White Noise Analysis on manifolds and the energy representation of a gauge group

Takahiro Hasebe
Research Institute for Mathematical Sciences, Kyoto University
Kyoto 606-8502, Japan

The energy representation of a gauge group on a Riemannian manifold has been discussed by several authors. In Ref. 12, Y. Shimada has shown the irreducibility with the use of White Noise Analysis, where the compactness of the Riemannian manifold is assumed. In this paper we extend its technique to the noncompact case.

1 Introduction

The energy representation of a gauge group on a Riemannian manifold has been discussed by several authors, for instance, in Refs. 1, 2, 4, 5, 7 and 13. Their methods are essentially to reduce the problem to the estimate of the support of a Gaussian measure in an infinite dimensional space. The best result in this direction seems to be the one in Ref. 13. After all these contributions, the irreducibility in two dimensions has remained unsettled yet. One of the difficulties is the conformal invariance of the energy representation, i.e., when we transform the Riemannian metric \( g(x) \) into \( e^{\rho(x)} g(x) \), the energy representation remains unchanged. For a historical survey of this line of research, we refer the reader to Ref. 1.

In contrast to the above, Y. Shimada has recently shown the irreducibility of a gauge group on a compact Riemannian manifold in Ref. 12, applying White Noise Analysis. Unfortunately, there is a mistake in the proof of Lemma 4.6 in Ref. 12. Shimada has used the relation \( d\Phi_{s,t} = \beta(\exp(\Phi_{s,t})) \) in the proof of the equation (4.28), but this relation does not hold when the Lie group \( G \) is non-abelian.

The author could not find a way to overcome this mistake. However, Shimada’s approach is still of importance when we want an analysis of white noise indexed by manifolds, or an analysis of the energy representation. Hence we extend this technique to include some class of noncompact manifolds in the presence of a weight function.

2 Preliminaries

2.1 Notation

In this section, we explain the notation frequently used throughout this paper.
• $(M, g)$ denotes a Riemannian manifold $M$ equipped with a Riemannian metric $g$

• $\nabla$ denotes the Levi-Civita connection on $(M, g)$

• $dv = \sqrt{|g|} dx$ is the Riemannian measure on $(M, g)$

• $G, g$ denote a compact, semisimple Lie group and its Lie algebra, respectively

• $B(\cdot, \cdot)$ means the Killing form of $g$

• $N := \{0, 1, 2, 3, \cdots\}$

• $\Gamma_c(T^∗M) := \text{the set of all smooth sections of the cotangent bundle on } M \text{ with supports compact}$

• $C^\infty_c(M) := \text{the set of all smooth real-valued functions with supports compact}$

• $\langle \cdot, \cdot \rangle_x$ is the natural bilinear form induced by $g_x$ or $g_x \otimes (-B)$ on tensor products of tangent and cotangent spaces at $x$, and Lie algebra $g$, depending on the context. When the complexification $g_C$ is considered, $\langle \cdot, \cdot \rangle_x$ is the natural inner product which is antilinear in the left and linear in the right

• $\langle \cdot, \cdot \rangle_0$ is an inner product on $\Gamma_e(T^∗M)$ or $\Gamma_c(T^∗M) \otimes g_C$ determined by $\langle f, g \rangle_0 := \int_M \langle f, g \rangle_x dv(x)$

• for each $n \geq 1$, $\nabla^* : \Gamma_c(T^∗M^{\otimes n}) \longrightarrow \Gamma_c(T^∗M^{\otimes n-1})$ is the adjoint operator of $\nabla$ with respect to the inner product $\langle \cdot, \cdot \rangle_0$

• for each $n \geq 0$, $\Delta := -\nabla^* \nabla$ is called the Bochner Laplacian on $\Gamma_c(T^∗M^{\otimes n})$

• $|\omega|_x := \langle \omega, \omega \rangle_x^{1/2}$

• $C^\infty_b(M) := \{ h \in C^\infty(M); \text{sup}_{x \in M} |(\nabla^m h)(x)|_x < \infty \text{ for all } m \in \mathbb{N} \}$

