Improved Lower Bounds on the Domination Number of Hypercubes and Binary Codes with Covering Radius One

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Abstract
A dominating set on an $n$-dimensional hypercube is equivalent to a binary covering code of length $n$ and covering radius 1. It is still an open problem to determine the domination number $\gamma(Q_n)$ for $n \geq 10$ and $n \neq 2^k, 2^k - 1$ ($k \in \mathbb{N}$). When $n$ is a multiple of 6, the best known lower bound is $\gamma(Q_n) \geq \frac{2^n}{n}$, given by Van Wee (1988). In this article, we present a new method using congruence properties due to Laurent Habsieger (1997) and obtain an improved lower bound $\gamma(Q_n) \geq \frac{(n-2)2^n}{n^2-3n-2}$ when $n$ is a multiple of 6.

1. Introduction
Determining the domination number is an important optimization problem in graph theory, as well as an NP-complete problem in computational complexity theory [23]. The domination problem on hypercubes is equivalent to the covering code problem. Generic introductions to domination problems and covering codes can be found in [21][22].

The $n$-dimensional hypercube $Q_n$ is defined recursively in terms of the cartesian product of graphs as follows,

$$Q_1 = K_2, \quad Q_n = K_2 \square Q_{n-1}. \tag{1.1}$$

Therefore, $Q_n$ can also be defined as

$$V(Q_n) = 2^{\{1,2,\ldots,n\}}, \quad E(Q_n) = \{uv : u, v \in V(Q_n), u \subset v, \text{ and } |v \setminus u| = 1\}. \tag{1.2}$$

To avoid confusion, we use small brackets to express vertices. For instance, the vertex $\{2,3,5\}$ is written as $(2,3,5)$. Note that the vertex $\emptyset$ is written as $(0)$. Moreover, we define $(a_1,\ldots,a_i,0) \equiv (a_1,\ldots,a_i)$ to simplify some of our arguments.

Given $S \subseteq V(Q_n)$, we define the function $g$ to express all different members in the union of the coordinates of the vertices in $S$. That is,

$$g(S) := \bigcup_{v \in S} v. \tag{1.3}$$

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Moreover, let $S[a]$ indicate the subset of $S$ in which the number $a$ is contained in the coordinates of every vertex. That is,

$$S[a] := \{ v : a \in v \text{ and } v \in S \}.$$  \hfill (1.4)

e.g. For $S = \{(1,2,3), (2,5), (3,5)\}$, $g(S) = \{1,2,3,5\}$ and $S[5] = \{(2,5), (3,5)\}$.

For some $v \in V(Q_n)$ and $S \subseteq V(Q_n)$, we define the neighborhood of $v$ and $S$ as follows.

$$N_i[v] = \{ u \in V(Q_n) : d(u,v) = i \}, N_i[S] = \bigcup_{v \in S} N_i[v].$$

definition 1. Given a dominating set $D$ and $S \subseteq V(Q_n)$, we denote the excess of $D$ on $S$ by $\delta_S(D)$, which is defined as $\sum_{v \in S} (|N[v] \cap D| - 1)$. When $D$ is clear, we briefly write $\delta_S := \delta_S(D)$. Also, if $S = \{v\}$, then $\delta_v := \delta_{\{v\}}$.

The term excess has been used in many previous works, such as [1][4]. We further define the symbols below. Likewise, the $D$ can be omitted if it’s clear.

$$V\delta^x(D) := \{ v : \delta_v(D) = x \text{ and } v \in V(Q_n) \},$$

$$V\delta(D) := \bigcup_{x \geq 1} V\delta^x(D), \quad C(D) := \bigcup_{x \geq 2} V\delta^x(D).$$  \hfill (1.6)

Previous studies came up with various congruence properties of $\delta_{N[v]}(D)$, which help to estimate $\delta_{V(Q_n)}(D)$, and thus obtained the lower bounds on $\gamma(Q_n)$ due to the relation $\delta_{V(Q_n)}(D) = (n + 1)|D| - |V(Q_n)|$ from [4]. Theorem 1 is a segment of the properties given by Laurent Habsieger [1], which we will apply.

**Theorem 1. (Habsieger)** When $n$ is a multiple of 6,

$$\delta_{N[v]}(D) \equiv 1 \pmod{2}, \text{ if } v \notin D.$$  \hfill (1.7)

$$\delta_{N[v]}(D) \equiv 0 \pmod{2}, \text{ if } v \in D.$$  \hfill (1.8)

$$\delta_{N_1[v]}(D) + \delta_{N_2[v]}(D) \equiv 0 \pmod{3}.$$  \hfill (1.9)
We then put forward the pivotal concept throughout this article, *surfeit*. Although it seems closely related to *excess*, we shall demonstrate that such further analysis is enough to improve the known bounds.

**Definition 2.** Given a dominating set $D$ and $S \subseteq V(Q_n)$, we denote the *surfeit* of $D$ on $S$ by $\zeta_S(D)$, which is defined as $\sum_{v \in S \setminus D}(\delta_{N[v]}(D) - 1)$. When $D$ is clear, we briefly write $\zeta_S := \zeta_S(D)$.

