SOLVING NORMALIZED STATIONARY POINTS OF A CLASS OF EQUILIBRIUM PROBLEM WITH EQUILIBRIUM CONSTRAINTS

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Abstract. This paper focuses on solving normalized stationary points of a class of equilibrium problem with equilibrium constraints (EPEC). We show that, under some kind of separability assumption, normalized C-/M-/S-stationary points of EPEC are actually C-/M-/S-stationary points of an associated mathematical program with equilibrium constraints (MPEC), which implies that we can solve MPEC to obtain normalized stationary points of EPEC. In addition, we demonstrate the proposed approach on competition of manufacturers for similar products in the same city.

1. Introduction. An equilibrium problem with equilibrium constraints (EPEC) is a member of a new class of mathematical programs that often arise in engineering and economics applications. In noncooperative game theory, the well-known Stackelberg game (single-leader-multi-follower game) can be formulated as an optimization problem called a mathematical program with equilibrium constraints (MPEC). More generally, an EPEC is a mathematical program to find equilibria that simultaneously solve several MPECs, each of which is parameterized by decision variables of other MPECs. In this paper, we consider the following EPEC with $K$ players, in which each player $\nu$ solves the following parametric MPEC with independent decision variables $x^\nu \in \mathbb{R}^{n^\nu}$ and shared decision variables $y \in \mathbb{R}^{n_0}$:

$$\text{MPEC}(x^{-\nu}) \begin{cases} \min_{x^\nu, y} & f^\nu(x^\nu, y; x^{-\nu}) \\ \text{s.t.} & g^\nu(x^\nu) \leq 0, \ g(x^\nu, y; x^{-\nu}) \leq 0, \\ & 0 \leq G(x^\nu, y; x^{-\nu}) \perp H(x^\nu, y; x^{-\nu}) \geq 0. \end{cases} \quad (1)$$

where $f^\nu : \mathbb{R}^n \to \mathbb{R}$, $g^\nu : \mathbb{R}^{n^\nu} \to \mathbb{R}^{p^\nu}$, $g : \mathbb{R}^n \to \mathbb{R}^q$, and $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable functions in both $x = (x^\nu)_{\nu=1}^{K} \in \mathbb{R}^{n-n_0}$ and $y \in \mathbb{R}^{n_0}$ with $n = \sum_{\nu=1}^{K} n^\nu$, and $x^{-\nu} = (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^K) \in \mathbb{R}^{n-n_0-n_0}$ is not a variable but a fixed vector.

The EPEC has recently been studied by some researchers and used to model several problems in applications. Optimality conditions for EPEC were studied...
(see, e.g., [17, 18]) and some numerical methods are also proposed (see, e.g., [6, 12, 15, 21]), respectively. In particular, Hu [12] presented a diagonalization method that solved sequentially parametric MPECs of each player until an approximation equilibrium point is found. Su [21] proposed a sequential nonlinear equilibrium method that was based on a relaxation technique in [23]. Leyffer and Munson [15] investigated several reformulations of EPEC and, based on them, proposed some numerical methods. Hu and Fukushima [10, 11] proposed a method for games with only linear equality constraints in followers problems. Guo and Lin [6, 8] reformulated various stationarities for EPECs as constrained equations and proposed a globally and superlinearly convergent algorithm to solve these constrained equations. In [13, 14, 15], Kulkarni et al. reformulated some EPEC models as MPECs by using potential games and shared constraints. This enables a way to explore the existence of equilibria in EPEC problems.

An equilibrium problem usually has many solutions and the uniqueness of an equilibrium point is expected only under very restrictive assumptions. If there are many equilibrium points, we could try to find all equilibrium points or as many equilibrium points as possible. An alternative approach is to single out an equilibrium point that has some special properties. The normalized equilibrium point [20] is such an equilibrium point that the Lagrange multipliers (shadow prices) associated with the shared constraints are equal among all players. Note that EPEC is highly nonconvex and hence we study its stationary point.

