Intersections and Distinct Intersections in Cross-intersecting Families

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Abstract

Let $F, G$ be two cross-intersecting families of $k$-subsets of \{1, 2, \ldots, n\}. Let $F \land G$, $I(F, G)$ denote the families of all intersections $F \cap G$ with $F \in F, G \in G$, and all distinct intersections $F \cap G$ with $F \neq G, F \in F, G \in G$, respectively. For a fixed $T \subset \{1, 2, \ldots, n\}$, let $S_T$ be the family of all $k$-subsets of \{1, 2, \ldots, n\} containing $T$. In the present paper, we show that $|F \land G|$ is maximized when $F = G = S_{\{1\}}$ for $n \geq 2k^2 + 8k$, while surprisingly $|I(F, G)|$ is maximized when $F = S_{\{1,2\}} \cup S_{\{3,4\}} \cup S_{\{1,4,5\}} \cup S_{\{2,3,6\}}$ and $G = S_{\{1,3\}} \cup S_{\{2,4\}} \cup S_{\{1,4,6\}} \cup S_{\{2,3,5\}}$ for $n \geq 100k^2$. The maximum number of distinct intersections in a $t$-intersecting family is determined for $n \geq 3(t + 2)^3k^2$ as well.

1 Introduction

Let $n, k$ be positive integers and let $[n] = \{1, 2, \ldots, n\}$ denote the standard $n$-element set. Let $\binom{n}{k}$ denote the collection of all $k$-subsets of $[n]$. Subsets of $\binom{n}{k}$ are called $k$-\textit{uniform hypergraphs} or $k$-\textit{graphs} for short. A $k$-graph $F$ is called intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in F$. For a fixed set $T \subset [n]$, define the $T$-star $S_T$ by $S_T = \{S \in \binom{n}{k}: T \subset S\}$. We often write $S_p, S_{pq}$ and $S_{pqr}$ for $S_{\{p\}}, S_{\{p,q\}}$ and $S_{\{p,q,r\}}$, respectively. One of the most fundamental theorems in extremal set theory is the following:

\textbf{Erdős-Ko-Rado Theorem (1)}. Suppose that $n \geq 2k$ and $F \subset \binom{n}{k}$ is intersecting. Then

\begin{equation}
|F| \leq \binom{n-1}{k-1}.
\end{equation}

Hilton and Milner \cite{Hilton-Milner} proved that $S_1$ is the only family that achieves equality in (1) up to isomorphism for $n > 2k$.

Two families $F, G \subset \binom{n}{k}$ are called cross-intersecting if any two sets $F \in F, G \in G$ have non-empty intersection. If $A \subset \binom{n}{k}$ is intersecting, then $F = A, G = A$ are cross-intersecting. Therefore the following result is a strengthening of (1).
Theorem 1.1 ([7]). Suppose that \( n \geq 2k \) and \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) are cross-intersecting. Then

\[
|\mathcal{F}| |\mathcal{G}| \leq \binom{n-1}{k-1}^2.
\]

Let us introduce the central notion of the present paper.

**Definition 1.2.** For \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) define

\[
\mathcal{F} \land \mathcal{G} = \{ F \cap G : F \in \mathcal{F}, G \in \mathcal{G} \}
\]

and

\[
I(\mathcal{F}, \mathcal{G}) = \{ F \cap G : F \in \mathcal{F}, G \in \mathcal{G}, F \neq G \}.
\]

Clearly \( \mathcal{F} \land \mathcal{G} = (\mathcal{F} \cap \mathcal{G}) \cup I(\mathcal{F}, \mathcal{G}) \). For \( \mathcal{F} = \mathcal{G} \), we often write \( I(\mathcal{F}) \) instead of \( I(\mathcal{F}, \mathcal{F}) \).

The first result of the present paper shows another extremal property of the full star.

**Theorem 1.3.** Suppose that \( n \geq 2k^2 + 8k \), \( \mathcal{F}, \mathcal{G} \subset \binom{[n]}{k} \) are cross-intersecting. Then

\[
|\mathcal{F} \land \mathcal{G}| \leq \sum_{0 \leq i \leq k-1} \binom{n-1}{i}
\]

where equality holds if and only if \( \mathcal{F} = \mathcal{G} = S_1 \) up to isomorphism.

**Corollary 1.4.** Suppose that \( n \geq 2k^2 + 8k \), \( \mathcal{F} \subset \binom{[n]}{k} \) is intersecting. Then

\[
|\mathcal{F} \land \mathcal{F}| \leq \sum_{0 \leq i \leq k-1} \binom{n-1}{i}
\]

where equality holds if and only if \( \mathcal{F} = S_1 \) up to isomorphism.

One would expect that both Theorem 1.3 and Corollary 1.4 hold for \( n > ck \) for some absolute constant \( c \). Unfortunately, we could not prove it. We can demonstrate the same results for \( n > c'k^2 / \log k \) with a more complicated proof.

Let us now consider the probably more natural quantity \( |I(\mathcal{F}, \mathcal{G})| \), namely the case that intersections of identical sets are not counted. Quite surprisingly the pairs of families maximizing \( |I(\mathcal{F}, \mathcal{G})| \) is rather peculiar. The fact that we can prove the optimality of such a pair shows the strength of our methods.

Let us define the two families

\[
\mathcal{A}_1 = S_{12} \cup S_{34} \cup S_{145} \cup S_{236} \quad \text{and} \quad \mathcal{A}_2 = S_{13} \cup S_{24} \cup S_{146} \cup S_{235}.
\]

One can check that \( \mathcal{A}_1, \mathcal{A}_2 \) are cross-intersecting.

**Proposition 1.5.**

\[
|I(\mathcal{A}_1, \mathcal{A}_2)| = 4 \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + 6 \sum_{0 \leq i \leq k-3} \binom{n-4}{i} + 4 \sum_{0 \leq i \leq k-4} \binom{n-4}{i} + 
\]

\[
+ \sum_{0 \leq i \leq k-5} \binom{n-4}{i} + 2 \sum_{i \leq k-3} \binom{n-6}{i} + \sum_{0 \leq i \leq k-4} \binom{n-6}{i}.
\]
Proof. For any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, there are $\binom{t}{1} \sum_{0 \leq i \leq k-2} \binom{n-4}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 1$. There are $\binom{t}{2} \sum_{0 \leq i \leq k-3} \binom{n-4}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 2$. There are $\binom{t}{3} \sum_{0 \leq i \leq k-4} \binom{n-4}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 3$. There are $\sum_{0 \leq i \leq k-5} \binom{n-4}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 4$. There are $2 \sum_{0 \leq i \leq k-3} \binom{n-6}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 0$ and $|A_1 \cap A_2 \cap \{5, 6\}| = 1$. There are $\sum_{0 \leq i \leq k-4} \binom{n-6}{i}$ distinct intersections for $|A_1 \cap A_2 \cap \{1, 2, 3, 4\}| = 0$ and $|A_1 \cap A_2 \cap \{5, 6\}| = 2$. Thus the proposition follows.

Our main result shows that $|\mathcal{I}(\mathcal{F}, \mathcal{G})|$ is maximized by $\mathcal{A}_1, \mathcal{A}_2$ over all cross-intersecting families $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ for $n \geq 100k^2$.

Theorem 1.6. If $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting families and $n \geq 100k^2$, then $|\mathcal{I}(\mathcal{F}, \mathcal{G})| \leq |\mathcal{I}(\mathcal{A}_1, \mathcal{A}_2)|$.

Let $n \geq k > t$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called $t$-intersecting if any two members of it intersect in at least $t$ elements. Note that for $n \leq 2k - t$ the whole set $\binom{[n]}{k}$ is $t$-intersecting. Thus we always assume that $n > 2k - t$ when considering extremal problems for $t$-intersecting families.

Define $\mathcal{A}(n, k, t) = \left\{ A \in \binom{[n]}{k} : |A \cap [t+2]| \geq t+1 \right\}$. This family was first defined in [2] and it is easily seen to be $t$-intersecting.

Proposition 1.7.

$$|\mathcal{I}(\mathcal{A}(n, k, t))| = \binom{t+2}{t} \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i} + \binom{t+2}{t+1} \sum_{0 \leq i \leq k-t-2} \binom{n-t-2}{i} + \sum_{0 \leq i \leq k-t-3} \binom{n-t-2}{i}.$$  

(5)

Proof. For any $A_1, A_2 \in \mathcal{A}(n, k, t)$, we have $|A_1 \cap A_2 \cap [t+2]| \geq t$. Note that $|A_i \cap [t+2]| \geq t+1$ for $i = 1, 2$. There are $\binom{t+2}{i} \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}$ distinct intersections for $|A_1 \cap A_2 \cap [t+2]| = t$. There are $\binom{t+2}{t+1} \sum_{0 \leq i \leq k-t-2} \binom{n-t-2}{i}$ distinct intersections for $|A_1 \cap A_2 \cap [t+2]| = t+1$. There are $\sum_{0 \leq i \leq k-t-2} \binom{n-t-2}{i}$ distinct intersections for $|A_1 \cap A_2 \cap [t+2]| = t+2$. Thus the proposition follows.

Our third result shows that $|\mathcal{I}(\mathcal{F})|$ is maximized by $\mathcal{A}(n, k, t)$ over all intersecting families $\mathcal{F} \subset \binom{[n]}{k}$ for $n \geq 3(t+2)^3k^2$.

Theorem 1.8. If $\mathcal{F} \subset \binom{[n]}{k}$ is a $t$-intersecting family and $n \geq 3(t+2)^3k^2$, then $|\mathcal{I}(\mathcal{F})| \leq |\mathcal{I}(\mathcal{A}(n, k, t))|$.

We should mention that this result was proved for the case $t = 1$ in [5].

