ON THE RIEMANN-HILBERT PROBLEM FOR A $q$-DIFFERENCE PAINLEVÉ EQUATION

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Abstract. A Riemann-Hilbert problem for a $q$-difference Painlevé equation, known as $qP_{IV}$, is shown to be solvable. This yields a bijective correspondence between the transcendental solutions of $qP_{IV}$ and corresponding data on an associated $q$-monodromy surface. We also construct the moduli space of $qP_{IV}$ explicitly.

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1. INTRODUCTION

Our aim is to prove a bijection between transcendental solutions of a $q$-difference Painlevé equation and monodromy data characterizing a corresponding $q$-difference Riemann-Hilbert problem (RHP). The classical theory is well known [1, 3] but is concerned with one side of this bijection – providing $q$-monodromy data from a given linear $q$-difference equation. It does not consider the properties of the inverse problem essential for studying solutions of a $q$-difference Painlevé equation. We do so here and also describe the monodromy surface parametrised by the corresponding monodromy data.

Many systems of great physical interest are solved through their formulation as Riemann-Hilbert problems. Given an oriented contour $\gamma$ in $\mathbb{C}$, jump conditions across $\gamma$, and asymptotic conditions at infinity, a Riemann-Hilbert problem seeks a function holomorphic on $\mathbb{C} \setminus \gamma$ that satisfies all three conditions. See Definition 2.7 for a more precise statement.

Modern developments in the theory of RHPs have their origin in the solution of the Korteweg-de Vries equation [5] through the inverse scattering method. Reductions of such integrable PDEs led to RHPs for isomonodromic systems associated with the Painlevé equations [11–14]. RHPs provide a method for deducing asymptotic properties of solutions of these nonlinear integrable systems and a key step was provided by a steepest descent method developed by Deift and Zhou [5, 6].
RHPs for integrable discrete equations are also known. They began with the study of recurrence equations for semi-classical orthogonal polynomials [34]. Their reappearance in a model of quantum gravity led to a nonlinear difference equation called a “string equation”, which was identified as a discrete Painlevé equation [10]. Its corresponding RHP is associated with a differential isomonodromic problem.

However, the string equation falls in one of three possible classes of discrete Painlevé equations. It is an “additive” or “d”-discrete Painlevé equation [2]. In contrast, in this paper, we are concerned with RHPs for the class of “multiplicative” or $q$-discrete Painlevé equations, which are not associated with any linear differential equation.

We assume throughout this paper that $q \in \mathbb{C}$ with $0 < |q| < 1$. Difference equations act on the ring $R$ of sequences $(w_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}$ equipped with an iteration operator $\sigma : R \rightarrow R$. In this paper, we take $\sigma = \sigma_q$, where $\sigma_q(w(z)) = w(qz)$.

Further background material is given in Section 1.2.

1.1. Main Results. The linear $q$-difference system we study is given by Equations (2.4). A transformed version of the first problem is given in Equation (2.10). This leads to the characterisation of connection matrices in Definition 2.4 as well as the criterion of admissibility for contours in Definition 2.5. We define the central Riemann-Hilbert problem of this paper in Definition 2.7, whose ingredients are an admissible contour and connection matrix. Our main results are contained in Theorems 2.8, 2.10 and 2.12.

1.2. Background. The six classical differential Painlevé equations $P_I$, $P_{II}$, $P_{III}$, $P_V$, $P_{VI}$, were identified more than a century ago, while the discrete Painlevé equations are a more recent discovery.

There are three types of discrete Painlevé equations, distinguished by the iteration operator $\sigma(z)$. We focus on one of these: $q$-difference equations, which are iterated on spirals in the complex plane parametrized by $\lambda = \lambda_0 q^n$, for some given complex $q \neq 0, 1$ and $\lambda_0 \neq 0$. See [22, 26, 32].

Every discrete Painlevé equation is a compatibility condition for a pair of associated linear problems called a Lax pair. A Lax pair involves two independent variables, denoted $z$ and $\lambda$ in Equations (2.1). We follow the convention that $\lambda$ denotes the independent variable of the associated (discrete) Painlevé equation, while $z$ is an auxiliary variable. $z$ is also often referred to as a “spectral” or “monodromy” variable.

In the differential case, the corresponding Lax pair consists (usually) of two linear systems of differential equations,

$$Y_z = A(z, \lambda)Y,$$
$$Y_\lambda = B(z, \lambda)Y.$$  

The monodromy data describe the behaviour of a fundamental solution near each of the singularities of $A(z, \cdot)$ in $z \in \mathbb{C}$ as well as at $z = \infty$. Under variation of $\lambda$, the Painlevé flow deforms the linear system in such a way that the monodromy data are left invariant. For this reason, $z$ is referred to as the monodromy variable and the first equation is often referred to as the spectral or monodromy equation. A Painlevé transcendent and its derivative provide coefficients of the spectral equation, and hence lead to a set of monodromy data. This is often called the direct problem.

Such monodromy data lie on explicitly defined algebraic varieties [36]. The latter are moduli spaces of the corresponding Painlevé equations. For example, the moduli space of the first Painlevé equation

$$y_{\lambda \lambda} = 6y^2 + \lambda$$
is given by the algebraic variety \[ x \in \mathbb{C}^3 : x_1 x_2 x_3 + x_1 + x_2 + 1 = 0. \]

Conversely, given prescribed monodromy data on such a variety, the inverse problem asks for a corresponding Painlevé transcendent. This problem can be recast into an RHP with suitable contours and jumps given in terms of the monodromy data. Deift and Zhou developed a method of steepest descent to analyse the solutions of RHPs, and this method has been extended to the Painlevé equations to provide global asymptotic information of their general solutions.

In the context of \( q \)-difference equations, the associated Lax pair no longer consists of differential equations, but instead becomes a pair of linear \( q \)-difference equations – see Equations (2.1). The spectral linear problem (2.1a) has singularities only at \( z = 0 \) and \( z = \infty \). Under certain conditions, called Fuchsian non-resonance in this paper (see Definition 2.1), Carmichael constructed fundamental solutions of Equation (2.1a) in neighbourhoods of each point and characterised the connection matrix relating them. The connection matrix embodies the monodromy data of this linear system.

Given a connection matrix, Birkhoff showed how the problem of reconstructing a Fuchsian system with that connection matrix can be recast into a Riemann-Hilbert problem, and proved that this inverse problem always has a solution. A modern extension of this theory (to include non-Fuchsian cases) has also been developed by Ramis et al.

However, to the best of our knowledge, such a Riemann-Hilbert formulation has not been used to obtain information about general solutions of any \( q \)-discrete Painlevé equations, except in the case of \( qP_{VI} \). Analysis of such equations in certain limits has been carried out in two cases, but questions such as a bijection between the coefficients of the linear problem and the solutions of the nonlinear equation, or the moduli space of “monodromy” data have not been considered. This motivates our present paper.

1.3. Notation. Define the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We define the \( q \)-Pochhammer symbol by means of the infinite product

\[
(z; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k z) \quad (z \in \mathbb{C}),
\]

which converges locally uniformly in \( z \) on \( \mathbb{C} \). In particular \((z; q)_\infty\) is an entire function, satisfying

\[
(qz; q)_\infty = \frac{1}{1-z} (z; q)_\infty,
\]

with \((0; q)_\infty = 1\) and simple zeros on the semi \( q \)-spiral \( q^{-N} \). The \( q \)-theta function is defined as

\[
\theta_q(z) = (z; q)_\infty (q/z; q)_\infty \quad (z \in \mathbb{C}^*),
\]

which is analytic on \( \mathbb{C}^* \), with essential singularities at \( z = 0 \) and \( z = \infty \) and simple zeros on the \( q \)-spiral \( q^z \). It satisfies

\[
\theta_q(qz) = -\frac{1}{z} \theta_q(z) = \theta_q(1/z).
\]

For \( n \in \mathbb{N}^* \) we denote

\[
\theta_q(z_1, \ldots, z_n) = \theta_q(z_1) \cdots \theta_q(z_n),
\]

\[
(z_1, \ldots, z_n; q)_\infty = (z_1; q)_\infty \cdots (z_n; q)_\infty.
\]
1.4. **Outline of the paper.** In Section 3, we analyse the direct and inverse monodromy problem concerning the spectral part (2.4a) of the Lax pair. In Section 4, we show how \(q\)-P\(_{IV}\) defines an isomonodromic deformation of the spectral part (2.4a) and prove Theorems 2.8 and 2.10. In Section 5 we study the monodromy surface and prove Theorem 2.12 and Remark 2.13.

2. Statement of results

Discrete Painlevé equations arise as compatibility conditions for an associated pair of linear problems, called a Lax pair. We consider \(q\)-difference Lax pairs of the form

\[
\begin{align*}
Y(qz, \lambda) &= A(z, \lambda)Y(z, \lambda), \\
Y(z, q\lambda) &= B(z, \lambda)Y(z, \lambda),
\end{align*}
\]

where \(A\) and \(B\) are \(2 \times 2\) matrices polynomial in \(z\) and \(A\) satisfies the conditions in Definition 2.1. We focus on the compatible case, i.e., where \(A\) and \(B\) satisfy the compatibility condition:

\[
A(z, q\lambda)B(z, \lambda) = B(qz, \lambda)A(z, \lambda).
\]

Equation (2.1a) is the key object of our study and we call \(z\) a monodromy variable, in analogy with the differential case. The corresponding monodromy data are defined in Definition 3.5. The variable \(\lambda\) deforms Equation (2.1a) in such a way that the monodromy data are left invariant.

The following definitions were formulated by Carmichael [3] and refined by Sauloy [33]. These properties provide essential hypotheses for our main results.

**Definition 2.1.** Equation (2.1a) is characterized as Fuchsian or non-resonant according the properties of its coefficient matrix \(A\), which we assume to be polynomial in \(z\), i.e., \(A(z) = A_0 + A_1z + \ldots + A_nz^n\), for some non-negative integer \(n\) with \(A_n \neq 0\).

(a) The linear \(q\)-difference equation (2.1a) is said to be Fuchsian if \(\det(A_0) \neq 0\) and \(\det(A_n) \neq 0\).

(b) Let the eigenvalues of \(A_0\) be \(\theta_1, \theta_2\) and those of \(A_n\) be \(\kappa_1, \kappa_2\). Moreover, let the zeroes of \(\det(A)\) be \(\{x_1, \ldots, x_{2n}\}\). These quantities are called critical exponents and the collection \(\{\theta_i, \kappa_i, x_j\}\) is called the critical data of (2.1a).

(c) In case Equation (2.1a) is Fuchsian, then it is called non-resonant if the following conditions hold:

(i) \(\theta_1/\theta_2 \neq q^m\), \(\kappa_1/\kappa_2 \neq q^m\), for any integer \(m\).

(ii) \(x_i/x_j \neq q^m\), for \(i, j = 1, \ldots, 2n\), \(i \neq j\) and any integer \(m\).

Note that the critical data are related by

\[
\theta_1\theta_2 = \kappa_1\kappa_2 \prod_{1 \leq i \leq 2n} x_i.
\]

For the remainder of the paper, we focus on a Lax pair [19] given by

\[
\begin{align*}
Y(qz, \lambda) &= A_{\text{IN}}(z; \lambda, f, u)Y(z, \lambda), \\
Y(z, q\lambda) &= B_{\text{IN}}(z; \lambda, f, u)Y(z, \lambda),
\end{align*}
\]
with

\[
A_{IJ} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i q^{j}z & 1 & -i a_{0}a_{2}q^{j}z \\ -1 & -i q^{j}z & -1 \\ -i a_{0}a_{2}q^{j}z & -1 & -i a_{0}a_{2}q^{j}z \end{pmatrix} \times \\
\begin{pmatrix} 0 & -bu & 0 \\ b^{-1}u^{-1} & 0 & 1 \end{pmatrix},
\]

\[
B_{IJ} = \begin{pmatrix} 0 & -bu \\ b^{-1}u^{-1} & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix},
\]

where

\[
b = \frac{\lambda(1 + a_{1}f_{1}(1 + a_{2}f_{2}))}{i(q\lambda^{2} - 1)f_{2}}.
\]

Here, \(f_{i}, i = 0, 1, 2, 3, 4\), and \(u\) are functions of \(\lambda\), independent of \(z\). Explicit calculations show that Equation (2.4a) is Fuchsian. Moreover, it is non-resonant if and only if the following condition holds.

\[
\lambda^{2}, \pm a_{0}, \pm a_{1}, \pm a_{2} \notin q^{\mathbb{Z}}.
\]

The compatibility condition (2.4a) is a polynomial equation in \(z\). Requiring that coefficients of monomials \(z^{j}, 1 \leq j \leq 4\), vanish identically in \(\lambda\) leads to an overdetermined system of equations, which are satisfied if and only if the following nonlinear q-difference equation holds:

\[
q_{P_{IV}}(a) : \begin{cases}
    \frac{\mathcal{T}_{0}}{a_{0}a_{1}f_{1}} = 1 + a_{2}f_{2}(1 + a_{0}f_{0}) \\
    \frac{\mathcal{T}_{1}}{1 + a_{0}f_{0}(1 + a_{1}f_{1})} = 1 + a_{0}f_{0}(1 + a_{1}f_{1}) \\
    \frac{\mathcal{T}_{2}}{a_{1}a_{2}f_{2}} = 1 + a_{1}f_{1}(1 + a_{2}f_{2}) \\
    \frac{\mathcal{T}_{3}}{a_{2}a_{0}f_{0}} = 1 + a_{2}f_{2}(1 + a_{0}f_{0})
\end{cases}
\]

where \(f = (f_{0}, f_{1}, f_{2})\) are functions of \(\lambda\), and \(a := (a_{0}, a_{1}, a_{2})\) are complex parameters, subject to

\[
f_{0}f_{1}f_{2} = \lambda^{2}, \quad a_{0}a_{1}a_{2} = q.
\]

and

\[
\frac{\mathcal{T}}{u} = \left[\frac{\lambda(1 + a_{1}f_{1}(1 + a_{2}f_{2}))}{i(q\lambda^{2} - 1)f_{2}}\right]^{2}.
\]

Here we have used bars to denote iterations in \(\lambda\) for conciseness. Given any function \(f : \mathbb{C} \rightarrow \mathbb{C}\), with values \(\lambda \mapsto f(\lambda)\), we denote \(f = f(\lambda), \mathcal{T} = f(q\lambda), \text{ and } \mathcal{L} = f(\lambda/q)\). Equation (2.7) is referred to as \(q_{P_{IV}}\).

