ASYMPTOTIC AVERAGE SOLUTIONS TO LINEAR SECOND ORDER SEMI-ELLIPTIC PDES: A PIZZETTI-TYPE THEOREM

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Abstract. By exploiting an old idea first used by Pizzetti for the classical Laplacian, we introduce a notion of asymptotic average solutions making pointwise solvable every Poisson equation \( \mathcal{L}u(x) = -f(x) \) with continuous data \( f \), where \( \mathcal{L} \) is a hypoelliptic linear partial differential operator with positive semi-definite characteristic form.

1. Introduction

The Poisson-type equations related to hypoelliptic linear second order PDE’s with nonnegative characteristic form cannot be studied in \( L^p \) spaces due to the lack of a suitable Calderon-Zygmund theory for the relevant singular integrals. Our paper presents a result allowing to satisfactory study such equations in spaces of continuous functions. We follow a procedure introduced by Pizzetti in his 1909’s paper [14] based on the asymptotic average solutions for the classical Poisson-Laplace equation.

1.1. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), and let \( f : \Omega \to \mathbb{R} \) be a continuous bounded function. Let us denote by \( u_f \) the Newtonian potential of \( f \), i.e.,

\[
u_f : \mathbb{R}^n \to \mathbb{R}, \quad u_f(x) := \int_{\Omega} \Gamma(y-x)f(y) \, dy.
\]

Here \( \Gamma \) denotes the fundamental solution of the Laplace equation, i.e.,

\[
\Gamma(x) = c_n|x|^{2-n}, \quad x \in \mathbb{R}^n \setminus \{0\},
\]

\( \omega_n \) being the volume of the unit ball in \( \mathbb{R}^n \) and \( c_n := \frac{1}{n(n-2)\omega_n} \).

It is well known that \( u_f \in C^1(\mathbb{R}^n, \mathbb{R}) \), while, in general, \( u_f|_{\Omega} \notin C^2(\Omega, \mathbb{R}) \). However, in the weak sense of distributions,

(1.1) \[ \Delta u_f = -f \text{ in } \Omega. \]

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As a consequence, if the continuous function \( f \) is such that
\[
(1.2) \quad u_f \notin C^2(\Omega, \mathbb{R}),
\]
then the Poisson equation
\[
(1.3) \quad \Delta v = -f
\]
has no classical solutions, i.e., there does not exist a function \( v \in C^2(\Omega, \mathbb{R}) \) satisfying
\[
\Delta v(x) = -f(x) \text{ for every } x \in \Omega.
\]
Indeed, assume by contradiction that such a function exists. Then, by \( (1.1) \),
\[
\Delta(u_f - v) = 0 \text{ in } \Omega
\]
in the weak sense of distributions, so that, by Caccioppoli–Weyl’s Lemma, there exists a function \( h, \) harmonic in \( \Omega, \) such that
\[
u_f(x) - v(x) = h(x)
\]
a.e. in \( \Omega. \) Therefore, \( u_f - v \) being continuous in \( \Omega, \)
\[
u_f = v + h \in C^2(\Omega, \mathbb{R}),
\]
in contradiction with \( (1.2). \) This proves the existence of continuous functions \( f \) such that the Poisson equation \( (1.3) \) is not pointwise solvable. In his paper \([14]\), Pizzetti introduced a notion of pointwise weak Laplacian, making pointwise solvable every Poisson equation with continuous data. Pizzetti started from the following remark. Given a function \( u \) of class \( C^2 \) in \( \Omega \) one has
\[
limit_{r \to 0} \frac{M_r(u)(x) - u(x)}{r^2} = \frac{1}{2(n + 2)} \Delta u(x)
\]
for every \( x \in \Omega. \) Here \( M_r \) denotes the Gauss average
\[
M_r(u)(x) := \frac{1}{|B(x, r)|} \int_{\partial B(x, r)} u(y) \, dy,
\]
\( |B(x, r)| \) being the volume of \( B(x, r), \) the Euclidean ball centered at \( x \) with radius \( r. \) Then, if \( u \in C(\Omega, \mathbb{R}) \) is such that the limit at the left hand side of \( (1.4) \) exists at a point \( x \in \Omega, \) Pizzetti defines
\[
\Delta_a u(x) := 2(n + 2) \lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{r^2}.
\]
We call \( \Delta_a u(x) \) the asymptotic average Laplacian of \( u \) at \( x. \) Keeping in mind \( (1.4), \)
if \( u \in C^2(\Omega, \mathbb{R}), \) then
\[
\Delta_a u(x) = \Delta u(x) \text{ for every } x \in \Omega.
\]
We denote by 
\[ A(\Omega, \Delta) \]
the class of functions \( u \in C(\Omega, \mathbb{R}) \), such that \( \Delta u(x) \) exists at any point \( x \in \Omega \). Obviously, \( A(\Omega, \Delta) \) is a (linear) sub-space of \( C(\Omega, \mathbb{R}) \). Moreover, by the previous remark,

