Charged scalar quasi-normal modes for higher-dimensional Born-Infeld dilatonic black holes with Lifshitz scaling

S. Sedigheh Hashemi, 1 Mahdi Kord Zangeneh, 2, 3, * and Mir Faizal 4, 5, †

1 School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM), P. O. Box: 19395-5531, Tehran, Iran
2 Physics Department, Faculty of Science, Shahid Chamran University of Ahvaz, Ahvaz 61357-43135, Iran
3 Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P. O. Box: 55134-441, Maragha, Iran
4 Department of Physics and Astronomy, University of Lethbridge, Lethbridge, AB T1K 3M4, Canada
5 Irving K. Barber School of Arts and Sciences, University of British Columbia - Okanagan, Kelowna, BC, V1V 1V7, Canada

We study quasi-normal modes for a higher dimensional black hole with Lifshitz scaling, as these quasi-normal modes can be used to test Lifshitz models with large extra dimensions. Furthermore, as the effective Planck scale is brought down in many models with large extra dimensions, we study these quasi-normal modes for a UV completion action. Thus, we analyze quasi-normal modes for higher dimensional dilaton-Lifshitz black hole solutions coupled to a non-linear Born-Infeld action. We will analyze the charged perturbations for such a black hole solution. We will first analyze the general conditions for stability analytically, for a positive potential. Then, we analyze this system for a charged perturbation as well as negative potential, using the asymptotic iteration method for quasi-normal modes. Thus, we analyze the behavior of these modes numerically.

I. INTRODUCTION

Mathematically ideal black holes are usually studied as isolated systems. However, this mathematical idealization cannot be actualized in any real world system. This is because black holes will always interact with matter surrounding it. Even if all the matter around a black hole could be removed, it would still interact with the vacuum around it, and thus cannot be considered an isolated system. So, in any realistic description of a black hole we would have to consider such perturbative effects on the basic parameters describing such a black hole. Such perturbations of a black hole lead to the emission of gravitational waves [1]. Initially there is an short outburst, which is followed by long period of damping oscillations in the form of quasi-normal modes (QNMs) [2–4], and finally these QNMs are suppressed at late times by power-law or exponential tails. A complex frequency called the quasi-normal frequency (QNF) is associated with these QNMs [5–7]. The real part of this complex frequency represents the frequency of oscillation, and its imaginary part represents the rate of damping of that oscillation.

In this paper, we will be analyzing the QNMs as they are the most interesting phase of radiation, which can have interesting physical consequences. This is because the study of gravitational wave has become important due to the detection of these wave in GW150914 [8], and the analysis of QNMs from the data obtained from gravitational antennas such as LIGO, VIRGO and LISA, can have important consequences [9–12]. The existence of extra dimensions in string theory has motivated the study of black hole in large extra-dimensions, such that the extra dimensions are large enough to suppress the Planck scale to the TeV scale [13–16]. These models have been generalized to Randall-Sundrum brane world models with warped extra dimensions, such that the effective Planck scale again reduce to a TeV scale [17, 18]. It has been argued that such models motivated by string theory can be detected by analyzing the QNMs obtained from gravitational antennas [19–22]. However, as the effective Planck scale is brought down in these models, it is important to consider a UV completion of the effective theory in four dimensions for analyzing such black holes. An important UV completion of the effective field theory, obtained as a low energy limit of string theory is the D-branes action, which can be described by a Born-Infeld (BI) action coupled to dilaton field [24–27]. So, in this paper, we will analyze the QNMs from a black hole in such a theory with a BI action coupled to a dilaton field.

It may be noted that another interesting UV completion of general relativity, motivated by a formalism developed in condensed matter physics, is the Horava-Lifshitz gravity [28, 29]. The space and time scale with different Lifshitz scaling in the Horava-Lifshitz gravity, such that it reduces to general relativity in the IR limit, but differs from general relativity in the UV limit. In fact, motivated by Horava-Lifshitz gravity, the consequences of such different Lifshitz scaling have been studied for various geometrical structures that occur in the string theory. Such a Lifshitz scale has been studied for both type IIA string theory [30] and type IIB string theory [31]. The behavior of dilaton black branes [32, 33] and dilaton black holes [34, 35] with Lifshitz scaling has also been discussed. Furthermore, such black solutions in a BI non-linear action with Lifshitz scaling have also been studied [36]. The brane world models with warped extra dimensions and Lifshitz scaling have also been constructed [37, 38].

