Research article

Refinements of bounds for the arithmetic mean by new Seiffert-like means

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Abstract: In the article, we present the sharp upper and lower bounds for the arithmetic mean in terms of new Seiffert-like means, which give some refinements of the results obtained in [1]. As applications, two new inequalities for the sine and hyperbolic sine functions will be established.

Keywords: Seiffert-like mean; tangent mean; hyperbolic sine mean; sine mean; hyperbolic tangent mean; arithmetic mean

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1. Introduction

For two positive numbers \(a\) and \(b\), four means

\[
M_{\sin}(a, b) = \begin{cases} \frac{a - b}{2 \sin \left( \frac{a-b}{a+b} \right)} & a \neq b \\ a & a = b \end{cases}, \quad \text{(sine mean)} \quad (1.1)
\]

\[
M_{\tan}(a, b) = \begin{cases} \frac{a - b}{2 \tan \left( \frac{a-b}{a+b} \right)} & a \neq b \\ a & a = b \end{cases}, \quad \text{(tangent mean)} \quad (1.2)
\]

\[
M_{\sinh}(a, b) = \begin{cases} \frac{a - b}{2 \sinh \left( \frac{a-b}{a+b} \right)} & a \neq b \\ a & a = b \end{cases}, \quad \text{(hyperbolic sine mean)} \quad (1.3)
\]
and
\[
M_{\tanh}(a, b) = \begin{cases} \frac{a - b}{2 \tanh \left( \frac{a - b}{2} \right)} & a \neq b \\ a & a = b \end{cases} \quad \text{(hyperbolic tangent mean)} \tag{1.4}
\]
are the so-called Seiffert-like means introduced by Witkowski [2], which are the means of the form
\[
M_f(a, b) = \begin{cases} \frac{|a - b|}{2 f \left( \frac{a - b}{a + b} \right)} & a \neq b, \\ f, & a = b, \end{cases}
\tag{1.5}
\]
where \(a, b > 0\) and the function \(f : (0, 1) \mapsto \mathbb{R}\) (called Seiffert function) satisfies
\[
\frac{x}{1 + x} \leq f(x) \leq \frac{x}{1 - x}.
\]

It is worth mentioning that these new Seiffert-like means have the Seiffert functions \(\sin, \tan, \sinh\) and \(\tanh\), the inverse counterparts of which can produce the first Seiffert mean [3], second Seiffert mean [4], Neuman-Sándor mean [5] and logarithmic mean [6] by (1.5). In fact, these Seiffert-like means belong essentially to those ones constructed by trigonometric and hyperbolic functions. Such methods to create new means first appeared in [7] by Yang and embodied in several papers [8–10]. For more informations on these means, we refer to the literature in [11–23].

Sharp bounds for the Seiffert-like means and their related special functions have attracted the attention of several researchers [24–26]. In particular, the following chain of inequalities
\[
M_{\tan}(a, b) < M_{\sinh}(a, b) < A(a, b) < M_{\sin}(a, b) < M_{\tanh}(a, b)
\]
had been established in [2] for all \(a, b > 0\) with \(a \neq b\), where \(A(a, b) = (a + b)/2\) is the arithmetic mean.

Very recently, Nowicka and Witkowski [1] proved that the double inequalities
\[
M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) < A(a, b) < \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b), \tag{1.6}
\]
\[
M_{\tanh}^{1/3}(a, b)M_{\sinh}^{2/3}(a, b) < A(a, b) < \frac{1}{3}M_{\tanh}(a, b) + \frac{2}{3}M_{\sinh}(a, b) \tag{1.7}
\]
hold for all \(a, b > 0\) with \(a \neq b\).

Motivated by (1.6) and (1.7), it makes sense to ask about the optimal parameters \(\lambda_1, \lambda_2, \mu_1\) and \(\mu_2\) satisfying the following inequalities
\[
\left[ \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b) \right]^{\lambda_1} \left[ M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) \right]^{1-\lambda_1} < A(a, b)
\]
\[
< \left[ \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b) \right]^{\mu_1} \left[ M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) \right]^{1-\mu_1},
\]
\[
\left[ \frac{1}{3}M_{\tanh}(a, b) + \frac{2}{3}M_{\sinh}(a, b) \right]^{\lambda_2} \left[ M_{\tanh}^{1/3}(a, b)M_{\sinh}^{2/3}(a, b) \right]^{1-\lambda_2} < A(a, b)
\]
\[
< \left[ \frac{1}{3}M_{\tanh}(a, b) + \frac{2}{3}M_{\sinh}(a, b) \right]^{\mu_2} \left[ M_{\tanh}^{1/3}(a, b)M_{\sinh}^{2/3}(a, b) \right]^{1-\mu_2}
\]
hold for all \(a, b > 0\) with \(a \neq b\). This paper aims to answer this question.
2. Lemmas

To prove our main results we need several lemmas, which we present in this section.

