On the graph of partial orders

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Abstract:

Any binary relation $\sigma \subseteq X$ (where $X$ is an arbitrary set) generates a characteristic function on the set $X^2$: If $(x, y) \in \sigma$, then $\sigma(x, y) = 1$, otherwise $\sigma(x, y) = 0$. In terms of characteristic functions on the set of all binary relations of the set $X$ we introduced the concept of a binary of reflexive relation of adjacency and determined the algebraic system consisting of all binary relations of a set $X$ and all unordered pairs of various adjacent binary relations. If $X$ is a finite set then this algebraic system is a graph “a graph of graphs” in this work we investigated some features of the structures of the graph $G(X)$ of partial orders.

1. Adjacency of binary relations

Definition 1.1 Let $B = \{0, 1\}$ –Boolean set, $X$ – arbitrary set, and $X^2 = X \times X$ – a direct product. The function $X^2 \rightarrow B$, will be called characteristic. Any subset $\sigma \subseteq X^2$, called a binary relation (or relation) on the set $X$, generates characteristic function

$$\chi_r : X^2 \rightarrow B, \quad \chi_r(x, y) = \begin{cases} 1, & \text{if } (x, y) \in R, \\ 0, & \text{if } (x, y) \notin R. \end{cases}$$

Next, the function $\chi_r(\cdot, \cdot) = R(x, y)$. On the other hand, any characteristic function $\chi : X^2 \rightarrow B$ generates a binary relation $R_\chi \subseteq X^2$ such that $(x, y) \in R_\chi$ if $\chi(x, y) = 1$. Obviously, the map
$R \rightarrow R(\cdot, \cdot)$ is a bijection between the set of binary relations and the set of characteristic functions.

On the set of $2^X$ all sets of binary relations on the set $X$ we introduce a binary reflexive adjacency.

**Definition 1.2** Let $X = Y \cup Z$ – the disjoint union of two subsets (allowed, that either $Y = \emptyset$ or $Z = \emptyset$). Suppose that the relation $\sigma \subseteq X^2$ such that $\sigma(x, y) = 0$ for all $(x, y) \in Y \times Z$. It generates the relation $\tau \subseteq X^2$ such that

1) $\tau(x, y) = 1 - \sigma(y, x)$ for all $(x, y) \in Y \times Z$,  \hspace{1cm} (1)

2) $\tau(x, y) = 0$ for all $(x, y) \in Z \times Y$, \hspace{1cm} (2)

3) $\tau(x, y) = \sigma(x, y)$ for all $(x, y) \in Y^2 \cup Z^2$. \hspace{1cm} (3)

The relation $\tau$ is called adjacent with a relation $\sigma$.

**Remark 1.3** From the definition it follows that if the relation $\tau$ adjacent with a relation $\sigma$, then $\sigma$ adjacent with a relation $\tau$, and this fact we write in the form of a diagram $\sigma \leftrightarrow_{Y \times Z} \tau$:

![Diagram](image-url)
Here and elsewhere in the diagrams we mark for the value of the characteristic functions at those points which are known a priori. For example, in the block $Y \times Z$ for the relation $\sigma$ we write $\sigma_{\text{generalized}} = 0$, and this means that

$$\sigma(x, y) = 0 \text{ for all } (x, y) \in Y \times Z,$$

And in the same block for the relation $\tau$ we write $1 - \sigma(x, y)$ for all $(x, y) \in Y \times Z$.

For example, $X = \{1, \ldots, 6\}$, $Y = \{1, 2\}$, $Z = \{3, 4, 5, 6\}$,

2. Adjacency of the partial orders

Let $V(X)$ is the collection of all partial orders set define on the set $X$. In the other words, the relation $\sigma \subseteq X^2$ belongs in the set $V(X)$, if satisfies the following axioms:

1) reflexivity: $(x, x) \in \sigma$;

2) transitivity: if $(x, y) \in \sigma$, $(y, z) \in \sigma$, then $(x, z) \in \sigma$;

3) antisymmetry: if $(x, y) \in \sigma$, $(y, x) \in \sigma$, then $x = y$.

