Complete monotonicity of a function involving the $p$-psi function and alternative proofs

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Abstract

In the paper, the authors prove that the function $x^\alpha \left[ \ln \frac{x^{p+1}}{F^p+1} - \psi_p(x) \right]$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$, where $p \in \mathbb{N}$ and $\psi_p(x)$ is the $p$-analogue of the classical psi function $\psi(x)$.

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MSC: Primary 33D05; Secondary 26A48, 33B15, 33E50

1. Introduction

Recall from [12, Chapter XIII], [16, Chapter 1] and [17, Chapter IV] that a function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and satisfies

$$0 \leq (-1)^n f^{(n)}(x) < \infty$$

for $x \in I$ and $n \geq 0$. The celebrated Bernstein-Widder’s Theorem (see [16, p. 3, Theorem 1.4] or [17, p. 161, Theorem 12b]) characterizes that a necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \, d \alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. This expresses that a completely monotonic function $f$ on $[0, \infty)$ is a Laplace transform of the measure $\alpha$. 

It is common knowledge that the classical Euler’s gamma function $\Gamma(x)$ may be defined for $x > 0$ by
\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt. \]

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called psi function or digamma function.

An alternative definition of the gamma function $\Gamma(x)$ is
\[ \Gamma(x) = \lim_{p \to \infty} \Gamma_p(x), \quad (1.3) \]
where
\[ \Gamma_p(x) = \frac{p! p^x}{x(x+1) \cdots (x+p)} = \frac{p^x}{x(1+x/1) \cdots (1+x/p)} \quad (1.4) \]
for $x > 0$ and $p \in \mathbb{N}$, the set of all positive integers. See [3, p. 250]. The $p$-analogue of the psi function $\psi(x)$ is defined as the logarithmic derivative of the $\Gamma_p$ function, that is,
\[ \psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \quad (1.5) \]

The function $\psi_p$ has the following properties:

1. It has the following representations
\[ \psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x+k} = \ln p - \int_0^\infty \frac{1 - e^{-(p+1)t}}{1 - e^{-t}} e^{-xt} \, dt. \quad (1.6) \]

2. It is increasing on $(0, \infty)$ and $\psi'_p$ is completely monotonic on $(0, \infty)$.

The very right hand side of the formula (1.6) corrects errors appeared in [8, p. 374, Lemma 5] and [10, p. 29, Lemma 2.3].

In [2, pp. 374–375, Theorem 1], it was proved that the function
\[ \theta_\alpha(x) = x^\alpha [\ln x - \psi(x)] \quad (1.7) \]
is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$. For the history, background, applications and alternative proofs of this conclusion, please refer to [4, 13, p. 8, Section 1.6.6] and closely related references therein.

The aim of this paper is to generalize [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1] to the case of the $p$-analogue $\psi_p(x)$ of the psi function $\psi(x)$ as follows.

**Theorem 1.1.** The function
\[ \theta_{p,\alpha}(x) = x^\alpha \left[ \ln \frac{p^x}{x+p+1} - \psi_p(x) \right] \quad (1.8) \]
for $p \in \mathbb{N}$ is completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$.

**Remark 1.1.** Letting $p \to \infty$ in Theorem 1.1, we obtain [2, pp. 374–375, Theorem 1] and [4, p. 105, Theorem 1].

2. **Proofs of Theorem 1.1**

**First Proof.** From the identity (1.6) and the integral expression
\[ \ln \frac{b}{a} = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} \, dt \quad (2.1) \]
in [1, p. 230, 5.1.32], we obtain
\[ \theta_{p,1}(x) = x \int_0^\infty [1 - e^{-(p+1)t}] \varphi(t)e^{-xt} \, dt, \quad (2.2) \]
where
\[ \varphi(t) = \frac{1}{1 - e^{-t}} - \frac{1}{t}. \]  
(2.3)

The function \( \varphi(t) \) is increasing on \((0, \infty)\) with
\[ \lim_{t \to 0^+} \varphi(t) = \frac{1}{2} \quad \text{and} \quad \lim_{t \to \infty} \varphi(t) = 1. \]  
(2.4)

