Constraint on pulsar wind properties from induced Compton scattering off radio pulses

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Pulsar winds have longstanding problems in energy conversion and pair cascade processes, which determine the magnetization $\sigma$, the pair multiplicity $\kappa$, and the bulk Lorentz factor $\gamma$ of the wind. We study induced Compton scattering by a relativistically moving cold plasma to constrain the wind properties by imposing that radio pulses from the pulsar itself are not scattered by the wind, as was first studied by Wilson and Rees [D. B. Wilson and M. J. Rees, Mon. Not. R. Astron. Soc., 185, 297 (1978)]. We find that relativistic effects cause a significant increase or decrease of the scattering coefficient depending on scattering geometry. Applying this to the Crab, we consider the uncertainties of the inclination angle of the wind velocity with respect to the radio beam $\theta_{pl}$ and the emission region size $r_e$, which determines the opening angle of the radio beam. We obtain the lower limit $\gamma \gtrsim 10^{1.7} r_e/3 \theta_{pl}^{-1/2} (1 + \sigma)^{-1/4}$ ($r_e = 10^3 r_e, 3$ cm) at the light cylinder $r_{LC}$ for an inclined wind $\theta_{pl} > 10^{-2.7}$. For an aligned wind $\theta_{pl} < 10^{-2.7}$, we require $\gamma > 10^{2.7}$ at $r_{LC}$ and an additional constraint $\gamma > 10^{3.4} r_e^{-1/3} (1 + \sigma)^{-1/10}$ at the characteristic scattering radius $r_e = 10^{6.6} r_e, 3$ cm, within which the ‘lack of time’ effect prevents scattering. Considering the lower limit $\kappa \gtrsim 10^{6.6}$ suggested by recent studies of the Crab Nebula, for $r_e = 10^3$ cm, we obtain the most optimistic constraint $10^{1.7} \lesssim \gamma \lesssim 10^{3.9}$ and $10^{6.6} \lesssim \kappa \lesssim 10^{8.8}$, which are independent of $r$ when $\theta_{pl} \sim 1$ and $1 + \sigma \sim 1$ at $r_{LC}$.

Subject Index E11, E15, E33, J21

1. Introduction

Pulsar magnetospheres create pulsar winds through pair creation and particle acceleration [1]. Because pulsar winds are radiatively inefficient, it is difficult to constrain their properties. However, their properties are inferred from observations of the surrounding pulsar wind nebula (PWN) and pulsed emissions of the pulsar itself. Interestingly, a secular increase of their pulse period tells us their total energy output, $L_{\text{spin}}$. Because most of the spin-down power is converted into the pulsar wind, $L_{\text{spin}}$ constrains its properties as (see also Eq. (22))

$$\kappa \gamma (1 + \sigma) = 1.4 \times 10^{10} \left( \frac{L_{\text{spin}}}{10^{38} \text{ erg s}^{-1}} \right)^{1/2},$$

where $\kappa$ is the pair multiplicity ($e^\pm$ number flux normalized by the Goldreich–Julian number flux $N_{\text{GJ}}$), $\gamma$ is the bulk Lorentz factor, and $\sigma$ is the magnetization parameter (the ratio of the Poynting to the kinetic energy fluxes) of the pulsar wind, respectively. We used

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\[ \dot{N}_{\text{GJ}} \equiv 2 \pi r_{\text{pc}}^2 c n_{\text{GJ}}(r_{\text{pc}}) = \sqrt{6 c L_{\text{spin}} / \varepsilon}, \]

where \( r_{\text{pc}} \) is the polar cap radius, \( n_{\text{GJ}}(r_{\text{pc}}) \) is the Goldreich–Julian density at a magnetic pole, and the numerical factor two comes from the north and south magnetic poles. Pair cascade models within the magnetosphere of the Crab pulsar \( (L_{\text{spin}} = 4.6 \times 10^{38} \text{ erg s}^{-1}) \) predict \( \kappa \sim 10^4 \) with \( \gamma \sim 10^2 \) in the vicinity of the light cylinder \( r_{\text{LC}} \) (e.g. Refs. [2–4]). On the other hand, magnetohydrodynamic (MHD) models of the Crab Nebula reproduce its non-thermal emission from optical to \( \gamma \)-rays with \( \kappa \sim 10^4, \gamma \sim 10^{-3} \sim 10^{-2}, \) and \( \gamma \sim 10^6 \) [5–8]. Although \( \kappa \sim 10^4 \) in both models is consistent with particle number conservation, \( \sigma \) (and also \( \gamma \)) differs by many orders of magnitude, which is called the ‘\( \sigma \)-problem’ (cf. Ref. [9]).

It is noted that there is an additional problem of the pulsar wind properties (cf. Refs. [10,11]). Because the MHD models of the Crab Nebula do not explicitly account for the origin of radio-emitting particles, they may underestimate the pair multiplicity. Recent studies of the spectral evolution of PWNe showed \( \kappa > 10^6 \) for the Crab Nebula and \( \kappa > 10^5 \) for other PWNe (e.g. Refs. [11–14]). Although the origin of the low-energy particles that are responsible for the radio emission of PWNe is still an open problem, they are most likely to originate from the pulsar because of the continuity of the broadband spectrum and because of the radio structures apparently originating from the pulsar [15–17]. Thus there arises another problem on \( \kappa \) besides the \( \sigma \)-problem, while only the combination of \( \kappa \gamma (1 + \sigma) \) in Eq. (1) is firm.

In view of the \( \sigma \)- and \( \kappa \)-problems, it is interesting to consider other independent constraints on the physical conditions of pulsar winds. Wilson and Rees (1978, hereafter WR78) [18] considered induced Compton scattering off radio pulses by a pulsar wind. So far, it is thought that we have not observed the signature of scattering in radio spectra of pulsars, although we do not fully understand how scattering changes the radio spectrum (e.g., scattering by a non-relativistic plasma was studied in Refs. [19,20]). Observations suggest that the optical depth to induced Compton scattering is less than unity, and the radio spectrum is not changed. Based on this consideration, WR78 obtained the lower limit of the bulk Lorentz factor of the Crab pulsar wind \( \gamma > 10^4 \) at \( 10^3 r_{\text{LC}} \sim 10^{11} \text{ cm} \) away from the pulsar. Substituting Eq. (1), only for \( (1 + \sigma) \sim 1 \) at \( 10^3 r_{\text{LC}}, \) their conclusion is marginally consistent with the conclusion of \( \kappa \geq 10^6 \) \( \equiv \kappa_{\text{PWN}} \) obtained from the study of the Crab Nebula spectrum by Tanaka and Takahara [11,13].

The induced Compton scattering process has been studied for the application to high-brightness-temperature radio sources, such as pulsars (e.g. Refs. [18,21–23]), active galactic nuclei (e.g. Refs. [19,24,25]), and other sources (e.g. Refs. [26–28]). Induced Compton scattering is about a factor of \( \theta_{\text{hm}}^4 k_B T_b(\nu) / m_e c^2 \) times more effective compared with the spontaneous version in the rest frame of the plasma, where \( \theta_{\text{hm}} \leq 1 \) and \( T_b \) are the half-opening angle and the brightness temperature of a radio beam, respectively (see Eq. (15)). Note that the value of \( k_B T_b(\nu) / m_e c^2 \) can be larger than \( 10^{15} \) for the Crab pulsar (see Eq. (24)). However, for scattering by relativistically moving electrons, the scattering coefficient is modified by relativistic effects and, as we will see below, either an increase or a decrease is possible depending on the situations considered, e.g., the velocity \( \mathbf{u} = \gamma \mathbf{B} \) of the electrons and the inclination between an electron motion \( \mathbf{u} \) and a radio beam \( \mathbf{k} \), where \( \mathbf{k} \) is the wavenumber vector.

In this paper, we reconsider induced Compton scattering by a relativistically moving plasma and reevaluate the lower limit of the bulk Lorentz factor. Despite the strong dependence on scattering geometry, WR78 considered a specific scattering geometry where the pulsar wind is completely aligned with respect to the radio pulse beam and where \( \theta_{\text{hm}} \) of the radio beam is the widest value inferred from the observations. We consider rather general geometries of the system, such as the direction of the wind being inclined with respect to the radio pulse beam. Even if the direction of
pulsed radio emission is almost radial, the pulsar wind is likely to have a significant toroidal velocity just outside \( r_{LC} \), or its motion in the meridional plane is not strictly radial. As already noted by WR78, the scattering coefficient may be significantly reduced if the pulsar wind inclines with respect to the radio beam. For \( \theta_{bm} \), the scattering coefficient is reduced when the radio beam is narrow in the rest frame of the plasma. If this is the case, the lower limit of the bulk Lorentz factor of the pulsar wind may be reduced so as to be consistent with recent studies of the Crab Nebula spectrum.

While we focus on geometrical effects in this paper, we ignore the effects of the magnetic field and background photons, following WR78. The magnetic field effect may be important when the frequency of the photon at the plasma rest frame \( \nu' \) is smaller than the electron cyclotron frequency \( \nu_{cc} \) (e.g. Refs. [21,29]). For the Crab pulsar, although the magnetic field in the observer frame is about \( B_{\text{obs}} \sim 10^6 \text{ G} \) at the light cylinder (\( \nu_{cc} = 5.8 \times 10^{12} \text{ Hz} \) for the magnetic field of \( B' = 10^6 \text{ G} \) in the plasma rest frame), \( \nu_{cc} \) strongly depends on the magnetic field configuration and the direction of plasma motion in the observer frame. For example, if \( B_{\text{obs}} \perp \mathbf{u} \), we find \( B' = B_{\text{obs}}/\gamma' \) and \( \nu' = \nu/\delta_D \), where \( \delta_D \) is the Doppler factor. Basically, the magnetic field effect reduces the scattering cross section, i.e., smaller \( \gamma \) would be allowed. For the effect of background photons, Lyubarsky and Petrova [21] discussed that scattering off the background photons induced by the beam photons may be important. They discussed that the occupation number of the background photons increases exponentially, i.e., the beam photons may decrease accordingly, when the scattering optical depth to the background photons greatly exceeds unity, say \( \sim \exp(\kappa) \). They discussed that the occupation number of the background photons increases exponentially, i.e., the beam photons may decrease accordingly, when the scattering optical depth to the background photons greatly exceeds unity, say \( \sim \exp(\kappa) \). In this paper, we ignore background photons (\( \theta_{bm} < \theta \leq \pi \)), assuming that the occupation number of the beam photons is much larger than that of the background photons. If scattering off the background photons is efficient, scattering would be more efficient and larger \( \gamma \) would be required. These processes will be discussed in a separate paper.

In Sect. 2, we describe the scattering coefficient of induced Compton scattering by a relativistically moving plasma in a general geometry. We also show simple analytic forms of the scattering coefficient in some specific geometries. In general geometry, the scattering coefficient is written in an integral form and is obtained numerically in Appendix 4. In Sect. 3, we consider induced Compton scattering at pulsar wind regions, specifically applying to the Crab pulsar. We show the resultant lower limits of \( \gamma \) and also discuss the corresponding upper limits of the pair multiplicity \( \kappa \). We summarize the present results in Sect. 4.

2. Induced Compton scattering off a photon beam

Here, we express the scattering coefficient at a certain position \( \mathbf{x} \) and see that the scattering coefficient strongly depends on the geometry of scattering. The kinetic equation for a photon occupation number \( n(\mathbf{x}, \mathbf{k}, t) \) is expressed as (e.g. Refs. [21,30])

\[
\left( \frac{\partial}{\partial (ct)} + \mathbf{\Omega} \cdot \nabla \right) n(\mathbf{k}) = \int d^3 p \frac{f(p)}{k_1^2} \frac{d\sigma}{d\Omega}(p, \mathbf{k}, k_1) \left[ n(k_1)(1 + n(k)) \left( \frac{k_1}{k} \right)^2 - \delta(k - g(p, k_1)) - n(k)(1 + n(k_1))\delta(k_1 - g(p, k)) \right],
\]

where \( \mathbf{\Omega} = k/k \), \( f(p) \) is the distribution function of the plasma, and \( d\sigma/d\Omega \) is the differential scattering cross section, respectively. Note that when the electron is initially at rest, the recoil \( g \) is expressed as \( g(k, \xi) = k/[1 + k\lambda_c(1 - \cos \xi)] \), where \( \lambda_c = \hbar/m_ec \) represents the Compton wavelength for an electron and \( \xi \) is the angle between incident and scattered photons. We omit arguments \( \mathbf{x} \) and \( t \) in
Eq. (2) and in this section. The terms 1 + n represent spontaneous and induced scattering terms, and we only consider the induced process below, assuming \( n \gg 1 \).

