Research Article

An improved lower bound for the degree Kirchhoff index of bipartite graphs

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Abstract

For a connected graph \( G \) with \( n \) vertices and \( m \) edges, the degree Kirchhoff index of \( G \) is defined as

\[
K_f^\ast (G) = 2m \sum_{i=1}^{n-1} (\gamma_i)^{-1},
\]

where \( \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0 \) are the normalized Laplacian eigenvalues of \( G \). In this paper, a lower bound on the degree Kirchhoff index of bipartite graphs is established. Also, it is proved that the obtained bound is stronger than a lower bound derived by Zhou and Trinajstić in [J. Math. Chem. 46 (2009) 283–289].

Keywords: topological indices; degree Kirchhoff index.

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1. Introduction

Let \( G = (V(G), E(G)) \) be a simple connected graph with \( n \) vertices and \( m \) edges, where \( V(G) = \{v_1, v_2, \ldots, v_n\} \). The degree of a vertex \( v_i \in V(G) \) is denoted by \( d_i \), where \( i = 1, 2, \ldots, n \). If \( v_i \) and \( v_j \) are two adjacent vertices of \( G \), then it is written as \( i \sim j \).

Denote by \( A(G) \) and \( D(G) = \text{diag}(d_1, d_2, \ldots, d_n) \) the adjacency and the diagonal degree matrix of \( G \), respectively. The Laplacian matrix of \( G \) is defined as \( L(G) = D(G) - A(G) \) (see [16]). Since \( G \) is assumed to be a connected graph, the matrix \( D(G)^{-1/2} \) exists. The normalized Laplacian matrix of \( G \) is the matrix defined [8] by

\[
L(G) = D(G)^{-1/2} L(G) D(G)^{-1/2}.
\]

The eigenvalues \( \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n-1} > \gamma_n = 0 \) of \( L(G) \) represent the normalized Laplacian eigenvalues of \( G \). Details on the spectra of \( L(G) \) can be found in [8].

Chen and Zhang [7] introduced the degree Kirchhoff index of a connected graph \( G \) as

\[
K_f^\ast (G) = \sum_{i<j} d_i d_j r_{ij},
\]

where \( r_{ij} \) is the effective resistance distance between the vertices \( v_i \) and \( v_j \) of \( G \). In [7], it was also demonstrated that the degree Kirchhoff index can be expressed in terms of normalized Laplacian eigenvalues as follows:

\[
K_f^\ast (G) = 2m \sum_{i=1}^{n-1} \frac{1}{\gamma_i}.
\]

Both of the definitions of the graph invariant \( K_f^\ast (G) \) given by (1) and (2) are much studied in the chemical and mathematical literature. For survey and details, see \([1,2,4,5,10–12,14,15,17,18,20,21]\).

In this paper, we present a lower bound on the degree Kirchhoff index of bipartite graphs. In addition, we show that our lower bound improves the lower bound obtained by Zhou and Trinajstić [21].

2. Lemmas

In this section, we recall a few well-known properties of the normalized Laplacian eigenvalues of graphs.

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Lemma 2.1. [8] Let $G$ be a connected graph with $n \geq 2$ vertices. Then, the following properties regarding the normalized Laplacian eigenvalues are valid:

1. $\sum_{i=1}^{n} \gamma_i = n$.
2. $\gamma_1 \leq 2$ with equality if and only if $G$ is a bipartite graph.
3. For each $1 \leq i \leq n$, $\gamma_i \in [0,2]$, $\gamma_n = 0$ and $\gamma_{n-1} \neq 0$.

Lemma 2.2. [9] Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then,

$$\prod_{i=1}^{n-1} \gamma_i = \frac{2m \cdot t(G)}{\prod_{i=1}^{n} d_i},$$

where $t(G)$ is the total number of spanning trees of $G$.

Lemma 2.3. [13] Let $G$ be a connected graph of order $n$. Then, $\gamma_2 \geq 1$ with equality if and only if $G$ is a complete bipartite graph.

3. A lower bound for the degree Kirchhoff index of bipartite graphs

We now give an improved lower bound on the degree Kirchhoff index of bipartite graphs.

Theorem 3.1. Let $G$ be a connected bipartite graph with $n \geq 2$ vertices, $m$ edges and $t(G)$ spanning trees. Then, for any real $\alpha$, $\gamma_2 \geq \alpha \geq 1$,

$$K_{f^*}(G) \geq 2m \left( \frac{1}{2} + \frac{1}{\alpha} + n - 3 - \ln \left( \frac{m \cdot t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \alpha \right).$$

Equality in (3) holds if and only if $\alpha = 1$ and $G \cong K_{p,q}$ ($p + q = n$).

