A Gauge-fixed Hamiltonian for Lattice QCD

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Abstract
We study the gauge fixing of lattice QCD in 2+1 dimensions, in the Hamiltonian formulation. The technique easily generalizes to other theories and dimensions. The Hamiltonian is rewritten in terms of variables which are gauge invariant except under a single global transformation. This paper extends previous work, involving only pure gauge theories, to include matter fields.


\section{Introduction}

It is straightforward to write a Hamiltonian for lattice QCD, as was originally derived by Kogut and Susskind in 1975.\textsuperscript{2} However, the Hamiltonian operator commutes with local gauge transformations at each site on the lattice. Since these operators commute among themselves, the Hilbert space is split into subspaces characterized by the color charge at each site. In particular, the vacuum subspace has zero color at every site.

Writing a general wave function belonging to a specific subspace is a nontrivial task. Müller and Rühl \textsuperscript{3} showed how to do this for the vacuum subspace of a pure gauge theory in 2+1 dimensions, and Bronzan \textsuperscript{1} extended their work to 3+1 dimensions and the two-charge subspace. In this paper we will further extend this work to include theories with matter fields. We will illustrate our technique using lattice QCD in 2+1 dimensions, but it can be easily generalized to other dimensions and other theories which have unitary gauge groups.

The layout of the rest of the paper is as follows. In Section 2 we will review the approach of Müller and Rühl. In Section 3 we will present our method for gauge fixing in theories with fermions. In Section 4, we will summarize and comment on the generalization of the technique to other theories.

\section{Gauge fixing pure SU(3)}

We use a lattice consisting of $N \times N$ links, with open boundary conditions. We label links by the pair $(s, \mu)$, where $s$ is the site from which the link starts, with $0 \leq s_x, s_y \leq N$, and $\mu$ is the direction of the link. Each link is associated with an element $U_{s,\mu}$ of the gauge group in the usual fashion. By $J_{L,\alpha}(s, \mu)$ we denote the generators of the gauge group in differential form. For the eight-dimensional group SU(3), on which we will focus our attention, $\alpha = 1, \ldots, 8$. The action of $J_{L,\alpha}$ on a matrix representation of SU(3) is to left-multiply it by the matrix form of the $\alpha$th generator. For example,

\begin{equation}
J_{L,\alpha} U_{\beta \gamma} = \frac{1}{2} (\lambda_{\alpha} U)_{\beta \gamma},
\end{equation}

where $\lambda_{\alpha}$ is the $\alpha$th Gell-Mann matrix. Similarly, we denote by $J_{R,\alpha}(s, \mu)$ the right-multiplying generators. The operators $-J_L$ and $J_R$ obey the usual Lie algebra of SU(3).
The Hamiltonian can be written as the sum of an “electric” piece and a “magnetic” piece:

\[ H = H_e + H_m, \]

(2.2)

where

\[ H_e = \frac{g^2}{2a} \sum_{s,\mu} \mathcal{J}_L^2(s, \mu), \]

(2.3)

\[ H_m = -\frac{1}{g^2 a} \sum_{s,\mu > \nu} \left[ \text{Tr} (U_{s,\mu} U_{s+\vec{\nu},\mu} U_{s+\vec{\mu},\nu} U_{s,\nu}^\dagger) + H.c. - 6 \right]. \]

(2.4)

The arrow symbolizes that \( \mathcal{J}_L \) is a vector on the SU(3) manifold.

The gauge-fixing procedure of Ref. [3] begins with a change of variables. First, define a “maximal tree” consisting of all \( y \) links, and all \( x \) links on the \( x \) axis. (See Figure 1 of Ref. [3].) All the links on the maximal tree remain in the new variable set. The other links of the lattice are replaced by “loop variables”:

\[ y_s = \prod_{l \in R_s} (U_l)^{\sigma_l}, \]

(2.5)

where we have used \( l = (s, \mu) \) as shorthand. Here, \( R_s \) represents an oriented path that follows \( (0, 0) \rightarrow (0, s_y) \rightarrow (s_x, s_y) \rightarrow (s_x, 0) \rightarrow (0, 0) \). (See Figure 3 of Ref. [3].) \( \sigma_l = \pm 1 \) depending on whether \( R_s \) passes through the link in the positive or negative direction. The choice of these particular paths is discussed in Ref. [3]. Since SU(3) is a non-Abelian group, it is important to note that the oriented path gives the order in which the group elements are multiplied from left to right. While it is convenient, let us also define \( V_s \) as the same ordered product, except that the path stops at \( s \) rather than returning to \( (0, 0) \).

