Quasiprobability Based Criterion for Classicality and Separability of States of Spin-1/2 Particles

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Abstract
A sufficient condition for a quantum state of a system of spin-1/2 particles (spin-1/2s) to admit a local hidden variable (LHV) description i.e. to be classical is the separability of the density matrix characterizing its state, but not all classical states are separable. This leads one to infer that separability and classicality are two different concepts. These concepts are examined here in the framework of a criterion for identifying classicality of a system of spin-1/2s based on the concept of joint quasiprobability (JQP) for the eigenvalues of spin components (R.R.Puri, J.Phys. A29, 5719 (1996)). The said criterion identifies a state as classical if a suitably defined JQP of the eigenvalues of spin components in suitably chosen three orthogonal directions is non-negative. In agreement with other approaches, the JQP based criterion also leads to the result that all non-factorizable pure states of two spin-1/2s are non-classical. In this paper it is shown that the application of the said criterion to mixed states suggests that the states it identifies as classical are also separable and that there exist states which, identified as classical by other methods, may not be identified as classical by the criterion as it stands. However, the results in agreement with the known ones are obtained if the criterion is modified to identify also those states as classical for which the JQP of the eigenvalues of the spin components in two of the three prescribed orthogonal directions is non-negative. The validity of the modified criterion is confirmed by comparing its predictions with those arrived at by other methods when applied to several mixed states of two spin-1/2s and the Werner like state of three spin-1/2s (G.Toth and A.Acin, Phys.Rev. A74, 030306(R) (2006)). The JQP based approach, formulated as it is along the lines of the $P$-function approach for identifying classical states of the electromagnetic field, offers a unified approach for systems of arbitrary number of spin-1/2s and the possibility of linking classicality with the nature of the measurement process.

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1 Introduction

In view of its importance in quantum computing and quantum information processing [1], identification of exclusively quantum correlations in a collection of identical two level systems, hereafter referred to as a system of spin-1/2 particles (spin-1/2s), continues to attract a great deal of attention [2, 3]. The correlations are classical if they can be described in terms of a purely classical local hidden variable (LHV) theory. A sufficient condition for a quantum state of a system consisting of two subsystems to admit a LHV description is that the density matrix $\hat{\rho}$ characterizing its state be separable i.e. it be expressible as

$$\hat{\rho} = \sum_i p_i \hat{\rho}^{(1)}_i \otimes \hat{\rho}^{(2)}_i, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1,$$

where $\hat{\rho}^{(1)}_i$ and $\hat{\rho}^{(2)}_i$ are the density matrices of the subsystems numbered 1 and 2. A necessary condition for separability defined in (1) was derived by Peres [5]. It was subsequently shown that the Peres condition is also sufficient if the Hilbert space of each of the two subsystems is two dimensional or if the dimension of the Hilbert space of one of the systems is two and that of the other is three [6]. Thus, for a system of two spin-1/2s, the Peres criterion constitutes necessary and sufficient condition for separability.

Starting with the work of Werner [4], the relationship between entangled and non-locality has been investigated extensively (see for example references [7]-[9] and the references therein). In particular, by constructing a non-separable mixed state of two subsystems which nevertheless admits LHV description under projective measurements, Werner [4] showed that separability is a sufficient but not a necessary condition for classicality. This result has since been generalized to general measurement processes [7] and to tripartite systems [8].

In this paper we formulate an approach to identifying classicality in a system of spin-1/2s along the lines of that followed for classifying the states of the electromagnetic (e.m.) field as classical and non-classical which is based on the concept of quasiprobability distribution (QPD) function corresponding to a quantum state. Recall that a QPD is constructed by considering the quantum observables as classical random variables whose probability distribution is expressed in terms of averages of products of observables. The observables may correspond to non-commuting operators. A QPD is constructed by replacing the averages of products of observables by the expectation values of the corresponding operators by choosing the order in which the non-commuting operators be placed in the products of corresponding observables. There is thus a QPD for each ordering of product of non-commuting operators. The classicality of a state may be identified as that of the QPD in a specified operator ordering. The specification of the ordering for the said purpose is based on physics considerations like the relation between the operator ordering and the measurement process or some other desirable physics aspect. For example, the classicality of
the states of the electromagnetic (e.m.) field is defined in terms of the QPD called Sudarshan-Glauber P-function. That function is the QPD corresponding to the ordering of the field creation and annihilation operators appropriate for those processes of measurement, like the ones by photodetectors, which measure averages of normally ordered operators \[10, 11\].

The physics consideration for constructing the QPD relevant to identifying quantum nature of correlations in a system of spin-1/2s could be that it should identify every state of a spin-1/2 as also every uncorrelated state of a collection of spin-1/2s as classical. The so called phase space QPDs for spin-1/2s, constructed in analogy with those for the e.m. field, do not fulfill the desired conditions \[11-13\]. The desired end is achieved by invoking the concept of joint quasiprobability (JQP) \[11, 14-18\] for the distribution of the eigenvalues of the components of spins in various directions. Based on that concept, and by demanding that any state of a single spin-1/2 and any product of single spin-1/2 states be classical, a criterion has been introduced in Ref. \[13\] to classify the states as classical and non-classical. The said criterion identifies a state as classical if a suitably defined JQP of the eigenvalues of spin components in suitably chosen three orthogonal directions is non-negative i.e. classical. It leads to the conclusion that any non-factorizable pure state of two spin-1/2s is non-classical. This result is arrived at also by other approaches \[19\].

When applied to mixed states, it is found that the said criterion as it stands may leave out from the set of states it identifies as classical some separable states. It also does not identify those non-separable states as classical which are so identified by other methods. In this paper we propose a modified JQP based criterion which is free of the abovementioned lacunae. The modified criterion identifies as classical, not only those states whose JQP for suitably chosen three orthogonal components of each spin is non-negative, but also those for which the JQP for any two of those three orthogonal components is non-negative. The validity of the modified criterion is confirmed by examining the classicality and separability of some of those mixed states of two spin-1/2s and a state of three spin-1/2s whose said properties are known following other approaches.

