Convergence of the conical Ricci flow on $S^2$ to a soliton

D.H. Phong*, Jian Song†, Jacob Sturm‡ and Xiaowei Wang ‡

Abstract

In our previous work [PSSW], we showed that the Ricci flow on $S^2$ whose initial metric has conical singularities $\sum_j \beta_j[p_j]$ converges to a constant curvature metric with conic singularities (in the stable and semi-stable cases) or to a gradient shrinking soliton with conical singularities (in the unstable case). The purpose of this note is to show that in the unstable case, that is, the case where $\beta_k > \beta'_k = \sum_{j<k} \beta_j$, that the limiting metric is the unique shrinking soliton with cone singularity $\beta_k[p_\infty] + \beta'_k[q_\infty]$. This verifies the prediction made in [PSSW].

1 Introduction

Let $g_{S^2}$ be the round metric on $S^2$ and $\omega_{S^2} = \sqrt{-1} g_{z\bar{z}} dz \wedge d\bar{z}$ the Kähler form, so that $[\omega_{S^2}] = 2[p]$ for any $p \in S^2$. Let $p_1, ..., p_k \in S^2$ be a finite collection of points and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_k \in (0,1)$. Let $\beta = \sum_{j=1}^k \beta_j[p_j]$.

A smooth metric $g$ on $S^2\{p_1, ..., p_k\}$ is a cone metric on $(S^2, \beta)$ if it can be written in the form

$$\omega = e^f \prod_j |\sigma_j|^{\beta_j} \cdot \omega_{S^2}$$

for some bounded function $f$ on $S^2$ where $\sigma_j$ is a section of $K_{S^2}^{-1}$ such that $[\sigma_j] = 2[p_j]$.

A constant curvature metric on $(S^2, \beta)$ is a metric $\omega_\phi = \omega_{S^2} + i \partial \bar{\partial} \phi$ with the property

$$\omega_\phi = e^{-\gamma \phi} \prod_j |\sigma_j|^{\beta_j} \cdot \omega_{S^2}$$

where $\gamma = 1 - \frac{1}{2} \sum \beta_j$.

An alternative form of (1.1) is

$$\text{Ric}(\omega_\phi) = \gamma \omega_\phi + \sum \beta_j[p_j].$$

The Ricci flow is given by

$$e^{-\phi \cdot \omega_\phi} = e^{-\gamma \phi} \prod_j |\sigma_j|^{\beta_j} \cdot \omega_{S^2}, \quad \phi(0) = \phi_0$$

where $\dot{\phi} = \partial_t \phi$. An alternative form of (1.3) is

---

*Work supported in part by National Science Foundation grants DMS-12-66033, DMS-0847524 and DMS-0905873 and a Collaboration Grants for Mathematicians from Simons Foundation.
\[ \partial_t g = -\text{Ric}(g) + \gamma g \, , \, g(0) = g_0 \text{ a cone metric on } (S^2, \beta) \, . \]  

(1.4)

Here, and in all that follows, we assume \( \gamma > 0 \). We shall also assume that \( g_0 \) is “regular” in the sense of [PSSW]. This means that \( g_0 \) is smooth on \( S^2 \setminus \{p_1, \ldots, p_k\} \) and that in a neighborhood of \( p_j \) there is a holomorphic coordinate \( z \) such that

\[ g_0 = e^{u_j} \frac{dz \wedge d\bar{z}}{|z|^{2\beta_j}} \]  

(1.5)

where

\[ u_0, \Delta_0 u_0, \Delta_0 (\Delta_0 u_0) \in C^{2, \alpha}(S^2, \beta) \cap W^{1,2} \, . \]

Here \( C^{2, \alpha}(S^2, \beta) \) is the Yin-Hölder space defined in [Y]. In particular, if \( u_0 \) is harmonic in a neighborhood of \( p_j \), then \( g_0 \) is regular.

