VOLUME BOUNDS OF THE RICCI FLOW ON CLOSED MANIFOLDS

CHIH-WEI CHEN AND ZHENLEI ZHANG

Abstract. Let \{g(t)\}_{t \in [0,T]} be the solution of the Ricci flow on a closed Riemannian manifold \(M^n\) with \(n \geq 3\). Without any assumption, we derive lower volume bounds of the form \(\text{Vol}_g(t) \geq C(T - t)^{\frac{2}{n}}\), where \(C\) depends only on \(n\), \(T\) and \(g(0)\). In particular, we show that

\[ \text{Vol}_g(t) \geq e^{T \lambda - \frac{2}{n}} \left( \frac{4}{(A - r + 4B)T} \right)^{\frac{n}{2}} (T - t)^{\frac{n}{2}}, \]

where \(r := \inf_{\|\phi\|_2 = 1} \int_M R \phi^2 \text{dvol}_g(0)\), \(\lambda := \inf_{\|\phi\|_2 = 1} \int_M 4|\nabla \phi|^2 + R \phi^2 \text{dvol}_g(0)\) and \(A, B\) are Sobolev constants of \((M, g(0))\). This estimate is sharp in the sense that it is achieved by the unit sphere with scalar curvature \(R_{g(0)} = n(n - 1)\) and \(A = \frac{4}{n(n - 2)} \omega_n^{\frac{n}{2}}\), \(B = \frac{n - 2}{n} \omega_n^{\frac{n}{2}}\).

On the other hand, if the diameter satisfies \(\text{diam}_g(t) \leq c_1 \sqrt{T - t}\) and there exist a point \(x_0 \in M\) such that \(R(x_0, t) \leq c_2(T - t)^{-1}\), then we have \(\text{Vol}_g(t) \leq C(T - t)^{\frac{n}{2}}\) for all \(t > \frac{T}{2}\), where \(C\) depends only on \(c_1, c_2, n, T\) and \(g(0)\).

1. Introduction

Let \((M^n, g)\) be a closed Riemannian manifold with dimension \(n \geq 3\) and \(A\) and \(B\) be any Sobolev constants of \((M^n, g)\), i.e.,

\[ \left( \int_M |u|^{\frac{2n}{n-2}} \text{dvol} \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|^2 \text{dvol} + B \int_M u^2 \text{dvol} \]

for all \(u \in W^{1,2}(M)\). In [ZZha07], one of us observed that the Ricci flow on a closed manifold has a volume lower bound in terms of Sobolev constants. Especially, when \(\lambda := \inf_{\|\phi\|_2 = 1} \int_M 4|\nabla \phi|^2 + R \phi^2 \text{dvol}_g(0)\) is non-positive, one obtains \(\text{Vol}_g(t) \geq C_1 e^{-C_2 t}\) for all \(t > 0\), where \(C_1\) and \(C_2\) depend only on \(n\) and \(g(0)\). It means that the manifold cannot extinct at finite time and every blow up limit must be non-compact.

On the other hand, for positive \(\lambda\), R. Ye [Ye07] derived the following volume lower bound by using the estimate of \(A(t)\) and \(B(t)\):

**Proposition 1** (R. Ye). Assume that \(\lambda\) is positive. Then we have for any time \(t \in [0, T)\)

\[ \text{Vol}_g(t) \geq e^{-\frac{t}{4} - C} \text{ when } \bar{R}(t) \leq 0 \]

and

\[ \text{Vol}_g(t) \geq e^{-\frac{t}{4} - C \bar{R}(t)^{-\frac{n}{2}}} \text{ when } \bar{R}(t) > 0, \]

where \(\bar{R}(t) := \int R \text{dvol}_g(t)\) and \(C\) depends on \(n, g_0, A, B, \lambda, \text{Vol}_g(0)\) and \(\max R_{g(0)}^-\).

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Note that the constants $C$’s in Zhang’s estimate depend only on the initial metric $g(0)$, while the constants in Ye’s estimate, namely for the case $\lambda > 0$, might depend on $R(t)$.

In this article, we find a unified way to derive several volume bounds, whose proofs do not rely on the definite sign of $\lambda$.

**Theorem 1.** Let $\{g(t)\}_{t \in [0, T]}$ be the solution of the Ricci flow on a closed Riemannian manifold $M^n$ with $n \geq 3$ and $A, B$ be Sobolev constants of $(M, g(0))$. Then

$$\Vol_{g(t)} \geq e^{\nu \lambda n^2 t} e^{\nu(A(\lambda - r) + 4B) a_\nu^2 \left( \frac{T - t}{T} \right)^{\frac{2}{n}}},$$

for all $a \in (0, \frac{8T}{nA})$, where $r := \inf_{\phi \in \mathcal{H}} \int_M R \phi^2 \, d\Vol_{g(0)}$ and $\lambda := \inf_{\phi \in \mathcal{H}} \int_M 4|\nabla \phi|^2 + R \phi^2 \, d\Vol_{g(0)}$. In particular, when $a = \frac{8T}{nA}$, we obtain a lower bound

$$\Vol_{g(t)} \geq e^{T(r - 4BA^{-1})} \left( \frac{8}{nA} \right)^{\frac{2}{n}} \left( T - t \right)^{\frac{2}{n}},$$

which does not depend on $\lambda$.

