GAMMA FACTORS FOR THE ASAI CUBE REPRESENTATION

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Abstract. We prove an equality between the gamma factors for the Asai cube representation of \( R_{E/F} \text{GL}_2 \) defined by the Weil–Deligne representations and the local zeta integrals of Ikeda and Piatetski-Shapiro–Rallis, where \( E \) is an \( \acute{e}tale \) cubic algebra over a local field \( F \) of characteristic zero. As an application we obtain the analytic properties of the automorphic \( L \)-functions for the Asai cube representation.

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1. INTRODUCTION

1.1. Main results. Let \( F \) be a local field of characteristic zero and \( E \) an \( \acute{e}tale \) cubic algebra over \( F \). Denote by \( \mathbb{K} \) the quadratic discriminant algebra of \( E \). Let \( \alpha \) be a basis of \( E \) over \( F \) and \( \psi \) a non-trivial additive character of \( F \). Let \( \Pi \) be an irreducible generic admissible representation of \( \text{GL}_2(E) \). Denote by \( \phi_\Pi : W'_E \to L(R_{E/F} \text{GL}_2) \) the \( L \)-parameter associated to \( \Pi \) via the local Langlands correspondence. Here \( W'_E \) is the Weil–Deligne group of \( F \) and \( L(R_{E/F} \text{GL}_2) \) is the Langlands dual group of \( R_{E/F} \text{GL}_2 \). We have the Asai cube representation (see §3 for the precise definition)

\[
\text{As} : L(R_{E/F} \text{GL}_2) \rightarrow \text{GL}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2).
\]

Let \( L(s, \Pi, \varepsilon) \), \( \varepsilon(s, \Pi, \psi) \), and \( \gamma(s, \Pi, \psi) \) be the \( L \)-factor, \( \varepsilon \)-factor, and \( \gamma \)-factor, respectively, associated to the admissible representation \( \text{As} \circ \phi_\Pi \) of \( W'_E \). On the other hand, associated to \( \Pi \) we can define the local factors \( L_{\text{PSR}}(s, \Pi, \varepsilon) \), \( L_{\text{PSR}}(s, \Pi, \psi, \alpha) \), and \( L_{\text{PSR}}(s, \Pi, \psi, \alpha) \) via the local zeta integrals of Ikeda and Piatetski-Shapiro–Rallis introduced in [Ike89] and [PSR87]. The local zeta integrals are of the form

\[
Z_{\tilde{\alpha}}(f_s, W) = \int_{F \times U_0(F) \backslash G(F)} f_s(t_{\tilde{\alpha}}(\eta g))W(g) \, dg.
\]

Here, \( \tilde{\alpha} \) is a symplectic basis of the symplectic space \( V(F) = E \oplus E \) equipped with the trace form, \( t_{\tilde{\alpha}} \) is the isomorphism between \( \text{GSp}(V)(F) \) and \( \text{GSp}_6(F) \) induced by \( \tilde{\alpha} \), \( G(F) = \{ g \in \text{GL}_2(E) \mid \det(g) \in F^\times \} \subseteq \text{GSp}(V)(F) \), \( U_0(F) \) is a unipotent subgroup of \( G(F) \), \( \eta \in \text{GSp}(V)(F) \) satisfies certain conditions, \( f_s \) is a section in certain degenerate principal series representation of \( \text{GSp}_6(F) \), and \( W \) is a Whittaker function of \( \Pi \) with respect to \( \psi_E = \psi \circ \text{tr}_{E/F} \). We refer to §4 for more detail.

The main results of this paper are as follows:

Theorem 1.1. We have

\[
\gamma_{\text{PSR}}(s, \Pi, \psi, \alpha) = \omega_\Pi(\Delta_{E/F}(\alpha)|\Delta_{E/F}(\alpha)|^{s-1}\omega_{\mathbb{K}/F}(-1)\gamma(s, \Pi, \psi).
\]

Here \( \omega_\Pi \) is the central character of \( \Pi \), \( \Delta_{E/F}(\alpha) \) is the relative discriminant of \( \alpha \) for \( E/F \), and \( \omega_{\mathbb{K}/F} \) is the quadratic character associated to \( \mathbb{K}/F \) by local class field theory.
Remark 1.2. When $E$ is unramified over $F$ and $\Pi$ is unramified, the identity follows from the explicit calculation due to Piatetski-Shapiro and Rallis in [PSR87]. When $E = F \times F \times F$, the identity was established by Ikeda and Ramakrishnan in [Ike89] and [Ram00], respectively. When $E = F' \times F$ for some quadratic extension $F'$ of $F$, the assertion was partially proved by Chen–Cheng–Ishikawa in [CCI18].

Corollary 1.3. Assume $\max \{ |L(\Pi)|, |L(\Pi')| \} < 1/2$. Then

$$L_{PSR}(s, As \Pi) = L(s, As \Pi),$$

$$\varepsilon_{PSR}(s, As \Pi, \psi, \alpha) = \omega_\Pi(\Delta_{E/F}(\alpha))|\Delta_{E/F}(\alpha)|^{\frac{2s}{P}}\omega_{K/F}(-1)\varepsilon(s, As \Pi, \psi).$$

Here $L(\Pi)$ and $L(\Pi')$ are defined in (2.3).

Now we switch to a global setting. Let $F$ be a number field and $E$ an étale cubic algebra over $F$. Denote by $\mathcal{A}_E$ and $\mathcal{A}_F$ the rings of adeles of $E$ and $F$, respectively. Let $\psi$ be a non-trivial additive character of $\mathcal{A}_F/\mathcal{A}_F$.

Let $\Pi$ be an irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathcal{A}_E)$ with central character $\omega_\Pi$. Write $\omega = \omega_\Pi|_{\mathcal{A}_F}$. Let

$$L(s, As \Pi) = \prod_v L(s, As \Pi_v), \quad \varepsilon(s, As \Pi) = \prod_v \varepsilon(s, As \Pi_v, \psi_v)$$

be the automorphic $L$-function and $\varepsilon$-factor associated to $\Pi$ and the Asai cube representation. We have the following three cases:

$$\begin{cases} E = F \times F \times F & \text{Case 1}, \\ E = F' \times F & \text{Case 2}, \\ E \text{ is a field} & \text{Case 3}. \end{cases}$$

Combining Corollary 1.3 and the results in [Ike92], [Ram00], and [KS02], we have the following description of the analytic properties of $L(s, As \Pi)$.

Theorem 1.4. The $L$-function $L(s, As \Pi)$ is absolutely convergent for $\Re(s) \geq 3/2$, admits meromorphic continuation to $s \in \mathbb{C}$, bounded in vertical strips of finite width, and satisfies the functional equation

$$L(s, As \Pi) = \varepsilon(s, As \Pi)L(1 - s, As \Pi').$$

If either $\omega^2$ is not principal or $\omega$ is principal, then $L(s, As \Pi)$ is entire. If $\omega^2$ is principal and $\omega$ is not principal, we may assume $\omega^2 = 1$ and $\omega \neq 1$ and let $K$ be the quadratic extension of $F$ associated to $\omega$ by class field theory. Then $L(s, As \Pi)$ is not entire if and only if $K \neq F'$ in Case 2 and there exists a unitary Hecke character $\chi$ of $\mathcal{A}_E/\mathcal{A}_K$ with $\chi|_{\mathcal{A}_F} = 1$ such that $\Pi = \text{Ind}_{\mathcal{A}_E/\mathcal{A}_K}^E(\chi)$. In this case, we have

$$L(s, As \Pi) = \zeta_K(s) \begin{cases} L(s, \chi_1^1 \chi_2^1 \chi_3^1 \chi_3^1) & \text{in Case 1 and } \chi = (\chi_1, \chi_2, \chi_3), \\ L(s, \chi_1^1 \chi_2^1 \chi_3^2) & \text{in Case 2 and } \chi = (\chi_1, \chi_2), \\ L(s, \chi_1^{-1} \chi_3^{-1}) & \text{in Case 3}. \end{cases}$$

Here $\text{Ind}_{\mathcal{A}_E/\mathcal{A}_K}^E(\chi)$ is the automorphic induction of $\chi$ from $\mathcal{A}_E/\mathcal{A}_K$ to $\mathrm{GL}_2(\mathcal{A}_E)$, $\zeta_K(s)$ is the completed Dedekind zeta function of $K$, and $\sigma$ is the generator of

$$\begin{cases} \text{Gal}(K/F) & \text{in Case 1}, \\ \text{Gal}(F'/F') & \text{in Case 2}, \\ \text{Gal}(E/K/E) & \text{in Case 3}. \end{cases}$$

1.2. An outline of the proof. We sketch the proof of Theorem 1.1. Since the assertion is known when $F$ is archimedean, we assume $F$ is non-archimedean. Let $f_s$ be a good section of $I(\omega, s)$ and $W$ a Whittaker function of $\Pi$ with respect to $\psi_E$. By the functional equation of the local zeta integrals, it suffices to prove

$$(1.1) \quad Z_\lambda(M'_w f_s, W') = \omega_\Pi(\Delta_{E/F}(\alpha))|\Delta_{E/F}(\alpha)|^{2s-1}\omega_{K/F}(-1)\gamma(s, As \Pi, \psi)Z_\lambda(f_s, W).$$

We extend $\Pi$ to a family of irreducible generic admissible representations $\Pi_\lambda$ of $\mathrm{GL}_2(E)$ defined in (2.2). Here $M'_w$ is the normalized intertwining operator (4.5), $W'(g) = \omega_\Pi(\det(g)^{-1}W(g)$, and $\lambda$ varies in the domain
\( \mathcal{D}(\Pi) \) defined in \((2.4)\). Write \( \omega_\lambda = \omega_{\Pi_\lambda, s} \). Let \( f_{s, \lambda} \) be a good section of \( I(\omega_\lambda, s) \) and \( W_\lambda \) a holomorphic family of Whittaker functions of \( \Pi_\lambda \) with respect to \( \psi_E \) extending \( f_s \) and \( W_t \), respectively (cf. \S 2). Let

\[
Z_1(s, \lambda) = \frac{Z_0(M_t^s f_{s+1/2, \lambda}, W_\lambda^\gamma)}{L(s+1/2, \text{As} \Pi_\lambda^\gamma)}
\]

and

\[
Z_2(s, \lambda) = \omega_{\Pi_\lambda}(\Delta_{E/F}(\alpha)) |\Delta_{E/F}(\alpha)|^{2s} \omega_{E/F}(-1) \varepsilon(s+1/2, \text{As} \Pi_\lambda, \psi, \alpha) \frac{Z_0(f_{s+1/2, \lambda}, W_\lambda)}{L(s+1/2, \text{As} \Pi_\lambda)}
\]

be meromorphic functions on \( \mathbb{C} \times \mathcal{D}(\Pi) \). By the uniform asymptotic estimate for \( W_\lambda \) proved in Lemma 2.2 we show in Lemma 4.2 below that \( Z_1 \) and \( Z_2 \) define holomorphic functions on the domain

\[
\{(s, \lambda) \in \mathbb{C} \times \mathcal{D}(\Pi) \mid \Re(s) > |\lambda|_\Pi - 1/2\}.
\]

Here \( |\lambda|_\Pi \in \mathbb{R}_{\geq 0} \) is the absolute value of \( \lambda \) with respect to \( \Pi \) defined in \((2.3)\). Now we use the limit multiplicity method, which is a global-to-local argument (for example, see [BP18 \S 3.8] and [CH19 \S 5.3]). More precisely, based on the following ingredients:

- the limit multiplicity property for the principal congruence subgroup subgroups of \( \text{GL}_2 \) proved in [FLM15],
- the known cases for \((1.1)\) recalled in Lemma 4.3 and Corollary 4.5,
- the equality between the Asai cube \( \gamma \)-factors defined by the Weil–Deligne representation and the Langlands–Shahidi method proved in [HL18],

we prove that the functional equation

\[
(1.2) \quad Z_1(-s, \lambda) = Z_2(s, \lambda)
\]

holds for \( s \in \mathbb{C} \) and \( \lambda \) in a dense subset of

\[
\{\lambda \in \mathcal{D}(\Pi) \mid \Pi_\lambda \text{ is tempered}\}.
\]

Note that \( \Pi_\lambda \) is tempered if and only if \( |\lambda|_\Pi = 0 \). It then follows from the continuity and holomorphicity that \( Z_1 \) and \( Z_2 \) define holomorphic functions on the domain

\[
\mathbb{C} \times \{\lambda \in \mathcal{D}(\Pi) \mid |\lambda|_\Pi < 1/2\}
\]

and satisfy the functional equation \((1.2)\). Finally, we are in the position to apply [BP18 Proposition 2.8.1], which together with Lemma 4.2 imply that \( Z_1 \) and \( Z_2 \) admit holomorphic continuation to \( \mathbb{C} \times \mathcal{D}(\Pi) \) and satisfy the functional equation \((1.2)\). In particular, \((1.1)\) follows.

1.3. Notation. Let \( F \) be a local field of characteristic zero. When \( F \) is non-archimedean, let \( \mathfrak{o}_F \) be the ring of integers of \( F \), \( \varpi_F \) a uniformizer of \( \mathfrak{o}_F \), \( q_F \) the cardinality of \( \mathfrak{o}_F/\varpi_F \mathfrak{o}_F \), \( |\cdot|_F \) the absolute value on \( F \) normalized so that \( |\varpi_F|_F = q_F^{-1} \), and \( \text{ord}_F \) the valuation on \( F \) normalized so that \( \text{ord}_F(\varpi_F) = 1 \). When \( F \) is archimedean, let \( |\cdot|_\mathbb{R} = \sqrt{\cdot} \) be the usual absolute value on \( \mathbb{R} \) and \( |\cdot|_\mathbb{C} = \sqrt{\cdot} \) on \( \mathbb{C} \).

An additive character \( \psi \) of \( F \) is a continuous homomorphism \( \psi : F \to \mathbb{C}^\times \). For \( a \in F^\times \), let \( \psi^a \) be the additive character defined by \( \psi^a(x) = \psi(ax) \).

