On the Caginalp for a Conserve Phase-Field with a Polynomial Potential of Order $2p - 1$

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Abstract

Our aim in this paper is to study on the Caginalp for a conserved phase-field with a polynomial potential of order $2p - 1$. In this part, one treats the conservative version of the problem of generalized phase field. We consider a regular potential, more precisely a polynomial term of the order $2p - 1$ with edge conditions of Dirichlet type. Existence and uniqueness are analyzed. More precisely, we precisely, we prove the existence and uniqueness of solutions.

Keywords

A Conserved Phase-Field, Polynomial Potential of Order $2p - 1$, Dirichlet Boundary Conditions, Maxwell-Cattaneo Law

1. Introduction

The Caginalp phase-field model

\[
\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \tag{1}
\]

\[
\frac{\partial \theta}{\partial t} - \Delta \theta = \frac{\partial u}{\partial t} \tag{2}
\]

proposed in [1], has been extensively studied (see, e.g., [2]-[7] and [8]). Here, $u$ denotes the order parameter and $\theta$ the (relative) temperature.

Furthermore, all physical constants have been set equal to one. This system models, e.g., melting-solidification phenomena in certain classes of materials.

The Caginalp system can be derived as follows. We first consider the (total) free energy

\[
\psi(u, \theta) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + f(u) - u\theta - \frac{1}{2} \theta^2 \right) \, dx, \tag{3}
\]
where \( \Omega \) is the domain occupied by the material.

We then define the enthalpy \( H \) as

\[
H = - \frac{\partial \psi}{\partial \theta}
\]

where \( \partial \) denotes a variational derivative, which gives

\[
H = u + \theta.
\]

The governing equations for \( u \) and \( \theta \) are then given by (see [9])

\[
\frac{\partial u}{\partial t} = - \frac{\partial \psi}{\partial u},
\]

\[
\frac{\partial H}{\partial t} + \text{div} q = 0,
\]

where \( q \) is the thermal flux vector. Assuming the classical Fourier Law

\[
q = -\nabla \theta,
\]

we find (1) and (2).

Now, a drawback of the Fourier Law is the so-called “paradox of heat conduction”, namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see, e.g. [10] and [11]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo Law.

\[
\frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \Delta \theta = 0,
\]

This model can also be derived by considering, as in [12] (see also [13]-[20]), the Caginalp phase-field model with the so-called Gurtin-Pipkin Law

\[
q(t) = -\int_0^\infty k(s)\nabla \theta(t-s)ds.
\]

for an exponentially decaying memory kernel \( k \), namely,

\[
k(s) = e^{-s}.
\]

Indeed, differentiating (11) with respect to \( t \) and integrating by parts, we recover the Maxwell-Cattaneo Law (9).

Now, in view of the mathematical treatment of the problem, it is more convenient to introduce the new variable

\[
\alpha = \int_0^t \theta(s)ds, \quad \theta = \frac{\partial \alpha}{\partial t},
\]

and we have, integrating (10) with respect to \( s \in [0,1] \).
\[ \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = \frac{\partial u}{\partial t} \]  

(14)

where

\[ \alpha(t,x) = \int_0^t T(\tau,x) \, d\tau + \alpha_0(x) \]  

(15)

is the conductive thermal displacement. Noting that \( T = \frac{\partial \alpha}{\partial t} \), we finally deduce from (33) and (36)-(37) the following variant of the Caginalp phase-field system (see [17]):

\[ \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t} \]  

(16)

\[ \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \]  

(17)

In this paper, we consider the following conserved phase-field model:

\[ \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} \]  

(18)

\[ \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \]  

(19)

These equations are known as the conserved phase-field model (see [21]-[30]) based on type II heat conduction and with two temperatures (see [3] and [4]), conservative in the sense that, when endowed with Neumann boundary conditions, the spatial average of the order parameter is a conserved quantity. Indeed, in that case, integrating (18) over the spatial domain \( \Omega \), we have the conservation of mass,

\[ \langle u(t) \rangle = \langle u(0) \rangle, \quad t \geq 0 \]  

