Microscopic cluster description of light nuclei

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Abstract. We present a short overview of different cluster models, and emphasize on three-cluster approaches. An application to $^8$He and $^7$He suggests the presence of an excited state below 1 MeV in $^7$He, as observed in a recent experiment. We also present recent results on the $^{12}$C nucleus, and discuss general properties of the 3$\alpha$ system in non-microscopic theories.

1. Introduction

Clustering is a well known effect in the spectroscopy of light nuclei [1, 2, 3]. A typical example is the $^8$Be nucleus, which can be seen as an $\alpha + \alpha$ cluster system. Owing to its strong binding energy, the $\alpha$ particle is a natural cluster. Clustering effects are expected to show up near the $\alpha$ threshold; this property is at the origin of the Ikeda diagram [4], which predicts the existence of cluster states in various $\alpha$ nuclei. More recently, evidence was found for more exotic cluster structures, such as $^6$He+$^6$He in some $^{12}$Be states [5].

On the theoretical side, cluster models take advantage of this specific structure. In a cluster model, the wave function of a nucleus is described from internal wave functions of the clusters [6]. This simplification is in general reliable, and provides an approximate solution of the many-body Schrödinger equation. On the other hand, assuming a cluster structure is the natural starting point for the description of continuum states. This property makes cluster models well adapted to nuclear astrophysics, where the low energies involved are responsible for very small cross sections [7].

Originally, cluster models were developed for two-cluster systems. A textbook example is the $\alpha + \alpha$ system [8], whose phase shifts can be accurately described by the Resonating Group Method (RGM). Later on it was realized that cluster models could be extended to multi-cluster approaches with, for example, the $^{12}$C nucleus described by three $\alpha$ particles. This development also allows the treatment of reactions where one of the colliding nuclei has itself a cluster structure. A typical example is the $^7$Be+\textit{p} system where $^7$Be is described by an $\alpha$+$^3$He cluster structure [9]. Recent applications deal with more than 3 clusters [10]. Let us point out that the applications of cluster models are not limited to nucleons. The same formalism can be used in other fields, such as hypernuclei [11] or multi-quark systems [12]. All these extensions are of course highly time-consuming, and are strongly associated with the development of the computer power.

Recent developments of \textit{ab initio} models (see for example Refs. [13, 14]) are quite successful for spectroscopic properties of low-lying states. These models make use of realistic interactions, fitted on many properties of the nucleon-nucleon system. However a consistent description of
bound and scattering states of an \( A \)-body problem remains a very difficult task [15]. A possible simplification is the use of a potential-model wave function for the relative motion [16], but the \textit{ab initio} character of the model is partially lost.

In Sect. 2, we present a short overview of cluster models, and emphasize on the 3-body variant, where different definitions of the cluster wave functions can be considered. Section 3 is devoted to two recent applications: the \(^8\)He and \(^7\)He nuclei, and a triple \( \alpha \) description of \(^{12}\)C in the hyperspherical formalism. Conclusions are given in Sect. 4.

\section{Microscopic cluster models}

Let us consider the \( A \)-body Hamiltonian

\[ H = \sum_i A T_i + \sum_{i<j} V_{ij}, \]  

where \( T_i \) is the kinetic energy of nucleon \( i \), and \( V_{ij} \) a nucleon-nucleon interaction. The goal of a cluster model is to find an approximate solution of the Schrödinger equation associated with (1). This solution is written, assuming an \( N \)-cluster approximation, as

\[ \Psi = A \Phi_1 \Phi_N g(\rho_1, \ldots, \rho_N), \]  

where \( A \) is the \( A \)-nucleon antisymmetrizer, \( \Phi_k \) are the internal wave functions, and \( g \) is a cluster-cluster function depending on the \( N-1 \) Jacobi coordinates. Historically, cluster models were essentially developed with \( N = 2 \) [6], but later on, multicluster approaches allowed to widen the range of applications [17]. Using the number of clusters \( N \) equal to the nucleon number \( A \), a cluster model is equivalent to an \( ab \)-initio model, where the cluster approximation is not used [18]. Notice that this approximation makes it necessary to use in the Hamiltonian (1) an effective nucleon-nucleon interaction.

