REFLECTION FUNCTORS FOR CONTINUOUS QUIVERS OF TYPE A

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Abstract. As generalizations of quivers of type $A$, Igusa-Rock-Todorov in [6] introduced continuous quivers of type $A$. In this paper, we shall generalize BGP reflection functors to continuous quivers of type $A$.

1. Introduction

In [5], Gabriel gave the classification of indecomposable representations of a finite type quiver. In [1], Bernstein, Gelfand and Ponomarev introduced reflection functors and gave a new proof of Gabriel’s theorem.

Representations of quivers of type $A$ play an important role in persistent homology, which have been widely used in topological data analysis in [7]. In [3], Carlsson and de Silva introduced zigzag persistent homology. They also introduced diamond principle as a calculational tool, which has a direct connection with BGP reflection functors in [7].

As a generalization of Gabriel’s theorem for quivers of type $A$, Crawley-Boevey in [4] and Botnan in [2] gave the classification of indecomposable representations of $\mathbb{R}$ and infinite zigzag, respectively.

In [6], Igusa-Rock-Todorov introduced continuous quivers of type $A$, which is a generalization of quivers of type $A$, $\mathbb{R}$ and infinite zigzag. They also classified indecomposable representations and proved a decomposable theorem.

In this paper, we shall introduce reflection functors for the continuous quivers of type $A$, as generalizations of BGP reflection functors and diamond principle.

In Section 2, we shall recall basic notations for BGP reflection functors. Reflection functors for continuous quivers of type $A$ will be introduced in Section 3. In Section 4, we shall study the properties of this functor and give the main result.

2. BGP Reflection Functors

2.1. Quivers. A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple, where:

(a) $Q_0$ is the set of vertices and $Q_1$ is the set of arrows;
(b) $s, t : Q_1 \to Q_0$ are two maps such that $s(\alpha)$ is the source and $t(\alpha)$ is the target of $\alpha$ for any $\alpha$ in $Q_1$. 

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2.2. **Representations of quivers.** Let $Q$ be a finite quiver and fix an algebraically closed field $K$. A representation of $Q$ is $M = (M_a, \varphi_\alpha)_{a \in Q_0, \alpha \in Q_1}$, where:

(a) $M_a$ is a $K$-vector space for any vertex $a$ in $Q_0$;
(b) $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is a $K$-linear map for any arrow $\alpha : s(\alpha) \to t(\alpha)$ in $Q_1$.

Let $M = (M_a, \varphi_\alpha)$ and $M' = (M'_a, \varphi'_\alpha)$ be two representations of $Q$. A $K$-linear map $f = (f_a)_{a \in Q_0} : M \to M'$ is called a homomorphism of representations, if the following diagram is commutative:

\[
\begin{array}{c}
M_{s(\alpha)} \xrightarrow{\varphi_\alpha} M_{t(\alpha)} \\
| \\
M'_{s(\alpha)} \xrightarrow{\varphi'_\alpha} M'_{t(\alpha)}
\end{array}
\]

And the category of finite dimensional representations of $Q$ over $K$ is denoted by $\text{rep}_K(Q)$.

2.3. **BGP reflection functors.** Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. The vertex $i \in Q_0$ is called a sink, if $s(\alpha) \neq i$ for any $\alpha \in Q_1$.

Consider a new quiver $\sigma_i Q = (Q_0, Q'_1, s, t)$ by reversing the direction of arrows $\alpha$ such that $s(\alpha) = i$ or $t(\alpha) = i$.

For instance, if $Q$ is the quiver (1), then $\sigma_2 Q$ is the quiver (2).

(1) \begin{center} • −→ • ←− • ←− • ←− • \end{center} 1 2 3 4 5

(2) \begin{center} • ←− • −→ • ←− • ←− • \end{center} 1 2 3 4 5

Next, we shall recall the definition of reflection functor $S^+_i : \text{rep}_K(Q) \to \text{rep}_K(Q')$ for any $i \in Q_0$.

For any $M = (M_a, \varphi_\alpha) \in \text{rep}_K(Q)$, define $S^+_i M = M' = (M'_a, \varphi'_\alpha)$ as follows. Let $M'_a = M_a$ for $i \neq p$,

and $M'_i$ be the kernel of

\[
\bigoplus_{\alpha, t(\alpha) = i} M_{s(\alpha)} \longrightarrow M_i,
\]

that is, we have the following exact sequence of vector spaces

\[
0 \longrightarrow M'_i \xrightarrow{\varphi'_\alpha} \bigoplus_{\alpha, t(\alpha) = i} M_{s(\alpha)} \xrightarrow{\varphi_\alpha} M_i.
\]

Let $f = (f_a)_{a \in Q_0} : M \to N$ be a morphism in $\text{rep}_K(Q)$, where $M = (M_a, \varphi_\alpha)$ and $N = (N_a, \psi_\alpha)$. There exists a morphism

\[
S^+_i (f) = f' = (f'_a)_{a \in Q'_0} : S^+_i M \to S^+_i N
\]
in $\text{rep}_K(Q')$ defined as follows. For any $a \neq i$, let

$$f'_a = f_a,$$

whereas $f'_i$ is the unique morphism, making the following diagram commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & M'_i \\
\big| & \quad & \big| \\
\exists f'_i & \Downarrow & \exists f_i \\
0 & \longrightarrow & N'_i
\end{array}
\]

\[
\begin{array}{ccc}
M'_i & \longrightarrow & \bigoplus_a M_{s(a)} \\
\quad & \downarrow \quad & \downarrow \quad f_i \\
N'_i & \longrightarrow & \bigoplus_a N_{s(a)}
\end{array}
\]

And Bernstein, Gelfand and Ponomarev studied the properties of reflection functors $S^+_i$ in [1].