See [5, 6, 7, 11, 14, 15, 18] and related references therein. Therefore, for \( x > 0 \) and \( n \in \mathbb{N} \), we have
\[ (-1)^n \theta_p^n(x) = x(-1)^n \frac{d^n}{dx^n} \int_0^\infty \left[ 1 - e^{-(p+1)t^2} \right] \varphi(t) e^{-xt} \, dt - (-1)^n n \frac{d^{n-1}}{dx^{n-1}} \int_0^\infty \left[ 1 - e^{-(p+1)t^2} \right] \varphi(t) e^{-xt} \, dt \]
\[ = x \int_0^\infty t^n \varphi(t) \left[ 1 - e^{-(p+1)t^2} \right] e^{-xt} \, dt - n \int_0^\infty t^{n-1} \varphi(t) \left[ 1 - e^{-(p+1)t^2} \right] e^{-xt} \, dt \]
\[ = \int_0^{n/x} t^{n-1} \left[ 1 - e^{-(p+1)t^2} \right] \varphi(t)(tx-n) e^{-xt} \, dt + \int_{n/x}^\infty t^{n-1} \left[ 1 - e^{-(p+1)t^2} \right] \varphi(t) (tx-n) e^{-xt} \, dt \]
\[ > \varphi \left( \frac{n}{x} \right) \int_0^{n/x} t^{n-1} \left[ 1 - e^{-(p+1)t^2} \right] (tx-n) e^{-xt} \, dt \]
\[ = \varphi \left( \frac{n}{x} \right) \left[ x \int_0^\infty t^n e^{-xt} \, dt - x \int_0^{n/x} t^n e^{-(x+p+1)t^2} \, dt - n \int_0^\infty t^{n-1} e^{-xt} \, dt + n \int_0^\infty t^{n-1} e^{-(x+p+1)t^2} \, dt \right] \]
\[ = \varphi \left( \frac{n}{x} \right) \left[ \frac{n!}{x^{n+1}} - \frac{x}{(x+p+1)^{n+1}} \right] \left[ \frac{1 - x}{x^{n+1}} + \frac{1}{(x+p+1)^{n+1}} \right] \]
\[ = \varphi \left( \frac{n}{x} \right) \left[ \frac{n!}{(x+p+1)^{n+1}} \right] \left( 1 - \frac{x}{x+p+1} \right) \]
\[ > 0, \]

where we used the formula
\[ \frac{1}{x^n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{\omega-1} e^{-xt} \, dt \]  
(2.5)

for real numbers \( x > 0 \) and \( \omega > 0 \), see [1, p. 255, 6.1.1]. So we obtain that the function \( \theta_{p,1}(x) \) is completely monotonic on \((0, \infty)\).

Since
\[ (-1)^n [u(x)v(x)]^{(n)} = \sum_{i=0}^n \binom{n}{i} (-1)^i u^{(i)}(x) [-1]^{n-i} v^{(n-i)}(x), \]
the product of any two completely monotonic function is also completely monotonic on their common domain. On the other hand, the function \( x^{\alpha-1} \) for \( \alpha < 1 \) is clearly completely monotonic on \((0, \infty)\). Consequently the function
\[ \theta_{p,\alpha}(x) = x^{\alpha-1} \theta_{p,1}(x) \]
for \( \alpha \leq 1 \) is completely monotonic on \((0, \infty)\).

Conversely, if \( \theta_{p,\alpha}(x) \) is completely monotonic on \((0, \infty)\), then
\[ \frac{d \theta_{p,\alpha}(x)}{dx} = x^{\alpha-1} \left\{ \alpha \left[ \ln \frac{px}{x+p+1} - \psi_p(x) \right] + \frac{p+1}{x+p+1} - x \psi_p'(x) \right\} \leq 0 \]
for $x > 0$, equivalently,
$$
\alpha \leq \frac{x\psi_p'(x) - \frac{p+1}{x+p+1}}{\ln \frac{p}{x+p+1} - \psi_p(x)}.
$$

Employing L'Hôpital's rule and (1.6) results in
$$
\lim_{x \to \infty} \frac{x\psi_p'(x) - \frac{p+1}{x+p+1}}{\ln \frac{p}{x+p+1} - \psi_p(x)} = \lim_{x \to \infty} \frac{x\psi_p'(x) + \psi_p'(x) + \frac{p+1}{(x+p+1)^2}}{\frac{1}{x} - \frac{1}{x+p+1} - \sum_{k=0}^{p} \frac{1}{(x+k)^2}} = 1,
$$
so it is necessary that $\alpha \leq 1$. The proof is complete.

Second Proof. From (2.2) and by integration by part lead to
$$
\theta_{p,1}(x) = -\int_{\infty}^{\infty} \left[1 - e^{-(p+1)t}\right] \varphi(t) \frac{d}{dt} e^{-xt} dt
= \int_{0}^{\infty} \left\{ \left[1 - e^{-(p+1)t}\right] \varphi(t) \right\}' e^{-xt} dt - \left\{ \left[1 - e^{-(p+1)t}\right] \varphi(t) e^{-xt} \right\}' \bigg|_{t=0}^{t=\infty}
= \int_{0}^{\infty} \left\{ \left[1 - e^{-(p+1)t}\right] \varphi'(t) + (p+1)e^{-(p+1)t} \varphi(t) \right\} e^{-xt} dt.
$$

Therefore, for showing that the function $\theta_{p,1}(x)$ is completely monotonic on $(0, \infty)$ for all $p \in \mathbb{N}$, it suffices to prove that the function
$$
\left[1 - e^{-(p+1)t}\right] \varphi'(t) + (p+1)e^{-(p+1)t} \varphi(t)
$$
is positive. Since the function $\varphi(t)$ is increasing on $(0, \infty)$, the derivative $\varphi'(t)$ is positive on $(0, \infty)$. Further considering the limits in (2.4), the positivity of $\varphi(t)$ follows. As a result, the function (2.6) is positive.

The rest of the proof is the same as the first proof.

Remark 2.1. This paper is a slightly modified version of the preprint [9].

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