CHARACTERIZING CATEGORICALLY CLOSED COMMUTATIVE SEMIGROUPS

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Abstract. Let $C$ be a class of $T_1$ topological semigroups which contains all zero-dimension al Hausdorff topological semigroups. A semigroup $X$ is called $C$-closed if $X$ is closed in each topological semigroup $Y \in C$ containing $X$ as a discrete subsemigroup; $X$ is projectively $C$-closed if for each congruence $\approx$ on $X$ the quotient semigroup $X/\approx$ is $C$-closed. A semigroup $X$ is called chain-finite if for any infinite set $I \subseteq X$ there are elements $x, y \in I$ such that $xy \notin \{x, y\}$. We prove that a semigroup $X$ is $C$-closed if it admits a homomorphism $h : X \to E$ to a chain-finite semilattice $E$ such that for every $e \in E$ the semigroup $h^{-1}(e)$ is $C$-closed. Applying this theorem, we prove that a commutative semigroup $X$ is $C$-closed if and only if $X$ is periodic, chain-finite, all subgroups of $X$ are bounded, and for any infinite set $A \subseteq X$ the product $AA$ is not a singleton. A commutative semigroup $X$ is projectively $C$-closed if and only if $X$ is chain-finite, all subgroups of $X$ are bounded and the union $H(X)$ of all subgroups in $X$ has finite complement $X \setminus H(X)$.

1. Introduction and Main Results

In many cases, completeness properties of various objects of General Topology or Topological Algebra can be characterized externally as closedness in ambient objects. For example, a metric space $X$ is complete if and only if $X$ is closed in any metric space containing $X$ as a subspace. A uniform space $X$ is complete if and only if $X$ is closed in any uniform space containing $X$ as a uniform subspace. A topological group $G$ is Raïkov complete if and only if it is closed in any topological group containing $G$ as a subgroup.

On the other hand, for topological semigroups there are no reasonable notions of (inner) completeness. Nonetheless we can define many completeness properties of semigroups via their closedness in ambient topological semigroups.

A topological semigroup is a topological space $X$ endowed with a continuous associative binary operation $X \times X \to X, (x, y) \mapsto xy$.

Definition 1.1. Let $C$ be a class of topological semigroups.

A topological semigroup $X$ is called $C$-closed if for any isomorphic topological embedding $h : X \to Y$ to a topological semigroup $Y \in C$ the image $h[X]$ is closed in $Y$.

A semigroup $X$ is called $C$-closed if so is the topological semigroup $X$ endowed with the discrete topology.

$C$-closed topological groups for various classes $C$ were investigated by many authors including Arhangel’skii, Banakh, Choban, Dikranjan, Goto, Lukašć and Uspenskij [1, 2, 11, 14, 22, 26]. Closedness of commutative topological groups in the class of Hausdorff topological semigroups was investigated by Keyantuo and Y. Zelenyuk [20, 27]. Semigroup compactifications of locally compact commutative groups were investigated by Zelenyuk [28, 29]; $C$-closed topological semilattices were investigated by Gutik, Repovš, Stepp and the authors in [3, 4, 5, 6, 8, 9, 17, 18, 23, 24]. For more information about complete topological semilattices and pospaces we refer to the recent survey of the authors [7].

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We shall be interested in the $C$-closedness for the classes:

- $T_2S$ of Hausdorff topological semigroups;
- $T_2S$ of zero-dimensional Hausdorff topological semigroups;
- $T_1S$ of topological semigroups satisfying the separation axiom $T_1$.

Recall that a topological space is zero-dimensional if it has a base of the topology consisting of clopen (= closed-and-open) subsets. A topological space $X$ satisfies the separation axiom $T_1$ if each finite subset is closed in $X$.

Since $T_2S \subseteq T_2S \subseteq T_1S$, for every semigroup we have the implications:

$$T_1S\text{-closed} \Rightarrow T_2S\text{-closed} \Rightarrow T_2S\text{-closed}.$$

From now on we assume that $C$ is a class of topological semigroups such that

$$T_2S \subseteq C \subseteq T_1S.$$

Now we recall two known characterizations of $C$-closedness for semilattices and groups.

Let us mention that a semigroup $X$ is called a semilattice if it is commutative and each element $x \in X$ is an idempotent, which means that $xx = x$. Each semilattice $E$ carries a partial order $\leq$ defined by $x \leq y$ if and only if $xy = x$. In this case $xy = \inf\{x, y\}$. Given two elements $x, y$ of a semilattice we write $x < y$ if $x \leq y$ and $x \neq y$.

A subset $C$ of a semigroup $X$ is called a chain if $xy \in \{x, y\}$ for any elements $x, y \in C$. A subset $C$ of a semilattice is a chain if and only if any elements $x, y \in C$ are comparable in the partial order $\leq$.

A semigroup $X$ is called chain-finite if $X$ contains no infinite chains. Observe that a commutative semigroup $X$ is chain-finite if and only if its maximal semilattice $E(X) = \{x \in X : xx = x\}$ is chain-finite. Lemma 2.6 from [2] implies that a chain-finite semilattice $S$ is down-complete with respect to the natural partial order $\leq$, that is every nonempty subset $A \subseteq S$ has the greatest lower bound $\inf A \in S$. In particular, every chain-finite nonempty semilattice contains the least element. Also every chain-finite semilattice contains a maximal element (which is not necessary the greatest element of the semilattice).

The following characterization of $C$-closed semilattices was proved in [3].

**Theorem 1.2.** A semilattice $X$ is $C$-closed if and only if $X$ is chain-finite.

A semigroup $X$ is defined to be

- periodic if for every $x \in X$ there exists $n \in \mathbb{N}$ such that the power $x^n$ is an idempotent;
- bounded if there exists $n \in \mathbb{N}$ such that for every $x \in X$ the power $x^n$ is an idempotent.

It is clear that each bounded semigroup is periodic (but not vice versa).

The following characterization of $C$-closed commutative groups was obtained by the first author in [2].

**Theorem 1.3.** A commutative group $X$ is $C$-closed if and only if $X$ is bounded.

In this paper we characterize $C$-closed commutative semigroups.

Our principal tool for establishing the $C$-closedness is the following theorem.

**Theorem 1.4.** A semigroup $X$ is $C$-closed if $X$ admits a homomorphism $h : X \to E$ to a chain-finite semilattice $E$ such that for every $e \in E$ the semigroup $h^{-1}(e)$ is $C$-closed.

Theorem 1.4 will be applied in the proof of the following theorem, which is one of the two main results of this paper.

**Theorem 1.5.** A commutative semigroup $X$ is $C$-closed if and only if $X$ is periodic, chain-finite, all subgroups of $X$ are bounded, and for every infinite subset $A \subseteq X$ the set $AA = \{xy : x, y \in A\}$ is not a singleton.

**Corollary 1.6.** Each subsemigroup of a $C$-closed commutative semigroup is $C$-closed.
Remark 1.7. By [2, Proposition 10], the semidirect product $\mathbb{Z} \rtimes \{-1, 1\}$ (endowed with the binary operation $(x, i) \ast (y, j) = (x + i \cdot y, i \cdot j)$) is a $C$-closed group which is not bounded. This example shows that Theorem 1.3 and Corollary 1.6 can not be generalized to non-commutative semigroups.

Example 1.8. Take any infinite cardinal $\kappa$ and endow it with the binary operation $\ast$ defined by
$$x \ast y = \begin{cases} 1 & \text{if } x \neq y \text{ and } x, y \in \kappa \setminus \{0, 1\}; \\ 0 & \text{otherwise.} \end{cases}$$

The semigroup $X = (\kappa, \ast)$ was introduced by Taimanov [25]. Gutik [16] proved that the semigroup $X$ is $T_1$-closed but the quotient semigroup $X/I$ by the ideal $I = \{0, 1\}$ is not $T_2S$-closed.

Example 1.8 shows that the $C$-closedness is not preserved by taking quotient semigroups. This observation motivates introducing the definitions of ideally and projectively $C$-closed semigroups.

Let us recall that a congruence on a semigroup $X$ is an equivalence relation $\approx$ on $X$ such that for any elements $x \approx y$ of $X$ and any $a \in X$ we have $ax \approx ay$ and $xa \approx ya$. For any congruence $\approx$ on a semigroup $X$, the quotient set $X/\approx$ has a unique semigroup structure such that the quotient map $X \to X/\approx$ is a semigroup homomorphism. The semigroup $X/\approx$ is called the quotient semigroup $X$ by the congruence $\approx$.

A subset $I$ of a semigroup $X$ is called an ideal if $IX \cup XI \subseteq I$. Every ideal $I \subseteq X$ determines the congruence $(I \times I) \cup \{(x, y) : x = y\} \subseteq X \times X$. The quotient semigroup of $X$ by this congruence is denoted by $X/I$ and called the quotient semigroup of $X$ by the ideal $I$. If $I = \emptyset$, then the quotient semigroup $X/\emptyset$ can be identified with the semigroup $X$.

A semigroup $X$ is called
- projectively $C$-closed if for any congruence $\approx$ on $X$ the quotient semigroup $X/\approx$ is $C$-closed;
- ideally $C$-closed if for any ideal $I \subseteq X$ the quotient semigroup $X/I$ is $C$-closed.

It is easy to see that for every semigroup the following implications hold:

$$\text{projectively } C\text{-closed} \Rightarrow \text{ideally } C\text{-closed} \Rightarrow C\text{-closed}.$$ 

For a semigroup $X$ the union $H(X)$ of all subgroups of $X$ is called the Clifford part of $X$. A semigroup $X$ is called
- Clifford if $X = H(X)$;
- almost Clifford if $X \setminus H(X)$ is finite.

The second main result of this paper is the following characterization.

**Theorem 1.9.** For a commutative semigroup $X$ the following conditions are equivalent:

1. $X$ is projectively $C$-closed;
2. $X$ is ideally $C$-closed;
3. the semigroup $X$ is chain-finite, almost Clifford, and all subgroups of $X$ are bounded.

We do not know whether the equivalence (1) $\iff$ (2) in Theorem 1.9 remains true for any (semi)group, see Question 9.2.

**Remark 1.10.** Theorem 1.9 implies that the projective $C$-closedness of commutative semigroups is inherited by subsemigroups and quotient semigroups.

Theorems 1.5 and 1.9 will be proved in Sections 6 and 8 after a preliminary work made in Sections 2–5 and 7.

### 2. The Topological Semigroup of Filters on a Semigroup

In this section for every semigroup $X$ we define the topological semigroup $\varphi(X)$ of filters on $X$, containing $X$ as a dense discrete subsemigroup. This construction is our principal tool in the proofs of non-$C$-closedness of semigroups.
We recall that a filter on a set $X$ is any family $\mathcal{F}$ of nonempty subsets of $X$, which is closed under finite intersections and taking supersets in $X$. A filter $\mathcal{F}$ is

- free if $\bigcap \mathcal{F} = \emptyset$;
- principal if $\{x\} \in \mathcal{F}$ for some $x \in X$.

A subfamily $\mathcal{B} \subseteq \mathcal{F}$ is called a base of a filter $\mathcal{F}$ if $\mathcal{F} = \{ A \subseteq X : \exists B \in \mathcal{B} \ (B \subseteq A) \}$. By $\varphi(X)$ we denote the set of all filters on $X$. The set $\varphi(X)$ is partially ordered by the inclusion relation. Maximal elements of the partially ordered set $\varphi(X)$ are called ultrafilters. It is well-known that a filter $\mathcal{F}$ on $X$ is an ultrafilter if and only if for any partition $X = U \cup V$ of $X$ either $U$ or $V$ belongs to $\mathcal{F}$. By $\beta(X) \subseteq \varphi(X)$ we denote the set of all ultrafilters on $X$.

