SCATTERING THEORY FOR JACOBI OPERATORS WITH QUASI-PERIODIC BACKGROUND

IRYNA EGOROVA, JOHANNA MICHOR, AND GERALD TESCHL

Abstract. We develop direct and inverse scattering theory for Jacobi operators which are short range perturbations of quasi-periodic finite-gap operators. We show existence of transformation operators, investigate their properties, derive the corresponding Gel’fand-Levitan-Marchenko equation, and find minimal scattering data which determine the perturbed operator uniquely.

1. Introduction

Classical scattering theory deals with the reconstruction of a given Jacobi operator
\begin{equation}
Hu(n) = a(n)u(n + 1) + a(n - 1)u(n - 1) + b(n)u(n),
\end{equation}
which is a short range perturbation of the free one $H_0$ associated with the coefficients $a(n) = \frac{1}{2}$, $b(n) = 0$. This case has been first developed on an informal level by Case in a series of papers [5]–[10]. The first rigorous results were established by Guseinov [18], who gave necessary and sufficient conditions for the scattering data to determine $H$ uniquely under the assumption
\begin{equation}
\sum_n |n| \left( |a(n) - \frac{1}{2}| + |b(n)| \right) < \infty.
\end{equation}
Further extensions were made by Guseinov [19], [20], and Teschl [27]. Additional details and further references can be found, e.g., in [28].

In addition to being of interest on its own, scattering theory can also be used to solve the initial value problem for the Toda equation via the inverse scattering transform. This has been formally developed by Flaschka [14] (see also [29] and [30] for the case of rapidly decaying sequences) who also worked out the inverse procedure in the reflection-less case. Further results and an extension of the method to the entire Toda hierarchy were given by Teschl in [26] and [27].

The next interesting problem is to replace the free Hamiltonian $H_0$ by one with a periodic potential. First results in the case of Sturm-Liouville operators have been obtained by Firsova in a series of papers (see [13]). For further results, including potentials with different spatial asymptotics, and additional references see Gesztesy et al. [16]. In the discrete case, the investigation has only recently been started by Boutet de Monvel and Egorova [2] and by Volberg and Yuditskii [31], who treat the case where $H$ has a homogeneous spectrum and is of Szegö class exhaustively.

1991 Mathematics Subject Classification. Primary 47B36, 81U40; Secondary 34L25, 39A11.

Key words and phrases. Inverse scattering, Jacobi operators.

Work supported by the Austrian Science Fund (FWF) under Grant No. P17762, the Austrian Academy of Sciences under DOC-21388, and INTAS Research Network NeCCA 63-51-6637.
from an operator point of view. Applications to the Toda lattice can be found in Bazargan and Egorova [11] and Boutet de Monvel and Egorova [3]. Finally, let us give a brief overview of the paper:

Section 2 collects some well-known facts from Riemann surfaces and introduces the necessary notation. Section 3 introduces the Baker-Akhiezer function and investigates the quasi-momentum map. In the periodic case, where the integrals can be explicitly computed, this was first done in [24]. In addition, we characterize the second solution at the band edges. In Section 4 we prove existence of Jost solutions and use them to characterize the spectrum of the perturbed operator. In the periodic case, existence of Jost solutions was first shown by Geronimo and Van Assche [17] and the fact that there are only finitely many eigenvalues in each gap was first proven in Cojuhari [12] and later rediscovered in Teschl [25]. Section 5 introduces the transformation operator and proves the crucial decay estimate on its coefficients. This was first done by Boutet de Monvel and Egorova [2] in the periodic case under the additional assumption that all spectral gaps are open. We fix a problem in the original proof and at the same time simplify and streamline the argument. Section 6 investigates the scattering matrix. Our main result here is the reconstruction of the transmission coefficient from the reflection coefficient, which was not known previously, even in the periodic case. Section 7 derives the Gel’fand-Levitan-Marchenko equation and proves positivity of the Gel’fand-Levitan-Marchenko operator. In addition, we formulate necessary conditions for the scattering data to uniquely determine our Jacobi operator. Our final Section 8 shows that our necessary conditions for the scattering data are also sufficient. It should be mentioned that, due to the lack of continuity with respect to the spacial variable $n$, a significant change in the strategy of the original proof in the continuous case from [22] is needed.

Our approach uses heavily the fact that the Baker-Akhiezer function is a meromorphic function on the Riemann surface associated with the problem. This strategy gives a more streamlined treatment and more elegant proofs even in the special cases which were previously known. In this respect it is important to emphasize that, in contradistinction to the constant background case, the upper sheet of our Riemann surface is not simply connected and in particular not isomorphic to the unit disc.

2. QUASI-PERIODIC FINITE-GAP OPERATORS AND RIEMANN SURFACES

To set the stage let $\mathcal{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z), \quad R_{2g+2}(z) = \prod_{j=0}^{2g+1} (z - E_j), \quad E_0 < E_1 < \cdots < E_{2g+1},$$

$g \in \mathbb{N}$. $\mathcal{M}$ is a compact, hyperelliptic Riemann surface of genus $g$. We will choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1} \sqrt{z - E_j},$$

where $\sqrt{\cdot}$ is the standard root with branch cut along $(-\infty, 0)$. 
A point on \( \mathbb{M} \) is denoted by \( p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm), \ z \in \mathbb{C}, \) or \( p = \infty, \) and the projection onto \( \mathbb{C} \cup \{ \infty \} \) by \( \pi(p) = z. \) The points \( \{(E_j, 0), 0 \leq j \leq 2g+1\} \subseteq \mathbb{M} \) are called branch points and the sets
\[
\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C} \setminus \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}] \} \subset \mathbb{M}
\]
are called upper, lower sheet, respectively.

Let \( \{a_j, b_j\}_{j=1}^{g} \) be loops on the surface \( \mathbb{M} \) representing the canonical generators of the fundamental group \( \pi_1(\mathbb{M}). \) We require \( a_j \) to surround the points \( E_{2j-1}, E_{2j} \) (thereby changing sheets twice) and \( b_j \) to surround \( E_0, E_{2j-1} \) counter-clock wise on the upper sheet, with pairwise intersection indices given by
\[
a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad 1 \leq i, j \leq g.
\]
The corresponding canonical basis \( \{\zeta_j\}_{j=1}^{g} \) for the space of holomorphic differentials can be constructed by
\[
\zeta = \sum_{j=1}^{g} \zeta(j) \frac{\pi_{j-1}^{(g)}}{R_{2g+2}^{1/2}},
\]
where the constants \( \zeta(\cdot) \) are given by
\[
c_j(k) = C_{jk}^{-1}, \quad C_{jk} = \int_{a_k} \pi_{j-1}^{(g)} \frac{dz}{R_{2g+2}^{1/2}} = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}.
\]
The differentials fulfill
\[
\int_{a_j} \zeta_k = \delta_{j,k}, \quad \int_{b_j} \zeta_k = \tau_{j,k}, \quad \tau_{j,k} = \tau_{k,j}, \quad 1 \leq j, k \leq g.
\]

Now pick \( g \) numbers (the Dirichlet eigenvalues)
\[
(\mu_j)_{j=1}^{g} = (\mu_j, \sigma_j)_{j=1}^{g}
\]
whose projections lie in the spectral gaps, that is, \( \mu_j \in [E_{2j-1}, E_{2j}] \). Associated with these numbers is the divisor \( D_\mu \) which is one at the points \( \hat{\mu}_j \) and zero else. Using this divisor we introduce
\[
\hat{\zeta}(p, n) = \hat{A}_{\mu_0}(p) - \hat{A}_{\mu_0}(D_\mu) - n \hat{A}_{\infty}(-\infty) - \hat{\zeta}_{\mu_0} \in \mathbb{C}^g, \quad \hat{\zeta}(n) = \hat{\zeta}(\infty, n),
\]
where \( \hat{\zeta}_{\mu_0} \) is the vector of Riemann constants
\[
\hat{\zeta}_{\mu_0} = \frac{1}{2} - \sum_{k=1}^{g} \frac{\tau_{j,k}}{2}, \quad p_0 = (E_0, 0),
\]
and \( \hat{A}_{\mu_0} (\hat{\zeta}_{\mu_0}) \) is Abel’s map (for divisors). The hat indicates that we regard it as a (single-valued) map from \( \hat{M} \) (the fundamental polygon associated with \( \mathbb{M} \)) to \( \mathbb{C}^g. \)

We recall that the function \( \theta(\hat{\zeta}(p, n)) \) has precisely \( g \) zeros \( \hat{\mu}_j(n) \) (with \( \hat{\mu}_j(0) = \hat{\mu}_j \)), where \( \theta(z) \) is the Riemann theta function of \( \mathbb{M}. \)

Then our Jacobi operator \( H_g \) is given by
\[
a(n)^2 = \hat{a}(\theta(\zeta(n+1)))\theta(\zeta(n-1)) \theta(\zeta(n))^2,
\]
\[
b(n) = \hat{b} + \sum_{j=1}^{g} c_j(g) \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(w + \zeta(n))}{\theta(w + \zeta(n-1))} \right) \bigg|_{w=0}.
\]
The constants $\tilde{a}, \tilde{b}$ depend only on the Riemann surface and will be defined in the next section.

It is well known that the spectrum of $H_q$ is purely absolutely continuous and consists of $g + 1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}].$$

For further information and proofs we refer to [28], Section 9.

3. The Baker-Akhiezer function and the quasi-momentum map

The Baker-Akhiezer function $\psi_q(p, n) = \psi_q(p, n, 0)$ is given by

$$\psi_q(p, n, n_0) = \sqrt{\frac{\theta(z(n_0 - 1))\theta(z(n_0))}{\theta(z(n - 1))\theta(z(n))}} \exp \left( (n - n_0) \int_{p_0}^{p} \omega_{\infty, \infty} \right),$$

where $\omega_{\infty, \infty}$ is the normalized Abelian differential of the third kind with simple poles at $\infty_{\pm}$ and residues $\pm 1$, respectively. They are normalized such that $\psi_q(p, n_0, n_0) = 1$.

The two branches

$$\psi_{q,\pm}(z, n) = \prod_{j=0}^{n-1} \phi_{q,\pm}(z, j),$$

where ([28], (8.87))

$$\phi_{q,\pm}(z, n) = \frac{1}{2a_q(n)} \left( z - b_q(n) + \sum_{j=1}^{g} \frac{R_j(n)}{z - \mu_j(n)} \pm \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z - \mu_j(n))} \right),$$

$$R_j(n) = \frac{R_{2g+1}^{1/2}(\mu_j(n))}{\prod_{k \neq j}(\mu_j(n) - \mu_k(n))}, \quad \hat{R}_j(n) = \sigma_j(n)R_j(n),$$

of the Baker-Akhiezer function are solutions of $\tau_q u = zu$, $z \in \mathbb{C}$, where $\tau_q$ is the difference expression associated with $H_q$. However, the Wronskian

$$W(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{R_{2g+2}^{1/2}(z)}{\prod_{j=1}^{g}(z - \mu_j)}$$

$(\mu_j = \mu_j(0))$ shows that they are linearly dependent at the band edge $E_j$, $0 \leq j \leq 2g + 1$.

The branch $\psi_{q,\sigma_j}(z, n)$ has a first order pole at $\mu_j$ if $\mu_j$ is away from the band edges

$$\lim_{z \to \mu_j} (z - \mu_j)\psi_{q,\sigma_j}(z, n) = \psi_{q,\sigma_j}(\mu_j, n, 1) \frac{\hat{R}_j(0)}{a_q(0)},$$

(use (3.3) and $\psi_{q,\pm}(z, n) = \psi_{q,\pm}(z, n, 1)\phi_{q,\pm}(z, 0)$) and both branches have a square root singularity if $\mu_j$ coincides with a band edge $E_j$

$$\lim_{z \to \mu_j} \sqrt{z - \mu_j}\psi_{q,\pm}(z, n) = \pm \frac{i}{2a_q(0)} \prod_{k \neq j} \sqrt{|E_j - E_k|} \psi_{q,+(E_j, n, 1)}$$
Lemma 3.1. The solutions of $\tau_q u = zu$ can be characterized as follows.

(i) If $R_{2g+2}(z) \neq 0$, there exist two solutions satisfying

$$
\psi_{q,\pm}(z,n) = \theta_{\pm}(z,n) w(z)^{\pm n}, \quad w(z) = \exp \left( \int_{p_0}^{(z,\pm)} \hat{\omega}_{\infty,+,-} \right),
$$

with $\theta_{\pm}(z,n)$ quasi-periodic.

