Notes on (0,2) Superconformal Field Theories

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In these lecture notes, I review the “linear \(\sigma\)-model” approach to (0,2) string vacua. My aim is to provide the reader with a toolkit for studying a very broad class of (0,2) superconformal field theories with the requisite properties to be candidate string vacua. These lectures were delivered at the 1994 Trieste Summer School.

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1. Invitation au voyage

The case for considering string compactifications with spacetime supersymmetry is often made in four dimensional terms, say, that this provides a solution to the hierarchy problem. But the real justification is that these are the only known perturbative solutions to the fermionic string with four flat noncompact directions. So if perturbative string theory can tell us anything, these are the starting points about which to do our perturbation theory. Now, it is clear that not all of the physics is captured by perturbation theory. However, some of it is (and we even have reason to believe that many of the crucial features are) captured by perturbation theory. Besides, you have to walk before you can run . . .

Spacetime supersymmetry is equivalent to $(0,2)$ superconformal symmetry on the worldsheet with an integrality condition on the right-moving $U(1)$ charges (see [2,3,4]). So if we want to study perturbative solutions to string theory, we should be studying $(0,2)$ SCFTs. Surprisingly little is known about the subject, given its evident importance. Almost all of the results we have to date are for the very special case of left-right symmetric $(2,2)$ superconformal field theories, or very simple orbifolds thereof. This is clearly a very special case of a $(0,2)$ SCFT, where we simply ignore the left-moving superconformal

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symmetry. For a long time, more general (0,2) SCFTs were considered simply too hard to study. I hope in these lectures to convince you that they are not too hard, and that, because there are so many open questions, this is fertile area to work on.

Mainly in these lectures, I will concentrate on giving you a “toolkit” for building and studying (0,2) models of the requisite sort. More details can be found in the references, as can some results that one can learn both about the general features of (0,2) theories and the specifics of particular models. Much of the work that I describe in these notes was done jointly with S. Kachru.

2. Generalities

As I said, $\mathcal{N}=1$ Spacetime supersymmetry is equivalent to (0,2) superconformal symmetry on the worldsheet, provided a certain integrality condition on the $U(1)$ charges holds. The origin of this condition is that it is required so that we may define a chiral GSO projection. You might think, therefore, that what we will look for is a theory whose symmetry algebra is $(\text{vir})_L \times (N=2 \text{ svir})_R$. In fact, we will require a somewhat larger symmetry. Namely, we will require as well that there exist a left-moving $U(1)$ current algebra of level $r$, that is,

$$J(z)J(w) = \frac{r}{(z-w)^2}$$

and the symmetry algebra will be $(\hat{U}(1) \times \text{vir})_L \times (N=2 \text{ svir})_R$, with $(c, \bar{c}) = (6+r, 9)$.

To turn this into a string theory, we add four free bosons $X^\mu$, and their $(0,1)$ superpartners, four free Majorana-Weyl fermions $\psi^\mu$. We also add $\lambda^I$, $I = 1, \ldots, 16-2r$, free left-moving Majorana-Weyl fermions which yield a linearly-realized $SO(16-2r)$ subgroup of the spacetime gauge group, and we add a left-moving $E_8$ current algebra.

The left-moving $\hat{U}(1)$ current algebra plays a dual role in the theory. First, it provides another linearly-realized piece of the spacetime gauge group (which, at this stage, appears to be $SO(16-2r) \times U(1) \times E_8$). Second, it provides a candidate for a chiral GSO projection for the left-movers:

$$g = e^{-i\pi J_0} (-1)^{F_{\lambda^I}}$$

where $F_{\lambda^I}$ is the fermion number for the left-moving free fermions.

In keeping with its role in forming the GSO projection, the left-moving $\hat{U}(1)$ also provides the left-moving spectral flow generator (ground state of the left-moving Ramond sector) which promotes the spacetime gauge group to $E_6$, $SO(10)$, or $SU(5)$, for $r = 3, 4, 5$.

1 In the Ramond sector, $J_0$ is replaced by $J_0 + r/2$ in the formula for the GSO projection.
We are all familiar with how the representations of \( SO(10) \times U(1) \) assemble themselves into representations of \( E_6 \) in \((2,2)\) compactifications \((r = 3)\). The situation for \( r = 4, 5 \) may be more unfamiliar, so I have summarized it in the following tables.

| Rep. of \( SO(10) \) | Rep. of \( SO(8) \times U(1) \) | Cohomology Group |
|-----------------------|-------------------------------|------------------|
| 45 \( 8_s' \oplus (28_0 \oplus 1_0) \oplus 8_2' \) | \( H^*(M, \mathcal{O}) \) |
| 16 \( 8_s \oplus 8^r_1 \) | \( H^*(M, V) \) |
| 10 \( 1_{-2} \oplus 8_0' \oplus 1_2 \) | \( H^*(M, \wedge^2 V) \) |
| 1 | \( 1_0 \) | \( H^*(M, \text{End } V) \) |

\( r = 4 \)

| Rep. of \( SU(5) \) | Rep. of \( SO(6) \times U(1) \) | Cohomology Group |
|---------------------|-------------------------------|------------------|
| 24 \( 4_{-5/2} \oplus (15_0 \oplus 1_0) \oplus 4_{5/2} \) | \( H^*(M, \mathcal{O}) \) |
| 10 \( 4_{-3/2} \oplus 6_1 \) | \( H^*(M, V) \) |
| \( 5 \) \( 4_{-1/2} \oplus 1_2 \) | \( H^*(M, \wedge^2 V) \) |
| 1 | \( 1_0 \) | \( H^*(M, \text{End } V) \) |

\( r = 5 \)

**Table 1:** Representations of the linearly realized part of the gauge group and how they assemble themselves.

Note that the representations which appear alternate between spinor and tensor representations of \( SO(16 - 2r) \), and the \( U(1) \) charge jumps by \( r/2 \) with each application of the spectral flow. One realization of this general setup is \((2,2)\) superconformal field theory. In this case, \( r = 3 \), since the left-moving \( N = 2 \) superconformal algebra with \( c = 3r \) contains a \( U(1) \) subalgebra at level \( r \). Clearly, though, this is a very special case. Phenomenologically, it may also be a relatively unattractive one, as \( r = 4, 5 \) seems to lead to more attractive phenomenology.

3. **Nonlinear \( \sigma \) Models**

The \((2,2)\) nonlinear sigma model can be written

\[
S = \frac{i}{2\pi} \int \frac{1}{2}g_{ij}(\partial X^i \partial X^j + \partial X^j \partial X^i) - \frac{1}{2}b_{ij}(\partial X^i \partial \overline{X}^j - \partial X^j \partial \overline{X}^i) + i(\psi_i \overline{D} \psi^i + \lambda_i \overline{D} \lambda^i) + R^k_{i j}(X)\lambda_k \lambda^l \psi_i \psi^j \tag{3.1}
\]
where $X : \Sigma \to M$ is the $\sigma$-model map from the worldsheet into a Kähler manifold $M$, which for reasons that will become clear shortly, we will assume has vanishing first Chern class, $c_1(T) = 0$. The left- and right-moving fermions couple to the appropriate pullback connections, $D\psi^\bar{j} = \partial\psi^\bar{j} + \partial X^\bar{j}\Gamma^k_{\bar{i}\bar{j}}(X)\psi^k$, etc.

The $(0,2)$ generalization of this is to replace the action for the left-moving fermions by

$$S = \frac{i}{2\pi} \int \ldots + i(\ldots + \lambda_a \bar{D}\lambda^a) + F^a_{b\bar{j}}(X)\lambda_a \lambda^b \psi^i \psi^{\bar{j}}$$

(3.2)

where now the $\lambda^a$ transform as sections of a holomorphic vector bundle $V \to M$ with

$$c_1(V) = 0, \quad c_2(V) = c_2(T)$$

(3.3)

The data specifying the $\sigma$-model now is: the Kähler metric, $g^{\bar{i}\bar{j}}(X)$, a closed 2-form $b^{\bar{i}\bar{j}}(X)$, and the holomorphic connection on $V$, $A^a_{b\bar{j}}(X)$, whose curvature is $F^a_{b\bar{i}\bar{j}}(X)$.

For string theory, of course, we are interested in a conformally invariant $\sigma$-model. Requiring conformal invariance imposes some conditions on the above data. For instance, demanding that the 1-loop $\beta$-function of (3.2) vanish requires that $g^{\bar{i}\bar{j}}$ be Ricci-flat. However, these conditions are corrected at higher orders in $\sigma$-model perturbation theory, and we don’t have the slightest idea what the “all orders” equation necessary for conformal invariance is.

