CONVERGENT SEQUENCES IN MINIMAL GROUPS

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Abstract. A Hausdorff topological group $G$ is minimal if every continuous isomorphism $f : G \to H$ between $G$ and a Hausdorff topological group $H$ is open. Clearly, every compact Hausdorff group is minimal. It is well known that every infinite compact Hausdorff group contains a non-trivial convergent sequence. We extend this result to minimal abelian groups by proving that every infinite minimal abelian group contains a non-trivial convergent sequence. Furthermore, we show that “abelian” is essential and cannot be dropped. Indeed, for every uncountable regular cardinal $\kappa$ we construct a Hausdorff group topology $\mathcal{T}_\kappa$ on the free group $F(\kappa)$ with $\kappa$ many generators having the following properties:

(i) $(F(\kappa), \mathcal{T}_\kappa)$ is a minimal group;
(ii) every subset of $F(\kappa)$ of size less than $\kappa$ is $\mathcal{T}_\kappa$-discrete (and thus also $\mathcal{T}_\kappa$-closed);
(iii) there are no non-trivial proper $\mathcal{T}_\kappa$-closed normal subgroups of $F(\kappa)$.

In particular, all compact subsets of $(F(\kappa), \mathcal{T}_\kappa)$ are finite, and every Hausdorff quotient group of $(F(\kappa), \mathcal{T}_\kappa)$ is minimal (that is, $(F(\kappa), \mathcal{T}_\kappa)$ is totally minimal).

We denote by $\mathbb{N}$ the set the set of natural numbers.

Let $X$ be a topological space. A convergent sequence in $X$ is a sequence $S = \{x_n : n \in \mathbb{N}\}$ of points of $X$ such that there exists a point $x \in X$ (called the limit of $S$) so that $S \setminus U$ is finite for every open subset $U$ of $X$ containing $x$. (We also say that $S$ converges to $x$.) A sequence $S$ is non-trivial provided that the set $S$ is infinite.

The identity element of a group $G$ is denoted by 1. When $G$ is abelian, the additive notation is used, and so 1 is replaced by the zero element 0 of $G$.

1. Results

Our starting point in this manuscript is the following folklore fact.

Fact 1.1. Every infinite compact Hausdorff group contains a non-trivial convergent sequence.

This result is a consequence of the theorem of Ivanovskii [14] and Kuz'minov [15] that every compact group is dyadic. We refer the reader to [20] for the proof of Fact 1.1 based on Michael’s selection theorem in the spirit of [25].

In Fact 1.1 compactness cannot be weakened to pseudocompactness or countable compactness, even in the abelian case. Indeed, there exists an example (in ZFC) of a pseudocompact abelian group without non-trivial convergent sequences [21]. Furthermore, there are numerous consistent examples of countably compact abelian groups

1991 Mathematics Subject Classification. Primary: 22A05; Secondary: 22C05, 54A10, 54A20, 54A25, 54D25, 54H11.

Key words and phrases. convergent sequence, minimal group, totally minimal group, compact group, abelian group, free group.

The author was partially supported by the Grant-in-Aid for Scientific Research no. 19540092 by the Japan Society for the Promotion of Science (JSPS).
without non-trivial convergent sequences, see \cite{12, 11, 16, 23, 3, 9, 24, 7}. However, the existence of a countably compact group without non-trivial convergent sequences in ZFC alone remains a major open problem in the area \cite[Problem 22]{8}.

Another well-known generalization of compactness in the class of topological groups is related to the fundamental fact that the topology of a Hausdorff compact space $X$ is a minimal element in the set of all Hausdorff topologies on the set $X$.

**Definition 1.2.** A Hausdorff group topology $\mathcal{T}$ on a group $G$ is called minimal provided that every Hausdorff group topology $\mathcal{T}'$ on $G$ such that $\mathcal{T}' \subseteq \mathcal{T}$ satisfies $\mathcal{T}' = \mathcal{T}$. In such a case the pair $(G, \mathcal{T})$ is called a minimal group.

