DEFORMATION INEQUIVALENT COMPLEX CONJUGATED COMPLEX STRUCTURES AND APPLICATIONS

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Abstract. We start from a short summary of our principal result from [KK]: an example of a complex algebraic surface which is not deformation equivalent to its complex conjugate and which, moreover, has no homeomorphisms reversing the canonical class. Then, we generalize this result to higher dimensions and construct several series of higher dimensional compact complex manifolds having no homeomorphisms reversing the canonical class. After that, we resume and broaden the applications given in [KK] and [KK2], in particular, as a new application, we propose examples of (deformation) non equivalent symplectic structures with opposite canonical classes.

1. Introduction

Many of achievements in real algebraic geometry appeared as applications of complex geometry. By contrary, the results of the present paper, which has grown from a solution of some questions from the real algebraic geometry, can be considered as applications in the opposite direction. They may be of an interest for symplectic geometry too.

To state the principal questions we need to fix some definitions. We choose the language of complex analytic geometry since in this setting the results sometimes look stronger than if we restricted ourselves to algebraic or Kähler varieties, though the Kähler hypothesis could simplify several proofs and could allow one to extend the set of examples.

Thus, we define a real structure on a complex manifold X as an antiholomorphic involution $c : X \to X$. By a deformation equivalence we mean the equivalence generated by local deformations of complex manifolds. Since we are interested only in compact manifolds, this, commonly used, local deformation equivalence relation can be defined in the following way: given a proper holomorphic submersion $f : W \to B_1, B_1 = \{|z| < 1\} \subset \mathbb{C}$, the complex varieties isomorphic to $X_t = f^{-1}(t), t \in B_1$, are said to be deformation equivalent.

Note that anti-holomorphic automorphisms (anti-automorphisms for short) of a complex manifold $X = (M, J)$, where $M$ is the underlying smooth manifold and $J$ is a complex structure on it, can be regarded as isomorphisms between $X$ and its complex conjugated twin, $\bar{X} = (M, -J)$. All the anti-automorphisms together with the automorphisms form a group which we denote by Kl and call Klein group. Clearly, the Klein group is either a
\[ \mathbb{Z}/2\text{-extension of the automorphism group } \text{Aut}, 1 \to \text{Aut} \to \text{Kl} \xrightarrow{k} \mathbb{Z}/2 \to 1, \text{ or coincides with it, } \text{Kl} = \text{Aut}. \]

Note also that the underlying smooth manifolds of deformation equivalent varieties are necessary diffeomorphic. Moreover, the diffeomorphisms coming from a deformation preserve the complex orientation and the canonical class. Certainly, in the same time, on a given differentiable manifold here may exist isomorphic complex structures with different canonical classes or even with different complex orientations.

Below are two closely related questions from real algebraic geometry, which initiated our study. \textit{Can any complex compact manifold be deformed to a manifold which can be equipped with a real structure or at least to a manifold with an anti-automorphism (not necessarily of order 2)?}

It is clear that the response is in the affirmative as long as the (complex) dimension of the manifold is \( \leq 1 \), i.e., for points and Riemann surfaces: obviously, any Riemann surface can be deformed to a real one. To our knowledge, starting from \( \dim_{\mathbb{C}} = 2 \) the both questions remained open. We have proved the following theorem.

**Theorem 1.1.** \textit{In any dimension} \( \geq 2 \) \textit{there exists a compact complex manifold} \( X \) \textit{which cannot be deformed to} \( \bar{X} \); \textit{in particular,} \( X \) \textit{cannot be deformed to a real manifold or to a manifold with an anti-automorphism.}

Explicit examples are given below in Sections 2 and 3. Many of them have a stronger property (Corollary 3.13):

\textit{In any dimension} \( \geq 2 \) \textit{there exists a projective complex manifold} \( X \) \textit{of general type which has no homeomorphisms} \( h : X \to X \) \textit{such that} \( h^*c_1(X) = -c_1(X) \).

It is worth noticing that in dimension 2 all our examples are rigid, i.e., they have no local nontrivial deformations at all. Moreover, they are strongly rigid, i.e., up to isomorphism and conjugation, they have only one complex structure. Their strong rigidity is not used in the proof of Theorem 1.1 in its dimension 2 part, but we use it in higher dimensions.

The examples mentioned above provide the promised applications, which are: new counter-examples to the Dif = Def problem for complex surfaces (and higher dimensional varieties); first (to our knowledge) counter-examples to the ambient Dif \( \Rightarrow \) Def problem for plane cuspidal curves; and first (to our knowledge) examples of symplectic structures \( \omega \) not equivalent, up to diffeomorphisms and deformations, to their reverse \( -\omega \).

Deformation equivalent complex manifolds are orientedly (\( \mathcal{C}^\infty \)-) diffeomorphic and it is the converse which is called the Dif = Def problem (the, probably, first mentioning of this problem is found in [FM]). Obviously, the response is in the affirmative in dim \( \leq 1 \). For complex surfaces the question remained open for quite a while. The first counter-examples were found by Manetti only a couple of years ago, see [Ma]. The surfaces \( X \) and \( \bar{X} \) from our Theorem 1.1 are different and, by our opinion, more simple counter-examples. At least, they have the following, useful for our further applications, properties: our inequivalent complex structures are opposite to each other (\( J_0 = -J_1 \); in other words, the complex structures are complex conjugated), so that their canonical classes are opposite (\( c_1(J_0) = -c_1(J_1) \)), and, as in Corollary 3.13, there is no homeomorphism transforming \( c_1(J_0) \) in \( c_1(J_1) \). Moreover, in our examples the structures \( J_0 \) and \( J_1 \) are not equivalent even in the class of almost-complex structures, see the concluding Remark in Section 4.

The counter-examples to the Dif = Def problem in dim \( \geq 3 \) were known before. In dimension 3, at least in the category of Kähler manifolds, one can take products of the
Riemann sphere with Dolgachev surfaces, see [R]; as it follows from the classification of 6-dimensional manifolds [Z], they are diffeomorphic, and their stability under deformations in the category of Kähler manifolds follows from [Fujiki] (note that, as is proven in [R], these counter-examples work as well in the symplectic category). In dimension 4 and in higher even dimensions one can take products of Dolgachev surfaces and argue on divisors of the canonical class which are invariant under deformation, see [Fujiki]; such products are diffeomorphic due to $h$-cobordance of Dolgachev surfaces, which follows from their homotopy equivalence, see [W]. Note that in our dim $\geq 3$ examples the non equivalence of the varieties is not related to the divisibility of the canonical class and, in each dimension, among the examples there are varieties of general type.

It is worth noticing also that the convention to fix the orientation in the statement of the Dif = Def problem is not necessary in the case of complex dimension 2. As it follows from Kotschick’s results, see [Ko], in this dimension the existence of two complex structures with opposite orientations (which is an extremely rare phenomenon in this dimension) implies the existence of an orientation-reversing diffeomorphism, and, thus, the set of equivalence classes does not change if the orientation convention is removed.

Now, let us discuss the second application, which concerns the ambient Dif $\Rightarrow$ Def problem for plane curves. As in the absolute case, (equisingular) deformation equivalent pairs $(P^2, C)$, where $C \subset P^2$ stands for a plane curve, are diffeomorphic whenever $C$ is a cuspidal curve, and the problem consists in the converse statement (for non cuspidal curves the diffeotopy should be replaced by an isotopy; in fact, in all cases one can speak already on Dif $\Rightarrow$ Iso problem; note that the isotopy coming from an equisingular deformation certainly always can be made smooth outside the singular points, and for some types of singular points, as in the case of ordinary nodes and cusps, the isotopy can be made smooth everywhere). If the curve is nonsingular or if it has only ordinary nodes, the response to the Dif $\Rightarrow$ Def problem is in the affirmative (for the nonsingular case it is trivial, for curves with ordinary nodes it follows from the Severi assertion on the irreducibility of the corresponding spaces). We have produced, using our surfaces from theorem 1.1, an infinite sequence of counter-examples in which the curves are cuspidal. Our curves are irreducible and in each example the diffeomorphism $P^2 \to P^2$ transforming one curve into another is, in fact, the standard complex conjugation. They are obtained as the visible contours of the surfaces from theorem 1.1 (see more details in Section 4).

To conclude the introduction, let us mention also an application to a Dif = Def problem in symplectic geometry, which may also be of a certain interest. Consider, on a given oriented smooth 4-manifold, all the possible symplectic structures compatible with the orientation. Call two structures equivalent if one can be obtained from the other by a deformation followed, if necessary, by a diffeomorphism. Our surfaces from theorem 1.1 provide examples where the number of equivalence classes is at least 2; namely, in these examples any Kähler symplectic form $\omega$ and its reverse, $-\omega$, are not equivalent to each other; it is due to the fact that there is no diffeomorphism (or even homeomorphism) reversing the canonical class of $\omega$, see section 4 for details. To our knowledge, in the previously known examples of nonequivalent symplectic structures $\omega_1, \omega_2$ (see [MT], [LeB], and [Sm]) the assertion states only that $\omega_1 \neq \pm \omega_2$.

It is probably worth to note that in our examples the nonequivalence can not be detected by the divisibility properties of the canonical class or by the Seiberg-Witten invariant: we deal with surfaces of general type, our complex and symplectic structures are opposite,
as well as their canonical classes, and thus in our examples the Seiberg-Witten functions coincide.

Remarks. In dimension two, in our counter-examples to the $\text{Diff} = \text{Def}$ problem the number of equivalence classes is equal to 2, and, moreover, the moduli space is merely a two point set. Additional dimension two examples of deformation inequivalent complex conjugated complex structures are worked out by F.Catanese in [Ca].

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2. Principal examples.