**Remark 2.2.** We have introduced an auxiliary variable \(u\) in the Lax pair for convenience. (It is not contained in the original definition of the Lax pair in \[19\].) It is based on a gauge freedom by constant diagonal matrices.

Several of our results are more conveniently expressed in terms of a transformed version of \(q_{P_{IV}}\), which arises from an expanded form of \(A_{IJ}\):

\[
A_{IJ}(z; \lambda, f, u) = A(z; \lambda, g, u),
\]

given by

\[
A(z; \lambda, g, u) = \begin{pmatrix} 0 & -u \\ u^{-1} & 0 \end{pmatrix} + z \begin{pmatrix} ig_{1}\lambda & 0 \\ 0 & ig_{2}\lambda^{-1} \end{pmatrix} + z^{2} \begin{pmatrix} 0 & -ug_{3} \\ u^{-1}g_{4} & 0 \end{pmatrix} + z^{3}qg_{5}a_{2}^{2}i \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\]

\[\text{This equation has alternative names in the literature and is also referred to as } q_{P_{IV}}(A_{1}^{(1)}), \text{ for its initial value space, or } q_{P_{IV}}((A_{2} + A_{1})^{(1)}), \text{ for its symmetry group – see } [22].\]
where \( g = (g_1, g_2, g_3, g_4) \) satisfy the algebraic equations

\[
\begin{align*}
g_1g_2 - (g_1 + g_4) &= a_0^1(1 + a_1^2a_2^2 + a_3^2), \\
g_3g_4 - qa_0^3a_2(g_1 + g_2) &= a_0^3a_2^2(1 + a_1^2a_2^2 + a_3^2).
\end{align*}
\]

For \( g = (g_1, g_2, g_3, g_4) \), the algebraic surface in \( \mathbb{C}^4 \) defined by equations (2.11) will be denoted by \( \mathcal{G}(a) \).

See Equation (A.1) for the transformation from \( f \) to \( g \), and Equation (A.2) for its inverse. With respect to the variables \( g \), the compatibility condition (2.2) is equivalent to the \( q \)-difference system:

\[
q_{IV}^{\text{mod}}(a) : \begin{cases}
\overline{f}_1 = q^{-1}a^2g_2 + a_0^3a_2\lambda^{-2}g_3(1q\lambda^2 - 1), \\
\overline{f}_2 = q\lambda^2g_1 - qa_0^3a_2g_3^{-1}(q\lambda^2 - 1), \\
\overline{f}_3 = g_3 + \lambda^{-2}g_3^{-1}(q\lambda^2g_1 - g_2)a_0^3a_2(q\lambda^2 - 1) \\
-qa_0^3a_2\lambda^{-2}g_3^{-2}(q\lambda^2 - 1)^2, \\
\overline{f}_4 = g_3,
\end{cases}
\]

on \( \mathcal{G}(a) \).

Note that while this system is apparently singular at \( g_3 = 0 \), the system is well-posed for neighbouring initial values in an annular region around the origin, punctured on the line \( g_3 = 0 \). Denote this region by \( \mathcal{D}_0 \). We denote the line \( g_3 = 0 \) by \( \mathcal{S} \).

The iteration of initial values in \( \mathcal{D}_0 \) is well-defined and the iteration after 3 steps is continuous on the whole domain \( \mathcal{D}_0 \cup \mathcal{S} \). That is, \( \overline{f}_i, 1 \leq i \leq 3 \) (and similarly backward iterates) are well defined for a domain of initial values including \( g_3 = 0 \) in its interior. (This is part of a property called singularity confinement in the literature. See Equation (A.3) in Appendix A for further detail.)

We define notation that incorporates such singularities below.

**Definition 2.3.** Let \( \lambda_0 \in \mathbb{C}^* \) and \( a \in \mathbb{C}^3 \) be such that \( \lambda_0^2 \notin q^2 \), with \( a_0a_1a_2 = q \).

We call a sequence \(( q^{m_i})_{m \in \mathbb{Z}} \) a solution of \( q_{IV}^{\text{mod}}(\lambda_0, a) \) if

(i) it satisfies Equation (2.12) with \( \lambda = q^m\lambda_0, g_3^m \neq 0 \); or

(ii) it satisfies the continuation Equations (A.3), for \( \lambda = q^m\lambda_0, g_3^{m-1} = 0 \) or \( g_3^{m-2} = 0 \), in which case we write \( g_3^m = s \).

The solutions take values in \( \mathcal{G}(a) \cup \{s\} \) and we refer to them as \( q_{IV}^{\text{mod}}(\lambda_0, a) \)-transcendents. Analogous notions are defined for \( q_{IV}^{\text{mod}}(\lambda_0, a) \)-transcendents by means of the birational equivalence given in (A.1) and (A.2).

Carmichael [3] constructed a fundamental solution of non-resonant Fuchsian systems in a domain with \( 0 \) in its interior and another fundamental solution in a domain containing \( \infty \) in its interior. He also characterised the connection matrix relating them. In the following definition, we recall the properties of such connection matrices in the case of our interest, namely for the \( q \)-difference linear system (2.1a).

**Definition 2.4.** Define \( \mathcal{C}(\lambda, a) \) as the set of all \( 2 \times 2 \) matrix functions \( C(z) \), satisfying

\[
\begin{align*}
(1.1) & \quad C(z) \text{ is analytic in } \mathbb{C}^*; \\
(1.2) & \quad C(qz) = \frac{1}{a_0a_1a_2z^{-3}c_3}C(z)\lambda^{-\sigma_1}; \\
(1.3) & \quad |C(z)| = c_3^0(a_0z, -a_0z, a_0a_2z, -a_0a_2z, qz, -qz), \text{ for some } c \in \mathbb{C}^*; \\
(1.4) & \quad C(-z) = -\sigma_1C(z)\sigma_3.
\end{align*}
\]

Birkhoff [11] defined contours for the corresponding Riemann problem, which are recalled below.
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Figure 2.1. Example of an admissible contour $\gamma$ in Definition 2.5, where the six red spirals are $q^k \cdot x_i$, $1 \leq i \leq 6$.

Definition 2.5. Denote

\begin{align*}
x_1 &= +a_0^{-1}, & x_2 &= +a_1/q, & x_3 &= +q^{-1}, \\
x_4 &= -a_0^{-1}, & x_5 &= -a_1/q, & x_6 &= -q^{-1}.
\end{align*}

A positively oriented Jordan curve $\gamma$ in $\mathbb{C}^*$ is called admissible if the following conditions are satisfied:

(i) It is an analytic curve, i.e., it admits local parametrization by analytic functions around each point;

(ii) It has the reflection-symmetry $\gamma = -\gamma$;

(iii) Letting the region on the left (respectively right) of $\gamma$ in $\mathbb{C}$ be $D_-$ and $D_+$, we have

\[ q^k x_i \in \begin{cases} 
D_- & \text{if } k > 0, \\
D_+ & \text{if } k \leq 0,
\end{cases} \]

for $k \in \mathbb{Z}$ and $1 \leq i \leq 6$.

See Figure 2.1 for an example of an admissible contour $\gamma$.

Remark 2.6. Note that $x_i$, $1 \leq i \leq 6$ are zeroes of $\det(A_{JN})$. Moreover, not all such contours are homotopically equivalent in $\mathbb{C}^* \setminus (q^\mathbb{Z} \cdot \{x_1, \ldots, x_6\})$.

We now define a Riemann-Hilbert problem, which provides the setting for our main results.

Definition 2.7 (Riemann-Hilbert problem). Suppose we are given $\lambda_0 \in \mathbb{C}^*$, $a \in \mathbb{C}^3$, $a = (a_0, a_1, a_2)$ satisfying the non-resonance conditions (2.13), a matrix $C(z) \in \mathcal{C}(\lambda_0, a)$ and an admissible curve $\gamma$.

For $m \in \mathbb{Z}$, a $2 \times 2$ complex matrix function $Y^{(m)}(z)$ is called a solution of the Riemann-Hilbert problem $\text{RHP}^{(m)}(\gamma, C)$ if it satisfies the following conditions.

(i) $Y^{(m)}(z)$ is analytic on $\mathbb{C} \setminus \gamma$. 

\[ q^k x_i \in \begin{cases} 
D_- & \text{if } k > 0, \\
D_+ & \text{if } k \leq 0,
\end{cases} \]
Definition 2.9. Connection matrices.

2.1. Resonance conditions

Theorem 2.8. Given a connection matrix $C(z)$, Theorem 2.8 associates to it a solution $g$ of $\text{RHP}^m(\gamma, C)$. However this association is not injective, due to the freedom of scaling $C(z)$ by right-multiplication by diagonal matrices, which leaves the solution $g$ invariant. To overcome this issue, we define a quotient space of connection matrices.

Definition 2.9. Define

$$\mathcal{M}_c(\lambda_0, a) = \mathcal{C}(\lambda_0, a)/\text{diag}(\mathbb{C}^*, \mathbb{C}^*).$$
where the quotient is taken by right multiplication by diagonal matrices and the
space $C(\lambda_0, a)$ is defined in Definition 2.4. Moreover, define the Riemann-Hilbert
mapping
\[ \mathcal{RH} : \mathcal{M}_c(\lambda_0, a) \to \{ q_{IV}^{\text{mod}}(\lambda_0, a) \text{ transcendents} \}, \] (2.18)
which assigns to any equivalence class, a unique corresponding $q_{IV}^{\text{mod}}(\lambda_0, a)$ tran-
scedent, via Theorem 2.8. We call $\mathcal{M}_c(\lambda_0, a)$ the monodromy surface.

Our second main result is given by the following theorem.

**Theorem 2.10.** Let $\lambda_0 \in \mathbb{C}^*$ and $a \in \mathbb{C}^3$, satisfying $a_0 a_1 a_2 = q$, such that the non-
resonant conditions \((2.14)\) are satisfied. Then the Riemann-Hilbert mapping \((2.15)\)
is a bijection.

Our third main result concerns the construction of a moduli space of the mon-
dromy surface and thus of $q_{IV}$ in the non-resonant parameter case. To define
coordinates on the monodromy surface, we use the following notation: for any
nonzero $2 \times 2$ matrix $R$ which is not invertible, let $r_1$ and $r_2$ be respectively its first
and second row, then we define
\[ \pi(2.19) \]
with $\pi(R) = 0$ if and only if $r_1 = (0, 0)$ and $\pi(R) = \infty$ if and only if $r_2 = (0, 0)$.

Take any equivalence class $M = \{ C \} \in \mathcal{M}_c(\lambda_0, a)$. Let $1 \leq k \leq 3$, then \(|C(z)|
has a simple zero at $z = x_k$, due to item \((2.13)\) in Definition 2.3 and thus $C(x_k)$ is
nonzero and not invertible. We define the coordinates
\[ \rho_k = \pi(C(x_k)), \quad (1 \leq k \leq 3). \]
Note that $(\rho_1, \rho_2, \rho_3)$ are invariant under right multiplication by diagonal matrices
and they are thus well-defined coordinates on $\mathcal{M}_c(\lambda_0, a)$. We denote the corre-
spending mapping by
\[ \rho : \mathcal{M}_c(\lambda_0, a) \to \mathbb{P}^1(\mathbb{C})^3, M \mapsto (\rho_1, \rho_2, \rho_3). \] (2.19)
The coordinates $(\rho_1, \rho_2, \rho_3)$ are not independent: they are elements of an alge-
braic surface. To define this surface we introduce the polynomial
\[ T(p_1, p_2, p_3; \lambda_0, a) = + \theta_4(+a_0, +a_1, +a_2) (\theta_4(\lambda_0)p_1p_2p_3 - \theta_4(-\lambda_0)) - \theta_4(-a_0, +a_1, -a_2) (\theta_4(\lambda_0)p_1 - \theta_4(-\lambda_0)p_2p_3) + \theta_4(+a_0, -a_1, -a_2) (\theta_4(\lambda_0)p_2 - \theta_4(-\lambda_0)p_1p_3) - \theta_4(-a_0, -a_1, +a_2) (\theta_4(\lambda_0)p_3 - \theta_4(-\lambda_0)p_1p_2) \]
and its homogenisation
\[ T_{\text{hom}}(p_1^x, p_1^y, p_2^x, p_2^y, p_3^x, p_3^y, \lambda_0, a) = p_1^x p_2^y p_3^z T \left( \frac{p_1^x}{p_1}, \frac{p_2^x}{p_2}, \frac{p_3^x}{p_3}; \lambda_0, a \right). \]

**Definition 2.11.** We define the algebraic surface
\[ \mathcal{P}(\lambda_0, a) = \{ (p_1, p_2, p_3) \in \mathbb{P}^1(\mathbb{C})^3 : T(p_1, p_2, p_3; \lambda_0, a) = 0 \} \]
\[ = \{ (p_1, p_2, p_3) \in \mathbb{P}^1(\mathbb{C})^3 : T_{\text{hom}}(p_1^x, p_1^y, p_2^x, p_2^y, p_3^x, p_3^y, \lambda_0, a) = 0 \}, \]
where we used the standard coordinates $p = [p^x : p^y] \in \mathbb{P}^1(\mathbb{C})$.

