ANOMALY OF LINEARIZATION AND AUXILIARY INTEGRALS.

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ABSTRACT. In this note we discuss some formal properties of universal linearization operator, relate this to brackets of non-linear differential operators and discuss application to the calculus of auxiliary integrals, used in compatibility reductions of PDEs.

INTRODUCTION

Commutator $[\Delta, \nabla]$ of linear differential operators $\Delta, \nabla \in \text{Diff}(\pi,\pi)$ in the context of non-linear operators $F, G \in \text{diff}(\pi,\pi)$ is upgraded to the higher Jacobi bracket $\{F,G\}$, which plays the same role in compatibility investigations and symmetry calculus.

The linearization operator relates non-linear operators on a bundle $\pi$ with linear operators on the same bundle, whose coefficients should be however smooth functions on the space of infinite jets. The latter space is the algebra of $C$-differential operators and we get the map

$$\ell : \text{diff}(\pi,\pi) \to \mathcal{C} \text{Diff}(\pi,\pi) = C^\infty(J^\infty\pi) \otimes_{C^\infty(M)} \text{Diff}(\pi,\pi),$$

defined by the formula $[\text{KLV}]$

$$\ell_F(s)h = \frac{d}{dt} F(s + th)|_{t=0}, \quad F \in \text{diff}(\pi,\pi), \quad s, h \in C^\infty(\pi).$$

However it does not respect the commutator:

$$[\ell_F, \ell_G] \neq \ell_{\{F,G\}}.$$

Example: Consider the scalar differential operators on $\mathbb{R}$, so that $\pi = 1$ and $J^\infty(\pi) = \mathbb{R}^\infty(x,u,p = p_1, p_2, \ldots)$. Choose

$$F = p^2, \quad G = p + c \cdot x; \quad \{F,G\} = 2c \quad \Rightarrow \quad \ell_{\{F,G\}} = 2c \mathcal{D}_x.$$

If we commute $\ell_F = 2p \mathcal{D}_x$ and $\ell_G = \mathcal{D}_x$, we get: $[\ell_F, \ell_G] = -2p_2 \mathcal{D}_x$, so that we observe an anomaly.

There are two reasons for this. The first is that the operator of linearization disregards non-homogeneous linear terms, which are important for the Jacobi bracket. The second is the non-linearity itself.

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The goal of this note is to discuss reasons and consequences of this anomaly (this also plays a significant role in investigation of coverings and non-local calculus [KKV]).

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1. **Anomaly via Hessian**

The Jacobi bracket of non-linear operators $F, G \in \text{diff} (\pi, \pi)$ is expressed via linearization as follows:

$$\{F, G\} = \ell_F G - \ell_G F.$$

We also consider the evolutionary operators defined by duality:

$$\mathcal{E}_F G = \ell_G F.$$

Since $\ell_G$ is a derivation in $G$, $\mathcal{E}_F$ is a derivation (satisfies the Leibniz rule) and their union can be treated as the module of vector fields. These operators have no anomaly, i.e. the map $\mathcal{E} : \mathcal{C}^\infty (J^\infty \pi) \rightarrow \text{Vect}(J^\infty \pi)$ is an anti-homomorphism:

$$[\mathcal{E}_F, \mathcal{E}_G] = -\mathcal{E}_{\{F, G\}}.$$

This instantly implies Jacobi identity for the bracket $\{F, G\}$, so that $(\text{diff}(\pi, \pi), \{, \})$ is a Lie algebra [KLV].

The operators of universal linearization and evolutionary differentiation do not commute and this leads to the following

**Definition.** The Hessian operator $\text{diff}(\pi, \pi) \times \text{diff}(\pi, \pi) \rightarrow \mathcal{C} \text{ Diff}(\pi, \pi)$ is defined by the formula

$$\text{Hess}_F G = [\mathcal{E}_G, \ell_F].$$

We will also write $\text{Hess}_F (G, H) = \text{Hess}_F G (H)$ for $F, G, H \in \text{diff}(\pi, \pi)$ and note that $\text{Hess}_F \equiv 0$ for linear operators $F$, because in this case $\ell_F = F$, which reduces the claim to the commutation of left and right multiplications.