It may be noted that even though the charged scalar perturbations are important [39–44], mostly QNMs in Lifshitz theories have been studied for neutral scalar

*Email: mzkangeneh@scu.ac.ir
†Email: mirfaizalmir@googlemail.com
perturbations [45–52]. However, QNMs for a charged dilaton-Lifshitz solutions in the four dimensions have also been studied [53]. As the QNMs can be used to test models with large extra dimensions with Lifshitz scaling, it is important to study the QNMs for higher dimensional analogs of such a black hole. Furthermore, as the effective Planck scale is brought down in many models with large extra dimensions, it is important to discuss such results using a UV completion of such a theory. Thus, we will analyze the QNMs from a higher dimensional dilaton-Lifshitz black hole solutions coupled to a nonlinear Born-Infeld action.

We organize this paper as follows: In section II, we will review the dilaton-Lifshitz black hole solutions in the presence of BI nonlinear electrodynamics. In section III, the wave equations of charged scalar perturbations around our black holes will be presented. In latter section, we will analyze the QNMs numerically in section IV. Finally, in the last section, we will summarize our results and discuss about them.

II. LIFSHITZ SOLUTIONS

In this section, we will analyze a higher dimensional black hole in a UV complete theory. Thus, we will consider a dilaton-Lifshitz black hole coupled to nonlinear BI action. The metric for such a \((n+1)\)-dimensional Lifshitz black holes can be expressed as [32, 46]

\[
\text{ds}^2 = -r^{2s} f(r) dr^2 + \frac{l^2}{r^2} dr^2 + \frac{l^2}{r^2} d\Omega^2_{n-1},
\]

where \(d\Omega^2_{n-1}\) is a \((n-1)\)-dimensional hypersurface with constant curvature \((n-1)-(n-2)\), such that

\[
d\Omega^2_{n-1} = d\theta^2 = \sum_{i=2}^{n-1} \sum_{j=1}^{n-2} \prod_{j=1}^{n-2} \sin^2 \theta_j.
\]

Here \(\omega_{n-1}\) is the volume and \(z(\geq 1)\) is dynamical critical exponent. Now it can be observed that in the limit, \(r \to \infty\), the metric given by Eq. (1) will asymptotically reduce to the Lifshitz metric given by

\[
\text{ds}^2 = -r^{2s} dt^2 + \frac{l^2}{r^2} dr^2 + r^2 d\Omega^2_{n-1}.
\]

In this paper, we will analyze the coupling of a Lifshitz spacetime to a non-linear BI action. So, first we observe that in the absence of dilaton field, the function \(L(F)\) for BI Lagrangian is given by [54]

\[
L(F) = \frac{4}{\beta^2} \left( 1 - \sqrt{1 + \frac{\beta^2}{2}} \right),
\]

where \(\beta\) is the BI non-linearity parameter, \(F = F_{\mu \nu} F^{\mu \nu}\) is the the Maxwell invariant, as \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) with \(A_\mu\) being the \(U(1)\) abelian gauge field. The dilaton field can couple to this \(U(1)\) abelian gauge field. So, in presence of a dilaton field, the Lagrangian for the BI coupled to a dilaton scalar field \(\Phi\) can be written as [55, 56]

\[
L(F, \Phi) = \frac{4}{\beta^2} e^{4\lambda \Phi/(n-1)} \left( 1 - \sqrt{1 + \frac{\beta^2}{2}} e^{-8\lambda \Phi/(n-1)} \right),
\]

where \(\lambda\) is a constant. In Einstein frame, the Lagrangian density of Einstein-dilaton model with string theory corrections [57], contains two \(U(1)\) gauge field’s [32]. So, the Lagrangian for this system can be written as

\[
16\pi \mathcal{L} = R - \frac{4}{n-1} (\nabla \Phi)^2 - 2\lambda - \sum_{i=1}^{2} e^{-4\lambda \Phi/(n-1)} H_i + L(F, \Phi),
\]

where \(\mathcal{R}\) is the Ricci scalar and \(\Lambda, \lambda, \lambda_i\) are constants, in this Lagrangian. In the Lagrangian (5), \(H_i = (H_i)_{\mu \nu} (H_i)^{\mu \nu}\), where \((H_i)_{\mu \nu} = 2\partial_\mu (B_i)_\nu\), and \((B_i)_{\mu}\) is a gauge field. In the limit, \(\beta \to 0\), this Lagrangian \(\mathcal{L}\) reduces to the Einstein-dilaton-Maxwell Lagrangian (at the leading order) [32, 33]

\[
\lim_{\beta \to 0} 16\pi \mathcal{L} = \cdots - e^{-4\lambda \Phi/(n-1)} F + \beta^2 e^{-12\lambda \Phi/(n-1)} F^2 + O (\beta^4).
\]

