Extended Calogero models: a construction for exactly solvable $kN$-body systems

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Abstract

We propose a systematic procedure for the construction of exactly solvable $kN$-body systems which are natural generalisations of Calogero models. As examples, we present two new $3N$-body models and determine explicit expressions for their eigenvalues and eigenfunctions.

Keywords: exactly solvable systems, Calogero models, pre-superpotential

1. Introduction

Exactly solvable (ES) quantum many-body problems attract considerable research activity due to their connections with many branches of physics, e.g. [1–8]. In 1969, Calogero obtained the exact solution for a three-particle system with pairwise interactions via square and inverse square potentials [9], and later generalized this result to the $N$-body case [10]. In 1974, Wolfes extended Calogero’s three-body problem by adding terms which are inverse squares of certain linear combinations of the three-particle coordinates [11]. In [12], Sutherland proposed ES models with trigonometric potentials [12]. In the 1980s, Olshanetsky and Perelomov carried out a survey and gave a classification of ES models according to the root systems of simple Lie algebras [13].

Over recent decades, models of the Calogero type (i.e. those where the potential is of the form ‘oscillator/inverse square’) have received considerable attention, and many interesting properties have been discovered [14–26]. There are also many works which have attempted to obtain new ES models by extending existing ones through separation of variables [27–29].
More complicated extensions, which have connections with orthogonal polynomials, can be obtained by \( PT \) (parity and time reversal) symmetric quantum mechanics [30–32]. In this work, we propose a systematic method for constructing ES \( kN \)-body systems in one dimension. Such models consist of \( N \) interacting blocks, each of which contains \( k \) particles. The blocks interact through their centres of mass, while particles in each block interact via \( A \) or \( G_2 \) type potential. As examples, we provide two new ES \( 3N \)-body models and obtain their corresponding eigenvalues and eigenfunctions.

The paper is organized as follows. In section 2, we describe a general procedure for constructing ES many-body quantum Hamiltonians in terms of pre-superpotentials. By choosing an appropriate form of pre-superpotential, we derive a rational ansatz whose solutions give rise to ES models. All Calogero type models associated with the root systems of simple Lie algebras satisfy this ansatz. We list the Hamiltonians of such Calogero systems, and their corresponding ground state energies and wave functions. In section 3, we combine distinct \( A \) or \( G_2 \) type models together to form a new family of ES models through a coupling function. Various types of coupling functions will be studied. We show that every member in this family satisfies the rational ansatz, thus proving these new models remain ES. As examples, in section 4 we present two new \( 3N \)-body systems. Applying appropriate coordinate transformations, we separate the \( 3N \)-body eigenvalue problem into equations for radial, angular, and center-of-mass parts coordinates. We solve these equations to give the eigenvalues and eigenfunctions of the \( 3N \)-body models. We summarize our work in the final section.

2. General discussion and results

Throughout this paper we set \( \hbar = 2m = 1 \). We start with the basic relation [26],

\[
\hat{p}^2 e^W = -\sum_{i=1}^{N} \left[ \left( \frac{\partial W}{\partial x_i} \right)^2 + \frac{\partial^2 W}{\partial x_i^2} \right] e^W, \quad \hat{\rho}^2 = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.
\]

This relation guarantees that \( e^W \) is an eigenfunction of the following Hamiltonian

\[
\hat{H} = \hat{p}^2 + \sum_{i=1}^{N} \left[ \left( \frac{\partial W}{\partial x_i} \right)^2 + \frac{\partial^2 W}{\partial x_i^2} \right],
\]

with zero eigenvalue, i.e.

\[
\hat{H} e^W = 0,
\]

provided that \( e^W \) is square-integrable. Such a function \( W \) is called a pre-superpotential. Now we set \( W \) to be of the form

\[
W = \sum_{i=1}^{M} \alpha_i \log |\vec{v}_i \cdot \vec{x}| - \frac{\omega}{2} \sum_{i=1}^{N} x_i^2,
\]

where \( \vec{x} = (x_1, x_2, \cdots, x_N) \), and \( \vec{v}_i \)'s are some distinct vectors. Then (1) becomes

\[
\left\{ \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{M} \frac{\alpha_i (\alpha_i - 1) |\vec{v}_i|^2}{(\vec{v}_i \cdot \vec{x})^2} + 2 \sum_{i<j}^{M} \frac{\alpha_i \alpha_j}{(\vec{v}_i \cdot \vec{x})(\vec{v}_j \cdot \vec{x})} (\vec{v}_i \cdot \vec{v}_j) - E_0 \right\} e^W = 0,
\]

where \( E_0 = 2\omega \sum_{i=1}^{M} \alpha_i + N\omega \). Thus if we choose \( \vec{v}_i \) and \( \alpha_i \) such that the following so-called rational ansatz is satisfied

\[
\left\{ \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{M} \frac{\alpha_i (\alpha_i - 1) |\vec{v}_i|^2}{(\vec{v}_i \cdot \vec{x})^2} + 2 \sum_{i<j}^{M} \frac{\alpha_i \alpha_j}{(\vec{v}_i \cdot \vec{x})(\vec{v}_j \cdot \vec{x})} (\vec{v}_i \cdot \vec{v}_j) - E_0 \right\} e^W = 0,
\]

where \( E_0 = 2\omega \sum_{i=1}^{M} \alpha_i + N\omega \). Thus if we choose \( \vec{v}_i \) and \( \alpha_i \) such that the following so-called rational ansatz is satisfied
\[
\sum_{i<j}^{M} \frac{\alpha_i \alpha_j}{(\vec{v}_i \cdot \vec{x})(\vec{v}_j \cdot \vec{x})} (\vec{v}_i \cdot \hat{\vec{v}}_j) = 0, \tag{3}
\]

and denote the corresponding Hamiltonian as \( \hat{H}_{\text{Cal}} \), we then have

\[
\hat{H}_{\text{Cal}} = \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{M} \alpha_i(\alpha_i - 1)|\vec{v}_i|^2, \]

\[
\hat{H}_{\text{Cal}} e^W = \left( 2\omega \sum_{i=1}^{M} \alpha_i + N\omega \right) e^W. \tag{4}
\]