In this paper, we use an MPEC reformulation to show the existence of equilibria and solve the EPEC model. The EPEC model is different from the ones considered in [13, 14] in which the decision variables (on the lower level) vary among different leaders decision problems, while it is assumed in our model that the leaders have a common knowledge on the solution of the lower level equilibrium problem. Moreover, from the modeling perspective, our research focuses on the leadership role which is different from the framework in [13, 14]. We study normalized stationary points of a class of EPEC where the multipliers associated with the shared constraints are proportional, which generalizes the work of Leyffer and Munson [15] in which the multipliers are identical for all players. We show that, under some kind of separability assumption, normalized C-/M-/S-stationary points of EPEC is standard C-/M-/S-stationary points of an associated mathematical program with equilibrium constraints (MPEC), which implies that we can solve MPEC to obtain normalized stationary points of EPEC. Moreover, for any given sequence of positive constants, we can solve an MPEC to obtain an equilibrium point of EPEC and thus we can obtain much more equilibria of EPEC than [15].

This paper is organized as follows. In the next section we briefly introduce C-/M-/S-stationarity condition for EPEC and show how equilibrium points can be computed reliably by solving its C-/M-/S-stationary points for EPEC. Section 3 present a special kind of C-/M-/S-stationarity conditions for a class of EPEC, which is called normalized C-/M-/S-stationarities and reformulate to C-/M-/S-stationary points of an associated MPEC. In the section 4, we consider a class of EPEC model for similar products market in the same city and demonstrate the proposed approach on the model. In addition, some concluding remarks are given in Section 5.

The following notations will be used. We denote by $\nabla h(x) \in \mathbb{R}^{s \times t}$ the transposed Jacobian matrix of a differentiable function $h : \mathbb{R}^s \to \mathbb{R}^t$ at a given point $x$. For a real-valued function $h(x, y)$ with the variable $x \in \mathbb{R}^s$ and $y \in \mathbb{R}^t$, the partial
gradients with respect to $x$ and $y$ are denoted by $\nabla_x h(x, y) \in \mathbb{R}^s$ and $\nabla_y h(x, y) \in \mathbb{R}^t$, respectively.

2. Stationarity conditions and formulation. Let $\mathcal{F}^\nu$ be the feasible region of MPEC($x^{-\nu}$) for each $\nu \in \{1, \ldots, K\}$ and $\mathcal{F}$ be the feasible region of the EPEC (1), i.e.,

\[ \mathcal{F} := \{(x, y) \mid (x, y) \in \mathcal{F}^\nu, \nu = 1, \ldots, K\}. \]

To facilitate the notation, we define the following index sets for a given $(x^*, y^*) \in \mathcal{F}$:

\[ \mathcal{I}^* := \{ i \mid G_i(x^*, y^*) = 0 < H_i(x^*, y^*) \}, \]
\[ \mathcal{J}^* := \{ i \mid G_i(x^*, y^*) = 0 = H_i(x^*, y^*) \}, \]
\[ \mathcal{K}^* := \{ i \mid G_i(x^*, y^*) > 0 = H_i(x^*, y^*) \}. \]

Obviously, $\{\mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*\}$ is a partition of $\{1, 2, \ldots, m\}$. For each $\nu = 1, \ldots, K$, the MPEC-Lagrangian of MPEC($x^{-\nu}$) is defined by

\[ L^\nu(x, y, \lambda^\nu, \mu^\nu, u^\nu, v^\nu) : = f^\nu(x, y) + g^\nu(x^\nu)T \lambda^\nu + g(x, y)T \mu^\nu - G(x, y)^T u^\nu - H(x, y)^T v^\nu. \]

**Definition 2.1.** We say that $(x^*, y^*)$ is a local (global) equilibrium point of the EPEC (1), if for each $\nu = 1, \ldots, K$, $(x^{*\nu}, y^*)$ is locally (globally) optimal for the MPEC($x^{-\nu}$).

In the following, we review some well-known stationarities for MPEC($x^{-\nu}$) (see, e.g., [3, 22, 24, 25]).

