More on Seiberg-Witten Theory and Monstrous Moonshine

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Abstract

We continue the study of a relationship between the instanton expansion of the Seiberg-Witten (SW) prepotential of $D = 4$, $\mathcal{N} = 2$ SU(2) SUSY gauge theory and the Monstrous moonshine. Extending the previous results, we show for the cases of $N_f = 2$ and $3$ that $q = e^{2\pi i \tau}$, where $\tau$ is the complex gauge coupling, again has an expansion whose coefficients are all integer-coefficient polynomials of the moonshine coefficients of the modular $j$-function in terms of an appropriate expansion variable. We also demonstrate that the new method of calculating the SW prepotential developed here is useful by performing some explicit computations.

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I. INTRODUCTION

Recently, an explicit relationship between the instanton expansion of the Seiberg-Witten (SW) prepotential of $D = 4, \mathcal{N} = 2$ SU(2) SUSY gauge theory and the Monstrous moonshine was studied (See [11] and [12] for reviews on the respective topics.). There, by a simple method to obtain SW prepotentials using the modular $j$-function, it was shown that the coefficients of the expansion of $q = e^{2\pi i \tau}$ in terms of $A^2 \equiv \frac{A^2}{16a^2}$ ($N_f = 0$) or $\frac{A^2}{16a^2}$ ($N_f = 1$) are all integer-coefficient polynomials of the moonshine coefficients of the modular $j$-function\(^1\). In this paper, we continue this study; we further examine whether the same phenomenon is also observed for $N_f = 2$ and 3 and show that it is indeed the case.

As already emphasized in [10], the relation between the SW theory and the Monstrous moonshine is not at all surprising in itself, since SW curves are elliptic curves and the modular $j$-function is a bijection between the fundamental domain and the complex plane, and thus describes the modulus of an elliptic curve. On the other hand, it has been known for some time that the $u$-plane integral for a certain “mock modular form” arising in the elliptic genus of K3 yields the SO(3) Donaldson invariants of $\mathbb{P}^2$ (See [15] and references therein for more recent discussions.). Since the former exhibits a similar moonshine phenomenon called the Mathieu moonshine, while the latter are certain correlation functions of a twisted 4D $\mathcal{N} = 2$ SO(3) supersymmetric gauge theory, this observation may be the first example of a connection between a moonshine phenomenon and a supersymmetric gauge theory (albeit a twisted one), which is nontrivial. In contrast, the correspondence we will use is not that somewhat indirect and intricate, but more direct and easy to understand. Nonetheless, there are other reasons to focus on the relationship.

One of them is as an effective tool for calculating the prepotential. This method can be applied to any theory as long as its SW curve is an elliptic curve, as in E-string theory. We will demonstrate that the method developed in [10] for calculating the prepotentials is useful by implementing it in SU(2) SW theory with $N_f = 2$ and 3. To date, various ways of computing the SW prepotential are available, but our method is more efficient than A-model like computations such as the Nekrasov instanton counting, and simpler than other

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\(^1\) Here, as in the standard literature on the SW theory, $a$ is the eigenvalue of the SU(2) adjoint scalar playing the role of the Higgs, and $\tau$ is the complex gauge coupling $\frac{\vartheta_2}{\vartheta_3} + \frac{\vartheta_4}{\vartheta_3}$ identified as the modulus of the elliptic fiber over the $u$-plane of the SW curve.
B-model like approaches (e.g. \[20–28\]) as we do not need to compute the dual period $a_D$. Our approach is simply to develop a series, so there is no way to go wrong (as long as one can use computational software like Mathematica). For example, by using our method we found four typos in the result of the $N_f = 3$ prepotential computed by \[24\] long time ago, as shown in the text \[2\].

Another nontrivial observation in the connection between the SW theory and Monstrous moonshine is a possible relationship between two different 2d conformal field theories (CFTs) with different central charges. We will show (as was done in \[10\]) that the quantity $q$ allows an expansion in which the coefficients are integer-coefficient polynomials of the moonshine Fourier coefficients. This means that $q$ is also expressed in terms of integer-coefficient polynomials of the dimensions of representations of the Monster group. As pointed out in \[10\], since the dimensions of representations of the Monster are related to the vertex operator CFT \[6–9\] whereas each instanton contribution of the SW prepotential is written as a correlation function of the Liouville theory \[31\], our observation suggests that those are related in some way. What exactly that is remains as a problem to be solved in the future.

II. SW PREPOTENTIAL FROM THE MODULAR $j$-FUNCTION

Let us consider a SW curve given in the Weierstrass form

$$Y^2 = X^3 + f(u)X + g(u).$$

We can obtain the value of the modular $j$-function as

$$j(\tau) = \frac{123 \cdot 4f(u)^3}{4f(u)^3 + 27g(u)^2}.$$  \[2\]

We expand the right hand side of this equation \[2\] by $u$ around $u = \infty$, thereby we get $\frac{1}{u}$-expansion of $\frac{1}{j}$.

On the other hand, the modular $j$-function has an expansion in terms of $q = e^{2\pi i \tau}$ as

$$j(\tau) = \frac{1}{q} + \sum_{n=0}^{\infty} a_n q^n,$$  \[3\]

2 A similar method for computing the prepotential was already used in \[29\]. The reference to the $j$-function in SW theory itself is of course much older, e.g., the $q$-expansion for the $j$-function can be found in \[30\] (but no reference to the Monstrous moonshine).
whose coefficients $a_n$ are integers. Specifically, the first few terms of the expansion (3) are:

Thus we find $\frac{1}{j}$-expansion of $q$ as

$$
q = \frac{1}{j} + a_0 \left( \frac{1}{j} \right)^2 + \left( a_0^2 + a_1 \right) \left( \frac{1}{j} \right)^3 + \left( a_0^3 + 3a_1a_0 + a_2 \right) \left( \frac{1}{j} \right)^4
$$

$$
+ \left( a_0^4 + 6a_0^2a_1 + 2a_1^2 + 4a_0a_2 + a_3 \right) \left( \frac{1}{j} \right)^5
$$

$$
+ \left( a_0^5 + 10a_0^3a_1 + 10a_0a_1^2 + 10a_0^2a_2 + 5a_1a_2 + 5a_0a_3 + a_4 \right) \left( \frac{1}{j} \right)^6 + O \left( \left( \frac{1}{j} \right)^7 \right)
$$

by solving inversely the equation (3).

