On congruences between Drinfeld modular forms

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Abstract

Let \( \mathbb{F}_q \) denote a finite field of characteristic \( p \) and let \( n \) be an effective divisor on the affine line over \( \mathbb{F}_q \) and let \( v \) be a point on the affine line outside \( n \). In this paper, we get congruences between \( \mathbb{Q}_\ell \)-valued weight two \( v \)-old Drinfeld modular forms and \( v \)-new Drinfeld modular forms of level \( vn \). In order to do this, we shall first construct a cokernel torsion-free injection from a full lattice in the space of \( v \)-old Drinfeld modular forms of level \( vn \) into a full lattice in the space of all Drinfeld modular forms of level \( vn \). To get this injection we use ideas introduced by Gekeler and Reversat on uniformization of jacobians of Drinfeld moduli curves.

Introduction

In the function field case, the information given by Hecke action on Jacobians of compactified Drinfeld moduli curves is richer than the Hecke action on the space of automorphic forms. As a consequence, multiplicity one fails to hold mod-p and Hecke action will no longer be semi-simple. This points to the analytic nature of these building blocks of the theory of elliptic modular forms, which are absent in the function field case. In view of this disorder, it is crucial to understand the relation between geometric and analytic theories.

The results of Gekeler and Reversat confirm the importance of understanding congruences between Drinfeld modular forms. They prove that double cuspidal Drinfeld modular forms of weight two and height one with \( \mathbb{F}_p \) residues are reduction modulo \( p \) of automorphic forms. And only double cuspidal harmonic cochains with \( \mathbb{F}_p \) coefficients can be lifted to harmonic cochains with \( \mathbb{Z} \) coefficients [Ge-Re]. Finding congruences between Drinfeld modular forms is also an important step towards formulation and development of Serre’s conjectures on weight and level of modular forms. This is the main application we emphasize on. We use uniformizations for the Drinfeld moduli spaces introduced by Gekeler and Reversat to obtain congruences between Drinfeld modular forms of rank 2. The computations of the congruence ideal are similar to the number field case due to Ribet [Ri]. We also calculate the congruence module for more complicated congruences in the final section. These congruences can be used to obtain towers of congruences between Hecke algebras.
1 Background on Drinfeld modular forms

In the number field case, the notion of modular form is based on the moduli space of principally polarized abelian varieties. In a series of papers, Drinfeld introduced the notion of Drinfeld module as a function field analogue to an abelian variety and defined the notion of Drinfeld modular form using the moduli space of Drinfeld modules. He succeeded to prove a special case of Langlands conjectures in the function field context (see [Dr1], [Dr2]). Our main references in this chapter are [Dr3], [Ge], [Ge-Re], [Go] and [Go-Ha-Ro].

1.1 Drinfeld moduli spaces

Let $X$ be a smooth projective absolutely irreducible curve of genus $g$ over $\mathbb{F}_q$. The field of rational functions $K$ of the curve $X$ is an extension of $\mathbb{F}_q$ of transcendence degree one. We fix a place $\infty$ of $K$ with associated normalized absolute value. We use the same notation for the places of the field $K$ and ideals of its ring of integers $A$ and effective divisors on $X$. Every place $v$ which is not equal to $\infty$ is called a finite place. Let $q_v$ denote the order of the residue field of the ring of integers $O_v$ of the completion $K_v$. The basic example will be the function field $\mathbb{F}_q(t)$. The completion of the algebraic closure of $K_\infty$, will respect to the unique extension of absolute value is denoted by $C$.

Let $L$ be a field extension of $\mathbb{F}_q$ with an $A$-algebra structure $\gamma : A \to L$ and $\tau$ denote the endomorphism $x \mapsto x^q$ of the additive group scheme underlying $L$. The endomorphism ring of the additive group scheme underlying $L$ is the twisted polynomial ring $L\{\tau_p\}$ where $\tau_p : x \mapsto x^p$ satisfies the commutation rule $\tau_p \circ x = x^p \circ \tau_p$ for all $x \in L$. Let $S$ be a scheme over $\text{Spec} \, A$, an $L$-Drinfeld module $(\mathbb{L}, \Phi)$ of rank $r \in \mathbb{N}$ over $S$ consists of a line bundle $\mathbb{L}$ over $S$ and a ring homomorphism $\Phi : A \to \text{End}_S(\mathbb{L}, +)$ into the endomorphism ring of the additive group scheme underlying $L$ satisfying the following property: For some trivialization of $\mathbb{L}$ by open affine subschemes $\text{Spec} \, B$ of $S$ and for each nonzero $f \in A$ we have $\Phi(f) \mid \text{Spec} \, B = \sum_{0 \leq i \leq N(f)} a_i \cdot \tau^i$ with $a_i \in B$, such that

(i) $A \to B$ takes $f$ to $a_0$

(ii) $a_{N(f)}$ is a unit

(iii) $N(f) = r \cdot \deg(f)$, where $q^{\deg(f)} = \sharp (A/f)$.

To summarize these conditions, we must have $\text{End}_{\mathbb{L}} \mathbb{L} = \bigoplus H^0(S, \mathbb{L}^{r-1}) \gamma$.

A morphism of Drinfeld modules $\Phi \to \Phi'$ over $L$ is an element $u \in L\{\tau\}$ such that for all $f \in A$ we have $u \circ \Phi(f) = \Phi'(f) \circ u$. The endomorphism ring $L\{\tau\}$ is the subring $L\{\tau_p\}$ which is generated by $\tau : x \mapsto x^q$. Let $n$ be an ideal of $A$. The group scheme $\Phi$ of $n$-division points $\cap_{f \in n} \ker(\Phi(f))$ is a finite flat subscheme of $(\mathbb{L}, +)$ of degree $\sharp (A/n)^r$ over $S$ which is etale outside support of $n$. An isomorphism $\alpha : \Phi \to (A/n)^r$ is called an $n$-level structure outside the support of $n$. Equivalently, an $n$-level structure is defined to be a morphism $\alpha$ from the constant scheme of $A$-modules $(n^{-1}/A)^r$ to $\Phi$ such that,
forms which uses Tate uniformization (see [Go]). Let \( R \) coincides with \( \Omega_{n}^{1}_{r} \) the maximal ideal (\( \pi \)). Let \( S = \text{Spec}(A) \) be an \( A \)-algebra. A modular form \( f \) of weight \( k \) with respect to \( \text{GL}(r, A) \) is a rule which assigns to each Drinfeld module \( (L, \Phi) \) of rank \( r \) over \( S \) together with an \( n \)-level structure \( \alpha \) will be denoted by \( M^{r}(S) \) or simply by \( M^{r}(n) \). The moduli space \( M^{r}(n) \) is a smooth affine scheme of finite type of dimension \( r - 1 \) over \( A \). The algebra \( M^{r}_{n} \) denotes the affine algebra of the affine scheme \( M^{r}(n) \). For \( n | m \) the natural morphism \( M^{r}(m) \to M^{r}(n) \) is finite flat and even etale outside the support of \( m \). We intend to work with \( M^{r}(n) \) only over \( \text{Spec}(A[n^{-1}]) \). For a more detailed discussion see [De-Hu] and [Dr1]. We would like to compactify the etale covers \( M^{r}(n) \) of \( M^{r}(1) \). We require that for \( m | n \) there be natural maps between the compactifications \( M^{r*}(m) \to M^{r*}(n) \). At the moment an analytic theory of Satake compactifications is available [Ge], but we need an arithmetic theory of compactifications.

In the case of rank-two Drinfeld modules, the moduli space is a curve and we can compactify it by addition of a few points. This compactification is functorial. This is the compactification which Gekeler-Reversat uniformize its Jacobian [Ge-RE].

1.2 Geometric Drinfeld modular forms

Let \( R \) be an \( A \)-algebra. A modular form \( f \) of weight \( k \) with respect to \( \text{GL}(r, A) \) is a rule which assigns to each Drinfeld module \( (L, \Phi) \) of rank \( r \) over an \( R \)-scheme \( S \) a section \( f(L, \Phi) \in \Gamma(L^{-k}) \) with the following property: For any map of \( R \)-schemes \( g : S' \to S \) the section \( f(L, \Phi) \) is functorial with respect to \( g \), i.e. for any nowhere-zero section \( \beta \) of \( g^*(L) \) the element \( f(g^*(L), g^*(\Phi)) \cdot \beta^{\otimes k} \in \Gamma(S', \mathcal{O}_{S'}) \) depends only on the isomorphism class \( (E, \Phi, \beta) \). The same definition works for an arbitrary level structure.

Let \( H^{r}_{\pi} \) denote the graded ring of Drinfeld modular forms of rank \( r \) over \( A \), and \( H^{r}_{\pi,n} \) denote the graded ring of Drinfeld modular forms of rank \( r \) and level \( n \) over \( A \). The map \( \text{Spec}(H^{r}_{\pi,n}) \to M^{r}(n) \) is a principal \( \mathbb{G}_{m} \)-bundle which represents isomorphism classes of Drinfeld modules with level structures, together with a nowhere-zero section of \( L \). Let \( \omega \) denote the top wedge of the sheaf of relative differential forms on \( \text{Spec}(H^{r}_{\pi}) \) restricted to \( M^{r}(n) \). In rank-two case, \( \omega^{\otimes 2} \) coincides with \( \Omega_{\pi}^{1}(2 \cdot \text{cusps}) \).

Here we develop an algebraic theory of \( q \)-expansions for Drinfeld modular forms which uses Tate uniformization (see [Go]). Let \( R \) be a d.v.r. over \( A \) with maximal ideal \( (\pi) \) and fraction field \( K' \). Let \( (M, \Phi) \) be a Drinfeld module of rank \( r \) over \( K \). We say that \( (M, \Phi) \) has stable reduction modulo \( (\pi) \) if for some \( c \in K' \) we have

(i) The module \( (M, c \Phi c^{-1}) \) has coefficients in \( R \).

(ii) The reduction modulo \( (\pi) \) is a Drinfeld module of rank \( \leq r \).
By definition, equality holds in the good reduction case. A Drinfeld module with an admissible \( n \)-level structure, has stable reduction. Over \( A[n^{-1}] \) any \( n \)-level structure gives us a Drinfeld module with stable reduction.

**Definition.** Let \( K^s \) denote the maximal abelian extension of \( K \) split totally at \( \infty \). A \( \Phi \)-lattice \( N \) over \( K \) is a finitely generated projective \( A \)-submodule of \( K^s \) such that:

(i) The group \( N \) is \( \text{Gal}(K^s/K) \)-stable.

(ii) In any ball, there are only finitely many elements of \( N \). \( \square \)

**Theorem 1.2.1 (Drinfeld).** The isomorphism classes of Drinfeld modules of rank \( r + r_1 \) with stable reduction, are in one-to-one correspondence with the isomorphism classes of \( (\mathcal{M}, \Phi, N) \), where \( (\mathcal{M}, \Phi) \) is a Drinfeld module of rank \( r \) over \( K \), with good reduction, and \( N \) is \( \Phi \)-lattice of rank \( r_1 \).

One can show that the proper scheme \( M_1(n) \rightarrow \text{Spec}(A) \) is Spectrum of the ring of integers of \( K^s \). Let \( m \) and \( n \) be ideals of \( A \), and \( p \) be a prime dividing \( m \).