(20)

\[ \langle \rangle = \frac{1}{vol\Omega} \int_\Omega dx \]  

(21)

denotes the spatial average. Furthermore, integrating (19) over, we obtain

\[ \langle \alpha(t) \rangle = \langle \alpha(0) \rangle, \quad t \geq 0 \]  

(22)

Our aim in this paper is to study the existence and uniqueness of solution of (17)-(39). We consider here only one type of boundary condition, namely, Dirichlet (see [31] [32] [33]).

2. Setting of the Problem

We consider the following initial and boundary value problem

\[ \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \frac{\partial \alpha}{\partial t} \]  

(23)

\[ \frac{\partial^2 \alpha}{\partial t^2} + \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial u}{\partial t} \]  

(24)

\[ u \big|_t = \Delta u \big|_t = \alpha \big|_t = 0, \quad \text{on } \partial \Omega, \]  

(25)
As far as the nonlinear term $f$ is concerned, we assume that

$$f \in C^\infty(R), f(0) = 0$$

Consider the following polynomial potential of order $2p - 1$

$$f(s) = \sum_{i=1}^{2p+1} a_i s^i, p \in \mathbb{N}^*, p \geq 2, a_{2p-1} = 2 \mathbb{p}_{2p} \geq 0$$

The function $f$ satisfies the following properties

$$\frac{1}{2} a_{2p-1} s^{2p} - c_i \leq f(s) s \frac{3}{2} a_{2p-1} s^{2p} + c_i,$$

$$f'(s) \geq \frac{1}{2} a_{2p-1} s^{2p-1} - c_2 \geq -k, \forall s \in R, k \geq 0$$

where

$$F(s) = \int_0^s f(\tau) d\tau$$

such as

$$\frac{1}{4p} a_{2p-1} s^{2p} - c_3 \leq F(s) \leq \frac{3}{4p} a_{2p-1} s^{2p} + c_3$$

**Remark 2.1.** We take here, for simplicity, Dirichlet Boundary Conditions. However, we can obtain the same results for Neumann Boundary Conditions, namely,

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} \text{ on } \Gamma$$

where $\nu$ denotes the unit outer normal to $\Gamma$. To do so, we rewrite, owing to (23) and (24), the equations in the form

$$\frac{\partial u}{\partial t} + \Delta^2 \varphi - \Delta\left(f(u) - \langle f(u) \rangle\right) = -\Delta \frac{\partial \varphi}{\partial t}$$

$$\frac{\partial \varphi}{\partial \nu} = \frac{\partial \varphi}{\partial \nu} - \Delta \varphi = \frac{\partial u}{\partial \nu},$$

where $\varphi = v - \langle v \rangle$, $\|v_0\| \leq M_1$, $\|v_0\| \leq M_2$, for fixed positive constants $M_1$ and $M_2$. Then, we note that

$$v \rightarrow \left(\|(-\Delta)^{-1} v\|^2 + \langle v \rangle^2\right)^{\frac{1}{2}}$$

where, here, $-\Delta$ denotes the minus Laplace operator with Neumann boundary conditions and acting on functions with null average and where it is understood that

$$\langle \cdot \rangle = \frac{1}{\text{vol}(\Omega)}\langle \cdot \rangle_{H^{-1}(\Omega), H^0(\Omega)}$$

Furthermore


\[ v \mapsto \left( \|v\|^2 + \langle v, v \rangle \right)^{\frac{1}{2}}, \]

\[ v \mapsto \left( \|\nabla v\|^2 + \langle v, v \rangle \right)^{\frac{1}{2}}, \]

\[ v \mapsto \left( \|\Delta v\|^2 + \langle v, v \rangle \right)^{\frac{1}{2}} \]

are norms in \( H^{-1}(\Omega), \ L^2(\Omega), \ H^1(\Omega) \) and \( H^2(\Omega) \), respectively, which are equivalent to the usual ones. We further assume that

\[ f(s) \leq \varepsilon F(s) + c_s, \ \forall \varepsilon > 0, \ s \in \mathbb{R}, \]

which allows to deal with term \( f(u) \).

