Charges and Coupling Strengths in Gauge Theories
with Direct Product Symmetry Groups

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Abstract

For gauge theories with direct product internal symmetry groups, the relationship between internal quantum numbers (charges) and interaction coupling strengths is examined. In these types of theories, the Lagrangian density may contain non-trivial factors multiplying the matter field terms, and these factors can modify the interaction coupling strengths i.e., the gauge/matter field vertex factors. Consequently, a matter field can carry a given internal charge yet couple to the associated gauge field with an apparent fractional charge. An example with $SU(3) \otimes U(2)$ symmetry is presented in which the matter fields can have integer $U(2)$ charges but fractional $U(2)$ coupling strengths.

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I. INTRODUCTION

This paper studies the relationship between internal quantum numbers—commonly called charges—and coupling strengths in gauge theories with direct product Lie groups. For the most part, the analysis reproduces what is already known since gauge field theory is, by now, well studied and understood (see e.g. [1]-[3]). However, the analysis uncovers two important points that may be of relevance in physical applications.

The first point is that the group theory mathematics (see e.g. [4], [5]) suggests the matter field irreducible representations (irreps) for the direct product group include all combinations of irreps of the component subgroups. The second point is that matter field terms in the Lagrangian density can contain non-trivial multiplicative factors that do not destroy the salient elements of the gauge theory. Taken together, these two results imply the possibility of scaled gauge/matter field vertex couplings. In particular, it implies the possibility of matter fields with integer charges but fractional coupling strengths.

Our analysis starts with the Lie algebra of a direct product group. There exists a special Lie algebra basis that implies neutral and paired, oppositely-charged gauge bosons. This basis corresponds to ‘physical’ gauge bosons and characterizes the gauge bosons’ intrinsic quantum numbers. Likewise, one can choose an associated basis in the vector space furnishing a faithful irrep of the direct product group. Relative to this associated basis, the eigenvalues of the diagonal Lie algebra elements characterize the intrinsic quantum numbers of the elementary matter fields. Thus the kinematical structure encodes the intrinsic quantum numbers of the elementary gauge and matter fields.

On the other hand, the gauge/gauge and gauge/matter field couplings are characterized by their respective covariant derivatives. There is enough scale ambiguity in the Lagrangian density to allow the identification of the eigenvalues of the neutral conserved charge operators with the intrinsic quantum numbers—thereby ensuring the fields appearing in the Lagrangian density are elementary fields [6]. The two sets of parameters—so identified—will be referred to as intrinsic charges. The (properly scaled) intrinsic charges multiply vertex factors in Feynman diagrams and contribute to the gauge/matter field coupling strengths. However, this is not the sole contribution to the coupling strengths.

The Lagrangian density admits a multiplicative factor for each matter field representation. For direct product groups, the representations may be related in such a way that the
factors are not trivial in the sense that the factors cannot all be absorbed into the matter field definition. Consequently, non-trivial factors may multiply the intrinsic charges in the conserved currents, and, therefore, affect the gauge/matter field vertex factors (or coupling strengths). These renormalized coupling strengths will be referred to as extrinsic charges since they are actual measured quantities; provided the associated particle is observable.

The presence of non-trivial multiplicative factors implies an inequality between the associated intrinsic and extrinsic charges of elementary matter fields. Although it is the extrinsic charge that is observed, the notion of intrinsic charge is theoretically useful. An example is presented in section III that exhibits matter fields with integer $SU(3) \otimes U(2)$ intrinsic charges—yet they couple to $U(2)$ gauge bosons with fractional extrinsic charges.

II. INTRINSIC VS. EXTRINSIC CHARGES

There are at least two starting points for characterizing quantum numbers of elementary particles and fields. One is kinematical and the other dynamical in nature.

The kinematical starting point stems from viewing an elementary particle state as furnishing a faithful irreps of some assumed internal symmetry group that commutes with the Poincare group. The representations are labelled by certain parameters (that must be scalars by Poincare invariance) which then serve to characterize physical properties (apart from momentum and spin/helicity) of the elementary particle state. We will refer to these parameters as kinematical (internal) quantum numbers.

The other starting point is the Lagrangian—hence dynamical. By way of Noether’s theorem, symmetries of the Lagrangian lead to conserved currents that in turn lead to time independent quantum charge operators. Their equal-time commutators with the field variables of the Lagrangian yield what we will refer to as dynamical (internal) quantum numbers. Importantly, scale ambiguity in the Lagrangian can be exploited to guarantee equality between the kinematical and dynamical quantum numbers.

A. Kinematical quantum numbers

Given that a physical system is invariant under some internal symmetry group, it is possible to deduce some general properties or attributes of the associated gauge and matter
fields based solely on the mathematics of the symmetry group and its representations \[4\], \[5\]. In particular, the mathematics identifies special bases and associated eigenvalues (quantum numbers) in the vector spaces furnishing the representations. For local symmetries, these special bases can be chosen at each spacetime point, essentially creating an unchanging structure by which to associate the unchanging internal quantum numbers of elementary particles.

