Skew-symmetric differential forms. Invariants. Realization of invariant structures.
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Abstract

Skew-symmetric differential forms play an unique role in mathematics and mathematical physics. This relates to the fact that closed exterior skew-symmetric differential forms are invariants. The concept of “Exterior differential forms” was introduced by E. Cartan for a notation of integrand expressions, which can create the integral invariants. (The existence of integral invariants was recognized by A. Poincare while studying the general equations of dynamics.)

All invariant mathematical formalisms are based on invariant properties of closed exterior forms. The invariant properties of closed exterior forms explicitly or implicitly manifest themselves essentially in all formalisms of field theory, such as the Hamilton formalism, tensor approaches, group methods, quantum mechanics equations, the Yang-Mills theory and others. They lie at the basis of field theory.

However, in this case the question of how the closed exterior forms are obtained arises. In present work it is shown that closed exterior forms, which possess the invariant properties, are obtained from skew-symmetric differential forms, which, as contrasted to exterior forms, are defined on nonintegrable manifolds. The process of generating closed exterior forms describes the mechanism of realization of invariants and invariant structures.

1 Closed exterior skew-symmetric differential forms: Invariants. Invariant structures.

Distinguishing properties of the mathematical apparatus of exterior differential forms were formulated by Cartan [1]: “...I wanted to build the theory, which contains concepts and operations being independent of any change of variables both dependent and independent; to do so it is necessary to change partial derivatives by differentials that have interior meaning.”

1.1 Some foundations of closed exterior differential forms

The exterior differential form of degree \( p \) (\( p \)-form) on integrable manifold can be written as [2,3]

\[
\theta^p = \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} \, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \quad 0 \leq p \leq n
\]  

(1)

Here \( a_{i_1 \ldots i_p} \) are functions of variables \( x^{i_1}, x^{i_2}, \ldots, x^{i_n} \), \( n \) is the dimension of space, \( \wedge \) is the operator of exterior multiplication, \( dx^i, dx^i \wedge dx^j, dx^i \wedge dx^j \wedge dx^k \),
... is the local basis which satisfies the condition of exterior multiplication:

\[ dx^i \wedge dx^i = 0 \]
\[ dx^i \wedge dx^j = -dx^j \wedge dx^i \quad i \neq j \]

[In further presentation the symbol of summing \( \sum \) and the symbol of exterior multiplication \( \wedge \) will be omitted. Summation over repeated indices is implied.]

An exterior differential form is called a closed one if its differential is equal to zero:

\[ d\theta^p = 0 \quad (2) \]

From condition (2) one can see that the closed form is a conservative quantity. This means that this can correspond to the conservation law, namely, to some conservative physical quantity.

The differential of the form is a closed form. That is

\[ dd\omega = 0 \]

where \( \omega \) is an arbitrary exterior form.

The form which is the differential of some other form:

\[ \theta^p = d\omega \quad (3) \]

is called the exact form. The exact forms prove to be closed automatically

\[ d\theta^p = dd\omega = 0 \quad (4) \]

Here it is necessary to pay attention to the following points. In the above presented formulas it was implicitly assumed that the differential operator \( d \) is a total one (that is, the operator \( d \) acts everywhere in the vicinity of the point considered). However, the differential may be internal. Such a differential acts on some structure with the dimension being less than that of the initial manifold.

If the exterior form is closed only on structure, the closure condition is written as

\[ d_\pi \theta^p = 0 \quad (5) \]

In this case the structure \( \pi \) obeys the condition

\[ d_\pi^* \theta^p = 0 \quad (6) \]

where \( ^* \theta^p \) is the dual form.

Such an exterior form is called the closed inexact form.

The structure, on which the exterior differential form may become a closed (inexact) form, is a pseudostructure with respect to its metric properties.

From conditions (5) and (6) one can see that the form closed on pseudostructure is a conservative object, namely, this quantity conserves on pseudostructure. This can also correspond to some conservation law, i.e. to conservative object.
Pseudostructures

As one can see from condition (6), the structure, on which a closed (inexact) form is defined, is described by dual form. The dual form is a closed metric form of this structure.

To understand the properties of such structure, one can use the correspondence between the exterior differential form and skew-symmetric tensor. It is known that the skew-symmetric tensors correspond to closed exterior differential forms, and the pseudotensors correspond to relevant dual forms. This points to the fact that the structures, on which closed inexact forms are defined, are pseudostructures.

The characteristics, integral surfaces, surfaces of potential (of simple layer, double layer), sections of cotangent bundles (Yang-Mills fields), cotangent manifold, eikons, cohomologies by de Rham, singular cohomologies, the pseudo-Riemann and pseudo-Euclidean spaces and others can be regarded as examples of pseudostructures and pseudospaces, on which closed inexact forms are defined.

It should be emphasized that the pseudostructure and corresponding closed inexact form make up a differential - geometrical structure. As it will be shown below, such a differential - geometrical structure proves to be an invariant structure.

