Application of the regular perturbation method for the solution of first-order initial value problems

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Abstract. Several analytical and numerical methods are globally used in the solution of linear and non-linear ordinary differential equations. In recent times, an analytical method called the perturbation method was proposed by JiHuan He. This technique involves the use of the perturbation parameter, $\varepsilon$. The types of perturbation method which are usually used in Mathematics are the regular perturbation method and the singular perturbation method. In this research, various numerical examples of first-order IVPs are solved using the regular perturbation method. This method gives an approximate analytical solution to the first order IVPs and are discovered to be efficient and easy to adopt.

Keywords: Perturbation Method, Initial Value Problem, Ordinary Differential Equation, Numerical Method

1. Introduction

Differential equation is one of the significant areas in mathematics. They are not just used in mathematics; various disciplines such as Physics, Astronomy, Biology, Meteorology, and Economics apply differential equations in their study. Some of these applications include: the use of differential equations: to describe various exponential decays and growths; to describe the change in investment return over time, in the medical science field to model the development of cancer or disease.

The numerical methods of finding the solution of Ordinary Differential Equations (ODEs) problems have been a matter of great deal and many researchers have brought about several and different approaches and methods to solve ODE problems [1-2]. One of such methods is the Perturbation method.

The perturbation approach is applied by obtaining an approximate solution to the ODE problem, starting with the exact solution of a related, clearer problem. Some applications of perturbation method include: the use of perturbation method for fractional Fornberg-Whitham equation [3]. The perturbation approach is now employed by several researchers for the application of several differential problems [4 - 7].

The homotopy perturbation method (HPM) is another important method that have been introduced to solve differential equations and it is based on the perturbation method. The HPM and its modifications have been used in the solution of generalized linear and non-linear Riccati differential equation. The modified homotopy perturbation method was said to have been thoroughly tested in the solving of several implementations and it was shown that the method was efficient and accurate [8]. Meaningful interpretable explanation of Black boxes in a study done by [9]. The Least Squares Homotopy
Perturbation Method (HPM) was used in the solution of nonlinear differential equations and the main feature of this method is the accelerated convergence which when compared to the regular perturbation method gives more accurate results [10]. In a study shown in [11], the HPM was well discussed as the problem of its convergence is not prominent work done by other researchers. The perturbation method was analyzed and the convergence of this method is discussed. It also showed that the homotopy perturbation approach converges to the exact solution of the ordinary or partial differential equations which are not linear [12 – 17].

2. Methodology
In this section, the regular perturbation method would be discussed in respect to the solution of first order ordinary differential equations. It would explain how it should be applied to the first order differential equation.

2.1. Regular Perturbation Method
The regular perturbation method is implemented by getting an estimated solution to the ODE problem, beginning with the specific solution of a similar, more straightforward problem.

A first order differential equation such as:

\[ \frac{dy}{dx} = f(x, y, \varepsilon) \]  

(1)

where \( y \) and \( f \) are vector functions, \( x \) is an independent scalar variable and \( \varepsilon \) is a small parameter, said to be regularly perturbed if

\[ f(x, y, \varepsilon) = \sum_{k=0}^{\infty} f_k(x, y)\varepsilon^k. \]  

(2)

In such a case, the solution of such system is found in the form of an asymptotic series which is

\[ y(x, \varepsilon) = \sum_{k=0}^{\infty} y_k(x)\varepsilon^k. \]  

(3)

Considering an ODE:

\[ f \left( \frac{\partial^n y}{\partial x^n}, \frac{\partial^{n-1} y}{\partial x^{n-1}}, \ldots, \frac{\partial y}{\partial x}, y, x \right) + \varepsilon \left( \frac{\partial^n y}{\partial x^n}, \frac{\partial^{n-1} y}{\partial x^{n-1}}, \ldots, \frac{\partial y}{\partial x}, y, x \right) = 0 \]  

(4)

Suppose that \( y(x, \varepsilon) \) is the solution of an ODE in which the ordinary differential equation and the initial conditions solely depends on a very small parameter \( \varepsilon \). We would need the Taylor Series expansion in order to compute the solution, \( y(x, \varepsilon) \) using perturbation.

