of order \( p \) are scalars on \( M \) obtained from the first \( p \) covariant derivatives of the Riemann–Christoffel tensor \( \nabla_a \cdots \nabla_d R_{bcd} \) by tensor products and complete contractions [19]. There are nontrivial spacetimes with a VSI property which have received some attention in the context of general relativity, see e.g. [20–22].

As a classical example of VSI spacetimes, we deal with a pp-wave spacetime which admits a covariantly constant null Killing vector \( V^a \) satisfying \( \nabla_a V^b = 0, V_a V^a = 0 \). In dimension 3, the general form of the pp-wave metric takes the following form

\[
g_{pp} = h(u, x) du^2 + 2dudv + dx^2,
\]

where \( h \) is a positive function.
where \( h \) is a function of \( u \) and \( x \). It is obvious that the covariantly constant null vector is given by \( V = K_1 = \partial_v \). Our aim here is to obtain a complete classification of KVs of \( g_{ab} \) based on the type of the function \( h \).

By the definition of VSI spacetimes, the metric (5.6) belongs to the class 3. After simple calculations, we find that the Segre type of the traceless Ricci operator \( S^a_b \) depends on whether the function \( h \) satisfies a PDE \( h_{xx} = \frac{\partial^2 h}{\partial x^2} = 0 \) or not. If \( h_{xx} = 0 \) holds, the Segre type is \([1, 11]\) and \( h \) takes the form

\[
h(u, x) = h_0(u) + 2x h_1(u),
\]

where \( h_0 \) and \( h_1 \) are arbitrary functions of \( u \). Consequently it follows from the result in figure 7 that 6 KVs exist and the spacetime is locally reduced to the Minkowski \( \mathbb{R}^1 \). By solving the Killing equation (1.1), the explicit expressions of the set of KVs can be written as

\[
\begin{align*}
K_1 &= \partial_v, \\
K_2 &= H_1 \partial_v - \partial_x, \\
K_3 &= (x + uH_1 - \mathcal{H}_1) \partial_v - u\partial_x, \\
K_4 &= \partial_u - \frac{1}{2} (h + H_1^2) \partial_x + H_1 \partial_x, \\
K_5 &= u\partial_u - \frac{1}{2} (2v + uh + H + uH_1^2 - \int du H_1^2) \partial_x + uH_1 \partial_x, \\
K_6 &= (x - \mathcal{H}_1) \partial_u + \frac{1}{2} \left( (x + \mathcal{H}_1)H_1^2 - (x - \mathcal{H}_1)h + 2vH_1 + H_0H_1 - H_1 \int du H_1^2 \right) \partial_x - \frac{1}{2} \left( H_0 + 2H_1^2 - 2v \right) \partial_x,
\end{align*}
\]

where

\[
\begin{align*}
H(u, x) &\equiv \int du h(u, x), & H_0(u) &\equiv \int du h_0(u), \\
H_1(u) &\equiv \int du h_1(u), & \mathcal{H}_1(u) &\equiv \int du \mathcal{H}_1(u).
\end{align*}
\]

The nonvanishing commutation relations for these KVs are

\[
\begin{align*}
[K_1, K_3] &= - K_1, & [K_1, K_6] &= K_2, & [K_2, K_3] &= - K_1, \quad (5.9a) \\
[K_2, K_6] &= - K_2, & [K_3, K_4] &= - K_2, & [K_3, K_5] &= - K_3, \quad (5.9b) \\
[K_3, K_6] &= - K_3, & [K_4, K_5] &= K_4, & [K_5, K_6] &= - K_6. \quad (5.9c)
\end{align*}
\]

These KVs constitute the 3-dimensional Poincaré algebra.

If \( h_{xx} = 0 \) fails to be fulfilled, the Segre type of \( S^a_b \) is \([21]\) with the invariants \( \{ \sigma, \rho \} = \{ \frac{\partial}{\partial u}, 0 \} \) (see figure 11 and equation (4.34) for notations). The number of KVs is controlled by either rank\( R_{[21]} \) if \( h_{xxx} \neq 0 \) or rank\( R_{[[21]]} \) if \( h_{xxx} = 0 \). In either case, any KV can be identified at the outset \( K^a = \omega_0 u^a + \omega_1 v^a + \omega_2 e^a \), where \( \{ u^a, v^a, e^a \} \) is the double-null basis defined as...
\[ u^\alpha \equiv (\partial_\nu)^\alpha, \quad \nu^\alpha \equiv (\partial_\nu)^\alpha - h^2(\partial_\nu)^\alpha, \quad e^\alpha \equiv (\partial_\nu)^\alpha, \] (5.10)

whose nonvanishing rotation coefficient consists only of \( \eta_\nu = -h_\nu/2 \).

5.2.1 The case \( \sigma \propto h_{xxx} = 0 \). The solution to the equation \( h_{xxx} = 0 \) leads to the general form of \( h \) as

\[ h(u, x) = h_0(u) + 2x h_1(u) + x^2 h_2(u), \] (5.11)

where \( \{h_0, h_1, h_2\} \) are arbitrary functions of \( u \). By combining equations (5.11) and (5.10), the obstruction matrix \( R_{([21])} \) acting on \( \omega = [\omega_\nu, \omega_\tau, \omega_\tau^u, \omega_\tau] \) reads

\[ R_{([21])} = \begin{bmatrix} 0 & e^\tau_1 \Sigma & e^\tau_1 \Sigma & 0 \\ e^\tau_1 \Sigma & e^\tau_1 \Sigma & e^\tau_1 \Sigma & 0 \end{bmatrix} \propto \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (h_{xxu}/(h_{xx})^{3/2})_u & 0 & 0 \end{bmatrix}. \] (5.12)

Thereby \( \text{rank}R_{([21])} = 0 \) if \( h_2 \) solves the ODE \( (h_{xxu}/(h_{xx})^{3/2})_u = (h_{2uu}/h_{2}^{3/2})_u = 0 \) whose solution is given by \( h_2(u) = 1/(c_1 u + c_2)^2 \), where \( \{c_1, c_2\} \) are constants. Notice that a coordinate shift \( u \to u + c_0 \) (\( c_0 \) const.) allows us to classify the solution as either \( h_2(u) = \text{const.} \) for \( c_1 = 0 \) or \( h_2(u) \propto u^{-2} \) for \( c_1 \neq 0 \).

For the former case \( h_2(u) = c = \text{const.} \neq 0 \), there are 4 KVs in the form

\[ K_1 = \partial_\nu, \] (5.13a)

\[ K_2 = \left( \sqrt{c} x e^{-\sqrt{c}u} - \langle h_1 \rangle^- \right) \partial_\nu + e^{-\sqrt{c}u} \partial_\nu, \] (5.13b)

\[ K_3 = \left( \sqrt{c} x e^{\sqrt{c}u} + \langle h_1 \rangle^+ \right) \partial_\nu - e^{\sqrt{c}u} \partial_\nu, \] (5.13c)

\[ K_4 = 2\partial_\nu - \left( h_0 + 2x h_1 + \sqrt{c} x \left( e^{\sqrt{c}u} \langle h_1 \rangle^- - e^{-\sqrt{c}u} \langle h_1 \rangle^+ \right) + \langle h_1 \rangle^- \langle h_1 \rangle^+ \right) \partial_\nu + \left( e^{\sqrt{c}u} \langle h_1 \rangle^- + e^{-\sqrt{c}u} \langle h_1 \rangle^+ \right) \partial_\nu, \] (5.13d)

where we have assumed \( c > 0 \) and defined

\[ \langle h_1 \rangle_{\pm} \equiv \int du e^{\pm\sqrt{c}u} h_1. \] (5.13e)

The nonzero commutators for these KVs are

\[ [K_2, K_3] = 2\sqrt{c} K_1, \quad [K_2, K_4] = 2\sqrt{c} K_2, \quad [K_3, K_4] = -2\sqrt{c} K_3. \] (5.13f)

These correspond to the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra. One can deduce the explicit expressions of KVs also for the \( c < 0 \) case.