### 3. Notations

We denote by \( \|\cdot\| \) the usual \( L^2 \)-norm (with associated product scalar \( \langle \cdot, \cdot \rangle \) and set \( \|\cdot\|_1 = \left\| (-\Delta)^{-\frac{1}{2}} \cdot \right\| \), where \(-\Delta\) denotes the minus Laplace operator with Dirichlet Boundary Conditions. More generally, \( \|\cdot\|_X \) denote the norm of Banach space \( X \).

Throughout this paper, the same letters \( c_1, c_2 \) and \( c_3 \) denote (generally positive) constants which may change from line to line, or even a same line.

### 4. A Priori Estimates

The estimates derived in this subsection will be formal, but they can easily be justified within a Galerkin scheme. We rewrite (23) in the equivalent form

\[ (-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial \alpha}{\partial t}. \]

We multiply (35) by \( \frac{\partial u}{\partial t} \) and have, integrating over \( \Omega \) and by parts;

\[ \frac{d}{dt} \left( \|\nabla u\| + 2 \int_\Omega F(u) \, dx \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|_1 = 2 \left( \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \]

We then multiply (24) by \( \frac{\partial \alpha}{\partial t} \) to obtain

\[ \frac{d}{dt} \left( \|\nabla \alpha\| + \left\| \frac{\partial \alpha}{\partial t} \right\| \right) + 2 \left\| \frac{\partial \alpha}{\partial t} \right\| = -2 \left( \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \]

Summing (36) and (37), we find the differential inequality of the form

\[ \frac{d}{dt} \left( \|\nabla u\|^2 + 2 \int_\Omega F(u) \, dx + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 = 0 \]

Integrating from 0 to \( t \) with \( t \in [0; T] \) we obtain
\[
\int_0^1 \left( \frac{d}{dt} \| \nabla u \|^2 + 2 \int_0^1 F(u) \, dx + \| \nabla \alpha(s) \|^2 + \left\| \frac{\partial \alpha(s)}{\partial t} \right\|^2 \right) \, ds \\
+ 2 \int \left\| \frac{\partial \alpha(s)}{\partial t} \right\|^2 \, ds + 2 \int \left\| \frac{\partial \alpha(s)}{\partial t} \right\|^2 \, ds = 0
\]

of (35) we deduce
\[
F(u_0) \leq \frac{3}{4} a_{2p-1} u_0^{2p} + c_3
\]
which involves
\[
2 \int_0^1 F(u_0) \, dx \leq \frac{3}{2p} a_{2p-1} \| u_0 \|_{L^p}^{2p} + 2c_3 |\Omega|
\]
still of (35) we have
\[
\frac{3}{4} a_{2p-1} u_0^{2p} - c_3 \leq F(u)
\]
which involves
\[
\frac{1}{2p} a_{2p-1} \| u_0 \|_{L^p}^{2p} - 2c_3 |\Omega| \leq F(u)
\]
where
\[
E(t) + 2 \int_0^t \left( \left\| \frac{\partial \alpha(s)}{\partial t} \right\|^2 + \left\| \frac{\partial \alpha(s)}{\partial t} \right\|^2 \right) \, ds \leq C
\]
with
\[
E(t) = \| \nabla u(t) \|^2 + \frac{1}{2p} a_{2p-1} \| u_0 \|_{L^p}^{2p} + \| \alpha_0 \|^2 + \| \nabla \alpha_0 \|^2 + C_3
\]
and \( C = \| \nabla u_0 \|^2 + \frac{3}{2p} a_{2p-1} \| u_0 \|_{L^p}^{2p} + \| \alpha_0 \|^2 + \| \nabla \alpha_0 \|^2 + C_3 \).

