FUSION RULES OF EQUIVARIANTIZATIONS OF FUSION CATEGORIES

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Abstract. We determine the fusion rules of the equivariantization of a fusion category \( C \) under the action of a finite group \( G \) in terms of the fusion rules of \( C \) and group-theoretical data associated to the group action. As an application we obtain a formula for the fusion rules in an equivariantization of a pointed fusion category in terms of group-theoretical data. This entails a description of the fusion rules in any braided group-theoretical fusion category.

1. Introduction

Throughout this paper we shall work over an algebraically closed field \( k \) of characteristic zero. Let \( C \) be a fusion category over \( k \), that is, \( C \) is a semisimple rigid monoidal category over \( k \) with finitely many isomorphism classes of simple objects, finite-dimensional Hom spaces, and such that the unit object \( 1 \) is simple.

Consider an action \( \rho : G \to \text{Aut}_k^\otimes C \) of a finite group \( G \) by tensor autoequivalences of \( C \) and let \( C^G \) be the equivariantization of \( C \) with respect to this action. Equivariantization under a finite group action, as well as its applications, generalizations and related constructions, have been intensively studied in the last years by several authors. See for instance \([1, 2, 3, 5, 6, 7, 13, 16]\).

In the sense of the notions introduced in \([2, 3]\), equivariantization gives rise in a canonical way to a central exact sequence of tensor categories

\[
\text{rep}G \to C^G \to C,
\]

where \( \text{rep} G \) is the category of finite-dimensional representations of \( G \). On the other hand, combined with the dual notion of (graded) group extension of a fusion category, equivariantization underlies the notion of solvability of a fusion category developed in \([6]\).

An important invariant of a fusion category \( C \) is its Grothendieck ring, \( \text{gr}(C) \). For instance, the knowledge of the structure of the Grothendieck ring allows to determine all fusion subcategories of \( C \), which correspond to the so-called based subrings.

Let \( \text{Irr}(\mathcal{C}) = \{ 1 = S_0, \ldots, S_n \} \) denote the set of isomorphism classes of simple objects of \( \mathcal{C} \). Then \( \text{Irr}(\mathcal{C}) \) is a basis of \( \text{gr}(\mathcal{C}) \) and, for all \( 0 \leq i, j \leq n \), we have a relation

\[
S_i S_j = \sum_{l=0}^{n} A_{i,j}^l S_l,
\]

Date: June 27, 2012.

1991 Mathematics Subject Classification. 18D10; 16T05.

The work of S. Burciu was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFSCDI, grant no. 88/ 05.10.2011. The work of S. Natale was partially supported by CONICET and SeCyT-UNC. This project has started while the first author was visiting Universidad Nacional de Córdoba. He thanks UNC for its hospitality during his stay.
where $N^i_{l,j}$ are non-negative integers given by $N^i_{l,j} = \dim \operatorname{Hom}_C(S_i, S_l \otimes S_j)$, $0 \leq l \leq n$. The relations (1.1) are known as the fusion rules of $C$. They are determined by the set $\operatorname{Irr}(C)$ and the multiplicities $N^i_{l,j} \in \mathbb{Z}_{\geq 0}$.

The main result of this paper is the determination of the fusion rules of $C^G$ in terms of the fusion rules of $C$ and certain canonical group-theoretical data associated to the group action. This is contained in Theorem 3.9. As it turns out, the structure of the Grothendieck ring of $C^G$ resembles that of the rings introduced by Witherspoon in [17].

As an example, consider a semisimple cocentral Hopf algebra extension $H$ of a Hopf algebra $A$ by a finite group $G$, that is, $H$ fits into a cocentral exact sequence

$$k \rightarrow A \rightarrow H \rightarrow kG \rightarrow k.$$ 

As shown in [12], the category $\operatorname{Rep} H$ of finite-dimensional representations of $H$ is an equivariantization ($\operatorname{Rep} A)^G$ with respect to an appropriate action of $G$ on $\operatorname{Rep} A$. Thus Theorem 3.9 implies that the fusion rules of the category $\operatorname{Rep} H$ can be described in terms of the fusion rules of $\operatorname{Rep} A$ and the action of $G$. In particular, Theorem 3.9 generalizes the results obtained for cocentral abelian extensions of Hopf algebras in [8] and [17].

We discuss in detail the case where $C$ is a pointed fusion category, that is, when all simple objects of $C$ are invertible. In this case the fusion rules of $C^G$ are described completely in terms of group-theoretical data. See Theorem 4.1.

It is known that every braided group-theoretical fusion category is an equivariantization of a pointed fusion category [10] [11]. Therefore, our results entail the determination of the fusion rules in any braided group-theoretical fusion category.

In order to establish Theorem 3.9 we give an explicit description of the simple objects of the equivariantization $C^G$. This is done, more generally, for any action $\rho : G \rightarrow \operatorname{Aut}_C$ of the group $G$ by autoequivalences of a $k$-linear finite semisimple category $C$. Such an action induces naturally an action of $G$ on the set $\operatorname{Irr}(C)$ of isomorphism classes of simple objects of $C$. Let $Y \in \operatorname{Irr}(C)$ and let $G_Y \subseteq G$ denote the inertia subgroup of $Y$, that is,

$$G_Y = \{g \in G \mid \rho^g(Y) \simeq Y\}.$$ 

We show that isomorphism classes of simple objects of $C^G$ are parameterized by pairs $(Y, \pi)$, where $Y$ runs over the orbits of the action of $G$ on $\operatorname{Irr}(C)$, and $\pi$ is the equivalence class of an irreducible projective representation of the inertia subgroup $G_Y$ with a certain factor set $\tilde{\alpha}_Y \in \mathbb{Z}^G(G_Y, k^*)$. This result is analogous to the parameterization of irreducible representations of a finite group in terms of those of a normal subgroup given by Clifford Theorem. It extends the description obtained in [13] for the case where $C^G$ is the category of representations of an (algebra) group crossed product (see [12] Subsection 3.1)).

In the case where $C$ is a fusion category, the duality in $C$ gives rise to a ring involution $^* : \operatorname{gr}(C) \rightarrow \operatorname{gr}(C)$. We describe this ring involution for the Grothendieck ring of $C^G$ in Subsection 3.4, more precisely, we use Theorem 3.9 in order to determine the dual of the simple object in $C^G$ corresponding to a pair $(Y, \pi)$ as above.

We may regard $\operatorname{rep} G$ as a fusion subcategory of $C^G$ under a canonical embedding. In this way $C^G$ becomes a $\operatorname{rep} G$-bimodule category under the action given by tensor product. As another consequence of Theorem 3.9, we give a decomposition of $C^G$ into indecomposable $\operatorname{rep} G$-module categories. See Theorem 3.14.

The paper is organized as follows. In Section 2 we recall the definition of the equivariantization of a semisimple abelian category over $k$ under a finite group action and give a parameterization of its simple objects. With respect to this parameterization, we determine in Section 3 the fusion rules in an equivariantization
of a fusion category. Apart from the main result, Theorem 3.9, we also present in
this section the above mentioned applications to the determination of the dual of a
simple object and the decomposition of $C^G$ as a rep $G$-bimodule category. In Section
4 we specialize our main result to the case of an equivariantization of a pointed
fusion category and in particular, to braided group-theoretical fusion categories.

We include an Appendix at the end of the paper, where we give an account of the
relevant facts about projective group representations needed throughout.

2. Simple objects of an equivariantization

The goal of this section is to describe a (mostly well-known) Clifford correspon-
dence entailing a classification of isomorphism classes of simple objects of $C^G$ in
terms of the action of $G$ on the set $\text{Irr}(C)$ of isomorphism classes of simple objects
of $C$. By abuse of notation, we often indicate an object of a category $C$ and its
isomorphism class by the same letter.

2.1. Equivariantization under a finite group action. Let $C$ be a finite semisimple
$k$-linear category and let $G$ be a finite group. Let also $\rho : G \to \text{Aut} C$ be an
action of $G$ on $C$ by $k$-linear autoequivalences. Thus, for every $g \in G$, we have a
$k$-linear functor $\rho^g : C \to C$ and natural isomorphisms

$$\rho^g_{a,b} : \rho^g a \to \rho^g b, \quad g, h \in G,$$

and $\rho_0 : \text{id}_C \to \rho^e$, subject to the following conditions:

\begin{align}
(\rho^g_{a,b,c})_X (\rho^g_{a,b})_c(x) &= (\rho^g_{a,b,c})_X x (\rho^g_{a,b,c})_X x, \\
(\rho^g_{a,b,c})_X (\rho^g_{a,b,c})_X &= (\rho^g_{a,b,c})_X x (\rho^g_{a,b,c})_X x,
\end{align}

for all objects $X \in C$, and for all $a, b, c \in G$. By the naturality of $\rho^g_{a,b}$, $g, h \in G$, we
have the following relation:

$$\rho^g_{a,b}(f) (\rho^g_{a,b})_X = (\rho^g_{a,b})_X \rho^g_{b,a}(f),$$

for every morphism $f : Y \to X$ in $C$. For simplicity, we shall assume in what follows
that $\rho^e = \text{id}_C$ and $\rho_0, \rho^g_{a,b}, \rho^g_{a,b}$ are identities.

Let $C^G$ denote the corresponding equivariantization. Recall that $C^G$ is a finite semisimple
$k$-linear category whose objects are $G$-equivariant objects of $C$, that is, pairs $(X, \mu)$, where $X$ is an object of $C$ and $\mu = (\mu^g)_{g \in G}$, such that $\mu^g : \rho^g X \to X$ is
an isomorphism, for all $g \in G$, satisfying

$$\mu^g \rho^g (\mu^h)_X = (\mu^h)_X \rho^g \mu^h(f), \quad g, h \in G, \quad \mu_{e \rho_0 X} = \text{id}_X.$$

A morphism $f : (X, \mu) \to (X', \mu')$ in $C^G$ is a morphism $f : X \to X'$ in $C$ such that

$$f \mu^g = \mu'^g \rho^g(f),$$

for all $g \in G$.

We shall also say that an object $X$ of $C$ is $G$-equivariant if there exists such a
collection $\mu = (\mu^g)_{g \in G}$ so that $(X, \mu) \in C^G$. Note that $\mu$ is not necessarily unique.

The forgetful functor $F : C^G \to C$, $F(X, \mu) = X$, is a dominant functor. The
functor $F$ has a left adjoint $L : C \to C^G$, defined by $L(X) = \bigoplus_{g \in G} \rho^g X$, where $(\mu X)_g : \bigoplus_{g \in G} \rho^g X \to \bigoplus_{g \in G} \rho^g X$ is given componentwise by the isomorphisms $(\rho^g_{a,b})_X$. The composition $T \mu = FL : C \to C$ is a faithful $k$-linear monad on $C$ such that

$C^G$ is equivalent to the category $C^{T \psi}$ of $T \psi$-modules in $C$. See [2] Subsection
5.3].
2.2. Frobenius-Perron dimensions of simple objects of $\mathcal{C}^G$. Let $X, Y \in \mathcal{C}$. Then $\text{Hom}_\mathcal{C}(\rho^g X, \rho^g Y) \cong \text{Hom}_\mathcal{C}(X, Y)$, for all $g \in G$. Therefore, for all $g \in G$, and for all objects $M$ of $\mathcal{C}^G$, we have
\begin{equation}
\text{Hom}_\mathcal{C}(F(M), \rho^g Y) \cong \text{Hom}_\mathcal{C}(F(M), Y).
\end{equation}

The action of the group $G$ on $\mathcal{C}$ permutes isomorphism classes of simple objects of $\mathcal{C}$. Let $Y \in \text{Irr}(\mathcal{C})$. We shall denote $G_Y := \text{St}_\mathcal{C}(Y) \subseteq G$ the inertia subgroup of $Y$, that is,
\[ G_Y = \{ g \in G | \rho^g(Y) \cong Y \}, \]
Then $Y$ has exactly $n = [G : G_Y]$ mutually nonisomorphic $G$-conjugates $Y = Y_1, \ldots, Y_n$. For every $1 \leq j \leq n$, we have $Y_j \cong \rho^j Y$, where $g_1 = e, \ldots, g_n$ is a complete set of representatives of the left cosets of $G_Y$ in $G$.

