Generalized diffusion equation with fractional derivatives within Renyi statistics

P. Kostrobij¹, B. Markovych¹, O. Viznovych¹, M. Tokarchuk¹,²

¹Lviv Polytechnic National University,
12 Bandera Str., 79013 Lviv, Ukraine
²Institute for Condensed Matter Physics of NAS of Ukraine,
1 Svientsitskii Str., 79011 Lviv, Ukraine

Abstract

By using the Zubarev nonequilibrium statistical operator method, and the Liouville equation with fractional derivatives, a generalized diffusion equation with fractional derivatives is obtained within the Renyi statistics. Averaging in generalized diffusion coefficient is performed with a power distribution with the Renyi parameter \( q \).

Keywords: generalized diffusion equation, nonequilibrium statistical operator, Renyi statistics, anomalous diffusion
I. INTRODUCTION

The fractional derivatives and integrals [1] are widely used to study anomalous diffusion in porous media [2–15], in disordered systems [16, 17], in plasma physics [18–23], in turbulent [24–26], kinetic and reaction-diffusion processes [26–34], in quantum mechanics [35–39], etc. [2, 40].

Experimental data on the different processes of anomalous diffusion show that not only the law distribution, but also form of diffusion package is significantly different from the normal diffusion [2, 17, 26, 40]. Approaches with variable diffusion coefficients [41], on the basis of a degree correlation of fractional order [42], fractional derivatives [24–26], generalized Fokker-Planck equation [6, 26, 43], generalization of statistical mechanics (extensive and non-extensive) based on the Tsallis [44–46] and Renyi [44, 47] entropy, and others were developed to describe the anomalous diffusion in different physical and chemical systems. Conducted researches show that mathematical basis of anomalous diffusion is equation with fractional derivatives [2, 26]. In particular, during the study of three-dimensional models of anomalous diffusion [2, 40, 48], the basic equation of anomalous diffusion is derived from the general principles of the stochastic theory of random processes based on the Chapman-Kolmogorov integral equations for transition probability. Solutions of these equations form a new class of distributions, called fractional stable distributions. These distributions are solutions of partial differential equations of fractional order. These equations are generalization of usual diffusion equation to the case of anomalous diffusion. A partial case of the fractional stable distribution is the Gaussian distribution, which corresponds to the normal diffusion. It is important to note that the equations for anomalous diffusion with fractional derivatives contain diffusion coefficient, which is a constant in time and space. On the other hand, the diffusion coefficients are related to time correlation functions (the Green-Kubo relations) containing diffusion transfer mechanisms from the perspective of nonequilibrium statistical mechanics.

Currently, together with the phenomenological approach for constructing of the Fokker-Planck equation, the diffusion equation and its generalization — the Cattaneo equation with fractional derivatives, there are two methods of constructing such equations, namely, (1) probabilistic method, based on the Chapman-Kolmogorov equation in stochastic theory of random processes [2, 26, 49], and (2) statistical method, based on the
Liouville equation with fractional derivatives \([50–63]\). In particular, by using this method, the BBGKY hierarchy equations with fractional derivatives \([51, 52, 58]\), transport equation, diffusion equation, and the Heisenberg equation with fractional derivatives \([54–56]\) are obtained. This approach is formulated for non-Hamiltonian systems. If the Helmholtz conditions for the coordinate and momentum derivatives are fulfilled, the Hamiltonian systems with the time-reversible Liouville equation with fractional derivatives are obtained from non-Hamiltonian systems. In Ref. \([64]\), time-irreversible equations of motion of Hamilton and Liouville for dynamic of classical particles in space with multifractal time are offered. By using the definition of fractional derivative and the Riemann-Liouville integral, the time-irreversible Liouville equation with fractional derivatives (where the time is given on multifractal sets with fractional dimensions) is obtained. In Refs. \([65, 66]\), kinetic equation for systems with fractal structure (in particular, for description of diffusion processes in space of coordinates and momenta) is obtained within the Klimontovich approach. A similar approach for constructing of time fractional generalization for the Liouville equation and the Zwanzig equation (within projection formalism) is proposed in Ref. \([67]\).

