HIGH ENERGY Solutions TO \( p(x) \)-LaplACIAN Equations of Schrödinger Type

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Abstract. In this paper, we investigate nonlinear Schrödinger type equations in \( \mathbb{R}^N \) under the framework of variable exponent spaces. We propose new assumptions on the nonlinear term to yield bounded Palais-Smale sequences and then prove the special sequences we find converge to critical points respectively. The main arguments are based on the geometry supplied by Fountain Theorem. Consequently, we show that the equation has a sequence of solutions with high energies.

Keywords: \( p(x) \)-Laplacian; Variable exponent Sobolev space; Critical point; Fountain Theorem

Mathematics Subject Classification(2000): 34D05; 35J20; 35J70

1. Introduction

Inspired by X.L. Fan [12, 13] and L. Jeanjean [24], we study the following nonlinear Schrödinger type equation:

\[
-\text{div}(|Du|^{p(x)-2} Du) + V(x)|u|^{p(x)-2} u = f(x, u), \quad x \in \mathbb{R}^N,
\]

where \(-\text{div}(|Du|^{p(x)-2} Du)\) is called minus \( p(x) \)-Laplacian and \( V(x) \) satisfies the following condition (\( V \)) : \( V(x) \in C(\mathbb{R}^N, \mathbb{R}), \inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0 \); For every constant \( M > 0 \), the Lebesgue measure of the set \( \{ x \in \mathbb{R}^N ; V(x) \leq M \} \) is finite. Here \( V_0 \) is a constant.

These equations involving the \( p(x) \)-Laplacian (also called \( p(x) \)-Laplacian equations) arise in the modeling of electrorheological fluids (see [2, 5, 31] and [27]) and image restorations among other problems in physics and engineering. Lots of classical equations, for example the classical fluid equations, are also studied in this general framework (see the new monograph [42] and the references therein). Different from the Laplacian \( \Delta := \sum_j \partial_j^2 \) (linear and homogeneous) and the \( p \)-Laplacian \( \Delta_p u(x) := \text{div}(|Du|^{p-2} Du) \) (nonlinear but homogeneous) where \( 0 < p < \infty \) is a positive number, the \( p(x) \)-Laplacian is nonlinear and nonhomogeneous. Besides the

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applications we mentioned at the beginning of this paragraph, the $p(x)$-Laplacian equations can be regarded as a nonlinear and nonhomogeneous mathematical generalization of the stationary Schrödinger equation $Hu(x) = 0$ where the Hamiltonian is usually given by $H := -\frac{h^2}{2m} \Delta + V(x)$. For these connections and potential further generalizations, see [45, 46, 47].

Next, we give the definitions of variable exponent spaces in order to describe our problem precisely.

Let $\Omega$ be an open domain in $\mathbb{R}^N$ and denote:

$$C^+(\Omega) := \{p(x) \in C(\Omega) : 1 < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \Omega} p(x) < \infty\}.$$ 

For $p(x) \in C^+(\Omega)$, we consider the set:

$$L^{p(x)}(\Omega) = \{u : u \text{ is real-valued measurable function, } \int_{\Omega} |u|^{p(x)} \, dx < \infty\}.$$ 

We can introduce a norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x),\Omega} := \inf\{k > 0 : \int_{\Omega} \frac{|u|}{k}^{p(x)} \, dx \leq 1\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x),\Omega})$ is a Banach Space and we call it a variable exponent Lebesgue space.

Consequently, $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)} ; |Du| \in L^{p(x)}(\Omega)\}$$

with the following norm

$$||u||_{p(x),\Omega} = \inf\{k > 0 ; \int_{\Omega} \frac{|Du|^{p(x)}}{k} + V(x)|u|^{p(x)} \, dx \leq 1\}.$$ 

Then $(W^{1,p(x)}(\Omega), ||\cdot||_{p(x),\Omega})$ also becomes a Banach space and we call it a variable exponent Sobolev space.

Let

$$E := \{u \in W^{1,p(x)}(\mathbb{R}^N) ; \int_{\mathbb{R}^N} V(x)|u|^{p(x)} \, dx < \infty\}.$$ 

Then $E$ is a Banach space with the following norm

$$||u|| = \inf\{k > 0 ; \int_{\mathbb{R}^N} \frac{|Du|^{p(x)}}{k} + V(x)|u|^{p(x)} \, dx \leq 1\}.$$ 

Of course, our working space is $E$. Under reasonable and proper assumptions, we shall show that $(\mathbb{L})$ has a sequence of high energy solutions $\{u_n\}$ in $E$ in this paper (Theorem 2.2).
Since the last two decades, literatures and studies on variable exponent spaces have sprung up like mushrooms (see [1, 2, 5, 7, 8, 9-20, 25, 31, 39-41]). These kinds of spaces are extensions of the usual Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{m,p}(\Omega)$ where $1 \leq p < \infty$ is a constant. And they are special Orlicz spaces (see [26]). A lot of mathematical work has been done under the framework of the variable exponent spaces (see [1 4 11 27 29 35]). Meanwhile, a number of typical and interesting problems have come into light (see [4, 6, 10, 15, 20, 22, 23, 28, 29, 33]). For example, local conditions on the exponent $p(x)$ can assure the multiplicity of solutions to $p(x)$-Laplacian equations (See [35]).

There is no doubt that there are mainly two characteristics when we work with variable exponent spaces. For one thing, these spaces are more complicated than the usual spaces (See [3, 8, 17, 25]). As a result, the related problems are more difficult to deal with. For the other thing, we will obtain more general results if we work under the framework of the variable exponent spaces because these spaces are natural generalizations of the usual Sobolev and Lebesgue spaces.

In X.L. Fan [12], the author considered a constrained minimization problem involving $p(x)$-Laplacian in $\mathbb{R}^N$. Under periodic assumptions, the author could elaborately deal with this unbounded problem by concentration-compactness principle of P. L. Lions. In a following paper, X.L. Fan [13], the author considered $p(x)$-Laplacian equations in $\mathbb{R}^N$ with periodic data and non-periodic perturbations. Under proper conditions, the author was able to show the existence of solutions and gave a concise description of the ground state solutions. It is worth noting that the periodicity assumptions are essential for the validity of concentration-compactness principle under the framework of variable exponent spaces (see the recent paper of Bonder and coworkers [43, 44] for the concentration-compactness theory in the variable exponent space framework involving critical exponents). In our paper, we also consider an unbounded problem. However, under condition (V), we could get some compact embedding theorems. In fact, lots of other tricks can be used to recover some kinds of compactness. For example, weight function method was used in [9]. In [36], we considered a combined effect of the symmetry of the space and the coerciveness of potential $V(x)$.