3
Let us list some notions and results that we need for the proofs. Define the family of \( t \)-transversals of \( F \subset \binom{[n]}{k} \):

\[
T_t(F) = \{ T \subset [n]: |T| \leq k, |T \cap F| \geq t \text{ for all } F \in \mathcal{F} \}.
\]

Clearly, if \( F \) is \( t \)-intersecting then \( F \subset T_t(F) \) and vice versa. The \( t \)-covering number \( \tau_t(F) \) is defined as follows:

\[
\tau_t(F) = \min\{|T|: |T \cap F| \geq t \text{ for all } F \in \mathcal{F} \}.
\]

For \( t = 1 \), we often write \( T(F), \tau(F) \) instead of \( T_1(F), \tau_1(F) \), respectively. If \( F, G \) are cross-intersecting, then clearly \( F \subset T(G) \) and \( G \subset T(F) \).

Let us recall the following common notations:

\[ F(i) = \{ F \setminus \{ i \}: i \in F \}, \quad F(i) = \{ F \in \mathcal{F}: i \notin F \}. \]

Note that \( |F| = |F(i)| + |F(i)| \).

Define \( \nu(F) \), the matching number of \( F \) as the maximum number of pairwise disjoint edges in \( F \). Note that \( \nu(F) = 1 \) iff \( F \) is intersecting. We need the following inequality generalising the Erdős-Ko-Rado Theorem.

**Proposition 1.9** ([3]). Suppose that \( F \subset \binom{[n]}{k} \) then

\[
|F| \leq \nu(F) \binom{n-1}{k-1}.
\]

An intersecting family \( F \) is called non-trivial if \( \cap_{F \in \mathcal{F}} F = \emptyset \). We also need the following stability theorem concerning the Erdős-Ko-Rado Theorem.

**Hilton-Milner Theorem** ([6]). If \( n > 2k \) and \( F \subset \binom{[n]}{k} \) is non-trivial intersecting, then

\[
|F| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.
\]

Let us list some inequalities that will be used frequently in the proof.

**Proposition 1.10.** Let \( n, k, \ell, t, p \) be positive integers with \( k > \ell, k > t \) and \( n > 2k + p \). Then

\[
\binom{n}{k} \leq \frac{n-p}{n-p(k+1)} \binom{n-p}{k},
\]

\[
\sum_{0 \leq i \leq k-\ell} \binom{n-t}{i} \leq \frac{n-t-p}{n-t-pk} \sum_{0 \leq i \leq k-\ell} \binom{n-t-p}{i},
\]

\[
\sum_{0 \leq i \leq k-\ell-1} \binom{n-t}{i} \leq \frac{k}{n-t-k} \sum_{0 \leq i \leq k-\ell} \binom{n-t}{i},
\]

\[
\text{for } \ell \geq t + 1, \sum_{t \leq j \leq \ell} \binom{\ell}{j} \geq \frac{1}{2t+2} \sum_{t \leq j \leq \ell+1} \binom{\ell+1}{j}.
\]

**Proof.** Note that

\[
\frac{(n-p)}{(n-k)(n-k-1) \cdots (n-k-p+1)} \geq \left( 1 - \frac{k}{n-p} \right)^p \geq 1 - \frac{pk}{n-p}.
\]
Then (8) holds. By (8), we have for $i < k$
\[
\binom{n-t}{i} \leq \frac{n-t-p}{n-t-p(i+1)} \binom{n-t-p}{i} \leq \frac{n-t-p}{n-t-pk} \binom{n-t-p}{i},
\]
and thereby (9) follows. Since
\[
\frac{\binom{n-t}{i-1}}{\binom{n-t}{i}} = \frac{i}{n-t-i+1} \leq \frac{k}{n-t-k},
\]
we obtain (10).

For $\ell \geq 2t$, since
\[
\sum_{t \leq j \leq \ell} \binom{\ell}{j} \geq 2^{\ell-1} \quad \text{and} \quad \sum_{t \leq j \leq \ell+1} \binom{\ell+1}{j} \leq 2^{\ell+1},
\]
we see that
\[
\frac{\sum_{t \leq j \leq \ell} \binom{\ell}{j}}{\sum_{t \leq j \leq \ell+1} \binom{\ell+1}{j}} \geq \frac{1}{4}.
\]

For $t + 1 \leq \ell \leq 2t$,
\[
\sum_{t \leq j \leq \ell} \binom{\ell}{j} \geq \sum_{t \leq j \leq \ell} \frac{\ell+1-j}{\ell+1} \binom{\ell+1}{j} \\
\geq \sum_{t \leq j \leq \ell-1} \frac{\ell+1-j}{\ell+1} \binom{\ell+1}{j} + \frac{1}{\ell+1} \binom{\ell+1}{\ell} \\
\geq \frac{1}{\ell+1} \sum_{t \leq j \leq \ell-1} \binom{\ell+1}{j} + \frac{1}{\ell+1} \left( \binom{\ell+1}{\ell} + \binom{\ell+1}{1} \right) \\
> \frac{1}{2t+2} \sum_{t \leq j \leq \ell+1} \binom{\ell+1}{j}.
\]
Thus (11) holds.

2 Intersections in cross-intersecting families

In this section, we determine the maximum size of $F \cap G$ over all cross-intersecting families $F, G \subset \binom{n}{k}$. We also determine the maximum size of $(F_1 \cap G_1) \cup (F_2 \cap G_2)$ over all families $F_1, F_2, G_1, G_2 \subset \binom{n}{k}$ with $F_1, G_1$ being cross-intersecting and $F_2, G_2$ being cross-intersecting. This result will be used in Section 3.

First we prove a key proposition to the proof of Theorem 1.3.

Proposition 2.1. Let $F, G \subset \binom{n}{k}$ be cross-intersecting families and set $\mathcal{H} = \mathcal{I}(F, G) \cap \binom{n}{k-1}$. Then $\nu(\mathcal{H}) \leq 4$.

Proof. Suppose that $F_i \cap G_i = D_i$ are pairwise disjoint $(k-1)$-sets, $0 \leq i \leq 4$. Define $x_i, y_i$ by $F_i = D_i \cup \{x_i\}, G_i = D_i \cup \{y_i\}$ and note that $x_i \neq y_i$. There are altogether $5 \times 4$ conditions $F_i \cap G_j \neq \emptyset$ to satisfy. Each of them is assured by either of the following three relations: $x_i \in D_j$, $y_j \in D_i$, $x_i = y_j$. From the first two types there are at most one for each $x_i$ and $y_j$. Altogether at most $5 + 5 = 10$. If no multiple equalities (e.g. $x_1 = y_2 = y_3$)
exist, we get only at most 5 more relations and \(10 + 5 < 20\). Thus there must be places of coincidence, say by symmetry that of the form \(x_i = x_i^\prime\). Thus, again by symmetry, we may assume that \(x_i \not\in D_0\) for \(0 \leq i \leq 4\). Note that \(y_0 \in D_i\) holds for at most one value of \(i\). Without loss of generality assume \(y_0 \not\in D_i\), \(1 \leq i \leq 3\). By \(F_i \cap G_0 \neq \emptyset\), \(y_0 = x_i\), \(i = 1, 2, 3\). Look at \(y_1\). By symmetry assume \(y_1 \not\in D_2\). Now \(G_1 \cap F_2 \neq \emptyset\) implies \(y_1 = x_2\). Hence \(y_1 = x_1\), a contradiction. \(\square\)

Let \(D_1, D_2, D_3, D_4\) be pairwise disjoint \((k-1)\)-sets. Pick an element \(d_i \in D_i\), \(i = 1, 2, 3, 4\). Define \(x_i, y_i\) by \(x_1 = x_2 = y_1 = d_3\), \(x_3 = y_1 = d_2\) and \(x_4 = y_2 = y_3 = d_1\). Setting \(F_i = D_i \cup \{x_i\}, G_i = D_j \cup \{y_j\}\). One can check easily that \(F_i \cap G_j \neq \emptyset\) for \(1 \leq i \neq j \leq 4\). This example shows that Proposition 2.1 is best possible.

**Proof of Theorem 1.3** We distinguish two cases. First we suppose that

\[
|F \cap G| > \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1. \tag{12}
\]

Since \(F, G\) are cross-intersecting, \(F \cap G\) is intersecting. By (7) and (12), without loss of generality, we assume that \(1 \in F\) for all \(F \in F \cap G\). We claim that \(1 \in H\) for all \(H \in F \cup G\). Indeed, if \(1 \notin H \in F\) for \(F \in F \cap G\) then \(H \cap F \neq \emptyset\) for \(F \in F \cap G\) yields

\[
|F \cap G| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1}
\]

contradicting (12). We proved that \(1 \in H\) for all \(H \in F \cup G\) and thereby (3) holds.