A character \( \chi \) of \( F^\times \) is a continuous homomorphism \( \chi : F^\times \to \mathbb{C}^\times \). For a character \( \chi \) of \( F^\times \), let \( \text{wt}(\chi) \in \mathbb{R} \) defined so that \( |\chi| = |\chi|_{\text{wt}(\chi)} \) and denote by \( L(s, \chi) \), \( \varepsilon(s, \chi, \psi) \), and \( \gamma(s, \chi, \psi) \) the \( L \)-factor, \( \varepsilon \)-factor, and \( \gamma \)-factor of \( \chi \), respectively, with respect to an additive character \( \psi \) of \( F \) defined in [Tat79].

Let \( B \) be the standard Borel subgroup of \( \text{GL}_2 \) consisting of upper triangular matrices and \( N \) its unipotent radical. We put

\[
a(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \quad d(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}, \quad m(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

for \( \nu, t \in \mathbb{G}_m \) and \( x \in \mathbb{G}_a \).
2. Holomorphic family of Whittaker functions

Let $F$ be a local field of characteristic zero and $\psi$ a non-trivial additive character of $F$. Let

$$K = \begin{cases} 
\text{GL}_2(\mathcal{O}_F) & \text{if } F \text{ is non-archimedean}, \\
\text{O}(2) & \text{if } F = \mathbb{R}, \\
\text{U}(2) & \text{if } F = \mathbb{C}, 
\end{cases}$$

be a maximal compact subgroup of $\text{GL}_2(F)$. Denote by $C_0^\infty(N(F) \backslash \text{GL}_2(F), \psi)$ the space of smooth functions $W : \text{GL}_2(F) \to \mathbb{C}$ such that

- For $x \in F$ and $g \in \text{GL}_2(F)$,
  $$W(n(x)g) = \psi(x)W(g).$$

- $W$ is right $K$-finite.

Let $\pi$ be an irreducible generic admissible representation of $\text{GL}_2(F)$ with central character $\omega_\pi$. We denote by $\pi^\vee$ the contragredient representation of $\pi$ and by $W(\pi, \psi)$ the Whittaker model of $\pi$ with respect to $\psi$. Recall that $W(\pi, \psi)$ is the image of a non-zero intertwining map $\pi \to C_0^\infty(N(F) \backslash \text{GL}_2(F), \psi)$. For $W \in W(\pi, \psi)$, we define $W^\vee \in W(\pi^\vee, \psi)$ by

$$W^\vee(g) = \omega_\pi(\det(g))^{-1}W(g).$$

We define $l(\pi) \in \mathbb{R}$ by

$$l(\pi) = \begin{cases} 
\frac{\text{wt}(\omega_\pi)}{2} & \text{if } \pi \text{ is essentially square-integrable}, \\
\min\{\text{wt}(\chi_1), \text{wt}(\chi_2)\} & \text{if } \pi = \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi_1 \otimes \chi_2).
\end{cases}$$

If $\pi$ is essentially square-integrable and $\lambda \in \mathbb{C}$, we define $\pi_\lambda = \pi \otimes |\lambda|^{-1}_F$. If $\pi = \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi_1 \otimes \chi_2)$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, we define

$$\pi_\lambda = \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi_1 |\lambda_1|^{-1}_F \otimes \chi_2 |\lambda_2|^{-1}_F).$$

Let $\mathcal{D}(\pi)$ be the domain associated to $\pi$ defined by

$$\mathcal{D}(\pi) = \begin{cases} 
\mathbb{C} & \text{if } \pi \text{ is essentially square-integrable}, \\
\{\lambda \in \mathbb{C}^2 | \pi_\lambda \text{ is irreducible} \} & \text{if } \pi \text{ is a principal series representation}.
\end{cases}$$

For $\lambda \in \mathcal{D}(\pi)$, define $|\lambda|_\pi \in \mathbb{R}_{\geq 0}$ by

$$|\lambda|_\pi = \begin{cases} 
\frac{\text{wt}(\omega_\pi)}{2} + \text{Re}(\lambda) & \text{if } \pi \text{ is essentially square-integrable}, \\
\max\{\text{wt}(\chi_1) + \text{Re}(\lambda_1), \text{wt}(\chi_2) + \text{Re}(\lambda_2)\} & \text{if } \pi = \text{Ind}_{B(F)}^{\text{GL}_2(F)}(\chi_1 \otimes \chi_2).
\end{cases}$$

We call a map

$$\mathcal{D}(\pi) \to C_0^\infty(N(F) \backslash \text{GL}_2(F), \psi), \quad \lambda \mapsto W_\lambda$$

a holomorphic family of Whittaker functions of $\pi_\lambda$ with respect to $\psi$ if it satisfies the following conditions:

- The map $(\lambda, g) \mapsto W_\lambda(g)$ is continuous.
- For each $g \in \text{GL}_2(F)$, the map $\lambda \mapsto W_\lambda(g)$ is holomorphic on $\mathcal{D}(\pi)$.
- For each $\lambda \in \mathcal{D}(\pi)$, the function $g \mapsto W_\lambda(g)$ belongs to $W(\pi_\lambda, \psi)$.
- $W_\lambda$ is right $K$-finite.

We recall a construction of holomorphic family of Whittaker functions. Let $\omega_\psi$ be the Weil representation of $\text{GL}_2(F)$ on $\mathcal{S}(F^2)$, the space of Schwartz functions on $F^2$, with respect to $\psi$ defined by the following rules:

- For $t \in F^\times$,
  $$\omega_\psi(m(t)) \varphi(x, y) = |t|_F \varphi(tx, ty).$$

- For $b \in F$,
  $$\omega_\psi(n(b)) \varphi(x, y) = \psi(bxy) \varphi(x, y).$$

- For $\nu \in F^\times$,
  $$\omega_\psi(a(\nu)) \varphi(x, y) = \varphi(\nu x, y).$$

where $du$ and $dv$ are self-dual with respect to $\psi$.
Lemma 2.1. Assume $\omega$ is non-archimedean. Let $\pi_\lambda$ be a holomorphic family of Whittaker functions of $\pi_\lambda$ with respect to $\psi$. If $\pi = \text{Ind}_{B(F)}^{GL_2(F)}(\chi_1 \otimes \chi_2)$ and $\varphi \in \mathcal{S}_\psi(F^2)$, then the map

$$
\lambda \mapsto W(\varphi, \lambda),
$$

(2.1)

is a holomorphic family of Whittaker functions of $\pi_\lambda$ with respect to $\psi$. If $\psi \in \mathcal{W}(\pi, \psi)$, then the map

$$
\lambda \mapsto W(\varphi, \lambda),
$$

(2.2)

is a holomorphic family of Whittaker functions of $\pi_\lambda$ with respect to $\psi$. Note that for any fixed $\lambda_0 \in \mathcal{D}(\pi)$, any holomorphic family of Whittaker functions can be written as a linear combination, with holomorphic functions of $\lambda$ as coefficients, of holomorphic families of the form (2.1) or (2.2) in a neighborhood of $\lambda_0$.

We have the following uniform asymptotic estimate for holomorphic families of Whittaker functions.

**Lemma 2.1.** Assume $F$ is non-archimedean. Let $W_\lambda$ be a holomorphic family of Whittaker functions of $\pi_\lambda$ with respect to $\psi$. Let $\epsilon > 0$. There exist an integer $n$ independent of $\epsilon, \lambda$ and a constant $C_{\lambda, \epsilon} > 0$ bounded uniformly as $\lambda$ varies in a compact set such that

$$
|W(\varphi, \lambda)| \leq C_{\lambda, \epsilon} \cdot \|\varphi\|^{|(\pi_\lambda)|+1/2-\epsilon}
$$

for $\nu \in F^\times$ and $k \in GL_2(\mathfrak{o}_F)$.

**Proof.** Since we only consider the convergence for $\lambda$ varying in a compact set, it suffices to consider holomorphic families of the form (2.1) or (2.2). We assume $\pi = \text{Ind}_{B(F)}^{GL_2(F)}(\chi_1 \otimes \chi_2)$ is a principal series representation. The other case follows from the asymptotic estimate for a single Whittaker function in [Jac72, Lemma 14.3]. By the formulae defining $\omega_\psi$, there exist an integer $n$ and a constant $C > 0$ such that

$$
|\omega_\psi(\varphi(\nu)k)(x, y)| \leq C \cdot \|\omega_\psi(\varphi(\nu)k)(x, y)|
$$

for $(x, y) \in F^2, \nu \in F^\times$, and $k \in GL_2(\mathfrak{o}_F)$. Let

$$
s_1 = \text{wt}(\chi_1), \quad s_2 = \text{wt}(\chi_2).
$$

Then we have

$$
|W(\varphi, \lambda)| \leq C \cdot |\nu|^{s_1 + \text{Re}(\lambda_1) + 1/2} \int_{F^\times} \|\omega_\psi(\varphi(\nu)k)(x, y)|
$$

$$
= C \cdot \sum_{m = -n}^{n} q_F^{-s_1 + s_2 - \text{Re}(\lambda_1 - \lambda_2)}
$$

$$
= C \cdot \left[ C^{(1)}_{\lambda} \cdot |\nu|^{s_1 + \text{Re}(\lambda_1) + 1/2} + C^{(2)}_{\lambda} \cdot |\nu|^{s_2 + \text{Re}(\lambda_2) + 1/2} \right].
$$

for $\nu \in F^\times$ and $k \in GL_2(\mathfrak{o}_F)$. Here

$$
C^{(1)}_{\lambda} = \sum_{m = -n}^{n} q_F^{-s_1 + s_2 - \text{Re}(\lambda_1 - \lambda_2)},
$$

$$
C^{(2)}_{\lambda}(\nu) = \begin{cases} 
 q_F^{s_1 + s_2 - \text{Re}(\lambda_1 - \lambda_2)} \left( \frac{1}{1 - \nu^{s_1 + s_2 - \text{Re}(\lambda_1 - \lambda_2)}} \right) & \text{if } s_1 + \text{Re}(\lambda_1) \neq s_2 + \text{Re}(\lambda_2), \\
 \text{ord}_F(\nu) & \text{if } s_1 + \text{Re}(\lambda_1) = s_2 + \text{Re}(\lambda_2).
\end{cases}
$$
It is clear that $C^{(1)}_\lambda$ is bounded uniformly as $\lambda$ varies in a compact set and there exists a constant $C^{(2)}_{\lambda, \epsilon} > 0$ bounded uniformly as $\lambda$ varies in a compact set such that

$$|C^{(2)}_{\lambda, \epsilon}(\nu)| \leq C^{(2)}_{\lambda, \epsilon} \cdot |\nu|_{F}^{-\epsilon}$$

for $\nu \in \varpi^{-2n} a_F$. This completes the proof. \hfill \Box

Let $E$ be a finite étale algebra of degree $d$ over $F$. Let $\psi_E$ the additive character of $E$ defined by $\psi_E = \psi \circ \text{tr}_{E/F}$. We denote by $W(\Pi, \psi_E)$ the Whittaker model of $\Pi$ with respect to $\psi_E$. Let $\Pi$ be an irreducible generic admissible representation of $GL_2(E)$ with central character $\omega_{\Pi}$. Assume $\Pi = F_1 \times \cdots \times F_r$ for some finite extension $F_i$ of degree $d_i$ over $F$. Then

$$\Pi = \pi_1 \times \cdots \times \pi_r$$

for some irreducible generic admissible representation $\pi_i$ of $GL_2(F_i)$. Define $L(\Pi) \in \mathbb{R}$ by

$$L(\Pi) = \sum_{i=1}^{r} d_i \cdot l(\pi_i).$$

Let $D(\Pi)$ be the domain associated to $\Pi$ defined by

$$D(\Pi) = D(\pi_1) \times \cdots \times D(\pi_r).$$

For $\lambda = (\lambda_1, \cdots, \lambda_r) \in D(\Pi)$, define

$$|\lambda|_\Pi = \sum_{i=1}^{r} d_i \cdot |\lambda_i|_{\pi_i}, \quad \Pi_\lambda = (\pi_1)^{\lambda_1} \times \cdots \times (\pi_r)^{\lambda_r}.$$

Note that by definition we have

$$|\lambda|_\Pi \geq \max\{|L(\Pi_\lambda)|, |L(\Pi_\lambda')|\}$$

and $\Pi_\lambda$ is tempered if and only if $|\lambda|_\Pi = 0$. Let

$$D(\Pi)^0 = \{ \lambda \in D(\Pi) \mid |\lambda|_\Pi = 0 \}.$$

Similar to the case $E = F$, we have the notion of holomorphic families of Whittaker functions of $\Pi_\lambda$ with respect to $\psi_E$.

**Lemma 2.2.** Assume $F$ is non-archimedean. Let $W_\lambda$ be a holomorphic family of Whittaker functions of $\Pi_\lambda$ with respect to $\psi_E$. Let $\epsilon > 0$. There exist an integer $n$ independent of $\epsilon, \lambda$ and a constant $C_{\lambda, \epsilon} > 0$ bounded uniformly as $\lambda$ varies in a compact set such that

$$|W_\lambda(a(\nu)m(t))| \leq C_{\lambda, \epsilon} \cdot \prod_{\varpi a_F \subset \cdots \subset \varpi^n a_F} \left( (\nu_1, \cdots, \nu_r) \right) \prod_{i=1}^{r} |\nu_i|_{F}^{-d_i \cdot l((\pi_i)_{\lambda_i}) + d_i/2 - \epsilon}$$

for $\nu = (\nu_1, \cdots, \nu_r) \in (F^\times)^r$, $t \in C$, and $k \in GL_2(a_F)$. Here $C$ is a complete set of coset representatives for $(F_1^\times \times \cdots \times F_r^\times) / (F^\times)^r$.