In a Taylor Series expansion, we suppose a function \( g(y) \) is differentiable at \( y = y_0 \), then we express it in the power series of \( (y = y_0) \) as:

\[ g(y) = x_0 + x_1(y - y_0) + x_2(y - y_0)^2 + \ldots \]  

(5)

Where,

\[ x_n = \frac{1}{n!} g^{(n)}(y_0). \]  

(6)

The Taylor Series expansion of \( y(x, \varepsilon) \) is given as:

\[ y(x, \varepsilon) = y(x, \varepsilon) + \frac{\partial y(x, \varepsilon)}{\partial x} \varepsilon + \frac{\partial^2 y(x, \varepsilon)}{2! \partial x^2} \varepsilon^2 + \frac{\partial^3 y(x, \varepsilon)}{3! \partial x^3} \varepsilon^3 + \ldots + \frac{\partial^n y(x, \varepsilon)}{n! \partial x^n} \varepsilon^n \]  

(7)

doing this we have
\[ y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \frac{\varepsilon^2}{2!} y_2(x) + \frac{\varepsilon^3}{3!} y_3(x) + \ldots + \frac{\varepsilon^n}{n!} y_n(x) \]  

(8)

which is the perturbation solution.

In most cases, \( \varepsilon \) is sufficiently small and then the perturbation parameter \( p \) which has the range \([0, 1]\), is the power of \( \varepsilon \) and it needs to be ignored when \( p \geq 2 \). Since we are ignoring the power of \( \varepsilon \) when \( p \geq 2 \), we therefore get:

\[ y(x, \varepsilon) = y(x) + \frac{\partial y(x, \varepsilon)}{\partial \varepsilon} \frac{\varepsilon}{1!} \]  

(9)

Therefore, we would know that

\[ y(x, \varepsilon) = y_0 + \varepsilon y_1. \]  

(10)

After the Taylor Series expansion, we then compute the steps involved to show an instance of how it goes.

**FIRST STEP:** The equation and the initial conditions are written down. Then we focus on the variable which the parameter \( \varepsilon \) appears. For instance:

\[ y' = y - \varepsilon y^2, \ y(0) = 3 \]  

(11)

In that equation, we focus on the variable that has \( \varepsilon \) in it. Also, we put down \( y(x, \varepsilon) \).

\[ y(x, \varepsilon) = y_0 + \varepsilon y_1 \]  

(12)

**SECOND STEP:** Substitute (12) in (11)

**THIRD STEP:** Thereafter, we set \( \varepsilon = 0 \) in the original equation and also the initial conditions or boundary equations in order to get an initial value problem which determines the unperturbed solution. For instance:

\[ y(x, \varepsilon)_{\varepsilon=0} = y(x, 0) := y_0(x) \]  

(13)

from this we solve for \( y_0(x) \).

**FOURTH STEP:** Differentiate the original equation and the initial or boundary conditions with respect to \( \varepsilon \) and then we set \( \varepsilon = 0 \) and then the solution for \( y_1(x) \) is obtained. Mathematically, it means that we apply \( \frac{\partial}{\partial \varepsilon} \) to both equations and from this we obtain an initial value problem for \( \frac{\partial y}{\partial \varepsilon} \) for all \((x, \varepsilon)\). After the results of \( y_0(x) \) and \( y_1(x) \) have been obtained. It is then substituted into the solution which is:

\[ y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) \]  

(14)

in which we obtain our approximate solution. If for any reason \( y_1(x) = 0 \), there is need to generate more terms in order to know the significant order effect of the perturbations.

**FIFTH STEP:** In order to get the approximate solutions for higher order terms, if the equations for \( \frac{\partial^n y}{\partial \varepsilon^n} (x, \varepsilon) \) have already been obtained, we then continue to differentiate the original equations and the initial or boundary conditions with respect to \( \varepsilon \). Then, we set \( \varepsilon = 0 \) with each successive derivative and solving the resulting the initial value problems to obtain the values of \( y_2, y_3, y_4, \ldots, y_n \).

Mathematically, this is written as:

\[ \frac{\partial^{n+1} y(x, \varepsilon)}{\partial \varepsilon^{n+1}} |_{\varepsilon=0} = \frac{\partial^{n+1} y(x, 0)}{\partial \varepsilon^{n+1}} := y_{(n+1)}(x) \]  

(15)
3. Numerical Examples

In this section, the regular perturbation method is applied to both linear and nonlinear IVPs.

Case I: Solve $y' = y - \epsilon y^2$ with initial conditions $y|_{\epsilon=0} = 1$ using regular perturbation method.

