Pseudospectral frequency-domain analysis of rectangular waveguides filled by dielectrics whose permittivity varies continuously along the broad dimension

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Abstract
The calculation of dispersion diagrams and field patterns of metallic rectangular waveguides filled with an inhomogeneous dielectric whose permittivity varies continuously along the broad size of the guide is considered. In general, this problem has no exact solution, thus numerical techniques are needed to obtain the propagation constants and field patterns of inhomogeneously filled rectangular waveguides.

KEYWORDS
Chebyshev polynomials, finite-difference frequency-domain method, inhomogeneous dielectric, pseudospectral frequency-domain method, rectangular waveguide

1 | INTRODUCTION
The recently introduced transformation electromagnetics (also referred to as transformation optics) is a powerful technique for designing novel microwave and optical devices such as electromagnetic cloaks, field concentrators and rotators, planar focusing antennas, waveguide bends and couplers, and so on. The implementation of transformation electromagnetics devices requires the use of spatially inhomogeneous materials that has boost a renewed interest in artificial materials.

The characterization of waveguiding structures is a fundamental problem in microwave engineering. In this work, we consider the analysis of uniform metallic rectangular waveguides filled with an inhomogeneous dielectric material whose permittivity varies continuously along the transverse direction x, as shown in Figure 1. In general, this problem has no exact solution. Hence, numerical techniques are needed to obtain the propagation constants and field patterns of inhomogeneously filled rectangular waveguides.

The numerical analysis of the waveguide depicted in Figure 1 has previously been addressed by means of the Galerkin method. A drawback of this approach is that the integrals arising in the formulation should be recalculated for each permittivity profile. Furthermore, only cutoff frequencies are computed. Besides, the formulation used involves the numerical searching for the zeros of a matrix determinant of large dimension, which is a complicated numerical task.

To overcome the aforementioned limitations, in this article we propose an alternative numerical technique for the analysis of the rectangular waveguide shown in Figure 1. This technique is based on the pseudospectral frequency-domain (PSFD) method, which has been successfully applied to several microwave and optical problems. The PSFD method is a global collocation technique that provides high accuracy and flexibility while being free of integral calculations. In addition, not only cutoff frequencies but also dispersion diagrams and field patterns are calculated. For comparison purposes, the results obtained by the PSFD method are compared with those computed by the finite-difference frequency-domain (FDFD) method.

As an alternative to the frequency domain, the dispersion characteristics of uniform waveguiding structures can be analyzed by using the pseudospectral time-domain method. However, due to the eigenvalue (resonant) nature of the problem, the spectral analysis of the transient response of the waveguide cross section becomes a difficult task.
2 | THEORY

2.1 | Differential problem

Consider a metallic rectangular waveguide of dimensions \( a \times b \), filled with an inhomogeneous dielectric material whose permittivity varies continuously along the \( x \) direction, as shown in Figure 1. The waveguide structure is assumed to be uniform along the propagation direction \( z \). Thus, the time-harmonic electromagnetic fields can be expressed as

\[
E(x, y, z) = e(x, y) \exp(-j\gamma z) \\
H(x, y, z) = h(x, y) \exp(-j\gamma z)
\]

where \( \gamma = \alpha + j\beta \) is the propagation constant, with \( \alpha \) and \( \beta \) being the attenuation and phase constants, respectively.

Since the permittivity varies along the \( x \) direction only, the waveguide under study supports two different sets of modes: the longitudinal section electric (LSE) modes and the longitudinal section magnetic (LSM) modes. The LSE modes are characterized by \( E_z \) modes with no field variation along the \( y \) direction and standard transverse electric (TE\(_z\)) modes. The LSM modes with no field variation along the \( y \) direction reduce to standard transverse electric (TE\(_z\)) modes.

The solution for the LSE modes can be obtained by solving the scalar Helmholtz equation for \( \epsilon_r \):

\[
\frac{\partial^2 \epsilon_r}{\partial x^2} + \frac{\partial^2 \epsilon_r}{\partial y^2} + [k_0^2 \epsilon_r(x) + \gamma^2] \epsilon_r = 0
\]

where \( k_0 \) is the free-space wavenumber and \( \epsilon_r(x) \) the relative permittivity. By applying the method of separation of variables, \( \epsilon_r \) is found to be of the form

\[
\epsilon_r(x, y) = X(x) \cos\left(\frac{\pi n y}{b}\right)
\]

with \( n = 0, 1, 2, 3, \ldots \), and where \( X(x) \) is the solution of

\[
\frac{d^2 X(x)}{dx^2} + \left[ k_0^2 \epsilon_r(x) - \left(\frac{\pi n}{b}\right)^2 + \gamma^2 \right] X(x) = 0
\]

subjected to the boundary condition \( X(0) = X(a) = 0 \).