**Definition 2.2.** (a) A vector $(x^{*\nu}, y^*)$ is called a Weakly stationary (W-stationary) point of the MPEC($x^{-\nu}$) if there exist multipliers $(\lambda^\nu, \mu^\nu, u^\nu, v^\nu)$ such that

\[ \begin{cases} 
\nabla_x L^\nu(x^*, y^*, \lambda^\nu, \mu^\nu, u^\nu, v^\nu) = 0, \\
\nabla_y L^\nu(x^*, y^*, \lambda^\nu, \mu^\nu, u^\nu, v^\nu) = 0, \\
\min(\lambda^\nu, -g(x^\nu)) = 0, \\
\min(\mu^\nu, -g(x^*, y^*)) = 0, \\
u_i^\nu = 0, & i \in \mathcal{K}^*, \\
u_i^\nu = 0, & i \in \mathcal{I}^*. 
\end{cases} \quad \text{(2)} \]

(b) A vector $(x^{*\nu}, y^*)$ is called a Clarke stationary (C-stationary) point of the MPEC($x^{-\nu}$) if there exist multipliers $(\lambda^\nu, \mu^\nu, u^\nu, v^\nu)$ such that (2) holds and

\[ u_i^\nu v_i^\nu \geq 0, \quad i \in \mathcal{J}^*. \]

(c) A vector $(x^{*\nu}, y^*)$ is called a Mordukhovich stationary (M-stationary) point of the MPEC($x^{-\nu}$) if there exist multipliers $(\lambda^\nu, \mu^\nu, u^\nu, v^\nu)$ such that (2) holds and either $u_i^\nu > 0, v_i^\nu > 0$ or $u_i^\nu v_i^\nu = 0, \quad i \in \mathcal{J}^*$.

(d) A vector $(x^{*\nu}, y^*)$ is called a strongly stationary (S-stationary) point of the MPEC($x^{-\nu}$) if there exist multipliers $(\lambda^\nu, \mu^\nu, u^\nu, v^\nu)$ such that (2) holds and

\[ u_i^\nu \geq 0, v_i^\nu \geq 0, \quad i \in \mathcal{J}^*. \]

**Definition 2.3.** We say that $(x^*, y^*)$ is C-stationary (M-stationary, S-stationary) to the EPEC (1) if for each $\nu = 1, \ldots, K$, $(x^{*\nu}, y^*)$ is C-stationary (M-stationary, S-stationary) to the MPEC($x^{-\nu}$).

By the definition, it is easy to see that

S-stationarity $\implies$ M-stationarity $\implies$ C-stationarity.
The following result follows from [22, Theorem 2] or [24, Theorem 3.2] immediately.

**Proposition 1.** [21] Let \((x^*, y^*)\) be a local equilibrium point of EPEC (1). If for each \(\nu = 1, \ldots, K\), the MPEC-LICQ holds at \((x^\nu, y^\nu)\) for MPEC\((x^\nu, y^\nu)\), then there exist multipliers \((\lambda^*, \mu^*, u^*, v^*)\) such that \((x^*, y^*)\) is an S-stationary point of the EPEC (1).

The following result follows from [7, Corrolary 4.1]. Note that the MPEC-relaxed constant positive linear dependence (MPEC-RCPLD) is a very weak qualification for M-stationarity.

**Proposition 2.** [7] Let \((x^*, y^*)\) be a local equilibrium point of EPEC (1). If for each \(\nu = 1, \ldots, K\), the MPEC-relaxed constant positive linear dependence (MPEC-RCPLD) holds at \((x^\nu, y^\nu)\) for MPEC\((x^\nu, y^\nu)\), then there exist multipliers \((\lambda^*, \mu^*, u^*, v^*)\) such that \((x^*, y^*)\) is an M-stationary point of the EPEC (1) and then it is also a C-stationary point.

