On graphs having maximal independent sets of exactly $t$ distinct cardinalities

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Abstract

For a given positive integer $t$ we consider graphs having maximal independent sets of precisely $t$ distinct cardinalities and restrict our attention to those that have no vertices of degree one. In the situation when $t$ is four or larger and the length of the shortest cycle is at least $6t - 6$, we completely characterize such graphs.

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1 Introduction

A well-covered graph (Plummer [6]) is one in which every maximal independent set of vertices is of one cardinality and is hence a maximum independent set. Finbow, Hartnell and Whitehead [5] defined the class $\mathcal{M}_t$ to consist of those graphs which have exactly $t$ different sizes of maximal independent sets. Finbow, Hartnell and Nowakowski [4] proved that the well-covered graphs (the $\mathcal{M}_1$ collection) of girth (the length of a shortest cycle) 6 or more, with the exceptions of $K_1$ and $C_7$, have the property that every vertex has degree one or has exactly one vertex of degree one in its neighborhood. Thus, $C_7$ is the unique graph in $\mathcal{M}_1$ with girth at least 6.

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that has minimum degree at least two. The graphs in $\mathcal{M}_2$ of girth 8 or more have also been characterized ([3]). There are precisely five graphs in $\mathcal{M}_2$ of girth at least 8 and minimum degree 2 or more, namely the cycles $C_8, C_9, C_{10}, C_{11}$ and $C_{13}$. This implies there are no $\mathcal{M}_1$ graphs of girth at least 8 with minimum degree 2 or more and no $\mathcal{M}_2$ graphs of girth 14 or more and having minimum degree at least 2. For related work on the class $\mathcal{M}_1$ see [1] and [2].

In this paper we investigate the graphs in $\mathcal{M}_t$ that have minimum degree at least 2 and higher girth and establish that the characterization of these in $\mathcal{M}_1$ and $\mathcal{M}_2$ is part of a general pattern. In particular, for $t \geq 3$ we show that among graphs with minimum degree at least 2, $\mathcal{M}_t$ does not contain a graph of girth at least $6t+2$ and that $C_{6t-4}, C_{6t-3}, C_{6t-2}, C_{6t-1}$ and $C_{6t+1}$ are the only exceptions for girth at least $6t-4$. Furthermore, if $t \geq 4$, then these cycles along with $C_{6t-6}$ are the only graphs in $\mathcal{M}_t$ that have minimum degree at least 2 and girth at least $6t-6$.

Let $G$ be a finite simple graph. A vertex of degree 1 is called a leaf and any vertex that is adjacent to a leaf is called a support vertex. If $C$ is a cycle in a graph $G$ and $u$ and $v$ belong to $C$, we let $uCv$ denote the shorter of the two $u,v$-paths that are part of $C$. For $A \subseteq V(G)$ and $u$ a vertex in $G$, $d(u, A)$ will denote the length of a shortest path in $G$ from $u$ to a vertex of $A$. We will use $\mathcal{M}(G)$ to denote the collection of all maximal independent sets of $G$ and we define the independence spectrum (spectrum for short) of $G$ to be the set $S(G) = \{|I| : I \in \mathcal{M}(G)\}$. The class $\mathcal{M}_t$ consists of those graphs $G$ for which $|S(G)| = t$. The spectrum is not necessarily a set of consecutive positive integers (e.g., $S(K_{2,4,5}) = \{2, 4, 5\}$), but for paths and cycles it is. We denote the set of positive integers between $p$ and $q$ inclusive by $[p,q]$. The following proposition is easy to establish.

**Proposition 1** For each positive integer $n$ at least 3,

$$S(C_n) = [\lceil n/3 \rceil, \lfloor n/2 \rfloor] \quad \text{and} \quad S(P_n) = [\lceil n/3 \rceil, \lceil n/2 \rceil].$$

Hence, $C_n \in \mathcal{M}_t$ and $P_n \in \mathcal{M}_s$ where $t = \lfloor n/2 \rfloor - \lceil n/3 \rceil + 1$ and $s = \lceil n/2 \rceil - \lceil n/3 \rceil + 1$.

The following lemma from [5] will be used throughout—often without mention.

**Lemma 2** [5] If the graph $G$ belongs to $\mathcal{M}_t$ and $I$ is an independent set of $G$, then for every component $C$ of $G - N[I]$ there exists $k \leq t$ such that $C \in \mathcal{M}_k$. In addition, $G - N[I] \in \mathcal{M}_r$ for some $r \leq t$.