### 2.1. Scattering coefficient

The scattering coefficient of induced Compton scattering is the right-hand side of Eq. (2) divided by \( n(k) \) (e.g. Ref. [31]). Equation (2) is simplified by the following three approximations. (I) Plasma is cold, and moves with the velocity \( u = \gamma \beta \) (the bulk Lorentz factor \( \gamma = (1 - \beta^2)^{-1/2} \)). (II) The magnetic field is weak enough to satisfy the condition \( \nu_{ce} < \nu' \), where \( \nu_{ce} \) and \( \nu' \) are the electron cyclotron frequency and the frequency of an incident photon in the plasma rest frame, respectively (e.g. Ref. [21]). (III) Photons are in the Thomson regime, i.e., \( k \lambda_e \ll 1 \) (cf. Ref. [30]). Condition (III) is a good approximation for scattering off radio photons by plasma of \( \gamma \ll 10^10 \). In the observer frame, Eq. (2) then becomes (e.g. Refs. [18,21])

\[
\left( \frac{\partial}{\partial (ct)} + \mathbf{\Omega} \cdot \nabla \right) n(k) = n(k) \frac{3}{8\pi} \sigma_T n_{pl} \int \frac{d\Omega_1}{\gamma^3 D_1^2} R(\mathbf{\Omega}, \mathbf{\Omega}_1, \mathbf{u})(1 - \mu) \frac{\partial k_{1}^2 n(k)}{\partial k_{1}} \bigg|_{k_1 = \frac{\beta}{\gamma} k} ,
\]

where

\[
1 - \mu = 1 - \mathbf{\Omega} \cdot \mathbf{\Omega}_1 ,
\]

\[
D = 1 - \beta \cdot \mathbf{\Omega} ,
\]

\[
D_1 = 1 - \beta \cdot \mathbf{\Omega}_1 ,
\]

\[
R(\mathbf{\Omega}, \mathbf{\Omega}_1, \mathbf{u}) = 1 + \left( 1 - \frac{1 - \mu}{\gamma^2 D_1} \right)^2 ,
\]

and \( n_{pl} \) is a number density of plasma. \( R(\mathbf{\Omega}, \mathbf{\Omega}_1, \mathbf{u}) \) is order unity (\( 1 \leq R \leq 2 \)) and \( \sigma_T \) is the Thomson scattering cross section. The scattering coefficient contains an integral that depends on the occupation number itself and on the scattering geometry at \( x \), i.e., the directions of photons (\( \mathbf{\Omega} \) and \( \mathbf{\Omega}_1 \)) and the velocity of the plasma \( \mathbf{u} \). While WR78 performed this integral on a specific scattering geometry, we reevaluate it in more general geometries.

### 2.2. Geometry

The scattering geometry at a certain position \( x \) in the observer frame is depicted in Fig. 1. The photon beam with a half-opening angle \( \theta_{bm} \) is directed to an observer on the \( z \)-axis. The inclination angle of the plasma velocity is \( \theta_{pl} \). Note that the plasma should be depicted as a line rather than a cone on Fig. 1, i.e., zero opening angle, because we assume that the plasma is cold. However, we will see that there is a characteristic angle \( \gamma^{-1} \) around the plasma velocity and thus we associate the plasma with the cone of its half-opening angle \( \gamma^{-1} \) in the figures in this paper.

For the plasma, we express the velocity \( \mathbf{u} \) as

\[
\mathbf{u} = \gamma \beta \left( \sin \theta_{pl} \mathbf{e}_x + \cos \theta_{pl} \mathbf{e}_z \right).
\]

We assume that the occupation number of photons is uniform inside the beam and is expressed as

\[
n(k) = n(v, \mathbf{\Omega}) = n(v) H(\theta_{bm} - \theta),
\]

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Fig. 1. A sketch of scattering geometry. The photon beam of a half-opening angle $\theta_{bm}$ is toward the observer on the $z$-axis. The inclination angle of the cold plasma is $\theta_{pl}$ ($\phi_{pl} = 0$). Although the cold plasma beam should be described as a line, we associate it with the cone of a half-opening angle of $\gamma^{-1}$ in all the sketches for explanatory convenience.

where $H$ is Heaviside’s step function. The spectrum $n(\nu)$ is assumed to be a broken power-law form

$$n(\nu) = n_0 \left( \frac{\nu}{\nu_0} \right)^{p_1} \left[ \frac{1}{2} \left( 1 + \frac{\nu}{\nu_0} \right) \right]^{p_2-p_1},$$

where $p_1$ and $p_2$ are power-law indices of low- and high-frequency parts and $n_0$ is the occupation number at a break frequency $\nu_0$, respectively. The observed pulsar radio spectra correspond to $-7 \lesssim p_2 \lesssim -3$, and we require $p_1 > -3$ for the number density of photons to be finite at $\nu \to 0$. For the application in Sect. 3, we take $p_2 = -5$ and $\nu_0 = 10$ MHz, considering the radio observations.

We consider scattering off photons toward the observer, i.e., $\Omega = e_z$. The scattering coefficient $\chi$ at $x$ is expressed as

$$\chi(\nu, e_z) \equiv -\frac{3}{8\pi} n_{pl}\sigma_T \int_0^{2\pi} d\phi_1 \int_0^{\theta_{bm}} \sin \theta_1 d\theta_1 \frac{1-\mu}{\gamma^3 D_1^2} R(e_z, \Omega_1, \nu) \left( \frac{k_B T_0(\nu_1)}{m_e c^2} \frac{\partial \ln n(\nu_1) \nu_1^2}{\partial \ln \nu_1} \right)_{\nu_1 = \frac{\nu}{\nu_0}}^{\nu_1 = \frac{\nu}{\nu_0}}.$$

As for the conventional definition of the optical depth $d\tau = \chi dl$ for a path $l$ along the $z$-axis, we include a minus sign, where the occupation number decreases along the path for a positive value of $\chi$ and vice versa. The sign of $\chi$ can change with the sign of the function

$$S(\nu) \equiv \frac{\partial \ln n(\nu) \nu^2}{\partial \ln \nu} \approx \begin{cases} p_1 + 2 & \text{for } \nu \ll \nu_0, \\ p_2 + 2 & \text{for } \nu \gg \nu_0. \end{cases}$$

2.3. Analytic estimates

It is convenient to rewrite Eq. (11) by introducing the normalization

$$\chi_0 \equiv n_{pl}\sigma_T \frac{k_B T_0(\nu_0)}{m_e c^2}.$$
The scattering coefficient becomes

$$
\chi(\nu, e_z) = -\frac{3\chi_0}{8\pi\gamma^3} \int_{0}^{2\pi} \int_{0}^{\theta_{bm}} \sin \theta_1 d\theta_1 d\phi_1 \frac{1 - \mu}{D_1^2} R(e_z, \Omega_1, \nu) \left( \frac{T_b(v_1)}{T_b(v_0)} S(v_1) \right) v_1 = \frac{p}{\gamma \nu}
$$

$$
\equiv \chi_0 \gamma^{-3} I(\nu, \theta_{bm}, \theta_{pl}, \gamma),
$$

(14)

where the integral $I(\nu, \theta_{bm}, \theta_{pl}, \gamma)$ represents a geometrical effect. Note that $\chi$ contains a factor of $\gamma^{-3}$, which is independent of scattering geometries. The value of $I(\nu, \theta_{bm}, \theta_{pl}, \gamma)$ is obtained numerically in general and can take a wide range of values even for a fixed frequency. The numerical results of the integral $I(\nu)$ for different parameter sets $(\theta_{bm}, \theta_{pl}, \gamma)$ are described in Appendix 4 and are also briefly summarized in the last paragraph of this section. Below, we describe simple analytic forms of the integral $I(\nu)$ for some special cases. They help in understanding the dependence on $(\theta_{bm}, \theta_{pl}, \gamma)$ and turn out to be useful for applications in the next section.

We first see the non-relativistic limit $\beta \ll 1$ ($D, D_1 \sim 1$) where the $\theta_{pl}$-dependence can be neglected. Considering $\theta_{bm} < 1$, we obtain

$$
I_{NR}(\nu) \approx -\frac{3}{4} T_b(v) S(v) \int_{0}^{\theta_{bm}} \theta_1^3 d\theta_1 = -\frac{3}{16} \theta_{bm}^4 T_b(v) S(v),
$$

(15)

where we use $R(e_z, \Omega_1, \nu) \approx 1 + (1 - \theta_1^2/2)^2 \approx 2$. When the photon beam is narrow ($\theta_{bm} \ll 1$), the scattering coefficient can be small. This is because the number of photons that stimulate the scattering process decreases with $\theta_{bm}^2$ and another factor $\theta_{bm}^2$ comes from the recoil term $\propto 1 - \mu \approx \theta_1^2/2$. For typical values of $p_1$ and $p_2$, $|I_{NR}(\nu)|$ (i.e., $|\chi_{NR}(\nu)|$) has a peak and changes sign at $\nu \approx \nu_0$.

To see the relativistic effects, we expand $\sin \theta, \cos \theta$, and $\beta$ to second-order in $\theta_1, \theta_{pl}$, and $\gamma^{-1}$, i.e., we concern ourselves with the situations $0 \leq (\theta_{pl}, \theta_{bm}) \lesssim 1$ and $\gamma \gg 1$. The integrand is composed of the following three factors. (I) The solid angle (and the recoil) factor originates from the solid angle element $d\Omega_1$ and from the recoil term $1 - \mu$, and is expressed as

$$
(1 - \mu) \sin \theta_1 d\phi_1 d\theta_1 \approx \frac{1}{2} \theta_1^3 d\theta_1 d\phi_1.
$$

(16)

This factor has already appeared in the non-relativistic case (Eq. (15)). (II) The aberration factor originates from the Lorentz transformation of a solid angle element from the plasma rest frame to the observer frame, and is expressed as

$$
\frac{1}{D_1^2} \approx \frac{4\gamma^4}{(1 + \gamma^2 \psi_1^2)^2},
$$

(17)

where we introduce an angle $\psi_1$ between $\beta$ and $\Omega_1$, given by the approximation $\psi_1^2 = \theta_1^2 - 2\theta_1 \theta_{pl} \cos \phi_1 + \theta_{pl}^2$. (III) The frequency shift factor also originates from the Lorentz transformation of a frequency, and is expressed as

$$
\frac{D}{D_1} \approx \frac{1 + \gamma^2 \theta_{pl}^2}{1 + \gamma^2 \psi_1^2}.
$$

(18)

Analytic forms of the integral $I(\nu)$ presented below are explained by a simple combination of these three factors. We also show numerical results of the integral $I(\nu)$ for these cases in Figs. 2–4, where we adopt $p_1 = 3, p_2 = -5$, and $\gamma = 10^2$. Introducing normalized angles $\Theta_{bm} \equiv \gamma \theta_{bm}$ and $\Theta_{pl} \equiv \gamma \theta_{pl}$, it is easy to find that the integral $I(\nu)$ depends on $(\Theta_{bm}, \Theta_{pl})$ rather than separately on $\theta_{bm}, \theta_{pl}$, and $\gamma$. 

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Fig. 2. Plot of the integral $I(\nu, \theta_{bm}, \theta_{pl}, \gamma)$ in Eq. (14) (left) and a sketch of scattering geometry (right) in the ‘Narrow’ case ($1 > \Theta_{bm}^2 + \Theta_{pl}^2$). The plot shows absolute values $|I(\nu)|$ versus $\nu$ with $\gamma = 10^2$, $p_1 = 3$, and $p_2 = -5$ (lines a, b, and c), together with $T_b(\nu)/T_b(\nu_0)$ for comparison. Each line is for a different value of $(\Theta_{bm}, \Theta_{pl})$: ‘line a’ for $(10^{-1}, 10^{-1})$, ‘line b’ for $(10^{-2}, 10^{-2})$, and ‘line c’ for $(10^{-2}, 10^{-1})$, respectively. Note that ‘line a’ and ‘line b’ overlap since $I(\nu)$ is primarily determined by $\Theta_{bm}$, as seen in Eq. (19). The discontinuity found in each line is the frequency where the sign of $I(\nu)$ changes and the high-frequency side has a positive sign, while the low-frequency side has a negative sign for all lines. Note also that, in the right panel, the opening angle of the plasma cone (red in color) represents the $\gamma^{-1}$ cone and does not represent the velocity distribution (see Fig. 1 and the text).

Fig. 3. Plot of the integral $I(\nu, \theta_{bm}, \theta_{pl}, \gamma)$ in Eq. (14) (left) and a sketch of scattering geometry (right) in the ‘Inclined’ case ($\Theta_{pl}^2 > \Theta_{bm}^2 + 1$). Each line is for a different value of $(\Theta_{bm}, \Theta_{pl})$: ‘line a’ for $(1, 10)$, ‘line b’ for $(1, 10^2)$, and ‘line c’ for $(10^{-1}, 10)$, respectively and the other parameters are the same as in Fig. 2. Note that ‘line b’ and ‘line c’ overlap since $I(\nu)$ is primarily determined by the ratio $\Theta_{bm}/\Theta_{pl}$, as seen in Eq. (20).

