Heavy quarkonia spectra using wave function with gluonic components

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We calculate the spectra of charmonium and bottomium in an approximation scheme which treats hard gluons perturbatively while soft gluons are expanded in a set of localized wave functions. Quark-antiquark and quark-antiquark-gluon sectors are included. Reasonable agreement with 2 parameters only is found but the spectra are too coulombic. Despite large coupling constant the admixture of the quark-antiquark-gluon sector is found to be remarkably small.

I. INTRODUCTION

Quantum Chromodynamics (QCD) is a well established theory describing hadron interactions. Due to asymptotic freedom its predictions could be verified in high energy collisions. The bound state problem, however, remains nearly intractable. The only ab initio actually available calculations are obtained on the lattice\(^1\). There is, of course, a large body of phenomenological models\(^2\) which are quite successful in describing the spectra of hadrons. They are invariably based on the assumption that a meson is a quark-antiquark state and a baryon a 3 quark state. Here we present an application of a new method applicable to the bound state problem in QCD\(^3\). It consists in splitting the gluon momenta into 2 parts: high momenta, treated perturbatively and low momenta, those gluons are expanded not into plane waves but in an orthogonal set of localized functions. As a consequence only a finite number of localized gluons interact with a heavy quark-antiquark system. High momenta gluons are treated perturbatively within one gluon exchange approximation. Here we apply this method to the heavy quarkonia (charmonium and bottomium), which allows us to use the non relativistic approximation. To find the eigenstates we postulate a variational wave function with 2 components: quark-antiquark and quark-antiquark-gluon. After integrating out gluonic degrees of freedom we obtain a set of two coupled equations describing quark-antiquark system without and in the presence of one gluon. The solution is compared with experimental data. The agreement is not particularly good, our effective potentials being too coulombic. However, we use 2 parameters only (quark mass and QCD coupling constant). Moreover the inclusion of quark-antiquark-gluon sector improves the agreement.

II. ZEROTH ORDER HAMILTONIAN IN TEMPORAL GAUGE

The Lagrange density of QCD is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a + \bar{\Psi} \left( iD - m \right) \Psi. \tag{1}
\]

For the bound state problem in hamiltonian formalism it is convenient to choose a gauge which avoids the problem of constraints, the temporal gauge

\[
A_0^a = 0. \tag{2}
\]

In this gauge we have a 3-vector field \((\vec{A}^a)\), its canonically conjugate momenta are simply the components of the chromoelectric field

\[
\hat{A}_i^a = E_i^a, \tag{3}
\]

where the Latin index \(i\) takes values 1,2,3 and SU(3) gauge index \(a\) takes values 1..8. Introducing chromomagnetic field

\[
B_i^a = \varepsilon_{ijk} \left( \partial_j A_k^a + gf^{abc} A_j^b A_k^c \right) \tag{4}
\]

the gluonic part of the Lagrange density is

\[
\mathcal{L} = \frac{E_i^a E_i^a}{2} - \frac{B_i^a B_i^a}{2} \tag{5}
\]

and the corresponding Hamilton density is simply

\[
\mathcal{H} = \frac{E_i^a E_i^a}{2} + \frac{B_i^a B_i^a}{2}. \tag{6}
\]
Fields $A_i^a$ and $E_i^a$ are canonically conjugate and fulfill the commutation relations

$$[A_i^a(x), E_j^b(y)]_{x_0=y_0} = i\delta_{ij}\delta^{ab}\delta_3(x-y).$$

It should be noted that (6) contains 3 degrees of freedom at each point. The reduction of supernumerary degrees of freedom is accomplished by the imposition of Gauss law on the state vector $G^a(x) | \Psi \rangle = 0.$

For the quarks we use the non relativistic approximation

$$H_{q\bar{q}} = \text{Tr}[\frac{1}{2m}(p_{1i} - g\lambda_1^a A_i^a)^2 + \frac{1}{2m}(p_{2i} + g\lambda_2^a A_i^a)^2],$$

where $\lambda_{1,2}^a$ are the usual Gell-Mann matrices operating on the color index of the first (second) particle.

