Reinforcement Learning in Deep Structured Teams: Initial Results with Finite and Infinite Valued Features

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Abstract—In this paper, we consider Markov chain and linear quadratic models for deep structured teams with discounted and time-average cost functions under two non-classical information structures, namely, deep state sharing and no sharing. In deep structured teams, agents are coupled in dynamics and cost functions through deep state, where deep state refers to a set of orthogonal linear regressions of the states. In this article, we consider a homogeneous linear regression for Markov chain models (i.e., empirical distribution of states) and a few orthonormal linear regressions for linear quadratic models (i.e., weighted average of states). Some planning algorithms are developed for the case when the model is known, and some reinforcement learning algorithms are proposed for the case when the model is not known completely. The convergence of two model-free (reinforcement learning) algorithms, one for Markov chain models and one for linear quadratic models, is established. The results are then applied to a smart grid.

I. INTRODUCTION

Recently, there has been a surge of interest in the application of reinforcement learning algorithms in networked control systems such as social networks, swarm robotics, smart grids and transportation networks. This type of systems often consist of many interconnected agents (decision makers) that wish to perform a common task with limited resources in terms of computation, information and knowledge.

When every agent has perfect information and complete knowledge of the entire network, the optimal solution can be computed by dynamic programming decomposition. The computational complexity of solving this dynamic programming problem increases with the number of agents (the so-called "curse of dimensionality"). The above complexity is drastically exacerbated when the information is imperfect. For the case of decentralized information structure with finite spaces, on the other hand, the computational complexity of the resultant dynamic program is NEXP [1], and for infinite spaces with linear quadratic model, the optimization problem is non-convex [2]. In addition, the underlying network model is not always known completely; this lack of knowledge further increases the above complexity. Subsequently, it is very difficult to solve a large-scale control problem with imperfect information of agents and incomplete knowledge of the network.

As an attempt to address the above shortcomings, we propose several reinforcement learning algorithms for a class of multi-agent control problems called deep structured teams, introduced in [3], [4], [5], [6], [7], [8], where the interactions between the decision makers are modelled by a number of linear regressions (weighted averages) of states and actions, which is similar to the interactions between the neurons of a feed-forward deep neural network. In general, deep structured teams are decentralized control systems whose solutions are amenable to the size of the problem. More precisely, the complexity of finding an optimal solution of Markov chain deep structured team is polynomial (rather than exponential) with respect to the number of agents and is linear (rather than exponential) with respect to the control horizon [3]. On the other hand, the complexity of a linear quadratic deep structured team is independent of the number of agents [4]. It is worth highlighting that deep structured teams are the generalization of the notion of mean-field teams initially introduced in [9] and showcased in [10], [11], [12], [13], [14], [15], [16], [17].

The remainder of the paper is organized as follows. In Section III two models of deep structured teams are formulated, one with finite state and action spaces and the other one with infinite spaces. In Section IV different methods are discussed for solving the planning problem for the case when the model is known. In Section V some reinforcement learning methods are presented for the case when the model is not known. An example of a smart grid is provided in Section VI to verify the effectiveness of the proposed algorithms, and the paper is then concluded in Section VII.

II. PROBLEM FORMULATION

In this paper, \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \) are the sets of natural numbers, real numbers and non-negative real numbers, respectively. For any \( k, t \in \mathbb{N} \), \( \mathbb{N}_k \) denotes the finite set \( \{1, \ldots, k\} \) and \( x_{1:t} \) denotes the vector \( (x_1, \ldots, x_t) \). For any vectors \( x, y, z \), short-hand notation \( \text{vec}(x, y, z) \) denotes the vector \( [x^\top, y^\top, z^\top]^\top \). For any matrices \( A, B, C \) that have the same number of columns, \( \text{row}(A, B, C) \) denotes the matrix \( [A^\top, B^\top, C^\top]^\top \). Given any square matrices \( A, B, C, \) \( \text{diag}(A, B, C) \) denotes the block diagonal matrix with matrices \( A, B, C \) on its main diagonal. In addition, \( \mathbb{P}(\cdot) \) is the probability of a random variable, \( \mathbb{E}[\cdot] \) is the expectation of an event, \( \mathbb{I}(\cdot) \) is the indicator function of a set, \( \text{Tr}(\cdot) \) is the trace of a matrix, \( \|\cdot\| \) is the infinity norm of a vector, and \( |\cdot| \) is the absolute value of a real number or the cardinality of a set. For any \( n \in \mathbb{N} \), \( \text{binopdf}(n, p) \) denotes the binomial probability distribution of \( n \) trials with success probability \( p \in [0, 1] \). For any finite set \( \mathcal{X} \), the space of probability measures on \( \mathcal{X} \) is denoted by: \( \mathcal{P}(\mathcal{X}) = \{\mathcal{A} \mid (a_1, \ldots, a_{|\mathcal{X}|}) | a_i \in [0, 1], i \in \mathbb{N}_k \} \).
For feature $j \in \mathbb{N}_z$, define the following linear regressions:

$$x_{t+1}^j = \frac{1}{n} \sum_{i=1}^{n} \alpha_i^j x_t^i + \sum_{j=1}^{n} \alpha_i^{j} u_t^i,$$

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In practice, there are various methods to share the deep state among agents. For example, one may use a central distributor (such as a cloud-based server) to collect the states, compute the deep state and broadcast it among the agents. Alternatively, one can utilize distributed techniques such as consensus-based algorithms based on the local interaction of each agent with its neighbours. However, it is sometimes infeasible to share the deep state, specially when the number of agents is very large. In such a case, we consider another information structure called no sharing (NS), where

\[ u_t^i \sim g_t \cdot| x_t^i ) \in \mathcal{P}(U), \quad \text{(Model I: NS)} \]

and

\[ u_t^i \sim g_t^i \cdot| x_t^i ) \in \mathcal{P}(U). \quad \text{(Model II: NS)} \]

It is to be noted that the strategy of agent \( i \in \mathbb{N}_n \) in Model II depends on its index \( i \) as well as its local state \( x_t^i \).

B. Objective function

Let \( g := \{ g_t \}_{t=1}^{\infty} \) denote the control strategy for the system. Two performance indexes are considered, namely, discounted cost and time-average cost. In particular, given any discount factor \( \beta \in (0, 1) \), the objective function of Model I is defined as follows:

\[ J_{n,\beta}^{(I)}(g) := (1 - \beta) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{\infty} \beta^{t-1} c(x_t^i, u_t^i, D_t) \right]. \]

Similarly, the following discounted cost function is defined for Model II:

\[ J_{n,\beta}^{(II)}(g) := (1 - \beta) \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{\infty} \beta^{t-1} c(x_t^i, u_t^i, \bar{x}_t, \bar{u}_t) \right]. \]

In this article, standard mild assumptions are imposed on the model to ensure that the total cost is always bounded. As a result, it is possible to obtain the time-average cost function as the limit of the discounted cost function. More precisely, the following holds for Model I:

\[ J_{n,1}^{(I)}(g) := \lim_{\beta \to 1} J_{n,\beta}^{(I)}(g) \]

\[ = \lim_{\beta \to 1} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \beta^{t-1} c(x_t^i, u_t^i, D_t) \right] \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=1}^{\infty} \beta^{t-1} \right] \]

\[ = \lim_{T \to \infty} \sup \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \beta^{t-1} c(x_t^i, u_t^i, D_t) \right] \]

\[ = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} c(x_t^i, u_t^i, D_t) \right]. \]

Analogously, one has the following for Model II:

\[ J_{n,1}^{(II)}(g) := \lim_{\beta \to 1} J_{n,\beta}^{(II)}(g) \]

\[ = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} \beta^{t-1} c(x_t^i, u_t^i, \bar{x}_t, \bar{u}_t) \right] \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=1}^{\infty} \beta^{t-1} \right] \]

\[ = \lim_{T \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{t=1}^{T} c(x_t^i, u_t^i, \bar{x}_t, \bar{u}_t) \right]. \]

It is to be noted that there is no general theory for infinite-space average cost functions.

C. Problem statement

Four problems are investigated.

Problem 1 (Planning with DSS). Given dynamics \( \{ f, A, B, \bar{A}, \bar{B} \} \), cost function \( \{ c, Q, R, \bar{R} \} \), number of agents \( n \), probability distribution function of the initial states \( P_X \), probability distribution function of noises \( P_W \) and discount factor \( \beta \in (0, 1) \), find the optimal strategy \( g^* \) such that under DSS information structure for every strategy \( g \):

\[ J_{n,\beta}^{(I)}(g^*) \leq J_{n,\beta}^{(I)}(g), \quad \ell \in \{ I, II \}. \]

Problem 2 (Planning with NS). Given dynamics \( \{ f, A, B, \bar{A}, \bar{B} \} \), cost function \( \{ c, Q, R, \bar{Q}, \bar{R} \} \), number of agents \( n \), probability distribution function of the initial states \( P_X \), probability distribution function of noises \( P_W \) and discount factor \( \beta \in (0, 1) \), find a sub-optimal strategy \( g_N \) such that under NS information structure for every strategy \( g \):

\[ J_{n,\beta}^{(I)}(g_N) \leq J_{n,\beta}^{(I)}(g) + \varepsilon(n), \quad \ell \in \{ I, II \}, \]

where \( \lim_{n \to \infty} \varepsilon(n) = 0 \).