**Lemma 2.1.** ([27, L’Hospital Monotone Rule]) Suppose \( f, g : (a, b) \to \mathbb{R} \) are differentiable with \( g’(x) \neq 0 \) such that \( f(a^+) = g(a^+) = 0 \) or \( f(b^-) = g(b^-) = 0 \). If \( f’/g’ \) is (strictly) increasing (decreasing) on \((a, b)\), then so is \( f/g \).

The following lemma is a useful tool for dealing with the monotonicity of the ratio of two power series. The first part of Lemma 2.2 is first established by Biernacki and Krzyz [28], while the second part comes from Yang et al. [29, Theorem 2.1]. But we cite the latest version of the second part [30, Lemma 2], where the authors have corrected a bug in the previous version [29, Theorem 2.1].

**Lemma 2.2.** ([30]) Suppose that the power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) have the radius of convergence \( r > 0 \) with \( b_n > 0 \) for all \( n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Let \( h(x) = f(x)/g(x) \) and \( H_{f,g} = (f'/g')g - f \). Then the following statements hold true:

1. If the non-constant sequences \( \{a_n/b_n\}_{n=0}^{\infty} \) is increasing (decreasing) for all \( n \geq 0 \), then \( h(x) \) is strictly increasing (decreasing) on \((0, r)\);

2. If for certain \( m \in \mathbb{N} \), the sequence \( \{a_k/b_k\}_{0 \leq k \leq m} \) and \( \{a_k/b_k\}_{k \geq m} \) both are non-constant, and they are increasing (decreasing), respectively. Then \( h(x) \) is strictly increasing (decreasing) on \((0, r)\) if and only if \( H_{f,g}(r^*) \geq (\leq)0 \). Moreover, if \( H_{f,g}(r^*) < (>)0 \), then there exists \( x_0 \in (0, r) \) such that \( h(x) \) is strictly increasing (decreasing) on \((0, x_0)\) and strictly decreasing (increasing) on \((x_0, r)\).

Let us recall the Taylor series expansions for cot \( x \) and csc \( x \), which can be found in [31].

**Lemma 2.3.** For \( |x| < \pi \), then we have the Taylor series formulas

\[
\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n} (2n)!}{2n(2n-1)!} B_{2n} x^{2n-1} \quad \text{and} \quad \csc x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} - 2}{(2n)!} B_{2n} x^{2n-1},
\]

where \( B_{2n} \) is the even-index Bernoulli numbers for \( n \in \mathbb{N} \).

For the readers’ convenience, recall from [31, p.804, 23.1.1] that the Bernoulli numbers \( B_n \) may be defined by the power series expansion

\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.
\]

The first few Bernoulli numbers \( B_{2k} \) are

\[
B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}
\]

with the property \((-1)^{k+1} B_{2k} > 0\) for \( k \geq 1 \).

**Lemma 2.4.** ([32]) For \( k \in \mathbb{N} \), the Bernoulli numbers \( B_{2k} \) satisfy

\[
\frac{2^{2k+1}}{2^{2k+1} - 1} - \frac{1}{\pi^2} < \frac{B_{2k+2}}{|B_{2k}|} < \frac{2^{2k+1} - 1}{2^{2k+2} - 1} - \frac{1}{\pi^2}.
\]
Some other Taylor series formulas for the functions involving \( \cot x \) and \( \csc x \) can be obtained from Lemma 2.3 by differentiation.