Our first, main, example is a suitably chosen ramified Galois $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of $\mathbb{C}P^2$ (its construction is inspired by some Hirzebruch’s one, cf., [Hirz]). The ramification locus is the configuration of 9 lines dual to the inflection points of a nonsingular cubic in the dual plane, $\mathbb{C}P^2^*$. What follows does not depend on a particular choice of the cubic, moreover, even the configuration of the 9 lines does not depend, up to projective transformation, from this choice. In proper chosen homogeneous coordinates it is defined by equation $(x_1^3 - x_2^3)(x_3^3 - x_1^3)(x_2^3 - x_3^3) = 0$ (as it follows, indeed, from the famous elementary geometry Céva theorem; the equation given corresponds to the Fermat cubic $x_1^3 + x_2^3 + x_3^3 = 0$). In these configuration there are 12 triple points and they are the only multiple points.

Birationally, our covering is given by equations

$$z^5 = l_1 l_2 l_4^{l_3} l_5^{l_6} l_7^{l_8} l_9, \quad w^5 = l_1 l_2^{l_3} l_4^{l_5} l_6^{l_7} l_8^{l_9},$$

where the lines $l_1 = 0, \ldots, l_9 = 0$ are numbered like in the table below following the identification of inflection points with the points of order 3 in the Jacobian of the cubic. In the homogeneous coordinates like above one can take

$$l_1 = x_1 - x_3, \quad l_2 = x_1 - \mu x_3, \quad l_3 = x_1 + \mu x_3,$$

$$l_4 = x_2 - \mu^2 x_3, \quad l_5 = x_2 - x_3, \quad l_6 = x_2 + \mu x_3 = 0,$$

$$l_7 = x_1 + \mu x_2, \quad l_8 = x_1 - \mu^2 x_2, \quad l_9 = x_1 - x_2.$$

where $\mu = e^{\pi i/3}$.

**Correspondence between the lines and the points of order 3 in the Jacobian.**

| $l_1$ | $l_4$ | $l_7$ |
|-------|-------|-------|
| $l_2$ | $l_5$ | $l_8$ |
| $l_3$ | $l_6$ | $l_9$ |

Table 1

As any Galois covering, our one is determined by the corresponding homomorphism $\phi = (\phi_1, \phi_2) : H_1(\mathbb{C}P^2 \setminus \cup l_r) \to \mathbb{Z}/5 \times \mathbb{Z}/5$ (here and further, we do not make difference
between the notation of the covering lines and their linear form expressions). The exponents in the equations of the covering encode the value of $\phi$ on the standard, dual to the lines, generators $\lambda_r \in H_1(\mathbb{C}P^2 \setminus \cup l_r)$:

\[
\begin{align*}
\phi(\lambda_1) &= (1, 1), & \phi(\lambda_2) &= (1, 0), & \phi(\lambda_3) &= (1, 1), \\
\phi(\lambda_4) &= (3, 3), & \phi(\lambda_5) &= (3, 0), & \phi(\lambda_6) &= (0, 1), \\
\phi(\lambda_7) &= (0, 1), & \phi(\lambda_8) &= (0, 2), & \phi(\lambda_9) &= (1, 1),
\end{align*}
\]

Thus defined $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of $\mathbb{C}P^2$, $X \to \mathbb{C}P^2$, has isolated singularities, which arise from the 12 triple points of the ramification locus. To get its (minimal) resolution it is sufficient to blow up $\mathbb{C}P^2$ in the triple points and to take the induced $(\mathbb{Z}/5 \times \mathbb{Z}/5)$-covering of the blown up plane, $\tilde{X} \to \mathbb{C}P^2(12)$.

Let denote by $D_{ijk} \subset \tilde{X}$ the full transform of the blown up curves $E_{ijk} \subset \mathbb{C}P^2(12)$ and by $C_r \subset \tilde{X}$ the strict transform of the lines $l_r \subset \mathbb{C}P^2$. All of them belong to the ramification locus of $\tilde{X} \to \mathbb{C}P^2(12)$ and they all have the ramification index equal to 5. For the intermediate strict transforms $L_r \subset \mathbb{C}P^2(12)$ of $l_r$ one has $L_r^2 = -3$. So, elementary pull back calculation shows that

\[
(1) \quad C_r^2 = -3, \quad D_{ijk}^2 = -1, \quad \text{and} \quad 3K = 7 \sum C_r + 12 \sum D_{ijk}.
\]

In addition,

\[
(2) \quad (C_r, K) = 9 \quad \text{and} \quad (D_{ijk}, K) = 3.
\]

**Proposition 2.1.** The surface $\tilde{X}$ is a strongly rigid surface of general type with ample canonical divisor. The group $\text{Kl}(\tilde{X})$ coincides with the covering transformations group $G = \mathbb{Z}/5 \times \mathbb{Z}/5$. In particular, there does not exist neither a real structure nor even an anti-holomorphic diffeomorphism on $\tilde{X}$.

Here and everywhere further, by the strong or Mostow-Siu rigidity of a compact complex manifold $M$ we mean the following property: whatever are a compact complex Kähler variety $Y$ and a continuous map $p : Y \to M$ with nonzero $p_* : H_{2m}(Y; \mathbb{Z}) \to H_{2m}(M; \mathbb{Z})$, $m = \dim \mathbb{C}M$, then $p$ is homotopic to a holomorphic or anti-holomorphic map $Y \to M$. According to Siu’s results [S], any non singular compact quotient of an irreducible bounded Hermitian symmetric domain of dimension $\geq 2$ has such a property. In addition, as is known, for any such quotient $M$ one has $H^1(M; \Theta) = 0$ ($\Theta$ states for the sheaf of holomorphic tangent fields), which implies, in particular, that $M$ is locally rigid, i.e., its complex structure has no non trivial local deformations.

**Sketch of the proof of Proposition 2.1** (see [KK] for calculations and combinatorial details). A direct calculation shows that $K^2_{\tilde{X}} = 333$ and $e(\tilde{X}) = 111$ ($e$ states for the Euler characteristic), so $\tilde{X}$ is a so-called Miyaoka-Yau surface; their universal cover is a complex ball, see, for example, [BPV], and, thus, their strong rigidity follows from [S] (all the complex structures on the underlying smooth manifold are Kähler, as it follows, for example, from $b_1 = 0$; the strong rigidity applied to the identity map implies also the absence of complex structures with opposite orientation). Due to (1), (2) and the Nakai-Moishezon criterion, $K$ is ample.
Since the ramification locus contains the strict transform of at least two different pairs of lines, to prove $Kl = G$ it is sufficient to show, first, that both the union of strict transforms $C_r$ of the lines and the union of the exceptional divisors $D_{ijk}$ are invariant under Klein transformations of $\hat{X}$, and, second, that there is no Klein transformation of $\mathbb{C}p^2$, different from id, which can be lifted from $\mathbb{C}p^2$ to $\hat{X}$ (respecting the covering for short).

Suppose that $\cup C_r$ is not preserved under some $h \in Kl$, i.e., there is $i$ such that $h(C_i) \not\subset C = \cup C_r$. Then, $(h(C_i), \sum C_r) = a \geq 0$, $(h(C_i), \sum D_{ijk}) = b \geq 0$, and $3K = 7\sum C_r + 12 \sum D_{ijk}$ implies $7a + 12b = 3(h(C_i), K) = 27$.

To get a contradiction it remains to note that the latter equation has no solutions in $a, b \in \mathbb{Z}_+$ (here, counting the intersection numbers we equip all the curves, including $h(C_i)$, with their complex orientation induced from $\hat{X}$). The proof of $h(D) \subset D$, $D = \cup D_{ijk}$, is completely similar.

Now, it follows that every $h \in Kl(\hat{X})$ is lifted from $\mathbb{C}p^2$ and thus it remains to prove that the only $g \in Kl(\mathbb{C}p^2)$ respecting the covering is $g = id$. Since $g$ respects the covering, it acts on the set of intermediate $\mathbb{Z}/5$-coverings $Y \to \mathbb{C}p^2$ and their deck transformations. Namely, such a covering with a marked deck transformation is given by an epimorphism $\psi : H_1(\mathbb{C}p^2 \cup l_r) \to \mathbb{Z}/5$ and $g$ transforms it into $(-1)^{klg} \psi \circ g_* \ (\text{recall that } klg = 0 \text{ if } g \text{ is holomorphic and } 1 \text{ otherwise})$. The following is the table of rows representing all the epimorphisms $\psi = x\phi_1 + y\phi_2 : H_1(\mathbb{C}p^2 \setminus l_r) \to \mathbb{Z}/5$ in the base $\lambda_i$:

\[
\begin{array}{cccccc}
(1,1,1,3,3,0,0,0,1) & (2,2,2,1,1,0,0,0,2) & (3,3,3,4,4,0,0,0,3) & (4,4,4,2,2,0,0,0,4) \\
(1,0,1,3,0,1,1,2,1) & (2,0,2,1,0,2,2,4,2) & (3,0,3,4,0,3,3,1,3) & (4,0,4,2,0,4,4,3,4) \\
(2,1,2,1,3,1,1,2,2) & (4,2,4,2,1,2,2,4,4) & (1,3,1,3,4,3,3,1,1) & (3,4,3,4,2,4,4,3,3) \\
(3,1,3,4,3,2,2,4,3) & (1,2,1,3,1,4,4,3,1) & (4,3,4,2,4,1,1,2,4) & (2,4,2,1,2,3,3,1,2) \\
(4,1,4,2,3,3,3,1,4) & (3,2,3,4,1,1,1,2,3) & (2,3,2,1,4,4,4,3,2) & (1,4,1,3,2,2,2,4,1) \\
(0,1,0,0,3,4,4,3,0) & (0,2,0,0,1,3,3,1,0) & (0,3,0,0,4,2,2,4,0) & (0,4,0,0,2,1,1,2,0).
\end{array}
\]

The elements in a row represent, in fact, the weights of the deck transformation on the components of the ramifications locus. Thus, they should be preserved up to permutation. In particular, the functions $a \to r_i(a)$ counting the number of coordinates of a row $a$ equal $i$ should be invariant under the action of $g$.

An easy running over, taking into account, in addition, the incidence relations between the lines $l_i$, shows that the only possibility for $g$ is to keep invariant each line. Hence, $g = id.$

Theorem 1.1 in dimension two is a straightforward consequence of the second statement of Proposition 2.1. In fact, our proof of this part of Proposition 2.1 uses only the local rigidity of $\hat{X}$, which is an easier result than the strong rigidity. As to the latter, it implies the following property which is crucial for the other applications and for the proof of Theorem 1.1 in higher dimensions.

