Asymptotic enumeration by Khintchine-Meinardus probabilistic method: Necessary and sufficient conditions for sub exponential growth.

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Abstract

In this paper we prove the necessity of the main sufficient condition of Meinardus for sub exponential rate of growth of the number of structures, having multiplicative generating functions of a general form and establish a new necessary and sufficient condition for normal local limit theorem for aforementioned structures. The latter result allows to encompass in our study structures with sequences of weights having gaps in their support.

Keywords: Asymptotic enumeration- Generating function- Partitions-Local limit theorem.

Mathematical subject Classification
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I. Introduction and Mathematical setting

The present work was motivated by papers [8] and [10]-[12] coauthored respectively with Gregory Freiman and Dudley Stark, and by the paper [25], by Yifan Yang. Our objective in this paper is the asymptotic behavior, as

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\( n \to \infty \), of the quantity \( c_n \) depicting the number of combinatorial structures of size \( n \). The mathematical setting below is based on [12]. We consider throughout combinatorial objects that decompose into sets of simpler objects called irreducibles, or primes or connected components. Roughly speaking, the present paper proves the necessity of the main Meinardus’ sufficient condition for the sub exponential rate of growth of \( c_n, n \to \infty \) and establishes quite new necessary and sufficient conditions for the normal local limit theorem. The latter allows to encompass in our study structures with sequences of weights having gaps in their support.

The paper contains four sections. In Section I we provide a background for our study and formulate the mathematical setting, in Section II we state the results, which are proven in Section III. The final Section IV contains three examples that hint on perspectives for future research.

Let \( f \) be a generating function of a nonnegative sequence \( \{c_n, n \geq 0, c_0 = \} \):

\[
  f(z) = \sum_{n \geq 0} c_n z^n, \quad |z| < 1,
\]

with radius of convergence 1 and such that

\[
  \lim_{z \to 1^-} f(z) = \infty.
\]

The assumption (2) implies

\[
  \sum_{n \geq 0} c_n = \infty,
\]

which is a necessary condition for \( c_n \to \infty, n \to \infty \), the property that features the models considered in this paper.

Our study is restricted to structures (=models) with generating functions \( f \) of the following multiplicative form:

\[
  f = \prod_{k \geq 1} S_k.
\]

It is appropriate to note that the infinite product in (4) often conforms to \( q \)-series common in number theory (see [1], [16]). In [16] an algorithm was suggested for derivation the asymptotics of \( c_n \) in the case of \( q \)-series, under assumption that factors of the product have asymptotics of exponential type.
We assume that the functions $S_k$, $k \geq 1$ in (4) have the following Taylor expansions:

$$S_k(z) = \sum_{j \geq 0} d_k(j) z^k, \quad d_k(j) \geq 0, \quad j \geq 0, \quad k \geq 1.$$  \hspace{1cm} (5)

By virtue of (11) and (4), the radius of convergence of each one of the series for $S_k$ in (5) should be $\geq 1$. In the literature one can find examples of multiplicative combinatorial structures with radius of convergence of $S_k$ ranging from 1 to $\infty$ (for references see [3]).

The above setting induces a sequence of multiplicative probability measures (=random structures) $\mu_n$, $n \geq 1$ on the sequence of sets $\Omega_n$, $n \geq 1$ of integer partitions of $n$, such that $c_n$ is a partition function of the measure $\mu_n$ (for more details see [12]). The multiplicative measures were introduced in the seminal paper of Vershik [23] in which he investigated a variety of problems related to limit shapes of the measures $\mu_n$ for classical models of statistical mechanics. Our subsequent asymptotic analysis of $c_n$, as $n \to \infty$, is based on the probabilistic representation of $c_n$, derived by Khintchine in [15] for classical models of statistical mechanics and then extended in [10], [11], [12] to the above defined multiplicative models. Khintchine’s representation of $c_n$, $n \geq 1$, which is identity in the free parameter $\delta > 0$, reads as follows:

$$c_n = e^{n\delta} f_n(e^{-\delta}) \mathbb{P}(Z_n = n), \quad n \geq 1,$$  \hspace{1cm} (6)

where

$$f_n = \prod_{k=1}^{n} S_k$$  \hspace{1cm} (7)

is the $n$- truncation of the generic generating function $f$, and

$$Z_n := \sum_{k=1}^{n} Y_k, \quad n \geq 1,$$  \hspace{1cm} (8)

where $Y_k$ are independent integer-valued random variables with distributions derived from (4) and (5) by setting $z = e^{-\delta}$, $\delta > 0$:

$$\mathbb{P}(Y_k = jk) = \frac{d_k(j) e^{-\delta kj}}{S_k(e^{-\delta})}, \quad j \geq 0, \quad k \geq 1.$$  \hspace{1cm} (9)

It is clear from (9) that the representation (6) is valid if and only if in (5) the coefficients $d_k(j) \geq 0$, $k \geq 1, j \geq 0$. 

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Khintchine’s asymptotic method is based on the choice of the free parameter $\delta$ in the representation (3) as a solution, denoted $\delta_n$, of the following equation (=Khintchine’s equation):

$$\left(-\log \mathcal{F}(\delta)\right)' = n,$$

(10)

where $\mathcal{F}(\delta) := f(e^{-\delta})$, $\delta > 0$. Our subsequent study is devoted to a special class of multiplicative models defined as follows.

**Definition** A multiplicative model (4) is called exponential if under $z = e^{-\tau}$, $\Re(\tau) = \delta > 0$ its generating function $\mathcal{F}(\delta)$ has the expansion:

$$\mathcal{F}(\tau) = \exp \left( \sum_{l=0}^{r} h_l \tau^{-\rho_l} - A_0 \log \tau + \Delta(\tau) \right),$$

(11)

where

- $r$ is a given integer;
- $\Delta(\tau) < \infty, \tau \in \mathbb{C}$ is the remainder term that admits expansion into Taylor series, converging in $\tau \in \mathbb{C}$;
- $h_l > 0, l = 1, \ldots, r$ and $h_0, A_0$ are real constants, while $\rho_0 = 0$ and $0 < \rho_1 < \ldots < \rho_r$ are positive powers.

**Proposition 1** The Khintchine’s equation (11) has a unique solution $\delta = \delta_n > 0$ for all $n$ sufficiently large, where

$$\delta_n \to 0, \ n \to \infty.$$  

(12)

Moreover, if the model is exponential, then

$$n\delta_n \to \infty, \ n \to \infty.$$  

(13)

**Proof.** We have

$$\left(-\log \mathcal{F}(\delta)\right)' = \frac{\sum_{k=0}^{\infty} kc_k e^{-k\delta}}{\sum_{k=0}^{\infty} c_k e^{-k\delta}}.$$  

(14)

The nominator and the denominator of the fraction in (14) tend to $+\infty$, as $\delta \to 0^+$, by virtue of the assumption (2), and it is easy to see that the
fraction itself also tends to $+\infty$, as $\delta \to 0^+$. Differentiating w.r.t. $\delta$ the RHS of (14) and then applying the Cauchy-Shwartz inequality gives

$$\left(-\log F(\delta)\right)^{''} < 0, \text{ for all } \delta > 0,$$

from which we derive that the fraction in (14) decreases in $\delta > 0$, from $+\infty$ to 0. This proves the existence and the uniqueness of the solution $\delta_n$, as well as (12). For the proof of (13), we use (11) with $\tau = \delta$, and the fact that in (11), $\rho_r > \rho_{r-1} > \ldots > \rho_1 > 0$, to rewrite the equation (10) in the case of exponential models as

$$r h_r \delta_n^{-r-1} \sim n, \ n \to \infty.$$

From the above, (13) follows immediately. ■

Khintchine ([15],p.160) showed that the solution $\delta = \delta_n$ of (10) is the point of minimum of the entropy of the corresponding model of statistical mechanics (see also [11] for some more details).

**Historical remark.** In the present paper, as well as in [10]-[12] a combination of Khinchine and Meinardus’ asymptotic analysis is employed. It is interesting to understand the interplay between the two methods that originated absolutely independently from each other, in the 50-s of the past century. Khintchine’s objective in [15] was calculation of mathematical expectations (rather than $c_n$), with respect to the above measure $\mu_n$, of such quantities common in statistical mechanics, as occupation numbers. Occupation numbers depict numbers of particles that are at a certain energy level $l$, $l \geq 1$. The expectations of occupation numbers can be expressed as functions of the ratios $c_{\lambda} / c_n$, which by virtue of the representation (6) do not depend, as $\delta \to 0$, $n \to \infty$, on the second factor in (6). Because of it, Khinchine did not need the asymptotic analysis of the second factor in (6). The latter asymptotic analysis (for the case of weighted partitions) was developed by Meinardus (see [1]) who proposed to use the Mellin transform. From the other hand, Meinardus used complicated technique of the saddle point method that was replaced by Khintchine with his elegant local limit theorem approach. ■

As in [12], we restrict the study to functions $S_k, \ k \geq 1$ of the specific form

$$S_k(z) = \left(S(a_k z^k)\right)^{b_k},$$

(17)
where the series
\[ S(z) = \sum_{j=0}^{\infty} d_j z^j, \quad d_j \geq 0, \quad j \geq 0 \] (18)
has a radius of convergence \( \geq 1 \) and where \( 0 < a_k \leq 1, \quad b_k \geq 0, \quad k \geq 1 \)
are given sequences of the two parameters of the model. Note that in (18) \( d_0 = 1 \), by virtue of (1), (4) and the fact that \( c_0 = 1 \).