We further define the symbols below. Likewise, the $D$ can be omitted if it’s clear.

$$V \zeta^*(D) := \{v : \delta_{N[v]}(D) = x + 1 \text{ and } v \in V(Q_n) \setminus D\}, \quad V \zeta(D) := \bigcup_{x \geq 1} V \zeta^x(D) \quad (1.10)$$

We look into the cases when $n$ is a multiple of 6. By calculating $\zeta_{V(Q_n)}(D)$ using two different methods, we show that it leads to a contradiction if $\gamma(Q_n)$ is too small.

2. Generalities

All the arguments are considered in the cases when $n$ is a multiple of 6.

Given an arbitrary value $\gamma^*$, we assume that there exists a dominating set $D$ satisfying $|D| = \gamma^*$. We calculate $\zeta_{V(Q_n)}$ using two different methods, and write the value we obtain as $\zeta_{m1}$ and $\zeta_{m2}$, respectively. A dominating set should lead to $\zeta_{m1} = \zeta_{m2}$. However, we will show that there must be $\zeta_{m2} > \zeta_{m1}$ when $\gamma^*$ is too small, implying that such dominating set $D$ cannot exist, so $\gamma(Q_n) > \gamma^*$.

For all $v \in V(Q_n) \setminus D$, we have $\delta_{N[v]} \geq 1$ by (1.7), so the first method to calculate $\zeta_{V(Q_n)}$ holds. Note that for all $u \in V \delta^x$, we have $|N[u] \setminus D| = n - x$, and $\sum_{x \in \mathbb{N}} x|V \delta^x| = \delta_{V(Q_n)}$. The value obtained this way is written as $\zeta_{m1}$:

$$\zeta_{V(Q_n)} = \sum_{x \in \mathbb{N}} \sum_{u \in V \delta^x} x(n - x) - |V(Q_n) \setminus D|$$

$$= \sum_{x \in \mathbb{N}} x(n - x)|V \delta^x| - 2^n + |D|$$

$$= (n - 1)|V(Q_n)| - 2^n + |D| - \sum_{x \in \mathbb{N}} x(x - 1)|V \delta^x| =: \zeta_{m1}. \quad (2.1)$$

$\zeta_{m1}$ attains its maximum when $C = \emptyset$. We write this value as $\zeta_{\text{max}}$.

$$\zeta_{m1} \leq \zeta_{\text{max}} := (n - 1)|V(Q_n)| - 2^n + |D|. \quad (2.2)$$

Let us consider another method to calculate $\zeta_{V(Q_n)}$. The value obtained this way is written as $\zeta_{m2}$:

$$\zeta_{V(Q_n)} = \sum_{i \geq 0} \left(2i|V \zeta^{2i}| + (2i + 1)|V \zeta^{2i+1}| \right) = \sum_{i \geq 1} 2i|V \zeta^{2i}| =: \zeta_{m2}. \quad (2.3)$$

Note that by (1.7) we have $V \zeta^{2i+1} = \emptyset$.

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Lemma 1. For all $u \in V^{\delta_1}$, if $d(u, C) \geq 3$, then $|N[u] \cap V\zeta| \geq 3$.

Proof. Assume without loss of generality that $u = (0)$, $N[u] \cap D = \{(1), (a)\}$, where $a \in \{0, 2, 3, \ldots, n\}$, then $\{(0), (1, a)\} \subset V^{\delta_1}$. For convenience we assume that $a \neq 0$, since the case $a = 0$ can be dealt with similarly. Let $S = V\delta \cap (N_1[u] \cup N_2[u]) \setminus \{(1, a)\}$. Applying (1.7) and (1.8) on the vertices in $N_1[u]$, we have the following:

For all $k \in \{1, 2, \ldots, n\}$, there is $|S[k]| \equiv 0 \pmod 2$. (2.4)

Moreover, by applying (1.9) on $u$, we have $|S| \equiv 2 \pmod 3$. In particular, $|S| \geq 5$, otherwise there exists $k \in \{1, 2, \ldots, n\}$ such that $|S[k]| = 1$, contradicting (2.4).

Suppose that $|N[u] \cap V\zeta| \leq 2$, $N[u] \cap V\zeta \subseteq \{(b), (c)\}$ where $b, c \in \{0, 2, 3, \ldots, n\} \setminus \{(a)\}$, then $g(S) \subseteq \{1, a, b, c\}$. So let $T = \{(1), (a), (b), (c), (1, b), (1, c), (a, b), (a, c), (b, c)\}$, then $S \subseteq T$. If $b = 0$, then $S = T \setminus \{(0)\} = \{(1), (a), (b), (c), (1, c), (a, c)\}$, contradicting (2.4). Hence $b, c \neq 0$, $(0) \notin N[u] \cap V\zeta$, $|S \cap N_1[u]| = 0$, $S = T \cap N_2[u] = \{(1, b), (1, c), (a, b), (a, c), (b, c)\}$, but this still contradicts (2.4). Therefore, $|N[u] \cap V\zeta| \geq 3$. \hfill \Box

In other words, for each $u \in V^{\delta_1}$, there must be $d(u, C) \leq 2$ if $|N[u] \cap V\zeta| \leq 2$. In Lemma 2 we will show that $|N[u] \cap V\zeta| \leq 2$ gives us more rigorous conditions. To simplify our arguments, given $v \in C$, we divide $(N_1[v] \cup N_2[v]) \cap V^{\delta_1}$ into the following vertex sets:

$T_1(v) := (N_1[v] \setminus D) \cap V^{\delta_1}$;
$T_2(v) := \{u \in N_2[v] \cap V^{\delta_1} : |N[u] \cap N[v] \cap D| = 0\}$;
$T_3(v) := \{u \in N_2[v] \cap V^{\delta_1} : |N[u] \cap N[v] \cap D| = 2\}$; (2.5)
$T_4(v) := \{u \in N_2[v] \cap V^{\delta_1} : |N[u] \cap N[v] \cap D| = 1\}$; and
$T_5(v) := (N_1[v] \cap D) \cap V^{\delta_1}$.

