The light-curves of the two large amplitude variable stars, R Scuti and AC Herculis, display irregular pulsation cycles with 'periods' of 75 and 35 days, respectively. We review the evidence that the observed time-series are generated by low dimensional chaotic dynamics. In particular, a global flow reconstruction technique indicates that the minimum embedding dimensions are 4 and 3, for R Sct and AC Her, respectively, whereas the fractal dimensions are inferred to be $d_L \approx 3.1$ and $2.05 \lesssim d_L \lesssim 2.45$, respectively. It thus appears that the dimensions of the dynamics themselves are also 4 and 3.

1. Introduction

For several centuries it has been known that a sizeable subset of the stars exhibits a variable energy output (luminosity). The best known and best studied of these variable stars are the Cepheids that have a clock-like periodicity. Their notoriety comes from the tight relation between their period and luminosity and their consequent role as cosmological distance indicators. In parallel, there exists a class of variable stars with light-curves that instead exhibit various degrees of irregularity. These objects undergo changes in luminosity that can be as large as a factor of forty, indicating violent radial (i.e. $\ell = 0$) pulsations. Very little theoretical attention had been devoted to these stars.

Why do we think that the dynamics underlying these irregular pulsations could be low dimensional and chaotic? One of the reasons is that decade old numerical hydrodynamical simulations showed low dimensional chaotic behavior and period doubling cascades in models. However, the observational confirmation of low dimensional chaos in these stars, that is the subject of this paper, had to wait a little longer for a number of reasons.

2. The observational data

Nonlinear techniques are new in variable star Astronomy, and it is therefore hard to find data sets that are suitable for a modern nonlinear time series analysis and the search for low dimensional chaos. Partially because of the long 'periods'
involved, typically tens to hundreds of days, rarely are enough pulsation cycles covered, or they are covered incompletely. Generally, professional astronomers have not seen any merit in dedicating the necessary telescope time to what they consider the gathering of ‘useless’ erratic data. In contrast, thanks to the observations of numerous amateur astronomers, the American Association of Variable Star Observers (AAVSO) has recently made available long and relatively well sampled light-curve data of two stars, R Scuti and AC Herculis. (In Astronomy variable stars were named after the constellation in which they appear preceded by one or two capital letters, until the letters were exhausted). Both objects are characterized by irregular cycles of $\approx 75\,\text{d}$ and $35\,\text{d}$, respectively, with alternating shallow and deep minima. The details of the analyses of these light-curves have been published elsewhere. Here we present a summary review.

![Fig. 1 Typical segments of observational lightcurve data (dots) and fits (lines), magnitude versus time [years]; R Scuti (top) and AC Herculis (bottom).](image)

Fig. 1 displays typical subsections of the two data sets (points) with the solid line showing our smoothed and interpolated fit (see below). These data have been taken visually by many amateur observers worldwide at essentially random times. Therefore there is considerable scatter in the data and there are some observational gaps. We have plotted the lightcurves in magnitudes (for historical reasons defined as $-2.5 \log \times$ luminosity) rather than the more ‘physical’ luminosity. For both data sets the error in the magnitudes is found to have a normal distribution, independent of the magnitude (which is not true for the luminosity itself). The reason is to be found in the visual nature of the observations and the logarithmic response of the human eye. We thus work with the magnitude. Although the individual observations have a large error associated with them (0.20 and 0.15 mag, in R Sc and AC Her, resp.) it is
possible to extract a reasonably accurate light-curve by averaging and interpolating. The individual errors are larger for R Sct, but there are about four times as many data points and the pulsation amplitude is about a factor of 4 larger, so that the error on the final interpolated curve is much smaller for R Sct than for AC Her.

![Fig. 2 Observational lightcurves of R Scuti and best synthetic signal (4D).](image1)

![Fig. 3 Observational lightcurves of AC Herculis (top) and best synthetic signal (5D).](image2)

The preparation of the data is a delicate process. We want to eliminate as much observational and other extrinsic noise as possible. However there is additional 'intrinsic noise' associated with turbulence and convection in the star that, strictly speaking, is part of the dynamics. It is well known that well developed turbulent motions constitute high-dimensional behavior, and a priori it may be objected this
would preclude us from uncovering any low dimensional dynamics. The success of our nonlinear analysis provides an a posteriori indication that the pulsation can be separated into a large amplitude ‘true’ pulsation, and a low amplitude pulsational jitter. Our pre-smoothing eliminates this high-dimensional jitter together with the true noise, leaving a signal that represents the ‘true’ low dimensional dynamics.