**Lemma 2.2.** The homeotopy group of $X$ (i.e., the group of homotopy classes of homeomorphisms $X \to X$) coincides with the covering transformations group $G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. In particular, for any homeomorphism $f : \hat{X} \to \hat{X}$ one has $f^*[K] = [K], [K] \in H^2(X; \mathbb{Z})$.

**Proof.** By strong rigidity, any homeomorphism $\hat{X} \to \hat{X}$ is homotopic to an automorphism or to an anti-automorphism. Due to Proposition 2.1, there are no anti-automorphisms...
and the group of automorphisms coincides with the covering transformations group $G$. It remains to note that the canonical class is preserved by any automorphism. □

Another important property of $\tilde{X}$ is that its irregularity $q = q(\tilde{X})$ is zero, see Appendix.

There are other surfaces which have similar properties and for which Proposition 2.1 and Lemma 2.2 hold as well. As is proved in [KK], $\tilde{X}$ can be replaced, in particular, by a fake projective plane. Recall that, by definition, a fake projective plane is a surface of general type with $p_g = q = 0$ and $K^2 = 9$. Such surfaces do really exist, see [Mu].

3. Higher-dimensional examples

In this section we prove that part of Theorem 1.1 which concerns the dimensions $\geq 3$, the case of surfaces is covered by Section 2. For this purpose, we develop several series of examples: the examples from Proposition 3.7 cover even dimensions $\geq 4$ and those from Proposition 3.8 odd dimensions $\geq 3$. When applying Propositions 3.7 and 3.8 to prove this part of Theorem 1.1, it is sufficient to take as $M_1$ the surface constructed in Section 2 and as $M_2$ a fake projective plane.

Another series of examples sufficient for the proof of Theorem 1.1 is given by Propositions 3.9. and 3.11, see Corollary 3.13.

We start from some auxiliary results which, with one exception, we did not find in the literature; they may be of an independent interest. We use them in the proof of Propositions 3.7, 3.8, 3.9 and 3.11.

Lemma 3.1 ([LS]). Let $X$ and $X_0$ be compact complex manifolds contained in a deformation family over irreducible base space. Suppose that the canonical class of $X_0$ is ample. Then, $X$ is a Moishezon variety, i.e., the transcendence degree of meromorphic function field on $X$ coincides with $\dim X$. □

Lemma 3.2. Let $V_1$ and $V_2$ be minimal nonsingular surfaces of general type. If there exists a regular map $f : V_1 \to V_2$ of degree $d \geq 1$, then $K^2_{V_1} \geq K^2_{V_2}$. In addition, $K^2_{V_1} = K^2_{V_2}$ if and only if $f$ is an isomorphism (i.e., $d = 1$).

Proof. According to the pull-back formula, $K_{V_1} = f^*(K_{V_2}) + R$, where $R = \sum r_i R_i$ is the ramification divisor of $f$, $R_i$ are irreducible components of $R$, $r_i \geq 0$ (by ramification divisor we mean here the locus of points where the jacobian matrix is not of maximal rank). On the other hand, $K^2_{V_1} > 0$, $K^2_{V_2} > 0$, and $(f^*(K_{V_2}), R) \geq 0$, since, by Bombieri theorem (see, f.e., [BPV]), the linear system $|5K_{V_2}|$ has no fixed components. Besides, $(K_{V_1}, R_i) = (f^*(K_{V_2}) + R, R_i) \geq 0$ for each irreducible curve $R_i$ lying on a minimal model. Therefore,

$$K^2_{V_1} = dK^2_{V_2} + 2(f^*(K_{V_2}), R) + R^2$$
$$= dK^2_{V_2} + (f^*(K_{V_2}), R) + (f^*(K_{V_2}) + R, R)$$
$$= dK^2_{V_2} + (f^*(K_{V_2}), R) + \sum r_i(f^*(K_{V_2}) + R, R_i) \geq K^2_{V_2}$$

if $d \geq 1$ and we have the equality iff $d = 1$ and $R = 0$. Indeed, any degree one regular map $f$ is an isomorphism, since $V_1$ and $V_2$ are minimal surfaces. □
Lemma 3.3. Let $U$ be a Moishezon variety and $V$ a projective variety having no rational curves. Then, any meromorphic map $f : U \to V$ is holomorphic.

Proof. By Hironaka theorem, there is a sequence $\sigma_i : U_i \to U_{i-1}$, $U_0 = U$, $1 \leq i \leq k$, of monoidal transformations with non-singular centers such that $f_k = f \circ \sigma : U_k \to V$ is holomorphic, where $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$. Note that for any $p \in U$ the preimage $\sigma^{-1}(p)$ is rationally connected, i.e., any two distinct points from $\sigma^{-1}(p)$ belong to a connected union of a finite number of rational curves. Therefore, $f_k(\sigma^{-1}(p))$ is a point, since there are no rational curves lying on $V$. Thus, $f_k$ factors through the holomorphic map $f$. □

Lemma 3.4. Let $Z$ be a compact Moishezon manifold and $X$ be a projective one. Then, every holomorphic map $f : Z \to X$ inducing an isomorphism in homology, $f_* : H_*(Z;\mathbb{Z}) \to H_*(X;\mathbb{Z})$, should be biholomorphic.

Proof. To show that $f$ is an isomorphism, it is sufficient to check that there is no complex subvariety $Y$ of $Z$ such that $\dim f(Y) < \dim Y$.

First, let us eliminate the case $\dim Y = \dim Z - 1$. If $\dim f(Y) < \dim Y$ then $Y$ is homological to 0, since $f_*$ is an isomorphism. But, since $X$ is a projective variety, one can find a curve $B \subset X$ meeting $f(Y)$ and not lying in $f(Y)$. Therefore, the strict preimage $f^{-1}(B)$ and $Y$ (we assume here that $Y = f^{-1}(f(Y))$) have positive intersection number, which contradicts to $Y \cong 0$.

Now, consider any $Y$ such that $\dim f(Y) < \dim Y$. By above, we have $\dim Y < \dim Z - 1 = \dim X - 1$. One can find a meromorphic volume form $\omega$ on $X$ such that $f(Y)$ does not belong to the support of the divisor ($\omega$). But it is impossible, since in this case in a neighborhood of generic point of $Y$ the form $f^*(\omega)$ would have zero along a submanifold of $X$ of codimension at least two. □

Lemma 3.5. Let $M_1$ and $M_2$ be nonsingular regular surfaces of general type. If $M_1$, $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_2$ are pairwise non isomorphic and contain no rational curves, then:

(i) the products $M_1^a \times M_1^b \times M_2^c \times M_2^d$ with $(a, b, c, d) \in \mathbb{N}^4$ are pairwise non isomorphic; in particular, such a product admits an anti-automorphism only if $a = b$ and $c = d$;

(ii) in a deformation $\pi : X \to B_1$ each fiber manifold $X_z = \pi^{-1}(z), z \in B_1$, is isomorphic to a product $M_1^j \times M_1^n \times M_2^k \times M_2^n$ with $j = j(z), m = m(z), k = k(z), n = n(z)$, only if the deformation is trivial.

Proof. Let us first prove (i). Assume that there is an isomorphism

$$h : M_1^m \times \bar{M}_1^m \times M_2^m \times \bar{M}_2^m \to M_1^m \times \bar{M}_1^m \times M_2^m \times \bar{M}_2^m.$$  

Without loss of generality, we can assume also that $K_{M_1}^2 \geq K_{M_2}^2$ and $j_1 \leq j_2$.

Denote by $S_l \subset M_1^m \times \bar{M}_1^m \times M_2^m \times \bar{M}_2^m$, $l \geq 1$, a fiber of the projection of the whole product to the partial product taken over all the factors except the one with index $l$. So, $S_l$ is canonically isomorphic to $M_1$ for $1 \leq l \leq j_1$, to $\bar{M}_1$ for $j_1 < l \leq j_1 + m_1$, to $M_2$ for $j_1 + m_1 < l \leq j_1 + m_1 + k_1$, and to $\bar{M}_2$ for $j_1 + m_1 + k_1 < l$. Denote also by $\Sigma_s$ the similar fibers of $M_1^m \times \bar{M}_1^m \times M_2^m \times \bar{M}_2^m$.

Consider, first, the restrictions $p_{kl}$ to $h_1(S_l)$ of the projections $p_s : M_1^m \times M_2^m \times \bar{M}_1^m \times \bar{M}_2^m \to \Sigma_s$. Let us show that for each $l \geq j_1 + 1$ the image $p_{kl}(h_1(S_l))$ is a point as soon as $s \leq j_2$. In fact, $p_{kl}(h_1(S_l))$ can not be a curve, since there are no rational curves
Proof. Since \( \alpha \) map \( m \) tional curves. Let suffices to apply the universal property of the Albanese map and Lemma 3.2. By Theorem, dim \( H \) implies that dim \( H \) theorem (see [G], Section 7, Theorem 4), it implies that the canonical map where \( \text{manifold underlying } \) the image of this map is countable, it is constant. □

Lemma 3.6. Let \( M \) and \( M_2 \) be nonsingular regular surfaces of general type and \( C_1 \) and \( C_2 \) be non-singular curves of genus \( g > 0 \). If \( M_1, M_2, M_2 \) are pairwise non isomorphic and contain no rational curves, then the products \( M_{a_1,b_1,c_1,d_1,c_1} = M_1^{a_1} \times M_1^{b_1} \times M_2^{c_1} \times \bar{M}_d \times C_1 \) and \( M_{a_2,b_2,c_2,d_2,c_2} = M_1^{a_2} \times M_1^{b_2} \times M_2^{c_2} \times \bar{M}_d \times C_2 \), \((a_i, b_i, c_i, d_i) \in \mathbb{N}^4 \), are isomorphic if and only if \((a_1, b_1, c_1, d_1) = (a_2, b_2, c_2, d_2) \) and \( C_1 \) is isomorphic to \( C_2 \). In particular, such a product admits an anti-automorphism only if \( a_1 = b_1 \) and \( c_1 = d_1 \).

