LIE SUPERBIALGEBRA STRUCTURES ON THE $N = 2$
SUPERCONFORMAL NEVEU-SCHWARZ ALGEBRA

DONG LIU, LIANGYUN CHEN, AND LINSHENG ZHU

Abstract. In this paper, Lie superbialgebra structures on the $N = 2$ superconformal Neveu-Schwarz algebra are considered by a very simple method. We prove that every Lie superbialgebra structure on the algebra is triangular coboundary.

Key words: Lie superbialgebras, Yang-Baxter equation, the $N = 2$ superconformal Neveu-Schwarz algebra.

Mathematics Subject Classification (2000): 17B05, 17B37, 17B62, 17B66.

1. Introduction

The superconformal algebras, closely related to the conformal field theory and the string theory, play important roles in both mathematics and physics supplying the underlying symmetries of string theory. It is well known that the $N = 2$ superconformal algebras include four sectors: the Neveu-Schwarz sector, the Ramond sector, the topological sector and the twisted sector. There are many researches about these algebras (see [1, 2, 5, 6, 11, 19] and the reference cited therein). All sectors are closely related to the Virasoro algebra and the super-Virasoro algebra which play great roles in the two-dimensional conformal filed.

Lie bialgebras was introduced in 1983 by Drinfeld (see [3, 4]) during the process of investigating quantum groups. There appeared several papers on Lie bialgebras and Lie superbialgebras (e.g., [14, 15, 17, 22]). In [14]–[17], the Lie bialgebra structures on Witt and Virasoro algebras were investigated, which are shown to be triangular coboundary. Moreover, the Lie bialgebra structures on the one-sided Witt algebra were completely classified.

In [7, 12, 23], the Lie superbialgebra structures on the $N = 2$ superconformal twisted, topological, Ramond algebras were investigated case by case with complicated computations. Clearly, the above superalgebras and the $N = 2$ superconformal Neveu-Schwarz algebra all include the centerless twisted Heisenberg-Virasoro algebra (see Section 2.2 for the definition) as a subalgebra. It is naturally to consider such questions based on the related results of the twisted Heisenberg-Virasoro algebra.

As the above considerations, we study the Lie superbialgebra structures on the $N = 2$ superconformal Neveu-Schwarz algebra $\mathcal{L}$ in this paper. Successfully, we provide a very simple method (no need some complicated calculations as in [7, 12, 23]) to determine the structure of $H^1(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})$ based on such results for the twisted Heisenberg-Virasoro algebra in [13]. With our considerations, the original proofs in [7, 12, 23] can be great simplified. Throughout the paper, we denote by $\mathbb{Z}$, $\mathbb{C}$ the set of all integers, complex numbers, $\mathbb{Z}_+$ (resp $\mathbb{Z}^*$) the set of all nonnegative (resp. nonzero) integers.
2. Preliminaries

2.1. Lie super-bialgebras. Firstly, let us recall some related definitions. Let $L = L_0 \oplus L_1$ be a vector space over $\mathbb{C}$. If $x \in L_{[x]}$, then we say that $x$ is homogeneous of degree $[x]$ and we write $\deg x = [x]$. Denote by $\tau$ the super-twist map of $L \otimes L$, i.e.,

$$\tau(x \otimes y) = (-1)^{[x][y]} y \otimes x, \quad \forall \ x, y \in L.$$ 

For any $n \in \mathbb{N}$, denote by $L^\otimes_n$ the tensor product of $n$ copies of $L$ and $\xi$ the super-cyclic map cyclically permuting the coordinates of $L^\otimes_3$, i.e.,

$$\xi = (1 \otimes \tau) \cdot (\tau \otimes 1) : x_1 \otimes x_2 \otimes x_3 \mapsto (-1)^{[x_1][x_2] + [x_2][x_3]} x_2 \otimes x_3 \otimes x_1, \quad \forall \ x_i \in L, \ i = 1, 2, 3,$$

where $1$ is the identity map of $L$. Then the definition of a Lie superalgebra can be described in the following way: A Lie superalgebra is a pair $(L, \varphi)$ consisting of a vector space $L = L_0 \oplus L_1$ and a bilinear map $\varphi : L \otimes L \to L$ satisfying:

$$\varphi(L_i, L_j) \subset L_{i+j},$$

$$\operatorname{Ker}(1 - \tau) \subset \operatorname{Ker} \varphi,$$

$$\varphi \cdot (1 \otimes \varphi) \cdot (1 + \xi + \xi^2) = 0.$$  

(2.1)

