HARMONIC UNIVALENT FUNCTIONS DEFINED BY POST
QUANTUM CALCULUS OPERATORS

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ABSTRACT. We study a family of harmonic univalent functions in the open unit disc
deﬁned by using post quantum calculus operators. We ﬁrst obtained a coefﬁcient
characterization of these functions. Using this, coefﬁcients estimates, distortion and covering
theorems were also obtained. The extreme points of the family and a radius result were
also obtained. The results obtained include several known results as special cases.

1. Introduction

Let \( A \) be the class of functions \( f \) that are analytic in the open unit disc \( D := \{ z : |z| < 1 \} \)
with the normalization \( f(0) = f'(0) - 1 = 0 \). A function \( f \in A \) can be expressed in the
form
\[
(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in D.
\]

The theory of \((p, q)\)-calculus (or post quantum calculus) operators are used in various
areas of science and also in the geometric function theory. Let \( 0 < q \leq p \leq 1 \). The
\((p, q)\)-bracket or twin-basic number \([k]_{p,q} \) for \( k \) is defined by
\[
[k]_{p,q} = \frac{p^k - q^k}{p - q} \quad (q \neq p), \quad \text{and} \quad [k]_{p,p} = kp^{k-1}.
\]

Notice that \( \lim_{q \to p} [k]_{p,q} = [k]_{p,p} \). For \( 0 < q \leq 1 \), \( q \)-bracket \([k]_q \) for \( k = 0, 1, 2, \cdots \) is given
by
\[
[k]_q = [k]_{1,q} = \frac{1 - q^k}{1 - q} \quad (q \neq 1), \quad \text{and} \quad [k]_1 = [k]_{1,1} = k.
\]
The \((p, q)\)-derivative operator \( D_{p,q} \) of a function \( f \in A \) is given by
\[
(1.2) \quad D_{p,q}f(z) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1}.
\]

For a function \( f \in A \), it can be easily seen that
\[
(1.3) \quad D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad (p \neq q, z \neq 0),
\]

\((D_{p,q}f)(0) = 1 \) and \((D_{p,p}f)(z) = f'(z) \). For deﬁnitions and properties of \((p, q)\)-calculus,
one may refer to \([7]\). The \((1, q)\)-derivative operator \( D_{1,q} \) is known as the \( q \)-derivative
operator and is denoted by \( D_q \); for \( z \neq 0 \), it satisﬁes
\[
(1.4) \quad (D_qf)(z) = \frac{f(z) - f(qz)}{(1 - q)z}.
\]

For deﬁnitions and properties of \( q \)-derivative operator, one may refer to \([8,10,13]\).
For a function $h$ analytic in $\mathbb{D}$ and an integer $m \geq 0$, we define the $(p, q)$-Salagean differential operator $L_{p,q}^m$, using $(p, q)$-derivative operator, by

$$L_{p,q}^0 h(z) = h(z) \quad \text{and} \quad L_{p,q}^m h(z) = zD_{p,q}(L_{p,q}^{m-1}(h(z)).$$

For analytic function $g(z) = \sum_{k=1}^{\infty} b_k z^k$, we have

$$L_{p,q}^m g(z) = \sum_{k=1}^{\infty} [k]^m_{p,q} b_k z^k.$$

In particular, for $h \in \mathcal{A}$ with $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, we have

$$L_{p,q}^m h(z) = z + \sum_{k=2}^{\infty} [k]^m_{p,q} a_k z^k.$$

Let $\mathcal{H}$ be the family of complex-valued harmonic functions $f = h + \overline{g}$ defined in $\mathbb{D}$, where $h$ and $g$ has the following power series expansion

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k.$$

Note that $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second dilatation $w$ of $f$ satisfies the condition $|g'(z)/h'(z)| < 1$ in $\mathbb{D}$. Let $\mathcal{S}_\mathcal{H}$ be a subclass of functions $f$ in $\mathcal{H}$ that are sense-preserving and univalent in $\mathbb{D}$. Clunie and Sheil-Small studied the class $\mathcal{S}_\mathcal{H}$ in their remarkable paper [5]. For a survey or comprehensive study of the theory of harmonic univalent functions, one may refer to the papers [12][18]. We introduce and study a new subclass of harmonic univalent functions by using $(p, q)$-Salagean harmonic differential operator $L_{p,q}^m : \mathcal{H} \to \mathcal{H}$. For the functions in the newly introduced family, a coefficient characterization is obtained (Theorem 2.3). Using this, coefficients estimates (Corollary 2.4), distortion (Theorem 2.6) and covering (Corollary 2.7) theorems were also obtained. The extreme points of the family (Theorem 2.5) and a radius result (Theorem 2.8) were also obtained. The results obtained include several known results as special cases.