### III. HIGH-LOW DECOMPOSITION

In order to reduce the problem to a numerically manageable form we have to split our Hamiltonian eqs (6)(9) into a zeroth order solvable Hamiltonian and the rest to be treated perturbatively. We split the gluons into 2 groups: high momentum (hard) gluons (with $p_i > \Lambda$), where $\Lambda$ is the cut off parameter, and low momentum (soft) gluons. In high momentum sector we use the conventional approach - the interactions are treated as a perturbation. In the lowest order we have free gluons and quarks interacting via truncated coulomb potential. Notice that we have dropped spin-spin and spin-orbit terms.

$$V_c = -\frac{\alpha_s}{(\vec{p}_1 - \vec{p}_2)^2}, \quad |\vec{p}_1 - \vec{p}_2| > \Lambda,$$

where $\vec{p}_1, \vec{p}_2$ are the momenta of quark and antiquark respectively and $\alpha_s = \frac{4}{3}\alpha$. For soft gluons we use a different approach. Instead of expanding the field into plane waves, we expand them into a complete set of localized functions in position space. We have chosen to use the Fourier expansion in a cube $-\Lambda > p_i > \Lambda$ in momentum space. In position space our wave functions are the product of 3 well known sinc functions

$$e_{\vec{n}}(x,y,z) = (\frac{\Lambda}{\pi})^\frac{3}{2} \text{sinc}(n_x\pi - x\Lambda) \text{sinc}(n_y\pi - y\Lambda) \text{sinc}(n_z\pi - z\Lambda),$$

where $\vec{n}=(n_x,n_y,n_z)$, $n_x, n_y, n_z$ are integers (positive and negative). These functions are distributed on a lattice with lattice spacing $\pi/\Lambda$. They overlap very little and, if $\Lambda$ is small enough, the width of the peak will be much larger than the rms radius of heavy quark-antiquark system. We have found that $\Lambda=0.15 - 0.3$ GeV was sufficient and the results vary very little with $\Lambda$. Thus for soft gluons we have the expansion

$$A_i^a(\vec{r}) = \sum_{\vec{n}} e_{\vec{n}}(\vec{r}) A_{i,\vec{n}}$$

and similarly for the chromoelectric field

$$E_i^a(\vec{r}) = \sum_{\vec{n}} e_{\vec{n}}(\vec{r}) E_{i,\vec{n}}$$

where the $A_i^a$ and $E_i^a$ are the operators of localized gluons and fulfill

$$[A_{i,\vec{n}}, E_{j,\vec{m}}] = i\delta_{ij}\delta^{ab}\delta_3(\vec{n} - \vec{m})$$

Since the wave function of quark-antiquark system "fits" within the wave function at one lattice site (chosen to have $\vec{n} = (0,0,0)$) all others gluons will interact very little with quark-antiquark system. Thus it is reasonable to take into account, in the zeroth approximation, only the gluons with $\vec{n} = (0,0,0)$. In fact the quark-antiquark wave function is so narrow that we can approximate $e_{\vec{n}}(\vec{r})$ by its value at the origin. Our zeroth order hamiltonian describes a quark-antiquark pair with a truncated Coulomb potential interacting with gluons localized at the origin

$$H_0 = \frac{1}{2m}\vec{p}_1 + g(\frac{\Lambda}{\pi})^\frac{3}{2}\vec{A}_1 \vec{r} + \frac{1}{2m}\vec{p}_2 - g(\frac{\Lambda}{\pi})^\frac{3}{2}\vec{A}_2 \vec{r} + V_c + \frac{E_i^a E_i^a}{2} + \frac{B_i^a B_i^a}{2},$$

where $\vec{A}_1, \vec{A}_2$ are the coulomb potentials and $V_c$ is the color confinement potential.
where we have used

$$\lambda_1^a A_1^a = \vec{A}_1$$

(16)

and similarly for the antiquark. We have also dropped the index $\vec{n} = (0,0,0)$. $B_0^a$ is the chromomagnetic field with derivative terms dropped (functions $e_g(r)$ are slowly varying within the range of quark-antiquark wave function)

$$B_0^a = g \varepsilon_{ijk} f^{abc} A_j^b A_k^c$$

(17)