We begin with a gauge field theory with an internal symmetry group that is a direct product group \(G = G_1 \otimes \cdots \otimes G_n =: \otimes G_n\) where \(n \in \mathbb{N}\) and the \(G_i\) with \(i \in \{1, \ldots, n\}\) are Lie groups that mutually commute. Associated with each subgroup \(G_i\) is a Lie algebra \(\mathfrak{g}_i\) with basis \(\{g_{a_i}\}_{a_i=1}^{\dim G_i}\). The full Lie algebra is \(\mathfrak{g} := \oplus G_n\) (in obvious notation). Recall that the Lie algebra does not uniquely determine the Lie group.

Consider the adjoint representation \(ad : \mathfrak{g}_i \to GL(\mathfrak{g}_i)\) of the complex extension of \(\mathfrak{g}_i\) on \(\mathfrak{g}_i\). For a given element \(c^{a_i}g_{a_i}\) (with \(c^{a_i} \in \mathbb{C}\)) in the Lie algebra, the adjoint representation yields a secular equation

\[
\prod_{k_i=0}^{r_i} (\lambda - \alpha_{k_i})^{d_{k_i}} = 0
\]

where the \(\alpha_{k_i}\) are the (complex) roots of the secular equation with multiplicity \(d_{k_i}\). Since \(\lambda = 0\) is always a solution, we put \(\alpha_0 = 0\). Note that \(\sum_{k_i=0}^{r_i} d_{k_i} = \dim G_i\). Associated with the roots \(\alpha_{k_i}\) (which may not all be distinct in general) are \(r_i\) independent eigenvectors.

The roots and their associated eigenvectors determine the well-known Jordan block form of the element \(ad(c^{a_i}g_{a_i})\). That is, there exists a non-singular transformation of \(ad(c^{a_i}g_{a_i})\) into Jordan canonical form. With respect to the Jordan canonical form, the vector space that carries the representation \(ad(\mathfrak{g}_i)\) (and hence the Lie algebra) decomposes into a direct sum of subspaces:

\[
\mathfrak{g}_i = \sum_{\alpha_{k_i}} \oplus V_{\alpha_{k_i}}
\]

with each \(V_{\alpha_{k_i}}\) containing one eigenvector and \(\dim V_{\alpha_{k_i}} = d_{k_i}\).

The above decomposition is with respect to any given element in the Lie algebra. Regular elements are defined by the conditions: (i) that they lead to a decomposition that maximizes the distinct roots \(\alpha_{k_i}\) (equivalently, minimize the dimension of \(V_{\alpha_{k_i}}\)), and (ii) they all determine the same \(V_0\). For decomposition associated with regular elements, the subspaces \(V_{\alpha_{k_i}}\) have potentially useful properties for describing physical gauge bosons:

- \([V_0, V_0] \subseteq V_0\) and hence \(V_0\) is a subalgebra. It is known as a Cartan subalgebra.
• The subspace \( V_0 \) carries a representation of the Cartan subalgebra. Since its rank is 0, the Cartan subalgebra is solvable; in fact nilpotent.

• \([V_0, V_{\alpha k_i}] \subseteq V_{\alpha k_i}\). Hence, each \( V_{\alpha k_i} \) is invariant with respect to the action of \( V_0 \) and so carries a representation for \( V_0 \). Moreover, since \( V_0 \) is solvable, it has a simultaneous eigenvector contained in \( V_{\alpha k_i} \). More specifically, associated with the secular equation for an element of the subalgebra \( V_0 \) with basis \( \{h_{s_i}\}_{s_i=1}^{d_0} \) is a set of \( d_0 = \dim V_0 \) roots, collectively denoted by \( q_i := (q_{1i}, \ldots, q_{d_0i}) \), and a corresponding eigenvector \( e_{\alpha k_i} \in V_{\alpha k_i} \) such that
  \[
  [h_{s_i}, e_{\alpha k_i}] = q_{s_i} e_{\alpha k_i} ,
  \]
  or more succinctly,
  \[
  [h_{s_i}, e_{k_i}] = q_{s_i} e_{k_i} ,
  \]
  In particular, this holds for \( \alpha_0 = 0 \). That is, there exists an \( e_0 \in V_0 \) such that
  \[
  [h_{s_i}, e_0] = 0 .
  \]

• If \( V_0 \) is contained in the derived algebra of \( \mathcal{G}_i \), then for \( V_{\alpha k_i} \), there is at least one \( V_{\beta k_i} \) such that \([V_{\alpha k_i}, V_{\beta k_i}] \subseteq V_0 \). This implies that, for \( q_i \) associated with each \( e_{\alpha k_i} \), there is at least one \( e_{\beta k_i} \) with roots \(-q_i\). Additionally, any \( q'_i \neq -q_i \) must be a rational multiple of \( q_i \neq 0 \).