• $\Gamma_b(X)$ denotes the boson Fock space on $X$, where $X$ is a Hilbert space

• $\mathcal{L}(F_1, F_2)$ is the set of all continuous linear operators from a topological vector space $F_1$ to a topological vector space $F_2$

2.2 White Noise Analysis

We explain White Noise Analysis needed in this paper. Let $X$ be a complex Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle_0$ and $H$ be a self-adjoint operator defined on a dense domain $D(H)$ in $X$. Assume that $H$ has $\{\lambda_j\}_{j=1}^{\infty}$ as eigenvalues, and $\{e_j\}_{j=1}^{\infty}$ as corresponding eigenvectors.

Hypothesis

• $\{e_j\}_{j=1}^{\infty}$ is a CONS of $X$

• $1 < \lambda_1 \leq \lambda_2 \leq \cdots \nearrow \infty$
Then we can construct a nuclear countably Hilbert space as follows (for details, the reader is referred to Ref. 9). For \( p \in \mathbb{R} \), we can define an inner product \( \langle x, y \rangle_p := \langle H^p x, H^p y \rangle_0 \) on \( D(H^p) \). Then \( D(H^p) \) becomes a Hilbert space, which we write as \( E_p \). Let \( E := \cap_{p \geq 0} E_p \) be a nuclear countably Hilbert space equipped with the projective limit topology and let \( E^* \) be its dual with the strong dual topology. Thus we obtain a Gelfand triple \( E \subset X \subset E^* \). In the same way, we construct a Gelfand triple \( (E) \subset \Gamma_b(X) \subset (E)^* \) in terms of the self-adjoint operator \( \Gamma_b(H) \).

2.3 The energy representation of a gauge group

First we define an inner product \( \langle f, g \rangle_{\rho,0} := \int_M \langle f, g \rangle_x e^{\rho(x)} dv \) with \( \rho \in C^\infty(M) \) for \( f, g \in \Gamma_c(T^* M) \) or \( \Gamma_c(T^* M) \otimes g_C \). We simply write \( \langle f, g \rangle_0 \) for \( \rho = 0 \) in accordance with the notation in section 2.

Let \( \mathcal{H}(M; g_C)_\rho \) be the completion of the space \( \Gamma_c(T^* M) \otimes g_C \) by the inner product \( \langle \cdot, \cdot \rangle_{\rho,0} \). This space is physically the one-particle state space.

For \( \psi \in C^\infty_c(M; G) \), the right logarithmic derivative \( \beta(\psi) \in \Gamma_c(T^* M) \otimes g_C \) is defined as

\[
(\beta(\psi))(x) := d\psi_x \psi(x)^{-1} = R_{\psi(x)^{-1}}d\psi_x.
\]

\( \beta \) satisfies

\[
\beta(\psi\phi) = V(\psi)\beta(\phi) + \beta(\psi).
\]

The latter equality is said to be the Maurer-Cartan cocycle condition.

For \( \psi \in C^\infty_c(M; G) \) and \( f \in \mathcal{H}(M; g_C)_\rho \), let

\[
(V(\psi)f)(x) := [id_{T_x M} \otimes \text{Ad}(\psi(x))]f(x), \quad x \in M,
\]

then \( V \) is a unitary representation of the gauge group \( C^\infty_c(M; G) \) on the Hilbert space \( \mathcal{H}(M; g_C)_\rho \).

Let \( U \) be a unitary representation of the gauge group on the boson Fock space \( \Gamma_b(\mathcal{H}(M; g_C)_\rho) \) determined by

\[
U(\psi) \exp(f) := \exp \left( -\frac{1}{2} |\beta(\psi)|^2_{\rho,0} - \langle \beta(\psi), V(\psi)f \rangle_{\rho,0} \right) \exp (V(\psi)f + \beta(\psi))
\]

for \( f \in \mathcal{H}(M; g_C)_\rho \) and \( \psi \in C^\infty_c(M; G) \). We call this representation the (weighted) energy representation or, if we emphasize the weight function, the energy representation with the weight function \( \rho \).

