Spinor Two-Point Functions and Peierls Bracket in de Sitter Space

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Abstract

This paper studies spinor two-point functions for spin-1/2 and spin-3/2 fields in maximally symmetric spaces such as de Sitter(dS) spacetime, by using intrinsic geometric objects. The Feynman, positive- and negative-frequency Green functions are then obtained for these cases, from which we eventually display the supercommutator and the Peierls bracket under such a setting in two-component-spinor language.
I. INTRODUCTION

The formulation of a theory of quantum gravity requires one to thoroughly understand the particle propagation in curved spacetimes. Maximally symmetric spaces such as de Sitter and anti-de Sitter provide one with an interesting backdrop to study quantum field theory in curved spacetimes. In this background geometry, if one needs to calculate basic quantities like scattering amplitudes, one should find out the correlation function which involves the propagators for various particles in this background. Thus, the problem of calculating the propagators has always been of much physical interest to several authors. This hunt also assumed much significance after the advent of the famous Maldacena conjecture or the AdS/CFT correspondence [1, 2, 3] which proposes a duality between a quantum gravity theory on the bulk $AdS_{d+1}$ and a strongly coupled conformal d-dimensional gauge theory at large $N$ on the boundary of it. Then there is the recently proposed dS/CFT correspondence [4] which might shed light on quantum gravity in de Sitter space. This conjecture, which is largely modeled on analogy with AdS/CFT [5], still lacks a clear relation to string theory which in turn hinders the explicit realization of the proposal made by Strominger. At the same time, a consistent formulation of all interactions in de Sitter space is also tempting because of the recent observational data in favor of the inflationary picture.

In field theory the Peierls bracket is a Poisson bracket which is invariant under the full infinite-dimensional invariance group of the action functional. Without invoking a definition of canonical coordinates and canonical momenta in advance, the Peierls bracket follows directly from the classical action, and is made out of the advanced and retarded Green’s functions. Hence it is necessary to build the spinor parallel propagator and the spinor Green function in order to write the Peierls bracket in de Sitter and anti-de Sitter spaces for spin-1/2 and spin-3/2 particles. Here we focus our calculation on de Sitter space.

This paper consists of two parts, the first one involves the case of ordinary spin-1/2 particles, and the second part extends the same physics to spin-3/2 fields, i.e. the gravitino. In this pedagogical paper we first introduce the idea of the Peierls bracket in Sec. II, while Sec. III contains an introduction to maximally symmetric bitensors. In Sec. IV we review a few elementary properties of the spinor parallel propagator, in Sec. V we calculate the massive spinor Green functions and hence the Feynman, positive- and negative-frequency two-point functions. Then we show how to build a Peierls bracket from this for the spin-1/2
case. In Sec. VI we summarize all techniques developed so far in this paper and apply them to evaluate the gravitino Green functions in four-dimensional de Sitter space in two-component spinor language, and finally conclude our paper with the explicit construction of a Peierls bracket for gravitinos, which has not been done so far to our knowledge. Concluding remarks are presented in Sec. VII, while relevant details are given in the Appendix.

II. THE PEIERLS BRACKET

Since the Peierls bracket is not quite a familiar concept, a brief review about it is presented here. For more details on the subject we refer the reader to [6, 7, 8] and the references therein.

It was in the early fifties when R.E. Peierls first noticed a similarity in algebraic structure between the Poisson bracket and the Peierls bracket as it is called today (for theories without gauge freedom the Peierls bracket is indeed a Poisson bracket, whereas for gauge theories it becomes a Poisson under restriction to the space of observables [6, 8]), and found that this new structure could be defined directly from the action principle without performing a canonical decomposition into coordinates and momenta. His essential insight was to consider the advanced and retarded “effect of one quantity \( \langle A \rangle \) on another \( \langle B \rangle \).” Here, \( A \) and \( B \) are functions on the space of histories \( \mathcal{H} \). The space-of-histories formulation using the DeWitt condensed-index notation often proves indeed very useful in the study of the generalized Peierls algebra and provides the opportunity to introduce in a concise way the relevant techniques. We first define the advanced and retarded effects of \( A \) and \( B \) on each other as functions on \( \mathcal{H} \), from which the Peierls bracket follows. This will be straightforward by using the machinery of [8] and indeed, much of what follows is implicit in that treatment.

Once the action functional \( S \) is replaced by a new action functional \( S + \epsilon A \) after the interaction with some external agent, the small disturbances \( \delta \phi^i \) are ruled by an inhomogeneous differential equation (see below) which is solved after inverting a differential operator \( F_{ij} \). On denoting by \( G^{\pm ij} \) the advanced (resp. retarded) Green functions of \( F_{ij} \), one can define

\[
\delta^\pm A B \equiv \epsilon B_i G^{\pm ij} A_j, \quad D_A B \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \delta^A \to B, \quad (2.1)
\]

and the Peierls bracket

\[
(A, B) \equiv D_A B - D_B A. \quad (2.2)
\]
To be more precise, following [8], recall that the undisturbed fields satisfy the equations of motion

\[ 0 = S_{i} \left( \phi^{j} \right) \]  \quad (2.3)

while the disturbed fields satisfy

\[ 0 = S_{\epsilon i} \left( \phi^{j}_{\epsilon} \right) = S_{i} \left( \phi^{j}_{\epsilon} \right) + \epsilon A_{i} \left( \phi^{j}_{\epsilon} \right). \]  \quad (2.4)

To first order, the perturbations \( \delta \phi^{i} \) are therefore governed by the equation

\[ S_{ij} \left( \phi^{k} \right) \delta \phi^{j} = -\epsilon A_{i} \left( \phi^{k} \right), \]  \quad (2.5)

and we see that both the boundary conditions (advanced or retarded) and any gauge fixing applies only to the inversion of the operator \( S_{ij} \left( \phi^{k} \right) \) in the above linear equation for \( \delta \phi^{j} \) and not to the solution of (2.3) for \( \phi^{i} \). In the case where there are no gauge symmetries, \( S_{ij} \) is invertible and has advanced and retarded Green’s functions \( G^{\pm jk} \) that satisfy [8]

\[ S_{ij} G^{\pm jk} = -\delta^{k}_{i}, \]  \quad (2.6)

so that the advanced and retarded solutions to the above equations are \( \delta^{\pm} \phi^{j} = \epsilon G^{\pm ji} A_{i} \) where both \( G^{\pm ji} \) and \( A_{i} \) depend on the unperturbed solution \( \phi^{i} \). From the definitions (2.1) and (2.2), the Peierls bracket is just

\[ (A, B) = A_{i} \tilde{G}^{ij} B_{j}, \]  \quad (2.7)

where

\[ \tilde{G}^{ij} \equiv G^{+ij} - G^{-ij} \]  \quad (2.8)

is called the supercommutator function, i.e. the difference of advanced and retarded Green functions.

