Local antimagic vertex total coloring on fan graph and graph resulting from comb product operation

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Abstract. Let \( G = (V, E) \) be a connected graph with \( |V| = n \) and \( |E| = m \). A bijection \( f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, n + m\} \) is called local antimagic vertex total coloring if for any two adjacent vertices \( u \) and \( v \), \( w_r(u) \neq w_r(v) \), where \( w_r(u) = \sum_{e \in R(u)} f(e) + f(u) \), and \( E(u) \) is a set of edges incident to \( u \). Thus any local antimagic vertex total labeling induces a proper vertex coloring of \( G \) where the vertex \( v \) is assigned the color \( w_r(v) \). The local antimagic vertex total chromatic number \( \chi_{vat}(G) \) is the minimum number of colors taken over all colorings induced by local antimagic vertex total. In this paper we investigate local antimagic vertex total coloring on fan graph \( (F_n) \) and graph resulting from comb product operation of \( F_n \) and \( F_3 \) which denoted by \( F_n \sqcup F_3 \). We get two theorems related to the local antimagic vertex total chromatic number. First, \( \chi_{vat}(F_n) = 3 \) where \( n \geq 3 \). Second, \( 3 \leq \chi_{vat}(F_n \sqcup F_3) \leq 5 \) where \( n \geq 3 \).

1. Introduction

There are many topics in graph theory, one of them is coloring. In general, graph coloring is the giving of color to elements on the graph so that neighboring elements have a different color. Based on its element, graph coloring is divided into three kinds, namely vertex coloring, edge coloring and regional coloring. The detail of this theory can be read in [7] and [8].

The simple concept of graph coloring then has been developed into graph labeling. In general, graph labeling is the giving of labels, which is natural numbers, to the elements on the graph such as vertices or edges, or both [12]. Graph labeling is divided into two, namely magic labeling and antimagic labeling. Magic labeling on graph \( G \) is labeling the elements on \( G \) such that the sum of the labels of all elements incident with any vertex is the same [9]. On the contrary, if the sum of the labels of all elements incident with any two neighboring vertex is different its called antimagic labeling.

The research on antimagic labeling continues to be carried out by many researches. Dafik, et al. built the concept of super edge-antimagic total labeling [5, 6]. Arumugam, et al. have the concept of local antimagic vertex coloring [4]. Agustin, et al. got the concept of local edge antimagic coloring [1] which then developed to super local edge antimagic total coloring [2, 3, 10, 11]. Next, Putri, et al. built the concept of local vertex antimagic total coloring [13] which then continued by Kurniawati, et al. [12].

The concept of local vertex antimagic total coloring is explained as follows. Let \( G = (V, E) \) be a connected graph with \( |V| = n \) and \( |E| = m \). A bijection \( f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, n + m\} \) is
called local antimagic vertex total coloring if for any two adjacent vertices \( u \) and \( v \), \( w_t(u) \neq w_t(v) \), where \( w_t(u) = \sum_{e \in E(G)} f(e) + f(u) \), and \( E(G) \) is a set of edges incident to \( u \). Thus any local antimagic vertex total labeling induces a proper vertex coloring of \( G \) where the vertex \( v \) is assigned the color \( w_t(v) \). The local antimagic vertex total chromatic number \( n_{lvat}(G) \) is the minimum number of colors taken over all colorings induced by local vertex antimagic total labelings of \( G \) [12].

In the previous research, we got the local antimagic vertex total coloring on some families tree [13] and graphs with homogeneous pendant vertex [12]. Based on this, the author conduct further research on local antimagic vertex total coloring on fan graph \((F_n)\) and graph resulting from comb product operation, namely \( F_n \rightarrow F_3 \).

**Lemma 1.1** [13] If \( \chi(G) \) is a vertex coloring chromatic number of graph \( G \) then \( n_{lvat}(G) \geq \chi(G) \)

2. Local Antimagic Vertex Total Coloring on Fan Graph

Before we got the local antimagic vertex total coloring of fan graph, we must know its vertex coloring chromatic number.

**Lemma 2.1** If \( F_n \) is fan graph then the vertex coloring chromatic number of \( F_n \) is \( \chi(F_n) = 3 \)

**Proof.** Vertex coloring of graph \( G \) is coloring the vertices in \( G \) such that any two adjacent vertices in \( V(G) \) have different color. We have vertex set \( V(F_n) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \) and edge set \( E(F_n) = \{xx_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \). So, the color of vertex \( x \) must be different than the color of vertex \( x_1 \) and the color of vertex \( x_i \) must be different than the color of vertex \( x_{i+1} \). If we gave a color for vertex \( x \), i.e: color 1, then the color of vertices \( x_i \), for \( i = 1, 2, ..., n \), can not be color 1. Next if we gave vertex \( x_1 \) color 2 then the color of vertex \( x_{i+1} \) can not be color 2, for \( i = 1, 2, ..., n-1 \). So, we have 3 colors for \( F_n \) and this is the minimum color. Therefore \( \chi(F_n) = 3 \).