### 2. Main Results

We define the $(p, q)$-Salagean harmonic differential operator $L_{p,q}^m$ of a harmonic function $f = h + \overline{g} \in \mathcal{H}$ by

$$L_{p,q}^m f(z) = L_{p,q}^m h(z) + (-1)^m L_{p,q}^m g(z) = z + \sum_{k=2}^{\infty} [k]^m_{p,q} a_k z^k + (-1)^m \sum_{k=1}^{\infty} [k]^m_{p,q} b_k z^k.$$

This last expression is obtained by using (1.6) and (1.5) and is motivated by Salagean [17]. Recall that convolution (or the Hadamard product) of two complex-valued harmonic functions

$$f_1(z) = z + \sum_{k=2}^{\infty} a_{1k} z^k + \sum_{k=1}^{\infty} b_{1k} z^k \quad \text{and} \quad f_2(z) = z + \sum_{k=2}^{\infty} a_{2k} z^k + \sum_{k=1}^{\infty} b_{2k} z^k$$

is defined by

$$f_1(z) \ast f_2(z) = (f_1 \ast f_2)(z) = z + \sum_{k=2}^{\infty} a_{1k} a_{2k} z^k + \sum_{k=1}^{\infty} b_{1k} b_{2k} z^k, \quad z \in \mathbb{D}.$$

We now introduce a family of $(p, q)$-Salagean harmonic univalent functions by using convolution and the $(p, q)$-Salagean harmonic differential operator $L_{p,q}^m$. 

Definition 2.1. Suppose $i, j \in \{0, 1\}$. Let the function $\Phi_i, \Psi_j$ given by

\begin{equation}
(2.2) \quad \Phi_i(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k + (-1)^i \sum_{k=1}^{\infty} \mu_k z^k,
\end{equation}

\begin{equation}
(2.3) \quad \Psi_j(z) = z + \sum_{k=2}^{\infty} u_k z^k + (-1)^j \sum_{k=1}^{\infty} v_k z^k
\end{equation}

be harmonic in $\mathbb{D}$ with $\lambda_k > u_k \geq 0$ ($k \geq 2$) and $\mu_k > v_k \geq 0$ ($k \geq 1$). For $\alpha \in [0, 1)$, $0 < q \leq p \leq 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in \mathbb{D}$, let $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ denote the family of harmonic functions $f$ in $\mathcal{H}$ that satisfy the condition

\begin{equation}
(2.4) \quad \text{Re} \left\{ \frac{(L^m_{p,q}f \ast \Phi_i)(z)}{(L^n_{p,q}f \ast \Psi_j)(z)} \right\} > \alpha,
\end{equation}

where $L^m_{p,q}$ is defined by $(2.7)$.

Using $(2.1)$, $(2.2)$ and $(2.3)$, we obtain

\begin{equation}
(2.5) \quad (L^m_{p,q}f \ast \Phi_i)(z) = z + \sum_{k=2}^{\infty} \lambda_k [k]_{p,q} a_k z^k + (-1)^{m+i} \sum_{k=1}^{\infty} \mu_k [k]_{p,q} b_k z^k,
\end{equation}

and

\begin{equation}
(2.6) \quad (L^n_{p,q}f \ast \Psi_j)(z) = z + \sum_{k=2}^{\infty} u_k [k]_{p,q} a_k z^k + (-1)^{n+j} \sum_{k=1}^{\infty} v_k [k]_{p,q} b_k z^k.
\end{equation}

Definition 2.2. Let $\mathcal{T}_S H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ be the family of harmonic functions $f_m = h + g_m \in \mathcal{T}_S H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ such that $h$ and $g_m$ are of the form

\begin{equation}
(2.7) \quad h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad \text{and} \quad g_m(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1.
\end{equation}

The families of $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and $\mathcal{T}_S H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ include a variety of well-known subclasses of harmonic functions as well as many new ones. For example,

