A Parameter Eigenstructure Assignment Design with Any Desired State Vector in Linear Time-Invariant System

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Abstract. The parametric design problem for eigenstructure assignment with arbitrary expected state vector is studied via state feedback in linear time invariant systems. The main idea of the considered problem is to design a parametric form of the state feedback, so that the designed closed-loop system has the desired eigenvalues, the desired eigenvectors and arbitrary expected state vector. Based on the parametric solution for a class of Sylvester matrix equation, the parametric expression of all the state feedback gain matrices is given in this paper. The free parameters including in the state feedback gain matrix can be further used to meet other performance requirements for the control system design. Finally, a simulation experiment is given to illustrate the effectiveness and simplicity of the proposed parametric method for the state feedback eigenstructure assignment problem with arbitrary expected state vector in the linear time invariant systems.

1. Introduction

Eigenstructure assignment is one of the important research problems in the field of linear control systems. And there are many research results on eigenstructure assignment in control field. The application of robust eigenstructure assignment based on μ analysis in flight control law design is studied [1]. The application of robust control based on eigenstructure assignment method in aircraft attitude control is studied [2]. Using Rosenbrock system matrix pencil to calculate the output zero subspace of linear time invariant system, the robust eigenstructure assignment problem is solved [3]. Based on the eigenstructure assignment method, a top-down control method for heterogeneous multi-agent formation is proposed [4]. Based on the eigenstructure assignment, a method is proposed to incorporate the measured modal data into the existing fine finite element model [5]. By establishing two general parameter solutions of this kind of matrix equation, two complete parameterization methods for eigenstructure assignment problems are given [6]. A method to determine the structure and minimum order of dynamic output feedback is proposed based on the given characteristic structure [7]. An effective multi-stage parameterization method is proposed for eigenstructure assignment of linear multivariable systems [8].

As one of the important achievements on eigenstructure assignment of linear control system, Duan’s papers present a complete parameterization method of eigenstructure assignment based on solving Sylvester matrix equation of linear system. Then, based on the proposed parametric eigenstructure assignment, its application for the control design problems is studied [9-14]. In this
paper, the design of parameter eigenstructure assignment for any desired state vector in linear time invariant systems is studied.

2. Problem Statement

Consider the following linear time invariant systems:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(1)

where the state vector \( x \in \mathbb{R}^n \), the input vector \( u \in \mathbb{R}^r \), and matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times r} \) are known; the following assumptions are meted:

1. Matrix pair \( (A, B) \) satisfies 
   \[ \text{rank}(A - sI - B) = n. \]

2. Matrix \( B \) satisfies \( \text{rank}B = r. \)

Apply the following state feedback to the system (1)

\[ u = Kx, \quad K \in \mathbb{R}^{r \times n} \]  

(2)

and get the closed-loop system of (1)

\[ \dot{x}(t) = (A + BK)u(t) \]  

(3)

Problem DSV (Desired State Vector): For system (1) and arbitrary expected vector \( f(\neq 0) \in \mathbb{R}^n \). Design a state feedback \( u = Kx \) to satisfy \( \lim_{t \to \infty} x(t) = cf \), for any initial state vector \( x(0) \) of system (3) and a constant \( c \in \mathbb{R}^n \).

3. Preliminaries

As the controllability of the matrix pair \( (A, B) \), two unimodular matrices \( P(s) \in \mathbb{R}^{n \times n} \) and \( Q(s) \in \mathbb{R}^{(n+r) \times (n+r)} \) satisfy

\[ P(s)[A - sI - B]Q(s) = [0 \quad I] \]  

(4)

Multiply the inverse of \( P(s) \) on both sides of equation (4), and get

\[ [A - sI - B]Q(s) = [0 \quad P^{-1}(s)] \]

Partition \( Q(s) \) as \( Q(s) = [Q_1(s) \quad Q_2(s)] \), where \( Q_i(s) = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} \). Thus there holds

\( (A - sI)N(s) + BD(s) = 0 \) and \( [A - sI - B]Q_2(s) = P^{-1}(s) \). Assume that \( s_1, s_2, \ldots, s_n \) are the distinct and complex eigenvalues of system (3), and \( v_1, v_2, \ldots, v_n \) are the corresponding eigenvectors associated with the eigenvalues \( s_1, s_2, \ldots, s_n \). Then there holds

\[ (A + BK)v_i = s_i v_i, \quad i = 1, 2, \ldots, n \]  

(5)

