Gravity of a static massless scalar field and a limiting
Schwarzschild-like geometry

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Abstract

We study a set of static solutions of the Einstein equations in presence of a massless scalar field
and establish their connection to the Kantowski-Sachs cosmological solutions based on some kind
of duality transformations. The physical properties of the limiting case of an empty hyperbolic
spacetime (pseudo-Schwarzschild geometry) are analyzed in some detail.

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The construction of spherically symmetric static solutions of Einstein equations has been attracting
the attention of researchers since the time of the classical works by Schwarzschild \cite{1}, Tolman \cite{2}
and Oppenheimer and Volkoff \cite{3}. In this note we study solutions describing a spacetime filled with a massless
scalar field and their corresponding limiting vacuum forms.

Consider first a static spherical metric of the following form

\[ ds^2 = b^2(r)dt^2 - a^2(r)(dr^2 + d\theta^2 + \sin^2\theta d\phi^2). \] (1)

Einstein’s equations are then written as follows

\[ G^t_t = \frac{a'(r)^2 + a(r)^2 - 2a(r)a''(r)}{a(r)^4} = \varepsilon, \] (2)

\[ G^r_r = \frac{a(r)^2b(r) - b(r)a'(r)^2 - 2a(r)a'(r)b'(r)}{a(r)^4b(r)} = -\varepsilon, \] (3)

\[ G^\theta_\theta = G^\phi_\phi = \frac{b(r)a'(r)^2 - a(r)b(r)a''(r) - a(r)^2b''(r)}{a(r)^4b(r)} = \varepsilon \] (4)
where
\[ \varepsilon = \frac{4\pi \phi'^2}{a^2}. \] (5)
and \( \phi = \phi(r) \) is a massless scalar field. It is convenient to introduce the notations:
\[ A \equiv \frac{a'}{a}, \quad B \equiv \frac{b'}{b}. \] (6)
Einstein’s equations then imply that
\[ A' + A^2 + AB - 1 = 0, \] (7)
\[ A' + A^2 - B' - B^2 - 1 = 0. \] (8)
By eliminating \( A' \) one gets the following relationship:
\[ A = -\frac{B'}{B} - B, \] (9)
From Eq. (7) we then see that \( B^{-1} \) satisfies the equation for an upside-down harmonic oscillator:
\[ \left( \frac{1}{B} \right)'' - \frac{1}{B} = 0. \] (10)
The general solution for Eq. (10) can be written as follows
\[ B = \frac{2}{ce^r + de^{-r}}, \]
which implies
\[ A = 1 - \frac{2 (d + e^r)}{d + ce^r}. \]
One can easily solve Eqs. (6) for \( a \) and \( b \) and compute the square of \( \phi' \), which is pointwise proportional to the energy density \( \varepsilon \):
\[ \phi'^2 = \frac{\varepsilon a^2}{4\pi} = -\frac{(1 + cd)}{\pi (ce^r + de^{-r})^2}. \] (11)
If we demand the energy density to be nonnegative, we obtain that the parameters \( c \) and \( d \) must satisfy the inequality \( cd \leq -1 \). By introducing
\[ \gamma^2 = -\frac{1}{cd}, \quad r_0 = -\log \sqrt{-\frac{c}{d}}, \] (12)
we may parametrize the physically acceptable solutions as follows:
\[ B = \frac{\gamma}{\sinh(r - r_0)}, \quad A = \coth(r - r_0) - \frac{\gamma}{\sinh(r - r_0)}, \] (13)
where the parameter \( \gamma \) must satisfy the restriction
\[ \gamma^2 \leq 1 \] (14)
in order to insure positivity of the energy density. On the other hand the parameter \( r_0 \) has no physical meaning and can be set equal to zero. Explicit expressions for \( a, b \) and \( \phi'^2 \) are easily written:
\[ ds^2 = \left( \frac{\tanh \frac{r}{2}}{\tanh \frac{r_r}{2}} \right)^{2\gamma} dt^2 - \frac{a_0^2 \sinh^2 r}{\left( \frac{\tanh \frac{r}{2}}{\tanh \frac{r_r}{2}} \right)^{2\gamma}} (dr^2 + d\theta^2 + \sin^2 \theta d\varphi^2), \] (15)
\[ \phi'^2 = \frac{\varepsilon a^2}{4\pi} = \frac{(1 - \gamma^2)}{4\pi \sinh^2 r}. \]  

The values \( \gamma = \pm 1 \) correspond to the limiting cases of empty spacetimes; the value \( \gamma = 0 \) corresponds to the maximum possible value of the spatial variation of the scalar field; in this case \( b(r) = 1 \) and the time coordinate \( t \) is everywhere ticking at the same rate.

