NONLINEAR DEFORMED su(2) ALGEBRAS INVOLVING TWO DEFORMING FUNCTIONS

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Abstract

The most common nonlinear deformations of the su(2) Lie algebra, introduced by Polychronakos and Roček, involve a single arbitrary function of $J_0$ and include the quantum algebra $\text{su}_q(2)$ as a special case. In the present contribution, less common nonlinear deformations of su(2), introduced by Delbecq and Quesne and involving two deforming functions of $J_0$, are reviewed. Such algebras include Witten’s quadratic deformation of su(2) as a special case. Contrary to the former deformations, for which the spectrum of $J_0$ is linear as for su(2), the latter give rise to exponential spectra, a property that has aroused much interest in connection with some physical problems. Another interesting algebra of this type, denoted by $A_q^+(1)$, has two series of $(N + 1)$-dimensional unitary irreducible representations, where $N = 0, 1, 2, \ldots$. To allow the coupling of any two such representations, a generalization of the standard Hopf axioms is proposed. The resulting algebraic structure, referred to as a two-colour quasitriangular Hopf algebra, is described.

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1 Introduction

Quantized universal enveloping algebras, also called $q$-algebras, refer to some specific deformations of (the universal enveloping algebra of) Lie algebras, to which they reduce when the deforming parameter $q$ (or set of deforming parameters) goes to one \[1\]. The simplest example of $q$-algebra, $\mathfrak{su}_q(2) \equiv U_q(\mathfrak{su}(2))$, was first introduced by Sklyanin, and by Kulish and Reshetikhin \[2\]. It has found a lot of applications in various branches of physics since its realization in terms of $q$-bosonic operators was proposed by Biedenharn and Macfarlane \[3\].

The $\mathfrak{su}_q(2)$ algebra is a special case of more general deformations of $\mathfrak{su}(2)$, independently introduced by Polychronakos and Roček \[4\]. They involve one arbitrary function $f(J_0)$ in the commutator of $J_+$ with $J_-$, and their representation theory is characterized by a rich variety of phenomena, whose interest in particle physics has been stressed.

More recently, deformations of $\mathfrak{su}(2)$ involving two deforming functions $F(J_0)$ and $G(J_0)$ in the commutator of $J_+$ with $J_-$ and in that of $J_0$ with $J_+$ or $J_-$, respectively, have been proposed by Delbecq and Quesne \[5, 6, 7\]. It is the purpose of the present contribution to review the construction and representation theory of such algebras, and to show how the problem of endowing some of them with a Hopf algebraic structure can be addressed \[8\].

2 Nonlinear deformed $\mathfrak{su}(2)$ algebras

Polychronakos-Roček algebras (PRA’s) are associative algebras over $\mathbb{C}$, generated by three operators $j_0 = (\mathfrak{j}_0)\dagger$, $j_+$, and $j_- = (\mathfrak{j}_+)^\dagger$, satisfying the commutation relations \[1\]

\[
[j_0, j_+] = j_+, \quad [j_0, j_-] = -j_-, \quad [j_+, j_-] = f(j_0),
\]

where $f(z)$ is a real, parameter-dependent function of $z$, holomorphic in the neighbourhood of zero, and going to $2z$ for some values of the parameters. These algebras have a Casimir operator given by $c = j_+j_+ + h(j_0) = j_+j_- + h(j_0) - f(j_0)$, in terms of another real function $h(z)$, related to $f(z)$ through the equation $h(z) - h(z - 1) = f(z)$. An explicit expression for $h(z)$ has been given by Delbecq and Quesne \[8\] in terms of Bernoulli polynomials and Bernoulli numbers.

For all PRA’s, the spectrum of $j_0$ is linear as in the special case of the $q$-algebra $\mathfrak{su}_q(2)$. The latter corresponds to $f(j_0) = [2j_0]_q$, and $h(j_0) = [j_0][j_0 + 1]_q$, with $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$, and $q \in \mathbb{R}^+$ (the case where $q$ is a phase will not be considered here, as throughout the present work we shall restrict the parameters to real values) \[9\].

