Higher dimensional solutions for a nonuniformly elliptic equation

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Abstract

We prove \(m\)-dimensional symmetry results, that we call \(m\)-Liouville theorems, for stable and monotone solutions of the following nonuniformly elliptic equation

\[
-\text{div}(\gamma(x')\nabla u(x)) = \lambda(x')f(u(x)) \quad \text{for} \quad x = (x', x'') \in \mathbb{R}^d \times \mathbb{R}^s = \mathbb{R}^n,
\]

where \(0 \leq m < n\) and \(0 < \lambda, \gamma\) are smooth functions and \(f \in C^1(\mathbb{R})\). The interesting fact is that the decay assumptions on the weight function \(\gamma(x')\) play the fundamental role in deriving \(m\)-Liouville theorems. We show that under certain assumptions on the sign of the nonlinearity \(f\), the above equation satisfies a 0-Liouville theorem. More importantly, we prove that for the double-well potential nonlinearities, i.e. \(f(u) = u - u^3\), the above equation satisfies a \((d+1)\)-Liouville theorem. This can be considered as a higher dimensional counterpart of the celebrated conjecture of De Giorgi for the Allen-Cahn equation. The remarkable phenomenon is that the \(\tanh\) function that is the profile of monotone and bounded solutions of the Allen-Cahn equation appears towards constructing higher dimensional Liouville theorems.

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1 Introduction

We study \(m\)-dimensional symmetry of solutions for the following semilinear elliptic equation with an advection term

\[
-\Delta u + a(x) \cdot \nabla u = b(x)f(u) \quad x \in \mathbb{R}^n
\]

where \(a : \mathbb{R}^n \to \mathbb{R}^n\) is a smooth vector field, \(b \in C^\infty(\mathbb{R}^n)\) and \(f \in C^1(\mathbb{R})\). Note that if \(a(x)\) is of gradient form, that is there exists a smooth \(c(x)\) such that \(a(x) = \nabla c(x)\), then one can rewrite (1) as

\[
-\Delta u + \nabla c(x) \cdot \nabla u = b(x)f(u) \quad x \in \mathbb{R}^n.
\]

If we set \(\gamma(x) = e^{-c(x)}\) and \(\lambda(x) = e^{-c(x)}b(x)\) then we can rewrite (2) as the following equation in divergence form

\[
-\text{div}(\gamma(x)\nabla u) = \lambda(x)f(u) \quad x \in \mathbb{R}^n.
\]

Therefore, we assume that \(\gamma(x)\) and \(\lambda(x)\), which we call \textit{weights}, are smooth positive functions (we allow \(\lambda\) to be zero at say a point) and which satisfy various growth conditions at infinity. Note that the assumption \(\gamma(x) > 0\) implies that the operator \(\text{div}(\gamma(x)\nabla \cdot)\) is a nonuniformly elliptic operator.

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Notation 1. Throughout the paper we use the following notations.

- The weight functions $\lambda$ and $\gamma$ are only functions of $d$-variables meaning that $\gamma(x) = \gamma(x')$ and $\lambda(x) = \lambda(x')$ where $x = (x', x''') \in \mathbb{R}^d \times \mathbb{R}^s = \mathbb{R}^n$ for $n = d + s$. Another representation for $x$ in $n$ dimensional space is $x = (x'', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

- The following class of nonlinearities appears in our results,

$$G := \left\{ g : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is nondecreasing and } \int_1^{\infty} \frac{1}{rg^r} dr = \infty \right\}.$$ 

Note that $G$ is not empty, e.g., $g(r) = \log(1 + r)$ is in $G$.

The class $G$ of nonlinearities was defined by Karp in [22, 23] and was used by Moschini in [25].

Definition 1. We say that (3) satisfies $m$-Liouville theorem if for certain $\lambda$ and $\gamma$ solutions of (3) are $m$-dimensional for $0 \leq m < n$, i.e., they exactly depend on $m$ variables. Similarly, we say that (3) satisfies at most $m$-Liouville theorem if solutions of (3) are at most $m$-dimensional for $0 \leq m < n$, i.e., they depend on at most $m$ variables.

Definition 2. We call a classical solution $u$ of (3) to be

(i) asymptotically convergent if

$$\lim_{x_n \to \pm \infty} u(x'''', x_n) \to \pm 1 \text{ for all } x'''' \in \mathbb{R}^{n-1}. \quad (4)$$

If this limit is uniform then we call it uniformly asymptotically convergent.

(ii) monotone if $\partial_{x_n} u(x) > 0$ for all $x \in \mathbb{R}^n$.

(iii) pointwise stable if there exists a function $0 < v$ that satisfies the linearized equation

$$-\text{div}(\gamma(x')\nabla v) = \lambda(x')f'(u)v \text{ for all } x \in \mathbb{R}^n$$

(iv) stable if for all $\psi \in C^1_0(\mathbb{R}^n)$ the following inequality holds,

$$\int_{\mathbb{R}^n} \lambda(x')f'(u)\psi^2 dx \leq \int_{\mathbb{R}^n} \gamma(x')|\nabla \psi|^2 dx. \quad (5)$$

Note that by taking derivative of (3) with respect to $x_n$ monotonicity implies pointwise stability and multiplying $\frac{x_n}{x_n}$ and doing integration by parts one can see that pointwise stability implies stability as it is given in [12]. The equation (1) is a perturbation of the following semilinear elliptic equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^n. \quad (6)$$

When $m = 0$ and 1, $m$-Liouville theorems for (3) known as Liouville theorems and one dimensional symmetry results, respectively, are extensively studied in the literature [2, 4, 6, 13, 15, 19, 20, 24, 26]. The most well-known 1-Liouville theorem is the following conjecture of De Giorgi in 1978.

Conjecture 1. Suppose that $u : \mathbb{R}^n \to [-1, 1]$ is a classical monotone solution of (1) for $a = 0$, $b = 1$ and $f(u) = u - u^3$. Then for at least $n \leq 8$ equation (3) satisfies 1-Liouville theorem.

