Some remarks and conjectures about Hankel determinants of polynomials
which are related to Motzkin paths

Johann Cigler

johann.cigler@univie.ac.at

Abstract

This note collects some results and conjectures for the generating functions of the Hankel determinants of certain polynomials which are related to Motzkin paths.

1. For non-negative integers \( n,k \) let \( M_{n,k} \) be the set of all Motzkin paths from \((0,0)\) to \((n,k)\), i.e. all lattice paths from \((0,0)\) to \((n,k)\) consisting of up-steps \( U = (1,1) \), down-steps \( D = (1,-1) \) and horizontal steps \( H = (1,0) \), which never run below the \( x \) - axis. For each path \( P \) we define the weight \( w(P) \) as the product of the weights of its steps, where the horizontal steps \( H \) have weight \( t \) and the up-steps \( U \) and down-steps \( D \) have weight 1.

Let \( M_{n,k}(t) = \sum_{P \in M_{n,k}} w(P) \) be the weight of all Motzkin paths from \((0,0)\) to \((n,k)\).

These weights satisfy

\[
M_{n,k}(t) = M_{n-1,k-1}(t) + tM_{n-1,k}(t) + M_{n-1,k+1}(t)
\]  

(1)

with \( M_{n,k}(t) = 0 \) for \( k < 0 \) and \( M_{0,k}(t) = [k = 0] \).

For \( k = 0 \) we get the Motzkin polynomials

\[
M_n(t) = M_{n,0}(t) = \sum_{j=0}^{[n/2]} \left( \begin{array}{c} n \\ 2j \end{array} \right) C_j t^{n-2j}.
\]

(2)

The right-hand side follows because there are \( \left( \begin{array}{c} n \\ 2j \end{array} \right) \) ways to choose \( 2j \) positions for the up- and down-steps and \( C_j = \frac{1}{j+1} \left( \begin{array}{c} 2j \\ j \end{array} \right) \) ways to construct Dyck paths on them.

For \( t = 1 \) we get the Motzkin numbers \( M_n = M_n(1) \), \( (M_n)_{n \geq 0} = (1,1,2,4,9,21,51,127,323,\ldots) \), (cf. [9], A001006), for \( t = 0 \) the aerated Catalan numbers \( (M_n(0))_{n \geq 0} = (1,0,1,0,2,0,5,0,\ldots) \), (cf. [9] A126120), and for \( t = 2 \) the shifted Catalan numbers \( (M_n(2))_{n \geq 0} = (1,2,5,14,\ldots) \), (cf. [9], A000108).

For positive \( k \) we get the formula

\[
M_{n,k}(t) = \sum_{j=0}^{[n-k/2]} a(k+2j,k) \left( \begin{array}{c} n \\ 2j+k \end{array} \right) t^{n-2j-k},
\]

(3)
where \( a(n,k) \) denotes the number of non-negative paths with up- and down-steps from \((0,0)\) to \((n,k)\). By the reflection principle we get
\[
a(k + 2j, k) = \binom{k + 2j}{j} - \binom{k + 2j}{j - 1},
\]
i.e. the number of all paths \((0,0) \to (k + 2j, k)\) minus the number of those paths which cross the \(x\)–axis. If we reflect the latter paths on the axis \(x = -1\) after the first crossing we get a bijection with all paths \((0,0) \to (k + 2j, -k - 2)\), which have \(j - 1\) up-steps.

The first terms of the matrix \((M_{n,k}(t))_{n,k=0}^{n+1}\) are
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 + t^2 & 1 & 0 & 0 & 0 & 0 \\
t(3 + t^2) & 2t & 1 & 0 & 0 & 0 \\
2 + 6t^2 + t^4 & 4t(2 + t^2) & 3(1 + 2t^2) & 4t & 1 & 0 \\
t(10 + 10t^2 + t^4) & 5t(1 + 4t^2 + t^4) & 3t(2 + 3t^2) & 2(2 + 5t^2) & 5t & 1
\end{pmatrix}
\]

For \(t = 0, 1, 2, 3\) these triangles occur in OEIS[9] as A053121, A064189, A039598, A091965.

We want to obtain some information about the Hankel determinants
\[
d_m^{(k)}(n, t) = \det \left( M_{n+i+j, k}(t) \right)_{i,j=0}^{n-1}
\]
of the columns of this triangle.

