Relative chaos in gravitating systems with massive centre

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A geometric method, namely the Ricci curvature criterion, is used to study the relative instability of evolving $N$-body gravitating systems with massive centres. The performed numerical experiments show that the Ricci curvature decreases with the increase in central concentration, thus indicating an increasing instability for systems with massive centre — either in the form of a point mass or a core formed during the evolution of single and multimass systems. Simulations of such systems do indeed suggest that they have faster gravothermal evolution than less concentrated ones.

\textit{Key words:} Stellar dynamics – Chaos – Galaxies: evolution

1 Introduction

It has been known for sometime \cite{1,2} that gravitational systems display dynamical behaviour that is closer to that of hyperbolic systems than to near integrable ones. In particular, it was shown that spherical systems display negative configuration space curvature for the majority of initial conditions. As is well known \cite{3}, the behaviour of hyperbolic dynamical systems is very different from that of near integrable ones. It follows therefore that the study of the stability properties of $N$-body gravitating systems may throw light on basic dynamical properties of star clusters and galaxies which may not be recovered by standard techniques of galactic dynamics, which assume that large spherical $N$-body gravitational systems can be treated as integrable (perhaps with the inclusion of \textit{additive} noise to model discreteness effects — even this however is usually neglected for systems with large standard two body relaxation times) \cite{4}.
A first step in the study of chaotic behaviour in numerical models is obviously to find proper descriptors of such behaviour and to determine what they tell us about a given system. The difficulties of the interpretation of computer results concerning $N$-body models are known since the pioneering numerical work of Richard Miller [5]. The rigorous study of statistical properties of many dimensional non-linear systems, such as the $N$-body gravitating systems, is also associated with difficulties concerning the application of certain criteria and methods. For example, principal difficulties are associated with the use of Lyapunov exponents, given the fact that they are discontinuous functions of bifurcation parameters [6], and due to their discontinuity with respect to the computer-created images of $N$-body systems [7]. Lyapunov exponents are also asymptotic quantities which are not particularly suitable for the study for the local (in time) chaotic behaviour of $N$-body gravitational systems.

In the present study of the stability properties of the phase space trajectories of certain $N$-body configurations we use the geometric approach, well known in the theory of dynamical systems [8]. In physical problems these methods have been introduced originally by Krylov [9]. In stellar dynamics they have been first used by Gurzadyan & Savvidy [1,2] (for further reviews see [10]). In the aforementioned work the sign indefiniteness of configuration space curvature of $N$-body gravitating systems was also shown — meaning that, in general, these systems are not, strictly speaking, Anosov systems. The latter fact dictates the necessity of searching for more weak but checkable criteria in characterising the statistical properties of $N$-body trajectories. One possible way is to find criteria of relative chaos, i.e. conditions which should describe the instability properties of a given system with respect to another compatible system. Such a criterion appears to be one involving the so-called Ricci curvature, which we will use in the present study.

The question we are interested in here is the effect of the regular field produced by a central mass concentration on the stability properties of an evolving spherical $N$-body system. The massive centre in real star clusters or galaxies can appear as a massive black hole or central dense stellar core (resulting from core collapse or mass segregation resulting from energy equipartition in multimass systems). However, various physical processes directly associated with both these phenomena (loss cone effects, for example) and which are crucial in the vicinity of the centre, are out of the scope of our analysis.

Motivation for our study stems in part from a series of results where a central mass concentration has been found to affect the dynamics of gravitational systems in an important way. The destabilizing role of a regular central field has been observed by Gurzadyan & Kocharyan [13] using the Ricci curvature as a chaos indicator; in that study only small static systems were investigated, while in the present paper we will investigate systems with far larger number of particles which start near detailed dynamical equilibria and which may evolve
in time. Even before it has been noticed in \( N \)-body simulations that asymmetric systems quickly evolve towards more isotropic shapes when a significant central mass is present ([11]; see also [12]) an effect due to the destabilization of the main periodic orbits. The increase in the rate of relaxation for systems with massive centre (black hole) has been studied also by Rauch & Tremaine [14] where the effect of resonant relaxation was investigated. It was also shown that the central mass makes spherical system more unstable with respect to the gravothermal catastrophe [15]; that result has been obtained by means of methods of catastrophe theory. The stability of systems with a central point mass has been studied also by [16]; it was shown that not only the neighbouring particles have role in the stability, but also the mean field.