Solutions of this kind have already been obtained in [4] in terms of isotropic coordinates. The metric (15) can be written in isotropic coordinates by the change of coordinates \( r = \log \tilde{r} \):

\[ ds^2 = \left(1 - \frac{1}{2\rho}\right)^{2\gamma} dt^2 - \frac{1}{4a_0^2} \left(1 - \frac{1}{\tilde{r}}\right)^{2-2\gamma} \left(1 + \frac{1}{\tilde{r}}\right)^{2\gamma+2} \left(\tilde{r}^2 + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2 \theta d\phi^2\right) \]  

This form is manifestly flat at infinity and allows for a direct physical interpretation of the parameter \( \gamma \); indeed, for large values of the rescaled radial variable \( 2\rho = a_0 \tilde{r} \)

\[ g_{00} = \left(1 - \frac{a_0}{\rho}\right)^{2\gamma} \simeq 1 - \frac{2a_0\gamma}{\rho} \]  

and therefore \( m = a_0\gamma \) may be interpreted as the mass seen by an observer at infinity. The standard (Schwarzschild-like) form of the metric is also of interest. We will obtain it in two steps: first, introduce a radial coordinate \( \tilde{\phi} \) proportional to the scalar field \( \phi \) itself, by the relation

\[ \tilde{\phi} = \log \coth \frac{r}{2} = \sqrt{\frac{4\pi}{1 - \gamma^2}} \phi(r) \]  

This change of coordinates recasts the metric in the following way:

\[ ds^2 = e^{-2\tilde{\phi}} dt^2 - \frac{a_0^2 e^{2\tilde{\phi}}}{\sinh^4 \phi} \left( d\tilde{\phi}^2 + \sinh \tilde{\phi}^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]  

A second radial coordinate defined as

\[ R = \frac{a_0 e^\tilde{\phi}}{\sinh \phi} \]  

allows to rewrite finally the metric in the standard form:

\[ ds^2 = g_{00} dt^2 - \frac{1}{f} dR^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

where the metric components have parametric expressions in terms of the scalar field \( \tilde{\phi} \):

\[ g_{00} = e^{-2\gamma \tilde{\phi}}, \]  

\[ f = \frac{\cosh^2 (\tilde{\phi} + \tilde{\phi}_0)}{\cosh^2 \tilde{\phi}_0} \equiv 1 - \frac{2m(R)}{R} \]  

with \( \tanh \tilde{\phi}_0 = -\gamma \).

When \( \gamma = \pm 1 \) the metric reproduces the standard Schwarzschild metric with mass \( \pm 2a_0 \):

\[ ds^2 = \left(1 - \frac{2a_0}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{2m}{R}} - R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

When \( \gamma = 1 \) the singularity at \( R = 0 \) is hidden behind an horizon. On the other hand, it is not difficult to check that when \( \gamma < 1 \) there is a naked singularity at \( R = 0 \) (see Fig. 1 and Fig. 2).
Figure 1: Plot of $f(R)$ for values of $\gamma$ between -1 and 1. For $\gamma = 1$ the Schwarzschild vacuum solution is recovered and this is the only solution having an horizon (at $R = 2a_0$, where the corresponding plot stops).

The other special case is $\gamma = 0$. In this case

$$ds^2 = dt^2 - \frac{dR^2}{1 + \frac{a_0^2}{R^2}} - R^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (26)

The corresponding manifold is a direct product of time by the three-dimensional static non-compact manifold with a positive scalar curvature

$$R = \frac{2a_0^2}{R^4}.$$  \hspace{1cm} (27)

At the point $R = 0$ the manifold (26) has a naked singularity.