**Theorem 2.1** For \( n \geq 3 \), \( n \) natural numbers, the local antimagic vertex total chromatic number of fan graph is \( n_{lvat}(F_n) = 3 \).

**Proof.** Fan graph, denoted by \( F_n \), have vertex set \( V(F_n) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \) and edge set \( E(F_n) = \{xx_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \). The cardinality of vertex set and edge set of fan graph, respectively, is \( |V(F_n)| = n + 1 \) and \( |E(F_n)| = 2n - 1 \). To prove this theorem we have 2 cases, namely when \( n \) is odd and when \( n \) is even.

**Case 1.** For \( n \) odd number, to prove that \( n_{lvat}(F_n) = 3 \), it must be proved that \( n_{lvat}(F_n) \geq 3 \) and \( n_{lvat}(F_n) \leq 3 \). Next, it will be proven that \( n_{lvat}(F_n) \leq 3 \) by labeling fan graph \( F_n \) using the function of \( f: V(F_n) \cup E(F_n) \rightarrow \{1, 2, 3, \ldots, 3n\} \).

The vertex labeling functions are

\[
f(x) = \begin{cases} 
3n \quad &\text{if } i \text{ is odd}, \\
3n - \frac{i}{2} \quad &\text{if } i \text{ is even}
\end{cases}
\]

The edge labeling functions are

\[
f(x_i x_{i+1}) = \begin{cases} 
\frac{i + 1}{2} \quad &\text{if } i \text{ is odd} \\
\frac{2n - i}{2} \quad &\text{if } i \text{ is even}
\end{cases}
\]

\[
f(xx_i) = \begin{cases} 
\frac{2n - 1}{2} \quad &\text{if } i = 1 \\
\frac{2n + i - 3}{2} \quad &\text{if } i \text{ is odd} \\
\frac{3n + i - 3}{2} \quad &\text{if } i \text{ is even}
\end{cases}
\]
It will be proven that function $f$ is local antimagic vertex total coloring by proving that $f$ is a bijection and neighboring vertices have different weights. First it will be proven that $f$ is a bijection. It is known that $R_f = \{1, 2, 3, \ldots, 2n - 1\} \cup \{2n, 2n + 1, 2n + 2, \ldots, 3n - 1\} \cup \{3n\}$ so range of function $f$ is $R_f = \{1, 2, 3, \ldots, 3n\}$. It is clear that range and codomain have the same cardinality so $f$ is a surjective function. Next, it will be proven that $f$ is an injection. For any $u, v \in V(F_n)$ and $u \neq v$ applies $f(u) \neq f(v)$, so $f$ is an injective function. Since $f$ is surjection and injection, then $f$ is a bijection.

By function $f$, we get the total vertex weight of fan graph $(F_n)$. The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2} n(3n + 5)$$

$$w_t(x_i) = \begin{cases} \frac{9n - 3}{2}, & \text{if } i \text{ is odd} \\ \frac{11n - 3}{2}, & \text{if } i \text{ is even} \end{cases}$$

Based on the total vertex weight function, it will be shown that neighboring vertices have different weights. Vertex $x$ is adjacent to vertex $x_i$ and vertex $x_i$ is adjacent to vertex $x_{i+1}$, so it must have different weights. First, we assume that vertex $x$ has the same weight as vertex $x_i$. It can be stated that $w_t(x) = w_t(x_i)$ or $\frac{1}{2} n(3n + 5) = \frac{9n - 3}{2} = \frac{11n - 3}{2}$. If the equation is solved, we get the value of $n = 0$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex $x$ has a different weight from vertex $x_i$. Next, we assume that vertex $x_i$ has the same weight as vertex $x_{i+1}$. If $i$ is odd then $i + 1$ is even and vice versa. So we have $w_t(x_i) = w_t(x_{i+1})$ or $\frac{9n - 3}{2} = \frac{11n - 3}{2}$ or $n = 0$. This is a contradiction with the value of $n \geq 3$ so the assumption is wrong. Thus vertex $x_i$ has a different weight from vertex $x_{i+1}$. It can be concluded that neighboring vertices have different weights or $w_t(x) \neq w_t(x_i)$ and $w_t(x_i) \neq w_t(x_{i+1})$.

