On $p$-Adic Sector of Adelic String

Branko Dragovich*

Institute of Physics
Pregrevica 118, P.O. Box 57, 11001 Belgrade, Serbia

Abstract

We consider construction of Lagrangians which are candidates for $p$-adic sector of an adelic open scalar string. Such Lagrangians have their origin in Lagrangian for a single $p$-adic string and contain the Riemann zeta function with the d’Alembertian in its argument. In particular, we present a new Lagrangian obtained by an additive approach which takes into account all $p$-adic Lagrangians. The very attractive feature of this new Lagrangian is that it is an analytic function of the d’Alembertian. Investigation of the field theory with Riemann zeta function is interesting in itself as well.

1 Introduction

The first notion of a $p$-adic string was introduced by I. V. Volovich in 1987 [1]. After that, various versions of $p$-adic strings were developed. The most interest have attracted strings whose only world sheet is $p$-adic and all other properties are described by real or complex numbers. Such $p$-adic strings are connected with ordinary ones by product of their scattering amplitudes and notion of an adelic string has been considered. Adelic string enables to treat ordinary and $p$-adic strings simultaneously and on an equal footing.

* e-mail address: dragovich@phy.bg.ac.yu
Adelic strings can be regarded as more fundamental than ordinary and $p$-adic ones (for a review of the early days developments, see e.g. [2, 3]). Some $p$-adic structures have been also observed in many other parts of modern mathematical physics (for a recent review we refer to [4]).

One of the greatest achievements in $p$-adic string theory is an effective field description of open scalar $p$-adic string tachyon [5, 6]. The corresponding Lagrangian is nonlocal, nonlinear, simple and exact. It describes four-point scattering amplitudes as well as all higher ones at the tree-level.

In the last decade the Lagrangian approach to $p$-adic string theory has been significantly advanced and many aspects of $p$-adic string dynamics have been investigated, compared with dynamics of ordinary strings and applied to nonlocal cosmology (see, e.g. [7, 8, 9, 10, 11] and references therein).

Adelic approach to the string scattering amplitudes connects $p$-adic and ordinary counterparts, eliminates unwanted prime number parameter $p$ contained in $p$-adic amplitudes and cures the problem of $p$-adic causality violation. Adelic quantum mechanics [12] was also successfully formulated, and it was found a connection between adelic vacuum state of the harmonic oscillator and the Riemann zeta function. There is also successful application of adelic analysis to Feynman path integral [13], quantum cosmology [14], summation of divergent series [15], and dynamical systems [16].

The present paper is a result of investigation towards construction of an effective field theory Lagrangian for $p$-adic sector of adelic open scalar string. At the beginning, we give a brief review of Lagrangian for $p$-adic string and also of our previous work on this subject. Then, we present a new Lagrangian, which also contains Riemann zeta function, but in such way that Lagrangian is now an analytic function of the d’Alembertian $\Box$. Note that $p$-adic sector of the four point adelic string amplitude contains the Riemann zeta function.

### 2 $p$-Adic and Adelic Strings

Let us recall the crossing symmetric Veneziano amplitude for scattering of two ordinary open strings:

\[
A_\infty(a, b) = g_\infty^2 \int_\mathbb{R} |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x = g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)},
\]

where $a = -\alpha(s) = -\frac{s}{2} - 1$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $a + b + c = 1$, i.e. $s + t + u = -8$. In $\mathbb{R}$, $| \cdot |_\infty$ denotes the ordinary absolute
value, \( \mathbb{R} \) is the field of real numbers, kinematic variables \( a, b, c \in \mathbb{C} \), and \( \zeta \) is the Riemann zeta function. The corresponding Veneziano amplitude for scattering of \( p \)-adic strings was introduced as \( p \)-adic analog of the integral in (1), i.e.

