Scaling dimension of fidelity susceptibility in quantum phase transitions

SHI-JIAN GU(a) and HAI-QING LIN

Department of Physics and ITP, The Chinese University of Hong Kong - Hong Kong, China

received 17 April 2009; accepted in final form 15 June 2009
published online 17 July 2009

PACS 03.67.-a – Quantum information
PACS 64.60.-i – General studies of phase transitions
PACS 75.10.Jm – Quantized spin models

Abstract – We analyze ground-state behaviors of fidelity susceptibility (FS) and show that the FS has its own distinct dimension instead of real system’s dimension in general quantum phases. The scaling relation of the FS in quantum phase transitions (QPTs) is then established on more general grounds. Depending on whether the FS’s dimensions of two neighboring quantum phases are the same or not, we are able to classify QPTs into two distinct types. For the latter type, the change in the FS’s dimension is a characteristic that separates two phases. As a non-trivial application to the Kitaev honeycomb model, we find that the FS is proportional to $L^2 \ln L$ in the gapless phase, while $L^2$ in the gapped phase. Therefore, the extra dimension of $\ln L$ can be used as a characteristic of the gapless phase.

Copyright © EPLA, 2009

Introduction. – Let us consider a general QPT [1] occurring in the ground state $|\Psi_0(\lambda)\rangle$ of a quantum many-body Hamiltonian

$$H = H_0 + \lambda H_I,$$

where $H_I$ is the driving Hamiltonian and $\lambda$ denotes its strength. The motivation of the fidelity approach [2] to QPTs is that the overlap between two ground states separated by a small amount $\delta \lambda$, i.e. $F(\lambda, \lambda + \delta \lambda) = |\langle \Psi_0(\lambda) | \Psi_0(\lambda + \delta \lambda) \rangle|$, is expected to show a minimum around the critical point $\lambda_c$ due to the dramatic change in structures of the ground-state wave function [3,4]. This interesting insight to QPTs from quantum information theory [2] had then been demonstrated in a few strongly correlated systems [5–8]. It was realized consequently that the leading term of the fidelity, called the FS [9] or the Riemannian metric tensor [10], should play a key role in such a new approach to QPTs. After that, various issues based on the fidelity or its leading term [11–27], including scaling and universality class [13,14], and its role in topological QPTs [23–26] etc, were raised and addressed.

However, it seems to us that all relevant studies took it for granted that the FS, in general quantum phases, has dimension $d$, i.e., $\chi_F \propto L^d$, where $d(L)$ is the dimension (length) of the system. In this work, we will show instead that the FS has its own dimension depending on both the scaling dimension of the driving Hamiltonian and long-range behaviors of their correlations. The distinct dimension of the FS means that, in a class of quantum phases, the adiabatic response of the ground state to driving parameter is no longer proportional to the system size. This property is different from our previous understanding of critical phenomena from statistical quantities, such as energy that is extensive in a thermodynamic system. Therefore, the observation not only provides a new angle to understand the role of FS in QPTs, but also is of fundamental importance to apply quantum adiabatic theorem to scalable systems.

The critical exponents of the FS and their scaling relation then will be proposed in a more general way. Clearly, the FS’s dimension can be changed, or not changed in QPTs, this property classifies all QPTs into two different types. For a class of QPTs, the change in the FS’s dimension is a characteristic that separates two phases. As a non-trivial application, we will show that, in the gapless phase of the Kitaev honeycomb model, the dimension of the FS becomes $L^2 \ln L$ rather than $L^2$ in the gapped phase. Therefore, the extra dimension of $\ln L$ becomes a characteristic of the gapless phase.

Scaling analysis revisited. – The FS is defined as the leading term of the fidelity $\chi_F = -\lim_{\lambda \to 0} 2 \ln F(\delta \lambda)^2$. 