From the definition of 1-Liouville theorem, $u$ depends only on one variable and therefore it has to be of the form

$$u(x) = \tanh \left( \frac{x \cdot y - c}{\sqrt{2}} \right) \quad \text{for all } x \in \mathbb{R}^n \quad (7)$$
for some \( c \in \mathbb{R} \) and some \( y \in \mathbb{R}^n \) where \( |y| = 1 \) and \( y_n > 0 \). Note that the function \( w(t) = \tanh(t/\sqrt{2}) \) is the unique solution up to translation of the following ordinary differential equation,

\[-w'' = w - w^3, \quad w' > 0, \quad w(\pm \infty) = \pm 1.\]

In 1997, Ghoussoub and Gui [20] proved the De Giorgi’s conjecture for \( n = 2 \). They used a linear 0-Liouville theorem for the ratio \( \sigma := \frac{\partial u}{\partial x_1} / \frac{\partial u}{\partial x_2} \) developed by Berestycki, Caffarelli and Nirenberg in [8] for the study of symmetry properties of positive solutions of semilinear elliptic equations in half spaces. Unfortunately, it is not known whether or not this 0-Liouville theorem is optimal, see Proposition 1 and what follows shortly after.

As shown by Gilbarg and Serrin in [21] (see P. 324) a 0-Liouville theorem holds for bounded solutions of the Lane-Emden equation with specific nonlinearities. To prove 0-Liouville theorem, we assumed either

\[-u'' = \sigma u \quad \text{or} \quad |\nabla \gamma(x)| \leq C \lambda(x) \quad \text{i.e.} \quad |\nabla c(x)| \text{ bounded}.\]

In this note we prove 0-Liouville theorem for (3) with a general nonlinearity \( f \in C^1(\mathbb{R}) \) as well as \( m \)-Liouville theorems for (3) when \( m \geq 1 \) under certain conditions on \( \lambda \) and \( \gamma \). To prove higher dimensional Liouville theorems, which are more challenging problems, we apply a standard linear 0-Liouville theorem given in [8, 20].

The organization of the paper is as follows. In Section 2 we state the main results of the paper and in particular the applications of the main results for the nonuniformly elliptic Allen-Cahn equation. In Section 3 we prove a linear 0-Liouville theorem and a geometry Poincaré inequality that are the essential tools in our proofs. Finally in Section 4 we provide \( m \)-Liouville theorems and the proof of main results.

2 Main results and related backgrounds

As shown by Gilbarg and Serrin in [21] (see P. 324) a 0-Liouville theorem holds for bounded solutions of the linear equation

\[-\Delta u + a(x) \cdot \nabla u = 0 \quad \text{in} \ \mathbb{R}^n\]
where $n \geq 2$ and $a(x) = O(|x|^{-1})$. The validity or the failure of 0-Liouville theorems for this equation under appropriate conditions have been also studied in [28, 29]. If we replace the equality with the inequality $\geq$ in [9], then it is straightforward to construct nonconstant bounded solutions satisfying specific $a(x) = O(|x|^{-1})$. This implies a natural question that under what assumptions on $a$, $b$ and solutions one can prove a 0-Liouville theorem for the nonlinear case, [10], with a general nonlinearity $f \geq 0$. In what follows, we prove a 0-Liouville theorem for bounded stable solutions of (1).

**Theorem 1.** Let $u$ be a bounded pointwise stable solution for (3) and let either $0 \leq f(t)$ or $tf(t) \leq 0$ for all $t$ in the range of $u$. If either

$$\int_{B_2R} \gamma(x')dx' \leq kg(R) \quad \text{and} \quad n \leq d + 4,$$

or

$$\int_{B_2R} \gamma(x')dx' \leq kRg(R) \quad \text{and} \quad n \leq d + 3,$$

where $g \in G$ and $k$ is a constant independent of $R$. Then, (3) satisfies 0-Liouville theorem.

The proof of the theorem is strongly motivated by the methods and ideas developed by Dupaigne-Farina in [14], where they examined the advection free equation that is (1) when $a = 0$ and $b = 1$. Note that the double-well potential nonlinearity $f(t) = t - t^3$ for $t \in [-1, 1]$ does not satisfy neither $0 \leq f(t)$ nor $tf(t) \leq 0$. Therefore, in what follows we focus on this type nonlinearity. Berestycki, Hamel and Monneau, Theorem 2 in [9], have shown that a 1-Liouville theorem holds for uniformly asymptotically convergent solutions of (1) under the assumption that $a$ is a constant vector, $b(x) = b(x_n)$ is bounded and $f$ is Lipschitz continuous on $[-1, 1]$ satisfying

(P) $f(\pm 1) = 0$ and there exists $\delta > 0$ such that $f$ is non-increasing on $[-1, -1 + \delta]$ and on $[1 - \delta, 1]$.

However, a counterexample given by Bonnet-Hamel in [10] shows that this result no longer holds if we drop the ”uniformly” assumption. In other words, they constructed a two dimensional monotone and asymptotically convergent solution such that for $\alpha \in (0, \frac{\pi}{2}]$

$$u(t \cos \theta, t \sin \theta) \to -1 \quad \text{as} \quad t \to \infty \quad \text{for} \quad -\frac{\pi}{2} - \alpha < \theta < -\frac{\pi}{2} + \alpha \quad \text{and} \quad \frac{\pi}{2} + \alpha < \theta < \frac{3\pi}{2} - \alpha$$

when $u$ is a solution of the following equation

$$-\Delta u + k\partial_{x_2} u = f(u) \quad \text{in} \quad \mathbb{R}^2$$

where $k$ is just a constant and for some particular $f$ that satisfies (P). The level sets of such a solution are parallel lines and cannot be one dimensional. Therefore, De Giorgi’s conjecture does not hold for (12). Note that this is a sharp result, since when $k = 0$, it follows from the result of Ghoussoub and Gui [20] that (12) satisfies a 1-Liouville theorem.

Moreover, Berestycki, Hamel and Monneau, Theorem 3 in [9], have proved that the 1-Liouville theorem no longer holds for (1) if $a$ is a non constant vector, even for uniformly asymptotically convergent solutions. More precisely, they proved that the following equation in two dimensions

$$-\Delta u + a(x_1)\partial_{x_2} u = f(u) \quad \text{in} \quad \mathbb{R}^2$$

admits both a solution depending on only $x_2$ and infinitely many nonplanar solutions, that is, solutions whose level sets are not parallel. The construction of nonplanar solutions is very technical and relies on the subsolution-supersolution method. As a conclusion, the Gibbons’ conjecture (and therefore De Giorgi’s conjecture) cannot be extended to (13) that is in dimension two.