For the latter case \( h_2(u) = c u^{-2}(c = \text{const.} \neq 0) \), 4 KVs exist in the form

\[ K_1 = \partial_\nu, \] (5.14a)

\[ K_2 = \left( 2c x u^{-\frac{1+k}{2}} - (1 + k) \langle h_1 \rangle_{1-k} \right) \partial_\nu + (1 + k) a_{1-k} u^{\frac{1-k}{2}} \partial_\nu, \] (5.14b)

\[ K_3 = \left( 2c x u^{-\frac{1+k}{2}} - (1 - k) \langle h_1 \rangle_{1+k} \right) \partial_\nu + (1 - k) a_{1+k} u^{\frac{1+k}{2}} \partial_\nu, \] (5.14c)
\[ K_4 = 2ku\partial_u - \left( 2kv + x\left( 2kah_1 + \frac{1}{2}(1 + k)^2u^{1/4}\langle h_1 \rangle_{1-k} - \frac{1}{2}(1 - k)^2u^{-1/4}\langle h_1 \rangle_{1+k} \right) \right. \]
\[ + k\left( ab_0 + H_0 \right) + (1 + k)\int du u^{1/4}\langle h_1 \rangle_{1-k} - (1 - k)\int du u^{-1/4}\langle h_1 \rangle_{1+k} \right) \partial_u \]
\[ + \left( (1 + k)u^{1/4}\langle h_1 \rangle_{1-k} - (1 - k)u^{-1/4}\langle h_1 \rangle_{1+k} \right) \partial_x, \]  
(5.14d)

where we have used abbreviations
\[ k \equiv \sqrt{1 + 4c}, \quad H_0 \equiv \int du h_0, \quad \langle h_1 \rangle_p \equiv \int du u^p h_1. \]  
(5.14e)

In the above expressions, we have tentatively assumed \( c > -1/4 \). The commutation relations are given by
\[ [K_1, K_4] = -2kK_1, \quad [K_2, K_4] = -k(1 - k)K_2, \]
\[ [K_2, K_3] = 4ckK_1, \quad [K_3, K_4] = -k(1 + k)K_3. \]
(5.15a, b)

The KVs for the \( c = -1/4 \) and \( c < -1/4 \) cases can be obtained in a similar fashion, but we shall not attempt to do this here.

If \( \sigma \propto h_{xxx} = 0 \) but \( \Sigma_\alpha \propto (h_{xxx}/(h_{xx}))^{3/2} \neq 0 \), it follows from equation (5.12) that \( \text{rank} \mathbf{R}_{[21]} \propto \omega_h \) has to be zero. As any KV takes the form \( K^a = \omega_0 u^2 + \omega_1 e^2 \), the results in section 3.2 are reusable. Since all the spin coefficients are vanishing except for \( \eta_h \), \( \text{rank} \mathbf{R}_{ck2} \) governs the number of KVs (see figure 6). For the function \( h \) in the form of equation (5.11), one can verify that \( \text{rank} \mathbf{R}_{ck2} \) is always zero. By solving equation (1.1) directly, it can be ascertained that 3 KVs exist in the form
\[ K_1 = \partial_x, \quad K_\pm = -\left( \int du h_1 \phi_{\pm} + x \int du h_2 \phi_{\pm} \right) \partial_u + \phi_{\pm} \partial_x, \]
(5.16)

where \( \phi_{\pm}(u) \) are the two linearly independent solutions to the following ODE
\[ \phi_{\pm,uu} = h_2 \phi. \]  
(5.17)

By the conservation of Wronskian \( \phi_{+},u\phi_{-} - \phi_{+}\phi_{-},u = \text{const.} \equiv W \), the only nonvanishing commutator is
\[ [K_+, K_-] = WK_1. \]  
(5.18)

5.2.2. The case \( \sigma \propto h_{xxx} \neq 0 \). For the case in question, the obstruction matrix \( \mathbf{R}_{[21]}^{\#2} \) controls the number of KVs primarily (see figure 11). As a strategy for the classification, we focus on the first row of \( \mathbf{R}_{[21]}^{\#2} \)
\[ \left[ \mathcal{L}_u (\sigma - \frac{3}{2} \tau_1), \mathcal{L}_v (\sigma - \frac{3}{2} \tau_1), \mathcal{L}_w (\sigma - \frac{3}{2} \tau_1) \right] \propto \begin{bmatrix} 0 & (h_{xxx}/h_{xx})_u & (h_{xxx}/h_{xx})_v \end{bmatrix}. \]  
(5.19)

It can be shown that if \( (h_{xxx}/h_{xx})_u = (h_{xxx}/h_{xx})_v = 0 \) all entries of \( \mathbf{R}_{[21]}^{\#2} \) are zero except for the fifth row
\[ \begin{bmatrix} \Phi_u & \Phi_v & \Phi_w \end{bmatrix} \propto \begin{bmatrix} 0 & \zeta_{hx} & \zeta_{hx} \end{bmatrix}, \]  
(5.20)
where \( \varsigma \equiv (h_{xxx}/h_{xx})_u - (h_{xx}/2)(h_x/h_{xx})_x \). The obstruction elements for \( \text{rank } R^{\#2}_{[[21]]} = 0 \) are therefore given by

\[
\left( \frac{h_{xxx}}{h_{xx}} \right)_x, \quad \left( \frac{h_{xxx}}{h_{xx}} \right)_u,
\]

and collaterally

\[
\left( \frac{h_{xxx}}{h_{xx}} \right)_u - \frac{h_{xx}}{2} \left( \frac{h_x}{h_{xx}} \right)_x.
\]

If these criteria (5.21) are all vanishing, \( \text{rank } R^{\#2}_{[[21]]} = 0 \) and we can parameterise \( h \) by a nonzero function \( h_1 \) as

\[
h(u, x) = h_0(u) + e^{c_1(x+h_1(u))} - 2xh_{1,uu}(u),
\]

where \( h_0 \) is an arbitrary functions of \( u \) and \( c_1 \) is a nonzero constant, thereby allowing us to obtain 3 KVs

\[
K_1 = \partial_v,
\]

\[
K_2 = \partial_u - \frac{1}{2}(h_0 + h_{1,uu} - 2xh_{1,uu})\partial_x - h_{1,u}\partial_x,
\]

\[
K_3 = u\partial_u - \left(v + \frac{uh_0 + \int du h_0}{2} - uh_{1,uu} + \frac{uh_{1,uu} - \int du h_{1,uu}^2}{2} + \left( \frac{2}{c_1} - x \right)h_{1,u} \right)\partial_v - \left( \frac{2}{c_1} + uh_{1,u} \right)\partial_x,
\]

together with their commutators

\[
[K_1, K_2] = 0, \quad [K_2, K_3] = K_2, \quad [K_3, K_1] = K_1.
\]

Let us next consider the case in which \( \text{rank } R^{\#2}_{[[21]]} \neq 0 \). Since \( \text{rank } R^{\#2}_{[[21]]} = 2 \) implies that there exists a single KV \( \partial_\iota \), we shall concentrate on the case \( \text{rank } R^{\#2}_{[[21]]} = 1 \). As a result, the conditions for which the metric (5.6) admits 2 KVs are identified as shown in figure 16. This will be achieved by separating our analysis into the four types (A, B, C, D) based on the nonzeroness of equation (5.21). We shall fix the explicit forms of \( h(u, x) \), the 2nd KV and its commutator in the rest of this subsection.

**Type A.** In this type, the function \( h \) solves the PDEs \( (h_{xxx}/h_{xx})_u = 0 \) and \( (h_{xxx}/h_{xx})_u = 0 \) simultaneously, whereas \( \varsigma = (h_{xxx}/h_{xx})_u - (h_{xx}/2)(h_x/h_{xx})_x \neq 0 \). Thus we have

\[
\text{class 2}
\]

- Either of two annihilators of \( R_u \) is null

\[
\text{class 2 null}
\]

\[
\text{class 2 non-null}
\]

Figure 4. The sub-algorithm for the class 2. For details, see the beginning of this section.
\[ h(u,x) = h_0(u) + e^{c_1(x+h_1(u))} - 2x (h_{1,uu}(u) - h_2(u)), \quad \text{(5.27)} \]

where \( \{h_0, h_1, h_2\} \) are functions of \( u \) and \( c_1 \) is a constant. \( h_2 \) and \( c_1 \) are nonvanishing. From equation (5.20), \( R_{[21]}^\# \omega = 0 \) imposes
\[
\omega_e = \gamma \omega_f, \quad \gamma \equiv -\frac{\varsigma h_{xxx} - \varsigma_u h_{xx}}{\varsigma h_{xxx}} = \frac{h_{2,uu} - c_1 h_2 h_{1,uu}}{c_1 h_2}. \quad \text{(5.28)}
\]

In a nod to equation (5.28) and the basis (5.10), we take an orthonormal basis \( \{e_1^a, e_2^a, e_3^a\} \) as
\[
e_1^a \equiv -\gamma u^{\prime} + e^a, \quad e_2^a \equiv \gamma^{-1} v^{\prime} + e^a, \quad e_3^a \equiv \gamma u^{\prime} - \gamma^{-1} v^{\prime} - e^a, \quad \text{for} \quad \gamma \neq 0, \quad \text{(5.29)}
\]
\[
e_1^a \equiv e^a, \quad e_2^a \equiv \frac{1}{\sqrt{2}} (u^{\prime} + v^{\prime}), \quad e_3^a \equiv \frac{1}{\sqrt{2}} (u^{\prime} - v^{\prime}), \quad \text{for} \quad \gamma = 0, \quad \text{(5.30)}
\]

whence any KV is reduced to the form \( K^a = \omega_2 e_1^a + \omega_3 e_3^a \). In either case, the basis satisfies \( g^{ab} = e_1^a e_1^b + e_2^a e_2^b - e_3^a e_3^b \) and \( \tau_2 = \tau_3 \) with \( \tau_2 \neq 0 \). Therefore from figure 5 rank \( R_{\text{ch}2}^\# \) determines the existence of the 2nd KV.