Our third main result is given by the following theorem.

**Theorem 2.12.** The range of the mapping $\rho$, defined in equation \((2.19)\), is given
by the algebraic surface $\mathcal{P}(\lambda_0, a)$. Upon restricting the co-domain of $\rho$ in equation
\((2.19)\), the mapping
\[ \mathcal{M}_c(\lambda_0, a) \to \mathcal{P}(\lambda_0, a), M \mapsto \rho(M) \]
is a bijection and in particular the algebraic surface $P(\lambda_0, a)$ is the moduli space of $\mathcal{M}_c(\lambda_0, a)$ and thus of $qP_{IV}^0(\lambda_0, a)$ and $qP_{IV}(\lambda_0, a)$.

Remark 2.13. It follows from Theorem 2.12 that $(\rho_1, \rho_2, \rho_3) \in P(\lambda_0, a)$ parametrise the solution $f = (f^{(|m|)})_{m \in \mathbb{Z}}$ of $qP_{IV}(\lambda_0, a)$. In the special case $a \in \mathbb{R}^3$, the transcendent $f$ is real-valued if and only if $(\rho_1, \rho_2, \rho_3) \in P(\lambda_0, a) \cap \mathbb{P}^1(\mathbb{R})^3$.

3. A class of Fuchsian systems

In this section, we study the direct and inverse monodromy problem concerning the spectral part of the Lax pair (2.4). We analyse linear systems of the form

\[ f = (f^{(|m|)})_{m \in \mathbb{Z}} \]

where $A$ is a bijection and in particular the algebraic surface $P(\lambda_0, a)$ and there exists a unique $f^{(|m|)}$ systems have solutions with convergent expansions near 0 and $\infty$.

3.1. Fundamental Solutions.

In Section 3.1, we consider the direct and inverse monodromy problem concerning the class of Fuchsian systems (3.1). We describe the monodromy problem concerning the class of Fuchsian systems (3.1).

3.1.1. Fundamental Solutions. Carmichael [3] showed that Fuchsian $q$-difference systems have solutions with convergent expansions near 0 and $\infty$. We restate Carmichael’s results here for the $q$-difference system (3.1). First, we define some terminology.

Definition 3.1. For $\lambda \in \mathbb{C}^*$ and $a = (a_0, a_1, a_2) \in \mathbb{C}^3$ with $a_0a_1a_2 = q$, we denote by $F(\lambda, a)$ the set of matrix polynomials of the form (3.2) satisfying (3.1)–(3.4).

Lemma 3.2. Suppose $\lambda \in \mathbb{C}^*$, $a \in \mathbb{C}^3$ with $a_0a_1a_2 = q$, and $A(z) \in F(\lambda, a)$ are given. Define $u = -A_{12}(0)$. Then, for any $d \in \mathbb{C}^*$, we have

\[ A(0) = M_0 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} M_0^{-1}, \text{ where } M_0 := d \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & -i \\ 0 & 1 \end{pmatrix}, \]  

(3.3)

and there exists a unique $2 \times 2$ matrix function $\Phi_0(z)$, meromorphic on $\mathbb{C}^*$, satisfying

\[ \Phi_0(qz) = A(z)\Phi_0(z) \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \]

(3.4)

\[ \Phi_0(z) = M_0 + \mathcal{O}(z) \quad (z \to 0). \]

(3.5)

Furthermore, $\Phi_0(z)$ has the following properties:

(z.1) $\Phi_0(z)^{-1}$ is analytic on $\mathbb{C}$ and $\Phi_0(0) = M_0$;

(z.2) $|\Phi_0(z)| = |M_0| |a_0z, a_0z, -a_0a_2z, qz, -qz; q\rangle |^{-1};$

(z.3) $\Phi_0(-z) = -\sigma_3 \Phi_0(z) \sigma_1.$
In particular,
\[ Y_0(z) = \Phi_0(z)E_0(z), \]
defines a fundamental solution of (3.1), for any meromorphic $2 \times 2$ matrix function $E_0(z)$ on $\mathbb{C}^*$, satisfying
\[ E_0(qz) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} E_0(z), \quad |E_0(z)| \neq 0. \]

**Proof.** Except for property (i.3), all the contents of the lemma can be inferred directly from [3, Theorem 1]. To prove (i.3), let
\[ \Phi(z) := \sigma_3 \Phi_0(-z) \sigma_3. \]
Then $\Phi(z)$ is analytic at $z = 0$, with $\Phi(0) = I$, and straightforward calculation, using symmetry (a.3), shows that $\Phi(z)$ also satisfies equation (3.4). By uniqueness, we must have $\Phi(z) = \Phi_0(z)$ and the lemma follows. \(\square\)

Similar to Lemma 3.2, the following lemma provides a fundamental solution with a convergent expansion at infinity.

**Lemma 3.3.** Suppose $\lambda \in \mathbb{C}^*$, $a \in \mathbb{C}^3$ are given with $\lambda^2 \notin q^\mathbb{Z}$ and $a_0a_1a_2 = q$. For each $A(z) \in \mathcal{F}(\lambda, a)$, there exists a unique $2 \times 2$ matrix function $\Phi_\infty(z)$, meromorphic on $\mathbb{C}^*$, satisfying
\[ \Phi_\infty(qz) = \frac{1}{q a_0^2 a_2 z} z^{-3} A(z) \Phi_\infty(z) \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \] (3.6)
\[ \Phi_\infty(z) = I + O(z^{-1}) \quad (z \to \infty). \] (3.7)

Furthermore, $\Phi_\infty(z)$ has the following properties:

(i.1) $\Phi_\infty(z)$ is analytic on $\mathbb{P}^1 \setminus \{0\}$ and $\Phi_\infty(\infty) = I$;

(i.2) $|\Phi_\infty(z)| = \frac{q}{a_0 a_2 z} - \frac{q}{a_0 a_2 z}, q/(a_0 a_2 z), q/(a_0 a_2 z), 1/z, -1/z; q)$;

(i.3) $\Phi_\infty(-z) = \sigma_3 \Phi_\infty(z) \sigma_3$.

In particular,
\[ Y_\infty(z) = \Phi_\infty(z)E_\infty(z), \]
defines a fundamental solution of (3.1), for any $2 \times 2$ matrix function $E_\infty(z)$, meromorphic on $\mathbb{C}^*$, satisfying
\[ E_\infty(qz) = q a_0^2 a_2 z^{-3} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} E_\infty(z), \quad |E_\infty(z)| \neq 0. \]

**Proof.** Except for property (i.3), all the contents of the lemma can be inferred directly from [3, Theorem 1]. To check this property, let
\[ \Phi(z) := \sigma_3 \Phi_\infty(-z) \sigma_3. \]
Then $\Phi(z)$ is analytic at $z = \infty$, with $\Phi(\infty) = I$, and straightforward calculation, using symmetry (a.3), shows that $\Phi(z)$ also solves equation (3.6). By uniqueness, we must have $\Phi(z) = \Phi_\infty(z)$ and the lemma follows. \(\square\)

Note that $\Phi_\infty(z)$ is uniquely defined in Lemma 3.3, whereas $\Phi_0(z)$ is only uniquely defined up to a constant multiplier in Lemma 3.2.
3.1.2. The Connection Matrix. In Lemmas 3.2 and 3.3 we have constructed fundamental solutions \( Y_0(z) \) and \( Y_\infty(z) \) of the Fuchsian system (3.1). They are related by

\[
Y_\infty(z) = Y_0(z)P(z),
\]

\[
P(z) = Y_0(z)^{-1}Y_\infty(z) = E_0(z)^{-1}C(z)E_\infty(z),
\]

where

\[
C(z) = \Phi_0(z)^{-1}\Phi_\infty(z). \tag{3.8}
\]

Note that there is a great deal of freedom in choosing \( E_0(z) \) and \( E_\infty(z) \), which in turn implies that \( P(z) \) is not rigidly defined. In contrast, the matrix \( C(z) \) is rigidly defined up to a constant multiplier. We thus define \( C(z) \) to be the connection matrix associated with \( A(z) \). This is in line with the Galoisian approach in [35], where \( E_0(z) \) and \( E_\infty(z) \) are considered merely as formal scalings.

**Lemma 3.4.** Let \( \lambda \in \mathbb{C}^* \) and \( a \in \mathbb{C}^3 \), satisfying \( \lambda^2 \not\in q^2 \) and \( a_0 a_1 a_2 = q \). Take an \( A(z) \in \mathcal{F}(\lambda, a) \), and let \( \Phi_0(z) \) and \( \Phi_\infty(z) \) be as defined in Lemma 3.2 and 3.3. Then the corresponding connection matrix \( C(z) = \Phi_0(z)^{-1}\Phi_\infty(z) \) is an element of the space \( \mathcal{C}(\lambda, a) \), defined in Definition 2.7.

**Proof.** Property (6.2) follows from the \( q \)-difference equations which \( \Phi_0(z) \) and \( \Phi_\infty(z) \) satisfy. The remaining properties (6.1), (6.2) and (6.4) are a direct consequence of the further properties of \( \Phi_0(z) \) and \( \Phi_\infty(z) \) listed in Lemmas 3.2 and 3.3. \( \square \)

3.1.3. The Monodromy Mapping. Since the Fuchsian system (3.1) only has two critical points, the connection matrix embodies the monodromy of the Fuchsian system (3.1). Recalling that the connection matrix is only uniquely defined up to scalar multiplication, we make the following definition.

**Definition 3.5.** Let \( \lambda \in \mathbb{C}^* \) and \( a \in \mathbb{C}^3 \), satisfying \( a_0 a_1 a_2 = q \), such that the non-resonant conditions (3.0) are satisfied. Then we define

\[
M_c(\lambda, a) = \mathcal{C}(\lambda, a)/\mathbb{C}^* \tag{3.9}
\]

where the quotient is taken with respect to scalar multiplication.

We define the monodromy mapping

\[
M_\mathcal{F} : \mathcal{F}(\lambda, a) \to M_c(\lambda, a),
\]

by attaching to every matrix polynomial \( A(z) \) in the space \( \mathcal{F}(\lambda, a) \), see Definition 2.7, the up to scalar multiplication unique connection matrix \( C(z) \) corresponding to the Fuchsian system (3.1) via Lemma 3.7.

The reason for requiring the non-resonant conditions (2.6) in the above definition, is that otherwise, the Fuchsian system can generally not be uniquely reconstructed from the connection matrix. The latter point is clear from the proof of the following proposition.

**Proposition 3.6.** The monodromy mapping \( M_\mathcal{F} \) defined in 3.5 is injective.

**Proof.** Let \( A(z), \tilde{A}(z) \in \mathcal{F}(\lambda, a) \), and denote corresponding \( \Phi_0(z), \Phi_\infty(z), C(z) \) and \( \tilde{\Phi}_0(z), \tilde{\Phi}_\infty(z), \tilde{C}(z) \) as defined in Lemmas 3.2, 3.3 and 3.4 respectively.

Suppose \( M_\mathcal{F}(A) = M_\mathcal{F}(\tilde{A}) \), then there exists a \( c \in \mathbb{C}^* \) such that \( \tilde{C}(z) = cC(z) \), hence

\[
G(z) := \tilde{\Phi}_\infty(z)\Phi_\infty(z)^{-1} \tag{3.9}
\]

\[
= \tilde{\Phi}_0(z)\tilde{C}(z)C(z)^{-1}\Phi_0(z)^{-1}
\]

\[
= c\Phi_0(z)\Phi_0(z)^{-1}. \tag{3.10}
\]
From equation (3.9) and Lemma 3.3 it is clear that $G(z)$ is analytic on
\[ \mathbb{C}^* \setminus \left( q^{N} \{x_1, \ldots, x_6\} \right), \]
where $x_1, \ldots, x_6$ as defined in (2.13). Similarly, from (3.10) and Lemma 3.2 it follows that $G(z)$ is analytic on
\[ \mathbb{C} \setminus \left( q^{-N} \{x_1, \ldots, x_6\} \right). \]
We conclude that $G(z)$ is analytic on the complement of
\[ \left( q^{N} \{x_1, \ldots, x_6\} \right) \cap \left( q^{-N} \{x_1, \ldots, x_6\} \right), \quad (3.11) \]
in $\mathbb{C}$. However, precisely because of the non-resonant conditions (2.6), the intersection in (3.11) is empty, so $G(z)$ is analytic on $\mathbb{C}$. Finally from equation (3.9) and Lemma 3.3 it follows that $G(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$, and hence $G(z) \equiv I$ by Liouville’s theorem. Therefore $\Phi_\infty(z) = \Phi(z)$, and hence $\tilde{A}(z) = A(z)$, by equation (3.6). The proposition follows. \hfill $\Box$

3.2. The Inverse Monodromy Problem. In this section we consider the surjectivity of the monodromy mapping, which is a more delicate issue than its injectivity (as it is the content of the $q$-analogue of Hilbert’s 21st problem). Birkhoff \cite{1} gave a comprehensive treatment of this problem in the generic non-resonant case.

Considering the class of Fuchsian systems (3.1), we formulate the main inverse problem as follows.

**Problem 3.7 (The Inverse Monodromy Problem).** Let $\lambda \in \mathbb{C}^*$ and $a \in \mathbb{C}^1$, satisfying $a_0a_1a_2 = q$, such that the non-resonant conditions (2.6) are satisfied. Given a monodromy datum $M = [C(z)] \in M_r(\lambda, a)$, construct a Fuchsian system (3.1), whose associated connection matrix (modulo a constant) is $M$.

