Malliavin calculus for the optimal estimation of the invariant density of discretely observed diffusions in intermediate regime.

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Abstract

Let \((X_t)_{t \geq 0}\) be solution of a one-dimensional stochastic differential equation. Assuming that a discrete observation of the process \((X_t)_{t \in [0,T]}\) is available, when \(T\) tends to \(\infty\), our aim is to study the convergence rate for the estimation of the invariant density in cases where the effect of sampling is non-negligible. This scenario is referred to as the 'intermediate regime'. We find the convergence rates associated to the kernel density estimator we proposed and a condition on the discretization step \(\Delta_n\) which plays the role of threshold between the intermediate regime and the continuous case. In intermediate regime the convergence rate is \(n^{-2\beta+1}\), where \(\beta\) is the smoothness of the invariant density. After that, we complement the upper bounds previously found with a lower bound over the set of all the possible estimator, which provides the same convergence rate: it means it is not possible to propose a different estimator which achieves better convergence rates. This is obtained by the two hypotheses method; the most challenging part consists in bounding the Hellinger distance between the laws of the two models. The key point is a Malliavin representation for a score function, which allows us to bound the Hellinger distance through a quantity depending on the Malliavin weight.

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Non-parametric estimation, Malliavin calculus, invariant density, discrete observations, convergence rate, local time, ergodic diffusion.

1 Introduction

In this work, we consider the process \((X_t)_{t \geq 0}\), solution to the following stochastic differential equation:

\[
X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dB_s,
\]

where \(B\) is a one dimensional Brownian motion. Starting from the discrete observation of the process at the times \(0 = t^n_0 \leq \cdots \leq t^n_n = T\), with \(T \to \infty\), we aim at discussing the effect of the discretization of the observations on the optimal rate of convergence for the nonparametric estimation of the invariant density.

The field of nonparametric statistics for diffusion processes has become more and more relevant, in statistics. Due to their practical relevance as standard models in many areas of applied science such as genetics, meteorology or financial mathematics, the statistical analysis of stochastic differential equations receives nowadays special attention. Inference for stochastic differential equations (SDEs) based on the observation of sample paths on a time interval \([0, T]\) has already been widely investigated in several different context. Moreover, these works have opened the field of inference for more complex stochastic differential equations

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such as diffusions with jumps, diffusions with mixed effects or McKean-Vlasov diffusion models; to name a few.

Regarding the issue of nonparametric invariant density estimation for stochastic processes, contributions in this context include [14], [11]; more recently [4] and [21] starting from the observation of diffusion with jumps; [8] and [18] for diffusions driven by a fractional Brownian motion, [20] for interacting particles system, [37] for iid copies of diffusions as in (1). The nonparametric estimation of the invariant density starting from the observation of a stochastic differential equation in an anisotropic context has recently been studied in [47] and [5] assuming that the continuous record of the process is available. Closer to the purpose of this work, [6] deals with the estimation of the invariant density from the discrete observation of a stochastic differential equation as in (1), for $d \geq 2$. In particular, the results in [6] provide a condition on the discretization step needed in order to recover the continuous convergence rates, (which are the convergence rate achieved in the case where the continuous trajectory of the process is available). Moreover, Theorems 2 and 3 in [6] state that, in the case where such a condition is not satisfied, the kernel density estimator achieves the convergence rate $(\frac{1}{n})^{\frac{2\beta}{d+2\beta}}$, where $\beta$ is the harmonic mean smoothness of the invariant density we want to estimate and $d$ the dimension of the process. Up to our knowledge, few results deals with the optimality in the context of a discrete sampling. A notable exception is [13], where the authors obtains lower bounds for the estimation of the invariant density associated to the same process as in (1) starting from the observation of $X_{\Delta}, \ldots, X_{n\Delta}$ for $n \to \infty$ where $\Delta = 1$ is fixed. However, their approach heavily relies on the fact the discretization step does not go to zero. For example their observation can be replaced by some independent random variable while in our context $\Delta = \Delta_n$ goes to 0 and so the data are definitively far from being independent. Hence, the technique proposed in [13] can not work in our framework and some other methods need to be introduced.

In this paper we identify, in dimension 1, the conditions on the sampling step that delineate between two regime for the estimation rate. Under a regime where the sampling step goes to 0 fast enough, it is possible to estimate the stationary distribution with the same rate as in the continuous observation case. We introduce the term 'intermediate regime' to refer to the high-frequency case where the discretization step $\Delta_n$ approaches 0, but not rapidly enough to achieve the same convergence rate as continuous observation. Precisely, we define the 'intermediate regime' as the scenario where the discretization step $\Delta_n := t_{i+1} - t_i$ is greater than $(\frac{1}{n})^{\frac{2\beta}{d+2\beta}}$ (according with Theorem 2 below). In this context, we observe that the kernel density estimators achieve a convergence rate of $(\frac{1}{n})^{\frac{2\beta}{d+2\beta}}$, which aligns with the findings presented in [6]. We complement this upper bound with the corresponding lower bound, demonstrating that the infimum over all possible estimators of the pointwise $L^2$ error is always larger than $(\frac{1}{n})^{\frac{2\beta}{d+2\beta}}$. This finding implies that the kernel density estimator we propose achieves the best possible convergence rate in the intermediate regime.

The lower bound presented in Theorem 3 represents the main result of this work and is based on the two hypotheses method. In particular, we introduce two models, denoted as $X$ and $\tilde{X}$, sharing the same drift but with different diffusion coefficients. It is worth noting that constructing the two hypotheses by disturbing the diffusion coefficient, instead of the drift, might seem unusual to readers familiar with lower bound proofs. However, we have chosen this approach because, even in the case of parameter estimation, the drift coefficient is estimated at a rate of $\sqrt{T}$, which hinders the ability to perceive the dependence on the discretization step. As our results confirm, perturbing the diffusion coefficient proves to be the appropriate strategy in this context, enabling the observation of the dependence on the discretization step.

In order to evaluate the total variation distance between these two models we use an interpolation argument which leads us to the introduction of the process $X^\epsilon$ for $\epsilon \in [0, 1]$ (see its definition in (22)). Then, the main proof of the lower bound is reduced to the research of a bound for the Fisher information $E[(\frac{\partial p_{\Delta_n}}{\partial x} - \frac{\partial \tilde{p}_{\Delta_n}}{\partial x})^2]$, where $p_{\Delta_n}(x_0, y)$ is the transition density of the process $X^\epsilon$ starting in $x_0$ and arriving in $y$ after a time $\Delta_n$, and $\frac{\partial p_{\Delta_n}}{\partial x}(x_0, y)$ its derivative with respect to $\epsilon$.

The central aspect of our proof relies on a Malliavin representation for the quantity $\frac{\partial p_{\Delta_n}}{\partial x}(x_0, y)$. This approach mirrors the one used to establish the LAMN (Local Asymptotic Mixed Normality) property of the process.

Regarding the literature concerning the LAMN property, it was initially proven for a statistical
model of one-dimensional diffusion processes with synchronous, equipossed observations by Dohmal [22]. Later, the results were extended to multidimensional diffusions by Gobet [27], who utilized a Malliavin calculus approach. Subsequently, in [28], Gobet demonstrated the LAN (Local Asymptotic Normality) property for ergodic diffusion processes as $T$ goes to infinity. This was further extended to the case of nonsynchronously observed diffusion processes in [43].

In particular, our methodology consists in estimating the local Hellinger distance at time $\Delta_n$ and then conclude by tensorization. Using Malliavin calculus, we can then bound the Hellinger distance by a quantity depending on the Malliavin weight. The approach we propose in this part is close to the one presented in [17]. However, in [17] the author can bound the Hellinger distance by the $L^2$ norm of the Malliavin weight, while in our case it appears to be not enough. A challenge, in our paper, consists indeed in proposing a sharp bound for the conditional expectation of the Malliavin weight (see Proposition 4 and Lemma 9 below).

This is achieved by obtaining some occupation formulas and some upper bounds for the conditional first moment for integrals of the local time.

The estimation of occupation time functionals is a well-studied topic in the literature: it appears in the study of numerical approximation schemes for stochastic differential equations ([29], [33], [39]) and in the analysis of statistical methods for stochastic processes ([16], [24], [32]). Furthermore, their smoothness properties play an important role for solving ordinary differential equations, for example in combination with the phenomenon of regularization by noise (see for example [15]). Some estimations for occupation time functionals of stationary Markov processes can be found in [2], while [1] applies also to non-Markovian processes.

Denoting as $\hat{B}$ a Brownian motion and as $(L_t^z(\hat{B}))_t$, the local time at level $z$ of the Brownian motion $\hat{B}$, we show in Lemma 11 some upper bounds of $E_y[\int_0^{\Delta_n} \frac{dL_t^z(\hat{B})}{\gamma}\mid \hat{B}_{\Delta_n} = x]$ for $\gamma \geq 0$, where $E_y[\cdot]$ is the conditional expectation given $\hat{B}_0 = y$. In particular, we deduce some controls which extend the results in [31], where this quantity has been studied in detail for $\gamma \leq \frac{1}{2}$. We remark that, for $\gamma = 0$, the conditional expectation above turns out being the expectation of the local time for the Brownian bridge, which has been intensively studied in [44].

Let us introduce the notation $E_{(a,b)}(\cdot)$ for the expectation under the law of $(X_t)_{t \in [0,T]}$, stationary solution of (1). Then, we are able to prove that, when $\Delta_n > \left(\frac{1}{n} \right)^\frac{1}{\beta}$, the following lower bound holds true:

$$\inf_{\hat{\pi}_{T_n}(a,b) \in \Sigma} \sup_{(a,b) \in \Sigma} E_{(a,b)}[(\hat{\pi}_{T_n}(x^*) - \pi(x^*))^2] \geq c \left( \frac{1}{n} \right)^\frac{2\beta}{\beta + 1}, \tag{2}$$

where the infimum is taken over all estimators of the invariant density based on $X_0, X_{\Delta_1}, \ldots, X_{n\Delta_n}$. By estimator we mean any real-valued random variable given as a measurable function of $X_0, X_{\Delta_1}, \ldots, X_{n\Delta_n}$. Here above $\Sigma$ is a class of coefficients for which the stationary density has some prescribed regularity and $\beta$ is the smoothness of the invariant density $\pi$.

The lower bound in (2) complements the upper bounds we show in our Theorem 2:

$$\sup_{(a,b) \in \Sigma} E_{(a,b)}[(\hat{\pi}_{n,a}(x^*) - \pi(x^*))^2] \leq \begin{cases} \frac{1}{n} \frac{1}{\beta + 1} & \text{if } \Delta_n \leq \left( \frac{1}{n} \right)^\frac{1}{\beta}, \\ \frac{1}{n} \frac{1}{\beta + 1} + \frac{1}{n} \frac{1}{\beta + 2} & \text{if } \Delta_n > \left( \frac{1}{n} \right)^\frac{1}{\beta}, \end{cases} \tag{3}$$

where $\hat{\pi}_{n,a}(x)$ is the kernel density estimator.

We observe that the convergence rate recovered above for $\Delta_n \leq \left( \frac{1}{n} \right)^\frac{1}{\beta}$ is the superoptimal rate $1/n$, which is the optimal rate when the continuous trajectory of the process is available and has already been deeply studied in the literature (see for example [35]). Here we also study in detail what happens in the intermediate regime, which is completely new.

We remark that the convergence rate we found in the intermediate regime is the same as for the estimation of a probability density belonging to an Hölder class, associated to $n$ iid random variables $X_1, \ldots, X_n$. To summarize our finding, the results (2)–(3) show that the optimal rate of estimation for $\pi(x^*)$ is the slowest rate between the super-efficient rate $1/T_n$ and the classical non parametric one $n^{-\frac{2\beta}{\beta + 2}}$. The condition $\Delta_n = (1/T_n)^{1/(2\beta)}$ is the critical value for which these two rates are equal, and defines the frontier between the two regimes.

While our findings are confined to dimension 1, it prompts curiosity about the potential existence of a similar dichotomy in higher dimensions. A comparison with Theorems 2 and 3 in [6] reveals a similar dichotomy in the upper bounds, even though the critical values exhibit different structures depending on the dimension. On the contrary, establishing a lower bound becomes notably more intricate in higher dimensions. In Remark 5, we discuss the additional challenges that must be addressed to extend the lower bound established in Theorem 3 to higher dimensions.
The outline of the paper is the following. In Section 2 we introduce the model and we list the assumptions we will need in the sequel, while Section 3 is devoted to the construction of the estimator and the statement of our main results. In Sections 4.1–4.2, we give the proof of the upper bound (3), while in Section 4.3, we construct the two hypotheses setting and deduce the lower bound (2). The Section 5 is devoted to the proof of the main control on the Malliavin weight used in the representation of the Fisher information. The proof of the technical results is delegated to the Appendix A, while Appendix B is devoted to an introduction of Malliavin calculus, presenting some helpful tools used along the manuscript.

2 Model Assumptions

We aim at proposing a non-parametric estimator for the invariant density associated to a monodimensional diffusion process $X$. The diffusion is a strong solution of the following stochastic differential equation:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t a(X_s)dB_s, \quad t \in [0, T], \quad (4)$$

where $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ and $B = (B_t, t \geq 0)$ is a standard Brownian motion. The initial condition $X_0$ and $B$ are independent.

**A1**: The functions $b(x)$ and $a(x)$ are globally Lipschitz functions of class $C^3$. For all $x \in \mathbb{R}$ the following hold true for the functions $a, b$ and their first three derivatives :

$$|b(0)| \leq b_0, \quad |a(x)| \leq a_0, \quad |b^{(l)}(x)| \leq b_l, \quad |a^{(l)}(x)| \leq a_l, \quad \text{with } l = 1, 2, 3,$$

for some positive constants $(a_l)_{0 \leq l \leq 3}$, $(b_l)_{0 \leq l \leq 3}$. Furthermore, for some $a_{\min} > 0$,

$$a^2_{\min} \leq a^2(x).$$

**A2** (Drift condition):

There exist $\tilde{C} > 0$ and $\tilde{\rho} > 0$ such that $xb(x) \leq -\tilde{C}|x|$, $\forall x : |x| \geq \tilde{\rho}$.

Under the assumptions A1 - A2 the process $X$ admits a unique invariant distribution $\mu$ and the ergodic theorem holds (see Theorem 1.4 in [36]). We suppose that the invariant probability measure $\mu$ of $X$ is absolutely continuous with respect to the Lebesgue measure and from now on we will denote its density as $\pi: d\mu = \pi dx$.

We want to estimated the invariant density $\pi$ belonging to the Hölder class $H(\beta, L)$ defined below.

**Definition 1.** Let $\beta > 0$, $L > 0$. A function $g : \mathbb{R} \to \mathbb{R}$ is said to belong to the Hölder class $H(\beta, L)$ of functions if,

$$\|g^{(k)}\|_{\infty} \leq L, \quad \forall k = 0, 1, \ldots, \lfloor \beta \rfloor,$$

$$\|g^{(\lfloor \beta \rfloor)}(\cdot + t) - g^{(\lfloor \beta \rfloor)}(\cdot)\|_{\infty} \leq L|t|^{\beta - \lfloor \beta \rfloor} \quad \forall t \in \mathbb{R},$$

for $g^{(k)}$ denoting the $k$-th order derivative of $g$ and $\lfloor \beta \rfloor$ denoting the largest integer strictly smaller than $\beta$.

This leads us to consider a class of coefficients $(a, b)$ for which the stationary density $\pi = \pi_{(a, b)}$ has some prescribed Hölder regularity.

**Definition 2.** Let $\beta > 0$, $L > 0$, $(a_l)_{0 \leq l \leq 3} \in (0, \infty)^4$, $(b_l)_{0 \leq l \leq 3} \in (0, \infty)^4$, $0 < a_{\min} < a_0$, $\tilde{C} > 0$, $\tilde{\rho} > 0$.

We define $\Sigma(\beta, L, a_{\min}, (a_l)_{0 \leq l \leq 3}, (b_l)_{0 \leq l \leq 3}, \tilde{C}, \tilde{\rho})$ the set of couple of functions $(a, b)$ where $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are such that

- the coefficients $a$ and $b$ satisfy A1 with the constants $a_{\min}, (a_l)_{0 \leq l \leq 3}, (b_l)_{0 \leq l \leq 3}$,
- $b$ satisfies A2 with the constants $(\tilde{C}, \tilde{\rho})$,
- the density $\pi_{(a, b)}$ of the invariant measure associated to the stochastic differential equation (4) belongs to $H(\beta, L)$. 

We know from Theorem 1.4 in [36] that the stationary density \( \pi = \pi_{(a,b)} \) is explicit as a function of the coefficients of the one dimensional SDE, and in consequence the conditions in Definition 2 are not independent. In particular, if the coefficients \( a \) and \( b \) of the diffusion are \( C^k \), then the stationary density is also of class \( C^3 \). Thus, for a diffusion satisfying A1–A2, the stationary density must be at least \( C^3 \), and we can check that its derivatives up to order three are bounded. Hence, \( \pi \) is at least of class \( \mathcal{H}(3,L) \) for some \( L > 0 \).

Let us emphasize that the status of the conditions in Definition 2 are different. Condition (A1) is a technical smoothness condition on the model, and we require up to order three regularity for the coefficients in order to apply results in [38] on the transition density of the process (see Theorem 1 below). Condition (A2) is a classical mean reverting condition on the drift, sufficient to get the existence of a stationary probability. The last condition on the H"older smoothness of \( \pi \) is a standard assumption on the regularity of the non-parametric function which is estimated.

In the subsequent discussion, our analysis heavily relies on well-established results pertaining to the transition density \( p_t(x,y) \) of diffusion processes. To ensure clarity, we deem it essential to explicitly outline these results. The bound on the transition density in the presence of an unbounded drift can be derived from [38]. Notably, when contrasting this bound with the known results for the transition density under a bounded drift, the primary distinction arises from substituting the primary distinctness arises from substituting the transition density

\[
\hat{\theta}_{t,s}(x) = b(\theta_{t,s}(x)), \quad t \geq 0, \quad \theta_{t,s}(x) = x.
\]

With this in mind, we recall the first point of Theorem 1.2 of [38] and the results of Section 4 of the same reference, in the theorem below.

**Theorem 1** ([38]). Under A1-A2, for any \( \tau > 0 \), \( (s,t) \in [0,\infty)^2 \), \( 0 < t-s < \tau \), the unique weak solution of (4) admits a transition density \( p_{t-s}(x,y) \) which is continuous in \( x,y \in \mathbb{R} \). Moreover, there exist \( \lambda_0 \in (0,1) \) and \( C_0,c_0 \geq 1 \) such that, for any \( (s,t) \), \( 0 < t-s < \tau \) and \( x,y \in \mathbb{R} \) it is

\[
p_{t-s}(x,y) \leq C_0(t-s)^{-1} \exp\left(-\lambda_0 \frac{|\theta_{t,s}(x)-y|^2}{t-s}\right)
\]

and the constants depend only on \( \tau \), \( a_{\min} \), \( b_0 \), \( a_0 \), \( a_1 \) and \( b_1 \).

Moreover, for \( k = 1,2 \), we have the control on the derivatives,

\[
\left| \frac{\partial^k p_{t-s}(x,y)}{\partial y^k} \right| \leq C_0(t-s)^{-1} \exp\left(-\lambda_0 \frac{|\theta_{t,s}(x)-y|^2}{t-s}\right)
\]

where the constants \( C_0 \) and \( \lambda_0 \) depends on \( \tau \), \( a_{\min} \), \( a_l \), \( b_l \) for \( l = 0,\ldots,3 \).

In the next section we will propose an estimator for the estimation of the invariant density \( \pi \) starting from a discrete observation of the process \( X \). In particular, we want to find the convergence rates in intermediary regime, i.e. when the discretization step goes to zero but the associated error is not negligible.

### 3 Construction estimator and main results

We suppose that we observe a finite sample \( X_{t_0}, \ldots, X_{t_n} \), with \( 0 = t_0 < t_1 < \cdots < t_n =: T_n \). The process \( X \) is solution of the stochastic differential equation (4). Every observation time point depends also on \( n \) but, in order to simplify the notation, we suppress this index. We assume the discretization scheme to be uniform which means that, for any \( i \in \{0, \ldots, n-1\} \), it is \( t_{i+1} - t_i =: \Delta_n \). We will be working in a high-frequency setting i.e. the discretization step \( \Delta_n \to 0 \) for \( n \to \infty \). We assume moreover that \( T_n = n\Delta_n \to \infty \) for \( n \to \infty \).

It is natural to estimate the invariant density \( \pi \in \mathcal{H}(\beta,L) \) by means of a kernel estimator. We therefore introduce some kernel function \( K : \mathbb{R} \to \mathbb{R} \) satisfying

\[
\int_{\mathbb{R}} K(x)dx = 1, \quad \|K\|_{\infty} < \infty, \quad \text{supp}(K) \subset [-1,1], \quad \int_{\mathbb{R}} K(x)x^l dx = 0, \quad (5)
\]

for all \( l \in \{0, \ldots, M\} \) with \( M \geq \beta \).

When the continuous trajectory of the process is available the convergence rates for the estimation
of the invariant density are known. In particular, in the mono-dimensional case, it is known that in our framework the proposed kernel estimator achieves the parametric rate $\frac{n}{h}$ and such a rate is optimal (see for example [35] or Theorem 1 in [34]).

We propose to estimate the invariant density $\pi \in \mathcal{H}(\beta, \mathcal{L})$ associated to the process $X$, solution to (4), disposing only of the discrete observation of the process. To do that, we propose the following kernel estimator: for $x^* \in \mathbb{R}$ we define

$$
\hat{\pi}_{h,n}(x^*) := \frac{1}{n} \sum_{i=0}^{n-1} K\left(\frac{x^* - X_i}{h}\right)(t_{i+1} - t_i)
$$

(6)

with $K$ a kernel function as in (5).

The asymptotic behaviour of the estimator proposed in (6) is based on the bias-variance decomposition. To find the convergence rates it achieves we need a bound on the variance, which heavily relies on Castellana and Leadbetter condition, as introduced in [14].

The work of Castellana and Leadbetter in [14] establishes that, subject to the condition CL described below, the density can be estimated using non-parametric estimators (including kernel estimators) with a parametric rate of $\frac{n}{h}$.

To introduce the condition CL, we require that the process $X$ belongs to a class of real processes with a common marginal density $\pi$ with respect to the Lebesgue measure on $\mathbb{R}$. Furthermore, we assume that the joint density of $(X_s, X_t)$ exists for all $s \neq t$, is measurable, and satisfies $\pi(X_s, X_t) = \pi(X_t, X_s) = \pi(X_s, X_{t-s})$. This joint density is denoted by $\pi_{t-s}$. Additionallly, we define the function $g_u$ as $g_u(x, y) = \pi_u(x, y) - \pi(x)\pi(y)$. The condition CL can be stated as follows:

**CL:** $u \mapsto \|g_u\|_\infty$ is integrable on $(0, \infty)$ and $g_u(\cdot, \cdot)$ is continuous for each $u > 0$.

It is important to remark that in our context it is $\pi_u(x, y) = \pi(x)p_u(x, y)$, where we recall that $p_u(x, y)$ is the transition density. Condition CL can be fulfilled by ergodic continuous diffusion processes (see [49] for sufficient conditions). In [19], the authors developed a projection estimator and established that its $L^2$-integrated risk achieves the parametric rate of $\frac{n}{h}$, but under a weaker condition known as WCL.

**WCL:** There exists a positive integrable function $k$ (defined on $\mathbb{R}$) such that

$$
\sup_{y \in \mathbb{R}} \int_0^\infty |g_u(x, y)| du \leq k(x), \quad \text{for all } x \in \mathbb{R}.
$$

The sufficient conditions can be separated into two components: a local irregularity condition referred to as WCL1, and an asymptotic independence condition referred to as WCL2. These conditions require the existence of two positive integrable functions $k_1$ and $k_2$ defined on $\mathbb{R}$, as well as a positive constant $u_0$, satisfying the following conditions:

**WCL1:** $\sup_{y \in \mathbb{R}} \int_0^{u_0} |g_u(x, y)| du < k_1(x), \quad \text{for all } x \in \mathbb{R},$

**WCL2:** $\sup_{y \in \mathbb{R}} \int_{u_0}^\infty |g_u(x, y)| du < k_2(x), \quad \text{for all } x \in \mathbb{R}.$

In this paper, which primarily focuses on estimation based on discrete observations, we introduce analogous conditions in the discrete framework. We refer to these conditions as WDCL1 and WDCL2, where the additional ‘D’ denotes ‘discrete’. There exist two positive functions $k_1$ and $k_2$ on $\mathbb{R}$, as well as a positive constant $u_0$, such that for all $n \geq 1$,

**WDCL1:** $\sup_{y \in \mathbb{R}} \sum_{i=1}^{n} |g_u(x, y)| < k_1(x), \quad \text{for all } x \in \mathbb{R},$

**WDCL2:** $\sup_{y \in \mathbb{R}} \sum_{i=n+1}^{\infty} |g_u(x, y)| < k_2(x), \quad \text{for all } x \in \mathbb{R},$
when the discretization step is equal to the critical value \( \Delta_n \), which is such that \( \tilde{i} := \sup \{ i \in [1, n-1] \mid i \Delta_n \leq u_0 \} \) for \( n \) large enough.

In order to show these conditions hold, the main tools consists in the bounds on the transition density \( p_t(x, y) \) and its derivatives obtained in [38] and recalled in Theorem 1. It leads us to the following bound on the variance.