The solution for the LSM modes can be determined by solving the scalar Helmholtz equation for \( h_y \). Analogously to the LSE case, after applying the method of separation of variables, \( h_y \) can be expressed as

\[
h_y(x, y) = X(x) \sin\left(\frac{\pi n y}{b}\right)
\]

with \( n = 1, 2, 3, \ldots \), and where \( X(x) \) is the solution of

\[
\frac{d^2 X(x)}{dx^2} - \frac{d\ln \epsilon_r(x)}{dx} \frac{dX(x)}{dx} + \left[ k_0^2 \epsilon_r(x) - \left(\frac{\pi n}{b}\right)^2 + \gamma^2 \right] X(x) = 0
\]

subjected to the boundary conditions

\[
\frac{dX(0)}{dx} = \frac{dX(a)}{dx} = 0.
\]

It is worth noting that if the permittivity varies with both transverse coordinates, that is, \( \epsilon_r = \epsilon_r(x, y) \), the waveguide solutions are no longer LSE and LSM modes, in general. Consequently, in such a case, the vector Helmholtz equation (instead of the scalar one) need to be solved.

2.2 | Numerical solution: The PSFD method

To illustrate the application of the Chebyshev PSFD method to the present problem, we consider the LSE case. Since the Chebyshev polynomials are defined for \( x \in [-1, 1] \), we initially consider this mathematical interval as the domain of solution.

The Chebyshev PSFD method is based on approximating the unknown function \( X(x) \) in (1) as a linear combination of Chebyshev polynomials as

\[
X(x) = \sum_{p=0}^{N} a_p T_p(x)
\]

with \( x \in [-1, 1] \), where \( a_p \) are unknown coefficients and \( T_p(x) = \cos[p \cos^{-1}(x)] \) is the \( p \)-th order Chebyshev polynomial.

The solution interval \([-1, 1]\) is discretized by considering \( N + 1 \) collocation points defined as

\[
x_i = \cos\left(\frac{\pi i}{N}\right)
\]

with \( i = 0, 1, \ldots, N \). These points are known as Chebyshev-Gauss-Lobatto points. They are composed of the extrema of \( T_N(x) \) along with the endpoints \(-1\) and 1.

An alternative and equivalent way to express (2) is as a linear combination of Chebyshev cardinal basis functions:

\[
X(x) = \sum_{i=0}^{N} X_i C_i(x)
\]

where

\[
C_i(x) = \frac{(-1)^i + 1}{c_i N^2 (x - x_i)} \frac{dT_N}{dx}
\]
with \( c_0 = c_N = 2 \) and \( c_i = 1 \) for \( i = 1, 2, \ldots, N - 1 \). The unknown coefficients \( X_i \) in (3) are the values of the function \( X \) at the collocation points, that is, \( X_i \equiv X(x_i) \).

By using (3), the first derivative of \( X \) at the collocation point \( x_k \) is calculated simply as

\[
\frac{dX(x)}{dx} \bigg|_{x=x_k} = \sum_{i=0}^{N} X_i \frac{dC_i(x)}{dx} \bigg|_{x=x_k}
\]

Then, the first-order derivative of \( X \) at the whole set of collocation points can be expressed in matrix-vector form as the product \( D_x X \), where \( X = [X_0, X_1, \ldots, X_N]^T \) and \( D_x \) is the first-order Chebyshev differentiation matrix of dimension \((N + 1) \times (N + 1)\). The elements of \( D_x \) are the derivatives of the cardinal functions at the grid points

\[
D_x(k,i) = \frac{dC_i(x)}{dx} \bigg|_{x=x_k}
\]

which are given by

\[
D_x(k,i) = \begin{cases} 
\frac{2N^2 + 1}{6} & k = i = 0 \\
- \frac{2N^2 + 1}{6} & k = i = N \\
- \frac{2}{x_k} & 0 < k = i < N \\
\frac{(-1)^i + k c_k}{c_i (x_k - x_i)} & k \neq i
\end{cases}
\]

where \( c_i = 2 \) for \( i = 0, N \) and \( c_i = 1 \) otherwise. The second-order differentiation matrix can be calculated simply as \( D_{xx} = D_x D_x \).

Now, the discretization of (1) is carried out by replacing each term of this equation by its pseudospectral counterpart, which leads to the following ordinary eigenvalue problem for \( \gamma^2 \):

\[
AX = -\gamma^2 X
\]

with

\[
A = \left( \frac{2}{a} \right)^2 D_{xx} + k_0^2 E_r - \left( \frac{n^2}{b} \right)^2 U
\]

where \( E_r = \text{diag}[\varepsilon_r(x_0), \varepsilon_r(x_1), \ldots, \varepsilon_r(x_N)] \) is a diagonal matrix with the relative permittivity values and \( U \) is the \((N + 1) \times (N + 1)\) identity matrix. Note that the solution interval for the physical problem is \([0,a]\). Thus, the differentiation matrix \( D_x \) calculated on \([-1,1]\) has been scaled by the factor \( 2/a \).