**Proof.** The assertion follows directly from Lemma 2.1 and the fact that $(F_1^\times \times \cdots \times F_r^\times) / (F^\times)^r$ is compact. \hfill \Box

3. **Asai cube factors via the Weil–Deligne representations**

Let $F$ be a local field of characteristic zero and $\psi$ a non-trivial additive character of $F$. Let $W'_F$ be the Weil–Deligne group of $F$. We identify characters of $F^\times$ with one-dimensional admissible representations of $W'_F$ by local class field theory.

Let $F'$ be a finite extension of degree $d$ over $F$. We identify the Langlands dual group $L(R_{F'/F}GL_2)$ of $R_{F'/F}GL_2$ with $GL_2(\mathbb{C})^d \rtimes \text{Gal}(F'/F)$ (cf. [Bor79, §5]), where the action of $\text{Gal}(F'/F)$ on $GL_2(\mathbb{C})^d$ is the permutation of components induced by the natural homomorphism $\text{Gal}(F'/F) \rightarrow \text{Gal}(F'/F)$. Let $K$ be the Asai representation of $L(R_{F'/F}GL_2)$ on $(\mathbb{C}^2)^{\otimes d} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ so that the restriction of $K$ to $GL_2(\mathbb{C})^d$ is defined by

$$K(g_1, \cdots, g_d) \cdot (v_1 \otimes \cdots \otimes v_d) = (g_1 \cdot v_1, \cdots, g_d \cdot v_d)$$

and the action of $\text{Gal}(F'/F)$ on $(\mathbb{C}^2)^{\otimes d}$ is the permutation of components induced by the natural homomorphism $\text{Gal}(F'/F) \rightarrow \text{Gal}(F'/F)$. Let $\pi$ be an irreducible admissible representation of $(R_{F'/F}GL_2)(F) = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. Let $\psi : \text{Gal}(F'/F) \rightarrow \{0, 1\}$ be the trivial character. Then $\text{Gal}(F'/F) / \ker(\pi) \cong \mathbb{Z}/d$. Let $e : \mathbb{Z}/d \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$. Let $\text{Gal}(F'/F) \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$. Let $\text{Gal}(F'/F) / \ker(\pi) \cong \mathbb{Z}/d$. Let $e : \mathbb{Z}/d \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$. Let $\text{Gal}(F'/F) / \ker(\pi) \cong \mathbb{Z}/d$. Let $e : \mathbb{Z}/d \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$. Let $\text{Gal}(F'/F) / \ker(\pi) \cong \mathbb{Z}/d$. Let $e : \mathbb{Z}/d \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$. Let $\text{Gal}(F'/F) / \ker(\pi) \cong \mathbb{Z}/d$. Let $e : \mathbb{Z}/d \rightarrow \text{Gal}(F'/F)$ be the natural map. Then $K = e(\psi)(\pi)$ is the restriction of $K$ to $R_{F'/F}GL_2$ and $\text{Gal}(F'/F)$.
GL₂(F′) with central character ωₚ. Denote by φₚ : W′ F → L(RF′/F GL₂) the L-parameter associated to π via the local Langlands correspondence. Then we have a 2ᵈ-dimensional admissible representation

\[ \text{As} \circ \phiₚ : W′ F \longrightarrow \text{GL}((\mathbb{C}²)^{⊗ d}). \]

Assume \( d = 2 \) and \( \pi = \text{Ind}_{B(F')}^{GL(2,F')}((χ₁ \otimes χ₂) \otimes (F')^×) \) for some characters \( χ₁ \) and \( χ₂ \) of \( (F')^× \). Then

\[ (3.1) \]

\[ \text{As} \circ \phiₚ = χ₁|_{F×} \oplus χ₂|_{F×} \oplus \text{Ind}_{W′_F}^{W_F}(χ₁χ₂^2), \]

where \( σ \) is the generator of \( \text{Gal}(F'/F) \) and \( χ₂^σ(a) = χ₂(σ(a)) \).

Assume \( d = 3 \) and \( \pi = \text{Ind}_{B(F')}^{GL(3,F')}((χ₁ \otimes χ₂) \otimes (F')^×) \) for some characters \( χ₁ \) and \( χ₂ \) of \( (F')^× \). When \( F' \) is Galois over \( F \), we have

\[ (3.2) \]

\[ \text{As} \circ \phiₚ = χ₁|_{F×} \oplus χ₂|_{F×} \oplus \text{Ind}_{W′_F}^{W_F}(χ₁χ₂^σ_2 \oplus χ₂χ₀\chi₂^σ^2), \]

where \( σ \) is the generator of \( \text{Gal}(F'/F) \) and \( χ₂^σ_0(a) = χ₂(σ_0(a)) \). When \( F' \) is not Galois over \( F \), we have

\[ (3.3) \]

\[ \text{As} \circ \phiₚ = χ₁|_{F×} \oplus χ₂|_{F×} \oplus \text{Ind}_{W′_F}^{W_F}(χ₁χ₂\chi_2^{-1}(χ₂ \circ N_{F'/F}) \oplus χ₂χ₀\chi_2^{-1}(χ₁ \circ N_{F'/F})). \]

Let \( E \) be a finite étale algebra of degree \( d \) over \( F \). Let \( ψ_E \) be the additive character of \( E \) defined by \( ψ_E = ψ \circ \text{tr}_{E/F} \). Let \( Π \) be an irreducible admissible representation of \( GL₂(E) \) with central character \( ω₁ \). Assume \( E = F₁ \times \cdots \times F_r \) for some finite extension \( F_i \) of degree \( d_i \) over \( F \). Then

\[ Π = π₁ \times \cdots \times π_r \]

for some irreducible generic admissible representation \( π_i \) of \( GL₂(F_i) \). Let

\( (\text{As} \circ \phi_{π₁}) \otimes \cdots \otimes (\text{As} \circ \phi_{π_r}) \)

be the admissible representation of \( W′_F \) obtained by composing \( (\text{As} \circ \phi_{π₁}) \times \cdots \times (\text{As} \circ \phi_{π_r}) \) with the tensor representation

\[ GL((\mathbb{C}²)^{⊗ d_1}) \times \cdots \times GL((\mathbb{C}²)^{⊗ d_r}) \longrightarrow GL((\mathbb{C}²)^{⊗ d}). \]

We denote by

\[ L(s, AsΠ), \quad ε(s, AsΠ, ψ) \]

the L-factor and ε-factor associated to the admissible representation \( (As \circ φ_{π₁}) \otimes \cdots \otimes (As \circ φ_{π_r}) \) defined as in [Lat79 § 3]. Let

\[ γ(s, AsΠ, ψ) = ε(s, AsΠ, ψ)L(s, AsΠ)L(1 − s, AsΠ^−) \]

be the associated γ-factor. Since \( (As \circ φ_{π₁}) \otimes \cdots \otimes (As \circ φ_{π_r}) \) has determinant \( (ω₁|_{F×})^{2d−1} \) and dimension \( 2^d \), we have

\[ ε(s, AsΠ, ψ^a) = ω₁(α)^{2d−1}|a|_{F}^{2d(s−1)/2}ε(s, AsΠ, ψ) \]

for all \( a \in F× \).

**Lemma 3.1.** Let \( E \) be an étale cubic algebra over \( F \) and \( Π \) an irreducible generic admissible representation of \( GL₂(E) \). Then the L-factor \( L(s, AsΠ) \) has no poles for

\[ \text{Re}(s) > −L(Π). \]

**Proof.** We assume \( F \) is non-archimedean and \( E \) is a field. The other cases can be proved in a similar way and we omit it. When \( Π \) is essentially square-integrable, the representation \( Π \otimes |E|^{wt(ω₁)/2} \) is square-integrable. Thus the L-factor \( L(s, AsΠ \otimes |E|^{−wt(ω₁)/2}) \) has no poles for \( \text{Re}(s) > 0 \). Since \( As \circ (φΠ \otimes |E|^{−wt(ω₁)/2}) = (As \circ φΠ) \otimes |E|^{−3wt(ω₁)/2} \), we deduce that \( L(s, AsΠ) \) has no poles for

\[ \text{Re}(s) > −3wt(ω₁)/2 = −L(Π). \]

When \( Π = \text{Ind}_{B(E)}^{GL₂(E)}(χ₁ \otimes χ₂) \), the L-factor \( L(s, AsΠ) \) is equal to

\[ L(s, χ₁|_{F×})L(s, χ₂|_{F×})L(s, χ₁χ₁^σ₁χ₂^σ) \]

or

\[ L(s, χ₁|_{F×})L(s, χ₂|_{F×})L(s, χ₁^−1(χ₂ \circ N_{F'/F}))L(s, χ₁^−1(χ₁ \circ N_{F'/F})), \]

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depending on whether $E/F$ is Galois or not. In any case, we see that $L(s, \text{As II})$ has no poles for
\[
\text{Re}(s) > - \min \{3\text{wt}(\chi_1), 3\text{wt}(\chi_2), \text{wt}(\chi_1) + 2\text{wt}(\chi_2), 2\text{wt}(\chi_1) + \text{wt}(\chi_2)\}
\]
\[
= - \min \{3\text{wt}(\chi_1), 3\text{wt}(\chi_2)\}
\]
\[
= -L(\Pi).
\]
This completes the proof. 

4. Asai cube factors via the local zeta integrals

4.1. Preliminaries. Recall the similitude symplectic group
\[
\text{GSp}_6 = \left\{ g \in \text{GL}_6 \mid g \begin{pmatrix} 0 & 1_3 \\ -1_3 & 0 \end{pmatrix} A g = \nu(g) \begin{pmatrix} 0 & 1_3 \\ -1_3 & 0 \end{pmatrix}, \nu(g) \in \mathbb{G}_m \right\}
\]
and its standard Siegel parabolic subgroup
\[
P = \left\{ \begin{pmatrix} \nu^t A^{-1} & * \\ 0 & A \end{pmatrix} \in \text{GSp}_6 \mid A \in \text{GL}_3, \nu \in \mathbb{G}_m \right\}.
\]
Denote by $U$ the unipotent radical of $P$.

Let $F$ be a field of characteristic zero and $E$ an étale cubic algebra over $F$. Let $K$ be the quadratic discriminant algebra of $E$. Let
\[
\mathcal{G} = \{ g \in R_{E/F} \text{GL}_2 \mid \det(g) \in \mathbb{G}_m \}
\]
be a linear algebraic group over $F$. Let $(V, \langle , \rangle)$ be the nondegenerate symplectic form over $F$ defined by
\[
V = (R_{E/F} \mathbb{G}_a)^2, \quad \langle x, y \rangle = \text{tr}_{E/F}(x_1 y_2 - x_2 y_1)
\]
for $x = (x_1, x_2), y = (y_1, y_2) \in V$. Let
\[
\text{GSp}(V) = \{ g \in R_{E/F} \text{GL}_2 \mid \langle xg, yg \rangle = \nu(g) \langle x, y \rangle \text{ for all } x, y \in V, \nu(g) \in \mathbb{G}_m \}
\]
be the similitude symplectic group associated to $(, )$. Then it is easy to verify that $\mathcal{G}$ is a subgroup of $\text{GSp}(V)$ and $\det(g) = \nu(g)$ for $g \in \mathcal{G}$.

Let $X$ and $X_0$ be two maximal isotropic subspaces of $V$ defined by
\[
X = \{ (0, y) \in V \mid y \in R_{E/F} \mathbb{G}_a \}, \quad X_0 = \{ (x, y) \in V \mid x \in \mathbb{G}_a, \text{tr}_{E/F}(y) = 0 \}.
\]
Define an isomorphism between $X(F)$ and $X_0(F)$ by
\[
X(F) \longrightarrow X_0(F), \quad (0, x) \longmapsto (3x, 0), \quad (0, y) \longmapsto (0, y)
\]
for $x \in F$ and $y \in E$ with $\text{tr}_{E/F}(y) = 0$. Fix $\eta \in \text{GSp}(V)(F)$ such that
\[
X(F) \cdot \eta = X_0(F), \quad \nu(\eta) = 1 \text{ and } \det(\eta|_{X(F)}) = 1 \text{ with respect to the isomorphism (4.1)}.
\]
We denote by $P_0$ and $R_0$ the stabilizers of $X_0$ in $\text{GSp}(V)$ and $\mathcal{G}$, respectively. Let $U_0$ be the unipotent radical of $R_0$. Note that
\[
R_0 = \{ a(t_1) d(t_2) n(x) \mid t_1, t_2 \in \mathbb{G}_m, x \in R_{E/F} \mathbb{G}_a, \text{tr}_{E/F}(x) = 0 \},
\]
\[
U_0 = \{ n(x) \mid x \in R_{E/F} \mathbb{G}_a, \text{tr}_{E/F}(x) = 0 \}.
\]
A symplectic basis of $V(F)$ is an ordered basis $\alpha = \{ e_1^*, e_2^*, e_3^*, e_1, e_2, e_3 \}$ of $V(F)$ such that
\[
\langle e_i^*, e_j \rangle = \delta_{ij}
\]
and $\{ e_1, e_2, e_3 \}$ is an ordered basis of $X(F)$. We write $\alpha_X = \{ e_1, e_2, e_3 \}$. With respect to a symplectic basis $\alpha$ of $V(F)$, we identify $V(F)$ with the space of row vectors $F^6$. The identification induces an isomorphism
\[
\tau_\alpha : \text{GSp}(V)(F) \longrightarrow \text{GSp}_6(F).
\]
Let $\beta$ be another symplectic basis of $V(F)$ and $A_{\alpha,\beta} \in \text{GL}_3(F)$ be the transition matrix from $\alpha_X$ to $\beta_X$. We recall that the transition matrix $A_{\alpha,\beta} = (a_{ij})_{1 \leq i, j \leq 3}$ is defined so that
\[
e_i' = a_{i1} \cdot e_1 + a_{i2} \cdot e_2 + a_{i3} \cdot e_3,
\]
where $\beta_X = \{e_1', e_2', e_3'\}$. Then we have
\begin{equation}
(4.4) \quad \iota_{\beta}(g) = p_{\alpha, \beta} \cdot \iota_{\alpha}(g) \cdot p_{\alpha, \beta}^{-1}
\end{equation}
for $g \in \text{GSp}(V)(F)$, where $p_{\alpha, \beta} \in P$ is of the form
\[ p_{\alpha, \beta} = \begin{pmatrix} tA_{\alpha, \beta}^{-1} & * \\ 0 & A_{\alpha, \beta} \end{pmatrix}. \]

4.2. Local zeta integrals and local factors. Let $F$ be a local field of characteristic zero and $\psi$ be a non-trivial additive character of $F$. Let
\[ K = \begin{cases} 
\text{GSp}_6(F) \cap \text{GL}_6(\mathfrak{g}_F) & \text{if } F \text{ is non-archimedean,} \\
\text{GSp}_6(\mathbb{R}) \cap \text{O}(6) & \text{if } F = \mathbb{R}, \\
\text{GSp}_6(\mathbb{C}) \cap \text{U}(6) & \text{if } F = \mathbb{C},
\end{cases} \]
be a maximal compact subgroup of $\text{GSp}_6(F)$.

