ON HELE-SHAW PROBLEMS ARISING AS SCALING LIMITS

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To the memory of Vladimir Markovich Entov

1. Introduction

This work is motivated by the paper [LP]. In this paper, the authors consider the scaling limits of three discrete aggregation models with multiple sources - the internal DLA (diffusion-limited aggregation), the rotor-router, and the divisible sandpile model. They show that for all three models, the scaling limit is the same, and it is a solution of the Hele-Shaw injection problem with multiple sources. In two dimensions, the latter problem can be solved explicitly using the theory of quadrature domains, which gives an explicit formula for the scaling limit.

They also consider the same aggregation models in two dimensions with just one source, but with an additional condition on the positive (horizontal) half-axis (namely, the condition that a particle hitting the positive half-axis is killed, or the condition that it is directed downward). In these cases, the existence of the scaling limit remains unknown, although it is expected to exist, and computer-generated pictures for its shape are given in Fig. 4 of [LP].

The goal of this paper is to study the Hele-Shaw problems which are expected to be the scaling limits of the above models with a condition. Namely, we provide an explicit solution for the Hele-Shaw problem corresponding to the killing condition, which is a close fit with the left shape at Fig. 4 of [LP]. We also describe moment properties of the solution of the Hele-Shaw problem corresponding to the downward condition (the right shape at Fig. 4 of [LP]), although we are unable to compute this shape explicitly.

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2. Flows with the killing/reflecting condition

2.1. The killing condition on a half-axis. Consider a 2-dimensional discrete aggregation model with injection at the origin and killing condition on a half-axis; we prefer to impose this condition on the negative half-axis (rather than the positive half-axis used in [LP]). To be more concrete, consider the internal DLA model. In this model, particles are emitted from the origin into a rectangular lattice, and each particle walks randomly on this lattice until it reaches an unoccupied node, where it stays, and a new particle is emitted. In addition, one stipulates that a particle that hits the negative half-axis is killed (after which a new particle is emitted). The scaling limit of this model is the limit in which the number \( N \) of surviving particles goes to infinity, and the spacing of the lattice goes to zero as \((A/N)^{1/2}\), where \( A \) is a constant.

If the scaling limit exists, it is a region \( \Omega \) in the plane of area \( A \), containing the origin in its interior. We would like to compute this region explicitly using conformal mappings, assuming the scaling limit exists and is the solution of the Hele-Shaw problem naturally associated to this model.

Now consider the Hele-Shaw problem for the asymptotic region \( \Omega \). We refer the reader to the books [VE] and [GV] for the basics on Hele-Shaw moving boundary problems.

**Problem 2.1.** Let \( \Phi \) be the harmonic potential in \( \Omega \) with deleted negative half-axis, with boundary conditions \( \Phi = 0 \) on the boundary \( \Gamma \) of \( \Omega \) and also on the negative half-axis (which is the continuous counterpart of the killing condition on the negative half-axis in the discrete model), and \( \Phi \sim -\text{Re}(z^{-1/2}) \) near zero (where \( z \) is the complex coordinate in the plane). The condition on \( \Omega \) is that the infinitesimal deformation of the boundary given by the normal derivative of \( \Phi \) is homothetic.

Let us find \( \Omega \) from this data (up to scaling). For this purpose, let \( f : D \to \Omega \) be the conformal map of the unit disk onto \( \Omega \) with \( f(0) = 0 \) and \( f'(0) > 0 \) (it is unique). Let \( \Psi(u) = \Phi(f(u)) \). Then \( \Psi \) is a

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1Note that most of the particles will not survive. Namely, as was pointed out to us by Lionel Levine, it follows from the Beurling estimate that for \( N \) particles to survive, the number of produced particles will have to be \( \sim CN^{5/4} \), where \( C \) is a constant.
harmonic potential on the disk with a cut along the negative half-axis, and \( \Psi \) behaves like \(-\text{Re}(u^{-1/2})\) at zero. Thus, up to scaling, 
\[
\Psi = \text{Re}(u^{1/2} - u^{-1/2}).
\]
(we choose the standard branch of the square root with a cut on the negative half-axis). Indeed, \( \Psi \) is uniquely determined by its properties, and the given \( \Psi \) satisfies them.

