Slice Energy and Conformal Frames in Theories of Gravitation

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Abstract

We examine and compare the behaviour of the scalar field slice energy in different classes of theories of gravity, in particular higher-order and scalar-tensor theories. We find a universal formula for the energy and compare the resulting conservation laws with those known in general relativity. This leads to a comparison between the inflaton, the dilaton and other forms of scalar fields present in these generalized theories. It also shows that all such conformally-related, generalized theories of gravitation allow for the energy on a slice to be invariably defined and its fundamental properties be insensitive to conformal transformations.
1 Introduction

There have recently been many investigations on various structural as well as evolutionary aspects of higher-order and scalar-tensor theories of gravity (see Refs. \[1\]-\[11\] for a partial list). While many of these analyses explore particular problems in these modified frameworks, others aim to compare such metric theories of gravitation from different points of view. Although different these theories share two important common characteristics: Firstly they can all be formulated in different conformal frames. It is well known (cf. \[12\], \[13\]) that the formulation of each of these modifications and extensions of general relativity can be given in different, conformally-related, spacetime manifolds, called conformal frames, and depending on the particular problem one is working on, one frame may prove more useful to all other conformally-related ones. Secondly, they all require for their proper formulation in at least one of the conformally-related frames the existence of scalar fields. In some of these theories scalar fields are mediators of the gravitational interaction while in others they emerge as by-products of the transformation which relates two different conformal frame representations of the same theory.

In fact these two characteristics turn out to be closely related: the conformal transformation that relates two different conformal representations of a theory is usually defined through the introduction of a scalar field. Further the existence of different conformal frames poses nontrivial relations between the scalar fields present in them, which would otherwise have no connection. It is therefore important to be able to state clearly such relationships: What is the precise relation between the scalar fields present in two different, conformally-related frames? Are two frame representations of the same theory mathematically and/or physically equivalent? Of course these questions are not new and some of the related previous work is contained in Refs. \[14\], \[15\], \[8\].

In this paper we analyze these questions from the viewpoint of a geometric quantity, the energy of fields on a slice in spacetime, and compare our findings about the behaviour of the slice energy in such theories with that known in general relativity. This comparison shows that slice energy is a kind of ‘universal invariant’ in metric theories of gravitation.
Further slice energy may be used to clarify possible relations between the different forms of scalar fields appearing in such theories, as well as help uncover and compare the physical content in different conformal frames in an invariant way.

In the next Section we write down the field equations which define and describe the different theories we study and give the conservation laws valid in each one of these frameworks to establish our notation. Section 3 is the heart of this paper. There we find how the slice energy behaves for the case of each one of the theories given in Section 2. We conclude with a discussion of how our results can be used to shed light on the two aforementioned issues, namely the differences between the scalar fields appearing in these theories and the possible physical equivalence of these generalized modifications of general relativity.

2 Field equations

We are interested below in a comparison of the conservation properties of slice energy of certain fields in general relativity, higher-order gravity theories and scalar-tensor theories of gravitation. We denote any matter field present by the letter $\psi$. In general relativity we take the field equations to be of the form

$$G_{\alpha\beta} = T_{\alpha\beta}(\phi) + T_{\alpha\beta}(\psi),$$

(1)

where $G_{\alpha\beta}$ is the Einstein tensor, $\phi$ is a scalar field with stress tensor

$$T^{\alpha\beta}(\phi) = \partial^\alpha \phi \partial^\beta \phi - \frac{1}{2} g^{\alpha\beta} (\partial^\lambda \phi \partial^\chi \phi - 2 V(\phi)),$$

(2)

$T(\psi)$ represents the stress tensor of a field $\psi$, and we assume the conservation identities $\nabla_{\alpha} T^{\alpha\beta}(\phi) = 0$ and $\nabla_{\alpha} T^{\alpha\beta}(\psi) = 0$.

