Triangleland: II. Quantum mechanics of pure shape

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Abstract
Relational particle models are of value in the absolute versus relative motion debate. They are also analogous to the dynamical formulation of GR, and as such are useful for investigating conceptual strategies proposed for resolving the problem of time in quantum general relativity. Moreover, to date there are few explicit examples of these at the quantum level. In this paper I exploit recent geometrical and classical dynamics work to provide such a study based on reduced quantization in the case of pure shape (no scale) in 2D for 3-particles (triangleland) with multiple harmonic oscillator type potentials. I explore solutions for these making use of exact, asymptotic, perturbative and numerical methods. An analogy to the mathematics of the linear rigid rotor in a background electric field is useful throughout. I argue that further relational models are accessible by the methods used in this paper, and for specific uses of the models covered by this paper in the investigation of the problem of time (and other conceptual and technical issues) in quantum general relativity.

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1. Introduction

In Euclidean relational particle mechanics [1–9], only relative times, relative angles and relative separations are meaningful. While, in similarity relational particle mechanics [4, 6, 7, 10–12], only relative times, relative angles and ratios of relative separations are meaningful. These mechanics are valuable models as regards the absolute versus relative motion debate [13]. It is then of interest what structure one gets when one quantizes such theories. There has not been much work on this to date as regards nontrivial explicit examples ([14] concerns formal quantum constraints, [15] is a toy of geometrical quantization, [5, 16] is a semiclassical toy (but with only explicit examples for Euclidean relational particle mechanics in 1D) and...
some simple solutions of the Dirac quantization scheme [5]). This paper goes further than
these works in being the first quantum treatment of explicit nontrivial examples of similarity
relational particle mechanics, while [17] goes further, rather, by considering more complicated
explicit examples of Euclidean relational particle mechanics than those considered before. We
get there using our recent understanding of reduced configuration spaces [11] as a path to
quantization.

In investigating conceptual strategies for the problem of time [3, 18–23, 25–27] in quantum
gravity, relational particle mechanics are useful analogues of GR [2, 3, 23, 24, 26]; I view
this as major long-term motivation for the current series of papers. This notorious problem
occurs because ‘time’ takes a different meaning in each of GR and ordinary quantum theory.
This incompatibility underscores a number of problems with trying to replace these two
branches with a single framework in situations in which the premises of both apply, namely
in black holes and in the very early universe. One facet of the problem of time that shows up
in attempting canonical quantization is that the lack of linear momentum dependence of the
GR Hamiltonian constraint leads to a frozen (i.e. timeless, or stationary) quantum equation
for the universe arises therein: the quantum counterpart of (I.11) is the Wheeler–DeWitt
equation

\[ \hat{H} \Psi = -\hbar^2 \left\{ \frac{1}{\sqrt{\mathcal{M}}} \frac{\delta}{\delta h^{\mu\nu}} \left( \sqrt{\mathcal{M}} N^{\mu\nu\rho\sigma} \frac{\delta}{\delta h^{\rho\sigma}} \right) - \frac{\xi}{2} \text{Ric}(\mathcal{M}) \right\} \Psi \]

\[ -\sqrt{h} \left( \text{Ric}(h) - 2\Lambda \right) \Psi + \hat{H}_{\text{matter}} \Psi = 0, \quad (1) \]

where \( \Psi \) is the wavefunction of the universe.

Before further consideration of the problem of time, a detour is first required about other
features of this equation and of the toy models of it that are the subject of this paper. The
inverted commas indicate that the Wheeler–DeWitt equation has, in addition to the problem
of time, various technical problems, including

(A) regularization problems—not at all straightforward for an equation for a theory of an
infinite number of degrees of freedom in the absence of background structure, while the
mathematical meaningfulness of functional differential equations is open to question. I
emphasize that this is not an issue in this paper’s toy models as these are for a finite
number of degrees of freedom, similarly to the situation in minisuperspace quantum
cosmology.

(B) There are operator-ordering issues, which this paper’s toy models do exhibit an analogue
of. To view the analogy, let \( Q_A \) be general coordinates (spanning spatial indices and
either particle labels or field species along with spatial dependence) with a corresponding
configuration space metric \( \mathcal{M}_{AB} \) with inverse \( N^{AB} \) and determinant \( \mathcal{M} \) (perhaps merely
at the formal level). Then let \( \nabla_{Q_A} \) be a partial derivative for finite systems or (perhaps
merely formally) a functional derivative for infinite systems. Then the Laplacian ordering
for the classical combination of configurations and their momenta \( N^{AB}(Q^C)P_A P_B \) is

\[ D^2 = \frac{1}{\sqrt{\mathcal{M}}} \nabla_{Q_A} \left\{ \sqrt{\mathcal{M}} N^{AB} \frac{\nabla_{Q_B}}{\nabla_{Q_B}} \right\}, \quad (2) \]

which has the desirable property of being independent of coordinate choice on the
configuration space [28]. This property is not, however, unique to this ordering:
one can include a Ricci scalar curvature term so as to have \( D^2 - \xi \text{Ric}(\mathcal{M}) \) [28–30].
There is then a unique conformally invariant choice [30, 31] among these orderings.
The conformal invariance here corresponds to retaining at the quantum level the banal
conformal invariance that is obvious and natural in the relational action at the classical
level [8, 31]. This ordering is
\[
D^2 = \frac{1}{\sqrt{\mathcal{M}} \psi} \nabla_x \left( \sqrt{\mathcal{M}} \nabla_{\psi} \right) - \frac{k-2}{4(k-1)} \text{Ric}(\mathcal{M}),
\]

where \( k \) is the configuration space dimension. Furthermore, for this to be conformal, it is required that \( \Psi \) itself transforms in general tensorially under the banal conformal transformation \( [32]^2 \)

\[
\Psi \rightarrow \Psi_\Omega = \Omega^{\frac{k-2}{2}} \Psi.
\]

Moreover, this paper focuses on a model with a 2D configuration space, for which the conformal \( \xi = \{k - 2\}/4(k - 1) \) collapses to zero, so that Laplacian ordering and conformally invariant wavefunctions suffice (but almost all other relational particle mechanics models, such as [17], have configuration space dimension \( \geq 3 \) for which this subtlety is required). Finally, if one sends \( H\Psi = E\Psi \) to \( H_\Omega\Psi_\Omega = E_\Omega\Psi_\Omega = (E/\Omega^2)|\Psi_\Omega| \), one has now an eigenvalue problem with a weight function \( \Omega^{-2} \) which then appears in the inner product

\[
\int_{\Sigma} (\Psi_1)_\Omega^* (\Psi_2)_\Omega \Omega^{-2} \sqrt{\mathcal{M}}_\Omega d^k x.
\]

This inner product additionally succeeds in being banal conformally invariant, being equal to

\[
\int_{\Sigma} \Psi_1^* \Omega^{\frac{k-2}{2}} \Psi_2 \Omega^{\frac{k-2}{2}} \Omega^{-2} \sqrt{\mathcal{M}} \Omega^k d^k x = \int_{\Sigma} \Psi_1^* \Psi_2 \sqrt{\mathcal{M}} d^k x
\]

in the ‘original or mechanically natural’ representation in which \( E \) comes with the trivial weight function, 1. I should caution however that, while this conformal choice of ordering is a nice choice which I am suggesting lies on a more solid principle than how it is usually presented (fully explained in [31]), it might nevertheless eventually be found to clash with other requirements such as existence and suitable behaviour of crucial operators.

(C) Next, and talking only in the context of finite models such as in this paper rather than for full geometrodynamics for which the mathematical machinery is lacking\(^3\), in general \( H_\Omega \) is not self-adjoint with respect to \( \Omega (\lambda) \Omega \), while the mechanically natural \( \hat{H} \) is, in a simple sense, with respect to \( \langle \rangle \). That is in the sense that \( \int \sqrt{\mathcal{M}} d^k x \Psi^* D^2 \Psi = \int \sqrt{\mathcal{M}} d^k x \{ D^2 \Psi^* \} \Psi + \) boundary terms, which amounts to self-adjointness if the boundary terms can be arranged to be zero (which is definitely not a problem in this paper as there are no boundaries) and in other cases involves such as suitable fall-off conditions on \( \Psi \). This is not shared by the \( \Omega \)-inner product as that has an extra factor of \( \Omega^{-2} \), which in general interferes with the corresponding move by the product rule (\( \sqrt{\mathcal{M}} \) does not interfere thus above, since the Laplacian is built out of derivatives that are covariant with respect to the metric \( \mathcal{M}_{\mu\nu} \)). However, solving \( H_\Omega \Psi_\Omega = E_\Omega \Psi_\Omega \) is equivalent to solving \( H\Psi = E\Psi \), so the banal conformal transformation might at this level be viewed as a sometimes-useful computational aid, with the answer then being placed in the mechanically natural representation for further physical interpretation. This is not an issue in this paper as \( \Omega \) is but 1/4 in my spherical calculations, thus not presenting any product-rule obstacles to self-adjointness.

\(^2\) This paper’s models, like minisuperspace, only require the finite configuration space dimension \( k \) case of these equations.