Meanwhile, the definition of a Lie super-coalgebra can be described in the following way: A Lie super-coalgebra is a pair $(L, \Delta)$ consisting of a vector space $L = L_0 \oplus L_1$ and a linear map $\Delta : L \to L \otimes L$ satisfying:

$$\Delta(L_i) \subset \sum_{j \in \mathbb{Z}_2} L_j \otimes L_{i-j},$$

$$\operatorname{Im} \Delta \subset \operatorname{Im}(1 - \tau),$$

$$(1 \otimes 1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0.$$  

(2.2)

Now one can give the definition of a Lie superbialgebra, which is a triple $(L, \varphi, \Delta)$ satisfying:

(i) $(L, \varphi)$ is a Lie superalgebra,

(ii) $(L, \Delta)$ is a Lie super-coalgebra,

(iii) $\Delta \varphi(x \otimes y) = x \ast \Delta y - (-1)^{[x][y]} y \ast \Delta x, \quad \forall \ x, y \in L$,

where the symbol “$\ast$” means the adjoint diagonal action

$$x \ast (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + (-1)^{[x][a_i]} a_i \otimes [x, b_i]), \quad \forall \ x, a_i, b_i \in L,$$  

(2.3)

and in general $[x, y] = \varphi(x \otimes y)$ for $x, y \in L$.

Denote by $\mathcal{U}(L)$ the universal enveloping algebra of $L$ and $A \setminus B = \{x \mid x \in A, x \notin B\}$ for any two sets $A$ and $B$. If $r = \sum_i a_i \otimes b_i \in L \otimes L$, then the following elements are in $\mathcal{U}(L) \otimes \mathcal{U}(L) \otimes \mathcal{U}(L)$

$$r^{12} = \sum_i a_i \otimes b_i \otimes 1 = r \otimes 1, \quad r^{23} = \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes r,$$

$$r^{13} = \sum_i a_i \otimes 1 \otimes b_i = (1 \otimes \tau)(r \otimes 1) = (\tau \otimes 1)(1 \otimes r),$$

2
while the following elements are in $L \otimes L \otimes L$

$$[r^{12}, r^{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j,$$

$$[r^{12}, r^{13}] = \sum_{i,j} (-1)^{|a_i| |b_j|} [a_i, a_j] \otimes b_i \otimes b_j,$$

$$[r^{13}, r^{23}] = \sum_{i,j} (-1)^{|a_i| |b_j|} a_i \otimes a_j \otimes [b_i, b_j].$$

**Definition 2.1.** (i) A coboundary superbialgebra is a quadruple $(L, \varphi, \Delta, r)$, where $(L, \varphi, \Delta)$ is a Lie superbialgebra and $r \in \text{Im}(1 - \tau) \subset L \otimes L$ such that $\Delta = \Delta_r$ is a coboundary of $r$, i.e.,

$$\Delta_r(x) = (-1)^{|x|} x \ast r, \ \forall \ x \in L. \quad (2.4)$$

(ii) A coboundary Lie superbialgebra $(L, \varphi, \Delta, r)$ is called triangular if it satisfies the following classical Yang-Baxter Equation

$$c(r) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \quad (\text{CYBE}) \quad (2.5)$$

Let $V = V_0 \oplus V_1$ be an $L$-module where $L = L_0 \oplus L_1$. A $\mathbb{Z}_2$-homogenous linear map $d : L \rightarrow V$ is called a homogenous derivation of degree $[d] \in \mathbb{Z}_2$, if $d(L_i) \subset V_{i+[d]} \ (\forall \ i \in \mathbb{Z}_2)$,

$$d([x, y]) = (-1)^{|d|[x]} x \ast d(y) - (-1)^{|y|([d]+|[x]|)} y \ast d(x), \ \forall \ x, y \in L. \quad (2.6)$$