Differentials

The exact form is, by definition, a differential (see condition (3)). In this case the differential is total. The closed inexact form is a differential too. And in this case the differential is an interior one defined on pseudostructure. Thus, any closed form is a differential. The exact form is a total differential. The closed inexact form is an interior (on pseudostructure) differential, that is

\[ \theta^p = d_\pi \omega \]

At this point it is worth noting that the total differential of the form closed on pseudostructure is nonzero, that is

\[ dd_\pi \omega \neq 0 \] (8)

1.2 Invariants. Invariant structures.

Since the closed form is a differential (a total one if the form is exact, or an interior one on the pseudostructure if the form is inexact), it is obvious that the closed form proves to be invariant under all transformations that conserve the differential. The unitary transformations (0-form), the tangent and canonical transformations (1-form), the gradient and gauge transformations (2-form) and so on are examples of such transformations.

These are gauge transformations for spinor, scalar, vector, tensor fields. It can be pointed out that just such transformations are used in field theory.
As mentioned above, from the closure conditions it follows that the closed form is a conservative quantity. As the result, the closed form is a conservative invariant quantity. This property of closed forms plays an essential role in describing the conservation laws and lies at the basis of field theory. The covariance of dual form is directly connected with the invariance of exterior closed inexact form.

**Invariant structures**

The closed inexact exterior forms are of most significance in mathematical formalisms and mathematical physics. This is due to the fact that the closed exterior form and relevant dual form describe the differential-geometrical structure, which is invariant one.

From the definition of closed inexact exterior form one can see that to this form there correspond two conditions:

1. condition (5) is a closure condition of exterior form itself, and
2. condition (6) is that of dual form.

Conditions (5) and (6) can be regarded as equations for a binary object that combines the pseudostructure (dual form) and the conservative quantity (the exterior differential form) defined on this pseudostructure. Such a binary object is a differential - geometrical structure. (The well-known G-Structure is an example of such differential-geometrical structure.)

As it has been already pointed out, the closed inexact exterior form is a differential (an interior one on pseudostructure), and hence it remains invariant under all transformations that conserve the differential. Therefore, the relevant differential-geometrical structure also remains invariant under all transformations that conserve differential. For the sake of convenience in subsequent presentation such differential - geometrical structures will be called the Inv. Structures.

To an unique role of such invariant structures in mathematics it points the fact that the transformations conserving the differential (unitary, tangent, canonical, gradient and gauge ones) lie at the basis of many branches of mathematics, mathematical physics and field theory. The differential-geometrical structures made up of characteristics and integral curves of differential equations and relevant conditions on those are examples of Inv. Structures.

As it will be shown in Section 3 of present paper, the Inv. Structures are of unique importance in mathematical physics and field theory. The physical structures, of which physical fields are made up, are such invariant structures.

It should be emphasized ones more that the Inv. Structure is a differential - geometrical structure. That is not a spatial structure. The spatial structure is described by *exact* exterior form, whereas the invariant structure is described by *inexact* exterior form.
1.3 Invariance as the result of conjugacy of elements of exterior or dual forms

The closure of exterior differential forms, and hence their invariance, results from the conjugacy of elements of exterior or dual forms.

From the definition of the exterior differential form one can see that exterior differential forms have complex structure. The specific features of the exterior form structure are the homogeneity with respect to the basis, skew-symmetry, the integration of terms each of which made up by two objects of different nature (the algebraic nature of the form coefficients, and the geometric nature of the base components). Besides, the exterior form depends on the space dimension and on the manifold topology. The closure property of exterior form means that any objects, namely, elements of exterior form, components of elements, elements of the form differential, exterior and dual forms and others, turn out to be conjugated. The variety of objects of conjugacy leads to the fact that closed forms can describe a great number of different invariant structures.

Let us consider some types of conjugacy.

One of the types of conjugacy is that for the form coefficients.

Let us consider the exterior differential form of first degree \( \omega = a_i dx^i \). In this case the differential will be expressed as \( d\omega = K_{ij} dx^i dx^j \), where \( K_{ij} = (\partial a_j/\partial x^i - \partial a_i/\partial x^j) \) are the components of the form commutator.

It is evident that the differential may vanish if the components of commutator vanish. One can see that the components of commutator \( K_{ij} \) may vanish if derivatives of the form coefficients vanish. This is a trivial case. Besides, the components \( K_{ij} \) may vanish if the coefficients \( a_i \) are derivatives of some function \( f(x^i) \), that is, \( a_i = \partial f/\partial x^i \). In this case, the components of commutator are equal to the difference of mixed derivatives

\[
K_{ij} = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right)
\]

and therefore they vanish. One can see that the form coefficients \( a_i \), that satisfy these conditions, are conjugated quantities (the operators of mixed differentiation turn out to be commutative).