\[ C^2(\Omega, \mathbb{R}) \subseteq A(\Omega, \Delta). \]

Pizzetti proved that the Newtonian potentials of continuous bounded functions are contained in \( A(\Omega, \Delta) \). Precisely he proved the following theorem.

**Theorem A** (Pizzetti Theorem). Let \( \Omega \subseteq \mathbb{R}^n, n \geq 3 \), be a bounded open subset of \( \mathbb{R}^n \) and let \( f : \Omega \rightarrow \mathbb{R} \) be a bounded continuous function. Then

\[ u_f \in A(\Omega, \Delta) \]

and

\[ \Delta u_f = -f \text{ in } \Omega. \]

The aim of this paper is to extend the notion of asymptotic average solution and Pizzetti’s Theorem to the class of linear second order semi-elliptic partial differential operators that we will introduce in the next subsection.

1.2. We will deal with partial differential operators of the type

\[
L = \sum_{i,j=1}^{n} \partial_{x_i} (\partial_{x_j} a_{ij}(x)), \quad x \in \mathbb{R}^n,
\]

where \( A(x) := (a_{ij} = a_{ji})_{i,j=1,...,n} \) is a symmetric nonnegative definite matrix,

\[ \begin{align*}
&x \mapsto a_{ij}(x), \quad i, j = 1, \ldots, n \\
&\text{are smooth functions in } \mathbb{R}^n \text{ and}
\end{align*} \]

\[ \sum_{i=1}^{n} a_{ii}(x) > 0 \text{ for every } x \in \mathbb{R}^n. \]

Together with these qualitative properties we assume that \( L \) is hypoelliptic in \( \mathbb{R}^n \) and endowed with a smooth fundamental solution

\[ \Gamma : \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \neq y\} \rightarrow \mathbb{R}, \]

such that

(i) \( \Gamma(x, y) = \Gamma(y, x) > 0, \text{ for every } x \neq y; \)
(ii) \( \lim_{x \to y} \Gamma(x, y) = \infty, \text{ for every } y \in \mathbb{R}^n; \)
(iii) \( \lim_{x \to \infty} \left( \sup_{y \in K} \Gamma(x, y) \right) = 0, \text{ for every compact set } K \subseteq \mathbb{R}^n; \)
(iv) $\Gamma(x, \cdot)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$.