3. Reflection Functors For Continuous Quivers Of Type $A$

In this section, firstly, we shall recall the definition of continuous quivers in [6].

3.1. Continuous quivers of type $A$. A quiver of continuous type $A$, denoted by $A_R$, is a triple $(R, S, \preceq)$, where:

(a) $S \subset R$ is a discrete subset;
(b) elements of $S \cup \pm \infty$ are indexed by a subset of $\mathbb{Z} \cup \pm \infty$;
(c) the partial order $\preceq$ on $R$ (which we call the orientation of $A_R$) does not change between consecutive elements of $S \cup \pm \infty$.

The element $S_i$ is called a sink, if $S_i \prec S_{i+1}$ and $S_i \prec S_{i-1}$, and $S_i$ is called a source, if $S_{i+1} \prec S_i$ and $S_{i-1} \prec S_i$ for any $i \in \mathbb{Z}$.

Remark 3.1. There is a minute difference for the definitions of continuous quivers between this paper and [6]. In [6], $S$ is regularly punctuated with sinks and sources, that is, sinks are surrounded only by sources. In this paper, we don't specify the places of sinks and sources.

3.2. Representations of continuous quivers. Let $A_R = (\mathbb{R}, S, \preceq)$ be a quiver of continuous type $A$. And $V = (V(x), V(x, y))$ is called a representation of $A_R$, if $V(x)$ is a vector space for any $x \in \mathbb{R}$, and $V(x, y) : V(x) \rightarrow V(y)$ is a linear map for any $x \preceq y$ satisfied $V(y, z) \circ V(x, y) = V(x, z)$ for any $x \preceq y \preceq z$.

Consider linear map $f_x : V(x) \rightarrow W(x)$ for any $x \in \mathbb{R}$, making the following diagram commutative

\[
\begin{array}{ccc}
V(x) & \xrightarrow{f_x} & V(y) \\
\downarrow & & \downarrow f_y \\
W(x) & \xrightarrow{f_x} & W(y)
\end{array}
\]

The collection $f = (f_x)_{x \in \mathbb{R}} : V \rightarrow W$ is called a morphism of representations of $A_R$. And we denote the category of finite dimensional $K$-linear representations of $A_R$ by $\text{rep}_K(A_R)$.
3.3. Reflection functors. In this subsection, we shall give the definition of reflection functors for continuous quivers of type $A$.

Let $A_R = (\mathbb{R}, S, \preceq)$ and $S_{k-1}, S_k, S_{k+1} \in S$ be three points satisfied $S_k \prec S_{k-1}$, $S_k \prec S_{k+1}$, and consider a new continuous quiver $\sigma_k A_R = A'_R = (\mathbb{R}, S', \preceq')$ defined as follows.

(a) The set $S' \setminus \{S'_k\} = S \setminus \{S_k\}$ and $S'_k$ satisfies that $S'_k + S_k = S_{k+1} + S_{k-1}$.

(b) The partial order $\preceq'$ on $\mathbb{R}$, which we call the orientation of $A_R$, is defined as follows.

For any $x \leq y \in (S'_{k-1} = S_{k-1}, S'_k)$, define $x \preceq' y$, and for any $x \leq y \in (S'_k, S'_{k+1} = S_{k+1})$, define $y \preceq' x$.

For instance, if $A_R$ is the quiver (3), then $A'_R = \sigma_k A_R$ is the quiver (4).

If we put $A_R$ and $A'_R$ together, we get the following diagram.

Now, we shall give the definition of reflection functors. Let $S_k$ be a sink, then we shall define the reflection functor

$$S^+_k : rep_K(A_R) \longrightarrow rep_K(A'_R).$$

For any object $V \in rep_K(A_R)$, we can give the definition of representation $S^+_k V = V'$. If $x \notin (S_{k-1}, S_{k+1})$, let $V'(x) = V(x)$. If $x \in (S_{k-1}, S_{k+1})$, let $x' = S_{k+1} + S_{k-1} - x$, and define $V'(x)$ as follows.

(a) For any $x \in (S_{k-1}, S'_k)$,
$V'(x)$ is defined as the kernel of

$$V(S_{k-1}) \oplus V(x') \to V(S_k),$$

that is, we have the following short exact sequence of vector spaces:

$$0 \to V'(x) \to V(S_{k-1}) \oplus V(x') \xrightarrow{D_1} V(S_k),$$

where $D_1 = (V(S_{k-1}, S_k), -V(x', S_k))$.

