Semistability of cubulated groups

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Abstract
We prove that all cubulated groups are semistable at infinity. In doing so we prove two further results about cubulations of groups. The first of these states that any one-ended cubulated group has a cubulation for which all halfspaces are one-ended. The second states that any cubulated group has a cubulation for which all quarterspaces are deep—analogous to the fact that passing to the essential core of a given cubulation ensures that all halfspaces are deep.

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1 Introduction

A connected, locally finite CW complex $X$ is semistable at infinity if any two proper rays $r, s : [0, \infty) \to X$ converging to the same end are properly homotopic. The terminology comes from the following connection with inverse systems of groups. An inverse systems of groups $\{H_n\}$ is semistable if, for each $n$, the images of the bonding homomorphisms $H_m \to H_n$ are the same for all but finitely many $m > n$. Given a connected, locally finite CW complex $X$, we consider the inverse system of groups $\{\pi_1(X - C_n, r)\}$, where $\{C_n\}$ is an exhausting sequence of compact subsets of $X$, $r$ is a proper ray in $X$, and the bonding maps are induced by inclusions of subsets. If $X$ is one-ended and simply connected then $\{\pi_1(X - C_n, r)\}$ is semistable if and only if $X$ is semistable at infinity [24]. In this case the inverse limit of $\{\pi_1(X - C_n, r)\}$ provides a well defined notion of fundamental group at infinity for $X$. Both semistability at infinity and the fundamental group at infinity are quasi-isometry invariants for simply connected, locally finite CW complexes [5, 12].

If a finitely presented group $G$ acts properly and cocompactly on a simply connected CW complex $X$, then we say that $G$ is semistable at infinity if $X$ is semistable at infinity. The quasi-isometry invariance from the preceding paragraph implies that semistability of $G$ is independent of the choice of complex $X$, and is a quasi-isometry invariant for groups. Various classes of groups are known to be semistable at infinity, including hyperbolic groups [3, 4, 13, 15, 21, 22, 35], Artin and Coxeter groups [25], one-relator groups [26], and certain graphs of groups [27]. It is unknown if all CAT(0) groups are semistable at infinity—and this is one of the more heavily studied problems in the field (see [15]). In fact it is even unknown if all finitely presented groups are semistable at infinity. In this paper we prove the following.

Theorem 1.1 Cubulated groups are semistable at infinity.

There is also a connection between semistability and group cohomology: if $G$ is semistable at infinity then $H^2(G, \mathbb{Z}G)$ is free abelian [14, 20], so we obtain the following corollary. It is an open question of Hopf whether $H^2(G, \mathbb{Z}G)$ is free abelian for all finitely presented groups.

Corollary 1.2 If $G$ is a cubulated group then $H^2(G, \mathbb{Z}G)$ is free abelian.

We say that a group $G$ is cubulated if it acts properly and cocompactly on a CAT(0) cube complex $X$—and we refer to such an action as a cubulation of $G$. Examples of cubulated groups include small cancellation groups, finite volume hyperbolic 3-manifold groups and many Coxeter groups—see [37] for a more extensive list. The geometry and combinatorics of CAT(0) cube complexes is a rich and dynamic theory, and it grants cubulated groups with many properties stronger than those of CAT(0) groups – such as bi-automaticity [36] and the Tits Alternative [31].

We remark that semistability at infinity for many cubulated groups, including virtually special groups (see [18]), can be deduced fairly directly from existing literature. Indeed, if $X$ is a finite non-positively curved cube complex whose hyperplanes are two-sided and do not self-intersect, then successively cutting $X$ along hyperplanes corresponds to successively splitting the fundamental group of $X$, terminating in trivial
groups. We deduce from [27] that the fundamental group of $X$ is semistable at infinity, and it follows that all virtually special groups are semistable at infinity because semistability is a quasi-isometry invariant. However, for general cubulated groups there is no (virtual) hierarchy that we can use, and indeed our proof of Theorem 1.1 uses a different argument.

The idea for our proof is as follows. Using results from the literature we can easily reduce to the case of a one-ended group; and if the cubulation is given by a CAT(0) cube complex $X$ then we can reduce to showing that, for any compact $C \subset X$, any loop sufficiently far from $C$ (and based on a proper base ray) can be pushed arbitrarily far from $C$ (relative to the base ray) by a homotopy that avoids $C$. The key step is to achieve this “pushing out” using two geometric (and cubical) properties of $X$. However, these properties do not hold for arbitrary cubulations, so we must first modify the cubulation (Theorem 1.5). In fact most of the work in this paper goes into proving Theorem 1.5. The first geometric property we need is one-ended halfspaces, which is obtained with the following theorem.

**Theorem 1.3** Let $G$ be a group acting cocompactly on a one-ended locally finite CAT(0) cube complex $X$. Suppose there exists a subgroup $\Gamma \leq G$ whose induced action on $X$ is proper and cocompact. Then there is a locally finite CAT(0) cube complex $Y$ with the following properties:

1. All halfspaces in $Y$ are one-ended.
2. $G$ acts cocompactly on $Y$.
3. There exists a $G$-equivariant quasi-isometry $\theta : X \to Y$.
4. The $G$-stabilizers of hyperplanes in $Y$ are subgroups of the $G$-stabilizers of hyperplanes in $X$.

The second geometric property we need is about quarterspaces. A *quarterspace* in a CAT(0) cube complex is an intersection $h_1 \cap h_2$ of transverse halfspaces (equivalently, halfspaces for which the corresponding hyperplanes intersect). The quarterspace $h_1 \cap h_2$ is *shallow* if it is contained in a bounded neighborhood of the opposite quarterspace $h^*_1 \cap h^*_2$, otherwise $h_1 \cap h_2$ is *deep*. These notions are analogous to the notions of halfspaces being shallow or deep (see Sect. 2.1). One can remove shallow halfspaces using the essential core of Caprace–Sageev (Proposition 2.5), and similarly the following theorem provides a way to remove shallow quarterspaces.

**Theorem 1.4** Let $G$ be a group acting cocompactly on a CAT(0) cube complex $X$. Then there is a CAT(0) cube complex $Y$ with the following properties:

1. All quarterspaces in $Y$ are deep.
2. $G$ acts cocompactly on $Y$.
3. There exists a $G$-equivariant quasi-isometry $\phi : Y \to X$.
4. $\phi$ maps each halfspace in $Y$ to within finite Hausdorff distance of a halfspace in $X$.
5. $Y$ is locally finite if $X$ is locally finite.

Applying Theorem 1.3 and then Theorem 1.4 to the case where $G$ acts properly as well as cocompactly yields the following theorem, which we use in the proof of
Theorem 1.1 as outlined above. Note that the halfspaces remain one-ended when applying Theorem 1.4 because of property (4).

**Theorem 1.5** Every one-ended cubulated group admits a cubulation in which all halfspaces are one-ended and all quarterspaces are deep.

**Remark 1.6** One can alternatively use panel collapse [17] to show that any cocompact action on a CAT(0) cube complex can be modified to make quarterspaces deep. Indeed, panel collapse yields a hyperplane-essential action, and one can easily deduce that the cube complex has no shallow quarterspaces in this case. However, there is no analogue to property (4) from Theorem 1.4 when performing panel collapse, so this does not give an alternative proof of Theorem 1.5.

In fact, a cocompact, essential and hyperplane-essential action on a locally finite CAT(0) cube complex satisfies a stronger version of having deep quarterspaces in the sense that no quarterspace $h_1 \cap h_2$ is contained in a bounded neighborhood of its complement $(h_1 \cap h_2)^* = h_1^* \cup h_2^*$. Indeed, suppose the cube complex is a product $X = X_1 \times \cdots \times X_n$ of irreducible cube complexes; if the halfspaces $h_1, h_2$ come from different factors $X_i$ then we can use the fact that $h_1, h_2$ are deep in their respective factors to deduce that $h_1 \cap h_2$ is not contained in a bounded neighborhood of its complement, and if $h_1, h_2$ come from the same factor $X_i$ then we can use [16, Proposition 1].

The idea for the proof of Theorem 1.3 is to take a halfspace $h_0$ with more than one end and chop it up using a $G_{h_0}$-orbit of finite subcomplexes. We then obtain a new CAT(0) cube complex $Y$ by replacing the halfspace $h_0$ with new halfspaces that correspond to the pieces leftover from the chopping, and we do the same for every $G$-translate of $h_0$. Formally speaking, $Y$ is constructed from the cubing of a certain pocset (see Sect. 2.2 for background on pocsets). The new cube complex $Y$ might still have halfspaces with more than one end, but if we iterate the construction then we argue that it must terminate after a finite number of steps by considering the accessibility of the $\Gamma$-stabilizers of hyperplanes. The halfspaces in the terminal cube complex have at most one end, and we can easily remove bounded halfspaces using the essential cores of Caprace–Sageev. The assumption in Theorem 1.3 about the existence of the subgroup $\Gamma < G$ is needed in the proof when we consider the accessibility of the $\Gamma$-stabilizers of hyperplanes, but we conjecture that Theorem 1.3 holds without this assumption.

The proof of Theorem 1.4 is similar to that of Theorem 1.3 in that it involves cubing a pocset to obtain a new cube complex with lower complexity, and then iterating until the desired cube complex $Y$ is obtained. In this case the idea behind the pocset is to take certain shallow quarterspaces $h_1 \cap h_2$ and pull apart the halfspaces $h_1$ and $h_2$. And this time the measure of complexity is just the number of $G$-orbits of vertices rather than anything to do with accessibility.

The structure of the paper is as follows. In Sect. 2 we provide some background on CAT(0) cube complexes, pocsets, cubings, group splittings and accessibility of groups. We prove Theorems 1.3, 1.4 and 1.1 in Sects. 3, 4 and 5 respectively. Finally, in Sect. 6 we give an example of a one-ended group with a cubulation given by a CAT(0) cube complex that is essential and contains an infinite-ended halfspace. This demonstrates that Theorem 1.3 is not vacuous, and that it requires more than simply passing to the essential core of a CAT(0) cube complex.
2 Preliminaries

2.1 CAT(0) cube complexes

In this section we recall some basic concepts and facts regarding CAT(0) cube complexes. See [23, 37] for further background and proofs, including the definitions of CAT(0) cube complex and halfspace.

Let \( X \) be a CAT(0) cube complex. We will write \( H = H(X) \) for the collection of halfspaces. In this paper halfspaces will be combinatorial, so we will consider \( h \in H \) as a collection of vertices in \( X \) rather than a convex subspace of \( X \). We will write \( h^* \) for the complementary halfspace, so \( h \sqcup h^* \) is a partition of the vertex set \( X^0 \). We will write \( \hat{h} \) for the hyperplane corresponding to \( h \) and \( \hat{H}(X) \) for the collection of hyperplanes in \( X \).

We will mostly work with the combinatorial metric on \( X \), i.e. the metric induced by the 1-skeleton \( X^1 \). We will denote this metric by \( d \). Occasionally we will refer to the CAT(0) metric on \( X \) (i.e. the metric induced by making every cube unit Euclidean), but we will always make this explicit. (Note that properties related to these two metrics often coincide, for instance a subcomplex \( Y \subseteq X \) is convex in the CAT(0) metric if and only if \( Y^1 \) is convex in \( X^1 \). These two metrics are also bi-Lipschitz equivalent if \( X \) is finite dimensional.) When describing properties of sets of vertices we will tacitly be referring to their induced subgraphs in \( X^1 \); for example we will say that \( a \subseteq X^0 \) is *convex* if its induced subgraph is convex in \( X^1 \), or \( b \subseteq a \) *separates* \( a \) if the induced subgraph of \( b \) separates the induced subgraph of \( a \). The *convex hull* \( \text{Hull}(a) \subseteq X^0 \) of \( a \subseteq X^0 \) is the smallest convex set containing \( a \)—equivalently \( \text{Hull}(a) \) is the intersection of all halfspaces containing \( a \) (with the empty intersection being \( X^0 \) by convention). We will also use the following definition.