**Lemma 2.5.** Let \( B_{2n} \) be the even-index Bernoulli numbers for \( n \in \mathbb{N} \). Then

\[
\csc^2 x = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}}{(2n)!} |B_{2n}| x^{2n-2},
\]

\[
\csc^2 x \cot x = \frac{1}{x^3} - \sum_{n=1}^{\infty} \frac{n(2n+1)2^{2n+2}}{(2n+2)!} |B_{2n+2}| x^{2n-1},
\]

\[
\csc x \cot x = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{(2n-1)(2^{2n} - 2)}{(2n)!} |B_{2n}| x^{2n-2}
\]

and

\[
\csc x \cot^2 x = -\frac{1}{2x} + \frac{1}{x^3} - \sum_{n=1}^{\infty} \frac{(2n+1)((n+1)(2^{2n} - 2)|B_{2n}| - n(2^{2n+2} - 2)|B_{2n+2}|)}{(2n+2)!} x^{2n-1}
\]

for \( |x| < \pi \).

**Proof.** Differentiation yields

\[
(\cot x)' = -\csc^2 x, \quad (\cot x)'' = 2 \csc^2 x \cot x,
\]

\[
(\csc x)' = -\csc x \cot x, \quad (\csc x)'' = 2 \csc x \cot^2 x + \csc x,
\]

which in conjunction with Lemma 2.3 gives the desired results. \( \square \)

**Lemma 2.6.** Let \( \sigma = [(2 + \cos 1)(1 + 3 \cot^2 1 - 3 \cot 1)]/[2(1 - \cos 1)] = 0.8581 \cdots \). Then the function

\[
f(x) = \frac{(2 + \cos x)(3x - 2x \sin^2 x - 3 \sin x \cos x)}{2x(1 - \cos x) \sin^2 x}
\]

is strictly increasing from \((0, 1)\) onto \((4/5, \sigma)\).

**Proof.** Let

\[
f_1(x) = 2x \csc x + 6x \csc x \cot^2 x - 2x \cot x + 3x \csc^2 x \cot x - 3 \csc^2 x - 6 \csc x \cot x + 3
\]

and

\[
f_2(x) = 2x \csc x - 2x \cot x.
\]

Then we clearly see that \( f(x) = f_1(x)/f_2(x) \).

By Lemma 2.3 and Lemma 2.5, we can rewrite \( f(x) \) in terms of power series as follows

\[
f(x) = \frac{\sum_{n=1}^{\infty} a_n x^{2n}}{\sum_{n=1}^{\infty} b_n x^{2n}}
\]

(2.1)

where

\[
a_n = \frac{(2^{2n} + 2)|B_{2n}| + 6(2^{2n} - 1)|B_{2n+2}|}{(2n)!}
\]
and
\[ b_n = \frac{4(2^{2n} - 1)}{(2n)!} |B_{2n}|. \]

It can be easily seen from (2.1) and Lemma 2.2(1) that Lemma 2.6 will be proved if we can show that the sequence
\[ \left\{ \frac{a_n}{b_n} = \frac{2^{2n-1} + 1}{2(2^{2n} - 1)} + \frac{3 |B_{2n+2}|}{2 |B_{2n}|} \right\} \tag{2.2} \]
is strictly increasing for \( n \geq 1 \).

Simple calculations with (2.2) and Lemma 2.4 yield
\[
\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \left[ \frac{2^{2n+1} + 1}{2(2^{2n+2} - 1)} + \frac{3 |B_{2n+4}|}{2 |B_{2n+2}|} \right] - \left[ \frac{2^{2n-1} + 1}{2(2^{2n} - 1)} + \frac{3 |B_{2n+2}|}{2 |B_{2n}|} \right]
\]
\[
> \left[ \frac{2^{2n+1} + 1}{2(2^{2n+2} - 1)} + \frac{3(n + 2)(2n + 3) 2^{2n+1} - 1}{\pi^2 \pi^2} \right] - \left[ \frac{2^{2n-1} + 1}{2(2^{2n} - 1)} + \frac{3(n + 1)(2n + 1) 2^{2n} - 1}{\pi^2 \pi^2} \right]
\]
\[
= \frac{3\alpha_n}{4\pi^2(2^{2n-1})(2^{2n+2} - 1)(2^{2n+3} - 1)}, \tag{2.3}
\]
where
\[ \alpha_n = 2^{4n+2}[(4n + 5)2^{2n+3} + 6n^2 - 47n - (67 + 6\pi^2)] - 2^{2n}[24n^2 - 76n - (128 + 3\pi^2)] - 4(4n + 5). \tag{2.4} \]

