Generating entanglement of photon-number states with coherent light via cross-Kerr nonlinearity

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Abstract
We propose a scheme for generating entangled states of light fields. This scheme only requires the cross-Kerr nonlinear interaction between coherent light beams, followed by a homodyne detection. Therefore, this scheme is within the reach of current technology. We study in detail the generation of the entangled states between two modes, and that among three modes. In addition to the Bell states between two modes and the W states among three modes, we find plentiful new kinds of entangled states. Finally, the scheme can be extended to generate the entangled states among more than three modes.

1. Introduction

Entanglement is a characteristic feature of quantum states and has important applications in quantum science and technology, for example, in quantum computation and quantum information [1]. There are a lot of schemes for generating various kinds of entanglement, for example, the entanglement between photons, the entanglement between atoms, the entanglement between trapped ions and the entanglement between different kinds of particles (for example, between photons and atoms). In addition to the entanglement between two parties, there is also entanglement of multiparties. Among these schemes many use single-photon has been sources and/or single-photon detectors. Although there has been great progress in the study on these single-photon devices, how to obtain them is still a challenging task. In this paper, we propose a simple scheme for generating entangled states of light fields. This scheme only requires the cross-Kerr nonlinear interaction between light fields in coherent states, followed by a homodyne detection. Therefore, this scheme is within the reach of current technology. In fact, Nemoto, Munro and their coworkers [5] proposed a scheme to construct a near deterministic optical controlled-NOT gate, and the key elements of this gate are non-demolition detectors that use the cross-Kerr nonlinearity. They also showed that this kind of detectors has the properties of high efficiency and of single-photon-number resolving power. Generally, it was thought that strong nonlinearities are required to realize near deterministic entanglers and near deterministic quantum gates; however, weak nonlinearities are enough for the schemes of [5]. The scheme of our present paper has some similar features with that of [5] in using weak nonlinearities. However, there are some differences between them; the main difference is that single-photon sources are needed in [5] while only coherent light beams are needed in our scheme. Considering that single-photon sources are difficult to realize in experiments while the coherent light source is an ordinary one, it has some advantages to replace single-photon sources with coherent light beams. The basic idea of our scheme is shown in figure 1. Mode $a$ is a bright beam which is in a coherent state $|\alpha\rangle$. Mode $b$ is a weak or bright beam which is also in a coherent state. BS is a 50/50 beam splitter. KM1 and KM2 are Kerr media. HD means homodyne detection [2].

This paper is organized as follows: in section 2 we briefly introduce the cross-Kerr nonlinear interaction between two field-modes. In sections 3 and 4, we study the generation of entanglement between two modes and that among three modes, respectively. Section 5 is a summary.

2. Cross-Kerr nonlinear interaction

First, let us briefly review the cross-Kerr nonlinear interaction between a mode $A$ and a mode $B$. The interaction Hamiltonian
3. Entanglement between two modes

Now let us study the generation of the entangled states between two modes. The scheme is shown in figure 1. Assume that mode a is in a coherent state $|\alpha\rangle$ [4]. Mode b is also in a coherent state which is divided by the 50/50 beam splitter BS into two beams b1 and b2, and both b1 and b2 are in coherent state $|\beta\rangle$.

We first consider the case of weak coherent state $|\beta\rangle$. In this case we have

$$|\beta\rangle \approx \frac{1}{\sqrt{1 + |\beta|^2}}(0 + \beta|1\rangle),$$

where $|0\rangle$ and $|1\rangle$ are the vacuum state and one-photon state, respectively. Let mode a interact with mode b1 and b2 successively. For simplicity, we assume that both the scaled interaction times are $\tau$, that is, $\tau_1 = K_1\tau = \tau_2 = K_2\tau = \tau$. The interactions change the state in the following way,

$$|\beta\rangle|\beta\rangle = \frac{1}{1 + |\beta|^2}(|0\rangle|0\rangle + \beta|1\rangle|1\rangle),$$

$$+ |0\rangle|1\rangle \langle 1| + \beta^*|1\rangle|0\rangle \langle 0| + \beta|1\rangle|1\rangle \langle 1| + \beta^*|1\rangle|0\rangle \langle 0|,$$

where the subscripts 1 and 2 denote modes b1 and b2, respectively. We note that the internal product of coherent states satisfies [4]

$$|\langle e^{-i\tau\hat{a}^\dagger}\hat{a}e^{-i\tau\hat{a}^\dagger}\rangle|^2 = e^{-4|\alpha|^2\sin^2(\tau/2)} \approx e^{-|\alpha|^2\tau^2},$$

