TOTALLY IN Variant DIVisors OF ENdOMORPHISMS 
OF PROJECTIVE SPACES

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Abstract. Totally invariant divisors of endomorphisms of the projective space are expected to be always unions of linear spaces. Using logarithmic differentials we establish a lower bound for the degree of the non-normal locus of a totally invariant divisor. As a consequence we prove the linearity of totally invariant divisors for $\mathbb{P}^3$.

1. Introduction

An endomorphism of a complex projective variety $X$ is a finite morphism $f : X \to X$ of degree at least two. A totally invariant subset of $f$ is a subvariety $D \subset X$ such that we have a set-theoretic equality $f^{-1}(D) = D$. The projective space $X = \mathbb{P}^n$ admits many endomorphisms (simply take $n+1$ homogeneous polynomials of degree $m$ without a common zero), and it is an interesting problem to understand their dynamics [FS94]. A well-known conjecture claims that totally invariant subvarieties of endomorphisms $f : \mathbb{P}^n \to \mathbb{P}^n$ are always linear subspaces. This conjecture is known for divisors of degree $n+1$ [HN11, Thm.2.1] and smooth hypersurfaces of any degree. In fact, by results of Beauville [Bea01, Thm.], Cerveau-Lins Neto [CLN00] and Paranjape-Srinivas [PS89, Prop.8] a smooth hypersurface $D$ of degree at least two does not admit an endomorphism, in particular it is not a totally invariant subset of $f : \mathbb{P}^n \to \mathbb{P}^n$. However there are examples of singular normal hypersurfaces $D \subset \mathbb{P}^n$ of degree $n$ that admit an endomorphism $g : D \to D$ [Zha14, Ex.1.9]. One should thus ask if $g$ is induced by an endomorphism of the projective space. The main result of this paper is a negative answer to this question:

1.1. Theorem. Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree at least two, and let $D \subset \mathbb{P}^n$ be a prime divisor of degree $d \geq 2$ that is totally invariant. Denote by $Z \subset D$ the non-normal locus of $D$. Then we have

\[ \deg(Z) > (d-1)^2 - \frac{n(n-1)}{2}. \] (1)

In particular if $d \geq 1 + \sqrt{\frac{n(n-1)}{2}}$, then $D$ is not normal.

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\[1\] This statement is claimed in [BCS04], but the proof has a gap.
Note that if \( d = n \), then the inequality (1) simplifies to
\[
\deg(Z) > \frac{1}{2}(n - 2)(n - 1).
\]
However, by a well-known result about singularities of irreducible plane curves \([\text{Fis}01, \text{3.8}]\), one has \( \deg(Z) \leq \frac{1}{2}(n - 1)(n - 2) \). Thus an irreducible divisor \( D \) of degree \( n \) is not totally invariant. This observation significantly improves \([\text{Zha}13, \text{Thm}1.1]\), combined with \([\text{NZ}10, \text{Thm}1.5(5) \text{(arXiv version)}]\) we obtain:

**1.2. Corollary.** Let \( f : \mathbb{P}^3 \to \mathbb{P}^3 \) be an endomorphism, and let \( D \subset \mathbb{P}^n \) be a prime divisor that is totally invariant. Then \( D \) is a hyperplane.

**Notation and terminology.**

We work over the complex field \( \mathbb{C} \). Let \( f : \mathbb{P}^n \to \mathbb{P}^n \) be an endomorphism, and let \( D \subset \mathbb{P}^n \) be a totally invariant prime divisor. Then (e.g. by \([\text{BH}14, \text{Lemma} \, 2.5]\)) there exists a unique effective divisor \( R \) such that the logarithmic ramification formula
\[
K_{\mathbb{P}^n} + D = f^*(K_{\mathbb{P}^n} + D) + R
\]
holds, and we call \( R \) the logarithmic ramification divisor. Since \( \rho(\mathbb{P}^n) = 1 \) one easily deduces that \( d := \deg D \leq \deg(-K_{\mathbb{P}^n}) = n + 1 \).