Since each neighboring vertex has different weights and $f$ is a bijection, it can be concluded that $f$ is local antimagic labeling. Then the vertices on fan graph $(F_n)$ are colored according to its total weight, which it is called local antimagic vertex total coloring. Because we have $3$ total weights, then also we have $3$ colors. Therefore, the local antimagic vertex total chromatic number of fan graph is $\chi tvat(F_n) \leq 3$.

Next it will be shown that $\chi tvat(F_n) \geq 3$. Based on Lemma 1.1. and Lemma 2.1, we have $\chi tvat(F_n) \geq \chi(F_n) = 3$. Then $\chi tvat(F_n) \geq 3$. Because $\chi tvat(F_n) \leq 3$ and $\chi tvat(F_n) \geq 3$, it is proven that $\chi tvat(F_n) = 3$.

As an illustration, it is presented by Figure 1 which is the local antimagic vertex total coloring of $F_7$.

![Figure 1. Illustration of local antimagic vertex total coloring on $F_7$.](image-url)
Case 2. For \( n \) even number, to prove that \( \chi_{vat}(F_n) = 3 \), it must be proved that \( \chi_{vat}(F_n) \geq 3 \) and \( \chi_{vat}(F_n) \leq 3 \). Next, it will be proven that \( \chi_{vat}(F_n) \leq 3 \) by labeling fan graph \( F_n \) using the function of \( f: V(F_n) \cup E(F_n) \rightarrow \{1, 2, 3, \ldots, 3n\} \).

The vertex labeling functions are
\[
\begin{align*}
    f(x) &= 3n \\
    f(x_i) &= \begin{cases} 
        \frac{5n - i - 1}{2}, & \text{if } i \text{ is odd} \\
        \frac{6n - i - 2}{2}, & \text{if } i \text{ is even, } i \neq n \\
        \frac{3n - 1}{2}, & i = n
    \end{cases}
\end{align*}
\]

The edge labeling functions are
\[
\begin{align*}
    f(x_i x_{i+1}) &= \begin{cases} 
        \frac{i + 1}{2}, & \text{if } i \text{ is odd} \\
        \frac{2n - i}{2}, & \text{if } i \text{ is even} \\
        \frac{2n - 1}{2}, & i = 1
    \end{cases} \\
    f(xx_i) &= \begin{cases} 
        \frac{2n + i - 3}{2}, & \text{if } i \text{ is odd} \\
        \frac{3n + i - 4}{2}, & \text{if } i \text{ is even}
    \end{cases}
\end{align*}
\]

It will be proven that function \( f \) is the local antimagic vertex total coloring by proving that \( f \) is a bijection and neighboring vertices have different weights. First it will be proven that \( f \) is a bijection. It is known that \( R_f = \{1, 2, 3, \ldots, 2n - 1\} \cup \{2n - 1\} \cup \{3n\} \) so the range of \( f \) is \( R_f = \{1, 2, 3, \ldots, 3n\} \). It is clear that range and codomain have the same cardinality so \( f \) is a surjective function. Next, it will be proven that \( f \) is an injection. For any \( u, v \in V(F_n) \) and \( u \neq v \) applies \( f(u) \neq f(v) \), so \( f \) is an injective function. Since \( f \) is surjection and injection, then \( f \) is a bijection.

By function \( f \) we get the total vertex weight of fan graph \( (F_n) \). The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:
\[
\begin{align*}
    w_t(x) &= \frac{1}{2} n(3n + 5) \\
    w_t(x_i) &= \begin{cases} 
        \frac{9n - 2}{2}, & \text{if } i \text{ is odd} \\
        \frac{11n - 6}{2}, & \text{if } i \text{ is even}
    \end{cases}
\end{align*}
\]

Based on total vertex weight function above, it will be shown that neighboring vertices have different weights. Vertex \( x \) is adjacent to vertex \( x_i \) and vertex \( x_i \) is adjacent to vertex \( x_{i+1} \), so it must have different weights. First, we assume that vertex \( x \) has the same weight as vertex \( x_i \). It can be stated that \( w_t(x) = w_t(x_i) \) or \( \frac{1}{2} n(3n + 5) = \frac{9n - 2}{2} = \frac{11n - 6}{2} \). If the equation is solved, we get the value of \( n = 2 \). This is a contradiction with the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( x \) has a different weight from vertex \( x_i \). Next, we assume that vertex \( x_i \) has the same weight as vertex \( x_{i+1} \). If \( i \) is odd then \( i + 1 \) is even and vice versa. So we have \( w_t(x_i) = w_t(x_{i+1}) \) or \( \frac{9n - 2}{2} = \frac{11n - 6}{2} \) or \( n = 2 \). This is a contradiction with the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( x_i \) has a different weight from vertex \( x_{i+1} \). It can be concluded that neighboring vertices have different weights or \( w_t(x) \neq w_t(x_i) \) and \( w_t(x_i) \neq w_t(x_{i+1}) \).