The content of this paper is as follows. In Section 2 we describe the geometric criterion for instability used in this paper, in Section 3 we discuss the numerical method, Section 4 presents the results which include simulations of single mass systems without initial central concentrations (Section 4.1) and systems with initial central concentrations or with different particle masses (Section 4.2). The results are discussed in Section 5.

2 Ricci curvature criterion

For an introduction to the basic ideas behind the use of geometric methods in the theory of dynamical systems the reader is referred to reference [8]. Here we briefly present the basic concepts needed in the interpretation of the numerical results. The Ricci curvature as a criterion for relative chaos has been introduced by Gurzadyan and Kocharyan in [17], where a kind of classification of various stellar configurations by their degree of chaos had been also achieved. The criterion is based on the computation of the Ricci curvature in the direction of the geodesic velocity \( \mathbf{u} \) of the phase space of the system, and is defined as follows [19,18]:

\[
   r_{\mathbf{u}}(s) = \frac{\text{Ric}(\mathbf{u}, \mathbf{u})}{\| \mathbf{u} \|^2} = \sum_{\mu=1}^{3N-1} k_{\mathbf{n}_\mu, \mathbf{u}}(s),
\]

where \( \text{Ric} \) is the Ricci tensor, \( s \) is the geodesic time, \( k_{\mathbf{n}, \mathbf{u}} \) is the two-dimensional curvature, and the deviation vectors are chosen as \( \mathbf{n}_\mu \perp \mathbf{u} \) and \( \mathbf{n}_\mu \perp \mathbf{n}_\nu \) for all \( \nu \neq \mu \).

This form of the Ricci curvature arises while averaging the Jacobi equation — which links the dynamics of the flow with the geometry of the phase space. One then obtains

\[
   \frac{d^2 \| \mathbf{n} \|}{ds^2} = -\frac{1}{3N}r_{\mathbf{u}} + \langle \| \nabla_{\mathbf{u}} \mathbf{n} \|^2 \rangle.
\]
Then, the criterion of relative instability reads: among two systems with \( r_1 \) and \( r_2 \) within an interval of affine parameter \( s - [0, s^*] \), the system with smaller negative \( r \) should typically be more unstable, i.e. unstable with higher probability, where

\[
r = \frac{1}{3N} \inf_{0 \leq s \leq s^*} [r_u(s)], \quad r_1 < 0; r_1 < r_2;
\]

Certain loss of information on configurations of the system is obvious, however the Ricci curvature does contain information on typical systems, as distinct to the scalar curvature which does not contain it.

The explicit form of the Ricci tensor readily follows from the Riemann tensor is given by \[1\]

\[
\text{Ric}_{\lambda\rho} = -(1/2W)[\Delta W g_{\lambda\rho} + (3N - 2)W_{\lambda\rho}]
\]

\[
+ (3/4W^2) [(3N - 2)W_\lambda W_\rho] - \left[ \frac{3}{4W^2} - \frac{(3N - 1)}{4W^2} \right] g_{\lambda\rho} \| dW \|^2.
\]

where

\[
W = E - V(q) = E - \sum_{b < a}^N Gm_am_b/r_{ab},
\]

and \( m_i \) are the masses of the particles of the \( N \)-body system.

3 Numerical method and choice of system

The numerical \( N \)-body routine used in this study is the NBODY2 code developed by Sverre Aarseth [20] and kindly made available by him. This is a direct summation code which uses individual time-steps for each particle in the simulation and speeds up the force calculation by using a neighbour scheme which separates the force calculations for neighbouring particles from those further off. These improvements take into account the very different natural times (\( \sim 1/\sqrt{\rho} \), \( \rho \) being the local density) in a gravitational system and the fact that, at a given point, the irregular force due to nearby neighbours varies much faster than the regular force due to particles further off. For highly inhomogeneous centrally concentrated systems these refinements can lead to a considerable increase in efficiency while essentially maintaining the high accuracy of direct summation method. The errors in the calculations (as measured by energy conservation) are controlled by an accuracy parameter \( \eta \) which determines the size of the integration time-steps. These errors are constant for values of \( \eta \) below 0.01 and increase as \( \eta^2 \). We have found that a value of \( \eta_{\text{irr}} = 0.02 \) (controlling the irregular time-step) and \( \eta_{\text{reg}} = 0.04 \) (for the regular time-step) gave reasonable results while maintaining efficient
running of the code. For \( N \leq 1000 \) the maximum number of particles allowed in the neighbour sphere was \( 10 + \sqrt{N} \) while for larger \( N \) it was taken to be \( \left( \sqrt{\frac{\eta_{\text{reg}}}{\eta_{\text{irr}}}} \frac{N}{4} \right)^{3/4} \).

