FRACTIONAL KIRCHHOFF EQUATION WITH A GENERAL CRITICAL NONLINEARITY

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Abstract. In this paper, we study the fractional Kirchhoff equation with critical nonlinearity

\[ \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right) (-\Delta)^{s} u + u = f(u) \quad \text{in} \quad \mathbb{R}^N, \]

where \( N > 2s \) and \((-\Delta)^{s/2}\) is the fractional Laplacian with \( 0 < s < 1 \). By using a perturbation approach, we prove the existence of solutions to the above problem without the Ambrosetti-Rabinowitz condition when the parameter \( b \) is small. What’s more, we obtain the asymptotic behavior of solutions as \( b \to 0 \).

1. Introduction and main result

In this paper, we are concerned with the following fractional Kirchhoff equation

\[ \left( a + b \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right) (-\Delta)^{s} u + u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1} \]

where \( N > 2s \) with \( 0 < s < 1 \), \( a, b \) are positive constants and \((-\Delta)^{s/2}\) is the fractional Laplacian which arises in the description of various phenomena in the applied science, such as the phase transition \([19]\), Markov processes \([1]\) and fractional quantum mechanics \([15]\). When \( a = 1 \) and \( b = 0 \), \( (1.1) \) becomes the fractional Schrödinger equations which have been studied by many authors. We refer the readers to \([2, 5-7]\) and the references therein for the details. When \( s = 1 \), the problem \((1.1)\) reduces to the well-known Kirchhoff equation

\[ - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2} \]

which has been studied in the last decade, see \([9, 12, 17]\). The equation \((1.2)\) is related to the stationary analogue of the Kirchhoff equation \( u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 \, dx) \Delta u = f(x, u) \) on \( \Omega \subset \mathbb{R}^N \) bounded, which was proposed by Kirchhoff \([13]\) in 1883 as a generalization the classic D’Alembert’s wave equation for free vibrations of elastic strings.

Recently, in bounded regular domains of \( \mathbb{R}^N \), Fiscella and Valdinoci \([11]\) proposed the following fractional stationary Kirchhoff equation

\[ \begin{cases} M \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, dx \right) (-\Delta)^{s} u = f(x, u), & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3} \]

which models nonlocal aspects of the tension arising from nonlocal measurements of the fractional length of the string. Also in bounded domains, Autuori et al. \([4]\) dealt with the existence and the asymptotic behavior of non-negative solutions of a class of fractional stationary Kirchhoff equation. In the whole of \( \mathbb{R}^N \), Pucci et al. \([18]\) established the existence and multiplicity of nontrivial non-negative entire solutions of a stationary Kirchhoff eigenvalue problem. In the
subcritical case, by using minimax arguments, Ambrosio et al. [3] obtained the multiplicity results for (1.2) in $H^s_t(\mathbb{R}^N)$ with $b$ small. Also in the subcritical case, without the (AR)-condition, the authors [20] investigated the existence of radial solutions by using the variational methods combined with a cut-off function technique. More recently, without the (AR)-condition and monotonicity assumptions, in low dimension ($N = 2, N = 3$), Z. Liu et al. [16] studied the existence of ground states in the critical case. To the best of our knowledge, there are few papers on the fractional Kirchhoff equations involving the critical growth in $\mathbb{R}^N$ with $N > 3$, because of the tough difficulties brought by the nonlocal term and the lack of compactness of the Sobolev embedding $H^s_t(\mathbb{R}^N) \rightarrow L^{2^*_s}(\mathbb{R}^N)$.

Motivated by the works above, we investigate the existence of the positive solutions of (1.1) in $\mathbb{R}^N (N > 2s)$ with the critical growth. Precisely, $f$ satisfies the following conditions:

$(f_1)$ \( f \in C^1(\mathbb{R}^+, \mathbb{R}) \), \( \lim_{t \to 0} f(t)/t = 0 \) and \( f(t) \equiv 0 \) for \( t \leq 0 \),

$(f_2)$ \( \lim_{t \to \infty} f(t)/t^{2^*_s - 1} = 1 \), where \( 2^*_s = \frac{2N}{N - 2s} \),

$(f_3)$ there exist $D > 0$ and $p < 2^*_s$ such that \( f(t) \geq t^{2^*_s - 1} + Dt^{p-1} \) for \( t \geq 0 \).