Let \( \beta_k' = \sum_{j<k} \beta_j \) and let

\[ \beta_\infty = \beta_k[p_\infty] + \beta_k'[q_\infty] \]

where \( p_\infty \) and \( q_\infty \) are the north and south pole respectively. Then we say \( \beta \) is stable, semi-stable or unstable if \( \beta_k' > \beta_k, \beta_k' = \beta_k, \) or \( \beta_k' < \beta_k \) respectively.

When \( \beta \) is stable it is known, by the work of [MRS], that the Ricci flow converges to the unique constant scalar metric on \( (S^2, \beta) \). In [PSSW] we give a new proof of this result. We also show that in the semi-stable case, the Ricci flow converges to the unique constant scalar curvature metric on \( (S^2, \beta_\infty) \).

We now assume that \( \beta \) is unstable and we let \( g_\infty \) the unique conic shrinking soliton on \( (S^2, \beta_\infty) \). This means that \( g_{\text{sol}} \) (which is rotationally symmetric by uniqueness) satisfies the following equation on \( S^2 \setminus \{p_\infty, q_\infty\} \):

\[ R(g_{\text{sol}}) = \gamma + \Delta_{g_{\text{sol}}} \theta_{\text{sol}}, \nabla_{g_{\text{sol}}}^2 \theta_{\text{sol}} = \frac{1}{2}(\Delta_{g_{\text{sol}}} \theta_{\text{sol}})g_{\text{sol}} \, , \, \int_{S^2} e^{\theta_{\text{sol}}} dg_{\text{sol}} = 2 \]  

(1.6)

for a unique \( \theta_{\text{sol}} \in C^0(S^2) \cap C^\infty(S^2 \setminus \{p_\infty, q_\infty\}) \). Here \( R(g_{\text{sol}}) \) is the scalar curvature of \( g_{\text{sol}} \).

We wish to prove the following:

**Theorem 1** For any initial regular metric \( g_0 \) on \( (S^2, \beta) \) the Ricci flow converges to \( g_\infty \).

Remark: In [PSSW] we proved that there is a partition \( \{1, 2, \ldots, k\} = I \cup J \) into disjoint subsets with the following property. The flow (1.4) converges to the unique Kähler-Ricci soliton \( g_{I,J} \) with cone structure

\[ \beta_{I,J} = \left( \sum_{i \in I} \beta_i \right)p_i + \left( \sum_{j \in J} \beta_j \right)p_j \, . \]

Thus the content of Theorem 1 is that \( I = \{p_k\} \) and \( J = \{p_1, \ldots, p_{k-1}\} \). In particular, \( p_k \to p_\infty \) and \( p_1, \ldots, p_{k-1} \to q_\infty \) as \( t \to \infty \).
2 The proof

Let $\beta = \sum_{j=1}^{k} \beta_j[p_j]$ and $g$ a cone metric on $(S^2, \beta)$. Let $f \in C^0(S^2) \cap W^{1,2}(S^2)$. We define the normalized $W$-functional for the pair $(g, f)$ by the same expression as in the smooth case,

$$W(g, f) = \int_{S^2 \setminus \beta} \left( \frac{1}{2\gamma} (R + |\nabla f|^2) + f \right) \frac{e^{-f}}{4\pi^2} dg .$$  \hspace{1cm} (2.1)

We also define

$$\mu(g) = \inf_f \{ W(g, f) : \int_{S^2} e^{-f} dg = 2 \} .$$

Let $\mu_1 = \max\{\mu(g_1, \beta)\}$ and $\mu_2 = \max\{\mu(g, \beta) : g_1 \notin g_\infty\}$ where, as above,

$$g_\infty = g_{1,j} , \text{ where } I = \{p_k\} \text{ and } J = \{p_1, \ldots, p_{k-1}\}$$

We shall need the following (Lemma 7.3) from [PSSW] which was proved by first showing $\mu(g_t)$ is increasing along the Ricci flow, and then using the toric structure of $g_{1,j}$ to compare $\mu$ invariants.