Substituting $\frac{1}{u}$-expansion of $\frac{1}{j}$ into this equation (4), we can get $\frac{1}{u}$-expansion of $q$. Furthermore, $\frac{1}{a}$-expansion of $\frac{1}{u}$ is known from the period integral on the SW curve [21] and we obtain $\frac{1}{a}$-expansion of $q$. Since the complex structure modulus $\tau$ and the SW prepotential $\mathcal{F}$ are related as

$$
\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2},
$$

we finally get the expression of the prepotential $\mathcal{F}$ by integrating $\frac{1}{a}$-expansion of $q$ with respect to $a$ twice.

This procedure was performed in [10] for $N_f = 0$ and 1, thereby an explicit relationship between the instanton expansion and the moonshine was obtained. Below we will do this for the cases $N_f = 2$ and 3.  

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3 It is known [14, 32] that the monodromy groups in these cases are the congruence subgroups $\Gamma(2)$ and $\Gamma_0(4)$ of $SL(2,\mathbb{Z})$, respectively. The Hauptmoduln for these reduced monodromy groups, that is, the weight-0 modular forms associated with these congruence subgroups, also allow expansions in term of $q$ known as the McKay-Thompson series, whose coefficients are a sum of traces of a particular element of the Monster group [3]. The analysis of such modular rational elliptic surfaces was recently worked out in [33, 34].
$N_f = 2 \text{ case}$

Let us consider the $\mathcal{N} = 2$ $SU(2)$ SUSY gauge theory with $N_f = 2$ hypermultiplets. The SW curve is given by

$$y^2 = \left(x^2 - u + \frac{\Lambda^2}{8}\right)^2 + \Lambda^2(x + m_1)(x + m_2)$$  \hspace{1cm} (6)

in the quartic-polynomial representation [21]. This is equivalent to the Weierstrass form

$$Y^2 = X^3 + f(u, m_1, m_2)X + g(u, m_1, m_2),$$  \hspace{1cm} (7)

$$f(u, m_1, m_2) = -\frac{\Lambda^4}{4} + 4\Lambda^2m_1m_2 - \frac{16}{3}u^2,$$  \hspace{1cm} (8)

$$g(u, m_1, m_2) = \Lambda^4(m_1^2 + m_2^2) - \frac{2}{3}(\Lambda^4 + 8\Lambda^2m_1m_2)u + \frac{128}{27}u^3.$$  \hspace{1cm} (9)

Substituting (8) and (9)) into the definition of the $j$-function [11], we have an expansion of $\frac{1}{j}$

$$\frac{1}{j} = U^2 - (\hat{m}_1^2 + \hat{m}_2^2)U^3 + [\hat{m}_1\hat{m}_2(\hat{m}_1\hat{m}_2 + 64) - 704]U^4$$

$$- 72(\hat{m}_1\hat{m}_2 - 16)(\hat{m}_1^2 + \hat{m}_2^2)U^5$$

$$- 16[27(\hat{m}_1^4 + \hat{m}_2^4) - 5\hat{m}_1^3\hat{m}_2^3 - 102\hat{m}_1^2\hat{m}_2^2 + 5376\hat{m}_1\hat{m}_2 - 18688]U^6$$

$$- 3456(\hat{m}_1\hat{m}_2 - 40)(\hat{m}_1\hat{m}_2 - 4)(\hat{m}_1^2 + \hat{m}_2^2)U^7$$

$$- 2304(\hat{m}_1\hat{m}_2 - 4)^2[27(\hat{m}_1^4 + \hat{m}_2^4) - 2\hat{m}_1^3\hat{m}_2^3 + 30\hat{m}_1^2\hat{m}_2^2 + 2880\hat{m}_1\hat{m}_2 - 11008]U^8$$

$$- 110592(\hat{m}_1\hat{m}_2 - 112)(\hat{m}_1\hat{m}_2 - 4)^2(\hat{m}_1^2 + \hat{m}_2^2)U^9$$

$$- 221184(\hat{m}_1\hat{m}_2 - 4)^3[27(\hat{m}_1^4 + \hat{m}_2^4) - \hat{m}_1^3\hat{m}_2^3 + 66\hat{m}_1^2\hat{m}_2^2 + 2112\hat{m}_1\hat{m}_2 - 8576]U^{10}$$

$$+ 95514880(\hat{m}_1\hat{m}_2 - 4)^3(\hat{m}_1^2 + \hat{m}_2^2)U^{11}$$

$$- 1769472(\hat{m}_1\hat{m}_2 - 4)^3[270(\hat{m}_1^4 + \hat{m}_2^4) - 5\hat{m}_1^3\hat{m}_2^3 + 816\hat{m}_1^2\hat{m}_2^2 + 17472\hat{m}_1\hat{m}_2 - 73984]U^{12}$$

$$+ 382205952(\hat{m}_1\hat{m}_2 - 4)^4(\hat{m}_1\hat{m}_2 + 176)(\hat{m}_1^2 + \hat{m}_2^2)U^{13}$$

$$- 254803968(\hat{m}_1\hat{m}_2 - 4)^4[27(\hat{m}_1^4 + \hat{m}_2^4) - \hat{m}_1^3\hat{m}_2^3 + 450\hat{m}_1^2\hat{m}_2^2 + 7680\hat{m}_1\hat{m}_2 - 33536]U^{14}$$

$$+ \mathcal{O}(U^{15}),$$  \hspace{1cm} (10)

where we have introduced $U \equiv \frac{\Lambda^2}{64u}$, $\hat{m}_i \equiv \frac{8m_i}{\Lambda}$.