The notion of a minimal group was introduced independently by Choquet (see Do¨ıtchinov \cite{10}) and Stephenson \cite{22}. We refer the reader to \cite{6, 4} for additional information on minimal groups.

While compactness cannot be replaced by pseudocompactness or countable compactness in the statement of Fact \ref{FA1}, our first result demonstrates that compactness can be weakened to minimality provided that the group in question is commutative.

**Theorem 1.3.** Every infinite minimal abelian group contains a non-trivial convergent sequence.

The particular version of Theorem \ref{TH1} for countably compact groups has been announced without proof on page 393 of \cite{8} (see the text preceding \cite[Problem 23]{8}).

As usual, we say that a group $G$ is non-trivial provided that $|G| \geq 2$.

Our second result shows that the word “abelian” in Theorem \ref{TH1} is essential and cannot be omitted.

**Theorem 1.4.** For every uncountable regular cardinal $\kappa$ there exists a Hausdorff group topology $\mathcal{T}_\kappa$ on the free group $F(\kappa)$ with $\kappa$ many generators having the following properties:

(i) $(F(\kappa), \mathcal{T}_\kappa)$ is a minimal group;
(ii) every subset of $F(\kappa)$ of size less than $\kappa$ is $\mathcal{T}_\kappa$-discrete (and thus also $\mathcal{T}_\kappa$-closed);
(iii) there are no non-trivial proper $\mathcal{T}_\kappa$-closed normal subgroups of $F(\kappa)$.

In particular,

(a) all compact subsets of $(F(\kappa), \mathcal{T}_\kappa)$ are finite, and
(b) every Hausdorff quotient group of $(F(\kappa), \mathcal{T}_\kappa)$ is minimal; that is, $(F(\kappa), \mathcal{T}_\kappa)$ is totally minimal.

2. Proof of Theorem \ref{TH1}

**Lemma 2.1.** An infinite subgroup of a compact metric group has a non-trivial convergent sequence.

**Proof.** Assume that $G$ is an infinite subgroup of a compact metric group $K$. Then $G$ cannot be discrete, and thus the identity 1 of $G$ is a non-isolated point of $G$. Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing local base at 1. By induction on $n \in \mathbb{N}$ choose $x_n \in U_n \setminus \{x_0, \ldots, x_{n-1}\}$. Then $\{x_n : n \in \mathbb{N}\}$ is a non-trivial sequence in $G$ converging to 1. \hfill \Box
Definition 2.2. [17] [22] A subgroup \( G \) of a topological group \( K \) is said to be essential in \( K \) provided that \( G \cap H \) is a non-trivial subgroup of \( K \) for every non-trivial closed normal subgroup \( H \) of \( K \).

If \( K \) is abelian, then every subgroup \( H \) of \( K \) is normal, and so the word “normal” can be omitted in the above definition.

The notion of an essential subgroup is a crucial ingredient of the so-called “minimality criterion”, due to Prodanov and Stephenson [17] [22], describing dense minimal subgroups of compact groups.

Fact 2.3. ([17] [22]; see also [5] [6]) A dense subgroup \( G \) of a Hausdorff compact group \( K \) is minimal if and only if \( G \) is essential in \( K \).

The straightforward proof of the following lemma is left to the reader.

Lemma 2.4. If \( G \) is an essential subgroup of an abelian topological group \( K \), then
\[
K[p] = \{ x \in K : px = 0 \} \subseteq G
\]
for every prime number \( p \).

Lemma 2.5. Let \( I \) be an infinite set, \( \{ K_i : i \in I \} \) a family of non-trivial topological groups and \( G \) an essential subgroup of the product \( K = \prod_{i \in I} K_i \). Then \( G \) has a non-trivial convergent sequence.

Proof. We identify each \( K_i \) with the closed normal subgroup
\[
\{ 1 \} \times \cdots \times \{ 1 \} \times K_i \times \{ 1 \} \times \cdots \times \{ 1 \}
\]
of \( K \), where \( K_i \) occupies the \( i \)th place. For each \( i \in I \), use essentiality of \( G \) in \( K \) to fix \( g_i \in G \cap K_i \) with \( g_i \neq 1 \). Since \( K_i \cap K_j = \{ 1 \} \) whenever \( i, j \in I \) and \( i \neq j \), it follows that \( \{ g_i : i \in I \} \) is a a faithfully indexed family of elements of \( G \). Choosing a faithfully indexed subset \( \{ i_n : n \in \mathbb{N} \} \) of \( I \), we obtain an infinite sequence \( \{ g_{i_n} : n \in \mathbb{N} \} \) of elements of \( G \) converging to \( 1 \).