The paper is organized as follows. The Sec.2 summarizes the main results of the theory of JQP for a system of spin-1/2s as formulated in \[17, 18\] and states the JQP based criterion of classicality as it stands and its proposed modified version. The conditions of classicality of certain mixed states of two spin-1/2s and that of a system of three spin-1/2s are derived using the modified JQP based criterion and the results are compared with those found by other methods. The Sec.4 summarizes the conclusions.

## 2 Joint Quasiprobability for System of Spin-1/2s

In local hidden variable (LHV) theory a spin-1/2 is visualized as a vector $\mathbf{S}$ whose components along any direction can assume two values, say $\pm 1/2$, and is assumed to be under the influence of some unknown "hidden" causes or
variables acting randomly. The random influence of the hidden variables results in the components of the spin in any direction acquiring randomly the values 1/2 and -1/2. The properties of the spin may then be described in terms of the probability distribution functions \( f(S_{a_1}, S_{a_2}, \ldots, S_{a_N}) \) for the components of the spin in various directions where

\[
S_{a_i} = \mathbf{S} \cdot \mathbf{a}_i, \tag{2}
\]

is the component of spin in the direction \( \mathbf{a}_i \). Now, let \( p(\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_N}) \) (\( \epsilon_{a_i} = \pm 1 \)) denote the joint probability for the components of the spin along the directions \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N \) to have the values \( \epsilon_{a_1}/2, \epsilon_{a_2}/2, \ldots, \epsilon_{a_N}/2 \) respectively so that

\[
f(S_{a_1}, S_{a_2}, \ldots, S_{a_N}) = \sum_{\epsilon_1, \epsilon_2, \ldots, \epsilon_N = \pm 1} \left[ \delta \left( S_{a_1} - \frac{\epsilon_{a_1}}{2} \right) \delta \left( S_{a_2} - \frac{\epsilon_{a_2}}{2} \right) \cdots \delta \left( S_{a_N} - \frac{\epsilon_{a_N}}{2} \right) \right] \times p(\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_N}). \tag{3}
\]

From this it is straightforward to see that [17] [18]

\[
p(\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_N}) = \left( \left( \frac{1}{2} + \epsilon_{a_1} S_{a_1} \right) \left( \frac{1}{2} + \epsilon_{a_2} S_{a_2} \right) \cdots \left( \frac{1}{2} + \epsilon_{a_N} S_{a_N} \right) \right), \tag{4}
\]

where the angular bracket denotes average with respect to the distribution function \( f(S_{a_1}, S_{a_2}, \ldots, S_{a_N}) \):

\[
\langle A \rangle = \int Af(S_{a_1}, S_{a_2}, \ldots, S_{a_N}) \prod_{i=1}^{N} dS_{a_i}. \tag{5}
\]

The joint probability distribution for two spins can be similarly defined and shown to be expressible as

\[
p(\epsilon_{a_1}^{(1)}, \epsilon_{a_2}^{(1)}, \ldots, \epsilon_{a_M}^{(1)}; \epsilon_{b_1}^{(2)}, \epsilon_{b_2}^{(2)}, \ldots, \epsilon_{b_N}^{(2)}) = \left( \left( \frac{1}{2} + \epsilon_{a_1}^{(1)} S_{a_1} \right) \left( \frac{1}{2} + \epsilon_{a_2}^{(1)} S_{a_2} \right) \cdots \left( \frac{1}{2} + \epsilon_{a_M}^{(1)} S_{a_M} \right) \right) \left( \left( \frac{1}{2} + \epsilon_{b_1}^{(2)} S_{b_1} \right) \left( \frac{1}{2} + \epsilon_{b_2}^{(2)} S_{b_2} \right) \cdots \left( \frac{1}{2} + \epsilon_{b_N}^{(2)} S_{b_N} \right) \right). \tag{6}
\]

The function \( p(\epsilon_{a_1}^{(1)}, \epsilon_{a_2}^{(1)}, \ldots, \epsilon_{a_M}^{(1)}; \epsilon_{b_1}^{(2)}, \epsilon_{b_2}^{(2)}, \ldots, \epsilon_{b_N}^{(2)}) \) stands for the probability of finding the components of spin number 1 to have values \( \epsilon_{a_1}^{(1)}/2, \epsilon_{a_2}^{(1)}/2, \ldots, \epsilon_{a_M}^{(1)}/2 \) in the directions \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M \) and the components of spin number 2 to have the values \( \epsilon_{b_1}^{(2)}/2, \epsilon_{b_2}^{(2)}/2, \ldots, \epsilon_{b_N}^{(2)}/2 \) in the directions \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \) with \( \epsilon_{a_i}^{(1)}, \epsilon_{b_i}^{(2)} = \pm 1 \).

Now, a spin-1/2 in quantum theory is described by the vector operator \( \hat{\mathbf{S}} \) and its state is characterized by a density matrix \( \hat{\rho} \) using which one can evaluate expectation values of relevant operators. As we know, there is no concept of
joint probability of assigning definite values for two non-commuting observables and hence there is no place for the joint probabilities, like the ones defined in (4) and (6), in the quantum theory.