Given a locally free sheaf \( E \to X \) over some manifold \( X \) and \( x \in X \) a point, we denote by \( E_x \) the \( \mathbb{C} \)-vector space \( E \otimes \mathcal{O}_X/m_x \) where \( m_x \subset \mathcal{O}_X \) is the ideal sheaf of \( x \). If \( \alpha : E_1 \to E_2 \) is a morphism of sheaves between locally free sheaves \( E_1 \) and \( E_2 \), we denote by \( \alpha_x : E_{1,x} \to E_{2,x} \) the linear map induced between the vector spaces.

## 2. The sheaf of logarithmic differentials

We consider the complex projective space \( \mathbb{P}^n \) of dimension \( n \geq 2 \).

**2.1. Assumption.** In this whole section we denote by \( D \subset \mathbb{P}^n \) a prime divisor of degree \( d \geq 2 \). We suppose that there exists a subset \( W \subset \mathbb{P}^n \) of codimension at least three such that \( D \setminus W \) has at most normal crossing singularities.

**2.A. Definition and Chern classes.** Since \( D \) has normal crossing singularities in codimension two, the sheaf of logarithmic differentials in the sense of Saito \([\text{Sai}80]\) and the sheaf of logarithmic differentials in the sense of Dolgachev \([\text{Dol}07, \text{Defn.}2.1]\) coincide by \([\text{Dol}07, \text{Cor.}2.2]\), we will denote this sheaf by \( \Omega_{\mathbb{P}^n}(\log D) \). The sheaf \( \Omega_{\mathbb{P}^n}(\log D) \) is reflexive (it is defined as a dual sheaf \([\text{Dol}07, \text{p.36, line -4}]\) and locally free in the points where \( D \) has normal crossing singularities. By \([\text{Dol}07, (2.8)]\) there exists a residue exact sequence
\[
0 \to \Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}(\log D) \to \nu_*(\mathcal{O}_D) \to 0,
\]
where $\nu : \tilde{D} \to D$ is the normalisation\footnote{The statement in \cite[2.8]{Dol07} is for a desingularisation, but since $\pi_*(\mathcal{O}_{D''}) = \mathcal{O}_{D'}$ for any birational morphism $\pi : D'' \to D'$ between normal varieties, the statement holds for the normalisation.}.

Our goal is to compute the first and second Chern class of the sheaf $\Omega_{\mathbb{P}^n}(\log D)$. Recall first that
\begin{equation}
\label{eq:chern_classes}
c_1(\mathcal{O}_D) = D, \quad c_2(\mathcal{O}_D) = D^2.
\end{equation}

Denote by $Z \subset D$ the non-normal locus of $D$. Since $D$ is Cohen-Macaulay, we know by Serre’s criterion that $Z \subset \mathbb{P}^n$ is empty or a projective set of pure dimension $n-2$. We have an exact sequence
\begin{equation}
0 \to \mathcal{O}_D \to \nu_*\mathcal{O}_{\tilde{D}} \to \mathcal{K} \to 0,
\end{equation}
where $\mathcal{K}$ is a sheaf with support on $Z$. Since $D$ has normal crossings on $D \setminus W$ the restriction of (4) to $D \setminus W$ is
\begin{equation}
0 \to \mathcal{O}_{D \setminus W} \to \nu_*\mathcal{O}_{\tilde{D}} \to \mathcal{O}_{\mathbb{P}^n(D \setminus W)} \to 0.
\end{equation}

Since $Z$ is empty or of pure dimension $n-2$ and $W$ has codimension at least three in $\mathbb{P}^n$, we see that $W$ does not contain any irreducible component of $Z$. The second Chern class $c_2(\nu_*(\mathcal{O}_{\tilde{D}}))$ is determined by intersecting with the class of a general linear 2-dimensional subspace $P \subset \mathbb{P}^n$. Since $P$ is disjoint from $W$, the sequence (5) combined with (3) yields
\begin{equation}
\label{eq:chern_classes1}
c_1(\nu_*(\mathcal{O}_{\tilde{D}})) = D, \quad c_2(\nu_*(\mathcal{O}_{\tilde{D}})) = D^2 - [Z].
\end{equation}

Recall now that $c_1(\Omega_{\mathbb{P}^n}) = (n + 1)H$, $c_2(\Omega_{\mathbb{P}^n}) = \frac{n(n+1)}{2}H^2$ where $H$ is the hyperplane class. Then the exact sequence (2) combined with (6) yields
\begin{equation}
\label{eq:chern_classes2}
c_2(\Omega_{\mathbb{P}^n}(\log D)) = \left(\frac{(n + 1)(n - 2d)}{2} + d^2\right)H^2 - [Z].
\end{equation}

