Factors of Alternating Convolution of the Gessel Numbers

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Abstract

The Gessel number \( P(n, r) \) is the number of the paths in plane with \((1, 0)\) and \((0, 1)\) steps from \((0, 0)\) to \((n + r, n + r - 1)\) that never touch any of the points from the set \( \{(x, x) \in \mathbb{Z}^2 : x \geq r\} \). We show that there is a close relationship between the Gessel numbers \( P(n, r) \) and the super Catalan numbers \( S(n, r) \). By using new sums, we prove that an alternating convolution of the Gessel numbers \( P(n, r) \) is always divisible by \( \frac{1}{2} S(n, r) \).

Keywords: Gessel Number, Super Catalan Number, Catalan Number, M sum, Stanley’s formula.

2020 Mathematics Subject Classification: 05A10, 11B65.

1 Introduction

Let \( n \) be a non-negative integer, and let \( r \) be a fixed positive integer. Let the number \( P(n, r) \) denote \( \frac{r}{2(n + r)} \binom{2n}{n} \binom{2r}{r} \). We shall call \( P(n, r) \) as the \( n \)th Gessel number of order \( r \).

It is known that \( P(n, r) \) is always an integer. They have an interesting combinatorial interpretation. The Gessel number \( P(n, r) \) [5, p. 191] counts all paths in the plane with unit horizontal and vertical steps from \((0, 0)\) to \((n + r, n + r - 1)\) that never touch any of the points \((r, r), (r + 1, r + 1), \ldots, \).

Recently [9, Theorem 5, p. 2], it is shown that \( P(n, r) \) is the number of all lattice paths in plane with \((1, 0)\) and \((0, 1)\) steps from \((0, 0)\) to \((n + r, n + r - 1)\) that never touch any of the points from the set \( \{(x, x) \in \mathbb{Z}^2 : 1 \leq x \leq n\} \); where \( n \) and \( r \) are positive integers.

Let \( C_n = \frac{1}{n+1} \binom{2n}{n} \) denote the \( n \)th Catalan number, and let \( S(n, r) = \frac{\binom{2n}{n} \binom{2r}{r}}{\binom{n+r}{n}} \) denote the \( n \)th super Catalan number of order \( r \). Obviously, \( P(n, 1) = C_n \). Furthermore, it is readily verified that

\[
P(n, r) = \binom{n + r - 1}{n} \frac{1}{2} S(n, r).
\]
It is known that $S(n, r)$ is always an even integer except for the case $n = r = 0$. See [1, Introduction] and [3, Eq. (1), p. 1]. For only a few values of $r$, there exist combinatorial interpretations of $S(n, r)$. See, for example [1, 3, 4, 14, 15]. The problem of finding a combinatorial interpretation for super Catalan numbers of an arbitrary order $r$ is an intriguing open problem.

By using Eq. (1), it follows that $P(n, r)$ is an integer. Note that Gessel numbers $P(n, r)$ have a generalization [7, Eq. (1.10), p. 2]. Also, it is known [5, p. 191] that, for a fixed positive integer $r$, the smallest positive integer $K_r$ such that $\frac{K_r}{n+r}(\binom{2n}{n})$ is an integer for every $n$ is $\frac{r}{2}\binom{2n}{r}$.

Let us consider the following sum:

$$\phi(2n, m, r - 1) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m P(k, r) P(2n - k, r);$$

where $m$ is a positive integer.

For $r = 1$, the sum in Eq. (2) reduces to

$$\phi(2n, m, 0) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m C_k C_{2n-k}.$$  

Recently, by using new method, it is shown [12, Cor. 4, p. 2] that the sum $\phi(2n, m, 0)$ is divisible by $\binom{2n}{n}$ for all non-negative integers $n$ and for all positive integers $m$. In particular, $\phi(2n, 1, 0) = C_n \binom{2n}{n}$. See [13, Th. 1, Eq. (2)]. Interestingly, Gessel numbers appear [12, Eq. (68), p. 17] in this proof.