However, the concept of quasiprobabilities has been found useful to understand the signatures of quantum behaviour. That concept in the case of single spin-1/2 may be introduced (i) by replacing the classical random variables \( S_a \) by the operator \( \hat{S}_a \) which obey the commutation relation

\[
[\hat{S}_a, \hat{S}_b] = i (a \times b) \cdot \hat{S},
\]

and the anti-commutation relation

\[
\hat{S}_a \hat{S}_b + \hat{S}_b \hat{S}_a = \frac{a \cdot b}{2},
\]

(ii) by assigning a rule, called the Chosen Ordering (CO), for ordering operators in a product of non-commuting operators, and (iii) by replacing the average therein as the quantum mechanical expectation value wherein the system is described by a density matrix \( \hat{\rho} \) and the expectation values of an operator \( \hat{A} \) is given by \( \text{Tr}(\hat{A} \hat{\rho}) \). The expression in quantum theory corresponding to (4) then reads

\[
p(\epsilon_{a_1}, \epsilon_{a_2}, \ldots, \epsilon_{a_N}) = \text{Tr}\left[\hat{\rho}\left\{\left(\frac{1}{2} + \epsilon_{a_1} \hat{S}_{a_1}\right)\left(\frac{1}{2} + \epsilon_{a_2} \hat{S}_{a_2}\right) \cdots \left(\frac{1}{2} + \epsilon_{a_N} \hat{S}_{a_N}\right)\right\}_\text{CO}\right],
\]

where ‘CO’ stands for Chosen Ordering of the operator product. In the same way, the quantum analog of the joint quasiprobability for two spins reads

\[
p(\epsilon_{a_1}^{(1)}, \epsilon_{a_2}^{(1)}, \ldots, \epsilon_{a_M}^{(1)}; \epsilon_{b_1}^{(2)}, \epsilon_{b_2}^{(2)}, \ldots, \epsilon_{b_N}^{(2)}) = \text{Tr}\left[\hat{\rho}\left\{\left(\frac{1}{2} + \epsilon_{a_1}^{(1)} \hat{S}_{a_1}\right)\left(\frac{1}{2} + \epsilon_{a_2}^{(1)} \hat{S}_{a_2}\right) \cdots \left(\frac{1}{2} + \epsilon_{a_M}^{(1)} \hat{S}_{a_M}\right)\right\}_\text{CO}\right].
\]

In what follows, it will be seen that an ordering of interest is the symmetric ordering in which the c-number product is replaced by the operator product by following correspondence:

\[
S_a S_b \rightarrow \frac{1}{2} [\hat{S}_a \hat{S}_b + \hat{S}_b \hat{S}_a] = \frac{a \cdot b}{4},
\]

\[
S_a S_b S_c \rightarrow \frac{1}{12} [\hat{S}_a (\hat{S}_b \hat{S}_c + \hat{S}_c \hat{S}_b) + (\hat{S}_b \hat{S}_c + \hat{S}_c \hat{S}_b) \hat{S}_a + \left( a \rightarrow b, b \rightarrow c, c \rightarrow a \right) + \left( a \rightarrow c, c \rightarrow b, b \rightarrow a \right)]
\]

\[
= \frac{1}{12} \left[ b \cdot c \hat{S}_a + a \cdot c \hat{S}_b + a \cdot b \hat{S}_c \right].
\]

In writing the equation above we have invoked the anti-commutation relation (8).
On using the correspondence above as the 'CO', the expression \( (9) \) for the probability distribution of the components of single spin-1/2 in three mutually orthogonal directions \( a_1, b_1, c_1 \) assumes the form

\[
p(\epsilon_{a_1}, \epsilon_{b_1}, \epsilon_{c_1}) = \frac{1}{2^2} \text{Tr} \left[ \hat{\rho} \left( \frac{1}{2} + \epsilon_{a_1} \hat{S}_{a_1} + \epsilon_{b_1} \hat{S}_{b_1} + \epsilon_{c_1} \hat{S}_{c_1} \right) \right].
\]

(12)

Similarly, the expression \( (10) \) for the JQP for the components of spin number 1 in the orthogonal directions \( a_1, b_1, c_1 \) and those of spin number 2 in the orthogonal directions \( a_2, b_2, c_2 \) in the 'CO' defined in \( (11) \) would read

\[
p(\epsilon^{(1)}_{a_1}, \epsilon^{(1)}_{b_1}, \epsilon^{(1)}_{c_1}; \epsilon^{(2)}_{a_2}, \epsilon^{(2)}_{b_2}, \epsilon^{(2)}_{c_2}) = \frac{1}{2^4} \text{Tr} \prod_{j=1}^{2} \left[ \hat{\rho} \left( \frac{1}{2} + \epsilon^{(j)}_{a_j} \hat{S}_{a_j} + \epsilon^{(j)}_{b_j} \hat{S}_{b_j} + \epsilon^{(j)}_{c_j} \hat{S}_{c_j} \right) \right].
\]

(13)

The generalization of the considerations above leads to the following expression for the JQP for the components of three spin-1/2s in the mutually orthogonal directions \( a_i, b_i, c_i \) \( (i = 1, 2, 3) \):

\[
p(\epsilon^{(1)}_{a_1}, \epsilon^{(1)}_{b_1}, \epsilon^{(1)}_{c_1}; \epsilon^{(2)}_{a_2}, \epsilon^{(2)}_{b_2}, \epsilon^{(2)}_{c_2}; \epsilon^{(3)}_{a_3}, \epsilon^{(3)}_{b_3}, \epsilon^{(3)}_{c_3})
= \frac{1}{2^6} \text{Tr} \prod_{j=1}^{3} \left[ \hat{\rho} \left( \frac{1}{2} + \epsilon^{(j)}_{a_j} \hat{S}_{a_j} + \epsilon^{(j)}_{b_j} \hat{S}_{b_j} + \epsilon^{(j)}_{c_j} \hat{S}_{c_j} \right) \right].
\]

(14)

The expressions \( (12) - (14) \) are central for classifying the states as classical or non-classical.

Now, recall that the states of the e.m. field are labelled classical or non-classical on the basis of the classicality or otherwise of the quasiprobability function, called Sudarshan-Glauber P-function. The choice of the P-function for the purpose is based on the fact that it serves as a probability distribution function for the averages of normally ordered field creation and annihilation operators and that the process of photo-detection measures the average of normally ordered field operators. Since in the case of a system of spins, it is the correlations between the spins which are the desired measure of the quantum nature, it would be appropriate to formulate the criterion for identifying the classicality of the states of a system of spin-1/2s by demanding that the chosen quasiprobability identifies any state of single spin-1/2 as also any product of single spin-1/2 states to be classical so that non-classicality, if any, is attributable to spin-spin correlations. It has been shown in [18] that the the said conditions are satisfied by the following criterion:

A quantum state of a system of \( N \) spin-1/2s is classical if the joint quasiprobability for the eigenvalues of the components of each spin in three mutually orthogonal directions, one of which is the average direction of that spin, is classical in the symmetric ordering of the operators. It is non-classical if any of those \( m \)-spin \( (m \leq N) \) joint quasiprobabilities is negative in the said ordering.

