Statistical features of the Stretched Exponentials Densities

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Abstract. A derivation of the Stretched Exponentials Probability Densities from a maximum entropy principle is given. The informational entropy is constrained such that the $\nu$-moment, $\langle |x|^{\nu} \rangle$, must be finite. Also higher order moments are computed. Moreover, conditions for a central limit theorem are satisfied. From the physical point of view, we discuss the role of the entropy defining stationary states associated to complex systems.

1. Introduction

Complex systems are characterized by the onset of power laws and by non Gaussian probability distributions. The stretched exponentials probability densities (SEPD) are an example of such behaviour. The SEPD appears in the study of many complex systems. There is no doubt about its physical significance, they are encountered for example, in recurrence time statistics [1], long-term correlations on rare events [2, 3], fractal rain distributions [4], Earth’s magnetic field fluctuations [5], distribution of shortest paths in percolation [6], catalytic activity of molecules [7], distribution of potential energy in granular avalanche [8], trap times distributions [9], random walks on fractals [10], enstrophy flux in two dimensional turbulence [11], velocity distributions in sedimentation [12], sleep state transitions in humans [13], activation energy barriers [14], noncolloidal suspensions [15], dynamical heterogeneities [16], distributions of relaxation times [17], time step distributions in a protein folding model [18], distributions of film earnings [19].

Also, they are relevant in the study of the tail of distributions, for example, the study of velocity fluctuations [20], in velocity fluctuations in granular flows [21], velocity distributions in inelastic hard sphere systems [22], velocity increments in turbulent flows [23], turbulent scaling [24], velocity increments in weakly turbulence flows [25], quantum diffusion [26].

Despite the evidence from a lot of experimental or numerical research, an explanation for such densities from physical grounds is still an unsolved problem. However, these densities are associated to an observable displaying a power law.

The SEPD are usually defined in two different settings by:

\begin{align}
    f_\nu(x) &= A e^{-\beta |x|^{\nu}}, \quad x \in (-\infty, \infty), \ \nu \in (1, 2), \\
    k_\nu(x) &= k_0 e^{-\beta x^{\nu}}, \quad x \in [0, \infty), \ \nu \in (0, 1),
\end{align}

where $A$, $k_0$ and $\beta$ are constants. In particular, $\beta$ is different in each case. Also, as we see below, such constants are depending only on $\nu$. In other words, Equation 1 and Equation 2 are families...
of densities parametrized by \(\nu\). Usually, in the applications the pre-factor \(A\) is not explicitly considered, and attention is centered in the argument \(\beta|\nu|\).

It is the aim of this paper to show that the SEPD obey a maximum entropy principle, hence \(A\) and \(\beta\) can be computed in each case, and to work out the higher order statistics of the SEPD. Moreover, concerning the dependence on \(\nu\), the global maximum of the entropy is attained at the Gaussian density for the SEPD given by Equation 1 and, at the exponential density for SEPD defined by Equation 2.

This paper is organized as follows, in section 2 we give a derivation of the SEPD, \(f_\nu(x)\), by means of a maximum entropy principle. In section 3, we determine \(A, \beta\) and \(\langle |x|^\nu \rangle\), we compute its generalized moments and the cumulative distribution of \(f_\nu\). In particular, we give a representation of \(\langle |x|^\nu \rangle\) as a function of the \(\sigma = \sqrt{\langle x^2 \rangle}\) and \(\nu\); in section 4, we study the properties of the entropy and its role defining stationary states; in section 5 we show that these densities satisfies a the Central Limit Theorem; in section 6, we compute the statistical properties of the Normalized Kohlraush functions, \(k_\nu\); finally in section 7, we give the conclusions. Also, we include 2 appendices containing the details about the computations involved in the text.

### 2. Maximum entropy principle

Let us consider a probability space \((\Omega, \mathcal{B}, \mu)\), where \(\Omega = (-\infty, \infty)\). The \(\mathcal{B}\) is the smallest Borel \(\sigma\)-algebra of subsets on \(\Omega\). The measure \(F(x)\) assigns the probabilities to the events \(E_i \in \mathcal{B}\), i.e. \(F(x)(E_i) := \text{Prob}(E_i)\). The relevant physical observable is represented by a random variable \(X\), which is distributed according to a probabilistic density \(f(x) = F'(x)\). For example \(X: \Omega \to U\), where we choice \(U = \mathbb{R}\) or \(\mathbb{Z}\). Now, we introduce the entropy of this system which is defined by:

\[
H[f] = -\int_{\Omega} f(x) \ln f(x) dx. \tag{3}
\]

We are looking for the probability density \(f\) that maximizes \(H[f]\), imposing the supplementary condition, that the \(\nu\)-moment:

\[
\langle |x|^\nu \rangle = \int_{\Omega} |x|^\nu f(x) dx, \tag{4}
\]

must be finite.