It is important that this representation is, as easily checked, not a projective representation since the Maurer-Cartan cocycle \( \beta \) is real.

**Note.** As we stated in Introduction, the energy representation is conformally invariant in two dimensions. This is understood as follows. Let \( M \) be a \( d \)-dimensional Riemannian manifold. If the Riemannian metric \( g \) is transformed into \( e^\phi g \), \( dv \) and \( \langle \cdot, \cdot \rangle_x \) on \( T^*_x M \) are transformed correspondingly:

\[
dv \rightarrow e^{\frac{\phi}{2}} dv
\]

\[
\langle \cdot, \cdot \rangle_x \rightarrow e^{-\phi(x)} \langle \cdot, \cdot \rangle_x.
\]

Hence, the inner product \( \langle \cdot, \cdot \rangle_{\rho,0} \) remains invariant if and only if \( d = 2 \). Because of the existence of this conformal invariance in two dimensions, the proof of irreducibility is difficult. The details are in Refs. 2 and 13.
3 Several conditions for a self-adjoint operator

In the following, we show several conditions in order to use White Noise Analysis on a Riemannian manifold. For this purpose we introduce a function $W$ which tends to infinity in infinite distances. This function and approximately constant functions make the manifold behave as if it is compact. Here the phrase “as if it is compact” means that we can use constant functions, which will be shown in propositions 2 and 3. We prove in Theorem 1 that the energy representation is irreducible for a Riemannian manifold diffeomorphic to some Riemannian manifold equipped with such a function $W$ and approximately constant functions.

Let $M$ be a Riemannian manifold equipped with a Riemannian metric $g$ and $W$ be a positive smooth function. Let $L^2(T^*M)$ denote the completion of the space $\Gamma_c(T^*M)$ with respect to the norm induced by $g$. Note that the quadratic form $Q(f,f) = \int_M (|\nabla f|^2 + W|f|^2) dv(x)$ with domain $Q = \{ f : Q(f,f) < \infty \}$ is a nonnegative, symmetric closed form. Hence there is a self-adjoint operator denoted by $H = -\Delta + W$ such that $Q(f,g) = \langle Hf, g \rangle_0$. First we consider the following condition on $(M,g)$ and $W$.

(a) $W \in C^\infty(M)$, $W \geq 1$;

the spectrum of $H = -\Delta + W$ is discrete (denoted by $\{ \lambda_n \}_{n=1}^\infty$) and satisfies $1 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \uparrow \infty$;

there exists $p \geq 0$ such that $H - p$ belongs to the Hilbert-Schmidt class.

The condition (a) suffices for compact manifolds. Using this, we can introduce the family of seminorms $\{| \cdot |_p \}_{p\geq 0}$ defined on $\Gamma_c(T^*M)$ (see section 2). In order to deal with noncompact manifolds, however, (a) is not sufficient. Below we introduce a few more conditions. Here we define an important family of seminorms $\{| \cdot |_m \}_{m \in \mathbb{N}}$ defined by $|f|'_m = \sum_{n=0}^m |W^m \nabla^n f|_0$, $f \in \Gamma_c(T^*M)$.

(b) the two families of seminorms $\{| \cdot |_p \}_{p\geq 0}$ and $\{| \cdot |_m \}_{m \in \mathbb{N}}$ define the same topology on $E$;

(c) there exists a sequence $\{ \psi_n \}_{n=1}^\infty$ of smooth functions with supports compact, which enjoys the following properties:

- $\psi_n(x) \to 1$ as $n \to \infty$, for all $x \in M$,
- for every $m \in \mathbb{N}$ there exists $C = C(m)$ independent of $n$ such that $\sup_{x \in M} |\nabla^m \psi_n(x)|_x \leq C(m)$ for all $n \geq 1$.