In Proposition 3.9 we show that this inverse problem is equivalent to Riemann-Hilbert problem $\text{RHP}^{(m)}(\gamma, C)$, defined in Definition 2.7 for any admissible curve $\gamma$. But first we prove that Riemann-Hilbert problem $\text{RHP}^{(m)}(\gamma, C)$ has at most one solution, for any $m \in \mathbb{Z}$.

**Lemma 3.8.** For any $m \in \mathbb{Z}$, if $\text{RHP}^{(m)}(\gamma, C)$ has a solution $Y^{(m)}(z)$, then it is unique. Furthermore, let $c \in \mathbb{C}^*$ be defined by (3.3), then the determinant $\Delta(z) = |Y^{(m)}(z)|$ equals
\[ \Delta(z) = \begin{cases} (q x_1/z, \ldots, q x_6/z; q)_\infty & \text{if } z \in D_+, \\ e^{-1} (z/x_1, \ldots, z/x_6; q)^{-1}_\infty & \text{if } z \in D_. \end{cases} \quad (3.12) \]
In particular $Y^{(m)}(z)$ is globally invertible on $\mathbb{C} \setminus \gamma$.

**Proof.** Note that the determinant $\Delta(z) = |Y^{(m)}(z)|$ solves the following scalar Riemann-Hilbert problem:
- $\Delta(z)$ is analytic on $\mathbb{C} \setminus \gamma$.
- $\Delta(z)$ has continuous boundary values $\Delta_-(z)$ and $\Delta_+(z)$ for $z \in \gamma$, related by the jump condition
  \[ \Delta_+(z) = \Delta_-(z) e^{\theta_q}(z/x_1, \ldots, z/x_6) \quad (z \in \gamma) \]
- $\Delta(z)$ has the following asymptotic behaviour near infinity,
  \[ \Delta(z) = 1 + \mathcal{O}(z^{-1}) \quad (z \to \infty). \]
It is easy to see that this problem has a unique solution, given by (3.12). In particular $|Y^{(m)}(z)|$ vanishes nowhere and hence $Y^{(m)}(z)^{-1}$ is an analytic function on $\mathbb{C} \setminus \gamma$. Therefore, given another solution $\Psi(z)$ of Riemann-Hilbert problem $\text{RHP}^{(m)}(\gamma, C)$, then
\[ H(z) := \Psi(z)Y^{(m)}(z)^{-1} \]
defines an analytic function on $\mathbb{C} \setminus \gamma$. Since $Y^{m}(z)$ and $\Psi(z)$ satisfy the same jump condition on $\gamma$, $H(z)$ has analytic continuation to $\mathbb{C}$. Furthermore the asymptotic behaviour near infinity of both solutions implies $H(z) = I + O(\frac{1}{z})$ as $z \to \infty$, and we conclude $H(z) \equiv I$ by Liouville’s theorem. So $Y^{m}(z) = \Psi(z)$ and uniqueness follows.

**Proposition 3.9.** Let $\lambda \in \mathbb{C}^*$ and $a \in \mathbb{C}^3$, satisfying $aq_0a_2 = q$, such that the non-resonant conditions (3.1) are satisfied. Given a monodromy datum $M = [C(z)] \in M_{r}(\lambda, a)$, the inverse monodromy problem (3.7) is equivalent to $\text{RHP}^{0}(\gamma, C)$, for any admissible curve $\gamma$, in the following sense:

(i) If $A(z) \in \mathcal{F}(\lambda, a)$ is a solution of the inverse monodromy problem (3.7), then there exists a unique value of $d \in \mathbb{C}^*$ in Lemma 3.8 for which the corresponding matrix function $\Phi_{0}^{m}(z)$, together with the matrix function $\Phi_{0}^{m}(z)$ constructed in Lemma 3.3, define a solution

$$
\Psi(z) := \begin{cases} 
\Phi_{\infty}(z) & \text{if } z \in D_{+}, \\
\Phi_{0}(z) & \text{if } z \in D_{-},
\end{cases}
$$

(3.13)

of Riemann-Hilbert problem $\text{RHP}^{0}(\gamma, C)$.

(ii) Conversely, suppose $\Psi(z)$ is a solution of Riemann-Hilbert problem $\text{RHP}^{0}(\gamma, C)$, defining

$$
\Psi_{\infty}(z) := \begin{cases} 
\Psi(z) & \text{if } z \in D_{+}, \\
\Psi(z)C(z) & \text{if } z \in D_{-},
\end{cases}
$$

(3.14)

then $\Psi_{\infty}(z)$ and $\Psi_{0}(z)^{-1}$ are related by

$$
\Psi_{\infty}(z) = \Psi_{0}(z)C(z),
$$

(3.15)

and

$$
A(z) := qa_{0}^{2}a_{2}z^{2}i^{3}\Psi_{\infty}(qz)\begin{pmatrix} \lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}\Psi_{\infty}(z)^{-1}
$$

(3.16)

$$
= \Psi_{0}(qz)\begin{pmatrix} i & 0 \\
0 & -i
\end{pmatrix}\Psi_{0}(z)^{-1}
$$

(3.17)

defines a solution $A(z) \in \mathcal{F}(\lambda, a)$ of the inverse monodromy problem (3.7).

**Proof.** Consider the first part of the proposition. Take $d \in \mathbb{C}^*$ and define $\Phi_{0}(z)$ and $\Phi_{\infty}(z)$ as in Lemma 3.2 and 3.3 such that $\Phi_{\infty}(z) = \Phi_{0}(z)C(z)$, see equation (3.8). Then equation (3.13) defines a matrix function $\Psi(z)$, which satisfies the jump condition of $\text{RHP}^{0}(\gamma, C)$. Furthermore, by Lemmas 3.2 and 3.3 it also satisfies the analyticity and asymptotic condition of $\text{RHP}^{0}(\gamma, C)$ and thus solves it.

Considering the converse, clearly $\Psi_{\infty}(z)$ and $\Psi_{0}(z)^{-1}$, defined by equations (3.14) are analytic on $\mathbb{C}^*$ and $\mathbb{C}$ respectively, related by (3.16). Hence, defining $A(z)$ by (3.16), equation (3.17) follows from property (3.2) of $C(z)$.

We proceed in showing that $A(z) \in \mathcal{F}(\lambda, a)$. From equations (3.10) and (3.17) we infer respectively that $A(z)$ is analytic on $q^{-1}D_{+}$ and $D_{-}$. Furthermore, note that

$$
A(z) = qa_{0}^{2}a_{2}z^{2}i^{3}\Psi_{0}(qz)C(qz)\lambda^{qz}\Psi_{\infty}(z)^{-1},
$$

from which it follows that $A(z)$ is also analytic on $\mathbb{C} \setminus (q^{-1}D_{+} \cup D_{-})$. Hence $A(z)$ is analytic on $\mathbb{C}$. It follows from the asymptotic behaviour of $\Psi(z)$ that

$$
A(z) = qa_{0}^{2}a_{2}z^{2}i^{3}\lambda^{qz}O(z^{-1}) \quad (z \to \infty),
$$

and thus $A(z)$ is a matrix polynomial of degree three, satisfying property (3.2).

Due to equation (3.17), we know that $A(0)$ has eigenvalues $\{ \pm i \}$, so $A(z)$ satisfies property (3.1). Furthermore, using the explicit expression for the determinant
computing the values of a particular $q$ solves the inverse problem 4.1 for a given monodromy datum is equivalent to

$$\lambda \rightarrow \tau$$

whose monodromy equals $F$ Fuchsian system of the class of Fuchsian systems (3.1) explicitly using the associated RHP, yielding Theorem 2.8 in Section 1.1. Finally, in Section 4.3, we prove Theorem 2.10.

In Definition 2.7. In Section 4.1 we show that the inverse problem 4.1 is equivalent given a monodromy datum $M$ satisfying $a \equiv \tau$ if it trivially deforms its monodromy as $\lambda$, we also denote by $\tau$. Furthermore we show that the RHP is always solvable, for at least one value of $m \in \mathbb{Z}$. In Section 4.2 we compute the isomonodromic deformation of the class of Fuchsian systems (3.1) explicitly using the associated RHP, yielding Theorem 2.8 in Section 1.1. Finally, in Section 1.3 we prove Theorem 2.10.

4. ISOMONODROMIC DEFORMATION

In this section, we consider isomonodromic deformation of Fuchsian systems of the form (3.1) as $\lambda \rightarrow q\lambda$. Clearly the space $C(\lambda, a)$, defined in Definition 2.4, is not invariant under $\lambda \rightarrow q\lambda$. However, we do have the following bijective mapping

$$\tau : \begin{cases} C(\lambda, a) \rightarrow C(q\lambda, a), \\
C(z) \rightarrow \sigma_3 C(z)z^{-\sigma_3}. \end{cases}$$

Note that this mapping commutes with scalar multiplication and right-multiplication by invertible diagonal matrices. It therefore induces bijective mappings

$$\tau : M_c(\lambda, a) \rightarrow M_c(q\lambda, a), \quad \tau : M_c(\lambda, a) \rightarrow M_c(q\lambda, a), \quad (4.1)$$

which we also denote by $\tau$, where we recall the notations $M_c(\lambda, a)$ and $M_c(\lambda, a)$ for the spaces defined in Definitions 3.5 and 2.11 respectively.

We call a deformation, as $\lambda \rightarrow q\lambda$, of the Fuchsian system (3.1) isomonodromic if it trivially deforms its monodromy as $\tau$. Correspondingly, we define the following inverse problem.

**Problem 4.1** (Generalised Inverse Monodromy Problem). Let $\lambda_0 \in \mathbb{C}^*$ and $a \in \mathbb{C}^3$, satisfying $a_0a_1a_2 = q$, such that the non-resonant conditions (2.4.0) are satisfied. Given a monodromy datum $M = [C(z)] \in M_c(\lambda_0, a)$ and an $m \in \mathbb{Z}$, construct a Fuchsian system

$$Y(qz) = A^m(z)Y(z), \quad A^m(z) \in \mathcal{F}(q^m, \lambda_0, a),$$

whose monodromy equals $\tau^m(M)$.

In this section we prove that the aforementioned isomonodromic deformation as $\lambda \rightarrow q\lambda$ is equivalent to the $q^{\text{mod}}$ time-evolution. In particular we show that solving the inverse problem 4.1 for a given monodromy datum is equivalent to computing the values of a particular $q^{\text{mod}}(\lambda_0, a)$ transcendental.

Associated with the inverse problem 4.1 is the Riemann-Hilbert Problem defined in Definition 2.7. In Section 4.4 we show that the inverse problem 4.1 is equivalent to this RHP. Furthermore we show that the RHP is always solvable, for at least one value of $m \in \mathbb{Z}$. In Section 4.4 we compute the isomonodromic deformation of the class of Fuchsian systems (3.1) explicitly using the associated RHP, yielding Theorem 2.8 in Section 1.1. Finally, in Section 1.3 we prove Theorem 2.10.
4.1. Solvability of the Generalised Inverse Monodromy Problem. In this Section, we prove the solvability of inverse problem (4.1) for at least one value of $m \in \mathbb{Z}$. Firstly, in the following proposition, we make the equivalence of the generalised inverse monodromy problem (4.1) and the RHP defined in Definition 2.7 explicit.

**Proposition 4.2.** Let $\lambda_0 \in \mathbb{C}^*$ and $a \in \mathbb{C}^3$, satisfying $a_0 a_1 a_2 = q$, such that the non-resonant conditions (2.16) are satisfied. Given a monodromy datum $M = [C(z)] \in M_c(\lambda, a)$ and an $m \in \mathbb{Z}$, then, for any choice of admissible curve, the generalised inverse monodromy problem (4.1) is equivalent to Riemann-Hilbert problem $\text{RHP}^m(\gamma, C)$, in the following sense.

(i) If $A^m(z) \in \mathcal{F}(q^m \lambda_0, a)$ is a solution of the inverse monodromy problem (4.1), then there exists a unique value of $d = d_m \in \mathbb{C}^*$ in Lemma 3.2 for which the corresponding matrix function $Φ_0^m(z)$, together with the matrix function $Φ_\infty^m(z)$ constructed in Lemma 3.2, define a solution

$$Y^m(z) := \begin{cases} Φ_\infty^m(z)z^m & \text{if } z \in D_+, \\ Φ_0^m(z) & \text{if } z \in D_-, \end{cases} \tag{4.2}$$

of $\text{RHP}^m(\gamma, C)$.

(ii) Conversely, suppose $Y^m(z)$ is a solution of $\text{RHP}^m(\gamma, C)$, writing

$$\Psi^m(z) := \begin{cases} Y^m(z)z^{-m\sigma_3} & \text{if } z \in D_+, \\ Y^m(z)C(z)z^{-m\sigma_3} & \text{if } z \in D_-, \end{cases} \quad \Psi_0^m(z) := \begin{cases} Y^m(z)\sigma_3^{-m}C(z)^{-1} & \text{if } z \in D_+, \\ Y^m(z)\sigma_3^m & \text{if } z \in D_-,$$

then $Ψ^m(z)$ and $Ψ_0^m(z)^{-1}$ are related by

$$Ψ^m(z) = Ψ_0^m(z)\sigma_3^mC(z)z^{-m\sigma_3}, \tag{4.3}$$

and, denoting

$$κ = qa_0^2a_2i, \quad λ_m = q^m \lambda_0, \tag{4.4}$$

the matrix polynomial

$$A^m(z) := z^3Ψ^m_\infty(qz) \begin{pmatrix} κλ_m & 0 \\ 0 & κ^{-1}λ_m \end{pmatrix} Ψ^m_\infty(z)^{-1} \tag{4.5}$$

$$= Ψ_0^m(qz) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} Ψ_0^m(z)^{-1}, \tag{4.6}$$

defines a solution $A^m(z) \in \mathcal{F}(q^m \lambda_0, a)$ of the inverse monodromy problem (4.1).

