Discrete analysis on non-cubic lattices

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Abstract. The paper proposes practical and computation methods for discrete analysis of functions defined on the weight lattices or model sets of semisimple Lie groups. They are entirely group theoretical, being based on finite groups and their duals. Numerical computations approximate target functions uniformly on the entire infinite space. Several examples are considered.

1. Introduction

Discrete Fourier transform is one of the common tools for the spectral analysis of signals and for the general analysis of functions. It is widely used in various branches of science and technology, e.g. for mp3 and jpg coding, in statistics, etc. There are many other discrete transformations (Hartley, cosine, wavelet, Lapped, finite Legendre, etc.), but there are a few methods for processing the rapidly increasing amount of 3D digital data gathered today and methods that take into account the non-cubic atomic structures of the solid bodies. Moreover, the existence of the materials with the structures that are shift non-invariant, Faraday wave experiments, potential field of a physical quasicrystal and other physical phenomena require even more sophisticated approaches to the Fourier analysis.

In this paper we consider analogs of multidimensional Fourier transforms for non-cubic translation-invariant lattices and model sets (quasicrystals).

First we explore families of special functions depending on the weight lattices of simple Lie algebras, which are not only cubic and point out their most valuable properties:

• The functions are defined for each compact semisimple Lie group $G$. There are several infinite families of functions per $G$. The number of continuous variables on which the functions depend, is equal to the rank of $G$.

• Orbit functions have well defined symmetries with respect to the affine Weyl group of $G$ and these functions (within each family) are orthogonal when integrated over the fundamental region of $F$ of the maximal torus $T$ of $G$.

• The functions can be sampled on the lattice fragment $F_M = F \cap L/M$ of the weight lattice $L$ refined by $M \in N$. There is a finite subset $\Lambda_M$ of such ‘digital’ functions that are pairwise orthogonal when summed up over the points of $F_M$.

• Each function is an eigenfunction of the Laplace operator appropriate for $G$ and their eigenvalues are known explicitly. Therefore orbit functions can be applied to the solution
of the corresponding Neumann and Dirichlet boundary-value problems on the fundamental domains of the Weyl groups.

- Each function can be transformed to the polynomial form in several ways.

Then we consider the problem of discrete methods for dealing with functions that are intrinsically almost periodic, but not actually periodic (the functions that depend on quasicrystals). Usually such functions are approximated as periodic ones with the loose of some information, but it is not entirely satisfactory.

The main idea (due to H. Bohr) is to build the analysis around the theory of quasicrystals using the local hulls and dynamical systems. The n-dimensional Fourier analysis for such functions was set in [5] and some examples of functions based on the standard Fibonacci quasicrystal were considered.

The key features of the method:

- retains the essential feature of the almost periodicity;
- uses all the spectral components of the natural Fourier module;
- creates approximations that are valid uniformly on the entire space of quasicrystal;
- involves methods of fast discrete Fourier analysis, namely use finite groups;
- takes advantage of internal symmetry.

2. Functions on non-cubic lattices

The main task of this section is to give all the tools for analog of the discrete Fourier transformation not with the complete system of exponents on cubic lattices, but with orbit functions on non-cubic lattices generated by fundamental region of a semisimple group.

There are seven compact semisimple Lie groups of rank 3 $SU(2) \times SU(2) \times SU(2)$, $SU(3) \times SU(2)$, $O(5) \times SU(2)$, $G(2) \times SU(2)$, $SU(4)$, $O(7)$ and $Sp(6)$ with the respective Lie algebras $A_1 \times A_1 \times A_1$, $A_2 \times A_1$, $C_2 \times A_1$, $G_2 \times A_1$, $A_3$, $B_3$ and $C_3$.

Let $R^n$ is the real Euclidean space spanned by the simple roots $\alpha$ of a simple Lie algebra and $\omega$-basis is the basis of fundamental weights.

In the weight lattice $P$, we define the cone of dominant weights $P^+$ and its subset of strictly dominant weights $P^{++}$: $P = Z\omega_1 + Z\omega_2 + \cdots + Z\omega_n \supset P^+ = Z^{\geq 0}\omega_1 + \cdots + Z^{\geq 0}\omega_n \supset P^{++} = Z^{>0}\omega_1 + \cdots + Z^{>0}\omega_n$.