In what follows, we provide a higher dimensional Liouville theorem for solutions of (3) under certain decay assumptions on $\gamma$ and $\lambda$, and in a particular case this can be applied to prove higher dimensional Liouville theorems for (13).
**Theorem 2.** Assume that \( f \in C^1([-1, 1]) \) and \( F(t) \leq \min\{F(-1), F(1)\} \) for all \( t \in (-1, 1) \), where \( F' = f \). Let \( u \) be a monotone and asymptotically convergent solution of (5). Moreover, suppose that there exists a positive constant \( k \) such that \( |\nabla \gamma(x')| \leq k \gamma(x') \) and \( \lambda(x') \leq k \gamma(x') \) for any \( x' \) outside a compact set in \( \mathbb{R}^d \) and either
\[
\int_{B_R} \gamma(x')dx' \leq k g(R) \quad \text{and} \quad n \leq d + 3,
\]
or
\[
\int_{B_R} \gamma(x')dx' \leq k Rg(R) \quad \text{and} \quad n \leq d + 2,
\]
where \( g \in \mathcal{G} \) and \( k \) is a constant independent of \( R \). Then, (3) satisfies at most \((d + 1)\)-Liouville theorem.

**Corollary 1.** Assume that \( d = 1 \), \( a \in L^\infty(\mathbb{R}) \) and
\[
\int_{-R}^R e^{-\int_0^1 a(t)dt}dx_1 \leq k R^{1-\epsilon} g(R)
\]
where \( k = k(n, a, g) \) is a constant independent of \( R \), \( 0 \leq \epsilon \leq 1 \) and any \( g \in \mathcal{G} \). Then monotone and asymptotically convergent solutions of
\[
- \Delta u + a(x_1) \partial_{x_1} u = u - u^3 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^{n-1}
\]
satisfy at most 2-Liouville theorem for \( n \leq 3 + \epsilon \).

In particular, this shows that monotone and asymptotically convergent solutions of (13) on \( \mathbb{R}^n \) and up to dimension \( n \leq 4 \) are at most two dimensional provided \( a \in L^\infty(\mathbb{R}) \) and
\[
\lim_{R \to \infty} \int_{-R}^R \gamma(x_1)dx_1 < \infty \quad \text{or equivalently} \quad \lim_{R \to \infty} \int_{-R}^R e^{-\int_0^1 a(s)ds}dx_1 < \infty.
\]
Note that \( a(x_1) \equiv k \) where \( k \) is just a constant does not satisfy this condition. However, either \( a(x_1) = \frac{2x_1}{1+x_1^2} \) or \( a(x_1) = t \tanh x_1 + s \) for any \( t > |s| \) can be chosen to fulfill the assumption (17). Note also that the double-well potential \( f(t) = t - t^3 \) and therefore \( F(t) = -\frac{1}{3}(1-t^2)^2 \) satisfies the assumptions of Theorem 2.

For \( \lambda = \gamma = 1 \) this result is given by Ambrosio-Cabré in [3] and Ghoussoub-Gui in [20].

The remarkable phenomenon is that according to the De Giorgi’s conjecture monotone and asymptotically convergent solutions of the Allen-Cahn equation, i.e., (3) with \( f(u) = u - u^3 \), are one dimensional solutions up to dimension eight and the profile solution is the \( \tanh \) function. Now, if we perturb the Allen-Cahn equation by \( \tanh \) function that is
\[
- \Delta u + \tanh(x_1) \partial_{x_1} u = u - u^3 \quad \text{in} \quad \mathbb{R}^n
\]
then according to Theorem 2 the monotone and asymptotically convergent solutions are at most two dimensional up to dimension four. Similarly, higher dimensional Liouville theorems can be constructed as following.

**Corollary 2.** Assume that \( d = 2 \), \( a_1, a_2 \in L^\infty(\mathbb{R}) \) and
\[
\int_t^R e^{-\int_0^t a_1(t)dt}e^{-\int_0^t a_2(t)dt}dx_1dx_2 \leq k R^{2-\epsilon} g(R)
\]
where \( k = k(n, a_1, a_2, g) \) is a constant independent of \( R \), \( 0 \leq \epsilon \leq 2 \) and any \( g \in \mathcal{G} \). Then monotone and asymptotically convergent solutions of
\[
- \Delta u + a_1(x_1) \partial_{x_1} u + a_2(x_2) \partial_{x_2} u = u - u^3 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}^{n-2}
\]
satisfy at most 3-Liouville theorem for \( n \leq 3 + \epsilon \).
Following ideas given in [16, 27, 30] we provide a geometric Poincaré inequality for stable solutions of (3). The interesting point is that both the weight function $\lambda$ and the nonlinearity $f$ in (3) do not appear in this geometric Poincaré inequality. However, the weight function $\gamma$ in (3) appears as a weight function for both sides of the inequality.

**Theorem 3.** Let $\sigma$ be a stable solution of (3). Then the following inequality holds for any $\phi \in C^1_c(\mathbb{R}^n)$,

$$
\int_{x' \in \mathbb{R}^d} \gamma(x') \int_{x'' \in \mathbb{R}^s \cap \{\nabla_{x''} u \neq 0\}} \phi^2 \left( |\nabla_{x''} u|^2 K^2 + |\nabla T| \nabla_{x''} u ||^2 \right) dx'' dx' \\
+ \int_{\mathbb{R}^n} \gamma(x') \phi^2 S \leq \int_{\mathbb{R}^n} \gamma(x') |\nabla_{x''} u|^2 |\nabla \phi|^2
$$

where $\nabla T$ denotes the orthogonal projection of the gradient along this level set and

$$
S := \sum_{j=d+1}^{n} \sum_{i=1}^{d} |\partial_i \partial_j u|^2 - |\nabla_{x''} |\nabla_{x''} u||^2
$$

and $K$ is the full curvature defined by

$$
K(x) = \sqrt{\sum_{j=1}^{k-1} \kappa_j(x)^2}
$$

when $\kappa_j$ are the principal curvatures of the level set of $u$ at $x$.

**Remark 1.** The function $S$ given in (21) is nonnegative. This can be seen by taking the gradient of $|\nabla_{x''} u|$ with respect to $x'$ and then applying the Cauchy inequality for the points that $|\nabla_{x''} u| \neq 0$.

In this context and for the case of $\gamma = \lambda = 1$, this type of geometric Poincaré inequality was introduced by Sternberg and Zumbrun in [30] to study semilinear phase transition problems. Later on and for the first time, Farina, Sciunzi and Valdinoci in [16] used and extended the inequality to prove very interesting results related to the De Giorgi’s conjecture. Then Cabrè used it (see Proposition 2.2 in [11]) to prove the boundedness of extremal solutions of semilinear elliptic equations with Dirichlet boundary conditions on a convex domain up to dimension four. Similar inequalities are proved by Savin and Valdinoci in [27] for (3) when $\gamma = 1$. Recently in [18], Ghoussoub and the author extended this inequality to elliptic systems and used it to prove De Giorgi type results for systems.

