Critical fluctuations at a many-body exceptional point

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Critical phenomena arise ubiquitously in various contexts of physics, from condensed matter, high energy physics, cosmology, to biological systems, and consist of slow and long-distance fluctuations near a phase transition or critical point. Usually, these phenomena are associated with the softening of a massive mode. Here we show that a novel, non-Hermitian-induced mechanism of critical phenomena that do not fall into this class can arise in generic driven-dissipative many-body systems with coupled binary order parameters such as exciton-polariton condensates and driven-dissipative Bose-Einstein condensates in a double-well potential. The criticality of this “critical exceptional point” is attributed to the coalescence of the collective eigenmodes that convert all the thermal and dissipative noise activated fluctuations to the Goldstone mode, leading to anomalously giant phase fluctuations that diverge at spatial dimensions \(d \leq 4\). Our dynamic renormalization group analysis shows that this anomalous feature gives rise to a strong-coupling fixed point at dimensions as high as \(d \leq 8\) associated with a new universality class.

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Understanding and manipulating dissipation effects in open quantum systems \cite{1} is increasing in importance, due to its crucial role in designing new optical devices and performing quantum computation. Particularly intense interest has recently emerged to the study of “exceptional points” (EPs) \cite{2,3}, which can arise in these dissipative devices owing to their non-Hermitian nature. An EP is a point where two (or more) eigenstates coalesce such that they lose their completeness, leading to a spectral singularity. It turns out that this singularity gives rise to a number of counter-intuitive phenomena in the vicinity of the EP, such as loss-induced transmission \cite{4}, unidirectional invisibility \cite{5}, enhanced quantum sensitivity \cite{6,7} and chiral behavior \cite{8}. These concepts have been proven to be applicable to a rich variety of many-body systems such as superconductivity \cite{9}, atomic gases \cite{10}, and correlated materials \cite{11}.

One of the most striking phenomena that arise in many-body systems are critical phenomena, which are collective many-body phenomena associated with the divergence of length and time scales near the destruction of a long-range order \cite{12}. Critical points (CPs) and EPs have a crucial feature in common: the occurrence of a gap closure. EPs can of course occur in low-degree-of freedom systems, but when they occur in a many-body context this raises the issue whether EPs can possess similar properties to CPs such as critical fluctuations. An important step forward in this direction was made by Tripathi \textit{et al.} \cite{13}, who identified the quantum critical point of the field-induced dynamic Mott transition as a many-body EP \cite{14}. Recently, Ref. \cite{15} identified a similar quantum critical point in a parity-time (PT) symmetric sine-Gordon model. In both of the works, however, the quantum critical point is found only in the ground state, limiting to situations where the thermal or the dissipation-induced noise is negligible or can be excluded by post-selection.

Here we show that a novel, non-Hermitian induced critical phenomena activated by the thermal and dissipative noise may occur at the many-body EP of generic driven-dissipative many-body systems with coupled binary order parameters. Examples in scope include exciton-polariton condensates \cite{16} (composed of excitons and photons) and driven-dissipative Bose-Einstein condensates (BECs) in a double-well potential \cite{17,18}. Our previous work \cite{19} has

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{Non-Hermitian phase transition and the critical exceptional point (CEP). a, Schematic phase diagram of a driven-dissipative many-body systems with coupled binary order parameters, in terms of the input parameters \((\alpha_1, \alpha_2)\). The solid line is the phase boundary of the non-Hermitian induced phase transition, which exhibits an endpoint at the critical exceptional point (CEP). b, Demonstration of the appearance of the CEP for systems described by the coupled driven-dissipative Gross-Pitaevskii equations. At a small decay rate \(\kappa/g < 1\), as the number of the loss component \(n_0 = |\Phi|^2\) increases, the steady state exhibit a phase transition signaled by a discontinuity in the condensate emission energy \(E\). A CEP found at \(\kappa/g = 1\), represented by the star, marks the endpoint of the phase boundary, where the discontinuity is absent at \(\kappa/g > 1\). We set \(\omega_1/g = -1.9, \omega_2/g = -2, \text{ and } U_\alpha/g = 0.1\).}
\end{figure}
shown that these systems exhibit a non-Hermitian phase transition with an endpoint of its phase boundary marked by a many-body EP (Fig. 1(a)), proposed as a new interpretation of the phase transition observed in some polaron experiments in the $U(1)$-broken phase (the so-called “second threshold” [22–27]). In this work, we show that this endpoint (which we refer to as a critical EP (CEP)) exhibits anomalous critical phenomena which have a fundamentally different origin from the conventional critical phenomena.

At conventional CPs, the critical fluctuations arise due to the softening of the longitudinal mode, caused by the flattening of the free energy landscape (Figs. 2(a)-(c)); longitudinal and transverse fluctuations are separable. Here we point out that the criticality of the CEP arises from the coalescence of collective modes owing to the non-Hermitian nature of the condensate; hence thermally activated fluctuations in both degenerate modes occupy the Goldstone mode (Figs. 2(d)-(f)). We show that this peculiar property results in anomalously giant phase fluctuations that diverge at spatial dimensions $d \leq 4$, which is to be compared to the conventional equilibrium case where the divergence only happens at $d \leq 2$ [28, 29]. We further perform a dynamic renormalization group analysis [30] and identify a strong-coupling fixed point at dimensions as high as $d \leq 8$ associated with a new universality class, which arises due to the anomalous phase fluctuations.

We emphasize that the rise of the CEP is a generic feature of a nonequilibrium steady state, as long as the many-body system is (1) driven-dissipative, (2) composed of two components, and (3) exhibits spontaneous symmetry breaking [21]. The two-component nature offers two branches of eigenstates that the system can condense into, letting us define two distinct phases of matter not distinguished by symmetry – analogous to liquid and gas in an equilibrium system. Because of the driven-dissipative feature, the CEP arises as a point where the two phases coalesce, making that the endpoint of the phase boundary. Considering the rapid experimental development in various driven-dissipative many-body interacting platforms such as ultracold atoms [31], circuit QED [32, 33], photon BEC [34], plasmonic-lattice polaron BEC [35], and strongly interacting photons [36], it seems promising to realize this point by making them binary.
NOISY DRIVEN-DISSIPATIVE
GROSS-PITAEVSKII EQUATION FOR BINARY
CONDENSATES

Below, we consider a driven-dissipative, repulsively interacting BEC composed of two components. To our knowledge, this is the simplest system that exhibits a non-Hermitian induced phase transition associated with the rise of the CEP [21].