Consider the universal module \( \Phi \) over \( M_1(m) \otimes K((q)) \) with trivial bundle. For any ring \( R \) we define \( R((q)) \) to be the ring of finite tailed Laurent series over \( R \). Theorem 1.1 associates a Drinfeld module \( T(m,n,p) \) over \( (M_1(m) \otimes K)((q)) \) to the triple \( (K, \Phi, \Phi(mn)(1/q)) \). The module \( T(m,n,p) \) can be extended over \( M_1(m) \otimes A[p^{-1}] \) with a no-where zero section “1” and a natural \( m \)-level structure \( \psi \). For more details see [Go]. We call

\[ f(T(m,n,p), "1", \psi) \in H^r \otimes M_1[p^{-1}]((q)) \]

the \( q \)-expansion of \( f \) at the cusp \( (m,n,p) \). \( f \) is holomorphic at the cusp \( (m,n,p) \) if the expansion contains no negative terms.

### 1.3 Analytic Drinfeld modular forms

We shall first give an analytic description of Drinfeld modules. We introduce a construction similar to a “Weierstrass-preparation” which starts with introducing lattices in \( C \). An \( A \)-lattice of rank \( r \) in \( C \) is a finitely generated and thus projective \( A \)-submodule \( \Lambda \) of \( C \) of projective rank \( r \), whose intersection with each bounded subset of \( C \) is finite. To an \( r \)-lattice we associate an exponential function \( e_\Lambda(z) = z \prod_{\lambda \in \Lambda \setminus \{0\}} (1 - z/\lambda) \). This product converges and defines a surjective and entire \( F_q \)-linear function \( e_\Lambda : C \rightarrow C \). For \( a \in A \) there exists \( \Phi^A_a \in C\{\tau\} \) such that \( e_\Lambda(az) = \Phi^A_a(e_\Lambda(z)) \). The endomorphism \( \Phi^A_a \) induces a \( A \)-module structure on the additive group scheme

\[ \mathbb{G}_a/C = C \leftarrow C/\ker(r_\Lambda) = C/\Lambda. \]

In fact \( A \mapsto \Phi^A_a \) is a Drinfeld module. This association is one-to-one. More precisely, the associated Drinfeld modules are isomorphic if the lattices are similar.

Let \( Y \) be a projective \( A \)-module of rank \( r \). Let \( \Gamma_Y \) be the automorphism group of \( Y \). Fixing a \( K \)-basis for \( Y \otimes K \) we can assume \( \Gamma_Y \) is a subgroup of
GL(r, K) commensurable with GL(r, A). Let \( \tilde{\Omega}^r \) denote \( C^r \) with all hyperplanes defined over \( K_{\infty} \) removed. One can see that \( \Gamma_Y \backslash \tilde{\Omega}^r \) is in one-to-one correspondence with \( A \)-lattices isomorphic to \( Y \). The quotient \( \Omega^r := \tilde{\Omega}^r / C^r \) is a subset of projective \( (r-1) \)-space. One can identify \( \cup_Y \Gamma_Y \backslash \tilde{\Omega}^r \) with \( M^r(1)(C) \). The union is over a finitely many isomorphism classes of projective \( A \)-modules of rank \( r \).

In the special case where \( A = \mathbf{F}_q[t] \) every finitely generated module is free. So we can assume \( \Gamma_Y = \text{GL}(r, A) \) and thus can identify \( M^r(1)(C) \) with \( \text{GL}(r, A) \backslash \tilde{\Omega}^r \). In rank two case, let \( \Lambda \) be the 2-lattice \( A \omega + A \). We write \( \Phi^2_1 = T + g(\omega) + \Delta(\omega) = 2 \). The \( g(\omega) \eta^2 + \Delta(\omega) \) defines a map from \( \Omega \) to \( C \) which gives us a \( j \)-invariant \( : \Gamma \backslash \Omega = \Gamma \backslash (C - K_{\infty}) \rightarrow M(1)(C) = C \).

Let \( k \) be a non-negative integer and let \( m \) be an integer class modulo \( q - 1 \). We define a Drinfeld modular form of rank two, weight \( k \) and type \( m \) with respect to the arithmetic group \( \Gamma \) to be a map \( f : \Omega \rightarrow C \) which is holomorphic in the rigid analytic sense, such that

\[
f(z \mid \gamma) = (\det \gamma)^{-m}(cz + d)^kf(z) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.
\]

We also require that \( f \) be holomorphic at the cusps of \( \Gamma \). The space of these modular forms is denoted by \( M_k^r(\Gamma) \) and the subspace of \( i \)-cusp forms is denoted by \( M_k(\Gamma) \). We shall give some cuspidal conditions to determine the exact subspace which comes from the geometric definition.

For general rank \( r \), we define a Drinfeld modular form of weight \( k \) and type \( Y \) and level \( n \) to be a rigid analytic holomorphic function satisfying the following transformation rule:

\[
f(\cdot g) = f(za + b / c \cdot z + d)(z - c / d)^{-k} \quad \text{for} \quad g \in \Gamma_Y(n)
\]

where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( a \) is the upper left \( (r-1) \times (r-1) \)-minor. Here we think of \( z \in \Omega^r \) as an element in \( A^{r-1} \). The dot product on the left hand side is the group action.

In case of rank 2 one can give cusp conditions on analytic Drinfeld modular forms which guarantee algebraicity. Let \( f \) denote an analytically defined Drinfeld modular form of weight \( k \) and level \( n \). The form \( f \) is algebraic if and only if at each cusp the \( q \)-expansion is finite tailed [Go]. If there are no negative terms at the cusps, then \( f \) comes from a section of \( \omega^k \). The space of Drinfeld modular forms holomorphic at cusps, is finite dimensional.

### 1.4 Harmonic cochains

First of all, we shall define the Bruhat-Tits tree \( \mathcal{I} \) of \( \text{PSL}(2, K_{\infty}) \). The vertices of \( \mathcal{I} \) are defined to be rank two \( \text{O}_\infty \)-lattices in \( K_{\infty}^2 \). For vertices \( L \) and \( L' \), we consider the similarity classes \([L]\) and \([L']\) adjacent, if for some representatives \( L_1 \in [L] \) and \( L'_1 \in [L'] \) we have \( L_1 \subset L'_1 \) with \( L'_1 / L_1 \) being a length one as \( \text{O}_\infty \)-module. The graph \( \mathcal{I} \) is a \((q_\infty + 1)\)-regular tree. Let \( G \) denote the group scheme \( \text{GL}(2) \) with center \( Z \). The group \( G(K_{\infty}) \) acts on \( \mathcal{I} \). The stabilizer of the
standard vertex $[O^2_\infty]$ is $G(O_\infty) \cdot Z(K_\infty)$. This gives the standard identifications of the vertices $V(I)$ with $G(K_\infty)/G(O_\infty) \cdot Z(K_\infty)$ and the edges $E(I)$ with $G(K_\infty)/\Gamma_0(O_\infty) \cdot Z(K_\infty)$. Here

$$\Gamma_0(O_\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(O_\infty) \mid c \equiv 0 \pmod{\infty} \right\}.$$ 

There exists a bijection between the ends of $I$ and $\mathbb{P}^1(K_\infty)$. Also by a theorem of Goldman and Iwahori [Go-Iw], there exists a $G(K_\infty)$-equivariant bijection between $I(\mathbb{Q})$ and the set of similarity classes of non-archimedean norms on $K^2_\infty$. Based on this bijection, one can find a $G(K_\infty)$-equivariant building map $\lambda : \Omega \rightarrow I(\mathbb{R})$ with $\lambda(\Omega) = I(\mathbb{Q})$ (for definition see [Ge-Re] p. 37). By constructing an explicit open covering of $\Omega$ (see [Ge-Re] p. 33) one can define $\Omega$ and an analytic reduction map $R : \Omega \rightarrow \Omega$ is a scheme locally of finite type over the residue field $k_\infty$. Each irreducible component of $\Omega$ is isomorphic with $\mathbb{P}^1(k_\infty)$ and meets exactly $q_\infty + 1$ other components. Via the building map one identifies $I$ with the intersection graph of $\Omega$.

Now we define harmonic cochains. Let $X$ be an abelian group. A harmonic cochain with values in $X$ is a map $\phi : V(I) \rightarrow X$ which satisfies

(i) $\phi(e) + \phi(\bar{e}) = 0$ for all $e \in E(I)$ where $\bar{e}$ denotes the reversely oriented $e$.

(ii) $\sum_{e \in E(I), \text{tail}(e) = v} \phi(e) = 0$ for all $v \in V(I)$.

The group of $X$-valued harmonic cochains is denoted by $H(I, X)$. Van der Put gives a map $r : O_\Omega(\Omega)^* \rightarrow H(I, \mathbb{Z})$ and obtains an exact sequence of $G(K_\infty)$-modules

$$0 \rightarrow C^* \rightarrow O_\Omega(\Omega)^* \rightarrow H(I, \mathbb{Z}) \rightarrow 0.$$ 

He defines $r$ using the standard covering of $\Omega$ (see [Ge-Re] p. 40). There is also a residue map from holomorphic differential forms on $\Omega$ to the space of harmonic cochains $\Omega^1(\Omega) \rightarrow H(I, C)$. The residue map is also defined using the open covering of $\Omega$. The fact that residue map is well-defined is a consequence of residue theorem in rigid analytic geometry. The map defined by Van der Put coincides with the residue map if we reduce these maps modulo $p$. In other words the following diagram is commutative (see [Ge-Re] p. 40):

$$\begin{array}{ccc}
O_\Omega(\Omega)^* & \longrightarrow & H(I, \mathbb{Z}) \\
d \log \downarrow & & |_{\text{res}} \downarrow \\
\Omega^1(\Omega) & \longrightarrow & H(I, C).
\end{array}$$

Here $d \log : f \mapsto df/f$ denotes the logarithmic differentiation. For an arithmetic subgroup $\Gamma$ of $G(K)$, we define $H(I, X)^\Gamma$ to be the subgroup of invariants under $\Gamma$. The compact support cohomology $H_c(I, X)^\Gamma$ is the subgroup of harmonic cochains which have compact support modulo $\Gamma$. For any commutative ring $B$ the image of the injective map $H(I, \mathbb{Z})^\Gamma \otimes B \rightarrow H(I, B)^\Gamma$ is denoted by $H_m(I, B)^\Gamma$. In fact $H(I, \mathbb{Z})^\Gamma$ is a free abelian group of rank $g$, where $g$ is
dim_\mathbb{Q}(\Gamma^a \otimes \mathbb{Q})$. Let $\Gamma^*$ denote $\Gamma$ divided by its torsion subgroup, and $\tilde{\Gamma}$ denote the quotient of $\Gamma$ by the finite subgroup $\Gamma \cap Z(K)$. There exists a map

$$H_1(\Gamma \setminus \mathcal{I}) \to H(\mathcal{I}, \mathbb{Z})^\Gamma$$

\[ \phi \mapsto \phi^* \]

defined by $\phi(e) = n(e) \phi(\tilde{e})$ where $n(e) : = 2(\Gamma \cap Z(K))^{-1} \gamma(e)$. We get an injection with finite cokernel (for definition of these maps see [Ge-Re] p. 49)

$$\tilde{\Gamma} = \Gamma^a \to (\Gamma^a)^{ab} \to H_1(\Gamma \setminus \mathcal{I}) \to H(\mathcal{I}, \mathbb{Z})^\Gamma.$$  

Let $c(-,-)$ denote the unique geodesic connecting two vertices. One can explicitly define the above map. We fix a vertex $v$ of $\mathcal{I}$. For $e \in E(\mathcal{I})$ and $\alpha, \gamma \in \Gamma$ we put

$$i(e, \alpha, \gamma, v) = 1, -1, 0, \text{ if } \gamma(e) \text{ belongs to } c(v, \alpha(v)), c(\alpha(v), v) \text{ and neither one, respectively.}$$

Now we get the function

$$\phi_\alpha = \phi_{\alpha,v} := z(\Gamma)^{-1} \sum_{\gamma \in \Gamma} i(e, \alpha, \gamma, v) \in H_1(\mathcal{I}, \mathbb{Z})^\Gamma$$

which is independent of the choice of $v$. One can show that $\phi_{\alpha, \beta} = \phi_\alpha + \phi_\beta$. We get an injection $j : \tilde{\Gamma} \to (\Gamma^*)^{ab} \to H_1(\Gamma \setminus \mathcal{I}) \to H(\mathcal{I}, \mathbb{Z})^\Gamma$. It is shown that $j$ is an isomorphism in case $K = \mathbb{F}_q(t)$. Also Gekeler Reversat prove that $j$ is an isomorphism whenever $\tilde{\Gamma}$ has only $p$-torsion as its torsion subgroup (see [Ge-Re] p. 74).

