On the approximation of weakly plurifinely plurisubharmonic functions

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Abstract

In this note, we study the approximation of singular plurifinely plurisubharmonic function \(u\) defined on a plurifinely domain \(\Omega\). Under some conditions, we prove that \(u\) can be approximated by an increasing sequence of plurisubharmonic functions defined on Euclidean neighborhoods of \(\Omega\).

Keywords: complex variables, plurifinely pluripotential theory, plurifinely plurisubharmonic functions

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1. Notation and main result

Let \(D\) be an open set in \(\mathbb{C}^n\) and let \(PSH^- (D)\) be the family of negative plurisubharmonic functions in \(D\). The plurifine topology \(\mathcal{F}\) on a Euclidean open set \(D\) is the smallest topology that makes all plurisubharmonic functions on \(D\) continuous. Notions pertaining to the plurifine topology are indicated with the prefix \(\mathcal{F}\) to distinguish them from notions pertaining to the Euclidean topology on \(\mathbb{C}^n\). For a set \(A \subset \mathbb{C}^n\) we write \(\overline{A}\) for the closure of \(A\) in the one point compactification of \(\mathbb{C}^n\), \(\overline{A}^{\mathcal{F}}\) for the \(\mathcal{F}\)-closure of \(A\) and \(\partial_{\mathcal{F}} A\) for the \(\mathcal{F}\)-boundary of \(A\).

Let \(\Omega\) be a bounded \(\mathcal{F}\)-domain in \(\mathbb{C}^n\). A function \(u : \Omega \to [-\infty, +\infty)\) is said to be \(\mathcal{F}\)-plurisubharmonic if \(u\) is \(\mathcal{F}\)-upper semicontinuous and for every complex line \(l\) in \(\mathbb{C}^n\), the restriction of \(u\) to any \(\mathcal{F}\)-component of the finely open subset \(l \cap \Omega\) of \(l\) is either finely subharmonic or \(\equiv -\infty\). El Kadiri, Fuglede and Wiegerinck \[16\] proved the most important properties of the \(\mathcal{F}\)-plurisubharmonic functions. El Kadiri and Wiegerinck \[18\] defined the complex Monge-Ampère operator for finite \(\mathcal{F}\)-plurisubharmonic functions on an \(\mathcal{F}\)-domain \(\Omega\). Recently, Hong and coauthors have been successfully pushing the theory of \(\mathcal{F}\)-plurisubharmonic functions (see \[12\], \[13\], \[14\], \[19\]). The aim of this note is to study the conditions on \(u\) and \(\Omega\) such that \(u\) can be approximated by an increasing sequence of plurisubharmonic functions defined on Euclidean neighborhoods of \(\Omega\).

When \(\Omega\) is a bounded Euclidean domain with \(C^1\)-boundary. Fornæss and Wiegerinck \[9\] proved that if \(u\) is continuous on \(\overline{\Omega}\) then \(u\) can be approximated uniformly on \(\overline{\Omega}\) by a sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of \(\Omega\).

When \(\Omega\) is a bounded hyperconvex domain. According to the results by \[4\], \[5\], \[8\], \[10\] and other authors, the approximation is possible if the domain \(\Omega\) has the \(\mathcal{F}\)-approximation property and \(u\) belongs to one of the Cegrell’s classes in \(\Omega\).

When \(\Omega\) is bounded \(\mathcal{F}\)-domain. In research \[19\], the authors gave the kind of \(\Omega\) and \(u\) that are in line with the \(\mathcal{F}\)-set up to make the approximation possible.

The purpose of this note is to extend the result of \[19\]. In analogy with the set up of the hyperconvex domain to make the approximation possible, we introduce the following.