These properties can be used to characterize the ‘physical’ gauge bosons if we make one restriction: for \( \alpha k_i \neq 0 \), \( \dim \oplus V_{\alpha k_i} = \dim \mathcal{G}_i - d_{0i} = r_i \). That is \( \dim V_{\alpha k_i} = 1 \) for all \( \alpha k_i \neq 0 \). Without this restriction, there would be no means (mathematically) to distinguish between basis elements, and hence gauge bosons, in a given \( V_{\alpha k_i} \). As a consequence of this restriction, we must have \([V_0, V_0] = 0 \) since otherwise \([ [V_0, V_0], V_{\alpha k_i} ] \) in the Jacobi identity leads to a contradiction.

The commutativity of \( V_0 \) is a necessary condition for \( \mathcal{G}_i \), and hence \( \mathcal{G} \), to be the direct sum of one-dimensional abelian and/or simple algebras. Moreover, eventually the Lie algebra elements will be promoted to quantum fields so the adjoint carrier space is required to be Hilbert. Therefore, the inner product on the Lie algebra is required to be positive definite. This implies the Lie algebra is the direct sum of \( u(1) \) and/or compact simple complex algebras. With this identification, all the results of root space analysis of compact, semisimple Lie algebras become applicable.
If the symmetry is not broken under quantization, then we can conclude that the quantized gauge fields associated with the Lie algebra \( G \) describe gauge bosons characterized by the set of roots \( q_i \). We refer to these as *kinematical (internal) quantum numbers* for the gauge bosons. They correspond to physical, i.e., measured, properties of the gauge bosons for broken symmetries, and for unbroken symmetries they are physically relevant once a choice of matter field representations has been made.

Evidently, gauge bosons associated with the \( h \) carry no kinematical ‘charge’ and those associated with the \( e^\alpha_\kappa i \) carry the \( q_i \) kinematical ‘charges’. Note that \( e^-\alpha_\kappa i \) carries \(-q_i\) charges. It is in this sense that the Lie algebra basis, defined by (properly restricted) decomposition (2.2), characterizes the physical gauge bosons. It should be kept in mind that the class of Lie algebras under consideration are *complex*. However, for the groups of interest in gauge theory applications, the basis elements are linearly independent over the complex numbers. Hence, the Lie algebra can be taken to be *real* (having the same basis as its complex counterpart) provided the gauge fields are allowed to be complex.

Turn now to the matter particle states. We will confine our attention to Dirac spinors. Let \( V_{R_i} \) be a vector space that furnishes a representation of \( G_i \) having basis \( \{ e^{(R_i)}_l \}^{d_{R_i}} \). And let \( \rho^{(R_i)} : G_i \to GL(V_{R_i}) \) denote a faithful irrep. The \( R_i \) is a collection \( (R_i^1, \ldots, R_i^{d_{R_i}}) \) of \( d_{R_i} \) numbers and serves to label the representation.

Given some set \( \{ R_i \} \), suppose the corresponding set of fields \( \{ \Psi^{(R_i)} \} \) furnish inequivalent irreps \( \rho^{(R_i)}(G_i) \) of the \( G_i \). The associated tensor product representation

\[
\rho^{(\times R_n)}(\otimes G_n) := \rho^{(R_1)}(G_1) \otimes \ldots \otimes \rho^{(R_n)}(G_n)
\]

of the direct product group is also irreducible (where \( \times R_n := (R_1, \ldots, R_n) \) denotes an element in the cartesian product \( \{ R_1 \} \times \ldots \times \{ R_n \} \)). In fact for the class of groups under consideration here, all irreps of \( G \) are comprised of all possible combinations of relevant \( \{ R_i \} \). That the irreps of \( \rho^{(\times R_n)}(\otimes G_n) \) include all relevant combinations of component irreps suggests including these representations in physical models. The idea is these relevant combinations of irreps can be identified with elementary fields.

The corresponding Lie algebra representation

\[
\rho_e^{(\times R_n)}(\oplus G_n) := \rho_e^{(R_1)}(G_1) \oplus \ldots \oplus \rho_e^{(R_n)}(G_n)
\]

(where \( \rho_e \) is the derivative map of the representation evaluated at the identity element) is likewise irreducible for all combinations of \( \{ R_i \} \) that are associated with irreps of the \( G_i \).
The representations $\rho^{(R_i)}(G_i)$ are largely a matter of choice depending on physical input. By assumption, the internal degrees of freedom associated with $G_i$ of elementary particles correspond to the basis elements $\{e_{l_i=1}^{(R_i)}\}^{d_{R_i}}_{l_i}$ spanning $V_{R_i}$. Hence, a given label $R_i$ (partially) characterizes the elementary particles (along with Lorentz labels). In particular, a basis is chosen such that the representation of the diagonal Lie algebra elements is (no summation implied)

$$\rho_{e_{R_i}}^{(R_i)}(h_{s_i}) e_{l_i}^{(R_i)} = q_{s_i,l_i}^{(m)} e_{l_i}^{(R_i)} . \quad (2.6)$$

where $q_{s_i,l_i}^{(m)}$ are $(d_0 \times d_{R_i})$ imaginary numbers since the Lie algebra generators must be anti-hermitian (which implies the $\rho_{e_{R_i}}^{(R_i)}(h_{s_i})$ must also be anti-hermitian). In an obvious short-hand notation,