For \(k = 0\) the determinants \(d_m(n, t) = d_m^{(0)}(n, t)\) have been considered in [1], [2] and [4] with different methods. For \(k > 0\) and small \(m\) the determinants \(d_m^{(k)}(n, t)\) have been obtained and proved in [2]. We give an overview of these results and state conjectures for the remaining cases. In order to obtain nice results we shall always set \(d_m^{(k)}(0, t) = 1\).

The approach with orthogonal polynomials shows that the Motzkin polynomials \(M_n(t)\) are the moments of the polynomials \(p_n(x, t)\) which satisfy
\[
p_n(x, t) = (x - t)p_{n-1}(x, t) - p_{n-2}(x, t)
\]
with initial values \(p_{-1}(x, t) = 0\) and \(p_0(x, t) = 1\). Thus
\[
p_n(x, t) = F_n(x - t),
\]
where \(F_n(x)\) are the Fibonacci polynomials defined by
\[
F_n(x) = xF_{n-1}(x) - F_{n-2}(x)
\]
with \(F_0(x) = 1\) and \(F_1(x) = x\).

This implies \(d_n(n, t) = 1\) and
\[ d_i(n,t) = (-1)^i p_n(0,t) = (-1)^i F_n(-t) = F_n(t). \] (8)

The first terms are \( d_i(0,t) = 1, \ d_i(1,t) = t, \ d_i(2,t) = t^2 - 1, \ d_i(3,t) = t^3 - 2t, \ d_i(4,t) = t^4 - 3t^2 + 1. \)

The higher determinants can be computed using Dodgson’s condensation.

The first terms of \( d_z(n,t) \) are 1, \( t^1 + 1, t^2 - t^2 + 2, t^4 - 3t^4 + 3t^2 + 2, t^8 - 5t^8 + 8t^4 - 3t^2 + 3. \)

There is no obvious regularity in the coefficients, but there are nice expressions in terms of Fibonacci polynomials.

In [1] it is shown that

\[ d_z(n,t) = \sum_{j=0}^{n} F_j(t)^2 \] (9)

and in [4] that

\[ d_z(n,t) = \det \begin{pmatrix} F_n(t) & F'_n(t) \\ F_{n+1}(t) & F'_{n+1}(t) \end{pmatrix}. \] (10)

For higher orders explicit expressions become more complicated, but there are nice generating functions.

Let

\[ D_m(x,t) = \sum_{n \geq 0} d_m(n,t)x^n \] (11)

denote the generating function of \( d_m(n,t) \).

For small \( m \) it can be proved that

\[ D_0(x,t) = \frac{1}{1-x}, \] (12)

\[ D_1(x,t) = \frac{1}{1-tx + x^2}. \] (13)

\[ D_2(x,t) = \frac{1+x}{(1-x)^2 \left( \frac{1}{2} - (t^2 - 2)x + x^2 \right)}, \] (14)

\[ D_3(x,t) = \frac{(1-x^2) \left( 1 + 3tx + x^2 \right)}{(1-(t^3 - 3t)x + x^2) \left( 1-tx + x^2 \right)}. \] (15)

It turns out that there is a close relation with the Lucas polynomials \( L_n(t) \). These are defined by

\[ L_n(t) = tL_{n-1}(t) - L_{n-2}(t) \] (16)
with initial values $L_0(t) = 2$ and $L_1(t) = t$.

Let

$$\alpha(t) = \frac{t + \sqrt{t^2 - 4}}{2},$$

$$\beta(t) = \frac{t - \sqrt{t^2 - 4}}{2}$$

be the roots of $x^2 - tx + 1 = 0$.

Binet’s formulas show that

$$L_n(t) = \alpha(t)^n + \beta(t)^n,$$

$$F_n(t) = \frac{\alpha(t)^{n+1} - \beta(t)^{n+1}}{\alpha(t) - \beta(t)}.$$  \hspace{1cm} (18)

Comparing the denominators $h_m(x, t)$ of $D_m(x, t)$ in the above examples we see that

$$h_0(x, t) = 1 - x,$$

$$h_1(x, t) = 1 - tx + x^2 = (\alpha(t) - x)(\beta(t) - x) = \left(1 - \frac{x}{\alpha(t)}\right)\left(1 - \frac{x}{\beta(t)}\right) = h_0\left(\frac{x}{\alpha(t)}, t\right)h_0\left(\frac{x}{\beta(t)}, t\right),$$

$$h_2(x, t) = h_1\left(\frac{x}{\alpha(t)}, t\right)h_1\left(\frac{x}{\beta(t)}, t\right) = (\alpha(t)\alpha(t) - x)(\alpha(t)\beta(t) - x)(\beta(t)\alpha(t) - x)(\beta(t)\beta(t) - x).$$

This leads to the conjecture that $D_m(x, t)$ can be written as a fraction with denominator

$$h_m(x, t) = \prod_{\gamma_i \neq \gamma_i'} \left(\gamma_i, \gamma_i', \ldots, \gamma_i, -x\right),$$

where the product is taken over all $(i_1, i_2, \ldots, i_m) \in \{0, 1\}^m$ with $\gamma_0 = \gamma_0(t) = \alpha(t), \quad \gamma_1 = \gamma_1(t) = \beta(t)$.