Next we note that the Hessian $\text{Hess}_F$ is symmetric:

**Lemma 1.** $\text{Hess}_F (G, H) = \text{Hess}_F (H, G)$.

Indeed:

$$\text{Hess}_F (G, H) = \mathcal{E}_G \ell_F H - \ell_F \mathcal{E}_G H = \mathcal{E}_G \mathcal{E}_H F - \ell_F \ell_H G,$$

so that

$$\text{Hess}_F (G, H) - \text{Hess}_F (H, G) = [\mathcal{E}_G, \mathcal{E}_H] F - \ell_F \{H, G\} = -\mathcal{E}_{\{G, H\}} F - \ell_F \{H, G\} = 0.$$
Now we can express the anomaly of linearization via the Hessian:

**Proposition 2.** \([\ell_F, \ell_G] - \ell_{\{F,G\}} = \text{Hess}_G F - \text{Hess}_F G.\)

Indeed we have:

\[
[\ell_F, \ell_G]H = \ell_F \mathcal{H} G - \ell_G \mathcal{H} F
= \mathcal{H}(\ell_F G - \ell_G F) - \text{Hess}_F(H, G) + \text{Hess}_G(H, F)
= \mathcal{H}\{F, G\} - \text{Hess}_F(G, H) + \text{Hess}_G(F, H)
= \ell_{\{F,G\}}H + (\text{Hess}_G F - \text{Hess}_F G)H.
\]

Finally let us express the Leibniz identity for non-linear operators and the Jacobi bracket. For linear operators it is well-known, but for non-linear ones there’s an anomaly:

**Proposition 3.** \(\{F, \ell_G H\} = \ell_{\{F,G\}}H + \ell_G\{F, H\} - \text{Hess}_F(G, H).\)

This is obtained as follows:

\[
\{F, \ell_G H\} = \ell_F \ell_G H - \mathcal{H}_F \ell_G H
= [\ell_F, \ell_G]H + \ell_G(\ell_F - \mathcal{H}_F)H - \text{Hess}_G(F, H)
= \ell_{\{F,G\}}H + \ell_G\{F, H\} - \text{Hess}_F(G, H).
\]

### 2. Coordinate Expressions

A local coordinate system \((x^i, u^j)\) on \(\pi\) induces the canonical coordinates \((x^i, p^j_\sigma)\) on the space \(J^\infty\pi\), where \(\sigma = (i_1, \ldots, i_n)\) is a multi-index of length \(|\sigma| = i_1 + \cdots + i_n\). The operator of total derivative of multi-order \(\sigma\) (and order \(|\sigma|\)) is \(D_\sigma = D_1^{i_1} \cdots D_n^{i_n}\), where \(D_i = \partial_{x^i} + \sum_{r=1}^n p^j_{r+1} \partial_{p^j_\sigma}\).

The linearization of \(F = (F_1, \ldots, F_r)\) is \(\ell_F = (\ell(F_1), \ldots, \ell(F_r))\) with

\[
\ell(F_i) = \sum (\partial_{p^j_\sigma} F_i) \cdot D_\sigma^{[j]},
\]

where \(D_\sigma^{[j]}\) denotes the operator \(D_\sigma\) applied to the \(j\)-th component of the section from \(C^\infty(\pi)\).

The \(i\)-th component of the evolutionary differentiation \(\mathcal{H}_G\) corresponding to \(G = (G_1, \ldots, G_n)\) equals

\[
\mathcal{H}^i_G = \sum (D_\sigma G_j) \cdot \partial_{p^j_\sigma}^{[i]},
\]

where \(\partial_{p^j_\sigma}^{[i]}\) denotes the operator \(\partial_{p^j_\sigma}\) applied to the \(i\)-th component of the section from \(C^\infty(\pi)\).
Then $i$-th components of the Jacobi bracket is given by

$$\{F, G\}_i = \sum (D_\sigma(G_j) \cdot \partial_{p_j} F_i - D_\sigma(F_j) \cdot \partial_{p_j} G_i).$$

These formulas are known [KLV]. It is instructive to demonstrate the Jacobi identity in coordinates. For this we need the following assertion.

**Lemma 4.** In canonical coordinates on $J^\infty\pi$:

$$\partial_{p_\sigma} D_\tau = \sum D_{\tau - \kappa} \partial_{p_{\kappa - \mu}}$$

(the difference of multi-indices $\sigma - \kappa$ is defined whenever $\kappa \subset \sigma$), the summation is by $\kappa$ counted with multiplicity. More generally for vector differential operators if $D_\sigma^{[j]}$ is the operator $D_\sigma$ acting on the $j$-th component, then the above formula holds true for such specification.

