Answering UCQs under updates
and in the presence of integrity constraints

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Abstract

We investigate the query evaluation problem for fixed queries over fully dynamic databases where tuples can be inserted or deleted. The task is to design a dynamic data structure that can immediately report the new result of a fixed query after every database update. We consider unions of conjunctive queries (UCQs) and focus on the query evaluation tasks testing (decide whether an input tuple $\pi$ belongs to the query result), enumeration (enumerate, without repetition, all tuples in the query result), and counting (output the number of tuples in the query result).

We identify three increasingly restrictive classes of UCQs which we call $t$-hierarchical, $q$-hierarchical, and exhaustively $q$-hierarchical UCQs. Our main results provide the following dichotomies: If the query’s homomorphic core is $t$-hierarchical ($q$-hierarchical, exhaustively $q$-hierarchical), then the testing (enumeration, counting) problem can be solved with constant update time and constant testing time (delay, counting time). Otherwise, it cannot be solved with sublinear update time and sublinear testing time (delay, counting time).

We also study the complexity of query evaluation in the dynamic setting in the presence of integrity constraints, and we obtain according dichotomy results for the special case of small domain constraints (i.e., constraints which state that all values in a particular column of a relation belong to a fixed domain of constant size).

*) To be precise: our lower bound for the enumeration problem is obtained only for queries that are self-join free, with sublinear we mean $O(n^{1-\varepsilon})$ for $\varepsilon > 0$ and where $n$ is the size of the active domain of the current database, and all our lower bounds rely on the OV-conjecture and/or the OMv-conjecture, two algorithmic conjectures on the hardness of the Boolean orthogonal vectors problem and the Boolean online matrix-vector multiplication problem.

1 Introduction

Dynamic query evaluation refers to a setting where a fixed query $q$ has to be evaluated against a database that is constantly updated [19]. In this paper, we study dynamic query evaluation for unions of conjunctive queries (UCQs) on relational databases that may be updated by inserting or deleting tuples. A dynamic algorithm for evaluating a query $q$ receives an initial database and performs a preprocessing phase which builds a data structure that contains a suitable representation of the database and the result of $q$ on this database. After every database update, the data structure is updated so that it suitably represents the new database $D$ and the result $q(D)$ of $q$ on this database.

To solve the counting problem, such an algorithm is required to quickly report the number $|q(D)|$ of tuples in the current query result, and the counting time is the time used to compute
this number. To solve the testing problem, the algorithm has to be able to check for an arbitrary input tuple $\vec{a}$ if $\vec{a}$ belongs to the current query result, and the testing time is the time used to perform this check. To solve the enumeration problem, the algorithm has to enumerate $q(D)$ without repetition and with a bounded delay between the output tuples. The update time is the time used for updating the data structure after having received a database update. We regard the counting (testing, enumeration) problem of a query $q$ to be tractable under updates if it can be solved by a dynamic algorithm with linear preprocessing time, constant update time, and constant counting time (testing time, delay).

This setting has been studied for conjunctive queries (CQs) in our previous paper [5], which identified a class of CQs called $q$-hierarchical that precisely characterises the tractability frontier of the counting problem and the enumeration problem for CQs under updates: For every $q$-hierarchical CQ, the counting problem and the enumeration problem can be solved with linear preprocessing time, constant update time, constant counting time, and constant delay. And for every CQ that is not equivalent to a $q$-hierarchical CQ, the counting problem (and for the case of self-join free queries, the enumeration problem) cannot be solved with sublinear update time and sublinear counting time (delay), unless the OMv-conjecture or the OV-conjecture (the OMv-conjecture) fails. The latter are well-known algorithmic conjectures on the hardness of the Boolean online matrix-vector multiplication problem (OMv) and the Boolean orthogonal vectors problem (OV) [18, 1], and “sublinear” means $O(n^{1-\epsilon})$, where $\epsilon > 0$ and $n$ is the size of the active domain of the current database.

Our contribution. We identify a new subclass of CQs which we call $t$-hierarchical, which contains and properly extends the class of $q$-hierarchical CQs, and which precisely characterises the tractability frontier of the testing problem for CQs under updates (see Theorem 3.4): For every $t$-hierarchical CQ, the testing problem can be solved by a dynamic algorithm with linear preprocessing time, constant update time, and constant testing time. And for every CQ that is not equivalent to a $t$-hierarchical CQ, the testing problem cannot be solved with arbitrary preprocessing time, sublinear update time, and sublinear testing time, unless the OMv-conjecture fails.

Furthermore, we transfer the notions of $t$-hierarchical and $q$-hierarchical queries to unions of conjunctive queries (UCQs) and identify a further class of UCQs which we call exhaustively $q$-hierarchical, yielding three increasingly restricted subclasses of UCQs. In a nutshell, our main contribution concerning UCQs shows that these notions precisely characterise the tractability frontiers of the testing problem, the enumeration problem, and the counting problem for UCQs under updates (see the Theorems 4.1, 4.2, 4.4): For every $t$-hierarchical (q-hierarchical, exhaustively q-hierarchical) UCQ, the testing (enumeration, counting) problem can be solved with linear preprocessing time, constant update time, and constant testing time (delay, counting time). And for every UCQ that is not equivalent to a $t$-hierarchical (q-hierarchical, exhaustively q-hierarchical) UCQ, the testing (enumeration, counting) problem cannot be solved with sublinear update time and sublinear testing time (delay, counting time); to be precise, the lower bound for enumeration is obtained only for self-join free queries, the lower bounds for testing and enumeration are conditioned on the OMv-conjecture, and the lower bound for counting is conditioned on the OMv-conjecture and the OV-conjecture.

Finally, we transfer our results to a scenario where databases are required to satisfy a set of small domain constraints (i.e., constraints stating that all values which occur in a particular column of a relation belong to a fixed domain of constant size), leading to a precise characterisation of the UCQs for which the testing (enumeration, counting) problem under updates is tractable in this scenario (see Theorem 5.3).

Further related work. The complexity of evaluating CQs and UCQs in the static setting (i.e., without database updates) is well-studied. In particular, there are characterisations of “tractable” queries known for Boolean queries [16, 15, 23] as well as for the task of counting the result tuples [11, 7, 12, 14, 8]. In [3], the fragment of self-join free CQs that can be enumerated
with constant delay after linear preprocessing time has been identified, but almost nothing is known about the complexity of the enumeration problem for UCQs on static databases. Very recent papers also studied the complexity of CQs with respect to a given set of integrity constraints \[13, 20, 4\]. The dynamic query evaluation problem has been considered from different angles, including descriptive dynamic complexity \[26, 27, 28\] and, somewhat closer to what we are aiming for, incremental view maintenance \[17, 9, 21, 22, 25\]. In \[19\], the enumeration and testing problem under updates has been studied for q-hierarchical and (more general) acyclic CQs in a setting that is very similar to our setting and the setting of \[5\]; the Dynamic Constant-delay Linear Representations (DCLR) of \[19\] are data structures that use at most linear update time and solve the enumeration problem and the testing problem with constant delay and constant testing time.

Outline. The rest of the paper is structured as follows. Section 2 provides basic notations concerning databases, queries, and dynamic algorithms for query evaluation. Section 3 is devoted to CQs and proves our dichotomy result concerning the testing problem for CQs. Section 4 concerns databases, queries, and dynamic algorithms for query evaluation. Section 3 is devoted to CQs and proves our dichotomy result concerning the testing problem for CQs. Section 4 focuses on UCQs and proves our dichotomies concerning the testing, enumeration, and counting problem for UCQs. Section 5 is devoted to the setting in which integrity constraints may cause a query whose evaluation under updates is hard in general to be tractable on databases that satisfy the constraints.

2 Preliminaries

Basic notation. We write \(\mathbb{N}\) for the set of non-negative integers and let \(\mathbb{N}_{\geq 1} := \mathbb{N} \setminus \{0\}\) and \([n] := \{1, \ldots, n\}\) for all \(n \in \mathbb{N}_{\geq 1}\). By \(2^S\) we denote the power set of a set \(S\). We write \(i_i\) to denote the \(i\)-th component of an \(n\)-dimensional vector \(\vec{i}\), and we write \(M_{i,j}\) for the entry in row \(i\) and column \(j\) of a matrix \(M\). By () we denote the empty tuple, i.e., the unique tuple of arity 0. For an \(r\)-tuple \(t = (t_1, \ldots, t_r)\) and indices \(i_1, \ldots, i_m \in \{1, \ldots, r\}\) we write \(\pi_{i_1, \ldots, i_m}(t)\) to denote the projection of \(t\) to the components \(i_1, \ldots, i_m\), i.e., the \(m\)-tuple \((t_{i_1}, \ldots, t_{i_m})\), and in case that \(m = 1\) we identify the 1-tuple \((t_{i_1})\) with the element \(t_{i_1}\). For a set \(T\) of \(r\)-tuples we let \(\pi_{i_1, \ldots, i_m}(T) := \{\pi_{i_1, \ldots, i_m}(t) : t \in T\}\).

Databases. We fix a countably infinite set \(\text{dom}\), the domain of potential database entries. Elements in \(\text{dom}\) are called constants. A schema is a finite set \(\sigma\) of relation symbols, where each \(R \in \sigma\) is equipped with a fixed arity \(\text{ar}(R) \in \mathbb{N}\) (note that here we explicitly allow relation symbols of arity 0). Let us fix a schema \(\sigma = \{R_1, \ldots, R_s\}\), and let \(r_i := \text{ar}(R_i)\) for \(i \in [s]\). A database \(D\) of schema \(\sigma\) (\(\sigma\)-db, for short), is of the form \(D = (R_1^D, \ldots, R_s^D)\), where \(R_i^D\) is a finite subset of \(\text{dom}^{r_i}\). The active domain \(\text{adom}(D)\) of \(D\) is the smallest subset \(A\) of \(\text{dom}\) such that \(R_i^D \subseteq A^{r_i}\) for all \(i \in [s]\).

Queries. We fix a countably infinite set \(\text{var}\) of variables. We allow queries to use variables as well as constants. An atomic formula (for short: atom) \(\psi\) of schema \(\sigma\) is of the form \(Rv_1 \cdots v_r\), with \(R \in \sigma\), \(r = \text{ar}(R)\), and \(v_1, \ldots, v_r \in \text{var} \cup \text{dom}\). A conjunctive formula of schema \(\sigma\) is of the form

\[
\exists y_1 \cdots \exists y_\ell \left( \psi_1 \land \cdots \land \psi_d \right)
\]

where \(\ell \geq 0, d \geq 1, \psi_j\) is an atomic formula of schema \(\sigma\) for every \(j \in [d]\), and \(y_1, \ldots, y_\ell\) are pairwise distinct elements in \(\text{var}\). For a conjunctive formula \(\varphi\) of the form \((*)\), we let \(\text{vars}(\varphi)\) (and \(\text{cons}(\varphi)\), respectively) be the set of all variables (and constants, respectively) occurring in \(\varphi\). The set of free variables of \(\varphi\) is \(\text{free}(\varphi) := \text{vars}(\varphi) \setminus \{y_1, \ldots, y_\ell\}\). For every variable \(x \in \text{vars}(\varphi)\) we let \(\text{atoms}_x(\varphi)\) (or \(\text{atoms}(x)\), if \(\varphi\) is clear from the context) be the set of all atoms \(\psi_j\) of \(\varphi\) such that \(x \in \text{vars}(\psi_j)\). The formula \(\varphi\) is called quantifier-free if \(\ell = 0\), and it is called self-join free if no relation symbol occurs more than once in \(\varphi\).
For $k \geq 0$, a $k$-ary conjunctive query (k-ary CQ, for short) is of the form
\[
\{ (u_1, \ldots, u_k) : \varphi \}
\]
where $\varphi$ is a conjunctive formula of schema $\sigma$, $u_1, \ldots, u_k \in \text{free}(\varphi) \cup \text{dom}$, and $\{u_1, \ldots, u_k\} \cap \text{var} = \text{free}(\varphi)$. We often write $q_\varphi(\overline{\pi})$ for $\varphi = (u_1, \ldots, u_k)$ (or $q_\varphi$ if $\overline{\pi}$ is clear from the context) to denote such a query. We let $\text{vars}(q_\varphi) := \text{vars}(\varphi)$, $\text{free}(q_\varphi) := \text{free}(\varphi)$, and $\text{cons}(q_\varphi) := \text{cons}(\varphi) \cup \{u_1, \ldots, u_k\} \cap \text{dom}$. For every $x \in \text{vars}(q_\varphi)$ we let $\text{atoms}_{q_\varphi}(x) := \text{atoms}_{\varphi}(x)$, and if $q_\varphi$ is clear from the context, we omit the subscript and simply write $\text{atoms}(x)$. The CQ $q_\varphi$ is called quantifier-free (self-join free) if $\varphi$ is quantifier-free (self-join free).

The semantics are defined as usual: A valuation is a mapping $\beta : \text{vars}(q_\varphi) \cup \text{dom} \rightarrow D$ with $\beta(a) = a$ for every $a \in \text{dom}$. A valuation $\beta$ is a homomorphism from $q_\varphi$ to a $\sigma$-db $D$ if for every relation $R_{v_1} \cdots v_r$ in $q_\varphi$ we have $(\beta(v_1), \ldots, \beta(v_r)) \in R^D$. The query result $q_\varphi(D)$ of a $k$-ary CQ $q_\varphi(u_1, \ldots, u_k)$ on the $\sigma$-db $D$ is defined as the set \( \{ (\beta(u_1), \ldots, \beta(u_k)) : \beta \text{ is a homomorphism from } q_\varphi \text{ to } D \} \). If $\overline{\pi} = (x_1, \ldots, x_k)$ is a list of the free variables of $\varphi$ and $\overline{x} \in \text{dom}^k$, we sometimes write $D \models \varphi[\overline{x}]$ to indicate that there is a homomorphism $\beta : q \rightarrow D$ with $\overline{\pi} = (\beta(x_1), \ldots, \beta(x_k))$, for the query $q = q_\varphi(x_1, \ldots, x_k)$.

A $k$-ary union of conjunctive queries (k-ary UCQ) is of the form $q_1(\overline{\pi}_1) \cup \cdots \cup q_d(\overline{\pi}_d)$ where $d \geq 1$ and $q_i(\overline{\pi}_i)$ is a $k$-ary CQ of schema $\sigma$ for every $i \in [d]$. The query result of such a k-ary UCQ $q$ on a $\sigma$-db $D$ is $q(D) := \bigcup_{i=1}^{d} q_i(D)$.