In the face of this obstacle, there are two attitudes one can adopt. The first is to imagine that we can construct the exact conformally invariant theory order by order in perturbation theory, starting with a solution to the 1-loop $\beta$-function equations. If the $\sigma$-model is weakly coupled, we might expect that the exact conformally invariant theory is “close” to its 1-loop approximation.

A more fruitful point of view is to accept that the $\sigma$-model (3.2) (or, at least any $\sigma$-model we can actually write down) is not conformally-invariant. However, it flows under the Renormalization Group to an infrared fixed point theory which is the desired conformally invariant theory.

This second point of view is very useful. It suggest several helpful ways of looking at the $\sigma$-model. First, the RG flow is dissipative. The data $g^{\bar{i}\bar{j}}, b^{\bar{i}\bar{j}}, A^a_{b\bar{i}}$ represent an infinite number of coupling constants in the two dimensional quantum field theory. All but a finite number of these are marginally irrelevant and flow to zero in the infrared. Thus the fixed-point theory is characterized by a finite number of parameters which are RG-invariant.
So what are the RG-invariant parameters characterizing (3.2)? They are

a) the complex structure of $M$

b) the holomorphic structure of the vector bundle $V$

c) the cohomology class of the complex Kähler form $\mathcal{J} = B + iJ$, where

$$J = i \ g_{ij} dX^i \wedge dX^\bar{j}, \quad B = b_{ij} dX^i \wedge dX^\bar{j}$$

The first two are automatic in this formalism. They are assured by the existence of a chiral $U(1)_L \times U(1)_R$ symmetry under which

$$\psi^{\bar{i}} \rightarrow e^{i\theta_R} \psi^{\bar{i}}, \quad \lambda^a \rightarrow e^{-i\theta_L} \lambda^a$$

$$\psi^{\bar{i}} \rightarrow e^{-i\theta_R} \psi^{\bar{i}}, \quad \lambda^a \rightarrow e^{i\theta_L} \lambda^a$$

Note that the conditions $c_1(T) = c_1(V) = 0$ are precisely what is needed to ensure that this chiral $U(1)_L \times U(1)_R$ is nonanomalous. In the conformal limit, the corresponding conserved $U(1)$ currents become the generators of the left-moving $\hat{U}(1)$ current algebra and the right-moving $\hat{U}(1)_R$ current algebra in the N=2 superconformal algebra.

The RG-invariance of the cohomology class of $\mathcal{J}$ is, by contrast, highly nontrivial. It was proven to all orders in perturbation theory in [6], where it was shown that all perturbative corrections to $J$ are exact two-forms. Beyond perturbation theory, one needs to worry about $\sigma$-model instantons, topologically nontrivial maps from the worldsheet into $M$. Naively, corrections to $g_{ij}$ are instanton-antiinstanton effects, and so rather hard to see. There are rather indirect arguments [7] which one might use to try to show that the cohomology class of $\mathcal{J}$ is unrenormalized, even when $\sigma$-model instantons are taken into account. But the necessary conditions are very hard to verify, and for a long time this pretty much stymied any progress on (0,2) $\sigma$-models.

Since nonlinear $\sigma$-models are so hard, we can invoke another great principle of the renormalization group, namely universality. There are many QFTs which renormalize to the same IR fixed point. If nonlinear $\sigma$-models are too hard, we should look for another, simpler family of QFTs which happen to be in the same universality class. This motivates us to look at linear $\sigma$-models [8].

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4. Linear $\sigma$-models

Since we will be interested in $(0,2)$ linear $\sigma$-models, we should first discuss $(0,2)$ superfields. Our $(0,2)$ superspace has coordinates $(z, \bar{z}, \theta^+, \theta^-)$. The spinor derivatives are

$$\bar{D}_\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\mp \partial_{\bar{z}}$$

Chiral (scalar) superfields $\Phi$ satisfy

$$\bar{D}_+ \Phi = 0$$

In components,

$$\Phi = \phi + \theta^- \psi + \theta^- \theta^+ \partial_{\bar{z}} \phi$$

A (chiral) fermi superfield $\Lambda$ also satisfies the chiral constraint $\bar{D}_+ \Lambda = 0$, but its lowest component is a left-handed fermion $\lambda$, and its upper component is an auxiliary field $l$:

$$\Lambda = \lambda + \theta^- l + \theta^- \theta^+ \partial_{\bar{z}} \lambda$$

The $(0,2)$ gauge multiplet actually consists of a pair of $(0,2)$ superfields $\mathcal{A}, V$, where $V$ is a superfield whose Minkowski continuation is a real superfield, and $\mathcal{A}$ is one whose Minkowski continuation is pure imaginary. The lowest component of $V$ is a real scalar, and the lowest component of $\mathcal{A}$ is the left-moving component of the gauge field, $a$ (which we take to be anti-Hermitian).

Super-gauge transformations act on $\mathcal{A}, V$ as

$$V \to V - i(\chi - \bar{\chi}), \quad \mathcal{A} \to \mathcal{A} - i(\chi + \bar{\chi})$$

where $\chi$ is a chiral scalar superfield, $\bar{D}_+ \chi = \bar{D}_- \bar{\chi} = 0$.

In Wess-Zumino gauge, the nonzero components of the gauge multiplet are

$$V = \theta^- \theta^+ \bar{a}$$

$$\mathcal{A} = a + \theta^+ \alpha - \theta^- \bar{\alpha} + \frac{1}{2} \theta^- \theta^+ D$$

(The residual gauge symmetry in WZ gauge is $\chi = \rho + \theta^- \theta^+ \partial_{\bar{z}} \rho$, with $\rho$ real.) $a, \bar{a}$ are the left- and right-moving components of the gauge field, $\alpha, \bar{\alpha}$ are the left-moving gauginos, and $D$ is a (real) auxiliary field.

Under a super gauge transformation,

$$\Phi \to e^{2iQ \chi} \Phi, \quad \bar{\Phi} \to e^{-2iQ \bar{\chi}} \bar{\Phi}$$
where \( Q \) is the charge of \( \Phi \), and similarly for \( \Lambda \). Let
\[
\tilde{\Phi} = e^{QV} \Phi, \quad \bar{\tilde{\Phi}} = e^{QV} \bar{\Phi}
\]
and similarly for \( \Lambda \).

A gauge invariant kinetic term for \( \Phi \) is
\[
S_\Phi = \int d^2z d^2\theta (\partial_z - Q A) \tilde{\Phi} \bar{\Phi} - (\partial_z + Q A) \tilde{\Phi} \bar{\Phi} + 2 \tilde{\psi}(\partial_z + Q a) \psi + Q(\bar{\alpha} \psi - \alpha \bar{\psi}) - Q \bar{\Phi} \bar{\Phi} D
\]
(4.1)

An invariant kinetic term for \( \Lambda \) is
\[
S_\lambda = \int d^2z d^2\theta \tilde{\Lambda} \bar{\Lambda}
\]
\[
= \int d^2z 2\tilde{\lambda}(\partial_z + Q \bar{a}) \lambda - \bar{l}
\]
(4.2)

Define the spinor covariant derivatives
\[
\bar{D}_\pm = \pm e^{\mp V} \bar{D}_\pm e^{\mp V}, \quad D = \partial_z + A
\]
and the corresponding gauge field strengths
\[
\mathcal{F} = 2[D, \bar{D}_+] = -\alpha + \theta^-(D + f) - \theta^- \theta^+ \partial_z \alpha
\]
\[
\bar{\mathcal{F}} = 2[D, \bar{D}_-] = -\bar{\alpha} + \theta^+(D - f) + \theta^- \theta^+ \partial_z \bar{\alpha}
\]
where \( f = 2(\partial_z \bar{a} - \partial_z a) \). Note that \( \mathcal{F} \) is chiral: \( \bar{D}_+ \mathcal{F} = 0 \). The kinetic term for the gauge fields is
\[
S_{\text{gauge}} = -\frac{1}{2e^2} \int d^2z d^2\theta \mathcal{F} \bar{\mathcal{F}}
\]
\[
= \frac{1}{2e^2} \int d^2z (f^2 - D^2 + 2\alpha \partial_z \bar{\alpha})
\]
(4.3)

and the Fayet-Iliopoulos D-term is
\[
S_D = -\frac{it}{2} \int d^2z d\theta^- \mathcal{F} + \frac{it}{2} \int d^2z d\theta^+ \bar{\mathcal{F}}
\]
\[
= r \int d^2z D - \frac{i\theta}{2\pi} \int d^2z f
\]
(4.4)

\[2 \text{ The normalization is such that } d^2z = \frac{i}{2} dz \wedge d\bar{z}, \text{ so } \frac{1}{2\pi} \int d^2z f = n_{\text{inst}} \in \mathbb{Z}. \]
where $t = \frac{\theta}{2\pi} + ir$.