We first consider the case $1 > \Theta_{bm}^2 + \Theta_{pl}^2$, where the narrow photon beam and $\boldsymbol{\Omega} = e_z$ are well inside the $\gamma^{-1}$ cone associated with the plasma, as shown in the right panel of Fig. 2. We call this case ‘Narrow’. In this case, we obtain $D_1^{-2} \approx 4\gamma^4$ and $D/D_1 \approx 1$, and then the integral $I(\nu)$ is approximated as

$$I_{\text{Narrow}}(\nu) \approx -\frac{3}{4} \frac{T_b(\nu)}{T_b(\nu_0)} S(\nu) \int_0^{\theta_{bm}} (4\gamma^4) \phi_1^2 d\phi_1 = -\frac{3}{4} \Theta_{bm}^2 \frac{T_b(\nu)}{T_b(\nu_0)} S(\nu), \quad (19)$$
where we use \( R(e_z, \Omega_1, u) \approx 1 + (1 - 2\Theta_1^2) \approx 2 \ (\Theta_1 \equiv \gamma \beta_1) \). This expression with \( \gamma \to 1 \) \((\Theta_{\text{bm}} \to \Theta_{\text{pl}})\) is almost the same as that of the non-relativistic case (Eq. (15)). For the ‘Narrow’ case, the aberration factor increases the integral \( I(\nu) \) by a factor of \( D_{1}^{-2} \sim 4\nu^{4} \) compared with \( I_{\text{NR}}(\nu) \) because the opening angle increases by a factor of \( \sim \gamma \) in the plasma rest frame, while the frequency shift is negligible \( (D/D_1 \approx 1) \). Note that \( \chi_{\text{Narrow}}(\nu) \) is a factor of \( \gamma \) larger than \( \chi_{\text{NR}}(\nu) \), accounting for the factor of \( \gamma^{-3} \) in Eq. (14). In the left panel of Fig. 2, we plot the numerical results of absolute values of the integral \( I(\nu) \) (Eq. (14)) as a function of \( \nu \), \( |I(\nu)| \) has a discontinuity because \( S(\nu) \) changes sign at \( \nu \sim \nu_{0} \), where \( I(\nu) > 0 \) (i.e., \( \chi(\nu) > 0 \)) for \( \nu > \nu_{0} \) and vice versa.

The next case is \( \Theta_{\text{pl}}^{2} > \Theta_{\text{bm}}^{2} + 1 \), where \( u \) is inclined with respect to \( \Omega \) and the associated cones do not overlap with \( \Omega \), as shown in the right panel of Fig. 3. We call this case ‘Inclined’. The integral \( I(\nu) \) also suffers from little frequency shift \( (D/D_1 \approx 1) \) and the aberration factor is approximated as \( D_{1}^{-2} \sim 4\Theta_{\text{pl}}^{-4} \). We obtain an approximated form of

\[
I_{\text{Inclined}}(\nu) \approx -\frac{3}{4} \frac{T_{b}(\nu)}{T_{b}(\nu_{0})} S(\nu) \int_{0}^{\Theta_{\text{bm}}} (4\Theta_{\text{pl}}^{-4})^{2} d\theta_{1} = -\frac{3}{4} \frac{\Theta_{\text{bm}}^{4}}{\Theta_{\text{pl}}^{4}} \frac{T_{b}(\nu)}{T_{b}(\nu_{0})} S(\nu),
\]

where we use \( R(e_z, \Omega_1, u) \approx 1 + (1 - 2\Theta_1^2)(\Theta_{\text{pl}}^{-4}) \approx 2 \). In the left panel of Fig. 3, we show the numerical results for the ‘Inclined’ case. The aberration factor decreases the integral \( I(\nu) \) by a factor of \( \Theta_{\text{pl}}^{-4} \) compared with \( I_{\text{NR}}(\nu) \). Note that \( \chi_{\text{Inclined}}(\nu) \) can be smaller than \( \chi_{\text{NR}}(\nu) \), as \( \chi_{\text{Inclined}}(\nu)/\chi_{\text{NR}}(\nu) \sim \gamma^{-3}\Theta_{\text{pl}}^{-4} \). For example, we find \( \chi_{\text{Inclined}}(\nu) \sim \gamma^{-3}\chi_{\text{NR}}(\nu) \) for \( \Theta_{\text{pl}} \sim 1 \), while \( \chi_{\text{Inclined}}(\nu) \sim \gamma \chi_{\text{NR}}(\nu) \) for \( \Theta_{\text{pl}} \sim 1 \).

The scattering geometry satisfying \( \Theta_{\text{bm}} > 1 > \Theta_{\text{pl}} \) is sketched in the right panel of Fig. 4, where the \( \gamma^{-1} \) cone of plasma contains \( \Omega \) and is well within the photon beam. We call this case ‘Wide’. Note that although we take \( \Theta_{\text{pl}} = 0 \) in Fig. 4 and in Eq. (21), we will find that the integral \( I(\nu) \) behaves in a similar way for \( \Theta_{\text{bm}} > 1 > \Theta_{\text{pl}} \neq 0 \) in Appendix 4. For \( \Theta_{\text{pl}} = 0 \), the frequency shift factor is approximated as \( D/D_1 \approx (1 + \Theta_1^2)^{-1} \leq 1 \). The aberration factor behaves as \( D_{1}^{-2} \sim 4\nu^{4}/(1 + \Theta_1^2)^2 \) and makes the angular distribution of the photon beam almost isotropic in the plasma rest frame. A simple analytic form is found for the frequency range \( \nu > (1 + \Theta_1^2)\nu_{0} \approx \Theta_{\text{bm}}^{2} \nu_{0} \), where we use the expressions \( T_{b}(\nu_{1}) \approx T_{b}(\nu_{0})/(1 + \Theta_1^2)^{p_{2}+1} \) and \( S(\nu_{1}) \approx p_{2} + 2 \). We obtain

\[
I_{\text{Wide}}(\nu) \approx -\frac{3}{4} \frac{T_{b}(\nu)}{T_{b}(\nu_{0})} S(\nu) \int_{0}^{\Theta_{\text{bm}}} (4\Theta_{\text{pl}}^{-4})^{2} d\theta_{1}
\]

\[
\approx -\frac{3}{4} \frac{\Theta_{\text{bm}}^{4}}{\Theta_{\text{pl}}^{4}} \frac{T_{b}(\nu)}{T_{b}(\nu_{0})} S(\nu),
\]

\[
(\text{for } \nu > (1 + \Theta_1^2)\nu_{0}).
\]
an approximated form

\[ I_{\text{Wide}}(v > \Theta_{\text{bm}}^2 v_0) \approx -\frac{3}{2} \left( \frac{v}{v_0} \right)^{p_2+1} (p_2 + 2) \int_0^{\Theta_{\text{bm}}} \frac{\Theta_1^2 d\Theta_1}{(1 + \Theta_1^2)^{p_2+3}} \]

where we take \( R(e_z, \Omega_1, u) \approx 1 + (1 - 2\Theta_1^2(1 + \Theta_1^2)^{-1})^2 \approx 1 \) because the value varies in the range between 1 \( \leq R(e_z, \Omega_1, u) \leq 2 \) for \( 0 \leq \theta_1 \leq \theta_{\text{bm}} \). \( I_{\text{Wide}}(v) \) is order unity at \( v \sim \Theta_{\text{bm}}^2 v_0 \). The numerical results are shown in Fig. 4. Figure 4 shows that \( I_{\text{Wide}}(v) \) is approximated as \(-1 \) (order unity) even for \( v_0 < v < \Theta_{\text{bm}}^2 v_0 \). \( I_{\text{Wide}}(v < v_0) \) is approximated as \((T(v)/T(v_0))S(v)\), corresponding to Eq. (15) with \( \theta_{\text{bm}} \sim 1 \), i.e., almost isotropic. It is important to note that \( I_{\text{Wide}} \approx -1 \) can be used for applications in Sect. 3 rather than Eq. (21). Note that \( \chi_{\text{Wide}}(v) \) can also be smaller than \( \chi_{\text{NR}}(v) \), depending on \( p_2 \) and \( \theta_{\text{bm}} \) in a somewhat complex way because of the frequency shift.

There remains the geometry \( \Theta_{\text{bm}} > \Theta_{\text{pl}} > 1 \) where the cone of the plasma does not contain \( \Omega \) but is within the photon beam. We do not find an analytic form of the integral \( I(v) \) in this case. The numerical calculation in Appendix 4 shows that \(|I(v)| \) takes between \(|I_{\text{Inclined}}(v)|\) and \(|I_{\text{Wide}}(v)|\) for the frequency range \( v > v_0 \) in which we are interested in Sect. 3. Note that \(|I_{\text{Inclined}}(v)|\) gives the smallest value and \(|I_{\text{Wide}}(v)|\) gives the largest value in any geometries (\( \Theta_{\text{bm}}, \Theta_{\text{pl}} \)) for \( v > v_0 \). We give a detailed discussion including this exceptional geometry in Appendix 4.

3. Application to the Crab pulsar

We evaluate the optical depth to induced Compton scattering applying to the Crab pulsar. We require that the optical depth \( |\tau(v)| \) is less than unity and then we constrain the Crab pulsar wind properties \( \kappa, \gamma, \) and \( \sigma \).

3.1. Setup

We describe assumptions to estimate the normalization \( \chi_0 \) for the Crab pulsar. For a pulsar wind, three assumptions are made. (I) Almost all of the spin-down power \( L_{\text{spin}} \) goes to the pulsar wind. (II) The pulsar wind is a cold magnetized \( e^\pm \) flow whose bulk Lorentz factor is \( \gamma \). (III) The number density of the pulsar wind decreases with \( r^{-2} \), and we ignore structures in the pulsar wind, such as the current sheet (e.g. Ref. [32]). Now, the number density of the pulsar wind in the observer frame is

\[ n_{\text{pl}}(r) = \frac{L_{\text{spin}}}{4\pi r^2 c\beta_r \gamma m_e c^2 (1 + \sigma)} \]

\[ \sim 3.2 \times 10^{16} \gamma^{-1} (1 + \sigma)^{-1} \left( \frac{r}{10^8 \text{ cm}} \right)^{-2} \left( \frac{L_{\text{spin}}}{10^{38} \text{ erg s}^{-1}} \right) \text{ cm}^{-3}, \]

where we assume the radial velocity \( \beta_r \sim 1 \). Note that we obtain Eq. (1) from Eq. (22) by normalizing \( 4\pi r^2 c\beta_r n_{\text{pl}}(r) \) with \( \dot{N}_{\text{GJ}} \). Note also that a product \( \gamma (1 + \sigma) \) does not depend on \( r \) because we expect no particle production outside the light cylinder \( r_{\text{LC}} \), i.e., \( n_{\text{pl}} \propto r^{-2} \).

For radio pulses, uncertainty of the brightness temperature arises from the opening angle of the radio emission \( \theta_{\text{bm}} \). Following WR78, we assume that the emission is isotropic at \( r = r_e \), where \( r_e \) is the emission region size. The opening angle \( \theta_{\text{bm}}(r) \) is written as

\[ \theta_{\text{bm}}(r) \approx \frac{r_e}{r} \text{ for } r > r_e. \]

We adopt Eq. (23) for the opening angle of the radio pulse throughout this paper.
Fig. 5. The observed spectrum of the Crab pulsar in radio. Note that the emission at $v < 100\,\text{MHz}$ is no longer observed to be pulsed, most probably because of interstellar scattering. Therefore, the apparently rising spectrum around 100 MHz is not real. Since the high-frequency radio flux of the Crab pulsar is $F_v = 646(v/400\,\text{MHz})^{-3.1}\,\text{mJy}$ for $v > 400\,\text{MHz}$ [33,36], there seems a spectral break around 100 MHz. The low-frequency spectrum extends down to at least 5.6 MHz with a spectral index $\alpha = -2.09$ [37,38]. The fitted line in this range is $F_v \sim 50(v/100\,\text{MHz})^{-2}\,\text{Jy}$ for $v < 100\,\text{MHz}$. Observational data are taken from Refs. [33,37,38].

The brightness temperature is expressed as (e.g. Ref. [34])

$$\frac{k_B T_b(v)}{m_e c^2} = 1.7 \times 10^{16} \left( \frac{F_v}{\text{Jy}} \right) \left( \frac{d}{\text{kpc}} \right)^2 \left( \frac{v}{100\,\text{MHz}} \right)^{-2} \left( \frac{r_e}{10^7\,\text{cm}} \right)^{-2}, \quad (24)$$

where $F_v$ and $d$ are the flux density at a frequency $v$ and the distance to the object, respectively. WR78 adopted $r_e = 10^7\,\text{cm}$, which is estimated from the integrated pulse width $W_{50} = 3\,\text{msec}$ [33,34]. We study dependence on $r_e$ in Sect. 3.5. In Sect. 3.5, we will take $r_e = 10^3\,\text{cm}$, considering the ‘microbursts’ of which individual pulses from the Crab pulsar show nano–microsecond duration structures [35]. Note that $r_e = 10^3\,\text{cm}$ would also be considered as almost the minimum size of plasma to emit a coherent electromagnetic wave of the frequency $v = 100\,\text{MHz}$ ($c/v = 3 \times 10^2\,\text{cm}$).

