COHERENT STATES, ENTANGLEMENT, 
AND GEOMETRIC INVARIANT THEORY

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INTRODUCTION

From a thought experiment for testing the very basic principles of quantum mechanics in its early years [EPR35, Schrö35], entanglement nowadays is growing into an important technical tool for quantum information processing [EPR96,...]. Surprisingly enough currently there is no agreement of opinion among experts on the very definition of entanglement, and its proper measure [Peres98, BarPho89, PleVed98, VedPle98, BZZ01, BZ01]. Here we propose a new approach to entanglement, based on dynamic symmetry group of a quantum system. A similar approach was applied by A. Perelomov [Perel86] to coherent states, which in many respects are opposite to entangled ones. The celebrated “unexpected efficiency” [Wig67] of group-theoretical methods in quantum mechanics was many times demonstrated by E. Wigner [Wig31, Wig39], whose centenary holds this year.

The main objective of the paper is to unveil an adequate mathematics hidden behind entanglement, that is Geometric Invariant Theory [MFK94]. More specifically relation between these two subjects can be described by the following theses.

(1) Total variance of completely entangled state $\psi$ is maximal.
(2) This distinguishes $\psi$ as a minimal vector in its orbit under action of complexified dynamic group $G^c$.

(3) An ultimate aim of Geometric Invariant Theory is a description of complex orbits and their minimal vectors. It suggests that noncompletely entangled states are just GIT semistable vectors.

This approach provides a powerful tool for treatment of entanglement, and shed new light on some old problems. We consider many classical and not so classical examples in support of these theses, and discuss their relation with conventional approach. Formal proofs are mostly skipped, and will be published elsewhere, since they help little in search for a proper definition of an unclear physical concept.

I first start think about the subject in Erwin Schrödinger Institute of Mathematical Physics in Vienna in January 2001, and would like to express here my gratitude for financial support and an exiting atmosphere.

This paper arouses from an attempt to unveil a hidden meaning of some words used by physicists. The following quotation from A. Grothendieck exposes the encountered difficulties

"Passer de la mécanique de Newton à celle d'Einstein doit être un peu, pour le mathématicien, comme de passer du bon vieux dialecte provençal à l'argot parisien dernier cri. Par contre, passer a la mécanique quantique, j'imagine, c'est passer du français au chinois."\footnote{To pass from Newton’s mechanics to that of Einstein must be as easy, for mathematician, as to pass from good old provincial dialect to the last cry of Paris slang. On the contrary, to pass to the quantum mechanics, I think, is to pass to Chinese.}

1. Coherent states

Coherent states, first introduced by Schrödinger [Schrö26], lapsed into obscurity for decades until Glauber [Glaub63] rediscovered them in connection with laser emission. Later on Perelomov [Perel86] put them into an adequate context of dynamic symmetry group. We’ll use a similar approach for entanglement, and to warm up recall here some basic facts about coherent states.

1.1. Glauber’s coherent states. Let’s start with quantum oscillator, described by canonical pair of operators $p, q, [p, q] = i\hbar$, generating Weil-Heisenberg algebra $\mathcal{W}$. This algebra has unique unitary irreducible representation, which can be realized in Fock space $\mathcal{F}$ spanned by orthonormal set of $n$-excitations states $|n\rangle$ on which dimensionless annihilation and creation operators

$$a = \frac{q + ip}{\sqrt{2\hbar}}, \quad a^\dagger = \frac{q - ip}{\sqrt{2\hbar}}, \quad [a, a^\dagger] = 1$$

act by formulae

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

A typical element from Weil-Heisenberg group $W = \exp \mathcal{W}$, up to a phase factor, is of the form $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ for some $\alpha \in \mathbb{C}$. Action of this operator on vacuum $|0\rangle$ produces state

$$|\alpha\rangle := D(\alpha)|0\rangle = \exp \left( -\frac{|\alpha|^2}{2} \right) \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$
known as Glauber coherent state. The number of excitations in this state has Poisson distribution with parameter $|\alpha|^2$. In many respects its behavior is close to classical [Perel86], e.g. Heisenberg’s uncertainty $\Delta p \Delta q = \hbar/2$ for this state is minimal. We can summarize this construction as follows:

$$G\text{lauber’s coherent states} = W\text{-orbit of vacuum.}$$

1.2. Dynamic group and general coherent states. Let’s now turn to arbitrary quantum system $S$ with dynamic symmetry group $G = \exp G$. By definition its Lie algebra $\mathcal{G}$ is generated by all essential observables of the system$^2$ (like $p, q$ in the above example). To simplify the underlying mathematics suppose in addition that state space $\mathbb{H} = \mathbb{H}(S)$ of the system is finite, and representation of $G$ in $\mathbb{H}$ is irreducible.

To extend (1.1) to this general setting we have to understand the special role of the vacuum, which primarily considered as a ground state of a system. For group-theoretical approach, however, another its property is more relevant:

$$\text{Vacuum is a state with maximal symmetry.}$$

This may be also spelled out that vacuum is a most degenerate state of a system. Symmetries of state $\psi$ are given by its stabilizers

$$G_\psi = \{ g \in G \mid g\psi = \lambda \psi \}, \quad \mathcal{G}_\psi = \{ X \in \mathcal{G} \mid X\psi = \lambda \psi \}$$

in the dynamic group $G$ or in its Lie algebra $\mathcal{G}$. Looking back to the quantum oscillator, we see that some symmetries are actually hidden, and manifest themselves only in complexified algebra $\mathcal{G}^c = \mathcal{G} \otimes \mathbb{C}$ and group $G^c = \exp \mathcal{G}^c$. For example, stabilizer of vacuum in Weyl algebra $\mathcal{W}$ consists of scalars, while in complexified algebra $\mathcal{W}^c$ it contains a non-scalar annihilation operator, $\mathcal{W}^c_0 = \mathbb{C} + \mathbb{C}a$. In the last case the stabilizer is big enough to recover the whole dynamic algebra

$$\mathcal{W}^c = \mathcal{W}^c_0 + \mathcal{W}^c_0 \dagger.$$  

This decomposition, called complex polarization, gives a precise meaning for the maximal degeneracy of a vacuum or a coherent state [Perel86]. It ensures that dimension of the symmetry group of such state is at least half of dimension the whole dynamic group.