For gauge fields, however, there exists on \( \Phi \) a set of vector fields \( Q_{\alpha} \) that leave the action \( S \) invariant, i.e.

\[ Q_{\alpha} S = 0. \]  \quad (2.9)

If \( A \) and \( B \) are two such gauge-invariant functionals:

\[ Q_{\alpha} A = Q_{\alpha} B = 0, \]  \quad (2.10)
then their Peierls bracket \((A, B)\) is defined as follows [9, 10]:

\[
(A, B) \equiv A_i \tilde{G}^{ij} B_j = \int \int dx \, dy \frac{\delta A}{\delta \varphi^i(x)} \tilde{G}^{ij}(x, y) \frac{\delta B}{\delta \varphi^j(y)},
\]

(2.11)

where the advanced and retarded Green functions used to define the supercommutator \(\tilde{G}^{ij}\) now pertain to the invertible gauge-field operator obtained from the gauge-fixing procedure. Since \(A\) and \(B\) are observables, Jacobi identity and gauge invariance hold for the Peierls bracket (for a detailed proof of these properties see, for example, [6, 8]).

**III. MAXIMALLY SYMMETRIC BITENSORS**

More than two decades ago Allen and co-authors used intrinsic geometric objects to calculate correlation functions in maximally symmetric spaces; their results, here exploited, were presented in a series of papers [11, 12]. In this section we would like to review the elementary maximally symmetric bi-tensors which have been discussed previously by Allen and Jacobson [11], although more recently the calculation of spinor parallel propagator has been carried out in arbitrary dimension [13].

A maximally symmetric space is a topological manifold of dimension \(n\), with a metric which has the maximum number of global Killing vector fields. This type of space looks exactly the same in every direction and at every point. The simplest examples are flat space and sphere, each of which has \(\frac{1}{2}n(n+1)\) independent Killing fields. For \(S^n\) these generate all rotations, and for \(\mathbb{R}^n\) they include both rotations and translations.

Consider a maximally symmetric space of dimension \(n\) with constant scalar curvature \(n(n-1)/R^2\). For the space \(S^n\), the radius \(R\) is real and positive, whereas for the hyperbolic space \(H^n\), \(R = il\) with \(l\) positive, and in the flat case, \(\mathbb{R}^n\), \(R = \infty\). Consider further two points \(x\) and \(x'\), which can be connected uniquely by a shortest geodesic. Let \(\mu(x, x')\) be the proper geodesic distance along this shortest geodesic between \(x\) and \(x'\). If \(n^a(x, x')\) and \(n^{a'}(x, x')\) are the tangents to the geodesic at \(x\) and \(x'\), the tangent vectors are then given in terms of the geodesic distance as follows:

\[
n_a(x, x') = \nabla_a \mu(x, x') \quad \text{and} \quad n^{a'}(x, x') = \nabla_{a'} \mu(x, x').
\]

(3.1)

Furthermore, on denoting by \(g^a_{b'}(x, x')\) the vector parallel propagator along the geodesic, one can then write \(n^{b'} = -g^b_{a'} n^a\). Tensors that depend on two points, \(x\) and \(x'\), are bitensors [14]. They may carry unprimed or primed indices that live on the tangent space at \(x\) or \(x'\).
These geometric objects \( n^a, n^{a'} \) and \( g_{\mu}^{b} \) satisfy the following properties \[11\]:

\[
\nabla_a n_b = A(g_{ab} - n_a n_b), \quad (3.2a)
\n\nabla_a n_{\nu} = C(g_{a\nu} + n_a n_\nu), \quad (3.2b)
\n\nabla_a g_{bc} = -(A + C)(g_{ab} n_c + g_{ac} n_b), \quad (3.2c)
\]

where \( A \) and \( C \) are functions of the geodesic distance \( \mu \) and are given by \[11\]

\[
A = \frac{1}{R} \cot \frac{\mu}{R} \quad \text{and} \quad C = -\frac{1}{R \sin(\mu/R)}, \quad (3.3)
\]

and thus they satisfy the relations

\[
dA/d\mu = -C^2, \quad dC/d\mu = -AC \quad \text{and} \quad C^2 - A^2 = 1/R^2. \quad (3.4)
\]

Last, our convention for covariant gamma matrices is

\[
\{\Gamma^\mu, \Gamma^\nu\} = 2Ig^{\mu\nu}. \quad (3.5)
\]

IV. THE SPINOR PARALLEL PROPAGATOR

In this paper we follow the conventions for two-component spinors, as well as all signature and curvature conventions, of Allen and Lutken \[12\], and hence we use dotted and undotted spinors instead of the primed and unprimed ones of Penrose and Rindler \[15\]. In our work a primed index indicates instead that it lives in the tangent space at \( x' \), while the unprimed ones live at \( x \). The fundamental object to deal with here is the bispinor \( D_A^{A'}(x, x') \) which parallel transports a two-component spinor \( \phi^A \) at the point \( x \), along the geodesic to the point \( x' \), yielding a new spinor \( \chi^{A'} \) at \( x' \), i.e.

\[
\chi^{A'} = \phi^A D_A^{A'}(x, x'). \quad (4.1)
\]

Complex conjugate spinors are similarly transported by the complex conjugate of \( D_A^{A'}(x, x') \), which is \( \overline{D_A^{A'}}(x, x') \). A few elementary properties of \( D_A^{A'} \) are listed below (some of them will
be used for later calculations) [12]:

\[ D^A_A(x, x') = -D^A_A(x', x), \]  
\[ D^A_A D^B_A = \varepsilon^B_A, \]  
\[ D_{AA'} D^{AA'} = 2, \]  
\[ \lim_{x \to x'} D^B_A = \varepsilon^B_A, \]  
\[ g^B_a = D^B_A D^B_A, \]  
\[ D^B_A n^{AA} = -n^{B'B'}, \]  
\[ n^{AC} D^A_B = -n^{B'B'} D^B_C, \]  
\[ \nabla_{AA'} D^A_A = \frac{3}{2}(A + C)n^{AA} D^A_A, \]  
\[ D^A_A n^{AA'} = -\frac{3}{2} C D^A_A, \]  
\[ \nabla_{AA'} n^A_B = \frac{3}{2} A \varepsilon^B_{AB}. \]  

Just to recall the previously defined notations and set up the two-component formalism, we note from (3.1) that \( n^{AA} = \nabla_{AA} \mu \) and \( n^{AA'} = \nabla_{AA'} \mu \), where \( \mu(x, x') \) is the geodesic separation of \( x \) and \( x' \). For completeness we should also find the covariant derivative of \( D^A_A \), which is formed out of the tangent \( n^{AA} \) to the geodesic and from \( D^A_A \) itself, i.e.

\[ \nabla_{AA'} D^B_A \equiv \alpha(\mu)n^{AA} D^B_A + \beta(\mu)n^{BA} D^A_A. \]  

Here \( \alpha \) and \( \beta \) are two arbitrary functions of the geodesic distance to be determined. But both of them are not independent and are related to each other because of the fact that \( D^B_A \) and \( n^{AA} \), by definition, satisfy the following relations [12]:

\[ n^a \nabla_a D^B_A = 0, \]  
\[ n^{AA} n^{BA} = \frac{1}{2} \delta^B_A. \]  

From the relations (4.4) and (4.5) it follows that \( \beta(\mu) = -2\alpha(\mu) \). One then determines \( \beta(\mu) \) by using the Ricci identity, i.e. the integrability condition for spinors [15], and after all dust gets settled one obtains the final form of the covariant derivative of the spinor parallel propagator as

\[ \nabla_{AA'} D^B_A = (A + C) \left[ \frac{1}{2} n^{AA} D^B_A - n^{BA} D^A_A \right], \]  

where \( A \) and \( C \) are defined as in the previous section.
V. THE SPINOR GREEN FUNCTION

