THE PRODUCT RULE IN $\kappa^*(\mathcal{M}_{g,n}^{ct})$

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Abstract. We describe explicit formulas for the product rule in $\kappa^*(\mathcal{M}_{g,n}^{ct})$.

1. Introduction

Let $\epsilon : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map, viewed as the universal curve over $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable curves of genus $g$ with $n$ marked points. Let $L_i \to \overline{\mathcal{M}}_{g,n+1}$ be the cotangent line bundle over $\overline{\mathcal{M}}_{g,n+1}$, whose fiber over a given curve is the cotangent space at the $i$th marked point. Define $\psi_i = c_1(L_i) \in A^1(\overline{\mathcal{M}}_{g,n+1})$ and $\kappa_i = \epsilon^*(\psi_{i+1}^{n+1}) \in A^i(\overline{\mathcal{M}}_{g,n})$.

By restriction, one can define the psi and kappa classes over $\overline{\mathcal{M}}_{g,n}$, the moduli space of smooth curves of genus $g$ with $n$ marked points, and $\mathcal{M}_{g,n}^{ct}$, the moduli space of curves of compact type. Consider the sub-ring of $A^*(\mathcal{M}_{g,n}^{ct})$ generated by the kappa classes, and call it the kappa ring. In [3] Pandharipande described an additive basis for the kappa ring.

Theorem. Given $D, n \in \mathbb{N}$, the set

$$\{ \kappa_p \mid p \in P(D, 2g + n - D - 2) \}$$

generates $\kappa^D(\mathcal{M}_{g,n}^{ct})$ as a $\mathbb{Q}$-vector space, and for $n > 0$ it is a $\mathbb{Q}$-basis.

The natural question to ask, as first raised by Pandharipande [3], is to determine explicit formulas for the product rule in the kappa ring of $\mathcal{M}_{g,n}^{ct}$, that is the main result of this paper.

Theorem 1. Let $A = \{a_1, \ldots, a_k\}$ be a multi-set of integers, and $n, g, d \in \mathbb{N}$ be integers such that $d = 2g + n - \sum a_i - 2$. In $A^*(\mathcal{M}_{g,n}^{ct})$ we have:

$$\kappa_{a_1} \cdots \kappa_{a_k} = \sum_{p \in SP(A, d)} x_p \kappa_p(A)$$

where

$$x_p = \sum_{\ell(t) \leq \ell(r) \leq \ell(p)} \frac{(-1)^{\ell(t) + \ell(t) + \ell(r) + M}}{(|t| + 1)!} \left(\frac{\ell(t) - \ell(r)}{M - \ell(r)}\right) \prod_{j=1}^{\ell(r)}(|r_j + 1|)! \prod_{l=1}^{\ell(p)}(\ell(r|p|)_l - 1)!$$

in which $M = \min\{\ell(t), d\}$, $SP(A)$ (resp. $SP(A, d)$) is the set of partitions of the multi-set $A$ (resp. into at most $d$ parts) and $\ell(p)$ is the number of components of $p$ (for notations see Definition [3] and Definition [9]).
Plan of the paper. In Section 2 we review some known results relating kappa classes and pushforwards of the psi classes. Section 3 contains the proof of the main theorem. The main idea is to use the following theorem of Pandharipande.

Theorem 2. [3] There is a canonical surjective map \( \iota_{g,n} : \kappa^*(\overline{\mathcal{M}}_{0,n+2g}) \to \kappa^*(\mathcal{M}_{g,n}^c) \) which is an isomorphism for \( n > 0 \).

This allows us to reduce the computation to the case of genus zero. In the genus zero case, by the work of Keel [2] we have a very good understanding of the Chow ring, and we can explicitly compute all the required classes. In Section 4 we prove the combinatorial identity used in the Section 3.

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2. Kappa and Psi classes

In this section we explain the relation between the kappa classes and pushforwards of the psi classes. Before stating the results, we need to fix some notations.

Definition 3. Let \( \pi_{g,n} : \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n} \) denote the forgetful map which forgets the last \( k \) marked points, and let \( \mathbf{q} \) be the multi-set \( \{q_1, \cdots, q_k\} \). We define:

- \( \psi(\mathbf{q}) = \psi(q_1, \cdots, q_k) := (\pi_{g,n})_* \left( \prod_{i=1}^{k} \psi_{q_i+1}^{q_i+1} \right) \).
- \( \kappa_{\mathbf{q}} = \kappa_{q_1, \cdots, q_k} := \prod_{i=1}^{r} \kappa_{q_i} \).

Definition 4. Let \( \mathbf{A} = \{a_1, \cdots, a_k\} \) be a multi-set. Given \( \sigma \in S_k \) with the cycle decomposition \( \sigma = \gamma_1 \cdots \gamma_r \) (including the 1-cycles). We define:

- \( \sigma_1(\mathbf{a}) := \sum_{j \in n} a_j \).
- \( \sigma(\mathbf{A}) = \{\sigma_1(\mathbf{A}), \cdots, \sigma_r(\mathbf{A})\} \) (as multi-set).