$f^{abc}$ are the structure constants of the SU(3) group. We have reduced the problem to a quark-antiquark system interacting with $3 \times 8 = 24$ gluons localized at the origin. This is numerically tractable by a method widely used in atomic and nuclear physics: Ritz variational method. Not all eigenstates of this hamiltonian are physically acceptable - the wave function must fulfill the Gauss law eq. (8), i.e. the state must be colorless. After elimination of center of mass motion we get finally

$$H_0 = \frac{1}{2m} \left( \vec{p} + g \frac{\Lambda}{\pi} \vec{A}_1 \right)^2 + \frac{1}{2m} \left( \vec{p} + g \frac{\Lambda}{\pi} \vec{A}_2 \right)^2 + V_c + \frac{E_1^a E_1^a}{2} + \frac{B_1^a B_1^a}{2}. \quad (18)$$

where $p$ is the relative momentum of the quark-antiquark.

IV. VARIATIONAL ANSATZ

The mass of the quark-antiquark system is the energy difference between the energy eigenvalue of Hamiltonian $H_0$ and the energy of vacuum fluctuations i.e. the lowest energy of gluonic part of $H_0$. We have chosen to work in Schrödinger representation with

$$E_i^a = \frac{1}{i} \partial_0 A_i^a, \quad (19)$$

For the vacuum part of the problem we have chosen a variational gaussian Ansatz

$$\Psi_{\text{vac}}(\vec{A}) = N \exp\{ - \beta_{\text{vac}} \frac{\text{Tr}[\vec{A} \cdot \vec{A}]}{4} \}, \quad (20)$$

$\beta$ being the variational parameter. We obtain

$$\beta_{\text{vac}} = \frac{4 \frac{2\pi}{\alpha} \frac{\Lambda}{\pi}}{\frac{3\pi}{\alpha}} = 0.168 \text{ GeV}, \quad (21)$$

$$E_{\text{vac}} = \frac{12 \frac{6\pi}{\alpha} \frac{\Lambda}{\pi}}{\frac{3\pi}{\alpha}} = 1.511 \text{ GeV}. \quad (22)$$

For the quark-antiquark system we postulate a 2 component wave function

$$\Psi(\vec{p}, \vec{A}) = \delta_{ij} f_1(\vec{p}) \exp\{ - \beta \frac{\text{Tr}[\vec{A} \cdot \vec{A}]}{4} \} + f_2(\vec{p}) \frac{\vec{p} \cdot \vec{A}}{p} \exp\{ - \beta \frac{\text{Tr}[\vec{A} \cdot \vec{A}]}{4} \} \quad (23)$$

The first part describes a quark antiquark system with no gluons (gaussian i.e. lowest energy state of the oscillator) while the second part represents a quark-antiquark-gluon system (the gluonic part is proportional to the first excited state of an oscillator). Both parts are colorless. The variation is to be done with respect to $\beta$, $f_1(\vec{p})$ and $f_2(\vec{p})$ with the condition that the norm of the wave function $\Psi(\vec{p}, \vec{A})$ be one. After solving the variational problem for $f_1(\vec{p})$ and $f_2(\vec{p})$ and integrating out gluonic degrees of freedom we have a system of two coupled equations

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} f_1(\vec{p}) \\ f_2(\vec{p}) \end{pmatrix} = \mathcal{E} \begin{pmatrix} f_1(\vec{p}) \\ f_2(\vec{p}) \end{pmatrix}, \quad (24)$$

where
\[ h_{11} = \frac{p^2}{2 \mu} + \tilde{V}_c(\vec{r}) + \frac{4 g^2 A^3}{\pi^3 \beta \mu} + 6 \beta + \frac{32 g^2 A^3}{3 \pi^3 \beta^2} \]  
(25)

\[ h_{22} = \frac{p^2}{2 \mu} + \tilde{V}_c(\vec{r}) + \frac{31 g^2 A^3}{6 \pi^3 \beta \mu} + \frac{13 \beta}{2} + \frac{112 g^2 A^3}{9 \pi^3 \beta^2} \]  
(26)

\[ h_{12} = \left( \frac{8 g A^2}{3 \beta \mu \pi^2} \right) p \]  
(27)

\[ h_{21} = \left( \frac{g A^2}{\mu \pi^2} \right) p. \]  
(28)

Notice that diagonal terms differ by a constant only and that the coupling terms are proportional to \( p \). This set of equations was solved numerically by expanding \( f_1(\vec{r}) \) and \( f_2(\vec{r}) \) in the basis of Laguerre polynomials. We found that a basis of 16 polynomials was sufficient.