For a k-ary query $q$ we write $\text{vars}(q)$ (and $\text{cons}(q)$) to denote the set of all variables (and constants) that occur in $q$. Clearly, $q(D) \subseteq (\text{dom}(D) \cup \text{cons}(q))^k$.

A Boolean query is a query of arity $k = 0$. As usual, for Boolean queries $q$ we will write $q(D) = \text{yes}$ instead of $q(D) \neq \emptyset$, and $q(D) = \text{no}$ instead of $q(D) = \emptyset$. Two k-ary queries $q$ and $q'$ are equivalent ($q \equiv q'$, for short) if $q(D) = q'(D)$ for every $\sigma$-db $D$.

Homomorphisms. We use standard notation concerning homomorphisms (cf., e.g., [2]). The notion of a homomorphism $\beta : q \rightarrow D$ from a CQ $q$ to a database $D$ has already been defined above. A homomorphism $g : D \rightarrow q$ from a database $D$ to a CQ $q$ is a mapping from $\text{dom}(D) \rightarrow \text{vars}(q) \cup \text{cons}(q)$ such that whenever $(a_1, \ldots, a_r)$ is a tuple in some relation $R^D$ of $D$, then $R^D(a_1) \cdots g(a_r)$ is an atom of $q$.

Let $q(u_1, \ldots, u_k)$ and $q'(v_1, \ldots, v_k)$ be two k-ary CQs. A homomorphism from $q$ to $q'$ is a mapping $h : \text{vars}(q) \cup \text{dom} \rightarrow \text{vars}(q') \cup \text{dom}$ with $h(a) = a$ for all $a \in \text{dom}$ and $h(u_i) = v_i$ for all $i \in [k]$ such that for every relation $R_{w_1} \cdots w_r$ in $q$ there is an atom $Rh(w_1) \cdots h(w_r)$ in $q'$. We sometimes write $h : q \rightarrow q'$ to indicate that $h$ is a homomorphism from $q$ to $q'$. Note that by [3] there is a homomorphism from $q$ to $q'$ if and only if for every database $D$ it holds that $q(D) \supseteq q'(D)$. A CQ $q$ is a homomorphic core if there is no proper subquery of $q$. Here, a subquery of a CQ $q_\varphi(\overline{\pi})$ where $\varphi$ is of the form $[\text{3}]$ is a CQ $q_{\varphi'}(\overline{\pi})$ where $\varphi'$ is of the form $\exists y_1 \cdots \exists y_m \psi_{j_1} \land \cdots \land \psi_{j_n}$ with $i_1, \ldots, i_m \in [\ell]$, $j_1, \ldots, j_n \in [d]$, and free($\varphi'$) = free($\varphi$).

We say that a UCQ is a homomorphic core, if every CQ in the union is a homomorphic core and there is no homomorphism between two distinct CQs. It is well-known that every UCQ and every UCQ is equivalent to a unique (up to renaming of variables) homomorphic core, which is therefore called the core of the query (cf., e.g., [2]).

Sizes and Cardinalities. The size $|\sigma|$ of a schema $\sigma$ is $|\sigma| + \sum_{R \in \sigma} |R|$. The size $|q|$ of a query $q$ of schema $\sigma$ is the length of $q$ when viewed as a word over the alphabet $\sigma \cup \text{var} \cup \text{dom} \cup \{ \land, \lor, \exists, (, ), [\cdot], : \} \cup \{, \}$. For a $k$-ary query $q$ and a $\sigma$-db $D$, the cardinality of the query result is the number $|q(D)|$ of tuples in $q(D)$. The cardinality $|D|$ of a $\sigma$-db $D$ is defined as the number of tuples stored in $D$, i.e., $|D| := \sum_{R \in \sigma} |R^D|$. The size $|D|$ of $D$ is defined as $|\sigma| + |\text{dom}(D)| + \sum_{R \in \sigma} |R| \cdot |R^D|$ and corresponds to the size of a reasonable encoding of $D$.

The following notions concerning updates, dynamic algorithms for query evaluation, and algorithmic conjectures are taken almost verbatim from [5].
Updates. We allow to update a given database of schema $\sigma$ by inserting or deleting tuples as follows. An insertion command is of the form $\text{insert } R(a_1, \ldots, a_r) \text{ for } R \in \sigma, r = \text{ar}(R),$ and $a_1, \ldots, a_r \in \text{dom}$. When applied to a $\sigma$-db $D$, it results in the updated $\sigma$-db $D'$ with $R' := R^D \cup \{(a_1, \ldots, a_r)\}$ and $S^D := S^D$ for all $S \in \sigma \setminus \{R\}$. A deletion command is of the form $\text{delete } R(a_1, \ldots, a_r) \text{ for } R \in \sigma, r = \text{ar}(R),$ and $a_1, \ldots, a_r \in \text{dom}$. When applied to a $\sigma$-db $D$, it results in the updated $\sigma$-db $D'$ with $R' := R^D \setminus \{(a_1, \ldots, a_r)\}$ and $S^D := S^D$ for all $S \in \sigma \setminus \{R\}$. Note that both types of commands may change the database’s active domain.

Dynamic algorithms for query evaluation. Following [10], we use Random Access Machines (RAMs) with $O(\log n)$ word-size and a uniform cost measure to analyse our algorithms. We will assume that the RAM’s memory is initialised to 0. In particular, if an algorithm uses an array, we will assume that all array entries are initialised to 0, and this initialisation comes at no cost (in real-world computers this can be achieved by using the lazy array initialisation technique, cf. e.g. [24]). A further assumption is that for every fixed dimension $k \in \mathbb{N}_{\geq 1}$, we have available an unbounded number of $k$-ary arrays $A$ such that for given $(n_1, \ldots, n_k) \in \mathbb{N}^k$ the entry $A[n_1, \ldots, n_k]$ at position $(n_1, \ldots, n_k)$ can be accessed in constant time.\footnote{While this can be accomplished easily in the RAM-model, for an implementation on real-world computers one would probably have to resort to replacing our use of arrays by using suitably designed hash functions.} For our purposes it will be convenient to assume that $\text{dom} = \mathbb{N}_{\geq 1}$.

Our algorithms will take as input a $k$-ary query $q$ and a $\sigma$-db $D_0$. For all query evaluation problems considered in this paper, we aim at routines $\text{preprocess}$ and $\text{update}$ which achieve the following. Upon input of $q$ and $D_0$, the $\text{preprocess}$ routine builds a data structure $D$ which represents $D_0$ (and which is designed in such a way that it supports the evaluation of $q$ on $D_0$). Upon input of a command $\text{update } R(a_1, \ldots, a_r)$ (with update $\in \{\text{insert, delete}\}$), calling $\text{update}$ modifies the data structure $D$ such that it represents the updated database $D$. The preprocessing time $t_p$ is the time used for performing $\text{preprocess}$. The update time $t_u$ is the time used for performing an $\text{update}$, and in this paper we aim at algorithms where $t_u$ is independent of the size of the current database $D$. By $\text{init}$ we denote the particular case of the routine $\text{preprocess}$ upon input of a query $q$ and the empty database $D$, where $R^D = \emptyset$ for all $R \in \sigma$. The initialisation time $t_i$ is the time used for performing $\text{init}$. In all algorithms presented in this paper, the $\text{preprocess}$ routine for input of $q$ and $D_0$ will carry out the $\text{init}$ routine for $q$ and then perform a sequence of $|D_0|$ update operations to insert all the tuples of $D_0$ into the data structure. Consequently, $t_p = t_i + |D_0| \cdot t_u$.

In the following, $D$ will always denote the database that is currently represented by the data structure $D$. To solve the enumeration problem under updates, apart from the routines $\text{preprocess}$ and $\text{update}$, we aim at a routine $\text{enumerate}$ such that calling $\text{enumerate}$ invokes an enumeration of all tuples, without repetition, that belong to the query result $q(D)$. The delay $t_d$ is the maximum time used during a call of $\text{enumerate}$:

- until the output of the first tuple (or the end-of-enumeration message $\text{EØE}$, if $q(D) = \emptyset$),
- between the output of two consecutive tuples, and
- between the output of the last tuple and the end-of-enumeration message $\text{EØE}$.

To $\text{test}$ if a given tuple belongs to the query result, instead of $\text{enumerate}$ we aim at a routine $\text{test}$ which upon input of a tuple $\pi \in \text{dom}^k$ checks whether $\pi \in q(D)$. The testing time $t_t$ is the time used for performing a $\text{test}$. To solve the counting problem under updates, we aim at a routine $\text{count}$ which outputs the cardinality $|q(D)|$ of the query result. The counting time $t_c$ is the time used for performing a $\text{count}$. To answer a Boolean query under updates, we aim at a routine $\text{answer}$ that produces the answer yes or no of $q$ on $D$. The answer time $t_a$ is the time used for performing $\text{answer}$. Whenever speaking of a dynamic algorithm, we mean an algorithm that has routines $\text{preprocess}$ and $\text{update}$ and, depending on the problem at hand, at least one of the routines $\text{answer}$, $\text{test}$, $\text{count}$, and $\text{enumerate}$.
Throughout the paper, we often adopt the view of data complexity and suppress factors that may depend on the query $q$ but not on the database $D$. E.g., “linear preprocessing time” means $t_p \leq f(q) \cdot |D_0|$ and “constant update time” means $t_u \leq f(q)$, for a function $f$ with codomain $\mathbb{N}$. When writing $\text{poly}(n)$ we mean $n^{O(1)}$, and for a query $q$ we often write $\text{poly}(q)$ instead of $\text{poly}(|q|)$.

**Algorithmic conjectures.** Similarly as in [5] we obtain hardness results that are conditioned on algorithmic conjectures concerning the hardness of the following problems. These problems deal with Boolean matrices and vectors, i.e., matrices and vectors over $\{0,1\}$, and all the arithmetic is done over the Boolean semiring, where multiplication means conjunction and addition means disjunction.

The orthogonal vectors problem (OV-problem) is the following decision problem. Given two sets $U$ and $V$ of $n$ Boolean vectors of dimension $d$, decide whether there are vectors $\vec{u} \in U$ and $\vec{v} \in V$ such that $\vec{u}^T \vec{v} = 0$. The OV-conjecture states that there is no $\epsilon > 0$ such that the OV-problem for $d = \lceil \log^2 n \rceil$ can be solved in time $O(n^{2-\epsilon})$, see [1].

The online matrix-vector multiplication problem (OMv-problem) is the following algorithmic task. At first, the algorithm gets a Boolean $n \times n$ matrix $M$ and is allowed to do some preprocessing. Afterwards, the algorithm receives $n$ vectors $\vec{v}^1, \ldots, \vec{v}^n$ one by one and has to output $M \vec{v}^t$ before it has access to $\vec{v}^{t+1}$ (for each $t < n$). The running time is the overall time the algorithm needs to produce the output $M \vec{v}^1, \ldots, M \vec{v}^n$. The OMv-conjecture [18] states that there is no $\epsilon > 0$ such that the OMv-problem can be solved in time $O(n^{3-\epsilon})$.

A related problem is the OuMv-problem where the algorithm, again, is given a Boolean $n \times n$ matrix $M$ and is allowed to do some preprocessing. Afterwards, the algorithm receives a sequence of pairs of $n$-dimensional Boolean vectors $\vec{u}^t, \vec{v}^t$ for each $t \in [n]$, and the task is to compute $(\vec{u}^t)^T M \vec{v}^t$ before accessing $\vec{u}^{t+1}, \vec{v}^{t+1}$. The OuMv-conjecture states that there is no $\epsilon > 0$ such that the OuMv-problem can be solved in time $O(n^{3-\epsilon})$. It was shown in [18] that the OuMv-conjecture is equivalent to the OMv-conjecture, i.e., the OuMv-conjecture fails if, and only if, the OMv-conjecture fails.

### 3 Conjunctive queries

This section’s aim is twofold: Firstly, we observe that the notions and results of [5] generalise to CQs with constants in a straightforward way. Secondly, we identify a new subclass of CQs which precisely characterises the CQs for which testing can be done efficiently under updates.

The definition of $q$-hierarchical CQs can be taken verbatim from [5]:

**Definition 3.1.** A CQ $q$ is $q$-hierarchical if for any two variables $x, y \in \text{vars}(q)$ we have

(i) $\text{atoms}(x) \subseteq \text{atoms}(y)$ or $\text{atoms}(y) \subseteq \text{atoms}(x)$ or $\text{atoms}(x) \cap \text{atoms}(y) = \emptyset$, and

(ii) if $\text{atoms}(x) \subset \text{atoms}(y)$ and $x \in \text{free}(q)$, then $y \in \text{free}(q)$.

Obviously, it can be checked in time $\text{poly}(q)$ whether a given CQ $q$ is $q$-hierarchical. It is straightforward to see that if a CQ is $q$-hierarchical, then so is its homomorphic core. Using the main results of [5], it is not difficult to show the following.

**Theorem 3.2.** (a) There is a dynamic algorithm that receives a $q$-hierarchical $k$-ary CQ $q$ and a $\sigma$-db $D_0$, and computes within $t_p = \text{poly}(q) \cdot O(|D_0|)$ preprocessing time a data structure that can be updated in time $t_u = \text{poly}(q)$ and allows to

(i) compute the cardinality $|q(D)|$ in time $t_c = O(1)$,

(ii) enumerate $q(D)$ with delay $t_d = \text{poly}(q)$,

(iii) test for an input tuple $\vec{\pi} \in \text{dom}^k$ if $\vec{\pi} \in q(D)$ within time $t_t = \text{poly}(q)$,
(iv) and when given a tuple $\pi \in q(D)$, the tuple $\pi'$ (or the message EOE) that the enumeration procedure of [ai] would output directly after having output $\pi$, can be computed within time $\text{poly}(q)$.

(b) Let $\epsilon > 0$ and let $q$ be a CQ whose homomorphic core is not $q$-hierarchical (note that this is the case if, and only if, $q$ is not equivalent to a $q$-hierarchical CQ).

(i) If $q$ is Boolean, then there is no dynamic algorithm with arbitrary preprocessing time and $t_u = O(n^{1-\epsilon})$ update time that answers $q(D)$ in time $t_a = O(n^{2-\epsilon})$, unless the OMv-conjecture fails.

(ii) There is no dynamic algorithm with arbitrary preprocessing time and $t_u = O(n^{1-\epsilon})$ update time that computes the cardinality $|q(D)|$ in time $t_c = O(n^{1-\epsilon})$, unless the OMv-conjecture or the OV-conjecture fails.