(ii) If $R_{2g+2}(z) = 0$, $z = E_l$, there are two solutions satisfying

$$
\psi_q(E_l,n) = \psi_{q,+}(E_l,n) = \psi_{q,-}(E_l,n), \quad \hat{\psi}_q(E_l,n) = \psi_q(E_l,n)(\hat{\theta}(n) + n),
$$

where $\hat{\theta}(n)$ is quasi-periodic.

Proof. (ii). We construct a second linearly independent solution at $z = E = E_l$ using (see [28], (1.50))

$$
s_q(E,n) = \lim_{z \to E} \frac{a_q(0)\psi_{q,+}(z,n) - \psi_{q,-}(z,n)}{W(\psi_{q,-}(z), \psi_{q,+}(z))},
$$

where $s_q(z,n)$ denotes the fundamental solution of $\tau_q u = zu$ with initial conditions $s_q(z,0) = 0$, $s_q(z,1) = 1$. W.l.o.g. we assume that $E_l$ does not coincide with one of the Dirichlet eigenvalues $\mu_j$ (otherwise shift the base point). To derive an expression for $\psi_{q,\pm}(z)$ at $z = E + \epsilon^2$ we start with

$$
R_{2g+2}^{1/2}(z) = \epsilon(\bar{R} + O(\epsilon^2)), \quad \bar{R} = -\prod_{j \neq l} \sqrt{E - E_j}.
$$

Moreover,

$$
W(\psi_{q,-}(z), \psi_{q,+}(z)) = \frac{\bar{R}}{\prod_{j=1}^{g}(E - \mu_j)}(1 + O(\epsilon^2))
$$

and for $p = (E + \epsilon^2, \pm)$ (see [22] below),

$$
\int_{p_0}^{p} \hat{\omega}_{\infty,+,-} = \int_{p_0}^{E} \hat{\omega} \pm \beta \epsilon + O(\epsilon^3), \quad \beta = \frac{2 \prod_{j=1}^{g}(E - \lambda_j)}{R},
$$

$$
z(p,n) = z(E,n) \pm \gamma \epsilon + O(\epsilon^3), \quad \gamma = \sum_{j=1}^{g} c(j) \frac{2E_{j-1}^{j-1}}{R},
$$

and

$$
\theta(z(p,n)) = \theta(z(E,n)) \pm \frac{\partial \theta}{\partial z}(z(E,n)) \gamma \epsilon + O(\epsilon^3).
$$

Using this to evaluate the limit $\epsilon \to 0$ shows

$$
s_q(E,n) = 2a_q(0) \prod_{j=1}^{g} \frac{E - \mu_j}{E - \lambda_j} \hat{\psi}_q(E,n) = \psi_q(E,n)(\hat{\theta}(n) + n),
$$

where

$$
\hat{\theta}(n) = \frac{1}{\prod_{j=1}^{g}(E - \lambda_j)} \sum_{j,k=1}^{g} E_{j} c_{k}(j) \frac{\partial}{\partial w_k} \ln \theta(z(E,n) + \tilde{w}),
$$

and finishes the proof. \qed
Remark 3.2. (i). Since \( \psi_{q}(z,n) \) has a singularity if \( z = \mu_{j} \) the solutions in Lemma 3.1 are not well-defined for those \( z \). However, you can either remove the singularities of \( \psi_{q}(z,n) \) or choose a different normalization point \( n_{0} \neq 0 \) to see that solutions of the above type exist for every \( z \).

(ii). In the periodic case Floquet theory tells you that there are two possible cases at a band edge: Either two (linearly independent) periodic solutions or one periodic and one linearly growing solution. The above lemma shows that the first case happens if the corresponding gap is closed and the second if the gap is open.

To understand the properties of \( \psi_{q, \pm}(z,n) \) we need to investigate the quasi-momentum map

\[
(3.10) \quad w(z) = \exp \left( \int_{p_{0}}^{p} \hat{\omega}_{\infty+, \infty-} \right), \quad p = (z,+).
\]

The differential \( \omega_{\infty+, \infty-} \) is given by

\[
(3.11) \quad \omega_{\infty+, \infty-} = \frac{\prod_{j=1}^{g}(\pi - \lambda_{j})}{R_{1/2}^{1/2}} d\pi,
\]

where the constants \( \lambda_{j} \) have to be determined from the normalization

\[
(3.12) \quad \int_{a_{j}}^{E_{2j}} \omega_{\infty+, \infty-} = 2 \int_{E_{2j-1}}^{E_{2j}} \frac{\prod_{j=1}^{g}(z - \lambda_{j})}{R_{1/2}^{1/2}} dz = 0,
\]

which shows \( \lambda_{j} \in (E_{2j-1}, E_{2j}) \).

Since \( \lambda_{j} \in (E_{2j-1}, E_{2j}) \) the integrand is a Herglotz function and admits the following representation (c.f. \[28\], Appendix B)

\[
(3.13) \quad \frac{\prod_{j=1}^{g}(z - \lambda_{j})}{R_{1/2}^{1/2}(z)} = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} d\tilde{\mu}(\lambda)
\]

with the probability measure

\[
(3.14) \quad d\tilde{\mu}(\lambda) = \frac{\prod_{j=1}^{g}(\lambda - \lambda_{j})}{\pi R_{1/2}^{1/2}(\lambda)} \chi_{\sigma(H_{q})}(\lambda) d\lambda.
\]

Hence

\[
(3.15) \quad g(z, \infty) = \int_{p_{0}}^{p} \omega_{\infty+, \infty-} = \int_{-\infty}^{z} \int_{-\infty}^{\infty} \frac{1}{\lambda - \zeta} d\tilde{\mu}(\lambda) d\zeta
\]

\[
= \int_{-\infty}^{\infty} \ln \left( \frac{\lambda - E_{0}}{\lambda - z} \right) d\tilde{\mu}(\lambda).
\]

In particular, note that \( -\text{Re}(g(z, \infty)) \) is the Green’s function of the upper sheet \( \Pi_{+} \) with pole at \( \infty_{+} \) and \( \tilde{\mu} \) is the equilibrium measure of the spectrum (see \[30\], Thm. III.37). We will abbreviate \( g(z) = g(z, \infty) \).

The asymptotic expansion of \( \exp(g(z)) \) is given by \([28], (9.42)\)

\[
(3.16) \quad \exp \left( \int_{p_{0}}^{p} \hat{\omega}_{\infty+, \infty-} \right) = -\frac{a}{z} \left( 1 + \frac{b}{z} + O \left( \frac{1}{z^{2}} \right) \right), \quad z \to \infty,
\]
where $\tilde{a}$ is the capacity of the spectrum and

$$
\tilde{b} = \frac{1}{2} \sum_{j=0}^{2g+2} \Delta - \sum_{j=1}^{g} \lambda_j.
$$

**Theorem 3.3.** The map \( g \) is a bijection from the upper (resp. lower) half plane \( \mathbb{C}^\pm = \{ z \in \mathbb{C} \mid \pm \text{Im}(z) > 0 \} \) to

$$
S^\pm = \{ z \in \mathbb{C} \mid \pm \text{Re}(z) < 0, 0 < \text{Im}(z) < \pi \} \setminus \bigcup_{j=1}^{g} [g(\lambda_j), g(E_{2j+1})]
$$

such that \( \sigma(H_q) = \{ z \mid \text{Re}(z) = 0 \} \).

**Proof.** By the Herglotz property of its integrand, the function \( g(z, \infty) \) satisfies the conditions of [24], Theorem 1(b) in Chapter VI, which shows that it is one-to-one.

To prove that \( g(z, \infty) \) is surjective, it suffices to show that the boundary of \( \mathbb{C}^+ \) is mapped to the boundary of \( S^+ \). Note that \( g(\lambda) \) is negative for \( \lambda < E_0 \) and purely imaginary for \( \lambda \in [E_0, E_1] \). At \( E_1 \), the real part starts to decrease from zero until it hits its minimum at \( \lambda_1 \) and increases again until it becomes 0 at \( E_2 \) (since all \( a \)-periods are zero), while the imaginary part remains constant. Proceeding like this we move along the boundary of \( S^+ \) as \( \lambda \) moves along the real line. For \( \lambda > E_{2g+1} \), \( g(\lambda) \) is again negative.

**Remark 3.4.** In the special case where \( H_q \) is periodic the quasi-momentum is given by \( w(z) = \exp(iN^{-1} \arccos \Delta(z)) \), where \( \Delta(z) \) is the Floquet discriminant, and our result is due to [24].

Therefore the map

$$
w: \mathbb{C}^\pm \rightarrow W^\pm = \{ w \in \mathbb{C} \mid |w| < 1, \pm \text{Im}(w) > 0 \} \setminus \bigcup_{j=1}^{g} [w(\lambda_j), w(E_{2j+1})]
$$

(3.19) \( z \mapsto \exp(g(z)) \)

is bijective. Denote \( W = W^+ \cup W^- \cup (-1, 1), W_0 = W \setminus \{0\} \). If we identify corresponding points on the slits \( [w(\lambda_j), w(E_{2j+1})] \) we obtain a Riemann surface \( \mathcal{W} \) which is isomorphic to the upper sheet \( \Pi^+ \).

**Remark 3.5.** In [24] the largest band edge \( E_{2g+1} \) is chosen for \( p_0 \) and \( w \) will map \( \mathbb{C}^\pm \rightarrow W^\pm \) in this case. Moreover, in the periodic case the slits \( [w(\lambda_j), w(E_{2j+1})] \) appear at equal angles \( \frac{2\pi}{N} \), where \( N \) is the period.

Since \( z \mapsto w(z) = \exp(g(z)) \) is a bijection, we consider the functions \( \psi_{q,\pm} \) as functions of the new parameter \( w \) whenever convenient. For notational simplicity we will write \( \psi_{q,\pm}(w, n) \) for \( \psi_{q,\pm}(\lambda(w), n) \) and similarly for other quantities. The functions \( \psi_{q,\pm}(w, n) \) are meromorphic in \( \mathcal{W} \) and continuous up to the boundary with the only possible singularities at the images of the Dirichlet eigenvalues \( w(\mu_j) \) and at 0. More precisely, denote by \( M_{\pm} \) the sets of poles (and square root singularities if \( \mu_j = E_i \)) of the Weyl \( m \)-functions \( \tilde{m}_{\pm}(\lambda) \), i.e. \( M_+ \cup M_- = \{ \mu_j \}_{j=1}^{g} \) (see (3.2) and [28], Section 2.1). Note that \( \mu_j \in M_+ \cap M_- \) if and only if \( \mu_j = E_i \). Then

(B1) \( \psi_{q,\pm}(w, n) \) are holomorphic in \( \mathcal{W} \setminus \{w(\mu_j)\}_{j=1}^{g} \cup \{0\} \) and continuous on \( \partial \mathcal{W} \setminus \{w(\mu_j)\} \).
\( \psi_{q, \pm}(w, n) \) has a simple pole at \( w(\mu_j) \) if \( \mu_j \in M_\pm \backslash \{E_l\} \), no pole if \( \mu_j \notin M_\pm \), and if \( \mu_j = E_l \),

\[
\psi_{q, \pm}(w, n) = \pm \frac{iC(n)}{w - w_l} + O(1),
\]

where \( C(n) \) is bounded and real.

(B3) \( \psi_{q, \pm}(w, n) = \psi_{q, \mp}(w, n) \) for \( |w| = 1 \).

(B4) At \( w = 0 \) the following asymptotics hold

\[
\psi_{q, \pm}(w, n) = (-1)^n \left( \prod_{g=0}^{n-1} a_q(m) \right)^{\pm 1} w^{\pm n}(1 + O(w)).
\]

By Section 2.5 of [28] the vector valued functions

\[
U(\lambda, n) = \sqrt{\frac{1}{4a_q(0)^2 \pi \text{Im}(\tilde{m}_+(\lambda))}} \begin{pmatrix} \psi_{q, +}(\lambda, n) \\ \psi_{q, -}(\lambda, n) \end{pmatrix}
\]

form an orthonormal basis for the Hilbert space \( L^2(\sigma(H_q), \mathbb{C}^2, d\lambda) \). The Weyl \( m \)-functions \( \tilde{m}_\pm(z) \) satisfy (see [28], eq. (8.27))

\[
\text{Im}(\tilde{m}_\pm(\lambda)) = \frac{\mp R_{2g+2}^{1/2}(\lambda)}{2ia_q(0)^2 \prod_{j=1}^g (\lambda - \mu_j)}, \quad \lambda \in \sigma(H_q).
\]

Using our map \( w(z) = \exp(\int_{p_0}^{z, (+)} \hat{\omega}_{\infty, +} \, dw) \) we can transform this into an orthogonal basis on the unit circle.

