THEORETICAL CHIRAL DYNAMICS

H. LEUTWYLER

Institute for Theoretical Physics, University of Bern
Sidlerstr. 5, CH-3012 Bern, Switzerland
E-mail: leutwyler@itp.unibe.ch

The reasons why a considerable effort is made to resolve the low energy structure of QCD are discussed. The effective field theory used for this purpose is illustrated with the recent progress made in the predictions for \( \pi \pi \) scattering and in understanding the low energy properties of the theory in the large \( N_c \) limit.

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1 Standard Model

In the Standard Model, the dynamical variables are the gauge bosons \( \gamma, W, Z, G \), the Higgs fields \( \phi_1, \phi_2, \phi_3, \phi_4 \), the quarks \( q \) and the leptons \( \ell \). Except for the mass term of the Higgs field, the Lagrangian does not contain mass parameters – the masses of the various particles are of dynamical origin: The ground state contains a condensate of neutral Higgs particles, \( \langle 0 | \phi_1 | 0 \rangle \neq 0 \). Neither the photon nor the gluons take notice – for these, the vacuum is transparent, because \( \phi_1 \) is electrically neutral and does not carry colour. For the gauge fields that mediate the weak interaction, however, this is not the case: The vacuum is not transparent for \( W \) and \( Z \) waves of low frequency – these particles do interact with those forming the condensate, because \( \phi_1 \) is not neutral with respect to flavour. As a consequence, the frequency of the \( W \) and \( Z \) waves tends to a nonzero value at large wavelength: The corresponding particles move at a speed that is smaller than the velocity of light – both the \( W \) and the \( Z \) pick up a mass.

The quarks and leptons also interact with the particles in the condensate and thus also pick up mass. It so happens that the interactions of \( \nu, e, \mu, u, d, s \) with the Higgs fields are weak, so that the masses \( m_\nu, m_e, m_\mu, m_u, m_d, m_s \) are small. The remaining fermion masses, as well as \( m_W, m_Z \) and \( m_H \) are not small. We do not know why the observed mass pattern looks like this, but we can analyze the consequences of this empirical fact.

At energies that are small compared to \( \{ m_W, m_Z, m_H \} = O(100 \text{GeV}) \), the weak interaction freezes out, because these energies do not suffice to bridge the mass gap and to excite the corresponding degrees of freedom. As a consequence, the gauge group of the Standard Model, \( \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \), breaks
down to the subgroup SU(3)×U(1) – only the photons, the gluons, the quarks and the charged leptons are active at low energies. Since the neutrini neither carry colour nor charge, they decouple.

2 Effective theory for $E \ll 100$ GeV

The Lagrangian relevant in the low energy domain is the one of QCD + QED, which is characterized by the two coupling constants $g$ and $e$. In contrast to the Standard Model, the SU(3)×U(1) Lagrangian does contain mass terms: the quark and lepton mass matrices $m_q$, $m_\ell$. Moreover, Lorentz and gauge invariance permit the occurrence of a term proportional to the operator

$$\omega = \frac{1}{16 \pi^2} \frac{1}{e^2} \text{tr} \ G_{\mu\nu} \tilde{G}^{\mu\nu}. \quad (1)$$

The corresponding coupling constant $\theta$ is referred to as the vacuum angle. The field basis may be chosen such that $m_q$ and $m_\ell$ are diagonal and positive. The fact that the electric dipole moment of the neutron is very small implies that – in this basis – $\theta$ must be tiny. This is called the strong CP-problem: We do not really understand why the neutron dipole moment is so small.

The two gauge fields involved in the effective low energy theory behave in a qualitatively different manner: While the photons do not carry electric charge, the gluons do carry colour. This difference is responsible for the fact that the strong interaction becomes strong at low energies, while the electromagnetic interaction becomes weak there, in fact remarkably weak: The photons and leptons essentially decouple from the quarks and gluons. The electromagnetic interaction can be accounted for by means of the perturbation series in powers of $e$. For the QCD part of the theory, on the other hand, perturbation theory is useful only at high energies. In the low energy domain, the strong interaction is so strong that it confines the quarks and gluons.