Let us consider the case when the exterior form is written as

\[
\theta = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy
\]

where \( f \) is the function of two variables \((x, y)\). It is evident that this form is closed because it is equal to the differential \( df \). And for the dual form

\[
*\theta = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy
\]

be also closed, it is necessary that its commutator be equal to zero

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \Delta f = 0
\]

where \( \Delta \) is the Laplace operator. As a result the function \( f \) has to be a harmonic one.

Assume the exterior differential form of first degree has the form \( \theta = u dx + v dy \), where \( u \) and \( v \) are the functions of two variables \((x, y)\). In this case, the closure condition of the form, that is, the condition under which the form commutator vanishes, takes the form

\[
K = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0
\]

One can see that this is one of the Cauchy-Riemann conditions for complex functions. The closure condition of the relevant dual form \(*\theta = -vdx + udy\) is the second Cauchy-Riemann...
condition. {Here one can see the connection between exterior differential form and the functions of complex variables. If we consider the function \( w = u + iv \) of complex variables \( z = x + iy \) and \( \overline{z} = x - iy \) that obeys the Cauchy-Riemann conditions, then the closed exterior and dual forms will correspond to this function. (The Cauchy-Riemann conditions are conditions under which the function of complex variables does not depend on the conjugated coordinate \( \overline{z} \). And the closed exterior differential form, whose coefficients \( u \) and \( v \) are conjugated harmonic functions, corresponds to the harmonic function of complex variables).}

It can exist the conjugacy that makes the interior differential on pseudostructure equal to zero, \( d\pi \theta = 0 \). Assume the interior differential is the first degree form (the form itself is a form of zero degree), and it can be presented as \( d\pi \theta = p_x dx + p_y dy = 0 \), where \( p \) is the form of zero degree (a certain function). In this case the closure condition of the form is

\[
\frac{dx}{dy} = -\frac{p_y}{p_x} \tag{9}
\]

This is a conjugacy of the basis and derivatives of the form coefficients. One can see that this formula is one of the formulas of canonical relations. The second formula of canonical relations follows from the condition that the dual form differential vanishes. This type of conjugacy is connected with canonical transformation. For the differential of the first degree form (in this case the differential is a form of second degree) the corresponding transformation has to be a gradient transformation. At this point it should be remarked that relation (9) is the condition of existence of implicit function. That is, the closed (inexact) form of zero degree is an implicit function.}

1.4 Identical relations of exterior differential forms, description of conjugacy and invariance

Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that the relations can be used to express conjugacy mathematically. Identical relations of exterior differential forms disclose also the properties of Inv. Structure.

At this point it should be emphasized the following. The relation is a comparison, i.e. a correlation of two objects. The relation may be identical or nonidentical. The basis of mathematical apparatus of exterior differential forms is made up of identical relations. (Below nonidentical relations will be presented, and it will be shown that identical relations for exterior differential forms are obtained from nonidentical relations. Also it will be shown that transitions from nonidentical relations to identical ones describe the realization of invariant structures.)

The identical relations of exterior differential forms reflect the closure conditions of differential forms, namely, vanishing the form differential (see formulas (2), (5) and (6)) and the condition that the closed differential form is a differential (see formulas (3) and (7)). All these conditions are the expression of conjugacy and invariance.

One can distinguish several types of identical relations.

1. \textit{Relations in differential forms.}

They correspond to formulas (3) and (7). The examples of such identical relations are

(a) the Poincare invariant \( ds = -H dt + p_j dq_j \),

(b) the second principle of thermodynamics \( dS = (dE + pdV)/T \),
(c) the vital force theorem in theoretical mechanics: \(dT = X_i dx^i\) where \(X_i\) are the components of potential force, and \(T = mV^2/2\) is the vital force,

(d) the conditions on characteristics in the theory of differential equations.

The requirement that the function is an antiderivative (the integrand is a differential of a certain function) can be written in terms of such an identical relation.

The existence of harmonic function is written by means of identical relation: the harmonic function is a closed form, that is, a differential (a differential on the Riemann surface).

In general form such an identical relation can be written as

\[
d\phi = \theta^p
\]

In this relation the form in the right-hand side has to be a **closed** one.

As it will be shown below, the identical relations are satisfied only on pseudostructures. That is, the identical relation can be written as

\[
d_\pi \phi = \theta^p_\pi
\]

Identical relations (10) and (11) are the proof that the closed exterior form is a differential, and hence, this form is an invariant with respect to all transformations that conserve the differential.

Identical relations occur in various branches of mathematics and mathematical physics. Identical relations can be of another type, namely, integral, tensor and others. And all identical relations are an analog to the identical relation in differential forms.

All identical relations correspond to invariant structures.

It would be noted some another types of identical relations.

2. **Integral identical relations.**

At the beginning of the paper it was pointed out that the exterior differential forms were introduced as integrand expressions possessing the following property: they can have integral invariants. This fact (the availability of integral invariant) is mathematically expressed as a certain identical relation.

