IDEAL CONTAINMENTS UNDER FLAT EXTENSIONS

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Abstract. Let \( \varphi : S = k[y_0, ..., y_n] \to R = k[y_0, ..., y_n] \) be given by \( y_i \to f_i \) where \( f_0, ..., f_n \) is an \( R \)-regular sequence of homogeneous elements of the same degree. A recent paper shows for ideals, \( I_\Delta \subseteq S \), of matroids, \( \Delta \), that \( I_\Delta^{(m)} \subseteq I^r \) if and only if \( \varphi_*(I_\Delta^{(m)}) \subseteq \varphi_*(I_\Delta)^r \) where \( \varphi_*(I_\Delta) \) is the ideal generated in \( R \) by \( \varphi(I_\Delta) \). We prove this result for saturated homogeneous ideals \( I \) of configurations of points in \( \mathbb{P}^n \) and use it to obtain many new counterexamples to \( I^{(rn-n+1)} \subseteq I^r \) from previously known counterexamples.

1. Introduction

Let \( R \) be a commutative Noetherian domain. Let \( I \) be an ideal in \( R \). We define the \( m \)th symbolic power of \( I \) to be the ideal

\[
I^{(m)} = R \cap \bigcap_{P \in \text{Ass}_R(I)} I^m R_P \subseteq R_{(0)}.
\]

In this note we shall be interested in symbolic powers of homogeneous ideals of 0-dimensional subschemes in \( \mathbb{P}^n \). In the case that the subscheme is reduced, the definition of the symbolic power takes a rather simple form by a theorem of Zariski and Nagata [11] and does not require passing to the localizations at various associated primes. Let \( I \subseteq k[\mathbb{P}^n] \) be a homogeneous ideal of reduced points, \( p_1, ..., p_k \), in \( \mathbb{P}^n \) with \( k \) a field of any characteristic. Then \( I = I(p_1) \cap \cdots \cap I(p_k) \) where \( I(p_i) \subseteq k[\mathbb{P}^n] \) is the ideal generated by all forms vanishing at \( p_i \), and the \( m \)th symbolic power of \( I \) is simply \( I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_k)^m \).

In [10], Ein, Lazarsfeld and Smith proved that if \( I \subseteq k[\mathbb{P}^n] \) is the radical ideal of a 0-dimensional subscheme of \( \mathbb{P}^n \), where \( k \) is an algebraically closed field of characteristic 0, then \( I^{(mr)} \subseteq (I^{(r+1-n)})^m \) for all \( m \in \mathbb{N} \) and \( r \geq n \). Letting \( r = n \), we get that \( I^{(mn)} \subseteq I^m \) for all \( m \in \mathbb{N} \). Hochster and Huneke in [15] extended this result to all ideals \( I \subseteq k[\mathbb{P}^n] \) over any field \( k \) of arbitrary characteristic.

In [5] Bocci and Harbourne introduced a quantity \( \rho(I) \), called the resurgence, associated to a nontrivial homogeneous ideal \( I \) in \( k[\mathbb{P}^n] \), defined to be \( \sup \{s/t : I^{(s)} \not\subseteq I^t \} \). It is seen immediately that if \( \rho(I) \) exists, then for \( s > \rho(I)t \), \( I^{(s)} \subseteq I^t \). The results of [10] guarantee that \( \rho(I) \) exists since \( I^{(mn)} \subseteq I^m \) implies that \( \rho(I) \leq n \) for an ideal \( I \) in \( k[\mathbb{P}^n] \). For an ideal \( I \) of points in \( \mathbb{P}^2 \), \( I^{(mn)} \subseteq I^m \) gives \( I^{(4)} \subseteq I^2 \). According to [5] Huneke asked if \( I^{(3)} \subseteq I^2 \) for a homogeneous ideal \( I \) of points in \( \mathbb{P}^2 \). More generally Harbourne conjectured in [3] that if \( I \subseteq k[\mathbb{P}^n] \) is a homogeneous ideal, then \( I^{(rn-(n-1))} \subseteq I^r \) for all \( r \). This led to the conjectures by Harbourne and Huneke in [13] for ideals \( I \) of points...
that \( I^{(mn-n+1)} \subseteq m^{(m-1)(n-1)}I^m \) and \( I^{(mn)} \subseteq m^{m(n-1)}I^m \) for \( m \in \mathbb{N} \).

