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Existence and Multiplicity
Results for Systems of Singular
Boundary Value Problems

– Monograph –

February 1, 2019

Springer
Singular boundary value problems (SBVPs) arise in various fields of Mathematics, Engineering and Physics such as boundary layer theory, gas dynamics, nuclear physics, nonlinear optics, etc. The present monograph is devoted to systems of SBVPs for ordinary differential equations (ODEs). It presents existence theory for a variety of problems having unbounded nonlinearities in regions where their solutions are searched for. The main focus is to establish the existence of positive solutions. The results are based on regularization and sequential procedure.

First chapter of this monograph describe the motivation for the study of SBVPs. It also include some available results from functional analysis and fixed point theory. The following chapters contain results from author’s PhD thesis, National University of Sciences and Technology, Islamabad, Pakistan. These results provide the existence of positive solutions for a variety of systems of SBVPs having singularity with respect to independent and/or dependent variables as well as with respect to the first derivatives of dependent variables.
## Contents

1 **Introduction and Preliminaries** ................................................. 1
   1.1 Some basic definitions and known results ........................... 2

2 **Singular Systems of ODEs with Nonlocal BCs** ....................... 5
   2.1 Sufficient conditions for the existence of at least one positive solution .......................................................... 10
   2.2 Sufficient conditions for the existence of at least one positive solution to a more general singular systems .................. 23
   2.3 Singular systems of ODEs with four-point coupled BCs ........... 28

3 **Singular Systems of Second-Order ODEs with Two-Point BCs** ...... 33
   3.1 Existence of $C^1$-positive solutions .................................. 35
   3.2 Existence of at least two positive solutions .......................... 44
   3.3 Existence of $C^1$-positive solutions with more general BCs ..... 57
   3.4 Existence of at least two positive solutions with more general BCs 65
   3.5 System of ODEs with two-point coupled BCs .......................... 79
   3.6 Existence of at least one $C^1$-positive solution via lower and upper solutions .......................................................... 86

4 **Singular Systems of Second-Order ODEs on Infinite Intervals** .... 89
   4.1 Systems of BVPs on infinite intervals .................................. 90
      4.1.1 Existence of positive solutions on finite intervals .......... 91
      4.1.2 Existence of positive solutions on an infinite interval .... 97
   4.2 Systems of BVPs on infinite intervals with more general BCs .. 103
      4.2.1 Existence of positive solutions on finite intervals ........ 103

vii
## Contents

| Section                                      | Page |
|----------------------------------------------|------|
| 4.2.2 Existence of positive solutions        | 110  |
| 5 Concluding Remarks                         | 115  |
| References                                   | 119  |
Chapter 1
Introduction and Preliminaries

Many problems in applied sciences are modeled by singular boundary value problems (SBVPs). For example, in the study of rotating flow [48], in the theory of viscous fluids [23], in the study of pseudoplastic fluids [25, 85], in boundary layer theory [24, 87, 94, 95], the theory of shallow membrane caps [18, 32, 57], in pre-breakdown of gas discharge [34], the turbulent flow of a gas in a porous medium [39], can be represented by SBVPs. Further, many mathematical models of various applications from nuclear physics, plasma physics, nonlinear optics, fluid mechanics, chemical reactor theory, predator-prey interactions [21, 40, 81, 93, 108] are systems of time dependent partial differential equations (PDEs) subject to initial and/or boundary conditions. In the investigation of stationary solutions, these models of systems of PDEs can be reduced to systems of SBVPs.

In the scenario of the above mentioned models of various phenomenon, the theory of SBVPs has become much more important. In this monograph, we present existence results for positive solutions to various systems of BVPs for nonlinear ODEs. We provide sufficient conditions for the existence of at least one and two solutions for the singular systems of nonlinear ODEs subject to various type of boundary conditions (BCs) both on finite and infinite domains. We use the classical tools of functional analysis including the fixed point theory and the theory of the fixed points index. The rest of this chapter is devoted to the basic study of these notions. In the following section, we present some definitions and notions from functional analysis. Moreover, some famous fixed point results such as Schauder’s fixed point theorem and the Guo-Krasnosel’skii fixed point theorem are also included, [6, 9, 33, 35, 42, 46, 53].
1.1 Some basic definitions and known results

**Definition 1.1.1** A subset $\Omega$ of a Banach space $\mathcal{B}$ is said to be compact if and only if every sequence $\{x_n\} \subset \Omega$ has a convergent subsequence with limit in $\Omega$. Moreover, $\Omega$ is relatively compact if $\overline{\Omega}$ is compact.

**Definition 1.1.2** Let $\Omega$ be a subset of a Banach space $\mathcal{B}$. A map $T : \Omega \to \mathcal{B}$ is compact if $T$ maps every bounded subset of $\Omega$ into a relatively compact subset of $\mathcal{B}$. Moreover, $T$ is completely continuous if $T$ is continuous and compact.

**Definition 1.1.3** A nonempty subset $K$ of a Banach space $\mathcal{B}$ is a retract of $\mathcal{B}$ if there exist a continuous map $r : \mathcal{B} \to K$, a retraction, such that $r|_K = I_K$, where $I_K$ is identity map on $K$.

**Definition 1.1.4** Let $\mathcal{B}$ be a real Banach space. A nonempty, closed and convex set $P \subset \mathcal{B}$ is said to be a cone if the following axioms are satisfied:

- $(P_1)$ $\alpha x \in P$ for all $x \in P$ and $\alpha \geq 0$,
- $(P_2)$ $-x \in P$ implies $x = 0$.

**Theorem 1.1.5** (Arzelà-Ascoli theorem) Let $\Omega$ be a compact subset of $\mathbb{R}^n$. A set $\mathcal{M}$ of continuous functions on $\Omega$ is relatively compact in $C(\Omega)$ if and only if $\mathcal{M}$ is a family of uniformly bounded and equicontinuous functions.

Now we recall the notion of degree for continuous maps. The degree of a map in finite dimensional spaces is known as the Brouwer degree. Let $\Omega$ be a bounded and open subset of a finite dimensional Banach space $(\mathcal{B}, \| \cdot \|)$. Let $T_0 : \overline{\Omega} \to \mathcal{B}$ be a $C(\overline{\Omega}) \cup C^1(\Omega)$ map, and $S_0 = \{x \in \Omega : J_0(x) = 0\}$ be the set of all critical points of the map $T_0$, where $J_0(x) := \det T_0'(x)$ is the Jacobian of $T_0$ at $x$. If $y \notin T_0(\partial \Omega \cup S_0)$, then the Brouwer degree is defined as

$$\deg_B(T_0, \Omega, y) = \sum_{x \in T_0^{-1}(y)} \text{sgn} J_0(x),$$

which corresponds to the number of solutions of $T_0(x) = y$ in $\Omega$. However, if $T_0 \in C(\overline{\Omega}) \cap C^2(\Omega)$, $y \notin T_0(\partial \Omega)$ and $y \in T_0(S_0)$, then $\deg_B(T_0, \Omega, y) = \deg_B(T_0, \Omega, z)$, where $z \notin T_0(S_0)$ such that $\|z - y\| < \text{dist}(y, T_0(\partial \Omega))$. Further, if $T_0 \in C(\overline{\Omega})$ and $y \notin T_0(\partial \Omega \cup S_0)$, then $\deg_B(T_0, \Omega, y) = \deg_B(T_1, \Omega, y)$, where $T_1 \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $\sup_{x \in \Omega} \|T_0(x) - T_1(x)\| < \text{dist}(y, T_0(\partial \Omega))$. 


In 1934, J. Leray and J. Schauder extended the notion of degree to infinite dimensional spaces. They proved that along with continuity some compactness condition for the map is required. However, this is more suitable for a map of the form \( I - T \), where \( I \) is the identity map and \( T \) is a completely continuous map. For defining the Leray-Schauder degree, the following theorem is helpful to approximate a compact map with a finite-dimensional map.

**Theorem 1.1.6** Assume that \( \Omega \) is an open and bounded subset of a real Banach space \( (\mathcal{B}, \|\cdot\|) \) and \( T : \overline{\Omega} \to \mathcal{B} \) is a completely continuous map. Then, for every \( \varepsilon > 0 \), there exist a finite-dimensional space \( \mathcal{B} \) and a continuous map \( T_\varepsilon : \overline{\Omega} \to \mathcal{B} \) such that \( \|T(x) - T_\varepsilon(x)\| < \varepsilon \) for every \( x \in \overline{\Omega} \).

Let \( \Omega \) be a bounded and open subset of a Banach space \( \mathcal{B} \). Let \( T : \overline{\Omega} \to \mathcal{B} \) be a completely continuous mapping and \( y \notin (I - T)(\partial \Omega) \). The Leray-Schauder degree of \( I - T \) over \( \Omega \) at point \( y \) is defined as

\[
\deg_{LS}(I - T, \Omega, y) = \deg_B(I - T_\varepsilon, \Omega, y),
\]

where \( T_\varepsilon \) is an approximation of \( T \) in a finite dimensional space such that \( \|T(x) - T_\varepsilon(x)\| < \varepsilon := \text{dist}(y, (I - T)(\partial \Omega)) \). When \( T : \overline{\Omega} \to K \) be a completely continuous map such that \( 0 \notin (I - T)(\partial \Omega) \), where \( K \) is a retract of a Banach space \( \mathcal{B} \) and \( \Omega \) is an open subset of \( K \), then for any retraction \( r : \mathcal{B} \to K \) the Leray-Schauder degree \( \deg_{LS}(I - T \circ r, r^{-1}(\Omega), 0) \) is known as the fixed point index of the map \( T \) over \( \Omega \) with respect to the retract \( K \) and is denoted by \( \text{ind}_{fp}(T, \Omega, K) \). The following are the most significant properties of the fixed point index for completely continuous maps.

**(F_1) Normalization:** For every constant map \( T \) mapping \( \overline{\Omega} \) into \( \Omega \),

\[\text{ind}_{fp}(T, \Omega, K) = 1.\]

**(F_2) Additivity:** For any disjoint open subsets \( \Omega_1 \) and \( \Omega_2 \) of \( \Omega \) such that \( T \) has no fixed point on \( \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2) \),

\[\text{ind}_{fp}(T, \Omega, K) = \text{ind}_{fp}(T, \Omega_1, K) + \text{ind}_{fp}(T, \Omega_2, K).\]

**(F_3) Homotopy:** For every compact interval \( [a, b] \subset \mathbb{R} \) and every compact map \( h : [a, b] \times \overline{\Omega} \to K \) such that \( h(\tau, x) \neq x \) for \( (\tau, x) \in [a, b] \times \partial \Omega \),

\[\text{ind}_{fp}(h(\tau, \cdot), \Omega, K) \]
is well defined and is independent of \( \tau \in [a, b] \).

\((F_4)\) **Solution:** If \( \text{ind}_{FP}(T, \Omega, K) \neq 0 \), then \( T \) has at least one fixed point in \( \Omega \).

\((F_5)\) **Permanence:** If \( K_1 \) is a retract of \( K \) and \( T(\overline{\Omega}) \subset K_1 \), then

\[
\text{ind}_{FP}(T, \Omega, K) = \text{ind}_{FP}(T, \Omega \cap K_1, K_1).
\]

\((F_6)\) **Excision:** For every open set \( \Omega_1 \subset \Omega \) such that \( T \) has no fixed point in \( \overline{\Omega} \setminus \Omega_1 \), then

\[
\text{ind}_{FP}(T, \Omega, K) = \text{ind}_{FP}(T, \Omega_1, K).
\]

**Theorem 1.1.7 (Schauder’s fixed point theorem)** Let \( X \) be a nonempty, closed, bounded and convex subset of a Banach space \( \mathcal{B} \) and \( T : X \rightarrow X \) be a completely continuous map. Then, \( T \) has a fixed point in \( X \).

**Proof.** For some \( x^* \in X \), consider a map \( h : [0, 1] \times X \rightarrow X \) defined by

\[
h(\tau, x) = (1 - \tau)x^* + \tau Tx.
\]

Then, the conclusion follows from the homotopy property \((F_4)\) together with the normalization property \((F_1)\).

**Lemma 1.1.8** Let \( \Omega \) be an open and bounded set in a real Banach space \( \mathcal{B} \), \( P \) be a cone of \( \mathcal{B} \), \( 0 \in \Omega \) and \( T : \overline{\Omega} \cap P \rightarrow P \) be a completely continuous map. Suppose \( x \neq \lambda Tx \), for any \( x \in \partial \Omega \cap P \), \( \lambda \in (0, 1] \). Then, the fixed point index \( \text{ind}_{FP}(T, \Omega \cap P, P) = 1 \).

**Lemma 1.1.9** Let \( \Omega \) be an open and bounded set in a real Banach space \((\mathcal{B}, \| \cdot \|)\), \( P \) be a cone of \( \mathcal{B} \), \( 0 \in \Omega \) and \( T : \overline{\Omega} \cap P \rightarrow P \) be a completely continuous map. Suppose there exist a \( v \in P \setminus \{0\} \) with \( x \neq Tx + \delta v \) for every \( \delta > 0 \) and \( x \in \partial \Omega \cap P \). Then, the fixed point index \( \text{ind}_{FP}(T, \Omega \cap P, P) = 0 \).

**Proof.** Let \( \mu = \sup \{ \|Tx\| : x \in \Omega \cap P \} \) and \( \rho = \sup \{ \|x\| : x \in \Omega \} \). Choose \( \delta_1 > (\mu + \rho) / \|v\| \) and define a map \( h : [0, 1] \times (\overline{\Omega} \cap P) \rightarrow P \) as

\[
h(\tau, x) = Tx + \tau \delta_1 v.
\]

Then, by the homotopy property \((F_4)\), we obtain

\[
\text{ind}_{FP}(T, \Omega \cap P, P) = \text{ind}_{FP}(T + \delta_1 v, \Omega \cap P, P).
\]
Now, if \( \text{ind}_{fp}(T, \Omega \cap P, P) \neq 0 \), then there exist an element \( x \in \Omega \cap P \) such that \( x = Tx + \delta v \). Consequently,
\[
\|x\| = \|Tx + \delta v\| \geq \|v\| - \|Tx\| \geq \|v\| - \mu > \rho,
\]
a contradiction. Hence, \( \text{ind}_{fp}(T, \Omega \cap P, P) = 0 \).

**Lemma 1.1.10** Let \( \Omega \) be a bounded and open set in a real Banach space \((\mathcal{B}, \| \cdot \|)\). \( P \) be a cone of \( \mathcal{B} \), \( 0 \in \Omega \) and \( T : \overline{\Omega} \cap P \to P \) be a completely continuous map. Suppose \( Tx \neq x \), for any \( x \in \partial \Omega \cap P \). Then, the fixed point index \( \text{ind}_{fp}(T, \Omega \cap P, P) = 0 \).

**Proof.** We claim that, for some \( v \in P \setminus \{0\} \), \( x \neq Tx + \delta v \) for every \( \delta > 0 \) and \( x \in \partial \Omega \cap P \). Suppose there exist some \( \delta_0 > 0 \) and \( x_0 \in \partial \Omega \cap P \) such that \( x_0 = Tx_0 + \delta_0 v \). Then, \( x_0 = Tx_0 + \delta_0 v > Tx_0 \), a contradiction as \( x_0 \) can not be mapped toward the origin under \( T \). Hence, by Lemma 1.1.9 \( \text{ind}_{fp}(T, \Omega \cap P, P) = 0 \).

**Theorem 1.1.11** (Guo-Krasnosel’skii fixed point theorem) Let \( P \) be a cone of a real Banach space \((\mathcal{B}, \| \cdot \|)\). Let \( \Omega_1, \Omega_2 \) be bounded and open neighborhoods of \( 0 \in \mathcal{B} \) such that \( \Omega_1 \subset \Omega_2 \). Suppose that \( T : (\overline{\Omega_2} \setminus \overline{\Omega_1}) \cap P \to P \) is completely continuous such that one of the following conditions holds:

(i) \( \|Tx\| \leq \|x\| \) for \( x \in \partial \Omega_1 \cap P \), \( \|Tx\| \geq \|x\| \) for \( x \in \partial \Omega_2 \cap P \).

(ii) \( \|Tx\| \leq \|x\| \) for \( x \in \partial \Omega_2 \cap P \), \( \|Tx\| \geq \|x\| \) for \( x \in \partial \Omega_1 \cap P \).

Then, \( T \) has a fixed point in \((\overline{\Omega_2} \setminus \overline{\Omega_1}) \cap P \).

**Proof.** Assume that (i) holds. If \( T \) has a fixed point on \( \partial \Omega_1 \cup \partial \Omega_2 \) then proof is complete. Suppose \( Tx \neq x \) for all \( x \in \partial \Omega_1 \cup \partial \Omega_2 \). First of all we show that \( x \neq \lambda Tx \) for \( \lambda \in (0, 1] \) and \( x \in \partial \Omega_1 \cap P \). Suppose, \( x_1 = \lambda_1 Tx_1 \) for some \( \lambda_1 \in (0, 1) \) and \( x_1 \in \partial \Omega_1 \cap P \). Then, \( \|x_1\| = \lambda_1 \|Tx_1\| < \|Tx_1\| \leq \|x_1\| \), a contradiction. Hence, by Lemma 1.1.8 the fixed point index \( \text{ind}_{fp}(T, \Omega_1 \cap P, P) = 1 \).

We claim that there exist a \( v \in P \setminus \{0\} \) with \( x \neq Tx + \delta v \) for every \( \delta > 0 \) and \( x \in \partial \Omega_2 \cap P \). Suppose, \( x_2 = Tx_2 + \delta_2 v \) for some \( \delta_2 > 0 \) and \( x_2 \in \partial \Omega_2 \cap P \). Then, \( \|x_2\| = \|Tx_2 + \delta_2 v\| > \|Tx_2\| \geq \|x_2\| \), a contradiction. Therefore, by Lemma 1.1.9 the fixed point index \( \text{ind}_{fp}(T, \Omega_2 \cap P, P) = 0 \).

Thus, by the additivity property of fixed point index \( (F_2) \), we obtain
\[
\text{ind}_{fp}(T, (\Omega_2 \setminus \Omega_1) \cap P, P) = \text{ind}_{fp}(T, \Omega_2 \cap P, P) - \text{ind}_{fp}(T, \Omega_1 \cap P, P) = 0 - 1 = -1.
\]
Thus, $T$ has a fixed point in $(\Omega_2 \setminus \Omega_1) \cap P$. The proof for (ii) is similar.

For each $x \in C[0,1] \cap C^1(0,1]$, we write $\|x\|_{0,1} = \max_{t \in [0,1]} |x(t)|$, $\|x\|_1 = \sup_{t \in (0,1]} |x(t)|$ and $\|x\|_2 = \max\{\|x\|, \|x\|_1\}$. Further, for each $x \in C^1[0,1]$ we write $\|x\|_3 = \max\{\|x\|, \|x'\|\}$. The following results are known \cite{7,76,96,99,101,102}.

**Lemma 1.1.12** \((\mathcal{E}, \|\cdot\|_2)\) is a Banach space.

**Lemma 1.1.13** If $x \in \mathcal{E}$, then $|x'(t)| \leq \frac{\|x\|_2}{T}$ for all $t \in (0,1]$.

**Lemma 1.1.14** If $x \in P := \{x \in \mathcal{E} : x(t) \geq t\|x\| \text{ for all } t \in [0,1], x(1) \geq \|x\|_1\}$, then $\|x\|_2 = \|x\|$.

**Lemma 1.1.15** Let $\sigma \in C(0,1)$ and $\sigma > 0$ on $(0,1)$ with $\int_0^1 \sigma(t)dt < +\infty$. Then,

\[
\sup_{t \in [0,1]} \int_0^1 G(t,s)\sigma(s)ds \leq \int_0^1 G(t,s)\sigma(s)ds \text{ for } t \in [0,1],
\]

where

\[
G(t,s) = \begin{cases} 
  s, & 0 \leq s \leq t \\
  t, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

**Lemma 1.1.16** If $x \in P := \{x \in C^1[0,1] : x(t) \geq \gamma\|x\| \text{ for all } t \in [0,1], x(0) \geq \frac{b}{a+b}\|x'\|\}$, then $x(t) \geq \gamma \rho \|x\|_3$ for all $t \in [0,1]$, where $\gamma = \frac{b}{a+b}$, $\rho = \frac{1}{\max\{1, \sigma\}}, a,b > 0$.

**Lemma 1.1.17** Let $\sigma \in C(0,1)$ and $\sigma > 0$ on $(0,1)$ with $\int_0^1 \sigma(t)dt < +\infty$. Then,

\[
\gamma \max_{t \in [0,1]} \int_0^1 G(t,s)\sigma(s)ds \leq \int_0^1 G(t,s)\sigma(s)ds \text{ for } t \in [0,1], \gamma = \frac{b}{a+b}, a,b > 0,
\]

\[
\frac{b}{a} \max_{t \in [0,1]} \int_0^1 \sigma(s)ds = \int_0^1 G(0,s)\sigma(s)ds,
\]

where

\[
G(t,s) = \frac{1}{a} \begin{cases} 
  b+as, & 0 \leq s \leq t \\
  b+at, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

**Lemma 1.1.18** Let $x \in C^1[0,1] \cap C^2(0,1)$ satisfies $x'' < 0$ on $(0,1), x(0) = 0, x'(1) = a \geq 0$. Then, $x(t) \geq tx(1)$ for $t \in [0,1]$. 

1.1 Some basic definitions and known results

**Lemma 1.1.19** The Green’s function

\[
H(t, s) = \begin{cases} 
\frac{t(1-s)}{1-\alpha \eta} - s \frac{\alpha(n-s)}{1-\alpha \eta} - (t-s), & 0 \leq s \leq t \leq 1, s \leq \eta, \\
\frac{t(1-s)}{1-\alpha \eta} - s \frac{\alpha(n-s)}{1-\alpha \eta}, & 0 \leq t \leq s \leq 1, s \leq \eta, \\
\frac{t(1-s)}{1-\alpha \eta}, & 0 \leq t \leq s \leq 1, s \geq \eta, \\
\frac{t(1-s)}{1-\alpha \eta} - (t-s), & 0 \leq s \leq t \leq 1, s \geq \eta. 
\end{cases} 
\]

satisfies

(i) \( H(t, s) \leq \mu s(1-s), \quad (t, s) \in [0, 1] \times [0, 1], \)

(ii) \( H(t, s) \geq \nu s(1-s), \quad (t, s) \in [\eta, 1] \times [0, 1], \)

(iii) \( H(t, s) \geq \nu r(1-t)s(1-s), \quad (t, s) \in [0, 1] \times [0, 1], \)

where

\[
\mu := \frac{\max\{1, \alpha\}}{1-\alpha \eta} > 0, \quad \nu := \frac{\min\{1, \alpha\} \min\{\eta, 1-\eta\}}{1-\alpha \eta} > 0.
\]
Chapter 2
Singular Systems of Ordinary Differential Equations with Nonlocal Boundary Conditions

Existence theory for nonlinear three-point boundary value problems (BVPs) was initiated by Gupta [47]. Since then the study of nonlinear regular multi-point BVPs has attracted the attention of many researchers; see for example, [22, 60, 69, 71, 79, 96, 73, 107] for scalar equations, and for systems of BVPs, see [28, 30, 59]. Recently, the study of SBVPs has also attracted much attention. An excellent resource with an extensive bibliography was produced by Agarwal and O’Regan [3]. In Sections 2.1 and 2.2, we study the following systems of SBVPs

\begin{align*}
-x''(t) &= f(t, y(t)), \quad t \in (0, 1), \\
-y''(t) &= g(t, x(t)), \quad t \in (0, 1), \\
\alpha x(\eta) &= x(0), \quad x(1) = \alpha y(\eta), \\
\alpha y(\eta) &= y(0), \quad y(1) = \alpha x(\eta),
\end{align*}

(2.1)

and

\begin{align*}
-x''(t) &= f(t, x(t), y(t)), \quad t \in (0, 1), \\
-y''(t) &= g(t, x(t), y(t)), \quad t \in (0, 1), \\
\alpha x(\eta) &= x(0), \quad x(1) = \alpha y(\eta), \\
\alpha y(\eta) &= y(0), \quad y(1) = \alpha x(\eta)
\end{align*}

(2.2)

where \( \eta \in (0, 1), \ 0 < \alpha \eta < 1 \). For the system of SBVPs (2.1), we assume that \( f, g : (0, 1) \times (0, \infty) \to (0, \infty) \) are continuous, \( f(t, 0) \) and \( g(t, 0) \) are not identically 0. For the system of SBVPs (2.2), we assume that \( f, g : (0, 1) \times (0, \infty) \times (0, \infty) \to (0, \infty) \) are continuous. Further, both \( f \) and \( g \) are allowed to be singular at \( t = 0, t = 1, x = 0 \)
and/or \( y = 0 \). By singularity we mean that the nonlinearities \( f \) and \( g \) are allowed to be unbounded at \( t = 0, t = 1, x = 0 \) and \( y = 0 \).

Further, in Section 2.3 we present existence result for the following coupled system of SBVPs subject to four-point coupled BCs

\[
\begin{align*}
-x''(t) &= f(t, x(t), y(t)), \quad t \in (0, 1), \\
-y''(t) &= g(t, x(t), y(t)), \quad t \in (0, 1), \\
x(0) &= 0, x(1) = \alpha y(\xi), \\
y(0) &= 0, y(1) = \beta x(\eta),
\end{align*}
\]

where the parameters \( \alpha, \beta, \xi, \eta \) satisfy \( \xi, \eta \in (0, 1), 0 < \alpha \beta \xi \eta < 1 \). We assume that the nonlinearities \( f, g : (0, 1) \times [0, \infty) \times [0, \infty) \to [0, \infty) \) are continuous and allowed to be singular at \( t = 0 \) or \( t = 1 \).

### 2.1 Sufficient conditions for the existence of at least one positive solution

In this section, we establish existence of positive solution to the system of SBVPs (2.1). We say \((x, y)\) is a positive solution to system of SBVPs (2.1) if \((x, y) \in (C[0, 1] \cap C^2(0, 1)) \times (C[0, 1] \cap C^2(0, 1)), x > 0 \) and \( y > 0 \) on \((0, 1] \), \((x, y)\) satisfies (2.1). Let \( n_0 > \max\{\frac{1}{\eta}, 1 - \frac{1}{\eta}, \frac{2 - \alpha}{\alpha \eta}\} \) be a fixed positive integer. For each \( x \in \mathcal{E}_n := C[\frac{1}{n}, 1 - \frac{1}{n}] \), we write \( \|x\|_{\mathcal{E}_n} = \max\{|x(t)| : t \in [\frac{1}{n}, 1 - \frac{1}{n}]\} \), where \( n \in \{n_0, n_0 + 1, n_0 + 2, \cdots \} \). Clearly, \( \mathcal{E}_n \) with the norm \( \| \cdot \|_{\mathcal{E}_n} \) is a Banach space. Define a cone \( K_n \) of \( \mathcal{E}_n \) as

\[
K_n = \{x \in \mathcal{E}_n : x(t) \geq 0 \text{ and } x''(t) \leq 0 \text{ for } t \in [\frac{1}{n}, 1 - \frac{1}{n}]\}.
\]

For any real constant \( r > 0 \), define an open neighborhood of \( 0 \in \mathcal{E}_n \) of radius \( r \) by

\[
\Omega_r = \{x \in \mathcal{E}_n : \|x\|_{\mathcal{E}_n} < r\}.
\]

**Lemma 2.1.1** For \( z \in \mathcal{E}_n \), the linear BVP

\[
\begin{align*}
-u''(t) &= z(t), \quad t \in [\frac{1}{n}, 1 - \frac{1}{n}], \\
u(\frac{1}{n}) &= 0, u(1 - \frac{1}{n}) = \alpha u(\eta),
\end{align*}
\]
2.1 Sufficient conditions for the existence of at least one positive solution

has integral representation

\[ u(t) = \int_{1/n}^{1-1/n} H_n(t, s)z(s)ds, \]  

(2.5)

where the Green's function \( H_n : \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \rightarrow [0, \infty) \) is defined by

\[
H_n(t, s) = \begin{cases} 
\frac{(t - \frac{1}{n})(1 - \frac{1}{n} - s - \alpha(s - s)) - (t - s)}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta}} - \alpha s, & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, s \leq \eta, \\
\frac{(t - \frac{1}{n})(1 - \frac{1}{n} - s - \alpha(s - s)) - (t - s)}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta}} - \alpha s, & \frac{1}{n} \leq t \leq s \leq 1 - \frac{1}{n}, s \leq \eta, \\
\frac{(t - \frac{1}{n})(1 - \frac{1}{n} - s - \alpha(s - s)) - (t - s)}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta}} - \alpha s, & \frac{1}{n} \leq t \leq s \leq 1 - \frac{1}{n}, s \geq \eta, \\
\frac{(t - \frac{1}{n})(1 - \frac{1}{n} - s - \alpha(s - s)) - (t - s)}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta}} - \alpha s, & \frac{1}{n} \leq s \leq t \leq 1 - \frac{1}{n}, s \geq \eta.
\end{cases}
\]

(2.6)

**Lemma 2.1.2** The Green's function \( H_n \) satisfies

(i) \( H_n(t, s) \leq \mu_n(s - \frac{1}{n})(1 - \frac{1}{n} - s), \quad (t, s) \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \)

(ii) \( H_n(t, s) \geq \nu_n(s - \frac{1}{n})(1 - \frac{1}{n} - s), \quad (t, s) \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \times \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \)

where

\[
\mu_n := \frac{\max\{1, \alpha\}}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta} - \alpha s} > 0, \quad \nu_n := \frac{\min\{1, \alpha\} \min\{\eta - \frac{1}{n}, 1 - \frac{1}{n} - \eta\}}{1 - \frac{1}{n} + \frac{\alpha n}{1 - \eta}} > 0.
\]

Now we consider the modified system of non-singular BVPs

\[
\begin{align*}
-x''(t) &= f(t, \max\{y(t) + \frac{1}{n}, \frac{1}{n}\}), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \\
-y''(t) &= g(t, \max\{x(t) + \frac{1}{n}, \frac{1}{n}\}), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \\
x\left(\frac{1}{n}\right) &= 0, x\left(1 - \frac{1}{n}\right) = \alpha x(\eta), \\
y\left(\frac{1}{n}\right) &= 0, y\left(1 - \frac{1}{n}\right) = \alpha y(\eta),
\end{align*}
\]

(2.7)

which in view of Lemma 2.1.1 can be expressed as a system of integral equations

\[
\begin{align*}
x(t) &= \int_{1/n}^{1-1/n} H_n(t, s)f(s, \max\{y(s) + \frac{1}{n}, \frac{1}{n}\})ds, \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \\
y(t) &= \int_{1/n}^{1-1/n} H_n(t, s)g(s, \max\{x(s) + \frac{1}{n}, \frac{1}{n}\})ds, \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right].
\end{align*}
\]

(2.8)

Thus, \((x_n, y_n) \in \mathcal{E} \times \mathcal{E}\) is a solution of (2.7) if and only if \((x_n, y_n)\) is a solution of (2.8). Define maps \(A_n, B_n, T_n : \mathcal{E} \rightarrow K\) by
Assume that uniformly bounded and equicontinuous. We introduce

\[ (A_n y)(t) = \int_{1/n}^{1-1/n} H_n(t, s) f(s, \max \left\{ y(s) + \frac{1}{n}, \frac{1}{n} \right\}) ds, \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \]

\[ (B_n x)(t) = \int_{1/n}^{1-1/n} H_n(t, s) g(s, \max \left\{ x(s) + \frac{1}{n}, \frac{1}{n} \right\}) ds, \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \quad (2.9) \]

\[ (T_n x)(t) = (A_n(B_n x))(t), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]. \]

If \( u_n \in K_n \) is a fixed point of \( T_n \); then the system of BVPs (2.7) has a solution \((x_n, y_n)\) given by

\[ x_n(t) = u_n(t), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right], \]

\[ y_n(t) = (B_n u_n)(t), \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]. \]

Assume that the following holds:

\[ (A_1) \quad \text{there exist } K, L \in C((0, 1), (0, \infty)) \text{ and } F, G \in C((0, \infty), (0, \infty)) \text{ such that} \]

\[ f(t, u) \leq K(t) F(u), \quad g(t, u) \leq L(t) G(u), \quad t \in (0, 1), u \in (0, \infty), \]

where

\[ a := \int_0^1 t(1-t)K(t)dt < +\infty, \quad b := \int_0^1 t(1-t)L(t)dt < +\infty. \]

**Lemma 2.1.3** Assume that \((A_1)\) holds. Then the map \( T_n : \Omega_n \cap K_n \to K_n \) is completely continuous.

**Proof.** Clearly, for any \( u \in K_n \), we have \((T_n u)(t) \geq 0\) and \((T_n u)'(t) \leq 0\) for \( t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \). Consequently, \( T_n u \in K_n \) for all \( u \in K_n \). Now, we show that \( T_n : \Omega_n \cap K_n \to K_n \) is uniformly bounded and equicontinuous. We introduce

\[ d_n = b \mu_n \max_{u \in [0, 1]} G(u + \frac{1}{n}), \]

\[ \omega_n = \int_{1/n}^{1-1/n} f(t, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(t, s) g(s, u(s) + \frac{1}{n}) ds dt. \quad (2.10) \]

For any \( u \in \Omega_n \cap K_n \), using (2.9), \((A_1)\) and (i) of Lemma 2.1.2 we have
2.1 Sufficient conditions for the existence of at least one positive solution

\[ (T_n u)(t) = \int_{1/n}^{1-1/n} H_n(t, s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \]

\[ \leq \int_{1/n}^{1-1/n} H_n(t, s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \]

\[ \leq \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) K(s) F(\frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds. \]

But, using \((A_1), (i)\) of Lemma 2.1.2 and (2.10),

\[ 0 \leq \int_{1/n}^{1-1/n} H_n(t, s) g(s, u(s) + \frac{1}{n}) ds \]

\[ \leq \int_{1/n}^{1-1/n} H_n(t, s) L(s) G(u(s) + \frac{1}{n}) ds \]

\[ \leq \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) L(s) G(u(s) + \frac{1}{n}) ds \]

\[ \leq \mu_n \max_{u \in [0, r]} G(u + \frac{1}{n}) \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) L(s) ds \]

\[ \leq b \mu_n \max_{u \in [0, r]} G(u + \frac{1}{n}) = d_n. \]

Therefore,

\[ (T_n u)(t) \leq \mu_n \max_{u \in [0, d_n]} F(u + \frac{1}{n}) \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) L(s) ds \]

\[ \leq a \mu_n \max_{u \in [0, d_n]} F(u + \frac{1}{n}), \]

which implies that

\[ \| T_n u \|_{d_n} \leq a \mu_n \max_{u \in [0, d_n]} F(u + \frac{1}{n}), \]

that is, \( T_n(\overline{S}_n \cap K_n) \) is uniformly bounded. To show \( T_n(\overline{S}_n \cap K_n) \) is equicontinuous, let \( t_1, t_2 \in [\frac{1}{n}, 1 - \frac{1}{n}] \). Since the Green’s function \( H_n \) is uniformly continuous on \( [\frac{1}{n}, 1 - \frac{1}{n}] \times [\frac{1}{n}, 1 - \frac{1}{n}] \), therefore, \( T_n(\overline{S}_n \cap K_n) \) is equicontinuous. By Theorem 1.1.5, \( T_n(\overline{S}_n \cap K_n) \) is relatively compact. Hence, \( T_n \) is a compact map. Now, we show that \( T_n \) is continuous. Let \( u_m, u \in \overline{S}_n \cap K_n \) such that

\[ \| u_m - u \|_{d_n} \to 0 \text{ as } m \to +\infty. \]

Using (2.9) and (i) of Lemma 2.1.2 we have
\[ |(T_n u_m)(t) - (T_n u)(t)| = \] 
\[ \int_{1/n}^{1-1/n} H_n(t, s) \left( f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u_m(\tau) + \frac{1}{n}) d\tau) - f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) \right) ds \] 
\[ \leq \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) \left( f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) \right) 
\[ g(\tau, u_m(\tau) + \frac{1}{n}) d\tau - f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) \right) ds. \]

Consequently,
\[ ||T_n u_m - T_n u||_{\delta_n} \leq \mu_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) \left( f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) \right) 
\[ g(\tau, u_m(\tau) + \frac{1}{n}) d\tau - f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) \right) ds. \]

By the Lebesgue dominated convergent theorem, it follows that
\[ ||T_n u_m - T_n u||_{\delta_n} \to 0 \text{ as } m \to +\infty, \]
that is, \( T_n : \overline{\Omega}_r \cap K_n \to K_n \) is a continuous. Hence, \( T_n : \overline{\Omega}_r \cap K_n \to K_n \) is completely continuous.

Assume that
\( (A_2) \)  
there exist \( \alpha_1, \alpha_2 \in (0, \infty) \) with \( \alpha_1 \alpha_2 \leq 1 \) such that
\[ \lim_{u \to 0} \frac{F(u)}{u^{\alpha_1}} = 0, \quad \lim_{u \to 0} \frac{G(u)}{u^{\alpha_2}} = 0, \]

\( (A_3) \)  
there exist \( \beta_1, \beta_2 \in (0, \infty) \) with \( \beta_1 \beta_2 \geq 1 \) such that
\[ \liminf_{u \to 0^+} \min_{t \in [0, 1]} \frac{f(t, u)}{u^{\beta_1}} > 0, \quad \liminf_{u \to 0^+} \min_{t \in [0, 1]} \frac{g(t, u)}{u^{\beta_2}} > 0. \]

**Theorem 2.1.4** Assume that \( (A_1) - (A_3) \) hold. Then the system of SBVPs (2.1) has a positive solution.

**Proof.** By \( (A_2) \), there exist real constants \( c_1, c_2, c_3, c_4 > 0 \) such that
\[ 2^{\alpha_1} \alpha_2 c_3 c_4 = \frac{\alpha_1 + \alpha_2}{2} < 1, \quad (2.11) \]
In view of (2.11), we choose a real constant $R > 0$ such that

$$R \geq \frac{a_1a_2 + 2^{2m_a}a_b^m \mu_2^{-\alpha_1}c_2^e + 2^{2m_a}a_b^m \mu_2^{-\alpha_1}c_1c_3^e}{1 - 2^{2m_a}a_b^m \mu_2^{-\alpha_1}c_1c_3^e}.$$  \hspace{1cm} (2.13)

For any $u \in \partial \Omega_n \cap K_n$, using (2.9), (A) and (2.12), it follows that

$$(T_n u)(t) = \int_{1/n}^{1-1/n} H_n(t, s) F(s) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds$$

$$\leq \int_{1/n}^{1-1/n} H_n(t, s) K(s) F(s) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds$$

$$\leq \int_{1/n}^{1-1/n} H_n(t, s) K(s) (c_1 + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds$$

$$= c_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) (1 + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds$$

Again using (A) and (2.12), we obtain

$$(T_n u)(t) \leq c_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) (1 + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) G(u(\tau) + \frac{1}{n}) d\tau) ds$$

$$+ c_2 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds$$

$$\leq c_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) (1 + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) (c_3 (u(\tau) + \frac{1}{n})^{a_2} + c_4 d\tau) ds$$

$$+ c_2 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds$$

$$\leq c_1 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds (1 + \int_{1/n}^{1-1/n} H_n(s, \tau) L(\tau) d\tau (c_3 (R + \frac{1}{n})^{a_2} + c_4) d\tau$$

$$+ c_2 \int_{1/n}^{1-1/n} H_n(t, s) K(s) ds.$$
Choose

Also,

Therefore,

But,

Therefore,

Also,

Consequently,

Using (2.13), we obtain

Now, by \((A_3)\), there exist constants \(c_5, c_6 > 0\) and \(\rho \in (0, R)\) such that

Choose
Using (2.16), we obtain

\[ r_n = \min \left\{ \rho, v_n^{\beta_1 + 1}n^{-\beta_1}c_5c_6^\beta_1 \left( \int_\eta^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_1+1} \right\}. \tag{2.16} \]

For any \( u \in \partial \Omega_n \cap K_n \) and \( t \in [\eta, 1 - \frac{1}{n}] \), using (2.9) and (2.15), we have

\[
(T_nu)(t) = \int_{1/n}^{1-1/n} H_n(t,s)f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau) + \frac{1}{n})d\tau)ds \geq c \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau) + \frac{1}{n})d\tau)^{\beta_1}ds
\]

\[
\geq c \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau) + \frac{1}{n})d\tau)^{\beta_1}ds
\]

\[
\geq c_n c_5 c_6^\beta_1 \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau) + \frac{1}{n})d\tau)^{\beta_1}ds
\]

\[
\geq n^{-\beta_1}c_5 c_6^\beta_1 \int_{1/n}^{1-1/n} H_n(t,s)(\int_{1/n}^{1-1/n} H_n(s, \tau)g(\tau, u(\tau) + \frac{1}{n})d\tau)^{\beta_1}ds
\]

Employing (ii) of Lemma 2.1.2 we get

\[
(T_nu)(t) \geq v_n^{\beta_1 + 1}n^{-\beta_1}c_5c_6^\beta_1 \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)d\tau)^{\beta_1}
\]

\[
= v_n^{\beta_1 + 1}n^{-\beta_1}c_5c_6^\beta_1 \left( \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)ds \right)^{\beta_1+1}.
\]

Using (2.16), we obtain

\[
\|T_nu\|_{\partial n} \geq \|u\|_{\partial n} \text{ for all } u \in \partial \Omega_n \cap K_n. \tag{2.17}
\]

In view of (2.14), (2.17) and by Theorem 1.1.11, \( T_n \) has a fixed point \( u_n \in (\bar{\Omega}_R \setminus \Omega_n) \cap K_n \). Note that

\[
r_n \leq \|u_n\|_{\partial n} \leq R \tag{2.18}
\]

and \( r_n \to 0 \) as \( n \to \infty \). Thus, we have exhibited a uniform bound for each \( u_n \in \partial \Omega_n \cap K_n \), and \( \{u_n\}_{m \geq n} \) is uniformly bounded on \( [\frac{1}{n}, 1 - \frac{1}{n}] \).