Finally, a (0,2) superpotential is

$$S_{W} = \int d^{2}d\theta^{-} \mathbf{m} \Lambda F(\Phi) + \int d^{2}d\theta^{+} \bar{\mathbf{m}} \bar{\Lambda} F(\bar{\Phi})$$

$$= \int d^{2}z \mathbf{m}(F(\phi) - \lambda \frac{\partial F}{\partial \phi} \psi) + \text{h.c.}$$

(4.5)

where $F$ is a homogeneous polynomial of the appropriate degree such that (4.5) is gauge invariant, and $\mathbf{m}$ is a coupling constant with dimensions of mass. It is commonplace to set $\mathbf{m} = 1$, which simplifies the notation, but it is useful to remember that it is there.

This is all that is required in order to discuss (0,2) supersymmetric linear $\sigma$-models. But if we wish to discuss (2,2) supersymmetric theories (and certain (0,2) generalizations), we actually need to enlarge the gauge multiplet. We introduce a complex fermionic superfield $\Sigma$ and its conjugate $\bar{\Sigma}$. N.B. these do not obey a chiral constraint! Correspondingly, we introduce a new gauge symmetry, under which

$$\begin{align*}
\Sigma &\rightarrow \Sigma + i \Omega \\
\bar{\Sigma} &\rightarrow \bar{\Sigma} - i \bar{\Omega} \\
\Lambda &\rightarrow \Lambda + 2iQ\Omega \Phi \\
\bar{\Lambda} &\rightarrow \bar{\Lambda} - 2iQ\bar{\Omega} \bar{\Phi}
\end{align*}$$

(4.6)

with $\Omega$ a chiral fermionic superfield, and all other fields being invariant.

$\Sigma$ has four independent components, but the $\Omega$ gauge symmetry allows us to gauge two of them away. We’ll call the ones that remain $\sigma, \beta$, and note that they appear in the gauge invariant quantity

$$\bar{D}^{+} \Sigma = \frac{1}{2} (\sigma + \theta^{-} \beta + \theta^{-} \theta^{+} \partial_{z} \sigma)$$

Unfortunately, the action (4.2) is not invariant under the $\Omega$ gauge symmetry. To correct this, we add

$$S_{\Sigma} = \int d^{2}z d^{2}\theta Q^{2} \bar{\Phi} \Phi \Sigma \Sigma - Q(\tilde{\Lambda} \Phi \Sigma - \tilde{\Phi} \Lambda \Sigma)$$

(4.7a)

$$+ \frac{1}{2e^{2}} (-\bar{D}^{-} \Sigma \partial_{z} \bar{D}^{+} \Sigma + \partial_{z} \bar{D}^{-} \bar{\Sigma} \bar{D}^{+} \Sigma)$$

(4.7b)

(4.7a) makes (4.2) invariant, while (4.7b) gives $\Sigma$ a kinetic term. In “Wess-Zumino” gauge,

$$S_{\Sigma} = \int d^{2}z Q^{2} |\phi|^{2} |\sigma|^{2} + Q(\tilde{\lambda} \psi \sigma - \lambda \bar{\psi} \bar{\sigma}) - Q(\beta \lambda \phi - \bar{\beta} \bar{\lambda} \bar{\phi})$$

$$+ \frac{1}{2e^{2}} (\partial_{z} \bar{\sigma} \partial_{z} \bar{\sigma} + \partial_{z} \sigma \partial_{z} \bar{\sigma}) + \frac{1}{e^{2}} \bar{\beta} \partial_{z} \beta$$

(4.8)

We also need to make (4.5) invariant under the $\Omega$ gauge transformations. We’ll see how to do that later. First, let’s start looking at some examples
5. Examples

**Example 1: (2,2) Linear $\sigma$-model on $\mathbb{C}P^N$**

Choose $\Phi^i, \Lambda^i, i = 1, \ldots, N + 1$ to all have charge $Q = 1$. After eliminating the auxiliary field $D$,

$$\mathcal{L} = (\text{kinetic terms for } \phi, \psi, \lambda, \alpha, \beta, \sigma) + (\bar{\alpha} \bar{\psi}^i \phi^i - \alpha \psi^i \bar{\phi}^i)$$

$$+ (\bar{\beta} \lambda^i \bar{\phi}^i - \beta \bar{\lambda}^i \phi^i) + (\bar{\lambda}^i \psi^i \sigma - \lambda^i \bar{\psi}^i \bar{\sigma})$$

$$+ \sum_i |\phi^i|^2 |\sigma|^2 + \frac{e^2}{2} \left( \sum_i |\phi^i|^2 - r \right)^2$$

$$+ \frac{1}{2e^2} f^2 - \frac{i \theta}{2\pi} f$$

We can analyse this theory semiclassically in the $r \gg 0$ limit. Supersymmetry requires that the scalar potential vanish, and hence that

$$\sum_i |\phi^i|^2 = r, \quad \sigma = 0 \quad (5.1)$$

The space of solutions to (5.1) is a big sphere $S^{2N+1}$, but we still must mod out by the action of the gauge transformations $\phi \to e^{i \theta} \phi$. So, after modding out, the $\phi$’s live on $\mathbb{C}P^N = S^{2N+1}/U(1)$. All of the degrees of freedom transverse to $\mathbb{C}P^N$ have masses of order $m^2 \sim e^2 r$. At energies well below this mass scale, we have an effective nonlinear $\sigma$-model with target space $\mathbb{C}P^N$.

Well, that’s what’s happening with the bosons. Let us see what happens to the fermions. Since $\phi$ has a VEV, one linear combination of the $\psi$’s gets a mass with $\alpha$. Which linear combination is it? Let $\psi^i = \psi \phi^i$.

$$-\alpha \phi^i \bar{\phi}^i = -\alpha |\phi|^2 = -r \alpha \psi$$

so it is precisely the linear combination represented by $\psi$ which becomes massive. The remaining $\psi^i$ transform as sections of the tangent bundle to $\mathbb{C}P^N$. Mathematically, the $\psi^i$ fit into the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus N+1} \to T_{\mathbb{C}P^N} \to 0 \quad (5.2)$$

Of course, exactly the same analysis holds for the left-moving fermions $\lambda^i$. 
Digression: Line bundles on $\mathbb{CP}^N$

Recall $\mathbb{CP}^N = \mathbb{C}^{N+1}/\sim$, where $(z_0, z_1, \ldots, z_N) \sim (\lambda z_0, \lambda z_1, \ldots, \lambda z_N)$, $\lambda \in \mathbb{C}^*$. $\mathcal{O}(-1)$, the “tautological” line bundle has, as fiber over the point $[z_0, z_1, \ldots, z_N] \in \mathbb{CP}^N$, the complex line through the origin in $\mathbb{C}^{N+1}$ which passes through $(z_0, z_1, \ldots, z_N)$. Powers of the tautological line bundle are denoted by $\mathcal{O}(-n) = \mathcal{O}(-1)^{\otimes n}$, with negative powers denoting powers of the dual line bundle $\mathcal{O}(1)$. $\mathcal{O}(1)$ is called the “hyperplane bundle” because it has a global holomorphic section which vanishes along a hyperplane, say $\{z_0 = 0\} \subset \mathbb{CP}^N$. The $\phi^i$ (and their superpartners $\psi^i$) in the above example can be viewed as sections of $\mathcal{O}(1)$.

Very nice. Unfortunately for our intended application, $r$ is not a renormalization group invariant in this model. Rather, there’s a one-loop log-divergent diagram Fig. 1 which contributes to the renormalization of $r$. The $\beta$-function is proportional to the sum of the scalar charges ($\sum Q_i = N + 1$ in this case), and the sign is such that $r(\mu)$ decreases in the infrared. So even if we start out at large $r$, where the theory is semiclassical, we don’t stay there.

But this is exactly the sort of behaviour we expect. The $\mathbb{CP}^N$ nonlinear $\sigma$-model also has a nonzero $\beta$-function and flows to strong coupling in the infrared. It develops a mass gap, and the infrared theory is a $c = 0$ CFT (a topological field theory) [9].

![Fig. 1: Log-divergent diagram leading to the renormalization of $r$. Charged scalars run around the loop, and the coefficient of the log-divergence is proportional to $\sum Q_i$, the sum of the scalar charges.]