First we define a four-component Dirac spinor by

$$\psi_\alpha = \begin{pmatrix} \phi_A \\ \chi_{\dot{A}} \end{pmatrix} = \begin{pmatrix} \phi_A \\ \chi_{\dot{A}} \end{pmatrix},$$ (5.1)

where $\phi_A$ and $\chi_{\dot{A}}$ are a pair of two-component spinors satisfying the Dirac equation [15]

$$\nabla_A \phi_A = -\frac{m}{\sqrt{2}} \chi_{\dot{A}},$$ (5.2)

$$\nabla_{\dot{A}} \chi_{\dot{A}} = \frac{m}{\sqrt{2}} \phi_A,$$ (5.3)

$m$ being the mass of the spin-1/2 field. We can define two basic massive two-point functions, which are

$$P^{A\dot{B}'} = \langle \phi^A(x) \bar{\phi}^{\dot{B}'}(x') \rangle = f(\mu)D_A^{\dot{B}'} n^{A'\dot{B}'},$$ (5.4)

$$Q_A^{\dot{B}'} = \langle \chi_{\dot{A}}(x) \bar{\phi}^{\dot{B}'}(x') \rangle = g(\mu)D^{\dot{B}'}_{\dot{A}}.$$. (5.5)

Here we temporarily assume the spacelike separation between the points $x$ and $x'$ such that the field operators in (5.4) and (5.5) anti-commute. On the right-hand side of (5.4) and (5.5) we have the most general maximally symmetric bispinor with the correct index structure. It is to be noted that the functions $f$ and $g$ appearing here in the structure, do depend only on the geodesic distance $\mu$, and other two-point functions like $\langle \chi_{\dot{A}} \chi^{B'} \rangle$ and $\langle \phi^A \chi^{B'} \rangle$ are entirely determined by $f$ and $g$ only. The equations of motion (5.2) and (5.3) now imply that

$$\nabla_{\dot{A}} P^{A\dot{B}'} = -\frac{m}{\sqrt{2}} Q_A^{\dot{B}'};$$ (5.6)

$$\nabla^A Q_A^{\dot{B}'} = \frac{m}{\sqrt{2}} P^{A\dot{B}'}.$$ (5.7)

If now we insert equations (5.4) and (5.5) into equations (5.6) and (5.7) we obtain, after a little gymnastics with two-spinor calculus, two coupled equations for the coefficients $f(\mu)$ and $g(\mu)$ as follows:

$$f' + \frac{3}{2} (A - C) f + \sqrt{2} m g = 0,$$ (5.8)

$$g' + \frac{3}{2} (A + C) g - \frac{m}{\sqrt{2}} f = 0.$$ (5.9)
where the prime stands for derivative with respect to $\mu$. On differentiating (5.8) with respect to $\mu$ once and then using (3.4) and (5.9) successively one finds a second-order equation for $f$:

$$f''(\mu) + 3Af'(\mu) + \left[m^2 - \frac{9}{4}R^{-2} + \frac{3}{2}C(A - C)\right] f(\mu) = 0.$$  \hfill (5.10)

Now to solve for $f(\mu)$ and $g(\mu)$, one makes a change of variable

$$Z \equiv \cos^2 \left(\frac{\mu}{2R}\right)$$  \hfill (5.11)

to write (5.10) as

$$Z(1 - Z) \frac{d^2}{dZ^2} f(Z) + 2(1 - 2Z) \frac{d}{dZ} f(Z) + \left[m^2R^2 - \frac{9}{4} - \frac{3}{4(1 - Z)}\right] f(Z) = 0.$$  \hfill (5.12)

On making further a redefinition

$$w(Z) \equiv [R^2(1 - Z)]^{-1/2} f(Z),$$  \hfill (5.13)

one rewrites (5.12) as a hypergeometric equation in the variable $w$, i.e.

$$H(a, b, c; Z) w(Z) = 0,$$  \hfill (5.14)

where $H(a, b, c)$ is the hypergeometric operator

$$H(a, b, c; Z) = Z(1 - Z) \frac{d^2}{dZ^2} + [c - (a + b + 1)Z] \frac{d}{dZ} - ab.$$  \hfill (5.15)

Following our source, the factor $R^2$ is included in the definition (5.13) of $w$ to ensure that the standard branch cut of the square root function lies along the timelike separations $\mu^2 > 0$. The parameters $a, b, c$ here take the values

$$a = 2 + \sqrt{m^2R^2},$$  \hfill (5.16a)

$$b = 2 - \sqrt{m^2R^2},$$  \hfill (5.16b)

$$c = 2.$$  \hfill (5.16c)

In the same way it can be shown that if we let $w(Z) = [R^2(Z)]^{-1/2} g(Z)$, then $w$ satisfies a hypergeometric equation with parameters $a, b$ and $c + 1$. Now one has to specify the boundary conditions to uniquely specify a solution to the hypergeometric equation. The correct solution to (5.14) in de Sitter space $R^2 < 0$ is obtained (following (11)) by demanding that it is only singular when $\mu = 0$, that is $Z = 1$, and not when $\mu = \pi R$, that is $Z = 0$. Two
independent solutions of the hypergeometric equations \[16, 17\] are therefore \(F(a, b; c; Z)\) and \(F(a, b; c + 1; Z)\), and this yields the following solutions:

\[
f_{DS} = N_{DS}(1 - Z)^{1/2} F(a, b; c; Z), \tag{5.17}
\]
\[
g_{DS} = -i N_{DS} 2^{-3/2} m |R| Z^{1/2} F(a, b; c + 1; Z). \tag{5.18}
\]

The short distance behavior \(\mu \to 0\) can now be used to fix the as yet undetermined constant \(N_{DS}\). The flat-space limit as \(\mu \to 0\) is

\[
f \sim -i \frac{1}{\sqrt{2 \pi^2}} (-\mu^2)^{-3/2}. \tag{5.19}
\]

Thus, from (5.17) it follows that

\[
N_{DS} = \frac{f_{DS}}{(1 - Z)^{1/2} F(a, b; c; Z)}. \tag{5.20}
\]

Furthermore, near \(Z = 1\) we have

\[
F(a, b; c; Z) \sim \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} (1 - Z)^{c - a - b}, \tag{5.21}
\]

and \((1 - Z) = (\mu/2R)^2\), hence one finds that

\[
N_{DS} = -i \frac{(-\mu^2)^{-3/2}}{\sqrt{2 \pi^2}} \frac{\Gamma(a) \Gamma(b)}{\Gamma(c) \Gamma(a + b - c)} (1 - Z)^{a + b - c - \frac{1}{2}} = -i \frac{(-\mu^2)^{-3/2}}{\sqrt{2 \pi^2}} \frac{\Gamma(a) \Gamma(b)}{\Gamma(2)} \frac{\mu^3}{8R^3}, \tag{5.22}
\]

where we have used the fact that \(\Gamma(a + b - c) = \Gamma(2) = 1\) and similarly \(\Gamma(c) = 1\). On using the values of \(a\) and \(b\) and putting them together in the expression (5.22) one gets

\[
N_{DS} = -i \frac{\Gamma(2 + \sqrt{m^2 R^2}) \Gamma(2 - \sqrt{m^2 R^2})}{8 \sqrt{2 \pi^2} |R|^3}. \tag{5.23}
\]