The point of all this machinery is the following fact (see [Ge-Re] p. 52, 72):

**Theorem 1.4.1 (Gekeler-Reversat)** Let $\Gamma$ be an arithmetic subgroup of $G(K)$. The map $\text{res} : f \mapsto \text{res}(f)$ induces an isomorphism

$$M^1_{2,1}(\Gamma)(C) \to H(\mathcal{I}, C)^\Gamma$$

and an isomorphism $M^2_{2,1}(\Gamma)(C) \to H_0(\mathcal{I}, C)^\Gamma$.

One can identify $M^1_{2,1}(\Gamma)(C)$ with the vector space $H^0(M_\Gamma, \Omega^1(\text{cusps}))$ and $M^2_{2,1}(\Gamma)(C)$ with the vector space $H^0(M_\Gamma, \Omega^1)$. Formulation of $H(\mathcal{I}, B)^\Gamma$ and $H_0(\mathcal{I}, B)^\Gamma$ commutes with flat ring extensions $B'/B$. Hence we have an $\mathbb{F}_p$-structure $M^1_{2,1}(\Gamma)(\mathbb{F}_p)$ on $M^2_{2,1}(\Gamma)(C)$.

### 1.5 Adelic automorphic forms

The basic references for automorphic modular forms are [Ja-La], [Gel]. Harmonic cochains are related to automorphic forms. To explain the automorphic theory, we shall first introduce the Adelic formulation of Drinfeld modular forms. Let $\mathbb{U} = \mathbb{U}_K = \mathbb{U}_f \times K_\infty$ denote the ring of Adeles over $K$. The Adele group is a locally compact ring containing $K$ as a discrete cocompact subring.

The ring of integers $\mathbb{O} = \mathbb{O}_K = \mathbb{O}_f \times O_\infty$, is the maximal compact subring of $\mathbb{U}$. We have a decomposition of idle group $I = I_K = I_f \times K_\infty^*$ where $I_f$ denotes the finite idles. The ring $\mathbb{O}_f$ is the completion with respect to ideal
topology of $A = K \cap \mathbb{D}_f$. The class group of $A$ can be identified with the quotient $K^* \backslash \mathbb{I}_f / \mathbb{O}_f^\ast$. Hence this quotient is finite. We have a bijection between $K^* \backslash \mathbb{I}_f / \mathbb{O}_f^\ast$ and $G(K) \backslash G(\mathbb{U}_f) / G(\mathbb{D}_f)$ where $G = \text{GL}(2)$. The latter is identified with isomorphism classes of rank-two $A$-lattices in $K^2$ in the following manner. We identify $g \in G(\mathbb{U}_f)$ with the $A$-lattice whose span in $\mathbb{O}_f^2$ is the same as $g^{-1} \cdot \mathbb{O}_f^2$. Here the action of $g$ is action of a matrix. More generally as a result of strong approximation theorem for the group $\text{SL}(2)$, for any open subgroup $\mathbb{K}_f$ in $G(\mathbb{U}_f)$ determinant induces a bijection

\[ G(K) \backslash G(\mathbb{U}_f) / \mathbb{K}_f \rightarrow K^* \backslash \mathbb{I}_f / \det(\mathbb{K}_f). \]

Choosing a measure $\mu$ on the locally compact group $G(\mathbb{U}) / Z(\mathbb{U})$ we always get a finite volume for $G(K) \backslash G(\mathbb{U}) / Z(\mathbb{U})$. So is the case for $G(K) \backslash G(\mathbb{U}) / Z(K_\infty)$.

Now consider the Hilbert space of complex valued square integrable functions on $G(K) \backslash G(\mathbb{U}) / Z(K_\infty)$ and denote it by $L^2(G(K) \backslash G(\mathbb{U}) / Z(K_\infty))$. The subspace $L^2(G(K) \backslash G(\mathbb{U}) / Z(K_\infty)) \times L^2(G(K) \backslash G(\mathbb{U}) / Z(K_\infty))$ consists of $G(\mathbb{U})$-stable functions $\phi$ which satisfy the cusp condition

\[ \int_{K \backslash U} \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot g \right) \, dx = 0. \]

Here $dx$ is a Haar measure on $K \backslash U$. We have a decomposition $L^2_1 = \oplus L^2_1(\psi)$, according to the character $\psi$ of the compact group

\[ Z(K) \backslash Z(\mathbb{U}) / Z(K_\infty) = K^* \backslash I / K^*_\infty. \]

For each $\psi$ we have $L^2_1(\psi) = \hat{\psi} V_Q$, where $V_Q$’s are irreducible unitary $G(\mathbb{U})$-submodules occuring with multiplicity one. The underlying unitary representation will be called cuspidal automorphic representations. In each $V_Q$ there exists a distinguished nonzero function $\phi_Q$, satisfying $\phi(gk) = \phi(g) \psi(k)$ for all $g \in G(\mathbb{U})$ and $k \in K_Q(n_Q)$. The open subgroup $K_Q(n_Q) \subset G(\mathbb{O})$ is defined by

\[ K_Q(n_Q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{O}) \mid c \equiv 0 \pmod{n_Q} \right\} \]

where $n_Q$ is an integer, the conductor of the representation. Here $\psi$ is extended to $K_Q(n_Q)$ by defining $\psi(k) = \prod_v \psi_v(a_v)$, where $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a = (a_v)$.

The support of automorphic cusp forms lives on the double coset space $G(K) \backslash G(\mathbb{U}) / K_Q(n_Q) \cdot Z(K_\infty)$. So the space $W(K_Q(n_Q))$ of automorphic cusp forms for $K_Q(n_Q)$ has finite dimension and is independent of $C$, i.e. for any field $F$ of characteristic zero $W(K_Q(n_Q), F) \otimes \mathbb{C} = W(K_Q(n_Q), \mathbb{C})$.

Only a subspace of $W(K_Q(n_Q))$ is related to harmonic cochains. Consider the subspace of representations transforming like the irreducible $G(K_\infty)$-module

\[ V_{sp,F} = \{ f : \mathbb{P}^1(K_\infty) \rightarrow F \mid f \text{ locally constant} \}/F. \]

The underlying representation is independent of $F$ and is denoted by $Q_{sp}$. Over $\mathbb{C}$ there is a unique (up to scaling) $G(K_\infty)$-invariant scalar product on $V_{sp}$
such that \( Q_{sp} \) extends to \( \hat{Q}_{sp} \) on \( \hat{V}_{sp} \). The conductor is the prime divisor \( \infty \).

An \( F \)-valued automorphic form is said to transform like \( Q_{sp} \) if the space of its \( G(K_{\infty}) \)-right translates generate a module isomorphic to a finite number of copies of \( Q_{sp} \).

**Theorem 1.5.1 (Drinfeld)** [Dr1] For an open subgroup \( \mathbb{K}_f \) of \( G(\mathbb{O}_f) \) and for \( \mathbb{K} = \mathbb{K}_f \times \Gamma_0(\infty) \) there exists a canonical bijection

\[
\bigoplus_{x \in S} H_v(I, F)^{T_x} \to W_{sp}(\mathbb{K}, F)
\]

where \( S \) is a choice of representatives for the finite set \( G(K)\backslash G(\mathbb{U}_f)/\mathbb{K}_f \).

### 1.6 Hecke operators

In this section we give an analytic description of Hecke operators and an algebraic description of Hecke correspondences. Again this picture is motivated by the traditional adelic definition of Hecke correspondences using double cosets.

Let \( \mathbb{K}_f \) be an open subset in \( G(\mathbb{O}_f) \) with conductor \( n \). The conductor is the least positive divisor coprime to \( \infty \) such that \( \mathbb{K}_f(n) \subset \mathbb{K}_f \). Let \( v \) be a finite place coprime to \( n \). Then \( \mathbb{K}_v := G(O_v) \) embeds in \( \mathbb{K}_f \). If \( \pi_v \) is a local uniformizer, and \( \tau_v = \text{diag}(\pi_v, 1) \in G(K_v) \), the group \( H_v = \mathbb{K}_v \cap \tau_v \mathbb{K}_v \tau_v^{-1} \) has index \( q_v + 1 \) in \( \mathbb{K}_v \). We define the Hecke operator acting on a function \( \phi \) on \( G(K)\backslash G(\mathbb{U})/K \cdot Z(K_{\infty}) \) by the integral

\[
(T_v \phi)(g) := \int_{\mathbb{K}_v} \phi(g k_v \tau_v) \, dk_v = \sum_{k_v \in \mathbb{K}_v \backslash H_v} \phi(g k_v \tau_v).
\]

Here \( k_v \) runs through a set of representatives of \( \mathbb{K}_v \backslash H_v \) and \( dk_v \) is a Haar measure chosen in a way that the volume of the quotient \( \mathbb{K}_v \backslash H_v \) be equal to one.

The Hecke operators preserve \( W_{sp} \). They commute. Together with their adjoints generate a commutative algebra of normal operators on \( W_{sp} \). So there exists a basis of simultaneous eigen-forms for all \( T_v \). The eigenvalues are algebraic integers. Drinfeld used Weil conjectures to prove analogue of Ramanujan’s conjecture, bounding the norms of eigenvalues \( |\lambda(\phi, v)| \leq 2 q_v^{1/2} \). The Hecke operators respect the decomposition \( W_{sp} = \oplus \psi W_{sp}(\psi) \). We know that

\[
T_v^* | W_{sp}(\psi) = \psi^{-1}(\pi_v) \cdot T_v | W_{sp}.
\]

Let \( K_{\infty}^{un} \) be the maximal unramified extension of \( K_{\infty} \). The Galois group \( \text{Gal}(K_{\infty}^{un}/K_{\infty}) \) is isomorphic to the profinite completion \( \hat{Z} \), where the canonical generator corresponds to the Frobenius element \( \text{Fr}_{\infty} \) of the extension \( K_{\infty}^{un} \) over \( K_{\infty} \). Let \( l \) be a prime different from \( p = \text{char}(\mathbb{F}_q) \) and let \( E_l/K_{\infty}^{un} \) be the extension obtained by adding all the \( l^v \)-th roots of the uniformizer \( \pi_{\infty} \) to \( K_{\infty}^{un} \). Then \( \text{Gal}(E_l/K_{\infty}^{un}) = \mathbb{Z}_l(1) \). Moreover \( E_l \) is Galois over \( K_{\infty} \) and we have

\[
\text{Gal}(E_l/K_{\infty}) = \text{Gal}(K_{\infty}^{un}/K_{\infty}) \times \text{Gal}(E_l/K_{\infty}^{un}) = \hat{Z} \times \mathbb{Z}_l(1).
\]
For any $u \in \mathbb{Z}_l$ the action of $F_{r,\infty} = 1 \in \hat{\mathbb{Z}}$ on $\mathbb{Z}_l$ is given by $F_{r,\infty} u F_{r,\infty} = u q_{\infty}$.