Each point $x \in X$ will be identified with the principal ultrafilter $\mathcal{U}_x = \{ U \subseteq X : x \in U \} \in \beta(X) \subseteq \varphi(X)$. So, $X$ can be identified with the subset of $\beta(X)$ consisting of all principal ultrafilters. Thus we get the chain of inclusions $X \subseteq \beta(X) \subseteq \varphi(X)$.

The set $\varphi(X)$ carries the canonical topology generated by the base consisting of the sets

$$\langle U \rangle = \{ U \in \varphi(X) : U \subseteq U \}$$

where $U \subseteq X$ runs over subsets of $X$. It can be shown that this topology satisfies the separation axiom $T_0$, i.e., for each distinct points $x, y \in \varphi(X)$ there exists an open set which contains precisely one of them. The set $X$ of principal ultrafilters is dense in $\varphi(X)$ and for each $x \in X$ the singleton $\{ x \} = \{ F \in \varphi(X) : \{ x \} \in F \} = (\{ x \})$ is an open set in $\varphi(X)$. So, $X$ is a dense discrete subspace of $\varphi(X)$. The subspace $\beta(X)$ of ultrafilters is compact, Hausdorff, zero-dimensional, and dense in $\varphi(X)$. Consequently, each subspace of $\beta(X)$ is zero-dimensional and Tychonoff.

If $X$ is a (commutative) semigroup, then $\varphi(X)$ has a natural structure of a (commutative) topological semigroup: for any filters $U, V \in \varphi(X)$ their product $UV$ is the filter generated by the base $\{ UV : U \in U, V \in V \}$, where $UV = \{ uv : u \in U, v \in V \}$. Every neighborhood of $UV$ in $\varphi(X)$ contains a basic neighborhood $\langle UV \rangle$ for some $U \in U$ and $V \in V$. Then $U$ and $V$ are basic neighborhoods of the filters $U, V$ in $\varphi(X)$ such that $\langle U \rangle \cdot \langle V \rangle \subseteq \langle UV \rangle$, which means that $\varphi(X)$ is a topological semigroup, containing $X$ as a dense discrete subsemigroup. Observe that the product of two ultrafilters is not necessarily an ultrafilter, so $\beta(X)$ is not necessarily a subsemigroup of $\varphi(X)$.

3. Some properties of periodic semigroups

In this section we establish some properties of periodic semigroups. Let us recall that a semigroup $S$ is periodic if for every $x \in S$ there exists $n \in \mathbb{N}$ such that $x^n$ is an idempotent of $S$.

For a subset $A$ of a semigroup $S$, let

$$\sqrt[n]{A} = \{ x \in X : \exists n \in \mathbb{N} \ (x^n \in A) \}.$$

For an element $e \in S$, the set $\sqrt[n]{e}$ will be denoted by $\sqrt[e]{\mathbb{N}}$. Observe that a semigroup $S$ is periodic if and only if $S = \bigcup_{e \in E(S)} \sqrt[e]{\mathbb{N}}$, where $E(S) = \{ e \in S : ee = e \}$ is the set of idempotents of $S$.

For an element $a$ of a semigroup $S$ the set

$$H_a = \{ x \in S : (xS^1 = aS^1) \land (S^1x = S^1a) \}$$

is called the $\mathcal{H}$-class of $a$. Here $S^1 = S \cup \{ 1 \}$ where $1$ is an element such that $1x = x = 1x$ for all $x \in S^1$. We shall assume that $x^0 = 1$ for every $x \in S^1$.

By Corollary 2.2.6 [19], for every idempotent $e \in E(S)$ its $\mathcal{H}$-class $H_e$ coincides with the maximal subgroup of $S$, containing the idempotent $e$. The union

$$H(S) = \bigcup_{e \in E(S)} H_e$$

is the Clifford part of $S$.

For two subsets $A, B$ of a semigroup $S$ their product in $S$ is defined as

$$A \cdot B = \{ ab : a \in A, b \in B \}.$$
The set $A \cdot B$ will be also denoted by $AB$.

**Lemma 3.1.** For any idempotent $e$ of a semigroup $S$ we have

$$(\sqrt{\Pi e} \cdot H_e) \cup (H_e \cdot \sqrt{\Pi e}) \subseteq H_e.$$

**Proof.** Fix any element $x \in \sqrt{\Pi e}$. First we prove that $xe \in H_e$. Since $x \in \sqrt{\Pi e}$, there exists $n \in \mathbb{N}$ such that $x^n \in H_e$ and hence $x^{2n} \in H_e$. Observe that $xeS^1 = xx^nS^1 \subseteq x^nS^1 = eS^1$ and $eS^1 = x^{2n-1}S^1 = xS^1$ and hence $xeS^1 = eS^1$. By analogy we can prove that $S^1xe = S^1e$. Then $xe \in H_e$ by the definition of the $H$-class $H_e$.

Now fix any element $y \in H_e$. Since $H_e$ is a group with neutral element $e$, we obtain $xy = (xe)y \in H_eH_e = H_e$. By analogy we can prove that $yx \in H_e$.

Let $S$ be a periodic semigroup. Then for any $x \in S$ there exists $n \in \mathbb{N}$ and idempotent $e \in S$ such that $e = x^n$. Assume that there exist $m \in \mathbb{N}$ and idempotent $f \in S$ such that $x^m = f$. Then $e = e^m = (x^n)^m = x^nm = (x^m)^n = f^n = f$ which implies that the monogenic subsemigroup $x^N = \{x^n : n \in \mathbb{N}\}$ of $S$ contains a unique idempotent. This allows us to define a function $\pi : S \to E(S)$ assigning to each $x \in S$ the unique idempotent of the semigroup $x^N$.

If some element $x \in S$ belongs to a maximal subgroup $H_e \subseteq S$, then $\pi(x) \in x^N \subseteq H_e$ coincides with the unique idempotent $e$ of the group $H_e$ and hence $\pi[H_e] = \{e\}$. Therefore, $\pi[H(S)] = E(S) = \pi[S]$.

For a semigroup $S$ let

$$Z(S) = \{z \in S : \forall x \in S \ (xz = zx)\}$$

be the center of $S$.

**Proposition 3.2.** If $S$ is a periodic semigroup with $E(S) \subseteq Z(S)$, then $\pi : S \to E(S)$ is a homomorphism and $H(S)$ is a subsemigroup of $S$.

**Proof.** Since $E(S) \subseteq Z(S)$, the set $E(S)$ is a semilattice and hence $E(S)$ carries the partial order $\leq$ defined by $x \leq y$ if and only if $xy = x$.

**Claim 3.3.** For any $x \in S$ and $y \in Z(S)$ we have $\pi(xy) = \pi(x)\pi(y)$.

**Proof.** Since $S$ is periodic, there exist numbers $n, m \in \mathbb{N}$ such that $\pi(x) = x^n$ and $\pi(y) = y^m$. Since $xy = yx$, $(xy)^{nm} = x^{nm}y^{mn} = \pi(x)^n\pi(y)^m = \pi(x)\pi(y) \in E(S)$ and hence $\pi(xy) = \pi(x)\pi(y)$.

The following claim implies that $H(S)$ is a subsemigroup of $S$.

**Claim 3.4.** For any $x, y \in E(S)$ we have $H_xH_y \subseteq H_{xy}$.

**Proof.** Take any elements $a \in H_x$ and $b \in H_y$ and observe that since $x, y \in E(S) \subseteq Z(S)$ we have $abS^1 = ayS^1 = yaS^1 = yxS^1 = xyS^1$ and $S^1ab = S^1xb = S^1bx = S^1xy = S^1xy$ and hence $ab \in H_{xy}$.

**Claim 3.5.** For any $x, y \in S$ we have $\pi(x)\pi(y) \leq \pi(xy)$.

**Proof.** By Lemma 3.1, $x\pi(x) \in \sqrt{\Pi \pi(x)} \pi(x) \in H_{\pi(x)}$ and $y\pi(y) \in H_{\pi(y)}$. Then $xy\pi(x)\pi(y) = x\pi(x)y\pi(y) \in H_{\pi(x)\pi(y)} \subseteq H_{\pi(x)\pi(y)}$ according to Claim 3.2. Hence $\pi(xy\pi(x)\pi(y)) = \pi(x)\pi(y)$.

By Claim 3.3,

$$\pi(x)\pi(y) = \pi(x\pi(x)y\pi(y)) = \pi(xy\pi(x)\pi(y)) = \pi(xy\pi(x)\pi(y)),$$

which means that $\pi(x)\pi(y) \leq \pi(xy)$.

**Claim 3.6.** For any $x \in S$ and $y \in H(S)$ we have $\pi(xy) = \pi(x)\pi(y)$.

**Proof.** It follows from $y \in H(S)$ that $y \in H_{\pi(y)}$ and hence $y = y\pi(y)$. Let $y^{-1}$ be the inverse element of $y$ in the group $H_{\pi(y)}$. By Claims 3.5 and 3.3,

$$\pi(x)\pi(y) \leq \pi(xy) = \pi(xy\pi(y)) = \pi(xy\pi(y)) = \pi(xy\pi(y)) \leq \pi(xy^{-1}) = \pi(x\pi(y)) = \pi(x)\pi(y)$$

and hence $\pi(xy) = \pi(x)\pi(y)$.
Claim 3.7. For every \( x, y \in S \) we have \( \pi(xy) = \pi(yx) \).

Proof. Since \( S \) is periodic, there exists \( n \in \mathbb{N} \) such that \( (xy)^n \) and \( (yx)^n \) are idempotents. Taking into account that \( E(S) \subseteq Z(S) \), we conclude that
\[
(xy)^n = ((xy)^n)^{n+1} = ((xy)^{n+1})^n = (x(yx)^ny)^n = ((yx)^n(xy)^n = (yx)^n(xy)^n =
\]
and hence \( \pi(xy) = (xy)^n = (yx)^n = \pi(yx) \).

Claim 3.8. For every \( x, y \in S \) we have \( \pi(xy) = \pi(x)\pi(y) \).

Proof. By Claim 3.7, \( \pi(xy) = \pi(yx) \). Let \( e = \pi(xy) = \pi(yx) \). By Claim 3.5, \( \pi(x)\pi(y) \leq \pi(xy) = e \). Since \( S \) is periodic, there exists \( n \in \mathbb{N} \) such that \( (xy)^n = e = (yx)^n \). Observe that \( xeS^1 = exS^1 \subseteq eS^1 \) and \( eS^1 = eeS^1 = (xy)^n eS^1 \subseteq xeS^1 \) and hence \( xeS^1 = eS^1 \). Similarly, \( S^1 xe \subseteq S^1 e \) and \( S^1 e = S^1 ee = S^1 e(xy)^n \subseteq S^1 e(xe)^n = S^1 xe \) and hence \( xe \in H_e \). By analogy we can prove that \( ye \in H_e \). By Claim 3.4 and the inequality \( \pi(x)\pi(y) \leq e \), we finally have
\[
\pi(x)\pi(y) = \pi(x)\pi(y)e = \pi(xe)\pi(ye) = \pi(xeye) = e = \pi(xy).
\]

4. Sufficient conditions of C-closedness

In this section we prove some sufficient conditions of the C-closedness of a semigroup. We start with the following lemma that implies Theorem 1.4 announced in the introduction.

Lemma 4.1. A subsemigroup \( X \) of a topological semigroup \( Y \) is closed in \( Y \) if \( X \) admits a continuous homomorphism \( h : X \to E \) to a chain-finite discrete topological semilattice \( E \) such that for every \( e \in E \) the set \( h^{-1}(e) \) is closed in \( Y \).

Proof. To derive a contradiction, assume that \( X \) is not closed in \( Y \). So, we can fix an element \( y \in \overline{X} \setminus X \subseteq Y \). Replacing \( Y \) by \( \overline{X} \), we can assume that \( X \) is dense in \( Y \).