In the present work, by using the Zubarev nonequilibrium statistical operator method \([68–71]\) and the maximum entropy principle for the Renyi entropies, we consider a way of obtaining generalized (non-Markovian) diffusion equation with fractional derivatives. Using the Liouville equation with fractional derivatives proposed by Tarasov in Refs. \([50–53]\) is an important and fundamental step for obtaining this equation. In the second section, by using the Zubarev nonequilibrium statistical operator method and the maximum entropy principle for the Renyi entropies, we found a solution of the Liouville equation with fractional derivatives at a selected set of observed variables. In the third section, we chose nonequilibrium average values of particle density as a parameter of reduced description, and then we received a generalized (non-Markovian) diffusion equation with fractional derivatives.
We use the Liouville equation with fractional derivatives obtained by Tarasov in Refs. [50–53] for non-equilibrium particle function $\rho(x^N;t)$ of classical system:

$$\frac{\partial}{\partial t}\rho(x^N;t) + \sum_{j=1}^{N} D_{\vec{r}_j}^\alpha \left( \rho(x^N;t)\vec{v}_j \right) + \sum_{j=1}^{N} D_{\vec{p}_j}^\alpha \left( \rho(x^N;t)\vec{F}_j \right) = 0, \quad (1)$$

where $x^N = x_1, \ldots, x_N$, $x_j = \{\vec{r}_j, \vec{p}_j\}$ are the phase variables (coordinate and momentum) of $j$-th particle, $\vec{v}_j$ is the fields of velocity, $\vec{F}_j$ is the force field acting on $j$-th particle.

$$D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^n(z)}{(x-z)^{\alpha+1-n}} dz \quad (2)$$

is the Riemann-Liouville fractional derivative [1, 72], $n - 1 < \alpha < n$, $f^n(z) = \frac{d^n}{dz^n} f(z)$. In the general case we have

$$D_{\vec{r}_j}^\alpha \left( \rho(x^N;t)\vec{F}_j \right) \neq \rho(x^N;t)D_{\vec{r}_j}^\alpha \vec{F}_j + \vec{F}_j D_{\vec{r}_j}^\alpha \rho(x^N;t).$$

If $\vec{F}_j$ does not depend on $\vec{p}_j$, and $\vec{v}_j$ does not depend on $\vec{r}_j$, we get

$$\frac{\partial}{\partial t}\rho(x^N;t) + \sum_{j=1}^{N} \vec{v}_j D_{\vec{r}_j}^\alpha \rho(x^N;t) + \sum_{j=1}^{N} \vec{F}_j D_{\vec{p}_j}^\alpha \rho(x^N;t) = 0,$$

$$\vec{v}_j = D_{\vec{p}_j}^\alpha H(\vec{r},\vec{p}), \quad \vec{F}_j = -D_{\vec{r}_j}^\alpha H(\vec{r},\vec{p}),$$

where $H(\vec{r},\vec{p})$ is a Hamiltonian of system with fractional derivatives. We get the Liouville equation in the form

$$\frac{\partial}{\partial t}\rho(x^N;t) + \sum_{j=1}^{N} D_{\vec{r}_j}^\alpha H(\vec{r},\vec{p}) D_{\vec{p}_j}^\alpha \rho(x^N;t) - \sum_{j=1}^{N} D_{\vec{r}_j}^\alpha H(\vec{r},\vec{p}) D_{\vec{p}_j}^\alpha \rho(x^N;t) = 0, \quad (3)$$

or

$$\frac{\partial}{\partial t}\rho(x^N;t) + iL_{\alpha}\rho(x^N;t) = 0, \quad (4)$$

where $iL_{\alpha}$ is the Liouville operator with the fractional derivatives,

$$iL_{\alpha}\rho(x^N;t) = \left( \sum_{j=1}^{N} D_{\vec{r}_j}^\alpha H(\vec{r},\vec{p}) D_{\vec{p}_j}^\alpha - \sum_{j=1}^{N} D_{\vec{r}_j}^\alpha H(\vec{r},\vec{p}) D_{\vec{p}_j}^\alpha \right) \rho(x^N;t). \quad (5)$$