In [12] it was described the combinatorial meaning of the parameters \( a_k, b_k \). In the sequel of this section we mention another interpretation of these parameters.

For multiplicative models, the aforementioned setting is also used for the study of another asymptotic problem, which is a limit shape, the topic which has a rich history. We mention below two recent papers on limit shapes. Developing the work [23], Yakubovich [24] derived limit shapes for models (17) in the case \( a_k = 1, \quad k \geq 1 \), under some analytic conditions on the function \( S \) and on the parameters \( b_k, \quad k \geq 1 \), while Bogachev in [4] developed a unified approach to derivation of limit shapes in the case of equiweighted parts: \( b_k = b > 0, \quad k \geq 1 \). In the sequel of the present paper, some other links to research on limit shapes will be indicated.

To simplify the exposition, we make the additional assumption on \( a_k \):
\[ a_k > 0, \quad k \geq 1. \] (19)

We note that the assumption \( d_j \geq 0, \quad j \geq 0 \) in (18) is not sufficient for \( c_n \geq 0, \quad n \geq 1 \). In fact, in the case of weighted partitions with distinct parts, we have \( S(z) = 1 + z \), so that not all coefficients of the binomial series \( (S(a_k z^k))^{b_k}, \quad k \geq 1 \) are nonnegative, unless \( b_k, \quad k \geq 1 \) are integers. In particular, in the case \( a_k = 1, \quad k \geq 1 \), it is not difficult to check that \( c_2 = \frac{b_1 (b_1 - 1)}{2} + b_2 < 0 \), for some values of \( 0 < b_1 < 1, \quad b_2 > 0 \). In this connection recall that for unrestricted weighted partitions, the property \( c_n \geq 0, \quad n \geq 1 \) holds for all \( b_k \geq 0, \quad k \geq 1 \).

By the assumptions made, \( \log S(z) \) can be expanded as
\[ \log S(z) = \sum_{j=1}^{\infty} \xi_j z^j, \] (20)
with the radius of convergence \( \geq 1 \). In view of [11], the function \( S(z) \), in the case of an exponential model, may have zeros and singularities on the unit circle \( |z| = 1 \) only. We also point that \( \xi_j \geq 0, \quad j \geq 1 \) is a necessary and
sufficient condition for $c_n \geq 0$, $n \geq 0$ to hold under all $b_k \geq 0$, $k \geq 1$, which explains the aforementioned dichotomy between unrestricted partitions and partitions into distinct parts.

We assume further on that all singular points of $S(z)$, if they exist, are poles $z_0 : |z_0| = 1$, which means that

$$S(z) \sim \frac{L(z)}{(z - z_0)^l}, \quad z \to z_0 : |z| < 1,$$

with a given integer $l \geq 1$ and with a function $L$ analytic in the unit disk and such that $0 < |L(z)| < \infty$, $|z| \leq 1$. The assumption (21) conforms to the one by Yakubovich in [24], in the particular case $z_0 = 1$ and $L$ is a slowly varying function. Assumption (21) extends our study to models with more general $S(z)$, e.g. $S(z) = \frac{1+z}{1-z}$ with an integer $p \geq 1$ and $|z| \leq 1$.

In connection with (21), it is appropriate to recall Vivanti-Pringsheim theorem (see e.g. [14]) which says that if the radius of convergence of the series (18) for $S(z)$ equals to 1, then the assumption $d_j \geq 0$, $j \geq 0$ implies that $z = 1$ is a singular point of the series (18).

**Remark** The example $S(z) = \exp\left(\frac{1}{1-z}\right)$, $|z| < 1$ demonstrates that a non-regular growth of $S(z)$, $z \to 1^-$ may lead to a non-exponential growth of $c_n$, $n \to \infty$.

Setting $\tilde{b}_k = lb_k$ and $\tilde{L}(z) = (L(z))^{1/l}$ in (17) we may assume without loss of generality that $l = 1$ in (21).

By (17) and (20), the following expansion of $\log f(z)$ is valid:

$$\log f(z) = \sum_{k \geq 1} b_k \log S(a_k z^k) := \sum_{k \geq 1} \Lambda_k z^k, \quad |z| < 1,$$

with

$$\Lambda_k = \sum_{j | k} b_j a_j^{k/j} \xi_{k/j}.$$

**Remark** In view of (22), the function $\log f(z)$ has a pattern of a harmonic sum with base functions $\log S(a_k z^k)$, $k \geq 1$, amplitudes $b_k$, $k \geq 1$ and frequencies $a_k$, $k \geq 1$. Harmonic sums are widely applied in computer science. For more details see [7] which studies the asymptotics of harmonic sums with the help of Mellin transform.

By virtue of (23), the Dirichlet generating function $D$ for the sequence $\Lambda_k$, $k \geq 1$

$$D(s) = \sum_{k=1}^{\infty} \Lambda_k k^{-s},$$

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can be written as the double Dirichlet series:

\[ D(s) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} b_k \xi_j \alpha_k^j (jk)^{-s}. \]  

(25)

It is known that if \( D(s) \) converges in the half-plane \( \Re(s) > \rho_r \), for some \( \rho_r > 0 \), then \( D(s) \) converges absolutely for \( \Re(s) > \rho^* \), \( \rho_r < \rho^* < \rho_r + 1 \). This amounts to say that \( \rho_r \) is the rightmost pole of the Dirichlet series \( D(s) \) and that \( \Lambda_k = o(k^{\rho_r}) \), \( k \to \infty \).

In the particular case, \( a_k \equiv a \), \( 0 < a \leq 1 \) the function \( D(s) \) can be factored as

\[ D(s) = D_b(s) D_{(\xi,a)}(s), \]

where

\[ D_{(\xi,a)}(s) = \sum_{j=1}^{\infty} a^j \xi_j j^{-s} \]

and

\[ D_b(s) = \sum_{k=1}^{\infty} b_k k^{-s}. \]

(27)

In the general case, consider the Dirichlet generating function for the sequence \{\( \xi_j a_k^j, j \geq 1, k \) is fixed\}:

\[ D_{(\xi,a_k)}(s) := \sum_{j \geq 1}^{\infty} \frac{\xi_j a_k^j}{j^s}, \quad 0 < a_k \leq 1, \quad k \geq 1, \]

which allows to rewrite (25) as

\[ D(s) = \sum_{k \geq 1} b_k \frac{D_{(\xi,a_k)}(s)}{k^s}. \]

(29)

We will assume throughout the rest of the paper that the function \( D(s) \), \( s = \sigma + it \) is of finite order in the whole domain of its definition, which means (see e.g. [21]) that

\[ D(s) = O(|t|^C), \quad t \to \infty, \]

(30)

for some constant \( C > 0 \), uniformly for \( \sigma \) from the domain of definition of \( D(s) \). (Here and throughout the paper

\[ f(x) = O(g(x)), \quad x \to a \in \mathbb{R}, \quad g(x) > 0 \]
means that $|f(x)| \leq C_1 g(x)$, with some constant $C_1 > 0$ and for all $x$ sufficiently close to $a$.

Recalling that a Dirichlet series is of finite order in any half-plane from the half-plane of its convergence, the assumption (30) requires that if the Dirichlet series (29) admits analytic continuation, then (30) holds also in the extended domain. Finally, recall that the assumption (30) appears in Meijer–-dus’ theorem (see [1]) as one of the sufficient conditions for sub exponential growth of $c_n$.

