Undecidability of a Theory of Strings, Linear Arithmetic over Length, and String-Number Conversion

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Abstract. In recent years there has been considerable interest in theories over string equations, length function, and string-number conversion predicate within the formal verification, software engineering, and security communities. SMT solvers for these theories, such as Z3str2, CVC4, and S3, are of immense practical value in exposing security vulnerabilities in string-intensive programs. Additionally, there are many open decidability and complexity-theoretic questions in the context of theories over strings that are of great interest to mathematicians.

Motivated by the above-mentioned applications and open questions, we study a first-order, many-sorted, quantifier-free theory $T_{s,n}$ of string equations, linear arithmetic over string length, and string-number conversion predicate and prove three theorems. First, we prove that the satisfiability problem for the theory $T_{s,n}$ is undecidable via a reduction from a theory of linear arithmetic over natural numbers with power predicate, we call power arithmetic. Second, we show that the string-numeric conversion predicate is expressible in terms of the power predicate, string equations, and length function. This second theorem, in conjunction with the reduction we propose for the undecidability theorem, suggests that the power predicate is expressible in terms of word equations and length function if and only if the string-numeric conversion predicate is also expressible in the same fragment. Such results are very useful tools in comparing the expressive power of different theories, and for establishing decidability and complexity results. Third, we provide a consistent axiomatization $\Gamma$ for the functions and predicates of $T_{s,n}$. Additionally, we prove that the theory $T_{r,\Gamma}$, obtained via logical closure of $\Gamma$, is not a complete theory.

1 Introduction

The satisfiability problem for theories over finite-length strings (aka words) has long been studied by mathematicians such as Quine [29], Post, Markov and Matiyasevich [24], Makanin [20], and Plandowski [14, 27, 28]. Post, Markov, and Quine were motivated by the connections between theories over word or string equations and Peano arithmetic, while Matiyasevich’s primary reason for studying string equations was their connection to Diophantine equations [24].

More recently there has been considerable interest in efficient solvers for theories over string equations from the formal verification, software engineering, and security research communities. Examples of such solvers include Z3str2 [38] and CVC4 [16], both of which support the quantifier-free first-order many-sorted theory $T_{s,n}$ of string equations, length, and string-integer conversions. The theory $T_{s,n}$ is expressive enough that many string-related library functions and programming constructs from languages such as C, C++, Java, PHP, and JavaScript can be easily encoded in terms of $T_{s,n}$-functions and predicates. The expressive power of $T_{s,n}$ and efficient practical implementation of solvers such as Z3str2 has already had significant impact and enabled many applications in program analysis and verification [19, 32, 38]. For example, an important sub-area of software engineering research where string solvers are of great interest is dynamic symbolic execution aimed at automated bug-finding [9, 32], and analysis of database/web applications [7, 19, 37].

Footnote: In this paper, we interchangeably use the terms word equations and string equations. The term “word equations” is the convention among logicians, while formal verification researchers tend to use the term “string equations”.

Given the fundamental nature of the theory $T_{s,n}$ and its fragments (e.g., note that word equations essentially form a free semigroup studied intensively by mathematicians over the last several decades [18]), it is no surprise that there is strong motivation from logicians to study their decidability and complexity. In the 1940’s, Post and Markov conjectured that the fully-quantified first-order theory of string equations (i.e., quantified sentences over Boolean combination of string equations) must be undecidable. In his 1946 paper, Quine [29] showed that this theory is indeed undecidable. In 1977, Makanin famously proved that the satisfiability problem for the quantifier-free theory of string equations is decidable [20]. This result is often considered as one of the most complex proofs in theoretical computer science. In recent years, Plandowski and others considerably improved Makanin’s results and showed that satisfiability problem for string equations is in PSPACE [28]. In 2012, Ganesh et al. showed that $\forall\exists$-fragment of positive string equations is undecidable, strengthening Quine’s result and essentially establishing the boundary between decidability and undecidability for string equations [10]. Ganesh et al. also proved conditional decidability results for the quantifier-free theory of string equations and linear arithmetic over the string length function.

As automated reasoning tools and algorithms for the satisfiability problem for the theory $T_{s,n}$ continue to be intensively researched and developed, it is but natural question to ask whether the theory is decidable. This question has been considered open over the last decade since interest in string solvers dramatically increased in the formal methods community, and is the primary focus of this paper.

Problem Statement:

1. Is the satisfiability problem for the quantifier-free fragment of a first-order two-sorted theory $T_{s,n}$ (the subscripts refer to the two sorts str and num) of finite-length strings over a finite alphabet, concatenation function, equality predicate over string terms, string to natural number conversion predicate, length function from string terms to natural numbers, and linear arithmetic over natural numbers and length function decidable? An answer to this question can lead us to answers to decidability questions to fragments of $T_{s,n}$ that remain open. For example, it is not known whether the quantifier-free theory of word equations and equality over the length function is decidable even though this problem has been open for many decades [24].

2. Is the string-numeric conversion predicate expressible in terms of string equations and length function?

3. What is a consistent (possibly minimal) axiomatization $\Gamma$ for the functions and predicates of $T_{s,n}$? Is the first-order many-sorted fully-quantified theory $T_\Gamma$ obtained as a logical closure of the axiom set $\Gamma$ complete? (Note that the existential closure of the quantifier-free first-order many-sorted theory $T_{s,n}$ is a subset of $T_\Gamma$.)

Brief Summary of Results: First, we show that the satisfiability problem for the theory $T_{s,n}$ is undecidable. We establish this via a reduction from the satisfiability problem for the first-order quantifier-free theory $T_p$ of natural numbers with a power predicate, we call power arithmetic, which was shown to be undecidable by Büchi [11]. Second, we show that the string-numeric conversion predicate, which asserts equivalence between the string representation of a natural number and the number itself, is expressible in terms of the power predicate, string equations, and length function. Third, we show that the first-order many-sorted theory $T_\Gamma$ is not a complete theory.