Now, we show that \( \{u_m\}_{m \geq n} \), is equicontinuous on \( [\frac{1}{n}, 1 - \frac{1}{n}] \). For \( t \in [\frac{1}{n}, 1 - \frac{1}{n}] \), consider the integral equation

\[
u_m(t) = u_m\left( \frac{1}{n} \right) + \frac{u_m(1 - \frac{1}{n}) - \alpha u_m(\eta) - (1 - \alpha)u_m(\frac{1}{n})}{1 - \frac{2}{n} + \frac{\alpha}{n} - \alpha \eta} \left( t - \frac{1}{n} \right) + \int_{1/n}^{1-1/n} H_n(t,s)f(s)ds,
\]
where \( \tilde{f}(t) = f(t, \frac{1}{n} + \int_{1/n}^{1} H_n(t, s)g(s, \frac{1}{n} + u_m(s)) ds) \).

Which can also be written as

\[
\begin{align*}
u_m(t) &= u_m(\frac{1}{n}) + \frac{u_m(1 - \frac{1}{n}) - \alpha u_m(\eta)}{1 - \frac{\alpha}{n} - \alpha \eta} (t - \frac{1}{n}) + \frac{t - \frac{1}{n}}{1 - \frac{\alpha}{n} - \alpha \eta} \\
\int_{1/n}^{1 - 1/n} (1 - \frac{1}{n} - s) \tilde{f}(s) ds - \frac{\alpha(t - \frac{1}{n})}{1 - \frac{\alpha}{n} - \alpha \eta} \int_{1/n}^{\eta} (\eta - s) \tilde{f}(s) ds - \int_{1/n}^{t} (t - s) \tilde{f}(s) ds.
\end{align*}
\]

Differentiating with respect to \( t \), we get

\[
u_m'(t) = \frac{u_m(1 - \frac{1}{n}) - \alpha u_m(\eta) - (1 - \alpha) u_m(\frac{1}{n})}{1 - \frac{\alpha}{n} - \alpha \eta} + \frac{1}{1 - \frac{\alpha}{n} - \alpha \eta} \int_{1/n}^{1 - 1/n} (1 - \frac{1}{n} - s) \tilde{f}(s) ds
\]
Moreover, Proof. For any \( \alpha = 2, \eta = \frac{1}{2} \). Choose

\[
K(t) = \frac{1}{t(1-t)} , \quad F(y) = \frac{1}{t(1-t)} , \quad G(x) = \frac{1}{t(1-t)} ,
\]

and \( \alpha_1 = \frac{1}{2}, \alpha_2 = 2, \beta_1 = \beta_2 = 1 \). Clearly, \( (A_1) - (A_3) \) are satisfied. Hence, by Theorem 2.1.4, the system of SBVPs (2.1) has a positive solution.

Assume that \( (A_4) \) \( f(t,u), G(u) \) are non-increasing with respect to \( u \) and for each fixed \( n \in \{n_0, n_0 + 1, n_0 + 2, \cdots \} \), there exists a constant \( \rho_n > 0 \) such that

\[
f(t, \frac{1}{n} + b \mu_n G(\frac{1}{n})) \geq \rho_n (\frac{1}{n} + b \mu_n G(\frac{1}{n}))^{-1}, \quad t \in \left[ \frac{1}{n}, 1 - \frac{1}{n} \right].
\]

Theorem 2.1.6 Assume that \( (A_1), (A_2) \) and \( (A_4) \) hold. Then the system of SBVPs (2.1) has a positive solution.

Proof. For any \( u \in \partial \Omega_{\rho_n} \cap K_n \), using (2.2), (i) of Lemma 2.1.2 and (A_1), we have
(\(T_n u\))(t) = \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) g(\tau, u(\tau) + \frac{1}{n}) d\tau) ds \\
\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)L(\tau)G(u(\tau) + \frac{1}{n}) d\tau) ds \\
\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + \mu_n G(\frac{1}{n}) \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)L(\tau) d\tau) ds \\
\geq \int_{1/n}^{1-1/n} H_n(t,s) f(s, \frac{1}{n} + b \mu_n G(\frac{1}{n}) ) ds.

Now in view of (\(A_4\)), we have

\[(T_n u)((t) \geq \rho_n \int_{1/n}^{1-1/n} H_n(t,s) ds \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) d\tau)^{-1} \]

\[\geq \rho_n v_n \int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s) ds \int_{1/n}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau) d\tau)^{-1} \]

\[= \rho_n v_n,\]

which implies that

\[\|T_n u\|_{\Omega_n} \geq \|u\|_{\Omega_n} \text{ for all } u \in \partial \Omega_n \cap K_n. \quad (2.19)\]

In view of (\(A_2\)), we can choose \(R > \rho_n\) such that (2.14) holds. Hence, in view of (2.14), (2.19) and by Theorem 1.1.11, \(T_n\) has a fixed point \(u_n \in (\Omega_K \setminus \Omega_{\rho_n}) \cap K_n\).

Now, following the same procedure as done in Theorem 2.1.4, the system of SBVPs (2.1) has a positive solution.

**Example 2.1.7** Let

\[f(t,y) = \frac{e^{\frac{1}{t(1-t)}}}{1} \quad g(t,x) = \frac{e^{\frac{1}{t(1-t)}}}{1}\]

and \(\alpha = 2, \eta = \frac{1}{4}\). Choose

\[K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = e^{\frac{1}{2}}, \quad G(x) = e^{\frac{1}{4}}, \]

\[\rho_n \leq \frac{(n-3)}{n} e^{\frac{1/4}{\pi}} \int_{1/3}^{1-1/n} (s - 1/n)(1 - 1/n - s) ds. \] Then (\(A_1\)), (\(A_2\)) and (\(A_4\)) are satisfied. Hence, by Theorem 2.1.4, the system of SBVPs (2.1) has a positive solution.

Assume that
2.1 Sufficient conditions for the existence of at least one positive solution

\((\mathbf{A}_5)\) \(F(u), g(t,u)\) are non-increasing with respect to \(u\) and for each fixed \(n \in \{n_0, n_0+1, n_0+2, \cdots\}\), there exists a constant \(M > 0\) such that

\[
a \mu_n F(v_n) \int_{\eta}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)g(s, M + \frac{1}{n})ds \leq M.
\]

**Theorem 2.1.8** Assume that \((\mathbf{A}_1), (\mathbf{A}_3)\) and \((\mathbf{A}_5)\) holds. Then the system of SBVPs \((\mathbf{A}_7)\) has a positive solution.

**Proof.** For any \(u \in \partial \Omega_M \cap K_n\), using \((\mathbf{A}_1), (\mathbf{A}_5)\) and \((\mathbf{A}_5)\), we obtain

\[
(T_n u)(t) = \int_{1/n}^{1-1/n} H_n(t, s) f(s, \frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}d\tau)ds
\]

\[
\leq \int_{1/n}^{1-1/n} H_n(t, s) K(s)f(\frac{1}{n}) + \int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}d\tau)ds
\]

\[
\leq \int_{1/n}^{1-1/n} H_n(t, s) K(s)f(\int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, u(\tau) + \frac{1}{n}d\tau)ds)
\]

\[
\leq \int_{1/n}^{1-1/n} H_n(t, s) K(s)f(\int_{1/n}^{1-1/n} H_n(s, \tau) g(\tau, M + \frac{1}{n}d\tau)ds)
\]

Employing \((ii)\) of Lemma 2.1.2 \((\mathbf{A}_5)\) leads to

\[
(T_n u)(t) \leq F(v_n) \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, M + \frac{1}{n})d\tau \int_{1/n}^{1-1/n} H_n(t, s) K(s)ds.
\]

Now, using \((i)\) of Lemma 2.1.2 \((\mathbf{A}_1)\) and \((\mathbf{A}_5)\), we obtain

\[
(T_n u)(t) \leq \mu_n F(v_n) \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, M + \frac{1}{n})d\tau
\]

\[
\int_{1/n}^{1-1/n} (s - \frac{1}{n})(1 - \frac{1}{n} - s)K(s)ds
\]

\[
\leq a \mu_n F(v_n) \int_{\eta}^{1-1/n} (\tau - \frac{1}{n})(1 - \frac{1}{n} - \tau)g(\tau, M + \frac{1}{n})d\tau
\]

\[
\leq M,
\]

which implies that

\[
\|T_n u\| \leq \|u\|_{\mathcal{A}_n} \text{ for all } u \in \partial \Omega_M \cap K_n.
\]  \((2.20)\)
22 2 Singular Systems of ODEs with Nonlocal BCs

By \((A_1)\), we can choose \(r_n \in (0, M)\) such that (2.17) holds. Hence, in view of (2.20), (2.17) and by Theorem 1.1.11, \(T_n\) has a fixed point \(u_n \in (\overline{\Omega}_M \setminus \Omega_{r_n}) \cap K_n\). Consequently, the system of SBVPs (2.1) has a positive solution.

**Example 2.1.9** Let

\[
f(t, y) = \begin{cases} 
  y^{\frac{1}{n}} & \text{if } y \leq 1, \\
  e & \text{if } y > 1,
\end{cases}
\]

\[
g(t, x) = \begin{cases} 
  x^{\frac{1}{n}} & \text{if } x \leq 1, \\
  e & \text{if } x > 1,
\end{cases}
\]

\(\alpha = 2, \eta = \frac{1}{3}\). Choose \(\beta_1 = \beta_2 = 1\),

\[
K(t) = L(t) = \frac{1}{t(1 - t)}, \quad F(y) = \begin{cases} 
  y^{\frac{1}{n}} & \text{if } y \leq 1, \\
  e & \text{if } y > 1,
\end{cases}
\]

\[
G(x) = \begin{cases} 
  x^{\frac{1}{n}} & \text{if } x \leq 1, \\
  e & \text{if } x > 1,
\end{cases}
\]

\[
M \geq \max \{1, 6 F(e(1 - \frac{3}{n}) \int_{1/3}^{1-1/n} \frac{(t - \frac{1}{n})(1 - \frac{1}{n} - t)}{t(1-t)} ds)\}.
\]

Then \((A_1), (A_3)\) and \((A_5)\) are satisfied. Hence, by Theorem 2.1.8 the system of SBVPs (2.1) has a positive solution.

**Theorem 2.1.10** Assume that \((A_1), (A_4)\) and \((A_5)\) hold. Then the system of SBVPs (2.1) has a positive solution.

**Proof.** By \((A_1)\) and \((A_4)\), we obtain (2.19). By \((A_5)\) we can choose a constant \(M > \rho_n\) such that (2.20) holds. Then by Theorem 1.1.11, \(T_n\) has a fixed point \(u_n \in (\overline{\Omega}_M \setminus \Omega_{\rho_n}) \cap K_n\). Consequently, the system of SBVPs (2.1) has a positive solution.

**Example 2.1.11** Let

\[
f(t, y) = \frac{1}{t(1-t)} \frac{1}{\sqrt[3]{y}}, \quad g(t, x) = \frac{1}{t(1-t)} \frac{1}{x^2}
\]

and \(\alpha = 2, \eta = \frac{1}{3}\). Choose

\[
K(t) = L(t) = \frac{1}{t(1-t)}, \quad F(y) = \frac{1}{\sqrt[3]{y}}, \quad G(x) = \frac{1}{x^2}.
\]

Choose constants \(\rho_n\) and \(M\) such that
2.2 Sufficient conditions for the existence of at least one positive solution to a more general singular system

\[ \rho_n \leq \frac{4(n-3)}{n(6n^2+1)} \int_0^{1/3} (t - \frac{1}{n})(1 - \frac{1}{n} - t) dt, \]
\[ M \geq \frac{1}{n} \left( (1 - \frac{3}{n}) \int_0^{1/3} \frac{(t - \frac{1}{n})(1 - \frac{1}{n} - t)}{t(1-t)} dt \right)^{1/2} - 1. \]

Then \((A_1), (A_4),\) and \((A_5)\) are satisfied. Hence, by Theorem 2.1.10, the system of BVPs (2.1) has a positive solution.

2.2 Sufficient conditions for the existence of at least one positive solution to a more general singular systems

In this section, we establish the existence of positive solution for the system of SBVPs (2.2). We say \((x, y)\) is a positive solution of the system of SBVPs (2.2) if \((x, y) \in (C[0, 1] \cap C^2(0, 1)) \times (C[0, 1] \cap C^2(0, 1)), x > 0 and y > 0\) on \((0, 1), (x, y)\) satisfies (2.2). For \(x \in C[0, 1]\), we write \(||x|| = \max_{t \in [0, 1]} |x(t)|\). For any real constant \(r > 0\), we define an open neighborhood of 0 \(\in C[0, 1]\) as

\[ \Omega_r = \{ x \in C[0, 1] : ||x|| < r \}. \]

Define a cone \(K\) of \(C[0, 1]\) as

\[ K = \{ x \in C[0, 1] : x(t) \geq t(1-t) \gamma ||x|| \text{ for } t \in [0, 1] \}, \]

where

\[ 0 < \gamma := \frac{\min\{1, \alpha\} \min\{\eta, 1-\eta\}}{\max\{1, \alpha\}} < 1. \]

For each \((x, y) \in C[0, 1] \times C[0, 1]\), we write \(||(x, y)|| = ||x|| + ||y||\). Clearly, \((C[0, 1] \times C[0, 1], ||\cdot||)\) is a Banach space and \(K \times K\) is a cone of \(C[0, 1] \times C[0, 1]\).

For \(n \in \{1, 2, \cdots\}\), consider the following system of SBVPs

\[ -x''(t) = f(t, \max\{x(t) + \frac{1}{n}, \frac{1}{n}\}, \max\{y(t) + \frac{1}{n}, \frac{1}{n}\}), \quad t \in [0, 1], \]
\[ -y''(t) = g(t, \max\{x(t) + \frac{1}{n}, \frac{1}{n}\}, \max\{y(t) + \frac{1}{n}, \frac{1}{n}\}), \quad t \in [0, 1], \]
\[ x(0) = y(0) = 0, x(1) = \alpha x(\eta), y(1) = \alpha y(\eta). \]
The system of BVPs (2.21) can be expressed as an equivalent system of integral equations

\[
\begin{align*}
x(t) &= \int_0^1 H(t,s) f(s, \max \{x(s) + \frac{1}{n}, \frac{1}{n}\}, \max \{y(s) + \frac{1}{n}, \frac{1}{n}\}) ds, \\
y(t) &= \int_0^1 H(t,s) g(s, \max \{x(s) + \frac{1}{n}, \frac{1}{n}\}, \max \{y(s) + \frac{1}{n}, \frac{1}{n}\}) ds,
\end{align*}
\] (2.22)

where the Green’s function \(H\) is represented by (1.1). By a solution of the system of BVPs (2.21), we mean a solution of the corresponding system of integral equations (2.22). Define a map \(T_n : C[0,1] \times C[0,1] \to K \times K\) by

\[
T_n(x,y) = (A_n(x,y), B_n(x,y)),
\] (2.23)

where the maps \(A_n, B_n : C[0,1] \times C[0,1] \to K\) are defined by

\[
A_n(x,y)(t) = \int_0^1 H(t,s) f(s, \max \{x(s) + \frac{1}{n}, \frac{1}{n}\}, \max \{y(s) + \frac{1}{n}, \frac{1}{n}\}) ds,
\]

\[
B_n(x,y)(t) = \int_0^1 H(t,s) g(s, \max \{x(s) + \frac{1}{n}, \frac{1}{n}\}, \max \{y(s) + \frac{1}{n}, \frac{1}{n}\}) ds.
\] (2.24)

Clearly, if \((x_n, y_n) \in C[0,1] \times C[0,1]\) is a fixed point of \(T_n\), then \((x_n, y_n)\) also a solution of the system of BVPs (2.21).

Assume that

\((A_6)\) for each \(t \in (0,1)\), \(f(t,x,y)\) and \(g(t,x,y)\) are non-increasing with respect to

\(x\) and \(y\), \(f(\cdot, 1, 1), g(\cdot, 1, 1) \in C((0,1),(0,\infty))\) and

\[
\begin{align*}
a &:= \int_0^1 t(1-t)f(t,t(1-t),t(1-t)) dt < +\infty, \\
b &:= \int_0^1 t(1-t)g(t,t(1-t),t(1-t)) dt < +\infty,
\end{align*}
\]

\((A_7)\) there exist real constants \(\alpha_i, \beta_i\) with \(\alpha_i \leq 0 \leq \beta_i, i = 1,2\), such that for all \(t \in (0,1), x,y \in (0,\infty),\)

\[
\begin{align*}
c^{\beta_1} f(t,x,y) &\leq f(t,cx,y) \leq c^{\alpha_1} f(t,x,y), & &\text{if } 0 < c \leq 1, \\
c^{\alpha_1} f(t,x,y) &\leq f(t,cx,y) \leq c^{\beta_1} f(t,x,y), & &\text{if } c \geq 1, \\
c^{\beta_2} f(t,x,y) &\leq f(t,x,cy) \leq c^{\alpha_2} f(t,x,y), & &\text{if } 0 < c \leq 1, \\
c^{\alpha_2} f(t,x,y) &\leq f(t,x,cy) \leq c^{\beta_2} f(t,x,y), & &\text{if } c \geq 1;
\end{align*}
\]
2.2 Sufficient conditions for the existence of at least one positive solution to a more general singular system

(A₈) there exist real constants γᵢ, ρᵢ with γᵢ ≤ 0 ≤ ρᵢ, i = 1, 2, such that for all t ∈ (0, 1), x, y ∈ (0, ∞),

\[ e^{ρ₁}g(t,x,y) ≤ g(t,ε x,y) ≤ e^{ρ₂}g(t,x,y), \quad \text{if } 0 < c ≤ 1, \]

\[ e^{ρ₃}g(t,x,y) ≤ g(t,c x,y) ≤ e^{ρ₄}g(t,x,y), \quad \text{if } c ≥ 1, \]

\[ e^{ρ₅}g(t,x,y) ≤ g(t,x,c y) ≤ e^{ρ₆}g(t,x,y), \quad \text{if } 0 < c ≤ 1, \]

\[ e^{ρ₇}g(t,x,y) ≤ g(t,x,c y) ≤ e^{ρ₈}g(t,x,y), \quad \text{if } c ≥ 1. \]

**Lemma 2.2.1** Assume that (A₆) – (A₈) holds. Then the map \( Tₙ : (\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K) \rightarrow K × K \) is completely continuous.

**Proof.** Clearly, \( Tₙ(x,y) ∈ K × K \) for all \( (x,y) ∈ K × K \). Now, we show that the map \( Aₙ : (\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K) \rightarrow K × K \) is uniformly bounded and equicontinuous. For \( (x,y) ∈ (\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K) \), using (2.24), (A₆), (A₇) and Lemma 1.1.19, \( Aₙ((\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K)) \) is uniformly bounded. Similarly, using (2.24), (A₆), (A₈) and Lemma 1.1.19 we can show that \( Bₙ((\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K)) \) is also uniformly bounded. Thus, \( Tₙ((\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K)) \) is uniformly bounded. Since the Green’s function \( H \) is uniformly continuous on \([0, 1] × [0, 1]\), therefore, \( Tₙ((\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K)) \) is equicontinuous. Thus by Theorem 1.1.5, it follows that \( Tₙ((\overline{Ω}_{r₁} × \overline{Ω}_{r₂}) \cap (K × K)) \) is relatively compact. Hence, \( Tₙ \) is a compact map.

Now, we show that \( Tₙ \) is continuous. Let \( (xₘ, yₘ), (x,y) ∈ K × K \) such that

\[ ||(xₘ, yₘ) - (x,y)|| → 0 \text{ as } m → +∞. \]

Using (2.24) and (i) of Lemma 2.1.2, we have

\[ |Aₙ(xₘ, yₘ)(t) - Aₙ(x,y)(t)| = \left| \int_{0}^{1} H(t,s) \left( f(s,xₘ(s) + \frac{1}{n}, yₘ(s) + \frac{1}{n}) - f(s,x(s) + \frac{1}{n}, y(s) + \frac{1}{n}) \right) ds \right| \]

\[ ≤ \int_{0}^{1} H(t,s) \left| f(s,xₘ(s) + \frac{1}{n}, yₘ(s) + \frac{1}{n}) - f(s,x(s) + \frac{1}{n}, y(s) + \frac{1}{n}) \right| ds \]

\[ ≤ \mu \int_{0}^{1} s(1-s) \left| f(s,xₘ(s) + \frac{1}{n}, yₘ(s) + \frac{1}{n}) - f(s,x(s) + \frac{1}{n}, y(s) + \frac{1}{n}) \right| ds. \]

Consequently,

\[ ||Aₙ(xₘ, yₘ) - Aₙ(x,y)|| ≤ \mu \int_{0}^{1} s(1-s) \left| f(s,xₘ(s) + \frac{1}{n}, yₘ(s) + \frac{1}{n}) - f(s,x(s) + \frac{1}{n}, y(s) + \frac{1}{n}) \right| ds. \]

By Lebesgue dominated convergence theorem, it follows that
\begin{align*}
\|A_n(x_m, y_m) - A_n(x, y)\| &\to 0 \text{ as } m \to +\infty, \quad (2.25) \\
\|B_n(x_m, y_m) - B_n(x, y)\| &\to 0 \text{ as } m \to +\infty, \quad (2.26)
\end{align*}

Similarly, by using (2.24) and (i) of Lemma 2.1.2, we have

\begin{align*}
\|T_n(x_m, y_m) - T_n(x, y)\| &\to 0 \text{ as } m \to +\infty,
\end{align*}

From (2.25), (2.26) and (2.23), it follows that

\[
\|T_n(x_m, y_m) - T_n(x, y)\| \to 0 \text{ as } m \to +\infty,
\]

that is, \(T_n : (\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K) \to K \times K\) is continuous. Hence, \(T_n : (\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K) \to K \times K\) is completely continuous.

**Theorem 2.2.2** Assume that \((A_6) - (A_8)\) hold. Then the system of SBVPs (2.2) has a positive solution.

**Proof.** Choose a real constants \(R_1 > 0\) and \(R_2 > 0\) such that

\[
R_1 \geq \max \left\{1, \left(\frac{\mu \gamma^{\alpha_2} + \alpha_2 c_7^{\alpha_1 + \alpha_2 - \beta_1 - \beta_2 R_{1 R^2}}}{\gamma^2} \right)^{\frac{1}{\gamma^2}} \right\},
\]

\[
R_2 \geq \max \left\{1, \left(\frac{\mu \gamma^{\beta_1} + \beta_1 c_7^{\gamma_1 + \gamma_2 - \rho_1 - \rho_2 R_1 R_2}}{\gamma^2} \right)^{\frac{1}{\gamma^2}} \right\}, \quad (2.27)
\]

where \(c_7 \in (0, 1]\) such that \(c_7 R_1 \leq 1\) and \(c_7 R_2 \leq 1\). For any \((x, y) \in \partial(\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K)\), using (2.24), (A_6), (A_7), (2.27) and (i) of Lemma 1.1.19, we have

\[
A_n(x, y)(t) = \int_0^1 H(t, s)f(s, x(s) + \frac{1}{n}y(s) + \frac{1}{n})ds \\
\leq \mu \int_0^1 s(1-s)f(s, s(1-s)\gamma\|x\|, s(1-s)\gamma\|y\|)ds \\
\leq R_1
\]

which implies that

\[
\|A_n(x, y)\| \leq \|x\| \text{ for all } (x, y) \in \partial(\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K). \quad (2.28)
\]

Similarly, using (2.24), (A_6), (A_8), (i) of Lemma 1.1.2, (2.27), we obtain

\[
B_n(x, y) \leq \|y\| \text{ for all } (x, y) \in \partial(\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K). \quad (2.29)
\]

From (2.28), (2.29) and (2.23), it follows that

\[
\|T_n(x, y)\| \leq \|x,y\| \text{ for all } (x, y) \in \partial(\Omega_{r_1} \times \Omega_{r_2}) \cap (K \times K). \quad (2.30)
\]
2.2 Sufficient conditions for the existence of at least one positive solution to a more general singular system

Choose a real constants \( r_n \in (0, R_1) \) and \( s_n \in (0, R_2) \) such that

\[
\begin{align*}
\frac{r_n}{c_8} &\leq \frac{r_n}{c_8} \int_0^1 (1 - s) f(s, 1, 1) ds \\
\frac{s_n}{c_8} &\leq \frac{s_n}{c_8} \int_0^1 (1 - s) g(s, 1, 1) ds
\end{align*}
\]

(2.31)

where \( c_8 \in (0, 1] \) is such that \( c_8 (r_n + \frac{1}{n}) \leq 1 \) and \( c_8 (s_n + \frac{1}{n}) \leq 1 \). For any \( (x, y) \in \partial (\Omega_{r_n} \times \Omega_{s_n}) \cap (K \times K) \), using (2.24), (A7) and Lemma 1.1.19 we have

\[
A_n(x, y)(t) = \int_0^1 H(t, s) f\left(s, x(s) + \frac{1}{n}, y(s) + \frac{1}{n}\right) ds
\]

(2.32)

and

\[
B_n(x, y)(t) = \int_0^1 H(t, s) \frac{x(s) + \frac{1}{n}}{c_8}, c_8, x(s) + \frac{1}{n}) ds
\]

(2.33)

From (2.32) and (2.33), we obtain

\[
\|T_n(x, y)\| \geq \|(x, y)\| \text{ for all } (x, y) \in \partial (\Omega_{r_n} \times \Omega_{s_n}) \cap (K \times K).
\]

(2.34)

In view of (2.30), (2.34), and by Theorem 1.1.11, \( T_n \) has a fixed point \( (x_n, y_n) \in (\Omega_{R_1} \times \Omega_{R_2} \setminus (\Omega_{r_n} \times \Omega_{s_n})) \cap (K \times K) \). Further

\[
r_n \leq \|x_n\| \leq R_1, s_n \leq \|y_n\| \leq R_2,
\]

(2.35)

where \( r_n, s_n \to 0 \) as \( n \to \infty \). Thus, \( \{x_n, y_n\}_{n=1}^\infty \) bounded uniformly on \([0, 1]\). Moreover, since the Green’s function \( H \) is uniformly continuous on \([0, 1] \times [0, 1]\), therefore \( \{x_n, y_n\}_{n=1}^\infty \) is equicontinuous on \([0, 1]\). Thus, there exists a subsequence \( \{x_{n_k}, y_{n_k}\} \) of \( \{x_n, y_n\} \) converging uniformly to \( (x, y) \in C[0, 1] \times C[0, 1] \). Now, for \( t \in [0, 1] \) consider the integral equations
\[ x_{n_k}(t) = \int_{0}^{1} H(t,s) f(t,x_{n_k}(s) + \frac{1}{n_k} y_{n_k}(s) + \frac{1}{n_k}) ds, \]
\[ y_{n_k}(t) = \int_{0}^{1} H(t,s) g(t,x_{n_k}(s) + \frac{1}{n_k} y_{n_k}(s) + \frac{1}{n_k}) ds, \]
as \( n_k \to \infty \), we have
\[ x(t) = \int_{0}^{1} H(t,s) f(t,x(s), y(s)) ds, \quad t \in [0,1], \]
\[ y(t) = \int_{0}^{1} H(t,s) g(s,x(s), y(s)) ds, \quad t \in [0,1]. \]
Moreover,
\[ x(0) = 0, x(1) = \alpha x(\eta), y(0) = 0, y(1) = \alpha y(\eta). \]
Hence, \((x, y)\) is a solution of the system of BVPs (2.2). Moreover, since \( f, g : (0, 1) \times (0, \infty) \times (0, \infty) \to (0, \infty) \) and the Green's function \( H \) is positive on \((0, 1) \times (0, 1)\), it follows that \( x > 0 \) and \( y > 0 \) on \([0, 1]\).

**Example 2.2.3** Let
\[ f(t,x,y) = \frac{1}{\sqrt{t(1-t)}xy}, \]
\[ g(t,x,y) = \frac{1}{\sqrt{t(1-t)}xy} \]
Clearly, \( f \) and \( g \) satisfy assumptions \((A_6) - (A_8)\). Hence, by Theorem 2.2.2 the system of SBVPs (2.2) has a positive solution.

### 2.3 Singular systems of ODEs with four-point coupled BCs

In this section, we establish the existence of positive solutions for the system of SBVPs (2.3). By a positive solution to the system of SBVPs (2.3), we mean that \((x, y) \in (C[0, 1] \cap C^2(0, 1)) \times (C[0, 1] \cap C^2(0, 1))\), \((x, y)\) satisfies (2.3), \( x > 0 \) and \( y > 0 \) on \([0, 1]\). For each \( x \in C[0, 1] \) we write \( \|x\| = \max_{t \in [0,1]} |x(t)| \). Let
\[ P = \{ x \in C[0, 1] : \min_{t \in [\max\{\xi, \eta\}, 1]} x(t) \geq \gamma\|x\| \}, \]
where
Lemma 2.3.2

\[ 0 < \gamma := \frac{\min\{1, \alpha \xi, \alpha \beta \xi, \beta \eta, \alpha \beta \eta\} \min\{\xi, \eta, 1 - \xi, 1 - \eta\}}{\max\{1, \alpha, \beta, \alpha \beta \xi, \alpha \beta \eta\}} < 1. \]

Clearly, \((C[0, 1], \| \cdot \|)\) is a Banach space and \(P\) is a cone of \(C[0, 1]\). Similarly, for each \((x, y) \in C[0, 1] \times C[0, 1]\) we write \(||(x, y)|| = ||x|| + ||y||\). Clearly, \((C[0, 1] \times C[0, 1], || \cdot ||)\) is a Banach space and \(P \times P\) is a cone of \(C[0, 1] \times C[0, 1]\). For any real constant \(r > 0\), define \(\mathcal{C}_r = \{(x, y) \in C[0, 1] \times C[0, 1] : \|(x, y)|| < r\}.

**Lemma 2.3.1** Let \(u, v \in C[0, 1]\), then the system of BVPs

\[ \begin{align*}
-x''(t) &= u(t), & t & \in [0, 1], \\
-y''(t) &= v(t), & t & \in [0, 1], \\
x(0) &= 0, x(1) = \alpha y(\xi), \\
y(0) &= 0, y(1) = \beta x(\eta),
\end{align*} \tag{2.36} \]

has integral representation

\[ \begin{align*}
x(t) &= \int_0^1 F_{\xi \eta}(t, s)u(s)ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s)v(s)ds, \\
y(t) &= \int_0^1 F_{\eta \xi}(t, s)v(s)ds + \int_0^1 G_{\beta \alpha \eta \xi}(t, s)u(s)ds,
\end{align*} \tag{2.37} \]

where

\[ F_{\xi \eta}(t, s) = \begin{cases} \\
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi (\eta - s)}{1-\alpha \beta \xi \eta} - (t - s), & 0 \leq s \leq t \leq 1, s \leq \eta, \\
\frac{t(1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \beta \xi (\eta - s)}{1-\alpha \beta \xi \eta}, & 0 \leq t \leq s \leq 1, s \leq \eta, \\
(t - s), & 0 \leq s \leq t \leq 1, s \geq \eta, \\
(t - s), & 0 \leq t \leq s \leq 1, s \geq \eta,
\end{cases} \tag{2.38} \]

\[ G_{\alpha \beta \xi \eta}(t, s) = \begin{cases} \\
\frac{\alpha \xi (1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi (\eta - s)}{1-\alpha \beta \xi \eta}, & 0 \leq s \leq t \leq 1, s \leq \xi, \\
\frac{\alpha \xi (1-s)}{1-\alpha \beta \xi \eta} - \frac{\alpha \xi (\eta - s)}{1-\alpha \beta \xi \eta}, & 0 \leq s \leq t \leq 1, s \geq \xi,
\end{cases} \tag{2.39} \]

**Lemma 2.3.2** The functions \(F_{\xi \eta}\) and \(G_{\alpha \beta \xi \eta}\) satisfies

(i) \( F_{\xi \eta}(t, s) \leq \frac{\max\{1, \alpha \xi\}}{1-\alpha \beta \xi \eta}s(1-s), \quad t, s \in [0, 1], \)

(ii) \( G_{\alpha \beta \xi \eta}(t, s) \leq \frac{\alpha}{1-\alpha \beta \xi \eta}s(1-s), \quad t, s \in [0, 1], \)

**Remark 2.3.3** In view of Lemma 2.3.2 we have
Lemma 2.3.4 The functions $F_{\eta \xi}(t,s)$ and $G_{\alpha \beta \xi \eta}$ satisfies

(i) $F_{\eta \xi}(t,s) \geq \frac{\min\{1, \alpha \beta \eta\} \min\{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1-s), \quad (t,s) \in [\eta, 1] \times [0,1],$

(ii) $G_{\alpha \beta \xi \eta}(t,s) \geq \frac{\beta \eta \min\{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1-s), \quad (t,s) \in [\xi, 1] \times [0,1].$

Remark 2.3.5 In view of Lemma 2.3.4 we have

$$F_{\eta \xi}(t,s) \geq \frac{\min\{1, \alpha \beta \eta\} \min\{\xi, 1 - \xi\}}{1 - \alpha \beta \xi \eta} s(1-s), \quad (t,s) \in [\xi, 1] \times [0,1],$$

$$G_{\alpha \beta \xi \eta}(t,s) \geq \frac{\beta \eta \min\{\eta, 1 - \eta\}}{1 - \alpha \beta \xi \eta} s(1-s), \quad (t,s) \in [\eta, 1] \times [0,1].$$

Remark 2.3.6 From Lemma 2.3.2 and Remark 2.3.3 for $t,s \in [0,1]$, we have

$$F_{\eta \xi}(t,s) \leq \mu s(1-s), F_{\eta \xi}(t,s) \leq \mu s(1-s),$$

$$G_{\alpha \beta \xi \eta}(t,s) \leq \mu s(1-s), G_{\alpha \beta \xi \eta}(t,s) \leq \mu s(1-s),$$

where $\mu = \max\{1, \alpha \beta \eta\} \min\{\xi, 1 - \xi\} / (1 - \alpha \beta \xi \eta).$ Similarly, from Lemma 2.3.4 and Remark 2.3.5 for $t,s \in [\max\{\xi, \eta\}, 1] \times [0,1]$, we have

$$F_{\eta \xi}(t,s) \geq \nu s(1-s), F_{\eta \xi}(t,s) \geq \nu s(1-s),$$

$$G_{\alpha \beta \xi \eta}(t,s) \geq \nu s(1-s), G_{\alpha \beta \xi \eta}(t,s) \geq \nu s(1-s),$$

where $\nu = \min\{1, \alpha \beta \beta \eta \eta, (\xi, \xi, 1 - \xi, 1 - \xi)\} / (1 - \alpha \beta \xi \eta).$

In view of Lemma 2.3.1 the system of BVPs (2.3) can be expressed as

$$x(t) = \int_0^t F_{\eta \xi}(t,s)f(s,x(s),y(s))ds + \int_0^t G_{\alpha \beta \xi \eta}(t,s)g(s,x(s),y(s))ds, \quad t \in [0,1],$$

$$y(t) = \int_0^t F_{\eta \xi}(t,s)g(s,x(s),y(s))ds + \int_0^t G_{\alpha \beta \xi \eta}(t,s)f(s,x(s),y(s))ds, \quad t \in [0,1].$$

(2.40)

By a solution of the system of BVPs (2.3), we mean a solution of the corresponding system of integral equations (2.40). Define a map $T : P \times P \to P \times P$ by
2.3 Singular systems of ODEs with four-point coupled BCs

\[ T(x, y) = (A(x, y), B(x, y)), \]

(2.41)

where the maps \( A, B : P \times P \to P \) are defined by

\[
A(x, y)(t) = \int_0^1 F_{\xi \eta}(t, s)f(s, x(s), y(s))ds + \int_0^1 G_{\alpha \beta \xi \eta}(t, s)g(s, x(s), y(s))ds, \quad t \in [0, 1],
\]

\[
B(x, y)(t) = \int_0^1 F_{\xi \eta}(t, s)g(s, x(s), y(s))ds + \int_0^1 G_{\beta \alpha \eta \xi}(t, s)f(s, x(s), y(s))ds, \quad t \in [0, 1].
\]

(2.42)

Clearly, if \((x, y) \in P \times P\) is a fixed point of \(T\), then \((x, y)\) is a solution of the system of BVPs (2.3).

Assume that

\((A_9)\) \quad \(f(\cdot, 1, 1), g(\cdot, 1, 1) \in C((0, 1), (0, \infty))\) and satisfy

\[
a := \int_0^1 t(1-t)f(t, 1, 1)dt < +\infty, \quad b := \int_0^1 t(1-t)g(t, 1, 1)dt < +\infty,
\]

\((A_{10})\) \quad there exist real constants \(\alpha_i, \beta_i\) with \(0 \leq \alpha_i \leq \beta_i < 1, i = 1, 2; \beta_1 + \beta_2 < 1\), such that for all \(t \in (0, 1), x, y \in [0, \infty),\)

\[
c^{\beta_i}f(t, x, y) \leq f(t, c x, y) \leq c^{\alpha_i}f(t, x, y), \quad 0 < c \leq 1,
\]

\[
c^{\alpha_i}f(t, x, y) \leq f(t, c x, y) \leq c^{\beta_i}f(t, x, y), \quad c \geq 1,
\]

\[
c^{\beta_2}f(t, x, y) \leq f(t, x, c y) \leq c^{\alpha_2}f(t, x, y), \quad 0 < c \leq 1,
\]

\[
c^{\alpha_2}f(t, x, y) \leq f(t, x, c y) \leq c^{\beta_2}f(t, x, y), \quad c \geq 1.
\]

\((A_{11})\) \quad there exist real constants \(\gamma_i, \rho_i\) with \(0 \leq \gamma_i \leq \rho_i < 1, i = 1, 2; \rho_1 + \rho_2 < 1\), such that for all \(t \in (0, 1), x, y \in [0, \infty),\)

\[
c^{\rho_i}g(t, x, y) \leq g(t, c x, y) \leq c^{\gamma_i}g(t, x, y), \quad 0 < c \leq 1,
\]

\[
c^{\gamma_i}g(t, x, y) \leq g(t, c x, y) \leq c^{\rho_i}g(t, x, y), \quad c \geq 1,
\]

\[
c^{\rho_2}g(t, x, y) \leq g(t, x, c y) \leq c^{\gamma_2}g(t, x, y), \quad 0 < c \leq 1,
\]

\[
c^{\gamma_2}g(t, x, y) \leq g(t, x, c y) \leq c^{\rho_2}g(t, x, y), \quad c \geq 1.
\]

**Lemma 2.3.7** Assume that \((A_9) - (A_{11})\) hold. Then the map \(T : \overline{Q} \cap (P \times P) \to P \times P\) is completely continuous.

**Theorem 2.3.8** Assume that \((A_9) - (A_{11})\) hold. Then the system of BVPs (2.3) has a positive solution.
Theorem 2.3.9 Assume that \( (A_6) - (A_8) \) hold. Then the system of BVPs with singularity at \( t = 0, t = 1, x = 0 \) and/or \( y = 0 \) has a positive solution.
Chapter 3

Singular Systems of Second-Order Two-Point Boundary Value Problems

The existence of positive solutions for a nonlinear second-order two-point BVPs has received much attention; see for example the case of regular nonlinearities, [37, 38, 52, 67], and the case of singular nonlinearities, see [1, 4, 29]. However, these results are for the case when nonlinear functions are independent of the first derivative. The BVPs involving the first derivative with regular nonlinear functions can be seen in [45, 51, 105].

In [3, Section 2.4], Agarwal and O’Regan studied the existence of at least one positive solution for the following BVP with $a = 1$ and $b = 0$,

$$
\begin{align*}
- y''(t) &= q(t)f(t, y(t), y'(t)), \quad t \in (0, 1), \\
ay(0) - by'(0) &= y'(1) = 0,
\end{align*}
$$

(3.1)

where $f : [0, 1] \times [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous and is allowed to be singular at $y' = 0$; $q \in C(0, 1)$ and $q > 0$ on $(0, 1)$. The existence of multiple positive solutions for second-order BVPs has also invited attention of many authors, [51, 56, 61, 62, 77, 82, 107, 66, 106]. B. Yan et al. [101] have studied the existence of multiple positive solutions of the BVP (3.1) with $a = 1$ and $b = 0$. Further, they generalized these results and established the existence of at least two positive solutions for BVP (3.1) with $a > 0$ and $b > 0$. [102].

In Sections 3.1, 3.2, 3.3 and 3.4 we study the existence and multiplicity of positive solutions to the following coupled systems of SBVPs
Singular Systems of Second-Order ODEs with Two-Point BCs

\[-x''(t) = p(t)f(t,y(t),y'(t)), \quad t \in (0,1),\]
\[-y''(t) = q(t)g(t,x(t),y'(t)), \quad t \in (0,1),\]
\[x(0) = y(0) = x'(1) = y'(1) = 0,\]  

(3.2)

and

\[-x''(t) = p(t)f(t,y(t),y'(t)), \quad t \in (0,1),\]
\[-y''(t) = q(t)g(t,x(t),y'(t)), \quad t \in (0,1),\]
\[a_1x(0) - b_1x'(0) = x'(1) = 0,\]
\[a_2y(0) - b_2y'(0) = y'(1) = 0,\]

where the functions \(f,g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)\) are continuous and are allowed to be singular at \(x' = 0, y' = 0\). Moreover, \(p,q \in C(0,1)\) and positive on \((0,1)\), and the real constants \(a_i (i = 1,2) > 0, b_i (i = 1,2) > 0\). By singularity of \(f\) and \(g\), we mean that the functions \(f(t,x,y)\) and \(g(t,x,y)\) are allowed to be unbounded at \(y = 0\).

Agarwal and O’Regan [3, Section 2.10] have developed the method of upper and lower solutions for the SBVP

\[-y''(t) = q(t)f(t,y(t),y'(t)), \quad t \in (0,1),\]
\[y(0) = y(1) = 0,\]

(3.4)

where \(f : [0,1] \times (0,\infty) \times \mathbb{R} \to \mathbb{R}\) is continuous and singular at \(y = 0\) and the function \(q \in C(0,1)\) is positive on \((0,1)\). Further, they have presented the method of upper and lower solutions for more general problems in [36].