**Lemma 3.6.** Both functions \( \psi_{q, +}(w, n) \) and \( \psi_{q, -}(w, n) \) form orthonormal bases in the Hilbert space \( L^2(S^1, d\omega) \), where

\[
d\omega(w) = \prod_{j=1}^g \frac{\lambda(w) - \mu_j}{\lambda(w) - \lambda_j} \, dw.
\]

**Proof.** Just use

\[
\frac{dw}{dz} = w \prod_{j=1}^g \frac{z - \lambda_j}{R_{2g+2}^{1/2}(z)}.
\]

Observe that \( d\omega \) is meromorphic on \( \mathbb{W} \) with a simple pole at \( w = 0 \). In particular, there are no poles at \( w(\lambda_j) \).

**Remark 3.7.** In the periodic case we have

\[
\|\psi_{p, \pm}(\lambda)\|^2_N := \sum_{n=1}^N |\psi_{p, \pm}(\lambda, n)|^2 = N \prod_{j=1}^{N-1} \frac{\lambda - \lambda_j}{\lambda - \mu_j}.
\]

### 4. Existence of Jost solutions

After we have these preparations out of our way, we come to the study of short-range perturbations \( H \) of \( H_q \) associated with sequences \( a, b \) satisfying \( a(n) \to a_q(n) \) and \( b(n) \to b_q(n) \) as \( |n| \to \infty \). More precisely, we will make the following assumption throughout this paper.
Hypothesis H.4.1. Let $H$ be a perturbation such that
\begin{equation}
\sum_{n \in \mathbb{Z}} |n| \left( |a(n) - a_q(n)| + |b(n) - b_q(n)| \right) < \infty.
\end{equation}

We first establish existence of Jost solutions, that is solutions of the perturbed operator which asymptotically look like the Baker-Akhiezer solutions.

Theorem 4.2. Assume (H.4.1). Then there exist solutions $\psi_{\pm}(z,.)$, $z \in \mathbb{C}$, of $\tau \psi = z \psi$ satisfying
\begin{equation}
\lim_{n \to \pm \infty} |w^{\mp n}(\psi_{\pm}(z,n) - \psi_{q,\pm}(z,n))| = 0,
\end{equation}
where $\psi_{q,\pm}(z,.)$ are the Baker-Akhiezer functions. Moreover, $\psi_{\pm}(z,.)$ are continuous (resp. holomorphic) with respect to $z$ whenever $\psi_{q,\pm}(z,.)$ are and inherit the properties (B1) and (B2), where now $\psi_{\pm}(z,n) = \frac{v_{\pm}(n)}{\sqrt{z-\mu_j}} + O(1)$ and (B4) has to be replaced by
\begin{equation}
\psi_{\pm}(z,n) = A_{\pm}(0) \left( \prod_{m=n}^{\infty} \frac{a_q(j)}{a(j)} \right)^{\pm 1} (1 + (B_{\pm}(0) \pm \sum_{j=1}^{n} b(j-\xi)) \frac{1}{z} + O(\frac{1}{z^2})),
\end{equation}
where
\begin{align*}
A_{+}(n) &= \prod_{j=n}^{\infty} \frac{a_q(j)}{a(j)}, \quad B_{+}(n) = \sum_{m=n+1}^{\infty} (b_q(m) - b(m)), \\
A_{-}(n) &= \prod_{j=-\infty}^{n-1} \frac{a_q(j)}{a(j)}, \quad B_{-}(n) = \sum_{m=-\infty}^{n-1} (b_q(m) - b(m)).
\end{align*}

Proof. The proof can be done as in the periodic case (see e.g., [17], [25] or [28], Section 7.5). The only problem is to show that the second solution at a band edge grows at most linearly. In the periodic case this follows from Floquet theory, here we just use Lemma 3.6. \qed

From this result we obtain a complete characterization of the spectrum of $H$.

Theorem 4.3. Assume (H.4.1). Then we have $\sigma_{ess}(H) = \sigma(H_q)$, the point spectrum of $H$ is finite and confined to the spectral gaps of $H_q$, that is, $\sigma_p(H) \subset \mathbb{R}\setminus \sigma(H_q)$. Furthermore, the essential spectrum of $H$ is purely absolutely continuous.

Proof. Again the proof can be done as in the periodic case (see e.g., [25] or [28], Section 7.5). \qed

5. The transformation operator

We define the kernel of the transformation operator as the Fourier coefficients of the Jost solutions $\psi_{\pm}(w,n)$ with respect to the orthonormal system given in Lemma 3.6 $\{\psi_{q,\pm}(w,n)\}_{n \in \mathbb{Z}}$,
\begin{equation}
K_{\pm}(n,m) := \frac{1}{2\pi i} \int_{|w|=1} \psi_{\pm}(w,n) \psi_{q,\pm}(w,m) d\omega(w).
\end{equation}
By the Cauchy theorem, this integral equals the residue at $w = 0$,
\begin{equation}
K_{\pm}(n,m) = \text{Res}_0 \frac{1}{w} \psi_{\pm}(w,n) \psi_{q,\pm}(w,m).
\end{equation}
In particular, since \( \psi_\pm(w, n) \psi_{q, \mp}(w, m) = O(w^{\pm(n-m)}) \), we conclude
\[
K_\pm(n, m) = 0, \quad \pm(m - n) < 0.
\]

**Lemma 5.1.** Assume \( H_4 \). The Jost solutions \( \psi_\pm(w, n) \) can be represented as
\[
\psi_\pm(w, n) = \sum_{m=n}^{\pm\infty} K_\pm(n, m) \psi_{q, \pm}(w, m), \quad |w| = 1,
\]
where the kernels \( K_\pm(n, \cdot) \) satisfy \( K_\pm(n, m) = 0 \) for \( \pm m < \pm n \) and
\[
|K_\pm(n, m)| \leq C \sum_{j=[(m+n)/2]}^{\pm\infty} \left( |a(j) - a_q(j)| + |b(j) - b_q(j)| \right), \quad \pm m > \pm n.
\]
The constant \( C \) depends only on \( H_4 \) and the value of the sum in \( (4.1) \).

**Proof.** We prove the estimate for \( K_+(n, m) \) and omit ”+” and ”z” whenever possible. Define \( \varphi(n) = \psi(n)K(n, n)^{-1} \), then \( \varphi \) fulfills
\[
\varphi(n) = \psi_q(n) + \sum_{m=n+1}^{\infty} J(n, m) \varphi(m),
\]
where
\[
J(z, n, m) = \tilde{a}(m-1) \frac{s_q(z, n, m-1)}{a_q(m-1)} + \tilde{b}(m) \frac{s_q(z, n, m)}{a_q(m)}
\]
with the abbreviation
\[
\tilde{a}(m) = \frac{a(m)^2}{a_q(m)} - a_q(m), \quad \tilde{b}(m) = b(m) - b_q(m).
\]
On the other hand, \( \varphi(n) \) is given by
\[
\varphi(n) = \sum_{m=n}^{\infty} \kappa(n, m) \psi_q(m), \quad \kappa(n, m) = \frac{K(n, m)}{K(n, n)},
\]
therefore
\[
\sum_{m=n}^{\infty} \kappa(n, m) \psi_q(m) = \sum_{m=n+1}^{\infty} J(n, m) \psi_q(m) + \sum_{m=n+1}^{\infty} \sum_{l=m+1}^{\infty} J(n, m) \kappa(m, l) \psi_q(l).
\]
Multiplying both sides of \( (5.9) \) by \( \psi_{q, -}(k) \) and integrating over the unit circle yields
\[
\sum_{m=n}^{\infty} \kappa(n, m) \psi_q(m) = \sum_{m=n+1}^{\infty} \sum_{l=m+1}^{\infty} \Gamma(n, m, l, k) \kappa(m, l),
\]
where
\[
\Gamma(n, m, l, k) = \frac{1}{2\pi i} \int_{|w|=1} J(w, n, m) \psi_{q, +}(w, l) \psi_{q, -}(w, k) d\omega(w).
\]
Using \( 28 \), \( (1.50) \),
\[
\frac{s_q(n, m)}{a(m)} = \frac{\psi_{q, +}(m) \psi_{q, -}(n) - \psi_{q, +}(n) \psi_{q, -}(m)}{W(\psi_{q, +}, \psi_{q, -})},
\]
we obtain
\[
\Gamma(n, m, l, k) = \tilde{b}(m) \Gamma_q(n, m, l, k) + \tilde{a}(m) \Gamma_q(n, m - 1, l, k)
\]
The order of the poles at \( n, m, l, k \) and \( \Gamma \) (which shows that \( \Gamma_0 \) on the other side including the spectrum (and thus \( \infty \)) given by

\[
\begin{align*}
\Gamma_0(n, m, l, k) &= -\text{Res}_{\infty +} \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} \\
&= \left( \text{Res}_{\infty -} + \sum_{j=0}^{2g+1} \text{Res}_{E_j} \right) \left( \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} \right).
\end{align*}
\]

(5.15)

Here \( \gamma \) is a path on the upper sheet encircling the spectrum. The integrand of \( \Gamma_0 \) is meromorphic on the Riemann surface \( \mathbb{M} \) with poles of order one at \( E_j \) and poles of order \( O(\frac{1}{(n-m+l-k)^{2g+1}}) \) near \( \infty \) (there are no poles at the Dirichlet eigenvalues \( \mu_j \)).

We apply the residue theorem twice, first on the side of \( \gamma \) including \( \infty_+ \), then on the other side including the spectrum (and thus \( \infty_- \))

\[
\begin{align*}
\Gamma_0(n, m, l, k) &= -\text{Res}_{\infty +} \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} \\
&= \left( \text{Res}_{\infty -} + \sum_{j=0}^{2g+1} \text{Res}_{E_j} \right) \left( \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} \right).
\end{align*}
\]

(5.15)

The order of the poles at \( \infty_\pm \) implies

\[
\Gamma_0(n, m, l, k) = \left\{ \begin{array}{ll}
\sum_{j=0}^{2g+1} \text{Res}_{E_j} \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} & n - m + l - k < 0 \\
0 & n - m + l - k \geq 0,
\end{array} \right.
\]

which shows that \( \Gamma_0(n, m, l, k) \) is real and bounded since \( \psi_{q, +}(E, \cdot) = \psi_{q, -}(E, \cdot) \) are (if \( \mu_j = E_j \), use (B2)). Together with \( 5.14 \) this yields

\[
\Gamma_0(n, m, l, k) = -\Gamma_0(m, n, k, l) = -\Gamma_0(m, n, k, l) = -\Gamma_0(n, m, k, l).
\]

Moreover,

\[
\begin{align*}
\Gamma_q(n, m, l, k) &= 0, \quad l - k \geq |m - n|, \\
\Gamma_q(n, m, l, k) &= -\Gamma_q(m, n, k, l) = \Gamma_q(n, m, k, l),
\end{align*}
\]

(5.17)

which then implies

\[
\Gamma_q(n, m, l, k) = \left\{ \begin{array}{ll}
\text{sign}(n - m) \sum_{j=0}^{2g+1} \text{Res}_{E_j} \frac{\psi_{q, +}(n)\psi_{q, -}(m)\psi_{q, +}(l)\psi_{q, -}(k)}{W(\psi_{q, +}, \psi_{q, -})^2} & |l - k| < |m - n| \\
0 & |l - k| \geq |m - n|
\end{array} \right.
\]

(5.18)

and \( \Gamma(n, m, l, k) = 0 \) for \( |l - k| \geq m - n \) if \( m > n \). Note that the residue at \( E_j \) is given by

\[
\begin{align*}
\frac{2}{\prod_{\ell \neq j}(E_j - E_{\ell})} \psi_q(E_j, n)\psi_q(E_j, m)\psi_q(E_j, l)\psi_q(E_j, k).
\end{align*}
\]