Contributions in Detail

In greater detail, the contributions of this paper are as follows:

1. We prove that the satisfiability problem for the quantifier-free theory of string equations, linear arithmetic over string length, and string-number conversion is undecidable. This problem has been open for some time, and is of great interest to formal verification researchers. The ability to model string

\[\text{Note that the theory } T_{s,n} \text{ is stronger than the quantifier-free theory of string equations and linear arithmetic over string length function, since } T_{s,n} \text{ additionally has the string-number conversion predicate.}\]
provide a list of open problems related to various extensions and fragments of the theory and establish that the theory is incomplete. In Section 3, we prove the undecidability of the satisfiability problem of $T_s,n$, whose existential closure is the existential fragment of $T_s,n$. More precisely, we encode $\pi$ using only the $\text{numstr}$ predicate, string equations, and string length function. In the above-mentioned undecidability theorem, we establish that $\text{numstr}$ can be encoded using only the $\pi$ predicate, string equations, and string length function. These two reductions put together suggest that the $\pi$ predicate is expressible using string equations and length function iff $\text{numstr}$ is. Expressibility results are very useful tools in constructing reductions, distinguishing the expressive powers of various theories, and in establishing (un)-decidability results. Additionally, our expressibility results suggest that the $\text{numstr}$ predicate is much more complex, both from a theoretical and a practical point-of-view, than it seems at first glance. (Section 4)

3. We establish a consistent finite axiomatization $\Gamma$ for the functions and predicates in the language $L$ of $T_{s,n}$. Additionally, we show that the first-order many-sorted $L$-theory $T_L$, that is the closure of the axioms $\Gamma$, is not a complete theory. That is, there are $L$-sentences $\phi$ such that $T_L$ does not entail either $\phi$ or its negation. (Section 5)

The paper is organized as follows: In Section 2, we provide the syntax and semantics of the theory $T_{s,n}$. In Section 3, we prove the undecidability of the satisfiability problem of $T_{s,n}$. In Section 4, we show a reduction from the power arithmetic theory to $T_{s,n}$. In Section 5, we discuss the consistency of the axiom system $\Gamma$, and establish that the theory $T_L$ is incomplete. In Section 6, we provide a comprehensive overview of the decidability/undecidability results for theories of strings over the last several decades, and also the practical relevance of this theory in the context of verification and security. Finally, we conclude in Section 7 and provide a list of open problems related to various extensions and fragments of the theory $T_{s,n}$ some of which have been open for many decades now.

2 Preliminaries

In this section, we define the syntax and semantics of the first-order, many-sorted, language $L$ of string (aka word) equations with concatenation, length function over string terms, linear arithmetic over natural numbers and the length function, and string-number conversion predicate. Additionally, we present a consistent axiomatization $\Gamma$ for the functions and predicates of this language $L$, present the theories $T_L$ which is the logical closure of the axiom system $\Gamma$, and $T_{s,n}$ whose existential closure is the existential fragment of $T_L$.

2.1 The Language $L$: Syntax for Theories over String Equations, Length, and String-Number Conversion

We first define the countable language $L$ below, i.e., its sorts, and constant, function, and predicate symbols.

1. **Sorts:** The language is many-sorted, with a string sort $\text{str}$ and a natural number sort $\text{num}$. The Boolean sort $\text{Bool}$ is standard. When necessary, we write the sort of an $L$-term $t$ explicitly as $t : \text{sort}$.

2. **Finite Alphabet:** We fix a finite alphabet $\Sigma = \{0, 1\}$ over which all strings are defined. As necessary, we may subscript characters of $\Sigma$ with an $s$ to indicate that their sort is str.

3. **String and Natural Number Constants:** We fix a two-sorted set of constants $\text{Con} = \text{Con}_{\text{str}} \cup \text{Con}_{\text{num}}$. The set $\text{Con}_{\text{str}}$ is a subset of $\Sigma^*$, the set of all finite-length string constants over the finite alphabet $\Sigma$. Elements of $\text{Con}_{\text{str}}$ will be referred to as *string constants* or simply *strings*. The empty string is represented by $\epsilon$. Elements of $\text{Con}_{\text{num}}$ are the natural numbers starting from 0. As necessary, we may subscript numbers by $n$ to indicate that their sort is num.
always represented in prenex-normal form, i.e., a block of quantifiers followed by a quantifier-free formula.

The symbol \( Q_x \) refers to a block of quantifiers over a set \( x \) of variables. We assume that formulas are always represented in prenex-normal form, i.e., a block of quantifiers followed by a quantifier-free formula.

### Terms and Formulas in the Language \( L \)

**Terms:** \( L \)-terms may be of string or numeric sort. A string term \( t_{str} \) in Figure 1 is inductively defined as either an element of \( \text{var}_{str} \), an element of \( \text{Con}_{str} \), or a concatenation of string terms (denoted by the function \( \text{concat} \) or interchangeably by the \( \cdot \) operator). A numeric or natural number term \( t_{num} \) in Figure 1 is an element of \( \text{var}_{num} \), an element of \( \text{Con}_{num} \), the length function applied to a string term, a constant multiple of a length term, or a sum of length terms. (Note that for convenience we may write concatenation and addition as \( n \)-ary functions, even though we define them as binary operators.)

**Atomic Formulas:** There are five types of atomic formulas as given in Figure 1: (1) word equations \( \text{A}_{\text{wordeq}} \), (2) linear arithmetic predicates over natural numbers and length constraints \( \text{A}_{\text{num}} \), (3) string-numeric conversion predicates \( \text{A}_{\text{numstr}} \), (4) equations over natural number terms, or (5) inequality over natural number terms.

**Quantifier-free Formulas:** Boolean combination of atomic formulas. When we use the term “quantifier-free” formulas, we mean that each free variable is implicitly existentially quantified and no explicit quantifiers may be written in the formula.

**Formulas and Prenex-normal Form:** \( L \)-Formulas are defined inductively over atomic formulas (see Figure 1). The symbol \( Q_x \phi \) refers to a block of quantifiers over a set \( x \) of variables. We assume that formulas are always represented in prenex-normal form, i.e., a block of quantifiers followed by a quantifier-free formula.