Denote \( V = [v_1 \quad v_2 \quad \ldots \quad v_n] \) and \( J = \text{diag}(s_1 \quad s_2 \quad \ldots \quad s_n) \), (5) can be changed as

\[ (A + BK)V = VJ \]  

(6)

Denote

\[ W = KV \]  

(7)

Equation (6) can be changed as

\[ AV + BW = VJ \]  

(8)
All the matrices $V$ and $W$ in equation (8) are obtained as $V = [v_1 \ v_2 \ \cdots \ v_n]$ and $W = [w_1 \ w_2 \ \cdots \ w_n]$, where

$$v_i = N(s_i)g_i, \ w_i = D(s_i)g_i$$

(9)

And $g_i \in \mathbb{C}, \ i = 1,2,\ldots,n$ are a group of free parameter vectors satisfying the assumption:

(c) If $s_i = \overline{s_i}$, there holds $g_i = \overline{g_i}$.

Because $V$ is invertible, from (11) we can obtain

$$K = WV^{-1}$$

(10)

Theorem 1 [9-14]. For system (1) satisfying assumptions (a) and (b), assume that $s_1, s_2, \ldots, s_n$ are the distinct and complex eigenvalues of system (3), and $v_1, v_2, \ldots, v_n$ are the corresponding eigenvectors. Therefore, all gain matrices in state feedback (2) can be obtained by (10), where matrix columns of $V$ and $W$ are given by (9), respectively.

4. Solution to Problem DSV

On the basis of Theorem 1, Problem DSV can be solved.

Theorem 2. Consider system (1) and let $f \neq 0 \in \Re^n$ be arbitrary expected vector. Then there exists a state feedback as $u = Kx$ such that for arbitrary expected system state vector $x(0)$, $\lim_{t \to \infty} x(t) = cf$ for a constant $c \in \Re$ in Problem DSV. And the gain matrix $K$ can be given by

$$K = WV^{-1}$$

(11)

where

$$V = [v_1 \ v_2 \ \cdots \ v_n], \ W = [w_1 \ w_2 \ \cdots \ w_n]$$

(12)

and

$$v_i = N(s_i)g_i, \ w_i = D(s_i)g_i$$

(13)

And $g_i \in \mathbb{C}, \ i = 1,2,\ldots,n$ are a group of free parametric vectors.

Proof. The state responses of system (3) is

$$x(t) = e^{(A+BK)t}x(0)$$

(14)

Because there holds $(A+BK)V = VJ$. From (14), we can obtain $x(t) = e^{(A+BK)t}x(0) = Ve^{BV^{-1}}x(0)$. Denote

$$V = [v_1 \ v_2 \ \cdots \ v_n], V^{-1} = \begin{bmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{bmatrix} \text{ and } J = \text{diag}(s_1 \ s_2 \ \cdots \ s_n).$$

If the first eigenvalue $s_1 = 0$ and other eigenvalues satisfy $re(s_i) < 0, \ i = 2,3,\ldots,n$, then

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} Ve^{BV^{-1}}x(0) = v_1^*x(0) = cf$$

(15)

where $c = v_1^*x(0)$ and $v_1 = \overline{f}$. From (14) and (15), the matrix $K$ in Problem DSV is one particular solution of the matrix equation (8) in Theorem 1. Thus the matrix $K$ can be given by (11)-(13).

5. A Numerical Example

Consider a system (1) with
Assume that the desired eigenvalues are \( s_1 = 0 \), \( s_2 = -2 + 3j \), \( s_3 = s_2 \), the initial state vector is \( x(0) = \begin{bmatrix} 9 \\ 12 \\ 3 \end{bmatrix} \) and the arbitrary expected state vector is \( f = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \). According to Theorem 1 and Theorem 2, we have the following steps:

**Step 1.** Obtain \( \text{rank} [A - sf B] = \text{rank} \begin{bmatrix} -s - 5 & 1 & 0 & 0 & 0 \\ 0 & 1 - s & 0 & 0 & 1 \\ 1 & 1 & 1 - s & 0 & 1 \end{bmatrix} = 3 \) and \( \text{rank} B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 2 \). Thus assumptions (a) and (b) are satisfied.