**Proposition 1.** Let \( \beta > 0 \), \( \mathcal{L} > 0 \), \( (a_i)_{0 \leq i \leq 3} \in (0, \infty)^4 \), \( (b_i)_{0 \leq i \leq 3} \in (0, \infty)^4 \), \( 0 < a_{\min} < a_0 \), \( \tilde{C} > 0 \), \( \tilde{p} > 0 \) and denote by \( \Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, (a_i)_{0 \leq i \leq 3}, (b_i)_{0 \leq i \leq 3}, \tilde{C}, \tilde{p}) \) the set of coefficients \((a, b)\) introduced in Definition 2. We assume that \( X \) is a stationary solution of (4) and let \( \hat{\pi}_{h,n} \) be the estimator proposed in (6). Then, there exist \( c > 0 \) and \( T_0 > 0 \) such that, for \( T_n \geq T_0 \),

\[
\text{Var}(\hat{\pi}_{h,n}(x^*)) \leq \frac{c}{T_n} + \frac{c \Delta_n}{T_n h}
\]

Moreover, the constants \( c, T_0 \) are uniform over the set of coefficients \((a, b) \in \Sigma \) and \( x^* \in \mathbb{R} \).

We deduce the following result on the risk of the estimator \( \hat{\pi}_{h,n} \).

**Theorem 2.** Let \( \beta > 0 \), \( \mathcal{L} > 0 \), \( (a_i)_{0 \leq i \leq 3} \in (0, \infty)^4 \), \( (b_i)_{0 \leq i \leq 3} \in (0, \infty)^4 \), \( 0 < a_{\min} < a_0 \), \( \tilde{C} > 0 \), \( \tilde{p} > 0 \) and denote \( \Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, (a_i)_{0 \leq i \leq 3}, (b_i)_{0 \leq i \leq 3}, \tilde{C}, \tilde{p}) \). Then, there exist \( c > 0 \) and \( T_0 > 0 \) such that for \( T_n \geq T_0 \), the following hold true.

- **If** \( \Delta_n \lesssim \left( \frac{1}{T_n} \right)^{\frac{\beta}{2}} \), **then** there exists a sequence \((h_n) \) such that

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}_{(a, b)}[|\hat{\pi}_{h,n}(x^*) - \pi(x^*)|^2] \leq \frac{c}{T_n}.
\]

- **If otherwise** \( \Delta_n \gtrsim \left( \frac{1}{T_n} \right)^{\frac{\beta}{2}} \), **then** there exists a sequence \((h_n) \) such that

\[
\sup_{(a, b) \in \Sigma} \mathbb{E}_{(a, b)}[|\hat{\pi}_{h,n}(x^*) - \pi(x^*)|^2] \leq c \left( \frac{1}{h_n} \right)^{2\frac{\beta}{2\beta+1}}.
\]

We use the notation \( \mathbb{E}_{(a, b)} \) to emphasize that we compute the expectation under the law of \((X_t)_{t \in [0, T_n]} \) stationary solution of (4).

**Remark 1.** The conditions \( \Delta_n \lesssim \left( \frac{1}{T_n} \right)^{\frac{\beta}{2}} \) and \( \Delta_n \gtrsim \left( \frac{1}{T_n} \right)^{\frac{\beta}{2}} \) separate the two different regimes under consideration. The first corresponds to the case where it is possible to recover the continuous convergence rate, while the second is what we refer to as the intermediate regime. When \( \Delta_n \) equals the threshold \( \left( \frac{1}{T_n} \right)^{\frac{\beta}{2}} \), and knowing that \( T_n = n \Delta_n \), we find that \( \Delta_n = n^{-\frac{\beta}{2\beta+1}} \). Substituting this value, we discover that \( \frac{1}{T_n} = \left( \frac{1}{n} \right)^{\frac{2\beta}{2\beta+1}} \). Consequently, there is no effective difference between the two cases in Theorem 2 when the discretization step is equal to the critical value \( \left( \frac{1}{n} \right)^{\frac{\beta}{2}} \).

From Theorem 2 it follows that, in the intermediate regime, the convergence rate achieved by the proposed estimator is \( \left( \frac{1}{n} \right)^{\frac{\beta}{2\beta+1}} \), where \( d \) is the dimension and it is here equal to 1. It has been shown in Theorems 2 and 3 of [6] that the convergence rates in the intermediate regime are the same also in higher dimension, up to replacing \( \beta \) with \( \tilde{\beta} \), the harmonic mean of the smoothness over the \( d \) different directions. It is interesting to remark that it is also the convergence rate for the estimation of a probability density belonging to an Hölder class, associated to \( n \) iid random variables \( X_1, \ldots, X_n \).

**Remark 2.** It can be seen in the proof of Theorem 2, that the optimal bandwidths \((h_n) \) depend on the unknown smoothness degree \( \beta \). Moreover, the condition which separates the two regimes depends on \( \beta \) as well. Consequently, it can be worthwhile to propose an adaptive procedure, akin to the one initially introduced by Goldenslager and Lepski in [30], which allows to choose the bandwidth using only the data without the prior knowledge of \( \beta \). This aspect has been analyzed in a context close to ours in [3], which studies the same model as in (4) but assumes that the continuous observation of the process is available. The analogous procedure, in the case where only a discrete sampling of the process is available, has not yet been considered and is left for future investigation.
Remark 3. Observe that we know the rate $1/T_n$ is optimal in a minimax sense starting from a discrete sampling. This is a consequence of Theorem 4.3 in [36], which establishes that the convergence rate $1/T$ is optimal in a minimax sense for the estimation of $\pi$ starting from the observation of the continuous trajectory of the process. Thus, in the case $\Delta_n \lesssim \frac{\pi}{T_n}$, our estimator based on a discrete sampling is optimal in a minimax sense. In the intermediate regime, we can also prove that the convergence rate is optimal, as demonstrated in Theorem 3 below.

The following theorem demonstrates the optimality of the convergence rate identified in the intermediate regime. Its proof relies on Malliavin calculus, which is extensively detailed in Appendix B. Particularly, the crucial element for our result is the Malliavin representation of a score function through a Malliavin weight (refer to Section 4.4 and Appendix B.2 below for more information).

Theorem 3. Let $\beta \geq 3$, $\mathcal{L} > 0$, $(a_l)_{0 \leq l \leq 3} \in (0, \infty)^4$, $(b_l)_{0 \leq l \leq 3} \in (0, \infty)^4$, $0 < a_{\min} < a_0 \tilde{C} > 0$. Assume also that $\left(\frac{1}{n}\right) \lesssim \Delta_n$, $\forall n \geq 1$, and $\Delta_n = O(n^{-\varepsilon})$, for some $\varepsilon > 0$. Then, there exist $\tilde{\rho}$, $c > 0$ and $T_0 > 0$ such that, for $T_n \geq T_0$,

$$\inf_{\tilde{\pi}_{T_n}(a,b) \in \Sigma} \sup_{x^*} \mathbb{E}[(\tilde{\pi}_{T_n}(x^*) - \pi(x^*))^2] \geq c\left(\frac{1}{n}\right)^{\frac{4\varepsilon}{1+\varepsilon}},$$

where $\Sigma = \Sigma(\beta, \mathcal{L}, a_{\min}, (a_l)_{0 \leq l \leq 3}, (b_l)_{0 \leq l \leq 3}, \tilde{C}, \tilde{\rho})$ and the infimum is taken over all estimators of the invariant density at point $x^*$.

Remark 4. Contrary to the upper bound, the lower bound is not stated over any set $\Sigma$ potentially given by Definition 2, but only with some sufficiently large value for $\tilde{\rho}$. Actually, it is impossible to get the lower bound for all values $\tilde{\rho}$ as the mean reverting condition A2 could conflict with the upper bounds on the drift defined by the constants $b_0, b_1$ appearing in A1. In such case, the set $\Sigma$ is empty, and the lower bound cannot hold true.

Theorem 3 here above implies that, on a sufficiently large class of diffusion $X$ discretely observed (with a uniform discretization step), whose invariant density belongs to $\mathcal{H}(\beta, \mathcal{L})$, it is not possible to find an estimator with rate of estimation better than $\left(\frac{1}{n}\right)^{\frac{4\varepsilon}{1+\varepsilon}}$. Comparing the results here above with the upper bound in Theorem 2 we observe that the convergence rate we found in the lower bound and in the upper bound in the intermediate regime are the same. It follows that the estimator $\tilde{\pi}_{h,n}$ we proposed in (6) achieves the best possible convergence rate. On the other side, however, it is worth underlining that the class $\Sigma$ considered in Theorem 3 is more restricted than the classes allowed in the upper bounds.

Remark 5. One may wonder if it possible to extend the lower bound gathered in Theorem 3 in higher dimension, proving in this way that the convergence rate in intermediate regime found in [6] is optimal for any $d$. On one side, the proof of Theorem 3 relies on Malliavin calculus, which can be easily extended for $d > 1$. On the other side, we need some controls on the local time to bound the main term coming from the Malliavin weight and this does not allow us to move to higher dimension. In particular, the result in Lemma 9 below holds true only for $d = 1$, and less sharp estimations on the conditional expectation in the left hand side of Lemma 9 are not enough to recover the wanted convergence rate. An idea to overcome the problem and to obtain similar controls in higher dimension could be to use the solution of the Poisson equation associated to the generator of the diffusion, in a similar way as in Lemma 1 of [40]. This will be object of further investigation.

4 Proofs

This section is devoted to the proof of our main results. Remark, that in the proofs, the constant $c$ may change from line to line, but remains uniform on the class of models with diffusion and drift coefficients in $\Sigma$. 

8
4.1 Proof of Proposition 1

Proof. We start by expanding the variance term in the following way

\[
Var(\hat{\pi}_{h,n}(x^*)) = Var\left(\frac{1}{n\Delta_n} \sum_{j=0}^{n-1} K_h(x^* - X_{t_j}) \Delta_n \right)
\]

\[
= \frac{\Delta_n^2}{T_n} \left\{ nk(t_0) + 2 \sum_{j=1}^{n-1} (n-j) k(t_j) \right\},
\]

with

\[k(t_j) = \text{Cov}(K_h(x^* - X_{t_j}), K_h(x^* - X_0)).\]

We have \(k(t_0) \leq \mathbb{E}[K_h(x^* - X_0)^2] = \int_{\mathbb{R}} K_h(x^* - y)^2 \pi(y)dy \leq c \int_{\mathbb{R}} K^2(y)dy \leq \frac{c}{h},\) where \(c\) is some constant independent of \((a, b) \in \Sigma,\) as we know \(\|\pi\|_{\infty} \leq \mathcal{L}\) by Definition 2. This leads us to write,

\[
Var(\hat{\pi}_{h,n}(x^*)) \leq \frac{c\Delta_n}{T_n h} + 2 |I|
\]

where \(I = \frac{\Delta_n^2}{T_n} \sum_{j=1}^{n-1} (n-j) k(t_j).\) On \(I\) we want to use the weak discrete Castellana and Leadbetter condition as formulated in WDCL1 and WDCL2. We observe it is

\[
k(t_j) = \text{Cov}(K_h(x^* - X_{t_j}), K_h(x^* - X_0)) = \int_{\mathbb{R}} \int_{\mathbb{R}} K_h(x^* - y) K_h(x^* - z) g_{t_j}(y, z) dy dz
\]

where we recall that \(g_{t_j}(y, z) = \pi(y) p_{t_j}(y, z) - \pi(y) \pi(z).\) Hence, we can write

\[
|I| \leq \frac{\Delta_n^2}{T_n} \sum_{j=1}^{n-1} (n-j) \int_{\mathbb{R}} \int_{\mathbb{R}} |K_h(x^* - y)| |K_h(x^* - z)| |g_{t_j}(y, z)| dy dz
\]

\[
\leq \frac{c \Delta_n}{T_n} \int_{\mathbb{R}} \int_{\mathbb{R}} |K_h(x^* - y)| |K_h(x^* - z)| \sup_{z \in \mathbb{R}} \left| \sum_{j=1}^{n-1} g_{t_j}(y, z) \right| dy dz
\]

\[
\leq \frac{c \Delta_n}{T_n} \int_{\mathbb{R}} |K_h(x^* - z)| \sup_{z \in \mathbb{R}} \left| \sum_{j=1}^{n-1} g_{t_j}(y, z) \right| dy,
\]

where we have used that

\[
\int_{\mathbb{R}} |K_h(x^* - z)| dz \leq c,
\]

for some \(c > 0.\) Now the result follows once we show that WDCL1 and WDCL2 hold true, as stated in next proposition and proved at the end of this section.

Proposition 2. Suppose that \((a, b) \in \Sigma.\) Then, conditions WDCL1 and WDCL2 are satisfied with two bounded functions \(k_1\) and \(k_2.\) Moreover, \(\|k_1\|_{\infty}\) and \(\|k_2\|_{\infty}\) are bounded independently of \((a, b) \in \Sigma.\)

It directly follows from the boundedness of \(\pi\) and the properties of \(K\)

\[
|I| \leq \frac{c \Delta_n}{T_n} \int_{\mathbb{R}} |K_h(x^* - y)| \left| \frac{k_1(y) + k_2(y)}{\Delta_n} \right| dy
\]

\[
\leq \frac{c(\|k_1\|_{\infty} + \|k_2\|_{\infty})}{T_n} \int_{\mathbb{R}} |K_h(x^* - y)| dy \leq \frac{c}{T_n},
\]

where we have used the fact that the kernel function has compact support and its \(L^1\) norm is bounded by a constant. We put all the pieces together in (7), which implies the following bound holds true:

\[
Var(\hat{\pi}_{h,n}(x^*)) \leq \frac{c \Delta_n}{T_n h} + \frac{c}{T_n},
\]

which gives the proposition.
Proof of Proposition 2. We start proving WDCL1. Because of Theorem 1 we have for 0 ≤ t ≤ 2,

\[ p_t(x, y) \leq ct^{-\frac{1}{4}} e^{-\lambda_0 \frac{\| |g_{t,0}(x,y)\|}{t}} \leq ct^{-\frac{1}{4}}. \]

As \( \sup_{y \in \mathbb{R}} \pi(y) < \infty \), it gives

\[
\sup \sum_{i=1}^{\tilde{i}} \left| g_{t_i}(x, y) \right| \leq \sup \sum_{i=1}^{\tilde{i}} c(t_i^{-\frac{1}{4}} + 1) \\
\leq \left( \frac{c}{\sqrt{\Delta_n}} \sum_{i=1}^{\tilde{i}} \frac{1}{\sqrt{i}} \right) + c \tilde{i} \leq \frac{c \sqrt{i} \Delta_n + i \Delta_n}{\Delta_n},
\]

where we chose \( \tilde{i} = \sup \{i \in [1, n-1] \text{ such that } i \Delta_n \leq 2 \} \). Then, Equation (8) here above provides WDCL1 with \( u_0 = 2 \) and \( k_1(x) = e(\sqrt{2} + 2) \).

We move to the proof of WDCL2. Let us set \( \varphi(x) := E[\exp(i\xi X_t)] \) and \( \varphi_x(\xi, t) := E[\exp(i\xi X_t)|X_0 = x] \) and claim that there exists a constant \( \hat{c} > 0 \) such that for all \( \xi \in \mathbb{R}, \)

\[
|\varphi(\xi)| \leq \hat{c}(1 + |\xi|)^{-2}.
\]

Moreover, there exists \( \hat{c} > 0 \), such that for all \( t \geq 2, x \in \mathbb{R}, \text{ and } \xi \in \mathbb{R}, \)

\[
|\varphi_x(\xi, t)| \leq \hat{c}(1 + |\xi|)^{-2}.
\]

We will now prove that, if the conditions (9) and (10) hold true, then the result follows. After that, we will conclude our proof by proving that the above mentioned conditions are satisfied in our context.

Using the inverse Fourier transform, we can write

\[
2\pi(p_t(x, y) - \pi(y)) = \int_{\mathbb{R}} \exp(-i\xi y)(\varphi_x(\xi, t) - \varphi(\xi))d\xi.
\]

We set \( \psi_x(\xi) = e^{i\xi y} - \int_{\mathbb{R}} e^{i\xi z} \pi(z)dz \) which is a centered function under the stationary probability, and remark that, with this notation, \( \psi_x(\xi, t) = \psi_x(\xi) = P_t(\psi_x)(x) \) where \( (P_t)_{t \geq 0} \) is the semi-group of the diffusion. \( P_t(\psi)(x) = \int_{\mathbb{R}} p_t(x, y)\psi(y)dy \). We know from Lemma 8 in [5] that we have the following semi-group contraction property, with a constant \( c \) uniform on \( \Sigma, \)

\[
\|P_t(\psi_x)\|_{L^2(\pi)} \leq ce^{-t/c} \|\psi\|_{L^1(\pi)}.
\]

Let us state the following lemma, whose proof is postponed to the Appendix A.

**Lemma 1.** There exists a constant \( c > 0 \) depending on the class \( \Sigma \), such that for all \( \psi \in L^1(\pi), s \in (0, 1), x \in \mathbb{R}, \) we have, \( |P_s(\psi)(x)| \leq \sup_{y \in \mathbb{R}} \pi(y)^s \|\psi\|_{L^1(\pi)} \).

Applying this lemma with \( s = 1, \) we can write for \( t \geq 2, \)

\[
|P_t(\psi)(x)| = |P_t(P_{t-1}(\psi_x))(x)| \leq \frac{1}{\pi(x)} \|P_{t-1}(\psi_x)\|_{L^1(\pi)} \\
\leq \frac{1}{\pi(x)} \|P_{t-1}(\psi_x)\|_{L^2(\pi)} \leq \frac{c}{\pi(x)} e^{-(t-1)/c} \|\psi_x\|_{\infty},
\]

where in the second line we used \( \|\psi\|_{L^1(\pi)} \leq \|\psi\|_{L^2(\pi)} \) and eventually the contraction property of the semi-group as in (12). Since the functions \( \psi_x \) are bounded by the constant 2, we get \( |P_t(\psi_x)(x)| \leq ce^{-t/c} \) for some \( c > 0 \).

Meanwhile, as we claimed that both (9) and (10) are satisfied, we have \( |P_t(\psi_x)(x)| = |\varphi_x(\xi, t) - \varphi(\xi)| \leq \frac{\|\psi_x\|_{\infty}}{1 + |\xi|} \).

Using (11), we deduce

\[
|p_t(x, y) - \pi(y)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} \exp(-i\xi y)P_t(\Psi_x)(x)d\xi \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |P_t(\Psi_x)(x)|^{1/4} |P_t(\Psi_x)(x)|^{1/4} d\xi \\
\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{\hat{c} + \hat{c}}{1 + |\xi|^2} \right|^{3/4} \left| \frac{ce^{-(t-1)/c}}{\pi(x)} \right|^{1/4} d\xi \leq \frac{c}{\pi(x)^{1/4}} e^{-t/c}.
\]
for some $c > 0$. Hence, we get that there exists a finite constant $c$ such that, for all $t \geq 2$ and $x, y \in \mathbb{R}$,

$$|g_t(x, y)| = \pi(x) |p_1(x, y) - \pi(y)| \leq c\pi(x)^{3/4} e^{-t/c}.$$ 

It is worth noting that the right hand side is independent of $y$.

Replacing it in the definition of WDCL2 it yields

$$\sup_{y \in \mathbb{R}} \Delta_2 \sum_{i=1}^{n-1} |g_t(x, y)| \leq \sup_{y \in \mathbb{R}} \Delta_2 \sum_{i=1}^{n-1} c\pi(x)^{3/4} e^{-t/c}$$

$$\leq c\pi(x)^{3/4} \Delta_2 \sum_{i=1}^{n-1} e^{-i\Delta_n/c} \leq c\pi(x)^{3/4} \int_2^\infty e^{-s/c} ds \leq c^2 \pi^{3/4}(x).$$

This implies WDCL2 with $u_0 = 2$ and $k_2(x) = c^2 \pi(x)^{3/4}$, and we remark that $\|k_2\|_\infty$ is bounded by $c^2 \mathcal{L}^{3/4}$, independently of $(a, b) \in \Sigma$.

To conclude, we need to demonstrate that the constraints presented in (9) and (10) are satisfied. The proof closely follows the argumentation provided on page 7 of [49]. It is based on the integrability of the derivatives of the transition density, which is well-established in the case of a bounded drift (refer to Theorem 7, Chapter 9, Section 6 in Friedman [25]). However, when dealing with an unbounded drift, the proof becomes more challenging and relies on some bounds obtained in [38]. We are ready to prove (9) and (10). Let us start with the proof of (10). Integrating by parts and using Fubini Theorem yields

$$|\varphi_\varepsilon(\xi, t)| = \int_{\mathbb{R}} \exp(i\xi y) p_1(x, y, dy) = |\xi|^{-2} \int_{\mathbb{R}} \exp(i\xi y) \partial^2_y p_1(x, y, dy)$$

$$= |\xi|^{-2} \int_{\mathbb{R}} \exp(i\xi y) |\partial^2_y| \int_{\mathbb{R}} p_{t-1}(x, z) p_1(z, y, dz)|dy| \leq |\xi|^{-2} \int_{\mathbb{R}} p_{t-1}(x, z) |\partial^2_y p_1(z, y)|dz dy$$

$$= |\xi|^{-2} \int_{\mathbb{R}} p_{t-1}(x, z) \int_{\mathbb{R}} |\partial^2_y p_1(z, y)|dy dz.$$

We want to prove that

$$\int_{\mathbb{R}} |\partial^2_y p_1(z, y)|dy < \infty.$$ 

(13)

From the results of Section 4 of [38] recalled in Theorem 1, we know that, under our hypothesis, it is

$$|\partial^2_y p_1(z, y)| \leq C_0 e^{-\lambda_0 |\theta_{1,0}(z) - y|^2}$$

where $\theta_{1,0}(z)$ is the deterministic flow as introduced above Theorem 1 and $C_0$, $\lambda_0$ depends only on the class $\Sigma$. Then, the change of variable $\tilde{y} := \theta_{1,0}(z) - y$ yields

$$\int_{\mathbb{R}} |\partial^2_y p_1(z, y)|dy \leq \int_{\mathbb{R}} C_0 e^{-\lambda_0 |\tilde{y}|^2} d\tilde{y},$$

which is bounded as we wanted. It follows

$$|\varphi_\varepsilon(\xi, t)| \leq c|\xi|^{-2} \int_{\mathbb{R}} p_{t-1}(x, z)dz \leq c|\xi|^{-2},$$

for some $c > 0$. This gives (10).

The same inequalities hold true for $\varphi$ and provide (9). Indeed,

$$|\varphi(\xi)| = \int_{\mathbb{R}} \exp(i\xi y) p(y, dy) = |\xi|^{-2} \int_{\mathbb{R}} \exp(i\xi y) |\partial^2_y| \int_{\mathbb{R}} p_1(z, y, dz)|dy|$$

$$\leq |\xi|^{-2} \int_{\mathbb{R}} p_1(z, y, dz)dy dz \leq c|\xi|^{-2},$$

where we have used (13) once again. The proof of Proposition 2 is therefore concluded. 

\[\square\]
4.2 Proof of Theorem 2

Proof. If $\Delta_n \leq (\frac{1}{n})^{1/4}$, then it is enough to choose $h(T_n) := (\frac{1}{n})^{1/4}$ to get

$$E[|\tilde{\pi}_{h,n}(x^*) - \pi(x^*)|^2] \leq ch^{2\beta} + \frac{c}{T_n} + \frac{c}{T_n} \frac{\Delta_n}{h} \leq \frac{c}{T_n},$$

which is the first result we aimed to show.

On the other side, when $\Delta_n > (\frac{1}{n})^{1/4}$, as $T_n = n\Delta_n$ it is also

$$\Delta_n > (\frac{1}{n})^{1/4}.$$

Using the bias-variance decomposition and Proposition 1 it follows

$$E[|\tilde{\pi}_{h,n}(x^*) - \pi(x^*)|^2] \leq ch^{2\beta} + \frac{c}{T_n} + \frac{c}{T_n} \frac{\Delta_n}{h} \leq ch^{2\beta} + c(\frac{1}{n})^{1-\frac{1}{3}} + \frac{c}{n}. \frac{1}{h}.$$

We take $h(n) := (\frac{1}{n})^{1/4}$, it yields

$$E[|\tilde{\pi}_{h,n}(x^*) - \pi(x^*)|^2] \leq c(\frac{1}{n})^{2\frac{1}{4}} + c(\frac{1}{n})^{2\frac{1}{4}} + c(\frac{1}{n})^{2\frac{1}{4}}$$

and so the balance is achieved with the convergence rate we wanted. \hfill \Box

4.3 Proof of Theorem 3

The proof of Theorem 3 relies on the two hypotheses method, as explained for example in Section 2.3 of [48]. In the sequel we will introduce the Hellinger distance and we will need to bound it. In order to do that we will use Malliavin calculus as it appears in Section 2.3 of [2].