Figure 5 shows the radio spectrum of the Crab pulsar. We assume $F_v \sim 50(v/100\,\text{MHz})^{p_2+3}\,\text{Jy}$ for $v_0 \leq v \leq 100\,\text{MHz}$ with $v_0 = 10\,\text{MHz}$ and $p_2 = -5$. Adopting $d = 2\,\text{kpc}$, $L_{\text{spin}} = 4.6 \times 10^{38}\,\text{erg}\,\text{s}^{-1}$ and the light cylinder radius $r_{LC} = 1.6 \times 10^8\,\text{cm}$ for the Crab pulsar, we obtain the normalization

$$\chi_{0,\text{Crab}}(v_0 = 10\,\text{MHz}, r) = 1.3 \times 10^{15} \gamma^{-1} (1 + \sigma)^{-1} \left( \frac{r}{r_{LC}} \right)^{-2} \left( \frac{r_e}{10^7\,\text{cm}} \right)^{-2}. \quad (25)$$

Although we used $v_0 = 10\,\text{MHz}$, we require $|\tau(v)| < 1$ at $v = 100\,\text{MHz}$ because the Crab pulsar spectrum (Fig. 5) is obviously unaffected by scattering in the range $v \geq 100\,\text{MHz}$.

On the assumptions made in this section, the scattering coefficient $\chi(v, r)$ is considered to be a rapidly decreasing function of $r$. We introduce the exponents $a$ and $b$ ($(a, b) > 0$) characterizing the $r$-dependence of the velocity $u(r)$ as $\gamma \propto r^a$ and $\theta_{\text{pl}} \propto r^{-b}$. Now, the $r$-dependence of $\chi(v, r)$
(Eq. (14)) is expressed as
\[
\begin{align*}
\chi_{\text{Narrow}} & \propto r^{-2} \theta_{\text{bm}}^4 \propto r^{-6}, \\
\chi_{\text{Inclined}} & \propto r^{-2} \gamma^{-4} \theta_{\text{pl}}^{-4} \theta_{\text{bm}}^4 \propto r^{-6+4(b-a)}, \\
\chi_{\text{Wide}} & \propto r^{-2} \gamma^{-4} \propto r^{-2-4a},
\end{align*}
\]  
\tag{26}
\]
where \( I_{\text{Wide}}(\nu) \approx -1 \) is used in this section because \( \nu_0 \lesssim \nu < \Theta_{\text{bm}}^2 \nu_0 \) (\( \nu_0 = 10 \text{ MHz} \) and \( \nu = 100 \text{ Hz} \)) is mostly attainable for the ‘Wide’ case (\( \Theta_{\text{bm}} > 1 \)). In Eq. (26), \( b - a < 1.25 \) is sufficient for \( \chi(\nu, r) \) to be considered as a rapidly decreasing function of \( r \). Otherwise we consider moderate values of \( a \) and \( b \), say, \( 0 < (a, b) \lesssim 1.25 \). Therefore, the choice of the innermost scattering radius is important for evaluating the optical depth.

Here, we consider scattering beyond the light cylinder \( r \geq r_{\text{LC}} \), because we do not know where the electron–positron plasma and the radio emission are produced inside the magnetosphere and because we do not take into account magnetic field effects, which may be important close to the pulsar. We evaluate the optical depth as
\[
\tau(\nu) = \int_{r_{\text{in}}}^{d} \chi(\nu, r) dr \\
\sim \chi(\nu, r = r_{\text{in}}) \Delta r,
\]
\tag{27}
\]
where \( r_{\text{in}} \) and \( \Delta r \) are the innermost scattering radius and the path length, respectively. In Eq. (27), we should not simply put \( r_{\text{in}} = \Delta r = r_{\text{LC}} \) because the path length \( \Delta r \) has a lower limit originating from the ‘lack of time’ effect, which we will discuss in the next subsection.

### 3.2. Characteristic scattering length

The ‘lack of time’ effect introduced by WR78 should be taken into account for the evaluation of \( r_{\text{in}} \) and \( \Delta r \) in Eq. (27). This is similar to the concept of the ‘coherence radiation length’ (e.g. Refs. [39, 40]). The normal treatment of scattering breaks down when an electron does not see one cycle of the electric field oscillation of radio waves. We determine this characteristic length \( l_c \) as follows. A cycle of the incident and scattered photons in the plasma rest frame is described as \( \Delta t' = \delta_D / \nu \), where \( \delta_D = (\gamma D)^{-1} \) or \( (\gamma D_1)^{-1} \) is the Doppler factor. The characteristic length \( l_c \) is the speed of light multiplied by the time interval \( \Delta t = \gamma \Delta t' \) in the observer frame. Using \( D^{-1} \approx 2\gamma^2 / (1 + \theta_{\text{pl}}^2) \) and \( D_1^{-1} \approx 2\gamma^2 / (1 + \psi_1^2) \) (\( \Psi_1^2 \equiv \gamma^2 \psi_1^2 \)), we obtain
\[
l_c(\nu, u, \Omega, \Omega_1) = \frac{c}{\nu} \max(D^{-1}, D_1^{-1})
\approx 2\gamma^2 \frac{c}{\nu} \times \begin{cases} 
\max(1, 1) & \text{for ‘Narrow’}, \\
\max(\theta_{\text{pl}}^{-2}, \theta_{\text{pl}}^{-2}) & \text{for ‘Inclined’}, \\
\max(1, (1 + \psi_1^2)^{-1}) & \text{for ‘Wide’},
\end{cases}
\]
\[
= 6 \times 10^2 \text{ cm} \left( \frac{\nu}{100 \text{ MHz}} \right)^{-1} \times \begin{cases} 
\gamma^2 & \text{for ‘Narrow’ and ‘Wide’}, \\
\theta_{\text{pl}}^{-2} & \text{for ‘Inclined’}.
\end{cases}
\]
\tag{28}
\]

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In this case, we take $\gamma_c$ is divided into two regions by the line $l_c(r) = r$, which satisfies $\gamma_{LC} < 10^{2.7}$ for $r \leq r_{LC}$, so that three different choices of $\gamma_{LC}$ and the exponent $a$ corresponding to points ‘A’, ‘B’, and ‘C’ in Fig. 6. Point ‘A’ corresponds to $\gamma_{LC} < 10^{2.7}$ with any values of the exponent $a$. Since $l_c(r_{LC}) < r_{LC}$ in this case, we take $r_{in} = \Delta r = r_{LC}$. Point ‘B’ corresponds to $\gamma_{LC} > 10^{2.7}$ with $a < 0.5$. The radio pulse is not scattered at $r_{LC}$ but beyond $r_{LC}$. Here, we introduce the characteristic scattering radius $r_c$, which satisfies $r_c = l_c(r_c) > r_{LC}$, so that we take $r_{in} = \Delta r = r_c = (10^{2.8} \gamma_{LC}^2 r_{LC}^{-2a})^{1/(1-2a)}$ cm.

$$l_c(r) \propto \frac{r}{r_{LC}} \left( \frac{r}{r_{LC}} \right)^{2a} \text{cm.} \quad (30)$$

In Fig. 6, we show the $l_c(r)$ vs. $r$ diagram. We do not consider the region $r < r_{LC}$. The region $r > r_{LC}$ is divided into two regions by the line $l_c(r) = r$, which corresponds to $\gamma_{LC} \approx 10^{2.7}$ and $a = 0.5$. Scattering off the radio pulse should be considered when $l_c(r) < r$ so that three different choices of $r_{in}$ are possible for different values of $\gamma_{LC}$ and the exponent $a$, corresponding to points ‘A’, ‘B’, and ‘C’ in Fig. 6. Point ‘A’ corresponds to $\gamma_{LC} < 10^{2.7}$ with any values of the exponent $a$. Since $l_c(r_{LC}) < r_{LC}$ in this case, we take $r_{in} = \Delta r = r_{LC}$. Point ‘B’ corresponds to $\gamma_{LC} > 10^{2.7}$ with $a < 0.5$. The radio pulse is not scattered at $r_{LC}$ but beyond $r_{LC}$. Here, we introduce the characteristic scattering radius $r_c$, which satisfies $r_c = l_c(r_c) > r_{LC}$, so that we take $r_{in} = \Delta r = r_c = (10^{2.8} \gamma_{LC}^2 r_{LC}^{-2a})^{1/(1-2a)}$ cm.

**Fig. 6.** The $l_c$-$r$ diagram for the ‘Narrow’ and ‘Wide’ cases (Eq. (30)). The region $r < r_{LC}$ (gray area) is not considered in this paper. When $l_c(r) > r$ (light blue area), scattering does not occur because of the ‘lack of time’ effect, while scattering should be considered in the region $l_c(r) < r$ (pink area). Three cases for $r_{in}$ (points ‘A’, ‘B’, and ‘C’) are possible by different behaviors of $l_c(r)$, i.e., $\gamma_{LC}$ and the exponent $a$ (see also Eq. (30)). $r_{in}$ becomes point ‘A’ when $\gamma_{LC} < 10^{2.7}$. For $\gamma_{LC} > 10^{2.7}$, $r_{in}$ is point ‘B’ when $a < 0.5$. While no $r_{in}$ exists for $a > 0.5$, because $\gamma$, i.e., $l_c(r)$, cannot be infinitely large, there must be point ‘C’ where $l_c(r) = r$ is satisfied.

$l_c$ is considered as a function of only $r$ through $\gamma(r)$ or $\theta_{pl}(r)$ for the given frequency $\nu = 100$ MHz. On the other hand, for the geometry $\Theta_{bm} > \Theta_{pl} > 1$, we obtain

$$\max(D^{-1}, D_1^{-1}) \approx \begin{cases} D^{-1} & \text{for } \theta^2_{pl} \leq \Psi^2_1, \\ D_1^{-1} & \text{for } \theta^2_{pl} > \Psi^2_1, \\ \geq D^{-1} \approx 2\theta_{pl}^{-2}. \end{cases} \quad (29)$$

We find that $l_c$ for this case is equal to or larger than that for the ‘Inclined’ case. Because $l_c$ depends on $\Omega_\text{1}$, we cannot separate integrals over $\Omega_\text{1}$ and $r$ in Eqs. (14) and (27). In this subsection, we limit the discussion about the ‘Narrow’, ‘Inclined’, and ‘Wide’ cases.

Now, we describe how we determine $r_{in}$ and $\Delta r$, taking into account the $r$-dependence of $l_c(r)$. Although we discuss only for the ‘Narrow’ and ‘Wide’ cases ($l_c \propto \gamma^2$), the same discussion is applicable to the ‘Inclined’ case ($l_c \propto \theta_{pl}^{-2}$) by replacing $\gamma$ with $\theta_{pl}^{-1}$. We set $\gamma(r) = \gamma_{LC}(r/r_{LC})^a$, where $\gamma_{LC}$ is the Lorentz factor at $r_{LC}$. Substituting this into Eq. (28), we obtain

$$l_c(r) = 6 \times 10^2 \gamma_{LC}^2 \left( \frac{r}{r_{LC}} \right)^{2a} \text{cm.} \quad (30)$$

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Table 1. The optical depth $|\tau(100\text{MHz})|$ at $r_{LC}$ ($\gamma_{LC} < 10^{-2.7}$ or $\theta_{pl,LC} > 10^{-2.7}$). Scattering geometries are classified by $u_{LC}$, i.e., $\gamma_{LC}$ and $\theta_{pl,LC}$. We take $r_c = 10^7 r_{e,7}$ cm.

| Geometry   | $\tau(100\text{MHz})$ | $u(r_{LC})$ |
|------------|------------------------|-------------|
| 'Narrow'   | $10^{4.9} r_{e,7}^2 \gamma_{e,7} (1 + \sigma_{LC})^{-1}$ | $\gamma_{LC} \lesssim 10^{1.2} r_{e,7}$ & $\gamma_{LC} < \theta_{pl,LC}$ |
| 'Inclined' | $10^{4.9} r_{e,7}^2 \gamma_{LC}^4 \theta_{pl,LC}^{-4} (1 + \sigma_{LC})^{-1}$ | $\theta_{pl,LC} \gtrsim 10^{-1} r_{e,7}$ & $\gamma_{LC} > \theta_{pl,LC}$ |
| 'Wide'     | $-10^{2.3} r_{e,7}^2 \gamma_{e,7} (1 + \sigma_{LC})^{-1}$ | $\gamma_{LC} \gtrsim 10^{1.2} r_{e,7}$ & $\gamma_{LC} < \theta_{pl,LC}$ |

Table 2. The optical depth $|\tau(100\text{MHz})|$ at $r_c$ ($\gamma_{LC} > 10^{-2.7}$ and $\theta_{pl,LC} < 10^{-2.7}$). Scattering geometries are classified by $u_c$. We take $r_c = 10^7 r_{e,7}$ cm.

| Geometry   | $\tau(100\text{MHz})$ | $u(r_c)$ |
|------------|------------------------|----------|
| 'Narrow'   | $10^{12} r_{e,7}^2 \gamma_{e,7}^{-10} (1 + \sigma_{e})^{-1}$ | $\gamma_{e} \gtrsim 10^{4.2} r_{e,7}$ & $\gamma_{e} < \theta_{pl,c}$ |
| 'Inclined' | $10^{12} r_{e,7}^2 \gamma_{c}^4 \theta_{pl,c}^{-4} (1 + \sigma_{e})^{-1}$ | $\theta_{pl,c} \lesssim 10^{-4} r_{e,7}$ & $\gamma_{c} > \theta_{pl,c}$ |
| 'Wide'     | $-10^{8.7} r_{e,7}^2 \gamma_{e,7}^{-6} (1 + \sigma_{e})^{-1}$ | $\gamma_{e} \lesssim 10^{4.2} r_{e,7}$ & $\gamma_{e} < \theta_{pl,c}$ |

For $\gamma_{LC} > 10^{-2.7}$ with $a \geq 0.5$, we obtain $l_{c}(r) > r$ everywhere beyond $r_{LC}$, i.e., the electron never sees one cycle of radio waves (red dot-dashed line). However, $\gamma(r)$ cannot be infinitely large, so there should exist a radius satisfying $r_{in} = l_{c}(r_{in}) > r_{LC}$ corresponding to point ‘C’. In this case, we also take $r_{in} = \Delta r = r_c$, the expression for which is different from that for $a < 0.5$. Therefore, $\gamma_{LC} = 10^{2.7}$ or $\theta_{pl,LC} = 10^{-2.7}$ is a critical value in determining which to adopt as $r_{in}$.