Delbecq-Quesne algebras (DQA’s) differ from PRA’s by the replacement of \[1\] by \[10\]

\[
[j_0, J_+] = G(J_0)J_+, \quad [J_0, J_-] = -J_-G(J_0), \quad [J_+, J_-] = F(J_0),
\]
where \( J_0 = (J_0)^\dagger \), \( J_- = (J_+)^\dagger \), and the commutators involve two real, parameter-dependent functions of \( z \), \( F(z) \) and \( G(z) \), holomorphic in the neighbourhood of zero, and going to \( 2z \) and 1 for some values of the parameters, respectively. These functions are further restricted by the assumption that the algebras have a Casimir operator given by \( C = J_- J_+ + H(J_0) = J_+ J_- + H(J_0) - F(J_0) \), in terms of some real function \( H(z) \), holomorphic in the neighbourhood of zero. The latter restriction implies that \( F(z) \), \( G(z) \), and \( H(z) \) satisfy the consistency condition 
\[
H(z) - H(z - G(z)) = F(z).
\]

Since for \( G(J_0) = 1 \), DQA’s reduce to PRA’s, the first significant case corresponds to \( G(J_0) = 1 + (1 - q)J_0 \), where \( q \in \mathbb{R}^+ \). In such a case, it has been shown that there exist \( (\lambda - 1) \)-parameter algebras \( \mathcal{A}_{\lambda}^+ \) and \( \mathcal{A}_\lambda \), for which the functions \( F(J_0) \) and \( H(J_0) \) are polynomials of degree \( \lambda \) in \( J_0 \). In particular, for \( \lambda = 2 \) and 3, one finds the algebras \( \mathcal{A}_2^+ \) and \( \mathcal{A}_3^+ \), for which 
\[
F(J_0) = 2J_0(1 + (1 - q)J_0), \quad H(J_0) = 2(1 + q)^{-1}J_0(J_0 + 1)\text{, and } F(J_0) = 2J_0(1 + (1 - q)J_0)(1 - (1 - p)J_0), \quad H(J_0) = 2((1 + q)^{-1}(1 + q^2))^2J_0(J_0 + 1)(1 + (p + q)q - (1 - p)(1 + q)J_0), \text{ respectively.}
\]
The representation theory of the DQA’s can be dealt with as that of \( su_q(2) \), or more generally of the PRA’s. Considering the case where \( G(J_0) = 1 + (1 - q)J_0 \), and denoting by \( |cm\) a simultaneous eigenvector of the commuting Hermitian operators \( C \) and \( J_0 \), associated with the eigenvalues \( c \) and \( m \) respectively, it can be proved that \( J_0^\dagger |cm\) (resp. \( J_0^\dagger |cm\) \( n \in \mathbb{N}^+ \), is either the null vector or a simultaneous eigenvector of \( C \) and \( J_0 \), corresponding to the eigenvalues \( c \) and \( mq^{-n} - (1 - q^{-n})/(1 - q) \) (resp. \( mq^n - (1 - q^n)/(1 - q) \)). Hence, the spectrum of \( J_0 \) is exponential, instead of linear as for the PRA’s.

Moreover, if the starting \( m \) value belongs to the interval \( ([q - 1]^{-1}, +\infty) \) (resp. \( (-\infty, (q - 1)^{-1}) \)), then all the \( J_0 \) eigenvalues obtained by successive applications of \( J_+ \) or \( J_- \) upon \( |cm\) do not belong to the same interval and \( J_+ \) (resp. \( J_- \)) is a raising generator, whereas if \( m = (q - 1)^{-1} \), then neither \( J_+ \) nor \( J_- \) change the \( J_0 \) eigenvalue. The unirreps therefore separate into two classes according to whether the eigenvalues of \( J_0 \) are contained in the interval \( (-\infty, (q - 1)^{-1}) \), or in the interval \( ([q - 1]^{-1}, +\infty) \).

In general, they may fall into one out of four categories: (i) infinite-dimensional unirreps with a lower bound \(-j\), (ii) infinite-dimensional unirreps with an upper bound \( J \), (iii) infinite-dimensional unirreps with neither lower nor upper bounds, and (iv) finite-dimensional unirreps with both lower and upper bounds, \(-j \) and \( J \) (where in general \( j \neq J \)). In addition, there is a trivial one-dimensional unirrep corresponding to \( m = (q - 1)^{-1} \).