We get a two dimensional Galois representation

$$sp = sp_l = \text{Gal}(K_{\text{sep}}^\infty/K_\infty) \to \text{Gal}(E_l/K_\infty) = \hat{\mathbb{Z}} \times \mathbb{Z}_l \to \text{GL}(2, \mathbb{Q}_l).$$

Here we have chosen an isomorphism between $\mathbb{Z}_l(1)$ and $\mathbb{Z}_l$. The last arrow maps $(1,0)$ to $\begin{pmatrix} 1 & 0 \\ 0 & q_{\infty} \end{pmatrix}$ and $(0,1)$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Now we can state the Drinfeld reciprocity law [Ge2]:

**Theorem 1.6.1 (Drinfeld).** Let $\bar{M}_{K_f}$ denote the compactified Drinfeld moduli space over $C$ associated to $K = K_f \times I_\infty$. There exists a Hecke invariant isomorphism between $\text{W}_{sp}(K, \mathbb{Q}_l) \otimes \text{sp}_l$ and $\text{H}^1(\bar{M}_{K_f}, \mathbb{Q}_l)$ which is compatible with the local Galois action of $\text{Gal}(K_{\text{sep}}^\infty/K_\infty)$.

### 1.7 Theta functions

A holomorphic theta function $u : \Omega \to C$ with respect to $\Gamma$ is a holomorphic nonzero function on $\Omega$ and on cusps, such that $u(\alpha z) = c_u(\alpha) u(z)$ for some $c_u(\alpha) \in \mathbb{C}^*$ and all $\alpha \in \Gamma$. For a meromorphic theta function, we allow poles and zeroes on $\Omega$ but not at cusps. The logarithmic derivative of a holomorphic theta function is in $M^2_2(\Gamma)$. Let $\Theta_m(\Gamma)$ denotes the multiplicative group of meromorphic theta functions and $\Theta_h(\Gamma)$ denotes the subspace of holomorphic theta functions. The family of maps $c_u : \Gamma \to C^*$ induces a map

$$c : \Theta_m(\Gamma) \to \text{Hom}(\Gamma, C^*) = \text{Hom}(\Gamma^\text{ub} \to C^*).$$

We have $\text{ker}(c) \cap \Theta_h(\Gamma) = C^*$. For fixed $\omega, \eta \in \Omega$ define the following holomorphic theta function

$$\theta_{\Gamma}(\omega, \eta, z) = \prod_{\gamma \in \bar{\Gamma}} z - \gamma \omega \gamma^{-1} \eta.$$ 

Here $\bar{\Gamma} = \Gamma/\Gamma \cap Z(K)$. For $\alpha \in \Gamma$ the constant $c(\omega, \eta, \alpha) \in C^*$ defined by

$$\theta_{\Gamma}(\omega, \eta, \alpha z) = c(\omega, \eta, \alpha) \cdot \theta_{\Gamma}(\omega, \eta, z)$$

introduces a group homomorphism $\Gamma \to C^*$ with factors through the quotient $\bar{\Gamma} = \Gamma^\text{ub}/\text{torsion}$. Now consider the holomorphic function

$$u_{\alpha}(z) = \theta(\omega, \alpha \omega, z).$$

One can see that this is a holomorphic function independent of $\omega$ and depending only on class of $\alpha$ in $\bar{\Gamma}$. We also have $u_{\alpha \beta}(z) = u_{\alpha}(z) \cdot u_{\beta}(z)$. Now the equality

$$c(\omega, \eta, \alpha) = u_{\alpha}(\eta)/u_{\alpha}(\omega)$$

helps us to define a symmetric bilinear map $\bar{\Gamma} \times \bar{\Gamma} \to C^*$ via $(\alpha, \beta) \mapsto c_u(\beta)$ where $c_u(\beta) := c(\omega, \alpha \omega, \beta) \in K_\infty$. See [Ge-Re], 65 for more details.
On the other hand the residue map we defined before \( r : O_\Omega(\Omega)^* \to H(I, \mathbb{Z}) \) satisfies \( r(u_\alpha) = \phi_\alpha \in H(I, \mathbb{Z})^\Gamma \). The Peterson pairing on \( H(I, \mathbb{Z})^\Gamma \) is compatible with pairing on \( \Gamma \) that we defined above. More precisely for \( \alpha, \beta \in \Gamma \) we have \( 2(\alpha, \beta) = (\phi_\alpha, \phi_\beta)_\mu \). We get injectivity of \( \bar{c} : \bar{\Gamma} \to \text{Hom}(\bar{\Gamma}, C^*) \) induced by \( \alpha \to c_\alpha \).

2 Congruences between Drinfeld modular forms

In this paper we prove an Ihara lemma which is a kind of cokernel torsion-freeness result for Drinfeld modular forms. This can be used to obtain congruences between Drinfeld modular forms. The idea of using Ihara lemma to get congruences is due to Ribet [Ri].

2.1 Uniformization of the Jacobian

Gekeler-Reversat in their important paper [Ge-Re] introduce a uniformization of the Jacobian of Drinfeld modular curve \( \text{Jac}(\bar{M}_\Gamma) \). Here we summarize the procedure of constructing this uniformization. More details can be found in section 7 of [Ge-Re]. A subgroup \( \Lambda \) of \( (C^*)^g \) is called a lattice, if \( \Lambda \cong \mathbb{Z}^g \) and the image of \( \Lambda \) under \( \log : (C^*)^g \to \mathbb{R}^g \) is a lattice in \( \mathbb{R}^g \). The logarithm depends on the choice of a basis for the character group of \( (C^*)^g \), but the lattice property is independent of this choice. Thus, we can define lattices in tori over arbitrary complete fields in particular over \( \mathbb{K}_\infty \). For any torus \( T \) over \( C \) of dimension \( g \) and any lattice \( \Lambda \) in \( T \), \( T/\Lambda \) is an analytic group variety which is compact in the rigid analytic sense. \( T/\Lambda \) is projective algebraic if and only if there exists a homomorphism from \( \Lambda \) to the character group \( \text{Hom}(T, G_m) \) which induces a symmetric positive definite pairing \( \Lambda \times \Lambda \to C^* \). Positive definiteness means that \( \log |(\alpha, \alpha)| > 0 \), for \( \alpha \in \Lambda \) which is not equal to 1. The requirement that \( C \) is algebraically closed is not necessary to get algebraicity of \( T/\Lambda \). For any symmetric positive definite pairing \( \Lambda \times \Lambda \to C_0^* \) with \( C_0 \) any complete subextension of \( C/K_\infty \) we get a uniformization of an abelian variety defined over \( C_0 \). Given an arithmetic subgroup \( \Gamma \subset \text{GL}(2, K) \), we have defined a pairing (see 1.7) \( \bar{\Gamma} \times \bar{\Gamma} \to C^* \) which induces an analytic uniformization of some abelian variety \( A_\Gamma(C) \). We get an exact sequence

\[
0 \to \bar{\Gamma} \xrightarrow{\bar{c}} \text{Hom}(\bar{\Gamma}, C^*) \to A_\Gamma(C) \to 0.
\]

Now fix \( \omega_0 \in \Omega \) and consider the map \( \psi : \Omega \to \text{Hom}(\bar{\Gamma}, C^*) \to A_\Gamma(C) \) which is induced by \( \omega \to \bar{c}(\omega_0, \omega, \cdot) \). The map \( \psi \) factors through \( \Omega \to C = M_{\Gamma}(C) \) and extends to an analytic embedding \( \tilde{M}_\Gamma(C) \to A_\Gamma(C) \). By GAGA theorems this is a morphism of algebraic varieties. Using theta series one can show that the abelian variety \( A_\Gamma(C) \) is \( K_\infty \)-isomorphic to the Jacobian of the compactified modular curve \( \text{Jac}(\bar{M}_\Gamma) \) (see [Ge-Re] p. 77). So in fact, we have obtained an analytic uniformization of the Jacobian which has a purely function field origin. The map \( \tilde{M}_\Gamma(C) \to A_\Gamma(C) \) factors through \( \text{Jac}(\bar{M}_\Gamma) \) and we use the induced
isomorphism $\text{Jac}(\bar{M}_\tau) \xrightarrow{\sim} A_\tau(C)$ identify $\text{Jac}(\bar{M}_\tau)$ and $A_\tau(C)$ from now on. Let $n$ be a divisor in $F_q[t]$. We define the following congruence groups:

$$\Gamma(n) = \left\{ \gamma \in \text{SL}(2, F_q[t]) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}$$

$$\Gamma_0(n) = \left\{ \gamma \in \text{SL}(2, F_q[t]) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{n} \right\}$$

$$\Gamma^0(n) = \left\{ \gamma \in \text{SL}(2, F_q[t]) \mid \gamma \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{n} \right\}.$$

Let $v$ be a point outside the divisor $n$. Conjugation by $\tau_v = \begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}$ gives an isomorphism $\Gamma_0(v) \cap \Gamma(n) \to \Gamma^0(v) \cap \Gamma(n)$. Here $\pi_v$ is a uniformizer for the prime ideal $v$. Conjugation of $\Gamma_0(v)$ by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives another isomorphism. Composing one with the inverse of the other induces an involution $w_v$ of the space $\Gamma_0(v) \cap \Gamma(n)$ which is called the Atkin-Lehner involution. This induces an involution $w_v'$ of $\text{Hom}(\Gamma_0(v) \cap \Gamma(n), C^*)$ and an involution $w_v''$ of $\text{Jac}(\bar{M}_\tau v)\cap \Gamma(n))$.

Let $I : \Gamma_0(v) \cap \Gamma(n) \to \Gamma(n)$ denote the map induced by inclusion. The transfer map has the reverse direction $V : \Gamma(n) \to \Gamma_0(v) \cap \Gamma(n)$. We have $I \circ V = [v_0 + 1]$. We define the Hecke operator $T_v : \Gamma(n) \to \Gamma(n)$ to be the composition $I \circ w_v \circ V$. The operator $T_v$ acting on $\Gamma(n)$ is self adjoint with respect to the pairing $(\alpha, \beta) \mapsto c_\alpha(\beta)$. The action of $T_v$ commutes with the isomorphisms $J_{\Gamma(n)} : \Gamma(n) \to H_1(\mathbb{Z}, \mathbb{Z})^{\Gamma(n)}$. The operator $T_v = I \circ w_v \circ I$ induces an action $T_v = \text{Hom}(\Gamma(n), C^*) \to \text{Hom}(\Gamma(n), C^*)$.

The Jacobian $\text{Jac}(\bar{M}_\tau(n))$ is also equipped with action of the Hecke operator $T_v$. Note that $\bar{M}_\tau(n)$ has several geometric components and $T_v$ does not respect them. Gekeler-Reversat prove that, this Hecke action is compatible with Hecke actions on $\Gamma(n)$ and $\text{Hom}(\Gamma(n), C^*)$. So uniformization of the Jacobian is Hecke equivariant (see [Ge-Re] p. 86). In fact, we have the following commutative diagram with exact rows, where the right hand vertical maps are induced by $V$ and $I$:

$$\begin{array}{ccc}
0 & \longrightarrow & \Gamma(n) \\
I \uparrow & & V \uparrow \\
\Gamma_0(v) \cap \Gamma(n) & \longrightarrow & \text{Hom}(\Gamma(n), C^*) \\
\downarrow & & \downarrow \\
\text{Jac}(\bar{M}_\tau(n)) \longrightarrow 0 \\
V \uparrow & & I \uparrow \\
\text{Jac}(\bar{M}_\tau v) \cap \Gamma(n)) \longrightarrow \text{Jac}(\bar{M}_\tau(n)) \longrightarrow 0.
\end{array}$$

This is because $c_{I(\alpha)}(\beta) = c_\alpha(V(\beta))$ and $c_{V(\alpha)}(\beta) = c_\alpha(I(\beta))$ (follows from [Ge-Re] 6.3.2).
Proposition 2.1.1. The map $\text{Jac}(\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}) (C) \rightarrow \text{Jac}(\bar{M}_{\Gamma(n)}) (C)$ and the map $\text{Jac}(\bar{M}_{\Gamma(n)}) (C) \rightarrow \text{Jac}(\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}) (C)$ in the above diagram are the same as the maps given by Albanese functoriality and Picard functoriality respectively.

