An upper limit on fermion mass spectrum in non-Hermitian models and its implications for studying of dark matter

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Abstract

The paper formulates a principal positions of non-Hermitian models with $\gamma_5$-mass extensions, which often be ignored in some investigations for this subject. In fact in this case Hamiltonians contain not only Hermitian masses $m_1$, but also contribution from anti-Hermitian components of fermion masses $m_2$. Main misunderstanding a number of papers is consist in using of this model for any values of fermion masses for fixed values of $m_1$ and $m_2$. However the basis appearing of two parameters masses may be undertaken a simple estimation for determination maximal permissible value of fermion mass $M = m_1^2 / 2m_2$ which may be used for this model. Easy to see that $M$ becomes infinite and hence experimentally doesn’t observable only in Dirac’s limit, when the non-Hermitian mass fully is disappearing. In particular the equality $m_1 = m_2$ can be realized only in two cases when $m_1 = m_2 = 0$ and $m_1 = m_2 = 2M$. Moreover in the second case the question is about the possibility existence a number of new fermions masses which are equal to the masses of particles Standard Model(SM) but when they have the non-Hermitian characteristics. In this case the paradox of the "two masses" takes place and its solution may be done only in suggested model with a maximal mass. Appearing particles can be considered as some new particles arising beyond SM. The unusual properties of these particles allow also to consider their as possible candidates in structure of dark matter.

1 Introduction

As it is now known, the defining feature of elementary particles is that their properties and interactions may be characterized properly in terms of local fields. The following question may be posed in the same terms: should the mass of elementary particles be limited from the above? To put this
another way, what is the maximal mass of a particle that still has the local field concept applicable to it? No experiments focused on finding particles with the maximal mass have been carried out yet. It is known only that a top quark is the heaviest particle in the Standard Model (SM). Naturally, the scope of experimental search for “maximons” is limited by the feasibility to construct very high-energy accelerators. However, a detailed study of models with a maximal mass may reveal new and unique opportunities for detecting the effects stemming from the mentioned limitation. This refers to various external influences that, when taken into account, may reveal a number of effects induced by the limited mass spectrum of elementary particles. For example, if the interaction with intense magnetic fields is considered, several effects may become observable. The emergence of the so-called exotic particles, which were predicted by V.G. Kadyshhevsky in the geometric approach [1], is one of the possible effects of finiteness of the mass spectrum.

It should be noted that the idea of finiteness of the mass spectrum of elementary particles has been proposed first in 1965 by M.A. Markov [2]. This finiteness was associated with Planck mass $M_{\text{Planck}} = \sqrt{\hbar c/G} \sim 10^{19}$ GeV, where $G$ is the gravitational constant, $\hbar$ is the Planck constant, and the $c$ is speed of light, and was written as [1]

$$m \leq M_{\text{Planck}}.$$  \hspace{1cm} (1)

The particles with maximal mass, which were called by the author maximons, hold a special place among elementary particles. In particular the concept of maximons formed the basis of the Markov scenario of the early Universe [3]. However, condition (1) was initially a purely phenomenological and was not taken into account in the development of the theory.

A new radical approach involving the actual introduction of the finiteness condition into the theory was proposed by Kadyshhevsky at the end of the 1970s [2]. Markov’s concept of the maximal particle mass was regarded in this approach as a new fundamental physical principle of quantum field theory (QFT). The condition of finiteness mass spectrum was postulated in the proposed theory in the form:

$$m \leq M,$$  \hspace{1cm} (2)

where maximal mass parameter (fundamental mass) was considered as a new physical constant. This quantity was regarded as the curvature radius of a five-dimensional hyperboloid when its surface is a realization of the curved
momentum 4-space. Using an anti-de Sitter space for this purpose, one easily
obtains the following relation:

\[ p_0^2 - p_1^2 - p_2^2 - p_3^2 + p_5^2 = M^2, \]  

from which it follows that at the mass surface for a free particle \( p_0^2 - \vec{p}^2 = m^2 \) inequality (2) is realized automatically.