When choosing $B \geq \frac{8A}{8T}$ and $a = 4(A\lambda - r + 4B)^{-1}$, we have the following theorem which shows that our estimate is sharp.

**Theorem 2.** Let $\{g(t)\}_{t \in [0, T]}$ be the solution of the Ricci flow on a closed Riemannian manifold $M^n$ with $n \geq 3$. Suppose that $A$ and $B \geq \frac{8A}{8T}$ are Sobolev constants of $(M, g(0))$. Then

$$\Vol_{g(t)} \geq e^{T \nu \lambda n^2 t} \left( \frac{4}{A(\lambda - r) + 4B} \right)^{\frac{2}{n}} \left( T - t \right)^{\frac{2}{n}}.$$

The bound is achieved when $(M, g(0))$ is the unit sphere with $A = \frac{4}{n(n-2)d_{n-1}}$ and $B = \frac{n-1}{n(n-2)}d_{n-1}^{-2}$.

The proof of Theorem 2 is based on the monotonicity of Perelman’s $\mu$-entropy. Recall that $\mu$-entropy is defined by

$$\mu(g(t), \tau(t)) := \inf_{\phi \in \mathcal{H}} \int_M \left[ \tau(4|\nabla \phi|^2 + R \phi^2) - \phi^2 \ln \phi^2 - n \frac{\phi^2}{\Vol_{g(t)}} \right] \, d\Vol_{g(t)},$$

for all $\phi(\cdot, t) \in W^{1,2}(M)$. When fixing $t$ and choosing $\phi^2$ to be the constant $(4\pi\tau)^{-\frac{n}{2}}\Vol_{g(t)}^{-1}$, one obtains

$$\mu(g(t), \tau(t)) \leq \int_M \tau R \, d\Vol_{g} - \ln \left( \frac{(4\pi\tau)^{\frac{n}{2}}}{\Vol_{g(t)}} \right) - n = -\tau \left( \ln \Vol_{g(t)} \right)' - \ln \left( \frac{(4\pi\tau)^{\frac{n}{2}}}{\Vol_{g(t)}} \right) + n$$

This shows that the evolution of volume is closely related to the $\mu$-entropy. The relationship between $\mu$ and $\Vol$ has been studied by one of the authors in [ZZha07], especially for manifolds with $\lambda < 0$. Here we derive results for generic manifolds.

**Theorem 3.** Let $\{g(t)\}_{t \in [0, T]}$ be the solution of the Ricci flow on a closed Riemannian manifold $M^n$ with $n \geq 2$. Denote $\mu = \inf_{\phi \in \mathcal{H}} \Vol(g(0), \phi, T)$. Then

$$\Vol_{g(t)} \geq (4\pi)^{\frac{n}{2}} e^{\mu + \frac{n}{2}(T - t)^{\frac{1}{2}}}.$$
As a consequence of the volume lower bound, for any closed Riemannian manifold \((M, g)\), one has
\[
\mu(g, T) \leq -\frac{n}{2} + \ln \text{Vol}_g(M) - \frac{n}{2} \ln 4\pi T.
\]
In particular, for any Ricci flow defined on a closed manifold, the maximal time \(T\) cannot exceed \((4\pi e)^{-1} (e^{-\mu(0, T)} \text{Vol}_g(0))^{\frac{2}{n}}\).

If we further assume some controls on diameter and curvature, we derive the following upper bound for volume.

**Proposition 2.** Let \(\{g(t)\}_{t \in [0, T]}\) be the solution of the Ricci flow on a closed Riemannian manifold \((M^n, g(0))\) with \(n \geq 3\). If the diameter satisfies \(\text{diam}_{g(t)} \leq c_1 \sqrt{T - t}\) and there exist a point \(x_0 \in M\) such that \(R(x_0, t) \leq c_2 (T - t)^{-1}\), then we have
\[
\text{Vol}_{g(t)} \leq C(T - t)^{\frac{2}{n}} \quad \text{for all} \ t > \frac{T}{2}, \text{where} \ C \text{ depends only on} \ c_1, c_2, n, T \text{ and} \ g(0).
\]

If the curvature condition in Proposition 2 is replaced by the stronger one that \(R(x, t) \leq c_2 (T - t)^{-1}\) for all \(x \in M\), then the theorem follows directly from Q. Zhang’s Theorem 1.1 in [QZha12]. However, by mimicking Zhang’s argument carefully, one can see that the Type I curvature assumption can be reduced as \(R(x_0, t) \leq c_1(T - t)^{-1}\) for some \(x_0 \in M\). For the reader’s convenience, we include an outline of Zhang’s proof in Section 6.

