STABLE COHOMOLOGY OF CONGRUENCE SUBGROUPS

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Abstract. We describe the \( F_p \)-cohomology of the congruence subgroups \( \text{SL}_n(\mathbb{Z}, p^m) \) in degrees \( * < p \), for all large enough \( n \), establishing a formula proposed by F. Calegari. Along the way, we also establish a formula for the stable cohomology of \( \text{SL}_n(\mathbb{Z}/p) \) with certain twisted coefficients.

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Introduction

Stable cohomology of congruence subgroups. Let \( p \) be an odd prime number. We will be concerned with the \( F_p \)-cohomology of the level \( p^m \) congruence subgroups

\[
\text{SL}_n(\mathbb{Z}, p^m) := \text{Ker}(\text{SL}_n(\mathbb{Z}) \to \text{SL}_n(\mathbb{Z}/p^m))
\]

with \( m \geq 1 \), in a range of cohomological degrees which is stable in two senses: certainly \( n \) should be large compared with the cohomological degree, but \( p \) should be too. Our main result is expressed in terms of the completed cohomology

\[
\tilde{H}^*(\text{SL}_n; \mathbb{F}_p) := \text{colim}_m H^*(\text{SL}_n(\mathbb{Z}, p^m); \mathbb{F}_p)
\]

of Calegari and Emerton [CE12], and should be considered as extending some of the ideas developed by Calegari [Cal15] (Corollary 3.4) to higher cohomological degrees.

Theorem A. Let \( p \) be odd. In degrees \( * < p \) and for all large enough \( n \) there is an isomorphism

\[
H^*(\text{SL}_n(\mathbb{Z}, p^m); \mathbb{F}_p) \cong \Lambda^*_p[\mathfrak{sl}(\mathbb{F}_p)^\vee] \otimes \tilde{H}^*(\text{SL}_n; \mathbb{F}_p)
\]

of \( F_p \)-algebras and of \( \text{SL}_n(\mathbb{Z}/p^m) \)-representations.

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Here \( sl_n(\mathbb{F}_p) \) denotes the vector space of traceless \( n \times n \) matrices, considered as an \( \text{SL}_n(\mathbb{Z}/p^m) \)-representation via \( \text{SL}_n(\mathbb{Z}/p^m) \to \text{SL}_n(\mathbb{Z}/p) \) and the adjoint action.

Crucial to the proof of this theorem, as well as to its applications, is that completed cohomology satisfies homological stability with respect to \( n \) \cite{CF16}, and the relation between the stable completed cohomology and the fibre of the \( p \)-adic completion map in algebraic \( K \)-theory \cite{C15}. More precisely, if

\[
\kappa : \text{SK}(\mathbb{Z}/p^m) \to \text{SK}(\mathbb{Z}_p; \mathbb{Z}_p)
\]

is the map induced by \( p \)-adic completion \( \mathbb{Z} \to \mathbb{Z}_p \) on (the 1-connected cover of) \( p \)-adic \( K \)-theory, then Calegari proves that

\[
\hat{H}^*(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p) \cong H^*(\Omega^\infty\text{hob}(\kappa); \mathbb{F}_p).
\]

Combined with deep results in algebraic \( K \)-theory, this can be used to evaluate completed cohomology for \( p \) a regular prime, leading to the following formula.

**Corollary B.** Let \( p \) be an odd regular prime. Then in degrees \( * < p \) and for all large enough \( n \) there is an isomorphism

\[
H^*(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p) \cong \Lambda^*_p \{ sl_n(\mathbb{F}_p)^* \} \otimes \mathbb{F}_p[x_2, x_6, x_{10}, \ldots] \]

of \( \mathbb{F}_p \)-algebras and of \( \text{SL}_n(\mathbb{Z}/p^m) \)-representations.

The relation to algebraic \( K \)-theory may also be used to obtain results at irregular primes. Here two essentially different phenomena can arise: there can be \( p \)-torsion in \( K_{4i+2}(\mathbb{Z}/p^m) \), or the map \( \kappa_* : K_{4i+1}(\mathbb{Z}/p^m) \to K_{4i+1}(\mathbb{Z}_p; \mathbb{Z}_p) \) can fail to be an isomorphism. We justify the following in Remark \[6.2\]

**Example.** For \( p = 37 \) we have

\[
H^*(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p) \cong \Lambda^*_p \{ sl_n(\mathbb{F}_p)^* \} \otimes \mathbb{F}_p[x_2, x_6, x_{10}, \ldots] \otimes \mathbb{F}_p[y_6] \otimes \Lambda^*_p[y_9]
\]

in degrees \( * < p \) for all large enough \( n \).

**Example.** For \( p = 16843 \) we have

\[
H^*(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p) \cong \Lambda^*_p \{ sl_n(\mathbb{F}_p)^* \} \otimes \mathbb{F}_p[x_2, x_6, x_{10}, \ldots] \otimes \mathbb{F}_p[y_4] \otimes \Lambda^*_p[y_5]
\]

in degrees \( * < p \) for all large enough \( n \).

**Example.** For \( p = 2124679 \) we have

\[
H^*(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p) \cong \Lambda^*_p \{ sl_n(\mathbb{F}_p)^* \} \otimes \mathbb{F}_p[x_2, x_6, x_{10}, \ldots] \otimes \mathbb{F}_p[y_4] \otimes \Lambda^*_p[y_5]
\]

\[
\otimes \mathbb{F}_p[y_{1403794}] \otimes \Lambda^*_p[y_{1403795}]
\]

in degrees \( * < p \) for all large enough \( n \).

**Strategy.** The general strategy for proving Theorem \[\ref{thm:main}\] is the same as \[C15\] Section 3. The theory of completed cohomology provides a spectral sequence

\[
E_2^{s,t} = H^s_{ct}(\text{SL}_n(\mathbb{Z}_p; p^m); \mathbb{F}_p) \otimes \hat{H}^t(\text{SL}_n(\mathbb{F}_p) \implies H^{s+t}(\text{SL}_n(\mathbb{Z}/p^m); \mathbb{F}_p),
\]

and the theory of \( p \)-adic analytic groups gives an identification

\[
H^{s+t}_{ct}(\text{SL}_n(\mathbb{Z}_p; p^m); \mathbb{F}_p) \cong \Lambda^*_p \{ sl_n(\mathbb{F}_p)^* \}
\]

of \( \mathbb{F}_p \)-algebras and of \( \text{SL}_n(\mathbb{Z}/p^m) \)-representations. One must then show that this spectral sequence collapses at \( E_2 \) in degrees \( * < p \), and that it has no non-trivial extensions either multiplicatively or as \( \text{SL}_n(\mathbb{Z}/p^m) \)-representations. This is what we shall do.

That a statement like Corollary \[\ref{cor:finite}\] could be true we learnt from a talk given by Calegari at BIRS in October 2021. Based on heuristics including that Corollary \[\ref{cor:finite}\] should be true, Calegari presented a conjectural formula for the cohomology of the finite groups \( \text{SL}_n(\mathbb{F}_p) \) with coefficients in certain modular representations.
(coming from representations of the algebraic group $\text{SL}_n$), and suggested that such formula could be useful in approaching results like Corollary $B$. The second thing we do in this paper is to prove this conjectural formula (in Theorem $C$ below). We will not directly use this formula to prove Theorem $A$, but we will use many of the same ingredients that go into proving it. We formulate it in the following section.

**Stable twisted cohomology of $\text{SL}_n(k)$**. In this section we work not just with $\mathbb{F}_p$ but with a finite field $k$ of characteristic $p$. We work throughout with $k$-modules, and in particular form all tensor products over $k$. If $V$ is a finite-dimensional $k$-module with dual $V^\vee$ and coevaluation map $\text{coev} : k \to V \otimes V^\vee$, then we may form the quotient $V_{[n,m]}$ of $V^\otimes n \otimes (V^\vee)^\otimes m$ by the subspace spanned by inserting coevaluations in all possible ways. The group $\Sigma_n \times \Sigma_m$ acts on $V^\otimes n \otimes (V^\vee)^\otimes m$ by permuting the factors, and this action descends to $V_{[n,m]}$. For partitions $\lambda \vdash n$ and $\mu \vdash m$ with associated Specht modules $S^\lambda$ and $S^\mu$, we define

$$S_{\lambda,\mu}(V) := \text{Hom}_k(\Sigma_n \times \Sigma_m)(S^\lambda \otimes S^\mu, V_{[n,m]}).$$

By functoriality in the vector space $V$, this is a $\text{GL}(V)$-representation.

The formula proposed by Calegari is then as follows. Form the graded algebra $\text{Sym}^\bullet(V \otimes V^\vee)$, where $V \otimes V^\vee$ is placed in degree 2, and let $X^\bullet$ denote its quotient by the ideal generated by the $\text{GL}(V)$-invariant element $1$.

**Theorem C.** For all partitions $\lambda \vdash n$ and $\mu \vdash m$ with $n + m \leq p + 1$, there is an isomorphism

$$H^i(\text{SL}(V); S_{\lambda,\mu}(V)) \cong [S_{\lambda,\mu}(V) \otimes X^i]^\text{SL}(V)$$

in degrees $i < 2p$, as long as $\dim(V)$ is large enough.

**Remark 0.1.** The analogous statement for $\text{GL}(V)$ instead of $\text{SL}(V)$ holds too, and is in fact what we shall focus on: the statement for $\text{SL}(V)$ will follow because the $k^\times$-action on $H^*(\text{SL}(V); V_{[n,m]})$ is trivial in a stable range.

**Leitfaden.** The reader interested only in the proof of Theorem $A$ may read Sections $1$–$3$ and then skip to Section $6$.

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**Part 1: Stable twisted cohomology of $\text{SL}_n(k)$**

1. **Stability considerations**

We will say that a cohomological statement about $\text{GL}(V)$ or $\text{SL}(V)$ holds “in a stable range of degrees” if it holds in a range of cohomological degrees tending
to infinity with \( \dim(V) \). For homological stability we will use the framework of Putman–Sam \([PS17]\). 

1.1. **Coefficient systems.** The modules \( V \) and \( V^\vee \) for \( \text{GL}(V) \) both fit into the general framework developed in \([PS17]\). We recall it out briefly as follows. There is a category called \( \text{VIC}(k) \) whose objects are the finite-dimensional \( k \)-modules, and whose morphisms from \( V \) to \( W \) are given by a linear injection \( f : V \to W \) along with a choice of subspace \( U \leq W \) complementary to \( f(V) \). A **coefficient system** will for us be defined to be a functor

\[
F : \text{VIC}(k) \to k\text{-Mod},
\]

abbreviated a \( \text{VIC}(k) \)-module. Here are two examples. The first is \( I(V) := V \), with its functoriality given on a morphism \( (f,U) : V \to W \) by \( I(f,U) = f : V \to W \). The second is \( I^\vee(V) := V^\vee \), given on a morphism \( (f,U) : V \to W \) by

\[
I^\vee(f,U) : V^\vee \to (f(V))^\vee \oplus U^\vee = W^\vee.
\]

From these, for finite sets \( S \) and \( T \) we can form coefficient systems \( I^{\otimes S} \otimes (I^\vee)^{\otimes T} \), whose values at \( V \) are of course \( V^{\otimes S} \otimes (V^\vee)^{\otimes T} \). (Here the tensor product is, of course, taken over \( k \).) Both \( I \) and \( I^\vee \) are finitely-generated \( \text{VIC}(k) \)-module in the sense of \([PS17]\), and it is easy to check that tensor products of finitely-generated \( \text{VIC}(k) \)-modules are again finitely-generated: thus \( I^{\otimes S} \otimes (I^\vee)^{\otimes T} \) is finitely-generated.

The evaluation maps yield commutative squares

\[
\begin{array}{ccc}
V \otimes V^\vee & \xrightarrow{ev_V} & k \\
\downarrow{id \otimes f^\vee} & & \downarrow{id} \\
V \otimes (f(V))^\vee & \xrightarrow{inc \otimes inc} & (f(V) \oplus U) \otimes (f(V)^\vee \oplus U^\vee) \xrightarrow{ev_{f(V) \oplus U}} k
\end{array}
\]

giving a morphism of coefficient systems \( ev : I \otimes I^\vee \to k \), the target being the constant coefficient system. We can therefore form the coefficient systems

\[
I^{[S,T]} := \text{Ker} \left( I^{\otimes S} \otimes (I^\vee)^{\otimes T} \to \bigoplus_{s \in S} I^{\otimes S-s} \otimes (I^\vee)^{\otimes T-t} \right).
\]

This is a sub-\( \text{VIC}(k) \)-module of the finitely-generated \( \text{VIC}(k) \)-module \( I^{\otimes S} \otimes (I^\vee)^{\otimes T} \), so by \([PS17]\) Theorem D) it too is finitely-generated.

**Remark 1.1.** In order to deal with \( \text{SL} \), there is an oriented version of the category \( \text{VIC}(k) \) called \( \text{VIC}(k,\{1\}) \). Its objects are finite-dimensional \( k \)-modules equipped with an orientation (i.e. a nonzero vector of its top exterior power) and the data of a morphism \( (f,U) : V \to W \) is equipped with an orientation of \( U \) so that the two orientations of \( f(V) \oplus U = W \) agree. The result \([PS17]\) Theorem D) in fact applies to \( \text{VIC}(k,\{1\}) \), so \( I, I^\vee, I^{\otimes S} \otimes (I^\vee)^{\otimes T}, \) and \( I^{[S,T]} \) are all finitely-generated over this category.

