Determination of Eigenvalues in Problems of Loss of Stability of Compressed Rods (Part II)

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Abstract. The tasks of determining eigenvalues in the design of compressed rods in mechanical engineering and building structures and in their study in technical disciplines are among the most difficult. The existing methods for their solution are of a private nature and are unsuitable for the modern, more complex structures of the digital economy. In this paper, it is proposed to use computer technologies to significantly simplify the solution of complex problems of the stability of the rod and at the same time to achieve greater clarity and versatility with the proposed methods. Using specific examples, the application of the proposed analytical-graphic and numerical-graphic methods is shown. Mathematical models and decision algorithms are verified using computational experiments. Based on the results obtained, a number of practical conclusions were drawn. Part II is a continuation of Part I of the single article.

1. Introduction
The tasks of determining eigenvalues in the design of compressed rods in mechanical engineering and building structures and in their study in technical disciplines are among the most difficult [1-3]. In part I of this work, when determining the eigenvalues, analytical and analytical-graphic methods were used [4-6]. They are quite effective for simple design schemes, but become unsuitable for more complex rods: with a variable cross section, with stepped sections, with several loads in different places, with supports not only at the ends, but also at arbitrary points between the supports, etc. The reason is that the boundary conditions used by these methods take into account the kinematic and static actions at the points of the bar that belong to the ends. In such cases, numerical methods turn out to be more effective, which cover in the mathematical model the structure of the rod and the applied actions along the entire length. Therefore, in this part of the article, attention will be focused on the application of numerical methods [7-10], which easily overcome the difficulties and disadvantages listed above. For this purpose, let us return to one of the calculation schemes of part I (Fig. 1) and repeat the calculations based on the finite difference method (FCD), i.e. using the analytical-numerical-graphic method (in short, the numerical-graphic). In this case, the design scheme will be used twice: the axial moment of inertia J = const, the axial moment of inertia is variable along the length J = J (x). This approach has a double purpose: in the first case, the correctness of the results obtained in part I of the work by the analytical-graphic method is checked; in the second case, the effectiveness of MCS in solving a complex problem is proved. In the Russian Federation, MCR was deeply developed and introduced in the works of the school of A.A. Samarsky. [11, 12].
2. Constant section bar
The basic equation and boundary conditions that are required to determine the eigenvalues of a constant section bar obtained earlier in Part I will be required in this case as well. Therefore, we will write them out again.

\[ v''(x) + kv''(x) = 0, \quad k = F/b, \quad b = EJ, \quad x \in (0,l). \]  

At the lower end: 
\[ v(0) = 0, \quad bv''(0) - c_1v'(0) = 0. \]  

At the top end: 
\[ bv''(l) - c_2v'(l) = 0, \quad bv'''(l) - c_3v(l) + Fv'(l) = 0. \]

Passing from differential operators to finite difference operators, to the finite difference method (FCD), the region of continuous variation of the argument of the function \( x \in [0,l] \) is replaced by uniform mesh with step \( h \).

\[ l_n = \{x_i = (i-1)h, \quad h = l/(n-1), \quad i = 1, 2, ..., n, \quad i = 1, 2, ..., n\}. \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Calculation diagram of the bar constant section.}
\end{figure}

Then the function \( v(x) \) is associated with the grid function \( v_i(X_i) \) at the nodes, i.e. \( v_i(x_i) = v(x_i) \).

In what follows, we will use finite-difference operators that ensure accuracy \( O(h^2) \) [16]. Their use and elementary transformations give instead of the basic equation (1) the algebraic system

\[ v_{i-2} + dv_{i-1} + av_i + dv_{i+1} + v_{i+2} = 0, \quad i = 3, 4, ..., n-2, \quad a = 6 - 2k h^2, \quad d = -4 + k h^2. \]  

(4)

Similar transformations under additional conditions give the following results:

At the lower end:
\[ v_1 = 0, \quad -5v_1 + 18v_2 - 24v_3 + 14v_4 - 3v_5 = 0, \]  

(5)

At the top end:

\[ -v_{n-3} + \beta v_{n-2} + \gamma v_{n-1} + \delta v_n = 0, \quad 3v_{n-4} - 14v_{n-3} + \eta v_{n-1} + \theta v_n = 0, \]  