Proposition 2.1. Let $M = (X, \mu)$ be a simple object of $\mathcal{C}^G$ and let $Y$ be a simple constituent of $X$ in $\mathcal{C}$. Let $Y = Y_1, \ldots, Y_n$, $n = [G : G_Y]$, be the mutually nonisomorphic $G$-conjugates of $Y$. Then $X \cong \bigoplus_{i=1}^n Y_i$, where $m = \text{dim } \text{Hom}_\mathcal{C}(X, Y)$.

Proof. Consider the object $T(Y) = FL(Y) = \bigoplus_{g \in G} \rho^g Y$. Let $Z$ be a simple object of $\mathcal{C}$ such that $Z \not\cong Y_j$, $j = 1, \ldots, n$. Then $\text{Hom}_\mathcal{C}(Z, T(Y)) = 0$. By adjointness, we have $\text{Hom}_\mathcal{C}(L(Y), M) \cong \text{Hom}_\mathcal{C}(Y, X) \neq 0$. Then $M$ is a simple direct summand of $L(Y)$ in $\mathcal{C}^G$. This implies that $X = F(M)$ is a direct sum of simple subobjects of $FL(Y) = T(Y)$. Therefore $\text{Hom}_\mathcal{C}(Z, X) = 0$.

Hence $X \cong \bigoplus_{i=1}^n m_i Y_i$, where $m_i = \text{dim } \text{Hom}_\mathcal{C}(Y_i, X)$, for all $i$. By (2.5), we have $m_i = m_1 = m$, for all $i = 1, \ldots, n$. This proves the proposition. □

Corollary 2.2. Let $M = (X, \mu)$ be a simple object of $\mathcal{C}^G$ and let $Y$ be a simple constituent of $X$ in $\mathcal{C}$. Then $\text{FPdim } M = m[G : G_Y] \text{FPdim } Y$, where $m = \text{dim } \text{Hom}_\mathcal{C}(Y, X)$. □

2.3. Equivariantization and projective group representations. Let $Y \in \mathcal{C}$ be a fixed simple object. The action $\rho$ of $G$ on $\mathcal{C}$ induces by restriction an action of $G_Y$ on $\mathcal{C}$ by autoequivalences. We may thus consider the equivariantization $\mathcal{C}^{G_Y}$.

By definition of $G_Y$, there exist isomorphisms $\mathcal{C}^g : \rho^g(Y) \to Y$, for all $g \in G_Y$. For all $g, h \in G_Y$, the composition $\mathcal{C}^g \rho^h(\mathcal{C}^h) = (\mathcal{C}^h)^{-1} \rho^h \mathcal{C}^g \rho^h$ defines an isomorphism $Y \to Y$. Since $Y$ is a simple object, there exist nonzero $\alpha_Y(g, h) \in k$ such that
\begin{equation}
\tilde{\alpha}_Y(g, h)^{-1} \text{id}_Y = \mathcal{C}^g \rho^h(\mathcal{C}^h) = (\mathcal{C}^h)^{-1} \rho^h \mathcal{C}^g \rho^h \mathcal{C}^h : Y \to Y.
\end{equation}
This defines a map $\tilde{\alpha}_Y : G_Y \times G_Y \to k^*$ which is a 2-cocycle on $G_Y$.

Remark 2.3. The cocycle $\tilde{\alpha}_Y$ measures the possible obstruction for $(Y, c)$ to be a $G_Y$-equivariant object, where $c = (\mathcal{C}^g)_{g \in G_Y}$.

Consider another choice of isomorphisms $\mathcal{C}^g : \rho^g(Y) \to Y$, $g \in G_Y$. Since $Y$ is a simple object, the composition $\mathcal{C}^g(\mathcal{C}^h)^{-1} : Y \to Y$ is given by scalar multiplication by some $f(g) \in k^*$, for all $g \in G_Y$. Denoting by $\tilde{\beta}_Y$ the 2-cocycle related to $(\mathcal{C}^g)_g$, it easy to see that $\tilde{\alpha}_Y$ and $\tilde{\beta}_Y$ differ by the coboundary of the cochain $f : G_Y \to k^*$. This implies that the cohomology class $\alpha_Y \in H^2(G_Y, k^*)$ of $\tilde{\alpha}_Y$ depends only on the isomorphism class of the simple object $Y$.

Lemma 2.4. Let $(X, \mu) \in \mathcal{C}^G$ and let $Y \in \text{Irr}(\mathcal{C})$. Consider, for every $g \in G_Y$, isomorphisms $\mathcal{C}^g : \rho^g(Y) \to X$ and let $\tilde{\alpha}_Y$ be the associated 2-cocycle on $G_Y$. Then the space $\text{Hom}_\mathcal{C}(Y, X)$ carries a projective representation of $G_Y$ with factor set $\tilde{\alpha}_Y$, defined in the form
\begin{equation}
\pi(g)(f) = \mu^g(\rho^g(f)(\mathcal{C}^g)^{-1} : Y \to X,
\end{equation}
for all $f \in \text{Hom}_\mathcal{C}(Y, X)$. 
Proof. The relation $\pi(g)\pi(h) = \tilde{\alpha}_Y(g,h)\pi(gh)$, $g, h \in G$, follows by straightforward computation, using the compatibility conditions for $\rho$.

\[\square\]

Remark 2.5. Suppose that $\phi : (X, \mu) \to (X', \mu')$ is an isomorphism in $C^G$. Then the induced isomorphism $\Hom_{\mathcal{C}}(Y, X) \to \Hom_{\mathcal{C}}(Y, X')$, $f \mapsto \phi f$, is an isomorphism of projective representations. Similarly, if $Y' \simeq Y$ is another representative of the isomorphism class of $Y$ and $c^\rho : \rho^\alpha(Y') \to Y'$, $g \in G_Y$, is a collection of isomorphisms, then the projective representations $\Hom_{\mathcal{C}}(Y, X)$ and $\Hom_{\mathcal{C}}(Y', X)$ are projectively equivalent.

Proposition 2.6. Let $Y \in \Irr(\mathcal{C})$. There is a bijective correspondence between isomorphism classes of simple objects $L = (N, \nu)$ of $C^{G_Y}$ such that $N \simeq \Hom_{\mathcal{C}}(Y, N) \otimes Y$ and equivalence classes of irreducible $G_Y$-projective representations of the group $G_Y$. If the simple object $L = (N, \nu)$ corresponds to the projective representation $\pi$, then $\pi \simeq \Hom_{\mathcal{C}}(Y, N)$ and $\FPdim L = \dim \pi \FPdim Y$.

Proof. Let $c^\rho : \rho^\alpha(Y) \to Y$, $g \in G_Y$, be any fixed choice of isomorphisms, and let $\tilde{\alpha}_Y$ be the associated 2-cocycle. Let also $\pi$ be a projective representation of $G_Y$ on the vector space $V$ with factor set $\tilde{\alpha}_Y$. Then the pair $(V \otimes Y, \nu)$ is a $G_Y$-equivariant object, where

\[\nu \equiv \pi(g) \otimes c^\rho : \rho^\alpha(V \otimes Y) = V \otimes \rho^\alpha(Y) \to V \otimes Y.\]

Conversely, if $L = (N, \nu)$ is an object of $C^{G_Y}$ with $N \simeq \Hom_{\mathcal{C}}(Y, N) \otimes Y$, then $V = \Hom_{\mathcal{C}}(Y, N)$ carries a projective representation $\pi$ of $G_Y$ with factor set $\tilde{\alpha}_Y$ defined by (2.7).

These assignments are functorial and mutually inverse up to isomorphisms. Then $L = (N, \nu)$ is a simple object of $C^{G_Y}$ if and only if $V = \Hom_{\mathcal{C}}(Y, N)$ is an irreducible projective representation. This implies the proposition.

2.4. The relative adjoint. Consider the forgetful functor $F_Y : C^G \to C^{G_Y}$. We discuss in this subsection the left adjoint $L_Y : C^{G_Y} \to C^G$ of the functor $F_Y$.

Let $\mathcal{R}$ be a set of representatives of the left cosets of $G_Y$ in $G$. So that $G$ is a disjoint union $G = \cup_{t \in \mathcal{R}} tG_Y$.

Set, for all $(N, \nu) \in C^{G_Y}$, $L_Y(N, \nu) = L^\mathcal{R}_Y(N, \nu) = (\oplus_{t \in \mathcal{R}} \rho^t(N), \mu)$, where, for all $g \in G$, $\mu^g : \oplus_{t \in \mathcal{R}} \rho^t(N) \to \oplus_{t \in \mathcal{R}} \rho^t(N)$ is defined componentwise by the formula

\[\mu^g = \rho^g(\rho^h)^{-1} \rho^{ht} : \rho^g(N) \to \rho^h(N),\]

where the elements $h \in G_Y$, $s \in \mathcal{R}$ are uniquely determined by the relation

\[gt = sh.\]

Remark 2.7. We shall show in Proposition 2.8 below that the functor $L^\mathcal{R}_Y$ thus defined is left adjoint of the functor $F_Y$. By uniqueness of the adjoint, it will follow that, up to isomorphism, $L^\mathcal{R}_Y$ does not depend on the particular choice of the set of representatives $\mathcal{R}$.

Lemma 2.8. Let $(N, \nu) \in C^{G_Y}$. Then $L_Y(N, \nu) \in C^G$.

Proof. For every $g \in G$, $t \in \mathcal{R}$, let $s(g, t) \in \mathcal{R}$, $h(g, t) \in G_Y$ be the elements uniquely determined by the relation $gt = s(g, t)h(g, t)$. Note that, for all $a, b \in G$, $t \in \mathcal{R}$, the following relations hold:

\[s(ab, t) = s(a, s(b, t)),\]

\[h(ab, t) = h(a, h(a, s(b, t)))h(b, t),\]

\[s(ab, t)h(a, s(b, t)) = as(b, t).\]
In order to prove the lemma we shall show that, for all objects \((N, \nu) \in C^G\), and for all \(a, b \in G, t \in \mathcal{R}\), the following diagram is commutative:

\[
\begin{array}{ccc}
\rho^a \rho^b \rho^f(N) & \xrightarrow{\left(\rho^a_{\mu, s(b,t)}\right)_{\mu, s(b,t)}} & \rho^a \rho^f(N) \\
\rho^a \rho^s(b,t)(N) & \xrightarrow{\mu_{s(b,t)}} & \rho^s(ab,t)(N).
\end{array}
\]

Using \((2.1)\), we see that commutativity of the diagram \((2.14)\) is equivalent and using \((2.13)\), we obtain the expression

\[
Hence we get
\]

Composing this resulting morphism with the inverse of

\[
\rho^s(ab,t) \left(\mu^{b,t}(\nu^{a,s(b,t)})\right) \left(\rho^b_{a,b, \nu^{a,s(b,t)}}\right)^{-1} \rho^b_{a,b,t} = \mu_{a,s(b,t)} \rho^s(\mu^{b,t}).
\]
Finally, we compute
\[
\rho_2^{a,b,t} = \rho_2^{a,s(b,t),h(b,t)} \\
= \rho_2^{a,s(b,t),h(b,t)} (\rho_2^{a,s(b,t)})_{\rho^a(h,b,t)}^{-1} \\
= \rho_2^{a,s(b,t),h(b,t)} (\rho^a(h,b,t))^{-1} (\rho_2^{a,s(b,t),h(b,t)})^{-1}. 
\]

Combining this with (2.17) we get relation (2.19). This shows that the diagram (2.13) is commutative, as claimed, and finishes the proof of the lemma. □

In view of Lemma 2.8 there is a well-defined functor \( L_Y : \mathcal{C}^G_Y \to \mathcal{C}^G \).