In other words, if (3) is satisfied, the Hamiltonian (4) admits a ground state \( e^W \) with corresponding energy \( E_0 \).

It is straightforward to show that \( e^{-\hat{W}H_{\text{Cal}}e^W} \) gives

\[
e^{-\hat{W}H_{\text{Cal}}e^W} = \hat{p}^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} x_i^2 x_j + 2\omega \sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + E_0 \tag{5}
\]

where \( v_{ij} \) denotes the \( i \)th component of the vector \( \vec{v}_j \). We find \( e^{-\hat{W}H_{\text{Cal}}e^W} P_n(t) \subset P_n(t) \) for any positive integer \( n \), where \( P_n(t) \) is defined by

\[
\mathcal{P}_n(t) = \text{span}\{1,t,t^2,\ldots,t^n\}, \quad t = \sum_{i=1}^{N} x_i^2.
\]

That is, \( e^{-\hat{W}H_{\text{Cal}}e^W} \) preserves the infinite flag of spaces

\[
\mathcal{P}_0(t) \subset \mathcal{P}_1(t) \subset \cdots \subset \mathcal{P}_n(t) \subset \cdots.
\]

It is not difficult to obtain

\[
e^{-\hat{W}H_{\text{Cal}}e^W} L_n^{(\alpha)}(4\omega t) = (4\omega n + E_0) L_n^{(\alpha)}(4\omega t), \quad \alpha = 4 \sum_{j=1}^{M} \alpha_j + 2N + 1,
\]

where \( L_n^{(\alpha)}(4\omega t) \) is the Laguerre polynomial of degree \( n \). Moreover, the results from [23–25] can be generalized to show the exact solvability of the Hamiltonian \( \hat{H}_{\text{Cal}} \). Indeed, from (5), we find

\[
\hat{g}^{-1} e^{-\hat{W}H_{\text{Cal}}e^W} \hat{g} = \hat{p}^2 + \omega^2 \sum_{i=1}^{N} x_i^2 + E_0 - N\omega, \]

\[
\hat{g} = \exp \left\{ \frac{1}{4\omega} \left[ \hat{p}^2 - 2 \sum_{i=1}^{N} \sum_{j=1}^{M} \alpha_i \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} x_i^2 x_j \right] \right\} \cdot \exp \left\{ - \frac{1}{4\omega} \hat{p}^2 \right\} \cdot \exp \left\{ \frac{\omega}{2} \sum_{i=1}^{N} x_i^2 \right\}.
\]

In terms of the ladder operators

\[
\hat{a}_j = \frac{\partial}{\partial x_j} + \omega x_j, \quad \hat{a}_j^\dagger = -\frac{\partial}{\partial x_j} + \omega x_j,
\]
we have
\[ \hat{g}^{-1} e^{-W} \hat{H}_{\text{Cal}} e^{W} \hat{g} = \frac{1}{2} \sum_{i=1}^{N} \{ \hat{a}_i, \hat{a}_i^\dagger \} + E_0 - N\omega. \]

We see that Hamiltonian \( \hat{H}_{\text{Cal}} \) can be mapped to independent harmonic oscillators and thus is ES. It is straightforward to show that the transformed ‘number operators’ \( \hat{J}_{ii} = e^{W} \hat{g}_i \hat{a}_i^\dagger \hat{a}_i \hat{g}_i^{-1} e^{-W} \), \( i = 1, 2, \ldots, N \) are conserved
\[ [\hat{J}_{ii}, \hat{H}_{\text{Cal}}] = 0, \]
as expected.

It can be shown that the rational ansatz (3) has non-trivial solutions. With appropriate \( \alpha_j \)'s, root vectors of simple Lie algebras satisfy (3) and the corresponding \( \hat{H}_{\text{Cal}} \) in (4) give the Hamiltonians of the Calogero type models. On the other hand, it is worth noting that [27] provides an example of an ES Hamiltonian corresponding to \( \hat{J}_{ij} \)'s in (3) which are not related to a root system of a Lie algebra.

We now list some known results, taken from [15, 16, 20, 26], for later use. While the general discussion above is valid for parameters \( M \) and \( N \) are independent, the fact that the results below are expressed in terms of root systems imposes a relation between \( M \) and \( N \).