$$\rho_{e_{R_i}}^{(R_i)}(h_{s_i}) e_{R_i} = q_{s_i}^{(m)} e_{R_i} . \quad (2.7)$$

where $q_{s_i}^{(m)} := (q_{s_i,1}^{(m)}, \ldots, q_{s_i,d_{R_i}}^{(m)})$ and $(e_{R_i})^T = (e_{1}^{(R_i)}, \ldots, e_{d_{R_i}}^{(R_i)})$. Hence, $q_{s_i}^{(m)}$ can serve to label the basis elements corresponding to elementary matter particle states for a given representation labelled by $R_i$. In this sense, the elementary matter particles carry the kinematical (internal) quantum numbers $q_{s_i}^{(m)}$.

Taking the complex conjugate of (2.6), gives

$$[\rho_{e_{R_i}}^{(R_i)}(h_{s_i})]^{*} e_{l_i}^{(R_i)} = -q_{s_i,l_i}^{(m)} e_{l_i}^{(R_i)} . \quad (2.8)$$

Hence,

$$e_{R_i}^{(R_i)^\dagger} [\rho_{e_{R_i}}^{(R_i)}(h_{s_i})]^{\dagger} = -q_{s_i}^{(m)} e_{R_i}^{(R_i)^\dagger} . \quad (2.9)$$

So $\{e_{l_i}^{(R_i)^\dagger}\}$ furnishes a conjugate representation of $G_i$ and is obverse to $\{e_{l_i}^{(R_i)}\}$. That is, $\{e_{l_i}^{(R_i)^\dagger}\}$ represents the internal degrees of freedom of the anti-$G_i$-particles associated with $\{e_{l_i}^{(R_i)}\}$ since they are characterized by opposite quantum numbers.

The analysis in this subsection has yielded two insights that may be useful in model building. First, the Lie algebra possesses a special basis that is particularly suited to model gauge bosons and their physical attributes. Second, the matter field irreps for the direct product group $G = \otimes G_n$ should include all relevant combinations of irreps of the subgroups $G_i$. This leads to all combinations of internal quantum numbers.
B. Dynamical quantum numbers

Since local internal symmetries can be used to model some attributes of elementary particles, it is natural to include them in dynamical models. The method has been known for a long time: define a covariant derivative that facilitates building an invariant Lagrangian density. The covariant derivative encodes the interactions and, hence, the dynamics associated with the internal symmetries.

Consider a principal fiber bundle with structure group $G_i$ and Minkowski space-time base space. Let $A_i(x) := A^a_i(x) \otimes g_{ai}$ be the local coordinate expression on the base space of the gauge potential (the pull-back under a local trivialization of the connection defined on the principal bundle). $A^a_i(x)$ is a complex one-form on the base space whose hermitian components $A^a_i(x)$ represent gauge fields. The gauge field self-interactions are encoded in the covariant derivative of the gauge potential

$$F_i(x) := DA_i(x) = dA_i(x) + \frac{1}{2} [A_i(x), A_i(x)] =: F^a_i(x) g_{ai}$$

where $F^a_i$ is a two-form on the base space. In the special basis determined by the decomposition of the previous section, the commutator term describes interactions between gauge fields characterized by the kinematical quantum numbers $q_i$ by virtue of (2.8).

Matter fields will be sections of a tensor product bundle $S \otimes V$. Here $S$ is a spinor bundle over space-time with typical fiber $\mathbb{C}^4$, and $V$ is a vector bundle associated to the gauge principal bundle with typical fiber $V_{\times R_n} := \otimes V_{R_i}$.

A basis element in $\mathbb{C}^4 \otimes V_{\times R_n}$ will be denoted $e^{(\times R_n)} := \otimes e^{(R_i)}_{l_i}$. (For clarity, we will not make the spinor index explicit.) Vector space $V_{\times R_n}$ furnishes the representation $\rho^{(\times R_n)}(\otimes G_n)$. It is this representation that determines the gauge/matter field interactions via the covariant derivative $\hat{D}$;

$$\hat{D} \Psi^{(\times R_n)}(x) = \left[ \hat{D} + \rho_{e}^{(\times R_n)}(\mathcal{A}) \right] \Psi^{(\times R_n)}(x)$$

where $\mathcal{A} := i_\gamma \mathcal{A} \in \oplus G_n$ and $\Psi^{(\times R_n)}(x) := \Psi^{(\times R_n)}(x) e^{(\times R_n)}_{l_n}$. There is a scale ambiguity that resides in the matter field covariant derivative. The inner product on $\rho_{e}(G_i)$ for any faithful representation of a simple or abelian $G_i$ is proportional to the inner product on $G_i$. This implies the matrices in the covariant derivative (2.11) are determined only up to overall constants $\kappa_{G_i}$ —relative to the scale of the gauge fields. These
constants are conventionally interpreted as coupling constants characterizing the matter field/gauge boson interaction. We choose the coupling constants so that, given gauge and matter field normalizations, the parameters in the matter field covariant derivative that characterize neutral gauge/matter field interactions coincide with the kinematical quantum numbers $\mathbf{q}_i^{(m)}$.