The systems are taken to start from Plummer sphere configurations with the initial conditions determined by the method described in [21]. Using the units of Heggie & Mathieu [22] (by setting the total mass and the gravitational constant to unity while keeping the total energy of the system at \( E = -0.25 \)) the Plummer model is defined completely: the virial radius is fixed to unity and the mean virial equilibrium crossing time is equal to \( 2\sqrt{2} \) time units.

The NBODY2 code is not designed to work with zero softening [20], therefore a small softening, again of Plummer form, has been added to avoid numerical overflow due to close encounters and large angle scatterings. The softening parameter was taken as \( \epsilon = \frac{2}{N} \) which scales as the ratio of the minimum to maximum impact parameters of standard relaxation theory [23].

Since the Ricci curvature determines the average or typical instability response to random perturbation, it requires not too large dispersions. It is therefore necessary to remove the larger contributions due to close encounters. In [24] the Ricci curvature time series corresponding to a given simulation have been averaged over small time intervals while filtering out terms that were much larger than the typical value. Here instead we simply not include contributions to the Ricci curvature due to interactions of particles within a (softened) radius of 0.05 units from each other (5% of the virial radius). The typical fraction of operations excluded by this procedure is about a few in ten thousand.

4 Results

4.1 Variation of the Ricci curvature as a result of core contraction

The diagrams on the left hand side of Fig. 1 show the time evolution of the Ricci curvature, as a function of the crossing time, for systems starting from different \( N \)-body realizations of the same Plummer model. The uppermost plot corresponds to the evolution of 200 particle system, while the following plots correspond to the evolution of systems consisting of 600, 1000 and 1400 particles respectively. Clearly the Ricci curvature is always negative and becomes more so as the systems evolve. This can be understood readily.

Since the pioneering work of Antonov [25] and Lynden-Bell and Wood [26], it is well known that gravitating systems in virial equilibrium have no entropy maxima. Instead they can continually evolve towards higher entropy states.
characterised by the appearance of progressively tighter cores surrounded by a diffuse halo.

It is therefore natural to take the evolution of the core radius as an indicator of the evolutionary state of a spherical system. Plummer spheres do indeed tend towards more concentrated configurations as they evolve, we must therefore assume that these higher entropy states are characterised by smaller values of the Ricci curvature, i.e. with higher dynamical entropy states. Note that here a correlation does exist between the dynamic entropy given by Ricci curvature and the thermodynamic one, which is not the case for any physical states of gravitating configurations because of their non-compact phase space.

NBODY2 contains a routine for finding the core radius of a spherical system using the prescription of Casertano and Hut [27]. Particles inside a given radius from the origin (usually the half mass radius) are selected. For each of these particles a list is made of the six nearest particles. The “mass density” for particle \(i\) is then defined as \(\rho_i = 3M_5/(4\pi r_6^3)\) where \(r_6\) is the distance to the particle farthest away from particle \(i\) in the list and \(M_5\) is the sum of the masses of the five other particles. The density centre of the system

\[
\mathbf{r}_d = \frac{\sum \mathbf{r}_i \rho_i}{\sum \rho_i} \quad (1)
\]

is then calculated. With the help of this quantity the core radius is then defined to be

\[
R_c = \sqrt{\frac{\sum \| \mathbf{r}_i - \mathbf{r}_d \|^2 \rho_i^2}{\sum \rho_i^2}} \quad (2)
\]

The plots on the right hand side of Fig. 1 show the evolution of the core radius as calculated from Eq. (2) for the runs which correspond to the Ricci curvature plots adjacent to them. These diagrams clearly show that the evolution of the Ricci curvature properly correlates with the evolution of the core radius, in the sense that they both decrease together, thus confirming the conjecture outlined above. Only the scaling in the Ricci curvature undergoes changes with the number of particles, but not its behaviour.