Lemma 2. For all $u \in V^{\delta_1}$, if $|N[u] \cap V\zeta| \leq 2$, then $|N[u] \cap V\zeta| = 2$, and the following four claims hold.

1. There exists $v \in C \setminus D$ such that $u \in T_1(v) \cup T_2(v)$, or there exists $v \in C \cap D$ such that $u \in T_2(v)$.

2. For all $v \in C \setminus D$ such that $u \in T_1(v)$, we have $|N_1[v] \cap V\delta| \leq 3$. (2.6)

3. For all $v \in C$ such that $u \in T_2(v) \setminus D$, if we rename the coordinates so that $v = (0)$, $u = (a, b)$, then

$(a), (b) \notin V\delta$ and $|(N_2[v] \cap V\delta)[a]|, |(N_2[v] \cap V\delta)[b]| \leq 3$. (2.7)

4. For all $v \in C$ such that $u \in T_2(v) \cap D$, if we rename the coordinates so that $v = (0)$, $u = (a, b)$, then

$|(N_2[v] \cap V\delta)[a]|, |(N_2[v] \cap V\delta)[b]| \leq 2$. (2.8)

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**Proof.** Given \( v \in C \), we will prove the following statements.

(A) If \( u \in T_1(v), v \notin D \) and \( |N[u] \cap V\zeta| \leq 2 \), then \( |N[u] \cap V\zeta| = 2 \) and (2.6) holds for \( v \).

(B) If \( u \in T_2(v) \setminus D \) and \( |N[u] \cap V\zeta| \leq 2 \), then \( |N[u] \cap V\zeta| = 2 \) and (2.7) holds for \( v \).

(C) If \( u \in T_2(v) \cap D \) and \( |N[u] \cap V\zeta| \leq 2 \), then \( |N[u] \cap V\zeta| = 2 \) and (2.8) holds for \( v \).

(D) If \( u \in T_1(v), v \in D \) and \( |N[u] \cap V\zeta| \leq 2 \), then \( |N[u] \cap V\zeta| = 2 \) and there exists some \( v' \in C \setminus D \) such that \( u \in T_1(v') \).

(E) If \( u \in T_3(v) \cup T_4(v) \cup T_5(v) \) and \( u \notin T_1(v') \cup T_2(v') \) for all \( v' \in C \), then \( |N[u] \cap V\zeta| \geq 3 \).

(A), (B), and (C) prove Claims 2, 3, 4 in Lemma 2. By Lemma 1 there exists \( v_0 \in C \) with \( u \in \bigcup_{1 \leq i \leq 5} T_i(v_0) \), so Claim 1 follows by an application of (D) and (E) with \( v = v_0 \).

Below, (A) and (D) follow from Case 1, (B) and (C) follow from Case 2, while (E) follows from Cases 3 to 5. We assume without loss of generality that \( v = (0) \).

**Case 1-(1)** \( u \in T_1(v) \) and \( v \notin D \).

\( \{u, v\} \subseteq N[u] \cap V\zeta \), so \( |N[u] \cap V\zeta| \geq 2 \). If equality holds, then \( N[u] \cap V\zeta = \{u, v\} \).

Let \( u = (a) \notin D \). Since \( u \in V\delta^1 \), we have \( |g(N[u] \cap D)| = 3 \). So if \( |N_1[v] \cap V\delta| \geq 4 \), then there exists \( (k) \in N_1[v] \cap V\delta \) such that \( k \notin g(N[u] \cap D) \), \( (a, k) \in N[u] \cap V\zeta \), which is a contradiction. Hence \( |N_1[v] \cap V\delta| \leq 3 \) and (A) is proved.

**Case 1-(2)** \( u \in T_1(v) \) and \( v \in D \).

If \( \delta_v \geq 3 \), then assume without loss of generality that \( (1), (2), (3) \in D \) and that \( u = (4) \). \( u \notin C \), so \( |\{(1, 4), (2, 4), (3, 4)\} \cap D| \leq 1 \). Note that \( \{(u, (1, 4), (2, 4), (3, 4)) \setminus D \subseteq N[u] \cap V\zeta \), so \( |N[u] \cap V\zeta| \geq 3 \). Therefore, if \( |N[u] \cap V\zeta| = 2 \), then \( \delta_v = 2 \).

Let \( N[v] \cap D = \{(0), (1), (2)\} \) and \( u = (a) \notin D \). We have \( (1), (2), (a) \in V\delta \), so for \( w \in \{(1, a), (2, a)\} \), there is \( w \notin N[u] \cap V\zeta \) if and only if \( w \in D \). Thus, we assume that \( \{(1, a), (2, a)\} \cap D = \{(1, a)\} \), \( N[u] \cap V\zeta = \{u, (2, a)\} \).

There does not exist \( k \in \{3, 4, \ldots, n\} \setminus \{a\} \) such that \( (a, k) \in N[u] \cap V\zeta \). Therefore, \( (N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1), (2), (1, a), (2, a), (1, 2, a)\} \).