**Example 1.2.** The coefficient system \( I^{[1,1]} \) associates to \( V \) the vector space

\[
\text{sl}(V) := \text{Ker}(ev_V : V \otimes V^\vee \to k).
\]

When \( V = k^n \) we will sometime denote this \( \text{sl}_n(k) \) instead.
1.2. Stability. By [PS17, Theorem L], and using Remark 1.1, it follows that for fixed finite sets $S$ and $T$ the maps
\[ H_k(\text{SL}(V); V \otimes (V^*)^\otimes T) \to H_k(\text{SL}(V \oplus k); (V \oplus k)^\otimes S \otimes ((V \oplus k)^*)^\otimes T) \]
\[ H_k(\text{SL}(V); V^{[S,T]}) \to H_k(\text{SL}(V \oplus k); (V \oplus k)^{[S,T]}) \]
are isomorphisms in a stable range of degrees.

We can obtain a similar result on cohomology, by dualising. Recalling that we defined $V^{[S,T]} = \text{Coker} \left( \bigoplus_{s \in S \atop t \in T} V \otimes s - V \otimes t \to V \otimes S - V \otimes T \right)$, we have $V^{\vee [S,T]} \cong V^{[T,S]}$. By the Universal Coefficient Theorem the maps
\[ H^k(\text{SL}(V \oplus k); (V \oplus k)^{\otimes T} \otimes (V^* \otimes S)) \to H^k(\text{SL}(V); V^\otimes S \otimes (V^* \otimes T)) \]
\[ H^k(\text{SL}(V \oplus k); (V \oplus k)^{[T,S]}) \to H^k(\text{SL}(V); V^{[T,S]}) \]
are also isomorphisms in a stable range of degrees.

1.3. SL vs. GL. The extension
\[ 1 \to \text{SL}(V) \to \text{GL}(V) \overset{\text{det}}{\to} k^\times \to 1 \]
induces an $k^\times$-action on $H^*(\text{SL}(V); V^{[S,T]})$.

Lemma 1.3. This action is trivial in a stable range of degrees.

Proof. Choose a decomposition $V = W \oplus k$. The cohomological stability result above applies to show that the natural map
\[ H^*(\text{SL}(W \oplus \mathbb{F}_p); (W \oplus \mathbb{F}_p)^{[S,T]}) \to H^*(\text{SL}(W); W^{[S,T]}) \]
is an isomorphism in a stable range of degrees. For the $k^\times$-action on the source given by conjugation with diag$(1_W, k^\times)$, and the trivial $k^\times$-action on the target, this map is equivariant, and so in the range in which this map is an isomorphism the $k^\times$-action on the domain is trivial.

Corollary 1.4. The natural map
\[ H^*(\text{GL}(V); V^{[S,T]}) \to H^*(\text{SL}(V); V^{[S,T]}) \]
is an isomorphism in a stable range of degrees.

Proof. As $|k^\times| = q - 1$ is coprime to $p$, the spectral sequence for the extension (1.1) collapses to give $H^*(\text{GL}(V); V^{[S,T]}) \cong \tilde{H}^*(\text{SL}(V); V^{[S,T]})^{k^\times}$. However, as the $k^\times$-action is trivial in a stable range of degrees, the latter is $H^*(\text{SL}(V); V^{[S,T]})$ in this range.

2. Functor homology

Our initial goal is to calculate $H^*(\text{GL}(V); V^{\otimes S} \otimes (V^*)^\otimes T)$, which we will do using methods of functor homology. We have attempted to keep the actual use of functor homology in the proofs, and to formulate statements only at the level of group (co)homology. There are no new ideas in the proofs, which simply combine results extracted from the functor homology literature. We are grateful to C. Vespa for explaining how to do so.
2.1. Defining cohomology classes. For a finite-dimensional \( k \)-vector space \( V \) there is an isomorphism
\[
H^*(\text{GL}(V); V \otimes V^\vee) = \text{Ext}_{\text{GL}(V)}^*(k, V \otimes V^\vee) \cong \text{Ext}_{\text{GL}(V)}^*(V, V),
\]
and the latter has an \( k \)-algebra structure by the Yoneda product. We define
\[
\text{Ext}_{\text{GL}}^*(I, I) := \lim_{\dim(V) \to \infty} \text{Ext}_{\text{GL}(V)}^*(V, V),
\]
which again has an \( k \)-algebra structure.

**Theorem 2.1.** There are classes \( x^{[i]} \in \text{Ext}_{\text{GL}}^{2i}(I, I) \) such that the map
\[
\Gamma_k[x] = k\{x^{[0]}, x^{[1]}, x^{[2]}, \ldots\} \longrightarrow \text{Ext}_{\text{GL}}^*(I, I)
\]
is an isomorphism of \( k \)-algebras from the free divided power algebra on \( x = x^{[1]} \).

For a finite set \( S \) and a function \( \ell : S \to \mathbb{N} \), we can produce cohomology classes
\[
\kappa(\ell) := \prod_{s \in S} x^{[\ell(s)]} \in \text{Ext}_{\text{GL}}^{\sum_s \ell(s)}(I \otimes S, I \otimes S).
\]
Given in addition a bijection \( f : T \to S \), we may compose with the isomorphism induced by \( I^\otimes f : I^\otimes T \to I^\otimes S \) to obtain classes
\[
\kappa(\ell, f) \in \text{Ext}_{\text{GL}}^{\sum_s \ell(s)}(I \otimes T, I \otimes S).
\]
Writing \( \text{Bij}(T, S) \) for the set of bijections from \( T \) to \( S \), and \( k\{\text{Bij}(T, S)\} \) for the free \( k \)-module on this set, this construction defines a map
\[
\Psi_{S,T} : \Gamma_k[x]^{\otimes S} \otimes k\{\text{Bij}(T, S)\} \longrightarrow \text{Ext}_{\text{GL}}^*(I \otimes T, I \otimes S).
\]
We will often write \( \Gamma_k[x]^{\otimes S} = \Gamma_k[x_s | s \in S] \).

**Theorem 2.2.** The map \( \Psi_{S,T} \) is an isomorphism for all finite sets \( S \) and \( T \).

By the stability discussion in Section 1.2 the map
\[
\text{Ext}_{\text{GL}}^*(I \otimes T, I \otimes S) \longrightarrow \text{Ext}_{\text{GL}(V)}^*(V \otimes T, V \otimes S) \cong H^*(\text{GL}(V); V \otimes S \otimes (V^\vee) \otimes T)
\]
is an isomorphism in a stable range of degrees, so Theorem 2.2 determines the latter cohomology groups in a stable range.

2.2. Proof of Theorems 2.1 and 2.2. Let \( F \) denote the category of functors from finite-dimensional \( k \)-modules to \( k \)-modules. This is an abelian category with a set of projective generators, so one may do homological algebra in this category.

For each \( k \)-module \( V \) there is a functor
\[
F \mapsto F(V) : F \longrightarrow k[\text{GL}(V)]-\text{mod}
\]
which is exact, so induces a map \( \text{Ext}_{F}^*(F, G) \to \text{Ext}_{\text{GL}(V)}^*(F(V), G(V)) \) for each \( V \). Taking the limit as \( \dim(V) \to \infty \), these assemble into a map
\[
\text{Ext}_{F}^*(F, G) \longrightarrow \text{Ext}_{\text{GL}}^*(F, G),
\]
which by [FTSS99, Theorem A.1] or [Bet99] is an isomorphism.

Taking \( F \) and \( G \) to be the “identity” functor \( I \), combining Théorème 7.3 and Section 11 of [FLS94] identifies \( \text{Ext}_I^*(I, I) \) with the divided power algebra \( \Gamma_k[x] = k\{x^{[0]}, x^{[1]}, x^{[2]}, \ldots\} \). Combined with the previous paragraph this gives Theorem 2.1.
Remark 2.3. The paper [FLS94] describes a specific choice of the generator $x$, but it will be convenient for us to (perhaps) change this choice by a unit. Let $\tilde{V}$ be a lift of $V$ from $k$ to the length 2 Witt vectors $W_2(k)$, and consider the extension
\begin{equation}
1 \longrightarrow SL(\tilde{V}, p) \longrightarrow SL(\tilde{V}) \longrightarrow SL(V) \longrightarrow 1,
\end{equation}
where the right-hand map is reduction modulo $p$. There is an isomorphism given by $I + pA \rightarrow A : SL(\tilde{V}, p) \to sl(V)$. The Euler class of this extension is an element $e \in H^2(SL(V); sl(V))$. The exact sequence $0 \to sl(V) \to V \otimes V' \mathrel{\xrightarrow{\cong}} k \to 0$ gives
\[H^1(SL(V); k) \mathrel{\xrightarrow{\partial}} H^2(SL(V); sl(V)) \longrightarrow H^2(SL(V); V \otimes V') \mathrel{\xrightarrow{\cong}} H^2(SL(V); k)\]
and the outer terms vanish (for dim($V$) large enough) by the theorem of Quillen [Qui72]. We define $x' \in H^2(SL(V); V \otimes V')$ to be the image of the Euler class $e$. One may check that this does not depend on the choice of lift $\tilde{V}$, and that these classes are compatible under stabilisation so give an $x' \in \operatorname{Ext}^2_{GL}(I, I)$. It is non zero.

If it were then the extensions (2.2) would be split for dim($V$) large enough, but they are not: see [Sah77, Proposition 0.3]. Thus $x' = u \cdot x$ for some $u \in k^\times$, so we can let $(x')^U := u^{-1} \cdot x^U$ so that $\operatorname{Ext}^2_{GL}(I, I) = \Gamma_k[x']$. In Theorem 2.1 we take this $x'$ for $x$.

If $|S| \neq |T|$ then Pirashvili’s cancellation lemma [BP94, Theorem A.1] shows that $\operatorname{Ext}^2_{GL}(I^T \otimes S, I^S) \cong \operatorname{Ext}^2_{GL}(I^S \otimes T, I^S)$ vanishes. If $|S| = |T|$ then [FFSS99, Corollary 1.8], using Pirashvili’s cancellation lemma to neglect most terms, gives an isomorphism
\[\operatorname{Ext}^2_T(I, I)^{\otimes S} \otimes k\{\operatorname{Bij}(T, S)\} \cong \operatorname{Ext}^2_T(I^T \otimes S, I^S).
\]
Combined with the previous two paragraphs, and after checking that the maps which induce this isomorphism agree with those that we have described above, this gives Theorem 2.2