In Section 3.6 we study the existence of \(C^1\)-positive solutions for the following system of SBVPs

\[-x''(t) = p_1(t)f_1(t,x(t),y(t),y'(t)), \quad t \in (0,1),\]
\[-y''(t) = p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in (0,1),\]
\[x(0) = x(1) = y(0) = y(1) = 0,\]

(3.5)

where \(f_1, f_2 : [0,1] \times (0,\infty) \times (0,\infty) \times \mathbb{R} \to \mathbb{R}\) are continuous. Moreover, \(f_1, f_2\) are allowed to change sign and may be singular at \(x = 0, y = 0\). Also, \(p_1, p_2 \in C(0,1)\) are positive on \((0,1)\).
3.1 Existence of $C^1$-positive solutions

Further in Section 3.5, we study more general coupled system of ODEs and prove the existence of $C^1$-positive solution to the following system of ODEs subject to two-point coupled BCs

$$
-x''(t) = p(t)f(t,x(t),y(t),x'(t)), \quad t \in (0,1),
-\psi(0) = q(t)g(t,x(t),y(t),y'(t)), \quad t \in (0,1),
$$

$$
a_1y(0) - b_1x'(0) = 0, y'(1) = 0,
-a_2x(0) - b_2y'(0) = 0, x'(1) = 0,
$$

where the nonlinearities $f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty)$ are continuous and are allowed to be singular at $x' = 0, y' = 0$. Moreover, $p, q \in C(0,1)$, $p > 0$ and $q > 0$ on $(0,1)$.

3.1 Existence of $C^1$-positive solutions

In this section, we establish sufficient conditions for the existence of $C^1$-positive solutions to the system of SBVPs \((3.2)\). By a $C^1$-positive solution to the system of SBVPs \((3.2)\), we mean that $(x,y) \in C^1[0,1] \cap C^2(0,1) \times (C^1[0,1] \cap C^2(0,1))$ satisfying \((3.2)\), $x > 0$ and $y > 0$ on $(0,1)$, $x' > 0$ and $y' > 0$ on $(0,1)$. For each $x \in C[0,1] \cap C^1(0,1]$, we write $\|x\| = \max_{t \in [0,1]} |x(t)|$ and $\|x\|_1 = \sup_{t \in (0,1)} |t| |x'(t)|$. Moreover, for each $x \in \mathcal{E} := \{x \in C[0,1] \cap C^1(0,1] : \|x\| < +\infty\}$, we write $\|x\|_2 = \max\{\|x\|, \|x\|_1\}$. By Lemma 1.1.12 $(\mathcal{E}, \|\cdot\|_2)$ is a Banach space. Moreover, for each $x \in C^1[0,1]$, we write $\|x\|_3 = \max\{\|x\|, \|x'\|\}$. Clearly, $(C^1[0,1], \|\cdot\|_3)$ is a Banach space.

Assume that

\( (B_1) \quad p, q \in C(0,1), p, q > 0 \text{ on } (0,1), \int_0^1 p(t)dt < +\infty \text{ and } \int_0^1 q(t)dt < +\infty; \)

\( (B_2) \quad f, g : [0,1] \times [0,\infty) \times (0,\infty) \to [0,\infty) \text{ are continuous with } f(t,x,y) > 0 \text{ and } g(t,x,y) > 0 \text{ on } [0,1] \times (0,\infty) \times (0,\infty); \)

\( (B_3) \quad f(t,x,y) \leq k_1(x)(u_1(y) + v_1(y)) \text{ and } g(t,x,y) \leq k_2(x)(u_2(y) + v_2(y)), \text{ where } u_i(i = 1,2) > 0 \text{ are continuous and nonincreasing on } (0,\infty), k_i(i = 1,2) \geq 0, \)

\( (B_4) \)
Remark 3.1.1 Since $I$, $J$ are continuous, $I^{-1}$, $J^{-1}$ are also monotonically increasing. Hence, $I$ and $J$ are invertible. Moreover, $I^{-1}$ and $J^{-1}$ are also monotonically increasing.

**Theorem 3.1.2** Assume that $(B_1) - (B_7)$ hold. Then the system of BVPs $(3.7)$ has a $C^1$-positive solution.

Proof. In view of $(B_4)$, we can choose real constants $M_1 > 0$ and $M_2 > 0$ such that

\[
\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds))\int_0^1 p(s)ds)} > 1,
\]

\[
\frac{M_2}{J^{-1}(k_2(J^{-1}(k_1(M_2)\int_0^1 p(s)ds))\int_0^1 q(s)ds)} > 1.
\]

From the continuity of $k_1$, $k_2$, $I$ and $J$, choose $\varepsilon > 0$ small enough such that

\[
\frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1)\int_0^1 q(s)ds + J(\varepsilon)))\int_0^1 p(s)ds + I(\varepsilon))} > 1, \quad (3.7)
\]

\[
\frac{M_2}{J^{-1}(k_2(J^{-1}(k_1(M_2)\int_0^1 p(s)ds + I(\varepsilon)))\int_0^1 q(s)ds + J(\varepsilon))} > 1. \quad (3.8)
\]

Choose real constants $L_1 > 0$ and $L_2 > 0$ such that

\[
I(L_1) > k_1(M_2)\int_0^1 p(s)ds + I(\varepsilon), \quad (3.9)
\]
Consider the modified system of BVPs

\[ J(L_2) > k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon). \] (3.10)

Choose \( n_0 \in \{1, 2, \cdots\} \) such that \( \frac{1}{n_0} < \varepsilon \). For each fixed \( n \in \{n_0, n_0 + 1, \cdots\} \), define retractions \( \theta_i : \mathbb{R} \to [0, M_i] \) and \( \rho_i : \mathbb{R} \to [\frac{1}{n}, L_i] \) by

\[
\theta_i(x) = \max\{0, \min\{x, M_i\}\} \quad \text{and} \quad \rho_i(x) = \max\{\frac{1}{n}, \min\{x, L_i\}\}, \quad i = 1, 2.
\]

Consider the modified system of BVPs

\[
\begin{align*}
-x''(t) &= p(t)f(t, \theta_2(y(t)), \rho_1(x'(t))), \quad t \in (0, 1), \\
y''(t) &= q(t)g(t, \theta_1(x(t)), \rho_2(y'(t))), \quad t \in (0, 1), \\
x(0) &= y(0) = 0, \quad x'(1) = y'(1) = \frac{1}{n}.
\end{align*}
\] (3.11)

Since \( f(t, \theta_2(y(t)), \rho_1(x'(t))), g(t, \theta_1(x(t)), \rho_2(y'(t))) \) are continuous and bounded on \([0, 1] \times \mathbb{R}^2\), by Theorem 1.1.7, it follows that the modified system of BVPs (3.11) has a solution \((x_n, y_n) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))\).

Using (3.11) and (B2), we obtain

\[
x_n''(t) \leq 0 \quad \text{and} \quad y_n''(t) \leq 0 \quad \text{for} \quad t \in (0, 1),
\]

which on integration from \( t \) to 1, using the BCs (3.11), implies that

\[
x_n(t) \geq \frac{1}{n} \quad \text{and} \quad y_n(t) \geq \frac{1}{n} \quad \text{for} \quad t \in [0, 1]. \] (3.12)

Integrating (3.12) from 0 to \( t \), using the BCs (3.11), we have

\[
x_n(t) \geq \frac{t}{n} \quad \text{and} \quad y_n(t) \geq \frac{t}{n} \quad \text{for} \quad t \in [0, 1]. \] (3.13)

From (3.12) and (3.13), it follows that

\[
\|x_n\| = x_n(1) \quad \text{and} \quad \|y_n\| = y_n(1).
\]

Now, we show that

\[
x_n'(t) < L_1, \quad y_n'(t) < L_2, \quad t \in [0, 1]. \] (3.14)

First, we prove \( x_n'(t) < L_1 \) for \( t \in [0, 1] \). Suppose \( x_n'(t_1) \geq L_1 \) for some \( t_1 \in [0, 1] \). Using (3.11) and (B3), we have
which can also be written as

\[-x_n''(t) \leq p(t)k_1(\theta_2(y_n(t)))(u_1(p_1(x_n'(t))) + v_1(p_1(x_n'(t)))) , \quad t \in (0, 1),\]

which implies that

\[
-\frac{x_n''(t)}{u_1(p_1(x_n'(t))) + v_1(p_1(x_n'(t)))} \leq k_1(M_2)p(t), \quad t \in (0, 1).
\]

Integrating from \(t_1\) to 1, using the BCs (3.11), we obtain

\[
\int_{t_1}^{1} \frac{dz}{u_i(z) + v_1(z)} \leq k_1(M_2) \int_{t_1}^{1} p(t)dt,
\]

which can also be written as

\[
\int_{\frac{t_1}{2}}^{L_1} \frac{dz}{u_i(z) + v_1(z)} + \int_{L_1}^{L_1} \frac{dz}{u_i(L_1) + v_1(L_1)} \leq k_1(M_2) \int_{0}^{1} p(t)dt.
\]

Using the increasing property of \(I\), we obtain

\[
I(L_1) + \frac{x_n'(t_1)}{u_1(L_1) + v_1(L_1)} \leq k_1(M_2) \int_{0}^{1} p(t)dt + I(\varepsilon),
\]

a contradiction to (3.9). Hence, \(x_n'(t) < L_1\) for \(t \in [0, 1]\). Similarly, we can show that \(y_n'(t) < L_2\) for \(t \in [0, 1]\).

Now, we show that

\[
x_n(t) < M_1, \quad y_n(t) < M_2, \quad t \in [0, 1]. \tag{3.15}
\]

Suppose \(x_n(t_2) \geq M_1\) for some \(t_2 \in [0, 1]\). From (3.11), (3.14) and (B_3), it follows that

\[-x_n''(t) \leq p(t)k_1(\theta_2(y_n(t)))(u_1(x_n'(t)) + v_1(x_n'(t))) , \quad t \in (0, 1),
\]

\[-y_n''(t) \leq q(t)k_2(\theta_1(x_n(t)))(u_2(y_n'(t)) + v_2(y_n'(t))) , \quad t \in (0, 1),
\]

which implies that

\[
-\frac{x_n''(t)}{u_1(x_n'(t)) + v_1(x_n'(t))} \leq k_1(\theta_2(\|y_n\|))p(t), \quad t \in (0, 1),
\]

\[
-\frac{y_n''(t)}{u_2(y_n'(t)) + v_2(y_n'(t))} \leq k_2(M_1)q(t), \quad t \in (0, 1).
\]

Integrating from \(t\) to 1, using the BCs (3.11), we obtain
Integrating (3.16) from 0 to \( t \) which implies that

\[ I(x_n'(t)) - I\left(\frac{1}{n}\right) \leq k_1(\theta_2(\|y_n\|)) \int_0^1 p(s)ds, \quad t \in [0, 1], \]

\[ J(y_n'(t)) - J\left(\frac{1}{n}\right) \leq k_2(M_1) \int_0^1 q(s)ds, \quad t \in [0, 1]. \]

The increasing property of \( I \) and \( J \) leads to

\[ x_n'(t) \leq I^{-1}\left(k_1(\theta_2(\|y_n\|))\right) \int_0^1 p(s)ds + I(\varepsilon), \quad t \in [0, 1], \]  \hspace{1cm} (3.16)

\[ y_n'(t) \leq J^{-1}\left(k_2(M_1)\right) \int_0^1 q(s)ds + J(\varepsilon), \quad t \in [0, 1]. \]  \hspace{1cm} (3.17)

Integrating (3.16) from 0 to \( t_2 \) and (3.17) from 0 to 1, using the BCs (3.11), we obtain

\[ M_1 \leq x_n(t_2) \leq I^{-1}(k_1(\theta_2(\|y_n\|))) \int_0^1 p(s)ds + I(\varepsilon), \]  \hspace{1cm} (3.18)

\[ ||y_n|| \leq J^{-1}(k_2(M_1)) \int_0^1 q(s)ds + J(\varepsilon). \]  \hspace{1cm} (3.19)

Either we have \( ||y_n|| < M_2 \) or \( ||y_n|| \geq M_2 \). If \( ||y_n|| < M_2 \), then from (3.18), we have

\[ M_1 \leq I^{-1}(k_1(\|y_n\|)) \int_0^1 p(s)ds + I(\varepsilon). \]  \hspace{1cm} (3.20)

Now, by using (3.19) in (3.20) and the increasing property of \( k_1 \) and \( I^{-1} \), we obtain

\[ M_1 \leq I^{-1}(k_1(J^{-1}(k_2(M_1))) \int_0^1 q(s)ds + J(\varepsilon)) \int_0^1 p(s)ds + I(\varepsilon), \]

which implies that

\[ \frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1))) \int_0^1 q(s)ds + J(\varepsilon)) \int_0^1 p(s)ds + I(\varepsilon)} \leq 1, \]

a contradiction to (3.7).

On the other hand, if \( ||y_n|| \geq M_2 \), then from (3.18) and (3.19), we have
We claim that

\[ M_1 \leq x_n(t_2) \leq I^{-1}(k_1(M_2) \int_0^1 p(s)ds + I(\varepsilon)), \quad (3.21) \]

\[ M_2 \leq J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon)). \quad (3.22) \]

Using (3.22) in (3.21) and the increasing property of \( k_1 \) and \( I^{-1} \), leads to

\[ M_1 \leq I^{-1}(k_1(J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon)), \]

which implies that

\[ \frac{M_1}{I^{-1}(k_1(J^{-1}(k_2(M_1) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon))} \leq 1, \]

a contradiction to (3.7). Hence, \( x_n(t) < M_1 \) for \( t \in [0, 1] \). Similarly, we can show that \( y_n(t) < M_2 \) for \( t \in [0, 1] \).

Thus, in view of (3.11)-(3.15), \((x_n, y_n)\) is a solution of the following coupled system of BVPs

\[ \begin{align*}
-x''(t) &= p(t)f(t, y(t), y'(t)), & t \in (0, 1), \\
y''(t) &= q(t)g(t, x(t), y'(t)), & t \in (0, 1), \\
x(0) &= y(0) = 0, & x'(1) = y'(1) = \frac{1}{n},
\end{align*} \quad (3.23) \]

satisfy

\[ \begin{align*}
\frac{t}{n} &\leq x_n(t) < M_1, \quad \frac{1}{n} \leq y_n'(t) < L_1, & t \in [0, 1], \\
\frac{t}{n} &\leq y_n(t) < M_2, \quad \frac{1}{n} \leq y_n'(t) < L_2, & t \in [0, 1].
\end{align*} \quad (3.24) \]

Now, in view of (B6), there exist continuous functions \( \varphi_{M_2L_1} \) and \( \psi_{M_2L_2} \) defined on \([0, 1]\) and positive on \((0, 1)\), and real constants \( 0 \leq \delta_1, \delta_2 < 1 \) such that

\[ \begin{align*}
f(t, y_n(t), y_n'(t)) &\geq \varphi_{M_2L_1}(t) (y_n(t))^\delta_1, & (t, y_n(t), y_n'(t)) \in [0, 1] \times [0, M_2] \times [0, L_1], \\
g(t, x_n(t), y_n'(t)) &\geq \psi_{M_2L_2}(t) (x_n(t))^\delta_2, & (t, x_n(t), y_n'(t)) \in [0, 1] \times [0, M_1] \times [0, L_2].
\end{align*} \quad (3.25) \]

We claim that

\[ x_n'(t) \geq \frac{\delta_1}{2} \int_t^1 s^\delta_1 p(s) \varphi_{M_2L_1}(s)ds, \quad (3.26) \]
3.1 Existence of $C^1$-positive solutions

\[ y_n'(t) \geq C_1^{\delta_2} \int_0^t s^{\delta_2} q(s) \psi_{M_1 L_2}(s) ds, \]  
\[ (3.27) \]

where

\[ C_1 = \left( \int_0^1 s^{\delta_2+1} q(s) \psi_{M_1 L_2}(s) ds \right)^{\frac{\delta_1}{1-\delta_1 \delta_2}} \left( \int_0^1 s^{\delta_1+1} p(s) \phi_{M_2 L_1}(s) ds \right)^{\frac{1}{1-\delta_1 \delta_2}}, \]

\[ C_2 = \left( \int_0^1 s^{\delta_1+1} p(s) \phi_{M_2 L_1}(s) ds \right)^{\frac{\delta_2}{1-\delta_1 \delta_2}} \left( \int_0^1 s^{\delta_2+1} q(s) \psi_{M_1 L_2}(s) ds \right)^{\frac{1}{1-\delta_1 \delta_2}}. \]

To prove (3.26), consider the following relation

\[ x_n(t) = t + \int_0^t s p(s) f(s, y_n(s), x_n'(s)) ds + \int_t^1 t p(s) f(s, y_n(s), x_n'(s)) ds, \]  
\[ (3.28) \]

which implies that

\[ x_n(1) \geq \int_0^1 s p(s) f(s, y_n(s), x_n'(s)) ds. \]

Using (3.25) and Lemma 1.1.18, we obtain

\[ x_n(1) \geq (y_n(1))^{\delta_1} \int_0^1 s^{\delta_1+1} p(s) \phi_{M_2 L_1}(s) ds. \]  
\[ (3.29) \]

Similarly, using (3.25) and Lemma 1.1.18, we obtain

\[ y_n(1) \geq (x_n(1))^{\delta_2} \int_0^1 s^{\delta_2+1} q(s) \psi_{M_1 L_2}(s) ds, \]

which in view of (3.29) implies that

\[ y_n(1) \geq (y_n(1))^{\delta_1 \delta_2} \left( \int_0^1 s^{\delta_1+1} p(s) \phi_{M_2 L_1}(s) ds \right) \int_0^1 s^{\delta_2+1} q(s) \psi_{M_1 L_2}(s) ds \]

Hence,

\[ y_n(1) \geq C_2. \]  
\[ (3.30) \]

Now, from (3.28), it follows that

\[ x_n'(t) \geq \int_0^1 p(s) f(s, y_n(s), x_n'(s)) ds. \]

Using (3.25), Lemma 1.1.18 and (3.30), we obtain (3.26). Similarly, we can prove (3.27).

Now, using (3.25), (B_3), (3.26), (3.30) and (3.27), we have
In view of (3.24), (3.31), (B1) and (B7), it follows that the sequences \( \{x_n^{(j)} , y_n^{(j)} \} \) \((j = 0, 1)\) are uniformly bounded and equicontinuous on \([0, 1]\). Hence, by Theorem 1.1.5 there exist subsequences \( \{x_{n_k}^{(j)} , y_{n_k}^{(j)} \} \) \((j = 0, 1)\) of \( \{x_n^{(j)} , y_n^{(j)} \} \) \((j = 0, 1)\) and \((x,y) \in C^1[0,1] \times C^1[0,1]\) such that \(x_{n_k}^{(j)} , y_{n_k}^{(j)}\) converges uniformly to \((x^{(j)} , y^{(j)})\) on \([0, 1]\) \((j = 0, 1)\). Also, \(x(0) = y(0) = x'(1) = y'(1) = 0\). Moreover, from (3.26) and (3.27), with \(n_k\) in place of \(n\) and taking \(\lim_{n_k \to +\infty}\), we have

\[
\begin{align*}
  x'(t) &\geq C_{\alpha}^0 \int_0^t s^{\alpha_1} p(s) \varphi_{M_1 L_1} (s) ds, \\
  y'(t) &\geq C_{\beta}^0 \int_0^t s^{\beta_1} q(s) \psi_{M_1 L_2} (s) ds,
\end{align*}
\]

which shows that \(x' > 0\) and \(y' > 0\) on \([0, 1]\), \(x > 0\) and \(y > 0\) on \((0, 1]\). Further, \((x_{n_k}, y_{n_k})\) satisfy

\[
\begin{align*}
  x'_{n_k}(t) &= x_{n_k}'(0) - \int_0^t p(s) f(s, y_{n_k}(s), x_{n_k}(s)) ds, \quad t \in [0, 1], \\
  y'_{n_k}(t) &= y_{n_k}'(0) - \int_0^t q(s) g(s, x_{n_k}(s), y_{n_k}(s)) ds, \quad t \in [0, 1].
\end{align*}
\]

Passing to the limit as \(n_k \to \infty\), we obtain

\[
\begin{align*}
  x'(t) &= x'(0) - \int_0^t p(s) f(s, y(s), x'(s)) ds, \quad t \in [0, 1], \\
  y'(t) &= y'(0) - \int_0^t q(s) g(s, x(s), y'(s)) ds, \quad t \in [0, 1],
\end{align*}
\]

which implies that

\[
\begin{align*}
  -x''(t) &= p(t) f(t, y(t), x'(t)), \quad t \in (0, 1), \\
  -y''(t) &= q(t) g(t, x(t), y'(t)), \quad t \in (0, 1).
\end{align*}
\]

Hence, \((x,y)\) is a \(C^1\)-positive solution of the system of SBVPs (3.2).

**Example 3.1.3** Consider the following coupled system of SBVPs
where $0 < \beta_1 < 1$ and $0 < \beta_2 < \frac{1}{2}$.

Choose $p(t) = t^{-\frac{1}{2}}(1 - t)^{-\frac{3}{2}}$, $q(t) = t^{-\frac{1}{2}}(1 - t)^{-\frac{3}{2}}$, $k_1(x) = x^\beta$, $k_2(x) = x^\alpha$, $u_1(x) = x^{-\beta_1}$, $u_2(x) = x^{-\beta_2}$ and $v_1(x) = v_2(x) = 0$.

Then, $I(z) = \beta_1 + 1$, $J(z) = \beta_2 + 1$, $I^{-1}(z) = (\beta_1 + 1)^{\frac{1}{\beta_1 + 1}}z^{\frac{1}{\beta_1 + 1}}$ and $J^{-1}(z) = (\beta_2 + 1)^{\frac{1}{\beta_2 + 1}}z^{\frac{1}{\beta_2 + 1}}$.

Also, $\int_0^1 p(t) dt = \int_0^1 q(t) dt = \frac{2\pi}{\sqrt{3}}$.

Moreover,

$$\sup_{c \in (0, \infty)} \frac{c}{\int_0^1 (k_1(J^{-1}(k_2(c) \int_0^1 q(s) ds)) \int_0^1 p(s) ds)} = \sup_{c \in (0, \infty)} \frac{c}{\left(\frac{2\pi}{\sqrt{3}}\right) \beta_1 + 1}^{\frac{1}{\beta_1 + 1}} \left(\beta_1 + 1\right)^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} \left(\beta_1 + 1\right)^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} c^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} c^{\frac{1}{\beta_2 + 1}} = \infty,$$

and

$$\sup_{c \in (0, \infty)} \frac{c}{\int_0^1 (k_2(I^{-1}(k_1(c) \int_0^1 p(s) ds)) \int_0^1 q(s) ds)} = \sup_{c \in (0, \infty)} \frac{c}{\left(\frac{2\pi}{\sqrt{3}}\right) \beta_1 + 1}^{\frac{1}{\beta_1 + 1}} \left(\beta_1 + 1\right)^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} \left(\beta_1 + 1\right)^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} c^{\frac{1}{\beta_1 + 1}} \left(\beta_2 + 1\right)^{\frac{1}{\beta_2 + 1}} c^{\frac{1}{\beta_2 + 1}} = \infty.$$
shows that \((B_7)\) also holds. Since, \((B_1) - (B_7)\) are satisfied. Therefore, by Theorem 3.2, the system of BVPs 3.2 has a \(C^1\)-positive solution.

### 3.2 Existence of at least two positive solutions

In this section, we establish sufficient conditions for the existence of at least two positive solutions of the system of SBVPs (3.2). By a positive solution \((x, y)\) of the system of BVPs (3.2), we mean that \((x, y) \in \mathcal{E} \times \mathcal{E}\) satisfies (3.2), \(x > 0\) and \(y > 0\) on \([0, 1]\), \(x' > 0\) and \(y' > 0\) on \([0, 1]\). Define a cone \(P\) of \(\mathcal{E}\) by

\[
P = \{ x \in \mathcal{E} : x(t) \geq t \| x \| \text{ for all } t \in [0, 1], x(1) \geq \| x \|_1 \}.
\]

For each \((x, y) \in \mathcal{E} \times \mathcal{E}\) we write \(\| (x, y) \|_4 = \| x \|_2 + \| y \|_2\). Clearly, \((\mathcal{E} \times \mathcal{E}, \| \cdot \|_4)\) is a Banach space and \(P \times P\) is a cone of \(\mathcal{E} \times \mathcal{E}\). We define a partial ordering in \(\mathcal{E}\), by \(x \leq y\) if and only if \(x(t) \leq y(t), t \in [0, 1]\). We define a partial ordering in \(\mathcal{E} \times \mathcal{E}\), by \((x_1, y_1) \preceq (x_2, y_2)\) if and only if \(x_1 \leq x_2\) and \(y_1 \leq y_2\). For any real constant \(r > 0\), we define an open neighborhood of \((0, 0) \in \mathcal{E} \times \mathcal{E}\) as

\[
\mathcal{O}_r = \{ (x, y) \in \mathcal{E} \times \mathcal{E} : \| (x, y) \|_4 < r \}.
\]

In view of \((B_4)\), there exist real constants \(R_1 > 0\) and \(R_2 > 0\) such that

\[
R_1 \frac{I^{-1}(k_1(J^{-1}(k_2(R_1) \int_0^1 q(s)ds) \int_0^1 p(s)ds))}{J^{-1}(k_2(k_1(R_1) \int_0^1 p(s)ds) \int_0^1 q(s)ds)} \geq 1, \quad (3.33)
\]

\[
R_2 \frac{J^{-1}(k_1(J^{-1}(k_2(R_2) \int_0^1 q(s)ds) \int_0^1 p(s)ds))}{J^{-1}(k_2(J^{-1}(k_1(R_2) \int_0^1 p(s)ds) \int_0^1 q(s)ds))} \geq 1, \quad (3.34)
\]

From the continuity of \(k_1, k_2, I\) and \(J\), we choose \(\varepsilon > 0\) small enough such that

\[
R_1 \frac{I^{-1}(k_1(J^{-1}(k_2(R_1 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)) + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon))}{J^{-1}(k_2(J^{-1}(k_1(R_1 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)) + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon))} \geq 1, \quad (3.35)
\]

\[
R_2 \frac{J^{-1}(k_1(J^{-1}(k_2(R_2 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)) + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon))}{J^{-1}(k_2(J^{-1}(k_1(R_2 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)) + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon))} \geq 1. \quad (3.36)
\]

Choose \(n_0 \in \{1, 2, \cdots\}\) such that \(\frac{1}{n_0} < \varepsilon\) and for each fixed \(n \in \{n_0, n_0 + 1, \cdots\}\), consider the system of non-singular BVPs...
3.2 Existence of at least two positive solutions

\[ -x''(t) = p(t)f(t,y(t) + \frac{t}{n}, |x'(t)| + \frac{1}{n}), \quad t \in (0,1), \]
\[ -y''(t) = q(t)g(t,x(t) + \frac{t}{n}, |y'(t)| + \frac{1}{n}), \quad t \in (0,1), \]
\[ x(0) = x'(1) = y(0) = y'(1) = 0. \] \hspace{1cm} (3.37)

We write (3.37) as an equivalent system of integral equations

\[ x(t) = \int_0^1 G(t,s)p(s)f(s,y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds, \quad t \in [0,1], \]
\[ y(t) = \int_0^1 G(t,s)q(s)f(s,x(s) + \frac{s}{n}, |y'(s)| + \frac{1}{n})ds, \quad t \in [0,1], \] \hspace{1cm} (3.38)

where the Green’s function is defined as

\[ G(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1. \end{cases} \]

By a solution of the system of BVPs (3.37), we mean a solution of the corresponding system of integral equations (3.38).

Define a map \( T_n : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E} \) by

\[ T_n(x,y) = (A_n(x,y), B_n(x,y)), \] \hspace{1cm} (3.39)

where the maps \( A_n, B_n : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \) are defined by

\[ A_n(x,y)(t) = \int_0^1 G(t,s)p(s)f(s,y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n})ds, \quad t \in [0,1], \]
\[ B_n(x,y)(t) = \int_0^1 G(t,s)q(s)f(s,x(s) + \frac{s}{n}, |y'(s)| + \frac{1}{n})ds, \quad t \in [0,1]. \] \hspace{1cm} (3.40)

Clearly, if \((x_n, y_n) \in \mathcal{E} \times \mathcal{E}\) is a fixed point of \( T_n \); then \((x_n, y_n)\) is a solution of the system of BVPs (3.37).

Assume that

\[ \text{(B)} \quad \text{for any real constant } E > 0, \text{ there exist continuous functions } \varphi_E \text{ and } \psi_E \]

\[ \text{defined on } [0,1] \text{ and positive on } (0,1), \text{ and constants } 0 \leq \delta_1, \delta_2 < 1 \text{ such that} \]
\[ f(t,x,y) \geq \varphi_E(t)x^{\delta_1}, \quad g(t,x,y) \geq \psi_E(t)y^{\delta_2} \]
\[ \text{on } [0,1] \times [0,E] \times [0,\infty). \]
Firstly, we show that \( A_n(x,y)(t) = \int_0^1 G(t,s)p(s)f(s,y(s) + \frac{s}{n}|x'(s)| + \frac{1}{n})ds \) \( \geq t \max_{\tau \in [0,1]} \int_\tau^1 G(\tau,s)p(s)f(s,y(s) + \frac{s}{n}|x'(s)| + \frac{1}{n})ds = t ||A_n(x,y)|| \) (3.41)

and

\[
||A_n(x,y)||_1 = \sup_{\tau \in [0,1]} \tau |A_n(x,y)'(\tau)| = \sup_{\tau \in [0,1]} \tau \int_\tau^1 p(s)f(s,y(s) + \frac{s}{n}|x'(s)| + \frac{1}{n})ds 
\leq \max_{\tau \in [0,1]} \int_0^1 G(t,s)p(s)f(s,y(s) + \frac{s}{n}|x'(s)| + \frac{1}{n})ds \leq A_n(x,y)(1).
\]

(3.42)

From (3.41) and (3.42), \( A_n(x,y) \in P \) for every \((x,y) \in P \times P\), that is, \( A_n(P \times P) \subseteq P \).

Similarly, by using (3.40) and Lemma 1.1.15, we can show that \( B_n(P \times P) \subseteq P \).

Hence, \( T_n(P \times P) \subseteq P \times P \).

Now, we show that \( T_n : \overline{O}_r \cap (P \times P) \to P \times P \) is uniformly bounded. For any \((x,y) \in \overline{O}_r \cap (P \times P)\), using (3.40), (B3), Lemma 1.1.15 (B1) and (B0), we have

\[
||A_n(x,y)|| = \max_{\tau \in [0,1]} \left| \int_0^1 G(t,s)p(s)f(s,y(s) + \frac{s}{n}|x'(s)| + \frac{1}{n})ds \right| 
\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n})ds 
\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x|^2 s + \frac{1}{n})ds 
\leq \int_0^1 p(s)k_1(y(s) + \frac{s}{n})(u_1 \left( \frac{1}{n} \right) + v_1 \left( \frac{r}{s} \right) + \frac{1}{n})ds 
\leq k_1(r + \frac{1}{n}) \int_0^1 p(s)(u_1 \left( \frac{1}{n} \right) + v_1((r + \frac{1}{n}) \frac{1}{s}))ds < +\infty.
\]

Also, for \((x,y) \in \overline{O}_r \cap (P \times P)\), using (3.40), Lemma 1.1.15 and (B1), we have
\[ \|A_n(x,y)\|_1 = \sup_{\tau \in (0,1]} \tau |A_n(x,y)'(\tau)| = \sup_{\tau \in (0,1]} \tau \int_{\tau}^{1} p(s)f(s,y(s) + \frac{s}{n},|x'(s)| + \frac{1}{n})ds \]
\[ \leq \max_{t \in [0,1]} \int_{t}^{1} G(t,s)p(s)f(s,y(s) + \frac{s}{n},|x'(s)| + \frac{1}{n})ds \leq \int_{0}^{1} p(s)f(s,y(s) + \frac{s}{n},|x'(s)| + \frac{1}{n})ds \]
\[ \leq \int_{0}^{1} p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds. \]

Now, using Lemma 1.1.13 (B₁) and (B₀), we obtain
\[ \|A_n(x,y)\|_1 \leq \int_{0}^{1} p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \]
\[ \leq k_1(r + \frac{1}{n}) \int_{0}^{1} p(s)(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds < +\infty. \]

From (3.43) and (3.44), it follows that \( A_n(\overline{\Omega}, \cap (P \times P)) \) is uniformly bounded under the norm \( \| \cdot \|_2 \). Similarly, by using (3.40), Lemma 1.1.13, Lemma 1.1.15 (B₁), (B₃) and (B₄), we can show that \( B_n(\overline{\Omega}, \cap (P \times P)) \) is uniformly bounded under the norm \( \| \cdot \|_2 \). Hence, \( T_n(\overline{\Omega}, \cap (P \times P)) \) is uniformly bounded.

Now, we show that \( T_n(\overline{\Omega}, \cap (P \times P)) \) is equicontinuous. For \((x,y) \in \overline{\Omega}, \cap (P \times P), t_1, t_2 \in [0,1]\), using (3.40), (B₃) and Lemma 1.1.13, we have
\[ |A_n(x,y)(t_1) - A_n(x,y)(t_2)| = \left| \int_{0}^{1} (G(t_1,s) - G(t_2,s))p(s)f(s,y(s) + \frac{s}{n},|x'(s)| + \frac{1}{n})ds \right| \]
\[ \leq \int_{0}^{1} |G(t_1,s) - G(t_2,s)|p(s)f(s,y(s) + \frac{s}{n},|x'(s)| + \frac{1}{n})ds \]
\[ \leq \int_{0}^{1} |G(t_1,s) - G(t_2,s)|p(s)k_1(y(s) + \frac{s}{n})(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \]
\[ \leq k_1(r + \frac{1}{n}) \int_{0}^{1} |G(t_1,s) - G(t_2,s)|p(s)(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \]
\[ \leq k_1(r + \frac{1}{n}) \int_{0}^{1} |G(t_1,s) - G(t_2,s)|p(s)(u_1(|x'(s)| + \frac{1}{n}) + v_1(r + \frac{1}{n})\frac{1}{s})ds, \]

(3.45)
From (3.45), (3.46), (B1) and (B9), it follows that \( A_n(\overline{O}_r \cap (P \times P)) \) is equicontinuous under the norm \( \| \cdot \|_3 \). But, the norm \( \| \cdot \|_3 \) is equivalent to the norm \( \| \cdot \|_2 \). Hence, \( A_n(\overline{O}_r \cap (P \times P)) \) is equicontinuous under \( \| \cdot \|_2 \).

Similarly, using (3.40), (B3) and Lemma 1.1.13, we can show that \( B_n(\overline{O}_r \cap (P \times P)) \) is equicontinuous under the norm \( \| \cdot \|_2 \). Consequently, \( T_n(\overline{O}_r \cap (P \times P)) \) is equicontinuous. Hence, by Theorem 1.1.5, \( T_n(\overline{O}_r \cap (P \times P)) \) is relatively compact which implies that \( T_n \) is a compact map.

Now, we show that \( T_n \) is continuous. Let \((x_m, y_m), (x, y) \in \overline{O}_r \cap (P \times P)\) such that \( \| (x_m, y_m) - (x, y) \|_4 \to 0 \) as \( m \to +\infty \). Using (B3) and Lemma 1.1.13 we have

\[
\left| f(t, y_m(t) + \frac{r}{n}, x_m(t)) + \frac{1}{n} \right| \leq k_1(y_m(t) + \frac{r}{n})(u_1(|x_m'(t)| + \frac{1}{n}) + v_1(|x_m'(t)| + \frac{1}{n}))
\]

\[
\leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1(|x_m'| + \frac{1}{n}))) \leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1(\frac{r}{n} + \frac{1}{n})))
\]

\[
\leq k_1(r + \frac{1}{n})(u_1(\frac{1}{n}) + v_1((r + \frac{1}{n}) \frac{1}{n}))).
\]

Using (3.40) and Lemma 1.1.15 we have

\[
\| A_n(x_m, y_m) - A_n(x, y) \| =
\]

\[
\max_{t \in [0,1]} \left| \int_0^t G(t, s) p(s) f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right| ds
\]

\[
\leq \int_0^1 p(s) \left| f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right| ds
\]

(3.47)
3.2 Existence of at least two positive solutions

\[ |A_n(x_m, y_m)' - A_n(x, y)'|_1 = \]
\[ \sup_{\tau \in (0,1]} \int_{\tau}^{1} p(s) \left( f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right) ds \leq \max_{t \in [0,1]} \int_{0}^{t} G(t,s) p(s) \left( f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right) ds \leq \int_{0}^{1} p(s) \left( f(s, y_m(s) + \frac{s}{n}, |x_m'(s)| + \frac{1}{n}) - f(s, y(s) + \frac{s}{n}, |x'(s)| + \frac{1}{n}) \right) ds. \]

(3.48)

From (3.47) and (3.48), using the Lebesgue dominated convergence theorem, it follows that

\[ |A_n(x_m, y_m) - A_n(x, y)|_2 \to 0 \quad \text{as} \quad m \to +\infty, \]
\[ |A_n(x_m, y_m)' - A_n(x, y)'|_1 \to 0 \quad \text{as} \quad m \to +\infty. \]

Hence, \( |A_n(x_m, y_m) - A_n(x, y)|_2 \to 0 \) as \( m \to \infty \).

Similarly, we can show that \( |B_n(x_m, y_m) - B_n(x, y)|_2 \to 0 \) as \( m \to \infty \). Consequently, \( |T_n(x_m, y_m) - T_n(x, y)|_4 \to 0 \) as \( m \to +\infty \), that is, \( T_n : \partial \mathring{\Omega} \cap (P \times P) \to P \times P \) is continuous. Hence, \( T_n : \partial \mathring{\Omega} \cap (P \times P) \to P \times P \) is completely continuous.

Assume that

(B10) there exist \( h_1, h_2 \in C([0, \infty) \times (0, \infty) ; [0, \infty)) \) with \( f(t, x, y) \geq h_1(x, y) \) and \( g(t, x, y) \geq h_2(x, y) \) on \([0, 1] \times [0, \infty) \times (0, \infty) \) such that

\[ \lim_{x \to +\infty} \frac{h_i(x, y)}{x} = +\infty, \quad \text{uniformly for} \quad y \in (0, \infty), \quad i = 1, 2. \]

**Theorem 3.2.2** Assume that (B1) – (B3) and (B8) – (B10) hold. Then the system of BVPs (3.2) has at least two positive solutions.