(5.19)
Now we obtain for $\kappa(n,k)$
\[
\kappa(n,k) = \sum_{m=n+1}^{\infty} \Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty} \sum_{l=m+1}^{\infty} \Gamma(n,m,l,k)\kappa(m,l)
\]
(5.20) \[
= \sum_{m=\left\lceil \frac{n+k}{2} \right\rceil +1}^{\infty} \Gamma(n,m,m,k) + \sum_{m=n+1}^{\infty} \sum_{l=m+k-n-1}^{m+k-n} \Gamma(n,m,l,k)\kappa(m,l),
\]

since $\Gamma(n,m,m,k) \neq 0$ only if $|m-k| < m-n$ implying $m > \frac{n+k}{2}$. In the third sum of (5.20) we need that $|m+\delta-k| < m-n$ for $\delta \geq 1$ which yields $\delta < k-n$ and $\delta > n+k-2m$. Two remarks might be in order: $m+k-n-1 \geq n+k-m+1$ since $m-n \geq n-m+2$, and the starting point $l = n+k-m+1$ of the third sum actually has a lower limit, namely $m \leq \frac{n+k}{2}$, since we require $l \geq m+1$ for $\kappa(m,l) \neq 0$. Note that
\[
\left| \sum_{m=\left\lceil \frac{n+k}{2} \right\rceil +1}^{\infty} \Gamma(n,m,m,k) \right| \leq D \sum_{m=\left\lceil \frac{n+k}{2} \right\rceil +1}^{\infty} |\tilde{b}(m) + \tilde{a}(m)| =: \hat{q}(\frac{n+k}{2}),
\]
(5.22)
\[
\left| \sum_{l=n+k-m+1}^{m+k-n-1} \Gamma(n,m,l,k) \right| \leq D (m-n-1)|\tilde{b}(m) + \tilde{a}(m)| =: \hat{c}(m) \in \ell^1(\mathbb{Z}),
\]
where $D$ is the estimate provided by (5.18), (5.19). We set up the following iteration procedure
\[
\kappa_0(n,k) = \sum_{m=\left\lceil \frac{n+k}{2} \right\rceil +1}^{\infty} \Gamma(n,m,m,k),
\]
(5.21) \[
\kappa_j(n,k) = \sum_{m=n+1}^{\infty} \sum_{l=n+k-m+1}^{m+k-n} \Gamma(n,m,l,k)\kappa_{j-1}(m,l).
\]
Then using induction one has
\[
|\kappa_j(n,k)| \leq \hat{q}(\frac{n+k}{2}) \left( \sum_{m=n+1}^{\infty} \hat{c}(m) \right)^j \frac{j!}{j!}
\]
and hence the iteration converges and implies the estimate
\[
|\kappa(n,k)| = \left| \sum_{j=0}^{\infty} \kappa_j(n,k) \right| \leq \hat{q}(\frac{n+k}{2}) \exp \left( \sum_{m=n+1}^{\infty} \hat{c}(m) \right).
\]

Associated with $K_\pm(n,m)$ is the operator
\[
(K_\pm f)(n) = \sum_{m=n}^{\pm \infty} K_\pm(n,m)f(m), \quad f \in \ell_\pm^{\infty}(\mathbb{Z}, \mathbb{C}),
\]
(5.24) \[
\text{which acts as a transformation operator for the pair } \tau, \tau_q.
\]

**Theorem 5.2.** Let $\tau_q$ and $\tau$ be the quasi-periodic and perturbed Jacobi difference expression, respectively. Then
\[
\tau K_\pm f = K_\pm \tau_q f, \quad f \in \ell_\pm^{\infty}(\mathbb{Z}, \mathbb{C}).
\]
(5.25)
Proof. It suffices to show that $HK_{\pm} = K_{\pm}H_q$.
\[
HK_{\pm}(n, m) = \frac{1}{2\pi i} \int_{|w|=1} H\psi_{\pm}(w, n)\psi_{q, \mp}(w, m)dw(w) \\
= \frac{1}{2\pi i} \int_{|w|=1} \lambda(w)\psi_{\pm}(w, n)\psi_{q, \mp}(w, m)dw(w) \\
= \frac{1}{2\pi i} \int_{|w|=1} \psi_{\pm}(w, n)H_q\psi_{q, \mp}(w, m)dw(w).
\]
(5.26)

Lemma 5.3. For $n \in \mathbb{Z}$ we have
\[
a(n) = \frac{K_{\pm}(n + 1, n + 1)}{K_{\pm}(n, n)} = \frac{K_{-}(n, n)}{K_{-}(n + 1, n + 1)}, \\
b(n) = \frac{a_q(n)K_{\pm}(n + 1, n)}{K_{\pm}(n, n)} - \frac{a_q(n - 1)K_{\pm}(n - 1, n)}{K_{\pm}(n - 1, n - 1)}.
\]
(5.27)

Proof. Consider the equation of the transformation operator $HK_{\pm} = K_{\pm}H_q$, which is equivalent to (c.f. (5.20))
\[
a(n - 1)K_{\pm}(n - 1, m) + b(n)K_{\pm}(n, m) + a(n)K_{\pm}(n + 1, m) = \]
\[
= a_q(m - 1)K_{\pm}(n, m - 1) + b_q(m)K_{\pm}(n, m) + a_q(m)K_{\pm}(n, m + 1).
\]
Evaluating at $m = n \mp 1$ we obtain the first equation and at $m = n + 1$ the second. □

In particular, observe
\[
K_{\pm}(n, n) = A_{\pm}(n), \quad K_{\pm}(n, n \pm 1) = \frac{A_{\pm}(n)}{a_q(n - 0)}B_{\pm}(n).
\]
(5.28)

6. The Scattering Matrix

Let $H_q$ be a given quasi-periodic Jacobi operator and $H$ a perturbation of $H_q$ satisfying Hypothesis H[4]. To set up scattering theory for the pair $(H, H_q)$ we proceed as usual.

The Wronskian of our Jost functions can be evaluated as $n \to \pm \infty$ and is given by
\[
W(\psi_{\pm}(\lambda), \overline{\psi_{\mp}(\lambda)}) = W_q(\psi_{q, \pm}(\lambda), \psi_{q, \mp}(\lambda)) = \frac{R^{1/2}_{2g+2}(\lambda)}{\prod_{j=1}^{g}(\lambda - \mu_j)}, \quad \lambda \in \sigma(H_q).
\]

Hence $\psi_{\pm}(\lambda), \overline{\psi_{\mp}(\lambda)}$ are linearly independent for $\lambda$ in the interior of $\sigma(H_q)$ and we consider the scattering relations
\[
\psi_{\pm}(\lambda, n) = \alpha(\lambda)\overline{\psi_{\mp}(\lambda, n)} + \beta(\lambda)\overline{\psi_{\mp}(\lambda, n)}, \quad \lambda \in \sigma(H_q),
\]
(6.2)
where
\begin{align}
\alpha(\lambda) &= \frac{W(\psi_{+}(\lambda), \psi_{\pm}(\lambda))}{W(\psi_{\pm}(\lambda), \psi_{+}(\lambda))} = \frac{\prod_{j=1}^{g}(\lambda - \mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_{-}(\lambda), \psi_{+}(\lambda)), \\
\beta_{\pm}(\lambda) &= \frac{W(\psi_{\pm}(\lambda), \psi_{\pm}(\lambda))}{W(\psi_{\pm}(\lambda), \psi_{\pm}(\lambda))} = \frac{\prod_{j=1}^{g}(\lambda - \mu_j)}{R_{2g+2}^{1/2}(\lambda)} W(\psi_{\mp}(\lambda), \psi_{\pm}(\lambda)).
\end{align}

While \( \alpha(\lambda) \) is only defined for \( \lambda \in \sigma(H_{q}) \), (6.8) may be used as a definition for \( \lambda \in \mathbb{C}\setminus\{E_j\} \). Therefore \( \alpha(w) \) can be continued as a holomorphic function on \( \mathbb{W} \) and it is continuous up to the boundary except possibly at the band edges.

**Remark 6.1.** Note that \( \alpha(\lambda) \) does not depend on the normalization of \( \psi_{\pm}(\lambda) \) at the base point \( n_0 = 0 \) whereas \( \beta_{\pm} = \beta_{\pm,0} \) does. Using \( \psi_{\pm}(z, n_0) = \psi_{q,\pm}(z, n_0)^{-1} \psi_{\pm}(z, n) \) and
\begin{equation}
W((\psi_{+}(\lambda), \psi_{-}(\lambda))) = \prod_{j=1}^{g} \frac{\lambda - \mu_j(n_0)}{\lambda - \mu_j} W((\psi_{+}(\lambda, n_0), \psi_{-}(\lambda, n_0))
\end{equation}
we see
\begin{equation}
\beta_{\pm,0}(\lambda) = \frac{\psi_{q,\pm}(\lambda, n_0)}{\psi_{q,\pm}(\lambda, n_0)} \beta_{\pm,0}(\lambda).
\end{equation}

A direct calculation shows
\begin{equation}
\alpha(w) = \overline{\alpha(w)}, \quad \beta_{\pm}(w) = \beta_{\pm}(w) = -\beta_{\mp}(w)
\end{equation}
and the Plücker identity (c.f. [28], (2.169)) implies
\begin{equation}
|\alpha(w)|^2 = 1 + |\beta_{\pm}(w)|^2, \quad |w| = 1.
\end{equation}

We will denote the eigenvalues of \( H \) by
\begin{equation}
\sigma_{p}(H) = \{\rho_j\}_{j=1}^{g}.
\end{equation}

Our next aim is to study the behavior of \( \alpha(\lambda) \) at the eigenvalues \( \rho_j \), therefore we modify the Jost solutions \( \psi_{\pm}(\lambda, n) \) according to their poles at \( \mu_j \) and define the following \( \hat{\psi}_{\pm}(\lambda, n) \)
\begin{equation}
\hat{\psi}_{+}(\lambda, \cdot) = \prod_{\mu_j \in M_{+}} (\lambda - \mu_j) \psi_{+}(\lambda, \cdot), \quad \hat{\psi}_{-}(\lambda, \cdot) = \prod_{\mu_j \in M_{-}\setminus\{E_j\}} (\lambda - \mu_j) \psi_{-}(\lambda, \cdot).
\end{equation}

Define \( \hat{\psi}_{q,\pm}(\lambda, \cdot) \) accordingly. Moreover, \( \hat{\psi}_{\pm}(\rho_j, n) = c_{j}^{\pm} \hat{\psi}_{\mp}(\rho_j, n) \) with \( c_{j}^{+} c_{j}^{-} = 1 \). The **norming constants** \( \gamma_{\pm, j} \) are defined by
\begin{equation}
\frac{1}{\gamma_{\pm, j}} := \sum_{m \in \mathbb{Z}} |\hat{\psi}_{\pm}(\rho_j, m)|^2.
\end{equation}

To compute the derivative of \( \alpha(\lambda) \) at \( \rho_j \), note that
\begin{equation}
\alpha(\lambda) = \frac{W(\psi_{-}(\lambda), \psi_{+}(\lambda))}{R_{2g+2}^{1/2}(\lambda)}.
\end{equation}

By virtue of [28], Lemma 2.4,
\begin{equation}
\left. \frac{d}{d\lambda} W(\psi_{-}(\lambda), \psi_{+}(\lambda)) \right|_{\rho_j} = -\sum_{k \in \mathbb{Z}} \hat{\psi}_{-}(\rho_j, k) \hat{\psi}_{+}(\rho_j, k) = -\frac{1}{c_{j}^{+} \gamma_{\pm, j}}.
\end{equation}
In addition, simple poles at $w$ at (6.20) $\gamma_{\pm,j} R_{2g+2}^{1/2} (\rho_j)$.

From (6.12) we obtain a connection between the left and right norming constants

$$\gamma_{+,j} \gamma_{-,j} = \frac{1}{(\alpha'(\rho_j))^2 R_{2g+2}(\rho_j)}.$$

As a last preparation, we study the behavior of $\alpha(w)$ as $w \to 0$. By (6.13),

$$W(\psi_-(w), \psi_+(w)) = A_-(0) A_+(0) \tilde{a} w^{-1} + O(w)$$

and

$$\frac{R_{1/2}^{1/2}(\lambda(w))}{\prod_{j=1}^{g} (\lambda(w) - \lambda_j)} = \tilde{a} w^{-1} + O(1),$$

therefore $\alpha^{-1}(w)$ is bounded at 0 with

$$\alpha(0) = \prod_{j=\infty}^{\infty} \frac{a_q(j)}{a(j)}.$$

We now define the scattering matrix

$$S(w) = \begin{pmatrix} T(w) & R_-(w) \\ R_+(w) & T(w) \end{pmatrix}, \quad |w| = 1,$$

where $T(w) := \alpha^{-1}(w)$ and $R_{\pm}(w) := \alpha^{-1}(w) \beta_{\pm}(w)$ are called transmission and reflection coefficients. Equations (6.18) and (6.16) imply

**Lemma 6.2.** The scattering matrix $S(w)$ is unitary. The coefficients $T(w), R_{\pm}(w)$ are bounded for $|w| = 1$, continuous for $|w| = 1$ except at possibly $w_1 = w(E_i)$, fulfill

$$|T(w)|^2 + |R_{-}(w)|^2 = 1, \quad |w| = 1,$$

$$T(w) R_+(\bar{w}) + T(\bar{w}) R_-(w) = 0, \quad |w| = 1$$

and $\overline{T(w)} = T(\bar{w}), \overline{R_{\pm}(w)} = R_{\pm}(\bar{w})$ for $|w| = 1$.