II. Two main theorems

**Theorem 1** A multiplicative model with functions $S_k$, $k \geq 1$ of the form (17) is exponential if and only if the following two conditions hold:

- **Condition I** The Dirichlet series $D(s) = \sum_{k=1}^{\infty} \Lambda_k k^{-s}$, $s > \rho_r$ admits meromorphic continuation to $\mathbb{C}$, where it is analytic except $r \geq 1$ simple poles $0 < \rho_1 < \ldots < \rho_r$ with respective residues $A_1 > 0, \ldots, A_r > 0$, and may be a simple pole at $s = 0$ with residue $A_0$. The Taylor expansion of the remainder term $\Delta(\tau)$ in (11) is given by

$$\Delta(\tau) = \sum_{l \geq 1} \frac{(-1)^l D(-l)}{l!} \tau^l, \quad \tau \in \mathbb{C}.$$  

- **Condition II**

  All $r \geq 1$ simple positive poles $\rho_1, \ldots, \rho_r$ in Condition I belong to the Dirichlet series $D_b$ in (27), while the Dirichlet series $D_{(\xi, 1)}$ in (28) may have only one simple pole at 0.

To formulate Theorem 2 below we need the following

**Definition** Let the random variable $Z_n$ be defined as in (8), (9), with $\delta = \delta_n$ given by (10). Then we say that for $Z_n$ the normal local limit theorem (NLLT) is in force if

$$P(Z_n = n) \sim \frac{1}{\sqrt{2\pi \text{Var}(Z_n)}}, \quad n \to \infty.$$  

**Theorem 2** For an exponential model, the NLLT (32) holds if and only if the following two conditions on coefficients $d_j$, $j \geq 0$ in (18) and weights
\( b_k, \ k \geq 1 \) in (17) are satisfied:
\[
gcd\{ j \geq 1 : d_j > 0 \} = 1 \tag{33}
\]
and for any integer \( q \geq 2 \) and \( n \to \infty \),
\[
\sum_{1 \leq k \leq n, k \neq k} b_k \geq \begin{cases} 
C \log n, \ C > 0, & \text{in Case (A);} \\
C \log^2 n, \ C > 0, & \text{in Case (B),}
\end{cases} \tag{34}
\]
where the cases (A), (B) are as defined below, in the course of proof (see (37) and the Remark after it).

Corollary 1 Let all conditions of Theorem 2 hold. Then the following asymptotic formula for \( c_n \) is valid:
\[
c_n \sim \frac{\delta_n^{\rho r \rho r + 1}}{\sqrt{2\pi}} \exp \left( \sum_{l=0}^{r} h_l \delta_n^{-\rho l} - A_0 \log \delta_n + \Delta(\delta_n) + n\delta_n \right), \ n \to \infty, \tag{35}
\]
where \( \delta_n \) is the unique solution of the Khintchine’s equation (14), while \( \Delta(\tau) \), as well as the constants are as defined in Condition I.

It is important to note that since \( \delta_n = O(\rho r^{-\rho r + 1}) \), \( n \to \infty \), by virtue of (16), the sub exponential rate of growth of \( c_n \), as determined by (35), is
\[
O(\rho r^{-\rho r + 1}) \exp(O(n^{\rho r \rho r + 1})), \ n \to \infty, \tag{36}
\]
whereas the assumption (1) requires \( c_n = o(e^n) \), \( n \to \infty \). Thus, for \( \rho r > 0 \) sufficiently large, the rate (36) of sub exponential growth of \( c_n, \ n \to \infty \) approaches the maximal possible one.

In this connection note that all models of harmonic sums treated in [7] exhibit non exponential rate of growth of \( c_n \).

III Proofs

- Necessity of Condition I of Theorem 1

The key ingredient in our proof of the necessity of Condition 1 is the forthcoming Lemma 1. The lemma is an obvious extension to our setting of Yifan Yang’s Lemma 2 in [25], where it is formulated for a special case of partitions into powers of primes.
Lemma 1  Let the Dirichlet series \( (25) \) converge for \( \Re(s) > \rho_r > 0 \) and let \( \log \mathcal{F}(\delta) \) satisfy \((22),(23)\). Then
\[
\int_0^1 \delta^{s-1} \log \mathcal{F}(\delta) d\delta = \Gamma(s) D(s) - W(s), \quad \Re(s) > \rho_r,
\] (37)
where \( W(s), s \in \mathcal{C} \) is an entire function.

Proof  Substituting \((23)\), gives
\[
\int_0^1 \delta^{s-1} \log \mathcal{F}(\delta) d\delta = \left( \int_0^{\infty} - \int_1^{\infty} \right) \delta^{s-1} \log \mathcal{F}(\delta) d\delta = 
\sum_{k=1}^{\infty} \Gamma(s) \left( \frac{b_k a_k^j \xi_j}{k^s j^s} \right) - \sum_{k,j=1}^{\infty} \Gamma(s,j) \left( \frac{b_k a_k^j \xi_j}{k^s j^s} \right),
\] (38)
where the first double series in \((38)\) converges to \( \Gamma(s) D(s) \), for \( \Re(s) > \rho_r \) and converges absolutely for \( \Re(s) > \rho^* \), where \( \rho^* \) is the abscissa of absolute convergence of the Dirichlet series \((25)\). We will show that the second double series in \((38)\) defines an entire function in \( s \in \mathcal{C} \). For this purpose we use the following bound on the incomplete Gamma function \( \Gamma(s,u) \) which itself is entire in \( s \in \mathcal{C} \) for all \( u > 0 \). Letting \( s = \Re(s) \), we have for any reals \( \sigma_1 < \sigma_2 \),
\[
|\Gamma(s,u)| \leq \int_u^{\infty} x^{\sigma-1} e^{-x} dx \leq \left( \max_{x \geq 1} \max_{\sigma \in [\sigma_1, \sigma_2]} x^{\sigma-1} e^{-\frac{x}{2}} \right) \int_u^{\infty} e^{-\frac{u}{2}} dx = 2C(\sigma_1, \sigma_2) e^{-\frac{u}{2}}, \quad u \geq 1,
\] (39)
uniformly for \( \sigma \in [\sigma_1, \sigma_2] \), where \( 0 < C(\sigma_1, \sigma_2) < \infty \) denotes the maximum in \((39)\).

Next, the absolute convergence in the half-plane \( \Re(s) > \rho^* > 0 \), of the double series \( \sum_{k \geq 1, j \geq 1} \frac{b_k \xi_j a_k^j}{k^\sigma j^\sigma} \) that represents the function \( D(s) \) in the above half-plane, implies
\[
\frac{b_k \xi_j a_k^j}{k^{\rho^*+\epsilon} j^{\rho^*+\epsilon}} \to 0, \quad k, j \to \infty, \quad \epsilon > 0.
\]
As a result, applying (39) we get

\[ |W(s)| = \left| \sum_{k,j=1}^{\infty} \Gamma(s, jk) \frac{b_k a_k^j \xi_j}{k^s j^s} \right| \leq 2C(\sigma_1, \sigma_2) \sum_{k,j=1}^{\infty} \frac{b_k a_k^j |\xi_j|}{(jk)^\sigma} e^{-\frac{ij}{s\sigma}} = \]

\[ 2C(\sigma_1, \sigma_2) \sum_{k,j=1}^{\infty} \frac{o((kj)^{\sigma+\epsilon})}{(jk)^\sigma} e^{-\frac{ij}{s\sigma}} < \infty, \]

uniformly for \( \Re(s) = \sigma \in [\sigma_1, \sigma_2] \), with any real \( \sigma_1, \sigma_2 \). This proves that the function \( W(s) \) is entire.

Assuming that the structure is exponential, it follows from the definition (11) with \( \tau = \delta_n > 0 \), the necessity of the asymptotic formula

\[ \mathcal{F}(\delta_n) = \exp \left( \sum_{l=0}^{r} h_l \delta_n^{-\rho_l} - A_0 \log(\delta_n) + \Delta(\delta_n) \right), \]

\[ \Delta(\delta_n) \to 0, \ n \to \infty. \quad (40) \]

By Lemma 1,

\[ D(s) = \frac{1}{\Gamma(s)} \left( \int_0^1 \delta^{s-1} \log \mathcal{F}(\delta) d\delta + W(s) \right), \quad \Re(s) > \rho_r. \quad (41) \]

Next, substituting (41) and the Taylor expansion of \( \Delta(\delta_n) \), into the integral in (11), we obtain

\[ D(s) = \frac{1}{\Gamma(s)} \left( \frac{h_0}{s} + \frac{A_0}{s^2} + \sum_{l=1}^{r} \frac{h_l}{s - \rho_l} + W(s) + \sum_{k=1}^{\infty} \frac{\Delta(k)(0)}{k!(s+k)} \right), \ s > \rho_r. \quad (42) \]

Since \( \frac{1}{\Gamma(s)} \) is an entire function, with zeros at \( s = -n, n \geq 1 \), (42) says that the function \( D \) is analytic in \( \mathbb{C} \), except the simple positive poles \( \rho_1, \ldots, \rho_r \), with the respective residues \( A_l = \frac{h_l}{\Gamma(\rho_l)} > 0, \ l = 1, \ldots, r \) and a simple pole at \( s = 0 \) if \( A_0 \neq 0 \) in (11). The latter, together with the fact that \( s = -n, n \geq 1 \) are simple poles of \( \Gamma(s) \) with residues \( (-1)^n/n! \), respectively, gives

\[ \Delta(k)(0) = (-1)^k D(-k), \ k \geq 1. \]

As a result, the proof of the necessity of Condition I is completed.
Necessity of Condition II of Theorem 1

Our first objective is to prove the remarkable fact that \( \rho_1, \ldots, \rho_r \) are poles of the Dirichlet generating series \( D_b(s) = \sum_{k \geq 1} b_k k^s \) for the weights \( b_k, \ k \geq 1 \).