By using the Bernoulli inequality, we obtain
\[
(4n + 5)2^{2n+3} + 6n^2 - 47n - (67 + 6\pi^2) > 8(4n + 5)(2n + 1) + 6n^2 - 47n - 127
\]
\[
= 70n^2 + 65n - 87 > 0
\]
for \( n \geq 1 \). According to this with (2.4), it follows that
\[ \alpha_n > 2^{4n+2}(70n^2 + 65n - 87) - 2^{2n}(24n^2 - 76n - 155) - 4(4n + 5) \]
\[
> 2^{2n}[2^4(70n^2 + 65n - 87) - (24n^2 - 76n - 155)] - 4(4n + 5)
\]
\[
= 2^{2n}(1096n^2 + 1116n - 1237) - 4(4n + 5)
\]
\[
> 2^2(1096n^2 + 1116n - 1237) - 4(4n + 5)
\]
\[
= 8(548n^2 + 556n - 621) > 0
\]
for \( n \geq 1 \). This together with (2.3) implies that the sequence \( \{a_n/b_n\} \) is strictly increasing for \( n \geq 1 \). So is \( f(x) \) from Lemma 2.2(1).

By L’Hopital rule, we obtain
\[
f(0^+) = \lim_{x \to 0^+} \frac{f_1'(x)}{f_2'(x)} = \frac{a_1}{b_1} = \frac{4}{5},
\]
\[
f(1^-) = \lim_{x \to 1^-} \frac{f(x)}{2(1 - \cos 1)} = \frac{(2 + \cos 1)(1 + 3 \cot^2 1 - 3 \cot 1)}{2(1 - \cos 1)} = \sigma.
\]

This completes the proof. \( \square \)
**Lemma 2.7.** Let \( \tau = [(2 + \cosh 1)(3 \coth^2 1 - 3 \coth 1 - 1)]/[2(\cosh 1 - 1)] = 0.7603 \ldots \). Then the function

\[
g(x) = \frac{(2 + \cosh x)(3x + 2x \sinh^2 x - 3 \sinh x \cosh x)}{2x(\cosh x - 1) \sinh^2 x}
\]

is strictly decreasing from \((0, 1)\) onto \((\tau, 4/5)\).

**Proof.** Let

\[
g_1(x) = 16x + 10x \cosh x + 8x \cosh(2x) + 2x \cosh(3x) - 3 \sinh(2x) - 3 \sinh(3x)
\]

and

\[
g_2(x) = 2x[2 - \cosh x - 2 \cosh(2x) + \cosh(3x)].
\]

Then it is easy to see that \(g(x) = g_1(x)/g_2(x)\).

Recall the Taylor series expansions of \(\sinh x\) and \(\cosh x\) are

\[
\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \quad \text{and} \quad \cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}.
\]

According to this, we can rewrite \(g(x)\), in terms of power series, as

\[
g(x) = \frac{\sum_{n=0}^{\infty} u_n x^{2n}}{\sum_{n=0}^{\infty} v_n x^{2n}},
\]

where

\[
u_n = \frac{2(3^{2n+4} - 2^{2n+5} - 1)}{(2n+4)!}.
\]

From (2.5) and Lemma 2.2(1), it suffices to consider the monotonicity of the sequence \([u_n/v_n]_{n=0}^{\infty}\). Simple calculations lead to

\[
\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = \frac{12\beta_n}{(2n+5)(2n+7)(3^{2n+4} - 2^{2n+5} - 1)(3^{2n+6} - 2^{2n+7} - 1)},
\]

where

\[
\beta_n = -(40n^2 + 170n + 127)6^{2n+4} + 3^{2n+4}[3^{2n+7} - 2(32n^2 + 176n + 237)]
\]

\[
-2^{2n+4}[2^{2n+10} - (24n^2 + 162n + 239)] - 1.
\]

It can be easily verified that

\[
\beta_0 = -38400 \quad \text{and} \quad \beta_1 = -2257920.
\]