### 3. Normalized stationary points

In this section, we consider a special kind of C-/M-/S-stationarity conditions for a class of EPEC, which is called normalized C-/M-/S-stationarities. We need the following separability assumption for objective functions.

\[
f^\nu(x, y) = \bar{f}^\nu(x^\nu) + \tilde{f}^\nu(x^{\nu}) + \hat{f}^\nu(y), \quad \nu = 1, \ldots, K. \tag{3}
\]

In the rest of the paper, we assume that the assumption (3) holds. Then the EPEC (1) becomes the following problem: For each \(\nu = 1, \ldots, K\), player \(\nu\) solves

\[
\begin{align*}
\min_{x^\nu, y} & \quad \bar{f}^\nu(x^\nu) + \tilde{f}^\nu(x^{-\nu}) + \hat{f}^\nu(y) \\
\text{s.t.} & \quad g^\nu(x^\nu) \leq 0, \quad g(x, y; x^{-\nu}) \leq 0, \\
& \quad 0 \leq G(x^\nu, y; x^{-\nu}) \perp H(x^\nu, y; x^{-\nu}) \geq 0,
\end{align*} \tag{4}
\]

**Definition 3.1.** We say that \((x^*, y^*)\) is a normalized C-(M-S-) stationary point of the EPEC (4) if \((x^*, y^*)\) is a C-(M-S-) stationary point of the EPEC (4) and there exist vector \(\beta\) with positive components and vector \((\mu^0, u^0, v^0)\) such that the following holds

\[
\beta^\nu(\mu^\nu, u^\nu, v^\nu) = (\mu^0, u^0, v^0), \quad \nu = 1, \ldots, K.
\]

The following theorem reveals the relations between normalized stationary points of the EPEC (4) and stationary points of an MPEC.

**Theorem 3.2.** The normalized C-(M-S-) stationary point of the EPEC (4) is the C-(M-S-) stationary point of the following MPEC, where \(k\) is any element in \(\{1, \ldots, K\}\):

\[
\begin{align*}
\min_{x,y} & \quad \sum_{\nu=1}^{K} \beta^\nu \bar{f}^\nu(x^\nu) + \beta^k \hat{f}^k(y) \\
\text{s.t.} & \quad g^\nu(x^\nu) \leq 0, \quad \nu = 1, \ldots, K, \\
& \quad g(x, y) \leq 0, \\
& \quad 0 \leq G(x, y) \perp H(x, y) \geq 0.
\end{align*} \tag{5}
\]
Conversely, if \((x^*, y^*)\) is the \((\alpha, \beta)\)-stationary point of \((5)\)
then \((x^*, y^*)\) is the normalized \((\alpha, \beta)\)-stationary point of the EPEC \((4)\).

Proof. Let \((x^*, y^*)\) be the normalized \((\alpha, \beta)\)-stationary point of the EPEC \((4)\). By
definition of \((\alpha, \beta)\)-stationary points of the EPEC \((4)\), we have that, for \(\nu = 1, \ldots, K,\)
\[
\nabla_x f'(x^*; y^*) + \nabla_y g'(x^*; y^*)\lambda^\nu + \nabla_x g(x^*, y^*)\mu^\nu - \nabla y H(x^*, y^*)\nu^\nu = 0,
\]
\[
\nabla_y f'(y^*) + \nabla_y g(x^*, y^*)\mu^\nu - \nabla y G(x^*, y^*)\nu^\nu - \nabla y H(x^*, y^*)\nu^\nu = 0,
\]
\[
\min(\lambda^\nu, -g'(x^*; y^*)) = 0, \quad \min(\mu^\nu, -g(x^*, y^*)) = 0,
\]
\[
u^\nu = 0, \quad i \in K^*, \quad v^\nu = 0, \quad i \in I^*,
\]
\[
u^\nu \geq 0, \quad i \in J^*.
\]

Since \((x^*, y^*)\) is normalized \((\alpha, \beta)\)-stationary, there exist positive vector \(\beta\) and \((\mu^0, u^0, v^0)\) such that
\[
\beta^\nu(\mu^\nu, u^\nu, v^\nu) = (\mu^0, u^0, v^0), \quad \nu = 1, \ldots, K.
\]