\[
A_p(a, b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} |1 - x|_p^{b-1} d_p x,
\]

where \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers, \( | \cdot |_p \) is \( p \)-adic absolute value and \( d_p x \) is the additive Haar measure on \( \mathbb{Q}_p \). In (2), kinematic variables \( a, b, c \) maintain their complex values with condition \( a + b + c = 1 \). After integration in (2) one obtains

\[
A_p(a, b) = g_p^2 \frac{1 - p^{-a-1}}{1 - p^{-a}} \frac{1 - p^{-b-1}}{1 - p^{-b}} \frac{1 - p^{-c-1}}{1 - p^{-c}},
\]

where \( p \) is any prime number. Recall the definition of the Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i \tau, \quad \sigma > 1,
\]

which has analytic continuation to the entire complex \( s \) plane, excluding the point \( s = 1 \), where it has a simple pole with residue 1. According to (4) one can take product of \( p \)-adic string amplitudes

\[
\prod_p A_p(a, b) = \frac{\zeta(a)}{\zeta(1 - a)} \frac{\zeta(b)}{\zeta(1 - b)} \frac{\zeta(c)}{\zeta(1 - c)} \prod_p g_p^2,
\]

what gives a nice simple formula

\[
A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2.
\]

To have infinite product of amplitudes (5) finite it must be finite product of coupling constants, i.e. \( g_\infty^2 \prod_p g_p^2 = const \). From (6) it follows that the ordinary Veneziano amplitude, which is rather complex, can be expressed as product of all inverse \( p \)-adic counterparts, which are much more simpler. Moreover, expression (6) gives rise to consider it as the amplitude for an adelic string, which is composed of the ordinary and \( p \)-adic ones.
2.1 Lagrangian for a $p$-Adic Open String

The exact tree-level Lagrangian of the effective scalar field $\varphi$, which describes the open $p$-adic string tachyon, is \cite{5, 6}

$$L_p = \frac{g_p^2}{D} \left[ -\frac{1}{2} \varphi p^{-\frac{\Box}{2m^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (7)$$

where $p$ is a prime, $\Box = -\partial_t^2 + \nabla^2$ is the $D$-dimensional d’Alembertian.

An infinite number of spacetime derivatives follows from the expansion

$$p^{-\frac{\Box}{2m^2}} = \exp \left( -\frac{1}{2m^2} \log p \Box \right) = \sum_{k=0}^{+\infty} \left( -\frac{\log p}{2m^2} \right)^k \frac{1}{k!} \Box^k.$$

The equation of motion for (7) is

$$p^{-\frac{\Box}{2m^2}} \varphi = \varphi^p, \quad (8)$$

and its properties have been studied by many authors (see, \cite{9} and references therein).

3 Lagrangians for $p$-Adic Sector

Now we want to consider construction of Lagrangians which are candidates to describe entire $p$-adic sector of an adelic open scalar string. In particular, an appropriate such Lagrangian should describe scattering amplitude (5), which contains the Riemann zeta function. Consequently, this Lagrangian has to contain the Riemann zeta function with the d’Alembertian in its argument. Thus we have to look for possible constructions of a Lagrangian which contains the Riemann zeta function and has its origin in $p$-adic Lagrangian (7).

We have found and considered two approaches: additive and multiplicative.

3.1 Additive approach

Prime number $p$ in (7) can be replaced by any natural number $n \geq 2$ and consequences also make sense.

Now we want to introduce a Lagrangian which incorporates all the above Lagrangians (7), with $p$ replaced by $n \in \mathbb{N}$. To this end, we take the sum of all Lagrangians $L_n$ in the form
whose explicit realization depends on particular choice of coefficients $C_n$ and coupling constants $g_n$. To avoid a divergence in $1/(n-1)$ when $n = 1$ one has to take that $C_n/g_n^2$ is proportional to $(n-1)/2$. Here we consider some cases when coefficients $C_n$ are proportional to $n-1$, while coupling constants $g_n$ do not depend on $n$, i.e. $g_n = g$. In fact, according to (6), in this case $g_n^2 = g^2 = 1$. Another possibility is that $C_n$ is not proportional to $n-1$, but $g_n^2 = n^2 - 1$ and then $\prod_p g_p^2 = \zeta(2) = \pi^2/6$, what is consistent with (6). To differ this new field from a particular $p$-adic one, we use notation $\phi$ instead of $\varphi$.

We have considered three cases for coefficients $C_n$ in (9): (i) $C_n = n^{-1} n^{2+h}$, where $h$ is a real parameter; (ii) $C_n = n^2 - 1/n^2$; and (iii) $C_n = \mu(n) n^{-1}$, where $\mu(n)$ is the M"obius function.