\textbf{Definition 1.1.} Let \(\Omega\) be a bounded \(\mathcal{F}\)-hyperconvex domain, i.e., it is a bounded, connected, and \(\mathcal{F}\)-open set such that there exist a negative bounded plurisubharmonic function \(\gamma_\Omega\) defined in a bounded hyperconvex domain \(\Omega\) such that \(\Omega = \Omega' \cap \{\gamma_\Omega > -1\}\) and \(-\gamma_\Omega\) is \(\mathcal{F}\)-plurisubharmonic in \(\Omega\). We say that \(\Omega\) has the \(\mathcal{F}\)-approximation property if there

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exist an increasing sequence of negative plurisubharmonic functions \(\rho_j\) defined on bounded hyperconvex domains \(\Omega_j\) such that \(\Omega \subset \Omega_{j+1} \subset \Omega\) and \(\rho_j \nearrow \rho \in E_0(\Omega)\) a.e. on \(\Omega\) as \(j \to +\infty\). Here

\[
E_0(\Omega) := \{ u \in \mathcal{F}^- PS H^-(\Omega) \cap L^\infty(\Omega) : \int_{\Omega} (dd^c u)^n < +\infty \text{ and } \forall \epsilon > 0, \exists \delta > 0, \overline{\Omega} \cap \{u < -\epsilon\} \subset \Omega' \cap \{\gamma_{\Omega} > -1 + \delta\} \}.
\]

Example 3.3 in [19] showed that there exists a bounded \(\mathcal{F}\)-hyperconvex domain \(\Omega\) that has the \(\mathcal{F}\)-approximation property, moreover, it has no Euclidean interior point exists. For the precise definition and properties of the class \(\mathcal{F}(\Omega)\) we refer the reader to the next section. Our main result is the following theorem.

**Theorem 1.2.** Let \(\Omega\) be a bounded \(\mathcal{F}\)-hyperconvex domain and let \(u \in \mathcal{F}(\Omega)\). Assume that \(\Omega\) has the \(\mathcal{F}\)-approximation property. Then, there exists an increasing sequence of plurisubharmonic functions \(u_j\) defined on Euclidean neighborhoods of \(\Omega\) such that \(u_j \nearrow u\) a.e. on \(\Omega\) as \(j \to +\infty\).

The note is organized as follows. In Section 2, we introduce and investigate the class \(\mathcal{F}(\Omega)\). Section 3 is devoted to prove Theorem 1.2.

### 2. The class \(\mathcal{F}(\Omega)\)

Some elements of pluripotential theory (plurifine potential theory) that will be used throughout the paper can be found in [1]-[22]. We denote by \(\mathcal{F}^- PS H^-(\Omega)\) the set of negative \(\mathcal{F}\)-plurisubharmonic functions defined in \(\mathcal{F}\)-open set \(\Omega\). First, we recall the definition of the complex Monge-Ampère measure for finite \(\mathcal{F}\)-plurisubharmonic functions.

**Definition 2.1.** Let \(\Omega\) be an \(\mathcal{F}\)-open set in \(\mathbb{C}^n\) and let \(QB(\Omega)\) be the trace of \(QB(\mathbb{C}^n)\) on \(\Omega\), where \(QB(\mathbb{C}^n)\) denotes the \(\sigma\)-algebra on \(\mathbb{C}^n\) generated by the Borel sets and the pluripolar subsets of \(\mathbb{C}^n\). Assume that \(u_1, \ldots, u_j \in \mathcal{F}^- PS H^-(\Omega)\) are finite. Using the quasi-Lindelöf property of the plurifine topology and Theorem 2.17 in [18], there exist a pluripolar set \(E \subset \Omega\), a sequence of \(\mathcal{F}\)-open subsets \(\{O_k\}\) and plurisubharmonic functions \(f_{j,k}, g_{j,k}\) defined in Euclidean neighborhoods of \(\overline{O}_k\) such that \(\Omega = E \cup \bigcup_{k=1}^{\infty} O_k\) and \(u_j = f_{j,k} - g_{j,k}\) on \(O_k\). We define \(O_0 := \emptyset\) and

\[
\int_{A} dd^c u_1 \wedge \ldots \wedge dd^c u_n := \sum_{k=1}^{\infty} \int_{\Lambda \cap (O_{k} \cup \bigcup_{j=k}^{\infty} O_j)} dd^c(f_{j,k} - g_{j,k}) \wedge \ldots \wedge dd^c(f_{j,k} - g_{j,k}), A \in QB(\Omega). \tag{2.1}
\]

Theorem 3.6 in [18] implies that the measure defined by (2.1) is independent on \(E\), \(\{O_k\}\), \(\{f_{j,k}\}\) and \(\{g_{j,k}\}\). This measure is called the complex Monge-Ampère measure.