By using the Eq. (1), the sum $\phi(2n, m, r - 1)$ can be rewritten, as follows:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{k + r - 1}{k} \binom{2n - k + r - 1}{2n - k} \frac{1}{2} S(k, r) \frac{1}{2} S(2n - k, r).$$  

Let $\Psi(2n, m, r - 1)$ denote the following sum

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m S(k, r) S(2n - k, r).$$

Recently, by using new method, it is shown [11, Th. 3, p. 3] that the sum $\Psi(2n, m, r - 1)$ is divisible by $S(n, r)$ for all non-negative integers $n$ and $m$. In particular, $\Psi(2n, 1, r - 1) = S(n, r) S(n + r, n)$. See [11, Th. 1, Eq. (1), p. 2].

Also, it is known [10, Th. 12] that the sum $\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{k + r - 1}{k} \binom{2n - k + r - 1}{2n - k}$ is divisible by $lcm(\binom{2n}{n}, \binom{n+r-1}{n})$ for all non-negative integers $n$ and all positive integers $m$ and $r$.

Our main result is, as follows:

**Theorem 1.** The sum $\phi(2n, m, r - 1)$ is always divisible by $\frac{1}{2} S(n, r)$ for all non-negative integers $n$ and for all positive integers $m$ and $r$. 


For proving Theorem 1, we use a new class of binomial sums [10, Eqns. (27) and (28)] that we call $M$ sums.

**Definition 2.** Let $n$ and $a$ be non-negative integers, and let $m$ be a positive integer. Let $S(n, m, a) = \sum_{k=0}^{n} \binom{n}{k}^m F(n, k, a)$, where $F(n, k, a)$ is an integer-valued function. Then $M$ sums for the sum $S(n, m, a)$ are, as follows:

$$M_S(n, j, t; a) = \binom{n-j}{j} \sum_{v=0}^{n-2j} \binom{n-2j}{v} \left( \binom{n}{j+v} t \right) F(n, j+v, a);$$  \hspace{1cm} (6)

where $j$ and $t$ are non-negative integers such that $j \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Obviously, the following equation [10, Eq. (29)],

$$S(n, m, a) = M_S(n, 0, m-1; a)$$  \hspace{1cm} (7)

holds.

Let $n, j, t,$ and $a$ be same as in Definition 2.

It is known [10, Th. 8] that $M$ sums satisfy the following recurrence relation:

$$M_S(n, j, t+1; a) = \binom{n-j}{j} \sum_{u=0}^{n-2j} \binom{n-j}{u} M_S(n, j+u, t; a).$$  \hspace{1cm} (8)

Moreover, we shall prove that $M$ sums satisfy another interesting recurrence relation:

**Theorem 3.** Let $n$ and $a$ be non-negative integers, and let $m$ be a positive integer. Let $R(n, m, a)$ denote

$$\sum_{k=0}^{n} \binom{n}{k}^m G(n, k, a);$$

where $G(n, k, a)$ is an integer-valued function. Let $Q(n, m, a)$ denote

$$\sum_{k=0}^{n} \binom{n}{k}^m H(n, k, a);$$

where $H(n, k, a) = \binom{a+k}{a} \binom{a+n-k}{a} G(n, k, a)$. Then the following recurrence relation is true:

$$M_Q(n, j, 0; a) = \binom{a+j}{a} \sum_{l=0}^{a} \binom{n-j+l}{l} \binom{n-j}{a-l} M_R(n, j+a-l, 0; a).$$  \hspace{1cm} (9)

Note that, by using substitution $k = v + j$, the Eq. (6) becomes, as follows:

$$M_S(n, j, t; a) = \binom{n-j}{j} \sum_{k=j}^{n-j} \binom{n-2j}{k-j} \binom{n}{k}^t F(n, k, a).$$  \hspace{1cm} (10)

From now on, we use the Eq. (10) instead of the Eq. (6).
2 Background

In 1998, Calkin proved that the alternating binomial sum \( S_1(2n, m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \) is divisible by \( \binom{2n}{n} \) for all non-negative integers \( n \) and all positive integers \( m \). In 2007, Guo, Jouhet, and Zeng proved, among other things, two generalizations of Calkin’s result \([6, \text{Thm. 1.2, Thm. 1.3, p. 2}]\). In 2018, Calkin’s result \([2, \text{Thm. 1}]\) is proved by using \( D \) sums \([8, \text{Section 8}]\). Note that there is a close relationship between \( D \) sums and \( M \) sums \([10, \text{Eqns. (9) and (19)}]\).