Moreover, $R_{1/2}^{1/2}(w)T(w)^{-1}$ is continuous (in particular $T(w)$ can only vanish at $w_1$) and

$$\lim_{w \to w_1} R_{1/2}^{1/2}(w) \frac{R_+(w) + 1}{T(w)} = 0, \quad w_1 \neq w(\mu_j)$$

$$\lim_{w \to w_1} R_{1/2}^{1/2}(w) \frac{R_+(w) - 1}{T(w)} = 0, \quad w_1 = w(\mu_j).$$

The transmission coefficient $T(w)$ has a meromorphic continuation to $\mathbb{W}$ with simple poles at $w(\rho_j),$

$$\left( \text{Res}_{\rho_j} T(\lambda) \right)^2 = \gamma_{+,j} \gamma_{-,j} R_{2g+2}(\rho_j).$$

In addition, $T(z) \in \mathbb{R}$ as $z \in \mathbb{R} \setminus \sigma(H_q)$ and

$$T(0) = \frac{1}{K_+(n,n) K_-(n,n)} = \prod_{m=0}^{\infty} \frac{a(m)}{a_q(m)},$$

where $K_{\pm}(n,n)$ are the coefficients of the transformation operators.
Proof. To show (6.20) we use the definition (6.3),
\[ R_{2g+2}^{1/2}(\lambda) \frac{R_+^{s}(\lambda) + 1}{T(\lambda)} = \prod_{j=1}^{g}(\lambda - \mu_j)(W(\psi_-(\lambda), \psi_+(\lambda)) + W(\psi_-(\lambda), \psi_-^{s})). \]
There are two cases to distinguish: If \( \mu_j \neq E_l \) then \( \psi_{\pm} \) are continuous and real at \( \lambda = E_l \) and the two Wronskians cancel. Otherwise, if \( \mu_j = E_l \) they are purely imaginary (by property (B2) of the Jost functions) and the two terms are equal in the limit and add up. \( \square \)

The sets
\[ S^{\pm}(H) = \{|R_{\pm}(w), |w| = 1; (\rho_j, \gamma_{\pm, j}), 1 \leq j \leq q\} \]
are called left/right scattering data for \( H \).

First we want to show that the transmission coefficient can be reconstructed from either left or right scattering data.

Let \( g(w, w_0) \) be the Green function associated with \( W \) and let
\[ \mu(w, w_0)dw_0 = \frac{\partial g}{\partial r}(w, re^{i\theta}) \bigg|_{r=1} e^{i\theta} d\theta, \quad w_0 = e^{i\theta}, \]
be the corresponding harmonic measure on the boundary (see, e.g., [30]). Since \( W_0 \) is simply connected, we can choose a function \( h(w, v) \) such that \( \hat{g}(w, w_0) = g(w, w_0) + ih(w, w_0) \) is analytic in \( W_0 \). Clearly \( \hat{g} \) is only well-defined up to an imaginary constant and it will not be analytic on \( W \setminus \{0\} \) in general. Similarly we can find a corresponding \( \nu(w, w_0) \) and set \( \hat{\mu}(w, w_0) = \mu(w, w_0) + \nu(w, w_0) \).

**Theorem 6.3.** Either one of the sets \( S^{\pm}(H) \) determines the other and \( T(w) \) via the Poisson-Jensen type formula
\[ T(w) = \exp \left( \sum_{j=1}^{q} \hat{g}(w, w(\rho_j)) \right) \exp \left( \frac{1}{2} \int_{|w|=1} \ln(1 - |R_{\pm}(w_0)|^2) \hat{\mu}(w, w_0)dw_0 \right), \]
where the constant of \( \hat{g} \) has to be chosen such that \( T(0) > 0 \), and
\[ \frac{R_{-}(w)}{R_{+}(w)} = \frac{T(w)}{\overline{T(w)}}, \quad \gamma_{+, j} \gamma_{-, j} = \frac{(\text{Res}_{\rho_j} T(\lambda))^2}{\prod_{l=0}^{2g+1} (\rho_j - E_l)}. \]

Proof. It suffices to prove the formula for \( T(w) \), since evaluating the residua provides \( \gamma_{\pm, j} \), together with \( \{\lambda_i\}, \{E_l\} \). The formula for \( T(w) \) holds by [32], Theorem 1 at least when taking absolute values. Since both sides are analytic, and have equal absolute values, they can only differ by a constant of absolute value one. But both sides are positive at \( w = 0 \) and hence this constant is one. \( \square \)

Note that neither the Blaschke factors nor the outer function in (6.25) are single valued on \( W \) in general. In particular, the eigenvalues cannot be chosen arbitrarily, which was first observed in [21].

7. The Gel’fand-Levitan-Marchenko equations

In this section we want to derive a procedure which allows the reconstruction of the Jacobi operator \( H \) with asymptotically quasi-periodic coefficients from its scattering data \( S^{\pm}(H) \). This will be achieved by deriving an equation for \( K^{(n, m)}(n, m) \) which is generally known as Gel’fand-Levitan-Marchenko equation.
Since $K_{\pm}(n, m)$ are essentially the Fourier coefficients of the Jost solutions $\psi_{\pm}(w, n)$ we compute the Fourier coefficients of the scattering relations (6.12). Therefore we multiply

$$T(w)\psi_{\pm}(w, n) = R_{\pm}(w)\psi_{\pm}(w, n) + \bar{\psi}_{\pm}(w, n)$$

by $(2\pi)^{-1}\psi_{q,\pm}(w, m)d\omega$, where $\pm m \geq \pm n$, and integrate around the unit circle.

First we evaluate the right hand side of (7.1) using (5.1)

$$\frac{1}{2\pi i} \int_{|w|=1} \psi_{q, +}(w, m)d\omega(w) = K_{+}(n, m),$$

$$\frac{1}{2\pi i} \int_{|w|=1} R_{+}(w)\psi_{q, +}(w, n)d\omega(w) = \sum_{l=n}^{\infty} K_{+}(n, l)\tilde{F}^{+}(l, m),$$

where

$$\tilde{F}^{+}(l, m) = \frac{1}{2\pi i} \int_{|w|=1} R_{+}(w)\psi_{q, +}(w, l)d\omega(w).$$

Note that $\tilde{F}^{+}(l, m) = \tilde{F}^{+}(m, l)$ is real.

To evaluate the left hand side of (7.1) we use the residue theorem. The only poles are at the eigenvalues and at 0 if $n = m$, hence

$$\frac{1}{2\pi i} \int_{|w|=1} T(w)\psi_{-}(w, n)\psi_{q, +}(w, m)d\omega(w)$$

$$= \frac{\delta(n, m)}{K_{+}(n, n)} + \sum_{j=1}^{q} \text{Res}_{\gamma_{j}} \left( \frac{T(\lambda)\hat{\psi}_{-}(\lambda, n)\hat{\psi}_{q, +}(\lambda, m)}{R_{2g+2}(\lambda)} \right).$$

Here $\delta(n, m)$ is one for $m = n$ and zero else. By (6.12) the residua at the eigenvalues are given by

$$\text{Res}_{\gamma_{j}} \left( \frac{T(\lambda)\hat{\psi}_{-}(\lambda, n)\hat{\psi}_{q, +}(\lambda, m)}{R_{2g+2}(\lambda)} \right) = -\gamma_{+j}\hat{\psi}_{+}(\rho_{j}, n)\hat{\psi}_{q, +}(\rho_{j}, m).$$

Collecting all terms yields

$$K_{\pm}(n, m) + \sum_{l=n}^{\pm\infty} K_{\pm}(n, l)\tilde{F}^{\pm}(l, m) = \frac{\delta(n, m)}{K_{\pm}(n, n)} - \sum_{j=1}^{q} \gamma_{+j}\hat{\psi}_{q, +}(\rho_{j}, n)\hat{\psi}_{q, +}(\rho_{j}, m).$$

and we have thus proved the following result.

**Theorem 7.1.** The kernel $K_{\pm}(n, m)$ of the transformation operator satisfies the Gel’fand-Levitan-Marchenko equation

$$K_{\pm}(n, m) + \sum_{l=n}^{\pm\infty} K_{\pm}(n, l)F^{\pm}(l, m) = \frac{\delta(n, m)}{K_{\pm}(n, n)}, \quad \pm m \geq \pm n,$$

where

$$F^{\pm}(l, m) = \tilde{F}^{\pm}(l, m) + \sum_{j=1}^{q} \gamma_{+j}\hat{\psi}_{q, +}(\rho_{j}, l)\hat{\psi}_{q, +}(\rho_{j}, m).$$
Defining the Gel’fand-Levitan-Marchenko operator

\[ (7.8) \quad \mathcal{F}_n^\pm f(j) = \sum_{l=0}^{\infty} \mathcal{F}_n^\pm (n \pm l, n \pm j) f(l), \quad f \in \ell^2(\mathbb{N}_0, \mathbb{C}), \]

yields that the Gel’fand-Levitan-Marchenko equation is equal to

\[ (7.9) \quad (1 + \mathcal{F}_n^\pm) K^\pm (n, n \pm .) = (K^\pm (n, n))^{-1} \delta_0. \]

Our next aim is to study the Gel’fand-Levitan-Marchenko operator \(\mathcal{F}_n^\pm\) in more detail. The structure of the Gel’fand-Levitan-Marchenko equation suggests that the estimate (5.5) for \(K^\pm (n,m)\) should imply a similar estimate for \(\mathcal{F}_n^\pm (n,m)\).

**Lemma 7.2.**

\[ (7.10) \quad |\mathcal{F}_n^\pm (n,m)| \leq C \sum_{j=\frac{n+m}{2}+1}^{\infty} \left( |a(j) - a_q(j)| + |b(j) - b_q(j)| \right), \]

where the constant \(C\) is of the same nature as in (5.5).

**Proof.** We abbreviate the estimate (5.5) for \(K^\pm (n,m)\) by

\[ (7.11) \quad |K^\pm (n,m)| \leq C \, C^\pm_n (n+m), \]

where

\[ C^\pm_n (n+m) = \sum_{j=\frac{n+m}{2}+1}^{\infty} c(j), \quad c(j) = |a(j) - a_q(j)| + |b(j) - b_q(j)|. \]

Note that \(C^\pm_n (n+1) \leq C^\pm_n (n)\). Moreover, \(C^\pm_n (n) \in \ell^1(\mathbb{Z})\) since the summation by parts formula (e.g. [28], (1.18))

\[ (7.12) \quad \sum_{m=n}^{N} g(m) (f(m+1) - f(m)) = g(N) f(N+1) - g(n-1) f(n) + \sum_{m=n}^{N} (g(m-1) - g(m)) f(m) \]

implies for \(g(m) = m, \ f(m) = C^\pm_n (m)\) that

\[ (7.13) \quad \sum_{m=n}^{\infty} m c(m) = (n-1) C^\pm_n (n) + \sum_{m=n}^{\infty} C^\pm_n (m), \]

where we used \(\lim_{n \to \infty} n C^\pm_n (n+1) \leq \lim_{n \to \infty} \sum_{m=n}^{\infty} m c(m) = 0\). Solving the GLM-equation (7.6) for \(F^\pm (n, m), \ m > n\), we obtain

\[ |F^\pm (n,m)| \leq \frac{1}{K^\pm (n,n)} \left( |K^\pm (n,m)| + \sum_{l=n+1}^{\infty} |K^\pm (n,l) F^\pm (l,m)| \right) \]

\[ \leq C_1(n) \left( C^\pm_n (n+m) + \sum_{l=n+1}^{\infty} C^\pm_n (n+l) |F^\pm (l,m)| \right), \]
where \( C_1(n) = C |K_+(n, n)|^{-1} \to C \) for \( n \to \infty \) (see \cite{GLM}). For \( n \) large enough, i.e. \( C_1(n)C_+(2n) < 1 \), we apply the discrete Gronwall-type inequality \cite{GLM}. Lemma 10.8,

\[
|F^+(n, m)| \leq C_1(n) \left( C_+(n + m) + \sum_{l=n+1}^{\infty} \frac{C_1(l)C_+(l + m)C_+(n + l)}{\prod_{k=n+1}^{l}(1 - C_1(k)C_+(n + k))} \right)
\]

(7.14) \[
\leq C_1(n)C_+(n + m) \left( 1 + \sum_{l=n+1}^{\infty} \frac{C_1(l)C_+(n + l)}{\prod_{k=n+1}^{l}(1 - C_1(n)C_+(n + k))} \right),
\]

which finishes the proof. \( \square \)