**Free and Bound Variables, and Sentences:** We say that a variable under a quantifier in a formula \( \phi \) is bound. Otherwise we refer to variables as free. A formula with no free variables is called a sentence.
2.3 Signature of the Theories $T_{s,n}$ and $T_f$

We define the signature of $T_{s,n} = \langle \Sigma^*, \mathbb{N}, 0_s, 1_s, \cdot, 0_n, 1_n, +, \text{len}, \text{numstr}, =_s, =_n, <_n \rangle$, where $\Sigma^*$ is the set of all string constants over a finite alphabet $\Sigma$, $\mathbb{N}$ is the set of natural numbers, $\cdot$ is the two-operand string concatenation function, $+$ is the two-operand addition function for natural numbers, $\text{len}$ is a function that takes a string and returns its length as a natural number, $=_s$ is the equality predicate over strings, $=_n$ and $<_n$ are the equality and less-than predicates over natural numbers, and $\text{numstr}$ is a two-argument predicate such that $\text{numstr}(i, s)$ is true for natural number $i$ and string $s$ if and only if $s$ is a valid binary representation of the natural number $i$. By a “valid binary representation” we mean that $s$ does not contain any characters other than ‘0’ and ‘1’, and interpreting the characters of $s$ as the digits of a numeral in base 2, where the last character of $s$ is the least significant digit, produces a natural number that is equal to $i$. (Hence we require that the alphabet $\Sigma$ contain characters ‘0’ and ‘1’.) Note that the signatures of all theories considered in this paper are countable.

2.4 $L$-Semantics and the Canonical Model $\mathcal{M}$

In this section, we provide semantics for the symbols in the language $L$ via what we call a canonical model $\mathcal{M}$. We take the finite alphabet $\Sigma$ to be the set $\{0, 1\}$. The results presented here can be easily extended to other finite alphabets. We assume standard definitions for the terms interpretation of symbols and model $\mathcal{M}$.

**Universe of Discourse for symbols in $L$:** The universe of discourse over which all symbols are interpreted is two-sorted disjoint sets. The first set $\Sigma^*$, of sort str, is the set of all finite-length strings over the alphabet $\Sigma = \{0, 1\}$ including the empty string (represented by $\epsilon$), and the second set $\mathbb{N}$, of sort num, is the set of natural numbers starting from 0.

**Interpretation of Natural Number Variables, Constants, Functions, and Predicates:** Variables of sort range over the set $\mathbb{N}$ of natural numbers, and constants represent corresponding natural numbers. Note that all natural number constants are represented as binary numbers, unless otherwise specified. The function $+$ and the predicates $=_n$, $\leq_n$ have the standard interpretations. (Multiplication by constant is also treated in the standard way as a shorthand for repeated addition.)

**Interpretation of String Variables, Constants, Functions, and Predicates:** String constants are interpreted as a finite concatenation of letters 0 and 1 and correspond to appropriate strings in $\Sigma^*$, and string variables range over values from $\Sigma^*$. The string concatenation function is inductively defined over elements of $\Sigma^*$ in the natural way.

**What is meant by the Length of a String:** For a string or a word, $w$, $\text{len}(w)$ denotes the length of $w$, or equivalently, the (natural) number of characters from $\Sigma$ in the interpretation of $w$ under a given assignment.

**The Meaning of $\text{numstr}$ Predicate:** The $\text{numstr}$ predicate asserts that the interpretation of its string argument is a valid binary representation of the natural number represented by its numeric argument. A string $s$ is a valid binary representation of a natural number $i$ iff the following properties hold:

1. $s$ does not contain any characters in $\Sigma$ other than ‘0’ and ‘1’.
2. Let $s[n]$ denote the $n$th character in $s$, where $n$ is a natural number between 0 and $\text{len}(s) - 1$ inclusive.

   Let $s'[n]$ denote the numeric value of $s[n]$, where $s'[n] = 1$ if $s[n]$ is ‘1’, and $s'[n] = 0$ if $s[n]$ is ‘0’. Then it must be the case that $\sum_{n=0}^{\text{len}(s)-1} s'[n]2^{\text{len}(s)-n} = i$. (Here we expand the characters of $s$ into a binary representation of $i$.)

**The Meaning of Equality between String Terms:** For a word equation of the form $t_1 = t_2$, we refer to $t_1$ as the left hand side (LHS), and $t_2$ as the right hand side (RHS). Two string terms are considered equal if their interpretations have the same characters appearing in the same order, i.e., the LHS and RHS evaluate
to the same string in $\Sigma^*$ under the appropriate interpretation for variables and constants in the LHS and RHS of the given equality.

**The Canonical Model:** The above described interpretation of $L$-symbols along with the universe of discourse defines the canonical $L$-model. (An interpretation of a set of symbols in a language $L$ along with universe of discourse is called an $L$-model.)

### 2.5 Standard Logic Definitions

Here we give some standard definitions such as assignment, satisfiability, validity, consistency of an axiom system, and completeness of a theory.

**Assignments, Satisfiability, Validity, and Equisatisfiability:** Given an $L$-formula $\theta$, an assignment for $\theta$ (with respect to $\Sigma$) is a map from the set of free variables in $\theta$ to $\Sigma^* \cup \mathbb{N}$ (where string variables are mapped to strings and natural number variables are mapped to numbers). Given such an assignment, $\theta$ can be interpreted as an assertion about $\Sigma^*$ and $\mathbb{N}$. If this assertion is true, then we say that $\theta$ itself is true under the assignment. If there is some assignment which makes $\theta$ true, then $\theta$ is called satisfiable. An $L$-formula with no satisfying assignment is called an unsatisfiable formula. We say two formulas $\theta, \phi$ are equisatisfiable if $\theta$ is satisfiable iff $\phi$ is satisfiable. Note that this is a broad definition: equisatisfiable formulas may have different numbers of assignments and, in fact, need not even be from the same language. We say a formula is valid if it is true under all possible assignments.

**The Satisfiability Problem:** The satisfiability problem for a set $S$ of formulas is the problem of deciding whether any given formula in $S$ is satisfiable or not. We say that the satisfiability problem for a set $S$ of formulas is decidable if there exists an algorithm (or satisfiability procedure) that solves its satisfiability problem. Satisfiability procedures must have three properties: soundness, completeness, and termination. Soundness and completeness guarantee that the procedure returns “satisfiable” if and only if the input formula is indeed satisfiable. Termination means that the procedure halts on all inputs. In a practical implementation, some of these requirements may be relaxed for the sake of improved typical performance. Analogous to the definition of the satisfiability problem for formulas, we can define the notion of the validity problem (aka decision problem) for a set $Q$ of sentences in a language $L$. The validity problem for a set $Q$ of sentences is the problem of determining whether a given sentence in $Q$ is true under all assignments.

**Logical Entailment:** We say that a set of sentences $C$ entails a sentence $\phi$, written as $C \models \phi$, if any model $A$ of $C$ is also a model of $\phi$. We say a model $A$ is a model of a set of sentences $C$, if all sentences of $C$ are true under some assignments in $A$, written as $A \models C$.