in which we have taken into account the fact that in practice $\tau$ is small [3] and therefore $\sin(\tau/2) \approx \tau/2$. However, if mode a is bright enough so that $|\alpha|^2\tau^2 \gg 1$, then the two coherent states will be approximately orthogonal. This condition can be easily satisfied in experiments. In following discussions we assume that this condition is satisfied. In this case, a homodyne detection can distinguish different coherent states [5]. Therefore, when we find that mode a is in the coherent state $|e^{-i\tau}\rangle$ then beam b1 and beam b2 will be projected into the entangled state

$$\frac{1}{\sqrt{2}}(|1\rangle|0\rangle|1\rangle + |0\rangle|2\rangle|1\rangle),$$

and the probability for getting this entangled state is $2|\beta|^2/(1 + |\beta|^2)^2$. This state is one of Bell states [1] and a special case of the NOON states [6].

Now let us consider the general situation in which beam b1 and beam b2 are normal coherent states [4]. In this situation,

$$|\beta\rangle = \exp \left(-\frac{1}{2}|\beta|^2 \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}}|n\rangle\right).$$

The cross-Kerr interactions transform the state as follows:

$$|\beta\rangle|\beta\rangle|\alpha\rangle \rightarrow e^{-|\beta|^2} \sum_{m,n} \frac{\beta^{m+n}}{\sqrt{m!n!}} |m\rangle|n\rangle|\alpha\rangle a\rangle \rightarrow e^{-|\beta|^2} \sum_{m,n} \frac{\beta^{m+n}}{\sqrt{m!n!}} |m\rangle|n\rangle|\alpha e^{-i(m+n)\tau}\rangle a\rangle.$$

If the homodyne detection finds mode a in the state $|\alpha e^{-i(m+n)\tau}\rangle a\rangle = |\alpha e^{-i\tau}\rangle a\rangle(k = m + n = 1, 2, \ldots)$, then mode b1 and mode b2 will collapse into the entangled state

$$\frac{1}{\sqrt{2}} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} |k-n\rangle|n\rangle|1\rangle (k = 1, 2, \ldots).$$

Since in this state the sum of photon numbers of the two modes is equal to $k$, we name this state the 2-mode k-photon entangled state. The probability for getting this state is $\exp(-2|\beta|^2)\frac{1}{k!}\sum_{n=0}^{k} \frac{k!}{n!(k-n)!} |k-n\rangle|n\rangle|1\rangle (k = 1, 2, \ldots)$. The entanglement property of the states expressed by equation (10) can be proved by using the following entanglement criteria [7]:

$$\langle [b_1^* b_2^*] \rangle > \langle N_0 N_0 \rangle,$$

where $N_0(N_0)$, $b_1(b_2)$ and $b_1^* (b_2^*)$ are the photon-number operator, the photon annihilation operator and the photon creation operator of mode b1(b2), respectively. For the states of equation (10), we can find $\langle [b_1^* b_2^*] \rangle = \frac{1}{2}k^2$ and $\langle N_0 N_0 \rangle = \frac{1}{2}k(k-1)$. Therefore the entanglement condition (11) is satisfied, and the states (10) are indeed entangled states. However, it should be noted that $\langle [b_1^* b_2^*] \rangle = \frac{1}{2}k^2$ and $\langle N_0 N_0 \rangle = \frac{1}{2}k(k-1)$ that the entanglement condition (11) is satisfied better for smaller $k$, and this shows that one can generate better entangled number-states with smaller photon-numbers by using the present scheme with the weaker coherent beams b1 and b2. For $k = 1$, equation (10) reduces to equation (7) and some other examples of the 2-mode k-photon
entangled states are listed below:

\[
\begin{align*}
(12) & : \frac{1}{\sqrt{2}}[(\frac{1}{2}|2\rangle_1 + |0\rangle_2 |2\rangle_1 ) + \frac{\sqrt{3}}{2}|1\rangle_2 |1\rangle_1 ] & (k = 2) \\
(13) & : \frac{1}{\sqrt{6}}[(|3\rangle_2 |0\rangle_1 + |0\rangle_2 |3\rangle_1 ) + \sqrt{3}(|2\rangle_2 |1\rangle_1 + |1\rangle_2 |2\rangle_1 )] & (k = 3).
\end{align*}
\]

Equations (12) and (13) are new kinds of entangled states. Equation (12) can be understood as a superposition of a NOON state \((|2\rangle_2 |0\rangle_1 + |0\rangle_2 |2\rangle_1 )\) and a product state \(|1\rangle_2 |1\rangle_1\), while equation (13) can be understood as a superposition of a NOON state \((|3\rangle_2 |0\rangle_1 + |0\rangle_2 |3\rangle_1 )\) and a NOON-like state \((|2\rangle_2 |1\rangle_1 + |1\rangle_2 |2\rangle_1 )\). We also note that in the superposition (13) the probability of getting the state \((|2\rangle_2 |1\rangle_1 + |1\rangle_2 |2\rangle_1 )\) is larger than that of getting the state \((|3\rangle_2 |0\rangle_1 + |0\rangle_2 |3\rangle_1 )\). That is, the photons tend to distribute between the two modes symmetrically. The properties and applications of these new kinds of entangled states will be studied in the future.