(iii) If $q$ is self-join free, then there is no dynamic algorithm with arbitrary preprocessing time and $t_u = O(n^{1-\epsilon})$ update time that enumerates $q(D)$ with delay $t_d = O(n^{1-\epsilon})$, unless the OMv-conjecture fails.

All lower bounds remain true, if we restrict ourselves to the class of databases that map homomorphically into $q$.

Proof. From [5] we already know that the theorem’s statements [ai] and [a ii] and [b i]–[b iii] are true for all CQs $q$ with $\text{cons}(q) = \emptyset$, and a close look at the dynamic algorithm provided in [5] shows that also the statements [a iii] and [a iv] are true for all CQs $q$ with $\text{cons}(q) = \emptyset$. Furthermore, a close inspection of the proofs provided in [5] for the statements [a ii]–[a iii] for constant-free CQs $q$ shows that with only very minor modifications these proofs carry over to the case of CQs $q$ with $\text{cons}(q) \neq \emptyset$.

All that remains to be done is to transfer the results [ai]–[a iv] from constant-free CQs to CQs $q$ with $\text{cons}(q) \neq \emptyset$. To establish this, let us consider an arbitrary CQ $q$ of schema $\sigma$ with $\text{cons}(q) \neq \emptyset$. Without loss of generality we can assume that

$$q = \{ (x_1, \ldots, x_k, b_1, \ldots, b_\ell) : \varphi \}$$

(1)

where $\varphi$ is a conjunctive formula of schema $\sigma$, $\text{free}(\varphi) = \{x_1, \ldots, x_k\}$, and $b_1, \ldots, b_\ell \in \text{dom}$. Let $\overline{x} := (x_1, \ldots, x_k)$ and $\overline{b} := (b_1, \ldots, b_\ell)$.

In the following, we construct a new schema $\hat{\sigma}$ (that depends on $q$) and a constant-free CQ $\hat{q}$ of schema $\hat{\sigma}$ and of size $\text{poly}(|q|)$ such that the following is true:

1. $\hat{q}$ is $q$-hierarchical $\iff$ $q$ is $q$-hierarchical.

2. A dynamic algorithm for evaluating $\hat{q}$ on $\sigma$-dbs with initialisation time $\hat{t}_i$, update time $\hat{t}_u$, counting time $\hat{t}_c$ (delay $\hat{t}_d$, testing time $\hat{t}_t$) can be used to obtain a dynamic algorithm for evaluating $q$ on $\sigma$-dbs with initialisation time $t_i$, update time $t_u$, $\text{poly}(|q|)$, counting time $t_c$ (delay $O(t_d) + \text{poly}(|q|)$, testing time $O(t_t) + \text{poly}(|q|)$).

For each atom $\psi$ of $q$ we introduce a new relation symbol $R_\psi$ of arity $|\text{vars}(\psi)|$, and we let $\hat{\sigma} := \{ R_\psi : \psi \text{ is an atom in } \varphi \}$. For each atom $\psi$ of $q$ let us fix a tuple $\overline{v}^\psi = (v_1, \ldots, v_m)$ of pairwise distinct variables such that $\text{vars}(\psi) = \{v_1, \ldots, v_m\}$. The CQ $\hat{q}$ is defined as

$$\hat{q} = \{ (x_1, \ldots, x_k) : \hat{\varphi} \},$$

where the conjunctive formula $\hat{\varphi}$ is obtained from $\varphi$ by replacing every atom $\psi$ with the atom $R_\psi(\overline{v}^\psi)$. Obviously, $\hat{q}$ is a CQ of schema $\hat{\sigma}$, $\text{cons}(\hat{q}) = \emptyset$, $\text{free}(\hat{q}) = \text{free}(q)$, and $\text{vars}(\hat{q}) = \text{vars}(q)$. Furthermore, for every variable $y \in \text{vars}(q)$ we have $\text{atoms}^q_y = \{ R_\psi : \psi \in \text{atoms}^q(y) \}$ (and, equivalently, $\text{atoms}^q_y = \{ \psi : R_\psi \in \text{atoms}^q_y \}$). Therefore, $\hat{q}$ is $q$-hierarchical if and only if $q$ is $q$-hierarchical.
With every σ-db \( D \) we associate a \( \sigma \)-db \( \hat{D} \) as follows: Consider an atom \( \psi \) of \( q \) and let \( \psi \) be of the form \( Sw_1 \cdots w_s \). Thus, \( \{w_1, \ldots, w_s\} \cap \text{var} = \text{vars}(\psi) = \{v_1, \ldots, v_m\} \) for \( (v_1, \ldots, v_m) := \bar{v}^\psi \). Then, the relation symbol \( R_\psi \) is interpreted in \( \hat{D} \) by the relation

\[
(R_\psi)^\hat{D} := \{ (\beta(v_1), \ldots, \beta(v_m)) : \beta \text{ is a valuation with } (\beta(w_1), \ldots, \beta(w_s)) \in S^D \}.
\]

It is straightforward to verify that for every σ-db \( D \) we have

\[
q(D) = \{ (\pi, \bar{b}) : \pi \in \hat{q}(\hat{D}) \}.
\]

Now, assume we have available a dynamic algorithm \( \mathcal{A} \) for evaluating \( \hat{q} \) on \( \sigma \)-dbs with preprocessing time \( t_p \), update time \( t_u \), counting time \( t_c \) (delay \( t_d \), testing time \( t_t \)). We can use this algorithm to obtain a dynamic algorithm \( \mathcal{B} \) for evaluating \( q \) on σ-dbs as follows.

The \textbf{init} routine of \( \mathcal{B} \) performs the \textbf{init} routine of \( \mathcal{A} \). The \textbf{update} routine of \( \mathcal{B} \) proceeds as follows. Upon input of an update command of the form \textit{update} \( S(c_1, \ldots, c_s) \) for some \( S \in \sigma \), we consider all atoms \( \psi \) of \( q \) of the form \( Sw_1 \cdots w_s \). For each such atom we check if

- for all \( i \in [s] \) with \( w_i \in \text{dom} \) we have \( w_i = c_i \), and
- for all \( i, j \in [s] \) with \( w_i = w_j \) we have \( c_i = c_j \).

If this is true, we carry out the \textbf{update} routine of \( \mathcal{A} \) for the command \textit{update} \( R_\psi(c_j, \ldots, c_j) \), where \((w_{j1}, \ldots, w_{jm}) = (v_1, \ldots, v_m) = \bar{v}^\psi \). Thus, one call of the \textbf{update} routine of \( \mathcal{A} \) performs \( \text{poly}(|q|) \) calls of the \textbf{update} routine of \( \mathcal{B} \). This takes time \( t_u \cdot \text{poly}(|q|) \) and ensures that afterwards, the data structure of \( \mathcal{B} \) has stored the information concerning the \( \sigma \)-db \( D \) associated with the updated σ-db \( D \).

The \textbf{count} routine of \( \mathcal{B} \) simply calls the \textbf{count} routine of \( \mathcal{A} \), and we know that the result is correct since \(|q(D)| = |\hat{q}(\hat{D})| \) due to (2). For the same reason, the \textbf{enumerate} routine of \( \mathcal{B} \) can call the \textbf{enumerate} routine of \( \mathcal{A} \) and output the tuple \((\pi, \bar{b})\) for each output tuple \( \pi \) of \( \mathcal{A} \). The \textbf{test} routine of \( \mathcal{B} \) upon input of a tuple \((c_1, \ldots, c_{k+\ell}) \in \text{dom}^{k+\ell} \) outputs \textit{yes} if \((c_{k+1}, \ldots, c_{k+\ell}) = \bar{b} \) and the \textbf{test} routine of \( \mathcal{A} \) returns \textit{yes} upon input of the tuple \((c_1, \ldots, c_k) \). For statement (aiv) of Theorem 3.2 when given a tuple \((\pi, \bar{b}) \in q(D) \) we know that \( \pi \in \hat{q}(\hat{D}) \). Thus, we can use \( \pi \) and call the according routine of \( \mathcal{A} \) for (aiv) and obtain a tuple \( \bar{\pi} \in \hat{q}(\hat{D}) \) (or the message \text{EDE} and know that \((\bar{\pi}, \bar{b}) \) is the next tuple that the \textbf{enumerate} routine of \( \mathcal{B} \) will output after having output the tuple \((\pi, \bar{b}) \) (or that there is no such tuple).

Note that this suffices to transfer the statements (aiv) from a q-hierarchical CQ \( \hat{q} \) with \( \text{cons}(\hat{q}) = \emptyset \) to the q-hierarchical CQ \( q \) with \( \text{cons}(q) \neq \emptyset \). This completes the proof of Theorem 3.2.

Note that neither the results of [5] nor Theorem 3.2 provide a precise characterisation of the CQs for which testing can be done efficiently under updates. Of course, according to Theorem 3.2 [a(vii)], the testing problem can be solved with constant update time and constant testing time for every q-hierarchical CQ. But the same holds true, for example, for the non-q-hierarchical CQ \( p_{S.E.T.} := \{ (x, y) : Sx \land Ey \land Ty \} \). The according dynamic algorithm simply uses 1-dimensional arrays \( A_S \) and \( A_T \) and a 2-dimensional array \( A_E \) and that for all \( a, b \in \text{dom} \) we have \( A_S[a] = 1 \) if \( a \in S^D \), and \( A_S[a] = 0 \) otherwise, \( A_T[a] = 1 \) if \( a \in T^D \), and \( A_T[a] = 0 \) otherwise, and \( A_E[a, b] = 1 \) if \((a, b) \in E^D \), and \( A_E[a, b] = 0 \) otherwise. When given an update command, the arrays can be updated within constant time. And when given a tuple \((a, b) \in \text{dom}^2 \), the \textbf{test} routine simply looks up the array entries \( A_S[a], A_E[a, b], A_T[b] \) and returns the correct query result accordingly. To characterise the conjunctive queries for which testing can be done efficiently under updates, we introduce the following notion of t-hierarchical CQs.

**Definition 3.3.** A CQ \( q \) is \textit{t-hierarchical} if the following is satisfied:
(i) for all \(x, y \in \text{vars}(q) \setminus \text{free}(q)\), we have
\[
\text{atoms}(x) \subseteq \text{atoms}(y) \text{ or } \text{atoms}(y) \subseteq \text{atoms}(x) \text{ or } \text{atoms}(x) \cap \text{atoms}(y) = \emptyset,
\]
and
(ii) for all \(x \in \text{free}(q)\) and all \(y \in \text{vars}(q) \setminus \text{free}(q)\), we have
\[
\text{atoms}(x) \cap \text{atoms}(y) = \emptyset \text{ or } \text{atoms}(y) \subseteq \text{atoms}(x).
\]

Obviously, it can be checked in time \(\text{poly}(q)\) whether a given CQ \(q\) is t-hierarchical. Note that every q-hierarchical CQ is t-hierarchical, and a Boolean query is t-hierarchical if and only if it is q-hierarchical. The queries \(p_{S,E,T}\) and \(p_{E,E,R} := \{(x,y) : \exists v_1 \exists v_2 \exists v_3 (E x v_1 \land E y v_2 \land R x v_3)\}\) are examples for queries that are t-hierarchical but not q-hierarchical. It is straightforward to verify that if a CQ is t-hierarchical, then so is its homomorphic core. This section’s main result shows that the t-hierarchical CQs precisely characterise the CQs for which the testing problem can be solved efficiently under updates:

**Theorem 3.4.** (a) There is a dynamic algorithm that receives a t-hierarchical \(k\)-ary CQ \(q\) and a \(\sigma\)-db \(D_0\), and computes within \(t_p = \text{poly}(q) \cdot O(|D_0|)\) preprocessing time a data structure that can be updated in time \(t_u = \text{poly}(q)\) and allows to test for an input tuple \(\pi \in \text{dom}^k\) if \(\pi \in q(D)\) within time \(t_t = \text{poly}(q)\).

(b) Let \(\epsilon > 0\) and let \(q\) be a \(k\)-ary CQ whose homomorphic core is not t-hierarchical (note that this is the case if, and only if, \(q\) is not equivalent to a t-hierarchical CQ). There is no dynamic algorithm with arbitrary preprocessing time and \(t_u = O(n^{1-\epsilon})\) update time that can test for any input tuple \(\pi \in \text{dom}^k\) if \(\pi \in q(D)\) within testing time \(t_t = O(n^{1-\epsilon})\), unless the OME-conjecture fails. The lower bound remains true, if we restrict ourselves to the class of databases that map homomorphically into \(q\).

**Proof.** To avoid notational clutter, and without loss of generality, we restrict attention to queries \(q_\varphi(u_1, \ldots, u_k)\) where \((u_1, \ldots, u_k)\) is of the form \((z_1, \ldots, z_k)\) for pairwise distinct variables \(z_1, \ldots, z_k\).

For the proof of [41], we combine the array construction described above for the example query \(p_{S,E,T}\) with the dynamic algorithm provided by Theorem 3.2[3] and the following Lemma 3.5.

To formulate the lemma, we need the following notation. A \(k\)-ary generalised CQ is of the form \(\{z_1, \ldots, z_k : \varphi_1 \land \cdots \land \varphi_m\}\) where \(k \geq 0\), \(z_1, \ldots, z_k\) are pairwise distinct variables, \(m \geq 1\), \(\varphi_j\) is a conjunctive formula for each \(j \in [m]\), \(\text{free}(\varphi_j) \cup \cdots \cup \text{free}(\varphi_m) = \{z_1, \ldots, z_k\}\), and the quantified variables of \(\varphi_j\) and \(\varphi_{j'}\) are pairwise disjoint for all \(j, j' \in [m]\) with \(j \neq j'\) and disjoint from \(\{z_1, \ldots, z_k\}\). For each \(j \in [m]\) let \(\overline{\sigma}(j)\) be the sublist of \(\overline{\sigma} := \{z_1, \ldots, z_k\}\) that only contains the variables in \(\text{free}(\varphi_j)\). I.e., \(\overline{\sigma}(j)\) is obtained from \(\overline{\sigma}\) by deleting all variables that do not belong to \(\text{free}(\varphi_j)\). Accordingly, for a tuple \(\overline{\sigma} = (a_1, \ldots, a_k) \in \text{dom}^k\) by \(\overline{\sigma}(j)\) we denote the tuple that contains exactly those \(a_i\) where \(z_i\) belongs to \(\overline{\sigma}(j)\). The query result of \(q\) on a \(\sigma\)-db \(D\) is the set
\[
q(D) := \{ \overline{\sigma} \in \text{dom}^k : D \models \varphi_j[\overline{\sigma}(j)] \text{ for each } j \in [m] \},
\]
where \(D \models \varphi_j[\overline{\sigma}(j)]\) means that there is a homomorphism \(\beta_j : q_j \to D\) for the query \(q_j := \{z_j : \varphi_j\}\), with \(\beta_j(z_i) = a_i\) for every \(i\) with \(z_i \in \text{free}(\varphi_j)\). For example, \(p_{E,E,R}' := \{(x,y) : \exists v_1 E x v_1 \land \exists v_2 E y v_2 \land \exists v_3 R x v_3\}\) is a generalised CQ that is equivalent to the CQ \(p_{E,E,R}\).