### 3 Linear 0-Liouville Theorem and a geometric Poincaré inequality

We start this section with the following linear 0-Liouville theorem that is given by Berestycki-Caffarelli-Nirenberg [8] and Ghoussoub-Gui [20] for bounded $h \sigma$ and then improved by Ambroso-Cabrè [6] and Moschini [25].

**Proposition 1.** Let $0 < h \in L^\infty_{loc}(\mathbb{R}^n)$ and $\sigma \in H^1_{loc}(\mathbb{R}^n)$. If $\sigma$ satisfies the following differential inequality

$$
\sigma \text{ div}(h(x) \nabla \sigma) \geq 0 \quad \text{in} \quad \mathbb{R}^n,
$$

such that for any $R > 1$,

$$
\int_{B_{2R} \setminus B_R} h(x) \sigma^2 \leq CR^2 g(R),
$$

where $g \in \mathcal{G}$. Then $\sigma$ is constant.
Note that in two dimensions Proposition 1 is sharp in the sense that the following example

\[
h \equiv 1 \quad \text{and for } R_0 > e^{3/4} \text{ set } \sigma(r) := \begin{cases} \log R_0 + \frac{r^2}{R_0} - \frac{r^4}{4R_0^2} - \frac{3}{4} & \text{for } r < R_0, \\ \log r & \text{for } r \geq R_0, \end{cases}
\]
given in [25] (Remark 5.4) shows that this proposition does not hold if \( g(R) = \log^2(R) \). Straightforward calculations show that \( \log^2(1 + r) \) is not in the class \( G \), however \( \log(1 + r) \) belongs to \( G \).

Ambrosio and Cabré in [6] and later on with Alberti in [2] proved the following energy estimate holds in any dimension regarding the De Giorgi’s conjecture

\[
\int_{B_R} |\nabla u|^2 \leq CR^{n-1}.
\]

Then applying Proposition 1 when \( g = 1 \) and equating the right hand sides of (24) and (23) they gave a positive answer to Conjecture 1 in three dimensions. Now, comparing (24) and (23) in any dimensions for the choice of \( g(R) = R^{n-3} \), one sees that the right-hand side of these integral estimates are the same. Therefore, potentially the function \( g(R) = R^{n-3} \) can play an important role in solving Conjecture 1 in dimensions \( 4 \leq n \leq 8 \).

Remark 2. Let \( n \geq 4 \), \( h(x) = (1 + |x|^2)^{\frac{3}{2n+4}} \) and \( \sigma(x) = (1 + |x|^2)^{\frac{3}{2n+4}} \). The functions \( h \) and \( \sigma \) are smooth functions and \( 0 < h \in L^\infty(\mathbb{R}^n) \). By a simple calculation one can see that (22) holds and moreover

\[
\int_{B_R} h(x)\sigma^2 \leq R^{n-1} = R^2 g(R)
\]

where \( g(R) = R^{n-3} \). Therefore, \( h \) and \( \sigma \) satisfy the assumptions of Proposition 1. But \( \sigma \) is not a constant even though \( h\sigma^2 \in L^\infty(\mathbb{R}^n) \). This means that to prove Conjecture 1 in dimensions \( 4 \leq n \leq 8 \) via using (24) and (23) when \( g(R) = R^{n-3} \), a counterpart of Proposition 1 is needed that assumes equality in (22) and allows a wider class of functions in \( G \).

For the rest of this section, we provide a proof for the geometric Poincaré inequality (20).

Proof of Theorem 3: Let \( u \) be a stable solution of (3). Test the stability inequality (5) with \( \psi = |\nabla_{x'}u|\phi \) where \( \phi \in C_c^1(\mathbb{R}^n) \) is a test function to get

\[
I := \int_{\mathbb{R}^n} \lambda(x')f'(u)|\nabla_{x'}u|^2\phi^2 \leq \int_{\mathbb{R}^n} \gamma(x')|\nabla (|\nabla_{x'}u|\phi)|^2 =: J
\]
In what follows we simplify $I$ and $J$. Let’s start with $I$.

\[
I = \int_{\mathbb{R}^n} \lambda(x') f'(u) \nabla_{x'} u \cdot \nabla_{x'} u \, \phi^2 = \int_{\mathbb{R}^n} \nabla_{x'} (\lambda(x') f(u)) \cdot \nabla_{x'} u \, \phi^2
\]

\[
= - \int_{\mathbb{R}^n} \nabla_{x'} (\text{div}(\gamma(x') \nabla u)) \cdot \nabla_{x'} u \, \phi^2 = - \int_{\mathbb{R}^n} \nabla_{x'} [\gamma(x') \Delta u + \nabla_x \gamma(x') \cdot \nabla_{x'} u] \cdot \nabla_{x'} u \, \phi^2
\]

\[
= - \sum_{i=d+1}^n \int_{\mathbb{R}^n} \gamma(x') \partial_i u \Delta (\partial_i u) \, \phi^2 - \sum_{i=d+1}^n \sum_{j=1}^d \int_{\mathbb{R}^n} \partial_j \gamma(x') \partial_i \partial_j u \partial_i u \, \phi^2
\]

\[
= - \sum_{i=d+1}^n \int_{\mathbb{R}^n} \gamma(x') \partial_i u \Delta (\partial_i u) \, \phi^2 + \sum_{i=d+1}^n \sum_{j=1}^d \int_{\mathbb{R}^n} \gamma(x') |\partial_j \partial_i u|^2 \, \phi^2
\]

\[
= \frac{1}{2} \sum_{i=d+1}^n \int_{\mathbb{R}^n} \gamma(x') \nabla (\partial_i u)^2 \cdot \nabla \phi^2 + \int_{\mathbb{R}^n} \gamma(x') |D_{x'}^2 u|^2 \phi^2 + \sum_{i=d+1}^n \sum_{j=1}^d \int_{\mathbb{R}^n} \gamma(x') |\partial_j \partial_i u|^2 \phi^2
\]  

(26)

