Abstract. We prove the local Kudla–Rapoport conjecture, which is a precise identity between the arithmetic intersection numbers of special cycles on unitary Rapoport–Zink spaces and the derivatives of local representation densities of hermitian forms. As a first application, we prove the global Kudla–Rapoport conjecture, which relates the arithmetic intersection numbers of special cycles on unitary Shimura varieties and the central derivatives of the Fourier coefficients of incoherent Eisenstein series. Combining previous results of Liu and Garcia–Sankaran, we also prove cases of the arithmetic Siegel–Weil formula in any dimension.

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1. Introduction

1.1. Background. The classical Siegel–Weil formula ([Sie51, Wei65]) relates certain Siegel Eisenstein series with the arithmetic of quadratic forms, namely expressing special values of these series as theta functions — generating series of representation numbers of quadratic forms. Kudla ([Kud97b, Kud04]) initiated an influential program to establish the arithmetic Siegel–Weil formula relating certain Siegel Eisenstein series with objects in arithmetic geometry, which among others, aims to express the central derivative of these series as the arithmetic analogue of theta functions — generating series of arithmetic intersection numbers of $n$ special divisors on Shimura varieties associated to $\text{SO}(n - 1, 2)$ or $\text{U}(n - 1, 1)$. These special divisors include Heegner points on modular or Shimura curves appearing in the Gross–Zagier formula ([GZ86, YZZ13]) ($n = 2$), modular correspondence on the product of two modular curves in the Gross–Keating formula ([GK93]) and Hirzebruch–Zagier cycles on Hilbert modular surfaces ([HZ76]) ($n = 3$).

The arithmetic Siegel–Weil formula was established by Kudla, Rapoport and Yang ([KRY99, Kud97b, KR00b, KRY06]) for $n = 1, 2$ (orthogonal case) in great generality. The archimedean component of the formula was also known, due to Liu ([Liu11a] (unitary case), and Garcia–Sankaran [GS19] in full generality (cf. Bruinier–Yang [BY18] for an alternative proof in the orthogonal case). However, the full formula (in particular, the nonarchimedean part) was widely open in higher dimension.

In the works [KR11, KR14] Kudla–Rapoport made the nonarchimedean part of the conjectural formula more precise by defining arithmetic models of the special cycles (for any $n$ in the unitary case), now known as Kudla–Rapoport cycles. They formulated the global Kudla–Rapoport conjecture for the nonsingular part of the formula, and explained how it would follow (at least at an unramified place) from the local Kudla–Rapoport conjecture, relating the derivatives of local representation densities of hermitian forms and arithmetic intersection numbers of Kudla–Rapoport cycles on unitary Rapoport–Zink spaces. They further proved the conjectures in the special case when the arithmetic intersection is non-degenerate (i.e., of the expected dimension 0). Outside the non-degenerate case, the only known result was due to Terstiege [Ter13], who proved the Kudla–Rapoport conjectures for $n = 3$. Analogous results were known in the orthogonal case, see [GK93, KR99, KR00a, BY18] (non-degenerate case) and [Ter11] ($n = 3$).

The main result of this paper settles the local Kudla–Rapoport conjecture for any $n$ in the unitary case. As a first application, we will be able to deduce the global Kudla–Rapoport conjecture, and prove the first cases of the arithmetic Siegel–Weil formula in any higher dimension. In a companion paper [LZ20], we will also use similar methods to prove analogous results in the orthogonal case.

As explained in [Kud97b] and [Liu11a], the arithmetic Siegel–Weil formula (together with the doubling method) has important application to the arithmetic inner product formula, relating the central derivative of the standard $L$-function of cuspidal automorphic representations on orthogonal or unitary groups to the height pairing of certain cycles on Shimura varieties constructed from arithmetic theta liftings. It can be viewed as a higher dimensional generalization of the Gross–Zagier formula, and an arithmetic analogue of the Rallis inner product formula. Further applications to the arithmetic inner product formula are investigated in [LL20]. We also mention that the local
1.2. The local Kudla–Rapoport conjecture. Let $p$ be an odd prime. Let $F_0$ be a finite extension of $\mathbb{Q}_p$ with residue field $k = \mathbb{F}_q$ and a uniformizer $\varpi$. Let $F$ be an unramified quadratic extension of $F_0$. Let $\bar{F}$ be the completion of the maximal unramified extension of $F$. For any integer $n \geq 1$, the unitary Rapoport–Zink space $\mathcal{N} = \mathcal{N}_n$ (§2.1) is the formal scheme over $S = \text{Spf} O_{\bar{F}}$ parameterizing hermitian formal $O_F$-modules of signature $(1, n - 1)$ within the supersingular quasi-isogeny class. Let $E$ and $X$ be the framing hermitian $O_F$-module of signature $(1, 0)$ and $(1, n-1)$ over $\bar{k}$. The space of quasi-homomorphisms $\mathcal{V} = \mathcal{V}_n := \text{Hom}_{O_F}(E, X)$ carries a natural $F/F_0$-hermitian form, which makes $\mathcal{V}$ the unique (up to isomorphism) nondegenerate non-split $F/F_0$-hermitian space of dimension $n$ (§2.2). For any subset $L \subseteq \mathcal{V}$, the local Kudla–Rapoport cycle $Z(L)$ (§2.3) is a closed formal subscheme of $\mathcal{N}$, over which each quasi-homomorphism $x \in L$ deforms to homomorphisms.

Let $L \subseteq \mathcal{V}$ be an $O_F$-lattice (of full rank $n$). We now associate to $L$ two integers: the arithmetic intersection number $\text{Int}(L)$ and the derivative of the local density $\partial \text{Den}(L)$.

Let $x_1, \ldots, x_n$ be an $O_F$-basis of $L$. Define the arithmetic intersection number

$$\text{Int}(L) := \chi(\mathcal{N}, O_{Z(x_1)} \otimes^{L} \cdots \otimes^{L} O_{Z(x_n)}),$$

where $O_{Z(x_i)}$ denotes the structure sheaf of the Kudla–Rapoport divisor $Z(x_i)$, $\otimes^{L}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and $\chi$ denotes the Euler–Poincaré characteristic (§2.4). By [Ter13, Proposition 3.2] (or [How18, Corollary D]), we know that $\text{Int}(L)$ is independent of the choice of the basis $x_1, \ldots, x_n$ and hence is a well-defined invariant of $L$ itself.

For $M$ another hermitian $O_F$-lattices (of arbitrary rank), define $\text{Rep}_{M,L}$ to be the scheme of integral representations of $M$ by $L$, an $O_{F_0}$-scheme such that for any $O_{F_0}$-algebra $R$, $\text{Rep}_{M,L}(R) = \text{Herm}(L \otimes_{O_{F_0}} R, M \otimes_{O_{F_0}} R)$, where $\text{Herm}$ denotes the set of hermitian module homomorphisms. The local density of integral representations of $M$ by $L$ is defined to be

$$\text{Den}(M, L) := \lim_{N \to +\infty} \frac{\# \text{Rep}_{M,L}(O_{F_0}/\varpi^N)}{q^N \dim(\text{Rep}_{M,L})_{F_0}}.$$

Let $(1)^k$ be the self-dual hermitian $O_F$-lattice of rank $k$ with hermitian form given by the identity matrix $1_k$. Then $\text{Den}((1)^k, L)$ is a polynomial in $(-q)^{-k}$ with $\mathbb{Q}$-coefficients. Define the (normalized) local Siegel series of $L$ to be the polynomial $\text{Den}(X, L) \in \mathbb{Z}[X]$ (Theorem 3.5.1) such that

$$\text{Den}((-q)^{-k}, L) = \frac{\text{Den}((1)^{n+k}, L)}{\text{Den}((1)^{n+k}, (1)^n)}.$$

It satisfies a functional equation relating $X \leftrightarrow \frac{1}{X}$,

$$\text{Den}(X, L) = (-X)^{\text{val}(L)} \cdot \text{Den}\left(\frac{1}{X}, L\right).$$
Since $\mathcal{V}$ is nonsplit, we know that $\text{val}(L)$ is odd and so the value $\text{Den}(1, L) = 0$. We thus consider the derivative of the local density

$$\partial \text{Den}(L) := -\frac{d}{dX} \bigg|_{X=1} \text{Den}(X, L).$$

Our main theorem in Part 1 is a proof of the local Kudla–Rapoport conjecture \cite[Conjecture 1.3]{KR11}, which asserts an exact identity between the two integers just defined.

**Theorem 1.2.1** (Theorem 3.4.1 local Kudla–Rapoport conjecture). Let $L \subseteq \mathcal{V}$ be an $O_F$-lattice of full rank $n$. Then

$$\text{Int}(L) = \partial \text{Den}(L).$$

We refer to $\text{Int}(L)$ as the geometric side of the identity (related to the geometry of Rapoport–Zink spaces and Shimura varieties) and $\partial \text{Den}(L)$ the analytic side (related to the derivative of Eisenstein series and $L$-functions).

Our main theorem in Part 2 proves a variant of the local Kudla–Rapoport conjecture in the presence of a minimal nontrivial level structure, given by the stabilizer of an almost self-dual lattice in a nonsplit $F/F_0$-hermitian space. The relevant Rapoport–Zink space on the geometric side is no longer formally smooth. See Theorems 10.3.1 and 10.4.4 for the precise statement.

1.3. The arithmetic Siegel–Weil formula. Next let us describe some global applications of our local theorems. We now switch to global notations. Let $F$ be a CM number field, with $F_0$ its totally real subfield of index 2. Fix a CM type $\Phi \subseteq \text{Hom}(F, \mathbb{Q})$ of $F$. Fix an embedding $\mathbb{Q} \hookrightarrow \mathbb{C}$ and identify the CM type $\Phi$ with the set of archimedean places of $F$, and also with the set of archimedean places of $F_0$. Let $V$ be an $F/F_0$-hermitian space of dimension $n$ and $G = \text{Res}_{F_0/\mathbb{Q}} U(V)$. Assume the signatures of $V$ are $\{(n-1,1)_{\phi_0}, (n,0)_{\phi \in \Phi - \{\phi_0\}}\}$ for some distinguished element $\phi_0 \in \Phi$. Define a torus $Z^\mathbb{Q} = \{z \in \text{Res}_{F/\mathbb{Q}} G_m : Nm_{F/F_0}(z) \in G_m\}$. Associated to $\tilde{G} := Z^\mathbb{Q} \times G$ there is a natural Shimura datum $(\tilde{G}, \{h_{\tilde{G}}\})$ of PEL type (§11.1). Let $K = K_{Z^\mathbb{Q}} \times K_G \subseteq \tilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\text{Sh}_K = \text{Sh}_K(\tilde{G}, \{h_{\tilde{G}}\})$ is of dimension $n-1$ and has a canonical model over its reflex field $E$.

Assume $K_{Z^\mathbb{Q}} \subseteq Z^\mathbb{Q}(\mathbb{A}_f)$ is the unique maximal open compact subgroup, and $K_{G, v} \subseteq U(V)(F_0,v)$ $(v$ a place of $F_0)$ is given by

- the stabilizer of a self-dual or almost self-dual lattice $\Lambda_v \subseteq V_v$ if $v$ is inert in $F$,
- the stabilizer of a self-dual lattice $\Lambda_v \subseteq V_v$ if $v$ is ramified in $F$,
- a principal congruence subgroup if $v$ is split in $F$. 

Then we construct a global regular integral model $\mathcal{M}_K$ of $\text{Sh}_K$ over $O_E$ following \cite[§14.1]{RSZ17b}. When $F_0 = \mathbb{Q}$, we have $E = F$ and the integral model $\mathcal{M}_K$ recovers that in \cite[BHK+17]{BHK} when $K_G$ is the stabilizer of a global self-dual lattice, which is closely related to that in \cite[KR14]{KR14}.

Let $\mathcal{V}$ be the incoherent $\mathbb{A}_F/\mathbb{A}_{F_0}$-hermitian space nearby $V$ such that $\mathcal{V}$ is totally positive definite and $\mathcal{V}_v \cong V_v$ for all finite places $v$. Let $\varphi_K \in \mathcal{S}(\mathcal{V}_f^n)$ be a $K$-invariant (where $K$ acts on $\mathcal{V}_f$ via the second factor $K_G$) factorizable Schwartz function such that $\varphi_{K,v} = 1_{(\Lambda_v)^n}$ at all $v$ inert in $F$. Let $T \in \text{Herm}_n(F)$ be a nonsingular hermitian matrix of size $n$. Associated to $(T, \varphi_K)$ we construct arithmetic cycles $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}_K$ (§14.3) generalizing the Kudla–Rapoport cycles.
Z(T) in [KRI14]. Analogous to the local situation (1.2.0.1), we may define its local arithmetic intersection numbers $\text{Int}_{T,v}(\varphi_K)$ at finite places $v$ (§13.4). Using the star product of Kudla’s Green functions, we also define its local arithmetic intersection number $\text{Int}_{T,v}(y, \varphi_K)$ at infinite places (§15.3), which depends on a parameter $y \in \text{Herm}_n(F_\infty) > 0$ where $F_\infty = F \otimes F_0 \mathbb{R}^\Phi \cong \mathbb{C}^\Phi$. Combining all the local arithmetic numbers together, define the global arithmetic intersection number, or the arithmetic degree of the Kudla–Rapoport cycle $Z(T, \varphi_K)$,

$$\hat{\deg}_T(y, \varphi_K) := \sum_{v|\infty} \text{Int}_{T,v}(\varphi_K) + \sum_{v\nmid \infty} \text{Int}_{T,v}(y, \varphi_K).$$

It is closely related to the usual arithmetic degree on the Gillet–Soulé arithmetic Chow group $\hat{\text{Ch}}_n^G(M_K)$ (§15.4).

On the other hand, associated to $\varphi = \varphi_K \otimes \varphi_\infty \in \mathcal{S}(\mathbb{V}_n^m)$, where $\varphi_\infty$ is the Gaussian function, there is a classical incoherent Eisenstein series $E(z, s, \varphi_K)$ (§12.4) on the hermitian upper half space

$$\mathbb{H}_n = \{ z = x + iy : x \in \text{Herm}_n(F_\infty), y \in \text{Herm}_n(F_\infty) > 0 \}.$$

This is essentially the Siegel Eisenstein series associated to a standard Siegel–Weil section of the degenerate principal series (§12.1). The Eisenstein series here has a meromorphic continuation and a functional equation relating $s \leftrightarrow -s$. The central value $E(z, 0, \varphi_K) = 0$ by the incoherence. We thus consider its central derivative

$$\partial \text{Eis}(z, \varphi_K) := \frac{d}{ds} \bigg|_{s=0} E(z, s, \varphi_K).$$

It has a decomposition into the central derivative of the Fourier coefficients

$$\partial \text{Eis}(z, \varphi_K) = \sum_{T \in \text{Herm}_n(F)} \partial \text{Eis}_T(z, \varphi_K).$$

Now we can state our first application to the global Kudla–Rapoport conjecture [KRI14, Conjecture 11.10], which asserts an identity between the arithmetic degree of Kudla–Rapoport cycles and the derivative of nonsingular Fourier coefficients of the incoherent Eisenstein series.

\textbf{Theorem 1.3.1} (Theorem 14.5.1, global Kudla–Rapoport conjecture). Let $\text{Diff}(T, V)$ be the set of places $v$ such that $V_v$ does not represent $T$. Let $T \in \text{Herm}_n(F)$ be nonsingular such that $\text{Diff}(T, V) = \{ v \}$ where $v$ is inert in $F$ and not above 2. Then

$$\hat{\deg}_T(y, \varphi_K) q^T = c_K \cdot \partial \text{Eis}_T(z, \varphi_K),$$

where $q^T := e^{2\pi i \text{tr}(Tz)} = \prod_{\phi \in \Phi} e^{2\pi i \text{tr}(Tz_\phi)}$, $c_K = \frac{(-1)^n}{\text{vol}(K)}$ is a nonzero constant independent of $T$ and $\varphi_K$, and $\text{vol}(K)$ is the volume of $K$ under a suitable Haar measure on $\tilde{G}(\mathbb{A}_f)$.

We form the generating series of arithmetic degrees

$$\hat{\deg}(z, \varphi_K) := \sum_{T \in \text{Herm}_n(F) \atop \det T \neq 0} \hat{\deg}_T(y, \varphi_K) q^T.$$

Now we can state our second application to the arithmetic Siegel–Weil formula, which relates this generating series to the central derivative of the incoherent Eisenstein series.
Theorem 1.3.2 (Theorem \[15.5.1\] arithmetic Siegel–Weil formula). Assume that \( F/F_0 \) is unramified at all finite places and split at all places above 2. Further assume that \( \varphi_K \) is nonsingular (\[12.3\]) at two places split in \( F \). Then

\[
\hat{\deg}(z, \varphi_K) = c_K \cdot \partial \text{Eis}(z, \varphi_K).
\]

In particular, \( \hat{\deg}(z, \varphi_K) \) is a nonholomorphic hermitian modular form of genus \( n \).

Remark 1.3.3. The unramifiedness assumption on \( F/F_0 \) forces \( F_0 \neq \mathbb{Q} \). To treat the general case, one needs to formulate and prove an analogue of Theorem \[1.2.1\] when the local extension \( F/F_0 \) is ramified. We remark that at a ramified place, in addition to the nonsingularity assumption on \( \varphi_K \), we may also consider the case of exotic good reduction with level associated to an (almost) \( \varpi \)-modular lattice. In a future work we hope to extend our methods to cover these cases, which in particular requires an extension of the local density formula of Cho–Yamauchi \[CY18\] to the ramified case.

Remark 1.3.4. The nonsingularity assumption on \( \varphi_K \) allows us to kill all the singular terms on the analytic side. Such \( \varphi_K \) exists for a suitable choice of \( K \) since we allow arbitrary Drinfeld levels at split places.

1.4. Strategy of the proof of the main Theorem \[1.2.1\]. The previously known special cases of the local Kudla–Rapoport conjecture (\[KR11\], \[Ter13\]) are proved via explicit computation of both the geometric and analytic sides. Explicit computation seems infeasible for the general case. Our proof instead proceeds via induction on \( n \) using the uncertainty principle.

More precisely, for a fixed \( O_F \)-lattice \( L^\flat \subseteq V = V_n \) of rank \( n-1 \) (we assume \( L^\flat_F \) is non-degenerate throughout the paper), consider functions on \( x \in V \setminus L^\flat_F \),

\[
\text{Int}_{L^\flat}(x) := \text{Int}(L^\flat + \langle x \rangle), \quad \partial \text{Den}_{L^\flat}(x) := \partial \text{Den}(L^\flat + \langle x \rangle).
\]

Then it remains to show the equality of the two functions \( \text{Int}_{L^\flat} = \partial \text{Den}_{L^\flat} \). Both functions vanish when \( x \) is non-integral, i.e., \( \text{val}(x) < 0 \). Here \( \text{val}(x) \) denotes the valuation of the norm of \( x \). By utilizing the inductive structure of the Rapoport–Zink spaces and local densities, it is not hard to see that if \( x \perp L^\flat \) with \( \text{val}(x) = 0 \), then

\[
\text{Int}_{L^\flat}(x) = \text{Int}(L^\flat), \quad \partial \text{Den}_{L^\flat}(x) = \partial \text{Den}(L^\flat)
\]

for the lattice \( L^\flat \subseteq V_{n-1} \cong \langle x \rangle_{\overline{F}} \) of full rank \( n-1 \). By induction on \( n \), we have \( \text{Int}(L^\flat) = \partial \text{Den}(L^\flat) \), and thus the difference function \( \phi = \text{Int}_{L^\flat} - \partial \text{Den}_{L^\flat} \) vanishes on \( \{ x \in V : x \perp L^\flat, \text{val}(x) \leq 0 \} \). We would like to deduce that \( \phi \) indeed vanishes identically.

The uncertainty principle (Proposition \[8.1.6\]), which is a simple consequence of the Schrödinger model of the local Weil representation of \( \text{SL}_2 \), asserts that if \( \phi \in C_c^\infty(V) \) satisfies that both \( \phi \) and its Fourier transform \( \hat{\phi} \) vanish on \( \{ x \in V : \text{val}(x) \leq 0 \} \), the \( \phi = 0 \). In other words, \( \phi, \hat{\phi} \) cannot simultaneously have “small support” unless \( \phi = 0 \). We can then finish the proof by applying the uncertainty principle to \( \phi = \text{Int}_{L^\flat} - \partial \text{Den}_{L^\flat} \), if we can show that both \( \text{Int}_{L^\flat} \) and \( \partial \text{Den}_{L^\flat} \) are invariant under the Fourier transform (up to the Weil constant \( \gamma_V = -1 \)). However, both functions have singularities along the hyperplane \( L^\flat_F \subseteq V \), which cause trouble in computing their Fourier transforms or even in showing that \( \phi \in C_c^\infty(V) \).
To overcome this difficulty, we isolate the singularities by decomposing

\[
\text{Int}_{L^\flat} = \text{Int}_{L^\flat, H} + \text{Int}_{L^\flat, V}, \quad \partial \text{Den}_{L^\flat} = \partial \text{Den}_{L^\flat, H} + \partial \text{Den}_{L^\flat, V}
\]

into “horizontal” and “vertical” parts. Here on the geometric side \(\text{Int}_{L^\flat, H}\) is the contribution from the horizontal part of the Kudla–Rapoport cycles, which we determine explicitly in terms of quasi-canonical lifting cycles (Theorem 4.2.1). On the analytic side we define \(\partial \text{Den}_{L^\flat, H}\) to match with \(\text{Int}_{L^\flat, H}\). We show the horizontal parts have logarithmic singularity along \(L^\flat_F\), and vertical parts are indeed in \(C^\infty_c(V)\) (Corollary 6.2.2 Proposition 7.3.4). We can then finish the proof if we can determine the Fourier transforms as

(1.4.0.1) \[
\widehat{\text{Int}}_{L^\flat, V} = -\text{Int}_{L^\flat, V}, \quad \widehat{\partial \text{Den}}_{L^\flat, V} = -\partial \text{Den}_{L^\flat, V}.
\]

On the geometric side we show (1.4.0.1) (Corollary 6.3.3) by reducing to the case of intersection with Deligne–Lusztig curves. This reduction requires the Bruhat–Tits stratification of \(N^{\text{red}}\) into certain Deligne–Lusztig varieties (§2.7 due to Vollaard–Wedhorn [VW11]) and the Tate conjecture for these Deligne–Lusztig varieties (Theorem 5.2.2 which we reduce to a cohomological computation of Lusztig [Lus76]).

On the analytic side we are only able to show (1.4.0.1) (Theorem 7.4.1) directly when \(x \perp L^\flat\) and \(\text{val}(x) < 0\). The key ingredient is a local density formula (Theorem 3.5.1) due to Cho–Yamauchi [CY18] together with the functional equation (1.2.0.2). We then deduce the general case by performing another induction on \(\text{val}(L^\flat)\) (§8.2).

We remark the extra symmetry (1.4.0.1) under the Fourier transform can be thought of as a local modularity, in analogy with the global modularity of arithmetic generating series (such as in [BHK+17]) encoding an extra global \(\text{SL}_2\)-symmetry. The latter global modularity plays a crucial role in the second author’s recent proof [Zha19] of the arithmetic fundamental lemma. In contrast to [Zha19], our proof of the local Kudla–Rapoport conjecture does not involve global arguments, thanks to a more precise understanding of the horizontal part of Kudla–Rapoport cycles. In other similar (non-arithmetic) situations, induction arguments involving Fourier transforms and the uncertainty principle are not unfamiliar: here we only mention the second author’s proof [Zha14] of the Jacquet–Rallis smooth transfer conjecture, and more recently Beuzart-Plessis’ new proof [BP19] of the Jacquet–Rallis fundamental lemma.

1.5. The structure of the paper. In Part 1 we review necessary background on the local Kudla–Rapoport conjecture and prove the main Theorem 1.2.1. In Part 2 we prove a variant of the local Kudla–Rapoport conjecture in the almost self-dual case (Theorems 10.3.1, 10.4.4), by relating both the geometric and analytic sides in the almost self-dual to the self-dual case (but in one dimension higher). In Part 3 we review semi-global and global integral models of Shimura varieties and Kudla–Rapoport cycles, and incoherent Eisenstein series. We then apply the local results in Parts 1 and 2 to prove the local arithmetic Siegel–Weil formula (Theorem 13.5.1), the global Kudla–Rapoport conjecture (Theorem 14.5.1), and cases of the arithmetic Siegel–Weil formula (Theorem 15.5.1).
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1.7. Notation and convention. Let \( p \) be a prime. In the local parts of the paper (Part 1 and 2), we let \( F_0 \) be a non-archimedean local field of characteristic \( p \), with ring of integers \( \mathcal{O}_{F_0} \), residue field \( k = \mathbb{F}_q \) of size \( q \), and uniformizer \( \varpi \). Unless otherwise specified, we let \( F \) be a quadratic extension of \( F_0 \), with ring of integers \( \mathcal{O}_F \) and residue field \( k_F \). Let \( \sigma \) be the nontrivial automorphism of \( F/F_0 \). Let \( \hat{F} \) be the completion of the maximal unramified extension of \( F \), and \( \mathcal{O}_{\hat{F}} \) its ring of integers.

Unless otherwise specified, we assume that \( F/F_0 \) is unramified (with an exception of §8.1.5 on the uncertainty principle). We further assume that \( F_0 \) has characteristic 0 and residue characteristic \( p > 2 \) (with exceptions of §3, §7, §8.1.5, §9 which concern only the analytic side).

For \( \mathbb{V} \) an \( F/F_0 \)-hermitian space with hermitian form \( (\ , \ ) \), we write \( \text{val}(x) := \text{val}((x, x)) \) for any \( x \in \mathbb{V} \), where \( \text{val} \) is the valuation on \( F_0 \).

For \( \Lambda \) an \( \mathcal{O}_F \)-lattice in \( \mathbb{V} \), we denote by \( \Lambda^\vee \) its dual lattice under \( (\ , \ ) \). We say that \( \Lambda \) is integral if \( \Lambda \subseteq \Lambda^\vee \). If \( \Lambda \) is integral, define its fundamental invariants to be the unique sequence of integers \( (a_1, \ldots, a_n) \) such that \( 0 \leq a_1 \leq \cdots \leq a_n \), and \( \Lambda^\vee/\Lambda \cong \oplus_{i=1}^n \mathcal{O}_F/\varpi^{a_i} \) as \( \mathcal{O}_F \)-modules; define its valuation to be \( \text{val}(\Lambda) := \sum_{i=1}^n a_i \); and define its type, denoted by \( t(\Lambda) \), to be the number of nonzero terms in its fundamental invariants \( (a_1, \ldots, a_n) \). We say \( \Lambda \) is minuscule or a vertex lattice if it is integral and \( \Lambda^\vee \subseteq \varpi^{-1} \Lambda \). Note that \( \Lambda \) is a vertex lattice of type \( t \) if and only if it has fundamental invariants \( (0^{(n-t)}, 1(t)) \), if and only if \( \Lambda \subseteq \Lambda^\vee \subseteq \varpi^{-1} \Lambda \), where \( \subseteq^t \) indicates that the \( \mathcal{O}_F \)-colength is equal to \( t \). The set of vertex lattices of type \( t \) is denoted by \( \text{Vert}^t \). We say \( \Lambda \) is self-dual if \( \Lambda = \Lambda^\vee \), or equivalently \( \Lambda \) is a vertex lattice of type 0. We say \( \Lambda \) is almost self-dual if \( \Lambda \) is a vertex lattice of type 1.

We denote by \( L^b \subseteq \mathbb{V} \) an \( \mathcal{O}_F \)-lattice of rank \( n - 1 \), and we always assume that \( L^b_F \) is non-degenerate. Here we use the subscript \( (-)_F \) to stand for the base change to \( F \), so \( L^b_F = L^b \otimes_{\mathcal{O}_F} F \).

Fix an unramified additive character \( \psi: F_0 \to \mathbb{C}^\times \). For an integrable function \( f \) on \( \mathbb{V} \), we define its Fourier transform \( \hat{f} \) to be

\[
\hat{f}(x) := \int_{\mathbb{V}} f(y)\psi(\text{tr}_{F/F_0}(x, y))dy, \quad x \in \mathbb{V}.
\]

We normalize the Haar measure on \( \mathbb{V} \) to be self-dual, so \( \hat{f}(x) = f(-x) \). For an \( \mathcal{O}_F \)-lattice \( \Lambda \) in \( \mathbb{V} \), we have (under the assumption that \( F/F_0 \) is unramified)

\[
\hat{1}_\Lambda = \text{vol}(\Lambda)1_{\Lambda^\vee}, \quad \text{and} \quad \text{vol}(\Lambda) = [\Lambda^\vee: \Lambda]^{-1/2} = q^{-\text{val}(\Lambda)}.
\]

Note that \( \text{val}(\Lambda) \) can be defined for any lattice \( \Lambda \) (not necessarily integral) so that the above equality for \( \text{vol}(\Lambda) \) holds.

For a regular formal scheme \( X \), and a closed formal subscheme \( Y \), let \( K^Y_0(X) \) denote the Grothendieck group of finite complexes of coherent locally free \( \mathcal{O}_X \)-modules which are acyclic.
outside $Y$, and $\text{Gr}^i K_0^Y(X)$ is the $i$-th graded piece under the (descending) codimension filtration on $K_0^Y(X)$ (see [Zha19, Appendix B]). For closed formal subschemes $Z_1, \ldots, Z_m$ of $X$, we denote by $Z_1 \cap_X \cdots \cap_X Z_m$ (or simply $Z_1 \cap \cdots \cap Z_m$) the derived tensor product $O_{Z_1} \otimes_{O_X} \cdots \otimes_{O_X} O_{Z_m}$, viewed as an element in $K_0^Z_{Z_1 \cap \cdots \cap Z_m}(X)$.

**Part 1. Local Kudla–Rapoport conjecture: the self-dual case**

2. **Kudla–Rapoport cycles**

2.1. **Rapoport–Zink spaces $\mathcal{N}$**. Let $n \geq 1$ be an integer. A hermitian $O_F$-module of signature $(1, n - 1)$ over a Spf $O_F$-scheme $S$ is a triple $(X, \iota, \lambda)$ where

1. $X$ is a formal $p$-divisible $O_{F_0}$-module over $S$ of relative height $2n$ and dimension $n$,
2. $\iota : O_F \to \text{End}(X)$ is an action of $O_F$ extending the $O_{F_0}$-action and satisfying the Kottwitz condition of signature $(1, n - 1)$: for all $a \in O_F$, the characteristic polynomial of $\iota(a)$ on $\text{Lie} X$ is equal to $(T - a)(T - \sigma(a))^{n-1} \in O_S[T]$,
3. $\lambda : X \xrightarrow{\sim} X^\vee$ is a principal polarization on $X$ whose Rosati involution induces the automorphism $\sigma$ on $O_F$ via $\iota$.

Up to $O_F$-linear quasi-isogeny compatible with polarizations, there is a unique such triple $(X, \iota_X, \lambda_X)$ over $S = \text{Spec} \bar{k}$. Let $\mathcal{N} = \mathcal{N}_n = \mathcal{N}_{\bar{F}/F_0,n}$ be the (relative) unitary Rapoport–Zink space of signature $(1, n - 1)$, parameterizing hermitian $O_F$-modules of signature $(1, n - 1)$ within the supersingular quasi-isogeny class. More precisely, $\mathcal{N}$ is the formal scheme over Spf $O_{\bar{F}}$ which represents the functor sending each $S$ to the set of isomorphism classes of tuples $(X, \iota, \lambda, \rho)$, where the additional entry $\rho$ is a framing $\rho : X \times_S \bar{S} \xrightarrow{\sim} X \times_{\text{Spec} \bar{k}} \bar{S}$ is an $O_F$-linear quasi-isogeny of height 0 such that $\rho^*((\lambda_X)_S) = \lambda_{\bar{S}}$. Here $\bar{S} := S_{\bar{k}}$ is the special fiber.

The Rapoport–Zink space $\mathcal{N} = \mathcal{N}_n$ is formally locally of finite type and formally smooth of relative formal dimension $n - 1$ over Spf $O_{\bar{F}}$ ([RZ96, Mih16 Proposition 1.3]).

2.2. **The hermitian space $\mathcal{V}$**. Let $\mathbb{E}$ be the formal $O_{F_0}$-module of relative height 2 and dimension 1 over $\text{Spec} \bar{k}$. Then $D := \text{End}_{O_{F_0}}(\mathbb{E})$ is the quaternion division algebra over $F_0$. We fix a $F_0$-embedding $\iota_{\mathbb{E}} : F \to D$, which makes $\mathbb{E}$ into a formal $O_F$-module of relative height 1. We fix an $O_{F_0}$-linear principal polarization $\lambda_{\mathbb{E}} : \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee$. Then $(\mathbb{E}, \iota_{\mathbb{E}}, \lambda_{\mathbb{E}})$ is a hermitian $O_F$-module of signature $(1, 0)$. We have $\mathcal{N}_1 \simeq \text{Spf} O_{\bar{F}}$ and there is a unique lifting (the canonical lifting) $\mathcal{E}$ of the formal $O_F$-module $\bar{\mathbb{E}}$ over Spf $O_{\bar{F}}$, equipped with its $O_F$-action $\iota_{\mathbb{E}}$, its framing $\rho_{\mathbb{E}} : \mathcal{E}_{\bar{k}} \xrightarrow{\sim} \mathbb{E}$, and its principal polarization $\lambda_{\mathbb{E}}$ lifting $\lambda_{\mathbb{E}}$ of $\rho_{\mathbb{E}}$. Define $\bar{\mathbb{E}}$ to be the same $O_{F_0}$-module as $\mathbb{E}$ but with $O_F$-action given by $\iota_{\mathbb{E}} := \iota_{\mathbb{E}} \circ \sigma$, and $\lambda_{\mathbb{E}} := \lambda_{\mathbb{E}}$, and similarly define $\bar{\mathbb{E}}$ and $\lambda_{\mathbb{E}}$.

Define $\mathcal{V} = \mathcal{V}_n := \text{Hom}_{O_F}(\bar{\mathbb{E}}, X)$ to be the space of special quasi-homomorphisms ([KR11 Definition 3.1]). Then $\mathcal{V}$ carries a $F/F_0$-hermitian form: for $x, y \in \mathcal{V}$, the pairing $(x, y) \in F$ is given by

$$(\bar{\mathbb{E}} \xrightarrow{x} X \xrightarrow{\lambda_{\mathbb{E}}} X^\vee \xrightarrow{y^\vee} \bar{\mathbb{E}}^\vee \xrightarrow{\lambda_{\mathbb{E}}^{-1}} \bar{\mathbb{E}}) \in \text{End}_{O_F}(\bar{\mathbb{E}}) = \iota_{\mathbb{E}}(F) \simeq F.$$ 

The hermitian space $\mathcal{V}$ is the unique (up to isomorphism) nondegenerate non-split $F/F_0$-hermitian space of dimension $n$. The space of special homomorphisms $\text{Hom}_{O_F}(\bar{\mathbb{E}}, X)$ is an integral hermitian
extends to a homomorphism \( \bar{Kudla–Rapoport cycles} \) 2.3.

and hence acts on the Rapoport–Zink space \( \mathcal{N} \) not locally nilpotent) on each of its irreducible components. Let \( O \) the homomorphism special cycle \( S \) each \( Z \) of the Kudla–Rapoport cycle of all vertical (resp. horizontal) closed formal subschemes of \( Z \) where each \( Z \) is a smooth projective variety of dimension \( d \) Grassmannian \( \text{Gr}(k) \) 2.5.

Generalized Deligne–Lusztig varieties \( x \) of the basis \( \chi \) arithmetic intersection numbers 2.4.

By \([\text{Ter13, Proposition 3.2}]\) (or \([\text{How18, Corollary D}]\)), we know \( \text{Int}(\mathcal{Z}(L)) \) is independent of the choice of the basis \( x_1, \ldots, x_n \) and hence is a well-defined invariant of \( L \) itself, justifying the notation.