Definition 5. Let \( \mathbf{A} = \{a_1, \cdots, a_k\} \) be a multi-set, we denote the set of partitions of \( \mathbf{A} \) by \( SP(\mathbf{A}) \). Given \( \mathbf{p} \in SP(\mathbf{A}) \), we use the following notations.

- \( |\mathbf{A}| = \sum_{j=1}^{k} a_j \).
- \( SP(\mathbf{A}; l) \) (resp. \( SP(\mathbf{A}, l) \)) denotes the set of partitions of \( \mathbf{A} \) with exactly (resp. at most) \( l \) parts.
- We denote the components of \( \mathbf{p} \) by \( \mathbf{p}_i \), i.e. \( \mathbf{p} = \bigsqcup \mathbf{p}_i \).
• \( \ell(p) \) denotes the number of components of \( p \). Also we write \( \ell(A) = k \) and from the context it should be clear whether we work with a multi-set or a partition, so there should be no confusions.
• \( |p| = \{ |p_1|, \ldots, |p_{\ell(p)}| \} \) (as multi-set)
• By abuse of notation we write \( \psi(p) \) instead of \( \psi(|p|) \), i.e.

\[
\psi(p) = \psi(|p|) = \left( \pi_{g,n}^{\ell(p)} \right) \prod_{i=1}^{\ell(p)} \psi_{n+i}^{p_i+1}.
\]

The relation between the kappa classes and pushforward of the psi classes, due to Faber, is (see [1]):

\[
\psi(a_1, \ldots, a_k) = \sum_{\sigma \in S_k} \kappa_{\sigma(a)}.
\]

**Lemma 6.** ([1, 3 Lemma 11]) The subset of \( A^d(M_{g,n}) \) defined by

\[
\{ \psi(A) \mid A \in \text{P}(d) \} \quad \text{and} \quad \{ \kappa_A \mid A \in \text{P}(d) \}
\]

are related by an invertible linear transformation independent of \( g \) and \( n \).

The map in one direction is given by the Faber’s formula. The inverse map is given by the following proposition.

**Proposition 7.** \( \kappa_A = \sum_{p \in SP(A)} (-1)^{n+\ell(p)} \psi(p) \).

**Proof.** Given \( q \in SP(A) \), we say that a pair \( (p, \sigma) \) with \( p \in SP(A) \) and \( \sigma \in S_{\ell(p)} \) splits \( q \) if \( \sigma(|p|) = q \). Using Faber’s formula we have:

\[
\sum_{p \in SP(A)} (-1)^{n+\ell(p)} \psi(p) = \sum_{p \in SP(A)} \sum_{\sigma \in S_{\ell(p)}} (-1)^{n+\ell(p)} \kappa_{\sigma(|p|)}
\]

\[
= (-1)^n \sum_{q \in SP(A)} \kappa_q \left( \sum_{(p, \sigma) \text{ splits } q} (-1)^{\ell(p)} \right).
\]

Let \( \left\{ \binom{n}{k} \right\} \) denote the number of partitions of a set of \( n \) elements into \( k \) parts (the Stirling numbers of the second kind). Given a splitting \( (p, \sigma) \) of \( q = \bigcup_{i=1}^{n} q_i \), by restriction of \( p \) to each \( q_i \) we obtain an element \( p_i \) of \( SP(q_i) \) and the restriction of \( \sigma \) is a permutation with only one cycle of length \( \ell(p_i) \). Using the identity

\[
\sum_{k=1}^{n} (-1)^k (k-1)! \left\{ \binom{n}{k} \right\} = -\delta_1^n
\]

...
for the Stirling numbers of the second kind (see [11]), we have:

\[
\sum_{(p,\sigma) \text{ splitting of } q} (-1)^\ell(p) = \prod_{i=1}^r \left( \sum_{p_i \in SP(q_i)} (-1)^\ell(p_i)(\ell(p_i) - 1)! \right)
\]

\[
= \prod_{i=1}^r \left( \sum_{k=1}^r (-1)^k (k - 1)! \left\{ \frac{\ell(q_i)}{k} \right\} \right)
\]

\[
= (-1)^r \prod_{i=1}^r \delta_1^{\ell(q_i)}.
\]

Hence if for some \( i \) we have \( \ell(q_i) > 1 \) then the coefficient of \( \kappa_q \) is zero. Therefore the only term with non-zero contribution is \( q_0 = \{a_1\} \cup \{a_2\} \cup \cdots \cup \{a_n\} \). Therefore we have:

\[
\sum_{p \in SP(A)} (-1)^{n+\ell(p)} \psi(p) = \kappa_{q_0}
\]

\[
= \kappa(A)
\]

\[
\square
\]

**Example 8.**

\[
\sum_{p \in SP(\{a,b,c\})} (-1)^{3+\ell(p)} \psi(p) = \psi(a, b, c) - \psi(a + b, c) - \psi(a + c, b) - \psi(b + c, a) + \psi(a + b + c)
\]

\[
= (\kappa_a\kappa_b\kappa_c + \kappa_{a+b}\kappa_c + \kappa_{a+c}\kappa_b + \kappa_{b+c}\kappa_a + 2\kappa_{a+b+c})
\]

\[
- (\kappa_{a+b}\kappa_c + \kappa_{a+c}\kappa_b + \kappa_{a+b+c}) - (\kappa_{a+c}\kappa_b + \kappa_{a+b+c}) - (\kappa_{b+c}\kappa_a + \kappa_{a+b+c})
\]

\[
+ \kappa_{a+b+c}
\]

\[
= \kappa_{a+b+c}
\]