\(^3\) The loop quantum gravity approach [33] is advantageous at this point in being equipped with a Hilbert space structure, as well as being better suited as regards (A) above. Its passage to Ashtekar variables renders operator-ordering issues there different to those of geometrodynamics and the toy model of this paper. Nevertheless, discussion of analogies with the geometrodynamical Wheeler–DeWitt equation as regards the problem of time remain reasonable and commonplace in the literature, e.g. [23, 34, 26] make use of such.
Now, returning to the problem of time, while many conceptual strategies have been put forward to resolve it, there is a long history of proposed resolutions not standing up to detailed examination [23], so that this remains an open problem for GR. Some of these strategies are as follows.

(1) Perhaps within GR at the classical level there is a fundamental hidden time. For example, one could seek for such a time by canonically transforming the geometrodynamical variables to new variables among which an explicit and genuinely time-like time variable is isolated out. One candidate time of this form is the York time, [20, 23, 35]. This is proportional to \( h^{\mu\nu}/\sqrt{h} \) so it is a ‘dilational object’.

(2) Perhaps instead there is no fundamental time in quantum GR but a notion of time emerges in the quantum regime, e.g. in regions of the universe that behave semiclassically [26, 36]. In situations in which the Born–Oppenheimer approximation \( \Psi = \psi(H_A) |\chi(H_A, L_A')\rangle \) for \( H_A \) ‘heavy, slow’ and \( L_A \) ‘light and fast’ degrees of freedom and the WKB approximation \( \psi(H_A) = e^{iF} e^{-i\hbar/\hbar} \) are applicable, an emergent time drops out of the WDE [19, 26, 36, 37]. For (e.g. using \( h_{\mu\nu} \) as ‘\( H \)’ and the matter as ‘\( L \)’, and requiring that \(|\chi\rangle\) depends nontrivially on \( H \) so that the QM is rendered nonseparable in \( H, L \) variables),

\[
\hbar^2 \mathcal{A}_{\mu\nu} \frac{\delta^2 \Psi}{\delta h^{\mu\nu} \delta h^{\rho\sigma}} \text{ contains } \hbar^2 \mathcal{A}_{\mu\nu} \frac{i}{\hbar} \frac{\delta F}{\delta h^{\mu\nu}} \frac{\delta |\chi\rangle}{\delta h^{\rho\sigma}} = i\hbar \mathcal{A}_{\mu\nu} \frac{\delta |\chi\rangle}{\delta h^{\rho\sigma}} \tag{7}
\]

(by identifying \( F \approx W/M \) where \( W = W(H_A) \) is Hamilton’s principal function and \( M \) is a generic \( H \)-mass, and then using the Hamilton–Jacobi relation for the momentum and

\[
\frac{\delta}{\delta t_{\text{WKB}}(\hbar_{\mu\nu})} = \frac{1}{I} \frac{\partial}{\partial \lambda} \tag{8}
\]

Then by the momentum–velocity relation (1.10), this expression contains

\[
i\hbar \frac{\delta h^{\sigma\tau}}{\delta t_{\text{WKB}}} \frac{\delta |\chi\rangle}{\delta h^{\rho\sigma}} = i\hbar \frac{\delta |\chi\rangle}{\delta t_{\text{WKB}}} \tag{9}
\]

by the chain rule in reverse so that one has a TDSE for the local L degrees of freedom with respect to a time standard that is (approximately) provided by the background H degrees of freedom. An issue here is that (semi)classical conditions need not always occur—guarantee of a classical ‘large’ as in the Copenhagen Interpretation of QM has been cast aside in considering the universe as a whole, and will then by no means be recovered in all possible situations. The above semiclassical approach is additionally a useful framework for discussing the origin [36] of galaxies and cosmic microwave background perturbations within the semiclassical scheme, for which one needs to study spatially located fast light degrees of freedom that are coupled to global slow heavy degrees of freedom such as the size of the universe.

(3) There are also timeless records strategies [2, 3, 21, 24, 38–40]. Here, the primary objects are records—information-containing subconfigurations of a single instant that are localized in both space and configuration space. One would then seek to construct a semblance of dynamics or history from the correlations between such records. QM probability density functions on shape space are here of interest as regards whether one can substantiate Barbour’s conjectures [3, 24] about a present populated with time capsules.

(4) There are also approaches in which it is the histories that are primary [38, 41].

The currently intended applications of this paper’s toy models are to approaches (2), (3), (4) and combinations thereof. However, for completeness and due to current interest, I also outline the following additional family of strategies.
Distinct timeless approach involve evolving constants of the motion (a Heisenberg rather than Schrödinger type approach) or partial observables [34], which is used in Loop Quantum Gravity’s Master Constraint program [33].

Quantum GR being technically difficult, toy models such as that being developed in the current paper have been useful towards developing strategies such as the above. Relational particle mechanics are useful such, due to geometrodynamics being parallely formulable in relational terms (see [42, 43] and [42, 44] for its robustness to the inclusion of matter, or for a summary, section 1.2.3–4). Specific problem of time applications and extensions of this analogy then feature in [2–6, 16, 23, 24, 37, 40, 45–47]. In particular, both relational particle mechanics and geometrodynamics have a constraint that depends quadratically but not linearly on the momenta, which feature underlies the frozen formalism aspect of the problem of time, and both have further nontrivial constraints that are linear in the momenta, which cause many of the complications with strategies proposed to resolve the problem of time. Relational particle mechanics have so far been useful toy models in possessing an analogue of the above-mentioned dilational York time [6, 37, 45], for the semiclassical emergent time approach [37, 46, 47] and for the timeless records theory approach [2, 3, 24, 40]. See the conclusions of this paper and [17] for brief discussion of useful applications of (extensions of) these papers’ models to these issues. In relational particle mechanics models, by their constraints and the subsequent appearance of more complicated reduced configuration spaces on which these models’ partial observables would live, the partial observables approach to these examples of nonrelativistic mechanics would be rather distinct from that for the mechanics models considered in [34], and may lead to a broader view than therein on what features distinguish nonrelativistic mechanics theories from relativistic ones.

Both the absolute versus relative motion debate and the study of conceptual strategies suggested towards resolving the problem of time in quantum gravity benefit from study of explicit examples of quantum relational particle mechanics. I provide such in this paper (similarity relational particle mechanics) and [17] (Euclidean relational particle mechanics), by applying my recent understanding of reduced configuration spaces [11] to carry out quantization. (The above-mentioned QM investigations of relational particle mechanics [5, 14–16], and the recent preprint [48] (path integral method), carry out their quantization by other means.) In this paper I provide Schrödinger equations for scalefree N-stop metroland (relational mechanics of N particles in 1D) and scalefree N-a-gonland (relational mechanics of N particles in 2D), concentrating on the latter’s triangleland case (3-particles in 2D). For this I discuss how relative angle independent potentials are a substantial simplification (in close analogy with how central potentials are in ordinary QM), and provide some simple solutions. I begin with relative angle independent multiple harmonic oscillator type potentials, for which I additionally use asymptotic, perturbative and numerical methods. It is important for this work that I identified my problem to have the same mathematics as the Stark effect for the linear rigid rotor. In section 3 I use rotated coordinates/normal modes/adapted bases to remove the relative angle independence restriction: any multiple harmonic oscillator for scalefree triangleland admits coordinates in which it takes the form of section 2’s special case, which amounts to the ‘electric field’ being in an arbitrary direction rather than in the simplifying adapted basis in which it points along the Z-axis (of the R³ embedding of the configuration space sphere). This is of importance as regards setting up models of the semiclassical approach and records theory. My conclusion (section 4) includes an outline of several further examples to which techniques of this paper can be applied and which are also useful as regards modelling the problem of time and quantum cosmology.
2. Quantum similarity relational particle mechanics

2.1. Time-independent Schrödinger equations for scalefree N-stop metroland and N-a-gonland

For scalefree $N$-stop metroland, the configuration space is $S^{n-1}$ and the conformal-ordered time-independent Schrödinger equation is (via appendix A.1 and the notation of section I.3.2 and with $E$ being redefined to absorb the conformal contribution, which is possible as spheres are spaces of constant curvature)

$$-rac{\hbar^2}{2} \sin^{n-1-\ell} \Theta \prod_{p=1}^{n-1} \sin^2 \Theta_p \frac{\partial}{\partial \Theta_p} \left( \sin^{n-1-\ell} \Theta \frac{\partial}{\partial \Theta} \right) + \nabla \Psi = E \Psi. \quad (10)$$

For scalefree $N$-a-gonland ($N \geq 3$), the configuration space is $\mathbb{C}P^{n-1}$ and the conformal-ordered time-independent Schrödinger equation is (via appendix A.1 and likewise being able to absorb the conformal contribution as complex projective spaces are Einstein and hence of constant Ricci-scalar curvature [11])

$$-rac{\hbar^2}{2} \prod_{p=1}^{n-1} \frac{\partial}{\partial R_p} \left( \prod_{p=1}^{n-1} \frac{R_p}{1 + \|R\|^2} ^{2n-3} \left[ \delta^{pq} + \frac{R_p R_q}{\|R\|^2} \right] \frac{\partial \Psi}{\partial \Theta} \right) + \nabla \Psi = E \Psi. \quad (11)$$

2.2. Time-independent Schrödinger equation for scalefree triangleland

In this paper we mostly consider the triangleland case for which there is the additional ‘accident’ $\mathbb{C}P^1 = S^2$, which permits use of both spherical type and complex projective type variables and carries stronger guarantees of good mathematical behaviour. In this case, in spherical coordinates $[\Theta, \Phi]$ and using the ‘barred banal conformal representation’ $\mathbb{T} = 4\mathbb{T}, \mathbb{E} + \mathbb{U} = [E + U]/4$, the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2} \tilde{J}^2 \Psi = -\frac{\hbar^2}{2} \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial \Psi}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Psi}{\partial \Phi^2} = \tilde{\mathbb{E}}(\Theta) + \tilde{\mathbb{U}}(\Phi, \Theta) \Psi. \quad (12)$$