Suppose next that (12) does not hold. By Proposition 2.1 and (6), we have for \(n \geq 5k\),

\[
|I(F, G) \cap \left[\binom{n}{k-1}\right]\right| \leq 4 \binom{n-1}{k-2} \leq 4 \frac{n-2}{n-k} \binom{n-2}{k-2} \leq 5 \binom{n-2}{k-2}.
\]

Since the remaining sets in \(I(F, G)\) are of size at most \(k - 2\), we have

\[
|I(F, G)| \leq 5 \binom{n-2}{k-2} + \sum_{0 \leq i \leq k-2} \binom{n}{i}.
\]

Moreover,

\[
|F \cap G| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 \leq k \binom{n-2}{k-2}.
\]

Thus, for \(n \geq 2k + 1\) we have

\[
|F \cap G| \leq (k + 5) \binom{n-2}{k-2} + \sum_{0 \leq i \leq k-2} \binom{n}{i}
\]

\[
\leq \frac{(k + 5)(k-1)}{n-1} \binom{n-1}{k-1} + \frac{k}{n-k} \sum_{0 \leq i \leq k-1} \binom{n}{i}
\]

\[
\leq \frac{(k + 5)(k-1)}{n-1} \binom{n-1}{k-1} + \frac{k}{n-k} \sum_{0 \leq i \leq k-1} \binom{n-1}{i}.
\]

Note that \(n \geq 2k^2 + 8k\) implies

\[
\frac{(k + 5)(k-1)}{n-1} \leq \frac{1}{2}
\]
and

\[
\frac{k}{n-k} - \frac{n-1}{n-k} < \frac{k}{n-k} \left(1 + \frac{k}{n-k}\right) < \frac{k}{2k^2 + 7k} \left(1 + \frac{k}{2k^2}\right) = \frac{2k+1}{2k(2k+7)} \leq \frac{1}{2}
\]

Thus,

\[
|\mathcal{F} \cap \mathcal{G}| \leq \frac{1}{2} \binom{n-1}{k-1} + \frac{1}{2} \sum_{0 \leq i \leq k-1} \binom{n-1}{i} < \sum_{0 \leq i \leq k-1} \binom{n-1}{i}.
\]

Lemma 2.2. Suppose that \( n \geq 2k^2 + 9k \), \( \mathcal{F}_1, \mathcal{G}_1 \subset \binom{[n]}{k} \) are cross-intersecting and \( \mathcal{F}_2, \mathcal{G}_2 \subset \binom{[n]}{k} \) are cross-intersecting. Then

\[
|\mathcal{F}_1 \cap \mathcal{G}_1| \cup (\mathcal{F}_2 \cap \mathcal{G}_2)| \leq 2 \sum_{0 \leq i \leq k-1} \binom{n-2}{i} + \sum_{0 \leq i \leq k-2} \binom{n-2}{i}
\]

with equality holding if and only if \( \mathcal{F}_1 = \mathcal{G}_1 = S_1 \) and \( \mathcal{F}_2 = \mathcal{G}_2 = S_2 \) up to isomorphism.

Proof. By Theorem 1.3 for \( j = 1, 2 \)

\[
|\mathcal{F}_j \cap \mathcal{G}_j| \leq \sum_{0 \leq i \leq k-1} \binom{n-1}{i}.
\]

By Proposition 2.1 and (6), for \( j = 1, 2 \)

\[
|\mathcal{I}(\mathcal{F}_j, \mathcal{G}_j) \cap \binom{[n]}{k-1}| \leq 4 \binom{n-1}{k-2} \leq \frac{4(n-2)}{n-k} \binom{n-2}{k-2} \leq \frac{5}{n-k} \binom{n-2}{k-2}.
\]

Since the remaining sets in \( \mathcal{I}(\mathcal{F}_j, \mathcal{G}_j) \) are of size at most \( k-2 \), for \( j = 1, 2 \)

\[
|\mathcal{I}(\mathcal{F}_j, \mathcal{G}_j)| \leq 5 \binom{n-2}{k-2} + \sum_{0 \leq i \leq k-2} \binom{n}{i}.
\]

If \( |\mathcal{F}_j \cap \mathcal{G}_j| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \leq k \binom{n-2}{k-2} \) for some \( j \in \{1, 2\} \), then for \( n \geq 2k + 2 \)

\[
|\mathcal{F}_1 \cap \mathcal{G}_1| + |\mathcal{F}_2 \cap \mathcal{G}_2| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} \leq k \binom{n-2}{k-2} \]

\[
\leq \binom{n-2}{k-2} + \sum_{0 \leq i \leq k-2} \binom{n}{i} + \sum_{0 \leq i \leq k-1} \binom{n-1}{i}
\]

\[
\leq \frac{(k+5)(k-1)}{n-k} \binom{n-2}{k-2} + \frac{n-2}{n-2k} \sum_{0 \leq i \leq k-2} \binom{n-2}{i} + \frac{n-2}{n-1-k} \sum_{0 \leq i \leq k-1} \binom{n-2}{i}.
\]

Note that \( n \geq 2k^2 + 9k \geq 10k \) implies

\[
\frac{(k+5)(k-1)}{n-k} \leq \frac{1}{2}, \quad \frac{n-2}{n-2k} \leq \frac{5}{4} \quad \text{and} \quad \frac{n-2}{n-1-k} \leq \frac{5}{4}.
\]

Thus,

\[
|\mathcal{F}_1 \cap \mathcal{G}_1| + (\mathcal{F}_2 \cap \mathcal{G}_2)| \leq \frac{1}{2} \binom{n-2}{k-1} + \frac{5}{4} \sum_{0 \leq i \leq k-2} \binom{n-2}{i} + \frac{5}{4} \sum_{0 \leq i \leq k-1} \binom{n-2}{i}
\]

\[
< 2 \sum_{0 \leq i \leq k-1} \binom{n-2}{i} + \sum_{0 \leq i \leq k-2} \binom{n-2}{i}.
\]
Thus we may assume that \(|\mathcal{F}_j \cap \mathcal{G}_j| \geq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\) for each \(j = 1, 2\). By Lemma 3.1, both \(\mathcal{F}_1 \cap \mathcal{G}_1\) and \(\mathcal{F}_2 \cap \mathcal{G}_2\) are trivial intersecting families. By the same argument as in Theorem 1.3, we see that there exist \(x, y\) such that \(\mathcal{F}_1 \cup \mathcal{G}_1 \subset S_x\) and \(\mathcal{F}_2 \cup \mathcal{G}_2 \subset S_y\). If \(x \neq y\), then we are done. If \(x = y\), then

\[
|\mathcal{F}_1 \cup \mathcal{G}_1| \leq |S_x \cap S_x|
\]

\[
= \sum_{0 \leq i \leq k-1} \binom{n-1}{i}
\]

\[
< 2 \sum_{0 \leq i \leq k-1} \binom{n-2}{i} + \sum_{0 \leq i \leq k-2} \binom{n-2}{i}.
\]

\(\Box\)

### 3 Distinct intersections in cross-intersecting families

In this section, we determine the maximum number of distinct intersections in cross-intersecting families.

For the proof, we need the following notion of basis. Two cross-intersecting families \(\mathcal{F}, \mathcal{G}\) are called saturated if any cross-intersecting families \(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}\) with \(\mathcal{F} \subseteq \tilde{\mathcal{F}}, \mathcal{G} \subseteq \tilde{\mathcal{G}}\) have \(\mathcal{F} = \tilde{\mathcal{F}}\) and \(\mathcal{G} = \tilde{\mathcal{G}}\). Since \(\mathcal{F} \subseteq \tilde{\mathcal{F}}\) and \(\mathcal{G} \subseteq \tilde{\mathcal{G}}\) imply \(I(\mathcal{F}, \mathcal{G}) \subseteq I(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})\), we may always assume that \(\mathcal{F}, \mathcal{G}\) are saturated when maximizing the size of \(I(\mathcal{F}, \mathcal{G})\). Let \(B(\mathcal{F})\) be the family of minimal (for containment) sets in \(T(\mathcal{G})\) and let \(B(\mathcal{G})\) be the family of minimal sets in \(T(\mathcal{F})\). Let us prove some properties of the basis.

**Lemma 3.1.** Suppose that \(\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}\) are saturated cross-intersecting families. Then (i) and (ii) hold.

(i) Both \(B(\mathcal{F})\) and \(B(\mathcal{G})\) are antichains, and \(B(\mathcal{F}), B(\mathcal{G})\) are cross-intersecting,

(ii) \(B(\mathcal{F}) = \left\{ F \in \binom{[n]}{k} : \exists B \in B(\mathcal{F}), B \subset F \right\}\) and \(B(\mathcal{G}) = \left\{ G \in \binom{[n]}{k} : \exists B \in B(\mathcal{G}), B \subset G \right\}\).

**Proof.** (i) Clearly, \(B(\mathcal{F})\) and \(B(\mathcal{G})\) are both anti-chains. Suppose for contradiction that \(B \in B(\mathcal{F}), B' \in B(\mathcal{G})\) but \(B \cap B' = \emptyset\). If \(|B| = |B'| = k\), then \(B \in \mathcal{F}, B' \in \mathcal{G}\) follows from saturatedness, a contradiction. If \(|B| < k\), then there exists \(F \supseteq B\) such that \(|F| = k\) and \(|F \cap B'| = |B \cap B'| = 0\). By definition \(F \in T(\mathcal{G})\). Since \(\mathcal{F}, \mathcal{G}\) are saturated, we see that \(F \in \mathcal{F}\). But this contradicts the assumption that \(B'\) is a transversal of \(\mathcal{F}\). Since \(\mathcal{F}, \mathcal{G}\) are saturated, (ii) is immediate from the definition of \(B(\mathcal{F})\) and \(B(\mathcal{G})\). \(\Box\)

Let \(r(\mathcal{B}) = \max\{|B| : B \in \mathcal{B}\}\) and \(s(\mathcal{B}) = \min\{|B| : B \in \mathcal{B}\}\). For any \(\ell\) with \(s(\mathcal{B}) \leq \ell \leq r(\mathcal{B})\), define

\[
\mathcal{B}(\ell) = \left\{ B \in \mathcal{B} : |B| = \ell \right\}\quad \text{and}\quad \mathcal{B}(\leq \ell) = \bigcup_{i = s(\mathcal{B})}^{\ell} \mathcal{B}(i).
\]

It is easy to see that \(s(\mathcal{B}(\mathcal{G})) = r(\mathcal{F})\).

By a branching process, we establish an upper bound on the size of the basis.

**Lemma 3.2.** Suppose that \(\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}\) are saturated cross-intersecting families. Let \(\mathcal{B}_1 = B(\mathcal{F})\) and \(\mathcal{B}_2 = B(\mathcal{G})\). For each \(i = 1, 2\), if \(s(\mathcal{B}_i) \geq 2\) and \(\tau(\mathcal{B}_i(\leq r_i)) \geq 2\) then

\[
\sum_{0 \leq \ell \leq k} \ell^{-2k-\ell+2}|\mathcal{B}_3(\leq \ell)| \leq 1.
\]
Proof. By symmetry, it is sufficient to prove the lemma only for \( i = 1 \). For the proof we use a branching process. During the proof a sequence \( S = (x_1, x_2, \ldots, x_\ell) \) is an ordered sequence of distinct elements of \([n]\) and we use \( \hat{S} \) to denote the underlying unordered set \( \{x_1, x_2, \ldots, x_\ell\} \). At the beginning, we assign weight 1 to the empty sequence \( S_0 \). At the first stage, we choose \( B_{1,1} \in B_1 \) with \( |B_{1,1}| = s(B_1) \). For any vertex \( x_1 \in B_{1,1} \), define one sequence \( (x_1) \) and assign the weight \( s(B_1)^{-1} \) to it.