**Step 2.** The matrices \( N(s) \in \mathbb{R}^{m \times n} \) and \( D(s) \in \mathbb{R}^{n \times n} \) satisfying \( (A - sf)N(s) + BD(s) = 0 \) are calculated as

\[
N(s) = \begin{bmatrix} 1 & 0 \\ s + 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad D(s) = \begin{bmatrix} -(s + 6) & s - 1 \\ (s + 5)(s - 1) & -1 \end{bmatrix}
\]

**Step 3.** Set \( g_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} \), \( g_2 = \begin{bmatrix} g_{21} \\ g_{22} \end{bmatrix} \), \( g_3 = g_2 \) and calculate the column vectors of the matrices

\[
V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}
\]

as follows \( v_1 = N(s_1)g_1, \quad v_2 = N(s_2)g_2, \quad v_3 = v_2 \).

**Step 4.** From equation \( v_1 = N(0)g_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} \) \( f = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \), obtain the solutions \( g_{11} = 1, \quad g_{12} = 10 \) in the parameter vector \( g_1 = \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix} \).

**Step 5.** Substitute \( g_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \) in Step 4 to obtain \( v_1 = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix} \) and \( w_1 = \begin{bmatrix} -16 \\ -15 \end{bmatrix} \).

**Step 6.** Calculate \( v_2 = \begin{bmatrix} g_{21} \\ (3 + 3i)g_{22} \end{bmatrix}, \quad v_3 = v_2, \quad w_2 = \begin{bmatrix} -(4 + 3i)g_{21} - (3 - 3i)g_{22} \\ 18g_{21} - 18g_{22} \end{bmatrix}, \quad w_3 = w_2 \). Thus we can obtain the eigenvector matrix \( V \) as follows \( V = \begin{bmatrix} 1 & g_{21} \\ 5 & (3 + 3i)g_{21} \\ 10 & (3 - 3i)g_{21} \end{bmatrix} \) and the matrix \( W \) as follows

\[
W = \begin{bmatrix} -16 & -(4 + 3i)g_{21} - (3 - 3i)g_{22} & -(4 + 3i)g_{21} - (3 - 3i)g_{22} \\ -15 & -18g_{21} - 18g_{22} & -18g_{21} - 18g_{22} \end{bmatrix}
\]
Particularly we assume that \( g_2 = \begin{bmatrix} 26+7i \\ 8-5i \end{bmatrix} \), \( g_3 = \begin{bmatrix} 26-7i \\ 8+5i \end{bmatrix} \). Thus

\[
V = \begin{bmatrix}
1 & 26+7i & 26-7i \\
5 & 57+99i & 57-99i \\
10 & 8-5i & 8+5i
\end{bmatrix},
W = \begin{bmatrix}
-16 & -92-67i & -92+67i \\
-15 & -476-121i & -476+121i
\end{bmatrix}
\]

Calculate the inverse matrix of \( V \) as follows

\[
V^{-1} = \begin{bmatrix}
-0.0499 & 0.0086 & 0.1007 \\
0.0235 + 0.0123i & -0.0017 - 0.0058i & -0.0015 + 0.0017i \\
0.0235 - 0.0123i & -0.0017 + 0.0058i & -0.0015 - 0.0017i
\end{bmatrix}
\]

And the parameter \( c = v_1^* x(0) = -0.0433 \).

Step 6. According to \( K = W V^{-1} \), obtain

\[
K = \begin{bmatrix}
-1.8811 & -0.5999 & -1.1119 \\
-18.6481 & 0.1119 & 0.3088
\end{bmatrix}
\]

Such that for the initial state vector \( x(0) = \begin{bmatrix} 9 \\ 12 \\ 3 \end{bmatrix} \), there holds \( \lim_{t \to \infty} x(t) = cf = \begin{bmatrix} -0.0433 \\ -0.2166 \\ -0.4333 \end{bmatrix} \).

In order to prove the effectiveness of the proposed algorithm DSV, the state response of the closed-loop system is shown in figure 1. As can be seen from figure 1, the closed-loop system is stable, and its state vector is close to the expected state vector, that is, \( \lim_{t \to \infty} x(t) = cf \) holds.

**Figure 1.** The three state responses of the system.

6. **Conclusion**

Based on the parametric solution of Sylvester matrix equation, a parameterization method for eigenstructure assignment of arbitrary expected state vectors via state feedback is proposed. The idea is to design a parametric state feedback so that the closed-loop system has both the expected eigenstructure pair and any expected final state vector. Free parameters included in the state feedback gain matrix can be further used to meet other performance requirements of control system design. Finally, a numerical example and its simulation results show that the proposed parametric design method is effective and simple for the eigenstructure assignment problems with arbitrary expected state vectors in linear systems.

**Acknowledgments**

This work is supported by National Nature Science Foundation under Grant (61663014).

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