Some features of a Schwarzchild Black Hole from a Snyder perspective

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In this paper, the consequences of introducing a deformed Snyder-Kepler potential in the Schwarzchild metric are investigated. After this modification, it is obtained a dynamically depending horizon with different penetration radius for massive particles and light rays in radial orbits. In the case of circular orbits, they all remain untouched.

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I. INTRODUCTION

The main stream research on very high energy physics and on possible candidates for a suitable quantum gravity theory has lead to the idea of the existence of a fundamental length. At energies corresponding to this length it is expected to find a totally new non commutative physics that even could break the causal principle. Among the main theories that back up this idea we can enumerate Loop Quantum Gravity, String Theory and all their modifications, proposals and variations, along with other efforts to give a suitable quantum version for gravity. The minimal fundamental length is usually identified with the minimal size of the elements of these theories and the proposal usually is that it approaches to the order of Planck length.

We can considerate different ways to introduce a minimal fundamental length, but many of them are incompatible with Lorentz symmetry that is a corner stone in modern Physics, and that is, of course, a very undesirable consequence. In order to avoid that issue it is usual to use a proposal made by H. Snyder, who in the 40’s postulated a modification

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of the Heisenberg algebra that implies discrete spectra of the spacetime operators. This modification can be seen as one of the $\kappa$-deformed spacetime modifications.

During decades the Snyder proposal was put aside due to the success of the renormalization program in the standard model. However, the emergence of candidates to quantum gravity that work on the base of finite elements with minimal length has brought back the idea and renewed interest on it.

One of the main challenges today, in this area, is to find ways to experimentally test the discreteness of spacetime or the associated non-commutativity of the canonical relations of the phase space variables. There have been some efforts in this direction using planetary data and different approaches from Newtonian Mechanic and General Relativity dynamics, for instance Mignemi [5] for the last case and Romero [6] and Leiva [7] for the former case. Some features can be predicted from works like these, but the main general conclusion is that the new scenario must be proved mainly in the presence of strong fields or in extraordinary energetic situations. Thats the reason to explore non commutativity nearby black holes.

The paper is organized as follows: in the next section a short review of the Snyder recipe is given and a modified Kepler potential is recalled from [7], in the third section Schwarzchild metrics is modified and the main features are discussed for massive particles and, finally, conclusions are given in the last section.

II. SNYDER SPACE

A. Quantum Snyder Algebra

Quantum Snyder algebra is characterized by the following set of canonical parenthesis for the phase space variables:

\[
\begin{align*}
[x_\mu, x_\nu] &= iM_{\mu\nu}, \\
[x_\mu, p_\nu] &= i\delta_{\mu\nu} - ilp_\mu p_\nu, \\
[p_\mu, p_\nu] &= 0.
\end{align*}
\]

(1)

(2)

(3)

Where $\mu, \nu = 0, 1, 2, 3$. 

This has been the initial point for many researches that have worked in many implications of the introduction of a non commutative parameter $l$ since the Snyder’s paper itself \[4\], others like \[1\], \[8\], \[9\] and in recent days \[10\].

B. Classical Snyder Algebra

Classical 3 dimensional Euclidean Snyder Space is characterized by its non linear commutation relations (now in the sense of Poisson brackets), between the non commutative variables of the phase space $\bar{x}_i, \bar{p}_i$. A simple realization of these variables in terms of standard commutative variables $x_i, p_i$ can be done after the recipe in \[8\]:

\[
\bar{x}_i = x_i + l(x)p_i, \quad \bar{p}_i = p_i
\]

With $i, j = 1, 2, 3$

This realization gives the following commutation relations:

\[
\{\bar{x}_i, \bar{x}_j\} = lL_{ij}, \quad \{\bar{x}_i, p_j\} = \delta_{ij} + lp_i p_j, \quad \{p_i, p_j\} = 0,
\]

where $l$ is as before, a parameter that measures the deformation introduced in the canonical Poisson brackets, and $L_{ij}$ is defined as a dimensionless matrix proportional to the angular momentum.