Note that from Theorem 2.17 in [18] and Lemma 4.1 in [18] we infer at \(dd^c u_1 \wedge \ldots \wedge dd^c u_n\) is a non-negative measure on \(QB(\Omega)\). We now give the following definition which is an extension of the class \(\mathcal{F}(\Omega)\) introduced and investigated by Cegrell [6] when \(\Omega\) is a bounded hyperconvex domain in \(\mathbb{C}^n\).

**Definition 2.2.** Let \(\Omega\) be a bounded \(\mathcal{F}\)-hyperconvex domain in \(\mathbb{C}^n\). We denote by \(\mathcal{F}(\Omega)\) the family of negative \(\mathcal{F}\)-plurisubharmonic functions \(u\) defined on \(\Omega\) such that there exist a decreasing sequence \(\{\varphi_j\} \subset E_0(\Omega)\) that converges pointwise to \(u\) on \(\Omega\) and

\[
\sup_{j \geq 1} \int_{\Omega} (dd^c \varphi_j)^n < +\infty.
\]

Furthermore, if \(p > 0\) satisfies

\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-\varphi_j)^p)(dd^c \varphi_j)^n < +\infty
\]

then we say that \(u \in \mathcal{F}_p(\Omega)\).

Note that \(\mathcal{F}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}_p(\Omega) \subset \mathcal{F}(\Omega)\) for all \(p > 0\).
Proposition 2.3. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $u \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$. Then, the statements are holds:

(i) If $\{\varphi_j\} \subseteq \mathcal{E}_0(\Omega)$ such that $\varphi_j \searrow u$ on $\Omega$ and $\sup_{j \geq 1} \int_{\Omega} (-\varphi_j)(dd^c u)^n < +\infty$ then

$$\int_{\Omega} (-\varphi)(dd^c u)^n = \sup_{j \geq 1} \int_{\Omega} (-\varphi_j)(dd^c u)^n, \quad \forall \varphi \in \mathcal{F} - \mathcal{PS} H^{-}(\Omega) \cap L^\infty(\Omega).$$

(ii) If $v \in \mathcal{F} - \mathcal{PS} H^{-}(\Omega)$ with $u \leq v < 0$ then $v \in \mathcal{F}(\Omega)$ and $\int_{\Omega} (dd^c v)^n \leq \int_{\Omega} (dd^c u)^n$.

Proof. The statement follows from Proposition 4.2 in [19] and Proposition 4.3 in [19].

Proposition 2.4. Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$ and let $u \in \mathcal{F}(\Omega)$. If $\{u_j\} \subseteq \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ such that $u_j \searrow u$ in $\Omega$ as $j \nearrow +\infty$ then

$$\sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n < +\infty,$$

and

$$\int_{\Omega} (dd^c \max(u, \rho))^n = \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n$$

for every $\rho \in \mathcal{F} - \mathcal{PS} H^{-}(\Omega) \cap L^\infty(\Omega)$ with $\sup_{\Omega} \rho < 0$. In particular,

$$\int_{\Omega} (dd^c \max(u, -1))^n < +\infty.$$

Proof. Let $\{\varphi_k\} \subseteq \mathcal{E}_0(\Omega)$ such that $\varphi_k \searrow u$ in $\Omega$ as $k \nearrow +\infty$ and

$$\sup_{k \geq 1} \int_{\Omega} (dd^c \varphi_k)^n < +\infty.$$

Since $\max(u, \rho_k) \searrow u_j$ in $\Omega$ as $k \nearrow +\infty$, by Proposition 3.4 in [19] and Proposition 4.2 in [19] we infer at

$$\int_{\Omega} (dd^c u_j)^n = \sup_{k \geq 1} \int_{\Omega} (dd^c \max(u_j, \varphi_k))^n \leq \sup_{k \geq 1} \int_{\Omega} (dd^c \varphi_k)^n.$$