The boundary conditions \( X_0 = X_N = 0 \) can easily be imposed by simply removing the first and last rows and columns of (4), which leads to

\[
\bar{A} \bar{X} = -\gamma^2 \bar{X}
\]

where \( \bar{A} \) denotes the restricted \( A \) matrix and \( \bar{X} \) contains the elements of \( X \) at the interior grid points only.

### 3 | RESULTS

To illustrate the accuracy of the PSFD method, the calculation of the cutoff wavelength, \( \lambda_c \), of the \( \text{TE}_{10}\), \( \text{TE}_{50} \), \( \text{TE}_{10,0} \), and \( \text{TE}_{15,0} \) modes of an empty rectangular waveguide is firstly considered. Figure 2 shows a log-log graph of the relative error in the cutoff wavelength as a function of \( N \). As it can be seen, the PSFD errors decrease exponentially until they reach values around \( 10^{-13}\% \). Beyond this level, round-off errors are dominant and convergence curves become nearly flat. For comparison purposes, the results obtained by using the FDFD method\(^1^0\) are also shown in Figure 2. As expected, the FDFD method exhibits a linear convergence rate with slope \(-2\), which is a much poorer behavior than the one obtained with the PSFD method.

Second, we consider a WR75 rectangular waveguide filled with a dielectric material whose relative permittivity varies linearly with the position as \( \varepsilon_r(x) = 1 - d(x/a) \), where \( d \) is a parameter ranging from \(-1\) to \(1\). Note that \( d = 0 \) corresponds to the empty case and for \( d > 1 \) the waveguide is partially filled with a negative permittivity. Figure 3 depicts the normalized cutoff wavelength, \( \lambda_c/\lambda_0 \), against the parameter \( d \) for the first \( \text{LS}_{em} \) modes. As expected, \( \lambda_c \) decreases as \( d \)
increases. The field pattern for the $E_y$ component of the LSE$_{10}$ (TE$_{10}$) mode is shown in Figure 4 for $d = 2$. In this case, the permittivity is positive for $x/a < 0.5$ and negative for $x/a > 0.5$. It can be seen that the electric field tends to concentrate in the region with positive permittivity.

As a third example, the dispersion curves for the first LSE$_{mn}$ modes of a WR75 rectangular waveguide filled with a dielectric with parabolic permittivity profile $\varepsilon_r(x) = 1 - (x/a)^2$ is shown in Figure 5. The normalized phase constant, $\beta/k_0$, is plotted for propagating modes and the normalized attenuation constant, $\alpha/k_0$, for modes under cutoff. The results obtained by the PSFD method with $N = 14$ are compared with those calculated by the FDFD formulation with 80 cells. Although both methods provide the same results within the scale of the plot, if, for instance, we focus on the LSE$_{40}$ mode at 15 GHz, it is found that the PSFD method computes its attenuation constant providing 6 exact figures ($\alpha = 607.766 \, m^{-1}$) while the FDFD method only provides 3 of them.

Finally, we consider a WR75 rectangular waveguide filled with a dielectric with Gaussian permittivity profile $\varepsilon_r(x) = \varepsilon_r1 + (\varepsilon_r2 - \varepsilon_r1)e^{-100(x/a)-0.5^2}$ where $\varepsilon_r1 = 1$ and $\varepsilon_r2 = 9$. Figure 6 illustrates the relative error in the cutoff wavelength of the TE$_{10}$ and the TE$_{20}$ modes as a function of $N$. For the sake of comparison, the results obtained by the PSFD and the FDFD methods are both shown. The relative error has been calculated by using the cutoff wavelength computed by the PSFD method with $N = 50$ as exact value. Even though we are now dealing with a nonpolynomial permittivity profile, it can be seen that the convergence curves exhibit the same behavior as in the homogeneous case shown in Figure 2 and discussed earlier.
4 | CONCLUSION

The PSFD method has been successfully applied to the analysis of rectangular waveguides filled with an inhomogeneous dielectric whose permittivity varies continuously. Several permittivity profiles such as linear, parabolic, and Gaussian profiles have been considered. Starting from the Helmholtz equation, a matrix eigenvalue problem has been obtained for computing cutoff frequencies, dispersion diagrams, and field patterns of the waveguide problem. The results obtained have been compared with those calculated by the conventional second-order FDFD method showing that the PSFD technique provides excellent convergence and accuracy.

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