Our goal is to reveal the dynamic critical properties of the CEP. For a one-component case, it has been shown by coarse-graining the Keldysh partition function [37, 38] that the critical properties of a driven-dissipative condensate can be captured by taking the noise average of the noisy driven-dissipative Gross-Pitaevskii (GP) equation [39, 40]. It is straightforward to extend this discussion to our two component case [21], to show that the stochastic equations of motion to consider is \( i\hbar \dot{\Psi}(r, t) = A_{GP}(\nabla^2)\Psi(r, t) + \eta(r, t) \) with \( \Psi = (\Psi_1, \Psi_2)^\text{T}, \eta = (\eta_1, \eta_2)^\text{T} \) and,

\[
A_{GP}(\nabla^2) = \left( \begin{array}{c}
\omega - i\kappa - K_1\nabla^2 \\
g \\
\omega_g + i\bar{P} - \bar{K}_g\nabla^2 + \bar{U}_g|\Psi_g|^2
\end{array} \right),
\]

where \( \alpha = (g) \) labels the loss (gain) component with its one-body decay (gain) rate given by \( \kappa(P) \). Here, \( \omega_{0}(g) \) and \( \Psi_{0}(g)(r, t) \) are the energy and macroscopic wave function of the loss (gain) component, respectively. \( g \) is the inter-component coupling and \( K_g = K - iD_g \bar{U}_g = U_g - i\nu_g \) are complex coefficients, where \( D_g, U_g, \nu_g \) are the diffusion constant, repulsive interaction strength, and two-body loss rate that gives nonlinear saturation of the gain component, respectively, and \( K_{1}(g) \) determines the kinetics of the loss (gain) component. We have assumed in this model that only the gain components interact and have a nonlinear saturation, which however, does not affect the critical properties of the CEP. White noise \( \eta_\alpha(r, t) \) that originates from thermal fluctuations and/or dissipation, is characterized by \( \langle \eta_\alpha(r, t)\eta_\beta(r', t') \rangle = \Delta_\alpha\delta^4(r - r')\delta(t - t') \), and \( \langle \eta_\alpha(r, t) \rangle = \langle \eta_\alpha(r, t)\eta_\beta(r', t') \rangle = 0 \). In general, the noise may not satisfy the dissipation-fluctuation theorem in driven-dissipative systems.

In the absence of noise \( \eta_\alpha(r, t) \), the macroscopic wave function in the uniform steady state, \( \Psi^0_\alpha(t) = \Phi^0_\alpha e^{-iEt} \) (\( E \) is the (real) condensate emission energy), follows the relation \( E\Phi^0_\alpha = \sum_\beta[A_{GP}^{0}\alpha_\beta]\Phi^0_\beta \), where \( A_{GP}^{0} \) is the matrix A GP with \( \Psi_\alpha \) replaced by the steady state value \( \Phi^0_\alpha \). Because of the non-Hermitian nature of \( A_{GP}^{0} \), the two eigenvectors of \( A_{GP}^{0} \) need not be orthogonal, giving rise to a point where two of the eigenvectors coalesce; the CEP. This point, found at \( \kappa = g = P - v_{1g}|\Phi^0_1|\) and \( \omega_1 = \omega_g + U_g|\Phi^0_1| \), has been shown to mark the endpoint of the phase boundary [21], as demonstrated in Fig. (b).

For the investigation of criticality, it is useful to rewrite the GP equation in terms of the noise-activated amplitude \( \delta \Phi_\alpha(r, t) \) and phase fluctuations \( \delta \theta_\alpha(r, t) \) by rewriting the macroscopic wavefunction as \( \Psi_\alpha(r, t) = [\Phi^0_\alpha + \delta \Phi_\alpha(r, t)]e^{-iEt} = (M^0_\alpha + \delta \Phi_\alpha(r, t))e^{-iEt} \), where \( \Phi^0_\alpha = M^0_\alpha e^{i\eta_\alpha} \). By integrating out \( \delta \Phi_\alpha(r, t) \) up to linear order and further assuming that the amplitude dynamics are overdamped [41], we arrive at a Kardar-Parisi-Zhang (KPZ)-like stochastic equation of motion (See the Supplemental Information (SI)),

\[
\partial_t \delta \theta_\alpha(r, t) = \sum_{\beta=1,g} W_{\alpha\beta}(\nabla\delta \theta_\beta(r, t) + t_\alpha(\Delta \delta \theta(r, t))^2 + \sum_{\beta=1,g} \lambda_{\alpha\beta}(\nabla\delta \theta_\beta(r, t))^2 + \xi_\alpha(r, t),
\]

with \( W(\nabla) = \left( \begin{array}{cc}
-s_1 + v_1\nabla^2 & s_1 + v_6\nabla^2 \\
-s_6 + v_6\nabla^2 & s_6 + v_6\nabla^2
\end{array} \right) \), and \( \Delta \delta \theta(r, t) = \delta \theta_1(r, t) - \delta \theta_g(r, t) \). Here, we have retained the most relevant nonlinear terms and the real parameters \( s_\alpha, \nu_\alpha, \lambda_\alpha \) are determined from the parameters used in \( A_{GP} \). \( \xi_\alpha(r, t) \) is a white noise for the phases, characterized by \( \langle \xi_\alpha(r, t)\xi_\beta(r', t') \rangle = \sigma_{\alpha\beta}\delta^4(r - r')\delta(t - t') \) and \( \langle \xi_\alpha(r, t) \rangle = 0 \).

ANOMALOUSLY GIANT PHASE FLUCTUATIONS AT THE CEP

We show below that an anomalous critical behavior appears at the CEP that satisfies \( s_1 = s_g = \gamma \). Let us start with a linearized theory (i.e. \( t_\alpha = \lambda_{\alpha\beta} = 0 \)). By solving the secular equation \( \det[-i\omega \mathbf{1} - W(\mathbf{k})] = 0 \), the eigenenergies are given by

\[
\omega_{\pm}(\mathbf{k}) = \frac{1}{2}i\left[-(\gamma + 2\mathbf{d}k^2) \pm \sqrt{\gamma^2 + 4v^2k^2}\right],
\]

where \( \gamma = s_1 - s_g, v^2 = [2(s_g\nu_g + s_{1g}) + (s_1 + s_g)(\nu_g - \nu_1)]/2 \) and \( D = (\nu_g + \nu_1)/2 \). We remark that \( v^2 \) can take a negative value, which may cause a dynamical instability, not uncommon for systems with loss and gain. In this paper, we assume that those situations are avoided by having a large enough nonlinear saturation \( v_g \) and diffusion constant \( D_g \) in the gain component, to have \( v^2 > 0 \). (See the discussions in the SI.)

When the system is away from the CEP, i.e. \( \gamma > 0 \), the eigenmodes are given by the diffusive Goldstone mode \( \omega_{-}(\mathbf{k}) \propto -i\mathbf{d}k^2 \) [39, 10] and a relaxational mode \( \omega_{\pm}(\mathbf{k}) = -i\gamma \). Since the relaxational mode would be gapped away as we further coarse-grain the system and thus play no role in the effective low-energy physics, the physics is essentially the same as the one-component case and recovers the dynamic scaling behavior [37, 41, 43, 45] known to obey the KPZ scaling [42].
The situation is dramatically different at the CEP $\gamma = 0$. In this case, we find both eigenmodes $\omega_{\pm}(k)$ to be gapless, which interestingly are sound modes, 

$$\omega_{\pm}(k) = \pm v|k| - iDK^2,$$  

(3)

demonstrating that both components play role and thus modifies the scaling properties.