4.2 Massive centre and multimass systems

As we have seen above, as the Plummer sphere models become more concentrated their dynamical entropy, as measured by the negative Ricci curvature, increases. Hence if there is any correlation between this quantity and the rate of evolution of a system towards a tighter core, then we expect centrally con-
centrated systems to evolve faster, thus outlining the self amplifying nature of gravothermal instability.

To see if the above argument is valid we have conducted simulations in which one of the particles, which is initially placed at the centre of the system, has a mass that is much larger than the other remaining 999 particles. The mass of the central particle was taken to be either ten, fifty or a hundred times the mass of the other particles in the simulation. Other system parameters like the total mass, energy, size etc were kept constant. The Plummer model is rescaled so as to start from virial equilibrium including the heavier particle. The systems stay near virial equilibrium throughout the simulations — in the case when the central particle is ten times more massive than the other particles, the virial ratio $T/W$ does not vary by more than 2%. This suggests that our models start from near a detailed dynamical equilibrium and that initial departure from such an equilibrium would not play a drastic role in determining the subsequent evolution. Because of its significantly larger mass the particle remains essentially within the inner 0.3% of the (tidal) radius of the system even for the case when it is only ten times the mass of the other particles.

Fig. 1. Evolution of the core radius and Ricci curvature for different realizations of the same Plummer model
Fig. 2. Evolution of core radius and Ricci curvature for a $N = 1000$ system in which of one of the particles, initially located at the centre, is ten times more massive than each of the remaining 999 particles.

| $M_p/M$ | $\tau$ | $\epsilon_{10}$ |
|---------|--------|------------------|
| 10      | 0.29   | 0.01             |
| 50      | 0.22   | 0.06             |
| 100     | 0.13   | 0.18             |

We show the corresponding plots for the Ricci curvature and core radius time evolutions in Fig. 2, which should be compared with the $N = 1000$ plots of Fig. 1. It is evident that the evolution of the core radius is much faster for the system with a central massive particle confirming that such system is more unstable. Thus, again the evolution of the Ricci curvature correlates with the evolution towards a denser core. This result is perhaps not that surprising since core contraction is related to the negative specific heat of open gravitational systems — which means that a subsystem losing energy increases its kinetic energy. This process is amplified when the velocity gradient in the mean field is steep, which is precisely what happens when there is a central density cusp or concentration.
Similar behaviour was found for the system with higher masses for the central particles. However, for $N = 1000$, the data for these systems were found to be more noisy. Both the Ricci curvature and the core radius calculations were found to be dominated by the position of the central massive body and by close encounters of this particle with the surrounding ones. Also, due to the continuing tightening of the core and the formation of hard binaries, numerical difficulties were encountered since the NBODY2 contains no provisions for handling such effects. The simulations had to be stopped because they had slowed down dramatically after about twenty two crossing times. It is interesting to note that, in the case when the central particle has a mass equal to a hundred times that of the other particles, the core radius had already contracted to 0.04 units after a few ($\sim 3$) crossing times. Table 1 shows the characteristic time-scales (in units of the crossing time) averaged over the first ten outputs (corresponding to an interval of about a crossing time) for the different models. These are derived from the values of the scalar curvature (which are less noisy). The use of scalar curvature is motivated by the analytical proof in references [1,2] that the relaxation time-scale of spherical $N$-body systems is determined by the scalar curvature.

Fig. 3 shows the behaviour of the Ricci curvature and core radii for systems consisting of two thousand particles when the central particle is ten, fifty and one hundred times more massive than the remaining particles. The estimate of the core radius was again found to be very noisy if the central particle is included. Moreover this estimate then reflected the large mass concentrated in the central particle rather than the overall contraction of the core due to the dynamical evolution. For these reasons, the contribution of this particle was removed when calculating the aforementioned quantity. This type of filtering was not applied in the calculation of the Ricci curvature (which is a characteristic of the self consistent dynamics). As a result the time evolution of this quantity contains larger fluctuations. Again, the larger the central mass the faster the evolution. While the presence of a central particle only ten times more massive than the remaining ones (a perturbation of 0.5%) was not found to have a marked effect on the evolution, systems with central masses of $1/40$ and $1/20$ the total mass of the system evolved significantly more rapidly. These results were also confirmed for systems with $N = 3500$. The results for these systems are shown in Fig. 4. Once more we see that when the mass of the central particle is sufficiently small, the evolution is relatively slow. It appears that (at least for the $N = 1000, 2000$ and $3500$) a central mass of $\sim 1\%$ is sufficient to significantly accelerate the evolution of spherical gravitational systems.