It is important that in the geometric model containing restrictions of fermion masses, may be observed some new particles, which were named by the "exotic fermions". At once there was suggested that arising of new fermions exclusively be attributed to the development of geometric representations. In particular this fact was explained by the appearance of additional degree of freedom which has been appearing thanks to presence of the fifth component of fermion momentum \( \varepsilon = p_5/|p_5| = \pm 1 \) in the anti-de Sitter space \([3]\).

However at present it is absolute clear that appearance the exotic fermions it is not prerogative of the geometric approach. In particular, in purely algebraic model, which contains both Hermitian \( (m_1) \) and anti-Hermitian massive components of fermions \( (m_2) \) when are provided that

\[ m^2 = m_1^2 - m_2^2, \]  

restriction of fermions mass and presence of exotic particles also exist \([3]\). This is becoming obvious on the basis of theorem about relation between arithmetic and geometric averages values for two positive numbers. If we take into account this inequality which in expanded form looks like as \( m_2^2 + m^2 \geq 2\sqrt{m_2^2 \times m^2} \) then from it follows restriction \( m \leq M = (m_1)^2/2m_2 \) and if to consider (4) we can obtain that "exotic fermions" really here also are appearing \([4]\).

The properties of exotic particles differ radically from the characteristics of their well-known partners. Besides it turned out that the geometric approach is not single prerequisite to the emergence of such particles in the theory. Indeed, the development of pseudo-Hermitian \( PT \)-symmetric quantum theory has shown that these particles emerge as a consequence by itself of finiteness of the mass spectrum of elementary particles. Thus, the experimental search for exotic particles may result in the discovery of the maximal mass values itself. In particular this approach becomes feasible owing to presence the calculation of the spectrum of energies of a neutral fermion with an anomalous magnetic moment (AMM) in theory with a maximal mass \([5, 6]\).
In our approach developed in [5, 6] is used procedure which differs considerably from the methods used earlier. In particular, we obtained that the heaviest fermions (maximons with mass $M$) may be a crucial component of dark matter and that they have to consist of pseudo-fermion components. They also should possess a modified nature of interactions. Therefore, both theoretical and experimental study of pseudo-Hermitian characteristics of massive fermions assumes a particular importance. In a result the development of guidelines the determination of constraints on the parameters of maximons one can’t improbable that new physical phenomena may be detected at an energy of several TeV. It should be noted that this is the centerpiece of the research program for the Large Hadron Collider (LHC) at CERN [7, 8]. It is already becoming widely understood that some of the basic principles of the SM require further refinement. Specifically, it concerns the description of the spectrum of fermion masses, which is not yet unlimited in the SM. However, a very wide range of masses of known elementary particles is found already in the Standard Model itself. For example, the mass of a top quark (the heaviest known elementary particle) is approximately 300000 times greater than that of an electron. All this suggests that late or early we simply will be forced to take requirements of restrictions of spectrum mass because it will become necessary.

Thus, the issue about neutral maximons with AMM is of considerable interest in the context of their probable inclusion into the structure of dark matter. Astrophysical studies may prove key factor here, since the clarification of unusual pseudo-Hermitian characteristics of the considered fermions is of paramount importance in such studies. It should be noted that several research groups throughout the world are already searching for marks of dark matter in cosmic rays. The most ambitious project of this kind is known as IceCube [9]. More than 5000 high-sensitivity IceCube sensors, which are installed within the Antarctic ice sheet at the Geographic South Pole, collect data on galactic fluxes of various particles. These detectors are deployed at depths ranging from 1500 to 2500 m. The sensors thus cover a cubic kilometer of ice, and the researchers hope that this galactic-ray experiment may help examine deep-space sources.

These studies will surely be instrumental in solving certain important problems related to both low-energy neutrinos and high-energy particles. Specifically, such experiments are concerned with the examination of the internal structure of supernovae, the processes of formation of black holes and gamma ray bursts. The current theory implies that interactions in these
objects are guarantee accelerate particles to exceptionally high energies. According to several astrophysical estimates, these energies may reach $10^6 - 10^9$ GeV [9]. It is likely that the masses of new particles, which could be regarded as dark matter candidates, also fall within this range. Thus, astrophysical estimates are way beyond the limit of masses probed at LHC, where a targeted search for heavy exotic particles is being also carried out [8].