The theorem above relates to the following conjecture. Roughly speaking, we suspect that \(R\) cannot be of Type II at every point on a manifold which shrinks to a point along the Ricci flow.

**Conjecture.** Let \(\{g(t)\}_{t \in [0, T]}\) be the solution of the Ricci flow on a closed Riemannian manifold \(M^n\) with \(n \geq 3\) and \(\text{diam}_{g(t)} \to 0\) as \(t \to T\). Then
\[
\liminf_{t \to T} (T - t)R(x, t) < \infty
\]

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and equivalently
\[
\sup_{\|\phi\|^2 = (4\pi \tau(t))^\frac{n}{2}} -\mathcal{W}(g(t), \phi, \tau(t)) \leq \sup_{\|\phi\|^2 = (4\pi \tau(0))^\frac{n}{2}} -\mathcal{W}(g(0), \phi, \tau(0)).
\]

Hence
\[
\sup_{\|\phi\|^2 = (4\pi \tau(t))^\frac{n}{2}} (4\pi \tau(t))^{-\frac{n}{2}} \int_M \left[ \phi^2 \ln \phi^2 + n\phi^2 - \tau(t)(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(t) \\
\leq \sup_{\|\phi\|^2 = (4\pi \tau(0))^\frac{n}{2}} (4\pi \tau(0))^{-\frac{n}{2}} \int_M \left[ \phi^2 \ln \phi^2 + n\phi^2 - \tau(0)(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(0).
\]

From now on, we denote \( \tau = \tau(t) \), \( \tau_0 = \tau(0) \) and \( V(t) = \text{Vol}_g(t) \). Considering the (spatial) constant function \( \phi^2 = (4\pi \tau)^\frac{n}{2} V^{-1} \) at time \( t \), one derives
\[
\int_M \left[ \ln \left( \frac{4\pi \tau}{V(t)} \right)^\frac{n}{2} + n - \tau R \right] d\text{vol}_g(t)
\]
\[
\leq \sup_{\|\phi\|^2 = (4\pi \tau_0)^\frac{n}{2}} (4\pi \tau_0)^{-\frac{n}{2}} \int_M \left[ \phi^2 \ln \phi^2 + n\phi^2 - \tau_0(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(0)
\]
\[
= \sup_{\|\phi\|^2 = 1} \int_M \left[ \phi^2 (\ln \phi^2 + \ln(4\pi \tau_0)^\frac{n}{2}) + n\phi^2 - \tau_0(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(0),
\]
i.e.,
\[
-\ln V(t) + \ln(4\pi \tau)^\frac{n}{2} - \tau \int_M R \ d\text{vol}_g(t)
\]
\[
\leq \sup_{\|\phi\|^2 = 1} \int_M \left[ \phi^2 \ln \phi^2 - \tau_0(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(0) + \ln(4\pi \tau_0)^\frac{n}{2}.
\]

On the other hand, since \( \tau' = -1 \) and \( \frac{d}{dt} V(t) = -\int_M R \ d\text{vol} \), one has
\[
\frac{d}{dt} (\tau \ln V(t)) = -\ln V(t) - \tau \int_M R \ d\text{vol}.
\]
Therefore,
\[
\frac{d}{dt} (\tau \ln V(t)) \leq \sup_{\|\phi\|^2 = 1} \int_M \left[ \phi^2 \ln \phi^2 - \tau_0(4|\nabla \phi|^2 + R\phi^2) \right] d\text{vol}_g(0) - \ln \left( \frac{\tau}{\tau_0} \right)^\frac{n}{2}.
\]

Inequalities (1) and the right hand side of (2) were observed and used before, see for example [QZha07, Ye07]. The left hand side of (2) also occurred in a more general form in [QZha07] and [QZha12]. However, they were used for tracing the evolution of Sobolev constants or fundamental solutions of the (conjugate) heat equation, instead of the global volume function \( V(t) \). In Ye’s Proposition (cf. Proposition 1 in the introduction), Ye needs the positivity assumption to make sure that \( A(t) \) and \( B(t) \) are under control along the Ricci flow. Indeed, as pointed out by Ye, when the assumption \( \lambda > 0 \) is removed, Hamilton-Isenberg’s example shows that local volume could collapse, which means that \( A(t) \) and \( B(t) \) must become wild. So Ye guessed that the positivity assumption of \( \lambda \) is indispensable [Ye07, p. 4]. However, we show that, although the Sobolev constants and the local volume could be bad, the global volume remains under control.
\textbf{Theorem 1.} Let \( \{g(t)\}_{t \in [0,T]} \) be the solution of the Ricci flow on a closed Riemannian manifold \( M^n \) with \( n \geq 3 \) and \( A, B \) be Sobolev constants of \( (M, g(0)) \). Then

\[ \text{Vol}_{g(t)} \geq e^{T \lambda - \frac{8n}{A} (A(\lambda - r) + 4B) a} (\frac{T - t}{T})^\frac{2}{n} \]