We also want to mention the celebrated paper of L. Jeanjean [24]. In this paper, the author illustrated a completely new idea to guarantee bounded $(PS)$ sequences...
for a given $C^1$ functional. Roughly speaking, we could consider a family of functionals which contains the original one we are interested in. When given additional structure assumptions, almost all the functional in the family have bounded ($PS$) sequences if the functionals enjoy specific geometry properties! In fact, the information of relevant functionals in the family can provide useful information for the original functional. Under our conditions (see Section 2), we could show the functional we consider satisfies the fountain geometry. Then following L. Jeanjean’s idea and Theorem 3.6 due to W. Zou [38], we could show equation (1) has a sequence of high energy solutions. We want to emphasize that our condition (C4) is somewhat mild and is first used in dealing with $p(x)$–Laplace equations. In addition, we do not need the Ambrosetti-Rabinowits type condition here.

For the reader’s convenience, we will recall some basic properties on the variable exponent spaces in the following part of this section.

**Proposition 1.1.** ([17] [18]) $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ are both separable, reflexive and uniformly convex Banach Spaces.

**Proposition 1.2.** ([17] [18]) Let $\rho(u) = \int_{\Omega} |u(x)|^{p(x)} dx$ for $u \in L^{p(x)}(\Omega)$, then we have

1. $|u|_{p(x),\Omega} = 1 \iff \rho(u) = 1$;
2. $|u|_{p(x),\Omega} \leq 1 \Rightarrow |u|_{p(x),\Omega}^{p^+} \leq \rho(u) \leq |u|_{p(x),\Omega}^{p^-}$;
3. $|u|_{p(x),\Omega} \geq 1 \Rightarrow |u|_{p(x),\Omega}^{p^-} \leq \rho(u) \leq |u|_{p(x),\Omega}^{p^+}$;
4. For $u_n \in L^{p(x)}(\Omega), \rho(u_n) \to 0 \iff |u_n|_{p(x),\Omega} \to 0$ as $n \to \infty$;
5. For $u_n \in L^{p(x)}(\Omega), \rho(u_n) \to \infty \iff |u_n|_{p(x),\Omega} \to \infty$ as $n \to \infty$.

**Proposition 1.3.** ([17] [18] [26]) Let $\rho(u) = \int_{\Omega} |Du(x)|^{p(x)} + |u(x)|^{p(x)} dx$ for $u \in W^{1,p(x)}(\Omega)$, then we have

1. $||u||_{p(x),\Omega} = 1 \iff \rho(u) = 1$;
2. $||u||_{p(x),\Omega} \leq 1 \Rightarrow ||u||_{p(x),\Omega}^{p^+} \leq \rho(u) \leq ||u||_{p(x),\Omega}^{p^-}$;
3. $||u||_{p(x),\Omega} \geq 1 \Rightarrow ||u||_{p(x),\Omega}^{p^-} \leq \rho(u) \leq ||u||_{p(x),\Omega}^{p^+}$;
4. For $u_n \in W^{1,p(x)}(\Omega), \rho(u_n) \to 0 \iff ||u_n||_{p(x),\Omega} \to 0$ as $n \to \infty$;
5. For $u_n \in W^{1,p(x)}(\Omega), \rho(u_n) \to \infty \iff ||u_n||_{p(x),\Omega} \to \infty$ as $n \to \infty$.

The following property can be easily verified:

**Proposition 1.4.** For $u \in E$, Let $\rho(u) = \int_{\mathbb{R}^N} |Du(x)|^{p(x)} + V(x)|u(x)|^{p(x)} dx$. Then we have the following relations:
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$$(1) \ ||u|| = 1 \Leftrightarrow \rho(u) = 1;$$
$$(2) \ ||u|| \leq 1 \Rightarrow ||u||^{p^+} \leq \rho(u) \leq ||u||^{p^-};$$
$$(3) \ ||u|| \geq 1 \Rightarrow ||u||^{p^-} \leq \rho(u) \leq ||u||^{p^+}.$$

From the above-mentioned properties, we can see that the norm and the integral (i.e. $\rho(u)$) don’t enjoy the equality relation, which is typical in variable exponent spaces and very different from the constant exponent case.

**Notation.** For $p(x) \in C_+(\Omega), p^*(x)$ refers to the critical exponent of $p(x)$ in the sense of Sobolev embedding, that’s, $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N; p^*(x) = \infty$, otherwise. For two continuous functions $a(x)$ and $b(x)$ in $C(\Omega)$, $a(x) \ll b(x)$ means that $\inf_{x \in \Omega}(b(x) - a(x)) > 0$. We will use the symbols $\rightharpoonup$, $\rightarrow$ to represent weak convergence and strong convergence in a Banach space respectively. While, $\hookrightarrow$, $\hookrightarrow$ will be used to denote continuous embedding and compact embedding between spaces respectively. Technically, we use $C$ to denote a generic positive constant.

**Proposition 1.5.** ([17], [18], [35])

1. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Assume that the boundary $\partial \Omega$ possesses cone property and $q(x) \in C(\Omega, \mathbb{R})$ with $1 \leq q(x) \ll p^*(x)$, then $W^{1,p(x)}(\Omega) \hookrightarrow L^q(x)(\Omega)$
2. $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^q(x)(\mathbb{R}^N)$ if $p^+ < N$ and $q(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \leq q(x) \ll p^*(x)$.

Following the spirit of [13], we have the following proposition:

**Proposition 1.6.** For $u \in E$, we define
$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})dx,$$
then $I \in C^1(E, \mathbb{R})$ and the derivative operator $L$ of $I$ is
$$\langle L(u), v \rangle = \int_{\mathbb{R}^N} (|Du|^{p(x)-2}Du \cdot Dv + V(x)|u|^{p(x)-2}uv)dx, \forall u, v \in E$$
and we have:

1. $L : E \rightarrow E^*$ (the dual space of $E$) is a continuous, bounded and strictly monotone operator;
2. $L$ is a mapping of type $(S_+), i.e. if u_n \rightharpoonup u$ in $E$ and $\limsup_{n \rightarrow \infty} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $E$. 
(3) $L : E \to E^*$ is a homeomorphism.

**Proposition 1.7.** ([17, 18, 35]) Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. If $f(x,t)$ is a Caratheodory function and satisfies

$$|f(x,t)| \leq a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}, \forall x \in \overline{\Omega}, t \in \mathbb{R}$$

where $p_1(x), p_2(x) \in C_+^{+}(\Omega), b \geq 0$ is a constant, $0 \leq a(x) \in L^{p_2(x)}(\Omega)$, then the superposition operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $Su = f(x,u(x))$ is a continuous and bounded operator. Moreover, if $\Omega$ is unbounded (e.g. $\Omega = \mathbb{R}^N$) and $a(x) \equiv 0$, the same conclusion is true.

In the variable Lebesgue space case, Hölder type inequality still holds.