Since each neighboring vertex has different weights and \( f \) is a bijection, it can be concluded that \( f \) is local antimagic labeling. Then the vertices on the fan graph \( (F_n) \) are colored according to its total weight, which it is called local antimagic vertex total coloring. Because we have 3 total weights, then
we also have 3 colors. Therefore, the local antimagic vertex total chromatic number of fan graph is $\chi_{\text{vat}}(F_n) \leq 3$.

Next it will be shown that $\chi_{\text{vat}}(F_n) \geq 3$. Based on Lemma 1.1. and Lemma 2.1, we have $\chi_{\text{vat}}(F_n) \geq \chi(F_n) = 3$. Then $\chi_{\text{vat}}(F_n) \geq 3$. Because $\chi_{\text{vat}}(F_n) \geq 3$ and $\chi_{\text{vat}}(F_n) \leq 3$, it is proven that $\chi_{\text{vat}}(F_n) = 3$.

As an illustration, it is presented by Figure 2 which is the local antimagic vertex total coloring of $F_8$.

![Illustration of local antimagic vertex total coloring on $F_8$.](image1)

**Figure 2.** Illustration of local antimagic vertex total coloring on $F_8$.

3. **Local Antimagic Vertex Total Coloring on Fan Graph with Comb Product Operation**

In this section, we got the local antimagic vertex total coloring of fan graph with comb product operation that is $F_n \triangleright F_3$. Based on the definition of comb product operation, the obtained result of observation is as follows:

**Observation 3.1** Let $F_n \triangleright F_3$ be a graph resulting from comb product operation by taking one copy of $F_n$ and $|V(F_3)|$ copies of $F_3$ and grafting the $i$-th copy of $F_3$ at the center vertex to $i$-th vertex of $F_n$. Then we have the vertex set of $F_n \triangleright F_3$ is $V(F_n \triangleright F_3) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq 3\} \cup \{y'_i; 1 \leq i \leq n, 1 \leq j \leq 3\}$ and edge set of $F_n \triangleright F_3$ is $E(F_n \triangleright F_3) = \{xx_i; 1 \leq i \leq n\} \cup \{x_ix_{i+1}; 1 \leq i \leq n - 1\} \cup \{xx_iy'_i; 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{x_iy_i; 1 \leq i \leq 3\} \cup \{y_iy'_i; 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{y_iy'_i; 1 \leq i \leq 2\}$. So, the vertex cardinality of $F_n \triangleright F_3$ is $4(n + 1)$ and edge set cardinality of $F_n \triangleright F_3$ is $7n + 4$.

As an illustration, it is presented by Figure 3 which is the example of fan graph with comb product operation $F_5 \triangleright F_3$.

![Illustration of fan graph with comb product operation $F_5 \triangleright F_3$.](image2)

**Figure 3.** Illustration of fan graph with comb product operation $F_5 \triangleright F_3$.

Before we got the local antimagic vertex total coloring of fan graph with comb product operation, we must know its vertex coloring chromatic number.

**Lemma 3.1** The vertex coloring chromatic number of $F_n \triangleright F_3$ is $\chi(F_n \triangleright F_3) = 3$

**Proof.** Vertex coloring of graph $G$ is coloring the vertices in $G$ such that any two adjacent vertices in $V(G)$ have different color. Based on Observation 3.1, we have vertex set $V(F_n \triangleright F_3) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq 3\} \cup \{y'_i; 1 \leq i \leq n, 1 \leq j \leq 3\}$ and edge set $E(F_n \triangleright F_3) = \{xx_i; 1 \leq i \leq n\} \cup \{x_ix_{i+1}; 1 \leq i \leq n - 1\} \cup \{xx_iy'_i; 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{x_iy_i; 1 \leq i \leq 3\} \cup \{y_iy'_i; 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{y_iy'_i; 1 \leq i \leq 2\}$. Therefore, the local antimagic vertex total coloring number of the graph $F_n \triangleright F_3$ is $3$.
Theorem 3.1 Let \( G \) be a fan graph with comb product operation \( F_n \bowtie F_3 \). For \( n \) natural numbers, \( n \geq 3 \), the local antimagic vertex total chromatic number of graph \( G \) is \( 3 \leq \chi_{\text{tvar}}(G) \leq 5 \).

