ON RATIONAL $p$-ADIC DYANAMICAL SYSTEMS

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Abstract

In the paper we investigate the behavior of trajectory of rational $p$-adic dynamical system in complex $p$-adic field $\mathbb{C}_p$. It is studied Siegel disks and attractors of such dynamical systems. We show that Siegel disks may either coincide or disjoin for different fixed points of the dynamical system. Besides, we find the basin of the attractor of the system. It is proved that such kind of dynamical system is not ergodic on a unit sphere with respect to the Haar measure.

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1 Introduction

The $p$-adic numbers were first introduced by the German mathematician K. Hensel. For about a century after the discovery of $p$-adic numbers, they were mainly considered objects of pure mathematics. Beginning with 1980's various models described in the language of $p$-adic analysis have been actively studied. More precisely, models over the field of $p$-adic numbers have been considered which is due to the assumption that $p$-adic numbers provide a more exact and more adequate description of micro-world phenomena. Numerous applications of these numbers to theoretical physics have been proposed in papers [4], [12], [20], [26], [27] to quantum mechanics [15], to $p$-adic-valued physical observable [15] and many others [16], [25].

The study of $p$-adic dynamical systems arises in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic vertices over a number field, as in [7]. In [17], [24] $p$-adic field have arisen in physics in the theory of superstrings, promoting questions about their dynamics. Also some applications of $p$-adic dynamical systems to some biological, physical systems has been proposed in [1], [2], [8], [17], [18]. Other studies of non-Archimedean dynamics in the neighborhood of a periodic and of the counting of periodic points over global fields using local fields appear in [13], [19], [21]. It is known that the analytic functions play important role in complex analysis. In the $p$-adic analysis the rational functions play a similar role to the analytic functions in complex analysis [23]. Therefore, naturally one arises a question to study the dynamics of these functions in the $p$-adic analysis. On the hand, such $p$-adic dynamical systems appear while studying $p$-adic Gibbs measures [9]. In [5], [6] dynamics on the Fatou set of a rational function defined over some finite extension of $\mathbb{Q}_p$ have been studied, besides, an analogue of Sullivan’s no wandering domains theorem for $p$-adic rational functions which have no wild recurrent Julia critical points were proved. In [3] the behavior of a $p$-adic dynamical system $f(x) = x^n$ in the fields of
$p$-adic numbers $\mathbb{Q}_p$ and complex $p$-adic numbers $\mathbb{C}_p$ was investigated. Some ergodic properties that dynamical system has been considered in [11].

The base of $p$-adic analysis, $p$-adic mathematical physics are explained in [10],[14],[25].

In the paper we will investigate the behavior of trajectory of rational $p$-adic dynamical systems in $\mathbb{C}_p$. We will study Siegel disks and attractors of such dynamical systems. In the final section we show the considered dynamical system is not ergodic.

2 Preliminaries

2.1 $p$-adic numbers

Let $\mathbb{Q}$ be the field of rational numbers. The greatest common divisor of the positive integers $n$ and $m$ is denotes by $(n,m)$. Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r,n \in \mathbb{Z}$, $m$ is a positive integer, $(p,n) = 1$, $(p,m) = 1$ and $p$ is a fixed prime number. The $p$-adic norm of $x$ is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

It has the following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,

2) $|xy|_p = |x|_p |y|_p$,

3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

3.1) if $|x|_p \neq |y|_p$ then $|x - y|_p = \max\{|x|_p, |y|_p\}$,

3.2) if $|x|_p = |y|_p$ then $|x - y|_p \leq |x|_p$,

this is a non-Archimedean one.

The completion of $\mathbb{Q}$ with respect to $p$-adic norm defines the $p$-adic field which is denoted by $\mathbb{Q}_p$.

The well-known Ostrovsky’s theorem asserts that norms $|x| = |x|_\infty$ and $|x|_p$, $p = 2,3,5...$ exhaust all nonequivalent norms on $\mathbb{Q}$ (see [?]). Any $p$-adic number $x \neq 0$ can be uniquely represented in the canonical series:

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \quad (2.1)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and $x_j$ are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0,1,2,...$ (see more detail [10],[14]). Observe that in this case $|x|_p = p^{-\gamma(x)}$.

The algebraic completion of $\mathbb{Q}_p$ is denoted by $\mathbb{C}_p$ and it is called complex $p$-adic numbers. For any $a \in \mathbb{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\}, \quad V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\},$$

$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$
A function \( f : U_r(a) \rightarrow \mathbb{C}_p \) is said to be analytic if it can be represented by
\[
f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n, \quad f_n \in \mathbb{C}_p,
\]
which converges uniformly on the ball \( U_r(a) \).

### 2.2 Dynamical systems in \( \mathbb{C}_p \)

In this section we recall some known facts concerning dynamical systems \((f, U)\) in \( \mathbb{C}_p \), where \( f : x \in U \rightarrow f(x) \in U \) is an analytic function and \( U = U_r(a) \) or \( \mathbb{C}_p \).