Furthermore, from the relations \(\Gamma(z + 1) = z \Gamma(z)\) and \(\Gamma(1 + i|mR|) \Gamma(1 - i|mR|) = \frac{\pi |Rm|}{\sinh(\pi |Rm|)}\)

one can rewrite the final answer for the constant \(N_{DS}\)

\[
N_{DS} = -i |Rm| (1 - m^2 R^2) \frac{(1 - m^2 R^2)}{8 \sqrt{2 \pi^2} |R|^3 \sinh \pi |Rm|}. \tag{5.24}
\]

Once we determine \(N_{DS}\), the Feynman Green function is obtained by evaluating \(f_{DS}(Z)\) and \(g_{DS}(Z)\) above the branch cut from \(Z = 1\) to \(\infty\), i.e. by taking \(f_{DS}(Z + i0)\) and \(g_{DS}(Z + i0)\). This is what happens in the de Sitter case. To conclude we have the following two-point functions:

\[
P_{(F)}^{AB'} = \lim_{\epsilon \to 0^+} f_{DS}(Z + i \epsilon) D_A n^A B', \tag{5.25}
\]
\[ Q_{(F)}^{AB'} = \lim_{\epsilon \to 0^+} g_{DS}(Z + i\epsilon)D^{AB'}, \] (5.26)

where \((F)\) stands for the Feynman Green functions.

It is now helpful to recall the discussion of various types of Green functions depending on the contours in the complex \(p^0\)-plane for the integral representation of the Green function for the simpler case of scalar fields, following [8]. From various contours the following relations among different Green’s functions can be easily established:

\[ G_F = G^- + G^(-) = G^+ - G^+, \] (5.27a)
\[ G^+(x, x') = -\theta(x, x')G_F(x, x') + \theta(x, x)G_F^+(x, x'), \] (5.27b)
\[ G^-(x, x') = \theta(x, x)G_F(x, x') - \theta(x, x')G_F^+(x, x'), \] (5.27c)
\[ \tilde{G} = (G^+ - G^-) = (G^+ + G^-) = -2\left(\theta(x, x') - \theta(x', x)\right)\text{Re}G_F. \] (5.27d)

With a standard notation, \(G^+\) and \(G^-\) are the advanced and retarded functions respectively, and their difference \(\tilde{G}\) is the supercommutator function. \(G_F\) is the Feynman Green function and \(G_F^*\) is its complex conjugate. \(G^+\) and \(G^-\) are the positive- and negative-frequency parts, respectively. The \(\theta(x, x')\) used above in the definition of advanced and retarded functions is the step function.

Now our approach to arrive at the Peierls bracket in the de Sitter case will be as follows: once we determine using (5.25) and (5.26) the Feynman Green function, instead of using the advanced and retarded functions, we can use (5.27b) and (5.27c) respectively to get \(G^+\) and \(G^-\), and then add them to get the supercommutator function \(\tilde{G}\). Then we use (2.11) to build the Peierls bracket \((\psi, \chi)_{P}\) which, in terms of the spinor fields

\[ \psi_\alpha = \begin{pmatrix} \phi_A \\ \bar{\chi}_A \end{pmatrix}, \quad \chi_{\beta'} = \begin{pmatrix} \rho_{B'} \\ \bar{\sigma}_{B'} \end{pmatrix} \] (5.28)

reads as

\[ (\psi, \chi)_P \equiv \int \int P(x, x')\sqrt{-g(x)}\sqrt{-g(x')}d^4xd^4x', \] (5.29)

where

\[ P(x, x') \equiv -2\left(\theta(x, x') - \theta(x', x)\right)\psi_\nabla(\text{Re}G_F)\chi_\nabla, \] (5.30)

having set

\[ \psi_\nabla(\text{Re}G_F)\chi_\nabla \equiv \left(\nabla_{AA'}\bar{\phi}^A\right)\text{Re}P_{(F)}^{AB'}\left(\nabla_{B'B'}\sigma^{B'}\right) + \left(\nabla_{AA}\chi^A\right)\text{Re}Q_{(F)}^{AB'}\left(\nabla_{B'B'}\rho^{B'}\right). \] (5.31)
VI. MASSIVE SPIN-3/2 PROPAGATOR

In this section we consider the propagator of the massive spin-3/2 field. Let us denote the gravitino field by $\Psi^\alpha_\lambda(x)$. In a maximally symmetric state $|s>$ the propagator is

$$S_{\lambda\nu}^{\alpha\beta'}(x,x') = <s|\Psi^\alpha_\lambda(x)\Psi^{\beta'}_{\nu'}(x')|s>.$$  \hspace{1cm} (6.1)

The field equations imply that $S$ satisfies

$$(\Gamma^{\mu\rho\lambda}D_\rho - m \Gamma^{\mu\lambda})^\alpha S_{\lambda\nu}^{\gamma\beta'} = \frac{\delta(x-x')}{\sqrt{-g}} g^{\mu\nu'} \delta^{\alpha\beta'}.$$ \hspace{1cm} (6.2)

A. The ten gravitino invariants in two-component-spinor language

It is very convenient to decompose the gravitino propagator in terms of independent structures constructed out of $n_\mu$, $n_{\nu'}$, $g_{\mu\nu'}$ and $\Lambda^{\alpha}_{\beta'}$. Thus, the propagator can be written in geometric way following Anguelova et al. \cite{18} (see also \cite{19}):

$$S_{\lambda\nu}^{\alpha}_{\beta'} = \alpha(\mu) g_{\lambda\nu'} \Lambda^{\alpha}_{\beta'} + \beta(\mu) n_\lambda n_{\nu'} \Lambda^{\alpha}_{\beta'} + \gamma(\mu) g_{\lambda\nu'} (n_\sigma \Gamma^{\sigma}\Lambda)^{\alpha}_{\beta'}$$
$$+ \delta(\mu) n_\lambda n_{\nu'} (n_\sigma \Gamma^{\sigma}\Lambda)^{\alpha}_{\beta'} + \varepsilon(\mu) n_\lambda (\Gamma_{\nu'}\Lambda)^{\alpha}_{\beta'} + \theta(\mu) n_{\nu'} (\Gamma_{\lambda}\Lambda)^{\alpha}_{\beta'}$$
$$+ \tau(\mu) (\Gamma_{\lambda}\Gamma_{\nu'}\Lambda)^{\alpha}_{\beta'} + \kappa(\mu) (n_\sigma \Gamma^{\sigma}\Gamma_{\lambda}\Lambda)^{\alpha}_{\beta'}.$$ \hspace{1cm} (6.3)