for all \( a \in (0, \frac{8n}{A}) \), where \( r := \inf_{\|\phi\|_2^2 = 1} \int_M R\phi^2 \ dvol_{g(0)} \) and \( \lambda := \inf_{\|\phi\|_2^2 = 1} \int_M 4|\nabla \phi|^2 + R\phi^2 \ dvol_{g(0)} \). In particular, when \( a = \frac{8n}{A} \), we obtain a lower bound

\[ \text{Vol}_{g(t)} \geq e^{T(r-4BA^{-1})} \left( \frac{8}{nA} \right)^\frac{n}{2} (T - t)^\frac{2}{n} \]

which does not depend on \( \lambda \).

\textbf{Proof.} For all \( \phi \in W^{1,2}(M) \), the Sobolev inequality implies that

\[ \int_M \phi^2 \ln \phi^2 \ dvol_{g(0)} \leq \frac{n}{2} \ln \left( A \int_M |\nabla \phi|^2 \ dvol_{g(0)} + B \int_M \phi^2 \ dvol_{g(0)} \right) \]

\[ \leq \frac{n}{2} a \left( A \int_M |\nabla \phi|^2 \ dvol_{g(0)} + B \int_M \phi^2 \ dvol_{g(0)} \right) - \frac{n}{2} \ln a - \frac{n}{2} \]

\[ = \frac{naA}{8} \int_M 4|\nabla \phi|^2 \ dvol_{g(0)} + \frac{naB}{2} \int_M \phi^2 \ dvol_{g(0)} - \frac{n}{2} \ln a - \frac{n}{2} \]

Note that the second inequality follows from the fact \( \ln x \leq ax - \ln a - 1 \). Hence

\[ \int_M \left[ \phi^2 \ln \phi^2 - \tau_0 (4|\nabla \phi|^2 + R\phi^2) \right] \ dvol_{g(0)} \]

\[ \leq \left( \frac{naA}{8} - \tau_0 \right) \int_M 4|\nabla \phi|^2 + R\phi^2 \ dvol_{g(0)} - \frac{naA}{8} \int_M R\phi^2 \ dvol_{g(0)} \]

\[ + \frac{naB}{2} \int_M \phi^2 \ dvol_{g(0)} - \frac{n}{2} \ln a - \frac{n}{2} \]

Since \( a \leq \frac{8n}{A} \), \( \frac{naA}{8} - \tau_0 \) is nonpositive and we have

\[ \sup_{\|\phi\|_2^2 = 1} \int_M \left[ \phi^2 \ln \phi^2 - \tau_0 (4|\nabla \phi|^2 + R\phi^2) \right] \ dvol_{g(0)} \]

\[ \leq \left( \frac{naA}{8} - \tau_0 \right) \lambda - \frac{naA}{8} r + \frac{naB}{2} - \frac{n}{2} \ln a - \frac{n}{2} \]

\[ = - \tau_0 \lambda + \frac{naA}{8} (\lambda - r) + \frac{naB}{2} - \frac{n}{2} \ln a - \frac{n}{2} \]

where \( \lambda := \inf_{\|\phi\|_2^2 = 1} \int_M 4|\nabla \phi|^2 + R\phi^2 \ dvol_{g(0)} \) and \( r := \inf_{\|\phi\|_2^2 = 1} \int_M R\phi^2 \ dvol_{g(0)} \).

Applying it to the key inequality (2), we obtain

\[ \frac{d}{dt} (\tau \ln V(t)) \leq \sup_{\|\phi\|_2^2 = 1} \int_M \left[ \phi^2 \ln \phi^2 - \tau_0 (4|\nabla \phi|^2 + R\phi^2) \right] \ dvol_{g(0)} - \ln \left( \frac{\tau}{\tau_0} \right)^\frac{2}{n} \]

\[ \leq - \tau_0 \lambda + \frac{naA}{8} (\lambda - r) + \frac{naB}{2} - \frac{n}{2} \ln a - \frac{n}{2} - \ln \left( \frac{\tau}{\tau_0} \right)^\frac{2}{n} \]

Taking \( \tau = T - t \) and integrating the inequality from \( t \) to \( T \), we have

\[- \ln V(t) \leq -T \lambda + \frac{naA}{8} (\lambda - r) + \frac{naB}{2} - \ln \left( \frac{a}{T} \right)^\frac{2}{n} - \ln (T - t)^\frac{2}{n} , \]
is easy to compute that 

\[ T = A \]

curvatures are 1. In [HV96], Hebey and Vaugon showed that one can always choose a manifold \( M \). Let \( T \) holds on the unit sphere when we choose \( e \). Theorem 2 becomes

\[ B \]

not necessarily given by the best Sobolev constants. Sobolev constants (cf. [DH02]) and show that this lower bound can be attained by \( e \). Theorem 1 becomes when \( B \) is chosen to be large, say \( B > \frac{8T}{nA} \). This fact shows that the best choice of \( a \) is not necessarily given by the best Sobolev constants.