Proof. We remind the reader that the set $\Sigma(\beta, L, a_{\text{min}}, (a)_{0 \leq l \leq 3}, (b)_{0 \leq l \leq 3}, \tilde{C}, \tilde{\rho})$ has been introduced in Definition 2. Using a scaling argument, which consists in replacing the process $X$ by $\lambda X$ with $\lambda > 0$, it is possible to assume that $a_{\text{min}} < 1 < a_0$. This choice will simplify some notations. In the proof we will lower bound the risk using in particular the following two models:

$$dX_t = b(X_t) dt + a(X_t) dB_t, \quad X_0 \sim \pi(x) dx, \text{ has stationary distribution},$$

$$d\tilde{X}_t = b(\tilde{X}_t) dt + a(\tilde{X}_t) dB_t, \quad \tilde{X}_0 \sim \tilde{\pi}(x) dx, \text{ has stationary distribution},$$

where we take $a(x) = 1$ and $\tilde{a}(x) = 1 + \frac{1}{M_n} \psi_n(x)$ with $\psi_n(x) = \psi(\frac{x-x^*}{M_n})$, $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^\infty$ function with support on $[-1, 1]$ such that $\psi(0) = 1$, $\int_1^1 \psi(z) dz = 0$ and $\|\psi\|_\infty \leq 1$. The quantities $M_n$ and $h_n$ will be calibrated later and satisfy $M_n \rightarrow \infty$, $h_n \rightarrow 0$ and $M_n h_n \rightarrow \infty$ for $n \rightarrow \infty$; moreover we impose $M_n > 2$, $h_n < 1$, $M_n h_n^3 \geq \frac{1}{a_0}$, where the constant $a_0 \in (0, 1)$ will be fixed later. The function $b$ is such that

$$b(x) = -\eta \frac{\text{sgn}(x)}{A}, \quad \tilde{\text{sgn}}(x) = \begin{cases} 0 & |x| \leq 1 \\ \text{sgn} x & |x| > 2 \\ \in (0, \text{sgn} x) & 1 < |x| \leq 2, \end{cases} \quad (14)$$

where $x \mapsto \tilde{\text{sgn}}(x)$ is $C^\infty$ and $\eta > 0$, $A > 1$. We build $b(x)$ such that it is a $C^\infty$ function satisfying A1-A2 with $(b_l)_{0 \leq l \leq 3}$, $\tilde{C}$ and some $\tilde{\rho}$. The constants $\eta$ and $A$ will be calibrated with that regard. First, we observe that both the couples of coefficients $(a, b)$ and $(\tilde{a}, \tilde{b})$ satisfy the Assumptions A1. Indeed, we have $0 < 1 - \frac{1}{M_n} < \tilde{a}(x) < 1 + \frac{1}{M_n}$, $|b(0)| = \eta \tilde{\text{sgn}}(-x^*/A)$,

$$\|a^{(k)}\|_\infty \leq \frac{|\psi^{(k)}|_{\infty}}{h_n^k M_n} \leq \frac{\|\psi^{(k)}\|_\infty}{h_n^k M_n} \leq \alpha_0 \|\psi^{(k)}\|_\infty, \|b^{(k)}\|_\infty \leq \eta \frac{\|\text{sgn}\|_\infty}{A}, \text{ for } k = 1, \ldots, 3.$$

Thus, by letting $A \geq |x^*|$ we get $|b(0)| = 0 \leq b_0$ and if $a_0$ and $\eta/A$ are sufficiently small, the coefficients
(a, b) and (ã, b) are satisfying Assumption A1 for any fixed \(a_{\min} < a_0, (a_i)_{1 \leq i \leq 3}, (b_i)_{0 \leq i \leq 3}\). We now check that the condition A2 holds with \(C = \eta/2\) and some \(\tilde{\rho} > 0\). For \(|x| \geq |x^*| + 2A\), we have \(\frac{|x - x^*|}{A} \geq 2\), and we can write

\[
x b(x) = -\eta x \text{sgn}(\frac{x - x^*}{A}) = -\eta (x - x^*) \text{sgn}(\frac{x - x^*}{A}) - \eta x^* \text{sgn}(\frac{x - x^*}{A}) \leq -\eta |x - x^*| + \eta |x^*| \leq -\eta |x| + 2\eta |x^*|.
\]

We deduce that if \(|x| \geq 4|x^*|\), we have \(xb(x) \leq -\eta |x|/2\), and the condition A2 follows with \(C = -\eta/2\) and \(\rho = \max(|x^*| + 2A, 4|x^*|)\). It entails to set \(\eta = 2\tilde{C}\) while \(a_0, 1/A\) can be chosen arbitrarily, up to guarantee they are small enough to ensure the constraints A1 on the coefficients \(a\) and \(b\).

As a consequence of A2, we know from Theorem 1.4 in [36] that both \(X\) and \(\tilde{X}\) admit a unique invariant distribution that we call \(\mu\) (and \(\mu\), respectively). We denote their densities as \(\pi\) and \(\tilde{\pi}\), respectively.

Having as a purpose to show the lower bound using the two hypotheses method, following Sections 2.2–2.4 in [48], the strategy consists in finding two densities \(\pi\) and \(\tilde{\pi}\) such that for \(n\) large enough,

1. \(\pi, \tilde{\pi} \in \mathcal{H}(\beta, 2\mathcal{L})\), so that \((a, b), (\tilde{a}, \tilde{b}) \in \Sigma(\beta, \mathcal{L}, a_{\min}, (a_i)_{0 \leq i \leq 3}, (b_i)_{0 \leq i \leq 3}, \tilde{C}, \tilde{\rho}) =: \Sigma.
2. \(|\tilde{\pi}(x^*) - \pi(x^*)| \geq \frac{c}{M_n}\) for some \(c > 0\).
3. \(\limsup_n H^2(\text{Law}((X_{1\Delta_n})_{i=0,...,n}), \text{Law}((\tilde{X}_{1\Delta_n})_{i=0,...,n}) < \epsilon_0 < 2\), where \(H^2\) is the squared Hellinger distance on probabilities.

Then, it follows from Theorem 2.2 and Section 2.2. in [48],

\[
\inf_{\hat{\pi}(x^*)} \sup_{\pi, \tilde{\pi} \in \Sigma} \mathbb{E}[(\hat{\pi}(x^*) - \pi(x^*))^2] \geq \tilde{c}(\frac{1}{M_n})^2,
\]

for some \(\tilde{c} > 0\). We now want to check that the three points here above hold true with a choice of calibration \(h_n = \left(\frac{1}{n}\right)^{\frac{1}{2}}\), \(M_n = 1/(a_0 h_n^2) = \frac{a_0}{\alph_a}\), where \(\alpha_0\) is some sufficiently small constant in \((0, 1]\). Let us stress that this choice is consistent with \(M_n h_n^2 \geq 1/\alpha_0\) as \(\beta \geq 3\) and \(h_n \leq 1\).

Proof of point 1. Regarding this point, we need to prove that \(\pi, \tilde{\pi} \in \mathcal{H}(\beta, \mathcal{L})\) as soon as \(\alpha_0 > 0\) and \(1/A > 0\) are fixed small enough. We know from Theorem 1.4 in [36] that

\[
\pi(x) = c_\pi e^{2 \int_{x^*}^x b(y) dy} = c_\pi e^{-2\eta \int_{x^*}^x \text{sgn}(\frac{x - x^*}{A}) dy}
\]

and

\[
\tilde{\pi}(x) = \frac{\tilde{c}_\pi}{(1 + \frac{1}{M_n} \psi_{h_n}(x))^\frac{2}{\alpha_0}} e^{2 \int_{x^*}^x \left(1 + \frac{1}{M_n} \psi_{h_n}(y)\right)^{-\alpha_0} dy} = \frac{\tilde{c}_\pi}{(1 + \frac{1}{M_n} \psi_{h_n}(x))^\frac{2}{\alpha_0}} e^{-2\eta \int_{x^*}^x \text{sgn}(\frac{x - x^*}{A}) dy},
\]

where we used that \(b = 0\) on the interval \([x^* - A, x^* + A]\) which contains the support of \(\psi_{h_n}\) as \(h_n \leq 1\) and \(A \geq 1\). We recall that \(A \geq 1\) in the definition of \(b\) can be chosen as large as we want.

Using the definition of \(\text{sgn}\) in (14), we have

\[
1 \geq \int_{x^* - A}^{x^* + A} \pi(x) dx = c_\pi \int_{x^* - A}^{x^* + A} \exp(-2\eta \int_{x^*}^x \text{sgn}(y - x^*)/A dy) dx = c_\pi \int_{x^* - A}^{x^* + A} \exp(0) dx \geq 2A c_\pi.
\]

We deduce \(c_\pi \leq 1/(2A)\), and analogously we have \(\tilde{c}_\pi \leq 2/A\). From (16) and the sign of \(b\), we infer that \(||\pi||_{\infty} \leq c_\pi \leq 1/(2A)\). And thus \(||\pi||_{\infty} \leq \mathcal{L}||\) if \(1/A\) is chosen small enough. In the same manner, and after differentiations of (16), we see that \(||\tilde{\pi}\|_{\infty} \leq \epsilon(k)/A\) for some constants \(\epsilon(k)\) and all \(k \geq 1\). Choosing \(1/A\) small enough, we obtain \(\tilde{\pi} \in \mathcal{H}(\beta, \mathcal{L})\). It is more delicate to see that \(\tilde{\pi} \in \mathcal{H}(\beta, \mathcal{L})\) under the condition

\[
\frac{1}{M_n} \frac{1}{h_n^2} = \alpha_0 \leq 1.
\]

By differentiation of (17) with \(\tilde{c}_\pi \leq 2/A\), we can prove that \(\left\| \frac{\partial^k}{\partial x^k} \tilde{\pi}\right\|_{\infty} \leq \frac{c(k)}{M_n h_n^k}\) for some constants \(c(k)\) and all \(k \geq 1\). Choosing \(A\) large enough, it is sufficient with (18) to imply that \(\left\| \frac{\partial^k}{\partial x^k} \tilde{\pi}\right\|_{\infty} \leq \mathcal{L}\)
for all $k \in \{0, 1, \ldots, [\beta]\}$. It remains to prove that $\frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (\beta - [\beta])$–Hölder. We proceed, as in Lemma 3 of [3],

$$\left| \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x + l) - \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x) \right| = \left| \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x + l) - \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x) \right| \frac{\beta - [\beta]}{0^{[\beta]}} \left| \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x + l) - \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x) \right| \frac{1 - \beta - [\beta]}{0^{[\beta]}}$$

$$\leq \frac{\beta - [\beta]}{0^{[\beta]}} \left| \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x + l) - \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x) \right| \frac{1 - \beta - [\beta]}{0^{[\beta]}}$$

$$\leq \frac{\beta - [\beta]}{0^{[\beta]}} \left| \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x + l) - \frac{\partial^{[\beta]}}{\partial x^{[\beta]}} \tilde{\pi} (x) \right| \frac{1 - \beta - [\beta]}{0^{[\beta]}}$$

where in the last line we used (18) and the fact that $A$ can be chosen large enough. Remark that the choices of $A$ and $\alpha_0$ depend only on the values $\beta$, $\mathcal{L}$, $|x^*|$ and is fixed from now on.

Proof of point 2. We observe it is $\pi(x^*) = c_\pi$ and $\tilde{\pi}(x^*) = \frac{\pi}{(1 + \frac{x}{M})^2}$. We know that

$$1 = \int_\mathbb{R} \pi(x) dx = \int_\mathbb{R} e^{2 \int_x^{x^*} b(y) dy} dx$$

and so

$$\frac{1}{c_\pi} = \int_\mathbb{R} e^{2 \int_x^{x^*} b(y) dy} dx.$$

In the same way

$$1 = \int_\mathbb{R} \tilde{\pi}(x) dx = \tilde{c}_\pi \int_\mathbb{R} \frac{1}{(1 + \frac{x}{M})^2} e^{2 \int_x^{x^*} b(y) dy} dx.$$

We can therefore write, recalling that the support of $\psi_{h_n}$ is $[x^* - h_n, x^* + h_n]$,

$$\frac{1}{\tilde{c}_\pi} = \int_\mathbb{R} \frac{1}{(1 + \frac{x}{M})^2} e^{2 \int_x^{x^*} b(y) dy} dx$$

$$= \int_{|x - x^*| > h_n} e^{2 \int_x^{x^*} b(y) dy} dx + \int_{|x - x^*| \leq h_n} \frac{1}{(1 + \frac{x}{M})^2} e^{2 \int_x^{x^*} b(y) dy} dx$$

$$= \frac{1}{\tilde{c}_\pi} + \int_{|x - x^*| \leq h_n} \left( 1 + \frac{x}{M} \right)^2 \psi_{h_n}(x) dx - 1 \right) e^{2 \int_x^{x^*} b(y) dy} dx.$$

Remarking that

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{(1 + \frac{x}{M})^2} \psi_{h_n}(x) \right| = \frac{2\pi}{(1 + \frac{x}{M})^2} \psi_{h_n}(x) = O(\frac{1}{M})$$

it is easy to see that

$$\frac{1}{\tilde{c}_\pi} - \frac{1}{c_\pi} = O(\frac{h_n}{M})$$

and so it is also

$$\tilde{c}_\pi - c_\pi = O(\frac{h_n}{M}).$$

(19)

It follows

$$|\tilde{\pi}(0) - \pi(0)| = \left| \frac{\tilde{c}_\pi}{(1 + \frac{x}{M})^2} - c_\pi \right|$$

$$= \left| \frac{\tilde{c}_\pi - c_\pi}{(1 + \frac{x}{M})^2} - c_\pi \frac{2\pi}{(1 + \frac{x}{M})^2} \right|$$

$$= \frac{2c_\pi}{M} + O(\frac{h_n}{M}) + O(\frac{M}{M})$$

$$\geq \frac{c}{M}.$$
which implies the point 2, as we wanted.

Proof of point 3. We first recall some properties of total variation and Hellinger distance (refer to Section 2.4 in [48]). Let P and Q be two probability measures on the probability space \((\Omega, \mathcal{F})\), dominated by \(\mu\). The Hellinger distance \(H\) is defined by

\[
H^2(P, Q) := \int_\Omega \left(\sqrt{\frac{dP}{d\mu}} - \sqrt{\frac{dQ}{d\mu}}\right)^2 d\mu.
\]

For a product measure we have the following tensorization property,

\[
H^2(\otimes_{i=1}^n P_i, \otimes_{i=1}^n Q_i) \leq \sum_{i=1}^n H^2(P_i, Q_i).
\]

Such a property has been extended to the distribution of Markov chains in Proposition 2.1 of [17]. In particular, adapting the results in [17] to our framework, we have \((X_{\Delta_n})_{i=0}^n\) and \((\tilde{X}_{\Delta_n})_{i=0}^n\) two homogeneous Markov chains on \(\mathbb{R}\) with transition densities \(p\) and \(q\) with respect to the Lebesgue measure. The conditional Hellinger distance given \(X_0 = \tilde{X}_0 = x\) is defined by

\[
H^2_x(p, q) = \int_{\mathbb{R}} \left(\sqrt{p(x,y)} - \sqrt{q(x,y)}\right)^2 dy.
\]

We denote by \(P^n\) and \(Q^n\) the laws of \(X_0, X_{\Delta_n}, \ldots, X_n\) and \(\tilde{X}_0, \tilde{X}_{\Delta_n}, \ldots, \tilde{X}_n\), respectively. We know that

\[
\frac{dP^n(dx)}{dx} = \prod_{i=0}^{n-1} p_{\Delta_n}(x_i, x_{i+1}) \quad \text{and} \quad \frac{dQ^n(dx)}{dx} = \prod_{i=0}^{n-1} q_{\Delta_n}(x_i, x_{i+1}),
\]

where

\[
p_{\Delta_n}(x, y)dy = P(X_{\Delta_n} \in dy | X_0 = x),
\]

\[
q_{\Delta_n}(x, y)dy = P(\tilde{X}_{\Delta_n} \in dy | X_0 = x).
\]

From a slight extension of Proposition 2.1 in [17], which allows for different initial laws and can be obtained by the same induction argument as in the proof of Proposition 2.1 in [17], we have that

\[
H^2(P^n, Q^n) \leq \frac{1}{2} \sum_{i=0}^{n-1} \left[ \mathbb{E}[H^2_{X_{\Delta_n}}(p_{\Delta_n}, q_{\Delta_n})] + \mathbb{E}[H^2_{\tilde{X}_{\Delta_n}}(p_{\Delta_n}, q_{\Delta_n})]\right] + H^2(\pi, \tilde{\pi}).
\]  

Using (16)–(17) with (19), one can show that

\[
H^2(\pi, \tilde{\pi}) \leq \frac{C h_n}{M^2}.
\]

Hence, we need to control the conditional squared Hellinger distance \(H^2_x(p_{\Delta_n}, q_{\Delta_n})\). With this purpose in mind, we interpolate between the two laws \(X\) and \(\tilde{X}\). For \(\epsilon \in [0, 1]\) we define

\[
dX'_t = b(X'_t)dt + (1 + \epsilon \frac{1}{M_n} \psi_n(X'_t))dB_t, \quad \tilde{X}_0 = x_0,
\]

and let \(p^\epsilon_{\Delta_n}(x_0, y)\) denote the density of \(X'_\Delta\). We observe that \(p^0_{\Delta_n} = p_{\Delta_n}\), while \(p^1_{\Delta_n} = q_{\Delta_n}\). As the coefficients are \(C^\infty\) and bounded, the function \((x, y, \epsilon) \mapsto p^\epsilon_{\Delta_n}(x, y)\) is smooth. Then, it is possible to bound the conditional Hellinger distance as below:

\[
H^2_{\Delta_n}(p_{\Delta_n}, q_{\Delta_n}) = \int_{\mathbb{R}} \left(\sqrt{p^0_{\Delta_n}(x_0, y)} - \sqrt{p^1_{\Delta_n}(x_0, y)}\right)^2 dy
\]

\[
= \int_{\mathbb{R}} \left(\int_0^1 \frac{p^\epsilon_{\Delta_n}(x_0, y)de}{\sqrt{p^\epsilon_{\Delta_n}(x_0, y)}}\right)^2 dy.
\]
We recall that, as the volatility coefficient is uniformly lower bounded, it is $p_{\Delta_n} > 0$. From Jensen inequality and Fubini–Tonelli’s theorem we get that the term here above is upper bounded in the following way

$$H^2_{\pi_n}(p_{\Delta_n}, q_{\Delta_n}) \leq \int_{\mathbb{R}} \int_0^1 \frac{1}{p_{\Delta_n}(x_0, y)} \left( \frac{\dot{p}_{\Delta_n}(x_0, y)}{p_{\Delta_n}(x_0, y)} \right)^2 dy dx + \int_0^1 \int_{\mathbb{R}} \frac{1}{p_{\Delta_n}(x_0, y)} \left( \frac{\dot{p}_{\Delta_n}(x_0, y)}{p_{\Delta_n}(x_0, y)} \right)^2 dy dx$$

$$= \int_0^1 \int_{\mathbb{R}} \frac{1}{p_{\Delta_n}(x_0, y)} \left( \frac{\dot{p}_{\Delta_n}(x_0, y)}{p_{\Delta_n}(x_0, y)} \right)^2 dy dx = \int_0^1 \int_{\mathbb{R}} \frac{1}{p_{\Delta_n}(x_0, y)} \left( \frac{\dot{p}_{\Delta_n}(x_0, y)}{p_{\Delta_n}(x_0, y)} \right)^2 dy dx$$

$$= \int_0^1 \mathbb{E}_{x_0}[\frac{1}{p_{\Delta_n}(x_0, X_{\Delta_n}^c)}]^2(dx_0) \leq \int_0^1 \mathbb{E}_{x_0}[\frac{1}{p_{\Delta_n}(x_0, X_{\Delta_n}^c)}]^2(dx_0) \leq \frac{h_n}{M_n^2},$$

having denoted as $\mathbb{E}_{x_0}[\cdot]$ the conditional expectation given $X_0^c = x_0$. We recognize in the above integral the Fisher information of the statistical model $\epsilon \rightarrow X_{\Delta_n}^c$. We now state the following control whose proof is delayed to Section 4.4.

**Proposition 3.** Assume that $h_n \leq \Delta_n$, $1/M_n = O(h_n^2)$ with $\beta \geq 3$ and $\Delta_n = O(n^{-\epsilon})$ for some $\epsilon > 0$. Then, there exists a constant $\hat{c} > 0$ such that for all $e \in [0, 1]$,

$$\int_{\mathbb{R}} \mathbb{E}_{x_0}[\frac{1}{p_{\Delta_n}(x_0, X_{\Delta_n}^c)}]^2(dx_0) \pi^c(x_0) dx_0 \leq \frac{h_n}{M_n^2},$$

where $\pi^c$ denotes the density of the stationary distribution of $X^c$.

Remark that the choices for $h_n$ and $M_n$ with the condition $\Delta_n \geq \left( \frac{1}{\pi^c_\beta} \right)^{1/(2\beta)}$ implies that $h_n \leq \Delta_n$ and $1/M_n = O(h_n^2)$. Hence, we can use the previous proposition. Before this, we note that in the same manner as (17) and (19) are obtained, the stationary distribution of $X^c$ is explicitly given by

$$\pi^c(x) = \frac{c_{\pi^c}}{(1 + \frac{c_{\pi^c}}{M_n} \psi_{h_n}(x))^2} e^{-2\eta f_{\pi^c}^{-\psi_{h_n}(x)}} dy,$$

with $c_{\pi^c} - c_\pi = O(h_n^2)$. Noting that $c_{\pi^c}$ is a constant independent of $\epsilon$ and $\eta$, and by comparison of the expressions (16), (17) and (24) we deduce that $\frac{\pi^c}{\pi}$ and $\frac{\pi^c}{\pi}$ are bounded by constants independent of $\epsilon$ and $\eta$. Now, we integrate (23) with respect to $\pi(x_0) dx_0$, and infer from the Proposition 3 that

$$\mathbb{E}[H^2_{X_{\Delta_n}}(p_{\Delta_n}, q_{\Delta_n})] = \int_{\mathbb{R}} H^2_{x_0}(p_{\Delta_n}, q_{\Delta_n}) \pi(x_0) dx_0 \leq \int_0^1 \int_{\mathbb{R}} \mathbb{E}_{x_0}[\frac{1}{p_{\Delta_n}(x_0, X_{\Delta_n}^c)}]^2(dx_0) \pi(x_0) dx_0 \leq \frac{h_n}{M_n^2}.$$ 

Similarly, integrating (23) with respect to $\pi(x_0) dx_0$ yields to $\mathbb{E}[H^2_{X_{\Delta_n}}(p_{\Delta_n}, q_{\Delta_n})] \leq c \frac{h_n}{M_n^2}$. Then, (20) and (21) provide

$$H^2(\mathbb{P}^n, Q^n) \leq c \frac{h_n}{M_n^2}.$$

It follows that the third point of the scheme for the lower bound through the two hypotheses method is satisfied, up to requiring the following condition:

$$\limsup_n \frac{h_n}{M_n^2} < \frac{\epsilon_0}{e},$$

with $\epsilon_0$ small enough. Recalling $M_n = 1/(\alpha_0 h_n^2)$ and $h_n = n^{-1/(1+2\beta)}$, we deduce $n \frac{h_n}{M_n^2} = \alpha_0^2 n^{1+2\beta} = \alpha_0^2$. Thus, (25) holds true as soon as $\alpha_0^2 < \epsilon_0/e$, which is a feasible choice as $\alpha_0 \in (0, 1]$ is arbitrary. This proves (15).
4.4 Proof of Proposition 3

The key point is that we have a Malliavin representation for the score function:

$$\frac{\hat{p}_{\Delta_n}(x_0, y)}{p_{\Delta_n}(x_0, y)} = E_{x_0}[W_{x_0, \Delta_n, \epsilon}]X_{\Delta_n}^\epsilon = y]$$

(26)

for some random variable $W_{x_0, \Delta_n, \epsilon}$ which is usually referred as Malliavin weight. The reader may find in Nualart [42] a detailed exposition on Malliavin calculus. In Appendix B.1, we also provide a recap of the notations and main properties of the Malliavin operators utilized in this paper. Specifically, the Malliavin operators are defined within the underlying Hilbert space $H = L^2([0, \Delta_n])$. We denote by $\delta$ the Skorohod integral, which is defined as the adjoint operator of the Malliavin operator $D$. Additionally, $\langle \cdot, \cdot \rangle$ represents the scalar product in $L^2([0, \Delta])$, and $\|\cdot\|_H$ signifies the corresponding norm. From Theorem 5 in [26] we know it is

$$W_{x_0, \Delta_n, \epsilon} = \delta\left(\frac{DX_{\Delta_n}^\epsilon X_{\Delta_n}^\epsilon}{< DX_{\Delta_n}^\epsilon, DX_{\Delta_n}^\epsilon>}\right),$$

(27)

and $X_{\Delta_n}^\epsilon = \frac{\partial X_{\Delta_n}}{\partial \epsilon}$ is solution to the SDE obtained by formal differentiation of (22):

$$X_{t}^\epsilon = \int_0^t b'(X_s^\epsilon)X_s^\epsilon ds + \int_0^t\left[\epsilon \psi_h(X_s^\epsilon)X_s^\epsilon + \frac{1}{M_n} \psi_h(X_s^\epsilon)\right] dB_s.$$

(28)

For the sake of readability, we give in Appendix B.2 a short proof of the formulae (26)–(27). We remark that the terms in (27) depend on $x_0$ as $X_{\Delta_n}^\epsilon$ is actually $X_{\epsilon, x_0}^\epsilon$. The Malliavin weight can be bounded as in the following proposition, which will be shown in the Section 5.

**Proposition 4.** Assume that $h_n \leq \Delta_n$, $1/M_n = O(h_n^\beta)$ with $\beta \geq 3$ and $\Delta_n = O(n^{-\epsilon})$ for some $\epsilon > 0$. Let $(X_{\Delta_n}^\epsilon)_{i=0, \ldots, n-1}$ be the discrete observations of the process solution to (22) and $W_{x_0, \Delta_n, \epsilon}$ the Malliavin weight as in (27). Then, there exists a constant $\tilde{c} > 0$ such that, for any $\epsilon \in [0, 1]$,

$$|\int \mathbb{E}_{x_0}[\mathbb{E}_{x_0}[W_{x_0, \Delta_n, \epsilon}]X_{\Delta_n}^\epsilon]]^{2} \pi^\epsilon(x_0) dx_0| \leq \tilde{c} \frac{h_n}{M_n^2},$$

(29)

where $\pi^\epsilon$ denotes the density of the stationary distribution of $X^\epsilon$.

Proposition 3 is then an immediate consequence of Proposition 4 with (26). □

5 Proof of Proposition 4

After translation of the process $X^\epsilon$ by a fixed value, it is possible to assume that $x^\epsilon = 0$. This will lighten the notations in the proofs appearing in this section and in the Appendix. The proof of the bound of the Malliavin weight is divided in several steps. In the first step below, we show that the main contribution in the integral of the LHS of (29) comes from a neighborhood of $x^\epsilon = 0$. In step 2, we compute a more tractable expression of the Malliavin weight (27) as sum of different terms, while in step 3–5 we study the contribution of each term.

**Step 1: Modifying the process.**

A first step is to prove that we can remove the contribution of the drift and that only the case where $x_0$ is such that $|x_0| \leq \Delta_n^{\frac{\gamma}{r}}$, for $\gamma > 0$ arbitrarily small matters. To do that, we need the following lemma, whose proof can be found in the Appendix A.