Furthermore,

**Lemma 7.3.** Let \( F^\pm(n, m) \) be solutions of the Gel’fand-Levitan-Marchenko equation. Then

\[
\sum_{n=n_0}^{\pm \infty} \sum_{n=0}^{\pm \infty} |n| |F^\pm(n, n) - F^\pm(n \pm 1, n \pm 1)| < \infty,
\]

(7.15) \[
\sum_{n=n_0}^{\pm \infty} |a_q(n)F^\pm(n, n + 1) - a_q(n - 1)F^\pm(n - 1, n)| < \infty.
\]

(7.16)

**Proof.** We first prove (7.14) for \( F^+ \). Lemma \ref{lemma10} implies

\[
b(n) - b_q(n) = a_q(n)\kappa_{+,1}(n) - a_q(n - 1)\kappa_{+,1}(n - 1),
\]

(7.17) where

\[
\kappa_{+,j}(n) := \kappa_+(n, n + j) := \frac{K_+(n, n + j)}{K_+(n, n)}.
\]

(7.18)

Abbreviate \( F^+_j(n) := F^+(n + j, n) \). With this notation, the GLM-equation (7.10) reads

\[
\kappa_{+,j}(n) + F^+_j(n) + \sum_{j=1}^{\infty} \kappa_{+,j}(n)F^+_j(n + l) = \frac{\delta(l, 0)}{K_+(n, n)^2}, \quad l \geq 0.
\]

(7.19) Insert the GLM-equation for \( F^+(n, n+1), F^+(n-1, n) \) (recall \( F^+(n, m) = F^+(m, n) \))

\[
a_q(n)F^+_1(n) - a_q(n - 1)F^+_1(n - 1)
\]

\[
= -a_q(n)\kappa_{+,1}(n) + a_q(n - 1)\kappa_{+,1}(n - 1)
\]

\[
(7.20) - \sum_{j=1}^{\infty} \left( a_q(n)\kappa_{+,j}(n)F^+_j(n + 1) - a_q(n - 1)\kappa_{+,j}(n - 1)F^+_j(n) \right).
\]
Since \(-a_q(n)\kappa_{+1}(n) + a_q(n-1)\kappa_{+1}(n-1) = b_q(n) - b(n)\) the only interesting part is the sum. For \(N, J < \infty\),
\[
\sum_{n=n_0}^{N} n \sum_{j=1}^{J} \left( a_q(n)\kappa_{+,j}(n)F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n) \right)
\]
\[
= \sum_{j=1}^{J} \sum_{n=n_0}^{N} \left( a_q(n)\kappa_{+,j}(n)F_{j-1}^+(n+1) - a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n) \right)
\]
\[
= \sum_{j=1}^{J} \left( Na_q(N)\kappa_{+,j}(N)F_{j-1}^+(N+1) - (n_0-1)a_q(n_0-1)\kappa_{+,j}(n_0-1)F_{j-1}^+(n_0) \right) + \sum_{n=n_0}^{N} (-1)a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n)
\]
(7.21)
where we used the summation by parts. Estimates (7.11), (7.14) imply for the first summand
\[
\left| \sum_{j=1}^{J} Na_q(N)\kappa_{+,j}(N)F_{j-1}^+(N+1) \right| \leq \sum_{j=1}^{J} |N|a_q(N)\tilde{C}C_+(2N+j)C_+(2N+j+1)
\]
\[
\leq |N|a_q(N)\tilde{C}C_+(2N+1),
\]
which holds uniformly in \(J\), and (compare (7.13))
(7.22)
\[
\lim_{N \to \infty} Na_q(N)\tilde{C}C_+(2N+1) = 0.
\]
Moreover,
\[
\lim_{N, J \to \infty} \left| \sum_{j=1}^{J} \sum_{n=n_0}^{N} a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n) \right|
\]
\[
\leq \lim_{N, J \to \infty} \sum_{j=1}^{J} \sum_{n=n_0}^{N} \left| a_q(n-1)\kappa_{+,j}(n-1)F_{j-1}^+(n) \right|
\]
\[
\leq \sum_{j=1}^{\infty} \sum_{n=n_0}^{\infty} a_q(n-1)\tilde{C}C_+(2n+j)C_+(2n+j+1) < \infty.
\]
Therefore \(|n|a_q(n)F^+(n, n+1) - a_q(n-1)F^+(n-1, n)| \in \ell_1^+(\mathbb{Z})\) as desired. To apply Lemma 5.3 for \(F^-\) use the symmetry property \(F^-(n, m) = F^-(m, n)\). For (7.16), inserting the GLM-equation yields
\[
F^+(n, n) - F^+(n+1, n+1) = K_{+}^{-2}(n, n) - K_{+}^{-2}(n+1, n+1)
\]
\[
+ \sum_{j=1}^{\infty} \left( \kappa_{+,j}(n+1)F_{j}^+(n+1) - \kappa_{+,j}(n)F_{j}^+(n) \right).
\]
By (5.28),
\[
|K_{+}^{-2}(n, n) - K_{+}^{-2}(n+1, n+1)| \leq \frac{|a(n) + a_q(n)|}{a(n)^2} \prod_{j=\max(n+1, n+1)}^{\infty} a(j)^2 |a(n) - a_q(n)|
\]
(7.23)
\[
\leq C|a(n) - a_q(n)|,
\]
and the same considerations as above imply □.
Remark 7.4. The Gel'fand-Levitan-Marchenko equation is symmetric in $K_{\pm}(n,m)$ and $F^\pm(n,m)$, therefore we can invert the analysis done in Lemma 8.1 and obtain estimates for $K_{\pm}(n,m)$ starting with an analogue of estimate (7.10) for $F^\pm(n,m)$ and the estimates (7.14), (7.16) (c.f. Lemma 8.1).

Theorem 7.5. For $n \in \mathbb{Z}$, the Gel'fand-Levitan-Marchenko operator $F_n^\pm : \ell^2 \rightarrow \ell^2$ is Hilbert-Schmidt. Moreover, $1 + F_n^\pm$ is positive and hence invertible.

In particular, the Gel'fand-Levitan-Marchenko equation (7.4) has a unique solution and $S_+(H)$ or $S_-(H)$ uniquely determine $H$.

Proof. That $F_n^\pm$ is Hilbert-Schmidt is a straight-forward consequence of our estimate Lemma 7.2

Let $f \in \ell^2(\mathbb{N}_0)$ be real (which is no restriction since $F^+(n,l)$ is real and the real and imaginary part of (7.24) could be treated separately) and abbreviate $f_n(w) = \sum_{j=0}^\infty f(j)\psi_{q,+}(w,n+j)$. Then

$$\sum_{j=0}^\infty f(j)F_n^+ f(j) = \sum_{j=0}^\infty f(j)\sum_{l=0}^\infty F^+(n+j,n+l)f(l)$$

$$= \frac{1}{2\pi i} \int_{|w|=1} R_+(w) \sum_{j=0}^\infty f(j)\psi_{q,+}(w,n+j)\psi_{q,+}(w,n+l)f(l) \, d\omega(w)$$

$$+ \sum_{k=1}^q \sum_{j,l=0}^\infty f(j)\gamma_{+,k}\hat{\psi}_{q,+}(\rho_k,n+j)\hat{\psi}_{q,+}(\rho_k,n+l)f(l)$$

$$= \frac{1}{2\pi i} \int_{|w|=1} \tilde{R}_+(w)f_n(w)\, d\omega(w) + \sum_{k=1}^q \gamma_{+,k}|f_n(\rho_k)|^2$$

(7.24)

where $\tilde{R}_+(w) = R_+(w)f_n(w)(\hat{f}_n(w))^{-1}$ with $|\tilde{R}_+(w)| = |R_+(w)|$ and $\hat{f}_n(w) = \sum_{j=0}^\infty f(j)\hat{\psi}_{q,+}(w,n+j)$. The integral over the imaginary part vanishes since $\tilde{R}_+(w) = \tilde{R}_+(\bar{w})$ and we replace the real part by $\text{Re}(\tilde{R}_+(w)) = \frac{1}{2}(1 + \tilde{R}_+(w)^2 - 1 - |\tilde{R}_+(w)|) = \frac{1}{2}(1 + \tilde{R}_+(w)^2 + |T(w)|^2) - 1$, (recall $|\tilde{R}_+(w)|^2 + |T(w)|^2 = 1$). This yields using $\sum |f(j)|^2 = \frac{1}{2\pi i} \int_{|w|=1} |f_n(w)|^2 \, d\omega$

$$\sum_{j=0}^\infty f(j)(1 + F_n^+) f(j) = \sum_{k=1}^q \gamma_{+,k}|\hat{f}_n(\rho_k)|^2$$

(7.25)

$$+ \frac{1}{2\pi i} \int_{|w|=1} (1 + \tilde{R}_+(w)^2 + |T(w)|^2)|f_n(w)|^2 \, d\omega(w)$$

which establishes $1 + F_n^+ \geq 0$. According to Lemma 7.2 $|T(w)|^2 > 0$ a.e., therefore $-1$ is not an eigenvalue and $1 + F_n^+ \geq \epsilon_n$ for some $\epsilon_n > 0$.

To finish the direct scattering step for the Jacobi operator $H$ with asymptotically quasi-periodic coefficients we summarize the properties of the scattering data $S_{\pm}(H)$. 


Hypothesis H.7.6. The scattering data

(7.26) \[ S_\pm(H) = \{ R_\pm(w), |w| = 1; (\rho_j, \gamma_{\pm,j}), 1 \leq j \leq q \} \]
satisfy the following conditions.

(i). The reflection coefficients \( R_\pm(w) \) are continuous except possibly at \( w_l = w(E_l) \) and fulfill

(7.27) \[ \overline{R_\pm(w)} = R_\pm(\overline{w}). \]

Moreover, \( |R_\pm(w)| < 1 \) for \( w \neq w_l \) and

(7.28) \[ 1 - |R_\pm(w)|^2 \geq C \prod_{l=0}^{2g+1} |w - w_l|^2. \]

The Fourier coefficients

(7.29) \[ \hat{F}^\pm(l, m) = \frac{1}{2\pi i} \int_{|w|=1} R_\pm(w) \psi_{q,\pm}(w, l) \psi_{q,\pm}(w, m) d\omega(w) \]
satisfy

\[ |\hat{F}^\pm(n, m)| \leq \sum_{j=n+m}^{\pm \infty} q(j), \quad q(j) \geq 0, \quad |j|q(j) \in \ell^1(\mathbb{Z}), \]

\[ \sum_{n=n_0}^{\pm \infty} |n||\hat{F}^\pm(n, n) - \hat{F}^\pm(n \pm 1, n \pm 1)| < \infty, \]

\[ \sum_{n=n_0}^{\pm \infty} |n|a_q(n)\hat{F}^\pm(n, n+1) - a_q(n-1)\hat{F}^\pm(n-1, n)| < \infty. \]

(ii). The values \( \rho_j \in \mathbb{R} \setminus \sigma(H_q), 1 \leq j \leq q, \) are distinct and the norming constants \( \gamma_{\pm,j}, 1 \leq j \leq q, \) are positive.

(iii). \( T(w) \) defined via equation (6.25) extends to a single valued function on \( \mathbb{W} \) (i.e., it has equal values on the corresponding slits).

(iv). Transmission and reflection coefficients satisfy and satisfies

(7.30) \[ \lim_{w \to w_l} (w - w_l) \frac{R_\pm(w)+1}{T(w)} = 0, \quad w_l \neq w(\mu_j) \]

and the consistency conditions

\[ \frac{R_-(w)}{R_+(w)} = \frac{T(w)}{T(\overline{w})}, \quad \gamma_{+,j} \gamma_{-j} = \frac{(\text{Res}_{\rho_j} T(\lambda))^2}{\prod_{l=0}^{2g+1} (\rho_j - E_l)}. \]

Remark 7.7. Note that (7.28) implies that \( \ln(1 - |R_\pm(w)|^2) \) is integrable and ensures that (7.26) is well-defined, at least as a multi valued function. Condition (iii), which is void in the constant background case, shows that the reflection coefficient and eigenvalues cannot be chosen independent of each other.
8. Inverse scattering theory

In this section we want to invert the process of scattering theory, that is, we want to reconstruct the operator $H$ from a given set $S_\pm$ and a given quasi-periodic Jacobi operator $H_q$.

If $S_\pm$ (satisfying Hypotheses H.7.6 (i)–(ii)) and $H_q$ are known, we can construct $F^\pm(l,m)$ via formula (7.7) and thus derive the Gel’fand-Levitan-Marchenko equation, which has a unique solution by Theorem 7.5. This solution is the kernel of the transformation operator. Since $1 + F^\pm_n$ is positive, $K_\pm(n,n)$ is positive and we can set in accordance with Lemma 5.3.