The following functional

\[
\Phi[f] = H[f] + \alpha \left( \int_{\Omega} f(x) dx - 1 \right) - \beta \left( \int_{\Omega} |x|^\nu f(x) dx - E_\nu \right),
\]

must be an extremal at such density. The parameters \(\alpha\) and \(\beta\) are the Lagrange multipliers associated to the normalization of \(f\) and to the \(\nu\)-moment, \(\langle |x|^\nu \rangle\), whose value is denoted as \(E_\nu\). For the exposition of the method see, for example, [27].

The solution of the variational equation \(\delta \Phi[f] = 0\) is given by:

\[
f_\nu(x) = e^{-1+\alpha-\beta|x|^\nu},
\]

where we define \(A = \exp(-1 + \alpha)\). The constants \(A\) and \(\beta\), as we see below, are depending only on the parameter \(\nu\).
3. Statistical characterization of SEPD
The normalization of $f_\nu$ allow us to determine $A$, as $A^{-1} = \int_{-\infty}^{\infty} e^{-\beta x^\nu} dx$, and obtaining

$$A = \frac{\nu \beta^{1/\nu}}{2 \Gamma(1/\nu)}.$$  \hspace{1cm} (5)

Here, $\Gamma(y)$ is the Gamma function, see section Appendix A for details.

Now, using Equation 5, we can compute the $\nu$–moment:

$$\langle|x|\rangle^\nu = \frac{1}{\beta^{\nu}}.$$  \hspace{1cm} (6)

This one is an implicit relation between $\beta, \nu$ and $\langle|x|\rangle^\nu$. Therefore, in order to determine $\beta$ we need an additional condition. To this end we proceed to compute the lowest non zero moment of $f_\nu$, i.e. the variance $\sigma^2 := \langle x^2 \rangle$. Explicitly, see Appendix A for details,

$$\beta = \frac{\sqrt{\Gamma(3/\nu)/\Gamma(1/\nu)}}{\sigma}.$$  \hspace{1cm} (7)

then, after its substitution on Equation 5, we have

$$A = \frac{\nu \Gamma(3/\nu)^{1/2}}{2 \sigma \Gamma(1/\nu)^{3/2}}.$$  \hspace{1cm} (8)

Now, introducing $b = \sqrt{\Gamma(3/\nu)/\Gamma(1/\nu)}$, i.e. $\beta = (b/\sigma)^\nu$, we get the final expression for $f_\nu$:

$$f_\nu(x) = \frac{\nu b}{2\sigma \Gamma(1/\nu)} e^{-(b|x|/\sigma)^\nu}.$$  \hspace{1cm} (9)

3.1. Higher order statistics
Also, the absolute moments, $\langle|x|^p\rangle$, are:

$$\langle|x|^p\rangle = \frac{\Gamma((p+1)/\nu)\Gamma(1/\nu)^{p/2-1}\sigma^p}{\Gamma(3/\nu)^{p/2}},$$  \hspace{1cm} (10)

which are well defined for any $p > -1$.

In particular, after substitution of Equation 7 in Equation 6, the $\nu$–moment is:

$$\langle|x|^\nu\rangle = \frac{\Gamma(1/\nu)^{\nu/2}\sigma^\nu}{\nu \Gamma(3/\nu)^{\nu/2}}.$$  \hspace{1cm} (11)

It is worth to mention that all the absolute moments $\langle|x|^p\rangle$ are proportional to $\sigma^p$ for any $p > -1$. In other words, for a given $\nu$, the standard deviation $\sigma$ gives the full characterization of $f_\nu$.