In a system consisting of particles with different masses, interactions between particles leading to equipartition of energy means that more massive particles tend to lose kinetic energy and reside near the centre. This leads to a growth in the central mass concentration and the processes described above will then
Fig. 3. Evolution of core radius and Ricci curvature for $N = 2000$ systems in which
one of the particles, initially located at the centre, is (from top to bottom) is 10, 50
and 100 times more massive than each of the remaining 1999 particles.
Fig. 4. Evolution of core radius and Ricci curvature for $N = 3500$ systems when one of the particles, initially located at the centre, is 10 (top) or 150 hundred times more massive than each of the remaining 3499 particles.

take place. As can be seen from Fig. 5 this is clearly the case. Actually the evolution in the multimass system is even much more rapid. This is due to the fact that here there are two effects — the exchange of energy leading to high central mass and the effect of the central mass itself — which reinforce each other and drive the evolution. After about thirty crossing times many particles have effectively escaped the system and many more are ejected from the central areas and, except for the tightening central core, the system becomes more diffuse. This is clear from the evolution of the tidal radius shown in Fig. 6. The core radius becomes very small and only a few particles are enclosed by it (rendering the calculation of that quantity to be itself inaccurate). Hard binaries start forming and become tighter as the system evolves. These effects cause the integration to practically stop at about forty five crossing times.
Fig. 5. Evolution of core radius and Ricci curvature for \( N = 1000 \) multimass system with Salpeter mass function and in which the highest mass particle is ten times more massive than the lowest mass one.

Fig. 6. Evolution of tidal radius of the system considered in Fig. 5.

Before that, the region surrounding the core reexpands as the negative specific heat is reversed by the energy stored in the binaries. This state of the system is actually more regular than the intermediate core collapse since many of the escaping particles are moving in regular orbits and the outer areas are very diffuse. This is one of the reasons for the increase of the Ricci curvature near the end of the simulation — the others being that at these later stages a large contribution to the Ricci curvature is due to the tight central binaries performing regular two body motion. At this stage we have reached the limits of the applicability of our geometric chaos indicator — an effect due to the non-compactness of the phase space.

Multimass systems with \( N = 2000 \) and \( N = 3500 \) were also found to evolve faster than their single mass counterparts with similar global properties. This can be seen from Fig. 7 which shows the evolution of Ricci curvatures and core radii for systems with Salpeter mass function and where the most massive particle has a mass ten (for the case when \( N = 2000 \)) and twenty (for \( N = 3500 \)) times larger than the least massive particle (respectively). Again
Fig. 7. Evolution of core radius and Ricci curvature for $N = 2000$ (top) and $N = 3500$ multimass systems with Salpeter mass functions. The most massive particles are 10 and 20 times more massive than the least massive ones in the $N = 2000$ and the $N = 3500$ respectively.

there is a clear general correlation between the evolution of the two quantities. However, as can be seen, there is some difference in the detailed behaviour. This might be expected since the Ricci curvature and the core radius are defined in rather different ways. Therefore fluctuations due to the finiteness of the system are expected to be different. In particular, the core radius is not a characteristic quantity of the dynamics of the system but is a (rather arbitrarily defined) indicator of its central concentration and depends only on the positions of the particles.
5 Discussion and conclusions

In the present study the statistical properties of certain $N$-body systems have been analysed, particularly the role of the field of a massive centre is revealed. The massive centre in stellar systems can be imagined either as a central point mass (massive black hole) or dense stellar core formed during the gravothermal evolution of the system. Therefore we have considered both cases, even though one can expect a priori that their resulting dynamical effect on the system should be similar (and indeed this turns out to be the case).