Our main result can read as

**Theorem 1.1.** Suppose that $f$ satisfies $(f_1)$ – $(f_3)$ and \( \max\{2, 2^*_s - 2\} < p < 2^*_s \), then for $b$ small, (1.1) admits a nontrivial positive radial solution $u_b$. What’s more, along a subsequence, $u_b$ converges to $u$ in $H^s_t(\mathbb{R}^N)$ as $b \to 0$, where $u$ is a radial ground state to the limit problem

$$a(-\Delta)^s u + u = f(u), \quad u \in H^s(\mathbb{R}^N).$$

Because of the presence of the Kirchhoff term, in high dimension $N > 4s$, for the energy functional $I_b(u)$ (see section 2), one has $I_b(tu) \rightarrow +\infty$ as $t \to +\infty$ for each $u \neq 0$. That means Mountain pass geometry may not holds and Mountain pass theorem may not be appropriate. To overcome this difficulty, we use the variational method combined with the perturbation approach [21, 22] to get a special bounded (PS)-sequence. On the other hand, because of the presence of the Kirchhoff term, for the bounded (PS)-sequence $\{u_n\}$, even $u_n \rightharpoonup u_0$ weakly, it doesn’t hold in general that $u_0$ is the critical point of the energy functional, which brings us more tough to get the compactness. We use the properties of the special (PS)-sequence and some results of the limit problem (1.4) to recover the compactness. Moreover, we obtain the asymptotic behavior of the solutions of (1.1) as $b \to 0$.

The paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we construct the min-max level. In Section 4, we complete the proof of Theorem 1.1.

2. Preliminaries and functional setting

2.1. Fractional order Sobolev spaces. The fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$ of a function $\phi : \mathbb{R}^N \to \mathbb{R}$ is defined by $\mathcal{F}((-\Delta)^s\phi)(\xi) = |\xi|^{2s}\mathcal{F}(\phi)(\xi)$, where $\mathcal{F}$ is the Fourier transform. If $\phi$ is smooth enough, it can be computed by the following singular integral

$$(-\Delta)^s \phi(x) = c_s \text{P.V.} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N + 2s}} \, dy, \quad x \in \mathbb{R}^N,$$

where $c_s$ is a normalization constant and P.V. stands the principal value. For any $s \in (0, 1)$, we consider the fractional order Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 \, d\xi < \infty \right\},$$

endowed with the norm $\|u\| = \left( \int_{\mathbb{R}^N} (1 + a|\xi|^{2s})|\hat{u}|^2 \, d\xi \right)^{1/2}$. $H^s(\mathbb{R}^N)$ denotes the space of radial functions in $H^s(\mathbb{R}^N)$, i.e. $H^s_t(\mathbb{R}^N) = \{ u \in H^s(\mathbb{R}^N) : u(x) = u(|x|) \}$. The homogeneous Sobolev
space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is defined by

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : |\xi|^s \hat{u} \in L^2(\mathbb{R}^N) \},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\|_{\mathcal{D}^{s,2}}^2 = \|(-\Delta)^{s/2}u\|_2^2 = \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 \, d\xi$.

For the further introduction on the fractional order Sobolev space, we refer to [10]. Now, we introduce the following Sobolev embedding theorems.

**Lemma 2.1** (see [8, 10, 14]). For any $s \in (0, 1)$, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, 2^*_s]$ and compactly embedded into $L^q_{\text{loc}}(\mathbb{R}^N)$ for $q \in [1, 2^*_s]$. $H^s(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_s}(\mathbb{R}^N)$, i.e., there exists $S_s > 0$ such that $S_s \left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{2/2^*_s} \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \, dx$.

### 2.2. The variational setting

We define the energy functional $I_0 : H^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( a|(-\Delta)^{s/2}u|^2 + u^2 \right) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \right)^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx,$$

with $F(t) = \int_0^t f(\zeta) \, d\zeta$. It is standard to show that $I_0$ is of class $C^1$.

**Definition 2.2.** We call $u \in H^s(\mathbb{R}^N)$ a weak solution of (1.1) if for any $\phi \in H^s(\mathbb{R}^N)$,

$$(a + b\|u\|_{\mathcal{D}^{s,2}}^2) \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}\phi \, dx + \int_{\mathbb{R}^N} u\phi \, dx = \int_{\mathbb{R}^N} f(u)\phi \, dx.$$

Obviously, the critical points of $I_0$ are the weak solutions of (1.1).