(b) implies that the space $\{ f \in L^2(T^*M); W^n \nabla^m f \in L^2(\otimes^m T^*M) \}$ for all $n, m \in \mathbb{N}$ coincides with the space $E$. (b) is true if the Riemannian manifold $M$ is $\mathbb{R}^d$ or compact, with the function $W$ taken as $|x|^2 + 1$ and $2$ respectively. Here $\Delta$ means the Bochner Laplacian $-\nabla^* \nabla$. For a proof of the compact case, we refer the reader to Ref. 11. The Euclidean case is well known. However, for convenience and in order to understand the reason why the condition (b) is nontrivial in a general Riemannian manifold, we prove this fact for the Euclidean case. This fact for the one-dimensional case can be seen in Ref. 10 without a proof.
Let \( A_j := \frac{1}{\sqrt{2}} \left( x_j + \frac{\partial}{\partial x_j} \right) \), \( A_j^* := \frac{1}{\sqrt{2}} \left( x_j - \frac{\partial}{\partial x_j} \right) \), \( N_j := A_j^* A_j \) and \( N := \sum_{j=1}^d N_j \). It holds that \( [A_j, A_k^*] = \delta_{jk} \) and \( -\Delta + |x|^2 + 1 = 2N + d + 1 \). Let \( A_j^\pm \) denote either \( A_j \) or \( A_j^* \), and let \( W = |x|^2 + 1 \). We show that there is some \( C = C(m) > 0 \) such that for \( f \in C_c^\infty(\mathbb{R}^d) \) and \( j_1, \ldots, j_m \in \{1, 2, \ldots, d\} \),

\[
|A_{j_1}^{\pm} \cdots A_{j_m}^{\pm} f|_0 \leq C(m)(2N + d + 1)^{\frac{m}{2}} |f|_0. \tag{7}
\]

The proof of (7) results from the canonical commutation relations. Once (7) is proved, the relations \( x_j = \frac{A_j + A_j^*}{\sqrt{2}} \) and \( \frac{\partial}{\partial x_j} = \frac{A_j - A_j^*}{\sqrt{2}} \) lead to the validity of condition (b).

The above argument depends on the properties special to the number operator and creation, annihilation operators on \( \mathbb{R}^d \). On a general Riemannian manifold, we do not know how to verify (b) (under some mild condition on the manifold), even if the function \( W \) is found to satisfy the condition (a).

**Proof of (7).** We show (7) by an example.

\[
|A_1 A_1^* A_2 A_2 f|_0^2 = \langle A_1 A_1^* A_1 A_1^* A_2 A_2 f, f \rangle_0
\]

\[
= \langle A_1 A_1^* A_1 A_2 A_2 A_2 f, f \rangle_0 + \langle A_1 A_1^* A_1^* A_2 A_2 A_2 f, f \rangle_0
\]

\[
= \langle A_1 A_1^* A_1 A_1^* A_2 A_2 A_2 f, f \rangle_0 + \langle A_1 A_1^* A_1^* A_2 A_2 A_2 f, f \rangle_0
\]

\[
+ \langle A_1 A_1^* A_1^* A_2^* A_2 A_2 f, f \rangle_0 + \langle A_1^* A_1^* A_2 A_2^* A_2 f, f \rangle_0
\]

\[
= \langle A_1 A_1^* A_1 A_2 A_2 A_2 f, f \rangle_0 - \langle A_1 A_1^* A_1 A_2 A_2 f, f \rangle_0
\]

\[
+ \langle A_1^* A_1^* A_2 A_2 A_2 f, f \rangle_0 - 2\langle A_1 A_1^* A_2 A_2 A_2 f, f \rangle_0
\]

\[
+ \langle A_2 A_2^* A_2 A_2 f, f \rangle_0 - \langle A_2^* A_2 A_2 f, f \rangle_0
\]