It is intuitively clear that studies which let covering the widest possible energy range should yield the most interesting results in this field and, that is no less important, could finally shed light on the mysteries of dark matter. It should be noted in this context that exotic particles emerging in theories with a maximal mass is noted by individual "code" (a set of pseudo-Hermitian characteristics), which differentiates them from common particles found in the SM. According to our estimates, the maximal mass may be bounded by $M = 2 \times 10^{14} GeV$ [10]; at this level, the magnetic moment of exotic neutrinos were of the order of $10^{-19} \mu_0$, where $\mu_0$ is the Bohr magneton, and the magnetic field was evaluated at 8000 G. However, if one assumes that the magnetic moment of a neutrino may be somewhat larger (see, for example, the GEMMA collaboration data [11]), the limit on the maximal mass may enter the domain of astrophysical observations ($M \sim 10^8$ GeV; see [9]).

In addition, one should consider the fact that giant magnetic fields produced by pulsars and magnetars may also be present in experiments on neutrino astrophysics and cosmic rays. This, the formulas for analytical and numerical evaluation of charactersitics of massive fermions propagating in magnetic fields with an intensity of $\sim 10^{15} - 10^{16}$ G [12] should be applicable to these enormous (relative to the scales familiar to us) values. Note that the exact solutions for the energy of neutral pseudo-fermions with AMM propagating in magnetic fields [5, 6] may help cover the range of ultrahigh energies (up to $\sim 10^9$ GeV) and intense magnetic fields (up to $\sim 10^{16}$ G).

Thus, further development of the theory established by Kadyshshevsky may provide specific guidelines for future experiments focused on the search for exotic fermions. Specifically, laboratory experiments with low-energy polarized neutrinos propagating in a magnetic field may be the least time-consuming and laborious. Arguably, precision experiments on the measurement of mass of neutrinos (e.g., the Troitsk tritium experiment), where weakly excited neutrino fluxes propagate in control and fairly intense magnetic fields, fit the requirements in this case.

It was already noted that a limited mass spectrum might be obtained not only in the geometric approach, but also in non-Hermitian (pseudo-
Hermitian) fermion systems, which have a direct application in neutrino physics. Systems of this kind are called - $PT$-symmetric models and are used in various areas of modern physics. Specifically, theoretical and experimental studies in non-Hermitian optics started more than ten years ago.

2 Modified Dirac model for non-Hermitian mass parameters

Let us now consider the modified Dirac equations for free massive particles using the $\gamma_5$-factorization of the ordinary Klein-Gordon operator. In this case we will make similar actions as for known Dirac procedure. As he himself wrote: "...get something like a square root from the equation Klein-Gordon" [12], [13]. And really if we shall not be restricted to only Hermitian operators then we can represent the Klein-Gordon operator in the form of a product of two commuting matrix operators with $\gamma_5$-mass extension:

$$\left(\partial_\mu^2 + m^2\right) = \left(i\partial_\mu \gamma^\mu - m_1 - \gamma_5 m_2\right)\left(-i\partial_\mu \gamma^\mu - m_1 + \gamma_5 m_2\right), \quad (5)$$

where the physical mass of particles $m$ is expressed through the parameters $m_1$ and $m_2$

$$m^2 = m_1^2 - m_2^2. \quad (6)$$

For the function would obey to the equations of Klein-Gordon

$$\left(\partial_\mu^2 + m^2\right)\tilde{\psi}(x,t) = 0 \quad (7)$$

one can demand that it also satisfies to one of equations of the first order

$$\left(i\partial_\mu \gamma^\mu - m_1 - \gamma_5 m_2\right)\tilde{\psi}(x,t) = 0; \quad \left(-i\partial_\mu \gamma^\mu - m_1 + \gamma_5 m_2\right)\tilde{\psi}(x,t) = 0 \quad (8)$$