**Proposition 1.8.** ([17]) Let $\Omega$ be a domain in $\mathbb{R}^N$ (either bounded or unbounded) and $u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega)$ where $p'(x) := \frac{p(x)}{p(x)-1}$ is the conjugate exponent of $p(x) \in C_+(\Omega)$. Then the following Hölder type inequality holds

$$\int_\Omega |uv|dx \leq \left( \frac{1}{p} + \frac{1}{p'} \right) |u|_{p(x),\Omega} |v|_{p'(x),\Omega}.$$  

We still use this inequality in the following section (for example, in Lemma 2.7).

This paper is divided into 3 sections. For the readers’ convenience, we have recalled some basic properties of the variable exponent spaces $W^{1,p(x)}(\Omega), L^{p(x)}(\Omega)$ in this section. In section 2, we will state our assumptions on the nonlinear term and our main result. Meanwhile, we prove some useful auxiliary results in this section. In our opinion, these results themselves are interesting and important when we study variable exponent problems. In Sections 3, we are devoted to proving the main result.

2. Main result

In this section, we will first specify our assumptions on the nonlinear term $f$ and give some comments on these assumptions. Then we state the main result.

We assume the following assumptions:

- $(C1) f \in C(\mathbb{R}^N \times R, R)$ satisfies
  
  $$|f(x,t)| \leq C(|t|^{p(x)-1} + |t|^{q(x)-1}), \forall t \in R, x \in \mathbb{R}^N$$

  $f(x,t)t \geq 0$, for $t \geq 0, x \in \mathbb{R}^N$

  $p(x) \leq q(x) \ll p^*(x), \forall x \in \mathbb{R}^N.$
(C2) There exists a constant $\mu > p^+$ such that
\[
\liminf_{|t| \to \infty} \frac{f(x, t)t}{|t|^\mu} \geq C_0 \text{ (a constant)} > 0, \text{ uniformly for } x \in \mathbb{R}^N.
\]

(C3)
\[
\limsup_{|t| \to 0} \frac{f(x, t)t}{|t|^{p^+}} = 0, \text{ uniformly for } x \in \mathbb{R}^N.
\]

(C4) Let $F(x, t) = \int_0^t f(x, s)ds$ and
\[
G(x, t) := f(x, t)t - p^- F(x, t);
\]
\[
H(x, t) := f(x, t)t - p^+ F(x, t).
\]

We assume $G$ and $H$ satisfy the monotonicity condition: there exist two positive constants $D_1$ and $D_2$ such that
\[
G(x, t) \leq D_1 G(x, s) \leq D_2 H(x, s), \text{ for } 0 \leq t \leq s.
\]

(C5)
\[
f(x, -t) = -f(x, t), \forall t \in \mathbb{R}, x \in \mathbb{R}^N.
\]

**Definition 2.1.** We say $u \in E$ is a solution to the equation (1.1) if for any $v \in E$, the following equality holds
\[
\int_{\mathbb{R}^N} |Du|^{p(x)-2} DuDv + V(x)|u|^{p(x)-2}uvdx = \int_{\mathbb{R}^N} f(x, u)vdx.
\]

Define a functional $\Phi$ from $E$ to $\mathbb{R}$:
\[
\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)})dx - \int_{\mathbb{R}^N} F(x, u)dx.
\]

Under our assumptions, we know the functional is $C^1$ (Proposition 1.6, Lemma 2.7 in the following) and for $v \in E$,
\[
\Phi'(u)v = \int_{\mathbb{R}^N} |Du|^{p(x)-2} DuDv + V(x)|u|^{p(x)-2}uvdx - \int_{\mathbb{R}^N} f(x, u)vdx.
\]

So the critical points of the functional $\Phi$ are corresponding to the solutions of the equation (1.1).

Next are some comments and analysis on the assumptions we give.

1. (C1) – (C4) are compatible. We give two examples. Let $f(x, t) = |t|^{q(x)-2}t$ with $q(x) \in C_+(\mathbb{R}^N)$ satisfying $q(x) \ll p^*(x), q_- > p^+$. Obviously, (C1), (C2), (C3), (C5)
hold. In order to verify (C4), we know that $F(x, t) = \frac{|t|^{q(x)}}{q(x)}$, $f(x, t) = |t|^{q(x)}$. Consequently, $G(x, t) = (1 - \frac{p}{q(x)})|t|^{q(x)}$, $H(x, t) = (1 - \frac{p}{q(x)})|t|^{q(x)}$. It’s easy to verify that $G(x, t)$ is nondecreasing in $t \geq 0$. Therefore, $G(x, t) \leq G(x, s)$ if $0 \leq t \leq s$. In view of $G, H \geq 0$, we know that $G(x, s) \leq D_2 H(x, s)$ when $s \geq 0$. On the whole, (C4) holds.

Next, we will illustrate another example. Let $f(x, t) = |t|^{q(x)} - 2t \ln(\lambda |t| + 1)$ where $q(x)$ satisfies $q(x) \ll p^+(x)$, $q^- > p^+$ and $\epsilon > a > 0$ is a real number. In view of the following relations:

$$\lim_{|t| \to \infty} \frac{\ln^a(|t| + 1)}{|t|^\epsilon} = 0 \quad \forall a \geq 0, \epsilon > 0;$$

$$\lim_{|t| \to 0} \frac{\ln^a(|t| + 1)}{|t|^\epsilon} = \infty \quad \forall a \geq 0, \epsilon > 0,$$

we can verify (C4) similarly. Evidently, (C1), (C2), (C3), (C5) hold.

From the two examples, we know lots of functions satisfy our assumptions. As a result, our main result is somewhat general.

2. Condition (C1) means that $f(x, t)$ is subcritical in the variable sense. Different from things in constant case (i.e. $p^+ = p^-$), we need $q(x) \ll p^+(x)$.

3. Condition (C4) is crucial for our proof. It’s because of this condition that we could obtain bounded Palais-Smale sequence (bounded (PS) sequences for short). We give this condition to $f$ other than the famous Ambrosetti-Rabinowitz type condition. However, we could still get bounded (PS) sequences via an indirect method. Lots of authors have tried to weaken the Ambrosetti-Rabinowitz type condition and they can only get weak type (PS) sequences (usual the Cerami Condition). It’s known that (C5) is much weaker than the Ambrosetti-Rabinowitz type condition in the constant exponent case ($p^+ = p^-$) (see [21]).

4. Condition (C5) assures the functional $\Phi$ we defined before is an even functional. So the condition is necessary for us to take advantage of the fountain geometry.
In this paper, we assume condition (V) always holds and $p^+ < N$. Hence, we know $E \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$. Consequently, $E \hookrightarrow L^{p(x)}(\mathbb{R}^N), E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ if $q(x) \in C_+(\mathbb{R}^N)$ satisfies $p(x) \leq q(x) \ll p^*(x)$.

Now we can state our main result clearly.

**Theorem 2.2.** Under condition (V) and (C1) – (C5), the equation (1.1) has a sequence of solutions $\{u_n\}$. Moreover, the solutions we get have high energies, i.e. $\Phi(u_n) \to \infty$ as $n \to \infty$.