Case (i) was considered in [17, 18]. Obtained Lagrangian is

$$L_h = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left( \frac{\Box}{2m^2} + h \right) \phi + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \phi^{n+1} \right],$$

(10)

where $\mathcal{AC}$ denotes analytic continuation.

Case (ii) was investigated in [19] and the corresponding Lagrangian is

$$L = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \left( \zeta \left( \frac{\Box}{2m^2} - 1 \right) + \zeta \left( \frac{\Box}{2m^2} \right) \right) \phi + \frac{\phi^2}{1-\phi} \right].$$

(11)

Case with the M"obius function $\mu(n)$ is presented in [20] and the corresponding Lagrangian is

$$L = \frac{m^D}{g^2} \left[ -\frac{1}{2} \phi \frac{1}{\zeta \left( \frac{\Box}{2m^2} \right)} \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right],$$

(12)

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$.

### 3.2 Multiplicative approach

In the multiplicative approach the Riemann zeta function emerges through its product form (14). Our starting point is again $p$-adic Lagrangian (7). It is
useful to rewrite (7), first in the form,
\[ \mathcal{L}_p = \frac{m^D}{g_p^2} \frac{p^2}{p^2 - 1} \left\{ -\frac{1}{2} \varphi \left[ p^{\frac{-m}{2m^2} + 1} + p^{\frac{-m}{2m^2}} \right] \varphi + \varphi^{p+1} \right\} \]  
(13)
and then, by addition and substraction of \( \varphi^2 \), as
\[ \mathcal{L}_p = \frac{m^D}{g_p^2} \frac{p^2}{p^2 - 1} \left\{ \frac{1}{2} \varphi \left[ (1 - p^{\frac{-m}{2m^2} + 1}) + (1 - p^{\frac{-m}{2m^2}}) \right] \varphi - \varphi^2 \left( 1 - \varphi^{p-1} \right) \right\} . \]  
(14)

Taking products
\[ \prod_p g_p^2 = C, \prod_p \frac{1}{1 - p^{-2}}, \prod_p (1 - p^{-\frac{m}{2m^2} + 1}), \prod_p (1 - p^{-\frac{m}{2m^2}}), \prod_p (1 - \varphi^{p-1}) \]  
(15)
in (14) at the relevant places one obtains Lagrangian
\[ \mathcal{L} = \frac{m^D}{C} \zeta(2) \left\{ \frac{1}{2} \varphi \left[ \zeta^{-1} \left( \frac{\Box}{2m^2} + 1 \right) + \zeta^{-1} \left( \frac{\Box}{2m^2} \right) \right] \varphi - \varphi^2 \prod_p \left( 1 - \varphi^{p-1} \right) \right\} , \]  
(16)
where \( \zeta^{-1}(s) = 1/\zeta(s) \). It is worth noting that from Lagrangian (16) one can easily reproduce its \( p \)-adic ingredient (13). Lagrangian (16) was introduced and considered in [21]. In particular, it was shown that very similar Lagrangian can be obtained from the additive approach with the Möbius function and that these two Lagrangians describe the same field theory in the weak field approximation.

### 3.3 A new Lagrangian with Riemann zeta function

Here we present a new Lagrangian constructed by additive approach taking \( C_n = (-1)^{n-1} \frac{n^2 - 1}{n^2} \) in (9). This choice of coefficients \( C_n \) is similar to the above case (ii) and distinction is in the sign \((-1)^{n-1}\). The starting \( p \)-adic Lagrangian is in the form (13) and it gives
\[ L = \sum_{n=1}^{+\infty} C_n \frac{m^D}{g_n^2} \frac{n^2}{n^2 - 1} \left[ -\frac{1}{2} \phi n^{\frac{-m}{2m^2} + 1} \phi - \frac{1}{2} \phi n^{\frac{-m}{2m^2}} \phi + \phi^{n+1} \right] . \]  
(17)
Recall that
\[ \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad s = \sigma + i\tau, \quad \sigma > 0 , \]  
(18)
which has analytic continuation to the entire complex $s$ plane without singularities. At point $s = 1$, one has $\lim_{s \to 1} (1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} = \log 2$. Applying (18) to (17) and using analytic continuation one obtains

$$L = -m^D \left[ \frac{1}{2} \phi \left\{ \left(1 - 2^{1-\frac{1}{2m^2}}\right) \zeta\left(\frac{\Box}{2m^2} - 1\right) + \left(1 - 2^{1-\frac{m^2}{2m^2}}\right) \zeta\left(\frac{\Box}{2m^2}\right) \right\} \phi - \frac{\phi^2}{1 + \phi} \right],$$

(19)

where it was taken $g_n^2 = g^2 = 1$.