A crucial observation is that, this gap closure is associated with the coalescence of the collective modes, which is crucially different from just being degenerate. The eigenmodes in the uniform limit $k \to 0$ are given by the in-phase motion $(\delta\theta_l, \delta\theta_g)^T \propto (1, 1)^T$ and $(\delta\theta_l, \delta\theta_g)^T \propto (s_1, s_g, 1)^T$, for the corresponding eigenenergies $\omega_{-}(k)$ and $\omega_{+}(k)$, respectively. The former mode, i.e. the Goldstone mode, is assured to be gapless by the global symmetry under the transformation $\Psi_a(r, t) \to \Psi_a(r, t)e^{i\theta}$.

These modes coalesce at the CEP $s_1 = s_g$ as schematically described in Figs. 2(a)-(c), giving rise to the gap closure. This gap-closing mechanism is fundamentally different from that in the conventional CPs, where the longitudinal mode itself softens by the flattening of the free energy but is still orthogonal to the Goldstone mode (Figs. 2(a)-(c)).

The above peculiar property at the CEP gives rise to anomalously giant phase fluctuations. To see this in a transparent way, it is useful to triangularize the kernel $W(k)$ by using an orthogonal basis $U^\dagger(k) \equiv (u_\perp(k), u_\parallel(k))^T$ (instead of diagonalizing $W(k)$, which is ill-defined at $k \to 0$ at the CEP) as,

$$\tilde{W}(k) = U(k)W(k)U^\dagger(k) = \begin{pmatrix} -i\omega_{-}(k) & \zeta \\ \zeta & -i\omega_{+}(k) \end{pmatrix},$$

Here, we have chosen one basis vector as the Goldstone mode, $W(k)u_\perp(k) = -i\omega_{-}(k)u_\perp(k)$, and the other basis, satisfying $W(k)u_\parallel(k) = -i\omega_{+}(k)u_\parallel(k) + \zeta u_\perp(k)$, is chosen as the longitudinal direction that is perpendicular to the Goldstone mode $u_\parallel(k) \cdot u_\parallel(k) = 0$. Importantly, the off-diagonal piece $\zeta = s_1 + s_g$ arises due to the non-Hermitian nature of $W$, which converts the longitudinal fluctuations to the Goldstone mode.

The Green’s function in this basis $\tilde{G}^0_{ss'}(k, \omega) = ([i\omega 1 - \tilde{W}(k)]^{-1})_{ss'}$ is given by,

$$\tilde{G}^0(k, \omega) = \begin{pmatrix} \omega_{-}(k) & \zeta \\ \zeta & \omega_{+}(k) \end{pmatrix}.$$  

(4)

Here, we find that the off-diagonal, non-Hermitian induced component $\tilde{G}^0_{\perp\parallel}$ exhibits a peculiarly strong singularity at the CEP, involving two gapless poles at $\omega = \omega_{\pm}(k)$. As a result, noting that the phase fluctuations are expressed in terms of Green’s functions as $\delta\theta_{s=\perp\parallel}(k, \omega) = \sum_{r'}^{r'} \tilde{G}^0_{ss'}(k, \omega)\tilde{G}^0_{\parallel}(k, \omega)$ (where $\tilde{\delta}\theta_{s}(k, \omega) = \sum_{\alpha=1, 2} U_{s\alpha}(k)\delta\theta_{\alpha}(k, \omega)$ and $\tilde{\xi}_s(k, \omega) = \sum_{\alpha=1, 2} U_{s\alpha}(k)\tilde{\xi}_{\alpha}(k, \omega)$), the most strongly diverging term in the equal-time correlation function involves two $\tilde{G}^0_{\perp\parallel}$’s,

$$\langle \delta\theta_{\alpha=(1,2)}(r)\delta\theta_{\beta=(1,2)}(r')\rangle \sim \int_0^{\Lambda_c} dk k^{d-1}e^{i\kappa(k-r-r')} \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{G}^0_{\perp\parallel}(k, \omega)\sigma_{\parallel}(k, -\omega) \sim \int_0^{\Lambda_c} dk k^{d-1}e^{i\kappa(k-r-r')} \frac{A}{k^2},$$  

(5)

which diverges for $d \leq 4$. Here, ‘$\sim$’ means that we have retained the most strongly diverging term and $A \equiv (\kappa^2\sigma_{\parallel}/Dv^2)$ with longitudinal noise $\sigma_{\parallel} = (\sigma_1 + \sigma_2 - \sigma_3 - \sigma_g)/2$. We have used the relation $\zeta = 2\kappa$ that holds at the CEP. $\Lambda_c$ is an ultraviolet cutoff. The obtained phase fluctuations are anomalously giant, in the sense that the phase fluctuations in the conventional case $\langle (\delta\theta_{\alpha})^2 \rangle \sim \int_0^{\Lambda_c} dk k^{d-1} \cdot k^{-2}$ diverges only for $d \leq 2$, as stated in the Mermin-Wagner-Hohenberg’s theorem. As is clear from this structure, the giant fluctuations are activated by the longitudinal noise $\sigma_{\parallel}$ that get converted to the Goldstone mode through the non-Hermitian-induced mixing $\zeta = 2\kappa$.

It is interesting to compare this result to an $O(N)$ model in a static random field studied by Imry and Ma [24], where the correlation function of the transverse magnetization also diverges at $d \leq 4$. In that case, the anomalous fluctuations emerge from coupling between a static random field and the Goldstone mode, causing the system to separate into domains. Our case may be viewed as a dynamical extension of this discussion, where the coupling between the longitudinal white noise and the Goldstone mode triggers the anomalous fluctuations.

**STRONG-COUPLING FIXED POINTS AT SPATIAL DIMENSION $d \leq d_c = 8$**

We now put back the nonlinear terms to analyze the dynamic critical behavior of the CEP. Following the standard procedure of the dynamic renormalization group method [20], we first compute the perturbative correction from the nonlinear couplings and then rescale space, time, and phase fluctuations according to,

$$r \to e^{-t}r, t \to e^{-z}t, \delta\theta_{\alpha} \to e^{x_{\alpha}\delta\theta_{\alpha}},$$  

(6)

to formulate the flow equations. Finding the fixed point of the flow equations provides the universal scaling features of the CEP, such as,

$$\langle \delta\theta_{\alpha}(r, t)\delta\theta_{\beta}(r', t')\rangle = |r - r'|x_{\alpha}x_{\beta}f_{\alpha\beta}\left(\frac{t - t'}{|r - r'|^2}\right),$$

where $f_{\alpha\beta}(x)$ is a scaling function.