From here on we will be trying to re-write each of the building blocks of the invariant structure in two-spinor language and finally construct the full invariant propagator in two-spinor form in harmony with the spin-1/2 propagator previously discussed. Following Allen and Lutken \cite{12} we can write the gamma matrix in two-spinor language as follows (Penrose and Rindler, on page 221 of \cite{15}, do not have the $-i$ factor since their $\gamma$-matrices satisfy the anti-commutation relation $\gamma_a \gamma_b + \gamma_b \gamma_a = -2I g_{ab}$, unlike the sign convention in our Eq. (3.5))

$$\gamma^\beta_a = -i\sqrt{2} \left( \begin{array}{cc} 0 & \varepsilon_{PA} \varepsilon^B \varepsilon_{F} \\ \varepsilon_{P} \varepsilon^A \varepsilon_{F} & 0 \end{array} \right),$$ \hspace{1cm} (6.4)

where $\varepsilon_{BC}$ is the curved epsilon symbol which raises and lowers indices within each spin-space, is skew-symmetric and encodes information on the curved spacetime metric. In the case of flat Minkowski spacetime it reduces to the well known form

$$\varepsilon_{BC} = \varepsilon_{BA} \varepsilon^A = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right).$$ \hspace{1cm} (6.5)
From the rules of two-spinor calculus and from the treatment of Allen and Lutken [12] we already know the following correspondences:

\[ n_\alpha \rightarrow n_{A\dot{A}}, \quad (6.6a) \]
\[ n_{\beta'} \rightarrow n_{B\dot{B}'}, \quad (6.6b) \]
\[ g_{\alpha\beta'} \rightarrow D_{AB'}\overline{D}_{\dot{A}\dot{B}'} \quad (6.6c) \]

We also know the form of the spinor parallel propagator, which acts according to

\[ \chi^A(x') = D^A_{\dot{A}'}(x,x')\phi^\dot{A}(x), \quad (6.7a) \]
\[ \chi^A(x) = D^A_{B'}(x,x')\phi^{B'}(x'), \quad (6.7b) \]
\[ \phi^{\dot{A}}(x) = \overline{D}^{\dot{A}}_{B'}(x,x')\chi^{B'}(x'). \quad (6.7c) \]

To translate the above set of equations, e.g. \( \phi^\alpha = \Lambda^\alpha_{\beta'}\phi^{\beta'} \), into two-spinor language, both left- and right-hand sides should involve a column vector, with upstairs indices at \( x \) and \( x' \) respectively. Written in matrix notation we can combine them into one reading, i.e.

\[
\begin{pmatrix}
\chi^A \\
\phi^{\dot{A}}
\end{pmatrix}
= 
\begin{pmatrix}
0 & D^A_{B'} \\
\overline{D}^{\dot{A}}_{B'} & 0
\end{pmatrix}
\begin{pmatrix}
\chi^{B'} \\
\phi^{B'}
\end{pmatrix}.
\] (6.8)

Therefore from now on we redefine \( \Lambda^\rho_{\beta'} \) to be a \((2x2)\) matrix, expressed in two-spinor language as \( \Lambda^{RR}_{B'B'} \) and satisfying the correspondence rule

\[
\Lambda^\rho_{\beta'} \rightarrow \begin{pmatrix}
0 & D^R_{B'} \\
\overline{D}^R_{B'} & 0
\end{pmatrix}.
\] (6.9)

Similarly, we go on translating each of the bits of the invariant structure into two-spinor notation. The next one is \((\Gamma^\sigma \Lambda)^\alpha_{\beta'}\). We note the following translation:

\[
(\Gamma^\sigma \Lambda)^\alpha_{\beta'} = (\Gamma^\sigma)^\alpha_{\rho} \Lambda^\rho_{\beta'} \rightarrow (\Gamma^{SS})^\alpha_{\rho} \Lambda^\rho_{\beta'}.
\] (6.10)

Therefore, on using the two-spinor form (6.4) of the gamma matrix and the two-spinor version (6.9) of the spinor parallel propagator we get

\[
(\Gamma^\sigma \Lambda)^\alpha_{\beta'} \rightarrow -i\sqrt{2}
\begin{pmatrix}
0 & \varepsilon^{S}_{R} \varepsilon^{SA} \\
\varepsilon^{S}_{R} \varepsilon^{SA} & 0
\end{pmatrix}
\begin{pmatrix}
0 & D^R_{B'} \\
\overline{D}^R_{B'} & 0
\end{pmatrix}
= -i\sqrt{2}
\begin{pmatrix}
\varepsilon^{S}_{R} \varepsilon^{SA} D^R_{B'} & 0 \\
0 & \varepsilon^{S}_{R} \varepsilon^{SA} D^R_{B'}
\end{pmatrix}.
\] (6.11)
Similarly, we find
\[
(n_\sigma \Gamma^\sigma \Lambda)^{\alpha \beta'} \rightarrow n_{SS}(\Gamma^{SSS})^\rho_{\beta'} & = -in_{SS}\sqrt{2} \begin{pmatrix}
\varepsilon^S_R \varepsilon^{SA}D^R_{B'} & 0 \\
0 & \varepsilon^S_R \varepsilon^{SA}D^R_{B'}
\end{pmatrix}.
\]
(6.12)

Now we use the antisymmetry property of the epsilon symbol, i.e. \(\varepsilon^{AB} = -\varepsilon^{BA}\), and the rules for raising and lowering spinor indices, i.e. \(\varepsilon^{AB}\varphi_B = \varphi^A\), \(\varphi^A\varepsilon_{AB} = \varphi_B\), to write \((n_\sigma \Gamma^\sigma \Lambda)^{\alpha \beta'}\) in matrix form as
\[
(n_\sigma \Gamma^\sigma \Lambda)^{\alpha \beta'} \rightarrow -i\sqrt{2} \begin{pmatrix}
n_R^A \overline{D}^R_{B'} & 0 \\
0 & n_R^A D^R_{B'}
\end{pmatrix}.
\]
(6.13)

Now let us start writing the invariants in two-spinor language. The first invariant structure (see (6.3) from now on) is
\[
g_{\lambda\nu'}\Lambda^{\alpha \beta'} \rightarrow D_{LN'}D_{L'N} \begin{pmatrix}
0 & D^A_{B'} \\
\overline{D}^A_{B'} & 0
\end{pmatrix}.
\]
(6.14)
The second one is
\[
n_\lambda n_{\nu'}\Lambda^{\alpha \beta'} \rightarrow n_{L'L}n_{N'N'} \begin{pmatrix}
0 & D^A_{B'} \\
\overline{D}^A_{B'} & 0
\end{pmatrix}.
\]
(6.15) Then the third reads as
\[
g_{\lambda\nu'}(n_\sigma \Gamma^\sigma \Lambda)^{\alpha \beta'} \rightarrow -iD_{LN'}D_{L'N'}\sqrt{2} \begin{pmatrix}
n_R^A \overline{D}^R_{B'} & 0 \\
0 & n_R^A D^R_{B'}
\end{pmatrix}.
\]
(6.16)

Next is the fourth invariant, i.e.
\[
n_\lambda n_{\nu'}(n_\sigma \Gamma^\sigma \Lambda)^{\alpha \beta'} \rightarrow -in_{L'L}n_{N'N'}\sqrt{2} \begin{pmatrix}
n_R^A \overline{D}^R_{B'} & 0 \\
0 & n_R^A D^R_{B'}
\end{pmatrix}.
\]
(6.17)
The subsequent invariant structure involves \((\Gamma^\nu'\Lambda)^{\alpha \beta'}\) and we know that
\[
(\Gamma^\nu'\Lambda)^{\alpha \beta'} \rightarrow (\Gamma_{N'N'})^{\alpha \beta'}.
\]
(6.18)