We consider whether the radio pulse can escape from scattering at the two radii $r_{LC}$ and $r_c$. Rather than using the exponents $a$ and/or $b$, it is convenient to introduce $\gamma_{c} \equiv \gamma(r_{c})$ and $\theta_{pl,c} \equiv \theta_{pl}(r_{c})$. We evaluate the optical depth by treating the velocities $u_{LC}$ and $u_c$, i.e., ($\gamma_{LC}$, $\theta_{pl,LC}$) and ($\gamma_{c}$, $\theta_{pl,c}$), as free parameters. The relationship between the exponent $a$ ($b$) and $\gamma_{c}$ ($\theta_{pl,c}$) will be discussed shortly in Sect. 3.3.3. Note that we indirectly obtain the characteristic scattering radius $r_c$ from Eq. (28) once $\gamma_{c}$ or $\theta_{pl,c}$ is obtained.

### 3.3. Constraints on Lorentz factor

The lower limits of $\gamma$ are obtained from the condition $|\tau(\nu)| < 1$ for a given $\theta_{pl}$. We evaluate the optical depth,

$$\tau(\nu) \sim \Delta r \chi_{0,\text{Crab}}(r_{in}) \gamma^{-3} I(\nu, \theta_{bm}, \theta_{pl}, \gamma),$$

at $\nu = 100\text{MHz}$. $\tau(100\text{MHz})$ strongly depends on $u_{LC}$ or $u_c$ (Tables 1 and 2). Below, we search the allowable region on the $\gamma$–$\theta_{pl}$ planes for $r_{in} = r_{LC}$ (Fig. 7) and for $r_{in} = r_c$ (Fig. 8), respectively. The results will be combined in Sect. 3.3.3.

For a given $r_c$, i.e., $\theta_{bm}(r_{in})$ (Eq. (23)), the scattering geometry is classified into four cases on the $\gamma$–$\theta_{pl}$ plane corresponding to the ‘Narrow’ ($1 > \Theta_{bm}^2 + \Theta_{pl}^2$), ‘Inclined’ ($\Theta_{pl}^2 > \Theta_{bm}^2 + 1$), and ‘Wide’ ($\Theta_{bm} > 1 > \Theta_{pl}$) cases, and the geometry satisfying $\Theta_{bm} > \Theta_{pl} > 1$. The first three geometries are studied in Sect. 2.3 and the expressions of $\tau(100\text{MHz})$ for them are obtained in Tables 1 and 2. For $\Theta_{bm} > \Theta_{pl} > 1$, $\tau(100\text{MHz})$ is not expressed by Eq. (31) because $l_{c}$ depends on $\Omega_{1}$, as already discussed in Eq. (29). Here, we infer the optical depth for $\Theta_{bm} > \Theta_{pl} > 1$ from the results of the other three cases. Thus, the $|\tau(100\text{MHz})| = 1$ lines at the $\Theta_{bm} > \Theta_{pl} > 1$ area in Figs. 7 and 8 (thick dashed lines) are not calculated but inferred.
Fig. 7. The $\gamma-\theta_{pl}$ diagram at $r_{LC}$ when $r_e = 10^7$ cm ($\theta_{bm}(r_{LC}) \approx 10^{-1.2}$). Choosing one point on the diagram specifies the pulsar wind velocity $u_{LC}$. Four areas divided by three lines $\gamma = \theta_{pl}^{-1}$, $\gamma = 10^{1.2}$, and $\theta_{pl} = 10^{-1.2}$ correspond to different scattering geometries, the ‘Narrow’ (lowermost area, outlined in red), ‘Inclined’ (rightmost area, outlined in green), and ‘Wide’ (left triangular area, outlined in yellow) cases and the same reason discussed in Eq. (26). Again, only the pulsar wind velocities also escape from scattering at $r_{in}$. The region above the lines $|\tau_{in}| = 1$ (light blue area) corresponds to $|\tau_{LC}| < 1$, i.e., where the pulsar wind does not scatter the radio pulses at and beyond $r_{LC}$. The upper left corner, which satisfies $\gamma > 10^{2.7}$ and $\theta_{pl} < 10^{-2.7}$ (gray area), corresponds to $l_c(r_{LC}) > r_{LC}$, i.e., the radio pulses are not scattered at $r_{LC}$ because of the ‘lack of time’ effect and we also require $|\tau_{in}| < 1$ in Fig. 8. The $|\tau_{LC}| = 1$ lines (thick lines) at the ‘Inclined’ and ‘Wide’ areas are determined by $\gamma_{LC} = 10^{3.7}\theta_{pl,LC}^{-1}(1 + \sigma_{LC})^{-1/4}$ and $\gamma_{LC} = 10^{5.3}(1 + \sigma_{LC})^{-1/4}$, respectively, and depend on $\sigma_{LC}$ (see also Table 1). We adopt $1 < 1 + \sigma_{LC} \ll 10^4$ in the diagram, e.g., the $\gamma$-intercept of the $|\tau_{LC}| = 1$ line in the ‘Inclined’ area is $\gamma \sim 10^{-2.2}$ for $1 + \sigma_{LC} \sim 10^2$. Note that the line in the geometry $\theta_{bm} > \theta_{pl} > 1$ is shown as a dashed line because it is an interpolated one (see text). On the other hand, the shaded region (pink area) is the forbidden region.

We adopt $r_e = 10^7 r_{e,7}$ cm to evaluate $\theta_{bm}(r_{in})$ and will use $r_e = 10^3$ cm in Sect. 3.5 ($r_e$-dependence is already included explicitly in Tables 1 and 2). We consider customarily used values of $\sigma$ ($\sigma_{LC}$ and $\sigma_c$) in a range of $1 < 1 + \sigma \lesssim 10^4$. We take $v_0 = 10$ MHz, $v = 100$ MHz, and $p_2 = -5$, i.e., $T_b(v)/T_b(v_0) \sim 10^{-4}$ in the integrals $I_{Narrow}(v)$ and $I_{Inclined}(v)$, while $I_{Wide}(v) \sim -1$ is used for the same reason discussed in Eq. (26). Again, only the pulsar wind velocities $u_{LC}$ and $u_c$ are the remaining parameters, i.e., we take $(\gamma_{LC}, \theta_{pl,LC})$ and $(\gamma_c, \theta_{pl,c})$ as the free parameters.

3.3.1. Escape from scattering at the light cylinder: Here, we are interested in whether the radio pulse can escape from scattering at $r_{LC}$. Figure 7 shows the resultant $\gamma-\theta_{pl}$ diagram, which tells us whether the radio pulses can escape from scattering or not at a given point on the diagram, i.e., a given velocity $u_{LC}$ of the pulsar wind (see also Table 1). Since we obtain $\theta_{bm}(r_{LC}) \approx 10^{-1.2}$ from Eq. (23), the scattering geometries are divided by the lines $\gamma = 10^{1.2}$ ($\theta_{bm,LC} = 1$), $\theta_{pl} = 10^{-1.2}$ ($\theta_{pl} = \Theta_{bm,LC}$), and $\gamma = \theta_{pl}^{-1}$ ($\theta_{pl} = 1$). Areas above the thick lines $|\tau_{LC}| = 1$ correspond to the pulsar wind structures that allow the radio pulses to escape, where $\tau_{LC}$ is the optical depth for $r_{in} = r_{LC}$. At the upper left corner on the diagram, the region satisfies $l_c(r_{LC}) > r_{LC}$ and the radio pulses also escape from scattering at $r_{LC}$ due to the ‘lack of time’ effect. The lines $|\tau_{LC}| = 1$ and $l_c(r_{LC}) = r_{LC}$ are different for different scattering geometries, as described below and summarized in Table 1.
Fig. 8. The $\gamma$–$\theta_{pl}$ diagram at $r_c > r_{LC}$ when $r_c = 10^7$ cm ($\Theta_{bm}(r_c) \approx 10^{4.2} \gamma^{-2} r_c$ or $10^{4.2} \theta_{pl}^2 r_c$). We show only the region that satisfies both $\gamma > 10^{2.7}$ and $\theta_{pl} < 10^{-2.7}$ because we consider the case $r_c > r_{LC}$. The pulsar wind velocity $u_c$ is specified by choosing one point on the diagram. Four areas divided by three lines $\gamma = \theta_{pl}^{-1}$, $\gamma = 10^{4.2}$, and $\theta_{pl} = 10^{-4.2}$ correspond to different scattering geometries, the ‘Narrow’ (left triangular area, outlined in red), ‘Inclined’ (upper triangle area: triangular area, outlined in green) and ‘Wide’ (lowermost area, outlined in yellow) cases and if the geometry $\Theta_{bm} > \Theta_{pl} > 1$ (rightmost area, outlined in blue). Note that each scattering geometry appears in a different layout compared with Fig. 7. The painted region (light blue area) satisfies $|\tau_c| < 1$, i.e., the radio pulses are not scattered at $r_c \gtrsim 10^{11.2}$ cm $\approx 10^3 r_{LC}$. The $|\tau_c| = 1$ line (thick dashed line) appears only in the geometry $\Theta_{bm} > \Theta_{pl} > 1$ and is an extrapolated one (see text). On the other hand, the shaded region (pink area) is a forbidden region because $|\tau_c| > 1$ or, in other words, $r_c < 10^{11.2}$ cm.

First, we consider the ‘Narrow’ case ($1 > \Theta_{bm}^2 + \Theta_{pl}^2$) corresponding to the lowermost area on the diagram. The optical depth of $\tau_{LC} \approx 10^{14.9}(1 + \sigma_{LC})^{-1}$ obtained from Eqs. (19), (25), and (31) is independent of both $\gamma_{LC}$ and $\theta_{pl,LC}$. Therefore, a region $|\tau_{LC}| < 1$ does not appear for $1 + \sigma_{LC} \lesssim 10^4$ and then we conclude that this case is not realized for the Crab pulsar.

Next, we consider the ‘Inclined’ case ($\Theta_{bm}^2 > \Theta_{pl}^2 + 1$) corresponding to the rightmost area on the diagram. In this case, the optical depth is expressed as $\tau_{LC} \approx 10^{14.9} \gamma_{LC}^{-4} \theta_{pl,LC}^{-4}(1 + \sigma_{LC})^{-1}$. The condition for $|\tau_{LC}| < 1$ is equivalent to $\gamma_{LC} \gtrsim 10^{3.7} \theta_{pl,LC}^{-1}(1 + \sigma_{LC})^{-1/4}$ with $\theta_{pl,LC} \gtrsim 10^{-1.2}$ where the painted area above the $|\tau_{LC}| = 1$ line in the ‘Inclined’ area on the diagram. We find that the radio pulses can escape for reasonable parameters when the pulsar wind has a significant non-radial motion. For example, the pulsar wind of $\gamma_{LC} > 10^{2.7}$ with $\theta_{pl,LC} \approx 1$ and $1 + \sigma_{LC} \approx 10^4$ can escape from scattering at $r_{LC}$.

The ‘Wide’ case ($\Theta_{bm} > 1 > \Theta_{pl}$) corresponds to the left triangular area on the diagram. For $|\tau_{LC}| \sim 10^{23.3} \gamma_{LC}^{-4}(1 + \sigma_{LC})^{-1}$ to be less than unity, we require $\gamma_{LC} > 10^{5.8}(1 + \sigma_{LC})^{-1/4}$ where the $|\tau_{LC}| = 1$ line in the ‘Wide’ area on the diagram. However, because the line is already above $\gamma_{LC} > 10^{2.7}$ for $1 < 1 + \sigma_{LC} \lesssim 10^4$, $\gamma_{LC} > 10^{2.7}$ (the ‘lack of time’ effect) is the condition for the radio pulses to escape from scattering at $r_{LC}$ in this case.