Let $\omega$ be a character of $F^\times$. For $s \in \mathbb{C}$, let $\chi_{\omega, s}$ be the character of $P(F)$ defined by
\[ \chi_{\omega, s}(p) = \omega(\nu \cdot \det(A)^{-1}) \cdot \delta_P(p)^{s/2 - 1/4} \]
for $p = \begin{pmatrix} \nu^t A^{-1} & * \\ 0 & A \end{pmatrix}$, where $\delta_P$ is the modulus character of $P$. Recall that $\delta_P(p) = |\nu^t \det(A)^{-1}|_F$. Denote by $I(\omega, s)$ the degenerate principal series representation
\[ \text{Ind}_{P(F)}^{\text{GSp}_6(F)}(\chi_{\omega, s}) \]
and by $\rho$ the right translation action of $\text{GSp}_6(F)$ on $I(\omega, s)$. Recall that $I(\omega, s)$ consisting of smooth functions $f : \text{GSp}_6(F) \to \mathbb{C}$ such that
- For $p \in P(F)$ and $g \in \text{GSp}_6(F)$,
\[ f(pg) = \chi_{\omega, s}(p)\delta_P(p)^{1/2} f(g). \]
- $f$ is right $K$-finite.

Let $w \in \text{GSp}_6(F)$ be the Weyl element defined by
\[ w = \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}, \]
where $J \in \text{GL}_3$ is the anti-diagonal matrix with non-zero entries all equal to 1. We define the intertwining operator
\[ M_w(\omega) : I(\omega, s) \to I(\omega^\prime, 1 - s), \]
\[ M_w(\omega)f(g) = \omega(\nu(g)) \int_{U(F)} f(w^{-1}ug) \, du. \]

The integral is absolutely convergent for $\text{Re}(s)$ sufficiently large and can be meromorphically continued to $s \in \mathbb{C}$. Let $M_{w, \psi}^*(\omega) : I(\omega, s) \to I(\omega^\prime, 1 - s)$ be the normalized intertwining operator defined by
\begin{equation}
(4.5) \quad M_{w, \psi}^*(\omega) = \gamma(2s - 2, \omega, \psi)\gamma(4s - 3, \omega^2, \psi)M_w(\omega).
\end{equation}

Then $M_{w, \psi}^*(\omega^{-1}) \circ M_{w, \psi}^*(\omega)$ is a scalar multiple of the identity map on $I(\omega, s)$. We normalize the Haar measure on $U(F)$ so that
\[ M_{w, \psi}^*(\omega^{-1}) \circ M_{w, \psi}^*(\omega) = \text{id}. \]

We write $M_{w, \psi}^*(\omega) = M_w^*$ if there is no cause of confusion. A map
\[ \mathbb{C} \times \text{GSp}_6(F) \to \mathbb{C}, \quad (s, g) \mapsto f_s(g) \]
is a holomorphic section of $I(\omega, s)$ if it satisfies the following conditions:
- For each $s \in \mathbb{C}$, the function $g \mapsto f_s(g)$ belongs to $I(\omega, s)$.
- For each $g \in \text{GSp}_6(F)$, the function $s \mapsto f_s(g)$ is holomorphic.
- $f_s$ is right $K$-finite.

A map
\[ \mathbb{C} \times \text{GSp}_6(F) \to \mathbb{C}, \quad (s, g) \mapsto f_s(g) \]
is a good section of $I(\omega, s)$ if it satisfies the following conditions:
• The map \((s, g) \mapsto L(2s + 1, \omega)^{-1}L(4s, \omega^2)^{-1}f_s(g)\) is a holomorphic section of \(I(\omega, s)\).
• The map \((s, g) \mapsto L(3 - 2s, \omega^{-1})^{-1}L(4 - 4s, \omega^{-2})^{-1}M_\alpha f_s(g)\) is a holomorphic section of \(I(\omega^{-1}, 1 - s)\).

By [Ke97, Lemma 1.3], every holomorphic section is a good section.

Let \(\Pi\) be an irreducible generic admissible representation of \(GL_2(E)\) with central character \(\omega_\Pi\). Write \(\omega = \omega_{\Pi|E} \cdot \chi\). Recall \(W(\Pi, \psi_E)\) is the space of Whittaker functions of \(\Pi\) with respect to \(\psi_E = \psi \circ \text{tr}_{E/F}\). Let \(f_s\) be a good section of \(I(\omega, s)\) and \(W \in W(\Pi, \psi_E)\). Define the local zeta integral

\[
Z_\alpha(f_s, W) = \int_{F^\times U_0(F) \backslash G(F)} f_s(\iota_\alpha(\eta g))W(g) \, dg,
\]

where \(\alpha\) is a symplectic basis of \(V(F)\), \(\iota_\alpha\) is the isomorphism \(A\), and \(\eta \in \text{GSp}(V)(F)\) satisfies \((4.2)\).

Note that the integral is independent of the choice of \(\eta\). Indeed, if \(\eta' \in \text{GSp}(V)(F)\) also satisfies \((4.2)\), then \(\iota_\alpha(\eta'\eta^{-1}) \in P(F)\) and \(\nu(\eta'\eta^{-1}) = \det(\eta'|_{X(F)}\eta_\alpha^{-1}_{X(F)}) = 1\).

By the results in [PSR87b] and [Ke89], the integral is absolutely convergent for \(\Re(s)\) sufficiently large and admits meromorphic continuation to \(s \in \mathbb{C}\).

Moreover, by [PSR87b, Proposition 3.1] and [Ke89, Proposition 4.2], there exists a unique meromorphic function \(\gamma_{PSR}(s, As\Pi, \psi, \alpha)\), called the \(\gamma\)-factor, such that we have the functional equation

\[
(4.6) \quad Z_\alpha(M_\alpha f_s, W^\vee) = \gamma_{PSR}(s, As\Pi, \psi, \alpha)Z_\alpha(f_s, W).
\]

Recall that \(W^\vee \in W(\Pi^\vee, \psi_E)\) is defined by

\[
W^\vee(g) = \omega_{\Pi}(\det(g))^{-1}W(g).
\]

By Lemma \(4.1(1)\) below, we see that \(\gamma_{PSR}(s, As\Pi, \psi, \alpha) = \gamma_{PSR}(s, As\Pi, \psi, \beta)\) if \(\alpha_X = \beta_X\). Therefore we also write \(\gamma_{PSR}(s, As\Pi, \psi, \alpha_X) = \gamma_{PSR}(s, As\Pi, \psi, \alpha)\). In particular, identifying \(E\) with \(X(F)\) via the isomorphism

\[
E \longrightarrow X(F), \quad x \mapsto (0, x),
\]

the notion \(\gamma_{PSR}(s, As\Pi, \psi, \alpha)\) is defined for any basis \(\alpha\) of \(E\) over \(F\).

**Lemma 4.1.** (1) Let \(\alpha, \beta\) be symplectic bases of \(V(F)\). We have

\[
\gamma_{PSR}(s, As\Pi, \psi, \beta) = \omega(\det(A_{\alpha, \beta})^2)^{-1} \gamma_{PSR}(s, As\Pi, \psi, \alpha).
\]

Here \(A_{\alpha, \beta} \in \text{GL}_3(F)\) is the transition matrix from \(\alpha_X\) to \(\beta_X\).

(2) Let \(a \in F^\times\) and \(\alpha\) be a symplectic basis of \(V(F)\). We have

\[
\gamma_{PSR}(s, As\Pi, \psi^a, \alpha) = \omega(a)^4|\det F_{\alpha, \beta}|^{-1} \gamma_{PSR}(s, As\Pi, \psi, \alpha).
\]

**Proof.** Let \(f_s\) be a good section of \(I(\omega, s)\) and \(W \in W(\Pi, \psi_E)\). First we prove assertion (1). By \(4.4\),

\[
(4.7) \quad Z_\beta(f_s, W) = \int_{F^\times U_0(F) \backslash G(F)} f_s(p_{\alpha, \beta} \cdot \iota_\alpha(\eta g) \cdot p_{\alpha, \beta}^{-1})W(g) \, dg
\]

\[
\quad = \omega(\det(A_{\alpha, \beta}))^{-1} \det(A_{\alpha, \beta})|F_{\alpha, \beta}|^{-1}Z_\alpha(p_{\alpha, \beta}^{-1}f_s, W).
\]

Similarly, we have

\[
Z_\beta(M_\alpha f_s, W^\vee) = \omega(\det(A_{\alpha, \beta})) \det(A_{\alpha, \beta})|F_{\alpha, \beta}|^{-1}Z_\alpha(p_{\alpha, \beta}^{-1}M_\alpha f_s, W^\vee).
\]

Assertion (1) then follows from the functional equation \((4.6)\).

To prove assertion (2), it follows from assertion (1) that we may assume

\[
\alpha = \{(1, 0), (\delta_1^+, 0), (\delta_2^+, 0), (0, 1/3), (0, \delta_1), (0, \delta_2)\}
\]

for some \(\delta_1, \delta_1^+, \delta_2, \delta_2^+ \in E\) with \(\text{tr}_{E/F}(\delta_1) = \text{tr}_{E/F}(\delta_2) = \text{tr}_{E/F}(\delta_1^+) = \text{tr}_{E/F}(\delta_2^+) = 0\) and \(\eta \in \text{GSp}(V)(F)\) satisfying \((4.2)\) is defined by

\[
(0, 1/3) \cdot \eta = (1, 0), \quad (\delta_1^+ \cdot 0) \cdot \eta = (\delta_1^+, 0), \quad (0, \delta_1) \cdot \eta = (0, \delta_1).
\]

Then it is easy to verify that \(\iota_\alpha(\eta a(\alpha^{-1})\eta^{-1}) = \text{diag}(1, a^{-1}, \alpha^{-1}, a^{-1}$, $1, 1)\). Let \(W' \in W(\Pi, \psi_E^\vee)\) defined by \(W'(g) = W(\alpha(a)g)\). We write \(Z_{\alpha, \nu}(f_s, W) = Z_{\alpha}(f_s, W)\) and \(M_{\alpha, \nu} = M_{\alpha}\) to emphasis the dependence on \(\nu\).
We have
\[
Z_{\alpha, \psi^s}(f_s, W') = \int_{F \times U_0(F) \backslash G(F)} f_s(\iota_\alpha(\eta g)) W(a(a)g) \, dg
= |a|_F \int_{F \times U_0(F) \backslash G(F)} f_s(\iota_\alpha(\eta a(a^{-1}) g)) W(g) \, dg
= |a|_F^{-s+1/2} Z_{\alpha, \psi}(f_s, W).
\]

Note that
\[
\gamma(2s - 2, \omega, \psi^s) \gamma(4s - 3, \omega^2, \psi^s) = \omega(a)^3 |a|_F^{6s - 6} \gamma(2s - 2, \omega, \psi) \gamma(4s - 3, \omega^2, \psi),
\]
\[
M_{\psi, \psi^s} = |a|_F^3 M_{\psi, \psi},
\]
\[
(W')^\vee(g) = \omega(a) W^\vee(a(a)g).
\]
Similarly, we have
\[
Z_{\alpha, \psi^s}(M_{\psi, \psi}^*, f_s, (W')^\vee) = \omega(a)^4 |a|_F^{7s - 7} \int_{F \times U_0(F) \backslash G(F)} M_{\psi, \psi}^* f_s(\iota_\alpha(\eta g)) W^\vee(a(a)g) \, dg
= \omega(a)^4 |a|_F^{7s - 7/2} Z_{\alpha, \psi}(M_{\psi, \psi}^*, f_s, W^\vee).
\]

Assertion (2) then follows from the functional equation (4.6). □

Assume $F$ is non-archimedean. By [PSR87b, Appendix 3 to §3], the $C[\mathfrak{q}_F^*, \mathfrak{q}_E^*]$-module generated by $Z_\alpha(f_s, W)$ for good sections $f_s$ of $I(\omega, s)$ and $W \in \mathcal{W}(\Pi, \psi_E)$ is a fractional ideal of $C[\mathfrak{q}_F^*, \mathfrak{q}_E^*]$ containing 1. Therefore, there is a unique generator of the form $(q_F^*)^\alpha$ with $P(X) \in \mathbb{C}[X]$ and $P(0) = 1$. Define the $L$-factor and $\varepsilon$-factor as follows:
\[
L_{PSR}(s, As \Pi) = P(q^{-s})^{-1},
\]
\[
\varepsilon_{PSR}(s, As \Pi, \psi, \alpha) = \gamma_{PSR}(s, As \Pi, \psi, \alpha) L_{PSR}(s, As \Pi) L_{PSR}(1 - s, As \Pi^\vee)^{-1}.
\]

Note that $\varepsilon$-factor is a unit in $C[\mathfrak{q}_F^*, \mathfrak{q}_E^*]$ by the functional equation (4.6).