By varying the action \(S = \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \mathcal{L}\) with respect to the metric \(g_{\mu \nu}\), the dilaton field \(\Phi\), and \(U(1)\) abelian gauge field \(A_\mu\), and \((B_i)_{\mu}\), the following field equations are obtained [58]

\[
\mathcal{R}_{\mu \nu} - \frac{g_{\mu \nu}}{n-1} \left( 2\lambda + 2LF - L(F, \Phi) - \sum_{i=1}^{2} e^{-4\lambda \Phi/(n-1)} H_i \right) - \frac{4}{n-1} \partial_\mu \Phi \partial_\nu \Phi + 2LF \mu \nu \lambda F^\lambda \mu
\]

\[
-2 \sum_{i=1}^{2} e^{-4\lambda \Phi/(n-1)} (H_i)_{\mu \lambda} (H_i)^{\nu \lambda} = 0,
\]

\[
\nabla^2 \Phi + \frac{n-1}{8} L \Phi + \sum_{i=1}^{2} \frac{\lambda_i}{2} e^{-4\lambda \Phi/(n-1)} H_i = 0,
\]

\[
\nabla_{\mu} \left( (L F) F^{\mu \nu} \right) = 0,
\]

\[
\nabla_{\mu} \left( e^{-4\lambda \Phi/(n-1)} (H_i)^{\mu \nu} \right) = 0,
\]

where we use the convention \(X_{\nu} = \partial X / \partial Y\). With metric (1), the Eqs. (9) and (10) can be solved for the \(U(1)\) abelian gauge field, and thus we obtain

\[
F_{\nu \tau} = \frac{q e^{4\lambda \Phi/(n-1)} r^{-n}}{\beta \Gamma},
\]

\[
(H_i)_{r \tau} = q r^{-n} e^{\lambda_i \Phi/(n-1)},
\]

where \(\Gamma = \sqrt{1 + q^2 (2s^2 - 2\beta^2)/(r^{2n-2})}\). By subtracting \((tt)\), and \((rr)\) components of Eq. (7), \(\Phi(r)\) can be obtained,
which is given by
\[ \Phi(r) = \frac{(n - 1)\sqrt{z - 1}}{2} \ln \left( \frac{r}{b} \right), \]  
(13)

where \( b \) is a constant. By substituting Eqs. (11), (12), and (13) in field equations (7), and (8), the \( f(r) \) function for this system is obtained, and this given by
\[ f(r) = \begin{cases} 1 - \frac{m}{r^{n+1}} + \frac{k(n-2)^2l^2}{(n+z-3)^2} + \frac{4l^2b^{2z-2}}{\beta^2(n-1)(n-z+1)} - \frac{4l^2b^{2z-2}}{\beta^2(n-1)r^{n+1}} \int \Gamma(r^n) dr, & \text{for } z \neq n + 1, \\ 1 - \frac{m}{r^{n+1}} + \frac{k(n-2)^2l^2}{4(n-1)^2r^2} - \frac{4l^2b^{2z-2}}{\beta^2(n-1)r^2} \left[ 1 - \Gamma + \ln \left( \frac{l+1}{2} \right) \right], & \text{for } z = n + 1. \]  
(14)

Here \( m \) is a constant, and it is related to the total mass of black brane. Now as this system has to satisfy the field equations, we can write
\[ \lambda = -\sqrt{z - 1}, \quad \lambda_1 = \frac{n - 1}{\sqrt{z - 1}}, \quad \lambda_2 = \frac{n - 2}{\sqrt{z - 1}}, \]
\[ q_1^2 = -\frac{\Lambda (z - 1)}{(z + n - 2)^2 (z - 1)^2}, \]
\[ q_2^2 = \frac{(n - 1)(n - 2)(z - 1)\beta^2(n-2)}{2(z + n - 3)^2 (z - 1)}, \]
\[ \Lambda = -\frac{(n + z - 1)(n + z - 2)}{2l^2}. \]  
(15)

Integrating the last term of \( f(r) \) for \( z \neq n + 1 \), we obtain
\[ f(r) = 1 - \frac{m}{r^{n+1}} + \frac{l^2(n-2)^2}{r^2(n+z-3)^2} + \frac{4l^2b^{2z-2} - 2z - 2z}{\beta^2(n-1)(n-z+1)} + \frac{4l^2b^{2z-2} - 2z}{(n-z+1)(n+z-3)} \Gamma \]
\[ \times F \left( \frac{2n + z - 4}{2n - 2}, \frac{3n + z - 5}{2n - 2}, 1 - \Gamma^2 \right). \]  
(16)

We can observe from this solution, that in the limit \( r \to \infty \), \( f(r) \) satisfies \( f(r) \to 1 \) (note that \( F(a, b, c, 0) = 1 \)). It may be noted that for \( z = n + 1 \), a Schwartzchild-like black hole does not exist in this system, because as \( r \to 0 \), \( f(r) \) will go to positive infinity. Nevertheless, for \( z \neq n + 1 \), the Schwartzchild-like black hole do exists in this system (apart from the extreme and non-extreme black holes and naked singularities). From here on, we fix \( l \) and \( b \) to unity.