Hence, according to this criterion, the JQP for classifying the states of two spin-1/2s as classical or non-classical is the one given in \( (13) \) where one of the
three orthogonal directions $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1$ is the average direction of spin number 1 and one of the three orthogonal directions $\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2$ is the average direction of spin number 2 with similar interpretation for the expression in (14) for the JQP for three spin-1/2s.

When applied to pure states of $N$ spin-1/2s, the criterion above identifies [18], (i) the spin coherent state of $N$ spin-1/2s as classical, (ii) any non-factorizable pure state of two spin-1/2s as non-classical which is in agreement with the conclusion in [19] for two spin-1/2s arrived at differently, (iii) any collective spin state $|N, m\rangle$ (with $\hat{S}_z|N, m\rangle = m|N, m\rangle$) as non-classical if $m \neq \pm N/2$, and (iv) a squeezed state of $N$ spin-1/2s as non-classical for all $N \geq 2$. Thus, except the states in which all the spins are uncorrelated, all the other pure states of spin-1/2s examined by applying the said criterion turn out to be non-classical. This is analogous to the result that, other than its coherent state, any pure state of the electromagnetic field is non-classical [10, 11].

However, when applied to mixed states, it is found that the criterion above may not identify all the separable states as classical and that it does not identify those non-separable states as classical which have been shown to be so by following other methods. Those deficiencies of the criterion can be remedied by modifying it to read as follows:

A quantum state of a system of $N$ spin-1/2s is classical (i) if joint quasiprobability (JQP) for the eigenvalues of the components of each spin in three mutually orthogonal directions, one of which is the average direction of that spin, is classical in the symmetric ordering of the operators. The state in this case is also separable. or (ii) if the JQP for two of the three orthogonal components prescribed as above is classical. The state in this case may or may not be separable.

The states identified as classical as per the condition in (i) in the criterion above will, of course, be so according to the condition in part (ii) as well. However, the states identified as classical as per part (i) are separable which need not be the case for those identified as classical as per part (ii). It should be emphasized that the criterion, based as it is on the non-negativity of suitably defined JQP, is meant basically to identify classical states. The separability is not related with non-negativity of the JQP. The separability conjectured in the criterion is based, not on any fundamental consideration, but on the inference drawn from the study of separability of some systems.

In the next section we demonstrate the validity of the criterion above by investigating the classicality and separability of some widely studied mixed states of two spin-1/2s and a state of three spin-1/2s.

3 JQP Based Classicality and separability of Some Mixed States

Let the spin number $j$ be described in the basis constituted by the eigenstates
1. Consider first Werner’s density matrix \[ 4 \] then conveniently by noting that if \( \hat{\epsilon} \) eigenstates of \( \hat{\epsilon} \) directions. The expectation values involving the operators \( \hat{a}, \hat{b}, \hat{c} \) each of the spins in the state described by \( \hat{\rho} \). We describe a combined state of the system in terms of the basis of direct product orthonormal states \( |\pm, \pm\rangle \equiv |\pm, 1 \rangle \otimes |\pm, 2 \rangle \) or their symmetrised orthonormal combinations,

\[
|\psi_{0,1}\rangle = \frac{1}{\sqrt{2}} \left( |+,-\rangle \mp |-,+\rangle \right) , \quad |\psi_{2,3}\rangle = \frac{1}{\sqrt{2}} \left( |+,+\rangle \pm |-,,-\rangle \right) .
\]

The state \( |\psi_0\rangle \), antisymmetric under the exchange of spins, is the singlet whereas the other three, symmetric under the exchange of spins, constitute the triplet. Similarly, the basis states of three spin-1/2s shall be denoted by \( |\pm, \pm, \pm\rangle \equiv |\pm, 1 \rangle \otimes |\pm, 2 \rangle \otimes |\pm, 3 \rangle \).

In the examples below, the average direction of the spins is identical. The JQPs in (13) and (14) then assume the form

\[
p \left( \epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)} ; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)} \right)
= \frac{1}{24} \text{Tr} \left[ \hat{\rho} \prod_{j=1}^{2} \left( \frac{1}{2} + \epsilon_a^{(j)} \hat{S}_a^{(j)} + \epsilon_b^{(j)} \hat{S}_b^{(j)} + \epsilon_c^{(j)} \hat{S}_c^{(j)} \right) \right] ,
\]

for the system of two spin-1/2s and, for the system of three spin-1/2s,

\[
p \left( \epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)} ; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)} ; \epsilon_a^{(3)}, \epsilon_b^{(3)}, \epsilon_c^{(3)} \right)
= \frac{1}{26} \text{Tr} \left[ \hat{\rho} \prod_{j=1}^{3} \left( \frac{1}{2} + \epsilon_a^{(j)} \hat{S}_a^{(j)} + \epsilon_b^{(j)} \hat{S}_b^{(j)} + \epsilon_c^{(j)} \hat{S}_c^{(j)} \right) \right] ,
\]

where \( \epsilon_{a,b,c} = \pm 1 \); \( \hat{S}^{(j)}_a, \hat{S}^{(j)}_b, \hat{S}^{(j)}_c \) are the components of the \( j \)-th spin in the mutually orthogonal directions \( \{\hat{a}, \hat{b}, \hat{c}\} \) in which \( \hat{a} \) is the average direction of each of the spins in the state described by \( \hat{\rho} \). In the examples in which the average value of the spin is zero, \( \{\hat{a}, \hat{b}, \hat{c}\} \) can be any set of mutually orthogonal directions. The expectation values involving the operators \( \hat{S}^{(j)}_b \) may be evaluated conveniently by noting that if \( \hat{S}^{(j)}_{a\pm} \) are the raising and lowering operators on the eigenstates of \( \hat{S}^{(j)}_a \) i.e. if

\[
\hat{S}^{(j)}_{a\pm} |+, j\rangle = |+, j\rangle , \quad \hat{S}^{(j)}_{a\pm} |+, j\rangle = 0 ,
\]

then \( \hat{S}^{(j)}_b = (\hat{S}^{(j)}_{a+} + \hat{S}^{(j)}_{a-})/2 \) and \( \hat{S}^{(j)}_c = (\hat{S}^{(j)}_{a+} - \hat{S}^{(j)}_{a-})/2i \).

