Entanglement generation in a double-Λ system

Ling Zhou, Yong-Hong Ma and Xin-Yu Zhao

School of Physics and Optoelectronic Technology, Dalian University of Technology, Dalian 116024, People Republic of China

Received 19 April 2008, in final form 26 August 2008
Published 13 October 2008
Online at stacks.iop.org/JPhysB/41/215501

Abstract
In this paper, we study the generation of entanglement in a double-Λ system. Employing a standard method of laser theory, we deduce the dynamic evolution equation of the two-mode field. We analyse the available entanglement criterion for the double-Λ system and the condition of entanglement existence. Our results show that, under proper parameters, the two-mode field can be entangled and amplified.

1. Introduction
Continuous variables entanglement (CVE), as an entanglement resource, has attracted much attention, because CVE not only has advantages in quantum-information science [1], but also can be prepared unconditionally, whereas the preparation of discrete entanglement usually relies on an event selection via coincidence measurements. Conventionally, CVE has been produced by nondegenerate parametric down-conversion (NPD) [2]. In order to improve the strength of the NPD, engineering the NPD Hamiltonian within cavity QED has also attracted much attention [3–5]. Besides NPD [2, 6, 7], Xiong et al [8] had shown that a two-photon correlated spontaneous emission laser can work as a CVE producer and amplifier, which opens a new and attractive research domain. A number of different schemes have been proposed [9–14]. Different from the gain medium atoms in [8–14], Lü et al [15] have studied a single-molecular-magnets system to produce CVE, where the physics process is similar to [11]. All of these works deal with a similar physics process where each of the two modes will create (annihilate) a photon in one loop respectively (similar to the NPD system).

In this paper, we propose a scheme to generate CVE where one mode creates a photon and the other annihilates one, which is different from [8–15]. The system consists of atoms in double-Λ configuration interacting with two-mode cavity fields. The atoms are driven into a coherent state of the upper two levels by two classical fields. We obtain the master equation of the two-mode fields. Through analysis of entanglement, we find that the criterion proposed in [16] can be used to judge entanglement. We show that in the double-Λ system, entanglement exists on the condition that the two-mode quantum field is tuned away from the atomic transition, and the initial field is in a quantum state. The strength of the Rabi frequency of the classical driving field determines the time region in which two-mode fields are entangled. Our study is helpful in understanding the entanglement characteristic within a system where the quantum field is in the ‘V’ configuration.

2. The model and theoretical calculation
We consider a system of atoms in a double-Λ configuration shown in figure 1. Two cavity fields interact with atomic transitions $|a⟩ ↔ |c⟩$ and $|b⟩ ↔ |c⟩$ with detunings $Δ_1$ and $Δ_2$, respectively. The two classical driving fields with Rabi frequencies $Ω_2$ and $Ω_1$ drive the atomic level between $|a⟩ ↔ |d⟩$ and $|b⟩ ↔ |d⟩$ with detunings $Δ_1$ and $Δ_2$, respectively. Our double-Λ system can be sodium atoms in a vapour cell [17], where the lower states are two hyperfine levels $|F = 1⟩$ and $|F = 2⟩$ of $3^2S_{1/2}$, and the upper states are $|F = 1⟩$ and $|F = 2⟩$ of $3^2P_{1/2}$, respectively. The double-Λ system also can be atomic Pb vapour [18]. The phase-dependent electromagnetically induced transparency [17] and efficient nonlinear frequency conversion [18] have been investigated experimentally in the double-Λ system. Chong and Soljačić [19] studied dark-state polaritons in a double-Λ system. Here, we are interested in producing a two-mode entangled laser via the double-Λ system. In the interaction picture, the Hamiltonian of the system can be written as

$$H = H_0 + H_1,$$  (1)

where

$H_0$:

$|a⟩⟨a| - |c⟩⟨c| + |b⟩⟨b| - |d⟩⟨d|$,  (2)