In order to make the exposition more concise, we will give some auxiliary results, some of which are very useful themselves.

**Lemma 2.3.** Let $\Omega$ be a nonempty domain in $\mathbb{R}^N$ which can be bounded or unbounded. We also allow $\Omega = \mathbb{R}^N$. Then $L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega) \subset L^{a(x)}(\Omega)$ if $p(x), q(x), a(x) \in C_+(\mathbb{R}^N)$ and $p(x) \leq a(x) \leq q(x)$. Moreover, if $p(x) \ll a(x) \ll q(x)$, the following interpolation inequality holds for $u \in L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega)$:

$$
\int_\Omega |u|^{a(x)}dx \leq 2||u|^{a_1(x)}|_{m(x),\Omega}||u|^{a_2(x)}|_{m'(x),\Omega},
$$

where

$$
a_1(x) = \frac{p(x)(q(x) - a(x))}{q(x) - p(x)}, a_2(x) = \frac{q(x)(a(x) - p(x))}{q(x) - p(x)};
$$

$$
m(x) = \frac{q(x) - p(x)}{q(x) - a(x)}, m'(x) = \frac{q(x) - p(x)}{a(x) - p(x)}.
$$

**Sketch of Proof.** For $L^{p(x)}(\Omega) \cap L^{q(x)}(\Omega)$, we have

$$
\int_\Omega |u|^{p(x)}dx < \infty, \int_\Omega |u|^{q(x)}dx < \infty.
$$

Obviously, $|u(x)|^{a(x)} \leq |u(x)|^{p(x)} + |u(x)|^{q(x)}$ for $x \in \Omega$. Hence, $\int_\Omega |u|^{a(x)} \leq \int_\Omega |u|^{p(x)}dx + \int_\Omega |u|^{q(x)}dx < \infty$, which means $u \in L^{a(x)}(\Omega)$. For the interpolation inequality, the readers can see [20].

**Lemma 2.4.** Under the condition (V), $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$.

**Proof.** We have known that $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$. Next, we assume $u_n \rightharpoonup 0$ in $E$. We need to show $u_n \to 0$ in $L^{p(x)}(\mathbb{R}^N)$ to complete the proof. By Proposition 1.2, it suffices to verify $\int_{\mathbb{R}^N} |u_n|^{p(x)}dx \to 0$ as $n \to \infty$. For any given $R > 0$, we write

$$
I(n) := \int_{\mathbb{R}^N} |u_n|^{p(x)}dx = \int_{B(0,R)} |u_n|^{p(x)}dx + \int_{\mathbb{R}^N \setminus B(0,R)} |u_n|^{p(x)}dx := I_1(n) + I_2(n).
$$
Since $E \hookrightarrow W^{1,p(x)}(R^N)$ and $W^{1,p(x)}(B(0,R)) \hookrightarrow \hookrightarrow L^{p(x)}(B(0,R))$, we have $I_1(n) \to 0$ as $n \to \infty$.

For any constant $M > 0$, let $A = \{x \in R^N \setminus B(0,R); V(x) > M\}$ and $B = \{x \in R^N \setminus B(0,R); V(x) \leq M\}$. Then $\int_A |u_n|^{p(x)}dx \leq \int_A \frac{V(x)}{M} |u_n|^{p(x)}dx \leq \frac{1}{M} \int_{R^N} V(x)|u_n|^{p(x)}dx \leq \frac{C}{M}$; since for the constant $M > 0$, $mes\{x \in R^N; V(x) \leq M\}$ is finite, we can choose $R > 0$ large enough such that $mes\{x \in R^N \setminus B(0,R); V(x) \leq M\} \to 0$. Consequently, $\int_B |u_n|^{p(x)}dx \to 0$.

Now let $M \to \infty$ and $R \to \infty$, we have $I(n) \to 0$ as $n \to \infty$.

**Lemma 2.5.** Under the condition $(V)$, $E \hookrightarrow \hookrightarrow L^{a(x)}(R^N)$ if $a(x) \in C_+(R^N)$ and $p(x) \leq a(x) \ll p^*(x)$.

**Proof.** Let $u_n \to 0$ in $E$. We need to show $u_n \to 0$ in $L^{a(x)}(R^N)$ to finish the proof.

First, we assume that $p(x) \ll a(x) \ll p^*(x)$. We can choose $q(x) \in C_+(R^N)$ such that $a(x) \ll q(x) \ll p^*(x)$. It’s obvious that $E \hookrightarrow L^{q(x)}(R^N)$. In view of $p(x) \ll a(x) \ll q(x)$, we use Lemma 2.3 with $\Omega = R^N$ and obtain

$$\int_{\Omega} |u_n|^{a(x)}dx \leq 2||u_n|^{a_1(x)}|_{m(x),\Omega}||u_n|^{a_2(x)}|_{m'(x),\Omega}, \quad (2.2)$$

where the symbols are the same as those of Lemma 2.3.

Let $\lambda_n := ||u_n|^{a_1(x)}|_{m(x),\Omega}$ and $\mu_n := ||u_n|^{a_2(x)}|_{m'(x),\Omega}$. By Proposition 1.2, we have

$$\int_{R^N} \frac{|u_n|^{a_1(x)}}{\lambda_n} m(x)dx = \int_{R^N} \frac{|u_n|^{p(x)}}{\lambda_n^{m(x)}} dx = 1;$$

$$\int_{R^N} \frac{|u_n|^{a_2(x)}}{\mu_n} m'(x)dx = \int_{R^N} \frac{|u_n|^{q(x)}}{\mu_n^{m'(x)}} dx = 1. $$

From the two equalities and Lemma 2.4, we know

$$\min\{\lambda_n^{-m_+}, \lambda_n^{-m_-}\} \leq \int_{R^N} |u_n|^{p(x)}dx \to 0,$$

$$\min\{\mu_n^{-m'_+}, \mu_n^{-m'_-}\} \leq \int_{R^N} |u_n|^{q(x)}dx \leq C.$$

Anyway, we have $\lambda_n \to 0$ as $n \to \infty$ and $0 \leq \mu_n \leq C$. So (2.2) yields that $\int_{R^N} |u_n|^{a(x)}dx \to 0$ as $n \to \infty$. 
Next, we assume \( p(x) \leq a(x) \ll p^*(x) \). We can choose \( q(x) \in C_+(R^N) \) such that \( a(x) \ll q(x) \ll p^*(x) \). By the arguments above, we have

\[
\int_{R^N} |u_n|^{q(x)} dx \to 0.
\]

By Lemma 2.6, Lemma 2.4, we have

\[
\int_{R^N} |u_n|^{a(x)} dx \leq \int_{R^N} |u_n|^{p(x)} dx + \int_{R^N} |u_n|^{q(x)} dx \to 0.
\]

\[\square\]

The following lemma can be considered as an extension of the result in M. Willem [34, Appendix A].