Denote by $\text{Der}_i(L, V) \ (i = 0, 1)$ the set of all homogenous derivations of degree $i$. Then the set of all derivations from $L$ to $V$ $\text{Der}(L, V) = \text{Der}_0(L, V) \oplus \text{Der}_1(L, V)$. Denote by $\text{Inn}_i(L, V) \ (i = 0, 1)$ the set of homogenous inner derivations of degree $i$, consisting of $a_{\text{inn}}, a \in V_i$, defined by

$$a_{\text{inn}} : x \mapsto (-1)^{|a|[x]} x \ast a, \ \forall \ x \in L. \quad (2.7)$$

Then the set of inner derivations $\text{Inn}(L, V) = \text{Inn}_0(L, V) \oplus \text{Inn}_1(L, V)$.

Denote by $H^1(L, V)$ the first cohomology group of $L$ with coefficients in $V$. Then

$$H^1(L, V) \cong \text{Der}(L, V)/\text{Inn}(L, V).$$

An element $r$ in a superalgebra $L$ is said to satisfy the modified Yang-Baxter equation if

$$x \ast c(r) = 0, \ \forall \ x \in L. \quad (\text{MYBE}) \quad (2.8)$$

The following result for the non-super case can be found in [17] while its super case can be found in [23].

**Lemma 2.2.** Let $L$ be a Lie superalgebra, $r \in \text{Im}(1 - \tau) \subset L \otimes L$ with $[r] = 0$. Then

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta_r) \cdot \Delta_r(x) = x \ast c(r), \ \forall \ x \in L. \quad (2.9)$$

Thus $(L, [\cdot, \cdot], \Delta_r)$ is a Lie superbialgebra if and only if $r$ satisfies (MYBE) (see(2.8)).
2.2. The $N = 2$ superconformal Neveu-Schwarz algebra. As a vector space over $\mathbb{C}$, the $N = 2$ superconformal Neveu-Schwarz algebra $\hat{\mathcal{L}}$ has a basis $\{L_m, I_m, G^+_r \mid m \in \mathbb{Z}, r \in \mathbb{Z} + 1/2\}$, with the following relations:

\[
[L_m, L_n] = (m - n)L_{n+m} + \frac{1}{12}(m^3 - m)C, \quad (2.10)
\]

\[
[I_m, I_n] = \frac{1}{3}m\delta_{m+n,0}C, \quad [L_m, I_n] = -nI_{m+n}, \quad (2.11)
\]

\[
[L_m, G^+_r] = (\frac{m}{2} - r)G^+_{m+r}, \quad [I_m, G^+_r] = \pm G^+_{m+r}, \quad (2.12)
\]

\[
[G^+_r, G^-_s] = 2L(r + s) + (r - s)I(r + s) + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}C, \quad (2.13)
\]

\[
[G^+_r, G^+_s] = 0, \quad (2.14)
\]

for $m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z}$.

Denote by $\mathcal{L}$ the centerless Lie superalgebra of $\hat{\mathcal{L}}$, then $\hat{\mathcal{L}}$ is the universal central extension of $\mathcal{L}$. Obviously, $\mathcal{L}$ is $\mathbb{Z}_2$-graded: $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$, with

\[
\mathcal{L}_0 = \text{span}_\mathbb{C}\{L_m, I_m \mid m \in \mathbb{Z}\}, \quad \mathcal{L}_1 = \text{span}_\mathbb{C}\{G^+_r \mid r \in \mathbb{Z} + \frac{1}{2}\}. \quad (2.15)
\]

Clearly, $\mathcal{L}_0$ is the centerless twisted Heisenberg-Virasoro algebra, and $W = \mathbb{C}\{L_m \mid m \in \mathbb{Z}\}$ is the Witt algebra. Moreover, $G^\pm = \mathbb{C}\{G^+_r \mid r \in \mathbb{Z} + \frac{1}{2}\}$ are $\mathcal{L}_0$-modules.