We now prove that \(\beta_n > 0\) for \(n \geq 2\).
By binomial theorem, elementary calculations lead to
\[
\frac{B_n}{3^{2n+4}} = -(40n^2 + 170n + 127)2^{2n+4} + 3^{2n+7} - 2(32n^2 + 176n + 237)
\]
\[
\quad - \left( \frac{2}{3} \right)^{2n+4} [2^{2n+10} - (24n^2 + 162n + 239)] - \frac{1}{3^{2n+4}}
\]
\[
> -(40n^2 + 170n + 127)2^{2n+4} + 3^{2n+7} - 2(32n^2 + 176n + 237)
\]
\[
\quad - \left( \frac{2}{3} \right)^{8} [2^{2n+10} - (24n^2 + 162n + 239)] - \frac{1}{3^{8}}
\]
\[
= 3^{2n+7} - \frac{1}{3^{8}} [(262440n^2 + 1115370n + 849631)2^{2n+4}
\quad + 418880n^2 + 2268000n + 3048731]
\]
\[
> 3^{7} \left[ 2^{2n} + (2n)2^{2n-1} - \frac{2n(2n - 1)}{2!}2^{2n-2} + \frac{2n(2n - 1)(2n - 2)}{3!}2^{2n-3} \right]
\]
\[
\quad - \frac{1}{3^{8}} [(262440n^2 + 1115370n + 849631)2^{2n+4}
\quad + 418880n^2 + 2268000n + 3048731]
\]
\[
> \frac{1}{3^{8}} \left[ \frac{1}{4}(9565938n^3 - 2447253n^2 - 23553990n + 3019244)(1 + 2n)
\quad - (418880n^2 + 2268000n + 3048731) \right]
\]
\[
> \frac{(n - 2)(42691728 + 34639615n + 42935184n^2 + 19131876n^3)}{4 \cdot 3^{8}} > 0
\]
for \( n \geq 2 \). Combining this with (2.6) and (2.7), it follows that \( u_n/v_n \) is strictly decreasing for \( 0 \leq n \leq 2 \) and strictly increasing for \( n \geq 2 \).

Further, differentiation yields
\[
H_{g_1,g_2}(x) = \frac{g'_1(x)}{g'_2(x)}g_2(x) - g_1(x)
\]
\[
= 4x \sinh^2 x(6x \sinh^2 x + 8x \cosh x - 7 \sinh x \cosh x - 8 \sinh x + 7x)
\]
\[
\quad + 3x \cosh x + \sinh x
\]
\[
\quad - 16x + 10x \cosh x + 8x \cosh(2x) + 2x \cosh(3x)
\]
\[
\quad - 3 \sinh x - 12 \sinh(2x) - 3 \sinh(3x),
\]
which gives
\[
H_{g_1,g_2}(1) = -0.06789 \cdots < 0. \tag{2.8}
\]

Lemma 2.2(2) and (2.8) together with the piecewise monotonicity of \( u_n/v_n \) lead to the conclusion that \( g(x) \) is strictly decreasing on \((0, 1)\). Finally, since
\[
g(0^+) = \frac{c_0}{d_0} = \frac{4}{5}, \quad g(1^-) = \frac{(2 + \cosh 1)(3 \coth^2 1 - 3 \coth 1 - 1)}{2(\cosh 1 - 1)} = \tau,
\]
the proof is completed. \( \square \)

AIMS Mathematics
3. Main results

**Theorem 3.1.** The double inequality

\[
\left[ \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b) \right]^{\lambda_1} \left[ M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) \right]^{1-\lambda_1} < A(a, b)
\]

\[
< \left[ \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b) \right]^{\mu_1} \left[ M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) \right]^{1-\mu_1}
\]

holds for all \(a, b > 0\) with \(a \neq b\) if and only if \(\lambda_1 \leq 4/5\) and \(\mu_1 \geq \mu_1^* := [3 \log(\sin 1) - \log(\cos 1)]/[3 \log (2+\cos 1) - \log(\cos 1)] = 0.8386 \cdots\).

**Proof.** Since \(M_{\sin}(a, b), M_{\tan}(a, b)\) and \(A(a, b)\) are symmetric and homogeneous of degree one, without loss of generality, we may assume that \(a > b > 0\).

