Conformal fields, restriction properties, degenerate representations and SLE *

Roland Friedrich †  Wendelin Werner ‡
Université Paris-Sud

Abstract

We show how to relate the Schramm-Loewner Evolution processes (SLE) to highest-weight representations of the Virasoro Algebra. The restriction properties of SLE that have been recently derived in [19] play a crucial role. In this setup, various considerations from conformal field theory can be interpreted and reformulated via SLE. This enables to make a concrete link between the two-dimensional discrete critical systems from statistical physics and conformal field theory.

1 Introduction

Conformal field theory has been remarkably successful in predicting the critical behaviour of two-dimensional systems from statistical physics (see [3, 4] and for instance the compilation of papers in [6]). One of the starting points [21, 3] for this theory is that each system should be (in its scaling limit) correspond to a conformal field (these fields are classified according to their central charge). The behaviour of the system should then be described by critical exponents which physicists identify as highest-weights of certain degenerate representations of infinite-dimensional Lie Algebras (this motivated also many mathematical papers on representation theory, one should probably cite here at least all the papers reprinted in [6], for the representation theory of the Virasoro Algebra and background, see [9, 8]). This applies for instance to Ising and Potts models, percolation, self-avoiding walks. However, as for instance pointed out in [11], many fundamental features have remained unclear at least for mathematicians. For instance, the actual relation between the discrete system and the fields ie.

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†rolandf@ihes.fr
‡wendelin.werner@math.u-psud.fr
the meaning of the field in terms of the discrete system, the interpretation of the
equations leading to the highest-weight representation were rather mysterious.

In 1999, Oded Schramm \[23\] defined a one-parameter family of random
curves based on Loewner’s differential equation, SLE_κ indexed by the positive
real parameter κ (SLE stands for Stochastic -or Schramm- Loewner Evolution).
These random curves are the only ones which combine conformal invariance and
and a Markovian-type property. Provided that the scaling limit of interfaces in the
above-mentioned models exist and are conformally invariant, then the limiting
objects must therefore be one of the SLE_κ curves. This has now been rigourously
proved in some cases (critical site percolation on the triangular lattice \[24\] and
uniform spanning trees \[17\]). For a general discussion of the conjectured relation
between the discrete models and SLE, see \[22\]. In this SLE setting, the
critical exponents are simply principal eigenvalues of some differential operators
\[13, 14, 15, 16\]. This led to a complete mathematical proof for the value of
critical exponents for those models that have been proved to be conformally
invariant. In order to confirm rigorously the conjectures for the other models,
the missing step is to derive their conformal invariance.

It is therefore natural to think that SLE should be related to conformal field
theory (as for instance hinted at in \[1\]) and to highest-weight representations of
the Virasoro Algebra. Our goal in the present note is to explain that it is indeed
the case. In order to make this exposition as clear and rigorous as possible (and
also to keep this note short), we will restrict ourselves to the simplest case: The
“boundary behaviour” of SLE_{8/3} that corresponds in the field theory language
to a zero central charge and conjecturally corresponds to the scaling limit of the
half-plane self-avoiding walk \[18\]. The other cases (non-zero central charge,
behaviour in the bulk) will be detailed in forthcoming papers.

2 SLE facts

The chordal SLE_κ curve γ is characterized as follows: The conformal maps g_t
from \( \mathbb{H} \setminus \gamma [0, t] \) onto \( \mathbb{H} \) such that \( g_t(z) = z + o(1) \) when \( z \to \infty \) solve the ordinary
differential equation \( \partial_s g_t(s) = 2/(g_t(z) - W_t) \) (and is started from \( g_0(z) = z \)),
where \( W_t = \sqrt{\kappa} \beta_t \) (here and in the sequel, \( (\beta_t, t \geq 0) \) is a standard real-valued
Brownian motion with \( \beta_0 = 0 \)). In other words, \( \gamma_t \) is precisely the point such
that \( g_t(\gamma_t) = W_t \). See e.g. \[13, 22\] for the definition and properties of SLE, or
\[12, 26\] for reviews.