For this purpose we firstly prove that for a given \( k \geq 1 \) and \( 0 < a_k \leq 1 \), the Dirichlet series \( D(\xi,a_k) \) given by (28) has no positive poles. The two cases \( 0 < a_k < 1 \) and \( a_k = 1 \) should be distinguished.

In the first case, \( D(\xi,a_k)(0) = \log S(a_k) < \infty, \ 0 < a_k < 1 \), since the radius of convergence of the series (20) is \( \geq 1 \). Consequently, the Dirichlet series \( D(\xi,a_k)(s), \ 0 < a_k < 1 \) converges in the half-plane \( \Re(s) \geq 0 \), and therefore it is analytic in this domain.

If \( a_k = 1 \) for a given \( k \), there are the following two possibilities:

(i) \( D(\xi,1)(0) = \sum_{j \geq 1} \xi_j < \infty \). This says that in the case considered the Dirichlet series \( D(\xi,1)(s) \) is analytic in the half-plane \( \Re(s) \geq 0 \).

Example

Partitions into distinct parts:

\[
S(z) = 1 + z, \quad D(\xi,1)(0) = \sum_{j \geq 1} \xi_j = \sum_{j \geq 1} \frac{(-1)^{j-1}}{j} < \infty.
\]

(ii) \( \sum_{j \geq 1} \xi_j = \infty \), which amounts to saying that \( z = 1 \) is the singular point of the series (20), with radius of convergence 1 and with \( \log S(z) |_{z \to 1^-} \to \infty \). Consequently, \( S(z) \to \infty, \quad z \to 1^- \), so that the assumption (21) is in force. Recalling that in (21) it can be taken \( l = 1 \), we have

\[
\log S(z) = \sum_{j \geq 1} \xi_j z^j \sim \log L(z) + \log \left( \frac{1}{1 - z} \right) \sim \log \left( \frac{1}{1 - z} \right), \quad z \to 1^-,
\]

where the second " \( \sim \)" is by the properties of the function \( L \) as stated in (21). Thus, we obtain from (43),

\[
\left( \log S(z) \right)_z' = \sum_{j \geq 1} j \xi_j z^{j-1} \sim \frac{1}{1 - z}, \quad z \to 1^-.
\]

We apply now Karamata’s tauberian theorem (see e.g. [6]) to the asymptotic relation in (44) to derive that \( j \xi_j \sim 1, \ j \to \infty \). Thus, \( \xi_j \sim \frac{1}{j}, \ j \to \infty \), which implies that in the case considered:

\[
D(\xi,1)(\rho) = \sum_{j \geq 1} \xi_j j^{-\rho} < \infty, \quad \text{for all } \rho > 0,
\]
while $D_{(\xi, 1)}(0) = \infty$. This says that in case (\textit{ii}), under the assumption (21) (with $l = 1$), the Dirichlet series $D_{(\xi, 1)}(s)$ has in the half-plane $\Re(s) \geq 0$ only one simple pole at $s = 0$.

\textbf{Example} Unrestricted partitions:

$$S(z) = \frac{1}{1-z}, \quad D_{(\xi, 1)}(0) = \sum_{j \geq 1} \frac{1}{j} = \infty.$$ 

The corresponding Dirichlet series $D_{(\xi, 1)}(s) = \zeta(1 + s)$ has a unique simple pole at $s = 0$.

From the above proven fact that for a given $k \geq 1$ the function $D_{(\xi, a_k)}$ has no positive poles we will derive now that all positive poles of $D(s)$ belong to the Dirichlet series $D_b(s)$. If $a_k \equiv 1$, then the claim follows immediately from (26). In the general case, $D_{(\xi, a_k)}(\rho) < \infty$, $k \geq 1$, $\rho > 0$ implies $\sup_{k \geq 1} D_{(\xi, a_k)}(\rho) := u(\rho) < \infty$, $\rho > 0$, because $0 < a_k \leq 1$. In view of this, taking $\rho = \rho_l > 0$, $1 \leq l \leq r$ we have

$$\infty = D(\rho_l) \leq u(\rho_l) \sum_{k \geq 1} \frac{b_k}{k^{\rho_l}},$$

which says that $\rho_l$, $l = 1, \ldots, r$ are indeed the poles of $D_b(s)$. 

\textbf{Remarks} (i) The property of $D_{(\xi, a_k)}$ stated in Condition II is shared by the three classic combinatorial structures, which are multisets, selections and assemblies (see [3],[10],[11]).

(ii) A Dirichlet series with real coefficients may have complex poles. For example,

$$\frac{\zeta(s + 1)}{\zeta(s)} = \zeta(s + 1) \frac{1}{\zeta(s)} = \zeta(s + 1) \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s}, \quad \Re(s) > 1,$$

where $\mu(k)$ is the Moebius function. The LHS of the above relation is a product of two Dirichlet series $\zeta(s + 1)$ and $1/\zeta(s)$ with real coefficients. However, the product has complex poles which are complex zeros of $\zeta(s)$ on the critical line $s = 1/2 + it$.

- \textbf{ Sufficiency of Conditions I and II of Theorem 1 }

Our proof of sufficiency of Condition I for (11) follows Meinardus’ scheme which is based on application of Mellin transform. Here we sketch the scheme,
assuming that the details can be found in [12]. We use the fact that $e^{-u}$, $\Re(u) > 0$, is the Mellin transform of the Gamma function:

$$e^{-u} = \frac{1}{2\pi i} \int_{v-i\infty}^{v+i\infty} u^{-s} \Gamma(s) \, ds, \quad \Re(u) > 0, \, v > 0. \quad (46)$$

Applying (46) with $u = \tau$ : $\Re(\tau) = \delta > 0$ and $v = \rho_r + \epsilon$, $\epsilon > 0$ we have

$$\log \mathcal{F}(\tau) = \sum_{k=1}^{\infty} b_k \log S(a_k e^{-r k})$$

$$= \frac{1}{2\pi i} \int_{\epsilon+\rho_r-i\infty}^{\epsilon+\rho_r+i\infty} \tau^{-s} \Gamma(s) D(s) \, ds, \quad (47)$$

where $D(s)$ is a meromorphic continuation to $C$ of the Dirichlet series (25).

Next, assuming that the Condition I holds and recalling (30), we apply the residue theorem for the integral (47) in the complex domain $\Re(s) \leq \rho_r + \epsilon$, to get the formula:

$$\log \mathcal{F}(\tau) = \sum_{l=0}^{r} h_l \tau^{-\rho_l} - A_0 \log \tau + \Delta(\tau), \quad (48)$$

where $h_l = A_l \Gamma(\rho_l)$, $l = 1, \ldots, r$ and where the expansion (31) of the remainder term $\Delta(\tau)$ follows from the fact that in the domain $\Re(s) < 0$ the integrand $\delta^{-s} D(s) \Gamma(s)$ has simple poles at $s = -k$, $k = 1, 2, \ldots$, only. Exponentiating (48) gives (11).

Remark In the previous research, started from the aforementioned seminal paper by Meinardus it was always assumed that $D$ admits meromorphic continuation to $-C_0 < \Re(s) < \rho_1$, for some $0 < C_0 < 1$. Accordingly, the error of the asymptotic expansion of $\log F(\delta)$ was $O(\delta^{C_0})$, $\delta \to 0^+$. The sufficiency of Condition II is obvious.

Two remarks regarding (48).

(i) By (48) with $\tau = \delta > 0$,

$$\frac{\log \mathcal{F}(\delta)}{\delta^{-\rho_r}} \to h_r = A_r \Gamma(\rho_r), \quad \delta \to 0^+,$$

since $\rho_r > 0$ is the rightmost pole. In the particular case of ordinary partitions the above asymptotics has deep meanings in statistical physics and
combinatorics, being related respectively, to the shape of a crystal at equilibrium and the limit shape of a random partition of large $n$. In both cases the interpretation is based on treating a limit shape as a solution of a certain variational problem (see [18] and references therein).