**Proof.** Fix any $m \in \mathbb{Z}$, then $Y^m(z)$ is a solution of $\text{RHP}^m(\gamma, C)$ if and only if $Ψ(z) = Y^m(z)S(z)^{-1}$ is a solution of $\text{RHP}^0(\gamma, τ^m(C))$, where

$$S(z) := \begin{cases} z^m & \text{if } z \in D_+, \\ \sigma_3^m & \text{if } z \in D_. \end{cases}$$

Therefore, statements (i) and (ii) are equivalent to the equally numbered statements in Proposition 3.9 after the substitutions $C(z) \mapsto τ^m(C(z))$, $λ \mapsto q^m λ_0$ and $Ψ(z) \mapsto Y^m(z)S(z)^{-1}$. The proposition is thus a direct corollary of Proposition 3.9. □

In the remainder of this section, we give a classical argument, going back to Birkhoff [11], showing that $\text{RHP}^m(\gamma, C)$ has a solution $Y^m(z)$, for at least one value of $m \in \mathbb{Z}$.  


Lemma 4.3. Let $\gamma$ be an oriented analytic Jordan curve in $\mathbb{P}^1$ and let $D_-$ and $D_+$ denote the inside and outside of $\gamma$ in $\mathbb{P}^1$ respectively. Let $C(z)$ be a $2 \times 2$ matrix function, analytic on $\gamma$, such that $|C(z)| \neq 0$ on $\gamma$. Then, for any $\alpha \in \mathbb{P}^1 \setminus \gamma$, there exists a $2 \times 2$ matrix function $Y(z)$, satisfying

- $Y(z)$ is analytic on $\mathbb{P}^1 \setminus (\gamma \cup \{\alpha\})$ and meromorphic at $z = \alpha$.
- $Y(z)$ has continuous boundary values $Y_+(z)$ and $Y_-(z)$ as $z$ approaches $\gamma$ from both the inside $D_+$ and outside $D_-$ respectively, related by the jump condition

$$Y_+(z) = Y_-(z)C(z).$$

- The determinant $|Y(z)|$ does not vanish on $\mathbb{P}^1 \setminus (\gamma \cup \{\alpha\})$, and both $|Y_+(z)|$ and $|Y_-(z)|$ do not vanish on $\gamma$.

Proof. This is a special case of the “Preliminary Theorem” in Birkhoff [1]. □

Lemma 4.4. RHP$_m^m(\gamma, C)$ has a solution $Y^m(z)$, for at least one value of $m \in \mathbb{Z}$.

Proof. For convenience of the reader, we paraphrase Birkhoff’s argument [1] for our special case. Firstly, note that the matrix $C(z)$ is analytic and $|C(z)|$ does not vanish on $\gamma$. We may thus apply Lemma 4.3 with $\alpha = \infty$, which gives a matrix function $Y(z)$ that satisfies the analyticity and jump condition in RHP$_m^m(\gamma, C)$, $m \in \mathbb{Z}$. It remains to normalise $Y(z)$ appropriately.

Firstly, we compare the determinant $|Y(z)|$ with $\Delta(z)$, defined in equation (3.12). Note that $d(z) = |Y(z)|/\Delta(z)$ defines a non-vanishing analytic function on $\mathbb{C}$ which is meromorphic at $z = \infty$, so $d(z) \equiv d_0 \in \mathbb{C}^*$ is constant. In particular

$$|Y(z)| = d_0 (qx_1/z, \ldots, qx_6/z)^\infty \quad (z \in D_+).$$

(4.8)

For any matrix function $H(z)$, replacing $Y(z) \mapsto \tilde{Y}(z) = H(z)Y(z)$ in the above, all analytic properties in Lemma 4.3 are conserved if and only if $H(z)$ is a matrix polynomial with $|H(z)| = h \in \mathbb{C}^*$ constant. It therefore suffices to find an appropriate such $H(z)$, so that

$$\tilde{Y}(z) = (U + O(z^{-1})) \begin{pmatrix} z^m & 0 \\ 0 & z^{-m} \end{pmatrix} \quad (z \to \infty).$$

(4.9)

for an $m \in \mathbb{Z}$ and $U \in GL_2(\mathbb{C})$. Indeed $U^{-1}\tilde{Y}(z)$ then defines a solution to Riemann-Hilbert Problem RHP$_m^m(\gamma, C)$.

To this end, determine the unique $m_1, m_2 \in \mathbb{Z}$ and $U \in \mathbb{C}^{2 \times 2}$, with both columns nonzero, such that

$$Y(z) = (U + O(z^{-1})) \begin{pmatrix} z^{m_1} & 0 \\ 0 & z^{m_2} \end{pmatrix} \quad (z \to \infty).$$

(4.10)

By equation (4.8), we must have $K(Y) := m_1 + m_2 \geq 0$. Furthermore, note that, again by (4.8), $K = 0$ if and only if $U$ is invertible, in which case we are done as $Y(z)$ has the desired form (4.9). We proceed in showing that, if $K(Y) > 0$, then there exists a matrix polynomial $G(z)$ with $|G(z)| \equiv 1$, such that $Y'(z) := G(z)Y(z)$ will have a strictly smaller $K$ value then $Y(z)$, i.e. $K(Y') < K(Y)$.

So assume $K(Y) > 0$, then $U$ is not invertible and has two nonzero columns, hence there exists an $M \in GL_2(\mathbb{C})$ such that

$$MU = \begin{pmatrix} 0 & 0 \\ u'_{21} & u'_{22} \end{pmatrix},$$

for some $u'_{21}, u'_{22} \in \mathbb{C}^*$. Let $l > 0$ be such that

$$MY(z) = \begin{pmatrix} z^{-l} & 0 \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix} + O(z^{-1}) \right) \begin{pmatrix} z^{m_1} & 0 \\ 0 & z^{m_2} \end{pmatrix} \quad (z \to \infty),$$

(4.10)
for some \( u_{11}', u_{12}' \in \mathbb{C} \) not both equal to zero. Next we multiply from the left by a matrix polynomial

\[
R(z) = \begin{pmatrix} 1 & 0 \\ rz' & 1 \end{pmatrix},
\]

which gives

\[
R(z)MY(z) = \left( z^{-1} \begin{pmatrix} u_{11}' & u_{12}' \\ u_{21}' + ru_{11}' & u_{22}' + ru_{12}' \end{pmatrix} + O(z^{-1}) \right) \begin{pmatrix} z_{m1} & 0 \\ 0 & z_{m2} \end{pmatrix},
\]

as \( z \to \infty \). Determine \( i \in \{1, 2\} \) such that \( u_{ii}' \neq 0 \) and choose \( r \in \mathbb{C} \) such that \( u_{ii}' + ru_{ii}' = 0 \). Set \( G(z) = M^{-1}R(z)M \), then \( G(z) \) is a matrix polynomial satisfying \( |G(z)| \equiv 1 \). Let \( Y'(z) := G(z)Y(z) \), then \( K(Y') < K(Y) \).

Applying the above argument recursively, we obtain matrix polynomials \( G_0(z), G_1(z), \ldots, G_k(z) \), each with unit determinant, for some \( k \in \mathbb{N} \), such that, setting

\[
Y_0(z) := Y(z), \quad Y_{s+1} = G_s(z)Y_s(z) \quad (0 \leq s \leq k - 1),
\]

we have \( K(Y_0) > K(Y_1) > \ldots > K(Y_k) = 0 \). Define

\[
\bar{Y}(z) = H(z)Y(z), \quad H(z) = G_{k-1}(z)G_{k-2}(z) \cdots G_0(z),
\]

then equation (4.9) holds true, yielding the lemma. \( \square \)

4.2. Isomonodromic Deformation and the Riemann-Hilbert Problem. In this section we prove Theorem 2.8. Firstly we make the relation between \( A^{(m)}(z) \) and \( Y^{(m)}(z) \) in Proposition 4.2 more explicit.

Let \( m \in \mathbb{Z} \) and suppose the solution \( Y^{(m)}(z) \) of the Riemann-Hilbert problem \( \text{RHP}^{(m)}(\gamma, C) \) exists. We know that there exist a unique \( g^{(m)} \in \mathcal{G}(a) \) and \( u_m \in \mathbb{C}^* \), such that \( A^{(m)}(z) \in \mathcal{F}(q^m\lambda_0, a) \) is given by

\[
A^{(m)}(z) = A(z; \lambda_m, g^{(m)}, u_m) \quad (4.11)
\]

where we again used the notation (4.4). Also there exist unique matrices \( U^{(m)}, V^{(m)}, W^{(m)} \in \mathbb{C}^{2\times2} \) such that

\[
Y^{(m)}(z) = (I + z^{-1}U^{(m)} + z^{-2}V^{(m)} + z^{-3}W^{(m)} + O(z^{-4}))z^{m\sigma_3} \quad (4.12)
\]

as \( z \to \infty \) and thus

\[
\Psi^{(m)}_\infty(z) = I + z^{-1}U^{(m)} + z^{-2}V^{(m)} + z^{-3}W^{(m)} + O(z^{-4}) \quad (z \to \infty).
\]

Due to Equation (3.18), we must have

\[
\sigma_3U^{(m)}\sigma_3 = -U^{(m)}, \quad \sigma_3V^{(m)}\sigma_3 = V^{(m)}, \quad \sigma_3W^{(m)}\sigma_3 = -W^{(m)},
\]

and hence these matrices take the form

\[
U^{(m)} = \begin{pmatrix} 0 & u_1^{(m)} \\ u_2^{(m)} & 0 \end{pmatrix}, \quad V^{(m)} = \begin{pmatrix} v_1^{(m)} & 0 \\ 0 & v_2^{(m)} \end{pmatrix}, \quad W^{(m)} = \begin{pmatrix} 0 & w_1^{(m)} \\ w_2^{(m)} & 0 \end{pmatrix}.
\]
Lemma 4.5. The variables \( \{g_1^{(m)}, g_2^{(m)}, g_3^{(m)}, g_4^{(m)}, u_m\} \) and \( \{u_1^{(m)}, u_2^{(m)}, v_1^{(m)}, v_2^{(m)}, w_1^{(m)}, w_2^{(m)}\} \) are completely determined in terms of one and another through the relations

\[
\begin{align*}
\kappa \lambda_m - q^{-1} \lambda_m^{-1} u_1^{(m)} &= u_m g_1^{(m)}, \\
\kappa(q^{-1} \lambda_m - \lambda_m^{-1}) u_2^{(m)} &= u_m^{-1} g_4^{(m)}, \\
(q^{-2} - 1) \kappa \lambda_m v_1^{(m)} &= -g_3^{(m)} u_2^{(m)} + i \lambda_m g_1^{(m)}, \\
(q^{-2} - 1) \kappa \lambda_m^{-1} v_2^{(m)} &= g_4^{(m)} u_1^{(m)} + i \lambda_m^{-1} g_2^{(m)}, \\
\kappa(\lambda_m - q^{-3} \lambda_m^{-1}) w_1^{(m)} &= u_m g_3^{(m)} v_2^{(m)} - i \lambda_m g_1^{(m)} u_1^{(m)} - u_m, \\
\kappa(q^{-3} \lambda_m - \lambda_m^{-1}) w_2^{(m)} &= u_m^{-1} g_4^{(m)} v_1^{(m)} + i \lambda_m^{-1} g_2^{(m)} w_2^{(m)} + u_m^{-1},
\end{align*}
\]

which in particular imply

\[
u_m = -\frac{a_0}{a_1}(1 + a_2^2(1 + a_2^2)) \lambda_m u_1^{(m)} + q^{-1} \kappa(q^{-1} \lambda_m - \lambda_m^{-1}) u_1^{(m)} (u_1^{(m)} u_2^{(m)} - v_2^{(m)}) + \kappa(\lambda_m - q^{-3} \lambda_m^{-1}) u_1^{(m)}.
\]

Proof. Firstly, note that, by equation (4.10),

\[
\Psi_{\infty}^{(m)}(q^2) \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} = z^{-3} A^{(m)}(z) \Psi_{\infty}^{(m)}(z),
\]

Equating the coefficients of \(z^{-1}, z^{-2}\) and \(z^{-3}\) of left and right-hand side gives respectively

\[
\begin{align*}
q^{-1} U^{(m)} \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} - \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} U^{(m)} &= A_2^{(m)}, \\
q^{-2} V^{(m)} \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} - \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} V^{(m)} &= A_2^{(m)} U^{(m)} + A_1^{(m)}, \\
q^{-3} W^{(m)} \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} - \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix} W^{(m)} &= A_2^{(m)} V^{(m)} + A_1^{(m)} U^{(m)} + A_0^{(m)},
\end{align*}
\]

where

\[
A^{(m)}(z) = A_0^{(m)} + z A_1^{(m)} + z^2 A_2^{(m)} + z^3 \begin{pmatrix} \kappa \lambda_m & 0 \\ 0 & \kappa \lambda_m^{-1} \end{pmatrix}.
\]

From these equations the relations follow directly.

In the following proposition we prove part (i) of Theorem 2.8.