3. **Product of Kappa Classes**

As we explained in the introduction, Theorem 2 shows that any relation in \( \kappa^*(\overline{M}_{0,n}) \) is also valid in \( \kappa^*(\overline{M}_{g,n}) \). Moreover for \( n > 1 \) all the relations in \( \kappa^*(\overline{M}_{g,n}) \) are obtained in this manner. Therefore in order to prove Theorem 11 it is enough to show that those relations hold in \( \kappa^*(\overline{M}_{0,n}) \). Thus we start by computing the product rule in the kappa ring of \( \overline{M}_{0,n} \), but we need to fix some more notations.

**Definition 9.** Let \( A = \{a_1, \cdots, a_k\} \) be a multi-set.

- \( A! = \prod_{i=1}^k a_i! \).
- \( A + 1 = \{a_i + 1 \mid i = 1, \cdots, k\} \).
- \( \left| A \right| = \left( \sum_{i=1}^k a_i \right) \).
- \( \left\lfloor A \right\rfloor = \left\lfloor \left( a_1, a_2, \cdots, a_k \right) \right\rfloor \).
- Given \( p, q \in SP(A) \), we write \( q \leq p \) if \( q \) is a refinement of \( p \). We use the following notations.
  - \( \left| p : q \right| \in SP(|q|) \) denotes the partition induced from \( p \) on the multi-set \( |q| \). Note that \( \ell(p) = \ell(|p : q|) \).
– $q|_{p_i} \in SP(p_i)$ denotes the partition induced from $q$ on the multi-set $p_i$ (by restriction).

- $\lambda_A = \lambda_{a_1, \ldots, a_n} := \sum_{p \in SP(A)} (-1)^{n+\ell(p)} \frac{(| \{p \} + 1|)}{|p| + 1}$.

In the following proposition we compute the product rule in the top degree of $\kappa^*_n(\overline{M}_{0,n})$.

**Proposition 10.** Let $A = \{a_1, \ldots, a_k\}$ be a multi-set of integers such that $\sum a_i = n - 3$. Then in $A^{n-3}(\overline{M}_{0,n})$ we have:

$$\kappa_{a_1, \ldots, a_l} = \lambda_{a_1, \ldots, a_l}^{k_{n-3}}.$$ 

**Proof.** Since $A^{n-3}(\overline{M}_{0,n}) = Q$, we have to integrate both sides over $\overline{M}_{0,n}$. Given $p = \prod_{i=1}^r p_i \in SP(A)$, we have:

$$\int_{\overline{M}_{0,n}} \psi(p) = \int_{\overline{M}_{0,n+r}} \prod_{i=1}^r \psi_{|p_i|+1}$$

$$= \left( \frac{|p_1| + \cdots + |p_r| + r}{|p| + 1} \right)$$

Therefore the result follows from Proposition 7. $\square$

**Definition 11.** Let $A = \{a_1, \ldots, a_k\}$ be a multi-set, $q \leq p \in SP(A)$.

$$N_A = N_{a_1, \ldots, a_n} := \sum_{r \in SP(A)} (-1)^{n+\ell(r)}(\ell(r) - 1)! \prod_{j=1}^{\ell(r)} \frac{|r_j + 1|!}{a_i + 1}$$

$$= \sum_{r \in SP(A)} (-1)^{n+\ell(r)}(\ell(r) - 1)! \prod_{j=1}^{\ell(r)} \left( \frac{|r_j + 1|}{r_j + 1} \right).$$

$$N_p := \prod_{i=1}^{\ell(p)} N_{p_i}.$$ 

Note that $N_{a_i} = 1$.

**Definition 12.** A genus zero $n$–pointed stable weighted graph is a connected graph $G$ together with a map $m : V(G) \to 2^\{1, \ldots, n\}$ that satisfies the following conditions.

- The sets $m(v)$ (for $v \in V(G)$) form a partition of $\{1, \cdots, n\}$. We call $m(v)$ the set of markings at the vertex $v$.
- For each $v \in V(G)$ we have $d(v) := \text{deg}(v) + \#m(v) \geq 3$.
- $H^1(G) = 0$, i.e. $G$ is a tree.
For any genus zero \( n \)-pointed stable weighted graph \( G \), we have a cycle \([G]\) \( \subset \mathcal{M}_{0,n} \) obtained as follows. We take \([G]\) to be the image of the map

\[
\prod_{v \in V(G)} \mathcal{M}_{0,d(v)} \to \mathcal{M}_{0,n}
\]

obtained by gluing along the edges of \( G \).