**Lemma 1** We have $\mu_1 > \mu_2$. Moreover, if there exists a regular cone metric $\tilde{g}_0$ on $(S^2, \beta)$ with the property $\mu(\tilde{g}_0) > \mu_2$, then the Ricci flow on $(S^2, \beta)$ converges in the Gromov-Hausdorff $C^\infty$ topology to $g_\infty$ for any initial metric $g_0$. Thus $(S^2, g_t) \rightarrow (S^2, g_\infty)$ as metric spaces in the Gromov-Hausdorff topology. Moreover, for any compact subset $K \subseteq S^2 \setminus \{p_\infty, q_\infty\}$ there exists a family of diffeomorphisms $f_t : S^2 \rightarrow S^2$ such that $f_t^*g_t \rightarrow g_\infty$ in $C^\infty(K)$.

To prove Theorem 1, we start by choosing coordinates on $\mathbb{P}^1$ in such a way that $p_\infty$ is the point at infinity and $q_\infty$ is the origin in $\mathbb{C}$. We define $g_\beta$ to be the conic metric on $(S^2, \beta)$ whose Kähler form is given by

$$\omega_\beta = c(\beta) \frac{\chi(z) dz \wedge d\bar{z}}{\prod_{j=1}^{k-1} |z - p_j|^{2\beta_j}} + c(\beta) \frac{(1 - \chi(z)) dz \wedge d\bar{z}}{(1 + |z|^2)^{2-\beta_k}} = F_\beta \omega_{FS} .$$ \hspace{1cm} (2.2)

Here $\chi$ is smooth with compact support on $\mathbb{C}$ and equal to one in a large ball $B$ centered at $0 \in \mathbb{C}$ which contains $p_1, \ldots, p_{k-1}$ zero on the ball $2B$. The constant $c(\beta)$ is chosen so that $\int dg_\beta = 2$. Thus $q_\infty = 0 \in \mathbb{C}$ and $p_\infty = \infty \in \mathbb{P}^1$.

We have

$$\text{Ric } g_\beta = \sum_{j=1}^{k-1} \beta_j[p_j] \text{ on the ball } B .$$

Note that $c(\beta)$ is a continuous function of $p_1, \ldots, p_{k-1}$ and is thus bounded from above and away from zero provided $p_1, \ldots, p_{k-1}$ remain in a bounded subset of $\mathbb{C}$.

Let $\rho(t) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the map $z \mapsto tz$ for $0 \leq t \leq 1$ and $\rho(p_j) = p_j(t)$. Thus $p_j(t) = tp_j$ for $j < k$ and $p_k(t) = p_\infty$. Let $\beta(t) = \sum_{j=1}^{k} \beta_j[p_j(t)]$. Then there exists $q > 1$ such that

$$\beta(t) \rightarrow \beta_\infty \text{ and } g_{\beta(t)} \rightarrow g_{\beta_\infty} \text{ in the GH topology and } F_{\beta(t)} \rightarrow F_{\beta_\infty} \text{ in } L^q .$$ \hspace{1cm} (2.3)

Here and in the following, when we use notation such as $W^{1,2}, L^p, \Delta$ etc., the background metric is always $g_{FS}$ unless otherwise specified.
We wish to construct a family of conic metrics \( g_t \) on \((S^2, \beta(t))\) such that

\[
g_t \to g_\infty \tag{2.4}
\]

in the Gromov-Hausdorff topology and \( \mu(g_t) \to \mu(g_\infty) \). Since \( \mu(g_\infty) = \mu_1 > \mu_2 \) we conclude that for \( t \) sufficiently large, \( \mu(g(t)) > \mu_2 \). For such a \( t \), we let \( g_0 = \rho(t)^rg(t) \). Then \( g_0 \) is a conic metric on \((S^2, \beta)\) with the property \( \mu(g_0) > \mu_2 \), and so Lemma 1 applies to give the desired conclusion.