The resulting effective low energy theory is mathematically more satisfactory than the Standard Model as such – it does not involve scalar degrees of freedom and has fewer free parameters. Remarkably, this simple theory must describe the structure of cold matter to a very high degree of precision, once the parameters in the Lagrangian are known. It in particular explains the size of the atoms in terms of the scale

$$a_B = \frac{4 \pi}{e^2 m_e},$$

which only contains the two parameters $e$ and $m_e$ – these are indeed known to an incredible precision. Unfortunately, our ability to solve the QCD part of the theory is rather limited – in particular, we are still far from being able to
demonstrate on the basis of the QCD Lagrangian that the strong interaction actually confines colour. Likewise, our knowledge of the magnitude of the light quark masses is still rather limited – we need to know these more accurately in order to test ideas that might lead to an understanding of the mass pattern, such as the relations with the lepton masses that emerge from attempts at unifying the electroweak and strong forces.

3 Massless QCD – a theoretical paradise

In the following, I focus on the QCD part and switch the electromagnetic interaction off. As mentioned already, $m_u, m_d$ and $m_s$ happen to be small. Let me first set these parameters equal to zero and, moreover, send the masses of the heavy quarks, $m_c, m_b, m_t$, to infinity. In this limit, the theory becomes a theoretician’s paradise: The Lagrangian contains a single parameter, $g$. In fact, since the value of $g$ depends on the running scale used, the theory does not contain any dimensionless parameter that would need to be adjusted to observation. In principle, this theory fully specifies all dimensionless observables as pure numbers, while dimensionful quantities like masses or cross sections can unambiguously be predicted in terms of the scale $\Lambda_{\text{QCD}}$ or the mass of the proton. The resulting theory – QCD with three massless flavours – is among the most beautiful quantum field theories we have. I find it breathtaking that, at low energies, nature reduces to this beauty, as soon as the dressing with the electromagnetic interaction is removed and the Higgs condensate is replaced by one that does not hinder the light quarks, but is impenetrable for $W$ and $Z$ waves as well as for heavy quarks.

The Lagrangian of the massless theory, which I denote by $\mathcal{L}_{\text{QCD}}^0$, has a high degree of symmetry, which originates in the fact that the interaction among the quarks and gluons is flavour-independent and conserves helicity: $\mathcal{L}_{\text{QCD}}^0$ is invariant under independent flavour rotations of the three right- and left-handed quark fields. These form the group $G = \text{SU}(3)_R \times \text{SU}(3)_L$. The corresponding 16 currents $V_{i\alpha}^\mu \gamma^\mu \frac{1}{2} \lambda_i q$ and $A_{i\alpha}^\mu = \gamma^\mu \gamma_5 \frac{1}{2} \lambda_i q$ are conserved, so that their charges commute with the Hamiltonian:

$$[Q_i^V, H_{\text{QCD}}^0] = [Q_i^A, H_{\text{QCD}}^0] = 0, \quad i = 1, \ldots, 8.$$ 

Vafa and Witten \(^1\) have shown that the state of lowest energy is necessarily invariant under the vector charges: $Q_i^V |0\> = 0$. For the axial charges, however, there are the two possibilities characterized in table 1.

The observed spectrum does not contain parity doublets. In the case of the lightest meson, the $\pi(140)$, for instance, the lowest state with the same spin and flavour quantum numbers, but opposite parity is the $a_0(980)$. 

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3
$$Q^a_i \langle 0 \rangle = 0$$