1. Consider first Werner’s density matrix \[ 4 \]

\[
\hat{\rho} = \frac{1}{1-x} I + x |\psi_0\rangle \langle \psi_0| , \quad 0 \leq x \leq 1 ,
\]
where $I$ is the identity operator, and $|\psi_0\rangle$ is as in [15]. The non-zero expectation values needed for evaluating the expression in (16) in this case are (with $\langle \hat{A} \rangle \equiv \text{Tr}[\hat{\rho} \hat{A}]$),

$$
\langle \hat{S}_\mu^{(1)} \hat{S}_\mu^{(2)} \rangle = -\frac{x}{4}, \quad (\mu = a, b, c).
$$

(20)

The JQP of (16) then assumes the form

$$
p \left( \epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)} \right)
= \frac{1}{26} \left[ 1 - x \left( \epsilon_a^{(1)} \epsilon_a^{(2)} + \epsilon_b^{(1)} \epsilon_b^{(2)} + \epsilon_c^{(1)} \epsilon_c^{(2)} \right) \right].
$$

(21)

The minimum value of $p$ is

$$
p_{\text{min}} = \frac{1}{26} \left[ 1 - 3x \right].
$$

(22)

This is non-negative if $x \leq 1/3$. Hence, according to the part (i) of the JQP based criterion, the state (19) is classical and also separable if $x \leq 1/3$. This is the same as the condition for separability of (19) according to the Peres criterion [5].

Next, we identify the classical states as per the part (ii) of the JQP based criterion. To that end, consider the reduced JQP $p^{(r)}(\epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)})$ for the probability of spin components in the directions $a$ and $b$ obtained by summing (21) over $\epsilon_c^{(1)}$ and $\epsilon_c^{(2)}$ so that

$$
p^{(r)} \left( \epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_a^{(2)}, \epsilon_b^{(2)} \right)
= \frac{1}{24} \left[ 1 - x \left( \epsilon_a^{(1)} \epsilon_a^{(2)} + \epsilon_b^{(1)} \epsilon_b^{(2)} \right) \right].
$$

(23)

The minimum value of $p^{(r)}$ is

$$
p^{(r)}_{\text{min}} = \frac{1}{24} \left[ 1 - 2x \right].
$$

(24)

This is non-negative if $x \leq 1/2$. This shows that the state (19) is classical, not only for the values of $x \leq 1/3$ for which it is separable, but also for the values $1/3 < x \leq 1/2$ for which it is not separable. The condition $x \leq 1/2$ for the classicality of the Werner state is the same as the one arrived at in [8] by another method.

2. Consider the system described by the density matrix [5]

$$
\hat{\rho} = x |\psi_0\rangle \langle \psi_0| + (1 - x)|+, +\rangle \langle +, +|, \quad 0 \leq x \leq 1.
$$

(25)

The non-zero expectation values needed for evaluating (16) are

$$
\langle \hat{S}_a^{(1)} \rangle = \langle \hat{S}_a^{(2)} \rangle = \frac{1 - x}{2}, \quad \langle \hat{S}_a^{(1)} \hat{S}_a^{(2)} \rangle = \frac{1 - 2x}{4},
$$

$$
\langle \hat{S}_b^{(1)} \hat{S}_b^{(2)} \rangle = \langle \hat{S}_c^{(1)} \hat{S}_c^{(2)} \rangle = -\frac{x}{4}.
$$

(26)
Consequently, the JQP \((16)\) is given by
\[
p(\epsilon^{(1)}_a, \epsilon^{(1)}_b, \epsilon^{(2)}_a, \epsilon^{(2)}_b, \epsilon^{(2)}_c) = \frac{1}{2^6} \left[ 1 + (1 - x)(\epsilon^{(1)}_a + \epsilon^{(2)}_a) + \epsilon^{(1)}_a \epsilon^{(2)}_a (1 - 2x) - x \left( \epsilon^{(1)}_b \epsilon^{(2)}_b + \epsilon^{(1)}_c \epsilon^{(2)}_c \right) \right].
\] (27)

The minimum value of the \(p\) above may easily be seen to be given by
\[
[p]_{\text{min}} = -\frac{x}{32}. \quad (28)
\]

For \(0 \leq x \leq 1\), the expression above is non-negative only for \(x = 0\) implying that the system admits a LHV description and is classical only for \(x = 0\). For this value of \(x\), the state \((25)\) is evidently separable. The test of Peres shows that \((25)\) is indeed separable only for \(x = 0\). Thus, the JQP based criterion gives results in agreement with those derived by applying Peres criterion.

3. Consider the density matrix \((20)\)
\[
\hat{\rho} = \sum_{i=0}^{3} x_i |\psi_i\rangle \langle \psi_i|, \quad \sum_{i=0}^{3} x_i = 1,
\] (29)

where \(|\psi_i\rangle\) are as in \((15)\). The non-zero expectation values in \((16)\) in this case are
\[
\langle \hat{S}^{(1)}_a \hat{S}^{(2)}_a \rangle = \frac{1}{4} (x_2 + x_3 - x_0 - x_1),
\]
\[
\langle \hat{S}^{(1)}_b \hat{S}^{(2)}_b \rangle = \frac{1}{4} (x_2 + x_1 - x_0 - x_3),
\]
\[
\langle \hat{S}^{(1)}_c \hat{S}^{(2)}_c \rangle = \frac{1}{4} (x_1 + x_3 - x_0 - x_2).
\] (30)