**Lemma 2.6.** Assume \( 1 \leq p_1(x), p_2(x), q_1(x), q_2(x) \in C(\Omega) \). Let \( f(x,t) \) be a Carathéodory function on \( \Omega \times R \) and satisfy

\[
|f(x,t)| \leq a|t|^{p_1(x)\frac{p_2(x)}{q_1(x)}} + b|t|^{p_2(x)\frac{p_2(x)}{q_2(x)}}, (x,t) \in \Omega \times R,
\]

where \( a,b > 0 \) and \( \Omega \) is either bounded or unbounded. Define a Carathéodory operator by

\[
B u := f(x,u(x)), u \in \mathcal{H} := L^{p_1(x)}(\Omega) \cap L^{p_2(x)}(\Omega)
\]

Define the space \( \mathcal{E} := L^{q_1(x)}(\Omega) + L^{q_2(x)}(\Omega) \) with a norm

\[
||u||_\mathcal{E} = \inf \{|v|_{q_1(x),\Omega} + |w|_{q_2(x),\Omega} : u = v + w, v \in L^{q_1(x)}(\Omega), w \in L^{q_2(x)}(\Omega)\}.
\]

If \( \frac{p_1(x)}{q_1(x)} \leq \frac{p_2(x)}{q_2(x)} \) for \( x \in \Omega \), then \( B = B_1 + B_2 \), where \( B_i \) is a bounded and continuous mapping from \( L^{p_i(x)}(\Omega) \) to \( L^{q_i(x)}(\Omega) \), \( i = 1, 2 \). In particular, \( B \) is a bounded continuous mapping from \( \mathcal{H} \) to \( \mathcal{E} \).

**Proof.** Let \( \psi : R \to [0,1] \) be a smooth function such that \( \psi(t) = 1 \) for \( t \in (-1,1) \); \( \psi(t) = 0 \) for \( t \not\in (-2,2) \). Let

\[
g(x,t) = \psi(t)f(x,t), h(x,t) = (1-\psi(t))f(x,t).
\]

Because \( \frac{p_1(x)}{q_1(x)} \leq \frac{p_2(x)}{q_2(x)} \) for \( x \in \Omega \), there are two constants \( d > 0, m > 0 \) such that

\[
|g(x,t)| \leq d|t|^{\frac{p_1(x)}{q_1(x)}}, |h(x,t)| \leq m|t|^{\frac{p_2(x)}{q_2(x)}}.
\]

Define

\[
B_1 u = g(x,u), u \in L^{p_1(x)}(\Omega); B_2 u = h(x,u), u \in L^{p_2(x)}(\Omega).
\]
Then by Proposition 1.7, $B_i$ is a bounded and continuous mapping from $L^{p_i(x)}(\Omega)$ to $L^{q_i(x)}(\Omega), i = 1, 2$. It’s readily to see that $B := B_1 + B_2$ is a bounded continuous mapping from $\mathcal{H}$ to $\mathcal{E}$.

From Lemma 2.4, Lemma 2.5, we know that the condition $(V)$ plays an important role. In enables $E$ to be compactly embedded into $L^{p(x)}(R^N)$ type spaces. Using Lemma 2.5 and Lemma 2.6, we can prove the following

Lemma 2.7. Under assumption $(V)$ and $(C1)$, the functional $J(u) = \int_{R^N} F(x, u) \, dx$ on $E$ is a $C^1$ functional. Moreover, $J'$ is compact.

Proof. The verification that $J$ is a $C^1$ functional is routine and we omit it here. We only show that $J'$ is compact. Because $E \hookrightarrow L^{p(x)}(R^N)$ (Lemma 2.4) and $E \hookrightarrow L^{q(x)}(R^N)$ (Lemma 2.5), any bounded sequence $\{u_k\}$ in $E$ has a renamed subsequence denoted by $\{u_k\}$ which converges to $u_0$ in $L^{p(x)}(R^N)$ and $L^{q(x)}(R^N)$. Using Lemma 2.6 with $p_1(x) = p(x), q_1(x) = \frac{p(x)}{p(x) - 1}, p_2(x) = q(x), q_2(x) = \frac{q(x)}{q(x) - 1}$ and $\Omega = R^N$, we have $J'(u)v = \int_{R^N} (B_1u + B_2u)v \, dx$ for $v \in E$. Hence, $B_1(u_k) \to B_1(u_0)$ in $L^{q_1(x)}(\Omega)$ and $B_2(u_k) \to B_2(u_0)$ in $L^{q_2(x)}(\Omega)$. Then Hölder type inequality (Proposition 1.8) and Sobolev embedding (Lemma 2.5) assure $J'(u_k) \to J'(u_0)$ in $E^*$, i.e. $J'$ is compact. This proves the Lemma. □

For convenience, we give the definition of $(PS)_c$ sequence for $c \in R$.

Definition 2.8. Let $\Pi$ be a $C^1$ functional defined on a real Banach space $X$. Any sequence $\{u_n\}$ satisfying $\Pi(u_n) \to c$ and $\Pi'(u_n) \to 0$ is called a $(PS)_c$ sequence. In addition, we call $c$ here a prospective critical level of $\Pi$.

Remark 2.9. (See also [14]) Under the assumption of Theorem 2.2, we have the following comments. $\Phi(u) = I(u) + J(u)$ and $\Phi'(u) = I'(u) + J'(u)$ for $u \in E$. Since $I'$ is of type $(S_+)$ (Proposition 1.6) and $J'$ is a compact (Lemma 2.7), we can easily derive that $\Phi'$ is of type $(S_+)$. It’s well-known that any bounded $(PS)_c$ sequence of a functional whose Fréchet derivative is of type $(S_+)$ in a reflexive Banach space has a convergent subsequence and so does $\Phi$ here.

3. Proof of Theorem 2.2

We will first state the Fountain Theorem before our proof.
Let $X$ be a Banach space with the norm $\| \cdot \|$ and let $\{X_j\}$ be a sequence of subspaces of $X$ with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Further, $X = \bigoplus_{j=1}^{\infty} X_j$, $W_k := \bigoplus_{j=1}^{k} X_j$, $Z_k := \bigoplus_{j=k}^{\infty} X_j$. Moreover, for $k \in \mathbb{N}$ and $\rho_k > r_k > 0$, we denote:

$$B_k = \{ u \in W_k : \| u \| \leq \rho_k \}; \quad S_k = \{ u \in Z_k : \| u \| = r_k \};$$

$$c_k := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi(\gamma(u)),$$

$$\Gamma_k := \{ \gamma \in C(B_k, X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = \text{id} \}.$$

**Theorem 3.1.** ([34], Fountain Theorem, Bartsch, 1992) Under the aforementioned assumptions, let $\Phi \in C^1(X, \mathbb{R})$ be an even functional. If for $k > 0$ large enough, there exists $\rho_k > r_k > 0$ such that

$$\text{a}_k := \max \{ \Phi(u) : u \in W_k, \| u \| = \rho_k \} \leq 0; \quad (3.1)$$

$$\text{b}_k := \inf \{ \Phi(u) : u \in Z_k, \| u \| = r_k \} \to \infty \text{ as } k \to \infty. \quad (3.2)$$

then $\Phi$ has a $(PS)_{c_k}$ sequence for each prospective critical value $c_k$ and $c_k \to \infty$ as $k \to \infty$.