For any $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma, \gamma^\dagger \in \mathbb{C}$, one can easily verify that the linear map $\varrho : \mathcal{L}_0 \to \mathcal{L}_0 \otimes \mathcal{L}_0$ defined below is a derivation:

\[
\varrho(L_n) = (n\alpha + \gamma)I_0 \otimes I_n + (n\alpha^\dagger + \gamma^\dagger)I_n \otimes I_0,
\]

\[
\varrho(I_n) = \beta I_0 \otimes I_n + \beta^\dagger I_n \otimes I_0, \quad n \in \mathbb{Z}.
\]

Denote $\mathcal{D}$ the vector space spanned by the such elements $\varrho$ over $\mathbb{C}$. From Theorem 3.2 and Corollary 4.5 in [13], we have following propositions.

**Proposition 2.3.** [13] $H^1(\mathcal{L}_0, \mathcal{L}_0 \otimes \mathcal{L}_0) = \mathcal{D}$.

**Proposition 2.4.** [13] $H^1(\mathcal{L}_0, G^\pm \otimes G^\pm) = H^1(\mathcal{L}_0, G^\pm \otimes G^\mp) = 0.$
The following lemma can be obtained by using the similar techniques of [20, Lemma 2.2].

**Lemma 2.5.** Regarding \( L^\otimes n \) as an \( L \)-module under the adjoint diagonal action of \( L \), if \( r \in L^\otimes n \) such that \( x \ast r = 0, \forall x \in L \), then \( r = 0 \).

**Proof.** It is easy to see that \( L^\otimes n \) is \( \frac{1}{2}\mathbb{Z} \)-graded by

\[
L_p^\otimes n = \sum_{p_1+p_2+\ldots+p_n = p} L_{p_1} \otimes L_{p_2} \otimes \cdots \otimes L_{p_n}, \quad \forall \; p, \; p_i \in \frac{1}{2}\mathbb{Z}, \; i = 1, 2, \ldots, n.
\]

Write \( r = \sum_{p \in \frac{1}{2}\mathbb{Z}} r_p \) as a finite sum with \( r_p \in L_p^\otimes n \). By hypothesis, \( L_0 \ast r = 0 \), which implies \( r \in L_0 \).

Then we can suppose that all terms in (2.16) are only some tensor products of \( G \) in (2.16) containing some \( G_j^1 \) for some \( j \in \mathbb{Z} + \frac{1}{2} \), then we can get an index sequence \((t_1, t_2, \ldots, t_n) \in (\frac{1}{2}\mathbb{Z})^n \) in which \( t_i = r_i \) if \( E_{r_i} = G_{r_i}^1 \) and zero's in otherwise. We arrange all such terms (containing some \( G_j^1 \)) by lexicographic order according to their index sequences and consider the first term (also called minimal term) \( c_{r_1, \ldots, r_n} E_{r_1} \otimes \cdots \otimes E_{r_n} \). Clearly the coefficient of the first term of all terms in \( I_{-1} \ast r \) which including some \( G_j^1 \) is also \( c_{r_1, \ldots, r_n} \). Then \( c_{r_1, \ldots, r_n} = 0 \). So all the coefficients of the terms containing \( G_j^1 \) for \( j \in \mathbb{Z} + \frac{1}{2} \) are zero. Similar we can consider the terms including \( G_j^- \).

Similarly, by \( I_i \ast r = 0 \) for all \( i \in \mathbb{Z} \) we can get that all the coefficients of the terms containing \( L_m \) for some \( m \in \mathbb{Z} \) are zero.

Then we can suppose that all terms in (2.16) are only some tensor products of \( I_m \) for some \( m \in \mathbb{Z} \). By \( G_q^+ \cdot r = 0 \) for all \( q \in \mathbb{Z} + \frac{1}{2} \) we can get \( r = 0 \). This proves the lemma.

As a conclusion of Lemma 2.5, one immediately obtains the following result for the Lie algebra \( L \).

**Lemma 2.6.** An element \( r \in \text{Im}(1 - \tau) \subset L \otimes L \) satisfies CYBE in (2.5) if and only if it satisfies MYBE in (2.8).

### 3. First Cohomology Group of \( L \) in \( L \otimes L \)

**Theorem 3.1.** \( \text{Der}(L, V) = \text{Inn}(L, V) \), where \( V = L \otimes L \), equivalently, \( H^1(L, V) = 0 \).