Another interesting quantity is the kurtosis, $\kappa$, which measures a kind of deviation from the Gaussian density, in the sense that $f_\nu$ has a sharp peak and fat tail for $\nu \neq 2$. The following definition is given to associate to a Gaussian density the value $\kappa = 0$. The kurtosis of $f_\nu$ is given by

$$\kappa = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 3 = \frac{\Gamma(5/\nu)\Gamma(1/\nu)}{\Gamma(3/\nu)^2} - 3.$$  \hspace{1cm} (12)

It is interesting to note that if we consider $\kappa$ as a function of $\nu$, then $\kappa$ is a monotonous decreasing function when $\nu \in (1, 2)$. $\kappa$ is running from 3 to 0 in that interval. As we see in the next section the entropy is a monotonous increasing function in the same interval.

We show in Figure 1 several examples of $f_\nu$ to compare with the Gaussian density.
3.2. Distribution of $f_\nu$

As stated in section 2 the probability distribution of $f_\nu$ is

$$F_\nu(x) = \int_{-\infty}^{x} f_\nu(y) dy = A \int_{-\infty}^{x} e^{-\beta |y|^\nu} dy,$$

(13)

which can be explicitly evaluated to give:

$$F_\nu(x) = \begin{cases} 
\frac{1}{2\sigma} \Gamma\left(\frac{1}{\nu}, x\right) & \text{if } x < 0, \\
\frac{1}{\sigma} + \frac{1}{2\sigma} \gamma\left(\frac{1}{\nu}, \beta x^{\nu}\right) & \text{if } x \geq 0,
\end{cases}$$

(14)

where $\Gamma(c, x)$ and $\gamma(c, x)$ are the upper and the lower Incomplete Gamma functions, respectively, [28].

4. Entropy

Now, substituting Equation 8 and 7 in Equation 3, we obtain:

$$H = - \int_{\Omega} f_\nu(x) \ln (f_\nu(x)) \, dx,$$

$$= \beta (|x|^\nu) - \ln A(\nu)$$

(15)

$$= \frac{1}{\nu} - \ln A(\nu)$$

(16)

In other words, the entropy is determined by the actual value of $\nu$ through the pre-factor $A$, such that $A = \exp(1/\nu - H)$. Hence, $H$ it can be estimated directly from data once $A$ is fitted from the estimated values of $\beta$ and $\nu$. 

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**Figure 1.** The SEPD $f_\nu$ as a function of $x$. The upper function corresponds to $\nu = 1$ and the lower one to the Gaussian density. The others are given by $\nu = 1.25, 1.5, 1.75$. 

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In spite of the fact that the entropy attains its maximum at \( f_\nu \), we ask the question about the maximum of the entropy considered as function of \( \nu \), i.e. among all set of the densities \( f_\nu \), at which one the entropy attains its maximum?. That density satisfy \( H'(\nu) = 0 \). Fortunately, this equation is easy to solve, see Appendix B for details. We show that it is the case when \( \nu = 2 \). These results implies that \( f_\nu(x) \) represents a stationary state with lower entropy than the the Gaussian state. In particular, \( f_\nu \) is an intermediate stationary state between the bilateral exponential \( f_{be} \), [29], and Gaussian densities, \( f_g \). In particular, we have:

\[
f_{be}(x) = \frac{1}{\sqrt{2\pi} \sigma_{be}} e^{-\frac{x^2}{2\sigma_{be}^2}}, \quad H_{be} = 1 + \log(\sqrt{2\pi} \sigma_{be}).
\]

\[
f_g(x) = \frac{1}{\sqrt{2\pi} \sigma_g} e^{-\frac{x^2}{2\sigma_g^2}}, \quad H_g = \frac{1}{2} + \log \sqrt{2\pi} \sigma_g.
\]

then for any \( 1 < \nu < 2 \), \( H_{be} < H_\nu < H_g \). We denote by \( \sigma_{be} \) and \( \sigma_g \) the corresponding standard deviations of \( f_{be} \) and \( f_g \).