Thus if we twist by $\mathcal{O}_{\mathbb{P}^n}(m)$ we obtain that
\begin{equation}
\label{eq:chern_classes3}
c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m)) = \left(\frac{(n + 1)(n - 2d)}{2} + d^2\right)H^2 - [Z] - (n - 1)(n + 1 - d)mH^2 + \frac{n(n - 1)}{2}m^2H^2.
\end{equation}

For $m = 1$ this formula simplifies to
\begin{equation}
\label{eq:chern_classes4}
c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = (d - 1)^2H^2 - [Z].
\end{equation}

2.B. \textbf{Global sections of $\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$}. We now choose homogeneous coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^n$. Since $D \subset \mathbb{P}^n$ is a prime divisor of degree $d \geq 2$, we have
\[H^0(D, \mathcal{O}_D(1)) = \langle X_0|_D, X_1|_D, \ldots, X_n|_D \rangle,
\]
and, for simplicity’s sake, we denote by $X_0|_D, X_1|_D, \ldots, X_n|_D$ also their images in $H^0(D, \nu_*(\mathcal{O}_{\tilde{D}}))$ under the natural inclusion $H^0(D, \mathcal{O}_D) \subset$
By Bott’s theorem we have $H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(1)) = 0$, so the cohomology sequence associated to the sequence (2) twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$ shows that $X_0|_D, X_1|_D, \ldots, X_n|_D$ lift to global sections of $\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$. In fact if we denote by $f$ an irreducible homogeneous polynomial defining the hypersurface $D$, these global sections can be written in homogeneous coordinates as

$$d\left(\frac{X_0 \cdot f}{f}\right), d\left(\frac{X_1 \cdot f}{f}\right), \ldots, d\left(\frac{X_n \cdot f}{f}\right).$$

The following elementary lemma is fundamental for our proof.

2.2. Lemma. Under the assumption 2.1, let

$$\alpha : \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)$$

be the morphism of sheaves defined by the global sections (9). Then $\alpha$ is surjective on $\mathbb{P}^n \setminus D_{\text{sing}}$. If $x \in D_{\text{sing}}$ is a point such that in local analytic coordinates $u_1, \ldots, u_n$ around $x$ the hypersurface $D$ is given by $u_1 \cdot u_2 = 0$, the linear map

$$\alpha_x : (\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1})_x \to (\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x$$

has rank at least $n-1$.

For the proof recall the well-known local description of logarithmic differentials in the points where $D$ is a normal crossings divisor: fix a point $x \in D$ and let $u_1, \ldots, u_n$ be holomorphic coordinates in an analytic neighbourhood of $x$. If $D$ is given by $u_1 = 0$ in these coordinates (so $x \in D_{\text{nons}}$), then $\Omega_{\mathbb{P}^n}(\log D)$ is locally generated by

$$du_1, du_2, \ldots, du_n.$$ 

If $D$ is given by $u_1 \cdot u_2 = 0$ a set of local generators is

$$du_1, du_2, du_3, \ldots, du_n.$$

Proof of the first statement. We prove the statement for $x \in D \setminus D_{\text{sing}}$, the (easier) case $x \in \mathbb{P}^n \setminus D$ is left to the reader. Up to linear coordinate change we can suppose that $x = (1 : 0 : \ldots : 0)$. The affine set $U_0 := \{x \in \mathbb{P}^n \mid x_0 \neq 0\}$ is isomorphic to $\mathbb{C}^n$ under the isomorphism

$$(X_0 : \ldots : X_n) \mapsto (X_1 \cdots X_n) = (Y_1, \ldots, Y_n).$$

In this affine chart the forms (9) can be written as

$$\frac{df_b}{f_b} + \frac{Y_1 df_b}{f_b} + \cdots + \frac{Y_n df_b}{f_b} + dY_n,$$