In a completely analogous way, we now consider a static hyperbolic metric of the following form

$$ds^2 = b^2(r)dt^2 - a^2(r)(dr^2 + d\chi^2 + \sinh^2 \chi d\phi^2).$$  \hspace{1cm} (28)

Einstein’s equations are then written as follows

$$G_t^t = \frac{a'(r)^2 - a(r)^2 + 2a(r)a''(r)}{a(r)^4} = \varepsilon,$$  \hspace{1cm} (29)

$$G_r^r = \frac{-a(r)^2 b(r) + b(r)a'(r)^2 + 2a(r)a'(r)b'(r)}{a(r)^4 b(r)} = -\varepsilon,$$  \hspace{1cm} (30)

$$G_\chi^\chi = G_\phi^\phi = \frac{b(r)a'(r)^2 - a(r)b(r)a''(r) - a(r)^2b''(r)}{a(r)^4 b(r)} = \varepsilon$$  \hspace{1cm} (31)

where again $\varepsilon = \frac{4\pi\phi'^2}{\phi^2}$. Now $1/B$, defined as in Eq. (20), behaves as an harmonic oscillator:

$$\left(\frac{1}{B}\right)'' + \frac{1}{B} = 0.$$  \hspace{1cm} (32)

Thus

$$B = \frac{\gamma}{\sin r}.$$  \hspace{1cm} (33)
Figure 2: Plot of the mass function $2m(R)$ for values of $\gamma$ between -1 and 1. For $\gamma = \pm 1$ the mass function is constant. Outside the horizon of the Schwarzschild solution, the mass function tends rapidly to $m = 2\gamma a_0$.

and

$$A = \cot r - \frac{\gamma}{\sin r}. \quad (34)$$

Solutions for $a$ and $b$ easily follow:

$$a = \frac{a_0 \sin r}{(\tan \frac{r}{2})^\gamma}, \quad b = b_0 \left(\tan \frac{r}{2}\right)^\gamma, \quad \phi' = \frac{1 - \gamma^2}{4\pi \sin^2 r}; \quad (35)$$

again the exponent $\gamma$ should satisfy the restriction $\gamma^2 \leq 1$. As before the values $\gamma = \pm 1$ correspond to the limiting case of empty spacetimes. When $\gamma = 0$, $b(r)$ is constant and the time $t$ is everywhere ticking at the same rate.

There is an interesting relationship linking the above static spherically symmetric or hyperbolically symmetric solutions with Kantowski-Sachs cosmologies \cite{5, 6}. Consider, for instance, the metric (28). By interchanging the time and the radial variables

$$t \leftrightarrow r \quad (36)$$

it becomes

$$ds^2 = -a^2(t)dt^2 + b^2(t)dr^2 - a^2(t)(d\chi^2 + \sinh^2 \chi d\varphi^2). \quad (37)$$

To remedy for the incorrect signs in front of $dr^2$ and $dt^2$ we make the replacement

$$g_{\alpha\beta} \rightarrow -g_{\alpha\beta} \quad (38)$$

and get

$$ds^2 = a^2(t)dt^2 - b^2(t)dr^2 + a^2(t)(d\chi^2 + \sinh^2 \chi d\varphi^2). \quad (39)$$

Now the last two terms at the right-hand side of Eq. (39) have wrong signs. To correct them, we make one more replacement:

$$\chi \rightarrow i\theta, \quad (40)$$

which finally produces the metric

$$ds^2 = a^2(t)dt^2 - b^2(t)dr^2 - a^2(t)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (41)$$
The indicated set of transformations also determines a map of the components (29), (30), (31) of the Einstein tensor into those of a Kantowski-Sachs spherical universe filled with the time-dependent massless scalar field $\phi(t)$, which is realized through the substitutions

$$G^t_t \rightarrow -[G^{KS}]^r_r,$$
$$G^r_r \rightarrow -[G^{KS}]^t_t,$$
$$G^\theta_\theta \rightarrow -[G^{KS}]^\theta_\theta,$$

A similar set of transformations performs the transition from the static spherically symmetric metric to the cosmological hyperbolic Kantowski-Sachs metric.

The non-triviality of transformations (36), (38), (40) consists in the fact that they not only exchange the radial and time variables among themselves, but also substitute the spherical symmetry by the hyperbolical one and vice versa.

In the hyperbolic static empty space ($\gamma = 1$) the metric becomes

$$ds^2 = b_0^2 \tan^2 r dt^2 - 4a_0^2 \cos^2 r (dr^2 + d\chi^2 + \sinh^2 \chi d\varphi^2).$$

In analogy with the spherical case, we introduce a new radial variable

$$\rho \equiv 2a_0 \cos^2 \frac{r}{2},$$

in terms of which, the metric takes the following “pseudo-Schwarzschild” form

$$ds^2 = \left(\frac{2a_0}{\rho} - 1\right) dt^2 - \left(\frac{2a_0}{\rho} - 1\right)^{-1} \rho^2 (d\chi^2 + \sin^2 \chi d\varphi^2),$$

where again we have set $b_0 = 1$, which amounts simply to a rescaling of the time parameter. The metric was written down by Harrison in [7], where he studied exact three-variable empty space solutions of the Einstein equations. There it was dubbed the degenerate solution III-9 but its properties were not discussed, nor have they been discussed anywhere so far to our knowledge. Here we would like to attract the reader’s attention to its peculiar properties.