2.1. A type Calogero model
For the Calogero model associated with A type root system, the positive root vectors are
\[ \hat{e}_i - \hat{e}_j = (\cdots, 1, \cdots, -1, \cdots) \quad 1 \leq i < j \leq N \]
where \( \hat{e}_i \) denotes a standard basis element in \( N \)-dimensional Euclidean space \( \mathbb{R}^N \), and the dots represent zeros. We set \( \vec{v}_1 = \hat{e}_1 - \hat{e}_2, \vec{v}_2 = \hat{e}_1 - \hat{e}_3, \ldots, \vec{v}_M = \hat{e}_N - \hat{e}_N \), where \( M = N(N - 1)/2 \). We also set \( \alpha_1 = \cdots = \alpha_M = \alpha \) in (2), such that
\[ W = \alpha \sum_{i<j}^{N} \log |x_i - x_j| - \frac{\omega}{2} \sum_{i=1}^{N} x_i^2, \]
\[ \hat{H}_A = \hat{p}^2 + \sum_{i<j}^{N} \frac{2\alpha(\alpha - 1)}{(x_i - x_j)^2} + \omega^2 \sum_{i=1}^{N} x_i^2, \]
\[ \hat{H}_A e^W = (N\omega + N(N - 1)\omega_\alpha)e^W. \] (6)

2.2. BC type Calogero model
For the Calogero model associated with BC type root system, the positive root vectors are
\[ \hat{e}_i \pm \hat{e}_j = (\cdots, 1, \cdots, \pm 1, \cdots) \quad 1 \leq i < j \leq N, \]
\[ \hat{e}_i = (\cdots, 1, \cdots), \quad i = 1, 2, \ldots, N. \]
We set \( \vec{v}_1 = \hat{e}_1 - \hat{e}_2, \vec{v}_2 = \hat{e}_1 - \hat{e}_3, \ldots, \vec{v}_M / 2 = \hat{e}_{N-1} - \hat{e}_N, \vec{v}_{M / 2 + 1} = \hat{e}_1 + \hat{e}_2, \ldots, \vec{v}_M = \hat{e}_{N-1} + \hat{e}_N \), where \( M' = N(N - 1) / 2 \). and \( \vec{v}_{M / 2 + i} = \hat{e}_i, i = 1, \ldots, N \). We also set \( \alpha_1 = \cdots = \alpha_{M'} = \beta_1, \alpha_{M' + 1} = \cdots = \alpha_M = \beta_2 \) in (2), where \( M = M' + N \). Then
\[
W = \beta_1 \sum_{i<j} \{ \log |x_i - x_j| + \log |x_i + x_j| \} + \beta_2 \sum_{i=1}^{N} \log |x_i| - \frac{\omega}{2} \sum_{i=1}^{N} x_i^2,
\]

\[
\hat{H}_{BC} = \hat{p}^2 + \sum_{i<j} \left\{ \frac{2 \beta_1 (\beta_1 - 1)}{(x_i - x_j)^2} + \frac{2 \beta_1 (\beta_1 - 1)}{(x_i + x_j)^2} \right\} + \omega^2 \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{N} \frac{\beta_2 (\beta_2 - 1)}{x_i^2},
\]

\[
\hat{H}_{BC} e^W = [2N(N-1)\omega \beta_1 + 2N\omega \beta_2 + N\omega] e^W.
\]

If \( \beta_2 = 0 \), then \( BC \) type reduces to \( D \) type.

### 2.3. \( E_8 \) type Calogero model

Before defining this model, we need to introduce some notation. For \( j = 1, \cdots, 7 \) let \( a_j \in \mathbb{Z}_2 \) and let \( \mathcal{I} \) denote the set of septuples \( \alpha = (a_1, \cdots, a_7) \) such that

\[
\sum_{j=1}^{7} a_j = 0.
\]

Then for the Calogero model associated with \( E_8 \) type root system, the positive root vectors are

\[
\hat{e}_i \pm \hat{e}_j = (\cdots, 1, \cdots, \pm 1, \cdots) \quad 1 \leq i < j \leq 8,
\]

\[
\hat{e}_8 + \sum_{i=1}^{7} (-1)^{a_i} \hat{e}_i = ((-1)^{a_1}, \cdots, (-1)^{a_7}, 1),
\]

\( (a_1, \cdots, a_7) \in \mathcal{I}. \) \hspace{1cm} \( (8) \)

We set \( \vec{v}_1 = \hat{e}_1 - \hat{e}_2, \) \( \vec{v}_{56} = \hat{e}_4 + \hat{e}_8, \) while the remaining \( \vec{v}_{57}, \cdots, \vec{v}_{120} \) have the form \( (8) \). We also set \( \alpha_1 = \cdots = \alpha_{120} = \beta \) in \( (2) \), so

\[
W = \beta \sum_{i<j} \{ \log |x_i - x_j| + \log |x_i + x_j| \} + \beta \sum_{\alpha \in \mathcal{I}} \log |x_{\alpha}| + \sum_{i=1}^{7} (-1)^{a_i} x_i - \frac{\omega}{2} \sum_{i=1}^{8} x_i^2,
\]

\[
\hat{H}_{E_8} = \hat{p}^2 + \sum_{i<j} \left\{ \frac{2 \beta (\beta - 1)}{(x_i - x_j)^2} + \frac{2 \beta (\beta - 1)}{(x_i + x_j)^2} \right\} + \sum_{\alpha \in \mathcal{I}} \left[ \frac{8 \beta (\beta - 1)}{(x_{\alpha} + \sum_{i=1}^{7} (-1)^{a_i} x_i)^2} \right] + \omega^2 \sum_{i=1}^{8} x_i^2,
\]

\[
\hat{H}_{E_8} e^W = (240 \omega \beta + 8 \omega) e^W.
\]