Applying (1.7) and (1.9) on \( u \), we get

\[ \delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1, a)} + \delta_{(2, a)} + \delta_{(1, 2, a)} \equiv 0 \mod 3, \]

and

\[ \delta_{(1, a)} + \delta_{(2, a)} \equiv 0 \mod 2. \]

Moreover, \( 1 \leq \delta_{(1, a)} \leq 2 \) since by (*) we know that \( N[(1, a)] \cap D \subseteq \{(1, a), (2, a)\} \).

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If $\delta_{(1,a)} = 2$, then $\{1, 2, a\} \in D$ and thus $\delta_{(2,a)} \geq 1$. By (2.10) we know $\delta_{(2,a)} \geq 2$.

If $\delta_{(1,a)} = 1$ and $(1, 2) \notin D$, then by (*) we know $\delta_{(1)} = 2$, $\delta_{(2)} = 1$, and $\delta_{(1, a)} = 0$. Now that $\delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1, a)} + \delta_{(1, 2, a)} = 6$, and by (2.10) we know $\delta_{(2,a)} \geq 1$, so (2.9) suggests that $\delta_{(2,a)} \geq 3$.

If $\delta_{(1,a)} = 1$ and $(1, 2) \in D$, then by (*) we know $\delta_{(1)} = 3$, $\delta_{(2)} = 2$, and $\delta_{(1, a)} = 1$. Now that $\delta_{(0)} + \delta_{(1)} + \delta_{(2)} + \delta_{(1, a)} + \delta_{(1, 2, a)} = 9$, and by (2.10) we know $\delta_{(2,a)} \geq 1$, so (2.9) suggests that $\delta_{(2,a)} \geq 3$.

Therefore, there must be $(2, a) \in C \setminus D$ and $u \in T_1((2, a))$. This shows that there exists $v' \in C \setminus D$ such that $u \in T_1(v')$, and (D) is proved.

**Case 2.** $u \in T_2(v)$.

Let $u = (a, b)$ where $(a), (b) \notin D$, then $\{(a), (b)\} \subseteq N[u] \cap V\zeta$, so $|N[u] \cap V\zeta| \geq 2$. Assume that equality holds. Consider the case $u \notin D$. We have $(a), (b) \notin V\delta$, for otherwise $\{u \in N[u] \cap V\zeta\}$, which is a contradiction. Also, $|g(N[u] \cap D)| = 4$, so if $|(N_2[v] \cap V\delta)[a]| \geq 4$, then there exists $w \in (N_2[v] \cap V\delta)[a]$, $\{b, k\} \subseteq N[u] \cap V\zeta$ such that

$$k \notin g(N[u] \cap D), \quad (a, b, k) \subseteq N[u] \cap V\zeta,$$

which is a contradiction. Likewise, we have $|(N_2[v] \cap V\delta)[b]| \leq 3$, so (B) is proved. The same argument can be applied to the case $u \in D$ and prove (C).

The following cases together prove (E).

**Case 3.** $u \in T_3(v)$ and $u \notin T_1(v') \cup T_2(v')$ for all $v' \in C$.

Let $(1), (2) \in D$ and $u = (1, 2)$. We prove $|N[u] \cap V\zeta| \geq 3$ by contradiction, assuming $|N[u] \cap V\zeta| \leq 2$. If $(a) \in D$ for some $a \in \{0, 3, 4, \ldots, n\}$, then $(1, 2) \in N[u] \cap V\zeta$. This implies $\delta_{(1)} \leq 3$, and since such an $a$ exists by $\delta_{(1)} \geq 2$, we have $|N[u] \cap V\zeta| \geq 1$.

If $|N[u] \cap V\zeta| = 1$, we let $N[u] \cap V\zeta = \{(1, 2, a)\}$, where $a \in \{0, 3, 4, \ldots, n\}$, then $(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1, a), (2, a)\}$ and $\delta_{(1,a)}, \delta_{(2,a)} \geq 1$. We know $\{(1, a), (2, a)\} \cap C \neq \emptyset$ by applying (1.9) on $u$. Let $(1, a) \in C$, then there exists $b \in \{0, 3, 4, \ldots, n\} \setminus \{a\}$ such that

$$(1, a, b) \in D, \quad (1, b) \in V\delta, \quad (1, 2, b) \in N[u] \cap V\zeta,$$

which is a contradiction. Therefore, $|N[u] \cap V\zeta| = 2$.

Now let $N[u] \cap V\zeta = \{(1, 2, 3), (1, 2, k)\}$, where $k \in \{0, 4, 5, \ldots, n\}$, then

$$(N_1[u] \cup N_2[u]) \cap V\delta \subseteq \{(0), (1, 3), (2, 3), (1, k), (2, k), (1, 2, 3, k)\}.$$  (**)