Assume $F$ is archimedean. Up to holomorphic functions without zeros, there exists a unique meromorphic function $L_{PR}(s, As \Pi)$ without zeros, called the $L$-factor, satisfying the following conditions:

- $L_{PR}(s, As \Pi)^{-1} Z(f_s, W)$ is holomorphic for any good section $f_s$ of $I(\omega, s)$ and $W \in \mathcal{W}(\Pi, \psi_E)$.
- For each $s_0 \in \mathbb{C}$, there exist a good section $f_s$ and $W \in \mathcal{W}(\Pi, \psi_E)$ such that $L_{PR}(s, As \Pi)^{-1} Z(f_s, W)$ is non-zero at $s = s_0$.

Define the $\varepsilon$-factor
\[
\varepsilon_{PR}(s, As \Pi, \psi, \alpha) = \gamma_{PR}(s, As \Pi, \psi, \alpha) L_{PR}(s, As \Pi) L_{PR}(1 - s, As \Pi^\vee)^{-1},
\]
which is well-defined up to holomorphic function without zeros. By the properties characterizing the $L$-factor above and the functional equation (4.6), the $\varepsilon$-factor is a holomorphic function without zeros.

Recall the domain $D(\Pi)$ associated to $\Pi$ defined in (2.4). Let $W_\lambda$ be a holomorphic family of Whittaker functions of $\Pi_\lambda$ with respect to $\psi_E$. Write $\omega_\lambda = \omega_{\Pi_\lambda}|_{F^\times}$. Similar to the case for $I(\omega, s)$, we define the notion of holomorphic sections and good sections of $I(\omega_\lambda, s)$ for $(s, \lambda)$ varying in $\mathbb{C} \times D(\Pi)$.

The following lemma is a variant of the estimation in [PSR87b, Proposition 3.2] and [Ike89, §3.1] by replacing a single Whittaker function with a holomorphic family of Whittaker functions.

**Lemma 4.2.** Assume $F$ is non-archimedean. Let $W_\lambda$ be a holomorphic family of Whittaker functions of $\Pi_\lambda$ with respect to $\psi_E$. Write $\omega_\lambda = \omega_{\Pi_\lambda}|_{F^\times}$. Let $f_{s, \lambda}$ be a good section of $I(\omega_\lambda, s)$. Then the integral $Z_\alpha(f_{s, \lambda}, W_\lambda)$ is absolutely convergent for
\[
\text{Re}(s) > -L(\Pi_\lambda),
\]
uniformly for $s$ and $\lambda$ varying in compact sets. In particular, the integral $Z_\alpha(f_{s, \lambda}, W_\lambda)$ defines a holomorphic function on the domain
\[
\{(s, \lambda) \in \mathbb{C} \times D(\Pi) \mid \text{Re}(s) > -L(\Pi_\lambda)\}.
\]
Here $L(\Pi_\lambda) \in \mathbb{R}$ is defined as in (2.3).
Proof. We have the following three cases:

\[\begin{align*}
E &= F \times F \times F & \text{Case 1,} \\
E &= F' \times F & \text{Case 2,} \\
E &\text{ is a field} & \text{Case 3.}
\end{align*}\]

Then \( \Pi = \pi_1 \times \pi_2 \times \pi_3 \) for some irreducible generic admissible representations \( \pi_i \) of \( \text{GL}_2(F) \) in Case 1, and \( \Pi = \pi_1 \times \pi_2 \) for some irreducible generic admissible representations \( \pi_1 \) and \( \pi_2 \) of \( \text{GL}_2(F') \) and \( \text{GL}_2(F) \), respectively, in Case 2. In Case 3, we write \( \Pi = \pi_1 \). Let

\[\begin{align*}
r &= \begin{cases}
3 & \text{Case 1,} \\
2 & \text{Case 2,} \\
1 & \text{Case 3.}
\end{cases}
\end{align*}\]

Define \((d_1, \cdots, d_r) \in \mathbb{Z}^r\) by

\[\begin{align*}
(d_1, \cdots, d_r) &= \begin{cases}
(1, 1, 1) & \text{Case 1,} \\
(2, 1) & \text{Case 2,} \\
3 & \text{Case 3.}
\end{cases}
\end{align*}\]

We write \( \lambda = (\lambda_1, \cdots, \lambda_r) \in D(\pi_1) \times \cdots \times D(\pi_r) = D(\Pi) \). Let \( \mathcal{C} \) be a complete set of coset representatives for \( E^\times / (F^\times)^r \). Let \( s_\lambda = \text{wt}(\omega_\lambda) \) and \( s_{i, \lambda} = \text{wt}(\omega(\pi_{i, \lambda})) \) for \( 1 \leq i \leq r \). Then \( s_\lambda = \sum_{i=1}^r s_{i, \lambda} \). Note that

\[L(\Pi_\lambda) \leq s_\lambda/2\]

by definition.

Let \( K = \text{G}(F) \cap \text{GL}_2(\mathfrak{o}_E) \). Let \( f_{s, \lambda}^0 \) be the \( K \)-invariant good section of \( I(\vert \omega_\lambda \vert, s) \) normalized so that \( f_{s, \lambda}^0(1) = 1 \). Since \( E^\times / (F^\times)^r \) is compact, by the \( K \)-finiteness of \( f_{s, \lambda} \), there exists a constant \( C_{s, \lambda} > 0 \) bounded uniformly as \( s \) and \( \lambda \) vary in a compact set such that

\[|f_{s, \lambda}(g \cdot t_\alpha (\mathbf{m}(t)k))| \leq C_{s, \lambda} \cdot |f_{s, \lambda}(g)|\]

for \( g \in \text{GSp}_6(F), t \in \mathcal{C}, \) and \( k \in K \). Let \( \epsilon > 0 \). By Lemma 22 there exist an integer \( n \) independent of \( \epsilon, \lambda \) and a constant \( C_{\lambda, \epsilon} > 0 \) bounded uniformly as \( \lambda \) varies in a compact set such that

\[|W_{\lambda}(a(\nu)\mathbf{m}(t)k)| \leq C_{\lambda, \epsilon} \cdot \varphi(\nu) \prod_{i=1}^r |\mathbf{m}_i(\pi_{i, \lambda})|^{d_i + d_{i-1}/2 - \epsilon/r} \]

for \( \nu = (\nu_1, \cdots, \nu_r) \in (F^\times)^r, t \in \mathcal{C}, \) and \( k \in K \), where \( \varphi \in \mathcal{S}(F^\times) \) is given by

\[\varphi((\nu_1, \cdots, \nu_r)) = \prod_{i=1}^r \|\mathbf{s}_{F, \alpha}(\nu_i)\|.
\]

We have

\[Z_\alpha(f_{s, \lambda}, W_\lambda) = Z_\alpha^{(0)}(f_{s, \lambda}, W_\lambda) + g_F^2 \cdot Z_\alpha^{(1)}(f_{s, \lambda}, W_\lambda),\]

where

\[Z_\alpha^{(0)}(f_{s, \lambda}, W_\lambda) = \int_F \int_{F^\times} \int_K f_{s, \lambda}(t_\alpha(\eta \mathbf{u}(x)\mathbf{m}(t)k))W_\lambda(\mathbf{u}(x)\mathbf{m}(t)k)|t|_E^{-2} dk dt dx,\]

\[Z_\alpha^{(1)}(f_{s, \lambda}, W_\lambda) = \int_F \int_{F^\times} \int_K f_{s, \lambda}(t_\alpha(\eta \mathbf{a}(\mathbf{w}_F)\mathbf{u}(x)\mathbf{m}(t)k))W_\lambda(\mathbf{a}(\mathbf{w}_F)\mathbf{u}(x)\mathbf{m}(t)k)|t|_E^{-2} dk dt dx.\]

Here \( \mathbf{u}(x) \in \text{G}(F) \) is defined by

\[\mathbf{u}(x) = \begin{cases}
(1, 1, \mathbf{n}(x)) & \text{Case 1,} \\
(1, \mathbf{n}(x)) & \text{Case 2,} \\
\mathbf{n}(x)/3 & \text{Case 3.}
\end{cases}\]

By (4.7), without lose of generality, we assume \( \alpha = \{e_1^*, e_2^*, e_3^*, e_1, e_2, e_3\} \) is given as follows:

- In Case 1,
  \[e_1^* = ((1, 0, 0), (0, 0, 0)), \quad e_2^* = ((0, 1, 0), (0, 0, 0)), \quad e_3^* = ((0, 0, 1), (0, 0, 0)), \quad e_1 = ((0, 0, 0), (1, 0, 0)), \quad e_2 = ((0, 0, 0), (0, 1, 0)), \quad e_3 = ((0, 0, 0), (0, 0, 1)).\]
In Case 2, 
\[ e_1^* = ((1, 0), (0, 0)), \quad e_2^* = ((\delta^*, 0), (0, 0)), \quad e_3^* = ((0, 1), (0, 0)), \]
\[ e_1 = ((0, 0), (1/2, 0)), \quad e_2 = ((0, 0), (\delta, 0)), \quad e_3 = ((0, 0), (0, 1)) \]
for some \( \delta, \delta^* \in F' \) with \( \text{tr}_{F'/F}(\delta) = \text{tr}_{F'/F}(\delta^*) = 0 \).

In Case 3, 
\[ e_1^* = (1, 0), \quad e_2^* = (\delta^*, 0), \quad e_3^* = (\delta^*, 0), \quad e_1 = (0, 1/3), \quad e_2 = (0, \delta_1), \quad e_3 = (0, \delta_2) \]
for some \( \delta_1, \delta_1^*, \delta_2, \delta_2^* \in E \) with \( \text{tr}_{E/F}(\delta_1) = \text{tr}_{E/F}(\delta_2) = \text{tr}_{E/F}(\delta_1^*) = \text{tr}_{E/F}(\delta_2^*) = 0 \).

We define \( \eta \in \text{GSp}(V)(F) \) as follows:

- We have  
  \[ e_1^* \cdot \eta = -e_1, \quad e_2^* \cdot \eta = e_2^*, \quad e_3^* \cdot \eta = e_3^*. \]

- In Case 1,  
  \[ e_1 \cdot \eta = e_1^* + e_2^* + e_3^*, \quad e_2 \cdot \eta = -e_1 + e_2, \quad e_3 \cdot \eta = -e_1 + e_3. \]

- In Case 2,  
  \[ e_1 \cdot \eta = e_1^* + e_3^*, \quad e_2 \cdot \eta = e_2, \quad e_3 \cdot \eta = -e_1 + e_3. \]

- In Case 3,  
  \[ e_1 \cdot \eta = e_1^*, \quad e_2 \cdot \eta = e_2, \quad e_3 \cdot \eta = e_3. \]

It is easy to verify that \( \eta \) satisfies \((12)\). Then

\[
\iota_0(\eta u(x)m(t)) = \begin{cases} 
\begin{pmatrix} * & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
t_1 & t_2 & t_3 & t_1^{-1}x & t_2^{-1}x \\
0 & 0 & -t_1^{-1} & t_2 & 0 \\
0 & 0 & -t_1^{-1} & 0 & t_3^{-1} 
\end{pmatrix} \quad & \text{Case 1,} \\
\begin{pmatrix} * & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
t_1 & 0 & t_2 & 0 & 0 \\
0 & 0 & 0 & t_1^{-1} & 0 \\
0 & 0 & -t_1^{-1} & 0 & t_2^{-1} 
\end{pmatrix} \quad & \text{Case 2,} \\
\begin{pmatrix} * & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
t_1 & 0 & t_1^{-1}x & 0 & 0 \\
0 & 0 & 0 & t_1^{-1} & 0 \\
0 & 0 & 0 & 0 & t_1^{-1} 
\end{pmatrix} \quad & \text{Case 3,} 
\end{cases}
\]

for \( x \in F \) and \( t = (t_1, \ldots, t_r) \in (F^\times)^r \). Therefore,

\[(4.10) \quad f_{s, \lambda}^0(\iota_0(\eta u(x)m(t))) = \max \{|x|_F, |t_1|_F^2, \ldots, |t_r|_F^2\} \prod_{i=1}^r |t_i|_{d_i}^{-2s-s_\lambda-1} \]

for \( x \in F \) and \( t = (t_1, \ldots, t_r) \in (F^\times)^r \). We deduce from \((13)\) and \((14)\) that the integral \( Z_{\alpha}(f_{s, \lambda}, W_\lambda) \) is majorized by

\[
C_{s, \lambda} \int_F \max \{|x|_F, 1\}^{-2\text{Re}(s)-s_\lambda-1} dx \\
\times \int_{(F^\times)^r} \varphi(t^2) \prod_{i=1}^r |t_i|_{d_i}^{2d_i \cdot \text{Re}(s)+2d_i \cdot l((t_i)_{s_i})+d_i \cdot s_\lambda-s_i-\lambda-2\epsilon/r} \max \{|t_1|_F^2, \ldots, |t_r|_F^2\}^{-2\text{Re}(s)-s_\lambda} dx. 
\]

The above integral is absolutely convergent for

\[
\text{Re}(s) > \max \{-s_\lambda/2, -L(\Pi_\lambda) + \epsilon\} = -L(\Pi_\lambda) + \epsilon. 
\]
Moreover, it is clear that the above integral is uniformly convergent as $s$ and $\lambda$ vary in compact sets. Therefore we obtain a uniform estimate for $Z^{(0)}_\alpha(f_{s,\lambda}, W_\lambda)$. We have a similar estimate for $Z^{(1)}_\alpha(f_{s,\lambda}, W_\lambda)$. This completes the proof.

In the following lemma, we recall the known cases of Theorem 4.1 in the literature.

**Lemma 4.3.** Assume one of the following assumptions is satisfied:

- $F$ is archimedean.
- $E = F \times F \times F$.
- $F$ is non-archimedean, $E$ is unramified over $F$, and $\Pi$ is unramified.
- $E$ is not a field and $\Pi$ is unramified.