Multiplying \((6)-(8)\) by \(\beta^\nu\) and \((9)\) by \(\beta^2\nu\), we have from the above formula that, for
\(\nu = 1, \ldots, K,\)
\[
\beta^\nu \nabla_x f'(x^*; y^*) + \nabla_y g'(x^*; y^*)\beta^\nu\lambda^\nu + \nabla_x g(x^*, y^*)\mu^\nu - \nabla y H(x^*, y^*)\nu^\nu = 0,
\]
\[
\nabla_y f'(y^*) + \nabla_y g(x^*, y^*)\mu^\nu - \nabla y G(x^*, y^*)\nu^\nu - \nabla y H(x^*, y^*)\nu^\nu = 0,
\]
\[
\min(\beta^\nu \lambda^\nu, -g'(x^*; y^*)) = 0, \quad \min(\mu^\nu, -g(x^*, y^*)) = 0,
\]
\[
u^\nu = 0, \quad i \in K^*, \quad v^\nu = 0, \quad i \in I^*,
\]
\[
u^\nu \geq 0, \quad i \in J^*.
\]

It follows from \((11)\) that
\[
\beta^1 \nabla_x f'(x^*) = \beta^2 \nabla_y f'(y^*) = \cdots = \beta^K \nabla_y f^K(y^*).
\]

Since \(\beta\) is positive, \((12)\) is equivalent to that for \(\nu = 1, \ldots, K,\)
\[
\min(\beta^\nu \lambda^\nu, -g'(x^*; y^*)) = 0, \quad \min(\mu^\nu, -g(x^*, y^*)) = 0.
\]

From \((10)-(16)\), it is not difficult to verify that \((x^*, y^*)\) is a \((\alpha, \beta)\)-stationary point of
\((5)\) with \(\alpha\)-multiplier \((\beta^1 \lambda^1, \ldots, \beta^K \lambda^K, \mu^0, u^0, v^0)\). In a similar way, we can show that
the normalized \((\alpha, \beta)\)-stationary points of the EPEC \((4)\) is the standard \((\alpha, \beta)\)-stationary points of \((5)\).
It is not hard to show the converse part in a similar way.

Example 1. There are two players and they solve the following problems:
\[
\begin{align*}
\min_{x_1, y} \ f_1(x_1) + \ f_1(x_2) + \frac{1}{2} y^2 & \quad \min_{x_2, y} \ f_2(x_2) + \ f_2(x_1) + \frac{1}{2} y^2 \\
\text{s.t.} \ g_1(x_1) & \leq 0, \quad \text{s.t.} \ g_2(x_2) \leq 0, \\
g(x_1, x_2) & \leq 0, \quad \text{g}(x_1, x_2) \leq 0, \\
0 \leq y - \alpha & \perp y H(x_1, x_2) \geq 0, \quad 0 \leq y - \alpha \perp y H(x_1, x_2) \geq 0, \\
\end{align*}
\]
where \(\alpha > 0\).
Since $\alpha > 0$, $y > 0$, we are easy to find $\beta^1 > 0$ and $\beta^2 > 0$ satisfying $\beta^1 y = \beta^2$. It is shown that EPEC (17) exists normalized equilibrium points. That is to say, the condition (15) is easy satisfied and if EPEC (17) exist stationary point then must exists normalized stationary point. The normalized C-(M-,S-) stationary point of the EPEC (17) is the standard C-(M-,S-) stationary point of the following MPEC:

$$\min_{x,y} \beta^1 \bar{f}_1(x_1) + \beta^2 \bar{f}_2(x_2) + \frac{\beta^1}{3} y^3$$

s.t. $g_1(x_1) \leq 0$, $g_2(x_2) \leq 0$, $g(x_1, x_2) \leq 0$, $0 \leq y - \alpha \perp yH(x_1, x_2) \geq 0$.

or

$$\min_{x,y} \beta^1 \bar{f}_1(x_1) + \beta^2 \bar{f}_2(x_2) + \frac{\beta^2}{2} y^2$$

s.t. $g_1(x_1) \leq 0$, $g_2(x_2) \leq 0$, $g(x_1, x_2) \leq 0$, $0 \leq y - \alpha \perp yH(x_1, x_2) \geq 0$.