Thus, $N[v] \cap D \subseteq \{(1), (2), (3), (k)\}$. We denote the excesses as $\delta_{(0)} =: o, \delta_{(1, 3)} =: p, \delta_{(2, 3)} =: q, \delta_{(1,k)} =: r, \delta_{(2,k)} =: s, \delta_{(1,2,3,k)} =: t$, respectively. By (1.7), (1.8) and (1.9) we
derive the following relations (note that \(\delta_{(1,2)} = 1\)):

\[
\begin{align*}
& \quad o + p + r \equiv 1 \pmod{2}, \quad \text{since } \delta_{(1)} + \delta_{N_1[(1)]} \equiv 0 \pmod{2} ; \\
& o + q + s \equiv 1 \pmod{2}, \quad \text{since } \delta_{(2)} + \delta_{N_1[(2)]} \equiv 0 \pmod{2} ; \\
& p + q + t \equiv 0 \pmod{2}, \quad \text{since } \delta_{(1,2,3)} + \delta_{N_1[(1,2,3)]} \equiv 1 \pmod{2} ; \\
& r + s + t \equiv 0 \pmod{2}, \quad \text{since } \delta_{(1,2,k)} + \delta_{N_1[(1,2,k)]} \equiv 1 \pmod{2} ; \\
& o + p + q + r + s + t \equiv 0 \pmod{3}, \quad \text{since } \delta_{N_1[u]} + \delta_{N_2[u]} \equiv 0 \pmod{3} .
\end{align*}
\]

If \(o = 3\), then \(N[v] \cap D = \{(1), (2), (3), (k)\}\), and the following relations hold. Note that the restrictions leading to these results are due to (**).

\[
\begin{align*}
1 \leq p \leq 2, & \quad \text{since } N[(1,3)] \cap D \subseteq \{(1), (3), (1,3, k)\} \quad \text{and } (1), (3) \in D; \\
1 \leq q \leq 2, & \quad \text{since } N[(2,3)] \cap D \subseteq \{(2), (3), (2,3, k)\} \quad \text{and } (2), (3) \in D; \\
1 \leq r \leq 2, & \quad \text{since } N[(1,k)] \cap D \subseteq \{(1), (k), (1,3, k)\} \quad \text{and } (1), (k) \in D; \\
1 \leq s \leq 2, & \quad \text{since } N[(2,k)] \cap D \subseteq \{(2), (k), (2,3, k)\} \quad \text{and } (2), (k) \in D; \\
t \leq 1, & \quad \text{otherwise } u \in T_1((1,2,3,k)) \cup T_2((1,2,3,k)).
\end{align*}
\]

Now if \(o + p + q + r + s + t = 12\), then \(p = q = r = s = 2\) and \(t = 1\), contradicting (III), so the only possibility left is \(o + p + q + r + s + t = 9\). However, this implies that \((p + q + t) + (r + s + t) - t = 6\). By (III) and (IV) we know \(t = 0\), and together with (I) and (II) we know that \(p, q, r, s\) have the same parity, which is impossible. Therefore, \(|N[u] \cap V| \geq 3\).

If \(o = 2\), then we assume without loss of generality that \(N[v] \cap D \subseteq \{(1), (2), (k)\}\). This time we obtain \(p \leq 1, q \leq 1, 1 \leq r \leq 2, 1 \leq s \leq 2, t \leq 1\). If \(o + p + q + r + s + t = 9\), then \(p = q = 1, r = s = 2, t = 1\), contradicting (III), so the only possibility left is \(o + p + q + r + s + t = 6\). This implies that \((p + q + t) + (r + s + t) - t = 4\). By (III) and (IV) we know \(t = 0\), and \(p, q\) as well as \(r, s\) have the same parity. Thus \(p = q\) and \(r = s\), implying \(p + r = 2\), which contradicts (I). Therefore, \(|N[u] \cap V| \geq 3\).

**Case 4.** \(u \in T_3(v)\) and \(u \notin T_1(v') \cup T_2(v')\) for all \(v' \in C\).

If \(v \in D\), then we can prove \(|N[u] \cap V| \geq 3\) using the same method as in Case 3. We sketch our arguments in a simplified version, for they are highly similar to those in Case 3:

Let \((1), (2) \in D\) and \(u = (1)\). We prove our claim by contradiction, assuming that \(|N[u] \cap V| \leq 2\). Like in case 3 we see \(\delta_r \leq 3\) and \(|N[u] \cap V| = 2\). Assume without loss of generality that \(N[u] \cap V = \{(1,2), (1,3)\}\), then

\[
(N_1[u] \cup N_2[u]) \cap V \delta \subseteq \{(0), (2), (3), (1,2), (1,3), (1,2,3)\} .
\]

(***)

We denote the excesses as \(\delta_{(0)} =: o, \delta_{(2)} =: p, \delta_{(3)} =: q, \delta_{(1,2)} =: r, \delta_{(1,3)} =: s, \delta_{(1,2,3)} =: t,\)
respectively. By (1.7), (1.8), and (1.9) we derive the following relations (note that \( \delta_{(1)} = 1 \)):

\[
\begin{align*}
o + r + s &\equiv 1 \pmod{2}, \quad \text{since } \delta_{(1)} + \delta_{N[(1)]} \equiv 0 \pmod{2}; \\
o + p + q &\equiv 1 \pmod{2}, \quad \text{since } \delta_{(0)} + \delta_{N[(0)]} \equiv 0 \pmod{2}; \\
p + r + t &\equiv 0 \pmod{2}, \quad \text{since } \delta_{(1,2)} + \delta_{N[(1,2)]} \equiv 1 \pmod{2}; \\
q + s + t &\equiv 0 \pmod{2}, \quad \text{since } \delta_{(1,3)} + \delta_{N[(1,3)]} \equiv 1 \pmod{2}; \\
o + p + q + r + s + t &\equiv 0 \pmod{3}, \quad \text{since } \delta_{N[u]} + \delta_{N[v]} \equiv 0 \pmod{3}.
\end{align*}
\]