**Consistency of an Axiom System:** A set of $L$-sentences may be designated as axioms. We say that an axiom system $A$ is consistent if for any $L$-formula $\phi$, the axiom system $A$ does not logically imply both $\phi$ and $\neg \phi$.

**Theory, Closure of an Axiom System, Completeness of a Theory:** A set of $L$-sentences is referred to as a theory. The closure $C$ of an axiom system $A$ is the set of sentences that are logically implied by $A$, i.e., every model of $A$ is a model of the set $C$. We say that a theory $T$ is complete if for every $L$-sentence $\phi$, $T$ logically entails either $\phi$ or its negation.

### 2.6 The Axiomatization $\Gamma$

In this section we present the axiom system $\Gamma$. In a subsequent section (Section 5) we show that this axiom system is consistent. For the sake of readability, we may not specify the sorts of various terms if they are clear from context.

**Axioms of Linear Arithmetic over the Natural Numbers** The following axioms follow from the ones for Presburger arithmetic. Note that both Presburger arithmetic and the linear arithmetic as part of $\Gamma$ include
only the addition symbol, and do not have full multiplication. (Multiplication by constants is simply a short-hand for repeated addition up to a known constant bound.)

- $0 \neq 1$
- $\forall x : \neg(0 = x + 1)$
- $\forall xy : x + 1 = y + 1 \rightarrow x = y$
- $\forall x : x + 0 = x$
- $\forall xy : x + (y + 1) = (x + y) + 1$
- $\forall xy : x < y \iff \exists c : \neg(c = 0) \land x + c = y$

**Axioms of Equality for Strings and Natural Numbers** It is assumed that the equality predicate for both string and numeric sorts is reflexive, symmetric, and transitive. In addition, we have the following axiom recursively defined over string terms. Below we present the axiom for string constants over the alphabet $\Sigma$.

- $\forall A B : A = B \rightarrow \text{len}(A) = \text{len}(B)$

**Axioms of Concatenation** Concatenation is associative, but not commutative.

- $\forall x : x \cdot \epsilon = \epsilon \cdot x = x$
- $\forall x y z : x \cdot (y \cdot z) = (x \cdot y) \cdot z$

**Axioms of the \text{len} Function**

- $\text{len}(\epsilon) = 0$
- $\text{len}(x \cdot y) = \text{len}(x) + \text{len}(y)$
- $\forall c : c \in \Sigma \rightarrow \text{len}(c) = 1$

**Axioms of numstr** The axioms for the numstr predicate essentially allow us to define a natural mapping between natural numbers, represented in binary, and strings over $\Sigma$.

- $\forall i : \neg\text{numstr}(i, \epsilon)$
- $\text{numstr}(0, "0")$
- $\text{numstr}(1, "1")$
- $\forall i s : \text{len}(s) = 1 \land s \neq "0" \land s \neq "1" \rightarrow \neg\text{numstr}(i, s)$
- $\forall i x z : \text{numstr}(i, x) \land \text{"0"}z = z\text{"0"} \rightarrow \text{numstr}(i, z)$
- $\forall i x z : \text{numstr}(i, z) \land \text{"0"}z = z\text{"0"} \rightarrow \text{numstr}(i, x)$
- $\forall x y z : (\exists u v : \text{numstr}(u, x) \land \text{numstr}(v, y)) \rightarrow (\text{numstr}(x, y) \iff x = u b v b)$, where $u b$ and $v b$ are the binary digits of $u$ and $v$ respectively. (This axiom enables a numstr conversion over a concatenation of two strings to be simplified.)
- $\forall x y z : \exists u v w : \text{numstr}(x + y, z) \iff \text{len}(u) = x \land \text{len}(v) = y \land w = u v \land \text{numstr}(\text{len}(w), z)$ (This axiom allows a numstr conversion over a sum of two numbers to be simplified.)

### 2.7 Relationship between $\Gamma_T$ and $\Gamma_{\forall n}$

We refer to the set of sentences logically entailed by the axiom system $\Gamma$ as the theory $\Gamma_T$. Note that this set contains sentences with arbitrary quantifiers in them. We assume that sentences are always written in prenex normal form.

The set $\Gamma_{\forall n}$ is a set of quantifier-free $L$-formulas. As discussed before, when we use the term “quantifier-free” formulas, we mean that each free variable is implicitly existentially quantified and there are no other explicit quantifiers in the formula. When the formulas in $\Gamma_{\forall n}$ are existentially quantified, we get the same set of sentences implied by $\Gamma$ that have a single set of existential quantifiers in prenex normal form. We also call this the existential fragment of $\Gamma_T$. 
2.8 The Theory of Power Arithmetic $T_p$, and Büchi’s Results

In this subsection, we present the syntax and semantics of the power arithmetic theory $T_p$, and discuss Büchi’s results for this theory.

Syntax, Semantics, and the Signature of theory $T_p$ We define the signature of $T_p = \langle \mathbb{N}, 0, 1, +, \pi, <, = \rangle$, where $\mathbb{N}$ is the set of natural numbers, 0 and 1 are constants, $+$ is the two-operand addition function, $<$ and $=$ are the two-operand less-than and equality predicates, and $\pi$ is a three-operand predicate defined as $\pi(p, x, y) \iff p = x \times 2^y$. Note that we only consider the satisfiability problem over the quantifier-free fragment of $T_p$ (equivalently the existential closure over quantifier-free formulas).

Büchi’s Undecidability Result Below we briefly present the necessary context for Büchi’s undecidability result for theory $T_p$. We note that Lemmas 1 and 2, as well as the statement of Theorem 1, are adapted from [1] where they were originally presented.

**Lemma 1. (Julia Robinson’s divisibility lemma)** If $m \leq n, l > 2n^2$, and $l + m, l - m \parallel 2 - n$, then $m^2 = n$. (Refer to Lemma 5 in [1].)

**Lemma 2. (Büchi’s Lemma)** In $T_p = \langle \mathbb{N}, 0, 1, +, \pi \rangle$ we can existentially define the operations of addition and multiplication on the natural numbers. (Refer to Lemma 6 in [1].)

**Theorem 1. (Büchi’s Undecidability Theorem)** The existential theory of $T_p = \langle \mathbb{N}, 0, 1, +, \pi \rangle$ is undecidable. (Corollary 5 in [1].)