(a)E-mail: sjgu@phy.cuhk.edu.hk
For the Hamiltonian system (eq. (1)), the ground-state FS can be evaluated from [9,10]

$$\chi_F(\lambda) = \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda) | H_I | \Psi_0(\lambda) \rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2},$$

(2)

where $H(\lambda)|\Psi_n(\lambda) = E_n(\lambda)|\Psi_n(\lambda)$. Here we would like to point out that, in the low-energy spectra of the Hamiltonian, only those excitations with nonzero $\langle \Psi_n(\lambda) | H_I | \Psi_0(\lambda) \rangle$ have contribution to the FS. So if we define $\Delta$ as the energy gap between the ground state and the lowest excitation with a nonzero $\langle \Psi_n(\lambda) | H_I | \Psi_0(\lambda) \rangle$, the FS satisfies the following inequalities:

$$\chi_F \leq \frac{1}{\Delta^2} \sum_{n \neq 0} \frac{|\langle \Psi_n(\lambda) | H_I | \Psi_0(\lambda) \rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2} = \frac{1}{\Delta^2} \frac{\partial^2 E(\lambda)}{\partial \lambda^2} \leq \Delta^{-2}[\langle \Psi_0(\lambda) | H_I^2 | \Psi_0(\lambda) \rangle - \langle \Psi_0(\lambda) | H_I | \Psi_0(\lambda) \rangle]^2].$$

Let us suppose $H_I = \sum_r V(\tau)$ and $N = L^d$. 1) If the system is gapped, the FS has the same (or smaller) dependence on the system size as the second derivative of the ground-state energy. 2) If $\chi_F/N$ still increases with the system size, the ground state must be gapless.

The above inequalities can only tell us some qualitative information. To find the scaling relation, we resort to the definition of the FS in terms of correlation functions [9],

$$\frac{\chi_F}{L^d} = \sum_r \int_r \tau G(r, \tau) \mathrm{d} \tau,$$

(3)

where $G(r, \tau) = \langle V(\tau)V(0,0) \rangle - \langle V(\tau) \rangle \langle V(0,0) \rangle$. Under the scaling transformation [28] $r' = s \cdot r$, $\tau' = s^\nu \tau$, $V(r') = s^{-\Delta} V(r)$, it has been found that the FS scales like $\chi_F/L^d \propto L^{d-2\Delta_{FS}}$ [13] where $\zeta$ is the dynamic exponent and $\Delta_V$ is the scaling dimension of $V(\tau)$ at the critical point, and $\chi_F$ is believed to be extensive away from the critical point. The latter “common sense” is also the reason that, in the most previous studied, the FS is believed to have the same critical exponent $\alpha$ on both sides of the critical point, that is $\chi_F/L^d \propto |\lambda - \lambda_c|^{-\alpha}$. However, it is not universally true.

For a general $d$-dimensional system, the correlation function $G(r, \tau)$ under the scaling transformation, becomes

$$G(r', \tau') = s^{2(\Delta_l - d)} G(r, \tau)$$

(4)

$$= G(s r, s^{\nu} \tau),$$

(5)

where $\Delta_l$ comes from the transformation $\lambda' = s^{-\Delta} \lambda$. Let $s = \tau^{-1/\zeta}$, we then have

$$G(r, \tau) = \tau^{2(\Delta_l - d)/\zeta} G(\tau^{-1/\zeta}, 1).$$

(6)

If we rearrange the expression,

$$G(\tau^{-1/\zeta}, 1) = [\tau^{-1/\zeta}]^2 (\Delta_l - d) f(\tau^{-1/\zeta}),$$

(7)

which defines the scaling function $f(\tau^{-1/\zeta})$. The correlation function becomes

$$G(r, \tau) = \frac{1}{\tau^{2(\Delta_l - d)} f(\tau^{-1/\zeta})},$$

(8)

where $\Delta_V = d - \Delta_l$ is just the scaling dimension of $V(\tau)$. If the correlation length is divergent, the correlation function decays algebraically, and

$$\chi_F \sim \left\{ \begin{array}{ll}
L^{d-2\zeta - 2\Delta_V} & 2\Delta_V - 2\zeta \neq d, \\
\ln L, & 2\Delta_V - 2\zeta = d.
\end{array} \right.$$

(9)

For some collective systems, such as the Lipkin-Meshkov-Glick (LMG) model [29,30], we will show below that the scaling dimension of $\sigma^2$ in the polarized phase is $N^{-1}$, hence $\chi_F$ is intensive. Therefore, the FS has its own distinct dimension instead of real system’s dimension. The dimension is related to the size-dependence of the fidelity of a given driving Hamiltonian, and as we show in our another paper that $L^{d_a}$ actually determines the duration time scale in the quantum adiabatic theorem [31], we will call it quantum adiabatic dimension (QAD), and use $d_a$ to denote it hereafter.