Proof. Since \( M_1 \) and \( M_2 \) are regular surfaces, the image \( \alpha_i(M_{a_i,b_i,c_i,d_i,C_i}) \) of the Albanese map \( \alpha_i : M_{a_i,b_i,c_i,d_i,C_i} \to \text{Alb}(M_{a_i,b_i,c_i,d_i,C_i}), \) \( i = 1, 2 \), coincides with \( C_i \). To conclude it suffices to apply the universal property of the Albanese map and Lemma 3.2.

Proposition 3.7. Let \( M_1, \bar{M}_1, M_2 \) and \( \bar{M}_2 \) be pairwise non-isomorphic compact regular nonsingular surfaces of general type satisfying the Mostow-Siu rigidity and having no rational curves. Let \( m \) and \( n \) be two non-negative integers and \( X_{0,0} \) be a product

\[ S_1 \times \cdots \times S_m \times S_{m+1} \times \cdots \times S_{m+n}. \]
where each $S_i$, $1 \leq i \leq m$, is isomorphic to $M_1$ and each $S_i$, $m + 1 \leq i \leq m + n$, is isomorphic to $M_2$. Then:

(i) there are at least $(m + 1)(n + 1)$ distinct deformation classes of complex structures on the underlying oriented smooth manifold $X_{0,0}$; these classes are represented by

$$X_{m-j,n-k} = M_1^i \times \tilde{M}_1^{m-j} \times M_2^j \times \tilde{M}_2^{n-k},$$

with $j = 0, \ldots, m$ and $k = 0, \ldots, n$ (here, $M_1^i$ states for the product of $j$ copies of $M_1$ and $\tilde{M}_1^i$ for the product of $s$ copies of $M_1$);

(ii) there are no anti-automorphisms on $X_{p,q}$ except the case $m = 2p$, $n = 2q$; for other values of $p, q$ the deformation class of $X_{p,q}$ is not invariant under reversing of complex structure, $X_{p,q} = X_{m-p, m-q}$.

(iii) a complex manifold is deformation equivalent to $X_{p,q}$ if and only if it is isomorphic to $X_{p,q}$.

Note that, as it follows from the proof below, the complex structures represented by $X_{p,q}$ are the only Moishezon complex structures on the smooth oriented manifold underlying $X_{0,0}$. Up to our knowledge, the existence of non Moishezon complex structures on this manifold is an open question. If all the complex structures on $X_{0,0}$ are Moishezon, the number of their deformation classes is exactly $(m + 1)(n + 1)$.

**Proof.** Clearly, (i) and (ii) are straightforward consequences of (iii). So, let us prove the part (iii) of the statement.

As it follows from Lemma 3.1 and Lemma 3.5 (ii), it is sufficient to show that any Moishezon complex structure on the underlying $X_{0,0}$ smooth manifold provides a complex manifold isomorphic to one of $X_{p,q}$.

So, consider a Moishezon complex compact variety $X$ having the same underlying smooth oriented manifold as $X_{0,0}$. Pick a diffeomorphism between $X$ and $X_{0,0}$ and consider the corresponding to it projections $p_i : X \to S_i$, $i = 1, \ldots, m + n$, (we use here notations from Lemma 3.5). By Moishezon Theorem, there is a sequence $\sigma_i : Z_i \to Z_{i-1}$, $Z_0 = X$, $1 \leq i \leq r$, of monoidal transformations with non-singular centers such that $Z = Z_r$ is a projective variety. Denote by $\sigma$ their composition. Every projection and every monoidal transformation induce an epimorphism in homology. So, according to the Mostow-Siu rigidity hypothesis, each $p_i \circ \sigma$, $i = 1, \ldots, m + n$, is homotopic to a map $\bar{p}_i : Z \to S_i$ which is either holomorphic or anti-holomorphic. Let $j$ be the number of holomorphic maps $\bar{p}_i$ for $i \leq m$ and $k$ be the number of holomorphic maps $\bar{p}_i$ for $i > m$. Thus, after a suitable renumbering of $\bar{p}_i$, we get a meromorphic map $f = \bar{p}_1 \circ \sigma^{-1} \times \cdots \times \bar{p}_{m+n} \circ \sigma^{-1} : X \to M_1^i \times \tilde{M}_1^{m-j} \times M_2^j \times \tilde{M}_2^{n-k}$. By Lemma 3.3 it is holomorphic, and, since $\sigma$ induces an epimorphism in homology, from the construction of $f$ it follows that $f$ induces an isomorphism in homology. Now, Lemma 3.4 applies and it shows that $X$ is isomorphic to $X_{j,k}$. $\square$

**Proposition 3.8.** Let $M_1$ and $M_2$ be as in Proposition 3.7, $C_0$ be a curve of genus $g > 1$, and $Y_{0,0}$ be a product $M_1^m \times M_2^m \times C_0$. Then:

(i) there are at least $(m + 1)(n + 1)$ distinct deformation classes of complex structures on the underlying oriented smooth manifold $Y_{0,0}$; these deformation classes are represented by

$$Y_{m-j,n-k} = M_1^i \times \tilde{M}_1^{m-j} \times M_2^j \times \tilde{M}_2^{n-k} \times C_0,$$
j = 0, \ldots, m and k = 0, \ldots, n;
(ii) there are no anti-automorphisms on Y_{p,q} except the cases when m = 2p, n = 2q and C_0 admits an anti-automorphism; for all the other values of p, q the deformation class of Y_{p,q} is not invariant under reversing of complex structure, Y_{p,q} \neq Y_{m-p, m-q}.

Here, the dimension of complex manifolds Y_{p,q} is odd, and, thus, reversing the complex structure in 3.8(ii) we change the complex orientation. However, since Y_{p,q} admit orientation reversing diffeomorphisms, there are as much deformation classes of complex structures on the underlying smooth manifold as with fixed or with any orientation.

Similarly to the situation of Proposition 3.7, if all the complex structures on Y_{0,0} are Moishezon, the number of their deformation classes is exactly (m+1)(n+1).

Proof. Let start of the first part of Proposition. As in the proof of Proposition 3.7, to prove the first part we should check that the complex structure which can be deformed to a Moishezon complex structure is isomorphic to some $M_1^j \times M_1^{n-j} \times M_2^k \times M_2^{-k} \times C$, where C is a curve of genus g, and then we should prove that a complex structure which is deformation equivalent to one of $Y_{p,q}$ is isomorphic to the complex structure on $M_1^p \times M_1^{n-p} \times M_2^q \times M_2^{-q} \times C$ for some curve C.

First, consider a Moishezon variety Y having the same underlying smooth oriented manifold as C_{00}. Consider the Albanese map $\alpha : Y \to \text{Alb}(Y)$. The Albanese map $\alpha_{0,0} : Y_{0,0} \to \alpha_{0,0}(Y_{0,0}) \subset \text{Alb}(Y_{0,0})$ coincides with the last factor projection $p_{m+n+1} : Y_{0,0} = M_1^m \times M_2^n \times C_0 \to C_0$. Since the irregularity $q(M_1^m \times M_2^n) = 0$, the subring of $H^1(Y_{0,0}; \mathbb{Z})$ generated by $H^1(Y_{0,0}; \mathbb{Z})$ coincides with $\alpha_{0,0}^*(H^*(C_0; \mathbb{Z}))$. In particular, $\bigwedge^2 H^1(Y_{0,0}; \mathbb{Z}) \subset H^2(Y_{0,0}; \mathbb{Z})$ is a one-dimensional subspace and $\bigwedge H^1(Y_{0,0}; \mathbb{Z}) = \{0\}$ for $i > 2$. Therefore, the image $\alpha(Y)$ is a curve C of genus g and $\alpha^*(H^*(\text{Alb}(Y), \mathbb{Z}))$ also coincides with the subring generated by $H^1(Y, \mathbb{Z}) = H^1(Y_{0,0}, \mathbb{Z})$.

Pick a diffeomorphism between Y and Y_{0,0} and consider the corresponding to it projections $p_i : Y \to S_i, i = 1, \ldots, m+n$, (we use here notations from Proposition 3.5), which are C^\infty-submersions, and $p_{m+n+1} : Y \to C$. By Moishezon Theorem, there is a sequence $\sigma_i : Z_i \to Z_{i-1}, Z_0 = Y, 1 \leq i \leq k$, of monoidal transformations with non-singular centers such that $Z = Z_k$ is a projective variety. Denote by $\sigma$ their composition. Every projection and every monoidal transformation induce an epimorphism in homology. So, according to the Mostow-Siu rigidity hypothesis, each $p_i \circ \sigma, i = 1, \ldots, m+n$, is homotopic to a map $\tilde{p}_i : Z \to S_i$ which is either holomorphic or anti-holomorphic. Let $f$ be the number of holomorphic maps $\tilde{p}_i$ for $i \leq m$ and $k$ be the number of holomorphic maps $\tilde{p}_i$ for $i > m$. Thus, we get a meromorphic map $f = \tilde{p}_i \circ \sigma^{-1} \times \cdots \times \tilde{p}_{m+n} \circ \sigma^{-1} \times \alpha : Y \to M_1^j \times M_1^{n-j} \times M_2^k \times M_2^{-k} \times \alpha(Y)$. By Lemma 3.3 it is holomorphic, and, since $\sigma$ induces an epimorphism in homology, from the construction of f it follows that f induces an isomorphism in homology. Now, Lemma 3.4 applies and it shows that $Y$ is isomorphic to $Y_{j,k}$.

To finish the proof of the first part of Proposition, let us consider a deformation $X \to B_1$ of $Y_{m-j, n-k}$. As it is already proved, for any $z \in B_1$ the manifold $X_z$ is isomorphic to $M_1^j \times M_1^{n-j} \times M_2^k \times M_2^{-k} \times C_z$ with some $j = j(z), k = k(z)$. Hence, as it follows, for example, from Grauert’s continuity theorem (see [G]) the associated Albanese varieties Alb$X_z$ of $X_z, z \in B_1$, form a deformation $W \to B_1$. The Albanese map $X \to W$ is of constant rank equal to 2 (one unit comes from the images $C_z \subset \text{Alb} X_z$ of the Albanese
map and another one from $B_1$, since as any deformation the composite map $X \to W \to B_1$ is a submersion). It is proper and defines a deformation of $M_1^p \times M_1^{m-p} \times M_2^k \times M_2^{n-k}$. By Proposition 3.7 (iii), all elements of such a deformation are isomorphic to each other.