We recall that when we say that $\Gamma$ is a fundamental solution of $\mathcal{L}$ we mean that, for every $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \Gamma(x, y) \mathcal{L}\varphi(y) \, dy = -\varphi(x).$$

1.3. Important examples of operators satisfying our assumptions are the “sum of squares” of homogeneous Hörmander vector fields. Precisely: let

$$X = \{X_1, \ldots, X_m\}$$

be a family of linearly independent smooth vector fields such that

(H1) $X_1, \ldots, X_m$ satisfy the Hörmander rank condition at $x = 0$, that is,

$$\dim \{ Y(0) \mid Y \in \text{Lie}\{X_1, \ldots, X_m\} \} = n;$$

(H2) $X_1, \ldots, X_m$ are homogeneous of degree 1 with respect to a group of dilations $(\delta_\lambda)_{\lambda > 0}$ of the following type

$$\delta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

$$\delta_\lambda(x) = \delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n),$$

where the $\sigma_j$’s are natural numbers such that $1 \leq \sigma_1 \leq \ldots \leq \sigma_n$.

Then,

$$\mathcal{L} = \sum_{j=1}^m X_j^2$$

satisfies all the assumptions listed in subsection 1.2 (see [1], [2]).

We stress that the sub-Laplacians on stratified Lie groups in $\mathbb{R}^n$ are particular cases of the operator $\mathcal{L}$ in (1.6).

1.4. The extension of Pizzetti’s Theorem to the operator $\mathcal{L}$ in (1.6) rests on some representation formulas on the superlevel set of $\Gamma$. If $x \in \mathbb{R}$ and $r > 0$, define

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^n : \Gamma(x, y) > \frac{1}{r} \right\}.$$ 

We will call $\Omega_r(x)$ the $\mathcal{L}$-ball centered at $x$ and with radius $r$. It is easy to recognize that $\Omega_r(x)$ is a nonempty bounded open set of $\mathbb{R}^n$. Moreover

$$\bigcap_{r > 0} \Omega_r(x) = \{x\}$$
and 
\[
\frac{\left| \Omega_r(x) \right|}{r} \to 0 \text{ as } r \to 0.
\]

**Remark 1.1.** If \( \mathcal{L} = \Delta \), then
\[
\Omega_r(x) = B(x, \rho), \quad \text{with } \rho = (c_n r)^{\frac{n}{n-2}}.
\]

Let \( \Omega \subseteq \mathbb{R}^n \) be open and let \( u \in C^2(\Omega, \mathbb{R}) \). Then, for every \( \mathcal{L} \)-ball, \( \Omega_r(x) \) such that \( \overline{\Omega_r(x)} \subseteq \Omega \) and for every \( \alpha > -1 \) we have

\[
(1.8) \quad u(x) = M_r(u)(x) - N_r(\mathcal{L}u)(x),
\]

where \( M_r \) and \( N_r \) are the following average operators:

\[
(1.9) \quad M_r(u)(x) := \frac{\alpha + 1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y) K(x, y) \, dy,
\]

where
\[
K(x, y) := \frac{\langle A(y) \nabla_y \Gamma(x, y), \nabla_y \Gamma(x, y) \rangle}{(\Gamma(x, y))^{\alpha+2}};
\]

\[
(1.10) \quad N_r(w)(x) := \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left( \int_{\Omega_r(x)} \left( \Gamma(x, y) - \frac{1}{\rho} \right) w(y) \, dy \right) \, d\rho.
\]

The proof of the representation formula (1.8) can be found in [4].

**Remark 1.2.** If \( \mathcal{L} = \Delta \) and \( \alpha = \frac{2}{n-2} \), then the kernel \( K \) is constant and \( M_r \) becomes the Gauss average on the Euclidean ball \( B(x, \rho) \), with \( \rho = (c_n r)^{\frac{n}{n-2}} \).