**Definition 2.1** Following [19, §4a], for \( S \subseteq X \) define the *cubical neighborhood* \( N(S) \subseteq X^0 \) to be the collection of vertices of cubes that intersect \( S \). For an integer \( R \geq 0 \), define the *cubical \( R \)-thickening* \( S^+R \) inductively by setting \( S^+0 \) to be the 0-skeleton of the smallest subcomplex of \( X \) that contains \( S \), and \( S^{+(R+1)} := N(S^+R) \). We will often be interested in cubical neighborhoods and thickenings of hyperplanes \( \hat{h} \); note that \( N(\hat{h}) = \hat{h}^+0 \) is convex and consists of the endpoints of edges that join \( h \) to \( h^* \).

**Remark 2.2** If \( a \subseteq X^0 \) is convex then \( a^+R \) is convex for all \( R \geq 0 \)—in particular \( R \)-thickenings of hyperplanes are convex.

**Remark 2.3** Cubical thickenings are not the same as metric neighborhoods, but they are related because

\[
\mathcal{N}_R(a) \subseteq a^+R \subseteq \mathcal{N}_{R \dim X}(a)
\]

for all \( a \subseteq X^0 \), where \( \mathcal{N}_r(a) \subset X^0 \) denotes the \( r \)-neighborhood of \( a \).

**Definition 2.4** A halfspace \( h \) is *shallow* if it is contained in a bounded neighborhood of \( h^* \), otherwise \( h \) is *deep*. A CAT(0) cube complex is *essential* if all of its halfspaces are deep.
The following proposition is due to Caprace–Sageev, it is a special case of [7, Proposition 3.5].

**Proposition 2.5** If $G$ is a group that acts cocompactly on an unbounded CAT(0) cube complex $X$, then there is a $G$-invariant convex subspace $Y \subseteq X$, which is either a subcomplex or a finite intersection of hyperplanes, and $Y$ is essential with respect to its induced cube complex structure. We call $Y$ the essential core of $X$.

We now recall the notion of quarterspace, and the concept of a quarterspace being shallow or deep. These notions are new to this paper, but are analogous to Definition 2.4.

**Definition 2.6** Halfspaces $h_1, h_2$ are transverse if $h_1 \cap h_2, h_1^* \cap h_2, h_1 \cap h_2^*, h_1^* \cap h_2^*$ are all non-empty (equivalently if we get a non-empty intersection of the bounding hyperplanes $\hat{h}_1 \cap \hat{h}_2 \neq \emptyset$). In this case $h_1 \cap h_2, h_1 \cap h_2^*, h_1^* \cap h_2, h_1^* \cap h_2^*$ are referred to as quarterspaces.

A quarterspace $h_1 \cap h_2$ is shallow if it is contained in a bounded neighborhood of $h_1^* \cap h_2^*$, otherwise $h_1 \cap h_2$ is deep. The depth of a quarterspace $h_1 \cap h_2$ is the least integer $r$ such that $h_1 \cap h_2$ is contained in the $(r + 2)$-neighborhood of $h_1^* \cap h_2^*$, with $r = \infty$ if $h_1 \cap h_2$ is deep. Note that $r \geq 0$ because any path from $h_1 \cap h_2$ to $h_1^* \cap h_2^*$ has length at least 2.

**Remark 2.7** By considering projection maps, one can show that a quarterspace $h_1 \cap h_2$ is shallow if and only if it is contained in a bounded neighborhood of $\hat{h}_1 \cap \hat{h}_2$. Furthermore, the depth of $h_1 \cap h_2$ is the least integer $r$ such that $h_1 \cap h_2$ is contained in the $r$-neighborhood of $N(\hat{h}_1) \cap N(\hat{h}_2)$.

The following three lemmas are well known, and will be used throughout the paper.

**Lemma 2.8** If $a_1, a_2 \subseteq X^0$ are convex then $d(a_1, a_2)$ is equal to the number of halfspaces $h \in H(X)$ with $a_1 \subseteq h$ and $a_2 \subseteq h^*$. By considering $a_1 = \{x_1\}$ and $a_2 = \{x_2\}$ for $x_1, x_2 \in X^0$, we deduce that any geodesic in $X^1$ has edges dual to distinct hyperplanes.

**Lemma 2.9** Any finite collection of pairwise intersecting convex sets in $X^0$ has non-empty intersection.

**Lemma 2.10** If $a \subseteq X^0$ is finite then $\text{Hull}(a)$ is finite.

**2.2 Pocsets and cubings**

We now recall the construction of a CAT(0) cube complex from a pocset or wallspace. This construction was originally due to Sageev [30], although our formulation will be closer to the version in [23]—see also [8, 28, 29, 37].

**Definition 2.11** A pocset is a poset $(P, \leq)$ together with an involution $A \mapsto A^*$ for all $A \in P$ satisfying:

1. $A$ and $A^*$ are incomparable.
(2) \( A \leq B \Rightarrow B^* \leq A^* \).

We define \( \hat{P} \) to be the set of pairs \( \{A, A^*\} \). Elements \( A, B \in P \) are transverse if neither \( A \) nor \( A^* \) is comparable with \( B \). The width of \( P \) is the maximum number of pairwise transverse elements, if such a maximum exists, otherwise we say the width is \( \infty \).

**Definition 2.12** An ultrafilter on a pocset \( P \) is a subset \( \omega \subseteq P \) satisfying:

1. (Completeness) For every \( A \in P \), exactly one of \( \{A, A^*\} \) is in \( \omega \).
2. (Consistency) If \( A \in \omega \) and \( A \leq B \), then \( B \in \omega \).

An ultrafilter \( \omega \) is DCC (descending chain condition) if it contains no strictly descending infinite chain \( A_1 > A_2 > A_3 > \cdots \).

**Proposition 2.13** Let \( P \) be a pocset of finite width that admits at least one DCC ultrafilter. Then there is a CAT(0) cube complex \( C = C(\hat{P}) \), called the cubing of \( P \), such that:

1. The vertices of \( C \) are the DCC ultrafilters on \( P \).
2. Two vertices \( \omega_1, \omega_2 \) in \( C \) are joined by an edge if and only if \( \omega_1 \triangle \omega_2 = \{A, A^*\} \) for some \( A \in P \).
3. The halfspaces of \( C \) take the form \( \{\omega \mid A \in \omega\} \) for \( A \in P \), so we have a pocset isomorphism \( (\hat{P}, \leq) \cong (\hat{C}(C), \subseteq) \), which also induces an identification \( \hat{P} \cong \hat{C}(C) \).
4. The dimension of \( C \) is equal to the width of \( P \).

**Definition 2.14** A wallspace \( (X, P) \) is a set \( X \) together with a family \( P \) of non-empty subsets that is closed under complementation, such that for any \( x, y \in X \) the set \( \{A \in P \mid x \in A, y \notin A\} \) is finite. \( P \) forms a pocset under inclusion, with the involution \( A \mapsto A^* \) given by complementation. Moreover, for any \( x \in X \) the set

\[
\omega_x := \{A \in P \mid x \in A\}
\]

is a DCC ultrafilter. Therefore, if \( P \) has finite width, we can form the cubing \( C = C(\hat{P}) \), and we have a map \( X \to C^0 \).

Any CAT(0) cube complex \( X \) forms a wallspace \( (X^0, \hat{C}(X)) \). We then get the following duality theorem between finite dimensional CAT(0) cube complexes and finite width pocsets.

**Theorem 2.15** If \( X \) is a finite dimensional CAT(0) cube complex then \( \hat{C}(X) \) has finite width, and the map \( X^0 \to C(\hat{C}(X)) \) extends to an isomorphism of cube complexes \( X \cong C(\hat{C}(X)) \).

Many geometric features of cubings can be interpreted using the pocset, for example distances, adjacent vertices and cocompactness.

**Lemma 2.16** \( d(\omega_1, \omega_2) = \frac{1}{2} |\omega_1 \triangle \omega_2| = |\omega_1 - \omega_2| \) for \( \omega_1, \omega_2 \in C(\hat{P})^0 \).

**Lemma 2.17** For \( \omega \in C(\hat{P})^0 \), the vertices adjacent to \( \omega \) are precisely the ultrafilters of the form \( (\omega \cup \{A^*\}) - \{A\} \) for \( A \in \omega \) that is \( \leq \)-minimal in \( \omega \).
Lemma 2.18 If a group $G$ acts on $\mathcal{P}$, then the induced action on $C(\mathcal{P})$ is cocompact if and only if there are finitely many $G$-orbits of collections of pairwise transverse elements of $\mathcal{P}$.

We will also make use of the following (nonstandard) definition and lemmas.

Definition 2.19 A partial ultrafilter on a poset $\mathcal{P}$ is a subset $\omega \subseteq \mathcal{P}$ (possibly empty) such that if $A \in \omega$ and $A \leq B$ then $B^* \not\in \omega$. (Note that $\omega$ contains at most one element from each pair $\{A, A^*\}$.) We will sometimes refer to ultrafilters as complete ultrafilters to stress that they satisfy the completeness property, which is what distinguishes ultrafilters from partial ultrafilters. A partial ultrafilter $\omega$ is $DCC$ (descending chain condition) if it contains no strictly descending infinite chain $A_1 > A_2 > A_3 > \cdots$. A partial ultrafilter $\omega$ is cofinite if $\omega \cap \{A, A^*\}$ is empty for only finitely many $A \in \mathcal{P}$.

Lemma 2.20 Any partial ultrafilter $\omega$ can be extended to a complete ultrafilter $\tilde{\omega}$. Moreover, if $\omega$ is $DCC$ and cofinite, then $\tilde{\omega}$ is $DCC$.

Proof Take $A \in \mathcal{P}$ with $\omega \cap \{A, A^*\} = \emptyset$. At least one of $\omega \cup \{A\}$ or $\omega \cup \{A^*\}$ must be a partial ultrafilter: indeed otherwise there exist $A \leq B_1$ and $A^* \leq B_2$ with $B_1^*, B_2^* \in \omega$, and this implies $B_1^* \leq B_2$, contradicting the fact that $\omega$ is a partial ultrafilter. The union of a chain of partial ultrafilters is clearly a partial ultrafilter, so it follows from Zorn’s lemma that $\omega$ can be extended to an ultrafilter $\tilde{\omega}$. If $\omega$ is cofinite, then any strictly descending infinite chain in $\tilde{\omega}$ contains an infinite subchain in $\omega$; hence $\tilde{\omega}$ is $DCC$ if $\omega$ is $DCC$ and cofinite.

Lemma 2.21 Let $\mathcal{P}$ be a poset of finite width that admits at least one $DCC$ ultrafilter. Let $\omega$ be a $DCC$ partial ultrafilter on $\mathcal{P}$ such that if $A \in \omega$ and $A \leq B$, then $B \in \omega$. Then for each $\omega_0 \in C(\mathcal{P})^0$, $\omega$ can be extended to a $DCC$ complete ultrafilter given by

$$\tilde{\omega} := \omega \cup \{A \in \omega_0 \mid \omega \cap \{A, A^*\} = \emptyset\}. \quad (2.1)$$

Moreover, the set of all possible $DCC$ complete extensions $\tilde{\omega}$ of $\omega$ is equal to the intersection of the halfspaces $\{\tilde{\omega} \in C(\mathcal{P})^0 \mid A \in \tilde{\omega}\}$ for $A \in \omega$, so it forms a convex subcomplex of $C(\mathcal{P})$. (If $\omega = \emptyset$ then this intersection is $C(\mathcal{P})^0$ by convention.)

Proof The completeness axiom is clearly satisfied by $\tilde{\omega}$. To check consistency, suppose $A \in \tilde{\omega}$ and $A \leq B$; we wish to show that $B \in \tilde{\omega}$. We have two cases. In the first case $A \in \omega$, so $B \in \omega \subseteq \tilde{\omega}$ by hypothesis of $\omega$. In the second case $A \in \omega_0$ and $\omega \cap \{A, A^*\} = \emptyset$; observe that $B^* \not\in \omega$, otherwise we would have $B^* \leq A^*$ and $A^* \in \omega$; so either $B \in \omega \subseteq \tilde{\omega}$, or $\omega \cap \{B, B^*\} = \emptyset$ and $B \in \omega_0$ by consistency of $\omega_0$, and we again have $B \in \tilde{\omega}$. So $\tilde{\omega}$ is a complete ultrafilter. Furthermore, we deduce that $\tilde{\omega}$ is $DCC$ because any strictly descending infinite chain in $\tilde{\omega}$ contains an infinite subchain in either $\omega$ or $\omega_0$.