Note that fortunately the term that includes the gradient of $\gamma$ cancels out in the fourth line of calculations where we have used integration by parts. Now we simplify the integral term given as $J$. First note that for $\psi = |\nabla_{x'} u| \phi$ we have

\[
|\nabla \psi|^2 = |\nabla \nabla_{x'} u|^2 \phi^2 + |\nabla_{x'} u|^2 |\nabla \phi|^2 + \frac{1}{2} |\nabla \phi|^2 \cdot |\nabla_{x'} u|^2
\]

\[
= |\nabla_{x'} |\nabla_{x'} u|^2 |\nabla \phi|^2 + |\nabla_{x'} u|^2 |\nabla \phi|^2 + \frac{1}{2} |\nabla \phi|^2 \cdot |\nabla_{x'} u|^2.
\]

Therefore,

\[
\int_{\mathbb{R}^n} \gamma(x') |\nabla (|\nabla_{x'} u| \phi)|^2 = \int_{\mathbb{R}^n} \gamma(x') \nabla_{x'} |\nabla_{x'} u|^2 \phi^2 + \int_{\mathbb{R}^n} \gamma(x') |\nabla_{x'} u|^2 |\nabla \phi|^2
\]

\[
+ \int_{\mathbb{R}^n} \gamma(x') |\nabla_{x'} u|^2 |\nabla \phi|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \gamma(x') \nabla \phi^2 \cdot |\nabla_{x'} u|^2
\]

(27)

The first term in the right-hand side of (26) and the last term in the right-hand side of (27) are the same. Substituting (26) and (27) in (25) we get

\[
\int_{\mathbb{R}^n} \gamma(x') \left( |D_{x'}^2 u|^2 - |\nabla_{x'} u|^2 |\nabla \phi|^2 \right) \phi^2 + \int_{\mathbb{R}^n} \gamma(x') S \phi^2 \leq \int_{\mathbb{R}^n} \gamma(x') |\nabla_{x'} u|^2 |\nabla \phi|^2
\]

(28)

According to formula (2.1) given in [31], the following geometric identity between the tangential gradients and curvatures holds. For any $w \in C^2(\Omega)$ where $\Omega$ is an open set in $\mathbb{R}^s$

\[
\sum_{i=1}^s |\nabla_{\partial_i} w|^2 - |\nabla w|^2 = \begin{cases} |\nabla w|^2 (\sum_{i=1}^{s-1} \kappa_i^2) + |\nabla T |\nabla w|^2 & \text{for } x \in \{ |\nabla w| > 0 \cap \Omega \}, \\
0 & \text{for } x \in \{ |\nabla w| = 0 \cap \Omega \}, 
\end{cases}
\]

(29)

where $\kappa_i$ are the principal curvatures of the level set of $w$ at $x''$ and $\nabla T$ denotes the orthogonal projection of the gradient along this level set. Setting $w(x'') = u(x', x'')$ and applying this formula together with (28), we finally get (27).

\[\square\]

4 $m$-Liouville theorems for the nonlinear equation

We now apply Proposition 4 the linear 0-Liouville theorem, to prove the following $(d+1)$-Liouville theorem under a strong assumption on the gradient of solutions.
Proposition 2. Let $u$ be a monotone solution of (3). If there exists $C(n, d) > 0$ such that
\[
\int_{B_{2R} \setminus B_R} \gamma(x') |\nabla u|^2 \, dx \leq CR^2 g(R),
\] (30)
for any $g \in \mathcal{G}$. Then, (3) satisfies at most $(d + 1)$-Liouville theorem.

Proof: Define $\phi_i(x) := \frac{\partial u}{\partial x_i}(x)$ for all $i = d + 1, \ldots, n$ and $x \in \mathbb{R}^n$. Taking derivative of (3), we get that $\phi_i$ satisfies the following linearized equation
\[- \divergence(\gamma(x') \nabla \phi_i) = \lambda(x') f'(u) \phi_i \quad \text{for all} \quad x \in \mathbb{R}^n.
\]
The straightforward calculations show that
\[\divergence(\gamma(x') \phi_n^2 \nabla \sigma_i) = 0 \quad \text{for all} \quad i = d + 1, \ldots, n
\]
where $\sigma_i := \frac{\partial u}{\partial x_i}$. Note that $\phi_n^2 \sigma_i^2 = |\partial_i u|^2$ and from (30) for all $i = d + 1, \ldots, n$ we have
\[\int_{B_{2R} \setminus B_R} \gamma(x') \phi_n^2 \sigma_i^2 \, dx = \int_{B_{2R} \setminus B_R} \gamma(x') |\partial_i u|^2 \, dx \leq \int_{B_{2R} \setminus B_R} \gamma(x') |\nabla u|^2 \, dx \leq CR^2 g(R).
\]
Applying Proposition 1 with $h(x) = \gamma(x') \phi_n^2(x)$, we get that $(\sigma_i)_{i=d+1}^n$ are all constant. Therefore, there exists $(k_i)_{i=d+1}^n$ such that $\sigma_i(x) = k_i$ for any $x \in \mathbb{R}^n$. Clearly $k_n = 1$.

From the definition of $\sigma_i$ we get $\frac{\partial^2 u}{\partial x_i^2}(x) = k_i \frac{\partial^2 u}{\partial x_n^2}(x)$ for all $i = d + 1, \ldots, n - 1$. Therefore, $\nabla_{x''} u(x) = \frac{\partial u}{\partial x_n}(x)(k_{d+1}, k_{d+2}, \ldots, k_{n-1}, 1)$. Since $u$ is monotone in $x_n$ direction that is $\frac{\partial u}{\partial x_n} > 0$ we conclude that $\nabla_{x''} u(x)$ does not change sign for all $x \in \mathbb{R}^n$. Also, note that $u$ is constant along the following directions:
\[(0, 0, \ldots, 0, 1, 0, \ldots, 0, -k_{d+1}), (0, 0, \ldots, 0, 0, 1, 0, \ldots, 0, -k_{d+2}), \ldots, (0, 0, \ldots, 0, 0, 0, \ldots, 0, 1, -k_{n-1}).
\]
Therefore, $u$ is a function of $(x', k \cdot x'')$ where $k = (k_{d+1}, \ldots, k_{n-1}, 1)$.

\[\square\]

Remark 3. Applying the geometric Poincaré inequality that is given as Theorem 3 when $\phi$ is the following standard test function
\[
\phi(x) := \begin{cases} 
\frac{1}{\log R - \log |x|}, & \text{if } |x| \leq \sqrt{R}, \\
\frac{1}{\log R}, & \text{if } \sqrt{R} < |x| < R, \\
0, & \text{if } |x| \geq R.
\end{cases}
\]
one can prove Proposition 2 for stable solutions as well. This test function is also used in [8, 18, 20] in order to prove certain results related to the De Giorgi’s conjecture.