To define \( g(t) \) we first write

\[
g_\infty = e^{u_\infty}g_{\beta_\infty}
\]

for some continuous function \( u_\infty \) which is smooth on \( S^2 \setminus \beta_\infty \). Theorem 1.1 of Datar-Guo-Song-Wang [DGSW] shows \( u_\infty \) is a “smooth \( S^1 \) invariant conic metric”. This implies the \( u_\infty \) is smooth on \( \mathbb{C} \setminus \{0\} \) and that there is a smooth \( S^1 \) invariant function \( \tilde{u}_\infty \) on \( B \) with the property \( u_\infty(z) = \tilde{u}_\infty(w) \) where \( |w|^2 = |z|^{2(1-\beta'_k)} \). In particular, \( u(z) \) has a Taylor expansion of the form

\[
u(x) = a_0 + a_1|z|^{2(1-\beta'_k)} + a_2|z|^{4(1-\beta'_k)} + \cdots a_m|z|^{2m(1-\beta'_k)} + O(|z|^{2(m+1)(1-\beta'_k)}) .
\]

In particular, there exist \( C > 0 \) such that on \( B \)

\[
|u_\infty(z) - a_0| + |z\partial_z u_\infty| + |z^2\partial_z^2 \tilde{u}_\infty| \leq C|z|^{2-2\beta'_k} . \tag{2.5}
\]

We would like to define \( g(t) := e^{u_\infty}g_{\beta(t)} \). This would satisfy (2.4) but doesn’t quite work since \( u_\infty \) is not \( C^2 \) on the complement of \( \beta(t) \) so \( e^{u_\infty}g_{\beta(t)} \) is not a regular metric in the sense of [PSSW]. Instead we proceed as follows. Let \( \psi = 1 - \chi \) which is zero on \( B \) and 1 outside \( 2B \). Define

\[
u(x) = a_0 + \psi(z/t)(u_\infty(z) - a_0) \quad \text{if} \quad t > 0 .
\]

Thus for each \( t \) we see \( u_\infty(t, z) \in C^\infty(S^2 \setminus \{p_\infty\}) \) and \( u_\infty(t, z) \) is constant on the ball \( tB \) and hence constant in a neighborhood of \( p_1(t), \ldots, p_{k-1}(t) \). Also,

\[
u(x) \to u_\infty(z) \quad \text{pointwise as} \quad t \to 0 ,
\]

\[
u(x) \to u_\infty \quad \text{and} \quad \Delta_g f \nu(x) \to \Delta_g f \nu(x) \quad \text{uniformly on compact subsets} \quad S^2 \setminus \beta_\infty .
\]

Define

\[
g(t) = e^{u_\infty(t)}g_{\beta(t)} .
\]

We see that for each \( t > 0 \) that \( g(t) \) satisfies (1.5) with \( u_j \) harmonic in a neighborhood of \( p_j \). In particular, \( g(t) \) is a regular metric. Moreover,

\[
\partial_z \tilde{u}_\infty(t, z) = \psi'(z/t)\frac{1}{t} [\partial_z u_\infty + \tilde{u}_\infty] + \psi''(z/t)\frac{1}{t^2}(u_\infty(z) - a_0) + \psi(z/t)\partial_z \tilde{u}_\infty .
\]

Since \( |t| \geq c|z| \) when the right side is non-zero, we conclude from (2.5) that on \( B \)

\[
|\partial_z \tilde{u}_\infty(t, z)| \leq \frac{C}{|z|^{2\beta'_k}}
\]

\]
for some \( C > 0 \) which is independent of \( t \). We conclude that there exists \( q > 1 \) such that 
\[
\|R(g(t))\|_{L^q} \leq C \text{ for all } t > 0.
\]
Moreover, decreasing \( q \) slightly if necessary, \( \|R(g)\|_{L^q} \to \|R(g_\infty)\|_{L^q} \).