(Now $E$ does not need redefining since the conformal term is zero because the configuration space dimension is 2.) While in plane polar coordinates $[R, \Phi]$ obtained by passing to stereographic coordinates on the sphere and then passing to the ‘tilded banal conformal representation’ $\tilde{T} = T(1 + R^2)^2$ and $\tilde{\mathbb{E}} + \tilde{\mathbb{U}} = [E + U]/[1 + R^2]^2$, the time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2} \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \Phi^2} = \tilde{\mathbb{E}}(R) + \tilde{\mathbb{U}}(R) \Psi. \quad (13)$$

2.3. Inner products for scalefree triangleland in various coordinate systems

The Jacobians in question are to be read off the metrics in section I.3.2. In the barred banal conformal representation, this gives in spherical coordinates the well-known $\sin \Theta$ factor, in stereographic coordinates it gives $R/[1 + R^2]^2$ (which combines the usual $R$ of plane polar coordinates with the requisite conformal factors), and, if one uses $I_1$ (or $I_2$ or $I_1^2$ or $I_2^2$) instead of $R$, it gives but a constant factor. As there is no nontrivial weight in this representation, the above are the entirety of the extra factors inside the inner product (up to proportion,
which one would then fix by normalization). In the tilded banal conformal representation, the Jacobians are the usual $R$ in plane polar coordinates $(R, \Phi)$, sec$^4\frac{\Theta}{2}\sin\Theta$ in spherical coordinates (which combines the usual sin$\Theta$ with the requisite conformal factors), if one uses $I_1$, it gives $1/(I - I_1)^2$ up to proportionality, and, if one uses $I_2$, it gives $1/I_2^2$. However this representation also has a nontrivial weight (cf (5)), so that one ends up with precisely the same inner product as for the barred banal conformal representation (as should be the case, for in configuration space dimension 2, the wavefunctions themselves do not scale).

2.4. Separability for $\Phi$-independent potentials

Then the above time-independent Schrödinger equations are separable under the separation ansätze
\[ \Psi(R, \Phi) = \xi(R)\eta(\Phi) \quad \text{or} \quad \Psi(\Theta, \Phi) = \xi(\Theta)\eta(\Phi), \]
and in each case one obtains simple harmonic motion solved by
\[ \eta = \exp(\pm i j/\Phi) \]
for $j$ an integer. This is a relative angular momentum quantum number corresponding to $J$ being classically conserved, in analogy with how there is an angular momentum quantum number $m$ in the central-potential case of ordinary QM corresponding to the angular momentum $L_z$ being classically conserved. The accompanying separated-out equation is, in the tilded banal conformal representation in $(R, \Phi)$ coordinates, the radial equation
\[ R^2 \xi'' + R \xi' - \left\{ 2 R^2 \tilde{V}(R) - \tilde{E}(R)/\hbar^2 + j^2 \right\} \xi = 0, \]
or, in the barred conformal representation in $(\Theta, \Phi)$ coordinates, the azimuthal equation
\[ \left\{ \sin\Theta \right\}^{-1} \left\{ \sin\Theta \xi_{\Theta} \right\}_{\Theta} - \left\{ 2 \tilde{V}(\Theta) - \tilde{E}/\hbar^2 + j^2 \left\{ \sin\Theta \right\}^{-2} \right\} \xi = 0. \]

2.5. The special harmonic oscillator quantum problem

The problem is, in the barred banal conformal representation in spherical coordinates, (12) with an harmonic oscillator type potential (I.59) inserted in, which I rearrange to the dimensionless form
\[ \frac{1}{\sin\Theta} \frac{\partial}{\partial \Theta} \left\{ \sin\Theta \frac{\partial\Psi}{\partial \Theta} \right\} + \frac{1}{\sin^2\Theta} \frac{\partial^2\Psi}{\partial \Phi^2} + \left\{ E - A - B \cos\Theta \right\} \Psi = 0 \]
for
\[ A \equiv 2A/\hbar^2, \quad B \equiv 2B/\hbar^2, \quad \tilde{E} \equiv 2\tilde{E}/\hbar^2. \]
This then separates into (15) and
\[ \left\{ \sin\Theta \right\}^{-1} \left\{ \sin\Theta \xi_{\Theta} \right\}_{\Theta} - \left\{ A - \tilde{E} + B\cos(\Theta) + j^2 \left\{ \sin\Theta \right\}^{-2} \right\} \xi = 0. \]

Alternatively, in the tilded banal conformal representation in $(R, \Phi)$ coordinates, it is (13) with (I.58) inserted in, which I rearrange to the dimensionless form
\[ \frac{1}{R} \frac{\partial R}{\partial R} \left\{ \frac{\partial^2\Psi}{\partial R^2} \right\} + \frac{1}{R^2} \frac{\partial^2\Psi}{\partial \Phi^2} - \left\{ K_1 + K_2 R^2 - \frac{E}{\left[ 1 + R^2 \right]^2} \right\} \Psi = 0, \]
for
\[ K_j = K_j/\hbar^2 \quad \text{and} \quad \tilde{E} \equiv 2\tilde{E}/\hbar^2. \]
This then separates into (15) and
\[ \mathcal{R}^2 \zeta_{\mathcal{R}} + \mathcal{R} \zeta_{\mathcal{R}} - \left( \frac{K_1 + K_2 \mathcal{R}^2}{(1 + \mathcal{R}^2)^2} - \frac{\tilde{E}}{(1 + \mathcal{R}^2)^2} + j^2 \right) \zeta = 0. \] (23)

Note that these equations are self-dual in the sense of [8], so that, again, direct study of only one of the two asymptotic regimes is necessary and then the other can be read off by mere substitution.

2.6. Useful mathematical analogue

It is next convenient to recollect my observation [8] that scalefree triangleland’s special triple harmonic oscillator like potential has the same mathematical form as the fairly well-known problem of the linear rigid rotor in a background homogeneous electric field in the symmetry-adapted basis for that problem. This then at the quantum level amounts to the present mathematics being analogous to that of the Stark effect for the linear rigid rotor, which is well documented (see, e.g. [49, 50] and sections 2.5, 2.7, 2.8, 3).

2.7. Exact solution for the ‘very special multiple harmonic oscillator’

In this case the analogy is with the linear rigid rotor itself [51, 52], and the spherical representation’s mathematics takes a familiar form, (20) now being the associated Legendre equation (B.6).

One then has a quantum number \( J \in \mathbb{N}_0 \) analogous to the total angular momentum quantum number \( l \) of the linear rigid rotor (the analogy is that \( J \) is a total angular momentum in a 3D space, but that 3D space is not ordinary 3D space but rather the scaled triangleland configuration space, cf appendix I.C). This obeys
\[ J(J+1) = \mathcal{E} - A = 2(\tilde{E} - A)/\hbar^2 \] (24)
and hence
\[ \tilde{E} = \hbar^2 J(J+1)/2 + A \] (25)
in the spherical representation, or
\[ E = 2\hbar^2 J(J+1) + (K_1 + K_2)/4 \] (26)
in the planar representation. One also has a quantum number \( j \in \mathbb{Z} \) such that \(|j| \leq J\); this is analogous to the magnetic quantum number of the linear rigid rotor, and here has the interpretation of being the 2D relative angular momentum (which is the \( Z \)-component of the above ‘total angular momentum’).

Moreover, if \( \tilde{E}, K_1, K_2 \) are to take the interpretation of being fixed, then there will either be 1 or 0 such \( J \)—a closed-universe type truncation of the number of allowed states. From (26) then, for there to be any chance of solutions one needs \( E \geq (K_1 + K_2)/4 \), so for harmonic oscillator type models, \( E > 0 \) is indispensable. If there is a \( J \) and it is not zero, there are various degenerate solutions corresponding to different values of \( j \). These correspond to states of different relative angular momentum between the particle 2, 3 subsystem and the particle 1 subsystem.

The wavefunctions are of the form
\[ \Psi(\Theta, \Phi) \propto P_j^l(\cos \Theta) \exp(i j \Phi). \] (27)

Thus, in the planar representation, the solution is
\[ \Psi_{jj}(\mathcal{R}, \Phi) \propto P_j^l \left( \frac{1 - \mathcal{R}^2}{1 + \mathcal{R}^2} \right)^j \exp(i j \Phi), \] (28)
and, in terms of the physically and visually useful partial moments of inertia-relative angle variables, it is

\[ \Psi_{jj}(I_1, I_2, \Phi) \propto P_j \left( \frac{I_2 - I_1}{I_1 + I_2} \right) \exp(i j \Phi) = P_j \left( \frac{1 - 2I_1}{I} \right) \exp(i j \Phi). \]  

\[ \exp(i j \Phi) = P_j \left( \frac{2I_2 - I}{I} \right) \exp(i j \Phi). \]  

(29)

While, in terms of the original variables of the problem,

\[ \Psi_{jj}(\iota_1, \iota_2) \propto P_j \left( \frac{\|\iota_2\|^2 - \|\iota_1\|^2}{\|\iota_1\|^2 + \|\iota_2\|^2} \right) \exp\left( i j \arccos \left( \frac{\iota_1 \cdot \iota_2}{\|\iota_1\| \|\iota_2\|} \right) \right). \]  

(30)

Note the following:

1) paralleling the classical working, the very special solution is unconditionally self-dual under the duality map;
2) while this case is simple and not general, its exact solution serves as something about which one can conduct a more general perturbative treatment (section 2.10).