1.2.1. Definition. State $\psi \in \mathbb{H}$ is said to be coherent$^3$ if

$$\mathcal{G}^c = \mathcal{G}^c_\psi + \mathcal{G}^c_\psi \dagger,$$

where $\mathcal{G}^c_\psi \dagger$ consists of operators conjugate to that of stabilizer $\mathcal{G}^c_\psi$.

In finite dimensional case all such decompositions come from Borel subalgebra, i.e. a maximal solvable subalgebra $\mathcal{B} \subset \mathcal{G}^c$. Algebra of all upper triangular matrices$^2$

$^2$To eliminate irrelevant phase factors we expect that the dynamic symmetry group acts on the state space by unimodular transformations. Equivalently, Lie algebra of observables consists of traceless operators.

$^3$This is what Perelomov called “coherent state closest to classical”. His generalized coherent states are defined as elements from $G$-orbit of an arbitrary initial vector $\psi_0$. They have no intrinsic meaning, and are useful mainly as a calculation tool.
\( T \subset G = \text{Mat}(n, \mathbb{C}) \) is a typical example. It is a basic structural fact that \( B + B^\dagger = G^c \), and therefore

\[
\psi \text{ is a coherent state } \iff \psi \text{ is an eigenvector of a Borel subalgebra } B.
\]

In representation theory eigenstate \( \psi \) of \( B \) is called highest vector, and the corresponding eigenvalue \( \lambda = \lambda(X) \),

\[
X \psi = \lambda(X) \psi, \quad X \in B
\]
is said to be highest weight. Here are their basic properties:

1. For irreducible space \( \mathbb{H} \) the highest vector \( \psi_0 \) (=vacuum) is unique.
2. There is only one irreducible representation \( \mathbb{H} = \mathbb{H}_\lambda \) with highest weight \( \lambda \).
3. All coherent states are of the form \( \psi = g \psi_0, \; g \in G \) (cf. with (1.1)).
4. Coherent state \( \psi \) in composite system \( \mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2 \) splits into product

\[
\psi = \psi_1 \otimes \psi_2 \text{ of coherent states of the components.}
\]

1.2.2. Remark. One can spell out these properties by saying that unitary irreducible representations of group \( G \) are parameterized by symmetry type of their coherent states or vacua. Coherent state theory, in the form given by Perelomov [Perel86], is a physical equivalent of Kirillov–Kostant orbit method in representation theory [Kiril76]. Notice that in many cases the complexified symmetry group is physically meaningful.

1.2.3. Example. For a particle of spin \( j > 0 \) the dynamic symmetry group is SU(2). Its complexification is group of unimodular matrices SL(2, \( \mathbb{C} \)), which, as first noted by Wigner [Wig39], is a double cover of Lorentz group. It is responsible for relativistic transformation of spin state in a moving frame [PST02]. Coherent states in this example are those with definite spin projection \( j \) onto some direction. Group of complex symmetries of such state is conjugate to group of triangular matrices (=Borel subgroup).

1.3. Total variance and extremal property. Let us define total variance of state \( \psi \) by equation

\[
\mathbb{D}(\psi) = \sum_i \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2,
\]

where \( X_i \) form an orthonormal basis in Lie algebra \( \mathcal{G} \) with respect to its invariant metric (for spin group SU(2) one can take moment operators \( J_x, J_y, J_z \) as \( X_i \)). The total variance is \( G \)-invariant \( \mathbb{D}(g\psi) = \mathbb{D}(\psi), \; g \in G \), and independent of the basis \( X_i \). It measures the total level of quantum fluctuations of a system in state \( \psi \).

The first sum in (1.7) contains well known Casimir operator

\[
C = \sum X_i^2,
\]

which acts as a scalar in every irreducible representation \( \mathbb{H} \) of \( G \). For spin \( j \) representation \( \mathbb{H}_j \) of SU(2) the Casimir is equal to square of moment \( j(j + 1) \), and in general \( C = \langle \lambda, \lambda + \rho \rangle \) in representation \( \mathbb{H}_\lambda \) with highest weight \( \lambda \) (here we use H. Weyl’s notation \( \rho \) for halfsum of positive roots). Hence

\[
\mathbb{D}(\psi) = \langle \lambda, \lambda + \rho \rangle - \sum \langle \psi | X_i | \psi \rangle^2.
\]
1.3.1. Theorem. State $\psi$ is coherent iff its total variance is minimal, and in this case

\begin{equation}
\mathbb{D}(\psi) = \langle \lambda, \rho \rangle.
\end{equation}

This theorem, in a slightly less precise form, belongs to Delbargo and Fox [DelFox70]. It supports a common believe, that coherent states are closest to classical ones. Note however that such simple characterization holds only for finite dimensional systems. The total variance, for example, makes no sense for quantum oscillator, for which we have minimal uncertainty $\Delta p \Delta q = \hbar/2$ instead.

1.3.2. Example. Recall that for spin $j$ representation of SU(2) coherent state $\psi$ has definite spin projection $j$ onto some direction, and by (1.10) $\mathbb{D}(\psi) = j$. The standard deviation $\sqrt{j}$ for such state is of smaller order then $j$, therefore for $j \to \infty$ it behaves classically [Perel86].

2. Entanglement

Everybody knows, and nobody understand what is entanglement. The very term was coined in the famous Schrödinger’s “cat paradox”\(^4\) paper [Schrö35], which in turn was inspired by the no less celebrated Einstein–Podolsky–Rosen gedanken experiment [EPR35]. While the authors were amazed by nonlocal nature of correlations between involved particles, J. Bell was the first to note that the correlations themselves, put aside the nonlocality, are inconsistent with “classical realism” [Bell65]. Since then Bell’s inequalities are produced in industrial quantities [CHSH69, GHZ89, GHSZ90, Merm90, WerWol01,...]. Neither of this effects, however, allows decisively distinguish entangled states from others. Therefore we develop another approach, based on the dynamic symmetry group.

2.1. EPR paradox. Decay of a spin zero state into two components of spin 1/2 subjects to a strong correlation between spin projections of the components, caused by conservation of moment. The correlation apparently creates an information channel between the components, acting beyond their light cones. This paradox, recognized in early years of quantum mechanics [EPR35], nowadays has many applications, but no explanation.