At the second stage, since \( \tau(B_1^{(\leq \ell_1)}) \geq 2 \), for each sequence \( S = (x_1) \) we may choose \( B_{1,2} \in B_1^{(\leq \ell_1)} \) such that \( x_1 \notin B_{1,2} \). Then we replace \( S = (x_1) \) by \( |B_{1,2}| \) sequences of the form \((x_1, y)\) with \( y \in B_{1,2} \) and weight \( \frac{w(S)}{|B_{1,2}|} \).

In each subsequent stage, we pick a sequence \( S = (x_1, \ldots, x_p) \) and denote its weight by \( w(S) \). If \( \hat{S} \cap B_1 \neq \emptyset \) holds for all \( B_1 \in B_1 \) then we do nothing. Otherwise we pick \( B_1 \in B_1 \) satisfying \( \hat{S} \cap B_1 = \emptyset \) and replace \( S \) by the \(|B_1|\) sequences \((x_1, \ldots, x_p, y)\) with \( y \in B_1 \) and assign weight \( \frac{w(S)}{|B_1|} \) to each of them. Clearly, the total weight is always 1.

We continue until \( \hat{S} \cap B_1 \neq \emptyset \) for all sequences and all \( B_1 \in B_1 \). Since \([n]\) is finite, each sequence has length at most \( n \) and eventually the process stops. Let \( S \) be the collection of sequences that survived in the end of the branching process and let \( S^{(\ell)} \) be the collection of sequences in \( S \) with length \( \ell \).

Claim 1. To each \( B_2 \in B_2^{(\ell)} \) with \( \ell \geq r_1 \) there is some sequence \( S \in S^{(\ell)} \) with \( \hat{S} = B_2 \).

Proof. Let us suppose the contrary and let \( S = (x_1, \ldots, x_p) \) be a sequence of maximal length that occurred at some stage of the branching process satisfying \( \hat{S} \subsetneq B_2 \). Since \( B_1, B_2 \) are cross-intersecting, \( B_1 \cap B_2 \neq \emptyset \), implying that \( p \geq 1 \). Since \( \hat{S} \) is a proper subset of \( B_2 \) and \( B_2 \in B_2 = B(G) \), it follows that \( \hat{S} \notin B(G) \subset T(F) \). Thereby there exists \( F \in F \) with \( \hat{S} \cap F = \emptyset \). In view of Lemma 3.1 (ii), we can find \( B_1' \in B_1 \) such that \( \hat{S} \cap B_1' = \emptyset \). Thus at some point we picked \( S \) and some \( B_1 \in B_1 \) with \( \hat{S} \cap B_1 = \emptyset \). Since \( B_1, B_2 \) are cross-intersecting, \( B_2 \cap B_1 \neq \emptyset \). Consequently, for each \( y \in B_2 \cap B_1 \), the sequence \((x_1, \ldots, x_p, y)\) occurred in the branching process. This contradicts the maximality of \( p \).

Hence there is an \( S \) at some stage satisfying \( \hat{S} = B_2 \). Since \( B_1, B_2 \) are cross-intersecting, \( \hat{S} \cap B_1' = B_2 \cap B_1' \neq \emptyset \) for all \( B_1' \in B_1 \). Thus \( \hat{S} \in S \) and the claim holds.

By Claim 1, we see that \( |B_2^{(\ell)}| \leq |S^{(\ell)}| \) for all \( \ell \geq r_1 \). Let \( S = (x_1, \ldots, x_\ell) \in S^{(\ell)} \) and let \( S_i = (x_1, \ldots, x_i) \) for \( i = 1, \ldots, \ell \). At the first stage, \( w(S_1) = 1/s(B_1) \). Assume that \( B_{1,i} \) is the selected set when replacing \( S_{i-1} \) in the branching process for \( i = 2, \ldots, \ell \). Clearly, \( x_i \in B_{i,i}, B_{1,2} \in B_1^{(\leq r_1)} \) and

\[
w(S) = \frac{1}{s(B_1)} \prod_{i=2}^{\ell} \frac{1}{|B_{1,i}|}.
\]

Note that \( s(B_1) \leq \ell, |B_{1,2}| = r_1 \leq \ell \) and \( |B_{1,i}| \leq k \) for \( i \geq 3 \). It follows that

\[
w(S) \geq \left( \ell^2 k^{\ell-2} \right)^{-1} = \ell^{-2} k^{-\ell+2}.
\]

Thus,

\[
\sum_{r_1 \leq \ell \leq k} \ell^{-2} k^{-\ell+2} |B_2^{(\ell)}| \leq \sum_{r_1 \leq \ell \leq k} \sum_{S \subseteq S^{(\ell)}} w(S) \leq \sum_{S \subseteq S} w(S) = 1.
\]

For the proof of Theorem 1.6 we also need the following lemma.
Lemma 3.3. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{G} \subset \binom{[n]}{k-1}$ are cross-intersecting. Then

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| \leq 2\binom{n-1}{k-2} + (2k + 1)\binom{n-1}{k-3} + \sum_{0 \leq i \leq k-3} \binom{n}{i}.$$

Proof. Let $\mathcal{H}_1 = \mathcal{I}(\mathcal{F}, \mathcal{G}) \cap \binom{[n]}{k-1}$. We claim that $\nu(\mathcal{H}_1) \leq 2$. Otherwise, let $G_i = F_i \cap G_i$, $i = 1, 2, 3$, be three pairwise disjoint members in $\mathcal{H}_1$ with $F_i \in \mathcal{F}$, $G_i \in \mathcal{G}$. Define $x_i$ by $F_i \setminus G_i = \{x_i\}$. By symmetry we may assume that $x_1 \notin G_3$. Then $F_1, G_3$ are disjoint, contradicting the fact that $\mathcal{F}, \mathcal{G}$ are cross-intersecting. Thus $\nu(\mathcal{H}_1) \leq 2$.

If $\nu(\mathcal{H}_1) \leq 1$, then (1) implies $|\mathcal{H}_1| \leq \binom{n-1}{k-2}$. Since the remaining sets in $\mathcal{I}(\mathcal{F}, \mathcal{G})$ are all of size at most $k - 2$, it follows that

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| \leq \binom{n-1}{k-2} + \sum_{0 \leq i \leq k-3} \binom{n}{i}.$$

If $\nu(\mathcal{H}_1) = 2$, let $G_1 = F_1 \cap G_1, G_2 = F_2 \cap G_2$ be two disjoint members in $\mathcal{H}_1$ and let $\mathcal{H}_2 = \mathcal{I}(\mathcal{F}, \mathcal{G}) \cap \binom{[n]\setminus(F_1\cup F_2)}{k-2}$. We claim that $\mathcal{H}_2$ is intersecting. Suppose not, let $D_3 = F_3 \cap G_3, D_4 = F_4 \cap G_4$ be two disjoint members in $\mathcal{H}_2$. Define $x_i$ by $F_i \setminus G_i = \{x_i\}$ for $i = 1, 2$ and define $x_i, y_i, z_i$ by $x_i \notin G_3$ and $y_i \notin G_2 = \emptyset$, by symmetry we may assume that $z_3 \notin G_1$ and $y_3 \notin G_2$. Similarly, assume that $x_4 \notin G_1$ and $y_4 \notin G_2$. Since $F_1 \cap G_3 \neq \emptyset$ and $F_2 \cap G_3 \neq \emptyset$, we see that $z_3 \notin F_1 \cap F_2$. It follows that $x_1 = x_2 = z_3$. Similarly we have $x_1 = x_2 = z_4$. But then $F_3, G_4$ are disjoint, contradicting the fact that $\mathcal{F}, \mathcal{G}$ are cross-intersecting. Thus $\mathcal{H}_2$ is intersecting. By (1) we have

$$|\mathcal{I}(\mathcal{F}, \mathcal{G}) \cap \binom{[n]}{k-2}| \leq |F_1 \cup F_2|\binom{n-1}{k-3} + \binom{n-2k}{k-3} \leq (2k + 1)\binom{n-1}{k-3}.$$

By (1) we obtain that

$$|\mathcal{I}(\mathcal{F}, \mathcal{G}) \cap \binom{[n]}{k-1}| \leq \binom{n-1}{k-2}.$$

Hence

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| \leq 2\binom{n-1}{k-2} + (2k + 1)\binom{n-1}{k-3} + \sum_{0 \leq i \leq k-3} \binom{n}{i}.$$ \hfill \Box

Corollary 3.4. Let $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ be cross-intersecting families. If $\mathcal{G}$ is a star, then

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| \leq 2\sum_{0 \leq i \leq k-2} \binom{n-1}{i} + \binom{n-2}{k-2} + (2k + 1)\binom{n-2}{k-3}.$$ (15)

Proof. Assume without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are saturated. Since $\mathcal{G}$ is a star, we may assume that $\mathcal{G} \subset S_1$. Then $\{1\} \in \mathcal{I}(\mathcal{G})$ whence $\{1\} \in \mathcal{B}(\mathcal{F})$. By Lemma 3.1 (ii) $S_1 \subset \mathcal{F}$. Note that $\mathcal{F}(1) \subset \binom{[2,n]}{k}$, $\mathcal{G}(1) \subset \binom{[2,n]}{k-1}$ are cross-intersecting. By Lemma 3.3 we have

$$|\mathcal{I}(\mathcal{F}(1), \mathcal{G}(1))| \leq 2\binom{n-2}{k-2} + (2k + 1)\binom{n-2}{k-3} + \sum_{0 \leq i \leq k-3} \binom{n-1}{i}$$

$$\leq \sum_{0 \leq i \leq k-2} \binom{n-1}{i} + \binom{n-2}{k-2} + (2k + 1)\binom{n-2}{k-3}.$$
Thus,
\[
|I(\mathcal{F}, \mathcal{G})| \leq |I(S_1, S_1)| + |I(\mathcal{F}(\bar{1}), \mathcal{G}(1))|
\leq 2 \sum_{0 \leq i \leq k-2} \left( \binom{n-1}{i} + \binom{n-2}{k-2} + (2k+1) \binom{n-2}{k-3} \right).
\]

Now we are in position to prove the main theorem.