The second part of Proposition is a straightforward consequence of the first part and Lemma 3.6. □

**Remark.** If one restricts himself to Kähler (or Moishezon) manifolds and there deformations constituted only of Kähler (or Moishezon) manifolds, then using the same arguments as in the proof of Propositions 3.7 and 3.8 one can prove that $X_{p,q} \times T_1$ and $X_{k,l} \times T_2$, where $T_1, T_2$ are tori of equal dimensions and $X_{p,q}, X_{k,l}$ are as in Proposition 3.7, are equivalent if and only if $p = q$ and $k = l$. In particular, $X_{p,q} \times T_1$ and $X_{p,q} \times T_2$ are not equivalent except the case $2p = m, 2q = n$ (for the definition of $m$ and $n$ see Proposition 3.7).

**Proposition 3.9.** Let $M$ be a fake projective plane or the surface constructed in section 2, and let $X$ be a Kähler manifold of Kodaira dimension $\leq 1$ with $c_1(X) \in H^2(X, \mathbb{Z})$ divisible by some integer $p > 3, p \neq 6$ (in particular, $c_1(X)$ can be zero). Then, there is no homeomorphism $h : Y \to Y, Y = M \times X$, such that $h^*(c_1(Y)) = -c_1(Y)$.

**Proof.** Assume that there is such a homeomorphism $h$. Since $M$ is a regular surface, the Kähnisch decomposition of $H^2(Y; \mathbb{Z})$ takes the form $H^2(Y, \mathbb{Z}) = H^2(M, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$. Let $h^*(c_1(M)) = \omega_1 + \nu_1$ and $h^*(c_1(X)) = \omega_2 + \nu_2$, where $\omega_i \in H^2(M, \mathbb{Z})$ and $\nu_i \in H^2(X, \mathbb{Z})$. Then, according to our assumption, $\omega_1 + \omega_2 = -c_1(M)$ and $\nu_1 + \nu_2 = -c_1(X)$.

Denote by $p_1 : Y \to M$ the projection to the first factor, by $p_2 : Y \to X$ the projection to the second one, by $i_2 : X \to Y$ the canonical isomorphism with one of the fibers of $p_1$, and by $i_1 : M \to Y$ the canonical isomorphism with one of the fibers of $p_2$. Consider the composition $p_1 \circ h$. By Mostow-Siu rigidity, $p_1 \circ h$ is homotopic to $p_1 : Y \to M$ which is either holomorphic or anti-holomorphic dominant map.

Since $M$ is regular and has no rational curves, its image $(\tilde{p}_1 \circ i_1)M \subset M$ cannot be a curve, and it can not be a point, since in this case the restriction of $\tilde{p}_1$ to $i_2(X)$ should be a dominant map, but the latter is impossible because of Kodim $X <$ Kodim $M$.

Therefore, Lemma 3.2 applies and it shows that $\tilde{p}_1 \circ i_1$ is a holomorphic or anti-holomorphic isomorphism. Since $M$ has no anti-automorphisms, $\tilde{p}_1 \circ i_1$ is a holomorphic map. Thus, $\omega_1 = (p_1 \circ h \circ i_1)^*c_1(M) = (\tilde{p}_1 \circ i_1)^*c_1(M) = c_1(M)$. It implies $(p_1 \circ h \circ i_2)^*c_1(X) = \omega_2 = -2c_1(M)$, and to complete the proof it remains to note that $2c_1(M)$ is not divisible by $p > 3, p \neq 6$, which follows from $c_2^1(M) = 9$ in the case of fake projective planes and from $c_2^1(M) = 333$ in the case of the surface constructed in Section 2. □

**Corollary 3.10.** Let $Y$ be a variety as in Proposition 3.9. Then, it is not deformation equivalent to its conjugate and, in particular, can not be deformed to a variety with anti-automorphisms. □

**Remark.** In the proof of Proposition 3.9 the hypothesis on Kodaira dimension of $X$ is used only to exclude existence of dominant maps from $X$ to $M$.

One more series of manifolds whose canonical class and its inverse are topologically distinct is given by the following proposition.

**Proposition 3.11.** Let $M_1$ be the surface constructed in section 2, $M_2$ a fake projective plane, $X$ as in Proposition 3.9, and $Y = M_1 \times M_2^2 \times X$. Then there is no homeomorphism $h : Y \to Y$ such that $h^*c_1(Y) = -c_1(Y)$. 
The proof of Proposition 3.11 is essentially the same as the proof of Proposition 3.9. The principal new element with respect to that proof (except the analysis of the first component of the homeomorphism \( h \), where we use arguments like in the proof of Proposition 3.7 in order to prove that the restriction to \( M_1 \) of the corresponding \( \tilde{p}_1 \) is an isomorphism) is the following lemma applied to \( Z = M_2 \).

**Lemma 3.12.** Let \( Z \) be a surface of general type such that for any holomorphic map \( f : Z \to Z \) either \( \text{codim} \, f(Z) \geq 2 \) or \( f \) is a biholomorphic. Then, the same is true for holomorphic maps from \( Z^a \) to \( Z^a \) for any \( a \geq 1 \).

**Proof.** Let \( f : Z^a \to Z^a \) be a holomorphic map. Denote by \( p : Z^a \to Z^{a-1} \) one of the canonical projections and by \( i : Z \to Z^a \) one of its fibers. If the projections of \( (f \circ i)Z \) to each of the factors are all not dominant, the image of this fiber is a point. Then, the image of all the parallel fibers is a point, which implies \( \text{codim} \, f(Z^a) \geq 2 \).

Otherwise, one of these projections provides an isomorphism \( Z \to Z \). Since the automorphism group of \( Z \) is discrete, the map \( f \) has, up to permutation of the factors in the source and the target, the following triangle form \( f(x,p) = (gx, h(x,p)) \), where \( g \) is an automorphism of \( Z \) and \( h \) is a holomorphic map \( Z^a \to Z^{a-1} \). By induction, we may assume that for some, and hence for any, \( x \in Z \) the map \( Z^{a-1} \to Z^{a-1} \) given by \( p \mapsto h(x,p) \) either is biholomorphic or has a codimension \( \geq 2 \) image. The above triangle form of \( f \) implies that it has the same property. □

**Corollary 3.13.** For any \( n \geq 2 \) there is a projective \( n \)-dimensional manifold \( Y \) of general type which has no homeomorphisms \( h \) with \( h^*c_1(Y) = -c_1(Y) \).

**Proof.** Apply Proposition 3.11 to a curve \( X \) of genus \( g > 4 \) in the odd-dimensional case and to a point \( X \) in the even-dimensional case. □

In fact, the same arguments which were used in the proof of Proposition 3.7 (or more traditional arguments base on the De Rham theorem of uniqueness of the decomposition in irreducible factors) allow to deduce from Lemma 2.3 a complete description of the homeotopy group \( \mathcal{H}(Y) \) in the case \( Y = M^n \), where \( M \) is the surface constructed in Section 2.

**Proposition 3.14.** Let \( M \) be the surface constructed in Section 2. Then, the group \( \mathcal{H}(M^n) = (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^n \rtimes S_n \) is the semi-direct product, with standard action of the symmetric group \( S_n \) on the factors \( (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}) \) of the direct product \( (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z})^n \). □

### 4. Applications

**A. Non connected moduli spaces.** Here, we discuss how Theorem 1.1 and our solution of the \( \text{Dif} = \text{Def} \) problem can be translated in the language of moduli spaces.

In most of our examples the varieties are of general type and, moreover, all varieties deformation equivalent to them are projective varieties of general type isomorphic to products of projective surfaces of general type and, possibly, some curve of genus \( > 1 \) (see Propositions 2.1, 3.7, and 3.8). So, here we may restrict ourselves to the case when the moduli space \( \mathcal{M}_{\text{Dif}} \) of complex structures on a given smooth orientable manifold \( M \) is well defined as a complex space.

In fact, we fix the orientation of \( M \) only if the complex dimension of \( M \) is even. Then, obviously, in even complex dimensions the \( \text{Dif} = \text{Def} \) problem becomes the question of
connectedness of $\mathcal{M}_{\text{Diff}}$. In addition, if the complex dimension is even, the involution $\text{conj}: X = (M, J) \mapsto \bar{X} = (M, -J)$ acts on $\mathcal{M}_{\text{Diff}}$ and defines a canonical real structure on it.

Since in all our odd-dimensional examples $M$ has orientation reversing diffeomorphisms, the $\text{Diff} = \text{Def}$ problem as it is stated in Introduction remains the question of connectedness of $\mathcal{M}_{\text{Diff}}$ even without introducing orientations in the definition of the moduli space. On the other hand, it allows still to consider the complex conjugation acting on $\mathcal{M}_{\text{Diff}}$ by $\text{conj}: X = (M, J) \mapsto \bar{X} = (M, -J)$.

To reinterpret Theorem 1.1 in its full content is more complicated, in general. Certainly, if $X$ can be equipped with a real structure then it defines a real point of $\mathcal{M}_{\text{Diff}}$, i.e., a fixed point of $\text{conj}$. But the reverse is not true, in general, since real points of $\mathcal{M}_{\text{Diff}}$ are given also by complex manifolds having antiholomorphic automorphisms of any (even) order. The simplest examples are given by the Shimura curves [Shi] which are double coverings of $\mathbb{C}P^1$ ramified in $4m + 2 \geq 6$ generic points invariant under the real structure of $\mathbb{C}P^1$ acting without fixed points. The real structure is lifted to an anti-automorphism $\text{conj}$ of order 4, and for generic points the curve has no other anti-automorphisms than $\text{conj}$ and the other lift, which is its composition with a deck transformation and which is also of order 4. In fact, in these examples the automorphism group $\text{Aut}$ is $\mathbb{Z}/2$ and the Klein group $\mathbb{K}l$, which includes all the anti-automorphisms and all the automorphisms, is $\mathbb{Z}/4$.