To relate $a$ and $u$, we use the fact [21] that $\frac{\partial a}{\partial u}$ is given by

$$\frac{\partial a}{\partial u} = \frac{F\left(\frac{1}{12}; \frac{5}{12}; 1; \frac{12^3}{j}\right)}{\sqrt{2}(-3f(u, m_1, m_2))^2},$$  \hspace{1cm} (11)
where $F(a, b, c; z)$ is a hypergeometric function. Since we have the $U$-expansion of $\frac{1}{j}$, we can obtain the $U$-expansion of $a$ after substituting (10) into (11) and integrating with respect to $U$. Therefore, defining $A \equiv \frac{\Lambda}{8\sqrt{2}a}$, $U$ is conversely expanded by this $A$ as

$$U = A^2 - 8(\hat{m}_1\hat{m}_2 + 1)A^6 + 24(\hat{m}_1^2 + \hat{m}_2^2)A^8 + 24(\hat{m}_1^2\hat{m}_2^2 - 8\hat{m}_1\hat{m}_2 + 1)A^{10}$$

$$+ 16(39 + 4\hat{m}_1\hat{m}_2)(\hat{m}_1^2 + \hat{m}_2^2)A^{12}$$

$$- 8\left[81(\hat{m}_1^4 + \hat{m}_2^4) + 56\hat{m}_1^3\hat{m}_2^3 + 660\hat{m}_1^2\hat{m}_2^2 + 1128\hat{m}_1\hat{m}_2 + 56\right]A^{14} + O(A^{15}).$$

By using this expression in (10), we obtain $A^2$-expansion of $\frac{1}{j}$. Thus, substituting this expansion into (4), we find $A^2$-expansion of $q$,

$$q = A^4 - (\hat{m}_1^2 + \hat{m}_2^2)A^6 + [(a_0 - 720) + 48\hat{m}_1\hat{m}_2 + \hat{m}_1^2\hat{m}_2^2]A^8$$

$$+ \left[(1224 - 2a_0)(\hat{m}_1^2 + \hat{m}_2^2) - 48\hat{m}_1\hat{m}_2(\hat{m}_1^2 + \hat{m}_2^2)\right]A^{10}$$

$$+ \left[(321648 - 1440a_0 + a_0^2 + a_1) + 4(a_0 - 120)\hat{m}_1^2\hat{m}_2^2 + 32(3a_0 - 2056)\hat{m}_1\hat{m}_2\right.$$  

$$+ 48\hat{m}_1^3\hat{m}_2 + (a_0 - 504)(\hat{m}_1^4 + \hat{m}_2^4)]A^{12}$$

$$+ \left[(-666024 + 3888a_0 - 3a_0^2 - 3a_1)(\hat{m}_1^2 + \hat{m}_2^2) - (744 + 2a_0)(\hat{m}_1^2\hat{m}_2^4 + \hat{m}_2^2\hat{m}_1^4)$$

$$+(114944 - 192a_0)(\hat{m}_1\hat{m}_2^3 + \hat{m}_2\hat{m}_1^3)]A^{14} + O(A^{16}),$$

whose coefficients are integer-coefficient polynomials of the moonshine Fourier coefficients $a_n$. 

$\text{6}$
Next, we take logarithm of \( q \) as the complex modulus \( \tau \) is given by \( q = e^{2\pi i \tau} \). We have
\[
2\pi i \tau = \log \left( A^4 - (\hat{m}_1^2 + \hat{m}_2^2) A^2 + \frac{1}{2} (2a_0 - 1440 - \hat{m}_1^4 - \hat{m}_2^4 + 96 \hat{m}_1 \hat{m}_2) A^4 \right)
- \frac{1}{3} (\hat{m}_1^2 + \hat{m}_2^2) (3a_0 - 1512 + \hat{m}_1^4 + \hat{m}_2^4 - \hat{m}_1^2 \hat{m}_2^2) A^6
+ \frac{1}{4} \left[ (2a_0^2 - 2880 a_0 + 4a_1 + 249792) - \hat{m}_1^8 - \hat{m}_2^8 + (4a_0 + 384) \hat{m}_1^2 \hat{m}_2^2 \right.
+ (192a_0 - 124928) \hat{m}_1 \hat{m}_2] A^8
- \frac{1}{5} (\hat{m}_1^2 + \hat{m}_2^2) \left[ (5a_0^2 - 6120 a_0 + 10a_1 - 92520) + \hat{m}_1^8 + \hat{m}_2^8 \right.
+ (240a_0 - 124800) \hat{m}_1 \hat{m}_2 + \hat{m}_1^4 \hat{m}_2^4 - \hat{m}_1^6 \hat{m}_2^6 - \hat{m}_1^2 \hat{m}_2^2 \right] A^{10}
+ \frac{1}{6} \left[ -\hat{m}_1^{12} - \hat{m}_2^{12} + (288a_0 - 87552) \hat{m}_1^3 \hat{m}_2^3 \right.
+ (288a_0^2 - 394752a_0 + 576a_1 + 22620672) \hat{m}_1 \hat{m}_2
+ (12a_0^2 - 2880a_0 + 24a_1 - 7466688) \hat{m}_1^2 \hat{m}_2^2
+ (3a_0^2 - 3024a_0 + 6a_1 - 390096) (\hat{m}_1^4 + \hat{m}_2^4)
\left. + (2a_0^3 - 4320a_0^2 + 12a_0a_1 + 1929888a_0 - 8640a_1 + 6a_2 - 53824512) \right] A^{12}
+ \mathcal{O}(A^{14}).
\]

Integrating (14) with respect to \( a \) twice, we can finally derive \( 2\pi i \) times the \( N_f = 2 \) prepotential.