### III. QUASI-NORMAL MODES

Now in this section, we will analyze the QNMs for the solution discussed in the previous section. So, first we consider a minimally coupled charged massive scalar field in such a spherically symmetric black hole background. It is assumed that this field satisfies the Klein-Gordon equation
\[ D^\nu D_\nu \Psi = m_s^2 \Psi, \]  
(17)

where \( D^\nu = \nabla^\nu - iq_s A^\nu \) is the gauge covariant derivative and \( m_s \) is the mass of the scalar field \( \Psi \). Now we decompose \( \Psi \) into the following standard form,
\[ \Psi(t, r, \text{angles}) = e^{-i\omega t} R(r) Y(\text{angles}), \]  
(18)

where \( Y(\text{angles}) \) is the spherical harmonic function related to the angular coordinates, and \( \omega \) is the frequency. As \( Y \) is the spherical harmonic function, it satisfies the following equation
\[ \nabla_\text{angles}^2 Y(\theta, \phi) = -QY(\text{angles}), \]  
(19)

where \( Q \) is the constant of separation. Moreover, the differential equation for the radial function in the \((n+1)\)-dimensional Lifshitz-dilaton background (1) is
\[ fR'' + \left( f' + \frac{(d + z - 1)f}{r} \right) R' + \left( \frac{q_s A_t + \omega}{r^{z+1}} \right)^2 \frac{R}{f} - \left( m_s^2 + \frac{Q}{r^2} \right) \frac{R}{r^2} = 0, \]  
(20)

where \( d = n + 1 \). So, using a tortoise coordinate \( r_s \) as \( dr_s/dr = [r^{z+1}f(r)]^{-1} \), and introducing a new radial function \( R(r) \) as \( R(r) = K(r)/r \), the Eq. (20) can be transformed into an equation resembling the Schrödinger equation,
\[ \frac{d^2K(r_s)}{dr_s^2} + \left[ (\omega + q_s A_t(r))^2 - V(r) \right] K(r_s) = 0, \]  
(21)

where the potential \( V(r) \) is given by
\[ V(r) = \left( \frac{d^2}{4} + \frac{dz}{2} + 1 - d - z \right) f^2 r^{2z} + f^{2z} \left( m_s^2 + \frac{Q}{r^2} \right) - ff' r^{2z+1} \left( 1 - \frac{d}{2} \right). \]  
(22)