Now, if \(|\zeta| = 1\) then the velocity vector of the boundary at the point \( f(\zeta) \) under the transformation by the normal derivative of \( \Phi \) is 
\[
\frac{\partial \Psi}{\partial n} \bigg|_{f'(\zeta)} - 1, \quad |\zeta| = 1
\]
while the outward unit normal to the boundary at \( f(\zeta) \) is 
\[
\frac{\zeta f'(\zeta) - 1}{|f'(\zeta)|}.
\]
Thus the condition of homothetic deformation has the form
\[
\text{Re} \left( \frac{\zeta f'(\zeta) f(\zeta)}{|f'(\zeta)|} \right) = \frac{\partial \Psi}{\partial n} \bigg|_{f'(\zeta)} - 1, \quad |\zeta| = 1,
\]
i.e.,
\[
(1) \quad \text{Re}(\zeta f'(\zeta) f(\zeta)) = \frac{\partial \Psi}{\partial n}, \quad |\zeta| = 1,
\]
or
\[
(2) \quad \text{Re}(\zeta f'(\zeta) f(\zeta)) = \frac{1}{2}(\zeta^{1/2} + \zeta^{-1/2}), \quad |\zeta| = 1.
\]

Now we would like to solve (2). We use the method similar to one described in [EEK], for evolution of polygons (Section 9).

Let \( D = \zeta \frac{\partial}{\partial \zeta} \). Also for any function \( g \) set \( g^*(\zeta) = g(1/\zeta) \), so \( (Dg)^* = -D(g^*) \).

Note that \( f \) is real, i.e. it has real Taylor coefficients at 0. Thus equation (2) can be rewritten in the form
\[
(3) \quad Df \cdot f^* + f \cdot (Df)^* = \zeta^{1/2} + \zeta^{-1/2}, \quad |\zeta| = 1.
\]
Differentiating this, we get
\[
(4) \quad D^2f \cdot f^* - f \cdot (D^2f)^* = \frac{1}{2}(\zeta^{1/2} - \zeta^{-1/2}).
\]

Let
\[
h = \frac{D^2f - \frac{1}{2}\zeta^{1/2} - \zeta^{-1/2}}{Df}.
\]
Then equations (3,4) imply that
\[
h - h^* = 0, \quad |\zeta| = 1.
\]
But by its definition, \( h \) extends to a function which is holomorphic everywhere in the unit disk except possibly the point \(-1\). Hence \( h \) is holomorphic in the Riemann sphere with the possible exception of the point \(-1\). Also, it’s clear that \( Df(-1) = 0 \), which by the removable
singularity theorem and Liouville’s theorem implies that in fact \( h \) is a constant.

Thus, we find that \( f \) satisfies the differential equation

\[
D^2 f + \frac{11 - \zeta}{2(1 + \zeta)} Df = hf
\]

The constant \( h \) is easy to find from the condition that \( f(\zeta) \) is proportional to \( \zeta + O(\zeta^2) \): namely, \( h = \frac{3}{2} \).

From this it is easy to solve the differential equation by the power series method. The answer is, up to scaling

\[
f(\zeta) = \frac{15}{32} \sum_{k \geq 1} \frac{k}{(k^2 - 1/4)(k^2 - 9/4)} (-\zeta)^k.
\]

(the normalization is such that \( f'(0) = 1 \)). Thus, we have

**Proposition 2.2.** The region solving Problem 2.1 is defined by the conformal mapping

\[
f(\zeta) = \frac{15}{32} (1 + \zeta)^2 (\zeta^{-1} - (1 - \zeta)\zeta^{-3/2} \arctan \zeta^{1/2}) - \frac{5}{8}.
\]

Thus we have a “logarithmic cusp” near \( \zeta = -1 \), i.e. \( f(-1 + u) \) behaves like \( cu^2 \log u \) (as opposed to the ordinary semicubic cusp, with local behavior \( cu^2 \)). We also see that the region is twice as thick in the positive direction as it is in the negative direction.

Formula (5) is a very close fit with the left shape on Fig. 4 of [LP] (when it is rotated by 180° to match the positive and the negative half-axes). This leads to a conjecture that the region defined by formula (5) is the scaling limit of the model.