I am indebted to Christian Krattenthaler for pointing out that a proof of this conjecture follows from his paper [8], Corollary 9, by setting there $s = t, \quad t = 1$ and $x_j = 0$.

For $2j \neq m$ there are \(m\) factors $\left(\alpha^{m-j} - x\right) = \left(\beta^{m-j} - x\right)$ and $\left(\beta^{m-j} - x\right) = \left(\beta^{m-j} - x\right)$.

Therefore $h_m(x, t)$ has \(m\) factors $\left(\alpha^{m-j} - x\right)\left(\beta^{m-j} - x\right) = x^2 - L_{m-2j}(t)x + 1$.

For even $m = 2\ell$ we get \(m\) factors $1 - x$. Setting

$$A_n(x, t) = x^2 - L_n(t)x + 1 = \left(x - \alpha(t)^n\right)\left(x - \beta(t)^n\right)$$

for $n > 0$, \hspace{1cm} (19)

$$A_0(x, t) = 1 - x.$$
we get
\[ h_m(x,t) = \prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{m-2j}(x,t)^{m} \]. This implies

**Theorem 1.1**

For \( m > 0 \)

\[ D_m(x,t) = \frac{R_m(x,t)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{m-2j}(x,t)^{m}} \]  

(20)

with \( R_m(x,t) \in \mathbb{Z}[x,t] \).

For \( t \in \{0,1,2\} \) some (conjured) simplifications are possible due to the periodicity of

\( \left( L_n(t) \right) \).

For example, for \( k = 0 \) and \( t = 1 \) we get

\[ D_{2m}(x,1) = \frac{r(2m,x)}{(1-x^3)^m(1-x)} \]  

(21)

where \( r(2m,x) \) is a palindromic polynomial with positive coefficients of degree

\( \text{deg}(r(2m,x)) = m(3m-2) \) and

\[ D_{2m+1}(x,1) = \frac{(1+(-1)^m x)(1+x)^2 r(2m+1,x,1)}{(1+x^3)^{m^2+m+1}} \]  

(22)

where \( r(2m+1,x) \) is a palindromic polynomial with degree \( 3m^2 + m + 1 \).

For example, \( r(1,x) = r(2,x) = 1+x \), \( r(3,x) = (1-x)(1+x)^2 \left( 1+3x+x^2 \right) \),

\( r(4,x) = 1+8x+9x^2+14x^3+32x^4+16x^5+9x^6+8x^7+x^8 \),

\( r(5,x) = (1+x)^3 \left( 1+18x+9x^2-115x^3-203x^4+132x^5+384x^6+132x^7-203x^8-115x^9+9x^{10}+18x^{11}+x^{12} \right) \).

Let us now consider \( d_{m}^{(k)}(n,t) \) and their generating functions

\[ D_m^{(k)}(x,t) = \sum_{n=0}^{\infty} d_{m}^{(k)}(n,t)x^n \]  

(23)

for \( k > 0 \).

In order to stress the analogy with the case \( k = 0 \) we consider generalized Fibonacci polynomials \( F_{n}^{(k)}(x) \) defined by
\[ F_n^{(k)}(x) = L_{k+1}(x)F_{n-1}^{(k)}(x) - F_{n-2}^{(k)}(x) \]  
(24)

with initial values \( F_{-1}^{(k)}(x) = 0 \) and \( F_0^{(k)}(x) = 1 \).

For \( k = 0 \) these reduce to \( F_n^{(0)}(x) = F_n(x) \).