Claim 4.2. If for some \( a \in X^1 \), \( e \in E \), and \( n \in \mathbb{N} \) we have \( ay^n \in h^{-1}(e) \), then \( ay \in h^{-1}(e) \).

Proof. Since \( h^{-1}(e) \) is an open subspace of \( X \), there exists an open subset \( W \subseteq Y \) such that \( W \cap X = h^{-1}(e) \). Assuming that \( ay^n \in h^{-1}(e) \subseteq W \), we can find a neighborhood \( V \subseteq Y \) of \( y \) such that \( aV^n \subseteq W \). Then for every \( v \in X \cap V \) we have \( h(a^p) = h(a^q) = h(a)h(v^n) = h(ay^n) = e \) and hence \( ay \in a(X \cap V) \subseteq a(X \cap V) \subseteq h^{-1}(e) = h^{-1}(e) \).

Claim 4.3. If \( ay \in h^{-1}(e) \) for some \( a \in X^1 \) and \( e \in E \), then the point \( y \) has a neighborhood \( U \subseteq Y \) such that \( aU \subseteq h^{-1}(e) \).

Proof. Since the set \( h^{-1}(e) \) is open in \( X \), there exists an open set \( W \) in \( X \) such that \( h^{-1}(e) = X \cap W \). Since \( ay \in h^{-1}(e) \subseteq W \), there exists a neighborhood \( U \subseteq Y \) of \( y \) such that \( aU \subseteq W \). Then \( a(U \cap X) \subseteq W \cap X = h^{-1}(e) \) and \( aU \subseteq a(U \cap X) \subseteq a(U \cap X) \subseteq h^{-1}(e) = h^{-1}(e) \).

Let \( \mathcal{T}_y \) be the family of all neighborhoods of \( y \) in \( Y \). In the semilattice \( E \) consider the subset
\[
E_y = \{ e \in E : \exists U \in \mathcal{T}_y \ h[X \cap U] \subseteq U \}, \quad \{ f \in E : e \leq f \}.
\]
The set \( E_y \) contains the smallest element of the chain-finite semilattice \( E \) and hence \( E_y \) is not empty. Let \( e \) be a maximal element of \( E_y \) (which exists, because \( E \) is chain-finite) and \( W \in \mathcal{T}_y \) be a neighborhood of \( y \) such that \( h[X \cap W] \subseteq U \).

By induction we shall construct a sequence \( (v_n)_{n \in \omega} \) of points of \( W \cap X \) such that for every \( n \in \omega \) the following conditions are satisfied:
(1) $v_0 \cdots v_n y \notin h^{-1}(e) \cup h^{-1}(h(v_0 \cdots v_n))$;
(2) $e < h(v_0 \cdots v_{n+1}) < h(v_0 \cdots v_n)$.

To start the inductive construction, observe that Claim 1.2 implies $y^2 \notin h^{-1}(e)$, so we can find a neighborhood $V \subseteq W$ of $y$ such that $V V \cap h^{-1}(e) = \emptyset$. Choose any element $v_0 \in V \cap X$ and observe that $v_0 y \in V V \subseteq Y \setminus h^{-1}(e)$. Taking into account that $h(v_0) = h(v_0 h v_0) = h(v_0 v_0) \in h[X \cap V V] \subseteq E \setminus \{e\}$ and $v_0 \in V \cap X \subseteq W \cap X \subseteq h^{-1}[\tau e]$, we conclude that $h(v_0) > e$.

We claim that $v_0 y \notin h^{-1}(h(v_0))$. Assuming that $v_0 y \in h^{-1}(h(v_0)) \subseteq X$, we can apply Claim 1.3 and find a neighborhood $U \subseteq V$ of $y$ such that $v_0 U \subseteq h^{-1}(h(v_0))$. By the maximality of $e$, the set $h[X \cap U]$ is not contained in $h^{-1}[\tau h(v_0)]$, so we can find an element $u \in X \cap U$ such that $h(u) \notin \tau h(v_0)$. Then $h(v_0) \neq h(v_0) \cdot h(u) = h(v_0 u) = h(v_0)$, which is a desired contradiction.

Now assume that for some $n \in \mathbb{N}$ the points $v_0, \ldots, v_{n-1} \in W \cap X$ with $v_0 \cdots v_{n-1} y \notin h^{-1}(e) \cup h^{-1}(h(v_0 \cdots v_{n-1}))$ and $h(v_0 \cdots v_{n-1}) > e$ have been constructed. Claim 1.2 implies $v_0 \cdots v_{n-1} y^2 \notin h^{-1}(e) \cup h^{-1}(h(v_0 \cdots v_{n-1})).$

Since the semigroups $h^{-1}(e)$ and $h^{-1}(h(v_0 \cdots v_{n-1}))$ are closed in $Y$, we can find a neighborhood $V \subseteq W$ of $y$ such that the set $v_0 \cdots v_{n-1} V V$ is disjoint with the closed set $h^{-1}(e) \cup h^{-1}(h(v_0 \cdots v_{n-1}))$. Choose any element $v_n \in V \cap X$. The choice of $W$ guarantees that $v_n \in V \cap X \subseteq W \cap X$ and hence $h(v_n) \in h[W \cap X] \subseteq \tau e$ and $h(v_0 \cdots v_n) = h(v_0 \cdots v_{n-1}) \cdot h(v_n) \in \tau e \cdot \tau e = \tau e$. On the other hand, $v_0 \cdots v_{n-1} y \in v_0 \cdots v_{n-1} V V$ and hence $v_0 \cdots v_n y$ does not belong to the set $h^{-1}(e) \cup h^{-1}(h(v_0 \cdots v_{n-1}))$. Also the idempotent

$$h(v_0 \cdots v_n) = h(v_0 \cdots v_{n-1} y^2) \in h[X \cap v_0 \cdots v_{n-1} V V]$$

does not belong to the set $\{e\} \cup \{h(v_0 \cdots v_{n-1})\}$, which implies that $e < h(v_0 \cdots v_n) < h(v_0 \cdots v_{n-1})$.

Finally, we show that $v_0 \cdots v_n y \notin h^{-1}(h(v_0 \cdots v_n))$. Assuming the opposite and using Claim 1.3, we can find a neighborhood $U \subseteq W$ of $y$ such that $v_0 \cdots v_n U \subseteq h^{-1}(h(v_0 \cdots v_n))$. Since $h(v_0 \cdots v_n) > e$, the maximality of the element $e$ guarantees that $h[X \cap U] \not\subseteq \tau h(v_0, \ldots, v_n)$, so we can choose an element $u \in X \cap U \setminus h^{-1}[\tau h(v_0, \ldots, v_n)]$ and conclude that $h(v_0 \cdots v_n u) = h(v_0 \cdots v_n) h(u) < h(v_0 \cdots v_n)$, which contradicts $v_0 \cdots v_n u \in v_0 \cdots v_n U \subseteq h^{-1}(h(v_0 \cdots v_n))$. This contradiction completes the inductive step.

After completing the inductive construction, we obtain a strictly decreasing sequence $(h(v_0 \cdots v_n))_{n \in \omega}$ of idempotents in $E$, which is not possible as $E$ is chain-finite.

Next we prove a sufficient condition of the $C$-closedness of a bounded semigroup.

Lemma 4.4. A bounded semigroup $X$ is $C$-closed if $E(X)$ is a $C$-closed semigroup and for every infinite set $A \subseteq X$ the set $AA$ is not a singleton.

Proof. Assuming that the semigroup $X$ is not $C$-closed, we can find an isomorphic topological embedding $h : X \rightarrow Y$ of $X$ endowed with the discrete topology to a topological semigroup $(Y, \tau) \in C \subseteq T_1 S$. By our assumption, the set $h[E(X)]$ is closed in $Y$, being a $C$-closed semigroup. Being discrete, the subspace $h[X]$ is open in its closure $\overline{h[X]}$. Identifying $X$ with its image $h[X]$ and replacing $Y$ by $\overline{h[X]}$, we conclude that $X$ is a dense open discrete subsemigroup of a topological semigroup $Y \in T_1 S$ such that $E(X)$ is closed in $Y$.

Since $X$ is bounded, there exists $n \in \mathbb{N}$ such that for every $x \in X$ the power $x^n$ is an idempotent of $X$. Pick any point $a \in Y \setminus X$. Note that $a^n \in \{x^n : x \in X\} \subseteq E(X) = E(X)$. Let $e = a^n \in E(X)$.

By the continuity of the semigroup operation, the point $a$ has a neighborhood $O_a \subseteq Y$ such that $O_a^n = \{e\}$. Let $H_e$ be the maximal subgroup of $Y$, containing $e$. Then $X \cap H_e$ is the maximal subgroup of $X$ containing $e$. For every $x \in O_a \cap X$ we get $x^n = e$ and hence $x^m \in X \cap H_e$ for all $m \geq n$ (see Lemma 3.1). We claim that for any $m \geq n$ the element $a^m$ belongs to the semigroup $X$. Taking into account that $(a^m)^n = (a^n)^m = e^m = e$, we can find a neighborhood $U \subseteq Y$ of $a^m$ such that $U^n = \{e\}$.

Next, find a neighborhood $V \subseteq O_a$ of $a$ such that $V^m \subseteq U$. It follows that $a^m$ is contained in the closure of the set $W := \{v^m : v \in V \cap X\} \subseteq X \cap H_e$. Assuming that $a^m \notin X$, we conclude that the
set $W \subseteq U \cap X \cap H_e$ is infinite. Since $W$ is a subset of the group $X \cap H_e$, the product $W^n \subseteq U^n$ is infinite and cannot be equal to the singleton $U^n = \{e\}$. This contradiction shows that $a^m \in X$ for all $m \geq n$. Then there exists a number $k \in \omega$ such that $a^{2k} \notin X$ but $a^{2k+1} \in X$. By the continuity of the semigroup operation, the point $b = a^{2k}$ has a neighborhood $O_b \subseteq Y$ such that $O_b^2 = \{b^2\} \subseteq X$. Since $b \in \overline{X} \setminus X$, the set $A = O_b \cap X$ is infinite and $AA \subseteq O_b^2 = \{b\}$ is a singleton.

Finally we establish a sufficient condition of the $C$-closedness of a periodic commutative semigroup with a unique idempotent.

Lemma 4.5. A periodic commutative semigroup $X$ with a unique idempotent $e$ is $T_1S$-closed if the maximal subgroup $H_e$ of $X$ is bounded and for every infinite set $A \subseteq X$ the set $AA$ is not a singleton.

Proof. Assume that the maximal subgroup $H_e$ of $X$ is bounded and for every infinite set $A \subseteq X$ the set $AA$ is not a singleton. To derive a contradiction, assume that $X$ is not $T_1S$-closed and hence $X$ is a non-closed discrete subsunigroup of some topological semigroup $(Y, \tau) \in T_1S$. Replacing $Y$ by the closure of $X$, we can assume that $X$ is dense and hence open in $Y$.

Claim 4.6. The semigroup $X$ is an ideal in $Y$.

Proof. Given any elements $x \in X$ and $y \in Y$, we should prove that $xy \in X$. Since $X$ is periodic, there exists $n \in \mathbb{N}$ such that $x^n = e$. Consider the set $\sqrt[n]{H_e} = \{b \in X : b^n \in H_e\}$. We claim that $\sqrt[n]{H_e}$ is an ideal in $X$. Indeed, for any $b \in \sqrt[n]{H_e}$ and $z \in X$ we have $(bz)^n = b^n z^n \in H_e z^n \subseteq H_e$ as $H_e$ is an ideal in $X$ (see Lemma 3.1). Since the group $H_e$ is bounded, the semigroup $\sqrt[E]{H_e}$ is bounded, too. By Lemma 4.4, the bounded semigroup $\sqrt[E]{H_e}$ is $T_1S$-closed and hence closed in $Y$.