Similar to the proof of Brezis-Lieb Lemma in [21], we can give the following lemma.

**Lemma 2.3.** For $s \in (0, 1)$, assume $(f_1) - (f_2)$ hold. Let $\{u_n\} \subset H^s(\mathbb{R}^N)$ such that $u_n \to u$ weakly in $H^s(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$ as $n \to \infty$, then $\int_{\mathbb{R}^N} F(u_n) \to \int_{\mathbb{R}^N} F(u)$.

When $b = 0$, problem (1.1) becomes the limit problem (1.4) which plays a crucial role in our paper. The energy functional of (1.4) is defined as

$$L(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( a|(-\Delta)^{s/2}u|^2 + u^2 \right) \, dx - \int_{\mathbb{R}^N} F(u) \, dx, \quad u \in H^s(\mathbb{R}^N).$$

With the same assumptions on $f$ in Theorem 1.1, it is not difficult to check that $L(u)$ satisfies the Mountain pass geometry. The Mountain pass value denoted by $c$ is defined by

$$c = \inf_{\gamma \in \Gamma_L} \max_{t \in [0,1]} L(\gamma(t)) > 0,$$

where $\Gamma_L = \{ \gamma \in C([0,1], H^s(\mathbb{R}^N)), \gamma(0) = 0, L(\gamma(1)) < 0 \}$. In the following, we present some results of the ground states of (1.4) and the proof is similar as that in [22].

**Proposition 2.4.** Suppose $f$ satisfies $(f_1) - (f_3)$ and $\max\{2, 2_s^* - 2\} < p < 2_s^*$. Let $S_r$ be the set of positive radial ground states of (1.4), then

(i) $S_r$ is not empty and $S_r$ is compact in $H^s(\mathbb{R}^N)$,

(ii) $c < \frac{s}{N}(aS_s)^{N-s}$ and $c$ agrees with the least energy level denoted by $E$, that is, there exists $\gamma \in \Gamma_L$ such that $u \in \gamma(t)$ and $\max_{t \in [0,1]} L(\gamma(t)) = E$, where $u \in S_r$,

(iii) $u \in S_r$ satisfies the Pohozaev identity

$$\frac{N - 2s}{2} \int_{\mathbb{R}^N} a|(-\Delta)^{s/2}u|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} u^2 \, dx = N \int_{\mathbb{R}^N} F(u) \, dx.$$
3. The minimax level

In order to get a bounded (PS)-sequence by the local deformation argument, a different mini-max level is needed. Take \( U \in S_r \) be arbitrary but fixed. By the definition of \( \bar{U} = \mathcal{F}(U) \), for \( U(x) = U(\frac{x}{\tau}), \tau > 0 \), we have \( \bar{U}(\cdot) = \tau^N \bar{U}(\tau) \). Thus \( \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 \, dx = \tau^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 \, dx \). From the Pohožáev identity (2.1), we obtain

\[
L(U_\tau) = \left( \frac{a \tau^{N-2s}}{2} - \frac{N - 2s}{2N} \tau^N \right) \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2.
\]

So, there exists \( \tau_0 > 1 \) such that \( L(U_\tau) < -2 \) for \( \tau \geq \tau_0 \). Set \( D_b = \max_{\tau \in [0, \tau_0]} I_b(U_\tau) \). Noting that \( I_b(U_\tau) = L(U_\tau) + \frac{b}{\tau^N} \| U_\tau \|^4_{L^{s,2}} \) and \( \max_{\tau \in [0, \tau_0]} L(U_\tau) = E \), we have \( D_b \to E \) as \( b \to 0^+ \).