We have

$$\gamma_{\text{PSR}}(s, \alpha, \Pi, \psi, \alpha) = \omega_{\Pi}(\Delta_{E/F}(\alpha))\Delta_{E/F}(\alpha)|_{E/F}^{2s-1}\omega_{K/F}(-1)\gamma(s, \alpha, \Pi, \psi)$$

for any basis $\alpha$ of $E$ and non-trivial additive character $\psi$ of $F$. Here $\Delta_{E/F}(\alpha)$ is the relative discriminant of $\alpha$ for $E/F$ and $\omega_{K/F}$ is the quadratic character associated to $K/F$ by local class field theory.

**Proof.** We assume that $F$ is non-archimedean, $E$ is unramified over $F$, and $\Pi$ is unramified. The rest of the cases follow from [Ke89 Theorem 3.1], [Ram00 Theorem 4.4.1], and [CC18 Theorem B]. By [3.3] and Lemma 4.1, we may further assume that $\alpha = \{x_1, x_2, x_3\}$ is an integral basis of $\sigma_F$ over $\sigma_F$ and $\psi$ is of conductor $\sigma_F$. Note that in this case $\omega_{\Pi}(\Delta_{E/F}(\alpha))|_{E/F}^{2s-1}\omega_{K/F}(-1) = 1$ by assumption. Let $f^o_s$ be the $K$-invariant good section of $I(\omega, s)$ and $W^o \in \mathcal{W}(\Pi, \psi)$ the $GL_2(\mathfrak{o}_E)$-invariant Whittaker function normalized so that $f^o_s(1) = 1$ and $W^o(1) = 1$. Let $\alpha^* = \{x_1^*, x_2^*, x_3^*\}$ be the dual basis of $\alpha$ and $\tilde{\alpha}$ the symplectic basis of $V(F)$ defined by

$$\tilde{\alpha} = \{(x_1^*, 0), (x_2^*, 0), (x_3^*, 0), (0, x_1), (0, x_2), (0, x_3)\}.$$ 

By [PSR76 Theorem 3.1], we have

$$Z_{\alpha}(f^o_s, W^o) = L(2s + 1, \omega)^{-1}L(4s, \omega^2)^{-1}L(s, \alpha, \Pi),$$

$$Z_{\alpha}(M^o_s f^o_s, (W^o)^\gamma) = L(2s + 1, \omega)^{-1}L(4s, \omega^2)^{-1}L(1 - s, \alpha, \Pi^\gamma).$$

Here the measure on $F^x U_0(F) \backslash G(F)$ is normalized so that $vol(\sigma^o_F U_0(\sigma_F) \backslash G(\sigma_F)) = 1$. It follows from the functional equation (3.6) that

$$\gamma_{\text{PSR}}(s, \alpha, \Pi, \psi, \alpha) = \varepsilon(s, \alpha, \Pi, \psi)^{-1}\gamma(s, \alpha, \Pi, \psi).$$

Since $\Pi$ is unramified and $\psi$ is of order $\sigma_F$, we have $\varepsilon(s, \alpha, \Pi, \psi) = 1$. This completes the proof. \hfill \Box

### 4.3. Unramified calculation.

In this section, we prove a case of Theorem 4.1 in Corollary 4.5. This case will be needed in the proof of general case in §5.2 below. Assume $F$ is non-archimedean, $E$ is a field, and there is a uniformizer $\varpi_E$ of $E$ such that $\varpi_E^2 = \varpi_F$ is a uniformizer of $\sigma_F$. In particular, the last assumption is satisfied if $E$ is totally tamely ramified over $F$. Note that $3\varpi_E^2$ is the relative different ideal for $E/F$.

Let $\alpha^o$ be the symplectic basis of $V(F)$ defined by

$$\alpha^o = \{(1, 0), (\varpi_E, 0), (\varpi_E^2, 0), (0, 1/3), (0, \varpi_E^{-1}/3), (0, \varpi_E^{-2}/3)\}.$$ 

**Lemma 4.4.** Let $\Pi$ be an irreducible generic admissible representation of $GL_2(E)$ with central character $\omega_{\Pi}$. Write $\omega = \omega_{\Pi}|_{F^x}$. Assume $\Pi$ is unramified and $\psi$ is of conductor $\sigma_F$. Let $f^o_s$ be the $K$-invariant good section of $I(\omega, s)$ and $W^o \in \mathcal{W}(\Pi, \psi_E)$ the $\mathfrak{a}(3^{-1}\varpi_E^2)\mathfrak{GL}_2(\mathfrak{o}_E)\mathfrak{a}(3\varpi_E^2)$-invariant Whittaker function normalized so that $f^o_s(1) = 1$ and $W^o(1) = 1$. We have

$$Z_{\alpha^o}(f^o_s, W^o) = L(2s + 1, \omega)^{-1}L(4s, \omega^2)^{-1}L(s, \alpha, \Pi).$$

Here the measure on $F^x U_0(F) \backslash G(F)$ is normalized so that

$$vol(\sigma^o_F U_0(\sigma_F) \cap N(\mathfrak{o}_E)) \backslash (G(F) \cap \mathfrak{a}(\varpi_E^2)GL_2(\mathfrak{o}_E)\mathfrak{a}(\varpi_E^2)) = 1.$$ 

**Proof.** Write $\varpi = \varpi_F$ and $q = q_F$ for brevity. Assume $\Pi = \text{Ind}_{E(E)}^{GL_2(E)}(\chi_1 \otimes \chi_2)$ for some unramified characters $\chi_1$ and $\chi_2$ of $E^\times$. Let $\alpha = \chi_1(\varpi_E), \beta = \chi_2(\varpi_E),$ and $\chi_1\chi_2 = |\chi_0|_{E^s}^s$ for some $s_0 \in \mathbb{C}$. We have

$$W^o(\alpha(\varpi_E^n)) = \begin{cases} q^{-n/2} \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} & \text{if } n \geq 0, \\
 \frac{\alpha - \beta}{\alpha} & \text{if } n < 0. 
\end{cases}$$

(4.11)
Also note that (cf. [PSR87b, p. 54])

\[
\int_F \max\{|x|_F, q^{-m}\}^{-s\psi(\varpi^n x)} \, dx
\]

(4.12)

\[
= \begin{cases} 
q^{m(s-1)}(1 - q^{-(m+n+1)(s-1)})\zeta_F(s-1)\zeta_F(s)^{-1} & \text{if } m + n \geq 0, \\
0 & \text{if } m + n < 0.
\end{cases}
\]

Since \{1, \varpi_E, \varpi_E^2\} is an integral basis of \(\mathfrak{o}_E\) over \(\mathfrak{o}_F\), one can easily verify that

\[\iota_{\alpha^o}(a(3^{-1}\varpi_E^{-2})) \text{GL}_2(\mathfrak{o}_E) a(3\varpi_E^2)) \subseteq K.\]

Therefore, by the invariance of \(f_s^o\) and \(W^o\), we have

\[Z_\alpha(f_s^o, W^o) = Z^{(0)}(s) + q^2 \cdot Z^{(1)}(s),\]

where

\[
Z^{(0)}(s) = \int_F \int_{E^\times} f_s^o(\iota_{\alpha^o}(\eta n(x/3) m(t))) W^o(\eta n(x/3) m(t)) |t|_{E}^{-2} d^x t \, dx,
\]

\[
Z^{(1)}(s) = \int_F \int_{E^\times} f_s^o(\iota_{\alpha^o}(\eta a(\varpi) n(x/3) m(t))) W^o(a(\varpi) n(x/3) m(t)) |t|_{E}^{-2} d^x t \, dx.
\]

Here the measures are normalized so that \(\text{vol}(\mathfrak{o}_F) = \text{vol}(\mathfrak{o}_E^\times) = 1\). We assume \(\eta \in \text{GSp}(V)(F)\) satisfying (4.2) is defined by

\[(0, 1/3) \cdot \eta = (1, 0), \quad (\varpi_E^i, 0) \cdot \eta = (\varpi_E^i, 0), \quad (0, \varpi_E^{-i}) \cdot \eta = (0, \varpi_E^{-i})\]

for \(i = 1, 2\). For \(x \in F, n \in \mathbb{Z}_{\geq 0}, \) and \(0 \leq i \leq 2\), we have

\[
\iota_{\alpha^o}(\eta n(x/3) m(\varpi^n \varpi_E^i)) = \begin{cases} 
\begin{pmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
\varpi^n & 0 & 0 & \varpi^{-n} x & 0 & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0
\end{pmatrix} & \text{if } i = 0, \\
\begin{pmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & \varpi^n & 0 & 0 & \varpi^{-n} x & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0
\end{pmatrix} & \text{if } i = 1, \\
\begin{pmatrix}
* & * & * & * & * & * \\
* & * & * & * & * & * \\
* & * & * & * & * & * \\
0 & \varpi^n & 0 & 0 & \varpi^{-n} x & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0 \\
0 & 0 & 0 & 0 & \varpi^{-n} & 0
\end{pmatrix} & \text{if } i = 2.
\end{cases}
\]

Therefore,

\[
f_s^o(\iota_{\alpha^o}(\eta n(x/3) m(\varpi^n \varpi_E))) = (\alpha^3 \beta^3 q^{-2s-1})^{3m+1} \max\{|x|_F, q^{-2n}\}^{-3s_0-2s-1}.
\]
It follows from \(\text{(1.1)}\) and \(\text{(1.3)}\) that
\[
Z^{(0)}(s) = L(2s, \omega)L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \sum_{i=0}^{2} \frac{\alpha^{2i} \beta^{2i}q^{-2i}}{1 - \alpha^2 \beta^2 q^{-2s}} \sum_{n=0}^{\infty} q^{-2ns}(\alpha^{6n+2i+1} - \beta^{6n+2i+1})(1 - (\alpha^3 \beta^3 q^{-2s})^{2n+1})
\]
\[
= L(2s, \omega)L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \left[ \frac{1 - \alpha^{12} \beta^6 q^{-6s}}{1 - \alpha^2 \beta^2 q^{-2s}} \left( \frac{\alpha}{1 - \alpha^6 q^{-2s}} - \frac{\alpha^4 \beta^3 q^{-2s}}{1 - \alpha^{12} \beta^6 q^{-6s}} \right)
- \frac{1 - \alpha^{6} \beta^{12} q^{-6s}}{1 - \alpha^2 \beta^4 q^{-2s}} \left( \frac{\beta}{1 - \beta^6 q^{-2s}} - \frac{\alpha^3 \beta q^{-2s}}{1 - \alpha^{6} \beta^{12} q^{-6s}} \right) \right].
\]

Note that \(f_s^{(\nu_s)}(\chi((\varpi))q) = q^{-s-1/2}f_s^{(\nu)}(g)\). Similarly,
\[
Z^{(1)}(s) = q^{-s-2}L(2s, \omega)L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \sum_{i=0}^{2} \frac{\alpha^{2i} \beta^{2i}q^{-2i}}{1 - \alpha^2 \beta^2 q^{-2s}} \sum_{n=0}^{\infty} q^{-2ns}(\alpha^{6n+2i+4} - \beta^{6n+2i+4})(1 - (\alpha^3 \beta^3 q^{-2s})^{2n+2})
\]
\[
= q^{-s-2}L(2s, \omega)L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \left[ \frac{1 - \alpha^{12} \beta^6 q^{-6s}}{1 - \alpha^2 \beta^2 q^{-2s}} \left( \frac{\alpha^4}{1 - \alpha^6 q^{-2s}} - \frac{\alpha^{10} \beta^6 q^{-4s}}{1 - \alpha^{12} \beta^6 q^{-6s}} \right)
- \frac{1 - \alpha^{6} \beta^{12} q^{-6s}}{1 - \alpha^2 \beta^4 q^{-2s}} \left( \frac{\beta^4}{1 - \beta^6 q^{-2s}} - \frac{\alpha^{3} \beta q^{-2s}}{1 - \alpha^{6} \beta^{12} q^{-6s}} \right) \right].
\]

By a direct calculation, we have
\[
\left( \frac{\alpha}{1 - \alpha^6 q^{-2s}} - \frac{\alpha^4 \beta^3 q^{-2s}}{1 - \alpha^{12} \beta^6 q^{-6s}} \right) + \left( \frac{\alpha^4 q^{-s}}{1 - \alpha^6 q^{-2s}} - \frac{\alpha^{10} \beta^6 q^{-4s}}{1 - \alpha^{12} \beta^6 q^{-6s}} \right)
= L(2s, \omega)^{-1}(1 - \alpha^3 q^{-s})^{-1}(1 - \alpha^6 \beta^3 q^{-3s})^{-1},
\]
\[
\left( \frac{\beta}{1 - \beta^6 q^{-2s}} - \frac{\alpha^3 \beta q^{-2s}}{1 - \alpha^{6} \beta^{12} q^{-6s}} \right) + \left( \frac{\beta^4 q^{-s}}{1 - \beta^6 q^{-2s}} - \frac{\alpha^{3} \beta q^{-2s}}{1 - \alpha^{6} \beta^{12} q^{-6s}} \right)
= L(2s, \omega)^{-1}(1 - \beta^3 q^{-s})^{-1}(1 - \alpha^3 \beta^6 q^{-3s})^{-1}.
\]

Therefore,
\[
Z^{(0)}(s) + q^2 \cdot Z^{(1)}(s) = L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \left[ \frac{\alpha(1 - \alpha^2 \beta q^{-s} + \alpha^4 \beta^2 q^{-2s})}{(1 - \alpha^3 q^{-s})(1 - \beta^3 q^{-s})} - \frac{\beta(1 - \alpha \beta^2 q^{-s} + \alpha^2 \beta^4 q^{-2s})}{(1 - \beta^3 q^{-s})(1 - \alpha^2 \beta^2 q^{-s})} \right]
\]
\[
= L(2s + 1, \omega)^{-1}(\alpha - \beta)^{-1}
\times \left[ \frac{(\alpha - \beta)(1 - \alpha^6 \beta^6 q^{-4s})}{(1 - \alpha^3 q^{-s})(1 - \beta^3 q^{-s})(1 - \alpha^2 \beta^2 q^{-s})(1 - \alpha \beta^2 q^{-s})} \right]
\]
\[
= L(2s + 1, \omega)^{-1}L(4s, \omega^2)^{-1}L(s, As \Pi).
\]

This completes the proof.