### 2.4. \( F_4 \) type Calogero model

For the Calogero model associated with \( F_4 \) type root system, the vectors are

\[
\hat{e}_i \pm \hat{e}_j = (\cdots, 1, \cdots, \pm 1, \cdots) \quad 1 \leq i < j \leq 4,
\]

\[
\hat{e}_i + \sum_{j=2}^{4} (-1)^{a_j} \hat{e}_j = (1, \pm 1, \pm 1, \pm 1),
\]

\[
\hat{e}_i = (\cdots, 1, \cdots), \quad i = 1, 2, 3, 4.
\]

We set \( \vec{v}_1 = \hat{e}_1 - \hat{e}_2, \cdots, \vec{v}_{12} = \hat{e}_3 + \hat{e}_4, \) \( \vec{v}_{13} = (1, 1, 1, 1), \cdots, \vec{v}_{20} = (1, -1, -1, -1) \) and \( \vec{v}_{20+i} = \hat{e}_i, \) \( i = 1, \cdots, 4. \) We also set \( \alpha_1 = \cdots = \alpha_{12} = \nu, \) \( \alpha_{13} = \cdots = \alpha_{24} = \mu \) in \( (2) \), so
\[ W = \nu \sum_{i<j}^{4} \{ \log |x_i - x_j| + \log |x_i + x_j| \} + \mu \sum_{i=1}^{4} \log |x_i| + \mu \sum_{a \in \mathbb{Z}_2} \log |x_1 + \sum_{i=2}^{4} (-1)^a x_i| - \frac{\omega}{2} \sum_{i=1}^{4} x_i^2, \]
\[ \hat{H}_{G_4} = p^2 + \sum_{i<j}^{4} \left\{ \frac{2\nu(\nu - 1)}{(x_i + x_j)^2} + \frac{2\nu(\nu - 1)}{(x_i - x_j)^2} \right\} + \sum_{i=1}^{4} \frac{\mu(\mu - 1)}{x_i^2} + \omega^2 \sum_{i=1}^{4} x_i^2 \]
\[ + \sum_{a \in \mathbb{Z}_2} \frac{4\mu(\mu - 1)}{(x_1 + \sum_{i=2}^{4} (-1)^a x_i)^2}. \]
\[ \hat{H}_{G_4}e^W = (4\omega + 24\omega \mu + 24\omega \nu)e^W. \]

### 2.5. \( G_2 \) type Calogero model

For the Calogero model associated with \( G_2 \) type root system, the vectors \( \vec{v}_i, i = 1, \ldots, 6 \) are given by

\[ \vec{v}_1 = (1, -1, 0), \quad \vec{v}_2 = (1, 0, -1), \quad \vec{v}_3 = (0, 1, -1), \]
\[ \vec{v}_4 = (1, 1, -2), \quad \vec{v}_5 = (1, -2, 1), \quad \vec{v}_6 = (-2, 1, 1). \]

We also set \( \alpha_1 = \alpha_2 = \alpha_3 = \beta_1, \alpha_4 = \alpha_5 = \alpha_6 = \beta_2 \) in (2), so

\[ W = \beta_1 \sum_{i<j}^{3} \log |x_i - x_j| + \beta_2 \sum_{l \neq i < j \neq l}^{3} \log |x_i + x_j - 2x_l| - \frac{\omega}{2} \sum_{i=1}^{3} x_i^2, \]
\[ \hat{H}_{G_2} = p^2 + \sum_{i<j}^{3} \frac{2\beta_1(\beta_1 - 1)}{(x_i - x_j)^2} + \sum_{l \neq i < j \neq l}^{3} \frac{6\beta_2(\beta_2 - 1)}{(x_i + x_j - 2x_l)^2} + \omega^2 \sum_{i=1}^{3} x_i^2, \]
\[ \hat{H}_{G_2}e^W = (3\omega + 6\beta_1 \omega + 6\beta_2 \omega)e^W. \] (10)

The \( E_6 \) and \( E_7 \) cases have constraints on the coordinates [20]. These do not lend to a convenient physical interpretation, so we omit them.

### 3. Construction of new models

In this section, we present a systematic approach for constructing ES \( kN \)-body systems in one dimension. Such models describe systems of \( N \) interacting blocks, each of which has \( k \) particles interacting via an \( A \) type or \( G_2 \) type potential.

The \( kN \)-body system is proposed to have a Hamiltonian \( \hat{H} \) given by

\[ \hat{H} = \sum_{i=1}^{N} \hat{H}_i + C(X_1, X_2, \cdots, X_N). \] (11)

Here \( X_i = \frac{1}{k} \sum_{j=1}^{k} x_j \), \( i = 1, 2, \cdots, N \), is the center-of-mass of the \( i \)th block, \( C(X_1, \cdots, X_N) \) is called the coupling function, and \( \hat{H}_i \) is the Hamiltonian for the \( i \)th block,
\[ \hat{H}_i = \hat{p}_i^2 + \omega^2 \sum_{j=1}^{k} x_j^2 + V_i, \quad \hat{p}_i^2 = -\sum_{j=1}^{k} \frac{\partial^2}{\partial x_j}, \quad i = 1, 2, \ldots, N. \] (12)

The potential \( V_i \) above is of the inverse square form:

\[ V_i = \sum_{l=1}^{m_i} \frac{g_i}{(\bar{v}_{il} \cdot \bar{x})^2}. \]