The main result of the present study is that a massive centre leads to an increase in the instability (i.e. mixing) properties of evolving $N$-body gravitating system. This increase in the mixing properties leads to faster gravothermal evolution for centrally concentrated systems underlining the self amplifying nature of the gravothermal catastrophe, in which central concentration leads to an increase in the temperature gradient which in turn leads to an acceleration of the process. Indeed, it was found that systems with central mass concentration, either in the form of a point mass initially placed at the centre or multimass systems which evolved a central concentration because of the accumulation of massive particles near the centre, evolved faster.

One prediction of this study, therefore, is that stellar clusters with initial central mass concentrations (such as primordial black holes for example) will evolve faster towards core collapse. The faster mixing rate also means that a perturbed system returns to its unperturbed state at a faster rate. This indicates that, for example, a star cluster with massive centre being shocked by Galactic tidal field, will recover its spherical shape faster than a cluster without a central mass. This problem is associated with that of the ellipticity of Galactic globular clusters, attracting now much attention. The relative rate of evolution should be proportional to the values of the Ricci curvature, which by definition describe not the absolute but relative degree of instability of different systems.

Since centrally concentrated states of gravitating systems have higher thermodynamic entropy, it follows that the dynamical entropy (which increases with decreasing curvature) is correlated with the thermodynamic entropy in gravitational systems (at least for the cases studied). In previous studies [24,28] it was shown that the existence of large scale ordered motion also decreased the dynamical entropy — thus also suggesting the above conclusion. In particular, El-Zant [24] has found that the dynamical entropy increased during the evolution of initially ordered systems with macroscopic (e.g., plasma type collective) instabilities. It is important to note however that this correlation between thermodynamical and dynamical entropy does not hold in the later stages of evolution of $N$-body systems, when effects due to the non-
compactedness of the phase space are prominent. This is because, strictly speaking, the methods we have employed here are applicable for systems with a compact phase space. The dynamics in these later stages are actually more regular as the system disintegrates. The peculiar thermodynamic properties of gravitational interaction allows such states to have higher entropy.

The correlation of the exponential instability with the various physical parameters of $N$-body systems suggests that the mixing properties predicted by the existence of this instability may play an important role in the evolution of gravitational systems. Of particular importance is its effect on the time-scales of relaxation phenomena as originally pointed out by Gurzadyan & Savvidy [1,2].

One has, however, to note the following. The characteristic time-scales, which are found to be a fraction of a crossing time (Table 1), need not be simply equal to the actual time scales of relaxation and physical evolution of a gravitational system. The latter properties are determined by the rate of diffusion in the action variables which may be much slower than the g time [29]. However, since no universal relaxation time scale does exist for all N-body systems and different physical quantities may have different relaxation time scales, this does not imply that the exponential divergence has no physical observable effect. Neither does it mean that the diffusion rate is determined for all action variables by classical relaxation theory since this assumes an integrable system which remains so under perturbations due to discreteness — and this contradicts the observed exponential divergence. Moreover, though the divergence is local, it takes place at almost all initial conditions so that the structure of the phase space is such that, for a given trajectory most nearby trajectories diverge almost everywhere. In this sense the divergence, although locally characterised, is a global phenomenon and must have an effect on the statistical properties of the dynamical system (see e.g. reference [29]). The criticism by Goodman, Heggie and Hut [30] of the linear theory of stability (Jacobi equation) and its relation to macroscopic relaxation, is therefore removed due to these results from the theory of dynamical systems. Cerruti-Sola and Pettini [31] suggest to add the first and second order time derivatives of the potential energy in the Jacobi equation, which perhaps can be of some interest for some systems, but not for stationary stellar systems where the potential energy does not undergo significant time fluctuations. Also, they state that due to Schur’s theorem, if the curvatures are constant, one can replace the two-dimensional curvatures with the averaged (Ricci or scalar) curvatures without loss of information. This, however, does not follow from that theorem. Finally, their use of the Eisenhart metric is also not justified, since that metric is not positively defined.

These examples once more show the difficulty of the problem of characterising and interpreting the instability of nonlinear many dimensional system like
$N$-body gravitating systems. Among recent results in this area one can also mention the indications for the possible existence of a universal scaling in $N$-body dynamics, either of fractal nature [32] or due to Feigenbaum period doubling bifurcations [33].

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