Let \(x = (a - b)/(a + b) \in (0, 1)\). Then from (1.1) and (1.2) we clearly see that

\[
\frac{M_{\sin}(a, b)}{A(a, b)} = \frac{x}{\sin x}, \quad \frac{M_{\tan}(a, b)}{A(a, b)} = \frac{x}{\tan x}.
\]

(3.1)

According to (3.1), we obtain

\[
\log A(a, b) - \log \left[ \frac{2}{3}M_{\sin}(a, b) + \frac{1}{3}M_{\tan}(a, b) \right] - \log \left[ M_{\sin}^{2/3}(a, b)M_{\tan}^{1/3}(a, b) \right]
\]

\[
= \frac{\log \left( \frac{\sin x}{x} \right) - \frac{1}{3} \log(\cos x)}{\log \left( \frac{2+\cos x}{3} \right) - \frac{1}{5} \log(\cos x)} := \varphi(x).
\]

(3.2)

Let

\[
\varphi_1(x) = \log \left( \frac{\sin x}{x} \right) - \frac{1}{3} \log(\cos x)\quad \text{and} \quad \varphi_2(x) = \log \left( \frac{2 + \cos x}{3} \right) - \frac{1}{3} \log(\cos x).
\]

Then we clearly see from (3.2) that \(\varphi(x) = \varphi_1(x)/\varphi_2(x)\).

Simple calculations lead to

\[
\varphi_1(0^+) = \varphi_2(0^+) = 0,
\]

(3.3)

\[
\frac{\varphi'_1(x)}{\varphi'_2(x)} = \frac{(2 + \cos x) \left( 3x - 2x \sin^2 x - 3 \sin x \cos x \right)}{2x(1 - \cos x) \sin^2 x} = f(x),
\]

(3.4)

where \(f(x)\) is defined as in Lemma 2.6.

Lemma 2.1 and Lemma 2.6 together with (3.3), (3.4) lead to the conclusion that \(\varphi(x)\) is strictly increasing on \((0, 1)\).

Therefore, Theorem 3.1 follows easily from (3.2) and the monotonicity of \(\varphi(x)\) together with

\[
\varphi(0^+) = \lim_{x \to 0^+} \varphi'_1(x)/\varphi'_2(x) = \frac{4}{5}, \quad \varphi(1^-) = \frac{3 \log(\sin 1) - \log(\cos 1)}{3 \log \left( \frac{2+\cos 1}{3} \right) - \log(\cos 1)} = \mu_1^*.
\]

\(\square\)
Theorem 3.2. The double inequality

\[
\left[ \frac{1}{3} M_{\tanh}(a, b) + \frac{2}{3} M_{\sinh}(a, b) \right]^2 < \left[ \frac{1}{3} M_{\tanh}(a, b) + \frac{2}{3} M_{\sinh}(a, b) \right]^{3 - \mu_2} < \left[ \frac{1}{3} M_{\tanh}(a, b) + \frac{2}{3} M_{\sinh}(a, b) \right]^{1 - \mu_2} < A(a, b)
\]

holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \lambda_2^* := [3 \log(\sinh 1) - \log(\cosh 1)] \\
[3 \log \left( \frac{2 + \cosh 1}{3} \right) - \log(\cosh 1)] = 0.7730 \cdots \) and \( \mu_2 \geq 4/5 \).

Proof. Since \( M_{\sinh}(a, b), M_{\tanh}(a, b) \) and \( A(a, b) \) are symmetric and homogeneous of degree one, without loss of generality, we may assume that \( a > b > 0 \).

Let \( x = (a - b)/(a + b) \in (0, 1) \). Then it is easy to see from (1.3) and (1.4) that

\[
\frac{M_{\tanh}(a, b)}{A(a, b)} = \frac{x}{\sinh x}, \quad \frac{M_{\tanh}(a, b)}{A(a, b)} = \frac{x}{\tanh x}. \tag{3.5}
\]

According to (3.1), it follows that

\[
\log A(a, b) - \log \left[ M_{\sinh}^{2/3}(a, b) M_{\tanh}^{1/3}(a, b) \right] \\
\log \left[ \frac{2}{3} M_{\sinh}(a, b) + \frac{1}{3} M_{\tanh}(a, b) \right] - \log \left[ M_{\sinh}^{2/3}(a, b) M_{\tanh}^{1/3}(a, b) \right] \\
= \frac{\log \left( \frac{\sinh x}{x} \right) - \frac{1}{3} \log(\cosh x)}{\log \left( \frac{2 + \cosh x}{3} \right) - \frac{1}{3} \log(\cosh x)} := \phi(x). \tag{3.6}
\]

Let

\[
\phi_1(x) = \log \left( \frac{\sinh x}{x} \right) - \frac{1}{3} \log(\cosh x) \quad \text{and} \quad \phi_2(x) = \log \left( \frac{2 + \cosh x}{3} \right) - \frac{1}{3} \log(\cosh x).
\]

Then we clearly see from (3.6) that \( \phi(x) = \phi_1(x)/\phi_2(x) \).