The corresponding equation of motion is

$$\left(1 - 2^{1-\frac{1}{2m^2}}\right) \zeta\left(\frac{\Box}{2m^2} - 1\right) + \left(1 - 2^{1-\frac{m^2}{2m^2}}\right) \zeta\left(\frac{\Box}{2m^2}\right) \phi = \frac{\phi^2 + 2\phi}{(1 + \phi)^2},$$

(20)

which in the week field approximation gives equation

$$\left(1 - 2^{1-\frac{M^2}{2m^2}}\right) \zeta\left(\frac{M^2}{2m^2} - 1\right) + \left(1 - 2^{1-\frac{m^2}{2m^2}}\right) \zeta\left(\frac{M^2}{2m^2}\right) \phi = \frac{\phi^2}{1 + \phi},$$

(21)

for the spectrum of masses $M^2$ as function of string mass $m^2$. Equation (20) has three $\phi = const.$ solutions, which are $\phi = 0, 1, -\frac{5}{3}$.

The potential can be obtained by equality $V(\phi) = -L(\Box = 0)$, i.e.

$$V(\phi) = m^D \frac{3\phi - 5}{8(1 + \phi)} \phi^2$$

(22)

which has two local minima at $\phi = 1$ and $\phi = -\frac{5}{3}$, and it has one local maximum $V(0) = 0$. These values of $\phi$ coincide with constant solutions of equation of motion (20). Potential (22) is singular at $\phi = -1$. Note that sign $(-1)^{p-1}$ in front of $\mathcal{L}_p$ in (17) is positive when $p$ is an odd prime and it has as a result that $V(\phi) \to +\infty$ when $\phi \to \pm \infty$.

### 4 Concluding remarks

The main result of this paper is construction of the Lagrangian (19). Unlike previously constructed Lagrangians, this one has no singularity with respect to the d’Alembertian $\Box$ and it enables to apply easier pseudodifferential approach. This analyticity of the Lagrangian should be also useful in its application to nonlocal cosmology, which uses linearization procedure (see, e.g. [22] and references therein).

It is worth mentioning that an interesting approach towards foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [23].
Acknowledgements

The work on this article was partially supported by the Ministry of Science and Technological Development, Serbia, under contract No 144032D. The author thanks organizers of the SFT’09 conference at the Steklov Mathematical Institute, Moscow, for a stimulating and useful scientific meeting.

References

[1] I. V. Volovich, “p-Adic space-time and string theory,” *Theor. Math. Phys.*, 71, 574–576 (1987); I. V. Volovich, “p-Adic string,” *Class. Quantum Grav.*, 4, L83–L87 (1987).

[2] L. Brekke and P. G. O. Freund, “p-Adic numbers in physics,” *Phys. Rep.*, 233, 1–66 (1993).

[3] V. S. Vladimirov, I. V. Volovich and Ye. I. Zelenov, *p-Adic Analysis and Mathematical Physics* [in Russian], Nauka, Moscow (1994); English transl. (Ser. Sov. East Eur. Math., Vol. 1), World Scientific, Singapore (1994).

[4] B. Dragovich, A. Yu. Khrennikov, S. V. Kozyrev and I. V. Volovich, “On p-adic mathematical physics,” *p-Adic Numbers, Ultrametric Analysis and Applications*, 1, 1–17 (2009); [arXiv:0904.4205v1](http://arxiv.org/abs/0904.4205v1) [math-ph].

[5] L. Brekke, P. G. O. Freund, M. Olson and E. Witten, “Non-archimedean string dynamics,” *Nucl. Phys. B* 302, 365–402 (1988).