2.5. Generalized Deligne–Lusztig varieties \( Y_V \). Let \( V \) be the unique (up to isomorphism) \( k_F/k \)-hermitian space of odd dimension \( 2d + 1 \). Define \( Y_V \) to be the closed \( k_F \)-subvariety of the Grassmannian \( \text{Gr}_d(V) \) parameterizing subspaces \( U \subseteq V \) of dimension \( d \) such that \( U \subseteq \sigma(U)^\perp \). It is a smooth projective variety of dimension \( d \), and has a locally closed stratification

\[
Y_V = \bigcup_{i=0}^{d} X_{P_i}(w_i),
\]

where each \( X_{P_i}(w_i) \) is a generalized Deligne–Lusztig variety of dimension \( i \) associated to a certain parabolic subgroup \( P_i \subseteq \text{U}(V) \) \([\text{Vol10, Theorem 2.15}]\). The open stratum \( Y^0_V := X_{P_d}(w_d) \) is a classical Deligne–Lusztig variety associated to a Borel subgroup \( P_d \subseteq \text{U}(V) \) and a Coxeter element \( w_d \). Each of the other strata \( X_{P_i}(w_i) \) is also isomorphic to a parabolic induction of a classical Deligne–Lusztig variety of Coxeter type for a Levi subgroup of \( \text{U}(V) \) \([\text{HLZ19, Proposition 2.5.1}]\).
2.6. Minuscule Kudla–Rapoport cycles \( \mathcal{V}(\Lambda) \). Let \( \Lambda \subseteq \mathbb{V} \) be a vertex lattice. Then \( V_\Lambda := \Lambda^\vee / \Lambda \) is a \( k_F \)-vector space of of dimension \( t(\Lambda) \), equipped with a nondegenerate \( k_F / k \)-hermitian form induced from \( \mathbb{V} \). Since \( \mathbb{V} \) is a non-split hermitian space, the type \( t(\Lambda) \) is odd. Thus we have the associated generalized Deligne–Lusztig variety \( Y_{V_\Lambda} \) of dimension \((t(\Lambda) - 1)/2\). The reduced subscheme of the minuscule Kudla–Rapoport cycle \( \mathcal{V}(\Lambda) := \mathcal{Z}(\Lambda)^{\text{red}} \) is isomorphic to \( Y_{V_{\Lambda, k}} \). In fact \( \mathcal{Z}(\Lambda) \) itself is already reduced ([LZ17 Theorem B]), so \( \mathcal{V}(\Lambda) = \mathcal{Z}(\Lambda) \).

2.7. The Bruhat–Tits stratification on \( \mathcal{N}^{\text{red}} \). The reduced subscheme of \( \mathcal{N} \) satisfies \( \mathcal{N}^{\text{red}} = \bigcup_{\Lambda} \mathcal{V}(\Lambda) \), where \( \Lambda \) runs over all vertex lattices \( \Lambda \subseteq \mathbb{V} \). For two vertex lattices \( \Lambda, \Lambda' \), we have \( \mathcal{V}(\Lambda) \subseteq \mathcal{V}(\Lambda') \) if and only if \( \Lambda \supseteq \Lambda' \); and \( \mathcal{V}(\Lambda) \cap \mathcal{V}(\Lambda') \) is nonempty if and only if \( \Lambda + \Lambda' \) is also a vertex lattice, in which case it is equal to \( \mathcal{V}(\Lambda + \Lambda') \). In this way we obtain a Bruhat–Tits stratification of \( \mathcal{N}^{\text{red}} \) by locally closed subvarieties ([VW11 Theorem B]),

\[
\mathcal{N}^{\text{red}} = \bigcup_{\Lambda} \mathcal{V}(\Lambda)^{\circ}, \quad \mathcal{V}(\Lambda)^{\circ} := \mathcal{V}(\Lambda) - \bigcup_{\Lambda \subseteq \Lambda'} \mathcal{V}(\Lambda').
\]

Each Bruhat–Tits stratum \( \mathcal{V}(\Lambda)^{\circ} \cong Y_{V_{\Lambda, k}}^{\circ} \) is a classical Deligne–Lusztig of Coxeter type associated to \( U(V_\Lambda) \), which has dimension \((t(\Lambda) - 1)/2\). It follows that the irreducible components of \( \mathcal{N}^{\text{red}} \) are exactly the projective varieties \( \mathcal{V}(\Lambda) \), where \( \Lambda \) runs over all vertex lattices of maximal type ([VW11 Corollary C]).

By [KR11 Proposition 4.1], the reduced subscheme \( \mathcal{Z}(L)^{\text{red}} \) of a Kudla–Rapoport cycle \( \mathcal{Z}(L) \) is a union of Bruhat–Tits strata,

\[(2.7.0.3) \quad \mathcal{Z}(L)^{\text{red}} = \bigcup_{L \subseteq \Lambda} \mathcal{V}(\Lambda). \]

2.8. Finiteness of \( \text{Int}(L) \). The following result should be well-known to the experts.

**Lemma 2.8.1.** Let \( L \subseteq \mathbb{V} \) be an \( O_F \)-lattice of rank \( n \). Then the formal scheme \( \mathcal{Z}(L) \) is a proper scheme over \( \text{Spf} \, \hat{O}_F \). In particular, \( \text{Int}(L) \) is finite.

**Proof.** The vertical part \( \mathcal{Z}(L)_\mathbb{V} \) is a scheme by Lemma [5.1.1] below. We show that the horizontal part \( \mathcal{Z}(L)_\mathbb{F} \) is empty. If not, there exists \( z \in \mathcal{Z}(L)(O_K) \) for some finite extension \( K \) of \( \hat{F} \). Let \( \mathcal{X} \) be the corresponding \( O_F \)-hermitian module of signature \((1, n - 1)\) over \( O_K \). Since \( L \) has rank \( n \), we know that \( \mathcal{X} \) admits \( n \) linearly independent special homomorphisms \( \tilde{x}_i : \tilde{E} \to \mathcal{X} \), which gives rise to an \( O_F \)-linear isogeny

\[(\tilde{x}_1, \ldots, \tilde{x}_n) : \tilde{E}^n \to \mathcal{X}. \]

It then follows that the \( O_F \)-action on \( \mathcal{X} \) satisfies the Kottwitz signature condition \((0, n)\) rather than \((1, n - 1)\) in characteristic 0, a contradiction. Thus \( \mathcal{Z}(L)_\mathbb{F} \) is empty, and so \( \mathcal{Z}(L) \) is a scheme.

Since \( L \) has rank \( n \), the number of vertex lattices \( \Lambda \subseteq \mathbb{V} \) such that \( L \subseteq \Lambda \) is finite. By \((2.7.0.3)\), we know that \( \mathcal{Z}(L)^{\text{red}} \) is contained in finitely many irreducible components of \( \mathcal{N}^{\text{red}} \). Since the scheme \( \mathcal{Z}(L) \) is a closed formal subscheme of \( \mathcal{N}_n \) and each irreducible component of \( \mathcal{N}^{\text{red}} \) is proper over \( \text{Spec} \, \hat{k} \), it follows that the scheme \( \mathcal{Z}(L) \) is proper over \( \text{Spf} \, \hat{O}_F \). \( \square \)

\(^1\)Notice that \( \mathcal{V}(\Lambda) \) in [VW11] and [KR11] is the same as our \( \mathcal{V}(\Lambda^\vee) \).
2.9. A cancellation law for $\text{Int}(L)$. Let $M \subset \mathbb{V}_n$ be a self-dual lattice of rank $r$. We have a natural embedding

$$(2.9.0.1) \quad \delta_M: \mathcal{N}_{n-r} \rightarrow \mathbb{N}_n,$$

which identifies $\mathcal{N}_{n-r}$ with the special cycle $Z(M)$. Let $\mathbb{V}_n = M_F \oplus \mathbb{V}_{n-r}$ be the induced orthogonal decomposition. For $u \in \mathbb{V}_n$, denote by $u^\flat$ the projection to $\mathbb{V}_{n-r}$. If $u^\flat \neq 0$, then the special divisor $Z(u)$ intersects transversely with $\mathcal{N}_{n-r}$ and its pull-back to $\mathcal{N}_{n-r}$ is the special divisor $Z(u^\flat)$. For later reference, we write this fact as follows:

$$(2.9.0.2) \quad \mathcal{N}_{n-r} \cap Z(u) = Z(u^\flat).$$

**Lemma 2.9.1.** Let $M \subset \mathbb{V}_n$ be a self-dual lattice of rank $r$ and $L^\flat$ an integral lattice in $\mathbb{V}_{n-r}$. Then

$$\text{Int}(L^\flat \oplus M) = \text{Int}(L^\flat).$$

**Proof.** This follows from the equation (2.9.0.2) and the definition of $\text{Int}$ by (2.4.0.2). $\Box$

3. Local densities

In this section (except §3.4) we allow $F_0$ to be a non-archimedean local field of characteristic not equal to 2 (but possibly with residue characteristic 2), and $F$ an unramified quadratic extension.

3.1. Local densities for hermitian lattices. Let $L, M$ be two hermitian $O_F$-lattices. Let $\text{Rep}_{M,L}$ be the scheme of integral representations of $M$ by $L$, an $O_{F_0}$-scheme such that for any $O_{F_0}$-algebra $R$,

$$(3.1.0.1) \quad \text{Rep}_{M,L}(R) = \text{Herm}(L \otimes_{O_{F_0} R} M \otimes_{O_{F_0} R}),$$

where Herm denotes the set of hermitian module homomorphisms. The local density of integral representations of $M$ by $L$ is defined to be

$$\text{Den}(M, L) := \lim_{N \to +\infty} \frac{\# \text{Rep}_{M,L}(O_{F_0}/\mathcal{O}^N)}{q^{N \cdot \dim(\text{Rep}_{M,L})}}.$$

Note that if $L, M$ have rank $n, m$ respectively and the generic fiber $(\text{Rep}_{M,L})_{F_0} \neq \emptyset$, then $n \leq m$ and

$$(3.1.0.2) \quad \dim(\text{Rep}_{M,L})_{F_0} = \dim U_m - \dim U_{m-n} = n \cdot (2m - n).$$

3.2. Local Siegel series for hermitian lattices. Let $k \geq 0$ be an integer. Let $\langle 1 \rangle^k$ be the self-dual hermitian $O_F$-lattice of rank $k$ with hermitian form given the identity matrix $1_k$. Let $L$ be a hermitian $O_F$-lattice of rank $n$. By [Hir98, Theorem II], $\text{Den}(\langle 1 \rangle^{n+k}, L)$ is a polynomial in $(-q)^{-k}$ with $\mathbb{Q}$-coefficients (zero if $L$ is not integral). A special case (see [KR11, p.677]) is

$$(3.2.0.3) \quad \text{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^n) = \prod_{i=1}^{n} (1 - (-q)^{-i}X) \bigg|_{X = (-q)^{-k}}.$$
Define the (normalized) **local Siegel series** of \( L \) to be the polynomial \( \text{Den}(X, L) \in \mathbb{Z}[X] \) (Theorem 3.5.1) such that

\[
\text{Den}((-q)^{-k}, L) = \frac{\text{Den}((1)^{n+k}, L)}{\text{Den}((1)^{n+k}, (1)^n)}.
\]

The local Siegel series satisfies a functional equation ([Hir12, Theorem 5.3])

(3.2.0.4) \[
\text{Den}(X, L) = (-X)^{\text{val}(L)} \cdot \text{Den}\left(\frac{1}{X}, L\right).
\]

**Definition 3.2.1.** Define the central value of the local density to be

\[
\text{Den}(L) := \text{Den}(1, L).
\]

In particular, if \( \text{val}(L) \) is odd, then \( \text{Den}(L) = 0 \). In this case, define the **central derivative of the local density** or derived local density by

\[
\partial \text{Den}(L) := -\left. \frac{d}{dX} \right|_{X=1} \text{Den}(X, L).
\]

Notice that by definition \( \text{Den}(M, L) \) only depends on the isometry classes of \( M \) and \( L \), and hence only depends on the fundamental invariants of \( M \) and \( L \). In particular, \( \text{Den}(X, L) \) and \( \partial \text{Den}(L) \) only depends on the fundamental invariants of \( L \).

### 3.3. Relation with local Whittaker functions.

Let \( \Lambda = \langle 1 \rangle^n \) be an self-dual hermitian \( O_F \)-lattice. Let \( L \) be a hermitian \( O_F \)-lattice of rank \( n \). Let \( T = ((x_i, x_j))_{1 \leq i, j \leq n} \) be the fundamental matrix of an \( O_F \)-basis \( \{x_1, \ldots, x_n\} \) of \( L \), an \( n \times n \) hermitian matrix over \( F \). Associated the standard Siegel–Weil section of the characteristic function \( \varphi_0 = 1_{\Lambda^n} \) and the unramified additive character \( \psi : F_0 \to \mathbb{C}^\times \), there is a local (generalized) Whittaker function \( W_T(g, s, \varphi_0) \) (see §12.2 for the precise definition). By [KR14, Proposition 10.1], when \( g = 1 \), it satisfies the interpolation formula for integers \( s = k \geq 0 \) (notice \( \gamma_p(V) = 1 \) in the notation there),

\[
W_T(1, k, \varphi_0) = \text{Den}(\langle 1 \rangle^{n+2k}, L).
\]

So its value at \( s = 0 \) is

\[
W_T(1, 0, \varphi_0) = \text{Den}(\langle 1 \rangle^n, L) = \text{Den}(L) \cdot \text{Den}(\langle 1 \rangle^n, \langle 1 \rangle^n),
\]

and its derivative at \( s = 0 \) is\(^2\)

\[
W'_T(1, 0, \varphi_0) = \partial \text{Den}(L) \cdot \text{Den}(\langle 1 \rangle^n, \langle 1 \rangle^n) \cdot \log q^2.
\]

Plugging in (3.2.0.3), we obtain

(3.3.0.1) \[
W_T(1, 0, \varphi_0) = \text{Den}(L) \cdot \prod_{i=1}^{n} (1 - (-q)^{-i}),
\]

(3.3.0.2) \[
W'_T(1, 0, \varphi_0) = \partial \text{Den}(L) \cdot \prod_{i=1}^{n} (1 - (-q)^{-i}) \cdot \log q^2.
\]

\(^2\)In [KR14, Proposition 9.3], the factor \( \log p \) should be \( \log p^2 \).
3.4. The local Kudla–Rapoport conjecture. Now we can state the main theorem of this article, which proves the Kudla–Rapoport conjecture on the identity between arithmetic intersection numbers of Kudla–Rapoport cycles and central derivatives of local densities. Recall that \( \mathcal{V} = \mathcal{V}_n \) is the hermitian space defined in \( \S 2.2 \).

**Theorem 3.4.1** (local Kudla–Rapoport conjecture). Let \( L \subseteq \mathcal{V} \) be an \( \mathcal{O}_F \)-lattice of full rank \( n \). Then

\[
\text{Int}(L) = \partial \text{Den}(L).
\]

This will be proved in \( \S 8.2 \).

**Remark 3.4.2.** In the notation of \( \S 3.3 \) it follows immediately from Theorem 3.4.1 and (3.3.0.2) that

\[
\text{Int}(L) = W_T'(1, 0; \varphi_0) \cdot \log q^2 \cdot \prod_{i=1}^n (1 - (-q)^{-i})^{-1}.
\]

3.5. Formulas in terms of weighted lattice counting: Theorem of Cho–Yamauchi. Define weight factors

\[
m(a; X) := \prod_{i=0}^{a-1} (1 - (-q)^iX), \quad m(a) := \left. \frac{d}{dX} \right|_{X=1} m(a; X) = \prod_{i=1}^{a-1} (1 - (-q)^i),
\]

where by convention \( m(0; X) = 1 \) and \( m(0) = 0, m(1) = 1 \). Then we have the following explicit formula for the local Siegel series.

**Theorem 3.5.1** (Cho–Yamauchi). The following identity holds:

\[
\text{Den}(X, L) = \sum_{L \subseteq L' \subseteq L^\vee} X^{2\ell(L'/L)} \cdot m(t(L'); X),
\]

where the sum runs over all integral lattices \( L' \supseteq L \). Here

\[
\ell(L'/L) := \text{length}_{\mathcal{O}_F} L'/L.
\]

**Proof.** This is proved the same way as in the orthogonal case [CY18, Corollary 3.11], using the following hermitian analogue of [Kit93, §5.6 Exercise 4]. Let \( U \) be an \( \mathbb{F}_{q^2}/\mathbb{F}_q \)-hermitian space of dimension \( n \) whose radical has dimension \( a \). Let \( V \) be a non-degenerate \( \mathbb{F}_{q^2}/\mathbb{F}_q \)-hermitian space of dimension \( m \geq n \). Then the number of isometries from \( U \) to \( V \) is equal to

\[
q^{n(2m-n)} \cdot \prod_{i=0}^{n+a-1} (1 - (-q)^{i-m}).
\]

Writing \( m = n + k \), this is equal to

\[
q^{n(2m-n)} \cdot \text{Den}(1)^{n+k} \cdot m(a; (-q)^{-k}),
\]

which explains the correct weight factor \( m(a; X) \) appearing in the theorem.

We remark that since \( F/F_0 \) is unramified, the analogue of the smoothness theorem [CY18, Theorem 3.9] is valid in the hermitian case even when the residue characteristic is \( p = 2 \), as [GY00, Lemma 5.5.2] is still valid for \( p = 2 \) by [GY00, §9]. \( \square \)
Example 3.5.2 (The case rank $L = 1$). If rank $L = 1$, the formula specializes to

$$\text{Den}(X, L) = \sum_{i=0}^{\text{val}(L)} (-X)^i.$$ 

Also note that if $L' \supset L$, then $\text{val}(L')$ and $\text{val}(L)$ has the same parity. In particular, if $\text{val}(L)$ is odd, then $t(L') > 0$ and hence $m(t(L'); 1) = 0$. Thus we obtain the following explicit formula for $\partial \text{Den}(L)$.

**Corollary 3.5.3.** If $\text{val}(L)$ is odd, then

$$\partial \text{Den}(L) = \sum_{L \subset L' \subset L'} \text{m}(t(L')).$$

3.6. Some special cases. Since $m(a; (-q)^{-k}) = 0$ if $0 \leq k \leq (a - 1)$, we also obtain

**Corollary 3.6.1.** For $k \geq 0$,

$$\text{Den}((-q)^{-k}, L) = \sum_{L \subset L' \subset L'} q^{-2\ell(L'/L)k} \cdot m(t(L'); (-q)^{-k})$$

In particular, for $k = 0$,

$$(3.6.1.1) \quad \text{Den}(L) = \text{Den}(1, L) = \sum_{L \subset L' \subset L', \ell(L') = 0} 1 = \#\{L' \text{ self-dual} : L \subset L'\}.$$ 

For $k = 1$,

$$(3.6.1.2) \quad \frac{1}{\text{vol}(L)} \text{Den}((-q)^{-1}, L) = \sum_{L \subset L' \subset L', \ell(L') = 0} 1 + \sum_{L \subset L' \subset L', \ell(L') = 1} (1 + q^{-1}) \frac{1}{\text{vol}(L')}$$

**Corollary 3.6.2.** The following identities hold:

$$(3.6.2.1) \quad \text{Den}(-q, L) = \sum_{L \subset L' \subset L'} [L' : L] \cdot m(t(L') + 1),$$

and

$$(3.6.2.2) \quad \text{Den}(-q, L) = \frac{1}{\text{vol}(L)} \text{Den}((-q)^{-1}, L).$$

**Proof.** The first part follows from Theorem 3.5.1 and the fact that

$$m(t(L'); -q) = m(t(L') + 1).$$

The second part follows from the functional equation (3.2.0.4). \qed

3.7. An induction formula.

**Proposition 3.7.1.** Let $L'$ be a lattice of rank $n - 1$ with fundamental invariants $(a_1, \cdots, a_{n-1})$. Let $L = L' + \langle x \rangle$ and $L' = L' + \langle \varpi^{-1} x \rangle$ where $x \perp L'$ with $\text{val}(x) > a_{n-1}$. Then

$$\text{Den}(X, L) = X^2 \text{Den}(X, L') + (1 - X) \text{Den}(-qX, L').$$

This is [Ter13, Theorem 5.1] in the hermitian case, and Katsurada [Kat99, Theorem 2.6 (1)] in the orthogonal case (see also [CY18]).
4. Horizontal components of Kudla–Rapoport cycles

4.1. Quasi-canonical lifting cycles. Let \( \langle y \rangle \subseteq \mathbb{V}_2 \) be a rank one \( O_F \)-lattice. By [KR11 Proposition 8.1], we have a decomposition as divisors on \( N_2 \),

\[
\mathcal{Z}(y) = \sum_{i=0}^{\lfloor \text{val}(y)/2 \rfloor} \mathcal{Z}_{\text{val}(y)-2i}.
\]

Here \( \mathcal{Z}_s \) (\( s \geq 0 \)) is the quasi-canonical lifting cycle of level \( s \) on \( N_2 \), the horizontal divisor corresponding to the quasi-canonical lifting of level \( s \) of the framing object \( (X, t_X, \lambda_X) \) of \( N_2 \) (the quasi-canonical lifting of level \( s = 0 \) is the canonical lifting). We denote

\[
\mathcal{Z}(y)^\circ := \mathcal{Z}_{\text{val}(y)} \subseteq \mathcal{Z}(y).
\]

Let \( O_{F,s} = O_{F_0} + x_s O_F \subseteq O_F \). Let \( \mathbb{F}_s \) be the finite abelian extension of \( \mathbb{F} \) corresponding to the subgroup \( O_{F,s}^\times \) of \( O_F^\times \). Let \( O_{F,s} \) be the ring of integer of the ring class field \( \mathbb{F}_s \). Then \( O_{F,0} = O_F \), and the degree of \( O_{F,s} \) over \( O_F \) is equal to \( q^s(1 + q^{-1}) \) when \( s \geq 1 \). We have

\[
\mathcal{Z}_s \cong \text{Spf} O_{F,s}.
\]

4.2. Horizontal cycles. Let \( L^\circ \subseteq \mathbb{V}_n \) be a hermitian \( O_F \)-lattice of rank \( n - 1 \). Let \( M^\circ \) be an integral hermitian \( O_F \)-lattice of rank \( n - 1 \) such that \( L^\circ \subseteq M^\circ \). When \( t(M^\circ) \leq 1 \), we can construct a horizontal formal subscheme in \( N_n \) using quasi-canonical lifting. In fact, since \( t(M^\circ) \leq 1 \), we may find a rank \( n - 2 \) \( O_F \)-lattice \( M_{n-2} \), which is self-dual in the hermitian space \( M_{n-2,F} \), and a rank one \( O_F \)-lattice \( \langle y \rangle \), such that we have an orthogonal direct sum decomposition

\[
M^\circ = M_{n-2} \oplus \langle y \rangle.
\]

Let \( M_{n-2,F}^\perp \subseteq \mathbb{V}_n \) be the orthogonal complement of \( M_{n-2,F} \) in \( \mathbb{V}_n \). Then we have an isomorphism \( M_{n-2,F}^\perp \cong \mathbb{V}_2 \), and thus an isomorphism (see \( \{2.9\} \))

\[
\mathcal{Z}(M_{n-2}) \cong N_2.
\]

Under this isomorphism, we can identify the divisor \( \mathcal{Z}(M^\circ) \subseteq \mathcal{Z}(M_{n-2}) \) with the divisor \( \mathcal{Z}(y) \subseteq N_2 \).

We denote by \( \mathcal{Z}(M^\circ)^\circ \subseteq \mathcal{Z}(M^\circ) \) the horizontal cycle corresponding to the quasi-canonical lifting cycle \( \mathcal{Z}(y)^\circ \subseteq \mathcal{Z}(y) \). It is independent of the choice of the self-dual lattice \( M_{n-2} \). In fact, since \( \text{val}(y) = \text{val}(M^\circ) \), we can characterize \( \mathcal{Z}(M^\circ)^\circ \) as the unique component of \( \mathcal{Z}(M^\circ) \) isomorphic to \( \mathcal{Z}_{\text{val}(M^\circ)} \). In particular, we have

\[
\deg_{O_F} (\mathcal{Z}(M^\circ)^\circ) = \begin{cases} 
1, & t(M^\circ) = 0, \\
\text{vol}(M^\circ)^{-1}(1 + q^{-1}), & t(M^\circ) = 1.
\end{cases}
\]

(4.2.0.1)

**Theorem 4.2.1.** As horizontal cycles on \( N \),

(4.2.1.1)

\[
\mathcal{Z}(L^\circ)_{\mathcal{D}} = \sum_{L^\circ \subset M^\circ \subset (M^\circ)^\vee \atop t(M^\circ) \leq 1} \mathcal{Z}(M^\circ)^\circ.
\]

**Lemma 4.2.2.** The cycles \( \mathcal{Z}(M^\circ)^\circ \) on the right-hand-side of (4.2.1.1) are all distinct.
4.3. Breuil modules. First let us review the (absolute) Breuil modules ([Bre00], [Kis06, Appendix], [BC, §12.2]). Let $W = W(\bar{k})$. Let $O_K$ be a totally ramified extension of $W$ of degree $e$ defined by an Eisenstein polynomial $E(u) \in W[u]$. Let $S$ be Breuil’s ring, the $p$-adic completion of $W[u][E(u)^i \overline{n}^{-1}]_{i \geq 1}$ (the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$). The ring $S$ is local and $W$-flat, and $S/uS \cong W$. Let $\mathrm{Fil}^1 S \subseteq S$ be the ideal generated by all $E(u)^i \overline{n}$. Then $S/\mathrm{Fil}^1 S \cong O_K$. By Breuil’s theorem, $p$-divisible groups $G$ over $O_K$ are classified by their Breuil modules $\mathcal{M}(G) = \mathbb{D}(G)(S)$ ([Kis06, Proof of A.6]), where $\mathbb{D}(G)$ is the Dieudonné crystal of $G$. It is a finite free $S$-module together with an $S$-submodule $\mathrm{Fil}^1 \mathcal{M}(G)$, and a $\phi_S$-linear homomorphism $\phi : \mathrm{Fil}^1 \mathcal{M}(G) \to G$ satisfying certain conditions. The classical Dieudonné module $M(G_\bar{k})$ of the special fiber $G_{\bar{k}}$ is given by $\mathbb{D}(G_{\bar{k}})(W) = \mathbb{D}(G)(S) \otimes_S W = \mathcal{M}(G)/u\mathcal{M}(G)$, with Hodge filtration $\mathrm{Fil}^1 M(G_\bar{k})$ equal to the image of $\mathrm{Fil}^1 \mathcal{M}(G)$. We also have $\mathbb{D}(G)(O_K) = \mathbb{D}(G)(S) \otimes_S O_K = \mathcal{M}(G) \otimes_S O_K$.

For $\varpi$-divisible $O_{F_0}$-modules, one has an analogous theory of relative Breuil modules (see [Hen16]) by replacing $W = W(\bar{k})$ with $O_F = W_{O_{F_0}}(\bar{k})$, and by defining $S$ to be the $\varpi$-adic completion of the $O_{F_0}$-divided power envelope (in the sense of [Fal02]) of $O_F[u]$ with respect to the ideal $(E(u))$.

4.4. Tate modules. Let $K$ be a finite extension of $\bar{F}$. Let $z \in \mathcal{N}_n(O_K)$ and let $G$ be the corresponding $O_F$-hermitian module of signature $(1, n - 1)$ over $O_K$. Let

$$L := \mathrm{Hom}_{O_F}(T_p \mathcal{E}, T_p G),$$

where $T_p(-)$ denotes the integral $p$-adic Tate modules. Then $L$ is a self-dual $O_F$-hermitian lattice of rank $n$, where the hermitian form $\{x, y\} \in O_F$ is defined to be

$$(T_p \mathcal{E} \xrightarrow{x} T_p G \xrightarrow{\lambda_G} T_p G^\vee \xrightarrow{y^\vee} T_p \mathcal{E}^\vee \xrightarrow{\lambda_E^\vee} T_p \mathcal{E}) \in \mathrm{End}_{O_F}(T_p \mathcal{E}) \cong O_F.$$
There are two injective $O_F$-linear homomorphisms (preserving their hermitian forms)

$$L = \text{Hom}_{O_F}(T_p\mathcal{E}, T_pG)$$

where the right map $i_\mathcal{F}$ is induced by the reduction to $\text{Spec } \bar{k}$ and the framings $\rho_{\mathcal{E}}$ and $\rho_z : G_{\mathcal{E}} \to \mathcal{X}_n$ corresponding to $\mathcal{E}$ and $z \in N_n(O_K)$ respectively. These extend to $F$-linear homomorphisms (still denoted by the same notation)

$$(4.4.0.1) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.2) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.3) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.4) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.5) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.6) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.7) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.8) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

$$(4.4.0.9) \quad \text{Hom}_{O_F}(\mathcal{E}, G)$$

where the intersection is taken inside the $F$-vector space $L_F$. By [Tat67, Theorem 4, Corollary 1], $i_K$ induces an isomorphism

$$\text{Hom}_{O_F}(\mathcal{E}, G) \cong \text{Hom}_{O_F}[\Gamma_K](T_p\mathcal{E}, T_pG),$$

where $\Gamma_K = \text{Gal}(\bar{K}/K)$, and so an isomorphism

$$\text{Hom}_{O_F}(\mathcal{E}, G) \cong \text{Hom}_{O_F}[\Gamma_K](V_p\mathcal{E}, V_pG),$$

where $V_p(-)$ denotes the rational $p$-adic Tate module. Thus we obtain

$$L \cap \text{Hom}_{O_F}(\mathcal{E}, G) \cong \text{Hom}_{O_F}(T_p\mathcal{E}, T_pG) \cap \text{Hom}_{O_F}[\Gamma_K](V_p\mathcal{E}, V_pG)$$

$$= \text{Hom}_{O_F}[\Gamma_K](T_p\mathcal{E}, T_pG)$$

$$\cong \text{Hom}_{O_F}(\mathcal{E}, G),$$

which proves the result. \hfill \Box

Let $M \subseteq \mathbb{V}_n$ be an $O_F$-lattice (of arbitrary rank). By definition we have $z \in \mathcal{Z}(M)(O_K)$ if and only if $M \subseteq i_{\mathcal{F}}(\text{Hom}_{O_F}(\mathcal{E}, G))$. It follows from Lemma 4.4.1 that $z \in \mathcal{Z}(M)(O_K)$ if and only if

$$(4.4.1.2) \quad M \subseteq i_{\mathcal{F}}(i_K^{-1}(L)).$$
4.5. **Proof of Theorem 4.2.1.** Let \( z \in \mathcal{Z}(L^b)(O_K) \) and let \( G \) be the corresponding \( O_F \)-hermitian module of signature \((1, n - 1)\) over \( O_K \). By (4.4.1.2), we know that
\[
L^b \subseteq i_K^1(i_K^{-1}(L)).
\]
Define \( M^b := L^b_F \cap i_K^1(i_K^{-1}(L)) \). By (4.4.1.2) again, we obtain that \( z \in \mathcal{Z}(M^b)(O_K) \). Moreover, the diagram (4.4.0.1) induces an isomorphism
\[
M^b \xrightarrow{\sim} L \cap i_K^1(i_K^{-1}(L^b_F)).
\]
Set \( \mathbb{W} = i_K^1(i_K^{-1}(L^b_F)) \). Then it has the same dimension as \( L^b_F \).

**Lemma 4.5.1.** Assume \( L \) is a self-dual \( O_F \)-hermitian lattice and \( \mathbb{W} \subset L_F \) is a sub-vector-space of dimension \( n - 1 \). Let \( M^b := \mathbb{W} \cap L \). Then \( t(M^b) \leq 1 \).

**Proof.** Since \( M^b = \mathbb{W} \cap L \), we may write \( L = M^b + (x) \) for some \( x \in L \) by Lemma 7.2.1 below. Choose an orthogonal basis \( \{e_1, \ldots, e_{n-1}\} \) of \( M^b \) such that \((e_i, e_i) = \varpi a_i \). The fundamental matrix of \( \{e_1, \ldots, e_{n-1}, x\} \) has the form
\[
T = \begin{pmatrix}
\varpi a_1 & (e_1, x) \\
\varpi a_2 & (e_2, x) \\
\vdots & \vdots \\
(x, e_1) & (x, e_2) & \cdots & (x, x)
\end{pmatrix}.
\]
If \( t(M^b) \geq 2 \) (i.e., at least two \( a_i \)'s are \( > 0 \)), then the rank of \( T \mod \varpi \) is at most \( n - 1 \), contradicting that \( L \) is self-dual. \( \square \)

It follows from Lemma 4.5.1 that \( z \in \mathcal{Z}(M^b)(O_K) \) is a quasi-canonical lifting supported on the right-hand-side of (4.2.1.1). By construction, \( M^b \) is the largest lattice in \( L^b_F \) contained in \( i_K^1(i_K^{-1}(L)) \), thus in fact we have \( z \in \mathcal{Z}(M^b)^0(O_K) \) by the equation (4.4.1.2).

It remains to check that each \( \mathcal{Z}(M^b)^0 \) has multiplicity one. Namely, we would like to show that for each \( z \in \mathcal{Z}(L^b)(O_K) \), there is a unique lift of \( z \) in \( \mathcal{Z}(L^b)(O_K[\varepsilon]) \) (where \( \varepsilon^2 = 0 \)). Let \( \mathbb{D}(G) \) be the (covariant) \( O_{K_0} \)-relative Dieudonné crystal of \( G \). Let \( \mathscr{A} = \text{Fil}_0 \mathbb{D}(G)(O_K) \) be the 0th graded piece of \( \mathbb{D}(G)(O_K) \) under the \( O_F \)-action, a free \( O_K \)-module of rank \( n \). By the Kottwitz signature condition, it is equipped with an \( O_K \)-hyperplane \( \mathcal{H} = \text{Fil}^1 \mathscr{A} \subseteq \mathscr{A} \) containing the image of \( L^b \). Let \( \widetilde{\mathcal{H}} = \text{gr}_0 \mathbb{D}(G)(O_K[\varepsilon]) \). Since the kernel of \( O_K[\varepsilon] \to O_K \) has a nilpotent divided power structure, by Grothendieck–Messing theory, a lift \( \tilde{z} \in \mathcal{Z}(L^b)(O_K[\varepsilon]) \) of \( z \) corresponds to an \( O_K[\varepsilon] \)-hyperplane \( \widetilde{\mathcal{H}} \) of \( \mathcal{A} \) lifting the \( O_K \)-hyperplane \( \mathcal{H} \) of \( \mathcal{A} \) and contains the image of \( L^b \) in \( \widetilde{\mathcal{H}} \) (cf. [LZ17, Theorem 3.1.3], [KR11, Proof of Proposition 3.5]). By Breuil’s theorem (§4.3), the image of \( L^b \) in \( \text{gr}_0 \mathbb{D}(G)(S) \) has rank \( n - 1 \) over \( S \) and thus its image in the base change \( \mathscr{A} \) has rank \( n - 1 \) over \( O_K \), we know that there is a unique choice of such hyperplane \( \widetilde{\mathcal{H}} \). Hence the lift \( \tilde{z} \) is unique, and thus each quasi-canonical lifting cycle \( \mathcal{Z}(M^b)^0 \) has multiplicity one.

4.6. **Relation with the local density.** Notice that \( \text{deg}_{O_F^\circ}(\mathcal{Z}(L^b), \mathcal{H}) \) is equal to the degree of the 0-cycle \( \mathcal{Z}(L^b) \) in the generic fiber \( \mathcal{N}_F \) of the Rapoport–Zink space, which may be interpreted as a geometric intersection number on the generic fiber. We have the following identity between this geometric intersection number and a local density.
Corollary 4.6.1. \( \deg_{\mathfrak{p}}(\mathcal{Z}(L^\flat)) = \text{vol}(L^\flat)^{-1}\text{Den}_{L^\flat}((-q)^{-1}) = \text{Den}_{L^\flat}(-q) \).

Proof. The first equality follows immediately from Theorem 4.2.1, Equation (4.2.0.1), and Equation (3.6.1.2). The second equality follows from the functional equation (3.6.2.2). \( \square \)

Remark 4.6.2. Using the \( p \)-adic uniformization theorem (Assumption 13.1) and the flatness of the horizontal part of the global Kudla–Rapoport cycles, one may deduce from Corollary 4.6.1, an identity between the geometric intersection number (i.e. the degree) of a special 0-cycle on a compact Shimura variety associated to \( U(n,1) \) and the value of a Fourier coefficient of a coherent Siegel Eisenstein series on \( U(n,n) \) at the near central point \( s = 1/2 \). This should give a different proof (of a unitary analogue) of a theorem of Kudla [Kud97a, Theorem 10.6] for compact orthogonal Shimura varieties.

5. Vertical components of Kudla–Rapoport cycles

5.1. The support of the vertical part. Let \( L^\flat \) be an \( O_F \)-lattice of rank \( n - 1 \) in \( \mathbb{V}_n \). Recall that \( \mathcal{Z}(L^\flat)_{\mathcal{V}} \) is the vertical part of the Kudla–Rapoport cycle \( \mathcal{Z}(L^\flat) \subseteq \mathcal{N}_n \).

Lemma 5.1.1. \( \mathcal{Z}(L^\flat)_{\mathcal{V}} \) is supported on \( \mathcal{N}^\text{red}_n \), i.e., \( O_{\mathcal{Z}(L^\flat)_{\mathcal{V}}} \) is annihilated by a power of the ideal sheaf of \( \mathcal{N}^\text{red}_n \subseteq \mathcal{N}_n \).

Proof. If not, we may find a formal integral curve \( C \subseteq \mathcal{Z}(L^\flat)_{\mathcal{V}} \) such that \( C^\text{red} \) consists of a single point \( z \in \mathcal{N}^\text{red}_n \). The universal \( p \)-divisible \( O_{F_0} \)-module \( X^\text{univ} \) over \( \mathcal{N}_n \) pulls back to a \( p \)-divisible \( O_{F_0} \)-module \( \mathcal{X}_\eta \) over the generic point \( \eta \) of \( C \). Since \( C^\text{red} = \{ z \} \), we know that the \( p \)-divisible \( O_{F_0} \)-module \( \mathcal{X}_\eta \) is not supersingular. On the other hand, if \( L^\flat = (x_1, \ldots, x_{n-1}) \), then \( \mathcal{X}_\eta \) admits \( n - 1 \) linearly independent special homomorphisms \( \tilde{\varphi} : \mathcal{E}_\eta \rightarrow \mathcal{X} \), which gives rise to a homomorphism \( (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}) : \mathcal{E}_\eta^{n-1} \rightarrow \mathcal{X}_\eta \).