A solution the Liouville equation (5) will be found with the Zubarev nonequilibrium statistical operator method [68, 69]. After choosing parameters of the reduced description,
taking into account projections we present the non-equilibrium particle function $\rho(x^N; t)$ (as a solution of the Liouville equation) in general form

$$\rho(x^N; t) = \rho_{\text{rel}}(x^N; t) - \int_{-\infty}^{t} e^{(t'-t)} T(t, t')(1 - P_{\text{rel}}(t')) i L_\alpha \rho_{\text{rel}}(x^N; t') dt', \quad (6)$$

where $T(t, t') = \exp_+ \left(-\int_{t'}^{t} (1 - P_{\text{rel}}(t')) i L_\alpha dt'\right)$ is the evolution operator containing the projection, $\exp_+$ is ordered exponential, $\varepsilon \to +0$ after taking the thermodynamic limit, $P_{\text{rel}}(t')$ is the generalized Kawasaki-Gunton projection operator depended on a structure of the relevant statistical operator (distribution function), $\rho_{\text{rel}}(x^N; t')$. By using the Zubarev nonequilibrium statistical operator method [68–70] and approach [71], $\rho_{\text{rel}}(x^N; t')$ will be found from the extremum of the Renyi entropy at fixed values of observed values $\langle \hat{P}_n(x) \rangle_{t, \alpha}$, taking into account the normalization condition $\langle 1 \rangle_{t, \alpha, \text{rel}} = 1$, where the nonequilibrium average values are found respectively [22, 23, 52, 53]

$$\langle \hat{P}_n(x) \rangle_{t, \alpha} = \hat{I}_\alpha(1, \ldots, N) \tilde{T}(1, \ldots, N) \hat{P}_n \rho(x^N; t). \quad (7)$$

$\hat{I}_\alpha(1, \ldots, N)$ has the following form for a system of $N$ particles

$$\hat{I}_\alpha(1, \ldots, N) = \hat{I}_\alpha(1), \ldots, \hat{I}_\alpha(N), \quad \hat{I}_\alpha(j) = \hat{I}_\alpha(\vec{r}_j) \hat{I}_\alpha(\vec{p}_j)$$

and defines operation of integration

$$\hat{I}_\alpha(x) f(x) = \int_{-\infty}^{\infty} f(x) d\mu_\alpha(x), \quad d\mu_\alpha(x) = \frac{|x|^\alpha}{\Gamma(\alpha)} dx. \quad (8)$$

The operator $\hat{T}(1, \ldots, N) = \hat{T}(1), \ldots, \hat{T}(N)$ defines the operation

$$\hat{T}(x_j) f(x_j) = \frac{1}{2} \left( f(\ldots, x'_j - x_j, \ldots) + f(\ldots, x'_j + x_j, \ldots) \right).$$

Accordingly, the average value, calculated with the relevant distribution function, is defined as

$$\langle (\ldots) \rangle_{t, \alpha, \text{rel}} = \hat{I}_\alpha(1, \ldots, N) \tilde{T}(1, \ldots, N)(\ldots) \rho_{\text{rel}}(x^N; t).$$

According to [71], the relevant distribution function has the form

$$\rho_{\text{rel}}(t) = \frac{1}{Z_R(t)} \left( 1 - \frac{q-1}{q} \beta \left( H - \sum_n \int d\mu_\alpha(x) F_n(x; t) \delta \hat{P}_n(x; t) \right) \right)_{q-1}, \quad (9)$$
where $Z_R(t)$ is the partition function of the Renyi distribution, which is determined from the normalization condition and has the form

$$Z_R(t) = \hat{I}^\alpha(1, \ldots, N) \hat{T}(1, \ldots, N) \times \left(1 - \frac{q - 1}{q} \beta \left(H - \sum_n \int d\mu(x) F_n(x; t) \delta \hat{P}_n(x; t)\right)\right)^{\frac{1}{q-1}}. \quad (10)$$

The parameters $F_n(x; t)$ are determined from the self-consistency conditions

$$\langle \hat{P}_n(x) \rangle^t_\alpha = \langle \hat{P}_n(x) \rangle^t_{\alpha,rel}. \quad (11)$$

In the next section we will consider a specific example of diffusion processes in dense gases and liquids.