2.3 Group splittings and accessibility

By a splitting of a group $G$ we mean an action on a tree $T$ without edge inversions. The splitting is finite if the action is cocompact and non-trivial if there is no fixed
point. The splitting is over finite subgroups if the edge groups are finite. A finitely generated group is accessible if it admits a splitting over finite subgroups in which each vertex group is either finite or one-ended. For such a splitting the vertex groups do not themselves admit non-trivial splittings over finite subgroups [34]. (One can also assume that the splitting is finite by passing to a minimal invariant subtree.) Dunwoody proved the following result.

**Theorem 2.22** [11] Every finitely presented group is accessible.

We can then deduce the following theorem using [10, Theorem 5.12].

**Theorem 2.23** Let $G$ be a finitely presented group and let $(G_i)$ be a sequence of groups such that $G_0 = G$, and $G_{i+1}$ is a vertex group in some non-trivial finite splitting of $G_i$ over finite subgroups. Then the sequence $(G_i)$ terminates.

### 3 Reducing to one-ended halfspaces

In this section we prove the following theorem.

**Theorem 1.3** Let $G$ be a group acting cocompactly on a one-ended locally finite $\text{CAT}(0)$ cube complex $X$. Suppose there is a subgroup $\Gamma < G$ whose induced action on $X$ is proper and cocompact. Then there is a locally finite $\text{CAT}(0)$ cube complex $Y$ with the following properties:

1. All halfspaces in $Y$ are one-ended.
2. $G$ acts cocompactly on $Y$.
3. There exists a $G$-equivariant quasi-isometry $\theta : X \to Y$.
4. The $G$-stabilizers of hyperplanes in $Y$ are subgroups of the $G$-stabilizers of hyperplanes in $X$.

We will deduce Theorem 1.3 from a repeated application of the following theorem. Recall that an action of a group $G$ on a $\text{CAT}(0)$ cube complex $X$ is without inversions in hyperplanes if there is no $g \in G$ and halfspace $h \in \mathcal{H}(X)$ with $g h = h^*$.

**Theorem 3.1** Let $G$ be a group acting cocompactly on a one-ended locally finite essential $\text{CAT}(0)$ cube complex $X$ without inversions in hyperplanes. Suppose there is a subgroup $\Gamma < G$ whose induced action on $X$ is proper and cocompact. If $X$ contains a halfspace with more than one end then there is a locally finite essential $\text{CAT}(0)$ cube complex $Y$ with the following properties:

1. $G$ acts cocompactly on $Y$ without inversions in hyperplanes, and the induced action of $\Gamma$ on $Y$ is proper and cocompact.
2. There exists a $G$-equivariant quasi-isometry $\theta : X \to Y$.
3. For each hyperplane $h \in \hat{\mathcal{H}}(X)$ there is a tree $T_h$, and there is an action of $G$ on $\sqcup_h T_h$ that is compatible with the action on $\hat{\mathcal{H}}(X)$. Furthermore, there is an injective $G$-equivariant map

$$\sigma : \hat{\mathcal{H}}(Y) \to \sqcup_{h \in \hat{\mathcal{H}}(X)} VT_h.$$
such that:

(a) $\Gamma$ acts on $\sqcup_{\h} T_\h$ with finite edge stabilizers.
(b) $\Gamma$ acts on $\sqcup_{\h} T_\h$ cocompactly.
(c) There exists $h_0 \in \mathcal{H}(X)$ with more than one end such that the action of $\Gamma_{\h_0}$ on $T_{\h_0}$ has no fixed point.

Let’s first see how to deduce Theorem 1.3.

Proof of Theorem 1.3 We may assume that $X$ is essential by Proposition 2.5, and we may assume that $G$ acts on $X$ without inversions in hyperplanes by passing to the first cubical subdivision. If all halfspaces in $X$ are one-ended then we can take $Y = X$, otherwise apply Theorem 3.1. For each hyperplane $\h \in \mathcal{H}(X)$, the action of $\Gamma_{\h}$ on $T_{\h}$ is a finite splitting of $\Gamma_{\h}$ over finite subgroups by (a) and (b) (subdivide $T_{\h}$ if there are edge inversions); and the $\Gamma$-stabilizers of hyperplanes in $\sigma^{-1}(T_{\h})$ are distinct vertex groups in this splitting. Moreover, it follows from (c) that this splitting is non-trivial for some hyperplane stabilizer $\Gamma_{\h}$.

If $Y$ also contains a halfspace with more than one end, then we may apply Theorem 3.1 again to $Y$, and the hyperplane stabilizers for the new cube complex will be vertex groups in splittings of the hyperplane stabilizers for $Y$. And we can keep applying Theorem 3.1 repeatedly, unless we obtain a cube complex $Y$ in which every halfspace is one-ended (there will be no bounded halfspaces since Theorem 3.1 always produces an essential cube complex). Each hyperplane stabilizer $\Gamma_{\h}$ for $X$ acts properly and cocompactly on $\mathcal{H}(Y)$, so in particular $\Gamma_{\h}$ is finitely presented; it then follows from Theorem 2.23 that the process of repeatedly applying Theorem 3.1 must terminate after a finite number of steps. The cube complex obtained at the final step is the desired cube complex $Y$ in Theorem 1.3. Note that properties (2)–(4) in Theorem 1.3 are satisfied because they hold for every application of Theorem 3.1.

We will spend the rest of this section proving Theorem 3.1. We will write $\mathcal{H} = \mathcal{H}(X)$ for the set of halfspaces of $X$.

3.1 Chopping up the halfspace $\h_0$

Let $\h_0 \in \mathcal{H}$ be a halfspace with more than one end, and let $c_0 \subset \h_0$ be a finite set that separates $\h_0$ into multiple unbounded components. Passing to the convex hull, we may assume that $c_0$ is convex (the convex hull is finite by Lemma 2.10). Also note that $c_0$ intersects the cubical neighborhood $N(\h_0)$, else $c_0$ would separate $X$ into multiple unbounded components, contradicting one-endedness of $X$.

Lemma 3.2 If $a$ is an unbounded component of $\h_0 - c_0$ then $a \cap N(\h_0)$ is unbounded.

Proof If not, then $c_0 \cup (a \cap N(\h_0))$ is finite and $a - N(\h_0)$ is a finite union of unbounded components of $X^0 - c_0 \cup (a \cap N(\h_0))$. The halfspace $\h_0^*$ is contained in another component of $X^0 - c_0 \cup (a \cap N(\h_0))$, and is itself unbounded because $X$ is essential. This contradicts $X$ being one-ended. \qed
We now consider $G_{h_0}$-translates of $c_0$, which also separate $h_0$ into multiple unbounded components. For convenience we will write $G_0$ in place of $G_{h_0} = G_{h_0}$. Define a wallspace $(h_0, P_0)$, where $P_0$ consists of the components of $h_0 - g c_0$ for $g \in G_0$, and their complements in $h_0$.

**Lemma 3.3** $P_0$ has finite width, and the cubing $C(P_0)$ has compact hyperplanes.

**Proof** As in Proposition 2.13(3), halfspaces in $C(P_0)$ correspond to elements of $P_0$, and intersecting hyperplanes in $C(P_0)$ come from transverse elements in $P_0$; so both assertions of the lemma follow if there is a bound on the number of elements of $P_0$ transverse to any given element of $P_0$. Local finiteness of $h_0$ implies that $h_0 - c_0$ has finitely many components, so there are finitely many $G_0$-orbits in $P_0$. Thus it suffices to consider $a \in P_0$ a component of $h_0 - c_0$, and show that it is transverse to finitely many elements of $P_0$. If $b$ is a component of $h_0 - g c_0$ with $c_0, g c_0$ disjoint, then $a$ is nested with either $b$ or $b^*$, so $a, b$ are not transverse. But $c_0$ is finite and $X$ is locally finite, so there are only finitely many sets $g c_0 (g \in G_0)$ with $c_0 \cap g c_0 \neq \emptyset$, hence only finitely many elements of $P_0$ are transverse to $a$, as required. □

**Lemma 3.4** $\Gamma_0 := \Gamma \cap G_0$ acts on $C(P_0)$ with finitely many orbits of hyperplanes and finitely many orbits of cubes.

**Proof** The action of $\Gamma_0$ on $h_0$ is cocompact, so there are finitely many $\Gamma_0$-orbits of the sets $g c_0$, hence finitely many $\Gamma_0$-orbits in $P_0$. It then follows from Proposition 2.13(3) that $C(P_0)$ has finitely many $\Gamma_0$-orbits of hyperplanes. Every cube of $C(P_0)$ is contained in the cubical neighborhood of a hyperplane, and these cubical neighborhoods are finite by Lemma 3.3, hence $C(P_0)$ has finitely many $\Gamma_0$-orbits of cubes. □

**Lemma 3.5** The $\Gamma_0$-stabilizer of any pair of cubes in $C(P_0)$ is finite.

**Proof** If $a \in P_0$ is a component of $h_0 - g c_0$, then the $\Gamma_0$-stabilizer of $a$ is contained in the $\Gamma_0$-stabilizer of the finite set of edges that join $a$ to $g c_0$, and this stabilizer is finite since $\Gamma$ acts properly on $X$. It follows from Proposition 2.13(3) that $\Gamma_0$ has finite hyperplane stabilizers in $C(P_0)$. The lemma then follows because the $\Gamma_0$-stabilizer of a pair of cubes in $C(P_0)$ stabilizes the finite set of hyperplanes that intersect or separate them. □

**Lemma 3.6** The action of $\Gamma_0$ on $C(P_0)$ has no fixed cube or pair of cubes.

**Proof** It suffices to find $a \in P_0$ and $g \in \Gamma_0$ with $g a \subsetneq a$ ($g$ skewers the hyperplane corresponding to $a$ in the language of [7]) as then $g$ will have no fixed vertex or pair of vertices. Indeed if $g$ fixes a vertex or pair of vertices, then $g^2$ fixes some vertex $\omega \in C(P_0)$, but then either $a \in \omega$ and $a \supseteq g^2 a \supseteq g^4 a \supseteq \cdots$ is a strictly descending infinite chain in $\omega$, or $a^* \in \omega$ and $a^* \supseteq g^{-2} a^* \supseteq g^{-4} a^* \supseteq \cdots$ is a strictly descending infinite chain in $\omega$—so either way we contradict $\omega$ being a DCC ultrafilter.

As $\Gamma_0$ acts cocompactly on the hyperplane $h_0$ and its cubical neighborhood $N(h_0)$, it follows from Lemma 3.2 that there exist $g_1, g_2 \in \Gamma_0$ with $g_1 c_0, g_2 c_0$ contained in distinct components $a_1, a_2$ of $h_0 - c_0$ respectively. If $g_1 a_1 \subsetneq a_1$ or $g_2 a_2 \subsetneq a_2$ then we are done, otherwise $g_1 a_1, g_2 a_2$ both contain $c_0$. But in that case $g_2 a_1$ is a component of $h_0 - g_2 c_0$ that doesn’t contain $c_0$, so $g_2 a_1 \subsetneq g_1 a_1$, and we are again done because $g_1^{-1} g_2 a_1 \subsetneq a_1$. □
3.2 The trees $T_{\mathring{h}}$

The action of $G_0$ on the cubical subdivision $\hat{C}(P_0)$ of $C(P_0)$ is without inversions in hyperplanes, and the hyperplanes of $\hat{C}(P_0)$ are finite since the hyperplanes of $C(P_0)$ are. Hence, we can repeatedly apply the panel collapse procedure of Hagen–Touikan [17, Theorem A] to $\hat{C}(P_0)$ to obtain an action of $G_0$ on a tree $T_0$. Moreover, there is a $G_0$-equivariant bijection between the vertex sets of $\hat{T}_0$ and $\hat{C}(P_0)$ (this is not stated explicitly in [17] but it follows from their construction). Thus, there is a $G_0$-equivariant bijection between the vertex sets of $\hat{T}_0$ and $\hat{C}(P_0)$ (this is not stated explicitly in [17] but it follows from their construction). Thus, there is a $G_0$-equivariant bijection between $V\hat{T}_0$ and the set of cubes of $C(P_0)$. It follows from Lemmas 3.5 and 3.6 that $\Gamma_0$ acts on $T_0$ with finite edge stabilizers and no fixed point. As $\Gamma_0$ is finitely generated, there is a $\Gamma_0$-invariant subtree $T_0' \subseteq T_0$ with finitely many $\Gamma_0$-orbits of edges. There is a $\Gamma_0$-equivariant bijection between the vertices in $T_0 - T_0'$ and the edges in $T_0 - T_0'$, where each vertex maps to the incident edge that points towards $T_0'$, so we conclude from Lemma 3.4 that $\Gamma_0$ acts cocompactly on $T_0$.