Taking into account that $\sqrt[E]{H_e}$ is an ideal in $X$ and $x \in \sqrt[E]{H_e}$, we conclude that $xY = xX \subseteq \sqrt[E]{H_e} \cdot X \subseteq \sqrt[E]{H_e} = \sqrt[E]{H_e} \subseteq X$.

Take any point $y \in Y \setminus X$ and consider its orbit $y^\mathbb{N} = \{y^n : n \in \mathbb{N}\}$.

Claim 4.7. $y^\mathbb{N} \cap X = \emptyset$.

Proof. To derive a contradiction, assume that $y^n \in X$ for some $n \in \mathbb{N}$. We can assume that $n$ is the smallest number with this property. It follows that $n \geq 2$ and hence $2n-2 \geq n + (n-2) \geq n$. Then $y^{n-2} \notin X$ and $y^{2n-2} = y^n y^{n-2} \subseteq XY^1 \subseteq X$, because $X$ is an ideal in $Y$. Since $X$ is an open discrete subspace of the topological semigroup $(Y, \tau)$, there exists a neighborhood $U \subseteq \tau$ of $y$ such that $U^{2n-2} = \{y^{2n-2}\}$. Consider the set $A = (U \cap X)^{n-1}$ and observe that $AA \subseteq U^{2n-2} = \{y^{2n-2}\}$ is a singleton. On the other hand, $y^{n-1} \in A \setminus X$ which implies that the set $A$ is infinite. But the existence of such set $A$ contradicts our assumptions.

By our assumption, the maximal subgroup $H_e$ of $X$ is bounded and hence there exists $p \in \mathbb{N}$ such that $x^p = e$ for every $x \in H_e$. Consider the subsemigroup $P = \{x^p : x \in X\}$ in the semigroup $X$. It follows from $y \in \overline{X}$ that the element $y^p$ belongs to the closure of the set $P$ in $Y$.

Claim 4.8. $Pe = \{e\}$.

Proof. Given any element $x \in P$, find an element $z \in X$ such that $x = z^p$. By Lemma 3.1, the subgroup $H_e$ is an ideal in $X$, which implies $ze \in H_e$. The choice of $p$ ensures that $xe = z^p e^p = (ze)^p = e$.

Claim 4.9. For any $x \in P$ there exists $n \in \mathbb{N}$ such that $x(y^p)^m = e$ for all $m \geq n$.

Proof. Since $X$ is an ideal in $Y$ we have $xy^p \in X$. Since $X$ is an open discrete subspace of the topological semigroup $(Y, \tau)$, there exists a neighborhood $U \subseteq Y$ of $y$ such that $xU^p = \{xy^p\}$. Choose any element $u \in U \cap X$ and observe that $xu^p = \{xy^p\}$. Let $V = \{v^p : v \in U \cap X\} \subseteq P$ and
observe that $xV = \{xu^p\}$. Then $xVV = xu^pV = u^pxV = u^pxu^p = xu^{2p}$. Proceeding by induction, we can show that $xV^n = xu^{np}$ for every $n \in \mathbb{N}$. Since the semigroup $X$ is periodic, there exists $n \in \mathbb{N}$ such that $u^m = e$. Then for every $m \geq n$, we obtain

$$xV^m = xu^{mp} = xu^{np}u^{p(m-n)} \in xeP = \{e\}$$

by Claim 4.8. Then

$$x(y^p)^m \in xV^m \subseteq \overline{xV^m} = \{e\} = \{e\}.$$

Now we are able to finish the proof of Lemma 4.10. Inductively we shall construct sequences of points $(x_k)_{k \in \mathbb{N}}$ in $X$, positive integer numbers $(n_k)_{k \in \mathbb{N}}$, $(m_k)_{k \in \mathbb{N}}$ and open neighborhoods $(U_k)_{k \in \mathbb{N}}$ of $y$ in $Y$ such that for every $k \in \mathbb{N}$ the following conditions are satisfied:

(i) $x_k \in U_{k-1}$;

(ii) $x_k^{pm_k} \notin \{e\} \cup \{x_i^{pm_i} : i < k\}$, $x_k^{2pm_k} = e$, and $m_k > n_{k-1}$;

(iii) $x_k(y^p)^{n_k} = e$, $x_k^{U_k^{pm_k}} = \{e\}$, and $n_k > n_{k-1}$;

(iv) $y \in U_k \subseteq U_{k-1}$.

To start the inductive construction, choose any neighborhood $U_0 \subseteq Y$ of $y$ such that $e \notin U_0^p$. Such neighborhood exists since $e \neq y^p$ by Claim 4.7. Also put $n_0 = 2$. Now assume that for some $k \in \mathbb{N}$ and all $i < k$ we have constructed a point $x_i$, a neighborhood $U_i$ of $y$ and two numbers $n_i$, $m_i$ satisfying the inductive conditions. Since $y^N \cap X = \emptyset$, there exists a neighborhood $W \subseteq Y$ of $y$ such that $e \notin W^l$ for any natural number $l \leq p(2+2k+k_{l-1})$. Choose any point $x_k \in U_{k-1} \cap W \cap X$. Then $e \neq x_k^l$ for any natural number $l \leq p(2+2k+k_{l-1})$. Since the semigroup $X$ is periodic and has a unique idempotent $e$, there exists a number $l_k$ such that $(x_k^l)^{l_k+1} = e$. We can assume that $l_k$ is the smallest number with this property. Then $(x_k^p)^{l_k+1} \neq e$. The choice of the neighborhood $W \ni x_k$ ensures that $l_k > n_{k-1} + k + 1$.

Claim 4.10. The set $\{(x_k^p)^{l_k+1} : 0 \leq i \leq k\}$ has cardinality $k+1$.

Proof. Assuming that this set has cardinality smaller than $k+1$, we can find two numbers $i, j$ such that $l_k \leq i < j \leq l_k + k$ and $x_k^{pa} = x_k^{pj}$. The equality $x_k^{pi} = x_k^{pj} = x_k^{pa}x_k^{p(j-i)}$ implies $x_k^{pi} = x_k^{pj}x_k^{np(j-i)}$ for all $n \in \mathbb{N}$. Find a (unique) number $n_k \in \mathbb{N}$ such that $i < n(j-i) < j$. Then

$$x_k^{pm(j-i)}x_k^{pm(j-i)} = x_k^{pm(j-i)-pi}x_k^{pm(j-i)} = x_k^{pm(j-i)-pi}x_k^{pi} = x_k^{pm(j-i)}$$

and hence $x_k^{pm(j-i)}$ is an idempotent. Since the semigroup $X$ contains a unique idempotent, $x_k^{pm(j-i)} = e$ and hence $j > n(j-i) \geq l_k + k + 1$, which contradicts the choice of $j$. □

By Claim 4.10 there exists a number $j$ such that $0 \leq j \leq k$ and $(x_k^p)^{l_k+j} \notin \{e\} \cup \{x_k^{pm_i} : 1 \leq i < k\}$. Put $m_k = l_k + j$ and observe that

$$(x_k^p)^{2m_k} = (x_k^p)^{l_k+1}(x_k^p)^{l_k+2j-1} \in eP = \{e\}$$

by Claim 4.8. By Claim 4.9 there exists a number $n_k > n_{k-1}$ such that $x_k^p(y^p)^{n_k} = e$. Since $X$ is an open discrete subspace of the topological semigroup $Y$, there exists a neighborhood $U_k \subseteq U_{k-1}$ of $y$ such that $x_k^p(U_k)^{pm_k} = \{e\}$. This completes the inductive construction.

Now consider the set $A = \{x_k^{pm_k} : k \in \mathbb{N}\}$ of $P$. The inductive condition (ii) guarantees that $A$ is infinite and $a^2 = e$ for every $a \in A$. Also for any $i < j$ we have

$$x_i^{pm_i}x_j^{pm_j} = x_i^{pm_i}x_j^{pm_i}x_j^{pm_i} = x_i^{pm_i}x_j^{pm_i}x_j^{pm_i} \in x_i^p(U_i)^{pm_i}P = eP = \{e\}.$$

Therefore, $AA = \{e\}$ is a singleton. But the existence of such set $A$ is forbidden by our assumption. □
5. Some properties of $T_2S$-closed semigroups

**Lemma 5.1.** For every $T_2S$-closed semigroup $X$, its center $Z(X)$ is chain-finite.

*Proof.* To derive a contradiction, assume that the semigroup $Z(X)$ contains an infinite chain $C$. Take any free ultrafilter $U \in \beta(X) \subseteq \varphi(X)$ containing the set $C$. Since $C$ is a chain, for every set $U \subseteq C$ we have $UU = U$, which implies that $UU = U$. Let $Y$ be the smallest subsemigroup of the semigroup $\varphi(X)$, containing the set $X \cup \{U\}$. Since the set $C$ is contained in the center of the semigroup $X$ and $UU = U$, the semigroup $Y$ is equal to the set $X \cup \{xU : x \in X^1\} \subseteq \beta(X)$. Then $X$ is not $T_2S$-closed, being a proper dense subsemigroup of the Hausdorff zero-dimensional topological semigroup $Y$. \(\Box\)

**Corollary 5.2.** For a semilattice $X$ the following conditions are equivalent:

1. $X$ is projectively $C$-closed;
2. $X$ is $C$-closed;
3. $X$ is chain-finite.

*Proof.* The implication (1) $\Rightarrow$ (2) is trivial, the implication (2) $\Rightarrow$ (3) follows from Lemma 5.1. To prove that (3) $\Rightarrow$ (1), assume that the semilattice $X$ is chain-finite. First we show that a homomorphic image of $X$ is chain-finite. Assuming the contrary, pick a homomorphism $h : X \to Y$ such that the semilattice $h[X]$ contains an infinite chain $L$. Let $\{y_n : n \in \omega\}$ be an infinite subset of $L$. For each $i \in \omega$, $h^{-1}(y_i)$ is a subsemilattice of $X$. Since $X$ is chain-finite, so is the semilattice $h^{-1}(y_i)$, $i \in \omega$. Since each chain-finite semilattice has the smallest element, for each $i \in \omega$ we can consider the element $x_i = \inf h^{-1}(y_i) \in h^{-1}(y_i)$. We claim that the set $K = \{x_i : i \in \omega\}$ is an infinite chain in $X$. Clearly, $K$ is infinite. Fix any $i, j \in \omega$. With no loss of generality we can assume that $y_i y_j = y_j$. Then $x_i x_j \leq x_j$ and $x_j = \inf h^{-1}(y_j)$ the formula above implies that $x_i x_j = x_j$, witnessing that $K$ is an infinite chain which contradicts the chain-finiteness of $X$. At this point the implication (3) $\Rightarrow$ (1) follows from Lemma 4.1. \(\Box\)

**Lemma 5.3.** If a semigroup $X$ is $T_2S$-closed, then for any infinite subset $A \subseteq Z(X)$ the set $AA$ is not a singleton.

*Proof.* Assume that for some infinite set $A \subseteq Z(X)$ the product $AA$ is a singleton. Choose any free ultrafilter $U \in \beta(X)$ containing the set $A$ and observe that $UU$ is a principal ultrafilter (containing the singleton $AA$). Then the subsemigroup $Y \subseteq \varphi(X)$ generated by the set $X \cup \{U\}$ is equal to $X \cup \{xU : x \in X^1\}$ and hence is contained in $\beta(X)$. Consequently, $X$ is a non-closed subsemigroup of Hausdorff zero-dimensional topological semigroup $Y$, which means that $X$ is not $T_2S$-closed. \(\Box\)

**Lemma 5.4.** The center $Z(X)$ of any $T_2S$-closed semigroup $X$ is periodic.

*Proof.* Assuming that $Z(X)$ is not periodic, find $x \in Z(X)$ such that the powers $x^n$, $n \in \mathbb{N}$, are pairwise distinct. On the set $X$ consider the free filter $\mathcal{F}$ generated by the base consisting of the sets $x^n \mathbb{N} = \{x^{nk} : k \in \mathbb{N}\}$, $n \in \mathbb{N}$.

