Long-range Correlation in Sheared Granular Fluids
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Abstract.
The spatial correlation function of the momentum density in the three-dimensional dilute sheared granular fluids is theoretically investigated. The existence of the long-range correlation is verified through both analytic calculation and numerical simulation.

Keywords: granular fluids, shear flow, long-range correlation
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INTRODUCTION
The time and the spatial correlations play important roles in non-equilibrium statistical physics [1-16]. The behaviors of the correlation functions in ordinary fluids are well understood. It is known that there exist long-time tails in the time correlation functions in fluids at equilibrium [1, 2, 3, 4]. In addition, long-range correlations exist in the spatial correlation functions in non-equilibrium ordinary fluids [5, 6, 7, 8].

On the other hand, interest in the correlations of granular fluids is rapidly growing. In the case of the freely cooling state, it is confirmed that long-time tails for the time-correlation of the velocity and the shear stress exist but the time-correlation function for the heat flux decays exponentially [10]. In addition, the spatial correlation functions are analytically calculated by using fluctuating hydrodynamics and the theoretical results are verified by numerical simulations [11]. See also the studies on the long-time tails in granular flows [12, 13]. Indeed, Kumaran predicted that the correlation function satisfies $t^{-3d/2}$ with the spatial dimension $d$ [14], while we obtained crossover from $t^{-d/2}$ to $t^{-(d+2)/2}$ for the velocity auto-correlation function of nearly elastic granular gases [15]. However, there is no corresponding theoretical or numerical argument on the spatial correlation functions in sheared granular fluids.

In this paper, thus, we investigate the spatial correlation functions in sheared granular fluids. In section 2.1, we will explain the set up. In section 2.2, we will present the analytic results for the spatial correlation of the momentum density. In section 2.3, the validity of the analytic results will be tested by our numerical simulation. Finally, we will discuss and conclude our results in section 3.

RESULT
Set up

Let us consider a three-dimensional dilute system consisting of $N$ identical smooth and hard spherical particles with mass $m$ and diameter $\sigma$ in the volume $V$. The position and the velocity of the $i$-th particle at time $t$ are denoted by $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$, respectively. The particles collide instantaneously with each other with a restitution constant $e$ which is less than unity for granular particles. When the particle $i$ with velocity $\mathbf{v}_i$ collides with the particle $j$ with $\mathbf{v}_j$, the post-collisional velocities $\mathbf{v}'_i$ and $\mathbf{v}'_j$ are respectively given by $\mathbf{v}'_i = \mathbf{v}_i - \frac{1}{2}(1+e)(\mathbf{n} \cdot \mathbf{v}_{ij})\mathbf{n}$ and $\mathbf{v}'_j = \mathbf{v}_j + \frac{1}{2}(1+e)(\mathbf{n} \cdot \mathbf{v}_{ij})\mathbf{n}$, where $\mathbf{n}$ is the unit vector parallel to the relative position of the two colliding particles at contact, and $\mathbf{v}_{ij} \equiv \mathbf{v}_i - \mathbf{v}_j$.

Let us assume that the uniform shear flow is stable and its velocity profile is given by $c_\alpha(r) = \dot{\gamma} \delta_{\alpha, x}$, where the Greek suffix $\alpha$ denotes the Cartesian component, and $\dot{\gamma}$ is the shear rate.

Let us consider the spatial correlation function of the momentum density defined by

$$C_{pp}(r) \equiv \frac{d}{V} \left\langle \left[ \mathbf{p}(\mathbf{r} + \mathbf{r}', t) - \rho(\mathbf{r} + \mathbf{r}', t)\mathbf{c}(\mathbf{r} + \mathbf{r}') \right] \cdot \left[ \mathbf{p}(\mathbf{r}', t) - \rho(\mathbf{r}', t)\mathbf{c}(\mathbf{r}') \right] \right\rangle, \quad (1)$$
where \( p(r,t) \equiv m \sum \delta (r - r_i(t)) \) and \( \rho (r,t) \equiv m \sum \delta (r - r_i(t)) \) are the momentum density and the density, respectively.