(VI) (VII) (VIII) (IX) (X)

If \( o = 3 \), then \( N[v] \cap D = \{(0), (1), (2), (3)\} \), and the following relations hold. Note that the restrictions leading to these results are due to (***)

\[
\begin{align*}
1 &\leq p \leq 2, \quad \text{since } N[(2)] \cap D \subseteq \{(0), (2), (2, 3)\} \text{ and } (0), (2) \in D; \\
1 &\leq q \leq 2, \quad \text{since } N[(3)] \cap D \subseteq \{(0), (3), (2, 3)\} \text{ and } (0), (3) \in D; \\
1 &\leq r \leq 2, \quad \text{since } N[(1, 2)] \cap D \subseteq \{(1), (2), (1, 2, 3)\} \text{ and } (1), (2) \in D; \\
1 &\leq s \leq 2, \quad \text{since } N[(1, 3)] \cap D \subseteq \{(1), (3), (1, 2, 3)\} \text{ and } (1), (3) \in D; \\
t &\leq 1, \quad \text{otherwise } u \in T_2((1, 2, 3)).
\end{align*}
\]

We have \( o + p + q + r + s + t = 9 \) or 12. The latter contradicts (VIII), while the former suggests that \((p + r + t) + (q + s + t) - t = 6\), and using (VI) to (IX) we know \( t = 0 \) and \( p, q, r, s \) have the same parity, which is impossible.

If \( o = 2 \), then \( N[v] \cap D \subseteq \{(0), (1), (2)\} \). This time we obtain \( 1 \leq p \leq 2, q \leq 1, 1 \leq r \leq 2, s \leq 1, t \leq 1 \), so we have \( o + p + q + r + s + t = 6 \) or 9, but again we can easily lead to contradictions using (VI) to (X). Therefore, \(|N[u] \cap V_\zeta| \geq 3\).

If \( v \notin D \), then let \( u = (1) \in D \). By \( \delta_v \geq 2 \) we may assume \( N[v] \cap D \supseteq \{(1), (2), (3)\} \). For \( w \in \{(1, 2), (1, 3)\} \), we have \( w \notin N[u] \cap V_\zeta \) if and only if \( w \in D \). Moreover, \( (0) \in N[u] \cap V_\zeta \). So if \(|\{(1, 2), (1, 3)\} \cap D| = 0\), then \(|N[u] \cap V_\zeta| \geq 3\). If not, let \((1, 2) \in D\), then \((1, 2) \in C \cap D\) and \( u \in T_5((1, 2)) \), implying \(|N[u] \cap V_\zeta| \geq 3\).

**Case 5.** \( u \in T_4(v) \) and \( u \notin T_1(v') \cup T_2(v') \) for all \( v' \in C \).

If \( v \in D \), then we let \((1), (2) \in N[v] \cap D, u = (1, a), \text{ where } (a) \notin D\). We have \( \{(0), (1), (1, 2), (1, a)\} \subset V_\delta \). Therefore, if \( \{(1, a), (1, 2, a)\} \cap D = \emptyset \), then \( \{(a), (1, a), (1, 2, a)\} \subseteq N[u] \cap V_\zeta \); if \( (1, 2, a) \in D \), then \( (1, 2) \in C \) and \( u \in T_3((1, 2)) \); if \( (1, a) \in D \), then \( (1) \in C \) and \( u \in T_5((1)) \).

On the other hand, if \( v \notin D \), then let \((1), (2), (3) \in N[v] \cap D, u = (1, a), \text{ where } (a) \notin D\). We have \( \{(1, 2, (1, 3)\} \subset V_\delta \) and \( (a) \in N[u] \cap V_\zeta \). Thus, if \( \{(1, 2, a), (1, 3, a)\} \cap D = \emptyset \), then \( \{(a), (1, 2, a), (1, 3, a)\} \subseteq N[u] \cap V_\zeta \); if \( (1, 2, a), (1, 3, a) \} \cap D \neq \emptyset \), then there exists \( w \in \{(1, 2), (1, 3)\} \subset C \) such that \( u \in T_5(w) \).

By Case 3 and Case 4, every possible condition above implies that \(|N[u] \cap V_\zeta| \geq 3\). □

Given \( v \in C \), we define \( S_1(v) := \{u \in T_1(v) : |N[u] \cap V_\zeta| = 2\} \). Lemma 3 gives an upper bound for \(|S_1(v) \cup S_2(v)|\) when \( v \notin D \) and an upper bound for \(|S_2(v)|\) when \( v \in D \).
Lemma 3. For \(v \in C\), we have \(|S_1(v) \cup S_2(v)| \leq \frac{3}{2}(n - \delta_v)\) if \(v \notin D\), and \(|S_2(v)| \leq \frac{3}{2}(n - \delta_v)\) if \(v \in D\).

Proof. Assume without loss of generality that \(v = (0)\).

If \(v \notin D\), then let \(\{(1), (2), \ldots, (\delta_v + 1)\} \subset D\).

Define \(A := g(S_2(v) \cap D)\) and \(B := g(S_2(v)) \setminus A\). By (2.7) and (2.8) we know that

\[
\forall k \in A, \quad |S_2(v)[k]| \leq 2; \quad \forall k \in B, \quad |S_2(v)[k]| \leq 3.
\]

So \(|S_2(v)| \leq \frac{1}{2}(2|A| + 3|B|)\).