Now let \( f : U \rightarrow U \) be an analytic function. Denote \( x_n = f^n(x_0) \), where \( x_0 \in U \) and \( f^n(x) = f \circ \cdots \circ f(x) \).

Recall some the standard terminology of the theory of dynamical systems (see for example [22]). If \( f(x_0) = x_0 \) then \( x_0 \) is called a fixed point. A fixed point \( x_0 \) is called an attractor if there exists a neighborhood \( V(x_0) \) of \( x_0 \) such that for all points \( y \in V(x_0) \) it holds \( \lim_{n \to \infty} y_n = x_0 \). If \( x_0 \) is an attractor then its basin of attraction is
\[
A(x_0) = \{ y \in \mathbb{C}_p : y_n \to x_0, \; n \to \infty \}.
\]

A fixed point \( x_0 \) is called repeller if there exists a neighborhood \( V(x_0) \) of \( x_0 \) such that \( |f(x) - x_0|^p > |x - x_0|^p \) for \( x \in V(x_0), \; x \neq x_0 \). Let \( x_0 \) be a fixed point of a function \( f(x) \). The ball \( V_\epsilon(x_0) \) (contained in \( U \)) is said to be a Siegel disk if each sphere \( S_\rho(x_0) \), \( \rho < r \) is an invariant sphere of \( f(x) \), i.e. if \( x \in S_\rho(x_0) \) then all iterated points \( x_n \in S_\rho(x_0) \) for all \( n = 1, 2, \ldots \). The union of all Siegel disks with the center at \( x_0 \) is said to a maximum Siegel disk and is denoted by \( SI(x_0) \).

**Remark.**[3] In complex geometry, the center of a disk is uniquely determined by the disk, and different fixed points cannot have the same Siegel disks. In non-Archimedean geometry, a center of a disk is nothing but a point which belongs to the disk. Therefore, in principle, different fixed points may have the same Siegel disk.

Let \( x_0 \) be a fixed point of an analytic function \( f(x) \). Put
\[
\lambda = \frac{d}{dx}f(x_0).
\]

The point \( x_0 \) is called attractive if \( 0 \leq |\lambda|^p < 1 \), indifferent if \( |\lambda|^p = 1 \), and repelling if \( |\lambda|^p > 1 \).

**Theorem 2.1.**[3] Let \( x_0 \) be a fixed point of an analytic function \( f : U \rightarrow U \). The following assertions hold

1. if \( x_0 \) is an attractive point of \( f \), then it is an attractor of the dynamical system \((f, U)\). If \( r > 0 \) satisfies the inequality
\[
q = \max_{1 \leq n < \infty} \left| \frac{1}{n!} \frac{d^n f}{dx^n}(x_0) \right|^p r^{n-1} < 1 \quad (2.2)
\]
and $U_r(x_0) \subset U$ then $U_r(x_0) \subset A(x_0)$;
2. if $x_0$ is an indifferent point of $f$ then it is the center of a Siegel disk. If $r$ satisfies the inequality
\[ s = \max_{2 \leq n < \infty} \left| \frac{1}{n!} \frac{d^n f}{dx^n}(x_0) \right| p^{n-1} < |f'(x_0)|_p \] (2.3)
and $U_r(x_0) \subset U$ then $U_r(x_0) \subset SI(x_0)$;
3. if $x_0$ is a repelling point of $f$ then $x_0$ is a repeller of the dynamical system $(f, U)$.

3 Rational $p$-adic dynamical systems

In this section we consider dynamical system associated with the function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by
\[ f(x) = \frac{x + a}{bx + c}, \quad b \neq 0, \ c \neq ab, \ a, b, c \in \mathbb{C}_p \] (3.1)
where $x \neq \hat{x} = \frac{-c}{b}$.

It is not difficult to check that fixed points of the function (3.1) are
\[ x_{1,2} = \frac{1 - c \pm \sqrt{(c - 1)^2 + 4ab}}{2b}. \] (3.2)

The following theorem is important in our investigation.

**Theorem 3.1.** Let $x_1$ and $x_2$ be the fixed points of (3.1) (see (3.2)). Then
1. the point $x_1$ (resp. $x_2$) is attractive if and only if the point $x_2$ (resp. $x_1$) is repelling.
2. the point $x_1$ is indifferent if and only if the point $x_2$ is one.