**Lemma 2.** Assume that $(A_n)_n$, $(B_n)_n$, $(A'_n)_n$, $(B'_n)_n$ are some sequences such that, on some set $\Omega_n$, it is

$$(A_n, B_n) = (A'_n, B'_n).$$

Moreover the complementary set of $\Omega_n$ is such that

$$\mathbb{P}(\Omega_n^c) \leq \kappa_r n^{-r},$$

(30)

for any $r > 1$ and constants $\kappa_r \geq 0$. Assume also that

$$\|A_n\|_{L^p} + \|A'_n\|_{L^p} \leq c_p r^\gamma$$

(31)
for some \( r_0 \geq 0, p > 1, c_p \geq 0 \). Then, for any \( 1 \leq p' < p \) and for any \( r > 1 \), we have

\[
\|||E[A_n|B_n] - E[A'_n|B'_n]|||_{L^{p'}} \leq cn^{-r},
\]

where the constant \( c \) depends on \( p', r_0, p, c_p \) and the \((\kappa_r)_r\).

We observe that, if in the lemma here above we can choose \( p' = 2 \) and so if \( p > 2 \), then it follows

\[
E[|E[A_n|B_n] - E[A'_n|B'_n]|^2] \leq cE[|E[A'_n|B'_n]|^2] + cn^{-r}.
\]

We want to apply this inequality to our context, we therefore need to define \( A_n := W_{x_0, \Delta_n, \epsilon} \), where \( W_{x_0, \Delta_n, \epsilon} \) has been defined in (27), and \( B_n = X_{\Delta_n} \). Having as a purpose to show that the contribution provided by the case where \( |x_0| \geq \Delta_n^{\frac{1}{2}-\gamma} \) is negligible, we introduce the set

\[
\Omega_n := \{ \omega : X'_n(\omega) \not\in [h_0, h_n] \quad \forall s \in [0, \Delta_n] \}.
\]

Let \( |x_0| \geq \Delta_n^{\frac{1}{2}-\gamma} \). We have assumed \( h_0 \leq \Delta_n \), hence, using Markov inequality, we get

\[
P_{x_0}(\Omega_n) \leq P_{x_0}(\exists s \in [0, \Delta_n] : |X'_n - x_0| \geq \Delta_n^{\frac{1}{2}-\gamma} - \Delta_n)
\]

\[
\leq P_{x_0}(\sup_{s \in [0, \Delta_n]} |X'_n - x_0| \geq \Delta_n^{\frac{1}{2}-\gamma} / 2), \quad \text{for } n \text{ large enough}
\]

\[
\leq 2r \cdot \frac{E_{x_0}[\sup_{s \in [0, \Delta_n]} |X'_n - x_0|^r]}{\Delta_n^{(1/2 - \gamma)r}}, \quad \text{with any } r' > 1.
\]

To control the expectation appearing in the last equation, we use Burkholder-Davis-Gundy's inequality, and the fact that the coefficients of the SDE (22) are bounded in the following way:

\[
\|b\|_\infty \leq 1, \quad 1 + \|a\|_\infty \leq 1 + \|a\|_M \leq 3/2.
\]

This implies that \( E_{x_0}[\sup_{s \in [0, \Delta_n]} |X'_n - x_0|^r] \leq c_r \Delta_n^{r/2} \). In turn, we get \( P_{x_0}(\Omega_n) \leq c_r \Delta_n^{\gamma r} \). As the constant \( 0 < \gamma < 1/2 \) is fixed, and the sampling step converges to zero at least with some polynomial rate in \( n \), it is possible to choose appropriately \( r' \) in order to satisfy the condition

\[
P_{x_0}(\Omega_n^c) \leq c_r n^{-r},
\]

with any \( r > 0 \). We remark that the bound we get in the right hand side here above is independent on \( x_0 \).

We let then \( A'_n = 0 \) and \( B'_n = B_n = X_{\Delta_n} \). We remark that, on \( \Omega_n \), by the definition of \( X' \) we know that its derivative with respect to \( \epsilon \) is going to be 0 and so \( W_{x_0, \Delta_n, \epsilon} = 0 \) as well. It follows that, on \( \Omega_n \), \( A'_n = A_n \) (and by definition \( B'_n = B_n \)). To apply Lemma 2 we are left to check that the \( L^p \) norm of the Malliavin weight \( W_{x_0, \Delta_n, \epsilon} \) satisfies the condition (31). We use the following lemma whose proof is given in Appendix A.

**Lemma 3.** We have \( \sup_{x_0 \in \mathbb{R}} E[|W_{x_0, \Delta_n, \epsilon}|^4] = O(\Delta_n^{-4}) \).

Using this lemma with the condition \( \Delta_n > (\frac{1}{\Delta_n^{1/2 - \gamma}} ) \) we deduce that (31) holds true for some \( r_0 > 0 \) and \( p = 4 \). Thus, we can choose \( p' = 2 \) in Lemma 2 and apply (32). It follows that, for \( |x_0| \geq \Delta_n^{\frac{1}{2}-\gamma} \),

\[
E_{x_0}[E_{x_0}[W_{x_0, \Delta_n, \epsilon}|X_{\Delta_n}^\epsilon|^2] \leq 2E_{x_0}[E_{x_0}[0|X_{\Delta_n}^\epsilon|^2]] + o(n^{-r})
\]

\[
= o(n^{-r})
\]

for any \( r > 0 \) and it is independent of \( x_0 \). We can therefore focus on the case where \( |x_0| \leq \Delta_n^{\frac{1}{2}-\gamma} \) and so the control (29) becomes a consequence of

\[
\int_{|x_0| \leq \Delta_n^{1/2 - \gamma}} E_{x_0}[E_{x_0}[W_{x_0, \Delta_n, \epsilon}|X_{\Delta_n}^\epsilon|^2]^2] \pi'(x_0)dx_0 \leq \frac{h_n}{M_n^2}.
\]
We now want to show that the drift function does not provide any contribution and so we can replace the models here above with the same ones, but with a drift coefficient which is now 0. Consider \( \hat{X}^\epsilon \) the same model as \( X^\epsilon \), but with \( b \equiv 0 \). Set now

\[
\Omega_n := \{ \omega, \ X^\epsilon_n(\omega) \in [-1,1], \ \forall s \in [0, \Delta_n] \}.
\]

Acting as we did in order to compute the probability of \( \Omega_n^\epsilon \) it is clearly, for \( n \) large enough,

\[
\sup_{|x| \leq \Delta_n^{1/2-\gamma}} P_{x_0}(\hat{\Omega}_n^\epsilon) \leq \sup_{|x| \leq \Delta_n^{1/2-\gamma}} P_{x_0}(\sup_{s \in [0, \Delta_n]} |X^\epsilon_s - x_0| \geq 1 - \Delta_n^{1/2-\gamma})
\]

\[
\leq \sup_{|x| \leq \Delta_n^{1/2-\gamma}} P_{x_0}(\sup_{s \in [0, \Delta_n]} |X^\epsilon_s - x_0| \geq 1/2) = o(n^{-r})
\]

for any \( r > 0 \). From the definition of the drift coefficient in (14) we can see that, on \( \Omega_n^\epsilon \), \( \hat{X}^\epsilon = X^\epsilon \) \( \forall \epsilon \) and \( W_{x_0, \Delta_n, \epsilon} = \hat{W}_{x_0, \Delta_n, \epsilon} \), where \( \hat{W}_{x_0, \Delta_n, \epsilon} \) is the Malliavin weight associated to the model with \( b \equiv 0 \). Hence, applying Lemma 2 with \( A_n = W_{x_0, \epsilon, \Delta_n} \), \( A'_n = \hat{W}_{x_0, \epsilon, \Delta_n} \) and \( B'_n = \hat{\Delta}_n \) we get

\[
E_{x_0}[E_{x_0}[W_{x_0, \Delta_n, \epsilon}|X^\epsilon_{\Delta_n}]] \leq cE_{x_0}[E_{x_0}[\hat{W}_{x_0, \Delta_n, \epsilon}|\hat{X}^\epsilon_{\Delta_n}]] + o(n^{-r}).
\]

Therefore, for \( |x_0| \leq \Delta_n^{1/2-\gamma} \), we can do as if \( b \equiv 0 \), which yields to some simplification in the computation of the Malliavin weight.

It follows our goal becomes to show that

\[
\int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[\mathbb{E}_{x_0}[\hat{W}_{x_0, \Delta_n, \epsilon}|\hat{X}^\epsilon_{\Delta_n}]]^2 \pi^\epsilon(x_0) dx_0 \leq c\frac{\hat{h}_n}{M_n}.
\]

This will be a consequence of (40), (42) and (52) that we show below.

We underline that, even if in the sequel in all our computations we will refer to the model where the drift function \( b \) is identically 0, the invariant density clearly refers to the stationary process, for which the drift is different from 0. Moving to the case where the drift is equal to 0, we need to prove some upper bounds which would have been straightforward with the original drift, thanks to the stationarity of the process. They are gathered in the following lemma, whose proof can be found in the Appendix A.

**Lemma 4.** Let \( X^\epsilon_{\Delta_n} \) and \( \hat{X}^\epsilon_{\Delta_n} \) be defined as above. Then, the following estimations hold true for \( r > 1 \) as in Lemma 2.

1. For any \( p \geq 1 \) there exists a constant \( c > 0 \) such that
   \[
   \sup_{0 < s \leq \Delta_n} \int_{B} \mathbb{E}_{x_0}[|\psi_{h_n}(\hat{X}^\epsilon_s)|^p] \pi^\epsilon(x_0) dx_0 \leq ch_n + o(n^{-r}).
   \]

2. For any \( p \geq 1 \) and \( k \geq 1 \) there exists a constant \( c > 0 \) such that
   \[
   \sup_{0 < s \leq \Delta_n} \int_{B} \mathbb{E}_{x_0}[|\psi_{h_n}^{(k)}(\hat{X}^\epsilon_s)|^p] \pi^\epsilon(x_0) dx_0 \leq ch_n^{1-kp} + o(n^{-r}).
   \]

3. There exists a constant \( c > 0 \) such that
   \[
   \int_{0}^{\Delta_n} \int_{B} \mathbb{E}_{x_0}[(\hat{X}^\epsilon - \hat{X}^\epsilon_0)^2] \psi_{h_n}^{(2)}(\hat{X}^\epsilon_0) dx_0 \pi^\epsilon(x_0) dx_0 \leq c\Delta_n^2 h_n + o(n^{-r}).
   \]

**Step 2: Formal derivation of the processes.**

We recall that the Malliavin weight \( \hat{W}_{x_0, \Delta_n, \epsilon} \) is explicit and it is as in (27), where the process \( X^\epsilon \) is replaced by \( \hat{X}^\epsilon \). We now want to formally derive all the processes that play a role in \( \hat{W}_{x_0, \Delta_n, \epsilon} \).

As

\[
\hat{X}^\epsilon_t = x_0 + \int_{0}^{t}(1 + \frac{1}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s))dB_s,
\]

we have that \( \hat{X}^\epsilon_t := \frac{\partial \hat{X}^\epsilon_t}{\partial x_0} \), by formal derivation (see Theorem 5.24 in [10]), is solution to

\[
\hat{X}^\epsilon_t = \int_{0}^{t} \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s)\hat{X}^\epsilon_s dB_s + \int_{0}^{t} \frac{1}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s)dB_s.
\]

Let \( Y^\epsilon_t := \frac{\partial \hat{X}^\epsilon_t}{\partial x_0} \) be the flow of SDE (34), which is the solution of

\[
Y^\epsilon_t = 1 + \int_{0}^{t} \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s)Y^\epsilon_s dB_s.
\]
Using variation of the constant method, we have
\[
\hat{X}^\epsilon_t = Y^\epsilon_t \int_0^t \psi_{h_n}(\hat{X}^\epsilon_s)(Y^\epsilon_s)^{-1} \frac{1}{M_n} dB_s - \frac{\epsilon Y^\epsilon_t}{M_n^2} \int_0^t \psi_{h_n}(\hat{X}^\epsilon_s)\psi_{h_n}(\hat{X}^\epsilon_s)(Y^\epsilon_s)^{-1} ds \tag{36}
\]
and \( Y^\epsilon_t \) is explicit as Doléans-Dade exponential:
\[
Y^\epsilon_t = \exp(\frac{\epsilon}{M_n} \int_0^t \psi_{h_n}(X^\epsilon_s)dB_s - \frac{\epsilon^2}{2M_n^2} \int_0^t (\psi_{h_n}(X^\epsilon_s))^2 ds). \tag{37}
\]
By standard computation (see Equation (2.59) in [42]) it is
\[
D_s \hat{X}^\epsilon_t = (Y^\epsilon_t)^{-1}(Y^\epsilon_t)(1 + \psi_{h_n}(\hat{X}^\epsilon_s)\frac{\epsilon}{M_n}). \tag{38}
\]
Recall the Malliavin weight is \( \hat{W}_{x_0,\Delta_n,\epsilon} = \delta(\frac{D_s \hat{X}^\epsilon_{\Delta_n}}{\langle D_s \hat{X}^\epsilon_{\Delta_n}, D_s \hat{X}^\epsilon_{\Delta_n} \rangle}) \), as proven in Appendix B. Then, we can compute it explicitly. It holds
\[
\hat{W}_{x_0,\Delta_n,\epsilon} = \hat{X}^\epsilon_{\Delta_n} \delta(\frac{D_s \hat{X}^\epsilon_{\Delta_n}}{\langle D_s \hat{X}^\epsilon_{\Delta_n}, D_s \hat{X}^\epsilon_{\Delta_n} \rangle}) - \frac{< D_s \hat{X}^\epsilon_{\Delta_n}, D_s \hat{X}^\epsilon_{\Delta_n} >}{< D_s \hat{X}^\epsilon_{\Delta_n}, D_s \hat{X}^\epsilon_{\Delta_n} >} \tag{39}
\]
\[
= \hat{W}_1^{x_0,\Delta_n,\epsilon} + \hat{W}_2^{x_0,\Delta_n,\epsilon} \tag{40}
\]
where we used Proposition 1.3.3 in Nualart [42]. This is possible as the process \( u \mapsto \frac{D_s \hat{X}^\epsilon_{\Delta_n}}{\langle D_s \hat{X}^\epsilon_{\Delta_n}, D_s \hat{X}^\epsilon_{\Delta_n} \rangle} \) can be shown, by the same computations as those in the proof of Lemma 3, to be an element of the space \( D^{1,2}(H) \), which is in the domain of \( \delta \) (see Appendix B).

**Step 3: Handling \( \hat{W}_1^{x_0,\Delta_n,\epsilon} \)**

We now study the two terms separately. Regarding the first one, the following proposition holds true. The proof of this proposition, given in Appendix A, is based on the fact that we can approximate \( Y^\epsilon_t \), \( (Y^\epsilon_s)^{-1} \) and \( D_s \hat{X}^\epsilon_{\Delta_n} \) by 1, omitting an error which is negligible.

**Proposition 5.** Let \( (\hat{X}^\epsilon_{\Delta_n})_{i=0,\ldots,n-1} \) be the discrete observations of the process solution to (34) and \( \hat{W}_1^{x_0,\Delta_n,\epsilon} \) the Malliavin weight defined as above. Then,
\[
\hat{W}_1^{x_0,\Delta_n,\epsilon} = \frac{1}{\Delta_n} \int_0^{\Delta_n} \frac{1}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s)dB_s B_{\Delta_n} + R^{(1)}_n,
\]
where \( R^{(1)}_n \) is a remainder term and it is such that
\[
\int_{\|x_0\| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0} \left[ |R^{(1)}_n|^2 \right] \pi^\epsilon(x_0)dx_0 \leq \frac{c}{M_n^4 h_n} = o\left(\frac{h_n}{M_n}\right). \tag{41}
\]

From Jensen inequality (41) implies that the contribution of the remainder term is negligible in (33). We now prove the result (33) for the principal term of \( \hat{W}_1^{x_0,\Delta_n,\epsilon} \). Let us introduce the following notation:
\[
\mathcal{M}_u := \int_0^u \frac{1}{M_n} \psi_{h_n}(\hat{X}^\epsilon_s)dB_s, \quad \mathcal{N}_u := \frac{1}{\Delta_n} B_u, \quad \text{for } u \in [0, \Delta_n].
\]

Then, we actually need to study in detail
\[
\int_{\|x_0\| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[\mathcal{M}_\Delta_n^2 \mathcal{N}_\Delta_n^2] \pi^\epsilon(x_0)dx_0.
\]
Using Itô’s formula it is
\[
\mathbb{E}_{x_0}[\mathcal{M}_\Delta_n^2 \mathcal{N}_\Delta_n^2] = \mathbb{E}_{x_0} \left[ \int_0^{\Delta_n} \mathcal{M}_s^2 d[\mathcal{N}_s, \mathcal{N}_s] + \int_0^{\Delta_n} \mathcal{N}_s^2 d[\mathcal{M}_s, \mathcal{M}_s] + \int_0^{\Delta_n} \mathcal{M}_s \mathcal{N}_s d[\mathcal{M}_s, \mathcal{N}_s] \right] \leq 2 \mathbb{E}_{x_0} \left[ \int_0^{\Delta_n} \mathcal{M}_s^2 d[\mathcal{N}_s, \mathcal{N}_s] + \int_0^{\Delta_n} \mathcal{N}_s^2 d[\mathcal{M}_s, \mathcal{M}_s] \right].
\]
where we used Kunita-Watanabe inequality (see Corollary 1.16 of Chapter IV in [45]). According to the decomposition here above we have

\[
\int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[\mathcal{M}_n^2 \mathcal{N}_n^2] \pi'(x_0) dx_0 \leq 2 \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[\mathcal{M}_n^2 d[N, N]_t] dx_0 \\
+ 2 \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[-\frac{\Delta_n}{2} N_\mathcal{S}^2 d[\mathcal{M}, \mathcal{M}]_t] dx_0 =: 2A_1 + 2A_2.
\]

Regarding \(A_1\), the first point of Lemma 4 ensures that

\[
A_1 \leq \frac{c}{M_n^2} \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \int_0^{\Delta_n} \mathbb{E}_{x_0}[\psi_{h_n}(\tilde{X}_u^\epsilon)]^2 du dx_0 \\
\leq \frac{c}{M_n^2} h_n.
\]

We now turn studying

\[
A_2 = \frac{2}{M_n^2 \Delta_n^{1/2-\gamma}} \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \int_0^{\Delta_n} \mathbb{E}_{x_0}[B_u^2 \psi_{h_n}(\tilde{X}_u^\epsilon)]^2 du dx_0.
\]

To bound this term we want to replace \(B_u^2\) and, in order to do that, we use the dynamics of our process \(\tilde{X}\). Indeed, it is

\[
dB_u = a_t(\tilde{X}_u^\epsilon)^{-1} d\tilde{X}_u^\epsilon \\
= d\tilde{X}_u^\epsilon - \frac{\psi_{h_n}(\tilde{X}_u^\epsilon)}{1 + \frac{M_n}{\psi_{h_n}(\tilde{X}_u^\epsilon)}} d\tilde{X}_u^\epsilon.
\]

It follows

\[
B_u = (\tilde{X}_u^\epsilon - \tilde{X}_0^\epsilon) - \frac{c}{M_n} \int_0^u \frac{\psi_{h_n}(\tilde{X}_u^\epsilon)}{1 + \frac{M_n}{\psi_{h_n}(\tilde{X}_u^\epsilon)}} d\tilde{X}_u^\epsilon.
\]

Hence, if we replace it in \(A_2\), we obtain

\[
A_2 \leq \frac{c}{M_n^2 \Delta_n^{1/2-\gamma}} \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \int_0^{\Delta_n} \mathbb{E}_{x_0}[(\tilde{X}_u^\epsilon - \tilde{X}_0^\epsilon)^2 \psi_{h_n}(\tilde{X}_u^\epsilon)]^2 du dx_0 \\
+ \frac{c}{M_n^2 \Delta_n^{1/2-\gamma}} \int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \int_0^{\Delta_n} \mathbb{E}_{x_0}[(\tilde{X}_u^\epsilon)^2 \psi_{h_n}(\tilde{X}_u^\epsilon)]^2 du dx_0 =: A_{2,1} + A_{2,2}.
\]

The bound on \(A_{2,1}\) is a direct consequence of the third point of Lemma 4 which yields to \(A_{2,1} \leq c h_n^2\). Regarding \(A_{2,2}\), from Cauchy-Schwarz, Burkholder-Davis-Gundy inequalities and the first point of Lemma 4 we have

\[
A_{2,2} \leq \frac{c h_n^2}{M_n^2}.
\]

It is negligible compared to \(c h_n^2\), since \(\frac{1}{h_n^2 M_n^2} = O(h_n^{2-1/2}) \xrightarrow{n \to \infty} 0\). Together with (41), it concludes the bound on the contribution of \(\tilde{W}_x^1_{x_0, \Delta_n, \epsilon}\) in (33):

\[
\int_{|x_0| \leq \Delta_n^{1/2-\gamma}} \mathbb{E}_{x_0}[(\mathbb{E}_{x_0}[\tilde{W}_x^1_{x_0, \Delta_n, \epsilon} \mathcal{X}_u^\epsilon])^2 \pi'(x_0) dx_0 \leq c h_n^2 M_n^2.
\]

**Step 4: Handling \(\tilde{W}_x^2_{x_0, \Delta_n, \epsilon}\), derivation leading term.**

It is more complicated to find a bound on \(\tilde{W}_x^2_{x_0, \Delta_n, \epsilon}\). We need first of all to derive the explicit expression for \(D_{\epsilon} \mathcal{X}_u^\epsilon\). We omit the details of the computations, which follow the same route. First, we apply Theorem 2.2.1 of Nualart [42], to obtain that \(t \mapsto D_{\epsilon} \mathcal{X}_u^\epsilon\) is solution of a linear
SDE obtained by formal differentiation of the dynamics (36). Second, we solve this linear SDE by the usual method of variation of parameters. It yields to the following explicit representation for \( D_u \hat{X}_{\Delta u} \):

\[
D_u \hat{X}_{\Delta u} = (D_u \hat{X}_{\Delta u})_1 + (D_u \hat{X}_{\Delta u})_2 + (D_u \hat{X}_{\Delta u})_3,
\]

with

\[
(D_u \hat{X}_t)_1 = \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_{t-}) \hat{X}_t + \psi_{h_n}(\hat{X}_{t-}) \frac{1}{M_n} (Y_t^{-1} Y_t^r),
\]

\[
(D_u \hat{X}_t)_2 = Y_t^r \int_0^t (Y_s^{-1})^{-1} \left[ \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_s) (D_u \hat{X}_s) \hat{X}_s \right] dB_s - Y_t^r \int_0^t (Y_s^{-1})^{-1} \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_s) \psi_{h_n}(\hat{X}_s) (D_u \hat{X}_s) \hat{X}_s ds
\]

\[
(D_u \hat{X}_t)_3 = Y_t^r \int_0^t (Y_s^{-1})^{-1} \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_s) (D_u \hat{X}_s) dB_s.
\]

\[
\text{It follows that}
\]

\[
\hat{W}^2_{x_0, \Delta u, c} = \frac{< D \hat{X}_{\Delta u}, D \hat{X}_{\Delta u} >}{< D X_{\Delta u}, D X_{\Delta u} >},
\]

consists in three terms. We therefore need to bound

\[
\int_{|x_0| \leq \Delta_n^{1/2}} E_{x_0} [E_{x_0} [\hat{W}^2_{x_0, \Delta u, c} | \hat{X}_{\Delta u}|] \pi^r(x_0)] dx_0
\]

\[
= \int_{|x_0| \leq \Delta_n^{1/2}} E_{x_0} [E_{x_0} \left[ \frac{< D \hat{X}_{\Delta u}, (D \hat{X}_{\Delta u})_1 + (D \hat{X}_{\Delta u})_2 + (D \hat{X}_{\Delta u})_3 >}{< D X_{\Delta u}, D X_{\Delta u} >} | \hat{X}_{\Delta u}| \right] \pi^r(x_0)] dx_0
\]

\[
= I_1 + I_2 + I_3.
\]

From Equation (43) and below it is easy to see that there are some terms negligible in \( I_1 \) and \( I_2 \), due to presence of \( \hat{X}_{\Delta u} \). Indeed, the following lemma on the process \( \hat{X}^r \) holds true. Its proof can be found in the Appendix A.

**Lemma 5.** Let \( \hat{X}^r \) be the process solution to (36). Then, for any \( p \geq 2 \), there exists a constant \( c > 0 \) such that

\[
\sup_{x_0 \in \mathbb{R}} \sup_{0 \leq \Delta u \leq \Delta_n} E_{x_0} [\hat{X}^r_p | \hat{X}_{\Delta u}|] \leq \Delta_n^{\Delta_n/2} \frac{h_n}{M_n^2}.
\]

Moreover, some rough estimation are enough to bound \( I_1 \) and \( I_2 \), while we need sharper bounds for the analysis of \( I_3 \). This is shown in Lemma 6 below, whose proof is in the Appendix A.

**Lemma 6.** Let \( I_1 \) and \( I_2 \) be as above. Then, there exists a constant \( c > 0 \) such that

\[
I_1 + I_2 \leq c \frac{h_n}{M_n^2}.
\]

Concerning \( I_3 \), approximating \( Y_t^r \), \( (Y_t^r)^{-1} \) and \( D_u \hat{X}_{\Delta u} \) by the constant \( 1 \), as in Proposition 5, leads us to the following decomposition.

**Lemma 7.** Let all the processes be as previously defined. Then, the following decomposition holds true:

\[
\frac{< D \hat{X}_{\Delta u}, (D \hat{X}_{\Delta u})_1 >}{< D X_{\Delta u}, D X_{\Delta u} >} = \frac{1}{M_n \Delta_n} \int_0^{\Delta_n} \int_0^{\Delta_n} \psi_{h_n}(\hat{X}_s) dB_s du + R_n^{(2)};
\]

where \( R_n^{(2)} \) is such that

\[
\int_{|x_0| \leq \Delta_n^{1/2}} E_{x_0} [R_n^{(2)} | \pi^r(x_0)] dx_0 \leq \frac{\Delta_n^{1/2}}{M_n^2}
\]

for some \( c > 0 \).
The proof of Lemma 7 is postponed to the Appendix A.

