Analytic Solution of a Delay Differential Equation Arising in Cost Functionals for Systems With Distributed Delays

Suat Gumussoy and Murad Abu-Khalaf, Members, IEEE

Abstract—The solvability of a delay differential equation arising in the construction of quadratic cost functionals, i.e., Lyapunov functionals, for a linear time-delay system with a constant and a distributed delay is investigated. We present a delay-free auxiliary ordinary differential equation system with algebraically coupled split-boundary conditions, which characterizes the solutions of the delay differential equation and is used for solution synthesis. A spectral property of the time-delay system yields a necessary and sufficient condition for existence and uniqueness of solutions to the auxiliary system, equivalently the delay differential equation. The result is a tractable analytic solution framework to the delay differential equation.

Index Terms—Cost functionals, delay differential equations (DDEs), distributed delay, Lyapunov functionals.

I. INTRODUCTION

This investigation considers the time-delay system

\[ \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \int_0^h A_2(\theta) x(t+\theta) d\theta \]  

(1)

with the initial function \( \phi(\cdot) \) such that at \( t = 0, x(t+\theta) = \phi(\theta), \ \forall \theta \in [-h,0] \). We study the solvability of the associated system of equations (2) abbreviated as (DDEc), a delay differential equation (DDE) (2a) with coupling conditions (2b) and (2c):

\[ P(\tau) = P(\tau)A_0 + P(\tau-h)A_1 + \int_0^h P(\tau+\theta)A_2(\theta) d\theta \]  

(2a)

\[ P(-\tau) = P(-\tau)^T \]  

(2b)

\[ -Q = A_3^T P(0)A_0 + A_4^T P(h) + P(-h)A_1 + \int_0^h [A_5(\theta)^T P(-\theta) + P(\theta)A_6(\theta)] d\theta \]  

(2c)

where \( h \geq 0, \tau \in [0,h] \). Equation (2b) is a symmetry constraint, and (2c) introduces algebraic and integral constraints involving the boundary values. The state \( x(t) \) is an \( n \times 1 \) function of time, and \( A_0(\cdot) \) and \( P(\cdot) \) are \( n \times n \) functions. Moreover, \( A_0, A_1, \) and \( Q = Q^T \) are \( n \times n \) constant matrices. \( A_0(\theta) = C_0 \exp(A_{\theta} \theta) B_0 \), with \( A_{\theta} \) being a matrix exponential. \( C_0, A_4, \) and \( B_0 \) are \( n \times n_d, n_d \times n_d, \) and \( n_d \times n \) constant matrices.

II. BACKGROUND

A. Origins of the Problem

In [1], the solvability of (2a)–(2c) was studied in the context of stability. In doing so, Karhunen [1] proposed solving the delay Lyapunov equation (2a)–(2c) by rewriting it as an ODE system with split-boundary conditions and solving the resulting equations instead. In [2], we show that the split-boundary conditions given in [1] are linearly dependent and do not yield a unique initial value problem. Thus, the auxiliary system proposed in [1] fails to characterize the solutions of the delay Lyapunov equation, and a numerical example therein incorrectly computes \( P(\tau) \). Nonetheless, Karhunen [1] provides great insights into solving such problems and inspires our work here.

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S. Gumussoy is with MathWorks, Natick, MA 01760 USA (e-mail: sgumusso@mathworks.com). M. Abu-Khalaf is with the MIT Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139 USA (e-mail: murad@mit.edu). Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org. Digital Object Identifier 10.1109/TAC.2019.2921658
We arrived at this problem due to our interest in optimizing and tuning the parameters of stable time-delay systems (cf., [3]) with a distributed delay. Distributed delays appear naturally in optimal control of time-delay systems [4]. Evaluating closed-loop performance could be done by integrating the cost over the system trajectories (4) or alternatively via (5a) provided \( P(\tau) \) (5b) is known.

For results earlier than [1], we refer the reader to a body of work on Lyapunov stability for different classes of time-delay systems resulting in related DDEs and auxiliary ODEs [5]–[15].

B. Notation

\( \mathbb{R} \) denotes the real line and \( \mathbb{C} \) the set of complex numbers. Given matrix \( Y \in \mathbb{R}^{p \times q} \), \( Y_{ij} \) denotes its \( i \)th column and \( y_{pq} \) the element at the \( p \)th row and \( q \)th column. \( \text{vec}(\cdot) \) stacks the columns of \( Y \in \mathbb{R}^{p \times q} \) into a single column. Operator \( \otimes \) is the Kronecker product [17], where for a matrix \( Z \)

\[
\text{vec}(Y) \triangleq \begin{bmatrix}
Y_{11} & y_{12} & \cdots & y_{1q} \\
Y_{21} & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
y_{p1} & \cdots & \cdots & y_{pq}
\end{bmatrix}, \quad Y \otimes Z \triangleq \begin{bmatrix}
y_{11}Z & y_{12}Z & \cdots & y_{1q}Z \\
y_{21}Z & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
y_{p1}Z & \cdots & \cdots & y_{pq}Z
\end{bmatrix}.
\]

An all-zero entry two-dimensional matrix of context-dependent size is denoted by \( 0 \), and \( I_{m \times m} \) for a fixed \( n \times m \) size. Similarly, \( I \) denotes an identity matrix of arbitrary size, and \( J_{n} \) a size-\( n \) identity matrix.

III. MAIN RESULT

In Section III-A, we introduce ODEc (6), an auxiliary ODE system with algebraically coupled split-boundary conditions, and study its solvability. In Section III-B, we show the relation between ODEc (6) and DDEc (2) systems.

A. ODE With Coupled Split-Boundary Conditions (ODEc)

The ODEc is defined by the dynamics (6a) and the split-boundary conditions (6b)–(6d) for the interval \([0, h]\), where \( \forall \tau \in \mathbb{R} \)

\[
\begin{align*}
\dot{\Omega}_1(\tau) &= \Omega_1(\tau)A_0 + \Omega_2(\tau)A_1 + \Omega_3(\tau)B_d + \Omega_4(\tau)B_d \\
\dot{\Omega}_2(\tau) &= -A_0^T\Omega_1(\tau) - A_1^T\Omega_2(\tau) - B_d^T\Omega_3(\tau) - B_d^T\Omega_4(\tau) \\
\dot{\Omega}_3(\tau) &= -\Omega_1(\tau)A_d - \Omega_2(\tau)C_d e^{-A_d h} \\
\dot{\Omega}_4(\tau) &= A_0^T\Omega_1(\tau) + (C_d e^{-A_d h})^T \Omega_1(\tau) \\
\dot{\Omega}_5(\tau) &= A_0^T\Omega_1(\tau) + (C_d e^{-A_d h})^T \Omega_1(\tau) \\
0 &= \Omega_1(0) - \Omega_2(0) \\
0 &= \Omega_3(0), \quad \Omega_4(0), \quad \Omega_\Delta(h), \quad \Omega_\Delta(h).
\end{align*}
\]

Note that \( \Omega_1(\cdot) \) and \( \Omega_2(\cdot) \) are \( n \times n \); \( \Omega_3(\cdot) \) and \( \Omega_4(\cdot) \) are \( n \times p \); and \( \Omega_5(\cdot) \) and \( \Omega_\Delta(\cdot) \) are \( n \times n \).