The Weyl group $W$ is the finite group generated by the reflections $r_\alpha x = x - \frac{2(x,\alpha)}{\langle\alpha,\alpha\rangle}\alpha$, $x \in R^n$.

For any simple Lie algebra there is a unique highest root $\xi = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n$ and a reflection through it’s middle is $r_0 x = x - \frac{2(x,\xi)}{\langle\xi,\xi\rangle}\xi + \frac{2\xi}{\langle\xi,\xi\rangle}$. The group generated by $n + 1$ reflections $r_0$, $r_1$, $\ldots$, $r_n$, is the affine Weyl group $W^{aff}$ and the even subgroup of it is $W_e$.

The fundamental region $F \subset R^n$ for any $W^{aff}$ is the convex hull of the vertices $F(G) = \{0, \frac{\alpha_1}{m_1}, \frac{\alpha_2}{m_2}, \ldots, \frac{\alpha_n}{m_n}\}$ (for the group $W_e$ we take a doubled fundamental region). All possible three-dimensional fundamental regions are presented on Figures 1–6, note, that only one of them $(A_1 \times A_1 \times A_1)$ is rectangular.

We define basic orbit functions $C$, $S$- and $E$-functions, specified by a given point $\lambda \in Z^n$ and a chosen semisimple Lie group $G$ with the corresponding Weyl group $W$, its even subgroup $W_e$ and fundamental region $F$:

- $C_\lambda(x) = \sum_{\mu \in W\lambda} e^{2\pi i (\mu, x)}$, where $x \in R^n$, $\lambda \in P^+$;
- $S_\lambda(x) = \sum_{\mu \in W\lambda} (-1)^{|\mu|} e^{2\pi i (\mu, x)}$, where $x \in R^n$, $\lambda \in P^{++}$ and $l(\mu)$ is the number of reflections from $\lambda$ to $\mu$;
- $E_\lambda(x) = \sum_{\mu \in W_{\lambda}} e^{2\pi i (\mu, x)}$, where $x \in R^n$, $\lambda \in P^+$.
For any complex square integrable functions $\phi(x)$ and $\psi(x)$, we define a continuous scalar product $\langle \phi(x), \psi(x) \rangle := \int_F \phi(x)\overline{\psi(x)} \, dx$. The integration is taken with respect to the Euclidean measure, the bar means complex conjugation, and $x \in F$.

\[ \langle C_\lambda(x), C_{\lambda'}(x) \rangle = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad \langle S_\lambda(x), S_{\lambda'}(x) \rangle = |W| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad \text{here } |W| \text{ is the order of Weyl group, } |W_\lambda| \text{ is the size of Weyl group orbit, and } |F| \text{ is the volume of fundamental region.} \]

The similar relation is true for $E$-functions as well, see the details [1].

Therefore, each family of orbit functions forms an orthogonal basis in the Hilbert space of squared integrable functions $L^2(F)$. Hence functions given on $F$ can be expanded in terms of linear combinations of $C$-, $S$- or $E$-functions.

Each continuous function on the fundamental region with continuous derivatives can be expanded as the sum of $C$-, $S$- or $E$-functions.

Let $f(x)$ be a function defined on $F$ (or $F_e$ for $E$-functions), then it may be written that
The fundamental region of the Lie algebra $B_3$.

$$f(x) = \sum_{\lambda \in P^+} c_{\lambda} C_{\lambda}(x), \quad c_{\lambda} = |W_\lambda|^{-1}|F|^{-1}(f(x), C_{\lambda}(x));$$

$$f(x) = \sum_{\lambda \in P^-} c_{\lambda} S_{\lambda}(x), \quad c_{\lambda} = |W|^{-1}|F|^{-1}(f(x), S_{\lambda}(x));$$

$$f(x) = \sum_{\lambda \in P^0} c_{\lambda} E_{\lambda}(x), \quad c_{\lambda} = |W_\lambda|^{-1}|F|^{-1}(f(x), E_{\lambda}(x)).$$

The respective discrete transform can be used for the continuous interpolation of values of a function $f(x)$ between its given values on some grid. There is a uniform way to introduce a grid for any dimension and any semisimple Lie algebra.