Wigner-Weyl realization of $G$

ground state is symmetric

$\langle 0 | \overline{q}_R q_L | 0 \rangle = 0$

ordinary symmetry

spectrum contains parity partners
degenerate multiplets of $G$

$$Q^a_i \langle 0 \rangle \neq 0$$

Nambu-Goldstone realization of $G$

ground state is asymmetric

$\langle 0 | \overline{q}_R q_L | 0 \rangle \neq 0$

spontaneously broken symmetry

spectrum contains Goldstone bosons
degenerate multiplets of SU(3) $\in G$

| Wigner-Weyl realization of $G$ | Nambu-Goldstone realization of $G$ |
|-------------------------------|-----------------------------------|
| ground state is symmetric     | ground state is asymmetric         |
| $\langle 0 | \overline{q}_R q_L | 0 \rangle = 0$ | $\langle 0 | \overline{q}_R q_L | 0 \rangle \neq 0$ |
| ordinary symmetry             | spontaneously broken symmetry      |
| spectrum contains parity partners | spectrum contains Goldstone bosons |
| degenerate multiplets of $G$  | degenerate multiplets of SU(3) $\in G$ |

Table 1. Alternative realizations of the symmetry group $G = SU(3)_R \times SU(3)_L$.

So, experiment rules out the first possibility. In other words, for dynamical reasons that yet remain to be understood, the state of lowest energy is an asymmetric state. Since the axial charges commute with the Hamiltonian, there must be eigenstates with the same energy as the ground state:

$$H^0_{\text{QCD}} Q^a_i \langle 0 \rangle = Q^a_i H^0_{\text{QCD}} | 0 \rangle = 0 .$$

The spectrum must contain 8 states $Q^a_i | 0 \rangle, \ldots , Q^8_i | 0 \rangle$ with $E = \vec{P} = 0$, describing massless particles, the Goldstone bosons of the spontaneously broken symmetry. Moreover, these must carry spin 0, negative parity and form an octet of SU(3).

### 4 Quark masses as symmetry breaking parameters

Indeed, the 8 lightest hadrons, $\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$, do have these quantum numbers, but massless they are not. This has to do with the deplorable fact that we are not living in paradise: The masses $m_u, m_d, m_s$ are different from zero and thus allow the left-handed quarks to communicate with the right-handed ones. The Lagrangian is of the form

$$L_{\text{QCD}} = L^0_{\text{QCD}} - \overline{q}_R m q_L - \overline{q}_L m^\dagger q_R , \quad m = \begin{pmatrix} m_u & m_d & m_s \end{pmatrix} .$$

The quark masses may be viewed as symmetry breaking parameters: The QCD-Hamiltonian is only approximately symmetric under independent rotations of the right- and left-handed quark fields, to the extent that these parameters are small. Chiral symmetry is thus broken in two ways:

- **spontaneously**
  \[ \langle 0 | \overline{q}_R q_L | 0 \rangle \neq 0 \]
- **explicitly**
  \[ m_u, m_d, m_s \neq 0 \]
The consequences of the fact that the explicit symmetry breaking is small may be worked out by means of an effective field theory. The various quantities of interest are expanded in powers of the momenta and quark masses. In the case of the pion mass, for instance, the expansion starts with

$$M_{\pi^+}^2 = (m_u + m_d) B_0 + O(m^2), \quad B_0 = \frac{1}{F_0} \langle 0 | \bar{u}u | 0 \rangle.$$  \hspace{1cm} (2)

$F_0$ is the value of the pion decay constant in the chiral limit, $m_u, m_d, m_s \to 0$. The formula shows that the square of the pion mass is proportional to the product of $m_u + m_d$ with the order parameter $\langle 0 | \bar{u}u | 0 \rangle$. The two factors represent quantitative measures for explicit and spontaneous symmetry breaking, respectively. If the explicit symmetry breaking is turned off, the pions do become massless, as they should: The symmetry is then exact, so that the spectrum must contain a massless Goldstone boson octet, while all other levels form massive, degenerate multiplets of SU(3). Actually, the excited mesonic states are unstable, because the strong interaction allows them to decay into the Goldstone bosons, but the symmetry ensures that both the mass and the lifetime of the members of a given multiplet are the same.