**Step 5: Handling the principal term of** $\hat{W}^2_{x_n, \Delta_n, \epsilon^*}$

We are left to study the main term of $I_3$ coming from the expansion given in the Lemma 7. The first step is to get rid of the stochastic integral by application of the Itô’s formula, as given by the next lemma whose proof is given in the Appendix A.

**Lemma 8.** We have

$$\frac{1}{\Delta_n M_n} \int_0^{\Delta_n} \int_u \psi_{h_n}'(\hat{X}_u^s) dB_u ds = -\frac{1}{2\Delta_n M_n} \int_0^{\Delta_n} \psi''_{h_n}(\hat{X}_u^s) ds + R_{n(3)}$$

where $R_{n(3)}$ is such that $\int_{[x_0]}^{\Delta_n^{1/2}} \mathbb{E}_{x_0}[|R_{n(3)}|^2] dx_0 \leq \frac{\epsilon \alpha}{M_n^2}$ for some $c > 0$.

As a consequence, the main term in $I_3$ is given by

$$I_3 := \int_{[x_0], \Delta_n^{1/2}} \mathbb{E}_{x_0}[(\mathbb{E}_{x_0}[\frac{1}{2\Delta_n M_n} \int_0^{\Delta_n} \psi''_{h_n}(\hat{X}_u^s) ds | \hat{X}_{\Delta_n}^s])^2] \pi'(x_0) dx_0.$$  \(48\)

Here, using Jensen’s inequality to get rid of the conditional expectation does not give the correct rate, and so we have to analyze in details the conditional expectation. Hence, we need a sharp bound for this quantity, as the one gathered in Lemma 9 stated below and shown in the Appendix A.

**Lemma 9.** For any $0 < \eta < 1/2$, there exists $C_\eta$ such that

$$|\mathbb{E}_x \left[ \frac{1}{\Delta_n} \int_0^{\Delta_n} (\psi_{h_n})''(\hat{X}_u^s) ds \mid \hat{X}_{\Delta_n}^s = y \right]| \leq C_\eta \left[ (1 + \frac{h_n}{\sqrt{\Delta_n}}) \mathbb{I}_{|y| \leq 2h_n} + \frac{h_n}{\sqrt{\Delta_n}} \Phi(\frac{1}{\eta} \frac{|y| - h_n}{\sqrt{\Delta_n}}) \mathbb{I}_{|y| > 2h_n} \right] e^{\frac{3\eta(\eta^2 - 1)}{\Delta_n}},$$  \(49\)

where $\Phi(u) = 1 + \frac{\epsilon_u^2}{u^2}$ for $u > 0$.

It follows using the Gaussian control \(81\), given in the Appendix A, on the transition density of the diffusion process, that

$$\int_{[x_0], \Delta_n^{1/2}} \mathbb{E}_{x_0}[(\mathbb{E}_{x_0}[\frac{1}{2\Delta_n M_n} \int_0^{\Delta_n} \psi''_{h_n}(\hat{X}_u^s) ds | \hat{X}_{\Delta_n}^s])^2] \pi'(x_0) dx_0$$

\(50\)

for some constant $c > 0$. Remark that $\eta \in (0, \frac{1}{2})$ can be chosen arbitrarily small, so that we have in particular $c^2 - 6 \eta > 0$, it is easy to see that

$$\int_{\mathbb{R}^2} (1 + \frac{h_n}{\sqrt{\Delta_n}})^2 \mathbb{I}_{|y| \leq 2h_n} e^{\frac{c^2 - 6 \eta (|x| - y)^2}{\Delta_n}} dy \pi'(x_0) dx_0 \leq (\sup_{x_0 \in \mathbb{R}, \epsilon \in [0,1]} \pi'(x_0)) \times \int_{\mathbb{R}} (1 + \frac{h_n}{\sqrt{\Delta_n}})^2 \mathbb{I}_{|y| \leq 2h_n} dy \leq c h_n,$$

where we used $h_n \leq \Delta_n \leq \sqrt{\Delta_n}$. It provides the wanted bound on the first term of \(50\). Regarding the second, we replace the function $\Phi$ obtaining

$$\frac{c}{M_n^2} \int_{\mathbb{R}^2} \frac{h_n^2}{\Delta_n} (1 + e^{\frac{c^2 - 6 \eta (|x| - y)^2}{\Delta_n}} \mathbb{I}_{|y| > 2h_n}) e^{\frac{c^2 - 6 \eta (|x| - y)^2}{\Delta_n}} dy \pi'(x_0) dx_0$$

$$\leq \frac{c}{M_n^2} \frac{h_n^2}{\Delta_n} \int_{\mathbb{R}^2} e^{\frac{c^2 - 6 \eta (|x| - y)^2}{\Delta_n}} dy \pi'(x_0) dx_0$$

$$+ \frac{c}{M_n^2} \frac{h_n^2}{\Delta_n} \int_{\mathbb{R}^2} e^{-\frac{3\eta|y| - h_n^2}{\Delta_n}} \mathbb{I}_{|y| > 2h_n} e^{\frac{c^2 - 6 \eta (|x| - y)^2}{\Delta_n}} dy \pi'(x_0) dx_0$$

$$\leq \frac{c}{M_n^2} \frac{h_n^2}{\Delta_n} (1 + \int_{|y| > 2h_n} e^{\frac{3\eta|y| - h_n^2}{\Delta_n}} dy).$$  \(51\)
On the last integral we apply the change of variable \( y := \frac{-h - n}{\sqrt{\Delta_n}} \) on the part \( y > 2h_n \), and use symmetry of the integrand on the part \( y < -2h_n \). We obtain it is smaller than

\[
\frac{c}{M_n^2}{\Delta_n^2} \int_{-\infty}^{h_n} \sqrt{\Delta_n} e^{-\frac{y^2}{2}} dy \leq \frac{c}{M_n^2}{\Delta_n^2} \frac{\sqrt{\Delta_n} \eta}{h_n} = \frac{ch_n}{M_n^2},
\]

where we have used that \( \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = O \left( \frac{1}{\sqrt{\pi}} \right) \) and in the last inequality we have included \( \eta \) in the constant \( c \).

The control on the principal term of \( \hat{W}_{z_0, \Delta_n, \epsilon} \) is concluded by replacing the estimation here above in (51) and remarking that, as \( h_n \leq \Delta_n \), it holds \( \frac{c}{M_n^2} \Delta_n^2 \leq \frac{ch_n}{M_n^2} \). We deduce \( \mathcal{I}_3 \leq \frac{ch_n}{M_n^2} \). Collecting (44) with Lemmas 6, 7, 8 and with (48) it follows

\[
\int_{|x_0| \leq \frac{1}{\sqrt{\Delta_n}}} \mathbb{E}_{x_0} [ (\mathbb{E}_{x_0} [ \hat{W}_{z_0, \Delta_n, \epsilon} | X_{\Delta_n} ] )^2 ] \pi^*(x_0) dx_0 \leq \frac{ch_n}{M_n^2}, \tag{52}
\]

From (40), (42) and (52) we obtain (33), which concludes the proof.

## A proof of technical results

This section is devoted to the proof of the results which are more technical and for which some preliminaries are needed.

### A.1 Proof of Lemma 1

**Proof.** For \( \psi \in L^1(\pi) \), we write \( \pi(x)p_\epsilon^* (\psi) (x) = \pi(x) \int \rho \sqrt{\epsilon} p_\epsilon(x, y) \psi(y) dy = \int \rho \sqrt{\epsilon} \frac{\pi(x)p_\epsilon(x, y)}{\pi(y)} \pi(y) \psi(y) dy. \) Using that the one dimensional diffusion \( X \) is reversible we have \( \pi(x)p_\epsilon(x, y) = \pi(y)p_\epsilon(y, x) \), and as a result, \( \pi(x)p_\epsilon^* (\psi) (x) = \int \rho \sqrt{\epsilon} p_\epsilon(x, y) \pi(y) \psi(y) dy. \) From Theorem 1, we have for \( s \in (0, 1] \), the upper bound \( p_\epsilon(x, y) \leq c/\sqrt{\epsilon} \), where the constant \( c \) is uniform on the class \( \Sigma \). We deduce \( \pi(x)|p_\epsilon^* (\psi) (x)| \leq \frac{c}{\sqrt{\epsilon}} \int \rho \sqrt{\epsilon} |\pi(y)| \psi(y) dy = \frac{c}{\sqrt{\epsilon}} \| \psi \|_{L^1(\pi)}. \)

### A.2 Proof of Lemma 2

**Proof.** Let us denote \( g_n(B_n) = \mathbb{E}[A_n | B_n] \) and \( g'_n(B'_n) = \mathbb{E}[A'_n | B'_n] \) where \( g_n \) and \( g'_n \) are some measurable functions. Let \( p' < p \), by duality, we have

\[
\mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right|^{p'} \right]^{1/p'} = \sup_{Z \in \mathbb{H}(L^{p'}, L^p)} \mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right| Z \right] \tag{53}
\]

where \( q' = \frac{p'}{p' - 1} \). For \( M > 0 \), we set \( Z^{(M)} = Z 1_{|Z| \leq M} \) and write

\[
\mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right| Z 1_{|Z| > M} \right] \leq \mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right| Z^{(M)} \right] + \mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right| Z 1_{|Z| > M} \right]. \tag{54}
\]

Using consecutively Hölder’s inequality where \( q = p/(p - 1) \) and Minkowski’s inequality in the first line below, Jensen’s inequality in the second line, and (31) in the third line, we can write

\[
\mathbb{E} \left[ \left| g_n(B_n) - g'_n(B'_n) \right| Z 1_{|Z| > M} \right] \leq \mathbb{E} \left[ \left| g_n(B_n) \right|^{1/p} + \mathbb{E} \left[ \left| g'_n(B'_n) \right| \right]^{1/p} \right] \times \mathbb{E} \left[ |Z|^{q} 1_{|Z| > M} \right]^{1/q'} \leq \kappa_p \mathbb{E} \left[ \left| g_n(B_n) \right|^{1/p} + \mathbb{E} \left[ \left| A'_n \right| \right]^{1/p} \right] \times \mathbb{E} \left[ |Z|^{q} 1_{|Z| > M} \right]^{1/q'} \leq \kappa_p \mathbb{E} \left[ |Z|^{q} 1_{|Z| > M} \right]^{1/q'}.
\]

As \( p > p' \) we have \( q < q' \) and using again Hölder inequality we deduce \( \mathbb{E} \left[ |Z|^{q} 1_{|Z| > M} \right]^{1/q'} \leq M^{(q - q')/(q' - q)} \leq (1/M)^{(q - q')/(q' - q)}, \) since \( \|Z\|_{q'} \leq 1 \). As a consequence, choosing
\[ M = \frac{\lvert g_n \rvert}{\lvert \partial \cdot \partial \rvert} \text{ for } r > 0, \text{ we deduce} \]
\[
\sup_{\|Z\|_{\beta} \leq 1} \left| \mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))Z \right] |Z| > M \right| \leq \kappa_n \frac{n^{\beta}}{M^{(q-q')/q}} \leq \kappa n^{-r}. \tag{55}
\]
We now focus on the the first term in the right hand side of \((54)\). Using that the \(L^p\) norm of 
\(g_n(B_n)\) and \(g_n'(B'_n)\) is upper bounded by \(n^{\beta_0}\), that \(|Z^{(M)}| \leq M\) and \((30)\), we have
\[
\mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))Z^{(M)} \right] \leq \mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))Z^{(M)} \mathbb{1}_{\Omega_n} \right] + O(n^{\beta_0}Mn^{-r'(1-1/p)}), \tag{56}
\]
where \(r' > 0\) can be chosen arbitrarily large. On \(\Omega_n\), we have \(Z^{(M)} = h(B_n, B'_n) \mathbb{1}_{\|h(B_n, B'_n)\| \leq M} = h(B_n, B'_n) \mathbb{1}_{h(B_n, B'_n) \leq M}\), and it follows
\[
\mathbb{E} \left[ g_n(B_n)Z^{(M)} \mathbb{1}_{\Omega_n} \right] = \mathbb{E} \left[ g_n(B_n)h(B_n, B'_n) \mathbb{1}_{h(B_n, B'_n) \leq M} \mathbb{1}_{\Omega_n} \right] = \mathbb{E} \left[ A_n h(B_n, B'_n) \mathbb{1}_{h(B_n, B'_n) \leq M} \right] + O(n^{\beta_0}Mn^{-r'(1-1/p)}),
\]
where in the last line we used \(g_n(B_n) = \mathbb{E}[A_n \mid B_n]\). In an analogous way, we have 
\(\mathbb{E} \left[ g_n'(B'_n)Z^{(M)} \mathbb{1}_{\Omega_n} \right] = \mathbb{E} \left[ A'_n h(B'_n, B'_n) \mathbb{1}_{h(B'_n, B'_n) \leq M} \right] + O(n^{\beta_0}Mn^{-r'(1-1/p)}).
\]
Now, we deduce from \((56)\),
\[
\mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))Z^{(M)} \right] \leq \mathbb{E} \left[ A_n h(B_n, B'_n) \mathbb{1}_{h(B_n, B'_n) \leq M} \mathbb{1}_{\Omega_n} \right] - \mathbb{E} \left[ A'_n h(B'_n, B'_n) \mathbb{1}_{h(B'_n, B'_n) \leq M} \mathbb{1}_{\Omega_n} \right] + O(n^{\beta_0}Mn^{-r'(1-1/p)}) = O(n^{\beta_0}Mn^{-r'(1-1/p)}). \tag{57}
\]
As \((A_n, B_n) = (A'_n, B'_n)\) on \(\Omega_n\), it implies
\[
\mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))Z^{(M)} \right] \leq \mathbb{E} \left[ A_n h(B_n, B'_n) \mathbb{1}_{h(B_n, B'_n) \leq M} \mathbb{1}_{\Omega_n} \right] - \mathbb{E} \left[ A'_n h(B'_n, B'_n) \mathbb{1}_{h(B'_n, B'_n) \leq M} \mathbb{1}_{\Omega_n} \right] + O(n^{\beta_0}Mn^{-r'(1-1/p)}) = O(n^{\beta_0}Mn^{-r'(1-1/p)}).
\]
Collecting \((53), (54), (55)\) and \((57)\) we deduce that
\[
\mathbb{E} \left[ (g_n(B_n) - g_n'(B'_n))^{r'} \right] \leq C n^{\beta_0}Mn^{-r'(1-1/p)} = C n^{\beta_0}n^{\lfloor \frac{\beta_0 - 1}{q-q'} \rfloor}n^{-r'(1-1/p)} \leq C n^{-r},
\]
as we can choose \(r'\) arbitrarily large. The lemma is shown. \[\square\]

### A.3 Proof of Lemma 3

We need to prove that \(\sup_{x \in \Omega} \mathbb{E} [W_{x, \Delta_n, \epsilon}] = O(\Delta_n^{-1})\). Using the expression \((27)\) and the fact that the operator \(\delta : D^{1,p}(H) \rightarrow L^p\) is bounded (see Proposition 1.5.8 in Nualart [42]), it is sufficient to prove that \(u \in D^{1,4}(H)\) and \(\|u\|_{D^{1,4}(H)} \leq c \Delta_n^{-1}\) with 
\(u_t = \frac{\partial}{\partial \Delta} X_{\Delta_n}^t\). Now, using the Leibniz rule for the Malliavin derivative and Hölder inequality, it is possible to extend the Proposition 1.5.6. in Nualart [42] and get \(\|u\|_{D^{1,4}(H)} \leq c \|D X_{\Delta_n}^t\|_{D^{1,16}(H)} \left\| X_{\Delta_n}^t \right\|_{D^{116}(B)} \left\| \frac{1}{<DX_{\Delta_n}^t, DX_{\Delta_n}^t>} \right\|_{D^{1,8}(B)}\).

It remains to bound the three norms in the right hand side of the last equation.

By recalling \((22)\) and \((28)\), we remark that the process \(t \mapsto (X_t^\epsilon, \dot{X}_t^\epsilon)\) is solution of the SDE
\[
\begin{align*}
X_t^\epsilon &= x_0 + \int_0^t b(X_s^\epsilon)ds + \int_0^t a_s(X_s^\epsilon)dB_s \\
\dot{X}_t^\epsilon &= \int_0^t b_s(X_s^\epsilon)\dot{X}_s^\epsilon ds + \int_0^t \left[ \dot{a}_s(X_s^\epsilon) + a_s'(X_s^\epsilon)\dot{X}_s^\epsilon \right] dB_s
\end{align*}
\]
where \(a_\epsilon(x) = 1 + \frac{\alpha_\epsilon}{M_{\alpha_\epsilon}} \psi_{\alpha_\epsilon}(x)\) and \(\dot{a}_\epsilon(x) = \frac{\alpha_\epsilon'}{M_{\alpha_\epsilon}} \psi_{\alpha_\epsilon}(x)\). Since \(\|a_\epsilon'\|_\infty \leq \frac{\alpha_\epsilon}{M_{\alpha_\epsilon}}, \|a_\epsilon''\|_\infty \leq \frac{\alpha_\epsilon}{M_{\alpha_\epsilon}}, \frac{1}{M_{\alpha_\epsilon}} = O(h_\alpha^{-2}) = O(1)\) using \(\beta \geq 3\), and the definition of \(b\) given in \((14)\), we see that the coefficients in the SDE satisfied by \((X_t^\epsilon)\) are bounded, together with their first and second order
derivatives. By Theorem 2.2.2 in [42], this implies that the Malliavin derivatives up to order 2 of $X_t^r$ are bounded (see (102) in Appendix B.1). It yields $\sup_{r \in [0, \Delta_n]} E[\max_{t \in [0, \Delta_n]} |D_r X_t^r|^p] \leq c(p)$, and $\sup_{r, r' \in [0, \Delta_n]} E[\max_{t \in [0, \Delta_n]} |D^2_{r, r'} X_t^r|^p] \leq c(p)$ for all $p \geq 2$, where the constant $c(p)$ does not depend on $r, n$. To get a control on the Malliavin derivative of $\tilde{X}_t^r$, we use Theorem 2.2.1 in [42], to obtain that the Malliavin derivatives of $(X_t^r, \tilde{X}_t^r)$ are solution of the following SDE, where $0 \leq r \leq t \leq \Delta_n$,

\[
\begin{align*}
[D_r X_t^r] = & \left[ a_r(X_t^r) \right] + \int_r^t \left[ b_r(X_u^r)D_r X_u^r + \partial_b A_r(X_v^r) + \partial_a B_r(X_v^r) + D_r \tilde{X}_s ds \right] ds \\
+ & \int_r^t \left[ \partial_a A_r(X_v^r) + \partial_b B_r(X_v^r) + D_r \tilde{X}_s ds \right] dB_s, \quad (58)
\end{align*}
\]

with $B(x, v) = b'(x)v$ and $A_r(x, v) = \tilde{a}_r(x) + a'_r(x)v = \frac{1}{M_r} \psi_n(x) + \frac{1}{M_r} \psi_n'(x)v$. Using $\frac{1}{M_r} = O(h_n^{\beta - 2}) = O(1)$, we have that $\|\partial_b A_r\|_\infty + \|\partial_a B\|_\infty \leq c$ and $|\partial_b A_r(x, v)| + |\partial_a B_r(x, v)| \leq c(1 + |v|)$ for some constant $c$ independent of $\epsilon, n$. We apply Lemma 2.2.1 in [42] on the second component of the SDE (58) and deduce $\sup_{r \in [0, \Delta_n]} E[\max_{t \in [0, \Delta_n]} |D_r X_t^r|^p] \leq c(p)$. It is sufficient to infer that $\|D X_{\Delta_n}^r\|_{L_{2, 16}(H)} \leq c$; $\|X_{\Delta_n}^r\|_{L_{2, 16}(R)} \leq c$ and $\|X_{\Delta_n}^r\|_{L_{2, 16}(R)} \leq c$.

It remains to prove that $\|\frac{1}{\Xi^+_r} \int_0^\Delta_n [D_r X_{\Delta_n}^r] |D_{\Xi^+_r}^a| ds \|_{L_{2, 16}(R)} = O(\Delta_n^{-1})$. Using Proposition 2.1 in [41], we write the explicit representation in the univariate case,

\[
D_r X_{\Delta_n} = a_r(X_{\Delta_n}^r) \exp \left( \int_0^{\Delta_n} [b'(X_u^r) - \frac{a'_r(X_u^r)}{a_r(X_u^r)} h(X_u^r) - \frac{1}{2} a''_r(X_u^r) du] \right), \quad \text{where } a_r = 1 + \frac{c}{M_r} \psi_n.
\]

Using the boundedness of $1/(M_r h_n^2)$ and $1/a_r$, we deduce that $1/c \leq |D_r X_{\Delta_n}^r| \leq c$ for some constant $c$. In turn, $< D X_{\Delta_n}^r, D X_{\Delta_n}^r > = \int_0^{\Delta_n} |D_r X_{\Delta_n}^r|^2 dr \geq c \Delta_n$ for some $c > 0$. By the chain rule property for the Malliavin derivative, see Proposition 1.2.3 in [42], we have $D(\int_{< D X_{\Delta_n}^r, D X_{\Delta_n}^r >} = -\int_{< D X_{\Delta_n}^r, D X_{\Delta_n}^r >} < D X_{\Delta_n}^r, D X_{\Delta_n}^r >$. Therefore, $\|\frac{1}{\Xi^+_r} \int_0^{\Delta_n} \frac{1}{D_r X_{\Delta_n}^r} D_r X_{\Delta_n}^r \|_{L_{2, 16}(R)} \leq \frac{c}{\Xi^+_r} \|< D X_{\Delta_n}^r, D X_{\Delta_n}^r >\|_{L_{2, 16}(R)}$.

We write

\[
D_r(< D X_{\Delta_n}^r, D X_{\Delta_n}^r >) = 2 \int_0^{\Delta_n} D^2_{r, r'} X_{\Delta_n}^r D_r X_{\Delta_n}^r dr
\]

and use that $|D_r X_{\Delta_n}^r| \leq c$, by the representation (59), to deduce

\[
\int_0^{\Delta_n} D_r(< D X_{\Delta_n}^r, D X_{\Delta_n}^r >)^2 dr \leq c \Delta_n \int_0^{\Delta_n} \int_0^{\Delta_n} |D^2_{r, r'} X_{\Delta_n}^r|^2 duds.
\]

It entails $\|< D X_{\Delta_n}^r, D X_{\Delta_n}^r >\|_{L_{2, 16}(R)} \leq c \Delta_n \|X_{\Delta_n}^r\|_{L_{2, 16}(R)} = O(\Delta_n)$ and in turn $\|\frac{1}{\Xi^+_r} \int_0^{\Delta_n} \frac{1}{D X_{\Delta_n}^r} D X_{\Delta_n}^r \|_{L_{2, 16}(R)} = O(1/\Delta_n)$. The lemma is proved.

A.4 Proof of Lemma 4

Proof. The proof of the three points relies on Lemma 2 and on the stationarity of the process $(X_t^r)_{t \geq 0}$. We recall that on $\Omega_n$ it is $\tilde{X}^r = X^r \forall \epsilon > 0$ while on the complement the following bound holds: $P(\Omega_n^c) = o(n^{-r})$. Hence, using also the boundedness of $\psi_n$, we have for any $p \geq 1$

\[
\begin{align*}
\sup_{0 < s \leq \Delta_n} \int_{\mathbb{R}} E_{x_0} [\psi_n(X_s^r)]^p |\pi^r(x_0)| dx_0 \\
\leq \sup_{0 < s \leq \Delta_n} \int_{\mathbb{R}} E_{x_0} [\psi_n(X_s^r)]^p |\pi^r(x_0)| dx_0 + o(n^{-r}) \\
= \sup_{0 < s \leq \Delta_n} E_{x_0} |\psi_n(X_s^r)]^p + o(n^{-r}).
\end{align*}
\]

Then from the stationarity of the process $X^r$ it is, for any $p \geq 1$,

\[
\forall s > 0, \quad E_{x_0} |\psi_n(X_s^r)]^p = \int_{-\Delta_n}^{\Delta_n} (\psi_n(y))^p \pi^r(y) dy \leq c_{n_0}.
\]

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We act similarly in order to show the second point of the lemma. We remark we can use Lemma 2 as both the $L^p$ norms of $|\psi_n^{(k)}(\hat{X}_s^\epsilon)|^p$ and $|\psi_n^{(k)}(X_s^\epsilon)|^p$ are upper bounded by $h_n^{-kp} = n^{-r}$ for some $r_0$ and some $p' > 1$. Hence, for $p \geq 1$ and $k \geq 1$ we have

$$\int \mathbb{E}_{x_0}|\psi_n^{(k)}(\hat{X}_s^\epsilon)|^p \pi^\epsilon(x_0)dx_0 \leq \mathbb{E}_{x_1}|\psi_n^{(k)}(X_s^\epsilon)|^p + o(n^{-r}).$$

Regarding the first term we observe it is

$$\mathbb{E}_{x_1}|\psi_n^{(k)}(X_s^\epsilon)|^p = \int_{-h_n}^{h_n} |\psi_n^{(k)}(y)|^p \pi^\epsilon(y)dy \leq c h_n^{-kp},$$

while the second is negligible, up to choose an $r$ which is large enough.