Example 2: Calabi-Yau hypersurface in $\mathbb{WP}^4$

As before, we consider $\Phi^i, \Lambda^i$, $i = 1, \ldots, 5$, but now, instead of taking them to all be of charge 1, we allow them to have (integer) charges $w_i > 0$. If we simply followed the analysis of example 1, we would obtain not $\mathbb{CP}^4$, but the weighted projective space $\mathbb{WP}^4 = \mathbb{C}^5/\sim$, where

$$(z_1, \ldots, z_5) \sim (\lambda^{w_1} z_1, \ldots, \lambda^{w_5} z_5)$$
The sum of the scalar charges is still nonzero, so let us add a chiral scalar superfield $P$, and fermionic superfield $\Gamma$ of charge $-d$, where $d = \sum w_i$. Under the $\Omega$-gauge transformation,
\[ \Gamma \rightarrow \Gamma - 2id\Omega P \tag{5.3} \]
The action is as before, but now we can add a gauge-invariant superpotential
\[ S_W = \int d^2z d\theta^- m(W(\Phi) + \Lambda^i P \frac{\partial W}{\partial \Phi^i}) + h.c. \tag{5.4} \]
where $W(\Phi)$ is a weighted homogeneous polynomial of degree $d$ in the $\Phi^i$. $m$ is a parameter with dimensions of mass. We will, for the most part, follow convention, and set it “equal to one”, but it is important to remember that it is really there, setting the scale for certain of the mass terms to be discussed below.

This is obviously invariant under the $\chi$-gauge transformations. Invariance under $\Omega$-gauge transformations follows from
\[ \sum_i w_i \Phi^i \frac{\partial W}{\partial \Phi^i} = d W(\Phi) \]
Adding the superpotential introduces new terms in the scalar potential from integrating out the auxiliary fields in $\Lambda^i$ and $\Gamma$:
\[ U = \frac{e^2}{2} \left( \sum w_i |\phi^i|^2 - d|p|^2 - r \right)^2 + |W|^2 + |p|^2 \left| \frac{\partial W}{\partial \phi^i} \right|^2 + |\sigma|^2 \left( \sum w_i^2 |\phi^i|^2 + d^2 |p|^2 \right) \]
We also get some new Yukawa couplings:
\[ \mathcal{L} = \ldots - (\gamma \psi + \lambda \pi) \frac{\partial W}{\partial \phi^i} - \lambda^i \psi^j p \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} + h.c. \]
We assume that $W$ is chosen to be transverse: $W = \frac{\partial W}{\partial \phi^i} = 0 \Rightarrow \forall \phi^i = 0$. The semiclassical analysis proceeds as before.

$r \gg 0$:
Minimizing the scalar potential requires $\sum w_i |\phi^i|^2 = r, p = \sigma = 0$ and $W(\phi) = 0$. So after modding out by $U(1)$, the massless fields live on the hypersurface $W(\phi) = 0$ in $\mathbb{C}P^4$.

The masses of the fields transverse to the hypersurface are of the order $m^2 \sim e^2 r$ or $m^2 \sim |m|^2$, depending on whether they get a mass from the D-term or from the superpotential. We will, for now, simply assume that these are of the same order.
For the fermions, as before, one linear combination of $\psi^i$ gets a mass with the gauge fermion. Another linear combination gets a mass from the Yukawa coupling $-\gamma \psi^i \frac{\partial W}{\partial \phi_i}$. Mathematically, the massless fermions form the cohomology of the sequence

$$0 \to \mathcal{O} \xrightarrow{f} \bigoplus_i \mathcal{O}(w_i) \xrightarrow{g} \mathcal{O}(d) \to 0$$

where $f(s) = (w_1 \phi_1 s, \ldots, w_5 \phi_5 s)$ and $g(u_1, \ldots, u_5) = \sum u_i \frac{\partial W}{\partial \phi_i}$. This is precisely the sequence which defines the tangent bundle of the Calabi-Yau hypersurface: $T = \ker(g)/\text{im}(f)$. So, as expected, the fermions transform as sections of the tangent bundle.

$r \ll 0$:

Here too, we can find a supersymmetric vacuum by minimizing the scalar potential. Set $|p|^2 = |r|/d$, $\phi = \sigma = 0$. $p$ and $\sigma$ are massive, while the $\phi^i$ are massless. The left-moving fermion $\gamma$ gets a mass with the gaugino $\bar{\beta}$ from $\mathcal{L} = \ldots - d\bar{\beta}\gamma \bar{p} + \ldots$. The right-moving fermion $\pi$ gets a mass with the gaugino $\alpha$ from $\mathcal{L} = \ldots + d\alpha \pi \bar{p} + \ldots$.

The low energy theory is described by the superpotential

$$\int d^2 z d\theta^- \ \text{const} \Lambda^i \frac{\partial W}{\partial \Phi^i} + \text{h.c.}$$

This should be recognizable as the superpotential for a (2,2) Landau-Ginzburg theory. Actually [8], it is a Landau-Ginzburg orbifold. Since $p$ has charge $-d$, its VEV doesn’t completely break the gauge symmetry. Gauge transformations by $d^{th}$ roots of unity are still unbroken and so we should still mod out our Landau-Ginzburg theory by this unbroken $\mathbb{Z}_d$ group. Projecting onto the $\mathbb{Z}_d$-invariant states requires, for modular invariance, that we introduce twisted sectors, with boundary conditions twisted by $\mathbb{Z}_d$ [3].

**Example 3: Deformations of (2,2) theories**

The superpotential (5.4) was not the most general one compatible with the $\Omega$ gauge transformations. More generally,

$$S_W = \int d^2 z d\theta^- \ (\Gamma W(\Phi) + \Lambda^i P F_i(\Phi)) + \text{h.c.} \quad (5.5)$$

where

$$\sum w_i \phi^i F_i(\phi) = d \ W(\phi) \quad (5.6)$$

Alternatively, on a higher genus Riemann surface, we sum over sectors where the boundary conditions on the fields around each cycle are twisted by $\mathbb{Z}_d$ gauge transformations.
is invariant under $\Omega$-gauge transformations. Generically, this breaks $(2,2)$ supersymmetry down to $(0,2)$. For example, if $W(\phi)$ is a quintic polynomial in $\mathbb{C}P^4$, there is a 224-dimensional space of polynomials $F_i$ satisfying (5.6).

For $r \gg 0$, we see that the right-moving fermions $\psi^i$ which remain massless again transform as sections of $T$, but the massless left-moving fermions $\lambda^i$ transform as sections of $V$ (a holomorphic deformation of $T$), which is the cohomology of the sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\otimes w_i \phi^i} \bigoplus_i \mathcal{O}(w_i) \xrightarrow{\otimes F_i(\phi)} \mathcal{O}(d) \rightarrow 0$$

These are, to be sure, $(0,2)$ theories, but the rank of $V$ remains $r = 3$, $E_6$ is unbroken and (one can show) the number of $27$s and $\overline{27}$s remains unchanged.

**Example 4: Dispensing with the $\Sigma$ multiplet**

We needed the $\Sigma$ multiplet and the accompanying $\Omega$ gauge transformations in order to describe $(2,2)$ supersymmetric theories. If we’re really interested in $(0,2)$ theories, why not dispense with them and consider

$$S_W = \int d^2zd\theta^- \left( \Gamma W(\Phi) + \Lambda_i^i P F_i(\Phi) \right) + h.c.$$ (5.7)

where, now, freed from the constraint of $\Omega$ gauge-invariance, the $F_i(\phi)$ are arbitrary polynomials of the appropriate degree.

Recall that previously there were two mass terms for the left-moving fermions

$$\mathcal{L} = \ldots + w_i \bar{\beta} \lambda^i \phi^i - \lambda^i \pi F_i(\phi) + h.c.$$  

Now there’s only one (since $\bar{\beta}$ is absent from the theory). So $V$ is defined by

$$0 \rightarrow V \rightarrow \bigoplus_i \mathcal{O}(w_i) \xrightarrow{\otimes F_i(\phi)} \mathcal{O}(d) \rightarrow 0$$ (5.8)

and now has rank $r = 4$.

It turns out that this theory is ill-behaved; we’ll see why later.

**Example 5: Arbitrary charges**

Since the $\Lambda$s are supposed to be unrelated to the $\Phi$s, it is silly to give them the same label and to assume that they have the same gauge charges. So let

$$S_W = \int d^2zd\theta^- \left( \Gamma W(\Phi) + \Lambda^a P F_a(\Phi) \right) + h.c.$$ (5.9)

where now we let the charges of the fields be given in the table below.
Field | $Q$ \\
---|---
$\Phi^i$ | $w_i$
$P$ | $-m$
$\Lambda^a$ | $n_a$
$\Gamma$ | $-d$

**Table 2:** $U(1)$ charges of the (bosonic and fermionic) chiral superfields.

We still require

$$
\sum_i w_i = d, \quad \sum_a n_a = m \quad (5.10)
$$

which in the semiclassical $r \gg 0$ analysis are equivalent to the conditions $c_1(T) = c_1(V) = 0$. But now, since we are dealing with a theory with chiral fermions coupled to a gauge field, we need to ensure that there is no gauge anomaly. This is a quadratic condition on the gauge charges:

$$
\sum \text{left-movers} + \sum \text{right-movers} \equiv 0
$$

Rearranging this,

$$
\frac{1}{2}(-m^2 + \sum n_a^2) = \frac{1}{2}(-d^2 + \sum w_i^2)
$$

which, in the $r \gg 0$ Calabi-Yau phase is simply the condition $c_2(V) = c_2(T)$! This is a general principle which holds in all of the (0,2) linear $\sigma$-models. The condition for the cancellation to worldsheet gauge anomalies translates into the $c_2(V) = c_2(T)$ condition for the vanishing of $\sigma$-model anomalies.