The grid $F_M \subset F$ depends on a natural number $M$ and consists of the points

$$F_M = \left\{ \frac{\omega_1}{M} s_1 + \cdots + \frac{\omega_n}{M} s_n \mid s_1, \ldots, s_n \in \mathbb{Z}^0, \sum_{i=1}^n s_i m_i \leq M \right\}.$$  

Then we have the following discrete transforms for the function $f(x)$:

$$f(x) = \sum_{\lambda \in \Lambda_M} b_{\lambda} C_{\lambda}(x), \quad x \in F_M, \quad b_{\lambda} = \frac{\langle f, C_{\lambda} \rangle_M}{\langle C_{\lambda}, C_{\lambda} \rangle_M};$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_{\lambda} S_{\lambda}(x), \quad x \in F_M, \quad b_{\lambda} = \frac{\langle f, S_{\lambda} \rangle_M}{\langle S_{\lambda}, S_{\lambda} \rangle_M};$$

$$f(x) = \sum_{\lambda \in \Lambda_M} b_{\lambda} E_{\lambda}(x), \quad x \in F_M, \quad b_{\lambda} = \frac{\langle f, E_{\lambda} \rangle_M}{\langle E_{\lambda}, E_{\lambda} \rangle_M}.$$  

Here $\langle \cdot, \cdot \rangle_M$ denotes the discrete scalar product, see [1]. Once the coefficients $b_{\lambda}$ are calculated, the function $f(x)$ smoothly interpolates the values of $f(x_i), i = 1, 2, \ldots, |F_M|$ and using the symmetry ($C$-, $E$-functions) or antisymmetry ($S$-functions) we get the approximation on the whole Euclidian space $R^n$.

Consider the orbit functions in the orthogonal bases (the orthogonal bases for these algebras are well known, see e.g. [2]).

The Laplace operator in the orthogonal coordinates gives the same eigenvalues on every exponential function summand of an orbit function with eigenvalue $-4\pi^2\langle \lambda, \lambda \rangle$. Therefore, the functions $C_{\lambda}(x), E_{\lambda}(x)$ and $S_{\lambda}(x)$ are eigenfunctions of the Laplace operator:

$$\Delta \begin{pmatrix} C_{\lambda}(x) \\ E_{\lambda}(x) \\ S_{\lambda}(x) \end{pmatrix} = -4\pi^2\langle \lambda, \lambda \rangle \begin{pmatrix} C_{\lambda}(x) \\ E_{\lambda}(x) \\ S_{\lambda}(x) \end{pmatrix}.$$  

Thereby, a Laplace operator for each Lie group is given in a different set of coordinates. In Table 1 the explicit forms of the 3d Laplacians are given in the $\omega$-basis for all semisimple Lie algebras.

On the boundary of $F$, the $C$-functions have a vanishing normal derivative, while $S$-functions reach zero at the boundary, what makes it possible to solve the following boundary value
Table 1. Three-dimensional Laplacians of semi-simple Lie algebras.

| Lie algebra | Laplace operator |
|-------------|------------------|
| $A_1 \times A_1 \times A_1$ | $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ |
| $A_2 \times A_1$ | $\Delta = \partial_{x_1}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_2}^2 + \partial_{x_3}^2$ |
| $C_2 \times A_1$ | $\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + \partial_{x_2}^2 + \partial_{x_3}^2$ |
| $A_3$ | $\Delta = \partial_{x_1}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_2}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_3}^2$ |
| $B_3$ | $\Delta = \partial_{x_1}^2 - \partial_{x_2} \partial_{x_3} + \partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + 2\partial_{x_3}^2$ |
| $C_3$ | $\Delta = 2\partial_{x_1}^2 - 2\partial_{x_1} \partial_{x_2} + 2\partial_{x_2}^2 - 2\partial_{x_2} \partial_{x_3} + \partial_{x_3}^2$ |

problems.

$C$-function is a solution of the Neumann boundary value problem on n-dimensional simplex $F$

$\Delta f(x) = \Lambda f(x), \ \frac{\partial f(x)}{\partial \nu} = 0 \text{ for } x \in \partial F.$

$S$-function is a solution of the Dirichlet boundary value problem on n-dimensional simplex $F$

$\Delta f(x) = \Lambda f(x), \ \ f(x) = 0 \text{ for } x \in \partial F.$

Let the continuous variable $x$ is given relative to the $\omega$-basis and $\Delta$ denotes the Laplace operator, where the differentiation $\partial_{x_1}$ is made with respect to the direction given by $\omega_i$. Then

$\Delta = \sum_{i,j=1}^{n} \frac{C_{ij}}{\omega_{i}, \omega_{j}} \partial_{x_i} \partial_{x_j},$ where $C$ is the Cartan matrix.