Denote the map from $V'(x) = \mathcal{S}_k^1 V(x)$ to $V(x')$ in the exact sequence by $C_V(x)$. In this case, the following diagram is a pull-back

$$
\begin{array}{ccc}
V'(x) & \xrightarrow{C_V(x)} & V(x') \\
\downarrow & & \downarrow \\
V'(x,S_{k-1}) & \xrightarrow{V(x',S_k)} & V(x',S_k)
\end{array}
$$

(b) For any $x \in [S'_k, S_{k+1}),$

$V'(x)$ is defined as the kernel of

$$V(x') \oplus V(S_{k+1}) \to V(S_k),$$

that is, we have the following short exact sequence of vector spaces:

$$0 \to V'(x) \to V(x') \oplus V(S_{k+1}) \xrightarrow{D_2} V(S_k),$$

where $D_2 = (V(x', S_k), -V(S_{k-1}, S_k))$. 
Denote the map from $V'(x) = S_k^x V(x)$ to $V(x')$ in the exact sequence by $C_V(x)$. In this case, the following diagram is a pull-back

$$
\begin{array}{ccc}
V'(S_k) & \xrightarrow{V'(S_k,x)} & V(S_{k+1}) \\
\downarrow & & \downarrow \\
V'(x) & \xrightarrow{C_V(x)} & V(x').
\end{array}
$$

For any $V \in \text{rep}_K(A_R)$, if $x \preceq y \not\in (S_{k-1}, S_{k+1})$, let $V'(x, y) = V(x, y)$. If $x \in (S_{k-1}, S_{k+1})$, let $x' = S_{k+1} + S_{k-1} - x$, $y' = S_{k+1} + S_{k-1} - y$, and define $V'(x, y)$ as follows.

(a) Let $x, y \in (S_{k-1}, S'_k)$ such that $x \prec y$. Note that $x' \prec y'$.

Since $V'(x)$ is the kernel of $V(S_{k-1}) \oplus V(x') \rightarrow V(S_k)$, and $V'(y)$ is the kernel of $V(S_{k-1}) \oplus V(y') \rightarrow V(S_k)$, there must exist a unique map $V'(x, y) : V'(x) \rightarrow V'(y)$ making the following diagram commutative

$$
\begin{array}{cccc}
0 & \longrightarrow & V'(x) & \longrightarrow & V(S_{k-1}) \oplus V(x') & \longrightarrow & V(S_k) & \\
& & \downarrow{V'(x,y)} & & \downarrow{(\text{id}_{V'(x',y')})} & & \downarrow{\text{id}} & \\
0 & \longrightarrow & V'(y) & \longrightarrow & V(S_{k-1}) \oplus V(y') & \longrightarrow & V(S_k) & 
\end{array}
$$

(b) Let $x, y \in [S'_k, S_{k+1})$ such that $x \prec y$. Note that $x' \prec y'$.

Since $V'(x)$ is the kernel of $V(x') \oplus V(S_{k+1}) \rightarrow V(S_k)$, and $V'(y)$ is the kernel of $V(y') \oplus V(S_{k+1}) \rightarrow V(S_k)$, there must exist a unique map $V'(x, y)$:
$V'(x) → V'(y)$ making the following diagram commutative

$$
\begin{array}{cccc}
0 & → & V'(x) & → \ V(x') \oplus V(S_{k+1}) \rightarrow V(S_k) \\
\downarrow{V'(x,y)} & & \downarrow{(V(x',y') \ id)} & \downarrow{id} \\
0 & → & V'(y) & → \ V(y') \oplus V(S_{k+1}) \rightarrow V(S_k).
\end{array}
$$

For any morphism $f : V → W$ in $rep_K(A_R)$, we can give the definition of a morphism $S_k' f = f' = (f'_x) : S_k^+ V → S_k^+ W$. If $x \notin (S_{k-1}, S_{k+1})$, let $f'_x = f_x$. If $x ∈ (S_{k-1}, S_{k+1})$, we can define $f'_x$ as follows. Denote $V' = S_k^+ V$, $W' = S_k^+ W$.

(a) Fix $x ∈ (S_{k-1}, S_k')$, since $V'(x)$ is the kernel of $V(S_{k-1}) \oplus V(x') → V(S_k)$, and $W'(x)$ is the kernel of $W(S_{k-1}) \oplus W(x') → W(S_k)$, there must be a map $f'_x$ making the following diagram commutative

$$
\begin{array}{cccc}
0 & → & V'(x) & → \ V(S_{k-1}) \oplus V(x') \rightarrow V(S_k) \\
\downarrow{f'_x} & & \downarrow{(f_{S_{k-1}} x', f)} & \downarrow{f_{S_k}} \\
0 & → & W'(x) & → \ W(S_{k-1}) \oplus W(x') \rightarrow W(S_k).
\end{array}
$$

(b) Similarly, fix $x ∈ [S_k', S_{k+1})$, since $V'(x)$ is the kernel of $V(x') \oplus V(S_{k+1}) → V(S_k)$, and $W'(x)$ is the kernel of $W(x') \oplus W(S_{k+1}) → W(S_k)$, there must be a map $f'_x$ making the following diagram commutative

$$
\begin{array}{cccc}
0 & → & V'(x) & → \ V(x') \oplus V(S_{k+1}) \rightarrow V(S_k) \\
\downarrow{f'_x} & & \downarrow{(f' x', f_{S_{k+1}})} & \downarrow{f_{S_k}} \\
0 & → & W'(x) & → \ W(x') \oplus W(S_{k+1}) \rightarrow W(S_k).
\end{array}
$$

In a similar way, if $S_k'$ is a source, we can give the definition of reflection functor $S_k^-$

$$
S_k^- : rep(A_R') → rep(A_R),
$$

$V' → V = S_k^- V'$,

$f' → f = S_k^- f'$.