1. $\mathcal{S}_H(m, n, \alpha) \equiv \mathcal{S}_H(m, n, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $\mathcal{T}_S H(m, n, \alpha) \equiv \mathcal{T}_S H(m, n, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $[?].$
2. $\mathcal{S}_H'(\alpha) \equiv \mathcal{S}_H(1, 0, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $\mathcal{T}_S H'(\alpha) \equiv \mathcal{T}_S H(1, 0, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $[?].$
3. $\mathcal{K}_H(\alpha) \equiv \mathcal{K}_H(2, 1, \frac{z}{(1-z)^2} + \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $\mathcal{T}_K H(\alpha) \equiv \mathcal{T}_K H(2, 1, \frac{z}{(1-z)^2} + \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, 1, \alpha)$, $[?].$
4. $\mathcal{S}_{H_q}(\alpha) \equiv \mathcal{S}_H(1, 0, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, q, \alpha)$, $\mathcal{T}_S H_q(\alpha) \equiv \mathcal{T}_S H(1, 0, \frac{z}{(1-z)^2} - \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, q, \alpha)$, $[?].$
5. $\mathcal{K}_{H_q}(\alpha) \equiv \mathcal{K}_H(2, 1, \frac{z}{(1-z)^2} + \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, q, \alpha)$, $\mathcal{T}_K H_q(\alpha) \equiv \mathcal{T}_K H(2, 1, \frac{z}{(1-z)^2} + \frac{z}{(1-z)^2}, \frac{z}{1-z} + \frac{z}{1-z}, 1, q, \alpha)$.
6. $\mathcal{S}_H(\Phi_i, \Psi_j, \alpha) \equiv \mathcal{S}_H(0, 0, \Phi_i, \Psi_j, 1, 1, \alpha)$, $\mathcal{T}_S H(\Phi_i, \Psi_j, \alpha) \equiv \mathcal{T}_S H(0, 0, \Phi_i, \Psi_j, 1, 1, \alpha)$, $[?].$

We first prove coefficient conditions for the functions in $\mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ and $\mathcal{T}_S H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$.
Theorem 2.3. Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by \([1.7]\). Also, let
\[
\sum_{k=2}^{\infty} \left( \frac{\lambda_k[k]_p^m - \alpha u_k[k]_p^n}{1 - \alpha} a_k \right) + \sum_{k=1}^{\infty} \left( \frac{\mu_k[k]_p^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_p^n}{1 - \alpha} b_k \right) \leq 1,
\]
be a given \((p, q)\)-coefficient inequality for \( \alpha \in [0, 1) \), \( 0 < q \leq p \leq 1 \), \( m \in \mathbb{N} \), \( n \in \mathbb{N}_0 \), \( m > n \), \( \lambda_k > u_k \geq 0 \) \( (k \geq 2) \) and \( \mu_k > v_k \geq 0 \) \( (k \geq 1) \). Then a function
\[(i) \ f = h + \overline{g} \text{ given by } [1.7] \text{ is a sense-preserving harmonic univalent functions in } \mathbb{D} \text{ and } \ f \in S_H(m, n, \Phi_i, \Psi_j, p, q, \alpha) \text{ if the inequality in } (2.8) \text{ is satisfied.}
\]
\[(ii) \ f_m = h + \overline{g}_m \text{ given by } (2.7) \text{ is in the } TS_H(m, n, \Phi_i, \Psi_j, p, q, \alpha) \text{ if and only if the inequality in } (2.8) \text{ is satisfied.}
\]
Proof. (i). Using the techniques used in [16], it is a routine step to prove that \( f = h + \overline{g} \)
given by \([1.7]\) is sense-preserving and locally univalent in \( \mathbb{D} \). Using the fact \( \text{Re}(\omega) > \alpha \) if
and only if \( |1 - \alpha + \omega| \geq |1 + \alpha - \omega| \), it suffices to show that
\[
(2.9) \quad |1 - \alpha + \frac{\left( L_{p,q}^m f \ast \Phi_i(z) \right)}{(L_{p,q}^n f \ast \Psi_j(z))} - 1 + \alpha - \frac{\left( L_{p,q}^m f \ast \Phi_i(z) \right)}{(L_{p,q}^n f \ast \Psi_j(z))} | \geq 0
\]
In view of \([2.3]\) and \([2.6]\), left side of \([2.9]\) yields
\[
|\left( L_{p,q}^m f \ast \Phi_i(z) \right) + (1 - \alpha)(L_{p,q}^n f \ast \Psi_j(z)) - (1 + \alpha)(L_{p,q}^m f \ast \Psi_j(z)) | = |(2 - 2\alpha)z| - 2 \sum_{k=2}^{\infty} \left( \lambda_k[k]_p^m - \alpha u_k[k]_p^n \right) a_k |z|^k
\]
\[
+ (-1)^{m+i} \sum_{k=1}^{\infty} \left( \mu_k[k]_p^m - (-1)^{n+j-(m+i)} (1 - \alpha) v_k[k]_p^n \right) b_k |z|^k
\]
\[
- |\alpha z + \sum_{k=2}^{\infty} \left( \lambda_k[k]_p^m - (1 + \alpha) u_k[k]_p^n \right) a_k |z|^k
\]
\[
+ (-1)^{m+i} \sum_{k=1}^{\infty} \left( \mu_k[k]_p^m - (-1)^{n+j-(m+i)} (1 + \alpha) v_k[k]_p^n \right) b_k |z|^k
\]
\[
\geq (2 - 2\alpha) |z| - 2 \sum_{k=2}^{\infty} \left( \lambda_k[k]_p^m - \alpha u_k[k]_p^n \right) |a_k||z|^k
\]
\[
- \sum_{k=1}^{\infty} \left( \mu_k[k]_p^m - (-1)^{n+j-(m+i)} (1 - \alpha) v_k[k]_p^n \right) |b_k||z|^k
\]
\[
- \sum_{k=1}^{\infty} \left( \mu_k[k]_p^m - (-1)^{n+j-(m+i)} (1 + \alpha) v_k[k]_p^n \right) |b_k||z|^k
\]
\[
\geq (1 - \alpha) |z| \left[ 1 - \sum_{k=2}^{\infty} \frac{\lambda_k[k]_p^m - \alpha u_k[k]_p^n}{1 - \alpha} |a_k||z|^{k-1}
\]
\[
- \sum_{k=1}^{\infty} \frac{\mu_k[k]_p^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_p^n}{1 - \alpha} |b_k||z|^{k-1}
\]
\[
+ (1 - \alpha) |z| \left[ 1 - \left( \sum_{k=2}^{\infty} \frac{\lambda_k[k]_p^m - \alpha u_k[k]_p^n}{1 - \alpha} |a_k|
\right.
\]
\[
\left. + \sum_{k=1}^{\infty} \frac{\mu_k[k]_p^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_p^n}{1 - \alpha} |b_k| \right]
\].
This last expression is non-negative because of the condition given in (2.8). This completes the proof of part (i) of theorem.