**Remark 13.** In [2] Keel proved that \( A^*(\mathcal{M}_{0,n}) \) is generated by cycles of the form \([G]\), and it has perfect pairing. Therefore given a Chow class \( \alpha \in A^*(\mathcal{M}_{0,n}) \), in order to show that \( \alpha \) vanishes it is enough to show that the pairing of \( \alpha \) against all the classes of the form \([G]\) are zero. Note that if \( \alpha \) belongs to the kappa ring, then \( \alpha.[G] \) depends only on the dimension of components of \([G]\) and not the distribution of the markings. Thus \( \alpha.[G] \) depends only on the dimension sequence of \( G \), i.e. \( \{d(v) - 3 : v \in V(G)\} \).

The following theorem describes an additive basis for the kappa ring of \( \mathcal{M}_{0,n} \).

**Theorem 14.** [3] Given \( D \in \mathbb{N} \), a \( \mathbb{Q} \)-basis of \( \kappa^D(\mathcal{M}_{0,n}) \) is given by

\[
\{\kappa_p | p \in P(D, n - D - 2) \}.
\]

**Theorem 15.** Let \( A = \{a_1, \cdots, a_k\} \) be a multi-set of integers, and \( n, d \in \mathbb{N} \) be such that \( d = n - \sum a_i - 2 \). In \( A^*(\mathcal{M}_{0,n}) \) we have:

\[
\kappa_{a_1, \cdots, a_k} = \sum_{p \in SP(A, d)} \left( \sum_{q \leq p, \ell(q) \leq d} \lambda_q N_{[p,q]} \right) \kappa_p.
\]

**Proof.** Using Remark 13 it is enough to check that the integral of both sides against cycles of the form \([G]\) are equal. Note that both sides belong to \( A^{n-d-2}(\mathcal{M}_{0,n}) \), so they can be paired with cycles with exactly \( (n - 3) - (n - d - 2) = d - 1 \) nodes. Thus we have to consider cycles that their dimension sequence have exactly \( d \) terms.

If the integral of the left (or right) hand side over \([G]\) is non-zero, then there is a way to put \( \kappa_{a_1}, \cdots, \kappa_{a_k} \) on irreducible components of \([G]\) such that the dimension of each component is equal to the sum of degree of kappa classes over it. Therefore there is a partition \( p \) of the multi-set \( \{a_1, \cdots, a_k\} \) (with at most \( d \) parts) such that the multi-set of dimension of irreducible components of \([G]\) is equal to \( |p| \). We call such \( G \) a \( p \)-graph, and call \([G]\) a \( p \)-cycle. Therefore it is enough to check our claim for all the strata of the form \([G]\), where \( G \) is a \( p \)-graph for some \( p \in SP(A, d) \).

By Theorem 14 kappa classes of the from \( \kappa_p \) for \( p \in SP(A, d) \) form an additive basis for the kappa ring, therefore for each \( p \) there exist (a unique) \( x_p \) such that

\[
\kappa_{a_1, \cdots, a_k} = \sum_{p \in SP(A, d)} x_p \kappa_p(A).
\]

Hence we have to prove that

\[
x_p = \sum_{q \leq p, \ell(q) \leq d} \lambda_q N_{[p,q]}.
\]
We prove this by induction on $\ell(p)$. Given a partition $p$ with $\ell(p) = d$, we integrate both sides of (3.1) against a $p$-cycle $[G]$. By Proposition 10, the integral of the left side is $\lambda_p$, and the integral of the right hand side is $x_p$. Hence we get $x_p = \lambda_p$, which confirms 3.2.

Fix $p \in SP(A)$ with $\ell(p) = e < d$. Similarly by Proposition 10, the integral of the left hand side of (3.1) is $\prod \lambda_{p_i} = \lambda_p$, and the integral of the right hand side is given by

$$\sum_{q \leq p, \ell(q) \leq d} x_q \lambda_{[p,q]}.$$  

Since any partition $q$ in this sum, except $p$, has length at least $e + 1$ by the induction hypothesis we know that

$$x_q = \sum_{r \leq q, \ell(r) \leq d} \lambda_r N_{[q,r]}.$$  

Thus we obtain

$$x_p = \lambda_p - \sum_{q < p, \ell(q) \leq d} \left\{ \sum_{r \leq q, \ell(r) \leq d} \lambda_r N_{[q,r]} \right\} \lambda_{[p,q]}$$

$$= \lambda_p - \sum_{r < p, \ell(r) \leq d} \lambda_r \left\{ \sum_{q < p} N_{[q,r]} \lambda_{[p,q]} \right\}.$$  

Let $p = p_1 \cdots p_e$, in order to simplify the formulas we denote $q_{[p_i]}$ (resp. $r_{[p_i]}$) by $q'_i$ (resp. $r'_i$), then $\lambda_{[p,q]} = \prod_{i=1}^e \lambda_{q'_i[i]}$ and $N_{[q,r]} = \prod_{i=1}^e N_{q'_i[r'_i,i]}$. Therefore we obtain:

$$x_p = \lambda_p - \sum_{r < p, \ell(r) \leq d} \lambda_r \left\{ \sum_{q < p} \prod_{i=1}^e N_{q'_i[r'_i,i]} \lambda_{q'_i[i]} \right\}$$

$$= \lambda_p - \sum_{r < p, \ell(r) \leq d} \lambda_r \left[ -N_{[p,r]} + \prod_{i=1}^e \left( \sum_{q'_i \in SP(p_i)} N_{q'_i[r'_i,i]} \lambda_{q'_i[i]} \right) \right].$$  

In the first line $q < p$ is a proper sub partition, and in the second identity we added the term $-N_{[p,r]}$ and took the sum over all partitions. Therefore in order to finish the proof, it is enough to check that the terms in the parenthesis are zero, which is the content of the following lemma.