Proof. Fixing $\omega_0 \in \Omega$ and using Gekeler-Reversat uniformization we get embeddings $M_{\Gamma_0(v)\cap \Gamma(n)} \rightarrow \text{Jac}(M_{\Gamma_0(v)\cap \Gamma(n)})$ and $M_{\Gamma(n)} \rightarrow \text{Jac}(M_{\Gamma(n)})$. We have to check that the map between Jacobians $\text{Jac}(\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}) (C) \rightarrow \text{Jac}(\bar{M}_{\Gamma(n)}) (C)$ induced by the above diagram restricts to a projection $M_{\Gamma_0(v)\cap \Gamma(n)} \rightarrow \bar{M}_{\Gamma(n)}$ between the embedded curves. Then automatically the projection will be the Albanese map. The fact that $C_{\Gamma_0}(\beta) = c_\alpha(V\beta)$ implies that $\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}$ projects to $\bar{M}_{\Gamma(n)}$. (For details see the proof of theorem 7.4.1 in [Ge-Re]). To show that $\text{Jac}(\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}) (C) \rightarrow \text{Jac}(\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}) (C)$ is given by Picard functoriality we should see if a divisor on $\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}$ pulls back to the appropriate set of points on $\bar{M}_{\Gamma(n)}$. We should check that $\prod c(\omega_0, \alpha_i \omega, \beta) = c(\omega_0, \omega, V\beta)$, where $\alpha_i$ are representatives of $\Gamma_0(v) \cap \Gamma(n) \setminus \Gamma(n)$. This is proven in proposition 6.3.2 of [Ge-Re]. $\square$

We have the following commutative diagrams (see [Ge-Re] p. 72):

\[
\begin{array}{cccc}
H^1(I, \mathbb{Z})^{\Gamma(n)} & \longrightarrow & \Gamma(n) & \longrightarrow & H^1(I, \mathbb{Z})^{\Gamma(n)} \\
\downarrow & & \downarrow \nu & & \downarrow \text{Norm}
\end{array}
\]

\[
\begin{array}{cccc}
H^1(I, \mathbb{Z})^{\Gamma_0(v)\cap \Gamma(n)} & \longrightarrow & \Gamma_0(v)\cap \Gamma(n) & \longrightarrow & H^1(I, \mathbb{Z})^{\Gamma_0(v)\cap \Gamma(n)}
\end{array}
\]

Assume $X - \{\infty\}$ is the affine line over $\mathbb{F}_q$. By [Ge-Re] p. 74 there is an isomorphism $\Gamma \cong \mathbb{H}_q(I, \mathbb{Z})^F$. By commutativity of the above diagrams, these isomorphisms are Hecke-equivariant. The non-degenerate pairing on $\Gamma$ is compatible with the Petersson pairing on $H^1(I, \mathbb{Z})^F$. Also from the injectivity of $H^1(I, \mathbb{Z})^{\Gamma(n)} \rightarrow H^1(I, \mathbb{Z})^{\Gamma_0(v)\cap \Gamma(n)}$ we get the injectivity of $V : \Gamma(n) \rightarrow \Gamma_0(v)\cap \Gamma(n)$.

2.2 $\text{SL}(2, \mathbb{F}_q[t])$ and congruences

In this section we assume that $q > 4$ and $X - \{\infty\}$ is the affine line over $\mathbb{F}_q$. Let $\Gamma$ denote an arbitrary arithmetic subgroup of $\text{GL}(2, \mathbb{F}_q(t))$ and $\bar{\Gamma}$ denote $\Gamma^{ab}$ divided by its torsion subgroup. We start with the analytic uniformization of $\text{Jac}(\bar{M}_{\Gamma})$ given by the exact sequence

\[
0 \rightarrow \bar{\Gamma} \rightarrow \text{Hom}(\bar{\Gamma}, C^*) \rightarrow \text{Jac}(\bar{M}_{\Gamma})(C) \rightarrow 0.
\]

Let $\pi : M_{\Gamma_0(v)\cap \Gamma(n)} \rightarrow \bar{M}_{\Gamma(n)}$ denote the natural projection induced by the injection of the corresponding congruence groups. Conjugation by the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

induces an involution $w_v$ of $\bar{M}_{\Gamma_0(v)\cap \Gamma(n)}$. We have the following
Namely $\Gamma = \Gamma_0(v) \cap \Gamma(n)$ are the conjugate injections $\Gamma_0(v) \rightarrow \Gamma(n)$. The two injections in the amalgamated product are the conjugate injections $\Gamma_0(v) \cap \Gamma(n) \hookrightarrow \Gamma(n)$ (see [Se] Ch II 1.4).

We define a map $\pi' : \pi \circ w_v : \tilde{M}_\Gamma(v) \rightarrow \tilde{M}_\Gamma(n)$. We get the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Gamma(n) & \rightarrow & \Gamma(n) & \rightarrow \\
& & (I, I \circ w_v) & \uparrow & (V, V \circ w'_v) & \uparrow (\pi, \pi \circ w''_v) \\
0 & \rightarrow & \Gamma_0(v) \cap \Gamma(n) & \rightarrow & \Gamma_0(v) \cap \Gamma(n) & \rightarrow \\
& & \rightarrow & \rightarrow & \rightarrow & \\
0 & \rightarrow & \Gamma(n) & \rightarrow & \Gamma(n) & \rightarrow \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & \Gamma(n) & \rightarrow & \Gamma(n) & \rightarrow \\
& & (I, I \circ w_v) & \uparrow & (V, V \circ w'_v) & \uparrow (\pi, \pi \circ w''_v) \\
0 & \rightarrow & \Gamma_0(v) \cap \Gamma(n) & \rightarrow & \Gamma_0(v) \cap \Gamma(n) & \rightarrow \\
& & \rightarrow & \rightarrow & \rightarrow & \\
0 & \rightarrow & \Gamma(n) & \rightarrow & \Gamma(n) & \rightarrow \\
\end{array}
\]

Here $\pi_1$ and $\pi_2$ denote the projections to the first and second component respectively. This diagram gives us control on the maps between Jacobians induced by the involution $w''_v$ and Albanese functoriality or Picard functoriality.  

**Theorem 2.2.1.** Let $n$ be an effective divisor on $X - \{\infty\}$ and let $v$ be a point on $X - \{\infty\}$ which does not intersect $n$. For $l$ not dividing $2q$ the map

$$\alpha = (I, I \circ w_v) : \Gamma_0(v) \cap \Gamma(n) \rightarrow \Gamma(n)$$

is a surjection modulo $l$.

**Proof.** Serre's machinery in geometric group theory discusses action of groups on trees [Se]. It can be used to understand the algebraic nature of $\alpha$. Serre proves that

$$\text{SL}(2, F_q[t]) = \text{SL}(2, F_q[t]) *_{\Gamma^v} \text{SL}(2, F_q[t]).$$

The congruence group $\Gamma^v$ is defined by

$$\Gamma^v = \left\{ \gamma \in \text{SL}(2, F_q[t]) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\pi_v} \right\}.$$  

Density of $F_q[t][v^{-1}]$ in $F_q(t)$ implies that

$$\text{SL}(2, F_q[t][v^{-1}]) = \text{SL}(2, F_q[t]) *_{\Gamma_0(v)} \text{SL}(2, F_q[t]).$$

Similarly we get an amalgamated structure on

$$\Gamma_n = \left\{ \gamma \in \text{SL}(2, F_q[t][v^{-1}]) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}.$$  

Namely $\Gamma_n = \Gamma(n) *_{\Gamma_0(v) \cap \Gamma(n)} \Gamma(n)$. The two injections in the amalgamated product are the conjugate injections $\Gamma_0(v) \cap \Gamma(n) \hookrightarrow \Gamma(n)$ (see [Se] Ch II 1.4).
Using the Lyndon exact sequence [Se], we get the exactness of the following sequence:

\[ H^1(\Gamma_n, \mathbb{Z}/l) \rightarrow H^1(\Gamma(n), \mathbb{Z}/l) \oplus H^1(\Gamma(n), \mathbb{Z}/l) \rightarrow H^1(\Gamma_0(v) \cap \Gamma(n), \mathbb{Z}/l). \]

We consider the group \( H^1(\Gamma_n, \mathbb{Z}/l) \). An element of this group is a homomorphism \( \Gamma_n \rightarrow \mathbb{Z}/l \). Serre has introduced an obstruction called the congruence group \( C(G_{D(S)}) \) to investigate the congruence subgroup property for a connected linear algebraic group defined over a global field \( K \) with \( S \) a non-empty set of places of \( K \) containing \( S_\infty \). Thus \( C(G_{D(S)}) \) vanishes if and only if the congruence subgroup property holds. Here

\[ \mathcal{D}(S) = \{ x \in K \mid v(x) \leq 1 \quad \text{for} \quad v \notin S \}. \]

Serre shows that for \( \text{SL}(2) \) the congruence group \( C(G_{D(S)}) \) is central (see [Se2] corollary of proposition 5). Central means that \( C(G_{D(S)}) \) is contained in the center of the completion \( \hat{G} \). The completion \( \hat{G} \) is with respect to the topology induced by subgroups of finite index in \( \text{SL}_2(A) \). Hence for \( G = \text{SL}(2) \) we have \( C(G_{D(S)}) \simeq \pi_1(G(S), G(K)) := \pi_1(G(S))/\text{im}(\pi_1(G(K)) \rightarrow \pi_1(G(S)) \) (see [Mo]). Moore shows that the fundamental group \( \pi_1(G(S), G_K) \) vanishes in the special case \( G = \text{SL}(2, \mathbb{F}_q(t)) \) and \( S = S_\infty \cup \{ v \} \) (see [Mo]). As a consequence \( \text{SL}(2, \mathbb{F}_q[t][v^{-1}]) \) has the congruence subgroup property and the kernel of the map \( \Gamma_n \rightarrow \mathbb{Z}/l \) contains a principal congruence subgroup. So we have \( \text{Hom}(\Gamma_n, \mathbb{Z}/l) = 0 \) whenever \( l \) does not divide \( 2q \), because \( \text{PSL}_2(\mathbb{F}_q) \) is simple for \( q > 5 \). The case of \( \text{PSL}_2(\mathbb{F}_9) \) which is exception can be checked directly. So we get an injection

\[ H^1(\Gamma(n), \mathbb{Z}/l) \oplus H^1(\Gamma(n), \mathbb{Z}/l) \rightarrow H^1(\Gamma_0(v) \cap \Gamma(n), \mathbb{Z}/l) \quad (\ast) \]

whenever \( l \) does not divide \( 2q \). We have an injection \( H^1(\overline{\Gamma(n)}, \mathbb{Z}/l) \rightarrow H^1(\Gamma(n), \mathbb{Z}/l) \). So we have shown that there exists an injection

\[ H^1(\overline{\Gamma(n)}, \mathbb{Z}/l) \oplus H^1(\overline{\Gamma(n)}, \mathbb{Z}/l) \rightarrow H^1(\overline{\Gamma_0(v) \cap \Gamma(n)}, \mathbb{Z}/l). \]

By duality the map \( \alpha : \overline{\Gamma_0(v) \cap \Gamma(n)} \rightarrow \overline{\Gamma(n)} \otimes \mathbb{Q} \) which is induced by the conjugate inclusions is a surjection modulo \( l \) for \( l \) not dividing \( 2q \) \( \square \)

**Remark 2.2.2.** Note that the injectivity of \( (\ast) \) implies that the \( l \)-torsion of the cokernel of \( H^1(\Gamma(n), \mathbb{Z}) \oplus H^1(\Gamma(n), \mathbb{Z}) \rightarrow H^1(\Gamma_0(v) \cap \Gamma(n), \mathbb{Z}) \) vanishes.