**Theoretical Analysis**

In order to obtain the analytic expression of \( C_{\rho\rho}(r) \), we assume that the time evolution of the hydrodynamic fields is described by fluctuating hydrodynamics \([18]\):

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \quad (2) \\
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{P}/\rho &= 0, \quad (3) \\
\partial_t T + \mathbf{u} \cdot \nabla T + 2m(\Pi : \nabla \mathbf{u} + \nabla \mathbf{q})/(3\rho) &= -\zeta T, \quad (4)
\end{align*}
\]

where \( \mathbf{u}(r,t) \equiv \mathbf{p}(r,t)/\rho(r,t) \) and \( T(r,t) \) are the velocity and the temperature, respectively. The heat flux \( \mathbf{q} \) and the pressure tensor \( \Pi_{ij} \) consist of two parts as 

\[
\mathbf{q} = q^s + q^k \quad \text{and} \quad \Pi_{ij} = \Pi^s_{ij} + \Pi^k_{ij}. \tag{15}
\]

Here, \( q^s \) and \( \Pi^s_{ij} \) represent systematic parts as 

\[
q^s = -k \nabla T - \mu \mathbf{v} \rho/m, \quad \text{and} \quad \Pi^s_{ij} = \rho T \delta_{ij} - \eta [\nabla u_j + \nabla u_i - (2\eta/3) \delta_{ij} \nabla \cdot \mathbf{u}], \tag{17}
\]

respectively, where \( \delta_{ij} \) is Kronecker delta. Note that the bulk viscosity disappears in fluids of dilute spherical particles. \( \zeta \), \( \kappa \), \( \mu \), and \( \eta \) are the cooling rate, the heat conductivity, the transport coefficient associated with the density gradient, and the viscosity, respectively. Here, \( \mu \) has a finite value when \( e \) is less than unity \([17]\). \( q^k \) and \( \Pi^k_{ij} \) are the random parts of the heat flux and the pressure tensor, respectively. \( q^k \) and \( \Pi^k_{ij} \) are respectively written as 

\[
q_i = \sqrt{\mathbf{f}_{ij} f_{ij}}, \quad \text{and} \quad \Pi^k_{ij} = \sqrt{\eta} f_{ij}, \tag{18}
\]

where \( f_i \) and \( f_{ij} \) satisfy 

\[
\langle f_i^* \rangle = 0, \quad \langle f_i f_j \rangle = 0, \quad \langle f_i^* \mathbf{r} f_j^* (\mathbf{r}', t') \rangle = 2\delta_{ij} \delta (\mathbf{r} - \mathbf{r'}) \delta (t - t'), \tag{19}
\]

and 

\[
\langle f_i^* \mathbf{r} f_j^* (\mathbf{r}', t') \rangle = 2\Delta_{ijkl} \delta (\mathbf{r} - \mathbf{r'}) \delta (t - t') \tag{20}
\]

with \( \Delta_{ijkl} = \delta_{il} \delta_{jk} + \delta_{ij} \delta_{kl} - 2 \delta_{ij} \delta_{kl} / 3 \).

Let the viscosity \( \eta \), the heat conductivity \( \kappa \), and the transport coefficient associated with the density gradient \( \mu \) be non-dimensionalized as

\[
\eta = \eta_0 \eta^*, \quad \zeta = \rho T \zeta^*/\eta_0, \quad \kappa = \kappa_0 \kappa^*, \quad \mu = mT \kappa_0 \mu^*/\rho, \tag{5}
\]

where \( \eta^*, \zeta^*, \kappa^*, \) and \( \mu^* \) are constants which depend only on \( e \) in dilute cases \([17]\). We note \( \eta^* \approx 1 \) and \( \zeta^* \approx 5\epsilon/12 \) with \( \epsilon = 1 - e^2 \) in the limit of small \( e \). Here, \( \eta_0 \) and \( \kappa_0 \) are the viscosity and the heat conductivity in the dilute elastic hard-core gas given by

\[
\eta_0 = a \sqrt{T}, \quad \kappa_0 = 15a \sqrt{T}/(4m), \tag{6}
\]

respectively. The explicit form of the constant \( a \) is given by 

\[
a = \sqrt{m/\pi/(16\sigma^2)}. \tag{7}
\]

Here, we introduce the average density \( \rho_0 \equiv \langle \rho(r,t) \rangle \), and the average temperature \( T_0 \equiv \langle T(r,t) \rangle \). These averages satisfy \( \rho_0 = mN/V \), and \( T_0 = 2m^2 a^2 \eta^* \rho_0^2 / (3\rho_0^3 \kappa^*) \).