We are left to show the third point of the lemma. The idea is once again to move back to the stationary process. To do that, Lemma 2 comes in handy one more time. Its applicability is ensured by the boundedness of $\psi_n$ and the fact that both $X_s^\epsilon - X_0^\epsilon$ and $X_s^\epsilon - X_0^\epsilon$ have bounded moments of any order. Then,

$$\int_0^{\Delta_n} \int \mathbb{E}_{x_0}|(\hat{X}_s^\epsilon - \hat{X}_0^\epsilon)^2| \psi_n^{(2)}(\hat{X}_s^\epsilon) du \pi(x_0)dx_0$$

$$\leq \int_0^{\Delta_n} \int \mathbb{E}_{x_0}|(X_s^\epsilon - X_0^\epsilon)^2| \psi_n^{(2)}(X_s^\epsilon) du \pi(x_0)dx_0 + o(n^{-r})$$

$$= \int_0^{\Delta_n} \mathbb{E}_{x_1}|(X_s^\epsilon - X_0^\epsilon)^2| \psi_n^{(2)}(X_s^\epsilon) du + o(n^{-r})$$

$$= \int_0^{\Delta_n} \mathbb{E}_{x_1}|(X_s^\epsilon - X_0^\epsilon)^2| \psi_n^{(2)}(X_s^\epsilon) du + o(n^{-r}),$$

where we have used that the diffusion is reversible. Introducing the conditional expectation with respect to $X_0^\epsilon$ we obtain that the integral here above is equal to

$$\int_0^{\Delta_n} \mathbb{E}_{x_1}|(X_s^\epsilon - X_0^\epsilon)^2| \psi_n^{(2)}(X_s^\epsilon) du + o(n^{-r})$$

$$\leq c \int_0^{\Delta_n} \mathbb{E}_{x_1}|u| \psi_n^{(2)}(X_s^\epsilon) du + o(n^{-r})$$

$$\leq c \Delta_n^2 h_n + o(n^{-r}),$$

as we wanted.

\[\square\]

### A.5 Proof of Proposition 5

In order to get an expansion for $\hat{W}_{x_0,\Delta_n,\epsilon}$ we need asymptotic controls on the Malliavin derivatives of the process $\hat{X}$. It is the purpose of the next Section to collect some properties on $D\hat{X}$, that will be useful for the proof of Proposition 5 and Lemmas 5–8.

#### A.5.1 Controls on $D\hat{X}$

First, we focus on $Y^\epsilon$ given explicitly by (37). Let us define

$$\mathcal{I}_t = \int_0^t \psi_n(\hat{X}_s^\epsilon)dB_s = \int_0^t \psi_n(\hat{X}_s^\epsilon)a_{\epsilon}(\hat{X}_s^\epsilon)dB_s,$$

where we recall that $a_{\epsilon}(\hat{X}_s^\epsilon)$ is the volatility of the process $\hat{X}_s^\epsilon$ appearing in (22), i.e. $a_{\epsilon}(X_s^\epsilon) = 1 + \frac{1}{\sqrt{\epsilon}} \psi_n(\hat{X}_s^\epsilon)$. We denote as $\Xi$ a primitive of the function $\frac{\psi_n(\hat{X}_s^\epsilon)}{a_{\epsilon}(\hat{X}_s^\epsilon)}$ which is null at 0. From Ito formula it follows

$$\mathcal{I}_t = \int_0^t \psi_n(\hat{X}_s^\epsilon)dB_s = \Xi(\hat{X}_t^\epsilon) - \Xi(\hat{X}_0^\epsilon) - \frac{1}{2} \int_0^t \left(\frac{\psi_n}{a_{\epsilon}}\right)'(\hat{X}_s^\epsilon)a_{\epsilon}^2(\hat{X}_s^\epsilon)ds.$$

(61)
We now observe that
\[
\|\Xi(u)\| = \int_0^u \frac{\psi'_{h_n}(y)}{(1 + \frac{y}{h_n})^2} dy \\
\leq \left| \int_0^u \frac{\psi'_{h_n}(y)}{h_n} dy \right| \\
= \int_0^u \frac{1}{h_n} \psi'\left(\frac{y}{h_n}\right) dy \\
\leq \int_\mathbb{R} |\psi'(y)| dy < \infty.
\]

In order to bound the last term in the right hand side of (61) we remark the following estimations hold true:
\[
|\psi'_{h_n}(y)| \leq \frac{c}{h_n}, \quad |\psi''_{h_n}(y)| \leq \frac{c}{h_n^2}, \quad |a''_0(y)| \leq \frac{1}{h_n M_n}.
\]

It follows, using also the fact that $Y^\epsilon$ is explicit as in (37)
\[
Y^\epsilon_t \leq \exp\left(\frac{c}{M_n} + \frac{c t}{h_n^2 M_n} + \frac{c t}{h_n^2 M_n^2}\right).
\]

In the same way we have an analogous upper bound for $(Y^\epsilon_t)^{-1}$, which provides
\[
\sup_{t \in [0,\Delta]} |Y^\epsilon_t| + |(Y^\epsilon_t)^{-1}| \leq 2 \exp\left(\frac{c \Delta_n}{h_n^2 M_n}\right) < \infty,
\]
where we used $h_n \to 0$, $M_n \to \infty$ and that $h_n \leq \Delta_n$, so that in consequence $\frac{\Delta_n}{h_n M_n}$ dominates the three terms in the exponential of (63).

Following the same reasoning we have used in order to get (64) and having in mind the explicit expression for $Y^\epsilon$ and $(Y^\epsilon)^{-1}$ given by (37) it is easy to see that
\[
|Y^\epsilon_{\Delta_n} - 1| \leq \frac{c \Delta_n}{M_n h_n^2},
\]
\[
\sup_{u \in [0,\Delta]} |(Y^\epsilon_u)^{-1} - 1| \leq \frac{c \Delta_n}{M_n h_n^2}.
\]

From (38), we deduce that for all $0 < u < s < \Delta_n$,
\[
0 < c \leq c(1 - \frac{1}{h_n M_n}) \leq |D_u \hat{X}_s^\epsilon| \leq c'(1 + \frac{1}{h_n M_n}) \leq c'.
\]

In turn, we have simple bounds on the Malliavin bracket, $\forall r \in (0, \Delta_n]$,
\[
cr \leq \langle D \hat{X}_r^\epsilon, D \hat{X}_r^\epsilon \rangle \leq c'r.
\]

### A.5.2 Proof of (41)

**Proof.** Let us denote
\[
\Phi_{\Delta_n} = \sqrt{\Delta_n} \delta \left( \frac{D \hat{X}_{\Delta_n}^\epsilon}{<D \hat{X}_{\Delta_n}^\epsilon, D \hat{X}_{\Delta_n}^\epsilon>} \right).
\]

- We start by proving for all $p \geq 2$
\[
\sup_{x \in \mathbb{R}} \mathbb{E}_{\mathbb{R}} \left[ \left| \Phi_{\Delta_n} - \frac{B_{\Delta_n}}{\sqrt{\Delta_n}} \right|^p \right] \leq \frac{c(p)}{(M_n h_n^2)^p}.
\]

Using Proposition 1.3.3. in Nualart [42] with the notation $L = -\delta \circ D$ for the so-called Ornstein-Uhlenbeck operator, we have
\[
\Phi_{\Delta_n} = -\Delta_n^{1/2} L(\hat{X}_{\Delta_n}^\epsilon) + \frac{\Delta_n^{1/2}}{<D \hat{X}_{\Delta_n}^\epsilon, D \hat{X}_{\Delta_n}^\epsilon>} D \hat{X}_{\Delta_n}^\epsilon > \mathbb{J}_{H}.
\]

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Application of the linear operator $L$ to the dynamic (34), recalling $a_c = 1 + \epsilon \frac{h_n}{M_n}$, yields to

$$L(\hat{X}^e_t) = \int_0^s \left\{ a'_c(\hat{X}^e_r) L(\hat{X}^e_r) + a''_c(\hat{X}^e_r) \right\} dB_r - \int_0^s a_c(\hat{X}^e_r) dB_r,$$

where we used the property $L(\int_0^t u_r dB_r) = \int_0^t L(u_r) dB_r - \int_0^t u_r dB_r$ for $(u_r)$. Then, an adapted process taking values in the space of smooth random variables in the Malliavin sense together with Proposition 1.4.5 in [42]. A rigorous justification of (72) is given by Theorem 10.3 of [10]. We deduce that for $p \geq 2$,

$$\mathbb{E}_{x_0}[|L(\hat{X}^e_t)|^p] \leq c_p \|a'_c\|_{\infty}^{p} s^{p/2-1} \int_0^s \mathbb{E}_{x_0}[|L(\hat{X}^e_r)|^p] dr + c_p \|a''_c\|_{\infty}^{p} s^{p/2}. $$

Remark that $\|a_c\|_{\infty} \leq c, \|a'_c\|_{\infty} \leq C/(M_n h_n) \leq C$ and $\|a''_c\|_{\infty} \leq C/(M_n h_n^2) \leq C$, as $1/M_n = a_0 h_n^\beta$ with $\beta \geq 3$. Using (68), $\mathbb{E}_{x_0}[|L(\hat{X}^e_t)|^p] \leq c p s^{p/2-1} \int_0^s \mathbb{E}_{x_0}[|L(\hat{X}^e_r)|^p] dr + c_p s^{p/2}$ and thus, by Gronwall inequality,

$$\mathbb{E}_{x_0}[|L(\hat{X}^e_t)|^p] \leq c_p s^{p/2}. $$

From (72) and the expression of $a_c$ we deduce, using Burkholder–Davis–Gundy inequality

$$\mathbb{E}_{x_0}\left[ |L(\hat{X}^e_{\Delta_n}) + B_{\Delta_n} |^p \right] \leq c \mathbb{E}_{x_0}\left[ \left| \int_0^{\Delta_n} a'_c(\hat{X}^e_r) L(\hat{X}^e_r) dB_r \right|^p \right] +$$

$$c \mathbb{E}_{x_0}\left[ \left| \int_0^{\Delta_n} a''_c(\hat{X}^e_r) \right| D(\hat{X}^e_r) \right| D(\hat{X}^e_r) > dB_r \right|^p \right] + c \mathbb{E}_{x_0}\left[ \left| \int_0^{\Delta_n} \epsilon M_n \psi(\hat{X}^e_r) dB_r \right|^p \right]$$

$$\leq c \|a'_c\|_{\infty} \|a''_c\|_{\infty} \Delta_n s^{p/2-1} \int_0^{\Delta_n} \mathbb{E}_{x_0}[|L(\hat{X}^e_r)|^p] dr +$$

$$c \|a''_c\|_{\infty} \Delta_n s^{p/2-1} \int_0^{\Delta_n} \mathbb{E}_{x_0}[\|D(\hat{X}^e_r)\|_{H}^{2p}] dr + c \frac{\|\psi\|_{H}^{p}}{M_n} \Delta_n s^{p/2}$$

$$\leq c \left( \frac{\Delta_n s^{p/2}}{M_n h_n^2} + \frac{\Delta_n s^{p/2}}{M_n h_n^4} + \frac{\Delta_n s^{p/2}}{M_n h_n^6} \right),$$

where in the last line we used $h_n \leq \Delta_n$. From (68), we deduce

$$\mathbb{E}_{x_0}\left[ \left| \frac{L(\hat{X}^e_{\Delta_n}) + B_{\Delta_n}}{< D(\hat{X}^e_{\Delta_n}), D(\hat{X}^e_{\Delta_n}) >} \right|^p \right] \leq c \left( \frac{1}{M_n h_n^2} + \frac{\Delta_n s^{p/2}}{M_n h_n^4} \right).$$

Using that from (65)–(66) we have,

$$\frac{\Delta_n}{< D(\hat{X}^e_{\Delta_n}), D(\hat{X}^e_{\Delta_n}) >} \geq 1 \leq c \frac{\Delta_n}{M_n h_n^2},$$

it is deduced

$$\mathbb{E}_{x_0}\left[ \left| \frac{\sqrt{\Delta_n} L(\hat{X}^e_{\Delta_n})}{< D(\hat{X}^e_{\Delta_n}), D(\hat{X}^e_{\Delta_n}) >} + \frac{B_{\Delta_n}}{\sqrt{\Delta_n}} \right|^p \right] \leq c \left( \frac{\Delta_n s^{p/2}}{M_n h_n^4} + \frac{\Delta_n s^{p/2}}{M_n h_n^6} \right).$$

We consider now the second term of (71). From (34) and Theorem 2.2.2. in Nualart [42], we derive the dynamics of the second Malliavin derivative of $X^e$,

$$D_{s_1, s_2} \hat{X}^e_t = \int_{s_1 \vee s_2}^t \left\{ a''_c(D_{s_1} \hat{X}^e_r) D_{s_2} \hat{X}^e_r + a'_c(D_{s_1} \hat{X}^e_r) D_{s_2} \hat{X}^e_r \right\} dB_r + a'_c(D_{s_2} \hat{X}^e_r) D_{s_1} \hat{X}^e_r + a'_c(D_{s_2} \hat{X}^e_r) D_{s_1} \hat{X}^e_r,$$
for $s_1 \vee s_2 < t$. As a consequence, we have $\sup_{s_1, s_2 \leq \Delta_n} \mathbb{E}_{x_0} \left[ |D_{s_1, s_2} \hat{X}_\Delta| \right] \leq c \|a'_t\|_{\infty}^p \leq c/(M_n h_n)^p$.

Next, using Cauchy-Schwarz inequality for the Malliavin bracket, we have

$$\Delta_n^{1/2} \left\| D \left( < D \hat{X}_\Delta, D \hat{X}_\Delta > \right), D \hat{X}_\Delta \right\|_H \leq \Delta_n^{1/2} \left\| D \left( < D \hat{X}_\Delta, D \hat{X}_\Delta > \right) \right\|_H \leq \frac{c}{\Delta_n^{1/2}} \left( \int_0^{\Delta_n} \int_0^{\Delta_n} |D_{s_1, s_2} \hat{X}_\Delta|^2 |D_{s_1, s_2} \hat{X}_\Delta|^2 \, ds_1 ds_2 \right)^{1/2},$$

where we used (68) and Jensen inequality in the second line. Using (67) and the upper bound on the second order Malliavin derivative of $X_\Delta$, we deduce

$$\mathbb{E}_{x_0} \left[ \frac{\Delta_n^{1/2} \left\| D \left( < D \hat{X}_\Delta, D \hat{X}_\Delta > \right), D \hat{X}_\Delta \right\|_H}{\left\| D \hat{X}_\Delta \right\|_H^4} \right] \leq c \frac{\Delta_n^{p/2}}{(M_n h_n)^p}. \quad (75)$$

Collecting (74)–(75) with (71), we deduce (70).

- We now prove (41). From (39) and (69) we have

$$R_n^{(1)} = \left( \frac{\hat{X}_\Delta}{\Delta_n^{1/2}} - \frac{1}{\Delta_n^{1/2} M_n} \int_0^{\Delta_n} \psi_{h_n}(\hat{X}_s) dB_s \right) \frac{B_{\Delta_n}}{\Delta_n^{1/2}} + \frac{\hat{X}_\Delta}{\Delta_n^{1/2}} \left( \Phi_{\Delta} - \frac{B_{\Delta_n}}{\Delta_n^{1/2}} \right) =: R_n^{(1,1)} + R_n^{(1,2)}.$$

By Cauchy–Schwarz and (35) we have

$$\mathbb{E}_{x_0} \left[ |R_n^{(1,1)}|^2 \right] \leq \Delta_n^{-1} \mathbb{E}_{x_0} \left[ \left( \int_0^{\Delta_n} \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_s) \hat{X}_s^\epsilon dB_s \right)^2 \right] \leq \frac{1}{\Delta_n^{1/2} M_n^2} \left( \int_0^{\Delta_n} \mathbb{E}_{x_0} \left[ |\psi_{h_n}(\hat{X}_s)|^4 |\hat{X}_s^\epsilon|^4 \right] \, ds \right)^{1/2},$$

Using that by Lemma 5 we have $\mathbb{E}_{x_0} \left[ |\hat{X}_s^\epsilon|^4 \right] \leq c \Delta_n^2 / M_n^4$ and $\|\psi_{h_n}\|_{\infty} \leq c/(h_n M_n)$, we deduce

$$\mathbb{E}_{x_0} \left[ |R_n^{(1,1)}|^2 \right] \leq c \Delta_n / (M_n h_n^4).$$

Using Cauchy-Schwarz and (70), we have $\mathbb{E}_{x_0} \left[ |R_n^{(1,2)}|^2 \right] \leq c(1/(\Delta_n M_n h_n^4)) \mathbb{E}_{x_0} \left[ |\hat{X}_s|^4 \right] / 2 \leq c/(M_n h_n^4).$

Collecting the upper bounds on $R_n^{(1,1)}$ and $R_n^{(1,2)}$, we deduce $\mathbb{E}_{x_0} \left[ |R_n^{(1)}|^2 \right] \leq c/(M_n h_n^4)$ and recalling (41), the proposition is proved as $1/(M_n h_n^4) = O(h_n^{2\beta - 3}) = o(1)$, using $\beta \geq 3$. □

A.6 Proof of Lemma 5

**Proof.** Using (35), it is, for any $p \geq 2$, and $u \in [0, \Delta_n],$

$$\mathbb{E}_{x_0} \left[ |\hat{X}_u^\epsilon|^p \right] \leq c \mathbb{E}_{x_0} \left[ \left( \int_0^u \frac{\epsilon}{M_n} \psi_{h_n}(\hat{X}_s) \hat{X}_s^\epsilon dB_s \right)^2 \right] + c \mathbb{E}_{x_0} \left[ \left( \int_0^u \frac{1}{M_n} \psi_{h_n}(\hat{X}_s)^2 dB_s \right)^2 \right] \leq c \Delta_n^{\frac{p-1}{2}} \left( \int_0^u \mathbb{E}_{x_0} \left[ |\psi_{h_n}(\hat{X}_s)|^4 \right] \, ds \right) + c \Delta_n^{\frac{p-1}{2}} \left( \int_0^u \mathbb{E}_{x_0} \left[ |\psi_{h_n}(\hat{X}_s)|^p \right] \, ds \right),$$

where we have used Burkholder-Davis-Gundy and Jensen inequalities with (62). Let $M_u = \sup_{s \leq u} \mathbb{E}_{x_0} \left[ |\hat{X}_s|^p \right]$. Then, from above it follows

$$M_u \leq c \Delta_n^{\frac{p-1}{2}} \left( \int_0^u \mathbb{E}_{x_0} \left[ |\psi_{h_n}(\hat{X}_s)|^p \right] \, ds \right) + c \Delta_n^{\frac{p-1}{2}} \left( \int_0^u \mathbb{E}_{x_0} \left[ |\psi_{h_n}(\hat{X}_s)|^p \right] \, ds \right).$$
Using Gronwall lemma, it yields

$$M_{\Delta_n} \leq c \exp(c \frac{\Delta_n^2}{(M_n h_n)^2}) \frac{\Delta_n^{j-1}}{M_n} \int_0^{\Delta_n} \mathbb{E}_{x_0} ||\psi_{h_n}(\hat{X}_s^\epsilon)||^p ds \leq c \frac{\Delta_n^{j-1}}{M_n} \int_0^{\Delta_n} \mathbb{E}_{x_0} ||\psi_{h_n}(\hat{X}_s^\epsilon)||^p ds,$$  \hspace{0.5cm} (76)

recalling that the constant $c$ may change value from line to line and that the quantity $\frac{\Delta_n^2}{(M_n h_n)^2}$ is bounded as $\Delta_n \to 0$ and $M_n h_n \to \infty$ for $n \to \infty$. As $\psi$ is a bounded function, this yields (45). Integrating (76) with respect to $\pi^c(x_0)dx_0$ and using the first point of Lemma 4 we deduce (46).

\[\square\]

### A.7 Proof of Lemma 6

**Proof.** To control $I_1$ and $I_2$, defined by (44) we use Cauchy–Schwarz inequality. It provides, for $j \in \{1,2\}$,

$$\left| \frac{< D \hat{X}_{\Delta_n}, (D \hat{X}_{\Delta_n})_j >}{< D \hat{X}_{\Delta_n}, D \hat{X}_{\Delta_n} >} \right| \leq \frac{|D \hat{X}_{\Delta_n}|_2 \|(D \hat{X}_{\Delta_n})_j\|_2}{< D \hat{X}_{\Delta_n}, D \hat{X}_{\Delta_n} >} = \frac{|(D \hat{X}_{\Delta_n})_j|_2}{\|D \hat{X}_{\Delta_n}\|_2}$$

Hence, using Jensen inequality, for $j = 1,2$ we have

$$I_j \leq \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \mathbb{E}_{x_0} \left[ \frac{< (D \hat{X}_{\Delta_n})_j, (D \hat{X}_{\Delta_n})_j >}{< D \hat{X}_{\Delta_n}, D \hat{X}_{\Delta_n} >} \right] \pi^c(x_0)dx_0$$

Using (68), we need to upper bound $\frac{1}{M_n} \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \mathbb{E}_{x_0} \left[ \frac{< (D \hat{X}_{\Delta_n})_j, (D \hat{X}_{\Delta_n})_j >}{< D \hat{X}_{\Delta_n}, D \hat{X}_{\Delta_n} >} \right] \pi^c(x_0)dx_0$.

We start considering what happens for $j = 1$. Using Cauchy-Schwarz inequality, the first two points of Lemma 4 and Lemma 5 we have

$$I_1 \leq \frac{c}{\Delta_n} \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \mathbb{E}_{x_0} \left[ \int_0^{\Delta_n} ((D_s \hat{X}_{\Delta_n})_1)^2 ds \right] \pi^c(x_0)dx_0$$

$$\leq \frac{c}{\Delta_n} \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \mathbb{E}_{x_0} \left[ \int_0^{\Delta_n} \frac{c}{M_n} \psi_{h_n}(\hat{X}_s^\epsilon)\hat{X}_s^\epsilon + \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^\epsilon) \right] ds \pi^c(x_0)dx_0$$

$$\leq \frac{c}{\Delta_n M_n^2} \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \left\{ \int_0^{\Delta_n} \mathbb{E}_{x_0} [\psi_{h_n}(\hat{X}_s^\epsilon)]^4 ds + \int_0^{\Delta_n} \mathbb{E}_{x_0} \left[ (\psi_{h_n}(\hat{X}_s^\epsilon))^2 \right] ds \right\} \pi^c(x_0)dx_0$$

$$\leq \frac{c}{\Delta_n M_n^2} \int_{|x_0| \leq \Delta_{n/2}^{-\gamma}} \left\{ \int_0^{\Delta_n} \mathbb{E}_{x_0} [\psi_{h_n}(\hat{X}_s^\epsilon)]^4 \frac{\Delta_n}{M_n^2} ds + \int_0^{\Delta_n} \mathbb{E}_{x_0} \left[ (\psi_{h_n}(\hat{X}_s^\epsilon))^2 \right] ds \right\} \pi^c(x_0)dx_0$$

$$\leq \frac{c}{\Delta_n M_n^2} \{ \Delta_n (h_n^{-1})^{\frac{1}{2}} \frac{\Delta_n}{M_n^2} + \Delta_n h_n \} \quad \text{[by the first two points of Lemma 4 with Jensen’s inequality]}$$

$$\leq \frac{c \Delta_n}{M_n^4 h_n^{3/2}} = \frac{c h_n}{M_n^2}$$

where the last line is a consequence of $1/M_n = O(h_n^\beta)$ with $\beta \geq 3$.