Equations (8) of course, are less common than (7), and although every solution of one of the equations (8) satisfies to (7), reverse approval has not designated. It is also obvious that the Hamiltonians, associated with the equations (8), are non-Hermitian, because in them the $\gamma_5$-dependent mass components appear ($H \neq H^+$):

$$H = -\alpha \hat{p} + \beta (m_1 + \gamma_5 m_2) \quad (9)$$
and

\[ H^+ = \mathbf{\alpha} \mathbf{p} + \beta (m_1 - \gamma_5 m_2). \]  

(10)

Here matrices \( \alpha_i = \gamma_0 \cdot \gamma_i, \beta = \gamma_0, \gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \). It is easy to see from (6) that the mass \( m \), appearing in the equation (7) is real, when the inequality

\[ m_1^2 \geq m_2^2. \]  

is accomplished, but this area contains descriptions not only ordinary particles, but also the the exotic particles which do not subordinate to the ordinary Dirac equation.

In this section, we will also want touch upon question of describing the motion of Dirac particles, if their own magnetic moment is different from the Bohr magneton. As it was shown by Schwinger [14] the equation of Dirac particles in the external electromagnetic field \( A^{ext} \) taking into account the radiative corrections may be represented in the form:

\[(\mathcal{P}\gamma - m) \Psi(x) - \int \mathcal{M}(x, y | A^{ext}) \Psi(y) dy = 0,\]  

(12)

where \( \mathcal{M}(x, y | A^{ext}) \) is the mass operator of the fermion in the external field and \( \mathcal{P}_\mu = p_\mu - A^{ext}_\mu \). From equation (12) by means of expansion of the mass operator in a series of according to \( eA^{ext} \) with precision not over then linear field terms one can obtain the modified equation. This equation preserves the relativistic covariance and consistent with the phenomenological equation of Pauli obtained in his early papers (see for example [15]).

3 Non-relativistic limit modified Dirac equation in the electromagnetic field with \( \gamma_5 \)-mass extension.

In a most cases no necessity in exact solutions of modified Dirac equation with non-Hermitian \( \gamma_5 \)-mass extensions. Really one can confine by non-relativistic amendments using expansions \( \sim v/c \) and \( \sim v^2/c^2 \). Below from exact relativistic equation in the electromagnetic field \( A^\mu \) we obtain this decomposition. Indeed, changing the \( \Psi \) function by the following way:

\[ \Psi(r, t) = \Psi(r) e^{-i(E+m)t} \]  

(13)
we obtain:

\[(E + m)\Psi(r) = [\overrightarrow{\alpha}(\overrightarrow{p} - e\overrightarrow{A}) + \beta(m_1 + \gamma_5m_2) - V]\Psi(r), \quad (14)\]

where \(E\) energy of fermion in which non included energy of rest, \(V\)-potential energy. Consider representations of the four-components function \(\Psi\) in a form \(\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}\), where \(\varphi\) and \(\chi\) in turn, two-component functions and taking into account, that in standard representation

\[\alpha\Psi = \begin{pmatrix} \sigma \chi \\ \sigma \varphi \end{pmatrix}, \quad \beta\Psi = \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}, \quad \beta\gamma_5\Psi = \begin{pmatrix} -\chi \\ \varphi \end{pmatrix},\]

we can write:

\[(E + m - m_1 + V)\varphi = [\overrightarrow{\sigma}(\overrightarrow{p} - e\overrightarrow{A}) - m_2]\chi; \quad (15)\]

\[(E + m + m_1 + V)\chi = [\overrightarrow{\sigma}(\overrightarrow{p} - e\overrightarrow{A}) + m_2]\varphi, \quad (16)\]

where \(\overrightarrow{\sigma}\) are matrix Pauli. Expressing \(\chi\) from \((16)\) we can obtain

\[\chi = \frac{\overrightarrow{\sigma}(\overrightarrow{p} - e\overrightarrow{A}) + m_2}{m + m_1} \left(1 + \frac{E + V}{m + m_1}\right)^{-1} \varphi. \quad (17)\]

Taking into account that in non relativistic limit

\[E + V \ll m + m_1,\]

with accuracy up to quadratic terms on velocity of fermion \(\sim v^2/c^2\) we have

\[\chi = \frac{\overrightarrow{\sigma}(\overrightarrow{p} - e\overrightarrow{A}) + m_2}{m + m_1} \varphi. \quad (18)\]