Therefore, to judge a quantum criticality from the FS, the scaling relation and conditions should be revised. If the dimension of $\chi_F$ is $d_a^{(-)}$ above (below) the critical point, then the rescaled FS, i.e. $\chi_F/L^{d_a^{(-)}},$ as an intensive quantity, scales like

$$\chi_F \sim \frac{1}{|\lambda - \lambda_c|^{\alpha^\pm}},$$

(10)

in the thermodynamic limit. Since the FS usually shows a maximum around the critical point and scales like $\chi_F |_{\lambda=\lambda_c} \propto L^\mu$ ($\mu = 2\Delta + 2\zeta - 2\Delta_V$) [13], the general scaling relation should be

$$\alpha^\pm = \frac{\mu - d_a^{\pm}}{\nu}.$$  

(11)

Here $\nu$ is the critical exponent of the correlation length, which defines the length scale of the system around the critical point. In case the exponent $\nu$ is also different (though we did not find such a case yet), a more general relation is $\alpha^\pm = (\mu - d_a^{\pm})/\nu^\pm$. Hence the condition of singularity in the FS around the critical point is

$$\mu \geq d_a^{\pm}$$

(12)

instead of $\mu \geq d$ [13]. A further remark is that, even if the equality condition of eq. (12) is satisfied, it is still possible for the system to undergo a logarithmic divergence at one side of the critical point, i.e.

$$\frac{\chi_F}{L^{d_a^{\pm}}} \propto \ln|\lambda - \lambda_c|,$$

(13)

if $\chi_F/L^{d_a} |_{\lambda=\lambda_c}$ divergences as $\ln L$. Therefore, if and only if the QAD (including logarithmic dependence on the system...
size) of the FS are the same above, at, and below the critical point \( \lambda_c \), it is firm to say that the FS does not have singular behavior around the critical point. 

Equations (10)–(13) represent our first main result of this work. These key relations define also a criteria to judge quantum phase transitions in perspective of the FS. They are a generalization of the results by Venuti and Zanardi [13]. In short, in order to study the scaling behavior of the FS, one should consider the QAD of the driving Hamiltonian in the corresponding quantum phases. The QAD can be generalized to a Hamiltonian defined in a high-dimensional parameter space. In this case, different driving Hamiltonian might have different QAD. In the following, we will check the validity of about analysis in a well studied QPT of Landau’s type and a topological QPT.

Examples: the LMG model and the Kitaev honeycomb model. – We first check the above scaling relations in an exactly solvable model [29,30], i.e. LMG model. Its Hamiltonian reads

\[
H_{\text{LMG}} = -\frac{1}{N} \sum_{i,j} \left( \sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y \right) - h \sum_i \sigma_i^z, \tag{14}
\]

where \( \gamma \) denotes the anisotropy. The prefactor \( 1/N \) is to ensure a finite energy per spin. In the thermodynamic limit, the ground state of the system for \( \gamma \neq 1 \) undergoes a second-order QPT at \( h_c = 1 \). If \( h > h_c \), the system is fully polarized, while if \( 0 < h < h_c \) it is a symmetry-broken state. The ground-state fidelity of the LMG model has been addressed previously by Kwok et al. [18]. They found that \( \chi_F \) is proportional to the system size \( N \) in the symmetric-broken phase, while is intensive in the fully polarized phase. This difference makes that critical exponents to be different at both sides of the critical point. Kwok et al. [18] obtained that \( d_a^+ = 0 \) for \( h > 1 \), \( d_a^+ = 1 \) for \( h < 1 \), \( \mu = 4/3 \) at \( h_c = 1 \), and the correlation length exponent \( \nu = 2/3 \), then the critical exponent above \( h_c \) is \( \alpha^+ = (\mu - d_a^+)/\nu = 2 \), while below \( h_c \) is \( \alpha^- = (\mu - d_a^-)/\nu = 1/2 \).