It is simple to see that the third, fourth and fifth rows of \( R_{\text{ch}2}^\# \) are left nonvanishing, yielding the condition to have rank \( R_{\text{ch}2}^\# = 0 \) as \( (h_{2,uu}/h_{2,xxx})_{,u} = 0 \). The solution to the ODE categorises into \( h_2 = \text{const.} \) or \( h_2 \propto u^{-2} \). For \( h_2 = c_2 = \text{const.} \), the 2nd KV arises in the form
\[
K_2 = \partial_u - \frac{1}{2} (h_0 + h_{1,uu} - 2xh_{1,uu} - 2c_2h_1) \partial_x - h_{1,uu} \partial_x, \quad \text{(5.31a)}
\]
with the commutator \( [K_1, K_2] = 0 \). For \( h_2 = c_2 u^{-2} (c_2 = \text{const.} \neq 0) \), the 2nd KV reads
\[
K_2 = u \partial_u - \left( \frac{\nu + uh_0}{2} + \frac{\int du h_0}{2} - u xh_{1,uu} + \frac{uh_{1,uu}^2}{2} - \frac{\int du h_{1,uu}^2}{2} \right) \partial_x - \left( \frac{2}{c_1} - x \right) h_{1,uu} - c_2 \int du \left( \frac{2}{c_1} + uh_{1,uu} \right) \partial_x \left( \frac{2}{c_1} + uh_{1,uu} \right) \partial_x, \quad \text{(5.31b)}
\]
with the commutator \( [K_1, K_2] = -K_1 \).
Type B. Here the function \( h \) is characterised by the two conditions \( (h_{xxx}/h_{xx})_x = 0 \) and \( (h_{xxx}/h_{xx})_u \neq 0 \), leading to the form
\[
h(u,x) = h_0(u) + x h_1(u) + e^{h_2(u)} + x h_3(u),
\]
where \( \{h_0, h_1, h_2, h_3\} \) are arbitrary functions of \( u \), and \( h_{3,u}(u) \neq 0 \) has to be true for the latter condition. It is required by \( R^{2}_{[[21]]} \) \( \omega = 0 \) that
\[
h_{3,u} \omega_v = 0, \quad h_{3,u} \omega_e = 0,
\]
concluding that \( \omega_v = \omega_e = 0 \), so there is no possibility of finding the 2nd KV.

Type C. Since the function \( h \) is the general solution to \( (h_{xxx}/h_{xx})_u = 0 \), we have
\[
h(u,x) = h_0(u) + x h_1(u) + h_2(u) h_3(x),
\]
where \( \{h_0, h_1\} \) are arbitrary functions of \( u \), and \( \{h_2, h_3\} \) are respectively nonzero functions of \( u \) and \( x \). As \( (h_{xxx}/h_{xx})_x = 0 \) is satisfied nowhere, it is stipulated that \( (h_{xxx}/h_{xx})_x \neq 0 \). From this and the first row of \( R^{2}_{[[21]]} \), it is inevitable that \( \omega_v = 0 \). The leftover components of \( R^{2}_{[[21]]} \) put the requirements to have \( \text{rank} R^{2}_{[[21]]} = 1 \) as
\[
(h_{2,u}/h_2)_u = 0, \quad h_{1,u}/h_1 = h_{2,u}/h_2.
\]
Assuming equations (5.35) and using new basis \( \{e_1^1, e_2^1, e_3^1\} \) defined by equation (5.30), it is easy to see that \( \text{rank} R^{2}_{[[21]]} = 0 \). For \( h_1 = c_1 = \text{const.} \) and \( h_2 = c_2 = \text{const.} \), the 2nd KVs is given by
\[
K_2 = \partial_u - \frac{a}{2} \partial_u,
\]
with the commutator \( [K_1, K_2] = 0 \). For \( h_1 = c_1 u^{-2}(c_1 = \text{const.}) \) and \( h_2 = c_2 u^{-2}(c_2 = \text{const.}) \), the 2nd KVs is expressed as
\[
K_2 = u \partial_u - \frac{1}{2} \left(u h_0 + \int du h_0 + 2 \nu \right) \partial_u,
\]
with the commutator \( [K_1, K_2] = -K_1 \).

Type D. It is immediately seen from the first row of \( R^{2}_{[[21]]} \) that
\[
\omega_v = -\frac{\sigma_x}{\sigma_y} \omega_v, \quad \sigma(u,x) = \frac{h_{xxx}}{2 h_{xx}},
\]
where the valuable \( \sigma(u,x) \) inherits from the definition (4.34). The remaining entries do not have illuminating expressions to be described here. Leaving aside the full implications of \( R^{2}_{[[21]]} \) we proceed to the analysis of class 2 to simplify the reasoning. By choosing an orthonormal basis \( \{e_1^1, e_2^1, e_3^1\} \) as
\[
e_1^1 \equiv \frac{\sigma_x}{\sigma_y} u^\alpha + e^\alpha, \quad e_2^1 \equiv - \frac{\sigma_x}{\sigma_y} v^\alpha + e^\alpha, \quad e_3^1 \equiv - \frac{\sigma_x}{\sigma_y} u^\alpha + \frac{\sigma_x}{\sigma_y} v^\alpha - e^\alpha,
\]
we have
\[
\kappa_1 = -\eta_1 = \frac{\sigma_{xx}}{\sigma_x} - \frac{\sigma_{yx}}{\sigma_y},
\]
Figure 6. The sub-algorithm for the class 2 null. See equations (3.18), (3.25)–(3.27) for notations.

Figure 7. The sub-algorithm for the class 3. The notation is as follows: The two indices on the traceless Ricci operator $S^a_b$ are dropped for short, e.g. $S^2$ denotes $S^a_bS^b_c$; $S^{(i)}$ denotes the $g$-trace of $S^i(i = 2, 3)$.

Figure 8. The sub-sub-algorithm for the class 3 type $[1,(11)]$. See equations (4.12) and (4.13) for notations.
From the result of subsection 3.1.1, equation (5.39) has to be zero so that the 2nd KV can exist. Solving a PDE
\[ \sigma_{xx}/\sigma - \sigma_{ux}/\sigma_u = 0, \]
we obtain
\[ \sigma(u,x) = \sigma(x + h_4(u)), \tag{5.40} \]
where \( h_4 \) is an arbitrary function of \( u \). Subsequently, the definition of \( \sigma \) (5.37) gives
\[ h(u,x) = h_0(u) + x(h_1(u) - 2h_{4,uu}(u)) + h_2(u)h_3(x + h_4(u)), \tag{5.41} \]
where \( \{h_0, h_1, h_2, h_3\} \) are arbitrary functions of one variable such that \( (h_3, x h_3, h_3)_x \neq 0 \) and \( h_{4,uu} \neq 0 \). Note that the existence of the 2nd KV is still not clear, so we go on to examining \( \text{rank} R_{(1,1)1}^{#2} \). It follows from the third and fourth rows of \( R_{(1,1)1}^{#2} \) that \( \text{rank} R_{(1,1)1}^{#2} = 0 \) if \( (h_{2,uu}/h_3^{1/2})_{uu} = 0 \) and \( h_1, h_2 = h_2, h_3 \).