The proof immediately follows from the easily checking equality
\[ f'(x_1) \cdot f'(x_2) = 1. \]

3.1 Case: $|f'(x_1)|_p = 1$

Let $|f'(x_1)|_p = 1$, then according to Theorem 3.1 we have $|f'(x_2)|_p = 1$. Observe that the considered case is equivalent to condition
\[ |f'(x_i)|_p = \left| \frac{c - ab}{(bx_i + c)^2} \right|_p = 1, \quad i = 1, 2. \] (3.3)

By Theorem 2.1 every fixed point $x_i$ is a center of Siegel disk. We now verify the condition (2.3). First of all we compute
\[ \left| \frac{1}{n!} \frac{d^n f}{dx^n}(x_i) \right|_p = \left| \frac{c - ab}{(bx_i + c)^2} \right|_p \left| \frac{nb^{n-1}}{(bx_i + c)^{n+1}} \right|_p \leq \]
\[
\leq \left| \left( \frac{b}{bx_i + c} \right)^{n-1} \right|_p = \left| \frac{b}{\sqrt{c - ab}} \right|_p^{n-1},
\]
here we have used the equality (3.3). Then the condition (2.3) is satisfied if
the following inequality holds
\[
\max_{2 \leq n < \infty} \left| \frac{b}{\sqrt{c - ab}} \right|_p^{n-1} r^{n-1} < 1. \tag{3.4}
\]
Let
\[
\left| \frac{b}{\sqrt{c - ab}} \right|_p < 1. \tag{3.5}
\]
If \( r \leq 1 \) then the condition (3.4) is satisfied, and hence \( U_1(x_i) \subset SI(x_i) \).

**Theorem 3.2.** Let the conditions (3.3) and (3.5) be satisfied. Then
\[
SI(x_i) = V_{1+\varepsilon_c}(x_i), \quad i = 1, 2,
\]
where \( \varepsilon_c = \left| \frac{\sqrt{c - ab}}{b} \right|_p - 1. \)

**Proof.** It suffices to prove that for any \( \varepsilon < \varepsilon_c \) the equality
\[
f(S_{1+\varepsilon}(x_i)) = S_{1+\varepsilon}(x_i) \tag{3.6}
\]
is valid. Let \( y \in S_{1+\varepsilon}(x_i) \), i.e. \( y = x_i + \gamma \), where \( |\gamma|_p = 1 + \varepsilon \). Then from (3.1) we get
\[
|f(y) - x_i|_p = \frac{1 + \varepsilon}{c - ab + \gamma b \left[ \frac{b}{bx_i + c} \right]_p^2 + b x_i + c}, \tag{3.7}
\]
The inequality \( \varepsilon < \varepsilon_c \) implies that \( \left| \frac{\gamma b}{bx_i + c} \right|_p < 1 \). It then follows from (3.7) and (3.3) that \( |f(y) - x_i|_p = 1 + \varepsilon \), which means that (3.6) is valid. Here we have used 3.1) property of the norm \( | \cdot |_p \). If \( \varepsilon > \varepsilon_c \) then \( \left| \frac{\gamma b}{bx_i + c} \right|_p > 1 \), consequently from (3.7) we infer \( |f(y) - x_i|_p = 1 + \varepsilon_c < 1 + \varepsilon \). Hence (3.6) does not hold. One remains to consider the case \( \varepsilon = \varepsilon_c \). We choose \( \gamma \) as follows
\[
\gamma = \tilde{\gamma} = (p-1) \frac{c - ab}{b(bx_1 + c)}.
\]
Then it easy to check that \( \tilde{y} = x_i + \tilde{\gamma} \) belongs \( S_{1+\varepsilon_c}(x_i) \), but
\[
|f(\tilde{y}) - x_i|_p = p(1 + \varepsilon_c) > 1 + \varepsilon_c.
\]
Thus the equality (3.6) is valid only at \( \varepsilon < \varepsilon_c \). This completes the proof.
Now we are interested in the relation between Siegel disks $SI(x_i), i = 1, 2$.

**Theorem 3.3.** Let the conditions (3.3) and (3.5) be satisfied.

(i) If $\left| \frac{\sqrt{(c-1)^2 + 4ab}}{b} \right|^{p} \geq 1 + \varepsilon_c$, then $SI(x_1) \cap SI(x_2) = \emptyset$;

(ii) otherwise $SI(x_1) = SI(x_2)$.

**Proof.** (i) From (3.2) we find

$$|x_1 - x_2|^{p} = \left| \frac{\sqrt{(c-1)^2 + 4ab}}{b} \right|^{p} \geq 1 + \varepsilon_c.$$

This means $x_1 \notin V_{1+\varepsilon_c}(x_2)$, hence by Theorem 3.2 we have $SI(x_1) \cap SI(x_2) = \emptyset$.

(ii) In this case we have $|x_1 - x_2|^{p} < 1 + \varepsilon_c$. Let $y \in SI(x_1)$, then by Theorem 3.2 we can write $|y - x_1|^{p} < 1 + \varepsilon_c$. Whence

$$|y - x_2|^{p} = |(y - x_1) + (x_1 - x_2)|^{p} < 1 + \varepsilon_c.$$

Consequently, we have $SI(x_1) \subset SI(x_2)$. So $SI(x_1) = SI(x_2)$ since balls with the same radius either coincide or disjoint in Non-Archimedean setting.