In higher-order gravity theories we consider the Jordan-frame equations

$$L_{\alpha\beta} \equiv f' R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} f - \nabla_{\alpha} \nabla_{\beta} f' + g_{\alpha\beta} \Box_g f' = T_{\alpha\beta}(\psi),$$

(3)
which, because $\nabla_\alpha L^{\alpha\beta} = 0$, imply the conservation identities $\nabla_\alpha T^{\alpha\beta}(\psi) = 0$. The Einstein frame representation of this theory is

$$\tilde{G}_{\alpha\beta} = T_{\alpha\beta}(\phi) + \tilde{T}_{\alpha\beta}(\tilde{\psi}),$$

(4)

where $\phi = \ln f'$ and $T_{\alpha\beta}(\phi)$ is of the form (2) with $V(\phi) = (1/2)(f')^{-2}(Rf' - f)$, cf. [12]. Here the whole tensor in the right-hand-side is conserved,

$$\tilde{\nabla}_\alpha \left( \tilde{T}^{\alpha\beta}(\phi) + \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right) = 0,$$

(5)

but the two components are not conserved separately, that is

$$\tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\phi) \neq 0, \quad \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) \neq 0.$$

(6)

The field $\phi$ appearing both in general relativity and in (the Einstein frame representation of) higher-order gravity theories is in certain contexts responsible for the existence of an inflationary period. For concreteness we call it the inflaton and distinguish it from a scalar field, say $\xi$, that may appear directly in the Jordan frame equations (3) in addition to the matterfield $\psi$.

Lastly we take the defining equations of our scalar-tensor theory to be the Brans-Dicke (BD) ones, with $\chi$ denoting the BD scalar field (everything we do below is valid if, instead of the BD theory assumed here only for brevity, we consider the most general scalar-tensor action having couplings of the form $h(\chi)$, where $h$ is any differentiable function of the field $\chi$),

$$S_{\alpha\beta} \equiv \chi G_{\alpha\beta} = T_{\alpha\beta}(\chi) + T_{\alpha\beta}(\psi).$$

(7)

The novel feature of this equation is the requirement that, if in accordance with the equivalence principle we assume that

$$\nabla_\alpha T^{\alpha\beta}(\psi) = 0,$$

(8)

only, then, because $\nabla_\alpha G^{\alpha\beta} = 0$ we find

$$\nabla_\alpha S^{\alpha\beta} = \nabla_\alpha T^{\alpha\beta}(\chi).$$

(9)
Here $T^{\alpha\beta}(\chi)$ is not given by (2) but by a different, more complicated, expression (cf. [16], pp. 159-60). For definiteness below we call the field $\chi$ the dilaton to distinguish it from the other scalar fields appearing in the $f(R)$ Eqs. (3), (4) and in general relativity, Eq. (1). Many currently popular string theories appear as special cases of the scalar-tensor equations.

3 Slice energy

Our starting point is the relation for the energies of a field on two end-slices, $M_{t_1}$ and $M_{t_0}$, of the time-oriented spacetime $(V, g)$ obtained in [17]:

$$E_{t_1} - E_{t_0} = \int_{t_0}^{t_1} \int_{M_t} T^{\alpha\beta} \nabla_{(\alpha} X_{\beta)} d\mu + \int_{t_0}^{t_1} \int_{M_t} X_{\beta} \nabla_{\alpha} T^{\alpha\beta} d\mu. \quad (10)$$

Here $V = M \times \mathbb{R}$, $M$ is a smooth manifold of dimension $n$, $g$ a spacetime metric and the spatial slices $M_t (= M \times \{t\})$ are spacelike submanifolds endowed with the time-dependent spatial metric $g_t$. (Greek indices run from 0 to $n$, while Latin ones from 1 to $n$ and the metric signature is $(+ − \cdots −)$.) For $X$ any causal vectorfield of $V$, we define the energy-momentum vector $P$ of a stress tensor $T$ relative to $X$ to be $P^\beta = X_{\alpha} T^{\alpha\beta}$ and the energy on $M_t$ with respect to $X$, called hereafter the slice energy, by the integral (when it exists) $E_t = \int_{M_t} P^{\alpha} n_\alpha d\mu_t$, where $n$ is the unit normal to $M_t$ and $d\mu_t$ is the volume element with respect to the spatial metric $g_t$. We call $P^{\alpha} n_\alpha$ the energy density and assume that $X$ and $T$ are smooth. Further, for the validity of the energy equation (10) on the spacetime slab $D = \Sigma \times [t_0, t_1]$, $\Sigma \subset M$ and with $T$ having support on $D$, we take $M$ to be compact or the field to satisfy appropriate fall-off conditions at infinity. Fundamental properties of the slice energy are proved in [17], Section 2.