**Definition 3.2.** Let $X$ be a Banach space, $\Phi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. The function $\Phi$ satisfies the $(PS)_c$ condition if any sequence $\{u_n\} \subset X$ such that

$$\Phi(u_n) \to c, \Phi'(u_n) \to 0 \quad (3.3)$$

has a convergent subsequence.

**Remark 3.3.** In fact, if the following condition $(C)$ holds

$$(C) \quad \Phi \text{ satisfies the } (PS)_c \text{ condition for every } c > 0, \quad (3.4)$$

the sequence $\{c_k\}$ in Theorem 3.1 is a sequence of unbounded critical values of $\Phi$. However, the condition $(C)$ isn’t necessary to guarantee $c_k$ is a critical level. We just need $(PS)_{c_k}$ condition.

In order to use the decomposition technique, we need a theorem on the structure of a reflexive and separable Banach space.

**Lemma 3.4.** (See [37, Section 17]) Let $X$ be a reflexive and separable Banach space, then there are $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that:

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, \text{ if } n = m \\ 0, \text{ if } n \neq m \end{cases}$$

$$X = \overline{\text{span}} \{ e_n : n = 1, 2, \cdots \}, \quad X^* = \overline{\text{span}}^{W*} \{ f_n : n = 1, 2, \cdots \}.$$
For \( k = 1, 2, \cdots, \) and \( X = E, \) we will choose:

\[
X_j = \text{span}\{e_j\}, W_k = \bigoplus_{j=1}^k X_j, Z_k = \bigoplus_{j=k}^{\infty} X_j.
\]

In the following, we identify the Banach space \( E \) and the functional \( \Phi \) as those we consider. Next, we will prove the main result step by step. First, we give a useful lemma. For simplicity, we write \( |u|_{p(x), \mathbb{R}^N} \) as \( |u|_{p(x)} \) when \( \Omega = \mathbb{R}^N \) for \( p(x) \in C_+(\mathbb{R}^N). \)

**Lemma 3.5.** Let \( q(x) \in C_+(\mathbb{R}^N) \) with \( p(x) \leq q(x) \ll p^*(x) \) and denote

\[
\alpha_k = \sup\{|u|_{q(x)} : ||u|| = 1, u \in Z_k\},
\]

(3.5)

then \( \alpha_k \to 0 \) as \( k \to \infty. \)

**Proof.** Obviously, \( \alpha_k \) is decreasing as \( k \to \infty. \) Noting that \( \alpha_k \geq 0, \) we may assume that \( \alpha_k \to \alpha \geq 0. \) For every \( k > 0, \) there exists \( u_k \in Z_k \) such that \( ||u_k|| = 1 \) and \( |u_k|_{q(x)} > \frac{\alpha_k}{2}. \) By definition of \( Z_k, \) \( u_k \to 0 \) in \( E. \) Then Lemma 2.5 implies that \( u_k \to 0 \) in \( L^{q(x)}(\mathbb{R}^N). \) Thus we have proved that \( \alpha = 0. \) \( \square \)

Using lemma 3.5, we can prove the following Lemma 3.6.

**Lemma 3.6.** Under the assumptions of Theorem 3.1, the geometry conditions of the Fountain Theorem hold, i.e. (A) and (B) hold.

**Proof.** By (C2) and (C3), for any \( \epsilon > 0, \) there exists a \( C(\epsilon) > 0 \) such that

\[
f(x, u)u \geq C(\epsilon)|u|^\mu - \epsilon|u|^{p^+}.
\]

In view of (C5), we have a constant, still denoted by \( C(\epsilon), \) such that

\[
F(x, u) \geq C(\epsilon)|u|^\mu - \epsilon|u|^{p^+}.
\]

When \( ||u|| > 1, \) we have

\[
\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{q(x)})dx - \int_{\mathbb{R}^N} F(x, u)dx \\
\leq \frac{1}{p}|u|^{p^+} - C(\epsilon) \int_{\mathbb{R}^N} |u|^{\mu}dx + \epsilon \int_{\mathbb{R}^N} |u|^{p^+}dx.
\]

(3.6)

Let \( u \in W_k, \) since \( \dim(W_k) < \infty. \) all norms on \( W_k \) are equivalent. Hence \( \Phi(u) \leq C||u||^{p^+} - C||u||^\mu. \) Because \( \mu > p^+, \) we can choose \( \rho_k > 0 \) large enough such that \( \Phi(u) \leq 0 \) when \( ||u|| = \rho_k. \) We have shown (A) holds.
To verify (B), we can still let \( ||u|| > 1 \) without loss of generality. By (C1) and (C3), for any \( \varepsilon > 0 \), there exists a \( C = C(\varepsilon) > 0 \) such that
\[
|F(x, u)| \leq \varepsilon|u|^{p^+} + C|u|^{q(x)},
\]
So
\[
\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}(|Du|^{p(x)} + V(x)|u|^{p(x)}\,dx) - \int_{\mathbb{R}^N} F(x, u)\,dx
\geq \frac{1}{p^+}||u||^{p^-} - \varepsilon|u|^{p^+} - C \max\{||u|^{q^-}, |u|^{q^+}\}. \tag{3.7}
\]
Let \( u \in Z_k \) with \( ||u|| = r_k > 0 \). We can choose uniformly an \( \varepsilon > 0 \) small enough such that \( \varepsilon|u|^{p^+} \leq \frac{1}{2p^+}||u||^{p^-} \). Hence
\[
\Phi(u) \geq \frac{1}{2p^+}||u||^{p^-} - C \max\{||u|^{q^-}, |u|^{q^+}\}.
\]
If \( \max\{||u|^{q^-}, |u|^{q^+}\} = ||u|^{q^-} \), we choose \( r_k = (2q^-C\alpha_k^{q^-})\frac{1}{p^-q^-} \), we get that
\[
\Phi(u) \geq \frac{1}{2p^+}||u||^{p^-} - C\alpha_k^{q^-}||u|^{q^-} \geq \frac{1}{2p^+} - C\alpha_k^{p^-}||u||^{q^-}. \tag{3.8}
\]
Since \( q^- > p^+ \) and \( \alpha_k \to 0 \), we obtain \( b_k \to \infty \).

If \( \max\{||u|^{q^-}, |u|^{q^+}\} = ||u|^{q^+} \), we can similarly derive that \( b_k \to \infty \). Hence we have shown (B) holds.