Taking into account that $(n+1)!\mathbb{N} \subseteq n!\mathbb{N} + n!\mathbb{N} \subseteq n!\mathbb{N}$ for all $n \in \mathbb{N}$, we conclude that $\mathcal{F} \mathcal{F} = \mathcal{F}$, so $\mathcal{F}$ is an idempotent of the semigroup $\varphi(X)$. Let $Y = X \cup \{a\mathcal{F} : a \in X^1\}$ be the smallest subsemigroup of $\varphi(X)$ containing the set $X \cup \{\mathcal{F}\}$. We endow $Y$ with the subspace topology inherited from $\varphi(X)$. Then $Y$ is a topological semigroup, containing $X$ as a proper dense discrete subsemigroup. Since the space $\varphi(X)$ is $T_0$ it is sufficient to show that the space $Y$ is zero-dimensional, because zero-dimensional $T_0$ spaces are Hausdorff.

By $I$ denote the set of all elements $a \in X^1$ such that the function $\mathbb{N} \to X$, $n \mapsto ax^n$, is injective. It is clear that for every $a \in I$ the filter $a\mathcal{F}$ is free and hence does not belong to the set $X \subseteq Y$ of principal ultrafilters.

**Claim 5.5.** For any $a \in X \setminus I$ the filter $a\mathcal{F}$ is principal.
Proof. By the definition of the set $I$, there are two numbers $n, k \in \mathbb{N}$ such that $ax^n = ax^{n+k} = ax^k x$ and hence $ax^n = ax^{n+k}$ for all $i \in \mathbb{N}$. Find a number $j \in \mathbb{N}$ such that $0 \leq kj - n < k$ and observe that for every integer number $i > j$ we get $ax^{kj} = ax^{kj-n} x^{k(i-j)} = x^{kj-n} ax^n x^{k(i-j)} = x^{kj-n} ax^n = ax^{kj}$. Consequently, for the set $F = \{x^{kj} : i > j\} \in F$ the set $aF = \{ax^{kj}\}$ is a singleton, which implies that the filter $aF$ is principal. 

Claim 5.6. The topological semigroup $Y$ is zero-dimensional.

Proof. We need to show that for any point $y \in Y$, any neighborhood $O_y \subseteq Y$ of $y$ contains a clopen neighborhood of $y$. If $y \in X$, then $y$ is an isolated point of the space $Y$ and $\{y\}$ is an open neighborhood of $y$, contained in $O_y$. To show that the set $\{y\}$ is closed in $Y$ fix any $z \in Y \setminus \{y\}$. Claim 5.5 implies that any element of $Y$ is either a principal ultrafilter or a free filter on $X$. Anyway, there exists $T \in z$ such that $y \notin T$. It is easy to see that $\langle T \rangle$ is an open neighborhood of $z$ disjoint with $\{y\}$.

Next, assume that $y \notin X$ and hence $y = aF$ for some $a \in X^1$. By Claim 5.5 $a \in I$. Find a set $F = x^{k\mathbb{N}} \in F$ such that $\langle aF \rangle \subseteq O_y$. We claim that the basic open set $\langle aF \rangle$ is closed in $Y$. Given any point $t \in Y \setminus \langle aF \rangle$, we should find a neighborhood $O_t \subseteq Y$, which is disjoint with $\langle aF \rangle$. If $t \in X$, then the neighborhood $O_t = \{t\}$ of $t$ is disjoint with $\langle aF \rangle$ and we are done. So, we assume that $t \notin X$. In this case $t = bF$ for some $b \in I$, according to Claim 5.5.

We claim that $aF \cap bF = \emptyset$. To derive a contradiction, assume that $aF \cap bF$ contains some common element $ax^{k_{\mathbb{N}}} = bx^{k_{\mathbb{N}}}$ where $n, m \in \mathbb{N}$. Then $ax^{k_{\mathbb{N}+j}} = bx^{k_{\mathbb{N}+j}}$ for all $i \in \mathbb{N}$ and hence the symmetric difference $aF \Delta bF = (aF \setminus bF) \cup (bF \setminus aF) \subseteq (ax^{k_{\mathbb{N}}})_{i \leq n} \cup \{bx^{k_{\mathbb{N}+j}}\}_{j \leq m}$ is finite. Since the filter $t$ is free and $bF \in t$ we obtain that $aF \notin t$, which contradicts the choice of $t$.

This contradiction shows that $aF \cap bF = \emptyset$ and hence $\langle bF \rangle$ is a neighborhood of the filter $t$, disjoint with the set $\langle aF \rangle$, which implies that $\langle aF \rangle$ is clopen and the space $Y$ is zero-dimensional. 

Therefore, $X$ is not $\mathbb{T}_2S$-closed.

Lemma 5.7. Assume that $X$ is a periodic $\mathbb{T}_2S$-closed semigroup with $H(X) \subseteq Z(X)$. If $X$ contains an unbounded subgroup, then for some $e \in E(X)$ and $x \in X$ there exists an infinite set $A \subseteq x \cdot H_e$ such that $AA$ is a singleton.

Proof. To derive a contradiction, assume that $X$ contains an unbounded subgroup but for any $e \in E(X)$, $x \in X$ and an infinite set $A \subseteq x \cdot H_e$ the set $AA$ is not a singleton.

Since $E(X) \subseteq H(X) \subseteq Z(X)$, the set of idempotents $E(X)$ is a semilattice. Let $\pi : X \rightarrow E(X)$ be the map assigning to each $x \in X$ the unique idempotent of the monogenic semigroup $x^\mathbb{N}$. By Proposition 3.2 $\pi$ is a homomorphism.

Since $X$ contains an unbounded subgroup, for some idempotent $e \in E(X)$ the maximal subgroup $H_e$ containing $e$ is unbounded. By Lemma 5.1 the semilattice $E(X)$ is chain-finite. Consequently, we can find an idempotent $e$ whose maximal group $H_e$ is unbounded but for every idempotent $f < e$ the group $H_f$ is bounded.

In the semigroup $X$, consider the set $T = \bigcup \{\sqrt{\langle f \rangle} : f \in E(X), fe < e\}$.

Claim 5.8. For every $a \in T$, the set $G_a = \{x \in H_e : ax = ae\}$ is a subgroup of $H_e$ such that the quotient group $H_e/G_a$ is bounded.

Proof. Observe that for any $x, y \in G_a$ we have $axy = aey = ay = ae$, which means that $G_a$ is a subsemigroup of the group $H_e$. Since the group $H_e$ is periodic, the subsemigroup $G_a$ is a subgroup of $H_e$. It remains to prove that the quotient group $H_e/G_a$ is bounded. To derive a contradiction, assume that $H_e/G_a$ is unbounded.

Let $f = \pi(a)$. It follows from $a \in T$ that $fe < e$. Now the minimality of $e$ ensures that the group $H_{fe}$ is bounded. Then there exists $p \in \mathbb{N}$ such that $x^p = fe$ for any $x \in H_{fe}$.
Claim 5.9. For every \( x \in H_f \) and \( h \in H_e \) we have \( x^{hp} = x \).

Proof. By Proposition 3.2, \( \pi(feh) = f \pi(e)(h) = fee = fe \) and by Lemma 3.1
\[
feh = feh \cdot fe = feh \cdot \pi(feh) \in H e \pi(feh) = H_f e.
\]
Then \((feh)^p = fe \) and \( x^{hp} = (x^f e)^p = x(fe)^p = xfe = x \). \( \square \)

In the group \( H_e \) consider the subgroup \( G = \{ h^p : h \in H_e \} \). By Proposition 3.2, \( \pi(ae) = \pi(a)\pi(e) = fe \) and hence \( (ae)^n \in H_f e \) for some \( n \in \mathbb{N} \). Claim 5.9 ensures that \( a^n G = a^n (e^n G) = (ae)^n G = \{(ae)^n \} = \{a^ne\} \) is a singleton and hence \( G \subseteq G_a^n \). Let \( k \leq n \) be the smallest number such that the subgroup \( G \cap G_{a^k} \) has finite index in \( G \). We claim that \( k \neq 1 \). Assuming that \( G \cap G_{a^k} \) has finite index in \( G \), we conclude that the quotient group \( G / (G \cap G_{a^k}) \) is finite and hence bounded. Since the quotient group \( H_e / G \) is bounded, the quotient group \( H_e / (G \cap G_{a^k}) \) is bounded and so is the quotient group \( H_e / G_a \). But this contradicts our assumption. This contradiction shows that \( k \neq 1 \). The minimality of \( k \) ensures that the subgroup \( G \cap G_{a^{k-1}} \) has infinite index in \( G \). Since the group \( G \cap G_{a^k} \) has finite index in \( G \), the subgroup \( G \cap G_{a^{k-1}} \) has infinite index in the group \( G \cap G_{a^{k-1}} \). So, we can find an infinite set \( I \subseteq G \cap G_{a^k} \) such that \( x \in (G \cap G_{a^{k-1}}) \cap y(G \cap G_{a^{k-1}}) = \emptyset \) for any distinct elements \( x, y \in I \). Observe that for any distinct elements \( x, y \in I \) we have \( a^k x = a^k e = a^k y \) and \( a^k - x \neq a^k - y \) (assuming that \( a^k - x = a^k - y \), we obtain that \( a^k - e = a^k - xx^{-1} = a^k - yx^{-1} \) and hence \( yx^{-1} \in G \cap G_{a^{k-1}} \) which contradicts the choice of the set \( I \)).

Then the set \( A = a^k - I \) is infinite. We claim that \( AA \) is a singleton. Indeed, for any \( x, y \in I \) we have \( a^k - x, a^k - y = a^k - xa^k - 2y = a^k - ea^k - 2y = a^k - e \) and \( a^k - x, a^k - y = a^k - e \). Therefore, \( AA = \{a^k - e\} \). But the existence of such set \( A \) contradicts our assumption. \( \square \)

Let \( Q_\infty = \{ z \in \mathbb{C} : \exists n \in \mathbb{N} (z^n = 1) \} \) be the quasi-cyclic group, considered as a dense subgroup of the compact Hausdorff group \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \).

Claim 5.10. There exists a homomorphism \( h : H_e \to Q_\infty \) whose image \( h[H_e] \) is infinite.

Proof. By Lemma 5.4 the group \( H_e \) is periodic, so for every \( x \in H_e \) we can choose the smallest number \( p(x) \in \mathbb{N} \) such that \( x^{p(x)} = e \). Since \( H_e \) is unbounded, there is a sequence \( (x_n)_{n \in \mathbb{N}} \) of elements such that \( p(x_n) > n \cdot \prod_{k<n} p(x_k) \) for all \( n \in \mathbb{N} \).

For every \( n \in \mathbb{N} \) let \( G_n \) be the subgroup of \( H_e \) generated by the elements \( x_1, \ldots, x_n \). Let \( G_0 = \{ e \} \) be the trivial group and \( h_0 : G_0 \to \{ 1 \} \subseteq Q_\infty \) be the unique homomorphism. By induction, for every \( n \in \mathbb{N} \) we shall construct a homomorphism \( h_n : G_n \to Q_\infty \) such that \( h_n[G_{n-1}] = h_{n-1} \) and \( |h_n[G_n]| > n \). Assume that for some \( n \in \mathbb{N} \) the homomorphism \( h_{n-1} \) has been constructed. Consider the cyclic subgroup \( x_n^n \) generated by the element \( x_n \). Consider the subgroup \( Z = x_n^n \cap G_{n-1} \subseteq x_n^n \) and let \( \varphi : x_n^n \to Q_\infty \) be a homomorphism such that \( \varphi(Z) = h_{n-1}[Z] \) and \( \varphi^{-1}(1) = (h_{n-1}[Z])^{-1}(1) \).