### III. GENERALIZED DIFFUSION EQUATION WITH FRACTIONAL DERIVATIVES

The nonequilibrium particle number density $n(\vec{r}_\alpha; t) = \langle \hat{n}(\vec{r}) \rangle^t_\alpha$ (where $n(\vec{r}) = \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)$ is the microscopic particle number density) is the main parameter of the reduced description for describing diffusion processes in classical gases and liquids. For such choice of parameter of the reduced description, the relevant distribution function has the form

$$\rho_{rel}(t) = \frac{1}{Z_R(t)} \left(1 - \frac{q - 1}{q} \beta \left(H - \int d\mu(\vec{r}) \nu(\vec{r}; t) \delta \hat{n}(\vec{r}_\alpha; t)\right)\right)^{\frac{1}{q-1}}, \quad (12)$$

where

$$Z_R(t) = \hat{I}^\alpha(1, \ldots, N) \hat{T}(1, \ldots, N) \left(1 - \frac{q - 1}{q} \beta \left(H - \int d\mu(\vec{r}) \nu(\vec{r}; t) \delta \hat{n}(\vec{r}_\alpha; t)\right)\right)^{\frac{1}{q-1}} \quad (13)$$

is the partition function of the relevant distribution function, $\beta = \frac{1}{k_B T}$, $k_B$ is the Boltzmann constant, $T$ is the equilibrium value of temperature, $\delta \hat{n}(\vec{r}_\alpha; t) = \hat{n}(\vec{r}) - \langle \hat{n}(\vec{r}) \rangle^t_\alpha$ is the fluctuations of the density, and parameter $\nu(\vec{r}; t)$ is determined by the self-consistency condition

$$\langle \hat{n}(\vec{r}) \rangle^t_\alpha = \langle \hat{n}(\vec{r}) \rangle^t_{\alpha,rel}. \quad (14)$$

It should be noted that the relevant distribution function (12) within the Renyi statistics for $q = 1$ coincides with one within the Gibbs statistics [68]. The distribution (12) can be
represented as
\[ \rho_{\text{rel}}(t) = \frac{1}{Z_R(t)} \left( 1 - \frac{q-1}{q} \beta \left( H - \int d\mu_\alpha(\vec{r}) \nu^s(\vec{r}; t) \hat{n}(\vec{r}) \right) \right)^{\frac{1}{q-1}}, \]  
where
\[ Z_R(t) = \hat{I}^\alpha(1, \ldots, N) \hat{T}(1, \ldots, N) \left( 1 - \frac{q-1}{q} \beta \left( H - \int d\mu_\alpha(\vec{r}) \nu^s(\vec{r}; t) \hat{n}(\vec{r}) \right) \right)^{\frac{1}{q-1}}, \]  
\[ \nu^s(\vec{r}; t) = \frac{\nu(\vec{r}; t)}{1 + \frac{q-1}{q} \int d\mu_\alpha(\vec{r}) \nu(\vec{r}; t) \langle \hat{n}(\vec{r}) \rangle^t}. \]
Substituting Eq. (15) in Eq. (6), we find the nonequilibrium statistical operator
\[ \rho(t) = \rho_{\text{rel}}(t) + \int_{-\infty}^t e^{\varepsilon(t'-t)} T(t, t') \int d\mu_\alpha(\vec{r}) I_n(\vec{r}^\alpha; t', t) \rho_{\text{rel}}(t) \nu^s(\vec{r}; t) \, dt', \]  
where
\[ I_n(\vec{r}^\alpha; t) = (1 - P(t)) \frac{1}{q} \psi^{-1}(t) iL_\alpha \hat{n}(\vec{r}) \]
is the generalized flow,
\[ \psi(t) = 1 - \frac{q-1}{q} \beta \left( H - \int d\mu_\alpha(\vec{r}) \nu^s(\vec{r}; t) \hat{n}(\vec{r}) \right). \]
\[ P(t) \] is the projection operator that has the following structure:
\[ P(t) = \int d\mu_\alpha(\vec{r}) \int d\mu_\alpha(\vec{r}') \langle \ldots \hat{n}(\vec{r}) \rangle^t_{\alpha,\text{rel}} \left\{ \hat{n}(\vec{r}) \delta \{ [q\psi(t)]^{-1} \hat{n}(\vec{r}') \} \right\}^t_{\alpha,\text{rel}} - 1 \delta \{ [q\psi(t)]^{-1} \hat{n}(\vec{r}') \}, \]
where \( \delta \{ A \} = A - \langle A \rangle^t_{\alpha,\text{rel}} \).
By using the nonequilibrium statistical operator (17), we get the generalized diffusion equation for the parameter of the reduced description
\[ \frac{\partial}{\partial t} \langle \hat{n}(\vec{r}) \rangle^t_{\alpha} = \int d\mu_\alpha(\vec{r}) \int_{-\infty}^t e^{\varepsilon(t'-t)} \phi_{nn}(\vec{r}, \vec{r}^\alpha; t, t') \beta \nu^s(\vec{r}; t') \, dt', \]  
where
\[ \phi_{nn}(\vec{r}, \vec{r}^\alpha; t, t') = \hat{I}^\alpha(1, \ldots, N) \hat{T}(1, \ldots, N) iL_\alpha \hat{n}(\vec{r}) T(t, t') I_n(\vec{r}^\alpha; t') \rho_{\text{rel}}(\vec{x}^N; t') \]
\[ = \frac{\partial}{\partial \vec{r}^\alpha} \cdot D_q(\vec{r}, \vec{r}^\alpha; t, t') \cdot \frac{\partial}{\partial \vec{r}^\alpha} \]
is the generalized transport kernel (memory function), in which the averaging is performed with the distribution function (15).
As a result, we get the non-Markovian diffusion equation with fractional derivatives