Lemma 2.6. An essential subgroup of a non-trivial Hausdorff compact torsion-free abelian group contains a non-trivial convergent sequence.

Proof. Assume that \( G \) is an essential subgroup of a non-trivial compact torsion-free abelian group \( K \). Since \( K \) is torsion-free, the Pontryagin dual of \( K \) is divisible, and from [13, Theorem 25.8] we conclude that there exists a sequence of cardinals \( \{ \sigma_p : p \in \mathbb{P} \cup \{ 0 \} \} \) such that
\[
K = \hat{\mathbb{Q}}^{\sigma_0} \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p^{\sigma_p},
\]
where \( \hat{\mathbb{Q}} \) denotes the Pontryagin dual of the discrete group \( \mathbb{Q} \) of rational numbers, \( \mathbb{P} \) is the set of all prime numbers, and \( \mathbb{Z}_p \) denotes the group of \( p \)-adic integers.

If the product (1) can be (re-)written as a product of infinitely many non-trivial topological groups, then the conclusion of our lemma follows from Lemma 2.5. In the remaining case \( K \) is metrizable being a finite product of compact metric groups. Since \( K \) is non-trivial and \( G \) is essential in \( K \), there exists \( g \in G \cap K \) with \( g \neq 0 \). Since \( K \) is torsion-free, \( g \) has an infinite order, and so \( G \) is an infinite group. Applying Lemma 2.1, we obtain a non-trivial convergent sequence in \( G \).
Proof of Theorem 1.3. Assume that $G$ is an infinite minimal abelian group. Then its completion $K$ is a compact Hausdorff abelian group [18] (see also [6]). Moreover, $G$ is essential in $K$ by Fact 2.3. We consider two cases, depending on the size of the torsion part

$t(K) = \{ x \in K : nx = 0 \text{ for some } n \in \mathbb{N} \setminus \{0\} \}$

of $K$.

Case 1. $t(K)$ is uncountable. Then the $p$-rank $r_p(K)$ of $K$ must be uncountable for some $p \in \mathbb{P}$. In particular, $K[p]$ is uncountable. Being a closed subgroup of the compact group $K$, the group $K[p]$ is compact. Hence $K[p]$ contains a non-trivial convergent sequence by Fact 1.1. Finally, $K[p] \subseteq G$ by Lemma 2.4.

Case 2. $t(K)$ is at most countable. Then $U = K \setminus (t(K) \setminus \{0\})$ is a $G_\delta$-subset of $K$ containing 0. Therefore, there exists a closed $G_\delta$-subgroup $N$ of $K$ satisfying $N \subseteq U$ (see, for example, [13, Chapter II, Theorem 8.7] or [1]). In particular, $N \cap t(K) = \{0\}$. This means that $N$ is torsion-free.

If $N \neq \{0\}$, then $N$ is a non-trivial compact abelian group. Since $G$ is essential in $K$, $G \cap N$ is essential in $N$. Since $N$ is torsion-free, Lemma 2.6 yields that $G \cap N$ (and thus $G$ as well) has a non-trivial convergent sequence.

If $N = \{0\}$, then $\{0\}$ is a $G_\delta$-subset of $K$, and so $K$ is metrizable. Applying Lemma 2.1 we obtain a non-trivial convergent sequence in $G$. \hfill \Box

3. Proof of Theorem 1.4

The construction in this section is inspired by an old construction of the author [19].

