Kink-antikink interactions in the double sine-Gordon equation and the problem of resonance frequencies

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Abstract

We studied the kink-antikink collision process for the ”double sine-Gordon” (DSG) equation in 1+1 dimensions at different values of the potential parameter $R > 0$. For small values of $R$ we discuss the problem of resonance frequencies. We give qualitative explanation of the frequency shift in comparison with the frequency of the discrete level in the potential well of isolated kink. We show that in this region of the parameter $R$ the effective long-range interaction between kink and antikink takes place.

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1. Introduction

Resonant energy exchange mechanism, that we shall consider in this our paper, was originally observed in the kink-antikink collisions for the $\lambda\phi^4_2$-theory. To examine such process one should consider initial configuration in the form of kink ($K$) and antikink ($\bar{K}$) placed at $x = \pm x_0$ ($x_0 \gg 1$) moving toward each other with some velocities $v_i$. It was found that there is critical value of the initial velocity $v_{cr} \approx 0.2598$ and at $v_i > v_{cr}$ inelastic $KK$ scattering takes place while at $v_i < v_{cr}$ kink and antikink form a bound state. This bound state then decays into small oscillations [1].

Later on, when the $KK$ collision process was studied more careful, so-called escape windows were found [2] in the range of the initial velocities $v_i < v_{cr}$. Escape windows are nothing more than some values of the initial velocity $v_i = v_n$ at which kinks escape to infinity after second collision instead of forming a bound state. This phenomenon was semiquantitative explained in Ref. [2]. The point is that the $\lambda\phi^4_2$-theory kink excitation spectrum has one zero (translational) and one non-zero (shape) mode with the frequency $\omega_1 = \sqrt{3}/2$. It was observed that the following condition is satisfied with a reasonable accuracy:

$$\omega_1 T_{12}(v_n) = \delta + 2\pi n,$$

where $T_{12}$ is the time interval between the two collisions of the kinks, $n$ is integer, $\delta$ is some constant phase. During the first $KK$ collision a part of their kinetic energy is transferred to excitation of the kink discrete mode $\omega_1$. Therefore, kinks cannot escape to infinity and only go away at some distance and collide again. If condition (1) is satisfied, part of energy which conserved in the mode $\omega_1$, is returned back to kinks translational mode (kinetic energy) and kinks can overcome the mutual attraction and go to infinity. Just this phenomenon was named "resonant energy transfer mechanism".

Note, that "higher orders" escape windows were also found. In these cases $K$ and $\bar{K}$ escape to infinity after three or more collisions. For more detailed information about solitary wave interactions in the classical field theory see Review [3].

Let's now turn our attention to the system which we shall investigate in the present paper. The double sine-Gordon (DSG) equation can be obtained from the Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi)$$

with the potential

$$V(\phi) = -\frac{4}{1 + 4|\eta|} \left( \eta \cos \phi - \cos \frac{\phi}{2} \right).$$

Parameter $\eta$ may be assigned any arbitrary real value ($-\infty < \eta < +\infty$). From Lagrangian (2) we get for the real scalar field $\phi(x, t)$ in (1+1) dimensions the following equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{2}{1 + 4|\eta|} \left( 2\eta \sin \phi - \sin \frac{\phi}{2} \right) = 0.$$
In the present work we shall consider the range $\eta > 0$. In this case it is suitable to introduce parameter $R$ related with $\eta$ by the equality:

$$\eta = \frac{1}{4} \sinh^2 R .$$

Equation (4) has static topological solution in the form of $4\pi$-kink (antikink):

$$\phi_{K(\tilde{K})}(x) = 4\pi n \pm 4 \arctan \frac{\sinh x}{\cosh R} . \quad (5)$$

The sign ”+” corresponds to the case of kink, ”–” – to the case of antikink, $n$ is integer. Eq. (5) can be rewritten in the form:

$$\phi_{K(\tilde{K})}(x) = 4\pi n \pm \left[ \phi_{SGK}(x + R) - \phi_{SGK}(R - x) \right] , \quad (5a)$$

where $\phi_{SGK}(x) = 4 \arctan \exp(x)$ is the sine-Gordon (SG) equation $2\pi$-soliton. From Eq. (5a) the physical meaning of the parameter $R$ becomes clear: DSG kink can be interpreted as a superposition of two SG solitons, separated by the distance $2R$.