In [17] we showed how Eq. (10) leads to relations describing the energy exchange between the scalar field $\phi$ and the matter component $\psi$, in general but also especially in the context of higher-order gravity. Here we follow a different route: Starting from Eq. (10) we derive relations showing the dependence of the total slice energy of the system on the special features of each one of the three theories given by Eqs. (1), (3) and (4).
and (7). Writing Eq. (10) for the scalar field $\phi$ and substituting from Eqs. (1) and the conservation identity for the terms $T(\phi)$ and $X \nabla T$ respectively, we find

$$E_{t_1}(\phi) - E_{t_0}(\phi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} [G^{\alpha\beta} - T^{\alpha\beta}(\psi)] \nabla_{(\alpha} X_{\beta)} d\mu$$

$$= \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} \nabla_{\alpha} X_{\beta} d\mu - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\psi) \nabla_{\alpha} X_{\beta} d\mu. \quad (11)$$

Using Stokes' theorem, the last term is just

$$\int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\psi) \nabla_{\alpha} X_{\beta} d\mu = \int_{\mathcal{M}_{t_1}} P^{\alpha\beta} n_{\alpha} d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} P^{\alpha\beta} n_{\alpha} d\mu_{t_0} = E_{t_1}(\psi) - E_{t_0}(\psi), \quad (12)$$

and so, setting $E_t(\phi + \psi) = E_t(\phi) + E_t(\psi)$, we find that in general relativity the total slice energy of a system comprised of the field $\phi$ and a matter field $\psi$ depends on the Einstein tensor as follows:

$$E_{t_1}(\phi + \psi) - E_{t_0}(\phi + \psi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} \nabla_{\alpha} X_{\beta} d\mu. \quad (13)$$

Further, since $\nabla_{\alpha} G^{\alpha\beta} = 0$, integrating by parts and using Stokes theorem we have

$$\int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} \nabla_{\alpha} X_{\beta} d\mu = \int_{\mathcal{M}_{t_1}} G^{\alpha\beta} X_{\alpha} N_{\beta} d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} G^{\alpha\beta} X_{\alpha} N_{\beta} d\mu_{t_0}, \quad (14)$$

where $N$ is the unit normal to the slices. Using this form we have the following result.

**Theorem 1** The total slice energy of the system comprised of the scalar field $\phi$ and a matterfield $\psi$ satisfying the Einstein equations (7), is given by

$$E_{t_1}(\phi + \psi) - E_{t_0}(\phi + \psi) = \int_{\mathcal{M}_{t_1}} G^{\alpha\beta} X_{\alpha} N_{\beta} d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} G^{\alpha\beta} X_{\alpha} N_{\beta} d\mu_{t_0}. \quad (15)$$

In particular, when $X$ is a Killing field of the metric $g$, the total slice energy of the system is conserved.

The terms of the form $\int_{\mathcal{M}_t} G^{\alpha\beta} X_{\alpha} N_{\beta} d\mu_t$ represent a gravitational flux through the slice $\mathcal{M}_t$. When $X$ is a Killing field, the right hand side of Eq. (13) is zero and we have an
integral conservation law given by the equality of the two terms in the right hand side of Eq. (15) and this agrees with the corresponding result originally given in \[18\], Chap. VI.

The situation in higher-order gravity is in fact, despite the different conservation laws, similar. In the Einstein frame we have

\[
E_{t_1}(\phi) - E_{t_0}(\phi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \left[ \tilde{G}^{\alpha\beta} - \tilde{T}^{\alpha\beta}(\tilde{\psi}) \right] \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu} \\
= \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{G}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\tilde{\psi}) \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} \\
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}. \tag{16}
\]