Given a set $X$, the symbol $S(X)$ denotes the symmetric group of $X$, i.e., the set of all bijections of the set $X$ with the composition of maps as the group operation. We equip $S(X)$ with the topology of pointwise convergence on $X$ whose base is given by the family

$\mathcal{W}(X) = \{ W(X, Z, \varphi) : Z \text{ is a finite subset of } X \text{ and } \varphi : Z \to X \text{ is an injection} \}$,

where

(2) $W(X, Z, \varphi) = \{ f \in S(X) : f \upharpoonright Z = \varphi \}$.

As usual, an ordinal $\alpha$ is considered to be the set consisting of all smaller ordinals; that is, $\alpha = \{ \beta : \beta < \alpha \}$. In what follows, $F(\alpha)$ denotes the free group with the alphabet $\alpha$. For special emphasis, we use $*_{\alpha}$ to denote the group operation of $F(\alpha)$ and $e_{\alpha}$ to denote the identity element of $F(\alpha)$.

Fix an uncountable regular cardinal $\kappa$. For $\gamma \in \kappa + 1$ define

(3) $T_\gamma = \{ (\alpha, \beta) \in (\kappa \setminus \omega) \times \gamma : \beta < \alpha \}$

and

(4) $X_\gamma = \bigcup_{\alpha \in \gamma} \{ \alpha \} \times F(\alpha)$.

For every $\gamma \in \kappa \setminus \omega$ we have $|T_{\gamma+1}| = |\gamma|$, so we can fix an injection $j_\gamma : T_{\gamma+1} \to \gamma$.

Claim 1. The unique homomorphism $J_\gamma : F(T_{\gamma+1}) \to F(\gamma)$ extending $j_\gamma$ is an injection.
For each $\gamma \in \kappa \setminus \omega$, the family $H_\gamma$ of all bijections of $X_\gamma$ that move only finitely many elements of $X_\gamma$, is dense in $S(X_\gamma)$ and has size $|X_\gamma| = |\gamma|$, so we can fix an enumeration

\[(5) \quad H_\gamma = \{h_{\gamma \beta} : \beta \in \gamma\}.
\]

For $(\alpha, \beta) \in T_\kappa$ define $f_{\alpha, \beta} \in S(X_\kappa)$ by

\[(6) \quad f_{\alpha, \beta}(\gamma, g) = \begin{cases} h_{\alpha \beta}(\gamma, g), & \text{for } \gamma \in \alpha \\ (\gamma, g \ast_\gamma j_\gamma(\alpha, \beta)), & \text{for } \gamma \in \kappa \setminus \alpha \end{cases} \text{ for } (\gamma, g) \in X_\kappa.
\]

Define

\[(7) \quad Y_\kappa = \{f_{\alpha, \beta} : (\alpha, \beta) \in T_\kappa\} \subseteq S(X_\kappa),
\]

and let $G_\kappa$ to be the subgroup of $S(X_\kappa)$ generated by $Y_\kappa$. Define the map $\theta : T_\kappa \to Y_\kappa$ by

\[(8) \quad \theta(\alpha, \beta) = f_{\alpha, \beta} \text{ for } (\alpha, \beta) \in T_\kappa,
\]

and let $\Theta : F(T_\kappa) \to G_\kappa$ be the unique homomorphism extending $\theta$.

**Claim 2.** $\Theta(g)(\gamma, e_\gamma) = (\gamma, J_\gamma(g))$ whenever $\gamma \in \kappa \setminus \omega$ and $g \in F(T_\gamma)$.

**Proof.** The conclusion of our claim obviously holds for the identity element of $F(T_\gamma)$, so we will assume that $g$ is not the identity of $F(T_\gamma)$. Then there exist $n \in \mathbb{N}$, $(\alpha_k, \beta_k) : k \leq n \in T_\gamma$ and $\{\varepsilon_k : k \leq n\} \subseteq \{-1, 1\}$ such that

$$g = \prod_{k=0}^{n} (\alpha_k, \beta_k)^{\varepsilon_k}.$$  

Together with (8) this yields

\[(9) \quad \Theta(g) = \Theta\left(\prod_{k=0}^{n} (\alpha_k, \beta_k)^{\varepsilon_k}\right) = \prod_{k=0}^{n} \theta(\alpha_k, \beta_k)^{\varepsilon_k} = \prod_{k=0}^{n} (f_{\alpha_k, \beta_k})^{\varepsilon_k} = f_{\alpha_n, \beta_n}^{\varepsilon_n} \circ f_{\alpha_{n-1}, \beta_{n-1}}^{\varepsilon_{n-1}} \circ \ldots \circ f_{\alpha_0, \beta_0}^{\varepsilon_0}.
\]