Its cokernel is a \( p \)-divisible \( O_{F_0} \)-module of relative height 2 and dimension 1 with an \( O_F \)-action, hence must be supersingular (note that \( \eta \) has characteristic \( p \)). It follows that \( \mathcal{X}_\eta \) itself is also supersingular, a contradiction. \( \square \)

Now we consider the derived tensor product

\[ ^L\mathcal{Z}(L^\flat) := O_{\mathcal{Z}(\mathcal{N}_n)} \otimes_L \cdots \otimes_L O_{\mathcal{Z}(\mathcal{N}_{n-1})} \]

viewed as an element in \( K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_n) \) (cf. Notation (1.7)), where \( x_1, \ldots, x_{n-1} \) is an \( O_F \)-basis of \( L^\flat \). There is a decomposition \( \mathcal{Z}(L^\flat) = \mathcal{Z}(L^\flat)_{\mathcal{W}} \cup \mathcal{Z}(L^\flat)_{\mathcal{V}} \) as formal schemes. Since \( \mathcal{Z}(L^\flat)_{\mathcal{W}} \) is one dimensional, the intersection \( \mathcal{Z}(L^\flat)_{\mathcal{W}} \cap \mathcal{Z}(L^\flat)_{\mathcal{V}} \) must be zero dimensional (if non-empty). It follows that there is a decomposition of the \((n-1)\)-th graded piece

\[ Gr^{n-1} K_0^{\mathcal{Z}(L^\flat)}(\mathcal{N}_n) = Gr^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathcal{W}}}(\mathcal{N}_n) \oplus Gr^{n-1} K_0^{\mathcal{Z}(L^\flat)_{\mathcal{V}}}(\mathcal{N}_n). \]

This induces a decomposition

\[ ^L\mathcal{Z}(L^\flat) = ^L\mathcal{Z}(L^\flat)_{\mathcal{W}} + ^L\mathcal{Z}(L^\flat)_{\mathcal{V}}. \]
Since $Z(L^\circ)_\mathcal{H}$ has the expected dimension, the first summand is represented by the structure sheaf of $Z(L^\circ)_\mathcal{H}$. Abusing notation we shall write the sum as

$$(5.1.1.2) \quad \mathcal{L}Z(L^\circ) = Z(L^\circ)_\mathcal{H} + \mathcal{L}Z(L^\circ)_\mathcal{H}.$$ 

By Proposition 5.1.1 we have a change-of-support homomorphism

$$\text{Gr}^{n-1} K_0^{Z(L^\circ)}(\mathcal{N}_n) \longrightarrow \text{Gr}^{n-1} K_0^{\mathcal{N}_n} (\mathcal{N}_n).$$

Abusing notation we will also denote the image of $\mathcal{L}Z(L^\circ)_\mathcal{H}$ in the target by the same symbol.

**Corollary 5.1.2.** There exist finitely many curves $C_i \subseteq \mathcal{N}_n^{\text{red}}$ and $\text{mult}_{C_i} \in \mathbb{Q}$ such that

$$\mathcal{L}Z(L^\circ)_\mathcal{H} = \sum_i \text{mult}_{C_i} [\mathcal{O}_{C_i}] \in \text{Gr}^{n-1} K_0^{\mathcal{N}_n} (\mathcal{N}_n).$$

**Proof.** Since $L^\circ \subseteq \mathcal{V}_n$ has rank $n-1$, the number of vertex lattices $\Lambda \subseteq \mathcal{V}_n$ such that $L \subseteq \Lambda$ is finite. By (2.7.0.3), we know that $Z(L^\circ)_\mathcal{H}^{\text{red}}$ is contained in finitely many irreducible components of $\mathcal{N}_n^{\text{red}}$. The finiteness of such curves $C_i$ then follows. \hfill $\square$

5.2. **The Tate conjecture for certain Deligne–Lusztig varieties.** Consider the generalized Deligne–Lusztig variety $Y_d := Y_V$ and the classical Deligne–Lusztig variety $Y_d^\circ := Y_V^\circ$ as defined in (2.5) where $V$ is the unique $k_F/k$-hermitian space of dimension $2d+1$. Recall that we have a stratification

$$Y_d = \bigsqcup_{i=0}^d X_P(w_i).$$

Let

$$X_i^\circ := X_P(w_i), \quad X_i := \bigsqcup_m X_i^\circ_m.$$  

Then $X_i^\circ$ is a disjoint union of the classical Deligne–Lusztig variety $Y_i^\circ$, and each irreducible component of $X_i$ is isomorphic to $Y_i$.

For any $k_F$-variety $S$, we write $H^j(S) = H^j(S_{k_F}, \mathcal{O}(i))$ ($\ell \neq p$ is a prime). Let $F = Fr_{k_F}$ be the $q^2$-Frobenius acting on $H^j(S)$. 

**Lemma 5.2.1.** For any $d, i \geq 0$ and $s \geq 1$, the action of $F^s$ on the following cohomology groups are semisimple, and the space of $F^s$-invariants is zero when $j \geq 1$.

(i) $H^{2j}(Y_d^\circ)(j)$.

(ii) $H^{2j}(X_i^\circ)(j)$.

(iii) $H^{2j}(Y_d - X_i)(j)$.

**Proof.** (i) By [Lus76, 7.3 Case $2A_{2n}$] (notice the adjoint group assumption is harmless due to [Lus76, 1.18]), we know that there are exactly $2d+1$ eigenvalues of $F$ on $H^j_c(Y_d^\circ)$, given by $(-q)^m$ where $m = 0, 1, \ldots, 2d$, and the eigenvalue $(-q)^m$ exactly appear in $H^j_c(Y_d^\circ)$ for $j = \lfloor m/2 \rfloor + d$. By the Poincare duality, we have a perfect pairing

$$H^{2d-j}_c(Y_d^\circ) \times H^j(Y_d^\circ)(d) \to H^{2d}(Y_d^\circ)(d) \simeq \mathbb{Q}_\ell.$$
Thus the eigenvalues of $F$ on $H^{2j}(V^\circ_d)(j)$ are given by $q^{2(d-j)}$ times the inverse of the eigenvalues in $H^{2(d-j)}(V^\circ_d)$, which is equal to $\{(q^{2j},-q^{2j})\}$ when $d \geq 2j > 0$, and $\{(-q^{2j})\}$ when $j = 0$. Hence the eigenvalue of $F^s$ is never equal to 1 when $j \geq 1$. The semisimplicity of the action of $F^s$ follows from [Lus76] 6.1.

(ii) It follows from [i] since $X_i$ is a disjoint union of $Y_i^\circ$.

(iii) It follows from [ii] since $Y_d - X_i = \bigsqcup_{m=i+1}^d X_m^\circ$. 

\[\square\]

**Theorem 5.2.2.** For any $0 \leq i \leq d$ and any $s \geq 1$, the space of Tate classes $H^{2i}(Y_d)(i)^{F^s=1}$ is spanned by the cycle classes of the irreducible components of $X_{d-i}$. In particular, the Tate conjecture holds for $Y_d$.

**Proof.** The assertion is clear when $i = 0$. Assume $i > 0$. Associated to the closed embedding $X_{d-i} \hookrightarrow Y_d$ we have a long exact sequence

\[\cdots \to H_{X_{d-i}}^j(Y_d) \to H^j(Y_d) \to H^j(Y_d - X_{d-i}) \to H_{X_{d-i}}^{j+1}(Y_d) \to \cdots\]

(5.2.2.1)

Take $j = 2i$. We have a Gysin isomorphism

\[\bigoplus_{Z \in \text{Irr}(X_{d-i})} H^0(Z) \xrightarrow{\sim} H_{X_{d-i}}^{2i}(Y_d)(i),\]

where the sum runs over all the irreducible components of $X_{d-i}$. Since the actions of $F^s$ on $H^{2i}(Y_d - X_{d-i})$ and $H_{X_{d-i}}^{2i}(Y_d)$ are semisimple (Lemma 5.2.1), taking the $i$-th Tate twist and taking the $F^s$-invariants of (5.2.2.1) we obtain a 3-term exact sequence

\[H_{X_{d-i}}^{2i}(Y_d)(i)^{F^s=1} \to H^{2i}(Y_d)(i)^{F^s=1} \to H^{2i}(Y_d - X_{d-i})(i)^{F^s=1}.\]

The last term is 0 by Lemma 5.2.1 (iii) as $i > 0$. Thus we have a surjection onto Tate classes

\[\bigoplus_{Z \in \text{Irr}(X_{d-i})} H^0(Z) \xrightarrow{\sim} H_{X_{d-i}}^{2i}(Y_d)(i)^{F^s=1} \to H^{2i}(Y_d)(i)^{F^s=1}.\]

So $H^{2i}(Y_d)(i)^{F^s=1}$ is spanned by the cycle classes of the irreducible components of $X_{d-i}$. 

\[\square\]

Let us come back to the situation of §5.1.

**Corollary 5.2.3.** There exists finitely many Deligne–Lusztig curves $C_i \subseteq \mathcal{N}^\text{red}_n$ and $\text{mult}_{C_i} \in \mathbb{Q}$, such that for any $x \in V_n \setminus L^\circ_F$, 

\[\chi(\mathcal{N}_n, \mathbb{L}Z(L^\circ)^{-1} \cap Z(x)) = \sum_i \text{mult}_{C_i} \cdot \chi(\mathcal{N}_n, C_i \cap \mathbb{L}Z(x)).\]

**Proof.** By the Bruhat–Tits stratification of $\mathcal{N}^\text{red}_n$ (§2.7), any curve in $\mathcal{N}^\text{red}_n$ lies in some Deligne–Lusztig variety $\mathcal{V}(\Lambda) \cong Y_d$. By Theorem 5.2.2 (for $i = d - 1$), the cycle class of such a curve can be written as a $\mathbb{Q}$-linear combination of the cycle classes of Deligne–Lusztig curves on $\mathcal{V}(\Lambda)$. The result then follows from Corollary 5.1.2. 

\[\square\]
5.3. The vertical cycle in the case \( n = 3 \), and Theorem 3.4.1 in the case \( n = 2 \). Now let \( n = 3 \), and let \( L^b \subset V_3 \) be a rank two lattice. Denote by \( \text{Vert}^t(L^b) \) the set of vertex lattices \( \Lambda \) of type \( t \) containing \( L^b \). For \( \Lambda \in \text{Vert}^t(L^b) \), we denote \( L^b_{\Lambda} := L^b_F \cap \Lambda \), an integral lattice in \( L^b_F \).

**Theorem 5.3.1.** (i) Let \( L^b \subset V_3 \) be a rank two lattice. Then the vertical cycle is a sum

\[
Z(L^b)_V = \sum_{\Lambda \in \text{Vert}^3(L^b)} \text{mult}_{L^b}(\Lambda) \cdot V(\Lambda),
\]

where

\[
\text{mult}_{L^b}(\Lambda) = \#\{ L^b | L^b \subset \Lambda \}.
\]

Similarly

\[
L Z(L^b)_V = \sum_{\Lambda \in \text{Vert}^3(L^b)} \text{mult}_{L^b}(\Lambda) \cdot [O_{V(\Lambda)}].
\]

(ii) Theorem 3.4.1 holds when \( n = 2 \), i.e., \( \text{Int}(L^b) = \partial \text{Den}(L^b) \) for all \( L^b \subset V_2 \).

**Remark 5.3.2.** (i) Part (ii) is known by [KR11, Theorem 1.1]. However, our proof is logically independent from loc. cit.

(ii) Later we will only need (in the proof of Lemma 6.2.1) a very special case of part (i) of Theorem 5.3.1, i.e., the minuscule case in the proof below.

We first establish a lemma.

**Lemma 5.3.3.** Fix \( \Lambda_0 \in \text{Vert}^3(L^b) \). Then there exists a vector \( e \) with unit norm such that

(i) \( \Lambda_e := \Lambda_0 + M \) is a vertex lattice of type 1 where \( M = \langle e \rangle \), and \( \Lambda_e = L^b_{\Lambda_e} \oplus M; \)

(ii) \( \Lambda_0 = L^b_{\Lambda_0} + \varpi M \) and \( L^b_{\Lambda_0} = L^b_{\Lambda_e}; \)

(iii) For any other \( \Lambda \neq \Lambda_0 \) in \( \text{Vert}^3(L^b + \varpi M) \), the lattice \( L^b_{\Lambda} \) is equal to \( L^b_{\varpi \Lambda, \varpi \Lambda_e} \), which is a sub-lattice of \( L^b_{\Lambda} = L^b_{\Lambda_e} \) of colength one;

(iv) For any lattice \( L^b \) such that \( L^b \subset L^b \subset L^b_{\Lambda_e} \), we have

\[
t(L^b \oplus M) = \begin{cases} 2, & \text{if } L^b \subset L^b_{\varpi \Lambda_e}, \\ 1, & \text{otherwise.} \end{cases}
\]

**Remark 5.3.4.** Before presenting the proof, we indicate the geometric picture of the lemma. The reduced scheme of \( Z(L^b) \) is a (connected, a fact we do not need) union of the curves \( V(\Lambda) \) for \( \Lambda \in \text{Vert}^3(L^b) \). The lemma implies that on any given connected component \( V(\Lambda_0) \), there exists a (superspecial) point \( V(\Lambda_e) \), such that among all the curves \( V(\Lambda) \subset Z(L^b)_{\text{red}} \) passing \( V(\Lambda_e) \), the given one \( V(\Lambda_0) \) has the (strictly) largest associated lattice \( L^b_{\Lambda_0} \). This suggests the possibility to determine the multiplicity \( \text{mult}_{L^b}(\Lambda) \) by induction on \( [L^b_{\Lambda_0} : L^b]\).

**Proof.** We pick a vector \( x \) of valuation one in \( L^b_{\Lambda_0} \) and denote by \( E \) the rank one lattice \( \langle x \rangle \). Denote by \( M' \) its orthogonal complement in \( L^b_{\Lambda_0} \), so that

\[
L^b_{\Lambda_0} = E \oplus M'.
\]

We claim that there exists a vector \( e \perp E \) such that
(i) The norm of $e$ is a unit;
(ii) Denoting $M = \langle e \rangle$, then the rank two lattice $M' \oplus M$ is self-dual;
(iii) $\Lambda_0 = E \oplus (M' \oplus \varpi M)$.

To show the claim, we consider the two dimensional subspace $\langle x \rangle_F^\perp$. From $\text{val}(x) = 1$, it follows that $\langle x \rangle_F^\perp$ is a split Hermitian space, and $\Lambda_0$ is an orthogonal direct sum $E \oplus E^\perp$ for a vertex lattice $E^\perp$ of type 2 in $\langle x \rangle_F^\perp$. The sublattice $M'$ is saturated in $E^\perp$. Consider the two dimensional $k_F$-vector space $V := \varpi^{-1}E^\perp/E^\perp$ with the induced hermitian form. The $q + 1$ isotropic lines in $V$ are bijective to self-dual lattices containing $E^\perp$. Since $q + 1 > 1$, there exists an isotropic line not containing the image of $\varpi^{-1}M'$ in $V$, or equivalently, there exists a self-dual lattice $\Xi \subset \langle x \rangle_F^\perp$ containing $E^\perp$ but not $\varpi^{-1}M'$ (i.e., $M'$ remains saturated in $\Xi$). Finally, we choose a unit-normed $e$ lifting a generator of the free $O_F$-module $\Xi/M'$ of rank one. It is easy to verify that such a vector $e$ satisfies all the conditions, which proves parts (i) and (ii).

Now let $\Lambda$ be a lattice in $\text{Vert}^3(L^\flat + \langle \varpi e \rangle)$. Then $\Lambda + \langle e \rangle$ is an integral lattice containing a unit-normed vector, hence a vertex lattice of type 1. Since $\Lambda + \langle e \rangle$ contains $L^\flat + \langle e \rangle$, it is unique (corresponding to the unique maximal integral lattice in the non-split two dimensional hermitian space $\langle e \rangle_F^\perp$), and hence $\Lambda + \langle e \rangle = \Lambda_e$. Now assume that $\Lambda \neq \Lambda_0$. Then we obtain the following diagram

\[
\begin{array}{ccc}
\Lambda_e & = & E \oplus (M' \oplus M) \\
\Lambda_0 & = & E \oplus (M' \oplus \varpi M) \\
\varpi \Lambda_e' & = & E \oplus (\varpi M' \oplus \varpi M) \\
\varpi \Lambda_e & = & \varpi E \oplus (\varpi M' \oplus \varpi M) \\
\end{array}
\]

It is easy to see that

$E \oplus \varpi M' \subset L^\flat_\Lambda \subset E \oplus M'$

and hence either $L^\flat_\Lambda = E \oplus M'$ or $L^\flat_\Lambda = E \oplus \varpi M'$. In the former case, we must have $\Lambda \supset E \oplus (M' \oplus \varpi M) = \Lambda_0$, contradicting $\Lambda \neq \Lambda_0$. This shows that $L^\flat_\Lambda = E \oplus \varpi M' = L^\flat_{\varpi \Lambda_e}$, and hence completes the proof of (iii).

Let $L^\flat \subset L^\flat_{\Lambda_e} = E \oplus M'$. Then the type of $L^\flat \oplus M$ is either 1 or 2. To show part (iv), we first assume that $L^\flat \subset E \oplus \varpi M'$. Then we have

\[
t(L^\flat \oplus M) \geq t(E \oplus (\varpi M' \oplus M)) = t(E) + t(\varpi M' \oplus M)
\]

and $t(E) = 1$. Now note that $M' \oplus M$ is self-dual, $\varpi M' \oplus M$ cannot be self-dual, hence $t(\varpi M' \oplus M) \geq 1$.  

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Now we let $L^b \subset E \oplus M'$ but $L^b \not\subset E \oplus \varpi M'$, then there must be a vector $u \in L^b$ whose projection to $M'$ is a generator of $M'$. It follows that $\langle u \rangle \oplus M$ is a rank-two self-dual sublattice of $L^b \oplus M$, forcing the type $t(L^b \oplus M) \leq 1$. This completes the proof of (iv). \hfill \square

**Proof of Theorem 5.3.1.** The formal scheme $Z(L^b)$ is the proper intersection of two divisors, hence $L^b \subset \text{Gr}^2 K_{3,1}^\flat (N_3)$ is represented by the class of $\mathcal{O}_{[Z(L^b)_\mathbf{r}]}$. So it is enough to prove the result about $Z(L^b)_\mathbf{r}$.

First of all, we note that both parts hold in the special case $t(L^b) \leq 1$. Note that part (iii) is then reduced to the case $n = 1$ by Lemma 2.9.1 and we have

$$\text{Int}(L^b) = \frac{\text{val}(L^b) + 1}{2} = \partial \text{Den}(L^b).$$

Then we consider the next simplest case of part (i), the minuscule case, i.e., the fundamental invariants of $L^b$ are $(1, 1)$. Then $\text{Vert}^3(L^b)$ consists of a single type 3 lattice $\Lambda = L^b \oplus \langle u \rangle$ for a vector $u$ of valuation one. By Theorem 4.2.1 the horizontal part is the sum of quasi-canonical lifting cycles $Z(L^b) \cong N_1$ corresponding to the $q + 1$ self-dual lattices $L^b$ containing $L^b$. Therefore we have an equality as 1-cycles

$$Z(L^b) = m \cdot \mathcal{V}(\Lambda) + \sum_{L^b \subset L^b_\mathbf{r}} Z(L^b),$$

where the multiplicity $m$ of $\mathcal{V}(\Lambda)$ is a positive integer to be determined. Now let $x_1, x_2$ be an orthogonal basis of $L^b$, so that $\text{val}(x_1) = \text{val}(x_2) = 1$. Now choose vector $e \perp x_1$ such that $e$ has unit norm and $\langle x_2 \rangle \oplus \langle e \rangle$ is a self-dual lattice. It follows that $L^b \oplus \langle e \rangle$ is a vertex lattice of type 1, and $Z(e)$ does not intersect with any of the quasi-canonical lifting cycles $Z(L^b)$. Now consider

$$\text{Int}(L^b \oplus \langle e \rangle) = \chi(N_3, Z(L^b) \cap Z(e)).$$

On one hand, this is equal to $\partial \text{Den}(L^b \oplus \langle e \rangle) = 1$ by Lemma 2.9.1. On the other hand, using the decomposition (5.3.4.2), we have

$$\text{Int}(L^b \oplus \langle e \rangle) = m \cdot \chi(N_3, \mathcal{V}(\Lambda) \cap Z(e)).$$

We deduce that the multiplicity $m = 1$ in (5.3.4.2), and

$$\chi(N_3, \mathcal{V}(\Lambda) \cap Z(e)) = 1.$$

We note that, choosing $L^b$ appropriately, the argument above shows that (5.3.4.3) holds for any $\Lambda \in \text{Vert}^3$ and a unit-normed $e$ such that $\Lambda + \langle e \rangle$ is an integral lattice (necessarily a vertex lattice of type 1). Obviously $\chi(N_3, \mathcal{V}(\Lambda) \cap Z(e)) = 0$ if $\Lambda + \langle e \rangle$ is not integral.

Next we show that part (iii) for $L^b$ (necessarily with odd $\text{val}(L^b)$) follows from part (i) with the same $L^b$. Here we have implicitly fixed an embedding of the form (2.9.0.1) induced by a self-dual lattice $M$ of rank one. Let $L^b$ be a type one lattice containing $L^b$, then by Lemma 2.9.1 and (5.3.4.1),

$$\text{Int}(L^b \oplus M) = \text{Int}(L^b) = \frac{\text{val}(L^b) + 1}{2}.$$
It follows that for the associated quasi-canonical lifting cycle $\mathcal{Z}(L^b)^o$ (cf. [4.1]),

\[ (5.3.4.4) \quad \chi(N_3, \mathcal{Z}(M) \cap \mathcal{Z}(L^b)^o) = 1. \]

Therefore by Theorem 4.2.1 we obtain

\[ \chi(N_3, \mathcal{Z}(M) \cap \mathcal{Z}(L^b)^\nu) = \# \{ L^b \mid L^b \subset L^b \subset (L^b)^\vee, t(L^b) = 1 \}. \]

By part [ii] for $L^b$, and by (5.3.4.3) we obtain

\[ \chi(N_3, \mathcal{Z}(M) \cap \mathcal{Z}(L^b)^\nu) = \sum_{\Lambda \in \text{Vert}^3(L^b)} \text{mult}_{L^b}(\Lambda) \]

\[ = \sum_{L^b \subset L^b \subset (L^b)^\vee \atop t(L^b) = 2} \# \{ \Lambda \in \text{Vert}^3(L^b) \mid L^b \subset \Lambda, M \subset \Lambda^\vee \}. \]

Here the condition $M \subset \Lambda^\vee$ is equivalent to $M + \Lambda$ being integral. There is a unique vertex lattice of type 1 containing $M$, i.e., the lattice $\Lambda_M := M \oplus \Lambda^2$ where $\Lambda^2$ is the unique maximal integral lattice in the two dimensional non-split hermitian space $M_{\mathbb{F}}$. The condition $M + \Lambda$ being integral (for $\Lambda \in \text{Vert}^3$) is therefore equivalent to $\Lambda \subset \Lambda_M$. If $L^b$ is of type 2, then $\langle L^b, \Lambda^2 \rangle \subset \varpi O_{\mathbb{F}}$ (we leave the proof to the reader), or equivalently $L^b \subset \varpi(\Lambda)^\vee$. Therefore any $L^b$ of type 2 is automatically contained in $\varpi\Lambda^\vee_M$, hence contained in any type 3 vertex lattice $\Lambda \subset \Lambda_M$. It follows that the condition $L^b \subset \Lambda$ is redundant in the sum above, and since there are $q + 1$ of type 3 lattices $\Lambda \subset \Lambda_M$, we obtain

\[ \chi(N_3, \mathcal{Z}(M) \cap \mathcal{Z}(L^b)^\nu) = (q + 1)\# \{ \text{integral } L^b \mid L^b \subset L^b, t(L^b) = 2 \}. \]

Then the desired assertion for part [ii] for $L^b$ follows, by the formula in Corollary 3.5.3

\[ \partial \text{Den}(L^b) = \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{m}(t(L^b)), \text{ where } \text{m}(t(L^b)) = \begin{cases} 1, & t(L^b) = 1, \\ q + 1, & t(L^b) = 2. \end{cases} \]

Finally, we prove part [ii] by induction on $\text{val}(L^b)$. We have proved it when $t(L^b) = 1$ or when $\text{val}(L^b) = 2$. Now fix $L^b$ of type 2 and by induction, we may assume that we have proved part [ii] for $L^b$ with $\text{val}(L^b) < \text{val}(L^b)$. Note that the induction hypothesis also implies that part [ii] holds for $L^b$ with $\text{val}(L^b) < \text{val}(L^b)$.

To determine the multiplicity, we fix $\Lambda_0 \in \text{Vert}^3(L^b)$. Choose $e$ as in Lemma 5.3.3 and follow the same notation. Then part [iii] of Lemma 5.3.3 implies that $L^b_{\Lambda_e} := \Lambda_e \cap L^b_{\mathbb{F}}$ has type 2, hence $\mathcal{Z}(M)$ does not intersect the horizontal part $\mathcal{Z}(L^b)^\nu$ (otherwise, $\Lambda_e$ must contain a type 1 lattice $L^b$ in Theorem 4.2.1). It follows that, by (5.3.4.3),

\[ (5.3.4.5) \quad \text{Int}(L^b \oplus M) = \text{mult}_{L^b}(\Lambda_0) + \sum_{\Lambda \subset \Lambda_e, \Lambda \neq \Lambda_0} \text{mult}_{L^b}(\Lambda). \]

By part [ii] of Lemma 5.3.3, we obtain $[\Lambda_e : L^b \oplus M] = [L^b_{\Lambda_e} : L^b]$. From $\text{val}(\Lambda_e) = 1$ and $\text{val}(L^b_{\Lambda_e}) \geq 2$, it follows that $\text{val}(L^b \oplus M) < \text{val}(L^b)$. By induction hypothesis on $L^b$, since $t(L^b \oplus$
Let \( L^b \subset \mathbb{V}_n \) be an integral lattice of rank \( n - 1 \) and type \( t(L^b) \leq 1 \). Then for any \( x \in \mathbb{V}_n \setminus L^b_F \),
\[
\chi(\mathcal{N}_n, \mathcal{Z}(x) \cap \mathcal{Z}(L^b)^\circ) = \sum_{L^b \cap \langle x \rangle \subset L^b \subset \mathcal{V}, L^b \not\subset \mathcal{V}^\circ} \m(t(L')).
\]

Proof. By assumption that \( t(L^b) \leq 1 \), there exists a self-dual lattice \( M \) of rank \( n - 2 \) such that \( L^b = M \oplus \langle u \rangle \). We then reduce the question to the case \( n = 2 \), in which case \( L^b = \langle u \rangle \). By Theorem 4.2.1 we have an equality of 1-cycles on \( \mathcal{N}_2 \),
\[
\mathcal{Z}(L^b) = \mathcal{Z}(\varpi^{-1} L^b) + \mathcal{Z}(L^b)^\circ.
\]

By Theorem 5.3.1 part (ii),
\[
\text{Int}(L^b \oplus \langle x \rangle) = \partial \text{Den}(L^b \oplus \langle x \rangle),
\]
and
\[
\text{Int}(\varpi^{-1} L^b \oplus \langle x \rangle) = \partial \text{Den}(\varpi^{-1} L^b \oplus \langle x \rangle).
\]

Therefore
\[
\chi(\mathcal{N}_2, \mathcal{Z}(x) \cap \mathcal{Z}(L^b)^\circ) = \text{Int}(L^b \oplus \langle x \rangle) - \text{Int}(\varpi^{-1} L^b \oplus \langle x \rangle)
\]
and the assertion follows from the formula for local density in Corollary 3.5.3.

6. Fourier transform: the geometric side

Let \( L^b \subset \mathbb{V}_n \) be an \( O_F \)-lattice of rank \( n - 1 \). Let \( L^b_F = L^b \otimes_{O_F} F \subset \mathbb{V}_n \) be the \( F \)-vector subspace of dimension \( n - 1 \). Assume that \( L^b_F \) is non-degenerate throughout the paper.
6.1. **Horizontal versus Vertical cycles.** Recall from (5.1.1.2) that there is a decomposition of the derived special cycle $^{L}Z(L^\flat)$ into a sum of vertical and horizontal parts

$$^{L}Z(L^\flat) = Z(L^\flat)_V + ^LZ(L^\flat)_H,$$

and by Theorem 4.2.1, the horizontal part is a sum of quasi-canonical lifting cycles

$$Z(L^\flat)_H = \sum_{L^\flat} Z(L^\flat)^o,$$

where the sum runs over all lattices $L^\flat$ such that

$$L^\flat \subset L^\flat \subset (L^\flat)^\vee \subset L^\flat_F, \quad t(L^\flat) \leq 1.$$

**Definition 6.1.1.** Define the *horizontal part of the arithmetic intersection number*

$$\text{Int}_{L^\flat,F}(x) := \chi(N_n, Z(x) \cap Z(L^\flat)_H), \quad x \in V \setminus L^\flat_F. \quad (6.1.1.1)$$

**Definition 6.1.2.** Analogously, define the *horizontal part of the derived local density*

$$\partial \text{Den}_{L^\flat,F}(x) := \sum_{L^\flat \subset L' \subset L_F \cap L'_F, \text{ type } t \leq 1} m(t(L')) \mathbf{1}_{L'}(x), \quad x \in \mathbb{V}_n \setminus L^\flat_F, \quad (6.1.2.1)$$

where we denote

$$L^\flat := L' \cap L^\flat_F \subset L^\flat_F. \quad (6.1.2.2)$$

**Theorem 6.1.3.** As functions on $V \setminus L^\flat_F$,

$$\text{Int}_{L^\flat,F} = \partial \text{Den}_{L^\flat,F}. \quad \square$$

**Proof.** By Corollary 5.3.5 for a fixed integral lattice $L^\flat \subset L^\flat_F$ of type $t \leq 1$, we have

$$\chi(N_n, Z(x) \cap Z(L^\flat)^o) = \sum_{L^\flat \subset L' \subset (L^\flat)^\vee, \text{ type } \leq 1} m(t(L')).$$

The assertion follows from Theorem 4.2.1 and the corresponding formula (6.1.2.1) for the horizontal part of the local density $\partial \text{Den}_{L^\flat,F}$.

**Definition 6.1.4.** Define the *vertical part of the arithmetic intersection number*

$$\text{Int}_{L^\flat,F}(x) := \chi(N_n, Z(x) \cap Z(L^\flat)_V), \quad x \in V \setminus L^\flat_F. \quad (6.1.4.1)$$

Then there is a decomposition

$$\text{Int}_{L^\flat}(x) = \text{Int}_{L^\flat,F}(x) + \text{Int}_{L^\flat,V}(x), \quad x \in \mathbb{V} \setminus L^\flat_F. \quad (6.1.4.2)$$

We will defer the vertical part of the derived local density to the next section (Definition 7.3.2).
6.2. Computation of \( \text{Int}_{V(\Lambda)} \). Let \( \Lambda \in \text{Vert}^3 \) and \( V(\Lambda) \) be the Deligne–Lusztig curve in the Bruhat–Tits stratification of \( N_n^\text{red} \). Define

\[
\text{Int}_{V(\Lambda)}(x) \coloneqq \chi(N_n, V(\Lambda) \cap Z(x)), \quad x \in V \setminus \{0\}.
\]

**Lemma 6.2.1.** Let \( \Lambda \in \text{Vert}^3 \). Then

\[
\text{Int}_{V(\Lambda)} = -q^2(1 + q)1_{\Lambda} + \sum_{\Lambda \subset \Lambda', t(\Lambda') = 1} 1_{\Lambda'}.
\]

**Proof.** We note that

\[
(6.2.1.1) \quad -q^2(1 + q)1_{\Lambda}(x) + \sum_{\Lambda \subset \Lambda', t(\Lambda') = 1} 1_{\Lambda'}(x) = \begin{cases} 
(1 - q^2), & x \in \Lambda, \\
1, & x \in \Lambda^\vee \setminus \Lambda, \text{and val}(x) \geq 0, \\
0, & \text{otherwise}.
\end{cases}
\]

We first consider the special case \( n = 3 \). If \( u \notin \Lambda \), then \( Z(u) \cap V(\Lambda) \) is non-empty only when \( u \) lies in one of the type 1 lattice nested between \( \Lambda \) and \( \Lambda^\vee \). Then the intersection number is equal to one by (5.3.4.3), and the desired equality follows.

Now assume \( u \in \Lambda \) and \( u \neq 0 \). Choose an orthogonal basis \( \{e_1, e_2, e_3\} \) of \( \Lambda \) (so the norm of them all have valuation one). Let \( L \) be the rank two lattice generated by \( e_1, e_2 \). Now we note that, by Theorem 4.2.1 and Theorem 5.3.1 part (i), as 1-cycles on \( N_3 \),

\[
Z(L) = V(\Lambda) + \sum_{L \subset M = M^\vee \subset L^\vee} Z(M),
\]

where each of \( Z(M) \simeq N_1 \) since \( M \) is self-dual. There are exactly \( q + 1 \) such \( M \).

Let \( u \in \Lambda \setminus \{0\} \), and write it in terms of the chosen basis

\[
u = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3, \quad \lambda_i \in O_F.
\]

Assume that \( \lambda_3 \neq 0 \), and let \( a_3 = 2 \text{val}(\lambda_3) + 1 \) (an odd integer). By [Ter13], we may calculate all of the intersection numbers

\[
\chi(N_3, Z(L) \cap Z(u)) = \frac{a_3 + 1}{2}(q + 1) + (1 - q^2),
\]

\[
\chi(N_3, Z(M) \cap Z(u)) = \frac{a_3 + 1}{2}.
\]

It follows that

\[
\chi(N_3, V(\Lambda) \cap Z(u)) = (1 - q^2).
\]

If \( \lambda_3 = 0 \), then we choose \( L \) to be the span of some other pairs of basis vectors, and we run the same computation. This proves the desired equality if \( u \in \Lambda \setminus \{0\} \) and completes the proof when \( n = 3 \).

Now assume that \( n > 3 \). Since \( \Lambda \) is a vertex lattice of type 3, it admits an orthogonal direct sum decomposition

\[
(6.2.1.2) \quad \Lambda = \Lambda^b \oplus M
\]

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where $\Lambda^b$ is a rank 3 vertex lattice of type 3, and $M$ is a type 0 (i.e., self-dual) lattice of rank $n - 3$. Then

$$\Lambda^\vee = \Lambda^{b,\vee} \oplus M$$

and any element $u \in \Lambda^\vee$ has a unique decomposition

$$u = u^b + u_M, \quad u^b \in \Lambda^{b,\vee}, \; u_M \in M.$$  

First assume that $u^b \neq 0$, i.e., $u \notin M$. Since $M$ is self-dual, we have a natural embedding

$$\delta_M : \mathcal{N}_3 \rightarrow \mathcal{N}_n$$

which identifies $\mathcal{N}_3$ with the special cycle $Z(M)$. Moreover, the Deligne–Lusztig curve $\mathcal{V}(\Lambda^b)$ on $\mathcal{N}_3$ is sent to $\mathcal{V}(\Lambda)$, and the special divisor $Z(u)$ intersects properly with $\mathcal{N}_3$ and its pull-back to $\mathcal{N}_3$ is the special divisor $Z(u^b)$, cf. (2.9.0.2).

We obtain (by the projection formula for the morphism $\delta_M$)

$$\chi(\mathcal{N}_n, \mathcal{V}(\Lambda) \cap Z(u)) = \chi(\mathcal{N}_3, \mathcal{V}(\Lambda^b) \cap Z(u^b)).$$

This reduces the case $u^b \neq 0$ to the case $n = 3$. In particular, when $u^b \in \Lambda^b \setminus \{0\},$

$$\chi(\mathcal{N}_n,\mathcal{V}(\Lambda)^{\mathcal{L}} \cap Z(u)) = 1 - q^2.$$  

Finally it remains to show that the intersection number is the constant $(1 - q^2)$ when $u \in (\Lambda^b \oplus M) \setminus \{0\}$. It suffices to show this when $u \in M \setminus \{0\}$. Choose an orthogonal basis $\{e_1, e_2, e_3\}$ for $\Lambda^b$, and $\{f_1, \cdots, f_{n-3}\}$ for $M$. Write

$$u = \mu_1 f_1 + \cdots + \mu_{n-3} f_{n-3}, \quad \mu_j \in \mathcal{O}_F.$$  

One of the $\mu_i$ is non-zero, and without loss of generality we assume $\mu_1 \neq 0$. Now define $\widetilde{M}$ to be the new lattice generated by $e_1 + f_1, f_2, \cdots, f_{n-3}$. It is self-dual, and its orthogonal complement $\widetilde{\Lambda}^b$ in $\Lambda$ is again a type 3-lattice. Now replace the decomposition $\Lambda = \Lambda^b \oplus M$ by the new one $\Lambda = \widetilde{\Lambda}^b \oplus \widetilde{M}$. Then $u \notin \widetilde{M}$, and hence we can apply (6.2.1.3). This completes the proof. \hfill $\square$

**Corollary 6.2.2.** The function $\text{Int}_{L^\vee, V} \in C_c^\infty(\mathcal{V})$, i.e., it is locally constant with compact support.