Problem 3 (Reinforcement learning with DSS). Given state and action spaces \( \{ X, U \} \) and discount factor \( \beta \in (0, 1) \), develop a reinforcement learning algorithm whose performance under the learned strategy \( g_k \), \( k \in \mathbb{N} \), converges to that under the optimal strategy \( g^* \), as the number of iterations \( k \) increases.

Problem 4 (Reinforcement learning with NS). Given state and action spaces \( \{ X, U \} \) and discount factor \( \beta \in (0, 1) \), develop a reinforcement learning algorithm whose performance under the learned strategy \( g_k \), \( k \in \mathbb{N} \), converges to that under the sub-optimal strategy \( g_N \), as the number of iterations \( k \) increases.

III. MAIN RESULTS FOR PROBLEMS 1 AND 2

Following [3], we define a local control law \( \gamma_t : X \to \mathcal{P}(U) \) for Model I such that under DSS information structure,

\[ \gamma_t := g_t(\cdot, d_t), \quad (6) \]

and under NS information structure,

\[ \gamma_t := g_t(\cdot). \quad (7) \]

From the definition of DSS and NS strategies given in Subsection 1.A, and the change of variable introduced in \( \{ g \} \) and \( \{ \gamma \} \), it follows that the action of agent \( i \) is selected randomly with respect to the probability mass function \( \gamma_t(x_t^i) \), i.e.

\[ u_t^i \sim \gamma_t(x_t^i). \]

Lemma 1. Given any \( n \in \mathbb{N}, \; d_t \in \mathcal{E}_n(X) \) and \( \gamma_t : X \to \mathcal{P}(U) \) at time \( t \in \mathbb{N} \), the following relations hold:

\[ nD_t(x, u) \sim \binopdf(nD_t(x), \gamma_t(x)(u), \; x \in X, \; u \in U, \]

and

\[ \mathbb{E} [D_t(x, u)] = d_t(x) \gamma_t(x)(u), \quad x \in X, \; u \in U. \]
Proof. The proof directly follows from (1) such that
\[ n\mathbf{D}_x(x,u) = \sum_{i=1}^{n} \mathbf{1}(x_i = x)\mathbf{1}(u_i = u), \]
where the above equation consists of \( nd(x) \) independent binary random variables with success probability \( \gamma_t(x)(u) \).

To ease the exposition of deep Chapman-Kolmogorov equation introduced in [3], define
\[ \mathbf{T} = \prod_{t=1}^{\infty} \mathbf{P}(x_{t+1} \mid x_t, u, \gamma_t, d_t). \]
Given any \( x, x' \in \mathcal{X}, \gamma : \mathcal{X} \to \mathcal{P}(\mathcal{U}) \) and \( d \in \mathcal{E}_n(\mathcal{X}) \), define the vector-valued function \( \phi(x', x, \gamma, d) \) \( (\mathcal{X}, \mathcal{P}(\mathcal{U}), (\mathcal{X}, \mathcal{U})) \) such that
\[ \phi(x', x, \gamma, d) := \delta_0(\mathbf{n}(d(x))) + \mathbf{1}(d(x) > 0) \text{binopdf}(\mathbf{n}(d(x)), \mathbf{T}(x', x, \gamma, d)), \]
where \( \delta_0(\mathbf{n}(d(x))) \) is the Dirac measure with the domain set \( \{0, 1, \ldots, nd(x)\} \) and a unit mass concentrated at zero. In addition, let \( \phi(x', \gamma, d) \in \mathcal{P}(\{0, 1, \ldots, n\}) \) be the convolution function of \( \phi(x', x, \gamma, d) \) over all states \( x \in \mathcal{X} \) := \( \{s_1, \ldots, s_{|\mathcal{X}|}\} \), i.e.,
\[ \bar{\phi}(x', \gamma, d) := \phi(x', s_1, \gamma, d) * \ldots * \phi(x', s_{|\mathcal{X}|}, \gamma, d), \]
where \( \bar{\phi}(x', \gamma, d) \) is a vector of size \( n + 1 \).