where $f_b(Y_1, \ldots, Y_n) := f(1, Y_1, \ldots, Y_n)$ is the deshomogenisation of $f$. Since $x \in D$ is a smooth point one of the partial derivatives $\frac{\partial f_b}{\partial Y_i}(x)$ is non-zero, so
up to renumbering the coordinates \(Y_1, \ldots, Y_n\) we can suppose that \(\frac{\partial f_b}{\partial Y_1}(x) \neq 0\). Thus \(f_b, Y_2, \ldots, Y_n\) form a set of holomorphic coordinates around \(x\) and

\[
\frac{df_b}{f_b} dY_2, \ldots, dY_n
\]

is a set of generators for \((\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))|_{U_0}\) in a neighbourhood of \(x\). Yet in the point \(x = (0, \ldots, 0)\) the global sections (10) are equal to \(\frac{df_b}{f_b} dY_1, \ldots, dY_n\), so they contain this generating set.

**Proof of the second statement.** Up to linear coordinate change we can suppose that \(x = (1 : 0 : \ldots : 0)\) and as before we consider the affine chart \(U_0 \cong \mathbb{C}^n, Y_i = \frac{X_i}{X_0}\) and the expression (10) of the global sections in these affine coordinates. Up to renumbering we can suppose that \(u_1, u_2, Y_3, \ldots, Y_n\) are coordinates in an analytic neighbourhood of \((0, \ldots, 0) \in \mathbb{C}^n\). Thus \((\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))|_{U_0}\) is generated in a neighbourhood of the origin by \(\frac{du_1}{u_1}, \frac{du_2}{u_2}, dY_3, \ldots, dY_n\).

The logarithmic forms \(Y_i \frac{df_b}{f_b} + dY_i\) are equal to \(dY_i\) in the origin, so they generate the subspace

\[
\langle dY_3, \ldots, dY_n \rangle \subset (\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x.
\]

In the coordinates \(u_1, u_2, Y_3, \ldots, Y_n\) the polynomial \(f_b\) is equivalent to \(u_1 \cdot u_2\), and

\[
\frac{d(u_1 \cdot u_2)}{u_1 u_2} = \frac{du_1}{u_1} + \frac{du_2}{u_2}
\]

is a non-zero element of \((\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1))_x\) which is not in the \((n - 2)\)-dimensional subspace \(\langle dY_3, \ldots, dY_n \rangle\). Thus the global sections generate a subspace of dimension at least \(n - 1\). \(\square\)

### 3. Proof of the main theorem

The proof of Beauville’s result [Bea01, Thm.] on endomorphisms of smooth hypersurfaces \(D \subset \mathbb{P}^n\) is based on the fact that a global section of \(\Omega_X(2)\) with isolated zeroes maps under the tangent map to a global section of \(\Omega_X(2m)\) which still has isolated zeroes [ARVdV99, Lemma 1.1]. The following technical statement gives an analogue for our setting:

**3.1. Lemma.** Let \(S\) be a smooth projective surface, and let \(E_1\) be a vector bundle on \(S\) of rank \(n \geq 2\). Suppose that there exists a linear subspace \(V \subset H^0(S, E_1)\) such that \(\dim V > r \text{rk} E_1\) and the evaluation morphism

\[
ev : V \otimes \mathcal{O}_S \to E_1
\]

is surjective in the complement of a finite set \(Z_S \subset S\). Suppose also that for every point \(x \in Z_S\) the linear map

\[
ev_x : (V \otimes \mathcal{O}_S)_x \to E_{1,x}
\]

has rank at least \(n - 1\).
Suppose that there exists a vector bundle $E_2$ on $S$ of rank $n$ and an injective morphism of sheaves

$$\varphi : E_1 \rightarrow E_2$$

such that the following holds:

(a) The linear map $\varphi_x : E_{1,x} \rightarrow E_{2,x}$ has rank at least $n-2$ in every point $x \in S$. The set $B_S$ where $\text{rk}(\varphi_x) = n-2$ is finite.

(b) Denote by $R_S \subset S$ the closed set such that $\text{rk}(\varphi_x) < n$. Then $R_S$ is disjoint from $Z_S$.

Then we have $c_2(E_1) \leq c_2(E_2)$.