**Proof.** We briefly reproduce Büchi’s proof here. This is a corollary of Lemma 2 as this lemma allows us to define addition and multiplication on the natural numbers, yielding a theory that is as expressive as Peano arithmetic. As Peano arithmetic is undecidable, it follows by reduction that this theory is also undecidable.

3 The Undecidability of the Satisfiability Problem for $T_{s,n}$

In this section we prove that the satisfiability problem for the first-order many-sorted quantifier-free theory $T_{s,n}$ over string equations and linear arithmetic over natural numbers extended with string length and a string-number conversion predicate is undecidable.

3.1 Proof Idea

We present a sound, complete, and terminating (recursive) reduction from the satisfiability problem for the theory of power arithmetic, $T_p$, which is an extension of arithmetic over natural numbers with a three-argument $\pi$ predicate defined as $\pi(p, x, y) \iff p = x \times 2^y$, to the satisfiability problem of the theory $T_{s,n}$. This theory $T_p$ (and its associated satisfiability problem) was shown by Büchi to be undecidable (in [1], as outlined in Section 2.8).

As the theory $T_{s,n}$ already has arithmetic over natural numbers, the only detail that is missing is an encoding of the $\pi$ predicate into $T_{s,n}$. We can see how to do this by recalling that in bit-vector arithmetic, an unsigned left shift corresponds to multiplication by a power of 2. Therefore, if we have a binary string that represents the natural number $x$ and we concatenate this string with a string of all zeroes of a given length $y$, the resulting string will be the binary representation of $x \times 2^y$. Once this encoding is provided, then it is easy to see that any quantifier-free formula in $T_p$ can be reduced equisatisfiably to a quantifier-free formula in $T_{s,n}$.

$^\dagger$ Representation of $\pi$ as a predicate is somewhat more natural given that string-number conversion is also represented as a predicate.
3.2 The Undecidability Theorem

Theorem 2. The satisfiability problem for the theory \( T_{s,n} \) is undecidable.

Proof. We prove this result via a recursive reduction from the theory \( T_p \) (Büchi’s power arithmetic) to theory \( T_{s,n} \), i.e., any quantifier-free formula in \( T_p \) can be equisatisfiably reduced to a quantifier-free formula in \( T_{s,n} \). Thus, if the satisfiability problem for \( T_{s,n} \) is decidable then so is the satisfiability problem for \( T_p \).

By Büchi’s theorem [1] the satisfiability problem for \( T_p \) is undecidable, and hence so is the satisfiability problem for \( T_{s,n} \).

The Reduction from \( T_p \) to \( T_{s,n} \). We reduce each constant, function, predicate, and atomic formula of \( T_p \) to \( T_{s,n} \) by applying the following rules recursively over the input formula:

1. Each natural number in \( \mathbb{N} \) is represented directly as a numeric constant in \( T_{s,n} \).
2. Variables in \( T_p \) are represented directly as variables of numeric sort in \( T_{s,n} \).
3. Addition of two terms \( t_1 + t_2 \) is represented directly as addition over natural numbers, \( t_1 + t_2 \), in \( T_{s,n} \).
4. Equality of terms in \( T_p \) is represented directly via a recursive reduction as equality \( t_1 = t_2 \) of terms of numeric sort.
5. The less-than predicate in \( T_p \) is represented directly as comparison of natural numbers, \( t_1 < t_2 \).
6. The predicate \( \pi(p, x, y) \) is expressible as follows: \( \exists z : \text{str}, \exists x_s : \text{str} : ("0" \cdot z = z \cdot "0" \land \text{len}(z) = y \land \text{numstr}(p, x_s \cdot z) \land \text{numstr}(x, x_s)) \). The interpretation of the \( \pi \) predicate is \( p = x \times 2^y \). The variables \( z \) and \( x_s \) are string variables, and \( z \) is a string of the “0” character of length equal to \( y \). The \( x_s \) variable is the string binary representation of the natural number \( x \). The concatenation of \( x_s \) followed by \( z \) is a binary representation of \( p \). It is easy to verify that the given formula over free numeric variables \( x, y, p \) is satisfiable iff \( \pi(p, x, y) \) is satisfiable.

The reduction can easily be extended to arbitrary quantifier-free formulas in \( T_p \). It is easy to verify that the reduction is sound, complete, and terminating for all inputs. □

3.3 Discussion

Recall that the satisfiability problem for the theory of quantifier-free string equations with string length remains open. Knowing whether that theory is decidable would be of value in many program analysis applications. The theory \( T_{s,n} \) we consider here is arguably more directly relevant to program analysis since many state-of-art solvers implement exactly this theory, as the extension of string-number conversion allows it to model similar operations which are present in almost all programming languages that have a data structure for strings. Examples of programming language operations/functions that could be modelled with the string-numeric conversion predicate include JavaScript’s \texttt{parseInt} and \texttt{toNumber} methods, which perform integer-string and string-integer conversion.

4 Expressibility Results

In this section we establish that the \( \pi(p, x, y) \) and \texttt{numstr} predicates are expressible in terms of each other. We define a new theory \( T_{\pi} \) (different from \( T_p \)), which is the same as \( T_{s,n} \) except that \texttt{numstr} is removed and replaced by the \( \pi(p, x, y) \) predicate. From the previous section it is clear that any formula involving the \( \pi(p, x, y) \) predicate can be reduced to some formula in \( T_{s,n} \) using some Boolean combination of \texttt{numstr} predicate, string equations, and length function. This shows us that a reduction exists from \( T_{\pi} \) to \( T_{s,n} \). We now show that a reduction in the opposite direction exists; that is, the \texttt{numstr} predicate can be expressed in terms of quantified formulas over the \( \pi(p, x, y) \) predicate, word equations, and length function.

The value of these two recursive reductions is that it suggests that the \( \pi \) predicate is expressible using string equations and length function iff \texttt{numstr} is. Expressibility results are very useful tools in constructing
reductions, distinguishing the expressive powers of various theories, and establishing (un)-decidability results. Additionally, our expressibility results suggest that the \( \text{numstr} \) predicate is much more complex, both from a theoretical and a practical point of view, than it seems at first glance.

