Finite generation of André–Quillen (co-)homology of F-finite algebras

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ABSTRACT
We prove that the André–Quillen homology and cohomology modules of F-finite \( \mathbb{Z}(p) \)-algebras are finitely generated.

1. Introduction and preliminaries

André–Quillen (co-)homology was invented independently by M. André and D. Quillen (see [1, 8] and [2]). The theory proved its importance over the years, for example by characterizing several classes of noetherian rings and morphisms between such rings (see [2, 8] and [6]). The finite generation of certain André–Quillen homology modules appears in many situations, maybe the most well-known being the proof of the celebrated André’s Theorem on the localization of formal smoothness (see [3]).

In a recent paper, Dundas and Morrow proved that for a F-finite \( \mathbb{Z}/p^e\mathbb{Z} \)-algebra \( A \), where \( e \geq 1 \), the \( A \)-modules \( H_i(\mathbb{Z}/p^e\mathbb{Z}, A, E) \) and \( H_i(\mathbb{Z}, A, E), i \geq 0 \) are finitely generated for any finitely generated \( A \)-module \( E \) and that for any F-finite \( \mathbb{Z}(p) \)-algebra \( A \) and any finitely generated \( A/pA \)-module \( E \), the \( A/pA \)-modules \( H_i(\mathbb{Z}(p), A, E) \) and \( H_i(\mathbb{Z}, A, E), i \geq 0 \) are finitely generated [4, Lemma 3.1, Theorem 3.6]. Actually they proved more, namely that also the higher André–Quillen homology modules are finitely generated, as their final goal was to prove the finite generation of the Hochschild homology modules. Our purpose is to give another proof of the finite generation of the above André–Quillen homology modules, as well as some similar results concerning the André–Quillen cohomology modules.

All the rings are commutative, with unit and Noetherian. All over the paper \( p > 0 \) will be a fixed prime number. We will denote by \( \mathbb{F}_p \) the prime field of characteristic \( p \) and by \( \mathbb{Z}(p) \) the ring of fractions of \( \mathbb{Z} \) with denominators not divisible by \( p \). For a ring \( A \) containing the field \( \mathbb{F}_p \), the Frobenius morphism of \( A \) will mean the ring morphism \( F: A \to A, F(a) = a^p \).

Remark 1.1. Let \( A \) be a \( \mathbb{Z}(p) \)-algebra and \( e \geq 1 \) be an integer. We have a canonical commutative diagram of ring morphisms...
There are two possibilities.

1. $pA = 0$. Then $A = A/pA$ contains $\mathbb{F}_p$.
2. $pA \neq 0$. Then $A$ does not contain $\mathbb{F}_p$, but $A/pA$ does.

Thus in both cases we can say that $A/pA$ contains $\mathbb{F}_p$ and the following definition makes sense and generalizes the usual one (see [7] and [4]):

**Definition 1.2.** A Noetherian $\mathbb{Z}_{(p)}$-algebra $A$ is called F-finite, if the Frobenius morphism of $A/pA$ is a finite morphism.

We will heavily rely on the following result of Gabber:

**Theorem 1.3.** [5, Rem. 13.6] Let $k$ be a field of characteristic $p > 0$ and $A$ be a Noetherian F-finite $k$-algebra. Then $A$ is the quotient of a regular F-finite $k$-algebra.

**Remark 1.4.** Theorem 1.3 does not extend to the case 2 of Remark 1.1, that is to algebras not containing a field of characteristic $p$.

### 2. Finiteness of André–Quillen (co-)homology

The next lemma is well-known, but we couldn’t find a precise reference.

**Lemma 2.1.** Let $A$ be a Noetherian ring and $E \rightarrowtail F \twoheadrightarrow G$ be an exact sequence of $A$-modules. If $E$ and $G$ are finitely generated, then $F$ is finitely generated.

**Proof.** We have the exact sequences

$$0 \rightarrow \ker(u) \rightarrow E \rightarrow E/\ker(u) \cong \text{im}(u) \rightarrow 0$$

and

$$0 \rightarrow \ker(v) \rightarrow F \rightarrow F/\ker(v) \cong \text{im}(v) \rightarrow 0.$$ 

From the first sequence it follows that $\text{im}(u) = \ker(v)$ is finitely generated and since $\text{im}(v)$ is finitely generated, from the second one we obtain the conclusion.

**Proposition 2.2.** Let $k$ be a perfect field of characteristic $p > 0$ and $A$ be a Noetherian F-finite $k$-algebra. Then $H_i(k, A, E)$ and $H^i(k, A, E)$ are finitely generated $A$-modules for any $i \geq 0$ and for any finitely generated $A$-module $E$.