In our two-component case, however, not all of the parameters can be fixed simultaneously under renormalization. For instance, the rescaling [1] changes $\kappa$, $v$ and
Here, we require \( A \) of the lowest-order kinetic term (i.e. the velocity \( v \)) to be fixed, \( \kappa \) flows to infinity as \( l \to \infty \) while \( D \to 0 \). Here, we require \( A = \kappa^2 \sigma_{\parallel\parallel}/(Dv^2) \), \( \gamma_{\parallel} \equiv \gamma/D \), and \( v \) to be fixed, since the correlation function within the linearized theory is determined solely by these parameters in the vicinity of the CEP in the limit \( D \to 0 \). We have implicitly assumed here that the strong-coupling fixed point is not very far away from the Gaussian fixed point \((\gamma_{\ast}, \Gamma_{\ast}) = (0, \gamma/D, 0)\), which will justify by restricting ourselves to spatial dimensions close to the upper critical dimension \( d_c \). The above assumption also requires \( \chi_{\parallel} = \chi_{\perp} = \chi \). In the linearized theory, we get the roughening exponent \( \chi = \chi_{G} = (4 - d)/2 \) and the dynamic exponent \( z = z_G = 1 \).

The property \( D \to 0 \) and \( \kappa \to \infty \) at the fixed point means that the system is infinitely sensitive to the noise that activates the fluctuations, which inevitably requires the longitudinal noise strength \( \sigma_{\parallel\parallel} \) to flow to zero to keep the parameter \( A = \kappa^2 \sigma_{\parallel\parallel}/(Dv^2) \) fixed. This anomalous sensitivity to noise may turn the perturbative nonlinear correction effectively relevant even at dimensions where the nonlinear couplings \( \lambda_{\alpha\beta}, t_{\alpha} \) themselves are irrelevant (which are \( d > 1 \) and \( d > 3 \), respectively), making the upper critical dimension \( d_c \) higher than that obtained from the trivial scaling.

Keeping the above discussions in mind, the flow equations within the one-loop order read, (See the SI for derivation.)

\[
\frac{d\gamma}{dt} = \left[ 2 - \left( \frac{C_8}{32d} + \frac{C_{10}}{16d} \right) \Gamma \right] \gamma, \tag{7}
\]

\[
\frac{dv}{dt} = (z - 1)v, \tag{8}
\]

\[
\frac{dA}{dt} = \left[ 4 - d - 2\chi - \left( \frac{C_8}{32d} + \frac{C_{10}}{16d} \gamma \right) \Gamma \right] A, \tag{9}
\]

where we have retained only the most relevant coupling. Here, \( C_{\gamma} = (S_d/(2\pi)^d)\Lambda_{\gamma}^{d-1-i} \) with \( S_d \) the surface area of a \( d \)-dimensional sphere. Since \( D \) is (dangerously) irrelevant and \( \kappa \) flows to infinity, the most relevant coupling is the term that have the lowest order on \( D \) and the highest order on \( \kappa \), which turns out to be,

\[ \Gamma \equiv \frac{t_{\parallel\parallel} \sigma_{\parallel\parallel}}{D^5} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}). \]

Here, \( t_{\parallel} = \sqrt{2}(t_1 + t_{\parallel}) \) is the massive out-of-phase nonlinearity that affects the longitudinal fluctuations, and \( \lambda_{\perp\perp} = (\lambda_0 + \lambda_{d0} + \lambda_{d1} + \lambda_{d2})/(2\sqrt{2}) \) is the KPZ-like nonlinear coupling that converts the two incoming Goldstone mode into the longitudinal mode. The effective coupling \( \Gamma \) follows the flow equation,

\[
\frac{d\Gamma}{dt} = \left[ 8 - d - \left( \frac{5C_8}{32d} + \frac{5C_{10}}{16d} \gamma \right) \Gamma \right] \Gamma, \tag{10}
\]

indicating that \( \Gamma \) is relevant at dimensions as high as \( d < d_c = 8 \).

This anomalously high upper critical dimension \( d_c = 8 \), which is to be compared to the conventional KPZ scaling with \( d_c = 2 \), is attributed again to the property that the eigenmodes coalesce to the Goldstone mode. In addition to the longitudinal noise \( \sigma_{\parallel\parallel} \), the nonlinear couplings \( t_{\parallel\parallel}, \lambda_{\perp\perp} \) induce additional longitudinal phase fluctuations. These excited longitudinal fluctuations are converted to the Goldstone mode through the non-Hermitian induced propagator \( G_{\perp\perp}^{(5)} \) with anomalously giant phase fluctuations, giving rise to the anomalously strong nonlinear corrections.

At spatial dimensions close to the upper critical dimension, \( d = d_c - \epsilon = 8 - \epsilon \), we find a strong-coupling fixed point at \((\gamma_{\ast}, \Gamma_{\ast}) \approx (0, \epsilon(32 \cdot 8)/(5C_8))\), associated with a new universality class with critical exponents,

\[ \chi \approx \chi_{G} - \frac{\epsilon}{10}; z = z_G. \]

We note that, as a consequence of the two-component nature, \( \Gamma \) may take either sign. Since \( \Gamma \) cannot change its sign during the flow because \( d\Gamma/dl = 0 \) at \( \Gamma = 0 \) as seen in Eq. (10), the system can only flow to the above-obtained strong-coupling fixed point when the flow initiates at positive value \( \Gamma(l = 0) > 0 \). Otherwise, when \( \Gamma(l = 0) < 0 \), the flow direct towards \( \Gamma \to -\infty \), implying the existence of another distinct phase of matter. The bare interaction parameter has a sign determined by the balance between the diffusion rate of the pump and the nonlinear loss rate of the field (see Eq. (S17)).

To summarize, we have proposed a novel type of dynamic critical phenomena that arise at the CEP due to the coalescence of the collective eigenmodes to the Goldstone mode. We showed that this property gives rise to anomalously giant phase fluctuations and strong nonlinear effects that survive at anomalously high spatial dimensions, but that nonetheless yields a stable strong coupling fixed point just below the upper critical dimension of 8. At dimensions less than 4, the linear theory is unstable to noise-induced roughening, though we cannot yet identify the lower critical dimension in the nonlinear theory. Direct numerical simulation of the stochastic coupled GP equation to extract the critical exponents of the strong-coupling fixed points at more realistic dimensions \( d = 1, 2, 3 \) is under progress.