Lemma 2 will most often be used in the following way. We will find an independent set $I$ in a graph $G$ and demonstrate that $G - N[I]$ has a component that is in the class $\mathcal{M}_s$ for some $s > t$ and conclude that $G \notin \mathcal{M}_t$. The following lemma will be used in that context with Lemma 2.

**Lemma 3** If a cycle $C$ is in $\mathcal{M}_t$ and a new vertex is added as a leaf adjacent to a single vertex of $C$, then the resulting graph belongs to $\mathcal{M}_{t+1}$.
Proof. Assume \( S(C) = [k, k + t - 1] \). Let \( H \) be the graph formed by adding a leaf \( x \) adjacent to \( y \). Let \( u \) and \( v \) be the neighbors of \( y \) on \( C \). Note that \( \{ I \in \mathcal{M}(H) : y \in I \} = \{ J \in \mathcal{M}(C) : y \in J \} \), and because of the symmetry of the cycle, \( S(C) = \{|J| : J \in \mathcal{M}(C), y \in J \} \). Also, \( \{ I \in \mathcal{M}(H) : u \in I \} = \{ J \cup \{ x \} : J \in \mathcal{M}(C), u \in J \} \). This shows that \([k, k + t] \subseteq S(H)\). If \( H \) has a maximal independent set \( A \) of size less than \( k \), then \( x \in A \) and neither \( u \) nor \( v \) is in \( A \), for otherwise \( A \cap C \) is a maximal independent set in \( C \) of cardinality less than \( k \). But now \( A' = (A - \{ x \}) \cup \{ y \} \in \mathcal{M}(C) \) and \(|A'| < k\), a contradiction. Therefore, \( S(H) = [k, k + t] \). We conclude that \( H \in \mathcal{M}_{t+1} \).

\[ \square \]

In the class of graphs with leaves there is no connection between girth and the size of the spectrum. This can be seen by the following general construction. Let \( t \geq 2 \) and \( g \geq 3 \) be integers. Let \( H \) be the graph formed by adding a single leaf adjacent to each vertex of a cycle of order \( g \). For a single vertex \( x \) on the cycle attach a path \( v_1, v_2, \ldots, v_{2t-3} \) to \( H \) by making \( x \) and \( v_1 \) adjacent. Then add two leaves adjacent to \( v_i \) if \( i \) is odd, and add one leaf adjacent to \( v_j \) if \( j \) is even. The resulting graph of order \( 2g + 5t - 7 \) has girth \( g \) and belongs to the class \( \mathcal{M}_t \). (The spectrum of this graph is \([g + 2t - 3, g + 3t - 4]\).) For this reason we will henceforth consider only graphs having minimum degree at least 2. For ease of reference we denote the class of graphs that are in \( \mathcal{M}_t \) and have no leaves (i.e., minimum degree at least 2) by \( \mathcal{M}_t^2 \). Note that \( \mathcal{M}_t^2 \subseteq \mathcal{M}_t \). In the course of several of our proofs we will show that some given graph is not in \( \mathcal{M}_t^2 \) by demonstrating it does not belong to \( \mathcal{M}_t \).

The remainder of this paper is devoted to verifying the entries in the following table.

| girth  | 6t - 6 | 6t - 5 | 6t - 4 | 6t - 3 | 6t - 2 | 6t - 1 | 6t | 6t + 1 | \( \geq 6t + 2 \) |
|--------|--------|--------|--------|--------|--------|--------|----|--------|----------------|
| \( t = 1 \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \Delta \) | \( \emptyset \) | \( C_7 \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) |
| \( t = 2 \) | \( C_{12} \) | \( \Delta \) | \( C_8 \) | \( C_9 \) | \( C_{10} \) | \( C_{11} \) | \( \emptyset \) | \( C_{13} \) | \( \emptyset \) |
| \( t = 3 \) | \( \emptyset \) | \( C_7 \) | \( C_8 \) | \( C_{14} \) | \( C_{15} \) | \( C_{16} \) | \( C_{17} \) | \( \emptyset \) | \( C_{19} \) | \( \emptyset \) |
| \( t = 4 \) | \( C_{21} \) | \( \emptyset \) | \( C_{22} \) | \( C_{23} \) | \( \emptyset \) | \( C_{24} \) | \( \emptyset \) | \( C_{25} \) | \( \emptyset \) |
| \( t \geq 5 \) | \( C_{6t-6} \) | \( \emptyset \) | \( C_{6t-4} \) | \( C_{6t-3} \) | \( C_{6t-2} \) | \( C_{6t-1} \) | \( \emptyset \) | \( C_{6t+1} \) | \( \emptyset \) |