See figure 1 for further discussion of these.

2.8. Small asymptotics solutions for the 'special multiple harmonic oscillator'

I work in the tilded Q-representation. In the first small approximation, \(2[\bar{E} + \bar{U}] = Q_0\) (a constant evaluated in section I.4.9). Thus, in terms of \(Q_0 = Q_0/h^2\), the time-independent Schrödinger equation is

\[ \frac{1}{\mathcal{R}} \frac{\partial^2 \Psi}{\partial \mathcal{R}^2} + \frac{1}{\mathcal{R}^2} \frac{\partial^2 \Psi}{\partial \Phi^2} + Q_0 \Psi = 0, \]  

(31)

which is a familiar problem, separating into simple harmonic motion (15) and a radial equation (16), which is now the Bessel equation of order \(j\) (B.1) in the rescaled variable \(S = \sqrt{Q_0} \mathcal{R}\). Thus there is just the one quantum number \(j \in \mathbb{Z}\) (relative angular momentum). The wavefunctions are then

\[ \Psi_j(\mathcal{R}, \Phi) \propto J_j(\sqrt{Q_0} \mathcal{R}) \exp(i j \Phi). \]  

(32)

So, in spherical variables,

\[ \Psi_j(\Theta, \Phi) \propto J_j \left( \sqrt{Q_0} \tan \frac{\Theta}{2} \right) \exp(i j \Phi). \]  

(33)

and in terms of the partial moments of inertia-relative angle variables, it is

\[ \Psi_j(I_1, I_2, \Phi) \propto J_j(\sqrt{Q_0} I_1/I_2) \exp(i j \Phi) = J_j(\sqrt{Q_0} I_1/[I - I_1]) \exp(i j \Phi) \]

\[ = J_j(\sqrt{Q_0}(I - I_2)/I_2) \exp(i j \Phi). \]  

(34)

While, in terms of the original variables of the problem,

\[ \Psi_j(\iota_1, \iota_2) \propto J_j \left( \sqrt{Q_0} \left\| \frac{I_1}{I_2} \right\| \right) \exp\left( i j \arccos \left( \frac{\iota_1 \cdot \iota_2}{\|\iota_1\| \|\iota_2\|} \right) \right). \]  

(35)

I sketch the probability density function in various of these variables referring forward in figure 2. Finally note that replacing the specific constant \(Q_0\) by some other constant \(Q\), this also provides the exact solution to the constant-potential problem.
Figure 1. For the first three values of $J$ (0, 1 and 2), I provide (a) the azimuthal probability density function on the sphere, which is standard. (b) The radial probability density function in the $(R, \Phi)$ plane. Unlike the usual situation with the radial probability density function in the atom, the inner peaks are the taller ones (in the atom it is more probable, e.g. that the 2s electron is ‘outside’ the 1s one). This is due to the unusual inner product of our planar problem. (c) These plots can then be interpreted in terms of straightforward relational variables such as $\iota_1, \iota_2, I_1$ or $I_2$. I plot in terms of $I_1$. Note the reflection symmetry about $I_2/2$ and also that plots in terms of $I_2$ coincide with those in terms of $I_1$ and so I do not provide them. From the various above plots, one can infer on which of the sorts of triangles defined in figure I.5(d)) the various wavefunctions peak. The cases depicted exhibit one of the following patterns of likelihood for configurations (triangles): a Jacobi-regular peak (if $|j| = J$), a Jacobi-tall and a Jacobi-flat peak with an interposed Jacobi-regular node (if $|j| = J - 1$), a Jacobi-tall peak, a Jacobi-regular peak and a Jacobi-flat peak with two interposed nodes of more moderate tallness and flatness ($|j| = J - 2$). Note that the higher $J$ is, the more pronounced the tallness and flatness involved is. Next, note that for $j = 0$ all the wavefunctions are surfaces of revolution. Higher values of $j$ have sinusoidal dependence on $\Phi$. Moreover the directions picked out by this have physical meaning. $|j| = 1$ has what would usually be $p_x'$ and $p_y$ (or $d_{xz}$ and $d_{yz}$) directionality in space, which in our problem signifies near-collinear peaking and near-isosceles peaking in the configuration space of triangles. While, $|j| = 2$ picks out both of the above equally ($d_{xy}$) or avoids both equally ($d_{x^2 - y^2}$).

In the second approximation, $2[\tilde{E} + \tilde{U}] = Q_0 - Q_2 R$, where $Q_2$ is another constant evaluated in section I.4.9, and which I consider here to be strictly positive. In this case the time-independent Schrödinger equation is

$$\frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \Phi^2} + \left( \frac{Q_0}{2 \hbar^2} - \frac{Q_2 R^2}{2} \right) \Psi = 0,$$

(36)

(where I use $Q_2 \equiv Q_2/\hbar^2$), which is also a familiar equation, straightforwardly mapping to the 2D isotropic harmonic oscillator [50, 53, 54] by the correspondence in appendix B. It separates into simple harmonic motion (15) and a radial equation (16) that this time can be mapped to the associated Laguerre equation (B.2). Thus this problem’s solution is as follows. In addition to the relative angular momentum quantum number $j \in \mathbb{Z}$, there is a ‘radial’, ‘principal’ or ‘node-counting’ quantum number $R \in \mathbb{N}_0$ such that

$$Q_0/2\hbar \sqrt{Q_2} = |j| + 2R + 1$$

(37)
holds. In terms of the original quantities of the problem, this corresponds to
\[ E = K_2/2 + \hbar \sqrt{4E + K_1 - 3K_2} |j| + 2R + 1 \] (38)
or, in the language of the spherical problem
\[ E = A + B + \hbar \sqrt{E - A - 2B} |j| + 2R + 1. \] (39)

Then
\[ E \geq K_2/2 + \hbar \sqrt{4E + K_1 - 3K_2} (> 0) \] (40)
is necessary for there to be any solutions at all. While, the non-standard interpretation that all of \(E, K_1, K_2\) are fixed will restrict the number of simultaneously relevant solutions in my closed-universe context; e.g. rational–irrational incompatibility can be used to construct cases with no solutions, but there are also cases for which there remain interesting degenerate possibilities such as \((R, j) = (0, \pm 2)\) and \((1, 0)\).

The wavefunctions are
\[ \Psi_{\alpha j}(R, \Phi) \propto R^{3/2} \exp(i \sqrt{Q_2 R^2}/2) L^{|j|}_R \left( \sqrt{Q_2} R \right) \exp(i j \Phi) \] (41)
or, in spherical variables,
\[ \Psi_{\alpha j}(\Theta, \Phi) \propto \tan \frac{\Theta}{2} \exp \left( \sqrt{Q_2} \tan \frac{\Theta}{2} \right) L^{|j|}_R \left( \sqrt{Q_2} \tan \frac{\Theta}{2} \right) \exp(i j \Phi). \] (42)

While in terms of the partial moments of inertia-relative angle variables, it is
\[ \Psi_{\alpha j}(I_1, I_2, \Phi) \propto \left| \begin{array}{c} I_1 \\ I_2 \end{array} \right|^{3/2} \exp \left( -\sqrt{Q_2} \frac{I_1}{2 I_2} \right) L^{|j|}_R \left( \sqrt{Q_2} \frac{I_1}{2 I_2} \right) \exp(i j \Phi) \]
\[ = \left| \begin{array}{c} I_1 \\ I_2 \end{array} \right|^{3/2} \exp \left( -\sqrt{Q_2} \frac{I_1}{I - I_1} \right) L^{|j|}_R \left( \sqrt{Q_2} \frac{I_1}{I - I_1} \right) \exp(i j \Phi) \]
\[ = \left| \begin{array}{c} I - I_2 \\ I_2 \end{array} \right|^{3/2} \exp \left( -\sqrt{Q_2} \frac{I - I_2}{2 I_2} \right) L^{|j|}_R \left( \sqrt{Q_2} \frac{I - I_2}{2 I_2} \right) \exp(i j \Phi). \] (43)

While, in terms of the original coordinates for the problem,
\[ \Psi_{\alpha j}(\xi_1, \xi_2) \propto \left| \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right|^{3/2} \exp \left( -\sqrt{Q_2} \frac{\xi_1^2}{2 \xi_2^2} \right) L^{|j|}_R \left( \sqrt{Q_2} \frac{\xi_1^2}{\hbar \xi_2^2} \right) \exp \left( i j \arccos \left( \frac{\xi_1 \cdot \xi_2}{\xi_1 \parallel \xi_2} \right) \right). \] (44)

That a 2D isotropic harmonic oscillator with eigenvalues (37) resides within the scalefree triangleland multiple harmonic oscillator like problem as a limit problem has counterpart in the literature on the linear rotor ([55], see also the figure in [56]). See figure 2 for this and the preceding subsection’s probability density functions.

Note the following:

1. As regards which inner product to use, in the first approximation, one can use the naïve \(\mathcal{R}\) in place of \(\mathcal{R}/[1 + \mathcal{R}^2]^2\) at the cost of an error of up to 2% over \(\mathcal{R} \in [0, 0.1]\). This however swamps the improvement in passing from first to second approximation; one should use at least \(\mathcal{R}[1 - 2\mathcal{R}^2]\) in the latter case so as to not compromise accuracy.