**Definition 1.** A predicate \( P \) is **expressible** in some theory \( T \) having language \( L_T \) if there exists an \( L_T \)-formula \( \phi(x_1, \ldots, x_n) \) such that for all interpretations \( m_1, \ldots, m_n \) of \( x_1, \ldots, x_n \) allowed by \( T \) and such that \( \phi(m_1, \ldots, m_n) \) is well-sorted, we have that \( P(m_1, \ldots, m_n) \) is true iff \( \phi(m_1, \ldots, m_n) \) is true.

The fact that \( \pi(p, x, y) \) is expressible in terms of \( \text{numstr}(i, s) \) in the theory \( T_{s,n} \) follows immediately from the reduction from \( T_p \) to the theory \( T_{s,n} \) used to establish the undecidability theorem in the previous section. We only have to show the reverse direction, i.e., that \( \text{numstr}(i, s) \) is expressible in terms of \( \pi(p, x, y) \).

**Theorem 3.** \( \text{numstr}(i, s) \) is expressible in terms of \( \pi(p, x, y) \) in \( T_p \).

**Proof.** We represent \( \text{numstr}(i, s) \) as a formula that asserts the non-existence of a witness for one of two kinds of error in the conversion. The first kind of error relates to the maximum possible value of \( i \). Suppose \( s \) is a binary string of length \( n \). Then \( s \) cannot represent a natural number greater than or equal to \( 2^n \). The second error is a discrepancy between the binary representation of \( i \) and the binary string \( s \). To check bit \( t \) of the number \( i \), decompose \( i \) into \( h2^{t+1} + x2^t + l \) where \( x \) is the \( t \)-th bit of \( i \) and \( x = 0 \lor x = 1 \), and \( l \) is the numeric representation of bits \( t-1 \) through 0 and so \( l < 2^t \). If \( x = 0 \) and \( s[len(s) - 1 - t] = “1” \), or if \( x = 1 \) and \( s[len(s) - 1 - t] = “0” \), there is an error. This gives us the following sentence:

\[
\text{numstr}(i, s) \iff \forall n \forall t \forall h \forall p \forall x \forall l \forall s_h \forall s_x \forall s_t : \\
\neg(len(s) = n \land \pi(p, 1, n) \land i \geq p) \\
\land \neg(\pi(p_h, h, t + 1) \land \pi(p_x, x, t) \land i = p_h + p_x + l \land \pi(l_u, 1, t) \land l < l_u) \\
\land s = s_h \cdot s_x \cdot s_t \land len(s_t) = t \land len(s_x) = 1 \land ((x = 0 \land s_x = “1”) \lor (x = 1 \land s_x = “0”))
\]

We can apply this rule recursively to the input formula, along with similar rules to the ones presented in the previous reduction, to obtain a reduction from \( T_{s,n} \) to \( T_p \).

5 **Consistency of \( \Gamma \)**

In this section we show that the axiom system \( \Gamma \) is consistent, and that the theory \( T_p \) is incomplete.

**Theorem 4.** The axiom system \( \Gamma \) presented in Section 2.4 is consistent.

**Proof.** It is well known that a theory or axiom system is consistent if it has a model \( \mathbb{M} \). We prove consistency by showing that the structure established in Section 2.4 is in fact a model of \( \Gamma \). The remainder of the proof is structured in sections corresponding to those in the description of \( \Gamma \).

1. **Axioms of arithmetic over natural numbers:** These are standard axioms for natural number arithmetic. Since we choose \( \mathbb{N} \) to model numeric terms, it follows that these axioms are true over the domain of natural numbers.

2. **Axioms of equality for strings and natural numbers:** This axiom states that if two strings \( A \) and \( B \) are equal, then \( A \) and \( B \) have the same length, in addition to the standard axioms of equality. Our model of string terms states that two strings are equal if they have the same characters appearing in the same order, and that the length of a string is the natural number of characters in that string. It follows that if two strings are equal, then they have the same characters, and therefore have the same length.

\[\text{Note that we do not present a reduction from } T_{s,n} \text{ to } T_p. \text{ However, we conjecture that one exists, due to the possibility of mapping the countably infinite set of string constants onto the countably infinite set of natural numbers and then constructing string functions and predicates as operators over natural numbers.}\]
3. **Axioms of concatenation:** The first axiom states that concatenating any string with the empty string, on either side, produces a result equal to the original string. Our model represents the result of concatenating $A$ and $B$ as a string having all of $A$’s characters (in the same order) followed by all of $B$’s characters (also in the same order). If one of $A$ or $B$ is empty, it follows that the resulting string has the same characters and in the same order as the other string, and therefore the two are equal.

The second axiom states that string concatenation is associative. Suppose strings $X$, $Y$, $Z$ are composed of characters $x_1 \ldots x_u$, $y_1 \ldots y_v$, $z_1 \ldots z_w$ respectively. Then by definition of concatenation in our model, we have the following:

$$ y \cdot z = y_1 \ldots y_v z_1 \ldots z_w $$
$$ x \cdot (y \cdot z) = x_1 \ldots x_o y_1 \ldots y_v z_1 \ldots z_w $$
$$ x \cdot y = x_1 \ldots x_o y_1 \ldots y_v $$
$$ (x \cdot y) \cdot z = x_1 \ldots x_o y_1 \ldots y_v z_1 \ldots z_w $$
$$ x \cdot (y \cdot z) = (x \cdot y) \cdot z $$

Therefore the axiom holds in this model.

4. **Axioms of the Length Function:** The first axiom states that the only string having length 0 is the empty string $\epsilon$, which follows trivially from the definition of the set of string constants $\Sigma^*$.

The second axiom states that the length of the concatenation of $A$ and $B$ is equal to the sum of the lengths of $A$ and $B$ taken separately. Our model represents the result of concatenating $A$ and $B$ as a string having all of $A$’s characters (in the same order) followed by all of $B$’s characters (also in the same order). Since characters are conserved by this process, it follows that the resulting string has length equal to the sum of the lengths of $A$ and $B$.

The third axiom states that all single-character strings have length 1, which holds trivially.

5. **Axioms of numstr string-numeric conversion predicate:** The first four axioms state some basic properties of string-number conversion: $\epsilon$ is not the binary representation of any number, “0” is the binary representation of 0, “1” is the binary representation of 1, and single-character strings that are not “0” or “1” are not the binary representation of any number. These axioms are true under this model by inspection.