**Proof.** Let \( R_0 = R_1 + R_2 \) and define \( \partial \mathring{\Omega} \cap (P \times P) \) where

\[ \Omega_{R_1} = \{ x \in E : ||x||_2 < R_1 \}, \quad \Omega_{R_2} = \{ x \in E : ||x||_2 < R_2 \}. \]

We claim that

\[ (x, y) \neq \lambda T_n(x, y), \quad \text{for} \quad \lambda \in (0, 1), \quad (x, y) \in \partial \mathring{\Omega} \cap (P \times P). \quad (3.49) \]

Suppose there exist \( (x_0, y_0) \in \partial \mathring{\Omega} \cap (P \times P) \) and \( \lambda_0 \in (0, 1) \) such that \( (x_0, y_0) = \lambda_0 T_n(x_0, y_0) \). Then,
\[-x''_0(t) = \lambda_0 p(t) f(t, y_0(t) + \frac{t}{n}, [x'_0(t)] + \frac{1}{n}), \quad t \in (0, 1), \]
\[-y''_0(t) = \lambda_0 q(t) g(t, x'_0(t) + \frac{t}{n}, [y'_0(t)] + \frac{1}{n}), \quad t \in (0, 1), \tag{3.50} \]
\[x_0(0) = x'_0(1) = y_0(0) = y'_0(1) = 0. \]

From (3.50) and (B2), we have \(x'''_0 \leq 0\) and \(y'''_0 \leq 0\) on \((0, 1)\), integrating from \(t\) to 1, using the BCs (3.50), we obtain \(x'_0(t) \geq 0\) and \(y'_0(t) \geq 0\) for \(t \in [0, 1]\). From (3.50) and (B3), we have

\[-x''_0(t) \leq p(t) k_1(y_0(t) + \frac{t}{n}) \left(u_1(x'_0(t) + \frac{t}{n}) + v_1(x'_0(t) + \frac{1}{n})\right), \quad t \in (0, 1), \]
\[-y''_0(t) \leq q(t) k_2(x_0(t) + \frac{t}{n}) \left(u_2(y'_0(t) + \frac{t}{n}) + v_2(y'_0(t) + \frac{1}{n})\right), \quad t \in (0, 1), \]

which implies that

\[-x''_0(t) \leq p(t) k_1(y_0(t) + \frac{t}{n}) \leq k_1(R_2 + \varepsilon) p(t), \quad t \in (0, 1), \]
\[-y''_0(t) \leq q(t) k_2(x_0(t) + \frac{t}{n}) \leq k_2(R_1 + \varepsilon) q(t), \quad t \in (0, 1). \]

Integrating from \(t\) to 1, using the BCs (3.50), we obtain

\[I(x'_0(t) + \frac{1}{n}) - I(\frac{1}{n}) \leq k_1(R_2 + \varepsilon) \int_t^1 p(s) ds, \quad t \in [0, 1], \]
\[J(y'_0(t) + \frac{1}{n}) - J(\frac{1}{n}) \leq k_2(R_1 + \varepsilon) \int_t^1 q(s) ds, \quad t \in [0, 1], \]

which implies that

\[x'_0(t) \leq I^{-1}(k_1(R_2 + \varepsilon) \int_0^t p(s) ds + I(\varepsilon)), \quad t \in [0, 1], \]
\[y'_0(t) \leq J^{-1}(k_2(R_1 + \varepsilon) \int_0^t q(s) ds + J(\varepsilon)), \quad t \in [0, 1], \]

which on integration from 0 to 1, using the BCs (3.50) and Lemma [1,1,1,1] leads to

\[R_1 \leq I^{-1}(k_1(R_2 + \varepsilon) \int_0^1 p(s) ds + I(\varepsilon)), \tag{3.51} \]
\[R_2 \leq J^{-1}(k_2(R_1 + \varepsilon) \int_0^1 q(s) ds + J(\varepsilon)). \tag{3.52} \]

Now, using (3.52) in (3.51) together with increasing property of \(k_1\) and \(I^{-1}\), we have
Now, choose a $t_0 \in (0, 1)$ and define

\[ N_1 = \left( t_0 \min_{t \in [0, 1]} \int_0^1 G(t, s)p(s)ds \right)^{-1} + 1 \quad \text{and} \quad N_2 = \left( t_0 \min_{t \in [0, 1]} \int_0^1 G(t, s)q(s)ds \right)^{-1} + 1. \] (3.54)

By (B10), there exist real constants with $R_1^+ > R_1$ and $R_2^+ > R_2$ such that

\[ h_1(x, y) \geq N_1x, \quad \text{for } x \geq R_1^+, y \in (0, \infty), \]
\[ h_2(x, y) \geq N_2x, \quad \text{for } x \geq R_2^+, y \in (0, \infty). \] (3.55)

Let $R^* = \frac{R_1^* + R_2^*}{t_0}$ and define $\partial \mathcal{R}^* = \Omega_{R_1^*} \times \Omega_{R_2^*}$, where

\[ \Omega_{R_1^*} = \{ x \in E : \|x\| < \frac{R_1^*}{t_0} \}, \quad \Omega_{R_2^*} = \{ x \in E : \|x\| < \frac{R_2^*}{t_0} \}. \]

We show that

\[ T_n(x, y) \not\in (x, y), \quad \text{for } (x, y) \in \partial \mathcal{R}^* \cap (P \times P). \] (3.56)

Suppose $T_n(x_0, y_0) \leq (x_0, y_0)$ for some $(x_0, y_0) \in \partial \mathcal{R}^* \cap (P \times P)$. Then,

\[ x_0(t) \geq A_n(x_0, y_0)(t) \quad \text{and} \quad y_0(t) \geq B_n(x_0, y_0)(t) \quad \text{for } t \in [0, 1]. \] (3.57)

By Lemma[1.1.14] we have

\[ x_0(t) \geq t \|x_0\| \geq t_0 \|x_0\|_2 = \frac{R_1^*}{t_0} = R_1^* \quad \text{for } t \in [t_0, 1]. \]

Similarly, $y_0(t) \geq R_2^*$ for $t \in [t_0, 1]$. Hence,

\[ |x_0(t)| + \frac{t}{n} \geq R_1^* \quad \text{and} \quad |y_0(t)| + \frac{t}{n} \geq R_2^* \quad \text{for } t \in [t_0, 1]. \]
Now, using (3.57), (3.55) and (B10), we have

\[ x_0(t) \geq A_n(x_0, y_0)(t) \]
\[ = \int_0^1 G(t, s)p(s)f(s, y_0(s)) + \frac{s}{n} |x'_0(s)| + \frac{1}{n} ds \]
\[ \geq \int_0^1 G(t, s)p(s)h_1(y_0(s)) + \frac{s}{n} |x'_0(s)| + \frac{1}{n} ds \]
\[ \geq \int_0^1 G(t, s)p(s)N_1(y_0(s)) + \frac{s}{n} ds \]
\[ \geq \int_0^1 G(t, s)p(s)dsN_1R_2^* \]
\[ \geq \min_{t \in [0, 1]} \int_0^1 G(t, s)p(s)dsN_1R_2^* \]
\[ > \frac{R_2^*}{t_0}, \]

which implies that \( \| x_0 \|_2 = \| x_0 \| > \frac{R_2^*}{t_0} \). Similarly, using (3.55), (3.57) and (B10), we have \( \| y_0 \|_2 > \frac{R_2^*}{t_0} \). Consequently, it follows that, \( \| (x_0, y_0) \|_4 = \| x_0 \|_2 + \| y_0 \|_2 > R^* \), a contradiction. Hence, (3.56) is true and by Lemma 1.1.10, the fixed point index

\[ \text{ind}_{pp}(T_n, \mathcal{O}_R \cap (P \times P), P \times P) = 0. \]  
(3.58)

From (3.53) and (3.58), it follows that

\[ \text{ind}_{pp}(T_n, (\mathcal{O}_R \setminus \mathcal{O}_R) \cap (P \times P), P \times P) = -1. \]  
(3.59)

Thus, in view of (3.53) and (3.59), there exist \( (x_n, y_n) \in \mathcal{O}_R \cap (P \times P) \) and \( (x_n, y_n, 2) \in (\mathcal{O}_R \setminus \mathcal{O}_R) \cap (P \times P) \) such that \( (x_n, y_n, j) = T_n(x_n, y_n, j), (j = 1, 2) \) which implies that

\[ x_{n,j}(t) = \int_0^1 G(t, s)p(s)f(s, x_n, j(s)) + \frac{s}{n} |y'_n(j)(s)| + \frac{1}{n} ds, \quad t \in [0, 1], \]
\[ y_{n,j}(t) = \int_0^1 G(t, s)q(s)g(s, x_n, j(s)) + \frac{s}{n} |y'_n(j)(s)| + \frac{1}{n} ds, \quad t \in [0, 1], j = 1, 2. \]  
(3.60)

Using (B8) there exist continuous functions \( \varphi_{R_2^*+\varepsilon} \) and \( \psi_{R_1^*+\varepsilon} \) defined on \([0, 1]\) and positive on \((0, 1)\) and real constants \( 0 \leq \delta_1, \delta_2 < 1 \) such that
3.2 Existence of at least two positive solutions

\[ f(t,x,y) \geq \varphi_{R_2 + \varepsilon}(t)x^{\delta_1}, \quad (t,x,y) \in [0,1] \times [0,R_2 + \varepsilon] \times [0,\infty), \]

\[ g(t,x,y) \geq \psi_{R_1 + \varepsilon}(t)x^{\delta_2}, \quad (t,x,y) \in [0,1] \times [0,R_1 + \varepsilon] \times [0,\infty). \]

By the definition of \( P \), we have \( x_{n,1}(t) \geq t \|x_{n,1}\| \) and \( y_{n,1}(t) \geq t \|y_{n,1}\| \) for \( t \in [0,1] \).

We show that

\[ x_{n,1}'(t) \geq C_4 \int_0^1 s^{\delta_1 - 1} p(s) \varphi_{R_2 + \varepsilon}(s) ds, \quad t \in [0,1], \]

\[ y_{n,1}'(t) \geq C_3 \int_0^1 s^{\delta_2 - 1} q(s) \psi_{R_1 + \varepsilon}(s) ds, \quad t \in [0,1], \]

where

\[ C_3 = \left( \int_0^1 s^{\delta_2 + 1} p(s) \varphi_{R_2 + \varepsilon}(s) ds \right)^{\frac{\delta_1}{1 - \delta_1 \delta_2}} \left( \int_0^1 s^{\delta_1 + 1} p(s) \varphi_{R_2 + \varepsilon}(s) ds \right)^{\frac{1}{1 - \delta_1 \delta_2}}, \]

\[ C_4 = \left( \int_0^1 s^{\delta_1 + 1} p(s) \psi_{R_1 + \varepsilon}(s) ds \right)^{\frac{\delta_2}{1 - \delta_1 \delta_2}} \left( \int_0^1 s^{\delta_2 + 1} q(s) \psi_{R_1 + \varepsilon}(s) ds \right)^{\frac{1}{1 - \delta_1 \delta_2}}. \]

In order to prove \( 3.62 \), using \( 3.60 \) and \( 3.61 \), we consider

\[ x_{n,1}(t) = \int_0^1 G(t,s)p(s)f(s,y_{n,1}(s) + \frac{s}{n} |x'_{n,1}(s)| + \frac{1}{n}) ds \]

\[ \geq \int_0^1 G(t,s)p(s)\varphi_{R_2 + \varepsilon}(s)(y_{n,1}(s) + \frac{s}{n}) \delta_1 ds \]

\[ \geq \|y_{n,1}\| \delta_1 \int_0^1 G(t,s)s^{\delta_1} p(s)\varphi_{R_2 + \varepsilon}(s) ds, \]

which shows that

\[ \|x_{n,1}\| \geq \|y_{n,1}\| \delta_1 \int_0^1 s^{\delta_1 + 1} p(s)\varphi_{R_2 + \varepsilon}(s) ds. \]

(3.64)

Similarly, from \( 3.60 \) and \( 3.61 \), we have

\[ \|y_{n,1}\| \geq \|x_{n,1}\| \delta_2 \int_0^1 s^{\delta_2 + 1} q(s)\psi_{R_1 + \varepsilon}(s) ds. \]

(3.65)

Using \( 3.65 \) in \( 3.64 \), we have

\[ \|y_{n,1}\| \geq \left( \|y_{n,1}\| \delta_1 \int_0^1 s^{\delta_1 + 1} p(s)\varphi_{R_2 + \varepsilon}(s) ds \right)^{\frac{\delta_2}{\delta_2 + 1}} \int_0^1 s^{\delta_2 + 1} q(s)\psi_{R_1 + \varepsilon}(s) ds, \]

which implies that
\[ ||y_{n,1}|| \geq \left( \int_0^1 s^{\delta_1+1} p(s) \phi_{R_2+\varepsilon}(s) ds \right)^{1/\delta_1} \left( \int_0^1 s^{\delta_2+1} q(s) \psi_{R_1+\varepsilon}(s) ds \right)^{1/\delta_2} = C_4. \]

Using (3.61) and (3.66) in the following relation
\[ x'_{n,1}(t) = \int_t^1 p(s)f(s,y_{n,1}(s)) + s^{-1} |x'_{n,1}(s)| + 1/n) ds, \]
we obtain (3.62). Similarly, we can prove (3.62).

Now, differentiating (3.60), using (B_3), (3.62), (3.63) and Lemma 1.1.13 we have
\[ 0 \leq -x''_{n,1}(t) \leq p(t)k_1(R_2+\varepsilon)(u_1(C_4^{\delta_1} s^{\delta_1} p(s) \phi_{R_2+\varepsilon}(s) ds + v_1(R_1 + 1/s)) \), \quad t \in [0,1], \]
\[ 0 \leq -y''_{n,1}(t) \leq q(t)k_2(R_1+\varepsilon)(u_2(C_3^{\delta_2} s^{\delta_2} q(s) \psi_{R_1+\varepsilon}(s) ds + v_2(R_2 + 1/s)), \quad t \in [0,1]. \]
Integration from t to 1, using the BCs (3.37), leads to
\[ x'_{n,1}(t) \leq k_1(R_2+\varepsilon) \int_t^1 p(s)(u_1(C_4^{\delta_1} s^{\delta_1} p(s) \phi_{R_2+\varepsilon}(s) ds + v_1(R_1 + 1/s)) ds, \quad t \in [0,1], \]
\[ y'_{n,1}(t) \leq k_2(R_1+\varepsilon) \int_t^1 q(s)(u_2(C_3^{\delta_2} s^{\delta_2} q(s) \psi_{R_1+\varepsilon}(s) ds + v_2(R_2 + 1/s)) ds, \quad t \in [0,1], \]
which implies that
\[ x'_{n,1}(t) \leq k_1(R_2+\varepsilon) \int_0^t p(s)(u_1(C_4^{\delta_1} s^{\delta_1} p(s) \phi_{R_2+\varepsilon}(s) ds + v_1(R_1 + 1/s)) ds, \quad t \in [0,1], \]
\[ y'_{n,1}(t) \leq k_2(R_1+\varepsilon) \int_0^t q(s)(u_2(C_3^{\delta_2} s^{\delta_2} q(s) \psi_{R_1+\varepsilon}(s) ds + v_2(R_2 + 1/s)) ds, \quad t \in [0,1]. \]

In view of (3.62), (3.63), (3.66), (3.67), (B_1) and (B_6), the sequences \{\{x_{n,1},y_{n,1}\}\} (j = 0, 1) are uniformly bounded and equicontinuous on [0,1]. Thus, by Theorem 1.1.5 there exist subsequences \{\{x_{n_{1,k}},y_{n_{1,k}}\}\} (j = 0, 1) of \{\{x_{n,k},y_{n,k}\}\} and functions \( (x_{0,1},y_{0,1}) \in \mathcal{C} \times \mathcal{C} \) such that \( x_{n_{1,k}},y_{n_{1,k}} \) converges uniformly to \( (x_{0,1},y_{0,1}) \) on [0,1]. Also, \( x_{0,1}(0) = y_{0,1}(0) = x_{0,1}'(1) = y_{0,1}'(1) = 0 \). Moreover, from (3.62) and (3.63), with \( n_k \) in place of \( n \) and taking \( \lim_{n_k \to +\infty} \), we have
3.2 Existence of at least two positive solutions

\[ x_0^1(t) \geq C_4^{\delta_1} \int_0^1 s^{\delta_1} p(s) \Phi_{R_2^+}(s) ds, \]
\[ y_0^1(t) \geq C_4^{\delta_2} \int_0^1 s^{\delta_2} q(s) \Psi_{R_1^+}(s) ds, \]

which implies that \( x_0^1 > 0 \) and \( y_0^1 > 0 \) on \([0, 1]\), \( x_0^1 > 0 \) and \( y_0^1 > 0 \) on \((0, 1]\).

Further,

\[
\left| f(t, y_{n,k}^1(t) + \frac{t}{n_k} y_{n,k}^1(t) + \frac{1}{n_k}) \right| \leq k_1(R_2 + \varepsilon)(u_1(C_4^{\delta_1}) \int_0^1 s^{\delta_1} p(s) \Phi_{R_2^+}(s) ds + v_1(R_1 + \frac{1}{t})),
\]
\[
\left| g(t, x_{n,k}^1(t) + \frac{t}{n_k} y_{n,k}^1(t) + \frac{1}{n_k}) \right| \leq k_2(R_1 + \varepsilon)(u_2(C_4^{\delta_2}) \int_0^1 s^{\delta_2} q(s) \Psi_{R_1^+}(s) ds + v_2(R_2 + \frac{1}{t})),
\]

(3.69)

\[
\lim_{n_k \to \infty} f(t, y_{n,k}^1(t) + \frac{t}{n_k} y_{n,k}^1(t) + \frac{1}{n_k}) = f(t, y_0^1(t), x_0^1(t)), \quad t \in [0, 1],
\]
\[
\lim_{n_k \to \infty} g(t, x_{n,k}^1(t) + \frac{t}{n_k} y_{n,k}^1(t) + \frac{1}{n_k}) = g(t, x_0^1(t), y_0^1(t)), \quad t \in (0, 1].
\]

(3.70)

Moreover, \((x_{n,k}^1, y_{n,k}^1)\) satisfies

\[
x_{n,k}^1(t) = \int_0^1 G(t, s) p(s) f(s, y_{n,k}^1(s) + \frac{s}{n_k} y_{n,k}^1(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],
\]
\[
y_{n,k}^1(t) = \int_0^1 G(t, s) q(s) g(s, x_{n,k}^1(s) + \frac{s}{n_k} y_{n,k}^1(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],
\]

which in view of (3.69), (B_9), (3.70), the Lebesgue dominated convergence theorem and taking \( \lim_{n_k \to +\infty} \), leads to

\[
x_0^1(t) = \int_0^1 G(t, s) p(s) f(s, y_0^1(s), x_0^1(s)) ds, \quad t \in [0, 1],
\]
\[
y_0^1(t) = \int_0^1 G(t, s) q(s) g(s, x_0^1(s), y_0^1(s)) ds, \quad t \in [0, 1],
\]

which implies that \((x_0^1, y_0^1) \in C^2(0, 1) \times C^2(0, 1)\) and

\[
-x_0''(t) = p(t) f(t, y_0^1(t), x_0^1(t)), \quad t \in (0, 1),
\]
\[
-y_0''(t) = q(t) g(t, x_0^1(t), y_0^1(t)), \quad t \in (0, 1).
\]

Moreover, by (3.32) and (3.34), we have \(\|x_0^1\|_2 < R_1\) and \(\|y_0^1\|_2 < R_2\), that is, \(\|x_0^1, y_0^1\|_3 < R_0\). By a similar proof the sequence \(\{(x_{n,2}, y_{n,2})\}\) has a convergent subsequence \(\{(x_{n,2}, y_{n,2})\}\) converging uniformly to \((x_0^2, y_0^2) \in E \times E\) on \([0, 1]\).
Moreover, \((x_{0.2}, y_{0.2})\) is a solution to the system \((3.2)\) with \(x_{0.2} > 0\) and \(y_{0.2} > 0\) on \((0, 1], x'_{0.2} > 0\) and \(y'_{0.2} > 0\) on \([0, 1), R_0 < \|x_{0.2}, y_{0.2}\|_4 < R^*\).

**Example 3.2.3** Consider the following coupled system of SBVPs

\[
\begin{aligned}
-x''(t) &= \mu_1 (1 + (y(t))^{\delta_1} + (y(t))^{\eta_1})(1 + (x'(t))^{\alpha_1} + (x'(t))^{-\beta_1}), \quad t \in (0, 1), \\
y''(t) &= \mu_2 (1 + (x(t))^{\delta_2} + (x(t))^{\eta_2})(1 + (y'(t))^{\alpha_2} + (y'(t))^{-\beta_2}), \quad t \in (0, 1), \\
x(0) &= y(0) = x'(1) = y'(1) = 0,
\end{aligned}
\tag{3.71}
\]

where \(0 \leq \delta_i < 1, \eta_i > 1, 0 < \alpha_i < 1, 0 < \beta_i < 1,\) and \(\mu_i > 0, i = 1, 2.\)

Choose \(p(t) = \mu_1, q(t) = \mu_2, k_i(x) = 1 + x^{\delta_i} + x^{\eta_i}, u_i(x) = x^{-\beta_i}\) and \(v_i(x) = 1 + x^{\alpha_i}, i = 1, 2.\) Also, \(\varphi_E(t) = \mu_1, \psi_E(t) = \mu_2\) and \(h_i(x, y) = \mu_i(1 + x^\eta_i), i = 1, 2.\)

Assume that \(\mu_1\) is arbitrary real constant and

\[
\mu_2 < \min\left\{ \inf_{c \in (0, \infty)} \frac{J(c)}{k_2 \left(I^{-1}(\mu_1 k_1(c))\right)} \cdot \frac{J((\mu_1^{-1}I(c))^{\delta_1^{-1}})}{k_2(c)}, \inf_{c \in (0, \infty)} \frac{J((\mu_1^{-1}I(c))^{\eta_1^{-1}})}{k_2(c)} \right\}.
\]

Then,

\[
\begin{aligned}
&\sup_{c \in (0, \infty)} \frac{c}{I^{-1}(k_1(J^{-1}(k_2(c) \int_0^1 q(s)ds)) \int_0^1 p(s)ds)} \\
&= \sup_{c \in (0, \infty)} \frac{c}{I^{-1}(\mu_1 k_1(J^{-1}(\mu_2 k_2(c)))))} \\
&\geq \frac{c}{I^{-1}(\mu_1 k_1(J^{-1}(\mu_2 k_2(c)))))}, \quad c \in (0, \infty) \\
&= \frac{c}{I^{-1}(\mu_1(1 + (J^{-1}(\mu_2 k_2(c)))^{\delta_1} + (J^{-1}(\mu_2 k_2(c)))^{\eta_1}))}, \quad c \in (0, \infty)
\end{aligned}
\]

\[
> 1
\]

and

\[
\begin{aligned}
&\sup_{c \in (0, \infty)} \frac{c}{I^{-1}(k_2(J^{-1}(k_1(c) \int_0^1 p(s)ds)) \int_0^1 q(s)ds)} \\
&= \sup_{c \in (0, \infty)} \frac{c}{I^{-1}(\mu_2 k_2(J^{-1}(\mu_1 k_1(c)))))} \\
&= \frac{c}{I^{-1}(\mu_2 k_2(J^{-1}(\mu_1 k_1(c)))))}, \quad c \in (0, \infty)
\end{aligned}
\]

\[
> 1.
\]

Moreover,
Theorem 3.3.1 Assume that (B_1) - (B_5), (B_6), (B_{11}) and (B_{12}) hold. Then the system of BVPs (3.3) has a $C^1$-positive solution.

Proof. In view of (B_{11}), we can choose real constants $M_3 > 0$ and $M_4 > 0$ such that

$$
\frac{M_4}{(1 + \frac{b_1}{a_1})J^{-1}(k_1((1 + \frac{b_2}{a_2})J^{-1}(k_2(c) J_0^1 p(t) dt))^\frac{1}{a_1})J_0^1 p(t) dt} > 1,
$$

where $I(\mu) = \int_0^\mu \frac{d\tau}{\tau(1 + \tau)^{1/2}}$, $J(\mu) = \int_0^\mu \frac{d\tau}{\tau(1 + \tau)^{1/2}}$, for $\mu \in (0, \infty)$;

$$(B_{12}) \quad \int_0^1 p(t)u_1(C, J_0^1 p(s) \psi_E(s) ds)dt < +\infty \text{ and } \int_0^1 p(t)u_2(J_0^1 q(s) \psi_E(s) ds)dt < +\infty \text{ for any real constant } C > 0.$$

Theorem 3.3.1 Assume that (B_1) - (B_5), (B_6), (B_{11}) and (B_{12}) hold. Then the system of BVPs (3.3) has a $C^1$-positive solution.

Proof. In view of (B_{11}), we can choose real constants $M_3 > 0$ and $M_4 > 0$ such that

$$
\frac{M_4}{(1 + \frac{b_1}{a_1})J^{-1}(k_1((1 + \frac{b_2}{a_2})J^{-1}(k_2(c) J_0^1 p(t) dt))^\frac{1}{a_1})J_0^1 p(t) dt} > 1,
$$
\[ M_1 = \frac{1}{1 + \frac{b_1}{a_1} I^{-1}(k_1(M_4) \int_0^1 q(t) dt)} > 1. \]

From the continuity of \( k_1, k_2, I \) and \( J \), we choose \( \varepsilon > 0 \) small enough such that
\[
\frac{1}{1 + \frac{b_1}{a_1} I^{-1}(k_1(M_4) \int_0^1 q(t) dt + J(\varepsilon)) I(\varepsilon)} > 1, \tag{3.72}
\]
\[
\frac{1}{1 + \frac{b_2}{a_2} I^{-1}(k_2(M_5) \int_0^1 q(t) dt + J(\varepsilon)) I(\varepsilon)} > 1. \tag{3.73}
\]

Choose real constants \( L_3 > 0 \) and \( L_4 > 0 \) such that
\[
I(L_3) > k_1(M_4) \int_0^1 p(t) dt + I(\varepsilon), \tag{3.74}
\]
\[
J(L_4) > k_2(M_5) \int_0^1 q(t) dt + J(\varepsilon). \tag{3.75}
\]

Choose \( n_0 \in \{1, 2, \ldots\} \) such that \( \frac{1}{n_0} < \varepsilon \). For each fixed \( n \in \{n_0, n_0 + 1, \ldots\} \), define retractions \( \theta_i : \mathbb{R} \to [0, M_i] \) and \( \rho_i : \mathbb{R} \to \left[\frac{1}{n}, L_i\right] \) by
\[
\theta_i(x) = \max\{0, \min\{x, M_i\}\} \text{ and } \rho_i(x) = \max\left\{\frac{1}{n}, \min\{x, L_i\}\right\}, i = 3, 4.
\]

Consider the modified system of BVPs
\[
-x''(t) = p(t) f(t, \theta_4(y(t)), \rho_3(x'(t))), \quad t \in (0, 1),
\]
\[
-y''(t) = q(t) g(t, \theta_3(x(t)), \rho_4(y'(t))), \quad t \in (0, 1),
\]
\[
a_1 x(0) - b_1 x'(0) = 0, x'(1) = \frac{1}{n},
\]
\[
a_2 y(0) - b_2 y'(0) = 0, y'(1) = \frac{1}{n}. \tag{3.76}
\]

Since \( f(t, \theta_4(y(t)), \rho_3(x'(t))), g(t, \theta_3(x(t)), \rho_4(y'(t))) \) are continuous and bounded on \([0, 1] \times \mathbb{R}^2\), by Theorem 1.1.7 it follows that the modified system of BVPs (3.76) has a solution \((x_n, y_n) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))\).

Using (3.76) and (B_2), we obtain
\[
x_n''(t) \leq 0 \text{ and } y_n''(t) \leq 0 \text{ for } t \in (0, 1),
\]
which on integration from \( t \) to 1, and using the BCs (3.76), yields
3.3 Existence of $C^1$-positive solutions with more general BCs

$$x_n(t) \geq \frac{1}{n} \text{ and } y_n(t) \geq \frac{1}{n} \text{ for } t \in [0,1]. \quad (3.77)$$

Integrating (3.77) from 0 to $t$, using the BCs (3.76) and (3.77), we have

$$x_n(t) \geq (t + \frac{b_1}{a_1}) \frac{1}{n} \text{ and } y_n(t) \geq (t + \frac{b_2}{a_2}) \frac{1}{n} \text{ for } t \in [0,1]. \quad (3.78)$$

From (3.77) and (3.78), it follows that

$$\|x_n\| = x_n(1) \text{ and } \|y_n\| = y_n(1).$$

Now, we show that

$$x_n'(t) < L_3, \quad y_n'(t) < L_4, \quad t \in [0,1]. \quad (3.79)$$

First, we prove $x_n'(t) < L_3$ for $t \in [0,1]$. Suppose $x_n'(t_1) \geq L_3$ for some $t_1 \in [0,1]$.

Using (3.76) and (B3), we have

$$-x_n''(t) \leq p(t)k_1(\theta_4(y_n(t)))(u_1(\rho_3(x_n'(t))) + v_1(\rho_3(x_n'(t)))) \quad t \in (0,1),$$

which implies that

$$-x_n''(t) \leq k_1(M_4)p(t), \quad t \in (0,1).$$

Integrating from $t_1$ to 1, using the BCs (3.76), we obtain

$$\int_0^{x_n'(t_1)} \frac{dz}{u_1(\rho_3(z)) + v_1(\rho_3(z))} \leq k_1(M_4) \int_{t_1}^1 p(t)dt,$$

which can also be written as

$$\int_0^{L_3} \frac{dz}{u_1(\rho_3(z)) + v_1(\rho_3(z))} + \int_{L_3}^{x_n'(t_1)} \frac{dz}{u_1(\rho_3(z)) + v_1(\rho_3(z))} \leq k_1(M_4) \int_0^1 p(t)dt.$$

Using the increasing property of $I$, we obtain

$$I(L_3) + \frac{x_n'(t_1) - L_3}{u_1(L_3) + v_1(L_3)} \leq k_1(M_4) \int_0^1 p(t)dt + I(\varepsilon),$$

a contradiction to (3.74). Hence, $x_n'(t) < L_3$ for $t \in [0,1]$. Similarly, we can show that $y_n'(t) < L_4$ for $t \in [0,1]$.

Now, we show that
\[ x_n(t) < M_3, \quad y_n(t) < M_4, \quad t \in [0, 1]. \]  

(3.80)

Suppose \( x_n(t_2) \geq M_3 \) for some \( t_2 \in [0, 1] \). From (3.76), (3.79) and (B₃), it follows that

\[
-x''_n(t) \leq p(t)k_1(\theta_4(y_n(t)))u_1(x'_n(t)) + v_1(x'_n(t)), \quad t \in (0, 1),
\]

\[
y''_n(t) \leq q(t)k_2(\theta_3(x_n(t)))u_2(y'_n(t)) + v_2(y'_n(t)), \quad t \in (0, 1),
\]

which implies that

\[
\frac{-x''_n(t)}{u_1(x'_n(t)) + v_1(x'_n(t))} \leq k_1(\theta_4(y_n(t)))p(t), \quad t \in (0, 1),
\]

\[
\frac{-y''_n(t)}{u_2(y'_n(t)) + v_2(y'_n(t))} \leq k_2(M_3)q(t), \quad t \in (0, 1).
\]

Integrating from \( t \) to 1, using the BCs (3.76), we obtain

\[
\int_{\frac{1}{2}}^{1} \frac{dz}{u_1(z) + v_1(z)} \leq k_1(\theta_4(y_n(t))) \int_{t}^{1} p(s)ds, \quad t \in [0, 1],
\]

\[
\int_{\frac{1}{2}}^{1} \frac{dz}{u_2(z) + v_2(z)} \leq k_2(M_3) \int_{t}^{1} q(s)ds, \quad t \in [0, 1],
\]

which implies that

\[
I(x'_n(t)) - I(\frac{1}{n}) \leq k_1(\theta_4(y_n(t))) \int_{0}^{1} p(s)ds, \quad t \in [0, 1],
\]

\[
J(y'_n(t)) - J(\frac{1}{n}) \leq k_2(M_3) \int_{0}^{1} q(s)ds, \quad t \in [0, 1].
\]

The increasing property of \( I \) and \( J \) leads to

\[
x'_n(t) \leq I^{-1}(k_1(\theta_4(y_n(t))) \int_{0}^{1} p(s)ds + I(\varepsilon)), \quad t \in [0, 1],
\]

(3.81)

\[
y'_n(t) \leq J^{-1}(k_2(M_3) \int_{0}^{1} q(s)ds + J(\varepsilon)), \quad t \in [0, 1].
\]

(3.82)

Integrating (3.81) from 0 to \( t_2 \) and (3.82) from 0 to 1, using the BCs (3.76), (3.81) and (3.82), we obtain

\[
M_3 \leq x_n(t_2) \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1(\theta_4(y_n(t))) \int_{0}^{1} p(s)ds + I(\varepsilon)),
\]

(3.83)

\[
\|y_n\| \leq (1 + \frac{b_2}{a_2})J^{-1}(k_2(M_3) \int_{0}^{1} q(s)ds + J(\varepsilon)).
\]

(3.84)
3.3 Existence of $C^1$-positive solutions with more general BCs

Either we have $\|y_n\| < M_4$ or $\|y_n\| \geq M_4$. If $\|y_n\| < M_4$, then from (3.83), we have

$$M_3 \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1(\|y_n\|) \int_0^1 p(s)ds + I(\varepsilon)), \quad (3.85)$$

Now, by using (3.84) in (3.85) and the increasing property of $k_1$ and $I^{-1}$, we obtain

$$M_3 \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1((1 + \frac{b_2}{a_2})J^{-1}(k_2(M_3) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$M_3 \leq \frac{M_3}{(1 + \frac{b_1}{a_1})I^{-1}(k_1((1 + \frac{b_2}{a_2})J^{-1}(k_2(M_3) \int_0^1 q(s)ds + J(\varepsilon))) \int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.72).

On the other hand, if $\|y_n\| \geq M_4$, then from (3.83) and (3.84), we have

$$M_4 \leq (1 + \frac{b_2}{a_2})J^{-1}(k_1(M_4) \int_0^1 p(s)ds + I(\varepsilon)). \quad (3.86)$$

Using (3.87) in (3.86) and the increasing property of $k_1$ and $I^{-1}$, we obtain

$$M_3 \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1(M_3) \int_0^1 q(s)ds + J(\varepsilon)) \int_0^1 p(s)ds + I(\varepsilon)),$$

which implies that

$$M_3 \leq \frac{M_3}{(1 + \frac{b_1}{a_1})I^{-1}(k_1(M_3) \int_0^1 q(s)ds + J(\varepsilon)) \int_0^1 p(s)ds + I(\varepsilon))} \leq 1,$$

a contradiction to (3.72). Hence, $x_n(t) < M_3$ for $t \in [0, 1]$. Similarly, we can show that $y_n(t) < M_4$ for $t \in [0, 1]$.

Thus, in view of (3.76)-(3.80), $(x_n, y_n)$ is a solution of the following coupled system of BVPs
which implies that

\[-x''(t) = p(t)f(t,y(t),x'(t)), \quad t \in [0,1],\]

\[-y''(t) = q(t)g(t,x(t),y'(t)), \quad t \in (0,1),\]

\[a_1 x(0) - b_1 x'(0) = 0, \quad x'(1) = \frac{1}{n},\]

\[a_2 y(0) - b_2 y'(0) = 0, \quad y'(1) = \frac{1}{n},\]

satisfy

\[(t + \frac{b_1}{a_1})\frac{1}{n} \leq x_n(t) < M_3, \quad \frac{1}{n} \leq x_n'(t) < L_3, \quad t \in [0,1],\]

\[(t + \frac{b_2}{a_2})\frac{1}{n} \leq y_n(t) < M_4, \quad \frac{1}{n} \leq y_n'(t) < L_4, \quad t \in [0,1].\]

Now, in view of \((B_5)\), there exist continuous functions \(\varphi_{M_4L_3}\) and \(\psi_{M_4L_4}\) defined on \([0,1]\) and positive on \((0,1)\), and real constants \(0 \leq \delta_1, \delta_2 < 1\) such that

\[f(t,y_n(t),x_n'(t)) \geq \varphi_{M_4L_3}(t)(y_n(t))^{\delta_1}, \quad (t,y_n(t),x_n'(t)) \in [0,1] \times [0,M_4] \times [0,L_3],\]

\[g(t,x_n(t),y_n'(t)) \geq \psi_{M_4L_4}(t)(x_n(t))^{\delta_2}, \quad (t,x_n(t),y_n'(t)) \in [0,1] \times [0,M_3] \times [0,L_4].\]

We claim that

\[x_n'(t) \geq C_6^{\delta_1} \int_0^1 p(s) \varphi_{M_4L_3}(s) ds,\]

\[y_n'(t) \geq C_6^{\delta_2} \int_0^1 q(s) \psi_{M_4L_4}(s) ds,\]

where

\[C_5 = \left(\frac{b_1}{a_1}\right)^{\frac{\delta_1}{1-\delta_1 \delta_2}} \left(\frac{b_2}{a_2}\right)^{\frac{\delta_2}{1-\delta_1 \delta_2}} \left(\int_0^1 p(t) \varphi_{M_4L_3}(t) dt\right)^{\frac{\delta_1}{1-\delta_1 \delta_2}} \left(\int_0^1 q(t) \psi_{M_4L_4}(t) dt\right)^{\frac{\delta_2}{1-\delta_1 \delta_2}},\]

\[C_6 = \left(\frac{b_1}{a_1}\right)^{\frac{\delta_1}{1-\delta_1 \delta_2}} \left(\frac{b_2}{a_2}\right)^{\frac{\delta_2}{1-\delta_1 \delta_2}} \left(\int_0^1 p(t) \varphi_{M_4L_3}(t) dt\right)^{\frac{\delta_1}{1-\delta_1 \delta_2}} \left(\int_0^1 q(t) \psi_{M_4L_4}(t) dt\right)^{\frac{\delta_2}{1-\delta_1 \delta_2}}.\]

To prove \((3.91)\), consider the following relation

\[x_n(t) = (t + \frac{b_1}{a_1})\frac{1}{n} + \frac{1}{a_1} \int_0^t (a_1 s + b_1) p(s)f(s,y_n(s),x_n'(s)) ds\]

\[+ \frac{1}{a_1} \int_t^1 (a_1 t + b_1) p(s)f(s,y_n(s),x_n'(s)) ds, \quad t \in [0,1],\]

which implies that
Using (3.90) and (3.89), we obtain
\[ x_n(0) \geq \frac{b_1}{a_1} \int_0^1 p(s)\Phi_{M}L_3(s)(y_n(s))^\delta_1 ds \geq (y_n(0))^\delta_2 \frac{b_1}{a_1} \int_0^1 p(s)\Phi_{M}L_3(s)ds. \] (3.94)

Similarly, using (3.90) and (3.89), we obtain
\[ y_n(0) \geq (x_n(0))^\delta_3 \frac{b_2}{a_2} \int_0^1 q(s)\Psi_{M}L_4(s)ds, \]
which in view of (3.94) implies that
\[ y_n(0) \geq (y_n(0))^\delta_4 \frac{b_2}{a_2} \int_0^1 q(s)\Psi_{M}L_4(s)ds. \]
Hence,
\[ y_n(0) \geq C_6. \] (3.95)

Now, from (3.93), it follows that
\[ x_n'(t) \geq \int_t^1 p(s)f(s, y_n(s), x_n'(s))ds, \]
and using (3.90) and (3.95), we obtain (3.91). Similarly, we can prove (3.92).

Now, using (3.88), (B3), (3.89), (3.91) and (3.92), we have
\[ 0 \leq -x_n''(t) \leq k_1(\Phi) p(t)(u_1(C_{3\delta}^1 \int_t^1 p(s)\Phi_{M}L_3(s)ds + v_1(L_3)), \] \[ t \in (0,1), \]
\[ 0 \leq -y_n''(t) \leq k_2(M_3) q(t)(u_2(C_{3\delta}^1 \int_t^1 q(s)\Psi_{M}L_4(s)ds + v_2(L_4)) \] \[ t \in (0,1). \] (3.96)

In view of (3.89), (3.96), (B1) and (B12), it follows that the sequences \{ (x_n^{(j)}, y_n^{(j)}) \} \((j = 0,1)\) are uniformly bounded and equicontinuous on \([0,1].\) Hence, by Theorem (1.1.1), there exist subsequences \{ (x_{n_k}^{(j)}, y_{n_k}^{(j)}) \} \((j = 0,1)\) of \{ (x_n^{(j)}, y_n^{(j)}) \} \((j = 0,1)\) and \((x,y) \in C^4[0,1] \times C^4[0,1]\) such that \((x_{n_k}^{(j)}, y_{n_k}^{(j)})\) converges uniformly to \((x^{(j)}, y^{(j)})\) on \([0,1] \quad (j = 0,1).\) Also,\( a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = x(1) = y'(1) = 0.\)

Moreover, from (3.91) and (3.92), with \(n_k\) in place of \(n\) and taking \(\lim_{n_k \to +\infty},\) we have
where

\[\frac{\partial}{\partial n} \int_0^1 p(s)\varphi_{M_kL_k}(s)ds,\]

\[\frac{\partial}{\partial n} \int_0^1 q(s)\varphi_{M_kL_k}(s)ds,\]

which shows that \(x' > 0\) and \(y' > 0\) on \([0, 1]\), \(x > 0\) and \(y > 0\) on \([0, 1]\). Further, \((x_n, y_n)\) satisfy

\[x_n'(t) = x_n(0) - \int_0^t p(s)f(s, y_n(s), x_n(s))ds, \quad t \in [0, 1],\]

\[y_n'(t) = y_n(0) - \int_0^t q(s)f(s, x_n(s), y_n(s))ds, \quad t \in [0, 1].\]

Passing to the limit as \(n_k \to \infty\), we obtain

\[x(t) = x(0) - \int_0^t p(s)f(s, y(s), x(s))ds, \quad t \in [0, 1],\]

\[y(t) = y(0) - \int_0^t q(s)f(s, x(s), y(s))ds, \quad t \in [0, 1],\]

which implies that

\[-x''(t) = p(t)f(t, y(t), x(t)), \quad t \in (0, 1),\]

\[-y''(t) = q(t)f(t, x(t), y(t)), \quad t \in (0, 1).\]

Hence, \((x, y)\) is a \(C^1\)-positive solution of the system of SBVPs (3.97).

**Example 3.3.2** Consider the following coupled system of singular BVPs

\[-x''(t) = (1 - t)^{-\frac{3}{4}}y(t)^{\frac{1}{3}}(x'(t))^{-\beta_1}, \quad t \in (0, 1),\]

\[-y''(t) = (1 - t)^{-\frac{3}{4}}(x(t))^{\frac{1}{3}}(y'(t))^{-\beta_2}, \quad t \in (0, 1),\]

\[x(0) - x'(0) = y(0) - y'(0) = x'(1) = y'(1) = 0,\]

where \(0 < \beta_1 < 1\) and \(0 < \beta_2 < 1\).

Choose \(p(t) = (1 - t)^{-\frac{3}{4}}, \quad q(t) = (1 - t)^{-\frac{3}{4}}, \quad k_1(x) = x^{\frac{1}{4}}, \quad k_2(x) = x^{\frac{1}{3}}, \quad u_1(x) = x^{-\beta_1}, \quad u_2(x) = x^{-\beta_2}\) and \(v_1(x) = v_2(x) = 0\). Also \(\delta_1 = \frac{1}{4}, \quad \delta_2 = \frac{1}{3}, \quad \varphi_{EF}(t) = F^{-\beta_1}\) and \(\varphi_{EF}(t) = F^{-\beta_2}\). Then, \(I(z) = \frac{\beta_1 + 1}{\beta_1 + 1}, \quad f(z) = \frac{\beta_1 + 1}{\beta_2 + 1}, \quad I^{-1}(z) = (\beta_1 + 1)^{\frac{1}{\beta_1 + 1}}, \quad I^{1}(z) = (\beta_1 + 1)^{\frac{1}{\beta_2 + 1}}\) and \(J^{-1}(z) = (\beta_2 + 1)^{\frac{1}{\beta_1 + 1}}, \quad J^{1}(z) = (\beta_2 + 1)^{\frac{1}{\beta_2 + 1}}\). Then, \(\int_0^1 p(t)dt = 4\) and \(\int_0^1 q(t)dt = \frac{3}{5}i\).

Clearly, \((B_1) - (B_3), \quad (B_4) \quad \text{and} \quad (B_5)\) are satisfied. Moreover,
3.4 Existence of at least two positive solutions with more general BCs

\[
\sup_{c \in (0, \infty)} \left( 1 + \frac{b_1}{a_1} \right) I^{-1}(k_1((1 + \frac{b_1}{a_1}) J^{-1}(k_1(c \int_0^1 q(t) \, dt) \int_0^1 p(t) \, dt) \right) = \sup_{c \in (0, \infty)} \left( 2^{\frac{3}{4}} (\frac{b_1}{a_1} + 1) \beta_1 + 1 \right) \beta_2 + 1 \beta_2 + 1 \int_0^1 q(t) \, dt) \int_0^1 p(t) \, dt) \right) = \infty,
\]

which shows that \((B_{11})\) and \((B_{12})\) also holds.

Since, \((B_1) - (B_3), (B_5), (B_6), (B_{11})\) and \((B_{12})\) are satisfied. Therefore, by Theorem 3.3.1, the system of BVPs (3.37) has at least one \(C^1\)-positive solution.