Now let us show the results of our calculations. The \( N_f = 2 \) prepotential \( \mathcal{F}^{N_f=2} \) takes the form
\[
\mathcal{F}^{N_f=2} = \frac{ia^2}{2\pi} \left[ \log \left( \frac{a}{\Lambda} \right)^2 + (5 \log 2 - 2 + i\pi) - \frac{\log a}{2a^2} \sum_{i} m_i^2 - \sum_{k=2}^{\infty} \mathcal{F}_k^{N_f=2} \left( \frac{\Lambda}{a} \right)^{2k} \right],
\]
where first few instanton expansion coefficients \( \mathcal{F}_k^{N_f=2} \) is given by
\[
\mathcal{F}_2^{N_f=2} = \frac{1}{4096} + \frac{1}{32} \hat{m}_1 \hat{m}_2 - \frac{1}{48} (\hat{m}_1^4 + \hat{m}_2^4),
\]
\[
\mathcal{F}_3^{N_f=2} = -\frac{3}{8192} (\hat{m}_1^2 + \hat{m}_2^2) - \frac{1}{480} (\hat{m}_1^6 + \hat{m}_2^6),
\]
\[
\mathcal{F}_4^{N_f=2} = \frac{5}{134217728} + \frac{5}{16384} \hat{m}_1^2 \hat{m}_2^2 + \frac{5}{196608} \hat{m}_1 \hat{m}_2 - \frac{1}{2688} (\hat{m}_1^8 + \hat{m}_2^8).
\]

\[
\text{Note that we define } \hat{m}_i = \frac{8a_i}{\Lambda} = 8 \tilde{m}_i \text{ and } \mathcal{F}_k^{N_f=2} \text{ in the above expression is related to } \mathcal{F}_k^2 \text{ in } [23] \text{ as } \mathcal{F}_k^{N_f=2}(\text{here}) = -\frac{2}{\Lambda^2} \mathcal{F}_k^2, \Lambda(\text{here}) = \Lambda_2.
\]

In this notation, we can obtain the coefficients of the instanton expansion \( \mathcal{F}_k^{N_f=2} \) which include the moonshine Fourier coefficients \( a_n \) from the integration of (14). Here we show
the coefficients of the instanton expansion of $N_f = 2$ prepotential up to order 4:

$$\mathcal{F}_2^{N_f=2} = \frac{1}{(8\sqrt{2})^4 \cdot 3 \cdot 2} \cdot \frac{1}{2} \left( (2a_0 - 1440) - (\hat{m}_1^4 + \hat{m}_2^4) + 96\hat{m}_1\hat{m}_2 \right)$$

(19)

$$\mathcal{F}_3^{N_f=2} = -\frac{1}{(8\sqrt{2})^6 \cdot 5 \cdot 4} \cdot \frac{1}{3} (\hat{m}_1^6 + \hat{m}_2^6) \left( (3a_0 - 1512) + (\hat{m}_1^4 + \hat{m}_2^4) - \hat{m}_1^2\hat{m}_2^2 \right)$$

(20)

$$\mathcal{F}_4^{N_f=2} = \frac{1}{(8\sqrt{2})^8 \cdot 7 \cdot 6} \cdot \frac{1}{4} \left[ (2a_0^2 - 2880a_0 + 4a_1 + 249792) - (\hat{m}_1^8 + \hat{m}_2^8) 
+ (4a_0 + 384)\hat{m}_1^2\hat{m}_2^2 + (192a_0 - 124928)\hat{m}_1\hat{m}_2 \right]$$

(21)

Substituting the coefficients of $j$-function $a_n$ into above expressions, we can derive explicit results of the instanton expansion and we have succeeded to reproduce all known results $\mathcal{F}_k^{N_f=2}$ for $k = 2, 3, 4$ from the coefficients of $A^4, A^6, A^8$ in [14], respectively.

Indeed, substituting $a_0 = 744$ and $\hat{m}_i = 8\hat{m}_i$ into (19), then we get

$$\mathcal{F}_2^{N_f=2} = \frac{1}{(8\sqrt{2})^4 \cdot 3 \cdot 2} \left[ \frac{1}{4096} + \frac{1}{32} \hat{m}_1\hat{m}_2 - \frac{1}{48} (\hat{m}_1^4 + \hat{m}_2^4) \right]$$

which is consistent with the known $k = 2$ expression of $\mathcal{F}^{N_f=2}$ in [13]. Also, the expansion coefficients $\mathcal{F}_k^{N_f=2}$ for larger $k$ can similarly be calculated from the coefficients of $A^{2k}$ in [14].

$N_f = 3$ case

A similar result can be obtained in the case of $N_f = 3$. The SW curve in this case is

$$y^2 = \left[ x^2 - u + \frac{\Lambda}{4} \left( x + \frac{m_1 + m_2 + m_3}{2} \right) \right]^2 + \Lambda(x + m_1)(x + m_2)(x + m_3)$$

(22)

in the quartic-polynomial representation [21]. This is equivalent to the Weierstrass form

$$Y^2 = X^3 + f(u, m_1, m_2, m_3)X + g(u, m_1, m_2, m_3),$$

(23)

$$f(u, m_1, m_2, m_3) = -\frac{\Lambda^4}{768} - \frac{\Lambda^2}{4} (m_1^2 + m_2^2 + m_3^2) + 4\Lambda m_1 m_2 m_3 - \frac{16u^2}{3} + \frac{\Lambda^2 u}{3},$$

(24)

$$g(u, m_1, m_2, m_3) = \frac{\Lambda^6}{55296} + \frac{\Lambda^4}{192} (m_1^2 + m_2^2 + m_3^2) + \frac{\Lambda^2}{3} (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2)$$

$$- u \left[ \frac{\Lambda^4}{144} + \frac{2}{3} \Lambda^2 (m_1^2 + m_2^2 + m_3^2) + \frac{16}{3} \Lambda m_1 m_2 m_3 \right]$$

$$- \frac{\Lambda^3}{12} m_1 m_2 m_3 + \frac{128}{27} u^3 + \frac{5\Lambda^2}{9} u^2.$$  

(25)
From the definition of the $j$-function (2) and then using (24) and (25), we get an expansion of $\frac{1}{j}$,

$$
\frac{1}{j} = -U + (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2 - 752)U^2
$$

$$
+ \left[ 1472(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - (\hat{m}_1^2 \hat{m}_2^2 + \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_3^2 \hat{m}_1^2) - 56\hat{m}_1 \hat{m}_2 \hat{m}_3 - 368640 \right] U^3
$$

$$
+ \left[ -704(\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) - 2656(\hat{m}_1^2 \hat{m}_2^2 + \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_3^2 \hat{m}_1^2) + \hat{m}_1^2 \hat{m}_2^2 \hat{m}_3^2