In the recent paper \[19\], it is shown how to measure the distortion of the
law of an SLE when its image is mapped conformally from a subdomain onto
some other domain. In particular, for the special value \( \kappa = 8/3 \), the SLE curve γ
(which is conjectured to be the scaling limit of a certain measure on self-avoiding
curves, see \[18\] for a discussion of this conjecture and some of its consequences
this conjecture is confirmed by numerical simulations [10]) has the conformal restriction property that we now briefly describe:

Suppose that $H$ is a simply connected open subset of the upper-half plane $\mathbb{H}$ such that $\mathbb{H} \setminus H$ is bounded and bounded away from 0. Assume also that $\gamma$ is chordal SLE$_{8/3}$. Then, the law of $\gamma$ conditioned to remain in $H$ is identical to the law of $\Phi(\gamma)$ where $\Phi$ is the conformal map from $\mathbb{H}$ onto $H$ that fixes the two boundary points 0 and $\infty$ and $\Phi(z) \sim z$ as $z \to \infty$. Actually [19], SLE$_{8/3}$ is the unique simple random curve with this property. As argued towards the beginning of the paper [19], if $\gamma$ satisfies the restriction property, then there exists $\alpha > 0$ such that

$$P[\gamma \subset H] = \Phi'(0)^{-\alpha}. \quad (1)$$

It is proved in [19] that the value of $\alpha$ corresponding to SLE$_{8/3}$ is $5/8$ (but we will not use this fact in this note, as one of our goals is to explain why one can recover it from algebraic considerations as was for instance predicted in [5]).

3 Boundary behaviour

When $\kappa < 8$ and $\epsilon \to 0$, the probability that chordal SLE$_\kappa$ intersects the $\epsilon$-neighbourhood of the real point $x \neq 0$ can be shown to decay (up to a multiplicative constant) like $\epsilon^s$ when $\epsilon \to 0$, where $s = (8/\kappa) - 1$. More generally, one defines for $x_1, \ldots, x_n \in \mathbb{R} \setminus \{0\}$,

$$B_n(x_1, \ldots, x_n) = \lim_{\epsilon_1, \ldots, \epsilon_n \to 0} \epsilon_1^{-s} \cdots \epsilon_n^{-s} P[\cap_{j=1}^n \{ \gamma \cap [x_j, x_j + i\epsilon_j \sqrt{2}] \neq \emptyset \}]$$

(the choice of these vertical slits enables one to have simple renormalizing constants later on). It is easy to see using (1) that if the restriction property holds, then $s = 2$ and $B_1(x) = \alpha/x^2$. The functions $B_n$ correspond to correlation functions in conformal field theory. They have the following properties:

**Proposition 1.**

1. Ward-type identity: If the restriction property holds (with the exponent $\alpha$), then for all $n \geq 1$,

$$B_{n+1}(x, x_1, x_2, \ldots, x_n) = \frac{\alpha}{x^2} B_n(x_1, \ldots, x_n)$$

$$- \sum_{j=1}^n \left\{ \left( \frac{1}{x_j - x} + \frac{1}{x} \right) \partial_{x_j} - \frac{2}{(x_j - x)^2} \right\} B_n(x_1, \ldots, x_n). \quad (2)$$