The exponential character of the \( J_0 \) spectrum in DQA representations may be of interest in various physical problems, wherein such spectra have been encountered, such as alternative Hamiltonian quantizations, exactly solvable potentials, \( q \)-deformed supersymmetric quantum mechanics, and \( q \)-deformed interacting boson models.
3 The algebra $A_q^+(1)$

Another example of DQA, for which the function $G(J_0)$ is linear, has been recently constructed [4]. Contrary to those considered in the previous section, this algebra, denoted by $A_q^+(1)$, is defined in terms of functions $F(J_0)$ and $H(J_0)$ that are not polynomials, but infinite series in $J_0$,

\[
\begin{align*}
F(J_0) &= \frac{-(G(J_0))^2 - (G(J_0))^{-2}}{q - q^{-1}}, \\
H(J_0) &= \frac{q^{-1}(G(J_0))^2 + q(G(J_0))^{-2} - q - q^{-1}}{(q - q^{-1})^2},
\end{align*}
\]

with $G(J_0) = 1 + (1 - q)J_0$. Since the transformation $q \rightarrow q^{-1}$, $J_0 \rightarrow -qJ_0$, $J_+ \rightarrow J_-$ is an automorphism of $A_q^+(1)$, the parameter values may be restricted to the range $0 < q < 1$.

$A_q^+(1)$ can be obtained from $su_q(2)$, a special case of PRA, by using a two-valued map $P_\delta : su_q(2) \rightarrow A_q^+(1)$, $\delta = \pm 1$, defined by

\[J_0 = p_\delta(j_0), \quad J_+ = j_+, \quad J_- = j_-, \quad \text{where} \quad p_\delta(z) \equiv \frac{1 - \delta q^{-z}}{q - 1}.\]

Such a generator map is well defined: it can indeed be easily checked that if $j_0$, $j_+$, $j_-$ satisfy the $su_q(2)$ commutation relations, then $J_0$, $J_+$, $J_-$, given in (4), fulfill those of $A_q^+(1)$. The functions $p_\delta(z)$, $\delta = \pm 1$, defined in (4), are entire and invertible functions, with $g(z) = p_\delta^{-1}(z) = \ln(G^2(z))\ln(q^{-2})$. If $z \in \mathbb{R}$, the range of $p_\delta$ (and consequently the domain of $p_\delta^{-1}$) is the interval $(-\infty, (q - 1)^{-1})$ or $((q - 1)^{-1}, +\infty)$ according to whether $\delta = -1$ or $\delta = +1$. The function $g(z)$ is well-behaved everywhere on $\mathbb{R}$, except in the neighbourhood of the point $z = (q - 1)^{-1}$.

It should be stressed that the use of $P_\delta$, $\delta = \pm 1$, implies an extension of the well-known deforming functional technique [3] for two reasons: first because here a map between two deformed algebras, $su_q(2)$ and $A_q^+(1)$, is considered instead of a map between a Lie algebra and a deformed one, as in the original method; and second because use is made of a two-valued functional, whose inverse is singular, instead of a single-valued one.

It can be easily shown [7] that $A_q^+(1)$ has no infinite-dimensional unirrep, but has, for any $N = 0, 1, 2, \ldots$, two $(N + 1)$-dimensional unirreps, which may be distinguished by $\delta = \pm 1$. The corresponding spectrum of $J_0$ is given by $m_\delta = (1 - \delta q^{-N})/(q - 1)$, for $n = 0, 1, \ldots, N$, with maximum and minimum eigenvalues $J_\delta^+ = (1 - \delta q^{-N/2})/(q - 1)$, and $-J_\delta^- = (1 - \delta q^{N/2})/(q - 1)$ respectively. The unirrep specified by $J_\delta^+$ (resp. $J_\delta^-$) is entirely contained in the interval $((-1)^{-1}, +\infty)$ (resp. $(-\infty, (q - 1)^{-1})$). For both unirreps, the eigenvalue of the Casimir operator is given by $\langle C \rangle = H(\gamma_\delta)$, where $\gamma_\delta = (1 - \delta q^{-N/2})/(q - 1)$.