Table 1: Graphs of given girth in \( \mathcal{M}_t^2 \)

The entry for a given girth (written as a function of \( t \)) and a given value of \( t \) should be interpreted as follows. If a specific graph is given, then this is the unique graph of that girth that belongs to \( \mathcal{M}_t^2 \). For example, \( C_{15} \) is the only graph of girth 15 in \( \mathcal{M}_3^2 \). If \( \emptyset \) appears, then there are no graphs of that girth in \( \mathcal{M}_t^2 \). When the
entry is $\Delta$, then it is known that $\mathcal{M}_t^2$ contains at least one graph of that girth (and it is not just a cycle). Some of these type of entries have been verified in previous papers. For example, see [4] and [5] for $\mathcal{M}_1^2$ and $\mathcal{M}_2^2$, respectively.

2 Establishing Table Entries

We begin by showing that for a given positive integer $t$ the only graphs in $\mathcal{M}_t$ with large enough girth must have leaves. The next result was proved for well-covered graphs ($t = 1$) in [3]. Proposition 1 shows it is sharp in terms of girth.

**Theorem 4** Let $t$ be a positive integer. If $g(G) \geq 6t + 2$ and $\delta(G) \geq 2$, then $G \in \mathcal{M}_r(G)$ for some $r > t$.

**Proof.** Assume $t \geq 2$. Let $G$ have girth at least $6t + 2$ and minimum degree at least two. We will show that $G$ has maximal independent sets of at least $t + 1$ different sizes. Choose a cycle $C = v_1, v_2, \ldots, v_s$ of minimum length in $G$.

Assume first that $s \geq 6t + 4$ and let $P$ denote the path $v_3, v_4, \ldots, v_{6t+1}$. Since $\delta(G) \geq 2$ and $g(G) = s$, each vertex $u \notin C$ that is adjacent to a vertex of $P$ has another neighbor $u'$ that does not belong to $P$ and is not adjacent to any vertex of $P$. Choose one such neighbor $u'$ for each $u$ and let $J$ denote the set of these neighbors. By the girth restriction it follows that the set $I = J \cup \{v_1, v_{6t+3}\}$ is independent. (If $s = 6t + 2$, then proceed as above except let $I = J \cup \{v_1\}$.) However, $P$ is a component of $G - N[I]$ and by Proposition 1 $P \in \mathcal{M}_{t+1}$. Similar to the proof of Lemma 2 this implies that $G$ has maximal independent sets of at least $t + 1$ different sizes.

If $s = 6t + 3$, let $P$ be the path $v_3, v_4, \ldots, v_{6t+2}$. The set $J$ is chosen as before, and now $G - N[J \cup \{v_1\}]$ has the path $P$ of order $6t$ as a component. By Proposition 1 it once again follows that $G$ has at least $t + 1$ distinct sizes of maximal independent sets.

□

For any positive integer $t$ it follows from Proposition 1 that $C_{6t+1} \in \mathcal{M}_t$. In [4] it was shown that $C_7$ is the only well-covered graph of girth 7 and minimum degree 2 or more. The following theorem shows the similar result is true for larger values of $t$.

**Theorem 5** Let $t \geq 2$ be an integer. The cycle $C_{6t+1}$ is the only graph of girth $6t + 1$ in $\mathcal{M}_t^2$, and $\mathcal{M}_t^2$ contains no graphs of girth $6t$.

**Proof.** By Proposition 1 the cycle of order $6t + 1$ belongs to $\mathcal{M}_t^2$. Suppose $G$ is a graph not isomorphic to $C_{6t+1}$ such that $g(G) = 6t + 1$ and $\delta(G) \geq 2$. Then $G$
Theorem 5. The set $Y$ contains a cycle since $t \geq 6$. For each integer $t \geq 6$ there is a cycle of the form $x_1, v_1, w_1, v_2, w_2, x_1$ that belongs to $M_t$. Let $X = \{u \in V(G) : d(u, a) = 2, d(u, w) = 3\}$. For any two vertices on $C$ there is a path using part of $C$ of length at most $3t$ joining them. Since $g(G) \geq 13$ it follows that $Y$ is independent. Suppose two vertices $x_1, x_2 \in X$ are adjacent. Let $x_1, v_1, w_1$ and $x_2, v_2, w_2$ be paths in $G$ with $w_1$ and $w_2$ on the cycle $C$. Then the cycle $x_1, v_1, w_1, v_2, x_2, x_1$ has length at most $3t + 5$. But then $3t + 5 \geq 6t + 1$, which implies that $t = 1$, a contradiction. Finally, if a vertex in $X$ is adjacent to a vertex in $Y$, then a similar argument shows that $G$ has a cycle of length at most $3t + 6$ which also leads to a contradiction.