$H_1$:

$$Ω_1(|a⟩⟨d| e^{−iΔ_1 t} + |d⟩⟨a| e^{iΔ_1 t} + |b⟩⟨d| e^{−iΔ_2 t} + |d⟩⟨b| e^{iΔ_2 t} + |c⟩⟨c| e^{−i(Δ_1 + Δ_2) t} + |c⟩⟨c| e^{i(Δ_1 + Δ_2) t},$$

with $Ω_1$ the Rabi frequency and $Δ_1, Δ_2$ the detunings.
We can see that the master equation has the term $\rho a_2 a_1^\dagger - a_1^\dagger a_2 \rho$, which means that one mode creates a photon and the other mode annihilates a photon. The detailed deduction of the equation is given in appendix A. In equation (5), we have included the loss of the two-mode cavity with loss rates $\kappa_1$ and $\kappa_2$. The coefficients are

$$
\begin{align*}
\alpha_1 &= \frac{g_1^2}{D}[A_{11} L_{bb} + A_{21} L_{ab} + A_{31} L_{db}], \\
\alpha_2 &= \frac{g_2^2}{D}[A_{12} L_{ba} + A_{22} L_{aa} + A_{32} L_{da}], \\
\alpha_{12} &= \frac{g_1 g_2}{D}[A_{11} L_{ba} + A_{21} L_{aa} + A_{31} L_{db}], \\
\alpha_{21} &= \frac{g_1 g_2}{D}[A_{12} L_{ab} + A_{22} L_{aa} + A_{32} L_{db}], \\
\end{align*}
$$

where $D = \{(\gamma - i\Delta_a)(\gamma - i\Delta_b)[\gamma - i(\Delta + \frac{2\Delta_1}{2} + \Delta_2)] + \Omega_1^2(\gamma - i\Delta_a) + \Omega_2^2(\gamma - i\Delta_b)$, and

$$
\begin{align*}
L_{aa} &= -i\Omega_2 \gamma y_3, \\
L_{bb} &= \frac{-i\Omega_1}{\gamma} y_2, \\
L_{ab} &= \frac{\gamma + i(\Delta_2 - \Delta_1)}{2\gamma} y_1 - i\Omega_2 \frac{y_2}{2\gamma} - \frac{-i\Omega_1}{2\gamma} y_3, \\
L_{db} &= \frac{-i\Delta_2}{2\gamma} y_2 - \frac{i\Omega_1}{2\gamma} y_1, \\
L_{da} &= \frac{-i\Delta_1}{2\gamma} y_3 + \frac{i\Omega_1}{2\gamma} y_1, \\
\end{align*}
$$

and

$$
\begin{align*}
A_{11} &= (\gamma - i\Delta_a) \left[ \gamma - i \left( \Delta + \frac{\Delta_1}{2} \right) \right], \\
A_{12} &= A_{21} = -i\Omega_1 \Omega_2, \\
A_{31} &= -i\Omega_1 (\gamma - i\Delta_a), \\
A_{32} &= -i\Omega_2 (\gamma - i\Delta_b), \\
\end{align*}
$$

with

$$
\begin{align*}
y_2 &= \frac{2i r_m s (a_1 \Omega_2 - b_2 \Omega_1)}{a_1 a_2 - b_2^2}, \\
y_3 &= \frac{2i r_m s (a_2 \Omega_1 - b_1 \Omega_2)}{a_1 a_2 - b_2^2}, \\
y_1 &= \frac{\Omega_2 (\Delta_1 - 2\Delta_2) y_2 + \Omega_1 (2\Delta_1 - \Delta_2) y_3}{s}, \\
\end{align*}
$$

in which

$$
\begin{align*}
s &= \gamma^2 + \Omega_1^2 + \Omega_2^2 + (\Delta_2 - \Delta_1)^2, \\
a_1 &= M_1 s - \Omega_2^2 (2\Delta_2 - \Delta_1)^2, \\
a_2 &= M_2 s - \Omega_1^2 (2\Delta_1 - \Delta_2)^2, \\
b &= \Omega_1 \Omega_2 [3s - (\Delta_1 - 2\Delta_2) (2\Delta_1 - \Delta_2)] \\
M_1 &= \gamma^2 + 4\Omega_1^2 + \Omega_2^2 + \Delta_2^2, \\
M_2 &= \gamma^2 + 4\Omega_1^2 + \Omega_2^2 + \Delta_1^2. \\
\end{align*}
$$

Although our four-level atom is similar to that in [11, 15], the physical process of the two-mode quantum fields is different, because the two quantum fields work in different atomic levels. In [8–15], each of the two modes will create or annihilate a photon in one loop. So the master equation is of the form $\rho a_1 a_1^\dagger - a_1^\dagger a_1 \rho$, which means that one mode creates a photon and the other mode annihilates a photon. The detailed deduction of the equation is given in appendix A. In equation (5), we have included the loss of the two-mode cavity with loss rates $\kappa_1$ and $\kappa_2$. The coefficients are
the spontaneous emissions from |a⟩ and |b⟩ to |c⟩ take place, entangled photons will be produced.