**Proposition 2.9.** The functor \( L_Y^R \) is left adjoint of the functor \( F_Y \).

**Proof.** We define natural transformations \( \eta : \text{id}_{\mathcal{C}^G_Y} \to F_Y L_Y^R \) and \( \epsilon : L_Y^R F_Y \to \text{id}_{\mathcal{C}^G_Y} \), in the form
\[
\eta(N, \nu) = i_e : N = \rho^e(N) \to F_Y L_Y^R(N, \nu) = F_Y(\bigoplus_{t \in \mathcal{R}} \rho^t(N), \mu), \\
\epsilon(M, \mu) = \bigoplus_{t \in \mathcal{R}} \rho^t : L_Y^R F_Y(M, \mu) = (\bigoplus_{t \in \mathcal{R}} \rho^t(M), \mu) \to (M, \mu),
\]
for every \((N, \nu) \in \mathcal{C}^G_Y\), \((M, \mu) \in \mathcal{C}^G\). It is straightforward to verify that \( \eta \) and \( \epsilon \) are well-defined and that the compositions
\[
F_Y \eta_F : F_Y L_Y^R \xrightarrow{\eta} L_Y^R F_Y F_Y \xrightarrow{\epsilon} F_Y, \quad L_Y^R \xrightarrow{\eta} L_Y^R F_Y L_Y^R \xrightarrow{\epsilon} L_Y^R 
\]
are identities. This implies the proposition. □

**Remark 2.10.** Note that the restriction of \( L_Y \) to the fusion subcategory \( \text{rep}\, G_Y \) of \( \mathcal{C}^G_Y \) is isomorphic to the induction functor \( \text{rep}\, G_Y \to \text{rep}\, G \subseteq \mathcal{C}^G \).

As pointed out in Remark 2.11, we have the following:

**Corollary 2.11.** The functor \( L_Y^R \) is, up to isomorphism, uniquely determined by the subgroup \( G_Y \). □

### 2.5. Parameterization of simple objects.

The following theorem is the main result of this section.

**Theorem 2.12.** Let \( Y \in \mathcal{C} \) be a simple object. Then the functor \( L_Y : \mathcal{C}^G_Y \to \mathcal{C}^G \) induces a bijective correspondence between isomorphism classes of:

(a) Simple objects \((N, \nu) \in \mathcal{C}^G_Y\) such that \( \text{Hom}_\mathcal{C}(Y, N) \neq 0 \), and

(b) Simple objects \((X, \mu) \in \mathcal{C}^G\) such that \( \text{Hom}_\mathcal{C}(Y, X) \neq 0 \).

If \((X, \mu) \in \mathcal{C}^G\) as in (b) corresponds to \((N, \nu) \in \mathcal{C}^G_Y\) as in (a), then we have \( \text{Hom}_\mathcal{C}(Y, N) \simeq \text{Hom}_\mathcal{C}(Y, X) \) as projective representations of \( G_Y \). Moreover, \( N \simeq \text{Hom}_\mathcal{C}(Y, N) \otimes Y \).

**Proof.** Let \((N, \nu) \in \mathcal{C}^G_Y\) be a simple object as in (a). By Proposition 2.1 applied to \( G_Y \), \( N \simeq mY \), where \( m = \dim \text{Hom}_\mathcal{C}(Y, N) \). Thus \( N \simeq \text{Hom}_\mathcal{C}(Y, N) \otimes Y \).

Let \((X, \mu) \in \mathcal{C}^G\) be a simple object such that \((N, \nu) \in \mathcal{C}^G_Y\) is a simple direct summand of \( F_Y(X, \mu) \) in \( \mathcal{C}^G_Y \). By adjunction, \((X, \mu) \in \mathcal{C}^G\) is a simple direct summand of \( L_Y(N, \nu) \) in \( \mathcal{C}^G \). Then \( X \) is a direct summand of \( \bigoplus_{t \in \mathcal{R}} \rho^t(N) \) in \( \mathcal{C} \). Since \( X \) is \( G \)-equivariant, then \( \text{Hom}_\mathcal{C}(X, N) \neq 0 \).

Therefore \( \text{Hom}_\mathcal{C}(Y, X) \neq 0 \) and \((X, \mu) \) satisfies the condition in (b).

Again by Proposition 2.1 we get that \( X \simeq e \bigoplus_{t=1}^n Y_t \), where \( Y = Y_1, \ldots, Y_n \), \( n = [G : G_Y] \), are the mutually nonisomorphic \( G \)-conjugates of \( Y \). Note that \( e = \dim \text{Hom}_\mathcal{C}(Y, X) \leq \dim \text{Hom}_\mathcal{C}(Y, N) = m \), because the multiplicity of \( Y \) in \( \rho^t(N) \) is 0, for all \( t \notin G_Y \).
Since \((N, \nu)\) is a direct summand of \(F_Y(X, \mu)\) in \(C^G Y\), comparing Frobenius-Perron dimensions, we obtain
\[
|G : G_Y| n \text{FPdim } Y = \text{FPdim } L_Y(N, \nu) \leq \text{FPdim } (X, \mu) = e|G : G_Y| \text{FPdim } Y.
\]
Therefore \(e = m\), and necessarily \((X, \mu) = L_Y(N, \nu)\). This implies surjectivity of the map induced by \(L_Y\) from (a) to (b).

Suppose \((N', \nu') \not\cong (N, \nu)\) is a simple summand of \(F_Y(X, \mu)\), with \((N', \nu')\) as in (b). Applying the forgetful functor \(C^G Y \to C\) and comparing the multiplicity of \(Y\) we get
\[
e = \dim \text{Hom}_C(Y, X) \geq \dim \text{Hom}_C(Y', N \oplus N') = m + \dim \text{Hom}_C(Y, N') > m.
\]
This is a contradiction since \(e = m\). Hence \((N, \nu)\) is the unique, up to isomorphisms, simple object as in (b) such that
\[
\text{Hom}_C(L_Y(N, \nu), (X, \mu)) \cong \text{Hom}_C(((N, \nu), F_Y(X, \mu)) \neq 0.
\]
This proves injectivity of the map induced by \(L_Y\). Thus this map is bijective, as claimed.

Now suppose that the class of the simple object \((X, \mu)\) of \(G\) as in (b) corresponds to the class of the simple object \((N, \nu)\) of \(C^G Y\) as in (a). The proof above shows that \(N \cong \text{Hom}_C(Y, N) \otimes Y\).

As we have shown, \(\dim \text{Hom}_C(Y, X) = \dim \text{Hom}_C(Y, N)\). Since \(N\) is a direct summand of \(X\) in \(C\), then \(\text{Hom}_C(Y, N) \subseteq \text{Hom}_C(Y, X)\), thus these spaces are equal. Furthermore, the corresponding projective representations given by Formula (2.7) clearly coincide on both spaces. This finishes the proof of the theorem.

Combining Theorem 2.12 with Proposition 2.6, we obtain the following:

**Corollary 2.13.** There is a bijective correspondence between the set isomorphism classes of simple objects \((X, \mu)\) of \(C^G\) and the set of pairs \((Y, \pi)\), where \(Y\) runs over the orbits of the action of \(G\) on \(\text{Irr}(C)\) and \(\pi\) runs over the equivalence classes of irreducible \(\alpha_Y\)-projective representations of the inertia subgroup \(G_Y \subseteq G\), where \(\alpha_Y \in H^2(G_Y, k^*)\) is the cohomology class of the cocycle \(\tilde{\alpha}_Y\) determined by (2.6).

Let \((X, \mu)\) be the simple object corresponding to the pair \((Y, \pi)\). Then we have \(X \cong \bigoplus_{\rho \in G/G_Y} \rho^Y(V_{\rho} \otimes Y)\). In particular, \(\text{FPdim}(X, \mu) = \dim \pi[G : G_Y] \text{FPdim } Y\).

Let \(Y \in \text{Irr}(C)/G\) and let \(\pi\) be an irreducible \(\alpha_Y\)-projective representation of the group \(G_Y\). We shall use the notation \(S_{Y, \pi}\) to indicate the isomorphism class of the simple object of \(C^G Y\) corresponding to the pair \((Y, \pi)\). We shall also say that such simple object \(S_{Y, \pi}\) lies over \(Y\).

**Remark 2.14.** For every set \(\mathcal{R}\) of left coset representatives of \(G_Y\) in \(G\) and for every collection of isomorphisms \(\{\rho^g : \rho^g(Y) \to Y\}_{g \in G_Y}\), the class \(S_{Y, \pi}\) is represented by the simple object \(S_{Y, \pi}^{\mathcal{R}} := L_{Y, \pi}^{\mathcal{R}}(\pi \otimes Y)\), with the \(G_Y\)-equivariant structure on \(\pi \otimes Y\) given by (2.3).

Let us describe more explicitly the dependence of the simple object \(S_{Y, \pi}^{\mathcal{R}, e}\) on the choice of the isomorphisms \(\rho^h : \rho^h(Y) \to Y\). Suppose we are given another collection of isomorphisms \(e' = \{\rho^{g'}\}\). Then, \(Y\) being simple, for any \(g \in G_Y\) we can write \(\rho^{g'} = d_{c, e'}(g)\rho^{g}\), for some scalar \(d_{c, e'}(g) \in k^*\). It follows from (2.8) that \(\pi \otimes Y = d_{c, e'}^{-1}\pi \otimes Y\) as objects of \(C^G Y\). Hence
\[
S_{Y, \pi}^{\mathcal{R}, e'} = L_{Y, \pi}^{\mathcal{R}}(\pi \otimes Y) = L_{Y, \pi}^{\mathcal{R}}(d_{c, e'}^{-1}\pi \otimes Y) = S_{Y, d_{c, e'}^{-1}\pi}^{\mathcal{R}, e'}.
\]

Theorem 2.12 implies the following:
Lemma 2.15. Let $Y \in \text{Irr}(C)$ and let $\pi$ be an $\alpha_Y$-projective representation of $G_Y$. Then

$$\pi \simeq \text{Hom}_C(Y, S_{Y,\pi})$$

as $G_Y$-projective representations.

Proof. Let $V_\pi$ denote the vector space of the representation $\pi$. We have $S_{Y,\pi} \simeq L_Y(V_\pi \otimes Y, (\pi(g) \otimes c^g)_{g \in G})$, where $c^g : \rho^\alpha(Y) \to Y$ is a collection of isomorphisms. It follows from Proposition 2.16 and Theorem 2.12 that $\pi \simeq \text{Hom}_C(Y, V_\pi \otimes Y) \simeq \text{Hom}_C(Y, S_{Y,\pi})$. □

As a consequence we now get:

Proposition 2.16. Let $Y \in \text{Irr}(C)$ and let $\pi$ be an irreducible $\alpha_Y$-projective representation of $G_Y$. Then, for all $(X, \mu) \in G^G$, we have

$$\dim \text{Hom}_C(S_{Y,\pi}, (X, \mu)) = m_{G_Y}(\pi, \text{Hom}_C(Y, X)).$$

In particular, the simple object $S_{Y,\pi}$ is a constituent of $(X, \mu)$ if and only if $\pi$ is a constituent of $\text{Hom}_C(Y, X)$.

Here, $m_{G_Y}(\pi, \text{Hom}_C(Y, X))$ denotes the multiplicity of $\pi$ in $\text{Hom}_C(Y, X)$. See Section 5.