Recently, Calkin’s result \([2, \text{Thm. 1}]\) is proved by using \( M \) sums \([10, \text{Section 5}]\). In particular, it is known \([10, \text{Eqns. (22) and (25)}]\) that:

\[
M_S(2n, j, 0) = \begin{cases} 
0, & \text{if } 0 \leq j < n; \\
(-1)^n, & \text{if } j = n. 
\end{cases}
\]

\[
M_S(2n, j, 1) = (-1)^n \binom{2n}{n} \binom{n}{j}.
\]

By using \( M \) sums, it is shown \([10, \text{Section 7}]\) that the sum \( S_2(2n, m, a) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{a+k}{2n-k} \) is divisible by \( \text{lcm}\left(\binom{a+n}{a}, \binom{2n}{n}\right) \) for all non-negative integers \( n \) and \( a \); and for all positive integers \( m \). In particular, it is known \([10, \text{Eqns. (52) and (57)}]\) that:

\[
M_S(2n, j, 0; a) = (-1)^{n-j} \binom{a+n}{a} \binom{a+j}{j} \binom{a}{n-j},
\]

\[
M_S(2n, j, 1; a) = (-1)^{n-j} \binom{2n}{n} \sum_{u=0}^{n-j} (-1)^u \binom{n}{j+u} \binom{j+u}{u} \binom{a+j+u}{j+u} \binom{a+n}{2n-j-u}.
\]

By using \( D \) sums, it is shown \([12, \text{Thm. 1}]\) that the sum \( S_3(2n, m) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \binom{2k}{2n-k} \) is divisible by \( \binom{2n}{n} \) for all non-negative integers \( n \), and for all positive integers \( m \). The same result also can be proved by using \( M \) sums. It can be shown that

\[
M_S(2n, j, 0) = (-1)^j \binom{2n}{n} \binom{2j}{j} \binom{2(n-j)}{n-j}.
\]

(11)

Note that Eq. (11) is equivalent with the \([12, \text{Eq. (12)}]\).

Recently, by using \( D \) sums, it is shown \([11, \text{Th. 3, p. 3}]\) that \( \Psi(2n, m, l-1) = \binom{2n}{n} \binom{2j}{j} \binom{2(n+l-j)}{n+l-j} \binom{2(n-j)}{n-j} \) for all non-negative integers \( n \) and \( m \). The same result also can be proved by using \( M \) sums. Let \( l \) be a positive integer, and let \( n \) and \( j \) be non-negative integers such that \( j \leq n \). It is readily verified \([10, \text{Eq. (91)}]\) that

\[
M_\Psi(2n, j, 0; l-1) = (-1)^j \binom{2j}{j} \binom{2n}{n} \binom{2(n+l-j)}{n+l-j} \binom{2(n-j)}{n-j}.
\]

(12)
Furthermore, by using [11, Eq. (103)] and [10, Eq. (33)], it can be shown that

\[ M_\Psi(2n, j, 1; l - 1) = (-1)^j S(n, l) \binom{n}{j} \sum_{v=0}^{n-j} (-1)^v S(n + l - j - v, n) \binom{2(j + v)}{j + v} \binom{n - j}{v}. \quad (13) \]

The rest of the paper is structured as follows. In Section 3, we give a proof of Theorem 3 by using Stanley’s formula. In Section 4, we give a proof of Theorem 1. Our proof of Theorem 1 consists from two parts. In the first part, we prove that Theorem 1 is true for \( m = 1 \). In the second part, we prove that Theorem 1 is true for all positive integers \( m \) such that \( m \geq 2 \).

### 3 A Proof of Theorem 3

We use three known binomial formulas.

Let \( a, b, \) and \( c \) be non-negative integers such that \( a \geq b \geq c \). The first formula is:

\[ \binom{a}{b} \binom{b}{c} = \binom{a}{c} \binom{a-c}{b-c}. \quad (14) \]

Let \( a, b, m, \) and \( n \) be non-negative integers. The second formula is Stanley’s formula:

\[ \sum_{k=0}^{\min(m,n)} \binom{a}{m-k} \binom{b}{n-k} \binom{a+b+k}{k} = \binom{a+n}{m} \binom{b+m}{n}. \quad (15) \]

The third formula is a symmetry of binomial coefficients.