Here we briefly review three-cluster variants. We refer the reader to Ref. [10] for multicluster extensions. Let us consider Fig. 1, where different variants of a three-cluster model are shown. Variant (\( a \)) is adapted to the spectroscopy only. A basis wave function depends on 3 generator coordinates \((R_1, R_2, \alpha)\) and is given by

\[ \Phi^{JM\pi}_{K\nu_i}(R_1, R_2, \alpha) = P^{JM}_K A \Phi_1^{\nu_1}(S_1) \Phi_2^{\nu_2}(S_2) \Phi_3^{\nu_3}(S_3), \]  

where \( \nu_i \) are the spin projections, \( K \) is the intrinsic projection and the coordinates \( S_i \) are defined from \((R_1, R_2, \alpha); P^{JM}_K \) is the usual spin projection operator [2]. The total wave function is given by a superposition of basis states (3). The calculation of matrix elements requires 3-dimensional integrals, which are computed numerically. More detail can be found in Ref. [19]; this model is well adapted to halo nuclei such as \(^{11}\)Li [20] or \(^8\)He [21].

Variant (\( b \)) was developed with the aim of investigating nucleus-nucleus reactions involving deformed nuclei, such as \(^7\)Be for example [22]. A double angular-momentum projection is performed, which provides correct asymptotic conditions for two-body scattering. A basis wave function depends on two generator coordinates \((R_1, R_2)\) and is written as

\[ \Phi^{JM\pi}_{\ell L}(R_1, R_2) = A \phi_1 \phi_2 \phi_3 \{Y_\ell(\Omega_{\rho_1}) \otimes Y_L(\Omega_{\rho_2})\}^{JM}_{\ell L}(\rho_1, R_1) \Gamma_L(\rho_2, R_2), \]  

where, for the sake of simplicity, we assume that the internal spins are zero, and where \( \Gamma_\ell \) and \( \Gamma_L \) are projected Gaussian functions. Owing to the multiple projection, matrix elements involving basis states (4) require 5-dimensional integrals (see Refs. [22, 23] for details). This model has been essentially used for reactions of astrophysical interest, such as \(^7\)Be(p,\( \gamma \)){^8}\)B [9] or \(^{19}\)Ne(p,\( \gamma \)){^20}\)Na [24]. In various cases it has been shown that a single \( R_2 \) value is sufficient to
have of fair description of the system. In practice, this value is chosen so as to minimize the binding energy of the 2-cluster subsystem.

Recently, variant (b) of the three-cluster model was extended to the hyperspherical formalism (variant (c), see Ref. [25]). This framework is known to be well adapted to 3-body systems [26]. The generator coordinates \((R_x, R_y)\) are transformed into new generator coordinates: the hyperradius \(R\), and the hyperangle \(\alpha\) [25]. Again, we assume that the clusters have a spin zero. In this case, a basis state is given by

\[
\Phi^{JM\pi}_{\ell_x \ell_y L K}(R) = A \phi_1 \phi_2 \phi_3 [Y_{\ell_x}(\Omega_{px}) \otimes Y_{\ell_y}(\Omega_{py})]^J M \frac{P_{n}^{\ell_{x}+1/2, \ell_{y}+1/2} \cos 2\alpha}{G_K(\rho, R)},
\]

where \(P^\alpha_{n}^{\beta}\) is a Jacobi polynomial, \(G_K(\rho, R)\) a radial function (see Ref. [25]), and \(K\) the hypermomentum \((K = 2n + \ell_x + \ell_y)\). Until now, this model has been applied to nuclear spectroscopy only. However, it is the starting point for a microscopic treatment of 3-body continuum states [27]. In all variants, the basis states (3-5) can be expressed as projected states determinants. In these conditions, the calculation of matrix elements (overlap, Hamiltonian, densities, etc...) is systematic and can be performed by using standard techniques [2].

With respect to ab-initio models, the main advantage of cluster models is their simple extension to continuum states. The asymptotic behaviour of GCM basis states has a gaussian form, which is corrected by using the Microscopic \(R\)-matrix method (MRM, see Ref. [28]). The MRM is a direct extension of the general \(R\)-matrix theory [29], which assumes that the configuration space is split in 2 regions: the internal region, where the cluster-cluster antisymmetrization and the nuclear force are important, and the external region, where only the Coulomb force remains. The matching of internal and external solutions provides the collision matrix. In non-microscopic models, this formalism has been extended to 3-body continuum states [27], but is more complicated owing to the long range of the potential terms in hyperspherical coordinates.

3. Applications
3.1. The \(^7\)He and \(^8\)He nuclei

The \(^8\)He nucleus has been studied in variant (a) of the three-cluster model [21], involving \(^6\)He+n+n configurations with excited states of \(^6\)He. One of the goals is to investigate the role of core excitations. The NN interaction is taken as the Minnesota force with a zero-range spin-orbit interaction (see Ref. [21] for more detail).