**Proof.** Note that \( V = \oplus_{i \in \frac{1}{2}\mathbb{Z}} V_i \) is also \( \frac{1}{2}\mathbb{Z} \)-graded with \( V_i = \sum_{j+k=i} L_j \otimes L_k \), where \( i, j, k \in \frac{1}{2}\mathbb{Z} \). We say a derivation \( d \in \text{Der}(L, V) \) is homogeneous of degree \( i \in \frac{1}{2}\mathbb{Z} \) if \( d(V_j) \subset V_{i+j} \) for all \( j \in \frac{1}{2}\mathbb{Z} \). Set \( \text{Der}(L, V)_i = \{ D \in \text{Der}(L, V) \mid \deg D = i \} \) for \( i \in \frac{1}{2}\mathbb{Z} \).
For any $D \in \text{Der}(\mathcal{L}, \mathcal{V})$, $i \in \frac{1}{2}\mathbb{Z}$, $u \in \mathcal{L}_j$ with $j \in \frac{1}{2}\mathbb{Z}$, we can write $D(u) = \sum_{k \in \mathbb{Z}} v_k \in \mathcal{V}$ with $v_k \in \mathcal{V}_k$, then we set $D_i(u) = v_{i+j}$. Then $D_i \in \text{Der}(\mathcal{L}, \mathcal{V})_i$ and

$$D = \sum_{i \in \frac{1}{2}\mathbb{Z}} D_i \text{ where } D_i \in \text{Der}(\mathcal{L}, \mathcal{V})_i,$$

which holds in the sense that for every $u \in \mathcal{L}$ only finitely many $D_i(u) \neq 0$, and $D(u) = \sum_{i \in \mathbb{Z}} D_i(u)$ (we call such a sum in (3.1) summable).

**Claim 1.** The sum in (3.1) is finite.

For any $i \in \mathbb{Z}$, suppose $d_i = (v_i)_{\text{inn}}$ for some $v_i \in \mathcal{V}_i$. If $|\{i \mid v_i \neq 0\}|$ is infinite, then $d(L_0) = \sum_{i \in \mathbb{Z}} L_0 * v_i = -\sum_{i \in \mathbb{Z}} iv_i$ is an infinite sum, which contradicts $d \in \text{Der}(\mathcal{L}, \mathcal{V})$. Thus the claim and proposition follow. □

**Claim 2.** If $i \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$, then $D_i \in \text{Inn}(\mathcal{L}, \mathcal{V})$.

Denote $u = -\frac{1}{i} D_i(L_0) \in \mathcal{V}_i$. For any $x_j \in \mathcal{L}_j$, $j \in \frac{1}{2}\mathbb{Z}$, applying $D_i$ to $[L_0, x_j] = -jx_j$, using $d_i(x_j) \in \mathcal{V}_{i+j}$ and the action of $L_0$ on $\mathcal{V}_{i+j}$ is the scalar $L_0|_{\mathcal{V}_{i+j}} = -(i+j)$, one has

$$-(i+j)D_i(x_j) - (-1)^{|d_i||x_j|}x_j \cdot D_i(L_0) = -jD_i(x_j),$$

(3.2)
i.e., $D_i(x_j) = u_{\text{inn}}(x_j)$, which implies $D_i$ is inner.

From the above we see that Theorem 3.1 follows from the following proposition.

**Proposition 3.2.** $\text{Der}(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})_0 = \text{Inn}(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})_0$.

**Proof.** For any $D_0 \in \text{Der}(\mathcal{L}, \mathcal{L} \otimes \mathcal{L})_0$, first we have $D_0(L_0) = 0$. In fact, using (3.2) with $i = 0$, we obtain $x \cdot D_0(L_0) = 0$, $\forall x \in \mathcal{L}_j$, $j \in \frac{1}{2}\mathbb{Z}$, which together with Lemma 2.5 gives $D_0(L_0) = 0$.

Now we shall consider Der($\mathcal{L}_0, \mathcal{L} \otimes \mathcal{L})_0$.