3. Entanglement criterion choice and the discussion of the entanglement condition

How to determine the entanglement is a key problem. In [8–15], employing the criterion (Δu)² + (Δv)² < 2 [23], an inequality of the sum of the quantum fluctuations of two operators u and v, they find the entanglement between the two-mode fields. However, the criterion inequality of the sum of the quantum fluctuations cannot be applied to measure the coherent state [24]. Although the entanglement criterion on a measure continuous variable has been developed [23–26], we still cannot find a criterion to judge all kinds of continuous variable entanglement. In order to clarify the kind of entanglement existing in our model, we now discuss the analytic solution in our system so as to choose an appropriate entanglement criterion as well as to know the condition of entanglement.

Now we analyse the entanglement condition. If g₁ = g₂, Ω₁ = Ω₂ and Δ = Δ₀ = Δ₁ > Ω₁, Ω₂, γ, through equations (6)–(10), one can obtain the relation a₁ = a₂ = a₁ = a₂ = iα (α is a real number). Usually, the loss of the cavity does not change the entanglement structure of the state. It just destroys the entanglement or sometimes enhances it a little. So, in the choice of entanglement criterion, we omit the loss of the cavity. Therefore, the master equation (5) can be simplified as

\[ \dot{\rho} = i\alpha [a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger + a_1^\dagger a_2 + a_2^\dagger a_1], \]

(11)

The effective Hamiltonian \( H_{\text{eq}} = -\alpha (a_1 a_2^\dagger + a_2 a_1^\dagger) \), does not meet with the criterion (Δu)² + (Δv)² < r² + 1/2 [23], if the initial field state is in the Fock state \(|n_1, n_2\rangle\). We recognize that the field Hamiltonian is the generator of the SU(2) coherent state [27]. The evolution of the state \(|\Psi(0)\rangle\) is

\[ |\Psi(t)\rangle = e^{-iH_{\text{eq}}t}|\Psi(0)\rangle = e^{i\alpha t, a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger} |\Psi(0)\rangle, \]

where \(x_0 = (\cosh \alpha t)^{1/2}, x_e = x_- = \tanh \alpha t, K_+ = a_1, K_0 = a_2, K_- = a_1^\dagger a_2 \). These operators satisfy the SU(2) commutation relations, i.e., [\(K_-, K_+\)] = −2K₀, [\(K_0, K_+\)] = K_+, [\(K_0, K_-\)] = −K_−, with \(K_0 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)\). If the initial field state is the two-mode Fock state \(|0, N\rangle\), then the evolution of the state is

\[ |\Psi(t)\rangle = \cos \alpha t)^{N/2} \sum_{n=0}^{N} \binom{N}{n} \frac{1}{\sqrt{n!}} (i \tan \alpha t)^n |n, N - n\rangle. \]

(12)

From the entanglement definition of the pure state, we know that the state \(|\Psi(t)\rangle\) is an entangled one.

Unfortunately, the Hamiltonian \( H_{\text{eq}} \) cannot entangle the initial coherent state, because the evolution of the system is

\[ |\Psi(t)\rangle = e^{i\alpha t, a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger} |\beta_1, \beta_2\rangle = |\tilde{\beta}_1, \tilde{\beta}_2\rangle, \]

(13)

with \(\tilde{\beta}_1 = \beta_1 \cos \alpha t + i\beta_2 \sin \alpha t, \tilde{\beta}_2 = \beta_2 \cos \alpha t + i\beta_1 \sin \alpha t\). So, it is not entangled.