In this model, \( \hat{H}_i \) is assigned \( m_i \) vectors, \( \bar{v}_{il} \) is the \( l \)th vector associated with \( \hat{H}_i \),

\[ \bar{v}_{il} = (0, 0, \ldots, 0, v_{il1}, v_{il2}, \ldots, v_{ilr}, 0, \ldots, 0), \]

and \( \bar{x} \) is the collection of coordinates of the form,

\[ \bar{x} = (x_{11}, \ldots, x_{1k}, \ldots, x_{l1}, \ldots, x_{lk}, \ldots, x_{N1}, x_{N2}, \ldots, x_{Nk}). \] (13)

We take the coupling function to be of the inverse square form, i.e.

\[ C(X_1, \ldots, X_N) = \frac{1}{k} \sum_{i=1}^{r} \frac{\beta_i (\beta_i - 1) |\bar{\mu}_i|^2}{(|\bar{\mu}_i \cdot \bar{X}|^2}, \]

where \( \bar{X} = (X_1, \ldots, X_N) \) and \( \bar{\mu}_i = (\mu_{i1}, \ldots, \mu_{iN}), \quad i = 1, 2, \ldots, r \). In fact, we have

\[ k \bar{\mu}_i \cdot \bar{X} = \bar{\mu}'_i \cdot \bar{x}, \]

where \( \bar{x} \) is given by (13), and \( \bar{\mu}'_i \) is some `expansion’ of \( \bar{\mu}_i \),

\[ \bar{\mu}'_i = (\mu_{i1}, \ldots, \mu_{i1}, \mu_{i2}, \ldots, \mu_{i2}, \ldots, \mu_{iK}, \ldots, \mu_{iK}), \]

with \( k \) copies of \( \mu_{i1} \), \( k \) copies of \( \mu_{i2} \), \( k \) copies of \( \mu_{iK} \).

It is then clear that the inner product \( \bar{\mu}'_i \cdot \bar{v}_{il} \) is well defined. Putting \( \bar{x}, \bar{v}_{il} \) and \( \bar{\mu}'_i \) into (3), and using the relations

\[ \bar{v}_{il} \cdot \bar{v}'_{il} = \delta_{il} \bar{v}_{il} \cdot \bar{v}'_{il}, \]
\[ \bar{v}_{il} \cdot \bar{\mu}'_i = \mu_{il} \sum_{j=1}^{k} v_{ilj}, \]
\[ \bar{\mu}_i \cdot \bar{\mu}_j = k (\bar{\mu}'_i \cdot \bar{\mu}'_j), \]

we arrive at \( S_1 + S_2 + S_3 = 0 \) where

\[ S_1 = \sum_{i=1}^{N} \sum_{l<j} \sum_{s<t} \frac{\alpha_{il}^s\alpha_{jt}^t}{(\bar{v}_{il} \cdot \bar{x}) (\bar{v}_{jt} \cdot \bar{x})} (\bar{v}_{il} \cdot \bar{v}_{jt}), \]
\[ S_2 = \sum_{i,j,l} \sum_{s<t} \frac{\beta_{il}^s\beta_{jt}^t}{(\bar{v}_{il} \cdot \bar{x}) (\bar{v}_{jt} \cdot \bar{x})} (\mu_{il} \sum_{j=1}^{k} v_{ilj}), \]
\[ S_3 = \frac{1}{k} \sum_{i,j,l} \sum_{s<t} \frac{\beta_{il}^s\beta_{jt}^t}{(\bar{\mu}_i \cdot \bar{x}) (\bar{\mu}_j \cdot \bar{x})} (\bar{\mu}_i \cdot \bar{\mu}_j). \] (14)

We want \( S_1, S_2 \) and \( S_3 \) in (14) to vanish individually. First, let us examine \( S_1 \), it is nothing but a sum of ansatzes (3) of each \( \hat{H}_i \). For the \( i \)th Hamiltonian \( \hat{H}_i \), we can choose \( \bar{v}_{il} \)'s to be the root
system of a simple Lie algebra, with appropriate $\alpha_i$’s such that $S_1$ vanishes. For $S_2$, we can make it vanish by choosing $\vec{\mu}_i$'s to be root vectors of Lie algebra $A$ or $G_2$, i.e. by choosing the potential $V_i$ to have the form

$$V_i = \sum_{j<i}^k \frac{2\lambda_i(\lambda_i - 1)}{(x_j - x_i)^2},$$

or

$$V_i = \sum_{j<i}^3 \frac{2\lambda_i(\lambda_i - 1)}{(x_j - x_i)^2} + \sum_{s=1}^3 \frac{6\lambda_i(\lambda_i - 1)}{(x_j + x_i - 2x_s)^2}.$$

To make $S_3$ vanish, we can just choose $\vec{\mu}_i$’s to be the root vectors of some Lie algebra. That is we choose the coupling function $C$ to be one of the $A$, $BC$, $F_4$ or $G_2$ types. It is seen from (6) and (10) that each $\hat{H}_i$ in (11) admits a ground state $e^{W_i}$ with ground energy $E_0^{(i)}$. So we can readily give the ground state wavefunction and energy of the Hamiltonian (11), for each choice of $C$, as follows.