3. The walled Brauer category

3.1. Functoriality on the upward walled Brauer category. The map (2.1) is, by construction, $\Sigma_S \times \Sigma_T$-equivariant, but has more structure. The upward walled Brauer category $\mathfrak{uwBr}$ is the category with objects given by pairs $(S, T)$ of finite sets, and with morphisms $\mathfrak{uwBr}((S, T), (U, V))$ given by a pair of injections $f : S \to U$, $g : T \to V$ and as a bijection $m : U \setminus f(S) \to V \setminus g(T)$. We visualise such morphisms as in the figure below, where the composition is given by gluing such 1-dimensional cobordisms.

```
   S  \quad \quad \quad \quad T
    \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Similarly, the construction \((S, T) \mapsto \Gamma_k[x]^\otimes_S \otimes k\{\text{Bij}(T, S)\}\) defines a functor from \(\text{uwBr}\) to graded \(k\)-modules: a morphism \((f, g, m)\) acts by sending the element \(\prod_{u \in S} x^{\ell(u)}_u \otimes \sigma\) to \(\prod_{u \in T} x^{\ell'(u)}_u \otimes \sigma'\) where \(\ell'(u) = \ell(s)\) if \(u = f(s)\) and is 0 otherwise, and the bijection \(\phi' : V \to U\) is equal to \(f \circ \phi \circ g^{-1}\) on \(g(T) \subset V\) and is equal to \(m^{-1}\) on \(V \setminus g(T)\).

As \(x^{[0]} = 1 \in \text{Ext}_G(I, I)\), it follows from these descriptions that the map \([2.1]\) we have described is a natural transformation

\[
\Gamma_k[x]^\otimes - \otimes k\{\text{Bij}(\cdot, -)\} \Rightarrow \text{Ext}_G^\bullet(I^\otimes -, I^\otimes -)
\]

of functors from \(\text{uwBr}\) to graded \(k\)-modules. Theorem \([2.2]\) then says that it is in fact a natural isomorphism.

### 3.2. Functoriality on the full walled Brauer category

The discussion in this section is not needed for the proof of Theorem \([\text{C}]\) but will be used in the proof of Theorem \([\text{A}]\).

For \(d \in k\) the \emph{walled Brauer category} \(\text{wBr}_d\) is the \(k\)-linear category with objects given by pairs \((S, T)\) of finite sets, and with morphisms \(\text{wBr}_d((S, T), (U, V))\) given by the \(k\)-vector space with basis given by tuples \((f, g, m, n)\) where \(f : S' \to U'\) is a bijection from a subset \(S' \subset S\) to a subset \(U' \subset U\), \(g : T' \to V'\) is a bijection from a subset \(T' \subset T\) to a subset \(V' \subset V\), and \(m : S \setminus S' \to T \setminus T'\) and \(n : U \setminus U' \to V \setminus V'\) are bijections. We visualise such morphisms as in the figure below, where as shown the composition is given by gluing such 1-dimensional cobordisms, and replacing any circles that are formed by the scalar \(d \in k\).

![Diagram](image.png)

For \(V\) a finite-dimensional vector space, the functor

\[(S, T) \mapsto H^\bullet(\text{GL}(V); V^\otimes S \otimes (V^\vee)^\otimes T)\]

from \(\text{uwBr}\) to graded \(k\)-modules extends to a functor on \(\text{wBr}_d\) with \(d := \dim(V)\). In Section \([2.1]\) we constructed maps

\[
\psi_{S,T} : \Gamma_k[x]^\otimes S \otimes k\{\text{Bij}(T, S)\} \to H^\bullet(\text{GL}(V); V^\otimes S \otimes (V^\vee)^\otimes T),
\]

which are isomorphisms in a stable range by Theorem \([2.2]\) are natural transformations of functors on \(\text{uwBr}\), and we wish to explain how the \(\text{wBr}_d\)-functoriality of the target translates to the source.

Let \(\varepsilon_{s,t} : V^\otimes S \otimes (V^\vee)^\otimes T \to V^\otimes S \setminus s \otimes (V^\vee)^\otimes T \setminus t\) be given by evaluation of the \(s\)th and \(t\)th terms, and define a map

\[
\delta_{s,t} : \Gamma_k[x]^\otimes S \otimes k\{\text{Bij}(T, S)\} \to \Gamma_k[x]^\otimes S \setminus s \otimes k\{\text{Bij}(T \setminus t, S \setminus s)\}
\]

by the formula

\[
\prod_{u \in S} x^{\ell(u)}_u \otimes \sigma \mapsto \begin{cases} 
\dim(V) \prod_{u \in S \setminus s} x^{\ell(u)}_u \otimes \sigma|_{T \setminus t} & \sigma(t) = s \text{ and } \ell(s) = 0 \\
0 & \sigma(t) = s \text{ and } \ell(s) > 0 \\
\ell(s) + \ell(\sigma^{-1}(t)) \prod_{u \in S \setminus s} x^{\ell'(u)}_u \otimes \sigma' & \sigma(t) \neq s
\end{cases}
\]
where $\ell'$ is given by $\ell'((\sigma^{-1}(t)) = \ell(s) + \ell((\sigma^{-1}(s))$ and by the restriction of $\ell$ on all other elements of $S \setminus s$, and $\sigma'$ is given by $\sigma'((\sigma^{-1}(s)) = \sigma(t)$, and by the restriction of $\sigma$ on all other elements of $T \setminus t$.

**Lemma 3.1.** The square

$$\begin{array}{ccc}
\Gamma_k[x]^\otimes S \otimes k\{\text{Bij}(T, S)\} & \xrightarrow{\delta_{s,s'}} & \Gamma_k[x]^\otimes S \otimes k\{\text{Bij}(T \setminus t, S \setminus s)\} \\
\psi_{S,T} & & \psi_{S \setminus s, T \setminus t} \\
H^*(\text{GL}(V); V^\otimes S \otimes (V^\vee)^\otimes T) & \xrightarrow{\psi_{S,T}} & H^*(\text{GL}(V); V^\otimes S \otimes (V^\vee)^\otimes T) \setminus t)
\end{array}$$

commutes in a stable range of degrees.

**Proof.** The top terms are zero unless $|S| = |T|$, so choose an identification $T \cong S$ sending $t$ to $s$: then we are considering an evaluation map $\varepsilon_{s,s'}$. By definition of the vertical maps we have

$$\varepsilon_{s,s'} \psi_{S,S} \left( \prod_{u \in S} x_u^{[\ell(u)]} \otimes \sigma \right) = \varepsilon_{s,s^{-1}(s)} \psi_{S,S} \left( \prod_{u \in S} x_u^{[\ell(u)]} \otimes \text{Id}_S \right).$$

If $s = \sigma^{-1}(s)$ then, by taking tensor products, we can reduce to the case $S = \{s\}$. In this case the result is given by applying the evaluation map $x^{[\ell(s)]} \in H^{2\ell(s)}(\text{GL}(V); V \otimes V^\vee)$. If $\ell(s) > 0$ then the result of this evaluation map is 0, as in the stable range $H^{2\ell(s)}(\text{GL}(V); k) = 0$ in that case. If $\ell(s) = 0$ then the element $x^{[0]} \in H^0(\text{GL}(V); V \otimes V^\vee)$ is the covevaluation, so applying the evaluation map to it gives $\dim(V) \in k = H^0(\text{GL}(V); k)$.

If $s \neq s' := \sigma^{-1}(s)$ then, by taking tensor products, we can reduce to the case $S = \{s, s'\}$. In this case

$$\psi_{S,S} \left( x_s^{[\ell(s)]} \right) x_{s'}^{[\ell(s')]}, \in H^*(\text{GL}(V); V^\otimes S \otimes (V^\vee)^\otimes S)$$

is the cup product of the classes

$$\psi_{\{s\},\{s\}} \left( x_s^{[\ell(s)]} \right) \in H^*(\text{GL}(V); (V \otimes V^\vee)^\otimes \{s\}) \cong \text{Ext}^*_\text{GL}(V)(V, V)$$

$$\psi_{\{s',s\},\{s'\}} \left( x_{s'}^{[\ell(s')]}} \right) \in H^*(\text{GL}(V); (V \otimes V^\vee)^\otimes \{s'\}) \cong \text{Ext}^*_\text{GL}(V)(V, V),$$

and applying $\varepsilon_{s,s'}$ corresponds to evaluating the Yoneda product. By the divided power algebra structure described in Theorem 2.1 the result is

$$\left( \ell(s) + \ell(s') \right) x_s^{[\ell(s)]} x_{s'}^{[\ell(s')]},$$

which agrees with $\psi_{\{s'\},\{s\}} \delta_{s,s'}$ applied to $x_s^{[\ell(s)]} x_{s'}^{[\ell(s')]}, \otimes \text{Id}_{\{s,s'\}}$. \hfill \square

Rather than the formulas given above, we can interpret the functoriality of $\Gamma_k[x]^\otimes \otimes k\{\text{Bij}(\bullet, -)\}$ on the walled Brauer category by interpreting elements of $\Gamma_k[x]^\otimes S \otimes k\{\text{Bij}(T, S)\}$ as given graphically as shown to the right. That is, an element of $w\text{Br}((\emptyset, \emptyset), (S, T))$ with each strand labelled by an $x^{[i]}$.

Then the functoriality is given by concatenating with an element of the walled Brauer category, multiplying labels which now lie on the same strand together using the divided power multiplication, then setting any closed components labeled by $x^{[i]}$ with $i > 0$ equal to zero, and setting any closed components labeled by $x^{[0]}$ equal to $\dim(V)$.  

\[x^{[0]} \quad \mathcal{S} \quad x^{[0]} \quad \mathcal{T} \quad \cdots \]
4. Proof of Theorem B tensor powers

In this section we prove Theorem B with the representations $S_{\lambda,\mu}(V)$ replaced by $V^\otimes n \otimes (V^\vee)^\otimes m$, and without conditions on the size of $n$ and $m$. That is, that

$$H^i(GL(V); V^\otimes n \otimes (V^\vee)^\otimes m) \cong [V^\otimes n \otimes (V^\vee)^\otimes m \otimes X]^{	ext{GL}(V)}$$

for $i < 2p$, as long as dim$(V)$ is large enough. In Section 5 we will explain how to deduce from this the statement of Theorem C for the $S_{\lambda,\mu}(V)$: it is there that the conditions on $n$ and $m$ will arise. Our proof of (4.1) will be by calculating both sides and comparing them. Given the homological stability results of Section 1, the conditions on $n$ and $m$ will be deduced from this the statement of Theorem C for the $S_{\lambda,\mu}(V)$, as long as dim$(V)$ is large enough.

The results of the last section provide isomorphisms

$$k[\text{Bij}(T, S)] \xrightarrow{\sim} H^0(\text{GL}; I^\otimes S \otimes (I^\vee)^\otimes T)$$

given by inserting copies of $\text{coev} : k \to I \otimes I^\vee$ and permuting the $I^\vee$ terms. Let $S = T = \Sigma : = \{1, 2, \ldots, r\}$, identify $I^\otimes S \otimes (I^\vee)^\otimes T = (I \otimes I^\vee)^\otimes r$, so that the above gives an isomorphism

$$k[\Sigma_r] \xrightarrow{\sim} H^0(\text{GL}; (I \otimes I^\vee)^\otimes r)$$

where the $\Sigma_r$-action on the right-hand side by permuting the tensor factors corresponds on the left-hand side to the action of $\Sigma_r$ on itself by conjugation; to avoid confusion we write $\Sigma^{ad}_r$ for this $\Sigma_r$-set. One should visualise elements of $\Sigma^{ad}_r$ as permutations presented as disjoint cycles. The class of an $r$-cycle defines an element $c_r \in H^0(\text{GL}; \text{Sym}^r(I \otimes I^\vee))$, which is independent of the choice of $r$-cycle.

**Lemma 4.1.** The map

$$k[c_1, c_2, c_3, \ldots] \to H^0(\text{GL}; \text{Sym}^\bullet(I \otimes I^\vee))$$

is an isomorphism of graded $k$-algebras in gradings $\bullet < p$.

**Proof.** For $r < p$ taking $\Sigma_r$-coinvariants is exact, giving an isomorphism

$$k[\Sigma^{ad}_r]_{\Sigma_r} \xrightarrow{\sim} H^0(\text{GL}; \text{Sym}^r(I \otimes I^\vee)).$$

For $\sum_i i \cdot c_i = r$ the image of the monomial $c_1^a c_2^b \cdots c_r^e \in k[c_1, c_2, c_3, \ldots]$ in $H^0(\text{GL}; \text{Sym}^r(I \otimes I^\vee)) \cong k[\Sigma^{ad}_r]_{\Sigma_r}$ is the class of any permutation having precisely $a_i$-many $i$-cycles. This is visibly a bijection.

As in the introduction, define a graded ring object $X^\bullet$ in coefficient systems by

$$\text{Sym}^\bullet(I \otimes I^\vee)/(c_1, c_2, \ldots)$$

with grading doubled, so that $c_i$ has degree $2i$. The following two lemmas are somewhat technical, but serve to say that we may commute $H^*(\text{GL}; -$) with dividing out the action of the $c_i$, at least in gradings $\bullet < p$.

**Lemma 4.2.** The sequence $c_1, c_2, \ldots$ acts regularly on $\text{Sym}^\bullet(I \otimes I^\vee)$ in gradings $\bullet < p$.

**Proof.** For any $k$-module $V$ the composition

$$k[c_1, c_2, c_3, \ldots] \to H^0(\text{GL}; \text{Sym}^\bullet(I \otimes I^\vee)) \to H^0(\text{GL}(V); \text{Sym}^\bullet(V \otimes V^\vee))$$