The second conjecture remains open. Cooper, Embree, Ha and Hoeful give a counterexample in [2] to the first for \( n = 2 = m \) for a homogeneous ideal \( I \subseteq k[\mathbb{P}^2] \). The ideal \( I \) in this case is \( I = (xy^2, y^2, x^2, xyz) = (x^2, y) \cap (y^2, z) \cap (z^2, x) \) whose zero locus in \( \mathbb{P}^2 \) is the 3 coordinate vertices of \( \mathbb{P}^2 \), \([0 : 0 : 1]\), \([0 : 1 : 0]\) and \([1 : 0 : 0]\) together with 3 infinitely near points, one at each of the vertices, for a total of 6 points. Clearly the monomial \( x^2y^2z^2 \in (x^2, y)^3 \cap (y^2, z)^3 \cap (z^2, x)^3 \) so \( x^2y^2z^2 \) is in \( I^{(3)} \). Note \( xyz \in I \) so \( x^2y^2z^2 \in I^2 \), but \( x^2y^2z^2 \notin I^2 \).

Shortly thereafter a counterexample to the containment \( I^{(3)} \nsubseteq I^2 \) was given by Dumnicki, Szemberg and Tutaj-Gasinska in [9]. In this case \( I \) is the ideal of the 12 points dual to the 12 lines of the Hesse configuration. The Hesse configuration consists of the 9 flex points of a smooth cubic and the 12 lines through pairs of flexes. Thus \( I \) defines 12 points lying on 9 lines. Each of the lines goes through 4 of the points, and each point has 3 of the lines going through it. Specifically \( I \) is the saturated radical homogeneous ideal \( I = (x(y^3-z^3), y(x^3-z^3), z(x^3-y^3)) \subset \mathbb{C}[\mathbb{P}^2] \). Its zero locus is the 3 coordinate vertices of \( \mathbb{P}^2 \) together with the 9 intersection points of any 2 of the forms \( x^3-y^3 \), \( x^3-z^3 \) and \( y^3-z^3 \). The form \( F = (x^3-y^3)(x^3-z^3)(y^3-z^3) \) defining the 9 lines belongs to \( I^{(3)} \) since for each point in the configuration, 3 of the lines in the zero locus of \( F \) pass through the point, but \( F \notin I^2 \) and hence \( I^{(3)} \nsubseteq I^2 \). (Of course this also means that \( I^{(3)} \nsubseteq mI^2 \).)

More generally, \( I = (x(y^n-z^n), y(x^n-z^n), z(x^n-y^n)) \) defines a configuration of \( n^2 + 3 \) points called a Fermat configuration [1]. For \( n \geq 3 \), we again have \( I^{(3)} \nsubseteq I^2 \) [14, 17] over any field of characteristic not 2 or 3 containing \( n \) distinct \( n \)th roots of 1.

Subsequent counterexamples to \( I^3 \subseteq I^2 \) were given in [11, 2, 14, 8] and [17] including related counterexamples to \( I^{(n-n+1)} \subseteq I^r \) for ideals of points in \( \mathbb{P}^n \) in positive characteristic given in [14]. All of the counterexamples to \( I^3 \subseteq I^2 \) are ideals of points where the points are singular points of multiplicity at least 3 of a configuration of lines. By considering flat morphisms \( \mathbb{P}^n \to \mathbb{P}^n \), we obtain many new counterexamples to \( I^{(rn-n+1)} \subseteq I^r \), taking \( I \) to be the ideal of the fibers over the points of previously known counterexamples.