In the carrier space $V^{J_\delta^+}$ of the unirrep characterized by $J_\delta^+$, whose basis vectors are specified by the values of $J_\delta^+$ and $m_\delta$, the $A_q^+(1)$ generators are represented by
some linear operators $\Phi^J (A) \in \mathcal{A}_q^\pm (1)$, defined by

$$
\Phi^J (J_0) |J^\delta, m^\delta\rangle = m^\delta |J^\delta, m^\delta\rangle = (\frac{q^n}{m}) |J^\delta, m^\delta\rangle.
$$

$$
\Phi^J (J_-) |J^\delta, m^\delta\rangle = \sqrt{H(\delta^\delta) - H(qm^\delta - 1)} |J^\delta, qm^\delta - 1\rangle
= \sqrt{[n+1][N-n]q} |J^\delta, qm^\delta - 1\rangle,
$$

$$
\Phi^J (J_+) |J^\delta, m^\delta\rangle = \sqrt{H(\delta^\delta) - H(m^\delta) |J^\delta, q^{-1}(m^\delta + 1)\rangle
= \sqrt{[n+1][N-n+1]q} |J^\delta, q^{-1}(m^\delta + 1)\rangle.
$$

The generator mapping $P_\delta$ can be used to transfer the quasitriangular Hopf structure of $\mathfrak{su}_q(2)$ to $\mathcal{A}_q^\pm (1)$. One gets in this way a double quasitriangular Hopf structure, with comultiplication, counit, antipode maps, and universal $R$-matrix given by

$$
\Delta^\delta (J_0) = (q - 1)^{-1} (1 \otimes 1 - \delta G(J_0) \otimes G(J_0)),
$$

$$
\Delta^\delta (J_\pm) = \delta (J_\pm \otimes (G(J_0))^{-1} + G(J_0) \otimes J_\pm),
$$

$$
\epsilon^\delta (J_0) = (1 - \delta)(q - 1)^{-1}, \quad \epsilon^\delta (J_\pm) = 0,
$$

$$
S^\delta (J_0) = -J_0(G(J_0))^{-1}, \quad S^\delta (J_+) = -qJ_+, \quad S^\delta (J_-) = -q^{-1}J_-,
$$

$$
R^\delta = q^{2\log_q(\delta G(J_0))\otimes \log_q(\delta G(J_0))} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q} q^n(n-1)/2
\times ((G(J_0))^{-1}J_+ \otimes G(J_0)J_-)^n,
$$

respectively. Both $(\Delta^\pm, \epsilon^\pm, S_\pm, R^\pm)$, and $(\Delta^-, \epsilon^-, S_-, R^-)$ satisfy the Hopf and quasitriangularity axioms, but the former are only valid for the representations of $\mathcal{A}_q^\pm (1)$ with eigenvalues of $J_0$ in the interval $((q - 1)^{-1}, +\infty)$, whereas the latter act in $(-\infty, (q - 1)^{-1})$.

4 Two-colour quasitriangular Hopf structure of $\mathcal{A}_q^\pm (1)$

The double Hopf structure considered in the previous section allows one to couple any two $\mathcal{A}_q^\pm (1)$ unirreps characterized by $J^\pm_1$ and $J^\pm_2$ (resp. $J^-_1$ and $J^-_2$), and with respective carrier spaces $V^{J^\pm_1}$ and $V^{J^\pm_2}$ (resp. $V^{J^-_1}$ and $V^{J^-_2}$), to represent a reducible representation of the same in $V^{J^\pm_1} \otimes V^{J^\pm_2}$ (resp. $V^{J^-_1} \otimes V^{J^-_2}$). No coupling of two unirreps of the types $J^\pm_1$ and $J^\pm_2$, or $J^-_1$ and $J^-_2$, is however possible.

To allow such types of couplings, it is necessary to extend the double Hopf structure of $\mathcal{A}_q^\pm (1)$. This can be accomplished by considering the ‘transmutation’ operators $T^J : V^J \to V^{J^\delta}$, which change the basis states of an $(N+1)$-dimensional
unchanged. The results can be written as

\[ \xi, \eta \in \mathcal{A}_q^+ \text{ specified by } \zeta, \eta, \delta \in \mathbb{Z} \text{ and the involutive automorphism of the algebra } \mathcal{A}_q^+ \text{ mapping } \sigma \mapsto \sigma \text{ of indices } \delta, \eta \in \mathbb{Z}, \delta \neq \eta. \]

where \( \zeta, \eta \) are the multiplication and unit maps of \( \mathcal{A}_q^+ \). Defining now \( \mathcal{A}_q^+ \) as the algebra of \( \mathcal{A}_q^+ \) and no sum-