On substituting these values in \((16)\), the JQP corresponding to \((20)\) reads
\[
p(\epsilon^{(1)}_a, \epsilon^{(1)}_b, \epsilon^{(1)}_c; \epsilon^{(2)}_a, \epsilon^{(2)}_b, \epsilon^{(2)}_c) = \frac{1}{2^6} \left[ 1 + \epsilon^{(1)}_a \epsilon^{(2)}_a (x_2 + x_3 - x_0 - x_1) + (\epsilon^{(1)}_b \epsilon^{(2)}_b - \epsilon^{(1)}_c \epsilon^{(2)}_c)(x_2 - x_3) + (\epsilon^{(1)}_b \epsilon^{(2)}_b + \epsilon^{(1)}_c \epsilon^{(2)}_c)(x_2 - x_0) \right].
\] (31)

It may be verified that
\[
p(\epsilon_a, \epsilon_b, \epsilon_c; \epsilon_a, \epsilon_b, \epsilon_c) = \frac{1}{2^5}(1 - 2x_0),
\]
\[
p(\epsilon_a, \epsilon_b, \epsilon_c; -\epsilon_a, -\epsilon_b, -\epsilon_c) = \frac{1}{2^5}(1 - 2x_1),
\]
\[
p(\epsilon_a, -\epsilon_b, -\epsilon_c; -\epsilon_a, -\epsilon_b, -\epsilon_c) = \frac{1}{2^5}(1 - 2x_2),
\]
\[
p(-\epsilon_a, \epsilon_b, -\epsilon_c; -\epsilon_a, -\epsilon_b, -\epsilon_c) = \frac{1}{2^5}(1 - 2x_3).
\] (32)
The $p'$s for other combinations of the $\epsilon'$s are non-negative for all values of the $x'_i$s. The $p'$s in the expressions above will also be non-negative i.e. the system will admit LHV description and will be separable if $x_i \leq 1/2 \ (i = 0, 1, 2, 3)$. It may be verified that the same condition is obtained for the system to be separable according to the Peres criterion.

4. Consider next the state \[21\]
\[\hat{\rho} = x|\psi\rangle\langle\psi| + \frac{1-x}{2}(|+;++|+|--\rangle\langle-|), \tag{33}\]
where
\[|\psi\rangle = \alpha|+;\rangle + \beta|\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{34}\]
For the sake of simplicity of illustration, we let $\alpha$ and $\beta$ to be real. It is then straightforward to show that non-zero expectation values needed for evaluating \[16\] are given by
\[\langle \hat{S}_{(1)} \rangle = -\langle \hat{S}_{(2)} \rangle = \frac{x}{2}(|\alpha|^2 - |\beta|^2), \quad \langle \hat{S}_{(1)} \hat{S}_{(2)} \rangle = \frac{1}{4}(1 - 2x), \tag{35}\]
\[\langle \hat{S}_{(1)} \rangle = \langle \hat{S}_{(2)} \rangle = \frac{x\alpha\beta}{2}. \]
On substituting these values in \[16\], the JQP corresponding to the state \[33\] reads
\[p \left( \epsilon_{(1)}^{(1)} , \epsilon_{(1)}^{(2)} , \epsilon_{(1)}^{(3)} , \epsilon_{(2)}^{(1)} , \epsilon_{(2)}^{(2)} \right) \]
\[= \frac{1}{2^6} \left[ 1 + x(|\alpha|^2 - |\beta|^2)(\epsilon_{(1)}^{(1)} - \epsilon_{(1)}^{(2)}) + \epsilon_{(2)}^{(1)} \epsilon_{(2)}^{(2)} (1 - 2x) + 2x\alpha\beta \left( \epsilon_{(1)}^{(1)} \epsilon_{(2)}^{(2)} + \epsilon_{(1)}^{(2)} \epsilon_{(2)}^{(1)} \right) \right]. \tag{36}\]
It is straightforward to see that the minimum of \[36\] with respect to $\epsilon_{(1)}^{(1)} , \epsilon_{(1)}^{(2)}$ is achieved when $\epsilon_{(1)}^{(1)} \epsilon_{(2)}^{(2)} = \epsilon_{(1)}^{(2)} \epsilon_{(2)}^{(1)} = -\alpha\beta/|\alpha||\beta| \equiv s$ so that
\[p \left( \epsilon_{(1)}^{(1)} , \epsilon_{(2)}^{(1)} , \epsilon_{(2)}^{(2)} , \epsilon_{(3)}^{(2)} , \epsilon_{(4)}^{(1)} , \epsilon_{(4)}^{(2)} \right) \]
\[= \frac{1}{2^6} \left[ 1 + x(|\alpha|^2 - |\beta|^2)(\epsilon_{(1)}^{(1)} - \epsilon_{(1)}^{(2)}) + \epsilon_{(2)}^{(1)} \epsilon_{(2)}^{(2)} (1 - 2x) + 4x|\alpha||\beta| \right]. \tag{37}\]
Now, for $\epsilon_{(1)}^{(1)} = \epsilon_{(2)}^{(2)} = \epsilon$,
\[p \left( \epsilon , \epsilon , \epsilon_{(3)}^{(2)} , s\epsilon_{(3)}^{(2)} , s\epsilon_{(3)}^{(2)} \right) = \frac{1}{2^6} (1 - x - 2x|\alpha||\beta|), \tag{38}\]
whereas for $\epsilon_{(1)}^{(1)} = -\epsilon_{(2)}^{(2)} = 1$,
\[p \left( 1 , \epsilon , \epsilon_{(3)}^{(2)} , -1 , s\epsilon_{(3)}^{(2)} , s\epsilon_{(3)}^{(2)} \right) = |\alpha| x(|\alpha| - |\beta|)/16, \tag{39}\]
and for $\epsilon^{(1)}_a = -\epsilon^{(2)}_a = -1$,

$$
p\left(-1, \epsilon_b, \epsilon_c; 1, s\epsilon_b, s\epsilon_c\right) = |\beta|x(|\beta| - |\alpha|)/16. \tag{40}
$$