Therefore, $X \cup Y$ is an independent set. One of the components of the graph $G - N[X \cup Y]$ is the cycle $C$ with a single leaf $a$ attached at the support vertex $w$. By Lemma 3 this component is in $M_{t+1}$. An application of Lemma 2 then shows that $G \notin M^2_t$.

Now let $G$ be a graph of girth $6t$, and as above find an induced cycle $C$ of length $6t$. This time let $X = \{u \in V(G) : d(u, C) = 2\}$. This set is independent unless there is a cycle of the form $x_1, v_1, w_1, v_2, w_2, x_1$ that has length at most $3t + 5$. But this means $3t + 5 \geq 6t$ contradicting our assumption that $t \geq 2$. Hence $X$ is independent. The cycle $C$ is one of the components of $G - N[X]$. Since $C_{6t} \in M_{t+1}$, Lemma 2 implies that $G \notin M^2_t$.

By following a line of reasoning similar to the first part of the proof of Theorem 5 one can prove the following result. The proof is omitted. As noted earlier, Theorem 6 also holds for $t = 2$. See [5].

**Theorem 6** Let $t \geq 3$ be a positive integer. For each integer $n$ such that $6t - 4 \leq n \leq 6t - 1$, the cycle $C_n$ is the unique graph of girth $n$ that belongs to $M^2_t$.

We now establish the uniqueness (for $t \geq 3$) of the table entry corresponding to those graphs with no leaves whose shortest cycle has length $6t - 6$ and which have maximal independent sets of exactly $t$ distinct cardinalities.

**Theorem 7** For each integer $t \geq 3$, the cycle $C_{6t-6}$ is the only graph of girth $6t - 6$ that belongs to $M^2_t$.

**Proof.** The cycle of order $6t - 6$ is in $M^2_t$ by Proposition 1. Suppose that $G$ is a graph of girth $6t - 6$ with no leaves. If $G$ is not $C_{6t-6}$, then we can find an induced cycle $C$ of length $6t - 6$ in $G$ with $w, a, b, c$, $X$ and $Y$ defined as in the proof of Theorem 5. The set $Y$ is independent because $g(G) \geq 12$, and $X$ is independent since $t \geq 3$. If some vertex of $X$ is adjacent to a vertex of $Y$, then $G$ contains a cycle...
of length at most $3t - 3 + 6$. It follows that $3t + 3 \geq g(G) = 6t - 6$, or equivalently $t \leq 3$.

If the set $X \cup Y$ is independent, then $G - N[X \cup Y]$ has a component isomorphic to a cycle of length $6t - 6$ with a single leaf attached at $w$. By Lemma 3 this component is in $\mathcal{M}_{t+1}$ and so it follows from Lemma 2 that $G \notin \mathcal{M}_t$.

Thus we may assume that $t = 3$ and that $X \cup Y$ is not independent. Without loss of generality we may assume that $c$ from $Y$ is adjacent to $x_1$ such that $x_1 \in X$ and $x_1, v_1, w_1$ is a path where $w_1$ is on the cycle $C$. See Figure 1. By using the fact that $C$ has length 12 and $g(G) = 12$ we infer that the length of $wCw_1$ is 6. Let $X' = X - N(v_1)$ and let $Z = \{u : d(u, v_1) = 2, d(u, w_1) = 3, ux \notin E(G)\}$. It is clear that $Z$ is independent.

As above, if a vertex of $Z$ is adjacent to a vertex $h$ of $X'$, then if $d(h, w) > 2$ a cycle of length at most 11 is present and if $d(h, w) = 2$ then $G$ contains a cycle of length 10, contradicting $g(G) = 12$. Suppose $z_1 \in Y \cap Z$, say $z_1 = y$ as in Figure 1. Then $z_1 \neq c$, and $a, b, c, x_1, v_1, x_2, z_1, u, a$ is a cycle, contradicting the girth assumption. Similarly, since $G$ has no cycles of length 9, it follows that $Z \cup Y$ is independent.

The set $X' \cup Y \cup Z$ is independent, and one of the components of the graph $G - N[X' \cup Y \cup Z]$ is the cycle $C$ with a single leaf attached at vertices $w$ and $w_1$. But this component has spectrum $\{4, 5, 6, 7, 8\}$ from which it follows that $G \notin \mathcal{M}_3$.