**Proof of Theorem 1.6.** Let \( B_1 = B(\mathcal{F}), B_2 = B(\mathcal{G}) \) and let \( s_1 = s(B_1), s_2 = s(B_2) \). Suppose first that \( \min\{s_1, s_2\} = 1 \). By symmetry let \( s_2 = 1 \), then \( \mathcal{G} \) is a star. By (15) and \( n \geq 2k+3 \), we have
\[
|I(\mathcal{F}, \mathcal{G})| 
\leq 2 \sum_{0 \leq i \leq k-2} \left( \binom{n-1}{i} + \binom{n-2}{k-2} + (2k+1) \binom{n-2}{k-3} \right)
\leq 2 \sum_{0 \leq i \leq k-2} \left( \frac{n-4}{n-2} \binom{n-4}{k-2} + \frac{(2k+1)(k-2)}{n-1} \binom{n-4}{k-2} \right)
\leq 2 \sum_{0 \leq i \leq k-2} \left( \binom{n-4}{i} + \frac{(2k+1)(k-2)}{n-1} \binom{n-4}{k-2} \right)
\leq \frac{2(n-4)}{n-1-3k} \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + \frac{(2k+1)(k-2)(n-4)}{(n-1)(n-3k)} \binom{n-2}{k-3}.
\]
Note that \( n \geq 63k \) implies
\[
\frac{2(n-4)}{n-1-3k} \leq \frac{21}{10}, \quad \frac{n-4}{n-2k} \leq \frac{11}{10}
\]
and \( n \geq 44k^2 \) implies
\[
\frac{(2k+1)(k-2)}{n-1} \leq \frac{11}{22}, \quad \frac{n-4}{n-3k} \leq \frac{11}{10}.
\]
Thus,
\[
|I(\mathcal{F}, \mathcal{G})| \leq \frac{21}{10} \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + \frac{11}{10} \binom{n-4}{k-2} + \frac{1}{20} \binom{n-4}{k-2}
\leq \frac{13}{4} \sum_{0 \leq i \leq k-2} \binom{n-4}{i} < |I(A_1, A_2)|.
\]

Thus, we may assume that \( s_1, s_2 \geq 2 \). Let us partition \( \mathcal{F} \) into \( \mathcal{F}(s_1) \cup \ldots \cup \mathcal{F}(k) \) where \( F \in \mathcal{F}(\ell) \) if \( \max\{|B| : B \in B_1, B \subset F\} = \ell \). Similarly, partition \( \mathcal{G} \) into \( \mathcal{G}(s_2) \cup \ldots \cup \mathcal{G}(k) \) where \( G \in \mathcal{G}(\ell) \) if \( \max\{|B| : B \in B_2, B \subset G\} = \ell \).

Fix an \( F \in \mathcal{F}(\ell) \) with \( B_1 \subset F, B_1 \in \mathcal{B}_1(\ell) \). For an arbitrary \( G \in \mathcal{G} \), we have
\[
F \cap G = (B_1 \cap G) \cup ((F \setminus B_1) \cap G),
\]
where \( B_1 \cap G \neq \emptyset \) and \( |(F \setminus B_1) \cap G| \leq |F \setminus B_1| = k - \ell \). It follows that for \( s_1 \leq \ell \leq k \)
\[
|I(\mathcal{F}(\ell), \mathcal{G})| \leq |B_1(\ell)| (2^\ell - 1) \sum_{0 \leq i \leq k-\ell} \binom{n-1}{i}.
\]

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Similarly, for \( s_2 \leq \ell \leq k \)

\[
\left| I(\mathcal{F}, \mathcal{G}^{(\ell)}) \right| \leq |\mathcal{B}^{(\ell)}_2| (2^\ell - 1) \sum_{0 \leq i \leq k - \ell} \binom{n - 1}{i}.
\]

Let \( \alpha \) be the smallest integer such that \( \tau(B^{(\ell)}_1) \geq 2 \) and let \( \beta \) be the smallest integer such that \( \tau(B^{(\ell)}_2) \geq 2 \). By symmetry, we may assume that \( \alpha \geq \beta \). We distinguish three cases.

**Case 1.** \( \beta \geq 3 \). Let \( \mathcal{F}' = \mathcal{F}(s_1) \cup \ldots \cup \mathcal{F}(\beta - 1) \). Note that \( \mathcal{F}' \) and \( \mathcal{G} \) are cross-intersecting and \( \mathcal{F}' \) is a star. By (15) and (16), we have

\[
\left| I(\mathcal{F}', \mathcal{G}) \right| \leq 2 \sum_{0 \leq i \leq k - 2} \binom{n - 1}{i} + \binom{n - 2}{k - 2} + (2k + 1) \binom{n - 2}{k - 3} < \frac{13}{4} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i}.
\]

Define

\[
f(n, k, \ell) = 2^\ell \ell^2 k^{\ell - 2} \sum_{0 \leq i \leq k - \ell} \binom{n - 1}{i}.
\]

and let

\[
\lambda_\ell = \ell^{-2} k^{-\ell + 2} |\mathcal{B}^{(\ell)}_1|.
\]

By (17), we see

\[
\sum_{\beta \leq \ell \leq k} \left| I(\mathcal{F}'^{(\ell)}, \mathcal{G}) \right| = \sum_{\beta \leq \ell \leq k} \lambda_\ell f(n, k, \ell).
\]

Since

\[
\frac{f(n, k, \ell)}{f(n, k, \ell + 1)} = \frac{\ell^2}{2k(\ell + 1)^2} \frac{\sum_{0 \leq i \leq k - \ell} \binom{n - 1}{i}}{\sum_{0 \leq i \leq k - \ell - 1} \binom{n - 1}{i}} \geq \frac{(n - 1 - k) \ell^2}{2k^2 (\ell + 1)^2} \geq 1 \text{ for } n \geq 5k^2,
\]

\( f(n, k, \ell) \) is decreasing as a function of \( \ell \). Moreover, by (14) we have

\[
\sum_{\beta \leq \ell \leq k} \lambda_\ell \leq 1.
\]

Hence,

\[
\sum_{\beta \leq \ell \leq k} \left| I(\mathcal{F}'^{(\ell)}, \mathcal{G}) \right| \leq f(n, k, \beta) \leq f(n, k, 3) = 72k \sum_{0 \leq i \leq k - 3} \binom{n - 1}{i}.
\]

Using (9) and (10), for \( n \geq 2k + 3 \) we have

\[
\sum_{\beta \leq \ell \leq k} \left| I(\mathcal{F}'^{(\ell)}, \mathcal{G}) \right| \leq 72k \sum_{0 \leq i \leq k - 3} \binom{n - 1}{i} \leq 72k^2 \frac{n - 1 - k}{n - 1 - k} \sum_{0 \leq i \leq k - 2} \binom{n - 1}{i} \leq 72k^2 \frac{n - 1 - 3}{n - 1 - k} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i}.
\]
Since $n \geq 100k^2 \geq 100k$, we infer
$$\frac{72k^2}{n - 1 - k} \leq \frac{8}{11} \text{ and } \frac{n - 1 - 3}{n - 1 - 3k} \leq \frac{33}{32}.$$ It follows that for $\beta \geq 3$
\begin{equation}
\sum_{\beta \leq \ell \leq k} |I(\mathcal{F}(\ell), \mathcal{G})| \leq \frac{3}{4} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i}.
\end{equation}
Using (19) and (21), we have
$$|I(\mathcal{F}, \mathcal{G})| \leq |I(\mathcal{F}', \mathcal{G})| + \sum_{\beta \leq \ell \leq k} |I(\mathcal{F}(\ell), \mathcal{G})| \leq 4 \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i} < |I(\mathcal{A}_1, \mathcal{A}_2)|.$$ 

**Case 2.** $\beta = 2$ and $\alpha > 2$.
By (21) we have
$$\sum_{3 \leq \ell \leq k} |I(\mathcal{F}(\ell), \mathcal{G})| \leq \frac{3}{4} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i}.$$ Since $\alpha > 2$, it follows that $\mathcal{F}^{(2)}$ is a star. By (19) and (19), we have
$$|I(\mathcal{F}^{(2)}, \mathcal{G})| < \frac{13}{4} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i}.$$ Thus,
$$|I(\mathcal{F}, \mathcal{G})| \leq |I(\mathcal{F}^{(2)}, \mathcal{G})| + \sum_{\beta \leq \ell \leq k} |I(\mathcal{F}(\ell), \mathcal{G})| \leq 4 \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i} < |I(\mathcal{A}_1, \mathcal{A}_2)|.$$ 

**Case 3.** $\beta = \alpha = 2$.
Since $\mathcal{B}_1^{(2)}, \mathcal{B}_2^{(2)}$ are cross-intersecting, we see that $\nu(\mathcal{B}_1^{(2)}) \leq 2$ and $\nu(\mathcal{B}_2^{(2)}) \leq 2$. Moreover, $\beta = \alpha = 2$ implies $\tau(\mathcal{B}_1^{(2)}) \geq 2$ and $\tau(\mathcal{B}_2^{(2)}) \geq 2$. It follows that $\mathcal{B}_1^{(2)}, \mathcal{B}_2^{(2)}$ are either both triangles or both subgraphs of $K_3$ with a matching of size two.