**Proof.** This follows from Corollary [5.2.3](#) and Lemma [6.2.1](#). \hfill $\square$

### 6.3. Fourier transform: the geometric side; “Local modularity”.

We compute the Fourier transform of $\text{Int}_{L^\vee, V}$ as a function on $\mathcal{V}$.

**Lemma 6.3.1.** Let $\Lambda \in \text{Vert}^3$. Then

$$\widehat{\text{Int}_{V(\Lambda)}} = \gamma_V \text{Int}_{V(\Lambda)}.$$  

Here $\gamma_V = -1$ is the Weil constant.
Proof. By Lemma 6.2.1, we obtain
\[
\hat{\text{Int}}_{\mathcal{V}(\Lambda)} = -\text{vol}(\Lambda) \cdot q^2(1 + q) \cdot 1_{\Lambda^\vee} + \sum_{\Lambda \subset \Lambda', t(\Lambda') = 1} \text{vol}(\Lambda') \cdot 1_{\Lambda'^\vee}
\]
\[= - (1 + q^{-1}) \cdot 1_{\Lambda^\vee} + \sum_{\Lambda \subset \Lambda', t(\Lambda') = 1} q^{-1} \cdot 1_{\Lambda'^\vee}.\]

Now we compute its value at \(u \in \mathcal{V}\) according to four cases.

(i) If \(u \in \Lambda\), there are exactly \(q^3 + 1\) type 1 lattices \(\Lambda'\) containing \(\Lambda\), and the value is
\[q^{-1}(q^3 + 1) - (1 + q^{-1}) = q^2 - 1.\]

(ii) If \(u \in \Lambda_1 \setminus \Lambda\) for some \(\Lambda_1 \in \text{Vert}^1\), i.e., the image of \(\bar{u}\) of \(u\) in \(\Lambda^\vee/\Lambda\) is an isotropic vector. Notice that \(u \in \Lambda'^\vee\) if and only if \(\pi\) is orthogonal to the line given by the image of \((\Lambda')^\vee\) in \(\Lambda^\vee/\Lambda\). So there is exactly one such \(\Lambda' \in \text{Vert}^1\), i.e., \(\Lambda' = \Lambda_1\), and we obtain the value
\[q^{-1} - (1 + q^{-1}) = -1.\]

(iii) If \(u \in \Lambda^\vee \setminus \Lambda\) but \(u \not\in \Lambda_1 \setminus \Lambda\) for any \(\Lambda_1 \in \text{Vert}^1\). Then \(\pi\) is anisotropic in \(\Lambda^\vee/\Lambda\). Notice that \(\pi \perp\) is a non-degenerate hermitian space of dimension two, and \(\Lambda'\) corresponds to an isotropic line in \(\pi \perp\). So there are exactly \(q + 1\) of such \(\Lambda' \in \text{Vert}^1\), and we obtain the value
\[q^{-1}(q + 1) - (1 + q^{-1}) = 0.\]

(iv) If \(u \not\in \Lambda\), then the value at \(u\) is
\[q^{-1} \cdot 0 - (1 + q^{-1}) \cdot 0 = 0.\]

This completes the proof by comparing with (6.2.1.1).

\[\square\]

Remark 6.3.2. It follows from Lemma 6.3.1 that \(\hat{\text{Int}}_{\mathcal{V}(\Lambda)}\) is \(\text{SL}_2(\mathcal{O}_{F_0})\)-invariant under the Weil representation. This invariance may be viewed as a “local modularity”, an analog of the global modularity of arithmetic generating series of special divisors (such as in \([\text{BHK}+17]\)).

Corollary 6.3.3. The function \(\text{Int}_{L^\vee, \gamma} \in C^\infty_c(\mathcal{V})\) satisfies
\[\hat{\text{Int}}_{L^\vee, \gamma} = \gamma^\vee \text{Int}_{L^\vee, \gamma}.\]

Proof. This follows from Corollary 5.2.3 and Lemma 6.3.1. \[\square\]

7. Fourier transform: the analytic side

In this section we allow \(F_0\) to be a non-archimedean local field of characteristics not equal to 2 (but possibly with residual characteristics 2), and \(F\) an unramified quadratic extension.
7.1. **Lattice-theoretic notations.** Recall that \( \mathbb{V} = \mathbb{V}_n \) is the hermitian space defined in \( \S 2.2 \) (in particular it is non-split). We continue to let \( L^b \subset \mathbb{V} = \mathbb{V}_n \) be an integral \( O_F \)-lattice of rank \( n - 1 \), such that \( L^b_F \) is non-degenerate. Define

\[
(L^b)^{\vee, 0} = \{ x \in (L^b)^{\vee} \mid (x, x) \in O_F \}.
\]

The fundamental invariants of \( L^b \) are denoted by

\[
(a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1},
\]

where \( 0 \leq a_1 \leq \cdots \leq a_{n-1} \). Denote the largest invariant by

\[
e_{\text{max}}(L^b) = a_{n-1}.
\]

Let

\[
(L^b)^{\vee, 0} = \{ x \in (L^b)^{\vee} \mid (x, x) \in O_F \}.
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The fundamental invariants of \( L^b \) are denoted by

\[
(a_1, \cdots, a_{n-1}) \in \mathbb{Z}^{n-1},
\]

where \( 0 \leq a_1 \leq \cdots \leq a_{n-1} \). Denote the largest invariant by

\[
e_{\text{max}}(L^b) = a_{n-1}.
\]

Let

\[
M = M(L^b) = L^b \oplus \langle u \rangle
\]

be the lattice characterized by the following condition: \( u \in \mathbb{V} \) is a vector satisfying \( u \perp L^b \) and with valuation \( a_{n-1} \) or \( a_{n-1} + 1 \) (only one of these two is possible due to the parity of \( \text{val} (\det(\mathbb{V})) \)). In other words, the rank one lattice \( \langle u \rangle \) is the set of all \( x \perp L^b \) with \( \text{val}(x) \geq a_{n-1} \). Then the fundamental invariants of \( M(L^b) \) are

\[
(a_1, \cdots, a_{n-1}, a_{n-1}), \quad \text{or} \quad (a_1, \cdots, a_{n-1}, a_{n-1} + 1).
\]

Finally we note that, if \( L'^b \subset L^b \) are two integral lattices of rank \( n - 1 \), then

\[
e_{\text{max}}(L'^b) \leq e_{\text{max}}(L^b)
\]

and \( M(L'^b) \subset M(L^b) \). The above inequality follows from the characterization of \( -e_{\text{max}}(L^b) \) as the minimal valuation of vectors in the lattice \( (L^b)^\vee \).

7.2. **Lemmas on lattices.** In this subsection, we do not require the lattice \( L^b \) to be integral.

**Lemma 7.2.1.** Let \( L'^b \subset L^b_F \) be an \( O_F \)-lattice (of rank \( n - 1 \)). Denote

\[
\text{Lat}(L^b) := \{ O_F \text{-lattices } L' \subset \mathbb{V} \mid \text{rank } L' = n, \ L'^b = L' \cap L^b_F \}.
\]

Then there is a bijection

\[
([\mathbb{V}/L^b] \setminus (L^b_F/L^b))/O_F^\times \xrightarrow{\sim} \text{Lat}(L^b) \quad \text{by } u \mapsto L^b + \langle u \rangle.
\]

**Proof.** The indicated map is well-defined and clearly injective. To show the surjectivity, we note that \( L'/L^b \) is free for any \( L' \in \text{Lat}(L^b) \). Choose any element \( u \in L' \) whose image in \( L'/L^b \) is a generator. Then it is clear that \( L' = L^b + \langle u \rangle \). \( \square \)

Let \( \langle x \rangle_F = F x \) be the \( F \)-line generated by \( x \in \mathbb{V} \setminus L^b_F \). Corresponding to the (not necessarily orthogonal) decomposition \( \mathbb{V} = L^b_F \oplus \langle x \rangle_F \), there are two projection maps

\[
\pi_b : \mathbb{V} \rightarrow L^b_F, \quad \pi_x : \mathbb{V} \rightarrow \langle x \rangle_F.
\]
Lemma 7.2.2. Let $L' \subset \mathbb{V}$ be an $O_F$-lattice (of rank $n$). Denote

$$L^b = L' \cap L_F^b, \quad L'_x = L' \cap \langle x \rangle_F.$$ 

The natural projection maps induce isomorphisms of $O_F$-modules

$$\pi_b(L')/L^b \xrightarrow{\sim} L'/(L^b \oplus L'_x) \xrightarrow{\sim} \pi_x(L')/L'_x.$$ 

In particular, all three abelian groups are $O_F$-cyclic modules.

Proof. Consider the map

$$\phi: L' \longrightarrow \pi_x(L')/L'_x.$$ 

We show that the kernel of $\phi$ is $L^b \oplus L'_x$; the other assertion can be proved similarly.

Let $u \in L'$ and write $u = u^b + u^x$ uniquely for $u^b \in L_F^b, u^x \in \langle x \rangle_F$. Then $\phi(u) = u^x \mod L'_x$. If $u \in \ker(\phi)$, then $u^x \in L'$. It follows that $u^x = u - u^x \in L'$, and hence $u^b \in L^b$. Therefore $u \in L^b \oplus L'_x$ and $\ker(\phi) \subset L^b \oplus L'_x$. Conversely, if $u \in L^b \oplus L'_x$, then $u^b \in L^b, u^x \in L'_x$, and clearly $\phi(u) = 0$. This completes the proof. \qed

Now we assume that $x \perp L'$. We rename the projection map to the line $\langle x \rangle_F = (L_F^b)^\perp$ as $\pi_\perp$. Then we have a formula relating the volume of $L'$ to that of $L^b = L' \cap L_F^b$ and of the image of the projection $\pi_\perp$ (the analog of “base \times height” formula for the area of a parallelogram)

$$(7.2.2.1) \quad \vol(L') = \vol(L^b) \vol(\pi_\perp(L')).$$

In fact, this is clear if $L' = L^b \oplus \pi_\perp(L')$ and in general we obtain the formula by Lemma 7.2.2

$$\frac{\vol(L')}{\vol(L^b \oplus L'_x)} = \frac{\vol(\pi_\perp(L'))}{\vol(L'_x)}.$$ 

7.3. Local constancy of $\partial\text{Den}_{F,x}$. We now resume the convention in §7.1. In particular, $L^b$ is now an integral lattice. When rank $L = n$ with $\text{val}(L)$ odd, recall that the derived local density is

(Corollary 3.5.3)

$$\partial\text{Den}(L) = \sum_{L \subset L' \subset L^{\vee}} \mathfrak{m}(t(L')),$$

where

$$\mathfrak{m}(a) = \begin{cases} (1 + q)(1 - q^2) \cdots (1 - (-q)^{a-1}), & a \geq 2, \\ 1, & a \in \{0, 1\}. \end{cases}$$

Definition 7.3.1. For $x \in \mathbb{V} \setminus L_F^b$, define

$$\partial\text{Den}_{L^b}(x) := \partial\text{Den}(L^b + \langle x \rangle).$$

Then

$$(7.3.1.1) \quad \partial\text{Den}_{L^b}(x) = \sum_{L' \subset L' \subset L^{\vee}} \mathfrak{m}(t(L')) \mathbf{1}_{L'}(x),$$

where the sum is over all integral lattices $L' \subset \mathbb{V}$ of rank $n$. Note that this is a finite sum for a given $x \in \mathbb{V} \setminus L_F^b$. However, when varying $x \in \mathbb{V} \setminus L_F^b$, infinitely many $L'$ can appear.
**Definition 7.3.2.** Recall that we have defined the horizontal part $\partial \text{Den}_{L^\vee, \mathcal{H}}$ in Definition 6.1.2. Now define the **vertical part of the derived local density**

\[ \partial \text{Den}_{L^\vee, \mathcal{H}}(x) := \partial \text{Den}_{L^\vee}(x) - \partial \text{Den}_{L^\vee, \mathcal{H}}(x), \quad x \in \mathcal{V} \setminus L^b_F. \]

Obviously the functions $\partial \text{Den}_{L^\vee, \mathcal{H}}$ and $\partial \text{Den}_{L^\vee}$ are locally constant on $\mathcal{V} \setminus L^b_F$. Hence $\partial \text{Den}_{L^\vee, \mathcal{H}}$ is also locally constant on $\mathcal{V} \setminus L^b_F$.

**Definition 7.3.3.**

(a) Let $L^1_c(\mathcal{V})$ be the space of integrable functions that are defined on a dense open subset of $\mathcal{V}$ and vanish outside a compact subset of $\mathcal{V}$.

(b) Let $\mathcal{W}$ be a non-degenerate co-dimension one subspace of $\mathcal{V}$. We say that a smooth function $f$ on $\mathcal{V} \setminus \mathcal{W}$ has **logarithmic singularity along $\mathcal{W}$** near $w \in \mathcal{V}$, there is a neighborhood $\mathcal{U}_w$ of $w$ in $\mathcal{V}$ such that

\[ f(u) = C_0 \log |(u^\perp, u^\perp)| + C_1 \]

holds for all $u \in \mathcal{U}_w \setminus \mathcal{W}$, where $u^\perp \in \mathcal{W}^\perp$ denotes the orthogonal projection of $u$ to $\mathcal{W}^\perp$, and $C_0, C_1$ are constants (depending on $w$). We say that a smooth function $f$ on $\mathcal{V} \setminus \mathcal{W}$ has **logarithmic singularity along $\mathcal{W}$** if it does so near every $w \in \mathcal{V}$.

**Proposition 7.3.4.**

(a) The functions $\partial \text{Den}_{L^\vee, \mathcal{H}}$ and $\partial \text{Den}_{L^\vee}$ lie in $L^1_c(\mathcal{V})$, and they have logarithmic singularity along $L^b_F$.

(b) The function $\partial \text{Den}_{L^\vee, \mathcal{H}}$ extends to a (necessarily unique) element in $C^\infty_c(\mathcal{V})$ (we will still denote this extension by $\partial \text{Den}_{L^\vee, \mathcal{H}}$), i.e., there exists an element in $C^\infty_c(\mathcal{V})$ whose restriction to the open dense subset $\mathcal{V} \setminus L^b_F$ is equal to $\partial \text{Den}_{L^\vee, \mathcal{H}}$.

**Proof.** Consider the set

\[ (7.3.4.1) \quad N(L^b) := \{ x \in \mathcal{V} \mid \langle x \rangle + L^b \text{ is integral} \}. \]

We claim that $N(L^b)$ is a compact open subset of $\mathcal{V}$. To show the claim, we rewrite the above set as

\[ N(L^b) = \{ x \in \mathcal{V} \mid (x, L^b) \subset O_F, \quad (x, x) \in O_F \}. \]

Write $x = x^b + x^\perp$ according to the orthogonal direct sum $\mathcal{V} = L^b_F \oplus (L^b_F)^\perp$. Then the condition $(x, L^b) \subset O_F$ is equivalent to $x^b \in (L^b)^\vee$. Since $(L^b)^\vee$ is a compact subset of $L^b_F$, $(x^b, x^\perp)$ is bounded in $F$. Together with the integrality of the norm $(x, x)$, it follows that $(x^\perp, x^\perp)$ is also bounded. Therefore $x^\perp$ lies in a bounded subset $\mathcal{L}$ of the $F$-line $(L^b_F)^\perp$. It follows that $N(L^b)$ is contained in a bounded set $(L^b)^\vee \cap \mathcal{L}$. Since $N(L^b)$ is open and closed in $\mathcal{V}$, it must be compact.

Note that all three function $\partial \text{Den}_{L^\vee}$, $\partial \text{Den}_{L^\vee, \mathcal{H}}$ and $\partial \text{Den}_{L^\vee, \mathcal{H}}$ vanish outside $N(L^b)$. It follows that all three vanish outside a compact subset of $\mathcal{V}$, and are smooth functions on $\mathcal{V} \setminus L^b_F$. To show part (a) it suffices to show both functions $\partial \text{Den}_{L^\vee}$, $\partial \text{Den}_{L^\vee, \mathcal{H}}$ have logarithmic singularity along $L^b_F$ near each $e \in L^b_F$ (then the integrability follows by the consideration of the support). To show part (b), it suffices to show that $\partial \text{Den}_{L^\vee, \mathcal{H}}$ is a constant near each $e \in L^b_F$. 

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We now consider the behavior of the three functions \( \partial \text{Den}_{L^\flat}, \partial \text{Den}_{L^\flat,H} \) and \( \partial \text{Den}_{L^\flat,Y} \) near each \( e \in L_F^\flat \). By the above discussion on the support, we may assume \( e \in L_F^\flat \cap N(L^\flat) \).

First we consider the case \( e \in L^\flat \), and we consider its neighborhood \( M(L^\flat) = L^\flat \oplus \langle u \rangle \), the lattice defined by (7.1.0.3). Obviously the three functions are all invariant under \( L^\flat \)-translation. By Lemma 7.3.5 both \( \partial \text{Den}_{L^\flat,H} \) and \( \partial \text{Den}_{L^\flat} \) have logarithmic singularity along \( L_F^\flat \) near such \( e \).

Again by Lemma 7.3.5 when \( x \in \langle \mathcal{w}u \rangle \) is non-zero, we have

\[
\partial \text{Den}_{L^\flat,Y}(x) - \partial \text{Den}_{L^\flat,Y}(\mathcal{w}^{-1}x) = \text{Den}(-q, L^\flat) - \frac{1}{\text{vol}(L^\flat)} \text{Den}((-q)^{-1}, L^\flat),
\]

which vanishes by the functional equation for \( \text{Den}(X, L^\flat) \) evaluated at \( X = -q \), cf. (3.6.2.2). It follows that \( \partial \text{Den}_{L^\flat,Y}(x) = \partial \text{Den}_{L^\flat,Y}(\mathcal{w}^{-1}x) \) when \( x \in \langle \mathcal{w}u \rangle \) is non-zero. Therefore \( \partial \text{Den}_{L^\flat,Y} \) is a constant in \( M(L^\flat) \setminus L_F^\flat \).

Next we consider the case \( e \in L_F^\flat \cap N(L^\flat) \) but \( e \notin L^\flat \). We denote \( \tilde{L}^\flat := \langle e \rangle + L^\flat \). Choose an orthogonal basis \( e_1, \cdots, e_{n-1} \) of the lattice \( L^\flat \) and write \( e = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} \). Up to re-ordering these basis vectors, we may assume that \( \lambda_1 \) attains the minimal valuation among all of the coefficients \( \lambda_i, 1 \leq i \leq n-1 \). Since we are assuming \( e \notin L^\flat \), we have \( \lambda_1 \notin O_F \). Let us denote

\[
\lambda := \lambda_1^{-1} \in O_F.
\]

Then we have

\[
e_1 = \lambda e - \lambda_1^{-1} \lambda_2 e_2 - \cdots - \lambda_1^{-1} \lambda_{n-1} e_{n-1} - \langle \lambda e, e_2, \cdots, e_{n-1} \rangle \in \langle \lambda e, e_2, \cdots, e_{n-1} \rangle.
\]

Fix a basis vector \( e_n \) of the line \( (L_F^\flat)^\perp \). Since all three functions are invariant under \( L^\flat \)-translation, it suffices to consider the behavior of the function:

\[
t \in O_F \longmapsto \partial \text{Den}_{L^\flat}(e + te_n)
\]

when \( t \) is near \( 0 \in O_F \), and the similar functions for \( \partial \text{Den}_{L^\flat,H}, \partial \text{Den}_{L^\flat,Y} \) respectively.

Set \( x_t := e + te_n \) and \( M_t := \langle e_2, e_3, \cdots, e_{n-1}, x_t \rangle \). Then we have

\[
L^\flat + \langle x_t \rangle = \langle e_1, e_2, e_3, \cdots, e_{n-1}, e + te_n \rangle = \langle \lambda te_n, e_2, e_3, \cdots, e_{n-1}, e + te_n \rangle = M_t + \langle \lambda te_n \rangle.
\]

Note that the vector space \( \mathcal{V} \) has an orthogonal basis \( \{ e_2, e_3, \cdots, e_{n-1}, e_1 + \lambda te_n, e'_n \} \), where \( e'_n := e_n - \sigma(t) \mu e_1 \) and \( \mu = \sigma(\lambda) \frac{\langle e_n, e_1 \rangle}{\langle e_2, e_1 \rangle} \in F \), where \( \sigma(t) \) denotes the Galois conjugation of \( t \). A straightforward computation shows that, when \( |t| \) is sufficiently small, with respect to \( \mathcal{V} = \langle e'_n \rangle_F \oplus \langle e'_n \rangle_{L_F^\flat} \), the projection of \( e_n \) to \( \langle e'_n \rangle_F \) lies in \( M_t \), and to the line \( \langle e'_n \rangle_F \) is \( u_t e'_n \) where \( u_t \in O_F^* \) is a unit (since \( u_t \to 1 \) as \( t \to 0 \)). It follows that, when \( |t| \) is sufficiently small,

\[
L^\flat + \langle x_t \rangle = M_t + \langle \lambda te'_n \rangle.
\]

Note that, when \( |t| \) is sufficiently small, the lattice \( M_t \) has the same fundamental invariants as \( \tilde{L}^\flat = \langle e \rangle + L^\flat = \langle e_2, e_3, \cdots, e_{n-1}, e \rangle \), and \( \langle e'_n, e'_n \rangle \) differs from \( \langle e_n, e_n \rangle \) by a unit. Since \( \partial \text{Den}_{L^\flat}(x_t) = \)
\[ \partial \text{Den}(L^b + \langle x_t \rangle) \text{ depends only on the fundamental invariants of the lattice } L^b + \langle x_t \rangle, \text{ we obtain that, when } |t| \text{ is sufficiently small,} \]

\[ (7.3.4.2) \quad \partial \text{Den}_{L^b}(x_t) = \partial \text{Den}_{L^b}(\lambda t e_n). \]

Now by Lemma 7.3.5, the function \( \partial \text{Den}_{L^b} \) has logarithmic singularity near \( e \).

Next we investigate the behavior of \( \partial \text{Den}_{L^b,\mathbb{F}}(x_t) \) when \( t \to 0 \). By (6.1.2.1), we have

\[ (7.3.4.3) \quad \partial \text{Den}_{L^b,\mathbb{F}}(x_t) - \partial \text{Den}_{L^b,\mathbb{F}}(x_t) = \sum_{L^b \subset L^b \cap (L^\prime)^\vee} \sum_{e \notin L^b, t(L^b) \leq 1} \sum_{L^\prime \cap L^b = L^\prime} \mathbb{m}(t(L')). \]

We claim that, when \( |t| \) is sufficiently small, the right hand side is a constant dependent on \( L^b \) and \( e \) but not on \( t \). The outer sum has only finitely many terms. Therefore, to show the claim, it suffices to show that each of the inner sums is a constant dependent on \( L^b \) and \( e \). Now by Lemma 7.3.5, the function \( \partial \text{Den}_{L^b} \) has logarithmic singularity near \( e \).

Then \( L^b \) must be an orthogonal sum \( M \oplus \langle f \rangle \) for some unimodular lattice \( M \) of rank \( n-2 \), and a rank one lattice \( \langle f \rangle \). Denote by \( e^* \) the orthogonal projection of \( e \) to the line \( \langle f \rangle \). Since \( L^b + \langle e \rangle \) is integral and \( e \notin L^b \), we must have \( e - e^* \in M \) and \( f = \xi e^* \) for some \( \xi \in O_F \) but \( \xi \notin O_F^\vee \). Let \( \mathbb{W} = \langle f, e_n \rangle_F \) be the orthogonal complement of \( M_F \). Then the inner sum associated to \( L^b \) is equal to

\[ \sum_{(f, e^* + t e_n) \in L^b \cap L^\prime \cap \mathbb{W}} \mathbb{m}(t(L')) = \partial \text{Den}(\langle f, e^* + t e_n \rangle) - \partial \text{Den}(\langle w^{-1} f, e^* + t e_n \rangle). \]

Now it is easy to see that, when \( |t| \) is sufficiently small, the lattice \( \langle f, e^* + t e_n \rangle = \langle \xi e^*, e^* + t e_n \rangle \) (resp., \( \langle w^{-1} f, e^* + t e_n \rangle = \langle w^{-1} \xi e^*, e^* + t e_n \rangle \)) has the same fundamental invariants as \( \langle \xi t e_n, e^* \rangle \) (resp., \( \langle w^{-1} \xi t e_n, e^* \rangle \)). By Lemma 7.3.5 when \( |t| \) is sufficiently small, the difference

\[ \partial \text{Den}(\langle f, e^* + t e_n \rangle) - \partial \text{Den}(\langle w^{-1} f, e^* + t e_n \rangle) \]

\[ = \partial \text{Den}(\langle \xi t e_n, e^* \rangle) - \partial \text{Den}(\langle w^{-1} \xi t e_n, e^* \rangle) \]

\[ = \partial \text{Den}(\xi t e_n - \partial \text{Den}(\xi t e_n) \]

\[ = \text{Den}(-q, \langle e^* \rangle) \]

is a constant independent of \( t \). This proves the claim. Note that \( \partial \text{Den}_{L^b,\mathbb{F}}(x_t) = \partial \text{Den}_{L^b,\mathbb{F}}(t e_n) \) and it has logarithmic singularity along \( L^b_F = L^b_F \) by Lemma 7.3.5. It follows from the claim that the function \( \partial \text{Den}_{L^b,\mathbb{F}} \) also has logarithmic singularity along \( L^b_F \) near \( e \). Now we have completed the proof of part (a).

By (7.3.4.2) and (7.3.4.3), let \( t \to 0 \) and denote by \( C \) the constant equal to (7.3.4.3):

\[ \partial \text{Den}_{L^b,\mathbb{F}}(x_t) = \partial \text{Den}_{L^b}(x_t) - \partial \text{Den}_{L^b,\mathbb{F}}(x_t) \]

\[ = \partial \text{Den}_{L^b}(\lambda t e_n) - \partial \text{Den}_{L^b,\mathbb{F}}(t e_n) - C \]

\[ = (\partial \text{Den}_{L^b}(\lambda t e_n) - \partial \text{Den}_{L^b}(t e_n)) + \partial \text{Den}_{L^b,\mathbb{F}}(t e_n) - C. \]

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By Lemma 7.3.5, the term \( (\partial \text{Den}_{L^b}(\lambda t e_n) - \partial \text{Den}_{L^b}(t e_n)) \) is a constant dependent on \( L^b \) and \( e \) but not on \( t \). By the previous case (replacing \( L^b \) by \( L^b \)) that we have considered, the term \( \partial \text{Den}_{L^b,\mathcal{F}}(t e_n) \) is a constant when \( t \to 0 \). This shows that \( \partial \text{Den}_{L^b,\mathcal{F}} \) is a constant near \( e \), and we have completed the proof of part (b). \( \square \)

**Lemma 7.3.5.** Assume that \( x \perp L^b \) and \( \text{val}(x) \geq 1 + e_{\text{max}}(L^b) \). Then

\[
\partial \text{Den}_{L^b}(x) - \partial \text{Den}_{L^b}(\mathcal{F}^{-1}) = \text{Den}(-q, L^b),
\]

and

\[
\partial \text{Den}_{L^b,\mathcal{F}}(x) - \partial \text{Den}_{L^b,\mathcal{F}}(\mathcal{F}^{-1}) = \frac{1}{\text{vol}(L^b)} \text{Den}((-q)^{-1}, L^b).
\]

**Proof.** The first part follows from the induction formula in Proposition 3.7.1.

\[
\text{Den}(X, L) = X^2 \text{Den}(X, L') + (1 - X) \text{Den}(-qX, L'),
\]

where

\[
L' = L^b \oplus \langle \mathcal{F}^{-1} \rangle, \quad L = L^b \oplus \langle x \rangle.
\]

Now we consider the second part. By the definition of the function \( \partial \text{Den}_{L^b,\mathcal{F}} \), we obtain

\[
\partial \text{Den}_{L^b,\mathcal{F}}(x) - \partial \text{Den}_{L^b,\mathcal{F}}(\mathcal{F}^{-1}) = \sum_{L^b \subset L' \subset L', t(L^b) \leq 1, L' \cap \langle x \rangle = \langle x \rangle} \text{m}(t(L')),
\]

where we recall that \( L^b = L' \cap L^b_F \), cf. (6.1.2.2). This sum can be rewritten as a double sum, first over all \( L' \) with a given \( L' \cap L^b_F = L^b \) then over all \( L^b \)

\[
(7.3.5.1) \quad \sum_{L^b \subset L'} \sum_{L' \subset L', t(L^b) \leq 1, L' \cap L^b_F = \langle x \rangle} \text{m}(t(L')).
\]

Fix \( L^b \) with \( t(L^b) \leq 1 \) and we consider the inner sum. Since \( t(L^b) \leq 1 \), we may assume that \( L^b \) has an orthogonal basis \( e'_1, \cdots, e'_{n-1} \) such that \( \text{val}(e'_1) = \text{val}(e'_2) = \cdots = \text{val}(e'_{n-2}) = 0 \), and

\[
a'_n := \text{val}(e'_{n-1}).
\]

By Lemmas 7.2.1 and 7.2.2, each lattice \( L' \) in the inner sum is of the form \( L^b + \langle u \rangle \) where \( u \) satisfies

\[
(u, L^b) \subset O_F, \quad \langle u, u \rangle \in O_F.
\]

Write \( u = u^b + u^\perp \) according to the orthogonal direct sum \( \mathbb{V} = L^b_F \oplus (L^b)^\perp \). We claim that \( \text{val}(u^b) \geq 0 \) and \( \text{val}(u^\perp) \geq 1 \).

To prove the claim, we first note that the condition \( (u, L^b) \subset O_F \) above is equivalent to \( u^b \in (L^b)^\perp \). Therefore we may write \( u^b = \lambda_1 e'_1 + \cdots + \lambda_{n-1} e'_{n-1} \) where \( \lambda_i \in O_F \) \( (1 \leq i \leq n - 2) \) and \( \lambda_{n-1} \in \mathcal{F}^{-1} \cap O_F \). By Lemma 7.2.2, we have

\[
(7.3.5.2) \quad \frac{\pi_{\perp}(L')}{L' \cap \langle x \rangle} \sim \frac{\pi_{\perp}(L^b)}{L^b} = \frac{L^b + \langle u^b \rangle}{L^b} \sim \frac{O_F + \lambda_{n-1} O_F}{O_F}.
\]

This isomorphism implies that

\[
(7.3.5.3) \quad \max\{0, -2 \text{val}(\lambda_{n-1})\} = -\text{val}(u^\perp) + \text{val}(x).
\]
Now if $\text{val}(u^\perp) < 0$, from $(u,u) \in O_F$ it follows that $\text{val}(u^\perp) = \text{val}(u^\perp) < 0$ and $\text{val}(u^\perp) = 2\text{val}(\lambda_{n-1}) + a'_{n-1}$. Hence $2\text{val}(\lambda_{n-1}) < -a'_{n-1}$ (in particular, $\text{val}(\lambda_{n-1}) < 0$). It follows from (7.3.5.3) that $\text{val}(x) = a'_{n-1}$, which contradicts $\text{val}(x) > a_{n-1} \geq a'_{n-1}$. Therefore we must have $\text{val}(u^\perp) \geq 0$ and $\text{val}(u^\perp) \geq 0$. It then follows that $\text{val}(\lambda_{n-1}L_{n-1}) \geq 0$, or equivalently $2\text{val}(\lambda_{n-1}) + a'_{n-1} \geq 0$. By (7.3.5.3), we have either $\text{val}(u^\perp) = \text{val}(x) \geq 1$ or

$$\text{val}(u^\perp) = \text{val}(x) + 2\text{val}(\lambda_{n-1}) \geq (1 + a_{n-1}) - a'_{n-1} \geq 1.$$ 

Here the last inequality follows from $e_{\text{max}}(L^b) = a'_{n-1} \leq e_{\text{max}}(L^b) = a_{n-1}$ by (7.1.0.4) applied to $L^b \subset L^\perp$. Now we define $\tilde{L}^\perp := \pi_\lambda(L' = L^\perp + \langle u^\perp \rangle)$. Then $\tilde{L}^\perp$ is an integral lattice. By $\text{val}(u^\perp) \geq 1$, we obtain

$$t(L') = t(\tilde{L}^\perp) + 1.$$ 

Moreover, for a given integral lattice $\tilde{L}^\perp \supset L^\perp$, the set of desired integral lattices $L'$ is bijective to the set of generators of the cyclic $O_F$-module $\tilde{L}^\perp/L^\perp$. Therefore the inner sum in (7.3.5.1) is equal to

$$(7.3.5.4) \quad \sum_{L^\perp \subset \tilde{L}^\perp} \text{m}(t(\tilde{L}^\perp) + 1)[\tilde{L}^\perp : L^\perp] \cdot \left\{ \begin{array}{ll} 1, & \text{if } \tilde{L}^\perp = L^\perp, \\ (1 - q^{-2}), & \text{if } \tilde{L}^\perp \neq L^\perp, \end{array} \right.$$ 

where the index $[\tilde{L}^\perp : L^\perp] = \frac{\text{vol}(\tilde{L}^\perp)}{\text{vol}(L^\perp)}$. For the sum (7.3.5.4), we distinguish three cases.

(1) If $t(\tilde{L}^\perp) = 0$, i.e., $a'_{n-1} = 0$, then the sum is equal to 1.

(2) If $a'_{n-1} > 0$ is odd, then the sum is equal to

$$(1 + q)(1 + (q^2 - 1) + \cdots + (q^{a'_{n-1}-1} - q^{a'_{n-1}-3})) = q^{a'_{n-1}-1}(1 + q).$$

(3) If $a'_{n-1} > 0$ is even, then the sum is equal to

$$(1 + q)(1 + (q^2 - 1) + \cdots + (q^{a'_{n-1}-2} - q^{a'_{n-1}-4})) + (q^{a'_{n-1}} - q^{a'_{n-1}-2}) = q^{a'_{n-1}-1}(1 + q).$$

Therefore the inner sum in (7.3.5.1) is equal to

$$(7.3.5.5) \quad \left\{ \begin{array}{ll} 1, & t(L^\perp) = 0, \\ (1 + q^{-1}) \frac{1}{\text{vol}(L^\perp)}, & t(L^\perp) = 1. \end{array} \right.$$ 

We obtain that (7.3.5.1) is equal to

$$\sum_{L^\perp \subset L^\omega, t(L^\perp) = 0} 1 + \sum_{L^\perp \subset L^\omega, t(L^\perp) = 1} (1 + q^{-1}) \frac{1}{\text{vol}(L^\perp)} = \frac{1}{\text{vol}(L^\perp)} \text{Den}((-q)^{-1}, L^\perp),$$

by (3.6.1.2), and hence

$$\partial\text{Den}_{L^\perp, \mathcal{H}}(x) - \partial\text{Den}_{L^\perp, \mathcal{H}}((w^{-1}x) = \frac{1}{\text{vol}(L^\perp)} \text{Den}((-q)^{-1}, L^\perp).$$

This completes the proof. □
We introduce an auxiliary function on $\mathbb{V} \setminus L_F^b$,

$$\widetilde{\partial \text{Den}}_{L^b}(x) = \sum_{L^b \subseteq L' \subseteq L^{\vee}} 1_{L'}(x).$$

Similar to Proposition 7.3.4, we have:

**Lemma 7.3.6.** The function $\widetilde{\partial \text{Den}}_{L^b}$ lies in $L^1_{\mathbb{V}}$, having logarithmic singularity along $L_F^b$.

**Proof.** It suffices to show the logarithmic singularity near $0 \in \mathbb{V}$. The behavior of $\widetilde{\partial \text{Den}}_{L^b}$ near an arbitrary $e \in \mathbb{V}$ is then reduced to this case by the same argument as the proof of part (a) in Proposition 7.3.4 for $\partial \text{Den}_{L^b}$. More precisely, the equality (7.3.4.2) also holds for the function $\widetilde{\partial \text{Den}}_{L^b}$, since $\partial \text{Den}_{L^b}(x)$ depends only on the fundamental invariants of the lattice $L^b + \langle x \rangle$.