**Lemma 2** (Deep Chapman-Kolmogorov equation [3]).

Given \( dt \) and \( \gamma_t \) at time \( t \in \mathbb{N} \), the transition probability matrix of the deep state can be computed as follows: for any \( x' \in \mathcal{X} \) and \( y \in \{0, 1, \ldots, n\} \).
\[ \mathbf{P}(x_{t+1} = y \mid d_t, \gamma_t) = \bar{\phi}(x', \gamma_t, d_t)(y+1). \]

Proof. The proof follows from [2] and the fact that the probability distribution of the sum of independent random variables can be described by the convolution of their individual probability distributions. See [3, Theorem 3] and [18, Theorem 1] for more details.

We now define a non-standard Bellman equation for Model I such that for every \( d_t \in \mathcal{E}_n(\mathcal{X}) \) and \( t \in \mathbb{N} \):
\[ V(d_t) = \min_{\gamma_t} \bar{\mathbf{c}}(d_t, \gamma_t) + \beta \mathbb{E}_x[V(d_{t+1} \mid d_t, \gamma_t)], \]
where
\[ \bar{\mathbf{c}}(d_t, \gamma_t) := \mathbb{E} \sum_{i=1}^{n} c(x_i, u_i, d_t) \mid d_t, \gamma_t \]
\[ = \mathbb{E} \sum_{x,u} c(x, u, \mathbf{D}_t) \mid d_t, \gamma_t \]
\[ = \sum_{x,u} \sum_{d} c(x, u, \mathbf{D}_t) \mathbf{D}_t(x, u) \mid d, \gamma_t \]
control systems and in [5] to dynamic games. From the orthogonality induced by the above gauge transformation, the dynamics (4) and cost function (5) can be decomposed into $n$ identical linear quadratic problems with states and actions $(\Delta x_i^t, \Delta u_i^t)$, $i \in \mathbb{N}_n$, and one linear quadratic problem with state and action $(\bar{x}_i, \bar{u}_i)$. The deep Riccati equation (7) gives the solution of the above problems. For the special case of weakly coupled systems, the Riccati equation associated with the state and action $(\bar{x}_i, \bar{u}_i)$ decomposes into $z$ smaller Riccati equations given by (10).

Remark 4 (Extended cost). The main results of this paper naturally extend to cost functions with post-decision states, i.e., $c(x_i^t, u_i^t, \bar{D}_t, x_{i+1}^t, d_{i+1})$ and $c(x_i^t, u_i^t, \bar{x}_i, \bar{u}_i, x_{i+1}^t, \bar{x}_{i+1})$. In addition, it is straightforward to consider cross terms between the states and actions in (5).

Remark 5 (Generalization). The main results of this paper naturally generalize to state-dependent discount factor and finite-horizon cost functions as well as multiple subpopulations with partially deep state sharing information structure [4] and intermittent deep state sharing [12].

A. NS information structure: mean-field approximation

When deep state is not observed, the above solutions are not practical. To overcome this shortcoming, one can approximate the deep state by mean field approximation [19], where the strong law of large numbers provides a simple asymptotic estimate.

Let $\mathfrak{M}_t \in \mathcal{P}(\mathcal{X} \times \mathcal{U})$ and $m_t \in \mathcal{P}(\mathcal{X})$ denote the mean-field approximations of $\mathfrak{D}_t \in \mathcal{E}_n(\mathcal{X} \times \mathcal{U})$ and $d_t \in \mathcal{E}_n(\mathcal{X})$, respectively, i.e.,

\[ M_t(x, u) = \lim_{n \to \infty} \mathfrak{D}_t(x, u) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1(x_i^t = x) 1(u_i^t = u), \]

\[ m_t(x) = \lim_{n \to \infty} d_t(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1(x_i^t = x). \]

We now define a non-standard Bellman equation for Model I such that for every $m_t \in \mathcal{P}(\mathcal{X})$ and $t \in \mathbb{N}$:

\[ \hat{V}(m_t) = \min_{\gamma_t} \left( \tilde{c}(m_t, \gamma_t) + \beta \mathbb{E}[\hat{V}(m_{t+1}) | m_t, \gamma_t] \right) = \min_{\gamma_t} \left( \tilde{c}(m_t, \gamma_t) + \beta \hat{V} (f(m_t, \gamma_t)) \right), \]