The method to calculate the spatial correlation function is parallel to that for the sheared ordinary fluids \([8]\). We, thus, show only the outline of the method in this paper. First, we introduce the fluctuations of the hydrodynamic fields 

\[
\delta \rho(r,t) \equiv \rho(r,t) - \rho_0, \quad \delta T(r,t) \equiv T(r,t) - T_0, \quad \text{and} \quad \delta \mathbf{u}(r,t) \equiv \mathbf{u}(r,t) - \mathbf{c}(r,t).
\]

Then, \( C_{\rho\rho}(r) \) can be approximated by

\[
C_{\rho\rho}(r) = \int \frac{d\mathbf{k}'}{V} \left( \langle \rho_0 + \delta \rho(r + \mathbf{r}', t) \rangle \delta \mathbf{u}(r + \mathbf{r}', t) \right) \cdot \left( \langle \rho_0 + \delta \rho(r', t) \rangle \delta \mathbf{u}(r', t) \right) \tag{8}
\]

where we have ignored nonlinear terms of the fluctuations. We have also introduced the correlation function \( \tilde{\Psi}_{uu}(\mathbf{k}) \), which satisfies

\[
\langle \delta \mathbf{u}_k(t) \cdot \delta \mathbf{u}_{k'}(t) \rangle = (2\pi)^3 \delta^3 (\mathbf{k} + \mathbf{k'}) \tilde{\Psi}_{uu}(\mathbf{k}). \tag{9}
\]

Let us introduce the vector \( \mathbf{z}(r,t) \) and its Fourier transform of the fluctuations as

\[
\tilde{\mathbf{z}}^* (\mathbf{k}, t) = (\delta n_k(t), \delta T_k(t), \delta u_k^{(1)}(t), \delta u_k^{(2)}(t), \delta u_k^{(3)}(t)), \tag{10}
\]
where we decompose $\delta u_k(t)$ as $\delta u_k(t) = \delta u_k^{(1)}(t)e^{(1)} + \delta u_k^{(2)}(t)e^{(2)} + \delta u_k^{(3)}(t)e^{(3)}$ with $e^{(1)} \equiv k/k$, $e^{(2)} \equiv \{e_y - (e^{(1)} \cdot e_y)e^{(1)}\} / N$, $e^{(3)} \equiv e^{(1)} \times e^{(2)}$, and $\delta u_k^{(i)}(t) = \delta u_k(t) \cdot e^{(i)}$. Here, $e_y = (0, 1, 0)$, and $N = |e_y - (e^{(1)} \cdot e_y)e^{(1)}|$. Then, the linearized time evolution equation for $\tilde{z}(k, t)$ is obtained from eqs. (2), (3), (4), and (10) as
\begin{equation}
(\partial_t - \gamma k_z \partial_z)\tilde{z}_a(k, t) + L_{a\beta} \tilde{z}_\beta(k, t) = \tilde{R}_a(k, t).
\end{equation}
Here, the matrix $L_{a\beta}$ is given by
\begin{equation}
L_{a\beta} = ikL^{(1)}_{a\beta} + k^2L^{(2)}_{a\beta} + \gamma L^{(3)}_{a\beta} + \gamma kL^{(4)}_{a\beta} + \gamma^2 L^{(5)}_{a\beta},
\end{equation}
where $L^{(i)}_{a\beta}$ is a matrix depending on $\zeta$, $\kappa$, $\mu$, and $\eta$. $\tilde{R}_a(k, t)$ is a random vector which is a function of $\bf{q}_R$ and $\Pi^R_j$. See the elements of the matrix $L^{(i)}_{a\beta}$ and the vector $\tilde{R}_a(k, t)$ in Appendix A.