The formulas by Newton, Leibnitz and Green, the integral relations by Stokes and Gauss-Ostrogradskii are examples of integral identical relations.

3. **Tensor identical relations.**

From the relations that connect exterior forms of consequent degrees one can obtain the vector and tensor identical relations that connect the operators of gradient, curl, divergence and so on.

From the closure conditions of exterior and dual forms one can obtain the identical relations such as the gauge relations in electromagnetic field theory, the tensor relations between connectednesses and their derivatives in gravitation (the symmetry of connectednesses with respect to lower indices, the Bianchi identity, the conditions imposed on the Christoffel symbols) and so on.

4. **Identical relations between derivatives.**

The identical relations between derivatives correspond to the closure conditions of exterior and dual forms. The examples of such relations are the above
presented Cauchi-Riemann conditions in the theory of complex variables, the transversality condition in the calculus of variations, the canonical relations in the Hamilton formalism, the thermodynamic relations between derivatives of thermodynamic functions, the condition that the derivative of implicit function is subject to, the eikonal relations and so on.

The importance of identical relations is manifested by the fact that practically in all branches of physics, mechanics, thermodynamics one faces such identical relations.

The functional significance of identical relations for exterior differential forms lies in the fact that they can describe the conjugacy of objects that have different mathematical meaning and different physical nature. This enables one to see internal connections between various branches of mathematics and physics. Due to these possibilities the exterior differential forms, and correspondingly, the Inv. Structures, have wide application in various branches of mathematics and mathematical physics.

Identical relations possess the duality that discloses the significance of invariant structures. The availability of differential in the left-hand side points to the availability of potential or state function, and the availability of closed inexact form points to that there is an invariant structure. Below it will be shown that such a relation has a deep physical sense.

2 Realization of invariant structures

The mechanism of realization of invariant structures is described by skew-symmetric differential forms, which, in contrast to exterior forms, are defined on deforming nonintegrable manifolds (see Appendix of work [4]). Such skew-symmetric differential forms possess the evolutionary properties. The evolutionary forms possess a peculiarity, namely, the closed inexact exterior forms are obtained from them. This elucidates the process of realization of invariant structures.

2.1 Some properties of evolutionary forms

The evolutionary skew-symmetric differential forms are obtained from differential equations that describe any processes.

Examples of nonintegrable manifolds, on which the evolutionary skew-symmetric differential forms are defined, are the tangent manifolds of differential equations, the Lagrangian manifolds, the manifolds constructed of trajectories of material medium particles and so on. These are manifolds with unclosed metric forms. The metric form differential, and correspondingly its commutator, are nonzero. (The commutators of metric forms of such manifolds describe the manifold deformation: torsion, curvature and others).

The specific feature of evolutionary forms, i.e skew-symmetric forms defined on deforming manifolds, is the fact that evolutionary forms are unclosed ones.
Since the basis of evolutionary form changes, the evolutionary form differential includes the nonvanishing differential of manifold metric form due to differentiating the basis. Therefore, the evolutionary form differential cannot be equal to zero. Hence, the evolutionary form, in contrast to the case of exterior form, cannot be closed. This leads to that in the mathematical apparatus of evolutionary forms there arise new nonconventional elements like nonidentical relations and degenerate transformations that allow to describe the generation of closed inexact exterior forms and the realization of invariant structures.

The nonidentical relations of evolutionary forms can be written as

\[ d\phi = \eta^p \]  

Here \( \eta^p \) is the \( p \)-degree evolutionary form being unclosed, \( \phi \) is some form of degree \((p-1)\), and the differential \( d\phi \) is a closed form of degree \( p \).

The form differential, i.e. a closed form being an invariant object, appears in the left-hand side of this relation. In the right-hand side it is appeared the unclosed form, which is not an invariant object. Such a relation cannot be identical one.

One can see the difference of relations for exterior forms and evolutionary ones. In the right-hand side of identical relation (see relation (10)) it is appeared the closed form, whereas the form in the right-hand side of nonidentical relation (12) is an unclosed one.

Nonidentical relations are obtained while describing any processes. A relation of such type is obtained while, for example, analyzing the integrability of the partial differential equation. The equation is integrable if it can be reduced to the form \( d\phi = dU' \). However, it appears that, if the equation is not subject to an additional condition (the integrability condition), it is reduced to the form (12), where \( \eta^p \) is an unclosed form and it cannot be written as a differential.

Nonidentical relations of evolutionary forms are evolutionary relations because they include the evolutionary form. Such nonidentical evolutionary relations appear to be selfvarying ones. The variation of any object of the relation in some process leads to variation of another object and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is a non-invariant (i.e. unmeasurable) quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot be completed.