\textbf{Corollary 4.5.} Let \(\Pi\) be an irreducible generic admissible representation of \(\text{GL}_2(E)\) with central character \(\omega_{\Pi}\). Assume \(\Pi\) is unramified. We have
\[
\gamma_{\text{PSR}}(s, As \Pi, \psi, \alpha) = \omega_{\Pi}(\Delta_{E/F}(\alpha))|\Delta_{E/F}(\alpha)|_{F}^{2s-1}\omega_{K/F}(-1)^{s} \gamma(s, As \Pi, \psi)
\]
for any basis \(\alpha\) of \(E\) over \(F\) and non-trivial additive character \(\psi\) of \(F\).

\textbf{Proof.} We write \(\omega = \omega_{\Pi}|_{F^\times}\). Assume \(\Pi = \text{Ind}_{B(E)}^{\text{GL}_2(E)}(\chi_1 \otimes \chi_2)\) for some unramified characters \(\chi_1\) and \(\chi_2\) of \(E^\times\). We first prove the assertion for \(\alpha = \alpha_X = \{0, 1/3, 0, \omega_{E}^{-1}/3, 0, \omega_{E}^{-2}/3\}\) and \(\psi\) is of conductor
Let $f_{\omega}^\circ$ be the $K$-invariant good section of $I(\omega, s)$ normalized so that $f_{\omega}^\circ(1) = 1$ and $W^\omega \in W(\Pi, \psi_E)$ the $a(3^{-1} \omega_E^{2}) GL_2(\sigma_E) a(3\omega_E^{2})$-invariant Whittaker function normalized so that $W^\omega(1) = 1$. By the Gindikin-Karpelevich formula (cf. [PSR87a Proposition 5.3]), we have

$$M_{\omega}^\circ f_{\omega}^\circ(1) = \frac{L(3-2s, \omega^{-1})L(4-4s, \omega^{-2})}{L(2s+1, \omega)L(4s, \omega^2)}.$$  

Note that $(W^\omega)^\vee \in W(\Pi, \psi_E)$ is the $a(3^{-1} \omega_E^{2}) GL_2(\sigma_E) a(3\omega_E^{2})$-invariant Whittaker function with $(W^\omega)^\vee(1) = 1$. By Lemma 4.3, we have

$$Z_{\omega}^\circ(f_{\omega}^\circ, W^\omega) = L(2s+1, \omega)^{-1}L(4s, \omega^2)^{-1}L(s, \Pi),$$

$$Z_{\omega}^\circ(M_{\omega}^\circ f_{\omega}^\circ, (W^\omega)^\vee) = L(2s+1, \omega)^{-1}L(4s, \omega^2)^{-1}L(1-s, \Pi).$$

It follows from the functional equation that

$$\gamma_{\psi SR}(s, \Pi, \psi, \alpha) = \varepsilon(s, \Pi, \psi)^{-1} \gamma(s, \Pi, \psi).$$

Since $\chi_1$ and $\chi_2$ are unramified, by (3.2) and (3.3), we have

$$\varepsilon(s, \Pi, \psi) = \varepsilon(s, \chi_1|F^s, \psi)\varepsilon(s, \chi_2|F^s, \psi)\varepsilon(s, \text{Ind}_W^{E}(\chi_1^2 \chi_2), \psi)\varepsilon(s, \text{Ind}_W^{E}(\chi_1^2 \chi_2), \psi).$$

Also note that $\varepsilon(s, \chi_1|F^s, \psi) = \varepsilon(s, \chi_2|F^s, \psi) = 1$ since $\psi$ is of conductor $\sigma_F$. On the other hand,

$$\varepsilon(s, \text{Ind}_W^{E}(\chi_1^2 \chi_2), \psi) = \lambda_{E/F}(\psi)\varepsilon(s, \chi_1^2 \chi_2, \psi_E),$$

$$\varepsilon(s, \text{Ind}_W^{E}(\chi_1^2 \chi_2), \psi) = \lambda_{E/F}(\psi)\varepsilon(s, \chi_1^2 \chi_2, \psi_E),$$

where $\lambda_{E/F}(\psi)$ is the Langlands constant for $E/F$ with respect to $\psi$ (cf. [BH06 §30]). Since $\chi_1$ and $\chi_2$ are unramified and $\psi_E^{3-1} \omega_E^2$ is of conductor $\sigma_E$, we have

$$\varepsilon(s, \chi_1^2 \chi_2, \psi_E) = \chi_1 \chi_2 (3\omega_E^2) 3\omega_E^2 |_{E}^{s-1/2},$$

$$\varepsilon(s, \chi_1^2 \chi_2, \psi_E) = \chi_1 \chi_2 (3\omega_E^2) 3\omega_E^2 |_{E}^{s-1/2}.$$ 

Note that $\Delta_{E/F}(\alpha) = 3^{-3} \omega_E^2$ and $\lambda_{E/F}(\psi)^2 = \omega_{K/F}(-1)$. Indeed, since $\omega_{K/F} = \det(\text{Ind}_W^{E}(1))$, we have $\lambda_{E/F}(\psi)^2 = \omega_{K/F}(-1)$ by [BH06 (30.4.3)]. We conclude that

$$\varepsilon(s, \Pi, \psi) = \omega_{\Pi}(\Delta_{E/F}(\alpha))^{-1} |\Delta_{E/F}(\alpha)|_{E}^{2s-1} \omega_{K/F}(-1).$$

The assertion for the special case is proved. It then follows from (5.1) and Lemma 4.1(2) that the assertion holds for any non-trivial additive character $\psi$ of $F$ and $\alpha = a_{\alpha}^\psi$. Let $\beta$ be any basis of $E$ over $F$ and $A_{\alpha, \beta} \in GL_2(F)$ the transition matrix from $\alpha$ to $\beta$. Then $\Delta_{E/F}(\beta) = \det(A_{\alpha, \beta})^2 \Delta_{E/F}(\alpha)$. The assertion for $\beta$ then follows from Lemma 4.1(1). This completes the proof.

5. **Proof of Main Results**

5.1. **Automorphic L-functions.** In §5.2 below, a key ingredient in the proof of Theorem 1.1 is a global-to-local argument. We recall in this section the automorphic $L$-functions for the Asai cube representation. Let $F$ be a number field and $E$ an étale cubic algebra over $F$. Let $\mathbb{K}$ be the quadratic discriminant algebra of $E$ and $\omega_{K/F}$ the quadratic Hecke character associated to $\mathbb{K}/F$ by class field theory. Denote by $\mathcal{A}_E$ and $\mathcal{A}_F$ the rings of adeles of $E$ and $F$, respectively. Let $\psi$ be a non-trivial additive character of $\mathcal{A}_F$. Let $\Pi$ be an irreducible cuspidal automorphic representation of $GL_2(\mathcal{A}_E)$ with central character $\omega_{\Pi}$. Write $\omega = \omega_{\Pi}|_{F^s}$. For each place $v$ of $F$, let

$$L(s, \Pi_v), \quad \varepsilon(s, \Pi_v, \psi_v), \quad \gamma(s, \Pi_v, \psi_v)$$

be the Asai cube factors associated to $\Pi_v$ defined in §4 and

$$\gamma_{LS}(s, \Pi_v, \psi_v)$$

the Asai cube $\gamma$-factor defined by the Langlands-Shahidi method [Sha90 (cf. [HL18 §5]). By [Sha90 Theorem 3.5-(1)] for $v$ archimedean and [HL18 Theorem 1.1] for $v$ non-archimedean (see also [Ram00 Theorem 4.4.1], [Kim03 §5], and [Kr03 §6]), we have

$$\gamma(s, \Pi_v, \psi_v) = \gamma_{LS}(s, \Pi_v, \psi_v).$$

(5.1)
The automorphic $L$-function and $\varepsilon$-factor associated to $\Pi$ and the Asai cube representation are defined by
\[ L(s, \text{As}\Pi) = \prod_v L(s, \text{As}\Pi_v), \quad \varepsilon(s, \text{As}\Pi) = \prod_v \varepsilon(s, \text{As}\Pi_v, \psi_v). \]

Note that the product defining the $L$-function is absolutely convergent for $\text{Re}(s)$ sufficiently large. For a set $S$ of places of $F$, we define the partial $L$-function
\[ L^S(s, \text{As}\Pi) = \prod_{v \in S} L(s, \text{As}\Pi_v). \]

We recall the integral representation of $L(s, \text{As}\Pi)$ due to Garrett [Gar87] (see also [PSR87b, §4.3] and [Ike89]). Let $\alpha$ be a symplectic basis of $V(F)$ and the isomorphism defined by $\iota_{\alpha, F} : \text{GSp}(V)(\mathbb{A}_F) \rightarrow \text{GSp}_6(\mathbb{A}_F)$
\[ \iota_{\alpha, v} : \text{GSp}(V)(F_v) \rightarrow \text{GSp}_6(F_v) \]
is the isomorphism (4.3) by regarding $\alpha$ as a symplectic basis of $V(F_v)$. Let $f_s$ be a good section of $I(\omega, s)$ and $\phi \in \Pi$ a cusp form. Let $E(f_s)$ be the Eisenstein series on $\text{GSp}_6(\mathbb{A}_F)$ defined by
\[ E(g; f_s) = \sum_{\gamma \in P(F) \setminus \text{GSp}_6(F)} f_s(\gamma g) \]
for $\text{Re}(s)$ sufficiently large, and by meromorphic continuation otherwise. Let $W_\phi$ be the Whittaker function of $\phi$ with respect to $\psi_F$ defined by
\[ W_\phi(g) = \int_{E(\mathbb{A}_F)} \phi(n(x)g)\overline{\psi_F(x)} \, dx. \]

By a standard unfolding argument (cf. [PSR87b, §2]), the integral
\[ \int_{\mathbb{A}_F G(F) \setminus G(\mathbb{A}_F)} E(t_{\alpha, F}(g); f_s)\phi(g) \, dg \]
is equal to
\[ Z_\alpha(f_s, W_\phi) = \int_{\mathbb{A}_F U(\mathbb{A}_F) \setminus G(\mathbb{A}_F)} f_s(t_{\alpha, F}(\eta g))W_\phi(g) \, dg \]
for $\text{Re}(s)$ sufficiently large, where $\eta \in \text{GSp}(V)(F)$ satisfies (4.2). By the functional equation of the Eisenstein series $E(f_s)$ and the properties of the local zeta integrals in §4.2, the integral $Z_\alpha(f_s, W_\phi)$ admits meromorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation
\[ (5.2) \quad Z_\alpha(f_s, W_\phi) = Z_\alpha(M^*_w f_s, W_\phi). \]

**Lemma 5.1.** Let $S$ be a finite set of places of $F$ containing all archimedean places such that for $v \notin S$, we have $E_v$ is unramified over $F_v$ and $\Pi_v$ is unramified. Then
\[ \prod_{v \in S} \gamma_{\text{PSR}}(s, \text{As}\Pi_v, \psi_v, \alpha) = \prod_{v \in S} \omega_{\Pi_v}(\Delta_{E_v/F_v}(\alpha))|\Delta_{E_v/F_v}(\alpha)|^{2s-1}\omega_{\mathbb{K}_v/F_v}(-1)\gamma(s, \text{As}\Pi_v, \psi_v) \]
for any basis $\alpha$ of $E$ over $F$.

**Proof.** Let $\alpha$ be a basis of $E$ over $F$. Let $T$ be a set of places containing $S$ such that for $v \notin T$, we have
- $\alpha$ is an integral basis of $\mathfrak{o}_{E_v}$ over $\mathfrak{o}_{F_v}$,
- $\psi_v$ is of conductor $\mathfrak{o}_{F_v}$,
- $E_v$ is unramified over $F_v$,
- $\Pi_v$ is unramified.

Write $\alpha = \{x_1, x_2, x_3\}$ and let $\alpha^* = \{x_1^*, x_2^*, x_3^*\}$ be the dual basis of $\alpha$. Let $\tilde{\alpha}$ be the symplectic basis of $V(F)$ defined by
\[ \tilde{\alpha} = \{(x_1^*, 0), (x_2^*, 0), (x_3^*, 0), (0, x_1), (0, x_2), (0, x_3)\}. \]

For each place $v$ of $F$, let $f_{s,v}$ be a good section of $I(\omega_v, s)$ and $W_v \in W(\Pi_v, \psi_{E_v})$ satisfying the following conditions:
For \( \nu \notin T \), \( f_{s, \nu} \) is the \( \text{GSp}_6(\mathfrak{O}_F) \)-invariant good section normalized so that \( f_{s, \nu}(1) = 1 \) and \( W_{\nu} \) the \( \text{GL}_2(\mathfrak{O}_{E_{\nu}}) \)-invariant Whittaker function normalized so that \( W_{\nu}(1) = 1 \).

For \( \nu \in T \), we have \( Z_{\alpha}(f_{s, \nu}, W_{\nu}) \neq 0 \).