We now deal with $I_2$. In order to bound it we will use several times (64), (67) and Lemma 5. It
follows

\[
\mathbb{E}_{x_0}[(D_s \hat{X}_{\Delta_n})^2] \leq \frac{c}{M_n^2} \int_s^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})\hat{X}_{s})^2]du + \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n \mathbb{E}_{x_0}[(\psi'_{h_n}(\hat{X}_{s})\psi''_{h_n}(\hat{X}_{s})\hat{X}_{s})^2]du \\
+ \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n \mathbb{E}_{x_0}[(\psi'_{h_n}(\hat{X}_{s})^2]du \\
\leq \frac{c}{M_n^2} \int_s^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]^4]^{1/2}\Delta_n du + \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n \|\psi_{h_n}\|_\infty^2 \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]^4]^{1/2}\Delta_n du \\
+ \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n \mathbb{E}_{x_0}[(\psi'_{h_n}(\hat{X}_{s})^2]du
\]

Integrating with respect to \(x_0\) the last equation and applying the first two points of Lemma 4, we find

\[
I_2 \leq \frac{c}{M_n^2} \int_s^\Delta_n (h_n^{-s})^{1/2}\Delta_n du + \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n (h_n^{-s})^{1/2}du + \frac{c\Delta_n}{M_n^4} \int_s^\Delta_n \frac{1}{h_n} du \\
\leq \frac{c\Delta_n^2}{M_n^4 h_n^{1/2}} + \frac{c\Delta_n^3}{M_n^4 h_n^{1/2}} + \frac{c\Delta_n^2}{M_n^4 h_n^{1/2}}.
\]

We remark that, as the choice of the calibration parameter \(M_n\) is such that \(1/M_n = O(h_n^3) = O(h_n^3)\), all the three terms here above are smaller than \(\frac{c\Delta_n}{M_n^2}\). It follows \(I_2 \leq \frac{c\Delta_n}{M_n^2}\), as we wanted. \(\square\)

### A.8 Proof of Lemma 7

**Proof.** We observe that, according to the definition of \((D \hat{X}_{\Delta_n})_3\), the rest term is

\[
R^{(2)}_n = \frac{1}{\Delta_n} \left[ \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})\hat{X}_{s})^2] - 1 \right] \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})\hat{X}_{s})] du \\
\times (1 + \frac{c}{M_n} \psi_{h_n}(\hat{X}_{s})) du \\
+ \frac{c}{\Delta_n} \int_0^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})^2 \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]) du \\
+ \frac{c}{\Delta_n} \int_0^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})^2 \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]) du \\
+ \frac{c}{\Delta_n} \int_0^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})^2 \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]) du \\
+ \frac{c}{\Delta_n} \int_0^\Delta_n \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})^2 \mathbb{E}_{x_0}[(\psi''_{h_n}(\hat{X}_{s})]) du \\
= \sum_{j=1}^4 \left[R^{(2)}_n(j)\right].
\]

Using the expression of \(D_u \hat{X}_{\Delta_n}\) as in (38) with (65)-(66), we have

\[
\sup_{u,s \in [0,\Delta_n]} |D_u \hat{X}_{\Delta_n} - 1| \leq \frac{c\Delta_n}{M_n h_n^2} + \frac{c}{M_n} \psi_{h_n}(\hat{X}_{\Delta_n}).
\]
It provides the following bounds, using also (64), (73), and the fact that $\psi_{h_n}$ is bounded

$$\begin{align*}
\mathbb{E}_{x_0}[[R_{n}^{(2,1)}]^2] & \leq \left( \frac{\Delta_n}{M_n h_n^2} \right)^2 \frac{1}{\Delta_n} \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 d\beta \right)^2] du \\
& \leq \left( \frac{c}{M_n h_n^2} \right)^2 \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 \psi_{h_n}(\hat{X}_s^*) \right) d\beta] du \\
\mathbb{E}_{x_0}[[R_{n}^{(2,2)}]^2] & \leq \left( \frac{c}{\Delta_n M_n^2} \right)^2 \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 \psi_{h_n}(\hat{X}_s^*) \right) d\beta] du \\
\mathbb{E}_{x_0}[[R_{n}^{(2,3)}]^2] & \leq \left( \frac{c}{h_n^2 M_n} \right)^2 \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 \psi_{h_n}(\hat{X}_s^*) \right) d\beta] du \\
\mathbb{E}_{x_0}[[R_{n}^{(2,4)}]^2] & \leq \left( \frac{c}{\Delta_n M_n^2} \right)^2 \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 \psi_{h_n}(\hat{X}_s^*) \right) d\beta] du \\
& \leq \left( \frac{c}{\Delta_n M_n^2} \right)^2 \Delta_n \int_0^{\Delta_n} \mathbb{E}_{x_0}[\left( \int_u^{\Delta_n} \left( Y_u^{-1} - \frac{1}{M_n} \psi_{h_n}(\hat{X}_s^*) (D_u \hat{X}_s^*) \right)^2 \psi_{h_n}(\hat{X}_s^*) \right) d\beta] du.
\end{align*}$$

We now use the second point of Lemma 4, and deduce

$$\int_{|x_0| \leq \Delta_n^{1/2}} \sum_{j=1}^{4} \mathbb{E}_{x_0}[[R_{n}^{(2,j)}]^2] \pi^j(x_0) dx_0 \leq c \left\{ \frac{\Delta_n^3}{M_n h_n^2} + \frac{\Delta_n^3}{M_n^2 h_n} + \frac{\Delta_n^3}{M_n^2 h_n^2} + \frac{\Delta_n^3}{M_n^2 h_n^2} \right\}.$$

It is easy to see that all these terms are smaller than $\frac{4h_n}{M_n^2}$, up to using $\frac{1}{M_n} = O(h_n^2)$ and remarking we have requested $\beta \geq 3$. It follows that

$$\int_{|x_0| \leq \Delta_n^{1/2}} \mathbb{E}_{x_0}[[R_{n}^{(2,2)}]^2] \pi^3(x_0) dx_0 \leq \frac{c h_n}{M_n^2}.$$

The proof is therefore concluded. \(\square\)

### A.9 Proof of Lemma 8

We want to use Ito’s formula, in order to get rid of the stochastic integral in the left hand side of (47). Hence, we write

$$\int_u^{\Delta_n} \psi_{h_n}(\hat{X}_s^*) dB_s = I_{\Delta_n} - I_u,$$

where $I_t$ is defined in (60). From (61), we deduce

$$\int_u^{\Delta_n} \psi_{h_n}(\hat{X}_s^*) dB_s = \Xi(\hat{X}_{\Delta_n}) - \Xi(\hat{X}_u) - \frac{1}{2} \int_u^{\Delta_n} \left( \frac{\psi_{h_n}'}{a_e}(\hat{X}_s^*) \right) a_e^2(\hat{X}_s^*) ds$$

Now, recall

$$\Xi(v) := \int_0^v \frac{\psi_{h_n}'}{a_e}(u) du = \int_0^v \frac{\psi_{h_n}'}{\left( \frac{1}{M_n} \psi_{h_n}(u) \right) du$$

$$= \psi_{h_n}(u) - \frac{1}{M_n} \psi_{h_n}(u) \psi_{h_n}(u) du - \psi_{h_n}(0)$$

$$=: \psi_{h_n}(u) + \psi_{h_n}(u) - \psi_{h_n}(0),$$

$$\Xi(v) = \psi_{h_n}(u) - \psi_{h_n}(0).$$
where we used $a_\varepsilon(u) = 1 + \frac{\varepsilon}{M_n}\psi_\varepsilon(u)$ in the first line. We have $\|n_{\varepsilon u}\|_\infty \leq \frac{e}{M_n}$. Then, by (78), we obtain

$$I_{\Delta_n} - I_u = \psi_{\varepsilon n}((\hat{X}_n^x) - \psi_{\varepsilon n}((\hat{X}_0^x)) + n_{\varepsilon n}(\hat{X}_{\Delta_n}^x) - n_{\varepsilon n}(\hat{X}_u^x)) - \frac{1}{2} \int_{\Delta_n} \frac{\psi_{\varepsilon n}'}{a_\varepsilon}(\hat{X}_n^x)a_\varepsilon(\hat{X}_n^x)dv.$$

We observe that $(\frac{\psi_{\varepsilon n}'}{a_\varepsilon})' = \psi_{\varepsilon n}'' + m_{\varepsilon n}$, where $\|m_{\varepsilon n}\|_\infty \leq \frac{e}{M_n}$. It yields

$$I_{\Delta_n} - I_u = \psi_{\varepsilon n}((\hat{X}_n^x) - \psi_{\varepsilon n}((\hat{X}_0^x)) + n_{\varepsilon n}(\hat{X}_{\Delta_n}^x) - n_{\varepsilon n}(\hat{X}_u^x)) - \frac{1}{2} \int_{\Delta_n} \psi_{\varepsilon n}''(\hat{X}_n^x)dv + o_{\|\cdot\|_\infty}(\Delta_n \frac{1}{h_n^2 M_n}),$$

where we have introduced the notation $o_{\|\cdot\|_\infty}(\cdot)$ for $o(\cdot)$ such that the control is uniform over $\mathbb{R}$. Using (77), we deduce

$$\frac{1}{\Delta_n M_n} \int_0^{\Delta_n} \int_{\Delta_n} \psi_{\varepsilon n}''(\hat{X}_n^x)dB_s du = \frac{1}{\Delta_n M_n} \int_0^{\Delta_n} (I_{\Delta_n} - I_u) du$$

and

$$= \frac{1}{\Delta_n M_n} \int_0^{\Delta_n} \left( \int_{\Delta_n} \psi_{\varepsilon n}''(\hat{X}_n^x)dv \right) du + \frac{1}{\Delta_n M_n} \int_0^{\Delta_n} (\psi_{\varepsilon n}(\hat{X}_{\Delta_n}) - \psi_{\varepsilon n}(\hat{X}_u)) du$$

$$+ \frac{1}{\Delta_n M_n} \int_0^{\Delta_n} (n_{\varepsilon n}(\hat{X}_{\Delta_n}) - n_{\varepsilon n}(\hat{X}_u)) du + \frac{c}{\Delta_n M_n} \int_0^{\Delta_n} o_{\|\cdot\|_\infty}(\Delta_n \frac{1}{h_n^2 M_n}) du.$$

The previous equation can be written

$$\frac{1}{\Delta_n M_n} \int_0^{\Delta_n} \int_{\Delta_n} \psi_{\varepsilon n}''(\hat{X}_n^x)dB_s du = - \frac{1}{\Delta_n M_n} \int_0^{\Delta_n} \psi_{\varepsilon n}''(\hat{X}_n^x)sdz + \sum_{j=1}^3 R_{n(3,j)}.$$

Comparing with (47), we need to prove $\int_{|x_0|\leq\Delta_n^{1/2-\gamma}} E_{x_0}([R_{n(3,j)}^2] |x_0) \pi^x dx_0 \leq \frac{c}{M_n^2}$ for $j = 1, 2, 3$.

In particular, using also the first point of Lemma 4, we have for $j = 1$,

$$\int_{|x_0|\leq\Delta_n^{1/2-\gamma}} E_{x_0}([R_{n(3,1)}^2] |x_0) \pi^x dx_0 \leq \frac{c}{M_n^2} \Delta_n \int_0^{\Delta_n} \left( E[\psi_{\varepsilon n}^2(\hat{X}_{\Delta_n}^x)] + E[\psi_{\varepsilon n}^2(\hat{X}_u^x)] \right) \pi^x dx_0 du$$

$$\leq \frac{c}{M_n^2}.$$

We have, using $\|\eta_{\varepsilon n}\| \leq C/M_n$,

$$\int_{|x_0|\leq\Delta_n^{1/2-\gamma}} E_{x_0}([R_{n(3,2)}^2] |x_0) \pi^x dx_0 \leq \frac{c}{M_n^2} (\frac{1}{M_n})^2 = c \frac{1}{M_n^4}.$$  

This is negligible with respect to $\frac{R_{n^2}}{M_n^2}$ as $1/M_n = O(h_n^3)$. Regarding $R_{n(3,3)}$ we have, $|R_{n(3,3)}| \leq c \Delta_n / (h_n^2 M_n^2)$ and thus,

$$\int_{|x_0|\leq\Delta_n^{1/2-\gamma}} E_{x_0}([R_{n(3,3)}^2] |x_0) \pi^x dx_0 \leq \frac{c}{M_n^2} \frac{\Delta_n^2}{h_n^2}.$$  

which is clearly negligible compared to $\frac{R_{n^2}}{M_n^2}$, acting as for $R_{n(3,2)}$.

**A.10 Proof of Lemma 9**

*Proof.* Let us denote by $p_{\varepsilon n}(\cdot \mid x, y)$ the density of the law of $\hat{X}_n^x$ conditional to $\hat{X}_0^x = x$ and $\hat{X}_{\Delta_n}^x = y$. Then, we have

$$\ell_{x, y, \Delta_n, n} := E_x \left[ \int_0^{\Delta_n} (\psi_{\varepsilon n})''(\hat{X}_n^x)sdz \mid \hat{X}_{\Delta_n}^x = y \right] = \int_0^{\Delta_n} \int_R (\psi_{\varepsilon n})'(z)p_{\varepsilon n}(z \mid x, y)dz dz (79)$$

$$= \int_0^{\Delta_n} \int_R \psi_{\varepsilon n}(z) \frac{\partial^2 p_{\varepsilon n}(z \mid x, y)}{\partial z^2} dz dz (80).$$  

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Recalling the expression for the density of the pinned diffusion
\[ p_{s,\Delta_n}^\delta(z \mid x, y) = \frac{p_s^\delta(x, z)p_{\Delta_n-s}^\delta(z, y)}{p_{\Delta_n}^\delta(x, y)}, \]
we have
\[ \frac{\partial^2 p_{s,\Delta_n}^\delta(z \mid x, y)}{\partial z^2} = \frac{\partial^2 p_s^\delta(x, z)p_{\Delta_n-s}^\delta(z, y) + 2\frac{\partial p_s^\delta(x, z)}{\partial x} \frac{\partial p_{\Delta_n-s}^\delta(z, y)}{\partial z} + p_s^\delta(x, z)\frac{\partial^2 p_{\Delta_n-s}^\delta(z, y)}{\partial z^2}}{p_{\Delta_n}^\delta(x, y)}. \]

Now, we use the Gaussian controls given in Sections A.2.2 - A.2.3 of Azencott [7] for the density of the transition of a diffusion and of its derivatives. Notice that the results of [7] apply here to the diffusion \( \hat{X} \) as the conditions given in Section A.2.1 of [7] are satisfied. Especially, the condition (3) p. 477 of [7] requires that at least the first three derivatives of the coefficients of the SDE are bounded, which is valid here from the condition \( 1/(M_n\Delta_n^2) \leq 1 \). An important feature of the controls given in [7] is that they sharply relate the variance of the Gaussian controls with the diffusion coefficient of the diffusion process. From (6) in Section A.2.2 of [7] (see also (75) p. 492 of [7]), we have
\[ \left| \frac{\partial^j p_s^\delta(u, v)}{\partial \nu^j} \right| \leq \frac{C}{\sqrt{s}} g_{s\lambda}(u - v), \quad j = 0, 1, 2, \]

where \( g_{s\lambda} \) is the density of the centered real Gaussian variable with variance \( s\lambda \) and \( \lambda > 0 \) is any real constant such that \( \lambda > (1 + e^{1/\Delta_n^2})^2 \) for all \( z \in \mathbb{R} \). As the diffusion process is symmetric with respect to the stationary probability, we know \( p_s^\delta(u, v) = p_s^\delta(v, u) \frac{\pi(x, u)}{\pi(x, v)} \) where \( \pi^\delta \) is the function \( u \mapsto e^{-2\eta \int_0^t \pi(x, u) \frac{\pi(x, u)}{\pi(x, v)} dw} \), and the constant \( c_{\pi^\delta} \) is bounded independently of \( n, \epsilon \) recalling (24) and the discussion below. Using the assumption \( \limsup_n h_n < \infty \), we can check that the first two derivatives of \( \pi^\delta \) are bounded and it follows from (81) that
\[ \left| \frac{\partial^j p_s^\delta(u, v)}{\partial \nu^j} \right| \leq \frac{C}{\sqrt{s}} g_{s\lambda}(u - v), \quad j = 0, 1, 2. \]

As \( \|\psi_{\Delta_n}/M_n\|_\infty \rightarrow 0 \), it is possible for \( n \) large enough to use (81)–(82) with \( \lambda = 1 + \eta \in (1/5, 1/4] \), we deduce
\[ \left| \frac{\partial^2 p_{s,\Delta_n}^\delta(z \mid x, y)}{\partial z^2} \right| \leq C \left[ \frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s}} \frac{1}{\Delta_n - s} + \frac{1}{\Delta_n - s} \right] g_{s(1+\eta)}(z-x)g_{s(1+\eta)}(y-z) p_{\Delta_n}^\delta(x, y). \]

Using Section A.2.3. in Azencott [7], with \( (1 + e^{1/\Delta_n^2})^2 > (1 - \eta) \) for \( n \) large enough, we have \( p_{\Delta_n}^\delta(x, y) \geq C g_{s(1-\eta)}(y-x) \). In turn, we deduce
\[ \left| \frac{\partial^2 p_{s,\Delta_n}^\delta(z \mid x, y)}{\partial z^2} \right| \leq C \left[ \frac{1}{s} + \frac{1}{\Delta_n - s} \right] g_{s(1+\eta)}(z-x)g_{s(1+\eta)}(y-z) g_{s(1-\eta)}(y-x) \]
\[ \leq C \left[ \frac{1}{s} + \frac{1}{\Delta_n - s} \right] g_{s(1+\eta)}(z-x)g_{s(1+\eta)}(y-z) g_{s(1-\eta)}(y-x) e^{2\eta(u-z)^2/\Delta_n}. \]

Remarking that the ratio of Gaussian densities in the previous display is the law at time \( s \in (0, \Delta_n) \) of a Brownian bridge from \( x \) to \( y \) with diffusion coefficient \( \sqrt{1+\eta} \), we deduce from (80),
\[ |\ell_{x,y,\Delta_n,h_n}| \leq C \int_0^{\Delta_n} s \left[ \frac{1}{s} + \frac{1}{\Delta_n - s} \right] \mathbb{E}_x \left[ \left| \psi_{\Delta_n}(\tilde{B}_s) \right| \mid \tilde{B}_{\Delta_n} = y \right] ds, \]

where \( \tilde{B} \) is a Brownian motion with diffusion coefficient \( \sqrt{1+\eta} \). Denoting the local time at level \( z \) of the Brownian motion \( \tilde{B} \) by \( (L^z_{\Delta_n}(\tilde{B}))_t \), we deduce using Fubini’s Theorem and the occupation time formula (see Corollary (1.6) and Exercise 1.15 of Chapter VI in [45]),
\[ |\ell_{x,y,\Delta_n,h_n}| \leq C \mathbb{E}_x \left[ \int_0^{\Delta_n} \left[ \frac{1}{s} + \frac{1}{\Delta_n - s} \right] \left| \psi_{\Delta_n}(\tilde{B}_s) \right| ds \mid \tilde{B}_{\Delta_n} = y \right] e^{2\eta(u-z)^2/\Delta_n} \]
\[ = C \frac{1}{1-\eta} \int_\mathbb{R} |\psi_{\Delta_n}(z)| \mathbb{E}_x \left[ \int_0^{\Delta_n} \left[ \frac{1}{s} + \frac{1}{\Delta_n - s} \right] dL^z_{\Delta_n}(\tilde{B}) \mid \tilde{B}_{\Delta_n} = y \right] dz e^{2\eta(u-z)^2/\Delta_n}. \]
Using that, after time reversal, the law of the Brownian bridge remains a Brownian bridge, we have
\[
\mathbb{E}_x \left[ \int_0^{\Delta_n} \left[ 1 + \frac{s}{\Delta_n - s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = y \right] = \mathbb{E}_y \left[ \int_0^{\Delta_n} \left[ 1 + \frac{\Delta_n - s}{s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = x \right].
\]
Using Lemma 11 with \(\beta = 0\) and \(\beta = 1\) on the right hand side of the previous equation, we deduce, for \(y \neq z\)
\[
\mathbb{E}_x \left[ \int_0^{\Delta_n} \left[ 1 + \frac{s}{\Delta_n - s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = y \right] \leq C \sqrt{\Delta_n} \left[ 1 + \left[ \frac{y - z}{\sqrt{\Delta_n}} \right] \right] + 1 e^{-\frac{(y-z)^2}{\Delta_n}} e^{\frac{\eta(y-z)^2}{\Delta_n}}.
\]
We now focus on the case \(|y| > 2h_n\). Using (84), (85) that the support of \(\psi_{h_n}\) is included in \([-h_n, h_n]\), we have,
\[
\|e_{x,y,\Delta_n,h_n}\| \leq C \sqrt{\Delta_n} \|\psi\|_\infty \int_{-h_n}^{h_n} \left[ 1 + \left| \frac{y - z}{\sqrt{\Delta_n}} \right| \right] e^{-\frac{(y-z)^2}{\Delta_n}} dze^{\frac{\eta(y-z)^2}{\Delta_n}}.
\]
Using that \(u \mapsto u^{-1}e^{-\frac{u^2}{\Delta_n}}\) is non increasing on \((0, \infty)\) and that \(|y| \geq 2h_n\), we deduce
\[
\|e_{x,y,\Delta_n,h_n}\| \leq 2C \sqrt{\Delta_n} h_n \left[ 1 + \left| \frac{y - h_n}{\sqrt{\Delta_n}} \right| \right] e^{-\frac{(y-h_n)^2}{\Delta_n}} e^{\frac{\eta(y-h_n)^2}{\Delta_n}}.
\]
Recalling the definition of \(e_{x,y,\Delta_n,h_n}\) given in (79), this entails (49) for \(|y| > 2h_n\).
We now treat the case \(|y| \leq 2h_n\). Recalling the definition of \(e_{x,y,\Delta_n,h_n}\) in (79) we write
\[
\int_0^{\Delta_n} (\psi_{h_n})''(\tilde{X}_s)ds = \int_0^{\Delta_n-\Delta_n^2} (\psi_{h_n})''(\tilde{X}_s)ds + O(\Delta_n),
\]
where we used \(\|\psi_{h_n}\|_\infty \leq C/h^2_n\). We deduce that
\[
e_{x,y,\Delta_n,h_n} = \mathbb{E}_x \left[ \int_0^{\Delta_n-\Delta_n^2} (\psi_{h_n})''(\tilde{X}_s)ds \mid \tilde{X}_{\Delta_n} = y \right] + O(\Delta_n).
\]
Then, we repeat the same computations that yields to (84), we deduce
\[
e_{x,y,\Delta_n,h_n} \leq \frac{C}{1 - \eta} \int_{\mathbb{R}} |\psi_{h_n}(z)| \mathbb{E}_x \left[ \int_0^{\Delta_n-\Delta_n^2} \left[ 1 + \frac{s}{\Delta_n - s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = y \right] dze^{\frac{\eta(y-z)^2}{\Delta_n}} + O(\Delta_n).
\]
By time reversal invariance of the law of the Brownian bridge, we have
\[
\mathbb{E}_x \left[ \int_0^{\Delta_n-\Delta_n^2} \left[ 1 + \frac{s}{\Delta_n - s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = y \right] = \mathbb{E}_y \left[ \int_0^{\Delta_n} \left[ 1 + \frac{\Delta_n - s}{s} \right] dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = x \right]
\]
\[
\leq \mathbb{E}_y \left[ \int_0^{\Delta_n} dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = x \right] + \Delta_n \mathbb{E}_y \left[ \int_0^{\Delta_n} \frac{s}{\Delta_n} dL^x_\Delta(\tilde{B}) \mid \tilde{B}\Delta_n = x \right]
\]
\[
\leq C \sqrt{\Delta_n} e^{\frac{\eta(y-z)^2}{\Delta_n}} + C \Delta_n e^{\frac{\eta(y-z)^2}{\Delta_n}}.
\]
where we used Lemma 11 with \(\beta = 0\), and (91) in Lemma 10. Since the support of \(\psi_{h_n}\) is \([-h_n, h_n]\), we deduce from (86) and (87)
\[
e_{x,y,\Delta_n,h_n} \leq C \sqrt{\Delta_n} h_n e^{\frac{\eta(y-z)^2}{\Delta_n}} + C \Delta_n e^{\frac{\eta(y-z)^2}{\Delta_n}}.
\]
This yields (49) for \(|y| \leq 2h_n\). \(\square\)
A.10.1 Conditional first moment for integrals of local time

In this section we obtain upper bound for conditional first moment of quantities of the form $\int_0^1 \varphi(s)d\mathbb{L}^0(B)$ where $B$ is a B.M. with diffusion coefficient $\sqrt{1+\eta} \in (1, \sqrt{2})$ and deduce controls useful in the proof of Lemma 9. In [31] the authors study the law of such quantities when $\varphi(s) = s^{-\beta}$ with $\beta \leq 1/2$. We follow some ideas from [31].

**Lemma 10.** Consider $(\hat{B}_s)_{s \in [0,1]}$ a Brownian motion with diffusion coefficient $\sqrt{1+\eta} \in (1, \sqrt{2})$.

1. Let $\psi : \mathbb{R} \rightarrow [0, \infty)$ be a bounded measurable function. Then,

$$\mathbb{E}_0 \left[ \int_0^1 \psi(s) d\mathbb{L}^0(\hat{B}) \right] \leq \frac{1}{\sqrt{\pi}} \int_0^1 \psi(u) \frac{1}{\sqrt{u}} du,$$

(88)

$$\mathbb{E}_0 \left[ \int_0^a \psi(s) d\mathbb{L}^0(\hat{B}) \mid \hat{B}_a = 0 \right] \leq \frac{\sqrt{a}}{\sqrt{\pi}} \int_0^a \psi(u) \left( \frac{1}{\sqrt{u}} + \frac{1}{\sqrt{a-u}} \right) du, \quad \forall a \in (0,1).$$

(89)

2. For $\beta \geq 0$, $\beta \neq 1/2$ and $(x,y) \in \mathbb{R}^2$, $y \neq 0$, we have

$$\mathbb{E}_y \left[ \int_0^1 \frac{d\mathbb{L}^0(\hat{B})}{s^{\beta}} \mid \hat{B}_1 = x \right] \leq C(1 + |y|^{(1-2\beta)\wedge 0} e^{-\frac{y^2}{\pi} + y^2}).$$

(90)

Moreover, the result holds true for $y = 0$ if $\beta \in [0,1/2)$.

3. We have, for all $(x,y,z) \in \mathbb{R}^3$, $0 < h_n^2 < \Delta_n < 1$,

$$\mathbb{E}_x \left[ \int_{h_n^2}^{\Delta_n} \frac{d\mathbb{L}^0(\hat{B})}{s} \mid \hat{B}_{\Delta_n} = y \right] \leq \frac{C}{h_n},$$

(91)

where the constant $C$ is independent of $x, y, z, h_n, \Delta_n$.

**Proof.** Point 1. We start with the proof of (88). We follow the ideas in Remark 1.25 iii) of [31]. Let us denote by $g_t = \sup \{s \in [0,t], \hat{B}_s = 0 \}$ which belongs to $[0,t]$ as $\hat{B}_0 = 0$. Applying the balayage formula (Theorem 4.2 in [45]), we have

$$\psi(g_t) | \hat{B}_1 | = \psi(0) | \hat{B}_0 | + \int_0^1 \psi(g_s) d|\hat{B}_s|$$

$$= \psi(0) | \hat{B}_0 | + \int_0^1 \psi(g_s) \text{sign}(\hat{B}_s) d\hat{B}_s + \int_0^1 \psi(g_s) d\mathbb{L}^0(\hat{B}).$$

Taking expectation, and remarking that on the support of the measure $d\mathbb{L}^0(\hat{B})$, we have $g_s = s$, we deduce

$$\mathbb{E}_0 \left[ \int_0^1 \psi(s) d\mathbb{L}^0(\hat{B}) \right] = \mathbb{E}_0 \left[ \psi(g_1) | \hat{B}_1 | \right].$$

From Section 3 of Chapter XII in [45] (see also [9]), we know that conditionally to $g_1$ the law of $|\hat{B}_1|$ is $\sqrt{1-g_1} \sqrt{1+\eta}M_1$ where $(M_t)_t$ is a Brownian meander. By [23], the variable $M_1$ follows the Rayleigh distribution, for which $\mathbb{E}[M_1] = \sqrt{\frac{\pi}{2}}$. It entails,

$$\mathbb{E}_0 \left[ \int_0^1 \psi(s) d\mathbb{L}^0(\hat{B}) \right] \leq \sqrt{\frac{(1+\eta)\pi}{2}} \mathbb{E}_0[\psi(g_1) \sqrt{1-g_1}].$$

As $g_1$ is distributed according to the arcsin law (e.g. see Paragraph 18 in Chapter 4 of [12]), with density $\frac{1}{\pi \sqrt{s(1-s)}}$ we deduce (88).