**3.2 Case: $|f'(x_1)|^{p} \neq 1$**

According to Theorem 3.1 without loss of generality we may assume that $|f'(x_1)|^{p} < 1$. In this case we have $|f'(x_2)|^{p} > 1$.

From Theorem 2.1 we obtain the following

**Proposition 3.4.** The fixed point $x_2$ is a repelling point of the dynamical system.

Now one remains to investigate the fixed point $x_1$. Observe that the condition $|f'(x_1)|^{p} < 1$ is equivalent to

$$\left| \frac{c - ab}{(bx_1 + c)^2} \right|^{p} < 1. \quad (3.8)$$

Suppose the following condition to be satisfied

$$\left| \frac{b}{bx_1 + c} \right|^{p} = \left| \frac{2b}{1 + c + \sqrt{(c-1)^2 + 4ab}} \right|^{p} \leq 1. \quad (3.9)$$

**Lemma 3.5.** Let the conditions (3.8) and (3.9) be satisfied. Then the inclusion

$$V_1(x_1) \subset A(x_1)$$

is valid.

**Proof.** We check the condition (2.2) of Theorem 2.1.

$$q = \max_{1 \leq n < \infty} \left| \frac{n(c - ab)}{(bx_1 + c)^{n+1}} \right|^{p} r^{n-1} < \max_{1 \leq n < \infty} \left| \frac{b}{bx_1 + c} \right|^{p} r^{n-1} < 1.$$
According to (3.9) this condition is fulfilled if $r < 1$. By Theorem 2.1 we infer the required assertion.

Denote

\[ \delta_1 = \left| \frac{(bx_1 + c)^2}{c - ab} \right|_p - 1, \quad \delta_2 = \left| \frac{bx_1 + c}{b} \right|_p - 1. \]

**Lemma 3.6.** Let the conditions (3.8) and (3.9) be satisfied. Then $\hat{x}, x_2 \in S_{1+\delta_2}(x_1)$, here

\[ \hat{x} = -\frac{c}{b}, \quad x_2 = \frac{1 - c - \sqrt{(c - 1)^2 + 4ab}}{2b}. \]

**Proof.** Consider

\[ |\hat{x} - x_1|_p = \left| 1 + \frac{c + \sqrt{(c - 1)^2 + 4ab}}{2b} \right|_p = \left| \frac{bx_1 + c}{b} \right|_p = 1 + \delta_2, \]

hence $\hat{x} \in S_{1+\delta_2}(x_1)$. It easy to check that

\[ |\hat{x} - x_2|_p = \left| \frac{c - ab}{b(x_1 + c)} \right|_p = \frac{1 + \delta_2}{1 + \delta_1} < 1 + \delta_2, \]

since $\delta_1 > 0$. So we have

\[ |x_1 - x_2|_p = |(x_1 - \hat{x}) + (\hat{x} - x_2)|_p = 1 + \delta_2, \]

here we have used 3.1) property of the norm. Lemma is proved.

**Theorem 3.7.** Let the conditions (3.8) and

\[ \left| \frac{b}{bx_1 + c} \right|_p < 1 \tag{3.10} \]

be satisfied. Then

\[ \bigcup_{0 \leq \delta \neq 1 + \delta_2} S_\delta(x_1) \subset A(x_1). \]

**Proof.** It suffices to prove that $S_{1+\delta}(x_1) \subset A(x_1)$ at $\delta \neq 1 + \delta_2$. Indeed, from the proof of Lemma 3.5 one easily sees that the condition (3.10) provides $U_1(x_1) \subset A(x_1)$. Let $x \in S_{1+\delta}(x_1)$, i.e. $x = x_1 + \gamma$, $|\gamma|_p = 1 + \delta$. From (3.1) and (3.2) we get

\[ |f(x) - x_1|_p = \left| \frac{(c - ab) \gamma}{(bx_1 + c)^2 + \gamma b(bx_1 + c)} \right|_p = \]
From (3.8) and \( \delta < \delta_n \)

The condition (3.8) implies that there is a positive integer

If the right side of the last equality is not greater than 1, then we can put \( f \)

this implies that \( \delta < \delta_2 \). Then from (3.11) we infer

If \( \delta \leq \delta_1 \) then the right side of (3.11) is not greater than 1. Hence, \( f(x) \in U_1(x_1) \). This yields that \( S_{1+\delta}(x_1) \subset A(x_1) \). If \( \delta > \delta_1 \), then the right side of (3.12) is greater than 1, denote it by \( 1 + \lambda \), i.e.

\[
1 + \lambda = \left| \frac{c - ab}{(bx_1 + c)^2} \right| p (1 + \delta), \quad \lambda > 0.
\]