Proposition 4.6. Considering Riemann-Hilbert Problems \(\text{RHP}^{(m)}(\gamma, C), m \in \mathbb{Z}\), for every \(n \in \mathbb{Z}\), the solution \(Y^{(m)}(z)\) exists for at least one \(m \in \{1, n + 1, n + 2\}\). Furthermore, let \(m \in \mathbb{Z}\) be such that \(Y^{(m)}(z)\) exists, then, using the notation in (4.12), either

(i) \(u_1^{(m)} \neq 0\), in which case \(Y^{(m+1)}(z)\) exists and

\[
Y^{(m+1)}(z) = R_+^{(m)}(z) Y^{(m)}(z),
\]

\[
R_+^{(m)}(z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u_1^{(m)} \\ 1/u_1^{(m)} & 0 \end{pmatrix},
\]

(ii) \(u_1^{(m)} = 0\), in which case \(Y^{(m+1)}(z)\) and \(Y^{(m+2)}(z)\) do not exist, whereas \(Y^{(m+3)}(z)\) does exist and

\[
Y^{(m+3)}(z) = S_+^{(m)}(z) Y^{(m)}(z),
\]

\[
S_+^{(m)}(z) = z^3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z \left( \begin{pmatrix} s_m & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u_1^{(m)} \\ 1/u_1^{(m)} & 0 \end{pmatrix} \right),
\]

\[
s_m := (q^{-2} - 1) \lambda_m v_1^{(m)} - q^{-3} \lambda_m^{-1} v_2^{(m)}.
\]
In particular we necessarily have $w_1^{(m)} \neq 0$.

Similarly either

(iii) $w_2^{(m)} \neq 0$, in which case $Y^{(m-1)}(z)$ exists and

$$
Y^{(m-1)}(z) = R_+^{(m)}(z)Y^{(m)}(z),
$$

(4.18)

$$
R_+^{(m)}(z) = z \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 1/u_2^{(m)} \\
-u_2^{(m)} & 0
\end{pmatrix},
$$

(4.19)

(iv) $w_3^{(m)} = 0$, in which case $Y^{(m-1)}(z)$ and $Y^{(m-2)}(z)$ do not exist, whereas $Y^{(m-3)}(z)$ does exist and

$$
Y^{(m-3)}(z) = S_-^{(m)}(z)Y^{(m)}(z),
$$

(4.20)

$$
S_-^{(m)}(z) = z^3 \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} + z \begin{pmatrix}
0 & 0 \\
0 & s_m
\end{pmatrix} + \begin{pmatrix}
0 & 1/w_2^{(m)} \\
-w_2^{(m)} & 0
\end{pmatrix}.
$$

(4.21)

In particular we necessarily have $w_2^{(m)} \neq 0$.

Proof. The first statement follows directly from the latter four parts, starting from any seed solution $Y^{(m)}(z)$, which is guaranteed to exist by Lemma 4.3.

Now suppose $Y^{(m)}(z)$ exists, let $R(z)$ be any matrix polynomial and set

$$
Y(z) = R(z)Y^{(m)}(z),
$$

then $Y(z)$ automatically satisfies the same analyticity and jump condition as $Y^{(m)}(z)$. Let $n \in \mathbb{Z}$. If we can choose $R(z)$ such that

$$
R(z)Y^{(m)}(z) = (I + O(z^{-1}))z^{n\sigma_3} \quad (z \to \infty),
$$

(4.22)

then $Y^{(n)}(z)$ exists and

$$
Y^{(n)}(z) = R(z)Y^{(m)}(z).
$$

(4.23)

Conversely, if $Y^{(n)}(z)$ exists, then, defining $R(z)$ by equation (4.22), $R(z)$ is a matrix polynomial satisfying (4.22).

To prove the theorem, it remains to study equation (4.22), which can essentially be reduced to linear algebra. Indeed, let us first consider the case $n = m + 1$. It is easy to see that $R(z)$ must take the form

$$
R_+^{(m)}(z) = z \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
r_{11}^0 & r_{12}^0 \\
r_{21}^0 & r_{22}^0
\end{pmatrix},
$$

and we find that, $Y^{(m+1)}(z)$ exists, if and only if equation (4.22) has a solution, which can be rewritten as

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & u_1^{(m)} & 0 \\
0 & 0 & 0 & u_1^{(m)}
\end{pmatrix} \begin{pmatrix}
r_{11}^0 \\
r_{12}^0 \\
r_{21}^0 \\
r_{22}^0
\end{pmatrix} = \begin{pmatrix}
-u_2^{(m)} \\
0 \\
0 \\
1
\end{pmatrix}.
$$

Clearly this system has a solution if and only if $u_1^{(m)} \neq 0$, in which case $R(z) = R_+^{(m)}(z)$ as defined in equation (4.11). In particular, if $u_1^{(m)} \neq 0$, then $Y^{(m+1)}(z)$ indeed exists and equation (4.13) holds true.

Now suppose $u_1^{(m)} = 0$, then we already know that $Y^{(m+1)}(z)$ cannot exist. As $u_1^{(m)} = 0$, we have, by Lemma 4.5

$$
g_1^{(m)} = i(1 - q^{-2})\kappa v_1^{(m)},
$$

$$
g_2^{(m)} = i(1 - q^{-2})\kappa v_2^{(m)},
$$

$$
g_3^{(m)} = 0,
$$

$$
g_4^{(m)} = \kappa(q^{-1}\lambda_m - \lambda_m^{-1})(\lambda_m - q^{-3}\lambda_m^{-1})w_2^{(m)}w_1^{(m)},
$$

$$
u_m = \kappa(\lambda_m - q^{-3}\lambda_m^{-1})w_1^{(m)},
$$

(4.24)
and in particular \( w_1^{(m)} \neq 0 \). We consider equation (4.22) with \( n = m + 2 \). It is easy to see that \( R(z) \) must take the form

\[
R(z) = z^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} r_{11}^1 & r_{12}^1 \\ r_{21}^1 & r_{22}^1 \end{pmatrix} + \begin{pmatrix} r_{11}^0 & r_{12}^0 \\ r_{21}^0 & r_{22}^0 \end{pmatrix},
\]

and we find that, \( Y^{(m+2)}(z) \) exists, if and only if equation (4.22) has a solution, which can be rewritten as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_2^{(m)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2^{(m)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
r_{12}^1 \\
r_{22}^1 \\
r_{12}^0 \\
r_{22}^0 \\
r_{11}^1 \\
r_{11}^0 \\
r_{11}^0 \\
r_{11}^0 \\
\end{pmatrix} = \begin{pmatrix}
-u_1^{(m)} \\
0 \\
0 \\
v_2^{(m)} \\
v_2^{(m)} \\
0 \\
0 \\
0 \\
\end{pmatrix}.
\]

As \( w_1^{(m)} \neq 0 \), but \( v_1^{(m)} = 0 \), this equation does not have a solution and hence \( Y^{(m+2)}(z) \) cannot exist.

We now show the existence of \( Y^{(m+3)}(z) \). We consider equation (4.22), with \( R(z) = S(z) \) and \( n = m + 3 \). To simplify the procedure, note that, by equation (3.18), we must have

\[
S(-z) = -\sigma_3 S(z) \sigma_3,
\]

and hence any solution \( S(z) \) must take the form

\[
S(z) = z^3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z^2 \begin{pmatrix} 0 & s_1^2 \\ s_2^1 & 0 \end{pmatrix} + z \begin{pmatrix} s_1^1 & 0 \\ 0 & s_2^1 \end{pmatrix} + \begin{pmatrix} 0 & s_1^0 \\ s_2^0 & 0 \end{pmatrix}.
\]

We extend on (4.12), by writing

\[
\Psi_{\infty}^{(m)}(z) = I + z^{-1}U^{(m)} + z^{-2}V^{(m)} + z^{-3}W^{(m)} + z^{-4}X^{(m)} + z^{-5}Z^{(m)} + \mathcal{O}(z^{-6}) \quad (z \to \infty),
\]

where \( X^{(m)} \) and \( Z^{(m)} \) take the form

\[
X^{(m)} = \begin{pmatrix} x_1^{(m)} \\ x_2^{(m)} \end{pmatrix}, \quad Z^{(m)} = \begin{pmatrix} 0 \\ x_2^{(m)} \end{pmatrix}.
\]

Then equation (4.22) with \( n = m + 3 \) for \( R(z) = S(z) \), is equivalent to

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
v_2^{(m)} & 0 & 1 & 0 & 0 & 0 \\
v_2^{(m)} & 0 & 0 & w_1^{(m)} & 0 & 0 \\
x_2^{(m)} & 0 & v_2^{(m)} & 0 & w_1^{(m)} & 0 \\
x_2^{(m)} & 0 & 0 & z_1^{(m)} & 0 & w_1^{(m)} \\
x_2^{(m)} & 0 & 0 & 0 & z_1^{(m)} & 0 \\
x_2^{(m)} & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
s_1^2 \\
s_2^1 \\
s_1^1 \\
s_2^1 \\
s_1^0 \\
s_2^0 \\
s_1^0 \\
s_2^0 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
w_2^{(m)} \\
v_2^{(m)} \\
v_2^{(m)} \\
v_2^{(m)} \\
0 \\
0 \\
\end{pmatrix}.
\]

We know \( w_1^{(m)} \neq 0 \), by equation (4.24), which implies that the above equation has a unique solution, given by

\[
s_1^2 = 0, \quad s_2^0 = 0, \quad s_1^1 = x_2^{(m)} - z_1^{(m)}/w_1^{(m)}, \quad s_1^0 = 0, \quad s_2^1 = -w_1^{(m)}, \quad s_2^0 = 1/w_1^{(m)},
\]

and hence \( Y^{(m+3)}(z) = S(z)Y^{(m)}(z) \) exists. It remains to be checked that

\[
v_2^{(m)} - z_1^{(m)}/w_1^{(m)} = s_m.
\]

Using the same method as in the proof of Lemma 4.15, we find

\[
z_1^{(m)} = (\lambda_m - q^{-5} \lambda_m^{-1})^{-1} \left[ (1 - q^{-2}) \lambda_m x_1^{(m)} + (\lambda_m - q^{-3} \lambda_m^{-1}) v_2^{(m)} \right] w_1^{(m)},
\]

from which equation (4.25) follows directly. We conclude that expression (4.16) is indeed correct. The second part of the theorem is proven analogously.

\[\square\]
Corollary 4.7. Considering the generalised inverse monodromy problem \[ \text{(4.1)} \] for every \( n \in \mathbb{Z} \), the solution \( A^{m}(z) \) exists for at least one \( m \in \{ n, n + 1, n + 2 \} \).

Furthermore, let \( m \in \mathbb{Z} \) be such that \( A^{m}(z) \) exists, then, using the notation in \[ \text{(4.11)} \], either

(i) \( g^{(m)}_{3} \neq 0 \), in which case \( A^{(m+1)}(z) \) exists and equals

\[
A(z; q\lambda, g^{(m+1)}, u_{m+1}) = R^{(m)}_{+}(q z) A(z; \lambda, g^{(m)}, u_{m}) R^{(m)}_{-}(z)^{-1},
\]

\[
R^{(m)}_{+}(z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u^{(m)}_{1} \\ 1/u^{(m)}_{1} & 0 \end{pmatrix},
\]

\[
u^{(m)}_{1} = \kappa(\lambda - q^{-1}\lambda^{-1})^{-1}u_{m}g_{3}^{(m)}.\]

(ii) \( g^{(m)}_{1} = 0 \), in which case \( A^{(m+1)}(z) \) and \( A^{(m+2)}(z) \) do not exist whereas \( A^{(m+3)}(z) \) does exist and equals

\[
A(z; q^{3}\lambda, g^{(m+3)}, u_{m+3}) = S^{(m)}_{+}(q z) A(z; \lambda, g^{(m)}, u_{m}) S^{(m)}_{-}(z)^{-1},
\]

\[
S^{(m)}_{+}(z) = z^{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} s_{m} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -u^{(m)}_{1} \\ 1/u^{(m)}_{1} & 0 \end{pmatrix},
\]

\[
u^{(m)}_{1} = \kappa(\lambda - q^{-3}\lambda^{-1})^{-1}u_{m}.\]

Here \( s_{m} \) is given by

\[
s_{m} = i\kappa^{-1}(\lambda - q^{-5}\lambda^{-1})^{-1}(\lambda g_{1}^{(m)} - q^{-3}\lambda^{-1}g_{2}^{(m)}).\]

Similarly, either

(iii) \( g^{(m)}_{4} \neq 0 \), in which case \( A^{(m-1)}(z) \) exists and equals

\[
A(z; q^{-1}\lambda, g^{(m-1)}, u_{m-1})(z) = R^{(m)}_{-}(q z) A(z; \lambda, g^{(m)}, u_{m}) R^{(m)}_{-}(z)^{-1},
\]

\[
R^{(m)}_{-}(z) = z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1/u^{(m)}_{2} \\ -u^{(m)}_{2} & 0 \end{pmatrix},
\]

\[
u^{(m)}_{2} = \kappa(q^{-1}\lambda - \lambda^{-1})^{-1}u_{m}^{-1}g_{4}^{(m)}.\]

(iv) \( g^{(m)}_{2} = 0 \), in which case \( A^{(m-1)}(z) \) and \( A^{(m-2)}(z) \) do not exist whereas \( A^{(m-3)}(z) \) does exist and equals

\[
A(z; q^{-3}\lambda, g^{(m-3)}, u_{m-3})(z) = S^{(m)}_{-}(q z) A(z; \lambda, g^{(m)}, u_{m}) S^{(m)}_{-}(z)^{-1},
\]

\[
S^{(m)}_{-}(z) = z^{3} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & s_{m} \end{pmatrix} + \begin{pmatrix} 0 & 1/u^{(m)}_{2} \\ -u^{(m)}_{2} & 0 \end{pmatrix},
\]

\[
u^{(m)}_{2} = \kappa(q^{-3}\lambda - \lambda^{-1})^{-1}u_{m}^{-1}.\]

Proof. This follows directly from Proposition \[ \text{(4.6)} \] and Lemma \[ \text{(4.5)} \] using the equivalence of the generalised inverse monodromy problem \[ \text{(4.1)} \] and RHP\[ \text{(4.26)} \], \( m \in \mathbb{Z} \).