In the terms of the characteristic we have: $\sigma \in V(X)$ if and only if
1) \( \sigma(x,x) = 1 \) for all \( x \in X \); (4)

2) \( \sigma(x,y) \sigma(y,z) \leq \sigma(x,z) \) for all \( x, y, z \in X \); (5)

3) \( \sigma(x,y) \sigma(y,x) = \delta_{xy} \) for all \( x,y \in X \) (where \( \delta_{xy} \) – Kronecker symbol). (6)

**Theorem 2.1** Let \( \sigma \) and \( \tau \) – are adjacent relations (i.e. \( \sigma \leftarrow y \times z \rightarrow \tau \)). Inclusion \( \sigma \in V(X) \) hold if and only if \( \tau \in V(X) \).

**Proof.** By symmetry, it suffices to prove this implication \( \sigma \in V(X) \Rightarrow \tau \in V(X) \).

Let \( \sigma \in V(X) \).

1. Since \( \tau(x,x) = \sigma(x,x) = 1 \), then the reflexivity relation \( \tau \) obviously.

2. Its clear that, \( \tau(x,y) \tau(y,x) = \sigma(x,y) \sigma(y,x) \) for any \( x, y \in X \), which proves that antisymmetry relations \( \tau \).

3. Transitivity. Let \( x,z,y \in X \) such that \( \tau(x,y) = \tau(y,z) = 1 \), in the first suppose that \( y \in Y \). Since \( \tau(\zeta,y) = 0 \) for all \( \zeta \in Z \), then \( x \in Y \). If \( z \in Y \), then \( \sigma(x,y) = \tau(x,y) = 1 \) and \( \sigma(y,z) = \tau(y,z) = 1 \), and since \( \sigma \in V(X) \), then \( \sigma(x,z) = 1 \), therefore \( \tau(x,z) = 1 \). If \( z \in Z \), then \( \sigma(x,y) = \tau(x,y) = 1 \) and \( \sigma(z,y) = 1 - \tau(y,z) = 0 \), and since \( \sigma \in V(X) \), then by (5) \( \sigma(z,x) = \sigma(z,x) \sigma(x,y) \leq \sigma(z,y) = 0 \), hence, \( \sigma(z,x) = 0 \), and therefore \( \tau(x,z) = 1 \).

Now suppose that \( y \in Z \). \( \tau(y,\eta) = 0 \) \( \eta \in Y \), then \( z \in Z \). If \( x \in Z \), then \( \sigma(x,y) = \tau(x,y) = 1 \) \( \sigma(y,z) = \tau(y,z) = 1 \), \( \sigma \in V(X) \), then \( \sigma(x,z) = 1 \), and since \( \sigma \in V(X) \), then by (5)
\( \sigma(z, x) = \sigma(y, z) \sigma(z, x) \leq \sigma(y, x) = 0 \), hence, \( \sigma(z, x) = 0 \), and therefore \( \tau(x, z) = 1 \). In all cases, we have the equality \( \tau(x, z) = 1 \).

Thus, the set \( X \) generates a pair \( \langle V(X), E(X) \rangle \), where \( V(X) \) is a set of vertices, consist of all partial orders of the set \( X \) and \( E(X) \) is a set of edges, consist of all unordered distinct pairs of adjacent partial orders of the set \( X \). The pair \( G(X) = \langle V(X), E(X) \rangle \) will be called (undirected) graph of partial orders of the set \( X \).

**Definition 2.2** The partial orders \( \sigma \) and \( \tau \) belong to the same connected component of the graph \( G(X) \), if there is a finite sequence of partial orders \( \sigma = \sigma_1, \sigma_2, \ldots, \sigma_m = \tau \), in which the relations \( \sigma_{k-1} \) and \( \sigma_k \) are adjacent for all \( k = 2, \ldots, m \). Let \( G_{\sigma}(X) \) is the connected component of the graph \( G(X) \), which contains the partial order \( \sigma \).