Here we first observe that the first and the second terms are positive (with \( m_s^2 > 0 \)). Now as \( d \geq 4 \), so the term \( (1 - d/2) \) is negative. Thus, by having positive \( f(r) \) and \( f'(r) \) out of the horizon \( r > r_h \), we can obtain \( V(r) > 0 \) out of horizon provided that \( m_s^2 > 0 \). Note that \( m_s^2 \) can be negative as long as it is above Breitenlohner-Freedman-(like) bound. One may read this bound from Eq. (35) in next section.
To perform a general stability analysis for this system, we first transform to Eddington-Finkelstein coordinates, such that \( v = t + r_+ \). In this new coordinate system, the metric can be written as
\[
 ds^2 = -r^2 f dv^2 + 2r^{-1} dv dr + r^2 d\Omega_{n-1}^2. 
\]
Using the ansatz
\[
 \Psi = e^{-i\omega u} \frac{\psi(r)}{r^{d/2-1}} Y(\text{angles}),
\]
the differential equation for radial function \( \psi(r) \) can be written as
\[
 (r^{d+z-3} f \psi')' - 2i (\omega + q_s A_t) \psi' - V \psi = 0. 
\]
Multiplying above equation by \( \tilde{\psi} \) (the complex conjugate of \( \psi \)) and performing the integration from \( r_+ \) to infinity, we obtain
\[
 \int_{r_+}^{\infty} dr \left[ r^{d+z-3} f |\psi'|^2 + 2i (\omega + q_s A_t) \tilde{\psi} \psi' + V |\psi|^2 \right] = 0, 
\]
where we used the integration by part and applied the Dirichlet boundary condition for \( \psi \). In this new coordinate system, the differential equation for radial function \( \eta \) can be written as
\[
 \frac{d^2 \eta}{dr^2} + \frac{2}{r} \frac{d \eta}{dr} + \left( \frac{d+z-3}{r^2} f \right) \eta = 0, 
\]
with boundary condition \( \eta = 0 \) at \( r = \infty \). In this case the Dirichlet boundary condition for \( \psi \) reduces to
\[
 0 = \int_{r_+}^{\infty} dr \left[ (\omega + q_s A_t) \tilde{\psi} \psi' + (\tilde{\omega} + \bar{q}_s \bar{A}_t) \tilde{\psi} \psi' \right] 
\]
\[
 = 2i \int_{r_+}^{\infty} dr \text{Im}[(\omega + q_s A_t) \tilde{\psi} \psi'] 
\]
\[
 - \bar{q}_s \int_{r_+}^{\infty} dr |\psi|^2 \bar{A}_t - \tilde{\omega} |\psi(r_+)|^2. 
\]
Here we have again used the integration by part and the Dirichlet boundary condition for \( \psi \). In the case of neutral scalar field \( (q_s = 0) \), Eq. (27) reduces to
\[
 \int_{r_+}^{\infty} dr |\psi|^2 \bar{A}_t - \tilde{\omega} |\psi(r_+)|^2.
\]
Now we can substitute \( \int_{r_+}^{\infty} dr \tilde{\psi} \psi' \) from Eq. (28) in Eq. (26) and obtain
\[
 \int_{r_+}^{\infty} dr \left[ r^{d+z-3} f |\psi'|^2 + V |\psi|^2 \right] = -\frac{\text{Im}[\omega]}{2} \frac{|\psi(r_+)|^2}{|\psi'(r_+)|^2}. 
\]
Now we observe from this equation that for the potential \( V \) to be positive outside the horizon, imaginary part of \( \omega \) is negative. So, for a neutral scalar field, the black hole is stable under scalar field perturbation, if the potential is positive outside the horizon (i.e. as long as \( m^2_s > 0 \)). In next section, we proceed more to study charged scalar fields numerically. We investigate the behavior of potential for \( m^2_s < 0 \), as well.

### IV. NUMERICAL ANALYSIS

In the previous section, we analyzed the stability of our black hole system, when the potential was positive outside the horizon and scalar perturbation was neutral. However, we cannot obtain such general results for charged scalar fields as well as negative potential. To understand the behavior of this system generally, we need to analyze it numerically. So, in this section, we will analyze the behavior of QNMs numerically using the asymptotic iteration method (AIM) [59]. Now we first change the variable \( u = 1 - r_+/r \), and write Eq. (20) as
\[
 \frac{r^z_h}{1-u} f(u) \frac{d^2 R''(u)}{du^2} + \left( \frac{d+z-3}{1-u} \right) f(u) R'(u) \right) 
\]
\[
 + \frac{r^z_h}{(1-u)^z} \left( q_s A_t (u) + \omega \right)^2 R(u) 
\]
\[
 - \frac{r^z_h}{(1-u)^{z+1}} \left( m^2_s + \frac{Q}{r^2_h} (1-u)^2 \right) R(u) = 0. 
\]
In order to propose an ansatz for (30), we are going to consider the behavior of the function \( R(u) \) at horizon \( u = 0 \), and boundary \( u = 1 \). At horizon \( u = 0 \), we have \( f(0) \approx uf'(0) \) and \( A_t(0) = 0 \), thus Eq. (30) can be written as
\[
 \frac{R''(u)}{u} + \frac{R'(u)}{u} + \frac{R(u)}{r^2_h u^2 f'(0) \omega} = 0. 
\]
The solution for this equation can be written as
\[
 R(u) \sim C_1 u^{-\xi} + C_2 u^{\xi}, \quad \xi = -\frac{i\omega}{r^2_h f'(0)}. 
\]
Here we have imposed the ingoing boundary condition at the horizon \( u = 0 \), and so we have required \( C_2 \) to vanish.

At infinity, where \( (u = 1) \), Eq. (30), can be written as
\[
 \frac{R''(u)}{1-u} + \frac{(z+d-3)R'(u)}{1-u} - \frac{m^2_s R(u)}{(1-u)^2} = 0. 
\]
The solution to this equation can be written as
\[
 R(u) \sim D_1 (1-u)^{1/2(z+2+\Pi)} + D_2 (1-u)^{1/2(z+2-\Pi)} \], 
\]
where
\[
 \Pi = \sqrt{(d+z-2)^2 + 4m^2_s}. 
\]
In order to impose Dirichlet boundary condition \( R(u \rightarrow 1) = 0 \), we can set \( D_2 = 0 \).