With this choice, the parameters characterizing couplings in both the gauge and matter field covariant derivatives are the kinematical quantum numbers.

The (bare) Lagrangian density is comprised of the usual Yang-Mills terms, spinor matter field terms, ghost terms, and gauge fixing terms. The Yang-Mills terms are

$$-\frac{1}{4} \sum_i \kappa_i \mathcal{F}_i \cdot \mathcal{F}_i$$

(2.12)

where $\kappa_i$ are positive real constants. The dot product represents the Minkowski metric and an $\text{Ad}(g_i)$ invariant inner product on each $\mathcal{G}_i$. Normalization of the Lie algebra basis elements is determined by a choice of Lie algebra inner product. Since the $\mathcal{G}_i$ are compact simple and/or $u(1)$ subalgebras, the inner product on each subspace is classified by a single constant; and, hence, the normalization can be conveniently fixed by taking $g_{a_i} \cdot g_{b_i} = \kappa_i^{-1} \delta_{a_i,b_i}$. This normalization effectively fixes the scale of $A^{a_i}(x)$ and hence also the gauge fields $A^a_{\mu}(x)$ given the standard Minkowski inner product.

The most general spinor matter field Lagrangian density consistent with the requisite symmetries is, according to the suggestion from the previous section, comprised of a sum over all the inequivalent faithful irreps of the elementary matter fields:

$$\mathcal{L}_m = i \sum_{\times R_n} \kappa_{\times R_n} \overline{\Psi}^{(\times R_n)} \cdot \mathcal{D} \Psi^{(\times R_n)} + \text{mass terms}$$

(2.13)

where $\overline{\Psi}^{(\times R_n)}$ is a section of the conjugate bundle $\mathcal{S} \otimes \nabla = \mathcal{S} \otimes \nabla$ and $\kappa_{\times R_n}$ are positive real constants that are constrained by various consistency conditions; for example, anomaly considerations and CPT symmetry. It is clear that $\delta \mathcal{L}_m = 0$ for $\Psi(x) \rightarrow \exp\{\theta(x)^{a_i} \rho_e(g_{a_i})\} \Psi(x)$ despite the presence of $\kappa_{\times R_n}$ (assuming appropriate mass terms).

The dot product here represents a Lorentz and $\rho(g)$ invariant hermitian matter field inner product. It is not true in general that $\kappa_{\times R_n}$ can be absorbed by a choice of matter field inner product: If the matter fields are functionally related, the associated factors cannot be absorbed by a choice of inner product (equivalently by a field redefinition). For example,
suppose some components of two matter fields, say $\Psi(\times R_n)$ and $\tilde{\Psi}(\times \tilde{R}_n)$, are related by $(x, \Psi_x(\times R_i)) \in S \otimes V_{R_i}$ and $(x, \tilde{\Psi}_x(\times \tilde{R}_i)) \in S \otimes \overline{V_{R_i}}$ in a given chart and trivialization. In other words, the $\tilde{R}_i$ representation is the conjugate representation of $R_i$. If the representation is not real, the ratio $\kappa_{\times R_n}/\kappa_{\times \tilde{R}_n}$ may be non-trivial because the inner products can’t be adjusted separately. This persists even after renormalization. Note that for $n = 1$ or if the fields are not functionally related, the $\kappa_{\times R_n}$ can always be absorbed into the scale of $e^{(\times R_n)}$, and so the $\kappa_{\times R_n}$ can be non-trivial only for direct product groups with functionally related matter fields. The possibility of non-trivial factors $\kappa_{\times R_n}$ in the matter field Lagrangian density is a key element in our analysis.

For each individual subgroup $G_i$, the gauge and matter field terms in the Lagrangian density give rise to the conserved currents

$$J^\mu_{(a_i)} = -F_i^{\mu\nu} \cdot [g_{a_i}, A_{i\nu}] + j^\mu_{(a_i)}$$

(2.14)

where

$$j^\mu_{(a_i)} = \sum_{\times R_n} \kappa_{\times R_n} \overline{\Psi}(\times R_n) \cdot \gamma^\mu R_e(\times R_n) (g_{a_i}) \Psi(\times R_n)$$

(2.15)

are the covariantly conserved matter field currents. In particular, the neutral conserved currents associated with $G_i$ are

$$J^\mu_{(s_i)} = -F_i^{\mu\nu} \cdot [h_{s_i}, A_{i\nu}] + j^\mu_{(s_i)}$$

$$= -q_{s_i} F_i^{\mu\nu} \gamma^\mu A_{i\nu} + \sum_{\times R_n} \kappa_{\times R_n} (q^{(m)}_{s_i}) \overline{\Psi}(\times R_n) \cdot \gamma^\mu \Psi(\times R_n).$$

(2.16)

The constants $\kappa_{\times R_n} (q^{(m)}_{s_i})$ will be termed coupling strengths since they represent the scale of the gauge/matter field couplings given matter field normalizations. Evidently, not all matter field currents contribute to interactions on an equal basis. This is significant because particles characterized by a set of internal quantum numbers will appear to have scaled internal quantum numbers when interacting with gauge bosons.