Proof. We have a decomposition $(X, \mu) \simeq \oplus_{(Z, \gamma)} \text{Hom}_C(S_{Z,\gamma}, (X, \mu)) \otimes S_{Z,\gamma}$, where $Z$ runs over a set of representatives of the orbits of the action of $G$ on $\text{Irr}(C)$ and $\gamma$ is an irreducible $\alpha_Z$-projective representation of $G_Z$. Since $\text{Hom}_C(Y, S_{Z,\gamma}) = 0$, for all $Z \neq Y$, then, as projective $G_Y$-representations,

$$\text{Hom}_C(Y, X) \simeq \oplus_{(Z, \gamma)} \text{Hom}_C(S_{Z,\gamma}, (X, \mu)) \otimes \text{Hom}_C(Y, S_{Z,\gamma})$$

$$\simeq \oplus_{\gamma} \text{Hom}_C(S_{Y,\gamma}, (X, \mu)) \otimes \gamma,$$

the last isomorphism by Lemma 2.15. This implies the proposition. □

2.6. On the choice of isomorphisms in a fixed orbit. Let $Y \in \text{Irr}(C)$ and let $t \in G$. Since $\rho^t(Y)$ is a constituent of $F(S_{Y,\pi})$, it follows from Theorem 2.12 that $S_{Y,\pi} \simeq S_{\rho^t(Y),\pi}$, for some irreducible projective representation $\delta$ of $G_{\rho^t(Y)}$. In this subsection we discuss the dependence of $\delta$ upon $\pi$ and the choice of the sets of isomorphisms $c^\gamma, c^{\rho^t(Y)}$.

Let $\pi$ be a projective representation of $G_Y$ with factor set $\tilde{\alpha}_Y$ and let $^t\pi$ be the conjugate projective representation of $G_{\rho^t(Y)} = tG_Y t^{-1} =: {^tG}_Y$. That is, $V_\pi = V_\pi$ and the action is defined as $^t\pi(h) = \pi(t^{-1}ht)$, for all $h \in G_Y$. Denote by $^t\tilde{\alpha}_Y$ the 2-cocycle of $G_{\rho^t(Y)}$ given by

$$^t\tilde{\alpha}_Y(th^{-1}, th't^{-1}) = \tilde{\alpha}_Y(h, h'), \quad h, h' \in G_Y.$$ 

Then $^t\pi$ is a projective representation of $G_{\rho^t(Y)}$ with factor set $^t\tilde{\alpha}_Y$.

Note that a given collection of isomorphisms $c^\gamma : \rho^\delta(Y) \to Y, g \in G_Y$, determines canonically a collection of isomorphisms $(^tc)^\gamma : \rho^t(\rho^\delta(Y)) \to \rho^\delta(Y), g \in {^tG}_Y$, in the form

$$(^tc)^\gamma := \rho^t(c_{\rho^\delta(Y)}^{-1}gt)(\rho_2^{-1}gt)^{-1}(\rho_2^{-1})_Y.$$ 

Indeed, $t^{-1}gt \in G_Y$ since $G_{\rho^t(Y)} = tG_Y t^{-1}$.

Remark 2.17. Assume that $Y$ is a simple object representing a class in a fixed orbit of the action of $G$. For the objects $\rho^t(Y)$, let the isomorphisms $^tc$ be given as in (2.22). Then formula (2.23) gives the 2-cocycle $^t\tilde{\alpha}_{\rho^t(Y)}(th^{-1}, th't^{-1}) = \tilde{\alpha}_Y(h, h')$ on the inertia subgroup $G_{\rho^t(Y)}$.
Lemma 2.18. Let $H$ be a subgroup of $G$ and let $(M, \nu) \in \mathcal{C}^H$. Then, for all $x \in G$, $(\rho^x(M), \nu) \in \mathcal{C}^H$ with equivariant structure $(t^x\nu)^{\rho^x} : \rho^{x\rho^x} \rightarrow \rho^x(M)$ defined, for every $h \in H$, as the composition

$$
(2.24) \quad \rho^{x\rho^x} \xrightarrow{(\rho^x(M), \nu)} \rho^x(M) \xrightarrow{t^x\nu} \rho^x(M) \xrightarrow{\rho^x} \rho^x(M).
$$

Proof. Consider the 2-cocycle $\tilde{\alpha}_{\rho^x(M)} \in Z^2(G, \rho^x(M), k^*)$ associated with the collection of isomorphisms $t^x\nu$. Since $\nu$ is an equivariant structure on $M$, it has a trivial associated 2-cocycle. It follows from Remark 2.17 that $\tilde{\alpha}_{\rho^x(M)} = 1$ and therefore $t^x\nu$ does define an equivariant structure on $\rho^x(M)$. \qed

Corollary 2.19. Let $t^c$ be the collection of isomorphisms given by equation (2.24) and let $V_{t^c} \otimes \rho^t(Y)$ be the associated object of $\mathcal{C}^{G, \rho^t(Y)}$. Then $\rho^t(V_{t^c} \otimes Y) = V_{t^c} \otimes \rho^t(Y)$ in $\mathcal{C}^{G, \rho^t(Y)}$.

In particular we have an isomorphism of $t^G_Y$-equivariant objects $\rho^t(V_{t^c} \otimes Y) \simeq V_{t^c} \otimes \rho^t(Y)$, where the $t^G_Y$-equivariant structure on $V_{t^c} \otimes \rho^t(Y)$ is induced by any choice of isomorphisms $c_{\rho^t(Y)}$ for $\rho^t(Y)$.

Proof. By Lemma 2.18, a $G, \rho^t(Y)$-equivariant structure on $\rho^t(V_{t^c} \otimes Y)$ is given by

$$
\rho^{ht^{-1}} \xrightarrow{\rho^{ht^{-1}}(Y)} \rho^{ht^{-1}}(\rho^t(Y)) \xrightarrow{\rho^{ht^{-1}}(Y)} \rho^{ht^{-1}}(\rho^t(Y)) \xrightarrow{\rho^{ht^{-1}}(Y)} \rho^{ht^{-1}}(\rho^t(Y)),
$$

for every $h \in H$. Since $t^c\nu(t^ht^{-1}) = \rho^t(h)$, for all $h \in H$, this coincides with the equivariant structure of $V_{t^c} \otimes \rho^t(Y)$ induced by $t^c$. \qed

Suppose now that for every $h \in G, \rho^t(Y)$ we have arbitrary isomorphisms

$$
c_{\rho^t(Y)}^{t^ht^{-1}} : \rho^{t^ht^{-1}}(\rho^t(Y)) \rightarrow \rho^t(Y)
$$

These give rise to isomorphisms

$$
\rho^t(Y) \xrightarrow{t^h\rho^t(Y)} \rho^{t^ht^{-1}}(\rho^t(Y)) \xrightarrow{t^h\rho^t(Y)} \rho^{t^ht^{-1}}(\rho^t(Y)) \xrightarrow{t^h\rho^t(Y)} \rho^{t^ht^{-1}}(\rho^t(Y)) \xrightarrow{t^h\rho^t(Y)} \rho^t(Y).
$$

Since $\rho^t(Y)$ is a simple object, there exist scalars $d_Y(t, h) \in k^*$ such that

$$
(2.25) \quad \rho^t(Y) \xrightarrow{\rho^t(Y)} \rho^t(Y) \xrightarrow{\rho^t(Y)} \rho^t(Y) \xrightarrow{\rho^t(Y)} \rho^t(Y) \xrightarrow{\rho^t(Y)} \rho^t(Y) = d_Y(t, h) c^{t^ht^{-1}}_{\rho^t(Y)},
$$

for all $h \in H$.

3. Fusion rules for $\mathcal{C}^G$

In this section we shall assume that $\mathcal{C}$ is a fusion category over $k$ and $\rho : G \rightarrow \text{Aut}_k \mathcal{C}$ is an action of $G$ on $\mathcal{C}$ by tensor autoequivalences, that is, $\rho^{ab}_g : \rho^a \rightarrow \rho^b$ are natural isomorphisms of tensor functors, for all $g, h \in G$. Thus, for all $g \in G$, $\rho^g$ is endowed with a monoidal structure $(\rho^g)_X \otimes Y : \rho^g(X \otimes Y) \rightarrow \rho^g(X) \otimes \rho^g(Y)$, $X, Y \in \mathcal{C}$, and the following relation holds:

$$
(3.1) \quad \rho^g_{X,Y} \rho^g_{X,Y} = \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y} \rho^g_{X,Y},
$$

for all $g, h \in G, X, Y \in \mathcal{C}$.

Then $\mathcal{C}^G$ is also a fusion category with tensor product $(X, \mu_X) \otimes (Y, \mu_Y) = (X \otimes Y, (\mu_X \otimes \mu_Y) \rho^g_{X,Y})$, where for all $g \in G, \rho^g_{X,Y} : \rho^g(X \otimes Y) \rightarrow \rho^g(X) \otimes \rho^g(Y)$ is the monoidal structure on $\rho^g$. 
Let \( \pi : G \to GL(V) \) be a finite dimensional representation of \( G \) on the vector space \( V \). Then the (trivial) object \( V \otimes 1 \in \mathcal{C} \) has a \( G \)-equivariant structure defined by \( \pi(g) \otimes \text{id}_1 : \rho^\pi(V \otimes 1) \to V \otimes 1 \). This induces an embedding of fusion categories \( \text{rep} \, G \to \mathcal{C}^G \) that gives rise to an exact sequence of fusion categories (3.2)

\[
\text{rep} \, G \to \mathcal{C}^G \to \mathcal{C}.
\]

See [2, Subsection 5.4].

**Remark 3.1.** Let \( \mathcal{G}(\mathcal{C}) \) be the set of isomorphism classes of invertible objects of \( \mathcal{C} \). The exact sequence (3.2) induces an exact sequence of groups

\[
1 \to \hat{G} \to \mathcal{G}(\mathcal{C}^G) \to \mathcal{G}_0(\mathcal{C}) \to 1,
\]

where \( \hat{G} \simeq G/[G, G] \) denotes the group of invertible characters of \( G \) and \( \mathcal{G}_0(\mathcal{C}) \) is the subgroup of \( \mathcal{G}(\mathcal{C}) \) consisting of isomorphism classes of invertible objects which are \( G \)-equivariant. Indeed, \( F \) preserves Frobenius-Perron dimensions, and thus it induces a group homomorphism \( F : \mathcal{G}(\mathcal{C}^G) \to \mathcal{G}(\mathcal{C}) \), which is clearly surjective. The kernel of \( F \) coincides with the invertible objects of \( \text{Ferr}_F = \text{rep} \, G \).

**Remark 3.2.** Note that if \( \pi \) is an irreducible representation of \( G = G_1 \) on \( V \), then the simple object \( (V \otimes 1, (\pi(g) \otimes \text{id}_1)_g) \) of \( \mathcal{C}^G \) is isomorphic to the simple object \( S_{1, \pi} \) corresponding to the pair \((1, \pi)\) as in Corollary 2.14.

### 3.1. Orbit formula for the tensor product of two simple objects

Let \( Y, Z, U \in \text{Irr}(\mathcal{C}) \) and let \( \pi, \gamma, \delta \) be projective representations of the corresponding inertia subgroups with factor sets determined by (2.6). Let also \( S_{Y, \pi}, S_{Z, \gamma} \) and \( S_{U, \delta} \) be the associated simple objects of \( \mathcal{C}^G \).

The multiplicity of \( S_{U, \delta} \) in the tensor product \( S_{Y, \pi} \otimes S_{Z, \gamma} \) is given by the dimension of the vector space \( \text{Hom}_{\mathcal{C}^G}(S_{U, \delta}, S_{Y, \pi} \otimes S_{Z, \gamma}) \). In view of Proposition 2.16 this multiplicity is the same as the multiplicity of \( \delta \) in the space (3.3)

\[
\text{Hom}_{\mathcal{C}}(U, S_{Y, \pi} \otimes S_{Z, \gamma}),
\]

regarded as an \( \alpha_U \)-projective representation of \( \text{G}_U \).

Consider the diagonal action of \( G \) on \( G/G_Y \times G/G_Z \) coming from the natural actions by left multiplication of \( G \) on \( G/G_Y \) and \( G/G_Z \). The stabilizer of a pair \((t, s)\) is the subgroup \( {}^tG_Y \cap {}^sG_Z \subseteq G \).