For \( h_1 = c_1 = \text{const.} \) and \( h_2 = c_2 = \text{const.} \), the 2nd KV is given by
\[ K_2 = \partial_u - \frac{1}{2} (h_0 + h_2^2 - 2xh_{4,uu} - c_1h_4) \partial_x - h_{4,uu}\partial_x, \tag{5.42} \]
with the commutator \([K_1, K_2] = 0\). For \( h_1 = c_1u^{-2}(c_1 = \text{const.}) \) and \( h_2 = c_2u^{-2}(c_2 = \text{const.}) \), the 2nd KV is expressed as

---

**Figure 9.** The sub-sub-algorithm for the class 3 type [(1, 1)1]. See equations (4.19) and (4.20) for notations.

**Figure 10.** The sub-sub-algorithm for the class 3 type [1, 11]. See equation (4.24) for notations.
\begin{figure}
\centering
\begin{tikzpicture}
    \node (class2) at (0,0) {class 2};
    \node (class1) at (0,-1) {class 1};
    \node (noKV) at (0,-2) {no KV};
    \node (3KVs) at (0,-3) {3 KVs};
    \node (class22) at (-1.5,-2) {class 2};
    \node (class12) at (-1.5,-3) {class 1};
    \node (noKV2) at (-1.5,-4) {no KV};
    \node (3KVs2) at (-1.5,-5) {3 KVs};
    \node (class23) at (1.5,-2) {class 2};
    \node (class13) at (1.5,-3) {class 1};
    \node (noKV3) at (1.5,-4) {no KV};
    \node (3KVs3) at (1.5,-5) {3 KVs};
    \draw (class2) -- (class22);
    \draw (class2) -- (class12);
    \draw (class2) -- (noKV2);
    \draw (class2) -- (3KVs2);
    \draw (class1) -- (class23);
    \draw (class1) -- (class13);
    \draw (class1) -- (noKV3);
    \draw (class1) -- (3KVs3);
    \draw (noKV) -- (noKV2);
    \draw (noKV) -- (noKV3);
    \draw (3KVs) -- (3KVs2);
    \draw (3KVs) -- (3KVs3);
\end{tikzpicture}
\caption{The sub-sub-algorithm for the class 3 type \([21]\). See equations (4.35)–(4.37) for notations.}
\end{figure}

\begin{figure}
\centering
\begin{tikzpicture}
    \node (class2) at (0,0) {class 2};
    \node (class1) at (0,-1) {class 1};
    \node (noKV) at (0,-2) {no KV};
    \node (3KVs) at (0,-3) {3 KVs};
    \node (class22) at (-1.5,-2) {class 2};
    \node (class12) at (-1.5,-3) {class 1};
    \node (noKV2) at (-1.5,-4) {no KV};
    \node (3KVs2) at (-1.5,-5) {3 KVs};
    \node (class23) at (1.5,-2) {class 2};
    \node (class13) at (1.5,-3) {class 1};
    \node (noKV3) at (1.5,-4) {no KV};
    \node (3KVs3) at (1.5,-5) {3 KVs};
    \draw (class2) -- (class22);
    \draw (class2) -- (class12);
    \draw (class2) -- (noKV2);
    \draw (class2) -- (3KVs2);
    \draw (class1) -- (class23);
    \draw (class1) -- (class13);
    \draw (class1) -- (noKV3);
    \draw (class1) -- (3KVs3);
    \draw (noKV) -- (noKV2);
    \draw (noKV) -- (noKV3);
    \draw (3KVs) -- (3KVs2);
    \draw (3KVs) -- (3KVs3);
\end{tikzpicture}
\caption{The sub-sub-algorithm for the class 3 type \([21]\). See equation (4.41a) for notations.}
\end{figure}

\begin{figure}
\centering
\begin{tikzpicture}
    \node (class2) at (0,0) {class 2};
    \node (class1) at (0,-1) {class 1};
    \node (noKV) at (0,-2) {no KV};
    \node (3KVs) at (0,-3) {3 KVs};
    \node (class22) at (-1.5,-2) {class 2};
    \node (class12) at (-1.5,-3) {class 1};
    \node (noKV2) at (-1.5,-4) {no KV};
    \node (3KVs2) at (-1.5,-5) {3 KVs};
    \node (class23) at (1.5,-2) {class 2};
    \node (class13) at (1.5,-3) {class 1};
    \node (noKV3) at (1.5,-4) {no KV};
    \node (3KVs3) at (1.5,-5) {3 KVs};
    \draw (class2) -- (class22);
    \draw (class2) -- (class12);
    \draw (class2) -- (noKV2);
    \draw (class2) -- (3KVs2);
    \draw (class1) -- (class23);
    \draw (class1) -- (class13);
    \draw (class1) -- (noKV3);
    \draw (class1) -- (3KVs3);
    \draw (noKV) -- (noKV2);
    \draw (noKV) -- (noKV3);
    \draw (3KVs) -- (3KVs2);
    \draw (3KVs) -- (3KVs3);
\end{tikzpicture}
\caption{The sub-sub-algorithm for the class 3 type \([3]\). See equation (4.50a) for notations.}
\end{figure}
**Figure 14.** The sub-sub-algorithm for the class 3 type $[z\bar{z}1]$. See equation (4.59) for notations.

- ![Diagram](image)

- $z = 1 \quad \rightarrow \quad 6$ KVs
- $z = 0 \quad \rightarrow \quad 4$ KVs
- No KVs

- 3 KVs

**Figure 15.** A flowchart to classify the number of KVs of the Lifshitz spacetime (5.1) in 3D.

- $h_{,xx} = 0 \quad \rightarrow \quad 6$ KVs
- $h_{,xxx} = 0 \quad \rightarrow \quad 4$ KVs
- $\left( \frac{h_{,xx}}{h_{,x}} \right)_{,x} = 0 \quad \rightarrow \quad 3$ KVs
- $1$ KV

**Figure 16.** A flowchart to classify the number of KVs of the pp-wave spacetime (5.6) in 3D.

- $\left( \frac{h_{,xx}}{h_{,x}} \right)_{,x} = 0 \quad \rightarrow \quad 2$ KVs
- $1$ KV
\[ K_2 = u \partial_u - \frac{1}{2} \left( uh_0 + \int du h_0 + 2v - \int du h_2^2 - 2x(uh_4,\nu) + uh_4^2 \right) \partial_x - uh_4,\nu \partial_x. \]  

(5.43)

with the commutator \([K_1, K_2] = -K_1\).

6. Conclusion

The basic questions we addressed in this paper are whether there exists a set of invariants associated with the existence of KVs, and if so, how to construct it for a given Lorentzian manifold. Our contribution is to give affirmative answers to such questions in dimension 3, extending the result for a Riemannian manifold [12]. According to our theorem in section 1, the number of linearly independent KVs can be counted using the algorithm described in figure 2, even if a given spacetime has a VSI property. As we have seen in section 5, the theorem can classify a given spacetime into a hierarchy based on their level of symmetry.

It would be instructive to mention the algorithmic efficiency of the Cartan–Karlhede and our formulations. Given a Lorentzian manifold of dimension 3, the Cartan–Karlhede algorithm uses Cartan scalars and requires at most six differentiations of \( R_{abcd} \). In that case, one must assess the functionally independent 336 Cartan scalars, whence it reveals the number of KVs in principle. On the other hand, our algorithm uses the Ricci rotation coefficients, their derivatives and the ratio thereof. In the worst case, it may be implemented in line with figure 17 and 66 differential invariants are required in total. In conjugation with this, our prescription requires up to the 3rd derivatives of the curvature in \( R^{[21]} \). Thus, our algorithm is more economic than that of the Cartan–Karlhede to count the number of KVs. The only price to pay for our method to work out is to solve the eigenvalue problem of the traceless Ricci operator \( S_{ab} \) in the class 3. Fortunately, this is not a demanding task in dimension 3.

As an application, our theorem may enable us to derive the canonical form of metrics with a high degree of symmetry. In fact, the calculations we have carried out in appendix B give the canonical form of metrics admitting 4 KVs, which is a reproduction and improvement of the classical result due to Kruchkovich [25]. As a side remark, we note that our algorithm does not ensure that local metrics endowed with a certain degree of symmetry exist in all branches of class 1–3. In other words, it seems that some values of \( \text{rank} R_{cls,d} \) are prohibited in principle, resembling Fubini’s theorem on the order of the isometry group.

The classification of metrics with 3 KVs makes this fact manifest. For instance, it can be shown that \( \text{rank} R^{[21]} \) cannot equal to zero. The detail of this extensive study will be reported in a forthcoming paper.

It is also noteworthy to comment that the local existence of KVs does not immediately give rise to the existence of the global isometry group due to topological restrictions. An emblematic sample is the black hole constructed by Bañados, Teitelboim and Zanelli (BTZ) [23]. The BTZ black hole solves the vacuum Einstein’s equations with a negative cosmological constant and is obtained by the identification of points in AdS by the discrete isometry. From the curvature points of view, the BTZ metric obviously admits 6 KVs generating \( so(2, 2) \) algebra. In
spite of this, only two of them are globally well-defined, since the rest of KVs is not single-valued under identifications [24]. It thus turns out that the global isometry group of the BTZ solution is broken from $SO(2, 2)$ down to $SO(1, 1) \times SO(2)$.