Finally, we conclude that
\[
u \in L^\infty \left( R^d; H_0^1(\Omega) \cap L^p(\Omega) \right) \cap H^{1,p}(\Omega) \cap L^2(\Omega) \cap L^2(0,T;H^{-1}(\Omega));
\]
\[
\frac{\partial u}{\partial t} \in L^2 \left( 0,T;H^{-1}(\Omega) \right); \frac{\partial \alpha}{\partial t} \in L^\infty \left( R^d; L^2(\Omega) \right) \cap L^2 \left( 0,T;L^2(\Omega) \right) \forall T > 0
\]

**Theorem 4.1.** (Existence) We assume
\[
(u_0, \alpha_0, \alpha_t) \in \left( H_0^1(\Omega) \cap L^p(\Omega) \right) \times H^1(\Omega) \times L^2(\Omega) \]
then the system (18)-(19) possesses at least one solution \((u, \alpha)\) such that
\[
u \in L^\infty \left( R^d; H_0^1(\Omega) \cap L^p(\Omega) \right) \cap H^{1,p}(\Omega) \cap L^2(\Omega) \cap L^2(0,T;H^{-1}(\Omega));
\]
\[
\frac{\partial u}{\partial t} \in L^2 \left( 0,T;H^{-1}(\Omega) \right); \frac{\partial \alpha}{\partial t} \in L^\infty \left( R^d; L^2(\Omega) \right) \cap L^2 \left( 0,T;L^2(\Omega) \right) \forall T > 0
\]

**Theorem 4.2.** (Uniqueness) Let the assumptions of Theorem 4.1 hold. Then, the system (18)-(19) possesses a unique solution \((u, \alpha)\) such that
Let \( u \) be a solution to the equation \( \frac{\partial u}{\partial t} + \Delta u - \Delta \left( f \left( u \right) - f \left( u^2 \right) \right) = 0 \) on \( \Gamma \) with initial data \( u(0) = u_0 \). Then, we have \( u \) satisfies the equation \( \frac{\partial u}{\partial t} + \Delta u - \Delta \left( f \left( u \right) - f \left( u^2 \right) \right) = 0 \) on \( \Gamma \) with initial data \( u(0) = u_0 \).
which involves
\[
\left\| f(u') - f(u^2) \right\| \leq \sum_{k=1}^{2p-1} |a_k| \left\| u^{(i)} - u^{(2)} \right\| + \frac{1}{k-1} \sum_{j=1}^{k-2} \left\| u^{(i)} \right\|^{k-j} + \left\| u^{(2)} \right\|^{k-1}.
\]

Based on Young’s inequality, we have
\[
\sum_{j=1}^{k-2} \left| u^{(i)} \right|^{k-j} \left| u^{(2)} \right|^j \leq \left( k-1 \right) \left| u^{(i)} \right|^{k-1} + \frac{1}{k-1} \sum_{j=1}^{k-2} \left( k-1 \right) \left| u^{(2)} \right|^{k-1}.
\]

As
\[
\sum_{j=1}^{k-2} j = \frac{(k-2)(k-1)}{2}
\]
then
\[
\sum_{j=1}^{k-2} \left| u^{(i)} \right|^{k-j} \left| u^{(2)} \right|^j \leq \left( k-2 \right) \left| u^{(i)} \right|^{k-1} + \frac{k-2}{2} \left| u^{(2)} \right|^{k-1} - \frac{k-2}{2} \left| u^{(i)} \right|^{k-1}
\]
\[
\leq \frac{k-2}{2} \left( \left| u^{(i)} \right|^{k-1} + \left| u^{(2)} \right|^{k-1} \right).
\]