The nonidentity of evolutionary relation is connected with the nonclosure of evolutionary form, that is, it is connected with the fact that the evolutionary form commutator is nonzero. The evolutionary form commutator includes two terms. The first term specifies the mutual variations of evolutionary form coefficients, and the second term (the metric form commutator) specifies the manifold deformation. These terms have a different nature and cannot make the commutator to be vanishing. In the process of selfvariation of nonidentical evolutionary relation the exchange between the terms of evolutionary relation proceeds and this is realized according to the evolutionary relation. The evolutionary form commutator describes the quantity that is a moving force of evolutionary process and leads to realization of differential-geometrical structures.
The process of the evolutionary relation selfvariation plays a governing role in description of evolutionary processes.

The significance of the evolutionary relation selfvariation consists in the fact that in such a process it can be realized conditions under which the closed inexact form is obtained from the evolutionary form and the identical relation is obtained from the nonidentical relation. These are conditions of degenerate transformation. Since the evolutionary form differential is nonzero, whereas the closed exterior form differential is zero, the transition from the evolutionary form to closed exterior form is allowed only under degenerate transformation. The conditions of vanishing the dual form differential are conditions of degenerate transformation.

These are such conditions that can be realized under selfvariation of the nonidentical evolutionary relation.

2.2 Realization of closed inexact exterior form. Derivation of invariant structures

To obtain the differential-geometrical structure, it is necessary to obtain the closed inexact exterior form, i.e. the form closed on pseudostructure.

To the pseudostructure it is assigned the closed dual form (whose differential vanishes). For this reason the transition from the evolutionary form to closed inexact exterior form proceeds only when the conditions of vanishing the dual form differential are realized, in other words, when the metric form differential or commutator becomes equal to zero.

The conditions of degenerate transformation are conditions of vanishing the dual form differential. That is, it is the condition of realization of pseudostructure. And this leads to realization of closed inexact exterior form.

As it has been already mentioned, the evolutionary differential form $\eta^p$ involved into nonidentical relation (12) is an unclosed one. The commutator, and hence the differential, of this form is nonzero. That is,

$$d\eta^p \neq 0$$  \hspace{1cm} (13)

If the conditions of degenerate transformation are realized, then from the unclosed evolutionary form one can obtain the differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition $d\eta^p \neq 0 \rightarrow$ (degenerate transformation) $\rightarrow d_{\pi}^* \eta^p = 0$, $d_{\pi} \eta^p = 0$.

The relations obtained

$$d_{\pi} \eta^p = 0, d_{\pi}^* \eta^p = 0$$  \hspace{1cm} (14)

are closure conditions for exterior inexact form, and this points to realization of exterior form closed on pseudostructure, that is, this points to origination of the differential-geometrical invariant structure.

Vanishing on pseudostructure the exterior form differential (that is, vanishing on pseudostructure the interior differential of the evolutionary form) points to that the exterior inexact form is a conservative quantity in the direction.
of pseudostructure. However, in the direction normal to pseudostructure this quantity exhibits the discontinuity. The value of such discontinuity is defined by the value of the evolutionary form commutator being nonzero. This argues to discreteness of the differential-geometrical structures.

Thus, while selfvariation of the evolutionary nonidentical relation the dual form commutator can vanish. This means that it is made up the pseudostructure on which the differential form turns out to be closed. The emergence of the form being closed on pseudostructure points out to origination of invariant structures.

On the pseudostructure \( \pi \) from evolutionary relation (12) it follows the relation

\[
d_\pi \psi = \omega_\pi
\]

which proves to be an identical relation. Indeed, since the form \( \omega_\pi \) is a closed one, on the pseudostructure this form turns out to be the differential of some differential form. In other words, this form can be written as \( \omega_\pi = d_\pi \theta \). Relation (15) is now written as

\[
d_\pi \psi = d_\pi \theta
\]

There are differentials in the left-hand and right-hand sides of this relation. This means that the relation is an identical one.

From evolutionary nonidentical relation (12) it is obtained the identical on pseudostructure relation. In this case the evolutionary relation itself remains to be nonidentical one. (At this point it should be emphasized that differential, which equals zero, is an interior one. The evolutionary form commutator becomes zero only on pseudostructure. The total evolutionary form commutator is nonzero. That is, under degenerate transformation the evolutionary form differential vanishes only on pseudostructure. The total differential of evolutionary form is nonzero. The evolutionary form remains to be unclosed.)

It can be shown that all identical relations of the exterior differential form theory are obtained from nonidentical relations (that contain evolutionary forms) by applying degenerate transformations.

[The conditions of degenerate transformation that lead to origination of invariant structures can be connected with any symmetries. While describing material system (see, Section 3), the symmetries can be conditioned, for example, by degrees of freedom of material system. Since the conditions of degenerate transformation are those of vanishing the interior differential of metric form, that is, vanishing the interior (rather then total) metric form commutator, the conditions of degenerate transformation can be caused by symmetries of coefficients of the metric form commutator (for example, it can be the symmetric connectedness).

Mathematically the requirement that some functional expressions become equal to zero is assigned to the conditions of degenerate transformation. Such functional expressions are Jacobians, determinants, the Poisson brackets, residues, and others.