(ii). Since $T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha) \subset \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$, the sufficient part of part (ii) follows from part (i).

In order to prove the necessary part of part (ii), we assume that $f_m \in T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$. We notice that

$$
\text{Re} \left\{ \frac{(L_{p,q} f \ast \Phi_i)(z)}{(L_{n,q} f \ast \Psi_j)(z)} - \alpha \right\}
= \text{Re} \left\{ \frac{(1 - \alpha)z - \sum_{k=2}^\infty (\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n) a_k z^k}{z - \sum_{k=2}^\infty u_k[k]_{p,q} a_k z^k + (-1)^{m+i+n+1} \sum_{k=1}^\infty v_k[k]_{p,q} b_k z^k} \right\}
= \text{Re} \left\{ \frac{(1 - \alpha)z - \sum_{k=2}^\infty (\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n) a_k z^k}{z - \sum_{k=2}^\infty u_k[k]_{p,q} a_k z^k + (-1)^{m+i+n+1} \sum_{k=1}^\infty v_k[k]_{p,q} b_k z^k} \right\}
\geq 1 - \sum_{k=2}^\infty u_k[k]_{p,q} a_k r^{k-1} - (-1)^{m+i+n+j} \sum_{k=1}^\infty v_k[k]_{p,q} b_k r^{k-1}
\geq 0,
$$

by (2.4). The above inequality must hold for all $z \in \mathbb{D}$. In particular, choosing the values of $z$ on the positive real axis and $z \to 1^-$, we obtain the required condition (2.8). This completes the proof of part (ii) of theorem.

The harmonic mappings

$$(2.10) \quad f(z) = z + \sum_{k=2}^\infty \frac{1 - \alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} x_k z^k + \sum_{k=1}^\infty \frac{1 - \alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_{p,q}^n} y_k z^k,$$

where $\sum_{k=2}^\infty |x_k| + \sum_{k=1}^\infty |y_k| = 1$, show that the coefficient bound given by (2.8) is sharp.

Theorem 2.3 also yields the following corollary.