**Proof.** Based on Observation 3.1, \( G \) has vertex set \( V(G) = \{x\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq 3\} \cup \{y_i^j; 1 \leq i \leq n, 1 \leq j \leq 3\} \) and edge set \( E(G) = \{xx_i; 1 \leq i \leq n\} \cup \{x_ix_{i+1}; 1 \leq i \leq n-1\} \cup \{x_iy_i; 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{xy_i; 1 \leq i \leq 3\} \cup \{y_i^j, y_i^{j+1}; 1 \leq i \leq n, 1 \leq j \leq 2\} \cup \{y_iy_{i+1}; 1 \leq i \leq n-1\} \). Vertex set and edge set cardinality of \( G \) are \( |V(G)| = 4(n+1) \) and \(|E(G)| = 7n+4\). To prove that \( \chi_{\text{tvar}}(G) \leq 5 \) we have 2 cases, namely when \( n \) is odd and when \( n \) is even.

**Case 1.** For \( n \) odd number, it will be proven that \( \chi_{\text{tvar}}(G) \leq 5 \) by labeling graph \( G \) using the function of \( f: V(G) \cup E(G) \to \{1, 2, 3, \ldots, 11n + 8\} \).

The vertex labeling functions are

\[
f(x) = 3n + \frac{4n + i - 2}{2}, \quad \text{if } i \text{ is even}
\]

\[
f(x_i) = \begin{cases} 
\frac{5n + i}{2}, & \text{if } i \text{ is odd, } i \neq n \\
\frac{5n - 1}{2}, & \text{if } i = n
\end{cases}
\]

\[
f(y_i) = \begin{cases} 
9n + 7, & \text{if } i = 1 \\
11n + 8, & \text{if } i = 2 \\
9n + 11, & \text{if } i = 3 \\
11n - i + 9, & \text{if } j = 1, i \text{ is even} \\
10n - i + 7, & \text{if } i \neq n, j = 1, i \text{ is odd} \\
10n + 8, & \text{if } i = n, j = 1 \\
9n + i + 6, & \text{if } j = 2, i \text{ is even} \\
10n + i + 8, & \text{if } i \neq n, j = 2, i \text{ is odd} \\
10n + 7, & \text{if } i = n, j = 2 \\
7n + i + 3, & \text{if } j = 3, i \text{ is even} \\
8n + i + 5, & \text{if } i \neq n, j = 3, i \text{ is odd} \\
8n + 4, & \text{if } i = n, j = 3
\end{cases}
\]

The edge labeling functions are

\[
f(x_ix_{i+1}) = \begin{cases} 
\frac{i + 1}{2}, & \text{if } i \text{ is odd} \\
\frac{2n - i}{2}, & \text{if } i \text{ is even}
\end{cases}
\]

\[
f(xx_i) = \begin{cases} 
\frac{2n - 1}{2}, & \text{if } i = 1 \\
\frac{2n + i - 3}{2}, & \text{if } i \text{ is odd, } i \neq 1 \\
\frac{3n + i - 3}{2}, & \text{if } i \text{ is even}
\end{cases}
\]
If $\sum_{i} w_i = 11 = j$, it will be proven that $R_{ij}$ is an injective function. By $f$ will be proven that $\{n, 2, 3, 4\}$ vertices have different weights. First it will be proven that $w_j = 2, i$ is odd

$$f(x_i y_i) = \left\{ \begin{array}{ll}
\frac{6n+3}{2}, & j = 1, i \text{ is even} \\
\frac{7n+6}{2}, & i \neq n, j = 1, i \text{ is odd} \\
\frac{6n+4}{2}, & i = n, j = 1 \\
\frac{10n+i+4}{2}, & j = 2, i \text{ is even} \\
\frac{11n+i+6}{2}, & i \neq n, j = 2, i \text{ is odd} \\
\frac{11n+5}{2}, & i = n, j = 2 \\
\frac{9n-i+8}{2}, & j = 3, i \text{ is even} \\
\frac{8n-i+8}{2}, & i \neq n, j = 3, i \text{ is odd} \\
\frac{8n+7}{2}, & i = n, j = 3 \\
\frac{14n-i+10}{2}, & j = 3, i \text{ is even} \\
\frac{13n-i+8}{2}, & i \neq n, j = 3, i \text{ is odd} \\
\frac{13n+9}{2}, & i = n, j = 3 \\
\end{array} \right.$$

$$f(y_{i+1}) = 4n+i$$

$$f(y_{j} y_{j+1}) = \left\{ \begin{array}{ll}
\frac{3n+i}{2}, & j = 1, i \text{ is even} \\
\frac{7n+i+2}{2}, & i \neq n, j = 1, i \text{ is odd} \\
\frac{2}{7n+1}, & i = n, j = 1 \\
\frac{10n-i+6}{2}, & j = 2, i \text{ is even} \\
\frac{2}{9n-i+4}, & i \neq n, j = 2, i \text{ is odd} \\
\frac{2}{9n+5}, & i = n, j = 2 \\
\end{array} \right.$$