Our algorithm can be extensible for a (semi-)Riemannian manifold $M$ of higher dimension. In this case, the Weyl tensor $W_{abcd}$ also comes into play. In particular, the first obstruction matrix in dimension $\dim M = n$ takes the form
\begin{equation}
R^n_a \equiv \left[ \nabla_a R \nabla_a S^{(2)} \ldots \nabla_a S^{(n)} \nabla_a W^{(2)} \ldots \nabla_a W^{(n(n-1)/2)} \right]^T , \tag{6.1}
\end{equation}
where $W^{(i)}$ are principal traces of the $i$-th powers of the Weyl operator $W_{abcd}$, considered as an endomorphism of $\Lambda^2 T(M)$. So the invariants associated with the existence of KVs can be specified based on $R^n_a$. This line of extension would be a promising way to improve the past-proposed schemes. Further consideration of this shall be done elsewhere.

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Appendix A. Relations amongst the Ricci rotation coefficients and their derivatives

In this appendix, we collect some relations amongst the Ricci rotation coefficients and their derivatives which are implicitly used in sections 3 and 4.

A.1. For an orthonormal frame

Given an orthonormal frame $\{e^a_i, i = 1, 2, 3\}$ satisfying
\begin{equation}
g^{ab} = \epsilon e^a_1 e^b_1 + e^a_2 e^b_2 - \epsilon e^a_3 e^b_3 , \tag{A.1}
\end{equation}
where $\epsilon = \text{sgn}(g_{ab} e^a_1 e^b_1)$ and its Ricci rotation coefficients (3.5), the following relations hold true.

The commutation relations:
\begin{align}
[e_1, e_2]^a &= -\epsilon \kappa_1 e^a_1 + \kappa_2 e^a_2 - \epsilon (\tau_1 - \tau_2) e^a_3 , \tag{A.2a} \\
[e_2, e_3]^a &= -\epsilon (\tau_2 - \tau_3) e^a_1 - \eta_2 e^a_2 - \epsilon \eta_3 e^a_3 , \tag{A.2b} \\
[e_3, e_1]^a &= \epsilon \eta_1 e^a_1 + (\tau_1 + \tau_3) e^a_2 + \epsilon \kappa_3 e^a_3 . \tag{A.2c}
\end{align}

The components of the Ricci tensor:
\begin{align}
R_{12} e^1_1 &= \eta_1^2 + \epsilon \eta_1 \eta_2 - \kappa_3^2 + \epsilon \eta_1 \kappa_1 - \kappa_1^2 - \kappa_2^2 - 2 \epsilon \tau_2 \tau_3 - \epsilon \xi_2 \eta_3 + \xi_2 \kappa_3 + \xi_1 \kappa_2 + \xi_2 \kappa_1 , \tag{A.3a} \\
R_{12} e^2_2 &= -\eta_1^2 + \eta_1 \eta_2 + \kappa_3^2 - \kappa_1^2 + \kappa_2 \kappa_3 - \kappa_1 \kappa_2 - 2 \tau_1 \tau_3 - \epsilon \xi_2 \eta_3 + \xi_1 \kappa_2 - \xi_2 \kappa_3 + \xi_2 \eta_1 , \tag{A.3b}
\end{align}
\[ R_{ab} e^a_b + \nu^2 - \eta_1^2 + \kappa_2 - \nu_3 \kappa_1 - \nu_3 \kappa_2 - 2 \nu_1 \tau_2 + \nu_3 \eta_1 + \nu_3 \eta_2 + \nu_1 \kappa_3 + \nu_2 \kappa_3 , \]  
\[ (A.3c) \]
\[ R_{ab} e^a_2 = - \eta_3 \kappa_3 - \nu_3 \kappa_2 - \eta_1 \tau_3 + \nu_2 \tau_3 + \eta_1 \tau_3 + \eta_2 \tau_3 - \nu_1 \eta_3 + \nu_2 \eta_3 , \]  
\[ (A.3d) \]
\[ R_{ab} e^b_3 = - \eta_3 \nu_1 - \eta_1 \eta_1 - \kappa_3 \tau_3 + \nu_2 \tau_3 + \kappa_1 \kappa_1 + \eta_2 \tau_2 - \nu_1 \eta_1 + \nu_2 \eta_1 + \nu_3 \kappa_1 - \nu_1 \kappa_1 , \]  
\[ (A.3e) \]
\[ R_{ab} e^b_1 = - \eta_2 \kappa_3 - \nu_2 \kappa_2 - \nu_1 \tau_3 - \kappa_1 \tau_3 + \nu_2 \tau_3 + \nu_1 \tau_2 - \nu_1 \kappa_2 + \nu_1 \kappa_3 + \nu_1 \kappa_3 - \nu_1 \kappa_2 - \nu_1 \kappa_2 - \nu_1 \kappa_2 , \]  
\[ (A.3f) \]
\[ R_{ab} e^b_1 = - \eta_1 \kappa_2 - \nu_1 \kappa_2 - \nu_1 \tau_3 + \kappa_1 \tau_3 + \nu_1 \tau_3 + \nu_1 \tau_2 + \nu_2 \tau_2 + \nu_1 \eta_1 - \nu_1 \eta_1 , \]  
\[ (A.3g) \]
\[ R_{ab} e^b_1 = - \eta_1 \kappa_2 - \nu_1 \kappa_2 - \nu_1 \tau_3 + \kappa_1 \tau_3 + \nu_1 \tau_3 + \nu_1 \tau_2 + \nu_2 \tau_2 + \nu_1 \eta_1 - \nu_1 \eta_1 . \]  
\[ (A.3h) \]
\[ \begin{align*}
A.2. & \text{ For a double-null frame} \\
& \text{Given an orthonormal frame } \{u^a, v^a, e^a\} \text{ satisfying} \\
& g^{ab} = u^a u^b + v^a v^b + e^a e^b , \\
& \text{and its Ricci rotation coefficients (3.18), the following relations hold true.} \\
& \text{The commutation relations:} \\
& [u, v]^a = \kappa_\nu u^a - \kappa_\alpha v^a + (\tau_\nu - \tau_\alpha)e^a , \\
& (A.5a) \\
& [v, e]^a = - \eta_\nu u^a - (\tau_\nu - \tau_\epsilon)e^a + \eta_\epsilon e^a , \\
& (A.5b) \\
& [e, u]^a = (\tau_\nu + \tau_\epsilon)u^a + \eta_\nu v^a - \kappa_\epsilon e^a . \\
& (A.5c) \\
& \text{The components of the Ricci tensor:} \\
& R_{ab}u^a u^b = - \kappa_\nu^2 - \kappa_\epsilon \kappa_\nu - 2 \eta_\nu \tau_\epsilon - \eta_\nu \tau_\nu - \eta_\nu \tau_\nu + \nu_\epsilon \eta_\nu + \nu_\epsilon \kappa_\nu , \] 
\[ (A.6a) \]
\[ R_{ab} v^a v^b = - \eta_\nu^2 + \eta_\epsilon \kappa_\nu + 2 \eta_\epsilon \tau_\nu - \eta_\nu \tau_\nu + \eta_\nu \tau_\nu + \nu_\epsilon \eta_\nu + \nu_\epsilon \kappa_\nu , \] 
\[ (A.6b) \]
\[ R_{ab}e^a e^b = - 2 \eta_\nu \eta_\epsilon - 2 \eta_\epsilon \kappa_\nu + \eta_\epsilon \kappa_\nu + \kappa_\nu \kappa_\nu - \tau_\epsilon^2 - \tau_\nu^2 - \nu_\epsilon \tau_\nu + \nu_\epsilon \tau_\nu - \eta_\epsilon \kappa_\nu + \eta_\epsilon \kappa_\nu - \nu_\epsilon \kappa_\nu , \] 
\[ (A.6c) \]
\[ R_{ab}u^a v^b = - \eta_\nu \kappa_\nu + \kappa_\nu \kappa_\nu - 2 \kappa_\nu \kappa_\nu + \nu_\epsilon \tau_\nu - \kappa_\nu \tau_\nu + \tau_\nu \tau_\nu - \kappa_\nu \tau_\nu + \nu_\epsilon \tau_\nu - \eta_\epsilon \kappa_\nu + \eta_\epsilon \kappa_\nu - \nu_\epsilon \kappa_\nu , \] 
\[ (A.6d) \]
\[ = - \eta_\epsilon \kappa_\nu + \eta_\epsilon \kappa_\nu - 2 \kappa_\nu \kappa_\nu + \nu_\epsilon \tau_\nu - \kappa_\nu \tau_\nu + \tau_\nu \tau_\nu - \kappa_\nu \tau_\nu + \nu_\epsilon \tau_\nu - \eta_\epsilon \kappa_\nu + \eta_\epsilon \kappa_\nu - \nu_\epsilon \kappa_\nu , \] 
\[ (A.6e) \]
\[ R_{ab}v^a e^b = - \eta_\nu \kappa_\nu - \nu_\epsilon \tau_\nu + \eta_\epsilon \tau_\nu - \kappa_\nu \tau_\nu + \nu_\epsilon \tau_\nu + \nu_\epsilon \tau_\nu - \nu_\epsilon \tau_\nu - \nu_\epsilon \tau_\nu , \] 
\[ (A.6f) \]
\[ = - 2 \eta_\nu \kappa_\nu - \eta_\nu \tau_\nu + \eta_\nu \tau_\nu + \eta_\nu \tau_\nu + \nu_\epsilon \eta_\nu + \nu_\epsilon \tau_\nu , \] 
\[ (A.6g) \]
\[ R_{ab}e^a u^b = - \eta_\nu \kappa_\nu - \kappa_\nu \kappa_\nu - \nu_\epsilon \tau_\nu + \kappa_\nu \tau_\nu + \nu_\epsilon \tau_\nu - \nu_\epsilon \kappa_\nu + \nu_\epsilon \kappa_\nu + \nu_\epsilon \kappa_\nu , \] 
\[ (A.6h) \]
\[ = - 2 \eta_\nu \kappa_\nu - \kappa_\nu \tau_\nu - \nu_\epsilon \tau_\nu + \nu_\epsilon \tau_\nu - \nu_\epsilon \kappa_\nu , \] 
\[ (A.6i) \]
Appendix B. Canonical form of metrics admitting 4Killing vectors