3.4 Existence of at least two positive solutions with more general BCs

In this section, we establish at least two \(C^1\)-positive solutions to the system of SBVPs (3.33). For each \((x, y) \in C^1[0, 1] \times C^1[0, 1]\), we write \(|(x, y)| = \|x\| + \|y\|\). Clearly, \((C^1[0, 1] \times C^1[0, 1], \| \cdot \|)\) is a Banach space. We define a partial ordering in \(C^1[0, 1]\), by \(x \leq y\) if and only if \(x(t) \leq y(t), t \in [0, 1]\). We define a partial ordering in \(C^1[0, 1] \times C^1[0, 1]\), by \((x_1, y_1) \leq (x_2, y_2)\) if and only if \(x_1 \leq x_2\) and \(y_1 \leq y_2\). Let

\[
P_i = \{ x \in C^1[0, 1] : x(t) \geq \gamma_i \| x \| \text{ for all } t \in [0, 1], x(0) \geq \frac{b_i}{a_i} \| x' \| \},
\]

where \(\gamma_i = \frac{b_i}{a_i + b_i}, i = 1, 2\). Clearly, \(P_i (i = 1, 2)\) are cones of \(C^1[0, 1]\) and \(P_1 \times P_2\) is a cone of \(C^1[0, 1] \times C^1[0, 1]\). For any real constant \(r > 0\), we define an open neighborhood of \((0, 0) \in C^1[0, 1] \times C^1[0, 1]\) as
\[ \mathcal{O}_r = \{(x,y) \in C^1[0,1] \times C^1[0,1] : \| (x,y) \|_5 < r \} . \]

In view of (B_{11}), there exist real constants \( R_3 > 0 \) and \( R_4 > 0 \) such that
\[
\begin{align*}
R_3 &> 1, \quad (1 + \frac{b_1}{a_1}) I^{-1}(k_1((1 + \frac{b_1}{a_1}) I^{-1}(k_2(R_3) \int_0^1 q(t)dt)) \int_0^1 p(t)dt) > 1, \\ R_4 &> 1. \quad (1 + \frac{b_2}{a_2}) I^{-1}(k_2((1 + \frac{b_2}{a_2}) I^{-1}(k_1(R_4) \int_0^1 p(t)dt)) \int_0^1 q(t)dt) > 1.
\end{align*}
\tag{3.98}
\tag{3.99}
\]

From the continuity of \( k_1, k_2, I \) and \( J \), we choose \( \varepsilon > 0 \) small enough such that
\[
\begin{align*}
R_3 &> 1, \quad (1 + \frac{b_1}{a_1}) I^{-1}(k_1((1 + \frac{b_1}{a_1}) I^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(t)dt + J(\varepsilon)) \int_0^1 p(t)dt + I(\varepsilon))) > 1, \\ R_4 &> 1. \quad (1 + \frac{b_2}{a_2}) I^{-1}(k_2((1 + \frac{b_2}{a_2}) I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(t)dt + I(\varepsilon)) \int_0^1 q(t)dt + J(\varepsilon))) > 1.
\end{align*}
\tag{3.100}
\tag{3.101}
\]

Choose \( n_0 \in \{1, 2, \cdots\} \) such that \( \max\{\frac{1}{n_0}(1 + \frac{b_1}{a_1}), \frac{1}{n_0}(1 + \frac{b_2}{a_2})\} < \varepsilon \) and for each fixed \( n \in \{n_0, n_0 + 1, \cdots\} \), consider the system of non-singular BVPs
\[
\begin{align*}
-x''(t) &= p(t)f(t,x,y) + \frac{1}{n}(t + \frac{b_2}{a_2}), |x'(t)| + \frac{1}{n}, \quad t \in (0, 1), \\
-y''(t) &= q(t)g(t,x,y) + \frac{1}{n}(t + \frac{b_1}{a_1}), |y'(t)| + \frac{1}{n}, \quad t \in (0, 1), \\
a_1 x(0) - b_1 x'(0) &= a_2 y(0) - b_2 y'(0) = x'(1) = y'(1) = 0.
\end{align*}
\tag{3.102}
\]

We write (3.102) as an equivalent system of integral equations
\[
\begin{align*}
x(t) &= \int_0^1 G_1(t,s) p(s)f(s,x(s),y(s)) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds, \quad t \in [0, 1], \\
y(t) &= \int_0^1 G_2(t,s) q(s)g(s,x(s),y(s)) + \frac{1}{n}(s + \frac{b_1}{a_1}), |y'(s)| + \frac{1}{n})ds, \quad t \in [0, 1],
\end{align*}
\tag{3.103}
\]

where
\[
G_i(t,s) = \frac{1}{a_i} \begin{cases} 
  b_i + a_i s, & 0 \leq s \leq t \leq 1, \\
  b_i + a_i t, & 0 \leq t \leq s \leq 1, i = 1, 2.
\end{cases}
\]

By a solution of the system of BVPs (5.102), we mean a solution of the corresponding system of integral equations (5.103).
Define a map $T_n : C^1[0,1] \times C^1[0,1] \to C^1[0,1] \times C^1[0,1]$ by
\[
T_n(x,y) = (A_n(x,y), B_n(x,y)),
\]
where the maps $A_n, B_n : C^1[0,1] \times C^1[0,1] \to C^1[0,1]$ are defined by
\[
A_n(x,y)(t) = \int_0^t G_1(t, s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2})), |x'(s)| + \frac{1}{n} ds, \quad t \in [0, 1],
\]
\[
B_n(x,y)(t) = \int_0^t G_2(t, s)q(s)f(s, x(s) + \frac{1}{n}(s + \frac{b_1}{a_1})), |y'(s)| + \frac{1}{n} ds, \quad t \in [0, 1].
\]

Clearly, if $(x_n, y_n) \in C^1[0,1] \times C^1[0,1]$ is a fixed point of $T_n$, then $(x_n, y_n)$ is a solution of the system of BVPs (3.102).

**Lemma 3.4.1** Assume that $(B_1) - (B_3)$ hold. Then the map $T_n : \overline{C} \cap (P_1 \times P_2) \to P_1 \times P_2$ is completely continuous.

**Proof.** Firstly, we show that $T_n(P_1 \times P_2) \subseteq P_1 \times P_2$. For $(x,y) \in P_1 \times P_2$, $t \in [0, 1]$, using (3.105) and Lemma 1.1.17 we obtain
\[
A_n(x,y)(t) \geq \gamma_1 \max_{\tau \in [0, 1]} \int_0^1 G_1(\tau, s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2})), |x'(s)| + \frac{1}{n} ds
\]
\[
= \gamma_1 ||A_n(x,y)||
\]
and
\[
A_n(x,y)(0) = \int_0^1 G_1(0, s)p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2})), |x'(s)| + \frac{1}{n} ds
\]
\[
= \frac{b_1}{a_1} \max_{\tau \in [0, 1]} \int_0^1 p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2})), |x'(s)| + \frac{1}{n} ds
\]
\[
= \frac{b_1}{a_1} ||A_n(x,y)||
\]
From (3.106) and (3.107), $A_n(x,y) \in P_1$ for every $(x,y) \in P_1 \times P_2$, that is, $A_n(P_1 \times P_2) \subseteq P_1$. Similarly, by using (3.105) and Lemma 1.1.17 we can show that $B_n(P_1 \times P_2) \subseteq P_2$. Hence, $T_n(P_1 \times P_2) \subseteq P_1 \times P_2$. 
Now, we show that \( T_n : \overline{\Omega} \cap (P_1 \times P_2) \to P_1 \times P_2 \) is uniformly bounded. For any \((x,y) \in \overline{\Omega} \cap (P_1 \times P_2)\), using (3.103), Lemma 1.1.17 (B_1) and (B_3), we have

\[
\|A_n(x,y)\| = \max_{\tau \in [0,1]} \left| \int_0^1 G_1(t,s) p(s) f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) ds \right|
\]

\[
\leq \frac{1}{a_1} \int_0^1 (a_1s + b_1)p(s)k_1(y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}))(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n})) ds
\]

\[
\leq \frac{1}{a_1} k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2}))(u_1(\frac{1}{n}) + v_1(r + \frac{1}{n})) \int_0^1 (a_1s + b_1)p(s) ds < +\infty.
\]

(3.108)

\[
\|A_n(x,y)\| = \max_{\tau \in [0,1]} |A_n(x,y)'(\tau)|
\]

\[
= \max_{\tau \in [0,1]} \int_0^1 p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) ds
\]

\[
= \int_0^1 p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) ds
\]

(3.109)

From (3.108) and (3.109), it follows that \( A_n(\overline{\Omega} \cap (P_1 \times P_2)) \) is uniformly bounded under \( \| \cdot \|_3 \). Similarly, by using (3.103), Lemma 1.1.17 (B_1) and (B_3), we can show that \( B_n(\overline{\Omega} \cap (P_1 \times P_2)) \) is uniformly bounded under \( \| \cdot \|_3 \). Hence, \( T_n(\overline{\Omega} \cap (P_1 \times P_2)) \) is uniformly bounded.

Now, we show that \( T_n(\overline{\Omega} \cap (P_1 \times P_2)) \) is equicontinuous. For any \((x,y) \in \overline{\Omega} \cap (P_1 \times P_2)\) and \( t_1, t_2 \in [0,1] \), using (3.105) and (B_3), we have

\[
|A_n(x,y)(t_1) - A_n(x,y)(t_2)|
\]

\[
= \left| \int_0^1 (G_1(t_1,s) - G_1(t_2,s)) p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) ds \right|
\]

\[
\leq \int_0^1 |G_1(t_1,s) - G_1(t_2,s)| p(s)f(s, y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) ds
\]

\[
\leq \int_0^1 |G_1(t_1,s) - G_1(t_2,s)| p(s)k_1(y(s) + \frac{1}{n}(s + \frac{b_2}{a_2}))(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n})) ds
\]

\[
\leq k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2}))(u_1(\frac{1}{n}) + v_1(r + \frac{1}{n})) \int_0^1 |G(t_1,s) - G(t_2,s)| p(s) ds,
\]

(3.110)
is equicontinuous. Hence, by Theorem 1.1.5,

\[ |A_n(x,y)'(t_1) - A_n(x,y)'(t_2)| = \left| \int_{t_1}^{t_2} p(s)f(s,y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \right| \leq \int_{t_1}^{t_2} p(s)k_1(y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}))(u_1(|x'(s)| + \frac{1}{n}) + v_1(|x'(s)| + \frac{1}{n}))ds \leq k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2})(u_1(\frac{1}{n}) + v_1(\frac{1}{n})) \int_{t_1}^{t_2} p(s)ds. \]  

(3.111)

From (3.110), (3.111) and (B1), it follows that \( A_n(\overline{O_r} \cap (P_1 \times P_2)) \) is equicontinuous under the norm \( \| \cdot \|_3 \). Similarly, using (3.105) and (B1), we can show that \( B_n(\overline{O_r} \cap (P_1 \times P_2)) \) is equicontinuous under \( \| \cdot \|_3 \). Consequently, \( T_n(\overline{O_r} \cap (P_1 \times P_2)) \) is equicontinuous. Hence, by Theorem 1.1.5, \( T_n(\overline{O_r} \cap (P_1 \times P_2)) \) is relatively compact which implies that \( T_n \) is a compact map. Further, we show that \( T_n \) is continuous.

Let \((x_m, y_m), (x, y) \in \overline{O_r} \cap (P_1 \times P_2)\) such that

\[ \| (x_m, y_m) - (x, y) \|_3 \to 0 \text{ as } m \to +\infty. \]

Using (B3), we have

\[ \left| f(t, y_m(t)) + \frac{1}{n}(t + \frac{b_2}{a_2}), |x'_m(t)| + \frac{1}{n} \right| \leq k_1(y_m(t) + \frac{1}{n}(t + \frac{b_2}{a_2})) \]

\[ (u_1(|x'_m(t)| + \frac{1}{n}) + v_1(|x'_m(t)| + \frac{1}{n})) \leq k_1(r + \frac{1}{n}(1 + \frac{b_2}{a_2}))(u_1(\frac{1}{n}) + v_1(\frac{1}{n})). \]

Using (3.105) and Lemma 1.1.17 we have

\[ \| A_n(x_m, y_m) - A_n(x, y) \| \]

\[ = \max_{t \in [0,1]} \left| \int_0^t G_1(t,s)p(s)f(s,y_m(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s,y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \right| \]

\[ \leq \frac{1}{a_1} \int_0^1 (a_1 s + b_1)p(s) \left| f(s,y_m(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s,y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) \right| ds \]

(3.112)

and

\[ \| A_n(x_m, y_m)' - A_n(x, y)' \| \]

\[ = \max_{t \in [0,1]} \left| \int_t^1 p(s)f(s,y_m(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s,y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n})ds \right| \]

\[ \leq \int_0^1 p(s) \left| f(s,y_m(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'_m(s)| + \frac{1}{n}) - f(s,y(s) + \frac{1}{n}(s+\frac{b_2}{a_2}), |x'(s)| + \frac{1}{n}) \right| ds. \]

(3.113)
Hence, \( \| A_n(x_m, y_m) - A_n(x, y) \| \to 0, \| A_n(x_m, y_m)' - A_n(x, y)' \| \to 0 \) as \( m \to +\infty \).

From (3.112) and (3.113), using Lebesgue dominated convergence theorem, it follows that

\[
\| A_n(x_m, y_m) - A_n(x, y) \|_3 \to 0 \quad \text{as} \quad m \to \infty.
\]

Similarly, we can show that \( \| B_n(x_m, y_m) - B_n(x, y) \|_3 \to 0 \) as \( m \to \infty \). Consequently, \( \| T_n(x_m, y_m) - T_n(x, y) \|_5 \to 0 \) as \( m \to +\infty \), that is, \( T_n : \overline{O}_r \cap (P_1 \times P_2) \to P_1 \times P_2 \) is continuous. Hence, \( T_n : \overline{O}_r \cap (P_1 \times P_2) \to P_1 \times P_2 \) is completely continuous.

Assume that

\[ (B_{13}) \quad \text{for any real constant } C > 0, \int_0^1 p(t)u_1(C^m p(s)\varphi_E(s))ds < +\infty \quad \text{and} \quad \int_0^1 q(t)u_2(C^m q(s)\psi_E(s))ds < +\infty. \]

**Theorem 3.4.2** Assume that \((B_1)\) - \((B_3)\), \((B_5)\), \((B_8)\), \((B_{10})\), \((B_{11})\) and \((B_{13})\) hold. Then the system of SBVPs (3.3) has at least two \( C^1 \)-positive solutions.

**Proof.** Let \( R_5 = R_3 + R_4 \) and define \( \partial R_5 = \Omega_{R_3} \times \Omega_{R_4} \) where

\[
\Omega_{R_3} = \{ x \in E : \| x \|_3 < R_3 \}, \quad \Omega_{R_4} = \{ x \in E : \| x \|_3 < R_4 \}.
\]

We claim that

\[
(x, y) \neq \lambda T_n(x, y), \quad \text{for } \lambda \in (0, 1], (x, y) \in \partial \Omega_{R_5} \cap (P_1 \times P_2).
\]

(3.114)

Suppose there exist \((x_0, y_0) \in \partial \Omega_{R_5} \cap (P_1 \times P_2)\) and \( \lambda_0 \in (0, 1] \) such that

\[
(x_0, y_0) = \lambda_0 T_n(x_0, y_0).
\]

Then,

\[
-x_0'(t) = \lambda_0 p(t)f(t, x_0, y_0(t)) + \frac{1}{n}(t + \frac{b_2}{a_2}), |x_0'(t)| > \frac{1}{n}, \quad t \in (0, 1),
\]

\[
y_0''(t) = \lambda_0 q(t)g(t, x_0, y_0(t)) + \frac{1}{n}(t + \frac{b_1}{a_1}), |y_0'(t)| > \frac{1}{n}, \quad t \in (0, 1),
\]

\[
a_1x(0) - b_1x'(0) = a_2y(0) - b_2y'(0) = x'(1) = y'(1) = 0.
\]

From (3.115) and \((B_2)\), we have \( x_0' \leq 0 \) and \( y_0' \leq 0 \) on \((0, 1)\), integrating from \( t \) to 1, using the BCs (3.115), we obtain \( x_0(t) \geq 0 \) and \( y_0(t) \geq 0 \) for \( t \in [0, 1] \). From (3.115) and \((B_1)\), we have
Integrating from (3.116)-(3.119), it follows that

\[-x_0'(t) \leq p(t)k_1(y_0(t) + \frac{1}{n}(t + \frac{b_2}{a_2}))u_1(x_0'(t) + \frac{1}{n}) + v_1(x_0'(t) + \frac{1}{n}), \quad t \in (0, 1),
\]

\[-y_0''(t) \leq q(t)k_2(x_0(t) + \frac{1}{n}(t + \frac{b_1}{a_1}))u_2(y_0'(t) + \frac{1}{n}) + v_2(y_0'(t) + \frac{1}{n}), \quad t \in (0, 1),
\]

which implies that

\[-x_0'(t) \leq p(t)k_1(y_0(t) + \frac{1}{n}(t + \frac{b_2}{a_2})) \leq k_1(R_4 + \varepsilon)p(t), \quad t \in (0, 1),
\]

\[-y_0''(t) \leq q(t)k_2(x_0(t) + \frac{1}{n}(t + \frac{b_1}{a_1})) \leq k_2(R_3 + \varepsilon)q(t), \quad t \in (0, 1).
\]

Integrating from \(t\) to 1, using the BCs (3.115), we obtain

\[I(x_0'(t) + \frac{1}{n}) - I(\frac{1}{n}) \leq k_1(R_4 + \varepsilon) \int_0^1 p(s)ds, \quad t \in [0, 1],
\]

\[J(y_0'(t) + \frac{1}{n}) - J(\frac{1}{n}) \leq k_2(R_3 + \varepsilon) \int_0^1 q(s)ds, \quad t \in [0, 1],
\]

which implies that

\[x_0'(t) \leq I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0, 1], \tag{3.116}
\]

\[y_0'(t) \leq J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1]. \tag{3.117}
\]

Integrating from 0 to \(t\), using the BCs (3.115), leads to

\[x_0(t) \leq \frac{b_1}{a_1}x_0(0) + I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0, 1],
\]

\[y_0(t) \leq \frac{b_2}{a_2}y_0(0) + J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1].
\]

Using (3.116) and (3.117), we have

\[x_0(t) \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0, 1], \tag{3.118}
\]

\[y_0(t) \leq (1 + \frac{b_2}{a_2})J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1]. \tag{3.119}
\]

From (3.116)-(3.119), it follows that

\[R_3 \leq (1 + \frac{b_1}{a_1})I^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)), \tag{3.120}
\]
We show that

\[ R_4 \leq (1 + \frac{b_2}{a_2}) J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)). \]  

(3.121)

Now, using (3.121) in (3.120) together with increasing property of \( k_1 \) and \( J^{-1} \), we have

\[ \frac{R_3}{(1 + \frac{b_2}{a_2}) J^{-1}(k_1((1 + \frac{b_2}{a_2}) J^{-1}(k_2(R_3 + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon)) + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon))} \leq 1, \]

a contradiction to (3.100). Similarly, using (3.121) in (3.120) together with increasing property of \( k_2 \) and \( J^{-1} \), we have

\[ \frac{R_4}{(1 + \frac{b_2}{a_2}) J^{-1}(k_2((1 + \frac{b_2}{a_2}) J^{-1}(k_1(R_4 + \varepsilon) \int_0^1 p(s)ds + I(\varepsilon)) + \varepsilon) \int_0^1 q(s)ds + J(\varepsilon))} \leq 1, \]

a contradiction to (3.101). Hence, (3.114) is true and by Lemma 1.1.8 the fixed point index

\[ \text{ind}_{\mathcal{F}}(T_n, \mathcal{O}_{R_3} \cap (P_1 \times P_2), P_1 \times P_2) = 1. \]  

(3.122)

Now, choose a \( t_0 \in (0, 1) \) and define

\[ N_3 = \frac{1 + \gamma_2^{-1} P_2^{-1}}{\max_{x \in [0, 1]} \int_0^1 G_1(t, s)p(s)ds} \quad \text{and} \quad N_4 = \frac{1 + \gamma_1^{-1} P_1^{-1}}{\max_{x \in [0, 1]} \int_0^1 G_2(t, s)q(s)ds}. \]  

(3.123)

By (B10), there exist real constants with \( R_3^* > R_3 \) and \( R_4^* > R_4 \) such that

\[ h_1(x, y) \geq N_3 x, \quad \text{for} \quad x \geq R_3^*, y \in (0, \infty), \]

\[ h_2(x, y) \geq N_4 x, \quad \text{for} \quad x \geq R_4^*, y \in (0, \infty). \]  

(3.124)

Let \( R^{**} = \frac{R_3^*}{\gamma_1 P_1} + \frac{R_4^*}{\gamma_2 P_2} \) and define

\[ \mathcal{O}_{R^{**}} = \mathcal{O}_{R_3^*} \times \mathcal{O}_{R_4^*}, \]

where

\[ \mathcal{O}_{R_3^*} = \{ x \in C^1[0, 1] : \| x \|_3 < \frac{R_3^*}{\gamma_1 P_1} \}, \quad \mathcal{O}_{R_4^*} = \{ x \in C^1[0, 1] : \| x \|_3 < \frac{R_4^*}{\gamma_2 P_2} \}. \]

We show that

\[ T_n(x, y) \not\in \mathcal{O}_{R^{**}} \quad \text{for} \quad (x, y) \in \mathcal{O}_{R^{**}} \cap (P_1 \times P_2). \]  

(3.125)

Suppose \( T_n(x_0, y_0) \preceq (x_0, y_0) \) for some \( (x_0, y_0) \in \partial \mathcal{O}_{R^{**}} \cap (P_1 \times P_2) \). Then,

\[ x_0(t) \geq A_n(x_0, y_0)(t) \quad \text{and} \quad y_0(t) \geq B_n(x_0, y_0)(t) \quad \text{for} \quad t \in [0, 1]. \]  

(3.126)

By Lemma 1.1.16 we have
3.4 Existence of at least two positive solutions with more general BCs

\[ x_0(t) \geq \gamma_1 \rho_1 \|x_0\|_3 = R_3^*, \quad t \in [0, 1]. \]

Similarly, \( y_0(t) \geq R_4^* \) for \( t \in [0, 1] \). Hence,

\[ |x_0(t)| + \frac{1}{n} (t + \frac{b_1}{a_1}) \geq R_3^* \quad \text{and} \quad |y_0(t)| + \frac{1}{n} (t + \frac{b_2}{a_2}) \geq R_4^* \quad \text{for} \quad t \in [0, 1]. \]

Now, using (3.126), (B10) and (3.124), we have

\[
\|x_0\| \geq x_0(t) \\
\geq A_n(x_0, y_0)(t) \\
= \int_0^1 G_1(t, s) p(s) f(s, y_0(s) + \frac{1}{n} (s + \frac{b_2}{a_2}), |x_0'(s)| + \frac{1}{n}) ds \\
\geq \int_0^1 G_1(t, s) p(s) h_1(y_0(s) + \frac{1}{n} (s + \frac{b_2}{a_2}), |x_0'(s)| + \frac{1}{n}) ds \\
\geq N_3 \int_0^1 G_1(t, s) p(s)(y_0(s) + \frac{1}{n} (s + \frac{b_2}{a_2})) ds \\
\geq N_3 R_3^* \int_0^1 G_1(t, s) p(s) ds, \quad t \in [0, 1],
\]

in view of (3.124) we have

\[
\|x_0\| \geq N_3 R_3^* \max_{t \in [0, 1]} \int_0^1 G_1(t, s) p(s) ds > \frac{R_3^*}{\gamma_2 \rho_2}.
\]

Thus \( \|x_0\|_3 \geq \|x_0\| > \frac{R_3^*}{\gamma_2 \rho_2} \). Similarly, using (3.126), (B10), (3.124) and (3.123), we have \( \|y_0\|_3 > \frac{R_4^*}{\gamma_3 \rho_3} \). Consequently, it follows that, \( \|(x_0, y_0)\| = \|x_0\|_3 + \|y_0\|_3 > R^{**} \), a contradiction. Hence, (3.125) is true and by Lemma 1.1.10, the fixed point index

\[
\text{ind}_{FP}(T_n, \partial_{R^{**}} \cap (P_1 \times P_2), P_1 \times P_2) = 0. \quad (3.127)
\]

From (3.122) and (3.127), it follows that

\[
\text{ind}_{FP}(T_n, (\partial_{R^{**}} \setminus \partial R) \cap (P_1 \times P_2), P_1 \times P_2) = -1. \quad (3.128)
\]

Thus, in view of (3.122) and (3.128), there exist \((x_{n,1}, y_{n,1}) \in \partial R \cap (P_1 \times P_2)\) and \((x_{n,2}, y_{n,2}) \in (\partial_{R^{**}} \setminus \partial R) \cap (P_1 \times P_2)\) such that \((x_{n,j}, y_{n,j}) = T_n(x_{n,j}, y_{n,j}), \quad (j = 1, 2)\) which implies that
We show that

\[ x_{n,j}(t) = \int_0^t G_1(t,s)p(s) f(t,y_{n,j}(s) + \frac{1}{n}(s + \frac{b_j}{a_j}), |x'_{n,j}(s)| + \frac{1}{n}) ds, \quad t \in [0, 1], \]

\[ y_{n,j}(t) = \int_0^t G_2(t,s) q(s) g(s,x_{n,j}(s) + \frac{1}{n}(s + \frac{b_j}{a_j}), |y'_{n,j}(s)| + \frac{1}{n}) ds, \quad t \in [0, 1], j = 1, 2. \] (3.129)

Using \((B_8)\) there exist continuous functions \(\varphi_{R_4+\varepsilon}\) and \(\psi_{R_3+\varepsilon}\) defined on \([0, 1]\) and positive on \((0, 1)\) and real constants \(0 \leq \delta_1, \delta_2 < 1\) such that

\[ f(t,x,y) \geq \varphi_{R_4+\varepsilon}(t)x^{\delta_1}, \quad (t,x,y) \in [0, 1] \times [0, R_4 + \varepsilon] \times [0, R_4 + \varepsilon], \]

\[ g(t,x,y) \geq \psi_{R_3+\varepsilon}(t)x^{\delta_2}, \quad (t,x,y) \in [0, 1] \times [0, R_3 + \varepsilon] \times [0, R_3 + \varepsilon]. \] (3.130)

By the Lemma [1.1.17] we have \(x_{n,1}(t) \geq y_1 \|x_{n,1}\|\) and \(y_{n,1}(t) \geq y_2 \|y_{n,1}\|\) for \(t \in [0, 1]\).

We show that

\[ x'_{n,1}(t) \geq C_8^{\delta_1} n^{\frac{\delta_1}{y_2}} \int_0^t p(s) \varphi_{R_4+\varepsilon}(s) ds, \quad t \in [0, 1], \] (3.131)

\[ y'_{n,1}(t) \geq C_8^{\delta_2} n^{\frac{\delta_2}{y_2}} \int_0^t q(s) \psi_{R_3+\varepsilon}(s) ds, \quad t \in [0, 1], \] (3.132)

where

\[ C_7 = y_{1} \frac{1+\delta_1}{y_2} \frac{\gamma_{1}}{\gamma_{2}} \left( \max_{t \in [0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\frac{1}{\gamma_{1}\gamma_{2}}} \]

\[ \left( \max_{t \in [0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds \right)^{\frac{\delta_1}{\gamma_{1}\gamma_{2}}}, \]

\[ C_8 = y_{1} \frac{1+\delta_2}{y_2} \frac{\gamma_{1}}{\gamma_{2}} \left( \max_{t \in [0,1]} \int_0^1 G_1(\tau,s)p(s)\varphi_{R_4+\varepsilon}(s)ds \right)^{\frac{\delta_2}{\gamma_{1}\gamma_{2}}} \]

\[ \left( \max_{t \in [0,1]} \int_0^1 G_2(\tau,s)q(s)\psi_{R_3+\varepsilon}(s)ds \right)^{\frac{1}{\gamma_{1}\gamma_{2}}}. \]

To prove (3.131), using (3.129) and (3.130), we have

\[ x_{n,1}(t) \geq \int_0^t G_1(t,s)p(s)f(s,y_{n,1}(s) + \frac{1}{n}(s + \frac{b_j}{a_j}), |x'_{n,1}(s)| + \frac{1}{n}) ds \]

\[ \geq \int_0^t G_1(t,s)p(s)\varphi_{R_4+\varepsilon}(s)(y_{n,1}(s) + \frac{1}{n}(s + \frac{b_j}{a_j})^{\delta_1}) ds \]

\[ \geq y_1^{\delta_1} \|y_{n,1}\|^\delta_1 \int_0^1 G_1(t,s)p(s)\varphi_{R_4+\varepsilon}(s) ds, \quad t \in [0, 1], \]

which implies that
\[ x_{n,1}(t) \geq \gamma_1^{\delta_1} \| y_{n,1} \|^{\delta_1} \max_{\tau \in [0,1]} \int_0^1 G_1(\tau, s) p(s) \varphi_{R_4 + \varepsilon}(s) ds, \quad t \in [0, 1]. \]

Hence,
\[ \| x_{n,1} \| \geq \gamma_1^{\delta_1} \| y_{n,1} \|^{\delta_1} \max_{\tau \in [0,1]} \int_0^1 G_1(\tau, s) p(s) \varphi_{R_4 + \varepsilon}(s) ds. \quad (3.133) \]

Similarly, from (3.129) and (3.130), we have
\[ \| y_{n,1} \| \geq \gamma_1^{\delta_2} \gamma_2^{\delta_2} \| x_{n,1} \|^{\delta_2} \max_{\tau \in [0,1]} \int_0^1 G_2(\tau, s) q(s) \psi_{R_3 + \varepsilon}(s) ds. \quad (3.134) \]

Using (3.133) in (3.134), we have
\[ \| y_{n,1} \| \geq \gamma_1^{\delta_2} \gamma_2^{\delta_2} \| x_{n,1} \|^{\delta_2} \left( \max_{\tau \in [0,1]} \int_0^1 G_1(\tau, s) p(s) \varphi_{R_4 + \varepsilon}(s) ds \right) \delta_2 \]
\[ \max_{\tau \in [0,1]} \int_0^1 G_2(\tau, s) q(s) \psi_{R_3 + \varepsilon}(s) ds, \]
which implies that
\[ \| y_{n,1} \| \geq \gamma_1^{\delta_2} \gamma_2^{\delta_2} \left( \max_{\tau \in [0,1]} \int_0^1 G_1(\tau, s) p(s) \varphi_{R_4 + \varepsilon}(s) ds \right) \delta_2 \]
\[ \left( \max_{\tau \in [0,1]} \int_0^1 G_2(\tau, s) q(s) \psi_{R_3 + \varepsilon}(s) ds \right)^{\frac{1}{\delta_2}} = C_8. \quad (3.135) \]

Using (3.130) and (3.135) in the following relation
\[ x'_{n,1}(t) = \int_0^1 p(s) f(s, y_{n,1}(s)) + \frac{1}{n} (s + \frac{b_2}{a_2}), |x'_{n,1}(s)| + \frac{1}{n} ds \]
we obtain (3.131). Similarly, we can prove (3.132).

Now, differentiating (3.129), using (B_3), (3.131) and (3.132), we have
\[ 0 \leq -x''_{n,1}(t) \leq p(t)k_1(R_4 + \varepsilon)(u_1(C_8^{\delta_1} \gamma_2^{\delta_1} \int_0^1 p(s) \varphi_{R_4 + \varepsilon}(s) ds + v_1(R_3 + \varepsilon)), \quad t \in (0, 1), \]
\[ 0 \leq -y''_{n,1}(t) \leq q(t)k_2(R_3 + \varepsilon)(u_2(C_8^{\delta_2} \gamma_2^{\delta_2} \int_0^1 q(s) \psi_{R_3 + \varepsilon}(s) ds + v_2(R_4 + \varepsilon)), \quad t \in (0, 1), \]

which on integration from \( t \) to 1, using the BCs (3.102), leads to
there exist subsequences
\[
\lim_{n \to \infty} y_n(t) = y(t),
\]
Further, which implies that
\[
x_n'(t) \leq k_1(R_4 + \varepsilon) \int_0^t p(s) \left( u_1(C_{n,k}^{\tilde{r} \tilde{r}} v_2^{\tilde{r} \tilde{r}}) + \int_s^t p(\tau) \Phi_{R_4+\varepsilon}(\tau)d\tau \right) + v_1(R_3 + \varepsilon)ds, \quad t \in [0, 1],
\]
and
\[
y_n'(t) \leq k_2(R_3 + \varepsilon) \int_0^t q(s) \left( u_2(C_{n,k}^{\tilde{r} \tilde{r}} v_1^{\tilde{r} \tilde{r}}) + \int_s^t q(\tau) \Psi_{R_3+\varepsilon}(\tau)d\tau \right) + v_2(R_4 + \varepsilon)ds, \quad t \in [0, 1],
\]
which implies that
\[
x_n'(t) \leq k_1(R_4 + \varepsilon) \int_0^1 p(s) \left( u_1(C_{n,k}^{\tilde{r} \tilde{r}} v_2^{\tilde{r} \tilde{r}}) + \int_s^t p(\tau) \Phi_{R_4+\varepsilon}(\tau)d\tau \right) + v_1(R_3 + \varepsilon)ds, \quad t \in [0, 1],
\]
and
\[
y_n'(t) \leq k_2(R_3 + \varepsilon) \int_0^1 q(s) \left( u_2(C_{n,k}^{\tilde{r} \tilde{r}} v_1^{\tilde{r} \tilde{r}}) + \int_s^t q(\tau) \Psi_{R_3+\varepsilon}(\tau)d\tau \right) + v_2(R_4 + \varepsilon)ds, \quad t \in [0, 1].
\]
(3.137)

In view of (3.131), (3.132), (3.137), (3.136), (B_1) and (B_13), the sequences \{\{x_n^{(j)}, y_n^{(j)}\}\} \quad (j = 0, 1) are uniformly bounded and equicontinuous on [0, 1]. Thus, by Theorem 1.1.3, there exist subsequences \{\{x_{n_{k},1}^{(j)}, y_{n_{k},1}^{(j)}\}\} \quad (j = 0, 1) of \{\{x_n^{(j)}, y_n^{(j)}\}\} and functions \(x_{0,1}, y_{0,1}\) \(\in C[0, 1] \times C[0, 1]\) such that \(x_{n_{k},1}^{(j)}, y_{n_{k},1}^{(j)}\) converges uniformly to \(x_{0,1}, y_{0,1}\) on [0, 1]. Also, \(a_1 x_{0,1}(0) - b_1 x_{0,1}(0) = a_2 y_{0,1}(0) - b_2 y_{0,1}(0) = x_{0,1}'(1) = y_{0,1}'(1) = 0\). Moreover, from (3.131) and (3.132), with \(n_k\) in place of \(n\) and taking \(\lim_{n_k \to +\infty}\), we have
\[
x_{0,1}'(t) \geq C_{k_1}^{\tilde{r} \tilde{r}} v_2^{\tilde{r} \tilde{r}} \int_t^1 p(s) \Phi_{R_4+\varepsilon}(s)ds,
\]
\[
y_{0,1}'(t) \geq C_{k_2}^{\tilde{r} \tilde{r}} v_1^{\tilde{r} \tilde{r}} \int_t^1 q(s) \Psi_{R_3+\varepsilon}(s)ds,
\]
which implies that \(x_{0,1} > 0\) and \(y_{0,1} > 0\) on [0, 1], \(x_{0,1} > 0\) and \(y_{0,1} > 0\) on [0, 1]. Further,
\[
\left| f(t, x_{n_k,1}(t)) + \frac{1}{n_k}(t + \frac{b_2}{a_2}) x_{n_k,1}'(t) + \frac{1}{n_k} \right| \leq p(t) k_1(R_4 + \varepsilon)(u_1(C_{n,k}^{\tilde{r} \tilde{r}} v_2^{\tilde{r} \tilde{r}}) + \int_t^1 p(s) \Phi_{R_4+\varepsilon}(s)ds) + v_1(R_3 + \varepsilon),
\]
\[
g(t, x_{n_k,1}(t)) + \frac{1}{n_k}(t + \frac{b_1}{a_1}) y_{n_k,1}'(t) + \frac{1}{n_k} \right| \leq q(t) k_2(R_3 + \varepsilon)(u_2(C_{n,k}^{\tilde{r} \tilde{r}} v_1^{\tilde{r} \tilde{r}}) + \int_t^1 q(s) \Psi_{R_3+\varepsilon}(s)ds) + v_2(R_4 + \varepsilon),
\]
(3.138)
Example 3.4.3

\[
\lim_{n_k \to \infty} f(t, y_{n_k}, 1(t) + \frac{1}{n_k} (t + \frac{b_2}{a_2}), x'_{n_k}, 1(t) + \frac{1}{n_k}) = f(t, y_{0,1}(t), x'_{0,1}(t)), \quad t \in (0, 1),
\]

\[
\lim_{n_k \to \infty} g(t, x_{n_k}, 1(t) + \frac{1}{n_k} (t + \frac{b_1}{a_1}), y'_{n_k}, 1(t) + \frac{1}{n_k}) = g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0, 1).
\]

Moreover, \((x_{n_k}, y_{n_k})\) satisfies

\[
x_{n_k}, 1(t) = \int_0^1 G_1(t, s) p(s) f(s, y_{n_k}, 1(s) + \frac{1}{n_k} (s + \frac{b_2}{a_2}), x'_{n_k}, 1(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],
\]

\[
y_{n_k}, 1(t) = \int_0^1 G_2(t, s) q(s) g(s, x_{n_k}, 1(s) + \frac{1}{n_k} (s + \frac{b_1}{a_1}), y'_{n_k}, 1(s) + \frac{1}{n_k}) ds, \quad t \in [0, 1],
\]

in view of (3.138), (B13), (3.139), the Lebesgue dominated convergence theorem and taking \(\lim_{n_k \to +\infty}\), we have

\[
x_{0,1}(t) = \int_0^1 G_1(t, s) p(s) f(s, y_{0,1}, 1(s), x'_{0,1}(s)) ds, \quad t \in [0, 1],
\]

\[
y_{0,1}(t) = \int_0^1 G_2(t, s) q(s) g(s, x_{0,1}, 1(s), y'_{0,1}(s)) ds, \quad t \in [0, 1],
\]

which implies that \((x_{0,1}, y_{0,1}) \in C^2(0, 1) \times C^2(0, 1)\) and

\[-x''_{0,1}(t) = p(t)f(t, x_{0,1}(t), x'_{0,1}(t)), \quad t \in (0, 1),
\]

\[-y''_{0,1}(t) = q(t)g(t, x_{0,1}(t), y'_{0,1}(t)), \quad t \in (0, 1).
\]

Moreover, by (3.98) and (3.99), we have \(\|x_{0,1}\|_3 < R_3\) and \(\|y_{0,1}\|_3 < R_4\), that is, \(\|(x_{0,1}, y_{0,1})\|_{L^3} < R_5\). By a similar proof the sequence \(\{(x_{n,2}, y_{n,2})\}\) has a convergent subsequence \(\{(x_{n,2}, y_{n,2})\}\) converging uniformly to \((x_{0,2}, y_{0,2}) \in C^1[0, 1] \times C^1[0, 1]\) on \([0, 1]\). Moreover, \((x_{0,2}, y_{0,2})\) is a solution to the system of SBVPs (5.5) with \(x_{0,2} > 0\) and \(y_{0,2} > 0\) on \([0, 1]\), \(x'_{0,2} > 0\) and \(y'_{0,2} > 0\) on \([0, 1]\), \(R_5 < \|(x_{0,2}, y_{0,2})\|_{L} < R^{**}\).

Example 3.4.3

Consider the following coupled system of SBVPs

\[
-x''(t) = \mu_1 (1 + (y(t))^{\delta_1} + (y(t))^{\eta_1})(1 + (x'(t))^{\alpha_1} + (x'(t))^{-\beta_1}), \quad t \in (0, 1),
\]

\[
y''(t) = \mu_2 (1 + (x(t))^{\delta_2} + (x(t))^{\eta_2})(1 + (y'(t))^{\alpha_2} + (y'(t))^{-\beta_2}), \quad t \in (0, 1),
\]

\[x(0) - x'(0) = y(0) - y'(0) = x'(1) = y'(1) = 0.
\]

where \(0 \leq \delta_i < 1, \eta_i > 1, 0 < \alpha_i < 1, 0 < \beta_i < 1,\) and \(\mu_i > 0, i = 1, 2,\).
Choose $p(t) = \mu_1$, $q(t) = \mu_2$, $k_i(x) = 1 + x^{\delta_i} + x^{\eta_i}$, $u_i(x) = x^{-\beta_i}$ and $v_i(x) = 1 + x^{\alpha_i}$, $i = 1, 2$. Assume that $\mu_1$ is arbitrary and

$$
\mu_2 < \min \left\{ \inf_{c \in (0, \infty)} \frac{J(2^{-1}(\mu_1^{-1}I(\xi_1)))}{k_2(c)} \right\},
$$

where $I(c) = \int_0^c x^{-1}x^{-1} \, dx$. We choose $\varphi_E(t) = \mu_1$, $\psi_E(t) = \mu_2$ and $h_i(x, y) = \mu_i(1 + x^{\eta_i})$, $i = 1, 2$. Then,

$$
\lim_{x \to +\infty} \frac{h_i(x, y)}{x} = \lim_{x \to +\infty} \frac{\mu_i(1 + x^{\eta_i})}{x} = +\infty, i = 1, 2.
$$

Moreover,

$$
\sup_{c \in (0, \infty)} \frac{c}{(1 + \frac{\mu_1}{\alpha_1})I^{-1}(k_1((1 + \frac{\mu_2}{\alpha_2})I^{-1}(k_2(c) f_0^1 q(t)dt) f_0^1 p(t)dt)} = \sup_{c \in (0, \infty)} \frac{c}{2I^{-1}(\mu_1 k_1(2I^{-1}(\mu_2 k_2(c)))}) \geq \frac{c}{2I^{-1}(\mu_1(1 + (2I^{-1}(\mu_2 k_2(c)))^{\delta_i} + (2I^{-1}(\mu_2 k_2(c)))^{\eta_i})}, c \in (0, \infty), c \geq 1,
$$

and

$$
\sup_{c \in (0, \infty)} \frac{c}{(1 + \frac{\mu_2}{\alpha_2})I^{-1}(k_2((1 + \frac{\mu_1}{\alpha_1})I^{-1}(k_1(c) f_0^1 p(t)dt) f_0^1 q(t)dt)} = \sup_{c \in (0, \infty)} \frac{c}{2I^{-1}(\mu_2 k_2(2I^{-1}(\mu_1 k_1(c)))}) \leq \frac{c}{2I^{-1}(\mu_2 k_2(2I^{-1}(\mu_1 k_1(c)))}, c \in (0, \infty)
$$

and

$$
> 1.
$$

Further,

$$
\int_0^1 p(t)u_1(C \int_t^1 p(s)\varphi_E(s)ds)dt = \mu_1^{-1-2\beta_1}C^{-\beta_1} \int_0^1 (1-t)^{-\beta_1} dt = \frac{\mu_1^{1-2\beta_1}C^{-\beta_1}}{1-\beta_1},
$$

$$
\int_0^1 q(t)u_2(C \int_t^1 q(s)\psi_E(s)ds)dt = \mu_2^{1-2\beta_2}C^{-\beta_2} \int_0^1 (1-t)^{-\beta_2} dt = \frac{\mu_2^{1-2\beta_2}C^{-\beta_2}}{1-\beta_2}.
$$

Clearly, $(B_1) - (B_3), (B_5), (B_8), (B_{10}), (B_{11})$ and $(B_{13})$ are satisfied. Hence, by Theorem 3.4.2, the system of BVPs 3.140 has at least two $C^1$-positive solutions.
3.5 System of ODEs with two-point coupled BCs

In this section, we establish existence of at least one $C^1$-positive solution for the system of BVPs (3.6). By a $C^1$-positive solution to the system of BVPs (3.6), we mean that $(x, y) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))$, $(x, y)$ satisfies (3.6), $x > 0$ and $y > 0$ on $[0, 1]$, $x' > 0$ and $y' > 0$ on $[0, 1)$.