We are not in position to comment this phenomenon, and confine ourself instead to less involved Bell’s approach [Bell65]. Henceforth we completely disregard the nonlocality, and turn to quantum correlations per se.

2.2. Bell’s paradox. Let $X_i$, $i \in I$ be observables of quantum system $S$, that is Hermitian operators $X_i \in \mathcal{G}$ from Lie algebra of the dynamic symmetry group $G$. According to quantum paradigm actual measurement of $X_i$ in state $\psi$ produces random quantity $x_i$, determined by expectations of all functions $f(x_i)$

\begin{equation}
\langle f(x_i) \rangle = \langle \psi | f(X_i) | \psi \rangle
\end{equation}

(the moments $\langle x_i^n \rangle$ are usually enough). If for some set of indices $J \subset I$ observables $X_j$, $j \in J$ commute, then the random quantities $x_J = \{x_i | i \in J\}$ have joint distribution given by

\begin{equation}
\langle f(x_J) \rangle = \langle \psi | f(X_J) | \psi \rangle,
\end{equation}

where $f(x_J)$ is a function of $x_j, j \in J$. The so called “classical realism” postulates existence of a hidden joint distribution of all variables, commuting or not. To test it we have to solve the following problem.

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\(^4\)As BBC puts it: *In quantum mechanic it is not so easy to be or not to be.*
2.2.1. Marginal problem. Under which conditions a system of marginal distributions of \( x_J, J \subset I \) can be extended to a joint distribution of all \( x_I \)?

This is a question about existence of a “body” (= probability density) in \( \mathbb{R}^I \) with given projections onto some coordinate subspaces \( \mathbb{R}^J, J \subset I \).

Note that univariant margins \( x_i \) are always compatible (one can take joint distribution of \( x_i \) as independent quantities). The following inequality is necessary for consistency of bivariant margins \( x_{ij} = (x_i, x_j) \)

\[
\mathbb{D}(x_i) + \mathbb{D}(x_j) + \mathbb{D}(x_k) + 2\text{Cov}(x_i, x_j) + 2\text{Cov}(x_j, x_k) + 2\text{Cov}(x_k, x_i) \geq 0,
\]

since LHS is equal to \( \mathbb{D}(x_i + x_j + x_k) \), provided a joint distribution of \( x_i, x_j, x_k \) exists. This is a simplest prototype of Bell’s inequalities.

2.2.1.1. Remark. The marginal problem has a long history, starting from works by W. Hoefling in Germany (1940), and a bit later by Freché in France. Springer Verlag published collected papers of Hoefling in 1994. Three conferences on the subject held in the last decade [Marg91, Marg96, Marg97]. None of the participants ever mentioned Bell’s problem, and apparently none of physicists was aware about these activities. This is a disturbing example of a split between mathematics and physics.

2.3. Ansatz for testing “classical realism”. The random quantity \( x_i, i \in I \) assumes values in \( \Lambda_i = \text{Spec} X_i \). For \( J \subset I \) put \( \Lambda_J = \prod_{j \in J} \Lambda_j \) and consider functions on \( \Lambda = \Lambda_I \) of the form

\[
F(\lambda) = \sum_{X_J \text{ commute}} f_J(\lambda_J), \quad \lambda \in \Lambda,
\]

where

\( J \subset I \) corresponds to commuting sets of operators \( X_J, j \in J \),
\( f_J \) is a real function on \( \Lambda_J \),
\( \lambda_J \) is projection of \( \lambda \in \Lambda_I \) onto \( \Lambda_J \).

Such function \( F \), by commutativity of \( X_J \), unambiguously determines Hermitian operator

\[
F(X) = \sum_{X_J \text{ commute}} f_J(X_J).
\]

2.3.1. Theorem. State \( \psi \) is consistent with classical realism iff

\[
F(\lambda) \geq 0 \Rightarrow \langle \psi | F(X) | \psi \rangle \geq 0
\]

for all functions \( F \) of form (2.1).

2.3.2. Corollary. Every state is compatible with classical realism iff

\[
F(\lambda) \geq 0 \Rightarrow F(X) \geq 0
\]

for all functions \( F \) of form (2.1).

The proof of the theorem is based on the above considerations and Kellerer’s criterion [Kell64] for solvability of the marginal problem.
2.3.3. Remark. The set of nonnegative functions $F$ of type (2.1) forms a convex cone $K$, which will be called Kellerer’s cone. It is enough to check (2.3) only for extremal functions $F \geq 0$ from $K$, i.e. for those which aren’t positive combinations of others. The corresponding Bell’s inequality $\langle \psi | F(X) | \psi \rangle \geq 0$ is also said to be extremal (it can’t be deduced from the others). The extremal functions generate edges of the Kellerer’s cone $K$.

2.3.4. Example. Let’s consider a system of two particles $a$ and $b$. The dynamic symmetry group in this case is $\text{SU}(2) \times \text{SU}(2)$, and the state space is tensor product $H_a \otimes H_b$ of spin spaces of the particles. Let $A_i$ and $B_j$ be spin projection operators for particles $a$ and $b$ onto directions $i$ and $j$. Operators $A_i$, $B_j$ commute, and for spin $1/2$ with two measurement per site Kellerer’s cone $K$ is given by

$$F(a_1, a_2, b_1, b_2) = f_{11}(a_1, b_1) + f_{12}(a_1, b_2) + f_{21}(a_2, b_1) + f_{22}(a_2, b_2) \geq 0,$$

where $a_i, b_j = \pm 1$ are eigenvalues of $A_i$ and $B_j$. All the edges of this cone can be obtained from Clauser-Horn-Shimony-Holt function [CHSH69]

$$a_1 b_1 + a_2 b_1 + a_2 b_2 - a_1 b_2 + 2 \geq 0$$

by permutation of particles $a \leftrightarrow b$ and switching eigenvalues $a_i \mapsto \pm a_i$, and $b_j \mapsto \pm b_j$. So we have essentially one Bell’s type inequality for testing “classical realism”

(CHSH) $\langle \psi | A_1 B_1 | \psi \rangle + \langle \psi | A_2 B_1 | \psi \rangle + \langle \psi | A_2 B_2 | \psi \rangle - \langle \psi | A_1 B_2 | \psi \rangle + 2 \geq 0$.