Conversely, if $(x^*, y^*)$ is the C-(M-,S-) stationary point of (18) or (19) then $(x^*, y^*)$ is the normalized C-(M-,S-) stationary point of the EPEC (17).

Since EPECS can be regarded as a special case of a generalized Nash equilibrium problems [9], we can generalize this result to the generalized Nash equilibrium problems.

**Corollary 1.** The normalized stationary point of the following generalized Nash equilibrium problems with $K$ players, in which each player $\nu$ solves the optimization problem:

$$\min_{x^{\nu}} \bar{f}^{\nu}(x^{\nu}) + \tilde{f}^{-\nu}(x^{-\nu})$$

s.t. $g^{\nu}(x^{\nu}) \leq 0$, $g(x) \leq 0$.

is the stationary point of the following optimization problem:

$$\min_{x} \sum_{\nu=1}^{K} \beta^{\nu} \bar{f}^{\nu}(x^{\nu})$$

s.t. $g^{\nu}(x^{\nu}) \leq 0$, $\nu = 1, \ldots, K$, $g(x) \leq 0$.

Conversely, if $(x^*, y^*)$ is the stationary point of (21) then $(x^*, y^*)$ is the normalized stationary point of the GNEP (20).

**Example 2.** (Facchinei’s example) This problem is taken from [1]. Consider the game with two players:

$$\min_{x_1} (x_1 - 1)^2$$

s.t. $x_1 + x_2 \leq 1$

and

$$\min_{x_2} (x_2 - \frac{1}{2})^2$$

s.t. $x_1 + x_2 \leq 1$.

Then it is easy to check that the solutions of this problem are given by $(\alpha, 1 - \alpha)$ for every $\alpha \in [\frac{1}{2}, 1]$. We consider its normalized equilibria with weights $(\beta^1, \beta^2)$ satisfying $\beta^1 \mu^1 = \beta^2 \mu^2 = \mu^0$, where $\beta^1 > 0$, $\beta^2 > 0$. By above result, the normalized
equilibria of the GNEP (22) is the optimal solution of the following optimization problem:

\[
\min_{x_1, x_2} \beta^1 (x_1 - 1)^2 + \beta^2 (x_2 - \frac{1}{2})^2
\]

s.t. \( x_1 + x_2 \leq 1. \)

This problem have the optimal solution \((1 - \frac{\alpha}{2}, \frac{\beta^2}{\beta^1} + \frac{\beta^2}{\beta^1})\). Obviously, if \(\alpha \in (\frac{1}{2}, 1)\), every equilibria of the GNEP (22) is the normalized equilibria.

4. Applications. In this section, we first discuss normalized equilibrium of EPEC satisfying the assumption (3) arising from competition of manufacturers for similar products in the same city.

4.1. Model in similar products market. Since similar products have some same function but not exactly the same, similar products have different price. We consider an oligopoly consisting of \(K + F\) manufacturers that produce similar products noncooperatively before the market demand is realized. The first \(K\) manufacturers (herein leaders) have no capacity installed and thus have to decide now what their future output will be before the demand function is realized. The remaining \(F\) manufacturers (followers) have sufficient capacity installed and thus do not have to make a decision today, but instead they can wait to observe the quantities supplied by the \(K\) leaders as well as the realized demand function before making a decision on their supply quantities.

The market demand is characterized by inverse demand functions \(p_{\nu}(x, y), \nu = 1, \ldots, K + F\), where \(p_{\nu}(x, y)\) is the market price of the product made by the manufacturer \(\nu\), \(x = (x_i)_{i=1}^K, x_i\) is the supply quantity of the leader \(i\), and \(y = (y_j)_{j=1}^F\), \(y_j\) is the supply quantity of the follower \(j\).