2.2. **The killing condition on the sides of an angle.** The analysis of the previous subsection can be generalized to the following more general problem. Let \( 0 < b \leq 1 \), and consider the angle \( \{ z \in \mathbb{C} | \arg z \in [-\pi b, \pi b] \} \).

Consider the internal DLA model as above inside the angle with a killing condition on the sides of the angle. Then the setting of the previous subsection is the special case \( b = 1 \).

The Hele-Shaw problem for the asymptotic region \( \Omega \) in this case is as follows.

**Problem 2.3.** Suppose that \( \Omega \) is a wedge-shaped region bounded by the sides of the angle and some curve \( \Gamma \) connecting them (symmetric with respect to the horizontal axis). Let \( \Phi \) be the harmonic potential in \( \Omega \) which vanishes on the boundary \( \Gamma \) and the sides of the angle, and behaves like \(-\text{Re}(z^{-1/2b})\) near zero. Then the defining condition for \( \Omega \)
is that \( \Gamma \) transforms homothetically under the infinitesimal deformation by the normal derivative of \( \Phi \).

**Remark 2.4.** Note that the condition \( b \leq 1 \) is not essential, as for \( b > 1 \) we can put the angle (which is \( > 2\pi \)) on the helical Riemann surface, which is the universal covering of the complex plane punctured at zero.

The region \( \Omega \) for any \( b \) can be found similarly to the case \( b = 1 \) considered above. Namely, let \( f \) be the conformal map of the circular sector defined by the inequalities \(|z| \leq 1, -\pi b \leq \arg z \leq \pi b \) (lying on the Riemann surface if \( b > 1 \)) onto \( \Omega \), such that \( f \) bijectively maps straight sides to straight sides (so \( f(0) = 0 \), and \( f'(0) > 0 \). Then we have an expansion

\[
f(\zeta) = \sum_{k \geq 0} a_k \zeta^{1+k/b},
\]

for some \( a_k \in \mathbb{R} \).

By a method analogous to the case \( b = 1 \), we derive a differential equation for \( f \), which has the form

\[
D^2 f + \frac{1}{2b} \frac{1 - \zeta^{1/b}}{1 + \zeta^{1/b}} Df = (1 + 1/2b) f.
\]

Then we can solve for \( f \) by the power series method, and up to scaling we get

**Proposition 2.5.** The region solving Problem 2.3 is defined by the conformal mapping

\[
f(\zeta) = \zeta F(\alpha, \beta, \gamma; -\zeta^{1/b}),
\]

where \( F \) is the Gauss hypergeometric function:

\[
F(\alpha, \beta, \gamma; z) = \sum_{n \geq 0} z^n \prod_{j=0}^{n-1} \frac{(\alpha + j)(\beta + j)}{(1 + j)(\gamma + j)},
\]

and

\[
\alpha = -1/2, \ \beta = 2b, \ \gamma = 2b + 3/2.
\]

**Example 2.6.** Consider the special case \( b = 1/2 \) (the angle is \( \pi \), i.e. its boundary is the imaginary axis). In this case the solution is

\[
f(\zeta) = \zeta F(-1/2, 1, 5/2; -\zeta^2),
\]

i.e.,

\[
f(\zeta) = \frac{3}{8}((\zeta - \zeta^{-1}) + (\zeta + \zeta^{-1})^2 \arctan \zeta).
\]
Remark 2.7. This analysis is similar to the analysis in [GV], subsection 2.2.3, where solutions are also expressed via the Gauss hypergeometric function.

Lionel Levine has simulated on a computer the corresponding discrete (rotor-router) model for $b = 1/2$ and $b = 1/4$ (the dynamics in the half-plane and the quarter-plane, respectively), and got a very close fit with formula (6). This gives rise to a conjecture that for these values of $b$ (and perhaps for all) the scaling limit is given by formula (6).

2.3. The killing and reflecting case. Another problem one can consider is the same as the previous one, except that the angle is $\{z \in \mathbb{C} | \arg z \in [0, \pi b] \}$ (i.e., the upper half of the angle considered before), and on the right side of the angle (the positive half-axis) we have not the Dirichlet boundary condition for $\Phi$, but the Neumann boundary condition, $\frac{\partial \Phi}{\partial n} = 0$. This can be conjecturely interpreted as the scaling limit of the random walk problem on a rectangular lattice in this angle with a killing condition on the left side and reflecting condition on the right side.