5. The Central Limit Theorem

In this section we prove that the SEPD satisfies a Central Limit Theorem (CLT), see for example [30] for details about such theorem. In particular, the SEPD satisfies the Berry-Essèen theorem which states that: let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with mean \( \langle X_i \rangle = 0 \), \( \langle X_i^2 \rangle > 0 \) and \( \langle |X_i|^3 \rangle = \rho < \infty \), and \( F_n(x) \) be the distribution of the normalized sum \( (X_1 + X_2 + \ldots + X_n)/\sqrt{n}\sigma \), then for all \( x \) and \( n \) we have \( |F_n(x) - \Phi(x)| \leq C\rho/(\sqrt{n}\sigma^3) \). Here \( \Phi(x) \) is the Normal density with zero mean and unit variance. Also, the current value for \( C \approx 0.7655 \). Using Equation 10 with \( p = 3 \),

\[
\rho = \frac{\nu \Gamma(4/\nu) \Gamma(1/\nu)^{3/2} \sigma^3}{\Gamma(3/\nu)^{3/2}},
\]

i.e. \( \rho < \infty \), because \( \nu \in (1, 2) \). The specific value of the upper bound in the inequality given above is independent of \( \sigma \), but only depending on \( \nu \) and \( n \).

Therefore, that expression tends to zero as \( n \to \infty \), hence, the probability of the inequality \( (X_1 + X_2 + \ldots + X_n)/\sqrt{n}\sigma < t \) tends uniformly to the limit \( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du \).

6. Normalized Kohlrausch functions

Now, we proceed to briefly describe the statistical properties of \( k_\nu \). Particularly related to our subject is the notion of Kohlrausch functions. Since its introduction in [31], and its rediscovery in [32, 33] in the study of relaxation processes, stretched exponentials, also called Kohlrausch functions, are playing an important role in the study of physical phenomena whose relaxation properties deviates from exponential behavior. The Kohlrausch functions are a subject of intensive study in recent years. We refer the reader to [34, 35] for a recent account of some of its mathematical properties and some applications.

The Kohlrausch functions are defined as:

\[
K(x) = K_0 e^{-(t/\tau)^\nu},
\]

where \( K_0, \tau > 0 \) and \( 0 < \nu < 1 \).

The main difference between the functions \( K \) and \( k_\nu \) comes from the fact that we use the parameter \( \beta \). We prefer the representation given by Equation 2 because if we normalize the function \( K(x) \), then we obtain

\[
K_0 = \frac{\nu}{\tau \Gamma(1/\nu)},
\]
such that
\[
\langle x \rangle = \frac{\tau \Gamma(2/\nu)}{\Gamma(1/\nu)},
\]
while, introducing \( \beta \), we have that \( \langle x \rangle = \tau \). In other words, the densities \( k_\nu \) give us an average \( \langle x \rangle \) which is independent of \( \nu \). This is the expected situation in applications.

Now, we proceed to determine \( A, \beta \) and \( \langle x^\nu \rangle \) obtaining
\[
A = \frac{\nu \beta^{\frac{3}{\nu}}}{\Gamma(\frac{1}{\nu})}, \quad \langle x^\nu \rangle = \frac{1}{\beta^{\nu}}, \quad \langle x \rangle = \frac{\Gamma(\frac{2}{\nu})}{\beta^{\frac{1}{\nu}} \Gamma(\frac{1}{\nu})},
\]
(20)
for any \( \beta, \nu > 0 \). Therefore we have
\[
\beta = \left[ \frac{\Gamma(\frac{2}{\nu})}{\Gamma(\frac{1}{\nu}) \langle x \rangle} \right]^{\nu}, \quad A = \frac{\nu \Gamma(\frac{2}{\nu})}{\Gamma(\frac{1}{\nu})^2 \langle x \rangle}, \quad \langle x^\nu \rangle = \frac{1}{\nu} \left[ \frac{\Gamma(\frac{1}{\nu}) \langle x \rangle}{\Gamma(\frac{2}{\nu})} \right]^\nu.
\]
(21)

Now, introducing \( b = \Gamma(\frac{2}{\nu})/\Gamma(\frac{1}{\nu}) \) we obtain the final form for \( k_\nu \) as
\[
k_\nu(x) = \frac{\nu b}{\Gamma(1/\nu) \langle x \rangle} e^{-(bx/\langle x \rangle)^\nu},
\]
(22)
The generalized moments of order \( p \), \( \langle x^p \rangle \), are given by:
\[
\langle x^p \rangle = \frac{\Gamma(p+1/\nu) \Gamma(\frac{1}{\nu})^{p-1} \langle x \rangle^p}{\Gamma(\frac{2}{\nu})^p},
\]
(23)
which, in particular, are finite for any \( p > -1 \) and \( \nu \in (1, 2) \). Hence, \( \langle x^3 \rangle \) is finite such that the conditions for a CLT are satisfied as explained in Section 5.