4. Entanglement among three modes

We can extend the above scheme to generate the entanglement among three modes. For this purpose we modify the scheme from figures 1 to 2, in which BS1 has the reflection/transmission = 1/2 and BS2 has the reflection/transmission = 1/1, so that the three beams b1, b2 and b3 have the same strength, and we assume all of them to be in the coherent state \(|\beta\rangle\). We let mode \(\alpha\) in a coherent state \(|\alpha\rangle\), interact with modes b1, b2 and b3 successively. And for simplicity, we assume that all of the scaled interaction times are equal, that is, \(\tau_1 = \tau_2 = \tau_3 = \tau\).

For the situation in which \(|\beta\rangle\) is weak and can be expressed as in equation (4), the interactions transform the states in the following way:

\[
\begin{align*}
|\beta\rangle_3 |\beta\rangle_2 |\beta\rangle_1 |\alpha\rangle_a & \rightarrow \frac{1}{1 + |\beta|^2} [\langle 0 |_3 |0\rangle_2 |0\rangle_1 |\alpha\rangle_a \\
& + \beta \langle 1 |_3 |0\rangle_2 |0\rangle_1 + \langle 0 |_3 |1\rangle_2 |0\rangle_1 + \langle 0 |_3 |0\rangle_2 |1\rangle_1 |\alpha e^{-i\tau}\rangle_a \\
& + \beta^2 \langle 1 |_3 |1\rangle_2 |0\rangle_1 + \langle 1 |_3 |2\rangle_2 |0\rangle_1 + \langle 1 |_3 |0\rangle_2 |1\rangle_1 + \langle 0 |_3 |1\rangle_2 |1\rangle_1 |\alpha e^{-i2\tau}\rangle_a \\
& + \beta^3 \langle 1 |_3 |1\rangle_2 |1\rangle_1 |\alpha e^{-i3\tau}\rangle_a ].
\end{align*}
\]

As discussed above, we assume that different coherent states in the above equation are approximately orthogonal, and we can use homodyne detection to distinguish them [5]. If we find that mode \(a\) is in state \(|\alpha e^{-i\tau}\rangle\), then modes b1, b2 and b3 will be projected to the entangled state

\[
\frac{1}{\sqrt{3}}[\langle 1 |_3 |0\rangle_2 |0\rangle_1 + \langle 0 |_3 |1\rangle_2 |0\rangle_1 + \langle 0 |_3 |0\rangle_2 |1\rangle_1],
\]

and the probability for obtaining this state is \(3|\beta|^2/(1 + |\beta|^2)^3\). On the other hand, if we find that mode \(a\) is in state \(|\alpha e^{-i2\tau}\rangle\), then modes b1, b2 and b3 will be projected to the entangled state

\[
\frac{1}{\sqrt{3}}[\langle 1 |_3 |0\rangle_2 |0\rangle_1 + \langle 1 |_3 |1\rangle_2 |0\rangle_1 + \langle 0 |_3 |1\rangle_2 |1\rangle_1],
\]

and the probability for getting this state is \(3|\beta|^2/(1 + |\beta|^2)^3\).

Equations (15) and (16) can be named as 1-photon W state [8] and 2-photon W state, respectively.

For the general case in which \(|\beta\rangle\) is not very weak we use equation (8). In this case the interactions transform the states as follows:

\[
|\beta\rangle_3 |\beta\rangle_2 |\beta\rangle_1 |\alpha\rangle_a = e^{-3|\beta|^2/2} \sum_{l,m,n} \frac{\beta^{l+m+n}}{\sqrt{l!m!n!}} \langle l |_3 |m\rangle_2 |n\rangle_1 |\alpha\rangle_a
\]

\[
\rightarrow e^{-3|\beta|^2/2} \sum_{l,m,n} \frac{\beta^{l+m+n}}{\sqrt{l!m!n!}} |l\rangle_3 |m\rangle_2 |n\rangle_1 |\alpha e^{-i(l+m+n)\tau}\rangle_a.
\]

If we find that mode \(a\) is in the state \(|\alpha e^{-i(l+m+n)\tau}\rangle_a = |\alpha e^{-i\tau}\rangle_a (k = l + m + n = 1, 2, \ldots)\), then modes b1, b2 and b3 will be projected to the entangled state

\[
\frac{1}{\sqrt{3^k}} \sum_{m=0}^{k} \sum_{n=0}^{k-m} \frac{k!}{(k-m-n)!} |k-m-n\rangle_3 |m\rangle_2 |n\rangle_1
\]

\[
(k = 1, 2, \ldots).
\]