defines elements $c_i \in \text{Sym}^i(V \otimes V^\vee)$, and it suffices to show that these are a regular sequence in $\text{Sym}^\bullet(V \otimes V^\vee)$ for $\bullet < p$ and all large enough dim$(V)$. This can be tested after base change to an algebraic closure of $k$, which we now implicitly do.

We identify $V \otimes V^\vee$ with $\text{End}(V)$ considered as an affine algebraic variety, and so identify $\text{Sym}^\bullet(V \otimes V^\vee)$ with the ring of regular functions on $\text{End}(V)$. There are
regular functions $\sigma_1, \sigma_2, \ldots, \sigma_{\dim(V)}$ given by the coefficients of the characteristic polynomial. The ideal $(\sigma_1, \ldots, \sigma_{\dim(V)})$ defines the subvariety of $\text{End}(V)$ consisting of those matrices with characteristic polynomial $t^{\dim(V)}$, i.e., the nilpotent matrices. This is well-known to have codimension $\dim(V) = \text{rk}(\text{GL}(V))$, see e.g. [Jan04] p. 64, so the sequence $\sigma_1, \ldots, \sigma_{\dim(V)}$ is regular.

The left-hand map in (1.2) is an isomorphism for $\bullet < p$, and in this range the right-hand map is an isomorphism for $\dim(V)$ large enough. As the $\sigma_i \in \text{Sym}^i(V \otimes V^*)$ are clearly $\text{GL}(V)$-invariant, they lie in the subring $k[c_1, c_2, \ldots] \subset \text{Sym}^\bullet(V \otimes V^*)$ in gradings $\bullet < p$, and as they form a regular sequence, in this range they must be equal to $c_i$ up to a unit and modulo decomposable$^2$. Thus the $c_i$ must also form a regular sequence in this range.

\[ \square \]

**Lemma 4.3.** The sequence $c_1, c_2, \ldots$ acts regularly on

\[ H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes \text{Sym}^\bullet(I \otimes I^\vee)) \]

in gradings $\bullet < p$.

**Proof.** As long as $r < p$ taking $\Sigma_r$-coinvariants is exact. Thus by Theorem 2.2 the cohomology of $\mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes \text{Sym}^\bullet(I \otimes I^\vee)$ is identified with the $\Sigma_r$-coinvariants of

\[ k[x_1^{(0)}, x_2^{(1)}, x_2^{(2)}, \ldots] \otimes \text{Sym}^{\otimes T}(1' \cup \Sigma_r S \cup \Sigma_2). \]

This is the vector space with basis the set $k[x_1^{(0)}, x_2^{(1)}, x_2^{(2)}, \ldots] \otimes \text{Sym}^{\otimes T}(1' \cup \Sigma_r S \cup \Sigma_2)$, so the coinvariants are identified with the vector space with basis the $\Sigma_r$-orbits of this set.

In this picture, multiplication by $c_i \in H^0(\text{GL}; \text{Sym}^\bullet(I \otimes I^\vee))$ corresponds to the map

\[ k\{(N^{S \cup T \cup \mathbb{R}}) \times \text{Bij}(T \cup (r-i, S \cup r-i))/\Sigma_r \}_r \rightarrow k\{(N^{S \cup T \cup (1,2,\ldots,r)}) \times \text{Bij}(T \cup (r, S \cup \mathbb{L}))/\Sigma_r \} \]

which adjoins the $i$-cycle $(r-1+1, r-1+2, \ldots, r)$ with all labels $0 \in \mathbb{N}$, so it is clear that the sequence of $c_i$’s acts regularly.

\[ \square \]

**Corollary 4.4.** The natural map

\[ H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes \text{Sym}^\bullet(I \otimes I^\vee))/(c_1, c_2, \ldots) \rightarrow H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes X^{2\bullet}) \]

is an isomorphism in gradings $\bullet < p$.

**Proof.** Let $X_i^\bullet$ denote $\text{Sym}^\bullet(I \otimes I^\vee)/(c_1, \ldots, c_i)$ with grading doubled, so that $X^\bullet = X_{\Sigma_r}$. We will show by induction over $i$ that the natural map

\[ H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes \text{Sym}^\bullet(I \otimes I^\vee))/(c_1, c_2, \ldots, c_i) \rightarrow H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes X_{i+1}^{2\bullet}) \]

is an isomorphism for $\bullet < p$; certainly it holds for $i = 0$. By Lemma 4.3 there are short exact sequences

\[ 0 \rightarrow X_i^{2\bullet}[2i] \rightarrow X_i^\bullet \rightarrow X_{i+1}^\bullet \rightarrow 0, \]

and as $c_{i+1}, c_{i+2}, \ldots$ acts regularly on $H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes X_i^\bullet)$, by the inductive assumption and Lemma 4.3 it follows that $H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes X_{i+1}^\bullet) = H^\bullet(\text{GL}; \mathcal{I}^S \otimes (I^\vee)^{\otimes T} \otimes X_{i+1}^\bullet)/(c_{i+1})$.

The class of an $(r+1)$-cycle in $k\{\Sigma_{i+r}^d\}$ gives an element

\[ d_r \in k\{\Sigma_{i+r}^d\}_r \rightarrow H^0(\text{GL}; I \otimes I^\vee \otimes \text{Sym}^\bullet(I \otimes I^\vee)), \]

independent of the choice of $(r+1)$-cycle, and hence in the quotient an element

\[ d_r \in H^0(\text{GL}; I \otimes I^\vee \otimes X^{2\bullet}). \]

$^2$Possibly they are simply equal up to a unit, but we prefer not to worry about this.
Using the $k$-algebra structure of $X^\bullet$ there is an induced map
\[ k\{d_0, d_1, d_2, \ldots \} \otimes S \to H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes S} \otimes X^\bullet) \]
and then acting on the $I^\vee$'s by bijections gives a map
\[ (4.3) \quad k\{d_0, d_1, d_2, \ldots \} \otimes k\{\text{Bij}(T, S)\} \to H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes T} \otimes X^\bullet). \]
This is in fact a natural transformation of functors from the upwards walled Brauer category to $k$-modules, as in Section 3.1.

**Lemma 4.5.** The map (4.3) is an isomorphism for $\bullet < 2p$ and all $S$ and $T$.

**Proof.** By Corollary 4.4 we have an identification
\[ H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes T} \otimes \text{Sym}(I \otimes I^\vee))/(c_1, c_2, \ldots) \cong H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes T} \otimes X^{2\bullet}) \]
for $\bullet < p$. For $r < p$ the object $\text{Sym}^r(I \otimes I^\vee)$ is a summand of $(I \otimes I^\vee)^{\otimes r}$, so the left-hand side is a subquotient of $H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes T} \otimes X^{2\bullet})$, and hence by Theorem 2.2 it vanishes unless $|S| = |T|$.

We therefore choose a bijection $S \cong T$. As in the proof of Lemma 4.3 the map
\[ k\{\Sigma^d_{S \sqcup \Sigma r}\} \to H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes S} \otimes \text{Sym}(I \otimes I^\vee)) \]
given by acting on the coevaluation element by permuting the $I^\vee$'s, is an isomorphism, and this induces a map
\[ k\{\Sigma^d_{S \sqcup \Sigma r}/\Sigma r\} \to H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes S} \otimes \text{Sym}^r(I \otimes I^\vee)) \]
which for $r < p$ is an isomorphism. Multiplying by $c_i \in H^0(\text{GL}; \text{Sym}^r(I \otimes I^\vee))$ on the right-hand side translates on the left-hand side to the map $\Sigma^d_{S \sqcup \Sigma r}/\Sigma r \to \Sigma^d_{S \sqcup \Sigma r}/\Sigma r$ which adds an $i$-cycle of elements in $r$. Thus the quotient by the $c_i$ on the right-hand side translates on the left-hand side to killing those basis elements which are represented by a permutation having a cycle of elements in $r$. Thus what remains are the permutations of $S \sqcup \Sigma r$ where every cycle contains an element of $S$ (let us call this $F_r \subset \Sigma^d_{S \sqcup \Sigma r}$), modulo relabelling the elements $\{1, 2, \ldots, r\}$, i.e. the induced map
\[ (4.4) \quad \bigoplus_{r \geq 0} k\{F_r/\Sigma r\} \to H^0(\text{GL}; I^{\otimes S} \otimes (I^\vee)^{\otimes S} \otimes X^\bullet) \]
is an isomorphism for $\bullet < 2p$.

For a function $\ell : S \to \mathbb{N}$ and a permutation $\sigma \in \Sigma^d_{S \sqcup \Sigma r}$ written in cycle form as
\[ \sigma = (s_1, s_2, \ldots, s_{c_1})(s_{c_1+1}, s_{c_1+2}, \ldots, s_{c_1+c_2}) \cdots (s_{c_1+c_{r-1}+1}, \ldots, s_{c_1+c_{r-1}+c_r}), \]
the image of the element $\otimes_{s \in S} d_{\ell(s)} \otimes \sigma$ under (4.3) agrees with the image of
\[ (s_1, s_2, \ldots, s_{c_1}, s_{c_1+1}, s_{c_1+2}, \ldots) \cdots (s_{c_1+c_{r-1}+1}, s_{c_1+c_{r-1}+2}, \ldots) \in F_r/\Sigma r \]
under (4.4), where the $\ast$ denote elements of $r$. This construction defines an isomorphism
\[ k\{d_0, d_1, d_2, \ldots \} \otimes k\{\Sigma^d_S\} \cong \bigoplus_{r \geq 0} k\{F_r/\Sigma r\}, \]
so the isomorphism (4.4) for $\bullet < 2p$ shows (4.3) is an isomorphism in this range. \(\square\)
Corollary 4.6. For \( i < 2p \) there is an identification

\[
H^i(\mathbb{L}; I^\otimes S \otimes (I^\vee)^\otimes T) \cong H^0(\mathbb{L}; I^\otimes S \otimes (I^\vee)^\otimes T \otimes X^i)
\]

of functors from the upward walled Brauer category to \( k \)-modules.

Proof. Identify the domain of \([4.3]\) with the domain of \([2.1]\) via \( d_i \mapsto x^i \). \( \square \)

5. Proof of Theorem C: the general case

5.1. Some semisimplicity. As long as \( p > n, m \) the algebra \( k[\Sigma_n \times \Sigma_m] \) is semisimple, and for partitions \( \lambda \vdash n \) and \( \mu \vdash m \) the modules \( S^\lambda \otimes S^\mu \) described in the introduction give a complete set of simple modules. Now \( \Sigma_n \times \Sigma_m \) acts on \( V_{[n,m]} \), and so defining

\[
S_{\lambda,\mu}(V) := \text{Hom}_{k[\Sigma_n \times \Sigma_m]}(S^\lambda \otimes S^\mu, V_{[n,m]})
\]

the evaluation map

\[
\bigoplus_{\lambda,\mu} S^\lambda \otimes S^\mu \otimes S_{\lambda,\mu}(V) \rightarrow V_{[n,m]}
\]

is an isomorphism.

As a final ingredient we should like to know that the quotient map

\[
q : V^\otimes n \otimes (V^\vee)^\otimes m \rightarrow V_{[n,m]}
\]

is split a a map of \( \text{GL}(V) \)-modules. Unfortunately this is not generally true: in the exact sequence

\[
0 \rightarrow k \xrightarrow{\text{coev}} V \otimes V^\vee \xrightarrow{\pi} V_{[1,1]} \rightarrow 0
\]

we have \( ev : H_0(\text{GL}(V); V \otimes V^\vee) \rightarrow k \), and \( ev \circ \text{coev} : k \rightarrow k \) is multiplication by \( \dim(V) \); thus there is an exact sequence

\[
\cdots \rightarrow H_1(\text{GL}(V); V_{[1,1]}) \xrightarrow{\partial} k \xrightarrow{-\dim(V)} k \xrightarrow{\pi} H_0(\text{GL}(V); V_{[1,1]}) \rightarrow 0
\]

and so if \( \dim(V) \equiv 0 \mod p \) then \( \partial \) is nontrivial and so \([5.2]\) cannot be \( \text{GL}(V) \)-equivariantly split. However, we have the following partial result, which will suffice.

Proposition 5.1. If \( \dim(V) \equiv \frac{p+1}{2} \mod p \) and \( n + m \leq \frac{p+1}{2} \) then the \( k[\text{GL}(V)] \)-module \( V^\otimes n \otimes (V^\vee)^\otimes m \) is semisimple. In particular, under these conditions the quotient map \([5.1]\) is split.

Proof. Let \( \delta := \dim(V) \). Under the stated conditions, by \([ AST17\) Theorem 6.1\]) the walled Brauer algebra \( B_{n,m}(\delta) \) is semisimple. Letting \( U := U_1 gl(V) \) denote the hyperalgebra associated to \( k[\text{GL}(V)] \), by \([ DDS14\) Corollary 7.2\]) the map

\[
B_{n,m}(\delta) \rightarrow \text{End}_U(V^\otimes n \otimes (V^\vee)^\otimes m)
\]

is an isomorphism (because \( \dim(V) \geq r + s \) under the stated conditions). By the Double Centraliser Theorem \([ EGH^\dag\) Theorem 5.18.1\]) it follows that \( U \) is semisimple, and that there is a decomposition

\[
V^\otimes n \otimes (V^\vee)^\otimes m \cong \bigoplus_{\alpha} M_\alpha \otimes \text{Hom}(M_\alpha, V^\otimes n \otimes (V^\vee)^\otimes m)
\]