Using the scheme developed in the present paper, we can obtain the canonical form of the metric admitting any number of KVs and the corresponding algebra. To make the discussion focused, we investigate in this appendix the case in which 4 KVs exist. As described in section 4, this occurs only for Segre types [1, (11)], [(1, 1), 1] and [(21)]. In each case, it turns out that we can actually obtain all the explicit metrics. Interestingly, these spacetimes are all homogeneous, in the sense that local isometry groups possess transitive actions on the manifold.

B.1. Type [1, (11)]

Let us begin with the case of Segre type [1, (11)]. Analysis in section 4.1.1 reveals that 4 KVs exist, provided
\[
\kappa_2 = \tau_2 + \tau_3 = 0, \quad \tau_2 = \text{const.},
\]

(B.1)
together with equation (4.7):
\[
\kappa_1 = 0, \quad \eta_1 = 0, \quad \kappa_3 = -\kappa_2.
\]

(B.2)

Here \( L_1 \tau_2 = 0 \) follows from \( R_{32} = R_{32} \). With these spin connections, the first derivative of \( \{e_1, e_2, e_3\} \) reads
\[
\nabla_b e_{1a} = \tau_2 e_{2b} e_{3a} - \tau_2 e_{3b} e_{2a},
\]

(B.3a)
\[
\nabla_b e_{2a} = -\tau_1 e_{1b} e_{3a} + \eta_2 e_{2b} e_{3a} + e_{3b} (-\tau_2 e_{1a} - \eta_1 e_{3a}),
\]

(B.3b)
\[
\nabla_b e_{3a} = \tau_1 e_{1b} e_{2a} + e_{2b} (\tau_2 e_{1a} - \eta_2 e_{2a}) + \eta_3 e_{3b} e_{2a}.
\]

(B.3c)

It follows that \( W_a = e_{2a} + i e_{3a} \) satisfies
\[
\nabla_b W_a = i (\tau_1 e_{1b} - \eta_2 e_{2b} + \eta_3 e_{3b}) W_a + i \tau_2 e_{1a} W_b,
\]

(B.4)
hence
\[
\nabla_{(ae_{1b})} = 0, \quad W_{[a} \nabla_b W_{c]} = 0.
\]

(B.5)

Then, there exist real functions \( t, x, y \) and \( \theta, \phi, \chi_1, \chi_2 \) such that
\[
e_{1a} = -f (\nabla_a \chi_1 + \chi_2 \nabla_a x + \chi_2 \nabla_a y), \quad W_a = e^{i \phi} (\nabla_a x + i \nabla_a y).
\]

(B.6)

By the redefinition \( t \to \int f^{-1} dt \), one can set \( f \equiv 1 \) without loss of generality. Exploiting the \( \text{SO}(2) \) gauge freedom which rotates \( (e_2, e_3) \), \( \theta = 0 \) is always achieved. The Killing equation \( \nabla_{(ae_{1b})} = 0 \) then demands that the metric is independent of \( t \). The condition \( \tau_2 (= \text{const.}) \) boils down to
\[
\partial_\chi_1 \chi_1 - \partial_\chi_2 \chi_2 = 2 \tau_2 e^{2 \phi}.
\]

(B.7)

Using this relation, the trace-free part of Ricci tensor gives rise to Liouville’s equation
\[
(\partial_x^2 + \partial_y^2) \phi = - k e^{2 \phi}, \quad k \equiv - \frac{1}{2} (3 \lambda_1 + 8 \tau_2^2).
\]

(B.8)

It follows that \( d \Sigma_k^2 = e^{2 \phi} (dx^2 + dy^2) \) corresponds to the space \( \Sigma_k \) with a constant sectional curvature \( k \), which can be normalised to be 0 or \( \pm 1 \), and the scalar curvature is given by \( R = 2(k + \tau_2^2) \). The local solution to Liouville’s equation can be chosen to be
\[ \phi = - \log \left( 1 + \frac{k}{4}(x^2 + y^2) \right), \quad \chi = \frac{\tau_2}{1 + \frac{4}{3}(x^2 + y^2)} (xdx - xdy), \] (B.9)

where \( \chi = \chi_1 dx + \chi_2 dy \). Defining \( x + iy = \frac{2}{\sqrt{k}} \tan \left( \frac{\sqrt{k}}{2} \theta \right) e^{i\phi} \), we therefore arrive at

\[ ds^2 = - \left[ dt - 4\tau_2 \left( \frac{\sin \left( \frac{\sqrt{k}}{2} \theta \right)}{\sqrt{k}} \right) d\phi \right]^2 + d\theta^2 + \left( \frac{\sin(\sqrt{k}\theta)}{\sqrt{k}} \right)^2 d\phi^2. \] (B.10)

If \( k = -4\tau_2^2 < 0 \), we have AdS3 for which the number of KVs is enhanced to 6. Otherwise, we have precisely 4 KVs

\[ K_1 = \partial_t, \] (B.11a)

\[ K_2 = \frac{2\tau_2}{\sqrt{k}} \cos \phi \tan \left( \frac{\sqrt{k}}{2} \theta \right) \partial_t - \sin \phi \partial_\theta - \frac{\sqrt{k}}{2} \cos \phi \left[ \cot^2 \left( \frac{\sqrt{k}}{2} \theta \right) - 1 \right] \tan \left( \frac{\sqrt{k}}{2} \theta \right) \partial_\phi, \] (B.11b)

\[ K_3 = \frac{2\tau_2}{\sqrt{k}} \sin \phi \tan \left( \frac{\sqrt{k}}{2} \theta \right) \partial_t + \cos \phi \partial_\theta - \frac{\sqrt{k}}{2} \sin \phi \left[ \cot^2 \left( \frac{\sqrt{k}}{2} \theta \right) - 1 \right] \tan \left( \frac{\sqrt{k}}{2} \theta \right) \partial_\phi, \] (B.11c)

\[ K_4 = \partial_\phi, \] (B.11d)

satisfying

\[ [K_2, K_3] = - (2\tau_2 K_1 + kK_4), \quad [K_2, K_4] = K_3, \quad [K_3, K_4] = - K_2. \] (B.12)

For \( \tau_2 = 0 \), the metric collapses to \( \mathbb{R} \times \Sigma_k \), which is locally symmetric. For \( \tau_2(k + 4\tau_2^2) \neq 0 \) with \( k = -1 \), the metric describes the 3-dimensional Gödel universe, which is sourced by a dust with a negative cosmological constant (see e.g. [26]).

### B.2. Type [(1, 1)]

In this case, we have \( \kappa_1 = \kappa_2 = \eta_2 = \eta_3 = 0 \) and \( \tau_1 = \tau_3 = \text{const.} \), for which \( e_{2a} \) is Killing and \( W_{\phi}^b = e_{1a} \pm e_{3b} \) are hypersurface-orthogonal. Since the rest of the derivation is parallel to the \([1, (11)] \) case, we only show the final results:

\[ ds^2 = \left[ dt + 4\tau_1 \left( \frac{\sin \left( \frac{\sqrt{k}}{2} \theta \right)}{\sqrt{k}} \right) d\phi \right]^2 + d\theta^2 - \left( \frac{\sin(\sqrt{k}\theta)}{\sqrt{k}} \right)^2 d\phi^2, \] (B.13)

where \( \tau_1 \) is a constant. This is the double Wick-rotated version of equation (B.10).