C. Classical Snyder Newton Potential

The Newtonian potential in terms of noncommutative variables is

\[
V = -\frac{MG}{\sqrt{\bar{x}^2}},
\]

This can be implemented then in terms of the commutative space variables $x, p$ and at first order in $l$:
\[ V(x) = -\frac{\kappa}{\sqrt{x^2 + 2l(x_p)^2}}, \]  \hspace{1cm} (10)

so, using spherical coordinates:

\[ x = \rho \dot{\rho}, \]  \hspace{1cm} (11)
\[ p = m(\dot{\rho} \dot{\rho} + \rho \dot{\theta}^2 + \rho \dot{\phi} \sin(\theta) \dot{\phi}), \]  \hspace{1cm} (12)

and assuming \( lm^2 \rho^2 \ll 1 \), the Snyder-Kepler potential for a particle can be written as

\[ V(\rho) = \frac{-MG}{\rho}(1 - lm^2 \dot{\rho}^2). \]  \hspace{1cm} (13)

III. SNYDER-SCHWARZCHILD PROPOSAL

Let’s introduce this new potential instead of the classical Newton potential in the Schwarzschild solution as a limit of weak gravitation:

\[ ds^2 = -\left[1 - \frac{2GM}{c^2 \rho}(1 - lm^2 \dot{\rho}^2)\right]c^2 dt^2 + \left[1 - \frac{2GM}{c^2 \rho}(1 - lm^2 \dot{\rho}^2)\right]^{-1}d\rho^2 + \rho^2 d\Omega^2. \]  \hspace{1cm} (14)

From simple inspection it’s easy to see that this proposal is a solution of vacuum Einstein field equation because the modification depends just on \( \dot{\rho} \). Indeed, Ricci tensor and Ricci scalar depend just on partial derivatives with respect to the variables \( \rho, \theta, \varphi \), so this proposal yields the Einstein equation \( G_{\mu\nu} = 0 \). On the other hand, the symmetries are still the same, rotational and temporal displacement.

One of the main features that can be detected is related to horizon. In standard Schwarzschild solution, does exist the well known radius \( \rho = \frac{2GM}{c^2} \) where the metric becomes singular and many facts happen from the point of view of an exterior observer. This horizon is modified in this formulation in the sense that the metric is singular at \( \rho = 1 - \frac{2MG}{c^2}(1 - lm^2 \dot{\rho}^2) \).

The radius where metric is singular and where the light cones tilt in that form that the inner sector is casually separated from the outer sector of the black hole becomes dependent on the velocity and mass of the particle.

Let’s study this condition:
\[ 1 - \frac{2MG}{\rho c^2} (1 - lm^2 \dot{\rho}^2) > 0. \]  

(15)

This condition imposes a constrain on the velocity of the particle that moves around the black hole:

\[ \dot{\rho}^2 > \frac{1}{lm^2} \left(1 - \frac{c^2 \rho}{2MG}\right). \]  

(16)

It is possible then study three cases:

1. \( \rho > \frac{2MG}{c^2} \):
   
   In this case \( 1 - \frac{\rho}{2MG} \) is always less than zero and there is no restrictions on the radial velocity.

2. \( \rho = \frac{2MG}{c^2} \):
   
   In this case (16) is always true and \( g_{tt} = -lm^2 \dot{\rho}^2 \) and \( g_{\rho\rho} = (lm^2 \dot{\rho}^2)^{-1} \)

   There is no horizon at \( \rho = 2MG \) because the metric is no singular at that point.

3. \( \rho < \frac{2MG}{c^2} \):
   
   While condition (16) remains true, there is no horizon, so it is possible to define a penetration radius:

\[ \rho_p = \frac{2MG}{c^2} \left(1 - lm^2 \dot{\rho}^2\right). \]  

(17)

Due to the inclusion of the terms depending on the Snyder deformation, for an external observer it is possible to "see" a particle penetrating to the interior of the horizon of a black hole. The penetration radius is as little as the factor \( lm^2 \dot{\rho}^2 \) that we supposed tiny ab initio, but can be significative for super massive particles with high speeds.

To see other effects, and to explicit velocity, specially when the particle approaches to \( \rho = 2MG \), let’s write the massive particle stipulation, \(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -1:\)

\[- \left[ 1 - \frac{2GM}{c^2 \rho} (1 - lm^2 \dot{\rho}^2) \right] c^2 \frac{dt^2}{d\lambda^2} + \left[ 1 - \frac{2GM}{c^2 \rho} (1 - lm^2 \dot{\rho}^2) \right]^{-1} \frac{d\rho^2}{d\lambda^2} + \rho^2 \frac{d\Omega^2}{d\lambda^2} = -1. \]  

(18)
Now, despite the modification, we still have the Killing vectors 

\[ A^\mu = (\partial_t)^\mu = (1, 0, 0, 0) \]

and \( B_\mu = (\partial_\phi)^\mu = (0, 0, 0, 1) \), that using our deformed metric lead to the conserved quantities:

\[
E = -\left[1 - \frac{2GM}{\rho} \left(1 - lm^2 \dot{\rho}^2\right)\right] \frac{dt}{d\lambda}, \quad (19)
\]

\[
L = \rho^2 \frac{d\phi}{d\lambda}. \quad (20)
\]

Where it has been chosen \( c = 1 \) and \( \theta = \frac{\pi}{2} \).