Since \( \alpha^{k+1} \) and \( \beta^{k+1} \) are roots of \( (x - \alpha^{k+1})(x - \beta^{k+1}) = x^2 - L_{k+1}(t)x + 1 \) we get as in (18)

\[ F_n^{(k)}(t) = \frac{\alpha^{(k+1)(n+1)} - \beta^{(k+1)(n+1)}}{\alpha^{k+1} - \beta^{k+1}}. \]  
(25)

By [2], Theorem 1, we know that \( d_0^{(k)}((k+1)n, t) = (-1)^{\frac{k+1}{2}} \) and \( d_0^{(k)}(n, t) = 0 \) else and by [2], Theorem 2, and (25)

\[ d_1^{(k)}((k+1)n, t) = (-1)^{\frac{k+1}{2}} F_n^{(k)}(t), \quad d_1^{(k)}((k+1)n + k, t) = (-1)^{\frac{k}{2}} F_n^{(k)}(t), \quad \text{and} \quad d_1^{(k)}(n, t) = 0 \) else.

Therefore we have

\[ D_0^{(k)}(x, t) = \sum_{n=0}^{\infty} (-1)^{\frac{k+1}{2}} x^{(k+1)n} = \frac{1}{1 - (-1)^{\frac{k+1}{2}} x^{k+1}} \]  
(26)

and

\[ D_1^{(k)}(x, t) = \frac{1 + (-1)^{\frac{k}{2}} x^k}{1 - (-1)^{\frac{k+1}{2}} L_{k+1}(t)x^{k+1} + x^{2(k+1)}}. \]  
(27)

Let \( h_m^{(k)}(x, t) \) be the denominator of \( D_m^{(k)}(x, t) \). Then we have

\[ h_0^{(k)}(x, t) = 1 - (-1)^{\frac{k+1}{2}} x^{k+1} \]  
and

\[ h_1^{(k)}(x, t) = 1 - (-1)^{\frac{k+1}{2}} L_{k+1}(t)x^{k+1} + x^{2(k+1)} = \left( \alpha(t)^{k+1} - (-1)^{\frac{k+1}{2}} x^{k+1} \right) \left( \beta(t)^{k+1} - (-1)^{\frac{k+1}{2}} x^{k+1} \right). \]

Further computations suggest that \( D_m^{(k)}(x, t) \) can always be written as a fraction with denominator \( h_m^{(k)}(x, t) = \prod_{i=1}^{n} \left( \gamma_i \cdot \ldots \cdot \gamma_n \right)^{k+1} \) where \( \gamma_i = (-1)^{\frac{k+1}{2}} x^{k+1} \).

Setting

\[ A_{k,0}(x, t) = 1 + (-1)^{\frac{k-1}{2}} x^{k+1}, \]  
(28)

\[ A_{k,a}(x, t) = 1 + (-1)^{\frac{k-1}{2}} L_n(t)x^{k+1} + x^{2(k+1)} \]
we get

Conjecture 1.2

\[
D_m^{(k)}(x,t) = \frac{R_m^{(k)}(x,t)}{\prod_{j=0}^{\lfloor m/2 \rfloor} A_{k,(k+1)(m-2j)}(x,t)}
\] (29)

with \( R_m^{(k)}(x,t) \in \mathbb{Z}[t,x] \).

A closer look at \( D_m^{(k)}(x,t) \) suggests that some factors can be cancelled. This leads to

Conjecture 1.3

\[
D_m^{(k)}(x,t) = \frac{r_m^{(k)}(x,t)}{\prod_{j=0}^{\lfloor m/2 \rfloor} A_{k,(k+1)(m-2j)}^{(1,(m-j))}(x,t)}
\] (30)

where \( r_m^{(k)}(x,t) \) is a polynomial of degree \( \deg r_m^{(k)}(x,t) = \left\lfloor \frac{m+1}{3} \right\rfloor + k \left( \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor \right) \).

For \( t = 2 \) the matrix \( \left( M_{n,k}^{(2)} \right)_{n,k \geq 0} \) is the Catalan triangle [9], A039598, whose first terms are

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\
14 & 14 & 6 & 1 & 0 & 0 & 0 & 0 \\
42 & 48 & 27 & 8 & 1 & 0 & 0 & 0 \\
132 & 165 & 110 & 44 & 10 & 1 & 0 & 0 \\
429 & 572 & 429 & 208 & 65 & 12 & 1 & 0
\end{pmatrix}.
\]