Before market demand is realized, leader \(i\) chooses his quantity \(x_i\). The leader’s profit can be formulated as

\[
R_i(x_i, X_{-i}, Y) = x_i P_i(x_i, X_{-i}, Y) - C_i(x_i),
\]

where \(X_{-i}\) denotes the total bids by the other leaders, \(Y = (y_j)_{j=1}^F\), where \(y_j\) is the strategies of the \(j\) follower. \(x_i P_i(x_i, X_{-i}, Y)\) means the total revenue for leader \(i\), and \(C_i(x_i)\) denotes the cost function of leader \(i\). The \(i\)th leader’s decision problem is to choose the supply quantity \(x_i\) that maximizes its profit; that is,

\[
\max_{x_i \in X_i} R_i(x_i, X_{-i}, Y) = x_i P_i(x_i, X_{-i}, Y) - C_i(x_i),
\]

where \(X_i := \{ x_i \in [0, +\infty) \mid g_i(x_i) \leq 0, g(x, y) \leq 0 \}\) is nonempty and bounded convex set, for each \(i = 1, \ldots, K\).

The \(j\)th follower chooses its supply quantity after observing the aggregate leaders supply \(X\). Thus, the total revenue of the \(j\)th follower is \(y_j P_j(X, y_j, Y_{-j})\), and its total cost is \(C_j(y_j)\). Consequently, the \(j\)th follower profit is

\[
R_j(X, y_j, Y_{-j}) = y_j P_j(X, y_j, Y_{-j}) - C_j(y_j),
\]

where \(Y_{-j}\) denotes the total bids by the other followers, \(X = (x_i)_{i=1}^K\), where \(x_i\) is the strategies of the leader \(i\). The \(j\)th follower decision problem is

\[
\max_{y_j \geq 0} R_j(X, y_j, Y_{-j}) = y_j P_j(X, y_j, Y_{-j}) - C_j(y_j). \quad (23)
\]
If (23) is convex and satisfies a constraint qualification for each follower, then the condition that each follower chooses an optimal strategy is equivalent to the following KKT conditions:

\[ 0 \leq y_j \perp -\nabla_{y_j} R_j(X, y_j, Y_{-j}) \geq 0, \quad j = 1, \ldots, F. \]

The aim of each leader \( i, \ i = 1, \ldots, K, \) is to choose a strategy \( x_i \) that solves the following MPEC:

\[
\begin{align*}
\min_{x_i \in X_i} & \quad -R_i(x_i, X_{-i}, Y) \\
\text{s.t} & \quad g_i(x_i) \leq 0, g(x, y) \leq 0, \\
& \quad 0 \leq y_j \perp -\nabla_{y_j} R_j(X, y_j, Y_{-j}) \geq 0, \quad j = 1, \ldots, F.
\end{align*}
\]

### 4.2. Preliminary numerical results.
Consider the above multi-leader-follower games with two leader and one follower and set the data as follows:

- The inverse demand functions are given by

\[
\begin{align*}
\text{Leader 1} & \quad P_1(x_1, x_2, y) := a_1 - b_1(x_1 + \frac{x_2 + y}{x_1}), \\
\text{Leader 2} & \quad P_2(x_1, x_2, y) := a_2 - b_2(x_2 + \frac{x_1 + y}{x_2}), \\
\text{follower} & \quad P_3(x_1, x_2, y) := a_3 - b_3(y + \frac{x_1 + x_2}{y}).
\end{align*}
\]

When \( x_1 \in (0, \sqrt{x_2 + y}) \) is bigger, \( P_1 \) is bigger. On the other hand, when \( x_1 \in (\sqrt{x_2 + y}, +\infty) \) is bigger, \( P_1 \) is smaller. Actually, the supply quantities of leaders are far more than the follower's and they are in the same order of magnitude. Consequently, \( x_1 \) is in \( (\sqrt{x_2 + y}, +\infty) \) in general, when there is not enough demand for its product, the price goes down. While the product is in short supply relative to the demand, the price will be bid up. The effective level of various variables was different.