5 Illustration: $\pi\pi$ scattering lengths

Above, I treated the masses of all three light quarks as expansion parameters. For the low energy analysis of the $\pi\pi$ scattering amplitude, however, there is no need to expand in powers of $m_s$. We can keep $m_s$ at its physical value and only expand in powers of $m_u$ and $m_d$. In the limit $m_u, m_d \to 0$ at fixed $m_s$, QCD already acquires an exact symmetry: The Hamiltonian becomes invariant under the group SU(2)$_R \times$SU(2)$_L$ of chiral rotations in the space spanned by the two massless flavours. The ground state spontaneously breaks that symmetry to the subgroup SU(2)$_V$ – the good old isospin symmetry discovered in the thirties of the last century. Only the pions then become massless, while the kaons and the $\eta$ remain massive. In the following, I consider this framework and ignore isospin breaking, setting $m_u = m_d = \hat{m}$. The partial wave decomposition then contains two independent $S$-waves, corresponding to $s$-channel isospin $I = 0$ and $I = 2$ (Bose statistics does not permit an $S$-wave with $I = 1$). The corresponding scattering lengths are denoted by $a^0_0$ and $a^2_0$ – the lower index specifies the total angular momentum, while the upper one refers to isospin.

As a general consequence of the symmetry, Goldstone bosons of zero momentum cannot interact with one another: $a^0_0$ and $a^2_0$ vanish for $m_u, m_d \to 0$. These quantities thus also measure the explicit symmetry breaking generated
by the quark masses, like $M^2_\pi$. In fact, Weinberg’s low energy theorem states that, to leading order of the expansion in powers of $m_u$ and $m_d$, the scattering lengths are proportional to $M^2_\pi$, the factor of proportionality being fixed by the pion decay constant. The low energy theorem may be written in the form

$$a_0^0 = \frac{7M^2_\pi}{32\pi F^2_\pi} R_0, \quad a^2_0 = -\frac{M^2_\pi}{16\pi F^2_\pi} R_2, \quad R_I = 1 + O(\hat{m}). \quad (3)$$

The two loop representation explicitly specifies the scattering lengths in terms of the effective coupling constants, up to and including contributions of $O(\hat{m}^3)$, so that the correction factors $R_0, R_2$ can be calculated to next-to-next-to leading order. Explicitly, the result reads

$$R_0 = 1 + x \left( \frac{9}{2} \ln \frac{M^2_\pi}{M^2} + x^2 \frac{769}{84} \left( \ln \frac{\hat{M}^2_\pi}{M^2} \right)^2 + x^2 k_0 + O(x^3) \right), \quad (4)$$

$$R_2 = 1 - x \left( \frac{3}{2} \ln \frac{M^2_\pi}{M^2} - x^2 \frac{17}{12} \left( \ln \frac{\hat{M}^2_\pi}{M^2} \right)^2 + x^2 k_2 + O(x^3) \right),$$

where $x \propto \hat{m}$ stands for the dimensionless quantity

$$x = \left( \frac{M^2_\pi}{4\pi F^2} \right)^2, \quad (5)$$

which measures the pion mass in units of the pion decay constant. More precisely, $M^2 = 2\hat{m}B$ is the first term in the expansion of $M^2_\pi$ and $F$ is the value of the pion decay constant for $m_u = m_d = 0$. Note that $B$ and $F$ differ from the quantities $B_0$ and $F_0$ in equation (2) by contributions of $O(m_u)$. 

### 6 Infrared singularities

The expansion in powers of the quark masses may be obtained by splitting the QCD Hamiltonian into two parts,

$$H_{\text{QCD}} = H_0 + H_1, \quad H_1 = \int d^3x \{ m_u \bar{u}u + m_d \bar{d}d \},$$

and treating $H_1$ as a perturbation. The formula (4) shows that the resulting expansion of the scattering lengths is not an ordinary Taylor series, but contains terms involving the logarithm thereof. This indicates that the straightforward perturbation series in powers of $H_1$ runs into infrared singularities,

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*aThe standard definition of the scattering length corresponds to $a_0/M_\pi$. It is not suitable in the present context, because it differs from the invariant scattering amplitude at threshold by a kinematic factor that diverges in the chiral limit.*
which occur because the spectrum of $H_0$ contains massless particles, pions. The infrared divergences are of logarithmic type. The coefficients of the logarithms are determined by the structure of the symmetry group and the transformation properties of $H_1$.