It will be proven that $f$ is the local antimagic vertex total coloring by proving that $f$ is a bijection and neighboring vertices have different weights. First it will be proven that $f$ is a bijection. It is known that $R_f = \{1, 2, 3, \ldots, 2n-1\} \cup \{2n, 2n+1, 2n+2, \ldots, 3n\} \cup \{3n+1, 3n+2, \ldots, 5n+2\} \cup \{5n+3, 5n+4, \ldots, 7n+4\} \cup \{7n+5, 7n+6, \ldots, 11n+8\}$ so the range of $f$ is $R_f = \{1, 2, 3, \ldots, 11n+8\}$. Next, it will be proven that $f$ is an injection. For any $u, v \in V(G)$ and $u \neq v$ applies $f(u) \neq f(v)$, so $f$ is an injective function. Since $f$ is surjection and injection, then $f$ is a bijection.

By function $f$ we get the total vertex weight of graph $G$. The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2} n(3n+43) + 13$$

$$w_t(x_i) = \left\{ \begin{array}{ll}
\frac{49n+25}{2}, & \text{if } i \text{ is odd} \\
\frac{51n+25}{2}, & \text{if } i \text{ is even} \\
\end{array} \right.$$

$$w_t(y_i) = \left\{ \begin{array}{ll}
\frac{19n+11}{2}, & \text{if } i \text{ is odd} \\
\frac{26n+17}{2}, & \text{if } i \text{ is even} \\
\end{array} \right.$$

$$w_t(y_{j}) = \left\{ \begin{array}{ll}
\frac{19n+11}{2}, & \text{if } j \text{ is odd} \\
\frac{26n+17}{2}, & \text{if } j \text{ is even} \\
\end{array} \right.$$
Based on the total vertex weight function it will be shown that neighboring vertices have different weights. Vertex \( x \) is adjacent to vertex \( x_i \) and vertex \( y_i \) so it must have different weight. We assume that \( w_t(x) = w_t(x_i) = w_t(y_i) \) or \( \frac{1}{2}n(3n + 43) + 13 = \frac{49n+25}{2} = 19n + 11 \) if \( i \) is odd and \( \frac{1}{2}n(3n + 43) + 13 = \frac{51n+25}{2} = 26n + 17 \) if \( i \) is even. If the equations is solved, we can not get the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( x \) has a different weight from vertex \( x_i \) and \( y_i \).

Next, vertex \( x_i \) is adjacent to vertex \( x_{i+1} \) and vertex \( y_{i+1} \). We assume that \( w_t(x_i) = w_t(x_{i+1}) = w_t(y) \) or \( \frac{49n+25}{2} = \frac{51n+25}{2} = 19n + 11 \) if \( i \) is odd and \( \frac{49n+25}{2} = \frac{51n+25}{2} = 26n + 17 \) if \( i \) is even. If the equations is solved, we can not get the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( x_i \) has a different weight from vertex \( x_{i+1} \) and \( y_{i+1} \).

Next, vertex \( y_i \) is adjacent to vertex \( y_{i+1} \). We assume that \( w_t(y_i) = w_t(y_{i+1}) \) or \( 19n + 11 = 26n + 17 \). If the equations is solved, we can not get the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( y_i \) has a different weight from vertex \( y_{i+1} \).

Last, vertex \( y_{i+1} \) is adjacent to vertex \( y_{i+1} \). We assume that \( w_t(y_{i+1}) = w_t(y_{i+1}) \) or \( 19n + 11 = 26n + 17 \). If the equations is solved, we can not get the value of \( n \geq 3 \) so the assumption is wrong. Thus vertex \( y_{i+1} \) has a different weight from vertex \( y_{i+1} \).

Since each neighboring vertex has different weights and \( f \) is a bijection so it can be concluded that \( f \) is local antimagic labeling. Then vertices on graph \( G \) are colored based on their total vertex weights. This is called the local antimagic vertex total coloring. Because we have 5 values of total weights, then we also have 5 colors. Therefore, the local antimagic vertex total chromatic number of graph \( G \) is \( \chi_{tvt}(G) \leq 5 \).