Lastly, we mention the geometry of $\Theta_{bm} > \Theta_{pl} > 1$, which appears in the upper triangular area on the diagram. The $l_c(r_{LC}) = r_{LC}$ and $|\tau_{LC}| = 1$ lines (dashed lines) are not calculated but interpolated ones. For escaping by the ‘lack of time’ effect ($l_c(r_{LC}) > r_{LC}$), we obtain at least $\theta_{pl,LC} < 10^{-2.7}$ from Eq. (29). The $|\tau_{LC}| = 1$ line is expected to be continuous at the boundaries on the $\gamma_{LC} = \theta_{pl,LC}^{-1}$ and
\(\theta_{\text{pl,LC}} = 10^{-1.2}\) lines because these boundaries just divide the approximated forms of Eq. (14). On the other hand, the \(|\tau_{\text{LC}}| = 1\) line would have at least one singular point because \(\tau_{\text{LC}}\) changes the sign at the left and right boundaries and a singular line (or curve) that satisfies \(\tau_{\text{LC}} = 0\) would be drawn on the diagram. Although a significantly small value of \(\gamma_{\text{LC}}\) might be allowed on the sides of the singular line, such a region on the \(\gamma - \theta_{\text{pl}}\) diagram would be as small as the dip around the discontinuity of \(I(v)\) in Figs. 2–4 because \(S(v_1)\), which appears in Eq. (14), controls the singularity \(\tau_{\text{LC}} = 0\). When we neglect such a singular region, the allowed region would be above the thick dashed line and the lower limit of \(\gamma_{\text{LC}}\) is clearly larger than the ‘Inclined’ case.

3.3.2. Escape from scattering beyond the light cylinder. We investigate whether the radio pulse can escape from scattering at \(r_c\) further than \(r_{\text{LC}}\). Because \(r_c > r_{\text{LC}}\), we have only to consider a region of \(\gamma > 10^{2.7}\) and \(\theta_{\text{pl}} < 10^{-2.7}\). The behaviors of \(\gamma(r)\) and \(\theta_{\text{pl}}(r)\) at \(r_{\text{LC}} < r < r_c\) will be discussed in Sect. 3.3.3. Figure 8 shows the resultant \(\gamma - \theta_{\text{pl}}\) diagram at \(r_c\). We set \(\theta_{\text{bm}}(r_c) \approx 10^{4.2}\gamma_c^{-2}\) for the ‘Narrow’ and ‘Wide’ cases and \(\theta_{\text{bm}}(r_{\text{c}}) \approx 10^{4.2}\theta_{\text{pl,c}}^2\) for the ‘Inclined’ case from Eqs. (23) and (28). The scattering geometries are divided by the lines \(\gamma = 10^{4.2}\) (\(\Theta_{\text{bm,c}} = 1\)), \(\theta_{\text{pl}} = 10^{-4.2}\) (\(\Theta_{\text{pl}} = \Theta_{\text{bm,c}}\), and \(\gamma = \theta_{\text{pl}}^{-1}\) (\(\Theta_{\text{pl}} = 1\)) (see Table 2). It should be noted that each scattering geometry appears in a different layout on the \(\gamma - \theta_{\text{pl}}\) diagram compared with Fig. 7 because \(\theta_{\text{bm}}(r_c)\) depends on \(\gamma_c\) or \(\theta_{\text{pl},c}\). The pulsar wind velocity \(u_c\) that allows the radio pulses to escape corresponds to the area satisfying \(\gamma_c \geq 10^{4.2}\) and \(\theta_{\text{pl},c} \leq 10^{-4.2}\) corresponding to the ‘Narrow’ or ‘Inclined’ cases. Except for the extrapolated line in the geometry \(\Theta_{\text{bm}} > \Theta_{\text{pl}} > 1\) (thick dashed line), the \(|\tau_c| = 1\) line is not drawn on the diagram as described below, where \(\tau_c\) is the optical depth for \(r_{\text{in}} = r_c\).

The ‘Narrow’ case \((1 > \Theta_{\text{bm}}^2 + \Theta_{\text{pl}}^2)\) corresponds to the left triangular area on the diagram. In this case, the optical depth is written as \(\tau_c \approx 10^{42.0}\gamma_c^{-10}(1 + \sigma_c)^{-1}\), i.e., we require \(\gamma_c \gtrsim 10^{4.2}(1 + \sigma_c)^{-1/10}\) to be \(|\tau_c| < 1\). The \(|\tau_c| = 1\) line is degenerate to or a bit lower than the \(\gamma_c = 10^{4.2}\) line for \(1 + \sigma_c > 1\). Therefore, all of the ‘Narrow’ geometry area \(\gamma_c \geq 10^{4.2}\) is allowed for radio pulses to escape. The corresponding characteristic scattering radius is \(r_c \gtrsim 10^{11.2}\) cm to \(10^{13}\) cm.

Next, we consider the ‘Inclined’ case \((\Theta_{\text{pl}}^2 > \Theta_{\text{bm}}^2 + 1)\) corresponding to the right triangular area on the diagram. For the optical depth, we require \(|\tau_c| \sim 10^{42.0}\gamma_c^{-4}\theta_{\text{pl,c}}^6(1 + \sigma_c)^{-1} = 10^{42.0}\gamma_c^{-10}\theta_{\text{pl,c}}^6(1 + \sigma_c)^{-1} < 1\) at \(r_c\). The \(|\tau_c| = 1\) line satisfies \(\gamma_c = 10^{4.2}\theta_{\text{pl,c}}^{3/5}(1 + \sigma_c)^{-1/10}\), which has the slope \(\gamma \propto \theta_{\text{pl,c}}^{3/2}\) and is continuous with the \(|\tau_c| = 1\) line for the ‘Narrow’ case on the boundary line \(\gamma = \theta_{\text{pl}}^{-1}\). Note that a large \(\theta_{\text{pl,c}}\) does not reduce \(|\tau_c|\) as \(|\tau_{\text{LC}}|\) is reduced by a large \(\theta_{\text{pl,LC}}\) (see the ‘Inclined’ area in Fig. 7) because \(r_c\) is a rapidly decreasing function of \(\theta_{\text{pl,c}}\). Therefore, all of the ‘Inclined’ geometry area \(\theta_{\text{pl,c}} \leq 10^{-4.2}\) is allowed for radio pulses to escape and we obtain \(r_c \gtrsim 10^{11.2}\) cm again.

The ‘Wide’ case \((\Theta_{\text{bm}} > 1 > \Theta_{\text{pl}})\) corresponds to the lowermost area on the diagram. The condition to be \(|\tau_c| < 1\) is \(\gamma_c \gtrsim 10^{4.8}(1 + \sigma_c)^{-1/6}\). In this case, a region \(|\tau_c| < 1\) does not appear in the ‘Wide’ area for \(1 + \sigma_c < 10^4\) and thus we conclude that this case is not realized for the Crab pulsar.

For the geometry of \(\Theta_{\text{bm}} > \Theta_{\text{pl}} > 1\) corresponding to the rightmost area on the diagram, we do not draw the \(|\tau_c| = 1\) line in the same manner as in Fig. 7 because no \(|\tau_c| = 1\) line appears in Fig. 8 for other geometries. One possibility is that the \(|\tau_c| = 1\) line emerges from the boundary \(\theta_{pl} = 10^{-4.2}\), such as the thick dashed line on the diagram. As implied from the \(|\tau_c| = 1\) line for the ‘Inclined’ case, the line has the slope \(\gamma \propto \theta_{\text{pl}}^q\) with \(q \geq 3/2\) because \(r_c\) rapidly decreases with increasing \(\theta_{\text{pl,c}}\).
3.3.3. Summary. There exist two possible cases of $u_{\text{LC}}$ where the radio pulses are not scattered at $r_{\text{LC}}$. First, when $u_{\text{LC}}$ is significantly inclined with respect to the radio pulses $10^{-1.2} < \theta_{\text{pl,LC}} \lesssim 1$ and has the Lorentz factor satisfying $\gamma_{\text{LC}} \sigma_{\text{pl,LC}} (1 + \sigma_{\text{LC}})^{1/4} \gtrsim 10^{3.7}$, we obtain $r_{\text{LC}} < 1$. In this case, the radio pulses reach the observer without scattering because $\chi(v, r)$ decreases rapidly with $r$ for $0 < (a, b) \lesssim 1.25$, as discussed in Eq. (26).

The second corresponds to the ‘lack of time’ effect, i.e., $u_{\text{LC}}$ is almost aligned with respect to the radio pulses $\theta_{\text{pl,LC}} < 10^{-2.7}$ with $\gamma_{\text{LC}} > 10^{2.7}$. In this case, $r_{\text{in}} = \Delta r = r_{\text{c}}$, we require $|\tau_{\text{c}}| < 1$ when an electron reaches $r_{\text{c}}$ and also require $\gamma_{\text{c}} > 10^{4.2}$ and $\sigma_{\text{pl}} < 10^{-4.2}$ for $|\tau_{\text{c}}| < 1 (r_{\text{c}} \gtrsim 10^{11.2} \text{ cm} \approx 10^{11.2} \text{ cm})$. $\gamma_{\text{c}} > 10^{4.2}$ and $\sigma_{\text{pl}} < 10^{-4.2}$ for $|\tau_{\text{c}}| < 1 (r_{\text{c}} \gtrsim 10^{11.2} \text{ cm} \approx 10^{11.2} \text{ cm})$ should be exchanged with $r$ as follows (see also Eq. (30) and Fig. 6). For the ‘Narrow’ and ‘Wide’ cases, we require that point ‘B’ ($a < 0.5$) or point ‘C’ ($a \gtrsim 0.5$) in Fig. 6 is more distant than $10^{11.2} \text{ cm}$. For example, if $\gamma$ has a constant value ($a = 0$), we require $\gamma > 10^{4.4}$ at $r_{\text{LC}}$. On the other hand, if $a \gtrsim 0.5$ with $\gamma_{\text{LC}} > 10^{2.7}$, $\gamma$ should have a terminal value of $\gamma > 10^{4.2}$. Although the ‘Inclined’ case is slightly complicated, we can constrain the behavior of $\gamma$ by replacing $\gamma$ with $\gamma_{\text{pl}}^{-1}$ in the above discussion and using the condition $\gamma > \theta_{\text{pl}}^{-1} (\theta_{\text{pl}} > 1)$ for the ‘Inclined’ case. The required values of the exponents $a$ and $b$ change with the values of $u_{\text{LC}}$, $\sigma_{\text{LC}}$, and $\sigma_{\text{c}}$.

Lastly, we mention the result obtained by WR78. Essentially, our ‘Wide’ geometry with scattering at $r_{\text{c}} \sim 10^{11.2} \text{ cm}$ corresponds to the situation that they considered, although their setup was not exactly the same as ours in the radial variations of $\gamma(r)$ and $n_{\text{pl}}(r)$. Our result of $\gamma_{\text{c}} \gtrsim 10^{4.8} (1 + \sigma_{\text{c}})^{-1/6}$ obtained in Sect. 3.3.2 is close to their result of $\gamma > 10^{4.4}$ (see their Eq. (10)). Note that we did not consider the ‘Wide’ case with scattering at $r_{\text{c}}$ because $\gamma_{\text{c}} < 10^{4.2}$ is also required for the geometry to be ‘Wide’. Also note that they did not account for the constraint at $r_{\text{LC}}$, although we require $\gamma_{\text{LC}} > 10^{2.7}$ and $\theta_{\text{pl,LC}} < 10^{-2.7}$ for $r_{\text{c}} > r_{\text{LC}}$.

3.4. Constraints on pair multiplicity
In the last section, we obtained lower limits of $\gamma$ for a given inclination angle $\theta_{\text{pl}}$ and a magnetization $\sigma$ of the pulsar wind. Here, we consider the corresponding upper limits of $\kappa$ using Eq. (1). Note that the combination of $\kappa \gamma (1 + \sigma) = 10^{10.5}$ is independent of $r$ from the energy conservation law and that $\kappa$ alone is also expected to be independent of $r$ from the law of particle-number conservation. Below, we consider the upper limits of $\kappa$ for the two possible values of $u_{\text{LC}}$ of the pulsar wind and we do not consider a constraint for the geometry $\theta_{\text{in}} > \theta_{\text{pl}} > 1$ for simplicity.

When the pulsar wind is inclined with respect to the radio pulses at $r_{\text{LC}} (10^{-1.2} < \theta_{\text{pl,LC}} \lesssim 1)$, we obtain an upper limit of $\kappa$ by eliminating $\gamma_{\text{LC}}$ from $\gamma_{\text{LC}} \theta_{\text{pl,LC}} (1 + \sigma_{\text{LC}})^{1/4} \gtrsim 10^{3.7}$ with the use of Eq. (1) ($\kappa \gamma(r)(1 + \sigma(r)) = 10^{10.5}$). We obtain

$$\kappa \lesssim 10^{6.8} \theta_{\text{pl,LC}} (1 + \sigma_{\text{LC}})^{-3/2}. \quad (32)$$

The upper limit is $\kappa < 10^{6.8}$ for both $1 + \sigma_{\text{LC}} \sim 1$ and $\theta_{\text{pl,LC}} \sim 1$. This upper limit of the pair multiplicity can satisfy $\kappa \gtrsim \kappa_{\text{PWN}} = 10^{6.6}$ obtained by Tanaka and Takahara [11,13]. However, for $\sigma_{\text{LC}} \sim 10^{4}$, the upper limit becomes $\kappa \lesssim 10^{3.8} \theta_{\text{pl,LC}}$ and $\gamma_{\text{LC}} \gtrsim 10^{2.7} \theta_{\text{pl,LC}}^{-1}$, which can be close to the customarily believed picture of the pulsar wind at the light cylinder [2,3]. In other words, $1 + \sigma_{\text{LC}} \lesssim 10^{0.2} \theta_{\text{pl,LC}}^{4/3}$ is required for $\kappa \gtrsim \kappa_{\text{PWN}}$.