Assume that

\[
\begin{align*}
(B_{14}) & \quad \sup_{c \in (0, \infty)} \frac{c}{(1 + \frac{b_2}{a_2})J^{-1}(h_1(c)k_1(c)\int_0^1 p(t)dt) + (1 + \frac{b_1}{a_1})J^{-1}(h_2(c)k_2(c)\int_0^1 q(t)dt)} > 1, \\
(B_{16}) & \quad \text{for real constants } M > 0 \text{ and } L > 0 \text{ there exist continuous functions } \varphi_{ML} \text{ and } \psi_{ML} \text{ defined on } [0, 1] \text{ and positive on } (0, 1), \text{ and constants } 0 \leq \gamma_1, \delta_1, \delta_2 < 1 \text{ satisfying } (1 - \gamma_1)(1 - \gamma_2) \neq \delta_1 \delta_2, \text{ such that } f(t, x, y, z) \geq \varphi_{ML}(t)x^{\gamma_1}y^{\delta_1} \text{ and } g(t, x, y, z) \geq \psi_{ML}(t)x^{\gamma_2}y^{\delta_2} \text{ on } [0, 1] \times [0, M] \times [0, M] \times [0, L]; \\
(B_{17}) & \quad \int_0^1 p(t)u_1(C, \int_1^s \delta p(s)\varphi_{ML}(s)ds)dt < +\infty \text{ and } \int_0^1 q(t)u_2(C, \int_1^s \delta q(s)\psi_{ML}(s)ds)dt < +\infty \text{ for any real constant } C > 0.
\end{align*}
\]

**Theorem 3.5.1** Under the hypothesis $(B_1) - (B_{17})$, the system of BVPs (3.6) has at least one $C^1$-positive solution.

**Proof.** In view of $(B_{14})$, we can choose real constant $M_5 > 0$ such that

\[
(1 + \frac{b_1}{a_1})J^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds) + (1 + \frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds) > 1.
\]

From the continuity of $I$ and $J$, we choose $\varepsilon > 0$ small enough such that

\[
(1 + \frac{b_1}{a_1})J^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(s)ds + I(\varepsilon)) + (1 + \frac{b_2}{a_2})J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(s)ds + J(\varepsilon)) > 1.
\]

Choose a real constant $L_5 > 0$ such that

\[
L_5 > \max\{J^{-1}(h_1(M_5)k_1(M_5)\int_0^1 p(t)dt + I(\varepsilon)), J^{-1}(h_2(M_5)k_2(M_5)\int_0^1 q(t)dt + J(\varepsilon))\}
\]

Choose $n_0 \in \{1, 2, \cdots\}$ such that $\frac{1}{n_0} < \varepsilon$. For each $n \in \{n_0, n_0 + 1, \cdots\}$, define retractions $\theta_n : \mathbb{R} \to [0, M_5]$ and $\rho_n : \mathbb{R} \to [\frac{1}{n_0}L_5]$ by
Since the system of BVPs (3.143) has a solution $x, y \in C^2[0, 1] \cap C^1(0, 1)$, consider the modified system of BVPs

\[
\begin{align*}
-x''(t) &= p(t)f(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(x'(t))), \quad t \in (0, 1), \\
-y''(t) &= q(t)g(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(y'(t))), \quad t \in (0, 1), \\
a_1y(0) - b_1x'(0) &= 0, \quad y'(1) = \frac{1}{n} \\
a_2x(0) - b_2y(0) &= 0, \quad x'(1) = \frac{1}{n},
\end{align*}
\tag{3.143}
\]

From (3.144) and (3.145), it follows that

\[
a_1y(0) - b_1x'(0) = 0, \quad y'(1) = \frac{1}{n}
\]

\[
a_2x(0) - b_2y(0) = 0, \quad x'(1) = \frac{1}{n},
\]

Since $f(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(x'(t)))$, $g(t, \theta_5(x(t)), \theta_5(y(t)), \rho_5(y'(t)))$ are continuous and bounded on $[0, 1] \times \mathbb{R}^3$, by Theorem 1.1.7, it follows that the modified system of BVPs (3.143) has a solution $(x_n, y_n) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1))$.

Using (3.143) and (B2), we obtain

\[
-x''_n(t) \leq 0 \quad \text{and} \quad y''_n(t) \leq 0 \quad \text{for} \quad t \in (0, 1),
\]

which on integration from $t$ to 1, using the BCs (3.143), implies that

\[
x'_n(t) \geq \frac{1}{n} \quad \text{and} \quad y'_n(t) \geq \frac{1}{n} \quad \text{for} \quad t \in [0, 1].
\tag{3.144}
\]

Integrating (3.144) from 0 to $t$, using the BCs (3.143) and (3.144), we have

\[
x_n(t) \geq (t + \frac{b_2}{a_2}) \frac{1}{n} \quad \text{and} \quad y_n(t) \geq (t + \frac{b_1}{a_1}) \frac{1}{n} \quad \text{for} \quad t \in [0, 1].
\tag{3.145}
\]

From (3.144) and (3.145), it follows that

\[
\|x_n\| = x_n(1) \quad \text{and} \quad \|y_n\| = y_n(1).
\tag{3.146}
\]

Now, we show that

\[
x'_n(t) < L_5, \quad y'_n(t) < L_5, \quad t \in [0, 1].
\tag{3.147}
\]

First, we prove $x'_n(t) < L_5$ for $t \in [0, 1]$. Suppose $x'_n(t_1) \geq L_5$ for some $t_1 \in [0, 1]$. Using (3.143) and (B3), we have

\[
-x''_n(t) \leq p(t)h_1(\theta_5(x_n(t)))k_1(\theta_5(y_n(t)))(u_1(\rho_5(x'_n(t))) + v_1(\rho_5(y'_n(t)))), \quad t \in (0, 1),
\]
which implies that
\[
-\frac{x_n''(t)}{u_1(p_s(x_n'(t))) + v_1(p_s(x_n'(t)))} \leq h_1(M_5)k_1(M_5) p(t), \quad t \in (0, 1).
\]

Integrating from \(t_1\) to 1, using the BCs (3.143), we obtain
\[
\int_{t_1}^{1} \frac{dz}{u_1(p_s(z)) + v_1(p_s(z))} \leq h_1(M_5)k_1(M_5) \int_{t_1}^{1} p(t) dt,
\]
which can also be written as
\[
\int_{L_5}^{1} \frac{dz}{u_1(z) + v_1(z)} + \int_{L_5}^{L} \frac{dz}{u_1(L_5) + v_1(L_5)} \leq h_1(M_5)k_1(M_5) \int_{0}^{1} p(t) dt.
\]

Using the increasing property of \(I\), we obtain
\[
I(L_5) + \frac{x_n'(t_1) - L_5}{u_1(L_5) + v_1(L_5)} \leq h_1(M_5)k_1(M_5) \int_{0}^{1} p(t) dt + I(\varepsilon),
\]
and using the increasing property of \(I^{-1}\), leads to
\[
L_5 \leq I^{-1}(h_1(M_5)k_1(M_5) \int_{0}^{1} p(t) dt + I(\varepsilon)).
\]

Which is a contradiction to (3.142). Hence, \(x_n'(t) < L_5\) for \(t \in [0, 1]\). Similarly, we can show that \(y_n'(t) < L_5\) for \(t \in [0, 1]\).

Now, we show that
\[
\|x_n\| + \|y_n\| < M_5.
\]  
Suppose \(\|x_n\| + \|y_n\| \geq M_5\). From (3.143), (3.144), (3.147) and (B3), it follows that
\[
-x_n''(t) \leq p(t)h_1(\theta_s(x_n(t)))k_1(\theta_s(y_n(t))) (u_1(x_n'(t)) + v_1(x_n'(t))), \quad t \in (0, 1),
\]
\[
-y_n''(t) \leq q(t)h_2(\theta_s(x_n(t)))k_2(\theta_s(y_n(t))) (u_2(y_n'(t)) + v_2(y_n'(t))), \quad t \in (0, 1),
\]
which implies that
\[
-\frac{x_n''(t)}{u_1(x_n'(t)) + v_1(x_n'(t))} \leq h_1(M_5)k_1(M_5) p(t), \quad t \in (0, 1),
\]
\[
-\frac{y_n''(t)}{u_2(y_n'(t)) + v_2(y_n'(t))} \leq h_2(M_5)k_2(M_5) q(t), \quad t \in (0, 1).
\]

Integrating from \(t\) to 1, using the BCs (3.143), we obtain
From (3.150) and (3.146), it follows that

which can also be written as

\[ I(x'_n(t)) - I(\frac{1}{n}) \leq h_1(M_5)k_1(M_5) \int_0^1 p(s)ds, \quad t \in [0, 1], \]
\[ J(y'_n(t)) - J(\frac{1}{n}) \leq h_2(M_5)k_2(M_5) \int_0^1 q(s)ds, \quad t \in [0, 1]. \]

The increasing property of \( I \) and \( J \) leads to

\[ x'_n(t) \leq I^{-1}(h_1(M_5)k_1(M_5) \int_0^1 p(s)ds + I(\varepsilon)), \quad t \in [0, 1], \]
\[ y'_n(t) \leq J^{-1}(h_2(M_5)k_2(M_5) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1]. \]  

(3.149)

Integrating from 0 to \( t \), using the BCs (3.143) and (3.149), we obtain

\[ x_n(t) \leq I^{-1}(h_1(M_5)k_1(M_5) \int_0^1 p(s)ds + I(\varepsilon)) + \frac{b_2}{a_2} J^{-1}(h_2(M_5)k_2(M_5) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1], \]
\[ y_n(t) \leq \frac{b_1}{a_1} I^{-1}(h_1(M_5)k_1(M_5) \int_0^1 p(s)ds + I(\varepsilon)) + J^{-1}(h_2(M_5)k_2(M_5) \int_0^1 q(s)ds + J(\varepsilon)), \quad t \in [0, 1]. \]  

(3.150)

From (3.150) and (3.146), it follows that

\[ M_5 \leq \|x_n\| + \|y_n\| \leq (1 + \frac{b_1}{a_1}) I^{-1}(h_1(M_5)k_1(M_5) \int_0^1 p(s)ds + I(\varepsilon)) \]
\[ + (1 + \frac{b_2}{a_2}) J^{-1}(h_2(M_5)k_2(M_5) \int_0^1 q(s)ds + J(\varepsilon)), \]

which implies that

\[ \frac{M_5}{(1 + \frac{b_1}{a_1}) I^{-1}(h_1(M_5)k_1(M_5) \int_0^1 p(s)ds + I(\varepsilon)) + (1 + \frac{b_2}{a_2}) J^{-1}(h_2(M_5)k_2(M_5) \int_0^1 q(s)ds + J(\varepsilon))} \leq 1, \]

a contradiction to (3.141). Hence, \( \|x_n\| + \|y_n\| < M_5 \).

Thus, in view of (3.143)-(3.148), \((x_n, y_n)\) is a solution of the following coupled system of BVPs
3.5 System of ODEs with two-point coupled BCs

\[ -x''(t) = p(t)f(t,x(t),y(t),x'(t)), \quad t \in (0, 1), \]
\[ -y''(t) = q(t)g(t,x(t),y(t),y'(t)), \quad t \in (0, 1), \]
\[ a_1y(0) - b_1x'(0) = 0, \quad x'(1) = \frac{1}{n}, \]
\[ a_2x(0) - b_2y'(0) = 0, \quad y'(1) = \frac{1}{n}, \]

satisfying

\[ (t + \frac{b_2}{a_2}) \frac{1}{n} \leq x_n(t) < M_S, \quad \frac{1}{n} \leq x_n'(t) < L_S, \quad t \in [0, 1], \]
\[ (t + \frac{b_1}{a_1}) \frac{1}{n} \leq y_n(t) < M_S, \quad \frac{1}{n} \leq y_n'(t) < L_S, \quad t \in [0, 1]. \]

We claim that

\[ x_n'(t) \geq C_9^9 C_{10}^5 \int_0^1 p(s)\varphi_{M_S L_S}(s)ds, \]
\[ y_n'(t) \geq C_9^9 C_{10}^5 \int_0^1 q(s)\psi_{M_S L_S}(s)ds, \]

where

\[ C_9 = \left( \frac{b_1}{a_1} \right) \frac{1}{n} \int_0^1 p(s)\varphi_{M_S L_S}(s)ds \left( \int_0^1 q(s)\psi_{M_S L_S}(s)ds \right)^{\frac{1}{1-\gamma_1}}, \]
\[ C_{10} = \left( \frac{b_1}{a_1} \right) \frac{1}{n} \int_0^1 p(s)\varphi_{M_S L_S}(s)ds \left( \int_0^1 q(s)\psi_{M_S L_S}(s)ds \right)^{\frac{1}{1-\gamma_2}}. \]

To prove (3.153), consider the following relation

\[ x_n(t) = (t + \frac{b_2}{a_2}) \frac{1}{n} + \int_0^t s p(s)f(s,x_n(s),y_n(s),x_n'(s))ds \]
\[ + \int_0^t t p(s)f(s,x_n(s),y_n(s),x_n'(s))ds + \frac{b_2}{a_2} \int_0^1 q(s)g(s,x_n(s),y_n(s),y_n'(s))ds, \]

which implies that

\[ x_n(0) = b_2 \frac{1}{a_2} \frac{1}{n} + \frac{b_2}{a_2} \int_0^1 q(s)g(s,x_n(s),y_n'(s))ds. \]

Using (B_16) and (3.152), we obtain

\[ x_n(0) \geq \frac{b_2}{a_2} \int_0^1 q(s)\psi_{M_S L_S}(s)(x_n(s))^{\gamma_1} ds \geq \int_0^1 (x_n(0))^{\gamma_1}(y_n(0))^{\delta_1} ds \geq \int_0^1 q(s)\psi_{M_S L_S}(s)ds, \]
which implies that
\[ x_n(0) \geq (y_n(0))^\frac{\delta_1}{1 - \eta} \left( \frac{b_2}{a_2} \int_0^1 q(s) \psi_{M_0L_0}(s) ds \right)^{\frac{1}{1 - \eta}}. \] (3.156)

Similarly, using (B_{16}) and (3.152), we obtain
\[ y_n(0) \geq (x_n(0))^\frac{\delta_1}{1 - \eta} \left( \frac{b_1}{a_1} \int_0^1 p(s) \psi_{M_0L_0}(s) ds \right)^{\frac{1}{1 - \eta}}. \] (3.157)

Now, using (3.157) in (3.156), we have
\[ x_n(0) \geq (x_n(0))^\frac{\delta_1}{1 - \eta} \left( \frac{b_1}{a_1} \int_0^1 p(s) \psi_{M_0L_0}(s) ds \right)^{\frac{1}{1 - \eta}} \left( \frac{b_2}{a_2} \int_0^1 q(s) \psi_{M_0L_0}(s) ds \right)^{\frac{1}{1 - \eta}}. \]

Hence,
\[ x_n(0) \geq C_9. \] (3.158)

Similarly, using (3.156) in (3.157), we obtain
\[ y_n(0) \geq C_{10}. \] (3.159)

Now, from (3.155), it follows that
\[ x_n'(t) \geq \int_t^1 p(s)f(s,x_n(s),y_n(s),x_n'(s))ds. \]

and using (B_{16}), (3.152), (3.158) and (3.159), we obtain (3.153). Similarly, we can prove (3.154).

Now, using (3.151), (B_{3}), (3.152), (3.153) and (3.154), we have
\[ 0 \leq -x_n''(t) \leq h_1(M_5)k_1(M_5)p(t)(u_1(C_0^{\frac{5}{3}} C_{10}^\frac{5}{3}) \int_0^1 p(s) \psi_{M_0L_0}(s) ds + v_1(L_5)), \quad t \in (0,1), \]
\[ 0 \leq -y_n''(t) \leq h_2(M_5)k_2(M_5)q(t)(u_2(C_0^{\frac{5}{3}} C_{10}^\frac{5}{3}) \int_0^1 q(s) \psi_{M_0L_0}(s) ds + v_2(L_5)), \quad t \in (0,1). \] (3.160)

In view of (3.152), (3.160), (B_{1}) and (B_{17}), it follows that the sequences \( \{(x_n^{(j),y_n^{(j)}})\} \) \( (j = 0,1) \) are uniformly bounded and equicontinuous on \([0,1] \). Hence, by Theorem (11.1.5), there exist subsequences \( \{(x_n^{(j),y_n^{(j)}})\} \) \( (j = 0,1) \) of \( \{(x_n^{(j),y_n^{(j)}})\} \) \( (j = 0,1) \) and \( (x,y) \in C^1[0,1] \times C^1[0,1] \) such that \( (x_n^{(j),y_n^{(j)}}) \) converges uniformly to \( (x^{(j),y^{(j)}}) \) on \([0,1] \) \( (j = 0,1) \). Also, \( a_2x(0) - b_2y'(0) = a_1y(0) - b_1x'(0) = x'(1) = y'(1) = 0. \)
Moreover, from (3.153) and (3.154), with \( n_k \) in place of \( n \) and taking \( \lim_{n_k \to +\infty} \), we have

\[
\begin{align*}
x'(t) & \geq C_0^\nu \int_0^1 p(s) \varphi_{M_k, L_k}(s) ds, \\
y'(t) & \geq C_0^\nu \int_0^1 q(s) \psi_{M_k, L_k}(s) ds,
\end{align*}
\]

which shows that \( x' > 0 \) and \( y' > 0 \) on \([0, 1]\), \( x > 0 \) and \( y > 0 \) on \([0, 1]\). Further, \((x_{n_k}, y_{n_k})\) satisfy

\[
\begin{align*}
x'_{n_k}(t) &= x'_{n_k}(0) - \int_0^t p(s)f(s, x_{n_k}(s), y_{n_k}(s), x'_{n_k}(s)) ds, \quad t \in [0, 1], \\
y'_{n_k}(t) &= y'_{n_k}(0) - \int_0^t q(s)g(s, x_{n_k}(s), y_{n_k}(s), y'_{n_k}(s)) ds, \quad t \in [0, 1].
\end{align*}
\]

Passing to the limit as \( n_k \to \infty \), we obtain

\[
\begin{align*}
x'(t) &= x'(0) - \int_0^t p(s)f(s, x(s), y(s), x'(s)) ds, \quad t \in [0, 1], \\
y'(t) &= y'(0) - \int_0^t q(s)g(s, x(s), y(s), y'(s)) ds, \quad t \in [0, 1],
\end{align*}
\]

which implies that

\[
\begin{align*}
-x''(t) &= p(t)f(t, x(t), y(t), x'(t)), \quad t \in (0, 1), \\
y''(t) &= q(t)g(t, x(t), y(t), y'(t)), \quad t \in (0, 1).
\end{align*}
\]

Hence, \((x, y)\) is a \( C^1 \)-positive solution of the system of BVPs (3.6).

**Example 3.5.2** Consider the following coupled system of SBVPs

\[
\begin{align*}
-x''(t) &= \nu^\beta_1 + (x(t))^{\gamma_1} (y(t))^{\delta_1} (x'(t))^{-\beta_1}, \quad t \in (0, 1), \\
y''(t) &= \nu^\beta_2 + (x(t))^{\gamma_2} (y(t))^{\delta_2} (y'(t))^{-\beta_2}, \quad t \in (0, 1), \\
x(0) - y'(0) &= y(0) - x'(0) = x'(1) = y'(1) = 0,
\end{align*}
\]

where \( 0 \leq \gamma_1, \gamma_2, \delta_1, \delta_2 < 1 \) satisfying \((1 - \gamma_1)(1 - \gamma_2) \neq \delta_1 \delta_2\), \( 0 < \beta_1 < 1, 0 < \beta_2 < 1 \) and \( \nu > 0 \) such that

\[
\nu < \sup_{c \in (0, 1)} \frac{c}{2 \sum_{i=1}^2 (\beta_i + 1)^{\frac{\gamma_i + \delta_i}{\beta_i + 1}}},
\]
Choose \( p(t) = q(t) = 1, h_1(x) = x^\beta_1, h_2(x) = x^\beta_2, k_1(x) = x^\delta_1, k_2(x) = x^\delta_2, u_1(x) = x^{-\beta_1}, u_2(x) = x^{-\beta_2} \) and \( v_1(x) = v_2(x) = 0 \). Assume that the following holds:

\[
L\left(\varphi_{ML}(t) = L^{-\beta_1}, \psi_{ML}(t) = L^{-\beta_2}\right).
\]

Then, \( I(v) = \frac{\nu_1}{\nu_2} I(v) + \frac{\nu_2}{\nu_1} I^{-1}(v) = (\beta_1 + 1) \frac{\nu_1}{\nu_2} \psi_{ML}(t) + \frac{\nu_2}{\nu_1} I^{-1}(v) = (\beta_2 + 1) \frac{\nu_1}{\nu_2} \psi_{ML}(t) + \frac{\nu_2}{\nu_1} I^{-1}(v).
\]

Also,

\[
\sup_{c \in (0, \infty)} \frac{c}{\nu_1 + 1} I^{-1}(h_1(c)k_1(c) I_0^1 p(t) dt) + (1 + \frac{\nu_2}{\nu_1}) I^{-2}(h_2(c)k_2(c) I_0^1 q(t) dt)\geq 1.
\]

Clearly, \( (B_1) - (B_{17}) \) are satisfied. Hence, by Theorem 3.5.1, the system of BVPs (3.16) has at least one \( C^1 \)-positive solution.

### 3.6 Existence of at least one \( C^1 \)-positive solution via lower and upper solutions

In this section, we establish existence of at least one \( C^1 \)-positive solution of the system of BVPs (3.3). By a \( C^1 \)-positive solution to the system of BVPs (3.3), we mean \((x, y) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1)) \) satisfying (3.3), \( x > 0 \) and \( y > 0 \) on \((0, 1)\).

Let \( \rho_n \to 0 \) be a nonincreasing sequence of real constants such that \( \lim_{n \to +\infty} \rho_n = 0 \). Assume that the following holds:

\( (B_{18}) \) \( p_i \in C(0, 1), p_i > 0 \) on \((0, 1)\) and \( \int_0^1 p_i(t) dt < +\infty, i = 1, 2; \)

\( (B_{19}) \) \( f_i : [0, 1] \times (0, \infty) \times (0, \infty) \to \mathbb{R} \) are continuous, \( i = 1, 2; \)

\( (B_{20}) \) there exist \( (\beta_1, \beta_2) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1)) \) and \( \rho_0 \in \{1, 2, \cdots\} \) such that \( \beta_i(t) \geq \rho_0 \) for \( t \in [0, 1] \) and

\[
-\beta_i'(t) \geq p_i(t)f_i(t, \beta_i(t), \beta_{i+1}(t)), \quad t \in (0, 1),
\]

\[
-\beta_i'(t) \geq p_{i+1}(t)f_{i+1}(t, \beta_i(t), \beta_{i+1}(t)), \quad t \in (0, 1);\]

\( (B_{21}) \) there exist \((\alpha_1, \alpha_2) \in (C^1[0, 1] \cap C^2(0, 1)) \times (C^1[0, 1] \cap C^2(0, 1)) \) with \( \alpha_1(0) = \alpha_2(0) = \alpha_2(0) = \alpha_2(1) = 0, \alpha_0 > 0 \) and \( \alpha_2 > 0 \) on \((0, 1)\) such that for \((t, x, y) \in (0, 1) \times \{x \in (0, \infty) : x < \alpha_1(t)\} \times \{y \in (0, \infty) : y \leq \beta_2(t)\}\),
3.6 Existence of at least one $C^1$-positive solution via lower and upper solutions

\[-\alpha_i''(t) < p_i(t)f_i(t,x,y,\alpha_i'(t)),\]

for $(t,x,y) \in (0,1) \times \{x \in (0,\infty) : x \leq \beta_1(t)\} \times \{y \in (0,\infty) : y < \alpha_2(t)\}$,

\[-\alpha_2''(t) < p_2(t)f_2(t,x,y,\alpha_2'(t));\]

\((B_{22}) \) for each $n \in \{n_0, n_0 + 1, \cdots\}$, $0 \leq t \leq 1$, $\rho_n \leq x \leq \beta_1(t)$, $\rho_n \leq y \leq \beta_2(t)$, we have $f_1(t,\rho_n,y,0) \geq 0$ and $f_2(t,x,\rho_n,0) \geq 0$;

\((B_{23}) \) $|f_i(t,x,y,z)| \leq (h_i(x) + k_i(x))(u_i(y) + v_i(y))\psi_i(|z|)$, where $h_i, u_i > 0$ are continuous and nonincreasing on $(0,\infty)$, $k_i, v_i \geq 0$, $\psi_i > 0$ are continuous on $[0,\infty)$ with $\frac{h_i}{u_i}$ nondecreasing on $(0,\infty)$, $i = 1, 2$;

\((B_{24}) \) $\int_0^1 p_i(t)h_i(\alpha_1(t))u_i(\alpha_2(t))dt < +\infty$, $i = 1, 2$;

\((B_{25}) \)

$$\int_0^\infty \frac{du}{\psi_i(u)} > \left[1 + \frac{k_i(b_1)}{h_i(b_1)}\right]\left[1 + \frac{v_i(b_2)}{u_i(b_2)}\right]\int_0^1 p_i(t)h_i(\alpha_1(t))u_i(\alpha_2(t))dt,$$

where $b_i = \max\{\beta_i(t) : t \in [0,1]\}$, $i = 1, 2$.

**Theorem 3.6.1** Assume that $(B_{18}) - (B_{25})$ hold. Then the system of BVPs (3.5) has a $C^1$-positive solution.
Chapter 4

Singular Systems of BVPs on Infinite Intervals

Recently, the theory on existence of solutions to nonlinear BVPs on unbounded domain has attracted the attention of many authors, see for example [5 36 58 68 74 50 100] and the references therein. For BVPs defined on half-line, an excellent resource is produced by Agarwal and O’Regan [5] that have been received considerable attentions.

Agarwal and O’Regan [3, Section 2.15] studied the existence of positive solutions to the following BVP

\[-x''(t) = \phi(t)f(t,x(t)), \quad t \in (0,\infty),\]
\[x(0) = 0, \lim_{t \to \infty} x'(t) = 0,\]  
(4.1)

where \(f(t,x)\) is singular at \(x = 0\). Further, in [5, Section 1.11] they establish the existence results for (4.1) when \(f\) includes first derivative also. However in [97, 27], it was assumed that the nonlinearities are positive which leads to concave solutions.

In Sections 4.1 and 4.2 we study the existence of \(C^1\)-positive solutions to the following coupled systems of SBVPs

\[-x''(t) = p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in \mathbb{R}_0^+,\]
\[-y''(t) = p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in \mathbb{R}_0^+,\]
\[x(0) = y(0) = \lim_{t \to \infty} y'(t) = \lim_{t \to \infty} x'(t) = 0\]  
(4.2)

and
\[-x''(t) = p_1(t) f_1(t, x(t), y(t), x'(t)), \quad t \in \mathbb{R}_0^+, \]
\[-y''(t) = p_2(t) f_2(t, x(t), y(t), y'(t)), \quad t \in \mathbb{R}_0^+, \]

\[a_1 x(0) - b_1 x'(0) = \lim_{t \to +\infty} x'(t) = 0, \]
\[a_2 y(0) - b_2 y'(0) = \lim_{t \to +\infty} y'(t) = 0, \tag{4.3}\]

where the functions \(f_i : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0 \to \mathbb{R}\) are continuous and allowed to change sign. Further, the nonlinearities \(f_i (i = 1, 2)\) are allowed to be singular at \(x' = 0\) and \(y' = 0\). Also, \(p_i \in C(\mathbb{R}_0^+), p_i (i = 1, 2) > 0\) on \(\mathbb{R}_0^+\) and the constants \(a_i, b_i (i = 1, 2) > 0\); here \(\mathbb{R} = (-\infty, \infty), \mathbb{R}_0 = \mathbb{R} \setminus \{0\}, \mathbb{R}^+ = [0, \infty), \mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}\).

### 4.1 Systems of BVPs on infinite intervals

In this section, we establish the existence of \(C^1\)-positive solutions for the system of BVPs (4.2). We say, \((x, y) \in (C^1(\mathbb{R}_0^+)) \times (C^2(\mathbb{R}_0^+))\) is a \(C^1\)-positive solution of the system of BVPs (4.2), if \((x, y)\) satisfies (4.2), \(x > 0\) and \(y > 0\) on \(\mathbb{R}_0^+\), \(x' > 0\) and \(y' > 0\) on \(\mathbb{R}^+\).

Assume that

\((C_1)\) \[p_i \in C(\mathbb{R}_0^+), p_i (i = 1, 2) > 0\) on \(\mathbb{R}_0^+\), \(\int_0^{+\infty} p_i(t)dt < +\infty, i = 1, 2;\)

\((C_2)\) \(f_i : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}_0 \to \mathbb{R}\) is continuous, \(i = 1, 2;\)

\((C_3)\) \[|f_i(t, x, y, z)| \leq h_i(|x|) k_i(|y|) (u_i(|z|) + v_i(|z|)), \text{ where } u_i > 0\text{ is continuous and nonincreasing on } \mathbb{R}_0^+, h_i, k_i, v_i \geq 0\text{ are continuous and nondecreasing on } \mathbb{R}^+, i = 1, 2;\]

\((C_4)\) there exist a constant \(M > 0\) such that \(\frac{M}{\omega(M)} > 1\), where \(\omega(M) = \lim_{\varepsilon \to 0} \omega_\varepsilon(M),\)

\[\omega_\varepsilon(M) = \sum_{i=1}^{2} \int_0^{+\infty} \left[ J_i^{-1} (h_i(M) k_i(M) \int_0^{+\infty} p_i(s)ds + J_i(\varepsilon)) \right] dt + \sum_{i=1}^{2} J_i^{-1} (h_i(M) k_i(M) \int_0^{+\infty} p_i(s)ds + J_i(\varepsilon)), \]

\[J_i(\mu) = \int_0^{\mu} \frac{d\tau}{u_i(\tau) + v_i(\tau)}, \text{ for } \mu > 0, i = 1, 2;\]

\((C_5)\) \(J_1(\infty) = \infty\) and \(J_2(\infty) = \infty;\)

\((C_6)\) \(f_i\) is positive on \(\mathbb{R}^+ \times (0, M]^3, i = 1, 2;\)
(C7) there exist continuous functions \( \varphi_M \) and \( \psi_M \) defined on \( \mathbb{R}^+ \) and positive on \( \mathbb{R}^+_0 \), and constants \( 0 \leq \gamma_1, \gamma_2, \delta_1, \delta_2 < 1 \) satisfying \((1 - \gamma_1)(1 - \gamma_2) \neq \delta_1 \delta_2\), such that \( f_1(t,x,y,z) \geq \varphi_M(t)x^\gamma_1 y^{\delta_1} \) and \( f_2(t,x,y,z) \geq \psi_M(t)x^{\gamma_2} y^{\delta_2} \) on \( \mathbb{R}^+ \times [0,M]^3 \).

### 4.1.1 Existence of positive solutions on finite intervals

Choose \( m \in N_0 \setminus \{0\} \), where \( N_0 := \{0, 1, \cdots\} \), and consider the following system of BVPs on finite interval

\[
\begin{align*}
-x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\
y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in (0,m), \\
x(0) &= y(0) = x'(m) = y'(m) = 0.
\end{align*}
\]

First, we show that system of BVPs (4.4) has a \( C^1 \)-positive solution. We say, \((x,y) \in (C^1[0,m] \cap C^2(0,m)) \times (C^1[0,m] \cap C^2(0,m))\), a \( C^1 \)-positive solution of the system of BVPs (4.4), if \((x,y)\) satisfies (4.4), \( x > 0 \) and \( y > 0 \) on \((0,m), x' > 0 \) and \( y' > 0 \) on \([0,m)\).

**Theorem 4.1.1** Assume that (C1) – (C7) hold. Then the system of BVPs (4.4) has a \( C^1 \)-positive solution.

**Proof.** In view of (C4), we choose \( \varepsilon > 0 \) small enough such that

\[
\frac{M}{\omega_x(M)} > 1.
\]

Choose \( n_0 \in \{1, 2, \cdots\} \) such that \( \frac{1}{n_0} < \varepsilon \). For each \( n \in N := \{n_0, n_0 + 1, \cdots\} \), define retractions \( \theta: \mathbb{R} \to [0,M] \) and \( \rho: \mathbb{R} \to [\frac{1}{n},M] \) as

\[
\theta(x) = \max\{0, \min\{x,M\}\} \text{ and } \rho(x) = \max\{\frac{1}{n}, \min\{x,M\}\}.
\]

Consider the modified system of BVPs

\[
\begin{align*}
-x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\
y''(t) &= p_2(t)f_2(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\
x(0) &= y(0) = 0, x'(m) = y'(m) = \frac{1}{n}.
\end{align*}
\]
Integrating from \( t = 0 \) to \( t = m \), the modified system of BVPs (4.6) has a solution \( x_{m,n}(t) \) and \( y_{m,n}(t) \). Clearly, \( x_{m,n}(t) \) and \( y_{m,n}(t) \) are continuous and bounded on \([0, m]\). Hence, by Theorem\( \ref{thm:existence} \), the modified system of BVPs (4.6) has a solution \((x_{m,n}, y_{m,n}) \in (C[0, m] \cap C^2(0, m)) \times (C^1[0, m] \cap C^2(0, m))\).

Using (4.6), (C1) and (C6), we obtain
\[
x''_{m,n} \leq 0 \quad \text{and} \quad y''_{m,n} \leq 0 \quad \text{on} \quad (0, m).
\]

Integrating from \( t = 0 \) to \( t = m \) and using the BCs (4.6), we obtain
\[
x'_{m,n}(t) \geq \frac{1}{n} \quad \text{and} \quad y'_{m,n}(t) \geq \frac{1}{n} \quad \text{for} \quad t \in [0, m]. \quad (4.7)
\]

Integrating (4.7) from \( t = 0 \) to \( t = m \), using the BCs (4.6), we have
\[
x_{m,n}(t) \geq \frac{t}{n} \quad \text{and} \quad y_{m,n}(t) \geq \frac{t}{n} \quad \text{for} \quad t \in [0, m]. \quad (4.8)
\]

From (4.7) and (4.8), it follows that
\[
\|x_{m,n}\|_{\gamma} = x_{m,n}(m) \quad \text{and} \quad \|y_{m,n}\|_{\gamma} = y_{m,n}(m), \quad \text{where} \quad \|u\|_{\gamma} = \max_{t \in [0, m]} |u(t)|.
\]

Now, we show that the following hold
\[
\|x'_{m,n}\|_{\gamma} < M \quad \text{and} \quad \|y'_{m,n}\|_{\gamma} < M. \quad (4.9)
\]

Suppose \( x'_{m,n}(t_1) \geq M \) for some \( t_1 \in [0, m] \). Using (4.6) and (C3), we have
\[
-x''_{m,n}(t) \leq p_1(t) h_1(\theta(x_{m,n}(t))) k_1(\theta(y_{m,n}(t))) (u_1(\rho(x'_{m,n}(t))) + v_1(\rho(y'_{m,n}(t)))) \quad \text{for} \quad t \in (0, m),
\]

which implies that
\[
-x''_{m,n}(t) u_1(\rho(x'_{m,n}(t))) + v_1(\rho(y'_{m,n}(t))) \leq h_1(M) k_1(M) p_1(t), \quad t \in (0, m).
\]

Integrating from \( t_1 \) to \( t = m \), using the BCs (4.6), we obtain
\[
\int_{t_1}^{m} \frac{dz}{u_1(\rho(z)) + v_1(\rho(z))} \leq h_1(M) k_1(M) \int_{t_1}^{m} p_1(t) dt,
\]

which can also be written as
\[
\int_{\frac{M}{2}}^{M} \frac{dz}{u_1(z) + v_1(z)} + \int_{M}^{\infty} \frac{dz}{u_1(M) + v_1(M)} \leq h_1(M) k_1(M) \int_{0}^{\infty} p_1(t) dt.
\]
Using the increasing property of \( J_1 \), we obtain
\[
J_1(M) + \frac{x'_{m,n}(t_1) - M}{u_1(M) + v_1(M)} \leq h_1(M)k_1(M) \int_0^\infty p_1(t)dt + J_1(\varepsilon),
\]
and the increasing property of \( J_1^{-1} \) yields
\[
M \leq J_1^{-1}(h_1(M)k_1(M) \int_0^\infty p_1(t)dt + J_1(\varepsilon)) \leq \omega(\varepsilon),
\]
a contradiction to (4.5). Hence, \( \|x'_{m,n}\|_{\gamma_m} < M \).

Similarly, we can show that \( \|y'_{m,n}\|_{\gamma_m} < M \).

Now, we show that
\[
\|x_{m,n}\|_{\gamma_m} < M \quad \text{and} \quad \|y_{m,n}\|_{\gamma_m} < M. \tag{4.10}
\]

Suppose \( \|x_{m,n}\|_{\gamma_m} \geq M \). From (4.6), (4.7), (4.9) and (C3), it follows that
\[
-x''_{m,n}(t) \leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))),
\]
which implies that
\[
\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0, m).
\]
Integrating from \( t \) to \( m \), using the BCs (4.6), we obtain
\[
\int_0^m \frac{x'_{m,n}(t)}{u_1(z) + v_1(z)} \leq h_1(M)k_1(M) \int_t^m p_1(s)ds, \quad t \in [0, m],
\]
which can also be written as
\[
J_1(x'_{m,n}(t)) - J_1(\frac{1}{n}) \leq h_1(M)k_1(M) \int_t^\infty p_1(s)ds, \quad t \in [0, m].
\]
The increasing property of \( J_1 \) and \( J_1^{-1} \), leads to
\[
x'_{m,n}(t) \leq J_1^{-1}(h_1(M)k_1(M) \int_t^\infty p_1(s)ds + J_1(\varepsilon)), \quad t \in [0, m].
\]
Now, integrating from 0 to \( m \), using the BCs (4.6), we obtain
\[
M \leq \|x_{m,n}\|_{\gamma_m} \leq \int_0^m [J_1^{-1}(h_1(M)k_1(M) \int_t^\infty p_1(s)ds + J_1(\varepsilon))]dt,
\]
which implies that
satisfying

\[ M \leq \int_0^\infty [T^{-1}(h_1(M)k_1(M) \int_0^\infty p_1(s)ds + J_1(\varepsilon))] dt \leq \omega_k(M), \]

a contradiction to (4.3). Therefore, \( \|x_{m,n}\|_{\gamma_m} < M \).

Similarly, we can show that \( \|y_{m,n}\|_{\gamma_m} < M \).

Hence, in view of (4.6)-(4.10), \((x_{m,n}, y_{m,n})\) is a solution of the following coupled system of BVPs

\[
\begin{align*}
-x''(t) &= p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in (0, m), \\
y''(t) &= p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in (0, m), \\
x(0) &= y(0) = 0, x'(m) = y'(m) = \frac{1}{n},
\end{align*}
\]

satisfying

\[
\begin{align*}
\frac{1}{n} \leq x_{m,n}(t) < M, \quad \frac{1}{n} \leq y_{m,n}(t) < M, & \quad t \in [0, m], \\
\frac{1}{n} \leq x'_{m,n}(t) < M, \quad \frac{1}{n} \leq y'_{m,n}(t) < M, & \quad t \in [0, m].
\end{align*}
\]

Now, we show that

\[
\{x'_{m,n}\}_{n \in \mathbb{N}} \text{ and } \{y'_{m,n}\}_{n \in \mathbb{N}} \text{ are equicontinuous on } [0, m]. \tag{4.13}
\]

From (4.11), (4.12) and (C3), it follows that

\[
\begin{align*}
-x''_{m,n}(t) &\leq p_1(t)h_1(M)k_1(M)(u_1(x_{m,n}(t)) + v_1(x_{m,n}(t))), \quad t \in (0, m), \\
y''_{m,n}(t) &\leq p_2(t)h_2(M)k_2(M)(u_2(y_{m,n}(t)) + v_2(y_{m,n}(t))), \quad t \in (0, m),
\end{align*}
\]

which implies that

\[
\begin{align*}
\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x_{m,n}(t))} &\leq h_1(M)k_1(M)p_1(t), \quad t \in (0, m), \\
\frac{-y''_{m,n}(t)}{u_2(y'_{m,n}(t)) + v_2(y_{m,n}(t))} &\leq h_2(M)k_2(M)p_2(t), \quad t \in (0, m).
\end{align*}
\]

Thus for \( t_1, t_2 \in [0, m] \), we have

\[
\begin{align*}
|J_1(x'_{m,n}(t_1)) - J_1(x'_{m,n}(t_2))| &\leq h_1(M)k_1(M) \left| \int_{t_1}^{t_2} p_1(t) dt \right|, \\
|J_2(y'_{m,n}(t_1)) - J_2(y'_{m,n}(t_2))| &\leq h_2(M)k_2(M) \left| \int_{t_1}^{t_2} p_2(t) dt \right|. \tag{4.14}
\end{align*}
\]
In view of (4.14), uniform continuity of \( J_i^{-1} \) over \([0,J_i(M)] (i = 1,2)\) and

\[
|y_m(t) - y_n(t)| = |J_1^{-1}(J_1(y_m(t))) - J_1^{-1}(J_1(y_n(t)))|,
\]

we obtain (4.13).

From (4.12) and (4.13), it follows that the sequences \( \{(x_{m,n},y_{m,n})\}_{n,N} \) are uniformly bounded and equicontinuous on \([0,m]\). Hence, by Theorem (1.1.5), there exist subsequence \( N \) of \( N \) and \((x_m,y_m) \in C^1[0,m] \times C^1[0,m]\) such that for each \( j = 0,1 \), the sequences \( \{x_{m,n},y_{m,n}\} \) converges uniformly to \((x_m,y_m)\) on \([0,m]\) as \( n \to \infty \) through \( N \). From the BCs (4.11), we have \( x_m(0) = y_m(0) = x'_m(0) = y'_m(0) = 0 \). Next, we show that \( x_m > 0 \) and \( y_m > 0 \) on \([0,m]\), \( x'_m > 0 \) and \( y'_m > 0 \) on \([0,m]\).