Fig. 2 shows the spectrum of \(^8\)He in different conditions: the full basis, including all \(^6\)He excited states (labeled by \(I_1\)), a basis limited to the \(^6\)He ground state, and to the first \(2^+\) excited

\[
\alpha
\]

\[
\gamma
\]

\[
\beta
\]

Figure 1. Three-cluster configurations (see text).
state only. Clearly the role of the $^6\text{He}(2^+) + n + n$ configuration is important. An analysis of the wave function shows that the $2^+$ component represents 81%, in agreement with an experimental estimate $5/6$ [30].

An $^6\text{He} + n$ two-cluster model, with the same NN interaction has been applied to $^7\text{He}$, which is unbound. Consequently, the energy levels shown in Fig. 3 have been obtained in the MRM framework. Again the role of the $^6\text{He}(2^+) + n$ channel is dominant: the $5/2^-$ excited state is absent with the $^6\text{He}(0^+) + n$ configuration only. The theoretical energy of the $1/2^-$ resonance is in good agreement with a recent experiment [31], which shows evidence for a $^7\text{He}$ excited state below 1 MeV.

![Figure 2. $^8\text{He}$ spectrum. $I_1$ corresponds to the spin of the $^6\text{He}$ core.](image)

![Figure 3. $^7\text{He}$ spectrum (see Fig. 2).](image)

### 3.2. The $^{12}\text{C}$ nucleus as a $3\alpha$ system

The $^{12}\text{C}$ nucleus is an ideal example for microscopic multicluster models, and is studied since many years [32]. Microscopic $3\alpha$ models [33, 22] succeed in reproducing the low energy spectrum of $^{12}\text{C}$. Here we extend these models to variant ($c$), which makes use of the hyperspherical formalism. The NN interaction is the Minnesota force, which provides $\alpha + \alpha$ phase shifts in very good agreement with experiment [8]. In Fig. 4, we show the $^{12}\text{C}$ energy curves, i.e. the energy of the system for a fixed value of the hyperradius $R$. The ground-state rotational band shows up with clustering effects decreasing with $J$. The model also suggests $1^-$ and $3^-$ states at higher energies. The corresponding $^{12}\text{C}$ spectrum is given in Fig. 5. The ground-state binding energy is too large (by 4 MeV) compared to experiment. This confirms that a simultaneous description of two-cluster and three-cluster properties is in general not possible with a good accuracy [34]. The $2^+$ and $4^+$ excitation energies are too low, due to the lack of spin-orbit force. On the contrary, the $0^+_2$ Hoyle state, just above the $3\alpha$ threshold, is well reproduced.

If microscopic $3\alpha$ models provide a fair description of $^{12}\text{C}$, the situation is more ambiguous for non-microscopic theories (see for example Ref. [35]). Although accurate $\alpha + \alpha$ potentials are available in the literature [36, 37], it is not clear whether they can be used for the $3\alpha$ system. The origin of the problem arises from $\alpha + \alpha$ forbidden states which must be removed in the 3-body system. Using the bare $\alpha + \alpha$ potential provides an underbinding of $^{12}\text{C}$, which requires, either the introduction of a phenomenological $3\alpha$ force, or a renormalization of the potential. This latter procedure gives rise to additional $^{12}\text{C}$ states, which have no physical meaning [38]. According to Fujiwara et al. [39], the lowest $0^+$ state is not the ground state. A recent progress has been done in this field by using the renormalized RGM [40], where a non-local $\alpha + \alpha$
Figure 4. Energy of the $3\alpha$ system for fixed hyperradius $R$. The energies are given with respect to the $3\alpha$ threshold.

Potential is deduced from the RGM kernel. This approach provides an exact treatment of the $\alpha + \alpha$ forbidden states, and could be used for the investigation of $3\alpha$ continuum states.

4. Conclusion
Cluster models have a long history. They have a wide range of applications in nuclear physics, from the spectroscopy of light nuclei to low-energy reactions. When dealing with exotic nuclei such as halo or molecular states, other models cannot compete with cluster approaches until now. The shell-model, for example, has tremendous problems to describe cluster states, or to reproduce the long-range behaviour of halo states. A strong effort is being done now to develop ab-initio models which, in principle, do not make any assumption on the structure of the wave functions. However, the difficulties associated with this approach makes it applicable essentially to the spectroscopy of light nuclei. Most likely cluster models will be still widely applied in the future.

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