Letting

\[
(1.11) \quad Q_r(x) := N_r(1) = \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left( \int_{\Omega_r(x)} \left( \Gamma(x, y) - \frac{1}{\rho} \right) \, dy \right) \, d\rho,
\]

an easy computation shows that
\[
Q_r(x) = \int_0^r \frac{\Omega_r(x)}{\rho^{\alpha+2}} \left( 1 - \left( \frac{\rho}{r} \right)^{\alpha+1} \right) \, d\rho.
\]

**Remark 1.3.** If \( \mathcal{L} = \Delta \) and \( \alpha = \frac{2}{n-2} \), then, letting \( \rho = (c_n r)^{\frac{1}{n-2}} \), we get
\[
\frac{M_r(u)(x) - u(x)}{Q_r(x)} = 2(n+2) \frac{1}{B(x, \rho)} \int_{B(x, \rho)} u(y) \, dy - u(0)
\]
\[
\frac{1}{\rho^2},
\]
so that, by [14],

\[\text{If } E \text{ is a measurable set of } \mathbb{R}^n, |E| \text{ denotes its Lebesgue measure.} \]
The limit in (1.12) extends to all the operators \( L \) in (1.5). Indeed, if \( u \) is a \( C^2 \) function in an open set \( \Omega \subseteq \mathbb{R}^n \), from the representation formula (1.8) and the identity (1.7), using Corollary 2.5 in Section 2, one immediately gets

\[
\lim_{r \to 0} \frac{M_r(u(x)) - u(x)}{Q_r(x)} = L u(x).
\]

Then, in analogy with the case \( L = \Delta \), we introduce the following definition.

**Definition 1.4.** Let \( L \) be a partial differential operator satisfying the assumptions of subsection 1.2 and let \( u \) be a continuous function in an open set \( \Omega \subseteq \mathbb{R}^n \). We say that \( u \in \mathcal{A}(\Omega, L) \), if

\[
\lim_{r \to 0} \frac{M_r(u(x)) - u(x)}{Q_r(x)}
\]

exists in \( \mathbb{R} \) at every point \( x \in \Omega \). In this case we define

\[
(L_a u)(x) := \lim_{r \to 0} \frac{M_r(u(x)) - u(x)}{Q_r(x)}.
\]

Furthermore, if \( f \in C(\Omega, \mathbb{R}) \) and there exists \( u \in \mathcal{A}(\Omega, L) \) such that

\[(L_a u)(x) = f(x) \text{ for every } x \in \Omega,\]

we say that \( u \) is an **asymptotic average solution to**

\[L_a u = f \text{ in } \Omega.\]

In the case \( f = 0 \) this definition was first introduced in the paper [6].

The main result of our paper is the following theorem which extends Pizzetti’s Theorem to the operators (1.5).

**Theorem 1.5.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a compactly supported continuous function. Define

\[u_f(x) := \int_{\mathbb{R}^n} \Gamma(x, y)f(y) \, dy, \quad x \in \mathbb{R}^n.\]

Then, \( u_f \in \mathcal{A}(\mathbb{R}^n, L) \) and

\[L_a u_f = -f \text{ in } \mathbb{R}^n.\]

We will prove this theorem in the next section. Here, by using a result in [6], we show a consequence of Theorem 1.5.
Theorem 1.6. Let \( f, u : \mathbb{R}^n \rightarrow \mathbb{R} \) be compactly supported continuous functions. Then,

\[
\mathcal{L} u = -f \text{ in } \mathbb{R}^n
\]

if and only if

\[
\mathcal{L} u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).
\]

Proof. By the previous Theorem 1.5,

\[
\mathcal{L} u = -f \text{ in } \mathbb{R}^n
\]

if and only if

\[
\mathcal{L} (u - uf) = 0 \text{ in } \mathbb{R}^n.
\]

Then, by Corollary 3.4 in \([6]\), \( u - uf \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) and

\[
\mathcal{L} (u - uf) = 0
\]

in the classical sense (and vice versa). Since \( \mathcal{L} \) is hypoelliptic, this is equivalent to say that

\[
\mathcal{L} (u - uf) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n),
\]

or that

\[
\text{(1.13)} \quad \mathcal{L}(u) = \mathcal{L}(uf) \text{ in } \mathcal{D}'(\mathbb{R}^n).
\]

On the other hand, \( \Gamma \) being a fundamental solution of \( \mathcal{L} \), \( \mathcal{L}(uf) = -f \) in \( \mathcal{D}'(\mathbb{R}^n) \). Then, \text{(1.13)} can be written as follows:

\[
\mathcal{L} u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).
\]

This completes the proof. \( \square \)

1.5. Bibliographical note. In recent years asymptotic mean value formulas characterizing classical or viscosity solutions to linear and nonlinear second order Partial Differential Equations have been proved by many authors; we refer to \([6, 12, 11, 8, 5, 10, 7, 13, 3]\). In those papers one can find quite exhaustive bibliography on this subject.