**3- On the features of the structure of the graph of partial orders.**

We fix the partial order \( \sigma \in V(X) \) and an element \( x \in X \). For \( \sigma \) we have the representation:

\[
\begin{array}{ccc}
I_x & K_x & J_x \\
I_x & | & 0 \\
\vdots & | & \vdots \\
\vdots & | & \vdots \\
J_x & | & 0 \\
\end{array}
\]

\[ \sigma = \begin{array}{cccc}
& 1 & & \\
1 & & 0 & \ldots \\
& \vdots & & \vdots \\
0 & \ldots & 1 & 0 \\
\end{array} \]

\[ x \uparrow \]

1451
Lemma 3.1 The following statements holds:

1) \( \sigma(y, z) = 1 \) for all \( (y, z) \in J_x \times I_x \);

2) \( \sigma(y, z) = 0 \) for all \( (y, z) \in I_x \times (K_x \cup J_x) \);

3) \( \sigma(y, z) = 0 \) for all \( (y, z) \in (K_x \cup I_x) \times J_x \).

Proof: Obviously, \( K_x = \{ y \in X : \sigma(x, y) = \sigma(y, x) = \delta_{xy} \} \),

\[ J_x = \{ y \in X : \sigma(x, y) = 0, \ \sigma(y, x) = 1 \}, \]
\[ I_x = \{ y \in X : \sigma(x, y) = 1, \ \sigma(y, x) = 0 \}. \]

1. Since \( y \in J_x \), then \( \sigma(y, x) = 1 \), and since \( z \in I_x \), then \( \sigma(x, z) = 1 \), therefore \( \sigma(y, z) = 1 \). In particular, \( (y, z) \in I_x \times J_x \) we have the equality \( \sigma(z, y) = 0 \).

2. Let \( (y, z) \in I_x \times K_x \).

If \( z = x \), then \( \sigma(y, z) = \sigma(y, x) = 0 \) (since \( y \in I_x \)).

Let \( z \neq x \), and \( z \in K_x \), \( \sigma(x, z) = 0 \). Since \( y \in I_x \), then \( \sigma(x, y) = 1 \), and then by (5) \( \sigma(y, z) = \sigma(x, y) \sigma(y, z) \leq \sigma(x, z) = 0 \) and therefore \( \sigma(y, z) = 0 \) for all \( (y, z) \in I_x \times K_x \).

3. Let \( (y, z) \in K_x \times J_x \).

If \( y = x \), then \( \sigma(y, z) = \sigma(x, z) = 0 \) (since \( z \in J_x \)).
Let \( y \neq x \), and \( y \in K_x \), then \( \sigma(y, x) = 0 \). Since \( z \in J_x \), then \( \sigma(z, x) = 1 \), and by (5), \( \sigma(y, z) = \sigma(z, x) \sigma(z, x) \leq \sigma(y, x) = 0 \) therefore \( \sigma(y, z) = 0 \) for all \((y, z) \in K_x \times J_x\)

Hence we can construct a sequence of adjacent of partial orders :

\[
\sigma \leftarrow I_x \times (K_x \cup J_x) \rightarrow \sigma' \leftarrow (K_x \cup J_x) \times J_x \rightarrow \sigma^x, \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (7)
\]

Which leads us to the partial order \( \sigma^x \in V(X) \), that \( \sigma^x(x, y) = \sigma^x(y, x) = \delta_{xy} \) for all \( y \in X \) (in other words, if we interpret the partial order as relation \( \leq \), then \( x \) is both a maximum and minimum element of a partial order \( \sigma^x \)).

Thus, for a fixed partial order \( \sigma \in V(X) \) defined a map \( X \rightarrow G_\sigma(X) \), associates to an element \( x \in X \) the partial order \( \sigma^x \in G_\sigma(X) \)(it may be that \( \sigma^x = \sigma^y \) at \( x \neq y \)). We also note that this map is uniquely defined in the algorithm (7) are used uniquely defined sets \( I_x(\sigma), K_x(\sigma), J_x(\sigma) \).
Lemma 3.2 Suppose that the partial orders $\sigma, \tau \in V(X)$ belong to the same connected component of the graph $G(X)$, Then $\sigma^x = \tau^x$ for any $x \in X$.