We claim that

\[
x_{m,n}(t) \geq C_{11}^2 C_{12}^2 \int_0^{\min\{t,1\}} t^{1+\eta+\delta} p_1(t) \varphi_M(t) d\tau \equiv \Phi_M(t), \quad t \in [0,m], \tag{4.15}
\]

\[
y_{m,n}(t) \geq C_{11}^2 C_{12}^2 \int_0^{\min\{t,1\}} t^{1+\eta+\delta} p_2(t) \psi_M(t) d\tau \equiv \Psi_M(t), \quad t \in [0,m], \tag{4.16}
\]

\[
x'_{m,n}(t) \geq \int_t^m p_1(s) \varphi_M(s) (\Phi_M(s))^{\delta_1} \delta_1 ds, \quad t \in [0,m], \tag{4.17}
\]

\[
y'_{m,n}(t) \geq \int_t^m p_2(s) \psi_M(s) (\Psi_M(s))^{\delta_2} \delta_2 ds, \quad t \in [0,m], \tag{4.18}
\]

where

\[
C_{11} = \left( \int_0^1 t^{1+\eta+\delta} p_1(t) \varphi_M(t) d\tau \right)^{\frac{1}{\delta_1}} \left( \int_0^1 t^{1+\eta+\delta} p_2(t) \psi_M(t) d\tau \right)^{\frac{1}{\delta_2}},
\]

\[
C_{12} = \left( \int_0^1 t^{1+\eta+\delta} p_1(t) \varphi_M(t) d\tau \right)^{\frac{1}{\delta_1}} \left( \int_0^1 t^{1+\eta+\delta} p_2(t) \psi_M(t) d\tau \right)^{\frac{1}{\delta_2}}.
\]

First we prove (4.15). Let \( z(t) = x_m(t) - tx_m(1) \) for \( t \in [0,1] \). Then, \( z(0) = z(1) = 0 \), \( z''(t) \leq 0 \) for \( t \in [0,1] \). So, \( z(t) \geq 0 \) for \( t \in [0,1] \), that is

\[
x_{m,n}(t) \geq tx_{m,n}(1), \quad t \in [0,1]. \tag{4.19}
\]

Similarly,

\[
y_{m,n}(t) \geq ty_{m,n}(1), \quad t \in [0,1]. \tag{4.20}
\]
Now, consider the following relation

\[ x_{m,n}(t) = \frac{t}{n} + \int_0^t sp(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s))ds + \int_t^m tp(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s))ds, \quad t \in [0,m]. \] (4.21)

In view of (C7), using (4.19) and (4.20), for \( t \in [0,m] \), we have

\[ x_{m,n}(t) \geq \int_0^t sp(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s))ds \]
\[ \geq \int_0^{\min\{1,1\}} sp(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s))ds \]
\[ \geq \int_0^{\min\{1,1\}} sp(s)(x_{m,n}(s))^{\gamma_1}(y_{m,n}(s))^{\delta_1} \varphi_M(s)ds \]
\[ \geq (x_{m,n}(1))^{\gamma_1}(y_{m,n}(1))^{\delta_1} \int_0^{\min\{1,1\}} s^{1+\gamma_1+\delta_1} p_1(s)\varphi_M(s)ds, \]

which implies that

\[ x_{m,n}(1) \geq (y_{m,n}(1))^{\frac{\delta_1}{1-\gamma_1}} \left( \int_0^1 s^{1+\gamma_1+\delta_1} p_1(s)\varphi_M(s)ds \right)^{\frac{1}{1-\gamma_1}}. \] (4.23)

Similarly,

\[ y_{m,n}(1) \geq (x_{m,n}(1))^{\frac{\delta_2}{1-\gamma_2}} \left( \int_0^1 s^{1+\gamma_2+\delta_2} p_2(s)\psi_M(s)ds \right)^{\frac{1}{1-\gamma_2}}. \] (4.24)

Now, using (4.24) in (4.23), we have

\[ (x_{m,n}(1))^{1-\frac{\delta_1\delta_2}{(1-\gamma_1)(1-\gamma_2)}} \geq \left( \int_0^1 s^{1+\gamma_1+\delta_1} p_1(s)\varphi_M(s)ds \right)^{\frac{1}{1-\gamma_1}} \]
\[ \left( \int_0^1 s^{1+\gamma_2+\delta_2} p_2(s)\psi_M(s)ds \right)^{\frac{\delta_1}{1-\gamma_1\gamma_2}}. \]

Hence,

\[ x_{m,n}(1) \geq C_{11}. \] (4.25)

Similarly, using (4.23) in (4.24), we obtain

\[ y_{m,n}(1) \geq C_{12}. \] (4.26)

Thus, from (4.22), using (4.23) and (4.26), we get (4.15).

Similarly, we can prove (4.16).
Now, we prove (4.17). From (4.21), it follows that
\[ x'_{m,n}(t) \geq \int_0^m p_1(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s)) \, ds. \]

Using (C_7), (4.15) and (4.16), we obtain (4.17).

Similarly, we can prove (4.18).

From (4.15), (4.18), passing to the limit \( n \to \infty \) through \( N_{\varepsilon} \), we obtain
\[
\begin{align*}
x_m(t) & \geq \Phi_M(t), \quad y_m(t) \geq \Psi_M(t), \quad t \in [0,m], \\
x'_m(t) & \geq \int_0^m p_1(s)\Phi_M(s)(\Phi_M(s))^{\delta_1}(\Psi_M(s))^{\delta_2} \, ds, \quad t \in [0,m], \\
y'_m(t) & \geq \int_0^m p_2(s)\Psi_M(s)(\Phi_M(s))^{\delta_1}(\Psi_M(s))^{\delta_2} \, ds, \quad t \in [0,m].
\end{align*}
\]

Consequently, \( x_m > 0 \) and \( y_m > 0 \) on \((0, m)\), \( x'_m > 0 \) and \( y'_m > 0 \) on \([0, m)\).

Moreover, \((x_{m,n}, y_{m,n})\) satisfy
\[
\begin{align*}
x'_{m,n}(t) & = x'_{m,n}(0) - \int_0^t p_1(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x'_{m,n}(s)) \, ds, \quad t \in [0,m], \\
y'_{m,n}(t) & = y'_{m,n}(0) - \int_0^t p_2(s)f_2(s,x_{m,n}(s),y_{m,n}(s),y'_{m,n}(s)) \, ds, \quad t \in [0,m].
\end{align*}
\]

Letting \( n \to \infty \) through \( N_{\varepsilon} \), we obtain
\[
\begin{align*}
x_m(t) & = x'_m(0) - \int_0^t p_1(s)f_1(s,x_m(s),y_m(s),x'_m(s)) \, ds, \quad t \in [0,m], \\
y_m(t) & = y'_m(0) - \int_0^t p_2(s)f_2(s,x_m(s),y_m(s),y'_m(s)) \, ds, \quad t \in [0,m],
\end{align*}
\]

which imply that
\[
\begin{align*}
x''_m(t) & = p_1(t)f_1(t,x_m(t),y_m(t),x'_m(t)), \quad t \in (0,m), \\
y''_m(t) & = p_2(t)f_2(t,x_m(t),y_m(t),y'_m(t)), \quad t \in (0,m).
\end{align*}
\]

Hence, \((x_m, y_m)\) is a \( C^1 \)-positive solution of the system of BVPs (4.4).

### 4.1.2 Existence of positive solutions on an infinite interval

**Theorem 4.1.2** Assume that \((C_1) - (C_7)\) hold. Then the system of BVPs (4.2) has a \( C^1 \)-positive solution.
Clearly, we claim that (In view of (4.32), the system of BVPs (4.2) has a \( C^1 \)-positive solution \((x_m, y_m)\) defined on \([0, m]\). By applying diagonalization argument we will show that the system of BVPs (4.2) has a \( C^1 \)-positive solution. For this purpose we define a continuous extension \((\bar{x}_m, \bar{y}_m)\) of \((x_m, y_m)\) by

\[
\bar{x}_m(t) = \begin{cases} 
  x_m(t), & t \in [0, m], \\
  x_m(m), & t \in [m, \infty),
\end{cases} \\
\bar{y}_m(t) = \begin{cases} 
  y_m(t), & t \in [0, m], \\
  y_m(m), & t \in [m, \infty).
\end{cases}
\]

(4.29)

Clearly, \( \bar{x}_m, \bar{y}_m \in C^1[0, \infty) \) and satisfy

\[
0 \leq \bar{x}_m(t) < M, \quad 0 \leq \bar{y}_m(t) < M, \quad t \in [0, \infty),
\]

\[
0 \leq \bar{y}_m(t) < M, \quad 0 \leq \bar{y}_m(t) < M, \quad t \in [0, \infty).
\]

(4.30)

We claim that

\[
\{\bar{x}_m\}_{m \in N_0} \text{ and } \{\bar{y}_m\}_{m \in N_0} \text{ are equicontinuous on } [0, 1].
\]

(4.31)

Using (4.28), (4.29), (4.30) and (C3), we obtain

\[
-\bar{x}_m'(t) \leq p_1(t)h_1(M)k_1(M)\left(u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))\right), \quad t \in (0, 1),
\]

\[
-\bar{y}_m'(t) \leq p_2(t)h_2(M)k_2(M)\left(u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))\right), \quad t \in (0, 1),
\]

which implies that

\[
\frac{-\bar{x}_m'(t)}{u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))} \leq \frac{h_1(M)k_1(M)p_1(t)}{u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))}, \quad t \in (0, 1),
\]

\[
\frac{-\bar{y}_m'(t)}{u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))} \leq \frac{h_2(M)k_2(M)p_2(t)}{u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))}, \quad t \in (0, 1).
\]

Hence, for \( t_1, t_2 \in [0, 1] \), we have

\[
\begin{align*}
  |J_1(\bar{x}_m(t_1)) - J_1(\bar{x}_m(t_2))| & \leq h_1(M)k_1(M)\int_{t_1}^{t_2} p_1(t)dt, \\
  |J_2(\bar{y}_m(t_1)) - J_2(\bar{y}_m(t_2))| & \leq h_2(M)k_2(M)\int_{t_1}^{t_2} p_2(t)dt.
\end{align*}
\]

(4.32)

In view of (4.32), (C1), uniform continuity of \( J_i^{-1} \) over \([0, J_i(L)]\) \((i = 1, 2)\) and

\[
\begin{align*}
  |\bar{x}_m(t_1) - \bar{x}_m(t_2)| &= |J_1^{-1}(J_1(\bar{x}_m(t_1))) - J_1^{-1}(J_1(\bar{x}_m(t_2)))|, \\
  |\bar{y}_m(t_1) - \bar{y}_m(t_2)| &= |J_2^{-1}(J_2(\bar{y}_m(t_1))) - J_2^{-1}(J_2(\bar{y}_m(t_2)))|,
\end{align*}
\]

Proof. By Theorem 4.1.1, for each \( m \in N_0 \setminus \{0\} \), the system of BVPs (4.2) has a \( C^1 \)-positive solution \((x_m, y_m)\) defined on \([0, m]\). By applying diagonalization argument we will show that the system of BVPs (4.2) has a \( C^1 \)-positive solution. For this purpose we define a continuous extension \((\bar{x}_m, \bar{y}_m)\) of \((x_m, y_m)\) by

\[
\bar{x}_m(t) = \begin{cases} 
  x_m(t), & t \in [0, m], \\
  x_m(m), & t \in [m, \infty),
\end{cases} \\
\bar{y}_m(t) = \begin{cases} 
  y_m(t), & t \in [0, m], \\
  y_m(m), & t \in [m, \infty).
\end{cases}
\]

(4.29)

Clearly, \( \bar{x}_m, \bar{y}_m \in C^1[0, \infty) \) and satisfy

\[
0 \leq \bar{x}_m(t) < M, \quad 0 \leq \bar{y}_m(t) < M, \quad t \in [0, \infty),
\]

\[
0 \leq \bar{y}_m(t) < M, \quad 0 \leq \bar{y}_m(t) < M, \quad t \in [0, \infty).
\]

(4.30)

We claim that

\[
\{\bar{x}_m\}_{m \in N_0} \text{ and } \{\bar{y}_m\}_{m \in N_0} \text{ are equicontinuous on } [0, 1].
\]

(4.31)

Using (4.28), (4.29), (4.30) and (C3), we obtain

\[
-\bar{x}_m'(t) \leq p_1(t)h_1(M)k_1(M)\left(u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))\right), \quad t \in (0, 1),
\]

\[
-\bar{y}_m'(t) \leq p_2(t)h_2(M)k_2(M)\left(u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))\right), \quad t \in (0, 1),
\]

which implies that

\[
\frac{-\bar{x}_m'(t)}{u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))} \leq \frac{h_1(M)k_1(M)p_1(t)}{u_1(\bar{x}_m(t)) + v_1(\bar{x}_m(t))}, \quad t \in (0, 1),
\]

\[
\frac{-\bar{y}_m'(t)}{u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))} \leq \frac{h_2(M)k_2(M)p_2(t)}{u_2(\bar{y}_m(t)) + v_2(\bar{y}_m(t))}, \quad t \in (0, 1).
\]

Hence, for \( t_1, t_2 \in [0, 1] \), we have

\[
\begin{align*}
  |J_1(\bar{x}_m(t_1)) - J_1(\bar{x}_m(t_2))| & \leq h_1(M)k_1(M)\int_{t_1}^{t_2} p_1(t)dt, \\
  |J_2(\bar{y}_m(t_1)) - J_2(\bar{y}_m(t_2))| & \leq h_2(M)k_2(M)\int_{t_1}^{t_2} p_2(t)dt.
\end{align*}
\]

(4.32)

In view of (4.32), (C1), uniform continuity of \( J_i^{-1} \) over \([0, J_i(L)]\) \((i = 1, 2)\) and

\[
\begin{align*}
  |\bar{x}_m(t_1) - \bar{x}_m(t_2)| &= |J_1^{-1}(J_1(\bar{x}_m(t_1))) - J_1^{-1}(J_1(\bar{x}_m(t_2)))|, \\
  |\bar{y}_m(t_1) - \bar{y}_m(t_2)| &= |J_2^{-1}(J_2(\bar{y}_m(t_1))) - J_2^{-1}(J_2(\bar{y}_m(t_2)))|,
\end{align*}
\]
we establish (4.31).

From (4.30) and (4.31), it follows that the sequences \( \{ (x_m^{(j)}, y_m^{(j)}) \} \) \( (j = 0, 1) \) are uniformly bounded and equicontinuous on \([0, 1]\). Hence, by Theorem 1.1.5 there exist subsequence \( N_1 \) of \( N_0 \setminus \{ 0 \} \) and \((u_1, v_1) \in C^1[0, 1] \times C^1[0, 1]\) such that for each \( j = 0, 1 \), the sequence \( (x_m^{(j)}, y_m^{(j)}) \) converges uniformly to \((u_1^{(j)}, v_1^{(j)})\) on \([0, 1]\) as \( m \to \infty \) through \( N_1 \). Also from BCs (4.4), we have \( u_1(0) = v_1(0) = 0 \).

Moreover, from (4.27) and (4.29), for each \( m \in N_0 \setminus \{ 0 \} \), we have

\[
\begin{align*}
\underline{x}_m(t) &\geq \Phi_M(t), \quad \underline{y}_m(t) \geq \Psi_M(t), \\
\overline{x}_m(t) &\geq \int_t^1 p_1(s) \Phi_M(s) (\Phi_M(s))^{\gamma_1} (\Psi_M(s))^{\delta_1} ds, \\
\overline{y}_m(t) &\geq \int_t^1 p_2(s) \Psi_M(s) (\Phi_M(s))^{\gamma_2} (\Psi_M(s))^{\delta_2} ds,
\end{align*}
\]

Passing to the limit \( m \to \infty \) through \( N_1 \), we obtain

\[
\begin{align*}
u_1(t) &\geq \Phi_M(t), \quad v_1(t) \geq \Psi_M(t), \\
u_1'(t) &\geq \int_t^1 p_1(s) \Phi_M(s) (\Phi_M(s))^{\gamma_1} (\Psi_M(s))^{\delta_1} ds, \\
v_1'(t) &\geq \int_t^1 p_2(s) \Psi_M(s) (\Phi_M(s))^{\gamma_2} (\Psi_M(s))^{\delta_2} ds,
\end{align*}
\]

which shows that \( u_1 > 0 \) and \( v_1 > 0 \) on \([0, 1]\), \( u_1' > 0 \) and \( v_1' > 0 \) on \([0, 1]\).

By the same process as above, we can show that

\[
\{ \underline{x}_m \}_{m \in N_1 \setminus \{ 1 \}} \quad \text{and} \quad \{ \overline{x}_m \}_{m \in N_1 \setminus \{ 1 \}} \quad \text{are equicontinuous families on} \quad [0, 2]. \quad (4.33)
\]

Further, in view of (4.30) and (4.33), it follows that the sequences \( \{ (x_m^{(j)}, y_m^{(j)}) \} \) \( (j = 0, 1) \) are uniformly bounded and equicontinuous on \([0, 2]\). Hence, by Theorem 1.1.5 there exist subsequence \( N_2 \) of \( N_1 \setminus \{ 1 \} \) and \((u_2, v_2) \in C^1[0, 2] \times C^1[0, 2]\) such that for each \( j = 0, 1 \), the sequence \( (x_m^{(j)}, y_m^{(j)}) \) converges uniformly to \((u_2^{(j)}, v_2^{(j)})\) on \([0, 2]\) as \( m \to \infty \) through \( N_2 \). Also from BCs (4.4), \( u_2(0) = v_2(0) = 0 \). Moreover, in view of (4.27) and (4.29), for each \( m \in N_1 \setminus \{ 1 \} \), we have
We can do this for each $\tau \in \mathbb{R}^+_0$ and $k \in N_0 \setminus \{0\}$ with $\tau \leq k$. Define $\tau_0(\tau) = u_k(\tau)$ and $\tau_0(\tau) = v_k(\tau)$. Then, $x$ and $y$ are well defined as $x(t) = u_k(t) > 0$ and $y(t) = v_k(t) > 0$ for $t \in [0, k]$. We can do this for each $\tau \in \mathbb{R}^+_0$. Thus, $(x, y) \in C^1(\mathbb{R}^+_0) \times C^1(\mathbb{R}^+_0)$ with $x > 0$ and $y > 0$ on $\mathbb{R}^+_0$, $x' > 0$ and $y' > 0$ on $\mathbb{R}^+$.

Now, we show that $(x, y)$ is a solution of system of BVPs (4.2). Choose a fixed $\tau \in \mathbb{R}^+$ and $k \in N_0 \setminus \{0\}$ such that $k \geq \tau$. Then, $(\tilde{x}_m(\tau), \tilde{y}_m(\tau))$ where $m \in N_k$, satisfy
4.1 Systems of BVPs on infinite intervals

\[ \begin{align*}
\varphi'_m(\tau) &= \varphi'_m(0) - \int_0^\tau p_1(s) f_1(s, \varphi_m(s), \varphi'_m(s)) ds, \\
\psi'_m(\tau) &= \psi'_m(0) - \int_0^\tau p_2(s) f_2(s, \psi_m(s), \psi'_m(s)) ds.
\end{align*} \]

Passing to the limit \( m \to \infty \) through \( N_k \), we obtain

\[ \begin{align*}
u'_k(\tau) &= u'_k(0) - \int_0^\tau p_1(s) f_1(s, u_k(s), v_k(s), u'_k(s)) ds, \\
v'_k(\tau) &= v'_k(0) - \int_0^\tau p_2(s) f_2(s, u_k(s), v_k(s), v'_k(s)) ds.
\end{align*} \]

Hence,

\[ \begin{align*}
x'(\tau) &= x'(0) - \int_0^\tau p_1(s) f_1(s, x(s), y(s), x'(s)) ds, \\
y'(\tau) &= y'(0) - \int_0^\tau p_2(s) f_2(s, x(s), y(s), y'(s)) ds,
\end{align*} \]

which implies that

\[ \begin{align*}
x''(\tau) &= p_1(\tau) f_1(\tau, x(\tau), y(\tau), x'(\tau)), \\
y''(\tau) &= p_2(\tau) f_2(\tau, x(\tau), y(\tau), y'(\tau)).
\end{align*} \]

We can do this for each \( \tau \in \mathbb{R}^+ \). Consequently,

\[ \begin{align*}
x''(t) &= p_1(t) f_1(t, x(t), y(t), x'(t)), & \tau \in \mathbb{R}^+_0, \\
y''(t) &= p_2(t) f_2(t, x(t), y(t), y'(t)), & \tau \in \mathbb{R}^+_0.
\end{align*} \]

Thus, \((x,y) \in C^2(\mathbb{R}^+_0) \times C^2(\mathbb{R}^+_0), x(0) = y(0) = 0\).

It remains to show that

\[ \lim_{t \to \infty} x'(t) = \lim_{t \to \infty} y'(t) = 0. \]

First, we show that \( \lim_{t \to \infty} x'(t) = 0 \). Suppose \( \lim_{t \to \infty} x'(t) = \varepsilon_0 \), for some \( \varepsilon_0 > 0 \). Then, \( x'(t) \geq \varepsilon_0 \) for all \( t \in [0, \infty) \). Choose \( k \in N_0 \setminus \{0\} \), then for \( m \in N_k \), in view of \( (4.29) \), we have

\[ x'(t) = u'_k(t) = \lim_{m \to \infty} \varphi'_m(t) = \lim_{m \to \infty} \varphi'_m(t), \quad t \in [0,k], \]

which leads to
\[ x'(k) = \lim_{m \to \infty} x'_m(k). \]

Thus for every \( \varepsilon > 0 \), there exist \( m^* \in N_k \) such that \( |x'_m(k) - x'(k)| < \varepsilon \) for all \( m \geq m^* \). Without loss of generality assume that \( m^* = k \), then \( |x'_k(k) - x'(k)| < \varepsilon \), that is, \( |x'(k)| < \varepsilon \). This is a contradiction whenever \( \varepsilon = \varepsilon_0 \). Hence, \( \lim_{t \to +\infty} x'(t) = 0 \).

Similarly, we can prove \( \lim_{t \to -\infty} y'(t) = 0 \). Thus, \((x, y)\) is a \( C^1 \)-positive solution of the system of BVPs (4.2).

**Example 4.1.3** Let

\[ f_i(t, x, y, z) = \nu^{\alpha_i+1}e^{-t}(M + 1 - x)(M + 1 - y)|x|^{\beta_i} |y|^{\gamma_i} |z|^{-\alpha_i}, i = 1, 2, \]

where \( \nu > 0 \), \( M > 0 \), \( \alpha_i > 0 \), \( 0 \leq \gamma_i, \delta_i < 1 \), \( i = 1, 2 \).

Assume that \((1 - \gamma_1)(1 - \gamma_2) \neq \delta_1 \delta_2 \) and

\[ \nu < \frac{M}{\sum_{i=1}^{2} (\alpha_i + 2)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}} (2M + 1)^{\frac{2}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}}. \]

Taking \( p_i(t) = e^{-t} \), \( h_i(x) = \nu^{\alpha_i+1}(M + 1 + x) x^{\gamma_i} \), \( k_i(y) = (M + 1 + y) y^{\delta_i} \), \( u_i(z) = z^{-\alpha_i} \)
and \( v_i(z) = 0 \), \( i = 1, 2 \). Choose \( \varphi_M(t) = \nu^{\alpha_1+1}M^{-\alpha_1}e^{-t} \) and \( \psi_M(t) = \nu^{\alpha_2+1}M^{-\alpha_2}e^{-t} \).

Then, \( J_i(\mu) = \mu^{\alpha_i+1} \alpha_i^{1+\frac{1}{\alpha_i+1}} \) and \( J_i^{-1}(\mu) = (\alpha_i + 1)^{\frac{1}{\alpha_i+1}} \mu^{\frac{1}{\alpha_i+1}}, i = 1, 2 \).

Also,

\[ \frac{M}{\omega(M)} = \frac{M}{\sum_{i=1}^{2} \int_{0}^{1} J_i^{-1}(h_i(M)k_i(M) J_i p_i(s) ds) \mu^{\frac{1}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}} = \frac{M}{\sum_{i=1}^{2} \int_{0}^{1} J_i^{-1}(\nu^{\alpha_i+1}(2M + 1)^{\frac{1}{\alpha_i+1}} e^{-t})^2 dt + \sum_{i=1}^{2} J_i^{-1}(\nu^{\alpha_i+1}(2M + 1)^{\frac{1}{\alpha_i+1}} e^{-t}) \delta_i M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}} = \frac{M}{\nu \sum_{i=1}^{2} (\alpha_i + 2)(\alpha_i + 1)^{\frac{1}{\alpha_i+1}} (2M + 1)^{\frac{2}{\alpha_i+1}} M^{\frac{\gamma_i+\delta_i}{\alpha_i+1}}} > 1. \]

Clearly, (C1) – (C7) are satisfied. Hence, by Theorem 4.1.2 the system of BVPs (4.2) has at least one \( C^1 \)-positive solution.
4.2 Systems of BVPs on infinite intervals with more general BCs

We say, $(x, y) \in (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)) \times (C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+))$ is a $C^1$-positive solution of the system of BVPs (4.3) if $(x, y)$ satisfies (4.3), $x > 0$, $y > 0$, $x' > 0$ and $y' > 0$.

Assume that

\[(C_8) \quad \text{there exist a constant } M > 0 \text{ such that } \frac{M}{\omega(M)} > 1, \text{ where } \omega(M) = \lim_{\varepsilon \to 0} \omega_\varepsilon(M),\]

\[
\omega_\varepsilon(M) = \sum_{i=1}^{2} \int_{0}^{\infty} \left[ J_{1}^{-1}(h_i(M)k_i(M)) \int_{t}^{\infty} p_i(s)ds + J_i(\varepsilon) \right] dt
+ \sum_{i=1}^{2} \left( 1 + \frac{b_i}{a_i} \right) J_{1}^{-1}(h_i(M)k_i(M)) \int_{0}^{\infty} p_i(s)ds + J_i(\varepsilon),
\]

\[J_i(\mu) = \int_{0}^{\mu} \frac{dz}{u_i(z) + v_i(z)}, \text{ for } \mu > 0, i = 1, 2.\]

4.2.1 Existence of positive solutions on finite intervals

Choose $m \in \mathbb{N}_0 \setminus \{0\}$, where $\mathbb{N}_0 := \{0, 1, \cdots\}$, and consider the system of BVPs on finite interval

\[-x''(t) = p_1(t)f_1(t, x(t), y(t), x'(t)), \quad t \in (0, m), \]
\[-y''(t) = p_2(t)f_2(t, x(t), y(t), y'(t)), \quad t \in (0, m), \]
\[a_1x(0) - b_1x'(0) = x'(m) = 0, \]
\[a_2y(0) - b_2y'(0) = y'(m) = 0.\]

First we show that the system of BVPs (4.34) has a $C^1$-positive solution. We say, $(x, y) \in (C^1[0, m] \cap C^2(0, m)) \times (C^1[0, m] \cap C^2(0, m))$, a $C^1$-positive solution of the system of BVPs (4.34), if $(x, y)$ satisfies (4.34), $x > 0$ and $y > 0$ on $[0, m]$, $x' > 0$ and $y' > 0$ on $[0, m]$.

**Theorem 4.2.1** Assume that $(C_1) - (C_3)$ and $(C_3) - (C_8)$ hold. Then the system of BVPs (4.34) has a $C^1$-positive solution.

**Proof.** In view of $(C_8)$, we choose $\varepsilon > 0$ small enough such that

\[
\frac{M}{\omega_\varepsilon(M)} > 1. \quad (4.35)
\]
Choose \( n_0 \in \{1, 2, \cdots \} \) such that \( \frac{1}{n_0} < \varepsilon \). For each \( n \in N := \{ n_0, n_0 + 1, \cdots \} \), define retractions \( \theta : \mathbb{R} \to [0, M] \) and \( \rho : \mathbb{R} \to \left[ \frac{1}{n}, M \right] \) as

\[
\theta(x) = \max \{0, \min \{x, M\}\} \quad \text{and} \quad \rho(x) = \max \left\{ \frac{1}{n}, \min \{x, M\} \right\}.
\]

Consider the modified system of BVPs

\[
\begin{align*}
-x''(t) &= p_1(t) f_1^*(t, x(t), y(t), x'(t)), & t \in (0, m), \\
-y''(t) &= p_2(t) f_2^*(t, x(t), y(t), x'(t)), & t \in (0, m), \\
a_1 x(0) - b_1 x'(0) &= 0, x'(m) = \frac{1}{n}, \\
a_2 y(0) - b_2 y'(0) &= 0, y'(m) = \frac{1}{n},
\end{align*}
\] (4.36)

where \( f_1^* (t, x, y, x') = f_1 (t, \theta(x), \theta(y), \rho(x')) \) and \( f_2^* (t, x, y, x') = f_2 (t, \theta(x), \theta(y), \rho(y')) \). Clearly, \( f_i^* (i = 1, 2) \) are continuous and bounded on \([0, m] \times \mathbb{R}^3\). Hence, by Theorem 1.1.7, the modified system of BVPs (4.36) has a solution \((x_{m,n}, y_{m,n}) \in (C^1[0, m] \cap C^2(0, m)) \times (C^1[0, m] \cap C^2(0, m))\).

Using (4.36), (C1) and (C5), we obtain

\[
x''_{m,n} \leq 0 \quad \text{and} \quad y''_{m,n} \leq 0 \quad \text{on} \quad (0, m).
\]

Integrating from \( t \) to \( m \) and using the BCs (4.36), we obtain

\[
x'_{m,n}(t) \geq \frac{1}{n} \quad \text{and} \quad y'_{m,n}(t) \geq \frac{1}{n} \quad \text{for} \quad t \in [0, m].
\] (4.37)

Integrating (4.37) from 0 to \( t \), using the BCs (4.36) and (4.37), we have

\[
x_{m,n}(t) \geq \left( t + \frac{b_1}{a_1} \right) \frac{1}{n} \quad \text{and} \quad y_{m,n}(t) \geq \left( t + \frac{b_2}{a_2} \right) \frac{1}{n} \quad \text{for} \quad t \in [0, m].
\] (4.38)

From (4.37) and (4.38), it follows that

\[
\|x_{m,n}\|_{\gamma, m} = x_{m,n}(m) \quad \text{and} \quad \|y_{m,n}\|_{\gamma, m} = y_{m,n}(m).
\]

Now, we show that the following hold

\[
\|x'_{m,n}\|_{\gamma, m} < M \quad \text{and} \quad \|y'_{m,n}\|_{\gamma, m} < M.
\] (4.39)

Suppose \( x'_{m,n}(t_1) \geq M \) for some \( t_1 \in [0, m] \). Using (4.36) and (C3), we have
\[-x''_{m,n}(t) \leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))) \quad \text{for } t \in (0,m),\]

which implies that
\[
\frac{-x''_{m,n}(t)}{u_1(\rho(x'_{m,n}(t))) + v_1(\rho(x'_{m,n}(t)))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0,m).\]

Integrating from \(t_1\) to \(m\), using the BCs (4.36), we obtain
\[
\int_{t_1}^{m} \frac{dz}{u_1(\rho(z)) + v_1(\rho(z))} \leq h_1(M)k_1(M) \int_{t_1}^{m} p_1(t)dt,
\]

which can also be written as
\[
\int_{t_1}^{M} \frac{dz}{u_1(z) + v_1(z)} + \int_{M}^{x'_{m,n}(t_1)} \frac{dz}{u_1(M) + v_1(M)} \leq h_1(M)k_1(M) \int_{0}^{\infty} p_1(t)dt.
\]

Using the increasing property of \(J_1\), we obtain
\[
J_1(M) + \frac{x'_{m,n}(t_1) - M}{u_1(M) + v_1(M)} \leq h_1(M)k_1(M) \int_{0}^{\infty} p_1(t)dt + J_1(\varepsilon),
\]

and the increasing property of \(J_1^{-1}\) yields
\[
M \leq J_1^{-1}(h_1(M)k_1(M) \int_{0}^{\infty} p_1(t)dt + J_1(\varepsilon)) \leq \omega_\varepsilon(M)
\]

a contradiction to (4.35). Hence, \(\|x'_{m,n}\|_{\gamma,m} < M\).

Similarly, we can show that \(\|y'_{m,n}\|_{\gamma,m} < M\).

Now, we show that
\[
\|x_{m,n}\|_{\gamma,m} < M \quad \text{and} \quad \|y_{m,n}\|_{\gamma,m} < M. \quad (4.40)
\]

Suppose \(\|x_{m,n}\|_{\gamma,m} \geq M\). From (4.36), (4.37), (4.39) and (C₃), it follows that
\[-x''_{m,n}(t) \leq p_1(t)h_1(\theta(x_{m,n}(t)))k_1(\theta(y_{m,n}(t)))(u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))) \quad \text{for } t \in (0,m),\]

which implies that
\[
\frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0,m).\]

Integrating from \(t\) to \(m\), using the BCs (4.36), we obtain
\[
\int_{1}^{x_{m,n}(t)} \frac{dz}{u_1(z) + v_1(z)} \leq h_1(M)k_1(M) \int_{t}^{m} p_1(s) ds, \quad t \in [0, m],
\]
which can also be written as
\[
J_1(x_{m,n}'(t)) - J_1(\frac{1}{n}) \leq h_1(M)k_1(M) \int_{t}^{m} p_1(s) ds, \quad t \in [0, m].
\]
The increasing property of \(J_1\) and \(J_1^{-1}\), leads to
\[
x_{m,n}'(t) \leq J_1^{-1}(h_1(M)k_1(M)) \int_{t}^{m} p_1(s) ds + J_1(\varepsilon), \quad t \in [0, m],
\]
Now, integrating from 0 to \(m\), using the BCs (4.36) and (4.41), we obtain
\[
M \leq \|x_{m,n}\|_{\gamma,m} \leq \int_{0}^{m} \left[ J_1^{-1}(h_1(M)k_1(M)) \int_{t}^{m} p_1(s) ds + J_1(\varepsilon) \right] dt
\]
\[
+ \frac{b_1}{a_1} J_1^{-1}(h_1(M)k_1(M)) \int_{0}^{m} p_1(s) ds + J_1(\varepsilon),
\]
which implies that
\[
M \leq \int_{0}^{m} \left[ J_1^{-1}(h_1(M)k_1(M)) \int_{t}^{m} p_1(s) ds + J_1(\varepsilon) \right] dt
\]
\[
+ \frac{b_1}{a_1} J_1^{-1}(h_1(M)k_1(M)) \int_{0}^{m} p_1(s) ds + J_1(\varepsilon) \leq \omega_\varepsilon(M),
\]
a contradiction to (4.35). Therefore, \(\|x_{m,n}\|_{\gamma,m} < M\).

Similarly, we can show that \(\|y_{m,n}\|_{\gamma,m} < M\).

Hence, in view of (4.36)-(4.41), \((x_{m,n}, y_{m,n})\) is a solution of the following coupled system of BVPs
\[
\begin{align*}
-x''(t) &= p_1(t)f_1(t,x(t),y(t),x'(t)), \quad t \in (0,m), \\
-y''(t) &= p_2(t)f_2(t,x(t),y(t),y'(t)), \quad t \in (0,m), \\
a_1x(0) - b_1x'(0) &= 0, \quad x'(m) = \frac{1}{n}, \\
a_2y(0) - b_2y'(0) &= 0, \quad y'(m) = \frac{1}{n},
\end{align*}
\]
satisfying
\[
\begin{align*}
(t + \frac{b_1}{a_1}) \frac{1}{n} &\leq x_{m,n}(t) < M, \quad \frac{1}{n} \leq x_{m,n}'(t) < M, \quad t \in [0, m], \\
(t + \frac{b_2}{a_2}) \frac{1}{n} &\leq y_{m,n}(t) < M, \quad \frac{1}{n} \leq y_{m,n}'(t) < M, \quad t \in [0, m].
\end{align*}
\]
Now, we show that

\[ \{x'_{m,n}\}_{n \in \mathbb{N}} \text{ and } \{y'_{m,n}\}_{n \in \mathbb{N}} \text{ are equicontinuous on } [0, m]. \]  \hfill (4.44)

From (4.42), (4.43) and \((C_3)\), it follows that

\[-x''_{m,n}(t) \leq p_1(t)h_1(M)k_1(M)(u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))), \quad t \in (0, m),\]
\[-y''_{m,n}(t) \leq p_2(t)h_2(M)k_2(M)(u_2(y'_{m,n}(t)) + v_2(y'_{m,n}(t))), \quad t \in (0, m),\]

which implies that

\[ \frac{-x''_{m,n}(t)}{u_1(x'_{m,n}(t)) + v_1(x'_{m,n}(t))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0, m),\]
\[ \frac{-y''_{m,n}(t)}{u_2(y'_{m,n}(t)) + v_2(y'_{m,n}(t))} \leq h_2(M)k_2(M)p_2(t), \quad t \in (0, m).\]

Thus for \(t_1, t_2 \in [0, m]\), we have

\[ |J_1(x'_{m,n}(t_1)) - J_1(x'_{m,n}(t_2))| \leq h_1(M)k_1(M) \left| \int_{t_1}^{t_2} p_1(t)dt \right|. \]  \hfill (4.45)
\[ |J_2(y'_{m,n}(t_1)) - J_2(y'_{m,n}(t_2))| \leq h_2(M)k_2(M) \left| \int_{t_1}^{t_2} p_2(t)dt \right|. \]

In view of (4.45), \((C_1)\), uniform continuity of \(J_i^{-1}\) over \([0, J_i(M)]\) \((i = 1, 2)\) and

\[ |x'_{m,n}(t_1) - x'_{m,n}(t_2)| = |J_1^{-1}(J_1(x'_{m,n}(t_1))) - J_1^{-1}(J_1(x'_{m,n}(t_2)))|,\]
\[ |y'_{m,n}(t_1) - y'_{m,n}(t_2)| = |J_2^{-1}(J_2(y'_{m,n}(t_1))) - J_2^{-1}(J_2(y'_{m,n}(t_2)))|,\]

we obtain (4.44).

From (4.42) and (4.44), it follows that the sequences \(\{(x_{m,n}^{(j)}, y_{m,n}^{(j)})\}_{n \in \mathbb{N}} \quad (j = 0, 1)\) are uniformly bounded and equicontinuous on \([0, m]\). Hence, by Theorem 4.1.5, there exist subsequence \(N_s\) of \(N\) and \((x_m, y_m) \in C^1[0, m] \times C^1[0, m]\) such that for each \(j = 0, 1\) the sequence \(\{(x_{m,n}^{(j)}, y_{m,n}^{(j)})\}\) converges uniformly to \((x_m^{(j)}, y_m^{(j)})\) on \([0, m]\) as \(n \to \infty\) through \(N_s\). From the BCs (4.42), we have \(a_1x_m(0) - b_1x_m'(0) = a_2y_m(0) - b_2y_m'(0) = x'_m(m) = y'_m(m) = 0\). Next, we show that \(x_m > 0\) and \(y_m > 0\) on \([0, m]\), \(x'_m > 0\) and \(y'_m > 0\) on \([0, m]\).

We claim that

\[ x'_{m,n}(t) \geq C_{13}^2 C_{14}^2 \int_t^m p_1(s) \phi_{M}(s)ds, \quad t \in [0, m], \]  \hfill (4.46)
Similarly, hence, using (4.50) in (4.49), we have

\[
y_m(t) \geq C_{13}^{\delta_2} C_{14} \int_{t}^{m} p_2(s) \psi_M(s) ds, \quad t \in [0, m],
\]

where

\[
C_{13} = \left( \frac{b_1}{a_1} \right) \int_0^1 p_1(s) \Phi_M(s) ds \right)^{\frac{1}{1-\gamma_1}} \left( \frac{b_2}{a_2} \right) \int_0^1 p_2(s) \psi_M(s) ds \right)^{\frac{\delta_2}{1-\gamma_2}},
\]

\[
C_{14} = \left( \frac{b_1}{a_1} \right) \int_0^1 p_1(s) \Phi_M(s) ds \right)^{\frac{\delta_2}{1-\gamma_2}} \left( \frac{b_2}{a_2} \right) \int_0^1 p_2(s) \psi_M(s) ds \right)^{1-\gamma_2}.
\]

To prove (4.46), consider the following relation

\[
x_{m,n}(t) = (t + \frac{b_1}{a_1}) \int_0^1 (a_1 s + b_1) p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s)) ds
\]

\[
+ \frac{1}{a_1} \int_0^m (a_1 t + b_1) p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s)) ds, \quad t \in [0, m], \quad (4.48)
\]

which implies that

\[
x_{m,n}(0) = \frac{b_1}{a_1} \int_0^1 p_1(s) f_1(s, x_{m,n}(s), y_{m,n}(s), x'_{m,n}(s)) ds.
\]

Using (C_7) and (4.48), we obtain

\[
x_{m,n}(0) \geq (x_{m,n}(0))^\gamma (y_{m,n}(0))^\delta \frac{b_1}{a_1} \int_0^1 p_1(s) \Phi_M(s) ds
\]

\[
\geq (x_{m,n}(0))^\gamma (y_{m,n}(0))^\delta \frac{b_1}{a_1} \int_0^1 p_1(s) \Phi_M(s) ds,
\]

which implies that

\[
x_{m,n}(0) \geq (y_{m,n}(0))^{\frac{\delta_1}{1-\gamma_1}} \left( \frac{b_1}{a_1} \right) \int_0^1 p_1(s) \Phi_M(s) ds \right)^{\frac{1}{1-\gamma_1}}. \quad (4.49)
\]

Similarly,

\[
y_{m,n}(0) \geq (x_{m,n}(0))^{\frac{\delta_2}{1-\gamma_2}} \left( \frac{b_2}{a_2} \right) \int_0^1 p_2(s) \psi_M(s) ds \right)^{\frac{1}{1-\gamma_2}}. \quad (4.50)
\]

Now, using (4.50) in (4.49), we have

\[
(x_{m,n}(0))^{\frac{\delta_1 \delta_2}{(1-\gamma_1)(1-\gamma_2)}} \geq \left( \frac{b_1}{a_1} \right) \int_0^1 p_1(s) \Phi_M(s) ds \right)^{\frac{1}{1-\gamma_1}} \left( \frac{b_2}{a_2} \right) \int_0^1 p_2(s) \psi_M(s) ds \right)^{\frac{\delta_2}{1-\gamma_2}}.
\]

Hence,
Similarly, using (4.49) in (4.50), we obtain
\[ y_{m,n}(0) \geq C_{14}. \]  
(4.52)

Now, from (4.48), it follows that
\[ x_{m,n}(t) \geq \int_t^{m} p_1(s)f_1(s,x_{m,n}(s),x'_{m,n}(s))ds. \]

Using (C_6), (4.43), (4.50) and (4.52), we obtain (4.46).