However, in order to conclude this, we must first confirm that the normalization freedom in the Lagrangian density allows us to maintain equality between the renormalized parameters $q_i$ and $q^{(m)}_{s_i}$ appearing in equations (2.17) and the kinematical quantum numbers. Moreover, we must verify that the $\kappa_{\times R_n}$ do not destroy the assumed local symmetries.

The associated neutral quantum charge operators $Q_{(s_i)} := -i \int J^0_{(s_i)} dV$ of currents (2.16)
encode the *dynamical (internal) quantum numbers* in the sense that

\[
[ Q(s_i), A^\alpha_{\perp} ] = q_{s_i} A^\alpha_{\perp} \delta_{ij}
\]

\[
[ Q(s_i), \Psi^{(\times R_n)} ] = q^{(m)}_{s_i} \Psi^{(\times R_n)}
\]  \hspace{1cm} (2.17)

where the gauge and matter fields have been promoted to quantum operators and \( A^\alpha_{\perp} \) are the transverse gauge fields. The second relation follows because the conjugate momentum of \( \Psi^{(\times R_n)} \) is \( \kappa \times R_n \Psi^{(\times R_n)} \) as determined from (2.13).

Equations (2.17) are in terms of bare quantities, but they are required to be valid for renormalized quantities as well. In particular, the *renormalized dynamical quantum numbers are required to coincide with the kinematical quantum numbers*. Under the renormalizations

\[
A_i^B \rightarrow Z_{A_i}^{1/2} A_i^R
\]  \hspace{1cm} (2.18)

and

\[
\Psi^{(\times R_n)} \rightarrow Z_{\Psi^{(\times R_n)}}^{1/2} \Psi^{(\times R_n) R},
\]  \hspace{1cm} (2.19)

the basis elements \( g_{a_i} \) (equivalently the \( \kappa_{i} \)) can be re-scaled so that \( q_{s_i}^B = Z_{\kappa_{i}}^{-1/2} q_{s_i}^R \). Likewise, the coupling constants \( \kappa_{G_{i}} \) can be chosen so that \( q^{(m)}_{s_i}^B = Z_{A_i}^{-1/2} q^{(m)}_{s_i}^R \). Consequently the relations (2.17) will be maintained under renormalization. The renormalized form of equations (2.17) are to be compared to (2.3) and (2.6). That they are consistent is a consequence of: (i) the covariant derivatives (2.10) and (2.11), (ii) our choice of Lie algebra inner product, and (iii) our choice of coupling constants. This consistency ensures the renormalized gauge and matter fields appearing in the Lagrangian density are the elementary fields associated with the quantum numbers \( q_i \) and \( q^{(m)}_{s_i} \). It should be emphasized that the coupling constants are implicit in \( q_i \) and \( q^{(m)}_{s_i} \), and the \( \kappa_{\times R_n} \) do not get renormalized; or, rather, non-trivial \( \kappa_{\times R_n} \) persist after renormalization of \( \Psi^{(\times R_n)} \).

We will refer to the two equivalent types of quantum numbers — renormalized dynamical quantum numbers and kinematical quantum numbers — by the common term *intrinsic charges*. The renormalized coupling strengths \( \kappa_{\times R_n} (q^{(m)}_{s_i}^R) \) in the renormalized currents (2.16) will be called *extrinsic charges*.

To maintain the assumed local symmetries of the Lagrangian density, \( Q(a_i) \) and \( \{ g_{a_i} \} \), along with their associated commutation relations, must determine isometric algebras. We
readily find
\[
[J^0_{(a)}, J^0_{(b)}] = \delta_{ij} C_{a_ib_j}^c \left\{ -\mathcal{F}^{\mu\nu}_i : [g_{cj}, A_{i\mu}] + \sum_{R_n} \kappa_{\times R_n} \Psi^{(\times R_n)}(g_{cj}) \Psi^{(\times R_n)}(\kappa_{\times R_n}) \right\}
\]
\[
= \delta_{ij} C_{a_ib_j}^c J^0_{(c)}
\]
where \(C_{a_ib_j}^c\) are the structure constants of \(G_i\).

It is crucial that the \(\kappa_{\times R_n}\) factors do not spoil the equality between the kinematical and dynamical internal quantum numbers or the local symmetries.

The analysis in this subsection leads to the conclusion that, in some cases at least, the intrinsic charges of matter fields do not fully determine their coupling strengths to gauge bosons. Stated otherwise, the intrinsic and extrinsic charges of matter fields are not necessarily equivalent.

III. AN EXAMPLE

It is useful to illustrate the generalities of the previous section with a concrete example. We choose the product group \(SU(3) \otimes U(2)\) for obvious reasons. The associated Lie algebra is \(su(3) \oplus u(2) \cong su(3) \oplus su(2) \oplus u(1)\).