We now show that when $t \geq 4$ there is a “gap” at girth $6t - 5$ among the leafless graphs. That is, if $G$ has minimum degree at least 2 and the shortest cycle of $G$ has order $6t - 5$, then $G$ does not belong to $\mathcal{M}_t$. 

![Figure 1: Part of $G$](image)
Theorem 8 For each integer \( t \) at least 4, the class \( M_t^2 \) contains no graphs of girth \( 6t - 5 \).

Proof. First observe that \( C_{6t-5} \in M_{t-1} \). Our approach will be similar as that pursued in earlier proofs, except that we will be attempting to isolate a cycle of length \( 6t - 5 \) with a path of order 5 attached as in Figure 2. It is easy to check, using either \( \{a, c, e\} \) or \( \{a, d\} \) together with all possible maximal independent sets of a path of order \( 6t - 6 \), that this component has spectrum \([2t, 3t]\) and hence belongs to \( M_{t+1} \). This in turn implies via Lemma 2 that \( G \notin M_t^2 \).

![Figure 2: The cycle C with attachments](image)

Suppose that \( G \) has girth \( 6t - 5 \) and has minimum degree at least 2. Let \( C \) be an induced cycle of length \( 6t - 5 \) in \( G \). There must exist a vertex \( w \) on \( C \) having degree at least 3. For any two vertices on \( C \) there is a path on \( C \) joining them whose length is at most \( 3t - 3 \). Because of the girth and minimum degree assumptions on \( G \) we can find a path \( w, a, b, c, d, e \) as in Figure 2. Let \( A = \{a, b, c, d, e\} \). Let \( X = \{u : d(u, C) = 2\} - N(a) \) and let \( Y = \{u : u \notin C, d(u, A) = 2, d(u, w) \geq 2\} \).

As in previous proofs it is straightforward to show that \( X \) is independent. Since \( g(G) = 6t - 5 \geq 19 \) no pair of vertices in \( Y \) can be adjacent. Suppose first that \( X \cup Y \) is independent. The graph in Figure 2 is a component of \( G - N[X \cup Y] \). As remarked at the outset, this shows that \( G \notin M_t^2 \). We note that for \( t \geq 5 \), the girth restriction ensures that \( X \cup Y \) is independent.

Now consider \( t = 4 \). Thus \( C \) is of length 19. Let \( s_1 \) and \( s_2 \) be the adjacent vertices on \( C \) that are at distance 9 from \( w \). If both \( s_1 \) and \( s_2 \) are of degree two, then \( X \cup Y \) is independent or else a cycle of length 18 would exist in \( G \). Assume then without loss of generality that \( s_1 \) has a neighbor \( r \) that is not on \( C \). Let \( U = N(r) - \{s_1\} \). For each \( u_i \in U \) choose a vertex \( v_i \in N(u_i) - \{r\} \), and set \( V = \{v_i : u_i \in U\} \). Similarly, let \( B = N(a) - \{w\} \). For each \( b_i \in B \) choose a vertex \( c_i \in N(b_i) - \{a\} \), and set \( D = \{c_i : b_i \in B\} \). Since \( g(G) = 19 \) the set \( V \cup D \cup (X - U) \) is independent, and one of the components of \( G - N[V \cup D \cup (X - U)] \) is a cycle of order 19 with a single leaf \( a \) adjacent to \( w \) and a single leaf \( r \) adjacent to \( s_1 \). This component
belongs to $\mathcal{M}_5$ which proves that $G \not\in \mathcal{M}_4^2$ and establishes the theorem.

\[ \square \]

3 Concluding Remarks

We have shown that for a positive integer $t \geq 4$ and for each possible value of girth at least $6t - 6$, the class $\mathcal{M}_t^2$ either contains exactly one graph of that girth (the cycle) or contains no graphs of that girth. It is interesting to note that as $t$ grows there is an ever increasing gap—in terms of girth—between the unique graph of girth $6t - 6$ in $\mathcal{M}_t^2$ and ones of smaller girth. For instance, we can show that $\mathcal{M}_{31}^2$ contains no graphs of girth $r$ for $131 \leq r \leq 179$. Hence the cycles $C_{180}, C_{182}, C_{183}, C_{184}, C_{185}$ and $C_{187}$ are the only leafless members of $\mathcal{M}_{31}$ that have girth at least 131. Thus the six cycles are quite special in $\mathcal{M}_t^2$.

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