**Proof.** By setting \( S := Q \) and \( t := 0 \) in the Eq. (10), it follows that

\[ M_Q(n, j, 0; a) = \binom{n-j}{j} \sum_{k=j}^{n-j} \binom{n-2j}{k-j} \binom{n}{k} H(n, k, a), \]

\[ = \sum_{k=j}^{n-j} \binom{n-j}{j} \binom{n-2j}{k-j} \binom{a+k}{a} \binom{a+n-k}{a} G(n, k, a). \quad (16) \]

By using symmetry of binomial coefficients and the Eq. (14), it follows that

\[ \binom{n-j}{j} \binom{n-2j}{k-j} = \binom{n-j}{k-j} \binom{n-k}{n-k-j}. \quad (17) \]

By using the Eq. (17), Eq. (16) becomes, as follows:

\[ M_Q(n, j, 0; a) = \sum_{k=j}^{n-j} \binom{n-j}{k-j} \binom{n-k}{n-k-j} \binom{a+k}{a} \binom{a+n-k}{a} G(n, k, a), \]

\[ = \sum_{k=j}^{n-j} \binom{n-j}{k-j} \binom{a+n-k}{a} \binom{n-k}{n-k-j} \binom{a+k}{a} G(n, k, a). \quad (18) \]
By using symmetry of binomial coefficients and the Eq. (18), it follows that
\[
\binom{a+n-k}{a} \binom{n-k}{n-k-j} = \binom{a+n-k}{a+j} \binom{a+j}{j},
\] (19)

By using the Eq. (19), the Eq. (18) becomes, as follows:
\[
M_Q(n, j, 0; a) = \left(\frac{a}{a+j} \right) \sum_{k=j}^{n-j} \binom{n-j}{k-j} \left(\frac{a+n-k}{a+j} \binom{a+k}{a} \right) G(n, k, a). (20)
\]

By setting \(m := a+j, n := a, a := n-k, b := k-j,\) and \(k := l\) in Stanley’s formula (15), we obtain that
\[
\left(\frac{a+n-k}{a+j} \right) \binom{a+k}{a} = \sum_{l=0}^{a} \binom{n-k}{a+j-l} \binom{k-j}{a-l} \binom{n-j+l}{l}. (21)
\]

By using Eqns. (20) and (21), it follows that \(M_Q(n, j, 0; a)\) is equal to
\[
\left(\frac{a+j}{a} \right) \sum_{k=j}^{n-j} \binom{n-j}{k-j} \left(\frac{n-k}{a+j-l} \binom{k-j}{a-l} \binom{n-j+l}{l} \right) G(n, k, a). (22)
\]

Note that, by the Eq. (21), we obtain the following inequality:
\[
\begin{align*}
n-k & \geq a+j-l, \text{ or } \\
k & \leq n-j-a+l.
\end{align*}
\] (23)

Similarly, by the Eq. (21), we obtain another inequality
\[
\begin{align*}
k-j & \geq a-l, \text{ or } \\
k & \geq a+j-l.
\end{align*}
\] (24)

By changing the order of sumation in the Eq. (22) and by using Inequalities (23) and (24), it follows that \(M_Q(n, j, 0; a)\) is equal to
\[
\left(\frac{a+j}{j} \right) \sum_{l=0}^{a} \binom{n-j+l}{l} \sum_{k=a+j-l}^{n-a} \binom{n-j}{k-j} \binom{n-k}{a+j-l} \binom{k-j}{a-l} G(n, k, a). (25)
\]

By using symmetry of binomial coefficients and the Eq. (14), it follows that
\[
\binom{n-j}{k-j} \binom{n-k}{a+j-l} = \binom{n-j}{a+j-l} \binom{n-a-2j+l}{k-j}. (26)
\]

By using Eqns. (22) and (26), it follows that \(M_Q(n, j, 0; a)\) is equal to
\[
(a + j) \sum_{l=0}^{a} \binom{n - j + l}{l} \binom{n - j}{a + j - l} \sum_{k=a+j-l}^{n-(a+j-l)} \binom{n - a - 2j + l}{k - j} \binom{k - j}{a - l} G(n, k, a). \tag{27}
\]

By using symmetry of binomial coefficients and the Eq. (14), it follows that
\[
\binom{n - a - 2j + l}{k - j} \binom{k - j}{a - l} = \binom{n - a - 2j + l}{a - l} \binom{n - 2a - 2j + 2l}{k - j - a + l}. \tag{28}
\]