The $K\bar{K}$ collision process at a variety of values of the parameter $\eta$ and the initial velocity $v_i$ was studied in details in Ref. [4]. As for the $\lambda\phi^4_2$-theory case, there is some critical velocity $v_{cr}$ below which kinks form a bound state decaying into small oscillations. Note, that in the DSG case this critical velocity is a function of the parameter ($\eta$ or $R$) [4].

It was found that in the DSG system the resonant energy exchange mechanism also takes place. As a consequence there is a system of escape windows at some values of $\eta$. Note, that there is one important difference in kinks collision processes between $\lambda\phi^4_2$ and DSG models. In the first case kinks cannot pass through each other at a valuable distance, while in the second they can travel to infinity after passing through each other. This difference is a consequence of the different structure of the potential $V(\phi)$.

In Ref. [4] different values of the parameter $R$ were studied. At $R = 1.2$ a typical picture of escape windows was similar to the $\lambda\phi^4_2$-theory case. However, at smaller $R$, namely at $R = 0.5$, a new phenomenon was observed in the $K\bar{K}$ collisions – so-called quasiresonances. The essence of the phenomenon is in the following. At all velocities $v_i < v_{cr}$ we get capture and formation of $K\bar{K}$ bound state. But the time between the second and third collisions $T_{23}$ as a function of the initial velocity $v_i$ has a series of well-defined maxima, see Fig. 1. Such behavior of $T_{23}(v_i)$ means that the resonant energy exchange mechanism appears in the system, but at the same time the energy returned to the translational mode during the second collision is not enough for kinks to escape to infinity after the second collision. Besides, it turned out that the frequency of oscillations in which a part of kinetic energy is transferred while kinks passing through each other for the first time, i.e. $\omega_1$ in Eq. (1), is smaller than the frequency of the localized DSG kink excitations at given $R$. 

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2. General approach

Up to now, investigating the resonant energy exchange mechanism, people involved the localized excitations over an isolated kink (or antikink). As it will be shown, in some cases of the $K\bar{K}$ scattering such approximation is not valid. In these cases one should consider the spectrum of small excitations for the $K\bar{K}$ system as a whole. For this purpose let’s look for the solution of Eq. (4) for the field $\phi$ in the form:

$$\phi(x, t) = 2\pi + \phi_K(x - x_0) + \phi_{\bar{K}}(x + x_0) + \delta\phi(x, t).$$

Such configuration corresponds to the kink and antikink placed at $x = \pm x_0$ plus some small perturbation $\delta\phi(x, t)$, $|\delta\phi| \ll 1$. Taking into account that $\phi_K$ and $\phi_{\bar{K}}$ are solutions of Eq. (4), we get for $\delta\phi$ the following linearized equation:

$$\delta\phi_{tt} - \delta\phi_{xx} + \delta\phi \cdot \frac{\partial^2 V}{\partial\phi^2}_{\phi=2\pi+\phi_K+\phi_{\bar{K}}} = Q(x, x_0),$$

(6)

where

$$Q(x, x_0) = \left. \frac{\partial V}{\partial\phi} \right|_{\phi=\phi_K} + \left. \frac{\partial V}{\partial\phi} \right|_{\phi=\phi_{\bar{K}}} - \left. \frac{\partial V}{\partial\phi} \right|_{\phi=2\pi+\phi_K+\phi_{\bar{K}}}.\tag{7}$$

The explicit form of the function $Q(x, x_0)$ is rather cumbersome (see Appendix), but nevertheless we can make several general notes. Inhomogeneity $Q(x, x_0)$ in Eq. (6) is a consequence of the fact that the configuration "kink+antikink" is not a solution of Eq. (4). The function $Q(x, x_0)$ characterizes overlapping of the kink and antikink, because $\phi_K(x - x_0)$ and $\phi_{\bar{K}}(x + x_0)$ are exact solutions of Eq. (4) when taken separately. Obviously, $Q(x, x_0)$ is an even function of $x$ and $x_0$ and it falls down exponentially when $x_0$ increases. At fixed $x_0$ as a function of $x$ $Q(x, x_0)$ looks like two bumps with maxima at $x = \pm x_0$.