If $\{g_i \mid i \in \Omega\}$ is a left transversal of $G_0$ in $G$ then we get an induced action of $G$ on the product $T_0 \times G/G_0$, explicitly this is given by

$$g \cdot (v, g_i G_0) := (g_0 v, g_j G_0), \quad (3.1)$$

where $g g_i = g_j g_0$ with $i, j \in \Omega$ and $g_0 \in G_0$, and $g_0 v$ refers to the action of $G_0$ on $T_0$. (This construction is essentially the same as the notion of induced representation from representation theory.) We may assume that the transversal $\{g_i\}$ includes the identity element, in which case the action of $G_0$ on $T_0 \times \{G_0\}$ recovers the original action of $G_0$ on $T_0$.

We can then define the trees $T_{\mathring{h}}$ from Theorem 3.1, and the action of $G$ on $\sqcup_{\mathring{h}} T_{\mathring{h}}$, by putting

$$T_{g,\mathring{h}} := T_0 \times \{g_i G_0\},$$

and letting $T_{\mathring{h}}$ be a single point for hyperplanes $\mathring{h} \notin G \cdot \mathring{h}_0$. Properties (a)–(c) from Theorem 3.1 hold because there are finitely many $\Gamma$-orbits of hyperplanes in $X$, and $\Gamma_0$ acts on $T_0$ cocompactly, with finite edge stabilizers, and with no fixed point.

3.3 The pocset $\mathcal{P}$

Let $R = \text{diam}(c_0) + 1$. We know that all $G_0$-translates of $c_0$ lie in the $R$-neighborhood of the halfspace $h_0^*$, so as $X$ is essential we deduce that

$$h_0' := h_0 - \bigcup_{g \in G_0} g c_0$$

is non-empty. Define an equivalence relation $\sim$ on $h_0'$ where $x \sim y$ if $x$ and $y$ are not separated in $h_0$ by any set $g c_0$ with $g \in G_0$. Let $[x]$ denote the equivalence class of $x$, and let $\mathcal{M}_0$ denote the set of equivalence classes. The equivalence relation is preserved by $G_0$, so $G_0$ acts on $\mathcal{M}_0$ (Fig. 1).
Fig. 1 Cartoon of an equivalence class \([x] \in M_0\). The elements \(g_1, g_2, g_3\) are in \(G_0\)

Now define \(\mathcal{P}\) to be the set of all pairs \((a, h)\), with \(a \subseteq X^0\) and \(h \in \mathcal{H}\), that arise in one of the following three ways:

\[
(a, h) = \begin{cases}
(h, h), & h \notin G \cdot \{h_0, h_0^\ast\}, \\
(g[x], gh_0), & g \in G, \ [x] \in M_0, \\
(g[x]^\ast, gh_0^\ast), & g \in G, \ [x] \in M_0,
\end{cases}
\]  

(3.2)

where \([x]^\ast\) is the complement of \([x]\) in \(X^0\)—in fact in this section we will always denote the complement of \(a \subseteq X^0\) by \(a^\ast\). Define an action of \(G\) on \(\mathcal{P}\) by \(g \cdot (a, h) := (ga, gh)\). Also define an involution on \(\mathcal{P}\) by \((a, h) \mapsto (a, h)^\ast := (a^\ast, h^\ast)\). Finally, we make \(\mathcal{P}\) into a poset with the ordering \((a_1, h_1) \leq (a_2, h_2)\) if \(a_1 \subseteq a_2\) or \((a_1, h_1) = (a_2, h_2)\).

Note that \(\mathcal{P}\) looks very much like a wallspace on \(X^0\) if one just considers the first coordinate of each pair \((a, h) \in \mathcal{P}\), but the second coordinate will be needed in order to relate \(\mathcal{P}\) and its cubing to \(X\).

### 3.4 The map \(\sigma\)

Each \(x \in h_0^\prime\) defines a DCC ultrafilter on \(\mathcal{P}_0\) given by

\[
\lambda_x := \{a \in \mathcal{P}_0 \mid x \in a\}.
\]

Let \(x \sim y\). For any \(g \in G_0\) we know that \(x\) and \(y\) lie in the same component of \(h_0 - gc_0\), so it follows that \(\lambda_x = \lambda_y\). Conversely, if \(x \sim y\) then there exists \(g \in G_0\) such that \(x\) and \(y\) are separated by \(gc_0\), so there exists a component \(a\) of \(h_0 - gc_0\) containing \(x\) but not \(y\), and it follows that \(\lambda_x \neq \lambda_y\). This yields the following lemma.

**Lemma 3.7** There is a \(G_0\)-equivariant injection \(M_0 \to C(\mathcal{P}_0)^0 = VT_0\) given by \([x] \mapsto \lambda_x\).
Recall from Sect. 3.2 that the action of $G_0$ on $T_0$ extends to an action of $G$ on $\bigcup_{\hat{h} \in \hat{H}(X)} VT_{\hat{h}}$, with $T_0 = T_{\hat{h}_0}$. Also recall that $T_{\hat{h}}$ consists of a single vertex if $\hat{h} \notin G \cdot \{\hat{h}_0, \hat{h}_0^*\}$. We can then extend the map $M_0 \to VT_0$ above to a $G$-equivariant map

$$\sigma : \mathcal{P} \to \bigsqcup_{\hat{h} \in \hat{H}(X)} VT_{\hat{h}},$$

by setting

$$\sigma(h, \hat{h}) = T_{\hat{h}}, \quad \hat{h} \notin G \cdot \{\hat{h}_0, \hat{h}_0^*\},$$

$$\sigma(g[x], g\hat{h}_0) = g\lambda_x, \quad g \in G, \ [x] \in \mathcal{M}_0,$$

$$\sigma(g[x]^*, g\hat{h}_0^*) = g\lambda_x, \quad g \in G, \ [x] \in \mathcal{M}_0.$$ 

We clearly have $\sigma(a, \hat{h}) = \sigma(a^*, \hat{h}^*)$ for all $(a, \hat{h}) \in \mathcal{P}$, and this is the only failure of injectivity since $[x] \mapsto \lambda_x$ is injective. Hence $\sigma$ descends to an injective $G$-equivariant map

$$\sigma : \hat{\mathcal{P}} \to \bigsqcup_{\hat{h} \in \hat{H}(X)} VT_{\hat{h}}.$$ 

We will soon construct $Y$ as the cubing of $\mathcal{P}$, and then $\hat{\mathcal{P}}$ will be identified with $\hat{H}(Y)$ by Proposition 2.13(3), thus making $\sigma$ the required map in Theorem 3.1(3).

### 3.5 The cubing $Y$

We will define the cube complex $Y$ to be the cubing of the pocset $(\mathcal{P}, \leq)$. So we must show that there exist DCC ultrafilters on $\mathcal{P}$ and that $\mathcal{P}$ has finite width.

For $x \in X^0$ let

$$\theta(x) := \{ (a, \hat{h}) \in \mathcal{P} \mid x \in a \}.$$ 

This is clearly an ultrafilter on $\mathcal{P}$, and we will show over the next four lemmas that it is DCC. As $X$ is locally finite and cocompact, there exists a function $k : \mathbb{N} \to \mathbb{N}$ such that the $r$-neighborhood of any vertex or edge in $X$ intersects at most $k(r)$ cubical neighborhoods of hyperplanes. We will use this function as a source of constants throughout this section.

**Lemma 3.8** If $(a, \hat{h}) \in \mathcal{P}$ and $e$ is an edge that joins a vertex in $a$ to a vertex in $a^*$, then

$$d(e, \hat{h}), d(e, \hat{h}^*) \leq R.$$ 

Moreover, for each edge $e$ in $X$ there are at most $2k(R)$ elements $(a, \hat{h}) \in \mathcal{P}$ such that $e$ joins a vertex in $a$ to a vertex in $a^*$. 

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Lemma 3.10  \( d(x, y) - 2k(R) \leq |\theta(x) - \theta(y)| \leq k(R)d(x, y) \) for all \( x, y \in X^0 \).

Proof  Fix \( x, y \in X^0 \) and \( y \) a geodesic between them (in \( X^1 \)). Let \( e \) be an edge on \( y \), and suppose it is dual to a hyperplane \( \hat{h} \) with \( x \in \hat{h} \) and \( y \in \hat{h}^* \). Further suppose that the cubical neighborhood \( N(\hat{h}) \) does not intersect the \( R \)-neighborhoods of \( x \) and \( y \), so \( d(x, \hat{h})^*, d(y, \hat{h}) > R \). By Lemma 3.9 there exists \( (a, \hat{h}) \in \mathcal{P} \) with \( x \in a \) and \( y \in a^* \), so \( (a, \hat{h}) \in \theta(x) - \theta(y) \) and \( d(x, \hat{h})^* \) and \( d(y, \hat{h})^* \) are dual to distinct hyperplanes, since \( y \) is a geodesic (Lemma 2.8), so this proves the first inequality.
For the second inequality, for each \((a, h) \in \theta(x) \Delta \theta(y)\) there is an edge \(e\) on \(y\) joining a vertex in \(a\) to a vertex in \(a^\ast\). By Lemma 3.8 we have \(|\theta(x) \Delta \theta(y)| \leq 2k(R)d(x, y)\), so \(|\theta(x) - \theta(y)| \leq k(R)d(x, y)\).

\[\Box\]

**Lemma 3.11** \(\theta(x)\) is a DCC ultrafilter for all \(x \in X^0\).

**Proof** Let \((a, h) \in \theta(x)\). Pick \(y \in a^\ast\). Any \((a', h') \in \theta(x)\) with \((a', h') \leq (a, h)\) satisfies \(a' \subseteq a\), so \((a', h') \in \theta(x) - \theta(y)\). We deduce from Lemma 3.10 that no strictly descending infinite chain in \(\theta(x)\) contains \((a, h)\). But \((a, h)\) was an arbitrary element of \(\theta(x)\), hence \(\theta(x)\) is DCC.

\[\Box\]

We now prove over the next three lemmas that \(\mathcal{P}\) has finite width. This involves the convex hull \(\text{Hull}(a)\) and the metric \(E\)-neighborhood \(C_E(a)\) of a set \(a \subseteq X^0\) (see Sect. 2.1).

**Lemma 3.12** There exists an integer \(E > 0\) such that \(\text{Hull}(a) \subseteq \mathcal{N}_E(a)\) for all \((a, h) \in \mathcal{P}\).

**Proof** If \(h \notin G \cdot \{h_0, h_0^\ast\}\) then \(a = h\) is already convex, so it suffices to find \(E > 0\) such that \(\text{Hull}([x]) \subseteq \mathcal{N}_E([x])\) and \(\text{Hull}([x]^\ast) \subseteq \mathcal{N}_E([x]^\ast)\) for all \([x] \in \mathcal{M}_0\).

Fix \([x] \in \mathcal{M}_0\). As in Sect. 3.4, \([x]\) defines a vertex \(\lambda_x \in C(\mathcal{P}_0)^0\). Let \([e_\tau]\) be the set of edges incident to \(\lambda_x\). Lemma 3.3 tells us that the hyperplanes of \(C(\mathcal{P}_0)\) are compact, and Lemma 3.4 tells us that \(C(\mathcal{P}_0)\) has finitely many \(G_0\)-orbits of hyperplanes, hence \(C(\mathcal{P}_0)\) has finitely many \(G_0\)-orbits of edges. It follows that the \(G_0\)-stabilizer of \([x]\) has finitely many orbits in \([e_\tau]\). Each edge \(e_\tau\) leaves a halfspace of \(C(\mathcal{P}_0)\) containing \(\lambda_x\), and such a halfspace corresponds to a component \(a_\tau\) of \(h_0 - g_\tau c_0\) containing \([x]\), for some \(g_\tau \in G_0\). Moreover, any \(y \in h_0^\ast - [x]\) defines a vertex \(\lambda_y \neq \lambda_x\) in \(C(\mathcal{P}_0)\), so \(\lambda_y\) is separated from \(\lambda_x\) by a hyperplane dual to some \(e_\tau\), and therefore \(y\) is separated from \([x]\) by one of the sets \(g_\tau c_0\).