The idea for this comes from [12]. Suppose \( \Delta \) is a matroid on \( s = \{1,..., s\} \) of dimension \( s-1-c \) and and let \( f_1,..., f_s \in R = k[y_0,..., y_n] \) be homogeneous polynomials that form an R-regular sequence, \( n \geq c \). Suppose now that \( \varphi : S = k[y_1,..., y_s] \to R \) is a \( k \)-algebra map defined by \( y_i \to f_i \). Then [12] shows that if \( I_\Delta \subseteq S \) is the ideal of the matroid and \( m \) and \( r \) are positive integers, then \( I^{(m)}_\Delta \subseteq I^{(r)}_\Delta \) if and only if \( \varphi_*(I^{(m)}_\Delta) \subseteq \varphi_*(I^{(r)}_\Delta) \) where \( \varphi_*(I^{(m)}_\Delta) \) denotes the ideal generated by \( \varphi(I^{(m)}_\Delta) \) in \( R \). Of course a natural question is whether \( I^{(m)} \subseteq I^{(r)} \) if and only if \( \varphi_*(I^{(m)} \subseteq \varphi_*(I^{(r)} \) for any saturated homogeneous ideal. The current note answers this question in the affirmative for ideals \( I \) of points in \( \mathbb{P}^n \), relying on the ideas in [12].

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2. Results

Throughout this note, let \( R = S = k[y_0, ..., y_n] \) and let \( \{f_0, ..., f_n\} \subseteq R \) be an \( R \)-regular sequence of homogeneous elements of \( R \) of the same degree. Let \( \varphi : S \to R \) be the \( k \)-algebra map given by \( y_i \mapsto f_i \). For an ideal \( I \subseteq S \), let \( \varphi_*(I) \subseteq R \) denote the ideal generated by \( \varphi(I) \).

**Lemma 1.** Let \( \varphi : S \to R \) be as above. Then \( R \) is a free graded \( S \)-module, hence \( R \) is faithfully flat as an \( S \)-module.

**Proof.** It suffices to show that \( R \) is free over \( S \) since free modules are faithfully flat modules. Note that \( \varphi \) is injective since \( \{f_0, ..., f_n\} \) is a regular sequence. It follows that \( S \cong k[f_0, ..., f_n] \subseteq R \). So we identify \( S \) with \( k[f_0, ..., f_n] \) and show that \( R \) is free over \( k[f_0, ..., f_n] \). Since \( \{f_0, ..., f_n\} \) is a maximal homogeneous \( R \)-regular sequence, it is a homogeneous system of parameters (sop). The reason is that every regular sequence is part of an sop and because \( R \) is Cohen-Macaulay (CM), every sop is a regular sequence (\( \text{depth} R = \dim R \)) and so if \( \{f_0, ..., f_n\} \) is a maximal regular sequence, then it is an sop. Since \( R = k[\mathbb{P}^n] \) is a positively graded affine \( k \)-algebra, the fact that \( \{f_0, ..., f_n\} \) is a homogeneous sop is equivalent to \( R \) being a finite \( S \)-module by [6, Theorem 1.5.17]. Since both \( R \) and \( S \) are CM, \( \text{depth} R = \dim R = n + 1 = \dim S = \text{depth} S \). By the Auslander-Buchsbaum formula [11, Exercise 19.8] [16, Theorem 15.3], \( \text{pd}_S R + \text{depth} R = \text{depth} S \). It follows that \( \text{pd}_S R = 0 \). So looking at the minimal free resolution of \( R \) as an \( S \)-module, we see that \( R \) is a free \( S \)-module. Therefore \( R \) is a faithfully flat \( S \)-module. \( \square \)

**Lemma 2.** Let \( I \subseteq S \) be a homogeneous saturated ideal defining a 0-dimensional subscheme of \( \mathbb{P}^n \). Then \( \varphi_*(I) \subseteq R \) also defines a 0-dimensional subscheme of \( \mathbb{P}^n \).