**Lemma 16.** Given a multi-set $A$ and a partition $s$ of $A$. We have

$$\sum_{s \leq p \in SP(A)} N_{[p,s]} \lambda_{[p]} = 0.$$
Proof. We have

\[ \sum_{s \leq p \in SP(A)} N_{[p:s]} \lambda_{[p]} = \sum_{p \in SP(|s|)} N_p \lambda_{[p]}. \]

Hence it is enough to check that the term in the left is zero. We denote the multi-set \( s \) by \( B \), and \( b := \sharp B \).

\[ \sum_{p \in SP(B)} N_p \lambda_{[p]} = \sum_{p \in SP(B)} \lambda_{[p]} \prod_{i=1}^{\ell(p)} N_{p_i} \]

\[ = \sum_{r \leq p \leq q \in SP(B)} (-1)^{\ell(p) + \ell(q)} \binom{|q + 1|}{q + 1}. \]

\[ (-1)^{b + \ell(r)} \prod_{i=1}^{\ell(r|p_i|)} \prod_{j=1}^{\ell(r_j|p_i|)} \left( \binom{|r_j| + 1}{r_j + 1} (\ell(r_j|p_i|) - 1)! \right) \]

(\text{where } r|p_i| = \prod_{j=1}^{\ell(r|p_i|)} r_j^i \text{ is the partition induced by } r \text{ on the set } p_i)

The data of a triple \( r \leq p \leq q \in SP(B) \) is equivalent to the data of \( r \leq q \in SP(B) \) plus the data of a partition \( p' := [p : r] \leq [q : r] \in SP(|r|) \). The data of a partition \( p' := [p : r] \leq [q : r] \in SP(|r|) \) is equivalent to the following.

For each \( 1 \leq l \leq \ell(q) = \ell([q : r]) \), the data of a partition

\[ p'_l = \prod_{j=1}^{\ell(p'_l)} p'^{ij}_l \]

of the multi-set \([q : r]|_l\).

Note that any multi-set \( p_i \) induces a unique \( p'^{ij}_l \) and we have

\[ \ell(r|p_i|) = \text{number of components or } r \text{ in } p_i \]
\[ = \text{number of elements of } p'^{ij}_l \text{ as a partition of the multi-set } |r| \]
\[ = \ell(p'^{ij}_l) \]

Also if we vary \( i \) and \( j \), \( r^i_j \) runs over all components of \( r \), hence

\[ \prod_{i=1}^{\ell(p)} \prod_{j=1}^{\ell(r|p_i|)} \left( \frac{|r_j^i| + 1}{r_j^i + 1} \right) = \prod_{i=1}^{\ell(r)} \left( \frac{|r_i + 1|}{r_i + 1} \right). \]

Therefore
\[ \sum_{p \in SP(B)} N_p \lambda_{|p|} = \sum_{r \leq q \in SP(B)} (-1)^{\ell(q) + b + \ell(r)} \left( |q + 1| \right) \prod_{i=1}^{\ell(r)} \left( |r_i + 1| \right) \tag{\ell(r)} \]

\[ \prod_{l=1}^{\ell(t)} \left( \sum_{p'_l \in SP(\{|q|r_i\})} (-1)^{\ell(p'_l)} \prod_{j=1}^{\ell(r_l)} \left( |p'_l| - 1 \right) \right) \]

Note that for a partition \( t \) of a set \( X \) there are exactly \( \prod_{j=1}^{\ell(t)} (\ell(t_j) - 1)! \) permutations \( \sigma \) of \( X \) such that the partition of \( X \) obtained from the cycle decomposition of \( \sigma \) is equal to \( t \). Hence we have

\[ \sum_{t \text{ a partition of } X} (-1)^{\ell(t)} \prod_{j=1}^{\ell(t)} (\ell(t_j) - 1)! = \sum_{\sigma \in S_{|X|}} (-1)^{|\sigma|}. \]

Therefore if \( r \neq q \) then the term

\[ \prod_{l=1}^{|q|} \left( \sum_{p_l \text{ is a partition of } s(r_l)} (-1)^{|p_l|} \prod_{j=1}^{\ell(r_l)} (|p'_l| - 1)! \right) \]

vanishes, and if \( r = q \) we get \((-1)^{\ell(q)}\). So

\[ \sum_p N_p \lambda_{s(p)} = \sum_q (-1)^{\ell(q) + b + \ell(r)} \left( \left| q + 1 \right| \right) \prod_{i=1}^{\ell(r)} \left( \left| r_i + 1 \right| \right) \]

\[ = \sum_{k=1}^b (-1)^{k+1} \binom{n-1}{k-1} \left( \left| B + 1 \right| \right) (-1)^{|B|} \]

\[ = 0. \]

\[ \blacksquare \]