**Case 3.1.** $\mathcal{B}_1^{(2)}, \mathcal{B}_2^{(2)}$ are both triangles.
Without loss of generality, assume that $\mathcal{B}_1^{(2)} = \mathcal{B}_2^{(2)} = \{(1,2), (1,3), (2,3)\}$. By saturatedness, we have
$$\mathcal{F} = \mathcal{G} = \mathcal{A}_3 = \left\{ A \in \binom{[n]}{k} : |A \cap \{1,2,3\}| \geq 2 \right\}.$$ Therefore,
$$|I(\mathcal{F}, \mathcal{G})| = |I(\mathcal{A}_3, \mathcal{A}_3)|$$
\begin{align*}
&= 3 \sum_{0 \leq i \leq k - 2} \binom{n - 3}{i} + 3 \sum_{0 \leq i \leq k - 3} \binom{n - 3}{i} + \sum_{0 \leq i \leq k - 4} \binom{n - 3}{i} \\
&\leq \frac{n - 4}{n - 3 - k} \left( 3 \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i} + 3 \sum_{0 \leq i \leq k - 3} \binom{n - 4}{i} + \sum_{0 \leq i \leq k - 4} \binom{n - 4}{i} \right).
\end{align*}
Since $n \geq 13k$ implies $\frac{n - 4}{n - 3 - k} \leq \frac{13}{12}$, we obtain that
$$|I(\mathcal{F}, \mathcal{G})| < \frac{13}{4} \sum_{0 \leq i \leq k - 2} \binom{n - 4}{i} + \frac{13}{4} \sum_{0 \leq i \leq k - 3} \binom{n - 4}{i} + 2 \sum_{0 \leq i \leq k - 4} \binom{n - 4}{i} < |I(\mathcal{A}_1, \mathcal{A}_2)|.$$
Case 3.2. $B^{(2)}_1, B^{(2)}_2$ are both subgraphs of $K_4$ with a matching of size two.
By symmetry, we may assume that $(1, 3), (2, 4) \in B^{(2)}_1$ and $(1, 2), (3, 4) \in B^{(2)}_2$. We further assume that $B^{(2)}_1 \geq |B^{(2)}_2|$. 

Case 3.2.1. $B^{(2)}_1 = \{(1, 3), (2, 4), (1, 4), (2, 3)\}$ and $B^{(2)}_2 = \{(1, 2), (3, 4)\}$.

Since $\mathcal{F}, \mathcal{G}$ are saturated, we have $\mathcal{F} = \mathcal{F}^{(2)}$ and $\mathcal{G} = \mathcal{G}^{(2)}$. Thus,

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| = 4 \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + 6 \sum_{0 \leq i \leq k-3} \binom{n-4}{i} + 4 \sum_{0 \leq i \leq k-4} \binom{n-4}{i}$$

$$+ \sum_{0 \leq i \leq k-5} \binom{n-4}{i} < |\mathcal{I}(A_1, A_2)|.$$ 

Case 3.2.2. $B^{(2)}_1 = \{(1, 3), (2, 4), (1, 4)\}$ and $B^{(2)}_2 = \{(1, 2), (3, 4), (1, 4)\}$.

Since $\mathcal{F}, \mathcal{G}$ are saturated, we have $\mathcal{F} = \mathcal{F}^{(2)}$ and $\mathcal{G} = \mathcal{G}^{(2)}$. Thus,

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| = 4 \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + 6 \sum_{0 \leq i \leq k-3} \binom{n-4}{i} + 4 \sum_{0 \leq i \leq k-4} \binom{n-4}{i}$$

$$+ \sum_{0 \leq i \leq k-5} \binom{n-4}{i} < |\mathcal{I}(A_1, A_2)|.$$ 

Case 3.2.3. $B^{(2)}_1 = \{(1, 3), (2, 4)\}$ and $B^{(2)}_2 = \{(1, 2), (3, 4)\}$.

By Lemma 3.1, we have $S_{13} \cup S_{24} \subset \mathcal{F}$ and $S_{12} \cup S_{34} \subset \mathcal{G}$. Let $\mathcal{F}' = \mathcal{F} \setminus (S_{13} \cup S_{24})$ and $\mathcal{G}' = \mathcal{G} \setminus (S_{12} \cup S_{34})$. Since $B^{(2)}_1, \mathcal{G}'$ are cross-intersecting, $G \cap \{1, 3\} \neq \emptyset$ and $G \cap \{2, 4\} \neq \emptyset$ for all $G \in \mathcal{G}'$. Moreover, $G \notin S_{12} \cup S_{34}$. It follows that $G \cap \{4\} = \{1, 4\}$ or $G \cap \{4\} = \{2, 3\}$ for all $G \in \mathcal{G}'$. Similarly, $F \cap \{4\} = \{1, 4\}$ or $F \cap \{4\} = \{2, 3\}$ for all $F \in \mathcal{F}'$. Let

$$\mathcal{F}'_{14} = \{F: F \in \mathcal{F}', F \cap \{4\} = \{1, 4\}\}, \mathcal{F}'_{23} = \{F: F \in \mathcal{F}', F \cap \{4\} = \{2, 3\}\}$$

and

$$\mathcal{G}'_{14} = \{G: G \in \mathcal{G}', G \cap \{4\} = \{1, 4\}\}, \mathcal{G}'_{23} = \{G: G \in \mathcal{G}', G \cap \{4\} = \{2, 3\}\}.$$ 

Since $\mathcal{F}'_{14}, \mathcal{G}'_{23}$ are cross-intersecting and $\mathcal{F}'_{23}, \mathcal{G}'_{14}$ are cross-intersecting, by (13) we have

$$|\mathcal{I}(\mathcal{F}'_{14} \cap \mathcal{G}'_{23}) \cup (\mathcal{F}'_{23} \cap \mathcal{G}'_{14})| \leq 2 \sum_{0 \leq i \leq k-3} \binom{n-6}{i} + \sum_{0 \leq i \leq k-4} \binom{n-6}{i}.$$ 

Note that $\mathcal{I}(\mathcal{F}'_{14}, \mathcal{G}' \setminus \mathcal{G}'_{23}) \subset \mathcal{I}(S_{13} \cup S_{24}, S_{12} \cup S_{34})$ and $\mathcal{I}(\mathcal{F}'_{23}, \mathcal{G}' \setminus \mathcal{G}'_{14}) \subset \mathcal{I}(S_{13} \cup S_{24}, S_{12} \cup S_{34})$. Thus,

$$|\mathcal{I}(\mathcal{F}, \mathcal{G})| = |\mathcal{I}(S_{13} \cup S_{24}, S_{12} \cup S_{34})| + |(\mathcal{F}'_{14} \cap \mathcal{G}'_{23}) \cup (\mathcal{F}'_{23} \cap \mathcal{G}'_{14})|$$

$$\leq 4 \sum_{0 \leq i \leq k-2} \binom{n-4}{i} + 6 \sum_{0 \leq i \leq k-3} \binom{n-4}{i} + 4 \sum_{0 \leq i \leq k-4} \binom{n-4}{i}$$

$$+ \sum_{0 \leq i \leq k-5} \binom{n-4}{i} + 2 \sum_{0 \leq i \leq k-3} \binom{n-6}{i} + \sum_{0 \leq i \leq k-4} \binom{n-6}{i}$$

$$= |\mathcal{I}(A_1, A_2)|.$$
Case 3.2.4. $B^{(2)} = \{(1, 3), (2, 4), (1, 4)\}$ and $B^{(2)}_2 = \{(1, 2), (3, 4)\}$.

By Lemma 3.1 (ii), we have $S_{13} \cup S_{24} \cup S_{14} \subset F$ and $S_{12} \cup S_{34} \subset G$. Let $F' = F \setminus (S_{13} \cup S_{24} \cup S_{14})$ and $G' = G \setminus (S_{12} \cup S_{34})$. Since $B^{(2)}_1$, $G'$ are cross-intersecting, $G \cap [4] = \{1, 4\}$ for all $G \in G'$. Similarly, $F \cap [4] = \{2, 3\}$ for all $F \in F'$. Since $F', G'$ are cross-intersecting, by (3) we have

$$|F'_{23} \cap G'_{14}| \leq \sum_{0 \leq r \leq k-3} \binom{n-5}{i} < 2 \sum_{0 \leq r \leq k-3} \binom{n-6}{i} + \sum_{0 \leq r \leq k-2} \binom{n-6}{i}.$$  

Note that $I(F'_{23}, G \setminus G'_{14}) \subset I(S_{13} \cup S_{24}, S_{12} \cup S_{34})$. Thus,

$$|I(F, G)| = |I(S_{13} \cup S_{24}, S_{12} \cup S_{34})| + |F'_{23} \cap G'_{14}| \leq |I(A_1, A_2)|.$$  

\[ \square \]

4 Distinct intersections in a $t$-intersecting family

In this section, we determine the maximum number of distinct intersections in a $t$-intersecting family.

Since $F \subset \bar{F}$ implies $I(F) \subset I(\bar{F})$, we may always assume that $F$ is saturated. Let $B = B_t(\bar{F})$ be the family of minimal (for containment) sets in $T_t(\bar{F})$.

Lemma 4.1. Suppose that $F \subset \binom{[n]}{k}$ is a saturated $t$-intersecting family. Then (i) and (ii) hold.

(i) $B$ is a $t$-intersecting antichain,

(ii) $F = \{ H \in \binom{[n]}{k} : \exists B \in B, B \subset H \}$.

Proof. (i) Clearly, $B$ is an anti-chain. Suppose for contradiction that $B, B' \in B$ but $|B \cap B'| < t$. If $|B| = |B'| = k$, then $B, B' \in F$ as $F$ is saturated, a contradiction. If $|B'| < k$, then there exists $F' \supset B'$ such that $|F'| = k$ and $|F' \cap B| = |B' \cap B| < t$. By definition $F' \in T_t(F)$. Since $F$ is saturated, we see that $F' \in F$. But this contradicts the assumption that $B$ is a $t$-transversal. Since $F$ is saturated, (ii) is immediate from the definition of $B$.