2. Evolution equation:

$$-2s \left( \sum_{j=1}^n \frac{1}{x_j} \right) B_n + \left( \sum_{j=1}^n \frac{2}{x_j} \partial_{x_j} \right) B_n + \frac{\kappa}{2} \left( \sum_{j=1}^n \partial_{x_j} \right)^2 B_n = 0, \quad (3)$$
Proof (sketch). The first identity is in fact a direct consequence of the restriction property: Suppose that the real numbers \( x_1, \ldots, x_n \) are fixed and let us focus on the event that \( \gamma \) does intersect all the slits \([x_j, x_j + i\varepsilon \sqrt{2}]\). Let us also choose another point \( x \in \mathbb{R} \) and a small \( \delta \). Now, either \( \gamma \) avoids \([x, x + i\delta \sqrt{2}]\) or it does also hit it. The probability of the first event can be expressed using the restriction property, while the second one involves the \((n+1)\)-points functions \(B_{n+1}\). The sum of the two probabilities remains independent of \( \delta \). Looking at the \( \delta^2 \) term in its expansion yields (2). Let us stress that these identities together with the knowledge of \(B_1(x_1) = \alpha/x_1^2\) fully determine all the functions \(B_n\). One can also write \(B_0 = 1\) as a function of 0 variables.

The Markovian property of the SLE curves (which is a straightforward consequence of the stationarity of the increments of the Brownian motion \( \beta \)) goes as follows: The law of \( \gamma|t, \infty) \) given \( \gamma[0, t] \) is the image of an independent copy \( \tilde{\gamma} \) of \( \gamma \) under a conformal map from \( \mathbb{H} \) on \( \mathbb{H} \setminus \gamma[0, t] \) which maps 0 onto \( \gamma(t) \) and \( \infty \) onto itself. In other words, the conditional law of \( g_t(\gamma|t, \infty)) - W_t \) is independent of \( \gamma[0, t] \) and identical to that of the initial SLE. This shows immediately that for all fixed reals \( x_1, \ldots, x_n \), the processes

\[|g_t'(x_1)|^s \ldots |g_t'(x_n)|^s B_n(g_t(x_1) - W_t, \ldots, g_t(x_n) - W_t)\]

are local martingales. Itô’s formula then yields immediately (3).

4 Representations

It is interesting to focus on the asymptotic expansion of \(B_{n+1}(x, x_1, \ldots, x_n)\) when \( x \to 0 \). It is natural to define the operators

\[\mathcal{L}_N = \sum_j \left\{ -x_j^{1-N} \partial_j + 2(N-1)x_j^{-N} \right\}.\]

Note that these operators satisfy the commutation relation

\[[\mathcal{L}_N, \mathcal{L}_M] = (N - M)\mathcal{L}_{N+M}.\]

The Algebra generated by vectors satisfying this commutation relation will be denoted by \( \mathcal{A} \) and is often viewed as the Lie algebra of vector fields of the circle i.e. the algebra generated by \( e_N = -z^{N+1}d/dz, N \in \mathbb{Z} \) (which satisfy the same commutation relation). One can rewrite the previous Proposition using the operators \( \mathcal{L} \): The Ward-type identity becomes

\[B_{n+1}(x, x_1, \ldots, x_n) = \frac{\alpha}{x^2}B_n(x_1, \ldots, x_n) + \sum_{N \geq 1} x^{N-2}\mathcal{L}_N B_n(x_1, \ldots, x_n)\]

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if $|x| < \min(x_1, \ldots, x_n)$. The evolution equation becomes (if $s = 2$ i.e. if the restriction property holds)

$$\left(\frac{\kappa}{2}\mathcal{L}_{-1}^2 - 2\mathcal{L}_{-2}\right)B_n = 0. \quad (6)$$

It is easy to make a little computation to see that $\kappa = \frac{8}{3}$ and $\alpha = \frac{5}{8}$ if the previous Proposition holds for some family of functions $B_n$. This computation has some similarities with the computation that shows that a highest-weight representation of $A$ that is degenerate at level two has a highest weight equal to $5/8$. We now show that this is not just a similarity. Note that scaling implies that $\mathcal{L}_0B_n = 0$, so that the representation is not simply given in terms of the $\mathcal{L}$’s.