(ii) We consider here the formula (48) in the special case of ordinary partitions: $D(s) = \zeta(s)\zeta(s+1)$, $r = 1$, $\rho_1 = 1$, $h_1 = \zeta(2)$, $A_0 = -\zeta(0)$. In the case considered (48) becomes:

$$\log \mathcal{F}(\delta) = \delta^{-1}\zeta(2) - \zeta(0) \log \delta + \zeta'(0) + \sum_{l=1}^{N} \delta^l \frac{(-1)^l}{l!} \zeta(-l+1)\zeta(-l) + \Delta I(C_N; \delta), \quad (50)$$

where we denoted

$$\Delta I(C_N; \delta) = \frac{1}{2\pi i} \int_{-C_N-i\infty}^{-C_N+i\infty} \delta^{-s} \Gamma(s)\zeta(s)\zeta(1+s)ds, \quad (51)$$

for a fixed integer $N$ and $C_N = N + \frac{1}{2}$. Next, from the functional equation for zeta-function and from the Gauss multiplication formula for the Gamma function (see Thm. 15.1 in [2]) we have:

$$\Gamma(s)\zeta(s)\zeta(s+1) = (2\pi)^2 \zeta(1-s)\zeta(-s)\Gamma(-s),$$

which gives

$$\Delta I(C_N; \delta) = \frac{1}{2\pi i} \int_{-C_N-i\infty}^{-C_N+i\infty} \delta^{-s} (2\pi)^2 \zeta(1-s)\zeta(-s)\Gamma(-s)ds. \quad (52)$$

Now we use the asymptotic bound

$$|\zeta(u)| \leq 1 + \sum_{n \geq 2} n^{-v} \leq 1 + \int_{2}^{\infty} x^{-v} dx = 1 + \frac{2^{1-v}}{v-1} = 1 + O(2^{-v}v^{-1}),$$

where we denoted

$$1 < v := \Re(u) \to +\infty.$$

This and the Mellin transform formula of the Gamma function allow to bound $\Delta I(C_N; \delta)$ in (52):

$$|\Delta I(C_N; \delta)| \leq e^{-(2\pi)^2\delta^{-1}}(1 + O(2^{-C_N}C_N^{-1})) \to e^{-(2\pi)^2\delta^{-1}}, \text{ as } 0 < C_N \to \infty.$$
The latter together with the fact that
\[ \zeta(-l + 1)\zeta(-l) = 0, \ l \geq 1 \]
allows to derive from (50),
\[
\log F(\delta) = \delta^{-1}\zeta(2) - \zeta(0) \log \delta + \zeta'(0) - \zeta(0)\zeta(-1)\delta + e^{-(2\pi)^2\delta^{-1}}, \ \delta \to 0^+.
\] (53)

Exponentiating the last expression and setting \( x = e^{-\delta} \) recovers the formulae (8.6.1), (8.6.2), p.117 in [13], for \( F(\delta) \). It is emphasized by Hardy([13]), that the remainder term \( \exp(e^{-(2\pi)^2\delta^{-1}}) \) in the expansion of the function \( F(\delta) \) goes to 1 very fast, as \( \delta \to 0^+ \). The formula (53), which is the key ingredient of the famous Hardy-Ramanujan expansion for the number of ordinary partitions was derived (see e.g. [13]) in a quite different way, based on the remarkable fact that the generating function for ordinary partitions is an elliptic function obeying a certain functional equation. Finally, note that the mysterious exponent 1/24 in the aforementioned Hardy-Ramanujan formula is equal to \( \zeta(0)\zeta(-1) \).

• Theorem 2

Three auxiliary facts. In (i) – (iii) below we assume that the structure is exponential. The detailed proofs and the history of (i) and (ii) can be found in [8], [9]. Lemma 2 in (iii) is new.

(i) Asymptotics of \( \delta_n \), as \( n \to \infty \).

Firstly we show that in the representation (6) with \( \delta = \delta_n \),
\[
f_n(e^{-\delta_n}) = F(\delta_n) + \epsilon_n, \ \epsilon_n \to 0, \ n \to \infty.
\] (54)

Recalling the expression (23) for \( \log f(z) \) and that in (18), \( d_0 = 1, d_j \geq 0, j \geq 1 \). we denote
\[
l_0 = \min\{j \geq 1 : d_j > 0\}.
\] (55)

To avoid the trivial case \( S(z) \equiv 1 \), we assume \( l_0 < \infty \). Now we have,
\[
S(a_k e^{-\delta_n k}) - 1 = O(a_k^{l_0} e^{-l_0 k\delta_n}) \to 0, \ n \to \infty, \ \text{for all } k \geq n,
\] (56)
since \( n\delta_n \to \infty \), by Proposition 1. Consequently,
\[
\log S(a_k e^{-k\delta_n}) = O(a_k^{l_0} e^{-l_0 k\delta_n}) \to 0, \ n \to \infty, \ \text{for all } k \geq n.
\] (57)
As a result,

$$
\sum_{k=n+1}^{\infty} b_k \log S(a_k e^{-k\delta_n}) = \sum_{k=n+1}^{\infty} b_k O(a_k^0 e^{-l_0 k\delta_n}) \to 0, \ n \to \infty, \quad (58)
$$

where the last step is because $b_k = o(k^{\rho_r})$, $k \to \infty$, since $\rho_r$ is the rightmost pole of $D_b(s)$ and because of (16). (58) proves (54). Next, substituting (40) into the LHS of the Khitchine’s equation (10), produces the asymptotic expansion of the solution $\delta_n$, $n \to \infty$ of the equation. For the case of multiple poles, i.e. $r > 1$, the expansion was firstly obtained in [12]. For our subsequent study we will need only the main term of the above expansion which is obtained from (16):

$$
\delta_n \sim (\rho_r h_r)\frac{1}{\rho+1} n^{\frac{1}{\rho+1}}, \ n \to \infty, \quad (59)
$$

where

$$ h_r = A_r \Gamma(\rho_r). $$

**Remark** In connection with (59) it is in order to note that in the theory of limit shapes the parameter $\delta$ (called there scaling) is taken to be equal $O(n^{-\frac{1}{\rho r + 1}})$, which is, roughly speaking, (59) (see e.g. [11], [23], [24]). The aforementioned coincidence is explained by the fact that the derivation of limit shapes consists of asymptotic approximation of probabilities with respect to the same multiplicative measure $\mu_n$ as in our setting.

(ii) Representation of $\mathbb{P}(Z_n = n)$.

We start from the formula

$$
\mathbb{P}(Z_n = n) = \int_{-1/2}^{1/2} \phi_n(\alpha) e^{-2\pi i n \alpha} d\alpha := I_1 + I_2, \quad (60)
$$

where the random variable $Z_n$ is as defined in (8), (9) and $\phi_n(\alpha)$ is the characteristic function of $Z_n$, while

$$
I_1(n) = \int_{-\alpha_0}^{\alpha_0} \phi_n(\alpha)e^{-2\pi i n \alpha} d\alpha, \quad \text{with} \quad \alpha_0 = \alpha_0(n) = (\delta_n)^{\frac{\rho_r + 2}{2}} \log n
$$

and

$$
I_2(n) = \int_{-1/2}^{-\alpha_0} \phi_n(\alpha)e^{-2\pi i n \alpha} d\alpha + \int_{\alpha_0}^{1/2} \phi_n(\alpha)e^{-2\pi i n \alpha} d\alpha. \quad (61)
$$
Our first goal will be to derive the asymptotics of the integral $I_1 = I_1(n)$, as $n \to \infty$. Let for a given $n$, $B_n^2$ and $T_n$ be defined by

$$Var Z_n := B_n^2 = \left( \log f_n(e^{-\delta}) \right)'''_{\delta=\delta_n}$$

and

$$T_n := -\left( \log f_n(e^{-\delta}) \right)'''_{\delta=\delta_n}.$$  

Due to the fact that

$$\phi_n(\alpha)e^{-2\pi i n \alpha} = \mathbb{E} \exp \left( 2\pi i \alpha(Z_n - n) \right), \quad \alpha \in \mathbb{R},$$

the following expansion in $\alpha$ is valid, when $n$ is fixed:

$$\phi_n(\alpha)e^{-2\pi i n \alpha} = \exp \left( 2\pi i \alpha(\mathbb{E}Z_n - n) - 2\pi^2 \alpha^2 B_n^2 + O(\alpha^3 T_n) \right) = \exp \left( -2\pi^2 \alpha^2 B_n^2 + O(\alpha^3 T_n) \right), \quad \alpha \to 0,$$

where the second equation is due to (10) and the fact that

$$\left( -\log F(\delta) \right)'_{\delta} = \mathbb{E}Z_n(\delta), \quad \delta > 0.$$ 

It follows from (10) that the main terms in the asymptotics for $B_n^2$ and $T_n$ depend on the rightmost pole $\rho_r > 0$ only:

$$B_n^2 \sim K_2(\delta_n)^{-\rho_r - 2}, \quad n \to \infty,$$

where $K_2 = h_r \rho_r(\rho_r + 1)$ and

$$T_n \sim K_3(\delta_n)^{-\rho_r - 3}, \quad n \to \infty,$$

where $K_3 = h_r \rho_r(\rho_r + 1)(\rho_r + 2)$. Therefore, by the choice of $\alpha_0$ as in (61),

$$B_n^2 \alpha_0^2 \to \infty, \quad T_n \alpha_0^3 \to 0, \quad n \to \infty.$$

Consequently, by the same argument as in the proof of the NLLT in [10],

$$I_1 \sim \frac{1}{\sqrt{2\pi B_n^2}} \sim (2\pi K_2)^{-1/2}(\delta_n)^{1+\frac{4}{\rho_r}}, \quad n \to \infty.$$ 

(iii) Bounding the integral $I_2$, as $n \to \infty$.