To proceed further, first observe that the metric has the property

$$g_{\rho\rho} = -g^{tt},$$

which it shares with the Schwarzschild metric, written in the standard form.

Write down the geodesics equations for the radial motion valid for both metrics, the metric and the Schwarzschild one:

$$\frac{d^2 t}{ds^2} + 2\Gamma^t_{t\rho} \frac{dt}{ds} \frac{d\rho}{ds} = 0,$$
$$\frac{d^2 \rho}{ds^2} + \Gamma^\rho_{tt} \left(\frac{dt}{ds}\right)^2 + \Gamma^\rho_{\rho\rho} \left(\frac{d\rho}{ds}\right)^2 = 0.$$
For the purpose of solving equation (50), we introduce a new variable:

\[ x \equiv \left( \frac{d\rho}{ds} \right)^2, \]  

(53)

and use the relation

\[ \frac{d^2\rho}{ds^2} = \frac{dp}{d\rho} \frac{d}{d\rho} \left( \frac{dp}{d\rho} \right) = \frac{1}{2} \frac{d}{d\rho} \left( \frac{dp}{d\rho} \right)^2 = \frac{1}{2} \frac{dx}{d\rho}. \]  

(54)

We need also the expressions for the Christoffel symbols:

\[ \Gamma^\rho_{tt} = -\frac{1}{2} g^\rho_{\rho \rho} \frac{d}{d\rho} g_{tt} = \frac{1}{2} \frac{d}{d\rho} g_{tt}, \]  

(55)

\[ \Gamma^\rho_{\rho \rho} = \frac{1}{2} g^\rho_{\rho \rho} \frac{d}{d\rho} g_{\rho \rho} = -\frac{1}{2} \frac{d}{d\rho} \log g_{tt}. \]  

(56)

Substituting Eqs. (55), (56), (52) and (53) into Eq. (50), we rewrite it in the following form:

\[ \frac{dx}{d\rho} - \frac{d}{d\rho} \log g_{tt} x + \frac{d}{d\rho} \log g_{tt} = 0. \]  

(57)

This equation can be immediately integrated to give

\[ x = 1 + D_0 g_{tt}. \]  

(58)

Combining Eqs. (58) and (52), one gets

\[ \left( \frac{d\rho}{dt} \right)^2 = g_{tt}^2 + D_0 g_{tt}^3. \]  

(59)

Formula (59) is valid for both the metrics: for the Schwarzschild and for the pseudo-Schwarzschild one (47). In both cases the values \( D_0 > 0, D_0 = 0, D_0 < 0 \) correspond to spacelike, lightlike and timelike geodesics respectively.

For the Schwarzschild metric the case \( D_0 > -1 \) corresponds to a situation when at spatial infinity \( \rho \to \infty \) the particle has some non vanishing velocity:

\[ \vec{v}^2 = 1 + D_0. \]  

(60)

The case \( D_0 < -1 \) describes the situation when the particle cannot reach spatial infinity having a turning point at

\[ \rho_{\text{turn}} = \frac{2a_0 D_0}{1 + D_0}. \]  

(61)

The value \( D_0 = -1 \) corresponds to the particle arriving at infinity with vanishing velocity.

We now turn to the study of the pseudo-Schwarzschild metric (47). This metric has a horizon at \( \rho = 2a_0 \), which is in a way analogous to the Schwarzschild horizon. The singularity in terms of the coordinate \( \rho \) occurs at \( \rho = 0 \) and is also analogous to the Schwarzschild singularity. The main difference lies in the fact that the metric (47) is defined at \( \rho < 2a_0 \), i.e. inside the horizon, in contrast to the Schwarzschild metric.