Finally, its cumulative distribution is given by
\[
F_k(x, \nu) = \frac{1}{\langle x \rangle \Gamma(\frac{1}{\nu})} \gamma \left( 1, \beta x^\nu \right),
\]
(24)
where, again, \( \gamma(c, x) \) is the incomplete Gamma function.

7. Conclusions
In this paper we have shown that the stretched exponentials probability densities, \( f_\nu \) and \( k_\nu \), obey a maximum entropy principle and a central limit theorem. Also, we have computed the higher order moments allowing us to characterize the full statistics in function of the respective lowest non zero moment. In particular, the \( \nu \)-moment \( \langle |x|^\nu \rangle \) is finite for the range of \( \nu \) values considered in applications.

From the physical point of view, according to the values attained by the entropy, the densities \( f_\nu, \nu \in (1, 2) \), can be interpreted as an intermediate stationary state between the bilateral exponential, \( f_1 \), and Gaussian, \( f_2 \), densities. Also, densities \( k_\nu \) are close related to the Kohlrausch functions which are very important in the study of non Gaussian relaxation processes.

Finally, due to the fact that a general derivation of such densities from physical basis is still an unsolved problem, the study of the statistical properties of these densities can help us to understand the behaviour of complex systems.

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Appendix A. Computation of $A$, $\beta$ and $\langle|x|^\nu\rangle$
If we make the change of variable $u = \beta x^\nu$ in
\[
A^{-1} = \int_0^\infty e^{-\beta x^\nu} \, dx = \frac{2}{\nu \beta^\frac{1}{\nu}} \int_0^\infty e^{-u} u^{\frac{1}{\nu}-1} \, du,
\] (A.1)
and using the Gamma function
\[
\Gamma(y) = \int_0^\infty e^{-u} u^{y-1} \, du, \quad y > 0,
\] (A.2)
we obtain the desired result for $A$ given in Equation 8. Now, computing $\langle x^2 \rangle$ we obtain
\[
\langle x^2 \rangle = \frac{\Gamma\left(\frac{3}{\nu}\right)}{\Gamma\left(\frac{1}{\nu}\right)} \beta^{\frac{2}{\nu}},
\] (A.3)
then denoting by $\sigma^2 = \langle x^2 \rangle$ and solving for $\beta$ we finally obtain $A$ and $\langle |x|^\nu \rangle$ as given in Equation 5 and Equation 11, respectively. In analogous way we can determine the corresponding expressions for $A, \beta$ and $\langle |x|^\nu \rangle$ for the densities $k_\nu$. The only differences are the domain of $k_\nu$ and that the lowest nonzero moment is the mean $\langle x \rangle$.

Appendix B. Solution of $\frac{dH}{d\nu} = 0$
From the analytical expression for the entropy of $f_\nu$,
\[
H = \frac{1}{\nu} - \ln(\nu) - \frac{1}{2} \ln \left( \Gamma\left(\frac{3}{\nu}\right) \right) + \frac{3}{2} \ln \left( \Gamma\left(\frac{1}{\nu}\right) \right) + \ln(2) + \ln(\sigma),
\] (B.1)
we take its derivative respect to $\nu$
\[
\frac{dH}{d\nu} = \frac{d}{d\nu} \left( \frac{1}{\nu} - \ln A(\nu) \right) = -\frac{1}{\nu} - \frac{A'(\nu)}{A(\nu)} = 0,
\] (B.2)
which has the form
\[
\frac{dS}{d\nu} = -\frac{1}{\nu^2} - \frac{1}{\nu} + \frac{3}{2\nu^2} \psi\left(\frac{3}{\nu}\right) - \frac{3}{2\nu^2} \psi\left(\frac{1}{\nu}\right),
\] (B.3)
where we are introducing the digamma function
\[
\psi(x) = \frac{d \ln \Gamma(x)}{dx}.
\] (B.4)
Now, by definition of the digamma function, we have $\psi(1/2) = -\gamma_{em} - 2 \ln(2)$ and $\psi(3/2) = -\gamma_{em} - 2 \ln(2) + 2$, where $\gamma_{em}$ is the Euler-Mascheroni constant, see [28]. Finally, we can see that Equation B.3 equals to zero when $\gamma = 2$. For the $k_\nu$ densities we obtain a similar result, but with $\nu = 1$.

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