**Lemma 3.5.** Every t-hierarchical CQ \(q_\varphi(z_1, \ldots, z_k)\) is equivalent to a generalised CQ \(q' = \{(z_1, \ldots, z_k) : \varphi_1 \land \cdots \land \varphi_m\}\) such that for each \(j \in [m]\) the CQ \(q_j := \{z_j : \varphi_j\}\) is q-hierarchical or quantifier-free. Furthermore, there is an algorithm which decides in time \(\text{poly}(q_\varphi)\) whether \(q_\varphi\) is t-hierarchical, and if so, outputs an according \(q'\).

**Proof.** Along Definition 3.3 it is straightforward to construct an algorithm which decides in time \(\text{poly}(q)\) whether a given CQ \(q\) is t-hierarchical.
Let \( q := q_\varphi(z_1, \ldots, z_k) \) be a given \( t \)-hierarchical CQ. Let \( A_0 \) be the set of all atoms \( \psi \) of \( q \) with \( \text{vars}(\psi) \subseteq \text{free}(q) \), and let \( \varphi_0 \) be the quantifier-free conjunctive formula

\[
\varphi_0 := \bigwedge_{\psi \in A_0} \psi.
\]

For each \( Z \subseteq \text{free}(q) \) let \( A_Z \) be the set of all atoms \( \psi \) of \( q \) such that \( \text{vars}(\psi) \supseteq \text{vars}(\psi) \cap \text{free}(q) = Z \). Let \( Z_1, \ldots, Z_n \) (for \( n \geq 0 \)) be a list of all those \( Z \subseteq \text{free}(q) \) with \( A_Z \neq \emptyset \). For each \( j \in [n] \) let \( A_j := A_{Z_j} \) and let \( Y_j := (\bigcup_{\psi \in A_j} \text{vars}(\psi)) \setminus Z_j \).

**Claim 3.6.** \( Y_j \cap Y_{j'} = \emptyset \) for all \( j, j' \in [n] \) with \( j \neq j' \).

**Proof.** We know that \( Z_j \neq Z_{j'} \). W.l.o.g. there is a \( z \in Z_j \) with \( z \notin Z_{j'} \).

For contradiction, assume that \( Y_j \cap Y_{j'} \) contains some variable \( y \). Then, \( y \in \text{vars}(\psi) \) for some \( \psi \in A_j \) and \( y \in \text{vars}(\psi') \) for some \( \psi' \in A_{j'} \). By definition of \( A_j \) we know that \( \text{vars}(\psi) \cap \text{free}(q) = Z_j \), and hence \( z \in \text{vars}(\psi) \). By definition of \( A_{j'} \) we know that \( \text{vars}(\psi') \cap \text{free}(q) = Z_{j'} \), and hence \( z \notin \text{vars}(\psi') \). Hence, \( \psi \in \text{atoms}(z) \) and \( \psi' \notin \text{atoms}(z) \). Since \( \psi \in \text{atoms}(y) \) and \( \psi' \in \text{atoms}(y) \), we obtain that \( \text{atoms}(y) \cap \text{atoms}(y) \neq \emptyset \) and \( \text{atoms}(y) \subseteq \text{atoms}(z) \). But by assumption, \( q \) is \( t \)-hierarchical, and this contradicts condition [iii] of Definition 3.3.

For each \( j \in [n] \) consider the conjunctive formula

\[
\varphi_j := \exists y_1^{(j)} \cdots \exists y_{\ell_j}^{(j)} \bigwedge_{\psi \in A_j} \psi,
\]

where \( \ell_j := |Y_j| \) and \( (y_1^{(j)}, \ldots, y_{\ell_j}^{(j)}) \) is a list of all variables in \( Y_j \). Using Claim 3.6, it is straightforward to see that

\[
q' := \{ (z_1, \ldots, z_k) : \varphi_0 \land \bigwedge_{j \in [n]} \varphi_j \}
\]

is a generalised CQ that is equivalent to \( q \). Furthermore, \( q' \) can be constructed in time \( \text{poly}(q) \).

To complete the proof of Lemma 3.5 we consider for each \( j \in [n] \) the CQ

\[
q_j := \{ \tau^{(j)} : \varphi_j \},
\]

where \( \tau^{(j)} \) is a tuple of length \( |Z_j| \) consisting of all the variables in \( Z_j \).

**Claim 3.7.** \( q_j \) is \( q \)-hierarchical, for each \( j \in [n] \).

**Proof.** First of all, note that \( q_j \) satisfies condition [ii] of Definition 3.1 since \( \text{free}(q_j) = Z_j \), \( \text{atoms}_{q_j}(z) = A_j \) for every \( z \in Z_j \), and \( \text{atoms}_{q_j}(y) \subseteq A_j \) for every \( y \in Y_j = \text{vars}(q_j) \setminus \text{free}(q_j) \).

For contradiction, assume that \( q_j \) is not \( q \)-hierarchical. Then, \( q_j \) violates condition [ii] of Definition 3.1. I.e., there are variables \( x, x' \in Z_j \cup Y_j \) and atoms \( \psi_1, \psi_2, \psi_3 \in A_j \) such that \( \text{vars}(\psi_1) \cap \{x, x'\} = \{x\} \), \( \text{vars}(\psi_2) \cap \{x, x'\} = \{x'\} \), and \( \text{vars}(\psi_3) \cap \{x, x'\} = \{x, x'\} \). Since \( \text{vars}(\psi) \cap \text{free}(q) = Z_j \) for all \( \psi \in A_j \), we know that \( x, x' \notin \text{free}(q) \). Therefore, \( x, x' \in \text{vars}(q_j) \setminus \text{free}(q) \), and hence \( \psi_1, \psi_2, \psi_3 \) are atoms of \( q \) which witness that condition [ii] of Definition 3.3 is violated. This contradicts the assumption that \( q \) is \( t \)-hierarchical.

This completes the proof of Lemma 3.5.

The proof of Theorem 3.4[a] now follows easily: When given a \( t \)-hierarchical CQ \( q_{\varphi}(z_1, \ldots, z_k) \), use the algorithm provided by Lemma 3.5 to compute an equivalent generalised CQ \( q' \) of the form \( \{ (z_1, \ldots, z_k) : \varphi_1 \land \cdots \land \varphi_m \} \) and let \( q_j := \{ \tau^{(j)} : \varphi_j \} \) for each \( j \in [m] \). W.l.o.g. assume that there is an \( m' \in \{0, \ldots, m\} \) such that \( q_j \) is \( q \)-hierarchical for each \( j \leq m' \) and \( q_j \) is quantifier-free for each \( j > m' \). We use in parallel, for each \( j \leq m' \), the data structures provided by Theorem 3.2[a] for the \( q \)-hierarchical CQ \( q_j \). In addition to this, we use an \( r \)-dimensional
array $A_R$ for each relation symbol $R \in \sigma$ of arity $r := \text{ar}(R)$, and we ensure that for all $\vec{b} \in \text{dom}^r$ we have $A_R[\vec{b}] = 1$ if $\vec{b} \in R^D$, and $A_R[\vec{b}] = 0$ otherwise. When receiving an update command $\text{update} R(\vec{b})$, we let $A_R[\vec{b}] := 1$ if $\text{update} = \text{insert}$, and $A_R[\vec{b}] := 0$ if $\text{update} = \text{delete}$, and in addition to this, we call the $\text{update}$ routines of the data structure for $q^{(j)}$ for each $j \leq m'$. Upon input of a tuple $\vec{a} \in \text{dom}^k$, the test routine proceeds as follows. For each $j \leq m'$, it calls the test routine of the data structure for $q^{(j)}$ upon input $\vec{a}$. And additionally, it uses the arrays $A_R$ for all $R \in \sigma$ to check if for each $j > m'$ the quantifier-free query $q^{(j)}$ is satisfied by the tuple $\vec{a}$. All this is done within time $\text{poly}(q)$, and we know that $\vec{a} \in Q(D)$ if, and only if, all these tests succeed. This completes the proof of part [a] of Theorem 3.4.

Let us now turn to the proof of part [b] of Theorem 3.4. We are given a query $q := q_\varphi(z_1, \ldots, z_k)$ and without loss of generality we assume that $q$ is a homomorphic core and $q$ is not t-hierarchical. Thus, $q$ violates condition [i] or [ii] of Definition 3.3. In case that it violates condition [i], the proof is virtually identical to the proof of Theorem 3.4 in [5]; for the reader’s convenience, the proof details are given in Appendix A.

Let us consider the case where $q$ violates condition [ii] of Definition 3.3. In this case, there are two variables $x \in \text{free}(q)$ and $y \in \text{vars}(q) \setminus \text{free}(q)$ and two atoms $\psi^{x,y}$ and $\psi^y$ of $q$ with $\text{vars}(\psi^{x,y}) \cap \{x, y\} = \{x, y\}$ and $\text{vars}(\psi^y) \cap \{x, y\} = \{y\}$. The easiest example of a query for which this is true is $q_{E,T} := \{(x) : \exists y (E x y \land T y)\}$. Here, we illustrate the proof idea for the particular query $q_{E,T}$: a proof for the general case is given in Appendix A.

Assume that there is a dynamic algorithm that solves the testing problem for $q_{E,T}$ with update time $t_u = O(n^{1-\epsilon})$ and testing time $t_t = O(n^{1-\epsilon})$ on databases whose active domain is of size $O(n)$. We show how this algorithm can be used to solve the OuMv-problem.

For the OuMv-problem, we receive as input an $n \times n$ matrix $M$. We start the preprocessing phase of our testing algorithm for $q_{E,T}$ with the empty database $D = (E^D, T^D)$ where $E^D = T^D = \emptyset$. As this database has constant size, the preprocessing is finished in constant time. We then apply $O(n^2)$ update steps to ensure that $E^D = \{(i, j) : M_{i,j} = 1\}$. All this takes time at most $O(n^2) \cdot t_u = O(n^{3-\epsilon})$. Throughout the remainder of the construction, we will never change $E^D$, and we will always ensure that $T^D \subseteq [n]$. When we receive two vectors $\vec{u}^t$ and $\vec{v}^t$ in the dynamic phase of the OuMv-problem, we proceed as follows. First, we perform the update commands $\text{delete} T(j)$ for each $j \in [n]$ with $\vec{u}^t = 0$, and the update commands $\text{insert} T(j)$ for each $j \in [n]$ with $\vec{v}^t_j = 1$. This is done within time $n \cdot t_u = O(n^{2-\epsilon})$. By construction of $D$ we know that for every $i \in [n]$ we have

$$i \in q_{E,T}(D) \iff \text{there is a } j \in [n] \text{ such that } M_{i,j} = 1 \text{ and } \vec{v}^t_j = 1.$$ 

Thus, $(\vec{u}^t)^T M \vec{v}^t = 1 \iff$ there is an $i \in [n]$ with $\vec{u}^t_i = 1$ and $i \in q_{E,T}(D)$. Therefore, after having called the test routine for $q_{E,T}$ for each $i \in [n]$ with $\vec{u}^t_i = 1$, we can output the correct result of $(\vec{u}^t)^T M \vec{v}^t$. This takes time at most $n \cdot t_u = O(n^{2-\epsilon})$. I.e., for each $t \in [n]$ after receiving the vectors $\vec{u}^t$ and $\vec{v}^t$, we can output $(\vec{u}^t)^T M \vec{v}^t$ within time $O(n^{2-\epsilon})$. Consequently, the overall running time for solving the OuMv-problem is bounded by $O(n^{3-\epsilon})$.

By using the technical machinery of [5], this approach can be generalised from $q_{E,T}$ to all queries $q$ that violate condition [ii] of Definition 3.3, see Appendix A for details. This completes the proof of Theorem 3.4.

4 Unions of conjunctive queries

In this section we consider dynamic query evaluation for UCQs. To transfer our notions of hierarchical queries from CQs to UCQs, we say that a UCQ $q(\vec{a})$ of the form $q_1(\vec{a}) \cup \cdots \cup q_d(\vec{a})$ is q-hierarchical (t-hierarchical) if every CQ $q_i(\vec{a})$ in the union is q-hierarchical (t-hierarchical). Note that for Boolean queries (CQs as well as UCQs) the notions of being q-hierarchical and
being t-hierarchical coincide, and for a k-ary UCQ \( q \) it can be checked in time \( \text{poly}(q) \) if \( q \) is q-hierarchical or t-hierarchical.

The following theorem generalises the statement of Theorem 3.4 from CQs to UCQs. Its proof follows easily from the Theorems 3.4 and 3.2.

**Theorem 4.1.** (a) There is a dynamic algorithm that receives a t-hierarchical k-ary UCQ \( q \) and a \( \alpha \)-db \( D_0 \), and computes within \( t_p = \text{poly}(q) \cdot O(|D_0|) \) preprocessing time a data structure that can be updated in time \( t_u = \text{poly}(q) \) and allows to test for an input tuple \( \bar{a} \in \text{dom}^k \) if \( \bar{a} \in q(D) \) within time \( t_1 = \text{poly}(q) \). Furthermore, the algorithm allows to answer a q-hierarchical UCQ within time \( t_a = O(1) \).

(b) Let \( \epsilon > 0 \) and let \( q \) be a k-ary UCQ whose homomorphic core is not t-hierarchical (note that this is the case if, and only if, \( q \) is not equivalent to a t-hierarchical UCQ). There is no dynamic algorithm with arbitrary preprocessing time and \( t_u = O(n^{1-\epsilon}) \) update time that can test for any input tuple \( \bar{a} \in \text{dom}^k \) if \( \bar{a} \in q(D) \) within testing time \( t_1 = O(n^{1-\epsilon}) \), unless the OMv-conjecture fails. Furthermore, if \( k = 0 \) (i.e., \( q \) is a Boolean UCQ), then there is no dynamic algorithm with arbitrary preprocessing time and \( t_u = O(n^{1-\epsilon}) \) update time that answers \( q(D) \) in time \( t_a = O(n^{2-\epsilon}) \), unless the OMv-conjecture fails.