$$\frac{\partial}{\partial t} \langle \hat{n}(\vec{r}) \rangle_{\alpha} = \frac{\partial^{\alpha}}{\partial r^{\alpha}} \cdot \int d\mu_{\alpha}(\vec{r}') \int_{-\infty}^{t} e^{\varepsilon(t'-t)} D_{q}(\vec{r}, \vec{r}'; t, t') \cdot \frac{\partial^{\alpha}}{\partial r'^{\alpha}} \beta^{\ast}(\vec{r}'; t') dt', \quad (20)$$

where

$$D_{q}(\vec{r}, \vec{r}'; t, t') = \langle \hat{v}(\vec{r}) T(t, t') \hat{v}(\vec{r}') \rangle_{\alpha, rel}$$

$$= \hat{I}^{\alpha}(1, \ldots, N) \hat{T}(1, \ldots, N)$$

$$\times \hat{v}(\vec{r}) T(t, t') \hat{v}(\vec{r}') \frac{1}{Z_{R}(t)} \left( 1 - \frac{q-1}{q} \beta \left( H - \int d\mu_{\alpha}(\vec{r}) \nu^{\ast}(\vec{r}; t) \hat{n}(\vec{r}) \right) \right)^{\frac{1}{q-1}}$$

$$\quad (21)$$

is the generalized diffusion coefficient within the Renyi statistics, in which the averaging is performed with the distribution function (15), where \( \hat{v}(\vec{r}) = \sum_{j=1}^{N} v_{j} \delta(\vec{r} - \vec{r}_{j}) \) is the microscopic particle number flux density. At \( q = 1 \) the generalized diffusion equation within the Renyi statistics goes over to one within the Gibbs statistics with fractional derivatives.

If \( q = 1 \) and \( \alpha = 1 \), we get the generalized diffusion equation within the Gibbs statistics [68]. In the Markov approximation for the general diffusion coefficient in time and space \( D_{q}(\vec{r}, \vec{r}'; t, t') \approx D_{q} \delta(t - t') \delta(\vec{r} - \vec{r}') \), after exclusion the parameter \( \nu^{\ast}(\vec{r}; t') \) by using the self-consistency condition, from Eq. (20) we obtain the diffusion equation with fractional derivatives

$$\frac{\partial}{\partial t} \langle \hat{n}(\vec{r}) \rangle_{\alpha}^{t} = D_{q} \frac{\partial^{2\alpha}}{\partial r^{2\alpha}} \langle \hat{n}(\vec{r}) \rangle_{\alpha}^{t}. \quad (22)$$

IV. CONCLUSION

By using the Zubarev nonequilibrium statistical operator [68–71], the Liouville equation with fractional derivatives [22, 23, 52, 53], and the principle of maximum entropy, we obtain the generalized (non-Markovian) diffusion equation with fractional derivatives. By using this approach, the generalized transfer equation with fractional derivatives can be obtained at some set of parameters of the reduced description \( \langle \hat{P}_{n}(x) \rangle_{\alpha, rel}^{t} \) of nonequilibrium state of the system In particular, if these parameters are nonequilibrium average values of particle density, momentum and energy, we obtain generalized hydrodynamic equations with
fractional derivatives, which are generalization of Tarasov’s results \[57\].