The best choice of \( a \), where \( f(a) \) attains it maximum, makes the lower bound in Theorem 1 becomes \( e^{T \lambda - \frac{\omega_n}{2} a^2} \left( \frac{T - t}{T} \right)^{\frac{n}{2}} \). We shall recall some facts from the theory of Sobolev constants (cf. [DH02]) and show that this lower bound can be attained by the shrinking sphere. From now on we consider closed Riemannian manifolds with dimension \( n \geq 3 \) and denote \( \omega_n \) as the volume of the unit sphere, whose sectional curvatures are 1. In [HV96], Hebey and Vaugon showed that one can always choose \( A = \frac{4}{n(n-2)} \omega_n^{-\frac{2}{n}} \) for a given \((M^n, g)\) so that the Sobolev inequality holds. Namely, there exists a constant \( B > 0 \) such that

\[
\left( \int_M |u|^{\frac{2n}{n-2}} \, d\text{vol} \right)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)} \omega_n^{-\frac{2}{n}} \int_M |\nabla u|^2 \, d\text{vol} + B \int_M u^2 \, d\text{vol}
\]

for all \( u \in W^{1,2}(M) \). The infimum of all the \( B \)'s which make the inequality valid is called the best \( B \)-constant and is denoted by \( B_0 \). For the unit sphere, a well-known result due to T. Aubin [Aus76] states that \( B_0 = \omega_n^{-\frac{2}{n}} \). Hence, the Sobolev inequality holds on the unit sphere when we choose \( B = \frac{n-1}{n-2} \omega_n^{-\frac{2}{n}} > B_0 \) and we have

**Theorem 2.** Let \( \{g(t)\}_{t \in [0,T]} \) be the solution of the Ricci flow on a closed Riemannian manifold \( M^n \) with \( n \geq 3 \). Suppose that \( A \) and \( B \geq \frac{nA}{2} \) are Sobolev constants of \((M, g(0))\). Then

\[
\text{Vol}_{g(t)} \geq e^{T \lambda - \frac{\omega_n}{2}} \left( \frac{4}{A(\lambda - r) + 4B} \right)^{\frac{n}{2}} \left( \frac{T - t}{T} \right)^{\frac{n}{2}} .
\]

The bound is achieved when \((M, g(0))\) is the unit sphere with \( A = \frac{4}{n(n-2)} \omega_n^{-\frac{2}{n}} \) and \( B = \frac{n-1}{n-2} \omega_n^{-\frac{2}{n}} \).

*Proof.* The first statement follows easily from Theorem 1 by taking \( a = 4(A(\lambda - r) + 4B)^{-1} \).

For the second statement, we consider the shrinking sphere with \( R_{g(0)} = n(n-1) \), it is easy to compute that \( T = \frac{1}{2(n-1)} \) and \( \text{Vol}_{g(t)} = (2(n-1))^{\frac{n}{2}} \omega_n (T - t)^{\frac{n}{2}} \). On the other
As a consequence of the volume lower bound, for any closed Riemannian manifold \( (M, g) \) with \( n \geq 2 \), the lower bound becomes
\[
e^{-\pi(n-1)/2} \left( 2(n-1) \frac{4}{A(\lambda - r) + 4B} \right)^{\frac{2}{n}} (T - t)^{\frac{2}{n}} = (2(n-1))^{\frac{n}{2}} \omega_n (T - t)^{\frac{n}{2}}
\]
because \( \lambda = r = R = n(n-1) \).

4. LOWER BOUNDS INVOLVING \( \mu \)

In this section, we derive lower and upper bounds of global volume in terms of \( \mu(g(0), T) \). Since we do not interpret \( \mu \) by using Sobolev constants, all the results in this section hold for \( n \geq 2 \), instead of \( n \geq 3 \).

**Theorem** Let \( \{g(t)\}_{t \in [0,T]} \) be the solution of the Ricci flow on a closed Riemannian manifold \( M^n \) with \( n \geq 2 \). Denote \( \mu = \inf_{\|\phi\|^2 = (4\pi T)^{n/2}} W(g(0), \phi, T) \). Then
\[
\text{Vol}_{g(t)} \geq (4\pi)^{\frac{n}{2}} e^{\mu + \frac{\mu}{2}} (T - t)^{\frac{n}{2}}.
\]
As a consequence of the volume lower bound, for any closed Riemannian manifold \( (M, g) \), one has
\[
\mu(g, T) \leq -\frac{n}{2} + \ln \text{Vol}_g(M) - \frac{n}{2} \ln 4\pi T.
\]
In particular, for any Ricci flow defined on a closed manifold, the maximal time \( T \) cannot exceed \( (4\pi e)^{-1} (e^{-\mu(g(0), T)} \text{Vol}_{g(0)})^{\frac{n}{2}} \).