The exact sequence defining the bundle $V$ is now

$$
0 \rightarrow V \rightarrow \oplus_i \mathcal{O}(n_a) \stackrel{F_a(\phi)}{\longrightarrow} \mathcal{O}(d) \rightarrow 0 \quad (5.11)
$$

instead of (5.8). And now, most importantly, the rank of $V$ and the number of generations independent of those of the tangent bundle.

Perhaps more surprising is the Landau-Ginzburg phase for $r \ll 0$. There, $p$ has a VEV, and since it has charge $-m$, this breaks $U(1) \rightarrow \mathbb{Z}_m$. But since $m \neq d$ in general, the orbifold group is different from that of the corresponding (2,2) model. This means, in particular, that the Kähler moduli space is topologically different from that of the (2,2) model. At large positive $r$, they are obviously the same, as they both describe the space of complexified Kähler forms for a weakly coupled Calabi-Yau $\sigma$-model. But globally, the CFT “knows” the difference between the “Kähler” degrees of freedom of the (2,2) and the (0,2) models.
**Example 6: Return of the $\Sigma$ field**

We can also construct models which include the $\Sigma$ multiplet. Let us simply *postulate* a transformation law under the $\Omega$-gauge transformations of the form

$$\Lambda^a \to \Lambda^a + 2i\Omega E^a(\Phi)$$  \hspace{1cm} (5.12)

where $E^a(\phi)$ is a polynomial of weighted degree $n_a$. Then the action

$$S_\Lambda + S_\Sigma = \int d^2z d^2\theta \; e^{2n_a} V(\bar{\Lambda}^a \Lambda^a + |E^a(\Phi)|^2 \bar{\Sigma} \Sigma - E^a(\Phi) \bar{\Lambda}^a \Sigma - \bar{E}^a(\Phi) \Lambda^a \bar{\Sigma})$$

$$+ \frac{1}{2e^2} (-D_- \Sigma \partial_+ \Sigma + \partial_+ D_- \Sigma D_+ \Sigma)$$

is invariant under the $\Omega$-gauge symmetry. The superpotential (5.5) is invariant, provided

$$E^a(\phi) F_a(\phi) = d W(\phi)$$

$V$ is now the cohomology of the sequence

$$0 \to \mathcal{O}^{\otimes E^a(\phi)} \bigoplus_a \mathcal{O}(n_a)^{\otimes F_a(\phi)} \to \mathcal{O}(m) \to 0$$  \hspace{1cm} (5.13)

This is not so obvious as before, when we were construction the tangent bundle $T$. It is clear that one linear combination of the $\lambda^a$ get a mass from the $-\lambda^a \pi F_a(\phi)$ term coming from the superpotential, and that this corresponds to their being in the kernel of the map to $\mathcal{O}(m)$ in (5.13). To see that the other linear combination of the $\lambda^a$ drop out as required, it is easiest to not work in W.Z. gauge, but instead to use the $\Omega$-gauge symmetry (5.12) to gauge them away. The remaining $\lambda^a$ are clearly in the quotient by the image of $\mathcal{O}$ in (5.13).

**Example 7: Complete intersection Calabi-Yau manifolds**

Our previous examples have, in the Calabi-Yau phase, corresponded to (0,2) models defined on hypersurfaces in $WP^4$. There is clearly no need to restrict ourselves to hypersurfaces. Complete intersection Calabi-Yau’s are just as easy to describe.

Instead of a single $\Gamma$, let there be several $\Gamma^\alpha$, and let the superpotential be

$$S_W = \int d^2z d^2\theta^- (\Gamma^\alpha W_\alpha(\Phi) + \Lambda^a PF_a(\Phi))$$  \hspace{1cm} (5.14)
As before, we require

\[- \sum d_\alpha + \sum w_i = 0 \iff c_1(T) = 0 \]

\[-m + \sum n_a = 0 \iff c_1(V) = 0 \]

\[\sum d_\alpha^2 - \sum w_i^2 = m^2 - \sum n_a^2 \iff c_2(T) = c_2(V) \quad (5.15)\]

For simplicity, we assume that we are working in a model without the \( \Sigma \) multiplet and the associated \( \Omega \)-gauge symmetry. In the semiclassical \( r \gg 0 \) analysis, this superpotential leads to a (0,2) model where the bundle \( V \) is defined by the sequence (5.11), living on the complete intersection \( W_\alpha(\phi) = 0 \).

### Table 3: \( U(1) \) charges of the (bosonic and fermionic) chiral superfields.

| Field | \( Q \) |
|-------|--------|
| \( \Phi^i \) | \( w_i \) |
| \( P \) | \( -m \) |
| \( \Lambda^a \) | \( n_a \) |
| \( \Gamma^\alpha \) | \( -d_\alpha \) |

### Spectators:

In the previous three examples, one generally finds that the sum of the scalar charges is nonzero. In a (2,2) model, this would be a fatal flaw. In the present context, we can always fix this (so that there is no perturbative renormalization of \( t \)) by adding a pair of chiral superfields \( S, \Xi \).

To the superpotential, we add a term

\[ S_W = \int d^2 z d\theta^- \ldots + m_\Xi \Xi S \quad (5.16) \]

Examining the scalar potential, \( U \), we see that \( s = 0 \), and all the fluctuations of \( S \) and \( \Xi \) are massive. So, naively, they do not affect the low-energy physics.

This is a little too slick. In the Landau-Ginzburg phase, we see that \( S, \Xi \) are charged under the unbroken \( Z_m \) symmetry, so their boundary conditions are twisted in the twisted
sectors of the orbifold. Nonetheless, since they appear only quadratically in the superpotential, when we compute the cohomology of the $\tilde{Q}_+$ operator in §7, there is a choice of representatives in which the $S, \Xi$ oscillators are not excited. In particular, this means that they do not appear in the massless states of the string theory. They also make no net contribution to the ground state energies or $U(1)$ charges of the twisted sectors.

This seems to pose a small paradox. What if we decided to make $m_s$ very large? Surely, below the scale of $m_s$, we should be able to describe the physics in terms of an effective theory with $\Xi, S$ absent. Aren’t we then back in the situation where $t = \frac{\theta}{2\pi} + ir$ is scale-dependent (as in Example 1)?

Actually, we might have asked a similar question already in the (2,2) theory we discussed as example 1. There we had two, a priori independent, mass scales $e$ and $m$ (where $m$ is the parameter introduced in (5.4)). If we take these to be disparate, we should work with an effective theory in the intervening regime, and in that effective theory, we expect that $t$ will run. Put another way, we might expect the low-energy Kähler class to depend on the dimensionless ratio $m/e$.

Is this, in fact, the case? The answer, of course, is no, and the reason is as follows. The Wilsonian coupling, $t_{LE}$, being the coefficient of a term in the superpotential, is an analytic function of $m$ \[10\]. So, though we’re really interested in the dependence on the magnitude of $m$, we can equally well inquire about its dependence on the phase of $m$. But the phase of $m$ can be rotated away by a common rotation of the superfields $\Gamma$ and $P$. This symmetry is nonanomalous, so the phase of $m$ is unphysical, and $t_{LE}$ must be independent of it (and hence of $m$ itself).

In the (0,2) theories under consideration, there is no nonanomalous symmetry which allows us to simultaneously remove both the phase of $m$ and of $m_S$. The ratio, $m/m_S$ is physical. More precisely, the combination

$$t_{phys} = t - \frac{i}{2\pi} Q_s \ln(m/m_s)$$

where $Q_s = m - \sum d_\alpha$ is the charge of the spectator scalar $S$, is physical. A change in the phase of $m_s$ can be compensated by a shift in the $\theta$-angle (the real part of $t$). Moreover, we can choose a basis in which this is the only anomalous $U(1)$.

If we choose to work with an effective theory in the range $|m| < \mu < |m_S|$, then, of course, $t(\mu)$ runs:

$$t(\mu) = \frac{i}{2\pi} Q_S \ln(\Lambda/\mu)$$  (5.18)
with \( \Lambda = m_se^{-2\pi it/Q_s} = m'e^{-2\pi it_{phys}/Q_s} \). \( t \) is no longer RG-invariant, but \( t_{phys} \) is.