\[
= \langle N_1^2 N_2^2 f, f \rangle_0 - \langle N_1^2 N_2 f, f \rangle_0 + 2\langle N_1 N_2^2 f, f \rangle_0
\]

\[
- 2\langle N_1 N_2 f, f \rangle_0 + \langle N_2 f, f \rangle_0 - \langle N_2 f, f \rangle_0
\]

\[
\leq \langle N_1^2 N_2^2 f, f \rangle_0 + \langle N_1^2 N_2 f, f \rangle_0 + 2\langle N_1 N_2^2 f, f \rangle_0
\]

\[
+ 2\langle N_1 N_2 f, f \rangle_0 + \langle N_2 f, f \rangle_0 + \langle N_2 f, f \rangle_0
\]

\[
\leq (2N + d + 1)^2 f_0^2
\]

\[
= |(2N + d + 1)^2 f|_0^2.
\]

**Remark.** If we replace the Schrödinger operator with an elliptic operator, propositions and theorems in this section still hold. However, for simplicity, we restrict ourselves to Schrödinger operators.

An example of manifolds where the condition (c) fails to hold is \( \mathbb{R}^2 \setminus (0, 0) \). In fact, if we try to make a sequence \( \{\psi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^2 \setminus (0, 0)) \) which tends to 1 pointwise, then derivatives of \( \psi_n \) tend to infinity near \( (0, 0) \) as \( n \) tends to infinity.

Next we consider the weighted representation case. Put \( \nabla_\rho := e^{-\frac{\rho}{2}} \circ \nabla \circ e^{\frac{\rho}{2}} \),
then we have $\nabla_p^* = e^{-\hat{\Psi}} \circ \nabla^* \circ e^{\hat{\Psi}}$, in fact for all $f, g \in \Gamma_c(T^*M)$,

$$\langle f, \nabla_p^* g \rangle_{\rho,0} = \langle \nabla_p f, g \rangle_{\rho,0}$$

$$= \int_M \langle \nabla_p f, g \rangle_x e^{\rho(x)} \, dv$$

$$= \int_M \langle e^{-\hat{\Psi}} \nabla (e^{\hat{\Psi}} f), g \rangle_x e^{\rho(x)} \, dv$$

$$= \int_M \langle \nabla (e^{\hat{\Psi}} f), e^{\hat{\Psi}} g \rangle_x \, dv$$

$$= \int_M \langle (e^{\hat{\Psi}} f), \nabla^* (e^{\hat{\Psi}} g) \rangle_x \, dv$$

$$= \langle f, e^{-\hat{\Psi}} \nabla^* (e^{\hat{\Psi}} g) \rangle_{\rho,0},$$

where $*$ on the left and right hand sides mean adjoints in $\Gamma_c(T^*M)_\rho$ and $\Gamma_c(T^*M)$ respectively. Let $H_\rho := e^{-\hat{\Psi}} \nabla_\rho + W$ and $\epsilon_{\rho,n} := e^{-\hat{\Psi} \epsilon_n}$, then we have $H_\rho^P \epsilon_{\rho,n} = \lambda_n \epsilon_{\rho,n}$. Correspondingly, $| \cdot |_p$ and $| \cdot |'_{m}$ are replaced with $|f|_{\rho,p} := |H_\rho^P f|_{\rho,0}$ and $|f|'_{\rho,m} := \sum_{n=m}^{\infty} |W^m \nabla^n f|_{\rho,0}$ respectively. With the above replacements, we can prove the following properties easily:

(a') the spectrum of $H_\rho = \nabla_\rho^* \nabla_\rho + W$ is $\{ \lambda_n \}_{n=1}^{\infty}$ and there exists $p \geq 0$ such that $H_\rho^P$ belongs to the Hilbert-Schmidt class,

