Finite W boson width effects in the top quark width

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Abstract

In the standard model, the top quark decay width $\Gamma_t$ is computed from the exclusive $t \rightarrow bW$ decay. We argue in favor of using the three-body decays $t \rightarrow bf_i\bar{f}_j$ to compute $\Gamma_t$ as a sum over these exclusive modes. As dictated by the S-matrix theory, these three-body decays of the top quark involve only asymptotic states and incorporate the width of the $W$ boson resonance in a natural way. The convolution formula commonly used to include the $W$ boson finite width effects is found to be valid in the limit of massless fermions $f_i\bar{f}_j$. The relation $\Gamma_t = \Gamma(t \rightarrow bW)$ is recovered by taking the limit of massless fermions followed by the $W$ boson narrow width approximation. Although both calculations of $\Gamma_t$ are different at the formal level, their results would differ only by tiny effects induced by first (second) order electroweak (QCD) radiative corrections.

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The precise calculation and measurement of the top quark decay width $\Gamma_t$ is important to provide a consistency check of the standard model between the mass and the width of the top quark. Moreover, the large mass of the top quark implies a large width, which makes the top to behave almost like a free particle. This feature makes attractive the application of perturbative QCD methods to evaluate the quantum corrections. In particular, one expects that the top quark decay would provide the QCD analog of muon decay at the level of radiative corrections. In this context, a definition of $\Gamma_t$ that satisfies general properties of the relativistic quantum theory is welcome.

As is known, theoretical predictions have to match an accuracy comparable to or better than the experimental error bars. On the experimental side, the mass of the top quark $m_t$ is expected to be measured with an uncertainty of up to 3 GeV in the Run II at the Tevatron Collider [1], while $\Gamma_t$ can eventually be measured with a precision of 5-7% from the forward-backward asymmetry at planned linear colliders [2]. On the theoretical side, it is customary to compute the top quark width from the electroweak $t \to bW$ process\(^1\) (see p. 385 in [3] and Refs. [4, 5, 6]). This is the so-called narrow width approximation because the W boson is considered as a stable particle. Different radiative corrections to this process have been reported within the standard model. The order $\alpha_s$ and $\alpha_s^2$ QCD corrections to this rate turn out to be at the 10% [4] and 1\(\sim\)2% level, respectively. On the other hand, the one-loop electroweak corrections are found to affect the decay rate at the 1\(\sim\)2% level [6]. Even some tiny effects in $\Gamma_t$ of $O(10^{-5})$ arising from the renormalization of the $V_{tb}$ matrix element, have been reported recently in the literature [7]. Finally, given the large mass of the top quark, virtual effects of hypothetical heavy particles might contribute to $\Gamma(t \to bW)$ at a few of percent level (see for example [8]).

Since measurements of the top quark width by different experiments can be intimately related to a particular definition of $\Gamma_t$, let us comment on the case where our discussion can be relevant. The Tevatron collider may eventually provide an indirect measurement of $\Gamma_t$ from the observation of single top quark production. In this case, the definition of $\Gamma_t$ based on the two-body decays can be more appropriate because the production mechanisms involve only the $tbW$ vertex. On the other hand, the measurement of $\Gamma_t$ from the threshold region for $t\bar{t}$ production at linear colliders or from the pole position of their resonant energy decay distribution, do not involve directly the $tbW$ vertex and a model-independent definition of $\Gamma_t$.

\(^1\)The decay modes $t \to sW, dW$ would be important only for a calculation aiming an accuracy below the 0.1% level.
would be more suitable.

As it was indicated above, most of these calculations of $\Gamma_t$ (with the exception of Refs.\[4\]) assume that the $W$ is a stable particle. The analysis of Refs.\[4\] has found that the finite width of the $W$ boson can induce an additional correction of $1\sim2\%$ to the $t \to bW$ decay rate. In practice, the $W$ boson is not a real particle that can be reconstructed from their decay products with a well defined invariant mass $m_W$. Instead, the $W$ boson is a resonance and it can not formally be considered as an asymptotic state to be used in the evaluation of S-matrix amplitudes. Furthermore, by cutting the production and decay mechanisms of a resonance can lead to miss important interference effects in the real observables which necessarily are related to the detection of (quasi)stable particles \[9\].

It is the purpose of the present paper to compute the top quark decay width by using the three-body decays $t \to bf_i\bar{f}_j$, such that the decay width is defined by the relation:

$$\Gamma_t \equiv \sum \Gamma(t \to bf_i\bar{f}_j) ,$$

(1)

where the sum is carried over flavors and colors of the pair of fermions $f_i\bar{f}_j$ that are allowed by kinematics. Since the lifetimes of fermions in the final states are much larger than the typical interaction time scales (the $W$ and $t$ decay times), they can be considered as asymptotic states of this process.