\right.

$$
+ 64\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) + 1057792(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - 149094400 \right] U^4
$$

$$
+ 8 \left[ -123392(\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) + 70123520(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - 7548928\hat{m}_1 \hat{m}_2 \hat{m}_3

\right.

$$
+ 144(\hat{m}_1^4 \hat{m}_2^2 + \hat{m}_2^4 \hat{m}_3^2 + \hat{m}_3^4 \hat{m}_1^2) - 337408(\hat{m}_1^2 \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_2^2 \hat{m}_3^2 \hat{m}_1^2)

$$

$$
+ 21376\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - 9\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^2 \hat{m}_2^2 + \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_3^2 \hat{m}_1^2)

$$

$$
- 6748635136 \right] U^5
$$

$$
+ 16 \left[ 18688(\hat{m}_1^6 + \hat{m}_2^6 + \hat{m}_3^6) - 48361472(\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) + 5\hat{m}_1^2 \hat{m}_2^2 \hat{m}_3^2

\right.

$$
+ 15678832640(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) + 116686848(\hat{m}_1^2 \hat{m}_2^2 + \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_3^2 \hat{m}_1^2)

$$

$$
+ 136704(\hat{m}_1^4 \hat{m}_2^2 + \hat{m}_2^4 \hat{m}_3^2 + \hat{m}_3^4 \hat{m}_1^2) - 27(\hat{m}_1^4 \hat{m}_2^4 + \hat{m}_2^4 \hat{m}_3^4 + \hat{m}_3^4 \hat{m}_1^4)

\right.

$$

$$
- 2016542720\hat{m}_1 \hat{m}_2 \hat{m}_3 + 11020288\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2)

\right.

$$

$$
- 5376\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) - 20544\hat{m}_1 \hat{m}_2 \hat{m}_3 (\hat{m}_1^2 \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_2^2 \hat{m}_3^2 \hat{m}_1^2)

\right.

$$

$$
+ 102\hat{m}_2^2 \hat{m}_3^2 (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - 1136991928320 \right] U^6 + O(U^7)
$$

where we have introduced $U \equiv \frac{\Lambda^2}{4996u}$, $\hat{m}_i \equiv \frac{64m_i}{\Lambda}$.

Similarly to the $N_f = 2$ case, we again use (11) and then get the $U$-expansion of $a$. 

Defining $A \equiv \frac{\Lambda}{64\pi^2 a_0}$, $U$ is conversely expanded by this $A$ as

$$U = A^2 - 8A^4 - 8\left[ (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) + \dot{m}_1 \dot{m}_2 \dot{m}_3 - 7 \right] A^6$$

$$+ 8\left[ 22(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) + 3(\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2) + 40 \dot{m}_1 \dot{m}_2 \dot{m}_3 - 48 \right] A^8$$

$$+ 8\left[ 3(\dot{m}_1^4 + \dot{m}_2^4 + \dot{m}_3^4) - 312(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) - 132(\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2)$$

$$- 840 \dot{m}_1 \dot{m}_2 \dot{m}_3 - 24 \dot{m}_1 \dot{m}_2 \dot{m}_3 (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) + 3 \dot{m}_1^2 \dot{m}_2^2 \dot{m}_3^2 + 323 \right] A^{10}$$

$$+ 16 \left[ -48(\dot{m}_1^4 + \dot{m}_2^4 + \dot{m}_3^4) + 1806(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) + 1616(\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2)$$

$$+ 39(\dot{m}_1^4 \dot{m}_2^2 + \dot{m}_1^2 \dot{m}_2^4 + \dot{m}_3^4 \dot{m}_1^2) + 6528 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 + 728 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)$$

$$+ 4 \dot{m}_1 \dot{m}_2 \dot{m}_3 (\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2) + 268 \dot{m}_1^2 \dot{m}_2^2 \dot{m}_3^2 - 1080 \right] A^{12}$$

$$- 8 \left[ 56(\dot{m}_1^6 + \dot{m}_2^6 + \dot{m}_3^6) - 1802(\dot{m}_1^4 + \dot{m}_2^4 + \dot{m}_3^4) + 37160(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)$$

$$+ 58704(\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2) + 5496(\dot{m}_1^4 \dot{m}_2^2 + \dot{m}_1^2 \dot{m}_2^4 + \dot{m}_3^4 \dot{m}_1^2) + 56 \dot{m}_1^2 \dot{m}_2^2 \dot{m}_3^2$$

$$+ 81(\dot{m}_1^4 \dot{m}_2^4 + \dot{m}_2^4 \dot{m}_3^4 + \dot{m}_3^4 \dot{m}_1^4) + 169928 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 + 45504 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)$$

$$+ 1128 \dot{m}_1 \dot{m}_2 \dot{m}_3 (\dot{m}_1^4 + \dot{m}_2^4 + \dot{m}_3^4) + 6176 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 \dot{m}_3 (\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2)$$

$$+ 41832 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 \dot{m}_3 (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) - 14344 \right] A^{14} + O(A^{16}).$$

(27)

By using this expression in (26), we obtain $A^2$-expansion of $\frac{1}{T}$. Thus, substituting this expansion into (4) and taking logarithm, we find $A^2$-expansion of $q$,

$$q = -A^2 + [(a_0 - 744) + (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)] A^4$$

$$+ [(356664 + 1488 a_0 - a_0^2 - a_1) + (1464 - 2a_0)(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)$$

$$+ (\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2) - 48 \dot{m}_1 \dot{m}_2 \dot{m}_3] A^6$$

$$+ [(140379008 + 1266864 a_0 - 2232 a_0^2 + a_0^3 - 2232 a_1 + 3a_0 a_1 + a_2)$$

$$+ (103496 - 4416 a_0 + 3a_0^2 + 3a_1)(\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2) + \dot{m}_1^2 \dot{m}_2^2 \dot{m}_3^2$$

$$+ (a_0 - 720)(\dot{m}_1^4 + \dot{m}_2^4 + \dot{m}_3^4) + (4 a_0 - 2688)(\dot{m}_1^2 \dot{m}_2^2 + \dot{m}_2^2 \dot{m}_3^2 + \dot{m}_3^2 \dot{m}_1^2)$$

$$+ (96 a_0 - 70144) \dot{m}_1 \dot{m}_2 \dot{m}_3 + 48 \dot{m}_1 \dot{m}_2 \dot{m}_3 \dot{m}_3 (\dot{m}_1^2 + \dot{m}_2^2 + \dot{m}_3^2)] A^8 + O(A^{10}).$$