(12)

where

\[ \mathfrak{M}_t(x, u) = m_t(x) \gamma_t(x)(u), \quad x \in \mathcal{X}, u \in \mathcal{U}, \]

\[ \tilde{c}(m_t, \gamma_t) := \sum_{x,u} c(x, u, \mathfrak{M}_t) m_t(x) \gamma_t(x)(u), \]

and for every $x' \in \mathcal{X}$,

\[ m_{t+1}(x') = \sum_{x \in \mathcal{X}, u \in \mathcal{U}} m_t(x) \gamma_t(x)(u) \mathbb{P}(x' | x, u, \mathfrak{M}) =: \hat{f}(m_t, \gamma_t)(x'). \]

Assumption 2. The initial states $\bar{x}_i$ are i.i.d. random variables with probability mass function $P_{\bar{x}}$.

Assumption 3. For any $x, x' \in \mathcal{X}$, $u \in \mathcal{U}$ and $\mathfrak{M}_1, \mathfrak{M}_2 \in \mathcal{P}(\mathcal{X} \times \mathcal{U})$, there exist positive real constants $H^p$ and $H^c$ (that do not depend on $n$) such that

\[ |\mathbb{P}(x' | x, \mathfrak{M}_1) - \mathbb{P}(x' | x, \mathfrak{M}_2)| \leq H^p \| \mathfrak{M}_1 - \mathfrak{M}_2 \|, \]

\[ |c(x, u, \mathfrak{M}_1) - c(x, u, \mathfrak{M}_2)| \leq H^c \| \mathfrak{M}_1 - \mathfrak{M}_2 \|. \]

The above assumption is mild because any polynomial function of $\mathfrak{M} \in \mathcal{P}(\mathcal{X} \times \mathcal{U})$ is Lipschitz on $\mathfrak{M}$ since $\mathcal{P}(\mathcal{X} \times \mathcal{U})$ is a confined space.

Theorem 3. Let Assumptions 2 and 3 hold. Let also $\tilde{\psi}(m)$, $m \in \mathcal{P}(\mathcal{X})$, be a minimizer of the right-hand side of equation (12). The following strategy is a solution for Problem 2 with Model I:

\[ \hat{u}_t^i \sim \tilde{\psi}(x_t^i, m_t) := \psi(m_t)(x_t^i), \]

where $m_{t+1} = \hat{f}(m_t, \gamma_t)$ with $m_1 = P_{\bar{x}}$.

Proof. The proof follows from the continuity and boundedness properties described in Assumption 3 and the strong law of large numbers. For more details, see [3, Theorem 4].

For Model II, we make the following assumption.

Assumption 4. All matrices in the agent dynamics (4) and cost function (5), as well as the covariance matrices of the initial states and driving noises are uniformly bounded with respect to $n \in \mathbb{N}$. In addition, initial states and driving noises are independent random variables across agents at any time instant $t \in \mathbb{N}$.

For coupled dynamics, an additional stability condition is required to ensure that the proposed infinite-population strategy is stable under NS information structure.

Assumption 5. Matrix $A + \tilde{B} \text{diag}(\theta^1, \ldots, \theta^z)$ is Hurwitz.

It is to be noted that Assumption 5 holds for decoupled dynamics, where $\tilde{A}^i$ and $\tilde{B}^j$ are zero, $\forall j \in \mathbb{N}_z$. Define mean field $\bar{m}_t := \text{vec}(m_t^1, \ldots, m_t^z)$ such that

\[ \bar{m}_{t+1} = (\bar{A} + \tilde{B}^* \gamma_t) \bar{m}_t, \]

where $\bar{m}_1 = E[\bar{x}_1]$.

Theorem 4. Let Assumptions 1, 4 and 5 hold. The following strategy is a solution of Problem 2 with Model II

\[ \bar{u}_t^i = \theta^* x_t^i + \sum_{j=1}^{z} \alpha^{i,j} (\tilde{B}^j \gamma_t - \theta^* m_t^j), \quad t \in \mathbb{N}. \]

Also, for the special case of weakly coupled matrices

\[ \bar{u}_t^i = \theta^* x_t^i + \sum_{j=1}^{z} \alpha^{i,j} (\tilde{B}^j \gamma_t - \theta^* m_t^j). \]

Proof. The proof follows from Assumptions 1 and 5, Theorem 2 the strong law of large numbers, and the fact that the optimal strategy and cost function are bounded and
continuous with respect to \( n \). For more details, see [4, Theorem 3] and [9, Theorem 3.8].