3.1. A type coupling

If $C$ is $A$ type, i.e.

$$C = \frac{1}{k} \sum_{j<s}^N \frac{2\alpha(\alpha - 1)}{(x_j - x_s)^2}, \quad \alpha > 0,$$

we have the pre-superpotential

$$W_A = \sum_{i=1}^N W_i + \alpha \sum_{j<s}^N \log |x_j - x_s|.$$

So $e^{W_A}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^N E_0^{(i)} + N(N - 1)\omega \alpha.$$

3.2. BC type coupling

If $C$ is $BC$ type, i.e.

$$C = \frac{1}{k} \sum_{j<s}^N \left( \frac{2\beta_1(\beta_1 - 1)}{(x_j - x_s)^2} + \frac{2\beta_1(\beta_1 - 1)}{(x_j + x_s)^2} \right) + \frac{1}{k} \sum_{j=1}^N \frac{\beta_2(\beta_2 - 1)}{x_j^2}, \quad \beta_1 > 0, \quad \beta_2 > 0,$$

we have the pre-superpotential

$$W_{BC} = \sum_{i=1}^N W_i + \beta_1 \sum_{j<s}^N \{\log |x_j - x_s| + \log |x_j + x_s|\} + \beta_2 \sum_{i=1}^N \log |x_i|.$$
Then $e^{W_{BC}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{N} E_0^{(i)} + 2N(N - 1)\omega_1 + 2N\omega_2.$$  

We remind that when $\beta_2 = 0$, $BC$ type reduces to $D$ type.

### 3.3. $E_8$ type coupling

If $C$ is $E_8$ type (so $N = 8$), i.e.

$$C = \frac{1}{k} \sum_{i < j} \left\{ \frac{2\beta(\beta - 1)}{(X_i - X_j)^2} + \frac{1}{k} \frac{2\beta(\beta - 1)}{(X_i + X_j)^2} \right\} + \frac{1}{k} \sum_{a \in \mathbb{Z}_2} \frac{8\beta(\beta - 1)}{(X_a + \sum_{i=1}^{a} (-1)^a X_i)^2}, \beta > 0,$$

we have the pre-superpotential

$$W_{E_8} = \sum_{i=1}^{8} W_i + \beta \sum_{i < j} \left\{ \log |X_i - X_j| + \log |X_i + X_j| \right\} + \beta \sum_{a \in \mathbb{Z}} \log |X_a + \sum_{i=1}^{7} (-1)^a X_i|.$$

Then $e^{W_{E_8}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{8} E_0^{(i)} + 240\omega\beta.$$

### 3.4. $F_4$ type coupling

If $C$ is $F_4$ type (so $N = 4$), i.e.

$$C = \frac{1}{k} \sum_{i < j} \left\{ \frac{2\nu(\nu - 1)}{(X_i + X_j)^2} + \frac{1}{k} \frac{2\nu(\nu - 1)}{(X_i - X_j)^2} \right\} + \frac{1}{k} \sum_{a \in \mathbb{Z}_2} \frac{4\mu(\mu - 1)}{(X_1 + \sum_{i=2}^{a} (-1)^a X_i)^2},$$

we have the pre-superpotential

$$W_{F_4} = \sum_{i=1}^{4} W_i + \nu \sum_{i < j} \left\{ \log |X_i - X_j| + \log |X_i + X_j| \right\} + \mu \sum_{a \in \mathbb{Z}_2} \log |X_a + \sum_{i=1}^{3} (-1)^a X_i|.$$

Then $e^{W_{F_4}}$ is the ground state of the Hamiltonian (11), with ground-state energy

$$E_0 = \sum_{i=1}^{4} E_0^{(i)} + 24\omega\nu + 24\omega\mu.$$

### 3.5. $G_2$ type coupling

If $C$ is $G_2$ type (so $N = 3$), i.e.

$$C = \frac{1}{k} \sum_{j < s} \frac{2\beta_1(\beta_1 - 1)}{(X_j - X_s)^2} + \frac{1}{k} \sum_{l \neq j \neq s} \frac{6\beta_2(\beta_2 - 1)}{(X_j + X_s - 2X_l)^2}, \beta_1 > 0, \beta_2 > 0,$$
we have the pre-superpotential

\[ W_{G_2} = \sum_{i=1}^{3} W_i + \beta_1 \sum_{j<k}^{3} \log |X_j - X_k| + \beta_2 \sum_{l \neq j<k}^{3} \log |X_l + X_j - 2X_k|. \]

Then \( e^{W_{G_2}} \) is the ground state of the Hamiltonian (11), with ground-state energy

\[ E_0 = \sum_{i=1}^{N} E_0^{(i)} + 6\omega (\beta_1 + \beta_2). \]

4. Exactly solvable 3N-body problems

As examples of the general results above, we consider the \( k = 3 \) case and construct two ES 3N-body systems.

**Model 1:** We choose all \( V_i \)'s in (12) to be \( G_2 \) type and set \( g_{il} = g_i \) for all \( l = 1, 2, 3 \), i.e. each \( \hat{H}_i \) is of the form

\[ \hat{H}_i = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} + \omega^2 \sum_{j=1}^{3} x_{ij}^2 + \frac{2}{9} \rho_i \sum_{j<k}^{3} (x_{ij} - x_{ik})^2 + \frac{2}{3} \sum_{s \neq j<k}^{3} \frac{\lambda_i}{(x_{sj} + x_{si} - 2x_{kj})^2}. \]

(18)