The problem of finding $\Omega$ in this case reduces to the previous one, by doubling the angle using reflection with respect to the positive half-axis. In particular, if $b = 2$ (the angle is the full plane, and we have the reflecting condition above the positive half-axis and the killing condition below), then the conformal map of the disk with a cut at the positive half-axis to the region $\Omega$ is

$$f(\zeta) = \zeta F(-1/2, 4, 11/2; -\zeta^{1/2}),$$

where $\zeta^{1/2} := r^{1/2}e^{\theta/2}$, $\theta \in [0, 2\pi)$.

3. Flows with the killing-passing-reflecting condition

Consider now the discrete model as above, with the condition on the positive half-axis saying that particles coming from above are reflected, while particles coming from below pass through with probability $p$ and are killed with probability $1 - p$. If $p = 0$, this is the killing-reflecting case considered in the previous section, and if $p = 1$, this is the passing-reflecting case, whose simulation is shown on the right side of Fig. 4 of [LP].

We expect that the scaling limit of this process exists for any $p \in [0, 1]$, and is described as follows.

Problem 3.1. Let $\Omega$ be the asymptotic region. Then the boundary of $\Omega$ consists of a curve $\Gamma$ and the interval $[a, b]$ of the positive half-axis (the curve $\Gamma$ goes counterclockwise from $b$ to $a$ around the origin).
Then the boundary conditions for the potential $\Phi$ in $\Omega$ are:

$$\Phi = 0 \text{ on } \Gamma;$$

$$(\Phi_y)_+ = 0 \text{ on } [a, b];$$

$$\Phi_- = 0 \text{ on } [0, a];$$

$$(\Phi_y)_+ = p(\Phi_y)_- \text{ on } [0, a].$$

Here $\Phi_y$ is the derivative of $\Phi$ with respect to $y$, and $+,-$ denote the one-sided limits from above and below ($\Phi$ and $\Phi_y$ are not assumed continuous on $(0,a)$).

The condition on $\Omega$, as before, is the homothetic transformation under the flow defined by the normal derivative of $\Phi$.

**Remark 3.2.** 1. To motivate these boundary conditions, it is convenient to use the following reformulation of the killing-passing-reflecting condition in the discrete model: all the particles hitting the positive half-axis from below are killed, but for each $k$ particles killed on a small interval $I$ we produce about $pk$ particles which originate at $I$ and move in the upward direction. This explains the second and third boundary conditions (reflecting condition on $[a,b]$ from above, killing condition on $[0,a]$ from below), and clarifies the meaning of the fourth boundary condition, which says that the flux through $[0,a]$ on the upper side is $p$ times the flux on the lower side.

2. Note that for $p=0$, Problem 3.1 reduces to the problem of Subsection 2.3 for $b=2$.

Unfortunately, we did not manage to compute the solution $\Omega$ of Problem 3.1 explicitly. However, here is an interesting moment property of this solution (which we expect to determine $\Omega$ uniquely up to scaling).

Define a continuous branch of $\log(z)$ in $\Omega \setminus \mathbb{R}_+$ by defining it to be real on the reflecting boundary (i.e. the limit of $\log(z)$ from above is real for $z \in \mathbb{R}_+, z \neq 0$). This also defines branches of $z^s = e^{s\log(z)}$ for any $s$.

**Proposition 3.3.** (i) If $\Omega$ is a solution of Problem 3.1, then one has

$$\int_{\Omega} \operatorname{Re}(z^{n+\alpha}) dx dy = 0, \int_{\Omega} \operatorname{Re}(z^{n-\alpha}) dx dy = 0, \ n \in \mathbb{N},$$

where $\alpha = \arccos(p)/2\pi$. In particular, $0 \leq \alpha \leq 1/4$.