As the \(G_0\)-stabilizer of \([x]\) has finitely many orbits in \([e_\tau]\), we see that there is an integer \(E_1 > 0\) such that all the sets \(g_\tau c_0\) are contained in the \(E_1\)-neighborhood of \([x]\). \(G_0\) acts cocompactly on \(h_0 - h_0^\ast\), so there is also an integer \(E_2 > 0\) such that \(h_0\) is contained in the \(E_2\)-neighborhood of \(h_0^\ast\). Let \(E := E_1 + 2E_2\). We now claim that every \(y \in h_0 - \mathcal{N}_E([x])\) is separated from \([x]\) by one of the sets \(g_\tau c_0\). Indeed, given such a \(y\) there exists \(y' \in h_0^\ast\) with \(d(y, y') \leq E_2\), and \(y' \notin \mathcal{N}_{E_1+E_2}([x])\), so in particular \(y' \notin [x]\), and there exists \(g_\tau c_0\) separating \(y'\) from \([x]\). But \(g_\tau c_0 \subseteq \mathcal{N}_{E_1}([x])\), so \(d(y', g_\tau c_0) > E_2\), hence \(y\) is also separated from \([x]\) by \(g_\tau c_0\).

If \(y \in h_0^\ast\) is separated from \([x]\) by \(g_\tau c_0\), then in particular \(y \notin g_\tau c_0\). As \(g_\tau c_0\) is convex, there exists a halfspace \(h \in \mathcal{H}\) with \(g_\tau c_0 \subseteq h\) and \(y \in h^\ast\) (Lemma 2.8). The intersection \(h_0 \cap h^\ast\) is convex (Lemma 2.9), so in particular connected, thus we must have \([x] \subseteq h\) (else there is a path in \(h_0 \cap h^\ast\) joining \(y\) to \([x]\) that avoids \(g_\tau c_0\)).

\(\text{Hull}([x])\) is the intersection of halfspaces containing \([x]\), so our arguments so far imply that any \(y \in h_0 - \mathcal{N}_E([x])\) lies outside \(\text{Hull}([x])\). We also know that \([x] \subseteq h_0\), so \(\text{Hull}([x]) \subseteq h_0\). This proves that \(\text{Hull}([x]) \subseteq \mathcal{N}_E([x])\).

Finally we turn to \(\text{Hull}([x]^\ast)\). Enlarging \(E\) so that \(E \geq R \dim X\), we claim that

\[\text{Hull}([x]^\ast) \subseteq \mathcal{N}_E([x]^\ast)\].
Suppose \( y \in [x] - \mathcal{N}_E([x]^*) \). Our task is to find a halfspace containing \([x]^*\) but not \( y \). Observe from Remark 2.2 that \((b_0^*)^+ \cap R\) is convex, and from Remark 2.3 that

\[
\mathcal{N}_R(b_0^*) \subseteq (b_0^*)^+ \subseteq \mathcal{N}_{R \dim X}(b_0^*).
\]

We know that \( b_0^* \subseteq [x]^* \), so \( y \notin \mathcal{N}_{R \dim X}(b_0^*) \) and \( y \notin (b_0^*)^+ \), so by convexity of \((b_0^*)^+ \cap R\) there is a halfspace \((b_0^*)^+ \subseteq b \in \mathcal{H} \) with \( y \in b^\ast \). The halfspace \( b^\ast \) is disjoint from \( \mathcal{N}_R(b_0^*) \), so lies in \( b_0^* \), and \( b^\ast \) is connected so it lies inside one of the classes in \( \mathcal{M}_0 \). As \( y \in b^\ast \cap [x] \) we deduce that \( b^\ast \subseteq [x] \), so \([x]^* \subseteq b \) as required.

**Lemma 3.13** For \((a, b) \in \mathcal{P}\) the intersection \( \text{Hull}(a)^{+1} \cap \text{Hull}(a^*)^{+1} \) is non-empty and is contained in the \((R + E + \dim X)\)-neighborhood of \( N(b) \).

**Proof** The intersection \( \text{Hull}(a)^{+1} \cap \text{Hull}(a^*)^{+1} \) is non-empty because it contains any edge that joins \( a \) to \( a^* \). Lemma 3.12 says that \( \text{Hull}(a) \subseteq \mathcal{N}_E(a) \), so \( \text{Hull}(a)^{+1} \subseteq \mathcal{N}_{E + \dim X}(a) \) by Remark 2.3. Similarly \( \text{Hull}(a^*)^{+1} \subseteq \mathcal{N}_{E + \dim X}(a^*) \). Thus any vertex in \( \text{Hull}(a)^{+1} \cap \text{Hull}(a^*)^{+1} \) is within distance \( E + \dim X \) of an edge that joins \( a \) to \( a^* \), so it lies in the \((R + E + \dim X)\)-neighborhood of \( N(b) \) by Lemma 3.8.

**Lemma 3.14** \( \mathcal{P} \) has finite width.

**Proof** Let \( \{(a_i, b_i)\} \) be a finite collection of pairwise transverse elements of \( \mathcal{P} \). It follows easily from the construction of \( \mathcal{P} \) that the \( b_i \) are distinct. Remark 2.2 implies that all the sets \( \text{Hull}(a_i)^{+1} \) and \( \text{Hull}(a_i^*)^{+1} \) are convex, and they pairwise intersect by Lemma 3.13 and the fact that \((a_i, b_i)\) are pairwise transverse. So Lemma 2.9 tells us that

\[
\bigcap_i (\text{Hull}(a_i)^{+1} \cap \text{Hull}(a_i^*)^{+1}) \neq \emptyset.
\]

But, by Lemma 3.13, any vertex in this intersection is in the \((R + E + \dim X)\)-neighborhood of \( N(b_i) \) for all \( i \), thus the size of the collection \( \{(a_i, b_i)\} \) is bounded by \( k(R + E + \dim X) \).

As promised, we can now define \( Y := C(\mathcal{P}) \). The construction also gives us a \( G \)-equivariant map \( \Theta : X^0 \to Y^0 \), which is a quasi-isometric embedding by Lemmas 2.16 and 3.10.

### 3.6 The quasi-isometries \( \Theta \) and \( \Phi \)

Next we will show that \( \Theta : X^0 \to Y^0 \) is a quasi-isometry by constructing a coarse inverse \( \Phi \). For \( \mu \in Y \), define the subset \( \omega_\mu \subseteq \mathcal{H} \) to consist of all halfspaces \( b \) with \( a \subsetneq b \) for some \( (a, b^\prime) \in \mu \).

**Lemma 3.15** \( \omega_\mu \) is a DCC partial ultrafilter on \( \mathcal{H} \).

**Proof** First let’s prove DCC, so suppose \( \mathcal{H}_1 \supsetneq \mathcal{H}_2 \supsetneq \mathcal{H}_3 \supsetneq \cdots \) is a strictly descending infinite chain in \( \omega_\mu \). For each \( \mathcal{H}_i \) pick \( (a_i, b_i^\prime) \in \mu \) with \( a_i \subsetneq b_i^\prime \). The intersection \( \bigcap_i \mathcal{H}_i \)
is empty, so each $a_j$ is contained in only finitely many $h_i$, hence the sequence $(a_j, b_j')$ contains infinitely many distinct elements. For $x \in h_i^*$ we have $x \not\in a_i$ for all $i$, so $(a_i, b_j') \in \mu - \theta(x)$ for all $i$, but then $\mu - \theta(x)$ is infinite, contradicting $\mu, \theta(x) \in Y^0$.

To see that $\omega_\mu$ is a partial ultrafilter, suppose that $h_1, h_2 \in \omega_\mu$ are disjoint. Then there exist $(a_1, h'_1), (a_2, h'_2) \in \mu$ with $a_i \subseteq h_i$. In turn this implies that $a_1 \subseteq \alpha_2^*$, so $(a_1, h'_1) \subseteq (a_2^*, (h'_2)^*)$, contradicting $\mu$ being an ultrafilter. \hfill $\square$

**Lemma 3.16** The intersection $b_\mu := \bigcap_{h \in \omega_\mu} h$ is non-empty, and $b_\mu \subseteq h \in \mathcal{H}$ implies $h \in \omega_\mu$.

**Proof** $\omega_\mu$ is a DCC partial ultrafilter by Lemma 3.15, and it is clear that $h_1 \in \omega_\mu$ and $h_1 \subseteq h_2 \in \mathcal{H}$ implies $h_2 \in \omega_\mu$. Hence we can apply Lemma 2.21 to deduce that $\omega_\mu$ can be extended to a DCC complete ultrafilter, and that the set of such extensions is precisely the set $b_\mu$ defined above (identifying $X$ with $C(\mathcal{H})$ by Theorem 2.15). Thus $b_\mu$ is non-empty.

For the second part, suppose $b_\mu \subseteq h \in \mathcal{H}$ with $h \not\in \omega_\mu$. Then we can pick $x \in h^*$, and define a DCC complete ultrafilter $\omega_x$ on $\mathcal{H}$ with $h^* \in \omega_x$. We know that $h^* \not\in \omega_\mu$ as that would contradict $b_\mu \subseteq h$, so we may apply Lemma 2.21 with $\omega_0 = \omega_x$ to extend $\omega_\mu$ to a DCC complete ultrafilter $\tilde{\omega}$ with $h^* \in \tilde{\omega}$. But we said above that such an extension corresponds to a vertex of $b_\mu$, so this implies $h^* \cap b_\mu \neq \emptyset$, once again contradicting $b_\mu \subseteq h$. \hfill $\square$

**Lemma 3.17** $b_\mu$ is contained in the intersection $\bigcap_{(a, h) \in \mu} N_1(\text{Hull}(a))$.

**Proof** Let $(a, h) \in \mu$ and suppose that $x \in X^0 - N_1(\text{Hull}(a))$. Lemma 2.8 implies that there are distinct $h_1, h_2 \in \mathcal{H}$ with $\text{Hull}(a) \subseteq h_i$ and $x \in h_i^*$. In particular, we either have $a \subseteq h_1$ or $a \subseteq h_2$, so at least one of $h_1, h_2$ is in $\omega_\mu$, hence $x \not\in b_\mu$. \hfill $\square$

**Lemma 3.18** The diameter of $b_\mu$ is at most $D := 2k(R + E + 1)$.

**Proof** Suppose $x, y \in b_\mu$ with $d(x, y) > D$. Then by considering the hyperplanes that separate $x$ and $y$, the definition of $k : N \rightarrow N$ tells us that there exists $h \in \mathcal{H}$ with $d(x, h^*)$, $d(y, h) > R + E + 1$. By Lemma 3.9, there exists $(a, h) \in \mathcal{P}$ with $d(x, a^*)$, $d(y, a) > E + 1$. We know that $\mu$ is an ultrafilter, so one of $(a, h)$ and $(a^*, h^*)$ is in $\mu$—say $(a, h) \in \mu$. Lemma 3.12 tells us that $\text{Hull}(a) \subseteq N_{E_1}(a)$, so $y \not\in N_1(\text{Hull}(a))$. But then Lemma 3.17 implies that $y \not\in b_\mu$, a contradiction. \hfill $\square$

We now have a coarsely well-defined map $\phi : Y^0 \rightarrow X^0$ by picking $\phi(\mu) \in b_\mu$ for each $\mu \in Y^0$. Our next task is to prove that $\phi$ is a coarse inverse to $\theta$.