From (9) and (6) we get

$$\Theta(g)(\gamma, e_\gamma) = f_{\alpha_n, \beta_n}^{\varepsilon_n} \circ f_{\alpha_{n-1}, \beta_{n-1}}^{\varepsilon_{n-1}} \circ \ldots \circ f_{\alpha_0, \beta_0}^{\varepsilon_0}(\gamma, e_\gamma) = f_{\alpha_n, \beta_n}^{\varepsilon_n} \circ f_{\alpha_{n-1}, \beta_{n-1}}^{\varepsilon_{n-1}} \circ \ldots \circ f_{\alpha_1, \beta_1}^{\varepsilon_1}(\gamma, j_\gamma(\alpha_0, \beta_0)^{\varepsilon_0}) \ldots$$

$$= f_{\alpha_n, \beta_n}^{\varepsilon_n}\left(\gamma, \prod_{k=0}^{n-1} j_\gamma(\alpha_k, \beta_k)^{\varepsilon_k}\right)$$

$$= \left(\gamma, \prod_{k=0}^{n} j_\gamma(\alpha_k, \beta_k)^{\varepsilon_k}\right) = (\gamma, J_\gamma(g)).$$

\[\square\]

**Claim 3.** $\Theta : F(T_\kappa) \to G_\kappa$ is an isomorphism.
Proof. Since \( Y_\kappa \) generates \( G_\kappa \) and \( \Theta(T_\kappa) = \theta(T_\kappa) = Y_\kappa \) by (7) and (8), it follows that \( \Theta \) is a surjection.

To prove that \( \Theta \) is an injection, assume that \( g \in F(T_\kappa) \) and \( \Theta(g) \) is the identity map of \( S(X_\kappa) \). From (3) we have
\[
(10) \quad F(T_\kappa) = \bigcup_{\alpha \in \kappa \setminus \omega} F(T_\alpha),
\]
and so there exists some \( \gamma \in \kappa \setminus \omega \) with \( g \in F(T_\gamma) \). Since \( \Theta(g) \) is the identity map of \( S(X_\kappa) \), from Claim 2 we get \( J_\gamma(g) = e_\gamma \). Then \( g = 1 \) by Claim 1. Therefore, \( \ker \Theta = \{1\} \), and so \( \Theta \) is an injection. \( \square \)

Claim 4. \( \Theta(F(T_\gamma)) \) is discrete for every \( \gamma \in \kappa \setminus \omega \).

Proof. For each \( g \in F(T_\gamma) \) let \( \varphi_g : \{ (\gamma, e_\gamma) \} \to X_\kappa \) be the map defined by
\[
(11) \quad \varphi_g(\gamma, e_\gamma) = (\gamma, J_\gamma(g)),
\]
so that \( W(X_\kappa, \{ (\gamma, e_\gamma) \}, \varphi_g) \in \mathcal{W} \).

(i) For every \( g \in F(T_\gamma) \) one has
\[
\Theta(g)(\gamma, e_\gamma) = (\gamma, J_\gamma(g)) = \varphi_g(\gamma, e_\gamma)
\]
by Claim 2 and (11), which yields
\[
\Theta(g) \in W(X_\kappa, \{ (\gamma, e_\gamma) \}, \varphi_g).
\]

(ii) Suppose that \( g_0, g_1 \in F(T_\gamma) \) and \( g_0 \neq g_1 \). Then \( J_\gamma(g_0) \neq J_\gamma(g_1) \) by Claim 4 which together with (11) yields \( \varphi_{g_0}(\gamma, e_\gamma) \neq \varphi_{g_1}(\gamma, e_\gamma) \), and thus
\[
W(X_\kappa, \{ (\gamma, e_\gamma) \}, \varphi_{g_0}) \cap W(X_\kappa, \{ (\gamma, e_\gamma) \}, \varphi_{g_1}) = \emptyset.
\]