The two-mode SU(2) cat state is sub-Poissonian distribution. We recall that the criterion proposed by Hillery and Zubairy [16] can be used for a non-Gaussian state. The criterion says, if

\[ (N_1N_2) < |\langle a_1a_2^\dagger \rangle|^2, \]

(14)

the two-mode field is entangled. If the field initially is in the Fock state \(|n_1, n_2\rangle\), using the differential equations (B.1)–(B.6) (let \(\kappa = 0\) and \(a_1 = a_2 = a_1 = a_2 = i\alpha \)), we finally obtain

\[ (N_1N_2) - |\langle a_1a_2^\dagger \rangle|^2 = n_1n_2 - \frac{1}{4}(n_1 + n_2 + 2n_1n_2) \sin^2 2\alpha. \]

(15)

The maximum value of \(\sin^2 2\alpha \) is 1; therefore, if

\[ 2n_1n_2 < n_1 + n_2, \]

(16)

the two-mode field will be entangled. Because \(n_1\) and \(n_2\) are integers, in order to meet with \(2n_1n_2 < n_1 + n_2\), the numbers \(n_1\) and \(n_2\) should not be equal. If either \(n_1\) or \(n_2\) is zero (the state is a standard SU(2) coherent state), we can see that \(\langle N_1N_2 \rangle - |\langle ab^\dagger \rangle|^2\) is always less than zero; thus, we say that the state is entangled. Therefore, the criterion equation (14) can be used to judge entanglement within our system.

However, for the resonant case \((\Delta_0 = \Delta_0 = \Delta = 0)\), if \(\gamma_b = \gamma_a\), \(g_1 = g_2\) and \(\Omega_1 = \Omega_2\), the coefficients \(a_1 = a_2 = a_1 = a_2 = \beta\) (real number). For the initial state \(|n_1, n_2\rangle\), after complicated calculation employing equations (B.1)–(B.6) for \(a_1 = a_2 = a_1 = a_2 = \beta\), we have

\[ (N_1N_2) - |\langle a_1a_2^\dagger \rangle|^2 = \frac{n_1 + n_2}{16} (3e^{βt} + 2e^{βt} - 5) \]

\[ + \frac{n_1n_2}{8} (1 + 6e^{βt} + e^{βt}) + \frac{1}{4} (1 + e^{8βt} - 2e^{6βt}) \geq 0. \]

(17)

If \(n_1\) or \(n_2\) is zero, then \(\langle N_1N_2 \rangle - |\langle a_1a_2^\dagger \rangle|^2\) is equal to zero at initial time. Except that \(\langle N_1N_2 \rangle - |\langle a_1a_2^\dagger \rangle|^2\) is larger than zero. It is obvious that we cannot obtain entanglement in the resonant case. This conclusion is consistent with the work in [12] where the authors show that for a two-level quantum beat laser, entanglement can be created only when the strong driving field is tuned away from the atomic transition.

4. The entanglement of the field

In the above section, we discuss a special case so as to choose the entanglement criterion and clarify the condition of entanglement. Although the above analysis is for the pure state (approximation of master equation (5)), the criterion \(\langle N_1N_2 \rangle < |\langle a_1a_2^\dagger \rangle|^2\) should be available in judging entanglement for the general case. Now, considering the loss of the cavity and the decay of the atomic levels, we numerically solve the differential equations (B.1)–(B.6), and plot the entanglement criterion \(\langle N_1N_2 \rangle - |\langle a_1a_2^\dagger \rangle|^2\) and the \(N_1\langle a_1^\dagger a_1 \rangle, N_2\langle a_1^\dagger a_2 \rangle\).

In figure 2, we plot the case that the initial field state is a number state \(|10, 0\rangle\) when \(\Delta_0 = \Delta_0 > \gamma_b = \gamma_a\) which
number state $|\Omega_1\rangle$. The quantum fields are in ‘V’ form. If the two-mode field also decreases under the large detuning, the entanglement gradually disappears, and photon number of means $|\Delta_1\rangle = |\Delta_2\rangle$. We see that due to the loss of the cavity, the entanglement gradually disappears, and photon number of the two-mode field also decreases under the large detuning case. Of course, if the cavity is ideal, one will observe the entanglement oscillation.