It can be easily shown that the generalized comultiplication, counit, and antipode maps, \( \Delta_\zeta^\eta, \epsilon_\delta, S_\zeta^\eta \), defined in (3) and (4), transform under \( \sigma_\delta \) as

\[ (\sigma_\mu \otimes \sigma_{\nu \eta}) \circ \Delta_\zeta^\eta = \Delta_\mu^\nu \circ \sigma_{\mu \delta}, \quad \epsilon_\delta \circ \sigma_{\delta \zeta} = \epsilon_\zeta, \quad \sigma_{\zeta \eta} \circ S_\zeta^\eta = S_\mu^\nu \circ \sigma_{\mu \delta}. \]

and satisfy the following generalized coassociativity, counit, and antipode axioms,

\[
\begin{align*}
(\Delta_\mu^\nu \otimes \text{id}) \circ \Delta_\zeta^\eta(A) &= (\text{id} \otimes \Delta_{\mu \nu}^\eta) \circ \Delta_\zeta^\eta(A), \\
(\epsilon_\zeta \otimes \sigma_{\delta \eta}) \circ \Delta_\zeta^\eta(A) &= (\sigma_{\zeta \delta} \otimes \epsilon_\eta) \circ \Delta_\zeta^\eta(A) = A, \\
m \circ (S_\zeta^\mu \otimes \sigma_{\mu \eta}) \circ \Delta_\zeta^\eta(A) &= m \circ (\sigma_{\mu \zeta} \otimes S_\mu^\eta) \circ \Delta_\zeta^\eta(A) = \iota \circ \epsilon_\delta(A),
\end{align*}
\]

where \( A \) denotes any element of \( \mathcal{A}_q^+ \), \( m \) and \( \iota \) are the multiplication and unit maps of \( \mathcal{A}_q^+ \), \( \delta, \zeta, \eta, \mu, \nu, \rho \) take any values in the set \( \{ -1, +1 \} \), and no sum-

\[ \Delta_\zeta^\eta(A) = (\sigma_{\zeta \delta} \otimes \epsilon_\eta) \circ \Delta_\zeta^\eta(A) = A, \]

where \( \zeta, \eta \) are the multiplication and unit maps of \( \mathcal{A}_q^+ \) and no sum-

The comultiplication and antipode maps, as well as the double \( \mathcal{R} \)-matrix of equation (6) can be extended by setting

\[
\begin{align*}
\Delta_\zeta^\eta(A) &= (\sigma_{\zeta \delta} \otimes \sigma_{\eta \delta}) \circ \Delta_\delta(A), \\
S_\zeta^\eta(A) &= \sigma_{\zeta \delta} \circ S_\delta(A), \\
\mathcal{R}^\zeta^\eta &= (\sigma_{\zeta \delta} \otimes \sigma_{\eta \delta}) \left( \mathcal{R}^\delta \right),
\end{align*}
\]

where \( \zeta, \eta, \delta = \pm 1 \), while the counit map \( \epsilon_\delta \), defined in the same equation, is left unchanged. The results can be written as

\[
\begin{align*}
\Delta_\zeta^\eta(J_0) &= (q - 1)^{-1}(1 \otimes 1 - \delta \zeta G(J_0) \otimes G(J_0)), \\
\Delta_\zeta^\eta(J_\pm) &= \eta J_\pm \otimes (G(J_0))^{-1} + \xi G(J_0) \otimes J_\pm, \\
S_\zeta^\eta(J_0) &= (q - 1)^{-1} \left( 1 - \xi \delta (G(J_0))^{-1} \right), \\
S_\zeta^\eta(J_\pm) &= -q^{-1} J_\pm, \\
\mathcal{R}^\zeta^\eta &= q^{2 \log_2(\xi G(J_0)) \otimes \log_2(\eta G(J_0))} \\
&\times \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{n!} \Delta^{n(n - 1)/2} \left( (\xi G(J_0))^{-1} J_+ \otimes \eta G(J_0) J_- \right)^n.
\end{align*}
\]

It can be easily shown that the generalized comultiplication, counit, and antipode maps, \( \Delta_\zeta^\eta, \epsilon_\delta, S_\zeta^\eta \), defined in (3) and (4), transform under \( \sigma_\delta \) as

and satisfy the following generalized coassociativity, counit, and antipode axioms,
mation over repeated indices is implied. Moreover, $\Delta^\zeta_\eta$ and $\epsilon_\delta$ are algebra homomorphisms, while $S^\delta_\zeta$ is both an algebra and a coalgebra antihomomorphism.