Existence and uniqueness of solutions to (6) corresponds to whether there exists a unique initial value \( \Omega_1(0) \), \( \Omega_2(0) \), \( \Omega_3(0) \), and \( \Omega_4(0) \) with \( 0 = \Omega_3(0) = \Omega_4(0) \) such that the solution of (6a) at \( \tau = h \) satisfies (6b)–(6d). This is addressed in Theorem 1 stated after the following lemmas.

Lemma 1: \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_\Delta(\tau)\} \) is a solution to ODEc (6) if and only if \( \forall \tau \in \mathbb{R}, \{\omega_1(\tau), \omega_2(\tau)\} \) satisfies

\[
\begin{align*}
\dot{\Omega}_1(\tau) &= \Omega_1(\tau)A_0 + \Omega_2(\tau)A_1 + f_0^\tau \Omega_1(\tau + \theta)A_D(\theta)d\theta \\
+ \int_{-h}^{h} \Omega_2(\tau + \theta + h)A_D(\theta)d\theta \\
\dot{\Omega}_2(\tau) &= -A_0^T\Omega_1(\tau) - A_1^T\Omega_2(\tau) \\
- \int_{-h}^{h+\tau} \Omega_2(\tau + \theta + h)A_D(\theta)d\theta \\
0 &= \Omega_1(0) - \Omega_2(h) \\
0 &= \Omega_1(0) - \Omega_2(h).
\end{align*}
\]

Proof: To show sufficiency, differentiating the terms in (9) with respect to \( \tau \), it follows that \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau)\} \) satisfies

\[
\begin{align*}
\dot{\omega}_1(\tau) &= -\omega_1(\tau)A_d + \omega_2(\tau)C_d e^{A_d h} \\
\dot{\omega}_2(\tau) &= -\omega_1(\tau)A_d - \omega_2(\tau)C_d e^{-A_d h} \\
\dot{\omega}_3(\tau) &= \omega_1(\tau)A_d + (C_d e^{-A_d h})^T \omega_1(\tau) \\
\dot{\omega}_4(\tau) &= -\omega_1(\tau)A_d - C_d^T \omega_1(\tau)
\end{align*}
\]

with boundary conditions \( 0 = \omega_2(0), \quad 0 = \omega_4(h), \quad 0 = \omega_5(h), \quad 0 = \omega_6(h). \) Moreover since \( \{\omega_1(\tau), \omega_2(\tau)\} \) satisfies (8), then \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau)\} \) satisfies ODEc (6).

To show necessity, if \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau)\} \) satisfies ODEc (6), then from the subsystem \( \dot{\omega}_1(\tau) = -\omega_1(\tau)A_d + \omega_2(\tau)C_d, \quad 0 = \Omega_1(0), \) it follows that \( \omega_1(\tau) \) satisfies

\[
\begin{align*}
\dot{\omega}_1(\tau)e^{A_d h} + \omega_1(\tau)A_d e^{A_d h} &= \omega_1(\tau)C_d e^{A_d h} \\
\int_{-h}^{0} d\theta \left( \omega_1(\tau)e^{A_d h} \right) d\theta &= \omega_1(\tau)C_d e^{A_d h}.
\end{align*}
\]

Therefore, \( \omega_1(\tau) \) satisfies (9a). Moreover, from the subsystem \( \dot{\omega}_2(\tau) = -\omega_1(\tau)A_d - \omega_2(\tau)C_d e^{-A_d h}, \quad 0 = \omega_2(h) \), it follows that \( \omega_2(\tau) \) satisfies (9b).

Similarly, from the subsystems for \( \Omega_1 \) and \( \Omega_2 \), it follows that \( \omega_3(\tau) \) satisfies (9c) and \( \omega_4(\tau) \) satisfies (9d). Moreover, we obtain (8a) and (8b) by noting that \( \{\omega_1(\tau), \omega_2(\tau)\} \) satisfies the first two equations of (6a), with \( \{\omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau)\} \) satisfying (9). Equation (8c) follows from evaluating at \( \tau = 0 \) the derivatives of the first two equations of (6a) and using (6b). Therefore, \( \{\omega_1(\tau), \omega_2(\tau)\} \) satisfies (8).

Lemma 2: If \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_\Delta(\tau)\} \) is a solution to ODEc (6), then the following is also a solution

\[
\{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_\Delta(\tau)\} = \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_\Delta(\tau)\} \tau T.
\]
Proof: ODE (6a) is satisfied by \(\{\omega_1(\tau), \omega_1(\tau), \omega_1(\tau), \omega_1(\tau)\}\):
\[
\dot{\omega}_1(\tau) = -\omega^T_1(h - \tau) + \omega_1(\tau)A_0 + \omega_1(\tau)A_1 + \omega_1(\tau)B_d + \omega_1(\tau)B_d,
\]
\[
\dot{\omega}_2(\tau) = -\omega^T_2(h - \tau) = -A^T\omega_2(\tau) - A_0\omega_2(\tau) - B_d\omega(\tau) - B_d\omega(\tau),
\]
\[
\dot{\omega}_3(\tau) = -\omega^T_3(h - \tau) = -A^T\omega_3(\tau) - A_0\omega_3(\tau) - B_d\omega(\tau) - B_d\omega(\tau),
\]
\[
\dot{\omega}_4(\tau) = -\omega^T_4(h - \tau) = -A^T\omega_4(\tau) - A_0\omega_4(\tau) - B_d\omega(\tau) - B_d\omega(\tau).
\]

From (6d), \(\forall \omega(\tau)\), we have
\[
\omega_1(\tau) = \omega_1(\tau)A_0 + \omega_1(\tau)A_1 + \omega_1(\tau)B_d + \omega_1(\tau)B_d.
\]

\[
0 = \omega(0) = \omega(0) + \omega(0), \quad 0 = \omega(0).
\]

Lemma 5: If \(\{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau)\}\) is a solution to ODE (6) for \(Q = 0\), then \(\forall \tau, \omega_1(\tau) = \omega_1(\tau + h)\).