For any object $V' ∈ rep_K(A_R)$, we can give the definition of representation $S_k^- V' = V$. If $x \notin (S_{k-1}, S_{k+1})$, let $V(x) = V'(x)$. If $x ∈ (S_{k-1}, S_{k+1})$, let $x' = S_{k+1} + S_{k-1} - x$, and define $V(x)$ as follows.

(a) For any $x ∈ (S_k, S_{k+1})$,
$V(x)$ is defined as the cokernel of

$$V'(S'_k) \rightarrow V'(S_{k+1}) \oplus V'(x'),$$

that is, we have the following short exact sequence of vector spaces:

$$V'(S'_k) \xrightarrow{D'_1} V'(S_{k+1}) \oplus V'(x') \rightarrow V(x) \rightarrow 0,$$

where $D'_1 = (V'(S'_k, S_{k+1})_{S'_k, x'}).$

Denote the map from $V'(x')$ to $V(x) = S'_k V'(x)$ in the exact sequence by $D_V(x).$ In this case, the following diagram is a push-out

$$
\begin{array}{ccc}
V'(S'_k) & \xrightarrow{V'(S'_k, S_{k+1})} & V'(S_{k+1}) \\
\downarrow V'(S'_k, x') & & \downarrow V(S_{k+1}, x) \\
V'(x') & \xrightarrow{D_V(x)} & V(x).
\end{array}
$$

(b) For any $x \in [S_{k-1}, S_k),$

$V(x)$ is defined as the cokernel of

$$V'(S'_k) \rightarrow V'(x') \oplus V'(S_{k-1}),$$

that is, we have the following short exact sequence of vector spaces:

$$V'(S'_k) \xrightarrow{D'_2} V'(x') \oplus V'(S_{k-1}) \rightarrow V(x) \rightarrow 0,$$
where $D_2' = \begin{pmatrix} V'(S'_k, x') \\ -V'(S'_k, s_{k-1}) \end{pmatrix}$.

Denote the map from $V'(x')$ to $V(x) = S_k^{-1} V'(x)$ in the exact sequence by $D_V(x)$. In this case, the following diagram is a push-out

$$
\begin{array}{c}
V'(S'_k) \\
V'(S'_k)
\end{array} \quad \begin{array}{c}
\rightarrow \ V'(x') \\
\rightarrow \ V'(x')
\end{array} \quad \begin{array}{c}
V'(S_{k-1}) \\
V'(S_{k-1})
\end{array} \quad \begin{array}{c}
\downarrow \quad V'(S_{k-1}, x') \\
\downarrow \quad V'(S_{k-1}, x')
\end{array} \quad \begin{array}{c}
D_V(x) \\
D_V(x)
\end{array} \quad \begin{array}{c}
\rightarrow \ V(x) \\
\rightarrow \ V(x)
\end{array}.
$$

For any $V' \in \text{rep}_K(A'_R)$, if $x \prec y \notin (S_{k-1}, S_{k+1})$, let $V(x, y) = V'(x, y)$. If $x \in (S_{k-1}, S_{k+1})$, let $x = S_{k+1} + S_{k-1} - x$, $y = S_{k+1} + S_{k-1} - y$, and define $V(x, y)$ as follows.

(a) Let $x, y \in (S_k, S_{k+1})$ such that $x \prec y$. Note that $x' \prec y'$.

Since $V(x)$ is the cokernel of $V'(S'_k) \longrightarrow V'(S_{k+1}) \oplus V'(x')$, and $V(y)$ is the cokernel of $V'(S'_k) \longrightarrow V'(S_{k+1}) \oplus V'(y')$, there must exist a unique map $V(x, y) : V(x) \rightarrow V(y)$ making the following diagram commutative

$$
\begin{array}{c}
V'(S'_k) \\
V'(S'_k)
\end{array} \quad \begin{array}{c}
\rightarrow \ V'(S_{k+1}) \oplus V'(x') \\
\rightarrow \ V'(S_{k+1}) \oplus V'(x')
\end{array} \quad \begin{array}{c}
V'(S'_k) \\
V'(S'_k)
\end{array} \quad \begin{array}{c}
\downarrow \quad \text{id} \\
\downarrow \quad \text{id}
\end{array} \quad \begin{array}{c}
\rightarrow \ V(x) \\
\rightarrow \ V(x)
\end{array} \quad \begin{array}{c}
\downarrow \quad (\text{id} \quad V'(x', y')) \\
\downarrow \quad (\text{id} \quad V'(x', y'))
\end{array} \quad \begin{array}{c}
0 \\
0
\end{array}.
$$