We have defined a \( \mathbb{Q} \)-valued bilinear pairing on \( \overline{\Gamma} \). This pairing is given by \( \log |c_0(\beta)| \). We know that \( c_0(\beta) \in K_\infty \) (see [Ge-Re] p. 67). So the image of this pairing is contained in \( \mathbb{Z} \). We get an injection of \( \overline{\Gamma} \rightarrow \text{Hom}(\overline{\Gamma}, \mathbb{Z}) \). We have the following commutative diagram with bijective vertical maps:

\[
\begin{array}{c}
\overline{\Gamma} \xrightarrow{\log |e|} \text{Hom}(\overline{\Gamma}, \mathbb{Z}) \\
\downarrow \quad \downarrow \\
H_1(I, \mathbb{Z})^\Gamma \longrightarrow \text{Hom}(H_1(I, \mathbb{Z})^\Gamma, \mathbb{Z}).
\end{array}
\]
The injection in the second row is defined by the following pairing

\[(\phi_\alpha, \phi_\beta)_\mu = \int_{E(\Gamma \setminus \mathcal{I})} \phi_\alpha(e) \cdot \phi_\beta(e) n(e)^{-1}.
\]

Using this diagram we can control cokernel of \(\bar{e}\).

**Lemma 2.2.3.** Let \(n\) be a nonzero effective divisor on \(X - \{\infty\}\) and let \(\Gamma \subset \Gamma(n)\) be a congruence subgroup. There exists a finite set of primes \(S_\Gamma\) which can be explicitly calculated in terms of the graph \(\Gamma \setminus \mathcal{I}\) such that for \(l\) not in \(M\) the cokernel of the injection \(\Gamma \to \text{Hom}(\bar{\Gamma}, \mathbb{Z})\) has no \(l\)-torsion.

**Proof.** Let \(T\) be a maximal tree in \(\Gamma \setminus \mathcal{I}\) and let \(\{\bar{e}_1, \ldots, \bar{e}_g\}\) be a set of representatives for the edges \(E(\Gamma \setminus \mathcal{I}) - E(T)\) modulo orientation. Let \(v_i\) and \(w_i\) denote the initial and terminal points of \(\bar{e}_i\) for \(i = 1, \ldots, g\). There exists a unique geodesic contained in \(T\) which connects \(w_i\) and \(v_i\). Let \(c_i\) be the closed path around \(v_i\) obtained by this geodesic and \(e_i\). Define the following harmonic cochains \(\phi_i \in H^1_{\text{I}}(\mathcal{I}, \mathbb{Z})^\Gamma:\)

\[\phi_i(\bar{e}) = \begin{pmatrix} n(e), & \text{if } \bar{e} \\ -n(e), & \text{if } \bar{e} \\ 0, & \text{otherwise} \end{pmatrix} \text{ appears in } c_i.\]

The inclusion \(\Gamma \subset \Gamma(n)\) implies that \(n(e) = 1\). Lift \(c_i\) to \(\mathcal{I}\) in order to get a path from \(v_i'\) to \(v_i''\) projecting to \(v_i\). There exists \(\alpha_i \in \Gamma\) such that \(\alpha_i(v_i') = v_i''\). In the special case where \(X = \{\infty\}\) is the affine line, we know that \(\alpha_i\) form a basis for \(\Gamma\) (see [Re]). The fact that \(n(e) = 1\) implies that \(\phi_i\) forms a basis for \(H^1_{\text{I}}(\mathcal{I}, \mathbb{Z})^\Gamma\) (see [Ge-Re] proposition 3.4.5). So \(\alpha_i \to \phi_i\) induces an isomorphism \(\Gamma \to H^1_{\text{I}}(\mathcal{I}, \mathbb{Z})^\Gamma\). Also we know that \(\mu(e) = n(e)^{-1}\) (see [Ge-Re] 4.8). Therefore \((\phi_i, \phi_j)_\mu = \sharp(c_i \cap c_j)\). Having this formula we have an explicit matrix form \((\sharp(c_i \cap c_j))_{i,j}\) for the map \(\bar{e}\) with respect to the basis \(\{\alpha_1, \ldots, \alpha_g\}\) for \(\Gamma\) and the dual basis for \(\text{Hom}(\bar{\Gamma}, \mathbb{Z})\). One can calculate the cokernel in terms of \(\sharp(c_i \cap c_j)\). We define \(S_\Gamma\) to be the set of primes dividing \(\det(\sharp(c_i \cap c_j))_{i,j}\). We have injection modulo a prime \(l \in S_\Gamma\). Therefore the cokernel does not have \(l\)-torsion. The graph \(\Gamma \setminus \mathcal{I}\) covers \(\text{SL}_2(\mathbf{F}_q[t])\) with degree \([\text{SL}_2(\mathbf{F}_q[t]) : \Gamma]\). So \(\sharp(c_i \cap c_j)\) is bounded by \((2 \text{diam}(\text{SL}_2(\mathbf{F}_q[t]) \setminus \mathcal{I}) + 1)\text{SL}_2(\mathbf{F}_q[t]) : \Gamma]\). This will help to bound primes in \(S_\Gamma\). □

**Theorem 2.2.4.** Let \(n\) be a nonzero effective divisor on \(X - \{\infty\}\) and let \(v\) be a point on \(X - \{\infty\}\) which does not intersect \(n\). For \(l\) not dividing \(2q(q_e + 1)\) and not contained in \(S_{\Gamma(n)}\) we get an injection

\[\text{Jac}(\bar{M}_{\Gamma(n)}[l]) \oplus \text{Jac}(\bar{M}_{\Gamma(n)}[l]) \to \text{Jac}(\bar{M}_{\Gamma_0(v) \cap \Gamma(n)}[l])\]

which is induced by \(\pi^* \circ \pi_1 + \pi'^* \circ \pi_2\).
Proof. We have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(\Gamma(n), \mu_1)^{\oplus 2} & \longrightarrow & \text{Jac}(\tilde{M}_{\Gamma(n)})/(C)[l]^{\oplus 2} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Hom}(\Gamma_0(v) \cap \Gamma(n), \mu_1) & \longrightarrow & \text{Jac}(\tilde{M}_{\Gamma_0(v) \cap \Gamma(n)})/(C)[l] & \longrightarrow & 0.
\end{array}
\]

Since \(\text{Hom}(\Gamma, \mu_1) = H^1(\Gamma, \mu_1)\) and \(\text{Hom}(\tilde{\Gamma}, \mu_1) \rightarrow H^1(\tilde{\Gamma}, \mu_1)\) is injective, (*) implies the injectivity of \(\text{Hom}(\Gamma(n), \mu_1)^{\oplus 2} \rightarrow \text{Hom}(\Gamma_0(v) \cap \Gamma(n), \mu_1)\). So it is enough to show that \(\alpha' = V \circ \pi_1 + w_v \circ V \circ \pi_2 : \Gamma_0(v) \cap \Gamma(n) \rightarrow \Gamma_0(v) \cap \Gamma(n)\) is an injection modulo \(l\) where \(l\) is a prime which does not divide \(2q\). The following diagram is commutative:

\[
\begin{array}{ccc}
\Gamma(n)^{\oplus 2} & \rightarrow & \text{Hom}(\Gamma(n), Z)^{\oplus 2} \\
V \circ \pi_1 + w_v \circ V \circ \pi_2 & \mid & (I, I_{ov})' \\
\Gamma_0(v) \cap \Gamma(n) & \rightarrow & \text{Hom}(\Gamma_0(v) \cap \Gamma(n), Z).
\end{array}
\]

The prime \(l\) is not contained in \(S_{\Gamma(n)}\). So the map \(\Gamma(n)^{\oplus 2} \rightarrow \text{Hom}(\Gamma(n), Z)^{\oplus 2}\) is injective modulo \(l\). We know that \(\text{Hom}(\Gamma(n), Z)^{\oplus 2} \rightarrow \text{Hom}(\Gamma_0(v) \cap \Gamma(n), Z)\) is injective modulo \(l\). So we get injectivity of \(\Gamma(n)^{\oplus 2} \rightarrow \Gamma_0(v) \cap \Gamma(n)\). \(\square\)

**Proposition 2.2.5.** Let \(n\) be an effective divisor on \(\mathcal{X} - \{\infty\}\) and let \(v\) be a point on \(\mathcal{X} - \{\infty\}\) which does not intersect \(n\). We have a surjection induced by \((\pi_*, \pi'_*):\)

\[
\text{Jac}(\tilde{M}_{\Gamma_0(v) \cap \Gamma(n)}) \rightarrow \text{Jac}(\tilde{M}_{\Gamma(n)}) \oplus \text{Jac}(\tilde{M}_{\Gamma(n)}).
\]

**Proof.** We will show that \(\text{Hom}(\Gamma_0(v) \cap \Gamma(n), C^*) \rightarrow \text{Hom}(\Gamma(n), C^*)^{\oplus 2}\) is surjective. We have an injection \(\Gamma(n)^{\oplus 2} \rightarrow \Gamma_0(v) \cap \Gamma(n)\). Let \(L\) be a lattice in \(\Gamma_0(v) \cap \Gamma(n)\) such that \(L \otimes \mathbb{Q}\) be the orthogonal complement of \(\Gamma(n)^{\oplus 2} \otimes \mathbb{Q}\) with respect to the non-degenerate pairing on \(\Gamma_0(v) \cap \Gamma(n) \otimes \mathbb{Q}\). Then we have an injection from \(L' = L + \Gamma(n)^{\oplus 2}\) into \(\Gamma_0(v) \cap \Gamma(n)\) with finite index. We get a surjective map \(\text{Hom}(L', C^*) \rightarrow \text{Hom}(\Gamma(n), C^*)^{\oplus 2}\). This proves the surjectivity of \(\text{Hom}(\Gamma_0(v) \cap \Gamma(n), C^*) \rightarrow \text{Hom}(\Gamma(n), C^*)^{\oplus 2}\). \(\square\)

Now we compute the action of Hecke correspondences on the two copies of \(v\)-old forms in order to compute the congruence module. We have a surjection

\[
T_l(\text{Jac}(\tilde{M}_{\Gamma_0(v) \cap \Gamma(n)})) \otimes \mathbb{Q}_l \rightarrow T_l(\text{Jac}(\tilde{M}_{\Gamma(n)})) \otimes \mathbb{Q}_l \oplus T_l(\text{Jac}(\tilde{M}_{\Gamma(n)})) \otimes \mathbb{Q}_l.
\]

The cohomology group \(H^1(\tilde{M}_\Gamma, \mathbb{Z}_l)\) is the dual of the \(l\)-adic Tate module \(T_l(\text{Jac}(\tilde{M}_\Gamma))\). Therefore, for \(l\) prime to \(p\) we get an injection

\[
H^1(\text{Jac}(\tilde{M}_{\Gamma(n)}), \mathbb{Z}_l) \oplus H^1(\text{Jac}(\tilde{M}_{\Gamma(n)}), \mathbb{Z}_l) \rightarrow H^1(\text{Jac}(\tilde{M}_{\Gamma_0(v) \cap \Gamma(n)}), \mathbb{Z}_l).
\]
By theorem 2.2.4 we still have an injection modulo \( l \) for \( l \nmid 2q(\nu_{c} + 1) \) and not inside \( \mathcal{T}_{\Gamma(n)} \). So we have an injection torsion-free cokernel. We shall point out that \( H^{1}(M_{\Gamma(n)}, \mathbb{Q}_{l}) = H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) and therefore \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) is a space of automorphic forms (see theorem 1.6.1).