Proof. For $x > 0$, the following inequality can be found in the monograph [19]

$$x \leq 1 + x \ln x,$$

where the equality holds if and only if $x = 1$. For $x > 0$, the above inequality can be considered as

$$\frac{1}{x} \geq 1 - \ln x$$

with equality if and only if $x = 1$. By Lemma 2.1, $\gamma_1 = 2$ and $\gamma_i > 0$, $i = 1, 2, \ldots, n-1$, since $G$ is a connected bipartite graph. Then, using these results and Lemma 2.2, we have

$$\sum_{i=1}^{n-1} \frac{1}{\gamma_i} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} \frac{1}{\gamma_i}$$

$$= \frac{1}{2} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} \frac{1}{\gamma_i}$$

$$\geq \frac{1}{2} + \frac{1}{\gamma_2} + \sum_{i=3}^{n-1} (1 - \ln \gamma_i)$$

$$= \frac{1}{2} + \frac{1}{\gamma_2} + n - 3 - \ln \prod_{i=3}^{n-1} \gamma_i$$

$$= \frac{1}{2} + \frac{1}{\gamma_2} + n - 3 - \ln \left( \frac{m \cdot t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \gamma_2. \quad (4)$$

Now, consider the function $f(x) = \frac{1}{x} + \ln x$. It can be easily seen that this function is increasing in the interval $1 \leq x \leq 2$. Then for any real $\alpha$, $\gamma_2 \geq \alpha \geq 1$, we have that

$$f(\gamma_2) \geq f(\alpha) = \frac{1}{\alpha} + \ln \alpha.$$

Bearing this fact in mind and using (2) and (4), we obtain that

$$K_{f^*}(G) \geq 2m \left( \frac{1}{2} + \frac{1}{\alpha} + n - 3 - \ln \left( \frac{m \cdot t(G)}{\prod_{i=1}^{n} d_i} \right) + \ln \alpha \right).$$
which is the required inequality (3). Now, assume that the equality holds in (3). Then

\[ \gamma_2 = \alpha \text{ and } \gamma_3 = \cdots = \gamma_{n-1} = 1. \]

Since \( G \) is bipartite, by Lemma 2.1, \( \sum_{i=2}^{n-1} \gamma_i = n - 2 \). Considering this with the above conditions, we get that \( \gamma_2 = \alpha = 1 \), which implies that \( G \cong K_{p,q} \).

Conversely, it is not difficult to show that the equality holds in (3) for the complete bipartite graph \( K_{p,q} \). Hence, the proof is completed. \( \Box \)

By Theorem 3.1 and Lemma 2.3, we have the following corollary.

**Corollary 3.1.** Let \( G \) be a connected bipartite graph with \( n \geq 2 \) vertices, \( m \) edges and \( t(G) \) spanning trees. Then,

\[ Kf^*(G) \geq m(2n - 3) - 2m \ln \left( \frac{m t(G)}{\prod_{i=1}^{n} d_i} \right). \]  

Equality in (5) holds if and only if \( G \cong K_{p,q} \) \((p + q = n)\).

**Remark 3.1.** For a connected bipartite graph \( G \) with \( n \geq 2 \) vertices and \( m \) edges, Zhou and Trinajstić [21] obtained that

\[ Kf^*(G) \geq m(2n - 3) \]  

with equality if and only if \( G \) is a complete bipartite graph. Furthermore, for connected bipartite graphs, the following inequality can be obtained from Theorem 3 of [3]:

\[ 0 < \frac{m t(G)}{\prod_{i=1}^{n} d_i} \leq 1. \]

From the above and (5), we conclude that

\[ Kf^*(G) \geq m(2n - 3) - 2m \ln \left( \frac{m t(G)}{\prod_{i=1}^{n} d_i} \right) \geq m(2n - 3). \]

This implies that the lower bound (5) improves the lower bound (6).

Recall that the general Randić index of a graph \( G \) is one of the graph topological indices defined by \( R_{-1}(G) = \frac{1}{\sum_{i<j} \frac{1}{d_i d_j}} \) (see [6]). The following lower bound was found in Theorem 3.2 of [5]

\[ \gamma_2 \geq 1 + \sqrt{\frac{2(R_{-1}(G) - 1)}{n-2}}. \]

**Remark 3.2.** Notice that the lower bound (5) can be improved by taking \( \alpha = 1 + \sqrt{\frac{2(R_{-1}(G) - 1)}{n-2}} \) in Theorem 3.1.

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