where the \( M_\alpha \) are a complete set of simple \( B_{n,m}(\delta) \)-modules, and each \( \text{Hom}(M_\alpha, V^\otimes n \otimes (V^\vee)^\otimes m) \) is either a simple \( U \)-module or zero. Using now that \( U \)- and \( k[\text{GL}(V)] \)-submodules agree \([ JKS1\) Theorem 8.2.13 (i)\]), this is a decomposition of \( V^\otimes n \otimes (V^\vee)^\otimes m \) as a sum of simple \( k[\text{GL}(V)] \)-submodules, showing that it is indeed semisimple. \( \square \)
5.2. Proof of Theorem C. To prove Theorem C it suffices to identify
\[ H^i(GL(V); V_{[n,m]}) \cong [V_{[n,m]} \otimes X^j]_{GL(V)} \]
as $k[\Sigma_n \times \Sigma_m]$-modules, for $i < 2p$ and $n + m \leq \frac{p+1}{2}$ and all large enough $\dim(V)$. Applying the exact functor $\text{Hom}$ we will often omit from the notation.

By the results of Section 4 the two sides stabilise with $\dim(V)$, so it suffices to establish this identity for all large enough $\dim(V)$ with $\dim(V) \equiv \frac{p+1}{2} \mod p$. In this case by Proposition 5.1 and the assumption $n + m \leq \frac{p+1}{2}$ the $GL(V)$-representation $V \otimes_n (V^\vee)^{\otimes m}$ is semisimple, and so the quotient map in the sequence
\[
\bigoplus_{i=1}^n \bigoplus_{j=1}^m V^{\otimes_{n-1}} \otimes (V^\vee)^{\otimes_{m-1}} \to V^{\otimes_n} \otimes (V^\vee)^{\otimes m} \to V_{[n,m]} \to 0
\]
is split and hence this sequence remains exact after applying $H^i(GL(V); -)$ or $[- \otimes X^j]_{GL(V)}$. Combining this with the natural isomorphism $H^i(GL(V); -) \cong [- \otimes X^j]_{GL(V)}$ (for $i < 2p$) of functors on the upward Brauer category, given by Corollary 4.6 yields the isomorphism \[5.3\] as required.

Part 2: Stable cohomology of congruence subgroups

For the rest of the paper all cohomology will be taken with $F_p$-coefficients, which we will often omit from the notation.

6. Recollections

6.1. Completed cohomology. Following [CE16] we consider the system of congruence subgroups defined as the kernels
\[ 1 \to \text{SL}_n(Z, p^r) \to \text{SL}_n(Z/p^r) \to 1 \]
as a pro-group, $\{\text{SL}_n(Z, p^r)\}_r$, and let the associated completed ($\mathbb{F}_p$-)cohomology\(^3\)
\[ \widetilde{H}^i(\text{SL}_n(Z)) := \varprojlim_{r \to \infty} H^i(\text{SL}_n(Z, p^r)) \]
be the cohomology of this pro-group. By the main theorem of [CE16] these groups enjoy homological stability with respect to $n$, and (hence) the outer action of the pro-(finite group) $\text{SL}_n(Z/p^r) := \{\text{SL}_n(Z/p^r)\}_r$ on these groups is trivial in the stable range. Taking colimits of the Leray–Hochschild–Serre spectral sequences of the \[6.1\] gives a spectral sequence
\[ E_2^{s,t} = H^s_{\text{cts}}(\text{SL}_n(Z/p^r); \widetilde{H}^t(\text{SL}_n(Z))) \Rightarrow H^{s+t}(\text{SL}_n(Z)), \]
and in the stable range the coefficient system is untwisted. More generally, considering the extension of pro-groups
\[ 1 \to \{\text{SL}_n(Z, p^{k+r})\}_r \to \text{SL}_n(Z, p^k) \to \{\text{SL}_n(Z/p^{r+k}, p^k)\}_r \to 1 \]
gives a spectral sequence
\[ E_2^{s,t} = H^s_{\text{cts}}(\text{SL}_n(Z, p^k); \widetilde{H}^t(\text{SL}_n(Z))) \Rightarrow H^{s+t}(\text{SL}_n(Z, p^k)), \]
again untwisted in a stable range, whose $E_2$-term is quite accessible for $k > 0$ as then $\text{SL}_n(Z, p^k)$ is a $p$-adic analytic group.

\[^3\]We use the notation $\widetilde{H}$ for completed cohomology, following Calegari and Emerton, and pre-emptively apologise for the confusion with reduced cohomology that it will no doubt cause topologists.
6.2. Relation to algebraic $K$-theory. We rephrase \cite[Section 2.3]{Cal15}. Let $K(-)$ denote the algebraic $K$-theory spectrum, so that there are acyclic maps $\text{BGL}(-) \to \Omega^{\infty} K(-)$. Write $K^{\text{top}}(\mathbb{Z}_p) := \underset{r \to \infty}{\text{holim}} K(\mathbb{Z}/p^r)$, so there are induced maps of pro-spectra

$$K(\mathbb{Z}) \to K(\mathbb{Z}_p) \to K^{\text{top}}(\mathbb{Z}_p) \to \{K(\mathbb{Z}/p^r)\}_r.$$  

Let us write $SK(-) = \tau_{\geq 2} K(-)$ for the 1-connected cover of the algebraic $K$-theory spectrum, and $\kappa : SK(\mathbb{Z}) \to SK(\mathbb{Z}_p)$ for the induced map. Using this we form the diagram of pro-spaces

$$\{\text{BSL}(\mathbb{Z}, p^r)\}_r \to \{\text{hofil}(\Omega^\infty \kappa'_r)\}_r \leftarrow \text{hofil}(\Omega^\infty \kappa') \leftarrow \text{hofil}(\Omega^\infty \kappa)$$

whose columns are fibration sequences. All columns but the first are fibrations of (pro-)infinite loop spaces, so the coefficient system given by the cohomology of the fibres is trivial. The same property holds for the first column by the main theorem of \cite{CE16}.

The map in the middle row is a cohomology isomorphism, as is the left-hand map in the bottom row, as they arise as covers of the acyclic maps $\text{BGL}(\mathbb{Z}) \to \Omega^{\infty}_0 K(\mathbb{Z})$ and $\text{BGL}(\mathbb{Z}/p^r) \to \Omega^{\infty}_0 K(\mathbb{Z}/p^r)$. For the middle map of the bottom row, the argument of \cite[Sublemma 2.18]{Cal15} shows that for each $i$ the dimension of $H_i(\Omega^\infty SK(\mathbb{Z}/p^r))$ is finite and bounded independently of $r$. By \cite[Theorem B]{Goe96} it then follows that the map

$$H_*(\Omega^\infty SK^{\text{top}}(\mathbb{Z}_p)) \to \lim_{r \to \infty} H_*(\Omega^\infty SK(\mathbb{Z}/p^r))$$

is an isomorphism (in principle the limit is taken in the category of $\mathbb{F}_p$-coalgebras, but as the corresponding limit in $\mathbb{F}_p$-modules has finite type, it agrees with the limit in $\mathbb{F}_p$-coalgebras), and so dualising shows that the middle map of the bottom row is a $\mathbb{F}_p$-cohomology isomorphism. The right-hand map of the bottom row is a $\mathbb{F}_p$-cohomology isomorphism by \cite[Theorem C]{HM94}.

Thus by the Zeeman comparison theorem the maps in the top row are all $\mathbb{F}_p$-cohomology isomorphisms: in particular

$$\tilde{H}^*(SL(\mathbb{Z})) \cong H^*(\text{hofil}(\Omega^\infty \kappa)),$$

relating the stable completed cohomology to the cohomology of the fibre of the completion map in $K$-theory.

6.3. Completed cohomology at odd regular primes. Let $p$ be an odd regular prime. As discussed in \cite[Section 3]{Rog03} it follows from the (affirmed) Quillen–Lichtenbaum conjecture that there is a $p$-adic equivalence $K(Z; \mathbb{Z}_p) \simeq j \oplus \Sigma^5 ko$ where $j$ is the $p$-adic image-of-$J$ spectrum and $ko$ is the $p$-adically completed real $K$-theory spectrum; similarly, by \cite{BM95} there is a $p$-adic equivalence $K(Z; \mathbb{Z}_p) \simeq j \oplus \Sigma j \oplus \Sigma^3 ku$, where $ku$ is the $p$-adically completed complex $K$-theory spectrum, and the completion map is the identity on the first summand and the map $\Sigma^5 ko \to \Sigma^3 ku \to \Sigma^3 ku$ given by complexification followed by multiplication by the Bott class on the second summand. As $j$ has the $(2p-3)$-type of $H\mathbb{Z}_p$, it follows that in degrees $* < 2p - 2$ the right-hand column in the large diagram above is equivalent to

$$\Omega^\infty \Sigma^2 ko \to \Omega^\infty \Sigma^5 ko \to \Omega^\infty \Sigma^3 ku.$$
In particular, in this range of degrees there are isomorphisms
\[
\begin{align*}
F_p[x_2, x_6, x_{10}, \ldots] &\cong H^*(\hocolim(\Omega^\infty \kappa)) \cong \tilde{H}^*(\mathbb{Z}) \\
\Lambda^*_p[y_3, y_9, y_{13}, \ldots] &\cong H^*(\Omega^\infty \mathbb{K}(\mathbb{Z})) \\
\Lambda^*_p[x_3, y_5, y_7, \ldots] &\cong H^*(\Omega^\infty \mathbb{K}(\mathbb{Z}_p))
\end{align*}
\]

and the Serre spectral sequence has the form
\[
\Lambda^*_p[y_3, y_5, y_7, \ldots] \otimes F_p[x_2, x_6, x_{10}, \ldots] \Rightarrow \Lambda^*_p[y_5, y_9, y_{13}, \ldots]
\]
with generating differentials \(d_{4k+3}(x_{4k+2}) = y_{4k+3}\), for an appropriate choice of generators \(x_i\) and \(y_i\). Using the large diagram above, this describes the spectral sequence \([6, 2]\) in the case \(k = 0\) and \(n = \infty\).

### 6.4. Completed cohomology at irregular primes.

The equivalence \(K(\mathbb{Z}_p; \mathbb{Z}_p) \cong j \oplus \Sigma j \oplus \Sigma^3 ku\) above holds for all odd primes, so in degrees \(1 < * < 2p - 2\) we have
\[
K_i(\mathbb{Z}_p; \mathbb{Z}_p) \cong \begin{cases} 
\mathbb{Z}_p & i \text{ odd} \\
0 & \text{else}
\end{cases}
\]

The structure of \(K_*(\mathbb{Z}; \mathbb{Z}_p)\) at irregular primes is more complicated. Following [Wei05], in degrees \(1 < * < 2(2p - 3)\) we have
\[
K_*(\mathbb{Z}; \mathbb{Z}_p) \cong \begin{cases} 
\mathbb{Z}_p/\text{Num}(B_{2k}/4k) & * = 4k - 2 \\
\mathbb{Z}_p & * = 4k - 1 \\
? & * = 4k \\
\mathbb{Z}_p & * = 4k + 1
\end{cases}
\]

where \(B_{2k}\) denotes the \(2k\)th Bernoulli number, and \(?\) is unknown but finite, and has no \(\ell\)-torsion assuming the Vandiver conjecture for the prime \(\ell\) (which has been checked for \(\ell \leq 2^{11}\)).

**Lemma 6.1.** If \(A\) and \(B\) are finitely-generated \(\mathbb{Z}_p\)-modules, then \([HA, \Sigma^i HB] = 0\) for \(1 < i < 2p - 2\).

**Proof.** We have \([HZ_p, \Sigma^i H\mathbb{Z}/p] \cong [HZ, \Sigma^i H\mathbb{Z}/p] \cong A_p/A_p\beta\), the quotient of the \(\mathbb{Z}/p\)-Steenrod algebra by the left ideal generated by the Bockstein. This vanishes in degrees \(0 < * < 2p - 2\), and in degree 0 is \(\text{Hom}(\mathbb{Z}_p, \mathbb{Z}/p) \cong \mathbb{Z}/p\).

Using the cofibre sequence \(HZ_p \xrightarrow{n} HZ_p \rightarrow HZ_p/n\) we find \([HZ_p/n, \Sigma^i H\mathbb{Z}/p] = 0\) for \(* > 1\), so the claim holds for \(B = \mathbb{Z}/p\) and for all \(A\). Furthermore \([HA, \Sigma^i H\mathbb{Z}/p]\) is identified with \(\text{Ext}^i_{\mathbb{Z}_p}(A, \mathbb{Z}/p)\). Bockstein sequences show that \([HA, \Sigma^i H\mathbb{Z}/p^r]\) = 0 for \(* > 1\) and all \(r\), and identify \([HA, \Sigma^i H\mathbb{Z}/p^r]\) with \(\text{Ext}_{\mathbb{Z}_p}(A, \mathbb{Z}/p^r)\). Finally, writing \(HZ_p = \text{holim}_r H\mathbb{Z}/p^r\) gives a Milnor sequence
\[
0 \rightarrow \lim_{r \downarrow} [HA, \Sigma^{i-1} H\mathbb{Z}/p^r] \rightarrow [HA, \Sigma^i H\mathbb{Z}_p] \rightarrow \lim_{r \downarrow} [HA, \Sigma^i H\mathbb{Z}/p^r] \rightarrow 0.
\]

The outer terms vanish for \(i > 2\). For \(i = 2\) the right-hand term vanishes, and the left-hand term is \(\text{lim}_{r \downarrow} \text{Ext}^1_{\mathbb{Z}_p}(A, \mathbb{Z}/p^r)\). As \(A\) is by assumption a finitely-generated \(\mathbb{Z}_p\)-module, the inverse system \(\{\text{Ext}^1_{\mathbb{Z}_p}(A, \mathbb{Z}/p^r)\}_{r}\) is eventually constant so the right-hand term vanishes too. \(\square\)

It follows that the truncations \(\tau_{[2,2p-3]}K(\mathbb{Z}; \mathbb{Z}_p)\) and \(\tau_{[2,2p-3]}K(\mathbb{Z}_p; \mathbb{Z}_p)\) are co-products of Eilenberg–MacLane spectra, because their Postnikov towers must split.
by the lemma: assuming a splitting \( \tau_{[2,i]} K(Z; \mathbb{Z}_p) \simeq \bigoplus_{j=2}^{i} \Sigma^j H K_j(Z; \mathbb{Z}_p) \) has been chosen, there is a pullback
\[
\begin{array}{ccc}
\tau_{[2,i+1]} K(Z; \mathbb{Z}_p) & \longrightarrow & \ast \\
\bigoplus_{j=2}^{i} \Sigma^j H K_j(Z; \mathbb{Z}_p) & \longrightarrow & \Sigma^{i+2} H K_{i+1}(Z; \mathbb{Z}_p)