Since $D_0(\mathcal{L}_0) \in (\mathcal{L}_0 \otimes \mathcal{L}_0) \oplus (\bigoplus_{X,Y \in \{G^+\}} X \otimes Y)$, where all direct sums are as $\mathcal{L}_0$-modules, from Proposition 2.3, 2.4, we see that

$$\text{Der}(\mathcal{L}_0, \mathcal{L} \otimes \mathcal{L})_0 = \text{Der}(\mathcal{L}_0, \mathcal{L}_0 \otimes \mathcal{L}_0)_0.$$  

For any $n \in \mathbb{Z}$, one can suppose that

$$D_0(L_n) = (n\alpha + \gamma)I_0 \otimes I_n + (n\alpha^\dagger - \gamma)I_n \otimes I_0, \quad (3.3)$$

$$D_0(I_n) = \beta I_0 \otimes I_n + \beta^\dagger I_n \otimes I_0, \quad n \in \mathbb{Z}, \quad (3.4)$$

for $\alpha, \alpha^\dagger, \beta, \beta^\dagger, \gamma \in \mathbb{C}$ since $D_0(L_0) = 0$.

Since $[L_n, G_r^\pm] = (\frac{1}{2}n - r)G_{n+r}^\pm$, $[I_n, G_r^\pm] = \pm G_{n+r}^\pm$ and (3.3),(3.4), then we can suppose that

$$D_0(G_r^\pm) = \sum_{i \in \mathbb{Z}} c_{r,i}^\pm I_i \otimes G_{r-i}^\pm + \sum_{i \in \mathbb{Z}} d_{r,i}^\pm G_{r-i}^\pm \otimes I_i,$$

where all sums are finite and all coefficients are in $\mathbb{C}$.  

6
Applying $D_0$ to $[L_{2r}, G^\pm_r] = 0$, we obtain

$$[L_{2r}, D_0(G^\pm_r)] = -(\pm(2r\alpha + \gamma)(I_0 \otimes G^\pm_{3r} + G^\pm_r \otimes I_{2r}) \pm (2r\alpha - \gamma)(G^\pm_{3r} \otimes I_0 + I_{2r} \otimes G^\pm_r)). \quad (3.5)$$

However, the $\max\{i \mid c_{r,i}d_{n,i} \neq 0\} \leq 0$ and $l=\min\{i \mid c_{r,i}d_{n,i} \neq 0\} \geq -2r$. Moreover if $l = -2r$, then $I_{-2r} \otimes G^\pm_{3r}$ is in the left side of (3.5). It is impossible. By similar consideration, we obtain $l = 0$. Then we have $D_0(G^\pm_r) = a^\pm I_0 \otimes G^\pm_r + b^\pm I_0 \otimes G^\pm_r$ for some $a^\pm, b^\pm \in \mathbb{C}$, and $\alpha = \alpha^\dagger = \gamma = 0$.

Applying $D_0$ to $[G^+_r, G^-_s] = 2L_{r+s} + (r - s)I_{r+s}$, we obtain $\beta = \beta^\dagger = 0$ and $a^+ = -a^- = b^+ = -b^-$. Then replaced $D_0$ by $D_0 + a^+ u_{inn}$ for $u = I_0 \otimes I_0$, $D_0 = 0$. \hfill $\square$

4. Lie super bialgebra structures on Lie superalgebras

In this section, we shall consider Lie super bialgebra structures on the $N = 2$ superconformal Neveu-Schwarz algebra. First we have

**Theorem 4.1.** Every Lie superbialgebra structure on the centerless $N = 2$ superconformal Neveu-Schwarz algebra $\mathcal{L}$ is triangular coboundary.

First we prove the following lemma.

**Lemma 4.2.** If $r \in \mathcal{V}$ satisfies $x \cdot r \in \text{Im}(1 - \tau)$ ($\forall x \in \mathcal{L}$), then $r \in \text{Im}(1 - \tau)$.

**Proof.** Note $\mathcal{L} \ast \text{Im}(1-\tau) \subset \text{Im}(1-\tau)$. Write $r = \sum_{i \in \frac{1}{2}\mathbb{Z}} r_i$ with $r_i \in \mathcal{V}_i$. Obviously, $r \in \text{Im}(1-\tau)$ if and only if $r_i \in \text{Im}(1-\tau)$ for all $i \in \frac{1}{2}\mathbb{Z}$. Thus without loss of generality, one can suppose $r = r_i$ is homogeneous.