By using Eqns. (27) and (28), it follows that \( M_Q(n, j; 0; a) \) is equal to
\[
(a + j) \sum_{l=0}^{a} \binom{n - j + l}{l} \binom{n - j}{a + j - l} \binom{n - a - 2j + l}{a - l} \sum_{k=a+j-l}^{n-(a+j-l)} \binom{n - 2a - 2j + 2l}{k - j - a + l} G(n, k, a). \tag{29}
\]

Again, by using symmetry of binomial coefficients and the Eq. (14), it follows that
\[
\binom{n - j}{a + j - l} \binom{n - a - 2j + l}{a - l} = \binom{n - j}{a - l} \binom{n - (j + a - l)}{j + a - l}. \tag{30}
\]

By using Eqns. (29) and (30), it follows that \( M_Q(n, j; 0; a) \) is equal to
\[
(a + j) \sum_{l=0}^{a} \binom{n - j + l}{l} \binom{n - j}{a - l} \binom{n - (j + a - l)}{j + a - l} \sum_{k=a+j-l}^{n-(a+j-l)} \binom{n - 2(a + j - l)}{k - (j + a - l)} G(n, k, a)). \tag{31}
\]

Note that, by setting \( S := R, F := G, j := j + a - l, \) and \( t := 0 \) in the Eq. (10), it follows that:
\[
M_R(n, j + a - l, 0; a) = \binom{n - (j + a - l)}{j + a - l} \sum_{k=a+j-l}^{n-(a+j-l)} \binom{n - 2(a + j - l)}{k - (j + a - l)} G(n, k, a). \tag{32}
\]

Hence, by using Eqns. (31) and (32), it follows that
\[
M_Q(n, j; 0; a) = \binom{a + j}{j} \sum_{l=0}^{a} \binom{n - j + l}{l} \binom{n - j}{a - l} M_R(n, j + a - l, 0; a).
\]

This completes the proof of Theorem 3.
4 A Proof of Theorem 1

Let the function $\varphi(2n, m, r - 1)$ be defined as in Eq. (4).

Let us consider the following sum

$$
\varphi(2n, m, r - 1) = \frac{1}{4} \psi(2n, m, r - 1).
$$

(33)

By Eq. (5), the Eq. (33) becomes

$$
\varphi(2n, m, r - 1) = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^m \left( \frac{1}{2} S(k, r) \right) \left( \frac{1}{2} S(2n - k, r) \right).
$$

(34)

Note that, since $r$ is a positive integer, both numbers $\frac{1}{2} S(k, r)$ and $\frac{1}{2} S(2n - k, r)$ are integers. Therefore, the sum $\varphi(2n, m, r - 1)$ is a sum from Definition 2.

Now we can apply Theorem 3. By setting $Q := \varphi$, $n := 2n$, $a := r - 1$, $R := \phi$ and $G(2n, k, r - 1) := \frac{1}{2} S(k, r) \frac{1}{2} S(2n - k, r)$ in Theorem 3, by the Eq. (9), it follows that

$$
M_{\varphi}(2n, j, 0; r - 1) = \sum_{l=0}^{r-1} \binom{j + r - 1 - l}{j} \binom{2n - j - r + 1 - l}{n - j - r + 1 - l} (2n - j)^{n - j - r + 1 - l}.
$$

(35)

By Eqns. (6) and (33), it follows that

$$
M_{\varphi}(2n, j, t; r - 1) = \frac{1}{4} M_{\psi}(2n, j, t; r - 1).
$$

(36)

By setting $t := 0$ and $l := r$ in the Eq. (12) and by using the Eq. (36) and the definition of super Catalan numbers, we obtain that

$$
M_{\varphi}(2n, j, 0; r - 1) = (-1)^{j} \frac{1}{2} S(n, r) \frac{(2j) (2n + r - j)}{2^{2n - r + j}}.
$$

(37)

By using Eq. (37), it follows that the sum $M_{\varphi}(2n, j + r - 1 - l, 0; r - 1)$ is equal to

$$
(-1)^{j+r-1-l} \frac{1}{2} S(n, r) \frac{2^{(2j+r-l-1)}}{2^{(2n+r-j)}}.
$$

(38)

By using a symmetry of binomial coefficients and the Eq. (14), it follows that

$$
\binom{2n - j}{r - 1 - l} \binom{2n - j - r + l + 1}{n - j - r + l + 1} = \binom{2n - j}{n} \binom{n - j}{j + r - l - 1}.
$$

(39)