Using Stokes’ theorem, the middle term is

\[
\int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{T}^{\alpha\beta}(\tilde{\psi}) \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu} = \int_{\mathcal{M}_{t_1}} \tilde{P}^{\alpha} \tilde{n}_\alpha d\tilde{\mu}_{t_1} - \int_{\mathcal{M}_{t_0}} \tilde{P}^{\alpha} \tilde{n}_\alpha d\tilde{\mu}_{t_0} \\
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu} \\
= E_{t_1}(\tilde{\psi}) - E_{t_0}(\tilde{\psi}) - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{X}_\beta \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta}(\tilde{\psi}) d\tilde{\mu}. \tag{17}
\]

and so, setting \(E_t(\phi + \tilde{\psi}) = E_t(\phi) + E_t(\tilde{\psi})\), we find that in higher-order gravity, because of the marvelous fact that the terms of the general form \(\int \tilde{X} \tilde{\nabla} \tilde{T}(\tilde{\psi})\) which were absent in general relativity now precisely cancel each other, the total slice energy of a system composed of the field \(\phi\) and a matter field \(\tilde{\psi}\) in the Einstein frame depends on the Einstein tensor in the same way as before:

\[
E_{t_1}(\phi + \tilde{\psi}) - E_{t_0}(\phi + \tilde{\psi}) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \tilde{G}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{X}_\beta d\tilde{\mu}. \tag{18}
\]

Hence we arrive at the following result.

**Theorem 2** The total slice energy of the system composed of the scalar field \(\phi\) and a matterfield \(\tilde{\psi}\) satisfying the Einstein equations (4), is given by

\[
E_{t_1}(\phi + \tilde{\psi}) - E_{t_0}(\phi + \tilde{\psi}) = \int_{\mathcal{M}_{t_1}} \tilde{G}^{\alpha\beta} \tilde{X}_\alpha \tilde{N}_\beta d\tilde{\mu}_{t_1} - \int_{\mathcal{M}_{t_0}} \tilde{G}^{\alpha\beta} \tilde{X}_\alpha \tilde{N}_\beta d\tilde{\mu}_{t_0}. \tag{19}
\]
In particular, when \( X \) is a Killing field of the metric \( \tilde{g} \), the total slice energy of the system is conserved.

Note that if we have a scalar field \( \xi \) in addition to the matter field \( \psi \) present in the original Jordan frame of the higher order gravity theory, then we obtain a result similar to that in general relativity but with \( L_{\alpha\beta} \) in place of the Einstein tensor, namely,

\[
E_{t_1}(\xi + \psi) - E_{t_0}(\xi + \psi) = \int_{\mathcal{M}_1} L^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{\mathcal{M}_0} L^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \tag{20}
\]

Then terms of the form \( \int_{\mathcal{M}_t} L^{\alpha\beta} X_\alpha N_\beta d\mu_t \) represent a higher-order gravitational flux through the slice \( \mathcal{M}_t \). When \( X \) is a Killing field, we again have an integral conservation law as before.

We now move to the analysis of the scalar-tensor theory (7). In this case Eq. (10) becomes

\[
E_{t_1}(\chi) - E_{t_0}(\chi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} S^{\alpha\beta} X^\beta_{\alpha} d\mu + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\chi) \nabla_\alpha X_\beta d\mu \tag{21}
\]

and therefore using Stokes' theorem and the scalar-tensor equation (7) we obtain

\[
E_{t_1}(\chi) - E_{t_0}(\chi) = \int_{\mathcal{M}_1} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_1} - \int_{\mathcal{M}_0} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_0}
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} S^{\alpha\beta} \nabla_\alpha X_\beta + \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} (S^{\alpha\beta} - T^{\alpha\beta}(\psi))(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)
\]

\[
= \int_{\mathcal{M}_1} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_1} - \int_{\mathcal{M}_0} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_0}
- \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} S^{\alpha\beta} (\nabla_\alpha X_\beta - \nabla_\beta X_\alpha)
\]

\[
- \frac{1}{2} \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\psi)(\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)
\]

\[
= \int_{\mathcal{M}_1} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_1} - \int_{\mathcal{M}_0} S^{\alpha\beta} X^\beta_{\alpha} d\mu_{t_0}
- \int_{t_0}^{t_1} \int_{\mathcal{M}_t} S^{\alpha\beta} \nabla_\alpha X_\beta|_{[\alpha} d\mu + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\psi) \nabla_\alpha X_\beta. \tag{22}
\]
The penultimate term in the last equality of this equation is zero, as the first tensor in the product is symmetric and the second antisymmetric, and so we find that

\[ E_{t_1}(\chi) - E_{t_0}(\chi) = \int_{\mathcal{M}_{t_1}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_0} - \int_{t_0}^{t_1} \int_{\mathcal{M}_t} T^{\alpha\beta}(\psi) \nabla_{(\alpha} X_{\beta)}. \]  