As objects of \( \mathcal{C} \), we have that

\[
S_{Y, \pi} \otimes S_{Z, \gamma} \simeq \bigoplus_{u \in G/G_Y} \rho^u(V_\pi \otimes Y) \otimes \bigoplus_{v \in G/G_Z} \rho^v(V_\gamma \otimes Z)
\]

\[
\simeq \bigoplus_{(u,v) \in G/G_Y \times G/G_Z} \rho^u(V_\pi \otimes Y) \otimes \rho^v(V_\gamma \otimes Z)
\]

(3.4)

\[
S_\mathcal{O} := \bigoplus_{(u,v) \in \mathcal{O}} \rho^u(V_\pi \otimes Y) \otimes \rho^v(V_\gamma \otimes Z).
\]

The subgroup \( G_U \) acts on \( G/G_Y \times G/G_Z \) by restriction and every \( G \)-orbit in \( G/G_Y \times G/G_Z \) is a disjoint union of \( G_U \)-orbits. Note that the stabilizer of \((t, s)\) in \( G/G_Y \times G/G_Z \) under the action of \( G_U \) is the subgroup \( T = G_U \cap tG_Y t^{-1} \cap sG_Z s^{-1} \).

**Lemma 3.3.** For every \( G \)-orbit (respectively, \( G_U \)-orbit) \( \mathcal{O} \subseteq \mathcal{G}/G_Y \times \mathcal{G}/G_Z \), \( S_\mathcal{O} \) is an equivariant subobject (respectively, a \( G_U \)-equivariant subobject) of the tensor product \( S_{Y, \pi} \otimes S_{Z, \gamma} \).
Proof. We shall prove the statement for $G$-orbits, the proof for $G_U$-orbits being analogous. Let $g \in G$. The equivariant structure $\mu^g := \mu_{gY,p \otimes \gamma}^g$ of $S_Y, \pi \otimes S_Z, \gamma$ is given componentwise by

$$
(\mu^g)^{u,v}(\mu^g)^u_{gS_Y, \pi \otimes S_Z, \gamma} = (\mu^g)^u_{gS_Y, \pi \otimes S_Z, \gamma}(V_u \otimes V_v \otimes (\mu^g)^u_{gS_Y, \pi \otimes S_Z, \gamma}(V_u \otimes V_v \otimes (\mu^g)^v_{gS_Y, \pi \otimes S_Z, \gamma}(Z)),
$$

where, for every $(u, v) \in G/G_Y \times G/G_Z$, $\mu^g_{gS_Y, \pi \otimes S_Z, \gamma}$ and $\mu^g_{gS_Y, \pi \otimes S_Z, \gamma}$ are given by formula (3.3). It follows that

$$
(\mu^g)^{u,v}(\mu^g)^u_{gS_Y, \pi \otimes S_Z, \gamma}(V_u \otimes V_v \otimes \rho^u(Y) \otimes \rho^v(Z)) \subseteq V_u \otimes V_v \otimes \rho^u(Z) \otimes \rho^v(Z),
$$

where $(u, v) \in G/G_Y \times G/G_Z$ are uniquely determined by the relations $gu = u'gY$ and $gv = v'gZ$, with $hY \in G_Y$ and $hZ \in G_Z$. Therefore, $\mu^g(S_O) \subseteq S_O$, for all $g \in G$. This implies the lemma.

The map $G \times G \to G$, $(a, b) \mapsto a^{-1}b$, induces a surjective map $p : G/G_Y \times G/G_Z \to G/Y' \times G/Z', \pi \otimes \gamma \quad \pi \otimes \gamma$, such that $p(tG_Y, sG_Z) = G_Y^{-1}tG_Y$.

Let $O_G(t, s)$ denote the $G$-orbit of an element $(t, s) \in G/G_Y \times G/G_Z$. Observe that for all $g \in G$, we have $p^{-1}(G_Y gG_Z) = O_G(e, g)$. Therefore, $p$ induces an identification between the orbit space of $G/G_Y \times G/G_Z$ under the action of $G$ and the space of double cosets $G/Y' \times G/Z'$. Combining this with (3.4), we obtain:

Corollary 3.4. We have a decomposition $S_Y, \pi \otimes S_Z, \gamma \simeq \bigoplus_{D \in G \backslash G/Y \times G/Z} S_D$, where $S_D := \bigoplus_{d \in D} \rho^d(V_u \otimes Y) \otimes \rho^d(Z)$. Moreover, for all $D \in G/Y \times G/Z$, $S_D$ is a $G$-equivariant subobject of $S_Y, \pi \otimes S_Z, \gamma$.

3.2. Projective representation on multiplicity spaces. It follows from Lemma 3.3 that for every $G$-orbit $O \subseteq G/G_Y \times G/G_Z$, the space $H_O := \text{Hom}_C(U, S_O)$ is an $\alpha_U$-projective representation of $G_U$.

Let $O = O_1 \cup \cdots \cup O_{n_O}$ be the decomposition of $O$ into disjoint $G_U$-orbits $O_1, \ldots, O_{n_O}$. Then, for all $1 \leq i \leq n_O$, the space $H(i) = \text{Hom}_C(U, S_O) \cap \bigoplus_{(t, s) \in O_i} S_{\mathbb{1}(t, s)} \otimes \rho^\alpha(U, V_u \otimes Y) \otimes \rho^\alpha(U, V_v \otimes Z)$.

Furthermore, as $G_U$-projective representations,

$$
\text{Hom}_C(U, S_Y, \pi \otimes S_Z, \gamma) \simeq \bigoplus_{O} \bigoplus_{O_1}^{n_O} H(i),
$$

where summation is understood to run over all orbits $O = O_G(t, s)$ such that $\text{Hom}_C(U, p(Y) \otimes \rho^\alpha(Z)) \neq 0$.

For every $(t, s) \in G/G_Y \times G/G_Z$, let

$$
H_{t,s} := \text{Hom}_C(U, V_u \otimes V_v \otimes \rho^\alpha(Y) \otimes \rho^\alpha(Z)).
$$

Lemma 3.5. Let $t, s \in G$. Then $H_{t,s}$ is an $\alpha_U|T$-projective representation of $T = G_U \cap (G_Y \cap G_Z)$. Moreover, for all $1 \leq i \leq n$, we have

$$
H(i) = \text{Hom}_C(U, S_O) = \bigoplus_{(t, s) \in O_i} H_{t,s},
$$

as projective representations of $T$.

Proof. Since $V_u \otimes \rho^\alpha(Y)$ and $V_v \otimes \rho^\alpha(Z)$ are $G_U \cap G_Z$-equivariant objects, so is their tensor product. Lemma 3.4 implies that $H_{t,s}$ is an $\alpha_U|T$-projective representation of $T$. The decomposition (3.3) follows from the definition of $S_O$.

Proposition 3.6. Let $U, Y, Z \in \text{Irr}(C)$ and let $t, s \in G$. Then the vector space $\tau_U^\alpha(Y, Z) := \text{Hom}_C(U, \rho^\alpha(Y) \otimes \rho^\alpha(Z))$ carries an $\alpha$-projective representation of the subgroup $T := G_U \cap G_{\rho^\alpha(Y)} \cap G_{\rho^\alpha(Z)}$, where $\alpha := \alpha_U[G_{\rho^\alpha(Y)}^{-1} | T \alpha_{\rho^\alpha(Y)}^{-1} | T \alpha_{\rho^\alpha(Z)}^{-1} | T]$. The action of $g \in T$ is given by

$$
g \cdot f = (c_{\rho^\alpha(Y)}^\alpha \otimes c_{\rho^\alpha(Z)}^\alpha)(\rho_{\rho^\alpha(Y)}^\alpha g, \rho_{\rho^\alpha(Z)}^\alpha g) \rho^\alpha(f) (c_{\rho^\alpha(Y)}^\alpha)^{-1},
$$

where $c_{\rho^\alpha(Y)}^\alpha$ and $c_{\rho^\alpha(Z)}^\alpha$ are left $G_{\rho^\alpha(Y)}$ and $G_{\rho^\alpha(Z)}$-equivariant objects, respectively.
for all $f \in \text{Hom}_C(U, \rho^f(Y) \otimes \rho^g(Z))$. Furthermore,
\[ \mathcal{H}_{t,s} \approx \iota|_T \otimes \iota^T \otimes \tau_U^{t,s}(Y, Z), \]
as projective representations of $T$.

Remark 3.7. Observe that the equivalence class of the projective representation $\tau_U^{t,s}(Y, Z)$ is independent on the choice of the isomorphism classes of $U, Y, Z$ as well as on the choice of isomorphisms $c^\rho_U(Y)$, $c^\rho(Z)$ and $c_U$.

Proof. Given $X \in \text{Irr}(C)$, we shall consider in what follows a fixed (but arbitrary) collection of isomorphisms $c_X = \{c_X^\rho : \rho(X) \to X\}_{\rho \in G_X}$. Let also $\tilde{\alpha}_X \in Z^2(G_X, k^\ast)$ be the associated 2-cocycles.

We first show that formula (3.6) does define a projective representation of $T$ with factor set $\tilde{\alpha}_U|T \tilde{\alpha}_U^{-1}(\gamma)|T \tilde{\alpha}_U^{-1}(\rho)|T$. Let $g, h \in T$, $f \in \text{Hom}_C(U, \rho^g(Y) \otimes \rho^h(Z))$. Using the definition of the cocycles $\tilde{\alpha}_U$ given by (2.6) and relation (3.3), we compute:
\[ g(h, f) = (c^g_Y \otimes c^h_Z)(\rho^g_Y \otimes \rho^h_Z) \left( (c^g_Y \otimes c^h_Z)(\rho^g_Y \otimes \rho^h_Z) \right)^{-1} (c^g_U \otimes c^h_U)^{-1} \]
and similar calculations for the other factors.

On the other hand, with respect to the given choice of isomorphisms $\{c_X^\rho\}_{\rho \in G_X}$, $X \in \text{Irr}(C)$, the $G_Y \cap G_Z$-equivariant structures on $V_{\pi} \otimes \rho^f(Y)$ and $V_{\gamma} \otimes \rho^g(Z)$ are given, respectively, by $\iota(\rho^f(Y) \otimes c^\rho(Y)) : \rho^f(V_{\pi} \otimes \rho^g(Y)) \to V_{\pi} \otimes \rho^f(Y)$, and $\iota^T(\rho^g(Z) \otimes c^\rho(Z)) : \rho^g(V_{\gamma} \otimes \rho^g(Z)) \to V_{\gamma} \otimes \rho^g(Z)$, for all $\rho \in T$.

Thus, the action of $g \in T$ on $f \in \mathcal{H}_{t,s} = \text{Hom}_C(U, V_{\pi} \otimes V_{\gamma} \otimes \rho^f(Y) \otimes \rho^g(Z))$ is determined by
\[ g(f) = \left( \iota(\rho(f) \otimes \rho^f(Y) \otimes c^\rho(U) \otimes c^\rho(Z)) \right)(V_{\pi} \otimes V_{\gamma} \otimes (\rho^g_Z) \otimes \rho^f(Y)) \rho^g(Z)(c^\rho(U) \otimes c^\rho(Z)). \]
In view of the $k$-linearity of the functors $\rho^f$, $g \in G$, this implies that the canonical isomorphism
\[ V_{\pi} \otimes V_{\gamma} \otimes \text{Hom}_C(U, \rho^f(Y) \otimes \rho^g(Z)) \approx \text{Hom}_C(U, V_{\pi} \otimes V_{\gamma} \otimes \rho^f(Y) \otimes \rho^g(Z)) \]
is indeed an isomorphism of projective representations of $T$. This finishes the proof of the proposition.

Proposition 3.8. Let $(t, s) \in O_i$, $1 \leq i \leq n$, and let $T_i = G_U \cap G_Y \cap G_Z$ be its stabilizer in $G_U$. Then $\mathcal{H}_i \simeq \text{Ind}^G_{T_i} \mathcal{H}_{t,s}$, as projective $G_U$-representations.