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Books on Mathematics (Dover Publications, 2006).

[2] V. Uchaikin, *Fractional Derivatives Method* (Artishock-Press, Uljanovsk, 2008).

[3] M. Sahimi, Physics Reports 306, 213 (1998).

[4] D. Korošak, B. Cvikl, J. Kramer, R. Jecl, and A. Prapotnik, Journal of Contaminant Hydrology 92, 1 (2007).

[5] R. K. Hobbie and B. J. Roth, *Intermediate Physics for Medicine and Biology* (Springer, New York, 2007).

[6] R. Metzler and J. Klafter, Physics Reports 339, 1 (2000).

[7] R. Hilfer, Chaos, Solitons & Fractals 5, 1475 (1995).

[8] R. Hilfer, The Journal of Physical Chemistry B 104, 3914 (2000).

[9] R. Hilfer, *Fractional Time Evolution* (World Scientific, Singapore, New Jersey, London, Hong Kong, 2000), chap. II, pp. 87–130.

[10] T. Kosztołowicz, K. Dworecki, and S. Mrówczyński, Phys. Rev. E 71, 041105 (2005).

[11] T. Kosztołowicz, Journal of Statistical Mechanics: Theory and Experiment 2015, P10021 (2015).

[12] J. Bisquert, G. Garcia-Belmonte, F. Fabregat-Santiago, N. S. Ferriols, P. Bogdanoff, and E. C. Pereira, The Journal of Physical Chemistry B 104, 2287 (2000).

[13] J. Bisquert and A. Compte, Journal of Electroanalytical Chemistry 499, 112 (2001).

[14] T. Kosztołowicz and K. D. Lewandowska, Journal of Physics A: Mathematical and Theoretical 42, 055004 (2009).

[15] Y. Pyanylo, M. Prytula, N. Prytula, and N. Lopuh, Mathematical Modeling and Computing 1, 84 (2014).

[16] B. Berkowitz and H. Scher, Phys. Rev. E 57, 5858 (1998).

[17] J.-P. Bouchaud and A. Georges, Physics Reports 195, 127 (1990).

[18] R. Balescu, Phys. Rev. E 51, 4807 (1995).

[19] M. Tribeche and P. K. Shukla, Physics of Plasmas 18, 103702 (2011).
[20] J. Gong and J. Du, Physics of Plasmas 19, 023704 (2012).
[21] B. A. Carreras, V. E. Lynch, and G. M. Zaslavsky, Physics of Plasmas 8, 5096 (2001).
[22] V. E. Tarasov, Physics of Plasmas 12, 082106 (2005).
[23] V. E. Tarasov, Physics of Plasmas 13, 052107 (2006).
[24] A. Monin, DAN SSSR, ser. geofiz. 2, 256 (1955).
[25] J. Klimontovich, Vvedenie v fiziku otkrytyh sistem (Moskva Janus, 2002).
[26] G. Zaslavsky, Physics Reports 371, 461 (2002).
[27] G. Zaslavsky, Physica D: Nonlinear Phenomena 76, 110 (1994).
[28] A. I. Saichev and G. M. Zaslavsky, Chaos 7, 753 (1997).
[29] G. Zaslavsky and M. Edelman, Physica D: Nonlinear Phenomena 193, 128 (2004).
[30] R. Nigmatullin, Physica A: Statistical Mechanics and its Applications 363, 282 (2006).
[31] A. V. Chechkin, V. Y. Gonchar, and M. Szydlowski, Physics of Plasmas 9, 78 (2002).
[32] V. V. Gafiychuk and B. Y. Datsko, Phys. Rev. E 75, 055201 (2007).
[33] T. Kosztołowicz and K. D. Lewandowska, Phys. Rev. E 78, 066103 (2008).
[34] V. P. Shkilev, Journal of Experimental and Theoretical Physics 117, 1066 (2013).
[35] N. Laskin, Chaos 10, 780 (2000).
[36] N. Laskin, Phys. Rev. E 62, 3135 (2000).
[37] N. Laskin, Physics Letters A 268, 298 (2000).
[38] N. Laskin, Phys. Rev. E 66, 056108 (2002).
[39] M. Naber, Journal of Mathematical Physics 45, 3339 (2004).
[40] V. V. Uchaikin, Physics-Uspekhi 56, 1074 (2013).
[41] B. O’Shaughnessy and I. Procaccia, Phys. Rev. Lett. 54, 455 (1985).
[42] B. B. Mandelbrot, The fractal geometry of nature (W. H. Freeman and Company, 1982), 1st ed.
[43] R. Metzler, E. Barkai, and J. Klafter, Europhysics Letters 46, 431 (1999).
[44] C. Essex, C. Schulzky, A. Franz, and K. H. Hoffmann, Physica A: Statistical Mechanics and its Applications 284, 299 (2000).
[45] C. Tsallis, Nonextensive Statistical Mechanics and Its Applications (Springer-Verlag Berlin Heidelberg, 2001), Lecture Notes in Physics 560, 1st ed.
[46] M. Gell-Mann and C. Tsallis, Nonextensive entropy: Interdisciplinary applications, Santa Fe Institute Studies on the Sciences of Complexity (Oxford University Press, USA, 2004).
[47] A. R. Vasconcellos, J. Galvão Ramos, A. Gorenstein, M. U. Kleinke, T. G. Souza Cruz, and R. Luzzi, International Journal of Modern Physics B 20, 4821 (2006).