We choose \( C \) in (11) to be \( A \) type, i.e. \( C \) is given by (17). Putting (18) and (17) into (11) gives the Hamiltonian

\[ \hat{H} = -\sum_{i=1}^{N} \sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} + \omega^2 \sum_{i=1}^{N} \sum_{j=1}^{3} x_{ij}^2 + \frac{2}{9} \sum_{i=1}^{N} \sum_{j<k}^{3} (x_{ij} - x_{ik})^2 + \frac{2}{3} \sum_{i=1}^{N} \sum_{l \neq j<k}^{3} \frac{\lambda_i}{(x_{ij} + x_{il} - 2x_{kl})^2} + \frac{2}{3} \sum_{i=1}^{N} \sum_{l \neq j<k}^{3} \frac{6\alpha(\alpha - 1)}{(x_{ij} + x_{il} + x_{kl} - x_{jl} - x_{jk} - x_{kl})^2}. \]

In order to solve the Schrödinger equation \( \hat{H}\Psi = E\Psi \), we first make a transformation for each triplet \( \{x_{i1}, x_{i2}, x_{i3}\} \)

\[ x_{i1} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho_i \left( -\frac{1}{2} \cos \theta_i + \frac{\sqrt{3}}{2} \sin \theta_i \right), \]

\[ x_{i2} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho_i \cos \theta_i, \]

\[ x_{i3} = \frac{Y_i}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho_i \left( -\frac{1}{2} \cos \theta_i - \frac{\sqrt{3}}{2} \sin \theta_i \right) \]

(19)

such that \( X_i = Y_i/\sqrt{3} \). Then \( \hat{H} \) becomes

\[ \hat{H} = -\sum_{i=1}^{N} \left( \frac{\partial^2}{\partial Y_i^2} + \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{\rho_i} \frac{\partial}{\partial \rho_i} + \frac{1}{\rho_i^2} \frac{\partial^2}{\partial \theta_i^2} \right) + \omega^2 \sum_{i=1}^{N} (Y_i^2 + \rho_i^2) \]

\[ + \sum_{i=1}^{N} \left( \frac{g_i}{\rho_i^2 \sin^2 \theta_i} + \frac{\lambda_i}{\rho_i^2 \cos^2 \theta_i} \right) + \sum_{i<j}^{N} \frac{2\alpha(\alpha - 1)}{(Y_i - Y_j)^2}. \]

This means we can partially factorize the eigenfunction \( \Psi \):
\[ \Psi = \psi_{(n_1, \ldots, n_0)}(Y_1, \ldots, Y_N) \prod_{i=1}^N R_i(r_i) \prod_{i=1}^N \Theta_n^{(i)}(\theta_i), \]

which leads to \(2N + 1\) independent equations:

\[
\left[ -\frac{\partial^2}{\partial \theta_i^2} + \frac{g_i}{\sin^2 \theta_i} + \frac{\lambda_i}{\cos^2 \theta_i} \right] \Theta_n^{(i)}(\theta_i) = (B_n^{(i)})^2 \Theta_n^{(i)}(\theta_i), \quad i = 1, 2, \ldots, N, \tag{20} \]

\[
\left[ -\frac{\partial^2}{\partial r_i^2} - \frac{1}{r_i} \frac{\partial}{\partial r_i} + \frac{(B_n^{(i)})^2}{r_i^2} + \omega^2 r_i^2 \right] R_i(r_i) = E_i R_i(r_i), \quad i = 1, 2, \ldots, N, \tag{21} \]

and

\[
\hat{H}_Y \psi_{(n_1, \ldots, n_0)} = E_Y \psi_{(n_1, \ldots, n_0)},
\]

\[
\hat{H}_Y = \sum_{i=1}^N \left[ -\frac{\partial^2}{\partial Y_i^2} + \omega^2 Y_i^2 \right] + \sum_{i<j}^N \frac{2\alpha(\alpha + 1)}{(Y_i - Y_j)^2}.
\]

The total energy \(E\) is given by

\[
E = E_Y + \sum_{i=1}^N E_i.
\]

For each \(i\), equation (20) has a known solution, with eigenvalue and eigenfunction given by

\[
\Theta_n^{(i)}(\theta_i) = \sin^{2\nu_i}(3\theta_i) \cos^{2\eta_i}(3\theta_i) P_n^{2\nu_i-1/2, 2\eta_i-1/2}(\cos 6\theta_i),
\]

\[
B_n^{(i)} = 6(n_i + \nu_i + \eta_i), \quad n_i = 0, 1, 2, \ldots, i = 1, 2, \ldots, N,
\]

\[
\nu_i = \frac{3 + \sqrt{9 + 4\omega^2}}{12}, \quad \eta_i = \frac{3 + \sqrt{9 + 4\lambda_i}}{12}, \tag{22}
\]

where \(P_n^{2\nu_i-1/2, 2\eta_i-1/2}(\cos 6\theta_i)\) is the Jacobi polynomial of degree \(n_i\).

Now we look at (21): for each \(i\), (21) is recognized from Calogero’s work [9], with solution given by

\[
R_i(r_i) = r_i^\frac{B_n^{(i)}}{2} L_n^{(i)}(\omega r_i^2) \times \exp \left\{ -\frac{\omega}{2} r_i^2 \right\},
\]

\[
E_i = 2\omega(2k_i + B_n^{(i)} + 1), \quad k_i = 0, 1, 2, \ldots, i = 1, 2, \ldots, N, \tag{23}
\]

where \(L_n^{(i)}\) is Laguerre polynomial of degree \(k_i\), with parameter \(B_n^{(i)}\).