**Lemma 3.19** $\sup_{\mu \in Y^0} d(\mu, \theta(\phi(\mu))) < \infty$

**Proof** Let $\mu \in Y^0$. Suppose $(a, h) \in \mu - \theta(\phi(\mu))$. Then $\phi(\mu) \not\in a$. But $\phi(\mu) \in b_\mu \subseteq N_1(\text{Hull}(a))$ by Lemma 3.17, so $\phi(\mu) \in N_{E+1}(a)$ by Lemma 3.12. We deduce that there exists an edge $e$ in the $(E + 1)$-neighborhood of $\phi(\mu)$ that joins $a$ to $a^*$. By Lemma 3.8, we can then bound $|\mu - \theta(\phi(\mu))|$ by the product of $2k(R)$ and the number of edges in the $(E + 1)$-neighborhood of $\phi(\mu)$, and this can be bounded independently of $\mu$ since $X$ is locally finite and cocompact. \hfill $\square$

It follows from Lemmas 3.10 and 3.19 that $\theta$ is a quasi-isometry with coarse inverse $\phi$. 

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3.7 The actions of \( G \) and \( \Gamma \) on \( Y \)

**Lemma 3.20** \( Y \) is locally finite.

**Proof** Let \( \mu \in Y^0 \). By Lemma 2.17, to show that \( Y \) is locally finite at \( \mu \) we must show that \( \mu \) has finitely many \( \leq \)-minimal elements.

Let \((a, h) \in \mu \). Lemma 3.17 tells us that \( \phi(\mu) \in N_1(\text{Hull}(a)) \), so \( d(\phi(\mu), a) \leq E + 1 \) by Lemma 3.12. Now suppose that \( d(\phi(\mu), a^*) > D + E \). Then Lemmas 3.12 and 3.18 imply that \( b_\mu \) and \( \text{Hull}(a^*) \) are disjoint, and since these are convex sets Lemma 2.8 provides us with \( b_\mu \subseteq h \in \mathcal{H} \) such that \( \text{Hull}(a^*) \subseteq h^* \) – in particular \( h \subseteq a \). Lemma 3.16 implies that \( h \in \omega_\mu \), so there exists \((a', h') \in \mu \) with \( a' \subset h \).

Then \( a' \subset a \) and \((a', h') \leq (a, h) \), so \((a, h)\) is not minimal in \( \mu \).

Our argument so far implies that any minimal \((a, h) \in \mu \) has \( d(\phi(\mu), a), d(\phi(\mu), a^*) \leq D + E \). But for such an \((a, h)\) there exists an edge \( e \) in the \((D + E)\)-neighborhood of \( \phi(\mu) \) joining \( a \) to \( a^* \). It follows from Lemma 3.8 and the local finiteness of \( X \) that \( \mu \) has finitely many minimal elements. \( \square \)

We now verify property (1) of Theorem 3.1, which concerns the actions of \( G \) and \( \Gamma \) on \( Y \). We already know that \( \theta : X \to Y \) is a \( G \)-equivariant quasi-isometry, and that \( G \) acts cocompactly on \( X \), so it follows from Lemma 3.20 that \( G \) acts cocompactly on \( Y \). It is also clear that \( G \) acts on \( Y \) without inversions in hyperplanes: we have a poset isomorphism \((\mathcal{H}(Y), \subset) \cong (\mathcal{P}, \leq)\), and \( g(a, h) = (a^*, h^*) \) for \((a, h) \in \mathcal{P} \) and \( g \in G \) implies that \( g\{a, h\} = h^* \), contradicting the fact that \( G \) acts on \( X \) without inversions in hyperplanes.

We also know that \( \theta : X \to Y \) is a \( \Gamma \)-equivariant quasi-isometry, and that \( \Gamma \) acts cocompactly on \( X \), so again it follows from Lemma 3.20 that \( \Gamma \) acts cocompactly on \( Y \). Similarly, \( \Gamma \) acts properly on \( Y \) because it acts properly on \( X \).

Finally, to ensure that \( Y \) is essential we can replace it with its essential core using Proposition 2.5 (noting that the closest point projection from \( Y \) to its essential core is a \( G \)-equivariant quasi-isometry, and that the set of hyperplanes of the essential core is a subset of the set of hyperplanes of \( Y \)). This completes the proof of Theorem 3.1.

4 Reducing to deep quarterspaces

In this section we prove the following theorem.

**Theorem 1.4** Let \( G \) be a group acting cocompactly on a CAT(0) cube complex \( X \). Then there is a CAT(0) cube complex \( Y \) with the following properties:

1. All quarterspaces in \( Y \) are deep.
2. \( G \) acts cocompactly on \( Y \).
3. There exists a \( G \)-equivariant quasi-isometry \( \phi : Y \to X \).
4. \( \phi \) maps each halfspace in \( Y \) to within finite Hausdorff distance of a halfspace in \( X \).
5. \( Y \) is locally finite if \( X \) is locally finite.

Theorem 1.4 follows from the following theorem by induction on the number of \( G \)-orbits of vertices in \( X \).
Theorem 4.1 Let $G$ be a group acting cocompactly on a CAT(0) cube complex $X$. If $X$ contains a shallow quarterspace then there is a CAT(0) cube complex $Y$ with the following properties:

1. $G$ acts cocompactly on $Y$.
2. $Y$ has fewer $G$-orbits of vertices than $X$.
3. There exists a $G$-equivariant quasi-isometry $\phi : Y \to X$.
4. $\phi$ maps each halfspace in $Y$ to within finite Hausdorff distance of a halfspace in $X$.
5. $Y$ is locally finite if $X$ is locally finite.

We will spend the rest of this section proving Theorem 4.1. If $X$ is bounded then we can take $Y$ to be a single vertex and $\phi(Y)$ to be the center of $X$ [6, Proposition II.2.7], so henceforth we assume that $X$ is unbounded. By Proposition 2.5 we may assume that $X$ is essential. Throughout this section we will write $\mathcal{H} = \mathcal{H}(X)$ for the set of halfspaces of $X$.

4.1 Depth-0 quarterspaces

We need the following lemma to characterize depth-0 quarterspaces in terms of inclusions of halfspaces (see also Fig. 3).

Lemma 4.2 Let $h_1 \cap h_2$ be a quarterspace. Then $h_1 \cap h_2$ has depth 0 if and only if any halfspace $h \subsetneq h_1$ (resp. $h \subsetneq h_2$) satisfies $h \subsetneq h_1^* \cap h_2^*$ (resp. $h \subsetneq h_2^*$).

Proof Suppose that $h_1 \cap h_2$ has depth greater than 0. Then there exists $x \in h_1 \cap h_2$ with $d(x, h_1^* \cap h_2^*) \geq 3$. Lemma 2.8 implies that there is a halfspace $x \in \mathcal{H} - \{h_1, h_2\}$ with $h_1 \cap h_1^* \cap h_2^* = \emptyset$. Using Lemma 2.9, we can then assume without loss of generality that $h_1 \cap h_1^* = \emptyset$, which means $h \subsetneq h_1$. Plus we know that $x \in h \cap h_2$, so $h \not\subsetneq h_2^*$.

Conversely, if there is a halfspace $h \subsetneq h_1$ that doesn’t satisfy $h \subsetneq h_2^*$, then $h \not= h_2^*$ (as $h_1, h_2$ are transverse) and $h \cap h_2 \not= \emptyset$. Therefore $h \cap h_1^* \cap h_2^* = \emptyset$ and there exists $x \in h \cap h_1 \cap h_2$, and such $x$ satisfies $d(x, h_1^* \cap h_2^*) \geq 3$, so that $h_1 \cap h_2$ has depth greater than 0. \hfill \Box

Next, we deduce that $X$ contains a depth-0 quarterspace by the following lemma.

Fig. 3 If $h_1 \cap h_2$ is a depth-0 quarterspace then Lemma 4.2 implies that there are no halfspaces as shown in blue.
Lemma 4.3 Any shallow quarterspace contains a depth-0 quarterspace.

Proof Let \( h_1 \cap h_2 \) be a shallow quarterspace of depth \( r > 0 \). By Lemma 4.2 we may assume there is a halfspace \( h \subseteq h_1 \) with \( h \cap h_2 \neq \emptyset \). We cannot have \( h \subseteq h_2 \), as then \( h \) being deep (since \( X \) is essential) would imply that \( h_1 \cap h_2 \) is deep, thus \( h \) and \( h_2 \) are transverse. We now claim that \( h \cap h_2 \) is a quarterspace of depth less than \( r \)—the lemma then follows by induction on depth. Indeed for any \( x \in h \cap h_2 \), we have an inclusion

\[
\{ h' \in \mathcal{H} \mid x \in h', \ h' \cap h^*_1 \cap h^*_2 = \emptyset \} \subseteq \{ h' \in \mathcal{H} \mid x \in h', \ h' \cap h^*_1 \cap h^*_2 = \emptyset \},
\]

which is strict because \( h_1 \) is in the second set but not the first (\( h_1 \cap h^*_1 \cap h^*_2 \neq \emptyset \) by Lemma 2.9). Lemma 2.8 then implies that

\[
d(x, h^*_1 \cap h^*_2) < d(x, h^*_1 \cap h^*_2) \leq r,
\]

so we conclude that \( h \cap h_2 \) has depth at most \( r - 1 \). \( \square \)

4.2 The pocset (\( \mathcal{H}/\sim, \leq \))

The cube complex \( Y \) will be constructed from a modified version of the pocset of halfspaces (\( \mathcal{H}, \subseteq \)). We first define a quasi-order \( \leq \) on \( \mathcal{H} \)—i.e. a binary relation that is reflexive and transitive but may have \( h_1 \leq h_2 \leq h_1 \) for \( h_1 \neq h_2 \). We define this by \( h_1 \leq h_2 \) if \( h_1 \subseteq h_2 \) or \( h_1 \cap h_2 \subseteq h_1 \) is a depth-0 quarterspace. We will make use of the following equivalent formulation.

Lemma 4.4 \( h_1 \leq h_2 \) if and only if any halfspace \( h \subseteq h_1 \) (resp. \( h \subseteq h^*_2 \)) satisfies \( h \subseteq h_2 \) (resp. \( h \subseteq h^*_1 \)).

Proof If \( h_1, h_2 \) are transverse then this equivalence reduces to Lemma 4.2. If \( h_1 \subseteq h_2 \) then both conditions are clearly satisfied. If \( h_2 \subseteq h_1 \) or \( h_1 \cap h_2 = \emptyset \) then it is easy to see that neither condition is satisfied (noting that \( \exists h \subseteq h_1 \) by Lemma 2.8 and essentialness of \( X \)). \( \square \)

Lemma 4.5 \( \leq \) is a quasi-order on \( \mathcal{H} \).

Proof Reflexivity is immediate from the definition. Transitivity follows easily from Lemma 4.4: if \( h_1 \leq h_2 \leq h_3 \) and \( h \subseteq h_1 \) then \( h_1 \leq h_2 \) implies \( h \subseteq h_2 \) and \( h_2 \leq h_3 \) implies \( h \subseteq h_3 \); similarly \( h \subseteq h^*_3 \) implies \( h \subseteq h^*_1 \). \( \square \)

To turn \( \leq \) into a partial order we quotient \( \mathcal{H} \) by the equivalence relation \( h_1 \sim h_2 \) if \( h_1 \leq h_2 \leq h_1 \). Let \([h]\) denote the equivalence class of \( h \). If \( h_1 \sim h_2 \), then we cannot have \( h_1 \subseteq h_2 \) as \( h_2 \leq h_1 \) would imply \( h_1 \subseteq h_2 \), similarly we cannot have \( h_2 \subseteq h_1 \). It follows that the elements within an equivalence class \([h]\) are pairwise transverse, and so the size of \([h]\) is bounded by the dimension of \( X \) (note that \( X \) is finite dimensional since it admits a cocompact group action) (Fig. 4).

It follows straight from the definition of \( \leq \) that \( h_1 \leq h_2 \) if and only if \( h^*_2 \leq h^*_1 \). It is also immediate that \( h_1, h_2 \) are \( \leq \)-transverse whenever \([h_1],[h_2]\) are \( \leq \)-transverse. Putting this all together we get the following lemma.
Lemma 4.6 \((\mathcal{H}/ \sim, \leq)\) is a poset with involution defined by \([h]^* := [h^*]\). Moreover, the width of \((\mathcal{H}/ \sim, \leq)\) is at most \(\dim X\); and the quotient map \(\mathcal{H} \to \mathcal{H}/ \sim\) defines a poset map \(q : (\mathcal{H}, \subseteq) \to (\mathcal{H}/ \sim, \leq)\), with sizes of fibers bounded by \(\dim X\).