**Proof.** We start by showing that \( R/\varphi_*(I) \) has the same Krull dimension as \( S/I \). By the graded Auslander-Buchsbaum formula, \( \text{pd}_S(R/\varphi_*(I)) + \text{depth}(R/\varphi_*(I)) = \text{depth}(S) = \text{pd}_S(S/I) + \text{depth}(S/I) \). By 3.1 in [12], \( S/I \) and \( R/\varphi_*(I) \) have the same graded Betti numbers so \( \text{pd}_S(S/I) = \text{pd}_S(R/\varphi_*(I)) \). Therefore \( \text{depth}(S/I) = \text{depth}(R/\varphi_*(I)) \). By 3.1 in [12] again, \( S/I \) is Cohen-Macaulay (CM) if and only if \( R/\varphi_*(I) \) is CM. Since \( I \) defines an ideal of points and is saturated, we have that \( S/I \) is CM. It follows that \( R/\varphi_*(I) \) is CM. For CM modules, the depth is the dimension so that \( \dim S/I = \dim R/\varphi_*(I) \). Now since \( S/I \) and \( R/\varphi_*(I) \) are both CM, \( \text{Ass}(R/\varphi_*(I)) \) and \( \text{Ass}(S/I) \) are both unmixed with their elements having height \( \text{ht}(\varphi_*(I)) \) and \( \text{ht}(I) \) respectively. But \( \text{ht}(\varphi_*(I)) = \text{ht}(I) \) since \( \dim S/I = \dim R/\varphi_*(I) \). It follows that the elements of \( \text{Ass}(R/\varphi_*(I)) \) are all ideals of points. It follows that \( \varphi_*(I) \) defines a 0-dimensional subscheme of \( \mathbb{P}^n \). \( \square \)

**Lemma 3.** Let \( I \subseteq S \) be a saturated homogeneous ideal such that the zero locus of \( I \) in \( \mathbb{P}^n \) is 0-dimensional. Let \( \varphi : S \to R \) be as above. Then \( \varphi_*(I^{(m)}) = \varphi_*(I)^{(m)} \).

**Proof.** By Lemma 2, \( \varphi_*(I) \) is the defining ideal of a 0-dimensional subscheme so that \( (\varphi_*(I))^{(m)} = \text{Sat}((\varphi_*(I))^{(m)}) \) where \( \text{Sat}((\varphi_*(I))^{(m)}) \) denotes the saturation of the ideal \( (\varphi_*(I))^{(m)} \). An ideal and its saturation have the same graded homogeneous components for high enough degree so that for \( t \gg 0 \), \( ((\varphi_*(I))^{(m)})_t = ((\varphi_*(I))^{(m)})_t \).

Using again that the symbolic power of an ideal of a 0-dimensional subscheme in \( \mathbb{P}^n \) is the saturation of the ordinary power, \( I^{(m)} = \text{Sat}(I^m) \), we have that \( (I^{(m)})_t = (I^m)_t \).
for \( t \gg 0 \). Therefore \((\varphi_*(I^{(m)}))_t = (I^{(m)} \otimes_S R)_t = (I^m \otimes_R R)_t = (\varphi_*(I^m))_t \) for \( t \gg 0 \). Since \( \varphi \) is a ring map, \( \varphi_*(I^m) = (\varphi_*(I))^m \). This gives that \((\varphi_*(I^{(m)}))_t = ((\varphi_*(I))^m)_t \) for \( t \gg 0 \).

The last two paragraphs imply that \((\varphi_*(I))^{(m)}_t = \varphi_*(I^{(m)})_t \) for \( t \gg 0 \). Recall that \((\varphi_*(I))^{(m)}_t \) is saturated since it is the saturation of \((\varphi_*(I))^m_t \) and \( \varphi_*(I^{(m)})_t \) is saturated by Lemma 3.1 in [12]. Two saturated graded homogeneous ideals that agree in degree \( t \) for \( t \gg 0 \), agree in all degrees. Hence \((\varphi_*(I))^{(m)}_t = \varphi_*(I^{(m)})_t \). \( \square \)

**Theorem 4.** Let \( I \subseteq S \) be a saturated homogeneous ideal such that \( V(I) \subseteq \mathbb{P}^n \) is a 0-dimensional subscheme. Let \( \varphi : S \to R \) be given by \( y_i \to f_i \), \( 0 \leq i \leq n \), where \( \{f_0, \ldots, f_n\} \) is an \( R \)-regular sequence of homogeneous elements of \( R \) of the same degree.