**Proof.** Part (a) follows immediately from Theorem 3.4(a) (and Theorem 3.2(a)) for the statement on Boolean UCQs), as we can maintain all CQs in the union in parallel and then decide whether at least one of them is satisfied by the current database and the given tuple.

For the proof of (b) let \( q' \) be the homomorphic core of \( q \), and let \( q' \) be of the form \( q_1 \cup \cdots \cup q_m \) for k-ary CQs \( q_1, \ldots, q_m \). We first consider the case that \( q \) is a Boolean UCQ. Then, by assumption, \( q' \) is not q-hierarchical, and hence there exists an \( i \in [m] \) such that the Boolean CQ \( q_i \) is not q-hierarchical. Suppose for contradiction that there exists a dynamic algorithm that evaluates \( q \) with \( t_u = O(n^{1-\epsilon}) \) update time and \( t_a = O(n^{2-\epsilon}) \) answer time. Since \( q' \) is equivalent to \( q \), it follows that the algorithm also evaluates \( q' \) with the same time bounds. Now consider the class of databases that map homomorphically into \( q_i \). For every database \( D \) in this class it holds that \( q(D) = \text{yes} \) if, and only if, \( q_i(D) = \text{yes} \). This is because every other CQ \( q_j \) in \( q' \) is not satisfied by \( D \), since otherwise there would be a homomorphism from \( q_j \) to \( D \) and therefore, since there is a homomorphism from \( D \) to \( q_i \), also from \( q_j \) to \( q_i \), contradicting that \( q_i \) is a core. Hence, the dynamic algorithm evaluates the non-q-hierarchical CQ \( q_i \) (which is a homomorphic core), contradicting Theorem 3.2(b).

The statement of (b) concerning non-Boolean UCQs and the testing problem follows along the same lines when using Theorem 3.4(b) instead of Theorem 3.2(b). \( \Box \)

It turns out that similarly as q-hierarchical CQs, also q-hierarchical UCQs allow for efficient enumeration under updates. This, and the according lower bound, is stated in the following Theorem 4.2, which will be proven at the end of this section. In contrast to Theorem 4.1 the result does not follow immediately from the tractability of the enumeration problem for q-hierarchical CQs, because one has to ensure that tuples from result sets of two different CQs are not reported twice while enumerating their union.

**Theorem 4.2.** (a) There is a dynamic algorithm that receives a q-hierarchical k-ary UCQ \( q \) and a \( \alpha \)-db \( D_0 \), and computes within \( t_p = \text{poly}(q) \cdot O(|D_0|) \) preprocessing time a data structure that can be updated in time \( t_u = \text{poly}(q) \) and allows to enumerate \( q(D) \) with delay \( t_d = \text{poly}(q) \).

(b) Let \( \epsilon > 0 \) and let \( q \) be a k-ary UCQ whose homomorphic core is not q-hierarchical and is a union of self-join free CQs. There is no dynamic algorithm with arbitrary preprocessing time and \( t_u = O(n^{1-\epsilon}) \) update time that enumerates \( q(D) \) with delay \( t_d = O(n^{1-\epsilon}) \), unless the OMv-conjecture fails.
Note that according to Theorem 3.2 for CQs the enumeration problem as well as the counting problem can be solved by efficient dynamic algorithms if, and (modulo algorithmic conjectures) only if, the query is q-hierarchical. In contrast to this, it turns out that for UCQs computing the number of output tuples can be much harder than enumerating the query result. To characterise the UCQs that allow for efficient dynamic counting algorithms, we use the following notation. For two k-ary CQs \( q_\varphi(u_1, \ldots, u_k) \) and \( q_\psi(v_1, \ldots, v_k) \) we define the intersection \( q := q_\varphi \cap q_\psi \) to be the following k-ary query. If there is an \( i \in [k] \) such that \( u_i \) and \( v_i \) are distinct elements from \( \text{dom} \), then \( q := \emptyset \) (and this query is q-hierarchical by definition). Otherwise, we let \( w_1, \ldots, w_k \) be elements from \( \text{var} \cup \text{dom} \) which satisfy the following for all \( i, j \in [k] \) and all \( a \in \text{dom} \):

\[
( w_i = a \iff u_i = a \text{ or } v_i = a ) \quad \text{and} \quad ( w_i = w_j \iff u_i = u_j \text{ or } v_i = v_j ).
\]

We obtain \( \varphi' \) from \( \varphi \) (and \( \psi' \) from \( \psi \)) by replacing every \( u_i \in \{u_1, \ldots, u_k\} \cap \text{free}(\varphi) \) (and \( v_i \in \{v_1, \ldots, v_k\} \cap \text{free}(\psi) \) by \( w_i \). Finally, we let \( q = \{ (w_1, \ldots, w_k) : \varphi' \land \psi' \} \), where we can assume w.l.o.g. that \( \varphi' \land \psi' \) is a conjunctive formula of the form \([\varphi] \). Note that for every database \( D \) it holds that \( q(D) = q_\varphi(D) \cap q_\psi(D) \).

**Definition 4.3.** A UCQ \( q \) of the form \( \bigcup_{i \in [d]} q_i(\pi_i) \) is *exhaustively q-hierarchical* if for every \( I \subseteq [d] \) the intersection \( q_I = \bigcap_{i \in I} q_i \) is equivalent to a q-hierarchical CQ.

It is not difficult to see that a Boolean UCQ is exhaustively q-hierarchical if and only if its homomorphic core is q-hierarchical. In the non-Boolean case, being exhaustively q-hierarchical is a stronger requirement than being q-hierarchical as the following example shows: the UCQ \( \{ (x, y) : Sx \land Ey \} \cup \{ (x, y) : E x \land Ty \} \) is q-hierarchical, but not exhaustively q-hierarchical. In contrast to the q-hierarchical property, the straightforward way of deciding whether a UCQ \( q \) is exhaustively q-hierarchical requires \( 2^{\text{poly}(q)} \) and it is open whether this can be improved. The next theorem shows that the exhaustively q-hierarchical queries are precisely those UCQs that allow for efficient dynamic counting algorithms.

**Theorem 4.4.** (a) There is a dynamic algorithm that receives an exhaustively q-hierarchical UCQ \( q \) and a \( \sigma \)-db \( D_0 \), and computes within \( t_p = 2^{\text{poly}(q)} \cdot |D_0| \) preprocessing time a data structure that can be updated in time \( t_u = 2^{\text{poly}(q)} \) and computes \( |q(D)| \) in time \( t_c = O(1) \).

(b) Let \( \epsilon > 0 \) and let \( q \) be a UCQ whose homomorphic core is not exhaustively q-hierarchical. There is no dynamic algorithm with arbitrary preprocessing time and \( t_u = O(n^{1-\epsilon}) \) update time that computes \( |q(D)| \) in time \( t_c = O(n^{1-\epsilon}) \), unless the OMv-conjecture or the OV-conjecture fails.

**Proof.** To prove part (a) we use the principle of inclusion-exclusion along with the upper bound of Theorem 3.2(a). Let \( q = \bigcup_{i \in [d]} q_i(\pi_i) \) be an exhaustively q-hierarchical UCQ. Our dynamic algorithm for solving the counting problem for \( q \) proceeds as follows. In the preprocessing phase we first compute for every non-empty \( I \subseteq [d] \) the homomorphic core \( \tilde{q}_I \) of the CQ \( q_I := \bigcap_{i \in I} q_i \). This can be done in time \( 2^{\text{poly}(q_I)} \). Since \( q \) is exhaustively q-hierarchical, every \( \tilde{q}_I \) is q-hierarchical and we can apply Theorem 3.2(a) to determine the number of result tuples \( |\tilde{q}_I(D)| = |q_I(D)| \) for every \( I \subseteq [d] \) with \( \sum_{I \subseteq [d]} \text{poly}(q_I) = 2^{\text{poly}(q)} \) update time. By the principle of inclusion-exclusion we have

\[
|q(D)| = | \bigcup_{i \in [d]} q_i(D)| = \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|+1} \cdot |\bigcap_{i \in I} q_i(D)| = \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|+1} \cdot |q_I(D)|.
\]

Therefore, we can compute the number of result tuples in \( q(D) \) by maintaining all \( 2^{d-1} \) numbers \( |\tilde{q}_I(D)| \) in parallel (for all non-empty \( I \subseteq [d] \)).

For the proof of part (b) let \( \tilde{q} = \bigcup_{i \in [d]} q_i(\pi_i) \) be the homomorphic core of \( q \). Consider the CQs \( q_I := \bigcap_{i \in I} q_i \) and their homomorphic cores \( \tilde{q}_I \) for all non-empty \( I \subseteq [d] \). First we take care
of equivalent queries and write \( I \cong J \) if \( q_I \equiv q_J \). Let \( I \) be a set of index sets \( I \) that contains one representative from each equivalence class \( I/\cong \). By the principle of inclusion-exclusion we have

\[
|q(D)| = |\bar{q}(D)| = \sum_{\emptyset \neq I \subseteq [d]} (-1)^{|I|+1} \cdot |q_I(D)| = \sum_{I \in \mathcal{I}} a_I \cdot |q_I(D)|,
\]

where \( a_I := \sum_{J : J \cong I} (-1)^{|J|+1} \). Because \( q \) is not exhaustively \( q \)-hierarchical, we can choose a set \( I \in \mathcal{I} \) such that \( \bar{q}_I \) is a non-\( q \)-hierarchical query, which is minimal in the sense that for every \( J \in \mathcal{I} \setminus \{I\} \) there is no homomorphism from \( q_J \) to \( q_I \). Note that such a minimal set \( I \) exists since otherwise we could find two distinct \( J, J' \in \mathcal{I} \) such that \( q_J \equiv q_{J'} \).

Now suppose that \( D \) is a database from the class of databases that map homomorphically into \( q_I \) and let \( h : D \to q_I \) be a homomorphism. For every \( J \in \mathcal{I} \setminus \{I\} \) it holds that there is no homomorphism \( h' : q_J \to D \), since otherwise \( h \circ h' \) would be a homomorphism from \( q_J \) to \( q_I \). Hence, \( q_J(D) = \emptyset \) for all \( J \in \mathcal{I} \setminus \{I\} \) and thus \( |q(D)| = a_I \cdot |q_I(D)| \). It follows that we can compute \( |q(D)| = |\bar{q}_I(D)| \) by maintaining the value for \( |q(D)| \) and dividing it by \( a_I \). Since \( \bar{q}_I \) is a non-\( q \)-hierarchical homomorphic core, the lower bound for maintaining \( |q(D)| \) follows from Theorem 3.2(bii). This completes the proof of Theorem 4.4.

The remainder of this section is devoted to the proof of Theorem 4.2. To prove Theorem 4.2(a), we first develop a general method for enumerating the union of sets. We say that a data structure for a set \( T \) allows to skip, if it is possible to test whether \( t \in T \) in constant time and for some ordering \( t_1, \ldots, t_n \) of the elements in \( T \) there is

- a function \texttt{start}, which returns \( t_1 \) in constant time and
- a function \texttt{next}(\( t_i \)), which returns \( t_{i+1} \) (if \( i < n \)) or \texttt{EOE} (if \( i = n \)) in constant time.

Note that a data structure that allows to skip enables constant delay enumeration of \( t_i, t_{i+1}, \ldots, t_n \) starting from an arbitrary element \( t_i \in T \) (but we do not have control over the underlying order). An example of such a data structure is an explicit representation of the elements of \( T \) in a linked list with constant access. Another example is the data structure of the enumeration algorithm for the result \( T := q(D) \) of a \( q \)-hierarchical CQ \( q \), provided by Theorem 3.2(aii)\&(aiv).

The next lemma states that we can use these data structures for sets \( T_j \) to enumerate the union \( \bigcup_{j} T_j \) with constant delay and without repetition.

**Lemma 4.5.** Let \( \ell \geq 1 \) and let \( T_1, \ldots, T_\ell \) be sets such that for each \( j \in [\ell] \) there is a data structure for \( T_j \) that allows to skip. Then there is an algorithm that enumerates, without repetition, all elements in \( T_1 \cup \cdots \cup T_\ell \) with \( O(\ell) \) delay.

**Proof.** For each \( i \in [\ell] \) let \texttt{start} and \texttt{next} be the start element and the iterator for the set \( T_i \). The main idea for enumerating the union \( T_1 \cup \cdots \cup T_\ell \) is to first enumerate all elements in \( T_1 \), and then \( T_2 \setminus T_1, T_3 \setminus (T_1 \cup T_2), \ldots, T_\ell \setminus (T_1 \cup \cdots \cup T_{\ell-1}) \). In order to do this we have to exclude all elements that have already been reported from all subsequent sets. As we want to ensure constant delay enumeration, we cannot just ignore the elements in \( T_i \cap (T_1 \cup \cdots \cup T_{\ell-1}) \) while enumerating \( T_i \). As a remedy, we use an additional pointer to jump from an element that has already been reported to the least element that needs to be reported next. To do this we use arrays \texttt{skip} for \( i \in [\ell] \) to jump over excluded elements: if \( t_r, \ldots, t_s \) is a maximal interval of elements in \( T_i \) that have already been reported, then \( \texttt{skip}[t_r] = t_{s+1} \) (if \( s \) is the last element in \( T_i \), then \( t_{s+1} := \texttt{EOE} \)). For technical reasons we also need the array \texttt{skipback} which represents the inverse pointer, i.e., \( \texttt{skipback}[t_{s+1}] = t_r \).

The enumeration algorithm is stated in Algorithm 1. It uses the procedure \texttt{EXCLUDE} described in Algorithm 2 to update the arrays whenever an element \( t \) has been reported. It is straightforward to verify that these algorithms provide the desired functionality within the claimed time bounds.
Algorithm 1 The enumeration algorithm for $T_1 \cup \cdots \cup T_\ell$

**Input:** Data structures for sets $T_j$ with first element $\text{start}^j$ and iterator $\text{next}^j$. Pointer $\text{skip}^j[t] = \text{skipback}^j[t] = \text{nil}$ for all $j \in [\ell]$ and $t \in T_j$.

for $i = 1, \ldots, \ell$ do
    $t = \text{start}^i$
    while $t \neq \text{EOE}$ do
        if $\text{skip}^i[t] == \text{nil}$ then
            Output element $t$
            for $j = i + 1 \to \ell$ do
                EXCLUDE$^j(t)$
                $t = \text{next}^i(t)$
            else
                $t = \text{skip}^i[t]$
        Output the end-of-enumeration message EOE.