Note what’s going on here. The dependence of the real part of \( t \) on the parameters in the superpotential is governed by the chiral anomaly (which, by the way, receives contributions only at one loop). But, by holomorphy, this constrains the dependence of the imaginary part of \( t \) as well. Thus (5.18) is an exact expression.

So ... the upshot is that there is one physical, RG-invariant parameter \((5.17)\) which parametrizes the low energy physics, rather than the two “naive” parameters \( t \) and \( m/m_S \).

Of course, if we are smart, we simply use our freedom to set \( m_S = m \), in which case \( t \) and \( t_{phys} \) coincide. In that case, the “spectators” should be thought of as being on exactly the same footing as the other massive particles in the linear \( \sigma \)-model.

Now, \( t_{phys} \) is the RG-invariant parameter in the linear \( \sigma \)-model which parametrizes the “Kähler moduli space”. However, if we wish to discuss the low energy physics in terms of the effective nonlinear \( \sigma \)-model, we need to perform a matching between \( t_{phys} \) and the complexified Kähler parameter of the nonlinear \( \sigma \)-model. At tree level, they are simply equal, and indeed this equality holds to all orders in perturbation theory. If we write \( z = e^{2\pi it_{phys}} \), and \( q = e^{2\pi iJ} \), then we have \( q = z \).

However, nonperturbatively, this relation is modified to

\[
q = z(1 + \sum_{n=1}^{\infty} a_n z^n)
\]

This is easily recognized as an instanton correction to the matching condition (a holomorphic \( n \)-instanton effect goes like \( z^n \)). Its origin is simply the fact that the linear \( \sigma \)-model possesses instanton solutions (dubbed “pointlike instantons” in [8]) which are not present in the nonlinear \( \sigma \)-model. To perform the matching, we need to integrate out the pointlike instantons, which introduces a nontrivial matching condition. Note that this is not a feature peculiar to \((0,2)\) theories. It occurs as well in \((2,2)\) theories, where it is called the relation between the algebraic and \( \sigma \)-model coordinates [13] (For a recent discussion from the point of view of linear \( \sigma \)-models, see [14]).

Onward!

There are numerous variations on the constructions described here. We can have more \( U(1) \) gauge groups, and thereby construct complete intersection Calabi-Yaus in more

\[\text{[12]}\]

Unlike in \( N=1 \) supersymmetric theories in four dimensions, where the analogous statement is true only in a very special renormalization scheme [11], here (because the theory is all but superrenormalizable) it is true without any delicacies in the argument. For a recent discussion of the scheme-dependence of the four dimensional result, see [12].
general toric varieties. We can also generalize the construction of the gauge bundle $V$. By mixing and matching the various constructions, one can produce a wide variety of different models (see [8,15]). This may still only be scratching the surface of the space of (0,2) models.

Having discussed the tools for writing down (0,2) models, we will now turn to computing some of their properties. The key, as discussed in §3, is to consider RG-invariant quantities which can, reliably, be calculated in the linear $\sigma$-model.

A rich class of interesting things to calculate can be found by considering the “twisted” model, or, equivalently, to compute on the cylinder with periodic boundary conditions on the right-movers. This is the sector of the String Hilbert space containing the spacetime fermions. In this sector, (0,2) supersymmetry is unbroken, and the supercharges, $\bar{Q}^\pm$, close into the generator of boosts along the lightcone,

$$\{\bar{Q}^+, \bar{Q}^-\} = \bar{L}_0$$

If we are interested in those states with $\bar{L}_0 = 0$, which includes all of the massless spacetime fermions, we can represent these as the cohomology of the $\bar{Q}^+$ operator. We will see that the $\bar{Q}^+$-cohomology is eminently computable, so that, in particular, we learn about the spectrum of massless spacetime fermions and (by spacetime supersymmetry) about the full massless spectrum of the string theory.

Similarly, we can compute the matrix elements of $\bar{Q}^+$-invariant operators between these fermion states, which, in particular, allows us to calculate the spacetime superpotential.

6. Landau-Ginzburg

As we saw, the semiclassical analysis becomes exact for $r \to \pm\infty$. $r \to +\infty$ is a weakly-coupled (0,2) nonlinear $\sigma$-model. Much of what we currently can say about such $\sigma$-models has been understood for many years [7]. Instanton corrections are also suppressed in the $r \to -\infty$ limit. Here, too, one can make some definite statements, at least in those cases where the conformal field theory that one obtains in the $r \to -\infty$ limit is understood. For the examples discussed in the previous section, the $r \to -\infty$ limit corresponds to what we might call a (0,2) Landau-Ginzburg orbifold. In this case, we actually have a fair handle on the conformal field theory, and can actually make some definite statements.
To describe the low energy theory, we need, in particular, two unbroken nonanomalous $U(1)$ symmetries. One will be the $U(1)_R \subset (\text{right-moving } N = 2)$. The other will become the left-moving $U(1)$ which we introduced in §2. Denoting the charges under the two $U(1)$, respectively, as $\overline{q}, q$, it is clear that in order for these to be symmetries of the superpotential

$$S_W = \int d^2z d\theta^+ \Gamma^\alpha W_\alpha(\Phi) + \Lambda^a F_a(\Phi) + \Xi S$$

the charges of the $\Phi_i$ must be proportional to the respective gauge charges, $q_i = w_i/a$, $\overline{q}_i = w_i/b$. Since the left $U(1)$ is an honest symmetry, and not an R-symmetry, we can, without loss of generality, rescale the left $U(1)$ charges s.t. $q_i = \overline{q}_i$, or $a = b$. We will see later that this corresponds to the “standard” normalization of the corresponding current, $J$. The charges of the fermions are now determined, up to this unknown constant $b$.

We need to check, first of all, that these $U(1)$’s are nonanomalous under the gauge symmetry. The anomaly, of course, is given by a one-loop diagram with one insertion of the current, one external gauge field, and fermions running around the loop:

$$J \bigotimes \bigotimes \sim \frac{1}{b} \left[ \sum (n_a - m)n_a + \sum d_\alpha^2 - \sum w_i^2 \right] = 0$$

$$\overline{J} \bigotimes \bigotimes \sim \frac{1}{b} \left[ \sum (n_a - m + b)n_a + \sum (d_\alpha - b)d_\alpha - \sum (w_i - b)w_i \right] - m = 0$$

and these symmetries are nonanomalous, precisely when the conditions (5.15) on the gauge charges are satisfied. Note that, since $\overline{J}$ is an R-symmetry, even though $p$ is neutral, its fermi superpartner, $\pi$, is not, and contributes to the anomaly. Note also that the “spectators” make no net contribution to the anomalies.

We also need to require that $J$ and $\overline{J}$ are pure left- and right-moving currents in the infrared, which means that the mixed anomaly vanishes:

$$J \bigotimes \bigotimes \overline{J} \sim \frac{1}{b^2} \left[ \sum (n_a - m)(n_a - m + b) + \sum d_\alpha(d_\alpha - b) - \sum w_i(w_i - b) \right] = - \frac{rm(m - b)}{b^2}$$

For this to vanish, we must have $b = m$. So the charges of the fields are listed in table 5.
We have cheated a bit. The anomaly considerations do not determine the charges of $\Xi, S$. But recall that, because of the unbroken $\mathbb{Z}_m$ discrete gauge symmetry of the model, we are really describing a Landau-Ginzburg orbifold. The generator of $\mathbb{Z}_m$ is simply $e^{-2\pi i q}$.

Since we know how $\Xi, S$ are supposed to transform under $\mathbb{Z}_m$, this fixes their charges, $q$, modulo 1.

In fact, for our purposes, it is not useful to separate the $\mathbb{Z}_m$ orbifolding from the $\mathbb{Z}_2$ orbifolding which implements the GSO projection. Together, they form a $\mathbb{Z}_{2m}$ group generated by

$$g = e^{-i\pi q}(-1)^F \lambda^i$$

Comparing with (2.1), we see that this is, indeed, the standard normalization of the left-$U(1)$ charge which we defined in §2.

Now we can calculate the central charge of the infrared $N = 2$ superconformal algebra. Recall that one of the OPEs in the $N = 2$ superconformal algebra is

$$\bar{J}(\bar{z})\bar{J}(\bar{w}) = \frac{\bar{c}/3}{(\bar{z} - \bar{w})^2}$$

So, by calculating this OPE, we get a direct measurement of the central charge. But this is, as we have seen, computable from the one loop anomaly diagram in the linear $\sigma$-model, $\bar{J} F \bar{J}$.

$$\frac{\bar{c}}{3} = \sum (\bar{q}_i - 1)^2 - \sum q^2_a - \sum q^2_\alpha = 3$$

So, indeed, we have $\bar{c} = 9$, as expected!