- The cost functions are given by

\[
\begin{align*}
\text{Leader 1} & \quad C_1(x_1) := c_1 x_1, \\
\text{Leader 2} & \quad C_2(x_2) := c_2 x_2, \\
\text{follower} & \quad C_3(y) := c_3 y.
\end{align*}
\]

- The constraint functions of the leaders are given by

\[
\begin{align*}
g_1(x_1) & := x_1 - d_1, \\
g_2(x_2) & := x_2 - d_2, \\
g(x_1, x_2, y) & := x_1 + x_2 + y - d_3.
\end{align*}
\]

where \( a_i \geq 0, b_i \geq 0, c_i \geq 0, \ i = 1, 2, 3, \ x_1 \geq 0, x_2 \geq 0, y \geq 0. \)

Then the multi-leader-follower game can be written as follows:

\[
\begin{align*}
\min_{x_1, y} & \quad b_1 x_1^2 - (a_1 - c_1) x_1 + b_1 x_2 + b_1 y \\
\text{s.t} & \quad x_1 - d_1 \leq 0, \\
& \quad x_1 + x_2 + y - d_3 \leq 0, \\
& \quad 0 \leq y \perp 2b_3 y - (a_3 - c_3) \geq 0.
\end{align*}
\]

\[
\begin{align*}
\min_{x_2, y} & \quad b_2 x_2^2 - (a_2 - c_2) x_2 + b_2 x_1 + b_2 y \\
\text{s.t} & \quad x_2 - d_2 \leq 0, \\
& \quad x_1 + x_2 + y - d_3 \leq 0, \\
& \quad 0 \leq y \perp 2b_3 y - (a_3 - c_3) \geq 0.
\end{align*}
\]
By Theorem 3.2, the normalized C-(M-,S-) stationary point of the EPEC (24) is the standard C-(M-,S-) stationary point of the following MPEC:

$$\begin{align*}
\min_{x,y} \quad & \beta_1 (b_1 x_1^2 - (a_1 - c_1) x_1) + \beta_2 (b_2 x_2^2 - (a_2 - c_2) x_2) + \beta_i b_i y \\
\text{s.t.} \quad & x_1 - d_1 \leq 0, \\
& x_2 - d_2 \leq 0, \\
& x_1 + x_2 + y - d_3 \leq 0, \\
& 0 \leq y - 2b_3 y - (a_3 - c_3) \geq 0,
\end{align*}$$

(25)

where $i$ can be 1 or 2. Conversely, if $(x^*, y^*)$ is the C-(M-,S-) stationary point of (25) and exists vector $\beta$ with positive components such that $\beta_1 b_1 = \beta_2 b_2$ then $(x^*, y^*)$ is the normalized C-(M-,S-) stationary point of the EPEC (24).

Obviously, if there no exists vector $\beta$ with positive components such that $\beta_1 b_1 = \beta_2 b_2$ then the EPEC (24) has not normalized C-(M-,S-) stationary point.

We set the parameters by

$$\begin{align*}
a_1 &= 68, b_1 = 4, a_2 = 78, b_2 = 5, a_3 = 66, b_3 = 5, \\
c_1 &= 24, c_2 = 26, c_3 = 28, d_1 = 7, d_2 = 6, d_3 = 14.
\end{align*}$$

Next we consider to solve the KKT stationarity system of MPEC(25). By introducing some slack and auxiliary variables, the KKT stationarity system of MPEC(25) becomes constrained equations, which can be solved in Matlab R2010a by the Levenberg-Marquardt-type methods given in [2]. The computational results are

$$(x^1, x^2, y) = (5.25, 4.95, 3.80)$$

and accordingly

$$(\beta_1, \beta_2) = (5, 4)$$

The results shown that the proposed methods were able to solve normalized S-stationary point of a class of special EPEC successfully.

5. Conclusions. We introduce a class of special EPEC and solve its normalized stationary point, that result in a standard MPEC. In this approach, the special equilibrium problem with equilibrium constraints is solved by a single optimization problem, unlike traditional approach that solve a sequence of related optimization problems. Additionally, we provide numerical results and application demonstrating that our new approach.

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