**Proof.** Recall that Perelman’s \( \mu \)-entropy
\[
\mu(g(t), \tau) := \inf_{\|\phi\|^2 = (4\pi T)^{n/2}} W(g(x, t), \phi(x, t), \tau(t))
\]
\[
= \inf_{\|\phi\|^2 = (4\pi T)^{n/2}} (4\pi T)^{-\frac{n}{2}} \int_M \left[ \tau(4|\nabla \phi|^2 + R\phi^2) - \phi^2 \ln \phi^2 - n\phi^2 \right] d\text{vol}_g,
\]
is non-decreasing along the Ricci flow for any \( \tau(t) \) and \( \phi \) such that \( \tau' = -1 \) and \( \frac{\partial}{\partial \tau} \phi^2 = -\Delta \phi^2 + (R - \frac{\partial}{\partial \tau}) \phi^2 \).

Denote \( V(t) = \text{Vol}_{g(t)}, \tau_0 = \tau(0), \tau = \tau(t) \), and consider \( \phi^2 = (4\pi T)^{\frac{n}{2}} V(t)^{-1} \) at time \( t \). So, by the monotonicity of \( \mu \), one has
\[
\mu(g(0), \tau_0) \leq \mu(g(t), \tau)
\]
\[
\leq (4\pi T)^{-\frac{n}{2}} \int_M \left[ (\tau R\phi^2 - \phi^2 \ln \phi^2 - n\phi^2) \right] d\text{vol}_{g(t)}
\]
\[
= \int_M \left[ \tau R + \ln V(t) \right] d\text{vol}_{g(t)} - \ln (4\pi T)^{\frac{n}{2}} - n
\]
\[
= -\tau \frac{d}{dt} (\ln V(t)) + \ln V(t) - \ln (4\pi T)^{\frac{n}{2}} - n
\]
and thus
\[
-\frac{d}{dt} (\tau \ln V(t)) \geq \frac{n}{2} \ln \tau + \frac{n}{2} \ln (4\pi) + n + \mu(g(0), \tau_0).
\]
Taking \( \tau = T - t \) and integrating the inequality from \( t \) to \( T \), we obtain
\[
V(t) \geq (4\pi)^{\frac{n}{2}} e^{\mu(g(0), T) + \frac{\mu}{2}} (T - t)^{\frac{n}{2}}.
\]

\[\square\]
One may compare the upper bound of $\mu$ with a former result given by one of the authors as follows.

**Proposition 3** ([ZZha07], cf. [CCG+10]). For any closed Riemannian manifold $(M^n, g)$, one has

$$\mu(g, T) \leq -n + T\lambda + e^{-1}\text{Vol}_g(M) - \frac{n}{2}\ln 4\pi T.$$ 

Moreover, if $\lambda \leq 0$, then

$$\mu(g, T) \leq -n + e^{-1} + \ln \text{Vol}_g(M) - \frac{n}{2}\ln 4\pi T.$$ 

For the reader’s convenience, we recall the proof.

**Proof.** The first inequality comes from the definition of $\mu$ and the fact $-x\ln x \leq e^{-1}$ for all $x \geq 0$. When $\lambda < 0$, we can simply remove the term $T\lambda$. However, a rescaling argument can do a better job. Indeed, since $\mu(g, T) = \mu(Qg, QT)$ for any $Q \in \mathbb{R}$, when choosing $Q = \sqrt[2]{\text{Vol}}_g$, one has $\sqrt[2]{\text{Vol}}_Q(M) = 1$,

$$\mu(Qg, QT) \leq -n + QT\lambda + e^{-1}\text{Vol}_Q(M) - \frac{n}{2}\ln 4\pi QT \leq -n + e^{-1} - \frac{n}{2}\ln 4\pi QT$$

and thus the proposition is proved. $\square$

More discussions about the behavior of $\mu$ and its applications can be found in [CCG+10, Chapter 17].

5. Upper bounds

Let $g(t), t \in [0, T)$, be the solution of the Ricci flow on a closed Riemannian manifold $(M^n, g(0))$. Along this flow, consider the heat kernel $G(x, t; y, s)$ for the heat operator $\partial_t - \Delta_x$. Namely, fixing $y$ and $s$, $u(x, t) := G(x, t; y, s)$ satisfies

$$\partial_t u(x, t) = \Delta_x u(x, t) \quad \text{and} \quad \lim_{t \to s} u(x, t) = \delta_y(x).$$

One can consult Chow et al.’s book [CCG+10, Ch. 24] for more details about the heat kernel. In [QZha12, pp. 247, 251], Q. Zhang derived the following two-sided bound for the integral heat kernel:

$$(1 + C(t - s))^2 \geq \int_M G(x, t; y, s) \, d\mu_{g(s)}(x) \geq \frac{C}{(t - s)^2} \exp \left( -C \frac{\text{dist}_{g(t)}^2(x, y)}{t - s} - \frac{1}{2}\sqrt{t - s} \int_s^t \sqrt{t - \tau} R(x_0, \sigma) d\sigma \right),$$

where $C$’s are constants depending on $n, T$ and $g(0)$. Hence, by our assumptions on the scalar curvature and the diameter, one obtains the following theorem, which is essentially due to Q. Zhang in [QZha12].