We know that
\[
\forall k \in N \; ; \; k - 2 \leq k \; \text{then} \; \frac{k-2}{2} \leq \frac{k}{2} \leq k
\]
\[
\sum_{j=1}^{k-2} \left| u^{(i)} \right|^{k-j} \left| u^{(2)} \right|^j \leq k \left( \left| u^{(i)} \right|^{k-1} + \left| u^{(2)} \right|^{k-1} \right)
\]
which gives
\[
\left\| f(u') - f(u^2) \right\| \leq \sum_{k=1}^{2p-1} |a_k| \left\| u^{(i)} - u^{(2)} \right\| \left( (k+1) \left| u^{(i)} \right|^{k-1} + (k+1) \left| u^{(2)} \right|^{k-2} \right)
\]
\[
\leq |u| \sum_{k=1}^{2p-1} \left( k+1 \right) \left| a_k \right| \left( \left| u^{(i)} \right|^{k-1} + \left| u^{(2)} \right|^{k-1} \right).
\]

\exists k > 0 \; \text{such as}
\[
(k+1) |a_k| \leq k \; ; \; \forall k \in 1, 2, \cdots, 2p-1
\]
so
\[
\left\| f(u') - f(u^2) \right\| \leq |u| k \sum_{k=1}^{2p-1} \left( \left| u^{(i)} \right|^{k-1} + \left| u^{(2)} \right|^{k-1} \right).
\]

Based on Young’s inequality, we have \forall k \geq 2
\[
\left| u^{(i)} \right|^{k-1} \leq \frac{k-1}{2p-2} \left( \left| u^{(i)} \right|^{k-1} + \left| u^{(2)} \right|^{k-1} \right).
\]

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and
\[
|u^{(2)}|^{\sigma-1} \leq \frac{k-1}{2p-2} \left( |u^{(0)}|^{\sigma-1} \right)^{\frac{2p-2}{k-1}} + \frac{2p-k-1}{2p-2}
\]
that involve
\[
\left| f(u') - f(u^2) \right| \leq |u| \frac{k}{2p-2} \sum_{i=1}^{2p-2} \left( (k-1) \left( |u^{(0)}|^{p-2} + |u^{(2)}|^{p-2} \right) + 2 \left( \frac{2p-k-1}{2p-2} \right) \right)
\]

\[
\leq c |\varepsilon| \left( |u^{(0)}|^{p-2} + |u^{(2)}|^{p-2} + 1 \right).
\]

We finally
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq c \int_\Omega |\varepsilon| \left( |u^{(0)}|^{p-2} + |u^{(2)}|^{p-2} + 1 \right) \left\| \frac{\partial u}{\partial t} \right\| dx.
\] (47)

The second member of (45) is increased in \( R^n \) for \( n = 1, 2, 3 \).

If \( n = 1 \); \( u' \in H^1_0(\Omega) \subset H^1(\Omega) = W^{1,2}(\Omega) \) for \( i = 1, 2 \).

Thanks to the continuous injection \( H^1(\Omega) \subset C(\overline{\Omega}) \), then \( C > 0 \), by applying Holder’s inequality, we get
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\] (48)

which involves using the compact injection \( H^1(\Omega) \subset L^2(\Omega) \), we have
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

If \( n = 2 \) then \( H^1(\Omega) \subset L^q(\Omega) \), \( \forall q \in [1, \infty[ \).

Based on Holder’s inequality, we have
\[
\int_\Omega |u| \left( |u^{(0)}|^{p-2} + |u^{(2)}|^{p-2} + 1 \right) \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

Finally
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

If \( n = 3 \), then \( H^1(\Omega) \subset L^q(\Omega) \) with \( q \in [1, 6] \).