**Corollary 2.4.** For $f_m = h + \overline{g}_m$ given by (2.7), we have

$$
|a_k| \leq \frac{1 - \alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n}, \quad k \geq 2 \quad \text{and} \quad |b_k| \leq \frac{1 - \alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_{p,q}^n}, \quad k \geq 1.
$$

The result is sharp for each $k$.

Using Theorem 2.3 (part ii), it is easily seen that the class $T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ is convex and closed with respect to the topology of locally uniform convergence so that the closed convex hulls of $T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ equals itself. The next theorem determines the extreme points of $T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$.

**Theorem 2.5.** Let $f_m = h + \overline{g}_m$ be given by (2.7). Then $f_m \in \text{clco} T \mathcal{S}_H(m, n, \Phi_i, \Psi_j, p, q, \alpha)$ if and only if $f_m(z) = \sum_{k=1}^\infty (x_k h_k(z) + y_k g_m(z))$, where

$$
h_1(z) = z, \quad h_k(z) = z - \frac{1 - \alpha}{\lambda_k[k]_{p,q}^m - \alpha u_k[k]_{p,q}^n} z^k, \quad (k \geq 2),
$$

and

$$
g_m(z) = z - \frac{1 - \alpha}{\mu_k[k]_{p,q}^m - (-1)^{n+j-(m+i)} \alpha v_k[k]_{p,q}^n} z^k, \quad (k \geq 1).
$$
Consequently, we obtain

\[ \frac{1 - \alpha}{\mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}}z^k, \quad (k \geq 1), \]

and \( \sum_{k=1}^{\infty} (x_k + y_k) = 1 \) where \( x_k \geq 0 \) and \( y_k \geq 0 \). In particular, the extreme points of \( \mathcal{T}_S_{m,n,\Phi_i,\Psi_j,p,q,\alpha} \) are \( \{h_k\} \) and \( \{g_{m_k}\} \).

Proof. For a function \( f_m \) of the form \( f_m(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z)) \), where \( \sum_{k=1}^{\infty} (x_k + y_k) = 1 \), we have

\[ f_m(z) = z - \sum_{k=2}^{\infty} \frac{1 - \alpha}{\lambda_k[k]_{p,q} - \alpha u_k[k]_{p,q}} x_k z^k + \sum_{k=1}^{\infty} \frac{1 - \alpha}{\mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}} y_k z^k. \]

Then \( f_m \in \text{clo} \mathcal{T}_S_{m,n,\Phi_i,\Psi_j,p,q,\alpha} \) because

\[ \sum_{k=2}^{\infty} \frac{1 - \alpha}{\lambda_k[k]_{p,q} - \alpha u_k[k]_{p,q}} x_k + \sum_{k=1}^{\infty} \frac{1 - \alpha}{\mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}} y_k = \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1. \]

Conversely, suppose \( f_m \in \text{clo} \mathcal{T}_S_{m,n,\Phi_i,\Psi_j,p,q,\alpha} \). Then

\[ |a_k| \leq \frac{1 - \alpha}{\lambda_k[k]_{p,q} - \alpha u_k[k]_{p,q}} \quad \text{and} \quad |b_k| \leq \frac{1 - \alpha}{\mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}}. \]

Set

\[ x_k = \frac{\lambda_k[k]_{p,q} - \alpha u_k[k]_{p,q}}{1 - \alpha} |a_k| \quad \text{and} \quad y_k = \frac{\mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q}}{1 - \alpha} |b_k|. \]

By Theorem 2.3 (ii), \( \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1 \). Therefore we define \( x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \geq 0 \). Consequently, we obtain \( f_m(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_{m_k}(z)) \) as required.

For functions in the class \( \mathcal{T}_S_{m,n,\Phi_i,\Psi_j,p,q,\alpha} \), the following theorem gives distortion bounds which in turns yields the covering result for this class.

**Theorem 2.6.** Let \( f_m \in \mathcal{T}_S_{m,n,\Phi_i,\Psi_j,p,q,\alpha} \), \( \gamma_k = \lambda_k[k]_{p,q} - \alpha u_k[k]_{p,q} \), \( \phi_k = \mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q} \), \( k \geq 2 \) and \( \phi_k = \mu_k[k]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_k[k]_{p,q} \), \( k \geq 1 \). If \( \{\gamma_k\} \) and \( \{\phi_k\} \) are non-decreasing sequences, then we have

\[ |f_m(z)| \leq (1 + |b_1|)|z| + \frac{1 - \alpha}{\beta} \left( 1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta} |b_1| \right) |z|^2 \quad \text{(2.11)} \]