In the case of $R_0$, the coefficients of the logarithms are positive and unusually large, while in the case of $R_2$ they are negative and of normal size. The qualitative difference can be understood on the basis of unitarity, which requires the partial waves to contain a branch cut for $s \geq 4M_2^2$, generated by the final state interaction. In the $I = 0$ channel, this interaction is attractive and very strong, while for $I = 2$, it is repulsive and weak, because that channel is exotic. As the scattering lengths represent the values of the partial waves at the branch point, we are in effect considering the expansion of the scattering amplitude at a point where it is singular. The fact that the coefficients of the logarithms occurring in the expansion of $R_0$ are extraordinarily large implies that, in this case, the series converges unusually slowly.

The logarithmic scales are not determined by the symmetry. In the language of the effective theory, these scales are related to the values of the effective coupling constants. Indeed, not only the scales $M_0, M_2$ that specify the corrections of order $\hat{m}$, but also those of the terms proportional to the square of a logarithm, are determined by the effective coupling constants that occur at first nonleading order in the derivative expansion of the effective Lagrangian, $\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots$. It is convenient to use the corresponding running couplings at the scale $\mu = M$, which are denoted by $\bar{\ell}_1, \ldots, \bar{\ell}_4$ and also depend logarithmically on $M$:

$$\bar{\ell}_n = \ln \frac{\Lambda_n^2}{M^2}.$$  

In terms of these quantities, the scales occurring in the expansion of the scattering lengths are given by

$$\ln \frac{M_0^2}{M^2} = \frac{1}{189} \left(40 \bar{\ell}_1 + 80 \bar{\ell}_2 - 15 \bar{\ell}_3 + 84 \bar{\ell}_4\right) + \frac{5}{9},$$

$$\ln \frac{M_2^2}{M^2} = \frac{1}{9} \left(8 \bar{\ell}_1 + 16 \bar{\ell}_2 - 3 \bar{\ell}_3 - 12 \bar{\ell}_4\right) + \frac{1}{3},$$

$$\ln \frac{\tilde{M}_0^2}{M^2} = \frac{1}{769} \left(132 \bar{\ell}_1 + 568 \bar{\ell}_2 - 267 \bar{\ell}_3 + 336 \bar{\ell}_4\right) + \frac{1361}{1538},$$

$$\ln \frac{\tilde{M}_2^2}{M^2} = \frac{1}{17} \left(-12 \bar{\ell}_1 + 8 \bar{\ell}_2 - 3 \bar{\ell}_3 + 24 \bar{\ell}_4\right) - \frac{125}{34}.$$  

The mass independent terms $k_0$ and $k_2$ account for the remaining contribu-
tions of $O(\hat{m}^2)$, in particular also for those from the effective couplings of $\mathcal{L}^{(6)}$.

7 Numerical discussion

Rough estimates for the coupling constants $\bar{\ell}_1, \ldots, \bar{\ell}_4$ were given long ago. In the meantime, the values of $\bar{\ell}_1$ and $\bar{\ell}_2$ have been determined more accurately. In order to analyze the corrections of order $\hat{m}^2$ in $R_0$ and $R_2$, we need phenomenological determinations of comparable accuracy -- one loop analyses are not suitable for the present purpose. One source of information, where the analysis has now been done to two loops, is $K_{\pi\pi}$ decay. The Roy equations for $\pi\pi$ scattering allow an entirely independent determination, which, moreover, only relies on SU(2)$_R \times$SU(2)$_L$. The constant $\bar{\ell}_4$ is related to the scalar charge radius, for which the calculation has also been done to two loops. The information about $\bar{\ell}_3$, on the other hand, is still very meagre -- the value of this constant makes the difference between the standard picture and “Generalized Chiral Perturbation Theory” (see below). In the following, I invoke the very crude estimate $\ln \Lambda^3_3/\Lambda^2_\pi = 2.9 \pm 2.4$, which amounts to $0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV}$. Concerning the values of $k_0$ and $k_4$, I rely on the estimates for the coupling constants of $\mathcal{L}^{(6)}$ in the literature.