Proof. The proposition follows from Lemma 3.6 in view of Lemma 5.1. Note that the group $G_U$ permutes the set $O_i$ transitively.
3.3. Fusion rules. The following theorem gives the fusion rules for the category \( C^G \).

**Theorem 3.9.** Let \( U, Y, Z \in \text{Irr}(C) \) and let \( \delta, \pi, \gamma \) be irreducible projective representations of the inertia subgroups \( G_U, G_Y, G_Z \) with factor sets determined by \( \mathbb{C}^G \). Then the multiplicity of \( S_{U, \delta} \) in the tensor product \( S_{Y, \pi} \otimes S_{Z, \gamma} \) is given by the formula

\[
(3.7) \quad \sum_{D \in G_Y \setminus G/G_Z} \sum_{\substack{1 \leq i \leq n \\ \text{if} \ t_i \in D \\ \text{and} \\ t_i^{-1}s_i \in D \\ \text{Hom}_C(U, \rho^j(Y) \otimes \rho^j(Z)) \neq 0}} m_{T_i}(\delta|_{T_i}, t_i \pi|_{T_i} \otimes s_i \gamma|_{T_i} \otimes \pi_{U|T_i}^k(Y, Z)),
\]

where \((t_1, s_1), \ldots, (t_n, s_n)\) are representatives of the distinct \( G_U\)-orbits \( \mathcal{O}_1, \ldots, \mathcal{O}_n \) in \( G/G_Y \times G/G_Z \) and, for all \( 1 \leq i \leq n \), \( T_i = G_U \cap t_i G_Y \cap s_i G_Z \), and \( m_{T_i} \) denotes the multiplicity form of projective \( T_i\)-representations.

**Proof.** It follows from Proposition \( \ref{prop:fusion} \) that

\[
\dim \text{Hom}_{C^G}(S_{U, \delta}, S_{Y, \pi} \otimes S_{Z, \gamma}) = m_{G_U}(\delta, \text{Hom}_C(U, S_{Y, \pi} \otimes S_{Z, \gamma})).
\]

In view of Corollary \( \ref{cor:decomposition} \), we have a decomposition

\[
\text{Hom}_C(U, S_{Y, \pi} \otimes S_{Z, \gamma}) = \bigoplus_{D \in G_Y \setminus G/G_Z} \mathcal{H}_D,
\]

as projective representations of \( G_U \), where

\[
\mathcal{H}_D := \bigoplus_{t^{-1}s_i \in D, \text{Hom}_C(U, \rho^j(Y) \otimes \rho^j(Z)) \neq 0} \text{Hom}_C(U, \rho^j(V_\pi \otimes Y) \otimes \rho^j(V_\gamma \otimes Z)).
\]

Consider a decomposition \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n \) of \( G/G_Y \times G/G_Z \) into disjoint \( G_U\)-orbits, and let \( \mathcal{H}(i) \simeq \text{Hom}_C(U, S_{\mathcal{O}_i}) \), \( 1 \leq i \leq n \), as in \( \ref{prop:fusion} \).

Let also \((t_i, s_i)\in \mathcal{O}_i\) be a representative of the orbit \( \mathcal{O}_i \) with stabilizer \( T_i = G_U \cap t_i G_Y \cap s_i G_Z \). By Proposition \( \ref{prop:decomposition} \) we have

\[
\mathcal{H}_D \simeq \bigoplus_{1 \leq i \leq n} \mathcal{H}(i) \simeq \bigoplus_{1 \leq i \leq n, t_i^{-1}s_i \in D} \text{Ind}_{T_i}^{G_U} \mathcal{H}_{t_i, s_i},
\]

Therefore, using Frobenius Reciprocity and Proposition \( \ref{prop:decomposition} \) we get

\[
\dim \text{Hom}_{C^G}(S_{U, \delta}, S_{Y, \pi} \otimes S_{Z, \gamma}) = \sum_{D \in G_Y \setminus G/G_Z} m_{G_U}(\delta, \mathcal{H}_D)
\]

\[
= \sum_{D \in G_Y \setminus G/G_Z} \sum_{\substack{1 \leq i \leq n, t_i^{-1}s_i \in D \\ \text{Hom}_C(U, \rho^j(Y) \otimes \rho^j(Z)) \neq 0}} m_{G_U}(\delta, \text{Ind}_{T_i}^{G_U} \mathcal{H}_{t_i, s_i})
\]

\[
= \sum_{D \in G_Y \setminus G/G_Z} \sum_{\substack{1 \leq i \leq n, t_i^{-1}s_i \in D \\ \text{Hom}_C(U, \rho^j(Y) \otimes \rho^j(Z)) \neq 0}} m_{T_i}(\delta|_{T_i}, \mathcal{H}_{t_i, s_i})
\]

\[
= \sum_{D \in G_Y \setminus G/G_Z} \sum_{\substack{1 \leq i \leq n, t_i^{-1}s_i \in D \\ \text{Hom}_C(U, \rho^j(Y) \otimes \rho^j(Z)) \neq 0}} m_{T_i}(\delta|_{T_i}, t_i \pi|_{T_i} \otimes s_i \gamma|_{T_i} \otimes \pi_{U|T_i}^k(Y, Z)).
\]

Thus we get formula \( \ref{eq:fusion} \). This finishes the proof of the theorem. \( \square \)

**Remark 3.10.** As a consequence of Theorem \( \ref{thm:fusion} \) we get that the structure of the Grothendieck ring of \( C^G \) is similar to that of the rings introduced by Witherspoon in \( \ref{witherspoon} \).
Corollary 3.11. A simple object $S_{Y,\pi}$ is a constituent of a tensor product of simple objects $S_Y \otimes S_Z$ if and only if there exist $t \in G/G_Y$ and $s \in G/G_Z$ such that
\begin{itemize}
  \item[(a)] $\text{Hom}_C(U, \rho^t(Y) \otimes \rho^s(Z)) \neq 0$
  \item[(b)] $m_T(\delta|_T, \pi|_T \otimes \tau^{-1}_Y(Y, Z)) \neq 0$, where $T = G_U \cap tG_Y t^{-1} \cap sG_Y s^{-1}$.
\end{itemize}

\[\square\]

3.4. The dual of a simple object. Let $Y \in \text{Irr}(C)$. Then the multiplicity of the unit object of $C$ in the tensor product $Y \otimes Y^*$ is one. Hence $\gamma_Y := \pi_1^e(Y, Y^*) = \text{Hom}_C(1, Y \otimes Y^*)$ is a one dimensional (linear) representation of $G = G_1$. In particular, it follows from Proposition 3.6 that the cohomology class of the product $\alpha_Y \alpha_Y^*$ is trivial on $G_Y = G_{Y^*}$.

Recall that the dual $\pi^*$ of the $G_Y$-projective representation $\pi$ is defined as $V^*_\pi$ with $\pi^*(h)(f) = f \circ \pi(h)^{-1}$. This is an $\alpha_Y^{-1}$-projective representation of $G_Y$.

Proposition 3.12. The dual object of $S_{Y,\pi} \in C^G$ is determined by
\[S_{Y,\pi}^* \simeq S_{Y^*,\pi^*}.\]

Proof. Observe that $S_{Y,\pi} \simeq S_{Z,\gamma}$, for some $Z \in \text{Irr}(C)/G$ and some $\alpha_Z$-projective representation of $G_Z$. On the other hand, $S_{Y,\pi} \simeq S_{Z,\gamma}$ if and only if the unit object is a constituent of $S_{Y,\pi} \otimes S_{Z,\gamma}$. Since the unit object of $C^G$ is isomorphic to $S_1$, where $\epsilon$ denotes the trivial representation of $G_1 = G$, it follows from Corollary 3.11 that $S_{Y,\pi} \simeq S_{Y^*,\pi^* \otimes \tau_Y^{-1}}$, where $\tau_Y = \text{Hom}_C(1, Y \otimes Y^*)$. Since $\tau_Y$ is a linear character of $G_Y$, then $\pi^* \otimes \tau_Y^{-1} \simeq \pi^*$ as projective $G_Y$-representations (see Section 3). Then $S_{Y^*,\pi^* \otimes \tau_Y^{-1}} \simeq S_{Y^*,\pi^*}$ and the proposition follows.

\[\square\]

Combining Proposition 3.12 with Frobenius Reciprocity we obtain:

Corollary 3.13. Let $Y, Z \in \text{Irr}(C)/G$ and let $\pi, \gamma$ be projective representations of $G_Y$ and $G_Z$ with factor sets determined by $2.6$. Let also $\delta$ be an irreducible representation of $G$. Then $S_{1,\delta}$ is a constituent of $S_{Y,\pi} \otimes S_{Z,\gamma}$ if and only if $Z = Y^*$ and $\delta$ is a constituent of $(\pi \otimes \gamma^*) \uparrow_{C^G}^{C^G}$.\[\square\]

3.5. $C^G$ as a $\text{rep } G$-bimodule category. Let us regard the category $\text{rep } G$ as a fusion subcategory of $C^G$ via the natural embedding $\pi \mapsto (\pi \otimes 1, \pi(g) \otimes \text{id}_1)$. So that the tensor product of $C^G$ makes $C^G$ into a $\text{rep } G$-bimodule category.

The results in Section 2 imply that there is an equivalence of $k$-linear categories $C^G \simeq \bigoplus_{Y \in \text{Irr}(C)/G} \text{rep}_{\alpha_Y} G_Y$, where $\text{rep}_{\alpha_Y} G_Y$ is the category of finite dimensional $\alpha_Y$-projective representations of $G_Y$. Under this equivalence, a simple object $\pi$ of $\text{rep}_{\alpha_Y} G_Y$, that is, an irreducible $\alpha_Y$-projective representation of $G_Y$, corresponds to the simple object $S_{Y,\pi}$ of $C^G$. In other words, $\text{rep}_{\alpha_Y} G_Y$ is identified with the full subcategory of $\mathcal{C}$ whose simple objects are lying over $Y$. An explicit equivalence is determined, for every $Y \in \text{Irr}(C)/G$, by the functors $L_Y : C^{G_Y} \to C^G$ and $F_Y : C^G \to C^{G_Y}$.

For each $Y \in \text{Irr}(C)$, the category $\text{rep}_{\alpha_Y} G_Y$ is in a canonical way an indecomposable $\text{rep } G$-bimodule category via tensor product of projective representations; see [15, Theorem 3.2]. As a consequence of Theorem 3.14 we obtain:

Theorem 3.14. There is an equivalence of $\text{rep } G$-bimodule categories
\begin{equation}
C^G \simeq \bigoplus_{Y \in \text{Irr}(C)/G} \text{rep}_{\alpha_Y} G_Y.
\end{equation}

Moreover, each $\text{rep}_{\alpha_Y} G_Y$ is an indecomposable $\text{rep } G$-bimodule category.
Proof. Let $\pi$ be an irreducible representation of $G$, so that $\pi$ corresponds to the simple object $S_{1,\pi} \in \text{rep} G$, and let $S_{Z,\gamma} \in \text{rep}_{G_Z} G_Z$ be another simple object, where $Z \in \text{Irr}(C)/G$. It follows from Corollary 3.11 that if the simple object $S_{U,\delta}$, $U \in \text{Irr}(C)/G$, is a constituent of $S_{1,\pi} \otimes S_{Z,\gamma}$, then $U \simeq \rho^s(Z)$, for some $s \in G/G_Z$. Hence $U = Z$ and thus the group $T = G_Z \cap G_1 \cap G_Z$ coincides with $G_Z$, $\tau_U(1, Z) \simeq \text{Hom}_C(Z, Z)$ is a one dimensional (linear) representation of $G_Z$.

Clearly, we have $S$ by the forgetful functor $F$ where, for all

$\pi \in \text{Irr}(C)/G$.

Therefore $\pi|_{G_Z} \otimes \gamma \otimes \tau_U(1, Z) \simeq \pi|_{G_Z} \otimes \gamma$ as projective representations of $G_Z$.