**Theorem 17.** Let \( A = \{a_1, \ldots, a_k\} \) be a multi-set of integers such that \( d = n - |A| - 2 \). In \( A^*(\mathcal{M}_{0,n}) \) we have:

\[ \kappa_{a_1, \ldots, a_k} = \sum_{p \in SP(A,d)} x_p \kappa_p(A) \]

where

\[ x_p = \sum_{t \leq r \leq p} \frac{(-1)^{\ell(A) + \ell(t) + \ell(r)}}{(|t| + 1)!} \prod_{j=1}^{\ell(r)} \left( |F_j + 1|! \right) \prod_{l=1}^{\ell(r)} \left( \ell(r_{l|p_i|}) - 1 \right)! \cdot (-1)^M \left( \ell(t) - \ell(r) \right) \]

and \( M = \min\{\ell(t), d\} \).
Proof. We define $C_k(\mathbf{A})$ as follows.

$$C_k(\mathbf{A}) = \sum_{\ell(q)=k} \lambda_q N_{|q|}$$

$$= \sum_{q \in SP(A;k)} \left( \prod_{i=1}^{k} \lambda_{q_i} \right) (-1)^{k+\ell(r)} \left( \prod_{j=1}^{\ell(r)} \left( \frac{|r_j| + 1}{r_j + 1} \right) \right) (\ell(r) - 1)!$$

The data of $q \in SP(A; k)$ and $r \in SP(|q|)$ is equivalent to the data of $q \leq r \in SP(A)$ with $q \in SP(A; k)$, and this is equivalent to the data of $r \in SP(A)$ plus the following. For each $1 \leq j \leq \ell(r)$ a partition

$$q'_j := q_{r_j} = \prod_{i=1}^{\ell(q'_j)}$$

of the multi-set $r_j$, such that $\sum \ell(q'_j) = k$. For such pairs $(q, r)$ and $(r, \{q'_j\}_j)$, we have

$$\left( \frac{|r_j + 1|}{r_j + 1} \right) = \left( \frac{|r_j| + d_j}{|q'_j| + 1} \right),$$

and as in the proof of Lemma 16 we have $\prod_i \lambda_q = \prod_{i,j} \lambda_{q'_{ij}}$.

$$C_k(\mathbf{A}) = \sum_{r \in SP(A)} \left( \prod_{j=1}^{\ell(r)} \left( \frac{|r_j| + d_j}{|q'_j| + 1} \right) \prod_{i=1}^{d_j} \lambda_{q'_{ij}} \right)$$

$$= \sum_{r \in SP(A)} \left( (-1)^{k+\ell(r)} (\ell(r) - 1)! \right)^{\ell(r)} \prod_{j=1}^{\ell(r)} \left( \prod_{i=1}^{d_j} \lambda_{q'_{ij}} \right)$$

$$= \sum_{r \in SP(A)} \left( (-1)^{k+\ell(r)} (\ell(r) - 1)! \right)^{\ell(r)} \prod_{j=1}^{\ell(r)} \left( \prod_{i=1}^{d_j} \lambda_{q'_{ij}} \right)$$

$$= \sum_{r \in SP(A)} \left( (-1)^{k+\ell(r)} (\ell(r) - 1)! \right)^{\ell(r)} \prod_{j=1}^{\ell(r)} \left( \prod_{i=1}^{d_j} \lambda_{q'_{ij}} \right)$$
In the second line we expanded $\lambda_{d^i_t}$, and in the third line we switch the role of $q^j_i$ and $t^i_j$ which by now is standard. In the fourth line we used Theorem 22.

Now we switch the role of $t$ and $r$, and take the sum over $t$ and $r$. We use the fact that $||t_i|+1|$ for a partition $t_i \in SP(r_i)$ is equal to $|\tilde{r}_i+1|$ for the corresponding component of $\tilde{r} \in SP(t)$ (as we explained in the proof of lemma 16), and also $(|t|+1)! = \prod(|t_i|+1)!$.

$$C_k(A) = \sum_{r \in SP(A)} (-1)^{k+\ell(r)} (\ell(r) - 1)! \cdot$$

$$\sum_{t_j \in SP(r)} (-1)^{\ell(r_j) + \ell(t_j)} \left( \frac{\ell(r)}{\ell(t_j)} \right)$$

$$= \sum_{t \in SP(A)} \frac{(-1)^{\ell(A)+k+|t|}}{(|t|+1)!} \cdot$$

$$\left( \sum_{\tilde{r} \in SP(|t|)} (-1)^{\ell(\tilde{r})} (\ell(\tilde{r}) - 1)! \left( \prod_{j=1}^{\ell(\tilde{r})} (|\tilde{r}_j| + 1)! \right) \left[ \sum_{d_j = k} \prod_{j=1}^{\ell(\tilde{r})} d_j - 1 \right] \right)$$

$$= \sum_{t \in SP(A)} \frac{(-1)^{\ell(A)+k+|t|}}{(|t|+1)!} \cdot$$

$$\left( \sum_{\tilde{r} \in SP(|t|)} (-1)^{\ell(\tilde{r})} (\ell(\tilde{r}) - 1)! \left( \prod_{j=1}^{\ell(\tilde{r})} (|\tilde{r}_j| + 1)! \right) \left( \frac{\ell(t) - \ell(\tilde{r})}{k - \ell(\tilde{r})} \right) \right)$$