Looking at Eq. (59) one can see that in the vicinity of the horizon, when \( g_{tt} \to 0 \), the second term in this equation can be omitted and the equation itself can be integrated:

\[ t = t_0 - \rho - 2a_0 \log(2a_0 - \rho). \]  

(62)

Thus, the approach to the horizon, described in terms of coordinates (47), takes an infinite time just as in the Schwarzschild case. It is easy to check that as in the Schwarzschild case the reaching of the horizon takes a finite interval of proper time.
We now study what is going on in the vicinity of the singularity. It is easy to see that at
\[ \rho_0 = \frac{2a_0}{1 - \frac{1}{2a_0}} \] (63)
the velocity of a massive particle vanishes and the latter cannot reach the singularity. Curiously, there is no obstruction for the light rays (massless particles) which fall to the singularity \( \rho = 0 \).

As has already been mentioned above the metric (47) is defined only at \( \rho < 2a_0 \). To describe the whole manifold one can construct Kruskal-type coordinates, analogous to those describing the Schwarzschild manifold [8]. Following a well-known algorithm (see e.g. [9]), one gets
\[ ds^2 = f^2(u, v)(dv^2 - du^2) - \rho^2(d\chi^2 + \sinh^2 \chi d\varphi^2), \] (64)
where
\[ f^2(u, v) = \frac{32a_0^3}{\rho} \exp \left(-\frac{\rho}{2a_0}\right). \] (65)

In the region \( \rho < 2a_0 \) below the horizon (see region I in Fig. 3), the relations between the Schwarzschild-type coordinates \( \rho \) and \( t \) and Kruskal-type coordinates \( u \) and \( v \) are
\[ u = \exp \left(\frac{\rho}{4a_0}\right) \sqrt{1 - \frac{\rho}{2a_0}} \cosh \frac{t}{4a_0}, \] (66)
\[ v = \exp \left(\frac{\rho}{4a_0}\right) \sqrt{1 - \frac{\rho}{2a_0}} \sinh \frac{t}{4a_0}, \] (67)
or
\[ u^2 - v^2 = \exp \left(\frac{\rho}{2a_0}\right) \left(1 - \frac{\rho}{2a_0}\right), \] (68)
\[ \frac{v}{u} = \tanh \frac{t}{4a_0}. \] (69)

Outside of horizon \( \rho > 2a_0 \) (see region II in Fig. 3) these relations take the following form:
\[ u = \exp \left(\frac{\rho}{4a_0}\right) \sqrt{\frac{\rho}{2a_0} - 1} \sinh \frac{t}{4a_0}, \] (70)
\[ v = \exp \left(\frac{\rho}{4a_0}\right) \sqrt{\frac{\rho}{2a_0} - 1} \cosh \frac{t}{4a_0}, \] (71)
or
\[ v^2 - u^2 = \exp \left(\frac{\rho}{2a_0}\right) \left(\frac{\rho}{4a_0} - 1\right), \] (72)
\[ \frac{v}{u} = \tanh \frac{t}{4a_0}. \] (73)

In the regions III and IV the Kruskal-type coordinates are, respectively,
\[ u = -\exp \left(\frac{\rho}{4a_0}\right) \sqrt{1 - \frac{\rho}{2a_0}} \cosh \frac{t}{4a_0}, \] (74)
\[ v = -\exp \left(\frac{\rho}{4a_0}\right) \sqrt{1 - \frac{\rho}{2a_0}} \sinh \frac{t}{4a_0}, \] (75)
and
\[ u = -\exp \left(\frac{\rho}{4a_0}\right) \sqrt{\frac{\rho}{2a_0} - 1} \sinh \frac{t}{4a_0}, \] (76)
\[ v = -\exp\left(\frac{\rho}{4a_0}\right) \sqrt{\frac{\rho}{2a_0} - 1} \cosh \frac{t}{4a_0}. \]  

(77)

The singularity \( \rho = 0 \) is located at the hyperbola \( u^2 - v^2 = 1 \), intersecting the axis \( v = 0 \) at the points \( u = \pm 1 \). Remember, that the Schwarzschild singularity is located at the hyperbola \( v^2 - u^2 = 1 \). Thus, the Kruskal diagram for the pseudo-Schwarzschild manifold is rotated by \( \pi/2 \) with respect to the familiar Kruskal diagram for the Schwarzschild manifold. In other words, in contrast to the Schwarzschild singularity which is spacelike, the pseudo-Schwarzschild singularity is timelike. When speaking of the Kruskal diagram for the pseudo-Schwarzschild manifold, it is necessary to keep in mind that its two-dimensional part, which is not drawn explicitly, is hyperbolic.