By (2.7) we also know that for all \(k \in g(S_1(v))\), we have \(k \notin g(S_2(v) \setminus D)\), \(k \notin B\). Hence, \(|B| \leq n - \delta_v - 1 - |S_1(v)|\). Since \(|A| + |B| \leq n - \delta_v - 1\), we have

\[
|S_1(v) \cup S_2(v)| \leq |S_1(v)| + \frac{1}{2}\left(2|A| + 3|B|\right)
\]

\[
\leq |S_1(v)| + \frac{1}{2}\left(2|S_1(v)| + 3(n - \delta_v - 1 - |S_1(v)|)\right)
\]

\[
= |S_1(v)| + \frac{1}{2}\left(3n - 3\delta_v - 3 - |S_1(v)|\right).
\]

By (2.6) we know that \(|S_1(v)| \leq 3\), so \(|S_1(v) \cup S_2(v)| \leq \frac{3}{2}(n - \delta_v)\).

On the other hand, if \(v \in D\), then let \(\{(0), (1), (2), \ldots, (\delta_v)\} \subset D\).

By (2.7), for all \(k \in \{\delta_v + 1, \delta_v + 2, \ldots, n\}\), we have \(|S_2(v)[k]| \leq 3\). Therefore,

\[
|S_2(v)| \leq \frac{3}{2}\left|\{\delta_v + 1, \delta_v + 2, \ldots, n\}\right| = \frac{3}{2}(n - \delta_v),
\]

and Lemma 3 is proved. \(\square\)

By definition, \(\sum_{i \geq 1}(2i + 1)|V \zeta^{2i}| = \sum_{x \in N} x \sum_{u \in V \delta^x} |N[u] \cap V |\zeta|,\) and we can now estimate its lower bound using the results in Lemma 3.

Lemma 4. For \(n \geq 12\), the following inequality holds:

\[
\sum_{i \geq 1}(2i + 1)|V \zeta^{2i}| \\
\geq 3\delta_{V(Q_n)} - |V \delta^2| - 4.5|V \delta^{n-3}| - (n + 1)|V \delta^{n-2}| - (2n - 0.5)|V \delta^{n-1}| - 3n|V \delta^n|.
\]
Proof. By Claim 1 in Lemma 2, we have

\[ \{ u : |N[u] \cap V \zeta| - 3 < 0, u \in V \delta^1 \} = \bigcup_{v \in C \setminus D} (S_1(v) \cup S_2(v)) \cup \bigcup_{v \in C \cap D} (S_2(v)) \].

Therefore,

\[
\sum_{i \geq 1} (2i + 1)|V\zeta^{2i}|
= \sum_{x \in N} \sum_{u \in V \delta^x} |N[u] \cap V \zeta|
= 3\delta_{V(Q_n)} + \sum_{x \in N} \sum_{u \in V \delta^x} \left( |N[u] \cap V \zeta| - 3 \right)
= 3\delta_{V(Q_n)} + \sum_{v \in C} \delta_v \left( |N[v] \cap V \zeta| - 3 \right) + \sum_{u \in V \delta^1} \left( |N[u] \cap V \zeta| - 3 \right)
\geq 3\delta_{V(Q_n)} + \sum_{v \in C} \delta_v (n - \delta_v - 3) + \sum_{v \in C \setminus D, u \in S_1(v) \cup S_2(v)} \left( |N[u] \cap V \zeta| - 3 \right)
+ \sum_{v \in C \cap D} \sum_{u \in S_2(v)} \left( |N[u] \cap V \zeta| - 3 \right)
= 3\delta_{V(Q_n)} + \sum_{v \in C \setminus D} \left( \delta_v (n - \delta_v - 3) - |S_1(v) \cup S_2(v)| \right) + \sum_{v \in C \cap D} \left( \delta_v (n - \delta_v - 3) - |S_2(v)| \right)
\geq 3\delta_{V(Q_n)} + \sum_{v \in C} \left( \delta_v (n - \delta_v - 3) - \frac{3}{2} (n - \delta_v) \right).
\]

Note that the last inequality is due to Lemma 3. A short calculation shows that for \( n \geq 12, \)

\[
\delta_v (n - \delta_v - 3) - \frac{3}{2} (n - \delta_v) \geq \begin{cases} 
0, & \text{if } 3 \leq \delta_v \leq n - 4, \text{ or } \delta_v = 2 \text{ and } n \geq 18; \\
-1, & \text{if } \delta_v = 2 \text{ and } n = 12; \\
-4.5, & \text{if } \delta_v = n - 3; \\
-n - 1, & \text{if } \delta_v = n - 2; \\
-2n + 0.5, & \text{if } \delta_v = n - 1; \\
-3n, & \text{if } \delta_v = n,
\end{cases}
\]

and Lemma 4 follows. \( \square \)

Finally, we can estimate \( \zeta_{m2} - \zeta_{m1}. \)
Lemma 5. When $n \geq 12$, $\zeta_{m2} - \zeta_{m1} \geq 2\delta_{V(Q_n)} - \zeta_{\text{max}}$.