Let $H_+$, $H_-$ be the associated Jacobi operators.

**Lemma 8.1.** Suppose a given set $S_\pm$ satisfies Hypotheses H.7.6 (i)–(ii). Then the sequences defined in (8.1) satisfy $|a_+(n) - a_q(n)|$, $n|b_+(n) - b_q(n)| \in \ell^1_+ (\mathbb{N})$.

Moreover, $\psi_\pm(\lambda, n) = \sum_{m=0}^{\pm\infty} K_\pm(n,m) \psi_{\lambda, \pm}(\lambda, m)$, where $K_\pm(n,m)$ is the solution of the Gel’fand-Levitan-Marchenko equation, satisfies $\tau_{\pm} \psi_\pm = \lambda \psi_\pm$.

**Proof.** We only prove the statements for the ”+” case. Define $F^+(n,m)$ by (c.f. 7.7.)

$$F^+(l,m) = \hat{F}^+(l,m) + \sum_{j=1}^{q} \gamma_{+j} \hat{\psi}_{q,+}(\rho_j,l) \hat{\psi}_{q,+}(\rho_j,m).$$

Hypothesis H.7.6 (i) implies

$$|F^+(n,m)| \leq C \sum_{j=1}^{\infty} q(j) =: C_+(n + m),$$

$$\sum_{n=0}^{\infty} |n||F^+(n,n) - F^+(n+1,n+1)| < \infty,$$

$$\sum_{n=0}^{\infty} |n||a_q(n)F^+(n,n+1) - a_q(n-1)F^+(n-1,n)| < \infty,$$

since $\hat{\psi}_{q,+}(\rho_j,n)$ decay exponentially as $n \to \infty$ and

$$\sum_{j} \gamma_{+j} \hat{\psi}_{q,+}(\rho_j,.) \hat{\psi}_{q,+}(\rho_j,.),$$

form a telescopic sum. Note that $C_+(n+1) < C_+(n)$. 

$$K_{\pm}(n,n) = \langle \delta_0, (1 + F^\pm_n)^{-1} \delta_0 \rangle^{1/2}$$

$$K_{\pm}(n,n \pm j) = \frac{1}{K_{\pm}(n,n)} \langle \delta_j, (1 + F^\pm_n)^{-1} \delta_0 \rangle$$

is the kernel of the transformation operator. Since $1 + F^\pm_n$ is positive, $K_\pm(n,n)$ is positive and we can set in accordance with Lemma 5.3.

$$a_+(n) = a_q(n) \frac{K_+(n+1,n+1)}{K_+(n,n)},$$

$$a_-(n) = a_q(n) \frac{K_-(n+1,n+1)}{K_-(n,n)},$$

$$b_+(n) = b_q(n) + a_q(n) \frac{K_+(n,n+1)}{K_+(n,n)} \frac{1}{K_+(n-1,n)} - a_q(n-1) \frac{K_+(n-1,n)}{K_+(n-1,n-1)} - a_q(n) \frac{K_+(n-1,n)}{K_+(n+1,n,1+1)}.$$
Set $\kappa_+(n, m) := K_+(n, m)K_+(n, n)^{-1}$. Then as in the proof of Lemma 8.2, we obtain
\begin{equation}
|\kappa_+(n, m)| \leq C_+(n + m)(1 + O(1)).
\end{equation}
Now we have all estimates at our disposal to prove $n|b_+(n) - b_q(n)| \in \ell^1(\mathbb{N})$. By definition (c.f. (8.2)),
\begin{equation}
b_+(n) - b_q(n) = a_q(n)\kappa_+(n, n + 1) - a_q(n - 1)\kappa_+(n - 1, n).
\end{equation}
We insert the GLM-equation for $\kappa_+(n, n + 1), \kappa_+(n - 1, n)$ and use estimate (8.3), the summation by parts formula, and estimates (8.6), (8.9) in the same way as in Lemma 7.3. Similarly using (8.4) we see
\begin{equation}
|\kappa(n)| = |\kappa_+(n, n)| = 1.
\end{equation}

The product converges and therefore $|n|a_+(n)^2 - a_q(n)^2| \in \ell^1(\mathbb{N})$.

Next we consider $\psi_+(\lambda, n)$. Abbreviate
\begin{equation}
(\Delta K_+)(n, m) = a_q(n - 1)\kappa_+(n - 1, m) + a_q(n)\kappa_+^{-1}(n)\kappa_+(n + 1, m)
- a_q(m - 1)\kappa_+(n, m - 1) - a_q(m)\kappa_+(n, m + 1) + (b_+(n) - b_q(n))\kappa_+(n, m).
\end{equation}
$\Delta K_+ = 0$ is equivalent to the operator equality $H_+K_+ = K_+H_q$, which in turn implies that $\psi_+(\lambda, n)$ satisfies $H_+\psi_+ = \lambda\psi_+$
\begin{equation}
H_+\psi_+ = H_+K_+\psi_+ = K_+H_q\psi_+ = K_+\lambda\psi_+ = \lambda K_+\psi_+ = \lambda\psi_+.
\end{equation}
To show that $\Delta K_+ = 0$ we insert the GLM-equation into (8.9) and obtain
\begin{equation}
(\Delta K_+)(n, m) + \sum_{l=n+1}^{\infty} (\Delta K_+)(n, l)F^+(l, m) = 0, \quad m > n + 1.
\end{equation}
In the calculations we used
\begin{equation}
a_q(n - 1)F^+(n - 1, m) + b_q(n)F^+(n, m) + a_q(n)F^+(n + 1, m) =
a_q(m - 1)F^+(n, m - 1) + b_q(m)F^+(n, m) + a_q(n)F^+(n, m + 1)
\end{equation}
which follows from (7.7). By Theorem 7.3, equation (8.11) has only the trivial solution $\Delta K_+ = 0$ and hence the proof is complete.

Now we can prove the main result of this section.

**Theorem 8.2.** Hypothesis $H(7.6)$ is necessary and sufficient for a sets $S_\pm$ to be the left/right scattering data of a unique Jacobi operator $H$ associated with sequences $a, b$ satisfying $H(7.4)$.

**Proof.** Necessity has been established in the previous section. By Lemma 8.1 we know existence of sequences $a_\pm, b_\pm$ and corresponding solutions $\psi_\pm(w, n)$ associated with $S_+$ (or $S_-$). Hence it remains to establish $a_+(n) = a_-(n)$ and $b_+(n) = b_-(n)$.

Consider the following part of the GLM-equation
\begin{equation}
\Phi_+(n, .) := \sum_{l=n}^{\infty} K_+(n, l)\tilde{F}^+(l, .) \in \ell^1(\mathbb{Z}).
\end{equation}
Then by use of (6.2) and Lemma 3.6,

$$\sum_{m \in \mathbb{Z}} \Phi_+(n, m) \psi_q-(w, m) = \sum_{m \in \mathbb{Z}} \left( \sum_{l=n}^{\infty} \delta(n, m) \Gamma^+_l(l, m) \right) \psi_q-(w, m)$$

$$= \sum_{m \in \mathbb{Z}} \left( \frac{1}{2\pi i} \int_{|w|=1} R_+(w) \psi_+(w, n) \psi_q+(w, m) d\omega(w) \right) \psi_q-(w, m)$$

$$= \sum_{m \in \mathbb{Z}} \langle \psi_q-(w, m), R_+(w) \psi_q+(w, n) \rangle \psi_q-(w, m)$$

(8.13) $$= R_+(w) \psi_+(w, n).$$

On the other hand, inserting the GLM-equation yields for $|w| = 1$

$$\sum_{m \in \mathbb{Z}} \Phi_+(n, m) \psi_q-(w, m) =$$

$$= \sum_{m=\infty}^{n-1} \Phi_+(n, m) \psi_q-(w, m) + \sum_{m=-\infty}^{n=1} \left[ \delta(n, m) K_+(n, n) - K_+(n, m) \right.$$

$$- \sum_{l=n}^{\infty} K_+(n, l) \sum_{j=1}^{\infty} \gamma_{+, j} \psi_q+(\rho_j, l) \psi_q+(\rho_j, m) \right] \psi_q-(w, m)$$

$$= \sum_{m=-\infty}^{n=1} \Phi_+(n, m) \psi_q-(w, m) + \psi_q-(w, n) K_+(n, n) - \psi_+(w, n)$$

(8.14) $$- \sum_{j=1}^{\infty} \gamma_{+, j} \psi_q+(\rho_j, n) \sum_{m=n}^{\infty} \tilde{\psi}_q+(\rho_j, m) \psi_q-(w, m),$$

(recall the definition of $\tilde{\psi}_q$ from (8.8)) and therefore

$$T(w) h_-(w, n) = \psi_+(w, n) + R_+(w) \psi_+(w, n), \quad |w| = 1,$$

where

$$h_-(w, n) = \frac{\psi_q-(w, n)}{T(w)} \left( \frac{1}{K_+(n, n)} + \sum_{m=-\infty}^{n=1} \Phi_+(n, m) \frac{\psi_q-(w, m)}{\psi_q-(w, n)} \right.$$

$$+ \sum_{j=1}^{\infty} \gamma_{+, j} \psi_q+(\rho_j, n) \frac{W_{n-1}(\psi_q+(\rho_j, \psi_q-(w) \lambda(w) - \rho_j))}{\psi_q-(w, n)} \right),$$

(8.16)

since Green’s formula (28, eq. (1.20)) implies for $\lambda \in \sigma(H_q)$

$$\sum_{m=n}^{\infty} \psi_q+(\rho_j, m) \psi_q-(\lambda, m) = -W_{n-1}(\psi_q+(\rho_j, \psi_q-(\lambda)).$$

Similarly, we obtain

$$h_+(w, n) = \frac{\psi_q+(w, n)}{T(w)} \left( \frac{1}{K_-(n, n)} + \sum_{m=n+1}^{\infty} \Phi_-(n, m) \frac{\psi_q+(w, m)}{\psi_q+(w, n)} \right.$$

$$- \sum_{j=1}^{\infty} \gamma_{-, j} \psi_q-(\rho_j, n) \frac{W_{n}(\psi_q-(\rho_j, \psi_q+(w) \lambda(w) - \rho_j))}{\psi_q+(w, n)} \right),$$

(8.17)
with

$$\Phi_-(n, m) = \sum_{l=-\infty}^{n} K_-(n, l) \tilde{F}^-(l, m).$$

For \(n \in \mathbb{Z}, |w| = 1\), we see that \(h_\pm(w^{-1}, n) = \hat{h}_\pm(w, n)\), since \(K_\pm(n, m)\) and \(\Phi_\pm(n, m)\) are real. The functions \(h_\pm(w, n)\) are continuous for \(|w| = 1, w \neq w(E_j)\), since \(T^{-1}(w)\) is continuous on this set by the Poisson-Jensen formula \(\|R_\pm(w)\| < 1\) for \(w \neq w(E_j)\) by Hyp. \(7.4\) (i) and \(\psi_{q, \pm}(w, m)\) are continuous on \(\partial W \setminus \{w(\mu_k)\}\). The functions \(h_\pm(w, n)\) have a meromorphic continuation to \(\mathbb{W} \setminus \{0\}\) with the only possible poles at \(w(\rho_j)\) and \(w(\mu_j)\). At \(w(\rho_j)\) there are no poles, due to the zeros of \(T^{-1}(w)\) at \(w(\rho_j)\). For \(w = w(\mu_j)\) we have the same type of singularity as \(\psi_{q, \pm}\). In summary, \(h_\pm(w, n)\) have simple poles at \(w(\mu_j)\) and are continuous at the boundary except possibly at \(w(E_j)\).