**Proof.** From 1.3 it follows that there exists a regular ring $R$ and a surjective morphism $R \rightarrow A$. Consider the Jacobi–Zariski exact sequence attached to $k \rightarrow R \rightarrow A$. For any $i \geq 1$ we get an exact sequence

$$H_i(k, R, E) \rightarrow H_i(k, A, E) \rightarrow H_i(R, A, E).$$
From [2, Prop. IV.55], it follows that \( H_i(R, A, E) \) is finitely generated. As \( k \) is perfect and \( R \) is regular, we get that \( k \to R \) is a regular morphism and from [6, Thm. 9.5], we have that \( H_i(k, R, E) = 0, \forall i \geq 1 \), whence \( H_i(k, A, E) \) is finitely generated. For \( i = 0 \) we have that \( H_0(k, A, E) = \Omega_{A/k} \otimes_A E = \Omega_{A/k[A]} \otimes_A E \) which is finitely generated.

As \( R \) is \( F \)-finite, the module of differentials \( \Omega_{R/k} \) is finitely generated. Since \( H_i(k, R, R) = 0 \) for all \( i \geq 1 \), by [2, Prop. 3.19] it follows that \( H_I(k, R, E) \simeq \text{Ext}_k(\Omega_{R/k}, E), j \geq 0 \), hence by [9, Th. 7.36] we obtain that \( H_i(k, R, E) \) is finitely generated. Now the Jacobi–Zariski exact sequence associated to the morphisms \( k \to R \to A \)

\[
H^i(R, A, E) \to H^i(k, A, E) \to H^i(k, R, E)
\]

and by [2, Prop. IV.55] the \( A \)-module \( H^i(R, A, E) \) is finitely generated, we apply 2.1 to end the proof.

**Remark 2.3.** Along the same lines one can prove that, if \( k \) is a perfect field (possibly of characteristic zero) and \( A \) is a \( k \)-algebra which is a quotient of a regular ring, then \( H_i(k, A, E) \) is a finitely generated \( A \)-module for any \( i \geq 1 \) and any finitely generated \( A \)-module \( E \).

**Proposition 2.4.** Let \( k \) be a field of characteristic \( p > 0 \), \( A \) be a Noetherian \( F \)-finite \( k \)-algebra and \( E \) a finitely generated \( A \)-module. Then:

a) \( H_i(k, A, E) \) and \( H^i(k, A, E) \) are finitely generated \( A \)-modules for any \( i \geq 0 \), \( i \neq 1 \);

b) If moreover \( k \) is \( F \)-finite, then \( H_1(k, A, E) \) and \( H^1(k, A, E) \) are finitely generated \( A \)-modules.

**Proof.** a) Let us consider the Jacobi–Zariski exact sequence in homology associated to the morphisms \( \mathbb{F}_p \to k \to A \), namely

\[
\cdots \to H_i(\mathbb{F}_p, k, E) \to H_i(\mathbb{F}_p, A, E) \to H_i(k, A, E) \to H_{i-1}(\mathbb{F}_p, k, E) \to \cdots
\]

But for \( i \geq 2 \), by the separability of \( k \) over \( \mathbb{F}_p \) and [2, Prop. VII.11 and VII.4] we have that \( H_i(\mathbb{F}_p, A, E) = H_i(k, A, E) \), hence by 2.2 we obtain the assertion. For \( i = 0 \) we have \( H_0(k, A, E) = \Omega_{A/k} \otimes_A E = \Omega_{A/k[A]} \otimes_A E \), hence it is finitely generated.

Let us now consider the Jacobi–Zariski exact sequence in cohomology associated to the same morphisms as above, that is

\[
\cdots \to H^{i-1}(\mathbb{F}_p, k, E) \to H^i(k, A, E) \to H^i(\mathbb{F}_p, A, E) \to H^i(\mathbb{F}_p, k, E) \to \cdots
\]

Again, for \( i \geq 2 \), by the separability of \( k \) over \( \mathbb{F}_p \) and [2, Prop. VII.4] we get the isomorphism \( H^i(\mathbb{F}_p, A, E) \simeq H^i(k, A, E) \), hence by 2.2 we obtain the assertion.

b) We have the exact sequences

\[
0 \to H_1(\mathbb{F}_p, A, E) \to H_1(k, A, E) \to \Omega_k \otimes_A E
\]

and

\[
\text{Der}_{\mathbb{F}_p}(k, E) \to H^1(k, A, E) \to H^1(\mathbb{F}_p, A, E) \to H^1(\mathbb{F}_p, k, E) = (0).
\]

But the hypothesis implies that \( \Omega_k \) is finitely generated, hence from 2.2 and 2.1 we get that \( H_1(k, A, E) \) is finitely generated. Moreover \( \text{Der}_{\mathbb{F}_p}(k, E) = \text{Hom}_k(\Omega_k/\mathbb{F}_p, E) \) which is finitely generated, since \( \Omega_k/\mathbb{F}_p \) is finitely generated. From the second exact sequence, 2.1 and 2.2 we obtain the desired conclusion.