Now from our previous discussion in this section we already have
\[
(\Gamma_{N'N'})^{\alpha \beta'} \rightarrow -i\sqrt{2} \begin{pmatrix}
0 & \varepsilon_{N'R'} \varepsilon^{A'}_{N'} \\
\varepsilon_{N'R'} \varepsilon^{A'}_{N'} & 0
\end{pmatrix}.
\]
(6.19)

The problem is that what is well defined is either \((\Gamma^\nu)^{\alpha \rho}_{\rho'}\) or \((\Gamma^\nu')^{\alpha' \rho'}\), where everything is evaluated at the same spacetime point (either \(x\) or \(x'\)). However, here the relevant invariant
consists of a mixed structure of the kind \((\Gamma_{\nu'})^\alpha_{\rho'}\), and, to understand what is meant by it, we should use the parallel displacement bi-vector. Eventually, with the help of some careful thought we can write

\begin{equation}
(\Gamma_{\nu'})^\alpha_{\rho} \rightarrow (\Gamma_{N'N'})^\alpha_{\rho'} g_{\rho\alpha} g_{\rho'\alpha'} = -i\sqrt{2} \left( \begin{array}{cc} 0 & \varepsilon_{N'R} \varepsilon_{N'}^A \\ \varepsilon_{N'R} \varepsilon_{N'}^A & 0 \end{array} \right) g_{A'A} \dot{A} \dot{R} \ddot{R}_R. \tag{6.20}
\end{equation}

Recalling the fact that \(g_{A'A} \dot{A} \dot{R} = D_A \dot{R} \ddot{R}_R = D_A R \ddot{R}_R\) and \(g_{\dot{R}R} \ddot{R}_R = D_{\dot{R}} \ddot{R}_R\) we can write the final form of the matrix \((\Gamma_{\nu'})^\alpha_{\rho}\) as follows:

\begin{equation}
(\Gamma_{\nu'})^\alpha_{\rho} \rightarrow -i\sqrt{2} \left( \begin{array}{cc} 0 & -D_{NN'} \ddot{R}_R D_{NN'} \ddot{R}_R^A D_{NN'}^A \\ -D_{NN'} \ddot{R}_R D_{NN'} \ddot{R}_R^A D_{NN'}^A & 0 \end{array} \right). \tag{6.21}
\end{equation}

Now we can build the fifth invariant quite easily as shown here,

\begin{equation}
n_{\lambda}(\Gamma_{\nu'} \Lambda)^\alpha_{\beta'} \rightarrow -in_{\ddot{L}L} \sqrt{2} \left( \begin{array}{cc} 0 & -D_{NN'} \ddot{R}_R D_{NN'} \ddot{R}_R^A D_{NN'}^A \\ -D_{NN'} \ddot{R}_R D_{NN'} \ddot{R}_R^A D_{NN'}^A & 0 \end{array} \right) \left( \begin{array}{cc} 0 & D_{B'}^{R} \\ \frac{D_{B'}^{R}}{D_{B'}^{R}} & 0 \end{array} \right). \tag{6.22}
\end{equation}

The sixth invariant is constructed as follows:

\begin{equation}
n_{\nu'}(\Gamma_{\Lambda} \Lambda)^\alpha_{\beta'} \rightarrow n_{NN'} \ddot{N'}(\Gamma_{\Lambda})^\alpha_{\rho} \Lambda^\rho_{\beta'} \rightarrow -n_{NN'} \ddot{N'} \sqrt{2} \left( \begin{array}{cc} 0 & \varepsilon_{LR} \varepsilon_{L}^A \\ \varepsilon_{LR} \varepsilon_{L}^A & 0 \end{array} \right) \left( \begin{array}{cc} 0 & D_{B'}^{R} \\ \frac{D_{B'}^{R}}{D_{B'}^{R}} & 0 \end{array} \right). \tag{6.23}
\end{equation}

Now we start building the last four invariants step by step. First we express the seventh invariant \(n_{\lambda}(\sigma \Gamma_{\nu'} \Lambda)^\alpha_{\beta'}\) in two-spinor language. We note that

\begin{equation}
n_{\lambda}(\sigma \Gamma_{\Lambda} \Lambda)^\alpha_{\beta'} = n_{\lambda} n_{\sigma} (\Gamma_{\Lambda})^\alpha_{\rho} (\Gamma_{\nu'})^\rho_{\tau'} g_{\rho\tau'} g_{\tau'\Lambda^\tau_{\beta'}} \\
\rightarrow -2n_{\ddot{L}L} n_{SS} \left( \begin{array}{cc} 0 & \varepsilon_{R} \varepsilon_{S}^A \\ \varepsilon_{R} \varepsilon_{S}^A & 0 \end{array} \right) \left( \begin{array}{cc} 0 & \varepsilon_{N'} \varepsilon_{N'}^R \\ \varepsilon_{N'} \varepsilon_{N'}^R & 0 \end{array} \right) \times D_{NN'} \ddot{N'}^{\ddot{R}} D_{NN'}^{T} D_{NN'}^{T'} \left( \begin{array}{cc} 0 & \ddot{N'}^{R} \\ \ddot{N'}^{R} & 0 \end{array} \right). \tag{6.24}
\end{equation}

After a little bit of algebra we arrive at the seventh invariant, i.e.

\begin{equation}
n_{\lambda}(n_{\sigma} \Gamma_{\nu'} \Lambda)^\alpha_{\beta'} \rightarrow -2n_{LL} \left( \begin{array}{cc} 0 & -n_{NN'} \ddot{N'}^{\ddot{R}} D_{NN'}^{R} D_{NN'}^{T} \ddot{N'}^{R} \\ -n_{NN'} \ddot{N'}^{\ddot{R}} D_{NN'}^{R} D_{NN'}^{T} \ddot{N'}^{R} & 0 \end{array} \right). \tag{6.25}
\end{equation}
Now let us write the eighth invariant term as follows:

\[ n_{\nu'}(n_{\sigma} \Gamma_{\lambda} \Lambda)^{\alpha_{\nu'}} = n_{\nu'} n_{\sigma}(\Gamma_{\lambda})^{\alpha_{\nu'}}(\Gamma_{\lambda})_{\alpha_{\nu'}} \Lambda^{\tau_{\nu'}} \]

\[ \rightarrow -2n_{N'N'} n_{SS} \left( \begin{array}{ccc} 0 & \varepsilon_{R'}^{S} \varepsilon_{S'}^{SA} & 0 \\ \varepsilon_{R'}^{S} \varepsilon_{S'}^{SA} & 0 & 0 \\ \varepsilon_{LT} \varepsilon_{LT} & 0 & \varepsilon_{LT} L_{B'} \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & D_{B'}^{T} \\ 0 & 0 & 0 \\ D_{B'}^{T} & 0 & 0 \end{array} \right). \] (6.26)

The final result for the eighth invariant is

\[ n_{\nu'}(n_{\sigma} \Gamma_{\lambda} \Lambda)^{\alpha_{\nu'}} \rightarrow -2n_{N'N'} \left( \begin{array}{ccc} 0 & n_{L} A_{\varepsilon_{LT}} D_{B'}^{T} \\ n_{L} A_{\varepsilon_{LT}} D_{B'}^{T} & 0 \end{array} \right). \] (6.27)