Here is one way to construct a highest-weight representation of $A$. Define the vector $B = (B_0, B_1, \ldots)$. Suppose that $w = (w_0, w_1, \ldots)$ is such that $w_n$ is a (rational) function of the $n$ variables $x_1, \ldots, x_n$. We define the operators $l_N$ is such a way that

$$w_{n+1}(x, x_1, \ldots, x_n) = \sum_{N \in \mathbb{Z}} x^{N-2}(l_{-N}(w))_n(x_1, \ldots, x_n) \quad (7)$$

when $x \to \infty$. In other terms the function of $n$ variables in $l_{-N}(w)$ is the $x^{N-2}$ term in the Laurent expansion of $w_{n+1}(x, x_1, \ldots, x_n)$. Equation (6) shows that

$$l_N(B) = \begin{cases} 
(0, 0, \ldots) & \text{if } N > 0 \\
(\alpha B_0, \alpha B_1, \ldots) & \text{if } N = 0 \\
(\mathcal{L}_N B_0, \mathcal{L}_N B_1, \ldots) & \text{if } N < 0
\end{cases} \quad (8)$$

We then define the vector space $V$ generated by $B$ and all the vectors $l_{N_1} \cdots l_{N_r}(B)$ for $N_1 \leq \cdots \leq N_r < 0$. It is then possible to prove (without using the evolution equation) that:

**Proposition 2.**
- For $N_1 \leq \cdots \leq N_r < 0$, the function of $n$ variables in $l_{N_1} \cdots l_{N_r}(B)$ is $\mathcal{L}_{N_1} \cdots \mathcal{L}_{N_r} B_n$.
- $V$ is stable under all $l_N$’s (for all $N \in \mathbb{Z}$, not only for $N \leq 0$).
- $[l_M, l_N]w = (M - N)[M+N]w$ for all $w \in V$ and all $N, M \in \mathbb{Z}$ (note again that $M$ and $N$ can be positive).

**Proof (sketch).** The first statement can be proved inductively using (8). It implies immediately the third statement for all negative $N, M$. In order to obtain the commutation relation for all $N, M$, and due to the fact that $l_N(B) = 0$ for all positive $N$, it in fact suffices to check it if only $M$ is positive. This can then be done again thanks to (8) and the first statement. Finally, note that the second statement is a consequence of the third one and of the fact that $l_N(B) = 0$ if $N > 0$. \qed
Equation (8) shows that this representation of the algebra $\mathcal{A}$ generated by the $l_N$’s is a highest-weight representation with highest weight $\alpha$. Equation (3) means that this representation is degenerate at level 2, and it is easy to verify that this implies that the highest weight $\alpha$ is equal to $5/8$ (see [8]). In other words, the fact that one obtains a representation of $\mathcal{A}$ is a consequence of the restriction property, and its degeneracy follows from the Markovian property.

5 Generalizations

Analogous arguments can be used to relate other chordal SLE’s to degenerate highest-weight representations of the Virasoro algebra (which is the central extension of $\mathcal{A}$). When $\kappa \neq 8/3$, SLE does not satisfy the conformal restriction property. However, it is possible to quantify how much the property fails to be true and this involves the integral of a certain constant $c(\kappa)$ times the Schwarzian derivative of some conformal maps (see also [19, 20] for another interpretation). This constant $c(\kappa)$ will turn out to be exactly the central charge of the corresponding representation. The highest-weight is the exponent of the restriction measure that is naturally associated to the SLE via the correspondence derived in [18]. This will be detailed in a forthcoming paper.

One can also make a similar analysis for the SLE behaviour in the bulk, and make the link to Ward-type identities and representation theory. However, in this case, there are some additional questions to solve if one looks for a completely rigorous link. For instance, the very definition of the “correlation function” (see [2] to see the difficulty of estimating precisely the probability that the SLE paths passes in the neighbourhood of $n$ given points in the upper half-plane) is problematic.

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Laboratoire de Mathématiques
Université Paris-Sud
91405 Orsay cedex, France