**Lemma 3.1.** There exist \( b_1 > 0 \) and \( C_0 > 0 \), such that for any \( 0 < b < b_1 \) there hold

\[
I_b(U_{\tau_0}) < -2, \quad \| U_\tau \| \leq C_0, \quad \forall \tau \in (0, \tau_0], \quad \| u \| \leq C_0, \quad \forall u \in S_r.
\]

**Proof.** Since \( S_r \) is compact, it is easy to verify that there exists \( C_0 > 0 \) such that the second and third part of the assertion hold. It follows from \( I_b(U_{\tau_0}) \leq L(U_{\tau_0}) + \frac{b}{\tau^N} \| U_\tau \|^4_{L^{s,2}} \) and \( L(U_{\tau_0}) < -2 \) that the first part holds for any \( 0 < b < b_1 \), where \( b_1 > 0 \) small. The proof is completed. \( \square \)

Now, for any \( b \in (0, b_1) \), we define a mini-max value \( C_b := \inf_{\gamma \in \Upsilon_b} \max_{\tau \in [0, \tau_0]} I_b(\gamma(\tau)) \), where

\[
\Upsilon_b = \{ \gamma \in C([0, \tau_0], H_r^s(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(\tau_0) = U_{\tau_0}, \| \gamma(\tau) \| \leq C_0 + 1, \tau \in [0, \tau_0] \}.
\]

**Proposition 3.2.** \( \lim_{b \to 0^+} C_b = E \).

**Proof.** For \( \tau > 0 \), by \( \| U_\tau \|^2 = a \tau^{N-2s} \| U \|_{L^{s,2}}^2 + \tau^N \| U \|_{L^\infty}^2 \), we can define \( U_0 \equiv 0 \). So \( U_\tau \in \mathcal{K}_b \). Moreover, \( \limsup_{b \to 0} C_b \leq \liminf_{b \to 0^+} D_b = E = c. \) On the other hand, for any \( \gamma \in \Upsilon_b \), it follows from \( L(U_{\tau_0}) < -2 \) that \( \tilde{\gamma}(\cdot) = \gamma(\tau_0) \in \Gamma_L \). Thus, from the definition of \( c \) and \( C_b \), we obtain \( C_b \geq E \) for any \( b \in (0, b_1) \). The proof is completed. \( \square \)

4. The Proof of Theorem 1.1

For \( \alpha, d > 0 \), we define

\[
I^b = \{ u \in H^s_r(\mathbb{R}^N) : I_b(u) \leq \alpha \}
\]

and

\[
S^d := \left\{ u \in H^s_r(\mathbb{R}^N) : \inf_{v \in S_r} \| u - v \| \leq d \right\}.
\]

**Proposition 4.1.** Let \( \{ b_n \}_{n=1}^\infty \) be such that \( \lim_{n \to \infty} b_n = 0 \) and \( \{ u_{b_n} \} \subset S^d \) with

\[
\lim_{n \to \infty} I_{b_n}(u_{b_n}) \leq E \quad \text{and} \quad \lim_{n \to \infty} I^b_{b_n}(u_{b_n}) = 0.
\]

Then for \( d \) small, there is \( u_0 \in S_r \), up to a subsequence, such that \( u_{b_n} \to u_0 \) strongly in \( H^s_r(\mathbb{R}^N) \).

**Proof.** For convenience, we write \( u_{b_n} \) for \( u_b \). Since \( u_{b_n} \in S^d \), there exists \( u_0 \in S_r \) such that \( \| u_{b_n} - u_0 \| \leq d \). Let \( v_b = u_{b_n} - u_0 \). By the fact that \( S_r \) is compact and \( \| v_b \| \leq d, \) up to a subsequence, there exist \( u_0 \in S_r \), and \( v_0 \in H^s_r(\mathbb{R}^N) \), such that \( u_b \to u_0 \) strongly in \( H^s_r(\mathbb{R}^N) \) and \( v_b \to v_0 \) weakly in \( H^s_r(\mathbb{R}^N) \). Denoting \( u_b = u_0 + v_b \), then \( u_b \in S^d \) and \( u_b \to u_0 \) weakly in \( H^s_r(\mathbb{R}^N) \). Next, we show \( u_b \to u_0 \) strongly in \( H^s_r(\mathbb{R}^N) \). Since \( \lim_{b \to \infty} I'_b(u_b) = 0 \), then for any \( \phi \in C_0^\infty(\mathbb{R}^N) \),

\[
I'_b(u_b)\phi = L'(u_b)\phi + b\| u_b \|_{L^{s,2}}^2 \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_b(-\Delta)^{s/2} \phi.
\]