(6)

\[ \alpha = \frac{hc_2}{2b}, \quad \beta = 4 - \alpha, \quad \gamma = -5 + 4\alpha, \quad \delta = 2 - 3\alpha, \quad \epsilon = 24 + \frac{ch^2}{b}, \]  

\[ \eta = -18 - \frac{4Fh^2}{b}, \quad \theta = 5 + \frac{-2c_3h^3 + 3Fh^2}{b}. \]  

The final result of the transition of the boundary value problem (1) - (3) to the grid domain using (4) - (6) is the homogeneous system of algebraic equations

\[ B V = 0, \]  

(7)
where $B$ is a square matrix of order $n$, $V$ is a column vector representing a continuous function $v(x)$ in the finite difference method. Let’s write them out

$$
B = \begin{pmatrix}
1 & & & & & & & & \\
\kappa & \lambda & \mu & -1 & & & & & \\
1 & d & a & d & 1 & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
1 & d & a & d & 1 & & & & \\
\end{pmatrix},
$$

$$
V = (v_1, v_2, \ldots, v_n)^T.
$$

Zero elements are not written out here. The right side of the system of equations (7) is a zero vector. Obviously, the system of equations (7) is trivial, i.e. zero solution $V = 0$, which is not of interest in this case. A nontrivial solution can exist only if the determinant of the matrix $B$ is equal to zero, that is,

$$
\det B(F) = 0. \tag{8}
$$

The characteristic equation (8) of the matrix $B$ is an algebraic equation and has a set of roots of cardinality $n$. You can do without drawing up this equation, using the ability to build a high-precision graph of the curve of the graph of the function $\det B(F)$. The points of intersection of the graph with the $F$ axis will determine the values of the critical forces.

**Example 1.** Vertical bar with the design scheme according to fig. 1 of a steel pipe has the initial data:

- $l = 7$ m,
- $E = 2,1 \cdot 10^{11}$ Pa,
- $D = 10,8$ cm,
- $t = 1,4$ mm,
- $c_1 = 2000$ Nm/rad,
- $c_2 = 1000$ Nm/rad,
- $c_3 = 1500$ N/m;
- $m = 1001$, $n = 1001$. $m$, $n$ - the number of points on the numerical axes $x$ and $F$.

It is required to determine the first three eigenvalues by the numerical-graphical method with the buckling of the bar.

To determine the eigenvalues, a small computer program was developed in the Matlab environment. She displayed a graph of the function $F = \det$ on the monitor screen (fig. 2). In the figure, bold dots mark the points of three eigenvalues on the $F$ axis. Their abscissas correspond to the three smallest eigenvalues. At the same time, note that Matlab has a "magnifying glass" that allows you to consider the neighborhoods of these points in a several thousand times enlarged form. Therefore, the eigenvalues, read with its help should be recognized as highly accurate, despite their graphic origin.
It is advisable to verify the results obtained in Part I by analytical-graphic and here by numerical-graphic methods for various finite-difference grids, i.e. for different m and n. For this purpose, calculations were performed using a computer program for two options m, n. The results of solving the problem by two methods are shown in the table.

| Number of eigenvalues | Analytical-graphical method. | Numerical-graphical method. |
|-----------------------|-------------------------------|-----------------------------|
|                       | n = 1001 m = 1001              | n = 10001 m = 101           |
| 1                     | 10,443                        | 10,619                      |
| 2                     | 25,097                        | 28,467                      |
| 3                     | 101,514                       | 112,164                     |

Some conclusions can be drawn from the data in the table. The first eigenvalues found by the two methods differ very insignificantly, despite the fact that only they are of practical importance and are used in real design solutions. The second and third eigenvalues differ from the first noticeably (∼4-11%). The noticeable excess of the results of the numerical-graphic method over the results of the analytical-graphic method is quite understandable. The reason is that the elastic axial smooth line of the pipe in the finite difference method is replaced by a solid one, but consisting of rigid rectilinear non-bendable segments.

Comparing the efforts spent on solving the same problem, it should be recognized that the analytical-graphical method has advantages over the numerical-graphical method when determining the eigenvalues for bars of constant cross-section. At the same time, it is clear that it is impossible to find the eigenvalues for bars of variable cross-section using the analytical-graphic method.