If $i \in \frac{1}{2}\mathbb{Z}^+$, then $r_i = -\frac{1}{i}L_0 \ast r_i \in \text{Im}(1-\tau)$. For the case $i = 0$, from Lemma 3.5 in [13] and $I_0 \ast v \in \text{Im}(1-\tau)$, one can write

$$r_0 = \sum_{p \in \mathbb{Z} + \frac{1}{2}} c_p G^+_p \otimes G^-_p + \sum_{p \in \mathbb{Z} + \frac{1}{2}} d_p G^-_p \otimes G^+_p,$$

where the sum are all finite. Since the elements of the form $u_{3,p} := G^+_p \otimes G^-_p - G^-_p \otimes G^+_p$ are all in $\text{Im}(1 - \tau)$, replacing $v$ by $v - u$, where $u$ is a combination of some $u_{3,r}$, one can rewritten as

$$r_0 = \sum_{p \in \mathbb{Z} + \frac{1}{2}} a_p G^+_p \otimes G^-_p, \quad (4.1)$$

Since the sum is finite, so there exist $s, t \in \mathbb{Z} + \frac{1}{2}$, such that

$$r_0 = \sum_{s \leq p \leq t} a_p G^+_p \otimes G^-_p, \quad (4.2)$$

with $a_s, a_t \neq 0$.

However by $I_1 \ast r_0 \in \text{Im}(1 - \tau) \subset \text{Ker}(1 + \tau)$, we have

$$(1 + \tau) \sum_{s \leq p \leq t} a_p (G^+_{p+1} \otimes G^-_p - G^+_p \otimes G^-_{p+1}) = 0. \quad (4.3)$$
It is a contradiction with the fact that \(a_t \neq 0\). Then one has \(a_p = 0, \forall p \in \mathbb{Z} + \frac{1}{2}\). Thus the lemma follows.

\[ \Box \]

**Proof of Theorem 4.1.** Let \((\mathcal{L}, [\cdot, \cdot], \Delta)\) be a Lie superbialgebra structure on \(\mathcal{L}\). Then \(\Delta = \Delta_r\) is defined by (2.4) for some \(r \in V_0\). By (2.2), \(\text{Im}\Delta \subset \text{Im}(1 - \tau)\). Thus by Lemma 4.2, \(r \in \text{Im}(1 \otimes 1 - \tau)\). Then (2.2), (2.9) and Corollary 2.6 show that \(c(r) = 0\). Thus \((\mathcal{L}, [\cdot, \cdot], \Delta)\) is triangular coboundary.

Now we return to consider Lie superbialgebra structures on \(\hat{\mathcal{L}}\). As that in Lie algebra case (see Lemma 5.2 in [17]), we have following lemma.

**Lemma 4.3.** Let \(L\) be a Lie superalgebra such that \(L^L, (L \otimes L)^L\) and \(H^1(L, L) = 0\). Then for any one dimensional central extension \(\hat{L}\) of \(L\), there is a linear embedding from \(H^1(\hat{L}, \hat{L} \otimes \hat{L})\) into \(H^1(L, L \otimes L)\). In particular, if \(H^1(L, L \otimes L) = 0\), then \(H^1(\hat{L}, \hat{L} \otimes \hat{L}) = 0\).

With Lemma 4.3 and Theorem 4.1, we obtain the following results.

**Theorem 4.4.** Every Lie superbialgebra structure on the \(N = 2\) superconformal Neveu-Schwarz algebra \(\hat{\mathcal{L}}\) is triangular coboundary.

**Remark 4.5.** The above methods can also be used to investigate such structures on the \(N = 2\) superconformal twisted, topological, Ramond algebras with no complicated calculations as in [7, 12, 23].

**ACKNOWLEDGMENTS**

Project is supported by the NNSF (No. 11071068, 10871057, 11171055), the ZJNSF (No. D7080080, Y6100148), Qianjiang Excellence Project (No. 2007R10031), the ”New Century 151 Talent Project” (2008), the ”Innovation Team Foundation of the Department of Education” (No. T200924) of Zhejiang Province, Natural Science Foundation of Jilin province (No.201115006), Scientific Research Foundation for Returned Scholars Ministry of Education of China and the Fundamental Research Funds for the Central Universities.