This follows from iteration of the formula $[\partial_{p_\sigma}, D_i] = \partial_{p_{\sigma - i}}$. Thus

$$\{F, \{G, H\}\} = \sum F_{p\sigma} D_{\sigma - \kappa}(G_{p\tau})D_{\tau + \kappa}(H) - F_{p\sigma} D_{\sigma - \kappa}(H_{p\tau})D_{\tau + \kappa}(G)$$

$$- G_{p\tau p\tau} D_{\tau}(H)D_\sigma(F) - H_{p\tau p\tau} D_{\tau}(G)D_\sigma(F)$$

$$- (G_{p\tau} D_{\sigma - \kappa}(H_{p\tau - \kappa}) - H_{p\tau} D_{\sigma - \kappa}(G_{p\tau - \kappa}))D_\tau(F),$$

which yields $\sum_{\text{cyclic}} \{F, \{G, H\}\} = 0$.

Now we write the Hessian:

$$\text{Hess}_F(G, H) = \sum F_{p\sigma p\tau} D_\sigma G \cdot D_\tau H,$$

and its symmetry in $G, H$ and vanishing for linear $F$ is obvious.

The compensated Leibniz formula can be written as follows:

$$\{F, \ell_G H\} - \ell_{\{F,G\}} H - \ell_G \{F, H\} =$$

$$\sum F_{p\sigma} D_{\sigma - \kappa}(G_{p\tau})D_{\tau + \kappa}(H) - (G_{p\tau p\tau} D_{\tau}(H)D_\sigma(F) + G_{p\tau} \partial_{p\tau} D_\tau(H))D_\sigma(F)$$

$$- (F_{p\tau p\tau} D_\sigma(G) + F_{p\tau} \partial_{p\tau} D_\sigma(G))D_\tau(H) + (G_{p\tau p\tau} D_\sigma(F) + G_{p\tau} \partial_{p\tau} D_\sigma(F))D_\tau(H)$$

$$- G_{p\tau} (D_{\sigma - \kappa}(F_{p\tau})D_{\tau + \kappa}(H) - D_{\sigma - \kappa}(H_{p\tau})D_{\tau + \kappa}(F)) = - \text{Hess}_F(G, H)$$

and the anomaly in commuting linearizations is:

$$[\ell_F, \ell_G] =$$

$$\sum F_{p\sigma} D_{\sigma - \kappa}(G_{p\tau})D_{\tau + \kappa}(H) - G_{p\tau} D_{\sigma - \kappa}(F_{p\tau})D_{\tau + \kappa}(H)$$

$$- (F_{p\tau p\tau} D_\sigma(G) + F_{p\tau} \partial_{p\tau} D_\sigma(G))D_\tau(H) + (G_{p\tau p\tau} D_\sigma(F) + G_{p\tau} \partial_{p\tau} D_\sigma(F))D_\tau(H)$$

$$= \text{Hess}_G(F, H) - \text{Hess}_F(G, H).$$

This gives an alternative proof of Propositions 4 and 2.
3. Auxiliary integrals

Definition. An operator \( G \in \text{diff}(\pi, \pi) \) is called an auxiliary integral for \( F \in \text{diff}(\pi, \pi) \) if

\[
\{F, G\} = \ell_\lambda F + \ell_\mu G
\]

for some operators \( \lambda \in \text{diff}(\pi, \pi) \) and \( \mu \in \mathcal{C} \cdot \text{Diff}(\pi, \pi) \setminus \{0\} \). The set of such \( G \) is denoted by \( \text{Aux}(F) \).

It is better to denote \( \text{Aux}_\mu(F) \) the space of \( G \) satisfying the above formula with some fixed \( \mu \in \text{diff}(\pi, \pi) \), because it is a vector space. Then \( \text{Aux}(F) = \bigcup_\mu \text{Aux}_\mu(F) \). We can assume \( \text{ord}(\mu) < \text{ord}(F) \) for scalar operators, i.e. \( \text{rank}\pi = 1 \).

With certain non-degeneracy condition for the symbols of \( F, G \) the following statement holds:

**Theorem 5.** A non-linear differential operator \( G \) is an auxiliary integral for another operator \( F \) iff the system \( F = 0, G = 0 \) is compatible (formally integrable).