The degenerate transformation is realized as the transition between nonequivalent frames of reference: the transition from the noninertial frame of reference to the locally inertial one. Evolutionary relation (12) and condition (13) are connected with the frame of reference being related to nonintegrable noninertial manifold, whereas condition (14) and identical relations (15) may be connected with only the locally inertial frame of reference being related to pseudostructure. For example, while studying the integrability of differential equations under degenerate transformation it occurs the transition from the tangent nonintegrable manifold to
cotangent integrable manifold. Here it can be underlined the connection between the degenerate transformation and nondegenerate one. The origination of the differential-geometrical structures (Inv. Structures) is connected with degenerate transformation that executes the transition from tangent space to cotangent one. And nondegenerate transformation executes the transition in cotangent space from any differential-geometrical structure to another.

Thus, the mathematical apparatus of evolutionary differential forms can describe the process of generation of closed inexact exterior differential forms, and this discloses the process of origination of invariant structures.

The process of generation of closed inexact exterior differential forms and the origination of invariant structures are processes of conjecting the operators. To the closed exterior form there correspond conjugated operators, whereas to the evolutionary form there correspond nonconjugated operators. The transition from evolutionary form to closed exterior form and the origination of differential-geometrical structures is a transition from nonconjugated operators to conjugated ones. This is expressed mathematically as the transition from nonzero differential (the evolutionary form differential is nonzero) to the differential that equals zero (the closed exterior form differential equals zero).

It can be seen that the process of conjugating the objects and obtaining the differential-geometrical structures is a mutual exchange between the quantities of different nature (for example, between the algebraic and geometric quantities, between the physical and spatial quantities) and vanishing some functional expressions (Jacobians, determinants and so on).

**Characteristics of Inv. Structure**

Since the closed exterior differential form, which corresponds to the Inv. Structure emerged, was obtained from evolutionary form that enters to the non-identical relation, it is evident that the Inv. Structure characteristics must be connected with those of the evolutionary form and of the manifold on which this form is defined, as well as the conditions of degenerate transformation and the values of commutators of the evolutionary form and the manifold metric form.

The conditions of degenerate transformation, as it was said before, determine the pseudostructures. The first term of the evolutionary form commutator determines the value of discrete change (the quantum), which the quantity conserved on the pseudostructure undergoes under transition from one pseudostructure to another. The second term of the evolutionary form commutator specifies the characteristics that fixes the character of initial manifold deformation, which took place before the Inv. Structure had been arisen. (Spin is such an example).

The discrete (quantum) change of a quantity proceeds in the direction that is normal (more exactly, transverse) to the pseudostructure. Jumps of the derivatives normal to potential surfaces are examples of such changes.

The connection of Inv. Structure with the skew-symmetric differential forms allows to introduce the classification of Inv. Structures in its dependence on parameters that specify the skew-symmetric differential forms and enter into nonidentical and identical relation of skew-symmetric differential forms. To
determine these parameters one has to consider the problem of integration of nonidentical evolutionary relation.

Under degenerate transformation from the nonidentical evolutionary relation one obtains the relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains the relation that can be integrated, and as the result one obtains the relation with differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

The resulting nonidentical relation of degree \( p - 1 \) (relation that includes the forms of degree \( p - 1 \)) can be integrated once again if the corresponding degenerate transformation has been realized and the identical relation has been formatted.

By sequential integrating the evolutionary relation of degree \( p \) (in the case of realization of corresponding degenerate transformations and formatting the identical relation), one can get closed (on the pseudostructure) exterior forms of degree \( k \), where \( k \) ranges from \( p \) to \( 0 \).

In this case one can see that after such integration the closed (on pseudostructure) exterior forms, which depend on two parameters, are obtained. These parameters are the degree of evolutionary form \( p \) (in the evolutionary relation) and the degree of created closed forms \( k \).

In addition to these parameters, another parameter appears, namely, the dimension of space. If the evolutionary relation generates the closed forms of degrees \( k = p, k = p - 1, \ldots, k = 0 \), to them there correspond the pseudostructures of dimensions \( n + 1 - k \), where \( n \) is the space dimension.

The invariant structures are of unique significance in mathematical physics and field theory. The physical structures that made up physical fields are such Inv. Structures.

As it will be shown below, the mechanism of realization of Inv. Structures, which correspond to physical fields, describes the mechanism of generation of physical structures. This discloses the physical meaning of Inv. Structures.

### 3 Physical meaning of invariant structures. Mechanism of generation of physical structures.

As it has been already pointed out, the invariant structures are realized while analyzing the integrability of differential equations. Their role in the theory of differential equations relates to the fact that they correspond to generalized solutions which describe measurable physical quantities. In this case the integral surfaces with conservative quantities (like the characteristics, the characteristic surfaces, potential surfaces and so on) are invariant structures. The examples of such studying the integrability of differential equations using the skew-symmetric differential forms are presented in paper [5].