The JQP’s in the equations (38)-(40) are non-negative for any $x$ if $\alpha\beta = 0$. The state (33) in this case is clearly separable. However, if $\alpha\beta \neq 0$ then the JQP’s (38)-(40) are non-negative if $|\alpha| = |\beta|$ and

$$
x \leq (1 + 2|\alpha||\beta|)^{-1}. \tag{41}
$$

Thus, according to the part (i) of the JQP based criterion, the system is classical and separable if $|\alpha| = |\beta|$ and $x$ obeys the condition in (41). The condition in the equation above is also the one for separability according to the Peres criterion [5] but without the additional condition $|\alpha| = |\beta|$. Since separable states are classical, the set of classical states predicted by part (i) of the JQP based criterion leaves out the states having the value of $x$ as in (41) but $|\alpha| \neq |\beta|$. Hence it is necessary to determine classical states as per part (ii) of that criterion to find whether the set of those states contains the ones identified as separable by Peres criterion. To that end, it may be seen from (36) that the JQP for the components along $b$ and $c$ for each of the two spins is given by

$$
p\left(\epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_b^{(2)}, \epsilon_c^{(2)}\right) = \frac{1}{24}\left[1 + 2x\alpha\beta \left(\epsilon_b^{(1)}\epsilon_b^{(2)} + \epsilon_c^{(1)}\epsilon_c^{(2)}\right)\right]. \tag{42}
$$

This is non-negative for all $\alpha, \beta$ if

$$
x \leq \frac{1}{4|\alpha||\beta|}. \tag{43}
$$

Keeping in mind the fact that $|\alpha|^2 + |\beta|^2 = 1$, it is seen that the condition (43) incorporates (39). Hence the JQP based criterion indeed identifies the set of all separable states as classical and, in addition, predicts classicality for non-separable states i.e. the states which satisfy (39) but not (38) as classical. One can construct other two-component JQPs and find the conditions under which they are non-negative. Those conditions need not be same as the one found above. However, it is sufficient to construct one non-negative two-component JQP to identify a state as classical.

5. Next, we consider a state described by [6]

$$
\hat{\rho} = (1 - x)|\phi\rangle\langle\phi| + x|\psi\rangle\langle\psi|, \tag{44}
$$

where $|\psi\rangle$ is as in (44) and

$$
|\phi\rangle = \alpha|+, +\rangle + \beta|-, -\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{45}
$$

For the sake of simplicity, we let $(\alpha, \beta)$ to be real. The non-zero expectation values needed for evaluating (16) are given by

$$
\langle \hat{S}^{(1)}_a \rangle = \frac{1}{2}(|\alpha|^2 - |\beta|^2), \quad \langle \hat{S}^{(2)}_a \rangle = \frac{(1 - 2x)}{2}(|\alpha|^2 - |\beta|^2),
$$
identifies all those states as classical which are found to be separable according to the criterion by examining the positivity of the JQPs for the components in the directions \( b \) and \( c \). To that end, note from (47) that the JQP for the components in the directions \( b \) and \( c \), given by

\[
\langle S^{(1)}_a S^{(2)}_a \rangle = \frac{1}{4}(1 - 2x), \\
\langle S^{(1)}_b S^{(2)}_b \rangle = \frac{\alpha \beta}{2}, \\
\langle S^{(1)}_c S^{(2)}_c \rangle = \frac{\alpha \beta(1 - 2x)}{2}.
\] (46)

Substitution of these values in (46) yields

\[
p\left( \epsilon_a^{(1)}, \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_b^{(2)}, \epsilon_c^{(2)} \right) = \frac{1}{64} \left[ 1 + \left( |\alpha|^2 - |\beta|^2 \right) \left( \epsilon_a^{(1)} + (1 - 2x) \epsilon_a^{(2)} \right) + \epsilon_a^{(1)} \epsilon_a^{(2)} (1 - 2x) \\
+ 2\alpha \beta \left( \epsilon_b^{(1)} \epsilon_b^{(2)} + (1 - 2x) \epsilon_c^{(1)} \epsilon_c^{(2)} \right) \right].
\] (47)

With \( s = -\alpha \beta / |\alpha| |\beta| \), it follows that

\[
p(1, \epsilon_b, \epsilon_c; 1, s \epsilon_b, s \epsilon_c) = (1 - x)|\alpha|(1 - |\beta|)/16, \\
p(1, \epsilon_b, \epsilon_c; 1, s \epsilon_b, -s \epsilon_c) = |\alpha|(1 - |\beta|)|\beta|/16, \\
p(1, \epsilon_b, \epsilon_c; -1, s \epsilon_b, -s \epsilon_c) = |\alpha| |\beta|/16, \\
p(1, \epsilon_b, \epsilon_c; -1, s \epsilon_b, s \epsilon_c) = |\alpha|(1 - x)|\beta|/16, \\
p(-1, \epsilon_b, \epsilon_c; 1, s \epsilon_b, s \epsilon_c) = |\beta| |\beta|/16, \\
p(-1, \epsilon_b, \epsilon_c; -1, s \epsilon_b, s \epsilon_c) = |\beta|(1 - x)/16, \\
p(-1, \epsilon_b, \epsilon_c; -1, s \epsilon_b, -s \epsilon_c) = |\beta|(1 - x)|\beta|/16. \\
p(-1, \epsilon_b, \epsilon_c; -1, s \epsilon_b, -s \epsilon_c) = |\beta|(1 - x)|\beta|/16.
\] (48)

The JQP (47) for combinations of the \( \epsilon \)'s other than those appearing in the equations above are non-negative for all values of \((x, \alpha, \beta)\). The JQPs in the equation above will also be non-negative for all \( 0 \leq x \leq 1 \) if \( \alpha \beta = 0 \). In this case the density matrix (44) is clearly separable.