Remark 4.8. Note that, not coincidently, equation \[ \text{(4.26)} \] agrees perfectly with the second bracket of the Lax pair \[ \text{(2.3)} \], namely \( R_{+}^{(m)}(z) = B_{\text{IN}}(z; \lambda, f^{(m)}, u_{m}) \).

Finally the following lemma shows that the isomonodromic deformation of the class of Fuchsian systems \[ \text{(3.1)} \] as \( \lambda \rightarrow q\lambda \) is equivalent to the PIV time-evolution.

Lemma 4.9. The time-evolution of \( g^{(m)} \in \mathcal{G}(a) \) and \( u_{m} \in \mathcal{C}^{*} \), induced by Corollary \[ \text{(4.4)} \], coincides with \( q^{m} \mathcal{P}_{IV}^{\text{mod}}(\lambda_{0}, a) \) plus its continuation formulae \[ \text{(A.3)} \], and the auxiliary equation \[ \text{(2.14)} \].

Proof. This follows by direct calculation. \( \square \)

We now have all the ingredients to prove Theorem \[ \text{(2.8)} \] in Section \[ \text{(1.1)} \].
Proof of Theorem 2.10. Firstly, note that part (i) follows from Proposition 4.10. As to part (ii), observe that the definition of $A^{(m)}(z)$ coincides with the one in Proposition 4.12, i.e. equation (4.29). So indeed $A^{(m)}(z) \in \mathcal{F}(q^m \lambda_0, a)$, by (ii) in Proposition 4.12. Finally part (iii) follows from Corollary 4.7 and Lemma 4.9. □

4.3. Bijectivity of the Riemann-Hilbert Mapping. In this section we prove Theorem 2.10.

Proof of Theorem 2.10. Note that Theorem 2.8 allows us to associate with any connection matrix $C(z) \in \mathcal{C}(\lambda_0, a)$, a unique $qP_{IV}^{mod}(\lambda_0, a)$ transcendent $g = (g^{(m)})_{m \in \mathbb{Z}}$ and solution $u = (u_m)_{m \in \mathbb{Z}}$ of the auxiliary equation (2.17). Upon scaling $C(z) \rightarrow \tilde{C}(z) = C(z)D$, where $D = \text{diag}(d_1, d_2)$ an invertible diagonal matrix, the solution of $RHP^{(m)}(\gamma, C)$ is scaled by $Y^{(m)} \rightarrow \tilde{Y}^{(m)}$, where

$$
\tilde{Y}^{(m)}(z) = \begin{cases} 
D^{-1}Y^{(m)}(z)D & \text{if } z \in D_+, \\
D^{-1}Y^{(m)}(z) & \text{if } z \in D_-.
\end{cases}
$$

(4.27)

In turn this scales the matrix $A^{(m)}(z)$ to $\tilde{A}^{(m)}(z) = D^{-1}A^{(m)}(z)D$, leaving the underlying $qP_{IV}^{mod}(\lambda_0, a)$ transcendent $g = \tilde{g}$ invariant whilst rescaling the solution of the auxiliary equation by $u \rightarrow \tilde{u} = \frac{1}{d_1}u$. Therefore the Riemann-Hilbert mapping (2.18) is well-defined. It also follows that both $g$ and $u$ remain invariant under scaling $C(z) \rightarrow \tilde{C}(z) = cC(z)$, for any $c \in \mathbb{C}^*$, yielding the mapping

$$
RH : M_c(\lambda_0, a) \rightarrow S(\lambda_0, a), M \mapsto (g, u),
$$

where $M_c(\lambda_0, a)$ is defined in Definition 3.3 and $S(\lambda_0, a)$ denotes the solution space of $qP_{IV}^{mod}(\lambda_0, a)$ plus the auxiliary equation (2.17). Furthermore, for $M \in M_c(\lambda_0, a)$, if $RH(M) = (g, u)$, then

$$
RH(M \cdot \text{diag}(1, d)) = (g, du), \quad M \cdot \text{diag}(1, d) := \{C(z) \cdot \text{diag}(1, d) : C(z) \in M\}.
$$

Thus, to prove the theorem, it suffices to show that the mapping $RH$ is bijective.

We proceed with constructing an inverse of $RH$. Let $g = (g^{(m)})_{m \in \mathbb{Z}}$ be any $qP_{IV}^{mod}(\lambda_0, a)$ transcendent and $u = (u_m)_{m \in \mathbb{Z}}$ be any solution of the auxiliary equation. Denote by $X \subseteq \mathbb{Z}$ the set of integers $m$ where $g^{(m)}$ is singular, i.e. $g^{(m)} = s$, recalling Definition 2.3.

For $m \in \mathbb{Z} \setminus X$, we write

$$
A^{(m)}(z) := \mathcal{A}(z, \lambda_0, g^{(m)}, u_m),
$$

(4.28)

denote corresponding monodromy by

$$
M^*_m := M_\mathcal{A}(A^{(m)}(z)) \in M_c(q^m \lambda_0, a),
$$

and set

$$
M_m := \tau^{-m}(M^*_m) \in M_c(\lambda_0, a),
$$

recalling the definition of $\tau$ in equation (4.1). Then we know, by Corollary 4.7 and Lemma 4.9 that the monodromy $M_m = M$ is independent of $m \in \mathbb{Z} \setminus X$. We write $M_{IV}(g, u) = M$, yielding a mapping

$$
M_{IV} : S(\lambda_0, a) \rightarrow M_c(\lambda_0, a).
$$

(4.29)

Due to Lemma 4.3 and the equivalence of the generalised inverse monodromy problem 4.1 and the main RHP defined in Definition 2.7, see Proposition 4.2, it is evident that $M_{IV}$ is an inverse of the mapping $RH$. In particular $RH$ is bijective and the theorem follows. □
5. The Moduli space

In this section we study the monodromy surface defined in Definition 2.11. In Section 5.1 we prove Theorem 2.12 and in Section 5.2 we classify those monodromy data corresponding to real-valued transcendents, yielding Remark 2.13.

5.1. Proof of Theorem 2.12 In order to study the monodromy surface $M_c(\lambda, a)$, defined in 2.11, we briefly recall some fundamental properties of theta functions. Following Rains [29], let $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{N}$, then we call an analytic function $c(z)$ on $\mathbb{C}^*$, satisfying

$$c(qz) = \alpha z^{-n}c(z),$$

(5.1)
a theta function of multiplier $\alpha z^{-n}$. Recalling Definition 2.2, note that for any $C(z) \in C(\lambda, a)$, all of its entries are theta functions. For $r \in \mathbb{R}_+$, we call

$$D_q(r) := \{|q| r \leq |z| < r\},$$
a fundamental annulus. Theta functions of a fixed multiplier are, up to scaling, completely determined by the location of their zeros within any fixed fundamental annulus. Indeed, we have the following

Lemma 5.1. Let $\alpha \in \mathbb{C}^*$, $n \in \mathbb{N}$ and $c(z) \neq 0$ be a theta function of multiplier $\alpha z^{-n}$. Then, within any fixed fundamental annulus, $c(z)$ has precisely $n$ zeros, counting multiplicity, say $\{a_1, \ldots, a_n\}$, and there exist unique $c \in \mathbb{C}^*$ and $s \in \mathbb{Z}$ such that

$$c(z) = cz^s\theta_q(z/a_1, \ldots, z/a_n), \quad \alpha = (-1)^n q^s a_1 \cdots a_n. \quad (5.2)$$

Conversely, for any choice of the parameters equation (5.2) defines a theta function of multiplier $\alpha z^{-n}$.

Proof. See for instance [30].

Furthermore, we recall that, for $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{N}^*$,

$$V = \{\text{theta functions of multiplier } \alpha z^{-n}\}$$
is a vector space of dimension $n$.

Proof of Theorem 2.12 We first prove that the mapping $\rho$ is injective and then prove that its range equals the algebraic surface $P(\lambda_0, a)$.

Let $M = [C], \tilde{M} = [\tilde{C}] \in M_c(\lambda_0, a)$ and suppose that corresponding coordinates $\rho_{1,2,3}$ and $\tilde{\rho}_{1,2,3}$ are equal. Set $D(z) = C(z)^{-1}\tilde{C}(z)$, then $D(z)$ is a meromorphic function on $\mathbb{C}^*$ satisfying

$$D(qz) = \lambda_0^n D(z)\lambda_0^{-\sigma_3}. \quad (5.3)$$

We know that $D(z)$ is analytic away from the $q$-spirals $\pm q^kx_k$, $1 \leq k \leq 3$. Let $1 \leq k \leq 3$, then $\pi(C(x_k)) = \pi(C(x_k))$ and thus $\pi(C(q^m x_k)) = \pi(C(q^m x_k))$ for all $m \in \mathbb{Z}$, which implies that $D(z)$ is analytic at the elements of the $q$-spirals $q^kx_k$. Furthermore, due to the symmetry (6.1), we have $D(-z) = \sigma_3 D(z)\sigma_3$ and thus $D(z)$ is also analytic at the elements of the $q$-spirals $-q^kx_k$. We conclude that $D(z)$ is analytic on $\mathbb{C}^*$. It thus follows immediately from Lemma 5.1 and equation (5.3) that

$$D(z) \equiv \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$
for some constants $d_1, d_2 \in \mathbb{C}^*$, since $\lambda_0^2 \notin q^\mathbb{Z}$. Therefore $\tilde{C} = CD$ and $C$ lie in the same equivalence class in $M_c(\lambda, a)$, i.e. $\tilde{M} = M$. It follows that the mapping $\rho$ is injective.
Next, we show that the range of $\rho$ is contained in the algebraic surface $\mathcal{P}(\lambda_0, a)$. Let $M = [C] \in \mathcal{M}_c(\lambda_0, a)$, then, because of the symmetry, $C(z)$ is of the form

$$C(z) = \begin{pmatrix} C_1(z) & C_2(z) \\ -C_1(-z) & C_2(-z) \end{pmatrix}.$$  \hfill (5.4)

Due to (5.1), (5.2) and $\mathcal{X}_3$, $C_1(z)$ and $C_2(z)$ are theta functions of multiplier respectively $\frac{qa^2_0}{qa^2_0} \lambda_0^{-1} z^{-3}$ and $\frac{qa^2_0}{qa^2_0} \lambda_0 z^{-3}$, satisfying the identity

$$C_1(z)C_2(-z) + C_1(-z)C_2(z) = cu(z),$$  \hfill (5.5)

for some $c \in \mathbb{C}^*$. We accordingly introduce the following two vector spaces

$$U = \{ \text{theta functions of multiplier } (qa^2_0 a_2)^{-1} \lambda^{-1} z^{-3} \},$$
$$V = \{ \text{theta functions of multiplier } (qa^2_0 a_2)^{-1} \lambda^{-3} \},$$

so that $C_1 \in U$ and $C_2 \in V$. It is helpful to fix explicit bases of $U$ and $V$. We define

$$u_1(z) = \theta_q(z/x_2, z/x_3, -z/x_1 \lambda_0), \quad v_1(z) = \theta_q(z/x_2, z/x_3, -z/x_1 \lambda_0^{-1}),$$
$$u_2(z) = \theta_q(z/x_1, z/x_3, -z/x_2 \lambda_0), \quad v_2(z) = \theta_q(z/x_1, z/x_3, -z/x_2 \lambda_0^{-1}),$$
$$u_3(z) = \theta_q(z/x_1, z/x_2, -z/x_3 \lambda_0), \quad v_3(z) = \theta_q(z/x_1, z/x_2, -z/x_3 \lambda_0^{-1}),$$

then $\{u_1, u_2, u_3\}$ is a basis of $U$ and $\{v_1, v_2, v_3\}$ is a basis of $V$. We have chosen these bases such that $u_k(x_l) = v_k(x_l) = 0$ if $k \neq l$ for $1 \leq k, l \leq 3$.

Let $\alpha \in \mathbb{C}^3$ and $\beta \in \mathbb{C}^3$ be such that

$$C_1(z) = \alpha_1 u_1(z) + \alpha_2 u_2(z) + \alpha_3 u_3(z),$$  \hfill (5.6)
$$C_2(z) = \beta_1 v_1(z) + \beta_2 v_2(z) + \beta_3 v_3(z).$$  \hfill (5.7)

We proceed in showing that $\rho = \rho(M)$ is an element of $\mathcal{P}(\lambda_0, a)$. We use the standard notation $p = [p^x : p^y]$ for elements $p \in \mathbb{P}^1(\mathbb{C})$ and accordingly write $\rho_k = [\rho_k^x : \rho_k^y]$ for $1 \leq k \leq 3$. Let $1 \leq k \leq 3$, then $\pi(C(x_k)) = \rho_k$ implies $\rho_k^c C_2(x_k) = \rho_k^x C_2(-x_k)$ and thus

$$\rho_k^c v_k(x_k) \beta_k = \rho_k^x (\beta_1 v_1(-x_k) + \beta_2 v_2(-x_k) + \beta_3 v_3(-x_k)),$$  \hfill (5.8)

for $k \in \{1, 2, 3\}$. This is a homogeneous linear system in $\beta_{1,2,3}$. Since $\beta$ is nonzero, this implies

$$\begin{vmatrix} \rho_1^c v_1(-x_1) - \rho_1^x v_1(x_1) & \rho_1^c v_2(-x_1) & \rho_1^c v_3(-x_1) \\ \rho_2^c v_1(-x_2) & \rho_2^c v_2(-x_2) - \rho_2^x v_2(x_2) & \rho_2^c v_3(-x_2) \\ \rho_3^c v_1(-x_3) & \rho_3^c v_2(-x_3) & \rho_3^c v_3(-x_3) - \rho_3^x v_3(x_3) \end{vmatrix} = 0.$$  \hfill (5.9)

This equation is precisely

$$T_{hom}(\rho_1^c, \rho_1^x, \rho_2^c, \rho_2^x, \rho_3^c, \rho_3^x; \lambda_0, a) = 0,$$

after some simplification. It follows that indeed the range of $\rho$ is contained in $\mathcal{P}(\lambda_0, a)$.