Let \( L = H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Z}_{l}) \) and \( L' = H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Z}_{l}) \). Also let \( V = L \otimes \mathbb{Q}_{l} \) and \( V' = L' \otimes \mathbb{Q}_{l} \) denote the associated vector spaces. The cokernel torsion-free injection map induced between lattices in the vector spaces \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) and \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) induces an injection

\[
\beta : (V'/L')^{2} \to V/L.
\]

We identify two copies of \( L' \) and \( V' \) with their images in \( L \) and \( V \) respectively. Let \( A \) denote the \( \mathbb{Q}_{l} \)-vector space generated by the image of \((L')^{2}\) and \( B \) denote the orthogonal subspace under the cup product. \( A \) is the same as \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l})^{v}\)-old. Since the Hecke operators \( T_{v} \) with \( v \) prime to the level are self-adjoint with respect to cup product and cup product restricted to \( A \) in non-degenerate as a result of Poincaré duality on \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) we deduce that \( B \) is stable under the given Hecke operators and is the same as \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l})^{v}\)-new. We define the congruence module by the formula

\[
\Omega = ((L + A) \cap (L + B))/L.
\]

We identify two copies of \( L' \) and \( V' \) with their images in \( L \) and \( V \) respectively. Let \( A \) denote the \( \mathbb{Q}_{l} \)-vector space generated by the image of \((L')^{2}\) and \( B \) denote the orthogonal subspace under the cup product. \( A \) is the same as \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l})^{v}\)-old. Since the Hecke operators \( T_{v} \) with \( v \) prime to the level are self-adjoint with respect to cup product and cup product restricted to \( A \) in non-degenerate as a result of Poincaré duality on \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) we deduce that \( B \) is stable under the given Hecke operators and is the same as \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l})^{v}\)-new. We define the congruence module by the formula

\[
\Omega = ((L + A) \cap (L + B))/L.
\]

One can show that \( \Omega := \text{image}(\beta) \cap \ker(\beta') = \ker(\beta' \circ \beta) \), where the map \( \beta' : V/L \to (V'/L')^{2} \) is the transpose of \( \beta \), induce by Poincaré duality. To get congruences one should compute the congruence module. We define the correspondence \( W_{v} \) on \( \tilde{M}_{\Gamma(n)} \) by the following formula \( W_{v} = \pi_{*} \pi^{*} w_{v} - w_{v} \), where \( \pi \) denotes the natural projection \( \tilde{M}_{\Gamma(n)} \to M_{\Gamma(n)} \).

**Proposition 2.2.6.** The correspondence \( W_{v}^{2} - \text{id} \) acts on \((V'/L')^{2}\). We have \( \Omega = \ker(W_{v}^{2} - \text{id}) \).

**Proof.** We calculate the action of \( W_{v} \) on \((V'/L')^{2}\). For a cohomology class \( f \) on \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) we have \( W_{v}f = \pi_{*} \pi^{*} w_{v} f - w_{v} f \). Therefore, we have \( W_{v} w_{v} f = \pi_{*} \pi^{*} f - f \). If \( f \) is pull back of a cohomology class in \( H^{1}(\text{Jac}(M_{\Gamma(n)}), \mathbb{Q}_{l}) \) then \( W_{v} w_{v} f = q_{v} f \). So %Wv preserves the subspace V' of V, and acts as the matrix \( \begin{pmatrix} T_{v} & q_{v} \\ -1 & 0 \end{pmatrix} \). For \( f \) in \( B \) we have \( \pi_{*} f = 0 \). So \( W_{v} \) acts on \( B \) as \( -w_{v} \). Therefore \( W_{v}^{2} - \text{id} \) vanishes on \( B \). Now we calculate action of the correspondence \( \beta' \circ \beta \) on \( V'^{2} \). We have

\[
\langle \pi^{*} f, h \rangle_{\Gamma(n)} = \langle f, \pi_{*} h \rangle_{\Gamma(n)} \quad \text{for} \quad f \in V', \ h \in V
\]

\[
\langle w_{v} \pi^{*} f, h \rangle_{\Gamma(n)} = \langle f, \pi_{*} w_{v} h \rangle_{\Gamma(n)} \quad \text{for} \quad f \in V', \ h \in V.
\]

The Hecke operator \( T_{v} \) is self adjoint. Therefore for \( f, g \in V \) we have

\[
\beta' \circ \beta(f, g) = \beta'((\pi^{*} f + w_{v} \pi^{*} g) = (\pi_{*} \pi^{*} f + \pi_{*} w_{v} \pi^{*} g + \pi_{*} w_{v} \pi^{*} f + \pi_{*} \pi^{*} g)^{t}.
\]
So the matrix of \( \beta' \circ \beta \) on \( V^2 \) is \( \begin{pmatrix} q_v + 1 & T_v \\ T_v & q_v + 1 \end{pmatrix} \). We have the equality of matrices

\[
W_v^2 - \mathrm{id} = \begin{pmatrix} T_v & q_v \\ -1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & T_v \\ 0 & -1 \end{pmatrix} \circ \beta' \circ \beta.
\]

Since \( \begin{pmatrix} -1 & T_v \\ 0 & -1 \end{pmatrix} \) acts as an automorphism, we have \( \ker(\beta' \circ \beta) = \ker(W_v^2 - \mathrm{id}) \).

We are interested in getting congruences between Hecke eigen-forms. Congruence conditions should be formulated in terms of the eigenvalues.

**Main Theorem 2.2.7.** Let \( \nu \) be a nonzero effective divisor on \( X - \{ \infty \} \) and let \( v \) be a point on \( X - \{ \infty \} \) which does not intersect \( \nu \). Let \( \ell \) be a prime not dividing \( 2q(q_v + 1) \) and not contained in \( S_{\Gamma(n)} \). Let \( f \) be a Hecke eigen-form of level \( \Gamma(n) \) and \( T_\nu f = t_\nu f \). If \( t_\nu^2 \equiv (q_v + 1)^2 \mod \ell \) then \( f \) is congruent to a new-form of level \( \Gamma_0(\nu) \cap \Gamma(n) \mod \ell \).

**Proof.** On the two copies of old forms we have \( W_\nu^2 - T_\nu W_\nu + q_v = 0 \). Let \( r \) and \( s \) denote the two roots of \( x^2 - t_\nu x + q_v = 0 \) and let \( K \) be a number field containing \( r \) and \( s \) with ring of integers \( \mathcal{O}_K \). Then \( f_r = f - sw_v f \) is an eigen-form of Hecke operators with eigenvalue \( r \) for \( W_\nu \) (see [Di]). The congruence module for \( f_r \) is \( (r^2 - 1)^{-1}\mathcal{O}_K/\mathcal{O}_K \). The equality \((r^2 - 1)(s^2 - 1) = -(t_\nu^2 - (q_v + 1)^2)\) finishes the argument. □

**Theorem 2.2.8.** Let \( \nu \) be a point on \( X - \{ \infty \} \) and let \( \nu \) be an effective divisor on \( X - \{ \infty \} \) which does not intersect \( \nu \). The projections \( \pi \) and \( \pi' \) induce an injection of weight two and type one modular forms

\[
M^2_{2,1}(\Gamma(n))(C) \oplus M^2_{2,1}(\Gamma(n))(C) \to M^2_{2,1}(\Gamma_0(\nu) \cap \Gamma(n))(C).
\]

**Proof.** We have obtained a surjection

\[
\text{Jac}(\overline{M}_{\Gamma_0(\nu) \cap \Gamma(n)}) \to \text{Jac}(\overline{M}_{\Gamma(n)}) \oplus \text{Jac}(\overline{M}_{\Gamma(n)})
\]

induced by \((\pi_\nu, \pi'_\nu)\). So we get a surjection of tangent spaces. This induces an injection \( \text{cot}(\text{Jac}(\overline{M}_{\Gamma(n)})) \oplus \text{cot}(\text{Jac}(\overline{M}_{\Gamma(n)})) \to \text{cot}(\text{Jac}(\overline{M}_{\Gamma_0(\nu) \cap \Gamma(n)})) \) between cotangent spaces. The cotangent space of \( \text{Jac}(\overline{M}_\Gamma) \) can be canonically identified with \( M^2_{2,1}(\Gamma)(C) \). therefore, the induced map

\[
M^2_{2,1}(\Gamma(n))(C) \oplus M^2_{2,1}(\Gamma(n))(C) \to M^2_{2,1}(\Gamma_0(\nu) \cap \Gamma(n))(C)
\]

is an injection. □

We know that the vector spaces \( M^2_{2,1}(\Gamma)(C) \) have \( \mathbf{F}_q \) structure. We are curious if the above injection is also defined over \( \mathbf{F}_q \). To show this we have to use the language of harmonic cochains.
Proposition 2.2.9. \( M_{2,1}^2(\Gamma(n))(C) \oplus M_{2,1}^2(\Gamma(n))(C) \) can be reduced to \( F_q \) to induce an injection

\[
M_{2,1}^2(\Gamma(n))(F_q) \oplus M_{2,1}^2(\Gamma(n))(F_q) \to M_{2,1}^2(\Gamma_0(v) \cap \Gamma(n))(F_q).
\]

Proof. The equalities \( M_{2,1}^2(\Gamma)(C) = H_r(I, C)^{\Gamma} \) and \( M_{2,1}^2(\Gamma)(F_q) = H_r(I, C)^{\Gamma} \)
induce a map \( H_r(I, C)^{\Gamma(n)}(C) \oplus H_r(I, C)^{\Gamma(n)}(F_q) \to H_r(I, C)^{\Gamma_0(v) \cap \Gamma(n)}(C) \) of the first copy maps
by the natural injection \( H_r(I, C)^{\Gamma(n)} \to H_r(I, C)^{\Gamma_0(v) \cap \Gamma(n)} \) and the second map is given by the first map composed with the involution \( w_v \) of \( H_r(I, C)^{\Gamma_0(v) \cap \Gamma(n)} \). We can restrict to \( F_q \) and obtain an injective map
\( H_r(I, F_q)^{\Gamma(n)}(C) \oplus H_r(I, F_q)^{\Gamma(n)}(F_q) \to H_r(I, F_q)^{\Gamma_0(v) \cap \Gamma(n)}(F_q) \). This means that by restriction we get \( M_{2,1}^2(\Gamma(n))(F_q) \oplus M_{2,1}^2(\Gamma_0(v) \cap \Gamma(n))(F_q) \). □

After tensoring \( M_{2,1}^2(\Gamma(n))(F_q) \oplus M_{2,1}^2(\Gamma_0(v) \cap \Gamma(n))(F_q) \) with \( F_q[t] \) or with the ring of integers \( A \) of any function field, we get a cokernel torsion-free injection. In particular we have a cokernel torsion-free injection of \( F_q[t] \)-modules

\[
M_{2,1}^2(\Gamma(n))(F_q[t]) \oplus M_{2,1}^2(\Gamma_0(v) \cap \Gamma(n))(F_q[t]).
\]

Unfortunately we don’t have a \( F_q[t] \)-valued pairing on these spaces. We can try to construct a pairing on \( M_{2,1}^2(\Gamma)(F_q) \) using the pairing on \( H_r(I, F_q)^{\Gamma} \). But it is not sensible to search for congruences between \( F_q[t] \)-valued Drinfeld modular forms using such a pairing.