However, with the same detuning $\Delta_0 = \Delta_0$ (when $g_1 = g_2$), the field cannot be amplified if it is entangled, shown in figure 2. The quantum fields are in ‘V’ form. If $\Delta_0 = \Delta_0$, the photon number in two modes only oscillates because of the symmetry. In our numerical simulation, we find that in order to have an amplified-entangled laser, $\Delta_0$ and $\Delta_0$ should be different. For the initial field state in the number state $|1, 0\rangle$, we plot entanglement and average photon number in figures 3 and 4 for several values of $\Omega_1(\Omega_2)$. One can see clearly that entanglement can be obtained without the preparation of atomic coherence before (here, atoms are injected in state $|d\rangle$). By adjusting the values of $\Omega_1(\Omega_2)$, we can adjust the time region of entanglement. Because we inject the atom in atomic state $|d\rangle$, it will need time to evolve into a coherence among the atomic levels $|a\rangle, |b\rangle$ and $|d\rangle$.

![Figure 2](image_url)

**Figure 2.** (a) The time evolution of the entanglement. (b) The time evolution of photon number of the two-mode fields where the red line is for $N_1$ and the blue line is for $N_2$. Initially, the field is in the number state $|10, 0\rangle$. The parameters are $g_1 = g_2 = 1$, $\Delta_0 = \Delta_0 = 50$, $\Delta = 4$, $r_m = 20$, $\gamma = 1$ and $\kappa_1 = \kappa_2 = 0.010$. $\Omega_2 = \Omega_1 = 5$.

![Figure 3](image_url)

**Figure 3.** The time evolution of entanglement. Initially, the atom is in the number state $|1, 0\rangle$. The parameters are $g_1 = g_2 = 1$, $\Delta_0 = 50$, $\Delta_0 = 20$, $\Delta = 10$, $r_m = 20$, $\gamma = 1$ and $\kappa_1 = \kappa_2 = 0.01$. $\Omega_2 = \Omega_1 = 4, 5, 6$ for the dotted, dashed and solid line, respectively.

So, we have no entanglement during the initial short time. With a large value of $\Omega_2(\Omega_2)$, the atoms will acquire their coherence quickly so that the entanglement appears quickly. However, with a large value of $\Omega_1(\Omega_2)$, the photon number also will be amplified quickly, as shown in figure 4. But the photon number in the two modes has a large difference. In our analytic calculation, section 3, we have known that the photon number in the two modes differs. Here, in order to amplify the photon number, the photon number not only should be different but also cannot afford to be very large. With the increase of photon number, the entanglement disappears. But the disentanglement does not result from the loss of the cavity, because we find even for $\kappa = 0$, entanglement also disappears. We conclude that the disentanglement results from the increase of photon number rather than from the loss of the cavity. As is pointed out in [28], in the high-gain limit the condition in equation (14) is no longer able to detect whether there is entanglement in the state.

Now, we show another function of the classical fields, i.e., the ability to overcome the loss of the cavity which is shown in figure 5. Let us compare the dotted line and the solid line. The two lines correspond to the loss rates of the
cavity $k_1 = k_2 = 0.01$ and 0.1, respectively, and all the other parameters are the same. Due to the increase of $k_1(k_2)$, the values of $(N_cN_b - |a_1a_2|^2)$ move up. If $k_1(k_2)$ keeps increasing, we will lose entanglement. However, with the help of classical fields, we still can obtain entanglement even though $k_1(k_2)$ is large, which can be observed by comparing the dashed line and the solid one. Although the loss rate $k_1 = k_2 = 0.1$, through increasing $\Omega_1(\Omega_2)$ to 6, we still have entanglement. Of course, because of the increase of $\Omega_1(\Omega_2)$, the time region moves left, which we analyse in figure 3.

5. Conclusion

In conclusion, we have studied the generation of entanglement in a double-$\Lambda$ system. We derive the theory of this system and analyse the available entanglement criterion for this double-$\Lambda$ system. When the atoms are injected in the ground state $|d\rangle$, the entangled laser can be achieved under the condition of suitable parameters. Due to the classical driving field introduction, we do not need to prepare atomic coherence, and the intensity of the quantum fields will be amplified. The classical fields can overcome the loss of the cavity. Our results show that the time for which the two modes remain entangled depends upon the strength of the Rabi frequency of the classical driving field. Our study is helpful in understanding the entanglement characteristic when the master equation contains the term $\rho_{a2a1} = -a_1\rho_{a2}$ such as the quantum beat laser and Hanle effect laser systems. The scheme produces CVE in a different way from similar NPD [8–15].