Similarly, we can prove (4.47).

From (4.46) and (4.47), passing to the limit \( n \to \infty \) through \( N_\epsilon \), we obtain
\[
\begin{align*}
x_m'(t) &\geq C_{13}^2 C_{14}^2 \int_t^{m} p_1(s)\phi_M(s)ds, \quad t \in [0,m], \\
y_m'(t) &\geq C_{13}^2 C_{14}^2 \int_t^{m} p_2(s)\psi_M(s)ds, \quad t \in [0,m].
\end{align*}
\]
(4.53)

Consequently, \( x_m' > 0, y_m' > 0 \) on \([0,m]\) and \( x_m > 0, y_m > 0 \) on \([0,m]\).

Moreover, \( x_{m,n}, y_{m,n} \) satisfy
\[
\begin{align*}
x_{m,n}'(t) &= x_{m,n}'(0) - \int_0^t p_1(s)f_1(s,x_{m,n}(s),y_{m,n}(s),x_{m,n}'(s))ds, \quad t \in [0,m], \\
y_{m,n}'(t) &= y_{m,n}'(0) - \int_0^t p_2(s)f_2(s,x_{m,n}(s),y_{m,n}(s),y_{m,n}'(s))ds, \quad t \in [0,m].
\end{align*}
\]

Letting \( n \to \infty \) through \( N_\epsilon \), we obtain
\[
\begin{align*}
x_m'(t) &= x_m'(0) - \int_0^t p_1(s)f_1(s,x_m(s),y_m(s),x_m'(s))ds, \quad t \in [0,m], \\
y_m'(t) &= y_m'(0) - \int_0^t p_2(s)f_2(s,x_m(s),y_m(s),y_m'(s))ds, \quad t \in [0,m],
\end{align*}
\]
which imply that
\[
\begin{align*}
-x_m''(t) &= p_1(t)f_1(t,x_m(t),y_m(t),x_m'(t)), \quad t \in (0,m), \\
y_m''(t) &= p_2(t)f_2(t,x_m(t),y_m(t),y_m'(t)), \quad t \in (0,m).
\end{align*}
\]
(4.54)

Hence, \( (x_m,y_m) \) is a \( C^1 \)-positive solution of (3.4.1).
4.2.2 Existence of positive solutions on an infinite interval

**Theorem 4.2.2** Assume that \((C_1) - (C_3)\) and \((C_5) - (C_8)\) hold. Then the system of BVPs (4.3) has a \(C^1\)-positive solution.

**Proof.** By Theorem 4.2.1, for each \(m \in \mathbb{N} \setminus \{0\}\), the system of BVPs (4.3) has a \(C^1\)-positive solution \((x_m, y_m)\) defined on \([0, m]\). By applying diagonalization argument we will show that the system of BVPs (4.3) has a \(C^1\)-positive solution. For this we define a continuous extension \((\overline{x}_m, \overline{y}_m)\) of \((x_m, y_m)\) by

\[
\overline{x}_m(t) = \begin{cases} 
  x_m(t), & t \in [0, m], \\
  x_m(m), & t \in [m, \infty) 
\end{cases}, \quad \overline{y}_m(t) = \begin{cases} 
  y_m(t), & t \in [0, m], \\
  y_m(m), & t \in [m, \infty) 
\end{cases} \tag{4.55}
\]

Clearly, \(\overline{x}_m, \overline{y}_m \in C^1[0, \infty)\) and satisfy,

\[
0 \leq \overline{x}_m(t) < M, \quad 0 \leq \overline{y}_m(t) < M, \quad t \in [0, \infty) 
\]

\[
0 \leq \overline{x}'_m(t) < M, \quad 0 \leq \overline{y}'_m(t) < M, \quad t \in [0, \infty) \tag{4.56}
\]

We claim that

\[
\{\overline{x}'_m\}_{m \in \mathbb{N} \setminus \{0\}} \text{ and } \{\overline{y}'_m\}_{m \in \mathbb{N} \setminus \{0\}} \text{ are equicontinuous on } [0, 1]. \tag{4.57}
\]

Using (4.54), (4.55), (4.56) and \((C_3)\), we obtain

\[
-\overline{x}'_m(t) \leq p_1(t)h_1(M)k_1(M)(u_1(\overline{x}_m(t)) + v_1(\overline{x}_m(t))), \quad t \in (0, 1) 
\]

\[
-\overline{y}'_m(t) \leq p_2(t)h_2(M)k_2(M)(u_2(\overline{y}_m(t)) + v_2(\overline{y}_m(t))), \quad t \in (0, 1) 
\]

which implies that

\[
\frac{-\overline{x}'_m(t)}{u_1(\overline{x}_m(t)) + v_1(\overline{x}_m(t))} \leq h_1(M)k_1(M)p_1(t), \quad t \in (0, 1) 
\]

\[
\frac{-\overline{y}'_m(t)}{u_2(\overline{y}_m(t)) + v_2(\overline{y}_m(t))} \leq h_2(M)k_2(M)p_2(t), \quad t \in (0, 1) 
\]

Hence, for \(t_1, t_2 \in [0, 1]\), we have

\[
|J_1(\overline{x}_m(t_1)) - J_1(\overline{x}_m(t_2))| \leq h_1(M)k_1(M) \left| \int_{t_1}^{t_2} p_1(t) dt \right|, \tag{4.58}
\]

\[
|J_2(\overline{y}_m(t_1)) - J_2(\overline{y}_m(t_2))| \leq h_2(M)k_2(M) \left| \int_{t_1}^{t_2} p_2(t) dt \right|.
\]
In view of (4.55), (C1), uniform continuity of $J_i^{-1}$ over $[0, J_i(L)]$ ($i = 1, 2$), and
\[
|x_m(t_1) - x_m'(t_2)| = |J_i^{-1}(J_i(x_m(t_1))) - J_i^{-1}(J_i(x_m'(t_2)))|,
\]
\[
|y_m(t_1) - y_m'(t_2)| = |J_i^{-1}(J_i(y_m(t_1))) - J_i^{-1}(J_i(y_m'(t_2)))|,
\]
we establish (4.57).

From (4.56) and (4.57), it follows that the sequences $\{(x_m^{(j)}, y_m^{(j)})\}$ ($j = 0, 1$) are uniformly bounded and equicontinuous on $[0, 1]$. Hence, by Theorem (1.1.5), there exist subsequence $N_1$ of $N_0 \setminus \{0\}$ and $(u_1, v_1) \in C^1[0, 1] \times C^1[0, 1]$ such that for each $j = 0, 1$, the sequence $(x_m^{(j)}, y_m^{(j)})$ converges uniformly to $(u_1^{(j)}, v_1^{(j)})$ on $[0, 1]$ as $m \to \infty$ through $N_1$. Also from BCs (4.34), we have $a_1u_1(0) - b_1u_1'(0) = a_2v_1(0) - b_2v_1'(0) = 0$.

Moreover, from (4.55) and (4.55), for each $m \in N_0 \setminus \{0\}$, we have
\[
\|\varphi_m(t)\| \leq C_{13}^2 C_{14}^2 \int_0^1 p_1(s) \phi_{rt}(s) ds, \quad t \in [0, 1],
\]
\[
\|\psi_m(t)\| \leq C_{13}^2 C_{14}^2 \int_0^1 p_2(s) \psi_{rt}(s) ds, \quad t \in [0, 1],
\]
as limit $m \to \infty$ through $N_1$, we obtain
\[
|u_1'(t)| \geq C_{13}^2 C_{14}^2 \int_0^1 p_1(s) \phi_{rt}(s) ds, \quad t \in [0, 1],
\]
\[
|v_1'(t)| \geq C_{13}^2 C_{14}^2 \int_0^1 p_2(s) \psi_{rt}(s) ds, \quad t \in [0, 1],
\]
which shows that $u_1' > 0$ and $v_1' > 0$ on $[0, 1)$, $u_1 > 0$ and $v_1 > 0$ on $[0, 1]$.

By the same process as above, we can show that
\[
\{\varphi_m\}_{m \in N_1 \setminus \{1\}} \text{ and } \{\psi_m\}_{m \in N_1 \setminus \{1\}} \text{ are equicontinuous families on } [0, 2]. \quad (4.59)
\]
Now, in view of (4.56) and (4.59), it follows that the sequences $\{(x_m^{(j)}, y_m^{(j)})\}$ ($j = 0, 1$) are uniformly bounded and equicontinuous on $[0, 2]$. Hence, by Theorem (1.1.5), there exist subsequence $N_2$ of $N_1 \setminus \{1\}$ and $(u_2, v_2) \in C^1[0, 2] \times C^1[0, 2]$ such that for each $j = 0, 1$, the sequence $(x_m^{(j)}, y_m^{(j)})$ converges uniformly to $(u_2^{(j)}, v_2^{(j)})$ on $[0, 2]$ as $m \to \infty$ through $N_2$. Also, $a_1u_2(0) - b_1u_2'(0) = a_2v_2(0) - b_2v_2'(0) = 0$.

Moreover, in view of (4.55) and (4.55), for each $m \in N_1 \setminus \{1\}$, we have
Passing to the limit

\[ \overline{\varphi}_m(t) \geq C_{13}^m C_{14}^{\delta j} \int_t^2 p_1(s) \varphi_M(s) ds, \quad t \in [0, 2], \]

\[ \underline{\varphi}_m(t) \geq C_{13}^m C_{14}^{\delta j} \int_t^2 p_2(s) \psi_M(s) ds, \quad t \in [0, 2]. \]

Now, the \( \lim_{m \to \infty} \) through \( N_2 \) leads to

\[ U^2(t) = C_{13}^m C_{14}^{\delta j} \int_t^k p_1(s) \varphi_M(s) ds, \quad t \in [0, 2], \]

\[ V^2(t) = C_{13}^m C_{14}^{\delta j} \int_t^k p_2(s) \psi_M(s) ds, \quad t \in [0, 2], \]

which shows that \( u_2 > 0 \) and \( v_2 > 0 \) on \( [0, 2) \), \( u_2 > 0 \) and \( v_2 > 0 \) on \([0, 2] \). Note that \( u_2 = u_1 \) and \( v_2 = v_1 \) on \([0, 1] \) as \( N_2 \subseteq N_1 \).

In general, for each \( k \in N_0 \setminus \{0\} \), there exists a subsequence \( N_k \) of \( N_{k-1} \setminus \{k-1\} \) and \( (u_k, v_k) \in C^1([0, k]) \times C^1([0, k]) \) such that \( (\overline{\varphi}_m(t), \underline{\varphi}_m(t)) \) converges uniformly to \( (u_j, v_j) \) \( (j = 0, 1) \) on \([0, k] \), as \( m \to \infty \) through \( N_k \). Also, \( a_1 u_k(0) - b_1 u_k(0) = a_2 v_k(0) - b_2 v_k(0) = 0, u_k = u_{k-1} \) and \( v_k = v_{k-1} \) on \([0, k-1] \) as \( N_k \subseteq N_{k-1} \). Moreover,

\[ U_k(t) = C_{13}^m C_{14}^{\delta j} \int_t^k p_1(s) \varphi_M(s) ds, \quad t \in [0, k], \]

\[ V_k(t) = C_{13}^m C_{14}^{\delta j} \int_t^k p_2(s) \psi_M(s) ds, \quad t \in [0, k], \]

which shows that \( u_k > 0 \) and \( v_k > 0 \) on \([0, k] \), \( u_k > 0 \) and \( v_k > 0 \) on \([0, k] \).

Define functions \( x, y : \mathbb{R}^+ \to \mathbb{R}^+ \) as:

For fixed \( \tau \in \mathbb{R}_0^+ \) and \( k \in N_0 \setminus \{0\} \) with \( \tau \leq k \), \( x(\tau) = u_k(\tau) \) and \( y(\tau) = v_k(\tau) \). Then, \( x \) and \( y \) are well defined as, \( x(t) = u_k(t) > 0 \) and \( y(t) = v_k(t) > 0 \) for \( t \in [0, k] \).

We can do this for each \( \tau \in \mathbb{R}_0^+ \). Thus, \( (x, y) \in C^1(\mathbb{R}_0^+) \times C^1(\mathbb{R}_0^+) \) with \( x > 0, y > 0, x' > 0 \) and \( y' > 0 \) on \( \mathbb{R}_0^+ \).

Now, we show that \( (x, y) \) is a solution of system of BVPs \( (4.3) \). Choose a fixed \( \tau \in \mathbb{R}^+ \) and \( k \in N_0 \setminus \{0\} \) such that \( k \geq \tau \). Then, \( (\overline{x}_m(\tau), \overline{\varphi}_m(\tau)) \) where \( m \in N_k \), satisfy

\[ \overline{x}_m(\tau) = \overline{x}_m(0) - \int_0^\tau p_1(s)f_1(s, \overline{x}_m(s), \overline{x}_m(s), \overline{\varphi}_m(s)) ds, \]

\[ \overline{\varphi}_m(\tau) = \overline{\varphi}_m(0) - \int_0^\tau p_2(s)f_2(s, \overline{x}_m(s), \overline{x}_m(s), \overline{\varphi}_m(s)) ds. \]

Passing to the limit \( m \to \infty \), we obtain
Hence,
\[
\begin{align*}
  u'_k(t) &= u'_k(0) - \int_0^\tau p_1(s) f_1(s, u_k(s), v_k(s), u'_k(s)) ds, \\
  v'_k(t) &= v'_k(0) - \int_0^\tau p_2(s) f_2(s, u_k(s), v_k(s), v'_k(s)) ds.
\end{align*}
\]
Thus for every \( k \), we have
\[
\begin{align*}
  x'(\tau) &= x'(0) - \int_0^\tau p_1(s) f_1(s, x(s), y(s), x'(s)) ds, \\
  y'(\tau) &= y'(0) - \int_0^\tau p_2(s) f_2(s, x(s), y(s), y'(s)) ds,
\end{align*}
\]
which implies that
\[
\begin{align*}
  -x''(\tau) &= p_1(\tau) f_1(\tau, x(\tau), y(\tau), x'(\tau)), \\
  -y''(\tau) &= p_2(\tau) f_2(\tau, x(\tau), y(\tau), y'(\tau)).
\end{align*}
\]
We can do this for each \( \tau \in \mathbb{R}^+ \). Consequently,
\[
\begin{align*}
  -x''(t) &= p_1(t) f_1(t, x(t), y(t), x'(t)), & t \in \mathbb{R}^+_0, \\
  -y''(t) &= p_2(t) f_2(t, x(t), y(t), y'(t)), & t \in \mathbb{R}^+_0.
\end{align*}
\]
Thus, \((x, y) \in C^2(\mathbb{R}^+_0) \times C^2(\mathbb{R}^+_0)\), \( a_1 x(0) - b_1 x'(0) = a_2 y(0) - b_2 y'(0) = 0 \).

It remains to show that
\[
\lim_{t \to 0^+} x'(t) = \lim_{t \to 0^+} y'(t) = 0.
\]
First, we show that \( \lim_{t \to 0^+} x'(t) = 0 \). Suppose \( \lim_{t \to 0^+} x'(t) = \varepsilon_0 \), for some \( \varepsilon_0 > 0 \). Then, \( x'(t) \geq \varepsilon_0 \) for all \( t \in [0, \infty) \). Choose \( k \in N_0 \setminus \{0\} \), then for \( m \in N_k \), in view of (4.55), we have
\[
x'(t) = u'_k(t) = \lim_{m \to \infty} x'_m(t) = \lim_{m \to \infty} x'_m(t), \quad t \in [0, k],
\]
which leads to
\[
x'(k) = \lim_{m \to \infty} x'_m(k).
\]
Thus for every \( \varepsilon > 0 \), there exist \( m^* \in N_k \) such that \( |x'_m(k) - x'(k)| < \varepsilon \) for all \( m \geq m^* \).

Without loss of generality assume that \( m^* = k \), then \( |x'_k(k) - x'(k)| < \varepsilon \), that is, \( |x'(k)| < \varepsilon \). Which is a contradiction whenever \( \varepsilon = \varepsilon_0 \). Hence, \( \lim_{t \to 0^+} x'(t) = 0 \).
Similarly, we can prove \( \lim_{r \to \infty} y'(t) = 0 \). Thus, \((x, y)\) is a \(C^1\)-positive solution of the system of BVPs (4.3).

**Example 4.2.3** Let

\[
f_i(t, x, y, z) = \nu^{\alpha_i+1} e^{-\gamma_i} (M + 1 - x)(M + 1 - y)|x|^\delta_i |y|^\gamma_i |z|^{-\alpha_i}, \quad i = 1, 2,
\]

where \(\nu > 0, M > 0, \alpha_i > 0, 0 \leq \gamma_i, \delta_i < 1, i = 1, 2\).

Assume that \((1 - \gamma_1)(1 - \gamma_2) \neq \delta_i \delta_2\) and

\[
\nu < \frac{M}{\sum_{i=1}^2 \left(\frac{M}{\nu^{\alpha_i}} + \alpha_i + 2\right)(\alpha_i + 1)^{\frac{1}{M+1}} (2M + 1)^{\frac{\gamma_i}{M+1}} M^{\frac{\delta_i}{M+1}}}
\]

Taking \(p_i(t) = e^{-\gamma_i}, h_i(x) = \nu^{\alpha_i+1}(M + 1 + x)x^\delta_i, k_i(y) = (M + 1 + y)y^\gamma_i, u_i(z) = z^{-\alpha_i}\)
and \(v_i(z) = 0, i = 1, 2\). Choose \(\theta_M(t) = \nu^{\alpha_i+1} M^{-\alpha_i} e^{-\gamma_i}\) and \(\psi_M(t) = \nu^{\alpha_2+1} M^{-\alpha_2} e^{-\gamma_2}\).

Then, \(J_i(\mu) = \frac{\nu^{\alpha_i+1}}{\alpha_i + 1} \) and \(J_i^{-1}(\mu) = (\alpha_i + 1)^{\frac{1}{\alpha_i + 1}} \mu^{\frac{1}{\alpha_i + 1}} \), \(i = 1, 2\).

Also,

\[
\frac{M}{\omega(M)} = \frac{\sum_{i=1}^2 \int_0^\infty J_i^{-1}(h_i(M)k_i(M)J_i^{-1}(h_i(M)k_i(M)J_i^{-1}(h_i(M)k_i(M)) p_i(s)ds) dt + \sum_{i=1}^2 (1 + \frac{M}{\nu^{\alpha_i}})J_i^{-1}(h_i(M)k_i(M)J_i^{-1}(h_i(M)k_i(M)) p_i(s)ds)}{M}
\]

\[
= \frac{\sum_{i=1}^2 \int_0^\infty (\nu^{\alpha_i+1}(2M + 1)^{\gamma_i + \delta_i} e^{-\gamma_i}) dt + \sum_{i=1}^2 (1 + \frac{M}{\nu^{\alpha_i}}) J_i^{-1}(\nu^{\alpha_i+1}(2M + 1)^{\gamma_i + \delta_i})}{\nu \sum_{i=1}^2 \left(\frac{M}{\nu^{\alpha_i}} + \alpha_i + 2\right)(\alpha_i + 1)^{\frac{1}{M+1}} (2M + 1)^{\frac{\gamma_i}{M+1}} M^{\frac{\delta_i}{M+1}}}
\]

Clearly, \(C_1\) – \(C_3\) and \(C_5\) – \(C_8\) are satisfied. Hence, by Theorem 4.2.2, the system of BVPs (4.3) has at least one \(C^1\)-positive solution.
Chapter 5
Concluding Remarks

In Chapter 2, Section 2.1, we have established four different results (Theorem 2.1.4, Theorem 2.1.6, Theorem 2.1.8, and Theorem 2.1.10) for the existence of at least one positive solution to the system of SBVPs (2.1) under the new assumption on the nonlinearities $f$ and $g$. In Theorem 2.1.4, we provide the existence of at least one positive solution for the system of SBVPs (2.1) under the assumptions $(A_1) - (A_3)$, where $(A_1)$ is about integrability of nonlinearities while $(A_2)$ and $(A_3)$ are natural assumptions satisfied by a class of singular nonlinearities. Our next result, Theorem 2.1.6, is obtained by replacing $(A_3)$ with $(A_4)$ in Theorem 2.1.4. Theorem 2.1.8 is obtained by replacing $(A_2)$ with $(A_5)$ in Theorem 2.1.4. Moreover, Theorem 2.1.10 can be obtained either by replacing $(A_2)$ with $(A_5)$ in Theorem 2.1.6 or by replacing $(A_3)$ with $(A_4)$ in Theorem 2.1.8. Further in Section 2.2, Theorem 2.2.2, the existence of positive solutions to SBVPs (2.2) is provided under the assumption $(A_6) - (A_8)$, where the assumption $(A_6)$ is integrability condition on nonlinearities while $(A_7)$ and $(A_8)$ are sublinear conditions. In Section 2.3, we discuss the four-point coupled system of SBVPs (2.3). In Theorem 2.3.8, by employing the Guo-Krasnosel'skii fixed point theorem for a completely continuous map on a positive cone, it is shown that the system (2.3) has a positive solution under the assumptions $(A_0) - (A_{11})$, where $(A_0)$ is integrability condition while $(A_{10})$ and $(A_{11})$ are sublinear conditions on nonlinearities $f$ and $g$.

In Chapter 3, Section 3.1, we establish the existence results for a coupled system of SBVPs (3.2). In Theorem 3.1.2, we prove the existence of at least one $C^1$-positive solution for the system of SBVPs (3.2) under the assumptions $(B_1) - (B_7)$. The assumptions $(B_1)$ and $(B_7)$ are some integrability conditions, $(B_2)$ is necessary
because, otherwise, positive solution \((x, y)\) will not satisfy the condition \(x' > 0\) and \(y' > 0\) on \([0, 1]\), and therefore, \((x, y) \notin C^2(0, 1) \cap C^2(0, 1)\). \((B_3)\) is a natural assumption when \(f(t, x, y)\) and \(g(t, x, y)\) have singularity at \(y = 0\). \((B_4)\) is required to bound the solution, whereas \((B_5)\) is necessary for invertibility of the maps \(I\) and \(J\), and the solution is positive due to \((B_6)\). By replacing the assumptions \((B_6)\) and \((B_7)\) of Theorem 3.1.2 with \((B_8)\) and \((B_9)\), and including \((B_{10})\), we obtained the existence of at least two positive solutions for the system of SBVPs (3.2), that is, Theorem 3.2.2 of Section 3.2. The assumption \((B_{10})\) is required for the existence of at least two solutions. By replacing the assumption \((B_4)\) and \((B_7)\) of Theorem 3.1.2 with \((B_{11})\) and \((B_{12})\) we get our next result, that is, Theorem 3.3.1 of Section 3.3 which provide existence of at least one \(C^1\)-positive solutions to the SBVPs (3.3). Theorem 3.4.2 of Section 3.4 is obtained by replacing the assumptions \((B_4)\) and \((B_8)\) of Theorem 3.2.2 with \((B_{11})\) and \((B_{13})\), which is a criteria for the existence of at least two \(C^1\)-positive solutions for the system of SBVPs (3.3). Moreover in Section 3.5 Theorem 3.5.1 we studied the existence of \(C^1\)-positive solutions to the system of SBVPs (3.6) under the assumptions \((B_1) - (B_3), (B_5), (B_{14}), (B_{16})\) and \((B_{17})\). The assumption \((B_{14})\) is a replacement of \((B_{11})\) in the case of two-point coupled BCs (3.6), \((B_{16})\) is a generalization of \((B_6)\) in case system of SBVPs (3.6) while \((B_{17})\) is just similar to \((B_7)\). In Section 3.6 we develop the notion of upper and lower solutions for the system of SBVPs (3.5). Theorem 3.6.1 guarantees the existence of \(C^1\)-positive solutions for the system (3.5) under the assumptions \((B_{18}) - (B_{25})\), where \((B_{18})\) is equivalent to \((B_1), (B_{19})\) is just a continuity condition on nonlinearities \(f_i (i = 1, 2), (B_{20})\) and \((B_{21})\) defines upper and lower solutions, \((B_{22})\) is a condition about the concavity of solutions, \((B_{23})\) is a natural assumption when the functions \(f, g\) are singular with respect to \(x = 0\) and \(y = 0\). \((B_{24})\) is about integrability condition and \((B_{25})\) is required to bound the derivative of solution.

In Chapter 4 Section 4.1 we establish the existence of \(C^1\)-positive solutions to the coupled system of SBVPs (4.2). Theorem 4.1.2 offer \(C^1\)-positive solutions to the system of SBVPs (4.2) under the assumption \((C_1) - (C_7)\), where \((C_1)\) is some integrability condition, \((C_2)\) and \((C_6)\) are weaker than \((B_2)\), \((C_3)\) is more general than \((B_3)\) when nonlinearities are sign-changing, \((C_4)\) is required to bound the solution which is much simpler than \((B_4)\), \((C_5)\) is just \((B_5)\), and \((C_7)\) is required to prove that the solution is positive. By replacing the assumption \((C_4)\) in Theorem
with (C₈) we obtain Theorem 4.2.2 of Section 4.2, which is a criteria for the existence of at least one $C^1$-positive solutions to the system of SBVPs (4.3).
References

1. R.P. Agarwal and D. O’Regan, Semipositive Dirichlet boundary value problems with singular dependent nonlinearities, *Houston J. Math.* 30 (2004) 297-308.
2. R.P. Agarwal and D. O’Regan, Infinite interval problems modeling phenomena which arise in the theory of plasma and electrical potential theory, *Stud. Appl. Math.* 111 (2003) 339-358.
3. R.P. Agarwal and D. O’Regan, *Singular Differential and Integral Equations with Applications*, Kluwer, Dordrecht, 2003.
4. R.P. Agarwal, D. O’Regan, Upper and lower solutions for singular problems with nonlinear boundary data, *Nonlinear Differ. Equ. Appl.* 9 (2002) 419-440.
5. R.P. Agarwal and D. O’Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer, Dordrecht, 2001.
6. R.P. Agarwal, M. Meehan and D. O’Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2004.
7. R.P. Agarwal and D. O’Regan, Right focal singular boundary value problems, *ZAMM* 79 (1999) 363-373.
8. R.P. Agarwal, D. O’Regan and P.K. Palamides, The generalized Thomas-Fermi singular boundary value problems for neutral atoms, *Math. Meth. Appl. Sci.* 29 (2006) 49-66.
9. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* 18 (1976) 620-709.
10. H. Amann, Parabolic evolution equations with nonlinear boundary conditions, in: *Nonlinear Functional Analysis and Its Applications*, Berkeley, (1983), in: Proc. Sympos. Pure Math. vol. 45, Amer. Math. Soc. Providence, RI, (1986) 17-27.
11. H. Amann, Parabolic evolution equations and nonlinear boundary conditions, *J. Differential Equations* 72 (1988) 201-269.
12. N.A. Asif, R.A. Khan and J. Henderson, Existence of positive solutions to a system of singular boundary value problems, *Dynam. Systems Appl.* 19 (2010) 395-404.
13. N.A. Asif and R.A. Khan, Positive solutions to singular system with four-point coupled boundary conditions *J. Math. Anal. Appl.* 386 (2012) 848-861. doi:10.1016/j.jmaa.2011.08.039
14. N.A. Asif and R.A. Khan, Multiplicity results for positive solutions of a coupled system of singular boundary value problems, *Communications on Applied Nonlinear Analysis* Volume 17 (2010), Number 2, 53-68.
15. N.A. Asif, R.A. Khan and P.W. Eloe, *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* 18 (2011) 353-361.
16. N.A. Asif, R.A. Khan and P.W. Eloe, Positive solutions for a system of singular second order nonlocal boundary value problems *J. Korean Math. Soc.* 47 (2010) No. 5, 985-1000.
17. N.A. Asif, Positive solutions of a singular system with two point coupled boundary conditions, *American J. Appl. Math.* Volume 3, (2015) 19-24.
18. J.V. Baxley and S.B. Robinson, Nonlinear boundary value problems for shallow membrane caps, II, *J. Computational and Appl. Math.* 88 (1998) 203-224.
19. A.V. Bitsadze, On the theory of nonlocal boundary value problems, *Soviet Math. Dokl.* **30** (1984) 8-10.
20. A.V. Bitsadze, On a class of conditionally solvable nonlocal boundary value problems for harmonic functions, *Soviet Math. Dokl.* **31** (1985) 91-94.
21. R. Blossey, *Computational Biology: A Statistical Mechanics Perspective*, Chapman & Hall/CRC, New York, 2006.
22. L.E. Bobisud, Existence of solutions for nonlinear singular boundary value problems, *Applied Analysis* **35** (1990) 43-57.
23. A. Callegari and Fiedman, An analytic solution of a nonlinear singular boundary value problem in the theory of viscous fluids, *J. Math. Anal. Appl.* **21** (1968) 510-529.
24. A. Callegari and A. Nachman, Some singular, nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* **64** (1978) 96-105.
25. A. Callegari and A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980) 275-281.
26. Y. Chen, B. Yan and L. Zhang, Positive solutions for singular three-point boundary value problems with sign changing nonlinearities depending on $x'$, *Elec. J. Diff. Equa.* **2007**(63)(2007), pp. 9.
27. X. Cheng and C. Zhong, Existence of positive solutions for a second-order ordinary differential system, *J. Math. Anal. Appl.* **312** (2005) 14-23.
28. W. Cheung and P. Wong, Fixed-sign solutions for a system of singular focal boundary value problems, *J. Math. Anal. Appl.* **329** (2007) 851-869.
29. J. Chu, X. Lin, D. Jiang, D. O'Regan and R.P. Agarwal, Positive solutions for second-order superlinear repulsive singular Neumann boundary value problems, *Positivity* **12** (2008) 555-569.
30. R. Dalmasso, Existence and uniqueness of positive solutions of semilinear elliptic systems, *Nonlinear Analysis* **57**(2004) 341-348.
31. R.W. Dickey, The plane circular elastic surface under normal pressure, *Arch. Rational Mech. Anal.* **26** (1967) 219-236.
32. R.W. Dickey, Rotationally symmetric solutions for shallow membrane caps, *Quart. Appl. Math.* **47** (1989) 571-581.
33. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
34. O. Dickmann, D. Hilhorst and L.A. Peletier, A Singular Boundary Value Problem Arising in a Pre-Breakdown Gas Discharge, *SIAM J. Appl. Math.* **39**(1)(1980) 48-66.
35. J. Dugundji, An extension of Tietze’s theorem, *Pacific J. Math.* **1**(1951) 353-367.
36. P.W. Eloe, E.R. Kaufmann and C.C. Tisdell, Multiple solutions of a boundary value problem on an unbounded domain, *Dynam. Systems Appl.* **15**(1)(2006) 53-63.
37. L.H. Erbe, Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems, *Math. Comput. Modelling* **32**(2000) 529-539.
38. L.H. Erbe, H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.* **120**(1994) 743-748.
39. J.R. Esteban and J.L. Vazquez, On the equation of turbulent filtration in one-dimensional porous media, *Nonlinear Analysis* **10**(1986) 1303-1325.
40. M. Farkas, *Dynamical Models in Biology*, Academic Press, New York, 2001.
41. E. Fermi, Un metodo statistico par la determinazione di alcune proprietá dell’ atoma, *Rend Accad. Naz. del Linci Cl. Sci. Fis. Mat. e Nat.* 6(1927) 602-607.
42. I. Fonseca and W. Gangbo, *Degree Theory in Analysis and Application*, Oxford University Press, New York, 1995.
43. J. Gatica, G. Hernandez and P. Waltman, Radially symmetric solutions of a class of singular elliptic equations, *Proc. Edin. Math. Soc.* 33(1990) 169-180.
44. R. Grief and J.A. Anderson, Mass transfer to a rotating disk in a non-Newtonian fluid, *Phys. Fluids* 16(1973) 1816-1817.
45. Y. Guo and W. Ge, Positive solutions for three-point boundary value problems with dependence on the first-order derivative, *J. Math. Anal. Appl.* 290(2004) 291-301.
46. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.
47. C.P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second-order ordinary differential equations, *J. Math. Anal. Appl.* 168(1988) 540-551.
48. T.G. Hallam and D.E. Loper, Singular boundary value problems arising in rotationg fluid flow, *Arch. Rational Mech. Anal.* 60(1976) 355-369.
49. J. Henderson and S.K. Ntouyas, Positive solutions for systems of nonlinear boundary value problems, *Nonlin. Studies* 15(2008) 51-60.
50. J. Henderson, S.K. Ntouyas and I.K. Ntouyas, Positive solutions for systems of second-order four-point nonlinear boundary value problems, *Comm. Appl. Anal.* 12(2008) 29-40.
51. J. Henderson, Existence of multiple solutions for second-order boundary value problems, *J. Differential Equations* 166(2000) 443-454.
52. J. Henderson and H. Wang, Positive solutions of nonlinear eigenvalue problems, *J. Math. Anal. Appl.* 208(1997) 252-259.
53. V. Hutson, J.S. Pym and M.J. Cloud, *Applications of Functional Analysis and Operator Theory*, Elsevier, New York, 2005.
54. V.A. Il’in and E.I. Moiseev, Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations* 23(7)(1987) 803-810.
55. V.A. Il’in and E.I. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Differential Equations* 23(8)(1987) 979-987.
56. D. Jiang, X. Xu, D. O’Regan and R.P. Agarwal, Multiple positive solutions to semipositive Dirichlet boundary value problems with singular dependent nonlinearities, *Fasc. Math.* 34(2004) 25-37.
57. K.N. Johnson, Circularly symmetric deformation of shallow elastic membrane caps, *Quart. Appl. Math.* 55(1997) 537-550.
58. P. Kang and Z. Wei, Multiple solutions of second-order three-point singular boundary value problems on the half-line, *Appl. Math. Comput.* 203(2008) 523-535.
59. P. Kang and Z. Wei, Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations, *Nonlinear Analysis* 70(2009) 444-451.
5 Concluding Remarks

60. P. Kelevedjiev, Nonnegative solutions to some singular second-order boundary value problems, *Nonlinear Analysis* **36**(1999) 481-494.

61. R.A. Khan and J.R.L. Webb, Existence of at least three solutions of a second-order three-point boundary value problem, *Nonlinear Analysis* **64**(2006) 1356-1366.

62. R.A. Khan and J.R.L. Webb, Existence of at least three solutions of nonlinear three-point boundary value problems with super-quadratic growth, *J. Math. Anal. Appl.* **328**(2007) 690-698.

63. E.H. Kim, Singular Gierer-Meinhardt systems of elliptic boundary value problems, *J. Math. Anal. Appl.* **308**(2005) 1-10.

64. M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, SIAM, Philadelphia, 2008.

65. J. Leray and J. Schauder, Topologie et équations fonctionnelles, *Ann. Sci. Ecole Norm. Sup* **51**(1934) 45-78.

66. J. Li and J. Shen, Multiple positive solutions for a second-order three-point boundary value problem, *Appl. Math. Comput.* **182**(2006) 258-268.

67. Y. Li, On the existence and nonexistence of positive solutions for nonlinear Sturm-Liouville boundary value problems, *J. Math. Anal. Appl.* **304**(2005) 74-86.

68. H. Lian and W. Ge, Solvability for second-order three-point boundary value problems on a half-line, *Appl. Math. Lett.* **19**(2006) 1000-1006.

69. B. Liu, Positive solutions of a nonlinear three-point boundary value problem, *Comput. Math. Appl.* **44**(2002) 201-211.

70. B. Liu, L. Liu and Y. Wu, Positive solutions for a singular second-order three-point boundary value problem, *Appl. Math. Comput.* **196**(2008) 532-541.

71. B. Liu, L. Liu and Y. Wu, Positive solutions for singular second-order three-point boundary value problems, *Nonlinear Analysis* **66**(2007) 2756-2766.

72. B. Liu, L. Liu and Y. Wu, Positive solutions for singular systems of three-point boundary value problems, *Comput. Math. Appl.* **53**(2007) 1429-1438.

73. X. Liu, J. Qiu and Y. Guo, Three positive solutions for second-order m-point boundary value problems, *Appl. Math. Comput.* **156**(2004) 733-742.

74. Y. Liu, Existence and unboundedness of positive solutions for singular boundary value problems on half-line, *Appl. Math. Comput.* **144**(2003) 543-556.

75. Y. Liu and B. Yan, Multiple solutions of singular boundary value problems for differential systems, *J. Math. Anal. Appl.* **287**(2003) 540-556.

76. Y. Liu and B. Yan, Multiple positive solutions for a class of nonresonant singular boundary-value problems, *Elec. J. Diff. Equa.* Vol. 2006(2006) No. 42, pp. 1-11.

77. Z. Liu, F. Li, Multiple positive solutions of nonlinear two-point boundary value problems, *J. Math. Anal. Appl.* **203**(1996) 610-625.

78. H. Lü, H. Yu and Y. Liu, Positive solutions for singular boundary value problems of a coupled system of differential equations, *J. Math. Anal. Appl.* **302**(2005) 14-29.

79. R. Ma, Positive solutions of a nonlinear three-point boundary value problem, *Elec. J. Diff. Equa.* **34**(1999), pp. 8.

80. R. Ma, Existence of positive solutions for second-order boundary value problems on infinity intervals, *Appl. Math. Lett.* **16**(2003) 33-39.
81. M. Marletta C. Tretter, Essential spectra of coupled systems of differential equations and applications in hydrodynamics, *J. Differential Equations* **243**(2007) 36-69.
82. R. Ma and B. Thompson, Multiplicity results for second-order two-point boundary value problems with nonlinearities across several eigenvalues, *Appl. Math. Lett.* **18**(2005) 587-595.
83. R. Ma, Multiple nonnegative solutions of second-order systems of boundary value problems, *Nonlinear Analysis* **42**(2000) 1003-1010.
84. M. Moshinsky, Sobre los problemas de condiciones a la frontera en una dimension de características discontinuas, *Bol. Soc. Mat. Mexicana* **7**(1950) 1-25.
85. A. Nachman and A. Callegari, A nonlinear boundary value problem in the theory of viscoelastic fluids, *SIAM J. Appl. Math.* **38**(1980) 275-281.
86. D. O’Regan and R.P. Agarwal, Singular problems: an upper and lower solution approach, *J. Math. Anal. Appl.* **251**(2000) 230-250.
87. H. Schlichting, *Boundary-layer Theory*, McGraw-Hill, New York, 1968.
88. P.J. Schmid, D.S. Henningson, Optimal energy density growth in Hagen-Poiseuille flow, *J. Fluid Mech.* **277**(1994) 197-225.
89. J.V. Shin, A singular nonlinear differential equation arising in the Homann flow, *J. Math. Anal. Appl.* **212**(1997) 443-451.
90. E. Soewono, K. Vajravelu and R.N. Mohapatra, Existence and nonuniqueness of solutions of a singular nonlinear boundary layer problem, *J. Math. Anal. Appl.* **159**(1991) 251-270.
91. S. Staněk, Existence of positive solutions to semipositone singular Dirichlet boundary value problems, *Acta Math. Sinica, English Series* **22**(6)(2006) 1891-1914.
92. T. Timoshenko, *Theory of Elastic Stability*, McGraw-Hill, New York, 1971.
93. P. Turchin, J.D. Reeve, J.T. Cronin, R.T. Wilkens, Spatial pattern formation in ecological systems: bridging theoretical and empirical approaches, in: J. Bascompte, R.V. Sole (Eds.), Modelling Spatiotemporal Dynamics in Ecology, Landes Bioscience, Austin, TX, 1997, 195-210.
94. K. Vajravelu, E. Soewono and R.N. Mohapatra, On solutions of some singular nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* **155**(1991) 499-512.
95. J. Wang and W. Gao, Singular nonlinear boundary value problems arising in boundary layer theory, *J. Math. Anal. Appl.* **233**(1999) 246-256.
96. J.R.L. Webb, Positive solutions of some three-point boundary value problems via fixed point index theory, *Nonlinear Analysis* **47**(2001) 4319-4332.
97. Z. Wei, Positive solution of singular Dirichlet boundary value problems for second-order differential equation system, *J. Math. Anal. Appl.* **328**(2007) 1255-1267.
98. X. Xian, Existence and multiplicity of positive solutions for multi-parameter three-point differential equations system, *J. Math. Anal. Appl.* **324**(2006) 472-490.
99. X. Xian, Positive solutions for singular semi-positone boundary value problems, *J. Math. Anal. Appl.* **273**(2002) 480-491.
100. B. Yan and Y. Liu, Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line, *Appl. Math. Comput.* **147**(2004) 629-644.
101. B. Yan, D. O’Regan and R.P. Agarwal, Multiple positive solutions via index theory for singular boundary value problems with derivative dependence, *Positivity* **11**(2007) 687-720.
102. B. Yan, D. O’Regan and R.P. Agarwal, Multiple positive solutions of singular second-order boundary value problems with derivative dependence, *Aequationes Mathematicae* 74(2007) 62-89.

103. Y. Yuan, C. Zhao and Y. Liu, Positive solutions for systems of nonlinear singular differential equations, *Elec. J. Diff. Equa.* 2008(74)(2008), pp. 14.

104. A. Zettl, *Sturm-Liouville Theory*, Math. Surveys Monogr. vol. 121, Amer. Math. Soc., Providence, RI, 2005.

105. G. Zhang, Positive solutions of two-point boundary value problems for second-order differential equations with the nonlinearity dependent on the derivative, *Nonlinear Analysis* 69(2008) 222-229.

106. Q. Zhang and D. Jiang, Multiple solutions to semipositone Dirichlet boundary value problems with singular dependent nonlinearities for second-order three-point differential equations, *Computers Math. Applic.* 59(2010) 2516-2527.

107. G. Zhang and J. Sun, Positive solutions of m-point boundary value problems, *J. Math. Anal. Appl.* 291(2004) 406-418.

108. Y. Zhao, Y. Wang and J. Shi, Exact multiplicity of solutions and S-shaped bifurcation curve for a class of semilinear elliptic equations, *J. Math. Anal. Appl.* 331(2007) 263-278.

109. Y. Zhou and Y. Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second-order ordinary differential equations, *J. Math. Anal. Appl.* 320(2006) 578-590.