4.3 The cubing \(Y\)

The cube complex \(Y\) will be the cubing of \((\mathcal{H}/ \sim, \leq)\). To show that this is well-defined we must find a DCC ultrafilter on \((\mathcal{H}/ \sim, \leq)\). Our arguments will mainly be on the level of ultrafilters for the remainder of this section, so we will consider vertices \(\omega \in X^0\) as DCC ultrafilters on \((\mathcal{H}, \subseteq)\) (using Theorem 2.15) and vertices \(\mu \in Y^0\) as DCC ultrafilters on \((\mathcal{H}/ \sim, \leq)\) (once we know that they exist!).

Given \(\omega \in X^0\), consider the partition \(\omega = \omega^0 \sqcup \omega^1\), where \(h_1 \in \omega^0\) if there exists \(h_2 \in \omega^1\) with \(h_1 \cap h_2\) a depth-0 quarterspace.

Lemma 4.7 \(|\omega^0| \leq \dim X\) for all \(\omega \in X^0\).

Proof We show that the halfspaces in \(\omega^0\) are pairwise transverse. Firstly, if \(h_1 \cap h_2\) is a depth-0 quarterspace for \(h_1, h_2 \in \omega\), then Lemma 4.2 together with the consistency of \(\omega\) implies that \(h_1, h_2\) are both \(\subseteq\)-minimal in \(\omega\). So every halfspace in \(\omega^0\) is \(\subseteq\)-minimal in \(\omega\). To prove the lemma it suffices to consider distinct \(h_1, h_2, h_3 \in \omega^0\) such that \(h_1 \cap h_2\) is a depth-0 quarterspace, and show that \(h_1, h_3\) are transverse. Indeed, \(\subseteq\)-minimality of \(h_1, h_3\) in \(\omega\) implies that we cannot have \(h_1 \not\subseteq h_3\) or \(h_3 \not\subseteq h_1\). And if \(h_3^* \not\subseteq h_1\) then \(h_3^* \not\subseteq h_2^*\) by Lemma 4.2, so \(h_2 \not\subseteq h_3\), contradicting \(\subseteq\)-minimality of \(h_3\).

If \(h_1, h_2 \in \omega^1\) then \(h_1 \not\subseteq h_2^*\) and \(h_1 \cap h_2\) is not a depth-0 quarterspace, hence \(h_1 \not\subseteq h_2^*\). Thus \(\omega^1\) pushes forward to a partial ultrafilter \(q_* \omega^1\) on \((\mathcal{H}/ \sim, \leq)\), given by

\[
q_* \omega^1 := \{[h] \mid h \in \omega^1\}.
\]

Lemma 4.8 \(q_* \omega^1\) is DCC and cofinite.
Proof Suppose for contradiction that \([h_1] > [h_2] > [h_3] > \cdots\) is a strictly descending infinite chain in \(q_\omega\omega^1\), with \(h_i \in \omega^1\). For each \(i < j\), either \(h_i \supsetneq h_j\) or \(h_j \cap h_i^\ast\) is a depth-0 quarterspace. In the latter case \(h_i, h_j\) are transverse, and a collection of pairwise transverse halfspaces can be no larger than \(\dim X\), so it follows from Ramsey’s Theorem that there is an infinite \(\subseteq\)-descending subsequence of the \(h_i\). This contradicts \(\omega\) being a DCC ultrafilter. Hence \(q_\omega\omega^1\) is DCC.

To see cofiniteness of \(q_\omega\omega^1\), consider a pair \([h], [h^\ast] \notin q_\omega\omega^1\). It follows that \(h, h^\ast \notin \omega^1\), so one of them must lie in \(\omega^0\), but then there are at most dim \(X\) possibilities for \(h, h^\ast\) by Lemma 4.7.

By Lemma 2.20, \(q_\omega\omega^1\) can be extended to a DCC complete ultrafilter \(\theta(\omega)\) on \((\mathcal{H}/ \sim, \leq)\). This shows the existence of DCC complete ultrafilters on \((\mathcal{H}/ \sim, \leq)\), and it also gives us a map \(\theta : X^0 \to Y^0\) (albeit not a canonical one!).

We will see later that \(\theta\) is actually a quasi-isometry. We now define a map \(\phi : Y^0 \to X^0\), which will turn out to be a coarse inverse of \(\theta\). Given \(\mu \in Y^0\), define

\[
\phi(\mu) := \{h \in \mathcal{H} \mid [h] \in \mu\}.
\]

Lemma 4.9 \(\phi(\mu)\) is a DCC ultrafilter on \((\mathcal{H}, \subseteq)\), and the map \(\phi : Y^0 \to X^0\) is injective.

Proof For a halfspace \(h \in \mathcal{H}\) we have exactly one of \([h], [h]^\ast = [h^\ast]\) in \(\mu\) by completeness of \(\mu\), so we have exactly one of \(h, h^\ast\) in \(\phi(\mu)\). If \(h_1 \subseteq h_2\) with \(h_1 \in \phi(\mu)\), then \([h_1] \in \mu\) and \([h_1] \leq [h_2]\), so \([h_2] \in \mu\) by consistency of \(\mu\). Thus \(h_2 \in \phi(\mu)\). This shows that \(\phi(\mu)\) is an ultrafilter on \((\mathcal{H}, \subseteq)\). Any strictly \(\leq\)-descending chain in \(\phi(\mu)\) projects to a strictly \(\leq\)-descending chain in \(\mu\), so \(\mu\) being DCC implies that \(\phi(\mu)\) is DCC. Finally, the map \(\phi : Y^0 \to X^0\) is injective, because for distinct \(\mu_1, \mu_2 \in Y^0\) there exists \([h] \in \mu_1 - \mu_2\), so \(h \in \phi(\mu_1) - \phi(\mu_2)\).

Apart from the map \(\theta\), all the constructions so far are entirely canonical, so the action of \(G\) on \(X\) induces actions of \(G\) on \((\mathcal{H}/ \sim, \leq)\) and \(Y\), and the map \(\phi\) is \(G\)-equivariant.

Lemma 4.10 \(G\) acts cocompactly on \(Y\).

Proof By Lemma 2.18 it is equivalent to show that there are finitely many \(G\)-orbits of collections of pairwise transverse elements in \((\mathcal{H}/ \sim, \leq)\). Now \(q : (\mathcal{H}, \subseteq) \to (\mathcal{H}/ \sim, \leq)\) is a surjective pocset map with pairwise transverse fibers, so the preimage of a collection of pairwise transverse elements is pairwise transverse. But \(G\) acts cocompactly on \(X\), so there are finitely many collections of pairwise transverse elements in \((\mathcal{H}, \subseteq)\). The result follows.

Lemma 4.11 \(\phi\) is not surjective. In particular \(Y\) has fewer \(G\)-orbits of vertices than \(X\).

Proof If \(h_1 \cap h_2\) is a depth-0 quarterspace then \(h_1 \leq h_2^\ast\), so no \(\mu \in Y^0\) has \([h_1], [h_2] \in \mu\), which means no \(\mu \in Y^0\) has \(h_1, h_2 \in \phi(\mu)\). But any vertex in the quarterspace \(h_1 \cap h_2\) is represented by an ultrafilter \(\omega \in X^0\) with \(h_1, h_2 \in \omega\). We know that \(X\) does contain depth-0 quarterspaces by Lemma 4.3, so we conclude that \(\phi\) is not surjective.
Remark 4.12 It is not hard to extend the arguments in Lemma 4.11 to show that $X^0 - \phi(Y^0)$ is precisely the union of depth-0 quarterspaces in $X$.

Now we turn to showing that $\phi$ is a quasi-isometry.

Lemma 4.13 $d(\mu_1, \mu_2) \leq d(\phi(\mu_1), \phi(\mu_2)) \leq (\dim X)d(\mu_1, \mu_2)$ for all $\mu_1, \mu_2 \in Y^0$.

Proof By Lemma 2.16 this is equivalent to showing

$$|\mu_1 \triangle \mu_2| \leq |\phi(\mu_1) \triangle \phi(\mu_2)| \leq (\dim X)|\mu_1 \triangle \mu_2|.$$

And this follows from

$$\phi(\mu_1) \triangle \phi(\mu_2) = \{h \in \mathcal{H} \mid [h] \in \mu_1 \triangle \mu_2\}$$

and the fact that the classes $[h]$ have size at most $\dim X$. □

We now show that $\theta$ is a coarse inverse to $\phi$, so we conclude from Lemma 4.13 that $\theta$ and $\phi$ are both quasi-isometries.

Lemma 4.14 $\theta \phi$ is the identity map on $Y^0$ and $d(\phi(\theta(\omega)), \omega) \leq 2 \dim X$ for all $\omega \in X^0$.

Proof Let $\mu \in Y^0$. We claim that $\phi(\mu)^0 = \emptyset$. Indeed $\phi(\mu)^0 \neq \emptyset$ would imply the existence of $h_1, h_2 \in \phi(\mu)$ such that $h_1 \cap h_2$ is a depth-0 quarterspace. But then $h_1 \leq h_2^*$, so $[h_1] \leq [h_2]^*$, contradicting the consistency of $\mu$. Hence $\phi(\mu) = \phi(\mu)^1$, so

$$q_\mu \phi(\mu)^1 = q_\mu \phi(\mu) = \mu$$

is already a complete ultrafilter, thus $\theta \phi(\mu) = \mu$.

For the second part of the lemma, we note that $q_\mu^1 \subseteq \theta(\omega)$, so $\omega^1 \subseteq \phi(\theta(\omega))$. Then applying Lemma 2.16 yields

$$d(\phi(\theta(\omega)), \omega) = 2|\omega - \phi(\theta(\omega))| \leq 2|\omega^0| \leq 2 \dim X.$$

□

Next we show that each halfspace in $Y$ is mapped by $\phi$ to within finite Hausdorff distance of a halfspace in $X$. By Proposition 2.13(3), halfspaces in $Y$ correspond to the elements of $\mathcal{H}/\sim$, and it is immediate from the construction of $\phi$ that each halfspace $[h]$ is mapped within the halfspace $h$ in $X$. Thus it remains to show that $h$ is contained in a bounded neighborhood of the image $\phi[h]$. This follows from Lemma 4.14 together with the following lemma.

Lemma 4.15 Let $h \in \mathcal{H}$. We can choose the map $\theta$ so that $\phi \theta : X^0 \to X^0$ preserves the halfspace $h$.

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Proof Suppose $h \in \omega \in X^0$. Our goal is to choose $\theta(\omega)$ so that $[h] \in \theta(\omega)$, as this would imply that $h \in \phi\theta(\omega)$. If $h \in \omega^1$ then $[h] \in \theta(\omega)$ is automatic because $[h] \in q_*\omega^1$, so suppose $h \in \omega^0$. Observe that $q_*\omega^1 \cup \{[h]\}$ is a partial ultrafilter on $(\mathcal{H}/\sim, \leq)$: indeed if there is $h' \in \omega^1$ with $h' \leq h^*$, then either $h' \subseteq h^*$, contradicting consistency of $\omega$, or $h' \cap h$ is a depth-0 quarterspace, contradicting $h' \in \omega^1$. Moreover, the fact that $q_*\omega^1$ is DCC and cofinite implies that $q_*\omega^1 \cup \{[h]\}$ is DCC and cofinite. Thus we may choose $\theta(\omega)$ to be an extension of $q_*\omega^1 \cup \{[h]\}$ (still using Lemma 2.20).

Finally, we prove that our construction preserves local finiteness.

Lemma 4.16 $Y$ is locally finite if $X$ is locally finite.

Proof Let $\mu \in Y^0$ and let $[h_1] \in \mu$ be $\leq$-minimal in $\mu$. If $h_2 \in \phi(\mu)$ satisfies $h_2 \subseteq h_1$ then $[h_2] \in \mu$ and $[h_2] \leq [h_1]$; so by minimality of $[h_1]$ we have $[h_2] = [h_1]$. But we saw earlier that elements of a $\sim$-equivalence class are pairwise transverse, so in fact $h_2 = h_1$. Hence $h_1$ is $\subseteq$-minimal in $\phi(\mu)$. It follows from Lemma 2.17 that the number of edges in $Y$ incident to $\mu$ is at most the number of edges in $X$ incident to $\phi(\mu)$, and there are finitely many such edges since $X$ is locally finite.