Similarly, the $J \cdot J$ anomaly computes the level of the left-moving $U(1)$ current algebra:

$$r = \sum q^2_\alpha + \sum q^2_a - \sum q^2_i$$

In fact, we can make an even stronger statement. The operators

$$T' = -\sum_i \left( \partial \phi_i \partial \bar{\phi}_i + \frac{q_i}{2} \partial (\phi_i \partial \bar{\phi}_i) \right) + \sum_a \left( \lambda_a \partial \bar{\lambda}_a + \frac{q_a}{2} \partial (\lambda_a \bar{\lambda}_a) \right) + \sum_\alpha \left( \gamma_\alpha \partial \bar{\gamma}_\alpha + \frac{q_\alpha}{2} \partial (\gamma_\alpha \bar{\gamma}_\alpha) \right)$$

$$J' = -\sum_i q_i \phi_i \partial \bar{\phi}_i - \sum_a q_a \lambda_a \bar{\lambda}_a - \sum_\alpha q_\alpha \gamma_\alpha \bar{\gamma}_\alpha$$

Table 5: Left-moving $U(1)$ and right-moving $U(1)_R$ charges of the fields.

| Field | $q$ | $\bar{q}$ |
|-------|-----|------------|
| $\Phi_i$ | $q_i = \frac{w_i}{m}$ | $\bar{q}_i = \frac{w_i}{m}$ |
| $\Lambda^a$ | $q_a = \frac{n_a}{m} - 1$ | $\bar{q}_a = \frac{n_a}{m}$ |
| $\Gamma^\alpha$ | $q_\alpha = -\frac{d_\alpha}{m} - 1$ | $\bar{q}_\alpha = 1 - \frac{d_\alpha}{m}$ |
| $P$ | 0 | 0 |
| $S$ | $\frac{Q_S}{m}$ | $\frac{Q_S}{m}$ |
| $\Xi$ | $-\frac{Q_S}{m}$ | $1 - \frac{Q_S}{m}$ |
commute with the $\bar{Q}^+$ operator, and generate a $U(1) \times \text{virasoro}$ algebra on the $\bar{Q}^+$ cohomology, with $\hat{U}(1)$ level $r$ and virasoro central charge $c = 6 + r$. To see this, note that rescaling the superpotential is a $\bar{Q}^+$-trivial operation. Thus, while working on the level of the $\bar{Q}^+$ cohomology, we can use free-fields to evaluate the OPEs of $T', J'$. 

Note, too, that $J'_0$ differs from the previously-defined $U(1)$ charge $q$ by $\bar{Q}^+$-trivial terms, as does $L'_0$ from the $L_0$ derived canonically from the Lagrangian. Thus, for instance, since we will be working on the $\bar{Q}^+$-cohomology, we can continue to use $q$ to label the $U(1)$ charge of physical states.

Not only does this “off-shell” virasoro algebra exist at the Landau-Ginzburg point, Silverstein and Witten have shown that one can construct the corresponding operators in the full linear $\sigma$-model [16]. So the fact that one gets the correct infrared virasoro central charge and $\hat{U}(1)$ level is a property of the linear $\sigma$-model for arbitrary $r$, not just for $r \to -\infty$. Indeed this can be used as the basis for an argument that the (0,2) linear $\sigma$-models do indeed give rise to (0,2) SCFTs in the infrared, and that deforming $t$ really is an exactly-marginal deformation of the SCFT [17].

**GSO projection:**

We have already said that to effect the left-moving GSO projection of this theory (while simultaneously implementing the discrete $\mathbb{Z}_m$ gauge symmetry), we orbifold by the $\mathbb{Z}_{2m}$ group generated by

$$g = e^{-i\pi q} (-1)^{F_{\chi, t}}$$

(All the charges in the theory are multiples of $1/m$, so $g^{2m} = 1$.) As usual, we must include $2m - 1$ twisted sectors, where the boundary conditions on the fields are twisted by powers of $g$. So the sectors of the theory are labeled by $k$, $k = 0, \ldots, 2m - 1$. $k$ even corresponds to the left-moving Ramond sector, $k$ odd corresponds to the left-moving Neveu-Schwarz sector. The right-mover will, for us, always be in the Ramond sector, so we are discussing states which are spacetime fermions. And the $\bar{Q}^+$ cohomology computes those states which are physical and have $\bar{L}_0 = 0$.

**Modular Invariance**

The usual level-matching conditions for orbifolds [18], which are certainly satisfied by our constructions are not obviously sufficient here to assure the consistency of the theory, unlike the case of the usual toroidal orbifolds.

There’s a general principle in string theory that there is a direct correspondence between worldsheet and spacetime anomalies. We will look for a spacetime anomaly in
the $\mathbb{Z}_{2m}$ quantum symmetry \cite{19} of the Landau-Ginzburg orbifold. This is a discrete R-symmetry in spacetime. As such, it may not be preserved in a nontrivial gauge or gravitational background. That is, it may suffer from an anomaly. But the quantum symmetry (which simply says that sector number must be conserved \textit{modulo} $2m$) in any correlation function) \textit{must} be a symmetry if the GSO-projected theory is to have a sensible interpretation.

Naively, the generator of the quantum symmetry is $\gamma = e^{2\pi i k/2m}$, where $k = 0, \ldots, 2m - 1$ is the sector number. However, it proves more convenient to compose this with a gauge transformation which lies in the $U(1)$ subgroup of the spacetime gauge group generated by $q$:

$$\gamma = e^{2\pi i (kr - 2q)/2mr}$$

With this choice, irreducible representations of the spacetime gauge group transform homogeneously under the quantum symmetry.

One can compute the anomaly in this discrete R-symmetry by embedding it in a continuous $U(1)$ R-symmetry (with generator $(kr - 2q)$) and computing the standard triangle diagram, with one insertion of this current and two external gauge bosons or gravitons. Of course, since we are really only interested in assuring that the discrete subgroup is nonanomalous, we need only require the anomaly coefficient to vanish \textit{mod} $2mr$, rather than actually vanish.

$$\begin{aligned}
A_1 &= \otimes & \text{G-gauge bosons} & \text{mod } 2mr \\
A_2 &= \otimes & \text{E}_8\text{-gauge bosons} & \text{mod } 2mr \\
A_3 &= \otimes & \text{gravitons} & \text{mod } 2mr
\end{aligned}$$

where $G$ is the “observable” gauge group ($E_6$, $SO(10)$, or $SU(5)$, for $r = 3, 4, 5$, respectively).
Each of these can be expressed in terms of a trace over the right-moving R-sector. For instance,

\[ A_1 = \sum_{k=0}^{2m-1} \text{Tr}_R \left( \frac{q^2}{2r} (r^k - 2q)(-1)^{F_R} \right) \mod 2mr \]

Now, you might think that we should demand that each of these anomaly coefficients vanish \((\mod 2mr)\) separately. In fact, this is too strong, and is not even true for many \((2,2)\) models. Instead, because of the Green-Schwarz mechanism, we can cancel the anomaly by assigning a nontrivial transformation under \(\gamma\) to the axion. But this only can succeed in canceling all three anomalies provided the respective anomaly coefficients obey the relations \[ A_1 - A_2 = 0 \\
A_3 - 24A_1 = 0 \\
A_3 - 24A_1 = 0 \] \mod 2mr \tag{6.5}

In particular, this means that the anomaly must vanish for any field configuration satisfying \(\text{Tr} R^2 = \text{Tr} F_1^2 + \text{Tr} F_2^2\).

For most of the models we discussed in §4, the anomaly in the “quantum” discrete R-symmetry does indeed cancel. But, as alluded to earlier, some models can be ruled out on this basis.

### 7. \(\bar{Q}^+\) Cohomology

Finally, let us get down to constructing the spectrum of spacetime fermions. We work on the cylinder, and as I said before, the states with \(\bar{L}_0^{(int)} = 0\) can be represented as elements of the cohomology of one of the right-moving supercharges \(\bar{Q}^\pm\), which we will take to be \(\bar{Q}^+\). The \(\bar{Q}^+\) operator has three terms:

\[ \bar{Q}^+ = \bar{Q}^+_R + \bar{Q}^+_L + \bar{Q}^+_{\text{spectator}} \]

\[ \bar{Q}^+_R = \oint i \bar{\psi}_i \partial \phi_i \]

\[ \bar{Q}^+_L = \oint \gamma^\alpha W_\alpha(\phi) + \lambda^a F_a(\phi) \]

\[ \bar{Q}^+_{\text{spectator}} = \oint i \bar{\psi}_s \partial s + \xi s \] \tag{7.1}

The contribution to the supercharge from the spectator fields, \(\bar{Q}^+_{\text{spectator}}\), is decoupled from the rest of the \(\bar{Q}^+\) operator, and is purely quadratic. This means that it has trivial cohomology, and all the physical states have representatives in which the \(\Xi, S\) oscillators
are in their ground states. (This may, to the reader, seem like a fancy way of noting that since $\Xi, S$ are massive, they cannot contribute to the states of the IR conformal field theory, but it is nice to see the same result emerging from this point of view as well.)