Now we are ready to provide the proof of Theorem 1. The idea is to apply the linear 0-Liouville theorem to prove a 0-Liouville theorem for the equation (3).

Proof of Theorem 1. Since $u$ is a pointwise stable solution, there exists $v > 0$ such that
\[- \divergence(\gamma(x') \nabla v) = \lambda(x') f'(u) v \quad \text{for all} \quad x \in \mathbb{R}^n.
\]
It is straightforward to see that
\[\divergence(\gamma(x') v^2 \nabla \sigma_i) = 0 \quad \text{for all} \quad i = d + 1, \ldots, n
\] (31)
where $\sigma_i := \frac{\partial u}{\partial x_i}/v$. Therefore, for all $i = d + 1, \ldots, n$ we have $(\sigma_i v)^2 \leq |\nabla u|^2$ that gives
\[\int_{B_R} \gamma(x') v^2 \sigma_i^2 \leq \int_{B_R} \gamma(x') |\nabla u|^2.
\] (32)
To apply Proposition 1 we need to find an upper bound for the right-hand side of the inequality (32). First, we assume that $f$ is a nonnegative nonlinearity. Multiply both sides of (3) with $(u - ||u||_\infty)\phi^2$ where $0 \leq \phi \leq 1$ is a test function. Since $\lambda(x')f(u)(u - ||u||_\infty)\phi^2 \leq 0$ in $\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}$, we have

$$-\operatorname{div}(\gamma(x')\nabla u)(u - ||u||_\infty)\phi^2 \leq 0 \quad \text{in} \quad \mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}.$$  

(33)

On the other hand, for the case $tf(t) \leq 0$ a similar differential inequality holds. Note that multiplying both sides of (3) with $u\phi^2$ we have

$$-\operatorname{div}(\gamma(x')\nabla u)u\phi^2 \leq 0 \quad \text{in} \quad \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}.$$  

(34) Now, integrating both sides of (33) and (34) and using the fact that $u$ is bounded we obtain

$$\int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2\phi^2 \leq k \int_{\mathbb{R}^n} \gamma(x')|\nabla u||\nabla \phi|\phi$$

$$\leq k \left( \int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2\phi^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \gamma(x')|\nabla \phi|^2 \right)^{\frac{1}{2}}$$

where $k$ is a constant that only depends on $||u||_\infty$. Set the test function $\phi$ to be the standard test function that is $\phi = 1$ in $B_R$ and $\phi = 0$ in $\mathbb{R}^n \setminus B_{2R}$ where $||\nabla \phi||_{L^\infty(B_{2R})} \leq kR^{-1}$. Then, we have

$$\int_{B_R} \gamma(x')|\nabla u|^2 \leq kR^{-2} \int_{B_{2R} \setminus B_R} \gamma(x')dx \leq kR^{-2} \int_{B_{2R}} \gamma(x')dx.$$  

(35)

Let (10) hold in dimensions $n \leq d + 4$, then $n - d - 2 = s - 2 \leq 2$ and

$$R^{s-2} \int_{B_{2R}} \gamma(x')dx' \leq R^2 \int_{B_{2R}} \gamma(x')dx' \leq kR^2 g(R).$$

Similarly, if (11) holds when $n \leq d + 3$ then $n - d - 2 = s - 2 \leq 1$ and

$$R^{s-2} \int_{B_{2R}} \gamma(x')dx' \leq R \int_{B_{2R}} \gamma(x')dx' \leq kR^2 g(R).$$

Therefore, from (35) and (32) we get

$$\int_{B_R} \gamma(x')v^2\sigma_i^2 \leq kR^2 g(R) \quad \text{for all} \quad i = d + 1, \cdots, n.$$

Set $h(x) = \gamma(x')v^2$ and $\sigma = \sigma_i$ in Proposition 1 for all $i = d + 1, \cdots, n$ to obtain that all $\sigma_i$ are constant. By similar discussions as in the proof of Proposition 2 we have $u(x', x'') = w(x', k \cdot x'')$ such that $k \in \mathbb{R}^s$ and $|k| = 1$. Note that $w$ satisfies

$$(w - ||w||_\infty)\nabla (w - ||w||_\infty) \geq 0 \quad \text{in} \quad \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R},$$  

(36)

where $f(w) \geq 0$ and similarly

$$w \nabla (\gamma(x')w) \geq 0 \quad \text{in} \quad \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R},$$  

(37)

where $wf(w) \leq 0$. The fact that $w$ is bounded and satisfies either (36) or (37) in dimension $d + 1$ and decay estimates (10) and (11) hold for $\gamma$ imply that

$$\int_{B_R} \gamma(x')(w - ||w||_\infty)^2dx \leq kR \int_{B_R} \gamma(x')dx' \leq R^2 g(R)$$

$$\int_{B_R} \gamma(x')w^2dx \leq kR \int_{B_R} \gamma(x')dx' \leq R^2 g(R)$$

10
where \( k \) is a positive constant independent of \( R \). Hence applying Proposition 1 again for (36) and (37) we obtain that \( w \) is constant. Therefore, \( u \) is constant.

\[ \square \]

Note that to apply Proposition 2 one needs to have a \( L^2(B_R) \) upper bound on \( |\nabla u| \) that we call the energy bound. In what follows we give such an energy bound in terms of weight functions \( \lambda \) and \( \gamma \). The following lemma holds for subsolutions of (3) as well. By subsolution we mean the inequality “\( \leq \)” holds in (4).

**Lemma 1.** Let \( u \) be a bounded solution of (3) with any \( f \in C^1(\mathbb{R}) \). Then

\[ \int_{B_R} \gamma(x')|\nabla u|^2 \, dx \leq kR^\gamma \int_{B_R} \{ \lambda(x') + R^{-2}\gamma(x') \} \, dx', \tag{38} \]

where the positive constant \( k \) is independent of \( R \).

