Harmonic Functions and Instanton Moduli Spaces on the Multi-Taub–NUT Space

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Abstract: Explicit construction of the basic SU(2) anti-instantons over the multi-Taub–NUT geometry via the classical conformal rescaling method is exhibited. These anti-instantons satisfy the so-called weak holonomy condition at infinity with respect to the trivial flat connection and decay rapidly. The resulting unit energy anti-instantons have trivial holonomy at infinity.

We also fully describe their unframed moduli space and find that it is a five dimensional space admitting a singular disk-fibration over \( \mathbb{R}^3 \).

On the way, we work out in detail the twistor space of the multi-Taub–NUT geometry together with its real structure and transform our anti-instantons into holomorphic vector bundles over the twistor space. In this picture we are able to demonstrate that our construction is complete in the sense that we have constructed a full connected component of the moduli space of solutions of the above type.

We also prove that anti-instantons with arbitrary high integer energy exist on the multi-Taub–NUT space.

1. Introduction

The aim of this paper is to construct the most relevant anti-instanton moduli space over the multi-Taub–NUT geometry by elementary means.

An important class of non-compact but complete four dimensional geometries is the collection of the so-called \emph{asymptotically locally flat (ALF)} spaces including several mathematically as well as physically important examples. The flat \( \mathbb{R}^3 \times S^1 \) plays a role in finite temperature Yang–Mills theories, the Euclidean Schwarzschild space \([20]\) deals with quantum gravity and Hawking radiation. If the metric is additionally hyper-Kähler then the space is also called an \emph{ALF gravitational instanton} (in the narrow sense). The flat \( \mathbb{R}^3 \times S^1 \) is a straightforward example; non-trivial ones for this restricted class are provided by the multi-Taub–NUT (or \( A_k \) ALF or ALF Gibbons–Hawking) spaces \([15]\) which carry supersymmetric solutions of string theory and supergravity models, the
Atiyah–Hitchin manifold (and its universal double cover) [3] describing the 2-monopole moduli space over \( \mathbb{R}^3 \) and last but not least the recently constructed \( D_k \) ALF spaces [9].

The ALF asymptotics is a natural generalization of the well-known ALE (asymptotically locally Euclidean) one including the multi-Eguchi–Hanson geometries [15]. Instanton theory over these later spaces possessing several phenomena related with non-compactness (e.g. existence of four dimensional moduli spaces isometric to the original space) is well-known due to the important paper of Kronheimer and Nakajima [27] in which a full ADHM construction was established. The existence of this construction is in some sense not surprising because the original ADHM construction was designed for the flat \( \mathbb{R}^4 \) and all ALE spaces arise by an algebro-geometric deformation of the flat quotients \( \mathbb{C}^2/\Gamma \), where \( \Gamma \subset \text{SU}(2) \) are various discrete subgroups [26].

The natural question arises: what about instanton theory over ALF spaces? Unlike the ALE geometries, these are essentially non-flat spaces in the sense above, therefore one may expect that instanton theory somewhat deviates from that over the flat \( \mathbb{R}^4 \). Some general questions have been answered recently [13] and these investigations pointed out that in spite of their more transcendental nature, open Riemannian spaces with ALF asymptotics rather resemble compact four manifolds at least from the point of view of instanton theory. To test this interesting observation more carefully in this paper we work out the simplest moduli space over the multi-Taub–NUT geometry: we will find that this moduli space is five dimensional, furthermore its part containing concentrated instantons looks like a “collar” of the original manifold, supporting the analogy with the compact case. We note that this apparent compactness is related to the existence of a smooth compactification of ALF spaces motivated by \( L^2 \) cohomology theory [19].

The paper is organized as follows. In Sect. 2 we quickly summarize two important tools for solving the self-duality equations over an anti-half-conformally flat space: the conformal rescaling method of Jackiw–Nohl–Rebbi [4,25] as well as the Atiyah–Ward correspondence [5]. Via conformal rescaling one constructs instantons out of positive harmonic functions with at most pointlike singularities and appropriate decay toward infinity while the Atiyah–Ward correspondence establishes a link between instantons and holomorphic vector bundles. Since these methods meet in Penrose twistor theory we shall also briefly outline it here.

Then we introduce ALF spaces. Referring to recent results on instanton theory over these spaces [13] we precisely define the class of anti-instantons which are expected to form nice moduli spaces. These solutions must obey two conditions: the so-called weak holonomy condition with respect to some smooth flat connection and the rapid decay condition (cf. Definition 2.1 here). The former condition, albeit looks like an analytical one, in fact deals with the topology of infinity of the ALF space only [13, Theorem 2.3] meanwhile the latter one controls the fall-off properties of an anti-instanton, hence is indeed analytical in its nature.

In Sect. 3, taking probably the most relevant ALF example namely the multi-Taub–NUT series, following Anderson–Kronheimer–LeBrun [1] and LeBrun [28] we review this geometry with special attention to the description of its generic complex structures as algebraic surfaces in \( \mathbb{C}^3 \). We also identify a finite number of distinguished complex structures that can be realized as blowups of these algebraic surfaces. The group \( S^1 \) acts on these spaces via isometries and this action has isolated fixed points, called NUTs.

Then in Sect. 4 we come to our main results. We shall prove via conformal rescaling that over the multi-Taub–NUT space \( (M_V, g_V) \) unframed \( L^2 \) moduli spaces of \( \text{SU}(2) \) anti-instantons obeying both the aforementioned weak holonomy condition with respect to the trivial flat connection \( \nabla_\Theta \) as well as the rapid decay condition are not empty.