Therefore:

$$x_p = \sum_{q \leq p, \ell(q) \leq d} \lambda_{q, N[p, q]}$$

$$= \sum_{k_1, \ldots, k_{\ell(p)}} \prod_{i=1}^{\ell(p)} \left( \sum_{q \in SP(p)} \lambda_{q_i, N[q_i]} \right)$$

$$= \prod_{\sum_{i=1}^{\ell(p)} k_i \leq d} C_k(p)$$

$$= \prod_{\sum_{i=1}^{\ell(p)} k_i \leq d} \sum_{t_i \in SP(p)} \frac{(-1)^{\ell(p_i)+k_i+|t_i|}}{(|t_i|+1)!} \cdot$$

$$\left( \sum_{\tilde{r}_i \in SP(|t_i|)} (-1)^{\ell(\tilde{r}_i)} (\ell(\tilde{r}_i) - 1)! \left( \prod_{j=1}^{\ell(\tilde{r}_i)} (|\tilde{r}_j| + 1)! \right) \left( \frac{\ell(t_i) - \ell(\tilde{r}_i)}{k_i - \ell(\tilde{r}_i)} \right) \right)$$
By reordering the terms in the expression and using Theorem 22, we get the required result.

\[
x_p = \sum_{t \leq r \leq p} \frac{(-1)^{\ell(A) + \ell(t) + \ell(r)}}{(|t| + 1)!} \prod_{j=1}^{\ell(r)} (|r_j + 1|)! \prod_{l=1}^{\ell(p)} (\ell(r_{|p|}) - 1)! 
\]

\[
\left( \sum_{k_i = k} \prod_{i=1}^{d} \left( \ell(t_{|p_i|}) - \ell(r_{|p_i|}) \right) \right) 
\]

\[
= \sum_{t \leq r \leq p} \frac{(-1)^{d+\ell(t)+\ell(r)}}{(|t| + 1)!} \prod_{j=1}^{\ell(r)} (|r_j + 1|)! \prod_{l=1}^{\ell(p)} (\ell(r_{|p|}) - 1)! \cdot (-1)^M \left( \frac{\ell(t) - \ell(r)}{M - \ell(r)} \right) \]

where \( M = \min\{\ell(t), d\} \).

\[\square\]

**Proof of Theorem 1**. As we mentioned in the introduction, Theorem 2 implies that the expression of Theorem 17 holds in \( \kappa^*(\mathcal{M}_{g,n}^d) \) for any genus \( g \). This completes the proof of Theorem 1.

\[\square\]

**Example 18.** We consider two special cases of Theorem 1. In both cases, in order to get a non-zero contribution from \( \left( \frac{\ell(t) - \ell(r)}{M - \ell(r)} \right) \), we have to consider special \( r \) and \( t \).

- \( d = 1 \) so \( M = 1 \). In order to get a non-zero contribution we must have \( \ell(r) = 1 \). There we obtain:

\[
x_p = \sum_{t \in SP(A)} \frac{(-1)^{\ell(A) + \ell(t)}}{|t| + 1)! (|A| + 1)!} 
\]

that is the result of Proposition 10.

- \( d = n \) which implies \( M = \ell(t) \). In order to get a non-zero contribution we must have \( r = t \). So

\[
x_p = \sum_{r \leq p \in SP(A)} (-1)^{\ell(A) + \ell(r)} \prod_{l=1}^{\ell(p)} (\ell(r_{|p_i|}) - 1)! 
\]

which vanishes unless \( \ell(p_i) = 1 \) for each \( i \).
4. Combinatorial identity

This section contains the statement and proof of the combinatorial identity (Theorem 22) that we used in the previous section. The methods are completely combinatorial and are independent from the rest of the paper.

Lemma 19. Let \( A = \{a_1, \cdots, a_n\} \) be a multi-set of integers.

\[
\sum_{P \in SP(A; k)} \prod_{i=1}^{k} |P_i|^{(\ell(P_i)-1)} = \binom{n-1}{k-1} (a_1 + \cdots + a_n)^{n-k}.
\]

Proof. Let \( K_{n+1} \) be the complete graph on \( n + 1 \) vertices with vertices labeled by \( 1 = a_0, a_1, \cdots, a_n \). For a spanning tree \( T \subset K_{n+1} \) we denote the Prüfer code of \( T \) by \( P(T) \), and

\[ p(T) := \text{the product of all the entries of } P(T) \]

which we call the Prüfer function of \( T \). For definition and basic properties of the Prüfer code see [4] page 13.

Let \( T_k \) be the set of spanning tree of \( K_{n+1} \) with \( \text{deg}(v_0) = k \). We compute the sum \( S = \sum_{T \in T_k} p(T) \) in two ways.