In general, if \( (X_t, g_t) \to (X, g_\infty) \) is any Gromov-Hausdorff limit of smooth manifolds, we know

(Chow, Lemma 6.28) that

\[
\mu(g_\infty) \geq \limsup \mu(g_t)
\]

Thus our goal is to show \( \lim_{t \to \infty} \inf_{t \geq T} \mu(g(t)) \geq \mu(g_\infty) \).

Assume not. Then there exist \( \delta > 0 \) and a sequence \( t_j \to \infty \) such that

\[
\mu(g_j) = \mu(g(t_j)) \leq \mu(g_\infty) - \delta
\]
(2.6)

Thus for each \( j \) there is a positive function \( \Phi_j = e^{-f_j/2} \) such that \( \|\Phi_j\|_{L^2(g_j)} = 1 \) and

\[
W(g_j, f_j) = \int_{S^2 \setminus \delta(t_j)} \left( \frac{2}{\gamma} |\nabla_j \Phi_j|^2 - \frac{\Phi_j^2}{2\gamma} - \Phi_j^2 \log \Phi_j^2 \right) dg_j = \mu(g_j) \leq \mu(g_\infty) - \delta
\]
(2.7)

where \( \gamma = 1 - \frac{1}{2}\sum_j \beta_j \).

**Lemma 2** We have the following bounds.

1. The \( \Phi_j \) are uniformly bounded in \( W^{1,2} \) that is, there exists \( C > 0 \) such that

\[
\int \Phi_j^2 dg_{FS} + \int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C \text{ for all } j
\]

2. There exists \( q > 1 \) such that \( \Delta u_j \to \Delta u_\infty \) in \( L^q \), that is

\[
\lim_{j \to \infty} \int_{S^2} |\Delta_{g_{FS}} u_j - \Delta_{g_{FS}} u_\infty|^q dg_{FS} = 0
\]

We postpone the proof for the moment and show how the lemma leads to a contradiction.

Part (1) implies there exists \( \Phi_\infty \in W^{1,2} \) such that \( \Phi_j \rightharpoonup \Phi_\infty \) that is, \( \Phi_j \) converges weakly to \( \Phi_\infty \) in \( W^{1,2} \). Since \( W^{1,2} \hookrightarrow L^p \) is a compact imbedding for all \( p > 1 \), we see that after passing to a subsequence, \( \Phi_j \to \Phi_\infty \) in \( L^p \) for all \( p > 1 \). Thus \( \|\Phi_j\|_{L^p} \leq C_p \) for all \( j \) and \( \Phi_j^2 \to \Phi_\infty^2 \) in \( L^p \) for all \( p \).

We claim

\[
\int \Phi_j^2 R_j dg_j = \int \Phi_j^2 (\Delta u_j + \gamma) dg_{FS} \to \int \Phi_\infty^2 (\Delta u_\infty + \gamma) dg_{FS} = \int \Phi_\infty^2 R_\infty dg_j
\]
(2.8)

\[
1 = \int \Phi_j^2 dg_j = \int \Phi_j^2 F_{\beta_j} dg_{FS} \to \int \Phi_\infty^2 F_{\beta_\infty} dg_{FS} = \int \Phi_\infty^2 dg_\infty = 1
\]
(2.9)

\[
\int \Phi_j^2 \log \Phi_j^2 dg_j = \int \Phi_j^2 \log \Phi_j^2 F_{\beta_j} dg_{FS} \to \int \Phi_\infty^2 \log \Phi_\infty^2 F_{\beta_\infty} dg_{FS} = \int \Phi_\infty^2 \log \Phi_\infty^2 dg_j
\]
(2.10)

\[
\liminf \int |\nabla_j \Phi_j|^2 dg_j \geq \int |\nabla_\infty \Phi_\infty|^2 dg_\infty
\]
(2.11)
To prove (2.8) we note that $\Delta u_j \to \Delta u_\infty$ in $L^q$ and $\Phi_j^2 \to \Phi_\infty^2$ in $L^p$ for all $p$. Similarly (2.9) follows from the fact that $F_{\beta j} \to F_{\beta \infty}$ in $L^q$ for some $q = q(\beta)$.