2.3.5. Remark. Finding of vertices or edges is a typical linear programming problem. Each time I decide to run my computer overnight, it finds a couple of new extremal Bell’s inequalities for three particles system of spin $1/2$. This amounts altogether to 27 nonequivalent classes, including five given in [WerWol01]. The list is probably still incomplete.

Notice that Scarani and Gisin [ScaGis01] relate violation of Bell’s inequalities to security of quantum communication. In the core of a conventional security system lies a “hard problem”, like prime decomposition of an integer $N \gg 1$, that complexity presumably grows faster then any power of $N$, while checking of a given solution takes only polynomial time. Currently, however, there is not a single problem, for which such widely expected behavior has been rigorously proven. This is a one million dollars Millennium Problem of Clay Mathematical Institute [Cook]. Quantum computers may drastically change the very notion of complexity [Shor97]. See also [Pitow89] on complexity of Bell’s type problems.

2.3.6. Theorem. An irreducible quantum system with dynamic group $G$ of rank at least two is incompatible with classical realism.

The rank of group $G$ is a maximal number of linear independent commuting operators (=observables) in its Lie algebra $\mathcal{G}$. For example, $\text{rk} \text{SU}(n) = n - 1$. A system of rank one can’t violates “classical realism”, since one dimensional margins are always consistent.

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5The title of this interesting paper may be misleading, since the authors confine themselves on functions of special form, which do not exhaust the whole Kellerer’s cone.

6Among 6 others, including Riemann Conjecture and Quantum Yang-Mills Theory.
A group of a greater rank, contains either \( SU(2) \times SU(2) \) or \( SU(3) \). The first case amounts to widely known violation of Bell’s inequalities in two particles systems. Below is a typical example of a nonclassical behavior in \( SU(3) \). Recall, that this is \emph{chromodynamic group} of internal states of hadrons. It has another physical incarnation as \emph{polarization group} of a massive quantum vector field.

2.3.7. \textit{Pentagonal inequality}. Let’s consider a cyclic quintuplet of orthogonal states

\begin{equation}
  e_i \in \mathbb{H}, \quad e_i \perp e_{i+1}, \quad i \mod 5
\end{equation}

in standard three dimensional representation of \( SU(3) \), and denote by \( S_i = 1 - 2|e_i\rangle\langle e_i| \) \emph{reflection operator} in a mirror orthogonal to \( e_i \). Operators \( S_i, S_{i+1} \) commute, and the corresponding Kellerer’s cone consists of functions of the form

\[
f_{12}(s_1, s_2) + f_{23}(s_2, s_3) + f_{34}(s_3, s_4) + f_{45}(s_4, s_5) + f_{51}(s_5, s_1) \geq 0
\]

where \( s_i \) assumes values \( \pm 1 = \text{Spec} S_i \). Here is an example of such function

\begin{equation}
  s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_5 + s_5 s_1 + 3 \geq 0.
\end{equation}

Indeed, each summand \( s_i s_{i+1} \) is equal to \( \pm 1 \), while their product is \( +1 \). Hence there exists at least one positive summand \( s_i s_{i+1} = +1 \), and (2.6) follows.

One can show that (2.6) is an extremal function from the Kellerer’s cone, and all such functions can be obtained from this one by switching the eigenvalues \( s_i \mapsto \pm s_i \).

Applying Theorem 3.3.1 we get extremal \emph{pentagonal inequality}

\begin{equation}
  \langle \psi | S_1 S_2 | \psi \rangle + \langle \psi | S_2 S_3 | \psi \rangle + \langle \psi | S_3 S_4 | \psi \rangle + \langle \psi | S_4 S_5 | \psi \rangle + \langle \psi | S_5 S_1 | \psi \rangle + 3 \geq 0,
\end{equation}

for testing “classical realism”. One can put it in geometric form as follows

\[
  \sum_i \cos^2 \alpha_i \leq 2, \quad \alpha_i = \hat{\psi} e_i.
\]

This inequality fails, for example, for a \emph{regular} configuration of vectors \( e_i \), and \( \psi \) directed along its axis of symmetry (of order 5). In this case

\[
  \sum_i \cos^2 \alpha_i = \frac{5 \cos \pi/5}{1 + \cos \pi/5} = 2.236067.
\]

In a smaller extent violation of the pentagonal inequality is almost inevitable in all settings: for every configuration (2.5) with no collinear vectors, operator \( \sum_i S_i S_{i+1} \) has an eigenvalue \( \lambda < -3 \), and the corresponding eigenstate \( \psi \) breaks classical law (2.7).

Notice that in this example all states are \emph{coherent}, and \emph{none} of them is compatible with “classical realism”.

2.3.8. \textit{Summary}.

(1) “Classical realism” fails whenever it is virtually possible (Theorem 2.3.6).

(2) It may fail for all states, including coherent ones (see n° 2.3.7).

(3) A state may be manifestly nonclassical, even if Bell’s approach fails to detect this (see below n° 2.5.3).

(4) Failure of “classical realism” is a basic fact of quantum mechanics, in noway specific for entanglement.

(5) Bell’s inequalities is a marginal problem indeed. I would say they lead into a dead end. See however n° 2.8, 2.5.4, 2.3.5.
2.4. Extremal property of completely entangled states. In the previous section we have seen how illusive may be connection between “classical realism” and entanglement. Instead of this ambiguous relation we put forward an extremal property of a completely entangled state, which can be checked in all known instances, namely the maximality of its total variance:

\[ D(\psi) := \sum_i (\langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle)^2 = \max. \]

One can see from equation (1.9)

\[ D(\psi) = \langle \lambda, \lambda + \rho \rangle - \sum_i \langle \psi | X_i | \psi \rangle^2 \]

that the maximum is attained for state \( \psi \) with zero expectation of all observables

\[ \langle \psi | X | \psi \rangle = 0, \quad \forall X \in \mathcal{G}, \]

and the maximum itself is equal to Casimir

\[ \max_{\psi} D(\psi) = \langle \lambda, \lambda + \rho \rangle, \]

Notice that (2.8) is opposite to the property of coherent states, for which the total variance is minimal and equal to \( \langle \lambda, \rho \rangle \) (Theorem 1.3.1). Therefore generically we have inequality

\[ \langle \lambda, \rho \rangle \leq D(\psi) \leq \langle \lambda, \lambda + \rho \rangle. \]

2.4.1. Remark. There is a minor discrepancy between conditions (2.8) and (2.9). They are equivalent, provided there exists at least one state with zero average of all observables. We’ll call a system degenerate if it has no such states. There are very few degenerate systems consisting of one component, i.e. with simple dynamic group [VinPop92]

1. \( n \)-dimensional representations of SU(\( n \)) and Sp(\( n \)).
2. For odd \( n \) representation of SU(\( n \)) in space of skew-symmetric bilinear forms.
3. A halfspinor representation of dimension 16 of Spin(10).