2. The unusual inner product in use here does cause these probability density function differ somewhat from the usual ones for a 2D constant-potential or 2D isotropic harmonic oscillator: the peaks are somewhat shifted inwards, and the inner peaks gain in relative importance; these are signs of greater confinement.
(3) How can few-peak functions be second approximations when infinite-peak functions are first approximations? The trick is that the few-peak functions have an extra node-counting quantum number \( R \), so that the first approximation can be seen as a superposition of many different values of \( R \) each with its own \( Q_L \) as dictated by (37), and thus build up a multiplicity of peaks. For the values considered, there is better than 1% accuracy between the two approximations up to some value of \( R \) of order of magnitude 0.01 to 0.1.
2.9. Large-asymptotics solutions for special harmonic oscillator and other problems

I now apply duality to write down first and second large approximation solutions. The first approximation gives

\[ \Psi_j(R, \Phi) \propto J_j(\sqrt{Q_4}/R) \exp(\imath j \Phi) \] or \[ \Psi_j(\Theta, \Phi) \propto J_j(\sqrt{Q_4} \cot \frac{\Theta}{2}) \exp(\imath j \Phi). \]

Thus, in terms of the physically and visually useful partial moments of inertia–relative angle variables, it is

\[ \Psi_j(I_1, I_2, \Phi) = J_j(\sqrt{Q_4}I_2/I_1) \exp(\imath j \Phi) = J_j(\sqrt{Q_4}(I - I_1)/I_1) \exp(\imath j \Phi) = J_j(\sqrt{Q_4}I_2/(I - I_2)) \exp(\imath j \Phi). \]
While, in terms of the original coordinates of the problem,

$$\Psi_{1j}(\iota_1, \iota_2) \propto J_j \left( \sqrt{\mathcal{Q}_4} \left\| \iota_1 \right\| \iota_2 \right) \exp \left( ij \arccos \left( \frac{\iota_1 \cdot \iota_2}{\| \iota_1 \| \| \iota_2 \|} \right) \right).$$

(45)

As happens classically, the above analysis also holds for $\tilde{\Psi}_{(\omega,0)}$ with constant of proportionality $\Lambda_1 \alpha$, one gets exactly the same large-asymptotics analysis as here, with $q_0 = 2E - \Lambda_1 \alpha$, $q_2 = 4E - \{4 + \alpha\} \Lambda_1 \alpha$. Thus the conditional duality map has put the curious large asymptotic region in firmly understood terms. See figure 3 for the probability density functions corresponding to this.

2.10. Special triple harmonic oscillator problem treated perturbatively for small $B$

Let us treat the special triple harmonic oscillator problem as a perturbation about the simple ‘very special triple harmonic oscillator’. For a perturbation $\mathcal{H}'$ to one’s rescaled Hamiltonian (rescaled as the other calligraphic quantities in (19) are) the first few objects of perturbation theory as used in this paper are as follows [52]. For $E_{(0)}^{(1)} = E_{(0)}^{(1)}$, $E_{(1)}^{(1)} = (J|\mathcal{H}'|J)$, $|\Psi_{(1)}^{(1)} = - \sum_{K \neq J} \frac{|K|\mathcal{H}'|J|}{\mathcal{E}_J - \mathcal{E}_K}$ at first order and $E_{(2)}^{(1)} = - \sum_{K \neq J} \frac{|K|\mathcal{H}'|J|^2}{\mathcal{E}_J - \mathcal{E}_K}$ at second order. For $E_{(0)}^{(2)}$ degenerate, one needs to solve at first order $\sum_{S} \langle J, j|\mathcal{H}'|J, s \rangle a_{(1)}^{(1)} = E_{(1)}^{(1)} a_{(1)}^{(1)}$, and, at second order, $\sum_{K \neq J} \sum_{S} \langle K, j|\mathcal{H}'|J, s \rangle a_{(2)}^{(1)} = E_{(2)}^{(1)} a_{(2)}^{(1)}$.

For us, the $|\Psi_{jj} = |J, j \rangle$ are (27) and $H' = B'X$ for $X$ the ‘Legendre variable’ related to $\Theta$ by $X = \cos \Theta$. Then the key integral underlying time-independent perturbation theory is $\langle \Psi_{jj}|V|\Psi_{jj}\rangle$ (here there’s no subtlety in the style of section 2.3 with the inner product as the banal conformal transformation here has a merely constant conformal factor and so cancels out upon performing normalization). The above has as its nontrivial factor $\int_{-1}^{1} P^J_j(X) X P^J_j(X) dX$, but there is a recurrence relation (B.11) enabling $X P^J_j(X)$ to be turned
into a linear combination of $P_{j'}^j(X)$, whereupon orthonormality of the associated Legendre functions (B.10) can be applied to evaluate it. This calculation parallels that in the derivation of selection rules for electric dipole transitions [51, 57]. Then $(J, j'|BX|J, j) = 0$ since the recurrence relation sends $XP_j^j$ to a sum of $P_{j'}^j$ for $j' \neq j$, so each contribution to the integral vanishes by orthogonality. While, for $(J', j'|BX|J, j)$, one similarly needs $J' = J \pm 1$ and $j' = j$ to avoid it vanishing by orthogonality. So two cases survive this ‘selection rule’. The first is, by direct computation,

$$
\langle J + 1, j|BX|J, j \rangle = B \left( \frac{(J + 1)^2 - j^2}{[2J + 1][2J + 3]} \right),
$$

(46)

while the second is

$$
\langle J - 1, j|BX|J, j \rangle = \langle J, j|BX|J - 1, j \rangle = B \left( \frac{J^2 - j^2}{[2J - 1][2J + 1]} \right),
$$

(47)

by using (46) with $J - 1$ in place of $J$ (paralleling, e.g. [50]).

Figure 3. Using the same ‘radial’ inner product as in figure 1, I obtain the following probability density functions in terms of $R$. (In terms of $I_1$, these have the same form as the corresponding small solution’s probability density function in terms of $I_2$ and vice versa, so I do not provide any new probability density functions as functions of $I_1, I_2$.) (a) For the first approximation, I provide an archetypal probability density function for $Q_1 = 1$ and $10^4$. (b) For the second approximation, I provide the probability density function’s for $(R, j) = (0, 0)$, $(0, 1)$ and $(1, 0)$ ($(0, 2)$ is qualitatively similar to $(0, 1)$) for $Q_4 = 1$ and then $10^4$. 


The eigenspectrum is thus
\[ E_{J,j} = J(J + 1) + A + \frac{B^2[J(J + 1) - 3j^2]}{2J(J + 1)[2J - 1][2J + 3]} + O(B^4) \]  
(48)

so
\[ E_{J,j} = 2\hbar^2 J[J + 1] + 4A + \frac{4B^2[J(J + 1) - 3j^2]}{\hbar^2 J(J + 1)[2J - 1][2J + 3]} + O(B^4). \]  
(49)

For \( J = 0 \), one needs a separate calculation, which gives
\[ E_{0,0} = 4A + \frac{4B^2}{3\hbar^2} + O(B^4). \]  
(50)

This calculation can then be checked against its rotor counterpart (originally in [58] and which can also be found in, e.g. [49, 50]). The corresponding eigenfunctions can be looked up (e.g. [59]) and reinterpreted in terms of the original problem’s mechanical variables.

2.11. Placing a closed-universe interpretation on the perturbed problem

Note well the above are universes not modes within a particular universe—one has just the one value of \( E \), \( E_{\text{univ}} \). Sometimes [19], this corresponds to no allowed \( J, j \), sometimes to one and at least sometimes to more than one. For example, \( E_{0,0} = E_{1,0} \) for \( B = \sqrt{15}/2\hbar^2 \), which is perturbatively acceptable provided that \( A \gg B \). From this, one can extract partition functions using [60] to take into account that the energies are only known perturbatively. Hence one can extract a classical notion of entropy and hence of information. Also, from knowing the wavefunctions, one can construct mixed states and then the quantum-mechanical von Neumann information corresponding to these.

But for records-theoretic purposes, it is information content of subsystems and relative information between subsystems that look to be more significant quantities. These are obtained from solving the subsystem quantum problems (subject to global restrictions such that the subsystem energies add up to a fixed energy of the universe [5]) and then proceeding to compute notions of information similar to the above (but now including relative information between pairs of subsystems that is a quantity of higher relevance as regards records-theoretic approaches). This would require considerable further study.

2.12. Exploiting further correspondences with the rotor problems

The following are available in the rotor literature (in the spherical presentation) and could thus be straightforwardly transcribed to the various presentations for this paper’s scalefree triangleland problem. Higher order corrections in \( B \) are in the literature for the rotor [61, 62] and so can be transcribed into the relational context (e.g. I was able to write \( O(B^4) \) and not \( O(B^3) \) in (49) due to this). Alternative variational methods (using the Hellmann–Feynman and hypervirial theorems) appear in [62]; these cover higher order terms too, and can be used to show generally that only even powers of \( B \) occur. The calculation for the large \( B \) regime has both been done [59] and matched to small \( B \) regime calculations.