Proof. By (2.3) and Lemma 4, we have

$$\zeta_{m2} - 2\delta_{V(Q_n)} = \frac{1}{3} \left( \sum_{i \geq 1} (6i|V\zeta^{2i}|) - 6\delta_{V(Q_n)} \right)$$

$$= \frac{1}{3} \left( \sum_{i \geq 1} (2i - 2)|V\zeta^{2i}| + 2 \sum_{i \geq 1} (2i + 1)|V\zeta^{2i}| - 6\delta_{V(Q_n)} \right)$$

$$\geq \frac{1}{3} \left( \sum_{i \geq 1} (2i - 2)|V\zeta^{2i}| - 2|V\delta^2| - 9|V\delta^{n-3}| - (2n + 2)|V\delta^{n-2}|$$

$$- (4n - 1)|V\delta^{n-1}| - 6n|V\delta^n| \right).$$

By (2.1) and (2.2) we have

$$\zeta_{m1} - \zeta_{\text{max}} = -\sum_{x \in N} x(x - 1)|V\delta^x|$$

$$= -2|V\delta^2| - 6|V\delta^3| - \ldots - (n - 3)(n - 4)|V\delta^{n-3}| - (n - 2)(n - 3)|V\delta^{n-2}|$$

$$- (n - 1)(n - 2)|V\delta^{n-1}| - n(n - 1)|V\delta^n|.$$

Therefore, $\zeta_{m2} - 2\delta_{V(Q_n)} \geq \zeta_{m1} - \zeta_{\text{max}}$ when $n \geq 12$ and Lemma 5 follows.

Theorem 2. If $n \equiv 0 \pmod{6}$, then $\gamma(Q_n) \geq \frac{(n - 2)2^n}{n^2 - 2n - 2}$.

Proof. Theorem 2 holds true for $n = 6$ by $\gamma(Q_6) = 12$, so assume $n \geq 12$. Consider a minimum dominating set of $Q_n$. We have

$$\delta_{V(Q_n)} = (n + 1)\gamma(Q_n) - 2^n; \quad \zeta_{\text{max}} = (n - 1)\delta_{V(Q_n)} - 2^n + \gamma(Q_n) = n^2\gamma(Q_n) - n2^n.$$ 

By Lemma 5, $\zeta_{m2} - \zeta_{m1} \geq 2\delta_{V(Q_n)} - \zeta_{\text{max}}$, so there must be $\zeta_{\text{max}} - 2\delta_{V(Q_n)} \geq 0$,

$$(n^2 - 2n - 2)\gamma(Q_n) - (n - 2)2^n \geq 0, \quad \gamma(Q_n) \geq \frac{(n - 2)2^n}{n^2 - 2n - 2}.$$

Corollary 1. $\gamma(Q_{12}) \geq 348$, $\gamma(Q_{18}) \geq 14666$, $\gamma(Q_{24}) \geq 701709$, $\gamma(Q_{30}) \geq 35876816$.  

October 23, 2023
3. Conclusion

Tables 1 and 2 in the appendix are due to Gerzson Kéri [2][3]. When $n$ is a multiple of 6, previously the best known result was $\gamma(Q_n) \geq \frac{2^n}{n}$, given by van Wee [4]. Our lower bound is higher, and several improvements are listed in Corollary 1.

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Appendix

| n  | bounds on $\gamma(Q_n)$ | n  | bounds on $\gamma(Q_n)$ | n  | bounds on $\gamma(Q_n)$ |
|----|--------------------------|----|--------------------------|----|--------------------------|
| 1  | 1                        | 12 | 342(i)                   | 23 | 352827(l)                |
| 2  | 2                        | 13 | 598(h)                   | 24 | 699051(j)                |
| 3  | 2                        | 14 | 1172(k)                  | 25 | 1298238(h)               |
| 4  | 4                        | 15 | 2048(c)                  | 26 | 2581111(i)               |
| 5  | 7(a)                     | 16 | 4096(i)                  | 27 | 4794174(q)               |
| 6  | 12(b)                    | 17 | 7419(l)                  | 28 | 9587084(m)               |
| 7  | 16(c)                    | 18 | 14564(i)                 | 29 | 17997161(l)              |
| 8  | 32(b)                    | 19 | 26309(m)                 | 30 | 35791395(i)              |
| 9  | 62(d)                    | 20 | 52618(m)                 | 31 | 67108864(c)              |
| 10 | 107(f)                   | 21 | 96125(h)                 | 32 | 134217728(i)             |
| 11 | 180(g)                   | 22 | 190651(i)                | 33 | 253523901(h)             |

Table 1: the latest results of the bounds on $\gamma(Q_n)$
| a | Taussky, Todd, 1948 [5] | j | Östergård, Weakly, 1999 [14] |
|---|------------------------|---|-------------------------------|
| b | Stanton, Kalbfleisch, 1968, 1969 [6][7] | k | Habsieger, 1997 [1] |
| c | perfect code | l | Haas, 2007-2008 [15] |
| d | Östergård, Blass, 2001 [8] | m | Habsieger, Plagne, 2000 [16] |
| e | Wille, 1990, 1996 [9][10] | n | Li, Chen, 1994 [17] |
| f | Bertolo, Östergård, Weakley, 2004 [11] | o | Kéri, 2006 [18] |
| g | Blass, Litsyn, 1998 [12] | p | Östergård, Kaikkonen, 1998 [19] |
| h | Cohen, Lobstein, Sloane, 1986 [13] | q | Plagne, 2009 [20] |
| i | van Wee, 1988 [4] | r | $\gamma(Q_{n+1}) \leq 2\gamma(Q_n)$ |

Table 2: References for Table 1