Replacing the constants quantities \( E \) and \( L \) in (18):

\[
-E^2 + \left(\frac{d\rho}{d\lambda}\right)^2 + \left(1 - \frac{2GM(1 - lm^2 \dot{\rho}^2)}{\rho}\right)\left(\frac{L^2}{r^2} + 1\right) = 0. \quad (21)
\]

The form of energy is no longer suitable of being divided into a dynamical term (depending just on velocities) and an effective potential (depending just on the coordinates), but it is noticeable that circular orbits are untouched by this proposal because in that case \( \dot{\rho} = 0 \), the deformation becomes zero and the standard case is recovered.

On the other hand, even though the effective potential \( (1 - \frac{2GM(1 - lm^2 \dot{\rho}^2)}{\rho}\right)\left(\frac{L^2}{r^2} + 1\right) \) contains terms that depend on radial velocity of the particle, its shape remains the same, just modulated by the modification introduced. In fact, the radial velocity can be expressed as:

\[
\dot{\rho}^2 = \left(\mathcal{E} + \frac{GM}{\rho} - \frac{L^2}{2r^2} + \frac{GML^2}{\rho^3}\right) \left(\frac{1}{2} + \frac{GM}{\rho} lm + \frac{GML^2}{\rho^3} l m^2\right). \quad (22)
\]

Where it has been identified \( \frac{d\rho}{d\lambda} = \dot{\rho} \) and defined \( \mathcal{E} = \frac{1}{2}(E^2 - 1) \)

It is possible to see that the condition \( \dot{\rho}^2 = 0 \) does not depend on the Planck term.

Obtaining \( \dot{\rho}^2 \) from (21) \( L = 0 \) for radial orbits and replacing in (17) (with \( c = 1 \)), we can see that the penetration radius of a particle is:

\[
\rho_p = 2MG(1 - E^2 lm^2), \quad (23)
\]

that can be considerably less than \( 2MG \), depending on the energy of the particle. So, for a enough energetic and massive particle an exterior observer can "see" how it falls into the singularity.

Let’s take a look on the penetration radius of a light ray in order of having a notion of what is really possible to see from the exterior. For that case we can replace \( m \dot{\rho} \) and use \( p_\rho \)
instead or, if we deal with radial orbits, just use \( p \). It is necessary also to let \( d\tau = 0 \). So we have:

\[
0 = [1 - \frac{2GM}{\rho}(1 - lp)]dt^2 + [1 - \frac{2GM}{\rho}(1 - lp)^{-1}d\rho^2.
\] (24)

Under the requirement that the penetration speed must be more than zero, we have a penetration radius of a ray light:

\[
\rho_{\text{light}}^p = 2MG(1 - lp^2).
\] (25)

So, light rays can penetrate (or escape), depending on its momentum. Anyway, to hypothetically reach the singularity the wave length should be of the order of the Planck scale, this is a condition that matches the idea that at that scale Physics can be totally different to the every day experience.

For the most part of high energy light rays it is possible to go beyond the Schwarzchild radius and it could be imaginable to get information from the interior of the black hole through a skin penetration like phenomenon.

**IV. FINAL REMARKS**

It has been introduced a mass/velocity depending deformation to the Schwarzchild metric that correspond to consider a minimal fundamental length. Under this deformation the metric conserves the symmetries and this fact is used to take a look on some variations of the horizon of the black hole. Effectively the introduction of the Snyder factor changes kinematically the Schwarzchild radius and allows particles and light rays to go beyond that limit. This result indicates that it could be possible to see some effects about the border of a black hole like light splitting depending on wave length, in the same way of a rainbow like situation. Furthermore, some effects are expectable for collisions with supermassive objects due to the different penetration radius for radial orbits. Altogether, these features can be considered as a skin effect. On the other hand, non radial orbits aren’t perturbed and no
effect seems to appear.

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