The generic position condition for the symbols of \( F, G \) is essential. If \( \pi = 1 \) is the trivial one-dimensional bundle, this condition is just the transversality of the characteristic varieties \( \text{Char}^C(F) \) and \( \text{Char}^C(G) \) in the bundle \( \mathbb{P}^C T^*M \) (after pull-back to the joint system \( F = G = 0 \) in jets); in this form it is a particular form of the statement proved in [KL2]. For rank \( \pi > 1 \) the condition is more delicate and will be presented elsewhere.

Notice that \( \text{Aux}_0(F) = \text{Sym}(F) \) is the space of symmetries of \( F \). This is a Lie algebra with respect to the Jacobi bracket. It can be represented as a union of spaces

\[
\text{Sym}_\theta(F) = \{H : \ell_F H = \ell_{\theta + H} F\}, \quad \theta \in \text{diff}(\pi, \pi),
\]

which are modules over \( \text{Sym}_0(F) \). More generally we have the graded group: \( \text{Sym}_\theta(F) + \text{Sym}_{\theta'}(F) \subseteq \text{Sym}_{\theta + \theta'}(F) \)

Let us assume \( G \in \text{Aux}_\mu(F), H \in \text{Sym}_\theta(F) \), i.e.

\[
\{F, G\} = \ell_\lambda F + \ell_\mu G, \quad \{F, H\} = \ell_\theta F.
\]

Then denoting \( \text{ad}_H = \{H, \cdot\} = \ell_H - \mathcal{E}_H \) we get:
\[ \text{ad}_F \{ G, H \} = \{ \text{ad}_F G, H \} + \{ G, \text{ad}_F H \} \]
\[ = \{ H, \ell_\lambda F + \ell_\mu G \} + \{ G, \ell_\theta F \} \]
\[ = \ell_{\{ \lambda, H \}} F + \ell_{\{ F, H \}} + \text{Hess}_H(\lambda, F) + \ell_{\{ \mu, H \}} G + \ell_{\{ G, H \}} + \text{Hess}_H(\mu, G) \]
\[ - \ell_{\{ \theta, G \}} F - \ell_{\theta \{ F, G \}} \]
\[ = (\ell_{\{ \lambda, H \}} + \ell_{\{ \mu, H \}} - \ell_{\{ \theta, G \}} + \text{Hess}_H(\lambda) - \text{Hess}_G(\theta)) F + \ell_{\{ G, H \}} \]
\[ + (\ell_{\{ \mu, H \}} - \ell_\theta \ell_\mu + \text{Hess}_H \mu) G. \]

Thus \( \{ G, H \} \) is an auxiliary integral for \( F \) if \( \ell_\theta \ell_\mu = \ell_{\{ \mu, H \}} + \text{Hess}_H \mu \) (the "iff" condition means the difference annihilates \( G \)), which can be written as
\[ \mu \in \text{Ker}[(\ell_\theta + \ell_{\text{ad}_H} - \text{Hess}_H) \circ \ell]. \]

Such a pair \( \theta \in \text{sym}^*(F) = \text{Sym}(F)/\text{Sym}_0(F) \), \( H \in \text{Sym}_\theta(F) \) determines the action of the second component
\[ \text{ad}_H: \text{Aux}_\mu(F) \rightarrow \text{Aux}_\mu(F). \]

Also since
\[ \ell_{\{ \mu, H \}} G = \mathcal{E}_G \{ \mu, H \} = \mathcal{E}_G \mathcal{E}_H(\mu) - \mathcal{E}_G \mathcal{E}_H(\mu) = (\mathcal{E}_H - \ell_H) \mathcal{E}_G(\mu) \]
\[ - \mathcal{E}_G(\mu) - \text{Hess}_H(\mu, G) = - \ell_{\mu} \mathcal{E}_G(\mu) - \text{Hess}_H(\mu, G) - \ell_{\mu} \{ G, H \}, \]
we have:
\[ \ell_{\mu} \{ G, H \} + (\ell_{\{ \mu, H \}} - \ell_\theta \ell_\mu + \text{Hess}_H \mu) G = -(\text{ad}_H + \ell_\theta) \ell_\mu G. \]

Thus if \( H \in \text{Sym}_\theta(F) \), i.e. \((\text{ad}_H + \ell_\theta) F = 0\), and \( \mu \in \text{Ker}[(\text{ad}_H + \ell_\theta) \circ \ell] \), i.e. \((\text{ad}_H + \ell_\theta) \ell_\mu = 0\), then
\[ \text{ad}_H: \text{Aux}_\mu(F) \rightarrow \text{Sym}(F). \]

4. Symmetries and compatibility

It has been a common belief that if \( G \in \text{Sym}(F) \), then the system \( F = 0, G = 0 \) is compatible, which forms the base of investigation for auto-model solutions. This is however not always true.