2.3 Towers of congruences

In order to get congruences between Hecke algebras we need to construct congruences between level \( \Gamma_0(rv) \cap \Gamma(n) \) and level \( \Gamma_0((r+2)v) \cap \Gamma(n) \). The first thing we need is an injection result on the \( l \)-torsion of Jacobians. The main reference in [Wi].

Conjugation by \( \tau_{rv} = \begin{pmatrix} \pi_v^r & 0 \\ 0 & 1 \end{pmatrix} \) gives an isomorphism between the abelianizations \( \Gamma_0(rv)(\Gamma(n)) \to \Gamma_0(rv)(\Gamma(n)) \). Conjugation of \( \Gamma_0(r, v) \) by \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)
gives another isomorphism. Composing one with the inverse of the other induces an involution \( w_{rv} : \Gamma_0(rv)(\Gamma(n)) \to \Gamma_0(rv)(\Gamma(n)) \). This induces an involution \( w_{rv}^\prime \) of \( \text{Hom}(\Gamma_0(rv)(\Gamma(n)), C^*) \) and an involution \( w_{rv}'' \) of \( \text{Jac}(\Gamma_0(rv)(\Gamma(n))) \). Now we have three maps from \( \tilde{M}_{\Gamma_0((r+2)v) \cap \Gamma(n)} \to \tilde{M}_{\Gamma_0(rv) \cap \Gamma(n)} \). The following non-commutative diagram helps to visualize these maps

\[
\begin{array}{ccc}
\tilde{M}_{\Gamma_0((r+2)v) \cap \Gamma(n)} & \xrightarrow{\pi_0} & \tilde{M}_{\Gamma_0((r+1)v) \cap \Gamma(n)} \\
\downarrow{w_{(r+2)v}} & & \uparrow{w_{(r+1)v}} \\
\tilde{M}_{\Gamma_0((r+2)v) \cap \Gamma(n)} & \xrightarrow{\pi_{00}} & \tilde{M}_{\Gamma_0((r+1)v) \cap \Gamma(n)} \\
& \downarrow{w_{rv}} & \\
\end{array}
\]

Here \( \pi_0, \pi_{00} \) denote the natural projections. The three maps are \( \alpha = \pi_0 \pi_{00} \) and \( \beta = w_{rv} \pi_0 \pi_{00} w_{(r+1)v} \pi_{00} \) and \( \gamma = w_{rv} \pi_0 \pi_{00} w_{(r+2)v} \). Let \( \pi_i \) denote projection to the \( i \)-th component.
Theorem 2.3.1. Let \( n \) be an effective divisor on \( X - \{ \infty \} \) and let \( v \) be a point on \( X - \{ \infty \} \) which does not intersect \( n \). For \( l \) not dividing \( q(v, + 1) \) which is not contained in \( S_{\Gamma_0((r+1)v)\cap \Gamma(n)} \) we get an exact sequence of \( l \)-torsion of Jacobians:

\[
0 \longrightarrow \text{Jac}(\tilde{M}_{\Gamma_0((r+1)v)\cap \Gamma(n)})[l] \xrightarrow{\zeta_1} \text{Jac}(\tilde{M}_{\Gamma_0((r+1)v)\cap \Gamma(n)})[l]^\oplus 2
\]

where \( \zeta_1 = (\pi_{00}^{-1}, -w_{(r+1)v}^* \pi_0^* w_{rv}^*) \) and \( \zeta_2 = \pi_{00}^* + w_{(r+2)v}^* \pi_0^* w_{(r+1)v}^* \).

**Proof.** See [Wi] lemma 2.5 for more details. Let \( B_0 \), \( B^0 \) and \( \Delta_{rv} \) be given by

\[
B_0 = \Gamma_0((r+1)v) \cap \Gamma(n)/\Gamma_0((r+1)v) \cap \Gamma(n),
\]

\[
B^0 = \Gamma_0(rv) \cap \Gamma_0^0(v) \cap \Gamma(n)/\Gamma_0(rv) \cap \Gamma(n),
\]

\[
\Delta_{rv} = \Gamma_0(rv) \cap \Gamma(n)/\Gamma_0((r+1)v) \cap \Gamma(n).
\]

Then \( \Delta_{rv} \simeq \text{SL}_2(\mathbb{F}_{q_v}) \) for \( r = 0 \) and is of order a power of \( q_v \) for \( r > 0 \). The groups \( B_0 \) and \( B^0 \) generate \( \Delta_{rv} \). The vanishing of \( H^2(\text{SL}_2(\mathbb{F}_{q_v}), \mathbb{Z}/l) \) can be checked by restriction to the Sylow \( l \)-subgroup which is cyclic. The following isomorphisms

\[
\lambda_0 : H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l) \xrightarrow{\cong} H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l)^{B_0},
\]

\[
\lambda^0 : H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l) \xrightarrow{\cong} H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l)^{B^0},
\]

\[
H^1(\Gamma_0(rv) \cap \Gamma(n), \mathbb{Z}/l) \xrightarrow{\cong} H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l)^{\Delta_{rv}}
\]

is induced by Inflation-Restriction exact sequences. We have the inclusion \( H^1(\Gamma)^{B^0} \cap H^1(\Gamma)^{B_0} \subset H^1(\Gamma)^{\Delta_{rv}} \). This implies exactness the following sequence:

\[
0 \rightarrow H^1(\Gamma_0(rv) \cap \Gamma(n), \mathbb{Z}/l)
\]

\[
\xrightarrow{\zeta_1} H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l) \oplus H^1(\Gamma_0(rv) \cap \Gamma_0^0(v) \cap \Gamma(n), \mathbb{Z}/l)
\]

\[
\xrightarrow{\zeta_2} H^1(\Gamma_0((r+1)v) \cap \Gamma(v) \cap \Gamma(n), \mathbb{Z}/l)
\]

where \( m_1 = (\text{res}_0, \text{res}^0) \) and \( m_2 = \lambda_0 + \lambda^0 \). By appropriate conjugation one gets the following exact sequence:

\[
0 \rightarrow H^1(\Gamma_0(rv) \cap \Gamma(n), \mathbb{Z}/l)
\]

\[
\xrightarrow{\zeta_1} H^1(\Gamma_0((r+1)v) \cap \Gamma(n), \mathbb{Z}/l)^{\oplus 2} \xrightarrow{\zeta_2} H^1(\Gamma_0((r+2)v) \cap \Gamma(n), \mathbb{Z}/l).
\]

As in Theorem 2.2.1 this gives a dual sequence which is exact modulo \( l \)

\[
\Gamma_0((r+2)v) \cap \Gamma(n) \rightarrow \Gamma_0((r+1)v) \cap \Gamma(n)^{\oplus 2} \rightarrow \Gamma_0(rv) \cap \Gamma(n) \rightarrow 0.
\]

As in the proof of Theorem 2.2.4 we can use the Gekeler-Reversat uniformization to get information about \( l \)-torsion of Jacobians. The condition that \( l \) is not
Let $n$ be an effective divisor on $X - \{\infty\}$ and let $v$ be a point on $X - \{\infty\}$ which does not intersect $n$. For $l$ not dividing $q(q_v + 1)$ which is not contained in $\Sigma_{\Gamma_0(4v)\cap \Gamma(n)}$, the map $\alpha^* \pi_1 + \beta^* \pi_2 + \gamma^* \pi_3$ induces an injection

$$\text{Jac}(\bar{M}_{\Gamma_0(4v)\cap \Gamma(n)})[l] \to \text{Jac}(\bar{M}_{\Gamma_0((r+2)v)\cap \Gamma(n)})[l].$$

Proof. Let $\pi'_0 = w^r\pi_0 w^{r+1}_v$ and $\pi'_0 = w^{(r+1)v}_v \pi_0 w^{(r+2)v}_v$, then we have $\pi'_0 = \pi_0 \cdot \pi'_0$. Therefore two of the four maps $\pi_0 \cdot \pi_0, \pi'_0 \pi_0, \pi_0 \cdot \pi'_0$ and $\pi'_0 \cdot \pi'_0$ below coincide

$$\text{Jac}(\bar{M}_{\Gamma_0(rv)\cap \Gamma(n)})[l] \to \text{Jac}(\bar{M}_{\Gamma_0((r+1)v)\cap \Gamma(n)})[l] \to \text{Jac}(\bar{M}_{\Gamma_0((r+2)v)\cap \Gamma(n)})[l]$$

and we get injection of the three copies which are left.

Let $L = H^1(\bar{M}_{\Gamma_0(rv)\cap \Gamma(n)}, \mathbb{Z})$ and $L' = H^1(\bar{M}_{\Gamma_0((r+2)v)\cap \Gamma(n)}, \mathbb{Z})$. Also let $V = L \otimes \mathbb{Q}$ and $V' = L' \otimes \mathbb{Q}$ denote the associated vector spaces. We have obtained an injection $\mu : (V'/L')^3 \to V/L$. We identify three copies of $L'$ with their images in $V$ and $L$ respectively. Let $A$ denote the $\mathbb{Q}$-vector space generated by the image of $(L')^3$ and $B$ denote the orthogonal subspace under the cup product. We define the congruence module by the formula

$$\Omega = ((L + A) \cap (L + B))/L.$$

One can show that $\Omega := \text{image}(\mu) \cap \ker(\mu') = \ker(\mu' \circ \mu)$, where the map $\mu' : V/L \to (V'/L')^3$ is the transpose of $\beta$, induced by Poincaré duality. To get congruences one should compute the congruence module. First we calculate action of the correspondence $\mu' \circ \mu$ on $V^3$. For $f \in V'$ and $h \in V$ we have

$$\langle \pi'_0 \pi_0^* f, h \rangle_{\Gamma_0((r+2)v)\cap \Gamma(n)} = \langle f, \pi_0 \pi_0^* h \rangle_{\Gamma_0(rv)\cap \Gamma(n)},$$

$$\langle w^{(r+1)v}_v \pi_0^* w^{(r+1)v}_v \pi_0^* f, h \rangle_{\Gamma_0((r+2)v)\cap \Gamma(n)} = \langle f, \pi_0 w^{(r+1)v} \pi_0^* w^{(r+2)v} h \rangle_{\Gamma_0(rv)\cap \Gamma(n)},$$

$$\langle w^{(r+2)v}_v \pi_0^* \pi_0^* w^{(r+2)v}_v f, h \rangle_{\Gamma_0((r+2)v)\cap \Gamma(n)} = \langle f, w_v \pi_0 w^{(r+2)v} \pi_0^* h \rangle_{\Gamma_0(rv)\cap \Gamma(n)}.$$

So the matrix of the action $\mu' \circ \mu$ on $V^3$ is given by

$$\begin{pmatrix}
q_v (q_v + 1) & w_{rv} T_v & q_v & w_{rv} T_{2v} \\
w_{rv} q_v & q_v (q_v + 1) & w_{rv} T_v & q_v \\
T_{2v} w_{rv} & T_v w_{rv} & q_v & q_v (q_v + 1)
\end{pmatrix}.
$$

The Hecke operators $T_v$ and $T_{2v}$ are self adjoint and defined by

$$T_v = w_{rv} \pi_0^* w^{(r+1)v} \pi_0^* w_{rv}$$

$$T_{2v} = \pi_0^* \pi_0 w^{(r+2)v} \pi_0^* \pi_0 w_{rv}.$$
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