There are many more such composite systems, and their classification is also known due to M. Sato and T. Kimura [SatKim77]. It tells which simple quantum systems can not be completely entangled into a composite one. See § 2.5.2 for examples.

2.5. Formal definition and examples. In what follows we assume (2.9), rather then (2.8), as a formal definition of a completely entangled state.

2.5.1. Definition. State \( \psi \in H \) is said to be completely entangled if all observables \( X \in \mathcal{G} \) have zero expectation in state \( \psi \)

\[ \langle \psi | X | \psi \rangle = 0, \quad \forall X \in \mathcal{G}. \]

Notice, that property (2.9) is \( G \)-invariant, i.e. the dynamic group transforms completely entangled state \( \psi \) into completely entangled one \( g\psi, \ g \in G \).
Recall also, that the total variance of a completely entangled state is maximal, as opposed to a coherent state, for which the variance is minimal. The total variance is a natural measures of quantum fluctuations in a system. Therefore one can informally think about coherent states as closest to classical, and completely entangled ones as manifestly nonclassical, see eqs. 2.5.3, 2.5.4 for examples. By this reason purely quantum effects, such as nonlocality, or violation of classical realism are most likely to happen for a completely entangled state.

All the states unanimously recognized as completely entangled conform with this definition, see examples 2.5.2 and conjecture 2.5.6 below. But the main argument in its favor comes from equation (2.9), which is mathematically meaningful, and connects entanglement to Geometric Invariant Theory to be discussed in the next section.

2.5.2. Completely entangled states in composite systems. Let’s consider composite system

\[ H = \bigotimes_{i=1}^{N} H_i \]

with components of dimension \( n_i \) and dynamic group \( G_i = \text{SU}(n_i) \). This scheme includes \( N \) qubits system of particles of spin 1/2. Choose orthonormal basis \( e_i \) in \( H_i \) and arrange components of tensor

\[ \psi = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_N} \psi_{\alpha_1, \alpha_2, \ldots, \alpha_N} e_1^{\alpha_1} \otimes e_2^{\alpha_2} \otimes \cdots \otimes e_N^{\alpha_N} \]

into \( N \) dimensional matrix \([\psi]\). Applying to \( \psi \) criterion (2.9) one can deduce

**2.5.2.1. Proposition.** State \( \psi \in H \) is completely entangled iff parallel slices of its matrix \([\psi]\) are orthogonal and have the same norm.

**2.5.2.2. Corollary.** Composite system (2.10) admits a completely entangled state iff information capacities \( \delta_i = \log n_i \), \( n_i = \dim H_i \) of the components satisfy polygonal inequalities

\[ \delta_i \leq \sum_{j(\neq i)} \delta_j. \]

The inequality follows from linear independence of the orthogonal parallel slices, which implies \( n_j \leq n_1, n_2, \ldots, n_N \).

### 2.5.2.3. Examples.

1) For completely entangled state \( \psi \in H_1 \otimes H_2 \) in a two components system, the matrix \([\psi]\) has format \( m \times n \), \( m = \dim H_1 \), \( n = \dim H_2 \) with orthogonal rows and columns of the same norm \( 1/\sqrt{m} \) and \( 1/\sqrt{n} \) respectively. This is possible only if \( n = m \), and in this case \([\psi]\) is proportional to a unitary matrix. This implies that completely entangled state is unique

\[ \psi = \frac{1}{\sqrt{n}} \sum_{i} e^i \otimes f^i, \]

up to action of dynamic group \( \text{SU}(n) \times \text{SU}(n) \). For \( n = 2 \) it is known as EPR or Bell state.
ii) Similarly in three qubits system there exists unique completely entangled state

\[(\text{GHZ}) \quad \psi = \frac{1}{\sqrt{2}}(e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2),\]

up to action of dynamic group SU(2)×SU(2)×SU(2). This is well known Greenberg-Horn-Zeilinger state [GHZ89,CHSZ90].

iii) The previous two rigid examples are actually exceptional. For \(N > 3\) completely entangled \(N\) qubits state, modulo action of the dynamic group, depends on \(2^N - 3N - 1\) complex parameters. The structure of this moduli space is not known neither in \(N\) qubits setting, nor for a generic three components system. See no 2.5.3 for description of a similar moduli space of spin entangled states.

iv) For composite system (2.10) coherent state \(\psi\) is just decomposable tensors \(\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_N\) (see no. 1.2.1). Such states for a long time where treated as completely disentangled.

2.5.3. Completely entangled spin states. Let’s consider completely entangled state \(\psi \in \mathbb{H}_j\) of a system of spin \(j\). According to Definition 2.5.1 this means that average spin projection onto every direction is zero. This certainly can’t happen for \(j = 1/2\), since in this case each state has definite spin projection 1/2 onto some direction. But for \(j \geq 1\) such states do exist. For example, one can take \(\psi = |0\rangle\) for integer \(j\), and in general

\[\psi = \frac{1}{\sqrt{2}}(|+j\rangle + |-j\rangle).\]

Up to a rotation this is the only possibility for \(j = 1\) or 3/2. All completely entangled states of arbitrary spin \(j\) can be constructed as follows [Kly94].

2.5.3.1. Ansatz. Start with a configuration of \(2j\) unit vectors \(p_i \in \mathbb{S}^2\) with zero sum (one can visualize it as a closed \(2j\)-gon with unit sides in \(\mathbb{R}^3\)). Then take their images \(\zeta_i \in \mathbb{C}\) under stereographic projection \(\pi: \mathbb{S}^2 \to \mathbb{C}\) and expand the product

\[
\prod_i (z - \zeta_i) = \sum_{\mu=-j}^{\mu=j} a_\mu \left( \begin{array}{c} 2j \\ j + \mu \end{array} \right) z^{j+\mu},
\]

to end up with completely entangled state

\[\psi = \sum_{\mu=-j}^{\mu=j} a_\mu \left( \begin{array}{c} 2j \\ j + \mu \end{array} \right)^{1/2} |\mu\rangle,
\]

possibly non normalized.