The solution of eq. (11) is expressed as
\begin{equation}
\tilde{z}_a(k, t) = \sum_{i} \int_{-\infty}^{t} ds \psi^{(i)}_a(k, t - s) F^{(i)}(k(\gamma(s - t)), s),
\end{equation}
where
\begin{equation}
\psi^{(i)}_a(k, t) \equiv \psi^{(i)}_a(k)e^{-\int_0^t ds \lambda^{(i)}(k(\gamma))}, \quad F^{(i)}(k, t) \equiv \phi^{(i)}_\beta(k) \tilde{R}_a(k, t)
\end{equation}
with $k(\tau) \equiv (k_x - \gamma \tau k_z, k_z)$. Here, we have introduced the linearly independent eigenvectors $\psi^{(i)}_a(k)$, the associated biorthogonal vectors $\phi^{(i)}_\alpha(k)$, and the eigenvalues $\lambda^{(i)}(k)$ satisfying
\begin{equation}
(-\gamma k_z \partial_z + L_{a\beta}) \psi^{(i)}_a(k) = \lambda^{(i)}(k) \psi^{(i)}_a(k),
\end{equation}
and $\psi^{(i)}_a(k)\phi^{(j)}_\alpha(k) = \delta_{ij}$.

In order to obtain an analytic expression of $\psi^{(i)}_a(k)$, $\phi^{(i)}_\alpha(k)$, and $\lambda^{(i)}(k)$, we assume [8]
\begin{equation}
\gamma \sim O(\eta^2 / \rho).
\end{equation}
Here, we note that eq. (16) is rewritten as $t_c^{-2} \sim O(k^2)$, where $l_c$ is the length scale defined by
\begin{equation}
l_c = \sqrt{\frac{2\eta}{mm_{\eta}m_\gamma}}.
\end{equation}
From eqs. (5), (6) and (7), the length scale $l_c$ becomes
\begin{equation}
l_c = \frac{5}{96} \nu^{-1} \left( \frac{2\eta^3}{\xi^*} \right)^{1/4} \sigma,
\end{equation}
where $\nu$ is the volume fraction. We, thus, find that $l_c$ does not depend on $\gamma$, but depends on the volume fraction $\nu$ and the restitution constant $e$ because $\eta^*$ and $\xi^*$ depend only on $e$. In the limit of small $e$, we obtain $l_c \propto \sigma \nu^{-1} e^{-1/4}$.

Substituting the solution (13) for small $k$ into eq. (8), we obtain an analytic expression of $C_{pp}(r)$ as
\begin{equation}
C_{pp}(r) = \frac{\rho_0 T_0}{l_c^2} \left\{ \Delta_1(r/l_c) + \Delta_2(r/l_c) + \Delta_3(r/l_c) \right\},
\end{equation}
where
\begin{align}
\Delta_1(\tilde{r}) &= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-ik \cdot \tilde{r}} \int_0^\infty dt k k(t) \left[ a_1 - a_2 \cos(c_1 k \alpha(t)) \right] \rho_0 T_0 e^{-\beta(k^2 + r^2 k^2 + r^2 k^2/3)}, \\
\Delta_2(\tilde{r}) &= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-ik \cdot \tilde{r}} \int_0^\infty dt \frac{k(t)^4}{k^2} e^{-\beta(k^2 + r^2 k^2 + r^2 k^2/3)}, \\
\Delta_3(\tilde{r}) &= \int \frac{d\mathbf{k}}{(2\pi)^3} e^{-ik \cdot \tilde{r}} \int_0^\infty dt \frac{k(t)^2}{k^2} e^{-\beta(k^2 + r^2 k^2 + r^2 k^2/3)} \times [M^2(k(t))^2 k^2 - 2M^2(k(t)) M^2(k) k_0(k(t)) + M^2(k)^2 k(t)^2 + k(t)^2],
\end{align}
where \( \mathbf{\hat{r}} = \mathbf{r}/l_c \). Here, \( M(\mathbf{k}) = -kk_c/(k_\perp \sqrt{k_\perp^2 - k_z^2})\tan^{-1}(k_z/\sqrt{k_\perp^2 - k_z^2}) \), \( a_1 = (2\eta^*/3 + \kappa^*/2)/\eta^* \), \( a_2 = (\kappa^*/2 - 2\eta^*/3)/\eta^* \), \( b = (2\eta^*/3 + \kappa^*/2 + 3\mu^*/4)/\eta^* \), and \( c_1 = 2\sqrt{3}v_0/\sqrt{3}\eta^* \) with \( v_0 = \rho_0T_0/(m\eta_0(T_0)) \). It should be noted that the expressions in eqs. (19) and (20) are the same as those for the sheared ordinary fluids by Lutsko and Duffy [8]. The difference between ours and theirs exists in the dependence on the restitution constant \( e \) through \( l_c \).