From (3.8) and \( \delta < \delta_2 \) we obtain \( \lambda < \delta_2 \), since in this case \( f(x) \in S_{1+\lambda}(x_1) \), so we can put \( f(x) \) instead of \( x \) in (3.12), namely

\[
|f^2(x) - x_1|_p = \left| \frac{c - ab}{(bx_1 + c)^2} \right| p |f(x) - x_1|_p = \left| \frac{c - ab}{(bx_1 + c)^2} \right| p (1 + \delta).
\]

If the right side of the last equality is not greater than 1, then \( f^2(x) \in U_1(x_1) \), and hence \( S_{1+\delta}(x_1) \subset A(x_1) \). Otherwise repeating the above argument we can prove the following equality

\[
|f^n(x) - x_1|_p = \left| \frac{c - ab}{(bx_1 + c)^2} \right| p^n (1 + \delta), \quad n \geq 1. \tag{13.13}
\]

The condition (3.8) implies that there is a positive integer \( n_0 \) such that \( f^n(x) \in U_1(x_1) \) for all \( n > n_0 \). This yields \( S_{1+\delta}(x_1) \subset A(x_1) \).

Case 2. Now suppose that

\[
\left| \frac{bx_1 + c}{b} \right| p^2 > \left| \gamma \right| p \left| \frac{bx_1 + c}{b} \right| p,
\]

this implies that \( \delta > \delta_2 \). It then follows from (3.11) that

\[
|f(x) - x_1|_p = \left| \frac{c - ab}{b(bx_1 + c)} \right| p. \tag{13.14}
\]

Observe that

\[
\left| \frac{c - ab}{b(bx_1 + c)} \right| p = \frac{1 + \delta_2}{1 + \delta_1}. \tag{13.15}
\]
If $\delta_2 \leq \delta_1$ then the equalities (3.14) and (3.15) provide that $f(x) \in U_1(x_1)$, and in this case we obtain the assertion of theorem. If $\delta_2 > \delta_1$, then from (3.15) we infer that the right side of (3.14) is greater than 1, and which is denoted by $1 + \mu$, $\mu > 0$. So $f(x) \in S_{1+\mu}(x_1)$. Show that $\mu < \delta_2$. Indeed,

\[
1 + \mu = \frac{1 + \delta_2}{1 + \delta_1} < 1 + \delta_2
\]

this implies $\mu < \delta_2$. Thus, we have reduced our consideration to the case 1. This completes the proof.

**Lemma 3.8.** Let $|x - x_2|_p > \frac{1 + \delta_2}{1 + \delta_1}$, then $f(x) \in S_{1+\delta_2}(x_2)$.

**Proof.** Denote $\gamma = x - x_2$, then the condition of lemma means that

\[
\frac{\gamma b}{b x_2 + c} > 1.
\]

we then have

\[
|f(x) - x_2|_p = \frac{|c - ab|_p |x - x_2|_p}{(|b x_2 + c|^2 + b^2)_{2}} = \frac{|c - ab|_p |x - x_2|_p}{1 + \frac{\gamma b}{b x_2 + c} _p}
\]

\[
= \frac{c - ab}{b^2} \frac{\gamma b}{b x_2 + c} _p = \frac{c - ab}{b} _p = 1 + \delta_2
\]

Lemma is proved.

From this lemma we obtain the following

**Corollary 3.9.** Let the condition of Lemma 3.8 be satisfied, then $f(S_{1+\delta_2}(x_2)) = S_{1+\delta_2}(x_2)$.

**Corollary 3.10.** Let $|x - x_2|_p \leq \frac{1 + \delta_2}{1 + \delta_1}$, then

\[
|f(x) - x_2|_p \geq |f'(x_2)|_p |x - x_2|_p
\]

The proof is similar to the proof of Lemma 3.8.

**Theorem 3.11.** Let the conditions of Theorem 3.7 be satisfied. Then

\[
A(x_1) = C_p \setminus \{\hat{x}, x_2\}.
\]
Proof. Since \( \hat{x} \) does not belong the domain of \( f \) and \( x_2 \) is a fixed point of one, therefore \( \hat{x} , x_2 \not\in A(x_1) \). According to Theorem 3.7 it suffices to prove that \( S_{1+\delta_2}(x_1) \setminus \{ \hat{x} , x_2 \} \subset A(x_1) \). Keeping in mind (3.15), \( x_1 - \hat{x} = \frac{bx_1 + c}{b} \) and \( \gamma = x - x_1 \) the equality (3.11) yields

\[
|f(x) - x_1|_p = \frac{1 + \delta_2}{1 + \delta_1} \cdot |x - x_1|_p. \tag{3.16}
\]

From \( |x - x_1|_p = 1 + \delta_2 \) we get

\[
|f(x) - x_1|_p = \frac{(1 + \delta_2)^2}{(1 + \delta_1)|x - \hat{x}|_p}. \tag{3.17}
\]