The explicit differences of \( d_a^+ \) and \( \alpha \) in two quantum phases of the LMG model has already been a straight-forward demonstration that the FS has its own dimension. According to eq. (4), the deep reason of such a difference is that \( \sigma_i^x \sim N^{-1} \) in region \( h > 1 \), while \( \sigma_i^z \sim N^{-1/2} \) if \( 0 < h < 1 \), which leads to that the FS is intensive in the polarized phase, while extensive in the symmetry-broken phase.

As a non-trivial application, we study the FS in a topological QPT [32] occurring in the ground state of the Kitaev honeycomb model [33]. The model is associated with a system of 1/2 spins which are located at the vertices of a honeycomb lattice. The Hamiltonian reads

\[
H = -J_x \sum_{x \text{bonds}} \sigma_x^x \sigma_x^x - J_y \sum_{y \text{bonds}} \sigma_y^x \sigma_y^y - J_z \sum_{z \text{bonds}} \sigma_z^y \sigma_z^y, \tag{15}
\]

where \( j, k \) denote two ends of the corresponding bond linked to a vertex, and \( J_{\kappa} \) (\( \kappa = x, y, z \)) are coupling constants. The ground state of the Kitaev honeycomb model consists of two different phases [33], i.e., a gapped phase with Abelian anyonic excitations and a gapless phase with non-Abelian anyonic excitations. The critical behavior of the fidelity in the model was previously addressed by two groups [24,25]. None of them addressed the QAD of the FS, but took it for granted that \( d_a = d \). We define the phase diagram on the plane \( J_x + J_y + J_z = 1 \), and consider a certain line \( J_x = J_y \) along which the ground state of the system evolves at zero temperature. In this case, \( J_x \) is the only driving parameter. The Hamiltonian can be explicitly diagonalized in the flux-free subspace [33,34] and the ground-state FS for a system of \( N = 2L^2 \) (odd \( L \)) sites is [25]

\[
\chi_F = \frac{1}{N} \sum_q \left[ \sin q_x \sin q_y \right]^2, \tag{16}
\]

where \( q_{x(y)} = 2\pi n/L, n = -(L-1)/2, \cdots, (L-1)/2, \) and

\[
\epsilon_q = J_x \cos q_x + J_y \cos q_y + J_z, \quad \Delta_q = J_z \sin q_x + J_y \sin q_y. \tag{17}
\]

In the thermodynamic limit, the FS becomes

\[
\chi_F \frac{L^2}{L^2} = \frac{1}{64\pi^2} \int_{-\pi}^{\pi} dq_x \int_{-\pi}^{\pi} dq_y \left[ \frac{\sin q_x + \sin q_y}{\epsilon_q^2 + \Delta_q^2} \right]^2. \tag{18}
\]

In the gapped phase \( J_x > 1/2 \), \( \epsilon_q^2 + \Delta_q^2 > 0 \). So there is no pole in the integrand of eq. (18). \( \chi_F / L^2 \) is intensive and the QAD of the FS is \( d_a^+ = 2 \). However, in the gapless phase, \( \epsilon_q^2 + \Delta_q^2 \) has zero points in \( k \) space. Expanding the integration around the pole of the integrand, we find...
\[ \chi_F/L^2 \propto \int_{\Lambda/L}^1 \frac{1}{x} \ln(x) \sim \ln L, \]
where \( \Lambda \) is a cut-off. Therefore, \( \chi_F/L^2 \) manifests distinct size dependence in the gapped and gapless phases. As a numerical demonstration, we show the FS \( \chi_F/L^2 \) as a function of \( \ln L \) in fig. 1 (left) for various \( J_z \). Obviously, \( \chi_F/L^2 \) does not change as the system size increases in the gapped phase, but is proportional to \( \ln L \) in the gapless phase. At the critical point \( J_z = 1/2 \), it has been shown that \( \chi_F \propto L^{5/2} \) [25]. Therefore, according to the scaling analysis, the critical exponents of the FS should be different at both sides of \( J_z = 1/2 \). At \( J_z = (1/2)^+ \) it has been already obtained \( \alpha^+ \approx 0.5 \), here we find that