The ninth invariant structure can be translated in two-spinor form according to the following rule:

\[ (\Gamma_{\lambda} \Gamma_{\nu} \Lambda)^{\alpha_{\nu'}} = (\Gamma_{\lambda})^{\alpha_{\nu'}}(\Gamma_{\nu})^{\alpha_{\nu'}} \Lambda^{\tau_{\nu'}} = (\Gamma_{\lambda})^{\alpha_{\nu'}}(\Gamma_{\nu})^{\alpha_{\nu'}} \Lambda^{\tau_{\nu'}} \]

\[ \rightarrow -2D_{R'}^{T} D_{R'}^{T} D_{T}^{T} D_{B'}^{T} \left( \begin{array}{ccc} 0 & \varepsilon_{L} R \varepsilon_{L} \\ \varepsilon_{L} R \varepsilon_{L} & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & \varepsilon_{N'N'} R \varepsilon_{N'} \\ \varepsilon_{N'N'} R \varepsilon_{N'} & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & D_{B'}^{R} \\ 0 & 0 & 0 \end{array} \right). \] (6.28)

Eventually, after a few algebraic steps with matrices, we get the ninth invariant

\[ (\Gamma_{\lambda} \Gamma_{\nu} \Lambda)^{\alpha_{\nu'}} \rightarrow -2 \left( \begin{array}{ccc} \varepsilon_{L} A D_{N'}^{R} \varepsilon_{L} R \varepsilon_{L} & \varepsilon_{N'N'} R \varepsilon_{N'} & 0 \\ \varepsilon_{L} A D_{N'}^{R} \varepsilon_{L} R \varepsilon_{L} & \varepsilon_{N'N'} R \varepsilon_{N'} & 0 \end{array} \right). \] (6.29)

Last, but not least, the tenth invariant is built as follows:

\[ (n_{\sigma} \Gamma_{\lambda} \Gamma_{\nu'} \Lambda)^{\alpha_{\nu'}} = n_{\sigma}(\Gamma_{\sigma})^{\alpha_{\nu'}}(\Gamma_{\lambda})^{\alpha_{\nu'}}(\Gamma_{\nu'})^{\alpha_{\nu'}} \Lambda^{\chi_{\nu'}} \]

\[ = n_{\sigma}(\Gamma_{\sigma})^{\alpha_{\nu'}}(\Gamma_{\lambda})^{\alpha_{\nu'}}(\Gamma_{\nu'})^{\alpha_{\nu'}} \Lambda^{\chi_{\nu'}} \]

\[ \rightarrow -2\sqrt{2} n_{SS} \left( \begin{array}{ccc} 0 & \varepsilon_{R}^{S} \varepsilon_{S} \varepsilon_{SA} & 0 \\ \varepsilon_{R}^{S} \varepsilon_{S} \varepsilon_{SA} & 0 & 0 \\ \varepsilon_{LT} \varepsilon_{LT} & 0 & \varepsilon_{LT} L_{B'} \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & D_{B'}^{K} \\ 0 & 0 & 0 \\ D_{B'}^{K} & 0 & 0 \end{array} \right). \] (6.30)

At the end of the day, when all dust gets settled we obtain the final invariant in the form

\[ (n_{\sigma} \Gamma_{\lambda} \Gamma_{\nu'} \Lambda)^{\alpha_{\nu'}} \rightarrow -2\sqrt{2} \left( \begin{array}{ccc} n_{L} A \varepsilon_{LT} D_{B'}^{K} & \varepsilon_{K'N'} D_{K'N'} \\ \varepsilon_{K'N'} D_{K'N'} & 0 \end{array} \right) \left( \begin{array}{ccc} 0 & n_{L} A \varepsilon_{LT} D_{B'}^{K} D_{K'N'} \\ n_{L} A \varepsilon_{LT} D_{B'}^{K} D_{K'N'} & 0 \end{array} \right). \] (6.31)
B. The weight functions multiplying the invariants

A rather tedious but straightforward calculation gives a system of 10 equations for the 10 coefficient functions $\alpha, ..., \kappa$ in (6.3) as found in (See equations (3.6)-(3.15) in [18]). It was also found there that one can easily express the algebraic solutions for $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \tau, \omega$ in terms of the ($\pi, \kappa$) pair in case of de Sitter space, i.e. (hereafter we set $n = 4$ in the general formulae of [18], since only in the four-dimensional case the two-component-spinor formalism can be applied)

\begin{align*}
\omega &= \frac{2mC\kappa + ((A + C)^2 - m^2)\pi}{(m^2 + R^{-2})}, \\
\theta &= \frac{((A - C)^2 - m^2)\kappa - 2mC\pi}{(m^2 + R^{-2})}, \\
\tau &= \frac{2mC\kappa + ((A + C)^2 - m^2)\pi}{(m^2 + R^{-2})}, \\
\varepsilon &= \frac{-((A - C)^2 + 2/R^2) + m^2)\kappa + 2mC\pi}{(m^2 + R^{-2})}, \\
\alpha &= -\tau - 4\pi, \\
\beta &= 2\omega, \\
\gamma &= \varepsilon - 2\kappa, \\
\delta &= 2\varepsilon + 4(\kappa - \theta),
\end{align*}

(6.32)

where we have used the relation $C^2 - A^2 = 1/R^2$.

Furthermore, from (6.32) we can immediately see that

\begin{equation}
\tau = \omega \quad \text{and} \quad \varepsilon + \theta = -2\kappa. \tag{6.33}
\end{equation}

On using (6.33) the differential equations for $\kappa$ and $\pi$, the equations (3.14) and (3.15) of [18] acquire the form

\begin{align*}
-(A+C)\theta + \kappa' + \frac{1}{2}(A-C)\kappa + m\pi &= 0, \\
(C-A)\omega + \pi' + \frac{1}{2}(A+C)\pi + m\kappa &= 0,
\end{align*}

(6.34)

where $\theta$ and $\omega$ are given in (6.32). Clearly one can solve algebraically the second equation for $\kappa$. By differentiating the result one obtains also $\kappa'$ in terms of $\pi$, $\pi'$ and $\pi''$, and substitution of these in the first equation yields a second order ODE for $\pi(\mu)$. Now let us look at the system (6.34) in case of de Sitter spacetime. On inserting $A$ and $C$ from (3.3) and passing to
the globally defined variable \( z = \cos^2 \frac{\theta}{2R} \) (see Sec. III), we obtain the following differential equation for \( \pi \):

\[
\left[ P_2 \frac{d^2}{dz^2} + P_1 \frac{d}{dz} + P_0 \right] \pi = 0,
\]  

(6.35)

where \( P_2 \) in (6.35) is a quartic polynomial in \( z \), i.e.

\[
P_2 = 4 \left[ m^2 R^2 + 1 \right] z^4 - 4(2m^2 R^2 + 3)z^3 + 4(m^2 R^2 + 2)z^2.
\]  

(6.36)

Similarly, \( P_1 \) in (6.35) is a cubic polynomial in \( z \),

\[
P_1 = 16 \left[ m^2 R^2 + 1 \right] z^3 - 12(2m^2 R^2 + 5)z^2 + 8(m^2 R^2 + 2)z.
\]  

(6.37)

Last, \( P_0 \) in (6.35) is a quadratic polynomial in \( z \), i.e.

\[
P_0 = \frac{(m^4 R^4 + 7m^2 R^2 + 10)}{(m^2 R^2 + 1)}.
\]  