**Proposition 2** (cf. Theorem 1.1 (a) in [QZha12]). Let $\{g(t)\}_{t \in [0, T)}$ be the solution of the Ricci flow on a closed Riemannian manifold $(M^n, g(0))$ with $n \geq 3$. If the diameter satisfies $\text{diam}_{g(t)} \leq c_1\sqrt{T - t}$ and there exist a point $x_0 \in M$ such that $R(x_0, t) \leq c_2(T - t)^{-1}$, then we have $\text{Vol}_{g(t)} \leq C(T - t)^{2/3}$ for all $t > \frac{T}{2}$, where $C$ depends only on $c_1, c_2, n, T$ and $g(0)$. 

Proof. The proof is adapted from Zhang’s local volume estimate in [QZha12]. The reader should be careful on tracing the dependence of the constant $C$, which varies line by line, in the following bounds.

Recall that $R(x, t)$ is either nonnegative, or negative somewhere and bounded below by the negative function $(\frac{1}{\min R_g(0)} - \frac{2t}{n})^{-1}$ for all $t > 0$. Moreover, since the manifold is closed and $\frac{d}{dt}d\mu_{g(t)} = -Rd\mu_{g(t)}$, one can derive

$$\frac{d}{dt}\int_M u(x, t) \, d\mu_{g(t)} = \int_M \Delta_x u(x, t) - Ru(x, t) \, d\mu_{g(t)} = -\int_M Ru(x, t) \, d\mu_{g(t)}$$
and thus either $\frac{d}{dt} \int_M u(x, t) \, d\mu_{g(t)} \leq 0$ or

$$\frac{d}{dt} \int_M u(x, t) \, d\mu_{g(t)} \leq \frac{n}{2} \left( t - \frac{n}{2 \min R_g(0)} \right)^{-1} \int_M u(x, t) \, d\mu_{g(t)}.$$

Integrating it from $s$ to $t$, we obtain either

$$\int_M u(x, t) \, d\mu_{g(t)} \leq \lim_{t \searrow s} \int_M u(x, t) \, d\mu_{g(t)} = 1$$
or

$$\int_M u(x, t) \, d\mu_{g(t)} \leq \left( t - \frac{n}{2 \min R_g(0)} \right)^{-n/2} \leq (1 + C_1(t-s))^n,$$

where $C_1 = (\frac{n}{2 \min R_g(0)})^{-1}$. Thus the upper bound for the integral heat kernel is obtained.

We claim that $G(x, t; x_0, s)$ is bounded pointwise from below by $C_2(t-s)^{-\frac{n}{2}}$ for all $t \geq \frac{T + s}{2}$, where $C_2$ depends on $c_1, c_2, n, T$ and $g(0)$. Therefore, combining with the upper bound above, we have

$$(1 + C_1(t-s))^{n/2} \geq \int_M G(x, t; x_0, s) \, d\mu_{g(t)}(x) \geq C_2(t-s)^{-\frac{n}{2}} \Vol_{g(t)}$$
and thus

$$\Vol_{g(t)} \leq C \left( (t-s) + (t-s)^2 \right)^{\frac{n}{2}} \leq C(t-s)^{\frac{n}{2}},$$

where the last $C$ depends only on $n, C_1, C_2$ and $T$. This upper bound holds for all fixed $s$ and all $t \in [\frac{T+s}{2}, T]$, so we may choose $t = \frac{T+s}{2}$ and derive the conclusion

$$\Vol_{g(t)} \leq C(T-t)^{\frac{n}{2}} \text{ for all } t > T/2.$$

Now we complete the proof by verifying the claim that $G(x, t; x_0, s) \geq C_2(t-s)^{-\frac{n}{2}}$. Note that, fixing $x$ and $t$, $v(y, s) := G(x, t; y, s)$ satisfies the backward conjugate heat equation $\partial_s v = -\Delta v + Rv$ and thus the function $f(y, s)$ defined by $(4\pi\tau)^{-\frac{n}{2}}e^{-f} = v$ satisfies $-f_s = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$, where $\tau = t-s$. Combining with Perelman’s estimate $\tau(2\Delta f - |\nabla f|^2 + R) + f - n \leq 0$ (cf. [Per02 Corollary 9.4]), one has

$$-f_s \leq \frac{1}{2} R - \frac{1}{2} |\nabla f|^2 + \frac{1}{2\tau} f \leq \frac{1}{2} R - \frac{1}{2\tau} f, \quad \text{i.e.,} \quad -(\sqrt{\tau} f)_s \leq \frac{1}{2} \sqrt{\tau} R.$$ Integrating from $s$ to $t$, one has