We name this state as the 3-mode \(k\)-photon entangled state. The probability for getting this state is \(\exp(-3|\beta|^2/2) |\beta|^2k^3\). The entanglement property of the states of equation (18) can be proved by using the following entanglement criteria [7]:

\[
[|b_1^* b_2^*|^2 > \langle N_{b1} N_{b2} \rangle \text{ and } |b_2^* b_3^*|^2 > \langle N_{b2} N_{b3} \rangle] \quad (19)
\]

For the states (18), we can find \(|b_1^* b_2^*|^2 = |b_2^* b_3^*|^2 = \frac{1}{8} k^2\), and \(|N_{b1} N_{b2} | = \langle N_{b2} N_{b3} \rangle = \frac{1}{8} k(k - 1)\). Therefore the entanglement condition (19) is satisfied and the states (18) are indeed entangled states of three modes. However, similar to the discussion under equation (11) for the case of two modes, the entanglement condition (19) is satisfied better for smaller \(k\). For \(k = 1\), equation (18) reduces to equation (15) and some other examples of the 3-mode \(k\)-photon entangled state are as follows:

\[
\frac{1}{\sqrt{3}}[\langle 1 |_3 |0\rangle_2 |0\rangle_1 + \langle 0 |_3 |1\rangle_2 |0\rangle_1 + \langle 0 |_3 |0\rangle_2 |1\rangle_1] + \sqrt{2} \langle 1 |_3 |1\rangle_2 |0\rangle_1 + \langle 1 |_3 |0\rangle_2 |1\rangle_1 + \langle 0 |_3 |1\rangle_2 |1\rangle_1]
\]

\[
(k = 2),
\]
\[
\frac{1}{\sqrt{5}}((|3\rangle_3|0\rangle_2|0\rangle_1 + |0\rangle_3|3\rangle_2|0\rangle_1 + |0\rangle_3|0\rangle_2|3\rangle_1) \\
+ \sqrt{3}(|2\rangle_3|1\rangle_2|0\rangle_1 + |2\rangle_3|0\rangle_2|1\rangle_1 + |1\rangle_3|2\rangle_2|0\rangle_1) \\
+ |1\rangle_3|0\rangle_2|2\rangle_1 + |0\rangle_3|2\rangle_2|1\rangle_1 + |0\rangle_3|1\rangle_2|2\rangle_1) \\
+ \sqrt{6}|1\rangle_3|1\rangle_2|1\rangle_1)_k. \quad (k = 3).
\]

Equation (20) is a superposition of two 2-photon W states while equation (21) is a superposition of a 3-photon W state (the first line), a product state (the third line) and a state (the second line) which can be expressed as

\[
\begin{align*}
&|2\rangle_i(|1\rangle_j|0\rangle_k + |0\rangle_j|1\rangle_k) + |1\rangle_i(|2\rangle_j|0\rangle_k + |0\rangle_j|2\rangle_k) \\
&+ |0\rangle_i(|2\rangle_j|1\rangle_k + |1\rangle_j|2\rangle_k), \quad (22)
\end{align*}
\]

where the subscripts \(i = 1, 2, \text{ or } 3\), and \(j, k\) are the other two, respectively. We also note that in the superposition (20) the probability of getting the state \(|1\rangle_3|1\rangle_2|0\rangle_1 + |1\rangle_3|0\rangle_2|1\rangle_1 + |0\rangle_3|2\rangle_2|0\rangle_1\rangle_3|0\rangle_2|2\rangle_1\rangle_0\rangle_3|1\rangle_2|1\rangle_1\rangle\) is larger than that of getting the state \(|2\rangle_3|0\rangle_2|1\rangle_1 + |0\rangle_3|2\rangle_2|0\rangle_1 + |0\rangle_3|1\rangle_2|2\rangle_1\rangle\). This shows again that the photons tend to distribute among different modes symmetrically.

5. Summary

In summary, we have proposed a scheme for generating entangled states of light fields. This scheme has the following advantages: first, the scheme only involves the cross-Kerr nonlinear interaction between coherent light-beams followed by a homodyne detection. It is not necessary that the cross-Kerr nonlinearity is very large, as long as the coherent light is bright enough. Therefore, in addition to the Bell states between two modes and the W states among three modes, plentiful new kinds of entangled states can be generated with this scheme. We also found that in the generated entangled states, the photons have a trend to distribute among different modes symmetrically. Finally, we would like to point out that the scheme can be extended to generate the entangled states among more than three modes.

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