We show in this paper that, in the limit of massless fermions, the tree-level expression of Eq. (1) give rise to a convolution formula commonly used (see for example Refs. \[10, 11\]) to include the finite width effects of the $W$ boson. Later, we let the $W$ boson width to vanish and we show that the r.h.s of Eq. (1) reduces to $\Gamma(t \to bW)$. Finally, we briefly comment on the implications of Eq. (1) for the calculation of $\Gamma_t$ at the level of the QCD radiative corrections.

Let us start our evaluation of the top decay width by computing of the $t \to bf_i\bar{f}_j$ decay rate. The tree-level amplitude for this process is given by:

$$\mathcal{M} = \frac{g}{2\sqrt{2}}V_{tb}\bar{u}(p_b)\gamma^\mu(1-\gamma_5)u(p_t) \cdot (-iD_{\mu\nu}(Q)) \frac{g}{2\sqrt{2}}V_{ij}^*\bar{v}(p_j)\gamma^\nu(1-\gamma_5)u(p_i) .$$

(2)

In this expression $V_{kl}$ denote the corresponding Kobayashi-Maskawa matrix elements associated to the $kl$ fermionic current ($V_{ij} = 1$ for lepton doublets), $g$ is the strength of the weak charged current and the momentum-transfer is defined by $Q = p_t - p_b = p_i + p_j$. In the unitary gauge, the full $W$ boson resonant propagator $D_{\mu\nu}(Q)$ \[12\] can be divided into its
spin-1 (transverse) and spin-0 (longitudinal) pieces according to:

\[ D_{\mu\nu}(Q) = \frac{g_{\mu\nu} - \frac{Q_{\mu}Q_{\nu}}{Q^2}}{Q^2 - m_{W}^2 + i\text{Im}\Pi_{T}(Q^2)} - \frac{Q_{\mu}Q_{\nu}}{Q^2} \frac{1}{m_{W}^2 - i\text{Im}\Pi_{L}(Q^2)}. \]  

(3)

As it was explained in Ref. \[12\], we consider \( m_{W} \) to be the renormalized mass of the \( W \) boson and we include only the (finite) absorptive corrections to the \( W \) propagator as done in the context of the fermion-loop scheme \[12, 13\]. In Eq. (3) we have defined \( \text{Im}\Pi_{T} \) and \( \text{Im}\Pi_{L} \) as the transverse and longitudinal projections of the \( W \) boson self-energy tensor, namely \[12\]:

\[ \text{Im}\Pi_{\alpha\beta}(Q) = \left( g^{\alpha\beta} - \frac{Q^{\alpha}Q^{\beta}}{Q^2} \right) \text{Im}\Pi_{T}(Q^2) + \frac{Q^{\alpha}Q^{\beta}}{Q^2} \text{Im}\Pi_{L}(Q^2). \]  

(4)

Using cutting rules techniques, we can compute these absorptive parts of the \( W \) boson self-energy. The expressions obtained at the lowest order in \( g \) are given by:

\[ \text{Im}\Pi_{T}(Q^2) = \sqrt{Q^2} \sum_{k,l} \Gamma^0(W(Q^2) \to f_{k}\bar{f}_{l})\theta(Q^2 - (m_{k} + m_{l})^2) \]

\[ \text{Im}\Pi_{L}(Q^2) = -\sum_{k,l} N_{C} \frac{g^2}{8} |V_{kl}|^2 \frac{Q^{2}}{4\pi} f(x_{k}^2, x_{l}^2) \lambda^{1/2}(1, x_{k}^2, x_{l}^2) \theta(Q^2 - (m_{k} + m_{l})^2), \]  

(5)

where the sum extends over flavors \( (k = u, c, e, \mu, \tau; \ l = s, d, b, \nu_{e}, \nu_{\mu}, \nu_{\tau}) \) in fermion loops that are allowed in the \( W \) boson decay when its mass \( \sqrt{Q^2} > m_{k} + m_{l} \). Let us emphasize that we have kept the masses of all fermions in these corrections in order to remain consistent when the masses of fermions in the final states of top decay are finite.

In the above expressions we have defined the (tree-level) partial decay width of off-shell \( W \) bosons as follows:

\[ \Gamma^0(W(Q^2) \to f_{k}\bar{f}_{l}) = N_{C} \left( \frac{g^2}{8} \right) \frac{|V_{kl}|^2}{12\pi} \sqrt{Q^2} \left[ 2 - f(x_{k}^2, x_{l}^2) \right] \lambda^{1/2}(1, x_{k}^2, x_{l}^2), \]  

(6)

where \( N_{C} \) is the number of colors, \( x_{i} \equiv m_{i}/\sqrt{Q^2} \) and we have defined the functions: \( f(r, s) \equiv r + s - (r - s)^2 \) and \( \lambda(a, b, c) \equiv a^2 + b^2 + c^2 - 2(ab + ac + bc) \).