Proof: Let \(\Delta(\tau) = \omega_1(\tau) - \omega_1(\tau + h)\), we then wish to show that \(\forall \tau \in \mathbb{R}, \Delta(\tau) = 0\). From (6b) and (6c), it follows that
\[
\Delta(0) = 0, \Delta(0) = 0.
\]

We next find the dynamics governing \(\Delta(\tau)\). For \(\omega_1(\tau)\), we have
\[
\dot{\omega}_1(\tau) = \omega_1(\tau)A_0 + \omega_1(\tau)A_1 + \omega_1(\tau)B_d + \omega_1(\tau)B_d.
\]

\[
0 = \omega(0) = \omega(0) + \omega(0), \quad 0 = \omega(0).
\]

Lemma 3: If \(\{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau)\}\) is a unique solution to ODE (6), then
\[
\Delta(\tau) = \Delta(\tau)A_0 + \Delta(\tau)A_1 + \Delta(\tau)B_d + \Delta(\tau)B_d.
\]

Lemma 4: Let \(\{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau)\}\) be a solution to ODE (6) for some \(Q\). If \(\forall \tau, \omega_1(\tau) = 0 \land \omega_2(\tau) = 0\), then \(Q = 0\).

Proof: From (6d), \(\omega_1(0) = 0\) and \(\omega_2(0) = 0\). Moreover, if \(\forall \tau, \omega_1(\tau) = 0\), then \(\forall \tau, \omega_2(\tau) = 0, \omega_3(\tau) = 0\). The following

\[
\omega_1(\tau) = \omega_1(\tau), \quad \omega_1(\tau) = \omega_1(\tau), \quad \omega_1(\tau) = \omega_1(\tau), \quad \omega_1(\tau) = \omega_1(\tau).
\]

This implies that
\[
\Delta(\tau) = \Delta(\tau)A_0 + \Delta(\tau)A_1 + \Delta(\tau)B_d + \Delta(\tau)B_d.
\]

From Lemma 1, \(\omega_3(\tau)\) to \(\omega_5(\tau)\) are function of \(\omega_1(\tau)\) and \(\omega_2(\tau)\). Therefore, (13) becomes
\[
\Delta(\tau) = \Delta(\tau)A_0 + \Delta(\tau)A_1 + \Delta(\tau)B_d + \Delta(\tau)B_d.
\]

where
\[
\delta_1(\tau) = \int_{\tau}^{\tau+h} \Delta(\theta + h)C_d\tau e^{A_d}\tau d\theta
\]
\[
\delta_2(\tau) = \int_{\tau-h}^{\tau} \left( C_d\tau e^{A_d}\tau \right) \Delta(\theta - \tau + h) d\theta
\]
\[
\delta_3(\tau) = \int_{\tau-h}^{\tau} \left( C_d\tau e^{A_d}\tau \right) \Delta(\theta - \tau + h) d\theta
\]
\[
\delta_4(\tau) = \int_{\tau-h}^{\tau} \left( C_d\tau e^{A_d}\tau \right) \Delta(\theta - \tau + h) d\theta
\]

(14)
and
\[ c(\tau) = \int_0^\tau \left[ \omega_2(\theta + \tau + h) + A_0^T \omega_2(\theta + \tau + h) + A_1^T \omega_1(\theta + \tau + h) \right] d\theta \]
\[ + \int_0^\tau A_0^T \omega_1(\theta + \tau + h) \omega_2(\theta + \tau + h) d\theta \]
\[ + \int_0^\tau A_0^T \omega_1(\theta + \tau + h) \omega_2(\theta + \tau + h) A_1 d\theta. \]

(16)

Note that from (15), \( \delta_1(0) = \cdots = \delta_6(0) = 0 \), and that
\[ \dot{\delta}_1(\tau) = -\delta_1(\tau) A_0 - \Delta_1(\tau) C_\delta \]
\[ \dot{\delta}_2(\tau) = A_0^T \delta_2(\tau) + (C_\delta e^{-A_k h})^T \Delta(\tau) \]
\[ \dot{\delta}_3(\tau) = -\delta_3(\tau) A_0 - \Delta_3(\tau) C_\delta \]
\[ \dot{\delta}_4(\tau) = A_0^T \delta_4(\tau) + (C_\delta e^{-A_k h})^T \Delta(\tau) \]
\[ \dot{\delta}_5(\tau) = -\delta_5(\tau) A_0 - \Delta_5(\tau) C_\delta e^{-A_k h} \]
\[ \dot{\delta}_6(\tau) = A_0^T \delta_6(\tau) + (C_\delta e^{-A_k h})^T \Delta(\tau). \]

Substitute for \( \omega_1(\tau - \theta) \) and \( \omega_2(\tau + \theta + h) \) in (16) using (8a) and (8b) from Lemma 1, respectively, to get \( c(\tau) = B^T \dot{\delta}_1(\tau) B_2 \), where
\[ \delta_1(\tau) = -f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \omega_1(\theta + \tau + \phi) d\phi C_\delta e^{A_k h} d\theta \]
\[ -f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \omega_2(\theta + \tau + \phi + h) d\phi C_\delta e^{A_k h} d\theta \]
\[ + f_0^\tau (C_\delta e^{A_k h})^T f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) d\phi C_\delta e^{A_k h} d\theta d\phi. \]

Note that \( \delta_1(0) = 0 \). The dynamics of \( \delta_1(\tau) \) is given by
\[ \dot{\delta}_1(\tau) = f_0^\tau \frac{d}{dt} f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta d\phi. \]

(17)

The expression \( \dot{\delta}_2(\tau) \) has two parts \( \delta_1 \) and \( \delta_2 \) that simplify each separately to get
\[ \delta_1 = -f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi \]
\[ -f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta \]
\[ -f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta \]
\[ -f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta d\phi. \]

and
\[ \delta_2 = f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi \]
\[ + f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta \]
\[ + f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta d\phi. \]

The dynamics for \( \delta_2(\tau) \) is, therefore, governed by
\[ \dot{\delta}_2(\tau) = -f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi \]
\[ + f_0^\tau f_0^\tau (C_\delta e^{A_k h})^T \Delta(\tau + \phi + \theta) C_\delta e^{A_k h} d\phi d\theta \]
\[ - f_0^\tau (C_\delta e^{A_k h})^T \delta_1(\tau) C_\delta e^{A_k h} d\phi. \]

Equations (14), (17), and (18) together with (12) and the fact that \( \delta_1(0) = \cdots = \delta_6(0) = 0 \) form a set of differential equations with zero initial conditions. This implies that \( \forall \tau \in \mathbb{R}, \Delta(\tau) = 0 \). □

Definition 1. Spectrum condition: The spectrum of the system (1) is
\[ \Lambda = \{ \lambda \in \mathbb{C} : \det (\lambda I - A_0 - e^{-A_k h} A_1 - f_0^\tau e^{A_k h} A_1(\theta) d\theta = 0 \} \]
and the spectrum states that \( \forall \lambda \in \Lambda, -\lambda \notin \Lambda \).