For the NLLT (32) to hold it is necessary and sufficient that

$$I_2 = o(I_1), \quad n \to \infty.$$
Lemma 2  For the NLLT (32) to hold it is necessary and sufficient that
\[ |\phi_n(\alpha)| = o(\delta_n^{1+\frac{1}{n}}), \quad n \to \infty, \quad \alpha \in [\delta_n, 1/2]. \]  

(68)

Proof  The following expression is valid for the multiplicative models considered:
\[ \log \phi_n(\alpha) = \sum_{k=1}^{n} b_k \left( \log(S(a_k e^{-\tau_n k}) - \log S(a_k e^{-\delta_n k}) \right), \quad \alpha \in \mathcal{R}, \]  

(69)

where \( \tau_n = \tau_n(\alpha) = \delta_n - 2\pi i \alpha \). It is easy to see that (54) holds with \( \delta_n \) replaced with \( \tau_n \). In view of (40) we thus have:
\[ \log \phi_n(\alpha) = \sum_{k=1}^{\infty} b_k \left( \log S(a_k e^{-\tau_n k}) - \log S(a_k e^{-\delta_n k}) \right) + \epsilon_n = \log F(\tau_n) - \log F(\delta_n) + \epsilon_n, \]

(70)

where for exponential structures
\[ F(\tau_n) = \exp \left( \sum_{l=0}^{r} h_l \tau_n^{-\rho_l} - A_0 \log \tau_n + \Delta(\tau_n) \right), \]

(71)

by (11). Applying (11), (54), (6), and (71) gives
\[ \phi_n(\alpha) = \prod_{k=1}^{n} \left( \frac{S(a_k e^{-k\tau_n})}{S(a_k e^{-k\delta_n})} \right)^{b_k} \sim \frac{F(\tau_n)}{F(\delta_n)}, \]

(72)

To bound \( |\phi_n(\alpha)|, \quad \alpha \in [\alpha_0(n), 1/2] \) from above, we use the formula
\[ \log |\phi_n(\alpha)| = \Re(\log \phi_n(\alpha)). \]

(73)

We see that for all \( \alpha \in [\alpha_0(n), 1/2] \),
\[ \Re(\tau_n^{-\rho_l}) = \Re\left((\delta_n - 2\pi \alpha i)^{-\rho_l}\right) = \Re\left(\delta_n^{-\rho_l}(1 - \frac{2\pi \alpha}{\delta_n i})^{-\rho_l}\right) = \]

20
\[
\delta_n^{-\rho_l} \left( 1 + \left( \frac{2\pi \alpha}{\delta_n} \right)^2 \right) \leq \delta_n^{-\rho_l} \left( 1 + (2\pi)^2 \delta_n^{\rho_r} \log^2 n \right) \frac{-\rho_l}{2}, l = 1, 2, \ldots, r, \tag{74}
\]

where \( w_n(\alpha) \) is a complex variable with \( |w_n(\alpha)| = 1 \).

Continuing the last inequality we have, for \( n \) sufficiently large and \( \alpha \in [\alpha_0(n), 1/2] \):

\[
\Re \left( (\delta_n - 2\pi \alpha i)^{-\rho_l} \right) \leq \delta_n^{-\rho_l} \left( 1 - \frac{\rho_l}{2} (2\pi)^2 \delta_n^{\rho_r} \log^2 n \right), \quad l = 1, \ldots, r. \tag{75}
\]

Consequently, for all \( \alpha \in [\alpha_0(n), 1/2] \) and \( l = 1, \ldots, r - 1, \)

\[
\Re (\tau_n^{-\rho_l}) - \delta_n^{-\rho_l} \leq -C \delta_n^{-\rho_l} \log^2 n \to 0, \quad n \to \infty, \quad C = \frac{\rho_l}{2} (2\pi)^2 > 0,
\]

while

\[
\Re (\tau_n^{-\rho_r}) - \delta_n^{-\rho_r} \leq -C \log^2 n, \quad n \to \infty, \quad C = \frac{\rho_r}{2} (2\pi)^2 > 0, \tag{76}
\]

with equality for \( \alpha = \alpha_0(n) \). Also, we have

\[
-A_0 \Re \left( \log \frac{\tau}{\delta_n} \right) = -\frac{A_0}{2} \log \left( 1 + \left( \frac{2\pi \alpha}{\delta_n} \right)^2 \right). \tag{77}
\]

Finally, it follows from (72), (76) and (77) that for all \( \alpha \in [\alpha_0(n), 1/2] \),

\[
\left| \log |\phi_n(\alpha)| \right| \geq C \log^2 n, \quad n \to \infty, \quad C > 0 \tag{78}
\]

and

\[
|\phi_n(\alpha)| \leq \left( 1 + \left( \frac{2\pi \alpha}{\delta_n} \right)^2 \right) \frac{-A_0}{2} \exp \left( -C \log^2 n \right) = o((\delta_n)^{1+\frac{\rho_r}{2}}), \quad n \to \infty, \quad C > 0. \tag{79}
\]

(79) yields (67) and, consequently (32), which proves that (11) implies (68). The sufficiency of (68) for LLT is immediate.

(iv) Proof of the conditions (33) and (34).

Our first goal is to bound from above the function \( \left| \log |\phi_n(\alpha)| \right| \) for rational \( \alpha \in [\alpha_0(n), 1/2] \). By virtue of (73) and (69) we write

\[
\left| \log |\phi_n(\alpha)| \right| = \frac{1}{2} \sum_{k=1}^{n} b_k \log \left( \frac{S^2(a_k e^{-k\delta_n})}{|S(a_k e^{-k\delta_n + 2\pi i k})|^2} \right), \quad \alpha \in \mathcal{R}. \tag{80}
\]
We firstly indicate the following two essential facts:

\[(a) \quad U_n(k; \alpha) := \log \left( \frac{S^2(a_k e^{-k\delta_n})}{|S(a_k e^{-k\delta_n + 2\pi i \alpha k})|^2} \right) \rightarrow 0, \quad (81)\]

for all \( \alpha \in \mathcal{R} \) and for all \( k = k(n) : k(n)\delta_n \geq C \log^2 n, C > 0, n \rightarrow \infty \), since \( S(0) = 1 \)

\[(b) \quad U_n(k; \alpha) = 0, \text{ if } \alpha k \text{ is an integer.} \quad (82)\]

As a particular case of \((a)\),

\[U_n(k; \alpha) \rightarrow 0, \quad n \rightarrow \infty, \quad \alpha \in \mathcal{R}, \quad \text{for all } k \geq \delta_n^{1-\epsilon}, \quad \text{with any } \epsilon > 0. \quad (83)\]

Next we show that for \( n \) sufficiently large, the main contribution to the sum in \((80)\) comes from the terms with \( k \in \kappa_v, \quad v = 1, \ldots, q-1 \), where, given an integer \( q > 1 \) and \( \epsilon > 0 \), the set of integers \( \kappa_v = \kappa_{v,q}(\epsilon) \) is defined by

\[\kappa_v := \{1 \leq k \leq \delta_n^{1-\epsilon}, \epsilon > 0 : k\delta_n < \infty, \text{ as } n \rightarrow \infty, \text{ and } k \equiv v(\text{mod } q)\}.\]

Let \( 0 < \alpha \leq 1/2 \) be a rational number, i.e. \( \alpha = \frac{p}{q} > 0, \gcd(p, q) = 1, \quad q > 1. \)

For \( n \) large enough, we can assume that \( \frac{p}{q} \in [\alpha_0(n), 1/2] \), by the above definition of \( \alpha_0(n) \). Then

\[U_n(k; \frac{p}{q}) \rightarrow \log \left( \frac{S^2(c_k)}{|S(c_k e^{2\pi i \frac{p}{q}})|^2} \right), \quad k \in \kappa_v, \quad n \rightarrow \infty, \quad (84)\]

where \( c_k := \lim \sup_{n \rightarrow \infty} (a_k e^{-\delta_n k}) \), for \( k \in \kappa_v \).