The unique results are obtained in studying the differential equations that
describe the conservation laws for material media. The Inv. Structures that correspond to physical structures are obtained under investigation of these equations.

The properties of conservation laws are at the basis of the process of physical structure emergence. Therefore it is necessary to call attention to some properties and peculiarities of conservation laws.

### 3.1 Properties and peculiarities of conservation laws.

From the closure condition of exterior form it follows that the closed inexact differential form is a conservative quantity on some pseudostructure. From this one can see that the closed inexact exterior differential form can correspond to conservation law. The conservation laws for physical fields are just such conservation laws. (The physical fields are a special form of the substance, they are carriers of various interactions such as electromagnetic, gravitational, wave, nuclear and other kinds of interactions. The conservation laws for physical fields are those that claim the existence of conservative physical quantities or objects. Such conservation laws can be named the exact conservation laws.)

One can see that Inv. Structures made up by closed inexact form and relevant dual form correspond to conservation laws for physical fields.

The evolutionary skew-symmetric forms, from which, as it has been shown, the closed inexact forms are obtained, correspond to conservation laws as well. However, these are conservation laws for material systems (material media). In contrast to conservation laws for physical fields, they are balance conservation laws (they establish the balance between the variation of physical quantities and external actions to the system) and are described by differential equations. (Material system is a variety of elements that have internal structure and interact to one another. As examples of material systems it may be thermodynamic, gas dynamical, cosmic systems, systems of elementary particles and others).

The conservation laws for material systems are conservation laws for energy, linear momentum, angular momentum, and mass.

The invariant structures corresponding to physical fields are obtained from the equations that describe balance conservation laws for material media.

### Analysis of the equations of conservation laws for material systems.

The balance conservation laws for energy, linear momentum, angular momentum, and mass are described by partial differential equations [6]. (On examination of the integrability of these equations it is obtained the nonidentical relation that includes evolutionary form. From such evolutionary form the closed inexact forms and invariant structures corresponding to physical structures are obtained.)

Let us analyze the equations that describe the balance conservation laws for energy and linear momentum.
In the accompanying frame of reference (this system is connected with the manifold made up by the trajectories of material system elements) the equations for energy and linear momentum are written as

\[ \frac{\partial \psi}{\partial \xi^1} = A_1 \]  

(16)

\[ \frac{\partial \psi}{\partial \xi^\nu} = A_\nu, \quad \nu = 2, ... \]  

(17)

Here \( \psi \) is the functional specifying the state of material system (the action functional, entropy, wave function can be regarded as examples of such a functional), \( \xi^1 \) is the coordinate along the trajectory, \( \xi^\nu \) are the coordinates in the direction normal to trajectory, \( A_1 \) is the quantity that depends on specific features of material system and on external energy actions onto the system, and \( A_\nu \) are the quantities that depend on specific features of material system and on external force actions.

Eqs. (16) and (17) can be convoluted into the relation

\[ d\psi = A_\mu d\xi^\mu, \quad (\mu = 1, \nu) \]  

(18)

where \( d\psi \) is the differential expression \( d\psi = (\partial \psi/\partial \xi^\mu) d\xi^\mu \).

Relation (18) can be written as

\[ d\psi = \omega \]  

(19)

here \( \omega = A_\mu d\xi^\mu \) is the skew-symmetric differential form of first degree.

The relation obtained is an evolutionary relation.

Relation (19) was obtained from the equations of balance conservation laws for energy and linear momentum. In this relation the form \( \omega \) is that of first degree. If the equations of balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form in the evolutionary relation will be a form of second degree. And in combination with the equation of balance conservation law for mass this form will be the form of degree 3.

Thus, in general case the evolutionary relation can be written as

\[ d\psi = \omega^p \]  

(20)

where the form degree \( p \) takes the values \( p = 0, 1, 2, 3, ... \). (The evolutionary relation for \( p = 0 \) is similar to that in differential forms, and it was obtained from the interaction of energy and time.)

The relations (19) and (20) are nonidentical evolutionary relations.

Let us show that the relation obtained from the equations of balance conservation laws proves to be nonidentical.

To do so we shall analyze relation (19).

In the left-hand side of relation (19) there is the differential that is a closed form. This form is an invariant object. The right-hand side of relation (20)
contains the differential form $\omega$, which is not an invariant object since in real processes, as it will be shown below, this form proves to be unclosed. The commutator of this form is nonzero. The components of commutator of the form $\omega = A_\mu d\xi^\mu$ can be written as follows:

$$K_{\alpha\beta} = \left( \frac{\partial A_\beta}{\partial \xi^\alpha} - \frac{\partial A_\alpha}{\partial \xi^\beta} \right)$$

(here the term connected with the manifold metric form has not yet been taken into account).

The coefficients $A_\mu$ of the form $\omega$ have been obtained either from the equation of balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form $\omega$ made up of the derivatives of such coefficients is nonzero. This means that the differential of the form $\omega$ is nonzero as well. Thus, the form $\omega$ proves to be unclosed and cannot be a differential like the left-hand side.