Note that the function $\widetilde{\partial \text{Den}}_{L^b}$ is invariant under $L^b$-translation. It suffices to show that, when $x \perp L_F^b$ and $\text{val}(x) > 2e_{\max}(L^b)$,

$$\widetilde{\partial \text{Den}}_{L^b}(x) = C_0 \text{val}(x) + C_1$$

for some constants $C_0, C_1$. Write the function as a double sum:

$$\widetilde{\partial \text{Den}}_{L^b}(x) = \sum_{L^b \subseteq L^\perp \subseteq (L^b)^\vee} \sum_{\nu \subseteq L^{\vee}} \sum_{L' \subseteq L^b} 1_{L'}(x).$$

Since the outer sum has only finitely many terms, it suffices to prove the desired logarithmic singularity for the inner sum associated to each integral lattice $L^b \supset L^b$. Fix such an $L^b$. It suffices to show that, when $x \perp L_F^b$ and $\text{val}(x) > 2e_{\max}(L^b)$ (we remind the reader that $e_{\max}(L^b) \leq e_{\max}(L^b)$ by (7.1.0.4)), the cardinality

(7.3.6.1) $$\#\{L' \mid L' \subset L^{\vee}, L' \cap L_F^b = L^b, L' \cap (L_F^b)^\perp = \langle x \rangle\}$$

is independent of $x$.

Following the proof of Lemma 7.3.5, each lattice $L'$ in the above set is of the form $L^b + \langle u \rangle$ where

$$(u, L^b) \subset O_F, \quad (u, u) \in O_F.$$ 

Write $u = u^b + u^\perp$ according to the orthogonal direct sum $\mathbb{V} = L_F^b \oplus (L_F^b)^\perp$. We **claim** that $\text{val}(u^\perp) \geq 1$. In fact, by $(u, L^b) \subset O_F$, we obtain $u^b \in (L^b)^\vee$, and hence $\text{length}_{O_F} \frac{L^b + \langle u^b \rangle}{L^b} \leq e_{\max}(L^b)$. Comparing the lengths of the $O_F$-modules in (7.3.5.2), we obtain

$$-\text{val}(u^\perp) + \text{val}(x) = 2 \text{length}_{O_F} \frac{L^b + \langle u^b \rangle}{L^b} \leq 2e_{\max}(L^b).$$

The claim follows.

It follows that the cardinality (7.3.6.1) is given by (7.3.5.4) without the weight factor $m(t(L^b)+1)$, hence independent of $x$. This completes the proof. □

By Proposition 7.3.4, the functions $\partial \text{Den}_{L^b}, \partial \text{Den}_{L^b, \mathcal{N}}$ and $\partial \text{Den}_{L^b, \mathcal{V}}$ are all in $L^1(\mathbb{V})$, hence Fourier transforms exist for all of them.

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Corollary 7.3.7. The Fourier transforms of $\partial \text{Den}_{L^\parallel}$ and $\partial \text{Den}_{L^\parallel, \mathcal{H}}$ are given by (point-wisely) absolutely convergent sums

\begin{equation}
\widehat{\partial \text{Den}}_{L^\parallel, \mathcal{H}}(x) = \sum_{L^\parallel \subset L' \subset L_{\mathcal{H}}, \ t(L') \leq 1} \text{vol}(L') \text{m}(t(L')) 1_{L_{\mathcal{H}}}(x),
\end{equation}

and

\begin{equation}
\widehat{\partial \text{Den}}_{L^\parallel}(x) = \sum_{L^\parallel \subset L' \subset L_{\mathcal{H}}} \text{vol}(L') \text{m}(t(L')) 1_{L_{\mathcal{H}}}(x),
\end{equation}

where $x \in \mathcal{V}$ in both equalities.

Proof. By Proposition 7.3.4 and Lemma 7.3.6, the two functions $\widehat{\partial \text{Den}}_{L^\parallel}$ and $\widehat{\partial \text{Den}}_{L^\parallel, \mathcal{H}}$ are $L^1$ and given by a sum of point-wisely non-negative functions. Since $|\text{m}(t(L'))|$ is bounded in the sum defining $\partial \text{Den}_{L^\parallel}$, the assertion follows from the dominated convergence theorem. □

7.4. Fourier transform of $\partial \text{Den}_{L^\parallel, \mathcal{F}}$.

Theorem 7.4.1. Assume that $x \perp L^\parallel$ and $\text{val}(x) < 0$. Then

\[ \widehat{\partial \text{Den}}_{L^\parallel, \mathcal{F}}(x) = 0. \]

Proof. This follows from Lemma 7.4.2 below, and the functional equation \[(3.6.2.2)\]

\[ \text{Den}(-q, L^\parallel + \langle u^b \rangle) = \frac{1}{\text{vol}(L^\parallel + \langle u^b \rangle)} \text{Den}((-q)^{-1}, L^b + \langle u^b \rangle). \]

Lemma 7.4.2. Assume that $x \perp L^b$ and $\text{val}(x) < 0$. Then

\[ \widehat{\partial \text{Den}}_{L^\parallel}(x) = (1 - q^{-2})^{-1} \text{vol}(\langle x \rangle^\vee) \int_{L^\parallel_F} \text{Den}(-q, L^b + \langle u^b \rangle) \, du^b, \]

and

\[ \widehat{\partial \text{Den}}_{L^\parallel, \mathcal{F}}(x) = (1 - q^{-2})^{-1} \text{vol}(\langle x \rangle^\vee) \int_{L^\parallel_F} \frac{1}{\text{vol}(L^\parallel + \langle u^b \rangle)} \text{Den}((-q)^{-1}, L^b + \langle u^b \rangle) \, du^b. \]

Recall that $\langle x \rangle^\vee$ denotes the dual lattice of $\langle x \rangle$ in the line $\langle x \rangle_F$. Here we use the self-dual measures on $L^\parallel_F$ and on $\langle x \rangle_F$ respectively, cf. §1.7.

Proof. First we consider the Fourier transform of $\partial \text{Den}_{L^\parallel}$. By (7.3.7.2), it is equal to the (point-wisely) absolutely convergent sum

\[ \widehat{\partial \text{Den}}_{L^\parallel}(x) = \sum_{L^\parallel \subset L' \subset L_{\mathcal{H}}, x \in L_{\mathcal{H}}'} \text{vol}(L') \text{m}(t(L')). \]

For an integral lattice $L^b \supset L^\parallel$, define

\begin{equation}
\Sigma(L^b, x) = \{ L' \subset \mathcal{V} | x \in L_{\mathcal{H}}', L' \subset L_{\mathcal{H}}', L^b = L' \cap L_F^b \}. \tag{7.4.2.1}
\end{equation}

Then

\begin{equation}
\widehat{\partial \text{Den}}_{L^\parallel}(x) = \sum_{L^\parallel \subset L^b \subset L_{\mathcal{H}}'} \sum_{L' \in \Sigma(L^b, x)} \text{vol}(L') \text{m}(t(L')). \tag{7.4.2.2}
\end{equation}
By Lemmas 7.2.1 and Lemma 7.2.2, we have a bijection

$$\text{(7.4.2.3)} \quad \frac{\left\{ \mu \in \langle x \rangle \setminus \mu \} \setminus (L^\flat)}{O_F^\times} \xrightarrow{\sim} \Sigma(L^\flat, x)$$

Thus we have

$$(\text{corresponding to } \langle u, u \rangle \text{ of the norm } (u \lvert u) \text{ lattice. By val}(u \lvert u) \text{ factor)}$$

The second factor $$(\text{is an integral } (u \lvert u) \text{ of the norm } (u \lvert u) \text{ lattice. By val}(u \lvert u) \text{ factor).}$$

Here, though $$(\text{is not necessarily a lattice, it is invariant under } L^\flat \text{ and } O_F^\times \text{ factor.)}$$

Hence the quotient on the left hand side makes sense.

Now we follow the same argument as in the proof of Lemma 7.3.5. Write $u = u^b + u^\perp$ according to the orthogonal direct sum $V = L'_F \oplus (L'_F)^\perp$. Then the condition $x \in L^\vee$ is equivalent to the projection $\pi_\perp(L') \subset \langle x \rangle$ (inside the line $(L'_F)^\perp = \langle x \rangle_F$), or equivalently, $(x, u^\perp) \in O_F$. Since $\text{val}(x) < 0$, we must have $\text{val}(u^\perp) > 0$ (due to $2 \text{val}((x, u^\perp)) = \text{val}(x) + \text{val}(u^\perp)$). It follows from the integrality of the norm $(u, u)$ and $(u^\perp, u^\perp)$ that $u^b$ also has integral norm and hence $u^b \in (L^\flat)^{\vee, o}$. Thus we have

$$\langle x \rangle + L^\flat \leftrightarrow (L^\flat)^{\vee, o} \oplus \langle x \rangle$$

and a bijection with the left hand side of (7.4.2.3)

$$\left(\frac{(x \vee \{0\})}{O_F^\times} \xrightarrow{\sim} \left(\frac{(\langle x \rangle \setminus \{0\})}{O_F^\times} \right) \right) \xrightarrow{\sim} \Sigma(L^\flat, x).$$

The second factor $$(\text{can be further identified with the set of lattices contained in } \langle x \rangle \text{ (corresponding to } \langle u \rangle = \pi_\perp(L')) \text{.) We write } \tilde{L}^\flat := \pi_\perp(L') = L^\flat + \langle u \rangle. \text{ Then } \tilde{L}^\flat \text{ is an integral lattice. By } \text{val}(u^\perp) \geq 1, \text{ we obtain }$$

$$t(L') = t(\tilde{L}^\flat) + 1,$$

and by [7.2.2.1],

$$\text{vol}(L') = \text{vol}(\tilde{L}^\flat) \text{ vol}(\pi_\perp(L')).$$

Therefore the inner sum in (7.4.2.2) is equal to

$$\text{vol}(L^\flat) \sum_{u^b \in (L^\flat)^{\vee, o} / L^\flat} \text{vol}(N)$$

$$= \text{vol}(L^\flat) \text{vol}(\langle x \rangle) \left( \sum_{i \geq 0} q^{-2i} \sum_{u^b \in (L^\flat)^{\vee, o} / L^\flat} \text{vol}(N) \right)$$

$$= \text{vol}(L^\flat) \text{vol}(\langle x \rangle) \left( 1 + \sum_{i \geq 0} q^{-2i} \right)$$

$$= \text{vol}(L^\flat) \text{vol}(\langle x \rangle) \left( 1 - q^{-2} \right)^{-1} \sum_{u^b \in (L^\flat)^{\vee, o} / L^\flat} \text{vol}(N).$$

The bijection depends on the choice of a basis vector of $\langle x \rangle$, and here we have simply chosen $x$. 


We now return to the sum (7.4.2.2), which is now equal to
\[ \bar{\text{Den}}_{L^b}(x) = \text{vol}(\langle x \rangle^\vee) \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(L^b)(1 - q^{-2})^{-1} \sum_{u^b \in \frac{(L^b)^\vee}{L^b}} \text{m}(t(\tilde{L}^b) + 1). \]

For a given integral lattice \( \tilde{L}^b \) such that \( \tilde{L}^b / L^b \) is a cyclic \( O_F \)-module, the number of \( u^b \in \frac{(L^b)^\vee}{L^b} \) such that \( L^b + \langle u^b \rangle = \tilde{L}^b \) is
\[
\left\{ \begin{array}{ll}
[\tilde{L}^b : L^b](1 - q^{-2}) = \frac{\text{vol}(\tilde{L}^b)}{\text{vol}(L^b)}(1 - q^{-2}), & \text{if } \tilde{L}^b \neq L^b, \\
1, & \text{if } \tilde{L}^b = L^b.
\end{array} \right.
\]

We thus obtain
\[
\bar{\text{Den}}_{L^b}(x) = \text{vol}(\langle x \rangle^\vee) \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(L^b) \sum_{L^b \subset \tilde{L}^b, \tilde{L}^b / L^b \text{ cyclic}} \frac{\text{vol}(\tilde{L}^b)}{\text{vol}(L^b)} \text{m}(t(\tilde{L}^b) + 1)
+ q^{-2}(1 - q^{-2})^{-1} \text{vol}(\langle x \rangle^\vee) \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(L^b) \text{m}(t(L^b) + 1).
\]

Here we split the contribution of the factor corresponding to \( \tilde{L}^b = L^b \) into two pieces \( q^{-2} + (1 - q^{-2}) \).
Interchanging the sum over \( L^b \) and \( \tilde{L}^b \), we obtain
\[(7.4.2.5) \quad \bar{\text{Den}}_{L^b}(x) = \text{vol}(\langle x \rangle^\vee) \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(\tilde{L}^b) \text{m}(t(\tilde{L}^b) + 1) \sum_{L^b \subset L^b \subset (L^b)^\vee} \frac{1}{\text{vol}(\tilde{L}^b)} \text{vol}(L^b) \text{m}(t(L^b) + 1)
+ q^{-2}(1 - q^{-2})^{-1} \text{vol}(\langle x \rangle^\vee) \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(L^b) \text{m}(t(L^b) + 1).
\]

Next we consider the integral
\[
\int_{L_F^b} \text{Den}(-q, L^b + \langle u^b \rangle) du^b.
\]
This can be written as a weighted sum over integral lattices \( M \subset L_F^b \) such that \( L^b \subset M \) and \( M/L^b \) is a cyclic \( O_F \)-module, with the weight factor
\[
\left\{ \begin{array}{ll}
\text{vol}(M)(1 - q^{-2}), & \text{if } M 
eq L^b, \\
\text{vol}(L^b), & \text{if } M = L^b.
\end{array} \right.
\]

Therefore we obtain
\[(7.4.2.6) \quad \int_{L_F^b} \text{Den}(-q, L^b + \langle u^b \rangle) du^b = q^{-2} \text{vol}(L^b) \text{Den}(-q, L^b)
+ (1 - q^{-2}) \sum_{L^b \subset M \subset (L^b)^\vee, M/L^b \text{ cyclic}} \text{vol}(M) \text{Den}(-q, M).
\]

Again here we split the contribution of the factor corresponding to \( M = L^b \) into two pieces \( q^{-2} + (1 - q^{-2}) \). By the formula (4.6.2.1), the first term is equal to
\[(7.4.2.7) \quad q^{-2} \text{vol}(L^b) \text{Den}(-q, L^b) = q^{-2} \sum_{L^b \subset L^b \subset (L^b)^\vee} \text{vol}(L^b) \text{m}(t(L^b) + 1).
\]

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Again by (3.6.2.1), the second term in (7.4.2.6) is equal to
\[
\sum_{L^\flat \subset M \subset M^\flat, M/L^\flat \text{ cyclic}} \text{vol}(M) \text{Den}(-q, M) = \sum_{L^\flat \subset M \subset L^\flat \in (L^\flat)^\flat, M/L^\flat \text{ cyclic}} \text{vol}(M) \frac{\text{vol}(L^\flat)}{\text{vol}(M)} m(t(L^\flat) + 1)
\]
\[
= \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} \text{vol}(L^\flat) m(t(L^\flat) + 1) \cdot \#\{M \mid L^\flat \subset M \subset L^\flat, M/L^\flat \text{ cyclic}\}.
\]

Now note that we have an equality
\[
\#\{M \mid L^\flat \subset M \subset L^\flat, M/L^\flat \text{ cyclic}\} = \#\{M \mid L^\flat \subset M \subset L^\flat, L^\flat/M \text{ cyclic}\}.
\]
In fact, the right hand side is the same as
\[
\#\{M^\flat \mid L^\flat \in (L^\flat)^\flat \subset M^\flat \subset L^\flat, M^\flat/L^\flat \text{ cyclic}\}.
\]
and this is equal to the left hand side, due to the (non-canonical) isomorphism of finite $O_F$-modules
\[
L^\flat / L^\flat \sim (L^\flat)^\flat / (L^\flat)^\flat.
\]
It follows that
\[
(7.4.2.8) \quad \sum_{L^\flat \subset M \subset L^\flat \in (L^\flat)^\flat, M/L^\flat \text{ cyclic}} \text{vol}(M) \text{Den}(-q, M) = \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} \text{vol}(L^\flat) m(t(L^\flat) + 1) \cdot \#\{M \mid L^\flat \subset M \subset L^\flat, L^\flat/M \text{ cyclic}\}.
\]
By (7.4.2.6), (7.4.2.7) and (7.4.2.8), we obtain
\[
(7.4.2.9) \quad \int_{L^\flat_F} \text{Den}(-q, L^\flat + \langle u^\flat \rangle) du^\flat = (1 - q^{-2}) \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} \text{vol}(L^\flat) m(t(L^\flat) + 1) \cdot \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} 1
\]
\[
+ q^{-2} \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} \text{vol}(L^\flat) m(t(L^\flat) + 1).
\]
Comparing (7.4.2.9) with (7.4.2.5) we obtain
\[
\hat{\partial} \text{Den}_{L^\flat}(x) = (1 - q^{-2})^{-1} \text{vol}(\langle x \rangle^\flat) \int_{L^\flat_F} \text{Den}(-q, L^\flat + \langle u^\flat \rangle) du^\flat,
\]
and this completes the proof of the first part concerning $\hat{\partial} \text{Den}_{L^\flat}$.
Similarly, let us consider the horizontal part $\partial \text{Den}_{L^\flat \in H}$. By (7.3.7.1), we have a (point-wisely) absolutely convergent sum
\[
(7.4.2.10) \quad \hat{\partial} \text{Den}_{L^\flat \in H}(x) = \sum_{L^\flat \subset L^\flat \in (L^\flat)^\flat} \sum_{L^\flat \in (L^\flat)^\flat, x} m(t(L^\flat)) \text{vol}(L^\flat).
\]
Then \( \phi \in C_c^\infty(\mathbb{V}) \) satisfies

\[
\begin{align*}
\text{supp}(\phi) & \subset \mathbb{V}^0, \quad \text{and} \\
\text{supp}(\hat{\phi}) & \subset \mathbb{V}^0.
\end{align*}
\]

\( \text{Then } \phi = 0. \)
Proof. If dim $V$ is odd, we consider the “doubling” quadratic space $V \oplus V$ and the function $\phi \otimes \phi \in C_c^\infty(V \oplus V)$. It is easy to see that $\text{supp}(\phi \otimes \phi) = \text{supp}(\phi) \times \text{supp}(\phi)$, $\hat{\phi} \otimes \hat{\phi} = \hat{\phi} \otimes \hat{\phi}$, $V^c \times V^c \subset (V \oplus V)^c$, and $V^\infty \times V^\infty \subset (V \oplus V)^\infty$. Therefore it suffices to consider the case when dim $V$ is even which we assume from now on. We use the Weil representation $\omega$ of SL$_2(F)$. The group SL$_2(F)$ acts on $C_c^\infty(V)$ by the following formula

$$\omega \begin{pmatrix} a & b \\ c & d \end{pmatrix} \phi(x) = \chi_V(a)|a|^{d/2} \phi(ax),$$

(8.1.2.1)

$$\omega \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \phi(x) = \psi \left( \frac{1}{2} b(x,x) \right) \phi(x),$$

$$\omega \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \phi(x) = \gamma_V \hat{\phi}(x),$$

where $\chi_V$ is a quadratic character of $F^\times$ associated to the quadratic space $V$, and $\gamma_V$ is the Weil constant.

By the assumption on the support, the functions $\phi$ and $\hat{\phi}$ are fixed by $N(\varpi^{-1}O_F)$ and $N(O_F)$ respectively, where $N$ denotes the unipotent subgroup of the standard Borel of SL$_2$ of upper triangular matrices. Therefore $\phi$ is fixed by $N(\varpi^{-1}O_F)$ and $N_{-}(O_{F})$ (the transpose of $N(O_{F})$). However, $N(\varpi^{-1}O_{F})$ and $N_{-}(O_{F})$ generate SL$_2(F)$. It follows that $\phi$ is fixed by SL$_2(F)$ and therefore $\text{supp}(\phi)$ is contained in the null cone $\{x \in V : (x,x) = 0\}$ (e.g., by using the invariance under the diagonal torus, or $N(F)$). Since $\phi$ is locally constant, it must vanish identically. \qed

**Remark 8.1.3.** The uncertainty principle is also used in the new proof by Beuzart-Plessis [BP19] of the Jacquet–Rallis fundamental lemma.

**Corollary 8.1.4.** Let $\phi \in C_c^\infty(V)$ satisfy

- $\text{supp}(\phi) \subset V^\infty$, and
- $\hat{\phi}$ is a multiple of $\phi$.

Then $\phi = 0$.

8.1.5. Hermitian case. Now we return to the case of hermitian space with respect to a (possibly ramified) quadratic extension $F/F_0$ where $F_0$ is non-archimedean local field with characteristic not equal to 2. Define $V^c$ and $V^\infty$.

$$V^c = \{x \in V \mid \text{val}(x) \geq 0\}, \quad V^\infty = \{x \in V \mid \text{val}(x) > 0\},$$

(8.1.5.1)

In fact, let us show that $N(\varpi^{-1}O_{F_0})$ and $N_{-}(O_{F_0})$ generate SL$_2(F)$. Using the following identity in SL$_2(F)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/c \\ c & 1 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}, \quad c \neq 0,$$

it is easy to show that the group SL$_2(F)$ is generated by $N(F)$ and any single element in SL$_2(F) \setminus B(F)$. Now we first apply the above equality to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ \varpi & 1 \end{pmatrix}$) to generate $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 1 & -1/\varpi \\ \varpi & 1 \end{pmatrix}$). Then we note that $\begin{pmatrix} 1 & -1 \\ \varpi & -1/\varpi \end{pmatrix}$ and this element together with $N(\varpi^{-1}O_{F_0})$ generate $N(F)$. 45
where we recall that $\text{val}(x) = \text{val}((x, x))$ for the hermitian form $(\ ,\ )$ on $V$, cf. §1.7

**Proposition 8.1.6.** Let $\phi \in C_c^\infty(V)$ satisfy

- $\text{supp}(\phi) \subset V^{oo}$, and
- $\text{supp}(\hat{\phi}) \subset V^o$.

Then $\phi = 0$.

**Proof.** Consider the symmetric bilinear form on the underlying $F_0$-vector space of $V$,

$$(x, y)_{F_0} := \text{tr}_{F/F_0}(x, y) \in F_0, \quad x, y \in V.$$ 

Then the Fourier transform on $C_c^\infty(V)$ defined in §1.7 using the hermitian form $(\ ,\ )$ on $V$ is the same as the one in (8.1.1.2) using $(\ ,\ )_{F_0}$ on the underlying $F_0$-vector space of $V$. Since $(x, x)_{F_0}/2 = (x, x)$ for any $x \in V$, the cones defined by (8.1.1.1) and (8.1.5.1) coincide. Therefore the desired assertion follows from Proposition 8.1.2. \qed

The uncertainty principle implies that, by Lemma 6.3.1, the function $\text{Int}_{L^*,Y}$ is determined by its restriction to

$$V^o \setminus V^{oo} = \{x \in V \mid \text{val}(x) = 0\}.$$ 

Ideally one would like to prove the same conclusion as Lemma 6.3.1 holds for the function $\partial\text{Den}_{L^*,Y}$. Then, by induction on $\dim V$, we can prove the main Theorem 3.4.1. However, we have not succeeded finding a direct proof the analog of Lemma 6.3.1 for $\partial\text{Den}_{L^*,Y}$. Nevertheless, a weaker version of the uncertainty principle suffices to prove the identity $\text{Int}_{L^*,Y} = \partial\text{Den}_{L^*,Y}$ and this is what we will actually do in the next subsection. A posteriori we can deduce that the function $\partial\text{Den}_{L^*,Y}$ also satisfies the same identity as $\text{Int}_{L^*,Y}$ does in Lemma 6.3.1.

**8.2. The proof of Theorem 3.4.1** We now prove the main Theorem 3.4.1. Fix a rank $n - 1$ lattice $L^\flat \subset V$ such that $L^\flat_{F}$ is non-degenerate. We want to prove an identity of functions on $V \setminus L^\flat_F$.

$$\text{Int}_{L^\flat} = \partial\text{Den}_{L^\flat}.$$ 

By Theorem 6.1.3, equivalently we need to show

**Theorem 8.2.1.** Let $L^\flat \subset V$ be a rank $n - 1$ lattice such that $L^\flat_F$ is non-degenerate. Then

$$(8.2.1.1) \quad \text{Int}_{L^\flat, Y} = \partial\text{Den}_{L^\flat, Y}$$

as elements in $C_c^\infty(V)$.

**Proof.** We prove the assertion by induction on $\text{val}(L^\flat)$. Let $(a_1, a_2, \ldots, a_{n-1})$ be the fundamental invariants of the lattice $L^\flat$, cf. §7.1. Let $M = M(L^\flat) = L^\flat \oplus \langle u \rangle$ be the lattice defined by (7.1.0.3).

**Lemma 8.2.2.** Let $x \in V \setminus L^\flat_F$ and let $(a'_1, a'_2, \ldots, a'_n)$ be the fundamental invariants of the lattice $L^\flat + \langle x \rangle$. Then the inequality

$$(8.2.2.1) \quad a'_1 + \cdots + a'_{n-1} \geq a_1 + \cdots + a_{n-1}$$

holds if and only if $x \in M$. 46
Proof. For $x \in M$, we write $x = x^b + x^\perp$ where $x^b \in L^b$ and $x^\perp \perp L^b$. Then $L^b + \langle x \rangle = L^b + \langle x^\perp \rangle$. Therefore we may assume that $x \perp L^b$. It follows that $\text{val}(x) \geq a_{n-1}$ by the definition of the lattice $M$, and $a_i^\prime = a_i$ for $1 \leq i \leq n-1$. Hence $a_1^\prime + \cdots + a_{n-1}^\prime = a_1 + \cdots + a_{n-1}$, and the equation (8.2.2.1) holds.

We now assume that the inequality (8.2.2.1) holds. We start with a special case. If $x \perp L^b$, the fundamental invariants of the lattice $L^b + \langle x \rangle$ is an re-ordering of $(a_1, a_2, \cdots, a_{n-1}, \text{val}(x))$. From the inequality (8.2.2.1), it follows that $\text{val}(x) \geq a_{n-1}$, and therefore $x \in M$.

Now we consider the general case. Let $\{e_1, \cdots, e_{n-1}\}$ be an orthogonal basis of $L^b$ such that $(e_i, e_i) = \varpi^{a_i}$. Write

$$x = \lambda_1 e_1 + \cdots + \lambda_{n-1} e_{n-1} + x^\perp,$$

where $\lambda_i \in F, 1 \leq i \leq n-1$ and $x^\perp \perp L^b$. The fundamental matrix of the basis $\{e_1, \cdots, e_{n-1}, x\}$ of $L^b + \langle x \rangle$ is of the form

$$T = \begin{pmatrix}
\varpi^{a_1} & (e_1, x) \\
\cdots & \vdots \\
\varpi^{a_{n-1}} & (e_{n-1}, x) \\
(x, e_1) & \cdots & (x, e_{n-1}) & (x, x)
\end{pmatrix}.$$

We now use the characterization of the sum $a_1^\prime + \cdots + a_{n-1}^\prime$ as the minimum among the valuations of the determinants of all $(n-1) \times (n-1)$-minors of $T$. The set of such minors is bijective to the set of $(i, j)$-th entry: removing $i$-th row and $j$-th column to get such a minor. The valuation of the determinant of the $(n, i)$-th minor is

$$\text{val}((e_i, x)) - a_i + (a_1 + \cdots + a_{n-1}).$$

From the inequality (8.2.2.1), it follows that

$$\text{val}((e_i, x)) \geq a_i,$$

or equivalently $\lambda_i \in O_F$, for all $1 \leq i \leq n-1$. Therefore $x - x^\perp \in L^b$, and $L^b + \langle x \rangle = L^b + \langle x^\perp \rangle$. Now we can assume that $x \perp L^b$ and by the special case above we complete the proof.

Now we assume that the equation (8.2.1.1)

$$\text{Int}_{L^b, \nu} = \partial \text{Den}_{L^b, \nu}$$

holds for $L^b$ such that $\text{val}(L^b) < \text{val}(L^b)$. We may further assume that $L^b + \langle x \rangle$ is integral and has a basis $(e_1', e_2', \cdots, e_n')$ such that $\text{val}(e_i') = a_i'. Let L^b = \langle e_1', \cdots, e_{n-1}' \rangle$. Then we have

$$\text{Int}_{L^b, \nu}(x) = \text{Int}_{L^b, \nu}(x'), \text{ and } \partial \text{Den}_{L^b, \nu}(x) = \partial \text{Den}_{L^b, \nu}(x'),$$

where $x' = e_n'$. By Lemma 8.2.2 if $x \notin M$, then we have a strict inequality

$$a_1' + \cdots + a_{n-1}' < a_1 + \cdots + a_{n-1}.$$ 

And so $\text{val}(L^b) < \text{val}(L^b)$. By induction hypothesis, we have

$$\text{Int}_{L^b, \nu}(x') = \partial \text{Den}_{L^b, \nu}(x').$$
It follows that the support of the difference
\[ \phi = \text{Int}_{L^\flat, V} - \partial \text{Den}_{L^\flat, V} \in C_c^\infty(V) \]
is contained in the lattice \( M \).

By Corollary 6.3.3 we know
\[ \hat{\text{Int}}_{L^\flat, V}(x) = - \hat{\text{Int}}_{L^\flat, V}(x). \]
We know a little less about \( \partial \text{Den}_{L^\flat, V} \): by Theorem 7.4.1 the same holds for \( x \perp L^\flat \) such that \( \text{val}(x) < 0 \). In particular, for \( x \perp L^\flat \) such that \( \text{val}(x) < 0 \),
\[ \hat{\phi}(x) = 0. \]

Obviously the function \( \phi \) is invariant under \( L^\flat \). By the constraints imposed by the support of \( \phi \) (being contained in \( M \)), it is of the form
\[ \phi = \mathbf{1}_{L^\flat} \otimes \phi_\perp, \]
where \( \phi_\perp \in C_c^\infty((L_F^b)^\perp) \) is supported on the (rank one) lattice \( M_\perp = \langle u \rangle \). Then
\[ \hat{\phi} = \text{vol}(L^b) \mathbf{1}_{L^\flat, V} \otimes \hat{\phi}_\perp. \]
Here \( \hat{\phi}_\perp \) is invariant under the translation by the dual lattice \( M_\perp^\vee = \langle u^\vee \rangle \), where \( u^\vee = \varpi^{-a_n} u \).
Note that \( \text{val}(u^\vee) = -a_n < 0 \). Now the Fourier transform \( \hat{\phi}_\perp \) vanishes at every \( x \perp L^\flat \) such that \( \text{val}(x) < 0 \). It follows that \( \hat{\phi}_\perp \) vanishes identically. Therefore \( \phi = 0 \). This completes the proof. \( \square \)

**Part 2. Local Kudla–Rapoport conjecture: the almost self-dual case**

9. LOCAL DENSITY FOR AN ALMOST SELF-DUAL LATTICE

9.1. **Local density for an almost self-dual lattice.** In this section we allow \( F_0 \) to be a non-archimedean local field of characteristic not equal to 2 (but possibly with residue characteristic 2), and \( F \) an unramified quadratic extension.

Recall that we have defined the local density for two hermitian \( O_F \)-lattices \( L \) and \( M \)
\[ \text{Den}(M, L) = \lim_{N \to +\infty} \frac{\# \text{Rep}_{M, L}(O_F/\varpi^N)}{N^{\dim(\text{Rep}_{M, L}F_0)}} \]
in terms of the scheme \( \text{Rep}_{M, L} \), cf. \( (3.1.0.1) \) in Section 3.1.

Let \( L \) be a hermitian \( O_F \)-lattice of rank \( n \). For \( k \geq 0 \), set
\[ M = \langle 1 \rangle^{n-1+k} \oplus \langle \varpi \rangle, \quad \widetilde{M} = \langle 1 \rangle^{n+1+k}, \]
and
\[ L^\flat = L \oplus \ell, \quad \ell = \langle u_0 \rangle, \quad (u_0, u_0) = \varpi. \]

We then have the following “cancellation law”.

**Lemma 9.1.1.** Let \( k \geq 0 \). Then
\[ \text{Den}(M, L) = \frac{\text{Den}(\widetilde{M}, L^\flat)}{\text{Den}(M, \ell)}. \]
Proof. For any hermitian $O_F$-lattice $L$, we denote
\[ L_i = L \otimes_{O_F} O_F/\varpi^i, \]
endowed with the reduction of the hermitian form.

Then the restriction to $\ell_i$ defines a map
\[ \text{Res}: \text{Herm}(L_i^\sharp, \widetilde{M}_i) \longrightarrow \text{Herm}(\ell_i, \widetilde{M}_i). \]
\[ \varphi \longmapsto \varphi|_{\ell_i}. \]

Let $\varphi \in \text{Herm}(L_i^\sharp, \widetilde{M}_i)$. Denote by $\varphi(\ell_i)\perp$ the orthogonal complement in $\widetilde{M}_i$ of the image $\varphi(\ell_i)$, i.e.,
\[ \varphi(\ell_i)\perp = \{ x \in \widetilde{M}_i \mid (x, \varphi(\ell_i)) = 0 \}. \]

Now let $i \geq 2$. We claim that there is an isomorphism of hermitian modules over $O_F/\varpi^i$:
\[ \varphi(\ell_i)\perp \cong M_i. \]

Since the norm of $u_0$ has valuation one, so is its image $w_0 := \varphi(u_0) \in \widetilde{M}_i$ (this makes sense when $i \geq 2$). Hence $w_0 \notin \varpi \widetilde{M}_i$, i.e., $w_0 \mod \varpi \neq 0 \in \widetilde{M}_i = M_i \otimes_{O_F/\varpi} O_F/\varpi$. By the non-degeneracy of the hermitian form on the reduction $\widetilde{M}_i$, the map
\[ \widetilde{M}_i \longrightarrow O_F/\varpi^i \]
\[ x \longmapsto (x, w_0) \]
is surjective, and its kernel is $\varphi(\ell_i)\perp$ by definition. The kernel is a free module over $O_F/\varpi^i$ (since it must be flat, being the kernel of a surjective morphism between finite free modules; alternatively, look at the reduction mod $\varpi$ and apply Nakayama’s lemma).

Now there exists $w'_0 \in \widetilde{M}_i$ such that $(w'_0, w_0) = 1$. Then $\{w_0, w'_0\}$ span a self-dual submodule of rank two, which must be an orthogonal direct summand of $\widetilde{M}_i$, again by non-degeneracy of the hermitian form on $\widetilde{M}_i$. This reduces the assertion $\varphi(\ell_i)\perp \cong M_i$ to the case $\text{rank} \widetilde{M}_i = 2$. In the rank two case, it is easy to verify the desired isomorphism, e.g., using the basis $\{w_0, w'_0\}$. This proves the claim.

Note that the fiber of the map Res above $\varphi|_{\ell_i}$ is the set $\text{Herm}(L_i, \varphi(\ell_i)\perp)$ (and $\varphi(\ell_i)\perp$ depends only on the restriction $\varphi|_{\ell_i}$). It follows from the claim that the fiber has a constant cardinality (in particular, the map Res is surjective), namely that of $\text{Herm}(L_i, M_i)$. Hence,
\[ \#\text{Herm}(L_i^\sharp, \widetilde{M}_i) = \#\text{Herm}(L_i, M_i) \cdot \#\text{Herm}(\ell_i, \widetilde{M}_i). \]

The result then follows from
\[ r(L^\sharp)(2r(M) - r(L^\sharp)) = r(L)(2r(M) - r(L)) + r(\ell)(2r(M) - r(\ell)), \]
where $r$ denotes the rank, cf. (3.1.0.2). \hfill \square

Recall that by (3.2.0.3)
\[ \text{Den}((1)^{n-1+k}, (1)^{n-1}) = \prod_{i=1}^{n-1} \left| 1 - (-q)^{-i} X \right|_{X = (-q)^{-k}}. \]
**Theorem 9.1.2.** Let \( \Lambda = \langle 1 \rangle^{n-1} \oplus \langle \varpi \rangle \). Let \( k \geq 0 \) and \( L \) be a hermitian \( O_F \)-lattice of rank \( n \). Then

\[
\frac{\text{Den}(\Lambda \oplus \langle 1 \rangle^k, L)}{\text{Den}(\langle 1 \rangle^{n-1+k}, \langle 1 \rangle^{n-1})} = \text{Den}(X, L^\sharp)|_{X=(-q)^{-k}}.
\]

**Proof.** By (3.2.0.3), we have

\[
\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^1) = (1 - (-q)^{-1}X)|_{X=(-q)^{-n-k}}.
\]
and

\[
\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^{n+1}) = \prod_{i=1}^{n+1} (1 - (-q)^{i}X)|_{X=(-q)^{-k}}.
\]

It follows that

\[
\frac{\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^{n+1})}{\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^1)} = \prod_{i=1}^{n} (1 - (-q)^{i}X)|_{X=(-q)^{-k}} = \text{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^{n}).
\]

(Alternatively, repeat the proof of Lemma 9.1.1 in the case \( \ell \) a self-dual lattice of rank one.)