Define the homomorphism \( h_n : G_n \to Q_\infty \) by the formula \( h_n(c x) = \varphi(c)h_{n-1}(x) \) where \( c \in x_n^n \) and \( x \in G_{n-1} \). To see that that \( h_n \) is well-defined, take any elements \( c, d \in x_n^n \) and \( x, y \in G_{n-1} \) with \( cx = dy \) and observe that \( d^{-1}c = yx^{-1} \in x_n^n \cap G_{n-1} \) and hence \( \varphi(d^{-1}c) = h_{n-1}(yx^{-1}) \), which implies the desired equality \( \varphi(c)h_{n-1}(x) = \varphi(d)h_{n-1}(y) \). So, the homomorphism \( h_n \) is well-defined. It is clear that \( h_n[G_{n-1}] = h_{n-1} \) and the image \( h_n[G_n] \) has cardinality
\[
|h_n[G_n]| \geq |h_n[x_n^n]| = |\varphi[x_n^n]| \geq \frac{|x_n^n|}{|Z|} = \frac{|x_n^n|}{|G_{n-1}|} \geq \frac{p(x_n)}{\prod_{k<n} p(x_k)} > n.
\]
After completing the inductive construction, consider the subgroup \( G = \bigcup_{n=1}^\infty G_n \subseteq H_e \) and the homomorphism \( h : G \to Q_\infty \) defined by \( h|G_n = h_n \) for all \( n \in \mathbb{N} \).

Taking into account that \( |h[G]| \geq |h_n[G_n]| > n \) for all \( n \in \mathbb{N} \), we conclude that the image \( h[G] \) is infinite. By a classical result of Baer [13, 21.1], the homomorphism \( h \) can be extended to a homomorphism \( \tilde{h} : H_e \to Q_\infty \). It is clear that the image \( \tilde{h}[H_e] \) is infinite. \( \square \)
Denote by \( \Phi \) the set of all homomorphisms from \( H_e \) to \( Q_\infty \). By the classical Baer Theorem [13, 21.1] on extending homomorphisms into divisible groups, the homomorphisms into \( Q_\infty \) separate points of \( H_e \), which implies that the homomorphism \( \varphi : H_e \rightarrow Q_\infty \), \( \varphi : x \mapsto (\varphi(x))_{\varphi \in \Phi} \), is injective. Identify the group \( H_e \) with its image \( \varphi[H_e] \subseteq Q_\infty \) in the compact topological group \( T^\Phi \) and let \( H_e \) be the closure of \( H_e \) in \( T^\Phi \).

By Claim [5, 10], the family \( \Phi \) contains a homomorphism \( h : H_e \rightarrow Q_\infty \) with infinite image \( h[H_e] \). The subgroup \( h[H_e] \), being infinite, is dense in \( T \). The homomorphism \( h \) admits a continuous extension \( \tilde{h} : H_e \rightarrow \tilde{T} \), \( \tilde{h} : (z_\varphi)_{\varphi \in \Phi} \mapsto z_h \). The compactness of \( \tilde{H}_e \) and density of \( h[H_e] = h[H_e] \) in \( T \) imply that \( \tilde{h}[H_e] = T \).

By Claim [5, 11] for every \( a \in T \) the quotient group \( H_e/G_a \) is bounded. So, we can find a number \( n_a \in N \) such that \( x^{n_a} \in G_a \) for all \( x \in H_e \). Moreover, for any non-empty finite set \( F \subseteq T \) and the number \( n_F = \prod_{a \in F} n_a \in N \), the intersection \( G_F = \bigcap_{a \in F} G_a \) contains the \( n_F \)-th power \( x^{n_F} \) of any element \( x \in H_e \).

Then for every \( y \in h[H_e] \subseteq Q_\infty \), we get \( y^{n_F} \in h[G_F] \), which implies that the subgroup \( h[G_F] \) is dense in \( T \). Let \( G_F \) be the closure of \( G_F \) in the compact topological group \( \tilde{H}_e \). The density of the subgroup \( h[G_F] \) in \( T \) implies that \( h[G_F] = h[G_F] = T \).

By the compactness, \( \tilde{h}\left[\bigcap_{F \in T^\omega} G_F \right] = \bigcap_{F \in T^\omega} \tilde{h}[G_F] = T \). So, we can fix an element \( s \in \bigcap_{F \in T^\omega} G_F \subseteq \tilde{H}_e \) whose image \( \tilde{h}(s) \in T \) has infinite order in the group \( T \). Then \( s \) also has infinite order and its orbit \( s^N \) is disjoint with the periodic group \( H_e \).

Consider the subsemigroup \( S \subseteq \tilde{H}_e \) generated by \( H_e \cup \{s\} \). Observe that \( S \subseteq \prod_{\varphi \in \Phi} Q_\varphi \) where \( Q_\varphi \) is the countable subgroup of \( T \) generated by the set \( Q_\infty \cup \{\varphi(s)\} \).

It is clear that the subspace topology \( \tilde{\tau} \) on \( S \), inherited from the topological group \( \prod_{\varphi \in \Phi} Q_\varphi \) is Tychonoff and zero-dimensional. Then the topology \( \tau' \) on \( S \) generated by the base \( \{U \cap aG_F : U \in \tilde{\tau}, a \in H_e, F \in [T]^{<\omega}\} \) is also zero-dimensional. It is easy to see that \( (S, \tau') \) is a topological semigroup and \( s \) belongs to the closure of \( H_e \) in the topology \( \tau' \). Finally, endow \( S \) with the topology \( \tau = \{U \cup D : U \in \tilde{\tau}, D \subseteq H_e\} \). The topology \( \tau \) is well-known in General Topology as the Michael modification of the topology \( \tau' \) (see [12, 5.1.22]). Since the (group) topology \( \tau' \) is zero-dimensional, so is its Michael modification \( \tau \) (see [12, 5.1.22]). Using the fact that \( S \setminus H_e \) is an ideal in \( S \), it can be shown that \( (S, \tau) \) is a zero-dimensional topological semigroup, containing \( H_e \) as a dense discrete subgroup. From now on we consider \( S \) as a topological semigroup, endowed with the topology \( \tau \).

Now consider the topological sum \( Y = S \cup (X \setminus H_e) \) of the topological space \( S \) and the discrete topological space \( X \setminus H_e \). It is clear that \( Y \) contains \( X \) as a proper dense discrete subspace.

It remains to extend the semigroup operation of \( X \) to a continuous commutative semigroup operation on \( Y \). In fact, for any \( a \in X, b \in H_e \) and \( n \in N \) we should define the product \( a(bs^n) \). By the periodicity of the semigroup \( X \), there is a number \( p \in N \) such that \( f := a^p \) is an idempotent. If \( fe < e \), then we put \( a(bs^n) = ab \). If \( fe = e \), then the element \( ae \) has power \( (ae)^p = a^pe^p = fe = e \) and hence \( ae \) belongs to the semigroup \( \sqrt[N]{H_e} \). By Lemma [5, 1] the subgroup \( H_e \) is an ideal in \( \sqrt[N]{H_e} \). Consequently, \( ae = ace \in H_e \). So, we can put \( a(bs^n) = (aebs^n) \). The choice of \( x \in \bigcap_{F \in T^\infty} G_F \) guarantees that the extended binary operation is continuous. Now the density of \( X \) in \( Y \) implies that the extended operation is commutative and associative. Since \( Y \in T_2S \), the semigroup \( X \) is not \( T_2S \)-closed, which is a desired contradiction completing the proof of Lemma [5, 1].

**Lemma 5.11.** If a \( T_2S \)-closed periodic semigroup \( X \) has \( X \cdot H(X) \subseteq Z(X) \), then each subgroup of \( X \) is bounded.

**Proof.** Assuming that \( X \) contains an unbounded subgroup, we can apply Lemma [5, 7] and find elements \( e \in E(X), x \in X \), and an infinite subset \( A \subseteq x \cdot H_e \) such that the set \( AA \) is a singleton. Since \( A \subseteq X \cdot H(X) \subseteq Z(X) \), we can apply Lemma [5, 3] and conclude that the semigroup \( X \) is not \( T_2S \)-closed. But this contradicts our assumption. \( \square \)
Lemma 5.12. Let $X$ be a $T_2S$-closed semigroup and $e \in E(X)$ be an idempotent such that the semigroup $H_e \cap Z(X)$ is bounded. Then for any sequence $(x_n)_{n \in \omega}$ in $(\sqrt[n]{e} \cap Z(X)) \setminus H_e$ there exists $n \in \omega$ such that $x_n \notin \{x_{n+1}^p : p \geq 2\}$.

Proof. To derive a contradiction assume that there exists a sequence $(x_n)_{n \in \omega}$ in $(\sqrt[n]{e} \cap Z(X)) \setminus H_e$ such that for every $n \in \mathbb{N}$ there exists $p_n \geq 2$ such that $x_{n-1} = x_{n+1}^{p_n}$. Since the semigroup $H_e \cap Z(X)$ is bounded, there exists $n_0 \in \mathbb{N}$ such that $z^{n_0}_x = e$ for every $z \in H_e \cap Z(X)$.

Consider the additive subsemigroup $Q_+ = \{x_n^{p_k} : k, n \in \mathbb{N}\}$ of the semigroup of positive rational numbers endowed with the binary operation of addition of rational numbers. Let $h : Q_+ \to \sqrt[e]{\cap} Z(X)$ be the (unique) homomorphism such that $h(\frac{1}{p_1}) = x_n$ for all $n \in \mathbb{N}$. Then $h(1) = h(\frac{p_1}{p_1}) = x_1 = x_0 \notin H_e$. By Lemma 5.1, the center $Z(X)$ of the $T_2S$-closed semigroup $X$ is periodic and hence $H_e \cap Z(X)$ is a periodic subgroup of $\sqrt[e]{\cap} Z(X)$. By Lemma 5.1, the subgroup $H_e \cap Z(X)$ is an ideal in $\sqrt[e]{\cap} Z(X)$. Consequently, the preimage $h^{-1}[H_e] = h^{-1}[H_e \cap Z(X)]$ is an upper set in $Q_+$, which means that for any points $q < r$ in $Q_+$ with $q \in h^{-1}[H_e]$ we get $r \in h^{-1}[H_e]$. Then $L = h^{-1}[\sqrt[e]{\cap} \setminus H_e]$ is a lower set, which contains 1 and hence contains the interval $Q_+ \cap (0, 1)$. We claim that the restriction $h|L$ injective. Assuming that $h(a) = h(b)$ for some distinct points $a \in L$, we can find natural numbers $k$ and $n < m$ such that $a = \frac{n}{p_1 \ldots p_k}$ and $b = \frac{m}{p_1 \ldots p_k}$. Then $x_k^n = h(a) = h(b) = x_k^m$. By Theorem 1.9 from [10], $x_k^n \in H_e$ and, therefore, $a = \frac{n}{p_1 \ldots p_k} \in h^{-1}[H_e]$. But this contradicts the choice of $a \in L \subseteq Q_+ \setminus h^{-1}[H_e]$.

Let $s = \sup L \in (0, +\infty)$ and $W = \{q \in n_e Q_+ : \frac{s}{2} < q < s\} \subseteq L$. The injectivity of $h|L$ guarantees that the set $h|[W]$ is infinite. Observe that for every points $a, b \in W$ we get $a + b > 2\frac{s}{2} = s$ and hence $h(a + b) = h_e \cap Z(X)$ and thus $h(a + b) = h(a) + e$. Find $z \in Q_+$ such that $a + b = n_e z$. Then $h(a + b) = h(n_e z) e = (h(z) e)^{n_e} e = e$ by the choice of $n_e$ and the inclusion $h(z) e \in \sqrt[e]{\cap} H_e \subseteq H_e$ (see Lemma 5.1). This implies that the infinite set $A = h|[W] \subseteq Z(X)$ has $AA = \{e\}$. Applying Lemma 5.3, we conclude that the semigroup $X$ is not $T_2S$-closed which contradicts our assumption.