$$\sqrt{t-s} f(y, s) \leq \lim_{\sigma \to t} \sqrt{t-\sigma} f(y, \sigma) + \frac{1}{2} \int_s^t \sqrt{t-\sigma} R(y, \sigma) \, d\sigma.$$
Because $G(x, t; y, \sigma)$ behaves like $(t - \sigma)^{-\frac{n}{2}}$ as $\sigma \to t$ whenever $x = y$ (cf. [CCG+10, Ch. 24]), $f(x, \sigma)$ is uniformly bounded as $\sigma \to t$, thus $\lim_{\sigma \to t} \sqrt{t - \sigma} f(x, \sigma) = 0$ and

$$\sqrt{t - s} f(x, s) \leq \frac{1}{2} \int_s^t \sqrt{t - \sigma} R(x, \sigma) d\sigma.$$  

Since $x$ can be chosen arbitrarily, one may take $x = x_0$ in the beginning and obtain

$$-f(x_0, s) \geq -\frac{1}{2\sqrt{T - s}} \int_s^t \sqrt{t - \sigma} R(x_0, \sigma) d\sigma \geq -c_2.$$  

So

$$G(x_0, t; x_0, s) = (4\pi(t - s))^{-\frac{n}{2}} e^{-c_2}.$$  

Moreover, by gradient estimate of heat equation along the Ricci flow (cf. [QZha06, (3.44)]), one can compare $v(x, s)$ with $v(x_0, s)$, i.e.,

$$G(x, t; x_0, s) \geq C_3 K^{-1} \exp \left( -2C_4 \frac{\text{dist}^2_{g(t)}(x, x_0)}{t - s} \right) (G(x_0, t; x_0, s))^2$$  

for all $x \in M$, where $K$ is the upper bound of $G$ and $C_3, C_4$ are universal constants. In [QZha12 (1.5)], it was proved that $K \leq C_5(t - s)^{-\frac{n}{2}}$. Therefore, using $\text{diam}_{g(t)} \leq c_1\sqrt{T - t}$ and $t \geq \frac{T + \epsilon}{2}$, one has

$$G(x, t; x_0, s) \geq C_3 K^{-1} \exp \left( -2C_4 \frac{\text{dist}^2_{g(t)}(x, x_0)}{t - s} \right) (G(x_0, t; x_0, s))^2$$  

$$\geq C_3(4\pi)^{-n} e^{-2c_2} C_5^{-1} \exp \left( -2C_4 \frac{2c_1^2(T - t)}{T - s} \right) (t - s)^{-\frac{n}{2}}$$  

$$\geq C_2(t - s)^{-\frac{n}{2}}, \text{ where } C_2 = (4\pi)^{-n} C_3 C_5^{-1} e^{-2c_2 - 4c_1^2 T}.$$  

The claim is verified for $C_2$ depending only on $c_1, c_2, n, T$ and $g(0).$  

**Remark 1.** Suppose that $\int R(t) \ dvol_{g(t)} \geq \frac{n}{2}(T - t)^{-1}$ for all $t > \frac{T}{2}$. Then

$$\frac{d}{dt} \text{Vol}_{g(t)} = -\int_M R \ dvol_{g(t)} \leq -\frac{n}{2}(T - t)^{-1} \text{Vol}_{g(t)}$$  

implies that $\text{Vol}_{g(t)} \leq C(T - t)^{\frac{n}{2}}$ for all $t > \frac{T}{2}$, where $C = \left(\frac{2}{T}\right)^{\frac{n}{2}} \text{Vol}_{g(\frac{T}{2})}$. This might help to remove the curvature assumption in Theorem 3. We remind the reader that, when $\int R(t) \ dvol_{g(t)} \geq \frac{n}{2}(T - t)^{-1}$ for some $t_k \to T$, there must exist a sequence of points $(x_k, t_k)$ with Type I blow-up scalar curvature. However, this is insufficient for us to apply Theorem 3 because we need a fixed point $x_0$.

**Remark 2.** For generic Ricci flows, it is not hard to see that volume grows at most polynomially. Indeed, this is trivial when $R_{g(0)} \geq 0$. For $\min R_{g(0)} < 0$, by using $\frac{d}{dt} R \geq \Delta R + \frac{2}{n} R^2$, one can show that $R_{g(t)} \geq \left(\frac{1}{\min R_{g(0)}} - \frac{2t}{n}\right)^{-1}$ and thus $\frac{d}{dt} \ln \text{Vol}_{g(t)} = -\int_M R \ dvol_{g(t)} \leq \left(\frac{2t}{n} - \frac{1}{\min R_{g(0)}}\right)^{-1}$. Hence

$$\text{Vol}_{g(t)} \leq \text{Vol}_{g(0)} \left(\min R_{g(0)}\right)^{\frac{n}{2}} \left(\frac{2t}{n} - \frac{1}{\min R_{g(0)}}\right)^{\frac{n}{2}}$$  

for all $t < T$.  

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