If the right side of (3.17) non equal to \( 1 + \delta_2 \) then according to Theorem 3.7 we infer that \( f(x) \in A(x_1) \), hence \( x \in A(x_1) \). One remains to consider a case when the right side of (3.17) is equal to \( 1 + \delta_2 \), i.e \( |f(x) - x_1|_p = 1 + \delta_2 \). In this case we find

\[
|x - \hat{x}|_p = \frac{1 + \delta_2}{1 + \delta_1}, \tag{3.18}
\]

From this we get

\[
|x - x_2|_p = |(x - \hat{x}) + (x - \hat{x})|_p \leq \frac{1 + \delta_2}{1 + \delta_1}, \tag{3.19}
\]

here it has been used the equality \( |\hat{x} - x_2|_p = \frac{1 + \delta_2}{1 + \delta_1} \). (see the proof of Lemma 3.6). According to Corollary 3.10 there exists a positive integer \( n_0 \) such that

\[
|f^{n_0}(x) - x_2|_p > \frac{1 + \delta_2}{1 + \delta_1},
\]

whence

\[
|f^{n_0}(x) - \hat{x}|_p = |(f^{n_0}(x) - x_2) + (x_2 - \hat{x})|_p > \frac{1 + \delta_2}{1 + \delta_1}, \tag{3.20}
\]

here we have used 3.1) property of the norm. From (3.16) we obtain

\[
|f^2(x) - x_1|_p = \frac{1 + \delta_2}{1 + \delta_1} \cdot |f(x) - x_1|_p = \frac{(1 + \delta_2)^2}{1 + \delta_1} \cdot \frac{1}{|f(x) - \hat{x}|_p}. \tag{3.21}
\]

Now we estimate \( |f(x) - \hat{x}|_p \):

\[
|f(x) - \hat{x}|_p = |(f(x) - x_1) + (x_1 - \hat{x})|_p \leq 1 + \delta_2.
\]

Hence the equality (3.21) implies

\[
|f^2(x) - x_1|_p \geq \frac{1 + \delta_2}{1 + \delta_1}. \tag{3.22}
\]
If the left side of (3.22) is not equal to $1 + \delta_2$, then $f^2(x) \in A(x_1)$, so $x \in A(x_1)$. If $|f^2(x) - x_1|_p = 1 + \delta_2$, then repeating this argument till a number $k$ such that

$$|f^k(x) - x_1|_p \neq 1 + \delta_2,$$

we conclude that $x \in A(x_1)$. Now we show that such number $k$ does exist. Let us assume that for all $m \leq n_0$ the following equality

$$|f^m(x) - x_1|_p = 1 + \delta_2,$$

is valid. Otherwise nothing to prove. Put $k = n_0 + 1$. According to (3.16),(3.20) and (3.24) we have

$$|f^{n_0+1}(x) - x_1|_p = 1 + \delta_2 \cdot \frac{|f^{n_0}(x) - x_1|_p}{1 + \delta_1} |f^{n_0}(x) - \hat{x}|_p < 1 + \delta_2,$$

i.e. (3.23) is valid. This completes the proof.

4 Dynamical system $f(x) = \frac{x}{bx + c}$ in $\mathbb{Q}_p$ is not ergodic

In this section we consider a dynamical system

$$f(x) = \frac{x}{bx + c}, \quad c \neq 0, \quad b, c \in \mathbb{Q}_p$$

in $\mathbb{Q}_p$. It is easy to that $x = 0$ is a fixed point for (4.1). A question about ergodicity of the considered system arises when the fixed point $x = 0$ is indifferent. This lead us to the condition $|c|_p = 1$. From condition (3.5) we find that $|b|_p < |c|_p$. Then it is not difficult to check that $f(S_1(0)) = S_1(0)$. From now we consider the dynamical system (4.1) on the sphere $S_1(0)$.

**Lemma 4.1.** For every ball $U_{p^{-l}}(a) \subset S_1(0)$ then the following equality holds

$$f(U_{p^{-l}}(a)) = U_{p^{-l}}(f(a))$$

**Proof.** From inclusion $U_{p^{-l}}(a) \subset S_1(0)$ we have $|a|_p = 1$. Let $|x - a|_p \leq p^{-l}$, then

$$|f(x) - f(a)|_p = \frac{|c|_p|x - a|_p}{|bx + c|_p|ba + c|_p} = |x - a|_p \leq p^{-l},$$

here we have used the equality $|bx + c|_p = 1$, which follows from $|b|_p < 1$. Lemma is proved.

Consider a measurable space $(S_1(0), \mathcal{B})$, here $\mathcal{B}$ is the algebra of generated by clopen subsets of $S_1(0)$. Every element of $\mathcal{B}$ is a union of some balls $U_{p^{-l}}(a)$. A measure $\mu : \mathcal{B} \to \mathbb{R}$ is said to be Haar measure if it is defined by

$$\mu(U_{p^{-l}}(a)) = \frac{1}{q^l},$$
here \( q \) is a prime number.