Solution

From Taylor Series expansion, it has been established that $y(x, \epsilon) = y_0 + \epsilon y_1$ is known as the solution to the perturbation equation. We then substitute $y = y_0 + \epsilon y_1$ in the original equation and the initial condition which would be:

$$y_0' + \epsilon y_1' = y_0 + \epsilon y_1 - \epsilon(y_0^2 + \epsilon y_1^2)$$

$$y_0 \bigg|_{\epsilon=0} + \epsilon y_1 \bigg|_{\epsilon=0} = 1$$

Expanding we have:

$$y_0' + \epsilon y_1' = y_0 + \epsilon y_1 - \epsilon y_0^2 - \epsilon^2 y_1^2$$

In every perturbation case, if for any reason in the equation the power of $\epsilon$ is greater than 2, we ignore it. Ignoring that in (16) we then have:

$$y_0' + \epsilon y_1' = y_0 + \epsilon y_1 - \epsilon y_0^2$$

Thereafter we set $\epsilon = 0$ in equations (17) and (19) respectively, in order to generate an equation in which we can obtain the value of $y_0$, in which we get:

$$y_0 \bigg|_{\epsilon=0} = 1$$

$$y_0 = y_0$$

$$y_0' - y_0 = 0$$

Solving equation (21), we get:

$$m - 1 = 0$$

$$\therefore m = 1$$

The exponential solution is then given as:

$$y_0 = Ae^{mx}$$

We have found out that $m = 1$, we then substitute it in $y_0 = Ae^{mx}$, doing that we then obtain:

$$y_0 = Ae^{(1)x}$$

$$\therefore y_0 = Ae^{x}$$

We have to get the solution of $A$, this is when the initial condition come in. Recall in (20) where $y_0 \bigg|_{\epsilon=0} = 1$, we then substitute in equation (22), doing that we get

$$1 = Ae^0$$

$$\therefore A = 1$$

Therefore, we substitute the value of $A$ in equation (22) to obtain the value of $y_0$. The value of $y_0$ is then given as:

$$y_0 = (1)e^x$$

$$\therefore y_0 = e^x$$

The next thing is to obtain the value of $y_1$, in order to do that, we differentiate equation (17) and (19) respectively with respect to $\epsilon$, doing that, we get:
\begin{align}
y_1 \mid_{x=0} &= 1 \\
y'_1 &= y_1 - y_0^2 \\
y'_1 - y_1 &= -y_0^2 \tag{24}
\end{align}

Recall, we already obtained the value of \( y_0 \), then we substitute \( y_0 \) in equation (24), we get:
\begin{align}
y'_1 - y_1 &= -(e^x)^2 \\
y'_1 - y_1 &= -e^{2x} \tag{25}
\end{align}

From (24), we know that we are meant to obtain the particular and complimentary solution for the equation. We should also know that \( y_1 = y_c + y_p \)

For the complimentary equation \( y_c \), we get
\begin{align}
y'_1 - y_1 &= 0 \tag{26}
\end{align}

Solving (26) we get:
\begin{align}
m - 1 &= 0 \\
\therefore \ m &= 1
\end{align}

The exponential solution is then given as:
\begin{align}
y_c &= Ae^{mx} \\
\therefore y_c &= Ae^x \tag{27}
\end{align}

The particular solution \( y_p \) from (25) is then given as:
\begin{align}
y_p &= ce^{2x} \\
\therefore y'_p &= 2ce^{2x} \tag{28}
\end{align}

We substitute both \( y'_p \) and \( y_p \) into (25), we obtain:
\begin{align}
2ce^{2x} - ce^{2x} &= -e^{2x} \\
ce^{2x} &= -e^{2x}
\end{align}

We then obtain the value for \( c \):
\begin{align}
\frac{ce^{2x}}{e^{2x}} &= \frac{-e^{2x}}{e^{2x}} \\
\therefore c &= -1
\end{align}

We then substitute the value of \( c \) in (28) to get the value of \( y_p \)
\begin{align}
y_p &= (-1)e^{2x} \\
\therefore y_p &= -e^{2x}
\end{align}

Recall that \( y_1 = y_c + y_p \), the next step is to get the value of \( y_1 \) by substituting the values of \( y_c \) and \( y_p \).

We then have that:
\begin{align}
y_1 = Ae^x - e^{2x} \tag{29}
\end{align}
We have to get the solution of $A$, this is when the initial condition come in. Recall in (23) where $y_1|_{x=0} = 1$, we then substitute in equation (29), doing that we get

$$1 = Ae^0 - e^0$$

$$1 = A - 1$$

$$\therefore A = 1 + 1 = 2$$

Therefore, we substitute the value of $A$ in equation (29) to obtain the value of $y_1$. The value of $y_1$ is then given as:

$$y_1 = 2e^x - e^{2x}$$

Now that we have gotten the values of $y_0$ and $y_1$, can now substitute in the solution which is $y(x, \varepsilon) = y_0 + \varepsilon y_1$.