Proof. We can assume that $\sigma$ and $\tau$ – adjacent partial orders, then there is a disjoint union $I \cup J = X$ such that $\sigma \leftarrow I \times J \rightarrow \tau$.

Without loss of generality, we can also assume that $x \in J$ (if $x \in I$ in the calculations presented below the relation $\sigma$ and $\tau$ changing places. For $\sigma$ have the representation

\[
\begin{array}{c|cc|cc|cc}
I_1 & I_2 & J_1 & J_2 & J_3 \\
\hline
I_1 & & & & \\
I_2 & & & & \\
J_1 & & & & \\
J_2 & & & & \\
J_3 & \cdots & 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \rightarrow x \\
\end{array}
\]

\[x \uparrow\]
\[ I_1 \subseteq \{ y \in I : \sigma(x,y) = 0 \}, \]
\[ I_2 \subseteq \{ y \in I : \sigma(x,y) = 1 \}, \]
\[ J_1 \subseteq \{ y \in J : \sigma(x,y) = 1, \sigma(y,x) = 0 \}, \]
\[ J_2 \subseteq \{ y \in J : \sigma(x,y) = 0, \sigma(y,x) = 1 \}, \]
\[ J_3 \subseteq \{ y \in J : \sigma(x,y) = \delta_{xy} \}. \]

It's clearly that \( x \in J_3 \).

1. We fix \( y \in I_2 \cup J_1 \), then \( \sigma(x,y) = 1 \), since \( z \in I_1 \), then \( \sigma(x,z) = 0 \) then by (5) we have \( \sigma(y,z) = \sigma(x,y) \sigma(y,z) \leq \sigma(x,z) = 0 \). Thus \( \sigma(y,z) = 0 \) for all \( (y,z) \in (I_2 \cup J_1) \times I_1 \).

2. Let \( (y,z) \in J_2 \times (I_2 \cup J_1) \). And since \( y \in J_2 \), then \( \sigma(y,x) = 1 \), and since \( z \in I_2 \cup J_1 \), then \( \sigma(x,z) = 1 \), therefore \( \sigma(y,z) = 1 \). Thus, \( \sigma(y,z) = 1 \) for all \( (y,z) \in J_2 \times (I_2 \cup J_1) \).

3. Due to the antisymmetry \( \sigma \) for all \( (y,z) \in J_1 \times J_2 \) have the equality \( \sigma(y,z) = 0 \).

4. Let \( (y,z) \in J_1 \times J_3 \). If \( z = x \), then \( \sigma(y,z) = \sigma(y,x) = 0 \) (since \( z \in J_3 \)).

Let \( z \neq x \), and \( z \in J_3 \) then \( \sigma(x,z) = 0 \). Since \( y \in J_1 \), then \( \sigma(x,y) = 1 \), then from (5) \( \sigma(y,z) = \sigma(x,y) \sigma(y,z) \leq \sigma(x,z) = 0 \). Thus \( \sigma(y,z) = 0 \) for all \( (y,z) \in J_1 \times J_3 \).

5. Let \( (y,z) \in J_3 \times J_2 \). If \( y = x \), then \( \sigma(y,z) = \sigma(x,z) = 0 \) (since \( z \in J_2 \)).

Let \( y \neq x \), and \( y \in J_3 \), to \( \sigma(y,x) = 0 \). Since \( z \in J_2 \), then \( \sigma(z,x) = 1 \), then from (5), \( \sigma(y,z) = \sigma(y,z) \sigma(z,x) \leq \sigma(y,x) = 0 \) therefore \( \sigma(y,z) = 0 \) thus \( \sigma(y,z) = 0 \) for all \( (y,z) \in J_3 \times J_2 \).
Thus, for the adjacent of partial orders $\sigma$ and $\tau$ we have the representation:

$$
\begin{array}{c|ccc|ccc|c}
& I^1 & I^2 & J^1 & J^2 & J^3 & \vdots \\
\hline
I^1 & * & 0 & 0 & 0 & 0 & \\
I^2 & 0 & 0 & 0 & 0 & 0 & \\
J^1 & 0 & * & 0 & 0 & 0 & \\
J^2 & * & 1 & 1 & & & \\
J^3 & \cdots & \cdots & 1 & 1 & \cdots & 0 & \\
\end{array}
$$
We construct a sequence of two adjacent of partial orders:

\[
\sigma \leftarrow \frac{I_2 \times (J_1 \cup J_2)}{J_1 \cup J_2} \rightarrow \sigma', \quad \sigma' \leftarrow \frac{J_1 \cup J_2}{J_1 \cup J_2} \rightarrow \sigma^x, \\
\tau \leftarrow \frac{J_1 \times (I_2 \cup J_2)}{I_2 \cup J_2} \rightarrow \tau', \quad \tau' \leftarrow \frac{J_1 \times (I_2 \cup J_2)}{I_2 \cup J_2} \rightarrow \tau^x.
\]
|   | $I_1$ | $I_2$ | $J_1$ | $J_2$ | $J_3$ |
|---|-------|-------|-------|-------|-------|
| $I_1$ | *     | 0     | 0     | 0     | 0     |
| $I_2$ | 0     | 0     | 0     | 0     | 0     |
| $J_1$ | 0     | *     | 0     | 0     | 0     |
| $J_2$ | *     | 1     | 1     |       |       |
| $J_3$ |       |       |       |       | 0     |

\[ \sigma = \begin{pmatrix} I_1 & I_2 & J_1 & J_2 & J_3 \\ I & I_2 & J_1 & J_2 & J_3 \\ J & J_1 & J_2 & J_3 \end{pmatrix} \]

\[ x \uparrow \]

\[ x \leftarrow \]
\[ \sigma' = \begin{array}{c|ccc|c|c}
\hline
 & I_1 & I_2 & J_1 & J_2 & J_3 \\
\hline
I_1 & 0 & 0 & 0 & 0 & 0 \\
I_2 & \begin{array}{c}
1 \star \ \\
0 & 0 \\
\end{array} & 0 & 0 & \vdots & 0 \\
J_1 & 1 & * & 0 & \vdots & 0 \\
J_2 & * & 0 & 0 & \vdots & \vdots \\
J_3 & \ldots \circ \ldots & 0 & 0 & 0 & \ldots \circ \ldots \\
\hline
\end{array} \]

\[ x \leftarrow x \uparrow \]
\[ \sigma^x = \begin{bmatrix}
I_1 & I_2 & J_1 & J_2 & J_3 \\
0 & 0 & 1^* & 0 & \\
1^* & 0 & 1 & \vdots & \\
1^* & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \\
J_3 & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \]

\[ x \leftarrow \]

\[ \sigma^x = \begin{bmatrix}
0 & 0 & 0 & \vdots & \\
0 & 0 & 0 & \vdots & \\
0 & 0 & 0 & \vdots & \\
\end{bmatrix} \]
\[ \tau = \begin{array}{ccc|ccc}
I & J & 1 \\
\hline
I_1 & * & 1 & 1-* \\
I_2 & 0 & 1-* & 0 \\
J_1 & 0 & 0 & 0 \\
J_2 & 0 & 0 & 1 \\
J_3 & 0 & 0 & 0 \\
\end{array} \]
|   | 1 | 2 | 3 |   |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | * |
| 3 | 0 | 0 | 0 | 0 |

\[ \tau = I - \star \]

\[ j = 1 \]

\[ k = 1 \]

\[ l = 1 \]
Visual comparison $\sigma^x$ and $\tau^x$ shows their equality.