It follows from Lemma 2.1 and \( u_b \in S^d \) that \( L'(u_0) = 0 \) as \( b \to 0 \). Obviously \( u_0 \neq 0 \) by \( u_0 \in S^d \) with \( d \) small. Thus \( L(u_0) \geq E \). Meanwhile, from Lemma 2.3, \( I_b(u_b) = L(u_b) + \frac{b}{\tau^N} \| u_b \|^4_{L^{s,2}} = \)

\[
\frac{a \tau^{N-2s}}{2} - \frac{N - 2s}{2N} \tau^N \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2.
\]
Remark 4.2. By Proposition 4.1, for small $d \in (0, 1)$, there exist $\omega > 0, b_0 > 0$ such that

$$\text{(4.1)} \quad \|I_b(u)\| \geq \omega \text{ for } u \in I^D_b \cap (S^d \setminus S^{d-\frac{2}{N}}) \text{ and } b \in (0, b_0).$$

Thus, we have the following proposition.

Proposition 4.3. There exists $\alpha > 0$ such that for small $b > 0$ and $\gamma(\tau) = U(\frac{\tau}{b}), \tau \in (0, \tau_0]$,

$$I_b(\gamma(\tau)) \geq C_b - \alpha \text{ implies that } \gamma(\tau) \in S^d_2.$$

Proof. By the Pohožaev identity (2.1), $I_b(\gamma(\tau)) = \left(\frac{a^{N-2s}}{2} - \frac{N-2s}{2N} \tau^N\right)\|U\|_{D^{s,2}}^2 + \frac{b^2}{4}\|U\|_{D^{s,2}}^4$. Then $\lim_{b \to 0^+} \max_{\tau \in [0, \tau_0]} I_b(\gamma(\tau)) = \max_{\tau \in [0, \tau_0]} \left(\frac{a^{N-2s}}{2} - \frac{N-2s}{2N} \tau^N\right)\|U\|_{D^{s,2}}^2 = E$. On the other hand, $\lim_{b \to 0^+} C_b = E$. The conclusion follows.

Thanks to (4.1) and Proposition 4.3, we can prove the following proposition, which assures the existence of a bounded (PS)-sequence for $I_b$. The proof is similar as that in [21, 22]. We omit the details here.

Proposition 4.4. For $b > 0$ small, there exist $\{u_n\}_n \subset I^D_b \cap S^d$ such that $I'_b(u_n) \to 0$ as $n \to \infty$.

The completion of Proof of Theorem 1.1

Proof. It follows from Proposition 4.4 that there exists $b_0 > 0$ such that for $b \in (0, b_0)$, there exists $\{u_n\} \in I^D_b \cap S^d$ with $I'_b(u_n) \to 0$ as $n \to \infty$. Thus, there exists $u_0 \in H^s_c(\mathbb{R}^N)$, up to a subsequence, such that $u_n \to u_0$ weakly in $H^s_c(\mathbb{R}^N)$, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*_s)$ and $u_n \to u_0$ a.e in $\mathbb{R}^N$. Next, we claim that $I'_b(u_0) = 0$ for $b$ small. Set $f(t) = g(t) + t^{2s-1}$. By Lemma 2.1, we have $\int_{\mathbb{R}^N} g(u_0) \varphi = \int_{\mathbb{R}^N} g(u_0) \varphi + o_n(1)$ for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} g(u_0) u_n = \int_{\mathbb{R}^N} g(u_0) u_0 + o_n(1)$. Let $v_n = u_n - u_0$ and $\|v_n\|_{D^{s,2}} \to A \geq 0$, then $\|u_n\|_{D^{s,2}}^2 = \|u_0\|_{D^{s,2}}^2 + A + o_n(1)$. From $I'_b(u_0) \to 0$, we have

$$\text{(4.2)} \quad (a + b\|u_0\|_{D^{s,2}}^2 + bA) \|u_0\|_{D^{s,2}}^2 + \|u_0\|_2^2 = \int_{\mathbb{R}^N} g(u_0) u_0 + \|u_0\|_{D^{s,2}}^2 + \|v_n\|_{D^{s,2}}^2 .$$

The corresponding Pohožaev identity is

$$\text{(4.3)} \quad \frac{N-2s}{2} (a + b\|u_0\|_{D^{s,2}}^2 + bA) \|u_0\|_{D^{s,2}}^2 + \frac{N}{2} \|u_0\|_2^2 = N \int_{\mathbb{R}^N} G(u_0) + \frac{N}{2s} \|u_0\|_{D^{s,2}}^2 .$$