This means that relation (19), as well as relation (20), cannot be identical ones. In such a way it can be shown that relation (20) is nonidentical as well.

Thus, the nonidentity of evolutionary relation means that the balance conservation law equations are inconsistent. And this indicates that the balance conservation laws are noncommutative. (If the balance conservation laws be commutative, the equations would be consistent and the evolutionary relation would be identical).

The noncommutativity of balance conservation laws is a moving force of evolutionary processes that proceed in material medium and lead to emergence of physical structures. This follows from the further analysis of the equations of balance conservation laws. The invariant structures obtained from these equations correspond to such physical structures.

### 3.2 Mechanism of generation of physical structures.

The relation obtained from the equations of balance conservation laws involves the functional that specifies the material system state. However, since this relation turns out to be not identical, from this relation one cannot get the differential $d\psi$ that could point out to the equilibrium state of material system. The absence of differential means that the system state is nonequilibrium. That is, in material system the internal force acts.

As it has been already shown, the nonidentical evolutionary relation turns out to be a selfvarying relation.

Selfvariation of the nonidentical evolutionary relation points to the fact that the nonequilibrium state of material system turns out to be selfvarying. It is evident that this selfvariation proceeds under the action of internal force whose quantity is described by commutator of the unclosed evolutionary form $\omega^p$. 

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(If the commutator be zero, the evolutionary relation would be identical, and this would point to the equilibrium state, i.e. the absence of internal forces.) Everything that gives the contribution into the commutator of the form $\omega^p$ leads to emergence of internal force.

Above it has been shown that under degenerate transformation from non-identical evolutionary relation it can be obtained the identical relation

$$d_\pi \psi = \omega^p$$

From such a relation one can obtain the state function and this corresponds to equilibrium state of the system. But identical relation can be realized only on pseudostructure (which is specified by the condition of degenerate transformation). This means that the transition of material system to equilibrium state proceeds only locally. In other words, it is realized the transition of material system from nonequilibrium state to locally equilibrium one. In this case the total state of material system remains to be nonequilibrium. The conditions of degenerate transformation can be caused by the degrees of freedom of material system.

As one can see from the analysis of nonidentical evolutionary relation, the transition of material system from nonequilibrium state to locally-equilibrium state proceeds spontaneously in the process of selfvarying nonequilibrium state of material system under realization of any degrees of freedom of this system. (Translational degrees of freedom, internal degrees of freedom of the system elements, and so on can be examples of such degrees of freedom).

As it has been already said above, the transition from nonidentical relation (21) obtained from balance conservation laws to identical relation (22) means the following. Firstly, the existence of state differential (left-hand side of relation (22)) points to the transition of material system from nonequilibrium state to locally-equilibrium state. And, secondly, the emergence of closed (on pseudostructure) inexact exterior form (right-hand side of relation (22)) points to the origination of physical structure. (Physical structures that are generated by material systems made up physical fields.)

Thus one can see that the transition of material system from nonequilibrium state to locally-equilibrium state is accompanied by originating the differential-geometrical structures, which are physical structures. The emergence of physical structures in the evolutionary process reveals in material system as the emergence of certain observable formations that develop spontaneously. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, creating massless particles and others. The intensity of such formations is controlled by a quantity accumulated by the evolutionary form commutator at the instant in time of originating physical structures. The transition from evolutionary forms to closed exterior forms describes such processes like the emergence of waves, vortices, turbulent pulsations, the origination of massless particles and others [7].

Since the closed exterior forms corresponding to physical structures are obtained from the evolutionary forms describing material systems, the characteristics of physical structures are determined by characteristics of material system
generating these structures, and this enables one to classify physical structures by the parameters of evolutionary forms and closed exterior forms.

As it has been shown above, the type of differential-geometrical invariant structures, and hence of physical structures (and, accordingly, of physical fields) generated by the evolutionary relation, depends on the degrees of differential form $p$ and $k$ and on the dimension of original inertial space $n$ (here $p$ is the degree of evolutionary form in nonidentical relation that is connected with the number of interacting balance conservation laws, and $k$ is the degree of closed form generated by nonidentical relation). Introducing the classification by numbers $p$, $k$, $n$ one can understand the internal connection between various physical fields.

The above described mechanism of generation of physical structures discloses an unique role of invariant structures in mathematical physics and field theory. It should be emphasized that such results were obtained due to using the skew-symmetric exterior and evolutionary differential forms. The mathematic apparatus of evolutionary forms, which describes the process of realization of closed exterior forms and invariant structures, enables one to investigate the integrability of differential equations (the conjugacy of the differential equations elements), discloses the mechanism of evolutionary processes, discrete transitions, quantum steps, transitions from nonconjugated operators to conjugated ones, and generation of various structures. There are no such possibilities in any mathematical formalism.

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