From lemma 4.1 we conclude that \( f \) preserves the measure \( \mu \), i.e.

\[
\mu(f(U_{p^{-1}}(a))) = \mu(U_{p^{-1}}(a)) \tag{4.2}
\]

Recall a dynamical system \((X, T, \lambda)\), where \( T : X \to X \) is a measure preserving transformation, is called ergodic if for every invariant set \( A \), i.e. \( T(A) = A \) the equalities \( \lambda(A) = 0 \) or \( \lambda(A) = 1 \) are valid. (see, [28])

**Proposition 4.2.** If there is some number \( N \in \mathbb{N} \) such that \(|f^N(a) - a| < 1\) for some \( a \in S_1(0) \) then the dynamical system (4.1) is not ergodic on \( S_1(0) \) with respect to the Haar measure.

**Proof.** Denote \( N = \min\{n \in \mathbb{N} : |f^n(a) - a| < 1 \text{ for some } a \in S_1(0)\} \). Because of the discreteness of the \( p \)-adic metric we can assume that \(|f^N(a) - a| \leq p^{-l}\) for some positive integer \( l \in \mathbb{N} \). Put

\[
A = \bigcup_{k=0}^{N-1} U_{p^{-1}}(f^k(a)).
\]

Then from Lemma 4.1 we find that \( f(A) = A \). It is clear that \( \mu(A) \neq 0 \) and \( \mu(S_1(0) \setminus A) \neq 0 \), hence \( f \) is not ergodic. This completes the proof.

**Corollary 4.3.** If \( p = 2 \) then the dynamical system (4.1) is not ergodic on \( S_1(0) \) with respect to the Haar measure.

**Proof.** Form the condition \(|c|_2 = 1\) using the property 3.2) of the norm we get \(|1 - c|_2 \leq \frac{1}{2}\). Then we have

\[
|f(a) - a|_2 = |1 - c - ba|_2 \leq \frac{1}{2},
\]

since \(|ba|_2 \leq \frac{1}{2}\). Hence the set \( A = U_{2^{-1}}(a) \) is invariant with respect to \( f \).

On the other hand \( \mu(A) = 1/2 \), that means \( f \) is not ergodic. The corollary is proved.

**Lemma 4.4.** For every \( N \in \mathbb{N} \) the following equalities hold

\[
f^2(x) = \frac{x}{bx(1 + c) + c^2}, \quad f^3(x) = \frac{x}{bx(1 + c^2 + c^3) + c^4}, \quad f^N(x) = \frac{x}{bx(1 + S_N) + c^{2N-1}},
\]

where \( S_N = \sum_{m=0}^{N-3} c^{2m(2^{N-2} - 1)} \).

The proof immediately follows from induction method.

From this Lemma we can prove the following

**Corollary 4.5.** For every integer \( N \in \mathbb{N} \) the following equalities hold

\[
f^{-1}(x) = -\frac{x}{bx/c + 1/c}, \quad f^{-2}(x) = -\frac{x}{bx/c(1 + 1/c) + 1/c^2},
\]

\[
f^{-3}(x) = -\frac{x}{bx/c(1 + 1/c^2 + 1/c^3) + 1/c^4}, \quad f^{-N}(x) = -\frac{x}{bx/c(1 + N) + 1/c^{2N-1}},
\]
where \( Z_N = \sum_{m=0}^{N-3} 1/c^{2m(2N-2-1)} \).

Using Lemma 4.4 consider the difference

\[
|f^N(x) - x|_p = |1 - bx(1 + S_N) - c^{2N-1}|_p
\]

form this we conclude that the condition \(|f^N(x) - x|_p = 1\) for all \( N \in \mathbb{N} \) and \( x \in S_1(0) \) is equivalent to the equality

\[
|1 - c^{2N-1}|_p = 1, \quad \forall N \in \mathbb{N}. \tag{4.3}
\]

Using Corollary 4.5 and analogous argument as above we can obtain that the condition \(|f^{-N}(x) - x|_p = 1\) for all \( N \in \mathbb{N} \) and \( x \in S_1(0) \) is also equivalent to the equality (4.3).

From (2.1) we have that every element \( c \in S_1(0) \) is represented in the form

\[
c = a_0 + a_1p + a_2p^2 + ..., \]

where \( a_0 \neq 0, \ a_k \in \{0, 1, ..., p - 1\}, \ k \in \mathbb{N} \). Then it is easy to see the condition (4.3) is equivalent to the following one

\[
a_0^{2N-1} \not\equiv 1(\mod p), \quad \forall N \in \mathbb{N}
\]

This condition is satisfied for example on \( p = 7 \) with \( a_0 = 2 \).