By Example 3.5.2, we have \( \text{Den}(X, \ell) = 1 - X \), and hence

\[
\text{Den}(\langle 1 \rangle^{n+1+k}, \ell) = \text{Den}((-q)^{-n-k}, \ell) = (1 - (-q)^{-n}X)|_{X=(-q)^{-k}}.
\]

It follows that

\[
\frac{\text{Den}(\langle 1 \rangle^{n+1+k}, \ell)}{\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^1)} = \prod_{i=1}^{n-1} (1 - (-q)^{i}X)|_{X=(-q)^{-k}} = \text{Den}(\langle 1 \rangle^{n-1+k}, \langle 1 \rangle^{n-1}).
\]

Finally, by Lemma 9.1.1, we obtain

\[
\frac{\text{Den}(\langle 1 \rangle^{n-1+k} \oplus \langle \varpi \rangle, L)}{\text{Den}(\langle 1 \rangle^{n-1+k}, \langle 1 \rangle^{n-1})} = \frac{\text{Den}(\langle 1 \rangle^{n+1+k}, L^\sharp)/\text{Den}(\langle 1 \rangle^{n+1+k}, \ell)}{\text{Den}(\langle 1 \rangle^{n+1+k}, \langle 1 \rangle^{n+1})/\text{Den}(\langle 1 \rangle^{n+1+k}, \ell)} = \text{Den}(X, L^\sharp)|_{X=(-q)^{-k}}.
\]

This completes the proof. \( \Box \)

**Example 9.1.3** (The case rank \( L = 2 \)). If rank \( L = 2 \), Theorem 9.1.2 above specializes to Sankaran’s formula [San17, Proposition 3.1] which we recall now. Let \( L = \langle \varpi^a \rangle \oplus \langle \varpi^b \rangle, a \leq b \), \( a + b \) even. Define

\[
\epsilon = \begin{cases} 
0, & \text{if } b \text{ is even} \\
1, & \text{if } b \text{ is odd}.
\end{cases}
\]
Then the formula *loc. cit.* asserts that the LHS of Theorem 9.1.2 is equal to
\[(1 - X)(X^2 - (q^2 - q)X + 1)^{t} + \frac{1 - X}{1 - q^{-1}X} \left\{ qX(1 - q) \frac{(qX)^{b} - (qX)^{c}}{qX - 1} \right\} \]
\[(9.1.3.1) + X^2(q - q^{-1}X) \frac{X^{2b} - X^{2c}}{X^2 - 1} + \left[ -q^{b+1}(X - 1) + qX^{b+1} - q^{-1}X^{b+2} \right] \frac{X^{a+1} - X^{b+1}}{X^2 - 1} \].

On the other hand, this is consistent with the explicit formula for \(\text{Den}(X, L^2)\) given by [Ter13, proof of Theorem 5.2].

\[(9.1.3.2) \quad \text{Den}(X, L^2) = \frac{1}{1 + X} \left\{ \sum_{l=0}^{b+1} X^{l}(q^l - q^{1+b-l}X^{a+1}) - \sum_{l=0}^{b-1} X^{1+l}(q^{2+l} - q^{1+b-l}X^{a+1}) \right\}. \]

In fact, two functions on \((a, b) \in (\mathbb{Z}_{\geq 0})^2\) (not only for \((a + b)\) such that \(2 \mid a + b\)) are characterized by the following properties:

- The value at \((0, 0)\) (resp., \((1, 1)\)) is \(1 - X\) (resp., \((1 - X)(X^2 - (q^2 - q)X + 1))\).
- The term involving \(a\) is
  \[\frac{1 - X}{1 - q^{-1}X} \cdot \left[ -q^{b+1}(X - 1) + qX^{b+1} - q^{-1}X^{b+2} \right] \cdot \frac{X^{a+1}}{X^2 - 1} = \frac{1}{1 + X}X^{a+1} \left\{ -\sum_{l=0}^{b+1} X^{l}q^{1+b-l} + \sum_{l=0}^{b-1} X^{1+l}q^{1+b-l} \right\}. \]

The two expressions come from (9.1.3.1) and (9.1.3.2) respectively.
- The term not involving \(a\) is a function \(\phi\) in one variable \(b \in \mathbb{Z}_{\geq 0}\), which satisfies a difference equation
  \[\phi(b) - \phi(b - 1) = \frac{1}{1 + X}q^{b+1}X^{b}(X - 1). \]

The difference equation is easy to see from (9.1.3.2), and from (9.1.3.1) by a straightforward calculation.

**Definition 9.1.4.** After Theorem 9.1.2, define the (normalized) *local Siegel series relative to \(\Lambda = \langle 1 \rangle^{n-1} \oplus \langle \varpi \rangle* as the polynomial \(\text{Den}_{\Lambda}(X, L) \in \mathbb{Z}[X]\) such that

\[(9.1.4.1) \quad \text{Den}_{\Lambda}((-q)^{-k}, L) = \frac{\text{Den}(\Lambda \oplus \langle 1 \rangle^{k}, L)}{\text{Den}(\langle 1 \rangle^{n-1} \oplus \langle 1 \rangle^{k}, \langle 1 \rangle^{n-1})}. \]

Then by Theorem 9.1.2,

\[(9.1.4.2) \quad \text{Den}_{\Lambda}(X, L) = \text{Den}(X, L^2) \in \mathbb{Z}[X]. \]

In particular, if \(\text{val}(L)\) is even, then \(\text{Den}_{\Lambda}(1, L) = 0\). In this case, we denote the central derivative of local density by

\[\partial \text{Den}_{\Lambda}(L) := \frac{d}{dX} \bigg|_{X=1} \text{Den}_{\Lambda}(X, L). \]
9.2. Relation with local Whittaker functions. Let \( \Lambda = \langle 1 \rangle^{n-1} \otimes (\varpi) \) be an almost self-dual hermitian \( O_F \)-lattice. Let \( L \) be a hermitian \( O_F \)-lattice of rank \( n \). Let \( T = ((x_i, x_j))_{1 \leq i, j \leq n} \) be the fundamental matrix of an \( O_F \)-basis \( \{x_1, \ldots, x_n\} \) of \( L \), an \( n \times n \) hermitian matrix over \( F \). Associated the standard Siegel–Weil section of the characteristic function \( \varphi_1 = 1_{\Lambda^*} \) and endow it with the hermitian form by the formula similar to \( V \).

Character

the standard Siegel–Weil section of the characteristic function

fundamental matrix of an

hermitian

equation that is indivisible by \( \varpi \).

Y

define special divisors, denoted by \( \mathcal{D} \), \( \mathcal{D} \) for integers \( s = k \geq 0 \) (notice \( \gamma_p(V) = -1 \) in the notation there),

\[
W_T(1, k, \varphi_1) = (-q)^n \cdot \text{Den}(\Lambda \otimes \langle 1 \rangle^{2k}, L).
\]

So its value at \( s = 0 \) is

\[
W_T(1, 0, \varphi_1) = (-q)^{-n} \cdot \text{Den}(\Lambda, L) = (-q)^{-n} \cdot \text{Den}_\Lambda(L) \cdot \text{Den}(\langle 1 \rangle^{n-1}, \langle 1 \rangle^{n-1}),
\]

and its derivative at \( s = 0 \) is

\[
W_T'(1, 0, \varphi_1) = (-q)^{-n} \cdot \frac{\partial}{\partial q} \text{Den}(\Lambda, L) \cdot \text{Den}(\langle 1 \rangle^{n-1}, \langle 1 \rangle^{n-1}) \cdot \log q^2.
\]

Plugging in \([3.2.0.3]\), we obtain

\[
W_T(1, 0, \varphi_1) = \text{Den}_\Lambda(L) \cdot (-q)^{-n} \prod_{i=1}^{n-1} (1 - (-q)^{-i}),
\]

\[
W_T'(1, 0, \varphi_1) = \frac{\partial}{\partial q} \text{Den}_\Lambda(L) \cdot (-q)^{-n} \prod_{i=1}^{n-1} (1 - (-q)^{-i}) \cdot \log q^2.
\]

10. Kudla–Rapoport cycles in the almost principally polarized case

10.1. Rapoport–Zink spaces \( \mathcal{N}_n^l \) with almost self-dual level. For a \( \text{Spf} \, O_F \)-scheme \( S \), we consider triples \((Y, \iota, \lambda)\) over \( S \) as in \([2.1]\) except that \( \lambda \) is no longer principal but satisfies \( \ker \lambda \subseteq Y[\iota(\varpi)] \) has order \( q^2 \). Up to \( O_F \)-linear quasi-isogeny compatible with polarizations, there is a unique such triple \((Y', \iota_Y, \lambda_Y)\) over \( S = \text{Spec} \, k \). Let \( \mathcal{N}^l = \mathcal{N}^l_n = \mathcal{N}^l_{F/F_0, n} \) be the formal scheme over \( \text{Spf} \, O_F \) which represents the functor sending each \( S \) to the set of isomorphism classes of tuples \((Y, \iota, \lambda, \rho)\), where the additional entry \( \rho \) is a framing \( \rho : Y \times_S \tilde{S} \to Y \times_{\text{Spec} \, k} \tilde{S} \) is an \( O_F \)-linear quasi-isogeny of height 0 such that \( \rho \circ (\lambda_Y)_S = \lambda_S \). Here \( \tilde{S} := S_k \) is the special fiber.

The Rapoport–Zink space \( \mathcal{N}^l_n \) is a formal scheme formally locally of finite type, regular, of relative formal dimension \( n - 1 \) and has semi-stable reduction over \( \text{Spf} \, O_F \) \([RSZ18, \text{Theorem 5.1}], \text{[Cho18, \text{Theorem 1.2}]}\). Denote

\[
\mathcal{W}_n = \text{Hom}_{O_F}(\mathcal{E}, \mathcal{Y}),
\]

and endow it with the hermitian form by the formula similar to \( V_n \). It is easy to see that \( \mathcal{W}_n \) is a split hermitian space of dimension \( n \). Then similar to the special divisors \( \mathcal{Z}(x) \)'s on \( \mathcal{N}^l_n \) one can define special divisors, denoted by \( \mathcal{Y}(x) \), on \( \mathcal{N}^l_n \) for every non-zero \( x \in \mathcal{W}_n \) (cf. \text{[Cho18 \text{§4}]}). Then the argument of \([KR11]\) extends to the current setup to show that \( \mathcal{Y}(x) \) is a locally defined by one equation that is indivisible by \( \varpi \).
Definition 10.1.1. Similar to (2.4.0.2), define

\[(10.1.1.1) \quad \text{Int}'(L) := \chi(N_1^\ell, O_{Y(x_1)} \otimes L \cdots \otimes L O_{Y(x_n)}),\]

where \(x_1, \ldots, x_n\) is an \(O_F\)-basis of \(L\). We have not justified the independence of the choice of the basis, which will be postponed.

However, we will not compute \(\text{Int}'(L)\) for now. Later we will see that, under a conjectural relation between \(N_1^\ell\) and some auxiliary Rapoport–Zink spaces, \(\text{Int}'(L)\) is not equal to the derived local density \(\partial \text{Den}_\Lambda(L)\) (cf. Theorem 10.4.4 below). This is a typical phenomenon in the presence of bad reductions, cf. [KR00b, San17, RSZ17a, RSZ18]. Therefore, we will instead define a variant \(\text{Int}(L)\), which will give an exact identity \(\text{Int}(L) = \partial \text{Den}_\Lambda(L)\) (Theorem 10.3.1).

10.2. Auxiliary Rapoport–Zink spaces. Before we present our variant, we need an auxiliary moduli space (cf. [KRSZ19]). Fix an \(O_F\)-linear isogeny of degree \(q\)

\[(10.2.0.2) \quad \alpha : Y \times E \to X_{n+1},\]

such that \(\ker \alpha \subset (Y \times E)[\varpi]\) and such that \(\alpha^*(\lambda_X) = \lambda_Y \times \varpi \lambda_E\). Let \(x_0 \in V_{n+1} = \text{Hom}_{O_F}(E, X_{n+1})\) correspond to the second factor of \(\alpha\). Then the assumption implies that the norm of \(x_0\) is \((x_0, x_0) = \varpi\), and we have an orthogonal decomposition

\[V_{n+1} = \mathbb{W}_n \oplus (x_0)_F.\]

We denote by

\[(10.2.0.3) \quad \tilde{N}_1^\ell \subset \mathcal{N}_n^\ell \times_{\text{Spf } O_F} \mathcal{N}_{n+1}^\ell\]

the closed formal subscheme consisting of tuples \((Y, \iota_Y, \lambda_Y, \rho_Y, X, \iota_X, \lambda_X, \rho_X)\) such that \(\alpha\) lifts to an isogeny \(\tilde{\alpha} : Y \times E \to X\). If \(\alpha\) lifts, then \(\tilde{\alpha}\) is unique and satisfies \(\ker \tilde{\alpha} \subset (Y \times E)[\varpi]\) and \(\tilde{\alpha}^*(\lambda_X) = \lambda_Y \times \varpi \lambda_E\).

We therefore obtain a diagram

\[(10.2.0.4) \quad \begin{array}{ccc} & \tilde{N}_1^\ell \downarrow \pi_1 & \\ \mathcal{N}_n^\ell & \rightarrow & \mathcal{Z}(x_0) \rightarrow \mathcal{N}_{n+1}^\ell, \end{array}\]

where \(\pi_1\), resp. \(\pi_2\), are the restrictions to \(\tilde{N}_{n-1}^\ell\) of the two projections from the product space.

All three formal schemes \(\tilde{N}_1^\ell\), \(\mathcal{N}_n^\ell\) and \(\mathcal{Z}(x_0)\) are regular. Both \(\pi_1\) and \(\pi_2\) are proper morphisms.

Remark 10.2.1. Let \(\Lambda = (1)^{n-1} \oplus (\varpi)\) be as before. Let \(\Lambda\) be a self-dual lattice of rank \(n+1\) containing \(\Lambda \oplus (\varpi)\); there are \(q+1\) such lattices in the vector space \(\Lambda_F \oplus (\varpi)_F\). Then we have a natural embedding of hermitian spaces

\[W_n := \Lambda \otimes_{O_F} F \hookrightarrow V_{n+1} := \Lambda \otimes_{O_F} F\]
and their isometry groups $U(W_n) \hookrightarrow U(V_{n+1})$. Let $K = \text{Aut}(\Lambda)$ be the stabilizer of $\Lambda$, and similarly let $K^\sharp = \text{Aut}(\Lambda^\sharp)$. Define $\bar{K} := K \cap K^\sharp$, where the intersection is taken inside the unitary group $U(V_{n+1})$:

$$\bar{K} = K \cap K^\sharp$$

$$K = \text{Aut}(\Lambda) \quad \quad K^\sharp = \text{Aut}(\Lambda^\sharp).$$

Then the Rapoport–Zink spaces $N_{1n}^1$, $\tilde{N}_{1n}^1$, and $N_{n+1}^1$ correspond to the level structure $K$, $\bar{K}$, and $K^\sharp$ respectively.

It is easy to see that the generic fiber of the map $\pi_1 : \tilde{N}_n^1 \to N_n^1$ is finite étale of degree $[K : \bar{K}] = q + 1$, and the generic fiber of the map $\pi_2 : \tilde{N}_n^1 \to \mathcal{Z}(x_0)$ is an isomorphism. Therefore, $\mathcal{Z}(x_0)$ is a regular integral model of a finite étale covering of the generic fiber of $N_n^1$.

Let $x \in W_n \subset V_{n+1}$. Denote by $\mathcal{Z}^\flat(x)$ the restriction of the special divisor $\mathcal{Z}(x)$ (on $N_{n+1}$) to $\mathcal{Z}(x_0)$, i.e.,

$$\mathcal{Z}^\flat(x) := \mathcal{Z}(x_0) \cap \mathcal{Z}(x)$$

viewed as a formal subscheme of $\mathcal{Z}(x_0)$.

**Remark 10.2.2.** It is clear that the generic fiber of $\mathcal{Z}^\flat(x)$ (viewed as a divisor on the generic fiber of $\tilde{N}_n^1$ since $\pi_2$ is an isomorphism on the generic fibers) is equal to the pull back along $\pi_1$ of the generic fiber of $\mathcal{Y}(x)$ on $N_n^1$. Therefore, we may use $\mathcal{Z}^\flat(x)$ as an integral model of the pull-back of the generic fiber of $\mathcal{Y}(x)$.

**Definition 10.2.3.** Now let $L \subset W_n$ be an $O_F$-lattice of rank $n$. Motivated by Remark 10.2.1, define a variant of $\text{Int}'(L)$:

$$\text{Int}(L) = \frac{1}{\deg \pi_1} \chi(\mathcal{Z}(x_0), \mathcal{Z}^\flat(x_1) \cap \cdots \cap \mathcal{Z}^\flat(x_n)),$$

where $x_1, \cdots, x_n$ is a basis of $L$, and the derived tensor product is taken as $O_{\mathcal{Z}(x_0)}$-sheaves. This is independent of the choice of the basis, as a consequence of similar independence for the rank $(n + 1)$ lattice $L^\sharp = L \oplus \langle x_0 \rangle$.

10.3. **The Int $=$ $\partial$Den theorem.** The following theorem justifies our definition of the variant of intersection numbers.

**Theorem 10.3.1.** Let $L \subseteq V$ be an $O_F$-lattice of full rank $n$. Then

$$\text{Int}(L) = \frac{1}{q+1} \partial\text{Den}_\Lambda(L).$$

**Proof.** Let $x \in W_n$ be non-zero. Then $x \perp x_0$. Since $\mathcal{Z}(x_0)$ is an irreducible subscheme in $N_{n+1}$, the two formal subschemes $\mathcal{Z}(x)$ and $\mathcal{Z}(x_0)$ of $N_{n+1}$ do not share common irreducible components (obviously $\mathcal{Z}(x)$ does not contain $\mathcal{Z}(x_0)$). It follows that the two divisors intersect properly and hence

$$O_{\mathcal{Z}^\flat(x)} = O_{\mathcal{Z}(x)} \otimes_{54}^L O_{\mathcal{Z}(x_0)}$$
as elements in \( K'_0(\mathcal{Z}(x_0)) \). Therefore,
\[
\chi(\mathcal{Z}(x_0), \mathcal{Z}^b(x_1) \hat{\cap} \cdots \hat{\cap} \mathcal{Z}^b(x_n)) = \chi(\mathcal{N}_{n+1}, \mathcal{Z}(x_0) \hat{\cap} \mathcal{Z}(x_1) \hat{\cap} \cdots \hat{\cap} \mathcal{Z}(x_n)),
\]
which is \( \text{Int}(L^v) \). By our main Theorem \( 3.4.1 \), this is equal to \( \partial \text{Den}(L^v) \). The proof is complete. \( \Box \)

**Remark 10.3.2.** In the notation of \([9.2]\) it follows immediately from Theorem \( 10.3.1 \) and \( 9.2.0.4 \) that
\[
\text{Int}(L) = \frac{W'_T(1,0,\varphi_1)}{\log q^2} \cdot \frac{(-q)^n - 1}{q + 1} \cdot \prod_{i=1}^{n} (1 - (-q)^{-i})^{-1}.
\]

### 10.4. The intersection number \( \text{Int}'(L) \)

The result in this subsection is not used in Part 3.

We now compute the intersection number \( \text{Int}'(L) \), conditional on the conjectural relation between \( \mathcal{N}_n, \mathcal{N}_n^{\text{ss}} \) and \( \mathcal{Z}(x_0) \). Recall from \([10.2.0.4]\) that there are two projections \( \pi_1 \) and \( \pi_2 \). Let \( \text{Vert}^0(\mathcal{W}_n) \) be the set of self-dual lattices \( \Lambda \) in \( \mathcal{W}_n \). Let \( \mathcal{Z}(x_0)^{\text{ss}} \subset \mathcal{Z}(x_0) \) be the zero-dimensional reduced subscheme consisting of the superspecial points corresponding to all type 1-lattices in \( \mathcal{V}_{n+1} \) of the form \( \Lambda \oplus \langle x_0 \rangle, \Lambda \in \text{Vert}^0(\mathcal{W}_n) \). Note that \( \mathcal{Z}(x_0)^{\text{ss}} \) does not contain all superspecial points on \( \mathcal{Z}(x_0) \). By the Bruhat–Tits stratification of the reduced locus of \( \mathcal{N}_n^{\text{ss}} \), there exist a family of (disjoint) projective spaces \( \mathbb{P}_\Lambda = \mathbb{P}^{n-1} \) indexed by \( \Lambda \in \text{Vert}^0(\mathcal{W}_n) \). Denote by \( \mathcal{N}_n^{\text{ss}} \) the (disjoint) union of them.

The following conjecture was observed by Kudla and Rapoport in an unpublished manuscript.

**Conjecture 10.4.1.** (1) The morphism \( \pi_1 \) is finite flat of degree \( q + 1 \), étale away from \( \mathcal{N}_n^{\text{ss}}, \) and totally ramified along \( \mathcal{N}_n^{\text{ss}} \).

(2) The morphism \( \pi_2 \) is the blow-up of \( \mathcal{Z}(x_0) \) along the zero-dimensional subscheme \( \mathcal{Z}(x_0)^{\text{ss}} \).

(3) The preimage of \( \mathcal{N}_n^{\text{ss}} \) under \( \pi_1 \) is exactly the exceptional divisor on \( \mathcal{N}_n^{\text{ss}} \).

In \([KRSZ19]\) the authors will prove this conjecture, which from now on we assume to hold.

**Lemma 10.4.2.** Let \( n \geq 2 \). Let \( x \in \mathcal{W}_n \) be non-zero vector. Define a locally finite divisor on \( \mathcal{N}_n^{\text{ss}} \)
\[
\text{Exp}(x) := \sum_{\Lambda \in \text{Vert}(x)} \mathbb{P}_\Lambda,
\]
where
\[
\text{Vert}(x) := \{ \Lambda \subset \mathcal{W}_n \mid \Lambda^x = \Lambda, x \in \Lambda \}.
\]
Then there is an equality of divisors on \( \mathcal{N}_n^{\text{ss}} \)
\[
\pi_1^* \mathcal{Y}(x) = \pi_2^* \mathcal{Z}^b(x) - \text{Exp}(x).
\]

Here and henceforth, the pull-back and the push-forward homomorphisms are always in the derived sense.

**Example 10.4.3** (The case \( n = 1 \)). Though Lemma \( 10.4.2 \) does not cover the case \( n = 1 \), we can still formulate an analog. It is easy to see that, \( \mathcal{N}_1^{\text{ss}} \cong \text{Spf} \mathcal{O}_F \), \( \pi_2 \) is an isomorphism \( \mathcal{N}_1^{\text{ss}} \cong \mathcal{Z}(x_0) \) where both \( \mathcal{N}_1^{\text{ss}} \) and \( \mathcal{Z}(x_0) \) are isomorphic to the quasi-canonical lifting, a degree \( q + 1 \) ramified cover \( \text{Spf} \mathcal{O}_{F,1} \) of \( \text{Spf} \mathcal{O}_F \). Let \( x \in \mathcal{W}_1 \), then \( \mathcal{Y}(x) \) is non-empty unless \( \text{val}(x) \geq 2 \) (note that \( \text{val}(x) \)
is even), in which case it has $O_p$-length \( \frac{\text{val}(x)}{2} \) by the theory of canonical lifting. By [KR11], we also know that the divisor \( \mathcal{Z}^\circ(x) = \mathcal{Z}(x) \cap \mathcal{Z}(x_0) \) has $O_{\mathcal{F},1}$-length \( 1 + (q + 1)\frac{\text{val}(x)}{2} \). Therefore we obtain an analogous equality of cycles on \( \tilde{N}_1^1 \):

\[
\pi_1^* \mathcal{Y}(x) = \pi_2^* \mathcal{Z}^\circ(x) - \tilde{N}_1^1,\text{red}.
\]

**Proof.** First of all we note that a point in \( \mathcal{Z}(x_0)^{ss} \) corresponding to \( \Lambda \in \text{Vert}^0(\mathbb{W}_n) \) lies on \( \mathcal{Z}^\circ(x) \) if and only if \( x \in \Lambda \).

When \( n = 2 \), the divisor \( \mathcal{Y}(x) \) is determined by [San17, Theorem 2.8]. The structure of the divisor \( \mathcal{Z}^\circ(x) = \mathcal{Z}(x) \cap \mathcal{Z}(x_0) \) can be deduced from [Ter13].

Now let \( n \geq 3 \). Then, the divisor \( \mathcal{Z}^\circ(x) \) (resp. the restriction of \( \mathcal{Y}(x) \) to \( \mathcal{N}^1_n - \mathcal{N}^{1,ss}_n \)) is flat over Spf \( O_{\mathcal{F}} \). In fact, \( \mathcal{Z}(x_0) - \mathcal{Z}(x_0)^{ss} \) and \( \mathcal{N}^1_n - \mathcal{N}^{1,ss}_n \) are smooth over Spf \( O_{\mathcal{F}} \) (e.g., by Grothendieck–Messing deformation theory), and their special fibers are connected (\( \mathcal{Z}(x_0)^{ss} \) is zero dimensional). It follows that the special fibers of both are irreducible. The divisor \( \mathcal{Z}^\circ(x) \) (resp. \( \mathcal{Y}(x) \)) does not contain the full special fibers (resp., the special fiber away from \( \mathcal{N}^{1,ss}_n \)). The flatness follows (and fails when \( n = 2 \)).

It is clear that \( \pi_1^* \mathcal{Z}(x) \subset \pi_2^* \mathcal{Z}^\circ(x) \) and they coincide on the generic fiber. By the flatness above, the difference is supported on the exceptional divisor on \( \tilde{N}_n^1 \). It follows that

\[
(10.4.3.1) \quad \pi_2^* \mathcal{Z}^\circ(x) - \pi_1^* \mathcal{Z}(x) = \sum_{\Lambda \in \text{Vert}(x)} \text{mult}_\Lambda(x) \mathbb{P}_\Lambda,
\]

where \( \text{mult}_\Lambda(x) \in \mathbb{Z}_{\geq 0} \) is to be determined.

To determine the multiplicity \( \text{mult}_\Lambda(x) \), we wish to intersect the divisors in the equation above with a carefully chosen special divisor \( \mathcal{Z}(e) \) on \( \mathcal{N}_{n+1} \) and its counterparts on the other moduli spaces in the diagram \( 10.2.0.4 \). To be precise, fix a \( \Lambda_0 \in \text{Vert}(x) \). Since \( \Lambda_0 \) is self-dual of rank \( n \geq 3 \), there exists a vector \( e \in \Lambda_0 \) such that \( e \perp x \) and \( \text{val}(x) = 0 \). The special divisor \( \mathcal{Z}(e) \subset \mathcal{N}_{n+1} \) is isomorphic to \( \mathcal{N}_n \). Denote by \( x_0^b \) (resp. \( x^b \)) the projection of \( x_0 \) (resp. \( x \)) to the orthogonal complement \( \mathbb{V}_n \) of \( e \) in \( \mathbb{V}_{n+1} \). We obtain a commutative diagram with the obvious maps

\[
\begin{array}{ccc}
\tilde{N}^1_n & \xrightarrow{\pi_1} & \mathcal{N}^1_n \\
\downarrow \delta & & \downarrow \pi_1 \\
\mathcal{N}^1_{n-1} & \xrightarrow{\pi_2} & \mathcal{Z}(x_0) & \xrightarrow{\pi_2} & \mathcal{N}_{n+1} \\
\downarrow \pi_1^b & & \downarrow \pi^b_2 & & \downarrow \mathcal{Z}(x_0^b) & \xrightarrow{\pi_2^b} & \mathcal{N}_n \simeq \mathcal{Z}(e),
\end{array}
\]

where the right-most square is cartesian. We consider the map \( \tilde{\delta} : \tilde{N}^1_{n-1} \to \tilde{N}^1_n \). The pull-back of \( 10.4.3.1 \) along \( \tilde{\delta} \) is

\[
\pi_2^b \mathcal{Z}^\circ(x^b) - \pi_1^b \mathcal{Y}(x^b) = \sum_{\Lambda \in \text{Vert}(x) \atop e \in \Lambda} \text{mult}_\Lambda(x) \mathbb{P}_\Lambda^b,
\]

where \( \Lambda^b \) (a self-dual lattice in \( \mathbb{W}_{n-1} \)) is the orthogonal complement of \( e \) in \( \Lambda \). By induction on \( n \), the left hand side is also equal to \( \text{Exp}(x^b) \), which is a sum over the same index set of \( \Lambda = \Lambda^b \oplus \langle e \rangle \),
but with known multiplicity one. We deduce mult$_A(x) = 1$ for $A \in \text{Vert}(x)$ such that $e \in A$. By varying $e$, the proof is complete. 

We are now ready to complete the computation of the intersection number Int$(L)$ defined by 

\[ \text{(10.1.1.1)} \]

**Theorem 10.4.4.** Let $L \subseteq V$ be an $O_F$-lattice. Then

\[ \text{Int}(L) = \frac{1}{q+1} (\partial \text{Den}_A(L) - \text{Den}(L)). \]

In particular, the definition \text{(10.1.1.1)} is independent of the choice of the basis.

**Remark 10.4.5.** The case $n = 2$ is due to \[San17\].

**Example 10.4.6** (The case $n = 1$). When $n = 1$, let $L = \langle x \rangle \subset W_1$. It is easy to see that, by Example 10.4.3

\[ \text{Int}(L) = \begin{cases} \frac{\text{val}(x)}{2}, & \text{val}(x) \geq 0, \\ 0, & \text{otherwise}. \end{cases} \]

On the other hand, the local density formula shows that

\[ \partial \text{Den}_A(L) = \begin{cases} 1 + (q+1)\frac{\text{val}(x)}{2}, & \text{val}(x) \geq 0, \\ 0, & \text{otherwise}. \end{cases} \]

This verifies the theorem in the case $n = 1$. 

**Proof.** We apply the projection formula to $\pi_2$:

\[ \pi_2^* (O_{F_\Lambda} \otimes_{O_{X}} L_\pi^* \mathcal{F}) = \pi_2^* O_{F_\Lambda} \otimes_{O_{Z(x_0)}} L \mathcal{F}, \]

where $\Lambda \in \text{Vert}_0(W_n)$, and $\mathcal{F}$ is any coherent sheaf on $Z(x_0)$. Since the first factor $(\pi_2^* O_{F_\Lambda})$ is supported on a zero-dimensional subscheme of $Z(x_0)$, we have

\[ \chi(O_{F_\Lambda} \otimes L_\pi^* \mathcal{F}) = 0, \]

for any $\mathcal{F}$ whose support has dimension smaller than $n$. It follows that the same vanishing result holds for $Z(x_2) \cap \cdots \cap Z(x_n)$ (or any $n-1$ of the $n$ divisors) in the place of $\mathcal{F}$. 

On the other hand, for $\Lambda_1, \cdots, \Lambda_n \in \text{Vert}_0(W_n)$, the intersection numbers between exceptional divisors are equal to

\[ \chi(\widetilde{N}_n, \mathbb{P}_{\Lambda_1} \cap \cdots \cap \mathbb{P}_{\Lambda_n}) = \begin{cases} (-1)^{n-1}, & \Lambda_1 = \cdots = \Lambda_n, \\ 0, & \text{otherwise}. \end{cases} \]

Therefore we obtain

\[ \chi\left(\widetilde{N}_n, \pi_2^* Z^\mathcal{F}(x_1) - \text{Exp}(x_1)\right) \cap \cdots \cap \left(\pi_2^* Z^\mathcal{F}(x_n) - \text{Exp}(x_n)\right) \]

\[ = \chi\left(\widetilde{N}_n, \pi_2^* Z^\mathcal{F}(x_1) \cap \cdots \cap \pi_2^* Z^\mathcal{F}(x_n)\right) + (-1)^n \sum_{\Lambda \in \text{Vert}_0(W_n)} (-1)^{n-1}. \]
Now, by $\text{(3.6.1.1)}$
\[
\# \{ \Lambda \in \text{Vert}^0(W_n) \mid L \subset \Lambda \} = \text{Den}(L).
\]

By the projection formula for $\pi_2$, and noting that $\pi_2*O_{N_n^1} - O_{z(x_0)}$ is supported on $Z(x_0)^{ss}$ which is zero-dimensional), we obtain
\[
\chi \left( \tilde{N}_n^1, \pi_2^*Z^\flat(x_1) \cap \cdots \cap \pi_2^*Z^\flat(x_n) \right)
= \chi \left( Z(x_0), Z^\flat(x_1) \cap \cdots \cap Z^\flat(x_n) \right)
= \chi \left( N_{n+1}, Z(x_0) \cap Z(x_1) \cap \cdots \cap Z(x_n) \right)
= \text{Int}(L^\flat) = \partial \text{Den}(L^\flat),
\]
where the last equality is by Theorem $\text{3.4.1}$.

Finally, by the projection formula for the finite flat map $\pi_1$, we obtain an equality in $K'_0(N^1_n)$
\[
\pi_1*(\pi_1^*O_{\mathcal{Y}(x_1)} \otimes \cdots \otimes \pi_1^*O_{\mathcal{Y}(x_n)}) = \deg(\pi_1)O_{\mathcal{Y}(x_1)} \otimes \cdots \otimes O_{\mathcal{Y}(x_n)},
\]
and hence
\[
\text{Int}'(L) = \chi \left( N_n^1, \mathcal{Y}(x_1) \cap \cdots \cap \mathcal{Y}(x_n) \right)
= \deg(\pi_1)\chi \left( \tilde{N}_n^1, \pi_1^*\mathcal{Y}(x_1) \cap \cdots \cap \pi_1^*\mathcal{Y}(x_n) \right).
\]
Combining the last equalities with Lemma $\text{10.4.2}$ the theorem follows. $\square$

\textbf{Remark 10.4.7.} In the notation of $\text{3.3}$ and $\text{9.2}$ it follows immediately from Theorem $\text{10.4.4}$ that
\[
\text{Int}'(L) = \left( \frac{W_T(1,0,\varphi_1)}{\log q^2} \cdot \frac{(-q)^n - 1}{q + 1} - W_T(1,0,\varphi_0) \cdot \frac{1}{q + 1} \right) \prod_{i=1}^n \frac{1}{1 - (-q)^{-i}}.
\]

\textbf{Part 3. Semi-global and global applications: arithmetic Siegel–Weil formula}

In this part we apply our main Theorem $\text{3.4.1}$ to prove an identity between the local intersection number of Kudla–Rapoport cycles on (integral models of) unitary Shimura varieties at an inert prime with hyperspecial level and the derivative of a Fourier coefficient of Siegel–Eisenstein series (also known as the \textit{local arithmetic Siegel–Weil formula}). This is achieved by relating the Kudla–Rapoport cycles on unitary Shimura varieties to those on unitary Rapoport–Zink spaces via the $p$-adic uniformization, and by relating the Fourier coefficients to local representation densities. This deduction is more or less standard (see $\text{[KR14]}$ and $\text{[Ter13]}$), and we will state the results for more general totally real base fields and level structures, making use of the recent advance on integral models of unitary Shimura varieties ($\text{[RSZ17b]}$). We will also apply the main Theorem $\text{10.3.1}$ in the almost self-dual case to deduce a similar identity at an inert prime with almost self-dual level. Finally, combining these semi-global identities with archimedean identities of Liu $\text{[Liu11a]}$ and Garcia–Sankaran $\text{[GS19]}$ will allow us to deduce the \textit{arithmetic Siegel–Weil
formula for Shimura varieties with minimal levels at inert primes, at least when the quadratic extension is unramified at all finite places.

11. Shimura varieties and semi-global integral models

11.1. Shimura varieties. We will closely follow [RSZ17b] and [RSZ19]. In this part we switch to global notations. Let $F$ be a CM number field, with $F_0$ its totally real subfield of index 2. We fix a CM type $\Phi \subseteq \text{Hom}(F, \overline{Q})$ of $F$ and a distinguished element $\phi_0 \in \Phi$. We fix an embedding $\overline{Q} \hookrightarrow \mathbb{C}$ and identify the CM type $\Phi$ with the set of archimedean places of $F$, and also with the set of archimedean places of $F_0$. Let $V$ be an $F/F_0$-hermitian space of dimension $n \geq 2$. Let $V_\phi = V \otimes_{F, \phi} \mathbb{C}$ be the associated $\mathbb{C}/\mathbb{R}$-hermitian space for $\phi \in \Phi$. Assume the signature of $V_\phi$ is given by

$$(r_\phi, r_\bar{\phi}) = \begin{cases} (n-1,1), & \phi = \phi_0, \\ (n,0), & \phi \in \Phi \setminus \{\phi_0\}. \end{cases}$$

Define a variant $G^Q$ of the unitary simulate group $\text{GU}(V)$ by

$$G^Q := \{g \in \text{Res}_{F_0/Q} \text{GU}(V) : c(g) \in G_m\},$$

where $c$ denotes the similitude character. Define a cocharacter

$$h_{G^Q} : \mathbb{C}^\times \to G^Q(\mathbb{R}) \subseteq \prod_{\phi \in \Phi} \text{GU}(V_\phi)(\mathbb{R}) \cong \prod_{\phi \in \Phi} \text{GU}(r_\phi, r_\bar{\phi})(\mathbb{R}),$$

where its $\phi$-component is given by

$$h_{G^Q, \phi}(z) = \text{diag}\{z \cdot 1_{r_\phi}, \bar{z} \cdot 1_{r_\bar{\phi}}\}.$$

Then its $G^Q(\mathbb{R})$-conjugacy class defines a Shimura datum $(G^Q, \{h_{G^Q}\})$. Let $E_r = E(G^Q, \{h_{G^Q}\})$ be the reflex field, i.e., the subfield of $\overline{Q}$ fixed by $\{\sigma \in \text{Aut}(\overline{Q}/Q) : \sigma^* (r) = r\}$, where $r : \text{Hom}(F, \overline{Q}) \to \mathbb{Z}$ is the function defined by $r(\phi) = r_\phi$.