(b) Let $x, y \in [S_{k-1}, S_k)$ such that $x \prec y$. Note that $x' \prec y'$.
Remark 3.2. By rephrasing the definition sequence of map $V(x, y)$, we have the following commutative diagram

\[
\begin{array}{ccc}
V'(x, S_{k-1}) & \xrightarrow{V'(x, y, S_{k-1})} & V'(y, S_{k-1}) \\
\downarrow V'(x, y) & & \downarrow V'(y, y) \\
V(x) & \xrightarrow{C_V(x)} & V(x') \\
\end{array}
\]

Since $V(x)$ is the cokernel of $V'(S_k') \longrightarrow V'(x') \oplus V'(S_{k-1})$, and $V(y)$ is the cokernel of $V'(S_k') \longrightarrow V'(y') \oplus V'(S_{k-1})$, there must exist a unique map $V(x, y) : V(x) \rightarrow V(y)$ making the following diagram commutative

\[
\begin{array}{ccc}
V'(S_k') & \xrightarrow{V'(x')} & V'(S_{k-1}) \\
\downarrow id & & \downarrow (V'(x, y)_{id}) \\
V'(S_k') & \xrightarrow{V'(y')} & V'(S_{k-1}) \\
\end{array}
\]

For any morphism $f' : V' \rightarrow W'$ in $\text{rep}_K(A_k')$, we can give the definition of a morphism $S_k f' = f = (f_x) : S_k V' \rightarrow S_k W'$. If $x \notin (S_{k-1}, S_{k+1})$, let $f_x = f'_x$. If $x \in (S_{k-1}, S_{k+1})$, we can define $f_x$ as follows. Denote $V = S_k V'$, $W = S_k W'$.

(a) Fix $x \in (S_k, S_{k+1})$, since $V(x)$ is the cokernel of $V'(S_k') \longrightarrow V'(S_{k+1}) \oplus V'(x')$, and $W(x)$ is the cokernel of $W'(S_k') \longrightarrow W'(S_{k+1}) \oplus W'(x')$, there must be a map $f_x : V(x) \rightarrow W(x)$ making the following diagram commutative

\[
\begin{array}{ccc}
V'(S_k') & \xrightarrow{V'(S_{k+1}) \oplus V'(x')} & V(x) \\
\downarrow f'_k & & \downarrow f_x \\
W'(S_k') & \xrightarrow{W'(S_{k+1}) \oplus W'(x')} & W(x) \\
\end{array}
\]

(b) Similarly, fix $x \in (S_{k-1}, S_k)$, since $V(x)$ is the cokernel of $V'(S_k') \longrightarrow V'(x') \oplus V'(S_{k-1})$, and $W(y)$ is the cokernel of $W'(S_k') \longrightarrow W'(y') \oplus W'(S_{k-1})$, there must be a map $f_x : V(x) \rightarrow W(x)$ making the following diagram commutative

\[
\begin{array}{ccc}
V'(S_k') & \xrightarrow{V'(x') \oplus V'(S_{k-1})} & V(x) \\
\downarrow f'_k & & \downarrow f_x \\
W'(S_k') & \xrightarrow{W'(x') \oplus W'(S_{k-1})} & W(x) \\
\end{array}
\]
According to the uniqueness of map, there must exist a map $V'(x, y) : V'(x) \to V'(y)$, in a similar way, we have the map $V(x', y') : V(x') \to V(y')$. Simplify the above diagram further, we have the final commutative diagram

\[
\begin{array}{ccc}
V'(x) & \xrightarrow{C_V(x)} & V'(y') \\
\downarrow V'(x, y) & & \downarrow V'(x', y') \\
V'(y) & \xrightarrow{C_V(y)} & V'(y') \\
\downarrow V'(y, S_{k-1}) & & \downarrow V'(y', S_k) \\
V(S_{k-1}) & \xrightarrow{V(S_{k-1}, S_k)} & V(S_k).
\end{array}
\]

4. Main Results

Let $A_\mathbb{R} = (\mathbb{R}, S, \preceq)$ be a continuous quiver of type $A$, and $S_k$ be a sink. The new quiver $\sigma_k A_\mathbb{R} = A'_\mathbb{R} = (\mathbb{R}, S', \preceq')$ has been defined in Section 3. The category $\text{rep}(A_{\mathbb{R}})$ and the reflection functors $S^+_k : \text{rep}(A_{\mathbb{R}}) \longrightarrow \text{rep}(A'_{\mathbb{R}})$, $S^-_k : \text{rep}(A'_{\mathbb{R}}) \longrightarrow \text{rep}(A_{\mathbb{R}})$ have been defined in Section 3.

**Definition 4.1.** Let $\text{rep}(A_{\mathbb{R}})$ be the subcategory consisting of the following representations of $A_{\mathbb{R}}$,

\[
\text{rep}(A_{\mathbb{R}}) = \{V \in \text{rep}(A_{\mathbb{R}}) | V(S_{k-1}) \oplus V(S_{k+1}) \xrightarrow{D'_1} V(S_k), D'_1 \text{ is surjective}\},
\]

where $D'_1 = (V(S_{k-1}, S_k), -V(S_{k+1}, S_k))$.