Using the above solutions at horizon and boundary, the general ansatz for Eq. (30), can be written as
\[
 R(u) = u^{-\xi} (1-u)^{1/2(z+2+\Pi)} \chi(u). 
\]

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1 Note that \( Im[2iab] = ab + \bar{a} \bar{b} \).
Now using the improved AIM method, Eq. (38) becomes

$$e^{-i\omega t} \text{ in scalar field} \text{ Eq. (18)) becomes } e^{-i\omega_R t} e^{\omega_I t}. \text{ Therefore, } \omega_R \text{ and } \omega_I \text{ determine the energy of scalar perturbations and the stability of the system under dynamical perturbations, respectively. If } \omega_I > 0, \text{ the scalar field grows with time and therefore the black hole is not stable under this perturbation. If } \omega_I < 0, \text{ the black hole system is stable under dynamical perturbation since the scalar field vanishes as time passes. For larger values of } |\omega_I|, \text{ the perturbation remains longer outside the black hole until it disappears. From holographic point of view, it means that it takes more time for the dual system to go back to equilibrium. As one can see from Figures 1 and 2, for both } d = z \text{ and } d \neq z \text{ cases, for fixed values of } \beta, |\omega_I| \text{ decreases as } q \text{ increases (or equivalently temperature } T \text{ decreases). According to what explained above, it means that by increasing } q, \text{ the complete decay of scalar perturbation...
outside the black hole takes more time. Holographically, it shows that dual system needs more time to return to equilibrium. As Figures 1(a) and 1(b) show, for \( d \neq z \) case, by increasing \( q \), \( \omega_R \) increases up to a maximum value first and then decreases. The value of maximum for a fixed \( d \) and \( z \), depends on nonlinear parameter \( \beta \). For larger values of \( \beta \), the maximum value of \( \omega_R \) is lower. It means that by increasing the effects of nonlinearity, the maximum energy of perturbation decreases. For specific values of \( d \) and \( z \), the influence of \( \beta \) on real and imaginary parts of quasi-normal frequencies disappears as \( q \) tends to smaller values (higher \( T \)). Moreover, for smaller values of \( \beta \) but fixed \( d \) and \( z \), the real part of quasi-normal frequencies is larger in general as \( q \) is further away from zero. The latter result shows that the perturbation has more energy in linear electrodynamics case. Furthermore, for larger values of \( z \), the maximum energy of perturbations is lower (Figures 1(a) and 1(b)). In \( d = z \) case, we have the same behaviors as \( d \neq z \) case (Figures 2(a) and 2(b)). However, as Figures 1(a) and 1(c) show, in the case of \( d \neq z \), for fixed \( z \), by increasing \( d \), the behavior of \( \omega_R \) with respect to \( q \) may change. For \( d = 5 \) and \( z = 2 \), \( \omega_R \) shows a decreasing trend as \( q \) increases.

In the case with \( z \neq d \) and \( m^2 > 0 \), the effective potential as a function of \( r^* \) are plotted in Figure 3. In Figure 3(a), with \( d = 4 \), and \( z = 2 \), each potential has a well, whose height depends on the value of \( m^2 \) and \( \beta \). With the same \( \beta \), the one with lower \( m^2 \), has a higher height of the well. The highest hight of the well in this case is obtained when \( \beta = 0.1 \) and \( m^2 = -26 \). Moreover, the same behaviors is obtained by going to higher dimension, such as \( d = 5 \), as depicted in Figure 3(b). This shows that the particle described by the scalar field is traped in the potential of the black hole in these cases. Another case is obtained by setting \( d = z \), where the dimension of spacetime equals the dynamical critical exponent \( z \) (Figure 4). For \( d = z \) cases, the same behavior is obtained as in Figure 3, in which the potential has a well. In order to see the effects of the existence of the extra dimension, in Figure 4(b), we consider the case \( d = z = 5 \).