### B.3. Type [(2, 1)]

A class of metrics with 4 KVs exists also for the type \([(2, 1)], \) for which

\[ \kappa_e = 0, \quad \eta_\mu = 0, \quad \sigma = 0. \] (B.14)
The second obstruction matrix $R_{[21]}$ given by equation (4.35) must vanish identically, yielding
\[ \tau_v = \text{const.}, \quad \Sigma = \text{const.}, \] (B.15)
where $\mathcal{L}_u \tau_v = 0$ follows from $S_{ab} e^a u^b = 0$. These are exhaustive information supplied from the condition for 4 KVs in type $[(2, 1)]$.

The definition of $\Sigma, \sigma = 0$ and Bianchi identity are combined to give 1st-order system for $\varphi = \frac{1}{2} \log(S_{vv})$ as
\[ 2e^\varphi \Sigma \tau_v = 0, \] (B.17)
which branches into (i) $\tau_v = 0$ and (ii) $\Sigma = 0$.

Before proceeding, let us note that the Segre type $[(2, 1)]$ allows the following gauge freedom for the choice of null basis $\{u_a, v_a, e_a\}$:
\[ u_a \rightarrow a u_a, \quad v_a \rightarrow a^{-1} v_a, \quad e_a \rightarrow e_a, \] (B.18)
and
\[ u_a \rightarrow u_a, \quad v_a \rightarrow v_a - \frac{1}{2} b^2 u_a + be_a, \quad e_a \rightarrow e_a - bu_a, \] (B.19)
where $a$ and $b$ are arbitrary functions. By these transformations, $\tau_v \equiv e^\varphi \nabla u^a u_a$ and $\Sigma \equiv \mathcal{L}_a (e^{-\varphi}) + \kappa_v e^{-\varphi}$ remain invariant. In contrast, $\tau_v \equiv -\nu^a u^b \nabla_{b} e_a$ and $\tau_v \equiv \nu^a e^\varphi \nabla_{b} u_a$ vary as $\tau_v \rightarrow \tau_v$ and $\tau_v \rightarrow \tau_v + e^\varphi \nabla_{b} \log a$ under equation (B.18), which permits us to set $\tau_v + \tau_v = 0$. Since $\{u.a\}' = 0$ is now satisfied because the condition $\kappa_v = \eta_b = 0$ does not change under equation (B.18), one can introduce local coordinates $(x, y, z)$ in such a way that
\[ u_a = \partial_a x, \quad v_a = \nabla_a \nu - \frac{V_2}{V_1} \nabla_a x, \quad e_a = \nabla_a \nu - \frac{V_3}{V_1} \nabla_a x. \] (B.21)
The hypersurface orthogonality of $u_a$ is a direct consequence of $\eta_b = 0$ and $\kappa_v = 0$. In this basis, $\tau_v$ is computed to be
\[ \tau_v = -\frac{1}{2} V_1 \left[ \partial_y \left( \frac{V_3}{V_1} \right) + \partial_z \left( \frac{1}{V_1} \right) \right]. \] (B.22)

The following discussion will be divided according to $\tau_v = 0$ or $\Sigma = 0$.

**B.3.1. $\tau_v = 0$ case.** Setting $\tau_v = 0$ in equation (B.22), one finds a local function $F = F(x, y, z)$ satisfying
\[ \frac{V_3}{V_1} = -\partial_z F, \quad \frac{1}{V_1} = \partial_y F. \] (B.23)
\[ \partial_j V_2 \partial_j F + 2V_2 \partial_j^2 F + \frac{\partial_x \partial_y F - \partial_y F \partial_x F}{\partial_x F} = f_1, \]  

(B.24)

which is further integrated to give

\[ V_2 = -\frac{1}{(\partial_x F)^2} \left[ \partial_x F - \frac{1}{2} (\partial_x F)^2 - f_1(x) F + \frac{1}{2} f_2(x, z) \right], \]  

(B.25)

where \( f_2 = f_2(x, z) \) is an arbitrary function of \( x \) and \( z \). Inspecting equations (B.23) and (B.25), one obtains

\[ R = 0, \quad \varphi = \frac{1}{2} \log \left( -\frac{\partial_x^2 f_2}{2(\partial_x F)^2} \right). \]  

(B.26)

Substitution of this expression of \( \varphi \) into \( \sigma \equiv L \varphi + \tau_v + \tau_e = 0 \), one gets \( \partial_x^2 f_2(x, z) = 0 \). Upon integration, we find

\[ f_2(x, z) = f_{20}(x) + f_{21}(x) z + f_{22}(x) z^2. \]  

(B.27)

The condition \( S_v \neq 0 \) asks for \( f_{22}(x) \neq 0 \). \( \Sigma \equiv L_x (e^{-\varphi}) + \kappa e^{-\varphi} \) is now computed to

\[ \Sigma = \frac{-2f_1(x)f_{22}(x) + f_{21}(x)}{2(-f_{22}(x))^{3/2}}. \]  

(B.28)

We have all ingredients in place to obtain the explicit metric form. Defining \( \tilde{y} = F(x, y, z) \), the metric becomes

\[ ds^2 = 2d\tilde{y}dx + dz^2 + dx^2 [f_{20}(x) + f_{21}(x)z + f_{22}(x)z^2 + \tilde{y}f_1(x)]. \]  

(B.29)

Further change of variable \( x = h(\tilde{x}) \), \( \tilde{y} = \tilde{y}/h'(\tilde{x}) \) renders the metric into

\[ ds^2 = 2d\tilde{x}d\tilde{y} + dz^2 + dx^2 \left[ \tilde{f}_{20} + \tilde{f}_{21} z + \tilde{f}_{22} z^2 + \frac{\tilde{y}}{h'(\tilde{x})} (\tilde{f}_1 + h''(\tilde{x})) \right]. \]  

(B.30)

where \( \tilde{f}_1(\tilde{x}) = h'(\tilde{x})^2 f_1(h(\tilde{x})) \) and \( \tilde{f}_{2i}(\tilde{x}) = h'(\tilde{x})^2 f_{2i}(h(\tilde{x})) \) \((i = 0, 1, 2)\). By choosing \( h''(\tilde{x}) = -f_1(\tilde{x}) \) and omitting hats, one obtains the metric of the following form

\[ ds^2 = 2dxdy + dz^2 + dx^2 [f_{20}(x) + f_{21}(x)z + f_{22}(x)z^2]. \]  

(B.31)

This amounts to setting \( f_1(x) = 0 \) in the metric (B.29). Thus, equation (B.28) is integrated to give

\[ f_{22}(x) = -\frac{1}{(\Sigma x - c_1)^2}, \]  

(B.32)

where \( c_1 \) and \( \Sigma \) are constants. This metric describes the pp-wave, whose 4 KVs were already obtained in section 5.2. The special case \( \Sigma = 0 \) corresponds to the locally symmetric space which admits a covariantly constant Ricci tensor \( \nabla_v R_{bc} = 0 \).

**B.3.2. \( \Sigma = 0 \) case.** We assume \( \tau_v \neq 0 \) henceforth, since the metric (B.31) with \( \Sigma = 0 \) in equation (B.32) is recovered for the \( \tau_v = 0 \) case. Equation (B.22) is solved as

\[ \partial_j V_3 = -2\tau_v + \frac{\partial_j V_1 + V_3 \partial_j V_1}{V_1}. \]  

(B.33)

Inserting this into \( S_{ab} e^b = 0 \), one finds a function \( k_1(x) \) satisfying

\[ s_{ab} e^b = 0, \]  

(B.34)
\[
\partial_{t} V_{2} = -k_{1} V_{1} + V_{3} \left(-2\tau_{v} + \frac{\partial_{e} V_{1}}{V_{1}}\right) + \frac{2V_{2}\partial_{e} V_{1}}{V_{1}} + \partial_{e} V_{1}. \quad (B.34)
\]