We would also like to quote the papers \([4]\) and \([9]\) where the notion of asymptotic sub-harmonic function is introduced in sub-Riemannian settings to extend classical results by Blaschke, Privaloff, Reade and Saks.

2. Proof of Theorem 1.5

For the readers’ convenience, we split this section in two subsections.
Let $G$ be a compact subset of $\mathbb{R}^n$ and let $r > 0$. Define

\begin{equation}
G_r := \bigcup_{x \in G} \Omega_r(x).
\end{equation}

Then, we have the following lemma.

**Lemma 2.1.** For every compact set $G \subseteq \mathbb{R}^n$ and for every $r > 0$, the set $G_r$ is compact.

**Proof.** It is enough to prove that $G_r$ is bounded. We argue by contradiction and assume that $G_r$ is not bounded. Then, there exists a sequence $(z_n)$ in $G_r$ such that $|z_n| \to \infty$.

By the very definition of $G_r$, for every $n \in \mathbb{N}$, there exists $x_n \in G$ such that $z_n \in \Omega_r(x_n)$. This means that

$$\Gamma(x_n, z_n) > \frac{1}{r}.$$ 

As a consequence,

$$\frac{1}{r} < \Gamma(x_n, z_n) \leq \sup_{x \in G} \Gamma(x, z_n),$$

so that, by the assumption (iii) related to $\Gamma$

$$0 < \frac{1}{r} \leq \lim_{n \to \infty} \left( \sup_{x \in G} \Gamma(x, z_n) \right) = 0.$$ 

This contradiction shows that $G_r$ is bounded. \qed

2.2. In this subsection we prove the following lemma.

**Lemma 2.2.** Let $G$ be a compact subset of $\mathbb{R}^n$ and let $r > 0$. Then, there exists a positive constant $C_r(G)$ such that

\begin{equation}
\sup_{x \in G} Q_r(x) \leq C_r(G).
\end{equation}

**Proof.** Keeping in mind the definition of $Q_r(x)$ (see (1.11)) for every $x \in G$ we get

\begin{equation}
Q_r(x) \leq \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \rho^{\alpha} \left( \int_{\Omega_r(x)} \Gamma(x, y) \, dy \right) \, d\rho.
\end{equation}

By (2.1),

\begin{equation}
Q_r(x) \leq \frac{\alpha + 1}{r^{\alpha+1}} \int_0^r \rho^{\alpha} \left( \int_{G_r} \Gamma(x, y) \, dy \right) \, d\rho.
\end{equation}
On the other hand, if \( \varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}) \) is such that \( \varphi = 1 \) on \( G_r \), \( \varphi \geq 0 \) (such a function exists thanks to Lemma 2.1), we have

\[
\int_{G_r} \Gamma(x, y) \, dy \leq \int_{\mathbb{R}^n} \varphi(y) \Gamma(x, y) \, dy \\
\leq \sup_{x \in G} \int_{\mathbb{R}^n} \varphi(y) \Gamma(x, y) \, dy \\
= C_{\varphi}(G).
\]

Using this estimate in (2.3) we obtain

\[
\sup_{x \in G} Q_r(x) \leq C_{\varphi}(G) \int_0^r \rho^\alpha \, d\rho = C_{\varphi}(G) := C_r(G).
\]

Remark 2.3. Since \( Q_\rho(x) \subseteq Q_r(x) \) for every \( \rho \in [0, r] \), we can assume

\[
C_\rho(G) \leq C_r(G)
\]

for every \( 0 < \rho < r \).