Proof. Denote by $|V|$ the projective space associated to the vector space $V$. Consider the projective set

$$B := \{(x, \sigma) \in X \times |V| \mid \varphi(ev(\sigma(x))) = 0\},$$

and denote by $p_1 : B \rightarrow X$ and $p_2 : B \rightarrow |V|$ the natural projections. If $x \in B_S \subset R_S$, then $x \not\in Z_S$ by hypothesis (b). Thus $(\varphi \circ ev)_x$ has rank $n - 2$ and $\dim p_1^{-1}(x) = \dim V - n + 1$. Analogously if $x \in R_S \setminus B_S$ (resp. $x \in Z_S$), then $\dim p_1^{-1}(x) = \dim V - n$. Finally for $x \in S \setminus (R_S \cup Z_S)$ we obviously have $\dim p_1^{-1}(x) = \dim V - n - 1$. Thus we see that all the irreducible components of $B$ have dimension at most $\dim V - n + 1$.

We will now argue by induction on the rank $n$.

Start of the induction: $n = 2$. Then all the irreducible components have dimension at most $\dim V - 1 = \dim |V|$, so the general fibre of $p_2$ is finite or empty. Hence for a general $\sigma \in |V|$, we have an induced section

$$\mathcal{O}_S \xrightarrow{\sigma} E_1 \xrightarrow{\varphi} E_2$$

of $E_2$ which vanishes at most in finitely many points (so it computes $c_2(E_2)$).

In particular the section $\mathcal{O}_S \xrightarrow{\sigma} E_1$ vanishes at most in finitely many points and clearly $c_2(E_1) \leq c_2(E_2)$.

Induction step: $n > 2$. In this case all the irreducible components have dimension at most $\dim V - 1 < \dim |V|$, so the general $p_2$-fibre is empty. Thus a general $\sigma \in |V|$ defines a morphism

$$\mathcal{O}_S \xrightarrow{\sigma} E_1 \xrightarrow{\varphi} E_2$$

that does not vanish, hence it defines a trivial subbundle of both $E_2$ and $E_1$. In particular the quotients $E_2/\mathcal{O}_S$ and $E_1/\mathcal{O}_S$ are locally free and it is easy to check that the space of global sections $V/\mathcal{O}_S$ and the induced map $\tilde{\varphi} : E_1/\mathcal{O}_S \rightarrow E_2/\mathcal{O}_S$ still satisfy the conditions of the lemma. Since $c_2(E_i) = c_2(E_i/\mathcal{O}_S)$ we can conclude. □

Proof of Theorem 1.1. Since $\mathbb{P}^n$ has Picard number one, the endomorphism $f$ is polarised, i.e. we have $f^*H \equiv mH$ for some $m \in \mathbb{N}$ and $H$ the hyperplane class. Since $D$ is totally invariant, we know by [BH14 Cor.3.3] (cf. also [HN11 Prop.2.4]) that the pair $(\mathbb{P}^n, D)$ is log-canonical. Since $D$
is Cohen-Macaulay its non-normal locus \( Z \) has pure dimension \( n - 2 \) and every irreducible component of \( Z \) is an lc centre of the pair \((X, D)\). Thus we know by [BH14, Cor.3.3] that (up to replacing \( f \) by some iterate \( f^l \)) every irreducible component of \( Z \) is totally invariant and not contained in the logarithmic ramification divisor \( R \). Since \( D \) is totally invariant for any iterate \( f^l \), we can suppose from now on that these properties hold for \( f \).

Since the pair \((X, D)\) is log-canonical there exists a subset \( W \subset \mathbb{P}^n \) of codimension at least three such that \( D \setminus W \) has at most normal crossing singularities. Thus we can use the logarithmic cotangent sheaf \( \Omega_{\mathbb{P}^n}(\log D) \) introduced in Section 2. Since \( D \) is a totally invariant divisor, the tangent map

\[
df : f^*\Omega_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}
\]

induces an injective morphism of sheaves

\[
df_{\log} : f^*\Omega_{\mathbb{P}^n}(\log D) \to \Omega_{\mathbb{P}^n}(\log D).
\]

Let \( P \subset \mathbb{P}^n \) be a general 2-dimensional linear subspace, and \( S := f^{-1}(P) \) its preimage. Then \( S \) is a smooth surface, and we claim that

\[
\varphi : f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S \to \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_S
\]

satisfies the conditions of Lemma 3.1.