**REFERENCES**

[1] S. L. Cheng, V.G. Kac, A new \(N = 6\) superconformal algebra, Commun. Math. Phys., 186 (1997), 219.

[2] V.K. Dobrev, Characters of the unitarizable highest weight modules over the \(N = 2\) superconformal algebras, Phys. Lett. B. 186(1) (1987), 43-51.

[3] V.G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations, Dokl. Akad. Nauk, 268 (1983), 285–287.

[4] V.G. Drinfeld, Quantum groups, Proceeding of the International Congress of Mathematicians, 1,2 (1986), 798–820.

[5] Matthias Dörrozapf, Beatriz Gato-Rivera, Singular Dimensions of the \(N = 2\) Superconformal Algebras II: The Twisted \(N = 2\) Algebra, Comm. Math. Phys., 220 (2001), 263–292.
[6] W. Eholzer, M.R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, *Comm. Math. Phys.*, **186** (1997), 61–85.

[7] H. Fa, J. Li, B. Xin, *Lie superbialgebra structures on the centerless twisted $N = 2$ superconformal algebra*, Algebra Colloquium, 18(3)(2011), 361-372.

[8] P. J. Hilton, U. Stammbach, *A Course in Homological Algebra.*, 2nd ed. New York: Springer-Verlag, (1997).

[9] E. Kiritsis, Character formula and the structure of the representations of the $N = 1, N = 2$ superconformal algebras, *J. Mod. Phys. A* **3** (1988), 1871–1906.

[10] V.G. Kac, Lie superalgebras, *Adv. Math.*, **26** (1977), 8–97.

[11] V.G. Kac, J.W. van de Leur, *On classification of superconformal algebras*. Strings 88, Singapore: World Scientific, (1988).

[12] L. Lin, H. Fa, J. Zhou, *Topological $N=2$ superconformal superbialgebras*, arXiv:0812.5003.

[13] D. Liu, Y. Pei, L. Zhu, *Lie bialgebra structures on the twisted Heisenberg-Virasoro algebra*, math.RA/0141049.

[14] W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, *Adv. Math.*, **107** (1994), 365–392.

[15] W. D. Nichols, The structure of the dual Lie coalgebra of the Witt algebra, *J. Pure Appl. Alg.*, **68** (1990), 395–364.

[16] A. Neveu, J. H. Schwarz, Factorizable dual model of pions, *Nucl. Phys. B*, **31** (1971), 86–112.

[17] S. H. Ng, E. J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, *J. Pure Appl. Alg.*, **151** (2000), 67–88.

[18] P. Ramond, Dual theory of free fermions, *Phys. Rev. D*, **3** (1971), 2451–2418.

[19] A. Schwimmer, N. Seiberg, Comments on the $N = 2,3,4$ superconformal algebras in two dimensions, *Phys. Lett. B*, **184** (1986), 191–196.

[20] Y. Wu, G. Song, Y. Su, Lie bialgebras of generalized Witt type, II, *Comm. Alg.* **35** **6** (2007), 1992–2007.

[21] E. J. Taft, Witt and Virasoro algebras as Lie bialgebras, *J. Pure Appl. Alg.*, **87** (1993), 301–312.

[22] H. Yang, *Lie superbialgebra structures on super-Virasoro algebra*, Front. Math. China 4 (2009), no. 2, 365–379.

[23] H. Yang, Y. Su, Lie bialgebras over the Ramond $N = 2$ super-Virasoro algebras, Chaos Solitons Fractals 40 (2009), no. 2, 661–671.

**Department of Mathematics, Huzhou Teachers College, Zhejiang Huzhou, 313000, China**  
*E-mail address: liudong@hutc.zj.cn*

**Department of Mathematics, Northeast Normal University, Changchun, 130024, China**  
*E-mail address: Corresponding author: chenly640@nenu.edu.cn*

**Department of Mathematics, Changshu Institute of Technology, Jiangsu Changshu, 215500, China**  
*E-mail address: lszhu@cslg.edu.cn*