We similarly define the group $Z^Q$ (a torus) associated to a totally positive definite $F/F_0$-hermitian space of dimension 1 (i.e., of signature $\{(1,0)_{\phi \in \Phi}\}$) and a cocharacter $h_{Z^Q}$ of $Z^Q$. The reflex field $E_\Phi = E(Z^Q, \{h_{Z^Q}\})$ is equal to the reflex field of the CM type $\Phi$, i.e., the subfield of $\overline{Q}$ fixed by $\{\sigma \in \text{Gal}(\overline{Q}/Q) : \sigma \circ \Phi = \Phi\}$.

Now define a Shimura datum $(\tilde{G}, \{h_{\tilde{G}}\})$ by

$$\tilde{G} := Z^Q \times_{G_m} G^Q = \{(z,g) \in Z^Q \times G^Q : \text{Nm}_{F/F_0}(z) = c(g)\}, \quad h_{\tilde{G}} = (h_{Z^Q}, h_{G^Q}).$$

Its reflex field $E$ is equal to the composite $E_r E_\Phi$, and the CM field $F$ becomes a subfield of $E$ via the embedding $\phi_0$. Let $K \subseteq \tilde{G}(\mathbb{A}_f)$ be a compact open subgroup. Then the associated Shimura variety $\text{Sh}_K = \text{Sh}_K(\tilde{G}, \{h_{\tilde{G}}\})$ is of dimension $n - 1$ and has a canonical model over $\text{Spec} E$. We remark that $E = F$ when $F/Q$ is Galois, or when $F = F_0 K$ for some imaginary quadratic $K/Q$ and the CM type $\Phi$ is induced from a CM type of $K/Q$ (e.g., when $F_0 = \mathbb{Q}$).
11.2. Semi-global integral models at hyperspecial levels. Let $p$ be a prime number. If $p = 2$, then we assume all places $v$ of $F_0$ above $p$ are unramified in $F$. Let $\nu$ be a place of $E$ above $p$. It determines places $v_0$ of $F_0$ and $w_0$ of $F$ via the embedding $\phi_0$. To specify the level $K$, notice that for $G := \text{Res}_{F_0/Q} U(V)$ we have an isomorphism

$$\bar{G} \simeq \mathbb{Z}^Q \times G, \quad (z, g) \mapsto (z, z^{-1}g).$$

We consider the open compact subgroup of the form

$$K \simeq K_{Z^Q} \times K_G$$

under the decomposition (11.2.0.1). We assume that $K_{Z^Q}$ is the unique maximal open compact subgroup of $Z^Q(\mathbb{A}_f)$ and

$$K_G = \prod_{v|p} K_{G,v} \times K_{G,v}^p.$$

In this subsection, we assume

(H1) $v_0$ is inert in $F$ (possibly ramified over $p$).

(H2) $V_{v_0}$ is split and we take $K_{G,v_0}$ to be the stabilizer of a self-dual lattice $\Lambda_{v_0} \subseteq V_{v_0}$, a hyperspecial subgroup of $U(V)(F_0,v_0)$.

(H3) For each place $v \neq v_0$ of $F_0$ above $p$, let $K_{G,v}^p$ be the maximal compact subgroup of $U(V)(F_0,v)$ given by the stabilizer of a vertex lattice $\Lambda_v \subseteq V_v$. We take $K_{G,v} = K_{G,v}^p$ if $v$ is nonsplit in $F$. We take $K_{G,v} \subseteq K_{G,v}^p$ to be any open compact subgroup if $v$ is split in $F$.

(H4) $K_G \subseteq G(\mathbb{A}_f^p)$ is any open compact subgroup.

Under these conditions, Rapoport–Smithling–Zhang [RSZ17, §4.1] and [RSZ19, §4–5] (see also [Lin18, Proposition C.20]) construct a smooth integral model $\mathcal{M}_K$ of $\text{Sh}_K$ over $O_{E,(\nu)}$. More precisely, for a locally noetherian $O_{E,(\nu)}$-scheme $S$, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \eta^p, \bar{\eta}_v^w)$, where

(M1) $A_0$ (resp. $A$) is an abelian scheme over $S$.

(M2) $\iota_0$ (resp. $\iota$) is an action of $\text{OF} \otimes \mathbb{Z}(p)$ on $A_0$ (resp. $A$) satisfying the Kottwitz condition of signature $\{(1,0)_{\phi \in \Phi}\}$ (resp. signature $\{(r_\phi, r_{\phi^\vee})_{\phi \in \Phi}\}$).

(M3) $\lambda_0$ (resp. $\lambda$) is a polarization of $A_0$ (resp. $A$) whose Rosati involution induces the automorphism given by the nontrivial Galois automorphism of $F/F_0$ via $\iota_0$ (resp. $\iota$).

(M4) $\eta^p$ is a $K_G$-orbit of $\mathbb{A}_{F,f}$-linear isometries between lisse $\mathbb{A}_{F,f}$-sheaves

$$\eta^p : \text{Hom}_F(\bar{V}^p(A_0), \bar{V}^p(A)) \simeq V \otimes_F \mathbb{A}_{F,f}^p.$$ 

Here $\bar{V}^p(\cdot)$ denotes the $\mathbb{A}_{F,f}$-Tate module.

(M5) $\bar{\eta}_v^w$ is a collection of $\{\bar{\eta}_v\}$, where $v \neq v_0$ runs over places of $F_0$ above $p$ such that $v$ is split in $F$, and each $\bar{\eta}_v$ is a $K_{G,v}$-orbit of $F_w$-linear isomorphisms between lisse $F_w$-sheaves

$$\eta_v : \text{Hom}_{O_{F_w}}(A_0[w^\infty], A[w^\infty]) \otimes_{O_{F_w}} F_w \simeq V \otimes_F F_w.$$ 

Here $w$ is the unique place over $v$ determined by the CM type $\Phi$, and we view $K_{G,v}$ as an open subgroup of $\text{GL}(V_w) \cong U(V_v)$ under the decomposition $V_v \cong V_w \oplus V_w^\perp$. Notice that
by the Kottwitz signature condition, both $A_0[w^\infty]$ and $A[w^\infty]$ are étale $O_{F_w}$-modules (cf. [Lin18 Definition C.19]).

Such a tuple is required to satisfy the following extra conditions:

(M6) $(A_0, \iota_0, \lambda_0) \in M_0^{a,\xi}(S)$. Here $M_0^{a,\xi}$ is an integral model of $\text{Sh}_{K_{\mathcal{O}}}((\mathbb{Q}, \{h_{Z_e}\})$ coming from an axillary moduli problem depending on a choice of nonzero coprime-to-$p$ ideal $a$ of $O_{F_0}$ and $\xi$ a certain similarity class of 1-dimensional hermitian $F/F_0$-hermitian spaces ([RSZ17b, §3.2]).

These axillary choices are made to ensure that the unitary group in 1-variable with $a$-level structure exists and so $M_0^{a,\xi}$ is non-empty. In particular, the polarization $\lambda_0$ is coprime-to-$p$. We remark that when $F_0/F$ is ramified at some finite place, one may choose $a$ to be the trivial ideal. Moreover, when $F_0 = \mathbb{Q}$, there is only one choice of $\xi$, and the condition $(A_0, \iota_0, \lambda_0) \in M_0^{a,\xi}(S)$ is nothing but requiring $\lambda_0$ to be principal.

(M7) For each place $v$ of $F_0$ above $p$, $\lambda$ induces a polarization $\lambda_v$ on the $p$-divisible group $A[v^\infty]$. We require $\ker \lambda_v \subseteq A[t(\varpi_v)]$ of rank equal to the size of $\Lambda_v^\vee/\Lambda_v$, where $\varpi_v$ is a uniformizer of $F_0,v$. In particular, we require $\lambda_v$ to be principal.

(M8) For the place $v_0$, we further require the Eisenstein condition in [RSZ19, §5.2, case (2)]. We remark the Eisenstein condition is automatic when $v_0$ is unramified over $p$.

(M9) For each place $v \neq v_0$ of $F_0$ above $p$, we further require the sign condition and Eisenstein condition as explained in [RSZ17b, §4.1]. We remark that the sign condition is automatic when $v$ is split in $F$, and the Eisenstein condition is automatic when the places of $F$ above $v$ are unramified over $p$.

A morphism $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{r}p, \tilde{v}^\lambda_\nu)$ → $(A'_0, \iota'_0, \lambda'_0, A', \iota', \lambda', \tilde{r}'p, \tilde{v}'^\lambda_\nu)$ in this groupoid is an isomorphism $(A_0, \iota_0, \lambda_0) \sim (A'_0, \iota'_0, \lambda'_0)$ in $M_0^{a,\xi}(S)$ and an $O_{F,(p)}$-linear quasi-isogeny $A \rightarrow A'$ inducing an isomorphism $A[p^\infty] \sim A'[p^\infty]$, pulling $\lambda'$ back to $\lambda$, pulling $\tilde{r}'p$ back to $\tilde{r}p$ and pulling $\tilde{v}'^\lambda_\nu$ back to $\tilde{v}^\lambda_\nu$.

By [RSZ17b Theorem 4.1], [RSZ19 Theorem 5.6 (c)], the functor $S \mapsto M_K(S)$ is represented by a Deligne–Mumford stack $M_K$ smooth over $\text{Spec} O_{E_{(\nu)}}$. For $K_{\mathcal{O}}^p$ small enough, $M_K$ is relatively representable over $M_0^{a,\xi}$, with generic fiber naturally isomorphic to the canonical model of $\text{Sh}_K$ over $\text{Spec} E$.

11.3. Semi-global integral models at almost self-dual parahoric levels. With the same set-up as §11.2 but replace the assumptions (H1) and (H2) by

(A1) $v_0$ is inert in $F$ and unramified over $p$.

(A2) $V_{v_0}$ is nonsplit and we take $K_{G,v_0}$ to be the stabilizer of an almost self-dual lattice $\Lambda_{v_0} \subseteq V_{v_0}$, a maximal parahoric subgroup of $U(V)(F_0,v_0)$.

For a locally noetherian $O_{E_{(\nu)}}$-scheme $S$, we consider $M_K(S)$ to be the groupoid of tuples $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \tilde{r}p, \tilde{v}^\lambda_\nu)$ satisfying (M1)–(M9). In particular, $\lambda_{v_0}$ is almost principal instead of principal in [M7].

By [RSZ17b Theorem 4.7], the functor $S \mapsto M_K(S)$ is represented by a Deligne–Mumford stack $M_K$ flat over $\text{Spec} O_{E_{(\nu)}}$. For $K_{\mathcal{O}}^p$ small enough, $M_K$ is relatively representable over $M_0^{a,\xi}$, with generic fiber naturally isomorphic to the canonical model of $\text{Sh}_K$ over $\text{Spec} E$. Moreover, when $\nu$ is
unramified over $p$, $\mathcal{M}_K$ has semi-stable reduction over $\text{Spec}O_{E,(\nu)}$ by [RSZ17b Theorem 4.7] and [Cho18 Proposition 1.4].

11.4. Semi-global integral models at split primes. With the same set-up as §11.2 but replace the assumption [H1] by

(S) $v_0$ is split in $F$ (possibly ramified over $p$).

For a locally noetherian $O_{E,(\nu)}$-scheme $S$, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \eta_0, \lambda_0, \nu_0, \nu_0^{\infty})$ satisfying [M1]–[M9] We further require

(MS) when $p$ is locally nilpotent on $S$, the $p$-divisible group $A[w_0^{\infty}]$ is a Lubin–Tate group of type $r|_{w_0}$ ([RZ17 §8]). We remark that this condition is automatic when $v_0$ is unramified over $p$.

By [RSZ17b Theorem 4.2], as in the hyperspecial case, the functor $S \mapsto \mathcal{M}_K(S)$ is represented by a Deligne–Mumford stack $\mathcal{M}_K$ smooth over $\text{Spec}O_{E,(\nu)}$. For $K_0^m$ small enough, $\mathcal{M}_K$ is relatively representable over $\mathcal{M}_0^{q,\xi}$, with generic fiber naturally isomorphic to the canonical model of $\text{Sh}_K$ over $\text{Spec} E$.

11.5. Semi-global integral models with Drinfeld levels at split primes. With the same set-up as §11.4 we may consider semi-global integral models with Drinfeld levels by further requiring

(D) the place $\nu$ of $E$ matches the CM type $\Phi$ (in the sense of [RSZ17b §4.3]): if $\phi \in \text{Hom}(F, \overline{Q})$ induces the $p$-adic place $w_0$ of $F$ (via $\nu : E \hookrightarrow \overline{Q}_p$), then $\phi \in \Phi$. We remark that this matching condition is automatic when $F = F_0K$ for some imaginary quadratic $K/\mathbb{Q}$ and the CM type $\Phi$ is induced from a CM type of $K/\mathbb{Q}$ (e.g., when $F_0 = \mathbb{Q}$), or when $v_0$ is of degree one over $p$.

For $m \geq 0$, we consider the open compact subgroup $K_G^m \subseteq K_G$ such that $K_{G,v_0}^m \subseteq K_{G,V_0}$ is the principal congruence subgroup modulo $\varpi_{v_0}^m$, and $K_{G,v}^m = K_{G,v}$ for $v \neq v_0$. Write $K^m = K_{Z_0}^m \times K_G^m$. Notice that $K^0 = K$. We define a semi-global integral model $\mathcal{M}_{K^m}$ of $\text{Sh}_{K^m}$ over $O_{E,(\nu)}$ as follows. For a locally noetherian $O_{E,(\nu)}$-scheme $S$, we consider $\mathcal{M}_{K^m}(S)$ to be the groupoid of tuples $(A_0, \eta_0, \lambda_0, A, \nu, \lambda, \eta_0^{\infty}, \eta_0^{w_0})$, where $(A_0, \eta_0, \lambda_0, A, \nu, \lambda, \eta_0^{\infty}, \eta_0^{w_0}) \in \mathcal{M}_K(S)$ and the additional datum $\eta_{w_0}$ is a Drinfeld level structure:

(MD) when $p$ is locally nilpotent on $S$, $\eta_{w_0}$ is an $O_{F,w_0}$-linear homomorphism of finite flat group schemes

$$\eta_{w_0} : \varpi_{w_0}^{-m}A_{w_0}/A_{w_0} \to \text{Hom}_{O_{F,w_0}}(A_0[w_0^m], A[w_0^m]).$$

By [RSZ17b Theorem 4.5], the functor $S \mapsto \mathcal{M}_{K^m}(S)$ is represented by a regular Deligne–Mumford stack $\mathcal{M}_K$, flat over $\text{Spec} O_{E,(\nu)}$ and finite flat over $\mathcal{M}_K$, with generic fiber naturally isomorphic to the canonical model of $\text{Sh}_{K^m}$ over $\text{Spec} E$.

11.6. Semi-global integral models at ramified primes. With the same set-up as §11.2 but replace the assumption [H1] by

(R) $v_0$ is ramified in $F$ (so $p \neq 2$) and unramified over $p$.

For a locally noetherian $O_{E,(\nu)}$-scheme $S$, we consider $\mathcal{M}_K(S)$ to be the groupoid of tuples $(A_0, \eta_0, \lambda_0, A, \nu, \lambda, \eta_0^{\infty}, \eta_0^{w_0})$ satisfying [M1]–[M9] We further require

(MR) when $p$ is locally nilpotent on $S$, the $p$-divisible group $A[w_0^{\infty}]$ satisfies the Pappas wedge condition ([KR14 Definition 2.4], [RSZ19 §5.2]).
By [RSZ19] Theorem 5.4, the functor \( S \mapsto \mathcal{M}_K(S) \) is represented by a Deligne–Mumford stack \( \mathcal{M}_K \) flat over \( \text{Spec} O_{E, (\nu)} \). For \( K_G \) small enough, \( \mathcal{M}_K \) is relatively representable over \( \mathcal{M}_0^\xi \), with generic fiber naturally isomorphic to the canonical model of \( \text{Sh}_K \) over \( \text{Spec} E \). By [RSZ19] Theorem 6.7, it has isolated singularities and we may further obtain a regular model by blowing up (the Krämer model, see [RSZ19] Definition 6.10) which we still denote by \( \mathcal{M}_K \).

12. Incoherent Eisenstein series

12.1. Siegel Eisenstein series. Let \( W \) be the standard split \( F/F_0 \) skew-hermitian space of dimension 2\( n \). Let \( G_n = \text{U}(W) \). Write \( G_n(\mathbb{A}) = G_n(\mathbb{A}_{F_0}) \) for short. Let \( P_n(\mathbb{A}) = M_n(\mathbb{A})N_n(\mathbb{A}) \) be the standard Siegel parabolic subgroup of \( G_n(\mathbb{A}) \), where

\[
M_n(\mathbb{A}) = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & t_a^{-1} \end{pmatrix} : a \in \text{GL}_n(\mathbb{A}_F) \right\},
\]

\[
N_n(\mathbb{A}) = \left\{ n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} : b \in \text{Herm}_n(\mathbb{A}_F) \right\}.
\]

Let \( \eta : \mathbb{A}_{F_0}^\times / F_0^\times \to \mathbb{C}^\times \) be the quadratic character associated to \( F/F_0 \). Fix \( \chi : \mathbb{A}_{F}^\times \to \mathbb{C}^\times \) a character such that \( \chi|_{\mathbb{A}_{F_0}^\times} = \eta^m \). We may view \( \chi \) as a character on \( M_n(\mathbb{A}) \) by \( \chi(m(a)) = \chi(\text{det}(a)) \) and extend it to \( \tilde{P}_n(\mathbb{A}) \) trivially on \( N_n(\mathbb{A}) \). Define the degenerate principal series to be the unnormalized smooth induction

\[
I_n(s, \chi) := \text{Ind}_{P_n(\mathbb{A})}^{G_n(\mathbb{A})} (\chi \cdot |\cdot|_F^{s+n/2}), \quad s \in \mathbb{C}.
\]

For a standard section \( \Phi(-, s) \in I_n(s, \chi) \) (i.e., its restriction to the standard maximal compact subgroup of \( G_n(\mathbb{A}) \) is independent of \( s \)), define the associated Siegel Eisenstein series

\[
E(g, s, \Phi) := \sum_{\gamma \in \tilde{P}_n(F_0) \backslash G_n(F_0)} \Phi(\gamma g, s), \quad g \in G_n(\mathbb{A}),
\]

which converges for \( \text{Re}(s) \gg 0 \) and admits meromorphic continuation to \( s \in \mathbb{C} \).

12.2. Fourier coefficients and derivatives. Fix a standard additive character \( \psi : \mathbb{A}_{F_0}^\times / F_0 \to \mathbb{C}^\times \). We have a Fourier expansion

\[
E(g, s, \Phi) = \sum_{T \in \text{Herm}_n(F)} E_T(g, s, \Phi),
\]

where

\[
E_T(g, s, \Phi) = \int_{N_n(F_0) \backslash N_n(\mathbb{A})} E(n(b)g, s, \Phi)\psi(- \text{tr}(Tb)) \, dn(b),
\]

and the Haar measure \( dn(b) \) is normalized to be self-dual with respect to \( \psi \). When \( T \) is nonsingular, for factorizable \( \Phi = \otimes_v \Phi_v \) we have a factorization of the Fourier coefficient into a product

\[
E_T(g, s, \Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v),
\]

where the local (generalized) Whittaker function is defined by

\[
W_{T,v}(g_v, s, \Phi_v) = \int_{N_v(F_0,v)} \Phi_v(w_n^{-1}n(b)v, s)\psi(- \text{tr}(Tb)) \, dn(b), \quad w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.
\]
and has analytic continuation to \( s \in \mathbb{C} \). Thus we have a decomposition of the derivative of a nonsingular Fourier coefficient at \( s = s_0 \),

\[
E'_T(g, s_0, \Phi) = \sum_v E'_{T,v}(g, s_0, \Phi),
\]

(12.2.0.2) where

\[
E'_{T,v}(g, s, \Phi) = W'_{T,v}(g_v, s, \Phi_v) \cdot \prod_{v' \neq v} W_{T,v'}(g_{v'}, s, \Phi_{v'}).
\]

(12.2.0.3)

12.3. **Incoherent Eisenstein series.** Let \( V \) be an \( \mathbb{A}_F/\mathbb{A}_{F_0} \)-hermitian space of rank \( n \). Let \( \mathcal{S}(V^n) \) be the space of Schwartz functions on \( V^n \). The fixed choice of \( \chi \) and \( \psi \) gives a Weil representation \( \omega = \omega_{\chi,\psi} \) of \( G_n(\mathbb{A}) \times \mathbb{U}(V) \) on \( \mathcal{S}(V^n) \). Explicitly, for \( \varphi \in \mathcal{S}(V^n) \) and \( x \in V^n \),

\[
\omega(m(a))\varphi(x) = \chi(m(a))|\det a|_{F}^{n/2} \varphi(x \cdot a), \quad m(a) \in M_n(\mathbb{A}),
\]

\[
\omega(n(b))\varphi(x) = \psi(\text{tr} b T(x))\varphi(x), \quad n(b) \in N_n(\mathbb{A}),
\]

\[
\omega_{\chi}(w_n)\varphi(x) = \gamma_n^0 \cdot \varphi(x), \quad w_n = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right),
\]

\[
\omega(h)\varphi(x) = \varphi(h^{-1} \cdot x), \quad h \in \mathbb{U}(V).
\]

Here \( T(x) = ((x_i, x_j))_{1 \leq i,j \leq n} \) is the fundamental matrix of \( x \), \( \gamma_n \) is the Weil constant (see [KR14 (10.3)]) , and \( \hat{\varphi} \) is the Fourier transform of \( \varphi \) using the self-dual Haar measure on \( V^n \) with respect to \( \psi \circ \text{tr}_{F/F_0} \).

For \( \varphi \in \mathcal{S}(V^n) \), define a function

\[
\Phi_\varphi(g) := \omega(g)\varphi(0), \quad g \in G_n(\mathbb{A}).
\]

Then \( \Phi_\varphi \in I_n(0, \chi) \). Let \( \Phi_\varphi(-, s) \in I_n(s, \chi) \) be the associated standard section, known as the standard Siegel–Weil section associated to \( \varphi \). For \( \varphi \in \mathcal{S}(V^n) \), we write

\[
E(g, s, \varphi) := E(g, s, \Phi_\varphi), \quad E_T(g, s, \varphi) := E_T(g, s, \Phi_\varphi), \quad E'_{T,v}(g, s, \varphi) := E'_{T,v}(g, s, \Phi_\varphi),
\]

and similarly for \( W_{T,v}(g_v, s, \varphi_v) \). We say \( V \) (resp. \( \Phi_\varphi, E(g, s, \varphi) \)) is coherent if \( V = V \otimes_{F_0} \mathbb{A}_{F_0} \) for some \( F/F_0 \)-hermitian space \( V \), and incoherent otherwise. When \( E(g, s, \varphi) \) is incoherent, its central value \( E(g, 0, \varphi) \) automatically vanishes. In this case, we write the central derivatives as

\[
\partial E_{T,v}(g, \varphi) := E'(g, 0, \varphi), \quad \partial E'_{T,v}(g, \varphi) := E'(g, 0, \varphi), \quad \partial E_{T,v}(g, \varphi) := E'(g, 0, \varphi)
\]

Let \( T \in \text{Herm}_n(F) \) be nonsingular. Then \( W_{T,v}(g_v, 0, \varphi_v) \neq 0 \) only if \( V_v \) represents \( T \), hence \( \partial E_{T,v}(g, \varphi) \neq 0 \) only if \( V_v \) represents \( T \) for all \( v' \neq v \). Let \( \text{Diff}(T, V) \) be the set of primes \( v \) such that \( V_v \) does not represent \( T \). Since \( V \) is incoherent, by (12.2.0.2) we know that \( \partial E_{T,v}(g, \varphi) \neq 0 \) only if \( \text{Diff}(T, V) = \{ v \} \) is a singleton, and in this case \( v \) is necessarily nonsplit in \( F \). Thus

(12.3.0.4) \( \partial E_{T,v}(g, \varphi) \neq 0 \Rightarrow \text{Diff}(T, V) = \{ v \}, \quad \partial E_{T,v}(g, \varphi) = \partial E_{T,v}(g, \varphi) \)

We say \( \varphi \in \mathcal{S}(V^n) \) is nonsingular if its support lies in \( \{ x \in V^n : \det T(x) \neq 0 \} \). By [Liu11b, Proposition 2.1], we have

(12.3.0.5) \( \varphi \) is nonsingular at two finite places \( \Rightarrow \partial E_{T,v}(g, \varphi) = 0 \) for any singular \( T \).
12.4. Classical incoherent Eisenstein series associated to the Shimura datum. Assume that we are in the situation in §11.1. Let \( \mathbb{V} \) be the incoherent hermitian space obtained from \( V \) so that \( \mathbb{V} \) has signature \((n,0)_{\phi \in \Phi}\) and \( \mathbb{V}_v \cong V_v \) for all finite places \( V_v \).

The hermitian symmetric domain for \( G_n = U(W) \) is the hermitian upper half space

\[
\mathbb{H}_n = \{ z \in \text{Mat}_n(F_\infty) : \frac{1}{2i} (z - i\bar{z}) > 0 \} = \{ z = x + iy : x \in \text{Herm}_n(F_\infty), y \in \text{Herm}_n(F_\infty) > 0 \},
\]

where \( F_\infty = F \otimes F_0 \mathbb{R}^\Phi \cong \mathbb{C}^\Phi \). Define the classical incoherent Eisenstein series to be

\[
E(z, s, \varphi) := \chi_\infty((\det(a))^{-1} \det(y)^{-n/2} \cdot E(g_z, s, \varphi), \quad g_z := n(x)m(a) \in G_n(\mathbb{A}),
\]

where \( a \in \text{GL}_n(F_\infty) \) such that \( y = a^t \bar{a} \). We write the central derivatives as

\[
\partial \text{Eis}(z, \varphi) := E'(z, 0, \varphi), \quad \partial \text{Eis}_T(z, \varphi) := E_T'(z, 0, \varphi), \quad \partial \text{Eis}_T,v(z, \varphi) := E_{T,v}'(z, 0, \varphi).
\]

Then we have a Fourier expansion

\[
(12.4.0.6) \quad \partial \text{Eis}(z, \varphi) = \sum_{T \in \text{Herm}_n(F)} \partial \text{Eis}_T(z, \varphi)
\]

By (12.3.0.4) we know that

\[
(12.4.0.7) \quad \partial \text{Eis}_T(z, \varphi) \neq 0 \Rightarrow \text{Diff}(T, \mathbb{V}) = \{ v \}, \quad \partial \text{Eis}_T(z, \varphi) = \partial \text{Eis}_{T,v}(z, \varphi).
\]

Then we have a Fourier expansion

\[
(12.4.0.6) \quad \partial \text{Eis}(z, \varphi) = \sum_{T \in \text{Herm}_n(F)} \partial \text{Eis}_T(z, \varphi)
\]

By (12.3.0.4) we know that

\[
(12.4.0.7) \quad \partial \text{Eis}_T(z, \varphi) \neq 0 \Rightarrow \text{Diff}(T, \mathbb{V}) = \{ v \}, \quad \partial \text{Eis}_T(z, \varphi) = \partial \text{Eis}_{T,v}(z, \varphi).
\]

For the fixed open compact subgroup \( K \subseteq \tilde{G}(\mathbb{A}_f) \), we will choose

\[
\varphi = \varphi_K \otimes \varphi_\infty \in \mathcal{S}(\mathbb{V}^m)
\]

such that \( \varphi_K \in \mathcal{S}(\mathbb{V}_f^m) \) is \( K \)-invariant (where \( K \) acts on \( \mathbb{V} \) via the second factor \( K_G \)) and \( \varphi_\infty \) is the Gaussian function

\[
\varphi_\infty(x) = e^{-2\pi \text{tr}(x)} := \prod_{\phi \in \Phi} e^{-2\pi \text{tr}(x_\phi)}.
\]

For our fixed choice of Gaussian \( \varphi_\infty \), we write

\[
E(z, s, \varphi_K) = E(z, s, \varphi_K \otimes \varphi_\infty), \quad \partial \text{Eis}(z, \varphi_K) = \partial \text{Eis}(z, \varphi_K \otimes \varphi_\infty)
\]

and so on for short. When \( T > 0 \) is totally positive definite, we have

\[
\partial \text{Eis}_T(z, \varphi_K) = \partial \text{Eis}_T(\varphi_K) \cdot q^T, \quad q^T := e^{2\pi i \text{tr}(Tz)}
\]

for some \( \partial \text{Eis}_T(\varphi_K) \in \mathbb{C} \) independent of \( z \).

13. The semi-global identity at inert primes

In this section we assume that we are in the situation of §11.2 (hyperspecial level) or §11.3 (almost self-dual level). We fix the level \( K \) as above and write \( \mathcal{M} = \mathcal{M}_K \) for short.
13.1. $p$-adic uniformization of the supersingular locus of $\mathcal{M}$. Let $\hat{\mathcal{M}}^{ss}$ be the completion of the base change $\mathcal{M}_{O_{\hat{E}_v}}$ along the supersingular locus $\mathcal{M}^{ss}_{k_v}$ of its special fiber $\mathcal{M}_{k_v}$. Here $E_v$ is the completion of $E$ at $v$ and $k_v$ is its residue field. Assume $p > 2$. Then we have a $p$-adic uniformization theorem ([RZ96], [Cho18, Theorem 4.3], see also the proof of [RSZ17b, Theorem 8.15]),

\[
\hat{\mathcal{M}}^{ss} \simeq \tilde{G}'(\mathbb{Q}) \backslash [\mathcal{N}' \times \tilde{G}(\mathbb{A}_F^p)/K^p].
\]

Here $\tilde{G}' = \mathbb{Z}^\times \times G^{\mathbb{Q}}$ is the group associated to a $F/F_0$-hermitian space $V'$ obtained from $V$ by changing the signature at $\phi_0$ from $(n-1,1)$ to $(n,0)$ and the invariant at $v_0$ from $+1$ (resp. $-1$) to $-1$ (resp. $+1$) (i.e., $V'_{v_0}$ is a non-split (resp. split) $F_{v_0}/F_{0,v_0}$-hermitian space) in the hyperspecial case (resp. the almost self-dual case). The relevant Rapoport–Zink space $\mathcal{N}'$ associated to $\tilde{G}'$ is given by

\[
\mathcal{N}' \simeq (Z^\mathbb{Q}(\mathbb{Q}_p)/K_{Z^\mathbb{Q},p}) \times \mathcal{N}_{O_{\hat{E}_v}} \times \prod_{v \neq v_0} U(V)(F_{0,v})/K_{G,v},
\]

where the product is over places $v \neq v_0$ of $F_0$ over $p$, and $\mathcal{N}$ is isomorphic to $\mathcal{N}_{F_{w_0}/F_{0,v_0},n}$, the Rapoport–Zink space defined in §10.1 in the hyperspecial case, or isomorphic to $\mathcal{N}_{F_{w_0}/F_{0,v_0},n}$, the Rapoport–Zink space defined in §10.1 in the almost self-dual case.

13.2. The hermitian lattice $\mathbb{V}(A_0, A)$. For a locally noetherian $O_{E,(\nu)}$-scheme $S$ and a point $(A_0, \iota_0, \lambda_0, A, \iota, \lambda, \iota') \in \mathcal{M}(S)$, define the space of special homomorphisms to be

\[
\mathbb{V}(A_0, A) := \text{Hom}_{O_F}(A_0, A) \otimes \mathbb{Z}(p),
\]

a free $O_{F,(p)} := O_F \otimes \mathbb{Z}(p)$-module of finite rank. Then $\mathbb{V}(A_0, A)$ carries a $O_{F,(p)}$-valued hermitian form: for $x, y \in \mathbb{V}(A_0, A)$, the pairing $(x, y) \in O_{F,(p)}$ is given by

\[
(A_0 \xrightarrow{x} A \xrightarrow{\lambda} A^\vee \xrightarrow{\nu} A_0 \xrightarrow{\lambda^{-1}_0} A_0) \in \text{End}_{O_F}(A_0) \otimes \mathbb{Z}(p) = \iota_0(O_{F,(p)}) \simeq O_{F,(p)}.
\]

Notice that $\lambda^{-1}_0$ makes sense as the polarization $\lambda_0$ is coprime-to-$p$ by (M6).

Let $m \geq 1$. Given an $m$-tuple $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{V}(A_0, A)^m$, define its fundamental matrix to be

\[
T(\mathbf{x}) := (x_i, x_j)_{1 \leq i, j \leq m} \in \text{Herm}_m(O_{F,(p)}),
\]

an $m \times m$ hermitian matrix over $O_{F,(p)}$.

---

\footnote{We use the convention $(1, n-1)$ for the signature of Rapoport–Zink spaces while the convention $(n - 1, 1)$ for Shimura varieties; each of these two conventions is more preferable in its respective setting. Strictly speaking, [Cho18, Theorem 4.3] and [RSZ17b, Theorem 8.15] assume that $v_0$ is unramified over $p$. This assumption can be dropped due to the Eisenstein condition in [M8]. In fact, [Mih16, Definition 2.8 (ii)] specializes to the Eisenstein condition in [M8] for signature $(1, n - 1)$, so Mihatsch’s theorem [Mih16, Theorem 4.1] is still applicable even when $v_0$ is ramified over $p$.}
13.3. Semi-global Kudla–Rapoport cycles $\mathcal{Z}(T, \varphi_K)$. We say a Schwartz function $\varphi_K \in \mathcal{S}(\mathcal{V}_f^n)$ is $v_0$-admissible if it is $K$-invariant and $\varphi_{K,v} = 1_{(\Lambda_v)^m}$ for all $v$ above $p$ such that $v$ is non-split in $F$. First we consider a special $v_0$-admissible Schwartz function of the form

$$\varphi_K = (\varphi_i) \in \mathcal{S}(\mathcal{V}_f^n), \quad \varphi_i = 1_{\Omega_i}, \quad i = 1, \ldots, m$$

where $\Omega_i \subseteq \mathcal{V}_f$ is a $K$-invariant open compact subset such that $\Omega_{i,v} = \Lambda_v$ for all $v$ above $p$ such that $v$ is non-split in $F$. Given such a special Schwartz function $\varphi_K$ and $T \in \text{Herm}_m(O_{F,(p)})$, define a semi-global Kudla–Rapoport cycle $\mathcal{Z}(T, \varphi_K)$ over $\mathcal{M}$ as follows. For a locally noetherian $O_{E,(\nu)}$-scheme $S$, define $\mathcal{Z}(T, \varphi_K)(S)$ to be the groupoid of tuples $(A_0, t_0, \lambda_0, A, \iota, \lambda, \bar{\eta}_p, \bar{\eta}_w^0, \mathbf{x})$ where

1. $(A_0, t_0, \lambda_0, A, \iota, \lambda, \bar{\eta}_p, \bar{\eta}_w^0) \in \mathcal{M}(S),$
2. $\mathbf{x} = [x_1, \ldots, x_m] \in \mathcal{V}(A_0, A)^m$ with fundamental matrix $T(\mathbf{x}) = T.$
3. $\eta_i(x) \in (\Omega_i^{(p)}) \subseteq (\mathcal{V}_f^{(p)})^m.$
4. $\eta_i(x) \in (\Omega_i, w) \subseteq \mathcal{V}_w^m$ for all $v \neq v_0$ split in $F$, where $\bar{\eta}_p^0, \bar{\eta}_w^0 = \{\bar{\eta}_v\}$ (cf. [M5]).

The functor $S \mapsto \mathcal{Z}(T, \varphi_K)(S)$ is represented by a (possibly empty) Deligne–Mumford stack which is finite and unramified over $\mathcal{M}$ ([K-R13, Proposition 2.9]), and thus defines a cycle $\mathcal{Z}(T, \varphi_K) \in \text{Ch}(\mathcal{M}).$ For a general $v_0$-admissible Schwartz function $\varphi_K \in \mathcal{S}(\mathcal{V}_f^n)$, by extending $\mathbb{C}$-linearly we obtain a cycle $\mathcal{Z}(T, \varphi_K) \in \text{Ch}(\mathcal{M})_{\mathbb{C}}.$

13.4. The local arithmetic intersection number $\text{Int}_{T,v_0}(\varphi_K)$. Assume $T \in \text{Herm}_n(O_{F,(p)})_{>0}$ is totally positive definite. Let $t_1, \ldots, t_n$ be the diagonal entries of $T$. Let $\varphi_K \in \mathcal{S}(\mathcal{V}_f^n)$ be a special Schwartz function as in (13.3.0.9).