**Proof of the claim.** Consider the \( n + 1 \)-dimensional subspace \( V \subset H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \) defined by the global sections of \( \mathcal{O}_{\mathbb{P}^n}(1) \). By Lemma 2.2 the evaluation morphism is surjective in the complement of the singular locus \( D_{\text{sing}} \), and if \( x \in Z \) is a general point, it has rank at least \( n - 1 \). Since \( P \) is general of dimension two, the intersection \( P \cap D_{\text{sing}} \) consists only of general points of \( Z \), so if we denote by

\[
ev_S : f^*(V \otimes \mathcal{O}_{\mathbb{P}^n}) \otimes \mathcal{O}_S \to f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S
\]

the restriction of the (pull-back of the) evaluation morphism to \( S \) it is surjective in the complement of the finite set \( Z_S := f^{-1}(P \cap Z) \) and has rank at least \( n - 1 \) in the points of \( Z_S \). Since \( Z \) is totally invariant, the finite set \( Z_S \) is contained in \( Z \cap S \). Since \( Z \) is not contained in the logarithmic ramification divisor \( R \) and \( P \) is general, the intersection \( Z \cap R \cap S \) is empty. This shows that the sets \( R_S := R \cap S \) and \( Z_S \) are disjoint.

Thus we are left to show that \( \text{rk} \varphi_x \geq n - 2 \) for every \( x \in S \) and the set \( B_S \) where equality holds is finite. For the tangent map \( df \) this is well-known: if \( W \subset \mathbb{P}^n \) is a variety of dimension \( d \) and \( x \in W \) is a general point, the finite map \( W \to f(W) \) is étale in \( x \), in particular the tangent map \( df \) has rank at least \( \dim W \) in \( x \). This shows that the sets

\[
\{x \in \mathbb{P}^n \mid \text{rk} \ df_x \leq n - k\}
\]

have codimension at least \( k \) in \( \mathbb{P}^n \). Since \( \Omega_{\mathbb{P}^n} \) and \( \Omega_{\mathbb{P}^n}(\log D) \) identify in the complement of \( D \) we are thus left to consider points of \( D \). Yet if \( x \in D_{\text{nons}} \) (resp. \( x \in Z \) general) the vector space \( \Omega_{\mathbb{P}^n}(\log D)_x \) contains a linear subspace that is naturally isomorphic to \( \Omega_{D,x} \) (resp. \( \Omega_{Z,x} \)), so we can reduce to the case of the tangent map of \( f|_D \) (resp. \( f|_Z \)). This proves the claim.
We can now finish the proof by comparing the Chern classes. Since \( f^*H \equiv mH \) we have \([S] = m^{n-2}H^{n-2}\) and \( f^*[Z] = m^2(\deg Z)H^2\). Thus it follows from (8) that
\[
c_2(f^*(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_S) = \left((d-1)^2 - \deg Z\right)m^n.
\]
By (7) and Lemma 3.1 this is less or equal than
\[
\tag{11}
c_2(\Omega_{\mathbb{P}^n}(\log D) \otimes \mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{O}_S) = \left(\frac{(n+1)(n-2d)}{2} + d^2 - \deg Z\right)m^{n-2} - (n-1)(n+1-d)m^{n-1} + \frac{n(n-1)}{2}m^n.
\]
Since we can replace \( f \) by some iterate the inequality holds for all sufficiently divisible \( m \in \mathbb{N} \). Thus by considering only the terms of order \( m^n \) we obtain
\[
\tag{12}
(d-1)^2 - \deg Z \leq \frac{n(n-1)}{2}.
\]
This inequality is always strict since otherwise we obtain
\[
0 \leq \left(\frac{(n+1)(n-2d)}{2} + d^2 - \deg Z\right)m^{n-2} - (n-1)(n+1-d)m^{n-1}
\]
for all sufficiently divisible \( m \in \mathbb{N} \). Now recall that \( d \leq n+1 \) and \( d = n+1 \) is excluded since we suppose that \( D \) is a prime divisor [HN11, Thm.2.1]. Hence we have \(-(n-1)(n+1-d) < 0\) which yields a contradiction. Thus the strict form of (12) holds, this is equivalent to our statement. \( \square \)

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