Numerical evaluation of eigenvalues was been done by Lamb’s [63] continued fraction method [55, 61, 64] (and otherwise, e.g. [56, 62]). Myself, I just simply used Maple’s\(^4\) rk45 solver alongside the iteration of the method in, e.g. [54] to locate the eigenvalues, which gives reasonable agreement with the formula at the end of section 2.10.

\[^4\] This was done using Maple 10 to 12.
The rotor literature also indicates how the $C$ perturbation can be transformed away with a new rotated choice of coordinates in which the maths is again that of a $B$ type perturbation (section 3—the quantum application of the trick in section I.5). While in the laboratory with a rotor one could choose one’s axial ‘$z$’ direction to be in the most convenient direction, there are various problem of time strategy modelling reasons not just to stay in these coordinates in our triangleland problem (see the next section).

3. General triple harmonic oscillator problem for scalefree triangleland

In the spherical coordinate representation, this takes the form

$$\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left\{ \sin \Theta \frac{\partial \Psi}{\partial \Theta} \right\} + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Psi}{\partial \Phi^2} + \{E - A - B \cos \Theta - C \sin \Theta \cos \Phi \} \Psi = 0,$$

(51)

where $A$, $B$, $C$ are given by (19) and

$$C = 2C/\bar{\hbar}^2.$$  

(52)

This is harder because the angle dependence of the potential results in nonseparability in these natural coordinates, which makes for a useful model of the semiclassical approach to the problem of time [47].

3.1. Solution via use of rotation/normal modes/adapted basis

As in [8], start again using rotated/normal coordinates at the classical level, or switch to such coordinates at the differential equation solving stage, amounting to a choice of basis in which the perturbation is in the (new) axial ‘$z_N$’ direction. While it can be viewed as before in the new rotated/normal coordinates, nevertheless studying the original coordinates’ $\Phi$-dependent $V$ term remains of interest as it may well be appropriate for the original coordinates to have mechanical attributes or the heavy-light subsystem distinction underlying the semiclassical approach$^5$. Moreover in that case, being able to proceed further in the rotated/normal coordinates can serve as a check on ‘standard’ procedures in the original coordinates (along the lines suggested in [37]), e.g. as a test of the validity and accuracy of assumptions and approximations made in the semiclassical approach. This section serves to begin to set up such a check point. Another reason why one might not adopt the normal coordinates as the physically significant ones is in order to use $J$ as a trackable non-conserved quantity as regards investigating the semblance of dynamics in a fundamentally timeless records theory approach.

There is now an issue in the projection that there is an additional factor due to the change in area in moving between each patch of sphere and each corresponding patch of plane. Namely, the probability density function on the sphere is $\sin \Theta |\Psi(\Theta, \Phi)|^2$ of which the $\sin \Theta$ pertains to the sphere itself, while the probability density function on the stereographic plane is $R/(1 + R^2)|\Psi(R, \Phi)|^2$ of which the $R/(1 + R^2)$ pertains to the stereographic plane itself. The very special solution now suffices as an illustration. The analytic form of its probability density function is

$$\text{PDF}(\Theta_N, \Phi_N) \propto \sin \Theta_N \left\{ P_j^N (\cos \Theta_N) \right\}^2$$

(53)

and so

$$\text{PDF}(\Theta, \Phi) \propto \sin \Theta \left\{ P_j^N \left( \frac{B \cos \Theta + C \sin \Theta \cos \Phi}{B_N} \right) \right\}^2$$

(54)

$^5$ In the laboratory, one might likewise not pick the normal coordinates of the rotor-electric field if there is, e.g. also a magnetic field that picks out a different direction.
and so

\[
\text{PDF}(R, \Phi) \propto \frac{R}{1 + R^2} \left\{ \frac{P^j_j \left( \frac{B(I - R^2) + 2C R \cos \Phi}{B_N(1 + R^2)} \right)}{B_N} \right\}^2. \tag{55}
\]

Then, in terms of the partial barycentric moments of inertia,

probability density function \((I_1, \Phi) \propto \left\{ p^j_j \left( \frac{B(I - 2I_1) + 2C \sqrt{I_1(I - I_1) \cos \Phi}}{B_N} \right) \right\}^2 \tag{56}\)

and

\[
\text{PDF}(I_2, \Phi) \propto \left\{ P^j_j \left( \frac{B(2I_2 - I) + 2C \sqrt{I_2(I - I_2) \cos \Phi}}{B_N} \right) \right\}^2. \tag{57}
\]

For \((J, j) = (0, 0)\), these are the same as for the \(C = 0\) case, so there is no need to provide a new plot. However for higher values of \((J, j)\), they are distinct (figure 4).

4. Conclusion

4.1. Results summary

Study of relational particle mechanics is motivated by the absolute versus relative motion debate and the analogy between relational particle mechanics and the canonical formulation of general relativity. Specific quantum similarity relational particle mechanics of \(N = n + 1\) particles in 2D can be constructed due to knowledge in this case that the classical configuration space is \(\mathbb{C}P^{n-1}\) \([7, 11, 65]\). Likewise, specific similarity relational particle mechanics of \(N\) particles in 1D can be constructed due to the knowledge that in this case the classical configuration space is \(S^{n-1}\). This paper provides a quantum study of similarity relational particle mechanics with harmonic oscillator like potentials, for which I use exact, asymptotic, perturbative and numerical methods. Throughout, a mathematical analogy with the linear rigid rotor in a background electric field is useful; I then transcribe this from spherical/stereographic plane terms to be in terms of (partial barycentric moments of inertia, relative angle) variables and the original similarity relational particle mechanics problem’s mass-weighted relative Jacobi variables. In particular,

1. In the spherical representation there is a constant-potential subcase soluble thereupon in terms of spherical harmonics.
2. I then consider relative angle \(\Phi\) independent potentials, corresponding to the existence of a relative angular momentum type conserved quantity whereby the classical and quantum theory is simplified. This permits solution of the large and small stereographic coordinate regimes in terms of Bessel functions, and, more accurately, in terms of associated Laguerre polynomials; the large regime is, moreover, universal rather than dependent on the choice of harmonic oscillator like potentials.
3. There are various theoretical reasons to wish to remove the \(\Phi\)-independent restriction—dynamical and quantum mechanical nontriviality and genericity, as well as toy-modelling semiclassical and records theory approaches to the problem of time in quantum gravity. Thus I also treat \(\Phi\)-dependent potentials by a coordinate rotation/normal modes/adapted basis construction.
4. In each case, I then interpret these solutions in terms of the underlying mechanical variables, including investigation of which triangles formed by the particles are more and less quantum-mechanically probable in a given state.
Figure 4. For $B = 1$ with $C = 0.1$, then 1 and then 10, I give the following triples of plots. PDF $(\mathcal{R}, \Phi)$ for $(j, j) = (1, 0)$. PDF $(\mathcal{R}, \Phi)$ for $(j, j) = (2, 0)$. PDF $(I_1, \Phi)$ for $(j, j) = (1, 0)$. PDF $(I_1, \Phi)$ for $(j, j) = (2, 0)$, the first of which has a shallower ring hidden inside the visible outer ring.
4.2. Further extensions

The present paper is furthermore useful in that many methods used in it can furthermore be used towards solving other concrete relational particle mechanics examples. Following the success of identifying the scalefree triangleland multiple harmonic oscillator like potential problem with the linear rigid rotor Stark effect, surveying the quantum chemistry literature for analogues of some of the below may be useful.

4.2.1. Scalefree 4-stop metroland also has the configuration space $S^2$. Then the conformal-ordered time-independent Schrödinger equation for multiple harmonic oscillator potentials is

$$\frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial \Psi}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Psi}{\partial \Phi^2} = A + B \cos(2\Theta) + C \sin^2 \Theta \cos(2\Phi),$$  

or, in Legendre variables,

$$\frac{\partial}{\partial X} \left( \frac{1 - X^2}{X} \frac{\partial \Psi}{\partial X} \right) + \frac{1}{1 - X^2} \frac{\partial^2 \Psi}{\partial \Phi^2} = D + X^2 \left[ 2B - C \cos(2\Phi) \right] + C \cos(2\Phi),$$  

where $A = 2(A - E)$, $B = 2B/\hbar^2$, $C = 2C$, and $D = A - B$. 

Figure 4. (Continued.)
Then for $B = C = 0$, one obtains the Legendre equation. Then for the $B$ perturbation, double use of the recurrence relation (B.10) gives rise to a $\Delta j = 0, \Delta J = 0, \pm 2$ 'selection rule'. While, the $C$ perturbation has $\int_{-1}^{1} P_{j+1/2}^{(0)}(X)(1 - X^2)P_{j}^{'}(X) dX$ contributions, for which double use of the recurrence relation (B.11) gives in this case the selection rule $\Delta j = 0, \Delta J = 0, \pm 2$. Note that normal mode trick does not work for this problem, so here one has to study the $C$ perturbation as well as the $B$ perturbation.

4.2.2. Higher N-stop metrolands. Here the conformal-ordered time-independent Schrödinger equation is, in terms of ultraspherical angles,

$$\prod_{j=1}^{A-1} \sin^2(\Theta_j) \sin^{n-1-A} \Theta_A \frac{\partial}{\partial \Theta_A} \left\{ \sin^{n-A} \Theta_A \frac{\partial \Psi}{\partial \Theta_A} \right\} = -E \Psi + \sum_{p=1}^{n} K_p n_p^2 \Psi$$

where $E = 2E/\hbar^2, K_p = K_p/\hbar^2, n_p$ is the unit vector of the embedding Euclidean configuration space $R(N, 1) = \mathbb{R}^n$ and $E$ has been redefined to incorporate the conformal term since (hyper)spheres are of constant Ricci scalar.