By contrary, the absence of deformation equivalence between $X$ and $\bar{X}$ has a simple meaning in terms of $\mathcal{M}_{\text{Diff}}$: it means that $\mathcal{M}_{\text{Diff}}$ contains two connected components interchanged by $\text{conj}$. Thus, the following statement is a straightforward corollary of Theorem 1.1 and Propositions 3.7, 3.8. (Let us recall our convention: in this theorem, as well as above, we fix the orientation only if the complex dimension of the manifold is even.)

**Theorem 4.1.** In any even real dimension $2n \geq 4$ there exist an oriented smooth compact manifold for which the moduli space of complex structures is disconnected and has at least $((\frac{n}{2}) + 1)((\frac{n}{2}) - (\frac{n}{4}) + 1) = N$ connected components. At most one of the components is invariant under complex conjugation.

This statement can be refined a bit. For example, in the case of surfaces, the moduli space consists of two connected components and the complex structures from different components have opposite canonical classes.

**B. Isotopy classes of cuspidal curves.**

The theorem below shows that diffeomorphic plane cuspidal curves can be not equivalent under equisingular deformations. In fact, in our examples the curves are not even isotopic. Thus, the following question remains open: does existence of an isotopy imply existence of an equisingular deformation.

**Theorem 4.3.** There are two infinite sequences, $\{C_{m,1}\}$ and $\{C_{m,2}\}$, of plane irreducible cuspidal curves of degree $\text{deg}(C_{m,1}) = \text{deg}(C_{m,2}) \to \infty$, such that the pairs $(\mathbb{C}P^2, C_{m,1})$ and $(\mathbb{C}P^2, C_{m,2})$ are diffeomorphic, but $C_{m,1}$ and $C_{m,2}$ are not isotopic and, in particular, they can not be connected by an equisingular deformation.

To get Theorem 4.3 it is sufficient to apply Proposition 4.4 stated below to any of the surfaces given in Section 2.

**Proposition 4.4.** Let $X$ be a surface of general type with ample canonical class $K$. Suppose that there is no homeomorphism $h$ of $X$ such that $h^*[K] = -[K]$, $[K] \in H^2(X; \mathbb{Q})$. 

Then the moduli space of $X$ consists of at least 2 connected components corresponding to $X$ and $\overline{X}$ (the bar states for reversing of complex structure, $J \mapsto -J$), and for any $m \geq 5$ these two connected components are distinguished by the isotopy types of the branch curves of generic coverings $f_m : X \to \mathbb{C}^2$ and $\overline{f}_m : \overline{X} \to \mathbb{C}^2$ given by $mK$ and $m\overline{K}$, respectively. In particular, the branch curves cannot be connected by an equisingular deformation.

When Proposition 4.4 is applied to the surface $X = \overline{X}$ constructed in Section 2 by means of the Céva configuration, one finds (see general formulae in [K], page 1155)

1. $\deg f_m = 333 m^2$;
2. $C_{m,1}$ is a plane cuspidal curve of $\deg C_{m,1} = 333 m(3m + 1)$;
3. the geometric genus $g_m$ of $C_{m,1}$ equals $g_m = 333 (3m + 2)(3m + 1)/2 + 1$;
4. $C_{m,1}$ has $c = 111 (36m^2 + 27m + 5)$ ordinary cusps.

If it is a fake projective plane which is taken as $X$, then

$$\deg f_m = 9 m^2, \quad \deg C_{m,1} = 9 m(3m + 1),$$

$$g_m = \frac{9}{2} (3m + 2)(3m + 1) + 1, \quad c = 3 (36m^2 + 27m + 5).$$

Sketch of the proof of Proposition 4.4 (see [KK2] for more details). Since $K$ is ample, from Bombieri theorem it follows that the map $X \to \mathbb{C}^2$, $r_m = \dim H^0(X, mK) - 1$, given by $mK$ is an imbedding if $m \geq 5$. Let $m \geq 5$ and denote by $X_m$ the image of $X$ under the imbedding in $\mathbb{C}^2$ given by $mK$, by $pr_m : \mathbb{C}^2 \to \mathbb{C}^2$ a linear projection generic with respect to $X_m$, by $f_m = pr_m|_{X_m}$ the restriction of $pr_m$ to $X_m$, and by $C_{m,1} \subset \mathbb{C}^2$ the branch curve of $f_m$. As soon as we identify $X_m$ and $\overline{X}_m$ as sets, the composition $\overline{f}_m = c \circ f_m : \overline{X}_m \to \mathbb{C}^2$ of $f_m$ with the standard complex conjugation $c : \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic generic covering with branch curve $C_{m,2} = c(C_{m,1})$. By construction, we have

$$\overline{f}_m^*(\Lambda) = -f_m^*(\Lambda) = -m[K],$$

where $\Lambda \in H^2(\mathbb{C}^2, \mathbb{Q})$ is the class of the projective line in $\mathbb{C}^2$.

The set of generic coverings $f$ of $\mathbb{C}^2$ branched along a cuspidal curve $C$ is in one-to-one correspondence with the set of epimorphisms from the fundamental group $\pi_1(\mathbb{C}^2 \setminus C)$ to the symmetric groups $S_{\deg f}$ (up to inner automorphisms) satisfying some additional properties (see [K]). By Theorem 3 in [K], for $C_{m,1}$ (respectively, $C_{m,2}$) there exists only one such an epimorphism $\varphi_m : \pi_1(\mathbb{C}^2 \setminus C_{m,1}) \to S_{\deg f_m}$ (respectively, $\overline{\varphi}_m$). Thus, if there exists an isotopy $F_t : \mathbb{C}^2 \to \mathbb{C}^2$ such that $F_0 = \text{id}$ and $F_1(C_{m,1}) = C_{m,2}$, then the epimorphism

$$\varphi_{m,t} : \pi_1(\mathbb{C}^2 \times [0, 1]) \setminus \{(F_t(C_{m,1}), t)\} \to S_{\deg f_m}$$

defines a locally trivial, with respect to $t$, family of generic coverings $f_{m,t} : Y \to \mathbb{C}^2$, $t \in [0, 1]$, which provides in its turn a homeomorphism $h$ of $X$ such that $h^*[K] = -[K] \in H^2(X; \mathbb{Q})$, as it follows from (3). Thus, the isotopy $F_t$ cannot exist. □

C. Two inequivalent classes of symplectic structures. Here, we treat the equivalence relation defined in Introduction.

Let $Y$ be a Kähler surface satisfying Lemma 2.2, or, more generally, a Kähler manifold $Y$ which has no diffeomorphisms $f : Y \to Y$ with $f^*[K] = -[K]$, $[K] \in H^2(Y; \mathbb{Q})$. One can take, for example, the surfaces given in Section 2 or any manifold $Y$ like in Propositions 3.9 or 3.11. Denote by $\omega$ the symplectic structure on $Y$ which is the imaginary part of its Kähler structure.
**Proposition 4.5.** The symplectic structures $\omega$ and $-\omega$ are not equivalent to each other. In particular, they are not symplectomorphic.

**Proof.** The class $[K]$ is the canonical class of $\omega$, while $-[K]$ is the canonical class of $-\omega$. So, the result follows from the invariance of the canonical class under deformations and the absence of diffeomorphisms transforming $[K]$ in $-[K]$. □

**Remark.** The above argument can be applied to deformations in the class of almost-complex structures. It shows that for any manifold $Y = (Y, J)$ as above the structures $J$ and $-J$ are not equivalent, where similar to equivalence of symplectic structures, we call two almost-complex structures equivalent if they can be obtained one from another by deformation followed, if necessary, by a diffeomorphism.

The following proposition shows that between manifolds studied in Section 3 there are more inequivalent symplectic and almost-complex structures. Its proof repeats word by word the proof of Proposition 3.7 (iii).

**Proposition 4.6.** Let $Y$ be the common underlying oriented smooth manifold of complex manifolds $X_{p,q}$ from Proposition 3.7. Then,

(i) there are no homeomorphisms $h : Y \to Y$ transforming $c_1(X_{p,q})$ in $c_1(X_{s,r})$ except if $p = s$ and $q = r$;

(ii) symplectic (respectively, almost-complex) structures on $X_{p,q}$ and $X_{r,s}$ are non-equivalent except if $p = s$ and $q = r$. □

**Appendix. Geometric genus calculation**

The aim of this Appendix is to show that the irregularity of the surface constructed in Section 2 equals zero. We start from explaining a general algorithm we use for this calculation and only after that apply it to the particular case in question. In fact, we calculate instead the geometric genus, which is sufficient, since their difference is a topological invariant, due to Noether's formula. In the calculation we use permanently the invariance of the geometric genus under birational transformations, which allows us at each step to use that nonsingular birational model which is more convenient for calculation.

The result is by no means new. It is contained, for example, in the results stated in [I]. For completeness of the proof of Theorem 4.1, we present a straightforward calculation and with as much details as possible.

**Algorithm of reduction to cyclic coverings.** Let $f : X_G \to \mathbb{C}p^2$, where $X_G$ is supposed to be a normal surface, be a Galois covering with abelian Galois group $G = (\mathbb{Z}/p\mathbb{Z})^m$, where $p$ is a prime number, and branched along curves $B_1, \ldots, B_t \subset \mathbb{C}p^2$. Such a covering is determined by an epimorphism $\phi : H_1(\mathbb{C}p^2 \setminus \cup B_i) \to G$. Write it in a form

$$\phi(\gamma_i) = k_1, \alpha_1 + \cdots + k_m, \alpha_m,$$

where $\alpha_j$ are standard generators of $G = \oplus(\mathbb{Z}/p\mathbb{Z})^m$, $\gamma_i$ are standard generators of $H_1(\mathbb{C}p^2 \setminus \cup B_i)$ dual to $B_i$ and $k_{i,j} \in \mathbb{Z}/\mathbb{Z}$, $0 \leq k_{i,j} < p$, are coordinates of $\phi(\gamma_i)$ with respect to $\alpha_j$. In this notation, $X_G$ is the normalization of the projective closure of the affine surface $Y_G \subset \mathbb{C}^{m+2}$ given by

$$x_j^p = \prod_{i=1}^t h_i^{k_{i,j}}(x, y), \quad j = 1, \ldots, m,$$
where \( h_i(x, y) \) are the equations of \( B_i \) in some chart \( \mathbb{C}^2 \subset \mathbb{C}p^2 \).