Let $r(B) = \max\{|B| : B \in B\}$ and $s(B) = \min\{|B| : B \in B\}$. For any $\ell$ with $s(B) \leq \ell \leq r(B)$ define

$$B^{(\ell)} = \{ B \in B : |B| = \ell \} \ \text{and} \ B^{(\leq \ell)} = \bigcup_{i=s(B)}^\ell B^{(i)}.$$  

It is easy to see that $s(B_t(F)) = \tau_t(F)$.

Lemma 4.2. Suppose that $F \subset \binom{[n]}{k}$ is a saturated $t$-intersecting family and $B = B_t(F)$. If $s(B) \geq t + 1$ and $\tau_t(B^{(\leq r)}) \geq t + 1$, then

$$\sum_{r \leq \ell \leq k} \left( \binom{k}{t} \ell k^{r-t-1} \right)^{-1} |B^{(\ell)}| \leq 1.$$  

(22)
Proof. For the proof we use a branching process. During the proof a sequence \( S = (x_1, x_2, \ldots, x_\ell) \) is an ordered sequence of distinct elements of \([n]\) and we use \( \hat{S} \) to denote the underlying unordered set \( \{x_1, x_2, \ldots, x_\ell\} \). At the beginning, we assign weight 1 to the empty sequence \( S_0 \). At the first stage, we choose \( B_1 \in F \) with \(|B_1| = s(B) \geq t + 1\). For any \( t \)-subset \( \{x_1, \ldots, x_\ell\} \subseteq B_1 \), define one sequence \((x_1, \ldots, x_\ell)\) and assign the weight \((s(B))^{-1}\) to it.

At the second stage, since \( \tau_t(B^{(\leq r)}) \geq t + 1 \), for each \( t \)-sequence \( S = (x_1, \ldots, x_\ell) \) we may choose \( B \in \mathcal{B}^{(\leq r)} \) such that \( |\hat{S} \cap B| < t \). Then we replace \( S = (x_1, \ldots, x_\ell) \) by \(|B \setminus \hat{S}|(t + 1)\)-sequences of the form \((x_1, \ldots, x_\ell, y)\) with \( y \in B \setminus \hat{S} \) and weight \( w(S) |B \setminus \hat{S}| \) to each of them. Clearly, the total weight is always 1.

We continue until \(|\hat{S} \cap B| \geq t \) for all sequences and all \( B \in \mathcal{B} \). Since \([n]\) is finite, each sequence has length at most \( n \) and eventually the process stops. Let \( \mathcal{S} \) be the collection of sequences that survived in the end of the branching process and let \( \mathcal{S}^{(\ell)} \) be the collection of sequences in \( \mathcal{S} \) with length \( \ell \).

**Claim 2.** To each \( B \in \mathcal{B}^{(\ell)} \) with \( \ell \geq r \) there is some sequence \( S \in \mathcal{S}^{(\ell)} \) with \( \hat{S} = B \).

**Proof.** Let us suppose the contrary and let \( S = (x_1, \ldots, x_\ell) \) be a sequence of maximal length that occurred at some stage of the branching process satisfying \( \hat{S} \subseteq B \). Since \( \mathcal{B} \) is \( t \)-intersecting, \(|B \cap B_1| \geq t \), implying that \( p \geq t \). Since \( \hat{S} \) is a proper subset of \( B \), there exists \( F \in \mathcal{F} \) with \(|\hat{S} \cap F| < t \). In view of Lemma 4.1 (ii) we can find \( B' \in \mathcal{B} \) such that \(|\hat{S} \cap B'| < t \). Thus at some point we picked \( S \) and some \( B \in \mathcal{B} \) with \(|\hat{S} \cap B| < t \). Since \( \mathcal{B} \) is \( t \)-intersecting, \(|B \cap B'| \geq t \). Consequently, for each \( y \in B \setminus (B \setminus \hat{S}) \) the sequence \((x_1, \ldots, x_\ell, y)\) occurred in the branching process. This contradicts the maximality of \( p \). Hence there is an \( S \) at some stage satisfying \( \hat{S} = B \). Since \( \mathcal{B} \) is \( t \)-intersecting, \(|\hat{S} \cap B'| \geq t \) for all \( B' \in \mathcal{B} \). Thus \( \hat{S} \in \mathcal{S} \) and the claim holds.

By Claim 2, we see that \(|\mathcal{B}^{(\ell)}| \leq |\mathcal{S}^{(\ell)}| \). Let \( S = (x_1, \ldots, x_\ell) \in \mathcal{S}^{(\ell)} \) and let \( S_i = (x_1, \ldots, x_i) \) for \( i = 1, \ldots, \ell \). At the first stage, \( w(S_1) = 1/((s(B))^{\ell}) \). Assume that \( B_i \) is the selected set when replacing \( S_{i-1} \) in the branching process for \( i = t + 1, \ldots, \ell \). Clearly, \( x_i \in B_i \), \( B_{t+1} \in \mathcal{B}^{(\leq r)} \) and

\[
    w(S) = \frac{1}{(s(B))^{\ell}} \prod_{i=t+1}^{\ell} \frac{1}{|B_i \setminus \hat{S}_{i-1}|}.
\]

Note that \( s(B) \leq r \leq \ell \), \(|B_{t+1} \setminus \hat{S}_t| \leq \ell \) and \(|B_i \setminus \hat{S}_{i-1}| \leq k \) for \( i \geq t + 2 \). It follows that

\[
    w(S) \geq \left( \left( \frac{\ell}{t} \right) \ell^{k-t-1} \right)^{-1}.
\]

Thus,

\[
    \sum_{r \leq \ell \leq k} \left( \left( \frac{\ell}{t} \right) \ell^{k-t-1} \right)^{-1} |\mathcal{B}^{(\ell)}| \leq \sum_{r \leq \ell \leq k} \sum_{S \in \mathcal{S}^{(\ell)}} w(S) \sum_{S \in \mathcal{S}} w(S) = 1.
\]

**Lemma 4.3.** Suppose that \( \tau_t(B^{(t+1)}) \geq t + 1 \). Then \( \mathcal{F} = \mathcal{A}(n, k, t) \).
Proof. Choose $B_1, B_2 \in \mathcal{B}^{(t+1)}$ and assume by symmetry that $B_i = [t] \cup \{t+i\}$ for $i = 1, 2$. Since $\tau_t(\mathcal{B}^{(t+1)}) \geq t+1$, we may choose $B_3 \in \mathcal{B}^{(t+1)}$ satisfying $[t] \not\subseteq B_3$. Now $|B_2 \cap B_3| \geq t$ implies $\{t+1, t+2\} \subseteq B_3$. Using $|B_3| = t+1$, by symmetry we may assume that $B_3 = [t+2] \setminus \{t\}$. Now take an arbitrary $F \in \mathcal{F}$. It is clear that $|F \cap B_i| \geq t$ can only hold for all $1 \leq i \leq 3$ if $|F \cap [t+2]| \geq t+1$. That is $\mathcal{F} \subseteq \mathcal{A}(n, k, t)$. Since $\mathcal{F}$ is saturated, $\mathcal{F} = \mathcal{A}(n, k, t)$. □

Proof of Theorem 1.8. By (5) and (9), we have

$$|I(S_k)| = \sum_{0 \leq i \leq k-t-1} \binom{n-t}{i}$$

$$\leq \frac{n-t-2}{n-t-2k} \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}$$

$$< \left(\frac{t+2}{t}\right) \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}$$

$$< |I(\mathcal{A}(n, k, t))|.$$}

Thus, we may assume that $s = s(B) \geq t+1$. Let us partition $\mathcal{F}$ into $\mathcal{F}^{(s)} \cup \ldots \cup \mathcal{F}^{(k)}$ where $F \in \mathcal{F}^{(t)}$ if $\max\{|B| : B \in \mathcal{B}, B \subseteq F\} = \ell$. Set

$$\mathcal{I}_\ell = \left\{ F \cap F' : F \in \mathcal{F}^{(t)}, F' \in \mathcal{F}^{(s)} \cup \ldots \cup \mathcal{F}^{(k)} \right\}.$$}

Then

$$|\mathcal{I}(\mathcal{F})| \leq \sum_{s \leq \ell \leq k} |\mathcal{I}_\ell|.$$}

The point is that for $F \in \mathcal{F}^{(t)}$ and $B \subseteq F, B \in \mathcal{B}^t$ for an arbitrary $F' \in \mathcal{F}$,

$$F \cap F' = (B \cap F') \cup ((F \setminus B) \cap F').$$}

Note that $|B \cap F'| \geq t$ and $|(F \setminus B) \cap F'| \leq |F \setminus B| = k-\ell$. It follows that for $s \leq \ell \leq k$

$$|\mathcal{I}_\ell| \leq \left(\sum_{\ell \leq j \leq \ell} \binom{\ell}{j}\right) |\mathcal{B}^{(t)}| \sum_{0 \leq i \leq k-\ell} \binom{n-t}{i}.$$}

(23)

Let $\alpha$ be the smallest integer such that $\tau_t(\mathcal{B}^{(\leq \alpha)}) \geq t+1$. The family $\mathcal{F}' = \cup_{i=1}^{\alpha-1} \mathcal{F}^{(i)}$ is a trivial $t$-intersecting family. By (9), we have for $n \geq 5k$

$$\left|\bigcup_{i=s}^{\alpha-1} \mathcal{I}_i \right| \leq |I(S_k)| = \sum_{0 \leq i \leq k-t-1} \binom{n-t}{i}$$

$$\leq \frac{n-t-2}{n-t-2k} \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}$$

$$\leq 2 \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}.$$}

(24)
If \( \alpha = s = t + 1 \), then \( B^{(t+1)} \) is a \( t \)-intersecting \( (t + 1) \)-uniform family with \( t \)-covering number \( t + 1 \). By Lemma \( \ref{lem:t-intersecting} \), \( \mathcal{F} = \mathcal{A}(n, k, t) \) and there is nothing to prove. Thus we may assume that \( \alpha \geq t + 2 \).