Let’s now find the excitation spectrum for $\delta\phi$. For this purpose we take Eq. (6) with zero right-hand side and look for $\delta\phi$ in the form:

$$\delta\phi(x, t) = e^{i\omega t}\chi(x).$$

Then for the function $\chi(x)$ we get the following differential equation of the Schrödinger type:

$$-\chi'' + U(x, x_0)\chi = \omega^2 \chi,$$

(8)

where

$$U(x, x_0) \equiv \left. \frac{\partial^2 V}{\partial\phi^2} \right|_{\phi=2\pi+\phi_K+\phi_{\bar{K}}}.\tag{9}$$

The explicit form of the potential $U(x, x_0)$ is rather complicated (see Appendix) and depends crucially on $x_0$. Note, that the shape of this potential depends on the parameter ($R$ or $\eta$)
and $U(x, x_0) \to 1$ when $x \to \pm \infty$. Hence, $\omega < 1$ form the discrete excitation spectrum, and $\omega > 1$ – the continuum one. In the limit $x_0 \gg 1 \ U(x, x_0)$ as a function of $x$ looks like two identical potential wells, separated by the distance $2x_0$. Each well contains one or more discrete levels which correspond to the localized excitations of the solitary kink (antikink). In the collision process DSG kinks pass through each other, i.e. $x_0$ decreases to zero and then starts to increase again. At small $x_0$ the distance between the wells is small and the discrete levels are not independent. With kinks moving toward each other from the infinity the levels begin to split and then at $x_0 \lesssim 1$ mutual potential of the system $K\bar{K}$ is quite different from one of the solitary kink (antikink).

It is worth mentioning that taking into account of both wells is also necessary in cases when in each potential well there is a discrete level with small binding energy situated near the continuum. In such cases one should take into account overlapping of the wave functions in both wells even at $x_0 \gg 1$. It means that under some conditions long-range interaction between kink and antikink appears in the system.

In what follows we will show, that within such approach the phenomenon of quasiresonances observed in the DSG system at $R = 0.5$ in Ref. [4] may be simply explained. We will also argue that the cause of the quasiresonances is just the resonant energy exchange mechanism, that leads to escape windows at some other values of $R$. Moreover, there is some intermediate region of $R$ where quasiresonances and escape windows appear together.

3. Small $R$

In Ref. [4] quasiresonances were observed at $R = 0.5$. We performed similar calculations and obtained analogous curve $T_{23}(v_i)$, see Fig. 1. Besides that, we have investigated the $K\bar{K}$ collision process at $R = 0.4$ and $R = 0.6$. At $R = 0.4$ (Fig. 2) we get a picture of quasiresonance peaks analogous to the case $R = 0.5$. At $R = 0.6$ (Fig. 3) there seem to exist escape windows in places of some peaks on the curve $T_{23}(v_i)$. It confirms that quasiresonances and escape windows are phenomena of the same nature and with the parameter $R$ increasing some quasiresonance peaks transform into the escape windows. At some intermediate values of $R$ both phenomena are presented, and in further increasing of $R$ only escape windows survive. At $R = 1.2$ in Ref. [4] a perfect picture of escape windows and no quasiresonances were observed.

To answer the question why at given $R$ quasiresonances or escape windows appear it is required, generally speaking, to solve Eq. (6) for $\delta \phi$ with the right-hand side. At the same time we can suggest some true-like hypothesis. Each bump of the source $Q(x, x_0)$ is localized on size of order of 1 (see Appendix). In the case of small $R \sim 0.4 – 0.6$ the first excited level in the well is not well-localized (binding energy is small). Therefore, integral of overlapping of $Q(x, x_0)$ and the wave function of the excited state is small. It corresponds to the fact that the part of the kinetic energy transferred to the discrete mode $\omega_1$ is small, and hence loss of energy
due to radiation is large. The situation changes with increasing of $R$. The binding energy of the first excited level is increasing, and for $R = 1.2$ the first excited level in the well is already well-localized. Because of this reason the character size of the wave function is of the same order as the source one. Hence the energy transfer mechanism is more effective in this case.