Let us explicitly demonstrate the existence of the long-range correlation in \( C_{pp}(\mathbf{r}) \). Let the angular average of any function \( f(\mathbf{r}) \) be denoted by \( \bar{f}(\mathbf{r}) \equiv \int d\Omega/(4\pi)f(\mathbf{r}) \). From the second equation in eq. (20), it is easy to show the asymptotic behavior

\[
\bar{\Delta}_2(\mathbf{\hat{r}}) \propto \mathbf{\hat{r}}^{-5/3}, \quad \mathbf{\hat{r}} \gg 1. \tag{21}
\]

Indeed, from the transformation of variables as \( k' = k\mathbf{\hat{r}} \) and \( s = t\mathbf{\hat{r}}^{-2/3} \), we obtain

\[
\bar{\Delta}_2(\mathbf{\hat{r}}) = \int \frac{dk}{(2\pi)^3} \frac{\sin(k\mathbf{\hat{r}})}{k\mathbf{\hat{r}}} \int_0^\infty dt \frac{(k^2 + 2tkk_x + \mathbf{r}^2k_y^2)^2}{k^2} e^{-(tk^2 + tk'k_y + \mathbf{r}^2k_y^2)/3} \]

\[
= \mathbf{\hat{r}}^{-5/3} \int \frac{dk'}{(2\pi)^3} \frac{\sin(k')}{k'} \int_0^\infty ds \frac{(\mathbf{r}^{-4/3}k_y^2 + 2\mathbf{r}^{-2/3}s^2k_yk_x' + s^2k_x'^2)^2}{k'^2} e^{-(\mathbf{r}^{-4/3}tk'^2 + \mathbf{r}^{-2/3}s^2k_x'k_y + s^2k_y^2)/3}. \tag{22}
\]

From this expression, we find that \( \bar{\Delta}_2(\mathbf{\hat{r}}) \) satisfies eq. (21) and the integrations in eq. (22) are reduced to a constant for \( \mathbf{\hat{r}} \gg 1 \). By using a parallel procedure to that for eq. (21), we find

\[
\bar{\Delta}_1(\mathbf{\hat{r}}) \propto \mathbf{\hat{r}}^{-11/3}, \quad \mathbf{\hat{r}} \gg 1, \tag{23}
\]

\[
\bar{\Delta}_3(\mathbf{\hat{r}}) \propto \mathbf{\hat{r}}^{-5/3}, \quad \mathbf{\hat{r}} \gg 1.
\]

Substituting eqs. (21) and (23) into eq. (19), we find the long-range correlation in \( \bar{C}_{pp}(r) \) as

\[
\bar{C}_{pp}(r) \propto \frac{\rho_0T_0}{l_c^3} \left( \frac{r}{l_c} \right)^{-5/3}, \quad r \gg l_c. \tag{24}
\]

Thus, we confirm that the threshold length \( l_c \) in eq. (17) plays an important role.

### Numerical Simulation

To verify our theoretical prediction, we perform simulation of three-dimensional hard spherical particles. In our simulation \( m, \sigma \) and the average temperature \( T_0 \) are set to be unity, and all quantities are converted to dimensionless forms, where the unit of time scale is \( \sigma/\sqrt{mT_0} \). We adopt the Lees-Edwards boundary condition [19]. The volume fraction \( \nu \) is \( \nu_c/8 \) with the closest packing fraction of particle \( \nu_c \). We use the parameters \( \gamma = 0.5 \) and \( e = 0.83 \) to keep the temperature unity. We examine the size of the system \( L = 28, 56, 112 \) in our simulation, which contain 4096, 32768, and 262144 particles, respectively.

Figure 1 shows the numerical result of \( \bar{C}_{pp}(r) \) for various system size \( L \). There is an apparent finite size effect, where \( \bar{C}_{pp}(r) \) decays faster than power-law function for \( r > 0.3L \). For the system with the largest size, we find the existence of a region where the spatial correlation function \( \bar{C}_{pp}(r) \) approximately satisfies \( r^{-5/3} \). We cannot confirm the existence of the region to obey \( r^{-5/3} \) in the wide range, but the results seem to be consistent with our theoretical result in eq. (24).