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Supplemental Material for “Non-Hermitian phase transition from a polariton Bose-Einstein condensate to a photon laser”

NOISY DRIVEN-DISSIPATIVE GROSS-PITAEVSKII EQUATION TO THE KARDAR-PARISI-ZHANG EQUATION

Here, we derive the Kardar-Parisi-Zhang (KPZ)-like stochastic equation of motion (1) from the coupled driven-dissipative Gross-Pitaevskii (GP) equation,

\[
\begin{align*}
\partial_t \left( \frac{\Psi_1(r,t)}{\Psi_k(r,t)} \right) &= \left( \omega_1 - i\kappa - K_1\nabla^2 \right) \frac{g}{\omega_g + iP - (K_g - iD_g)\nabla^2 + (U_g - i\nu_g)|\Psi_g(r,t)|^2} \left( \frac{\Psi_1(r,t)}{\Psi_k(r,t)} \right) + \left( \frac{\eta_1(r,t)}{\eta_g(r,t)} \right), \\
&= \left( \omega_1 - i\kappa - K_1\nabla^2 \right) \frac{g}{\omega_g + iP - (K_g - iD_g)\nabla^2 + (U_g - i\nu_g)|\Psi_g(r,t)|^2} \left( \frac{\Psi_1(r,t)}{\Psi_k(r,t)} \right) + \left( \frac{\eta_1(r,t)}{\eta_g(r,t)} \right). \\
\end{align*}
\]

(S1)

In the absence of noise, the uniform steady state described by the ansatz \(\Psi_0^s(t) = \Phi_0^s e^{-iEt}\) is determined from the relation,

\[
E \left( \frac{\Phi_1^0}{\Phi_k^0} \right) = A_{GP} \left( \frac{\Phi_1^0}{\Phi_k^0} \right) = \left( \frac{\omega_1 - i\kappa}{g} \right) \left( \omega_g + U_g|\Phi_0^0|^2 + i(P - \nu_g)|\Phi_0^0|^2 \right) \left( \frac{\Phi_1^0}{\Phi_k^0} \right). \\
\]

(S2)

To study the dynamics, we rewrite the GP equations (S1) in terms of the amplitude and phase fluctuations around this uniform steady state. Here, we define the amplitude and phase fluctuations by \(\Psi_\alpha(r,t) = [\Phi_0^\alpha + \delta\Phi_\alpha(r,t)]e^{-iEt} = (M_\alpha + \delta\Phi_\alpha(r,t))[e^{i\theta_\alpha + \delta\theta_\alpha(r,t)}]e^{-iEt}\), where \(\Phi_0^\alpha = M_\alpha e^{i\theta_\alpha}\). Assuming that the amplitude fluctuations are small such that the terms with \(O((\delta\Phi_\alpha)^2)\) are negligible, we obtain,

\[
\begin{align*}
-M_1^0 &\partial_1\delta\theta_1(r,t) = (\omega_1 - E)|\Phi_1(r,t)| + gM_0^0|\cos(\Delta\theta_0 + \delta\theta_1(r,t) - \delta\theta_g(r,t)) - \cos\Delta\theta_0| \\
&\quad + g|\cos(\Delta\theta_0 + \delta\theta_1(r,t) - \delta\theta_g(r,t))\delta(\Phi_1^0\Phi_k^0)| + K_1M_1^0(\nabla\theta_1(r,t))^2 + \eta_1^0(r,t), \\
\partial_1\delta\theta_1(r,t) &= -\kappa\delta(\Phi_1(r,t)| - K_1M_1^0\nabla\theta_1(r,t) - gM_0^0|\sin(\Delta\theta_0 + \delta\theta_1(r,t) - \delta\theta_g(r,t)) - \sin\Delta\theta_0| \\
&\quad - g\sin(\Delta\theta_0 + \delta\theta_1(r,t) - \delta\theta_g(r,t))\delta(\Phi_1^0\Phi_k^0) + \eta_1^0(r,t), \\
-M_0^0 &\partial_\alpha\delta\theta_\alpha(r,t) = gM_0^0|\cos(\Delta\theta_0 + \delta\theta_\alpha(r,t) - \delta\theta_g(r,t)) - \cos\Delta\theta_0| + g|\cos(\Delta\theta_0 + \delta\theta_\alpha(r,t) - \delta\theta_g(r,t))\delta(\Phi_\alpha(r,t)| \\
&\quad + (\omega_g + 3U_g|\Phi_0^0|^2 - E)|\delta(\Phi_\alpha^0\Phi_k^0)| - D_gM_0^0\nabla\theta_\alpha(r,t) + K_gM_0^0(\nabla\theta_\alpha(r,t))^2 + \eta_g^0(r,t), \\
\partial_\alpha\delta\theta_\alpha(r,t) &= -K_gM_0^0\nabla\theta_\alpha(r,t) + gM_0^0|\sin(\Delta\theta_0 + \delta\theta_\alpha(r,t) - \delta\theta_g(r,t)) - \sin\Delta\theta_0| \\
&\quad + g\sin(\Delta\theta_0 + \delta\theta_\alpha(r,t) - \delta\theta_g(r,t))\delta(\Phi_\alpha(r,t)| + (P - 3\nu_g|\Phi_0^0|^2)\delta(\Phi_\alpha(r,t)| - D_gM_0^0(\nabla\theta_\alpha(r,t))^2 + \eta_g^0(r,t). \\
\end{align*}
\]

(S3)

(S4)

(S5)

(S6)

where \(\Delta\theta_0 = \theta_0 - \theta_g^0\), and \(\eta_0(r,t) = \eta_1^0(r,t) + i\eta_1^\alpha(r,t)\). Following Ref. [1], we further assume that the amplitude fluctuations are overdamped (i.e., we neglect the time-derivative in the left-hand side of Eqs. (S5) and (S6)). This brings us to the desired form,

\[
\partial_\alpha\delta\theta_\alpha(r,t) = \sum_{\beta=1,g} W_{\alpha\beta}(\nabla)\delta\theta_\beta(r,t) + t_\alpha(\delta\theta_\alpha(r,t) - \delta\theta_g(r,t))^2 + \sum_{\beta=1,g} \lambda_{\alpha\beta}(\nabla\delta\theta_\beta(r,t))^2 + \xi_\alpha(r,t),
\]

with

\[
W(\nabla) = \begin{pmatrix} -s_1 + \nu_1 \nabla^2 & s_1 + \nu_g \nabla^2 \\ -s_g + \nu_g \nabla^2 & s_g + \nu_g \nabla^2 \end{pmatrix},
\]

where we have neglected the higher-order massive term \(O((\delta\theta_1 - \delta\theta_g)^3)\) that does not contribute to the critical properties within the one-loop order.