Algorithm 2 Procedure EXCLUDE$^j$ for excluding $t$ from $T_j$

if $t \in T_j$ then
    if $\text{skipback}^j[t] \neq \text{nil}$ then
        $t^- = \text{skipback}^j[t]$
        $\text{skipback}^j[t] = \text{nil}$
    else
        $t^- = t$
    if $\text{skip}^j[\text{next}^i(t)] \neq \text{nil}$ then
        $t^+ = \text{skip}^j[\text{next}^i(t)]$
        $\text{skip}^j[\text{next}^i(t)] = \text{nil}$
    else
        $t^+ = \text{next}^j(t)$
    $\text{skip}^j[t^-] = t^+$; $\text{skipback}^j[t^+] = t^-$

Lemma 4.5 enables us to prove the upper bound of Theorem 4.2 and the lower bound is proved by using Theorem 3.2[bii].

Proof of Theorem 4.2. The upper bound follows immediately from combining Lemma 4.5 with Theorem 3.2[aiv]. For the lower bound let $q_i$ be a self-join free non-$q$-hierarchical CQ in the homomorphic core $q’$ of the UCQ $q$. For every database $D$ that maps homomorphically into $q_i$ it holds that $q_j(D) = \emptyset$ for every other CQ $q_j$ in $q’$ (with $j \neq i$), since otherwise there would be a homomorphism from $q_j$ to $D$ and hence to $q_i$, contradicting that $q’$ is a homomorphic core. It follows that every dynamic algorithm that enumerates the result of $q$ on a database $D$ which maps homomorphically into $q_i$ also enumerates $q_i(D) = q(D)$, contradicting Theorem 3.2[bii].

5 CQs and UCQs with integrity constraints

In the presence of integrity constraints, the characterisation of tractable queries changes and depends on the query as well as on the set of constraints. When considering a scenario where databases are required to satisfy a set $\Sigma$ of constraints, we allow to execute a given update command only if the resulting database still satisfies all constraints in $\Sigma$. When speaking of $(\sigma, \Sigma)$-dbs we mean $\sigma$-dbs $D$ that satisfy all constraints in $\Sigma$. Two queries $q$ and $q’$ are $\Sigma$-equivalent (for short: $q \equiv_\Sigma q’$) if $q(D) = q’(D)$ for every $(\sigma, \Sigma)$-db $D$. 

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We first consider small domain constraints, i.e., constraints \( \delta \) of the form \( R[i] \subseteq C \) where \( R \in \sigma, i \in \{1, \ldots, ar(R)\} \), and \( C \subseteq \text{dom} \) is a finite set. A \( \sigma \)-db \( D \) satisfies \( \delta \) if \( \pi_i(R^D) \subseteq C \).

For these constraints we are able to give a clear picture of the tractability landscape by reducing CQs and UCQs with small domain constraints to UCQs without integrity constraints and applying the characterisations for UCQs achieved in Section 4. We start with an example that illustrates how a query can be simplified in the presence of small domain constraints.

**Example 5.1.** Consider the Boolean query \( q_{S,E,T} := \{ () : \exists x \exists y ( Sx \land Eyx \land Ty ) \} \), which is not \( q \)-hierarchical. By Theorem 3.2 it cannot be answered by a dynamic algorithm with sublinear update time and sublinear answer time, unless the OMv-conjecture fails. But in the presence of the small domain constraint \( \delta_{sd} := S[1] \subseteq C \) for a set \( C \subseteq \text{dom} \) of the form \( C = \{a_1, \ldots, a_c\} \), the query \( q_{S,E,T} \) is \( \{\delta_{sd}\}\)-equivalent to the \( q \)-hierarchical UCQ

\[
q' := \bigcup_{a_i \in C} \{ () : \exists y ( Sa_i \land Ea_iy \land Ty ) \}.
\]

Therefore, by Theorem 4.4 \( q' \) and hence \( q_{S,E,T} \) can be answered with constant update time and constant answer time on all databases that satisfy \( \delta_{sd} \).

For handling the general case, assume we are given a set \( \Sigma \) of small domain constraints and an arbitrary \( k \)-ary CQ \( q \) of the form \([\varphi] \) where \( \varphi \) is of the form \([\psi] \). We define a function \( \text{Dom}_{q,\Sigma} \) that maps each \( x \in \text{vars}(q) \) to a set \( \text{Dom}_{q,\Sigma}(x) \subseteq \text{dom} \) as follows. As an initialisation let \( f(x) = \text{dom} \) for each \( x \in \text{vars}(q) \). Consider each constraint \( \delta \) in \( \Sigma \) and let \( S[\delta] \subseteq C \) be the form of \( \delta \). Consider each atom \( \psi_j \) of \( \varphi \) and let \( Rv_1 \cdots v_{r}\) be the form of \( \psi_j \). If \( R = S \) and \( v_i \in \text{var} \), then let \( f(v_i) := f(v_i) \cap C \). Let \( \text{Dom}_{q,\Sigma} \) be the mapping \( f \) obtained at the end of this process. Note that \( rvars_{\Sigma}(q) := \{ x \in \text{vars}(q) : \text{Dom}_{q,\Sigma}(x) \neq \text{dom} \} \) consists of the variables of \( q \) that are restricted by \( \Sigma \).

Let \( M_{q,\Sigma} \) be the set of all mappings \( \alpha : V \to \text{dom} \) with \( V = rvars_{\Sigma}(q) \) and \( \alpha(x) \in \text{Dom}_{q,\Sigma}(x) \) for each \( x \in V \). Note that \( M_{q,\Sigma} \) is finite; and it is empty if, and only if, \( \text{Dom}_{q,\Sigma}(x) = \emptyset \) for some \( x \in \text{vars}(q) \).

For an arbitrary mapping \( \alpha : V \to \text{dom} \) with \( V \subseteq \text{var} \) we let \( q_\alpha \) be the \( k \)-ary CQ obtained from \( q \) as follows: for each \( x \in V \), if present in \( q \), the existential quantifier “\( \exists x \)” is omitted, and afterwards every occurrence of \( x \) in \( q \) is replaced with the constant \( \alpha(x) \). It is straightforward to check that \( q_\alpha(D) \subseteq q(D) \) for every \( \sigma \)-db \( D \). With these notations, we obtain the following lemma.

**Lemma 5.2.** Let \( q \) be a CQ and let \( \Sigma \) be a set of small domain constraints. Let \( M := M_{q,\Sigma} \).

If \( M = \emptyset \), then \( q(D) = \emptyset \) for every \( (\sigma,\Sigma) \)-db \( D \).

Otherwise, \( q \) is \( \Sigma \)-equivalent to the UCQ \( q_{\Sigma} := \bigcup_{\alpha \in M} q_{\alpha} \).

**Proof.** For a set \( \Sigma \) of constraints and a \( \sigma \)-db \( D \) we write \( D \models \Sigma \) to indicate that \( D \) satisfies every constraint in \( \Sigma \).

Let \( V := rvars_{\Sigma}(q) \) and \( M := M_{q,\Sigma} \). If \( V = \emptyset \), then \( M = \{ q_\emptyset \} \) where \( q_\emptyset \) is the unique mapping with empty domain. Thus, \( q_{\Sigma} = q_{q_\emptyset} = q \) and we are done. It remains to consider the case where \( V \neq \emptyset \).

Consider an arbitrary \( \sigma \)-db \( D \) with \( D \models \Sigma \) and let \( q \) be of the form \([\varphi] \). Consider an arbitrary tuple \( \bar{b} = (b_1, \ldots, b_k) \in q(D) \). By definition of the semantics of CQs, there is a valuation \( \beta : \text{var} \to \text{dom} \) such that \( (b_1, \ldots, b_k) = (\beta(u_1), \ldots, \beta(u_k)) \) and for every atomic formula \( Rv_1 \cdots v_r \) in \( q \) we have \( (\beta(v_1), \ldots, \beta(v_r)) \in R^D \). If \( x = v_i \) then \( \beta(x) \in \pi_i(R^D) \); and if \( \Sigma \) contains a constraint of the form \( R[i] \subseteq C \) then, since \( D \models \Sigma \), we have \( \pi_i(R^D) \subseteq C \), and hence \( \beta(x) \in C \). This holds true for every occurrence of \( x \) in an atom of \( q \), and hence \( \beta(x) \in \text{Dom}_{q,\Sigma}(x) \) for every \( x \in \text{vars}(q) \). In other words, the restriction \( \beta|_V \) of \( \beta \) to \( V \) belongs to \( M \), and \( \bar{b} \in q_{\beta|_V}(D) \subseteq q_{\Sigma}(D) \). In particular, this implies that the following is true.
1. If \( q(D) \neq \emptyset \) for some \( \sigma \)-db \( D \) with \( D \models \Sigma \), then \( M \neq \emptyset \). Hence, by contraposition, if \( M = \emptyset \) then \( q(D) = \emptyset \) for every \( \sigma \)-db \( D \) with \( D \models \Sigma \).

2. If \( M \neq \emptyset \), then \( q(D) \subseteq q_{\Sigma}(D) \) for every \( \sigma \)-db \( D \) with \( D \models \Sigma \). On the other hand, since \( q_{\alpha}(D) \subseteq q(D) \) for every \( \alpha \) and every \( \sigma \)-db \( D \), we have \( q_{\Sigma}(D) \subseteq q(D) \) for every \( \sigma \)-db \( D \). Hence, \( q \) is \( \Sigma \)-equivalent to \( q_{\Sigma} \).

This completes the proof of Lemma 5.2.

This reduction from a CQ \( q \) to a UCQ \( q_{\Sigma} \) directly translates to UCQs: if \( q \) is a union of the CQs \( q_1, \ldots, q_\alpha \), then we define the UCQ \( q_{\Sigma} := \bigcup_{i \in [q]} (q_i)_{\Sigma} \). It is not hard to verify that if the UCQ \( q \) is a homomorphic core, then so is \( q_{\Sigma} \). Therefore, the following dichotomy theorem for UCQs under small domain constraints is a direct consequence of Lemma 5.2 and the Theorems 4.1, 4.2 and 4.4.

**Theorem 5.3.** Let \( q \) be a UCQ that is a homomorphic core and \( \Sigma \) a set of small domain constraints with \( M_{q, \Sigma} \neq \emptyset \). Suppose that the OMV-conjecture and the OV-conjecture hold.

(1a) If \( q_{\Sigma} \) is \( t \)-hierarchical, then \( q \) can be tested on \((\sigma, \Sigma)\)-dbs in constant time with linear preprocessing and constant update time.

(1b) If \( q_{\Sigma} \) is not \( t \)-hierarchical, then on the class of \((\sigma, \Sigma)\)-dbs testing in time \( O(n^{1-\epsilon}) \) is not possible with \( O(n^{1-\epsilon}) \) update time.

(2a) If \( q_{\Sigma} \) is \( q \)-hierarchical, then there is data structure with linear preprocessing and constant update time that allows to enumerate \( q(D) \) with constant delay on \((\sigma, \Sigma)\)-dbs.

(2b) If \( q_{\Sigma} \) is not \( q \)-hierarchical and in addition self-join free, then \( q(D) \) cannot be enumerated with \( O(n^{1-\epsilon}) \) delay and \( O(n^{1-\epsilon}) \) update time on \((\sigma, \Sigma)\)-dbs.

(3a) If \( q_{\Sigma} \) is exhaustively \( q \)-hierarchical, then there is data structure with linear preprocessing and constant update time that allows to compute \( |q(D)| \) in constant time on \((\sigma, \Sigma)\)-dbs.

(3b) If \( q_{\Sigma} \) is not exhaustively \( q \)-hierarchical, then computing \( |q(D)| \) on \((\sigma, \Sigma)\)-dbs in time \( O(n^{1-\epsilon}) \) is not possible with \( O(n^{1-\epsilon}) \) update time.

In particular, this shows that the tractability of a UCQ \( q \) on \((\sigma, \Sigma)\)-dbs only depends on the structure of the query \( q_{\Sigma} \). Note that while the size of \( q_{\Sigma} \) might be \( c^{O(q)} \), where \( c \) is largest number of constants in a small domain, it can be checked in time \( poly(q) \) whether \( q_{\Sigma} \) is \( t \)-hierarchical or \( q \)-hierarchical.

Let us take a brief look at two other kinds of constraints: inclusion dependencies and functional dependencies, which both can also cause a hard query to become tractable.

An *inclusion dependency* \( \delta \) is of the form \( R[i_1, \ldots, i_m] \subseteq S[j_1, \ldots, j_m] \) where \( R, S \in \sigma \), \( m \geq 1 \), \( i_1, \ldots, i_m \in \{1, \ldots, \text{ar}(R)\} \), and \( j_1, \ldots, j_m \in \{1, \ldots, \text{ar}(S)\} \). A \( \sigma \)-db \( D \) satisfies \( \delta \) if \( \pi_1, \ldots, i_m(D) \subseteq \pi j_1, \ldots, j_m(S[D]) \). As an example consider the query \( q_{S-E-T} \) from Example 5.1 and the inclusion dependency \( \delta_{\text{ind}} := E[2] \subseteq T[1] \). Obviously, \( q_{S-E-T} \) is \( \{\delta_{\text{ind}}\} \)-equivalent to the \( q \)-hierarchical (and hence easy) CQ \( q' := \{() : \exists x \exists y \ (S x \land E x y)\} \). To turn this into a general principle, we say that an inclusion dependency \( \delta \) of the form \( R[i_1, \ldots, i_m] \subseteq S[j_1, \ldots, j_m] \) can be applied to a CQ \( q \) if \( q \) contains an atom \( \psi_1 \) of the form \( R v_1 \cdots v_r \) and an atom \( \psi_2 \) of the form \( Sw_1 \cdots w_s \) such that

1. \((v_1, \ldots, v_m) = (w_j, \ldots, w_{jm})\),
2. for all \( j \in [s] \setminus \{j_1, \ldots, j_m\} \) we have \( w_j \in \text{var} \), \( w_j \not\in \text{free}(q) \), \( \text{atoms}(w_j) = \{\psi_2\} \), and
3. for all \( j, j' \in [s] \setminus \{j_1, \ldots, j_m\} \) with \( j \neq j' \) we have \( w_j \neq w_{j'} \).
and applying $\delta$ to $q$ at $(\psi_1, \psi_2)$ then yields the CQ $q'$ which is obtained from $q$ by omitting the atom $\psi_2$ and omitting the quantifiers $\exists z$ for all $z \in \text{vars}(\psi_2) \setminus \{w_j, \ldots, w_m\}$. By this construction we have $\text{vars}(q') = \text{vars}(q) \setminus \{w_j : j \in [s] \setminus \{j_1, \ldots, j_m\}\}$.