Due to our assumption \((19)\) on \( a_k, \quad k \geq 1 \), and the definition of the set \( \kappa_v \), the constants \( 0 \leq c_k \leq 1, \quad k \in \kappa_v \). Also,

\[U_n(k; \frac{p}{q}) \geq 0, \quad k \in \kappa_v, \quad (85)\]

with equality if and only if in the expansion \((18)\), \( d_j = 0, \) for all \( j : q \mid j, \) for some integer \( q \geq 2. \) This says that in the case of equality, the condition \((33)\) of Theorem 2 does not hold, which leads to \( |\phi_n(\frac{p}{q})| = 1, \) in contradiction to the condition \((68)\). Hence, the condition \((33)\) is necessary for LLT to hold, which means that \((33)\) guarantees the strict inequality in \((85)\).

We now prove the necessity of the bounds \((34)\). Let \( k = ql + v \in \kappa_v : \delta_n k =
$c + \epsilon_n$, with $c \geq 0$, $\epsilon_n \to 0$, as $n \to \infty$, and let $z_0 = e^{-c}a_k e^{2\pi i \frac{u}{v}}$ be a zero of some order $m \geq 1$ of the function $S(z)$, $z = e^{-\delta_n k}a_k e^{2\pi i \frac{u}{v}}$:

$$S(z) = (z - z_0)^m \bar{S}(z), \quad \bar{S}(z_0) \neq 0.$$ 

Recalling (see the discussion after (20)) that it should be $|z_0| = 1$, it follows that in the above representation of $z_0$, $c = 0$, $a_k = 1$, and therefore, in the case discussed,

$$|S(e^{-\delta_n k} e^{2\pi i \frac{u}{v}})| \sim (\delta_n k)^m |\bar{S}(z_0)| = O((\delta_n k)^m) \leq C\delta_n, \quad C > 0, \quad k \in \kappa_v : \delta_n k \to 0, \quad n \to \infty. \quad (86)$$

Finally, if $z_0 \neq 1$ is a pole of $S(z)$ in the circle $|z| \leq 1$, then again it should be $|z_0| = 1$, by virtue of the aforementioned remark, and therefore, $c = 0$, $a_k = 1$. Consequently, applying (21) with $l = 1$, we get $|S(e^{-\delta_n k} e^{2\pi i \frac{u}{v}})| \sim C(\delta_n k)^{-1}$, $k \in \kappa_v : \delta_n k \to 0, \quad n \to \infty$, for some $C > 0$. Taking into account that $|S(z_0)| \leq S(1)$, we conclude that in the case considered $z = 1$ should be a pole as well, which implies

$$U_n(k; \frac{p}{q}) \sim \log \frac{|L(1)|}{|L(e^{2\pi i \frac{u}{v}})|} = O(1), \quad n \to \infty.$$ 

In light of the above reasoning it is clear that the following two cases should be broadly distinguished, as $n \to \infty$:

$$A : 0 < U_n(k; \frac{p}{q}) \leq C > 0$$

$$B : 0 < U_n(k; \frac{p}{q}) \leq -C \log \delta_n, \quad C > 0, \quad (87)$$

where $\frac{p}{q} \in [\alpha_0(n), 1/2]$ and where in both cases the bounds hold for all integers $q \geq 2$ and for all

$$k \in \kappa(q; \epsilon) := \{ 1 \leq k \leq \delta_n^{-1-\epsilon} : q \mid k, \quad k\delta_n < \infty, \quad n \to \infty \}.$$ 

Recalling that $S(z)$ may have only a finite number of zeros and poles on the unit disk, it follows that the above bounds on $U_n$ are valid for all poles and zeros with the same constant.

**Remark** The case (A) is in force when $S(z)$ has a complex pole on the boundary of the unit disk, while the case (B) holds when $S(z)$ has a complex zero on the boundary of the unit disk.
Combining (78), (83), (87) and (80) we derive the required bound:

\[ |\log |\phi_n(p/q)|| \leq \left( \sum_{1 \leq k \leq n,q\nmid k} b_k \right) \begin{cases} C, & \text{in Case A} \\ -C \log \delta_n, & \text{in Case B} \end{cases} \tag{88} \]

as \( n \to \infty \). (88) together with the lower bound (78) proves the necessity of conditions (33) and (34) for NLLT.

Note that the inequality (88) holds for all \( \alpha \in [\alpha_0, 1/2) \), because the characteristic function \( \phi_n(\alpha) \) is continuous in \( \alpha \in R \) for any given \( n \geq 1 \).

The sufficiency of (34) and (33) for NLLT follows from the fact that the bound (78) is necessary and sufficient for (67).

**Remark** Condition (33) was motivated by Example 2 in [10], which says that if the sequence of weights \( \{b_k, k \geq 1\} \) is supported on the set of \( k = lq, l = 1, 2, \ldots \), with some integer \( q > 1 \), then NLLT does not hold, though Conditions I and II may hold. In this regard, the condition (33) determines the minimal "total mass" of weights \( b_k \) that should be concentrated on integers \( k \leq n \) that are not divisible by a given \( q > 1 \), in order that NLLT be in force.

• **Corollary**

The proof is simple. Substituting into (6) the relation (54), the formula (11) with \( \delta = \delta_n \) and the asymptotic formula (32) for NLLT, gives the asymptotic formula (35).

**IV. Examples**

**Example 1: Partitions into primes.** This example demonstrates situation when the asymptotic expansion (40) of \( \log F(\delta) \), \( \delta \to 0 \), is not possible. For partitions into primes,

\[ f(z) = \prod_{k \geq 1} (1 - z^k)^{-b_k}, \quad |z| < 1, \]

where

\[ b_k = \begin{cases} 1, & \text{if } k \text{ is a prime} \\ 0, & \text{otherwise}. \end{cases} \]

Correspondingly, (see [22], p.117),

\[ D_b(s) = \sum_p \frac{1}{p^s} := P(s), \tag{89} \]
where the summation is over the set of all primes, is the so-called Prime zeta function that admits the representation

$$P(s) = \sum_{k \geq 1} \mu(k) \frac{\log \zeta(ks)}{k}, \quad (90)$$

where $\mu$ is the M"obius function. It is clear from (90) that $P(s)$ is analytic in the half-plane $\Re(s) > 1$ and that it has a meromorphic continuation to the strip $0 < \Re(s) < 1$. Also, it follows from (90) that $P(s)$ has logarithmic singularities at the following points: (i) $s = 1$, where

$$P(s) \sim \log \zeta(s) \sim \log \frac{1}{s - 1}, \quad s \to 1,$$

(ii) $s = \frac{1/2 + i\rho}{k}, \quad k \geq 1$ which are induced by non-trivial zeros $\rho$ of $\zeta(s)$ and (iii) $s = \frac{1}{k}, \quad k$ with square-free integers $k$, which are induced by those trivial zeros $-k$ of $\zeta(s)$ at which $\mu(k) \neq 0$.

As a result, the line $\Re(s) = 0$ is the natural barrier for the Dirichlet series for $P(s)$, which means that the series cannot be continued analytically to the left of the line $\Re(s) = 0$. The latter together with the fact that the singularities of $P(s)$ are not poles prevents the application of Meinardus’ approach. Roth and Szekeres [20] were able to obtain the principal term of the asymptotics of $\log c_n, \quad n \to \infty$, with the help of a complicated analysis adapted for the model considered (see also [22]).

**Example 2: Partitions into powers of primes.**

Though the structure in the example below is not exponential, because the condition $I$ of Theorem 1 does not hold, it is possible to apply the technique of Meinardus to obtain the main term in the expansion of $\log F(\delta)$. The generating function for the model considered is

$$f(z) = \prod_{k \geq 1} (1 - z^k)^{-b_k}, \quad |z| < 1, \quad (91)$$

$$b_k = \begin{cases} \log p, & \text{if } k = p^r, \text{ where } p \text{ is a prime and } r \text{ is an integer} \\ 0, & \text{otherwise.} \end{cases} \quad (92)$$

The weighted generating function in (92) was suggested in 1950 by Brigham (for references see [23]).
It is known that $b_k = \Lambda(k)$, where $\Lambda(k)$, $k \geq 1$ is the von Mangoldt function, the Dirichlet generating function of which is

$$D_b(s) = -\frac{\zeta'(s)}{\zeta(s)}.$$  \hspace{1cm} (93)

Thus, in the case considered $D(s) = \zeta(s + 1)D_b(s)$, where $D_b(s)$ is a meromorphic function in $C$ having poles at all trivial and non-trivial zeros of $\zeta(s)$. Since the non-trivial zeros are known to be complex numbers, which location depends on the solution of the Riemann hypothesis, the condition $I$ of our Theorem 1 does not hold. However, we will show that the technique of the present paper applied to the function $Q(s) := \delta^{-s}\Gamma(s)D(s)$ allows to find the main term in the asymptotic expansion for the log $F(\delta)$, as $\delta \to 0^+$, recovering the results of Richmond [19], and Yang [25].