Next it will be shown that \( \chi_{tvt}(G) \geq 3 \). Based on Lemma 1.1. and Lemma 3.1, we have \( \chi_{tvt}(G) \geq \chi(G) = 3 \). Then \( \chi_{tvt}(G) \geq 3 \). Because \( \chi_{tvt}(G) \leq 5 \) and \( \chi_{tvt}(G) \geq 3 \), it is proven that \( 3 \leq \chi_{tvt}(G) \leq 5 \).

As an illustration, it is presented by Figure 4 which is the local antimagic vertex total coloring of \( F_5 \triangleright F_3 \).

**Figure 4.** Illustration of the local antimagic vertex total coloring of \( F_5 \triangleright F_3 \).

**Case 2.** For \( n \) even number, it will be proven that \( \chi_{tvt}(G) \leq 5 \) by labeling graph \( G \) using the function of \( f: V(G) \cup E(G) \rightarrow \{1,2,3,\ldots,11n + 8\} \).

The vertex labeling functions are
\[ f(x) = 3n \]
\[
\begin{cases} 
\frac{4n + i - 1}{2}, & \text{if } i \text{ is odd} \\
\frac{5n + i}{2}, & \text{if } i \text{ is even}, \ i \neq n \\
\frac{5n}{2}, & \text{if } i = n 
\end{cases}
\]
\[
\begin{align*}
f(x_i) &= \\
&= \begin{cases} 
\frac{4n + i - 1}{2}, & \text{if } i \text{ is odd} \\
\frac{5n + i}{2}, & \text{if } i \text{ is even}, \ i \neq n \\
\frac{5n}{2}, & \text{if } i = n 
\end{cases} \\
&= \begin{cases} 
9n + 7, & i = 1 \\
11n + 8, & i = 2 \\
9n + 11, & i = 3 
\end{cases}
\]
\[
\begin{align*}
f(y_i) &= \\
&= \begin{cases} 
11n - i + 8, & j = 1, \ i \text{ is odd} \\
10n - i + 7, & i \neq n, j = 1, i \text{ is even} \\
10n + 7, & i = n, j = 1 \\
9n + i + 7, & j = 2, \ i \text{ is odd} \\
10n + i + 8, & i \neq n, j = 2, i \text{ is even} \\
10n + 8, & i = n, j = 2 \\
7n + i + 4, & j = 3, \ i \text{ is odd} \\
8n + i + 5, & i \neq n, j = 3, i \text{ is even} \\
8n + 5, & i = n, j = 3 
\end{cases}
\]
\]

The edge labeling functions are
\[
f(x_i x_{i+1}) = \begin{cases} 
\frac{i + 1}{2}, & \text{if } i \text{ is odd} \\
\frac{2n - 1}{2}, & \text{if } i \text{ is even} \\
\frac{2n + i - 3}{2}, & \text{if } i \text{ is odd}, \ i \neq 1 \\
\frac{3n + i - 4}{2}, & \text{if } i \text{ is even} \\
\frac{6n + 3}{2}, & i = 1 \\
\frac{7n + 6}{2}, & i = 2 \\
\frac{6n + 4}{2}, & i = 3 \\
\frac{10n + i + 5}{2}, & j = 1, \ i \text{ is odd} \\
\frac{11n + i + 6}{2}, & i \neq n, j = 1, i \text{ is even} \\
\frac{11n + 6}{2}, & i = n, j = 1 \\
\frac{9n - i + 7}{2}, & j = 2, \ i \text{ is odd} \\
\frac{8n - i + 6}{2}, & i \neq n, j = 2, i \text{ is even} \\
\frac{8n + 6}{2}, & i = n, j = 2 \\
\frac{14n - i + 9}{2}, & j = 3, \ i \text{ is odd} \\
\frac{13n - i + 8}{2}, & i \neq n, j = 3, i \text{ is even} \\
\frac{13n + 8}{2}, & i = n, j = 3 
\end{cases}
\]
\[
f(y_i y_{i+1}) = 4n + i
\]
It will be proven that $f$ is the local antimagic vertex total coloring by proving that $f$ is a bijection and neighboring vertices have different weights. First it will be proven that $f$ is a bijection. It is known that $R_f = \{1,2,3,\ldots,2n-1\} \cup \{2n,2n+1,2n+2,\ldots,3n\} \cup \{3n+1,3n+2,\ldots,5n+2\} \cup \{5n+3,5n+4,\ldots,7n+4\} \cup \{7n+5,7n+6,\ldots,11n+8\}$ so the range of $f$ is $R_f = \{1,2,3,\ldots,11n+8\}$. It is clear that range and codomain have the same cardinality so $f$ is a surjective function. Next, it will be proven that $f$ is an injection. For any $u,v \in V(G)$ and $u \neq v$ applies $f(u) \neq f(v)$, so $f$ is an injective function. Since $f$ is surjection and injection, then $f$ is a bijection.