When $\Lambda_{v_0}$ is self-dual, define

$$\text{Int}_{T,v}(\varphi_K) := \chi(\mathcal{Z}(T, \varphi_K), \mathcal{O}_Z(t_1, \varphi_i) \otimes^L \cdots \otimes^L \mathcal{O}_Z(t_n, \varphi_i)) \cdot \log q_\nu,$$

where $q_\nu$ denotes the size of the residue field $k_\nu$ of $E_\nu$, $\mathcal{O}_Z(t, \varphi_i)$ denotes the structure sheaf of the semi-global Kudla–Rapoport divisor $\mathcal{Z}(t, \varphi_i)$, $\otimes^L$ denotes the derived tensor product of coherent sheaves on $\mathcal{M}$, and $\chi$ denotes the Euler–Poincaré characteristic (an alternating sum of lengths of $\mathcal{O}_{E,(\nu)}$-modules).

When $\Lambda_{v_0}$ is almost self-dual, we consider a diagram of Shimura varieties

$$\begin{array}{ccc}
\text{Sh}_{K \cap K^1} & \xrightarrow{\pi_1} & \text{Sh}_K \\
\pi_2 & \downarrow & \\
\text{Sh}_{K^1} & \xleftarrow{\pi_1} & \\
\end{array}$$

where the level at $v_0$ is modified as in Remark 10.2.1. Analogous to Remark 10.2.2, we obtain a cycle $\mathcal{Z}^\#(t_i, \varphi_i)$ on an integral model $\mathcal{M}_{K \cap K^1}$ of $\text{Sh}_{K \cap K^1}$, which can serve as an integral model of the pullback along $\pi_1$ of the generic fiber of $\mathcal{Z}(t_i, \varphi_i)$ on $\text{Sh}_K$. Similarly, we obtain a cycle $\mathcal{Z}^\#(T, \varphi_K)$ on $\mathcal{M}_{K \cap K^1}$, which can serve as an integral model of the pullback of the generic fiber of $\mathcal{Z}(T, \varphi_K)$. Define

$$\text{Int}_{T,v}(\varphi_K) := \frac{1}{\deg \pi_1} \chi(\mathcal{Z}^\#(T, \varphi_K), \mathcal{O}_{Z\#(t_1, \varphi_i)} \otimes^L \cdots \otimes^L \mathcal{O}_{Z\#(t_n, \varphi_i)}) \cdot \log q_\nu,$$

where $\mathcal{O}_{Z\#}$ denotes the structure sheaf of the pullback along $\pi_1$ of the generic fiber of $\mathcal{Z}(t_i, \varphi_i)$ on $\text{Sh}_K$. Similarly, we obtain a cycle $\mathcal{Z}^\#(T, \varphi_K)$ on $\mathcal{M}_{K \cap K^1}$, which can serve as an integral model of the pullback of the generic fiber of $\mathcal{Z}(T, \varphi_K)$. Define

$$\text{Int}_{T,v}(\varphi_K) := \frac{1}{\deg \pi_1} \chi(\mathcal{Z}^\#(T, \varphi_K), \mathcal{O}_{Z\#(t_1, \varphi_i)} \otimes^L \cdots \otimes^L \mathcal{O}_{Z\#(t_n, \varphi_i)}) \cdot \log q_\nu,$$
Finally, when $\Lambda_{v_0}$ is self-dual or almost self-dual, define
\[
\text{Int}_{T,v_0}(\varphi_K) := \frac{1}{[E:F_0]} \cdot \sum_{v|v_0} \text{Int}_{T,v}(\varphi_K).
\]
We extend the definition of $\text{Int}_{T,v_0}(\varphi_K)$ to a general $v_0$-admissible $\varphi_K \in \mathcal{S}(\mathcal{W}^n_f)$ by extending $\mathbb{C}$-linearly.

13.5. The semi-global identity. Recall that we are in the situation of §11.2 (hyperspecial level) or §11.3 (almost self-dual level).

**Theorem 13.5.1.** Assume $p > 2$. Assume $\varphi_K \in \mathcal{S}(\mathcal{W}^n_f)$ is $v_0$-admissible (§13.3). Then for any $T \in \text{Herm}_n(O_F,p)_{>0}$,
\[
\text{Int}_{T,v_0}(\varphi_K)q^T = c_K \cdot \partial \text{Eis}_{T,v_0}(z,\varphi_K),
\]
where $c_K = (-1)^n \frac{\text{vol}(K)}{\text{vol}(K)}$ is a nonzero constant independent of $T$ and $\varphi_K$, and $\text{vol}(K)$ is the volume of $K$ under a suitable Haar measure on $\tilde{G}(\mathbb{A}_f)$.

**Proof.** As explained in [Ter13, Remark 7.4], this follows routinely from our main Theorem 3.4.1 in the hyperspecial case. We briefly sketch the argument. The support of $Z(T)$ lies in the supersingular locus $M_{ss,k}$ by the same proof of [KR14, Lemma 2.21]. We may then compute the left-hand-side via $p$-adic uniformization §13.1 to reduce to the arithmetic intersection numbers on the Rapoport–Zink space $\mathcal{N}$ and a point-count. The arithmetic intersection number is equal to $W_{T,v_0}(1,0,\varphi_K,v_0)$ up to a nonzero constant independent of $T$ by our main Theorem 3.4.1 and Remark 3.4.2 (as $p > 2$). The point-count gives a theta integral of $\varphi_{K,f}$ which can be evaluated using the Siegel–Weil formula (due to Ichino [Ich04, §6] in our case) and becomes $\prod_{v \neq v_0} W_{T,v}(1,0,\varphi_{K,v}) \cdot e^{-2\pi i T}$ up to a constant independent of $T$. The result then follows from the factorization (12.2.0.3) of Fourier coefficients.

The identity follows in a similar way from our main Theorem 10.3.1 and Remark 10.3.2 in the almost self-dual case. In fact, by the same proof of [San17, Theorem 4.13], it remains to check that for $\Lambda = \langle 1 \rangle^{n-1} \oplus \langle \pi \rangle$ an almost self-dual lattice and $L \subseteq V$ any $O_F$-lattice of full rank $n$, we have the following identity
\[
(13.5.1.1) \quad \frac{\text{Den}(\Lambda)}{\text{Den}(\langle 1 \rangle^{n-1},\langle 1 \rangle^{n-1})} = \frac{\partial \text{Den}_A(L)}{\text{Int}(L)}.
\]
By Theorem 9.1.2 the left-hand-side of (13.5.1.1) is equal to $\text{Den}(\Lambda^2)$. By (3.6.1.1), $\text{Den}(\Lambda^2)$ is equal to the number of self-dual lattices containing $\Lambda^2$. Since $\Lambda^2$ is a vertex lattice of type 2, the latter is equal to the number of isotropic lines in a 2-dimensional nondegenerate $k_F$-hermitian space, which is $q + 1$ (cf. Remark 10.2.1). By Theorem 10.3.1 the right-hand-side of (13.5.1.1) is also equal to $q + 1$, and thus the desired identity (13.5.1.1) is proved. $\square$

14. Global integral models and the global identity

14.1. Global integral models at minimal levels. In this subsection we will define a global integral model over $O_E$ of the Shimura variety $\text{Sh}_K$ introduced in §11.1. We will be slightly more general than [RSZ17b §5], allowing $F/F_0$ to be unramified at all finite places. 

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We consider an $O_F$-lattice $\Lambda \subseteq V$ and let
\[ K^\circ_G = \{ g \in G(\mathbb{A}_f) : g(\Lambda \otimes O_F, \hat{O}_F) = \Lambda \otimes O_F, \hat{O}_F \}. \]
Assume that for any finite place $v$ of $F_0$ (write $p$ its residue characteristic),

(G0) if $p = 2$, then $v$ is unramified in $F$.

(G1) if $v$ is inert in $F$ and $V_v$ is split, then $\Lambda_v \subseteq V_v$ is self-dual.

(G2) if $v$ is inert in $F$ and $V_v$ is nonsplit, then $v$ is unramified over $p$ and $\Lambda_v \subseteq V_v$ is almost self-dual.

(G3) if $v$ is split in $F$, then $\Lambda_v \subseteq V_v$ is self-dual.

(G4) if $v$ is ramified in $F$, then $v$ is unramified over $p$ and $\Lambda_v \subseteq V_v$ is self-dual.

We take $K^\circ = K_{Z^0} \times K^\circ_G$, where $K_{Z^0}$ is the unique maximal open compact subgroup of $Z^0(\mathbb{A}_f)$ as in \[11.2\].

Notice the assumptions (G0)–(G4) ensure that each finite place $\nu$ of $E$ (write $p$ its residue characteristic), the triple $(A, \iota, \lambda)$ (resp. signature $(\phi, r_{\phi})$ for $\Phi [G]$) when $\nu$ is ramified in $F$. Moreover for each finite place $\nu$ of $E$, the base change $\mathcal{M}_{K^\circ, O_E(\nu)}$ is canonically isomorphic to the semi-global integral models defined in \[11.2\] \[11.3\] \[11.4\] \[11.5\] \[11.6\].

14.2. Global integral models at Drinfeld levels. With the same set-up as \[14.1\] but now we allow Drinfeld levels at split primes. Let $m = (m_v)$ be a collection of integers $m_v \geq 0$ indexed by finite places $v$ of $F_0$. Further assume

(G5) if $m_v > 0$, then $v$ satisfies \[S\] and each place $\nu$ of $E$ above $v$ satisfies \[D\].

We take $K^m_G \subseteq K^\circ_G$ such that $(K^m_G)_v = (K^\circ_G)_v$ if $m_v = 0$ and $(K^m_G)_v = (K^\circ_G)^{m_v}$ to be the principal congruence subgroup mod $\varepsilon^m$ if $m_v > 0$. Write $K^m = K_{Z^0} \times K^m_G$. Define $\mathcal{M}_{K^m}$ to be the normalization of $\mathcal{M}_{K^\circ}$ in $\text{Sh}_{K^\circ}(\tilde{G}, h_G)$.

Then $\mathcal{M}_{K^m}$ is a Deligne–Mumford stack finite flat over $\mathcal{M}_{K^\circ}$. Moreover for each finite place $\nu$ of $E$, the base change $\mathcal{M}_{K^m, O_E(\nu)}$ is canonically isomorphic to the semi-global integral models defined in \[11.2\] \[11.3\] \[11.4\] \[11.5\] \[11.6\]. Thus $\mathcal{M}_{K^m}$ is smooth at places over $v_0$ in \[G1\] \[G3\].
2.12. The test function \( \phi \). Then the global integral model \( M \) we take

\[ \text{Proof.} \]

Corollary 14.5.2. Kudla–Rapoport's global conjecture \[\text{[KR14, Conjecture 11.10]}\] holds.

We also take the axillary ideal \( a \) to be trivial (see (M6)). Then the global integral model \( M_{K^0} \) of \( M \) denoted taking fiber product over \( M_{a,\xi}^0 \).

14.3. Global Kudla–Rapoport cycles \( Z(T, \varphi_K) \). We continue with the same set-up as \[\text{[14.2]}\] From now on write \( K = K^m \) and \( M = M_{K^m} \) for short. Let \( \varphi_K = (\varphi_i) \in \mathcal{S}(V_f^m) \) be \( K \)-invariant.

Let \( t_1, \ldots, t_m \in F \). Let \( Z(t_i, \varphi_i) \) be the (possibly empty) Kudla–Rapoport cycle on the generic fiber of \( M \) (defined similarly as in \[\text{[13.3]}\]) and let \( Z(t_i, \varphi_i) \) be its Zariski closure in the global integral model \( M \). Then we have a decomposition into the global Kudla–Rapoport cycles \( Z(T, \varphi_K) \) over \( M \),

\[
Z(t_1, \varphi_1) \cap \cdots \cap Z(t_m, \varphi_m) = \bigsqcup_{T \in \text{Herm}_n(F)} Z(T, \varphi_K),
\]

here \( \cap \) denotes taking fiber product over \( M \), and the indexes \( T \) have diagonal entries \( t_1, \ldots, t_m \).

\[
\text{14.4. The arithmetic intersection number } \text{Int}_T(\varphi_K). \text{ For nonsingular } T \in \text{Herm}_n(F), \text{ define }
\text{Int}_T(\varphi_K) := \sum_v \text{Int}_{T,v}(\varphi_K)
\]

to be the sum over all finite places \( v \) of \( F \) of local arithmetic intersection numbers defined as in \[\text{[13.4]}\] By the same proof of \[\text{[KR14, Lemma 2.21]}\], this sum is nonzero only if \( \text{Diff}(T, \mathcal{V}) = \{v\} \) is a singleton, and in this case \( v \) is necessarily nonsplit in \( F \). Hence

\[
(14.4.0.2) \quad \text{Int}_T(\varphi_K) \neq 0 \implies \text{Diff}(T, \mathcal{V}) = \{v\} \text{ and } \text{Int}_T(\varphi_K) = \text{Int}_{T,v}(\varphi_K).
\]

14.5. The global Kudla–Rapoport conjecture for nonsingular Fourier coefficients. Assume that we are in the situation of \[\text{[14.2]}\]. We say \( \varphi_K \in \mathcal{S}(V_f^m) \) is inert-admissible if it is \( v \)-admissible at all \( v \) inert in \( F \) (\[\text{[13.3]}\]). When \( \varphi_K \) is inert-admissible, the base change of the global Kudla–Rapoport cycle \( Z(T, \varphi_K) \) to \( \text{Spec} O_{E, (v)} \) above an inert prime agrees with the semi-global Kudla–Rapoport cycle defined in \[\text{[13.3]}\]. We say a nonsingular \( T \in \text{Herm}_n(F) \) is inert if \( \text{Diff}(T, \mathcal{V}) = \{v\} \) where \( v \) is inert in \( F \) and not above 2.

**Theorem 14.5.1.** Assume \( \varphi_K \in \mathcal{S}(V_f^m) \) is inert-admissible. Let \( T \in \text{Herm}_n(F) \) be inert. Then

\[
\text{Int}_T(\varphi_K)q^T = c_K \cdot \partial \text{Eist}(z, \varphi_K),
\]

where \( c_K = \frac{(-1)^n}{\text{vol}(K)} \) as in Theorem \[\text{[13.5.1]}\].

**Proof.** Since \( T \) is inert, we know that \( T > 0 \), and moreover by \[\text{[14.4.0.2]}\] and \[\text{[12.4.0.7]}\] both sides are contributed non-trivially only by the term at \( \text{Diff}(T, \mathcal{V}) = \{v\} \). Since \( \varphi_K \) is inert-admissible, both sides are zero unless \( T \in \text{Herm}_n(O_{E,(p)}) \) \( (p \text{ the residue characteristic of } v) \). So we can apply Theorem \[\text{[13.5.1]}\] to obtain \( \text{Int}_{T,v}(\varphi_K)q^T = c_K \cdot \partial \text{Eist}_{T,v}(z, \varphi_K) \). \( \square \)

**Corollary 14.5.2.** Kudla–Rapoport's global conjecture \[\text{[KR14, Conjecture 11.10]}\] holds.

**Proof.** We take \( F_0 = \mathbb{Q} \) and \( K = K^c \). We also take the axillary ideal \( a \) to be trivial (see (M6)). Then the global integral model \( M_{K^c} \) agrees with the moduli stack \( M^V \) in \[\text{[KR14, Proposition 2.12]}\]. The test function \( \varphi \) in \[\text{[KR14]}\] satisfies \( \varphi_K = 1_{(\mathbb{A})^n} \) and \( \varphi_{\infty} \) is the Gaussian function, so \( \varphi_K \)
under the standard basis of $D$. \hfill \Box

\section{15. The arithmetic Siegel–Weil formula}

15.1. **Complex uniformization.** Assume we are in the situation of \[11.1\]. Under the decomposition (11.2.0.1), we may identify the the $\tilde{G}(\mathbb{R})$-conjugacy class $\{h_{\tilde{G}}\}$ as the product $\{h_{\mathbb{Z}^Q}\} \times \prod_{\phi \in \Phi} \{h_{G,\phi}\}$. Notice $\{h_{\mathbb{Z}^Q}\}$ is a singleton as $\mathbb{Z}^Q$ is a torus, and $\{h_{G,\phi}\}$ is also a singleton for $\phi \neq \phi_0$ as $h_{G,\phi}$ is the trivial cocharacter. For $\phi = \phi_0$ the cocharacter is given by $h_{G,\phi_0}(z) = \text{diag}\{1_{n-1}, \bar{z}/z\}$, and $\{h_{G,\phi_0}\}$ is the hermitian symmetric domain

$$D_{n-1} \cong U(n-1,1)/(U(n-1) \times U(1)).$$

We may identify $D_{n-1} \subseteq \mathbb{P}(V_{\phi_0})(\mathbb{C})$ as the open subset of negative $\mathbb{C}$-lines in $V_{\phi_0}$, and $\tilde{G}(\mathbb{R})$ acts on $D_{n-1}$ via its quotient $\text{PU}(V_{\phi_0})(\mathbb{R})$. We may also identity it with the open $(n-1)$-ball

$$D_{n-1} \cong \{ z \in \mathbb{C}^{n-1} : |z| < 1 \}, \quad \begin{array}{c} (z_1, \ldots, z_n) \mapsto (z_1/z_n, \ldots, z_{n-1}/z_n), \end{array}$$

under the standard basis of $V_{\phi_0}$. In this way we obtain a complex uniformization (via $\phi_0$),

\begin{equation}
\text{Sh}_K(\mathbb{C}) = \tilde{G}(\mathbb{Q}) \backslash [D_{n-1} \times \tilde{G}(K_f)/K].
\end{equation}

15.2. **Green currents.** Write $\mathcal{D} = D_{n-1}$ for short. Let $x \in V_{\phi_0}$ be a nonzero vector. For any $z \in D$, we let $x = x_z + x_{z\perp}$ be the orthogonal decomposition with respect to $z$ (i.e., $x_z \in z$ and $x_{z\perp} \perp z$). Let $R(x, z) = -(x_z, x_z)$. Define

$$\mathcal{D}(x) = \{ z \in D : z \perp x \} = \{ z \in D : R(x, z) = 0 \}.$$

Then $\mathcal{D}(x)$ is nonempty if and only if $(x, x) > 0$, in which case $\mathcal{D}(x)$ is an analytic divisor on $\mathcal{D}$. Define *Kudla’s Green function* to be

$$g(x, z) = -\text{Ei}(-2\pi R(x, z)),$$

where $\text{Ei}(u) = -\int_1^\infty \frac{e^{-tu}}{t} \, dt$ is the exponential integral. Then $g(x, -)$ is a smooth function on $\mathcal{D} \setminus \mathcal{D}(x)$ with a logarithmic singularity along $\mathcal{D}(x)$. By [Liu11a, Proposition 4.9], it satisfies the $(1,1)$-current equation for $\mathcal{D}(x)$,

$$dd^c[g(x)] + \delta_{\mathcal{D}(x)} = [\omega(x)],$$

where $\omega(x, -) = e^{2\pi i(x, x)} \varphi_{KM}(x, -)$, and $\varphi_{KM}(-, -) \in (\mathcal{O}(V_{\phi_0}) \otimes A^{1,1}(\mathcal{D}))^{U(V_{\phi_0})(\mathbb{R})}$ is the *Kudla–Millson Schwartz form* ([KM86]). Here we recall $d = \partial + \bar{\partial}$, $d^c = \frac{1}{4\pi i} (\partial - \bar{\partial})$ and $dd^c = -\frac{1}{2\pi i} \partial \bar{\partial}$.

More generally, let $x = (x_1, \ldots, x_m) \in V_{\phi_0}^m$ such that its fundamental matrix $T(x) = ((x_i, x_j))_{1 \leq i, j \leq m}$ is nonsingular. Define

$$\mathcal{D}(x) = \mathcal{D}(x_1) \cap \cdots \cap \mathcal{D}(x_m),$$

which is nonempty if and only if $T(x) > 0$. Define Kudla’s Green current by taking star product

$$g(x) := [g(x_1)] \ast \cdots \ast [g(x_m)].$$

It satisfies the $(m, m)$-current equation for $\mathcal{D}(x)$,

$$dd^c(g(x)) + \delta_{\mathcal{D}(x)} = [\omega(x_1) \wedge \cdots \wedge \omega(x_m)].$$
Here we recall that
\[ [g(x)] * [g(y)] := [g(x)] \land \delta_D(y) + [\omega(x)] \land [g(y)]. \]

15.3. The local arithmetic Siegel–Weil formula at archimedean places. Let \( T \in \text{Herm}_m(F) \) be nonsingular. Let \( \varphi_K \in \mathcal{S}(\mathbb{V}_p^n) \) be \( K \)-invariant. Let \( Z(T, \varphi_K) \) be the (possibly empty) Kudla–Rapoport cycle on the generic fiber \( \text{Sh}_K \). Then
\[ Z(T, \varphi_K)(\mathbb{C}) = \sum_{(x, \tilde{g}) \in \tilde{G}(\mathbb{Q})\backslash [\text{Vol}(x) \times \tilde{G} / K]} \varphi_K(\tilde{g}^{-1}x) \cdot Z(x, \tilde{g})_K, \]
where we define the cycle on \( \text{Sh}_K(\mathbb{C}) \) via the complex uniformization \([15.1.0.1]\),
\[ Z(x, \tilde{g})_K = \tilde{G}_x(\mathbb{Q}) / \tilde{G}(x) \tilde{G} / K, \]
and \( \tilde{G}_x \subseteq \tilde{G} \) is the stabilizer of \( x \). Define a Green current for \( Z(T, \varphi_K)(\mathbb{C}) \) by
\[ g(y_{\phi_0}, T, \varphi_K, z, \tilde{g}) := \sum_{x \in \text{Vol}(x) / T \langle x \rangle = T} \varphi_K(\tilde{g}^{-1}x) \cdot g(x \cdot a, z), \quad (z, \tilde{g}) \in \mathcal{D} \times \tilde{G}(A_f), \]
where \( a \in \text{GL}(V_{\phi_0}) \cong \text{GL}_n(\mathbb{C}) \) and \( y_{\phi_0} = a^t \bar{a} \). Define the archimedean arithmetic intersection number (depending on the parameter \( y_{\phi_0} \)) to be
\[ \text{Int}_{T, \phi_0}(y_{\phi_0}, \varphi_K) \deq \frac{1}{2} \int_{\text{Sh}_K(\mathbb{C})} g(y_{\phi_0}, T, \varphi_K). \]
Replacing the choice of \( \phi_0 \) by another \( \phi \in \Phi \) \([11.1]\) gives rise to a Shimura variety \( \text{Sh}_K^\phi \) conjugate to \( \text{Sh}_K \), associated to a hermitian space \( V^\phi \) whose signature at \( \phi_0, \phi \) are swapped compared to \( V \). Thus we can define in the same way the archimedean intersection number for any \( \phi \in \Phi \),
\[ \text{Int}_{T, \phi}(y, \varphi_K) = \frac{1}{2} \int_{\text{Sh}_K^\phi(\mathbb{C})} g(y, T, \varphi_K). \]

**Theorem 15.3.1.** Assume \( \varphi_K \in \mathcal{S}(\mathbb{V}_p^n) \) is \( K \)-invariant. Let \( T \in \text{Herm}_n(F) \) be nonsingular and \( \phi \in \Phi \). Then
\[ \text{Int}_{T, \phi}(y, \varphi_K)q^T = c_K \cdot \partial \text{Eis}_{T, \phi}(z, \varphi_K), \]
where \( c_K = (-1)^n \text{vol}(K) \) as in Theorem \([13.5.1]\).

**Proof.** By the main archimedean result of \([\text{Liu11a}]\) Proposition 4.5, Theorem 4.17) (the archimedean analogue of our main Theorem \([3.4.1]\)) and the standard unfolding argument, we can express the integral \([15.3.0.2]\) as a product involving the derivative \( W_{T, \phi}(g_2, 0, \varphi_K)q^T \) and the product of values \( \prod_{v \neq \phi} W_{T, v}(g_2, 0, \varphi_K) \) from the Siegel–Weil formula, up to a nonzero constant independent of \( T \). The result then follows from the factorization \([12.2.0.3]\) of Fourier coefficients and comparing the constant with that of Theorem \([13.5.1]\). See the proof of \([\text{Liu11a}]\) Theorem 4.20] and the proof in the orthogonal case \([\text{BY18}]\) Theorem 7.1] for details. When \( V \) is anisotropic (e.g., when \( F_0 \neq \mathbb{Q} \)), the result also follows from \([\text{GS19}]\) (1.19) for \( r = p + 1 = n \) in the notation there. \(\square\)
15.4. **Arithmetic degrees of Kudla–Rapoport cycles.** Let us come back to the situation of §14.2. Let $T \in \text{Herm}_n(F)$ be nonsingular. Let $\varphi_K = (\varphi_i) \in \mathcal{S}(\mathcal{V}^n_f)$ be $K$-invariant. Define the arithmetic degree (depending on the parameter $y = (y_\phi)_{\phi \in \Phi}$)

$$\hat{\text{deg}}_T(y, \varphi_K) := \text{Int}_T(\varphi_K) + \sum_{\phi \in \Phi} \text{Int}_{T, \phi}(y_\phi, \varphi_K)$$

(15.4.0.1)

to be the sum of all nonarchimedean and archimedean intersection numbers. Define the generating series of arithmetic degrees of Kudla–Rapoport cycles to be

$$\hat{\text{deg}}(z, \varphi_K) := \sum_{T \in \text{Herm}_n(F)} \hat{\text{deg}}_T(y, \varphi_K) q^T.$$

It is related to the usual arithmetic degree on arithmetic Chow groups as we now explain. For nonzero $t_1, \ldots, t_n \in F$, we have classes in the Gillet–Soulé arithmetic Chow group (with $\mathbb{C}$-coefficients) of the regular Deligne–Mumford stack $\mathcal{M}_K$ ([GS90], [G09]),

$$\hat{Z}(y, t_i, \varphi_i) := (Z(t_i, \varphi_i), g(y, t_i, \varphi_i)) \in \hat{\text{Ch}}^1_C(\mathcal{M}_K).$$

We have an arithmetic intersection product on $n$ copies of $\hat{\text{Ch}}^1_C(\mathcal{M}_K)$,

$$\langle \cdot, \cdots, \cdot \rangle_{\text{GS}} : \hat{\text{Ch}}^1_C(\mathcal{M}_K) \times \cdots \times \hat{\text{Ch}}^1_C(\mathcal{M}_K) \to \hat{\text{Ch}}^n_C(\mathcal{M}_K),$$

and when $\mathcal{M}_K$ is proper over $O_E$, a degree map on the arithmetic Chow group of 0-cycles,

$$\hat{\text{deg}} : \hat{\text{Ch}}^n_C(\mathcal{M}_K) \to \mathbb{C}.$$

We may compose these two maps and obtain a decomposition

$$\hat{\text{deg}}(\hat{Z}(y, t_1, \varphi_1), \cdots, \hat{Z}(y, t_n, \varphi_n))_{\text{GS}} = \sum_{T} \hat{\text{deg}}_T(y, \varphi_K),$$

where the matrices $T$ have diagonal entries $t_1, \ldots, t_n$. The terms corresponding to nonsingular $T$ agree with (15.4.0.1), at least in the hyperspecial case at inert primes.

15.5. **The arithmetic Siegel–Weil formula when $F/F_0$ is unramified.** Assume that we are in the situation of §14.2.

**Theorem 15.5.1.** (Arithmetic Siegel–Weil formula). Assume that $F/F_0$ is unramified at all finite places and split at all places above 2. Assume that $\varphi_K \in \mathcal{S}(\mathcal{V}^n_f)$ is inert-admissible (§14.5) and nonsingular (§12.3) at two places split in $F$. Then

$$\hat{\text{deg}}(z, \varphi_K) = c_K \cdot \partial \text{Eis}(z, \varphi_K),$$

where $c_K = \left(-\frac{1}{\text{vol}(K)}\right)^n$ as in Theorem 13.5.1.

**Remark 15.5.2.** The assumption that $F/F_0$ is unramified at all finite places implies that $F_0 \neq \mathbb{Q}$ and hence the Shimura variety $\text{Sh}_K$ is projective and the global integral model $\mathcal{M}_K$ is proper over $O_E$. Moreover, this assumption forces that the hermitian space $V$ to be nonsplit at some inert place, and thus it is necessary to allow almost self-dual level at some inert place (as we did in [G2]).
Remark 15.5.3. The Schwartz function $\varphi_K$ satisfying the assumptions in Theorem 15.5.1 exists for a suitable choice of $K$ since we allow arbitrary Drinfeld levels at split places.

Proof. Since $\varphi_K$ is nonsingular at two places, by (12.3.0.5) we know that only nonsingular $T$ contributes non-trivially to the sum (12.4.0.6). For a nonsingular $T$, by (12.4.0.7) we know that $\text{Diff}(T,V) = \{v\}$ for $v$ nonsplit in $F$. By the assumption on $F/F_0$, we know that either $T$ is inert or $v$ is archimedean. The result then follows from Theorem 14.5.1 and Theorem 15.3.1 depending on $T$ is inert or $v$ is archimedean.

\[\square\]

References

[BC] Olivier Brinon and Brian Conrad. Cmi summer school notes on p-adic hodge theory. http://math.stanford.edu/~conrad/papers/notes.pdf

[BHK+17] Jan Bruinier, Benjamin Howard, Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. Modularity of generating series of divisors on unitary Shimura varieties. arXiv e-prints, page arXiv:1702.07812, Feb 2017.

[BP19] Raphaël Beuzart-Plessis. A new proof of Jacquet-Rallis’s fundamental lemma. arXiv e-prints, arXiv:1901.02653, 2019.

[Bre00] Christophe Breuil. Groupes p-divisibles, groupes finis et modules filtrés. Ann. of Math. (2), 152(2):489–549, 2000.

[BY18] Jan Hendrik Bruinier and Tonghai Yang. Arithmetic degrees of special cycles and derivatives of Siegel Eisenstein series. arXiv e-prints, arXiv:1802.09489, Feb 2018.

[Cho18] Sungyoon Cho. The basic locus of the unitary Shimura variety with parahoric level structure, and special cycles. arXiv e-prints, arXiv:1805.01666, 2018.

[CY18] Sungmun Cho and Takuya Yamauchi. A reformulation of the Siegel series and intersection numbers. arXiv e-prints, arXiv:1807.09997, 2018.

[CY00] Wee Teck Gan and Jiu-Kang Yu. Group schemes and local densities. Duke Math. J., 105(3):497–524, 2000.

[GS90] Benedict H. Gross and Christophe Soulé. Arithmetic intersection theory. Inst. Hautes Études Sci. Publ. Math., (72):93–174 (1990), 1990.

[GS91] Benedict H. Gross and Don B. Zagier. Heegner points and derivatives of L-series. Invent. Math., 84(2):225–320, 1986.
[HLZ19] Xuhua He, Chao Li, and Yihang Zhu. Fine Deligne-Lusztig varieties and arithmetic fundamental lemmas. *Forum Math. Sigma*, 7:e47, 55, 2019.

[How18] Benjamin Howard. Linear invariance of intersections on unitary Rapoport-Zink spaces. *arXiv e-prints, arXiv:1811.01482*, 2018.

[HZ76] F. Hirzebruch and D. Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976.

[Ich04] Atsushi Ichino. A regularized Siegel-Weil formula for unitary groups. *Math. Z.*, 247(2):241–277, 2004.

[Kat99] Hidenori Katsurada. An explicit formula for Siegel series. *Amer. J. Math.*, 121(2):415–452, 1999.

[Kis06] Mark Kisin. Crystalline representations and $F$-crystals. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 459–496. Birkhäuser Boston, Boston, MA, 2006.

[Kit93] Yoshiyuki Kitaoka. *Arithmetic of quadratic forms*, volume 106 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[KM86] Stephen S. Kudla and John J. Millson. The theta correspondence and harmonic forms. I. *Math. Ann.*, 274(3):353–378, 1986.

[KR99] Stephen S. Kudla and Michael Rapoport. Arithmetic Hirzebruch-Zagier cycles. *J. Reine Angew. Math.*, 515:155–244, 1999.

[KR00a] Stephen S. Kudla and Michael Rapoport. Cycles on Siegel threefolds and derivatives of Eisenstein series. *Ann. Sci. École Norm. Sup. (4)*, 33(5):695–756, 2000.

[KR00b] Stephen S. Kudla and Michael Rapoport. Height pairings on Shimura curves and $p$-adic uniformization. *Invent. Math.*, 142(1):153–223, 2000.

[KR11] Stephen S. Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties I. Unramified local theory. *Invent. Math.*, 184(3):629–682, 2011.

[KR14] Stephen S. Kudla and Michael Rapoport. Special cycles on unitary Shimura varieties II: Global theory. *J. Reine Angew. Math.*, 697:91–157, 2014.

[KRSZ19] Stephen S. Kudla, Michael Rapoport, Brian Smithling, and Wei Zhang. *in preparation*, 2019.

[KRY99] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. On the derivative of an Eisenstein series of weight one. *Internat. Math. Res. Notices*, (7):347–385, 1999.

[KRY06] Stephen S. Kudla, Michael Rapoport, and Tonghai Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006.

[Kud97a] Stephen S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.

[Kud97b] Stephen S. Kudla. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)*, 146(3):545–646, 1997.

[Kud04] Stephen S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin $L$-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004.

[Liu11a] Yifeng Liu. Arithmetic theta lifting and $L$-derivatives for unitary groups, I. *Algebra Number Theory*, 5(7):849–921, 2011.

[Liu11b] Yifeng Liu. Arithmetic theta lifting and $L$-derivatives for unitary groups, II. *Algebra Number Theory*, 5(7):923–1000, 2011.

[Liu18] Yifeng Liu. Fourier-Jacobi cycles and arithmetic relative trace formula. https://gauss.math.yale.edu/~yl2269/FJcycle.pdf August 2018.

[LL20] Chao Li and Yifeng Liu. Chow groups and $L$-derivatives of automorphic motives for unitary groups. *preprint*, May 2020.

[Lus76] G. Lusztig. Coxeter orbits and eigenspaces of Frobenius. *Invent. Math.*, 38(2):101–159, 1976.

[LZ17] Chao Li and Yihang Zhu. Remarks on the arithmetic fundamental lemma. *Algebra Number Theory*, 11(10):2425–2445, 2017.
[LZ20] Chao Li and Wei Zhang. On the arithmetic Siegel–Weil formula for GSpin Shimura varieties. in preparation, May 2020. 2

[Mih16] Andreas Mihatsch. Relative unitary RZ-spaces and the Arithmetic Fundamental Lemma. arXiv e-prints, arXiv:1611.06520, November 2016. 9, 66

[RSZ17a] M. Rapoport, B. Smithling, and W. Zhang. On the arithmetic transfer conjecture for exotic smooth formal moduli spaces. Duke Math. J., 166(12):2183–2336, 2017. 53

[RSZ17b] Michael Rapoport, Brian Smithling, and Wei Zhang. Arithmetic diagonal cycles on unitary Shimura varieties. arXiv e-prints, arXiv:1710.06962, Oct 2017. 4, 55, 59, 60, 61, 62, 66, 68

[RSZ18] Michael Rapoport, Brian Smithling, and Wei Zhang. Regular formal moduli spaces and arithmetic transfer conjectures. Math. Ann., 370(3-4):1079–1175, 2018. 52, 53

[RSZ19] Michael Rapoport, Brian Smithling, and Wei Zhang. On Shimura varieties for unitary groups. arXiv e-prints, arxiv:1906.12346, June 2019. 59, 60, 61, 62, 63

[RZ96] Michael Rapoport and Thomas Zink. Period spaces for p-divisible groups, volume 141 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1996. 9, 66

[RZ17] Michael Rapoport and Thomas Zink. On the Drinfeld moduli problem of p-divisible groups. Camb. J. Math., 5(2):229–279, 2017. 62

[San17] Siddarth Sankaran. Improper intersections of Kudla-Rapoport divisors and Eisenstein series. J. Inst. Math. Jussieu, 16(5):899–945, 2017. 59, 60, 61, 62, 63

[Sie51] Carl Ludwig Siegel. Indefinite quadratische Formen und Funktionentheorie. I. Math. Ann., 124:17–54, 1951. 2

[Tat67] J. T. Tate. p-divisible groups. In Proc. Conf. Local Fields (Driebergen, 1966), pages 158–183. Springer, Berlin, 1967. 18

[Ter11] Ulrich Terstiege. Intersections of arithmetic Hirzebruch-Zagier cycles. Math. Ann., 349(1):161–213, 2011. 2

[Ter13] Ulrich Terstiege. Intersections of special cycles on the Shimura variety for GU(1,2). J. Reine Angew. Math., 684:113–164, 2013. 2, 3, 6, 10, 15, 29, 51, 55, 58, 68

[Vol10] Inken Vollaard. The supersingular locus of the Shimura variety for GU(1,s). Canad. J. Math., 62(3):668–720, 2010. 10

[VW11] Inken Vollaard and Torsten Wedhorn. The supersingular locus of the Shimura variety of GU(1, n − 1) II. Invent. Math., 184(3):591–627, 2011. 7, 11

[Wei65] André Weil. Sur la formule de Siegel dans la théorie des groupes classiques. Acta Math., 113:1–87, 1965. 2

[YZZ13] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. The Gross-Zagier formula on Shimura curves, volume 184 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2013. 2

[Zha14] Wei Zhang. Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups. Ann. of Math. (2), 180(3):971–1049, 2014. 7

[Zha19] Wei Zhang. Weil representation and arithmetic fundamental lemma. preprint, May 2019. 7, 9

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