From analysis of the quasiresonance peaks of the $T_{23}(v_i)$ plot for $R = 0.5$ (Fig. 1) it follows that the frequency of the discrete mode being excited is approximately equal to $\tilde{\omega}_1 = 0.945$. At the same time in the well corresponding to one kink excitations there is discrete level with frequency $\omega_1 = 0.967$. As it will be shown, this deviation is not incidental and may be easily interpreted within our approach.

At small $R$ the system is close to the pure sine-Gordon. Therefore, the critical velocity $v_{cr}$ is small ($v_{cr} = 0$ corresponds to the pure sine-Gordon) and potential (9) in the Schrodinger equation (8) has one discrete level situated near the continuum (in the pure sine-Gordon case there is only zero mode). Presence of a shallow level implies that the corresponding wave function falls down slowly with the distance from the well. Due to this reason while studying the $K\bar{K}$ collision process it is necessary to take into account the fact that the wells affect each other even at large distance. It leads to changes in the excitation spectrum. In Fig. 4 we show how the excitation frequency of the $K\bar{K}$ system depends on the distance between $K$ and $\bar{K}$ (this distance is equal to $2x_0$). From the plot it is seen that even at $x_0 \gg 1$ there present some visible splitting of the higher discrete level. In presence of the second well this level $\omega_1 = 0.967$ splits into two sublevels: the higher with $\tilde{\omega}_1^{\text{odd}} > 0.967$ and the lower with $\tilde{\omega}_1^{\text{even}} < 0.967$. In the collision process the lower one is excited because the corresponding wave function is even. Moreover, the higher level may disappear during the collision. From Fig. 4 one can see, that this higher level appears from the continuum $\omega > 1$ at some critical distance $2(x_0)_{cr} \gg 1$ between kink and antikink. So, if the original level in a single well lies near the continuum (as it happens at small $R$), then $(x_0)_{cr}$ is very large. At the same time the level with $\tilde{\omega}_1^{\text{even}}$ exists in a relatively wide interval of distances $x_0$ between kink and antikink. Namely this level is excited during the $K\bar{K}$ collision because of the resonant energy exchange mechanism. In this case the frequency $\omega_1$ in expression (1) is indeed the averaged over different $x_0$ the frequency $\tilde{\omega}_1^{\text{even}}$. It is smaller than the frequency of the discrete mode of isolated kink, what is in correspondence with the numerical simulations.

4. Numerical calculations

We solved the second order partial differential equation (4) numerically on the lattice with $\Delta x = 0.01$. Initial conditions were taken in the form of kink and antikink (5) situated at $x = \pm 20$ moving towards each other with velocities $\mp v_i$ respectively. Moments of kink and antikink passing through each other were fixed via field behavior at the origin $x = 0$.

To find discrete levels in the potential (9) we used the fact that the wave function falls down exponentially at large distances. We took solution of the Schrodinger equation in the
form $\chi \sim \exp (x\sqrt{1-\omega^2})$ at $x = -50$ and solved numerically stationary equation (8). As a result we got $\chi$ at $x = 50$ as a function of $\omega$. When $\omega$ does not correspond to the discrete level, $\chi$ grows exponentially with $x$ at positive $x$’s. But if $\omega$ coincides with a discrete level of the potential, then $\chi$ is exponentially suppressed at large $x$. In real computations we observed that $\chi(x = 50)$ changed its sign when $\omega$ passed a discrete level. Note, that this method being applied to searching of a shallow level does not return a good result. In such a case one should take more distant starting and ending points. The origin of the problem is in the following: for a shallow level $\omega \to 1$, and exponents fall and grow very slowly with increasing of $x$.

Conclusion

This our paper presents qualitative and semiquantitative explanation of the phenomenon of quasiresonances in collisions of kink and antikink of the double sine-Gordon equation at small $R$. It is shown that the resonant energy exchange mechanism being applied in its previous form gives not satisfactory results for frequencies.