By using Eqns. (38) and (39), it follows that the sum $\binom{2n-j}{r-1-l} M_{\varphi}(2n, j + r - 1 - l, 0; r - 1)$ is equal to
(-1)^{j+r-1} \frac{1}{2} S(n, r) \left(\binom{2n-j}{n} \frac{2(2n-j-l)}{n} \binom{2n-j-l}{j-r-1} \frac{2(n-j)}{n} \frac{2(n-j+l+1)}{n} \right) \frac{2^{(j+r-1)-l}}{2^{(2n-j+l+1)}}. \quad (40)

By using Eqns. (35) and (40), it follows that the sum \( M_{\varphi}(2n, j, 0; r - 1) \) is equal to

\[
(-1)^{j+r-1} \binom{j+r-1}{j} \frac{1}{2} S(n, r) \left(\binom{2n-j}{n} \sum_{l=0}^{r-1} (-1)^{l} \binom{2n-j+l}{l} \frac{2(2n-j-l)}{(n+l+1)} \frac{2(n-j)}{n} \frac{2(n-j+l+1)}{n} \right) \frac{2^{(j+r-1)-l}}{2^{(2n-j+l+1)}}. \quad (41)
\]

By setting \( j := 0 \) in the Eq. (41), we obtain that

\[
M_{\varphi}(2n, 0, 0; r - 1) = (-1)^{r-1} \frac{1}{2} S(n, r) \left(\binom{2n}{n} \sum_{l=0}^{r-1} (-1)^{l} \binom{2n+l}{l} \frac{2(2n+l)}{n} \frac{2(n+l+1)}{n} \right) \frac{2^{(r-1)-l}}{2^{(2n+l+1)}}
\]

\[
= (-1)^{r-1} \frac{1}{2} S(n, r) \sum_{l=0}^{r-1} (-1)^{l} \binom{2n+l}{l} \frac{2(n+l+1)}{n} \binom{n}{r-l-1} \frac{2^{n}}{2^{(2n+l+1)}}
\]

\[
= (-1)^{r-1} \frac{1}{2} S(n, r) \sum_{l=0}^{r-1} (-1)^{l} \binom{2n+l}{l} \frac{2(n+l+1)}{n} \binom{n}{r-l-1} \frac{1}{2} \frac{S(n, n+l+1)}{2}. \quad (42)
\]

By using the Eq. (41), it follows that the sum \( M_{\varphi}(2n, 0, 0; r - 1) \) is divisible by \( \frac{1}{2} S(n, r) \).

By setting \( n := 2n, m := 1, S := \varphi, \) and \( r := r - 1 \) in the Eq. (7), we obtain that

\[
\varphi(2n, 1, r - 1) = M_{\varphi}(2n, 0, 0; r - 1). \quad (43)
\]

By using Eqns. (42) and (43), it follows that the sum \( \varphi(2n, 1, r - 1) \) is divisible by \( \frac{1}{2} S(n, r) \).

This completes the proof of Theorem 1 for the case \( m = 1 \).

Let us calculate the sum \( M_{\varphi}(2n, j, 1; r - 1) \), where \( n \) and \( j \) are non-negative integers such that \( j \leq n \).

By setting \( n := 2n, S := \varphi, t := 0, \) and \( a := r - 1 \) in the Eq. (8), we obtain that

\[
M_{\varphi}(2n, j, 1; r - 1) = \sum_{u=0}^{n-j} \binom{2n}{j} \binom{2n-j}{u} M_{\varphi}(2n, j + u, 0; r - 1). \quad (44)
\]

By using a symmetry of binomial coefficients and the Eq. (14), it follows that

\[
\binom{2n}{j} \binom{2n-j}{u} = \binom{2n}{2n-j-u} \binom{j+u}{u}. \quad (45)
\]

By using Eqns. (44) and (45), it follows that

\[
M_{\varphi}(2n, j, 1; r - 1) = \sum_{u=0}^{n-j} \binom{j+u}{u} \binom{2n}{2n-j-u} M_{\varphi}(2n, j + u, 0; r - 1). \quad (46)
\]
By using the Eq. (41), it follows that the sum \((\binom{2n}{2n-j-u})M_\varphi(2n, j + u; 0; r - 1)\) equals
\[
\left(\binom{2n}{2n-j-u}\right)\binom{2n-j-u}{n}(\binom{2n-j-u}{n})^{-1}\binom{j+u+r-1}{j+u}1_{2}S(n, r).
\]
\[
\sum_{l=0}^{r-1}(-1)^l\binom{2n-j-u+l}{l}\binom{2(n-j-u+l+1)}{n-j-u+l+1}\binom{n-j-u}{j+u+r-l-1}2_{2n-j-u+l+1}.\tag{47}
\]