This implies that

$$\sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n \leq \sup_{k \geq 1} \int_{\Omega} (dd^c \varphi_k)^n < +\infty.$$

Now, assume that $\rho \in \mathcal{F} - \mathcal{PS} H^{-}(\Omega) \cap L^\infty(\Omega)$, $\sup_{\Omega} \rho < 0$. Thanks to Proposition 3.4 in [19] and Proposition 2.3 we have

$$\int_{\Omega} (dd^c \max(u, \rho))^n = \sup_{k \geq 1} \int_{\Omega} (dd^c \max(\varphi_k, \rho))^n$$

$$= \sup_{k \geq 1} \int_{\Omega} (dd^c \varphi_k)^n$$

$$= \sup_{k \geq 1} \sup_{j \geq 1} \int_{\Omega} (dd^c \max(u_j, \varphi_k))^n = \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n.$$

The proof is complete.

Proposition 2.5. Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain in $\mathbb{C}^n$. Assume that $u \in \mathcal{F}(\Omega) \cap L^\infty(\Omega)$ and $v \in \mathcal{F} - \mathcal{PS} H^{-}(\Omega)$ such that $(dd^c u)^n \leq (dd^c v)^n$ in $\Omega \cap \{v > -\infty\}$. Then, $u \geq v$ in $\Omega$. 

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Proof. Without loss of generality we can assume that \(-1 \leq u \leq 0\) on \(\Omega\). Let \(j \in \mathbb{N}^*\) and define
\[
v_j := (1 + \frac{1}{j})(v - \frac{1}{j}) \text{ in } \Omega.
\]
Choose \(p > 0\) such that \(j^p < 1 + \frac{1}{j}\). It is easy to see that
\[
(1 + (-u)^p)(dd^c u)^p \leq 2(dd^c v)^p \leq (1 + v_j)^p(dd^c v_j)^p \text{ on } \Omega \cap \{v_j > -\infty\}.
\]
Proposition 4.4 in \([19]\) implies that \(u \geq v\) in \(\Omega\). Letting \(j \to +\infty\) we conclude that \(u \geq v\) in \(\Omega\). The proof is complete.

3. Proof of Theorem 1.2

We need the following.

Lemma 3.1. Let \(\Omega\) be a bounded \(F\)-hyperconvex domain in \(\mathbb{C}^n\) and let \(u, v \in F(\Omega)\) be such that
\begin{enumerate}[(i)]
    \item \(u \geq v\) in \(\Omega\);
    \item \((dd^c u)^p \leq (dd^c v)^p\) on \(\Omega \cap \{v > -\infty\}\);
    \item \(\int_{\Omega} (dd^c \max(u, -1))^p \geq \int_{\Omega} (dd^c \max(v, -1))^p\).
\end{enumerate}
Then, \(u = v\) in \(\Omega\).

Proof. Let \(R > 0\) be such that \(\Omega \subset B(0, R)\) and define \(\rho(z) := |z|^2 - R^2, z \in \mathbb{C}^n\). Let \(\varepsilon, \delta \in (0, 1)\). We set
\[
u_{\varepsilon, \delta} := \max((1 - \varepsilon)u_{\varepsilon, \delta} + \delta \rho, v).
\]
Since \(\sup_{\Omega} \rho < 0\) and \(u \geq v\) in \(\Omega\), by Proposition 2.3 and Proposition 2.4 we conclude by (iii) that
\[
\int_{\Omega} (dd^c v_{\varepsilon, \delta})^p = \int_{\Omega} (dd^c \max(v, -1))^p \\
= \int_{\Omega} (dd^c \max(u, -1))^p = \int_{\Omega} (dd^c u_{\varepsilon, \delta})^p.
\]  \hspace{1cm} (3.1)