To be specific, the bare data were first averaged in 5.0 day bins for R Sct and in 2.5 day bins for AC Her, followed by a cubic spline smoothing, where the smoothing parameter $\sigma$ was a free parameter in the flow reconstruction below. The result is then a lightcurve data set that is sampled at equal time-intervals.

Figs. 2 and 3 in the top rows show the smoothed and interpolated observational data of R Sct and AC Her, respectively. In Fig. 4 we display the (normalized) amplitude Fourier spectra of the two observed lightcurves. The FS of R Sct has much broader features than the more regular AC Her as is to be expected from the appearance of the lightcurves.

![Fig. 4 Amplitude Fourier spectrum of R Sct and AC Her lightcurve data.](image)

It is also of interest to look at the Broomhead–King (BK) projections of a signal. These projections onto the eigenvectors of the correlation matrix are optimal for spreading out the features of the signal. The lowest BK projections for R Sct and AC Her are thus shown in columns 1 and 5 in Fig. 5.

Before proceeding with a nonlinear analysis aimed at discovering and describing low dimensional chaos we briefly mention further tests to ascertain that the signals cannot be the result of a multi-periodic (n-torus) dynamics.

For R Sct we have performed the following tests. Taking the frequencies of the highest 35 peaks in the FS (Fig. 2) we have made a multi-periodic Fourier fit for the 35 amplitudes and phases. The fit is quite good over the temporal range of the data set, but when it is extrapolated in time the signal totally fails to resemble the R Sct lightcurve. Similarly, a linear autoregressive process with white noise (AR or ARMA) bears little resemblance to the data. Both tests indicate that the process that
produces the lightcurve is strongly nonlinear.

For AC Her, the situation is not so clear-cut because of the high noise level. We note here prewhitening with the frequencies of the 24 highest FS peaks leaves a noise level that is at least 3–4 times the noise that is expected from the observational error.

Fig. 5 Broomhead-King projections for the R Sct and AC Her data, as well as for the respective best synthetic signals; from top to bottom, coordinates ($\xi_2,\xi_1$),($\xi_3,\xi_1$) and ($\xi_3,\xi_2$).

There are theoretical reasons for not thinking that so many frequencies could be observed. First these stars are believed to be radial pulsators, and the radial modal spectrum does not have a structure similar to that of the highest peaks of the FS. Further, even if one insisted on nonradial modes, the visual observations are necessarily whole disk observations which cannot possibly resolve nonradial modes with $\ell > 2$. Again the required modal spectrum would not be compatible with stellar structure.

The stars cannot be evolving (time-dependent) multiperiodic systems either. Otherwise there should be a correlation between the multitude of smaller peaks in the Fourier spectrum when subsequent sections of the data are Fourier analyzed, which is not the case.

We conclude that neither the R Sct nor the AC Her lightcurve can reasonably be explained as being periodic or multi-periodic.

3. **The global flow reconstruction**

We now make the assumption that the observed lightcurve $s_n = \{s(t_n)\}$ has been generated by a low dimensional dynamics in some 'physical' phase-space of a
priori unknown properties, i.e.

\[ \frac{dY}{dt} = G(Y) \quad (1) \]

where \( Y(t) \in \mathbb{R}^d \) is the \( d \) dimensional state vector of the system. We could equivalently assume a stroboscopic description of the dynamics

\[ Y_{n+1} = G(Y_n) \quad (2) \]

where \( Y_n = Y(t_n) \). It is natural to assume that \( s(t_n) \) is a smooth function of the phase-space variable \( Y \).

A standard reconstruction method starts with the construction of the standard \( d_e \)-dim delay vectors

\[ X_n = \{s_n-1, s_n-2, \ldots, s_{nd_e}\} \quad (3) \]

The global flow reconstruction method then assumes that there is a single function (map) \( F \) that describes the dynamics all over,

\[ X_{n+1} = F(X_n) \quad (4) \]

Embedding theorems guarantee that for sufficiently large \( d_e \) there is a diffeomorphism between the physical phase-space and the reconstruction space (the latter is then called an embedding space, \textit{stricto sensu}). Following Giona et al. and Brown we assume that the function \( F(X) \) is of a multivariate polynomial form,

\[ F(X) = \sum_i \alpha_i P_i(X) \quad (5) \]

where the sum runs over all polynomials up to degree \( p \). More specifically we use natural polynomials, i.e. polynomials that are constructed to be orthogonal on the data set. However, instead of constructing these polynomials explicitly by recursion and then determining the coefficients \( \alpha_i \) it is equivalent and much faster to expand the function in terms of all monomials up to the same order \( p \)

\[ F(X) = \sum_i \beta_i M_i(X) \quad (6) \]

and then find the expansion coefficients \( \beta_i \) through a least squares minimization of the variance

\[ S = \sum_n \|X_n - \sum_i \beta_i M_i(X_{n-1})\|^2. \quad (7) \]

For this purpose a SVD (singular value decomposition) approach has been found to be most stable and efficient.