Using the identity

\[(\overrightarrow{\sigma}\overrightarrow{a})(\overrightarrow{\sigma}\overrightarrow{b}) = \hat{a} + i\overrightarrow{\sigma} [\hat{a} \times \hat{b}],\]

considering here

\[\hat{a} = \overrightarrow{b} = (\overrightarrow{p} - e\overrightarrow{A})^2,\]

in a result we obtain

\[\frac{[\overrightarrow{\sigma}(\overrightarrow{p} - e\overrightarrow{A})]^2}{(\overrightarrow{p} - e\overrightarrow{A})^2} = e\overrightarrow{\sigma}\overrightarrow{H}.\]
Thus in non-relativistic limit for two-components wave function we can write

\[ E\varphi(r) = i\frac{\partial}{\partial t}[\varphi(r)\exp(-iEt)] = \hat{H}\varphi(r) = \left[ \frac{\left(\hat{p} - e\vec{A}\right)^2}{m + m_1} - \frac{e\sigma\hat{H}}{m + m_1} + V \right] \varphi(r). \tag{19} \]

Thus, we see that in the obtained equations the role mass plays parameter

\[ m^* = \frac{(m + m_1)}{2} \geq m. \tag{20} \]

This value essentially different from the relativistic mass \( m \) defined in (6). We emphasize that this distinction has nothing to do with the value of the particles velocity.

First, contrary to the claims of [16, 19], not always Hermitian and pseudo-Hermitian components of the mass are contained in universal mass extension (6) needs to correspond relativistic model (8).

Second, pseudo-Hermitian extensions of the usual Dirac equation itself to another format can contain not only to some small differences but sometimes and very significant values of relativistic and non-relativistic masses. Here we can see that masses \( m \) and \( m^* \) is equal only in the limit of the usual Dirac’s equations \( m_2 \rightarrow 0 \) and \( m_1 \rightarrow m \).

4 An upper limit on fermion mass spectrum in non-Hermitian models

As already mentioned, according to elementary physical considerations the parameters \( m, m_1 \) and \( m_2 \) can not be completely arbitrary in this non-Hermitian model. In other words, the modified Dirac equation can describe real fermions only in a limited area of change \( m, m_1 \) and \( m_2 \). First, the positive definiteness of the observed relativistic mass requires obvious limit \( m_2 \leq m_1 \). Secondly, the mass of the fermion should also be limited the condition \( m \leq m_1 \) and finally from conditions (6), the masses \( m_1, m_2 \) and \( m \) can be directly linked with a right triangle.

At first sight seems that these boundaries are completely sufficient that to satisfy of any questions? However nobody with help of this constraints can’t answer on the simple question: can under fixed parameters \( m_1 \) and \( m_2 \) whether exist maximal value mass of fermions, which may be considered in this models?
In fact, the expression (6), which is obtained from generalization of the notion of Hermiticity and $\gamma_5$-factorization of the Klein-Gordon’s operator is accomplished automatically for any from right triangles constructed on this masses. However also it is obviously that areas of these triangles may be found are not equal to each other. Thus one may find the triangle with the maximal area. Maximal area can allow to define and the maximal mass $M$ which we must considered as maximal possible limiting value of fermion mass in this model. We also see, that non-Hermitian $\gamma_5$ mass extension in Dirac equation may be basis to develop the models with constraints spectrum mass of fermions.

Really this limitation is not arise being so obvious, but bears a fundamental limitations in the model. If continue geometric interpretation of the considered connection of masses, we can set another important ratio. We are talking about comparing different of triangles according to their square. In particular, if one consider an area of the two neighboring triangles forming rectangle $S_1 = m \ast m_2$ its total area may not exceed the area of half a square, built on the hypotenuse is $S_2 = m_1^2$. For visualization consider the function(see Fig.1)