**Proof:** Multiply both sides of (3) with \((|u|_\infty + u)\phi^2\) when \( 0 \leq \phi \leq 1 \) is a test function. Then, integrating by parts we get

\[ \int_{\mathbb{R}^n} \gamma(x')\nabla u \cdot \nabla (\phi^2(|u|_\infty + u)) \leq \int_{\mathbb{R}^n} \lambda(x')f(u)(|u|_\infty + u)\phi^2. \]

Simplifying this inequality and keeping the square of gradient of \( u \) in the left hand side, we end up with

\[ \int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2 \phi^2 \leq \int_{\mathbb{R}^n} \lambda(x')f(u)(|u|_\infty + u)\phi^2 + 4|u|_\infty \int_{\mathbb{R}^n} \gamma(x')|\nabla u||\nabla \phi|\phi \tag{39} \]

We now define the positive constants \( k \) and \( \epsilon \) such that \( 2|f(u)||u|_\infty \leq k < \infty \) and \( 0 < \epsilon < (4|u|_\infty)^{-1} \). Applying the Young’s inequality,\(^2\) for the last term in right hand side of (39) we get

\[ (1 - 4|u|_\infty \epsilon) \int_{\mathbb{R}^n} \gamma(x')|\nabla u|^2 \phi^2 \leq k \int_{\mathbb{R}^n} \lambda(x')\phi^2 + |u|_\infty \epsilon \int_{\mathbb{R}^n} \gamma(x')|\nabla \phi|^2. \tag{40} \]

Finally, set \( \phi \) to be the standard smooth test function that is \( \phi = 1 \) in \( B_R \) and \( \phi = 0 \) in \( \mathbb{R}^n \setminus B_{2R} \) with \( ||\nabla \phi||_{L^\infty(B_{2R})} < kR^{-1} \). This proves (38).

\[ \square \]

In the statement of Lemma 1 there is no assumption on the monotonicity of the solutions. However, monotonicity is a crucial assumption to derive \( m \)-Liouville theorems when \( m \geq 1 \). In other words, assuming the monotonicity of solutions we get a stronger upper bound on the energy of solutions. Before we discuss the new upper bound on the energy \( E_R \), let us mention that applying some standard elliptic estimates to bounded solutions of (3) gives us \( |\nabla u| \in L^\infty(\mathbb{R}^n) \). Indeed, assume that \( u \) is a bounded solution of either (3) when \( |\frac{\nabla u}{u}|, \frac{u}{\nabla u} \in L^\infty(\mathbb{R}^n) \) or equivalently (1) when \( a, b \in L^\infty(\mathbb{R}^n) \). Then applying interior \( W^{2,p} \) estimates with \( p > n \) to \( -\Delta u + a(x') \cdot \nabla u = b(x')f(u) \in L^p(B_2(y)) \) for every \( y \in \mathbb{R}^n \), we get

\[ ||u||_{W^{2,p}(B_1(y))} \leq k(||u||_{L^p(B_2(y))} + ||f(u)||_{L^p(B_2(y))}) \leq k, \]

where \( k \) is independent of \( y \). Using the Sobolev embedding \( W^{2,p}(B_1(y)) \subset C^1(\overline{B_1(y)}) \) for \( p > n \) and any \( y \in \mathbb{R}^n \), we have \( u \in C^1(\mathbb{R}^n) \) and \( |\nabla u| \in L^\infty(\mathbb{R}^n) \).

\[ \square \]

**Lemma 2.** Let \( u \) be a bounded monotone solution of (3) for any \( f \in C^1(\mathbb{R}) \) and

\[ \lim_{x_n \to \infty} u(x''', x_n) = 1 \quad \text{for all} \quad x = (x''', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}. \]

Then

\[ E_R(u) \leq k \int_{\partial B_R} \gamma(x')dS(x), \tag{41} \]

\(^2\)For any positive \( \epsilon \) and any \( a, b \in \mathbb{R} \), \( ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2 \).
where the positive constant $k$ is independent of $R$ and $E_R(u)$ is the energy functional defined by

$$E_R(u) := \frac{1}{2} \int_{B_R} \gamma(x') \left| \nabla u^t \right|^2 dx - \int_{B_R} \lambda(x')(F(u^t) - F(1)) dx.$$

**Proof:** Define $u^t(x) = u(x'', x_n + t)$ for $t \in \mathbb{R}$. Note that the shifted function $u^t$ satisfies (3) that is

$$- \text{div}(\gamma(x') \nabla u^t) = \lambda(x') f(u^t) \quad \text{in} \quad \mathbb{R}^n. \quad (42)$$

Moreover, the following monotonicity and decay conditions hold.

$$\begin{cases}
\partial_t u^t(x) > 0 \quad \text{in} \quad \mathbb{R}^n \\
\lim_{t \to \infty} u^t(x) = 1 \quad \text{in} \quad \mathbb{R}^n \\
|\nabla u^t| \in L^\infty(\mathbb{R}^n).
\end{cases} \quad (43)$$

Step 1: We claim that the following decay estimate holds for any $R > 1$

$$\lim_{t \to \infty} E_R(u^t) = 0. \quad (44)$$

To prove (44) we apply the properties of $u^t$ given in (43). Since $\lim_{t \to \infty} u^t(x) = 1$ for any $x \in \mathbb{R}^n$, for any $R > 1$ we get

$$\lim_{t \to \infty} \int_{B_R} \lambda(x')(F(u^t) - F(1)) = 0.$$

From definition of the energy functional $E_R$ we only need to prove that for any $R > 1$

$$\lim_{t \to \infty} \int_{B_R} \gamma(x') \left| \nabla u^t \right|^2 \to 0 \quad \text{for any} \quad R > 1. \quad (45)$$

Multiply both sides of (42) with $u^t - 1$ and do integration by parts on $B_R$ to end up with

$$\int_{B_R} \gamma(x') \left| \nabla u^t \right|^2 = \int_{\partial B_R} \gamma(x') (u^t - 1) \partial_n u^t + \int_{B_R} \lambda(x') f(u^t)(u^t - 1).$$

Taking the limit of both sides as $t \to \infty$ and using again the fact that $\lim_{t \to \infty} u^t(x) = 1$ for any $x \in \mathbb{R}^n$, one can get (45). This finishes the proof of (44).