Remark 6. When the system matrices are independent of the number of agents \( n \), the solution of the deep Riccati equation is also independent of \( n \) for the risk-neutral cost minimization [9, Chapter 3] and minmax optimization [12]. However, this is a rather special case, and more generally, such as in risk-sensitive cost minimization problem [4] and linear quadratic game [5], the solution depends on \( n \).

### B. Numerical solutions

To numerically solve the Bellman equations in Theorems 1 and 3, one can quantify the space of probability measures on the local state and local action (i.e. \( \mathcal{P}(X) \) and \( \mathcal{P}(U) \)), and use standard methods such as value iteration, policy iteration and linear programming, according to [20].

Remark 7. In practice, value iteration, policy iteration and linear programming suffer from the curse of dimensionality when the size of state and action spaces is large, unless some special structures are imposed such as a linear quadratic model. In Section IV, we discuss more practical approaches that provide more efficient solutions at the cost of losing the optimality. It is worth mentioning that the convergence in policy space is often faster than that in value space.

For Theorems 3 and 4, one can use different techniques to solve the algebraic deep Riccati equation \( \bar{\psi} \). Some common approaches include invariant subspaces such as Schur method, Newton-type iteration and Krylov subspaces for large-scale problems. See [21] for more advanced methods.

### C. Decentralized implementation

**Model I with DSS:** Every agent \( i \in \mathbb{N}_n \) can independently solve the dynamic program (8) upon the observation of deep state. Since the resultant optimization problem is the same for all agents, the agents commonly choose the control law \( \psi^* \). Then, agent \( i \) chooses its action \( u^i_t \) based on its local (private) state \( x^i_t \) and deep state \( d_t \) with respect to the probability distribution function \( \psi^*(d_t|x^i_t) = \gamma^i_t(x^i_t) \) at any time \( t \in \mathbb{N} \) (see Theorem 1 for more details).

**Model I with NS:** For NS information structure, deep state is replaced by mean field and the resultant solution asymptotically converges to the optimal solution as the number of agents goes to infinity.

### IV. Main results for problems 3 and 4

In the previous section, it was assumed that the model is completely known. In this section, we provide various techniques to approximate the proposed dynamic programs for the case when the model is not known completely.

In general, there are two fundamental approaches to learning the solution of the proposed dynamic programs. The first one is called model-based approach which uses supervised learning techniques to find the parameters of the models described in Section II and then solve the planning problems presented in Section III. In short, this approach is an indirect method that obtains the solution by constructing a model. The second approach, however, finds the solution directly without identifying the model. This approach is called reinforcement learning (RL) (also called model-free or approximate dynamic program).

The advantage of the model-based approaches is that they are more intuitive because they not only provide a solution but also construct a model. However, for large-scale problems, it is more efficient to use reinforcement learning methods as they directly search for the solution. In general, there are two types of RL algorithms: off-line and on-line. In the former type, the exploration step is not a major concern as all states and actions can be visited sufficiently often, given a rich set of data and/or simulator. In the latter type, however, it is critical to explore the model in such a way that all states and actions are visited sufficiently often.

#### A. Model I

In what follows, we briefly present the main idea behind approximate value iteration, approximate policy iteration and approximate linear programming. For the approximate value iteration, consider a one-step ahead update of Bellman equation (8) as follows:

\[
\min_{\tilde{c}} \tilde{c}(d, \gamma) + \beta \sum_{d'} \mathbb{P}(d' | d, \gamma) \tilde{V}(d')
\]

(14)

where \( \gamma : \mathcal{X} \rightarrow \mathcal{P} ($U$) \) can be approximated as:

\[
\gamma(x)(u) \approx \frac{\theta(x, u)}{\sum_{u' \in U} \theta(x, u')}, \quad \theta(x, u) > 0, \ x \in \mathcal{X}, u \in \mathcal{U},
\]

such that \( \sum_{u' \in U} \gamma(x)(u) = 1 \). In particular, one can use a normalized exponential distribution with the following form:

\[
\gamma(x)(u) \approx \frac{e^{-\theta(x, u)}}{\sum_{u' \in U} e^{-\theta(x, u')}}.
\]