6. Proof of Theorem 1.3

We should prove that a commutative semigroup $X$ is $C$-closed if and only if $X$ is periodic, chain-finite, all subgroups of $S$ are bounded and for every infinite set $A \subseteq X$ the set $AA$ is not a singleton.

The “only if” part follows from Lemmas 5.1, 5.3, 5.4 and 5.11. To prove the “if” part, assume that $X$ is periodic, chain-finite, all subgroups of $S$ are bounded and for every infinite set $A \subseteq X$ the set $AA$ is not a singleton. By the periodicity, $X = \bigcup_{e \in E(X)} \sqrt[e]{e}$. Consider the map $\pi : X \to E(X)$ assigning to each $x \in X$ the unique idempotent in the monogenic semigroup $x^\mathbb{N}$. By Proposition 3.2, the map $\pi$ is a semigroup homomorphism. By Lemma 4.5, for every idempotent $e \in E(X)$ the semigroup $\sqrt[e]{e}$ is $C$-closed. Since $X$ is chain-finite, so is the semilattice $E(X)$. Applying Lemma 4.1, we conclude that the semigroup $X$ is $C$-closed.

7. C-closedness of quotient semigroups

In this section we prove some lemmas that will be used in the proof of Theorem 1.9.

Lemma 7.1. Let $X$ be a periodic semigroup, $e \in E(X)$ and $Z_n := \{z \in Z(X) : z^n \in H_e\}$ for $n \in \mathbb{N}$. If for some $\ell \in \mathbb{N}$ the set $Z_\ell \setminus H_e$ is infinite, then there exist a finite set $F \subseteq Z_\ell$ and an infinite set $A \subseteq Z_\ell \setminus FX^1$ such that $AA \subseteq F \cup H_e \subseteq FX^1$.

Proof. Lemma 3.1 implies that $Z_n \subseteq Z_{n+1}$ for all $n \in \omega$. Let $Z_\infty = \bigcup_{n \in \mathbb{N}} Z_n = Z(X) \cap \sqrt[e]{\cap}$. By our assumption, there exists a number $\ell \in \mathbb{N}$ such that the set $Z_\ell \setminus H_e$ is infinite. We can assume that $\ell$ is the smallest number with this property. The obvious equality $Z_1 = Z(X) \cap H_e$ implies that $\ell \geq 2$ and hence the set $Z_{\ell-1} \setminus H_e$ is finite by the minimality of $\ell$.

Choose any sequence $(z_n)_{n \in \omega}$ of pairwise distinct elements of the infinite set $Z_\ell \setminus Z_{\ell-1}$. 

Claim 7.2. For every \( z \in \mathbb{Z}_\ell \) we have \( z^2 \in \mathbb{Z}_{\ell-1} \).

Proof. In the case \( \ell = 2 \), by the definition of \( \ell \), we get that \( z^2 \in Z(X) \cap H_e = Z_1 \). Assuming \( \ell > 2 \), we have that

\[
(z^2)^{\ell-2} = z^{2\ell-2} = z^\ell z^{\ell-2} \in H_e \cdot \sqrt{H_e} \subseteq H_e,
\]

as \( H_e \) is an ideal in \( \sqrt{H_e} \) by Lemma 3.1. Then \( z^2 \in \mathbb{Z}_{\ell-1} \) according to the definition of \( \mathbb{Z}_{\ell-1} \).

\[\square\]

Claim 7.3. For every \( n \in \mathbb{N} \) we have \( \mathbb{Z}_\infty \cap \mathbb{Z}_n X^1 = \mathbb{Z}_n \).

Proof. For any \( z \in \mathbb{Z}_n \) and \( x \in X^1 \), with \( zx \in \mathbb{Z}_\infty \), we should prove that \( (zx)^n \in H_e \). The inclusion \( z \in \mathbb{Z}_n \subseteq Z(X) \) implies \( z^n \in H_e \cap Z(X) \) and \( \pi(z) = e \in Z(X) \). By Claim 7.2, \( e = \pi(zx) = \pi(z) \pi(x) = e \pi(x) \) and \( \pi(e^n) = \pi(e) \pi(x) = e \) and thus \( ex^n \in \sqrt{H_e} \). Then

\[
(zx)^n = z^n x^n = (z^n e) x^n = z^n (ex^n) \in H_e \cdot \sqrt{H_e} \subseteq H_e.
\]

\[\square\]

If for some \( z \in \mathbb{Z}_\ell \), the set \( A = (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap (z X^1) \) is infinite, then for the finite set \( F = (\mathbb{Z}_{\ell-1} \setminus H_e) \cup \{ e \} \) we have

\[
A \cap F X^1 \subseteq (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap \mathbb{Z}_{\ell-1} X^1 = \emptyset
\]

by Claim 7.3. On the other hand, applying Claims 7.2 and 7.3, we obtain

\[
AA \subseteq \mathbb{Z}_\ell \cap z^2 X^1 \subseteq \mathbb{Z}_\ell \cap \mathbb{Z}_{\ell-1} X^1 \subseteq \mathbb{Z}_{\ell-1} \subseteq F \cup H_e \subseteq F X^1.
\]

Therefore, the finite set \( F \) and the infinite set \( A \) have the properties required in Lemma 7.1.

So, we assume that for every \( z \in \mathbb{Z}_\ell \), the set \( (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap (z X^1) \) is finite.

Let \( T = \{(i, j, k) \in \omega \times \omega \times \omega : i < j < k \} \) and \( \chi : T \to \{0, 1, 2\} \) be the function defined by the formula

\[
\chi(i, j, k) = \begin{cases} 
0 & \text{if } z_i z_j \in \mathbb{Z}_{\ell-1}; \\
1 & \text{if } z_i z_j \notin \mathbb{Z}_{\ell-1} \text{ and } z_i z_j \neq z_i z_k; \\
2 & \text{if } z_i z_j \notin \mathbb{Z}_{\ell-1} \text{ and } z_i z_j = z_i z_k.
\end{cases}
\]

By the Ramsey Theorem 5 in [15], there exists an infinite set \( \Omega \subseteq \omega \) such that \( \chi[T \cap \Omega^3] = \{c\} \) for some \( c \in \{0, 1, 2\} \).

If \( c = 0 \), then the infinite set \( A = \{z_n : n \in \Omega\} \) has \( AA \subseteq \mathbb{Z}_{\ell-1} \subseteq F \cup H_e \subseteq F X^1 \) for the finite set \( F = (\mathbb{Z}_{\ell-1} \setminus H_e) \cup \{ e \} \). On the other hand,

\[
A \cap F X^1 \subseteq (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap (\mathbb{Z}_{\ell-1} X^1) = \emptyset
\]

by Claim 7.3.

If \( c = 1 \), then for any \( i \in \Omega \) the set \( \{z_i z_j : i < j \in \Omega\} \) is an infinite subset of the set \( z_i Z_\ell \setminus Z_{\ell-1} \subseteq (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap (z_i X^1) \), which is finite by our assumption. Therefore, the case \( c = 1 \) is impossible.

If \( c = 2 \), then \( z_i z_j = z_i z_k \notin \mathbb{Z}_{\ell-1} \) for any numbers \( i < j < k \) in \( \Omega \). Then for every \( i < k \) in \( \Omega \) we have \( z_i z_k = z_i z^+ \) where \( i^+ = \min\{j \in \Omega : i < j\} \).

Now consider two cases.

1) The set \( A = \{z_i z^+ : i \in \Omega\} \subseteq \mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1} \) is infinite. Observe that for any numbers \( i < j \) in \( \Omega \), applying Claims 7.2 and 7.3, we obtain

\[
z_i z^+ z_j z^+ = z_i z_j z_j z^+ \in z_i^2 Z_\ell \subseteq \mathbb{Z}_{\ell-1} Z_\ell \subseteq \mathbb{Z}_{\ell-1}
\]

and \( z_i z^+ z_i z^+ \in z_i^2 Z_\ell \subseteq \mathbb{Z}_{\ell-1} Z_\ell \subseteq \mathbb{Z}_{\ell-1} \). Then \( AA \subseteq \mathbb{Z}_{\ell-1} \subseteq F \cup H_e \) for the finite set \( F = (\mathbb{Z}_{\ell-1} \setminus H_e) \cup \{ e \} \). Also \( A \cap F X^1 \subseteq (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap \mathbb{Z}_{\ell-1} X^1 = \emptyset \) by Claim 7.3.

2) The set \( C = \{z_i z^+ : i \in \Omega\} \) is finite. In this case there exists an element \( c \in C \) such that the set \( A = \{i \in \Omega : c = z_i z^+\} \) is infinite. By our assumption, the set \( (\mathbb{Z}_\ell \setminus \mathbb{Z}_{\ell-1}) \cap c X^1 \) is finite. Then, taking into account Claim 7.2, the infinite set \( A = \{z_i : i \in \Lambda\} \setminus c X^1 \) satisfies

\[
AA = \{z_i^2 : i \in \Lambda\} \cup \{z_i z_j : i, j \in \Lambda, i < j\} \subseteq \mathbb{Z}_{\ell-1} \cup \{z_i z^+ : i \in \Lambda\} = \mathbb{Z}_{\ell-1} \cup \{c\} \subseteq F \cup H_e \subseteq F X^1
\]
for the finite set $F = (Z_{\ell-1} \setminus H_e) \cup \{c, e\}$. Also

$$A \cap FX^1 \subseteq \left( (Z_e \setminus Z_{\ell-1}) \cap (Z_{\ell-1}X^1) \right) \cup (A \cap cX^1) = \emptyset.$$  

In all cases we have constructed an infinite set $A \subseteq Z_\ell$ and a finite set $F \subseteq Z_\ell$ such that $AA \subseteq F \cup H_e \subseteq FX^1$ and $A \cap FX^1 = \emptyset$. \hfill \Box

**Remark 7.4.** The finite set $F$ from the previous lemma can be forced to have at most two elements. For this we need to use one more time the Ramsey Theorem. Let $A$ be the infinite set constructed in the previous lemma. Recall that $AA \subseteq F \cup H_e$ and $A \cap FX^1 = \emptyset$. Write the set $F \cup \{e\}$ as \{f_0, f_1, \ldots, f_n\} where $f_0 = e$. The Pigeonhole Principle implies that there exists $k \leq n$ and an infinite subset $B \subseteq A$ such that $\{x^2 : x \in B\} \subseteq H_e$ if $k = 0$ and $\{x^2 : x \in B\} = \{f_k\}$ if $k > 0$. By $|B|^2$ we denote the set of all two-element subsets of $B$. Consider the function $\chi : [B]^2 \to \{0, \ldots, n\}$ defined by the formula:

$$\chi((a, b)) = \begin{cases} 0 & \text{if } ab \in H_e; \\ i & \text{if } ab = f_i \text{ for some } i \in \{1, \ldots, n\}. \end{cases}$$

By the Ramsey Theorem, there exist a number $i \in \{0, \ldots, n\}$ and an infinite subset $A' \subseteq B$ such that $\chi\{x, y\} = i$ for any distinct elements $x, y \in A'$. Then for the set $F' = \{f_k, f_i\}$ we have $A'A' \subseteq F' \cup H_e$ and $A' \cap F'X^1 \subseteq A \cap FX^1 = \emptyset$.