The regular perturbative solution of this differential equation is:

$$y = e^x + \varepsilon (2e^x - e^{2x})$$

**Case 2:** Using regular perturbation method, obtain the approximate solution to the first order differential equation: $xy' = y + \varepsilon y^2$ with initial condition $y(1) = 1$.

**Solution**

Following every step as shown in **Case 1**:

We then substitute $y = y_0 + \varepsilon y_1$ in the original equation and the initial condition which would be:

$$x[y_0' + \varepsilon y_1'] = y_0 + \varepsilon y_1 + \varepsilon y_0^2 + \varepsilon^2 y_1^2$$

(30)

$$y_0(1) + \varepsilon y_1(1) = 1$$

(31)

Recall that, if for any reason in the equation the power of $\varepsilon$ is greater than 2, we ignore it. Therefore, we get:

$$xy_0' + x\varepsilon y_1' = y_0 + \varepsilon y_1 + \varepsilon y_0^2$$

(32)

We then set $\varepsilon = 0$ in (30) and (31) order to obtain the value for $y_0$.

$$y_0(1) = 1$$

(33)

$$xy_0 = y_0$$

$$xy_0 - y_0 = 0$$

(34)

Solving (32), we get:

$$mx - 1 = 0$$

$$mx = 1$$

$$m = \frac{1}{x}$$

Substituting $m = \frac{1}{x}$, the exponential solution is given by:

$$y_0 = Ae^{\frac{(\frac{1}{x})x}{}}$$
\[ y_0 = Ae^{(1)} \]
\[ \therefore \ y_0 = Ae \]

We then obtain the value of \( A \) in order to get the value of \( y_0 \). We would then need the initial condition in (33),
\[ y_0(1) = 1, \text{ substituting this in } y_0 = Ae, \text{ we get:} \]
\[ 1 = Ae \]
\[ \therefore A = \frac{1}{e} = e^{-1} \]

We then substitute the value of \( A \) in \( y_0 \).
\[ y_0 = (e^{-1})(e) \]
\[ y_0' = e^{-1+1} \]
\[ \therefore y_0 = e^0 = 1 \]

Now, we solve for the value of \( y_1 \), the first step is to differentiate (31) and (32) with respect to \( x \).

Doing this, we get:
\[ y_1(1) = 1 \] (35)
\[ xy_1' = y_1 + y_0^2 \]
\[ xy_1' = -y_1 = y_0^2 \] (36)

From (36), it is seen that we have to obtain the complimentary and particular solution in order to obtain the value for \( y_1 \). Obtaining the complimentary solution, we have:

\[ mx - 1 = 0 \]
\[ mx = 1 \]
\[ m = \frac{1}{x} \]

Substituting \( m = \frac{1}{x} \), the exponential solution is given by:
\[ y_c = Ae^{\frac{1}{x}} \]
\[ y_c = Ae^{(1)} \]
\[ \therefore y_c = Ae \]

The particular solution is given as:
\[ y_p = c \] (37)
\[ y_p' = 0 \]

We substitute both \( y_c \) and \( y_p \) into (36) knowing that we have obtained \( y_0 \), we obtain:
\[ x(0) - c = (1)^2 \]
\[ -c = 1 \]
\[ \therefore c = -1 \]

Then we substitute the value of \( c \) in (37), we get that:
\[y_p = -1\]

After obtained the values for the complimentary and particular solution, we then substitute the values into \[y_i = y_c + y_p\].

\[y_i = Ae - 1\]

In order to get the value of \(y_i\), we need to get the value of \(A\). Using our initial condition in (35), we get:

\[1 = Ae - 1\]
\[Ae = 1 + 1\]
\[A = \frac{2}{e} = e^{-2}\]

Substituting \(A = e^{-2}\) into \(y_i\), we get:

\[y_i = (e^{-2})(e) - 1\]
\[y_i = e^{-2-1} - 1\]
\[y_i = e^{-3} - 1\]

Since we have gotten the values of \(y_0\) and \(y_1\), we can now substitute in the solution which is

\[y(x, \varepsilon) = y_0 + \varepsilon y_1\]

Thus, the regular perturbative solution of this differential equation is:

\[y = 1 + \varepsilon(e^{-1} - 1)\]

**Case 3:** Find the approximate solution of \(y' = (2-x)y - \varepsilon\) with initial condition \(y\big|_{x=0} = e^{-1}\) using perturbation method.