It is also possible to approximate the value function linearly using feature-based architecture as follows:

\[
\tilde{V}(d) \approx \tilde{V}(d) = \sum_{\ell=1}^{L} \tilde{r}_\ell \phi_\ell(d).
\]
Algorithm 1 Proposed Q-Learning Procedure

1: Set $Q_k(d, \gamma) = 0$ and $\eta_k(d, \gamma) = 1, \forall d \in E_n(X), \forall \gamma \in \mathcal{G}$.
2: At iteration $k \in \mathbb{N}$, given any deep state $d \in E_n(X)$ and any local law $\gamma \in \mathcal{G}$, update the corresponding Q-function and learning rate as follows:

$$
\begin{align*}
Q_{k+1}(d, \gamma) &= (1 - \eta_k(d, \gamma))Q_k(d, \gamma) \\
&\quad + \eta_k(d, \gamma)((c' + \beta \min_{\gamma' \in \mathcal{G}} Q_k(d', \gamma')), \\
\eta_{k+1}(d, \gamma) &= \lambda(d, \gamma, \gamma'),
\end{align*}
$$

where $c'$ is the immediate cost, $d'$ is the next deep state, and $\lambda$ determines proper learning rates $\eta_k \in [0, 1]$, $k \in \mathbb{N}$, such that $\sum_{k=1}^{\infty} \eta_k(d, \gamma) = \infty$ and $\sum_{k=1}^{\infty} (\eta_k(d, \gamma))^2 < \infty$.
3: Set $k = k + 1$, and go to step 2 until the algorithm terminates.

Moreover, one can use a multi-step ahead update in [18]. Similar function approximations can be used in policy iteration and linear programming. For more details, see [20], [22].

We present Algorithm 1 a (model-free) Q-learning algorithm, wherein attention is restricted to deterministic strategies, i.e., $\gamma : \mathcal{X} \rightarrow \mathcal{U}$ for Model I. Let $\mathcal{G}$ denote the set of mappings from the local state space $\mathcal{X}$ to the local action space $\mathcal{U}$. It is also possible to use various function approximations to provide a more practical algorithm, albeit at the cost of reduced performance.

Theorem 5. Suppose that every pair of deep state and local law $(d, \gamma) \in E_n(X) \times \mathcal{G}$ is visited infinitely often in Algorithm 1. Then, the following results hold:

(a) For any $(d, \gamma) \in E_n(X) \times \mathcal{G}$, the Q-function $Q_k(d, \gamma)$ converges to $Q^*(d, \gamma)$ with probability one, as $k \rightarrow \infty$.
(b) Let $g_k(\gamma, d) \in \arg \min_{\gamma \in \mathcal{G}} Q_k(d, \gamma)$ be a greedy strategy; then, the performance of $g_k$ converges to that of the optimal strategy $g^*$ given in Theorem 7 when attention is restricted to deterministic strategies.

Proof. The proof follows from the proof of convergence of the Q-learning algorithm and Theorem 1 which exploits the fact that the Bellman operator is contractive with respect to the infinity norm. See [23, Theorem 4] for more details on the convergence proof of the Q-learning algorithm.

Algorithm 2 Proposed Policy Gradient Procedure

1: Initialize the number of agents $n$, number of trajectories $L$, control horizon $T$, number of features $z$, feedback gains $(\theta_1, \theta_1)$, smoothing parameter $r$, and step size $\eta$.
2: At iteration $k \in \mathbb{N}$, run the following steps:

- initialize states $x_1 = \text{vec}(x_1^1, ..., x_t^1)$;
- given any agent $i \in \mathbb{N}$, use strategy (11) with perturbed feedback gains: $\theta_k + \eta \theta_k + \theta$, where $\theta \sim P_r$.
- compute the cost trajectories $c^i_{l,T} = \frac{1}{n} \sum_{i=1}^{n} c^i_{l,T}(i)$;
- compute $\nabla \tilde{C} = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{n} \beta^{l-1} c^i_{l,T} \theta$ and $\nabla \tilde{C} = \frac{1}{L} \sum_{l=1}^{L} \sum_{i=1}^{n} \beta^{l-1} c^i_{l,T} \theta$;
- Update feedback gains: $\theta_{k+1} = \theta_k - \eta \nabla \tilde{C}_k$ and $\theta_{k+1} = \theta_k - \eta \nabla \tilde{C}_k$.
3: Let $k = k + 1$, and go to step 2 until the algorithm terminates.