In this paper, our study is limited to the initial state $|1, 0\rangle$. One can research other initial field state. The initial field should be easy to obtain. Allowing an excited two-level atom with transition frequency $\nu 1(\nu 2)$ to pass through the vacuum two-mode cavity, when we detect the output atom in the ground state, we will have the field state $|1, 0\rangle$ (or $|0, 1\rangle$).

Acknowledgments

The authors thank Professor M S Zubairy and M Ikram for their critical reading. The project was supported by NSFC under grant no 10774020, and also supported by SRF for ROCS, SEM.

Appendix A. Calculation details of density matrix of a two-mode field

The classical field will be treated to all orders in the Rabi frequency. The transitions $|a\rangle-|c\rangle$ and $|b\rangle-|c\rangle$ are treated fully quantum mechanically but only up to second order in the corresponding coupling constants. By partially tracing the global state of the Schrödinger equation over the atomic variables, we have the formal reduced fields

$$\dot{\rho}_f = -i[\hat{H}_{cb}, \rho_{bc}] + [\hat{H}_{ca}, \rho_{ac}] + [\hat{H}_{ac}, \rho_{ca}] + [\hat{H}_{bc}, \rho_{cb}].$$

(A.1)

Equation (A.1) reveals that we need to obtain the density matrix elements $\rho_{ca}$, $\rho_{bc}$, etc. Inserting the Hamiltonian equation (1) into (A.1), from the Schrödinger equation, we have

$$\dot{\rho}_{bc} = -(\gamma - i\Delta_b)\rho_{bc} - i\Omega_1\rho_{dc} + (-ig_1a_1\rho_{ac} + ig_2\rho_{ba}a_2),$$

$$\dot{\rho}_{ac} = -(\gamma - i\Delta_a)\rho_{ac} - i\Omega_2\rho_{dc} + (ig_1a_1\rho_{ac} - ig_2\rho_{ba}a_2),$$

(A.2)

$$\dot{\rho}_{dc} = -[\gamma - i\Delta_a]\rho_{dc} - i\Omega_1\rho_{bc} - i\Omega_2\rho_{ac} + (ig_1a_1\rho_{ac} + ig_2\rho_{ba}a_2).$$

In the last equations (A.2), we have consider the spontaneous emission of the atomic level. We rewrite it in a matrix form as

$$\dot{\rho} = -M\rho + A,$$

where

$$\rho = \begin{pmatrix} \rho_{bc} & \rho_{ac} \\ \rho_{ac}^* & \rho_{dc} \end{pmatrix},$$

(A.4)

$$M = \begin{pmatrix} (\gamma - i\Delta_b) & i\Omega_1 \\ i\Omega_1 & (\gamma - i\Delta_a) \end{pmatrix},$$

(A.5)

$$A = \begin{pmatrix} ig_1a_1\rho_{ac} + ig_2\rho_{ba}a_2 \\ ig_1a_1\rho_{ac} - ig_2\rho_{ba}a_2 \end{pmatrix}.$$ (A.6)

When we write matrix $A$, we let $\rho_{ac} = 0$ and will explain the reason later. A solution of equation (A.3) which is linear in the coupling constant $g_{1(2)}$ can be obtained [20–22]. Here, we only care for the matrix elements $\rho_{bc}$ and $\rho_{ac}$, so we just write the solution of the two terms as

$$\rho_{bc} = \frac{i}{D}\left[\left(A_{11}\rho_{bb}^0 + A_{21}\rho_{ba}^0 + A_{31}\rho_{bc}^0\right)\rho_{a1} + \left(A_{11}\rho_{ba}^0 + A_{21}\rho_{ab}^0 + A_{31}\rho_{ca}^0\right)\rho_{a2}\right],$$

(A.7)

$$\rho_{ac} = \frac{i}{D}\left[\left(A_{12}\rho_{bb}^0 + A_{22}\rho_{ba}^0 + A_{32}\rho_{bc}^0\right)\rho_{a1} + \left(A_{12}\rho_{ba}^0 + A_{22}\rho_{ab}^0 + A_{32}\rho_{ca}^0\right)\rho_{a2}\right].$$