Let \( \tilde{X}_G \) be the minimal desingularization of \( X_G \). As is known, it exists, it is unique and the action of \( G \) lifts, in an unique way, to a regular action on \( \tilde{X}_G \).

Consider the action of \( G \) on the space \( H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G}) \) of regular 2-forms. It provides a decomposition

\[
H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G}) = \bigoplus_{(s_1, \ldots, s_m)} H(s_1, \ldots, s_m)
\]

into the direct sum of eigenspaces \( H(s_1, \ldots, s_m) \), where

\[
\omega \in H(s_1, \ldots, s_m) \iff \alpha_j(\omega) = e^{2\pi s_j/p} \cdot \omega \quad \text{for any} \quad j = 1, \ldots, m.
\]

Let \( H \subset G \) be a subgroup and \( G_1 = G/H \). We have the following commutative diagram

\[
\begin{array}{ccc}
X_G & \longrightarrow & X_{G_1} \\
f \downarrow & & \downarrow f_1 \\
\mathbb{C}p^2 & \xrightarrow{id} & \mathbb{C}p^2
\end{array}
\]

where \( f_1 : X_{G_1} \rightarrow \mathbb{C}p^2 \) is the Galois covering corresponding to \( \phi_1 = i \circ \phi \) with \( i : G \rightarrow G_1 = G/H \) being the canonical epimorphism. The map \( h \) induces a rational dominant (i.e., whose image is everywhere dense) map \( \tilde{X}_G \rightarrow \tilde{X}_{G_1} \), and the latter, as any rational dominant map between nonsingular varieties, transforms holomorphic \( p \)-forms in holomorphic \( p \)-forms. Thus, the subspace \( h^*(H^0(\tilde{X}_{G_1}, \Omega^2_{\tilde{X}_{G_1}})) \subset H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G}) \) is well defined, and it coincides with the subspace \( H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G})^H \subset H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G}) \) of the elements invariant under the action of \( H \). On the other hand, an eigenspace \( H(s_1, \ldots, s_m) \) is invariant under \( x_1 \alpha_1 + \cdots + x_m \alpha_m \) if and only if \( x_1 s_1 + \cdots + x_m s_m = 0 \) (p). Hence, the sum \( \bigoplus_{\theta \in \mathbb{Z}/p\mathbb{Z}} H(\theta s_1, \ldots, \theta s_m) \) taken over \( \theta \in \mathbb{Z}/p\mathbb{Z} \) coincides with \( H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G})^H \), where

\[
H = \{ x_1 s_1 + \cdots + x_m s_m = 0 \}.
\]

So, this sum is isomorphic to \( H^0(\tilde{X}_{G/H}, \Omega^2_{\tilde{X}_{G/H}}) \). These considerations give rise to the following result.

**Proposition 5.1.** The geometric genus \( p_\beta(\tilde{X}_G) = \dim H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G}) \) of \( \tilde{X}_G \) is equal to

\[
p_\beta(\tilde{X}_G) = \sum H p_\beta(\tilde{X}_{G/H}),
\]

where the sum is taken over all subgroups \( H \) of \( G \) of \( \text{rk} \, H = \text{rk} \, G - 1 \).

**Cyclic coverings.** Now, let \( G = \mathbb{Z}/p\mathbb{Z} \) be a cyclic group. To compute \( p_\beta(\tilde{X}_G) \), let us choose homogeneous coordinates \( (x_0 : x_1 : x_2) \) in \( \mathbb{C}p^2 \) such that the line \( x_0 = 0 \) does not belong to the branch locus of \( f : X_G \rightarrow \mathbb{C}p^2 \). As above, \( X_G \) is the normalization of the projective closure of the hypersurface in \( \mathbb{C}^3 \) given by equation

\[
z^p = h(x, y),
\]
where \( x = \frac{a}{x_0}, y = \frac{b}{x_0}, \)

\[
h(x, y) = \prod_{i=1}^{\ell} h_i^k(x, y),
\]

\( h_i(x, y) \) are irreducible equations in \( \mathbb{C}^2 \subset \mathbb{C}p^2 \) of the curves \( B_i \) constituting the branch locus, and \( 0 < k_i < p \). Note that the degree

\[
\deg h(x, y) = \sum k_i \deg h_i(x, y) = np
\]
is divisible by \( p \), since the line \( x_0 = 0 \) does not belong to the branch locus.

It is easy to see that over the chart \( x_1 \neq 0 \) the variety \( X_G \) coincides with the normalization of the hypersurface in \( \mathbb{C}^3 \) given by equation

\[
w^p = \tilde{h}(u, v),
\]

where \( u = \frac{1}{x}, v = \frac{y}{x}, \tilde{h}(u, v) = u^{np}h\left(\frac{1}{u}, \frac{v}{u}\right), \) and \( w = zu^n \).

Regularity condition over a generic point of the base. Consider

\[
\omega \in H^0(X_G \setminus \text{Sing}X_G, \Omega^2_{X_G \setminus \text{Sing}X_G})
\]

and find a criterion of its regularity outside the ramification and singular loci.

Over the chart \( x_0 \neq 0 \) the form \( \omega \) can be written as

\[
\omega = \left(\sum_{j=0}^{p-1} z^j g_j(x, y)\right)\frac{dx \wedge dy}{z^{p-1}},
\]

where \( g_j(x, y) \) are rational functions in \( x \) and \( y \). The form

\[
\frac{dx \wedge dy}{z^{p-1}}
\]

has neither poles or zeros outside of the preimage of the branch locus. Therefore, \( \omega \) is regular at such a point iff all \( g_j(x, y) \) are regular at each point \( (a, b) \notin \sum B_i \).

In fact, if some \( g_j(x, y) \) is not regular at \( (a, b) \), then the sum

\[
\sum_{j=0}^{p-1} z^j g_j(x, y)
\]
can be written as

\[
\sum_{j=0}^{p-1} z^j P_j(x, y) P_p(x, y),
\]

where \( P_j(x, y), j = 0, \ldots, p, \) are polynomials such that \( P_j(a, b) \neq 0 \) for some \( j < p \) and \( P_p(a, b) = 0 \). Therefore,

\[
\sum_{j=0}^{p-1} z^j P_j(a, b) = 0
\]
at all the $p$ points belonging to $f^{-1}(a, b)$, since otherwise $\omega$ would not be regular over $(a, b)$. On the other hand, it is impossible, since a non-trivial polynomial of degree less than $p$ can not have $p$ roots.

Regularity condition over the line at infinity. Consider the form $\omega$ over the chart $x_1 \neq 0$,

$$\omega = - \left( \sum_{j=0}^{p-1} w^j \frac{\bar{g}_j(u, v)}{w^j u^{n+\deg g_j}} \frac{1}{u^{n-(p-1)}} \frac{du \wedge dv}{w^{p-1}} \right).$$

The similar arguments as above show that the regularity criterion is equivalent to the following bound on the degree of the rational functions $g_j$

$$\deg g_j(x, y) \leq (p - j - 1)n - 3.$$  

Regularity conditions over a nonsingular point of the branch curve. Consider our form

$$\omega = \left( \sum_{j=0}^{p-1} z^j g_j(x, y) \right) \frac{dx \wedge dy}{z^{p-1}}$$

over a nonsingular point $(a, b)$ of one of the components, $B_{i_0}$, of the branch curve. Let $r_j$ be the order of zero (or of the pole if $r_j < 0$) of the function $g_j$ along the curve $B_{i_0}$, i.e., $g_j = h_{i_0}^{r_j}$ with $h_{i_0}$ having neither poles nor zeros along $B_{i_0}$. Since $(a, b)$ is a nonsingular point of $B_{i_0}$, we can assume that $h_{i_0}(x, y)$ and some function $g(x, y)$ are local analytic coordinates in some neighborhood $U$ of $(a, b)$ (denote them by $u$ and $v$). So, over $U$ the surface $X_G$ (after analytic change of variables) is isomorphic to the normalization $X_{G, \text{loc}}$ of the surface in $\mathbb{C}^3$ given by

$$z^p = u^{k_{i_0}}.$$ 

There is an analytic function $w$ in $X_{G, \text{loc}}$ such that $u = w^p$ and $z = w^{k_{i_0}}$, and such that $w$ and $y$ are analytic coordinates in $X_{G, \text{loc}}$. The differential 2-form $\omega$ considered above has the following form in the new coordinates

$$\omega = \left( \sum_{j=0}^{p-1} u^{j k_{i_0}} h_{i_0}^{j r_j} (x, y) u^{p r_j} \right) \frac{p w^{p-1} dw \wedge dv}{w^{p-1} u^{k_{i_0}}}.$$ 