The top quark decay width reported in the literature is calculated from the two-body \( t \to bW \) process. If the \( W \) boson is off its mass-shell, with a mass \( \sqrt{Q^2} \), the decay width of the top quark becomes:

\[ \Gamma^0(t \to bW(Q^2)) = \frac{C_{F}g^0m_{t}^3}{8\pi\sqrt{2}} |V_{tb}|^2 \frac{m_{W}^2}{Q^2} \left[ (1 - x^2)(1 + 2x^2) - y(2 - x^2 - y^2) \right] \lambda^{1/2}(1, x^2, y^2), \]  

(7)
where $x \equiv \sqrt{Q^2}/m_t$ and $y \equiv m_b/m_t$. Note that we have replaced $g^2 \rightarrow 8G^0_Fm_W^2/\sqrt{2}$, where $G^0_F$ is the Fermi constant at the tree-level.

An straightforward calculation of the top quark decay width using the formula (1) leads to the following result:

$$\Gamma^0_t = \sum_{i,j} \Gamma^0(t \rightarrow bf_i\bar{f}_j)$$

$$= \frac{1}{\pi} \sum_{i,j} \int_{(m_i+m_j)^2}^{(m_t-m_b)^2} dQ^2 \frac{\Gamma^0(t \rightarrow bW(Q^2)) \sqrt{Q^2} \Gamma^0(W(Q^2) \rightarrow f_i\bar{f}_j)}{(Q^2-m_W)^2 + (\text{Im} \Pi_T(Q^2))^2}$$

$$+ \frac{G^2_Fm_W^4m_t^2}{64\pi^5} \sum_{i,j} |V_{ib}|^2 \int_{(m_i+m_j)^2}^{(m_t-m_b)^2} dQ^2 \lambda^{1/2}(1,x_i^2,y^2)\lambda^{1/2}(1,x_j^2,y^2)$$

$$\times \left[ (1-y^2)^2 - x^2(1+y^2) \right] \frac{f(x_i^2,x_j^2)}{m_W^4 + (\text{Im} \Pi_L(Q^2))^2},$$

(8)

The first term in Eq.(8) arises from the spin-1 degrees of freedom in the $W$ boson propagator and the second one comes from its spin-0 component. The interference term in Eq.(8) vanishes due to the orthogonality of the amplitudes arising from the decomposition in Eqs.(2,3). Observe that the second term in the r.h.s of Eq.(8) vanishes in the limit of massless fermions ($x_i, x_j = 0$); if we keep the finite masses of these fermions, the second term in Eq.(8) will give a contribution of order $10^{-5} \sim 10^{-6}$ GeV to $\Gamma_t$.

Let us now consider the massless limit for the fermions that participate in the self-energy correction and the decay of the $W$ boson. In this case we have, from Eqs.(3), $\text{Im} \Pi_L(Q^2) = 0$ and:

$$\text{Im} \Pi_T(Q^2) = \sum_{k,l} \sqrt{Q^2} \Gamma(W(Q^2) \rightarrow f_k\bar{f}_l)$$

$$= \frac{Q^2}{m_W} \Gamma^0_W,$$

(9)

where $\Gamma^0_W = \sum_k N_C g^2 m_W/48\pi$ is the on-shell decay width of the $W$ boson in the limit of massless fermions. Thus, in the limit of massless (light) fermions, Eq.(8) is:

$$\Gamma^0_t = \int_{0}^{(m_t-m_b)^2} dQ^2 \frac{\Gamma^0(t \rightarrow bW(Q^2)) \rho_W(Q^2)}{\rho_W(Q^2)},$$

(10)

where,

$$\rho_W(Q^2) = \frac{1}{\pi} \cdot \frac{Q_W^2 \Gamma^0_W}{(Q^2-m_W^2)^2 + \left(\frac{Q^2}{m_W} \Gamma^0_W\right)^2}.$$
A few comments about Eqs. (10)-(11) can make more clear our point. The convolution kernel $\rho_W(Q^2)$ in Eq. Eq.(11) coincides with the one used in Refs.[10, 11] to include the finite width effects of gauge bosons in final states. The factor $Q^2$ in the numerator of the convolution kernel serves to cancel the $Q^2$ factor appearing in the denominator of the $t \rightarrow bW(Q^2)$ decay rate –see Eq.(7)– and avoids that the integrand in Eq.(11) diverges in the limit $Q^2 \rightarrow 0$. This result is consistent with the fact that in the limit $Q^2 \rightarrow 0$, the $W$ boson produced in the $t \rightarrow bW(Q^2)$ decay has only two degrees of freedom and some care must be taking when using Eq.(7) in that limit. On the other hand, Eq.(11) (and, as a matter of fact, the results of Refs. [10, 11]) can be viewed as a factorization of the production and decay subprocesses of the $W$ gauge-boson. As already explained, this approximation (which can be justified on probabilistic grounds when the production and decay mechanisms of the $W$ boson are independent) be introduced on statistical grounds) can be valid only in the limit of massless fermions.