**Corollary 3.3.** In each connected component $G_\sigma(X)$ of the graph $G(X)$ for any $x \in X$ there exists a unique $\sigma^x \in V(X)$, having the property, that $\sigma^x(x, y) = \sigma^x(y, x) = \delta_{xy}$ for all $y \in X$.

**Remark 3.4** We fix $x \in X$.

$G_\sigma(X)$ there unique partial order $\sigma^x$ such that $\sigma^x(x, y) = \sigma^x(y, x) = \delta_{xy}$ for all $y \in X$ therefore the component $G_\sigma(X)$
tial order \( \sigma_x \), define on the set \( X \setminus \{x\} \), such that \( \sigma_x(y,x) = \sigma^y(x) \) for all \( y, z \in X \setminus \{x\} \).

**Remark 3.5** If \( \text{card } X < \infty \) then there exist a one-to-one between the set \( V_0(X) \) and the set of all labeled of transitive graph define on the set \( X \) (see example [1, p28]) and there exist a one-to-one between these set and the set of \( T_0 \)-topology define on the set \( X \) (see example [2, p256]) and the number of these topology denoted by \( T_0(n) \) and in the particular \( \text{card } V_0(X) = T_0(n) \)

**Definition 3.6** For a partial order \( \sigma \in V(X) \).

The set \( S(\sigma) \equiv \{y \in X : \sigma(y,x) = \delta_{xy} \text{ for all } x \in X\} \) is called (support of partial order) \( \sigma \) (or support set) a fact that we write in the form.

\[
\begin{array}{c|c|c|c}
S(\sigma) & 0 & 1 \\
\hline
I & 0 & 0 \\
J & 0 & 0
\end{array}
\]
\[ S(G_\sigma) = \{ S(\tau) \subseteq X : \tau \in G_\sigma(X) \} \] the set of support of the partial order belong to the component \( G_\sigma(X) \) then:

1- \( \emptyset \notin S(G_\sigma) \).

2- if \( \emptyset \neq \alpha \subseteq \beta \subseteq X \) and \( \beta \in S(G_\sigma) \), then \( \alpha \in S(G_\sigma) \).

3- if \( \alpha \subseteq X \) and \( |\alpha| \leq 2 \), then \( \alpha \in S(G_\sigma) \).

**Remark 3.8** Suppose that \( \text{card} \ X = n \) then:

1- \( nT_0(n - 1) \) different support sets of partial orders which contain one element.

2- \( \frac{1}{2} n(n - 1)T_0(n - 1) \) different support sets of partial orders which contain two element.

The proof of the following theorem in \([3,4]\)

**Theorem 3.9** For any \( n \geq 2 \) then

\[ T_0(n) = \frac{1}{2} n(n + 1)T_0(n - 1) + \text{card} \{ \sigma \in V(\{1, \ldots, n\}) : |S(\sigma)| \geq 3 \}. \]

**Example 3.10 In the graph** \( G(\{1,2\}) \) which have unique component which contains the partial order:

\[
\begin{array}{c|c|c}
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{array}
\]
We denote the graphs of the components $K_1$, $K_2$, and $K_3$. It is clear that the component $K_2$ and $K_3$ are isomorphic if applied, for example, substitution \[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\] to the elements of the component $K_2$ we get the component $K_3$. 

$G\{1,2,3\}$ contains 19 partial order $T_0(2) = 3$, and $T_0(3) = 19$: 

We denote the graphs of the components $K_1$, $K_2$ and $K_3$. It is clear that the component $K_2$ and $K_3$ are isomorphic if applied, for example, substitution 

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\]

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\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\]

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\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\]

$G\{1,2,3\}$ contains 19 partial order $T_0(2) = 3$, and $T_0(3) = 19$: 

We denote the graphs of the components $K_1$, $K_2$ and $K_3$. It is clear that the component $K_2$ and $K_3$ are isomorphic if applied, for example, substitution 

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3
\end{pmatrix}
\]
and $S(K_1) = \{1, 2, 3\}$, $S(K_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, in the graph there is only one partial order, which $|S(\sigma)| \geq 3$. 
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