The next theorem establishes a necessary and sufficient condition for the existence and uniqueness of solutions for ODEc (6). Its proof is influenced by the single delay case; cf., [18, Ch. 2].

Theorem 1: A unique solution to ODEc (6) exists for all \( Q \) if and only if the spectrum condition in Definition 1 is satisfied.

Proof: Since (6a) is a linear dynamical system with \( n_z \) states (7), the existence and uniqueness of solutions to (6) is equivalent to whether there exists a unique \( \Omega_1(0), \Omega_2(0), \Omega_3(0), \Omega_4(0), \Omega_5(0), \Omega_6(0) \), and \( \Omega_0(0) \) such that the solution of (6a) satisfies the constraints (6b)–(6d). This results in \( n_z \) scalar linear algebraic equations with \( n_z \) unknowns such as formulated in (38). Therefore, the proof utilizes the fact that the solution to (6) is unique if and only if for the case \( Q = 0 \), the trivial solution is the only solution, thus having a trivial kernel for the underlying linear system of unknowns.

Necessity: We show that if the spectrum condition is not satisfied, then there exists a nontrivial solution for (6) when \( Q = 0 \). Therefore, assume that the spectrum condition is not satisfied; then, either (a) \( \exists \lambda_1, \lambda_2 \in \Lambda : \lambda_2 = -\lambda_1 \neq 0 \) or (b) \( \exists \lambda_0 \in \Lambda : \lambda_0 = 0 \). For (a), this implies \( \exists v_1, v_2 \in \mathbb{C}^n : v_1 \neq 0 \), \( v_2 \neq 0 \) and
\[ v_1^T \left[ \lambda_1 I - A_0 - e^{-A_k h} A_1 - f_0^\tau e^{A_k h} A_1(\theta) d\theta \right] v_2 \]
\[ = 0_{n_1 \times n_1} \]
\[ v_1^T \left[ -\lambda_1 I - A_0 - e^{-A_k h} A_1 - f_0^\tau e^{A_k h} A_1(\theta) d\theta \right] v_2 \]
\[ = 0_{n_2 \times n_2} \]
\[ v_1^T \left[ \lambda_1 e^{A_k (\tau - h)} v_1 \right] \]
\[ = 0 \]
\[ \text{and} \]
\[ v_2(\tau + h) = v_1(\tau - h) \]
\[ \text{then, we have} \]
\[ \lambda_1 e^{A_k (\tau - h)} v_1 \]
\[ = 0 \]
\[ \text{it follows from Lemma 1 that there exists a nontrivial solution to (6) for} \]
\[ Q = 0. \]

Similarly, for (b), \( \lambda_0 = 0 \), and this implies \( \exists v_0 \in \mathbb{C}^n : v_0 \neq 0 \), \( \lambda_0 \neq 0 \), and
\[ v_0^T \left[ A_0 + A_1 + \int_0^\tau e^{A_k h} A_1(\theta) d\theta \right] v_0 \]
\[ = 0_{n_2 \times n_2} \]
\[ v_0^T \left[ A_0 + A_1 + \int_0^\tau e^{A_k h} A_1(\theta) d\theta \right] v_0 \]
\[ = 0 \]
\[ \text{and} \]
\[ v_0^T \left[ A_0 + A_1 + \int_0^\tau e^{A_k h} A_1(\theta) d\theta \right] v_0 \]
\[ = 0 \]
\[ \text{and a nontrivial solution to (6) for} \]
\[ Q = 0 \text{ follows from Lemma 1.} \]

Sufficiency: By contraposition, we show that if there exists a nontrivial solution to ODEc (6) when \( Q = 0 \), then the spectrum condition
is not satisfied. Let \( \{ \omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau) \} \) be a nontrivial solution to ODE (6) with \( Q = 0 \). Since ODE (6a) is a linear time-invariant finite-dimensional system, \( \omega_1(\tau) \) is written in terms of the eigenmotions of ODE (6a)\( \omega_1(\tau) = \sum_{j=1}^{j_{\text{max}}} e^{\lambda_j \tau} \varphi_{1,j}(\tau) \) \( (22) \)

where \( \lambda_j \) are the distinct eigenvalues of ODE (6a), \( j_{\text{max}} \leq n_s \), and \( \varphi_{1,j}(\cdot) \) are polynomial matrices of degree \( k_{\text{max}}(j) \) given by

\[
\varphi_{1,j}(\tau) = \sum_{k=0}^{k_{\text{max}}(j)} W_{1,j,k} \tau^k.
\] \( (23) \)

The dimension of \( \varphi_{1,j}(\cdot) \) and \( \varphi_{2,j}(\cdot) \) is \( n \times n \); \( \varphi_{3,j}(\cdot) \) and \( \varphi_{4,j}(\cdot) \) are \( n \times n_d \); and \( \varphi_{5,j}(\cdot) \) and \( \varphi_{6,j}(\cdot) \) are \( n_d \times n \). Finally, \( W_{1,j,k} \) are constant complex matrices with compatible dimensions to \( \varphi_{1,j}(\cdot) \).

Lemma 4 requires the nontrivial solution (22) to be such that \( \omega_1(\tau) \neq 0 \) and \( \omega_2(\tau) \neq 0 \); otherwise, \( \forall \in [1, \ldots, 6], \omega_i(\tau) = 0 \).