Simple calculations lead to

\[
\phi_1(0^+) = \phi_2(0^+) = 0, \tag{3.7}
\]

\[
\phi_1'(x) = \frac{(2 + \cosh x) \left( \frac{3x + 2x \sinh^2 x - 3 \sinh x \cosh x}{x} \right)}{2x(\cosh x - 1) \sinh^2 x} = g(x), \tag{3.8}
\]

where \( g(x) \) is defined as in Lemma 2.7.

Lemma 2.1 and Lemma 2.7 together with (3.7), (3.8) lead to the conclusion that \( \phi(x) \) is strictly decreasing on \((0, 1)\).

Moreover, by L’Hopital rule and (3.8), one has

\[
\phi(0^+) = \lim_{x \to 0^+} \frac{\phi_1'(x)}{\phi_2'(x)} = \frac{4}{5}, \quad \phi(1^-) = \frac{3 \log(\sinh 1) - \log(\cosh 1)}{3 \log \left( \frac{2 + \cosh 1}{3} \right) - \log(\cosh 1)} = \lambda_2^*. \tag{3.9}
\]

Therefore, Theorem 3.2 follows easily from (3.6) and (3.9). \( \square \)
As a consequence of Theorem 3.1 and Theorem 3.2, new bounds for the sine and hyperbolic sine function are given in the following corollary.

**Corollary 3.3.** Let \( \mu^*_1 \) and \( \lambda^*_2 \) be defined as in Theorem 3.1 and Theorem 3.2 respectively. Then the double inequalities

\[
\left( \frac{2 + \cos x}{3} \right)^{4/5} (\cos x)^{1/15} < \frac{\sin x}{x} < \left( \frac{2 + \cos x}{3} \right)^{\mu^*_1} (\cos x)^{1-\mu^*_1/3},
\]

\[
\left( \frac{2 + \cosh x}{3} \right)^{\lambda^*_2} (\cosh x)^{1-\lambda^*_2/3} < \frac{\sinh x}{x} < \left( \frac{2 + \cosh x}{3} \right)^{4/5} (\cosh x)^{1/15}
\]

hold for all \( x \in (0, 1) \).

4. Conclusions

In the paper, we establish sharp upper and lower bounds for the arithmetic mean in terms of new Seiffert-like means, more precisely, the double inequalities

\[
\left[ \frac{2}{3} M_{\sin}(a, b) + \frac{1}{3} M_{\tan}(a, b) \right]^{\frac{4}{5}} \left[ M_{\sin}^{2/3}(a, b) M_{\tan}^{1/3}(a, b) \right]^{1/5} < A(a, b)
\]

\[
< \left[ \frac{2}{3} M_{\sin}(a, b) + \frac{1}{3} M_{\tan}(a, b) \right]^{\mu^*_1} \left[ M_{\sin}^{2/3}(a, b) M_{\tan}^{1/3}(a, b) \right]^{1-\mu^*_1}
\]

and

\[
\left[ \frac{1}{3} M_{\tanh}(a, b) + \frac{2}{3} M_{\sinh}(a, b) \right]^{\lambda^*_2} \left[ M_{\tanh}^{1/3}(a, b) M_{\sinh}^{2/3}(a, b) \right]^{1-\lambda^*_2} < A(a, b)
\]

\[
< \left[ \frac{1}{3} M_{\tanh}(a, b) + \frac{2}{3} M_{\sinh}(a, b) \right]^{4/5} \left[ M_{\tanh}^{1/3}(a, b) M_{\sinh}^{2/3}(a, b) \right]^{1/5}
\]

hold for all \( a, b > 0 \) with \( a \neq b \), where \( \mu^*_1 \) and \( \lambda^*_2 \) are given as in Theorem 3.1 and Theorem 3.2, respectively.

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**Conflict of interest**

The authors declare that they have no competing interests.


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