By far, we have shown the geometry conditions of the Fountain Theorem hold. In fact, in order to use the Fountain Theorem to get our main result, we needn’t verify the functional \( \Phi \) satisfies the \((PS)_c\) condition for every \( c > 0 \). It’s enough if we could find a special \((PS)\) sequence for each \( c_k \) and verify the sequence we find has a convergence subsequence. Of course, the first step is to show the \((PS)_{c_k}\) sequence is bounded. Because there is no Ambrosetti-Rabinowits type condition, we couldn’t give a direct proof. Following the ideas in L. Jeanjean [21] and W. Zou [38], we consider \( \Phi \) as a member in a family of functional. We will show almost all the functional in the family have bounded \((PS)\) sequences. The following result due to W. Zou and M. Schechter [38] is crucial for this purpose.

Let the notions be the same as in Theorem 3.1. Consider a family of real \( C^1 \) functional \( \Phi_\lambda \) of the form: \( \Phi_\lambda(u) := I(u) - \lambda J(u) \), where \( \lambda \in \Lambda \) and \( \Lambda \) is a compact interval in \([0, \infty)\). We make the following assumptions:

\((A_1)\) \( \Phi_\lambda \) maps bounded sets into bounded sets uniformly for \( \lambda \in \Lambda \). Moreover, \( \Phi_\lambda(-u) = \Phi_\lambda(u) \) for all \( (\lambda, u) \in \Lambda \times X \).
Let
\[ a_k(\lambda) := \max \{ \Phi_\lambda(u) : u \in W_k, ||u|| = \rho_k \} , \]
\[ b_k(\lambda) := \inf \{ \Phi_\lambda(u) : u \in Z_k, ||u|| = r_k \} . \]

Define
\[ c_k(\lambda) = \inf \max_{\gamma \in \Gamma_k} \Phi_\lambda(\gamma(u)), \]
where
\[ \Gamma_k := \{ \gamma \in C(B_k, X) : \gamma \text{ is odd and } \gamma|_{\partial B_k} = id \}. \]

**Theorem 3.7.** Assume that \((A_1)\) and \((A_2)\) hold. If \(b_k(\lambda) > a_k(\lambda)\) for all \(\lambda \in \Lambda\), then \(c_k(\lambda) \geq b_k(\lambda)\) for all \(\lambda \in \Lambda\). Moreover, for almost every \(\lambda \in \Lambda\), there exists a sequence of \(\{u^k_n(\lambda)\}_{n=1}^\infty\) such that \(\sup_n ||u^k_n(\lambda)|| < \infty\), \(\Phi'_\lambda(u^k_n(\lambda)) \to 0\) and \(\Phi_\lambda(u^k_n(\lambda)) \to c_k(\lambda)\) as \(n \to \infty\).

Next, we let \(I(u) = \int_{R^n} \frac{1}{p(x)} (|Dv|^{p(x)} + V(x)|u|^{p(x)}) \, dx\), \(J(u) = \int_{R^n} F(x, u) \, dx\) for \(u \in E\) and \(\Lambda = [1, 2]\). Under these terminologies, \(\Phi(u) = \Phi_1(u)\). Under the assumptions of Theorem 3.1, it’s easy to see \((A_1)\) and \((A_2)\) hold.

**Lemma 3.8.** Under the assumptions of Theorem 3.7, \(b_k(\lambda) > a_k(\lambda)\) for all \(\lambda \in [1, 2]\) when \(k\) is large enough.

**Sketch of Proof.** Let \(\rho_k > r_k > 0\) large enough. Using same reasoning, we can show that \(a_k(\lambda) \leq 0\) and \(b_k(\lambda) \to \infty\) uniformly for \(\lambda \in [1, 2]\) as \(k \to \infty\). Hence, we have shown the Lemma. Moreover, \(c_k(\lambda) \leq \sup_{u \in B_k} \Phi_\lambda(u) \leq \sup_{u \in B_k} \Phi(u) = \max_{u \in B_k} \Phi_1(u) = \max_{u \in B_k} \Phi(u) := c_k < \infty. \)

**Note 3.9.** Since \(\Phi'_\lambda(u)\) is of type \((S^*_+)\) (Remark 2.9), we know that any bounded \((PS)_{c(\lambda)}\) sequence of \(\Phi_\lambda\) has a convergent subsequence which converges to a critical point of \(\Phi_\lambda\) with critical level \(c(\lambda)\).

Now, using Theorem 3.7, we obtain that for almost every \(\lambda \in [1, 2]\), there exists a sequence of \(\{u^k_n(\lambda)\}_{n=1}^\infty\) such that \(\sup_n ||u^k_n(\lambda)|| < \infty\), \(\Phi'_\lambda(u^k_n(\lambda)) \to 0\) and \(\Phi_\lambda(u^k_n(\lambda)) \to c_k(\lambda)\) as \(n \to \infty\). Denote the set of these \(\lambda\) by \(\Lambda_0\). If 1 \(\notin\) \(\Lambda_0\), we have found bounded \((PS)_{c_k(\lambda)}\) sequence for the functional \(\Phi\).

If 1 \(\notin\) \(\Lambda_0\), we can choose a sequence \(\{\lambda_n\} \subset \Lambda_0\) such that \(\lambda_n \to 1\) decreasing. In view of Note 3.9, for each \(\lambda \in \Lambda_0\), the bounded \((PS)_{c_k(\lambda)}\) sequence has a convergent
subsequence. We denote the limit by \( u^k(\lambda) \). Accordingly, \( u^k(\lambda) \) is the critical point of the functional \( \Phi_\lambda \) with critical level \( c_k(\lambda) \). Next, we are going to show the sequence \( \{ u^k(\lambda_n) \}_{n=1}^{\infty} \) is a bounded \((PS)_{c_k}\) sequence of \( \Phi \). For simplicity, we write \( \{ u^k(\lambda_n) \} \) as \( \{ u(\lambda_n) \} \).

In fact, we only need to show \( \{ u(\lambda_n) \} \) is bounded. Indeed, if \( \{ u(\lambda) \} \) is bounded, we have

\[
\Phi(u(\lambda_n)) = \Phi_{\lambda_n}(u(\lambda_n)) + (1 - \lambda_n)J(u(\lambda_n)) \to c_k,
\]

\[
\Phi'(u(\lambda_n)) = \Phi'_{\lambda_n}(u(\lambda_n)) + (1 - \lambda_n)J'(u(\lambda_n)) \to 0.
\]

We have used the fact that \( \Phi_{\lambda_n} \), \( J \) map bounded sets into bounded sets under the assumptions of Theorem 2.2.

**Lemma 3.10.** Under the assumption of Theorem 2.2, the aforementioned \( \{ u(\lambda_n) \} \) is bounded.

**Proof.** By contradiction. We assume \( ||u(\lambda_n)|| \to \infty \) and consider \( w_n = \frac{u(\lambda_n)}{||u(\lambda_n)||} \).

Then up to a subsequence, we get that \( w_n \rightharpoonup w \) in \( E \), \( w_n \to w \) in \( L^{q(x)}(R^N) \) for \( p(x) \leq q(x) \ll p^*(x) \), \( w_n \to w \) a.e. in \( R^N \).