2.3. Now, we show a kind of continuity property of the \( \Omega_r(x) \) balls with respect to the Euclidean topology. Precisely, we prove the following lemma.

**Lemma 2.4.** For every \( x \in \mathbb{R}^n \) and for every \( R > 0 \) there exists \( r > 0 \) such that

\[
\Omega_r(x) \subseteq B(x, R).
\]

**Proof.** We still argue by contradiction and assume the existence of \( R > 0 \) such that \( \Omega_r(x) \notin B(x, R) \) for every \( r > 0 \). Then, if \( (r_n) \) is a sequence of real positive numbers such that \( r_n \searrow 0 \), for every \( n \in \mathbb{N} \) there exists \( y_n \in \Omega_{r_n}(x) \) such that

\[
y_n \notin B(x, R).
\]

This means

\[
y_n \notin B(x, R) \quad \text{and} \quad \Gamma(x, y_n) > \frac{1}{r_n}.
\]

Since \( \Gamma(x, y) \to 0 \) as \( y \to \infty \) and \( \frac{1}{r_n} \to \infty \), the sequence \( (y_n) \) is bounded. As a consequence, we may assume

\[
\lim_{n \to \infty} y_n = y^*
\]

for a suitable \( y^* \in \mathbb{R}^n \). Then \( y^* \notin B(x, R) \). In particular \( y \neq x \) so that \( \Gamma(x, y) < \infty \). On the other hand,

\[
\Gamma(x, y^*) = \lim_{n \to \infty} \Gamma(x, y_n) \geq \lim_{n \to \infty} \frac{1}{r_n} = \infty.
\]
This contradiction proves the lemma. \hfill \Box

From the previous lemma we obtain the following corollary.

**Corollary 2.5.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Then, for every $x \in \mathbb{R}^n$,

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \rightarrow 0 \text{ as } r \rightarrow 0.$$  

**Proof.** Since $f$ is continuous at $x$, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{y \in B(x,r)} |f(y) - f(x)| < \varepsilon.$$  

By the previous lemma, there exists $r_0 > 0$ such that $\Omega_{r_0}(x) \subseteq B(x,r)$. Then, for every $r < r_0$,

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \leq \sup_{y \in \Omega_{r_0}(x)} |f(y) - f(x)| \leq \sup_{y \in B(x,r)} |f(y) - f(x)| < \varepsilon.$$  

We have so proved that for every $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| < \varepsilon$$  

for every $r < r_0$. Hence,

$$\lim_{r \rightarrow 0} \left( \sup_{y \in \Omega_r(x)} |f(y) - f(x)| \right) = 0.$$  

\hfill \Box

2.4. Let $f$ as in Theorem 1.5 and, to simplify the notation, let us denote $u_f$ by $u$. The aim of this subsection is to prove the following identity:

$$(2.4) \quad u(x) = M_r(u)(x) + N_r(f)(x) \quad \forall x \in \mathbb{R}^n.$$  

To this end we choose a sequence $(f_p)$ in $C^\infty_0(\mathbb{R}^n, \mathbb{R})$ with the following properties:

(i) there exists a compact set $K \subseteq \mathbb{R}^n$ such that $\text{supp } f \subseteq K$ and $\text{supp } f_p \subseteq K$ for every $p \in \mathbb{N}$;

(ii) $\sup_K |f_p - f| \rightarrow 0$ as $p \rightarrow \infty$.