(6.38)

On making the substitution \( \pi(z) = \sqrt{z} \tilde{\pi}(z) \), (6.35) becomes an equation of the type

\[
z(z - 1)(z - a)y''(z) + \left\{ (b + c + 1)z^2 - [b + c + 1 + a(d + e) - e]z + ad \right\} y'(z) + (bcz - q)y(z) = 0,
\]  

(6.39)

where the parameters in (6.39) take the values

\[
a = \frac{(m^2 R^2 + 2)}{(m^2 R^2 + 1)},
\]

\[
b = 2 + imR,
\]

\[
c = 2 - imR,
\]

\[
d = e = 3,
\]

\[
q = -\frac{(m^4 R^4 + 7m^2 R^2 + 10)}{(m^2 R^2 + 1)}.
\]  

(6.40)

The equation (6.39) is known as Heun’s differential equation [20, 21]. Its solutions, here denoted by Heun\((a, b, c, d, e, q; z)\), have in general four singular points, i.e. \( z_0 = 0, 1, a, \infty \). Near each singularity the function behaves as a combination of two terms that are powers of \((z - z_0)\) with the following exponents: \( \{0, 1 - d\} \) for \( z_0 = 0 \), \( \{0, 1 - e\} \) for \( z_0 = 1 \), \( \{0, d + e - b - c\} \) for \( z_0 = a \), and \( \{b, c\} \) (that is, \( z^{-b} \) or \( z^{-c} \)) for \( z \to \infty \).

We now insert into the second of Eq. (6.34) the first of Eq. (6.32), finding eventually

\[
\kappa = f^{-1} \left\{ \left[ (A - C)((A + C)^2 - m^2) - \frac{1}{2}(A + C)(m^2 + R^{-2}) \right] \pi - (m^2 + R^{-2})\pi' \right\},
\]  

(6.41)
where
\[ f \equiv m(m^2 + R^{-2} + 2C(C - A)), \tag{6.42} \]
and \( \pi \) and \( \pi' \) are meant to be expressed through the Heun function \( \text{Heun}(a, b, c, d, e, q; z) \).
Eventually, all weight functions can be therefore expressed through such Heun function, although the calculational details are a bit cumbersome.

C. Peierls bracket for gravitinos

The expression (6.3) for the gravitino propagator can be written, concisely, in the form
\[ S_{\alpha'\beta'}^{\lambda\nu'} \rightarrow \sum_{k=1}^{10} w_k S_{AAB'BB'}^{LLN'N'}, \tag{6.43} \]
where, as \( k \) ranges from 1 through 10, \( w_k = \alpha, \beta, ..., \kappa \) in (6.3), while the \( kS_{AAB'BB'}^{LLN'N'} \) are the 10 spinor invariants written down in subsection 6.1. Two further indices are needed to characterize each \( w_k \) function, i.e. \( j \) which labels the four singular points at \( z = 0, 1, a, \infty \) and the subscript \( F \) to denote the Feynman prescription to approach such singular points, i.e. from the above along the positive real axis. Thus, the definition of Peierls bracket that we propose bears analogies with Eqs. (5.29) and (5.30), with \( \psi \nabla \) and \( \chi \nabla \) obtained from the covariant derivative of the Rarita–Schwinger potential (see appendix), while
\[ (\Re G_F) \rightarrow \Re \left( \sum_{k=1}^{10} w_k^{(jF)} S_{AAB'BB'}^{LLN'N'} \right). \tag{6.44} \]

VII. CONCLUDING REMARKS

Our paper has been devoted to geometric constructions of current interest in theoretical physics. Its original contributions, of structural nature, are as follows:

(i) A two-component-spinor analysis of geometric invariants contributing to the gravitino propagator in four-dimensional de Sitter spacetime.

(ii) A Peierls bracket for massive spin-1/2 and spin-3/2 fields in de Sitter spacetime has been proposed, by relying upon the same tools as in item (i) above.

Our use of positive- and negative-frequency Green functions to re-express the Peierls bracket is also of some interest, by virtue of the more direct link with the Feynman Green
function. It now remains to be seen whether our brackets can be exploited to study quantum field theories in de Sitter spacetime from a modern perspective. At a technical level, it would be also interesting to exploit the work in Ref. [22] to re-express all weight functions in subsection 6.B through hypergeometric functions.

APPENDIX A: RARITA–SCHWINGER POTENTIALS

The gravitinos of supergravity are described by spinor-valued one-forms $\psi^A_\mu$, where $\mu$ is the Greek index used to denote the one-form nature. Bearing in mind that the soldering form is obtained by contracting the tetrad $e^\hat{c}_a$ with the Infeld-van der Waerden symbols $\tau_{\hat{c}BB}$ according to

$$e^B_{a} = e^\hat{c}_a \tau_{\hat{c}BB}, \quad (A1)$$

one can write the spatial components of the gravitino in the form

$$\psi_{A i} = \Gamma_{AB}^C e^B_{C i}, \quad (A2)$$

where $\Gamma$, the Rarita–Schwinger potential, can be obtained from a spinor field $\alpha$ according to [23]

$$\Gamma^A_{BB} = \nabla_{BB} \alpha^A. \quad (A3)$$

It obeys the equations [24] ($\Lambda$ being the cosmological constant, and $\Phi$ being the trace-free part of Ricci)

$$\varepsilon^{BC} \nabla_{(A} \Gamma^A_{B)C} = -3\Lambda \alpha_A, \quad (A4)$$

$$\nabla^{B(B} \Gamma^A_{B)C} = \Phi^{ABL} C \alpha_L, \quad (A5)$$

and the gauge-transformation law

$$\hat{\Gamma}^A_{BC} = \Gamma^A_{BC} + \nabla^A_{B} \nu_C. \quad (A6)$$

In the Peierls bracket proposed in Sec. VI.C, the role of $\psi_\nabla$ and $\chi_\nabla$ in (5.30) will be played by covariant derivatives of such spinor-valued one-forms, or, in purely two-component-spinor language, by spinor covariant derivatives of the Rarita–Schwinger potential occurring in (A2)–(A6). A part of the existing literature on supergravity prefers instead to omit spinor indices, writing simply $\psi_\mu$ for gravitinos. With this notation, one can say that, to the
functional derivative $A_i$ in the definition (2.11) there corresponds the covariant derivative
\[ D_\nu \psi_\rho (x) = \partial_\nu \psi_\rho (x) - \Gamma^\sigma_{\nu\rho} \psi_\sigma (x) + \frac{1}{2} \omega_{\nu ab} \sigma^{ab} \psi_\rho (x), \]
where $\Gamma^\sigma_{\nu\rho}$ are the Christoffel symbols, $\omega_{\nu ab}$ is the spin-connection, and $\sigma^{ab}$ is proportional to the commutator of “flat” $\gamma$-matrices, i.e.
\[ \sigma^{ab} = \frac{1}{4} [\gamma_a, \gamma_b]. \]

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**Note added in proof:** In a forthcoming paper in General Relativity and Gravitation by us ([arXiv:0907.3634](http://arxiv.org/abs/0907.3634) [hep-th]) we have extended the analysis of the gravitino propagator in four-dimensional de Sitter space to the classification of the $z$ region, in the sense that we find two ranges of values of $z$, in which the weight functions can be divided into dominant and subdominant families.