For the second case when the pulsar wind is aligned with respect to the radio pulse at $r_{\text{LC}}$, we require both $\gamma_{\text{LC}} > 10^{2.7}$ ($\theta_{\text{pl,LC}} < 10^{-2.7}$) and $\gamma_{\text{c}} > 10^{4.2}$ ($\theta_{\text{pl,LC}} < 10^{-4.2}$). Using $\kappa \gamma(r) (1 + \sigma(r)) = 10^{10.5}$, we require both

$$\kappa \lesssim 10^{7.8} (1 + \sigma_{\text{LC}})^{-1} \text{ and } \kappa \lesssim 10^{6.3} (1 + \sigma_{\text{c}})^{-1}. \quad (33)$$
Because $\kappa$ conserves along the flow, $\kappa$ should satisfy both of the two inequalities. Even for $1 + \sigma c \sim 1$, $\kappa \lesssim 10^{6.3}$ at $r_c \sim 10^3 r_{PLC}$ is marginal for $\kappa > \kappa_{PWN}$. For the customarily used magnetization $\sigma_{LC} \sim 10^4$, an upper limit is $\kappa \lesssim 10^{3.8} \ll \kappa_{PWN}$. The results are summarized in Table 3. A slightly larger $\kappa$ is allowed for the inclined $u_{LC}$ ($\theta_{pl,LC} < 1$) than for the aligned $u_{LC}$ with respect to the radio pulse beam.

### 3.5. Dependence on the size of the emission region

We assume $r_c = 10^7$ cm in the above calculations. Here, we discuss the constraints on $\gamma$ and $\kappa$ assuming Eq. (23) with e.g. $r_c = 10^3$ cm. The dependence on $r_c$ ($10^3 \leq r_c \leq 10^7$ cm) is described explicitly in Tables 1 and 2. When we take a different value of $r_c$, the brightness temperature $T_b$ (Eq. (24)) and the integrals $I_{Narrow}$ and $I_{Inclined}$ (Eqs. (19) and (20)) are changed. In Tables 1 and 2, we find that the optical depth for the ‘Narrow’ and ‘Inclined’ cases is proportional to $r_c^2$. This is because $I_{Narrow}$ and $I_{Inclined}$ are proportional to $r_c^4$ and $T_b$ is proportional to $r_c^{-2}$. On the other hand, for the ‘Wide’ case, the optical depth is proportional to $r_c^{-2}$ because $I_{Wide}(v) \sim -1$, the value of which does not depend on $\theta_{bm}$ in the range of $v_0 \lesssim v < \Theta_{bm}^2 v_0$. Note that the layout of the scattering geometry in the $\gamma-\theta_{pl}$ diagrams (Fig. 9) has also changed where the ‘Narrow’ and ‘Inclined’ areas occur on the planes compared with those in Figs. 7 and 8.

We obtain the lower limits of $\gamma$ and the upper limits of $\kappa$ in the same manner as in the case of $r_c = 10^7$ cm. Figure 9 shows the resultant $\gamma-\theta_{pl}$ diagrams both at $r_{LC}$ (left) and $r_c$ (right). The obtained lower limits of $\gamma$ and upper limits of $\kappa$ are summarized in Table 4.

At $r_{LC}$ ($\theta_{bm}(r_{LC}) \approx 10^{-5.2}$), we find two allowed regions in the diagram in the left panel of Fig. 9. The first is when the pulse wind has a significant non-radial motion $10^{-2.7} < \theta_{pl,LC} \lesssim 1$. We require $\gamma_{LC} \theta_{pl,LC}(1 + \sigma_{LC})^{1/4} \sim 10^{0.1} r_{c,3}^{1/2}$ for $|\tau_{LC}| < 1$ and no scattering occurs beyond $r_{LC}$ for moderate values of the exponents $a$ and $b$. We also find that the non-relativistic pulsar wind $\beta_{LC} \ll 1$ is unfavorable even for such a small opening angle of the radio beam $\theta_{bm,LC} = 10^{-5.2}$ with $1 + \sigma_{LC} \approx 10^4$.

Secondly, the region that satisfies $\gamma_{LC} > 10^{2.7}$ and $\theta_{pl,LC} < 10^{-2.7}$ is also allowed to escape from scattering at $r_{LC}$ due to the ‘lack of time’ effect. In this case, in addition, we require $|\tau_c| < 1$ at $r_c$ (>$r_{LC}$). The right panel of Fig. 9 shows the $\gamma-\theta_{pl}$ diagram at $r_c$. We do not find the ‘Wide’ and $\Theta_{bm} > \Theta_{pl} > 1$ geometries on the diagram because $\theta_{bm}(r_c)$ for $r_c = 10^3$ cm is much smaller than that for $r_c = 10^7$ cm. The region that satisfies $|\tau_c| < 1$ is $\gamma_c \gtrsim 10^{3.4} r_{c,3}^{1/5} (1 + \sigma_c)^{-1/10}$ for the ‘Narrow’ case and $\gamma_c \gtrsim 10^{3.4} r_{c,3}^{1/5} \theta_{pl,c}^{3/5} (1 + \sigma_c)^{-1/10}$ for the ‘Inclined’ case. The corresponding $r_c$ is larger than $10^{6.6} r_{c,3}^{2/5} \mathrm{cm} = 10^{1.4} r_{c,3}^{2/5} r_{LC}$. It is important to note that the constraint at $r_c$ very weakly depends on $r_c$ as $r_c^{1/5}$.
Fig. 9. The $\gamma - \theta_{pl}$ diagrams at $r_{1c}$ (left) and $r_c$ (right). We take different emission region sizes of $r_c = 10^3$ cm from Figs. 7 and 8 (see also Tables 1 and 2). The ‘lack of time’ region (gray area) on the left panel has the same extent as that in Fig. 7. The shaded region (pink area) is a forbidden region for both panels. The ‘Narrow’ and ‘Inclined’ areas expand compared with Figs. 7 and 8 because $\theta_{bm} \propto r_c$ in Eq. (23). For the left panel, three lines $\gamma = \theta_{pl}^{-1}$, $\gamma = 10^{-5.2}$, and $\theta_{pl} = 10^{-5.2}$ divide the scattering geometries, while we do not find the ‘Wide’ and $\Theta_{bm} > \Theta_{pl} > 1$ areas for the right panel. The $|\tau| = 1$ lines are also different, and the $|\tau| < 1$ region becomes wider than that in Figs. 7 and 8. The $|\tau_{1c}| = 1$ line in the ‘Inclined’ region on the left panel corresponds to $\gamma_{1c} \gtrsim 10^{1.7} \theta_{pl,1c} (1 + \sigma_{1c})^{-1/4}$. We adopt $1 < 1 + \sigma_{1c} \lesssim 10^4$ in the figure, e.g., the $\gamma$-intercept of the $|\tau_{1c}| = 1$ line on the left panel is $10^{0.7} \lesssim \gamma < 10^{1.7}$. The $|\tau_c| = 1$ lines on the right panel correspond to $\gamma_c \gtrsim 10^{1.4} (1 + \sigma_c)^{-1/10}$ for the ‘Narrow’ area and $\gamma_c \gtrsim 10^{3.4} \theta_{pl,c}^{-3/5} (1 + \sigma_c)^{-1/10}$ for the ‘Inclined’ area.

### Table 4. Lower limits of the Lorentz factor and corresponding upper limits for the pair multiplicity for the two possible structures of the pulsar wind at $r_{1c}$ when $10^3 \lesssim r_c \lesssim 10^7$ cm.

| $\gamma$                         | $\kappa$                               |
|----------------------------------|----------------------------------------|
| Inclined $u_{1c}$ ($\max(10^{-2.7}, 10^{-5.2} r_{c,3}) < \theta_{pl,1c} \lesssim 1$) | $\kappa \lesssim 10^{8.8} r_{c,3}^{-1/2} \theta_{pl,1c} (1 + \sigma_{1c})^{-3/4}$ |
| Aligned $u_{1c}$ ($\theta_{pl,1c} < 10^{-2.7}$) and $\theta_{pl,c} < \gamma_{1c}^{-1}$ | $\kappa \lesssim 10^{7.8} (1 + \sigma_{1c})^{-1}$ |
| $\gamma_{1c} \gtrsim 10^{2.7}$    | $\kappa \lesssim 10^{7.1} r_{c,3}^{-1/5} (1 + \sigma_c)^{-9/10}$ |
| $\gamma_c \gtrsim 10^{3.4} r_{c,3}^{-1/5} (1 + \sigma_c)^{-1/10}$ | $\kappa \lesssim 10^{7.1} r_{c,3}^{-1/5} \theta_{pl,c}^{-3/5} (1 + \sigma_c)^{-9/10}$ |
| $\gamma_{1c} \gtrsim 10^{2.7}$    | $\kappa \lesssim 10^{7.8} (1 + \sigma_{1c})^{-1}$ |
| $\gamma_c \gtrsim 10^{3.4} r_{c,3}^{-1/5} \theta_{pl,c}^{-3/5} (1 + \sigma_c)^{-1/10}$ | $\kappa \lesssim 10^{7.1} r_{c,3}^{-1/5} \theta_{pl,c}^{-3/5} (1 + \sigma_c)^{-9/10}$ |

Accordingly, we obtain upper limits of $\kappa$ with the help of Eq. (1). When the pulsar wind is inclined with respect to the radio pulse at $r_{1c}$ ($10^{-2.7} < \theta_{pl,1c} \lesssim 1$), we obtain

$$\kappa \lesssim 10^{8.8} r_{c,3}^{-1/2} \theta_{pl,1c} (1 + \sigma_{1c})^{-3/4}. \quad (34)$$

We require $\sigma_{1c} \lesssim 10^3 \ll 10^4$ for $\kappa > \kappa_{\text{PWN}}$. When the pulsar wind is aligned with respect to the radio pulse at $r_{1c}$ ($\theta_{pl,1c} < 10^{-2.7}$ and $\gamma_{1c} > 10^{2.7}$), we obtain

$$\kappa \lesssim 10^{7.8} (1 + \sigma_{1c})^{-1} \quad \text{and} \quad \left\{ \begin{array}{ll} \kappa \lesssim 10^{7.1} r_{c,3}^{-1/5} (1 + \sigma_c)^{-9/10} & \text{for ‘Narrow’}, \\ \kappa \lesssim 10^{7.1} r_{c,3}^{-1/5} \theta_{pl,c}^{-3/5} (1 + \sigma_c)^{-9/10} & \text{for ‘Inclined’}. \end{array} \right. \quad (35)$$

$\kappa > \kappa_{\text{PWN}}$ is attainable for both the ‘Narrow’ and ‘Inclined’ cases again.
We obtain the lower limits of $\gamma$ and the upper limits of $\kappa$ for different sizes of the emission region $r_e$. Basically, as is found from Table 4, the smaller the emission region size becomes, the more easily the radio pulses escape from scattering, i.e., small $\gamma$ and large $\kappa$ are allowed. We obtain the most optimistic constraint for large $\kappa$ ($\kappa \lesssim 10^{8.8}$ at the uppermost row of Table 4), when $\theta_{pl,LC} \sim 1$ (inclined $u_{1,L}$), $1 + \sigma_{1,L} \sim 1$, and $r_e = 10^{-3}$ cm. Combined with $\kappa \gtrsim \kappa_{PWN} = 10^{6.6}$, we can write the pulsar wind properties as $10^{1.7} \lesssim \gamma \lesssim 10^{3.9}$ and $\kappa_{PWN} \lesssim \kappa \lesssim 10^{8.8}$. Although all these constraints are at $r_{1,L}$, the radio pulse can escape from scattering and $\kappa \gtrsim \kappa_{PWN}$ is satisfied beyond $r_{1,L}$ because $\gamma(r)(1 + \sigma(r)) \approx \gamma(r) = \text{constant}$ beyond $r_{1,L}$ for $1 + \sigma_{1,L} \sim 1$ from Eq. (1) and particle-number conservation ($\kappa = \text{constant}$). Note that we obtain $10^{1.2} \lesssim \gamma_{LC} \lesssim 10^{1.9}$ and $\kappa_{PWN} \lesssim \kappa \lesssim 10^{7.3}$ for $1 + \sigma_{1,L} \sim 10^{-2}$, and we require $\gamma(r)(1 + \sigma(r)) = \text{constant}$ and also $\kappa = \text{constant}$ beyond $r_{1,L}$.

4. Summary

To constrain the pulsar wind properties, we study induced Compton scattering by a relativistically moving cold plasma. Induced Compton scattering is $\theta_{bm}^4 k_B T_b(\nu)/m_e c^2$ times more significant compared with spontaneous scattering for the non-relativistic case. However, by scattering by a relativistically moving plasma, the scattering geometry of the system changes the scattering coefficient significantly. We consider fairly general geometries of scattering in the observer frame and obtain the scattering coefficient for induced Compton scattering off the photon beam. On the other hand, we do not take into account the magnetic field effects and the scattering off the background photons in this paper.