Let \( \varphi_*(I) \) denote the ideal in \( R \) generated by \( \varphi(I) \). Then \( I^{(m)} \subseteq I^r \) if and only if \((\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r \).

**Proof.** (\( \implies \)) Suppose that \( I^{(m)} \subseteq I^r \). Then \( \varphi(I^{(m)}) \subseteq \varphi(I^r) \) and so \( \varphi(I^{(m)}) \subseteq \varphi_*(I^r) \).

Since \( \varphi \) is a homomorphism, \( \varphi(I^r) = (\varphi(I))^r \). Note that \( \varphi(I^r) \) generates \( \varphi_*(I^r) \) in \( R \) and \((\varphi(I))^r \) generates \((\varphi_*(I))^r \) in \( R \). It follows that \( \varphi_*(I^r) = (\varphi_*(I))^r \) since they have the same generating set. Now applying Lemma 3 we have that \((\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r \) concluding the forward direction.

(\( \impliedby \)) Suppose now that for some homogeneous ideals \( I \) and \( J \) of \( S \), \( I \not\subseteq J \) but \( \varphi_*(I) \subseteq \varphi_*(J) \). Then there is a homogeneous element \( f \in I \setminus J \) such that \( \varphi(f) \in \varphi_*(J) \). We may assume with no loss in generality that \( I = (f) \). We have the sequence

\[
0 \to I \cap J \to I \oplus J \to I + J \to 0
\]

with the first map given by \( g \mapsto (g, -g) \) and the second map given by \( (h, r) \mapsto h + r \). It is clear that the sequence is exact. Since \( \varphi \) is faithfully flat, we get an exact sequence

\[
0 \to \varphi_*(I \cap J) \to \varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(I + J) \to 0.
\]

Since \( \varphi_*(I) \subseteq \varphi_*(J) \), \( \varphi_*(I + J) = \varphi_*(J) \). Then the map \( \varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(J) \) has kernel \( \varphi_*(I) \). It follows that \( \varphi_*(I \cap J) = \varphi_*(I) \). This is impossible since the generators of \( \varphi_*(I \cap J) \) are the images of the generators of \( I \cap J \) and thus have degree greater than degree \( f \) and hence greater than degree of \( \varphi(f) \) which generates \( \varphi_*(I) = I \otimes_S R \neq 0 \).

So it is the case that \( \varphi(f) \not\in \varphi_*(J) \). Hence \( \varphi_*(I) \not\subseteq \varphi_*(J) \). Therefore if \( I^{(m)} \not\subseteq I^r \), then by Lemma 3, \( (\varphi_*(I))^{(m)} = \varphi_*(I^{(m)}) \not\subseteq (\varphi_*(I))^r \). Hence \((\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r \) if and only if \( I^{(m)} \subseteq I^r \). \( \square \)

3. Examples

Using the above result, we obtain many new counterexamples to the containment \( I^{(3)} \subseteq I^2 \) of ideals in \( k[\mathbb{P}^2] \) and more generally counterexamples to the containment

\[
I^{(nr-n+1)} \subseteq I^r \quad (*)
\]