Let us also pay attention to the fact that the numerical values of the parameters m and n of the grid region, which differ significantly in the two versions of the solutions, did not significantly affect the eigenvalues. At the same time, the computer experiments presented here revealed a significant increase in computer time at large values of m and n.

3. Rod of variable (stepped) section
Design schemes of the bar in the initial and deformed states are shown in Fig. 3. The rod consists of two sections with lengths \( l_1 \) and \( l_2 \). The basic differential equations for them will be different.
\( v^IV(x) + k_1 v''(x) = 0, \ k_1 = \frac{F}{b_1}, \ b_1 = EJ_1, \ x \in (0, l_1) \) \hspace{1cm} (9)

\( v^IV(x) + k_2 v''(x) = 0, \ k_2 = \frac{F}{b_2}, \ b_2 = EJ_2, \ x \in (l_1, l_2). \) \hspace{1cm} (10)

**Figure 3.** Calculation diagram of the bar variable section.

Boundary conditions (2), (3) given above for the lower and upper ends are retained with amendments corresponding to equations (9), (10).

Bottom end: \( v(0) = 0, \ b v''(0) - c_1 v'(0) = 0. \) \hspace{1cm} (11)

Top end: \( b_2 v''(l) - c_2 v'(l) = 0, \ b_2 v''(l) - c_3 v(l) + F v'(l) = 0. \) \hspace{1cm} (12)

To the boundary conditions it is necessary to add the conditions of conjugation of two sections at point A. Part of these conditions of conjugation are kinematic and consist in the fact that the function \( v(x) \) and its first derivative \( v'(x) \) must be smooth at point A with coordinate \( x_A \):

\[ v(x_A-0) = v(x_A+0), \quad v'(x_A-0, t) = v'(x_A+0). \] \hspace{1cm} (13)

The need to satisfy conditions (13) is obvious from considerations of the continuity and continuity of the structure. It follows from two equalities that \( v(x) \) belongs to the space of continuously differentiable functions \( C^1 \).

**Figure 4.** Pairing parcels.

Other conditions are static and arise due to the presence of force factors at the junction of adjacent sections (Fig. 4): bending moments and shear forces from below and from above point A. If we take into account that bending moments and shear forces are applied at one point, their balance is obvious.

\[ b_1 v^*(x_A-0) = b_2 v^*(x_A+0). \] \hspace{1cm} (14)
It follows from (14) that the second derivative of the function \( v(x) \) has a discontinuity at the point A. Likewise, the shear forces \( Q_+ \) act in the horizontal direction and are balanced

\[
b_1 v''(x_A-0) = b_2 v''(x_A+0).
\]  

(15)

It is also obvious that at point A the third derivative of the function \( v(x) \) has a discontinuity and this must be taken into account in the forthcoming calculations. Equations (9), (10), boundary conditions (11), (12) and conjugation conditions (13) - (15) form a mathematical model that makes it possible to determine the eigenvalues.

Further, the replacement of differential operators in (9) - (15) by finite-difference operators will lead to homogeneous system of algebraic equations. Instead of (9), (10) there will be

\[
v_{i-2} + d_1 v_{i-1} + a_1 v_i + a_2 v_{i+1} + v_{i+2} = 0, \quad i = 3, 4, \ldots, n-2, \quad a_1 = 6 - 2k_1h^2, \quad d_1 = -4 + k_1h^2. \quad (16)
\]

\[
v_{i-2} + d_2 v_{i-1} + a_2 v_i + a_3 v_{i+1} + v_{i+2} = 0, \quad i = 3, 4, \ldots, n-2, \quad a_2 = 6 - 2k_2h^2, \quad d_2 = -4 + k_2h^2. \quad (17)
\]

Let us carry out similar changes of the boundary conditions (11), (12).