To prove (2.10) we need only show $\Phi_j^2 \log \Phi_j^2 \to \Phi_\infty^2 \log \Phi_\infty^2$ in $L^p$ for all $p$. To see this, first note that if $x, y > 0$ there exists $\theta$ between $x$ and $y$ such that

$$|x^2 \log x^2 - y^2 \log y^2| = |4\theta \log \theta + 2\theta| \cdot |x - y| \leq C_\delta (1 + |x|^2 + |y|^2) \cdot |x - y| \quad (2.12)$$

by the mean value theorem (c.f. [R]). Now substitute $x = \Phi_j^2$ and $y = \Phi_\infty^2$ and apply Hölder’s inequality.

Finally (2.11), which is equivalent to

$$\liminf_j \int \partial \Phi_j \wedge \bar{\partial} \Phi_j \geq \int \partial \Phi_\infty \wedge \bar{\partial} \Phi_\infty, \quad (2.13)$$

Since $\Phi_j \to \Phi_\infty$ in $W^{1,2}$ we know

$$\liminf_j \|\Phi_j\|_{W^{1,2}} \geq \|\Phi_\infty\|_{W^{1,2}} \quad (2.14)$$

But $\Phi_j \to \Phi_\infty$ strongly in $L^2(g_{FS})$. Thus (2.14) follows from (2.11).

Taking the $\lim_{j \to \infty}$ of both sides of (2.7) and applying (2.8), (2.9), (2.10) and (2.11), we obtain

$$\int \left[ \frac{2}{\gamma} |\nabla \Phi_\infty|^2 - \Phi_\infty^2 \frac{R_\infty}{2\gamma} - \Phi_\infty^2 \log \Phi_\infty \right] e^{u(t_\infty)} g_{\beta_\infty} \leq \mu(g_\infty) - \delta \quad (2.15)$$

which contradicts the definition of $\mu(g_\infty)$.

Thus we have reduced the proof of Proposition 1 to the proof of the lemma.

To prove the lemma, note that (2.7) implies

$$\int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C\|\Phi_j\|_{L^q}^2$$

for some $q > 2$. On the other hand

$$\|\Phi_j\|_{L^q}^2 - C_{q'} \|\Phi_j\|_{L^2}^2 \leq C_{q'} \int \partial \Phi_j \wedge \bar{\partial} \Phi_j$$

for any $q' > q$. This implies $\|\Phi_j\|_{L^q} \leq C$ and hence $\int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C$.

This concludes the proof first part of the lemma. The second follows from the fact that $\Delta u_j \to \Delta u_\infty$ pointwise almost everywhere and $\|\Delta u_j\|_{L^q}$ is uniformly bounded for some $q = q(\beta) > 1$.

References

[Chow] B. Chow et.al, “The Ricci Flow: Techniques and Applications, Part I”, Math. Surveys and Monographs, AMS (2007)

[DGSW] V. Datar, B. Guo, J. Song and X. Wang, “Connecting toric manifolds by conical Kahler-Einstein metrics”. [arXiv:1308.6781]
[MRS] Mazzeo, R., Y. Rubinstein and N. Sesum, “Ricci flow on surfaces with conic singularities”, arXiv:1306.6688

[PSSW] Phong, D.H., J. Sturm, J. Song and X. Wang, “The Ricci flow on the sphere with marked points”, arXiv:1407.1118

[R] Rothaus, O.S., “Logarithmic Sobolev Inequalities and the Spectrum of Schrödinger Operators”, J. of Fun. Anal. 43 110–120 (1981)

[Y] Yin, Hao, “Ricci flow on surfaces with conical singularities” J. Geom. Anal. 20 (2010), no. 4, 970–995.