Looking at formula (47) we note an important analogy of the pseudo-Schwarzschild metric to the Schwarzschild one. Whereas the Schwarzschild metric the crossing of the horizon requires an infinite coordinate time. By contrast, in the pseudo-Schwarzschild manifold there exists only one type of timelike geodesics: a test particle travels along the geodesics from spatial infinity of region III (white hole), (see Fig. 3). The Schwarzschild case, as is well known one has two types of timelike radial geodesics, depending on the value of the parameter \( D_0 \): those whose motion is finite \( (D_0 < -1) \) and those whose motion is infinite \( (D_0 \geq -1) \). In terms of the Kruskal diagram of Fig. 3, the former travel from region IV through region I (or III) to region II, i.e. they begin their motion from the (white hole) singularity \( S_{IV} \), cross the horizon, reach a turning point where the radial coordinate acquires a maximum value, then cross the horizon again and fall onto the (black hole) singularity \( S_{II} \). Instead, the geodesics corresponding to \( D_0 > -1 \) either arrive from spatial infinity of region I, cross the horizon and fall onto the singularity \( S_{II} \) (black hole), or leave the singularity \( S_{IV} \), cross the horizon and travel to spatial infinity in region III (white hole), (see Fig. 3).

In analogy to the Schwarzschild case the equation of motion for the radial timelike geodesic in the pseudo-Schwarzschild geometry can be integrated parametrically. To achieve this, using the relation (69) we rewrite equation (59) in the form

\[
\frac{d\rho}{dt} = \left(\frac{2a_0}{\rho} - 1\right) \sqrt{1 - \frac{(2a_0 - \rho)\rho}{(2a_0 - \rho)\rho}} \tag{78}
\]

and introduce the parameter \( \eta \) as

\[
\rho = \rho_0 \cosh^2 \eta. \tag{79}
\]

Then Eq. (78) can be integrated to give, up to an additive constant,

\[
t = -\rho_0 \sqrt{1 - \frac{\rho_0}{2a_0} \eta - 4a_0 \sqrt{1 - \frac{\rho_0}{2a_0} \eta - \frac{\rho_0}{2} \sqrt{1 - \frac{\rho_0}{2a_0} \sinh 2\eta}}}
+ 2a_0 \log \left| \frac{(e^{2\eta} - e^{-2\eta})(e^{2\eta}H - 1)}{(e^{2\eta}H - e^{2\eta})(1 - e^{-2\eta})} \right|, \tag{80}
\]

where

\[
\eta_H = \arccosh \sqrt{\frac{2a_0}{\rho_0}}. \tag{81}
\]

is the value of \( \eta \) which corresponds to the crossing of the horizon.

It is easy to see that when \( \eta \to \eta_H \), the parameter \( t \) tends to infinity. Thus, with the growth of the parameter \( \eta \) the parameter \( t \) is decreasing and at \( \eta \to \infty, t \to -\infty \). Thus, just like in the case of the Schwarzschild metric the crossing of the horizon requires an infinite coordinate time.
We can also find the dependence of the proper time $\tau$ on the parameter $\eta$. Using Eq. (78) and the relation $d\tau^2 = g_{tt}dt^2 + g_{\rho\rho}d\rho^2$ we get

$$\frac{d\rho}{d\tau} = \frac{2a_0}{\rho_0} - \frac{2a_0}{\rho}.$$  \hspace{1cm} (82)

Using the parametrization (79) one finds

$$\tau = \frac{\rho_0}{2} \sqrt{\frac{\rho_0}{2a_0}} (\sinh 2\eta + 2\eta).$$ \hspace{1cm} (83)

The proper time which a massive particle needs to cover the distance from the point $\rho_0$ to the horizon is finite and given by the formula

$$\tau_H = \frac{\rho_0}{2} \sqrt{\frac{\rho_0}{2a_0}} (\sinh 2\eta_H + 2\eta_H)$$

$$= \rho_0 \sqrt{\frac{2a_0}{\rho_0} - 1} + \rho_0 \sqrt{\frac{\rho_0}{2a_0} - \text{arccosh} \frac{2a_0}{\rho_0}}.$$ \hspace{1cm} (84)

We note that

$$\tau_H(0) = \tau_H(2a_0) = 0.$$
as expected. Therefore there exists a value of $\rho_0$ at which $\tau_H$ is maximum. The equation $\frac{d\tau_H(\rho_0)}{d\rho_0} = 0$ is transcendental and cannot be solved analytically. The graph of the function $\tau_H(\rho_0)$ is depicted in Fig. 4. One sees that the maximum value of $\tau_H$ is achieved at $\rho_0 \approx 0.61 \times 2a_0$.