The global polynomial reconstruction method was first tested on the Rössler attractor. Using as input a short time-series of only the first one of the 3 Rössler variables one obtains a very good estimation of the Lyapunov exponents and the
fractal dimension of the attractor. Furthermore it is possible to determine that this scalar signal was generated by a 3-D flow (Rössler’s three first order ODEs), in other words it is possible to determine not only the lowest embedding dimension, but also the dimension of the actual physical phase-space of the dynamics.

Encouraged by these benchmark results we have applied the reconstruction method to the observational data of stellar lightcurves shown in Figs. 2 and 3. Because the observational data sets are so small we have used all the points in constructing the map in Eq. 7. We note in passing that in Astronomy we are not interested in prediction, but rather in uncovering the properties of the dynamics that underlies the pulsation. In Fig. 6 we display the behavior of the error as a function of $d_e$, which therefore is an in-sample error. The vertically stacked points are for values of $p$ ranging from 3 to 6. This suggests strongly that for R Sct the minimum embedding dimension is $d_e = 4$. For AC Her, because of the much larger observational noise, the lowest achievable error level is seen to be much higher. Nevertheless Fig. 6 can be seen to indicate that we may get away with a minimum $d_e$ of 3. The behavior of the error norm with dimension by itself is of course insufficient to establish the minimum embedding dimension.

It is worthwhile noting that the results obtained with the ’false nearest neighbor’ method are consistent with these respective minimum dimensions of 4 and 3.

How do we decide that a map is ’good’? For that we have to compare the ’synthetic’ signals that a map yields to the original data. (By synthetic signal we mean data that have been generated by iteration from some seed values with a map generated according to Eq. 7). Of course, since we have chaotic behavior this comparison has to be done in an average or statistical sense. Unfortunately the observational data set is too small to make statistical comparison tests very meaningful. Instead we rely on three rather ad hoc criteria, viz. (a) the signals must have the same appearance, (b) their Fourier spectra must have the same envelope structure and (c) their lowest BK projections should be similar. We have found these simple criteria to be quite
useful. For example in our flow reconstructions it is quite clear that any 3D maps for R Sct can be rejected for not being able to capture the dynamics of the star; the synthetic lightcurves, while they may be chaotic, they bear essentially no resemblance to the observational data.

In Figs. 2 and 3, bottom rows, we exhibit the synthetic signals we could produce from what we consider the best of these maps for R Sct and AC Her, respectively. In the BK projections of Fig. 6 we display the same two signals, Cols. 2 and 7, respectively. For comparison we also show synthetic signals for R Sct that were obtained with a 5D map (Col. 3). Col. 6 shows a 3D reconstruction of AC Her. For a comparison of the Fourier spectra we refer to the original papers.

For further details concerning the sensitivity to the delay $\Delta$, the maximum polynomial order $p$ and the data preparation can be found in the original papers.

The reconstructions are seen to be very good and robust for R Sct. Col. 4 of Fig. 6 shows that this remains true when we reconstruct the dynamics with a 4D flow instead of the 4D map of Col. 2.

The AC Her data are inferior. Their signal to noise ratio is much lower and as a result the reconstruction is much less robust. It is nevertheless interesting that we are able to reconstruct a map that can produce synthetic signals with similar properties to the observational data.

It is not possible to extract Lyapunov exponents and a fractal dimension directly from the observational data because of their short length. However we can obtain these quantities once we have constructed the map or flow. In Table 1 we show the Lyapunov exponents $\{\lambda_k\}$, ordered from largest to smallest, and of the Lyapunov dimension $d_L$ for some of the reconstructions for both R Sct and AC Her. The latter is defined as

$$d_L = K + \sum_{i=1}^{K} \frac{\lambda_i}{|\lambda_{K+1}|}$$

where $K$ is the largest value such that the sum is positive. The coefficients have been obtained from the maps that have been constructed with the listed parameters $d_e$, $\Delta$ and $p$.

4. Results

We only note the most important points:

(1) While there is some scatter in the exponents and in the associated fractal dimension $d_L$, for all successful reconstructions $d_L < 4$ for R Sct and 3 for AC Her, independently of the embedding dimension. This is an indication that we have achieved an embedding of the dynamics.