and

\[ |f_m(z)| \geq (1 - |b_1|)|z| - \frac{1 - \alpha}{\beta} \left( 1 - \frac{\mu_1 - (-1)^{n+j-(m+i)}\alpha v_1}{\beta} |b_1| \right) |z|^2, \quad \text{(2.12)} \]

for all \( z \in \mathbb{D} \), where \( b_1 = f_m^{(0)} \) and

\[ \beta = \min\{\gamma_2, \phi_2\} = \min\{\lambda_2[2]_{p,q} - \alpha u_2[2]_{p,q}, \mu_2[2]_{p,q} - (-1)^{n+j-(m+i)}\alpha v_2[2]_{p,q}\}. \]
Proof. Let \( f_m \in \mathcal{T}_\mathcal{S}_H(m, n, \Phi, \Psi, p, q, \alpha) \). Taking the absolute value of \( f_m \), we obtain
\[
|f_m(z)| \leq (1 + |b_1|)|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^k
\]
\[
\leq (1 + |b_1|)|z| + \sum_{k=2}^{\infty} (|a_k| + |b_k|)|z|^2
\]
\[
\leq (1 + |b_1|)|z| + \frac{1 - \alpha}{\beta} \sum_{n=2}^{\infty} \left( \frac{\beta}{1 - \alpha} |a_k| + \frac{\beta}{1 - \alpha} |b_k| \right) |z|^2
\]
\[
\leq (1 + |b_1|)|z| + \frac{1 - \alpha}{\beta} \sum_{k=2}^{\infty} \left( \frac{\lambda_k[k]^m_p q - \alpha u[k]^n_p q}{1 - \alpha} |a_k| + \frac{\mu_k[k]^m_p q - (-1)^{n+j-(m+i)} \alpha v[k]^n_p q}{1 - \alpha} |b_k| \right) |z|^2
\]
\[
\leq (1 + |b_1|)|z| + \frac{1 - \alpha}{\beta} \left( 1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1]}{1 - \alpha} |b_1| \right) |z|^2
\]
This proves (2.11). The proof of (2.12) is omitted as it is similar to the proof of (2.11). \( \square \)

The following covering result follows from the inequality (2.12).

**Corollary 2.7.** Under the hypothesis of Theorem 2.6, we have
\[
\left\{ w : |w| < \frac{1}{\beta} \left( \beta - 1 + \alpha + (\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1] - \beta) |b_1| \right) \right\} \subset f(\mathbb{D}).
\]

**Theorem 2.8.** If \( f_m \in \mathcal{T}_\mathcal{S}_H(m, n, \Phi, \Psi, p, q, \alpha) \), then \( f_m \) is convex in the disc
\[
|z| \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1]}{1 - \alpha} |b_1|]} \right\}^{1/k}, \quad k \geq 2.
\]

**Proof.** Let \( f_m \in \mathcal{T}_\mathcal{S}_H(m, n, \Phi, \Psi, p, q, \alpha) \) and let \( r, 0 < r < 1 \), be fixed. Then \( r^{-1} f_m(rz) \in \mathcal{T}_\mathcal{S}_H(m, n, \Phi, p, q) \) and we have
\[
\sum_{k=2}^{\infty} k^2 (|a_k| + |b_k|) = \sum_{k=2}^{\infty} k(|a_k| + |b_k|)kr^{k-1}
\]
\[
\leq \sum_{k=2}^{\infty} \left( \frac{\lambda_k[k]^m_p q - \alpha u[k]^n_p q}{1 - \alpha} |a_k| + \frac{\mu_k[k]^m_p q - (-1)^{n+j-(m+i)} \alpha v[k]^n_p q}{1 - \alpha} |b_k| \right) kr^{k-1}
\]
\[
\leq \sum_{k=2}^{\infty} \left( 1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1]}{1 - \alpha} |b_1| \right) kr^{k-1}
\]
\[
\leq 1 - b_1
\]
provided
\[
k^r k^{r-1} \leq \frac{1 - b_1}{1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1]}{1 - \alpha} |b_1|}
\]
which is true if
\[
r \leq \min_k \left\{ \frac{1 - b_1}{k[1 - \frac{\mu_1 - (-1)^{n+j-(m+i)} \alpha v[1]}{1 - \alpha} |b_1|]} \right\}^{1/k}, \quad k \geq 2.
\]

**Remark 2.9.** Our results naturally includes several results known for those subclasses of harmonic functions listed after Definition 2.2.
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