It is known in Lie theory that the matrix of scalar products of the simple roots is positive definite, moreover our definition makes matrix $\frac{C_{ij}}{\omega_{i}, \omega_{j}}$ symmetric, hence it can be diagonalized and the Laplace operator could be transformed to the sum of second derivatives by an appropriate change of variables.

Note, that the discretization procedure works well for the both Neumann and Dirichlet boundary value problems.

One more feature that can be exploit in discrete analysis by means of orbit functions is the relation to orthogonal polynomials. Polynomials can be obtained by the exponential substitution, that is rather straightforward [3, 4], namely $X_j := e^{2\pi i x_j}, \ x_j \in R, \ j = 1, 2, \ldots, n$. Two other possibilities are the polynomials obtained by trigonometric substitution and polynomials obtained by the recursive method, see [7].

3. Discrete methods for almost periodic functions

Our key objective is to describe a method for discrete computational Fourier analysis of a function defined on aperiodic atomic arrangement with highly structured long-range order (quasicrystal).

Let $R^d$ is a real Euclidean space of finite dimension $d$ and $D_r(R^d)$ are all point sets $\Lambda \subset R^d$ for which the distance $|x - y| \geq r > 0 \ \forall x, y \in \Lambda$.

The local topology on $D_r(R^d)$ is introduced as follows. Two sets $\Lambda_1$ and $\Lambda_2$ of $D_r(R^d)$ are ‘close’ if, for some large $R$ and some small $\epsilon$, one has $\Lambda_1 \cap B_R \subset \Lambda_2 + B_\epsilon$, and $\Lambda_2 \cap B_R \subset \Lambda_1 + B_\epsilon$.

For $\Lambda \in D_r(R^d)$ the local hull of $\Lambda$ is $X(\Lambda) = \{t + \lambda : t \in R^d \} \subset D_r(R^d)$, i.e. take all translates of $\Lambda$ and take their closure in the local topology.

The translation action of $R^d$ on $\Lambda$ lifts to a translation action on $X(\Lambda)$. The local hull $X(\Lambda)$ is compact and the $R^d$-action on it is continuous.

Consider a function $F : X(\Lambda) \rightarrow \mathbb{C}$. We can define from it a function $f : R^d \rightarrow \mathbb{C}$ by $f(t) = F(t + \Lambda)$.

If $F$ is continuous then we note that for all $t_1, t_2 \in R^d, t_1 + \Lambda$ and $t_2 + \Lambda$ are close $\implies F(t_1 + \Lambda) and F(t_2 + \Lambda)$ are close $\implies f(t_1)$ and $f(t_2)$ are close. Thus continuity of $F$ implies...
continuity of \( f \).

A function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) is called local with respect to a set \( \Lambda \in \mathcal{D}_r(\mathbb{R}^d) \), or \( \Lambda \)-local, if for all \( \epsilon' > 0 \) there exist \( R \) and \( \epsilon \) so that whenever \( t_1, t_2 \in \mathbb{R}^d \) satisfy that \( t_1 + \Lambda \) and \( t_2 + \Lambda \) are \((R, \epsilon)\)-close then \( |f(t_1) - f(t_2)| < \epsilon' \).

Using locality, we can go in the opposite direction.

Let \( \Lambda \in \mathcal{D}_r(\mathbb{R}^d) \) and let \( f : \mathbb{R}^d \rightarrow \mathbb{C} \) be local with respect to \( \Lambda \). Define \( F : \{ t + \Lambda : t \in \mathbb{R}^d \} \rightarrow \mathbb{C} \) (so that \( F \) is a function on a part of \( \mathcal{D}_r(\mathbb{R}^d) \)) by \( F(t + \Lambda) = f(t) \). Then \( F \) is continuous on \( \{ t + \Lambda : t \in \mathbb{R}^d \} \) w.r.t. the local topology.

For each local function \( f \) with respect to \( \Lambda \) there is a unique continuous function \( F \) on the local hull, whose restriction to the orbit of \( \Lambda \) is \( f \). Every continuous function on the local hull of \( \Lambda \) arises in this way. Thus a locality with respect to \( \Lambda \) and the existence and continuity of an extension function on \( X(\Lambda) \) amount to the same thing.