**Definition 4.2.** Let $\text{rep}(A'_{\mathbb{R}})$ be the subcategory consisting of the following representations of $\text{rep}(A'_{\mathbb{R}})$,

\[
\text{rep}(A'_{\mathbb{R}}) = \{W \in \text{rep}(A_{\mathbb{R}}) | W(S'_k) \xrightarrow{D'_2} W(S_{k+1}) \oplus W(S_{k-1}), D'_2 \text{ is injective}\},
\]

where $D'_2 = (w(S'_k, S_{k+1}), -w(S'_k, S_{k-1}))$.

**Theorem 4.3.** The functor $S^+_k : \text{rep}(A_{\mathbb{R}}) \longrightarrow \text{rep}(A'_{\mathbb{R}})$ is an equivalence of categories.

For the proof of Theorem 4.3, we need the following lemmas.

**Lemma 4.4.** With above notations, for any $x \in (S_{k-1}, S'_k)$, the commutative diagram (5) is a pull-back

\[
\begin{array}{ccc}
V'(S'_k) & \xrightarrow{V'(S'_k, S_{k+1})} & V(S_{k+1}) \\
\downarrow V'(S'_k, x) & & \downarrow V(S_{k+1}, x') \\
V'(x) & \xrightarrow{C_V(x)} & V(x').
\end{array}
\]
and for any \( x \in V(S_k', S_{k+1}) \), the commutative diagram (6) is also a pull-back

\[
\begin{array}{ccc}
V'(S_k') & \xrightarrow{V'(S_k', x)} & V'(x) \\
\downarrow V'(x, S_{k-1}) \cdot V'(S_k', x) & & \downarrow C_V(x) \\
V(S_{k-1}) & \xrightarrow{V(S_{k-1}, x')} & V(x')
\end{array}
\]

**Proof.** Firstly, assume that \( x \in (S_{k-1}, S_k') \), the commutative diagram (7) is a pull-back

\[
\begin{array}{ccc}
V'(x) & \xrightarrow{C_V(x)} & V(x') \\
\downarrow V'(x, S_{k-1}) & & \downarrow V(x', S_k) \\
V(S_{k-1}) & \xrightarrow{V(S_{k-1}, S_k)} & V(S_k)
\end{array}
\]

and the commutative diagram (8) is both a pull-back and a push-out

\[
\begin{array}{ccc}
V'(S_k') & \xrightarrow{V'(S_k', S_{k+1})} & V(S_{k+1}) \\
\downarrow V'(S_k', x) & & \downarrow V(S_{k+1}, x') \\
V'(x) & \xrightarrow{C_V(x)} & V(x') \\
\downarrow V'(x, S_{k-1}) & & \downarrow V(x', S_k) \\
V(S_{k-1}) & \xrightarrow{V(S_{k-1}, S_k)} & V(S_k)
\end{array}
\]