At the lower end: \( v_1 = 0 \)

\[
\kappa v_1 + \lambda v_2 + \mu v_3 - v_4 = 0, \quad (18)
\]

\[
\kappa = 2 + \frac{3c_1h}{2b_1}, \quad \lambda = -5 - \frac{2hc_1}{b_1}, \quad \mu = 4 + \frac{c_1h}{2b_1}
\]

At the top end: \( -v_{n-3} + \beta v_{n-2} + \gamma v_{n-1} + \delta v_n = 0 \)

\[
3v_{n-4} - 14v_{n-3} + \epsilon v_{n-2} + \eta v_{n-1} + \theta v_n = 0, \quad (19)
\]

\[
\alpha = \frac{h^2c_2}{2b_2}, \quad \beta = 4 - \alpha, \quad \gamma = -5 + 4\alpha, \quad \delta = 2 - 3\alpha, \quad \epsilon = 24 + \frac{fh^2}{b_2}, \quad \eta = -18 - \frac{4fh^2}{b_2}, \quad \theta = 5 + \frac{2c_3h^3 + 3fh^2}{b_2}
\]

At the point where the parcels meet \( A(x_k = x_\lambda) \):

\[
v_{k-2} - 6v_{k-1} + 4v_k + 4v_{k+1} + v_{k+2} = 0, \quad (20)
\]

\[
-5v_{k-3} + 4v_{k-2} + c v_k + a_1 v_{k+1} + g v_{k+2} + p v_{k+3} = 0, \quad (21)
\]

\[
3v_{k-4} - 14v_{k-3} + 24v_{k-2} - 18v_{k-1} + q v_k + r v_{k+1} + s v_{k+2} + f v_{k+3} + t v_{k+4} = 0. \quad (22)
\]

\[
p = \frac{b_2}{b_1}, \quad c = 2 - 2p, \quad e = 5p, \quad g = -4p, \quad q = 5 + 5p, \quad r = -8p, \quad s = 24p, \quad f = -14p, \quad t = 3p.
\]

Equations (13) - (19) form a homogeneous algebraic system. In matrix-vector form, it has the form

\[
Bv = 0.
\]  

(23)

Unknown eigenvalues will be determined from the characteristic equation

\[
\det B(F) = 0,
\]  

(24)

where
\[
B = \begin{pmatrix}
1 & \kappa & \lambda & \mu & -1 \\
1 & d_1 & a_1 & d_1 & 1 \\
1 & d_1 & a_1 & d_1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & d_1 & a_1 & d_1 & 1 \\
3 & -14 & 24 & -18 & q & r & s & f & t \\
-1 & 4 & -5 & c & e & g & p \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & d_2 & a_2 & d_2 & 1 \\
1 & d_2 & a_2 & d_2 & 1 \\
3 & -14 & \varepsilon & \eta & \theta & \beta & \gamma & \delta \\
\end{pmatrix}
\]

\[
V = (v_1, v_2, \ldots, v_n)^T.
\]

Zero elements are not written out here. Typical lines are indicated by dots.

**Example 2.** A vertical bar with a stepped cross-section, consisting of two sections of different lengths and diameters with a design scheme according to Fig. 3, has original data: \(l = 7\) m, \(l_1 = 3\) m, \(l_2 = 4\) m, \(E = 2,1 \times 10^{11}\) Pa, \(D_1 = 10.8\) cm, \(D_2 = 9.2\) cm, \(t = 1.4\) mm, \(c_1 = 2000\) N/m/rad, \(c_2 = 1000\) Nm/rad, \(c_3 = 1500\) N/m. \(D_1, D_2\) – diameters of the lower and upper sections of the pipe, \(t\) - wall thickness.

It is required to determine the first three eigenvalues by the numerical-graphical method with the buckling of the bar.

According to the developed mathematical model and the solution algorithm, the computer program displayed the graph of the function \(F \cdot \text{det}\) (fig. 5) on the monitor screen, where the eigenvalues \(\{9.813; 57.273; 82.145\}\). In general, the eigenvalues have decreased in comparison with those obtained in example 1. The reason is that in the upper part of the bar in example 2 the diameter is smaller than in example 1.

![Figure 5. Eigenvalues of a constant section bar.](image)

The results obtained for the most important first eigenvalue using the proposed technique and using exact formulas agree with a satisfactory degree of accuracy. This allows us to assert that the finite difference method can be used to solve complex problems, in contrast to analytical methods.
4. Conclusions
1. According to the two algorithms discussed above, one important conclusion can be drawn, which is that the numerical method can be easily adapted to the determination of eigenvalues for bars of variable cross-section, while the analytical method for determining the critical force does not have such a possibility.

2. The closeness of the results obtained by analytical-graphic and numerical-graphic methods confirms the reliability of mathematical models, decision algorithms and computer programs used in parts I and II of this work.

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