(b') the two families of seminorms $\{ | \cdot |_{\rho,p} \}_{p \geq 0}$ and $\{ | \cdot |'_{m} \}_{m \in \mathbb{N}}$ define the same topology on $E_\rho$,

where $E_\rho$ is defined in the same way as $E$. (a') is obvious. The remaining (b') is easily checked; for instance,

$$|W^m \nabla^n f|_{\rho,0} = |e^{-\hat{\Psi}} W^m \nabla^n (e^{\hat{\Psi}} f)|_{\rho,0}$$

$$= |W^m \nabla^n (e^{\hat{\Psi}} f)|_0$$

$$\leq C|e^{\hat{\Psi}} f|_p \quad (\exists C > 0, \exists p \geq 0)$$

$$= C|H_\rho^P f|_{\rho,0}$$

$$= C|f|_{\rho,p}.$$

The next theorem is essentially due to Ref. 12, but the proof is changed slightly in the present case. This theorem allows us to differentiate the representation of the gauge group.

**Theorem 1.** Let $\rho$ be a smooth function on $M$. Assume that the conditions (a) and (b) hold. Let $\psi_t(x) := \exp(t \Psi(x))$ for $\Psi \in C_0^\infty(M; g)$. Then $\{ V(\psi_t) \}_{t \in \mathbb{R}}$ is a regular one-parameter subgroup of $GL(E_\rho)$, namely, for any $p \geq 0$ there exists $q \geq 0$ such that

$$\lim_{t \to 0} \sup_{f \in E_\rho, |f|_{\rho,q} \leq 1} \frac{|V(\psi_t)f - f|}{t} \to 0,$$

where

$$(V'(\Psi)f)(x) := [\text{id}_{T_x M} \otimes \text{ad}(\Psi(x))] f(x), \quad x \in M, \quad f \in E_\rho.$$
Proof. We only prove the case $\rho = 0$. The proof for nonzero $\rho$ is the same. By the condition (b), it is sufficient to prove for seminorms $| \cdot |'_{m}, m \in \mathbb{N}$. First we show that for $\Psi \in C_{c}^{\infty}(M; g)$ and $m \in \mathbb{N}$, there exists $C = C(\Psi, m) > 0$ such that

$$|V'(\Psi)f|'_{m} \leq C(\Psi, m)|f|'_{m}, \quad f \in E_{\rho}. \quad (13)$$

The above inequality is obtained by the direct calculation. We frequently use the same $C$ or $C(\Psi, m)$ in different lines.

$$|V'(\Psi)f|'_{m} = \sum_{n=0}^{m} |W^{m}\nabla^{n}(\Psi f - f\Psi)|'_{0} \leq C(m) \sum_{n=0}^{m} \sum_{k=0}^{n} (|W^{m}\nabla^{k}\Psi\nabla^{n-k}f|'_{0} + |W^{m}\nabla^{n-k}f|^{k}|\Psi|'_{0}) \leq C(\Psi, m) \sum_{n=0}^{m} \sum_{k=0}^{n} (|W^{m}\nabla^{n-k}f|'_{0} + |W^{m}\nabla^{n-k}f|'_{0}) \leq C(\Psi, m)|f|'_{m}. \quad (14)$$

In particular, $V'(\Psi)$ belongs to $L(E_{\rho}, E_{\rho})$. Hence,

$$V(\psi_{t})f(x) = [\text{id}_{T_{x}M} \otimes \text{Ad}(\exp(t\Psi(x)))]f(x) = \sum_{k=0}^{\infty} \left[ \frac{1}{k!} (t\mathcal{V}(\Psi))^{k} f \right](x),$$

and it holds that

$$\left| \frac{V(\psi_{t})f - f}{t} - V'(\Psi)f \right|'_{m} \leq t \sum_{k=2}^{\infty} \frac{1}{k!} t^{k-2} |V'(\Psi)^{k}f|'_{m} \leq t \sum_{k=2}^{\infty} \frac{1}{k!} t^{k-2} C(\Psi, n)^{k} |f|'_{m} \leq t \exp(C(\Psi, m)) |f|'_{m}. \quad (15)$$

Then the conclusion of the proposition follows immediately, q.e.d.