in \( \mathbb{P}^n \). In particular if \( I \subseteq k[\mathbb{P}^n] \) gives a counterexample to \((*)\), then \( \varphi_*(I) \) is a counterexample for any choice of homogeneous regular sequence \( \{f_0, \ldots, f_n\} \) of elements of the same degree. We illustrate this below with a few examples.
Example 1. In this example, we work over \( \mathbb{C} \). In [9], the Fermat configuration, for \( n = 3 \), was considered and its ideal \( I = (x(y^2 - z^3), y(x^3 - z^3), z(x^3 - y^3)) \subseteq \mathbb{C}[x, y, z] \) was found to be a counterexample to the containment \( I^{(3)} \subseteq I^2 \). Recall the configuration consists of the 3 coordinate vertices and the 9 intersection points of \( y^3 - z^3 \) and \( x^3 - z^3 \). The ideal \( I \) is radical and all of the points in the configuration are reduced points. Now let \( \varphi: \mathbb{C}[\mathbb{P}^2] \rightarrow \mathbb{C}[\mathbb{P}^2] \) be induced by \( x \mapsto f = x^2 + y^2, y \mapsto g = y^2 + z^2 \) and \( z \mapsto h = x^2 + z^2 \). One easily checks that \( \{ x^2 + y^2, y^2 + z^2, x^2 + z^2 \} \) is a \( \mathbb{C}[\mathbb{P}^2] \)-regular sequence. Then \( \varphi \) induces a map of schemes \( \varphi^#: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) which is faithfully flat. Consider the scheme-theoretic fibers of \( \varphi^# \) over the Fermat configuration and call it the fibered Fermat configuration. Note that the fibered Fermat configuration is 0-dimensional. Since \( \varphi^# \) has degree 4, the fibers consist of 48 points of \( \mathbb{P}^2 \) where we count with multiplicity. The fibered Fermat configuration gives rise to the radical ideal \( \varphi_*(I) = (f(g^3 - h^3), g(f^3 - h^3), h(f^3 - g^3)) \subseteq \mathbb{C}[\mathbb{P}^2] \) and by analyzing the ideal we see that the configuration consists of 4 multiplicity 1 points over each of the 3 coordinate vertices, given by \( f = 0 = g, f = 0 = h \) and \( g = 0 = h \). The remaining 36 points, each of multiplicity 1, in the configuration are the zero locus of \( f^3 - h^3 \) and \( f^3 - g^3 \). Since \( I^{(3)} \not\subseteq I^2 \), we have by Theorem 3 that \( \varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2 \).

Example 2. We give another example of a fibered Fermat configuration whose ideal also gives a counterexample to the containment \( I^{(3)} \subseteq I^2 \). The difference here is that 36 of the points in the configuration have multiplicity 1 while the remaining 3 points each have multiplicity 4. So there are still 48 points counting with multiplicity. Let \( \varphi: \mathbb{C}[\mathbb{P}^2] \rightarrow \mathbb{C}[\mathbb{P}^2] \) be induced by \( x \mapsto f = x^2, y \mapsto g = y^2 \) and \( z \mapsto h = z^2 \). This faithfully flat ring map induces a morphism of schemes \( \varphi^#: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) that is also flat. The fibers of \( \varphi^# \) over the Fermat configuration gives the fibered Fermat configuration that consists of the 36 points, each of multiplicity 1, of intersection of the degree 6 forms \( f^3 - g^3 \) and \( g^3 - h^3 \). The configuration has 3 more points each of multiplicity 4 over the 3 coordinate points. They are the zero loci of \( f = 0 = g, f = 0 = h \) and \( g = 0 = h \). So the fibered Fermat configuration here has points that are not all reduced. By Theorem 3, its nonradical ideal \( \varphi_*(I) \) is a counterexample to the containment \( \varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2 \).