There is also a highly special constant-potential case within the multiple harmonic oscillator like potentials. This is now a more complicated sequence of associated Gegenbauer equations as explained in appendices A and B; these are nevertheless also fairly standard and well documented [66, 67]). One can then study perturbations about this. Then, if the associated quantum number is not zero, one does not get the Gegenbauer pairings or the right weights straight away (see appendix B). Computation of first-order perturbations in this case requires the Gegenbauer parameter converting recurrence relation (B.14) as well as the polynomial order reducing recurrence relation (B.13), making the calculation somewhat more complicated in this case.

4.2.3. Yet further extensions. As regards $(N > 3)$-a-gonland’s ‘genuine’ $\mathbb{C}P^d$ mathematics (rather than $\mathbb{C}P^3$ mathematics that is re-expressible as $S^2$ mathematics) is required, placing this beyond the scope of the present paper.

An alternative type of extension is to keep a given geometry but consider a variety of other potentials, e.g. Coulomb-like potentials or a mixture of Coulomb-like and harmonic oscillator like potentials.

4.3. Applications to classical and quantum geometrodynamics

The semiclassical approach to the problem of time in quantum gravity requires nonseparability so that the crucial cross-term in (7) exists. To set up such a model, one could have, e.g. two heavy ($H$) particles and one light ($L$) one, leading to one $H$ relative Jacobi separation and one $L$ one. The nonseparability requirement would mean requiring the interpretation that the physics picks out $\Theta$ and $\Phi$ variables unaligned with the simplifying normal modes ones (in terms of which there is separability), and that the potential be $\Phi$-dependent. Another way of setting up a semiclassical approach model using the material in this paper is if $\Theta$ and $\Phi$ are considered to be $H$ and $L$ respectively (the opposite assignment is impossible as $\sin^2 \Theta$ cannot be $\gg 1$). The above two situations would be a significant improvement on the example in [37] because that has no nontrivial linear constraints; it remains to be seen how far such a calculation could be taken analytically (and, if needs be, numerically and/or subject to further approximations). While, the [37] example’s ability to be solved by usually unavailable means

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6 An alternative computational scheme is to proceed by the formulae for integrals of products of three spherical harmonics in, e.g. [57].
external to the semiclassical approach is retained by the above examples—it is the work in the present paper. Thereby one can assess whether the semiclassical ansätze and associated (and any other non-associated but unavoidable) approximations are sensible for these new semiclassical approach models. To have a *shape-scale aligned* \( H \text{--} L \) split, which in some ways more closely parallels GR cosmology \((H \text{ scalefactor versus } L \text { inhomogeneities and/or anisotropies})\), one needs to study the Euclidean relational particle mechanics counterpart of this paper \([17]\). This is also the case if one wishes to obtain a dilational (York-like) internal time model \([6, 37]\).

As regards building concrete relational particle mechanics examples of the records theory approach to quantum gravity, a notion of distance on configuration space readily follows \([40]\) from the metrics in \([11]\). As regards computing a notion of information/negentropy, given an explicitly solved QM, one can (cf section 2.11) build a statistical mechanics from that and extract the negentropy/information. Moreover, this continues to be the case when the QM is only known perturbatively \([60]\) (with correction terms up to the corresponding perturbative order \([60]\)), which is one reason for the relevance of the perturbative calculations in this paper. Further questions then are: does this paper’s model have a notion of information storage? (In parallel with \([39]\), might a heavy particle passing by—the signal—imprint separation/motion on the other two particles—the record? \([39]\) proceeds to study this via path integrals and associated objects such as the influence functional; the extent to which these more specialized computations have been carried out for the analogous rotor is an interesting question, whose investigation might substantially cut down on how much work is needed to complete the parallel calculation to \([39]\) for the present paper’s models.)

As regards semblance of dynamics emerging from timeless records theory, the relative angular momentum \( J \) that was a conserved quantity for \( B = 0 \) becomes a changing quantity for \( C \neq 0 \), so tracking and explaining that may be of significance. I can track this classically by, e.g. computing it at each stage in the rkf45 routine. An interesting question then is whether and how this could be tracked quantum-mechanically? As regards whether evidence can be found for/against Barbour’s conjecture of time capsules \([3, 24]\), the probability density functions plotted in this paper are the right output for addressing that. The fairly standard maths I obtain (at least in my simple specific example of harmonic oscillator like potentials) suggests that, if time capsules do occur, they ought to be findable also within standard QM (for perturbed linear rigid rotors). However, further scaled triangleland or trihaedronland \( (=3\text{-haedronland}) \) work would be necessary as regards more specific conjectures Barbour makes about time capsules that were specifically made about 3-particle Euclidean relational particle mechanics and whether its triple collision or uniform (i.e. equilateral) configurations play highly dominant roles (where the probability density function ‘mist’ might be highly concentrated). For the moment, my simple similarity relational particle mechanics model would not seem to exhibit very heavy peaking around its equilateral configuration.

This paper’s model is also conceivably an interesting one from the perspective of histories theory and as regards the problem of finding (partial) observables for quantum gravity.

On the whole, this paper downplays the suggestion \([3, 16]\) that (Barbour’s) relationalism requires radically different QM theory, though one would need to check further examples (including more complicated ones) to be more sure of this conclusion (some of the further examples above can be motivated on such grounds). Some closed-universe and finite-universe effects are, however, manifest. As regards the issue \([3, 19]\) of whether stationary quantum universes have single or multiple states, I comment that this paper’s models do exhibit some degenerate states (both among simple exact solutions and perturbatively to second order); however one needs a much more extensive study of relational particle mechanics model universes before one can begin to say whether these are, however, non-generic. There is also
a certain amount of tension between results obtained by reduced quantization as in this paper and by Dirac-like quantization as in [5], though I postpone discussion of this to [17] (one of the problems being that my method of Dirac-like quantization in [5] for Euclidean relational particle mechanics does not directly extend to similarity relational particle mechanics, so that checks between the two methods and lessons drawn from them are best left to the Euclidean relational particle mechanics arena).

Treating cases with more than three degrees of freedom will be necessary to get a grip of some aspects. This is the case first for investigation whether there are kinetic effects of the kind that Barbour conjectures [3, 24] as regards a semblance of time arising from timeless configurations. All of the $S^{n−1}$ being conformally flat, one would need to extend to such as the non-conformally flat $\mathbb{CP}^2$ so that effects that are irreducibly kinetic occur (rather than cases for which a kinetic and potential re-definition leaves one with a flat kinetic term in the end by passing the conformal factor into a re-defined potential). Second, for ($N \geq 4$)-stop metroland, the relative angle dependence comes from writing the potential in coordinates dictated by the kinetic term, and no longer in such a way that the standard rotation to normal coordinates removes this complication. A third such point, which constitutes an interesting further investigation in its own right, is that $N$-particle models also have a robustness application: is the QM of $N − 1$ particles stable to the inclusion of a further particle? This parallels the situation of whether Taub space is stable within the mixmaster solution in minisuperspace quantum cosmology (found to be unstable in [68]). Fourth, upgrading to models with $>3$ particles is likely to improve one’s capacity to use particle clumps to model inhomogeneities. A final question that such models can be used to investigate is whether the conformal ordering succeeds in avoiding conflict with other necessary technical conditions.

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Appendix A

Following from appendix I.A.2, the energy constraint gives, in conformal ordering, the time-independent Schrödinger equation

$$\hat{H}\psi = -\hbar^2/2 \left\{ \frac{\partial}{\partial Q^A} \left\{ \sqrt{M} N^{AB} \frac{\partial \psi}{\partial Q^B} \right\} - \frac{k-2}{4(k-1)} \text{Ric}(M)\psi \right\} + V\psi = E\psi. \quad (A.1)$$

A.1. Scalefree $N$-stop metroland

Here, by p 1269-70 of [69], an appropriate finite subalgebra acting on the corresponding cotangent space is then $\text{SO}(nd) \text{S}\mathbb{R}^n$, where $\text{S}$ stands for semidirect product. This can be considered to be generated by angular momenta $J_i$ and coordinates $u_i$ such that $\sum_i u_i^2 = 1$. 

Now, the Laplacian corresponding to line element (I.20) is
\[ D^2 = \prod_{j=1}^{n-2} \sin^{1-j} \Theta_j \frac{\partial}{\partial \Theta_j} \left\{ \frac{\prod_{\ell=1}^{n-2} \sin^{1-\ell} \Theta_\ell}{\prod_{i=1}^{n-2} \sin^2 \Theta_i} \frac{\partial}{\partial \Theta_\ell} \right\} \]
\[ = \frac{1}{\sin^{n-1-\Lambda} \Theta_n} \prod_{i=1}^{n-1} \sin^2 \Theta_i \frac{\partial}{\partial \Theta_n} \left\{ \sin^{n-1-\Lambda} \Theta_n \frac{\partial}{\partial \Theta_n} \right\} . \quad \text{(A.2)} \]

This situation applies to preshape space and for shape space \([65, 11]\). Hence QM on preshape space for \(N\) particles in dimension \(d\) has the time-independent Schrödinger equation
\[ -\frac{\hbar^2}{2} \frac{1}{\sin^{n-1-\Lambda} \Theta_n} \prod_{i=1}^{n-1} \sin^2 \Theta_i \frac{\partial}{\partial \Theta_n} \left\{ \sin^{n-1-\Lambda} \Theta_n \frac{\partial \Psi}{\partial \Theta_n} \right\} + V\Psi = E\Psi \quad \text{(A.3)} \]
(where the energy has been displaced by a constant term from the constant-curvature contribution to the conformal ordering). The \(d = 1\) case of this is, additionally, the time-independent Schrödinger equation for scalefree \(N\)-stop metroland, i.e. equation (10). These Hamiltonians involve suitable quantum operators (see, e.g. p 160 of [70] for an account of the properties of the constituent Laplacian operator on \(S^{n-1}\), see also [71, 72]).