By function $f$ we get total vertex weight of graph $G$. The total vertex weight is obtained from the sum of vertex labels and edge labels. The total vertex weights are as follows:

$$w_t(x) = \frac{1}{2}n(3n+43) + 13$$

$$w_t(x_i) = \begin{cases} 25n + 13, & \text{if } i \text{ is odd} \\ 25n + 11, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y_i) = \begin{cases} 19n + 11, & \text{if } i \text{ is odd} \\ 26n + 17, & \text{if } i \text{ is even} \end{cases}$$

$$w_t(y'_i) = \begin{cases} 19n + 11, & \text{if } j \text{ is odd} \\ 26n + 17, & \text{if } j \text{ is even} \end{cases}$$

Based on the total vertex weight function it will be shown that neighboring vertices have different weights. Vertex $x$ is adjacent to vertex $x_i$ and vertex $y_i$ so it must have different weight. We assume that $w_t(x) = w_t(x_i) = w_t(y_i)$ or $\frac{1}{2}n(3n+43) + 13 = 25n + 13 = 19n + 11$ if $i$ is odd and $\frac{1}{2}n(3n+43) + 13 = 25n + 11 = 26n + 17$ if $i$ is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex $x$ has a different weight from vertex $x_i$ and $y_i$.

Next, vertex $x_i$ is adjacent to vertex $x_{i+1}$ and vertex $y'_i$. We assume that $w_t(x_i) = w_t(x_{i+1}) = w_t(y'_i)$ or $25n + 13 = 25n + 11 = 19n + 11$ if $i$ is odd and $25n + 13 = 26n + 17$ if $j$ is even. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex $x_i$ has a different weight from vertex $x_{i+1}$ and $y'_i$.

Next, vertex $y_i$ is adjacent to vertex $y_{i+1}$. We assume that $w_t(y_i) = w_t(y_{i+1})$ or $19n + 11 = 26n + 17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex $y_i$ has a different weight from vertex $y_{i+1}$.

Last, vertex $y'_i$ is adjacent to vertex $y'_{i+1}$. We assume that $w_t(y'_i) = w_t(y'_{i+1})$ or $19n + 11 = 26n + 17$. If the equations is solved, we can not get the value of $n \geq 3$ so the assumption is wrong. Thus vertex $y'_i$ has a different weight from vertex $y'_{i+1}$.

Since each neighboring vertex has different weights and $f$ is a bijection so it can be concluded that $f$ is local antimagic labeling. Then vertices on graph $G$ are colored based on their total vertex weights. This is called the local antimagic vertex total coloring. Because we have 5 values of total
weights, then we also have 5 colors. Therefore, the local antimagic vertex total chromatic number of graph \( G \) is \( \chi_{lvat}(G) \leq 5 \).

Next it will be shown that \( \chi_{lvat}(G) \geq 3 \). Based on Lemma 1.1. and Lemma 3.1, we have \( \chi_{lvat}(G) \geq \chi(G) = 3 \). Then \( \chi_{lvat}(G) \geq 3 \). Because \( \chi_{lvat}(G) \leq 5 \) and \( \chi_{lvat}(G) \geq 3 \), it is proven that \( 3 \leq \chi_{lvat}(G) \leq 5 \).

As an illustration, it is presented by Figure 5 which is the local antimagic vertex total coloring of \( F_4 \bowtie F_3 \).

\[\text{Figure 5. Illustration of the local antimagic vertex total coloring of } F_4 \bowtie F_3.\]

4. Conclusion
In this study there are two results related to local antimagic vertex total coloring. The resulting theorem is related to the local antimagic vertex total chromatic number of fan graph \( F_n \) and fan graph resulting from comb product operation \( F_n \bowtie F_3 \). The local antimagic vertex total chromatic number on \( F_n \), for \( n \geq 3 \) is \( \chi_{lvat}(F_n) = 3 \). The local antimagic vertex total chromatic number of \( F_n \bowtie F_3 \) for \( n \geq 3 \) is \( 3 \leq \chi_{lvat}(F_n \bowtie F_3) \leq 5 \).

This result is not yet complete because we don’t get the local antimagic vertex total chromatic number of \( F_n \bowtie F_m \) and graph resulting from comb product operation by grafting the \( i \)-th copy of \( F_m \) at the center vertex (or non center vertex) to \( i \)-th vertex of \( F_n \). It could be an idea to the next research.

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