(2) The largest of the Lyapunov exponents is always positive. This is a confirmation that the signal is indeed chaotic.

(3) We have not shown the second Lyapunov exponent because it is always close to zero. This result is reassuring, because it indicates that we are actually close to describing a flow for which one of the exponents would be exactly zero (autonomous
Table 1. Lyapunov exponents and Lyapunov dimensions for R Scuti and AC Herculis

| R Scuti       | AC Herculis |
|---------------|-------------|
| \( d_e \) | \( \Delta \) | \( p \) | \( \lambda_1 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( d_L \) | \( d_e \) | \( \Delta \) | \( p \) | \( \lambda_1 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( d_L \) |
| 4 | 4 | 4 | 0.0019 | -0.0016 | -0.0061 | 3.05 | 3 | 5 | 6 | 0.0033 | -0.034 | 2.10 |
| 4 | 5 | 4 | 0.0017 | -0.0014 | -0.0054 | 3.06 | 3 | 10 | 6 | 0.0045 | -0.026 | 2.17 |
| 4 | 6 | 4 | 0.0019 | -0.0009 | -0.0051 | 3.19 | 3 | 13 | 6 | 0.0069 | -0.030 | 2.23 |
| 4 | 7 | 4 | 0.0020 | -0.0011 | -0.0052 | 3.18 | 4 | 5 | 5 | 0.0073 | -0.016 | -0.054 | 2.46 |
| 4 | 8 | 4 | 0.0014 | -0.0010 | -0.0049 | 3.07 | 5 | 6 | 4 | 0.0045 | -0.023 | -0.025 | 2.19 |
| 5 | 7 | 3 | 0.0016 | -0.0005 | -0.0041 | 3.27 | 5 | 7 | 4 | 0.0075 | -0.009 | -0.025 | 2.85 |
| 6 | 8 | 3 | 0.0022 | -0.0003 | -0.0018 | 3.52 | 3 | 13 | 6 | 0.0015 | -0.029 | 2.05 |

(4) For R Sct |\( \lambda_3 \)| < \( \lambda_1 \), whereas the inequality is reversed for AC Her, indicating a larger dissipation in the dynamics of AC Her.

As physicists we would like to infer more than just Lyapunov exponents and fractal dimensions, because as such they are not very useful. Thus it is interesting that from the linearization of the R Sct map about its fixed point we found two spiral roots, one unstable and the second stable with approximately twice the frequency. This led to the interesting physical interpretation that the chaotic dynamics is generated by the nonlinear interaction of two radial modes of pulsation that are in a close resonance condition. Such a physical picture is supported by further theoretical considerations. As for AC Her, it is quite likely that two strongly interacting vibrational modes also underlie this dynamics, but that the stronger dissipation alluded to just above compresses the dynamics down to 3D, in the same way the 2D Hénon map shrinks to a 1D logistic map for small values of \( b \).

5. Conclusions

The flow reconstruction has been applied to real data, viz. the light-curve of the irregular variable stars R Scuti and AC Herculis. Not only are these data contaminated with a good amount of observational and other extrinsic noise, but we have no a priori information about the dynamics that has generated them. The analysis has shown that the dynamics which generates the light-curve of R Scuti has a dimension of four, i.e. the signal can be generated by a flow in 4-D. Furthermore the attractor has a fractal dimension \( \approx 3.1 \). For the star AC Her we find a phase-space dimension of 3 and a fractal dimension \( 2.05 \leq d_L \leq 2.5 \).

There are no observational data sets available on the even less irregular W Vir stars. However, numerical hydrodynamical simulations and their analysis with the global flow reconstruction\(^ {11} \) indicate also a 3-D phase-space and a fractal dimension of \( d_L \approx 2.01 \sim 2.05 \). A systematic trend seems to exist, viz. that as we go from the low luminosity, weakly irregular W Vir stars to the high luminosity, strongly irregular
RV Tau stars, the fractal dimension gradually increases.

The fact that these stars have such a low dimensional dynamics must be considered remarkable. They undergo large luminosity fluctuations, up to a factor of 40 for Sct, with strong shock waves and ionization fronts criss-crossing the stellar envelope. Yet, the overall pulsational behavior is found to be relatively simple.

As a result of this nonlinear analysis the nature of the irregular variability of these types of stars is no longer a mystery, but it has been shown to be simply the manifestation of an underlying low dimensional chaotic dynamics.

Acknowledgements

It is a great pleasure to acknowledge the contributions of my collaborators, Zoltan Kolláth and Thierry Serre. This work has been supported by NSF (AST95–18068).

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