\end{array}
\]
but the lower map is nullhomotopic by the lemma. However, this splitting is not completely canonical: when \( i = 4k \) the nullhomotopy of the lower map may not be unique (though it is if the Vandiver conjecture holds). The analogous discussion goes through for \( \tau_{[2,2p-3]} K(Z; \mathbb{Z}_p) \), though now the splitting is canonical.

Using the lemma again it follows that the map \( \kappa \) is a coproduct of maps
\[
\begin{align*}
\kappa_{4k-2} : & \Sigma^{4k-2} H K_{4k-2}(Z; \mathbb{Z}_p) \longrightarrow \Sigma^{4k-1} H K_{4k-1}(Z; \mathbb{Z}_p) \\
\kappa_{4k} : & \Sigma^{4k} H K_{4k}(Z; \mathbb{Z}_p) \longrightarrow \Sigma^{4k+1} H K_{4k+1}(Z; \mathbb{Z}_p) \\
\kappa_{4k+1} : & \Sigma^{4k+1} H K_{4k+1}(Z; \mathbb{Z}_p) \longrightarrow \Sigma^{4k+2} H K_{4k+2}(Z; \mathbb{Z}_p)
\end{align*}
\]
The first two are Bockstein-type maps, and cannot be detected on homotopy groups; furthermore, in principle they depend on the choices of splitting we have made above. The last, however, is determined by its effect on homotopy groups, i.e. the map
\[
\mathbb{Z}_p \cong K_{4k+1}(Z; \mathbb{Z}_p) \longrightarrow K_{4k+1}(Z; \mathbb{Z}_p) \cong \mathbb{Z}_p.
\]
This is given by multiplication by the value \( L_p(1+2k, \omega^{-2k}) \) of the \( p \)-adic \( L \)-function, up to a \( p \)-adic unit; see [Hes13] for a concise discussion.

**Remark 6.2.** We may use this as follows. The \( \mathbb{F}_p \)-cohomology of the fibre of
\[
\Omega^\infty \kappa_{4k+1} : K(K_{4k+1}(Z; \mathbb{Z}_p), 4k+1) \longrightarrow K(K_{4k+1}(Z; \mathbb{Z}_p), 4k+1)
\]
in degrees \( < 4k+1 + 2p = 2p - 2 \) depends only on whether \( L_p(1+2k, \omega^{-2k}) \) is a \( p \)-adic unit: if so then the fibre has trivial \( \mathbb{F}_p \)-cohomology, and if not then the fibre has \( \mathbb{F}_p \)-cohomology \( \mathbb{F}_p[y_{4k}] \otimes \Lambda^2_{\mathbb{F}_p}[y_{4k+1}] \) in this range.

By [Coh07] Proposition 11.3.12 (1) we have
\[
2k \cdot L_p(1+2k, \omega^{-2k}) = \lim_{r \to \infty} B_{\phi(p^r)-2k},
\]
where the latter denote Bernoulli numbers. Assuming that \( 4k + 1 < p \), so certainly \( 2k < p - 1 \), then as \( \phi(p^r)-2k \equiv p - 1 - 2k \) mod \( p - 1 \) we have the Kummer congruences
\[
\frac{B_{\phi(p^r)-2k}}{\phi(p^r)-2k} \equiv \frac{2p-1-2k}{p-1-2k} \mod p
\]
and so
\[
L_p(1+2k, \omega^{-2k}) \equiv B_{p-1-2k} \mod p,
\]
where \( \equiv \) denotes congruence up to a \( p \)-adic unit. Thus we may determine whether or not \( L_p(1+2k, \omega^{-2k}) \) is a \( p \)-adic unit by calculating this residue class.

In particular, for \( p = 37 \) we find that \( B_{p-1-2k} \neq 0 \) mod \( p \) for all \( 1+2k < p \) except for \( 1+2k = 5 \). Certainly the Vandiver conjecture holds for \( p = 37 \). This Bernoulli number contributes to torsion in \( K_{62}(Z; \mathbb{Z}_p) \), so in degrees \( * < p \) there is no torsion in \( K_* (Z; \mathbb{Z}_p) \). It follows that
\[
H^* (\text{hofib}(\Omega^\infty \kappa); \mathbb{F}_p) \cong \mathbb{F}_p[x_2, x_6, x_{10}, \ldots] \otimes \mathbb{F}_p[y_0] \otimes \Lambda^2_{\mathbb{F}_p}[y_0]
\]
in degrees \( * < 37 \). With Theorem [1] this justifies the first example in the introduction.

For \( p = 16843 \) we find that \( B_{p-1-2k} \neq 0 \) mod \( p \) for all \( 1+2k < p \) except for \( 1+2k = 3 \). If \( p = 2124679 \) we find that \( B_{p-1-2k} \neq 0 \) mod \( p \) for all \( 1+2k < p \) except for \( 1+2k = 3 \) and \( 1+2k = 1422781 \). The Vandiver conjecture holds for these primes, so taking into account the contribution of these Bernoulli numbers to
the torsion in $K_{2(p−1−2k)−2}(\mathbb{Z};\mathbb{Z}_p)$ too, this justifies the second and third examples in the introduction.

6.5. Transgressive fibrations. Let us say that a fibration $\pi : E → B$ with 0-connected fibre $F$ is transgressive (with $\mathbb{F}_p$-coefficients) if $τ_1(B)$ acts trivially on $H^*(F;\mathbb{F}_p)$, and if $H^*(F;\mathbb{F}_p)$ is freely generated as a graded-commutative $\mathbb{F}_p$-algebra by a set of classes which are transgressive in the Serre spectral sequence for $\pi$. We say $\pi$ is transgressive in degrees $* < N$ if the above two conditions hold in cohomological degrees $* < N$. The class of such fibrations is closed under forming pullbacks, and by the Künneth theorem is closed under forming products of fibrations.

Lemma 6.3. The following fibrations of Eilenberg–MacLane spaces are transgressive in degrees $* < 2p − 2$:

(i) $π : K(\mathbb{Z}_p, n) → K(\mathbb{Z}_p, n)$ given by multiplication by some $N ∈ \mathbb{Z}_p$;
(ii) $π : K(B, n) → K(\mathbb{Z}_p, n + 1)$ obtained by delooping an extension $0 → \mathbb{Z}_p → A → B → 0$ with $B$ a finitely-generated $\mathbb{Z}_p$-module.

Proof. First note that the path fibration $PK(A, k) → K(A, k)$ is transgressive in degrees $* < k + 1 + 2p − 2$ for $A$ either $\mathbb{Z}_p$ or $\mathbb{Z}/p^r$. By the calculation of the cohomology of Eilenberg–MacLane spaces in [Car55] (and the fact that $K(\mathbb{Z}, k) → K(\mathbb{Z}_p, k)$ is a $\mathbb{F}_p$-cohomology isomorphism). In case (i), if $N = 0$ then $π$ is the product of the fibration $K(\mathbb{Z}_p, n) → *$ and the path fibration $PK(\mathbb{Z}_p, n) → K(\mathbb{Z}_p, n)$. The first is clearly transgressive, and the second is transgressive in degrees $* < n − 1 + 2p − 2$ by the paragraph above. If $N ≠ 0$ then $π$ is pulled back from the path fibration $PK(\mathbb{Z}_p/N, n) → K(\mathbb{Z}_p/N, n)$, which is again transgressive in degrees $* < n − 1 + 2p − 2$.

In case (ii) $π$ is pulled back from the path fibration $PK(A, n + 1) → K(A, n + 1)$, and $A$ is a finitely-generated $\mathbb{Z}_p$-module so this is a product of path fibrations over $K(\mathbb{Z}/p^r, n + 1)$’s and $K(\mathbb{Z}_p, n + 1)$’s, each of which are transgressive in degrees $* < n + 2p − 2$ by the first paragraph above.

Corollary 6.4. The fibration $$τ_{[2, 2p−3]}^{K} : τ_{[2, 2p−3]}^{K}(\mathbb{Z};\mathbb{Z}_p) → τ_{[2, 2p−3]}^{K}(\mathbb{Z}_p;\mathbb{Z}_p)$$ is transgressive in degrees $* < 2p − 2$.

Proof. By the discussion in Section 6.4 it suffices to show that each of $$\Omega^{∞}_{k4k−3} : K(4k−3(Z;\mathbb{Z}_p), 4k − 3) → K(4k−3(Z_p;\mathbb{Z}_p), 4k − 3)$$ $$\Omega^{∞}_{k4k−2} : K(4k−2(Z;\mathbb{Z}_p), 4k − 2) → K(4k−2(Z_p;\mathbb{Z}_p), 4k − 1)$$ $$\Omega^{∞}_{k4k} : K(4k(Z;\mathbb{Z}_p), 4k) → K(4k+1(Z_p;\mathbb{Z}_p), 4k + 1)$$ are transgressive. These fibrations are the kind to which Lemma 6.3 applies.

6.6. Cohomology of $p$-adic analytic groups. For $m ≥ 1$ and $p$ odd the groups $SL_n(\mathbb{Z}_p, p^m)$ are uniform (= uniformly powerful) pro-$p$-groups [DaS00, Theorem 5.2] and hence their continuous $\mathbb{F}_p$-cohomology may be described as the exterior algebra on their first $\mathbb{F}_p$-cohomology (cf. [Laz65, V.2.7.2] [SW00, Theorem 5.1.5]). On the other hand the map $$I + p^mA → A \mod p : SL_n(\mathbb{Z}_p, p^m) → sl_n(\mathbb{F}_p)$$

—

4This formulation needs to be unwrapped quite a bit. The first step is to realise that “équ-p-valués” is “uniformly powerful”.

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induces an isomorphism \( SL_n(\mathbb{Z}, p^m) / SL_n(\mathbb{Z}, p^{m+1}) \to sl_n(\mathbb{F}_p) \) and this is the maximal \( p \)-elementary abelian quotient for \( n \geq 2 \) by [DaSiMS99] Lemma 5.1. Thus there is an identification \( H^1(SL_n(\mathbb{Z}, p^m) ; \mathbb{F}_p) \cong sl_n(\mathbb{F}_p)^\vee \), and so an isomorphism

\[
\Lambda^*_p [sl_n(\mathbb{F}_p)^\vee] \cong H^*_\text{cts}(SL_n(\mathbb{Z}, p^m) ; \mathbb{F}_p).
\]

7. Proof of Theorem 7.1

The map of fibrations of pro-spaces

\[
\begin{array}{ccc}
\{B\text{SL}(\mathbb{Z}, p^{r+m})\}_r & \longrightarrow & \{B\text{SL}(\mathbb{Z}, p^r)\}_r \\
\downarrow & & \downarrow \\
B\text{GL}(\mathbb{Z}, p^m) & \longrightarrow & B\text{GL}(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\{B\text{GL}(\mathbb{Z}/p^{r+m}, p^m)\}_r & \longrightarrow & \{B\text{GL}(\mathbb{Z}/p^r)\}_r
\end{array}
\]

gives a map of spectral sequences

\[
\begin{array}{ccc}
^1E^\ast_2 \longrightarrow H^*_\text{cts}(\text{SL}(\mathbb{Z}_p)) \otimes \tilde{H}^\ast(\text{SL}(\mathbb{Z})) & \longrightarrow & H^{\ast+1}(\text{SL}(\mathbb{Z})) \\
\downarrow & & \downarrow \\
^1H^*_\text{cts}(\text{SL}(\mathbb{Z}_p, p^m)) \otimes \tilde{H}^\ast(\text{SL}(\mathbb{Z})) & \longrightarrow & H^{\ast+1}(\text{SL}(\mathbb{Z}, p^m))
\end{array}
\]

and in total degrees \( \ast < 2p - 2 \) the discussion in Section 6.3 (if \( p \) is regular) or in Sections 6.4 and 6.5 (in general) shows that \( \tilde{H}^\ast(\text{SL}(\mathbb{Z})) \) is freely generated as a graded-commutative \( \mathbb{F}_p \)-algebra by elements \( \{x_\alpha\}_{\alpha \in I} \) which are transgressive in the first spectral sequence, i.e. they survive until \( ^1E^{|x_\alpha|+1}_2 \) and then \( ^1d^{|x_\alpha|+1}(x_\alpha) = y_\alpha \) for certain \( y_\alpha \in H^{|x_\alpha|+1}_\text{cts}(\text{SL}(\mathbb{Z}_p)) \). The map of spectral sequences means that each \( x_\alpha \) also transgresses in the second spectral sequence, and furthermore transgress to the image of the corresponding \( y_\alpha \) under the restriction map

\[
H^*_\text{cts}(\text{SL}(\mathbb{Z})) \longrightarrow H^*_\text{cts}(\text{SL}(\mathbb{Z}_p, p^m)).
\]

We will prove that this map is zero in degrees \( 0 < \ast < p \), so that the second spectral sequence collapses:

**Theorem 7.1.** The restriction maps

\[
H^*_\text{cts}(SL_n(\mathbb{Z}_p)) \longrightarrow H^*_\text{cts}(SL_n(\mathbb{Z}_p, p^m))
\]

are zero in degrees \( 0 < \ast < p \), for all \( m > 0 \), and all large enough \( n \).

7.1. Proof of Theorem 7.1 for \( m > 1 \). The proof in this case is essentially trivial. There is a factorisation

\[
H^*_\text{cts}(SL_n(\mathbb{Z}_p)) \longrightarrow H^*_\text{cts}(SL_n(\mathbb{Z}_p, p)) \longrightarrow H^*_\text{cts}(SL_n(\mathbb{Z}_p, p^m))
\]

of the restriction map and by the discussion in Section 6.3 we know the latter two cohomology rings: both are given by \( \Lambda^*_p [sl_n(\mathbb{F}_p)]^\vee \). However, the composition

\[
SL_n(\mathbb{Z}_p, p^m) \longrightarrow SL_n(\mathbb{Z}_p, p) \longrightarrow SL_n(\mathbb{Z}_p, p^m) / SL_n(\mathbb{Z}_p, p^2) = sl_n(\mathbb{F}_p)
\]

is trivial for \( m > 1 \), so the second map in the factorisation of the restriction map is trivial in degrees \( \ast > 0 \), and so the restriction map is too.
7.2. Proof of Theorem 7.1 for \( m = 1 \). The fibration of pro-spaces

\[
\{ \text{BSL}_n(\mathbb{Z}/p^{r+1}, p) \}_{r} \to \{ \text{BSL}_n(\mathbb{Z}/p) \}_{r} \to \text{BSL}_n(\mathbb{Z}/p)
\]
yields a spectral sequence

\[
\text{III} E_2^{s,t} = H^*(\text{SL}_n(\mathbb{Z}/p); H^*_c(\text{SL}_n(\mathbb{Z}/p), p)) \implies H^*_{\text{cts}}(\text{SL}_n(\mathbb{Z}/p)),
\]
which has the map of Theorem 7.1 as an edge homomorphism. The following proposition describes the \( d_2 \)-differential on this edge.

**Proposition 7.2.** In degrees \( s < p \) and for all large enough \( n \) the differential

\[