To obtain the explicit expression of the parameters in the coupled KPZ-like equations (S7) in terms of those in the coupled driven-dissipative GP equations (S1), which is crucial for determining the stability condition, we firstly need to solve the steady state condition Eq. (S2). This requires numerical computation in general, but for the special case where the “effective detuning” \(\delta = \omega_1 - \omega_g - U_g|\Phi_0^0|^2\) is “on resonance”, \(\delta = 0\), an analytic form can be obtained. This includes the critical exceptional point (CEP), our main focus of this paper, which is found at \(\tilde{\delta} = 0\) and \(\kappa = g = P - \nu_g|\Phi_0^0|^2\).
As shown in Ref. 2, we can classify the solutions into two types; the “−” solution with \( E = E_− \) and the “+” solution with \( E = E_+ \), where

\[
E_± = \frac{1}{2} \left[ \omega_1 + \omega_0 + U_g |\Phi_0|^2 \right] \pm i \left( \kappa - P + v_g |\Phi_0|^2 \right) \pm \sqrt{4g^2 - (\kappa + P - v_g |\Phi_0|^2)^2},
\]

are the eigenvalues with \( \text{Re} E_+ > \text{Re} E_- \). These physically correspond to the condensate in the lower and the upper branches, respectively. For simplicity, let us restrict ourselves to the case \( g \geq \kappa \). Since \( E \) is real, \( \text{Im} E_{-(+)} = 0 \) for the “− (+)” solution, giving the condition \( \kappa = P - v_g |\Phi_0|^2 \).

Plugging these into Eqs. \((S3)-(S6)\), we obtain the explicit form of the parameters in Eq. \((S7)\). Here, we provide the list of the parameters for the “−” solution:

\[
\begin{align*}
s_1 &= \frac{g^2}{\kappa}, \quad s_g = 2\kappa - \frac{g^2}{\kappa} - \frac{2U_g \sqrt{g^2 - \kappa^2}}{v_g}, \\
\nu_{11} &= \frac{K_1 \sqrt{g^2 - \kappa^2}}{\kappa}, \quad \nu_{gg} = 0, \quad \nu_{gl} = -\frac{K_1 (U_g \kappa + v_g \sqrt{g^2 - \kappa^2})}{v_g \kappa}, \quad \nu_{gg} = D_g + \frac{K_g U_g}{v_g}, \\
\lambda_{11} &= -K_1, \quad \lambda_{gg} = \lambda_{gl} = 0, \quad \lambda_{gg} = -K_g - \frac{D_g \left( \sqrt{g^2 - \kappa^2} - 3U_g \right)}{2v_g}, \\
t_1 &= -\frac{g^2 \sqrt{g^2 - \kappa^2}}{v_g \kappa |\Phi_0|^2}, \quad t_g = \frac{2\nu_g}{U_g \kappa + v_g \sqrt{g^2 - \kappa^2}}, \\
\xi_{11}(r, t) &= -\frac{\kappa \eta'(r, t) + \sqrt{g^2 - \kappa^2} \eta''(r, t)}{|\Phi_1|}, \quad \xi_{gg}(r, t) = -\frac{v_g (\kappa \eta'(r, t) - \sqrt{g^2 - \kappa^2} \eta''(r, t)) + \kappa U_g (\eta''(r, t) - \eta''(r, t))}{\kappa v_g |\Phi_1|}.
\end{align*}
\]

Especially at the CEP \( g = \kappa \), we find,

\[
\begin{align*}
s_1 &= s_g = \kappa, \quad \nu_{11} = \nu_{gg} = 0, \quad \nu_{gl} = -\frac{K_1 U_g}{v_g}, \quad \nu_{gg} = D_g + \frac{K_g U_g}{v_g}, \\
\lambda_{11} &= -K_1, \quad \lambda_{gg} = \lambda_{gl} = 0, \quad \lambda_{gg} = -K_g + \frac{3D_g U_g}{2v_g}, \\
t_1 &= 0, \quad t_g = \frac{U_g \kappa}{v_g}, \\
\xi_{11}(r, t) &= -\frac{\eta'(r, t)}{|\Phi_1|}, \quad \xi_{gg}(r, t) = -\frac{\kappa \eta'(r, t) + U_g (\eta''(r, t) - \eta''(r, t))}{v_g |\Phi_1|}.
\end{align*}
\]

As discussed in the main text, since the collective mode is given by (Eq. (2)),

\[
\omega_{±}(k) = \frac{1}{2} \left[ -i (\gamma + 2DK^2) \right] \left( \pm \sqrt{-\gamma^2 + 4v^2 K^2} \right),
\]

the stability condition can be determined by \( \gamma = s_1 - s_g, v^2 = \frac{1}{2} \left( 2(s_g v_g + s_1 v_{gl}) + (s_1 + s_g) (v_{gg} - v_{gl}) \right) / 2, D = (\nu_{11} + v_{gg}) / 2 \geq 0 \). For the above-obtained “−” solution, we get,

\[
\begin{align*}
\gamma &= \frac{2(g^2 - \kappa^2)}{\kappa} + \frac{2U_g \sqrt{g^2 - \kappa^2}}{v_g} \geq 0, \\
v^2 &= \frac{\kappa - U_g \sqrt{g^2 - \kappa^2}}{v_g} \left( D_g + \frac{K_g U_g}{v_g} \right) - \frac{g^2 K_1 (U_g \kappa + v_g \sqrt{g^2 - \kappa^2})}{v_g \kappa^2}, \\
D &= \frac{1}{2} \left[ D_g + \frac{K_g U_g}{v_g} \right] > 0.
\end{align*}
\]

Especially, at the CEP,

\[
v^2 = \frac{\kappa (D_g v_g + K_g U_g - K_1 U_g)}{v_g}.
\]

This shows that the stability condition \( v^2 > 0 \) is satisfied when the nonlinear saturation \( v_g \) and the diffusion constant of the gain component \( D_g \) are large enough.
For completeness, we also provide the explicit form of the nonlinear couplings \( t_\| = \sqrt{2}(t_1 + t_\perp) \) and \( \lambda_{\perp\perp}^\| = (\lambda_{\parallel} + \lambda_{g\perp} + \lambda_{g\parallel})/(2\sqrt{2}) \) at the CEP:

\[
\begin{align*}
t_\| &= \frac{\sqrt{2}U_g \xi}{v_g} (> 0), \\
\lambda_{\perp\perp}^\| &= \frac{3D_g U_g - 2(K_1 + K_\perp)v_g}{4\sqrt{2}v_g}.
\end{align*}
\]  

(S16)  

(S17)

Note how the sign of the KPZ-like coupling \( \lambda_{\perp\perp}^\| \) depends on the details of the parameters in the original GP equation. This makes it possible for the effective coupling \( \Gamma \propto t_\| (t_\| v^2 + 4\kappa^2 \lambda_{\perp\perp}^\|) \) to take either sign, leading to two distinct phases of matter, as we have discussed in the main text.