\[ \frac{\chi_F}{L^2} \sim \ln |J - J_{z,c}|, \]

(19)

at \( J_z = (1/2)^- \) (simplified as 1/2-ln for QAD and shown in fig. 1 (right)). Moreover, differ from both the quantum Ising model and the LMG model, the FS has higher dimension \( 2 + \ln \) (for \( \chi_F \propto L^2 \ln L \)) than the system’s dimension 2 in the gapless phase. The extra dimension of \( \ln L \) has a special meaning because it only exists in the gapless phase with non-Abelian anyonic excitations. Therefore, it can be used as a characteristic of the gapless phase in the Kitaev honeycomb model.

From the point view of eq. (4), the deep reason behind the extra dimension \( \ln L \) is that the bond-bond correlation function of the \( z \)-bond in eq. (15) decays exponentially in the gapped phase, while algebraically, i.e., \( G(r,0) \sim 1/r^4 \), in the gapless phase [25]. Therefore, the FS in eq. (4) becomes

\[ \frac{\chi_F}{L^D} - \sum_r \frac{r^{1-2\zeta}}{r^{D-2\zeta}}. \]

(20)
The \( \ln L \) divergence of the FS \( \chi_F/L^D \) means that the dynamic exponent should be \( \zeta = 1 \).

Finally, we summarize the critical exponents of the FS in the above two typical models, the quantum Ising model, and the Kitaev toric model (KTM) in table 1. The critical exponents of the FS in the latter two models are referred from refs. [4] and [26], respectively. The table can be prolonged as one consider more and more QPTs. No matter what transitions are included, the QPTs can be classified into two distinct classes. For the first class, the FS has the same QAD in the both quantum phases, such as the quantum Ising model, while for the another class, the FS has different QAD, such the LMG model and the Kitaev honeycomb model. Moreover, for the Kitaev honeycomb model, we can see that the QAD actually plays a role of characteristic that separates two quantum phases. These conclusions constitute our second main result.

**A brief summary and a challenge.** – In summary, we have analyzed the dimension of the FS in quantum phase transitions. We have shown that the QAD of the driving Hamiltonian is not always the same as the system’s dimension. The scaling relation of the FS in various QPTs is established on more general grounds. The FS might have distinct critical exponent at both sides of the critical point. So the QPTs can be divided into two classes based on the criteria if the QAD is changed or not during the phase transition. Our results also show that the QAD can be used as a characteristic that separates two topological phases in the Kitaev honeycomb model. Therefore, the QAD provides a quite distinct tool instead of the traditional order parameter to study QPTs.

Clearly, the QAD provides a unique classification of quantum phases. From this point of view, a challenging problem might be that, for a \( d \)-dimensional system, does there exist such a quantum phase with a QAD of \( d+1 \)?

---

**REFERENCES**

1. Sachdev S., Quantum Phase Transitions (Cambridge University Press, Cambridge, UK) 2000.
2. Nilesen M. A. and Chuang I. L., Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK) 2000.
3. Quan H. T., Song Z., Liu X. F., Zanardi P. and Sun C. P., Phys. Rev. Lett., 96 (2006) 140604.
4. Zanardi P. and Paunkovic N., Phys. Rev. E, 74 (2006) 031123.
5. Zanardi P., Cozzini M. and Giorda P., J. Stat. Mech. (2007) L02002.
6. Cozzini M., Giorda P. and Zanardi P., Phys. Rev. B, 75 (2007) 014439.
7. Cozzini M., Ionicioiu R. and Zanardi P., Phys. Rev. B, 76 (2007) 104420.
Scaling dimension of fidelity susceptibility in quantum phase transitions