From Lemma 5, \( \omega_1(\tau) = \omega_2(\tau + h) \), which implies that

\[
\sum_{j=1}^{j_{\text{max}}} e^{\lambda_j \tau} \left[ \varphi_{1,j}(\tau) - e^{-\lambda_j h} \varphi_{2,j}(\tau + h) \right] = 0_{n \times n}.
\]

which, by substituting (23) results in

\[
0_{n \times n} = \sum_{j=1}^{j_{\text{max}}} \left[ \lambda_j e^{\lambda_j \tau} \varphi_{1,j}(\tau) + e^{\lambda_j \tau} \varphi_{2,j}(\tau) \right] - e^{\lambda_j \tau} \varphi_{1,j}(\tau) A_0 - e^{\lambda_j (\tau - h)} \varphi_{1,j}(\tau - h) A_1 + \int_{\tau}^{\tau + h} e^{\lambda_j \theta} \varphi_{1,j}(\theta) A_D(\theta) d\theta.
\] \( (24) \)

Collecting the terms associated with \( e^{\lambda_j \tau} \), it follows that

\[
0_{n \times n} = \lambda_j \varphi_{1,j}(\tau) + \varphi_{2,j}(\tau - h) A_1 - e^{\lambda_j h} \varphi_{1,j}(\tau - h) A_1 - \int_{\tau}^{\tau + h} e^{\lambda_j \theta} \varphi_{1,j}(\theta + \tau) A_D(\theta) d\theta.
\] \( (25) \)

and hence

\[
\det \left( -\lambda_j I - A_1 - \int_{\tau}^{\tau + h} A_D(\theta) e^{-\lambda_j \theta} d\theta \right) = 0_{n \times n}.
\] \( (26) \)

Not that \( \omega_1(\tau) \neq 0 \Rightarrow \exists j_0 : \varphi_{1,j_0}(\tau) \neq 0 \.

Collecting the \( e^{\lambda_j \tau} \) terms

\[
0_{n \times n} = \sum_{k=0}^{k_{\text{max}}(j_0)} \left[ \lambda_{j_0} W_{1,j_0,k} \tau^k + k W_{1,j_0,k} \tau^{k-1} - W_{1,j_0,k} A_0 \tau^k - e^{\lambda_{j_0} h} W_{1,j_0,k} A_1 (\tau - h)^k - f_{\tau}^0 e^{\lambda_{j_0} \theta} W_{1,j_0,k} A_D(\theta)(\theta + \tau)^k d\theta \right].
\] \( (27) \)

Collecting the \( e^{\lambda_{j_0} \tau} \) terms, it must be that

\[
0_{n \times n} = \lambda_{j_0} W_{1,j_0,h_{\text{max}}(j_0)} - W_{1,j_0,h_{\text{max}}(j_0)} A_0 - e^{\lambda_{j_0} h} W_{1,j_0,h_{\text{max}}(j_0)} A_1 - \int_{\tau}^{\tau + h} e^{\lambda_{j_0} \theta} W_{1,j_0,h_{\text{max}}(j_0)} A_D(\theta) d\theta = 0_{n \times n}.
\] \( (28) \)

Note that \( W_{1,j_0,h_{\text{max}}(j_0)} \neq 0_{n \times n} \) because (23) is of degree \( k_{\text{max}}(j_0) \).

Equation (28) implies that there exists a nonzero row of \( W_{1,j_0,h_{\text{max}}(j_0)} \), which is orthogonal to every column of \( \lambda_{j_0} I - A_0 - e^{-\lambda_{j_0} h} A_1 - \int_{\tau}^{\tau + h} e^{\lambda_{j_0} \theta} A_D(\theta) d\theta \).

This requires the dimension of the column space of (29) to be less than \( n \), and it must be that

\[
\det \left( -\lambda_{j_0} I - A_0 - e^{-\lambda_{j_0} h} A_1 - \int_{\tau}^{\tau + h} e^{\lambda_{j_0} \theta} A_D(\theta) d\theta \right) = 0.
\] \( (29) \)

Hence, \( \lambda_{j_0} \) is an eigenvalue of (1) in addition to being an eigenvalue of ODE (6a). Next, it is shown that \( -\lambda_{j_0} \) is also an eigenvalue of (1).

Substituting \( \omega_1(\tau + h) = \omega_1(\tau) \) (in (88), Lemma 1 yields

\[
\omega_2(\tau) = A_0 \omega_2(\tau) - A_1 \omega_2(\tau) + \int_{\tau}^{\tau + h} A_D(\theta) \omega_2(\tau + \theta) d\theta.
\]

Note that from (24) and (26), we have for \( j_0 \)

\[
\varphi_{1,j_0}(\tau) = e^{\lambda_{j_0} h} \varphi_{2,j_0}(\tau + h)
\]

\[
W_{1,j_0,h_{\text{max}}(j_0)} = e^{\lambda_{j_0} h} W_{1,j_0,h_{\text{max}}(j_0)} \neq 0_{n \times n}.
\]

Similar to (25), we get

\[
0_{n \times n} = \lambda_{j_0} W_{2,j_0,h_{\text{max}}(j_0)} - W_{2,j_0,h_{\text{max}}(j_0)} A_0 - e^{\lambda_{j_0} h} W_{2,j_0,h_{\text{max}}(j_0)} A_1 - \int_{\tau}^{\tau + h} e^{\lambda_{j_0} \theta} W_{2,j_0,h_{\text{max}}(j_0)} A_D(\theta)(\theta + \tau)^k d\theta.
\] \( (27) \)

and hence

\[
\det \left( -\lambda_{j_0} I - A_0 - e^{-\lambda_{j_0} h} A_1 - \int_{\tau}^{\tau + h} A_D(\theta) e^{-\lambda_{j_0} \theta} d\theta \right) = 0_{n \times n}.
\] \( (28) \)

Hence, \( -\lambda_{j_0} \) is an eigenvalue of (1), and the spectrum condition is not satisfied. Note that if \( \lambda_{j_0} = 0 \neq -\lambda_{j_0} \), then we get

\[
0_{n \times n} = W_{1,j_0,h_{\text{max}}(j_0)} (A_0 - A_1 - \int_{\tau}^{\tau + h} A_D(\theta) d\theta)
\]

and hence 

\[
\det (A_0 - A_1 - \int_{\tau}^{\tau + h} A_D(\theta) d\theta) = 0_{n \times n},
\]

and the spectrum condition is satisfied.

\[ \blacktriangleleft \]

B. Relation to the DDEs

DDE (2a) requires \( \forall \tau \in [0, h] \) values of \( P(\cdot) \) from \([-h, 0] \) acting as an initial function. Our objective is to eliminate the requirement for an initial function and rely on initial conditions only. A step in that direction is to use the symmetry property (2b) to rewrite DDE (2a) such that it reads from the positive time interval \([0, h] \), only.

This introduces counterflow terms that we eliminate by introducing auxiliary variables. The problem then becomes an initial value problem for a linear time-invariant system. This is the spirit of the next results.

Lemma 6: If a solution to DDEc (2) exists, then a solution to ODEc (6) exists.

Proof: We show that from an existing DDEc (2) solution, we construct a solution to ODEc (6) on \([0, h] \) and then extend it to \( \forall \tau \in \mathbb{R} \).