Here we guess that

\[
D_m^{(k)}(x,2) = \frac{A_m^{(k)}(x,2)}{\left[ \left( 1 - (-1)^{\frac{k+1}{2}} \right) A^{(m+1)}_m \right]^{\frac{(m+1)}{2}}}
\] (31)

where \( A_m^{(k)}(x,2) \) is a polynomial of degree \( \deg A_m^{(k)}(x,2) = \frac{m(k+1)(m+k-1)}{2} \) which satisfies

\[
x^{-\frac{m(k+1)(m+k-1)}{2}} A_m^{(k)}(\frac{1}{x},2) = \varepsilon(k,m) A_m^{(k)}(x,2)
\] (32)
with \(\varepsilon(k,m) = 1\) if \(k \equiv 0(\text{mod } 4)\), \(\varepsilon(k,m) = (-1)^m\) if \(k \equiv 1(\text{mod } 4)\), \(\varepsilon(k,m) = (-1)^{\frac{m+1}{2}}\) if \(k \equiv 1(\text{mod } 4)\), and \(\varepsilon(k,m) = (-1)^{\frac{m-1}{2}}\) if \(k \equiv 2(\text{mod } 4)\).

For \(k = 0\) formula (31) is a known result. The Hankel determinants of the shifted Catalan numbers satisfy

\[
\det \left( C_{m+i+j} \right)_{i,j=0}^{n-1} = \prod_{1 \leq i \leq j \leq m-1} \frac{2n+i+j}{i+j}.
\]

I owe to Sam Hopkins [5] the observation that the right-hand side can be interpreted as the number of plane partitions of the form \((m-1,m-2,\ldots,1)\) of non-negative integers \(\leq n\). (A simple proof due to Christian Krattenthaler can be found in [3]). Formula (31) can be deduced from [10], Theorem 3.15.8.

The polynomials \(A_m^{(0)}(x,2)\) are palindromic with positive coefficients of degree \(\frac{m}{2}\). For example, \(A_0^{(0)}(x,2) = 1\), \(A_1^{(0)}(x,2) = 1 + x\), \(A_2^{(0)}(x,2) = 1 + 7x + 7x^2 + x^3\), \(A_3^{(0)}(x,2) = 1 + 31x + 187x^2 + 330x^3 + 187x^4 + 31x^5 + x^6\).

For \(k > 0\) we get for example

\[
A_1^{(k)}(x,2) = 1 + (-1)^{\frac{k}{2}} x^k,
\]
\[
A_2^{(k)}(x,2) = 1 + (-1)^{\frac{k}{2}} x^{k-1} + (-1)^{\frac{k}{2}} (k+1)x^k \left( 1 + (-1)^{\frac{k+1}{2}} x^{k+1} \right) - x^{2k+2} \quad \text{for } k \geq 2, \text{ and}
\]
\[
A_2^{(1)}(x,2) = \left(1 - x^2\right) \left(1 + 4x + x^2\right).
\]

Apparently there are also analogs of (9) and (10):

**Conjecture 1.4**

The determinants \(d_2^{(k)}(n,t)\) satisfy

\[
(-1)^{\frac{k+1}{2}} d_2^{(k)}((k+1)n,t) = F_n^{(k)}(t)^2,
\]
\[
(-1)^{\frac{k+1}{2}} d_2^{(k)}((k+1)n+k+1,t) = F_n^{(k)}(t)^2,
\]
\[
(-1)^{\frac{k+1}{2}} d_2^{(k)}((k+1)n+k,t) = L_{n+1}(t) \sum_{j=0}^{n} F_j^{(k)}(t)^2 = \det \begin{pmatrix} F_n^{(k)}(t) & F_n^{(k)}(t)' \\ F_{n+1}^{(k)}(t) & F_{n+1}^{(k)}(t)' \end{pmatrix},
\]
\[
d_2^{(k)}(n,t) = 0 \quad \text{else}.
\]

2. Finally let us consider a slight generalization by changing the weight of the horizontal steps \(H\) on height 0 to \(s\) instead of \(t\). Let \(M_{n,k}(t,s)\) denote these weights of the Motzkin paths.
Here we get \( d_0(n,t,s) = 1 \) and

\[
\sum_{n \geq 0} d_1(n,t,s)x^n = \frac{1+(s-t)x}{1-tx+x^2}
\] (34)

\[
\sum_{n \geq 0} d_2(n,t,s)x^n = \frac{1+(1+s^2-t^2)x+(s-t)^2x^2}{(1-x)^2(1+(2-t^2)x+x^2)}.
\] (35)