d_2 : \text{III} E_2^{0,s} \to \text{III} E_2^{2,s-1}
\]
is given by

\[
\Lambda^*_p[c_3, c_5, c_7, \ldots] \to \mathbb{F}_p\{e_2, e_6, \ldots\} \otimes \Lambda^*_p[c_3, c_5, c_7, \ldots]
\]
and the (graded) Leibniz rule.

We defer the (quite involved) proof of this proposition to Section 8.

To finish the proof of Theorem 7.1 in this case, we claim that the map \( d_2 \) is injective in degrees \( 0 < s < p \). To see this, suppose that \( x \in \Lambda^*_p[c_3, c_5, c_7, \ldots] \) has \( d_2(x) = 0 \), then for each \( i < p \) write \( x = a + c_i \cdot b \) with \( a \) and \( b \) not containing \( c_i \).

As \( d_2(x) \) has \( ib \) as the coefficient of \( e_i \cdot 1 \), and \( d_2(x) = 0 \), it follows that \( b = 0 \), and so \( c_i \) does not occur in \( x \). This goes for all \( c_i \). Now, by Proposition 7.2 this means that the differential \( d_2 : \text{III} E_2^{0,s} \to \text{III} E_2^{2,s-1} \) is injective for \( 0 < t < p \), and hence that the edge homomorphism

\[
H^*(\text{SL}_n(\mathbb{Z}/p)) \to H^0(\text{SL}(\mathbb{Z}/p); H^*(\text{SL}_n(\mathbb{Z}/p), p))
\]
is trivial in degrees \( 0 < s < p \), proving Theorem 7.1.

7.3. Resolving extensions. By Theorem 7.1 the spectral sequence

\[
\text{II} E_2^{s,t} = H^s_c(\text{SL}_n(\mathbb{Z}/p); \mathbb{P}(\mathbb{Z}/p)) \otimes \tilde{H}^t(\text{SL}_n(\mathbb{Z})) \implies H^{s+t}(\text{SL}_n(\mathbb{Z}, p^m))
\]
collapses in total degrees \( s < p \): that is, there is a multiplicative filtration of \( H^*(\text{SL}(\mathbb{Z}, p^m)) \) with

\[
\text{Gr}^* H^*(\text{SL}_n(\mathbb{Z}, p^m)) \cong \Lambda^*_p[\mathbb{F}_p[\mathbb{Z}/p]].
\]
for \( s < p \) and all large enough \( n \). This is an isomorphism of bigraded \( \mathbb{F}_p \)-algebras and of \( \text{SL}(\mathbb{Z}/p^m) \)-representations.

To show that the latter formula in fact describes \( H^*(\text{SL}(\mathbb{Z}, p^m)) \) as an \( \mathbb{F}_p \)-algebra and as a \( \text{SL}(\mathbb{Z}/p^m) \)-representation in this range, and not just up to filtration, it suffices to show that for each of the free generators \( x_\alpha \in \tilde{H}^*_{\text{cts}}(\text{SL}_n(\mathbb{Z})) \) of degree \( |x_\alpha| < p \) there exists an \( \text{SL}(\mathbb{Z}/p^m) \)-invariant element \( \tilde{x}_\alpha \in H^*(\text{SL}(\mathbb{Z}, p^m)) \) which restricts to

\[
x_\alpha \in \frac{H^{ix_\alpha}(\text{SL}(\mathbb{Z}, p^m))}{F[x_\alpha] F^{ix_\alpha}(\text{SL}(\mathbb{Z}, p^m))} = H^0_{\text{cts}}(\text{SL}_n(\mathbb{Z}/p^m); \tilde{H}^{ix_\alpha}(\text{SL}_n(\mathbb{Z}))).
\]
By naturality, it suffices to produce such \( \tilde{x}_\alpha \)'s for \( m = 1 \).

To do so, consider the filtration of the \( \mathbb{F}_p[\text{SL}(\mathbb{Z}/p)] \)-module \( H^{ix_\alpha}_c(\text{SL}_n(\mathbb{Z}, p^m)) \), whose associated graded is

\[
\text{Gr}^t H^{ix_\alpha}_c(\text{SL}_n(\mathbb{Z}, p^m)) \cong \Lambda^*_p[\mathbb{F}_p[\mathbb{Z}/p]].
\]
where the second factor has trivial \( \text{SL}(\mathbb{Z}/p) \)-action. The spectral sequence of this filtered module takes the form

\[
\text{II} E_1^{s,t} = H^s(\text{SL}_n(\mathbb{Z}/p); \text{Gr}^t H^{ix_\alpha}_c(\text{SL}_n(\mathbb{Z}, p^m))) \Rightarrow H^s(\text{SL}_n(\mathbb{Z}/p); H^{ix_\alpha}_c(\text{SL}_n(\mathbb{Z}, p^m)))
\]
Lemma 7.3. As long as \(|x_α| < p\) we have \(IV E_{s,t}^0 = 0\), for all large enough \(n\).

Proof. As a \(\text{SL}_n(\mathbb{Z}/p)\)-representation \(\text{Gr}^t H^{[x_α]}(\text{SL}(\mathbb{Z}, p))\) is a direct sum of copies of \(\Lambda_p^t[s_\ell(\mathbb{F}_p)^\vee]\), so we must show that these representations have trivial cohomology for \(t < p\). As in Section 5.1, by homological stability we may suppose that \(n \neq 0 \mod p\), so that with \(V = \mathbb{F}_p\) the standard representation the sequence \(0 \rightarrow \mathbb{F}_p \rightarrow V \otimes V^\vee \rightarrow s_\ell(\mathbb{F}_p)^\vee \rightarrow 0\) has a cosplitting given by \(\frac{1}{2}ev : V \otimes V^\vee \rightarrow \mathbb{F}_p\), and hence \(s_\ell(\mathbb{F}_p)^\vee \otimes \mathbb{F}_p \cong V \otimes V^\vee\). Then \(\Lambda_p^t[s_\ell(\mathbb{F}_p)^\vee]\) is a summand of \(\Lambda_p^t[V \otimes V^\vee]\) so it suffices to show that the latter representation has trivial first cohomology. On the other hand, as \(t < p\) we have that \(\Lambda_p^t[V \otimes V^\vee]\) is a summand of \((V \otimes V^\vee)^{\otimes t}\), so it suffices to show that the latter representation has trivial first cohomology. This follows from Theorem 7.2.

It follows that for \(|x_α| < p\) the class \(x_α \in H^0(\text{SL}_n(\mathbb{Z}/p); \text{Gr}^0 H^{[x_α]}(\text{SL}_n(\mathbb{Z}, p))) = IV E_{0,0}^0\) is a permanent cycle in this spectral sequence, so that we may find a \(\bar{x}_α \in H^0(\text{SL}_n(\mathbb{Z}/p); H^{[x_α]}(\text{SL}_n(\mathbb{Z}, p))\) restricting to \(x_α\), as required.

8. Proof of Proposition 7.2

8.1. Tensor powers of the dual adjoint representation. Recall that we write \(s\ell(V)\) for the kernel of \(ev : V \otimes V^\vee \rightarrow \mathbb{F}_p\), so there is a short exact sequence

\[
0 \rightarrow \mathbb{F}_p \xrightarrow{\text{coev}} V \otimes V^\vee \rightarrow s\ell(V)^\vee \rightarrow 0.
\]

We may form the composition

\[
\Gamma_{\mathbb{F}_p}[x]^{\otimes T} \otimes \mathbb{F}_p\{\Sigma^{\text{nd}}_{\ell}\} \xrightarrow{\otimes \mathbb{T}} H^*(\text{SL}(V); (V \otimes V^\vee)^{\otimes T})
\rightarrow H^*(\text{SL}(V); (s\ell(V)^\vee)^{\otimes T})
\]

and then have the following analogue of Theorem 7.2.

Lemma 8.1. In a stable range of degrees, the kernel of this composition is spanned by those \(\prod_{t \in T} x_t^{[t]} \otimes \sigma\) such that there is an \(j \in T\) with \(\ell(j) = 0\) and \(\sigma(j) = j\).

Proof. Considering (8.1) as a resolution of \(s\ell(V)^\vee\), and tensoring together \(T\) copies of it, gives a hyperhomology spectral sequence of the form

\[
E_{1,t}^s = \bigoplus_{S \subseteq T} H^*(\text{SL}(V); (V \otimes V^\vee)^{\otimes T \setminus S} \otimes \mathbb{F}_p^{S}) \Rightarrow H^{t+s}(\text{SL}(V); (s\ell(V)^\vee)^{\otimes T}).
\]

The \(d_1\)-differential is induced by \(\text{coev} : \mathbb{F}_p \rightarrow V \otimes V^\vee\). In a stable range we have

\[
E_{1,t}^{*,s} = \bigoplus_{S \subseteq T} \Gamma_{\mathbb{F}_p}[x]^{\otimes T \setminus S} \otimes \mathbb{F}_p\{\Sigma^{\text{nd}}_{\ell}\},
\]

where the differential is given by summing over ways to add an element \(j \in S\) to \(T \setminus S\), then extending permutations by the identity on \(j\) multiplying by \(x_j^{[0]}\). It follows that in this range \(E_{2,t}^s\) is supported along the line \(s = 0\), and is given by the quotient of \(\Gamma_{\mathbb{F}_p}[x]^{\otimes T} \otimes \mathbb{F}_p\{\Sigma^{\text{nd}}_{\ell}\}\) by the span of the terms described in the statement of the lemma.

Corollary 8.2. There are isomorphisms

\[
H^0(\text{SL}(V); \Lambda^*[s\ell(V)^\vee]) \cong \Lambda_p^*[c_3, c_5, c_7, \ldots]
\]

\[
H^1(\text{SL}(V); \Lambda^*[s\ell(V)^\vee]) \cong \mathbb{F}_p[c_2, c_4, c_6, \ldots] \otimes \Lambda_p^*[c_3, c_5, c_7, \ldots]
\]

in degrees \(s < p\) for \(\dim(V)\) sufficiently large.
Proof. Recall that we write \( t \coloneqq \{1, 2, \ldots, t\} \). Let \( \tilde{c}_t \in H^0(SL(V); \Lambda^t[V \otimes V^\vee]) \) denote the image of the class

\[
1 \otimes (1, 2, 3, \ldots, t) \in \Gamma_{F_p}[x][\otimes]^2 \otimes F_p[\Sigma^T_z]
\]

under the maps

\[
(8.2) \quad \Gamma_{F_p}[x][\otimes]^2 \otimes F_p[\Sigma^T_z] \rightarrow H^*(SL(V); (V \otimes V^\vee)[\otimes]) \rightarrow H^*(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee])
\]

and \( c_t \) its further image in \( H^*(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \). The discussion above shows that \( c_1 = 0 \). Writing \( F_p \) for the sign representation, we have

\[
H^0(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \cong F_p[\Sigma^T_z] \otimes_{\Sigma^T_z} F_p^{-}. 
\]

This is given by the way conjugacy classes split in the alternating group: if a conjugacy class contains an even cycle or two odd cycles of the same length, then it becomes trivial in these coinvariants; otherwise it contributes a 1-dimensional space. This identifies this space with the degree \( t \) part of \( \Lambda^t_{F_p}[\tilde{c}_1, \tilde{c}_3, \tilde{c}_5, \ldots, ] \), and hence identifies \( H^0(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \) with this graded-commutative algebra.

As \( \Lambda^t_{F_p}[sl(V)^\vee] \) is a summand of \((sl(V)^\vee)[\otimes] \) for \( t < p \), using Lemma 8.1, we obtain the claimed formula for \( H^0(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \), and also for \( H^1(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \).

Let \( \tilde{e}_t \in H^2(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \) denote the image of the class

\[
x^{[1]}_t \otimes (1, 2, 3, \ldots, t) \in \Gamma_{F_p}[x][\otimes]^2 \otimes F_p[\Sigma^T_z]
\]

under the maps (8.2) and \( e_t \) its further image in \( H^*(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \). We have

\[
H^2(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \cong \left( F_p[x^{[1]}_t] \mid i \in \{1\} \otimes F_p[\Sigma^T_z] \right) \otimes_{\Sigma^T_z} F_p^{-}.
\]

We think of the first factor as being the space of permutations of \( t \) written as disjoint cycles, with one entry marked, on which \( \Sigma^T_z \) acts by conjugation. As above we find that to contribute the unmarked cycles must all be of different odd lengths, and the marked cycle must be of even length. As a module over \( H^0(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \) we therefore have that \( H^2(SL(V); \Lambda^t_{F_p}[V \otimes V^\vee]) \) is free on the basis of the marked cycles of even length, i.e. \( \tilde{e}_2, \tilde{e}_4, \tilde{e}_6, \ldots \). Using as above that \( \Lambda^t_{F_p}[sl(V)^\vee] \) is a summand of \((sl(V)^\vee)[\otimes] \) for \( t < p \), we find the same description for \( H^2(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \) as a module over \( H^0(SL(V); \Lambda^t_{F_p}[sl(V)^\vee]) \). \( \square \)

8.2. Multiplicative structure on the \( E_2 \)-page of the Serre spectral sequence. We will need to use some details of the \( d_2 \)-differentials in the spectral sequence for a group extension

\[
1 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 1,
\]

with coefficients in a commutative ring \( k \): it has the form

\[
E_2^{s,t} = H^s(Q; H^t(K; k)) \Longrightarrow H^{s+t}(G; k).
\]

Lemma 8.3. There are canonical elements \( d_2^s \in \text{Ext}^2_{sl}[Q](H^t(K; k), H^{t-1}(K; k)) \) for \( t \geq 1 \) such that:

(i) The differential \( d_2 : E_2^{s,t} \to E_2^{s+2, t-1} \) is given by the Yoneda product with \( d_2^s \).

(ii) The square

\[
\begin{array}{c}
\begin{array}{ccc}
H^t(K; k) \otimes H^{t+1}(K; k) & \overset{d_2^t \otimes Id + (-1)^t Id \otimes d_2^t}{\longrightarrow} & H^{t-1}(K; k) \otimes H^{t+1}(K; k)[2] \\