To solve the last equation \(\hat{H}_Y \psi_a = E_Y \psi_a\), we adopt the approach of [17] involving Dunkl operators. Define

\[
\hat{D}_j = -i \frac{\partial}{\partial Y_j} + \im \sum_{i \neq j}^N \frac{1}{Y_j - Y_i} \sigma_j, \quad \hat{A}_j^\pm = D_j \pm \im \omega Y_j, \quad j = 1, 2, \ldots, N,
\]

\[
\hat{A}_m^\pm \equiv \sum_{j=1}^N (\hat{A}_j^\pm)^m, \quad m = 1, 2, \ldots, N,
\]

\[
[H_Y, \hat{A}_m^\pm] = \pm 2\omega \hat{A}_m^\pm.
\]
where the $\sigma_{ij}$ interchange coordinates, i.e. $\sigma_{if}(\cdots x_i, \cdots, x_j, \cdots) = f(\cdots x_i, \cdots, x_j, \cdots)$. The solutions are given by
\[
\psi(n_i, \cdots, n_i) = \prod_{i=1}^{N} (A_i^+)^{n_i} \psi_0, \quad \psi_0 = \prod_{i<j} (Y_i - Y_j)|\exp\left\{ -\frac{\omega}{2} \sum_{i=1}^{N} Y_i^2 \right\},
\]
where
\[
n = \sum_{i=1}^{N} n_i, \quad n_i = 0, 1, 2, \cdots.
\]
The total energy $E$ for $\hat{H}$ is then
\[
E = 2n\omega + N\omega + N(N-1)\alpha\omega + 2\omega \sum_{i=1}^{N} [2\kappa_i + 6(n_i + \nu_i + \eta_i) + 1].
\]

**Model 2:** We again choose $V_i$ to be $G_2$ type but choose $C$ to be $D$ type. This gives rise to the following Hamiltonian:
\[
\hat{H} = -\sum_{i=1}^{N} \sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} + \omega^2 \sum_{i=1}^{N} \sum_{j=1}^{3} x_{ij}^2 + \frac{2}{9} \sum_{i=1}^{N} \sum_{j<l} \frac{g_{ij}}{(x_{ij} - x_{il})^2}
\]
\[
+ \frac{2}{3} \sum_{i=1}^{N} \sum_{j<l} \frac{\lambda_i}{(x_{ij} + x_{il} - 2x_{il})^2} + \sum_{i<j} \frac{6\beta(\beta - 1)}{(x_{ij} + x_{il} - x_{ji} - x_{ij})^2}
\]
\[
+ \sum_{i<j} \frac{6\beta(\beta - 1)}{(x_{ij} + x_{il} + x_{jl} + x_{ij} + x_{il} + x_{jl})^2}.
\]

(24)

It can be seen that when transformation (19) is applied again, the equations for $r_i$’s and $\theta_i$’s are the same as (20) and (21), as well as the solutions (22) and (23). The equation for $Y_t$ is
\[
\hat{H}_Y \psi_n = E_Y \psi_n,
\]
\[
\hat{H}_Y = \sum_{i=1}^{N} \left[ -\frac{\partial^2}{\partial Y_i^2} + \omega^2 Y_i^2 \right] + \sum_{i<j} \frac{2\beta(\beta - 1)}{(Y_i - Y_j)^2} + \sum_{i<j} \frac{2\beta(\beta - 1)}{(Y_i + Y_j)^2}.
\]

In order to solve this equation, we again use results from [17]:
\[
\hat{D}_j = -i \frac{\partial}{\partial Y_j} + i\beta \sum_{i<j} \left\{ \frac{1}{Y_j - Y_s} \sigma_{js} + \frac{1}{Y_j + Y_s} t_{js} \sigma_{js} \right\},
\]
\[
\hat{a}^\pm_j = \hat{D}_j \pm i\omega Y_j, \quad j = 1, 2, \cdots, N,
\]
\[
A^\pm = \sum_{j=1}^{N} (a^\pm_j)^2,
\]
where the $t_i$ change coordinate signs, i.e. $t_i f(\cdots Y_i, \cdots) = f(\cdots - Y_i, \cdots)$. Eigenfunctions and eigenvalues of (24) are given by
\[ \psi_n = (\hat{A}^\dagger)^n \psi_0, \quad \psi_0 = \prod_{i<j} |Y_i - Y_j|^\beta |Y_i + Y_j|^\beta \exp \left\{ -\frac{\omega}{2} \sum_{i=1}^N Y_i^2 \right\}, \]

\[ E_Y = 4n\omega + 2\beta N(N-1)\omega + N\omega, \quad n = 0, 1, 2, \ldots. \]

In this case, the total energy \( E \) is then

\[ E = 4n\omega + 2\beta N(N-1)\omega + N\omega + 2\omega \sum_{i=1}^N [2k_i + 6(n_i + \nu_i + \eta_i) + 1]. \]

5. Conclusion

In this work, we have presented a general approach for constructing ES \( kN \)-body systems in one dimension. In our construction the coupling function \( C \) plays a crucial role. We give examples which demonstrate that, in some instances, these can be chosen in relation to the root system of a simple Lie algebra. For each listed choice of \( C \), we give the ground state and ground-state energy of the corresponding ES model. As non-trivial examples, we have presented two \( 3N \)-body systems. We have solved the two models by separating their Schrödinger equations into the centers-of-mass, radial, and angular parts. The equations for radial and angular parts are familiar ones, and can be solved analytically. The equation for the centers-of-mass is not generally separable, but can be solved by using Dunkl operators [17].

For more general \( kN \)-body systems with \( k > 3 \), we have found that the procedure for separating variables does not generalise in an obvious manner. The solution to this problem will be the subject of future investigations.

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