The decomposition of \(su(3)\) with respect to its Cartan subalgebra is well-known and need not be reproduced here. Suffice it to say that it contains two neutral bosons, two bosons characterized by a single non-zero quantum number, and the remaining four bosons characterized by two quantum numbers. They come in pairs with opposite charges.

Remark: This decomposition has interesting implications for QCD. The conventional view is that all eight gluons carry color charges. Our point-of-view differs substantially; indeed, from our standpoint there are two neutral gluons (with concomitant neutral QCD currents). In particular, this suggests that the salient feature of color confinement is related to the representation and not the color charge. That is, presumably, QCD gauge bosons are confined not because they carry color charge but because they furnish the adjoint representation of \(SU_C(3)\).
The decomposition of \( u(2) \) is

\[
[h_s, h_r] = 0 \quad (3.1)
\]

and

\[
[h_s, e_\pm] = \pm q_s e_\pm \quad (3.2)
\]

where \( r, s \in \{1, 2\} \). It follows that there are two neutral bosons and two oppositely charged bosons characterized by two quantum numbers.

We will consider only Dirac matter fields in the fundamental representation of \( SU(3) \) and \( U(2) \). Consequently, the matter fields are sections of an associated fiber bundle with typical fiber \( \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 \). Since \( su(3) \) and \( u(2) \) are both rank two algebras, these matter fields can be labelled by four internal quantum numbers; two associated with \( SU(3) \) and two with \( U(2) \). According to the remark at the end of section II A, inequivalent irreps of the direct product group are postulated to include the \((3, 2)\) and \((3, -2)\) and their anti-fields \((0, 2)\) and \((3, -2)\).

The field furnishing the \((3, 2)\) is a section of the bundle \( S \otimes V_{SU(3)} \otimes V_{U(2)} \). Let \( \{e_A\} \) span \( V_3 \cong \mathbb{C}^3 \) and \( \{e_a\} \) span \( V_2 \cong \mathbb{C}^2 \). Internal degrees of freedom of elementary matter fields are associated with the basis \( \{e_{Aa}\} := \{e_A \otimes e_a\} \) that spans \( \mathbb{C}^3 \otimes \mathbb{C}^2 \). Explicitly, given a trivialization and coordinate chart, the matter field is \( \Psi = \psi^{Aa} e_{Aa} \) with the spinor index implicit. The most general two-dimensional representation of \( u(2) \) furnished by \( \Psi \) consistent with decomposition \((3.1)\) is given by

\[
\rho'_{e}(h_1) = i \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \quad \rho'_{e}(e_+) = i \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \quad \rho'_{e}(e_-) = i \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}, \quad \rho'_{e}(h_2) = i \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix},
\]

where \( R, S, T, U, V \) are real constants. Evidently, the elementary matter fields \( \psi_{A1} \) and \( \psi_{A2} \) have \( U(2) \) kinematical quantum numbers \((R, U)\) and \((S, V)\) respectively. The choice of Lie algebra inner product and the orthogonality of the basis elements determine to some extent the real constants. Similarly, the three-dimensional representation can be derived based on the analogous decomposition of \( SU(3) \).

The \((3, \bar{2})\) field \( \tilde{\Psi} \) is a section of \( S \otimes V_{SU(3)} \otimes \overline{V_{U(2)}} = S \otimes V_{SU(3)} \otimes \overline{V_{U(2)}} \). Bundle \( S \otimes V_{SU(3)} \otimes \overline{V_{U(2)}} \) is the image under the bundle morphism

\[
F : S \otimes V_{SU(3)} \otimes V_{U(2)} \rightarrow S \otimes V_{SU(3)} \otimes \overline{V_{U(2)}}
\]

\[
(x, \psi^{Aa}(x)e_{Aa}) \mapsto (x, [i \tau_2]^a_\bar{a} \psi^{A\bar{a}}(x)(e_A \otimes e_\bar{a})) =: (x, \psi^{A\bar{a}}(x)e_{A\bar{a}}) \quad (3.4)
\]
in a given chart and trivialization. The corresponding conjugate \( u(2) \) representation is 
\[
\bar{\sigma}_e := (i\tau_2)^t(\rho_e^*)^t(i\tau_2).
\]
Then, in particular,
\[
\bar{\sigma}_e(h_s) \bar{\Psi} = (i\tau_2)\rho_e(h_s)^* \Psi^A(e_A \otimes e_a^*) = -q_{s,a}^{(m)}(i\tau_2) \Psi^A(e_A \otimes e_a^*) = -q_{s,a}^{(m)} \bar{\Psi}.
\] (3.5)
So \( \bar{\Psi} \) indeed transforms by the conjugate representation of \( U(2) \). (We emphasize that there is no conjugation associated with the \( SU(3) \) or Dirac index here.)