(A.8)

with

$$A_{11} = (\gamma - i\Delta_a)(\gamma - i\Delta) + \Omega_1^2,$$

$$A_{12} = A_{21} = -\Omega_1\Omega_2,$$

$$A_{31} = -i\Omega_1(\gamma - i\Delta_a),$$

$$A_{32} = -i\Omega_2(\gamma - i\Delta),$$

(A.9)

where $D = (\gamma - i\Delta_a)(\gamma - i\Delta_b)(\gamma - i\Delta) + \Omega_1^2(\gamma - i\Delta_a) + \Omega_2^2(\gamma - i\Delta_b)$ in equations (A.7) and (A.8). As an approximation, the density matrix elements on the right side of equations (A.7) and (A.8) such as $\rho_{bb}^0, \rho_{ba}^0$, etc will be determined by the steady state of classical fields. In other words, the density matrix elements $\rho_{bb}, \rho_{ba}, \rho_{ab}, \rho_{ab}$, etc of classical fields, as a zero-order approximation, are substituted into the right side of equations (A.7) and (A.8). And then, we can obtain a first-order approximation of density matrix elements $\rho_{ca}, \rho_{bc}$ in terms of couplings $g_1(g_2)$.  

5.
Now, we just consider classical fields to determine the zero-order approximation of the density matrix elements $\rho_{0b}$ and $\rho_{0c}$, etc. The differential equations of density matrix elements only with classical fields and atomic decay are

\[
\begin{align*}
\dot{\rho}_{bb} &= -\gamma_s \rho_{bb}^0 - i\Omega (\rho_{bb}^0 - \rho_{bd}^0), \\
\dot{\rho}_{aa} &= -\gamma_s \rho_{aa}^0 - i\Omega (\rho_{aa}^0 - \rho_{da}^0), \\
\rho_{bc} &= \rho_{cb} = -[\gamma - i(\Delta_2 - \Delta_1)]\rho_{0b} + i\Omega_2 \rho_{bd}^0 - i\Omega_1 \rho_{da}^0, \quad \text{(A.10)} \\
\dot{\rho}_{bd} &= -[\gamma + i\Delta_1] \rho_{bd}^0 - i\Omega_2 \rho_{db}^0 - i\Omega_1 (\rho_{bb}^0 - \rho_{dd}^0), \\
\dot{\rho}_{dd} &= -\gamma_s \rho_{dd}^0 - i\Omega_1 (\rho_{db}^0 - \rho_{bd}^0) - i\Omega_2 (\rho_{da}^0 - \rho_{ad}^0) + r_{in}\rho,
\end{align*}
\]

where the atoms are injected into the cavity with a rate $r_{in}$. For $\rho_{bc}$, we have $\rho_{bc}^0 = 0$ (it is the reason why we let $\rho_{bc} = 0$ in equation (A.6)). Substituting the steady state solution of (A.10) into (A.7) and (A.8), we obtain $\rho_{bb}$ and $\rho_{ac}$. And then, inserting $\rho_{bc}$ and $\rho_{ac}$ back into (A.1), one can have the master equation (5) with coefficients (6)–(10).

**Appendix B. Differential equation for various moments**

In order to numerically calculate the entanglement criterion $\langle N_1|N_2 - |\langle a|a\rangle|^2 \rangle$ and the $N_i(N_2)$, we need to deduce a series differential equations for various moments from master equations (5) which are listed below:

\[
\frac{d[a_i^1 a_i^1]}{dr} = (\alpha_1 + \alpha_2^* - 2\kappa_1)|a_i^1 a_i^1| + \alpha_1 + \alpha_2,
\]

\[
\frac{d[a_i^2 a_i^2]}{dr} = (\alpha_1 + \alpha_2^* - \kappa_2 - \kappa_3)|a_i^2 a_i^2| + \alpha_1 \alpha_2 + \alpha_1 \alpha_2^* + \alpha_2 \alpha_2^* + \alpha_1 \alpha_2 + \alpha_2 \alpha_2^* + \alpha_1 \alpha_2 + \alpha_2 \alpha_2^*,
\]

\[
\frac{d[a_i^3 a_i^3]}{dr} = (\alpha_1 + \alpha_2 + 2\kappa_2 - \kappa_3)|a_i^3 a_i^3| + \alpha_1 \alpha_2 (\alpha_1 a_i^2 a_i^2 a_i^2| + \alpha_2 \alpha_2^* (\alpha_2 a_i^2 a_i^2 a_i^2|)
\]