In any case, we need to concentrate on the cohomology of $\bar{Q}_R^+ + Q_L^+$. Again, we note that $\bar{Q}_R^+$ is quadratic, and so has trivial cohomology. However, it is coupled to $Q_L^+$. Still, a standard spectral sequence argument \[21\] says that we can construct the cohomology of $\bar{Q}_R^+ + Q_L^+$ starting with the cohomology of $\bar{Q}_R^+$ as a “first approximation”. Actually applying the spectral sequence argument is a little delicate here, because the complex is infinite-dimensional. However, the result is basically correct \[22\], and the cohomology of $\bar{Q}_R^+$ is

$$H^*_Q = H^*_{Q_L^+} \cap H^*_{Q_{spectator}^+}$$  \(7.2\)

The cohomology of $\bar{Q}_R^+$ is trivial to calculate.

$$H^*_{Q_R^+} = \text{independent of } \psi_i, \bar{\psi}_i \text{ oscillators and}$$

(in untwisted sector) holomorphic in the zero mode of $\phi_i$

We simply have to compute the cohomology of $\bar{Q}_L^+$ on this smaller Hilbert space. This is made easier by the fact that, for these purposes, we can use free-field OPEs – the corrections from including the effect of the superpotential interactions are $\bar{Q}_R^+$-trivial \[23\]. So, more or less, we have reduced the problem to a free-field orbifold calculation, where the boundary conditions on the fields are twisted by the action of $g^k$, where $g$ is given by \(6.1\).

As usual in orbifold calculations, we expand the fields in oscillators which, because the boundary conditions are twisted by $g^k$, are fractionally moded. Also, the ground states of the twisted sectors carry fractional fermion number, and hence fractional values of the charges $q, \bar{q}$. In our case \[22\13\14\5\]

$$q = q_i \theta_{ik} + q_a \theta_{ak} + q_\alpha \theta_{\alpha k}$$

$$\bar{q} = \bar{q}_i \theta_{ik} + \bar{q}_a \theta_{ak} + \bar{q}_\alpha \theta_{\alpha k}$$  \(7.3\)

\[5\] Note that in Table 5, and here, we have made a slight change in our notation for the $U(1)$ charges of the fields from that of \[13\].
where
\[
\begin{align*}
\theta_{ik} &= \frac{k(q_i - 1)}{2} + \left[\frac{k(1 - q_i)}{2}\right] + \frac{1}{2} \\
\theta_{ak} &= \frac{k(q_a - 1)}{2} + \left[\frac{k(1 - q_a)}{2}\right] + \frac{1}{2} \\
\theta_{\alpha k} &= \frac{k(q_\alpha - 1)}{2} + \left[\frac{k(1 - q_\alpha)}{2}\right] + \frac{1}{2}
\end{align*}
\] (7.4)

Similarly, one calculates the ground state energy (eigenvalue of $L_0 - c/24$) of the $k^{th}$ twisted sector. The answer depends on whether $k$ is even, which gets paired with the R-sector for the $(16 - 2r)$ free left-moving fermions, or $k$ is odd, which gets paired with the NS-sector for the $(16 - 2r)$ free left-moving fermions

\[
\begin{align*}
E_{NS} &= -\frac{5}{8} + \sum \frac{\theta_{a,k}^2}{2} + \sum \frac{\theta_{\alpha,k}^2}{2} - \sum \frac{\theta_{i,k}^2}{2} \\
E_R &= -\frac{(r - 3)}{8} + \sum \frac{\theta_{a,k}^2}{2} + \sum \frac{\theta_{\alpha,k}^2}{2} - \sum \frac{\theta_{i,k}^2}{2}
\end{align*}
\] (7.5)

From here, constructing the spectrum is straightforward, but a little boring. One simply goes through, sector by sector, and computes the cohomology of $\bar{Q}_L^+$ on the twisted fock space. The spacetime quantum numbers of the states that we find are correlated with the $\bar{q}$ charge. For the massless states,

- $\bar{q} = -1/2$ right-handed spacetime fermions (chiral multiplet)
- $\bar{q} = +1/2$ left-handed spacetime fermions (antichiral multiplet)
- $\bar{q} = +3/2$ right-handed gauginos (vector multiplet)
- $\bar{q} = -3/2$ left-handed spacetime fermions (vector multiplet)

The left spectral flow maps us between adjoining twisted sectors. So, e.g. the left-handed gauginos are assembled, as in Table 1, from the ground state of the $k = 0$ sector, $|k = 0\rangle$, (a spinor of $SO(16 - 2r)$), two states from the $k = 1$ sector:

\[
\begin{align*}
\lambda_{-1/2}^j\lambda_{-1/2}^j|k = 1\rangle \\
\left[\sum q_i\phi_i^{i/2}\bar{\phi}_i^{i/2+q_i/2} - \sum q_a\lambda_{q_a/2}^\alpha\bar{\lambda}_{1-q_a/2}^\alpha - \sum q_\alpha\gamma_{q_\alpha/2}(\phi)\gamma_{1-q_\alpha/2}\right]|k = 1\rangle
\end{align*}
\]

and the ground state of the $k = 2$ sector, $|k = 0\rangle$, (again a spinor of $SO(16 - 2r)$). Each of these is evidently annihilated by $\bar{Q}_L^+ = f(\gamma^a W_\alpha(\phi) \lambda^a F_\alpha(\phi))$, and cannot be written as $\bar{Q}_L^+$ of some other state.
Similarly, we can go through and construct the rest of the states in the spectrum of massless fermions. Some examples are worked out in detail in [15].

As an exercise, I recommend that the reader work out the $\bar{Q}^+$ cohomology for the model which, in the Calabi-Yau phase, corresponds to a complete intersection of two sextics in $\mathbb{P}^5_{1,1,2,2,3,3}$ with the left-moving fermions coupling to a certain rank-4 vector bundle on it. The $U(1)$ charges of the fields in the linear $\sigma$-model (including the spectators) are listed in Table 6.

Since $m = 8$, there are 16 twisted sectors. You need not go through all of them, as the states you find in sectors 9-15 are simply the CTP conjugates of the states in sectors 1-7. You should find, for instance, that the right-handed 16s of $SO(10)$ arise in this model as states in the $k = 0$ and $k = 1$ sectors, corresponding respectively to the $8^+_{-1}$ and $8^+_{1}$ of $SO(8) \times U(1)$. Explicitly, these states are given by octic polynomials:

$$P_8(\phi^i_0)|k = 0\rangle$$

$$\lambda_{-1/2}^I P_8(\phi^i_{-Q_i/16})|k = 1\rangle$$

For these to be in the $\bar{Q}^+_L$ cohomology, we must mod out the space of octic polynomials by the ideal generated by the $F_a(\phi)$ and the $W_a(\phi)$. This yields 74 states in the $\bar{Q}^+_L$ cohomology.

### Interactions

We can also compute (unnormalized) Yukawa couplings of these fermions by sandwiching an operator (the vertex operator for a physical spacetime boson in a chiral multiplet), which commutes with $\bar{Q}^+$, between two of the fermion states we have just constructed. The results, at least in certain cases, [24][25] agree with those computed at large radius in the sigma model, leading one to hope that there might actually be a theorem, analogous to the one which holds in (2,2) theories [26], which states that these couplings are independent of the Kähler moduli.

| Field | $Q$ |
|-------|-----|
| $\Phi^{1,2}$ | 1 |
| $\Phi^{3,4}$ | 2 |
| $\Phi^{5,6}$ | 3 |
| $P$ | $-8$ |
| $\Lambda^{1,2,3,4}$ | 1 |
| $\Lambda^{5}$ | 4 |
| $\Gamma^{1,2}$ | $-6$ |
| $S$ | $-4$ |
| $\Xi$ | 4 |

**Table 6:** $U(1)$ charges of the (bosonic and fermionic) chiral superfields in the $\mathbb{P}^5_{1,1,2,2,3,3}$ models.
8. Envoi

Much, clearly, remains to be explored here. What are the analogues of mirror symmetry? Can one compute the spacetime Kähler potential for the fields? Is there indeed a theorem along the lines of [26] for some of the Yukawa couplings in these (0,2) theories? And what about the corrections to those couplings which do get corrected? Can one compute them in the linear sigma model, along the lines of [14]? Are there exactly-soluble conformal field theories (the analogues, perhaps, of the Kazama-Suzuki models [27] for (2,2) theories) which lie at special points in the moduli spaces of (0,2) theories that we have been exploring? These, and many other questions are waiting to be answered.

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