Next we focus in the narrow $W$ boson width approximation. In the limit $\Gamma^0_W \rightarrow 0$, Eq.(11) reduces to:

$$\Gamma^0_t = \Gamma^0(t \rightarrow bW)$$

(12)

(with the $W$ boson on its mass-shell) by virtue of the representation of the Dirac delta function given by $\lim_{\epsilon \rightarrow 0} \epsilon/(x^2 + \epsilon^2) = \pi \delta(x^2)$. Thus, we consistently recover the tree-level decay rate of the two-body decay, Eq.(7), which is used as the starting point to implement the radiative corrections to $\Gamma_t$ in other calculations [4]-[8]. This result reinforces our arguments that the top quark decay width and its radiative corrections must be computed using Eq.(1) as the starting point instead of Eq.(12).

Let us now briefly comment on the effects of radiative corrections. QCD corrections introduce two important differences between the calculation of $\Gamma_t$ based on Eqs. (1) and (12): the energy scale to evaluate $\alpha_s$ and the role of box diagrams. The $O(\alpha_s)$ QCD radiative corrections to the $t \rightarrow bW$ decay are reported in Ref. [4] and turns out to be of the order of 10%. These corrections include quark self-energies and gluon exchanges between quark lines of the $tbW$ and $Wq_i\bar{q}_j$ vertices [4]. Since $t \rightarrow bW$ is an exclusive process, the QCD energy scale used to evaluate the one-loop corrected width is the mass of the top quark, namely $\alpha_s(m_t)$ [4]. According to Eq. (1), $\Gamma_t$ is an inclusive quantity which is obtained from a sum over exclusive three-body decays. Thus, for an invariant mass $\mu$ of the fermion pair $f_i\bar{f}_j$, there are indeed two mass scales involved in the problem: $m_t$ and $\mu$. However, when $\mu$ is peaked around the $W$ boson mass (as it is the case of the dominant contributions in
three-body top quark decays) no large logarithms will be induced. Thus, in order to have an estimate of the higher order contributions one should compare the QCD corrections to $\Gamma_t$ at the scales $\mu = m_W$ and $\mu = m_t$. For the case of the $O(\alpha_s)$ corrections, the top decay rate decreases by 1.03% when the scale $\mu$ changes from $m_t$ to $m_W$.

On the other hand, the one-loop electroweak and QCD box diagrams will appear in three-body decays of the top quark, but will be absent in $t \rightarrow bW$. Fortunately, these corrections do not contribute to the $O(\alpha_s)$ in $t \rightarrow bf_i\bar{f}_j$. This happens because the interference of the box diagram and tree-level amplitudes involves the trace over the product of a color-singlet and a color-octet fermionic currents, which vanishes identically. However, the four (one-loop) box diagrams for the $t \rightarrow bf_i\bar{f}_j$ decay will contribute to the order $\alpha_s^2$ corrections in $\Gamma_t$. On the other hand, the interference of the one-loop electroweak box diagram and the tree-level amplitudes will not vanish and they will contribute to the order $\alpha$ corrections. Their calculation are, however, beyond the scope of the present paper.

In summary, the purpose of the present paper is to call the attention on the fact the top quark decay width must be evaluated from its three-body decays $(t \rightarrow bf_i\bar{f}_j)$ instead of using the decay $t \rightarrow bW$. The former modes involve only final particles that can be considered as asymptotic states required to correctly define S-matrix amplitudes. On the other hand, their amplitudes incorporate the finite-width effects of the $W$ boson in a natural way. Although, we do not find significant numerical differences for the top decay width at the tree-level, some differences can appear at the level of the $O(\alpha_s^2)$ and $O(\alpha)$ radiative corrections.

Before closing this paper, let us comment that the convolution integral used to include the finite width effects of massive gauge bosons in final states (see for example) turns out to be an approximation valid in the limit of massless fermions produced in the decays of these gauge bosons. On another hand, the validity of the convolution formula is based on the independence of the production and decay probabilities of the gauge bosons. In the present case, this independence can be justified, at the tree-level, on the fact that the spin-0 component of the $W$ propagator decouples in the limit of massless fermions.

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