\[
\frac{d[a_i^{1*} a_i^{1*}]}{dr} = 2(\alpha_1^* - \alpha_2 - \kappa_2 - \kappa_3)|a_i^{1*} a_i^{1*}| + \alpha_1 \alpha_2^* a_i^1 a_i^1 + \alpha_2 \alpha_2^* a_i^1 a_i^1,
\]

\[
\frac{d[a_i^1 a_i^{1*}]}{dr} = 2(\alpha_1 + \alpha_2^* - 2\kappa_1)|a_i^1 a_i^{1*}| + \alpha_1 + \alpha_2^* + 6\kappa_1)|a_i^1 a_i^{1*}| + \alpha_1 \alpha_2^* a_i^1 a_i^1 + \alpha_2 \alpha_2^* a_i^1 a_i^1 + 4\kappa_1 + \alpha_1 + \alpha_2^*,
\]

\[
\frac{d[a_i^2 a_i^{1*}]}{dr} = (\alpha_1 + \alpha_2 + 2\kappa_2 - \kappa_3)|a_i^2 a_i^{1*}| + \alpha_1 \alpha_2 (\alpha_1 a_i^2 a_i^1 a_i^1| + \alpha_2 \alpha_2^* (\alpha_2 a_i^2 a_i^1 a_i^1|)
\]

Substituting the subscript 1 (2) with 2 (1) and then making their Hermitian conjugate through (B.1) to (B.5), we can obtain the other seven differential equations. Total 13 differential equations will be a closed set. We can numerically solve it.

**References**

[1] Braunstein S L and van Look P 2005 Rev. Mod. Phys. 77 513
[2] Bouwmeester D, Pan J W and Mattle K et al 1997 Nature 390 575
[3] Serra R M, Villas-Boas C J, de Almeida N G and Moussa M H Y 2005 Phys. Rev. A 71 045802
[4] Zhou L, Xiong H and Zubairy M S 2006 Phys. Rev. A 74 022321
[5] Guzmán R, Retamal J C, Solano E and Zagury N 2006 Phys. Rev. Lett. 96 010502
[6] Zhang Y, Wang H, Li X Y, Jing J T, Xie C D and Peng K C 2000 Phys. Rev. A 62 023813
[7] Simon C and Bouwmeester D 2003 Phys. Rev. Lett. 91 053601
[8] Xiong H, Scully M O and Zubairy M S 2005 Phys. Rev. Lett. 94 023601
[9] Tan H T, Zhu S Y and Zubairy M S 2005 Phys. Rev. A 72 022305
[10] Abelow E 2007 Phys. Rev. A 76 023808
[11] Kiffner M, Zubairy M S, Evers J and Keitel C H 2007 Phys. Rev. A 75 033816
[12] Ikram M, Li C X and Zubairy M S 2007 Phys. Rev. A 76 042317
[13] Raymond Ooi C H 2007 Phys. Rev. A 76 013809
[14] Li G X, Tan H T and Macovei M 2007 Phys. Rev. A 76 053827
[15] Lü X Y, Liu J B, Tian Y, Song P J and Zhan Z M 2008 Euro. Phys. Lett. 82 64003
[16] Hillery M and Zubairy M S 2006 Phys. Rev. Lett. 96 050503
[17] Korschuky E A, Leinfellner N, Huss A, Baluschev S and Windholz L 1999 Phys. Rev. A 59 2302
[18] Merriam A J, Sharpe S J, Shervin D, Manuzsk D, Yin G Y and Harris S E 2000 Phys. Rev. Lett. 84 5308
[19] Chong Y D and Soljačić M 2008 Phys. Rev. A 77 013823
[20] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[21] Scully M O and Zubairy M S 1987 Phys. Rev. A 35 752
[22] Blockey C A and Walls D F 1991 Phys. Rev. A 43 5049
[23] Duan L M, Giedke G, Cirac J I and Zoller P 2000 Phys. Rev. Lett. 84 2722
[24] Shchukin E and Vogel W 2005 Phys. Rev. Lett. 95 230502
[25] Simon R 2000 Phys. Rev. Lett. 84 2726
[26] Miranowicz A, Piani M, Horodecki P and Horodecki R 2006 arXiv:quant-ph/06050001
[27] Gerry C C and Grobe R 1997 J. Mod. Opt. 44 41
[28] Hillery M and Zubairy M S 2006 Phys. Rev. A 74 032333