For constant potential, this time-independent Schrödinger equation is an equation of form
\[ D^2 \Psi = \Lambda \Psi. \]
Then the separation ansatz \(\Psi = \prod_{\ell=1}^{n-1} \psi_\ell(\Theta_\ell)\) yields the simple harmonic motion equation for \(\Theta_{n-1}\) and \(n-2\) equations of form
\[ (1 - X_{n-p}^2) \frac{d^2 \psi_{n-p}}{dX_{n-p}^2} - (p-1)X_{n-p} \frac{d\psi_{n-p}}{dX_{n-p}} + j_{p-1}(j_{p-1} + p - 2)\psi_{n-p} \]
\[ - \frac{j_{p-2}(j_{p-2} + p - 3)}{1 - X_{n-p}^2} \psi_{n-p} = 0 \quad \text{(A.4)} \]
under the transformations \(X_{n-p} = \cos \Theta_{n-p}, \Theta_{n-p} = 1 \to n - 2\). These are associated Gegenbauer equations (B.15) with parameter \(\lambda_p = (p-2)/2\), where the integers \(j_{p-1}, j_{p-2} \in \mathbb{N}_0\) are picked out as eigenvalues, so that the \((n-p)\)th equation is solved by \(C_j^{\lambda_p-2}(\cos \Theta_{n-p}; (p-2)/2)\). Then one gets a sequence of integer quantum numbers beginning with the familiar \(|j_i| \leq j_2\).

\[ \text{A.2. Scalefree } N\text{-a-gonland} \]

In this case, as regards kinematic quantization for scalefree \(N\)-a-gonland, use that \(S(N, 2) = \mathbb{C}^{\mathbb{N}_0^{n-1}} = SU(n) / U(n-1)\), which is of the general form \(Q = G/H\) considered in [69]. Thus a suitable finite algebra acting on the corresponding cotangent space is \(SU(n) \otimes \mathbb{R}^{2n}\).

The Fubini–Study Laplacian for \(S(N, 2)\) is then
\[ D^2 = \frac{1 + \|R\|^2}{\prod_{p=1}^{n-2} R_p} \frac{\partial}{\partial R_p} \left\{ \frac{\prod_{p=1}^{n-1} R_p}{[1 + \|R\|^2]^{2n-3}} \left[ \frac{\delta R_p}{\partial R_p} + \frac{\delta R_q}{\partial R_q} \right] \frac{\partial}{\partial R_q} \right\} \]
\[ + \frac{\partial}{\partial \Theta_p} \left\{ \frac{\prod_{p=1}^{n-1} R_p}{[1 + \|R\|^2]^{2n-3}} \left[ \frac{\delta R_p}{\partial R_p} + 1 \right] \frac{\partial}{\partial \Theta_q} \right\} . \quad \text{(A.5)} \]
so that the conformal-ordered time-independent Schrödinger equation for the 2D shape space of \(N\) particles is (11).
A.3. Scalefree triangleland case

In the tilded banal conformal representation, the (3, 2) case’s Laplacian in the flat coordinates ($R, \Phi$) takes the familiar form

$$D^2 = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2}{\partial \Phi^2}.$$  \hfill (A.6)

While, in the barred banal conformal representation in spherical coordinates ($\Theta, \Phi$), it takes the also-familiar form

$$D^2 = \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2}{\partial \Phi^2}.$$  \hfill (A.7)

While I provide a parallel scheme for general $\mathbb{CP}^k$ above, it is the $\mathbb{CP}^1 = S^2$ version actually used in this paper’s calculations for which I currently have guarantees of good behaviour as an operator.

Appendix B. Special functions and mappings of ordinary differential equations

The Bessel equation of order $\nu$,

$$v^2 w_{vv} + vw_v + [v^2 - \nu^2] w = 0,$$  \hfill (B.1)

is solved by the Bessel functions. I denote Bessel functions of the first kind by $J_{\nu}(v)$.

The associated Laguerre equation,

$$xy_{xx} + [\alpha + 1 - x]y_x + ny = 0,$$  \hfill (B.2)

is solved by the associated Laguerre polynomials $L_{\alpha}^{\nu}(x)$ (and unbounded second solutions). The 2D quantum isotropic harmonic oscillator’s radial equation for a particle of mass $\mu$ and classical oscillator frequency $\omega$,

$$-\hbar^2 \frac{1}{2\mu} \left( \frac{R_{rr}}{R} + \frac{R_r}{r} - \frac{m^2 R}{r^2} \right) + \frac{\mu \omega^2 r^2 R}{2} = ER,$$  \hfill (B.3)

maps to the associated Laguerre equation under the asymptotically motivated transformations

$$R = (hx/\mu \omega)^{|m|/2} e^{-x/2} y(x), x = \mu \omega r^2/\hbar$$  \hfill (B.4)

and so is solved by

$$R \propto r^{|m|} e^{-\mu \omega r^2/2\hbar} L_{\alpha+1}^{\nu}(\mu \omega r^2/\hbar),$$  \hfill (B.5)

corresponding to the discrete energies $E = (|m| + 2r + 1)\hbar \omega$ for radial quantum number $r \in \mathbb{N}_0$ [53, 54].

The associated Legendre equation

$$\{1 - X^2\} Y_{XX} - 2XY_X + \{J[J + 1] - J^2\{1 - X^2\}^{-1}\} Y = 0,$$  \hfill (B.6)

(which is equivalent to the equation

$$\sin^{-1}\Theta [\sin \Theta Y_\Theta]_{\Theta} + \{J[J + 1] - J^2 \sin^{-2}\Theta\} Y = 0,$$  \hfill (B.7)

under the transformation $X = \cos \Theta$), is solved by the associated Legendre functions $P_j^{|j|}(X)$ (and unbounded second solutions), for $J \in \mathbb{N}_0, j \in \mathbb{Z}, |j| \leq J$. We use the standard convention that

$$P_j^{|j|}(X) = \{(-1)^j\{1 - X^2\}^{\frac{j}{2}} \frac{d^j}{dX^j}\left\{ \frac{1}{2J^j} \frac{d^j}{dX^j} [X^2 - 1]^j \right\},$$  \hfill (B.8)
whereupon
\[
\left\{ \frac{2J + 1}{2} \frac{\{J - [j]|\}}{[J + |j|]!} P^{|j|}_J(X) \right\}
\]

is a complete set of orthonormal functions for \( X \in [-1, 1] \). We also require the recurrence relations [67, 66]
\[
XP^{|j|}_J(X) = \frac{\{J - [j]|\} + [J + |j|]}{2J + 1} P^{|j|+1}_J(X) + (J + |j|)P^{|j|}_J(X)
\]

\[
\sqrt{1 - X^2} P^{|j|+1}_J = \frac{P^{|j|+1}_J - P^{|j|}_J}{2J + 1}.
\]

The Gegenbauer equation
\[
[1 - X^2]Y_{XX} - (2\lambda + 1)XY_X + J(J + 2\lambda)Y = 0
\]

is solved boundedly by the Gegenbauer Polynomials \( C_J(X; \lambda) \). Normalization for these is provided in, e.g. [66, 67]; the weight function is \( (1 - X^2)^{\frac{\lambda}{2}} \) between equal-\( \lambda \) Gegenbauer polynomials. These furthermore obey the recurrence relations [66, 67]
\[
XC_J(X; \lambda) = \frac{(J + 1)C_{J+1}(X; \lambda) + [2\lambda + J - 1]C_{J-1}(X; \lambda)}{2J + \lambda},
\]

\[
C_{J+1}(X; \lambda) = \frac{\lambda[C_{J+1}(X; \lambda + 1) - C_{J-1}(X; \lambda + 1)]}{J + \lambda + 1}.
\]

The associated Gegenbauer equation
\[
[1 - X^2]Y_{XX} - (2\lambda + 1)XY_X + J(J + 2\lambda)Y - j(j + 2\lambda - 1)[1 - X^2]^{-2}Y = 0
\]

is solvable boundedly by the associated Gegenbauer functions \( C^J_J(X; \lambda) \). These are re-expressible in terms of Gegenbauer polynomials via [71]
\[
C^J_J(X; \lambda) \propto (1 - X^2)^{\frac{\lambda}{2}} C_{J-j}(X; \lambda/2 - 1 + j).
\]

With this conversion, the recurrence relations between Gegenbauer polynomials (B.13) and (B.14) turn out to suffice for this paper. For \( \lambda = 1/2 \), (B.12) is the Legendre equation solved by \( P^j_J(X) = C_J(X; 1/2) \), (B.13) becomes (B.10), and (B.15) becomes the associated Legendre equation (B.6) solved by \( P^j_J \propto C^J_J(X; 1/2) \), by (B.16) and [66]
\[
C_J(X; \lambda) \propto (X^2 - 1)^{-\frac{\lambda}{2}} P^{1-\lambda}_{\lambda+1/2}(X).
\]

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