First, consider the terms of (2a) and apply the symmetry property (2b) to write the counterflow backward-in-time term...
\[ P(\tau - h) = P(h - \tau)^T. \] Similarly, for
\[ \int_0^\tau P(\tau + \theta)A_D(\theta)d\theta = \int_0^\tau P(\tau + \theta)^T A_D(\theta)d\theta \]
and noting that
\[ \int_0^\tau P(\tau + \theta)A_D(\theta)d\theta = \int_0^\tau P(-\tau + \theta)^T A_D(\theta)d\theta \]
one can write
\[ \int_0^\tau P(\tau + \theta)A_D(\theta)d\theta = \int_0^\tau P(\tau + \theta)^T A_D(\theta)d\theta \]
Equation (2a) now includes two counterflow terms becoming
\[ \dot{P}(\tau) = P(\tau)A_0 + P(h - \tau)^T A_1 + \int_0^\tau P(\tau + \theta)A_D(\theta)d\theta \]
\[ + \int_0^\tau P(h - (\tau + \theta + h))^T A_D(\theta)d\theta. \]
(30)
The dynamics governing both \( \dot{P}(\tau) \) and \( P(\tau) \) is
\[ \dot{P}(\tau) = P(\tau)A_0 + P(h - \tau)^T A_1 \]
\[ + \int_0^\tau P(\tau + \theta)C_d e^{A_d \theta}d\theta B_d \]
\[ - B^T_1 \int_0^{\tau+h} (C_d e^{A_d \theta})^T P(\tau + \theta - h)d\theta \]
\[ - B^T_1 \int_0^{\tau+h} (C_d e^{A_d \theta})^T P(h - (\tau + \theta - h))^T d\theta. \]
(31a)
If \( p(\tau) \) is a solution to (2), then it also satisfies (31). Let \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau)\} \) be such that
\[ \omega_1(\tau) = p(\tau), \quad \omega_2(\tau) = p(h - \tau)^T \]
\[ \omega_3(\tau) = \int_0^\tau p(\tau + \theta)C_d e^{A_d \theta}d\theta \]
\[ \omega_4(\tau) = \int_0^{\tau+h} (C_d e^{A_d \theta})^T (\tau + \theta - h)d\theta \]
\[ \omega_5(\tau) = \int_0^{\tau+h} (C_d e^{A_d \theta})^TP(h - (\tau + \theta - h))^T d\theta. \]
We then have the following dynamics:
\[ \dot{\omega}_3(\tau) = -\omega_3(\tau)A_d + \omega_4(\tau)C_d \]
\[ \dot{\omega}_4(\tau) = -\omega_4(\tau)A_d + \omega_5(\tau)C_d e^{-A_d h} \]
\[ \dot{\omega}_5(\tau) = A_1^T \omega_5(\tau) + (C_d e^{-A_d h})^T \omega_1(\tau) \]
\[ \dot{\omega}_6(\tau) = A_1^T \omega_6(\tau) - C_d e^{-A_d h} \omega_2(\tau). \]
(32a)
(32b)
(32c)
(32d)
The dynamics (31) and (32) together give
\[ \dot{\omega}_1(\tau) = \omega_1(\tau)A_0 + \omega_2(\tau)A_1 + \omega_3(\tau)B_d + \omega_4(\tau)B_d \]
\[ \dot{\omega}_2(\tau) = -A_1^T \omega_1(\tau) - A_0^T \omega_2(\tau) - B^T_2 \omega_3(\tau) - B^T_2 \omega_6(\tau) \]
\[ \dot{\omega}_3(\tau) = -\omega_3(\tau)A_d + \omega_4(\tau)C_d \]
\[ \dot{\omega}_4(\tau) = -\omega_4(\tau)A_d - \omega_5(\tau)C_d e^{-A_d h} \]
\[ \dot{\omega}_5(\tau) = A_1^T \omega_5(\tau) + (C_d e^{-A_d h})^T \omega_1(\tau) \]
\[ \dot{\omega}_6(\tau) = A_1^T \omega_6(\tau) - C_d e^{-A_d h} \omega_2(\tau). \]
(33)
From (2c), \(-Q = \dot{p}(0^+) - \dot{p}(0^-) = \dot{\omega}_1(0) - \dot{\omega}_2(0)\), and we have
\[ -Q = \omega_1(0)A_0 + \omega_2(0)A_1 + \omega_3(0)B_d + \omega_4(0)B_d \]
\[ + A_1^T \omega_1(0) + A_0^T \omega_2(0) + B^T_2 \omega_3(0) + B^T_2 \omega_6(0) \]
\[ 0 = \omega_5(0) - \omega_5(0) = \omega_6(0) \]
(34a)
(34b)
(34c)
Equations (33) and (34) satisfy ODEc (6) restricted on \([0, h]\). This can be extended for \(\forall \tau \in \mathbb{R} \) by integrating (33) forward from \(\tau = h\) and backward from \(\tau = 0\). The result is a solution to ODEc (6).