To finish the proof, it remains to be shown that any element of $P(\lambda_0, a)$ can be realised as the coordinates of an equivalence class $M = [C(z)] \in \mathcal{M}_c(\lambda_0, a)$.

Take any $(\rho_1, \rho_2, \rho_3) \in \mathcal{P}(\lambda_0, a)$. Then we know that the determinant in (5.9) vanishes. Thus there exists a nonzero solution $\beta \in \mathbb{C}^3$ of the homogeneous linear system (5.8).

We define $C_2(z)$ by equation (5.7), then $C_2 \in V$ and

$$\rho_k^c C_2(x_k) = \rho_k^x C_2(-x_k)$$  \hfill (5.10)

for $1 \leq k \leq 3$. 

Similarly, to ensure that \( \pi(C(x_k)) = \rho_k \), we must have
\[
\rho_0^2 C_1(x_k) = -\rho_1^2 C_1(-x_k), \tag{5.11}
\]
which, using the notation in (5.6), is equivalent to
\[
-\rho_1^2 u_k(x_k) \omega_k = \rho_0^2 (\alpha_1 u_1(x_k) + \alpha_2 u_2(-x_k) + \alpha_3 u_3(-x_k)), \tag{5.12}
\]
for \( k \in \{1, 2, 3\} \). This homogeneous linear system has a nonzero solution \( \alpha \in \mathbb{C}^3 \) if and only if the determinant
\[
\begin{vmatrix}
\rho_1^2 u_1(-x_1) + \rho_0^2 u_1(x_1) & \rho_1^2 u_2(-x_1) & \rho_1^2 u_3(-x_1) \\
\rho_2^2 u_1(-x_2) & \rho_2^2 u_2(-x_2) + \rho_0^2 u_2(x_2) & \rho_2^2 u_3(-x_2) \\
\rho_3^2 u_1(-x_3) & \rho_3^2 u_2(-x_3) & \rho_3^2 u_3(-x_3) + \rho_0^2 u_3(x_3)
\end{vmatrix}
\]
vanishes. Direct computation gives that this determinant equals \(-\lambda_0^2 \Delta\), where \( \Delta \) is the determinant in (5.9). Since \( \Delta = 0 \), the linear system (5.12) has a nonzero solution \( \alpha \in \mathbb{C}^3 \), and with this choice of \( \alpha \) in (5.6), we know that \( C_1 \in U \) satisfies equation (5.11).

Define the matrix function \( C(z) \) by equation (5.4), then, by construction, it satisfies properties (c.1), (c.2) and (c.4). It only remains to be checked that equation (5.5) holds true. To this end, let us write
\[
W(z) := |C(z)| = C_1(z)C_2(-z) + C_1(-z)C_2(z). 
\]
Then \( W(z) \), just like \( w(z) \), is a theta function of multiplier \( -(q a_0^2 a_2)^{-2} z^{-6} \). Thus, to show equation (5.5), all we have to do is check that \( W(z) \) and \( w(z) \) have the same zeros, due to Lemma 5.1. Namely we have to check that \( W(\pm x_k) = 0 \) for \( 1 \leq k \leq 3 \). However the latter follows trivially from equations (5.11) and (5.10).

We conclude that \( C(z) \in \mathcal{C}(\lambda_0, a) \), and \( \pi(C(x_k)) = \rho_k \), \( 1 \leq k \leq 3 \), due to equations (5.11) and (5.10). The theorem follows.

5.2. Real-valued Transcendents. In this section we characterise those monodromy data which yield real solutions. Take \( a \in \mathbb{R}^3 \) and \( \lambda_0 \in \mathbb{R}^* \) such that \( q = a_0 a_1 a_2 \in (-1, 1) \setminus \{0\} \) and the non-resonant conditions (2.9) are satisfied. Then a q-PIV(\( \lambda_0, a \)) transcendent \( f \) is real-valued, if and only if its via (3.9) associated qPIV(\( \lambda_0, a \)) transcendent \( g \) is real-valued.

If \( g \) is real-valued, then we can choose a solution \( u \) of the auxiliary equation which is purely imaginary. Then the corresponding matrix
\[
A^{(m)}(z) := \mathcal{A}(z; q^m \lambda_0, g^m, u_m), \tag{5.13}
\]
satisfies
\[
\overline{A^{(m)}(z)} = -A^{(m)}(z). \tag{5.14}
\]
It follows that the fundamental solutions \( \Phi_0^{(m)}(z) \) and \( \Phi_\infty^{(m)}(z) \), defined in Lemmas 3.2 and 3.3 with \( d \in \mathbb{R} \), are real analytic. Thus the corresponding connection matrix \( C(z) \) is real analytic, that is
\[
\overline{C(z)} = C(z). \tag{5.15}
\]

Conversely, suppose \( C(z) \in \mathcal{C}(\lambda, a) \) is real analytic. Choose an admissible Jordan curve \( \gamma \) such that \( \gamma = \gamma \), then the solution \( Y^{(m)}(z) \) of RHP\((m)(\gamma, C) \) satisfies \( Y^{(m)}(\overline{z}) = -Y^{(m)}(z) \) for \( z \in \mathbb{C} \setminus \gamma \). Therefore \( A^{(m)}(z) \) satisfies (5.14) from which it follows that \( g \) and \( f \) are real-valued.

We conclude that a monodromy datum \( M \in \mathcal{M}_c(\lambda, a) \) corresponds to a real solution \( f \), via the Riemann-Hilbert mapping in Theorem 2.10 if and only if there exists a representative \( C(z) \in M \) which is real analytic. In turn it is easy to see that the latter holds true if and only if \( \rho(M) \in \mathbb{P}(\mathbb{R})^3 \). Indeed, the forward implication is trivial and its converse follows from the fact that, if \( \rho \in \mathbb{P}(\mathbb{R})^3 \), then
the homogeneous linear systems (5.12) and (5.8) have real nonzero solutions \(\alpha \in \mathbb{R}^3\) and \(\beta \in \mathbb{R}^3\) respectively. Remark 2.13 follows.

6. Conclusion

In this paper we have derived a Riemann-Hilbert representation for the general solution of \(q\text{P}_{IV}\) in the non-resonant parameter case. We have shown that the mapping, associating to any \(q\text{P}_{IV}\) transcendent corresponding equivalence class of connection matrices in the monodromy surface, is a bijection. Furthermore we have constructed an explicit algebraic surface which is the moduli space of the monodromy surface and thus of \(q\text{P}_{IV}\).

This lays the groundwork for analysis of the global asymptotics of solutions of \(q\text{P}_{IV}\). In particular, in our forthcoming paper, analogous to the differential theory [6, 25], by studying Riemann-Hilbert problem RHP\(^{\text{res}}\)(\(\gamma, C\)) in the limits \(m \to +\infty\) and \(m \to -\infty\), we derive corresponding asymptotics for solutions of \(q\text{P}_{IV}\) and associated connection formulae.

We anticipate that the Riemann-Hilbert theory developed here will extend to the resonant regime. We intend to use this approach to study special solutions, as has been done for the differential fourth Painlevé equation [15–18].

Finally, it is an intriguing question whether our Riemann-Hilbert representation of \(q\text{P}_{IV}\) can be used to derive convergence results of solutions with regard to the continuum limit \(q \to 1\).

Appendix A. A birational transformation and singularities

Define

\[
\begin{align*}
g_1 &= qf_2^{-1} + a_0a_2f_1^{-1}f_2^{-1} + a_0f_1^{-1}, \\
g_2 &= qf_2 + a_0a_2f_1f_2 + a_0f_1, \\
g_3 &= qa_0a_2f_1 + qa_0f_1f_2^{-1} + a_0^2a_2f_2^{-1}, \\
g_4 &= qa_0a_2f_1^{-1} + qa_0f_2f_1^{-1} + a_0^2a_2f_2.
\end{align*}
\]

(A.1)

then \(g = (g_1, g_2, g_3, g_4)\) satisfies the algebraic equations (2.11) and the rational inverse of (A.1) is given by

\[
\begin{align*}
f_1 &= \frac{a_0^2 + g_3}{a_0(qa_2 + g_1)} = \frac{a_0(qa_2 + g_2)}{a_0^2 + q_4}, \\
f_2 &= \frac{q^2 + g_4}{a_0a_2(a_0 + a_1g_1)} = \frac{a_0a_2(a_0 + a_1g_2)}{q^2 + g_3}.
\end{align*}
\]

(A.2)

We denote the algebraic surface obtained by cutting \(\{g \in \mathbb{C}^4\}\) with respect to (2.11) by \(\mathcal{G}(a)\). The \(f\) and \(g\) variables are bi-rationally equivalent, and in particular \(q\text{P}_{IV}(a)\) induces the time-evolution given by Equation (2.12) on \(\mathcal{G}(a)\).

While the forward iteration of Equation (2.12) is singular on \(\mathcal{G}(a)\), only when \(g_3 = 0\), we show its continuation is possible by means of singularity confinement. It is also possible to regularize these singularities by lifting to the initial value space \((A_2 + A_1)^{(1)}\) following Sakai [20].
Namely, if \( g_3 = 0 \), then \( \mathcal{F} \) and \( \mathcal{F}^* \) do not exist whereas \( \mathcal{F} \) does and is given explicitly by
\[
\mathcal{F}_1 = \frac{(1 - q^2 t^2) g_1 + (1 - q^2) g_2}{1 - q^4 t^2}, \quad \text{(A.3a)}
\]
\[
\mathcal{F}_2 = \frac{q t^2 (1 - q^2) g_1 + (1 - q^3 t^2) g_2}{1 - q^4 t^2}, \quad \text{(A.3b)}
\]
\[
\mathcal{F}_3 = g_4 + (q^{-2} - 1) g_1 g_2 + q^{-4} (q^2 - 1) g_1^2 + \frac{(1 - q^2) (g_1 - q^2 g_2) ((2 - q^2) g_1 - q^2 g_2)}{q^4 (1 - q^4 t^2)}
+ \frac{(1 - q^2)^2 (g_1 - q^2 g_2)^2}{q^4 (1 - q^4 t^2)}. \quad \text{(A.3c)}
\]
\[
\mathcal{F}_4 = 0. \quad \text{(A.3d)}
\]

Similarly the inverse time-evolution is singular only when \( g_4 = 0 \), in which case the first and second inverse iterates do not exist, whereas the third one does. We say that \( q(t) \) is singular at \( t_0 \) when it does not exist at \( t = t_0 \). The continuation formulae of \( q_{IV}^{\text{cond}}(a) \) can be obtained by means of direct calculation.

Considering the forward iteration, \( \mathcal{F} \) is ill-defined if and only if \( g_3 = 0 \). So let us take any \( g^* \in \mathcal{G}(a) \) with \( g_3^* = 0 \) and perturb around it within \( \mathcal{G}(a) \), setting
\[
g_1 = g_1^* + O(\epsilon), \quad g_2 = g_2^* + O(\epsilon), \quad g_3 = \epsilon + O(\epsilon^2), \quad g_4 = g_4^* + O(\epsilon),
\]
in particular \( g = g^* + O(\epsilon) \), as \( \epsilon \to 0 \). Then direct calculation gives
\[
\mathcal{F}_1 = a_0^2 a_2 (q t^2 - 1) t^{-2} \epsilon^{-1} + O(1),
\]
\[
\mathcal{F}_2 = -a_0^2 a_2 (q t^2 - 1) \epsilon^{-1} + O(1),
\]
\[
\mathcal{F}_3 = a_0^2 a_2 q (q t^2 - 1)^2 t^{-2} \epsilon^{-2} + O(\epsilon^{-1}),
\]
\[
\mathcal{F}_4 = O(\epsilon),
\]
which diverges, as \( \epsilon \to 0 \). Similarly
\[
\overline{g}_1 = -a_0^2 a_2 (q t^2 - 1) q^{-2} t^{-2} \epsilon^{-1} + O(1),
\]
\[
\overline{g}_2 = a_0^2 a_2 q (q t^2 - 1)^2 \epsilon^{-1} + O(1),
\]
\[
\overline{g}_3 = O(\epsilon),
\]
\[
\overline{g}_4 = -a_0^2 a_2 q (q t^2 - 1)^2 t^{-2} \epsilon^{-2} + O(\epsilon^{-1}),
\]
which diverges, as \( \epsilon \to 0 \). However, upon calculating the third iteration, we find
\[
\overline{g}_1 = \frac{(1 - q^2 t^2) g_1^* + (1 - q^2) g_2^*}{1 - q^4 t^2} + O(\epsilon),
\]
\[
\overline{g}_2 = \frac{q t^2 (1 - q^2) g_1^* + (1 - q^3 t^2) g_2^*}{1 - q^4 t^2} + O(\epsilon),
\]
\[
\overline{g}_3 = g_4^* + (q^{-2} - 1) g_1^* g_2^* + q^{-4} (q^2 - 1) g_1^* + \frac{(1 - q^2) (g_1^* - q^2 g_2^*) ((2 - q^2) g_1^* - q^2 g_2^*)}{q^4 (1 - q^4 t^2)}
+ \frac{(1 - q^2)^2 (g_1^* - q^2 g_2^*)^2}{q^4 (1 - q^4 t^2)} + O(\epsilon),
\]
\[
\overline{g}_4 = O(\epsilon),
\]
which converges to \( (A.3) \), as \( \epsilon \to 0 \). We conclude that the singularity is confined within three iterations. The singularity analysis of the inverse time evolution follows by similar arguments.
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