Define

\[
    f(n, k, \ell) = \left( \sum_{t \leq j \leq t^+} \binom{t}{j} \right) \left( \binom{\ell}{t} \right) \ell k^{\ell-t-1} \sum_{0 \leq i \leq k-\ell} \binom{n-t}{i}
\]

and let

\[
    \lambda_\ell = \left( \left( \binom{\ell}{t} \ell k^{\ell-t-1} \right) \sum_{0 \leq i \leq k-\ell} \binom{n-t}{i} \right)^{-1} |B^{(\ell)}|.
\]

Then by \( \ref{eq:lambda_ell} \)

\[(25) \quad \sum_{\alpha \leq \ell \leq k} |\mathcal{I}_\alpha| = \sum_{\alpha \leq \ell \leq k} \lambda_\ell \cdot f(n, k, \ell).\]

By \( \ref{eq:bound} \) and \( \ref{eq:lambda_ell} \), we have

\[
    \frac{f(n, k, \ell)}{f(n, k, \ell + 1)} = \frac{\sum_{t \leq j \leq t^+} \binom{t}{j}}{\sum_{t \leq j \leq t^+} \binom{t+1}{j} (t+1)k^t} \cdot \frac{\binom{\ell}{t} \ell k^{\ell-t-1}}{\sum_{0 \leq i \leq k-\ell} \binom{n-t}{i}} \cdot \frac{\sum_{0 \leq i \leq k-\ell-1} \binom{n-t}{i}}{\sum_{0 \leq i \leq k-\ell} \binom{n-t}{i}} \geq \frac{1}{2(t+1)} \cdot \frac{(\ell + 1 - t)\ell}{(\ell + 1)^2 k} \cdot \frac{n - t - k}{k}.
\]

By \( \ell \geq t + 1 \geq 3 \), we have

\[
    \frac{\ell + 1 - t}{\ell + 1} \cdot \frac{\ell}{t + 2} \geq \frac{3}{4} \geq \frac{3}{2(t + 2)}.
\]

Then by \( n \geq \frac{4}{3}(t + 2)^2k^2 \)

\[
    \frac{f(n, k, \ell)}{f(n, k, \ell + 1)} \geq \frac{3(n - t - k)}{4(t + 1)(t + 2)k^2} \geq 1.
\]

Hence \( f(n, k, \ell) \) is decreasing as a function of \( \ell \). Moreover, \( \ref{eq:bound} \) implies \( \sum_{\alpha \leq \ell \leq k} \lambda_\ell \leq 1 \).

From \( \ref{eq:bound} \) we see

\[
    \sum_{\alpha \leq \ell \leq k} |\mathcal{I}_\ell| \leq f(n, k, \alpha) \leq f(n, k, t + 2).
\]

Therefore,

\[
    \sum_{\alpha \leq \ell \leq k} |\mathcal{I}_\ell| \leq \left( \binom{t+2}{t} + \binom{t+2}{t+1} + \binom{t+2}{t+2} \right) \left( \binom{t+2}{t} \right) (t+2)k \sum_{0 \leq i \leq k-t-2} \binom{n-t}{i}
\]

\[
    \leq \frac{(t+2)^2(t+1)(t^2 + 5t + 8)k}{4} \cdot \frac{k}{n - t - k} \sum_{0 \leq i \leq k-t-1} \binom{n-t}{i}
\]

\[
    \leq \frac{(t+2)^2(t+1)(t+2)(t+4)k^2(n-t-2)}{4(n-t-k)(n-t-2k)} \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i}.
\]

Note that \( n \geq 5k \) implies

\[
    \frac{n - t - 2}{n - t - 2k} \leq 2
\]
and \( n \geq 3(t+2)^3k^2 \), \( t \geq 2 \) imply

\[
\frac{(t+2)^3(t+1)(t+4)k^2}{4(n-t-k)} \leq \frac{1}{2} \left( \binom{t+2}{2} - 2 \right).
\]

It follows that

\[
(26) \sum_{0 \leq i \leq k-t-1} \binom{n-t-2}{i} \leq \binom{n-t-2}{k-2}.
\]

By (24) and (26), we obtain that

\[
|I(F)| \leq \left| \bigcup_{s=0}^{\alpha-1} I_s \right| + \sum_{0 \leq \ell \leq k} |I_\ell| < |I(A(n,k,t))|,
\]

concluding the proof of the theorem. \( \square \)

5 Further problems and results

In their seminal paper [1] Erdős, Ko and Rado actually proved their main result for antichains. Namely, instead of considering \( k \)-graphs \( F \subset \binom{[n]}{k} \) they suppose that \( F \) is an antichain of rank \( k \), that is \( |F| \leq k \) for all \( F \in F \). The reason that this tendency has all but disappeared from recent research is that a \( t \)-intersecting antichain \( F \) of rank \( k \) which is not \( k \)-uniform can always be replaced by a \( t \)-intersecting family \( \tilde{F} \subset \binom{[n]}{k} \) with \( |\tilde{F}| > |F| \).

The way to do is to apply an operation on antichains discovered already by Sperner [8].

For a family \( A \subset \binom{[n]}{a} \) define its shade \( \sigma^+(A) \) by

\[
\sigma^+(A) = \left\{ B \in \binom{[n]}{a+1} : \exists A \in A, A \subset B \right\}.
\]

Sperner [8] proved that for \( a < n/2 \), \( |\sigma^+(A)| \geq |A| \) with strict inequality unless \( a = \frac{n-1}{2} \) and \( A = \binom{[n]}{\frac{n}{2}} \). Let \( F \subset \binom{[n]}{k} \) be a \( t \)-intersecting antichain of rank \( k \), \( n \geq 2k-t \). Suppose that \( a = \min\{|F| : F \in F\} \) and \( a < k \). Define

\[
F^{(a)} = \{ F \in F : |F| = a \} \quad \text{and} \quad \tilde{F} = (F \setminus F^{(a)}) \cup \sigma^+(F^{(a)}).
\]

Then not only is \( \tilde{F} \) a \( t \)-intersecting antichain of rank \( k \) with \( |\tilde{F}| > |F| \) but \( I(\tilde{F}) \supset I(F) \) can be checked easily as well. This shows that it was reasonable to restrict our attention to \( k \)-uniform families.

However there is a related, very natural problem.

**Problem 5.1.** Determine or estimate \( \max |I(A)| \) over all antichain \( A \subset 2^{[n]} \).

**Example 5.2.** Let \( \ell \leq \frac{n}{2} \) and define \( A = \binom{[n]}{n-\ell} \). Clearly,

\[
I(A) = \{ B \subset [n] : n - 2\ell \leq |B| < n - \ell \}.
\]

Choosing \( \ell = \lfloor n/3 \rfloor \), we have

\[
|I(A)| = 2^n - \sum_{0 \leq i \leq \lfloor n/3 \rfloor} \binom{n}{i} - \sum_{0 \leq j \leq n-2\lfloor n/3 \rfloor} \binom{n}{j}.
\]
Proposition 5.3. If $A \subset 2^{[n]}$ is an antichain, then $|I(A)| < 2^n - \sqrt{2^n}$.

Proof. Note that

$$|I(A)| \leq \binom{|A|}{2}.$$  

Consequently, if $|A| \leq \sqrt{2^n}$ then $|I(A)| < 2^n/2 < 2^n - \sqrt{2^n}$. Thus we can assume $|A| > \sqrt{2^n}$. Since $A \cap I(A) = \emptyset$, we have

$$|I(A)| \leq 2^n - |A| < 2^n - \sqrt{2^n}.$$  

Two families $A, B$ are called cross-Sperner if $A \not\subset B$ and $B \not\subset A$ hold for all $A \in A$, $B \in B$. Set

$I(A, B) = \{A \cap B: A \in A, B \in B\}$.

Define

$$m(n) = \max\{|I(A, B)|: A, B \subset 2^{[n]} \text{ are cross-Sperner}\}.$$

Example 5.4. Let $[n] = X \cup Y$ be a partition. Define

$$A = \{A \cup Y: A \subset X\}, \quad B = \{X \cup B: B \subset Y\}.$$  

Then

$I(A, B) = \{A \cup B: A \subset X, B \subset Y\}$

and

$$|I(A, B)| = 2^n - 2^{|X|} - 2^{|Y|} + 1.$$

Theorem 5.5. $m(n) = 2^n - 2 \cdot 2^{n/2} + 1$ holds for $n = 2d$ even.

Proof. The lower bound comes from the example with $|X| = \lfloor n/2 \rfloor$, $|Y| = \lceil n/2 \rceil$. Note that for $A, A' \in A, B, B' \in B$ the cross-Sperner property implies $A \not\subset A' \cap B', B \not\subset A' \cap B'$. In particular, $A \cap I(A, B) = \emptyset = B \cap I(A, B)$.

Cross-Sperner property implies $A \cap B = \emptyset$ and $[n] \not\in A \cup B \cup I(A, B)$. Thus

$$|A| + |B| + |I(A, B)| \leq 2^n - 1$$

or equivalently

$$|I(A, B)| \leq 2^n - |A| - |B| - 1.$$  

(27)

Obviously,

$$|I(A, B)| \leq |A| \cdot |B|.$$  

(28)

Suppose that $n = 2d$ (even). If $|A| + |B| \geq 2(2^d - 1)$, then (27) implies

$$|I(A, B)| \leq 2^n - 2 \cdot 2^d + 1.$$  

If $\frac{|A| + |B|}{2} \leq 2^d - 1$ then the inequality between arithmetic and geometric mean yields via (28):

$$|I(A, B)| \leq \left(2^d - 1\right)^2 = 2^n - 2 \cdot 2^d + 1.$$  

However the proof only gives $I(A, B) \leq 2^n - 2 \cdot 2^{n/2} + 1$ for $n = 2d + 1$.

Problem 5.6. For $n = 2d + 1$, does $m(n) = 2^n - 2^{d+1} - 2^d + 1$ hold?
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