Step 2: The following upper bound holds for the energy of $u$

$$E_R(u) \leq E_R(u^t) + k \int_{\partial B_R} \gamma(x') dS(x) \quad \text{for all} \quad t \in \mathbb{R}^+, \quad (46)$$

where $k$ is a contact and independent of $R$. Differentiating the energy functional of $u^t$ gives us

$$\partial_t E_R(u^t) = \int_{B_R} \gamma(x') \nabla u^t \cdot \nabla(\partial_t u^t) - \int_{B_R} \lambda(x') f(u^t) \partial_t u^t. \quad (47)$$

Now, multiply (42) with $\partial_t u^t$ and perform integration by parts on $B_R$ to get

$$\int_{B_R} \gamma(x') \nabla u^t \cdot \nabla(\partial_t u^t) - \int_{\partial B_R} \gamma(x') \partial_n u^t \partial_t u^t = \int_{B_R} \lambda(x') f(u^t) \partial_t u^t. \quad (48)$$

Note that the integral terms $\int_{B_R} \lambda(x') f(u^t) \partial_t u^t$ and $\int_{B_R} \gamma(x') \nabla u^t \cdot \nabla(\partial_t u^t)$ are common in (47) and (48). So, combining these two integral equalities we get a simplified form for the derivative of the energy of $u^t$

$$\partial_t E_R(u^t) = \int_{\partial B_R} \gamma(x') \partial_n u^t \partial_t u^t. \quad (49)$$
Note that the directional derivative of \( u^i \) is \( \partial_{\nu} u^i(x) = \nu(x) \cdot \nabla u^i(x) = \nu(x) \cdot \nabla u(x'', x_n + t) \) when \( ||\nu|| = 1 \). Therefore, \(-||\nabla u||_{L^\infty(\mathbb{R}^n)} \leq \partial_{\nu} u^i(x) \leq ||\nabla u||_{L^\infty(\mathbb{R}^n)} \). From this and the fact that \( \partial_{i} u^i(x) > 0 \) for all \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R}^+ \), we get

\[
\partial_i E_R(u^i) \geq -||\nabla u||_{L^\infty(\mathbb{R}^n)} \int_{\partial B_R} \gamma(x') \partial_t u^i dS(x).
\]  (50)

On the other hand, basic integration shows that

\[
E_R(u) = E_R(u^i) - \int_0^t \partial_s E_R(u^s) ds.
\]

From this and (50) we get

\[
E_R(u) \leq E_R(u^i) + ||\nabla u||_{L^\infty(\mathbb{R}^n)} \int_0^t \int_{\partial B_R} \gamma(x') \partial_s u^s dS(x) ds.
\]

Therefore,

\[
E_R(u) \leq E_R(u^i) + ||\nabla u||_{L^\infty(\mathbb{R}^n)} \int_{\partial B_R} \gamma(x')(u^i - u) dS(x).
\]

Note that from the definition of \( u^i \), we get \( u(x) < u^i(x) \) for all \( x \in \mathbb{R}^n \) and \( t \in \mathbb{R}^+ \) and then \( 0 < u^i(x) - u(x) < ||u||_{L^\infty(\mathbb{R}^n)} \). Set \( k = ||\nabla u||_{L^\infty(\mathbb{R}^n)}||u||_{L^\infty(\mathbb{R}^n)} \), this finishes the proof of (16). To complete the proof, just take the limit of (16) as \( t \to \infty \) in the light of (11).

\[ \Box \]

Now, we prove an elementary inequality that compares the surface integral with the volume integral.

**Lemma 3.** Let \( s \geq 2 \), \( d \geq 1 \) and \( \gamma \in C^\infty(\mathbb{R}^d) \) be positive. Then

\[
\int_{\partial B_R} \gamma(x') dS(x) \leq k R^{s-1} \int_{B_R} \gamma(x') dx'
\]

where \( k \) is independent of \( R \).

**Proof:** For a general surface \( x_n = \phi(x'') \), the surface area element is \( dA = \sqrt{1 + |D\phi|^2} dx_1 \cdots dx_{n-1} \). For the sphere \( \phi(x'') = (R^2 - |x_1|^2 - |x_2|^2 - \cdots - |x_{n-1}|^2)^{1/2} \) and therefore

\[
dA = \sqrt{1 + |D\phi|^2} dx_1 \cdots dx_{n-1} = \frac{R}{\phi} dx_1 \cdots dx_{n-1}.
\]

Integrating out the \( x'' \)-variable, we have

\[
\int_{\partial B_R} \gamma(x') dS(x) = \int_{B_R} \gamma(x') w(R, x') dx'
\]

for some weight function \( w(R, x') \geq 0 \). We now prove that

\[
w(R, x') = k_s R(R^2 - |x'|^2)^{\frac{s-2}{2}}, \quad (51)
\]

where \( k_s \) is a constant independent of \( R \). This proves the lemma since \( w(R, x') \leq k_s R^{s-1} \) whenever \( s \geq 2 \). Rewrite \( \phi = (\rho^2 - |y|^2)^{1/2} \), where \( \rho^2 = R^2 - |x'|^2 \) and \( x'' = (y, x_n) \in R^s \). The weight function is then

\[
w(R, x') = \int_{|y|<\rho} \frac{R}{\phi} dy = R \int_{|y|<\rho} \frac{dy}{(\rho^2 - |y|^2)^{1/2}} = k_s R \rho^{s-2}, \quad (52)
\]

where \( k_s := \int_{B_1^{s-1}} \frac{dx}{(1-|z|^2)^{s/2}} \) and \( B_1^{s-1} \) is the unit ball in \( \mathbb{R}^{s-1} \). From the definition of \( \rho \), this proves (51).

\[ \Box \]
We are now ready to see the proof of Theorem 2.

**Proof of Theorem 2** Without loss of generality we assume that $F(-1) \geq F(1)$. Therefore, from the assumptions $F(u) - F(1) \leq 0$ that gives us

$$\int_{BR} \lambda(x')(F(u) - F(1))dx \leq 0.$$  

From this and Lemma 2 we get the following bound on the gradient of solutions

$$\int_{BR} \gamma(x')|\nabla u|^2dx \leq k \int_{\partial BR} \gamma(x')dS(x),$$

where $k$ is a constant independent of $R$. Applying Lemma 3 we change the upper bound to a volume integral of $\gamma$ that is

$$\int_{BR} \gamma(x')|\nabla u|^2dx \leq k R^{s-1} \int_{BR} \gamma(x')dx'.$$

(53)

The key point is to apply Proposition 2 to show that solutions are at most $(d+1)$-dimensional. So, we need to make sure that (50) holds. From (53) we only need

$$\int_{BR} \gamma(x')dx' \leq k R^{3-s}g(R),$$

(54)

for any $g \in \mathcal{G}$. Note that for a positive $\gamma$ to satisfy (54) we need to assume that $s \leq 3$ and also we assumed $s \geq 2$ to prove Lemma 3. Therefore, for $s = 2$ that is $n = d + 2$ we assume that

$$\int_{BR} \gamma(x')dx' \leq k Rg(R),$$

and for $s = 3$ that is $n = d + 3$ we assume that

$$\int_{BR} \gamma(x')dx' \leq kg(R).$$

This finishes the proof for the case $F(-1) \geq F(1)$. Note that if $F(-1) < F(1)$, replace $u(x'',x_n)$ with $-u(x'',-x_n)$ and apply the same argument.

\[\square\]

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