Claim 5.4. $q' \equiv_{(s)} q$, and if $q$ is $q$-hierarchical, then so is $q'$.

Proof. For a set $\Sigma$ of constraints and a $\sigma$-db $D$ we write $D \models \Sigma$ to indicate that $D$ satisfies every constraint in $\Sigma$. For a constraint $\delta$ we write $D \models \delta$ instead of $D \models \{\delta\}$.

Obviously, $q(D) \subseteq q'(D)$ for every $\sigma$-db $D$. For the opposite direction, let $q$ be of the form $(\ast \ast)$, and consider a $\sigma$-db $D$ with $D \models \delta_{\text{ind}}$ and a tuple $t \in q'(D)$. Our goal is to show that $t \in q(D)$. Since $t \in q'(D)$, there is a valuation $\beta'$ such that $t = (\beta'(u_1), \ldots, \beta'(u_k))$ and $(D, \beta') \models \psi$ for each atom $\psi$ of $q'$. In particular, $(D, \beta') = Rv_1 \cdots v_r$, i.e., $(\beta'(v_1), \ldots, \beta'(v_r)) \in R^D$. To show that $t \in q(D)$ it suffices to modify $\beta'$ into a valuation $\beta$ which coincides with $\beta'$ on all variables in $\text{vars}(q')$ and which also ensures that $(D, \beta) = Sw_1 \cdots w_s$, i.e., that $(\beta(w_1), \ldots, \beta(w_s)) \in S^D$.

Since $D \models \delta_{\text{ind}}$ we obtain from $(\beta'(v_1), \ldots, \beta'(v_r)) \in R^D$ that $(\beta'(v_1), \ldots, \beta'(v_m)) \in \pi_{i_1, \ldots, i_m}(R^D) \subseteq \pi_{i_1, \ldots, i_m}(S^D)$. Since $(v_1, \ldots, v_m) = (w_{j_1}, \ldots, w_{j_m})$, this implies that $(\beta(w_{j_1}), \ldots, \beta(w_{j_m})) = (a_{j_1}, \ldots, a_{j_m})$.

We let $\beta$ be the valuation obtained from $\beta'$ by letting $\beta(w_j) := a_j$ for every $j \in [s] \setminus \{j_1, \ldots, j_m\}$. With this choice we have $(D, \beta) = Sw_1 \cdots w_s$. Note that $\beta$ differs from $\beta'$ only in variables $w_j$ for which we know that $\text{atoms}(w_j) = \{\psi_2\} = \{Sw_1 \cdots w_s\}$, i.e., variables that occur in no other atom of $q$ than the atom $Sw_1 \cdots w_s$. Therefore, $(D, \beta) \models \psi$ for each atom $\psi$ of $q$, and hence $t = (\beta(u_1), \ldots, \beta(u_k)) \in q(D)$.

In summary, we obtain that $q'(D) \subseteq q(D)$ for every $\sigma$-db $D$ with $D \models \delta_{\text{ind}}$. This completes the proof showing that $q' \equiv_{(\text{ind})} q$.

To verify the claim’s second statement, let $W := \{w_j : j \in [s] \setminus \{j_1, \ldots, j_m\}\}$ and note that $\text{vars}(q') = \text{vars}(q) \setminus W$. For all $x \in W$ we have $\text{atoms}_q(x) = \{\psi_2\}$, and for all $x \in \text{vars}(q')$ we have $\text{atoms}_{q'}(x) = \text{atoms}_q(x) \setminus \{\psi_2\}$. Using this, we obtain that if $q$ satisfies condition (i) of Definition 3.1 then so does $q'$.

It remains to show that if $q$ is $q$-hierarchical, then $q'$ also satisfies condition (i) of Definition 3.1. Assume for contradiction that $q'$ does not satisfy this condition. Then, there are $x \in \text{free}(q')$ and $y \in \text{vars}(q') \setminus \text{free}(q')$ with $\text{atoms}_q(x) \subsetneq \text{atoms}_{q'}(y)$.

Case 1: $\text{atoms}_q(x) = \text{atoms}_{q'}(x)$. Then, $\text{atoms}_q(x) \subsetneq \text{atoms}_{q'}(y)$, and thus $q$ does not satisfy condition (i) of Definition 3.1 and hence is not $q$-hierarchical.

Case 2: $\text{atoms}_q(x) = \text{atoms}_{q'}(x) \cup \{\psi_2\}$. If $\text{atoms}_q(y) = \text{atoms}_{q'}(y) \cup \{\psi_2\}$, then we are done by the same reasoning as in Case 1. On the other hand, if $\text{atoms}_q(y) = \text{atoms}_{q'}(y)$, then $\psi_2 \in \text{atoms}_q(x) \setminus \text{atoms}_q(y)$. Furthermore, since $\text{atoms}_q(x) \subsetneq \text{atoms}_q(y)$, there are atoms $\psi$ and $\psi'$ such that $\psi \in \text{atoms}_q(x) \cap \text{atoms}_q(y)$ and $\psi' \in \text{atoms}_q(y) \setminus \text{atoms}_q(x)$. Thus, $q$ violates condition (i) of Definition 3.1 and hence is not $q$-hierarchical.

From the claim it follows that we can simplify a given query by iteratively applying inclusion dependencies to pairs of atoms of the query. In some cases, this transforms queries that are hard in general into $\Sigma$-equivalent queries that are $q$-hierarchical and hence easy for dynamic evaluation. For example, an iterated application of $\delta_{\text{ind}} := E[2] \subseteq E[1]$ transforms the non-t-hierarchical query \{ $(x, y) : \exists z_1 \exists z_2 \ (E_{xy} \wedge Ey_{z_1} \wedge Ez_{z_2})$ \} into the $q$-hierarchical query \{ $(x, y) : E_{xy}$ \}. However, the limitations of this approach are documented by the query $q := \{ () : \exists z \exists y \exists z' (Sx \wedge E_{xy} \wedge Ty \wedge Rzz') \}$, which is $\Sigma$-equivalent to the $q$-hierarchical query $q' := \{ () : \exists z \exists z' (Rzz') \}$, for $\Sigma := \{ R[1, 2] \subseteq E[1, 2], \ R[1] \subseteq S[1], \ R[2] \subseteq T[1] \}$, but where $q'$ cannot be obtained by iteratively applying dependencies of $\Sigma$ to $q$.

The presence of functional dependencies can also cause a hard query to become tractable: Consider the functional dependency $\delta_{\text{id}} := E[1 \rightarrow 2]$, which is satisfied by a database $D$ iff for every $a \in \text{dom}$ there is at most one $b \in \text{dom}$ such that $(a, b) \in E^D$. On databases that satisfy
the query $q_{S,E,T}$ from Example 5.1 can be evaluated with constant answer time and constant update time as follows: One can store for every $b$ the number $m_b$ of elements $(a,b) \in E_D$ such that $a \in S_D$ and in addition the number $m = \sum_{b \in T_D} m_b$, which is non-zero if and only if $q_{S,E,T}(D) = \text{yes}$. The functional dependency guarantees that every update affects at most one number $m_b$ and one summand of $m$. Using constant access data structures, the query result can therefore be maintained with constant update time.

The nature of this example is somewhat different compared to the approaches for small domain constraints or inclusion constraints described above: We can show that the query becomes tractable, but we are not aware of any $\{\delta^b\}$-equivalent q-hierarchical CQ or UCQ that would explain its tractability via a reduction to the setting without integrity constraints. To exploit the full power of functional dependencies for improving dynamic query evaluation, it seems therefore necessary to come up with new algorithmic approaches that go beyond the techniques we have for (q- or t-)hierarchical queries.

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APPENDIX

A Full proof of Theorem 3.4 (b)

Proof of Theorem 3.4 for the case that \( q \) violates condition (i) of Definition 3.3

Assume we are given a query \( q := q_d(z_1, \ldots, z_k) \) that is a homomorphic core and that is not t-hierarchical because it violates condition (i) of Definition 3.3. Thus, there are two variables \( x, y \in \text{vars}(q) \setminus \text{free}(q) = \text{vars}(q) \setminus \{z_1, \ldots, z_k\} \) and three atoms \( \psi^x, \psi^{x,y}, \psi^y \) of \( q \) with \( \text{vars}(\psi^x) \cap \{x, y\} = \{x\} \), \( \text{vars}(\psi^{x,y}) \cap \{x, y\} = \{x, y\} \), and \( \text{vars}(\psi^y) \cap \{x, y\} = \{y\} \).

We show how a dynamic algorithm that solves the testing problem for \( q \) can be used to solve the \( \text{OnMv} \)-problem.

Without loss of generality we assume that \( \text{vars}(q) = \{x, y, z_1, \ldots, z_{\ell}\} \) for some \( \ell \geq k \), and \( |\text{vars}(q)| = \ell + 2 \). For a given \( n \times n \) matrix \( M \) we fix a domain \( \text{dom}_n \) that consists of \( 2n + \ell \) elements \( \{a_i, b_i : i \in [n]\} \cup \{c_s : s \in [\ell]\} \) from \( \text{dom} \setminus \text{cons}(q) \). For \( i, j \in [n] \) we let \( \iota_{i,j} \) be the injective mapping from \( \text{vars}(q) \cup \text{cons}(q) \) to \( \text{dom}_n \cup \text{cons}(q) \) with

- \( \iota_{i,j}(x) = a_i \),
- \( \iota_{i,j}(y) = b_j \),
- \( \iota_{i,j}(z_s) = c_s \) for all \( s \in [\ell] \), and
- \( \iota_{i,j}(d) = d \) for all \( d \in \text{cons}(q) \).

We tacitly extend \( \iota_{i,j} \) to a mapping from \( \text{vars}(q) \cup \text{dom} \) to \( \text{dom} \) by letting \( \iota_{i,j}(d) = d \) for every \( d \in \text{dom} \).

For the matrix \( M \) and for \( n \)-dimensional vectors \( \vec{u} \) and \( \vec{v} \), we define a \( \sigma \)-db \( D = D(q, M, \vec{u}, \vec{v}) \) with \( \text{dom}(D) \subseteq \text{dom}_n \cup \text{cons}(q) \) as follows (recall our notational convention that \( \vec{u}_i \) denotes the \( i \)-th entry of a vector \( \vec{u} \)). For every atom \( \psi = Rw_1 \cdots w_r \) in \( q \) we include in \( R^D \) the tuple \( (\iota_{i,j}(w_1), \ldots, \iota_{i,j}(w_r)) \)

- for all \( i, j \in [n] \) such that \( \vec{u}_i = 1 \), if \( \psi = \psi^x \),
- for all \( i, j \in [n] \) such that \( \vec{v}_j = 1 \), if \( \psi = \psi^y \),
- for all \( i, j \in [n] \) such that \( M_{i,j} = 1 \), if \( \psi = \psi^{x,y} \), and
- for all \( i, j \in [n] \), if \( \psi \notin \{\psi^x, \psi^{x,y}, \psi^y\} \).

Note that the relations in the atoms \( \psi^x, \psi^y, \) and \( \psi^{x,y} \) are used to encode \( \vec{u}, \vec{v}, \) and \( M, \) respectively. Moreover, since \( \psi^x, \psi^y \) does not contain the variable \( y \) (\( x \)), two databases \( D = D(q, M, \vec{u}, \vec{v}) \) and \( D' = D(q, M, \vec{u}', \vec{v}') \) differ only in at most \( 2n \) tuples. Therefore, \( D' \) can be obtained from \( D \) by \( 2n \) update steps. It follows from the definitions that \( \iota_{i,j} \) is a homomorphism from \( q \) to \( D \) if and only if \( \vec{u}_i = 1, \vec{v}_j = 1, \) and \( M_{i,j} = 1 \). Therefore, \( \vec{u}^T M \vec{v} = 1 \) if and only if there are \( i, j \in [n] \) such that \( \iota_{i,j} \) is a homomorphism from \( q \) to \( D \).

We let \( g \) be the (surjective) mapping from \( \text{dom}_n \cup \text{cons}(q) \) to \( \text{vars}(q) \cup \text{cons}(q) \) defined by \( g(d) = d \) for all \( d \in \text{cons}(q) \) and \( g(c_s) := z_s, g(a_i) := x, g(b_j) := y \) for all \( i, j \in [n] \) and \( s \in [\ell] \). Clearly, \( g \) is a homomorphism from \( D \) to \( q \). Obviously, the following is true for every mapping \( h \) from \( \text{vars}(q) \) to \( \text{dom}(D) \) and for all \( w \in \text{vars}(q) \):

- if \( h(w) = c_s \) for some \( s \in [\ell] \), then \((g \circ h)(w) = z_s \),
- if \( h(w) = a_i \) for some \( i \in [n] \), then \((g \circ h)(w) = x \),
- if \( h(w) = b_j \) for some \( j \in [n] \), then \((g \circ h)(w) = y \),

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• if $h(w) = d$ for some $d \in \text{cons}(q)$, then $(g \circ h)(w) = d$.

We define the partition $P = \{\{c_1\}, \ldots, \{c_\ell\}, \{a_i : i \in [n]\}, \{b_j : j \in [n]\}\}$ of $\text{dom}_n$ and say that a mapping $h$ from $\text{vars}(q) \cup \text{dom}$ to $\text{dom}$ respects $P$, if for each set from the partition there is exactly one element in the image of $\text{vars}(q)$ under $h$, i.e., the set $\{h(w) : w \in \text{vars}(q)\}$.

Claim A.1. $\vec{u}^T M \vec{v} = 1 \iff$ There exists a homomorphism $h : q \to D$ that respects $P$.

Proof. For one direction assume that $\vec{u}^T M \vec{v} = 1$. Then there are $i,j \in [n]$ such that $\iota_{i,j}$ is a homomorphism from $q$ to $D$ that respects $P$. For the other direction assume that $h : q \to D$ is a homomorphism that respects $P$. Thus, there are elements $w_a, w_b, w_1, \ldots, w_\ell$ in $\text{vars}(q)$ such that $h(w_a) \in \{a_i : i \in [n]\}, h(w_b) \in \{b_j : j \in [n]\}$, and $h(w_s) = c_s$ for each $s \in [\ell]$. It follows that $(g \circ h)$ is a bijective homomorphism from $q_\varphi(z_1, \ldots, z_k)$ to $(g \circ h)(z_1), \ldots, (g \circ h)(z_k))$. Therefore, it can easily be verified that $h \circ (g \circ h)^{-1}$ is a homomorphism from $q$ to $D$ which equals $\iota_{i,j}$ for some $i,j \in [n]$. This implies that $\vec{u}^T M \vec{v} = 1$.