then we shall prove the commutative diagram (5) is a pull-back. That is to say, for \( h : W \to V'(x) \) and \( k : W \to V(S_{k+1}) \), if \( C_V(x) \cdot h = V(S_{k+1}, x') \cdot k \), we need to find a homomorphism \( l : W \to V'(S_k') \), such that \( h = V'(S_k', x) \cdot l \), \( k = V'(S_k', S_{k+1}) \cdot l \), as the following diagram shows

```
\[
\begin{array}{ccc}
W & \xrightarrow{l} & V'(S_k') \\
\downarrow h & & \downarrow V'(S_k', S_{k+1}) \\
V'(S_k) & \xrightarrow{V'(S_k', x)} & V(S_{k+1}) \\
\downarrow V'(x, S_{k-1}) & & \downarrow V(S_{k+1}, x') \\
V'(x) & \xrightarrow{C_V(x)} & V(x') \\
\downarrow V'(x, S_{k-1}) & & \downarrow V(x', S_k) \\
V(S_{k-1}) & \xrightarrow{V(S_{k-1}, S_k)} & V(S_k)
\end{array}
\]```
Since the commutative diagram (8) is a pull-back, for homomorphism $V(S_{k-1}) \xleftarrow{m} W \xrightarrow{k} V(S_{k+1})$ such that $V(S_{k-1}, S_k) \cdot V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l = V(x', S_k) \cdot V(S_{k+1}, x') \cdot k$, we can find a unique homomorphism $l : W \rightarrow V'(S'_k)$ satisfies the following diagram is commutative, that is, $m = V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l$, $k = V'(S'_k, S_{k+1}) \cdot l$, and $V(S_{k-1}, S_k) \cdot V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l = V(x', S_k) \cdot V(S_{k+1}, x') \cdot k$, as the following diagram shows

![Diagram](image)

Since the commutative diagram (7) is a pull-back, for homomorphism $V(S_{k-1}) \xleftarrow{j} W \xrightarrow{k} V'(x')$ such that $V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l = V(x', S_k) \cdot V(S_{k+1}, x') \cdot k$, we can find a unique homomorphism $h : W \rightarrow V'(x)$ satisfies the following diagram is commutative, that is, $j = V'(x, S_{k-1}) \cdot h$, $V(S_{k+1}, x') \cdot k = C_V(x) \cdot h$, and $V'(x, S_{k-1}) \cdot h = V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l$, as the following diagram shows

![Diagram](image)

By the uniqueness of homomorphism $W \rightarrow V(S_{k-1})$, we have $k = V'(S'_k, S_{k+1}) \cdot l$, $V'(x, S_{k-1}) \cdot V'(S'_k, x) \cdot l = V(x, S_{k-1}) \cdot h$. Then we shall prove $V'(S'_k, x) \cdot l = h$. For the homomorphism $h : W \rightarrow V'(x)$, the following diagram is commutative

![Diagram](image)

For the homomorphism $V'(S'_k, x) \cdot l : W \rightarrow V'(x)$, we have $C_V(x) \cdot V'(S'_k, x) \cdot l = V(S_{k+1}, x') \cdot V'(S'_k, S_{k+1}) \cdot l = V(S_{k+1}, x') \cdot k$, and the following diagram is commutative

![Diagram](image)
By the uniqueness of map $W \rightarrow V'(x)$, we have $h = V'(S'_k, x) \cdot l$, and $k = V'(S'_k, S_{k+1}) \cdot l$, that is, the commutative diagram (5) is a pull-back. Similarly, the commutative diagram (6) is also a pull-back. □

**Lemma 4.5.** With above notations, for any $x \in (S_{k-1}, S_k)$, the commutative diagram (9) is a push-out

(9) \[
\begin{array}{ccc}
V'(x') & \xrightarrow{D_V(x)} & S_k V'(x) \\
\downarrow & & \downarrow \\
S_k V'(S_{k-1}) & \xrightarrow{S_k V'(S_{k-1}, S_k)} & S_k V'(S_k),
\end{array}
\]

and for any $x \in (S_k, S_{k+1})$, the commutative diagram (10) is also a push-out

(10) \[
\begin{array}{ccc}
V'(x') & \xrightarrow{V'(x, S_{k+1})} & S_{k+1} V'(S_k) \\
\downarrow & & \downarrow \\
S_k V'(x) & \xrightarrow{S_k V'(x, S_{k+1})} & S_k V'(S_k).
\end{array}
\]

**Proof.** Similarly to the proof of Lemma 4.4, we can get Lemma 4.5. □

**Proposition 4.6.** With above notations, we have $S_k^- S_k^+ \cong 1_{\operatorname{rep}(A_k)}$, and $S_k^+ S_k^- \cong 1_{\operatorname{rep}(A'_k)}$.

**Proof.** (1) We shall first prove $S_k^- S_k^+ V \cong V$. Assume that $x \in (S_k, S_{k+1})$. For any $V \in \operatorname{rep}(A_k)$,

$$
(S_k^+ V(S_k), S_k^+ V(S_{k+1}), S_k^+ V(x, S_{k-1}) \cdot S_k^+ V(S_k', x'))
$$

is the pull-back of

$$
(V(S_k), V(S_{k+1}, x) \cdot V(x, S_k), V(S_{k-1}, S_k)),
$$

as the following diagram shows
For any $S_k^+ V \in \text{rep} (A_{\mathbb{R}})$, $(S_k^- S_k^+ V(x), S_k^- S_k^+ V(S_{k+1}, x), D_V(x))$ is the push-out of 
\[
(S_k^+ V(S_k'), S_k^+ V(S_k', S_{k+1}), S_k^+ V(S_k', x')),
\]