**Example 2.5.** If \( [k : k^p] = \infty \) the module \( H_1(k, A, E) \) is not necessarily a finitely generated \( A \)-module. Indeed, let \( k \) be a field such that \( [k : k^p] = \infty \) and let \( \bar{k} \) be the algebraic closure of \( k \). The Jacobi–Zariski sequence associated to \( \mathbb{F}_p \to k \to \bar{k} \) is
0 \to H_1(k, \bar{k}, \bar{k}) \to \Omega_{\bar{k}} \otimes_{k} \bar{k} \to \Omega_{\bar{k}} \to \Omega_{k/k} \to 0.

Then \( \Omega_{\bar{k}} \) is a finite \( \bar{k} \)-vector space and if \( H_1(k, \bar{k}, \bar{k}) \) is a finite \( \bar{k} \)-vector space too, by 2.1 it follows that \( \Omega_{k} \otimes_{k} \bar{k} \) is a finite \( \bar{k} \)-vector space. Hence \( \Omega_{k} \) is a finite \( k \)-vector space, contradicting the hypothesis that \([k : k^p] = \infty\).

**Corollary 2.6.** Let \( p \) be a prime number, \( A \) be a Noetherian \( F \)-finite \( \mathbb{F}_p \)-algebra and \( E \) a finitely generated \( A \)-module. Then for any \( i \geq 0 \) and any \( e \geq 1 \), the \( A \)-modules \( H_i(\mathbb{Z}, A, E), H_i(\mathbb{Z}/p^e\mathbb{Z}, A, E), H^i(\mathbb{Z}, A, E) \) and \( H^i(\mathbb{Z}/p^e\mathbb{Z}, A, E) \) are finitely generated.

**Proof.** Considering the Jacobi–Zariski exact sequence associated to the morphisms \( \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \to A \) we get an exact sequence

\[ H_i(\mathbb{Z}, \mathbb{F}_p, E) \to H_i(\mathbb{Z}, A, E) \to H_i(\mathbb{F}_p, A, E). \]

By 2.2 we know that \( H_i(\mathbb{F}_p, A, E) \) is finitely generated and by \([2, \text{Prop. IV.55}]\) we obtain that \( H_i(\mathbb{Z}, \mathbb{F}_p, E) \) is finitely generated. For the second assertion let us remark first that we can assume \( e \geq 2 \). Consider the morphisms \( \mathbb{Z}/p^e\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \to A \) and the Jacobi–Zariski associated sequence

\[ H_i(\mathbb{Z}/p^e\mathbb{Z}, \mathbb{F}_p, E) \to H_i(\mathbb{Z}/p^e\mathbb{Z}, A, E) \to H_i(\mathbb{F}_p, A, E). \]

Then applying 2.2, 2.1, and \([2, \text{Prop. IV.55}]\) we get that \( H_i(\mathbb{Z}/p^e\mathbb{Z}, A, E) \) is finitely generated. In the same way, using the Jacobi–Zariski exact sequence in cohomology, we can prove that the cohomology modules \( H^i(\mathbb{Z}, A, E) \) and \( H^i(\mathbb{Z}/p^e\mathbb{Z}, A, E) \) are finitely generated.

**Remark 2.7.** Looking at Remark 2.3 and Corollary 2.6, one can see that if \( k \) is a perfect field (possibly of characteristic zero) and \( A \) is a \( k \)-algebra which is a quotient of a regular ring, then \( H_i(\mathbb{Z}, A, E) \) is a finitely generated \( A \)-module for any \( i \geq 1 \) and any finitely generated \( A \)-module \( E \).

**Corollary 2.8.** Let \( p \) be a prime number, \( A \) be a Noetherian \( F \)-finite \( \mathbb{F}_p \)-algebra and \( E \) a finitely generated \( A \)-module. Then \( H_i(\mathbb{Z}_{(p)}, A, E) \) and \( H^i(\mathbb{Z}_{(p)}, A, E) \) are finitely generated \( A \)-modules, for any \( i \geq 0 \).