To sum up: completely entangled states of spin \(j\) are parameterized by closed polygonal strings in \(\mathbb{R}^3\) of length \(2j\). Evolution and decay of the states may be viewed as evolution and decay of the string. This is a typical example of description of coherent states from a perspective of Geometric Invariant Theory.

Every such state is manifestly nonclassical, since average projection of moment onto any direction is zero, while the standard deviation \(\sqrt{j(2j+1)}\) exceeds maximum of the projection \(j\). This kind of nonclassical behavior can’t be detected by
Bell’s approach, which needs at least two independent commuting observables, see n° 2.3.6. But in no way it is less nonclassical then EPR.\footnote{Well, except the nonlocality, which has nothing to do with Bell’s inequalities, and remains enigmatic anyway.}

2.5.4. Entangled states in other systems. The previous arguments can be literally extended onto arbitrary system, using inequality $\langle \lambda, \lambda \rangle < \langle \lambda, \lambda + \rho \rangle$ instead of $j^2 < j(j+1)$:

2.5.4.1. Claim. Every completely entangled state is manifestly nonclassical.

It is easily seen that zero weight vector $\psi$, that is a vector annihilated by Cartan subalgebra (see n° 2.8.1), is always completely entangled. For spin group SU(2) this amounts to state $|0\rangle$ with zero spin projection. For chromodynamic group SU(3) this includes hadrons composed of equal number of all three quarks $u, d, s$ (antiquark is counted with coefficient -1). For example $\pi^0$ is an entangled state in octet (=adjoint representation) of spin 0 mesons, while $\pi^\pm$ are coherent states in this octet. Big quantum fluctuations in entangled state may be responsible for instability of $\pi^0$, which life time nine orders smaller then $\pi^\pm$.

2.5.5. Remark. Product $\psi = \psi_1 \otimes \psi_2$ of two completely entangled states is a completely entangled state, although very untypical one. For example completely entangled state of two particles of spin \( \geq 1 \) may decay into two components, each being entangled onto itself. The degenerated representations listed in n° 2.4.1 may be used as building blocks for stable systems, for which such decay is forbidden.

We close this section with the following conjecture, motivated by Theorem 2.3.6 and the previous remark.

2.5.6. Conjecture. Indecomposable completely entangled state of a system with dynamic symmetry group of rank at least two is incompatible with classical realism.

This can be checked in many cases, but general proof is still missing. The conjecture is primary designed to convert the perplexed.

2.6. Kempf-Ness unitary trick and GIT stability. The extremal property (2.8-9) of a completely entangled state is closely related to concept of stability in Geometric Invariant Theory (GIT). The later emerges from the classical, mostly algebraic, invariant theory of 19th century enhanced with innovating geometric insight by D. Hilbert. Later on it was transformed by D. Mumford into a powerful universal formalism, which infinite dimensional version is familiar to physicists from gauge theory. The third edition of his book [MFK94] includes a bibliography about 1000 titles.

Vector $\psi \in \mathbb{H}$ is said to be semistable if it can be separated from zero by a $G$-invariant function $I$, that is $I(\psi) \neq I(0)$. Invariant function $I(g\psi) = I(\psi)$, $g \in G$ is just a conservation law or an integral of the system. We expect the invariant $I$ to be holomorphic, in which case it retains the invariance with respect to complexified group $G^c$. Notice that Hermitian metric $\langle \psi | \psi \rangle$ is a $G$-invariant, but not a holomorphic function.

Nonvanishing invariant $I(\psi) \neq 0$ prevents $\psi$ from falling to zero under action of the complexified group $G^c$. This implies existence of a nonzero vector $\psi_0 = g\psi$, $g \in G^c$ of minimal length, provided complex orbit $G^c\psi$ of $\psi$ is closed. In the last
case state $\psi$ is said to be stable.\footnote{In GIT stable state supposed to have at most finite symmetry group, while condition (2.9) ensures only that its dimension is as small as possible. We’ll not pay much attention to the distinction between stability and semistability.} from $g\psi, g \in G^c$ as a limit. The following theorem [KemNes78] identify the minimal vector with a completely entangled state.

**2.6.1. Kempf-Ness unitary trick.** Orbit $G^c\psi$ is closed iff it contains vector $\psi_0 = g\psi$ of minimal length. Then the minimal vector is unique up to (unitary) action of $G$, and can be defined by equation

\begin{equation}
\langle \psi | X | \psi \rangle = 0, \quad X \in G.
\end{equation}

**2.6.2. Corollary.** Every stable vector belongs to a complex orbit of a completely entangled state.

This is a crucial observation for our approach, which unveils that an adequate mathematics hidden behind entanglement is Geometric Invariant Theory.

**2.6.3. Example.** Let’s consider completely entangled state $\psi_0 \in H_j$ of a particle of spin $j$ (see $\S$ 2.5.3). In moving frame it takes form $\psi = g\psi_0$ for some $g \in G^c = SL(2, \mathbb{C})$ (see Example 1.2.3). Notice that matrix $g = g(\Psi_0) \in SL(2, \mathbb{C})$ depends on the whole wave function $\Psi_0$ of the particle [PST02], i.e. a state vector in an irreducible representation of Lorentz group $SL(2, \mathbb{C})$, see for details [Wig39]. Nobody believes that Lorentz transformation can completely destroy an entangled state. Therefore, the set of (partially) entangled states must be closed under action of the complexified group $G^c$, hence by Corollary 2.6.2 it includes all stable states. By logical and technical reasons semistable states also must be included.

This example suggests equivalence between two apparently very different concepts.

**2.6.4. Definition.** Entangled state $\psi$ is just a semistable vector, that is it can be separated from zero $I(\psi) \neq I(0)$ by some holomorphic $G$-invariant function $I$.

Below we consider a number of other examples in support of conformity and significance of this formal definition.