By Theorem 3.9 the multiplicity of $S_{Z,\gamma}$ in the tensor product $S_{1,\pi} \otimes S_{Z,\gamma}$ equals $m_{G_Z}(\delta, \pi \otimes \gamma)$. Therefore we obtain

$$S_{1,\pi} \otimes S_{Z,\gamma} \simeq \bigoplus_{\delta} m_{G_Z}(\delta, \pi|_{G_Z} \otimes \gamma) S_{Z,\delta},$$

where $\delta$ runs over the equivalence classes of $\alpha_Z$-projective representations of $G_Z$.

Clearly this object corresponds to $\pi|_{G_Z} \otimes \gamma \in \text{rep}_{G_Z} G_Z$.

Similar arguments apply for the tensor product $S_{Z,\gamma} \otimes S_{1,\pi}$. This completes the proof of the theorem. \hfill $\square$

For any $U \in \text{Irr}(C)$, let us extend the notation $S_{U,\delta} = L_U(\delta \otimes U)$ to indicate the object of $C^G$ corresponding to an arbitrary $\alpha_U$-projective representation $\delta$ of $U$.

Remark 3.15. Let $Y, Z \in \text{Irr}(C)$ and let $S_{Y,\pi}$, $S_{Z,\gamma}$ be simple objects of $C^G$ lying over $Y$ and $Z$, respectively. So that $\pi$ is an irreducible $\alpha_Y$-projective representation of $G_Y$ and $\gamma$ is an irreducible $\alpha_Z$-projective representation of $G_Z$.

According to Theorem 3.11 the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$ has a decomposition

$$S_{Y,\pi} \otimes S_{Z,\gamma} \cong \bigoplus_{U \in \text{Irr}(C)/G} S_{U,\delta},$$

where, for all $U \in \text{Irr}(C)/G$, $S_{U,\delta} \in C^G$ is the sum of simple constituents of $S_{Y,\pi} \otimes S_{Z,\gamma}$ lying over $U$. It follows from Proposition 2.16 that $\delta \cong \text{Hom}_C(U, S_{Y,\pi} \otimes S_{Z,\gamma})$.

Remark 3.16. The action of $G$ on $C$ induces an action of $G$ on $\text{gr}(C)$ by algebra automorphisms. Let $\text{gr}(C)^G \subseteq \text{gr}(C)$ be the subring of $G$-invariants in $\text{gr}(C)$. For every $Y \in \text{Irr}(C)$, let us consider the element

$$(3.9) \quad S(Y) := \sum_{t \in G/G_Y} \rho^t(Y) \in \text{gr}(C).$$

Clearly, we have $S(Y) \in \text{gr}(C)^G$ and $S(Y) = S(\rho(Y))$, for all $Y \in \text{Irr}(C)$. Observe that $F^1(S_{Y,\pi}) = (\dim \pi)S(Y)$, where $F^1 : \text{gr}(C)^G \rightarrow \text{gr}(C)$ is the ring map induced by the forgetful functor $F : C^G \rightarrow C$. Moreover, the set $\{S(Y) : Y \in \text{Irr}(C)/G\}$ is a basis for $\text{gr}(C)^G$ and, for all $Y, Z \in \text{Irr}(C)/G$, we have

$$S(Y) S(Z) = \sum_{U \in \text{Irr}(C)/G} m_{U,Y,Z}^Z S(U),$$

for some nonnegative integers $m_{U,Y,Z}^Z$.

Let $Y, Z, U \in \text{Irr}(C)/G$. Consider any fixed simple objects $S_{Y,\pi}$ and $S_{Z,\gamma}$ of $C^G$ lying over $Y$ and $Z$, respectively. Applying the map $F^1$ in formula (3.9), we obtain that $m_{U,Y,Z}^Z = \dim \delta/ (\dim \pi)(\dim \gamma)$, where $\delta = \text{Hom}_C(U, S_{Y,\pi} \otimes S_{Z,\gamma}) \simeq V_\pi \otimes V_\gamma \otimes \bigotimes_{(t,s) \in G_\pi \times G_\nu \times G_\gamma} \text{Hom}_C(U, \rho^t(Y) \otimes \rho^s(Z))$. Therefore, for all $Y, Z, U \in \text{Irr}(C)/G$, the integers $m_{U,Y,Z}^Z$ are given by the formula

$$m_{U,Y,Z}^Z = \sum_{(t,s) \in G_\pi \times G_\nu \times G_\gamma} \dim \text{Hom}_C(U, \rho^t(Y) \otimes \rho^s(Z)).$$

4. Application to equivariantizations of pointed fusion categories

We shall consider in this section a pointed fusion category $\mathcal{C}$, that is, all simple objects of $\mathcal{C}$ are invertible. Then there is an equivalence of fusion categories $\mathcal{C} \simeq \mathcal{C}(\Gamma, \omega)$, where $\Gamma = G(\mathcal{C})$ is the group of isomorphism classes of invertible objects in $\mathcal{C}$, $\omega : \Gamma \times \Gamma \times \Gamma \rightarrow k^*$ is an invertible normalized 3-cocycle and $\mathcal{C}(\Gamma, \omega) = \text{Vec}_G^\omega$ is the category of finite dimensional $\Gamma$-graded vector spaces with associativity constraint induced by $\omega$.

4.1. Group actions on $\mathcal{C}(\Gamma, \omega)$ and equivariantizations. Let $\mathcal{C} = \mathcal{C}(\Gamma, \omega)$ and let $G$ be a finite group. An action $\rho : G \rightarrow \text{Aut}_G^\omega \mathcal{C}$ of $G$ on $\mathcal{C}$ is determined by an action by group automorphisms of $G$ on $\Gamma$, that we shall indicate by $x \mapsto g_x$, $x \in \Gamma$, $g \in G$, and two maps $\tau : G \times \Gamma \times \Gamma \rightarrow k^*$, and $\sigma : G \times G \times \Gamma \rightarrow k^*$, satisfying

$$\omega(x, y, z) = \omega(g_x g_y, g_z) \tau(g; x, y) \tau(g; x, y) \frac{\sigma(h, l; x) \sigma(g h l; x)}{\sigma(g h l; x) \sigma(g h; l; x)^{-1}} = 1,$$

for all $x, y, z \in \Gamma$, $g, h, l \in G$.

We shall also assume that $\tau$ and $\sigma$ satisfy the additional normalization conditions $\tau(g; x, y) = \sigma(g, h; x) = 1$, whenever some of the arguments $g$, $h$, $x$ or $y$ is an identity.

The action $\rho : G \rightarrow \text{Aut}_G^\omega \mathcal{C}$ determined by this data is defined by letting $\rho^\omega(x) = g_x$, for all $g \in G$, $x \in \Gamma$, and $\rho^\omega = \text{id}$ on arrows, together with the following constraints:

$$\tau(g; x, y) = \sigma(g, h; x)^{-1} \text{id}_{g_y g_x}, \quad (\rho^\omega)^2(x, y) = \tau(g; x, y)^{-1} \text{id}_{g_y g_x}, \quad \rho^\omega_0 = \text{id},$$

for all $g, h \in G$, $x, y \in \Gamma$. See [16, Section 7].

4.2. Fusion rules for $\mathcal{C}(\Gamma, \omega)^G$. Let us denote $\sigma_x(g, h) := \sigma(g, h; x)$ and $\tau_{x, y}(g) := \tau(g; x, y)$, $x, y \in \Gamma$, $g \in G$.

For all $x \in \Gamma$ and $g \in G_x$ we let the isomorphism $\epsilon_x : g_x = x \rightarrow x$ to be the identity of $x$. Therefore, the cocycle $\tilde{\alpha}_x : G_x \times G_x \rightarrow k^*$ defined by (2.1) is given by

$$\tilde{\alpha}_x(g, h) = \sigma_x(g, h)^{-1},$$

for all $g, h \in G_x$.

It follows from Corollary 2.13 that the set of isomorphism classes of simple objects of $\mathcal{C}^G$ is parameterized by isomorphism classes of pairs $(g, \pi)$, where $g$ runs over the orbits of the action of $G$ on $\Gamma$ and $\pi$ is an irreducible projective representation of the inertia subgroup $G_y \subseteq G$ with factor set $\sigma_y$.

Let $\mathcal{O}$ be a $G$-orbit in $G/G_y \times G/G_z$ corresponding to a double coset $D \in G_y \backslash G \backslash G_z$. Then $\mathcal{O} = \mathcal{O}(x, y, g)$, for any $g \in G$, and $\mathcal{O}$ contains at most one $G_U$-orbit, $\mathcal{O}(x, y, t, s)$, $t^{-1} s \in D$, such that $\text{Hom}_G(x, t x \otimes s z) \neq 0$. Indeed, the condition $\text{Hom}_G(x, t x \otimes s z) \neq 0$ amounts in this case to $x = t y z$. Thus, for all $e \neq g \in G/G_U$, $x \neq g y z = g t y z$.

In addition, if $x = t y z$, then $G_y \cap s G_z = G_x$, therefore $G_x \cap G_y \cap s G_z = G_y \cap s G_z$. 

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In the projective representation of $\text{Rep}_G \cong \text{Hom}_C(x, y^* z) \cong k$, defined in Lemma 3.3, the action of an element $g \in y^* z$ is nothing but scalar multiplication by $\tau_{y^* z}(g)^{-1}$.

As a consequence of Theorem 3.3, the following theorem gives the fusion rules for the category $C(\xi, \omega)^G$.

**Theorem 4.1.** Let $x, y, z \in \Gamma$ and let $\delta, \pi, \gamma$ be irreducible projective representations of the inertia subgroups $G_x, G_y, G_z$ with factor sets $\sigma_x, \sigma_y, \sigma_z$, respectively. Then the multiplicity of $S_{x, \delta}$ in the tensor product $S_{y, \pi} \otimes S_{z, \gamma}$ is given by the formula

$$\sum_{D \in G_y \setminus G_x} \sum_{x = y^* z} m_{\xi(G_x, G_y, G_z)}(\delta | G_x \cap G_y) \cdot (\pi | G_y \cap G_z) \cdot (\tau_{y^* z}^{-1} | G_z).$$

**Example 4.2.** Consider a cocentral abelian exact sequence of Hopf algebras $k \to k^\Gamma \to H \to kG \to k$, where $\Gamma$ and $F$ are finite groups. As special case of [12] Proposition 3.5], there is an action of $G$ on the category $C = C(\Gamma, 1)$ of finite dimensional representations of $k^\Gamma$ such that $\text{Rep} H \cong C^G$ as fusion categories (see [12] Remark 2.1]). In this situation, the formula for the fusion rules of $\text{Rep} H$ given by Theorem 4.1 specializes to the formula obtained by C. Goff in [8, Theorem 4.5]. See Section 5.

4.3. Braided group-theoretical fusion categories. Recall that a fusion category is called group-theoretical if it is Morita equivalent to a pointed fusion category. In view of [10] Theorem 7.2, a braided fusion category is group-theoretical if and only if it is an equivariantization of a pointed fusion category.

More precisely, it was shown in [11] Theorem 5.3] that every braided group-theoretical fusion category is equivalent to an equivariantization $C(\xi)^G$ of a crossed pointed fusion category $C(\xi)$ associated to a quasi-abelian 3-cocycle $\xi$ on a finite crossed module $(G, X, \partial)$, under a canonical action of $G$ on $C(\xi)$.

Recall that a finite crossed module $(G, X, \partial)$ consists of a finite group $G$ acting by automorphisms on a finite group $X$, and a group homomorphism $\partial : X \to G$ such that

$$\partial(y) = x y x^{-1}, \quad \partial(\gamma) = g \partial(x)g^{-1}, \quad g \in G, \ x, y \in X,$$

where $x \mapsto x^g, x \in X, g \in G$, denotes the action of $g$ on $X$.