The fifth and sixth axioms show that leading zeroes can be added to and removed from a string without changing its value. We can show that this is true in our model by demonstrating that if $y$ is a binary string and $z$ is a string of all zeroes, the binary expansions of $y$ and $zy$, denoted $y_b$ and $(zy)_b$ respectively, both represent the same natural number:
\[
y_b = y[0]2^{\text{length}(y)-1} + y[1]2^{\text{length}(y)-2} + \ldots + y[\text{length}(y)]\]
\[
+ y[\text{length}(y)-2]2^1 + y[\text{length}(y)-1]2^0
\]
\[
(zy)_b = (zy)[0]2^{\text{length}(zy)-1} + (zy)[1]2^{\text{length}(zy)-2} + \ldots + (zy)[\text{length}(zy)-1]2^0
\]
\[
+ (zy)[\text{length}(zy)]2^{\text{length}(zy)-\text{length}(z)-1} + \ldots + (zy)[\text{length}(zy)-1]2^{\text{length}(zy)-\text{length}(zy)}
\]
\[
(zy)[0] = 0
\]
\[
(zy)[1] = 0
\]
\[
\vdots
\]
\[
(zy)[\text{length}(z)-1] = 0
\]
\[
(zy)_b = 0 + 0 + \ldots + 0 + (zy)[\text{length}(z)]2^{\text{length}(zy)-\text{length}(z)-1} + \ldots + (zy)[\text{length}(zy)-1]2^{\text{length}(zy)-\text{length}(zy)}
\]
\[
(zy)[\text{length}(z)] = y[0]
\]
\[
(zy)[\text{length}(z)+1] = y[1]
\]
\[
\vdots
\]
\[
(zy)[\text{length}(z)+\text{length}(y)-1] = y[\text{length}(y)-1]
\]
\[
\text{length}(zy) - \text{length}(z) = \text{length}(y)
\]
\[
(zy)_b = y[0]2^{\text{length}(y)-1} + \ldots + y[\text{length}(y)-1]2^0 = y_b
\]

Hence adding or deleting leading zeroes has no effect on what number is represented by a given binary string, and so these axioms hold.

The seventh axiom holds if we assume that all numbers are written in binary; concatenating the binary digits of two numbers is equivalent to concatenating the string representations of those numbers.

The eighth axiom illustrates how to perform string-number conversion on an addition of two numeric terms \(x + y\). It suffices to show that \(\text{len}(w) = x + y\):

\[
w = uv
\]
\[
\text{len}(w) = \text{len}(u) + \text{len}(v)
\]
\[
= x + y
\]

This completes the proof. ☐

We are now ready to establish the incompleteness of the theory \(T_{\Gamma}\).

6 Incompleteness of the Theory \(T_{\Gamma}\)

We first state a number of useful definitions and theorems related to completeness of first-order theories from the standard model theory literature [11].
Definition 2. A first-order theory $T$ in language $L$ is complete if for all $L$-formulas $\phi$, exactly one of $\phi$ and $\neg \phi$ is a consequence of $T$.

Definition 3. Two models $A$, $B$ of a first-order theory are elementarily equivalent if for all first order $L$-sentences $\phi$, $A \models \phi \iff B \models \phi$.

Theorem 5. A first-order theory $T$ is complete if and only if all of its models are elementarily equivalent [11].

We are now in a position to prove the following result.

Theorem 6. $T_F$ is incomplete.

Proof. Consider two models $A$, $B$ of the theory $T_F$, defined as follows: $A$ is the canonical model given in Section 2.4 and $B$ is a restricted version of the canonical model where the only string constants that are allowed are nonempty string constants with no leading zeroes. (In other words, the only string constant in $B$ that starts with ‘0’ is “0”.) It is easy to see that both of these are models of $T_F$.

Now consider the first-order sentence $J$ which states “the numstr predicate describes a bijection between strings and natural numbers”.

We state this sentence $J$ formally as follows:

$$\forall i : \exists s : (\text{numstr}(i, s) \land \forall t : \text{numstr}(i, t) \rightarrow s = t)$$

$$\land \forall s : \exists i : (\text{numstr}(i, s) \land \forall j : \text{numstr}(j, s) \rightarrow i = j)$$

It follows that due to the restrictions on string constants imposed in $B$, numstr clearly defines a bijection between strings and natural numbers, where each integer is mapped to the unique string that is its minimal binary representation, and so $J$ is true in the model $B$. However, in the model $A$, numstr does not define a bijection, as by counterexample, numstr(3, “11”) and numstr(3, “0011”) are both true. Therefore $J$ is false in the model $A$.

From this we conclude that $J$ is able to distinguish between $A$ and $B$, and hence $A$ is not elementarily equivalent to $B$; by Theorem 5, therefore, $T_F$ is incomplete.

7 Related Work

We provide a relatively comprehensive overview of both theoretical and practical work done by researchers in the context of theories over strings.

7.1 Theoretical Results over Theories of Strings

In his original 1946 paper, Quine [29] showed that the first-order theory of string equations (i.e., quantified sentences over Boolean combination of word equations) is undecidable. Due to the expressibility of many key reliability and verification questions within this theory, this work has been extended in many ways.

One line of research studies fragments and modifications of this base theory which are decidable. Notably, in 1977, Makanin proved that the satisfiability problem for the quantifier-free theory of word equations is decidable [20]. In a sequence of papers, Plandowski and co-authors showed that the complexity of this problem is in PSPACE [28]. Stronger results have been found where equations are restricted to those

---

Note that as long as the alphabet $\Sigma$ is finite and string constants are concatenations of a finite number of characters, in general there exists a bijection between strings and natural numbers. This follows from the fact that the set $\Sigma^*$ of strings is countably infinite. The argument made in the proof above deals with a very particular bijection as defined by numstr.
where each variable occurs at most twice \[30\] or in which there are at most two variables \[4,5,12\]. In the first case, satisfiability is shown to be NP-hard; in the second, polynomial (which was improved further in the case of single variable word equations). Concurrently, many researchers have looked for the exact boundary between decidability and undecidability. Durnev \[6\] and Marchenkov \[21\] both showed that \(\forall \exists\) sentences over word equations is undecidable. Despite decades of effort, however, the satisfiability problem for the quantifier-free theory of word equations and numeric length remains open \[10,20,24,28\]. More recently, Artur Jéz presents a technique called recompression that gives more efficient algorithms for many fragments of theory of word equations \[13\].