**2.6.5. Invariants of a composite system.** Let us return to the settings of $\S$ 2.5.2 and consider composite system

\begin{equation}
H = \bigotimes_{i=1}^{N} H_i
\end{equation}

with dynamic group of $i$-th component $SU(H_i)$.

**2.6.5.1. Two component system.** As we know from Corollary 2.5.2.2 system $H_1 \otimes H_2$ with components of dimensions $m, n$ can’t be entangled, except $n = m$. Hence for $m \neq n$ there are no nontrivial invariants. For $m = n$ there exists unique basic invariant $I = \det[\psi]$, where $[\psi]$ is a matrix of tensor $\psi \in H_1 \otimes H_2$. In two qubits setting $n = 2$ this invariant separates entangled states $\det[\psi] \neq 0$ from coherent ones $\det[\psi] = 0 \iff \psi = \psi_1 \otimes \psi_2$ (see Example 2.5.2.3.iv).
2.6.5.2. Hyperdeterminant. For general system $N$ component system (2.10) under certain conditions there exists a similar invariant, called hyperdeterminant $\text{Det}[\psi]$ of $N$ dimensional matrix $[\psi]$, introduced by Gelfand, Kapranov, and Zelevinsky [GKZ94]. It is nontrivial only if projective dimensions $\pi_i = \dim \mathbb{H}_i - 1$ satisfy polygonal inequalities

\begin{equation}
\pi_i \leq \sum_{j \neq i} \pi_j.
\end{equation}

For $N = 2$ this condition confines us to square matrices, where $\text{Det}[\psi] = \det[\psi]$. The hyperdeterminant is invariant under elementary transformations of parallel slices, and shares many other properties of conventional determinant. For matrix of format $2 \times 2 \times 2$ it looks as follows

\begin{equation}
\text{Det} A = (a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{011}^2a_{100}^2)
- 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111}
+ a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{100}a_{110})
+ 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).
\end{equation}

Every such matrix can be diagonalized by elementary slice transformations. Therefore three qubits state is either coherent $\psi = \psi_1 \otimes \psi_2 \otimes \psi_3$, if $\text{Det}[\psi] = 0$, or entangled, if $\text{Det}[\psi] \neq 0$. In the later case it can be transformed by complex dynamic group into diagonal GHZ state n° 2.5.2.3.ii).

2.6.5.3. Conjecture. Invariants of $N$ qubits system

\[ \mathbb{H}_I = \bigotimes_{i \in I} \mathbb{H}_i, \quad \dim \mathbb{H}_i = 2 \]

are generated by hyperdeterminants of $\mathbb{H}_I = \mathbb{H}_{I_1} \otimes \mathbb{H}_{I_2} \otimes \cdots \otimes \mathbb{H}_{I_k}$ of format $2^{|I_1|} \times 2^{|I_2|} \times \cdots \times 2^{|I_k|}$ for all decompositions of $I$ into disjoint components $I_\alpha$.

For binary tensor of valency four $\psi_{ijkl}$ the hyperdeterminants are $\text{Det}[\psi_{ijkl}]$ of format $2 \times 2 \times 2 \times 2$, and three conventional $4 \times 4$ determinants like $\det[\psi_{ijkl}]$. If one of these hyperdeterminants is nonzero, then $\psi$ is entangled. The conjecture claims the inverse.

2.6.6. Invariants of spin states. Invariants of spin $j$ representation $\mathbb{H}_j$ of SU(2) is a classical subject, known as Binary Quantics. Recall, that a standard model for spin $j$ representation $\mathbb{H}_j$ is the space of binary forms

\[ f(x, y) = \sum_{\mu=-j}^{j} a_{\mu} \binom{2j}{j+\mu} x^{j+\mu} y^{j-\mu} \]

of degree $2j$ in which SU(2) acts by unitary transformations of $(x, y)$. In this model state $|\mu\rangle$ with spin projection $\mu$ corresponds to monomial

\[ |\mu\rangle = \left( \binom{2j}{j+\mu} \right)^{1/2} x^{j+\mu} y^{j-\mu}. \]
One of commonly known invariants of binary form \( f \) is discriminant \( \Delta(f) \) which vanishes iff the form has multiple factors in its decomposition into linear factors

\[
f(x, y) = \prod_i (\alpha_i x - \beta_i y).
\]

By Definition 2.6.4 every state \( \psi \) for which \( \Delta(\psi) \neq 0 \) is entangled. Coherent states from this point of view are most degenerate ones. They correspond to binomials \((ax - by)^{2j}\).

For spin \( j = 1 \) and \( 3/2 \) there are no other independent invariants, so in these cases \( \Delta \neq 0 \) is a criterion of entanglement. For \( j = 2 \) there is an extra invariant, called catalectican, which may be defined for all integer \( j \)

\[
C(f) = \det \begin{pmatrix}
a_j & a_{j-1} & \cdots & a_0 \\
-a_{j-1} & a_{j-2} & \cdots & a_{-1} \\
\vdots & \vdots & \vdots & \vdots \\
a_0 & a_{-1} & \cdots & a_{-j}
\end{pmatrix}.
\]

It has a transparent physical meaning: \( C(\psi) = 0 \) iff state \( \psi \) is a linear combination of \( j \) coherent states (\( j + 1 \) is always enough). For \( j = 2 \) discriminant \( \Delta \) and catalectican \( C \) are all the basic invariants. Hence in this case state \( \psi \) is entangled iff one of them is nonzero.

The complexity of the problem increases drastically with \( j \). This is an amazingly difficult job, done by classics for \( j \leq 3 \), and by modern authors for \( j = 4 \) [Shi67], and partially for \( j = 7/2 \) [Dix83].

For all \( j \) entangled states can be easily described geometrically, see n° 2.8.2.2. The difficulties come from a perverse desire to put geometry into Procrustean bed of algebra.

2.7. Density matrix and measure of entanglement. One can associate with entangled state \( \psi \in \mathbb{H} \) a density matrix, or operator, as follows. Let for simplicity \( \psi \) be a stable state with no symmetries. Then \( \psi \) can be transformed into a completely entangled state \( \psi_0 = g\psi \) by element \( g \in G^c \) of complex dynamic group. By Kempf-Ness theorem 2.6.1 such \( g \) is unique up to left multiplication by an element of dynamic group \( G \) acting in \( \mathbb{H} \) by unitary transformations. Therefore product \( g^\dagger g \) is a well defined positive unimodular operator independent of the above ambiguity in \( g \), and we define density matrix just by rescaling it to trace one

\[
\rho(\psi) = \frac{1}{\text{Tr} g^\dagger g} g^\dagger g.
\]

We define also entropy of entangled state in usual way

\[
S(\psi) = -\text{Tr} \left( \rho(\psi) \log \rho(\psi) \right).
\]

Below are some straightforward implications of these definitions.