\downarrow & & \downarrow \\
H^{t+1}(K; k) & \overset{d_2^{t+1} \otimes Id}{\longrightarrow} & H^{t+1}(K; k)[2]
\end{array}
\end{array}
\]

is commutative.
commutes in the derived category of \(k[Q]\)-modules.

It is surely possible to prove this working completely algebraically, but we prefer to generalise the statement to the setting of a Serre fibration

\[ \pi : E \to B \]

and use parameterised stable homotopy theory over \(B\). The precise model of parameterised homotopy theory that one uses does not affect our arguments, which are completely formal: one can take for instance the \(\infty\)-categorical model of \([ABG18]\), and write \(\mathcal{S}p_{/B}\) for the category of parameterised spectra over \(B\), i.e. the category of functors from \(B\) to spectra.

Writing \(H_{B^k}\) for the constant parameterised spectrum with value the Eilenberg–MacLane spectrum for \(k\), which is a commutative algebra in \(\mathcal{S}p_{/B}\), we form the parameterised function spectrum \(C := F_B(E, H_{B^k})\) of fibrewise \(k\)-cochains; the algebra structure on \(H_{B^k}\) and the coalgebra structure given by the diagonal map \(E \to E \times_B E\) makes \(C\) into an \(H_{B^k}\)-algebra in \(\mathcal{S}p_{/B}\).

The category \(H_{B^k}\)-mod has a natural \(t\)-structure, where an object is connective (or co-connective) if and only if all of its fibres are. The heart of this \(t\)-structure is the category of local systems of \(k\)-modules over \(B\), and we write \(\text{loc}(B)\) for this category (in our application \(B = BQ\), and this is equivalent to the category of \(k[Q]\)-modules).

Let \(h_t^i\) denote the local system \(b \mapsto H_{\pi^{-1}(b)}(k)\) over \(B\). The Postnikov tower of the co-connective \(H_{B^k}\)-module \(C\) with respect to this \(t\)-structure has the form

\[
\begin{array}{ccccccccc}
H_{B^k}^0 & \to & \tau_{\leq 0} C \\
\downarrow & & \\
S^{-1} \otimes H_{B^k}^1 & \to & \tau_{\leq -1} C & \to & S^1 \otimes H_{B^k}^0 \\
\downarrow & & \\
S^{-2} \otimes H_{B^k}^2 & \to & \tau_{\leq -2} C & \to & S^1 \otimes S^{-1} \otimes H_{B^k}^1 \\
\downarrow & & \\
& & & & \vdots
\end{array}
\]

Taking the parameterised homotopy groups \([S^*, -]_B\) of this tower gives a spectral sequence with

\[ E_2^{s,t} := [S^{s-t} \otimes H_{B^k}^t]_B \cong H^s(B; h^t), \]

which can be identified with the Serre spectral sequence for the fibration \(\pi\). (It is more usual to obtain the Serre spectral sequence by filtering \(B\) by skeleta: that these two spectral sequences are isomorphic is as in \([Mau63]\).)

In particular the horizontal compositions define a sequence of elements

\[ d_2^s \in [S^{s-t} \otimes H_{B^k}^t, S^1 \otimes S^{-(t-1)} \otimes H_{B^k}^{t-1}]_{H_{B^k}\text{-mod}} \cong \text{Ext}_{\text{loc}(B)}^2(h^t, h^{t-1}), \]

such that

\[ E_2^{s,t} = [S^{-(s+t)} \otimes H_{B^k}^t]_B \xrightarrow{d_2^{s,t}} [S^{-(s+t)} \otimes S^{-(t-1)} \otimes H_{B^k}^{t-1}]_B \]

\[ \cong [S^{-(s+t+1)} \otimes S^{-(t-1)} \otimes H_{B^k}^{t-1}]_B = E_2^{s+2,t-1} \]

is the \(d_2\)-differential of the spectral sequence.
The $H_\mathbb{k}$-algebra structure on $C$ gives a commutative diagram

$$
\begin{array}{ccc}
S^{-t'} \otimes H_\mathbb{k}^{t'} \otimes S^{-t''} \otimes H_\mathbb{k}^{t''} & \xrightarrow{d'_{2} \otimes \text{id} \otimes d''_{2}} & S^{1} \otimes S^{-t'(t'-1)} \otimes H_\mathbb{k}^{t'-1} \otimes S^{-t''} \otimes H_\mathbb{k}^{t''} \oplus S^{-t'} \otimes H_\mathbb{k}^{t'} \otimes S^{1} \otimes S^{-(t''-1)} \otimes H_\mathbb{k}^{t''-1} \\
\downarrow \text{swap} & & \downarrow \text{swap} \\
S^{-t'-t''} \otimes H_\mathbb{k}^{t'+t''} & \xrightarrow{d'_{2} + d''_{2}} & S^{1} \otimes S^{-(t'+t''-1)} \otimes H_\mathbb{k}^{t'+t''-1},
\end{array}
$$

which is the Leibniz rule in this setting. The swap map incurs the sign $(-1)^t$ on the second summand.

Specialised to the Serre fibration $\pi : B\Gamma \to BQ$ with fibres $BK$, the last two paragraphs provide (i) and (ii) of Lemma 8.3.

### 8.3. Proof of Proposition 7.2

Recall that we wish to understand the differentials $d_{2} : \text{III} E_{2}^{0,*} \to \text{III} E_{2}^{2,*+1}$ in the spectral sequence

$$
\text{III} E_{2}^{s,t} = H^{s}(\text{SL}_{n}(\mathbb{Z}/p); H_{c\text{ts}}(\text{SL}_{n}(\mathbb{Z}/p))),
$$

associated to the fibration (7.1). We have $H_{c\text{ts}}^{*}(\text{SL}_{n}(\mathbb{Z}/p)) = \Lambda_{\mathbb{F}_{p}}^{*}[sl_{n}(\mathbb{F}_{p})^{\vee}]$ as an $\text{SL}(\mathbb{Z}/p)$-module, so by Corollary 8.2 we have calculated that

$$
\text{III} E_{2}^{0,*} = \Lambda_{\mathbb{F}_{p}}^{*}[c_{3}, c_{5}, c_{7}, \ldots] \\
\text{III} E_{2}^{2,*} = \mathbb{F}_{p}[c_{2}, c_{4}, c_{6}, \ldots] \otimes \Lambda_{\mathbb{F}_{p}}^{*}[c_{3}, c_{5}, c_{7}, \ldots]
$$

for $* < p$ and all large enough $n$. To evaluate the differential we apply the discussion in Section 8.2. We have

$$
d_{2}^{t} \in \text{Ext}_{\mathbb{F}_{p}[\text{SL}_{n}(\mathbb{Z}/p)]}^{2}(\Lambda_{\mathbb{F}_{p}}^{*}[sl_{n}(\mathbb{F}_{p})^{\vee}], \Lambda_{\mathbb{F}_{p}}^{t-1}[sl_{n}(\mathbb{F}_{p})^{\vee}]).
$$

For each $t > 1$ odd, we wish to evaluate the composition

$$
\mathbb{F}_{p} \xrightarrow{c_{t}} \Lambda_{\mathbb{F}_{p}}^{t}[sl_{n}(\mathbb{F}_{p})^{\vee}] \xrightarrow{d_{2}^{t}} \Lambda_{\mathbb{F}_{p}}^{t-2}[sl_{n}(\mathbb{F}_{p})^{\vee}][2]
$$

as a morphism in the derived category of $\mathbb{F}_{p}[\text{SL}_{n}(\mathbb{Z}/p)]$-modules, i.e. an element of $H^{2}(\text{SL}(\mathbb{Z}/p); \Lambda_{\mathbb{F}_{p}}^{t-1}[sl_{n}(\mathbb{F}_{p})^{\vee}])$. The Leibniz rule as described in Lemma 8.3 (ii), applied $t$-many times, gives a commutative diagram

$$
\begin{array}{ccc}
(sln(\mathbb{F}_{p})^{\vee})^{\otimes t} & \xrightarrow{\sum_{i=1}^{t}(-1)^{i-1}d^{i-1}\otimes d_{2}^{i}\otimes d^{i-1}} & (sln(\mathbb{F}_{p})^{\vee})^{\otimes t-1}[2] \\
\Lambda_{\mathbb{F}_{p}}^{t}[sln(\mathbb{F}_{p})^{\vee}] & \xrightarrow{d_{2}^{t}} & \Lambda_{\mathbb{F}_{p}}^{t-1}[sln(\mathbb{F}_{p})^{\vee}][2],
\end{array}
$$

in the derived category of $\mathbb{F}_{p}[\text{SL}_{n}(\mathbb{Z}/p)]$-modules. Writing $V = \mathbb{F}_{p}^{n}$, and using the map $\text{8.1}$, Lemma 8.1 and the proof of Corollary 8.2, we have a commutative diagram

$$
\begin{array}{ccc}
(V \otimes V^{\vee})^{\otimes t} & \xrightarrow{\sum_{i=1}^{t}(-1)^{i-1}d^{i-1}\otimes d_{2}^{i}\otimes d^{i-1}} & (V \otimes V^{\vee})^{\otimes t-1}[2] \\
\mathbb{F}_{p} \xrightarrow{c_{t}} (sln(\mathbb{F}_{p})^{\vee})^{\otimes t} & \xrightarrow{\sum_{i=1}^{t}(-1)^{i-1}d^{i-1}\otimes d_{2}^{i}\otimes d^{i-1}} & (sln(\mathbb{F}_{p})^{\vee})^{\otimes t-1}[2].
\end{array}
$$
in the derived category of $\mathbb{F}_p[\mathrm{SL}_n(\mathbb{Z}/p)]$-modules, where $d_1^2$ is defined to be the composition

$$d_1^2 : V \otimes V^\vee \longrightarrow \mathfrak{sl}(V)^\vee \xrightarrow{d_1^1} \mathbb{F}_p[2].$$

**Lemma 8.4.** The element $d_1^2 \in H^2(\mathrm{SL}_n(\mathbb{Z}/p), \mathrm{Hom}(V \otimes V^\vee, \mathbb{F}_p)) \cong H^2(\mathrm{SL}_n(\mathbb{Z}/p), V \otimes V^\vee) \cong \mathbb{F}_p \{x_1^{[1]} \otimes \mathbb{F}_p \{\Sigma_1^{ad}\}\}$

corresponds to $x_1^{[1]} \otimes (1)$.

**Proof.** There is a map of fibrations of pro-spaces

$$\{\mathrm{BSL}_n(\mathbb{Z}/p^r, p)\}_r \longrightarrow \{\mathrm{BSL}_n(\mathbb{Z}/p^r)\}_r \longrightarrow \mathrm{BSL}_n(\mathbb{Z}/p),$$

and an isomorphism $I + pA \leftrightarrow A : \mathrm{SL}_n(\mathbb{Z}/p^2, p) \cong \mathrm{SL}_n(\mathbb{F}_p)$. Thus the map on fibres induces an isomorphism on (continuous) cohomology in degrees $s \leq 1$, so the element $d_1^2 \in \mathrm{Ext}^2_{\mathbb{Z}/p}[\mathrm{SL}_n(\mathbb{Z}/p), \mathbb{F}_p] \cong \mathrm{Ext}^2_{\mathbb{Z}/p}[\mathrm{SL}_n(\mathbb{Z}/p), \mathbb{F}_p]$ coming from the top fibration sequence is identified with that coming from the bottom sequence, and is its Euler class. By the convention we made in Remark 2.3, the image of this Euler class under $\mathrm{Ext}^2_{\mathbb{Z}/p}[\mathrm{SL}_n(\mathbb{Z}/p), \mathbb{F}_p, V \otimes V^\vee]$ is the class $x$.

**□**

**Lemma 8.5.** The composition $(\mathrm{Id}^{\otimes i-1} \otimes d_1^2 \otimes \mathrm{Id}^{\otimes t-i}) \circ \tilde{c}_t$ corresponds to $x_1^{[1]} \otimes (1, 2, \ldots, t-1) \in \mathbb{F}_p \{x_1^{[1]} \mid j \in (t-1)\} \otimes \mathbb{F}_p \{\Sigma_1^{ad}\} \cong H^2(\mathrm{SL}(V); (V \otimes V^\vee)^{\otimes t-1}).$

**Proof.** We wish to evaluate the map

$$H^*(\mathrm{SL}(V); (V \otimes V^\vee)^{\otimes 2}) \otimes H^*(\mathrm{SL}(V); \mathrm{Hom}((V \otimes V^\vee)^{\otimes 2}, (V \otimes V^\vee)^{\otimes t-1}))$$

$$\longrightarrow H^*(\mathrm{SL}(V); (V \otimes V^\vee)^{\otimes t-1})$$

given by cup product followed by evaluation. This may be re-written as the map

$$H^*(\mathrm{SL}(V); V^{\otimes i} \otimes (V^\vee)^{\otimes 2}) \otimes H^*(\mathrm{SL}(V); V^{\otimes i-1} \otimes (V^\vee)^{\otimes t-1})$$

$$\longrightarrow H^*(\mathrm{SL}(V); V^{\otimes i-1} \otimes (V^\vee)^{\otimes t-1})$$

given by cup product followed by evaluation of the $V^{\otimes 2}$ from the first factor on the $(V^\vee)^{\otimes 2}$ from the second factor, and evaluation of the $(V^\vee)^{\otimes 2}$ from the first factor on the $V^{\otimes 2}$ from the second factor.

Let us identify $t \cup t-1 = 2t-1$ by the monotonic bijection which is the identity on elements of $\mathbb{I}_t$. In the first factor the element $\tilde{c}_t$ corresponds to

$$1 \otimes (1, 2, \ldots, t) \in \Gamma_{\overline{\mathbb{F}_p}}[x_1^{[2]} \otimes \mathbb{F}_p \{\mathrm{Bij}(\mathbb{I}_t, \mathbb{I}_t)\}]$$

and in the second factor, the element $\mathrm{Id}^{\otimes i-1} \otimes d_1^2 \otimes \mathrm{Id}^{\otimes t-i}$ corresponds to the element

$$x_1^{[1]} \otimes (1, i+1)(2, i+2) \cdots (i-1, 2(i-1))(i) \cdot 2i+1) \cdots (t, 2t-1) \in \Gamma_{\overline{\mathbb{F}_p}}[x_1^{[2]} \otimes t-1] \otimes \mathbb{F}_p \{\mathrm{Bij}(\mathbb{I}_t \cup t-1, \mathbb{I}_t \cup t-1)\}.$$
In particular, for an odd $t \geq 3$ the class
\[
\sum_{i=1}^{t} (-1)^{i-1} (\text{Id} \otimes 1 \otimes d_{2} \otimes \text{Id} \otimes \tilde{d}) \circ \tilde{c}_{t} \in H^{2}(\text{GL}(V); (V \otimes V^{\vee}) \otimes t^{-1})
\]
mapped further to
\[
H^{2}(\text{GL}(V); \Lambda^{t-1}(V \otimes V^{\vee})) = \left( \mathbb{F}_{p} \{ x_{i}^{[1]} \mid i \in \{1, \ldots, t-1\} \} \otimes \Sigma_{t-1} \right) \otimes \Sigma_{t-1} \mathbb{F}_{p}^{-}
\]
is
\[
\left[ \sum_{i=1}^{t} (-1)^{i-1} x_{i-1}^{[1]} \otimes (1, 2, \ldots, i-1, i, \ldots, t-1) \right] \otimes \Sigma_{t-1} 1
\]
(with $x_{0}^{[1]} := x_{t-1}^{[1]}$) which, by acting on the ith term by the element $(1, 2, \ldots, t-1)^{t-i}$ of sign $(-1)^{t-i}$, and using that $t-1$ is even, is the same as
\[
t \cdot \left[ x_{t-1}^{[1]} \otimes (1, 2, \ldots, t-1) \right] \otimes \Sigma_{t-1} 1.
\]
When we map from $(V \otimes V^{\vee}) \otimes t^{-1}$ to $(sl(V)^{\vee}) \otimes t^{-1}$, according to the construction in Corollary 8.2 this is the element called $t \cdot e_{t-1}$.

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