(28)
Integrating (29) with respect to potential. Its logarithm yields

\[
2\pi i\tau = \log (-A^2) + \left[(744 - a_0) - (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2)\right] A^2 \\
+ \frac{1}{2} \left[(159792 - 1488a_0 + a_0^2 + 2a_1) + (2a_0 - 1440)(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - (\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) + 96\hat{m}_1\hat{m}_2\hat{m}_3\right] A^4 \\
+ \frac{1}{3} \left[(36893760 - 1069992a_0 + 2232a_0^2 - a_0^3 + 4464a_1 - 6a_0a_1 - 3a_2) + (4392a_0 - 3a_0^2 - 6a_1 - 426456)(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - (\hat{m}_1^6 + \hat{m}_2^6 + \hat{m}_3^6) \\
+ (1512 - 3a_0)(\hat{m}_1^2\hat{m}_2^2 + \hat{m}_2^2\hat{m}_3^2 + \hat{m}_3^2\hat{m}_1^2) + (103296 - 144a_0)\hat{m}_1\hat{m}_2\hat{m}_3\right] A^6 \\
+ \frac{1}{4} \left[(8515094496 - 561516032a_0 + 2533728a_0^2 - 2976a_0^3 + a_0^4 + 5067456a_1 - 17856a_0a_1 + 12a_0^2a_1 + 6a_1^2 - 8928a_2 + 12a_0a_2 + 4a_3 + 4a_0\hat{m}_1^2\hat{m}_2^2\hat{m}_3^2 \\
+ (4137984a_0 - 8832a_0^2 + 4a_0^3 - 17664a_1 + 24a_0a_1 + 12a_2 - 132871168)(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2) - (\hat{m}_1^8 + \hat{m}_2^8 + \hat{m}_3^8) \\
+ (249792 - 2880a_0 + 2a_0^2 + 4a_1)(\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) \\
+ (26941440 - 280576a_0 + 192a_0^2 + 384a_1)\hat{m}_1\hat{m}_2\hat{m}_3 + (454656 - 10752a_0 + 8a_0^2 + 16a_1)(\hat{m}_1^2\hat{m}_2^2 + \hat{m}_2^2\hat{m}_3^2 + \hat{m}_3^2\hat{m}_1^2) \\
+(192a_0 - 124928)\hat{m}_1\hat{m}_2\hat{m}_3(\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2)\right] A^8 + O(A^{10}).
\]

Integrating (29) with respect to \(a\) twice, we can finally derive \(2\pi i\) times the \(N_f = 3\) prepotential.

In this case, the \(N_f = 3\) prepotential \(\mathcal{F}^{N_f=3}\) is

\[
\mathcal{F}^{N_f=3} = \frac{ia^2}{2\pi} \left[\frac{1}{2} \log \left(\frac{a}{\Lambda}\right)^2 + \frac{1}{2}(9\log 2 - 2 - i\pi) - \frac{\log a}{2a^2} \sum_{i=1}^{2} m_i^2 - \sum_{k=2}^{\infty} \mathcal{F}_{k=3}^{N_f=3} \left(\frac{\Lambda}{a}\right)^{2k}\right],
\]
where first few instanton expansion coefficients $F_{N_f = 3}^{N_f = 3}$ is given in \[23\] by

$$F_{N_f = 3}^{N_f = 3} = \frac{1}{33554432} + \frac{1}{4096} \left( \hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4 \right) + \frac{1}{32} \hat{m}_1 \hat{m}_2 \hat{m}_3 + \frac{1}{48} \left( \hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4 \right), \quad (31)$$

$$F_{N_f = 3}^{N_f = 3} = -\frac{3}{33554432} \left( \hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4 \right) - \frac{1}{32768} \hat{m}_1 \hat{m}_2 \hat{m}_3$$

$$- \frac{3}{8192} \left( \hat{m}_1^2 \hat{m}_2^2 + \hat{m}_2^2 \hat{m}_3^2 + \hat{m}_3^2 \hat{m}_1^2 \right) - \frac{1}{480} \left( \hat{m}_1^6 + \hat{m}_2^6 + \hat{m}_3^6 \right) \quad (32)$$

$$F_{N_f = 3}^{N_f = 3} = \frac{5}{4503599627370496} + \frac{5}{103079215104} \left( \hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2 \right) + \frac{7}{268435456} \hat{m}_1 \hat{m}_2 \hat{m}_3$$

$$+ \frac{5}{134217728} \left( \hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4 \right) + \frac{25}{33554432} \left( \hat{m}_1 \hat{m}_2^2 + \hat{m}_2 \hat{m}_3^2 + \hat{m}_3 \hat{m}_1^2 \right)$$

$$+ \frac{5}{196608} \hat{m}_1 \hat{m}_2 \hat{m}_3 \left( \hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2 \right) + \frac{5}{16384} \hat{m}_1 \hat{m}_2 \hat{m}_3^2 - \frac{1}{2688} \left( \hat{m}_1^8 + \hat{m}_2^8 + \hat{m}_3^8 \right). \quad (33)$$

Note that we define $\hat{m}_i = \frac{64m_i}{A} = 64\hat{m}_i$ and $F_{N_f = 3}^{N_f = 3}$ in the above expression is related to $F_k^{N_f = 3}$ in \[23\] as $F_{N_f = 3}^{N_f = 3}(here) = -\frac{2}{A^{16}}F_3^{N_f = 3}$, $A(here) = \Lambda_3$.