A related result was shown by Furia \[8\], wherein he proved that the quantifier-free theory of integer sequences is decidable. The framework he establishes in this paper is closely related to the theory of concatenation and word equations, but weaker than either strings plus numeric length or the theory of arrays due to its inability to express facts relating the index of an element to the element itself.

Word equations augmented with additional predicates yield richer structures which are relevant to many applications, as we have considered here. In the 1970s, Matiyasevich formulated a connection between string equations augmented with integer coefficients whose integers are taken from the Fibonacci sequence and Diophantine equations \[22,24\]. In particular, he showed that proving undecidability for the satisfiability problem of this theory would suffice to solve Hilbert’s Tenth Problem in a novel way.

Schulz \[33\] extended Makanin’s satisfiability algorithm to the class of formulas where each variable in the equations is specified to lie in a given regular set. This is a strict generalization of the solution sets of word equations. Further work in \[14\] shows that the class of sets expressible through word equations is incomparable to that of regular sets. Matiyasevich extends Schulz’s result further to decision problems involving trace monoids and free partially commutative monoids \[4,5,23\].

Möller \[26\] studies word equations and related theories as motivated by questions from hardware verification. More specifically, Möller proves the undecidability of the existential fragment of a theory of fixed-length bit-vectors, with a special finite but parameterized concatenation operation, the extraction of substrings and the equality predicate. Although this theory is related to the word equations that we study, it is more powerful because of the finite but possibly arbitrary concatenation.

The question of whether the satisfiability problem for the quantifier-free theory of word equations and length constraints is decidable has remained open for several decades. Our decidability results are a partial and conditional solution. Matiyasevich \[25\] observed the relevance of this question to a novel resolution of Hilbert’s Tenth Problem. In particular, he showed that if the satisfiability problem for the quantifier-free theory of word equations and length constraints is undecidable, then it gives us a new way to prove Matiyasevich’s Theorem (which resolved the famous problem) \[24,25\].

Büchi et al. \[1\] consider extensions of the quantifier-free theory of word equations with various length predicates. They find that a predicate \(Elg\) that asserts that two strings have equal length is not existentially definable in this theory, and that by augmenting the theory with two stronger functions, \(Lg_1\) and \(Lg_2\) which count the number of occurrences of the characters ‘1’ and ‘2’ respectively, the resulting theory is undecidable.

The source of undecidability here, as the authors identify, is the ability for these functions to match the number of occurrences of certain subsequences, which allows them to encode addition and multiplication into the resulting theory. Our result is similar to this one; Büchi proposes an encoding of arithmetic into word equations, while we assume an extension of word equations that already contains the \(len\) function and natural number arithmetic (as well as \(numstr\)), and encode an arithmetic operation into operations on strings.

7.2 String Solvers and their application in Program Analysis, Bug-finding, and Verification

Formulas over strings became important in the context of automated bug-finding \[9,32\] and analysis of database/web applications \[7,19,37\]. These program analysis and bug-finding tools read string-manipulating programs and generate formulas expressing their outputs. These formulas contain equations over string constants and variables, membership queries over regular expressions, inequalities between string lengths, and
in some cases the string-integer conversion predicate/functions. In practice, formulas of this form have been solved by off-the-shelf solvers such as HAMPI [9, 15], Z3str2 [38], CVC4 [16], or Kaluza [32]. All these solvers are based on sound algorithms, but are incomplete in different ways.

Zheng et al. [38] present the Z3str2 solver for the quantifier-free many-sorted theory $T_{\text{wlr}}$ over word equations, membership predicate over regular expressions, and length function, which consists of the string (str) and numeric (num) sorts.

S3 [35] is another tool that supports word equations, length function, and regular expression membership predicate. S3 internally uses a version of Z3str2 to handle word equations and length functions.

CVC4 [16] handles constraints over the theory of unbounded strings with length and RE membership. Solving is based on multi-theory reasoning backed by the DPLL($T$) architecture combined with existing SMT theories. The Kleene star operator in RE formulas is dealt with via unrolling as in Z3str2.

In a separate paper, Liang et al. [17] give a decision procedure for regular language membership and numeric length constraints over unbounded strings. However, their decision procedure does not consider word equations, and hence is many ways weaker than the theory $T_{\text{s,n}}$ we consider in this paper. Hence the algorithm they propose, while useful in some contexts, is weaker than the full theory of strings, and their result does not yet resolve the question of whether the quantifier-free theory of strings and numeric length constraints is decidable.

It must be stressed that all the solvers (including Z3str2, CVC4, and S3) that purportedly solve the satisfiability problem for the theory $T_{\text{s,n}}$ or the word equation and length function fragment of $T_{\text{s,n}}$ are incomplete. Solvers such as HAMPI are limited by the fact that they reason only over a bounded string domain, where the bound is given as part of the input.

Pex [34] is a parameterized unit testing tool for .NET that observes program behaviour and uses a constraint solver in order to produce test inputs which exercise new program behaviour. It integrates a specialized string solver in order to generate string inputs that cause the desired branch conditions to be satisfied.

8 Conclusions and Future Work

In recent years there has been considerable interest in satisfiability procedures (aka solvers) for theories over string equations, length, and string-number conversions in the verification and security communities [16, 38]. These theories are also of great interest to logicians, since there are many open problems related to their decidability and complexity. We showed that a first-order many-sorted quantifier-free theory $T_{\text{s,n}}$ of string equations, linear arithmetic over length function, and string-number conversion predicates, variations of which have been implemented in Z3str2 and CVC4 solvers, is undecidable. We establish expressibility results for numstr predicate that suggest that this predicate is far more complex than appears at first glance. Finally, we also provide a consistent axiomatization $\Gamma$ for the symbols of $T_{\text{s,n}}$, and show that the theory $T_{\Gamma}$ is incomplete.

There are many decidability, complexity and efficient encoding questions related to fragments of $T_{\text{s,n}}$ that remain open. For example, it is not known whether the theory of word equations and arithmetic over length functions is decidable [24]. The satisfiability problem for the quantifier-free theory of string equations by itself is known to be in PSPACE; however, it is not known whether it is PSPACE-complete [28]. Yet another open question concerns efficient encoding of functions such as “Replace” that are heavily used in many programming languages, and predicates such as string comparison. More generally, efficient encoding of common programming language string-intensive functions and predicates in terms of $T_{\text{s,n}}$ functions and predicates can be of great value to practitioners, and remains a challenging practical problem.
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