(23)

Now, since \( \nabla_\alpha S^{\alpha\beta} = G^{\alpha\beta} \nabla_\alpha \phi \), we find that the first two terms can be expressed more simply as follows

\[ \int_{\mathcal{M}_{t_1}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} S^{\alpha\beta} X_\beta N_\alpha d\mu_{t_0} = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} S^{\alpha\beta} \nabla_\alpha X_\beta + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} X_\beta \partial_\alpha \chi. \]  

(24)

Therefore using Eq. (12) we find that

\[ E_{t_1}(\chi + \psi) - E_{t_0}(\chi + \psi) = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} \chi G^{\alpha\beta} \nabla_\alpha X_\beta + \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} X_\beta \partial_\alpha \chi \]

\[ = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta}(\chi \nabla_\alpha X_\beta + X_\beta \partial_\alpha \chi) \]

\[ = \int_{t_0}^{t_1} \int_{\mathcal{M}_t} G^{\alpha\beta} \nabla_\alpha (\chi X_\beta) \]

and we are led to the following result.

**Theorem 3** The total slice energy of the dilaton-matter system satisfying the scalar-tensor equations (7) is given by

\[ E_{t_1}(\chi + \psi) - E_{t_0}(\chi + \psi) = \int_{\mathcal{M}_{t_1}} S^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} S^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \]  

(25)

### 4 Discussion

In conclusion we have found the different forms that slice energy takes in various classes of generalized theories of gravitation which include higher-order gravity theories and scalar-tensor ones. These forms may be described symbolically as follows:

\[ E_{t_1}(\lambda + \psi) - E_{t_0}(\lambda + \psi) = \int_{\mathcal{M}_{t_1}} \Lambda^{\alpha\beta} X_\alpha N_\beta d\mu_{t_1} - \int_{\mathcal{M}_{t_0}} \Lambda^{\alpha\beta} X_\alpha N_\beta d\mu_{t_0}. \]  

(26)
Here $\lambda$ denotes either an inflaton field, $\phi$, which couples to matter in general relativity or in the Einstein frame in higher-order gravity, the scalar field $\xi$ which may appear in the Jordan frame of higher-order gravity, or the dilaton $\chi$ in scalar-tensor theory, while $\Lambda$ is a gravitational operator defining the left-hand-sides of the associated field equations, that is, $\Lambda$ is $G^{\alpha\beta}$, $\tilde{G}^{\alpha\beta}$, $L^{\alpha\beta}$ or $S^{\alpha\beta}$ respectively.

The above analysis allows for some conclusions to be drawn concerning the scalar fields present in the metric theories considered so far. Firstly Eq. (26) gives a universal formula for the scalar field energies appearing in different metric theories of gravitation and may provide an answer to the issue discussed in the Introduction and first posed by Brans in [14] of the relative behaviour of these fields. According to the results of this paper we may conclude that an observer who moves from slice to slice in spacetime finds that the total slice energy of the system composed of a scalar field and a matter field depends solely on the tensor field $\Lambda$ which defines the gravitational sector of the theory. We may therefore use Eq. (26) as a definition of the scalar field we wish to couple in a given theory and conclude that this equation provides a clue to the relative differences of the scalar fields discussed above.

Secondly it follows that in all these conformally related frames slice energy maintains the same structural definition, namely, Eq. (26), and its conservation properties are structurally the same in each conformal frame. Although its form obviously depends on the particular theory considered, as in Eqs. (13), (18), (20) and (25), its meaning and basic properties do not depend upon the particular conformal frame representation used. In this sense it may be called an invariant. This is true despite the existence of several different ‘measures of physical inequivalence’ sometimes used in the literature to show an inequivalence of the conformally related frames, such as conservation laws, positivity of energy, the existence of a stable ground state etc. Here we have a quantity of obvious physical meaning the properties of which are in the above sense insensitive to conformal transformations. We may therefore conclude that with respect to slice energy these different conformal frames are physically equivalent and this conclusion reinforces
that reached in \[.\]

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