In this notation, we have succeeded to reproduce all known results $F_{k = 2, 3, 4}^{N_f = 3}$ for $k = 2, 3, 4$ from the coefficients of $A^4, A^6, A^8$ in \[23\], respectively.

For example, we take the third term of \[32\] and integrating with respect to $a$ twice, we get the coefficient of $A^4$ term as

$$F_{N_f = 3}^{N_f = 3} = \frac{1}{(64\sqrt{2})^{12}} \cdot \frac{1}{32} \left( (159792 - 1488a_0 + a_0^2 + 2a_1) + (2a_0 - 1440) (\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) \right)$$

$$- (\hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4) + 96\hat{m}_1 \hat{m}_2 \hat{m}_3 \right]$$

$$= \frac{1}{33554432} + \frac{1}{4096} \left( \hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2 \right) + \frac{1}{32} \hat{m}_1 \hat{m}_2 \hat{m}_3 + \frac{1}{48} \left( \hat{m}_1^4 + \hat{m}_2^4 + \hat{m}_3^4 \right)$$

which corresponds to $F_{N_f = 3}^{N_f = 3}$ in \[31\] correctly.

### III. COMPARISON WITH NEKRASOV PARTITION FUNCTION

As is well known, the SW prepotential is also obtained by Instanton counting by Nekrasov \[18, 19\]. Our results are of course in full agreement with those obtained that way. We will see this briefly below.

$$N_f = 3$$

The instanton partition function is calculated as a sum over all possible Young tableaus parametrized as $Y = (\lambda_1 \geq \lambda_2 \geq \cdots )$, where $\lambda_\ell$ is the height of the $\ell$-th column. In the case
of $D = 4, \mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 3$, the instanton partition function is

$$Z_{\text{inst}}^{N_f = 3} = \sum_{(Y_1,Y_2)} \Lambda_3^{3Y_1} Z_{\text{vector}}(\vec{a}, \vec{Y}) Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_1) Z_{\text{antifund}}(\vec{a}, \vec{Y}, \mu_2) Z_{\text{fund}}(\vec{a}, \vec{Y}, -\mu_3).$$

(34)

Here

$$Z_{\text{vector}}(a, \vec{Y}) = \prod_{i,j=1,2} \prod_{s \in Y_i} (2a\delta_{ij} - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_j}(s) + 1))^{-1} \times \prod_{t \in Y_j} (-2a\delta_{ij} + \epsilon_1 L_{Y_j}(t) - \epsilon_2 (A_{Y_j}(t) + 1) + \epsilon_+)^{-1},$$

$$Z_{\text{fund}}(a, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a\delta_{i} + \epsilon_1 (l - 1) + \epsilon_2 (m - 1) - \mu + \epsilon_+),$$

$$Z_{\text{antifund}}(a, \vec{Y}, \mu) = \prod_{i=1,2} \prod_{s \in Y_i} (a\delta_{i} + \epsilon_1 (l - 1) + \epsilon_2 (m - 1) + \mu),$$

where we define $\epsilon_+ = \epsilon_1 + \epsilon_2$, $\delta_1 = +1$, $\delta_2 = -1$ and

$$\delta_{ij} = \begin{cases} 0 & \text{for } i = j, \\ 1 & \text{for } i = 1 \text{ and } j = 2, \\ -1 & \text{for } i = 2 \text{ and } j = 1. \end{cases}$$

(36)

For a box $s$ at the coordinate $(\ell, m)$, the leg-length $L_{Y}(s) = \lambda'_m - \ell$ and the arm-length $A_{Y}(s) = \lambda'_m - m$ where $\lambda'_m$ is the length of the $m$-th row. The prepotential can be obtained in the limit where the deformation parameters go to zero (with a fixed ratio $\epsilon_1/\epsilon_2$):

$$\mathcal{F}_{\text{inst}} = \lim_{\epsilon_1,\epsilon_2 \to 0} \left( -\epsilon_1 \epsilon_2 \right) \log Z_{\text{inst}}.$$ 

(37)

Therefore, $\mathcal{F}_{\text{inst}}^{N_f = 3}$ is expanded as

$$\mathcal{F}_{\text{inst}}^{N_f = 3} = \left( \frac{1}{33554432} + \frac{\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2}{4096} + \frac{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3}{32} \right) \Lambda^4 \frac{a^4}{\Lambda^4}$$

$$- \left( \frac{3}{33554432} (\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2) + \frac{\tilde{m}_1 \tilde{m}_2 \tilde{m}_3}{32768} + \frac{3}{8192} (\tilde{m}_1^2 \tilde{m}_2^2 + \tilde{m}_2^2 \tilde{m}_3^2 + \tilde{m}_3^2 \tilde{m}_1^2) \right) \Lambda^6 \frac{a^4}{\Lambda^4}$$

$$+ \left( \frac{5}{4503599627370496} + \frac{5}{134217728} (\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2) + \frac{7}{268435456} \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \right)$$

$$+ \left( \frac{25}{33554432} (\tilde{m}_1^4 + \tilde{m}_2^4 + \tilde{m}_3^4) + \frac{25}{16384} (\tilde{m}_1^2 \tilde{m}_2^2 + \tilde{m}_2^2 \tilde{m}_3^2 + \tilde{m}_3^2 \tilde{m}_1^2) \right)$$

$$+ \left( \frac{5}{196608} \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 (\tilde{m}_1^2 + \tilde{m}_2^2 + \tilde{m}_3^2) + \frac{5}{16384} \tilde{m}_1^2 \tilde{m}_2^2 \tilde{m}_3^2 \right) \Lambda^8 \frac{a^4}{\Lambda^4} + \mathcal{O}(\Lambda^9).$$

(38)

---

4 We use the notation used in [26], where the terms $\epsilon_1 L_{Y_j}(t)$ and $-\epsilon_2 A_{Y_j}(t)$ in the second line of eq. (A.3) in [26] should read $\epsilon_1 L_{Y_j}(t)$ and $-\epsilon_2 A_{Y_j}(t)$, respectively, and are corrected in [35].
Note that we have defined \( \tilde{m}_i \equiv \frac{A}{\sqrt{2}} \mu_i \) \((i = 1, 2, 3)\) and \( \Lambda = \frac{A}{4\sqrt{2}} \).