**Lemma 7.5.** Let $X$ be an ideally $T_\infty S$-closed semigroup such that for some $e \in E(X) \cap Z(X)$ the semigroup $H_e \cap Z(X)$ is bounded. Then the set $\left( \sqrt[n]{I_e} \cap Z(X) \right) \setminus H_e$ is finite.

**Proof.** To derive a contradiction, assume that the set $\left( \sqrt[n]{I_e} \cap Z(X) \right) \setminus H_e$ is infinite. For every $n \in \mathbb{N}$ consider the set

$$Z_n = \{z \in Z(X) : z^n \in H_e\}$$

and let $Z_\infty = \bigcup_{n \in \mathbb{N}} Z_n = Z(X) \cap \sqrt[n]{I_e}$. If for some $\ell \in \mathbb{N}$ the set $Z_\ell \setminus H_e$ is infinite, then we can apply Lemma 7.6 and find a finite set $F \subseteq Z_\ell \subseteq Z(X)$ and infinite set $A \subseteq Z_\ell$ such that $AA \subseteq F \cup H_e \subseteq FX^1$ and $A \cap FX^1 = \emptyset$. Consider the ideal $I = FX^1$ and the quotient semigroup $X/I$. Then the quotient image $q[A]$ of $A$ in $X/I$ is an infinite set in $Z(X/I)$ such that $q[A]q[A] = q[I]$ is a singleton. By Lemma 5.3 the semigroup $X/I$ is not $T_\infty S$-closed, which contradicts our assumption. This contradiction shows that for every $n \in \mathbb{N}$ the set $Z_n \setminus H_e$ is finite.

Since the semigroup $Z(X) \cap H_e$ is bounded, there exists $p \in \mathbb{N}$ such that $x^p = e$ for all $x \in Z(X) \cap H_e$. Consider the subsemigroup $P = \{z^p : z \in Z_\infty\}$ in $Z_\infty$.

**Claim 7.6.** $P \cap H_e = \{e\}$.

**Proof.** Given any element $x \in P \cap H_e$, find $z \in Z_\infty$ such that $x = z^p$. Lemma 3.1 implies that $ze \in \sqrt[n]{e} \cdot e \subseteq H_e$ and hence $(ze)^p = e$ by the choice of $p$. Then $x = xe = z^p e = (ze)^p = e$. \hfill \Box

**Claim 7.7.** For every $n \in \omega$ the set $P \setminus Z_n$ is not empty.

**Proof.** Assuming that $P \setminus Z_n = \emptyset$, we conclude that $P \subseteq Z_n$ and hence $Z_\infty \subseteq Z_{pn}$. Then the set $Z_\infty \setminus H_e \subseteq Z_{pn} \setminus H_e$ is finite, which contradicts our assumption. \hfill \Box

Consider the tree

$$T = \bigcup_{n \in \omega} \{(t_k)_{k \in n} \in P^n : t_0 = e \land (\forall k \in n \setminus \{0\} \ (t(k) \in Z_{2k} \setminus H_e \land t(k)^2 = t(k - 1)))\}$$

endowed with the partial order of inclusion of functions. Since the sets $Z_n \setminus H_e$ are finite, the tree $T$ has finitely many branching points at every vertex. On the other hand, this tree has infinite height. This follows from the fact that for every element $z \in P \setminus Z_{2k}$, there exists $n > k$ such that $z^{2n} \in P \cap H_e = \{e\}$ but $z^{2n-1} \notin H_e$. Then the sequence $(z^{2n-1})_{i \in k}$ belongs to the tree $T$. By König’s
Lemma 5.7 [21], the tree $T$ has an infinite branch which is a sequence $(z_n)_{n \in \omega}$ in $P$ such that $z_0 = e$ and $z_n^2 = z_{n-1}$, $z_n \in Z_{2^n} \setminus H_e$, for all $n \in \mathbb{N}$. But the existence of such a sequence contradicts Lemma 5.12. \qed

8. Proof of Theorem 1.9

Given a commutative semigroup $X$, we should prove the equivalence of the following conditions:

1. $X$ is projectively $\mathcal{C}$-closed;
2. $X$ is ideally $\mathcal{C}$-closed;
3. $X$ is chain-finite, almost Clifford, and all subgroups are bounded.

The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Assume that $X$ is ideally $\mathcal{C}$-closed. Then $X$ is $\mathcal{C}$-closed and by Lemmas 5.1, 5.4, 5.11, $X = Z(X)$ is chain-finite, periodic, and all subgroups of $X$ are bounded. It remains to prove that $X$ is almost Clifford. By Lemma 7.3, for every $e \in E(X)$ the set $\sqrt{\mathcal{P}_e} \setminus H_e$ is finite. Assuming that the semigroup $X$ is not almost Clifford, we conclude that the set $B = \{ e \in E(X) : \sqrt{\mathcal{P}_e} \neq H_e \}$ is infinite. Since the semilattice $E(X)$ is chain-finite, we can apply the Ramsey Theorem and find an infinite antichain $C \subseteq B$ (the latter means that $xy \notin \{x, y\}$ for any distinct elements $x, y \in C$).

Let $R = \{ e \in E(X) : \exists C \in (e < c) \}$. It is straightforward to check that $R$ is an ideal in $E(X)$. Lemma 4.1 and Proposition 5.2 imply that $J = \bigcup_{e \in R} \sqrt{\mathcal{P}_e}$ is an ideal in $X$. Let us show that the set $I = J \cup \bigcup_{e \in C} H_e$ is an ideal in $X$. Indeed, fix any $x \in X$ and $y \in I$. If $y \notin J$, then $xy \in J \subseteq I$, because $J$ is an ideal in $X$. Assume that $y \in H_e$ for some $e \in C$. Then Proposition 5.2 implies that either $xy \in \sqrt{\mathcal{P}_e} \subseteq I$ for some idempotent $f < e$, or $xy \in \sqrt{\mathcal{P}_e}$. In the latter case, since $H_e$ is an ideal in $\sqrt{\mathcal{P}_e}$, we get that $xy = yxe \in H_e \subseteq I$. So, $I$ is an ideal in $X$. Since $X$ is ideally $\mathcal{C}$-closed, the quotient semigroup $X/I$ is $\mathcal{C}$-closed. By the choice of the set $C$, for every $e \in C$ the set $\sqrt{\mathcal{P}_e} \setminus H_e$ is not empty. Let us show that the set $\sqrt{\mathcal{P}_e} \setminus H_e$ contains an element $a_e$ such that $a_e^2 \in H_e$. Pick any $b \in \sqrt{\mathcal{P}_e} \setminus H_e$. If $b^2 \in H_e$, then put $a_e = b$. Otherwise, let $n$ be the smallest integer such that $b^n \in H_e$, which exists by the periodicity of $X$. Note that $n > 2$ and $b^{n-1} \in \sqrt{\mathcal{P}_e} \setminus H_e$. Put $a_e = b^{n-1}$ and observe that

$$a_e^2 = (b^{n-1})^2 = b^{2n-2} = b^n b^{n-2} \in H_e b^{n-2} \subseteq H_e.$$ 

We claim that for the set $A = \{ a_e : e \in C \} \subseteq X \setminus I$ we have $AA \subseteq I$. Since the set $C \subseteq E(X)$ is an antichain, for every distinct $e, f \in C$ we have $ef \neq e$ and hence $ef \in R \subseteq I$. Then Proposition 5.2 implies that for every distinct $e, f \in C$, $a_e a_f \in \sqrt{\mathcal{P}_e} \vee \sqrt{\mathcal{P}_f} \subseteq \sqrt{\mathcal{P}_{ef}} \subseteq I$. Hence $AA \subseteq I$. Moreover, the image $q[A]$ of $A$ in the quotient semigroup $X/I$ is an infinite subset of $X/I$ such that $q[A] = q(A)$ is a singleton, which contradicts Lemma 5.3.

(3) $\Rightarrow$ (1) Assume that $X$ is chain-finite, almost Clifford and all subgroups of $X$ are bounded. Let us show that $X$ is periodic. Elements of $H(X)$ are periodic, since maximal subgroups are bounded. If $x \in X \setminus H(X)$, then the finiteness of $X \setminus H(X)$ and the fact that $E(X) \subseteq H(X)$ imply that for some $n \in \mathbb{N}$ the element $x^n \in H(X)$. Consequently $x^{nm} = e \in E(X)$ for some large enough $m$, because maximal subgroups are bounded. By Theorem 1.5, the projective $\mathcal{C}$-closedness of $X$ will be proved as soon as we check that for any congruence $\approx$ on $S$ the quotient semigroup $Y = X/\approx$ is periodic, chain-finite, all subgroups of $X/\approx$ are bounded and for any infinite subset $A \subseteq X/\approx$ the set $AA$ is not a singleton.

Let $q : X \rightarrow Y = X/\approx$ be the quotient homomorphism. The periodicity of $Y$ implies the periodicity of $X$. To see that $X$ is chain-finite, observe that for every $e \in E(Y)$ the (periodic) semigroup $q^{-1}(e)$ contains an idempotent. This implies that $E(Y) = q[E(X)]$. Since $X$ is chain-finite, its maximal semilattice $E(X)$ is chain-finite. By Corollary 5.2, $E(X)$ is projectively $\mathcal{C}$-closed and then so is its homomorphic image $E(Y)$. Using Corollary 5.2 one more time, we obtain that the semilattice $E(Y)$ is chain-finite. The following claim implies that the semigroup $Y$ is almost Clifford and all subgroups of $Y$ are bounded.
Claim 8.1. For any idempotent $e \in E(Y)$ there exists an idempotent $s \in E(X)$ such that $q[H_s] = H_e$.

Proof. Since $X$ is chain-finite, the semilattice $E(X) \cap q^{-1}(e)$ contains the smallest idempotent $s$. We claim that $H_e = q[H_s]$. In fact, the inclusion $q[H_s] \subseteq H_e$ is trivial. To see that $H_e \subseteq q[H_s]$, take any element $y \in H_e \subseteq Y$ and find $x \in X$ with $q(x) = y$. Find $n \in \mathbb{N}$ such that $x^n \in E(X)$ and $y^n = e$. It follows from $y = q(x)$ that $e = y^n = q(x^n)$ and hence $s = sx^n$ by the minimality of $s$. By Proposition 8.2, $\pi(sx) = \pi(s)\pi(x) = sx^n = s$ and then

$$sx = (ss)x = s(sx) = \pi(sx)sx \in H_{\pi(sx)} = H_s$$

by Lemma 8.1. Finally, $y = ey = q(s)q(x) = q(sx) \in q[H_s]$ and hence $H_e = q[H_s]$. □

It remains to prove that for any infinite subset $A \subseteq Y$ the set $AA$ is not a singleton. This follows from the next lemma.

Lemma 8.2. For any infinite set $A$ in an almost Clifford semigroup $S$ the set $AA$ is infinite.

Proof. Since $S$ is almost Clifford, the set $A \cap H(S)$ is infinite. If for some idempotent $e \in E(S)$ the intersection $A \cap H_e$ is infinite, then for any $a \in A \cap H_e$ the set $a(A \cap H_e) \subseteq (AA) \cap H_e \subseteq AA$ is infinite (since shifts in groups are injective) and hence $AA$ is not a singleton. So, assume that for every $e \in E(S)$ the intersection $A \cap H_e$ is finite.

Then the set $E = \{e \in E(S) : A \cap H_e \neq \emptyset\}$ is infinite. For every $e \in E$, fix an element $a_e \in A \cap H_e$ and observe that $a_e^2 \in AA \cap H_e$, which implies that the set $AA \supseteq \{a_e^2 : e \in E\}$ is infinite. □

9. Some open problems

In this section we ask two open problems, motivated by the results, obtained in this paper.

Question 9.1. Does there exist a $T_2S$-closed semigroup which is not $T_2S$-closed?

Question 9.2. Is every ideally $C$-closed (semi)group projectively $C$-closed?

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