Since \(u_{\varepsilon, \delta} = u\) on \(\{v > -2\delta R^2 \varepsilon^{-1}\}\), by Theorem 4.8 in \([18]\) and using (ii), we get
\[
(dd^c v)^p \geq (dd^c u)^p = (dd^c u_{\varepsilon, \delta})^p \text{ on } \{v > -2\delta R^2 \varepsilon^{-1}\}.
\]
Hence, Proposition 2.6 in \([19]\) implies that
\[
(dd^c v_{\varepsilon, \delta})^p \geq (1 - \varepsilon)^p(dd^c u_{\varepsilon, \delta})^p \text{ on } \{v > -2\delta R^2 \varepsilon^{-1}\}. \hspace{1cm} (3.2)
\]
Because \(v_{\varepsilon, \delta} = (1 - \varepsilon)u_{\varepsilon, \delta} + \delta \rho\) in \(\{v < -\delta R^2 \varepsilon^{-1}\} \cup \{v < u - \delta R^2\}\), by Theorem 4.8 in \([18]\) we infer that
\[
(dd^c v_{\varepsilon, \delta})^p \geq (1 - \varepsilon)^p(dd^c u_{\varepsilon, \delta})^p + \delta^p(dd^c \rho)^p \text{ on } \{v < -\delta R^2 \varepsilon^{-1}\} \cup \{v < u - \delta R^2\}.
\]
Combining this with (3.2) we arrive at
\[
(dd^c v_{\varepsilon, \delta})^p \geq (1 - \varepsilon)^p(dd^c u_{\varepsilon, \delta})^p + \delta^p1_{\{v < u - \delta R^2\}}(dd^c \rho)^p \text{ on } \Omega.
\]
It follows that
\[
\int_{\Omega} (dd^c v_{\varepsilon, \delta})^p \geq (1 - \varepsilon)^p \int_{\Omega} (dd^c u_{\varepsilon, \delta})^p + \delta^p \int_{\{v < u - \delta R^2\}} (dd^c \rho)^p.
\]
Letting \(\varepsilon \to 0\) we conclude by (3.1) that
\[
\int_{\{v < u - \delta R^2\}} (dd^c \rho)^p = 0, \forall \delta > 0.
\]
Therefore, by Proposition 2.3 in \([19]\) we infer that \(v \geq u\) in \(\Omega\), and hence, \(u = v\) in \(\Omega\). The proof is complete.
We now claim that Proposition 4.3 in [20] implies that the function \( u \) is \( (dd^c \max(u, -k))^n \) a.e. in \( \Omega \). Indeed, fix \( k \in \mathbb{N} \) such that \( k \geq 1 \). Proposition 2.4 implies that

\[
\int_\Omega (dd^c \max(u, -k))^n = \int_\Omega (dd^c \max(u, -1))^n < +\infty.
\]

Since the measure \( 1_\Omega (dd^c \max(u, -k))^n \) vanishes on all pluripolar subsets of \( \Omega_j \), by Lemma 5.14 in [6] there exists \( u_{j,k} \in \mathcal{F}(\Omega_j) \) such that

\[
(dd^c u_{j,k})^n = 1_\Omega (dd^c \max(u, -k))^n \text{ in } \Omega_j.
\]

Theorem 3.7 in [20] states that the function \( u := (\limsup_{k \to +\infty} u_{j,k})^* \) belongs to \( \mathcal{F}(\Omega_j) \), where * denotes the upper semi-continuous regularization. By Theorem 5.5 in [6] and Proposition 2.3 we infer that \( u_{j,k} \leq u_{j+1,k} \leq \max(u, -k) \) on \( \Omega_j \), and hence,

\[
u_j \leq u_{j+1} \leq u \text{ on } \Omega.
\]

We now claim that

\[
(dd^c u_j)^n \geq (dd^c u)^n \text{ on } \Omega \cap \{ u > -\infty \}
\]

and

\[
\int_\Omega (dd^c u_j)^n \leq \int_\Omega (dd^c \max(u, -1))^n.
\]