By using the generalized coproduct $\Delta^\zeta_\eta$, it is now possible to couple any $(N_1 + 1)$- and $(N_2 + 1)$-dimensional unirreps of $A_q^+(1)$, specified by $J^\zeta_1$ and $J^\eta_2$ respectively, to provide two reducible representations in $V^{J^\zeta_1} \otimes V^{J^\eta_2}$, which are characterized by $\delta = +1$ and $\delta = -1$, respectively. They can be decomposed into a direct sum of $(N + 1)$-dimensional unirreps, specified by $J^\delta$, by using some Wigner coefficients $\langle J^\zeta_1 m^\zeta_1, J^\eta_2 m^\eta_2 | J^\delta m^\delta \rangle_{DQ}$, given in terms of $\text{su}_q(2)$ Wigner coefficients by the relation

$$\langle J^\zeta_1 m^\zeta_1, J^\eta_2 m^\eta_2 | J^\delta m^\delta \rangle_{DQ} = \langle \frac{N_1}{2}, \frac{N_2}{2} | n_1, \frac{N_2}{2} | n_2, \frac{N_2}{2} - n \rangle_q.$$  

(11)

The carrier space of the unirep $J^\delta$ in $V^{J^\zeta_1} \otimes V^{J^\eta_2}$ is therefore spanned by the states

$$|J^\zeta_1 J^\eta_2 J^\delta m^\delta\rangle = \sum_{m^\zeta_1, m^\eta_2} \langle J^\zeta_1 m^\zeta_1, J^\eta_2 m^\eta_2 | J^\delta m^\delta \rangle_{DQ} |J^\zeta_1, m^\zeta_1\rangle \otimes |J^\eta_2, m^\eta_2\rangle.$$  

(12)

Turning now to the generalized universal $R$-matrix defined in (9) or (8), it can be easily shown \[8\] that its four pieces $R^\zeta_\eta$, $\zeta, \eta = \pm 1$, are invertible and satisfy the properties

\[
\begin{align*}
(\sigma_{\mu \zeta} \otimes \sigma_{\nu \eta}) \left( R^{\zeta_\eta} \right) &= R^{\mu_\zeta}, \\
(\Delta^\zeta_\eta \otimes \sigma_{\mu \eta}) \left( R^{\zeta_\eta} \right) &= R^{\lambda_\zeta} R^{\mu_\eta} = (R^{\lambda_\zeta} R^{\mu_\eta})^{-1}, \\
(\Delta^\zeta_\eta \otimes \sigma_{\mu \eta}) \left( R^{\zeta_\eta} \right) &= R^{\lambda_\zeta} R^{\mu_\eta}, \quad (\sigma_{\lambda \zeta} \otimes \Delta^\mu_\eta) \left( R^{\zeta_\eta} \right) = R^{\lambda_\zeta} R^{\mu_\eta} \quad (13)
\end{align*}
\]

for any $A \in A_q^+(1)$. From these results, or more simply from the corresponding properties fulfilled by $R^\delta$, one obtains the relations

\[
\begin{align*}
R^{\zeta_\eta} R^{\zeta_\eta} R^{\mu_\eta} &= R^{\eta_\mu} R^{\zeta_\eta} R^{\zeta_\eta}, \\
(\epsilon_\zeta \otimes \text{id}) \left( R^{\zeta_\eta} \right) &= (\text{id} \otimes \epsilon_\eta) \left( R^{\zeta_\eta} \right) = 1, \\
(S^\lambda_\zeta \otimes \sigma_{\mu \eta}) \left( R^{\zeta_\eta} \right) &= (\sigma_{\lambda \zeta} \otimes (S^\mu_\eta)^{-1}) \left( R^{\zeta_\eta} \right) = (R^{\lambda_\mu})^{-1}. \quad (14)
\end{align*}
\]

The first relation in (14) shows that the generalized universal $R$-matrix is a solution of the coloured YBE \[12\], where the ‘colour’ parameters $\zeta, \eta, \mu$ take discrete values in the set \{-1, +1\}. We may therefore call $(A_q^+(1), +, m, \epsilon, \Delta^\zeta_\eta, \epsilon_\delta, S^\delta_\zeta, R^{\zeta_\eta}; \mathbb{C})$ a two-colour quasitriangular Hopf algebra over $\mathbb{C}$ \[8\]. As will be shown elsewhere \[13\], this type of algebraic structure admits generalizations, which will be referred to as coloured quasitriangular Hopf algebras.

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