1. If we forget the condition \( \text{deg}(v_0) = k \), then any sequence of length \( n - 1 \) of \( a_i \)'s would appear exactly once as the Prüfer code of a spanning tree of \( K_{n+1} \). For trees in \( T_k \) the label \( a_0 = 1 \) appears exactly \( k - 1 \) times, and the remaining entries are filled with the rest of \( a_i \)'s with no extra conditions. In order to specify the code for a tree \( T \in T_k \), we have to chose the location of the \( k - 1 \) \( a_0 \)-entries, where we have \( \binom{n-1}{k-1} \) choices, and the remaining \( n - k \) entries are filled arbitrary with the remaining \( a_i \)'s, which contributes \( (a_1 + \cdots + a_n)^{n-k} \) to \( S \). Therefore

\[
S = \binom{n-1}{k-1} (a_1 + \cdots + a_n)^{n-k}.
\]

2. For a tree \( T \in T_k \) the vertex \( v_0 \) is connected to exactly \( k \) disjoint sub-trees \( \tilde{G}_1, \ldots, \tilde{G}_k \) of \( T \). \( v_0 \) is connected to a unique vertex in each \( \tilde{G}_i \), and we denote by \( G_i \) the union of \( \tilde{G}_i, v_0 \) and the edge connecting them. The vertex set of \( \tilde{G}_i \)'s gives a partition \( p \) of \( A \) into exactly \( k \) parts. The Prüfer code of each \( G_i \) uniquely determines \( G_i \), and if we know all the \( G_i \)'s then \( T \) and the Prüfer code of it are uniquely determined. Therefore we have:

\[
S = \sum_{T \in T_k} p(T)
\]

\[
= \sum_{P \in SP(A; k)} \prod_{i=1}^{k} \sum_{\substack{G_i: \text{spanning tree on} \\
\text{the vertex set } p_i \cup v_0 \\
\text{deg}_{G_i}(v_0) = 1}} p(G_i).
\]
Note that the Prüfer code of each $G_i$ is a sequence of length $\ell(p_i) - 1$, and the entries are from $p_i$. If we take the sum over all such trees any sequence of length $\ell(p_i) - 1$ with entries in $p_i$ appears exactly once, so

$$\sum_{G_i: \text{spanning tree on the vertex set } p_i \cup v_0 \atop \deg_{G_i}(v_0) = 1} p(G_i) = |p_i|^{\ell(p_i) - 1}. $$

Thus we have

$$S = \sum_{p \in SP(A;k)} \prod_{i=1}^k |p_i|^{\ell(p_i) - 1}. $$

\[ \square \]

**Definition 20.** For $x \in \mathbb{R}$ we denote the falling factorial of $x$ by

$$(x)_n := x(x - 1) \cdots (x - (n - 1)).$$

Let $P(x_1, \cdots, x_n) \in \mathbb{R}[x_1, \cdots, x_n]$ be a polynomial in $n$ variables. If

$$P = \sum c_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

we define the falling factorialization of $P$ to be

$$FF(P) = \sum c_{i_1, \cdots, i_n} (x_1)_{i_1} \cdots (x_n)_{i_n}.$$

**Remark 21.** It is straight forward to see that we have:

$$(x + y)_n = \sum_{i=0}^n \binom{n}{i} (x)_i (y)_{n-i}.$$

Using induction on the number of variables, we see that for falling factorials we have the multinomial coefficient theorem.

$$(x_1 + \cdots + x_n)_k = \sum_{k_1, \cdots, k_n = -k}^k \binom{k}{k_1, \cdots, k_n} (x_1)_{k_1} \cdots (x_n)_{k_n}$$

**Theorem 22.** Let $A = \{a_1, \cdots, a_n\}$ be a multi-set of integers.

$$\sum_{p \in SP(A;k)} \left( |p| + 1 \right) \prod_{i=1}^k \left( \frac{|p_i + 1|}{|p| + 1} \right) = \binom{n-1}{k-1} \binom{|A| + 1}{k}.$$

**Proof.** We can rewrite the identity as

$$\sum_{p \in SP(A;k)} (|p_i + 1|)^{\ell(p_i) - 1} = \binom{n-1}{k-1} (|A| + 1)^{n-k},$$

or equivalently, if we set $B = A + 1$ as

$$\sum_{p \in SP(B;k)} \prod_{i=1}^k (|p_i|)^{\ell(p_i) - 1} = \binom{n-1}{k-1} (|B|)^{n-k}. \quad (4.1)$$
Note that by Remark 21 the left hand side of (4.1) is the falling factorialization of
\[ \sum_{p \in SP(B;k)} k \prod_{i=1}^{k} |p_i|^{\ell(p_i)-1}, \]
and similarly the right hand is the falling factorialization of
\[ \binom{n-1}{k-1} \binom{|B|}{n-k}. \]
By Lemma 19
\[ \sum_{p \in SP(B;k)} k \prod_{i=1}^{k} |p_i|^{\ell(p_i)-1} = \binom{n-1}{k-1} \binom{|B|}{n-k}, \]
so their falling factorializations are also equal, which completes the proof. □

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