For simplicity reasons, let us put $u_p = u_{f_p}$, i.e.,

$$u_p(x) = \int_{\mathbb{R}^n} \Gamma(x,y) f_p(y) \, dy = \int_K \Gamma(x,y) f_p(y) \, dy.$$  

Then, by Lebesgue’s dominated convergence Theorem,

$$u(x) = \lim_{p \rightarrow \infty} u_p(x) = \int_K \Gamma(x,y) \lim_{p \rightarrow \infty} f_p(y) \, dy,$$  

for every \( x \in \mathbb{R}^n \). Actually, we have a stronger result. For every compact set \( G \subseteq \mathbb{R}^n \),
\[
\sup_{G} |u_p - u| \leq \sup_{x \in G} \left| \int_K \Gamma(x, y) (f_p(y) - f(y)) \, dy \right|
\leq \sup_K |f_p - f| \sup_{x \in G} \int_K \Gamma(x, y) \, dy
= C(G, K) \sup_K |f_p - f|.
\]
We explicitly observe that \( C(G, K) \) is a strictly positive finite constant.

Hence,
\[
(2.5) \quad \sup_{G} |u_p - u| \to 0 \text{ as } p \to \infty.
\]
Moreover, for every \( p \in \mathbb{N} \),
\[
u_p \in C^\infty(\mathbb{R}^n, \mathbb{R}) \quad \text{and} \quad \mathcal{L}u_p = -f_p.
\]
Then, by identity (1.3),
\[
u_p(x) = M_r(u_p)(x) - N_r(\mathcal{L}u_p)(x)
\leq M_r(u_p)(x) + N_r(f_p)(x)
\]
for every \( p \in \mathbb{N} \).

We have already noticed that \( u_p(x) \to u(x) \) as \( p \to \infty \).

To prove (2.4) we now show that
\[
(2.6) \quad \lim_{p \to \infty} M_r(u_p)(x) = M_r(u)(x)
\]
and
\[
(2.7) \quad \lim_{p \to \infty} N_r(f_p)(x) = N_r(f)(x).
\]
For every \( x \in \mathbb{R}^n \) we have
\[
|M_r(u_p)(x) - M_r(u)(x)| = |M_r(u_p - u)(x)|
\leq \sup_{\Omega_r(x)} |u_p - u| M_1(1)(x)
= \sup_{\Omega_r(x)} |u_p - u|.
\]
Since \( \Omega_r(x) \) is compact (see Lemma 2.1), and keeping in mind (2.5), the last right hand side goes to zero as \( p \to \infty \). Then,
\[
|M_r(u_p)(x) - M_r(u)(x)| \to 0 \text{ as } p \to \infty,
\]
proving (2.6).
Let us now prove (2.7). For every $x \in \mathbb{R}^n$, we have

$$|N_r(f_p)(x) - N_r(f)(x)| \leq |N_r(|f_p - f|)(x)| \leq \sup_K |f_p - f|Q_r(x).$$

Then, for every compact set $G \subseteq \mathbb{R}^n$,

$$\sup_G |N_r(f_p) - N_r(f)| \leq \sup_K |f_p - f| \sup_{x \in G} |Q_r(x)| \leq (\text{by (2.2)}) \ C_r(G) \sup_K |f_p - f|.$$

So we have proved that $(N_r(f_p))$ is uniformly convergent to $N_r(f)$ on every compact subset of $\mathbb{R}^n$. This, in particular, implies (2.7).

2.5. In this subsection we complete the proof of Theorem 1.5. To this end we first remark that, thanks to (2.4), for every $x \in \mathbb{R}^n$, we have

$$M_r(u)(x) - u(x) = -N_r(f)(x)Q_r(x),$$

so that, as $f(x)$ is constant with respect to $y \in \Omega_r(x)$,

$$\frac{|M_r(u)(x) - u(x)|}{Q_r(x)} + f(x) \leq \frac{1}{Q_r(x)} |N_r(f(x) - f)(x)| \leq \sup_{y \in \Omega_r(x)} |f(u) - f(y)|Q_r(x).$$

By Corollary 2.5 and Remark 2.3, the left hand side of the previous inequality goes to zero as $r \to 0$. Hence,

$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = -f(x)$$

for every $x \in \mathbb{R}^n$. This completes the proof of Theorem 1.5.

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