**DYNAMIC RENORMALIZATION GROUP**

We extend the perturbative dynamic renormalization group method [3] (See also Ref. [4] for the case on the KPZ equation.) to our coupled KPZ-like equation of motion (1). This is performed conveniently in the in-phase and out-of-phase basis,

\[
\delta \tilde{\theta}_s(k, \omega) = \sum_{s=\parallel,\perp} U_s \delta \theta_s(k, \omega), \tilde{\xi}_s(k, \omega) = \sum_{s=\parallel,\perp} U_s \xi_s(k, \omega).
\]  

(S18)

with

\[
U^\dagger \equiv U^\dagger(k = 0) = (u_\perp(k = 0), u_\parallel(k = 0))^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

(S19)

transforming the kernel as,

\[
\tilde{W}(k) = U W(k) U^\dagger = \begin{pmatrix} -D_{\perp\parallel} k^2 & \xi - D_{\parallel\parallel} k^2 \\ -\frac{1}{v_g^2} k^2 & -\gamma - D_{\parallel\parallel} k^2 \end{pmatrix},
\]

(S20)

with \( \zeta = s_1 + s_\parallel, D_{\perp\parallel} = (\nu_\parallel + \nu_{\perp\parallel} + \nu_{g\parallel} + \nu_{g\perp})/2, D_{\parallel\parallel} = (\nu_\parallel + \nu_{g\parallel} - \nu_{g\perp} - \nu_{g\perp})/2, \) and \( D_{\parallel\parallel} = (-\nu_\parallel + \nu_{g\parallel} + \nu_{g\perp} - \nu_{g\perp})/2 \).

The equation of motion (1) is expressed in this basis as \((s = \perp, \parallel)\),

\[
-i \omega \delta \tilde{\theta}_s(k, \omega) = \sum_{s'} \tilde{W}_{s,s'}(k) \delta \tilde{\theta}_{s'}(k, \omega) + t_s \sum_{k'} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \delta \tilde{\theta}_{s'}(k', \omega') \delta \tilde{\theta}_{s}(k - k', \omega - \omega')
\]

\[
- \sum_{s',s''} \lambda_{s',s''} \sum_{k'} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} |k'\cdot(k' - k)| \delta \tilde{\theta}_{s'}(k', \omega') \delta \tilde{\theta}_{s''}(k - k', \omega - \omega') + \tilde{\xi}_s(k, \omega).
\]

(S21)

We briefly note that the in-and-out-of phase basis used here is slightly different from the triangular basis introduced in the main text, where the present basis \( U^\dagger \) is momentum-independent. The former is more convenient (in our opinion) for technical reasons (since the massive nonlinear term only involves the out-of-phase fluctuations \( \delta \tilde{\theta}_s(k, \omega) \)), but we emphasize that the two representations are identical in the uniform limit \( k \to 0 \).

A simple power-counting analysis tells us that the diffusion constants \( D_{\perp\parallel}, D_{\parallel\parallel}, \) and \( D_{\parallel\parallel} \) are formally irrelevant when the velocity \( v \) is fixed because the former are coefficients of higher spatial derivatives. However, the average of \( D_{\perp\parallel} \) and \( D_{\parallel\parallel} \), \( D = (D_{\perp\parallel} + D_{\parallel\parallel})/2 = (\nu_\parallel + \nu_{g\parallel})/2 \), turns out to be dangerously irrelevant, in the sense that \( D \) still affects the critical properties, as we have discussed in the main text. From here on, we ignore the parts that do not affect the critical properties, by putting \( D_{\perp\parallel} = D_{\parallel\parallel} = D \) and \( D_{\parallel\parallel} = 0 \).

Equation (S21) can be rewritten in terms of Green’s function \( \tilde{G}^0 \) as,

\[
\delta \tilde{\theta}_s(k, \omega) = \sum_{s'} \tilde{G}_{s,s'}^{\dagger}(k, \omega) \tilde{\xi}_{s'}(k, \omega) + \sum_{s'} \tilde{G}_{s,s'}(k, \omega) t_s \sum_{k'} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \delta \tilde{\theta}_{s'}(k', \omega') \delta \tilde{\theta}_{s}(k - k', \omega - \omega')
\]

\[
- \sum_{s',s'',s'''} \tilde{G}_{s,s'}^{\dagger}(k, \omega) \lambda_{s',s''} \sum_{k'} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} |k'\cdot(k' - k)| \delta \tilde{\theta}_{s'}(k', \omega') \delta \tilde{\theta}_{s''}(k - k', \omega - \omega'),
\]

(S22)
where the non-perturbative Green’s function $\tilde{G}_{ss'}^0(k,\omega) = \left([-i\omega 1 - \hat{W}(k,\omega)]^{-1}\right)_{ss'}$ is given by,

$$
\tilde{G}_{ss'}^0(k,\omega) = \frac{1}{(\omega - \omega_-(k))(\omega - \omega_+(k))} \left( \frac{i\omega - \gamma - Dk^2}{\nu^2k^2/(2\kappa)} - 2\kappa \frac{i\omega - Dk^2}{\nu^2k^2/(2\kappa)} \right). \tag{S23}
$$

Here, we have used the relation $\zeta = 2\kappa$ that holds at the CEP. By iteratively substituting $\delta\hat{\theta}_s(k,\omega)$ into the right-hand side and taking the noise average, the perturbative corrections to the linearized theory can be conveniently computed by diagrammatic techniques. The obtained corrections are formalized into a dynamic renormalization group by integrating out only the fast modes with $\Lambda_c e^{-t} < |\mathbf{k}| < \Lambda_c$ and then rescaling the valuables according to

$$
\mathbf{r} \to e^{-t}\mathbf{r}, \ t \to e^{-\frac{1}{2}t}, \ \delta\hat{\theta}_s \to e^{\frac{1}{2}t}\delta\hat{\theta}_s. \tag{S24}
$$

The self energy $\Sigma_{ss'}$, defined as the correction to the Green’s function (where $\delta\hat{\theta}_s(k,\omega) \equiv \sum_{s'} G_{ss'}^0(k,\omega)\xi_{s'}(k,\omega)$) by $G_{ss'}^{-1}(k,\omega) = [G_{ss'}^0(k,\omega)]^{-1} - \Sigma_{ss'}(k,\omega)$, is given diagrammatically by Fig. S1(a) within the one-loop order. Here, we have dropped the terms that are obviously less relevant than the ones retained, by using the following properties at the CEP: (1) The most relevant propagator is the non-Hermitian-induced off-diagonal component $G_{\perp\perp}^0$, hence, for the KPZ-like nonlinear couplings $\lambda_{s',\nu}$, the diagrams associated with $G_{\parallel\parallel}^0$’s are the most relevant. (2) For the massive out-of-phase coupling $t_s$, the two out-going lines should be labeled “$\parallel$”. Thus, they may only be connected to propagators $G_{\parallel\parallel}^0$ and $G_{s\nu}^0$ (3) Since $G_{\parallel\parallel}^0$ is more relevant than $G_{\parallel\perp}$, for massive nonlinearity $t_s$, the diagrams associated with $G_{\parallel\parallel}^0$’s are the most relevant.