**Proof.** For \( i = 0 \) we have

\[ H_0(\mathbb{Z}_{(p)}, A, E) \cong \Omega_{A/\mathbb{Z}_{(p)}} \otimes E \cong \Omega_{A/\mathbb{Z}} \otimes E \cong H_0(\mathbb{Z}, A, E) \]

which is finitely generated by 2.6. For \( i \geq 1 \), from the morphisms \( \mathbb{Z} \to \mathbb{Z}_{(p)} \to A \) and applying \([2, \text{Prop. V.25}]\) we have the exact sequence

\[ 0 = H_1(\mathbb{Z}, \mathbb{Z}_{(p)}, E) \to H_1(\mathbb{Z}, A, E) \to H_1(\mathbb{Z}_{(p)}, A, E) \to H_{i-1}(\mathbb{Z}, \mathbb{Z}_{(p)}, E) = 0. \]

Now we apply again 2.6. The proof for the cohomology modules is similar.

We shall consider now the case of a \( \mathbb{Z}_{(p)} \)-algebra.

**Proposition 2.9.** Let \( A \) be a Noetherian \( F \)-finite \( \mathbb{Z}_{(p)} \)-algebra and \( E \) a finitely generated \( A/pA \)-module. Then \( H_i(\mathbb{Z}_{(p)}, A, E) \) and \( H^i(\mathbb{Z}_{(p)}, A, E) \) are finitely generated \( A \)-modules, for all \( i \geq 1 \).

**Proof.** The case \( pA = 0 \), that is \( A \) contains \( \mathbb{F}_p \), was considered above.

If \( pA \neq 0 \), consider first the morphisms \( \mathbb{Z}_{(p)} \to \mathbb{F}_p \to A/pA \). We have the Jacobi–Zariski exact sequence

\[ H_i(\mathbb{Z}_{(p)}, \mathbb{F}_p, E) \to H_i(\mathbb{Z}_{(p)}, A/pA, E) \to H_i(\mathbb{F}_p, A/pA, E). \]
But $H_i(\mathbb{Z}(p), \mathbb{F}_p, E)$ is finitely generated by [2, Prop. IV.55] and $H_i(\mathbb{F}_p, A/\mathbb{F}_p, E)$ is finitely generated by 2.2. By 2.1 we obtain that $H_i(\mathbb{Z}(p), A/pA, E)$ is finitely generated. Consider now the morphisms $\mathbb{Z}(p) \to A \to A/pA$. We have the exact sequence

$$H_{i+1}(A, A/pA, E) \to H_i(\mathbb{Z}(p), A, E) \to H_i(\mathbb{Z}(p), A/pA, E).$$

By the previous assertion $H_i(\mathbb{Z}(p), A/pA, E)$ is finitely generated and by [2, Prop. IV.55] $H_{i+1}(A, A/pA, E)$ is finitely generated. Now apply 2.1 to get the assertion.

For the cohomology modules, consider first the morphisms $\mathbb{Z}(p) \to \mathbb{F}_p \to A/pA$. We have the exact sequence

$$H^i(\mathbb{F}_p, A/pA, E) \to H^i(\mathbb{Z}(p), A/pA, E) \to H^i(\mathbb{Z}(p), \mathbb{F}_p, E).$$

Then $H^i(\mathbb{Z}(p), \mathbb{F}_p, E)$ is finitely generated by [2, Prop. IV.55] and $H^i(\mathbb{F}_p, A/pA, E)$ is finitely generated by 2.2. By 2.1 we obtain that $H^i(\mathbb{Z}(p), A/pA, E)$ is finitely generated. Consider now the morphisms $\mathbb{Z}(p) \to A \to A/pA$. We have the associated Jacobi–Zariski exact sequence

$$H^i(\mathbb{Z}(p), A/pA, E) \to H^i(\mathbb{Z}(p), A, E) \to H^{i+1}(A, A/pA, E).$$

As in the proof of 2.9 it follows that $H^i(\mathbb{Z}(p), A, E)$ is finitely generated.

**Corollary 2.10.** Let $A$ be a Noetherian $\mathbb{F}$-finite $\mathbb{Z}(p)$-algebra and $E$ a finitely generated $A/pA$-module. Then $H_i(\mathbb{Z}, A, E)$ and $H^i(\mathbb{Z}, A, E)$ are finitely generated $A$-modules, for all $i \geq 0$.

**Proof.** It follows at once from 2.9 and the Zariski–Jacobi exact sequence associated to $\mathbb{Z} \to \mathbb{Z}(p) \to A$, taking account of the fact that $H_i(\mathbb{Z}, \mathbb{Z}(p), E) = H^i(\mathbb{Z}, \mathbb{Z}(p), E) = (0), \forall i \geq 0$, cf. [2, Prop. V.25].

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