### CONCLUSION AND DISCUSSION

Let us discuss our results. From the condition \( l_c^{-2} \sim O(k^2) \), we could obtain the analytic expression of \( C_{pp}(r) \). This condition can be rewritten as \( l_c \sim O(L/(2\pi)) \). For the case of our numerical simulation, \( l_c \) is estimated as 1.6\( \sigma \). On the other hand, \( L/(2\pi) \) is estimated as 18\( \sigma \) in the largest system in our numerical simulation. Hence, it might be suspicious that the condition \( l_c \sim O(L/(2\pi)) \) is satisfied in our numerical simulation. However, the result of the simulation seems to be consistent with our analytic result. We need more careful consideration on the validity of our analytic method.
It should be noted that our method is not valid in the true hydrodynamic limit. Indeed, we have assumed eq. (16), which can be rewritten as $k \sigma \sim \varepsilon^{1/4}$. Thus, if $\varepsilon$ is finite, the applicable range of our analysis is limited for smaller systems. We may expect different features in the region of $k \sigma \ll \varepsilon^{1/4}$, which will be our future task.

The theoretical method we have used is applicable only to the dilute system. In general, sheared granular systems in experiments are not dilute ones. Hence, it is not clear that the long-range correlation predicted in this paper is experimentally observable. In order to theoretically understand the spatial correlation in such dense sheared granular systems, we must improve our theoretical method, which will be also our future work.

In conclusion, we have analytically calculated the behavior of $C_{pp}(r)$ in three-dimensional dilute sheared granular fluids. Based on fluctuating hydrodynamics, we find that there is a long-range correlation in $C_{pp}(r)$. The results are verified by numerical simulations.

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APPENDIX : THE ELEMENTS OF THE MATRIX $L_{\alpha\beta}^{(i)}$ AND THE VECTOR $R_\alpha(k,t)$

In this appendix, we explicitly show the elements of the matrix $L_{\alpha\beta}^{(i)}$ and the vector $R_\alpha(k,t)$. Here, we non-dimensionalize the variables by the mass $m$, the characteristic time $\nu_0 = \rho_0 T_0/\langle m \eta_0(T_0) \rangle$, and the characteristic length $l_0 = 2\sqrt{T_0/m/\nu_0}$. The matrix $L^{(i)}$ and the vector $R_\alpha(k,t)$ are explicitly given by

$$L_{\alpha\beta}^{(i)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2/3} & 0 & 0 \\ 1 & \sqrt{2/3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
\begin{align}
L^{(2)}_{\alpha \beta} &= \begin{bmatrix}
5\sqrt{3}\mu^*/(4\sqrt{2}) & 5\lambda^*/4 & 0 & 0 & 0 \\
0 & 0 & 2\eta^*/3 & 0 & 0 \\
0 & 0 & 0 & \eta^*/2 & 0 \\
0 & 0 & 0 & 0 & \eta^*/2
\end{bmatrix}, \\
L^{(3)}_{\alpha \beta} &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_xk_y/k^2 & 2k_xk_y/k^2 & 0 \\
0 & 0 & -k_x/k_\perp & -k_y/k_\perp & 0 \\
0 & 0 & -k_xk_y/k_\perp & -k_z/k & 0
\end{bmatrix}, \tag{26}
\end{align}

and
\begin{align}
\vec{R}_1 &= 0, \\
\vec{R}_2 &= -\sqrt{3}/2\alpha_hk_\perp f_i^h - \sqrt{3}2\alpha_s f_{ij}^s/\nu_0, \\
\vec{R}_{2+l} &= \alpha_s i \epsilon_i^{(l)} k_f f_{ij}^s, \tag{27}
\end{align}

where \( k_\perp^2 = k^2 - k_z^2 \), \( \alpha_h = \{ m\eta^*/(\rho_0 l_0^3) \}^{1/2} \), and \( \alpha_s = \{ 5m\kappa^*/(3\rho_0 l_0^3) \} \). \( f_i^h \) and \( f_{ij}^s \) are the Fourier transforms of \( f_i^h \) and \( f_{ij}^s \), respectively. We do not describe the explicit form of \( L^{(4)} \) and \( L^{(5)} \) in eq. (12) because they affect only the higher order corrections in eq. (19).

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