Lemma 7: If \( \{\omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau)\} \) is a solution to ODEc (6), then there exists a solution \( p(\cdot) \) to DDEc (2) on \([-h, h] \) given by
\[ \forall \tau \in [0, h], \begin{cases} p(\tau) = \frac{1}{2} [\omega_1(\tau) + \omega_1^*(h - \tau)] \\ p(-\tau) = \frac{1}{2} [\omega_1(\tau) + \omega_1^*(h - \tau)]^T \end{cases} \]
(35)
Proof: Restrict the solution of ODEc (6) to \([0, h]\) and let \( \forall \tau \in [0, h], \eta(\tau) = \frac{1}{2} [\omega_1(\tau) + \omega_1^*(h - \tau)] \). Observe from (6c) that \( \eta(0) = \frac{1}{2} [\omega_1(0) + \omega_1^*(0)] = \eta(0)^T \). Construct \( p(\cdot) \) on \((0, h]\) and \([-h, 0)\) as follows:
\[ \forall \tau \in (0, h], \begin{cases} p(\tau) \triangleq \eta(\tau) \\ p(-\tau) \triangleq \eta(\tau)^T \end{cases} \]
(36)
giving
\[ p(-\tau) = p(\tau)^T \forall \tau \in [0, h] \]
(36)
which is the symmetry property (2b). Note that (36) is not saying that \( p(\tau) = \frac{1}{2} [\omega_1(\tau) + \omega_1^*(h - \tau)] \) on \([-h, h] \). Note that
\[ 2 \dot{p}(\tau) = \dot{\omega}_1(\tau) - \dot{\omega}_2(\tau)^T \]
\[ = \omega_1(\tau)A_0 + \omega_2(\tau)A_1 + \omega_3(\tau)B_d + \omega_4(\tau)B_d \]
\[ + \int_0^\tau \omega_5(\tau + \theta)A_D(\theta)d\theta + \int_0^\tau \omega_6(\tau + \theta)A_D(\theta)d\theta \]
\[ + \int_0^\tau \omega_5(-\tau - \theta)^T A_D(\theta)d\theta \]
\[ + \int_0^\tau \omega_6(-\tau - \theta)^T A_D(\theta)d\theta \]
and hence
\[ \dot{p}(\tau) = p(\tau)A_0 + p(h - \tau)^T A_1 \]
\[ + \int_0^\tau p(\tau + \theta)A_D(\theta)d\theta \]
\[ + f^0_h p(h - \tau - \theta)^T A_D(\theta)d\theta \]
Using (36), we get
\[ \dot{p}(\tau) = p(\tau)A_0 + p(h - \tau)A_1 + f^0_h A_D(\theta)^T p(-\theta) + p(\theta)A_D(\theta)d\theta \]
which is the dynamical relation (2a). It remains to show that
\[ A_1^T p(0) + p(0)A_0 + A_1^T p(h) + p(-h)A_1 \]
\[ + f^0_h [A_D(\theta)^T p(-\theta) + p(\theta)A_D(\theta)d\theta] \]
This follows from the relations
\[ A_0^T p(0) + p(0)A_0 + A_0^T p(h) + p(-h)A_1 \]
\[ + A_0^T \omega_1(0) + A_0^T \omega_1(0) + \frac{1}{2} [A_0^T \omega_2(0) + A_0^T \omega_2(0)] \]
\[ + \omega_1^*(h)A_1 + \omega_1^*(h)A_0 \]
\[ + f^0_h A_D(\theta)^T p(-\theta) + p(\theta)A_D(\theta)d\theta \]
\[ + \omega_1^*(h + \theta) + \omega_1^*(h + \theta)A_D(\theta)d\theta \]
\[ + f^0_h A_D(\theta)^T p(-\theta) - A_1^T \omega_1(\theta) - A_0^T \omega_2(\theta) - B^T_2 \omega_3(\theta) - B^T_2 \omega_6(\theta) \]
\[ + A_1^T \omega_5(\theta) + (C_d e^{-A_d h})^T \omega_1(\theta) \]
\[ + A_1^T \omega_6(\theta) - C_d e^{-A_d h} \omega_2(\theta). \]

and using (6b) and (6c) to get
\[
\frac{1}{2} [\omega_1(0)A_0 + \omega_2(0)A_1 + \omega_3(0)B_d + A_1^T \omega_1(0) + A_0^T \omega_3(0) + B_d^T \omega_2(0)]
+ \omega_1^2(h)A_1 + \omega_3^2(0)A_0 + \frac{1}{2} \omega_2^2(h)B_d = -Q.
\]

**Theorem 2:** For all \( Q \), DDEc (2) has a unique solution if and only if ODEc (6) has a unique solution.

**Proof:** Sufficiency: Assume ODEc (6) has a unique solution denoted by \( \{ \omega_1(\tau), \omega_2(\tau), \omega_3(\tau), \omega_4(\tau), \omega_5(\tau), \omega_6(\tau) \} \). From Lemma 7, there exists a solution to DDEc (2) on \([a, b] \) given by
\[
\forall \tau \in [0, h], \left\{ \begin{array}{l}
\omega_1(\tau) = \omega_1^2(h - \tau) + \omega_1(0) \\
\omega_2(\tau) = \omega_2^2(h - \tau) + \omega_2(0) \\
\omega_3(\tau) = \omega_3^2(h - \tau) + \omega_3(0)
\end{array} \right.
\]
From Lemma 3, namely, \( \omega_1(\tau) = \omega_1^2(h - \tau) \), we simplify to have
\[
\forall \tau \in [0, h], \left\{ \begin{array}{l}
p_1(\tau) = p_1^2(h - \tau) + p_1(0) \\
p_2(\tau) = p_2^2(h - \tau) + p_2(0) \\
p_3(\tau) = p_3^2(h - \tau) + p_3(0)
\end{array} \right.
\]
Moreover, \( p_1(\tau) \) is a unique solution to DDEc (2). To see this, assume that there exists an arbitrary solution to DDEc (2) denoted by \( p_1(\tau) \). From Lemma 6, a solution to ODEc (6) on \([0, h] \) is given by
\[
\left\{ p_1^2(\tau), p_2(\tau - \tau)^T, \ldots, p_n(\tau) \right\} = \left( C_d e^{A_d h} \right)^T \left( h - (\tau - \theta) \right)^T d\theta.
\]
Uniqueness of solutions to ODEc (6) requires that on \([0, h] \), \( p_1^2(\tau) = \omega_1(\tau) = p_1^2(\tau) \), and thus, DDEc (2) has a unique solution.

**Necessity:** This requires showing that if DDEc (2) has a unique solution, then ODEc (6) has a unique solution. By contradiction, assume that ODEc (6) has a nonunique solution. This means a nontrivial solution to the ODEc exists for \( Q = 0 \), which from Lemma 7 can generate a solution to DDEc (2) with \( Q = 0 \) denoted by \( p_1(\tau) \). This implies that given any arbitrary symmetric \( Q \) and an associated solution \( p_1(\tau) \), then \( p_1(\tau) + p_2(\tau) \) is also a solution for the same \( Q \) resulting in DDEc (2) having a nonunique solution.

**Corollary 1:** For all \( Q \), DDEc (2) has a unique solution if and only if the spectrum condition in Definition 1 is satisfied.