The covariant derivative acting on the matter fields in the \((3, 2)\) and \((3, \bar{3})\) representation is
\[
(D \bar{\Psi}) = \left[ \mathbb{0}[1]_{A}^{A_{a}} + G^{A} \left[ \lambda_{a}\right]_{B} B \otimes [1]_{a}^{b} + [1]_{B}^{A} \right] \psi^{A_{a}} e_{A_{a}}
\] (3.6)
and
\[
(D \bar{\Psi}) = \left[ \mathbb{0}[1]_{A}^{A_{a}} + G^{A} \left[ \lambda_{a}\right]_{B} B \otimes [1]_{a}^{b} + [1]_{B}^{A} \right] \psi^{A_{a}} e_{A_{a}}
\] (3.7)
respectively. These yield the matter field Lagrangian density;
\[
L_{m} = i\kappa \bar{\Psi}^{A_{a}} \left[ \mathbb{0}[1]_{A}^{A_{a}} + G^{A} \left[ \lambda_{a}\right]_{B} B \otimes [1]_{a}^{b} + [1]_{B}^{A} \right] \psi^{A_{a}} e_{A_{a}}
\] + mass terms \] (3.8)

Note that it is not possible to absorb both \( \kappa \) and \( \bar{\kappa} \) by a field redefinition because \( \Psi^{A_{a}} = [i\tau_2]_{A}^{a} \Psi^{A_{a}} \) and \( e_{a} \cdot e_{a} = e_{a}^{\ast} \cdot e_{a}^{\ast} \); implying that the ratio \( \bar{\kappa}/\kappa \) is non-trivial in this example.

The corresponding \( U(2) \) and \( SU(3) \) currents are
\[
J_{\mu}^{(a)} = \kappa \bar{\psi}^{A_{a}} \gamma_{\mu} \left[ \lambda_{a}\right]_{B} B \psi^{A_{a}} + \bar{\kappa} \bar{\psi}^{A_{a}} \gamma_{\mu} \left[ \lambda_{a}\right]_{B} B \psi^{A_{a}}
\] (3.9)
and
\[
J_{\mu}^{(a)} = \kappa \bar{\psi}^{A_{a}} \gamma_{\mu} \left[ \lambda_{a}\right]_{B} B \psi^{A_{a}} + \bar{\kappa} \bar{\psi}^{A_{a}} \gamma_{\mu} \left[ \lambda_{a}\right]_{B} B \psi^{A_{a}} = (\kappa + \bar{\kappa}) \bar{\psi}^{A_{a}} \gamma_{\mu} \left[ \lambda_{a}\right]_{B} B \psi^{A_{a}}
\] (3.10)
respectively. Evidently, if \( (\kappa + \bar{\kappa}) = 1 \) the original \( SU(3) \) coupling strength is preserved, i.e., the \( SU(3) \) intrinsic and extrinsic charges are equivalent. However, in this case, the \( U(2) \) external charges are fractional relative to the internal charges since \( \kappa, \bar{\kappa} \neq 0 \) by assumption.

In a realistic model, the ratio is ultimately fixed by anomaly considerations.

**IV. CONCLUSIONS**

We have studied the relation between quantum numbers and coupling strengths for internal symmetry groups that are direct product groups. It was argued that the Lagrangian
density can have non-trivial factors multiplying matter field terms that do not spoil the local invariance or the equality between kinematical and dynamical quantum numbers. However, these non-trivial factors do affect the coupling strengths of gauge/matter field interactions. This relationship is summarized by the statement that intrinsic and extrinsic charges are not necessarily equivalent.

In our specific example, we found that non-trivial factors in the Lagrangian density can preserve the $SU(3)$ coupling strength if the factors sum to unity. On the other hand, the $U(2)$ coupling strengths are modified. The example shows that elementary particles can have intrinsic $U(2)$ charges that, nevertheless, appear to be fractional extrinsic charges due to dynamics. This has obvious implications for the Standard Model and will be explored further in a separate paper.

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[7] We use the qualifiers ‘ ’ around physical because, although for a broken symmetry the physical gauge bosons correspond to a definite basis in the Lie algebra, for an unbroken symmetry all bases are equivalent by virtue of the local symmetry. On the other hand, once one chooses a matter field representation, a natural choice of Lie algebra basis for unbroken symmetry corresponds to gauge bosons that exchange a quantum of charge with the matter fields. This special basis can be regarded as ‘physical’.
[8] The assumption that the state is elementary dictates an irreducible representation since otherwise the state would have subcomponents that transformed irreducibly. These subcomponents would then be considered “elementary”. Faithful representations are stipulated so that all as-
pects of the abstract symmetry group are carried by the representation.

[9] Semisimple Lie algebras uniquely decompose into a direct sum of simple Lie algebras.

[10] We will not display the spinor components of the matter fields since we work in Minkowski space-time and they play no (apparent) role in internal symmetries.

[11] The \((m)\) superscript indicates "matter".

[12] Since the Lie algebra is a direct sum, the inner product between subspaces vanishes.

[13] From this perspective, \(U(3)\) would not imply a new long-range massless gauge boson, and we postulate that \(U(3)\) is the symmetry group for the strong interaction. There are good reasons which will be presented elsewhere to advocate \(U(3)\) as the symmetry group of strong interactions.