Computing the self-energy shown in Fig. S1(a) at $\omega = 0$ (which is all we need for our purpose), we obtain,

- $\Sigma_{\perp\perp}(k,\omega = 0) = \left[\frac{C_6}{4d} - \frac{3C_8}{8d} \gamma\right] \Xi_{\perp}Dk^2$,
- $\Sigma_{\parallel\parallel}(k,\omega = 0) = \left[\frac{C_8}{4d} - \frac{C_6}{4}\gamma\right] \Pi_{\parallel} + O(k^2)$,
- $\Sigma_{\perp\parallel}(k,\omega = 0) = \left[\frac{C_6}{4} - \frac{C_8}{16} \frac{2d}{4d} \gamma\right] \Xi_{\parallel} \frac{\nu^2k^2}{2\kappa}$,
- $\Sigma_{\parallel\perp}(k,\omega = 0) = \left[\frac{C_4}{4} - \frac{C_6}{4} \gamma\right] \Pi_{\parallel} - \left[\frac{C_8}{16} + \frac{C_{10}}{8d} \gamma\right] \Gamma Dk^2$,
where

\[
\Gamma = \frac{t_{\parallel\parallel}}{D^2} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}), \\
\Xi_{\parallel} = \frac{\kappa^2 \lambda_{\perp\perp}}{D^2 v^2} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}), \\
\Xi_{\perp} = \frac{\kappa^2 \lambda_{\perp\perp}}{D^2 v^2} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}), \\
\Pi_{\parallel} = \frac{t_{\parallel\parallel}}{D^2 v^2} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}), \\
\Pi_{\perp} = \frac{t_{\parallel\parallel}}{D^2 v^2} (t_{\parallel\parallel} v^2 + 4\kappa^2 \lambda_{\perp\perp}).
\]  

(S25)

(S26)

(S27)

(S28)

(S29)

Here, we have retained the lowest order correction in terms of \(D\), since it is a (dangerous) irrelevant parameter that flows to zero as \(l \to \infty\). Since the dressed Green’s function \(G\) is given by,

\[
G^{-1}(k, \omega) = \begin{pmatrix}
-\omega + DK^2 - \Sigma_{\perp\perp}(k) & -2\kappa - \Sigma_{\perp\perp}(k) \\
\frac{1}{2\pi} v^2 k^2 - \Sigma_{\parallel\parallel}(k) & -\omega + \gamma + DK^2 - \Sigma_{\parallel\parallel}(k)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\omega + (D + \Delta D_{\perp\perp}) k^2 & -2(\kappa + \Delta \kappa) \\
\frac{1}{2\pi} (v^2 + \Delta v^2) k^2 & -\omega + (\gamma + \Delta \gamma) + (D + \Delta D_{\parallel\parallel}) k^2
\end{pmatrix},
\]

(S30)

these self energy corrections give the nonlinear correction \((\Delta v, \Delta \kappa, \Delta D, \Delta \gamma)\) to the valuables \((v, \kappa, D, \gamma)\). The correction to the diffusion constant is given by \(\Delta D = (\Delta D_{\perp\perp} + \Delta D_{\parallel\parallel})/2\).

Now, let us examine which effective couplings are the most relevant. Since the transformation \(\sigma_{\parallel\parallel} \to \sigma_{\parallel\parallel} e^{(z-1)l}\), \(D \to De^{(z-2)l}\), \(\sigma_{\parallel\parallel} \to \sigma_{\parallel\parallel} e^{(z-d-2)l}\), \(\lambda_{\parallel\parallel} \to \lambda_{\parallel\parallel} e^{(z-2)l}\), \(t_s \to t_se^{(z)l}\),

the effective couplings change as,

\[
\Gamma \to e^{(8-d)l} \Gamma, \\
(\Xi_{\parallel}, \Xi_{\perp}) \to e^{(6-d)l} (\Xi_{\parallel}, \Xi_{\perp}), \\
(\Pi_{\parallel}, \Pi_{\perp}) \to e^{(4-d)l} (\Pi_{\parallel}, \Pi_{\perp}).
\]

(S32)

(S33)

(S34)

This tells us that the effective coupling \(\Gamma\) is more relevant than all other effective couplings, with upper critical dimension \(d_c = 8\). Below, we restrict ourselves to spatial dimensions close to the upper critical dimension \(d_c = 8\), where \(\Gamma\) is the only relevant term. In such case, the self energy reduces to,

\[
\Sigma_{\parallel\parallel}(k, \omega = 0) = - \left[ \frac{C_8}{16d} + \frac{C_{10}}{8d} \right] \Gamma Dk^2,
\]

(S35)

and \(\Sigma_{\perp\perp} = \Sigma_{\perp\perp} = \Sigma_{\perp\perp} = 0\), which gives

\[
\Delta D = \left[ \frac{C_8}{32d} + \frac{C_{10}}{16d} \right] \Gamma,
\]

(S36)

and \(\Delta \gamma = \Delta \kappa = \Delta v = 0\). The flow equations are thus given by,

\[
\frac{d\gamma}{dl} = z\gamma, \\
\frac{d\kappa}{dl} = z\kappa, \\
\frac{dv}{dl} = (z-1)v, \\
\frac{dD}{dl} = \left[ z - 2 + \left( \frac{C_8}{32d} + \frac{C_{10}}{16d} \right) \Gamma \right] D.
\]

(S37)

(S38)

(S39)

(S40)

The noise kernel correction \(\Delta \sigma_{s's'}\) and the three-point vertex corrections \(\Delta \lambda_{s's''}^s, \Delta t_s\) can be analyzed similarly by computing the diagrams shown in Figs. \(\ref{fig:3.1}\)(b), (c), and (d), respectively. It turns out that these diagrams only give
nonlinear effective couplings that are less relevant than $\Gamma$, with the maximum upper critical dimension being $d_c = 6$. Ignoring such less-relevant terms, the flow equations for the noise kernel and the three-point vertices are given by,

$$\frac{d\sigma_{ss'}}{dl} = (z - d - 2\chi)\sigma_{ss'},$$

$$\frac{d\lambda_{s_1s_2s_3}}{dl} = (z - 2 + \chi)\lambda_{s_1s_2s_3},$$

$$\frac{dt_s}{dl} = (z + \chi)t_s.$$

Combining the flow equations for $\gamma, \kappa, v, D, \sigma_\parallel, \lambda_\parallel, \lambda_\perp, \lambda_\perp$, and $t_\parallel$, we obtain the flow equations for $\tilde{\gamma} = \gamma/D$ (Eq. (7)), $A = \kappa^2\sigma_\parallel/(Dv^2)$ (Eq. (9)), and the effective coupling $\Gamma$ (Eq. (10)) presented in the main text.

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