[8] Buonsante P. and Vezzani A., Phys. Rev. Lett., 98 (2007) 110601.
[9] You W. L., Li Y. W. and Gu S. J., Phys. Rev. E, 76 (2007) 022101.
[10] Zanardi P., Giorda P. and Cozzini M., Phys. Rev. Lett., 99 (2007) 100603.
[11] Zhou H. Q. and Barjaktarevic J. P., J. Phys. A: Math. Theor., 41 (2008) 412001.
[12] Zhou H. Q., Zhao J. H. and Li B., J. Phys. A: Math. Theor., 41 (2008) 492002.
[13] Venuti L. C. and Zanardi P., Phys. Rev. Lett., 99 (2007) 095701.
[14] Gu S. J., Kwok H. M., Ning W. Q. and Lin H. Q., Phys. Rev. B, 77 (2008) 245109.
[15] Chen S., Wang L., Hao Y. and Wang Y., Phys. Rev. A, 77 (2008) 032111.
[16] Zanardi P., Quan H. T., Wang X. G. and Sun C. P., Phys. Rev. A, 75 (2007) 032109.
[17] Yang M. F., Phys. Rev. B, 76 (2007) 180403(R); Tzeng Y. C. and Yang M. F., Phys. Rev. A, 77 (2008) 012311; Tzeng T. C., Hung H. H., Chen Y. C. and Yang M. F., Phys. Rev. A, 77 (2008) 062321.
[18] Kwok H. M., Ning W. Q., Gu S. J. and Lin H. Q., Phys. Rev. E, 78 (2008) 032103.
[19] Paunković N., Sacramento P. D., Nogueira P., Vieira V. R. and Dugaev V. K., Phys. Rev. A, 77 (2008) 052302.
[20] Wang X., Sun Z. and Wang Z. D., Phys. Rev. A, 79 (2009) 012105.
[21] Kwok H. M., Ho C. S. and Gu S. J., Phys. Rev. A, 78 (062302) 2008.
[22] Ma J., Xu L., Xiong H. and Wang X., Phys. Rev. E, 78 (2008) 051126; Lu X. M., Sun Z., Wang X. and Zanardi P., Phys. Rev. A, 78 (2008) 032309.
[23] Hamma A., Zhang W., Haas S. and Lidar D. A., Phys. Rev. B, 77 (2008) 155111.
[24] Zhao J. H. and Zhou H. Q., Phys. Rev. B, 80 (2009) 014403.
[25] Yang S., Gu S. J., Sun C. P. and Lin H. Q., Phys. Rev. A, 78 (2008) 012304.
[26] Abasto D. F., Hamma A. and Zanardi P., Phys. Rev. A, 78 (2008) 010301(R).
[27] Quan H. T. and Cucchietti F. M., Phys. Rev. E, 79 (2009) 031101.
[28] Continentino M. A., Quantum Scaling in Many-Body Systems (World Scientific Publishing, Singapore) 2001.
[29] Lipkin H. J., Meshkov N. and Glick A. J., Nucl. Phys., 62 (1965) 188; Meshkov N., Glick A. J. and Lipkin H. J., Nucl. Phys., 62 (1965) 199; Meshkov N., Lipkin H. J. and Glick A. J., Nucl. Phys., 62 (1965) 211.
[30] Dusuel S. and Vidal J., Phys. Rev. Lett., 93 (2004) 237204; Phys. Rev. B, 71 (2005) 224420.
[31] Gu S. J., Phys. Rev. E, 79 (2009) 061125.
[32] Wen X. G., Quantum Field Theory of Many-Body Systems (Oxford University, New York) 2004.
[33] Kitaev A., Ann. Phys. (N.Y.), 303 (2003) 2; 321 (2006) 2.
[34] Feng X. Y., Zhang G. M. and Xiang T., Phys. Rev. Lett., 98 (2007) 087204.