Fig.1 indicates the behaviour of the correction factors as functions of the quark mass $\hat{m}$, which is varied from 0 (where the symmetry is exact) to about 20 MeV. The variable shown on the horizontal axis is the corresponding pion mass. The figure shows that $R_0$ very rapidly grows with the strength of the symmetry breaking. In fact, the chiral perturbation theory formulae
underlying the figure are meaningful only in the range where the corrections are small (the shaded regions exclusively account for the uncertainties in the values of the coupling constants). If the quark masses are taken at their physical values, dropping the terms of $O(x^3)$ and using the central values for the coupling constants leads to $R_0 = 1.36$.

It is advantageous to replace the expansion of the scattering amplitude at threshold by one in the unphysical region, where the series converges much more rapidly. As discussed in detail by G. Colangelo at this meeting, that method yields a remarkably precise prediction for the scattering lengths: $a_0^0 = 0.220 \pm 0.005$, $a_0^2 = -0.0444 \pm 0.0010$. In the language used above, these numbers correspond to $R_0 = 1.38 \pm 0.03$, $R_2 = 0.98 \pm 0.02$. The lattice result, $a_0^2 = -0.0374 \pm 0.0049$ or $R_2 = 0.82 \pm 0.11$, is on the low side, but not inconsistent with the prediction. The current situation concerning the scattering lengths is depicted in fig.2, which is taken from the reference quoted above. 

![Figure 2](image_url)

Figure 2. Scattering lengths: theory versus experiment. The shaded region represents the intersection of the domains allowed by the old data and by the Olsson sum rule. The ellipse indicates the impact of the new, preliminary $K_{e4}$ data. The three diamonds illustrate the convergence of the chiral perturbation series at threshold (the one at the left corresponds to Weinberg’s leading order formulae) and the cross shows the result mentioned in the text.
8 Is the quark condensate the leading order parameter?

The preceding discussion of the scattering lengths relies on the standard hypothesis, according to which the quark condensate represents the leading order parameter of the spontaneously broken symmetry. This framework is natural, because among the various operators that give rise to order parameters – $\bar{q}q$, $\bar{q}Rq$ – is the one of lowest dimension. As emphasized by Stern and collaborators, experimental evidence for this to be the case is not available – yet. The picture advocated by these authors may be motivated by an analogy with spontaneous magnetization. In that context, spontaneous symmetry breakdown occurs in two quite different modes: ferromagnets and antiferromagnets. For the former, the magnetization develops a nonzero expectation value, while for the latter, this does not happen. In either case, the symmetry is spontaneously broken. The example illustrates that operators which the symmetry allows to pick up an expectation value may, but need not do so.

The issue concerns the validity of the Gell-Mann-Oakes-Renner relation, equation (2). At first non-leading order and in the isospin limit $m_u = m_d = \hat{m}$, the correction is determined by the coupling constant $\bar{\ell}_3$:

$$M^2_\pi = 2\hat{m} B - \frac{\hat{m}^2 B^2}{8\pi^2 F^2} \bar{\ell}_3 + O(\hat{m}^3).$$

(8)

In this context, the standard hypothesis amounts to the assumption that the first term dominates over the second, so that $M^2_\pi$ is approximately linear in $\hat{m}$. Since the structure of the ground state of QCD is not understood, it is conceivable that the quark condensate is small, so that the Gell-Mann-Oakes-Renner-relation fails, the “correction” of order $\hat{m}^2$ being comparable to or even larger than the first term. In the language of the effective theory, this would imply that the estimate used above for the coupling constant $\bar{\ell}_3$ is entirely wrong. In the generalized framework, this constant is treated as a free parameter, so that there is no prediction for the scattering lengths.

A sufficiently accurate measurement of the $\pi\pi$ scattering lengths would decide the issue, because it amounts to a determination of the coupling constant $\bar{\ell}_3$. An outcome like $a_0^3 = 0.26$, for example, would be totally incompatible with the standard framework – it would imply a value like $\bar{\ell}_3 \simeq -20$. I am confident that the forthcoming results from Brookhaven, CERN and Frascati will provide a very significant test.