**Proof:** Follows directly from Theorems 1 and 2.

IV. ANALYTIC SOLUTION

Since a solution to DDEc (2) can be derived from a solution to ODEc (6), the objective of this section is to solve the ODEc analytically by writing it as an initial value problem. This requires solving a linear system of \( n_s \) scalar equations. We assume in this section that the spectrum condition holds. Using \( \text{vec}(A + B) = \text{vec}(A) + \text{vec}(B) \), \( \text{vec}(AB) = (B^T \otimes A) \text{vec}(D) \) from [17], (6a) is equivalently written as
\[
\begin{bmatrix}
\text{vec}(\Omega_1(\tau)) \\
\vdots \\
\text{vec}(\Omega_n(\tau))
\end{bmatrix} = E
\begin{bmatrix}
\text{vec}(\Omega_1(\tau)) \\
\vdots \\
\text{vec}(\Omega_n(\tau))
\end{bmatrix}
\]
and (6b)–(6d) are written as
\[
\begin{bmatrix}
-\text{vec}(Q) \\
\vdots \\
0
\end{bmatrix} = F_1
\begin{bmatrix}
\text{vec}(\Omega_1(0)) \\
\vdots \\
\text{vec}(\Omega_n(0))
\end{bmatrix} + F_2
\begin{bmatrix}
\text{vec}(\Omega_1(h)) \\
\vdots \\
\text{vec}(\Omega_n(h))
\end{bmatrix}
\]
where \( E, F_1, \) and \( F_2 \) are \( n_s \times n_s \) constant matrices, written as follows:
\[
E = \begin{bmatrix}
A_0^T \otimes I_n & A_1^T \otimes I_n & B_1^T \otimes I_n \\
\vdots & \vdots & \vdots \\
-\text{vec}(Q) & \text{vec}(\Omega_1(0)) & \text{vec}(\Omega_n(0))
\end{bmatrix}
\]
\[
F_1 = \begin{bmatrix}
A_0^T \otimes I_n & A_1^T \otimes I_n & B_1^T \otimes I_n \\
\vdots & \vdots & \vdots \\
-\text{vec}(Q) & \text{vec}(\Omega_1(0)) & \text{vec}(\Omega_n(0))
\end{bmatrix}
\]
\[
F_2 = \begin{bmatrix}
A_0^T \otimes I_n & A_1^T \otimes I_n & B_1^T \otimes I_n \\
\vdots & \vdots & \vdots \\
-\text{vec}(Q) & \text{vec}(\Omega_1(0)) & \text{vec}(\Omega_n(0))
\end{bmatrix}
\]
Note that using ODE (37), the linear system of unknowns (38) becomes
\[
\begin{bmatrix}
-\text{vec}(Q) \\
\vdots \\
0
\end{bmatrix} = (F_1 + F_2 e^{E h})
\begin{bmatrix}
\text{vec}(\Omega_1(0)) \\
\vdots \\
\text{vec}(\Omega_n(0))
\end{bmatrix}
\]
where by the following Corollary 2, \( F_1 + F_2 e^{E h} \) is nonsingular. Note that
\[
\begin{bmatrix}
\text{vec}(\Omega_1(\tau)) \\
\vdots \\
\text{vec}(\Omega_n(\tau))
\end{bmatrix} = e^{E \tau}
\begin{bmatrix}
\text{vec}(\Omega_1(0)) \\
\vdots \\
\text{vec}(\Omega_n(0))
\end{bmatrix}
\]
generates \( \Omega_1(\tau) \) and, hence, solves \( P(\tau) \) analytically.

**Corollary 2:** The matrix \( F_1 + F_2 e^{E h} \) in (39) is nonsingular if and only if the spectrum condition in Definition 1 is satisfied.

**Proof:** To show necessity, assume that the spectrum condition is not satisfied. Then, by Theorem 1, there exists a nontrivial solution for (6) when \( Q = 0 \). From the Kronecker representation (37) and (38) of ODEc (6), the nontrivial solution satisfies (39) for \( Q = 0 \). Therefore, \( \exists \nu \neq 0 : v \in \ker(F_1 + F_2 e^{E h}) \), and hence, \( F_1 + F_2 e^{E h} \) is singular.

Conversely, to show sufficiency, assume that \( \exists \nu \neq 0 : v \in \ker(F_1 + F_2 e^{E h}) \). Then, from (37) and (38), there exists a nontrivial solution for (6) when \( Q = 0 \), and by Theorem 1, the spectrum condition is not satisfied.

An example incorrectly handled in [1], as noted in Section II-A, is treated next.
Example 1: Consider the following linear time-delay system:
\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - 1) + \int_0^\infty A_D(\theta)x(t + \theta)d\theta
\]
\[
A_0 = -I_2, A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B_0 = 0.3I_2, B_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} B_0,
\]
\[
A_D(\theta) = \sin(\pi\theta) B_0 + \cos(\pi\theta) B_1.
\]
The system is stable and the spectrum condition holds, which can be shown by plotting the spectrum of the time-delay system. To use ODEc (6), put \(A_D(\theta)\) in the form \(A_D(\theta) = C_d e^{A_d \theta} B_d\) as follows:
\[
A_d = \pi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C_d e^{A_d \theta} = \begin{bmatrix} \cos(\pi\theta) & -\sin(\pi\theta) \\ \sin(\pi\theta) & \cos(\pi\theta) \end{bmatrix},
\]
\[
B_d = I_2, B_d = B_1.
\]
From (39), and letting \(Q = I_2\), we get \(\text{vec}(\Omega_1(0)) = \text{vec}(\Omega_2(0)) = 0\) for the resulting ODEc via (40), we get \(P(\tau)\) on \([0, h]\), as shown in Fig. 1. Note that \(P(\tau) = \Omega_2(0) = 0.7072 x I_2\). From (5a), \(V(\phi) = (\phi(0))^{\top} P(0) \phi(0)\) is the cost over the trajectories of (1) assuming an initial function such that \(\phi(0) \neq 0\) and \(\phi(0) = 0\) for \(h \leq \theta < 0\). This is validated by writing \(y(t) = \int_{t-\theta}^{t} e^{A_d \theta} B_d x(t + \theta) d\theta\) to get
\[
\dot{x}(t) = A_0 x(t) + C_d y(t) + A_1 x(t - 1)
\]
\[
\dot{y}(t) = B_d x(t) - A_d y(t) - e^{A_d \theta} B_d x(t - 1).
\]
One can then use a DDE solver, such as dde23 of MATLAB in our case, to compute the cost-to-go of (41) for different \(\phi(0)\) to validate \(V(\phi)\), and thus implicitly \(P(0)\).

Remark 1: Equation (1) can be written as (41) for \(A_D(\theta) = C_d e^{A_d \theta} B_d\). One may use existing results for a constant delay system (see [15]) to obtain an analytic solution \(P(\tau)\) for the resulting DDEc system whose DDE is double the rows and double the columns of the original DDEc (2a). However, writing \(P(\tau)\) in terms of \(P(\tau)\) is not obvious.

Remark 2: In recent work [19] (see also [20]) the author comments on the insufficiency of the boundary conditions provided in [1]. To address existence and uniqueness, the author adds a new group of boundary conditions in the form of integral constraints. The result, therefore, is no longer an ODE system with algebraically related split-boundary conditions.

V. Conclusion
This work provides a method to compute the quadratic cost functional (5a) for systems with distributed delays. Future directions include extensions to multiple commensurate delays and generalized quadratic cost functionals suitable for optimal control applications.

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