Now, denote by \( \mathcal{J}_{L\alpha}(y_s) \) and \( \mathcal{J}_{R\alpha}(y_s) \) the differential generators for the corresponding loop variable. Also, denote by \( D_{\beta\alpha}(V_s) \) the adjoint representation of SU(3) in the basis where the rows and columns are labeled by the indices of the generators. We use Eq. (4.13) of Ref. [1] to write the old generators in terms of the new:

\[ \mathcal{J}_{L\alpha}(U_{s,\mu}) = \epsilon_{s,\mu} \mathcal{J}_{L\alpha}(U_{s,\mu}) + D_{\beta\alpha}(V_s) \left[ \sum_{s'}^{P} \mathcal{J}_{L\beta}(y_{s'}) - \sum_{s'}^{N} \mathcal{J}_{R\beta}(y_{s'}) \right], \]

(2.6)

where the first sum is over loops which make transits of the link \( U_{s,\mu} \) in the positive direction, and the second sum is over negative transits. \( \epsilon_{s,\mu} = 1 \) if
the link is on the maximal tree, and 0 if it is not. Note also that there is an implicit sum over $\beta$.

With this expression in hand, we are now able to write the electric Hamiltonian.

$$H_e = \frac{g^2}{2a} \sum_{s,\mu} [\sum_{s'} \vec{J}_L(y_{s'}) - \sum_{s'} \vec{J}_R(y_{s'})]^2 \quad (2.7)$$

Note that we are able to leave out the generators of the maximal tree variables, in accordance with the discussion in Sec. V of Ref. [1].

The magnetic Hamiltonian can also be written solely in terms of the loop variables:

$$H_m = -\frac{1}{g^2a} \sum_{s_x,s_y=1}^N [Tr(y_{s_x-1,s_y}^\dagger y_{s_x,s_y-1}^\dagger y_{s_x,s_y} - 1)] + H.c. - 6 \quad (2.8)$$

In this expression, if either subscript of $y_{s}$ is zero, then $y_{s}$ is defined to be the unit element.

This completes the gauge fixing of the SU(3) Hamiltonian. The general state in the zero-charge subspace is independent of the maximal tree variables, and we have also eliminated these variables from the Hamiltonian. The sole vestige of local gauge invariance is the invariance under a transformation performed at the origin. Since this transforms all the loop variables, it is in fact a global SU(3) invariance. The Hamiltonian is invariant under this transformation, and physical wave functions in the subspace must also be invariant under it. Once we have imposed this condition, we have eliminated all gauge arbitrariness from the problem.

### 3 Gauge fixing QCD

In this section, we add quark fields to the Hamiltonian and carry out the same prescription. We will see that this requires an additional change of variables to banish the unphysical degrees of freedom from the Hamiltonian.

We add to the Hamiltonian a naive Dirac term:

$$H = H_e + H_m + H_q, \quad (3.1)$$

$$H_q = \frac{a^3}{2} \sum_{s,\mu}[\bar{\psi}(s)\gamma_\mu U_{s,\mu}\psi(s + \hat{\mu}) - \bar{\psi}(s + \hat{\mu})\gamma_\mu U_{s,\mu}^\dagger\psi(s)] + ma^4 \sum_s \bar{\psi}(s)\psi(s) \quad (3.2)$$
We need not be concerned here with the doubling problem. The technique will work if either Wilson or staggered fermions are implemented.

The initial variable set consists of the link variables $U_{s,\mu}$ and the quark variables $\psi(s)$ and $\psi^\dagger(s)$. The first step is to change to a new set of quark variables,

\begin{align}
\psi'(s) &= V_s \psi(s), \\
\psi'^\dagger(s) &= \psi^\dagger(s)V_s^\dagger,
\end{align}

where $V_s$ is defined as it was in Section 2. Note that since $V_s$ is unitary, $\psi'(s)$ and $\psi'^\dagger(s)$ are still canonical fermions. The motivation behind this change of variables, as we will see explicitly at the end of this section, is that the gauge element in $H_q$ will now be either a product of loop variables, or the unit element. Thus, we have banished the maximal tree elements from these terms, as desired.

With this change of variables, the gauge generators and the quark operators are no longer “independent,” by which we mean that in general

\[ [\mathcal{J}_{L\alpha}(s_1, \mu_1), \psi'(s_2)] \neq 0. \] (3.4)

We can eliminate this difficulty. First, define the operator

\[ \mathcal{F}_\alpha(s) = -\frac{1}{2} \psi'^\dagger(s)\lambda_\alpha \psi'(s), \] (3.5)

where $\lambda_\alpha$ is the $\alpha$th Gell-Mann matrix. Now, define a new set of operators

\[ \mathcal{J}'_{L\alpha}(s, \mu) = \mathcal{J}_{L\alpha}(s, \mu) - D_{\beta\alpha}(V_s) \sum_{s'} \mathcal{F}_\beta(s'), \] (3.6)

where we have indicated that the sum is over paths to $s'$ which make a positive transit across $U_{s,\mu}$. Recall that $D_{\alpha\beta}(V_s)$ is the adjoint representation of SU(3). The following properties of these new operators are easily verified:

1. $\mathcal{J}'_L$ obeys the same algebra as $\mathcal{J}_L$:

\[ [\mathcal{J}'_{L\alpha}(s_1, \mu_1), \mathcal{J}'_{L\beta}(s_2, \mu_2)] = -i\delta_{s_1s_2}\delta_{\mu_1\mu_2} \sum_{\gamma} f_{\alpha\beta\gamma} \mathcal{J}_{L\gamma}(s_1, \mu_1); \] (3.7)

2. $\mathcal{J}'_L$ commutes with the primed quark operators:

\[ [\mathcal{J}'_{L\alpha}(s_1, \mu), \psi'(s_2)] = [\mathcal{J}'_{L\alpha}(s_1, \mu), \psi'^\dagger(s_2)] = 0; \] (3.8)
3. the commutators of $\mathcal{J}'_L$ with an arbitrary function of group elements is the same as that of $\mathcal{J}_L$:

$$[\mathcal{J}'_{L\alpha}(s, \mu), f(U_{s,\mu})] = [\mathcal{J}_{L\alpha}(s, \mu), f(U_{s,\mu})].$$  \hspace{3cm} (3.9)

This last point is important, because then the primed analogue of Eq. (2.4) will hold.

We now proceed with the second change of variables, which is the link-to-loop change used in the pure gauge case. As a result, the operators in the electric Hamiltonian are

$$\mathcal{J}_{L\alpha}(s, \mu) = \epsilon(s, \mu)\mathcal{J}'_{L\alpha}(s, \mu) + D_{\beta\alpha}(V_s)\left(\sum_{s'} \mathcal{J}'_{L\beta}(y_{s'})\right) - \sum_{s'} \mathcal{J}'_{R\beta}(y_{s'}) + \sum_{s'} \mathcal{F}_{\beta}(s').$$ \hspace{3cm} (3.10)

We can thus write the electric Hamiltonian,

$$H_e = \frac{g^2}{2a} \sum_{s,\mu} \left[\sum_{s'} \mathcal{J}'_{L}(y_{s'}) - \sum_{s'} \mathcal{J}'_{R}(y_{s'}) + \sum_{s'} \mathcal{F}_{\beta}(s')\right]^2.$$ \hspace{3cm} (3.11)

The magnetic Hamiltonian is unchanged from the pure-gauge case:

$$H_m = -\frac{1}{g^2a} \sum_{s,x,y=1}^N \left[Tr(y_{s_{x-1},s_{y-1},s_{x-1},s_{y-1}} y_{s_{x},s_{y}}) + H.c. - 6\right].$$ \hspace{3cm} (3.12)

The quark Hamiltonian is

$$H_q = \frac{a^2}{2} \sum_{s,\mu} \left[\overline{\psi}(s)\gamma_{\mu}V_{s,\mu}U_{s+\hat{\mu}}V_{s+\hat{\mu}}\psi'(s) - \overline{\psi}'(s + \hat{\mu})\gamma_{\mu}V_{s+\hat{\mu}}U_{s,\mu}^\dagger V_{s}^\dagger \psi'(s)\right] + m\alpha^4 \sum_{s} \overline{\psi}(s)\psi'(s).$$ \hspace{3cm} (3.13)

When $\hat{\mu} = \hat{x}$, the gauge element in $H_q$ is

$$V_{s}^\dagger U_{s,\mu}V_{s+\hat{\mu}} = V_{s+\hat{\mu}}^\dagger V_{s+\hat{\mu}} = 1;$$ \hspace{3cm} (3.14)

when $\hat{\mu} = \hat{y}$, the gauge element is

$$V_{s}^\dagger U_{s,\mu}V_{s+\hat{\mu}} = y_{s}y_{s+\hat{\mu}}^\dagger.$$ \hspace{3cm} (3.15)
We see, then, that $H_{q}$ does not depend on maximal tree variables. We have thus eliminated the maximal tree variables from all terms of the Hamiltonian. Again, the general wave function in the zero-charge subspace can be written in terms of only the loop variables and the new quark operators. Here, too, we must impose global SU(3) invariance on the wave function. Once this is done, we have again eliminated all gauge arbitrariness from the problem, as desired.

4 Summary

We have presented a technique to fix the gauge of the lattice QCD Hamiltonian in $2+1$ dimensions, in the zero-charge subspace. As we stated in the introduction, this method can be generalized. The generalization to $3+1$ dimensions and the two-charge subspace is conceptually trivial, but does involve a more intricate definition of the paths in the various changes of variables. These paths are described in Ref. [1]. One of us has used the scheme in $3+1$ dimensions in a calculation of the glueball spectrum. [4]

Generalization to other gauge groups is easily accomplished. A restriction is that the group must be unitary, so that the transformed quarks defined in Eq. (3.3) remain canonical fermions.

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