To study the behavior of \(h_\pm(w, n)\) as \(w \to 0\), we recall \(z^{-1} = -w/\bar{a}(1 + O(w))\). Then

$$\frac{w}{\bar{a}} + O(w^2)W_{n-1}(\hat{\psi}_{q, +}(\rho_j), \psi_{q, -}(w))$$

$$= (-1)^n \frac{\tilde{a}^{n-1}}{P_{j=0}^{n-2} a_q(j)} w^{n+1} (\hat{\psi}_{q, +}(\rho_j, n - 1) + O(w)),$$

$$\frac{w}{\bar{a}} + O(w^2)W_n(\hat{\psi}_{q, -}(\rho_j), \psi_{q, +}(w))$$

$$= (-1)^n \frac{\prod_{j=0}^{n} a_q(j)}{\tilde{a}^{n+1}} w^{n+1} (\hat{\psi}_{q, -}(\rho_j, n + 1) + O(w)),$$

and property (B4) implies

$$\sum_{m=\pm 1}^{\infty} \Phi_\pm(n, m) \psi_{q, \pm}(w, m)\psi_{q, \pm}^{-1}(w, n) = O(w), \quad w \to 0.$$  

We conclude that

$$\lim_{w \to 0} h_\pm(w, n)\psi_{q, \pm}(w, n) = \frac{1}{T(0) K_\pm(n, n)}.\quad (8.19)$$

Hyp. \(7.4\) (iv) and Hyp. \(6.1\) imply the following behavior of \(\hat{h}_\pm(\lambda, n)\) as \(\lambda \to \rho_j\)

$$\lim_{\lambda \to \rho_j} \hat{h}_\pm(\lambda, n) = \pm \gamma_{\pm, j} \hat{\psi}_\pm(\rho_j, n) \lim_{\lambda \to \rho_j} W_{n-1}(\hat{\psi}_{q, +}(\rho_j), \hat{\psi}_{q, \mp}(\lambda)) (\lambda - \rho_j)^{-1} \prod_{l=0}^{2g+1} \sqrt{\rho_j - E_l},$$

where \(\hat{h}_\pm\) are defined as in \(8.8\).

By virtue of the consistency condition \(T(w)R_+(w) = -T(w)R_-(w)\) we obtain

$$\frac{1}{T(w)} \left( \psi_{q, \pm}(w, n) + R_\pm(w)\psi_{q, \pm}(w, n) \right) + \frac{R_\pm(w)}{T(w)} \left( \psi_{q, \pm}(w, n) + R_\pm(w)\psi_{q, \pm}(w, n) \right)$$

$$= \psi_{q, \pm}(w, n) \left( 1 + \frac{R_\pm(w)}{T(w)} \right) + \psi_{q, \pm}(w, n) \left( \frac{R_\pm(w)}{T(w)} + \frac{R_\pm(w)}{T(w)} \right)$$

$$= \psi_{q, \pm}(w, n) T(w), \quad |w| = 1.$$
If we eliminate \( R_\pm (w) \) from the last equation and \( (8.15) \) we see

\[
T(w)R_{2g+2}(w)(\hat{\psi}_+(w,n)\hat{\psi}_-(w,n) - \hat{h}_+(w,n)\hat{h}_-(w,n)) = \frac{\prod_j (\lambda(w) - \mu_j)}{R_{2g+2}(w)} \left( \overline{h_\pm(w,n)}\psi_\pm(w,n) - \overline{\psi_\pm(w,n)}h_\pm(w,n) \right) =: G(w,n),
\]

for \( |w| = 1 \). Observe that \( G(\omega,n) = \overline{G(w,n)} = G(w,n), |w| = 1 \), since \( \overline{h_\pm\psi_\pm - \psi_\pm\overline{h_\pm}} \) and \( R_{2g+2}^{-1}(w) \) are odd functions for \( |w| = 1 \). The function \( G(w,n) \) can be continued analytically on \( \mathbb{W} \) since the difference \( \hat{\psi}_+\hat{\psi}_- - \hat{h}_+\hat{h}_- \) vanishes at the poles \( w(\mu_j) \) of \( T(w) \) by \( (8.20) \). Note that the product \( \hat{\psi}_+\hat{\psi}_- \) and hence also \( \hat{h}_+\hat{h}_- \) do not have poles at \( w(\mu_j) \). Moreover, since \( \mathbb{W} \) is just the image of the upper sheet, we can extend it to a compact Riemann surface \( \tilde{\mathbb{W}} \) by adding the image of the lower sheet. Now by \( G(\omega,n) = G(w,n) \) we can extend \( G \) to \( \tilde{\mathbb{W}} \) by setting \( G(w,n) = G(w^{-1},n) \) for \( |w| > 1 \).

Now let us investigate the behavior at the band edges: If \( w_l \neq w(\mu_j) \), we obtain by \( (7.30), (8.22) \), and real-valuedness of \( \hat{\psi}_\pm \) at the band edges that

\[
\lim_{w \to w_l} R_{2g+2}^{1/2}(w) \prod_j (\lambda(w) - \mu_j)h_\pm(w,n)\overline{\psi_\pm(w,n)} = \lim_{w \to w_l} \frac{R_{2g+2}^{1/2} \prod_j (\lambda(w) - \mu_j)}{T} (\overline{\psi_\pm + R_\pm\psi_\pm}) \overline{\psi_\pm} = \lim_{w \to w_l} \frac{R_{2g+2}^{1/2} \prod_j (\lambda(w) - \mu_j)}{T} (R_\pm + 1)\overline{\psi_\pm + \overline{\psi_\pm} - \psi_\pm} = 0.
\]

If \( w_l = w(\mu_j) \), the same calculation shows that

\[
\lim_{w \to w_l} R_{2g+2}^{1/2}(w) \prod_j (\lambda(w) - \mu_j)h_\pm(w,n)\overline{\psi_\pm(w,n)} = (-1)^{l+1}C_+(n)C_-(n) \lim_{w \to w_l} R_{2g+2}^{1/2}(w) \frac{R_\pm(w) - 1}{T(w)} = 0
\]

by \( (8.20) \), where we used \( \psi_\pm(w,n) = iC_\pm(n)(\lambda(w) - \mu_j)^{-1/2} + O(1) \).

Consequently \( R_{2g+2}(w)G(w,n) \) is continuous at \( w = w_l \) and vanishes at the band edges. Thus the singularities of \( R_{2g+2}^{1/2}(w)G(w,n) \) at \( w_l \) are removable. Furthermore, \( R_{2g+2}(w)G(w,n) \) is purely imaginary for \( |w| = 1 \) and real on the slits and hence must vanish at \( w_l \) by continuity. So the singularities of \( G(w,n) \) at \( w_l \) are removable as well. Thus \( G \) is holomorphic on all of \( \tilde{\mathbb{W}} \) and vanishes at \( w = 0 \), that is, \( G(w,n) \equiv 0 \) which implies (compare \( (B4) \))

\[
\lim_{w \to 0} \left( \psi_+(w,n)\psi_-(w,n) - h_+(w,n)h_-(w,n) \right) = K_+(n,n)K_-(n,n) - (T(0)^2K_+(n,n)K_-(n,n))^{-1} = 0.
\]

Using \( (8.22) \) we finally obtain from \( T(0)^2 = (K_+(n,n)K_-(n,n))^{-2} \) that

\[
(8.22) \quad a_+(n) = a_-(n) \equiv a(n), \quad \forall n \in \mathbb{Z}.
\]
It remains to prove \( b_+(n) = b_-(n) \). Proceeding as for \( G(w,n) \) we can show that

\[
T(w) R_{2g+2}^{-1/2}(w) \left( \hat{\psi}_+(w,n) \hat{\psi}_-(w,n+1) - \hat{h}_+(w,n+1) \hat{h}_-(w,n) \right) = \prod_{j} \frac{(\lambda(w) - \mu_j)}{R_{2g+2}^{1/2}(w)} \left( \hat{h}_+(w,n+1) \hat{\psi}_+(w,n) - \hat{\psi}_+(w,n) \hat{h}_+(w,n+1) \right)
\]

is a constant equal to \(-1/a(n)\). Thus

\[
W(w,n) := a(n) \left( \psi_+(w,n) \psi_-(w,n+1) - h_+(w,n+1) h_-(w,n) \right)
\]

and computing the asymptotics at \( w = 0 \) (compare (4.3)) we see

\[
0 = W(w,n) - W(w,n-1) = A_+(0) A_-(0) (b_+(n) - b_-(n))
\]

and in particular \( b_+(n) = b_-(n) \equiv b(n) \).

Our operator \( H \) has the correct norming constants since as in (6.12) it follows

\[
\sum_{n \in \mathbb{Z}} \hat{\psi}_+(\rho_j,n) \hat{\psi}_-(\rho_j,n) = (\text{Res}_\rho_j T(\lambda))^{-1} \prod_{l=0}^{2g+1} \sqrt{\rho_j - E_l}
\]

and by (8.20),

\[
\sum_{n \in \mathbb{Z}} \hat{\psi}_\pm(\rho_j,n) \hat{\psi}_\pm(\rho_j,n) = \gamma_{\pm,j}^{-1}.
\]

\[\square\]

**Acknowledgments**

I.E. thanks A. Boutet de Monvel for the kind hospitality of University Paris-7, where part of this work was done. G.T. thanks Peter Yuditskii for several helpful discussions and hints with respect to the literature. We thank Mark Losik for help with respect to literature.

**References**

[1] J. Bazargan and I. Egorova, *Jacobi operator with step-like asymptotically periodic coefficients*, Mat. Fiz. Anal. Geom. **10**, no. 3, 425–442 (2003).

[2] A. Boutet de Monvel and I. Egorova, *Transformation operator for Jacobi matrices with asymptotically periodic coefficients*, J. Difference Equ. Appl. **10**, 711-727 (2004).

[3] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16**, no. 4, 955-977 (2000).

[4] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Memoirs of the Amer. Math. Soc. **135/641**, (1998).

[5] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[6] K. M. Case, *The discrete inverse scattering problem in one dimension*, J. Math. Phys. **15**, 143–146 (1974).

[7] K. M. Case, *Orthogonal polynomials II*, J. Math. Phys. **16**, 1435–1440 (1975).

[8] K. M. Case, *On discrete inverse scattering problems. II*, J. Math. Phys. **14**, 916–920 (1973).

[9] K. M. Case and S. C. Chiu *The discrete version of the Marchenko equations in the inverse scattering problem*, J. Math. Phys. **14**, 1643–1647 (1973).

[10] K. M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).
[11] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[12] P. A. Cojuhari, *Finiteness of the discrete spectrum of Jacobi matrices* (Russian), Investigations in differential equations and mathematical analysis 173, “Shtiintsa”, Kishinev, 80–93 (1988).

[13] N.E. Firsova, *The direct and inverse scattering problems for the one-dimensional perturbed Hill operator*, Math. USSR, Sb. 58, 351–388 (1987).

[14] H. Flaschka, *On the Toda lattice. II*, Progr. Theoret. Phys. 51, 703–716 (1974).

[15] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Letters 19, 1095–1097 (1967).

[16] F. Gesztesy, R. Nowell and W. Pötz, *One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics*, Differ. Integral Equ. 10, No.3, 521–546 (1997).

[17] J. S. Geronimo and W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. App. Th. 46, 251–283 (1986).

[18] G.S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl., 17, 1684–1688 (1976).

[19] G.S. Guseinov, *The determination of an infinite Jacobi matrix from the scattering data*, Soviet Math. Dokl., 17, 596–600 (1976).

[20] G.S. Guseinov, *Scattering problem for the infinite Jacobi matrix*, Izv. Akad. Nauk Arm. SSR, Mat. 12, 365–379 (1977).

[21] E.A. Kuznetsov and A.V. Mikhailov, *Stability of stationary waves in nonlinear weakly dispersive media*, Z. Eksper. Teoret. Fiz. 67 no. 5, 1717–1727 (1974). (in Russian)

[22] V.A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[23] T. Parthasarathy, *On Global Univalence Theorems*, LNM 577, Springer, Berlin, 1983.

[24] L. Percolab, *The inverse problem for the periodic Jacobi matrix*, Theor. funk., funk. an., pril. 42, 107–121 (1984), in Russian.

[25] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. 129, 532–558 (1996).

[26] G. Teschl, *Inverse scattering transform for the Toda hierarchy*, Math. Nach. 202, 163–171 (1999).

[27] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, AMS/IP Studies in Advanced Mathematics 16, 375–384 (2000).

[28] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Math. Surv. and Mon. 72, Amer. Math. Soc., Rhode Island, 2000.

[29] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

[30] M. Tsuji, *Potential Theory in modern Functional Analysis*, Maruzen, Tokyo, 1959.

[31] A. Volberg and P. Yuditskii, *On the inverse scattering problem for Jacobi Matrices with the Spectrum on an Interval, a finite systems of intervals or a Cantor set of positive length*, Commun. Math. Phys. 226, 567–605 (2002).

[32] V. Veitchick and L. Zakman, *Inner and outer functions on Riemann surfaces*, Proc. Amer. Math. Soc. 16, 1200-1204 (1965).

Kharkiv National University, Ukraine

E-mail address: egorova@ilt.kharkov.ua

Fakultät für Mathematik, Nordbergstrasse 15, 1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria

E-mail address: Johanna.Michor@esi.ac.at

URL: [http://www.mat.univie.ac.at/~gerald/](http://www.mat.univie.ac.at/~gerald/)