**Example:** Let \( F, G \) be two linear diagonal operators with constant coefficients. Then \( \{ F, G \} = 0 \) (in this case the Jacobi bracket is the standard commutator), so that \( G \) is a symmetry of \( F \). However the system \( F = 0, G = 0 \) is usually incompatible: for generic \( F, G \) of the considered type the only solution will be the trivial zero vector-function.

More complicated non-diagonal operators are possible, but it would be better to consider non-homogeneous linear operators. Then if the
coefficients are constant and generic, the linear matrix part commute, but the system $F = 0, G = 0$ may have no solutions at all.

For instance if we take

\[
F = \begin{bmatrix}
(D_x^2 - D_y) & 0 \\
0 & (D_x D_y + 1)
\end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
G = \begin{bmatrix}
(D_x D_y - 1) & 0 \\
0 & (D_y^2 - D_x)
\end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

then $\{F, G\} = 0$, so that $G \in \text{Sym}(F)$, while the system $F = 0, G = 0$ is not compatible, and moreover its solutions space is empty.

Thus the flow $u_t = G(u)$ on the equation $F = 0$ has no fixed points (no auto-model solutions). Here $t$ is an additional variable ($x$ is the base multi-variable for PDEs $F = 0$ and $G = 0$), so that $G \in \text{Sym}(F)$ can be expressed as compatibility of the system

\[
F(u) = 0, \quad u_t = G(u),
\]

while symmetric solutions correspond to the stationary case $u_t = 0$, i.e. compatibility of the system $F(u) = 0, G(u) = 0$.

However if the non-degeneracy condition assumed in Theorem 5 is satisfied, then auto-model (or invariant) solutions exist in abundance, namely they have the required functional dimension and rank as Hilbert polynomial (or Cartan test [C]) predicts, see [KL4].

**Remark.** Symmetric solutions are the stationary points of the evolutionary fields and they are similar to the fixed points of smooth vector fields on $\mathbb{R}^n$, which must exist provided the vector field is Morse at infinity. The non-degeneracy condition plays a similar role.

Many examples of auto-model solutions and their generalizations can be found in [BK, Ol, Ov], non-local analogs use the same technique and similar theory [KLV, KK, KKV].

Compatible systems correspond to reductions of PDEs and are sometimes called conditional symmetries by analogy with finite-dimensional integrable systems on one isoenergetic surface [FZ]. But the rigorous result must rely on certain general position property for the symbol of differential operators, otherwise it can turn wrong [KL2, KL3]. The method based on this approach makes specification of the general idea of differential constraint and is described in [KL1].

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5. Conclusion

In this note we described the higher-jets calculus corresponding to symmetries and compatible constraints, basing on the Jacobi brackets. Another approach to integrability of vector systems is given by minimal overdetermination and it uses multi-brackets of differential operators

\[ \{ \cdots \} : \Lambda^{m+1} \text{diff}(m \cdot 1, 1) \rightarrow \text{diff}(m \cdot 1, 1) \]

introduced in [KL3], which are governed by the non-commutative Plücker identity.

Following this approach a minimal generalization of symmetry for \( F = (F_1, \ldots, F_m) \in \text{diff}(\pi, \pi) \) with \( \pi = m \cdot 1 \) is such \( G \in \text{diff}(\pi, 1) \) that

\[ \{ F_1, \ldots, F_m, G \} = \ell_{\theta_1} F_1 + \cdots + \ell_{\theta_n} F_m. \]

With certain non-degeneracy assumption [KL3] this implies that the overdetermined system \( F = 0, G = 0 \) is compatible (formally integrable).

A more advanced algebraic technique would yield another higher-jets calculus producing anomaly that manifests in non-vanishing of the expression

\[ \{ \ell_{F_1}, \cdots, \ell_{F_{m+1}} \} - \ell_{\{ F_1, \cdots, F_{m+1} \}}. \]

Implications for vector auxiliary integrals and generalized Lagrange-Charpit method follow the same scheme.

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