It follows that the scalar curvature is a negative constant \( R = -6\tau_{v}^{2} < 0 \). The first and third conditions in equation (B.16) give rise to
\[
\varphi = -2\tau_{v} + \log(k_{2}(x)V_{2}), \quad (B.35)
\]
where \( k_{2} = k_{2}(x) \) is an arbitrary function. Setting \( \Sigma = 0 \) in the second condition of equation (B.16), \( k_{1} \) is subjected to
\[
k_{1}(x) = -\frac{k_{2}'(x)}{k_{2}(x)}. \quad (B.36)
\]
Comparison of \( S_{v} = e^{2\varphi} \) in a coordinate basis with the one given by equation (B.35) assures the existence of a function \( k_{3} = k_{3}(x, y) \) such that
\[
k_{3}(x, y) = -\frac{\partial_{e} V_{3}}{V_{1}} + \frac{-2V_{2} + V_{3}^{2}}{V_{1}} \partial_{e} V_{1} - \frac{V_{3}}{V_{1}} \partial_{e} V_{3} + \frac{V_{3}}{V_{1}} \partial_{e} V_{1} + \frac{\partial_{e} V_{2}}{V_{1}} \partial_{e} V_{1} + \frac{V_{3}}{V_{1}} (-k_{1} V_{1} - \tau_{v} V_{3}) + \frac{2\tau_{v} V_{2}}{V_{1}} + \frac{e^{-4\tau_{v}}}{4\tau_{v}} k_{2}',
\]
which is arranged into
\[
\partial_{t} \left[ -\frac{V_{3} e^{2\tau_{v} e_{z}}}{V_{1}} k_{2} \right] = \partial_{t} \left[ \frac{e^{2\tau_{v} e_{z}}}{2k_{2}} \left( -\frac{2V_{2} + V_{3}^{2}}{V_{1}} + \frac{k_{3}}{\tau_{v}} + \frac{e^{-4\tau_{v} e_{z}}}{4\tau_{v}} k_{2}' \right) \right]. \quad (B.38)
\]
This implies the existence of a function \( F_{1} = F_{1}(x, y, z) \) such that the terms in the square bracket on the left-hand side is \( \partial_{t} F_{1} \) and the terms in the square bracket on the left-hand side is \( \partial_{t} F_{1} \). This condition is simplified to
\[
V_{2} = \frac{e^{-4\tau_{v} e_{z}}}{8\tau_{v}^{3}} V_{1}^{2} \left[k_{2}'^{2} + 4e^{4\tau_{v} e_{z}} k_{3} + 4\tau_{v}^{2} k_{2}'^{2} (\partial_{t} F_{1})^{2} - 8e^{2\tau_{v} e_{z}} k_{2}' \partial_{t} F_{1} \right], \quad (B.39a)
\]
\[
V_{3} = -e^{-2\tau_{v} e_{z}} k_{2}' \partial_{t} F_{1} \cdot \quad (B.39b)
\]
The compatibility of equations (B.33) and (B.39a) gives
\[
V_{1} = -\frac{e^{2\tau_{v} e_{z}}}{k_{4}(x, y) - k_{2}(x) \partial_{t} F_{1}(x, y, z)}, \quad (B.40)
\]
where \( k_{4}(x, y) \) is a function independent of \( z \). Together with equation (B.40), the compatibility of equations (B.34) and (B.39a) gives
\[
k_{3} = k_{3}(x), \quad k_{4}(x, y) = k_{2}(x) k_{41}(y). \quad (B.41)
\]
We have exhausted the constrains coming from the vanishing of second obstruction matrix in class 3 of Segre (121).

Let us now move on to obtaining the metric form. Defining \( \bar{x} = \int k_{2}(x) dx \), \( \bar{y} = F_{1}(x, y, z) - \int k_{41}(y) dy \), we get
\[
ds^{2} = 2e^{-2\tau_{v} e_{z}} d\bar{x} d\bar{y} + dz^{2} + dx^{2} \left( -\frac{e^{-4\tau_{v} e_{z}}}{4\tau_{v}^{3}} \frac{\bar{k}(\bar{x})}{\tau_{v}} \right), \quad (B.42)
where $\tilde{k}(x) = k_{31}(x)/k_2(x)^2$. We change coordinates further to

$$\tilde{x} = \int e^{-2\tau, h(i)} dx, \quad \tilde{y} = \tilde{y} - \frac{1}{2\tau_v} e^{2\tau, (\tilde{x} + h(i)) h'(\tilde{x})}, \quad z = \tilde{z} + h(\tilde{x}),$$

(B.43)

and choose $h(\tilde{x})$ to satisfy $\tilde{k}(\tilde{x}) + \tau_v h'(\tilde{x}) = 0$. Then, the resulting metric is equation (B.42) replaced by $(\tilde{x}, \tilde{y}, z) \rightarrow (\tilde{x}, \tilde{y}, \tilde{z})$ and a vanishing $\tilde{k}(\tilde{x})$. Namely, we obtain

$$ds^2 = 2e^{-2\tau_v} d\tilde{x} d\tilde{y} + dz^2 - \frac{1}{4\tau_v^2} e^{-4\tau_v} d\tilde{x}^2,$$

(B.44)

where we have dropped hats from the variables. This metric represents the plane-wave (or equivalently the Kundt class), but not the pp-wave. The KVs are given by

$$K_1 = \partial_x, \quad K_2 = \partial_y, \quad K_3 = y\partial_y + \frac{1}{2\tau_v}\partial_z, \quad K_4 = -\frac{e^{2\tau_v}}{2\tau_v}\partial_x + y^2\partial_y + \frac{y}{\tau_v}\partial_z, \quad (B.45)$$

whose nonvanishing commutation relations are

$$[K_2, K_3] = K_2, \quad [K_2, K_4] = 2K_3, \quad [K_3, K_4] = K_4. \quad (B.46)$$

This subalgebra is $sl(2, \mathbb{R})$. One can easily find that the above metric (B.44) recovers equation (B.47d) below.

**B.3.3. Remarks.** We conclude that the Segre ([21]) allows two metrics (B.31) and (B.44) with 4 KVs. Let us compare our results with those in the literature. The classification of spacetimes admitting 4 KVs for Segre ([21]) has been addressed by [25]. In this work, Kruchkovich contrastively obtained the following four classes of metrics

$$ds^2 = 2(2 - c)e^{\alpha_1} dx_1 dx_2 + e^{\alpha_2} dx_3^2, \quad c \neq 1, 2, \quad (B.47a)$$

$$ds^2 = e^{\alpha_1} (2dx_1 dx_2 - dx_3^2), \quad (B.47b)$$

$$ds^2 = e^{-\alpha_1} \left[ 2dx_1 dx_2 - \frac{4}{\omega} \cos^2 \left( \frac{\omega x_3}{2} \right) dx_3^2 \right] + kdx_3^2, \quad \omega = \sqrt{4 - q^2}, \quad q^2 < 4, \quad (B.47c)$$

$$ds^2 = e^{2\eta_0} dx_1^2 + 2n e^{\eta_0} dx_1 dx_2 + \varepsilon dx_3^2, \quad n \neq 0, \quad \varepsilon = \pm 1. \quad (B.47d)$$

However, this classification turns out to be redundant and consistent with our results. Indeed, one can bring the metrics (B.47a)–(B.47c) into a universal form:

$$ds^2 = 2 dx dy + \frac{a_0 + a_2 z^2}{x^2} dx^2 \pm dz^2, \quad (B.48)$$

where $a_0$ and $a_2$ are constants. This is nothing but the spacetime (B.31) up to the metric signature. The desired coordinate transformations are: For the metric (B.47a), plus sign in equation (B.48) with

$$x_1 = \frac{1}{c} \log \left( \frac{c x}{2 - c} \right), \quad x_2 = y + \frac{z^2}{2cx}, \quad x_3 = \left( \frac{cx}{2 - c} \right)^{-1/c} z,$$

$$a_2 = \frac{1 - c}{c^2}, \quad a_0 = 0. \quad (B.49a)$$
For the metric (B.47b), the minus sign in equation (B.48) with
\[ x_1 = \frac{1}{2} \log(2x), \quad x_2 = y - \frac{z^2}{4x}, \quad x_3 = \frac{z}{\sqrt{2x}}, \quad a_2 = \frac{1}{4}, \quad a_0 = 0. \quad (B.49b) \]

For the metric (B.47c), the minus sign in equation (B.48) with
\[ x_1 = -\frac{1}{q} \log(qx), \quad x_3 = \frac{\sqrt{4 - q^2} z}{2 \sqrt{x}} \sec \left( \frac{\sqrt{4 - q^2}}{2q} \log(qx) \right), \quad a_0 = \frac{k}{q^2}, \quad a_2 = \frac{1}{q^2}. \quad (B.49c) \]

It follows that the metrics admitting 4 KVs in Segre [[21]] type are classified into two: one is the pp-wave (B.31) and the other is the plane-wave (B.44), both of which are homogeneous. This refines the analysis in [25].

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