9 QCD at large $N_c$

As pointed out by ’t Hooft, it is very instructive to vary the number of quark colours, in particular to let it become large. The magnitude of the running
coupling \( g \) of QCD is given by the familiar leading logarithmic formula
\[
\frac{g^2}{(4\pi)^2} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{\text{QCD}}^2)} , \quad \beta_0 = \frac{1}{3} (11N_c - 2N_f) , \tag{9}
\]
where \( \mu \) and \( \Lambda_{\text{QCD}} \) denote the running and intrinsic scales, respectively. The formula shows that the coupling constant tends to zero in proportion to \( 1/\sqrt{N_c} \) if \( N_c \) is sent to infinity at a fixed value of the ratio \( \mu/\Lambda_{\text{QCD}} \). This implies that graphs with the smallest possible number of quark loops then dominate, so that the Okubo-Iizuka-Zweig rule becomes exact and the constituent quark picture applies. In the large \( N_c \) limit, the scattering amplitudes are of \( O(1/N_c) \) and the various resonances become stable – the width also shrinks in proportion to \( 1/N_c \). In the following, I discuss a few consequences for the low energy properties of the theory, drawing from work done in collaboration with R. Kaiser. We are by no means the first to study the subject – a review may be found in that reference.

The Ward identity obeyed by the singlet axial current contains an anomaly, proportional to the operator \( \omega \) defined in equation (1). As is well-known, the anomaly term is suppressed when \( N_c \) becomes large, because it arises from graphs containing an extra quark loop: In the large \( N_c \) limit, the ninth axial current is conserved, so that the theory acquires an additional symmetry, whose spontaneous breakdown gives rise to a ninth Goldstone boson – if the quark masses are turned off, the mass of the \( \eta' \) disappears when the number of colours is sent to infinity.

10 Vacuum angle

The correlation functions of \( \omega \) are conveniently collected in the effective action \( S_{\text{eff}} \{ \theta \} \) that results if the QCD–Lagrangian is perturbed with this operator,
\[
\mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{QCD}} - \theta \omega ,
\]
In the present context, the \( \theta \)–term plays a role of technical nature: \( \theta = \theta(x) \) is treated as an external field – the value of physical interest is \( \theta = 0 \).

The expansion of the effective action in powers of \( 1/N_c \) starts with a term of order \( N_c^2 \) that arises from graphs without any quark lines. The large \( N_c \) counting rules of perturbation theory imply that the correlation function \( \langle 0 | T \omega(x_1) \cdots \omega(x_n) | 0 \rangle \) is of \( O(N_c^{2-n}) \). Accordingly, the leading term of the expansion depends on \( \theta \) only through the ratio \( \theta/N_c \):
\[
S_{\text{eff}} \{ \theta \} = N_c^2 S_0 \{ \vartheta \} + N_c S_1 \{ \vartheta \} + \ldots , \quad \vartheta = \frac{\theta}{N_c} . \tag{10}
\]
The formula states that the dependence on the vacuum angle is suppressed. It is not difficult to understand why that is so: The contribution from the $\theta$–term is to be compared with the one from $L_{QCD}^0$. In the relevant combination,

$$\frac{1}{2g^2} \text{tr} \left( G_{\mu\nu} G^{\mu\nu} + \frac{g^2 \theta}{8\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu} \right),$$

the weight of the $\theta$–term is smaller than the one that governs the dynamics of the gluon field by $g^2 \theta \propto \theta/N_c$.