2.7.1. Properties of the density matrix.

(1) \( G \)-invariance: \( \rho(g\psi) = \rho(\psi) \), \( g \in G \).

(2) \( \psi \) is completely entangled \( \iff \rho(\psi) \) is a scalar matrix.

(3) \( \psi \) is completely entangled \( \iff \) its entropy \( S(\psi) \) is maximal.

(4) The density matrix of entangled state \( \psi \in \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \cdots \otimes \mathbb{H}_N \) in composite system splits into product \( \rho(\psi) = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \) of some density matrices of the components. Hence in this case \( S(\psi) = S(\rho_1) + S(\rho_2) + \cdots + S(\rho_N) \).
2.7.2. Comments. If Hamiltonian $H$ is included in algebra of observables $G$, then time evolution $\psi(t) = e^{itH}\psi(0)$ of isolated system is governed by a one parametric subgroup of $G$. In this case the first property tells that the density matrix is an integral of motion. This is one of the reasons why an external device is required for cooking an entangled state [CKS02,...].

The next two properties essentially tell that von Neumann entropy $S(\psi)$, and density matrix $\rho(\psi)$ itself, are natural measures of entanglement. However, precise definition of the density matrix for entangled states with symmetries is expectedly more involved, since the symmetries produce singularities in the orbit space. Because of the symmetries quantum entropy of a generic $N$ qubits state is well defined only for $N \geq 4$. This may looks not so bad if compared with classical entropy which makes sense only for $N \to \infty$.

Simple systems in which every state has a nontrivial symmetry group are all known [KPV76, Sch78]. These are exactly the systems with functionally independent basic invariants. For spin systems this happens for $j \leq 2$.

Kempf–Ness theorem provides another measure of entanglement, not so sensitive to the symmetries, namely length of minimal vector $\psi_0 = g\psi$ in complex orbit of entangled state $\psi$. For two components system it looks a bit strange

$$|\psi_0|^2 = n |\det[\psi]|^{2/n},$$

while the density matrix in this case is something like $[\psi]^\dagger[\psi]$ modulo an ambiguity caused by symmetries of $\psi$.

A precise physical meaning of all these invariants still needs to be clarified. Notice also that none of the measures of entanglement is relativistic invariant [PST02]. For a spin system every entangled state looks as completely entangled in an appropriate moving frame, see Example 2.6.3.

2.8. Hilbert-Mumford criterion. Until now we have two means to distinguish entangled states from others:

1. Produce $\psi = g\psi_0$ from a completely entangled state $\psi_0$ by complex dynamic transformation $g \in G^c$.
2. Find a holomorphic invariant $I$ which separates $\psi$ from zero.

Both approaches have some troubles. The first needs a description of all completely entangled states, and the second one assumes knowledge of all basic invariants. Getting either of this prerequisites is a challenge problem, see nn° 2.5.2.1, 2.5.3.1, 2.6.5, 2.6.6.

Hilbert-Mumford criterion [MFK94], provides a more practical way for such characterization, using so called stability inequalities. This approach bear a similarity to that of Bell, especially in the role played by Cartan subalgebras. Although the nature of stability inequalities is quiet different from that of Bell, they retain the main idea that entangled states may be characterized by some inequalities. One may expect a close connection between these two subjects.

2.8.1. Cartan subalgebras. By definition Cartan subalgebra $C \subset G$ is a maximal commutative subalgebra. Its dimension $\dim C = r$ is equal to the rank of group $G$, see $n°$ 2.3.6. A typical example is algebra of diagonal matrices in Lie algebra of all (skew) Hermitian matrices. Action of $C$ splits state space $H$ into orthogonal sum of eigenspaces spanned by eigenvectors $|e_i\rangle$

\begin{equation}
H = \bigotimes_i C|e_i\rangle, \quad X|e_i\rangle = \langle \omega_i, X|e_i\rangle, \quad X \in C.
\end{equation}
Elements $\omega_i \in C$ are said to be weights of $H$. Let now decompose state $\psi$ over the eigenbasis

$$\psi = \sum a_i |e_i\rangle$$

and define its $C$-support $\text{Supp}_C \psi \subset C$ as convex hull of those weights $\omega_i$ for which $a_i \neq 0$.

**2.8.2. Hilbert-Mumford criterion.** State $\psi$ is stable iff zero is an interior point of $C$-support $\text{Supp}_C \psi$ for every Cartan subalgebra $C \subset G$, and semistable iff it is never outside of the support.

Returning back to entanglement we may spell out this as follows

$$\text{(2.17) State } \psi \text{ is entangled } \iff 0 \in \text{Supp}_C \psi, \quad \forall C.$$  

Moreover if zero is always an internal point of the support then state $\psi$ is stable with finite symmetry group. In the last case the density matrix (2.13) is well defined.

**2.8.2.1. Entangled states in $N$ qubits system.** Applying Hilbert-Mumford criterion (2.17) to $N$-qubit state

$$\psi = \sum \psi_{s_1s_2...s_N} |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_N\rangle, \quad s_i = \pm$$

we find out that $\psi$ is entangled iff zero is contained in the convex hull of points $\{ (s_1, s_2, \ldots, s_N) \in \mathbb{R}^N \mid \psi_{s_1s_2...s_N} \neq 0 \}$ whichever directions are used for spin projections $s_i$.

**2.8.2.2. Entangled states in spin system.** In this case Hilbert-Mumford criterion tells that state $\psi \in H_j$ of spin $j$ is not entangled iff $\psi$ is a linear combination of states with positive spin projections onto some direction.

$$\psi = \sum_{0 \leq j - \mu < j} a_\mu |\mu\rangle$$

3. Conclusion

Group theoretical approach is inevitably more kinematic then dynamic. But this shortage may turn into an advantage in searching for the very basic concepts.

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