\( N_f = 2 \)

Next, let us consider the case of \( N_f = 2 \). We take a limit where \( \mu_2 \to \infty \) while keeping \( \mu_2 \Lambda_3 \equiv \Lambda_2^2 \) fixed. In this limit, the partition function becomes:

\[
Z_{\text{inst}}^{N_f=2} = \sum_{(Y_1, Y_2)} \Lambda_2^{2\tilde{Y}} Z_{\text{vector}}(a, \tilde{Y})Z_{\text{antifund}}(a, \tilde{Y}, \mu_1)Z_{\text{fund}}(a, \tilde{Y}, -\mu_3). 
\]  

Therefore, the expansion of \( \mathcal{F}_{N_f=2} \) is

\[
\mathcal{F}_{N_f=2} = \frac{1}{16} \Lambda^2 + \left( \frac{1}{4096} + \frac{1}{32} \tilde{m}_1 \tilde{m}_2 \right) \frac{\Lambda^4}{a^2} - \frac{3}{8192} \left( \tilde{m}_1^2 + \tilde{m}_2^2 \right) \frac{\Lambda^6}{a^4} + \left( \frac{5}{134217728} + \frac{5}{196608} \tilde{m}_1 \tilde{m}_2 + \frac{5}{16384} \tilde{m}_1^2 \tilde{m}_2 \right) \frac{\Lambda^8}{a^6} + \mathcal{O}(\Lambda^9), 
\]  

where \( \tilde{m}_1 \equiv \frac{A}{\sqrt{2}} \mu_1, \tilde{m}_2 \equiv \frac{A}{\sqrt{2}} \mu_3 \) and \( \Lambda \equiv \frac{A}{2\sqrt{2}} \).

Comparing (38) with (31)\(\sim\)33 and (40) with (16)\(\sim\)18, they perfectly agree except for \( \sum_i m_i^{2k} \). The first term of eq[40]: \( \frac{\Lambda^2}{16} \) is known to be required from U(1) factor, but this is beyond the scope of our study and will not be discussed further here.

As we can see above, we need to compute to the 8th order term in the instanton expansion in order to derive all terms of our 4th order \( \mathcal{F}_4 \) in our method. This is because Nekrasov’s instanton partition function is a \( \Lambda \)-expansion while our expansion is in terms of \( a \), where in the latter the coefficients are homogeneous polynomials of \( \Lambda \) and \( m_i \)’s, as is always the case in such B-model-like approaches.

IV. CONCLUSIONS

To summarize, we have confirmed that for both cases of \( N_f = 2 \) and 3, \( q \) allows an expansion in terms of \( A \equiv \frac{\Lambda}{8\sqrt{2}a} \) \((N_f = 2)\) or \( \frac{\Lambda}{64\sqrt{2}a} \) \((N_f = 3)\) in which the coefficients are integer-coefficient polynomials of the moonshine coefficients. As we mentioned at the beginning of this paper, this fact suggests an unknown relationship between the vertex operator algebra CFT and the Liouville CFT. The details of that relationship would be worth studying and will be discussed elsewhere.
On the other hand, the method to compute the instanton expansion of the SW prepotential developed in [10] and this paper is efficient and simple. We have not only re-derived all the known expansions of the $N_f = 0, \ldots, 3$ prepotentials but found some typos in the literature. For instance, in the $N_f = 3$ prepotential presented in section 6 of [24]:

1. $-a^2(m_1^2 m_2^2 + \cdots)$ in \( \mathcal{F}^{(2)} \) should read $-3a^2(m_1^2 m_2^2 + \cdots)$. (2) In \{ \cdots \} of \( \mathcal{F}^{(5)} \), the term $-6600a^6m_1^2 m_2^2 m_3^2$ is missing. (3) $330(m_1^4 m_2^2 + \cdots)$ in \{ \cdots \} of \( \mathcal{F}^{(6)} \) should read $3300(m_1^4 m_2^2 + \cdots)$. (4) Also in \( \mathcal{F}^{(6)} \), $2310(m_1^2 m_2^6 + \cdots)$ should be $2310(m_1^6 m_2^6 + \cdots)$. It is also easy and straight-forward to derive higher order terms by our method, and will also be applied to E-string theory.

There are several possible directions to extend the present study. One direction is to consider a theory with $SU(3)$ or a higher rank gauge group instead of $SU(2)$. This would generally lead to a high genus SW curve, which would not fit with the elliptic functions involving the Monstrous moonshine. However, it should be mentioned that ref. [35] considers special loci of $SU(3)$ Coulomb branch that parametrize a family of elliptic curves.

The Hauptmoduln for various rational elliptic surfaces (see e.g. [34]) could also be similarly used as an analogue of the $j$-function to compute prepotentials of the corresponding SW theories. For example, the $N_f = 2$ theory we considered in this paper has the congruence subgroup $\Gamma(2)$ as the monodromy group, whose corresponding Hauptmodul is known to be the modular $\lambda$-function. Thus, instead of the $j$-function, we could use it to expand $q$ or the prepotential in some variable, in which the coefficients would now be polynomials of the coefficients of the McKay-Thompson series, and hence of the characters of the Monster.

Finally, let us comment on other moonshine phenomena [16, 36] than the Monstrous moonshine. Since we used the special properties of the modular $j$-function to compute the instanton partition functions, we were naturally led to think about the Monstrous moonshine. On the other hand, in other moonshine phenomena, the Mathieu moonshine for example, the expansion of the elliptic genus of K3 in terms of $\mathcal{N} = 4$ superconformal characters, or equivalently the $q$-expansion of a certain mock modular form, gives coefficients which are related to dimensions of representations of $M_{24}$, the Mathieu group. As already mentioned in Introduction, this connection between the mock modular form and the (twisted) SUSY gauge theory is very different from that between the $j$-function and SUSY gauge theories in our discussion.
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