Claim A.2. If $q$ is a homomorphic core, then every homomorphism $h : q \to D$ respects $P$.

Proof. Assume for contradiction that $h : q \to D$ is a homomorphism that does not respect $P$. Then $(g \circ h)$ is a homomorphism from $q$ into a proper subquery of $q$, contradicting that $q$ is a homomorphic core.

Claim A.3. If $q$ is a homomorphic core, then $\vec{u}^T M \vec{v} = 1 \iff (c_1, \ldots, c_\ell) \in q(D)$.

Proof. We already know that $\vec{u}^T M \vec{v} = 1$ if and only if there are $i,j \in [n]$ such that $\iota_{i,j}$ is a homomorphism from $q$ to $D$. Furthermore, $\iota_{i,j}(z_s) = c_s$ for all $s \in [\ell]$ and all $i, j \in [n]$. Thus, if $\vec{u}^T M \vec{v} = 1$, then there exist $i,j \in [n]$ such that $(\iota_{i,j}(z_1), \ldots, \iota_{i,j}(z_k)) = (c_1, c_\ell) \in q(D)$. This proves direction $\implies$ of the claim. For the opposite direction, note that if $(c_1, \ldots, c_\ell) \in q(D)$, then there is a homomorphism $h$ from $q$ to $D$. By Claim A.2 $h$ respects $P$, and hence by Claim [A.1] $\vec{u}^T M \vec{v} = 1$.

We are now ready for proving Theorem [3.1][b] for the case that $q$ violates condition (i) of Definition [3.3]. Assume for contradiction that the testing problem for $q$ can be solved with update time $t_u = O(n^{1-\epsilon})$ and testing time $t_t = O(n^{2-\epsilon})$. We can use this algorithm to solve the OuMv-problem in time $O(n^{3-\epsilon})$ as follows.

In the preprocessing phase, we are given the $n \times n$ matrix $M$ and let $\vec{u}^0$, $\vec{v}^0$ be the all-zero vectors of dimension $n$. We start the preprocessing phase of our testing algorithm for $q$ with the empty database. As this database has constant size, the preprocessing phase finishes in constant time. Afterwards, we use $O(n^2)$ insert operations to build the database $D(q, M, \vec{u}^0, \vec{v}^0)$. All this is done within time $O(n^2 t_u) = O(n^{3-\epsilon})$.

When a pair of vectors $\vec{u}^t$, $\vec{v}^t$ (for $t \in [n]$) arrives, we change the current database $D(q, M, \vec{u}^{t-1}, \vec{v}^{t-1})$ into $D(q, M, \vec{u}^t, \vec{v}^t)$ by using at most $2n$ update steps. By Claim A.3 we know that $(\vec{u}^t)^T M \vec{v}^t = 1$ if, and only if, $(c_1, \ldots, c_\ell) \in q(D)$, for $D := D(q, M, \vec{u}^t, \vec{v}^t)$. Hence, after running the test routine with input $(c_1, \ldots, c_\ell)$ in time $t_t = O((\text{adom}(D))^{1-\epsilon}) = O(n^{2-\epsilon})$ we can output the value of $(\vec{u}^t)^T M \vec{v}^t$.

The time we spend for each $t \in [n]$ is bounded by $O(2nt_u + t_t) = O(n^{2-\epsilon})$. Thus, the overall running time for solving the OuMv-problem sums up to $O(n^{3-\epsilon})$, contradicting the OuMv-conjecture and hence also the OMv-conjecture.

This completes the proof of Theorem [3.1][b] for the case that $q$ violates condition (i) of Definition [3.3].
Proof of Theorem 3.4(b) for the case that \( q \) violates condition (ii) of Definition 3.3

Assume we are given a query \( q := q_\varphi(z_1, \ldots, z_k) \) that is a homomorphic core and that is not t-hierarchical because it violates condition (ii) of Definition 3.3. Thus, there are two variables \( x \in \text{free}(q) \) and \( y \in \text{vars}(q) \setminus \text{free}(q) \) and two atoms \( \psi^{x,y} \) and \( \psi^y \) of \( q \) with \( \text{vars}(\psi^{x,y}) \cap \{ x, y \} = \{ x, y \} \) and \( \text{vars}(\psi^y) \cap \{ x, y \} = \{ y \} \).

We show how a dynamic algorithm that solves the testing problem for \( q \) can be used to solve the OuMv-problem.

Without loss of generality we assume that \( \text{vars}(q) = \{ z_1, \ldots, z_\ell \} \) with \( \ell > k \), \( x = z_1 \), and \( y = z_\ell \).

For a given \( n \times n \) matrix \( M \) we fix a domain \( \text{dom}_n \) that consists of \( 2n + \ell - 2 \) elements \( \{ a_i, b_i : i \in [n] \} \cup \{ c_s : s \in \{ 2, \ldots, \ell - 1 \} \} \) from \( \text{dom} \setminus \text{cons}(q) \). For \( i, j \in [n] \) we let \( \iota_{i,j} \) be the injective mapping from \( \text{vars}(q) \cup \text{cons}(q) \) to \( \text{dom}_n \cup \text{cons}(q) \) with

- \( \iota_{i,j}(x) = a_i \),
- \( \iota_{i,j}(y) = b_j \),
- \( \iota_{i,j}(z_s) = c_s \) for all \( s \in \{ 2, \ldots, \ell - 1 \} \), and
- \( \iota_{i,j}(d) = d \) for all \( d \in \text{cons}(q) \).

We tacitly extend \( \iota_{i,j} \) to a mapping from \( \text{vars}(q) \cup \text{dom} \) to \( \text{dom} \) by letting \( \iota_{i,j}(d) = d \) for every \( d \in \text{dom} \).

For the matrix \( M \) and for an \( n \)-dimensional vector \( \vec{v} \), we define a \( \sigma \)-db \( D = D(q, M, \vec{v}) \) with \( \text{dom}(D) \subseteq \text{dom}_n \cup \text{cons}(q) \) as follows (recall our notational convention that \( \vec{v}_j \) denotes the \( j \)-th entry of a vector \( \vec{v} \)). For every atom \( \psi = R_{w_1} \cdots w_r \) in \( q \) we include in \( R^D \) the tuple \( (\iota_{i,j}(w_1), \ldots, \iota_{i,j}(w_r)) \)

- for all \( i, j \in [n] \) such that \( \vec{v}_j = 1 \), if \( \psi = \psi^y \),
- for all \( i, j \in [n] \) such that \( M_{i,j} = 1 \), if \( \psi = \psi^{x,y} \), and
- for all \( i, j \in [n] \), if \( \psi \notin \{ \psi^{x,y}, \psi^y \} \).

Note that the relations in the atoms \( \psi^y \) and \( \psi^{x,y} \) are used to encode \( \vec{v} \) and \( M \), respectively. Moreover, since \( \psi^y \) does not contain the variable \( x \), two databases \( D = D(q, M, \vec{v}) \) and \( D' = D(q, M, \vec{v}') \) differ only in at most \( n \) tuples. Therefore, \( D' \) can be obtained from \( D \) by \( n \) update steps. It follows from the definitions that

\[ \iota_{i,j} \text{ is a homomorphism from } q \text{ to } D \iff M_{i,j} = 1 \text{ and } \vec{v}_j = 1. \]

We let \( g \) be the (surjective) mapping from \( \text{dom}_n \cup \text{cons}(q) \) to \( \text{vars}(q) \cup \text{cons}(q) \) defined by \( g(d) = d \) for all \( d \in \text{cons}(q) \) and \( g(c_s) := z_s \), \( g(a_i) := x \), \( g(b_j) := y \) for all \( i, j \in [n] \) and \( s \in \{ 2, \ldots, \ell - 1 \} \). Clearly, \( g \) is a homomorphism from \( D \) to \( q \). Obviously, the following is true for every mapping \( h \) from \( \text{vars}(q) \) to \( \text{dom}(D) \) and for all \( w \in \text{vars}(q) \):

- if \( h(w) = c_s \) for some \( s \in \{ 2, \ldots, \ell - 1 \} \), then \( (g \circ h)(w) = z_s \),
- if \( h(w) = a_i \) for some \( i \in [n] \), then \( (g \circ h)(w) = x \),
- if \( h(w) = b_j \) for some \( j \in [n] \), then \( (g \circ h)(w) = y \),
- if \( h(w) = d \) for some \( d \in \text{cons}(q) \), then \( (g \circ h)(w) = d \).

We define the partition \( \mathcal{P} = \{ \{ c_2, \ldots, c_{\ell-1} \}, \{ a_i : i \in [n] \}, \{ b_j : j \in [n] \} \} \) of \( \text{dom}_n \) and say that a mapping \( h \) from \( \text{vars}(q) \cup \text{dom} \to \text{dom} \) respects \( \mathcal{P} \), if for each set from the partition there is exactly one element in the set \( h(\text{vars}(q)) := \{ h(w) : w \in \text{vars}(q) \} \).
Claim A.4. For every \( i \in [n] \), the following are equivalent:

- There is a \( j \in [n] \) such that \( M_{i,j} = 1 \) and \( \vec{v}_j = 1 \).
- There is a homomorphism \( h : q \to D \) that respects \( P \) such that \( a_i \in h(\text{vars}(q)) \).

Proof. Consider a fixed \( i \in [n] \). For one direction assume that there is a \( j \in [n] \) such that \( M_{i,j} = 1 \) and \( \vec{v}_j = 1 \). Then, \( h \) respects \( P \) and \( a_i \in h(\text{vars}(q)) \).

For the other direction assume that \( h : q \to D \) is a homomorphism that respects \( P \) and \( a_i \in h(\text{vars}(q)) \). Thus, there are elements \( w_{a_i}, w_2, \ldots, w_{k-1} \) in \( \text{vars}(q) \) such that \( h(w_{a_i}) = a_i, h(w_j) \in \{b_j : j \in [n]\} \), and \( h(w_s) = c_s \) for each \( s \in \{2, \ldots, k-1\} \). It follows that \( (g \circ h) \) is a bijection from \( q \) to \( D \) which equals \( t_{i,j} \) for some \( j \in [n] \).

Therefore, it can easily be verified that \( h \circ (g \circ h)^{-1} \) is a homomorphism from \( q \) to \( D \) which equals \( t_{i,j} \) for some \( j \in [n] \).

Claim A.5. If \( q \) is a homomorphic core, then every homomorphism \( h : q \to D \) respects \( P \).

Proof. Assume for contradiction that \( h : q \to D \) is a homomorphism that does not respect \( P \). Then \( (g \circ h) \) is a homomorphism from \( q \) into a proper subquery of \( q \), contradicting that \( q \) is a homomorphic core.

Claim A.6. If \( q \) is a homomorphic core, then for every \( i \in [n] \) the following are equivalent:

- There is a \( j \in [n] \) such that \( M_{i,j} = 1 \) and \( \vec{v}_j = 1 \).
- \((a_i, c_2, \ldots, c_k) \in q(D)\).

Proof. Consider a fixed \( i \in [n] \). For one direction assume that there is a \( j \in [n] \) such that \( M_{i,j} = 1 \) and \( \vec{v}_j = 1 \). Then, \( t_{i,j} \) is a homomorphism from \( q \) to \( D \), and thus \((t_{i,j}(x), t_{i,j}(z_2), \ldots, t_{i,j}(z_k)) \in q(D)\). By definition of \( t_{i,j} \) we have \((t_{i,j}(x), t_{i,j}(z_2), \ldots, t_{i,j}(z_k)) = (a_i, c_2, \ldots, c_k)\), and hence we are done (recall that \( y = z_k \) and \( j > k \)).

For the other direction assume that \((a_i, c_2, \ldots, c_k) \in q(D)\). Thus, there exists a homomorphism \( h : q \to D \) such that \((a_i, c_2, \ldots, c_k) = (h(x), h(z_2), \ldots, h(z_k))\). According to Claim A.5 \( h \) respects \( P \). Furthermore, \( a_i \in h(\text{vars}(q)) \), since \( h(x) = a_i \). From Claim A.4 we obtain that there is a \( j \in [n] \) such that \( M_{i,j} = 1 \) and \( \vec{v}_j = 1 \).

We are now ready for proving Theorem B.2 for the case that \( q \) violates condition B.3 of Definition B.3. Assume for contradiction that the testing problem for \( q \) can be solved with update time \( t_u = O(n^{1-\epsilon}) \) and testing time \( t_t = O(n^{1-\epsilon}) \). We can use this algorithm to solve the OnMv-problem in time \( O(n^{3-\epsilon}) \) as follows.

In the preprocessing phase, we are given the \( n \times n \) matrix \( M \) and let \( \vec{v}^0 \) be the all-zero vectors of dimension \( n \). We start the preprocessing phase of our testing algorithm for \( q \) with the empty database. As this database has constant size, the preprocessing phase finishes in constant time. Afterwards, we use \( O(n^2) \) insert operations to build the database \( D(q, M, \vec{v}^0) \). All this is done within time \( O(n^2 t_u) = O(n^{3-\epsilon}) \).

When a pair of vectors \( \vec{u}^t, \vec{v}^t \) (for \( t \in [n] \)) arrives, we change the current database \( D(q, M, \vec{v}^{t-1}) \) into \( D(q, M, \vec{v}^t) \) by using at most \( n \) update steps.

By assumption, \( q \) is a homomorphic core. Thus, Claim A.6 tells us that for \( D := D(q, M, \vec{v}^t) \) and for every \( i \in [n] \) we have

\[(a_i, c_2, \ldots, c_k) \in q(D) \iff \text{there is a } j \in [n] \text{ such that } M_{i,j} = 1 \text{ and } \vec{v}_j = 1.\]

Hence, after running the test routine with input \((a_i, c_2, \ldots, c_k)\) for each \( i \in [n] \) with \( \vec{u}_i = 1 \), we can output the value of \((\vec{u}^t)^T M \vec{v}^t \). For this, we use at most \( n \) calls of the test routine, and each such call is executed within time \( t_t = O(\text{adm}(D)[1-\epsilon]) = O(n^{1-\epsilon}) \). The time we spend to
compute \((\bar{u}^t)^T M \bar{v}^t\) for a fixed \(t \in [n]\) is therefore bounded by \(O(nt_u + nt_v) = O(n^{2-\epsilon})\). Thus, the overall running time for solving the OuMv-problem sums up to \(O(n^{3-\epsilon})\), contradicting the OuMv-conjecture and hence also the OMv-conjecture.

This completes the proof of Theorem 3.4(b) for the case that \(q\) violates condition (iii) of Definition 3.3.