(ii) For $p = 1$ ($\alpha = 0$), one has

$$\int_{\Omega} \operatorname{Re}(z^n) dx dy = 0, \int_{\Omega} \operatorname{Re}(z^n \log(z)) dx dy = 0, \ n \in \mathbb{N}.$$
Proof. (i) Let \( u \) be a function on \( \Omega \setminus \mathbb{R}_+ \). Then the time derivative of the moment \( \int_\Omega u \, dx \, dy \) under the Hele-Shaw flow is given by
\[
\frac{d}{dt} \int_\Omega u \, dx \, dy = \int_\Gamma u \frac{\partial \Phi}{\partial n} \, dl = \int_{\partial_\Omega} u \frac{\partial \Phi}{\partial n} \, dl - \int_0^b u_+ (\Phi_y)_+ \, dx + \int_a^a u_- (\Phi_y)_- \, dx.
\]
By Green’s formula, this equals
\[
\int_\Omega (u \Delta \Phi - \Phi \Delta u) \, dx \, dy + \int_{\partial_\Omega} \Phi \frac{\partial u}{\partial n} \, dl - \int_0^b u_+ (\Phi_y)_+ \, dx + \int_a^a u_- (\Phi_y)_- \, dx.
\]
If \( u \) is harmonic and vanishes to sufficient order at 0 then the first summand is zero. So, using the first three boundary conditions, we have
\[
(7) \quad \frac{d}{dt} \int_\Omega u \, dx \, dy = \int_0^b \Phi_+ (u_y)_+ \, dx - \int_0^a \Phi_+ (u_y)_+ \, dx + \int_0^a u_- (\Phi_y)_- \, dx.
\]
Now suppose that \( u = \text{Re}(z^{n\pm\alpha}) = r^{n\pm\alpha}\cos((n \pm \alpha)\theta), \ z = re^{i\theta}, \ \theta \in [0, 2\pi) \). Then \( (u_y)_+ = 0 \), so the first summand in (7) vanishes. Thus, we have
\[
(8) \quad \frac{d}{dt} \int_\Omega u \, dx \, dy = \int_0^a u_- (\Phi_y)_- \, dx - \int_0^a u_+ (\Phi_y)_+ \, dx + \cdot
\]
But we have \( u_+ = x^\alpha, \ u_- = x^\alpha \cos(2\pi\alpha) = px^\alpha \). So by the fourth boundary condition, we get from (7) that \( \frac{d}{dt} \int_\Omega u \, dx \, dy = 0 \). But since the time evolution of \( \Omega \) is homothetic, and \( \int_\Omega u \, dx \, dy \) is homogeneous of nonzero degree under dilations, we get \( \int_\Omega u \, dx \, dy = 0 \), as desired.

(ii) follows from (i) by considering the difference of the two equations in (i) and computing the first term of the Taylor expansion in \( \alpha \) at \( \alpha = 0 \). \( \square \)

Remark 3.4. Let \( \Omega \) be a region solving Problem 3.1 for \( p = 1 \) (which is expected to match the right shape at Fig. 4 of [LP]), and \( \overline{\Omega} \) be the complex conjugate region. Proposition 3.3(ii) implies that
\[
\int_{\Omega} z^n \, dx \, dy + \int_{\overline{\Omega}} z^n \, dx \, dy = 0, \ n \in \mathbb{N}.
\]
This implies that the “balayage sum” \( E \) of \( \Omega \) and \( \overline{\Omega} \) (i.e., the region whose moments are equal to the sum of the moments of \( \Omega \) and of \( \overline{\Omega} \), see [GV]) satisfies the condition \( \int_E z^n \, dx \, dy = 0, \ n \in \mathbb{N} \), i.e. is a disk (see [VE] and references therein).
Note that for $p = 0$ ($\alpha = 1/4$), i.e. the killing-reflecting case discussed in Subsection 2.3, the two arithmetic progressions in Proposition 3.3(i) combine into a single one, and yield

$$\int_{\Omega} \text{Re}(z^{n/2+1/4})dx dy = 0, \ n \in \mathbb{N}.$$ 

This can be generalized to the case of any angle (i.e., any $b$). Namely, we have the following proposition.

**Proposition 3.5.** Let $\Omega$ be a region solving Problem 2.3. Then

$$\int_{\Omega} z^{2n+1}b dx dy = 0, \ n \in \mathbb{N}.$$ 

Therefore, the solution of the killing-reflecting problem with parameter $b$ (which is the upper half of the region $\Omega$) satisfies the equality

$$\int_{\Omega} \text{Re}(z^{2n+1})dx dy = 0, \ n \in \mathbb{N}.$$ 

The proof of this proposition is similar to the proof of Proposition 3.3, using Green’s formula.

A similar property was established by Lionel Levine for the discrete model (for $b = 1$).

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