It is important to notice that the large $N_c$ counting rules only hold for generic momenta. The two-point-function, for instance,

$$i \int dx e^{ip \cdot x} \langle 0 | T \omega(x) \omega(0) | 0 \rangle = \frac{|\langle 0 | \omega(\eta') \rangle|^2}{M_{\eta'}^2 - p^2} + \ldots$$

picks up a pole term from $\eta'$ exchange, which arises from graphs containing at least one quark loop. If the quark masses are turned off, both $M_{\eta'}$ and $\langle 0 | \omega(\eta') \rangle$ are quantities of $O(1/\sqrt{N_c})$, so that the value of the pole term at $p = 0$ represents a contribution of $O(1)$, despite the counting rules, which state that, in $\langle 0 | T \omega(x) \omega(0) | 0 \rangle$, contributions from quark loops start showing up only at $O(1/N_c)$. If the vacuum angle is taken constant, we are in effect summing up the correlation functions of $\omega$ at zero momentum – relations like (11) are not in general valid in that case. Also, there are paradoxical aspects in connection with periodicity in $\theta$ – for a detailed discussion, I refer to a paper written together with A. Smilga.

11 Effective theory at large $N_c$ and KM–transformation

The standard form of the effective theory only accounts for the singularities generated by the exchange of the particles contained in the pseudoscalar octet. The contributions from all other states are described only summarily, through their contributions to the effective coupling constants. If the number of colours is allowed to become large, the effective theory must be extended, introducing an additional field to describe the low energy singularities due to $\eta'$ exchange. The effective Lagrangian then contains a new low energy scale: $M_{\eta'} \propto 1/\sqrt{N_c}$. The low energy structure of QCD can be analyzed within this extended framework, by means of a simultaneous expansion in powers of momenta, quark masses and $1/N_c$. This machinery is particularly suited for analyzing the dependence on the vacuum angle, because it explicitly accounts for the low energy singularities that upset the large $N_c$ counting rules at exceptional momenta. Indeed, the suppression of the $\theta$-dependence manifests itself in a remarkable manner: To any finite order of the $1/N_c$ expansion, the
effective Lagrangian is a polynomial in the vacuum angle, with coefficients that are suppressed by powers of $1/N_c$.

As an illustration, I briefly discuss the ambiguity pointed out by Kaplan and Manohar. The matrix

$$m' = m + \lambda e^{-i\theta} (m^+)^{-1} \det m^\dagger,$$

transforms in the same manner as the quark mass matrix $m$, under the full group $U(3)_R \times U(3)_L$ of chiral rotations. Symmetry does therefore not distinguish $m'$ from $m$. Since the effective theory exclusively exploits the symmetry properties of QCD, the above transformation of the quark mass matrix does not change the form of the effective Lagrangian relevant for $N_c = 3$: The transformation may be absorbed in a suitable change of the effective coupling constants, for any value of the vacuum angle, in particular also for the case of physical interest, $\theta = 0$. This implies, however, that the expressions for the masses of the pseudoscalars, for the scattering amplitudes or for the matrix elements of the vector and axial currents, which follow from this Lagrangian, are invariant under the operation $m \to m'$. Conversely, the experimental information on these observables does not distinguish $m$ from $m'$.

Since the KM-transformation mixes the quark flavours, it is evident that the parameter $\lambda$ violates the Okubo-Iizuka-Zweig rule and is therefore suppressed in the large $N_c$ limit. Actually, a much stronger result can be established: The transformation $m' \to m$ preserves the large $N_c$ properties of the theory only if $\lambda$ vanishes to all orders in $1/N_c$. This is a consequence of the conservation law obeyed by the singlet axial current. The fact that the anomaly in this conservation law disappears in the large $N_c$ limit implies that the dependence on $\theta$ is suppressed. The KM-transformation is in conflict with this property, because the factor $e^{-i\theta}$ introduces a dependence on $\theta$ that does not disappear when the number of colours tends to infinity. For precisely the same reason, an extra dynamical variable – a field that describes the low energy singluarities generated by the $\eta'$ – is needed to cover the large $N_c$ limit: The constraint $\det U = e^{-i\theta}$, which is imposed on the meson field in the effective theory relevant for $N_c = 3$, cannot be maintained in the large $N_c$ limit.

In the remainder of the talk, I discussed a few aspects of chiral dynamics in the context of the $\pi N$ interaction. That material is covered in the contribution by T. Becher. There are many further developments in chiral dynamics, however, which I did not have the time to discuss. For a more comprehensive picture, I refer to the review articles listed in the bibliography.

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6See also the contribution by R. Kaiser in these proceedings.
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