A quasi-abelian 3-cocycle $\xi$ on $(G, X, \partial)$ is a quadruple $\xi = (\omega, \gamma, \mu, c)$, where $\omega : X \times X \times X \to k^\times$ is a 3-cocycle, $\gamma : G \times X \times X \to k^\times, \mu : G \times X \times X \to k^\times$ and $c : X \times X \to k^\times$ are maps satisfying the compatibility conditions in [11] Definition 3.4].

As a fusion category $C(\xi) = C(X, \omega)$, and the action of $G$ on $C(\xi)$ is determined by the action of $G$ on $X$ and formulas [11], with respect to $\sigma_x(g, h) := \gamma(g, h; x), \tau_{x, y}(g) := \mu(g; x, y)^{-1}, x, y \in X, g, h \in G$. See [11] Subsection 4.1].

Theorem 4.1 gives thus the fusion rules in the category $C(\xi)^G$ in terms of group-theoretical data determined by the crossed module $(G, X, \partial)$ and the quasi-abelian 3-cocycle $\xi$, entailing the determination of the fusion rules in any braided group-theoretical fusion category.

**Example 4.3.** Let $\omega : G \times G \times G \to k^\times$ be a 3-cocycle on $G$. Consider the crossed module $(G, G, \text{id})$ with respect to the adjoint action of $G$ on itself. The quadruple $\xi = (\omega^{-1}, \gamma^{-1}, \mu^{-1}, 1)$ is a quasi-abelian 3-cocycle on $(G, G, \text{id})$, where $\gamma$ and $\mu$ are
defined in the form
\[ \gamma(g, h; x) = \frac{\omega(g, h, x)\omega(ghx^{-1}g^{-1}, g, h)}{\omega(g, h, x^{-1}, y)}, \]
\[ \mu(g; x, y) = \frac{\omega(g, xg^{-1}, g, y)}{\omega(g, xg^{-1}, g, y)} \]
for all \( g, h, x, y \in G \).

The equivariantization \( \mathcal{C}(\xi)^G \) is equivalent to the category \( \text{Rep} D^{\omega}G \) of finite dimensional representations of the twisted quantum double \( D^{\omega}G \) introduced in \cite{Furusawa}. See \cite{Furusawa} Lemma 6.3]

Simple objects of \( \mathcal{C}(\xi)^G \) are parameterized by \( S_{x, \pi} \), where \( x \) runs over a set of representatives of conjugacy classes of \( G \) and \( \pi \) is an irreducible projective representation of the centralizer \( Z(x) \) of \( x \) in \( G \) with factor set \( \gamma_x \). Theorem \cite{Furusawa} gives the following formula for the multiplicity of \( S_{x, \pi} \) in the tensor product \( S_{y, \pi} \otimes S_{z, \gamma} \):

\[
\sum_{D \in Z(y) \otimes Z(s)} \sum_{t^{-1} \in D \atop z = t y t^{-1} s z s^{-1}} m_{1Z(y) t^{-1} \cap a Z(z) s^{-1}} (\delta, t) \pi \otimes \gamma \mu (-t y t^{-1}, s z s^{-1})^{-1}).
\]

We point out that the fusion rules for the category \( \text{Rep} D^{\omega}G \) were also determined in \cite{Furusawa} Section 5.

5. Appendix

In this Appendix we give a brief account of the results on projective representations used in the paper. See for instance \cite{Furusawa}.

Let \( G \) be a finite group and let \( \tilde{\alpha} : G \times G \to k^* \) be a (normalized) 2-cocycle on \( G \), that is,
\[
\tilde{\alpha}(g, h)\tilde{\alpha}(gh, t) = \tilde{\alpha}(g, ht)\tilde{\alpha}(h, t), \quad \tilde{\alpha}(g, e) = 1 = \tilde{\alpha}(e, g), \quad \forall g, h, t \in G.
\]

A projective representation \( \pi \) of \( G \) with factor set \( \tilde{\alpha} \) on a vector space \( V \) is a map \( \pi : G \to \text{GL}(V) \), such that
\[
\pi(e) = \text{id}_V, \quad \pi(gh) = \tilde{\alpha}(g, h)\pi(g)\pi(h), \quad \forall g, h \in G.
\]

In other words, \( \pi \) is a representation of the twisted group algebra \( k\tilde{\alpha}G \) on the vector space \( V \). We shall also use the notation \( V_\pi = V \) to indicate such a projective representation.

Two projective representations \( \pi \) and \( \pi' \) of \( G \) are called (projectively) equivalent if there is a linear isomorphism \( \phi : V_\pi \to V_{\pi'} \) and a map \( f : G \to k^* \) such that \( \phi\pi(g) = f(g)\pi'(g)\phi \), for all \( g \in G \). In this case we shall use the notation \( \pi' \simeq \pi \).

If \( \pi' \simeq \pi \), then the associated cocycles \( \tilde{\alpha} \) and \( \tilde{\alpha}' \) are related by
\[
\tilde{\alpha}(g, h) = \tilde{\alpha}'(g, h)f(h)f(gh)^{-1}, \quad g, h \in G,
\]
that is, \( \tilde{\alpha} \) and \( \tilde{\alpha}' \) are cohomologous cocycles, and thus they belong to the same cohomology class \( \alpha \in H^2(G, k^*) \). We shall also call \( \pi \) an \( \alpha \)-projective representation. Note that the map \( f : G \to k^* \) induces an algebra isomorphism \( \tilde{f} : k\tilde{\alpha}G \to k\tilde{\alpha}'G \) in the form \( \tilde{f}(g) = f(g)g \), for all \( g \in G \). Thus \( \pi \) and \( \pi' \) are equivalent projective representations if and only if \( V_\pi \simeq f^*(V_{\pi'}) \) as \( k\tilde{\alpha}G \)-modules.

Let \( \pi \) and \( \pi' \) be projective representations of \( G \) with factor sets \( \tilde{\alpha} \) and \( \tilde{\alpha}' \), respectively. The tensor product \( \pi \otimes \pi' \) is the projective \( \tilde{\alpha}\tilde{\alpha}' \)-representation on the vector space \( V_\pi \otimes V_{\pi'} \) defined by \( (\pi \otimes \pi')(g)(u \otimes v) = \pi(g)u \otimes \pi'(g)v \). In particular, if \( \pi \) is a representation of \( G \), then \( \pi \otimes \pi' \) is again a projective representation with factor set \( \tilde{\alpha} \).
Lemma 5.1. Let $\tilde{\alpha} : G \times G \to k^*$ be a cocycle and let $V$ be a $k_G$-module. Suppose $V = \bigoplus_{x \in X} V_x$ is a grading of $V$ by a set $X$ and assume that there is a transitive action of $G$ on $X$, $G \times X \to X$, $(g, x) \mapsto g x$, such that $g.V_x = V_{g x}$, for all $g \in G$, $x \in X$. 

If $\pi_1$ and $\pi_1'$ are projective representations projectively equivalent to $\pi$ and $\pi'$, respectively, then the tensor products $\pi_1 \otimes \pi_1'$ and $\pi \otimes \pi'$ are projectively equivalent. Further, suppose that $\pi'$ is a one-dimensional representation, that is, a linear character of $G$. Then $\pi$ and $\pi \otimes \pi'$ are projectively equivalent via the canonical isomorphism $\phi : V_\pi \to V_\pi \otimes k$, $v \mapsto v \otimes 1$, and the map $f : G \to k^*$ given by $f(g) = \pi'(g)^{-1}$, for all $g \in G$.

A nonzero projective representation $\pi : G \to GL(V)$ of $G$ is called irreducible if $0$ and $V$ are the only subspaces of $V$ which are invariant under $\pi(g)$, for all $g \in G$. Hence, $\pi$ is irreducible if and only if it is not projectively equivalent to a projective representation $\rho$ of the form

$$\rho(g) = \begin{pmatrix} \pi_1(g) & * \\ 0 & \pi_2(g) \end{pmatrix}, \quad g \in G,$$

where $\pi_1$ and $\pi_2$ are nonzero projective representations, or, equivalently, if $V$ is a simple $k_G G$-module, where $\tilde{\alpha}$ is the factor set of $\pi$ [9, Theorem 3.2.5].

Let $\pi : G \to GL(V)$ be a projective representation of $G$ with factor set $\tilde{\alpha}$. Since the group algebra $k_G G$ is semisimple, then $V = V_\pi$ is completely reducible, that is, $V_\pi \cong V_{\pi_1} \oplus \cdots \oplus V_{\pi_n}$, where $V_i$ is a simple $k_G G$-module, for all $i = 1, \ldots, n$. If $\pi'$ is an irreducible projective representation with factor set $\tilde{\alpha}'$, then $\pi'$ is called a constituent of $\pi$ if $\pi'$ is projectively equivalent to $\pi_1$ for some $1 \leq i \leq n$. In this case, the multiplicity (or intertwining number) of $\pi'$ in $\pi$ is defined as

$$m_G(\pi', \pi) := \dim \text{Hom}_{k_G G}(V_{\pi_1}, V_{\pi'}).$$

Observe that if $\pi'$ is a constituent of $\pi$, then the cocycles $\tilde{\alpha}'$ and $\tilde{\alpha}$ belong to the same class in $H^2(G, k^*)$. Letting $\tilde{\alpha}'df = \tilde{\alpha}$, with $f : G \to k^*$, we have that $m_G(\pi', \pi) := \dim \text{Hom}_{k_G G}(\tilde{f}(V_{\pi'}), V_{\pi})$, where $\tilde{f} : k_G G \to k_G G$ is the isomorphism associated to $f$.

The character of a projective representation $\pi : G \to GL(V)$ is defined as the map $\chi = \chi_V : G \to k^*$ given by $\chi(g) = \text{Tr}(\pi(g))$, for all $g \in G$. Let $\tilde{\alpha}$ be the factor set of $\pi$. If $\pi'$ is an irreducible projective representation of $G$ with factor set $\tilde{\alpha}$ and character $\chi'$, then the multiplicity of $\pi'$ in $\pi$ can be computed by the formula

$$m_G(\pi', \pi) = \langle \chi', \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\tilde{\alpha}(g^{-1}, g)} \chi'(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G^0} \frac{1}{\tilde{\alpha}(g^{-1}, g)} \chi'(g) \chi(g^{-1}),$$

$G^0 \subseteq G$ is the subset of $\tilde{\alpha}$-regular elements of $G$. See [9] Chapter 5.

Let $\tilde{\alpha} : G \times G \to k^*$ be a 2-cocycle and let $H \subseteq G$ be a subgroup. Consider a projective representation $W$ of $G$ with factor set $\tilde{\alpha}|_H$. The induced projective representation of $G$ is defined as $\text{Ind}_{k_G H}^G W = k_G G \otimes_{k_G H} W$. This is a projective representation of $G$ with factor set $\tilde{\alpha}$. By Frobenius reciprocity, we have natural isomorphisms

$$\text{Hom}_{k_G G}(\text{Ind}_{k_G H}^G W, V) \simeq \text{Hom}_{k_G H}(W, V|_H),$$

for every projective representation $V$ of $G$ with factor set $\tilde{\alpha}$, where $V|_H$ denotes the restricted projective representation of $H$.

The following lemma gives a characterization of those projective representations which are induced from a subgroup.

Lemma 5.1. Let $\tilde{\alpha} : G \times G \to k^*$ be a cocycle and let $V$ be a $k_G$-module. Suppose $V = \bigoplus_{x \in X} V_x$ is a grading of $V$ by a set $X$ and assume that there is a transitive action of $G$ on $X$, $G \times X \to X$, $(g, x) \mapsto g x$, such that $g.V_x = V_{g x}$, for all $g \in G$, $x \in X$. 


Let also \( y \in X \) and \( G_y \subseteq G \) the inertia subgroup of \( y \). Then \( V_y \) is a \( kG_y \)-module and \( V \simeq \text{Ind}^G_{G_y} V_y \) as \( kG \)-modules.

Proof. See [9, Theorem 5.2.1].

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