Let us assume that the condition (4.2) is satisfied. Then for all \( n \in \mathbb{N} \), we get

\[
U_{p^{-l}}(f^{-n}(a)) \cap U_{p^{-l}}(a) = \emptyset
\]

for all \( a \in S_1(0), \ l \in \mathbb{N} \). Then according to Theorem 1.5[28] we conclude that the set the dynamical system (4.1) is not ergodic. So we have proved the following

**Theorem 4.6.** The dynamical system (4.1) is not ergodic on \( S_1(0) \) with respect to the Haar measure.

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**References**

[1] V.A.Avetisov, A.H. Bikulov, S.V.Kozyrev and V.A.Osipov, *p*-adic modes of ultrametric diffusion constrained by hierarchical energy landscapes, J. Phys. A: Math.Gen. 35(2002), 177-189.

[2] S.Albeverio, A.Khrennikov, and P.E.Koloeden, *Memory retrieval as a p-adic dynamical system*, BioSys. 49(1999), 105-115.
S. Albeverio, A. Khrennikov, B. Tirozzi and S. De Smedt, \textit{p-adic dynamical systems}, Theor. Math. Phys. \textbf{114} (1998), 276-287.

I. Ya. Araf'eva, B. Dragovich, P. H. Frampton and I. V. Volovich, \textit{Wave function of the universe and p-adic gravity}, Int. J. Mod. Phys. A. \textbf{6} (1991) 4341-4358.

R. Benedetto, \textit{Hyperbolic maps in p-adic dynamics}, Ergod. Th. & Dynam. Sys. \textbf{21} (2001), 1-11.

R. Benedetto, \textit{p-Adic dynamics and Sullivan's no wandering domains theorem}, Composito Math. \textbf{122} (2000), 281-298.

G. Call and J. Silverman, \textit{Canonical height on varieties with morphisms}, Composito Math. \textbf{89} (1993), 163-205.

D. Dubischer, V. M. Gundlach, A. Khrennikov and O. Steinkamp, \textit{Attractors of random dynamical system over p-adic numbers and a model of 'noisy' cognitive process}, Physica D. \textbf{130} (1999), 1-12.

N. N. Ganikhodjaev, F. M. Mukhamedov and U. A. Rozikov, \textit{Existence of phase transition for the Potts p-adic model on the set \mathbb{Z}}, Theor. Math. Phys. \textbf{130} (2002), 425-431.

F. Q. Gouvea, \textit{p-adic numbers}, Springer, Berlin 1991.

V. M. Gundlach, A. Khrennikov and K. O. Lindahl, \textit{On ergodic behavior of p-adic dynamical systems}, Infin. Dimen. Anal. Quantum Probab. Relat. Top. \textbf{4} (2001), 569-577.

P. G. O. Freund, and E. Witten, \textit{Adelic string amplitudes}, Phys. Lett. \textbf{B199} (1987) 191-194.

M. Herman and J.-C. Yoccoz, \textit{Generalizations of some theorems of small divisors to non-Archimedean fields}, In: Geometric Dynamics (Rio de Janeiro, 1981), Lec. Notes in Math. 1007, Springer, Berlin, 1983, pp. 408-447.

N. Koblitz, \textit{p-adic numbers, p-adic analysis and zeta-function} Springer, Berlin, 1977.

A. Yu. Khrennikov, \textit{p-adic quantum mechanics with p-adic valued functions}, J. Math. Phys. \textbf{32} (1991), 932-936.

A. Yu. Khrennikov, \textit{p-adic Valued Distributions in Mathematical Physics} Kluwer, Netherlands, 1994.

A. Yu. Khrennikov, \textit{Non-Archimedean analysis: quantum paradoxes, dynamical systems and biological models}, Kluwer, Netherlands, 1997.

A. Yu. Khrennikov, \textit{The description of Grain's functioning by the p-adic dynamical system}, Preprint Ruhr Univ. Bochum, SFB-237, N. 355.
[19] J.Lubin, *Nonarchimedean dynamical systems*, Composito Math. 94(3)(1994), 321-346.

[20] E.Marinary and G.Parisi, *On the $p$-adic five point function*, Phys.Lett. 203B(1988) 52-56.

[21] T.Pezda, *Polynomial cycles in certain local domains*, Acta Arith. 66 (1994), 11-22.

[22] H.-O.Peitgen, H.Jungers and D.Saupe, *Chaos Fractals*, Springer, Heidelberg-New York, 1992.

[23] A.M.Robert, *A course of $p$-adic analysis*, Springer, New York, 2000.

[24] E.Thiran, D.Verstegen and J.Weters, *$p$-adic dynamics*, J.Stat. Phys. 54(3/4)(1989), 893-913.

[25] V.S.Vladimirov and I.V.Volovich and E.I.Zelenov, *$p$-adic Analysis and Mathematical Physics*, World Scientific, Singapur, 1994.

[26] I.V.Volovich, *Number theory as the ultimate physical theory*, Preprint, TH, 4781/87.

[27] I.V.Volovich, *$p$-adic strings*, Class. Quantum Grav. 4(1987) L83-L87.

[28] P.Walters, *An introduction to ergodic theory*, Springer, Berlin-Heidelberg-New York, 1982.

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