[6] P. H. Frampton and Y. Okada, “Effective scalar field theory of p-adic string,” *Phys. Rev. D* 37, 3077–3079 (1988).

[7] D. Ghoshal and A. Sen, “Tachyon condensation and brane descent relations in p-adic string theory,” *Nucl. Phys. B* 584, 300–312 (2000); [arXiv:hep-th/0003278](http://arxiv.org/abs/hep-th/0003278).

[8] N. Moeller and B. Zwiebach, “Dynamics with infinitely many time derivatives and rolling tachyons,” *JHEP*, 0210, 034 (2002); [arXiv:hep-th/0207107](http://arxiv.org/abs/hep-th/0207107).
[9] V. S. Vladimirov, “On the equations for \( p \)-adic closed and open strings,” \( p \)-Adic Numbers, Ultrametric Analysis and Applications \textbf{1}, 79–87 (2009).

[10] I. Ya. Aref’eva, “Nonlocal string tachyon as a model for cosmological dark energy,” (AIP Conf. Proc. \textbf{826}), AIP, New York (2006), pp. 301–311; \texttt{arXiv:astro-ph/0410443}.

[11] N. Barnaby, T. Biswas and J.M. Cline, “\( p \)-Adic inflation,” \textit{JHEP}, \textbf{0704}, 056 (2007); \texttt{arXiv:hep-th/0612230}.

[12] B. Dragovich, “Adelic model of harmonic oscillator,” \textit{Theor. Math. Phys.} \textbf{101}, 1404–1412 (1994); \texttt{arXiv:hep-th/0402193}.

[13] G. Djordjević, B. Dragovich and Lj. Nešić, “Adelic path integrals for quadratic Lagrangians,” \textit{Infin. Dimens. Anal. Quan. Prob. Relat. Topics} \textbf{6}, 179–195 (2003); \texttt{arXiv:hep-th/0105030}.

[14] G. S. Djordjević, B. Dragovich, Lj. D. Nešić and I. V. Volovich, “\( p \)-Adic and adelic minisuperspace quantum cosmology,” \textit{Int. J. Mod. Phys. A} \textbf{17}, 1413–1433 (2002); \texttt{arXiv:gr-qc/0105050}.

[15] B. Dragovich, “\( p \)-Adic perturbation series,” \textit{Phys. Lett. B} \textbf{256}, 392–396 (1991).

[16] B. Dragovich, A. Khrennikov and D. Mihajlović, “Linear fractional \( p \)-adic and adelic dynamical systems,” \textit{Rep. Math. Phys.} \textbf{60}, 55 (2007); \texttt{arXiv:math-ph/0612058}.

[17] B. Dragovich, “Zeta strings,” \texttt{arXiv:hep-th/0703008}.

[18] B. Dragovich, “Zeta-nonlocal scalar fields,” \textit{Theor. Math. Phys.} \textbf{153}, 1671–1677 (2008).

[19] B. Dragovich, “Some Lagrangians with zeta function nonlocality,” in: \textit{Problems of Modern Theoretical Physics}, a volume in honour of Prof. I.L. Buchbinder on the occasion of his 60th birthday (Tomsk State Pedagogical University, Tomsk, Russia 2008) pp. 146-153; \texttt{arXiv:0805.0403v1[hep-th]}.

[20] B. Dragovich, “Lagrangians with Riemann zeta function,” \textit{Rom. Journ. Phys.} \textbf{53}, 1105–1110 (2008); \texttt{arXiv:0809.1601v1[hep-th]}. 
[21] B. Dragovich, “Towards effective Lagrangians for adelic strings,” *Fortschr. Phys.* **57**, 546–551 (2009); [arXiv:0902.0295](http://arxiv.org/abs/0902.0295) [hep-th].

[22] A. S. Koshelev and S. Yu. Vernov, “Cosmological perturbations in SFT inspired non-local scalar field models,” [arXiv:0903.5176v2](http://arxiv.org/abs/0903.5176v2).

[23] I.Ya. Aref’eva and I.V. Volovich, “Quantization of the Riemann zeta-function and cosmology,” *Int. J. Geom. Meth. Mod. Phys.* **4**, 881–895 (2007); [arXiv:hep-th/0701284v2](http://arxiv.org/abs/hep-th/0701284v2).