![Figure 1: Comparison of squares different figures constructed on using masses $m, m_1$ and $m_2$](image-url)
However, in addition it is arise another limitation, which is not being so obvious, but bears a fundamental limitations in the model. Indeed, if continue geometric interpretation of the considered connection of masses, we can set another important ratio. We are talking about comparing different of triangles according to their square. In particular, if one consider an area of the two neighboring triangles forming rectangle $S_1 = m_1 m_2$ its total area may not exceed the area of half a square, built on the hypotenuse is $S_2 = m_1^2$. For visualization consider the function(see Fig.2)

$$F(m_1, m_2) = -\frac{m_1^2}{2} + m_1 m_2 \leq 0$$

From Fig.2 we can easily see that the equality sign observed in the point $m = m_1/\sqrt{2}$. Thus, this single point, which is establishes the equality of the areas of two squares. At the same time it means in the model there is formed the value of the maximum possible mass of the fermion. Denoting the maximal mass of the fermion in the form $m_{max} = M = m_1^2/2m_2$, we now can express the mass parameters $m_1$ and $m_2$ as functions $M$ and $m$ (see [11]).

Of course the question arises - to what extent the above formulation the definition of maximal mass is the unambiguous definition? It is clear that the
presence in the non-Hermitian theory of two components of units of mass $m_1$ and $m_2$ gives our in principle the ability to build different variants expressions containing $m_1$ and $m_2$. However, it is obvious that making physically reasonable choices here is in substance only one. This is result which contains non-Hermitian mass $m_2$ in the denominator and itself expressions the desired type have the form $M = m_1^2/2m_2$. Thus, we come the expression is of the form $M = cm_1^2/m_2$ with an arbitrary constant $c$. The value $c = 1/2$ gives also the above-mentioned Cauchy’s theorem on the average.

5 The paradox of the "two masses" and its solution in suggested model. Exotic particles

Let us now discuss the paradox which arises between two masses in this model. It is important that relativistic and non-relativistic masses have absolutely different properties. It is inevitable when anti-Hermitian mass $m_2 \neq 0$. Using the relations (11), (20) and the expression for the maximal mass, we can express all the components of using relation of relativistic and maximal masses. In turn, we write it in the form $m = \nu M$. After this one may easy to see that for non-relativistic mass there are two options

$$\nu^* = \frac{m^*}{M} = \frac{\nu}{2} + \sqrt{1 \mp (1 - \nu^2)^{1/2}}$$  \hspace{1cm} (21)

It is clear (see expressions) that the upper sign here corresponds to ordinary particles and lower sign corresponds to its exotic partners. Appropriate the graphs shown in Fig. 3. We see that for normal particles the difference between relativistic and non-relativistic mass is negligible. Under such conditions, the normal particle equation (21) gives

$$\nu^* \approx \nu + \frac{\nu^3}{16} + \frac{7\nu^5}{256} + O(\nu^7)$$  \hspace{1cm} (22)

This is the key point of non-Hermitian theory. In our approach when there is the limit of the parameters $m$, $m_1$ and $m_2$, this allows to us to solve the paradox of two masses, when we have on one hand the value of relativistic mass $m$ and non-relativistic mass $m^*$ from the other. The existing of value maximal mass in a natural way allows to find the right results. Although in the absence of this limit is the difference between the relativistic and
non-relativistic masses of ordinary particles can to be arbitrarily large. It is
definitely a serious problem for modified theory of Dirac. In our opinion, to
find any other alternative explanation for this paradox besides restriction of
mass highly problematic.

Besides, for the particles which corresponding to the upper sign in (21) we
have in our model also fairly unusual properties. Their non-relativistic mass
comparable to the maximal masses. Such particles are extremely difficult to
get to move with acceleration, even in very strong electromagnetic fields and
therefore they practically do not create electromagnetic waves. At the same
time, their "gravitational charge" is determined by their relativistic mass
and hence comparable with masses of \( m \) ordinary particles. Obviously, this
masses determines the intensity of birth and annihilation of particles. Thus,
the described particles practically do not interact with electromagnetic waves,
however are born and are involved in gravitational interactions the level of
the ordinary particles. All this, in our opinion, makes them the obvious
candidates to the structure of dark matter particles.

Figure 3: The character of variation of the relativistic mass (straight line)
and non-relativistic mass for ordinary particles (dash-dot line) and exotic
particles (dash line). For the case of observed particles with small masses
curves corresponding to non-relativistic and relativistic masses are practically
identical.
6 The chiral representation for the modified Dirac equation. The mass spectrum of the left and right particles. Is it possible to exist "massless" neutrinos?