In the situation that \( X(\Lambda) \) is equipped with an \( \mathbb{R}^d \)-invariant probability measure \( \mu \) the action of \( \mathbb{R}^d \) on \( X(\Lambda) \) leads to unitary action \( T \) of \( \mathbb{R}^d \) on \( L^2(X(\Lambda), \mu) \). Namely for all \( F \in L^2(X(\Lambda), \mu) \) and for all \( t \in \mathbb{R}^d \), \( T_t F \) is the function defined by \( T_t F(\Gamma) = F(-t + \Gamma) \) and with \( \langle F | G \rangle := \int_{X(\Lambda)} F \overline{G} \, d\mu \) we have \( \langle T_t F | T_t G \rangle = \langle F | G \rangle \).

\[
\mathbb{R}^d \leftarrow (||) \quad \mathbb{R}^d \times \mathbb{R}^d \cup (\perp) \rightarrow \mathbb{R}^d \\
L \leftarrow (1-1) \quad \tilde{L} \quad \text{(dense image)} \rightarrow \quad L'
\]

Here \( \tilde{L} \) is a lattice in \( \mathbb{R}^d \times \mathbb{R}^d \) which is oriented so that the projections into \( \mathbb{R}^d \) are 1 and dense respectively.

The left-hand \( \mathbb{R}^d \) is physical space (in which \( \Lambda \) is going to lie).

The right-hand \( \mathbb{R}^d \) is internal space (is used to control the projection of the lattice \( \tilde{L} \) into physical space).

\( L \) denotes the image of \( \tilde{L} \) under projection into physical space. Since this projection is one-one, \( L \) and \( \tilde{L} \) similar groups, so \( L \) is a free Abelian group of rank \( 2d \), i.e. it has a \( Z \)-basis of \( 2d \) elements. However, it necessarily has accumulation points, and the typical situation is that \( L \) is dense in physical space.

Choose a subset \( \Omega \) in internal space. This window is assumed to be compact, equal to the closure of its interior, and to have boundary of measure 0. Using it we define \( \Lambda = \Lambda(\Omega) = \{ x : \tilde{x} \in L, \ x' \in \Omega \} \).

Sets of the form \( t + \Lambda(\Omega), t \in \mathbb{R}^d \), are called cut and project sets or model sets. In particular, for each \( (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \) we define \( \Lambda(x, y) = x + \Lambda(-y + \Omega) \).

If \( (x, y) = (x', y') \mod \tilde{L} \), then \( x + \Lambda(-y + \Omega) = x' + \Lambda(-y' + \Omega) \). Thus these model sets are parameterized by the torus of dimension \( 2d \) \( (\mathbb{R}^d \times \mathbb{R}^d)/\tilde{L} =: T \).

Denote \( (x, y)_L \) the congruence class \( (x, y) \mod \tilde{L} \).

There is a natural measure, the Haar measure, \( \theta_T \) on \( T \). It is invariant under the \( \mathbb{R}^d \)-action.

Suppose \( f \) is a local function with respect to the model set \( \Lambda \). From the local function \( f \) we have its extension \( F \in L^2(X(\Lambda), \mu) \) which is continuous. Then we obtain \( \tilde{F} \in L^2(T, \theta) \), where \( \tilde{F}(\beta(t + \Lambda)) = \hat{F}(t + \Lambda) = \hat{F}(t) \), and we can write

\[
\tilde{F}(\cdot) = \sum_{k \in L^0} a_k e^{2\pi i \langle k, \cdot \rangle}, \text{ where } L^0 \text{ is the dual of } L.
\]

The Fourier-Bohr expansion of the local function is

\[
f(t) = F(t + \Lambda) = \tilde{F}((t, 0)_L) = \sum_{k \in L^0} a_k e^{2\pi i \langle k, t(0) \rangle} = \sum_{k \in L^0} a_k e^{2\pi i \langle k, t \rangle}.
\]

Unfortunately, we don’t have total control over \( \tilde{F} \). We know it only on \( (\mathbb{R}^d, 0)_L \). To compute \( a_k \) out of \( f \) alone, we use the analog of Birkhoff ergodic theorem (transformed from Keller 1998) for all continuous functions \( \tilde{G} \) on \( T \),

\[
\frac{1}{\text{vol}_{B_R}} \int_{B_R} \tilde{G}((t, 0)_L) \, dt.
\]
Thus \( a_k = \lim_{R \to \infty} \frac{1}{\text{vol} B_R} \int_{B_R} e^{-2\pi i (k(t,0))} \bar{F}((t, 0)_L) \, dt = \lim_{R \to \infty} \frac{1}{\text{vol} B_R} \int_{B_R} e^{-2\pi i (k(t))} f(t) \, dt. \)