It follows from $I'_b(u_n) u_n \to 0$ and Brezis-Lieb Lemma that

$$\text{(a + b\|u_0\|_{D^{s,2}}^2 + bA) (\|u_0\|_{D^{s,2}}^2 + A) + (\|u_0\|_2^2 + \|v_n\|_2^2) = \int_{\mathbb{R}^N} g(u_0) u_0 + \|u_0\|_{D^{s,2}}^2 + \|v_n\|_{D^{s,2}}^2 + o_n(1).}$$

Together with (4.2), we have

$$\text{(4.4)} \quad (a + b\|u_0\|_{D^{s,2}}^2 + bA) A + \|v_n\|_2^2 = \|v_n\|_{D^{s,2}}^2 + o_n(1).$$
It follows from Lemma 2.1 that $A \leq \frac{1}{2a} \left(\frac{4}{a} S_s^2\right)^{2s} + o(1)$. If $A = 0$, we have done. If $A > 0$, then $A \geq a^{\frac{N-2s}{2s}} S_s^N$. By the Pohožaev identity (4.3) and (4.4),

$$I_b(u_n) = \left(\frac{1}{4} - \frac{1}{2s}\right) a(\|u_b\|^2_{D^{s,2}} + A) + \left(\frac{1}{4} - \frac{1}{2s}\right) b(\|u_b\|^2_{D^{s,2}} + A)^2 + \left(\frac{1}{2} - \frac{1}{2s}\right) \|v_n\|^2 + o(1)
$$

\begin{align*}
\geq \left(\frac{1}{2} - \frac{1}{2s}\right) aA + b \left(\frac{1}{4} - \frac{1}{2s}\right) \left(\|u_b\|^2_{D^{s,2}} + A\right)^2 + o(1).
\end{align*}

On the other hand, from \(\{u_n\} \subset S_r^d\), for $d$ small, there exist $\tilde{u}_n \in S_r$ and $\tilde{v}_n \in H^s(\mathbb{R}^N)$ such that $u_n = \tilde{u}_n + \tilde{v}_n$ with $\|\tilde{v}_n\| \leq d$. Thus $\|u_n\|^2_{D^{s,2}} \leq \|\tilde{v}_n\|^2_{D^{s,2}} + \|\tilde{u}_n\|^2_{D^{s,2}} + 1 \sup_{v \in S_r} \|v\|^2_{D^{s,2}} \triangleq B$ which implies that $\|u_b\|^2_{D^{s,2}} + A \leq 2B$, where $B$ is independent of $b, n$ and $d$. So

$$I_b(u_n) \geq \left(\frac{1}{2} - \frac{1}{2s}\right) aA - 4b \left|\frac{1}{4} - \frac{1}{2s}\right| B^2 + o(1).$$

Meanwhile, from $\limsup_{n \to \infty} I_b(u_n) \leq D_b$, we get

$$\left(\frac{1}{2} - \frac{1}{2s}\right) aA \leq D_b + b \left|\frac{1}{4} - \frac{1}{2s}\right| B^2.$$ 

Together with $A \geq a^{\frac{N-2s}{2s}} S_s^N$, we have $\frac{a}{N}(aS_s)^{\frac{N}{2}} \leq D_b + b \left|\frac{1}{4} - \frac{1}{2s}\right| B^2 \to E$, as $b \to 0$, which is a contradiction with $E < \frac{a}{N}(aS_s)^{\frac{N}{2}}$. So, the claim is true. Since $u_n \in S_r^d$, then for $d$ small, $u_b \neq 0$. Thus, for $b$ and $d$ small, there exists $u_b \in H^s_r(\mathbb{R}^N)$ which is a nontrivial solution of (1.1). In the following, we investigate the asymptotic behavior of $u_b$ as $b \to 0$. Noting that $D_b \to E$ as $b \to 0$, the similar proof as that in Proposition 4.1, we obtain that there exist $u \neq 0$ such that $u_b \to u$ strongly in $H^s_r(\mathbb{R}^N)$ with $L'(u) = 0$ and $L(u) = E$. The proof is finished. \(\square\)

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