![Figure 4: $\tau_H$ dependence on the ratio $\frac{\rho_0}{2a_0}$.](image)

Now consider the motion outside of the horizon. In this region the time and radial coordinates change their roles. Introducing the notations $\tilde{t}$ and $\tilde{\rho}$ for new timelike and spacelike coordinates one writes the metric outside the horizon in the following form:

$$ds^2 = \frac{d\tilde{t}^2}{1 - \frac{2a_0}{\tilde{\rho}}} - d\tilde{\rho}^2 \left( 1 - \frac{2a_0}{\tilde{\rho}} \right) - \tilde{t}^2 (d\chi^2 + \sinh^2 \chi d\varphi^2).$$  \hspace{1cm} (85)

This metric is non-stationary and when $\tilde{t} \to \infty$ it becomes the Minkowski metric. To be precise it asymptotically tends to the Minkowski metric, written in a rather particular way: it represents a direct product of the line $\tilde{r}$ times the $(2+1)$ - dimensional Milne manifold.

The geodesic equation for massive particles can be reduced to the following form:

$$\left( \frac{d\tilde{\rho}}{d\tilde{t}} \right)^2 = \frac{v^2 g_{\tilde{r}\tilde{t}}}{1 - v^2 + v^2 g_{\tilde{t}\tilde{t}}},$$  \hspace{1cm} (86)

where $v$ is the asymptotic value of the velocity $d\tilde{\rho}/d\tilde{t}$ when $\tilde{t} \to \infty$.

One can find the relation between the velocity parameter $v$ and the parameter $\rho_0$ of the minimal distance between a particle under consideration and the singularity, using Eqs. (79) and (80). Indeed,

$$v = \lim_{\tilde{t} \to \infty} \frac{d\tilde{\rho}}{d\tilde{t}} = \lim_{\tilde{\rho} \to \infty} \frac{d\tilde{t}}{d\tilde{\rho}} = \sqrt{1 - \frac{\rho_0}{2a}}.$$  \hspace{1cm} (87)

We now come back to the Kantowski-Sachs cosmological solutions for both the spherically symmetric and the hyperbolic universes and consider their empty space limits $\gamma = 1$. The corresponding metrics are respectively

$$ds^2 = 4a_0^2 \cos^4 \frac{t}{2} dt^2 - \tan^2 \frac{t}{2} dr^2 - 4a_0^2 \cos^4 \frac{t}{2} (d\theta^2 + \sin^2 \theta d\varphi^2),$$  \hspace{1cm} (88)

$$ds^2 = 4a_0^2 \cosh^4 \frac{t}{2} dt^2 - \tanh^2 \frac{t}{2} dr^2 - 4a_0^2 \cosh^4 \frac{t}{2} (d\chi^2 + \sinh^2 \chi d\varphi^2).$$  \hspace{1cm} (89)

The metrics of the empty Kantowski-Sachs universes, written in the form (88) and (89) do not describe complete manifolds just like their static analogs. Making the transformation

$$\tilde{t} = 2a_0 \cos^2 \frac{t}{2}$$  \hspace{1cm} (90)

one sees that the metric (88) describes the internal part of the Schwarzschild world (as was noticed in paper [3]). Completing the manifold we construct the external part of the Schwarzschild world, which is static.
Similarly, making the transformation

\[ t = 2a_0 \cosh^2 \frac{t}{2} \]  \hspace{1cm} (91)

we get the metric, describing the external part of the pseudo-Schwarzschild world which is nothing but (86), while its completion gives the static pseudo-Schwarzschild geometry (47) below the horizon.

The interrelations amongst different cosmological and static solutions of the Einstein equations in the presence of massless scalar field and their empty space limits are represented in the scheme above.

We conclude by noting that the pseudo-Schwarzschild metric can be obtained by applying an analogue of Birkhoff theorem to a vacuum hyperbolically symmetric solution of the Einstein equations. Indeed,
by mimicking the proof of Birkhoff theorem one can show that the most general vacuum hyperbolically symmetric metric is static and is equivalent to the internal pseudo-Schwarzschild solution (47).

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