5 Semistability at infinity

In this section we prove Theorem 1.1, that all cubulated groups are semistable at infinity. Dunwoody’s accessibility (Theorem 2.22) together with the following two theorems allow us to reduce to the case of one-ended cubulated groups (noting that finite groups are semistable).

Theorem 5.1 [27, Theorem 1]
If $G = A \ast_H B$ is an amalgamated product where $A$ and $B$ are finitely presented and semistable at infinity, and $H$ is finitely generated, then $G$ is semistable at infinity. If $G = A \ast_H$ is an HNN-extension where $A$ is semistable at infinity, and $H$ is finitely generated, then $G$ is semistable at infinity.

Theorem 5.2 Let $G$ be a cubulated group that admits a finite splitting over finite subgroups. Then each vertex group is cubulated.

Proof $G$ is hyperbolic relative to its infinite vertex groups, so each infinite vertex group is cubulated by [32, Theorem 1.1]. The finite vertex groups are also cubulated because they admit proper cocompact actions on a point.

We will also make use of the following theorem, which follows straight from the proof of [24, Theorem 2.1] (replacing $\tilde{X}$ by $X$). This provides a characterization for semistability in the one-ended case in terms of “pushing out” loops along a fixed proper ray.

Theorem 5.3 Let $X$ be a one-ended locally finite CW-complex and let $r : [0, \infty) \rightarrow X$ be a proper ray. Then the following are equivalent:
(1) \( X \) is semistable at infinity.

(2) For any compact set \( C \), there is a compact set \( D \) such that for any third compact set \( E \) and loop \( \alpha \) based on \( r \) with image in \( X - D \), \( \alpha \) is homotopic rel\( \{r\} \) to a loop in \( X - E \), by a homotopy in \( X - C \).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Cubulated groups are finitely presented, so by Dunwoody’s accessibility (Theorem 2.22) we know that any cubulated group admits a finite splitting over finite groups with vertex groups that are either finite or one-ended. If these vertex groups are semistable at infinity then Theorem 5.1 implies that the whole group is semistable at infinity. Theorem 5.2 tells us that the vertex groups in such a splitting are cubulated, so we reduce to proving semistability for finite or one-ended cubulated groups. Finite groups are automatically semistable at infinity, so it suffices to consider the one-ended case.

Let \( G \) be a one-ended cubulated group, acting properly and cocompactly on a \( \text{CAT}(0) \) cube complex \( X \). By applying Theorem 1.5, we may assume that all halfspaces in \( X \) are one-ended and that all quarterspaces in \( X \) are deep. We now prove that \( X \) (and hence \( G \)) is semistable at infinity using the characterization in Theorem 5.3. For this proof we will work with the \( \text{CAT}(0) \) metric on \( X \) rather than the combinatorial metric, so we will not consider halfspaces as collections of vertices but instead we will consider them as the \( \text{CAT}(0) \) convex subspaces of \( X \) that arise as complementary components of hyperplanes; and we will refer to the union of a halfspace with its bounding hyperplane as a closed halfspace. Let \( r \) be a geodesic ray in \( X \) and let \( C \subseteq X \) be compact. Let \( D \) be the intersection of all closed halfspaces containing \( C \), which is compact by Lemma 2.10. Let \( E \subseteq X \) be a third compact set, and let \( \alpha \) be a loop in \( X - D \) based on \( r \). Passing to the cubical neighborhood, we may assume that \( E \) is a subcomplex of \( X \). Since \( D \) is an intersection of closed halfspaces, \( \alpha \) can be written as a concatenation of paths \( \alpha_0, \alpha_1, \ldots, \alpha_n \), such that \( \alpha_i \) is contained in a halfspace \( h_i \) disjoint from \( D \) (see Fig. 5). Let \( x \) be the point at the beginning of \( \alpha_0 \) and the end of \( \alpha_n \), and assume that \( \alpha \) is based on \( r \) at \( x \). Also assume that \( h_0 = h_n \).

For \( 0 \leq i < n \), let \( x_i \) be the point at the end of the segment \( \alpha_i \) and the beginning of the segment \( \alpha_{i+1} \). Observe that \( x_i \in h_i \cap h_{i+1} \) and \( D \subseteq h_i^* \cap h_{i+1}^* \), so \( h_i, h_{i+1} \) are transverse and \( h_i \cap h_{i+1} \) is a quarterspace. (Note that a pair of halfspaces intersect in the \( \text{CAT}(0) \) setting if and only if they intersect in the combinatorial setting). Each \( h_i \) is one-ended, so only one of the finitely many components of \( h_i \cap E \) is unbounded, call this component \( E_i^* \). Each quarterspace \( h_i \cap h_{i+1} \) is deep, so in particular unbounded, and \( E_i^* \cap E_{i+1}^* \subseteq h_i \cap h_{i+1} \) is unbounded.

We now construct a loop \( \beta \) in \( X - E \), and a homotopy rel\( \{r\} \) in \( X - D \) from \( \alpha \) to \( \beta \) (in particular this homotopy is in \( X - C \)). Let \( y \) be a point on \( r \) in \( E_0^* \), and let \( y_i \) be a point in \( E_i^* \cap E_{i+1}^* \) for \( 0 \leq i < n \). Let \( \beta_0, \beta_1, \ldots, \beta_n \) be paths in \( E_0^*, E_1^*, \ldots, E_n^* \) respectively that join the points \( y, y_0, y_1, \ldots, y_{n-1}, y \). Let \( \beta \) be the concatenation of the \( \beta_i \), which lies in \( X - E \). For \( 0 \leq i \leq n \), we know that \( \alpha_i \) and \( \beta_i \) both lie in the halfspace \( h_i \), so we can homotope \( \alpha_i \) to \( \beta_i \) via geodesics, and this homotopy will be in \( h_i \) since halfspaces are convex. By uniqueness of geodesics in \( \text{CAT}(0) \) spaces, these homotopies will fit together to give a homotopy from \( \alpha \) to \( \beta \) in \( X - D \). Moreover, the
Fig. 5 The proof of Theorem 1.1

homotopy will move the point $x$ along a subsegment of $r$ to $y$, so the homotopy is rel{$r$} as required.

\[\square\]

Remark 5.4 The above proof only requires quarterspaces to be unbounded rather than deep, so Theorem 1.5 is actually stronger than what we need to prove Theorem 1.1.

6 Example with an infinite-ended halfspace

In this section we give an example of a one-ended group with a cubulation given by a CAT(0) cube complex that is essential and contains an infinite-ended halfspace. This demonstrates that Theorem 1.3 is not vacuous, and that it requires more than simply passing to the essential core of a CAT(0) cube complex.
Consider the following cyclic amalgam of free groups $F_m$ and $F_n$

$$G = F_m *_{\mathbb{Z}} F_n = \langle F_m, F_n \mid w_1 = w_2 \rangle,$$

(6.1)

where $w_1 \in F_m$ and $w_2 \in F_n$. If $w_1, w_2$ are cyclically reduced and have the same length $L$ with respect to the standard generators of $F_m$ and $F_n$, then we may construct a non-positively curved square complex $X$ with fundamental group $G$ as follows. Take graphs $R_1, R_2$ with one vertex each and $m, n$ edges respectively (the *roses with $m$ or $n$ petals*); $R_1, R_2$ have fundamental groups $F_m, F_n$ respectively, where each edge corresponds to a generator. Take an annulus $A$ formed by identifying the top and bottom of a $2 \times L$ square grid. Then form the square complex $X$ by attaching the left-hand boundary of $A$ to $R_1$ along the word $w_1$ and attaching the right-hand boundary of $A$ to $R_2$ along $w_2$ (see Fig. 6). $X$ is non-positively curved because $w_1, w_2$ are cyclically reduced. $X$ has the structure of a graph of spaces with vertex spaces $R_1, R_2$ and edge space $A$; and this structure corresponds to the splitting (6.1) for $G$, so $G = \pi_1(X)$.

The group $G$ might be one-ended or infinite-ended. For example, if one of the generators $a$ of $F_m$ does not appear in the word $w_1$ then $G$ admits a free splitting with $\langle a \rangle$ as one of the factors, so $G$ is infinite-ended. As an example where $G$ is one-ended, we can let $F_m, F_n$ be rank-2 free groups with generating sets $\{a_1, b_1\}, \{a_2, b_2\}$ respectively and take the elements $w_1, w_2$ to be the commutators $[a_1, b_1], [a_2, b_2]$; in this case $X$ is homeomorphic to the oriented surface of genus 2. As a further example, if $w_1, w_2$ are sufficiently generic elements then $G$ is one-ended and (6.1) is a JSJ splitting for $G$ over cyclic subgroups [33, Example 2.27]. Henceforth we will assume that $G$ is one-ended.

The universal cover $\tilde{X}$ of $X$ is a CAT(0) cube complex, and the action of $G$ on $\tilde{X}$ by deck transformations is a cubulation of $G$. In particular $\tilde{X}$ is one-ended. However, $\tilde{X}$ might not contain an infinite-ended halfspace. To exhibit a cubulation of $G$ with an infinite-ended halfspace we will modify $X$ to obtain another non-positively curved square complex $X'$, and then consider the universal cover of $X'$.
Proof

The horizontal and vertical edge paths in the annulus $A$ map to closed local geodesics in $X'$, and every edge of $X'$ is contained in one of these local geodesics. Lifting these local geodesics to $\hat{X}'$, we see that every edge $\hat{e}'$ in $\hat{X}'$ is contained in a bi-infinite geodesic $\hat{y}'$ (in a CAT(0) space local geodesics are geodesics). Furthermore, $\hat{y}'$ meets the hyperplane $\hat{h}'$ dual to $\hat{e}'$ at right-angles, so it follows from basic CAT(0) geometry that $\hat{y}'$ goes arbitrarily far from $\hat{h}'$ in both the halfspaces bounded by $\hat{h}'$. It follows that every halfspace in $\hat{X}'$ is deep, so $\hat{X}'$ is essential.

It remains to show that $\hat{X}'$ contains an infinite-ended halfspace. Let $y_1, y_2$ be the right-hand endpoints of the edges $e_1, e_2$ (shown in Fig. 6). In $X'$ the edges $e_1, e_2$ are glued together to form a single edge $e'$, and the vertices $y_1, y_2$ are identified to give a single vertex $y'$. Let $\tilde{y}'$ be a lift of $y'$ to $\tilde{X}'$ and let $\tilde{e}'$ be the lift of $e'$ incident at $\tilde{y}'$. Let $\hat{h}'$ be the hyperplane dual to $\tilde{e}'$ and let $h'$ be the halfspace containing $\tilde{y}'$ that is bounded by $\hat{h}'$. (For this proof we consider halfspaces as complementary components of hyperplanes rather than taking the combinatorial viewpoint from Sect. 2.1.) The local picture of $\tilde{X}'$ at $\tilde{y}$ is shown in Fig. 7. Observe that $\tilde{e}'$ cuts $\hat{h}'$ into two components in this local picture. We claim that $\tilde{e}'$ also cuts $h'$ into two components globally. Indeed, if $h' - \tilde{e}'$ was connected then $h'$ would be obtained from $h' - \tilde{e}'$ by amalgamating the two sides of $\tilde{e}' \cap h'$, so $h'$ would have non-trivial fundamental group (given by a HNN extension of $\pi_1(h' - \tilde{e}')$). But this cannot happen since $\hat{h}'$ is a convex subspace of the CAT(0) space $\hat{X}'$. The other lifts of $\tilde{e}'$ dual to $\hat{h}'$ also cut $h'$ into two components, so we conclude that $h'$ is infinite-ended.

\[\square\]

Remark 6.2

The folding together of $e_1$ and $e_2$ to produce $X'$ is an example of a cubical Stallings fold, as studied in [1, 2, 9]. One can also think of $\hat{X}'$ as being obtained from $\hat{X}$ by infinitely many Stallings folds corresponding to the lifts of $e_1$ and $e_2$. In fact, this folding is the reverse of the procedure in Sect. 3 that proves Theorem 3.1. More precisely, if we apply the procedure in Sect. 3 to $\hat{X}'$, with $h_0$ being the halfspace $h'$ from
the proof of Proposition 6.1 and $c_0$ consisting of just the vertex $\tilde{y}'$, then we recover the cube complex $\tilde{X}$.

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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