**Conjecture 2.1**

\[
\sum_{n \geq 0} d_m(n,t,s)x^n = \frac{R_m(x,t,s)}{\prod_{j=0}^{\lfloor m/2 \rfloor} A_{0,m-2j}^1(x,t)}
\] (36)

where \( R_m(x,t,s) \) is a polynomial in \( x,s,t \) with integer coefficients with

\[
\text{deg}_x R_m(x,t,s) = \left( \frac{m+1}{3} \right) + 1.
\]

**Conjecture 2.2**

For all \( s \) the sequences \( d^{(k)}_m(n,t,0)_{n \geq 0} \) and \( d^{(k)}_m(n,t+s,s)_{n \geq 0} \) satisfy the same recurrence of order \( 2^{k+m} \) for all \( k,m \geq 0 \). In the special case \( m = 0 \) we even get

\[
d^{(k)}_0(n,t+s,s) = d^{(k)}_0(n,t,0).
\] (37)

Let us mention some explicit formulas for some small \( m \) and \( k \):

\[
\sum_{n \geq 0} d^{(1)}_0(n,t,0)x^n = \frac{1-tx}{1-tx+x^2}
\] (38)

\[
\sum_{n \geq 0} d^{(2)}_0(n,t,0)x^n = \frac{1+x+t^2x^2}{1+x+t^2x^2+x^3+x^4}
\] (39)

\[
\sum_{n \geq 0} d^{(3)}_0(n,t,0)x^n = \frac{1+tx+6\binom{t+1}{3}x^3+(t^4+t^2-1)x^4+t^3x^5}{1+tx+6\binom{t+1}{3}x^3+(t^2-1)(t^2+2)x^4+6\binom{t+1}{3}x^5+tx^7+x^8}.
\] (40)

The denominator for \( k = 4 \) is

\[
1-x+t^2x^3-t^2(2t^2-1)x^4+(t^2-1)(t^2+3)x^5-(2t^2-3)x^6-t^2(t^4-t^2+1)x^7+(t^2-1)t^2(t^4+t^2+2)x^8
\]

\[-t^2(t^4+t^2+1)x^9-(2t^2-3)x^{10}+(t^2-1)(t^2+3)x^{11}-t^2(2t^2-1)x^{12}+t^2x^{13}-x^{15}+x^{16}\]
\[
\sum_{n=0}^{\infty} d_1^{(1)}(n,t,s)x^n = \frac{1 + (1 + t(s-t))x + (s-t)^2x^2}{1 + (s-t)x + \left(L_2(t) + (s-t)^2\right)x^3 + (s-t)x^3 + x^4}
\]  
(41)

\[
\sum_{n=0}^{\infty} d_1^{(2)}(n,t,0)x^n
= \frac{1 + tx + t^2(t^2 - 2)x^2 + t(2t^2 - 3)x^3 + (t^4 + t^2 - 1)x^4 + t^2(t^2 - 2)x^5}{1 + tx + t^2(t^2 - 2)x^2 + t(2t^2 - 3)x^3 + t^2(2t^2 - 3)x^4 + t^2(t^2 - 2)x^6 + tx^7 + x^8}.
\]  
(42)

For \((t,s) = (2,1)\) we get another Catalan triangle (cf. [9], A039599), whose first terms are
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 1 & 0 & 0 & 0 & 0 \\
5 & 9 & 5 & 1 & 0 & 0 & 0 \\
14 & 28 & 20 & 7 & 1 & 0 & 0 \\
42 & 90 & 75 & 35 & 9 & 1 & 0 \\
132 & 297 & 275 & 154 & 54 & 11 & 1 
\end{pmatrix}.
\]

In this case we get more information about the generating functions:
\[
\sum_{n=0}^{\infty} d_m^{(k)}(n,2,1)x^n = \frac{a_m^{(k)}(x)}{(1 - (-1)^kx^{2k+1})^{[m]!}},
\]  
(43)

with \(\deg a_m^{(k)}(x) = (m(m-1)+1)k + \binom{m-1}{2}\)

and

\[
\text{sgn} \left( \frac{a_m^{(k)}\left(\frac{1}{x}\right)}{a_m^{(k)}(x)} \right) = (-1)^{\binom{k}{2}} \text{ if } k \equiv 1, 2(\text{mod } 4) \text{ and } \text{sgn} \left( \frac{a_m^{(k)}\left(\frac{1}{x}\right)}{a_m^{(k)}(x)} \right) = (-1)^{\binom{k+1}{2}} \text{ if } k \equiv 3, 0(\text{mod } 4).
\]

The computations have been made with Mathematica and the Mathematica packages Guess by Manuel Kauers [6] and RATE by Christian Krattenthaler [7].

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