[48] V. V. Uchaikin, Journal of Experimental and Theoretical Physics 97, 810 (2003).

[49] A. A. Stanislavsky, Theoretical and Mathematical Physics 138, 418 (2004).

[50] V. E. Tarasov, Chaos 14, 123 (2004).

[51] V. E. Tarasov, Journal of Physics: Conference Series 7, 17 (2005).

[52] V. E. Tarasov, Phys. Rev. E 71, 011102 (2005).

[53] V. E. Tarasov, Chaos 16, 033108 (2006).

[54] V. E. Tarasov, International Journal of Modern Physics B 20, 341 (2006).

[55] V. E. Tarasov, Nonlinear Dynamics 71, 663 (2013).

[56] V. E. Tarasov, Physics Letters A 372, 2984 (2008).

[57] V. E. Tarasov, Annals of Physics 318, 286 (2005).

[58] V. E. Tarasov, Modern Physics Letters B 21, 237 (2007).

[59] V. E. Tarasov, Modern Physics Letters B 21, 163 (2007).

[60] V. E. Tarasov, Theoretical and Mathematical Physics 158, 179 (2009).

[61] V. E. Tarasov, Annals of Physics 327, 1719 (2012).

[62] V. E. Tarasov, Physics of Plasmas 20, 102110 (2013).

[63] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Nonlinear Physical Science (Springer Berlin Heidelberg, 2010), 1st ed.

[64] L. Y. Kobelev, The multifractal time and irreversibility in dynamic systems (2000), arXiv:physics/0002002.

[65] Y. L. Kobelev, L. Y. Kobelev, and E. P. Romanov, Doklady Physics 45, 194 (2000).

[66] Y. L. Kobelev, L. Y. Kobelev, V. L. Kobelev, and E. P. Romanov, Doklady Physics 47, 580 (2002).

[67] S. Y. Lukashchuk, Central European Journal of Physics 11, 740 (2013).

[68] D. N. Zubarev, Journal of Soviet Mathematics 16, 1509 (1981).

[69] D. Zubarev, V. Morozov, and G. Röpke, Statistical mechanics of nonequilibrium processes, vol. 1 (Fizmatlit, 2002).

[70] D. Zubarev, V. Morozov, and G. Röpke, Statistical mechanics of nonequilibrium processes, vol. 2 (Fizmatlit, 2002).
[71] B. Markiv, R. Tokarchuk, P. Kostrobij, and M. Tokarchuk, Physica A: Statistical Mechanics and its Applications 390, 785 (2011).

[72] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications (Gordon and Breach Science Publishers, 1993), 1st ed.