The next result is useful in the analysis of energy representation. This is a consequence of the conditions (b) and (c).

**Proposition 2.** Let $\rho$ be a smooth function on $M$. Let $\Psi$ be an element in $C_{b}^{\infty}(M; g)$. The operator

$$f \mapsto V'(\Psi)f$$

belongs to $L(E_{\rho}, E_{\rho})$ and there exists a sequence $\{\Psi_{n}\}_{n=1}^{\infty}$ of $g$-valued smooth functions with supports compact such that
\[ V'(\Psi - \Psi_n)f \rightarrow 0 \text{ in } E_\rho \text{ as } n \rightarrow \infty, \text{ for all } f \in E_\rho. \]

**Remark.** Proposition 2 enables us to make use of constant functions in \( C_0^\infty(M; \mathfrak{g}) \), which are useful in calculations of commutants of the representation.

**Proof.** Again we prove only for \( \rho = 0 \). Let \( \Psi \) be a fixed element in \( C_0^\infty(M; \mathfrak{g}) \) and \( \{\psi_n\}_{n=1}^\infty \) be the sequence in the condition (c). We define \( \Psi_n := \psi_n \Psi \in C_c^\infty(M; \mathfrak{g}) \).

\[
|V'(\Psi - \Psi_n)f|_p = |H^p((\Psi - \Psi_n)f - f(\Psi - \Psi_n))|_0 \\
\leq |H^p((\Psi - \Psi_n)f)|_0 + |H^p(f(\Psi - \Psi_n))|_0 \\
= \langle H^{2p}((\Psi - \Psi_n)f), (\Psi - \Psi_n)f \rangle_0^{\frac{1}{p}} + \langle H^{2p}(f(\Psi - \Psi_n)), f(\Psi - \Psi_n) \rangle_0^{\frac{1}{p}} \\
\leq \langle (\Psi - \Psi_n)f\rangle_0^{\frac{1}{2p}} + \langle f(\Psi - \Psi_n)\rangle_0^{\frac{1}{2p}}.
\]

Schwarz’s inequality was used in the last line. Remembering that there exist \( C > 0 \) and \( m \in \mathbb{N} \) such that \( |h|^{2p} \leq C|h'|_m \) for all \( h \in E_\rho \) and, by the condition (c), that for every \( l \in \mathbb{N} \) there exists \( C = C(l) \) independent of \( n \) such that

\[
\sup_{x \in M} |\nabla^l(\Psi - \Psi_n)(x)|_x \leq C(l)
\]

for all \( n \geq 1 \), we have

\[
|V'(\Psi - \Psi_n)f|_p \leq C\langle (\Psi - \Psi_n)f \rangle_0^{\frac{1}{2p}} + C\langle f(\Psi - \Psi_n)\rangle_0^{\frac{1}{2p}} \\
= C\left( \sum_{k=0}^m |W^m\nabla^k((\Psi - \Psi_n)f)|_0 \right)^{\frac{1}{2p}} \langle (\Psi - \Psi_n)f \rangle_0^{\frac{1}{2p}} \\
+ C\left( \sum_{k=0}^m |W^m\nabla^k(f(\Psi - \Psi_n))|_0 \right)^{\frac{1}{2p}} \langle f(\Psi - \Psi_n) \rangle_0^{\frac{1}{2p}} \\
\leq C\langle (\Psi - \Psi_n)f \rangle_0^{\frac{1}{2p}} + C\langle f(\Psi - \Psi_n)\rangle_0^{\frac{1}{2p}}.
\]

(17)

Applying Lebesgue’s bounded convergence theorem, we get the desired result, q.e.d.

### 4 Summary

We have considered how to apply White Noise Analysis to the energy representation of a gauge group. An interesting, expected development in the future is a construction of a function \( W \) on a general Riemannian manifold.

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