Since \( \mathcal{W} \) is a base for the topology of \( S(X_\kappa) \), from (i) and (ii) we conclude that the family
\[
\{ W(X_\kappa, \{ (\gamma, e_\gamma) \}, \varphi_g) : g \in F(T_\gamma) \}
\]
witnesses that the set \( \Theta(F(T_\gamma)) \) is discrete. \( \square \)

Claim 5. If \( D \subseteq G_\kappa \) and \( |D| < \kappa \), then \( D \) is discrete.

Proof. Since \( \Theta \) is a bijection by Claim 3, \( |\Theta^{-1}(D)| = |D| < \kappa \). By (3), \( F(T_\lambda) \subseteq F(T_\mu) \) whenever \( \omega \leq \lambda < \mu < \kappa \), so using (10) and regularity of \( \kappa \) we can find \( \gamma \in \kappa \setminus \omega \) such that \( \Theta^{-1}(D) \subseteq F(T_\gamma) \) and so \( D \subseteq \Theta(F(T_\gamma)) \). Now Claim 4 applies. \( \square \)

Claim 6. \( Y_\kappa \) is dense in \( S(X_\kappa) \). In particular, \( G_\kappa \) is dense in \( S(X_\kappa) \).

Proof. Let \( Z \) be a finite subset of \( X_\kappa \) and \( \varphi : Z \to X_\kappa \) be an injection. From (4) we get \( X_\kappa = \bigcup_{\gamma \in \kappa} X_\gamma \), and \( X_\lambda \subseteq X_\mu \) whenever \( \omega \leq \lambda < \mu < \kappa \). Since \( \kappa \) is uncountable and \( Z \cup \varphi(Z) \) is a finite subset of \( X_\kappa \), we have
\[
(12) \quad Z \cup \varphi(Z) \subseteq X_\alpha
\]
for some \( \alpha \in \kappa \setminus \omega \). The bijection \( \varphi \) between two finite subsets \( Z \) and \( \varphi(Z) \) of \( X_\alpha \) can be extended to a bijection of the whole \( X_\alpha \). Therefore, \( W(X_\alpha, Z, \varphi) \) is a non-empty open subset of \( S(X_\alpha) \). Since \( H_\alpha \) is dense in \( S(X_\alpha) \), using (5) we can find \( \beta \in \alpha \) such that \( h_{\alpha\beta} \in W(X_\alpha, Z, \varphi) \). That is, \( h_{\alpha\beta}(\gamma, g) = \varphi(\gamma, g) \) for every \( (\gamma, g) \in Z \). Combining
this with (12) and (6), we conclude that $f_{\alpha,\beta}(\gamma, g) = h_{\alpha,\beta}(\gamma, g) = \varphi(\gamma, g)$ whenever $(\gamma, g) \in Z$. Together with (2) and (7) this yields

$$f_{\alpha,\beta} \in W(X_\kappa, Z, \varphi) \cap Y_\kappa \neq \emptyset.$$  

Since $\mathcal{W}(X_\kappa)$ is a base for the topology of $S(X_\kappa)$, we conclude now that $Y_\kappa$ is dense in $S(X_\kappa)$. □

Proof of Theorem 1.4. Since $|T_\kappa| = \kappa$, the groups $F(\kappa)$ and $F(T_\kappa)$ are isomorphic. Combining this with Claim 3, we conclude that the groups $F(\kappa)$ and $G_\kappa$ are isomorphic. Let $\mathcal{T}_\kappa$ be the topology on $F(\kappa)$ obtained by transferring the subgroup topology that $G_\kappa$ inherits from $S(X_\kappa)$ via the isomorphism between $F(\kappa)$ and $G_\kappa$.

Every dense subgroup of an infinite symmetric group $S(X)$ is minimal and has no proper non-trivial closed normal subgroups [2]. Combining this with Claim 4, we get (i) and (iii). Claim 5 yields (ii). Finally, (a) follows from (ii), and (b) follows from (iii). □

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