In this case, we also
\[
\int_\Omega |u| \left( |u^{(0)}|^{p-2} + |u^{(2)}|^{p-2} + 1 \right) \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

So
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

We notice that in \( R^n \) for \( n = 1, 2, 3 \), we have
\[
\int_\Omega |f(u') - f(u^2)| \left\| \frac{\partial u}{\partial t} \right\| dx \leq C |\varepsilon| \left\| \frac{\partial u}{\partial t} \right\|
\]

Using Young’s inequality, we have
Inserting (49) into (46), we find
\[
\frac{d}{dt} E_2 + 2 \left\| \frac{\partial u}{\partial t} \right\|^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \leq C \left\| u \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2
\]
and recalling the interpolation inequality
\[
\left\| \nabla u \right\|^2 \leq C \left( \left\| \nabla u \right\|^2 + \left\| \nabla \alpha \right\|^2 \right)
\]
with \( E_2 = \left\| \nabla u \right\|^2 + \left\| \nabla \alpha \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \)

Finally
\[
\frac{d}{dt} E_2 + c \left\| \frac{\partial u}{\partial t} \right\|^2 + 2 \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \leq CE_2, \quad C > 0
\]

**Theorem 4.3.** (Second theorem of the solution’s existence) The existence and uniqueness of the solution of the (23)-(25) problem being proven, now we seek the solution of (23)-(25) with more regularity.

Assume
\[
(u_0, a_0) \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega)
\times (u_0, a_0) \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^p(\Omega) \times H^1_0(\Omega),
\]
then the (23)-(24) system admits a unique \((u, \alpha)\) solution such as
\[
u \in L^\infty \left(0, T; H^2(\Omega) \cap H^1_0(\Omega) \right), \alpha \in L^\infty \left(0, T; H^2(\Omega) \cap H^1_0(\Omega) \right),
\]
\[
\frac{\partial \alpha}{\partial t} \in L^\infty \left(0, T; H^2(\Omega) \cap H^1_0(\Omega) \right) \cap L^\infty \left(0, T; H^2(\Omega) \cap H^1_0(\Omega) \right),
\]
and
\[
\frac{\partial u}{\partial t} \in L^2 \left(0, T; H^{-1}(\Omega) \right)
\]

Theorems of existence (23) and uniqueness (24) being proven then
\[
\frac{\partial \alpha}{\partial t} \in L^\infty \left(0, T; H^2(\Omega) \cap H^1_0(\Omega) \right) \cap L^\infty \left(0, T; L^2(\Omega) \right) \quad \text{and} \quad \frac{\partial u}{\partial t} \in L^\infty \left(0, T; H^{-1}(\Omega) \right), \quad \forall T > 0.
\]

We multiply (23) by \((-\Delta)^{-1} \frac{\partial u}{\partial t}\) and have, integrating over \(\Omega\), we have
\[
\frac{d}{dt} \left( \left\| \nabla u \right\|^2 + 2 \int_\Omega F(u) \, dx \right) + 2 \frac{\partial u}{\partial t} \frac{\partial \alpha}{\partial t} = 2 \left( \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right)
\]

Multiplying (24) by \(\frac{\partial \alpha}{\partial t}\), we have
\[
\frac{d}{dt} \left( \left\| \nabla \alpha \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \frac{\partial \alpha}{\partial t} \frac{\partial \alpha}{\partial t} = -2 \left( \frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t} \right)
\]

Now summing (51) and (52) we obtain
\[
\frac{d}{dt} \left( \left\| \nabla u \right\|^2 + 2 \int_\Omega F(u) \, dx + \left\| \nabla \alpha \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) + 2 \frac{\partial u}{\partial t} \frac{\partial \alpha}{\partial t} + 2 \frac{\partial \alpha}{\partial t} \frac{\partial \alpha}{\partial t} = 0
\]
where

\[ E_2 = \|\nabla u\|^2 + 2 \int_{\Omega} F(u) \, dx + \|\nabla \alpha\|^2 + \frac{\partial \alpha}{\partial t} \int_{\Omega} \]
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