We first consider the case \( w \neq 0 \) in \( E \). Since \( \Phi'_{\lambda_n}(u(\lambda_n)) = 0 \), we have

\[
\int_{R^N} |Du(\lambda_n)|^{p(x)} + V(x)|u(\lambda_n)|^{p(x)}dx = \lambda_n \int_{R^N} f(x, u(\lambda_n))u(\lambda_n)dx.
\]

Assume \( ||u(\lambda_n)|| > 1 \). Dividing both sides by \( ||u(\lambda_n)||^{p^*} \), we get

\[
\int_{R^N} \frac{f(x, u(\lambda_n))u(\lambda_n)}{||u(\lambda_n)||^{p^*}}dx \leq \frac{1}{\lambda_n} \leq 1.
\]

Further, by Fatou’s Lemma and (C2), we have

\[
\int_{R^N} \frac{f(x, u(\lambda_n))u(\lambda_n)}{||u(\lambda_n)||^{p^*}}dx = \int_{R^N} \frac{f(x, u(\lambda_n))u(\lambda_n)|w_n(x)|^{p^*}}{||u(\lambda_n)||^{p^*}}dx \to \infty,
\]
a contradiction.

For the case \( w = 0 \) in \( E \), we define \( \Phi_{\lambda_n}(tu(\lambda_n)) = \max_{t \in [0, 1]} \Phi_{\lambda_n}(tu(\lambda_n)) \). Then for any \( C > 1 \), \( \overline{w_n} := \frac{C u(\lambda_n)}{||u(\lambda_n)||} \) and \( n \) large enough, we have

\[
\Phi_{\lambda_n}(tu(\lambda_n)) \geq \Phi_{\lambda_n}(\overline{w_n}) = \int_{R^N} \frac{1}{p(x)}(|CDw_n|^{p(x)} + V(x)|Cw_n|^{p(x)})dx - \lambda_n \int_{R^N} F(x, Cw_n)dx \\
\geq \lambda_n \int_{R^N} F(x, Cw_n)dx.
\]

Since \( w_n \to 0 \) a.e. in \( R^N \) and \( \lambda_n \in [1, 2] \), we have \( \lambda_n \int_{R^N} F(x, Cw_n)dx \to 0 \) as \( n \to \infty \). Since \( C \) is arbitrary, we have \( \Phi_{\lambda_n}(tu(\lambda_n)) \to \infty \) as \( n \to \infty \). Consequently,
we know $t_n \in (0, 1)$ when $n$ is large enough, which implies $\Phi'_{\lambda_n}(t_n u(\lambda_n)) t_n u(\lambda_n) = 0$. Thus,

$$\Phi_{\lambda_n}(t_n u(\lambda_n)) - \frac{1}{p} \Phi'_{\lambda_n}(t_n u(\lambda_n)) t_n u(\lambda_n) \to \infty,$$

which implies

$$\int_{\mathbb{R}^N} \left( \frac{1}{p(x)} - \frac{1}{p'} \right) \left( |t_n Du(\lambda_n)|^{p(x)} + V(x) |t_n u(\lambda_n)|^{p(x)} \right) dx + \lambda_n \int_{\mathbb{R}^N} \frac{1}{p'} f(x, t_n u(\lambda_n)) t_n u(\lambda_n) - F(x, t_n u(\lambda_n)) dx \to \infty.$$

So

$$\int_{\mathbb{R}^N} \frac{1}{p'} f(x, t_n u(\lambda_n)) t_n u(\lambda_n) - F(x, t_n u(\lambda_n)) dx \to \infty.$$

However,

$$\Phi_{\lambda_n}(u(\lambda_n)) = \Phi_{\lambda_n}(u(\lambda_n)) - \frac{1}{p'} \Phi'_{\lambda_n}(u(\lambda_n)) u(\lambda_n)$$

$$= \int_{\mathbb{R}^N} \left( \frac{1}{p(x)} - \frac{1}{p'} \right) \left( |Du(\lambda_n)|^{p(x)} + V(x) |u(\lambda_n)|^{p(x)} \right) dx + \lambda_n \int_{\mathbb{R}^N} \frac{1}{p'} f(x, u(\lambda_n)) u(\lambda_n) - F(x, u(\lambda_n)) dx$$

$$\geq \lambda_n \int_{\mathbb{R}^N} \frac{1}{p'} f(x, u(\lambda_n)) u(\lambda_n) - F(x, u(\lambda_n)) dx.$$

In view of (C4), there exist two positive constants $C_1$ and $C_2$ such that

$$\Phi_{\lambda_n}(u(\lambda_n)) \geq \lambda_n \int_{\mathbb{R}^N} \frac{1}{p} f(x, u(\lambda_n)) u(\lambda_n) - F(x, u(\lambda_n)) dx$$

$$\geq \lambda_n C_1 \int_{\mathbb{R}^N} \frac{1}{p} f(x, u(\lambda_n)) u(\lambda_n) - F(x, u(\lambda_n)) dx$$

$$\geq \lambda_n C_1 C_2 \int_{\mathbb{R}^N} \frac{1}{p} f(x, t_n u(\lambda_n)) t_n u(\lambda_n) - F(x, t_n u(\lambda_n)) dx$$

$$\geq C \int_{\mathbb{R}^N} \frac{1}{p} f(x, t_n u(\lambda_n)) t_n u(\lambda_n) - F(x, t_n u(\lambda_n)) dx \to \infty.$$

However, for each $k$ large enough, $\Phi_{\lambda_n}(u(\lambda_n)) = c_k(\lambda_n) \leq \overline{c_k} < \infty$ (See Lemman 3.8), a contradiction. \hfill \Box

**Proof of Theorem 2.2.** By now, whether $1 \in \Lambda_0$ or not, we have found a special bounded $(PS)_{c_k}$ sequence $\{u^k(\lambda_n)\}_{n=1}^\infty$ for each $c_k$ in the Fountain Theorem when $k$ is large enough. In view of Note 3.9 we know $\{u^k(\lambda_n)\}_{n=1}^\infty$ has a convergent subsequence and $c_k$ is indeed an critical level of $\Phi$ and Theorem 2.2 follows. \hfill \Box

**Remark 3.11.** We prove Theorem 2.2 in such a way because we want to emphasize the procedure of finding critical points. First, we consider the original functional and verify the functional satisfies some geometry properties (e.g. Mountain Pass Geometry in [24], Fountain geometry in this paper, general linking geometry, etc) to ensure prospective critical levels. Then, we consider our functional as a member in a family of functionals. Some given structure conditions on the family yield bounded
(PS) sequences for almost all the functionals. Using the information supplied by these functionals, we find special bounded (PS) sequences for those prospective critical levels. At last, we prove the special (PS) sequences we find converge to critical points respectively up to subsequences.

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