V. SUMMARY AND CONCLUSION

In this paper, we analyze the behavior of quasi-normal modes (QNMs) for a higher dimensional black hole with Lifshitz scaling. This is important as the QNMs can be used to test models with large extra dimensions with Lifshitz scaling. In fact, as the effective Planck scale is lowered in such models with large extra dimensions, we study these QNMs for a UV completion action. This UV completion action is motivated by Born-Infeld action for a D-brane action, which is a UV completion of the ordinary action for linear actions of such systems. So, in this paper, the QNMs for higher dimensional dilaton-Lifshitz black hole solutions coupled to a non-linear Born-Infeld
action have been studied. As it is important to study the charged perturbations for such a black hole solution, such charged perturbations were studied. In fact, we first analyzed general conditions for stability analytically, for a positive potential. However, as it was not possible to perform such an analysis for a charged perturbation as well as a negative potential, we analyze this system for these cases numerically. This was done using the asymptotic iteration method for quasi-normal modes. This is done by analyzing this system for two cases, i.e., when \( d \neq z \), and when \( d = z \). For both cases, it is observed that the absolute value of imaginary parts of quasi-normal frequencies decreases as charge of black hole increase to extremal value. It means that it takes more time for QNMs to completely decay outside the black hole. If one wants to interpret this in \( \text{AdS/CFT} \) language, it shows that it needs more time for the system to go back to equilibrium under the perturbation. Increasing the charge of black hole, the real part of quasi-normal frequencies first increase up to a maximum and then decreases. The value of this maximum is dependent on the nonlinearity parameter \( \beta \) and is higher for lower values of this. In the case of \( d \neq z \), we have a different behavior for real part of quasi-normal frequencies as the dimension of space-time increases. For \( d = 5 \neq z \), we observed that increase in value of black hole charge (equivalently decrease in black hole temperature) cause the real part of quasi-normal frequencies to decrease. To study the behavior of potential, we depicted the potential in tortoise coordinate with \( m_2^2 < 0 \), for both \( d \neq z \) and \( d = z \) cases. We observed that potential is partly negative and has a well. For \( d = 4 \), each potential has a well whose height depends on the \( \beta \) and \( m_2^2 \). The same behavior is observed for \( d = 5 \). It is observed that for the second case, the high of well depends on \( \beta \), and for a fixed \( \beta \), the high depends on the value of \( m_2^2 \). Thus, the behavior of this system depends both on \( \beta \) and \( m_2^2 \) for a fixed dimension.

The QNMs have also become important in string theory, due to the development of \( \text{AdS/CFT} \) correspondence \[60, 61\]. It has been observed that the poles of the retarded Green function in a conformal field theory on the boundary of an \( \text{AdS} \) space is related to the QNMs of a asymptotically \( \text{AdS} \) black hole. Thus, QNMs of a higher dimensional \( \text{AdS} \) black hole can be used to describe the behavior of strongly coupled quark-gluon plasmas \[62, 63\]. As the \( \text{AdS/CFT} \) correspondence has been used to analyze the CFT dual to a Lifshitz \( \text{AdS} \) spacetime \[64, 65\], it would be interesting to repeat the analysis done in this paper, for an asymptotically \( \text{AdS} \) black hole. Then such QNMs can be used to analyze various properties of CFT dual to such a black hole in the Lifshitz \( \text{AdS} \) spacetime. It would also be interesting to analyze the effect that UV completion of the theory has on the dual CFT. It may be noted that in this paper, we have used an abelian Born-Infeld action, in which a \( U(1) \) gauge field was coupled in a non-linear action. The abelian Born-Infeld action has been generalized to a non-abelian Born-Infeld action \[66, 67\]. Furthermore, black hole solution in such a theory with non-abelian Born-Infeld have also been constructed \[68, 69\]. It would be interesting to calculate the QNMs for such black holes with a non-abelian Born-Infeld action. It would also be interesting to analyze the QNMs for such black holes in an asymptotically \( \text{AdS} \) spacetime, and then use the \( \text{AdS/CFT} \) correspondence to analyze the CFT dual to such a system.