If \( \alpha \beta \neq 0 \) then the \( p \)'s in (48) are non-negative if \( |\alpha| = |\beta| \) and \( x = 1/2 \) in which case the given state is classical and separable as per the part (i) of the JQP based criterion. However, the separability criterion of Peres leads to the condition \( x = 1/2 \) [6] without the restriction \( |\alpha| = |\beta| \). Since separable states are classical, we look for the left out classical states by the part (i) of the criterion by examining the JQP for two orthogonal components as per its part (ii). To that end, note from (47) that the JQP for the components in the directions \( b \) and \( c \) for spin 1 and those in the directions \( a \) and \( c \) for spin 2 is given by

\[
p\left( \epsilon_b^{(1)}, \epsilon_c^{(1)}; \epsilon_a^{(2)}, \epsilon_c^{(2)} \right) = \frac{1}{16} \left[ 1 + (1 - 2x) \left( |\alpha|^2 - |\beta|^2 \right) \epsilon_a^{(2)} + 2\alpha \beta \epsilon_c^{(1)} \epsilon_c^{(2)} \right].
\] (49)

This is non-negative if \( x = 1/2 \) for all \( \alpha, \beta \). The JQP based criterion thus identifies all those states as classical which are found to be separable according to Peres criterion. As stated before, one may find additional classical states by examining the positivity of the JQPs corresponding to other combinations of two of the three components.
6. Lastly, we show that the condition for classicality predicted by the JQP based criterion of the following state of a system of three spin-1/2s, introduced in [8], is the same as the one obtained in that reference by another method:

\[
\hat{\rho} = \frac{I}{8} + \frac{1}{6} \sum_{\mu=x,y,z} \hat{S}^{(2)}_\mu \hat{S}^{(3)}_\mu - \frac{C}{4} \sum_{\mu=x,y,z} \left[ \hat{S}^{(1)}_\mu \hat{S}^{(3)}_\mu + \hat{S}^{(1)}_\mu \hat{S}^{(2)}_\mu \right]. \tag{50}
\]

Note that \(\langle \hat{S}^{(i)}_\mu \rangle = 0\). We choose the \(z\)-direction as the average direction \(a\) of the spins. Recall the expression (17) for the JQP for three spins and evaluate required averages to get

\[
p\left(\epsilon^{(1)}_a, \epsilon^{(2)}_b, \epsilon^{(3)}_c; \epsilon^{(1)}_a, \epsilon^{(2)}_b, \epsilon^{(3)}_c\right) = \frac{1}{2^3} \left[ 1 + \frac{1}{3} \left\{ \epsilon^{(2)}_a \epsilon^{(3)}_a + \epsilon^{(2)}_b \epsilon^{(3)}_b + \epsilon^{(2)}_c \epsilon^{(3)}_c \right\} \right] - \frac{C}{2} \left\{ \epsilon^{(1)}_a \left( \epsilon^{(2)}_b + \epsilon^{(3)}_b \right) + \epsilon^{(1)}_b \left( \epsilon^{(2)}_c + \epsilon^{(3)}_c \right) \right\}. \tag{51}
\]

It is straightforward to verify that \(p\) above is non-negative if \(C \leq 2/3\). This shows that, according to the part (i) of the JQP based criterion, the state is classical and separable if \(C \leq 2/3\).

To find the other set of classical states, we invoke the part (ii) of the criterion. To that end, we construct from (51) the reduced distribution for the directions \(b, c\) for spin 1 and the directions \(a, b\) for spins 2 and 3 to get

\[
p^{(r)}\left(\epsilon^{(1)}_a, \epsilon^{(2)}_b, \epsilon^{(3)}_c; \epsilon^{(1)}_a, \epsilon^{(2)}_b, \epsilon^{(3)}_c\right) = \frac{1}{2^3} \left[ 1 + \frac{1}{3} \left\{ \epsilon^{(2)}_a \epsilon^{(3)}_a + \epsilon^{(2)}_b \epsilon^{(3)}_b + \epsilon^{(2)}_c \epsilon^{(3)}_c \right\} \right] - \frac{C}{2} \left\{ \epsilon^{(1)}_b \left( \epsilon^{(2)}_c + \epsilon^{(3)}_c \right) \right\}. \tag{52}
\]

The probability above is non-negative if \(C \leq 1\). It may be verified that the upper bound on the value of \(C\) for which the JQPs corresponding to other combinations of two of the three components are non-negative is less than 1. Hence the JQP based criterion predicts classicality of the three spins-1/2 state (50) if \(C \leq 1\). This is in agreement with the result reported in [8] that \(\hat{\rho}\) in (50) admits LHV description if \(C \leq 1\).

It is also of interest to study the properties of the reduced density matrix corresponding to the spins numbered 1 and 2. It is obtained by summing (51) over \(\epsilon^{(3)}_\mu (\mu = a, b, c)\) and reads

\[
p\left(\epsilon^{(1)}_a, \epsilon^{(1)}_b, \epsilon^{(2)}_c; \epsilon^{(1)}_a, \epsilon^{(1)}_b, \epsilon^{(2)}_c\right) = \frac{1}{2^6} \left[ 1 - \frac{C}{2} \left( \epsilon^{(1)}_a \epsilon^{(2)}_a + \epsilon^{(1)}_b \epsilon^{(2)}_b + \epsilon^{(1)}_c \epsilon^{(2)}_c \right) \right]. \tag{53}
\]

The probability in the expression above is evidently positive if \(C \leq 2/3\). This implies that the reduced density matrix is classical and separable for \(C \leq 2/3\). This is in agreement with the result reported in [8] that the reduced density matrix corresponding to spins 1 and 2 exhibits entanglement if \(C > 2/3\). The same results hold for the system of spins 1 and 3 described by the corresponding reduced density matrix.
4 Conclusions

Based on the concept of joint quasiprobability distribution of the eigenvalues of the components of spins, a criterion has been proposed to identify the classical states of a system of spin-1/2s. Its validity has been demonstrated by applying it to mixed states of such systems of two spin-1/2s and a system of three spin-1/2s whose classicality and separability properties are known by other methods and showing that the proposed criterion gives results in agreement with the known ones. The criterion offers a unified approach to study classicality of a system of any number of spin-1/2s and also offers the possibility of identifying the processes of measurement \[22, 26\] under which the predicted classicality can be observed.

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