Indeed, fix \( a > 0 \) and let \( k \in \mathbb{N}^* \) be such that \( k \geq a \). Since

\[
(dd^c u_{j,k+1})^n \geq 1_{\Omega \setminus \{ u > -a \}}(dd^c u)^n \text{ in } \Omega_j, \forall s \geq 0,
\]

Proposition 4.3 in [20] implies that

\[
(dd^c \max(u_{j,k}, \ldots, u_{j, k+1}))^n \geq 1_{\Omega \setminus \{ u > -a \}}(dd^c u)^n \text{ in } \Omega_j, \forall s \geq 0.
\]

Main Theorem in [7] states that

\[
(dd^c (\sup_{k \geq 0} u_{j,k+1}))^n \geq 1_{\Omega \setminus \{ u > -a \}}(dd^c u)^n \text{ in } \Omega_j
\]

because \( \max(u_{j,k}, \ldots, u_{j, k+1}) \) is \( \sup_{k \geq 0} u_{j,k+1} \) a.e. in \( \Omega_j \) as \( s \to +\infty \). Moreover, since \( \sup_{k \geq 0} u_{j,k+1} \) belongs to \( \Omega_j \), we infer that

\[
(dd^c u)^n \geq 1_{\Omega \setminus \{ u > -\infty \}}(dd^c u)^n \text{ in } \Omega_j.
\]

Letting \( a \to +\infty \), we get

\[
(dd^c u)^n \geq (dd^c u)^n \text{ on } \Omega \cap \{ u > -\infty \}.
\]

Now, by Lemma 3.3 in [1] and Corollary 3.4 in [1] we have

\[
\int_{\Omega_j} (dd^c u_j)^n = \lim_{k \to +\infty} \int_{\Omega_j} (dd^c (\sup_{k \geq 0} u_{j,k+1}))^n \leq \sup_{k \geq 1} \int_{\Omega_j} (dd^c u_{j,k})^n = \int_{\Omega} (dd^c \max(u, -1))^n.
\]

This proves the claim. Let \( v \) be the least \( \mathcal{F} \)-upper semicontinuous majorant of \( \sup_{j \geq 1} u_j \) in \( \Omega \). Then, \( v \in \mathcal{F}-PSH^- (\Omega) \) and \( v \leq u \) on \( \Omega \). By Theorem 4.5 in [17] and using (3.3) we infer that

\[
(dd^c v)^n \geq (dd^c u)^n \text{ on } \Omega \cap \{ v > -\infty \}.
\]

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We claim that $v \in \mathcal{F}(\Omega)$. Indeed, put $v_k := \max(v, k\rho)$, where $k \in \mathbb{N}^*$. Proposition 3.4 in [19] implies that $v_k \in E_0(\Omega)$. Since $\max(u_j, k\rho) \not\nearrow v_k$ a.e. in $\Omega$ as $j \not\nearrow +\infty$, by Proposition 2.7 in [19] and Lemma 3.3 in [1] we obtain by (3.4) that

$$\int_\Omega (dd^c v_k)^n \leq \liminf_{j \to +\infty} \int_\Omega (dd^c \max(u_j, k\rho))^n \leq \liminf_{j \to +\infty} \int_\Omega (dd^c u_j)^n \leq \int_\Omega (dd^c \max(u, -1))^n.$$

Since $v_k \not\nearrow v$, by Proposition 2.4 we obtain $v \in \mathcal{F}(\Omega)$. This proves the claim. Now, again by Proposition 2.7 in [19] and Proposition 3.4 in [19] we have

$$\int_\Omega (dd^c \max(v, -1))^n \leq \liminf_{k \to +\infty} \int_\Omega (dd^c \max(v_k, -1))^n \leq \liminf_{k \to +\infty} \int_\Omega (dd^c v_k)^n \leq \int_\Omega (dd^c \max(u, -1))^n.$$

Combining this with (3.5) and using Lemma 3.1 we conclude that $v = u$ in $\Omega$. Thus, $u_j \not\nearrow u$ a.e. in $\Omega$ as $j \not\nearrow +\infty$. The proof is complete.

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