For Model II, we use a model-free policy-gradient method proposed in [25], and present Algorithm 2. Given a smoothing parameter $r > 0$, let $P_r$ denote the uniform probability distribution over the matrices of size $h_u \times h_x$, whose Frobenius norm is $r$. Similarly, let $P_r$ denote the uniform probability distribution over the matrices of size $zh_u \times zh_x$, whose Frobenius norm is $r$.

Assumption 6. The covariance matrices of initial states and driving noises are positive definite (i.e., they are invertible).

Theorem 6. Let Assumptions 7, 8, and 9 hold. The performance of the learned strategy $\{\theta_k, \tilde{\theta}_k\}$, given by Algorithm 2, converges to that of the optimal strategy $\{\theta^*, \tilde{\theta}^*\}$ in Theorem 2 with probability one, as $k \rightarrow \infty$.

Proof. The proof follows from [25, Theorem 9] and the decomposition proposed in Theorem 2.

Analogous to Theorem 6, one can devise an approximate policy gradient algorithm under NS information structure, where deep state is approximated by mean field. Note that Theorem 6 holds for arbitrary probability distribution (not necessarily Gaussian).

V. NUMERICAL EXAMPLE

The power grid is a complex large-scale network consisting of many decision makers such as users and service providers. Due to some fundamental challenges such as global warming, limited fossil fuel and intermittent nature of renewable energy sources, there is an inevitable need for smart grid wherein the decision makers intelligently interact with each other and use limited resources efficiently. As a result, there has been a growing interest recently in network management of smart grid [26]. In what follows, we provide a simple example showcasing the application of our results in learning the optimal resource allocation strategy.

Example 1. Consider a smart grid with $n \in \mathbb{N}$ users. Let $x_i^t \in \mathbb{R}$ denote the consumed energy of user $i \in \mathbb{N}$, at time $t \in \mathbb{N}$ and $\bar{x}_t$ denote the weighted average of the total energy consumption of users, i.e.,

$$
\bar{x}_t = \frac{1}{n} \sum_{i=1}^{n} \alpha^i x_i^t,
$$
where $\alpha^i$ indicates the relative importance (priority) of user $i$ compared to others. The linearized dynamics of user $i$ is:

$$x_{t+1}^i = x_t^i + u_t^i + w_t^i,$$

where $w_t^i$ reflects the uncertainty in energy consumption of user $i$ at time $t$. The objective is to find a resource allocation strategy such that the following cost is minimized:

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{E} \left[ \frac{1}{n} \sum_{t=1}^{T} \sum_{i=1}^{n} (x_t^i)^T Q x_t^i + (u_t^i)^T R u_t^i + \bar{x}_t^i \bar{Q} \bar{x}_t^i \right],$$

where the first two terms are the operational cost of each user and the third term is the cost associated with purchasing energy from a utility.

Suppose that the information structure is deep state sharing, and let all users run Algorithm 2 as their energy management strategy. Consider the following numerical parameters:

- $n = 10$, $A = 1$, $B = 1$, $\bar{Q} = 4$, $R = 1$, $Q = 1$, $\bar{R} = 1$, $r = 0.15$, $\eta = 0.3$, $T = 10$, $L = 100$,
- $\beta = 1$, $\alpha^1 = \sqrt{0.5}$, $\alpha^7 = \sqrt{1.5}$, $\alpha^8 = 1$, $\alpha^9 = \sqrt{2}$,
- $\alpha^{10} = \sqrt{2.5}$, $w_t^i \sim \text{norm}(0,0.02)$, $x_t^i \sim \text{unif}(0,1)$,

where the initial states and local noises are assumed to be i.i.d. random variables. The simulation results are provided in Figure 1, which gives the evolution of learning gains obtained by Algorithm 2 along with the optimal gains. The figure shows that the learned strategy reaches a sufficiently small neighbourhood of the optimal one obtained by Theorem 2 after a few thousand iterations.

**VI. CONCLUSIONS**

In this paper, the application of reinforcement learning algorithms in deep structured teams was studied for Markov chain and linear quadratic models with discounted and time-average cost functions. Two non-classical information structures were considered, namely, deep state sharing and no sharing. Different planning and reinforcement learning algorithms were proposed. In particular, it was shown that the solution of a (model-free) Q-learning algorithm and a (model-free) policy gradient algorithm converge to the optimal solution of the Markov chain model and linear quadratic model, respectively. Finally, the obtained results were applied to smart grid in a simulation environment.

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