**Solution**

Substituting \(y = y_0 + \varepsilon y_1\) in the original equation and the initial condition would be given as:

\[y_0' + \varepsilon y_1' = (2-x)(y_0 + \varepsilon y_1) - \varepsilon\]

Expanding this we have:

\[y_0' + \varepsilon y_1' = 2y_0 + 2\varepsilon y_1 - xy_0 - x\varepsilon y_1 - \varepsilon\]

If for any reason in the equation the power of \(\varepsilon\) is greater than 2, we ignore it. Therefore, we get:

\[y_0' + \varepsilon y_1' = 2y_0 + 2\varepsilon y_1 - xy_0 - x\varepsilon y_1 - \varepsilon\]  \hspace{1cm} (38)

The initial condition is given as:

\[y_0\big|_{x=0} + \varepsilon y_1\big|_{x=0} = e^{-1}\]  \hspace{1cm} (39)

Setting \(\varepsilon = 0\) in (38) and (39) respectively in order to obtain the value for \(y_0\), we get:

\[y_0' = 2y_0 - xy_0\]
\[y_0\big|_{x=0} = e^{-1}\]  \hspace{1cm} (40)

Solving for the value of \(y_0\), we get: \(m = 2 - x\)

Substituting \(m = 2 - x\), the exponential solution is given by:
\[ y_0 = Ae^{(2-x)x} \]
\[ y_0 = Ae^{(2x-x^2)} \]  

(42)

In order to obtain the value of \( y_0 \), we need to get the value of \( A \), we make use of the initial condition as given in (41).
\[ e^{-1} = Ae^{(2-x^2)} \]
\[ e^{-1} = Ae^{(2-x^2)} \]
\[ A = \frac{e^{-1}}{e} \]
\[ A = e^{-2} \]

Substituting the value of \( A \) in (42), we have:
\[ y_0 = (e^{-2})(e^{(2-x^2)}) \]
\[ y_0 = e^{-x^2+x^2-2} \]

In order to obtain the value of \( y_1 \), we differentiate (38) and (39) with respect to \( \varepsilon \). We get:
\[ y_1' = 2y_1 - xy_1 - 1 \]
\[ y_1' = (2-x)y_1 - 1 \]
\[ y_1' = (2-x)y_1 = -1 \]
\[ y_1|_{\varepsilon=0} = e^{-1} \]  

(43)

(44)

Solving for the value of \( y_1 \), we get: \( m - (2-x) = -1 \)

In this case, we have to solve for the complimentary and particular solution. For the complimentary solution \( y_c \), we get: \( m = (2-x) \)

Substituting \( m = 2-x \), the exponential solution is given by:
\[ y_c = Ae^{(2-x)x} \]
\[ y_c = Ae^{(2x-x^2)} \]

The particular solution is given as:
\[ y_p = c \]
\[ y_p' = 0 \]

Substituting \( y_p' \) and \( y_p \) in (44), we get:
\[ 0 - (2-x)c = -1 \]
\[ (x-2)c = -1 \]
\[ \therefore c = \frac{-1}{x-2} \]

After obtaining the values for the complimentary and particular solution, we then substitute the values into \( y_i = y_c + y_p \).
\[ y_i = Ae^{(2x-x^2)} + \frac{-1}{x-2} \]

(45)

In order to get the value of \( y_i \), we need to get the value of \( A \). Using our initial condition in (44), we get:
\[ e^{-1} = Ae^{2-1} + \frac{-1}{1-2} \]
\[ e^{-1} = Ae + 1 \]
\[ A = \frac{e^{-1} - 1}{e} \]
\[ A = e^{-2} - e^{-1} \]

Substituting \( A \) in (45), we get:
\[ y_1 = (e^{-2} - e^{-1})(e^{2x-x^2}) + \frac{-1}{x-2} \]
\[ y_1 = e^{2x-x^2-2} - e^{2x-x^2-1} - \frac{1}{x-2} \]

Since the values of \( y_0 \) and \( y_1 \) have been obtained, can now substitute in the solution which is
\[ y(x, \epsilon) = y_0 + \epsilon y_1. \]

Therefore, the regular perturbative solution of this differential equation is:
\[ y = e^{-x^2+x-2} + \epsilon \left( e^{2x-x^2-2} - e^{2x-x^2-1} - \frac{1}{x-2} \right). \]

4. Conclusion

The regular perturbation method was applied to three initial value problems (linear and nonlinear). Using this method to solve differential equations of first order, it is found that the method can provide numerical approximations to ordinary differential equations of any order. From the examples solved, the regular perturbation method is reliable and efficient. The major advantage of this method is the fact that it can be used to solve both linear and non-linear equations and the major disadvantage is that several and many iterations are needed in order to solve those equations.

Acknowledgement

The authors wish to acknowledge the financial support of Covenant University and also express sincere thanks for the provision of good working environment by the institution.

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