The original Dirac equation can be modified and in a slightly different way. To this end, in his equations \( \text{(5)} \) and \( \text{(8)} \) is enough to make a linear transformation

\[
m_{1(2)} = m_r \pm m_l
\]

Due to the fact that

\[
P_{r(l)} = (I \pm \gamma_5)/2
\]

are the projective operators (\( I \) is the identity matrix), the Dirac equation naturally splits into a system of two equations. If you imagine Dirac's bispinor, like section 3, in the form of a column of two-component spinors \( \Psi = \left( \begin{array}{c} \phi \\ \chi \end{array} \right) \) and use the standard representation for \( \gamma \)-matrices, will receive

\[
i \frac{\partial \phi}{\partial t} = i \vec{\sigma} \nabla \chi + (m_r + m_l)\phi - (m_r - m_l)\chi;
\]

\[
i \frac{\partial \chi}{\partial t} = i \vec{\sigma} \nabla \phi + (m_r - m_l)\phi - (m_r + m_l)\chi.
\]

Acting as projective operators on the initial equation written in the form

\[
i \frac{\partial \psi}{\partial t} = \left[ \alpha \vec{p} + 2m_l \beta P_r + 2m_r \beta P_l \right] \psi
\]

or folding and subtracting the resulting equations and introducing the notation \( \xi(\eta) = \phi \pm \chi \), come to the system

\[
i \frac{\partial \xi}{\partial t} - i \vec{\sigma} \nabla \xi = 2m_r \eta;
\]

\[
i \frac{\partial \eta}{\partial t} + i \vec{\sigma} \nabla \eta = 2m_l \xi.
\]

The resulting equations differ from the equations Weyl by only presence of non zero value the "left" and "right" of the masses. Naturally, the temptation
arises, using the arbitrariness of the initial parameters $m_1$ and $m_2$, to describe thus a massless Weyl particles. This is way was used by the authors of the paper [20]. In particular, one argue that when $m_1 = \pm m_2$ the system similar to (28-29) describes a massless right-wing or left neutrinos. In a sense it is the limit of the original equations of motion.

Indeed in this approach we are faced with quite a difficult situation. For example, let $m_1 = m_2$. Then becomes zero not only "left mass" $m_l = m_1 - m_2$, and also relativistic mass $m = 2\sqrt{m_l \cdot m_r}$, and the equation (29) really takes the form of a conventional two-component Weyl equation. However, the non-relativistic mass $m^*$ thus, generally speaking, in zero is not drawn, because mass $m_1$ can have a nonzero value. Apparently, to search physical meaning in such a situation under any values of masses $m_1$ and $m_2$ is quite difficult. And really relativistic mass becomes zero only if $m_1 = m_2 = 0$, which corresponds to the well-known case of pure Dirac neutrinos. Note also that case $m_1 = m_2$ maybe realized and for exotic particles when $m_1 = m_2 = 2M$.

![Figure 4: The character of variation of the relativistic mass (straight line), left-mass (dash-dot line) and the right-mass (dash line) for the case of ordinary particles. For light particles, the left and right masses practically coincide, which corresponds to the usual Dirac equations.](image-url)
Figure 5: The character of variation of the relativistic mass (straight line), left-mass (dash-dot line) and the right-mass (dash line) for the case of exotic particles. It is seen that right-mass have a huge values $m_r > M$, and the left-masses of particles is very small, much smaller than the mass of the known leptons.