(i) By the functional equation for $\zeta$-function,

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s), \ s \in C.$$  \hspace{1cm} (94)

Expanding $\sin\left(\frac{\pi s}{2}\right)$ around the trivial zeros $\{-2k, k \geq 1\}$ of $\zeta(s)$ and taking into account that the function $2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)$ is analytic at these points and does not equal to 0, shows that the function $\frac{1}{\zeta(s)}$ has simple poles at $s = -2k, \ k \geq 1$. Consequently, the function $Q(s)$ defined above, has at each of the above points a pole of the second order, with residue $O\left(\delta^2\log\delta\right) \to 0, \delta \to 0^+$, so that these poles influence the remainder term $\Delta(\delta)$ only, and the same is also true for the simple poles $\{-2k - 1, k \geq 0\}$ with the residues $O(\delta^{2k+1}) \to 0, \delta \to 0^+, \ k = 0, 1, 2, \ldots$, induced by $\Gamma(s)$;

(ii) Recalling the Laurent series expansion for the Riemann zeta function:

$$\zeta(s) = \frac{1}{s - 1} + \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \gamma_l (s - 1)^l,$$

where $\gamma_0 = \gamma$ is the Euler constant and $\gamma_l, l = 1, 2, \ldots$ are the Stiltjes constants, we have from (93):

$$D_b(s) = \frac{1}{s - 1} + O(s - 1), \ s \to 1.$$  \hspace{1cm} (95)

This shows that $D_b(s)$ has also a simple pole at $s = 1$, with the residue 1. As a result, the residue of $Q(s)$ at $s = 1$ is equal to $\delta^{-1}\zeta(2)$;
(iii) $s = 0$ is the second order pole of $Q(s)$ with the residue

$$-rac{\zeta'(0)}{\zeta(0)} \log \delta + \text{const};$$

(iv) Non-trivial zeros of $\zeta(s)$. These zeros are known to belong to the critical strip $0 \leq \Re(s) < 1$. We adopt the argument in [25], Section 5, which is based on the preceding works of Richmond, Brigham, Ingham and other researchers. The key observation is that denoting by $\rho$ the non-trivial zeros of $\zeta(s)$, the sum of residues of $Q(s)$ at these points is equal to

$$\sum_{\rho} \delta^{-\rho} \Gamma(\rho) \zeta(1 + \rho),$$

(96)

where the sum is taken over all $\rho$ counted with their multiplicities. Next, using the known bound for the number of the non-trivial zeros $\rho$: $\Im(\rho) \in [T, T + 1]$ one gets that the sum in (96) is $\ll \delta^{-\theta}$ for some $1/2 \leq \theta < 1$.

Summarizing (i) - (iv) it follows that the rightmost pole in the model is $s = 1$, so that $\delta_n \sim (\zeta(2))^{1/2} n^{-1/2}$, by (59). Consequently, by the same argument as in (47), (48) we deduce from (48) the asymptotic formula by Richmond (see Theorem B from [25]):

$$\log \mathcal{F}(\delta_n) = 2(\zeta(2))^{1/2} n^{1/2} + O(n^{\theta/2}), \quad 1/2 \leq \theta < 1, \quad n \to \infty,$$

(97)

where $\theta = 1/2$, if the Riemann Hypothesis is true.

Finally, applying the known asymptotic relation for the von Mangoldt function $\sum_{k=1}^{x} \Lambda(k) \sim x$, $x \to \infty$ shows that the condition (34) for the weights $b_k = \Lambda(k)$ holds, so that NLLT is in force.

**Example 3** In the example below we build a model that satisfies conditions $I$ and $II$, but disobeys condition (34) of Theorem 2. So, for this model the generating function grows exponentially, while the local limit theorem does not hold. The idea of the construction is motivated by Example 3 in [10]. Let $S(z) = (1 - z)^{-1}$, $|z| < 1$, $a_k \equiv 1$, $k \geq 1$ and

$$b_k = \begin{cases} (k \log^\epsilon k)^{-1}, & 0 < \epsilon < 1, \quad \text{if } 4 \nmid k, \quad k \geq 2, \\ (k \log^\epsilon k)^{-1} + 1, & 0 < \epsilon < 1, \quad \text{if } 4 \mid k. \end{cases}$$

(98)

Thus, in the case considered, the function $D_\theta(s)$ can be written as

$$D_\theta(s) = D_\theta(s; \epsilon) = D^{(1)}_\theta(s; \epsilon) + 4^{-s} \zeta(s),$$
where

\[ D_b^{(1)}(s; \epsilon) := \sum_{k=2}^{\infty} \frac{1}{k^{s+1} \log^\epsilon k}. \]

Denoting \( f(x; s, \epsilon) = (x^{s+1} \log^\epsilon x)^{-1} \), we apply the Euler-Maclaurin summation formula (see e.g. [2]) to get:

\[ D_b^{(1)}(s; \epsilon) = \int_{2}^{\infty} f(x; s, \epsilon) dx + \frac{f(2; s, \epsilon) + f(\infty; s, \epsilon)}{2} + \int_{2}^{\infty} f'(x; s, \epsilon)(x - [x]) dx, \quad (99) \]

where \([x]\) is the integer part of \( x \). In the representation (99), \( f(\infty; s, \epsilon) = 0 \), for \( \Re(s) > -1 \), \( 0 < \epsilon < 1 \), and \( f'(x; s, \epsilon) = -x^{-s-2} \log^{-\epsilon} x (s + 1 + \epsilon \log^{-1} x) \). The latter implies that the second integral in (99) converges absolutely for \( \Re(s) > -1 \), \( \epsilon > 0 \). Since the first integral in (99), which we denote \( Q(s) \), converges for \( s > 0 \), it is left to show that \( Q(s) \) admits meromorphic continuation to all \( \C \). We have for \( \Re(s) > 0 \), \( 0 < \epsilon < 1 \),

\[ Q(s) := \int_{2}^{\infty} f(x; s, \epsilon) dx = -s^{-1}(x^{-s} \log^{-\epsilon} x)|_{2}^{\infty} - \epsilon s^{-1} \int_{2}^{\infty} x^{-s-1} \log^{-\epsilon-1} x \, dx = \]

\[ s^{-1} 2^{-s} \log^{-\epsilon} 2 - \epsilon s^{-1} \int_{2}^{\infty} x^{-s-1} \log^{-\epsilon-1} x \, dx, \]

which can be rewritten as a differential equation with respect to \( Q(s) \):

\[ (sQ(s))' = -2^{-s} \log^{1-\epsilon} 2 + \epsilon Q(s). \quad (100) \]

Here we made use of the fact that \( (\int_{2}^{\infty} x^{-s-1} \log^{-\epsilon-1} x \, dx)'_s = -Q(s) \).

The solution of (100) can be found explicitly:

\[ Q(s) = s^{\epsilon-1} \left( C + (\log^{1-\epsilon} 2) \int_{0}^{s} 2^{-u} u^{-\epsilon} du \right), \quad 0 < \epsilon < 1, \ \Re(s) \neq 0, \quad (101) \]

where \( C = C(\epsilon) \) is a constant. We conclude from (101) that \( Q(s) \) allows analytic continuation to all \( s \in \C/0 \), with residue 0 at \( s = 0 \), because

\[ \lim_{s \to 0^+} sQ(s) = 0, \quad 0 < \epsilon < 1. \]
In view of (99), the residue of $D_1^b(s; \epsilon)$, $0 < \epsilon < 1$ at $s = 0$ is zero, as well. Thus, conditions I, II hold, while the NLLT is not in force, since the condition (34) is violated:

\[
\sum_{1 \leq k \leq \delta^{-1}} b_k = \sum_{1 \leq k \leq \delta^{-1}} (k \log^\epsilon k)^{-1} = O(-\log^{1-\epsilon} \delta_n), \ n \to \infty, \ 0 < \epsilon < 1.
\]

**Remark on models with non exponential rate of growth.** In Applications, mainly in number theory, one meets models for which $c_n$ does not grow exponentially with $n$. Such cases are known long ago, two typical examples are the number of square-free integers and Goldbach partitions of an even integer into sum of two odd primes. For recent developments in the study of such models see [5] and [7]. Note that the second model is a multiplicative one, while the first is not.

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