V. RESULTS AND DISCUSSION

We have calculated spectra of charmonium and bottomium. In both cases we used

\[ \Lambda = 0.15 \text{ GeV} \]  
(29)

In our \textit{ab initio} calculation we have 2 free parameters only: the mass of the quark and the coupling constant. For charmonium we found

\[ m_c = 1.8604 \text{ GeV}, \]  
(30)

\[ \alpha_{sc} = 0.973, \]  
(31)

by fitting the 2 lowest \( L=0 \) states. Since we have no spin-spin interaction in our non relativistic approximation \( ^1S_0 \) and \( ^3S_1 \) are degenerate, we used weighted the average. All the experimental data are from ref\textsuperscript{6}. The results are summarized in Table 1 and Table 2 and shown in Fig. 1 and Fig. 2. We show the results for a calculation with quark-antiquark sector only (\( E_{q\bar{q}} \)), with quark-antiquark-gluon only (\( E_{q\bar{q}g} \)) and full coupled channels (\( E_{\text{full}} \)). We consider the agreement reasonable considering that only 2 parameters were used. In our approximation the system is too coulombic but the inclusion of quark-antiquark-gluon sector improves the agreement with experimental data. To our surprise the admixture of quark-antiquark-gluon sector is very small; for the ground state we have

\[ p_{q\bar{q}g} = 0.0017 \]  
(32)

despite using a large value for \( \alpha_{sc} \approx 1 \). This could be an explanation why phenomenological models with constituent quarks only are so successful\textsuperscript{2}. For bottomium the results are similar. We find

\[ m_b = 5.1896 \text{ GeV}, \]  
(33)

\[ \alpha_{sb} = 0.66114. \]  
(34)

Notice that the coupling constant is 30% lower. Again the spectrum is too close to a purely coulombic one but the inclusion of quark-antiquark-gluon sector improves the agreement (Table 3-4 and Fig 3-4). The admixture of quark-antiquark-gluon sector is very small

\[ p_{q\bar{q}g} = 0.0004. \]  
(35)

The main feature of this calculation is the surprisingly small admixture of quark-antiquark-gluon sector. However the long range part of the effective potential is clearly wrong. The obvious improvement would be to take into account space variation of gluon wave function and the inclusion of gluons located on neighbour sites on the lattice. While the computation of spectra would be vastly more complicated it is certainly feasible. In conclusion it is seen that it is possible to perform bound state calculations within QCD and improve them step by step.
| Exp. | $\varepsilon_{q\bar{q}}$ | $\varepsilon_{q\bar{q}g}$ | $\varepsilon_{full}$ |
|------|-----------------|-----------------|-----------------|
| 3.068| 3.0955          | 3.8565          | 3.068           |
| 3.663| 3.6785          | 3.9659          | 3.663           |
| 4.040| 3.7687          | 4.0106          | 3.761           |
| 4.415| 3.7977          | 4.0106          | 3.761           |

Table 1: S levels in $c\bar{c}$ (in GeV)

| Exp. | $\varepsilon_{q\bar{q}}$ | $\varepsilon_{q\bar{q}g}$ | $\varepsilon_{full}$ |
|------|-----------------|-----------------|-----------------|
| 9.460| 9.469           | 10.381          | 9.460           |
| 10.023| 10.224         | 10.517          | 10.218          |
| 10.355| 10.361         | 10.57           | 10.355          |
| 10.580| 10.427         | 10.648          | 10.385          |
| 10.865| 10.555         | 10.805          | 10.421          |
| 11.019| 10.863         | 11.141          | 10.519          |

Table 3: S levels in $b\bar{b}$ (in GeV)

| Exp. | $\varepsilon_{q\bar{q}}$ | $\varepsilon_{q\bar{q}g}$ | $\varepsilon_{full}$ |
|------|-----------------|-----------------|-----------------|
| 9.888| 10.224          | 10.518          | 10.215          |
| 10.253| 10.361         | 10.564          | 10.356          |
|   | 10.414          | 10.613          | 10.408          |
| 10.492| 10.703         | 10.479          |
| 10.649| 10.868         | 10.523          |

Table 4: P levels in $b\bar{b}$ (in GeV)

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