It is easy to see that

$$j_1 k_{i_0} + pr_j + p - 1 - (p - 1)k_{i_0} \neq j_2 k_{i_0} + pr_j + p - 1 - (p - 1)k_{i_0}$$

if $0 < k_{i_0} < p$, $0 \leq j_1, j_2 \leq p - 1$, and $j_1 \neq j_2$. Therefore, all the rational functions $g_j(x, y)$ from (4) are regular functions over a nonsingular point $(a, b)$ of $B_{i_0}$ iff

$$j k_{i_0} + pr_j + p - 1 - (p - 1)k_{i_0} \geq 0$$

Moreover, if $\omega$ is a regular form over $B_{i_0}$ then $r_j$ must be greater than 0, since for $0 < k_{i_0} < p$, $0 \leq j \leq p - 1$, and $r_j \leq -1$, we obtain that

$$j k_{i_0} + pr_j + p - 1 - (p - 1)k_{i_0} < 0.$$
From this it follows that if $\omega$ is a regular form then all the rational functions $g_j(x, y)$ are regular functions everywhere in $\mathbb{C}^2$ outside codimension 2, and thus $g_j(x, y)$ should be polynomials in $x$ and $y$. Moreover, the polynomials $g_j(x, y)$ must be divisible by $h_i^j(x, y)$, where $r_j$ is the smallest integer satisfying the inequality

$$p r_j \geq (p - j - 1)k_i - p + 1. \tag{6}$$

**Regularity conditions over singular points of the branch curve.** This is the only step where the singular points of $X_G$ are concerned. Let $\nu : \tilde{X}_G \to X_G$ be the minimal resolution of singularities of $X_G$ and $E$ be the exceptional divisor of $\nu$. Pick a composition $\sigma : Y \to \mathbb{CP}^2$ of $\sigma$-processes with centers at singular points of $B$ (and their preimages) such that $\sigma^{-1} \circ f \circ \nu(E_i)$ is a curve for each irreducible component $E_i$ of $E$. Let $Z$ be the normalization of $Y \times_{\mathbb{CP}^2} X_G$. Denote by $g : \tilde{X}_G \to Z$ the birational map induced by $\nu$ and $\sigma$. It follows from the above choice of $\sigma$ that for any $\omega \in H^0(Z \setminus \text{Sing}Z, \Omega^2_{Z \setminus \text{Sing}Z})$ its pull-back $g^*(\omega)$ is regular at generic points of $E_i$ and, thus, extends to a regular form on the whole $\tilde{X}_G$. Hence, $H^0(\tilde{X}_G, \Omega^2_{\tilde{X}_G})$ is isomorphic to $H^0(Z \setminus \text{Sing}Z, \Omega^2_{Z \setminus \text{Sing}Z})$.

Therefore, it remains to consider a 2-form $\omega$ written as in (4) and to find a criterion of its regularity on $Z \setminus \text{Sing}Z$. It can be done by performing, step by step, the $\sigma$-processes chosen above. Let us accomplish only the first step, since it is sufficient for the calculation in our particular example which follows.

Represent, once more, $X_G$ as normalization of the surface given by

$$z^p = h(x, y).$$

Denote by $r$ the order of zero of $h(x, y)$ at the point $(0, 0)$, $r = sp + q$, $0 \leq q < p$, and perform a $\sigma$-process with center at this point. In a suitable chart, this $\sigma$-process $\sigma : \mathbb{C}^2_{(u,v)} \to \mathbb{C}^2_{(x,y)}$ is given by $x = u$, $y = uv$. The normalization $Z_1$ of $X_G \times_{\mathbb{C}^2_{(x,y)}} \mathbb{C}^2_{(u,v)}$ is birational to the normalization of the surface given by

$$w^p = u^q h(u, v),$$

where $w = z/u^s$ and $h(u, v) = h(u, uv)/u^r$. We have

$$\omega = \left( \sum_{j=0}^{p-1} z^j g_j(x, y) \right) \frac{dx \wedge dy}{z^{p-1}} =$$

$$= \left( \sum_{j=0}^{p-1} w^j g_j(u, v) u^{sj+s(j+1-s(p-1))} \right) \frac{du \wedge dv}{w^{p-1}},$$

where $s_j$ is the order of zero of $g_j(x, y)$ at $(0, 0)$. Applying (6), we get necessary conditions for the regularity of the pull-back of $\omega$ at generic points of the exceptional divisor: the order of zero $s_j$ of each $g_j(x, y)$ at singular point of the branch locus $B$ of order $r$ is the smallest integer satisfying the inequality

$$p s_j \geq (p - j - 1)r - 2p + 1. \tag{7}$$
Principal calculation. Here, we consider the Galois $G = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ covering of $\mathbb{C}p^2$ constructed in Section 2. In this special case, the minimal desingularization $\hat{X}_G$ of $X_G$ is the induced covering of $\mathbb{C}p^2$ blown up in the twelve multiple points of the configuration.

According to the above algorithm, we should examine one by one the six subgroups $H_1 = ((0,1)), H_2 = ((1,0)), H_3 = ((4,1)), H_4 = ((3,1)), H_5 = ((2,1))$, and $H_6 = ((1,1))$ of $G$ such that $G_i = G/H_i = \mathbb{Z}/5\mathbb{Z}$ (here we denote by $\alpha$ the cyclic subgroup of $G$ generated by $\alpha \in G$). The cyclic Galois coverings $X_{G_i} \rightarrow \mathbb{C}p^2$ can be represented as normalizations of the surfaces in $\mathbb{C}^3$ given respectively by

\begin{align*}
    z^5 &= l_1l_2l_3l_4l_5l_9, \\
    z^5 &= l_1l_3l_4l_5l_6l_9, \\
    z^5 &= l_1l_2l_3l_4l_5l_6l_8^2l_9, \\
    z^5 &= l_1l_2l_3l_4l_5l_6l_9^2l_8^2, \\
    z^5 &= l_1l_2l_3l_4l_5l_6l_7l_8, \\
    z^5 &= l_1l_2l_3l_4l_5l_6l_7l_8l_9,
\end{align*}

To calculate the geometric genus of each of $\hat{X}_G$, we find explicitly all the regular 2-forms, which we write as in (4). We use the criteria (5) – (7) (for convenience, we reproduce in Tables 2 - 4 below, using notations involved in (5) – (7), the exact values of these bounds evaluated in the case of $\mathbb{Z}/5$-coverings). For $G_1$ we get the forms

\[ cl_1l_5l_9 \frac{dx \wedge dy}{z^4}, \quad c \in \mathbb{C}; \]

for $G_2$ we get

\[ (P_2t_1^2l_8 + P_1l_4l_8z + cl_4z^2) \frac{dx \wedge dy}{z^4} \]

where $P_i$ are polynomials in $x, y$ of degree $i$, $c \in \mathbb{C}$, and

\[ p_{349}, p_{789}, p_{168}, p_{147} \in \{P_2 = 0\}, \quad p_{147} \in \{P_1 = 0\}; \]

for $G_3$ we get

\[ (P_1l_1l_3l_5l_6l_9 + Q_1l_4l_5l_6l_9z + c_1l_3l_9z^2 + c_2z^3) \frac{dx \wedge dy}{z^4} \]

where $P_i, Q_i$ are polynomials of degree $i$, $c_k \in \mathbb{C}$, and

\[ p_{123}, p_{789} \in \{P_1 = 0\}, \quad p_{456} \in \{Q_1 = 0\}; \]

for $G_4$ we get

\[ (P_2l_2l_4l_5l_6l_7l_9 + Q_2l_4l_5l_6l_7l_8z + P_1l_4l_5l_6l_7l_8z^2 + Q_1z^3) \frac{dx \wedge dy}{z^4} \]

with

\[ p_{159}, p_{123} \in \{P_2 = 0\}, \quad p_{348} \in \{Q_2 = 0\}, \quad p_{267} \in \{P_1 = 0\}, \quad p_{267} \in \{P_1 = 0\}; \]
for $G_5$ we get

$$\left(P_l^2 l_3 l_6 l_7 l_8^3 + P_l^2 l_4 l_5 l_6 l_7^2 z + Q_2 l_2 l_4 l_8 z^2 + Q_1 z^3\right) \frac{dx \wedge dy}{z^4}$$

with

$$p_{159}, p_{369}, p_{357} \in \{ P_l^2 = 0 \}, p_{357}, p_{267}, p_{258} \in \{ Q_2 = 0 \}, p_{258} \in \{ Q_1 = 0 \};$$

and, finally, for $G_6$,

$$2 l_4 l_7 z \frac{dx \wedge dy}{z^4}.$$

Therefore, $p_g(\tilde{X}_{G_1}) = 1$, $p_g(\tilde{X}_{G_2}) = 5$, $p_g(\tilde{X}_{G_3}) = 5$, $p_g(\tilde{X}_{G_4}) = 13$, $p_g(\tilde{X}_{G_5}) = 11$, and $p_g(\tilde{X}_{G_6}) = 1$.

| $\deg g_j \leq$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|------------------|--------|--------|--------|--------|
| $j = 0$          | 1      | 5      | 8      | 13     |
| $j = 1$          | 0      | 3      | 6      | 9      |
| $j = 2$          | 0      | 1      | 3      | 5      |
| $j = 3$          | 0      | 0      | 0      | 1      |
| $j = 4$          | 0      | 0      | 0      | 0      |

**Table 2**

| $r_j \geq$ | $k_1 = 1$ | $k_1 = 2$ | $k_1 = 3$ | $k_1 = 4$ |
|------------|----------|----------|----------|----------|
| $j = 0$    | 0        | 1        | 2        | 3        |
| $j = 1$    | 0        | 1        | 1        | 2        |
| $j = 2$    | 0        | 0        | 1        | 1        |
| $j = 3$    | 0        | 0        | 0        | 0        |
| $j = 4$    | 0        | 0        | 0        | 0        |

**Table 3**

| $s_j \geq$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ | $r = 6$ | $r = 7$ | $r = 8$ | $r = 9$ | $r = 10$ | $r = 11$ | $r = 12$ |
|------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $j = 0$    | 0       | 1       | 2       | 3       | 3       | 4       | 5       | 6       | 7       | 7       | 8       |
| $j = 1$    | 0       | 0       | 1       | 1       | 2       | 2       | 3       | 3       | 4       | 5       | 5       |
| $j = 2$    | 0       | 0       | 0       | 1       | 1       | 1       | 2       | 2       | 3       | 3       | 3       |
| $j = 3$    | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 1       | 1       | 1       |
| $j = 4$    | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |

**Table 4**

Now, from Proposition 5.1 it follows that $p_g(\tilde{X}_G) = 36$. On the other hand, by Noether formula, $(c_1^2 + c_2)/12 = 37$ and, thus,

$$q(\tilde{X}_G) = \dim H^0(\tilde{X}_G, \Omega^1_{\tilde{X}_G}) = (c_1^2 + c_2)/12 - 1 - p_g = 0,$$

i.e., $\tilde{X}_G$ is a regular surface.
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