In our model with the maximal mass appearing of such a paradox in impossible in principle. In Fig.4 shows the change of the "starboard mass" and the "left mass" in the same scale as in Fig.3. Appropriate expressions are obtained from the same formulas (4), (20) and the parameter $m = \nu M$, the same as in the previous section:

\[
\nu_l = \frac{m_l}{M} = \mu(1 - \mu), \tag{30}
\]
\[
\nu_r = \frac{m_r}{M} = \mu(1 + \mu), \tag{31}
\]

where

\[
\mu = \sqrt{\frac{1 \mp (1 - \nu^2)^{1/2}}{2}}. \tag{32}
\]

The upper signs correspond to ordinary, and the lower - for the exotic particles. For observed particles with $\nu \ll 1$, it is easy to see that the left and right of the mass are practically identical.
\[ \nu_{l(r)} \approx \nu/2 \mp \nu^2/4, \quad (33) \]

ie they correspond with great accuracy to the usual Dirac equations. On the contrary, for exotic particles with the same values of \( \nu \ll 1 \) the difference between the left and right masses is huge:

\[ \nu_l \approx \nu^2/8 \to 0; \nu_r \approx 2 - 3\nu^2/8. \quad (34) \]

If, as we assume, the exotic particles are relating to dark matter, its "background" obvious should associate with left polarized. However, far-reaching conclusions to do while early.

In conclusion, we note the following. The chiral representation gives possibility to have a somewhat another understanding the meaning of extensions of the Dirac equation. In fact, the introduction to it additional term \( \sim m_2 \gamma_5 \), is equivalent to the introduction of the difference between the right and the left mass of the particle. But this difference is absolutely natural element of the Standard Model (e.g. the electro-weak interaction). The only difference is that it is now refers to "more compact" spinor multiplet. The restriction in the spectrum mass of fermions associated with a fundamental mass in a natural way makes this difference is practically unobservable, or giant.

7 Conclusions and confinement

According to the results of the study of the modified Dirac equation can be done the following conclusions. Introduction pseudo-Hermitian contribution in this equation is equivalent to the introduction of the difference between right and left masses of the particles. Its inevitable consequence is the paradox of differences between relativistic and non-relativistic mass, which are different combinations Hermitian and non-Hermitian components of the mass. If someone want to ignore any restrictions for mass parameters and nevertheless are going to continue investigation of non-Hermitian \( \gamma_5 \)-mass extension for fermion systems must be obtained much more complicated physical situation may arise. Particularly unpleasant situation in which the relativistic mass of ordinary particles becomes zero and non-relativistic component of mass stays is not zero.

In essence, this means a departure from the principle of equivalence, which is inevitable in such kind of theories. This is the root cause of the difficulties
described. The same time, the limitations of the mass spectrum by the condition \( M = m_1^2/2m_2 \), associated with the introduction of a fundamental mass, naturally this paradox is allowed.

The ratio expressing the observed mass through the fundamental particles, is possible in two variants. In the first difference of the relativistic and non-relativistic mass (as the difference between the left and right of the masses) to the lungs (compared to maximal mass \( M \)) of the particles is practically unobservable, so that the ordinary Dirac equation is performed almost exactly. It is clear that this variant describes the usual particles.

Moreover, it becomes possible to estimate the value of the fundamental mass using this difference. The equivalence of the heavy and inert mass for ordinary matter was verified experimentally. According to the last known data [21], [22], the difference between the gravitational and inertial masses does not exceed the value \( \sim 1.4 \times 10^{-13} \). Associating this difference with the second term on the right-hand side (22), we obtain \( \nu \sim 1.308 \times 10^{-4} \). Assuming a proton with a mass of 0.94 GeV as the structural unit of the ordinary substance, we obtain a rough estimate for the fundamental mass \( M \sim 7.2 TeV \). Note that the energy of this level is attainable at the Large Hadron Collider. An experimental confirmation of our theory for ordinary matter would be the detection of the difference in particle masses measured from the threshold of its production, on the one hand, and its motion in the electromagnetic field, on the other.

In the second embodiment, on the contrary, the difference between relativistic and non-relativistic, and the left and right of the masses is huge and comparable to the fundamental mass. Such particles have non-relativistic the mass of the mass of maximal, but their relativistic mass is comparable with the mass of ordinary particles. Therefore, they do not participate in electromagnetic interactions, but gravity show themselves as ordinary particles. Thus, the introduction of fundamental mass as by-product gives clear candidates for the particles dark matter.

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\(^1\)Moreover, this condition makes the considered algebraic 4D-theory with a pseudo-Hermitian extension dual geometric 5D-theory of Kadyshevsky’s.
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