By [PSR87b] Theorem 3.1], for \( \nu \notin T \) we have
\[
Z_{\alpha}(f_{s, \nu}, W_{\nu}) = L(2s + 1, \omega_{\nu})^{-1}L(4s, \omega_{\nu}^2)^{-1}L(s, \text{As}_{\Pi_{\nu}}),
\]
\[
Z_{\alpha}(M_{s, \nu}f_{s, \nu}, W_{\nu}) = L(2s + 1, \omega_{\nu})^{-1}L(4s, \omega_{\nu}^2)^{-1}L(1 - s, \text{As}_{\Pi_{\nu}^\vee}).
\]

We conclude from the functional equations \((5.3)\) and \((5.2)\) that the partial \( L \)-function \( L^T(s, \text{As}_{\Pi}) \) admits meromorphic continuation to \( s \in \mathbb{C} \) and satisfies the functional equation
\[
L^T(s, \text{As}_{\Pi}) = \prod_{\nu \in T} \gamma_{\text{PSR}}(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu}, \alpha)L^T(1 - s, \text{As}_{\Pi_{\nu}^\vee}).
\]

On the other hand, by [Sha90] Theorem 3.5-[4]) and \((5.1)\), we have the functional equation
\[
L^T(s, \text{As}_{\Pi}) = \prod_{\nu \in T} \gamma(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu})L^T(1 - s, \text{As}_{\Pi_{\nu}^\vee}).
\]

We deduce that
\[
\prod_{\nu \in T} \gamma_{\text{PSR}}(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu}, \alpha) = \prod_{\nu \in T} \gamma(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu}).
\]

By Lemma 4.3, for \( \nu \in T \setminus S \), we have
\[
\gamma_{\text{PSR}}(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu}, \alpha) = \omega_{\Pi_{\nu}}(\Delta_{E_{\nu}/F_{\nu}}(\alpha)|\Delta_{E_{\nu}/F_{\nu}}(\alpha)|^{2s-1}\omega_{K_{\nu}/F_{\nu}}(-1)^{-1})\gamma(s, \text{As}_{\Pi_{\nu}}, \psi_{\nu}).
\]

By assumption, for \( \nu \notin T \), we have
\[
\omega_{\Pi_{\nu}}(\Delta_{E_{\nu}/F_{\nu}}(\alpha)|\Delta_{E_{\nu}/F_{\nu}}(\alpha)|^{2s-1}\omega_{K_{\nu}/F_{\nu}}(-1)^{-1} = 1.
\]

The assertion then follows from \((5.3)\) and the product formula
\[
\prod_{\nu} \omega_{\Pi_{\nu}}(\Delta_{E_{\nu}/F_{\nu}}(\alpha)|\Delta_{E_{\nu}/F_{\nu}}(\alpha)|^{2s-1}\omega_{K_{\nu}/F_{\nu}}(-1) = 1.
\]

This completes the proof.

5.2. Proof of Theorem 1.1.] Let \( F \) be a non-archimedean local field of characteristic zero and \( E \) an étale cubic algebra over \( F \). Let \( \mathbb{K} \) be the quadratic discriminant algebra of \( F \). Let \( \Pi \) be an irreducible generic admissible representation of \( \text{GL}_2(E) \) with central character \( \omega_\Pi \). Let \( \psi \) be a non-trivial additive character of \( F \) and \( \alpha \) a basis of \( E \) over \( F \). Recall the domains \( D(\Pi) \) and \( D(\Pi)^\circ \) defined in \((2.4)\) and \((2.6)\), respectively. Consider the family of irreducible generic admissible representations \( \Pi_\lambda \) of \( \text{GL}_2(E) \) for \( \lambda \) varying in \( D(\Pi) \).

We are going to prove that the identity
\[
\gamma_{\text{PSR}}(s, \text{As}_{\Pi_{\lambda}}, \psi, \alpha) = \omega_{\Pi_\lambda}(\Delta_{E/F}(\alpha)|\Delta_{E/F}(\alpha)|^{2s-1}\omega_{K/F}(-1)^{-1})\gamma(s, \text{As}_{\Pi_{\lambda}}, \psi)
\]
holds for all \( \lambda \in D(\Pi) \). In particular, \((5.3)\) holds for \( \Pi_{\lambda} = \Pi \). We divide the proof into three steps as follows:

Step 1. Establish \((5.4)\) for a dense subset \( \mathcal{U} \) of \( D(\Pi)^\circ \).

Step 2. Establish \((5.4)\) for all \( \lambda \in D(\Pi) \) with \( |\lambda|_\Pi < 1/2 \).

Step 3. Establish \((5.4)\) for all \( \lambda \in D(\Pi) \).

Fix a symplectic basis \( \tilde{\alpha} \) of \( V(F) \) such that \( \tilde{\alpha}_\lambda = \alpha \). Write \( \omega_\lambda = \omega_{\Pi_\lambda}|_{F^\times} \).

**Step 1.** By Kranser’s lemma, there exist an étale cubic algebra \( E \) over a number field \( F \) and a place \( \nu_1 \) of \( F \) such that
- \( E_{\nu_1} = E \) and \( F_{\nu_1} = F \),
- for any non-archimedean place \( \nu \neq \nu_1 \) and \( \nu \) divides 3, \( E_{\nu} \) is unramified over \( F_{\nu} \).

Let \( \alpha \) be a basis of \( E \) over \( F \), \( K \) the quadratic discriminant algebra of \( E \), and \( \psi \) a non-trivial additive character of \( K/F \). By \((3.4)\) and Lemma 4.11 we may assume \( \alpha = \alpha \) regarding as a basis of \( E \) over \( F \) and \( \psi = \psi_{\nu} \). Fix a non-archimedean place \( \nu_2 \) of \( F \) such that \( E_{\nu_2} = F_{\nu_2} \times F_{\nu_2} \times F_{\nu_2} \). Let \( U \) be an open subset of \( D(\Pi)^\circ \). By the limit multiplicity property for the principal congruence subgroups of \( \text{GL}_2 \) proved in [FLM13] Theorem 1.3], there exists an irreducible cuspidal automorphic representation \( \Pi \) of \( \text{GL}_2(\mathbb{A}_E) \) (see [BPT18] Theorem 3.7.1] for the deduction) such that
- \( \Pi_{\nu_1} = \Pi_\lambda \) for some \( \lambda \in U \),
By Lemma 4.2, we can take $C$ for all $\lambda \in U$. Let $S$ be the finite set of places $v$ of $F$ such that $v$ is archimedean or $v \in \{v_1, v_2\}$ or $v$ is non-archimedean and $E_v$ is ramified over $F_v$. By Lemma 5.1, we have

$$
\prod_{v \in S} \gamma_{\text{PSR}}(s, \Lambda v, \psi_v, \alpha) = \prod_{v \in S} \omega_{\text{I}}(\Delta_{E_v/F_v}(\alpha))|\Delta_{E_v/F_v}(\alpha)|_{F_v}^{\frac{2s}{v} - 1} \omega_{K_v/F_v}(-1)^{\gamma}(s, \Lambda v, \psi_v).
$$

On the other hand, by Lemma 4.3 and Corollary 4.5 for $v \in S$ with $v \neq v_1$, we have

$$
\gamma_{\text{PSR}}(s, \Lambda v, \psi_v, \alpha) = \omega_{\text{I}}(\Delta_{E_v/F_v}(\alpha))|\Delta_{E_v/F_v}(\alpha)|_{F_v}^{\frac{2s}{v} - 1} \omega_{K_v/F_v}(-1)^{\gamma}(s, \Lambda v, \psi_v).
$$

It follows that (5.4) holds for some $\lambda \in U$.

**Step 2.** Let $\lambda_0 \in D(\Pi)$ with $|\lambda_0|_\Pi < 1/2$. Let $0 < \epsilon < 1/2 - |\lambda_0|_\Pi$. Let $f_s$ be a good section of $I(\omega_{\lambda_0}, s)$ and $W \in \mathcal{W}(\Pi_{\lambda_0}, \psi)$. We extend $f_s$ to a good section $f_s,\lambda$ of $I(\omega_{\lambda}, s)$ and $W$ to a holomorphic family of Whittaker functions $W_\lambda$ of $\Pi_\lambda$ with respect to $\psi_\lambda$. By **Step 1,**

$$
Z(\lambda, s + 1, \lambda, W_\lambda) = \omega_{\Pi_\lambda}(\Delta_{E/F}(\alpha))|\Delta_{E/F}(\alpha)|_{E/F}^{\frac{2s}{2} - 1} \omega_{E/F}(-1)^{\epsilon}(s + 1/2, \Lambda_\lambda, \psi_\lambda, \alpha)\frac{Z(f_s, \Psi_\lambda, W_\lambda)}{(s + 1/2, \Lambda_{\lambda})}
$$

holds for $s \in \mathbb{C}$ and $\lambda \in \mathcal{U}$. On the other hand, by Lemma 4.2 the right-hand side and the left-hand side of (5.5) define holomorphic functions on

$$
\{s \in \mathbb{C} \mid \Re(s) > 1/2 - \epsilon\} \times \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < 1/2 - \epsilon\}
$$

and

$$
\{s \in \mathbb{C} \mid \Re(s) < 1/2 + \epsilon\} \times \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < 1/2 - \epsilon\},
$$

respectively. Since the set $\mathcal{U}$ is dense in $D(\Pi)^0$, it follows that (5.5) holds for $(s, \lambda)$ in the domain

$$
\{s \in \mathbb{C} \mid 1/2 - \epsilon < \Re(s) < 1/2 + \epsilon\} \times \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < 1/2 - \epsilon\}.
$$

We then deduce form the holomorphicity that (5.5) holds for all $s \in \mathbb{C}$ and $|\lambda|_\Pi < 1/2 - \epsilon$. In particular, it holds for $\lambda = \lambda_0$. Since $f_s$ and $W$ are arbitrary chosen, we see that (5.4) holds for $\lambda = \lambda_0$.

**Step 3.** Let $f_s,\lambda$ be a good section of $I(\omega_{\lambda}, s)$ and $W_\lambda$ a holomorphic family of Whittaker functions of $\Pi_\lambda$ with respect to $\psi_\lambda$. Let

$$
Z_1(s, \lambda) = \frac{Z(\lambda, s + 1, \lambda, W_\lambda)}{(s + 1/2, \Lambda_\lambda)}
$$

and

$$
Z_2(s, \lambda) = \omega_{\Pi_\lambda}(\Delta_{E/F}(\alpha))|\Delta_{E/F}(\alpha)|_{E/F}^{\frac{2s}{2} - 1} \omega_{E/F}(-1)^{\epsilon}(s + 1/2, \Lambda_\lambda, \psi_\lambda, \alpha)\frac{Z(f_s + 1/2, \lambda, W_\lambda)}{(s + 1/2, \Lambda_{\lambda})}
$$

be partially defined functions on $\mathbb{C} \times D(\Pi)$. By Lemma 4.2 and **Step 2,** $Z_1$ and $Z_2$ are holomorphic functions on the domain

$$
\mathbb{C} \times \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < 1/2\}
$$

and satisfy the functional equation

$$
Z_1(-s, \lambda) = Z_2(s, \lambda).
$$

Note that it is clear that $Z_1$ and $Z_2$ are periodic of period $\log(q_F)^{-1}2\pi\sqrt{-1}$ in the variable $s$. We claim that $Z_1$ and $Z_2$ admit holomorphic continuation to $\mathbb{C} \times D(\Pi)$ and satisfy the above functional equation. To prove the claim, by [BP18 Proposition 2.8.1] with $M = D(\Pi)$ and $U = \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < 1/2\}$, it suffices to show that for every $C' > 1/2$, there exists $C > 0$ such that $Z_1$ and $Z_2$ admit holomorphic continuation to the domain

$$
\{s \in \mathbb{C} \mid \Re(s) > C\} \times \{\lambda \in D(\Pi) \mid |\lambda|_\Pi < C'\}.
$$

By Lemma 4.2, we can take $C = C' - 1/2$. As $f_s,\lambda$ and $W_\lambda$ are arbitrary chosen, we conclude that (5.4) holds for all $\lambda \in D(\Pi)$. This completes the proof of Theorem 1.1.

**Remark 5.2.** In [BP18 Proposition 2.8.1], $U$ and $U'$ are connected relatively compact open subsets of $M$. It is clear that any connected relatively compact open subset of $D(\Pi)$ is contained in $\{\lambda \in D(\Pi) \mid |\lambda|_\Pi < C\}$ for some $C > 0$. 

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5.3. Proof of Corollary 1.3. By [Ike92] Lemma 2.1 and Lemma 3.1, \( L(s, \text{As} \Pi) \) and \( L_{PSR}(s, \text{As} \Pi) \) have no poles for \( \text{Re}(s) \geq 1/2 \). Similarly, \( L(1 - s, \text{As} \Pi^\vee) \) and \( L_{PSR}(1 - s, \text{As} \Pi^\vee) \) have no poles for \( \text{Re}(s) \leq 1/2 \). Also note that the \( L \)-functions have no zeros. We deduce that the poles of \( L(s, \text{As} \Pi) \), count with multiplicities, are equal to the poles of the meromorphic function \( L(s, \text{As} \Pi) L(1 - s, \text{As} \Pi^\vee)^{-1} \). We have a similar conclusion for \( L_{PSR}(s, \text{As} \Pi) \). Since the \( \varepsilon \)-factors have neither poles nor zeros, the assertion then follows from Theorem 1.1.

5.4. Proof of Theorem 1.4. By [KS02] Theorem C, we have \( \max\{|L(\Pi_v)|, |L(\Pi_v^\vee)|\} < 1/2 \) for all places \( v \) of \( F \). Therefore, by Corollary 1.3, we have \( L_{PSR}(s, \text{As} \Pi_v) = L(s, \text{As} \Pi_v) \) for all places \( v \) of \( F \). It follows from [Ike92] Propositions 2.3-2.5 and the integral representation recalled in [Ike92] that \( L(s, \text{As} \Pi) \) is absolutely convergent for \( \text{Re}(s) \geq 3/2 \), admits meromorphic continuation to \( s \in \mathbb{C} \), entire if either \( \omega^2 \neq 1 \) or \( \omega = 1 \), and satisfies the functional equation

\[
L(s, \text{As} \Pi) = \varepsilon(s, \text{As} \Pi)L(1 - s, \text{As} \Pi^\vee).
\]

Proceeding as in the proof of [Ram00] Theorem 3.4.1, we can prove that \( L(s, \text{As} \Pi) \) is bounded in vertical strips of finite width. Note that in our case the assumption that \( F \) is totally complex in [Ram00] Theorem 3.4.1 is not necessary. Finally, the description of the poles at \( s = 0, 1 \) were established in [Ike92] Theorems 2.6-2.8.

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