We obtain approximate expressions of the scattering coefficient for three geometries corresponding to the ‘Narrow’ ($1 > \Theta_{bm}^2 + \Theta_{pl}^2$), ‘Inclined’ ($\Theta_{pl}^2 > 1 + \Theta_{bm}^2$), and ‘Wide’ ($\Theta_{bm} > 1 > \Theta_{pl}$) cases, while the scattering coefficient for $\Theta_{bm} > \Theta_{pl} > 1$ is obtained numerically. The behavior of the scattering coefficient against a given scattering geometry is governed by a simple combination of four factors. In addition to the solid angle factor $\theta_{bm}^4$ appearing even for the non-relativistic case, there exist three relativistic effects: the factor independent of scattering geometry $\gamma^{-3}$ and the other two factors depending on geometry, the aberration factor $D_1^{-2}$ and the frequency shift factor $D/D_1$.

When the photon beam is inside the $\gamma^{-1}$ cone of the plasma beam (the ‘Narrow’ case), the aberration factor increases the scattering coefficient by a factor of $\sim \gamma^4$ (up to $\gamma \theta_{bm} \sim 1$). On the other hand, when the plasma velocity is significantly inclined with respect to the photon beam (the ‘Inclined’ case), this factor of $\gamma^4$ does not appear. The frequency shift factor is important when the photon beam is wider than the $\gamma^{-1}$ cone of the plasma beam (the ‘Wide’ case) and is rather complex and mostly increases the absolute value of the scattering coefficient compared with the non-relativistic case. Basically, the ‘Inclined’ case gives the smallest and the ‘Wide’ case gives the largest scattering coefficient, i.e., the $\Theta_{bm} > \Theta_{pl} > 1$ case is in between.

We apply induced Compton scattering to the Crab pulsar, where the high $T_b(\nu)$ radio pulses go through the relativistic pulsar wind and constrain the pulsar wind properties by imposing the condition of the optical depth being smaller than unity. We introduce the characteristic scattering radius $r_c$ of the ‘lack of time’ effect prevents scattering at $r < r_c$. We evaluate the scattering optical depth for both the $r_{in} = r_{1,L}$ and $r_{in} = r_c$ cases. We consider more general scattering geometries than WR78 and also study the dependence on the size of the emission region $10^3 \leq r_e \leq 10^7$ cm, which directly affects the opening angle of the radio pulses $\theta_{bm}(r)$. Allowable pulsar wind velocities at $r_{1,L}$ ($u_{1,L}$) and at $r_c$ ($u_c$) are explored assuming the canonical value of the magnetization $1 < 1 + \sigma \lesssim 10^4$.

Two pulsar wind velocities $u_{1,L}$ are allowed for radio pulses to escape from scattering at $r_{1,L}$. One is that the plasma velocity is inclined with respect to the photon beam ($\theta_{pl,1,L} \sim 1$). When $\gamma_{1,L} \gtrsim 20/24
$10^{1.7} r_{e3}^{1/2} \rho_{pl}^{-1} (1 + \sigma_{LC})^{-1/4}$ is satisfied, the radio pulses reach the observer without scattering for moderate radial variation of $\gamma(r)$ and $\theta_{pl}(r)$, where $\gamma \propto r^a$ and $\theta_{pl} \propto r^{-b}$ with $0 < (a, b) \lesssim 1.25$. The other is when the plasma velocity is aligned with respect to the photon beam ($\theta_{pl,LC} < 10^{-2.7}$). We require the lower limit $\gamma_{LC} \gtrsim 10^{2.7}$ for the ‘lack of time’ effect to prevent scattering at $r_{LC}$. In this case, we also require the optical depth at $r_c \gtrsim 10^{0.6} r_{e3}^{2/5} \approx 10^{1.4} r_{e3}^{2/5} r_{LC}$ to be less than unity, where $r_c (= l_c)$ depends on $\gamma_c$ or $\theta_c$ (Eq. (28)). For example, we require $\gamma_c \gtrsim 10^{3.4} r_{e3}^{1/5} (1 + \sigma_c)^{-1/10}$ for the completely aligned case $\theta_{pl} = 0$. Basically, the smaller the emission region size and the larger the inclination angle of the pulsar wind become, the smaller the allowed $\gamma$ is.

We have discussed the upper limits of the pair multiplicity using obtained constraints on the velocities of the Crab pulsar wind and Eq. (1). In principle, $\kappa \gtrsim \kappa_{PWN} \equiv 10^6 [11,13]$ is possible although we require $1 + \sigma_{LC} \ll 10^4$, i.e., the customarily used value $1 + \sigma_{LC} \approx 10^4$ contradicts $\kappa > \kappa_{PWN}$. The most optimistic constraint that allows large $\kappa$ is obtained when $\theta_{pl,LC} \sim 1$ and $r_c = 10^3$ cm (Eq. (34)). In this case with $\kappa \gtrsim \kappa_{PWN}$, we can write the pulsar wind properties as $10^{1.7} \lesssim \gamma \lesssim 10^{3.9}$ and $\kappa_{PWN} \lesssim \kappa \lesssim 10^{8.8}$ for $1 + \sigma_{LC} \sim 1$ and $10^{1.2} \lesssim \gamma \lesssim 10^{1.9}$ and $\kappa_{PWN} \lesssim \kappa \lesssim 10^{3.3}$ for $1 + \sigma_{LC} \sim 10^2$. Note that all these constraints are at $r_{LC}$ and we also require moderate radial variation of $\theta_{pl}(r)$ and $\gamma(r) (\propto (1 + \sigma(r))^{-1})$ beyond $r_{LC}$.

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Appendix: Numerical integration

We show the results of numerical integration of $I(\nu, \gamma, \theta_{bm}, \theta_{pl})$ (Eq. (14)). We focus on the situation $0 \leq (\theta_{pl}, \theta_{bm}) \lesssim 1$ and $\gamma \gg 1$, and then the integral $I(\nu)$ depends on the normalized angles $\Theta_{bm} \equiv \gamma \theta_{bm}$ and $\Theta_{pl} \equiv \gamma \theta_{pl}$ rather than on $\theta_{bm}$, $\theta_{pl}$, and $\gamma$ separately. As seen in Sect. 2.3, the behavior of $I(\nu)$ is very different for the values of $\Theta_{bm}$ and $\Theta_{pl}$, i.e., different scattering geometries. To obtain the results in Figs. A1 and A2, we set $\gamma = 10^2$ and adopt the broken power-law spectrum with $p_1 = 3$ and $p_2 = -5$ (Eq. (10)). The figures show absolute values of the integral $I(\nu)$ versus frequency $\nu$ for different sets of parameters $\Theta_{bm}$ and $\Theta_{pl}$. All the lines in these figures have a discontinuity where the sign of the integral $I(\nu)$ changes. The sign of the integral $I(\nu)$ is positive at the high-frequency side where the photon number decreases and vice versa.

Before describing the details of Figs. A1 and A2, we mention that the approximated forms studied in Sect. 2.3 can describe the behaviors of most of the lines in the figures. The behaviors of lines with no frequency shift are described by $I_{Narrow}$ and $I_{Inclined}$ and the behaviors of lines whose discontinuity point shifts to $\nu > \nu_0$ are described by $I_{Wide}$. Only the behaviors of ‘line d’ and ‘line e’ in the bottom-left panel in Fig. A1 and of ‘line e’ in the bottom-left panel in Fig. A2 are not explained by these three approximated forms corresponding to $\Theta_{bm} > \Theta_{pl} > 1$, which we will discuss later.

Figure A1 shows how the integral $I(\nu)$ changes with $\Theta_{pl}$ (0 $\leq \Theta_{pl} \leq 10$) for fixed $\Theta_{bm}$. Three panels in Fig. A1 correspond to different fixed values of $\Theta_{bm}$, e.g. the bottom-right sketch describes the scattering geometry when $\Theta_{bm} = 10$, corresponding to the bottom-left panel in Fig. A1. It is common for all the panels that ‘line a’ is very close to ‘line b’, i.e., we can approximate that the photon and plasma are completely aligned ($\theta_{pl} = 0$) even for $\Theta_{pl} < 1$. It is also common for all the panels that ‘line a’ is larger than other lines for $\nu > \nu_0$ and $|I(\nu)|$ decreases in order from ‘line a’
Fig. A1. Plots of the integral $I(\nu, \Theta_{bm}, \Theta_{pl}, \gamma)$ and a sketch of the scattering geometry (bottom right). To see the dependence on $\Theta_{pl}$, we fix $\Theta_{bm}$ for each panel, where (top left) $\Theta_{bm} = 10^{-1}$, (top right) $\Theta_{bm} = 1$, and (bottom left) $\Theta_{bm} = 10$, respectively. Each line is for a different value of $\Theta_{pl}$, where ‘line a’ $\Theta_{pl} = 0$, ‘line b’ $\Theta_{pl} = 0.3$, ‘line c’ $\Theta_{pl} = 1$, ‘line d’ $\Theta_{pl} = 3$, and ‘line e’ $\Theta_{pl} = 10$, respectively. We set $\gamma = 10^2$, $p_1 = 3$, and $p_2 = -5$. To ‘line e’, i.e., $|I(\nu)|$ is large when the photons and the plasma are aligned, at least at the frequency range $\nu > \nu_0$. The top-left panel ($\Theta_{bm} = 0.1$) shows the case when the photon beam is considered as narrow (compared with the $\gamma^{-1}$ cone associated with the plasma) and shows little frequency shift $D/D_1 \approx 1$ corresponding to $I_{\text{Narrow}}$ and $I_{\text{Inclined}}$, studied in Sect. 2.3. The bottom-left panel in Fig. A1 is the case when the photon beam is considered as wide ($\Theta_{bm} = 10$: the bottom-right sketch of Fig. A1). In this case, the frequency shift effect is extreme and the absolute value $|I(\nu)|$ is almost unity at a broad frequency range.

Figure A2 shows how the integral $I(\nu)$ changes with $\Theta_{bm}$ ($0.1 \leq \Theta_{bm} \leq 10$) for fixed $\Theta_{pl}$. Three panels in Fig. A2 correspond to different fixed values of $\Theta_{pl}$; e.g. the bottom-right sketch describes the scattering geometry when $\Theta_{pl} = 1$, corresponding to the top-right panel in Fig. A2. Note that some lines have the same parameter set as Fig. A1. It is common for all the panels that $|I(\nu)|$ decreases with smaller values of $\Theta_{bm}$. ‘Line d’ and ‘line e’ in the top-left panel ($\Theta_{pl} = 0$) and top-right panel ($\Theta_{pl} = 1$) show $I_{\text{Wide}}$, studied in Sect. 2.3.

Lastly, we discuss the behaviors of ‘line d’ and ‘line e’ in the bottom-left panel in Fig. A1 and of ‘line e’ in the bottom-left panel in Fig. A2. These lines satisfy $\Theta_{bm} \geq \Theta_{pl} > 1$ and show two notable features. One is the discontinuity point shifting toward $\nu < \nu_0$ (‘feature one’) and the other is $|I(\nu)|$ being significantly greater than unity at $\nu < \nu_0$ (‘feature two’). We can discuss these features qualitatively. To simplify the explanation, we take $\Theta_{bm}^2 = \Theta_{pl}^2 \gg 1$, corresponding to ‘line e’ in the bottom-left panel in both Figs. A1 and A2 ($\Theta_{bm} = \Theta_{pl} = 10$). For ‘feature one’, we obtain from Eq. (18) that the frequency shift factor has a peak value $D/D_1 \sim \Theta_{pl}^2$ at $\Theta_1 = \Theta_{bm}$ and $\phi_1 = 0$;
this value corresponds to the frequency that gives the peak of $|I(\nu)|$. For ‘feature two’, we try to evaluate $|I(\nu \approx \Theta_{pl}^{-2} v_0)|$. For $\nu \approx \Theta_{pl}^{-2} v_0$, we obtain $v_1 = (D/D_1)\nu \approx (\Theta_{pl}^2/(1 + \Psi_1^2))\nu \approx v_0/(1 + \Psi_1^2) \leq v_0$ so that we take $S(v_1) \sim p_1 + 2$ and $T_b(v_1) \approx T_b(v_0)(1 + \Psi_1^2)^{-p_1-1}$. Assuming that $R$ is a constant of order unity, we obtain

$$I(\nu \approx \Theta_{pl}^{-2} v_0) \approx -\frac{3RS\Theta_1^3}{16\pi} \int_0^{2\pi} d\phi_1 \int_0^{\Theta_{pl}} d\theta_1 \frac{4\Psi_1^4}{(1 + \Psi_1^2)^{p_1+3}}.$$

Although this integral cannot be performed analytically, we find that the integrand has a peak value $\Theta_{pl}^3$ at $\phi_1 = 0$ and $\theta_1 = \Theta_{pl}$. A crude estimate may be obtained by taking a peak value of the integrand $\Theta_{pl}^3$ with $\int_0^{2\pi} d\phi \sim 2\pi$ and $\int_0^{\Theta_{pl}} d\theta_1 \sim \Theta_{pl}$. This must be overestimated and gives $3RS\Theta_{pl}^4/2 \sim 10^4$ for $\Theta_{pl} = 10$. Although the value does not fit the numerical calculation ($I(\Theta_{pl}^{-2} v_0) \sim 10^2$ from Figs. A1 and A2), we find $I(\Theta_{pl}^{-2} v_0)$ can be much greater than unity.

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