By using a symmetry of binomial coefficients and the Eq. (14), it follows that
\[
\binom{2n}{2n-j-u}\binom{2n-j-u}{n} = \binom{2n}{n}\binom{n}{j+u}.\tag{48}
\]

By using Eqns. (47) and (48), it follows that the sum \((\binom{2n}{2n-j-u})M_\varphi(2n, j + u; 0; r - 1)\) equals
\[
\frac{1}{2}S(n, r)\binom{n}{j+u}(-1)^{j+u+r-1}\binom{j+u+r-1}{j+u}\sum_{l=0}^{r-1}(-1)^l\binom{2n-j-u+l}{l}.
\]
\[
\cdot\left(\binom{2(j+u+r-1-l)}{j+u+r-1-l}\binom{n-j-u}{j+u+r-l-1}\right)\frac{1}{2}S(n, n-j-u+l+1).\tag{49}
\]

Hence, by using the Eq. (49), we obtain that
\[
\binom{2n}{2n-j-u}M_\varphi(2n, j + u; 0; r - 1) = \frac{1}{2}S(n, r) \cdot c(n, j + u, r - 1);\tag{50}
\]
where \(c(n, j + u, r - 1)\) is always an integer.

By using the Eq. (50), the Eq. (46) becomes, as follows:
\[
M_\varphi(2n, j, 1; r - 1) = \frac{1}{2}S(n, r) \sum_{u=0}^{n-j}\binom{j+u}{u}c(n, j + u, r - 1).\tag{51}
\]

By Eq. (52), it follows that the sum \(M_\varphi(2n, j, 1; r - 1)\) is divisible by \(\frac{1}{2}S(n, r)\) for all non-negative integers \(n\) and \(j\) such that \(j \leq n\); and for all positive integers \(r\). By using the Eq. (8) and the induction principle, it can be shown that the sum \(M_\varphi(2n, j, t; r - 1)\) is divisible by \(\frac{1}{2}S(n, r)\) for all non-negative integers \(n\) and \(j\) such that \(j \leq n\); and for all positive integers \(r\) and \(t\).

By setting \(S = \varphi\), \(n := 2n\), \(m := t + 1\), and \(a := r - 1\) in the Eq. (7), it follows that
\[
\varphi(2n, t + 1, r - 1) = M_\varphi(2n, 0, t; r - 1).\tag{52}
\]

Since \(t \geq 1\), it follows that \(t + 1 \geq 2\). By Eq. (52), it follows that the sum \(\varphi(2n, m, r - 1)\) is always divisible by \(\frac{1}{2}S(n, r)\) for all non-negative integers \(n\); and for all positive integers \(m\) and \(r\) such that \(m \geq 2\). See [10, Section 4]. This completes the proof of Theorem 1.
Remark 4. For \( r = 1 \), by using Theorem 1 and the fact \( \frac{1}{2} S(n, 1) = C_n \), it follows that the sum \( \varphi(2n, m, 0) \) is divisible by \( C_n \). Therefore, for \( r = 1 \), the result of Theorem 1 is weaker than the result that sum \( \varphi(2n, m, 0) \) is divisible by \( \binom{2n}{n} \) [12, Cor. 4, p. 2]. However, for \( n = 3, r = 2, \) and \( m = 1 \), the sum \( \varphi(2n, m, r - 1) \) is neither divisible by \( \binom{2n}{n} \) nor by \( S(n, r) \).

5 Conclusion

Theorem 1 and Eq. (1) show that there is a close relationship between Gessel’s numbers \( P(n, r) \) and super Catalan numbers \( S(n, r) \). Gessel’s numbers have, at least, two combinatorial interpretations [5, 9], while super Catalan numbers do not have a combinatorial interpretation in general. We think that some combinatorial properties of Gessel’s numbers can be used for obtaining a combinatorial interpretation for super Catalan numbers in general.

Acknowledgments

I want to thank professor Tomislav Došlić for inviting me on the third Croatian Combinatorial Days in Zagreb.

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