It was shown that the resonant energy exchange between kinks’ translational mode and the discrete excitations of the $K\bar{K}$ system as a whole takes place. At small $R$ it is essential because of long-range interaction in the system caused by the presence of a shallow level in the discrete spectrum of excitations of an isolated kink (antikink).

The proposed mechanism explains qualitatively the decrease of the resonance frequency $\omega_1$ in Eq. (1) at small $R$ in comparison with the discrete frequency of an isolated kink.

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Appendix

In equation
\[ \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial V}{\partial \phi} = 0, \quad (A1) \]
where \( V(\phi) \) has the form (3) let us substitute \( \phi = 2\pi + \phi_K + \phi_{\bar{K}} + \delta \phi \). Taking into account that \( \phi_K \) and \( \phi_{\bar{K}} \) are exact solutions of (A1) and linearizing with respect to \( \delta \phi \) we get:
\[ \frac{\partial^2 \delta \phi}{\partial t^2} - \frac{\partial^2 \delta \phi}{\partial x^2} + \left. \frac{\partial^2 V}{\partial \phi^2} \right|_{\phi=2\pi+\phi_K+\phi_{\bar{K}}} \delta \phi = Q(x, x_0), \quad (A2) \]
where \( Q(x, x_0) \) is given by Eq. (7). If we substitute explicit expressions for kink and antikink situated at \( x = \pm x_0 \) respectively, then we get:
\[ Q(x, x_0) = \frac{8}{1 + 4\eta} \left[ \frac{s_+ - s_-}{(1 + s_+^2)(1 + s_-^2)} \right] + 4\eta \left[ \frac{s_-(1 - s_-^2)}{(1 + s_-^2)^2} \left( 1 - \left( \frac{1 - s_-^2}{1 + s_+^2} \right)^2 \right) - \frac{s_+(1 - s_+^2)}{(1 + s_+^2)^2} \left( 1 - \left( \frac{1 - s_+^2}{1 + s_-^2} \right)^2 \right) \right] \]
(\( A3 \))

here
\[ s_{\pm} = \frac{\sinh (x \pm x_0)}{\cosh R} \]
(\( \eta \) and \( R \) are related by \( \eta = (1/4) \sinh^2 R \).)

Let us use \( Q(x, x_0) = 0 \) in (A2) and substitute \( \delta \phi(x, t) = \chi(x) \exp (i\omega t) \), then we obtain:
\[ -\chi'' + U(x, x_0) \chi = \omega^2 \chi, \]
where
\[ U(x, x_0) \equiv \frac{\partial^2 V}{\partial \phi^2} \big|_{\phi=2\pi+\phi_K+\phi_{\bar{K}}}. \]
Insert here explicit expressions for kink and antikink. As a result we have
\[ U(x, x_0) = \frac{1}{1 + 4\eta} \left[ \frac{1 - s_-^2}{1 + s_-^2} \frac{1 - s_+^2}{1 + s_+^2} + \frac{4s_-s_+}{(1 + s_-^2)(1 + s_+^2)} \right]
+ 8\eta \left[ \frac{1 - s_-^2}{1 + s_-^2} \frac{1 - s_+^2}{1 + s_+^2} + \frac{4s_-s_+}{(1 + s_-^2)(1 + s_+^2)} \right]^2 - 4\eta. \quad (A4) \]
Figure captions

Fig. 1. The time $T_{12}$ between the first two $K\bar{K}$ collisions (dashed curve) and the time $T_{23}$ between their second and third collisions (solid curve) as functions of the initial velocity $v_i$ for $R = 0.5$.

Fig. 2. The time $T_{12}$ between the first two $K\bar{K}$ collisions (dashed curve) and the time $T_{23}$ between their second and third collisions (solid curve) as functions of the initial velocity $v_i$ for $R = 0.4$.

Fig. 3. The time $T_{12}$ between the first two $K\bar{K}$ collisions (dashed curve) and the time $T_{23}$ between their second and third collisions (solid curve) as functions of the initial velocity $v_i$ for $R = 0.6$. Arrows denote probable positions of the escape windows.

Fig. 4. The excitation frequency as a function of the initial half-distance between $K$ and $\bar{K}$. $R = 0.5$.

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