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[1] R. A. Konoplya and A. Zhidenko, “Quasi-normal modes of black holes: from astrophysics to string theory,” Rev. Mod. Phys. 83, 793 (2011) [arXiv:1102.4014].
[2] K. D. Kokkotas and B. G. Schmidt, “Quasi-normal modes of stars and black holes,” Living Rev. Rel. 2, 2 (1999) [arXiv:gr-qc/9909058].
[3] H. P. Nollert, “Quasi-normal modes: the characteristic sound of black holes and neutron stars,” Class. Quant. Grav. 16, R159 (1999).
[4] B. Wang, “Perturbations around black holes,” Braz. J. Phys. 35, 1029 (2005) [arXiv:gr-qc/0511133].
[5] T. Regge and J. A. Wheeler, “Stability of a Schwarzschild Singularity,” Phys. Rev. 108, 1063 (1957).
[6] F. J. Zerilli, “Gravitational Field of a Particle Falling in a Schwarzschild Geometry Analyzed in Tensor Harmonics,” Phys. Rev. D 2, 2141 (1970).
[7] F. J. Zerilli, “Effective Potential for Even-Parity Regge-Wheeler Gravitational Perturbation Equations,” Phys. Rev. Lett. 24, 737 (1970).
[8] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116, 061102 (2016) [arXiv:1602.03837].
[9] R. Brito, A. Buonanno and V. Raymond, “Black Hole Spectroscopy by Making Full Use of Gravitational-Wave Modeling,” Phys. Rev. D 98, 084038 (2018) [arXiv:1805.00293].
[10] T. Assumpcao, V. Cardoso, A. Ishibashi, M. Richartz and M. Zilhao, “Black hole binaries: ergoregions, photon surfaces, wave scattering, and quasi-normal modes,” Phys. Rev. D 98, 064036 (2018) [arXiv:1806.07909].
[11] L. Manfredi, J. Mureika and J. Moffat, “Quasi-normal Modes of Modified Gravity (MOG) Black Holes,” Phys. Lett. B 779, 492 (2018) [arXiv:1711.03199].
[51] R. Becar, P. A. González and Y. Vásquez, Quasi-normal modes of non-Abelian hyperscaling violating Lifshitz black holes, General Relativity and Gravitation, 49, 26 (2017) [arXiv:1510.04605].

[52] C. Park, A Massive Quasi-normal Mode in the Holographic Lifshitz Theory, Phys. Rev. D 89, 066003 (2014) [arXiv:1312.0826].

[53] M. Kord Zangeneh, B. Wang, A. Sheykhi, and Z. Y. Tang, Charged scalar quasi-normal modes for linearly charged dilaton-Lifshitz solutions, Phys. Lett. B 771, 257 (2017) [arXiv:1701.03644].

[54] M. Born and L. Infeld, Foundation of the new field theory, Proc. R. Soc. A 144, 425 (1934).

[55] A. Sheykhi, Thermodynamical properties of topological Born-Infeld-dilaton black holes, Int. J. Mod. Phys. D 18, 25 (2009) [arXiv:0801.4112].

[56] A. Sheykhi, Thermodynamics of charged topological dilaton black holes, Phys. Rev. D 76, 124025 (2007) [arXiv:0709.3619].

[57] J. Polchinski, String Theory, Cambridge University Press, Cambridge United Kingdom (1998).

[58] M. Kord Zangeneh et al. Thermodynamics, phase transitions and Ruppeiner geometry for Einstein-dilaton-Lifshitz black holes in the presence of Maxwell and Born-Infeld electrodynamics, The Eur. J. Phys. B 77, 6, 423 (2017) [arXiv:1610.06352].

[59] H. T. Cho, A. S. Cornell, J. Doukas and W. Naylor, Black hole quasi-normal modes using the asymptotic iteration method, Class. Quant. Grav. 27, 155004 (2010) [arXiv:0912.2740].

[60] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. 38, 1113 (1999) [Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200].

[61] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large N field theories, string theory and gravity, Phys. Rept. 323, 183 (2000) [hep-th/9905111].

[62] P. Kovtun, D. T. Son and A. O. Starinets, Viscosity in strongly interacting quantum field theories from black hole physics, Phys. Rev. Lett. 94, 111601 (2005) [hep-th/0405231].

[63] A. Peshier and W. Cassing, The Hot non-perturbative gluon plasma is an almost ideal colored liquid, Phys. Rev. Lett. 94, 172301 (2005) [hep-ph/0502138].

[64] E. I. Buchbinder and A. Buchel, Relativistic Conformal Magneto-Hydrodynamics from Holography, Phys. Lett. B 678, 135 (2009) [arXiv:0902.3170].

[65] Y. C. Ong and P. Chen, Stability of Horava-Lifshitz Black Holes in the Context of AdS/CFT, Phys. Rev. D 84, 104044 (2011) [arXiv:1106.3555].

[66] J. de Boer, K. Schalm and J. Wijnhout, General covariance of the non-Abelian DBI action: Checks and balances, Annals Phys. 313, 425 (2004) [hep-th/0310150].

[67] V. V. Dyadichev, D. V. Gal’tsov and P. Vargas Moniz, Chaos-order transition in Bianchi I non-Abelian Born-Infeld cosmology, Phys. Rev. D 72, 084021 (2005) [hep-th/0412334].

[68] M. Wirsching, A. Sood and J. Kunz, Non-Abelian Einstein-Born-Infeld-Higgs model, Phys. Lett. B 458, 252 (1999) [hep-th/9904186].