and $(S_k^- S_k^+ V(S_k), S_k^+ V(x, S_k) \cdot S_k^- S_k^+ V(S_{k+1}, x), S_k^- S_k^+ V(S_{k+1}, S_k))$ is the push-out of 
\[
(S_k^+ V(S_k'), S_k^+ V(S_k', S_{k+1}), S_k^+ V(x', S_{k-1}) \cdot S_k^+ V(S_k', x')),
\]
as the following diagram shows
\[
\begin{array}{ccc}
S_k^+ V(S_k') & \overset{S_k^+ V(S_k')}{\longrightarrow} & S_k^+ V(S_{k+1}) \\
\downarrow S_k^+ V(S_k', x') & & \downarrow S_k^+ V(S_k', x') \\
S_k^+ V(x') & \overset{S_k^-}{\longrightarrow} & S_k^- S_k^+ V(x)
\end{array}
\]
\[
\begin{array}{ccc}
S_k^+ V(S_k') & \overset{S_k^+ V(S_k')}{\longrightarrow} & S_k^+ V(S_{k+1}) \\
\downarrow S_k^+ V(S_k', S_{k+1}) & & \downarrow S_k^+ V(S_k', S_{k+1}) \\
S_k^+ V(S_{k-1}) & \overset{S_k^- S_k^+ V(S_{k-1}, S_k)}{\longrightarrow} & S_k^- S_k^+ V(S_k)
\end{array}
\]
That is, by the definition of $S_k^- S_k^+ V(x)$, the diagram (11) is a push-out
\[
\begin{array}{ccc}
S_k^+ V(S_k') & \overset{S_k^+ V(S_k')}{\longrightarrow} & S_k^+ S_k^+ V(S_{k+1}) \\
\downarrow S_k^+ V(S_k', x') & & \downarrow S_k^+ V(S_k', x') \\
S_k^+ V(x') & \overset{D_V(x)}{\longrightarrow} & S_k^- S_k^+ V(x)
\end{array}
\]
\[
\begin{array}{ccc}
S_k^+ V(S_k') & \overset{S_k^+ V(S_k')}{\longrightarrow} & S_k^+ V(S_{k+1}) \\
\downarrow S_k^+ V(S_k', S_{k+1}) & & \downarrow S_k^+ V(S_k', S_{k+1}) \\
S_k^+ V(x') & \overset{C_V(x')}{\longrightarrow} & V(x)
\end{array}
\]
By Lemma 4.4, the following diagram is a pull-back
\[
\begin{array}{ccc}
S_k^+ V(S_k') & \overset{S_k^+ V(S_k', S_{k+1})}{\longrightarrow} & V(S_{k+1}) \\
\downarrow S_k^+ V(S_k', x') & & \downarrow V(S_{k+1}, x) \\
S_k^+ V(x') & \overset{C_V(x')}{\longrightarrow} & V(x).
\end{array}
\]
Because $S_k^- S_k^+ V(S_{k+1}) = S_k^+ V(S_{k+1}) = V(S_{k+1})$, by the uniqueness of pull-back, $V(x) \cong S_k^- S_k^+ V(x)$ for $x \in (S_k, S_{k+1})$.
If $x \in (S_{k-1}, S_k)$, the conclusion is also true. That is, for any $x \in \mathbb{R}$, there exist a map $\eta_V(x) : V(x) \rightarrow S_k^- S_k^+ V(x)$ such that $V(x) \cong S_k^- S_k^+ V(x)$.
Next, we need to prove the diagram (12) is commutative

\[
\begin{array}{ccc}
V(x) & \xrightarrow{\eta(x)} & S_k^+S_k^+V(x) \\
\downarrow & & \downarrow \\
V(y) & \xrightarrow{\eta(y)} & S_k^-S_k^+V(y).
\end{array}
\]

For \( y \prec x \in (S_k, S_{k+1}) \), the following diagrams are commutative, respectively. By diagram (11), we have \( \eta_V(y) \cdot V(S_{k+1}, y) = (S_k^-S_k^+V)(x, y) \cdot (S_k^-S_k^+V)(S_{k+1}, x) \).

Since \((S_k^+V(S_k'), S_k^+V(S_k', S_{k+1}), S_k^+V(S_k', x'))\) is the push-out of \((V(x), V(S_{k+1}, x), C_V(x'))\), we have \( \eta_V(y) \cdot V(x, y) = (S_k^-S_k^+V)(x, y) \cdot \eta_V(x) \), and the diagram (12) is commutative. Similarly, for any \( x \in (S_{k-1}, S_k) \), we have the same conclusion. Thus we have \( S_k^-S_k^+V \cong V \) for any \( x \in (S_{k-1}, S_{k+1}) \).
(2) For maps $f_x : V(x) \to W(x)$ and $S_k^- S_k^+ f_x : S_k^- S_k^+ V(x) \to S_k^- S_k^+ W(x)$, we need to prove the diagram (13) is commutative.

$$\begin{array}{ccc}
V(x) & \xrightarrow{\eta_V(x)} & S_k^- S_k^+ V(x) \\
& f_x \downarrow & \downarrow S_k^- S_k^+ f_x \\
W(x) & \xrightarrow{\eta_W(x)} & S_k^- S_k^+ W(x).
\end{array}$$

Firstly, assume that $x \in (S_k, S_{k+1})$. For $V, W \in \text{rep}(A_\infty)$, the following diagrams are commutative, respectively. Note that $\eta_W(x) \cdot f_x \cdot V(S_{k+1}, x) = S_k^- S_k^+ f_x \cdot (S_k^- S_k^+ V)(S_{k+1}, x)$.

Since $(S_k^+ V(S_k'), S_k^+ V(S_k', S_{k+1}), S_k^+ V(S_k', x'))$ is the push-out of

$$(V(x), V(S_{k+1}, x), C_V(x')),$$

we have $\eta_W(x) \cdot f_x = S_k^- S_k^+ f_x \cdot \eta_V(x)$, and the diagram (13) is commutative.

If $x \in (S_{k-1}, S_k)$, we have the same conclusion. By the arbitrariness of $x$, the following diagram is commutative.
\[
\begin{align*}
V & \xrightarrow{\eta_V} S_k^- S_k^+ V \\
& \downarrow f \\
W & \xrightarrow{\eta_W} S_k^- S_k^+ W.
\end{align*}
\]

To sum up, we have \( S_k^- S_k^+ \cong 1_{\text{rep}(A_k)} \). Similarly, we can get \( S_k^+ S_k^- \cong 1_{\text{rep}(A_k')}. \) □

As the corollary of Proposition 4.6, we get Theorem 4.3.

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