We now prove (89). We split the left hand side of (89) as

$$\mathbb{E}_0 \left[ \int_0^{a/2} \psi(s) d\mathbb{L}^0(\hat{B}) \mid \hat{B}_a = 0 \right] + \mathbb{E}_0 \left[ \int_{a/2}^a \psi(s) d\mathbb{L}^0(\hat{B}) \mid \hat{B}_a = 0 \right].$$
The law of the Brownian bridge is absolutely continuous with respect to the law of the Brownian motion on $F_u = \sigma\{\hat{B}_s, s \leq u\}$, for $u < a$, and the Radon–Nikodym density as given in Paragraph 23 of Chapter 4 of [12] is
\[
\frac{\mathcal{G}_{(1-a)(a-u)}(\hat{B}_u)}{\mathcal{G}_{(1-a)a}(0)}.
\]
This enables us to write,
\[
\mathbb{E}_0 \left[ \int_0^{a/2} \psi(s) dL_0^y(\hat{B}) \mid \hat{B}_a = 0 \right] = \mathbb{E}_0 \left[ \int_0^{a/2} \psi(s) dL_0^y(\hat{B}) \frac{\mathcal{G}_{(1-a)a/2}(\hat{B}_{a/2})}{\mathcal{G}_{(1-a)a}(0)} \right]
\]
\[
\leq \sqrt{2} \mathbb{E}_0 \left[ \int_0^{a/2} \psi(s) dL_0^y(\hat{B}) \right] \leq \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{a/2} \frac{\psi(u)}{\sqrt{u}} du,
\]
where the first inequality on the second line follows from the explicit expression for the Gaussian density, and the last inequality comes from (88). Using invariance by time reversal of the law of the Brownian bridge, we get,
\[
\mathbb{E}_0 \left[ \int_0^a \psi(s) dL_0^y(\hat{B}) \mid \hat{B}_a = 0 \right] = \mathbb{E}_0 \left[ \int_0^{a/2} \psi(a-s) dL_0^y(\hat{B}) \mid \hat{B}_a = 0 \right] \leq \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{a/2} \frac{\psi(a-u)}{\sqrt{u}} du
\]
\[
= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{a/2}^a \frac{\psi(u)}{\sqrt{a-u}} du.
\]
We deduce (89).

**Point 2.** We now prove (90). Let us denote $\tau_0 = \inf\{t \geq 0, \hat{B}_t = 0\}$ and remark that $g_1 = \sup\{t \leq 1, \hat{B}_t = 0\}$ is finite on the event $\tau_0 < 1$. Using that $\int_0^1 dL_0^y(\hat{B}) = 0$ on $\tau_0 \geq 1$, we have
\[
\mathbb{E}_y \left[ \int_0^1 \frac{dL_0^y(\hat{B})}{s^3} \mid \hat{B}_1 = x \right] = \mathbb{E}_y \left[ \int_0^1 \frac{dL_0^y(\hat{B})}{s^3} 1_{\{\tau_0 < 1\}} \mid \hat{B}_1 = x \right]
\]
\[
= \mathbb{E}_y \left[ \int_{\tau_0}^1 \frac{dL_0^y(\hat{B})}{s^3} 1_{\{\tau_0, g_1, \hat{B}_1 = x\}} 1_{\{\tau_0 < 1\}} \mid \hat{B}_1 = x \right].
\]
Conditionally to $\tau_0$, the law of the process $(\hat{B}_{\tau_0+s})_{s \in [0, g_1-\tau_0]}$ is on the event $\tau_0 < 1$, the law of a Brownian bridge from 0 to 0. Moreover, this Brownian bridge is independent of $g_1$ and of $(\hat{B}_{g_1+s})_{s \in [0, 1-g_1]}$ (see Section 3 of Chapter XII in [45]). We deduce that, on $\tau_0 < 1$,
\[
G(\tau, g) := \mathbb{E}_y \left[ \int_{\tau_0}^1 \frac{dL_0^y(\hat{B})}{s^3} \mid \tau_0 = \tau, g_1 = g, \hat{B}_1 = x \right]
\]
\[
= \mathbb{E}_0 \left[ \int_0^{g_1-\tau} \frac{dL_0^y(\hat{B})}{(\tau+s)^3} \mid \hat{B}_{g_1-\tau} = 0 \right].
\]
Using (89), we obtain
\[
G(\tau, g) \leq \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{1}{(\tau+s)^3} \left( \frac{1}{\sqrt{g}} + \frac{1}{\sqrt{g} - \tau - s} \right) ds
\]
\[
\leq \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{2}{(\tau+s)^3} \sqrt{s} ds + \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{2}{(\tau+s)^3} \sqrt{g - \tau - s} ds
\]
\[
\leq \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{2}{(\tau+s)^3} \sqrt{s} ds + \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{2}{(g-s')^3} \sqrt{s'} ds', \quad \text{setting } s' = g - \tau - s.
\]
For $s' \leq \frac{g_1-\tau}{2}$, we have $g - s' \geq \tau + s'$, and thus
\[
G(\tau, g) \leq \frac{4\sqrt{2}}{\sqrt{\pi}} \int_0^{g_1-\tau} \frac{1}{(\tau+s)^3} \sqrt{s} ds.
\]
From this equation we can see that for \( \beta \in [0, 1/2) \), \( G(\tau, g) \leq \left(4\sqrt{2}/\sqrt{\pi}\right) \int_0^1 s^{-1/2-\beta} ds \leq C < \infty \) for some constant \( C \). And for \( \beta > 1/2 \), \( G(\tau, g) \leq \left(4\sqrt{2}/\sqrt{\pi}\right) \int_0^\infty \frac{ds}{(s+\tau)^{1/2}} = \frac{4\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \frac{du}{(u+1)^{1/2}} \leq C^{-1/2-\beta} \). Collecting this with (92) and (93), we deduce,

\[
E_y \left[ \int_0^{\tau} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq C E_y \left[ (\tau_0)^{1/2-\beta} \right] \leq C y_0 < 1.
\]

If \( \beta \in [0, 1/2) \), we deduce \( E_y \left[ \int_0^{\tau} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq C \) and (90) follows in the case \( \beta \in [0, 1/2) \).

Assume now that \( \beta > 1/2 \) and \( y \neq 0 \). We use that the Radon-Nikodym ratio between the law of the Brownian bridge and the Brownian motion restricted to the sigma field \( F_{\tau_0} \), and the event \( \tau_0 < 1 \) is \( \frac{g(1-\tau_0)(1+\eta)(x-\tilde{B}_{\tau_0})}{\theta_1(\tau_0)(x-y)} \), and we have for any \( 0 < \varepsilon < 1/2 \),

\[
E_y \left[ \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq \varepsilon \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \leq \varepsilon \left( \tau_0 \right)^{1/2-\beta} \left( \tau_0 \right)^{1/2-\beta} \leq C \varepsilon \left( \tau_0 \right)^{1/2-\beta} \left( \tau_0 \right)^{1/2-\beta},
\]

where in the last line we used \( \frac{g(1-\tau_0)(1+\eta)(x-\tilde{B}_{\tau_0})}{\theta_1(\tau_0)(x-y)} = \frac{g(1-\tau_0)(1+\eta)(x-\tilde{B}_{\tau_0})}{\theta_1(\tau_0)(x-y)} \leq C \frac{g(1+\eta)(x)}{\theta_1(\tau_0)(x-y)} \) for \( \tau_0 < \varepsilon \leq 1/2 \). On the other hand, we compute using the explicit expression for the law of the first hitting time of a Brownian motion (e.g. see page 107 in [45]),

\[
E_y \left[ \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq \int_0^{\tau_0} \frac{y e^{-\frac{y^2}{2(1+\eta)}}}{t^{3/2}} dt = \int_0^{\infty} \frac{|y|^{1-2\beta} u^{3/2}}{2(1+\eta)} du \chi(x(1+\eta))^\beta,
\]

where we have set \( u = \frac{y^2}{2(1+\eta)} \). Using that \( \beta > 0 \), we deduce

\[
E_y \left[ \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq C|y|^{1-2\beta} e^{-\frac{y^2}{2(1+\eta)}}.
\]

(95)

On the other hand, \( \frac{g(1-\tau_0)(1+\eta)(x-\tilde{B}_{\tau_0})}{\theta_1(\tau_0)(x-y)} = e^{-\frac{2x_0^2}{2(1+\eta)}} \). Using \( 2|x_0| \leq 2\eta(1+\eta) x_0^2 + \frac{y^2}{2(1+\eta)} \) we have \( \frac{g(1-\tau_0)(1+\eta)(x-\tilde{B}_{\tau_0})}{\theta_1(\tau_0)(x-y)} \leq e^{2x_0^2+y^2 / 2(1+\eta)} \). Choosing \( \varepsilon \) such that \( \frac{1}{(1+\eta)x_0} > \frac{1}{(1+\eta)x_0} + \frac{1}{\eta} \), we deduce from (94) and (95) that

\[
E_y \left[ \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \right] \leq \varepsilon \int_0^{\tau_0} \frac{dL^0(\tilde{B})}{s^{\beta}} \leq C|y|^{1-2\beta} e^{x_0^2+y^2 / 2(1+\eta)}.
\]

Hence (90) is proved in the case \( \beta > 1/2 \).

Point 3. By translation it is sufficient to consider the case \( z = 0 \). The proof follows the scheme of the Point 2. Exactly as we obtain (92)–(93), we prove

\[
E_y \left[ \int_0^{\Delta_n} \frac{dL^0(B)}{s} | \tilde{B}_{\Delta_n} = x \right] = E_y \left[ \tilde{G}(\tau_0, g_{\Delta_n}) \right] \leq \int_0^{\Delta_n} \frac{dL^0(B)}{s} \left| \tilde{B}_{\Delta_n} = x \right],
\]

where \( g_{\Delta_n} = \sup_{s \in [0, \Delta_n]} \tilde{B}_t = 0 \) and \( \tilde{G}(\tau, g) = \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right) \).

Using (89) with \( \psi(s) = \xi(s), \psi(h^2 - \tau) \leq \frac{1}{\pi} \), we have

\[
\tilde{G}(\tau, g) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right)
\]

Using (89) with \( \psi(s) = \xi(s), \psi(h^2 - \tau) \leq \frac{1}{\pi} \), we have

\[
\tilde{G}(\tau, g) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right)
\]

Using (89) with \( \psi(s) = \xi(s), \psi(h^2 - \tau) \leq \frac{1}{\pi} \), we have

\[
\tilde{G}(\tau, g) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right) \leq \frac{1}{\pi} \int_{(h^2 - \tau)^{1/2}} (h^2 - \tau)^{1/2} \frac{dL^0(B)}{s} \left( \tilde{B}_{(h^2 - \tau)^{1/2}} = 0 \right)
\]
where in the first integral we have set \( u = \tau h \), and in the second integral we made the change of variable \( s' = s + \tau - h^2 \) in the case \( h^2 > \tau \), and \( s' = s \) if \( h^2 \leq \tau \) and then used \( 2\sqrt{s'}\sqrt{\tau} \sqrt{h^2} \leq s' + \tau \sqrt{h^2} \). After some computations, using \( \int_0^1 \frac{ds}{\sqrt{s' + \gamma}} = \int_0^1 \frac{du}{\sqrt{u + \gamma}} \) for any \( \gamma > 0 \), and discussing according to the relative positions of \( h^2 \) and \( \tau \), we can get that \( G(\tau, g) \leq C/h \).

Then, the point 4 of the lemma follows from (96).

**Lemma 11.** Assume that \( \beta \geq 0 \), \( \beta \neq 1/2 \) and \( \tilde{B} \) is a B.M. with diffusion coefficient \( \sqrt{1 + z} \in (1, \sqrt{2}) \). Then, there exists \( C_\eta \) such that for \( \forall (x, y, z) \in \mathbb{R}^3 \), \( y \neq z \)

\[
\mathbb{E}_y \left[ \int_0^{\Delta_n} \frac{dL_s^0(\tilde{B})}{s^{\beta}} \mid \tilde{B}_{\Delta_n} = x \right] \leq C_\eta \Delta_n^{1/2-\beta} \left( 1 + \frac{|y-z|}{\sqrt{\Delta_n}} \right)^{(1-2\beta)\Lambda_0} e^{-|y-z|^2/\eta \Delta_n} e^{\eta |y-z|^2/\Delta_n}.
\]

(97)

Moreover, for \( \beta \in [0, 1/2) \) the inequality holds true also for \( y = z \).

**Proof.** If \( (\tilde{B}_u)_{u \in [0, \Delta_n]} \) is a Brownian motion with initial value \( y \), we set \( \tilde{B}_u = \Delta_n^{-1/2} \tilde{B}_u \Delta_n \) which is a Brownian motion starting from \( y \Delta_n^{-1/2} \). We have by change of variable,

\[
\int_0^{\Delta_n} \frac{dL_s^0(\tilde{B})}{s^{\beta}} = \Delta_n^{1/2-\beta} \int_0^1 \frac{dL_s^0(\tilde{B})}{s^{\beta}}.
\]

We deduce

\[
\mathbb{E}_y \left[ \int_0^{\Delta_n} \frac{dL_s^0(\tilde{B})}{s^{\beta}} \mid \tilde{B}_{\Delta_n} = x \right] = \Delta_n^{1/2-\beta} \mathbb{E}_{\tilde{B}_1} \left[ \int_0^1 \frac{dL_s^0(\tilde{B})}{s^{\beta}} \mid \tilde{B}_1 = \frac{x}{\Delta_n^{1/2}} \right],
\]

and the lemma follows in the case \( z = 0 \) by using (90) in Lemma 10. The case \( z \neq 0 \) is obtained by translation.

**B Malliavin calculus, some basic tools**

We have seen that the main idea to derive the lower bound in Theorem 3 is to use Malliavin calculus technique to represent a score function in a treatable way. For this reason we introduce some basic facts on Malliavin calculus, needed for our computations. We refer to Nualart [42] for more details.

**B.1 Notations and basic properties**

Consider a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}) \), associated with a 1- dimensional Brownian motion \( \{B_t, t \in [0, \Delta]\} \) on a finite interval \( [0, \Delta] \), where \( 0 < \Delta \leq 1 \). The underlying Hilbert space we consider is \( H := L^2([0, \Delta], \mathbb{R}) \). For any \( h \in H \) we denote as \( B(h) \) the It\' integral \( \int_0^\Delta h(t)dB_t \). Moreover, we denote as \( C^\infty_c(\mathbb{R}^m, \mathbb{R}) \) the set of all infinitely continuously differentiable functions \( f : \mathbb{R}^m \to \mathbb{R} \) such that \( f \) and all its partial derivatives have polynomial growth. Let \( \mathcal{S} \) denote the class of smooth random variables such that a random variable \( F \in \mathcal{S} \) has the form

\[
F = f(B(h_1), \ldots, B(h_m)),
\]

where \( f \) belongs to \( C^\infty_c(\mathbb{R}^m, \mathbb{R}) \) and \( h_1, \ldots, h_m \in H = L^2([0, \Delta], \mathbb{R}) \) and \( m \geq 1 \). The derivative of a smooth random variable \( F \in \mathcal{S} \) of the form (98) is the \( H \)-valued random variable given by

\[
DF = (D_t F)_{t \in [0, \Delta]},
\]

(99)

Roughly speaking, the scalar product \( <DF, h>_H \) is the derivative at \( \epsilon \) of the random variable \( F \) composed with the shifted process \( \{W(g) + \epsilon < g, h > g \in H\} \).

The following integration-by-parts formula holds true for any \( F \) smooth random variable and \( h \in H \):

\[
\mathbb{E}[<DF, h>] = \mathbb{E}[FB(h)],
\]

40
Moreover by Chapter 1.2 in [42], the operator $D$ is closable from $L^p(\Omega)$ to $L^p(\Omega, H)$ for any $p \geq 1$. We denote the domain of $D$ in $L^p(\Omega)$ by $\mathbb{D}^{1,p}(\mathbb{R})$, which means that $\mathbb{D}^{1,p}(\mathbb{R})$ is the closure of the class of smooth random variables $S$ with respect to the norm

$$
\|F\|_{\mathbb{D}^{1,p}(\mathbb{R})} := (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|^p])^{\frac{1}{p}}.
$$

A crucial property is the chain rule formula for the Malliavin derivative (Proposition 1.2.3. in [42]). If $\varphi : \mathbb{R} \to \mathbb{R}$ is $C^1$ with a bounded derivative and $F \in \mathbb{D}^{1,p}(\mathbb{R})$, $p \geq 1$, then $\varphi(F) \in \mathbb{D}^{1,p}(\mathbb{R})$ and

$$D_t \varphi(F) = \varphi'(F) D_t F. \quad (100)$$

It is possible to define higher order derivatives. The second order derivative $D^2_{s,t} F$ of the simple functional $F$ is obtained by differentiating the expression (99), considering that $D_t F$ is a simple functional taking values in $H$, which yields to the expression

$$D^2_{s,t} F := \sum_{1 \leq i,j \leq m} \frac{\partial^2 F}{\partial x_i \partial x_j}(B(h_1), \ldots, B(h_m))h_i(t)h_j(s).$$

By iteration, the derivative $D^k_{s_1,\ldots,s_k} F$ is defined for a simple functional $F \in \mathcal{S}$ and the operator $D^k$ from $\mathcal{S} \subset L^p(\Omega)$ to $L^p(\Omega, H^{\otimes k})$ is closable and can be extended from $\mathcal{S}$ to $\mathbb{D}^{k,p}(\mathbb{R})$ by closure under the norm defined by

$$\|F\|_{\mathbb{D}^{k,p}(\mathbb{R})} := \mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}[\|D^j F\|^p_{H^{\otimes s_j}}] = \mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E} \left[ \int_{[0,\Delta]} |D^j_{s_1,\ldots,s_j} F|^2 ds_1 \ldots ds_j \right]^{p/2}.$$ 

The space of infinitely differentiable variables, in the Malliavin sense, is defined by $\mathbb{D}^\infty(\mathbb{R}) = \cap_{k \geq 1} \cap_{p \geq 1} \mathbb{D}^{k,p}(\mathbb{R})$.

We can now introduce the divergence operator, defined as the adjoint of the derivative operator. As the underlying Hilbert space $H$ is an $L^2$ space we interpret the divergence operator as a stochastic integral and we call it Skorohod integral. Indeed, in the Brownian motion case, it coincides with the generalization of the Ito stochastic integral to anticipating integrands introduced by Skorohod [46].

**Definition 3.** The Skorohod integral $\delta$ is a linear operator on $L^2(\Omega, H)$ with values in $L^2(\Omega)$ such that

1. The domain of $\delta$, denoted by Dom($\delta$), is the set of $H$-valued square integrable random variables $u \in L^2(\Omega, H)$ such that

$$\mathbb{E}[<DF,u>] = \mathbb{E} \int_0^\Delta D_t Fu dt \| \leq c \|F\|_2$$

for all $F \in \mathbb{D}^{1,2}$; where $c$ is some constant depending on $u$.

2. If $u$ belongs to Dom($\delta$), then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[<DF,u>]=(= \mathbb{E}(\int_0^\Delta D_t Fu dt)) \quad (101)$$

for any $F \in \mathbb{D}^{1,2}(\mathbb{R})$.

We observe that, taking $F = 1$ in (101), we obtain $\mathbb{E}[\delta(u)] = 0$ for any $u \in \text{Dom}(\delta)$.

It is possible to extend the definition of the Malliavin derivative for $H$-valued variables $u$, in a way analogous to the case of real valued random variables $F$. This enables to define a set $\mathbb{D}^{k,p}(H) \subset L^p(\Omega, H)$ of $H$-valued variables admitting $k$ derivatives in $L^p$, endowed with the norm

$$\|u\|_{\mathbb{D}^{k,p}(H)} = \mathbb{E}[\|u\|^p_{L^p}] + \sum_{j=1}^k \mathbb{E}[\|D^j u\|^p_{H^{\otimes s_j}}] = \mathbb{E}[\|u\|^p_H] + \sum_{j=1}^k \mathbb{E} \left[ \int_{[0,\Delta]^{j+1}} |D^j_{s_1,\ldots,s_j} u_r|^2 dr ds_1 \ldots ds_j \right]^{p/2}.$$ 

The following proposition gives a smoothness criteria on $u$ which ensures that $u \in \text{Dom}(\delta)$, and provides useful formulae.
Proposition 6. 1. For any \( p > 1 \) the space of the Malliavin differentiable variables \( \mathbb{D}^{1,p}(H) \) is included in \( \text{Dom}(\delta) \) and it is
\[
\|\delta(u)\|_p \leq c_p \|u\|_{\mathbb{D}^{1,p}(H)}.
\]

More generally, the operator \( \delta \) is bounded from \( \mathbb{D}^{k,p}(H) \) to \( \mathbb{D}^{k-1,p}(\mathbb{R}) \), for \( k \geq 1, p > 1 \).

2. If \( u \) is an adapted process belonging to \( L^2([0,\Delta] \times \Omega, \mathbb{R}) \), then the Skorohod integral coincides with the Itô integral, i.e.
\[
\delta(u) = \int_0^\Delta u_t dB_t.
\]

3. If \( F \) belongs to \( \mathbb{D}^{1,2} \) then for any \( u \in \text{Dom}(\delta) \) for which \( \mathbb{E}[F^2 \int_0^\Delta u_t^2 dt] < \infty \), it is
\[
\delta(Fu) = F\delta(u) - \int_0^\Delta D_tFu dt
\]
whenever the right hand side belongs to \( L^2(\Omega) \).

The first point in the proposition above can be found in Proposition 1.5.7 of [42], the second point in Proposition 1.3.11 of [42], and the third point in Proposition 1.3.3. [42].

We recall that solutions of SDE with smooth coefficients are smooth variables in the Malliavin sense. Assume that \( X \) is solution to the SDE \((4)\), that the coefficients \( a, b \) are \( C^\infty \) and that \( \alpha \), \( \beta \) are bounded functions for any \( k \geq 1 \). Then, by Theorem 2.2.2 in [42], \( X_t \in \mathbb{D}^{\infty}(\mathbb{R}) = \cap_{p \geq 1, k \geq 1} \mathbb{D}^{k,p}(\mathbb{R}) \) and we have
\[
\sup_{(r_1,\ldots,r_k) \in [0,\Delta]^k} \mathbb{E} \left[ \sup_{0 \leq t \leq \Delta} \left| D_{r_1,\ldots,r_k}^p(X_t) \right|^p \right] \leq c_{p,\Delta,k}
\]
where the constant \( c_{p,\Delta,k} \), for bounded \( \Delta \), is upper bounded by a constant \( c_{p,k} \) that only depends on \( p, |a(0)|, |b(0)| \), \( \sup_{l=1,\ldots,k} \|a(l)\|_{\infty} \), and \( \sup_{l=1,\ldots,k} \|b(l)\|_{\infty} \). The control \((102)\) is the property (P1) in the proof of Theorem 2.2.2 in [42], where the dependence of \( c_{p,T,k} \) on the first \( k \) derivatives of the coefficient is a consequence of the expression \((P2)\) in the proof of Theorem 2.2.2. in [42].

B.2 Proof of formula \((26)\)

We follow closely the proof of Theorem 5 in [26], in our simpler one-dimensional situation, and relying on the properties recalled in Section B.1. As the coefficients of the SDE \((22)\) are \( C^\infty \) we know that \((x,y,\epsilon) \mapsto p_{\Delta,\epsilon}(x,y)\) is smooth. For \( \varphi : \mathbb{R} \to \mathbb{R} \) a \( C^\infty \) bounded function, we differentiate with respect to \( \epsilon \) the relation \( \mathbb{E}_{\epsilon|X_0} [\varphi(X_{\Delta,\epsilon})] = \int_{\mathbb{R}} p_{\Delta,\epsilon}(x_0,y) \varphi(y) dy \). It yields to
\[
\mathbb{E}_{x_0} [\varphi'(X_{\Delta,\epsilon}) \tilde{X}_{\Delta,\epsilon}] = \int_{\mathbb{R}} [D_\varphi(X_{\Delta,\epsilon})) (X_{\Delta,\epsilon})] \varphi(y) dy.
\]
We now use the chain rule formula \((100)\) to write
\[
\varphi'(X_{\Delta,\epsilon}) \mapsto \frac{D(X_{\Delta,\epsilon})X_{\Delta,\epsilon}}{<D_{X_{\Delta,\epsilon}},X_{\Delta,\epsilon}>}
\]
and thus
\[
\mathbb{E}_{x_0} [\varphi'(X_{\Delta,\epsilon}) \tilde{X}_{\Delta,\epsilon}] = \mathbb{E}_{x_0} \left[ <D (\varphi(X_{\Delta,\epsilon})), D X_{\Delta,\epsilon}^2 > \right] = \mathbb{E}_{x_0} \left[ <D (\varphi(X_{\Delta,\epsilon})), \frac{D X_{\Delta,\epsilon}^2 X_{\Delta,\epsilon}}{<D_{X_{\Delta,\epsilon}},X_{\Delta,\epsilon}>} > \right].
\]
We now apply \((101)\), with \( u = \frac{D X_{\Delta,\epsilon} X_{\Delta,\epsilon}}{<D_{X_{\Delta,\epsilon}},X_{\Delta,\epsilon}>} \), and deduce \( \mathbb{E}_{x_0} [\varphi'(X_{\Delta,\epsilon}) \tilde{X}_{\Delta,\epsilon}] = \mathbb{E}_{x_0} [\varphi'(X_{\Delta,\epsilon}) W_{x_0,\Delta,\epsilon}] \) with \( W_{x_0,\Delta,\epsilon} \) given by \((27)\). This is possible, as we show in the proof of Lemma 3 that \( u \in \mathbb{D}^{1,4}(H) \) which implies \( u \in \text{Dom}(\delta) \). Now, by conditioning on \( X_{\Delta,\epsilon} \), we get
\[
\int_{\mathbb{R}} p_{\Delta,\epsilon}(x_0,y) \varphi(y) dy = \mathbb{E}_{x_0} [\varphi'(X_{\Delta,\epsilon}) W_{x_0,\Delta,\epsilon}] = \int_{\mathbb{R}} p_{\Delta,\epsilon}(x_0,y) \mathbb{E}_{x_0} [|W_{x_0,\Delta,\epsilon}| X_{\Delta,\epsilon} = y] \varphi(y) dy.
\]
The equation \((26)\) follows. \( \square \)

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