Example 3. Similarly for the Fermat configurations considered in [14] for \( n \geq 3 \), we can construct new configurations of points, that may or may not be reduced in \( \mathbb{P}^2 \), that are the fibers of a morphism of schemes \( \varphi^#: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \). The morphism \( \varphi^# \) is induced by the ring map \( \varphi: \mathbb{C}[\mathbb{P}^2] \rightarrow \mathbb{C}[\mathbb{P}^2] \) given by \( x \mapsto f, y \mapsto g \) and \( z \mapsto h \) where \( \{ f, g, h \} \) is a homogeneous \( \mathbb{C}[\mathbb{P}^2] \)-regular sequence of the same degree. The Fermat configuration gives rise to a radical ideal \( I = (x(y^i - z^j), y(x^j - z^i), z(x^j - y^i)) \subseteq \mathbb{C}[\mathbb{P}^2], j \geq 3, \) and for a choice of \( \{ f, g, h \} \), the fibered Fermat configuration gives rise to an ideal \( \varphi_*(I) = (f(g^j - h^i), g(f^j - h^i), h(f^j - g^i)), j \geq 3, \) not necessarily radical, that is also a counterexample to \( \varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2 \). Here the Fermat configuration consists of the reduced \( j^2 \) points of intersection of \( y^i - z^j \) and \( x^j - y^i \) together with the 3 coordinate vertices for a total of \( j^2 + 3 \) points. If the degree of the homogeneous elements in \( \{ f, g, h \} \) is \( d \), then the fibered configuration consists of the \( d^2 j^2 \) points of intersection of \( g^i - h^j \) and \( f^j - h^i \) together with the 3\( d^2 \) fiber points over the three coordinate vertices that are the solutions of the three equations \( f = 0 = g, f = 0 = h \) and \( g = 0 = h \), counted with multiplicity. Again the points in the fibered configuration may or may not be reduced.
Example 4. Now we consider an example given in [4] that is inspired by the example of the Fermat configuration. Let \( k = \mathbb{Z}/3\mathbb{Z} \) and let \( K \) be an algebraically closed field containing \( k \). Note that \( \mathbb{P}^2_k \) has 13 \( k \)-points and 13 \( k \)-lines such that each line contains 4 of the points and each point is incident to 4 of the lines. The forms \( xy(x^2 - y^2), xz(x^2 - z^2) \) and \( yz(y^2 - z^2) \) vanish at all 13 points of \( \mathbb{P}^2_k \) but the form \( x(x^2 - y^2)(x^2 - z^2) \) does not vanish at the point \([1 : 0 : 0]\). One checks easily that the ideal \( I = (xy(x^2 - y^2), xz(x^2 - z^2), yz(y^2 - z^2), x(x^2 - y^2)(x^2 - z^2)) \subseteq k[\mathbb{P}^2_k] \) is radical and its zero locus is the 13 \( k \)-points of \( \mathbb{P}^2_k \). Then \( F = x(x-z)(x+z)(x^2-y^2)((x-z)^2-y^2)((x+z)^2-y^2) \) defines 9 lines meeting at 12 points with each point incident to 3 of the lines. It is not hard to see that \( F \in I^{(3)} \) but \( F \notin I^2 \). So the reduced configuration that comes from \( \mathbb{P}^2_k \) with the point \([1 : 0 : 0]\) removed together with all its incident lines gives rise to an ideal that is a counterexample to the containment \( I^{(3)} \subseteq I^2 \). Let \( \varphi : k[\mathbb{P}^2_k] \to k[\mathbb{P}^2_k] \) be the ring map \( x \to f = x^2, y \to g = y^2 \) and \( z \to h = z^2 \). Applying the degree 4 morphism of schemes \( \varphi^# : \mathbb{P}^2_k \to \mathbb{P}^2_k \), induced by \( \varphi \), and taking its fibers over the \( k \)-points, we get a configuration of 48 points. For each point in the original configuration, we get 4 points in the fibred configuration. The points in this new configuration are not all reduced. For instance over the point \([0 : 0 : 1]\), the fiber of \( \varphi^# \) is a point of multiplicity 4 in \( \mathbb{P}^2_k \) given by the vanishing of \( y^2 \) and \( x^2 \). The ideal of the fibred configuration as schemes is the ideal \( \varphi_*(I) = (fg(f^2 - g^2), fh(f^2 - h^2), gh(g^2 - h^2), f(f^2 - g^2)(f^2 - h^2)) \). This ideal is not radical and since \( \{f, g, h\} \subseteq \mathbb{P}^2_K \) is a regular sequence, we have by Theorem 3 that \( \varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2 \). If instead we take \( f = x^2 + y^2, g = y^2 + z^2 \) and \( h = x^2 + z^2 \) in the above example, then the fibred configuration we obtain is a reduced configuration and the ideal \( \varphi_*(I) \) is a radical ideal satisfying \( \varphi_*(I)^{(3)} \not\subseteq \varphi_*(I)^2 \).

Variations of the above example are considered in \( \mathbb{P}^n \) for various \( n \) in [4], giving counterexamples for the more general conjecture \( I^{(nr-n+1)} \subseteq I^r \). We can apply our result to these to obtain new counterexamples to the more general containment.

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