Here we use \( F((t, 0)_L) = f(t) \) and \( \tilde{k} = (k, k') \), so \( \langle \tilde{k} | (t, 0) \rangle = \langle k, t \rangle + \langle k', 0 \rangle = \langle k, t \rangle. \)

For discretization we begin with the cut and project scheme with torus \( T \) and note the natural extension of the mapping \( (\cdot)' \) to the rational span of the module \( L \):

\[
QL \leftarrow (1 - 1) \quad Q\tilde{L} \quad \rightarrow \quad QL'
\]

\( x \leftarrow \tilde{x} = (x, x') \rightarrow x' \)

We assume that \( R^{2d} \) is supplied with the standard dot product (denoted \( \langle \cdot | \cdot \rangle \)), and then define the dual lattice: \( \tilde{L}^o = \{ Y \in R^d \times R^d : \langle Y | \tilde{x} \rangle \in Z \text{ for all } \tilde{x} \in \tilde{L} \} \).

Then \( \tilde{L}^o \) is a \( Z \)-module of the same rank as \( \tilde{L} \), namely \( 2d \). There is a cut and project scheme of which \( \tilde{L}^o \) is the lattice:

\[
R^d \leftarrow (||) \quad R^d \times R^d \quad (\perp) \quad R^d
\]

\[
L^o \leftarrow (1 - 1) \quad \tilde{L}^o \quad \text{(dense image)} \quad (L^o)'.
\]

The data points in \( R^d \) at which computations of our functions will be made come by projection into physical space of a suitable set of coset representatives of \( \tilde{L}^o_N \) modulo \( \tilde{L} \). The corresponding frequencies (wave vectors) \( k \) are chosen from the dual lattice \( L^o \). The choice of values of \( k \) at which we should evaluate the Fourier coefficients \( a_k \) come by selecting suitable representatives of \( \tilde{L}^o \) modulo \( L^o_N \). The key point is the duality \( \langle \cdot, \cdot \rangle : \tilde{L}^o / L^o_N \times \tilde{L}^o / L \rightarrow \frac{1}{d} Z / Z \).

The main effort required is the creation of the data points. Once this is done, the same set of data points and Fourier frequencies will work for any almost periodic function arising from the same cut and project scheme.

The approximating functions are not just local approximations, they are also \textit{global} approximations, in the sense that they provide finite Fourier series that approximate \( f \) throughout its entire domain (namely \( R^d \)).

Below on Figures 7 and 8 we show two examples of approximation, which use the one-dimensional Fibonacci model set. The functions to be approximated are: a distance function \( f(t) = \text{the distance of } t \text{ to the nearest point of } \Lambda \) and a step function \( f(x) = 1, \text{ if } x \text{ is in a long interval, and } f(x) = -1, \text{ if } x \text{ is in a short interval}. \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{The distance function and approximants are drawn for the data sampled on 121 points}
\end{figure}

4. Final Comments

Concerning orbit functions, we insisted throughout this paper that the underlying group is semisimple. The extension to the finite Coxeter groups that are not Weyl groups of a simple Lie algebra is straightforward. Many of the properties of orbit functions extend to these cases. Only their orthogonality, continuous or discrete, has not been shown so far.
Figure 8. Here we adduce the step function and approximants are drawn on the interval [200, 215] for 49 data points.

Let $\Lambda_1 \in \mathbb{R}^{d_1}$ and $\Lambda_2 \in \mathbb{R}^{d_2}$ are two cut-and-project sets (quasicrystals), then their cartesian product $\Lambda = \Lambda_1 \times \Lambda_2$ is non-periodic as well. Therefore one-dimensional Fibonacci quasicrystals can be used for the construction of higher-dimensional ones. Unfortunately, to apply the proposed approximation we have to construct all cut-and-project ingredients, torus parametrization and discretization from the every beginning. As the initial lattice for the cut-and-project method the direct sum of the initial lattices of the quasicrystals $\Lambda_1$ and $\Lambda_2$ is to be taken.

Finally, the way in which finite groups and their duals are used for orbit functions and model sets makes the Fourier approximations amenable to the technique of the fast Fourier transform.

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