Volumetric maxima to be attained by a nonstatic black hole

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Abstract – Christodoulou and Rovelli have calculated the maximal interior volume of a Schwarzschild black hole which linearly grows with time. Recently, the entropy of the interior volume in a Schwarzschild black hole has also been calculated. In this article, the Eddington-Finkelstein metric is slightly modified. This modified metric satisfies Einstein’s equations. The interior volume of a black hole is also calculated with the modified metric. The volume explicitly depends on a function of time, different from the Christodoulou and Rovelli volume. Also entropy is calculated corresponding to the volume which is proportional to the square of a function of time and the thermodynamics is studied.

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A black hole (BH) is a central singularity at \( r = 0 \) and some singular horizon known as Cauchy horizon covered/wrapped by another singular horizon which is called event horizon. Generally, if we consider Planck’s unit, i.e., \( G = c = h = \kappa = 1 \), the radius of the event horizon in our curved space time is twice that of the mass of the central gravitating compact object. There arises an obvious question as to how much volume of a black hole is occupied inside the event horizon. If we assume the problem in flat space, the volume inside a sphere with radius \( r = 2m \) is \( \frac{4}{3}\pi (2m)^3 \). But this intuition of flat space time becomes invalid for the curved geometry inside a black hole. The volume of the surface where the time coordinate \( t \) is constant depends on the arbitrary choice of coordinates. This is discussed by various authors in different references like [1–8]. Recently, a different way of thinking about the volume inside a black hole has been suggested by Christodoulou and Rovelli [9] based on a simple observation that the exterior of the Schwarzschild metric is static but the interior is not. This leads us to the notion that volume can be time dependent. The horizon is naturally decorated by two spheres. The definition of “interior volume” is associated to the idea of single two-sphere. Let us consider a two-sphere \( S \) in Minkowski spacetime. The volume inside the two-sphere \( S \) is equal to the volume of the largest space-like spherically symmetric surface \( \Sigma \) bounded by \( S \), where \( \Sigma \) lies on the simultaneity surface determined by \( S \). This characterisation of the volume inside the two-sphere in flat spacetime also remains valid for the case of spherical black holes. By using this notion of volume inside a sphere Christodoulou and Rovelli [9] have found a simple expression for volume (CR volume hereafter), given by

\[
V(v)_{v \to \infty} = 3\sqrt{3}\pi m^2 v
\]  

by determining the maximal volume surface \( \Sigma \) when \( v \) is large enough with respect to \( m \).

After one gets the specific form of volume inside a black hole the requirement to know the physical significance of this volume is observed. Is it relevant to some information in the black hole? It is inevitable to involve Hawking radiation [10] and black-hole thermodynamics [11,12] in such studies. Recently, Zhang [13] has made some quantitative calculations to estimate the entropy associated with the CR volume. If we consider a black hole with mass \( m \), due to the emission of thermal radiation, the lifetime of the black hole becomes \( \sim m^3 \) and the CR volume inside the black hole has an extra ordinary form \( \sim m^5 \). So one may pose the query on the number of field modes included in such a large volume when radiation happens and the background geometry is also changed according to Einstein’s equation, which is actually the first law of thermodynamics.

In this letter, we slightly modify the Eddington-Finkelstein metric by including time in the lapse function,
Schwarzschild geometry. The Schwarzschild black hole in cross this surface, the line element follows the standard Spacetime is found to be flat before this surface. After we shell of energy Sandip Dutta et al. surface. Then we calculate entropy corresponding to the volume. Next, we try to determine about the volume inside the black hole and the entropy f. The partial derivative of f, where r has been studied. Here, we have to note some important evaporation in a non-commuting charged Vaidya metric +··· that mass of the BH is dinates increment is very small. So, we consider that the mass increases gradually. Because of this, we assume that the density is considered to be very small. So, the black hole’s t is taken as a parametric form, where v and t are coordinate singularity at the event horizon. So, it can be of the coordinate on the static surface is that there is no description of the geometry of the collapsed matter. γ is an affine parameter. On the horizon (r = 2m), we assume λ to vanish. λf (f for “final”) is the value of λ at r = 0.

So, the primary and the last end points of γ are defined by
\[ r(0) = 2ma(t), \quad r(\lambda_f) = 0, \quad (5) \]
\[ v(0) = v, \quad v(\lambda_f) = v_f. \quad (6) \]

λ, θ, φ are the coordinates of the surface Σ. The metric on Σ is defined by
\[ ds^2 = \{-f(r, t)\dot{v}^2 + 2\dot{v}\dot{r}\}d\lambda^2 + r^2d\theta^2 + r^2\sin^2 \theta d\phi^2, \quad (7) \]
where the differentiation with respect to λ is indicated by the “dot”. For all coordinate values, we have to maintain that \( \lambda \) is extremized by the curve. The volume is like which looks like
\[ -f(r, t)\dot{v}^2 + 2\dot{v}\dot{r} > 0. \quad (8) \]

If \( V_\Sigma[\gamma] \) is the proper volume of Σ, then it is defined as
\[ V_\Sigma[\gamma] = \int_0^{\lambda_f} \int d\lambda \int_{S^2} r^4(-f(r, t)\dot{v}^2 + 2\dot{v}\dot{r}) \sin^2 \theta. \quad (9) \]
The volume is extremized by the surface Σv. Also, the above integral is extremized by the curve. The volume is derived by the curve γv and by the particularization of v and v_f.

The Lagrangian equations of motion are
\[ L = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta} = r^4(-f(r, t)\dot{v}^2 + 2\dot{v}\dot{r}). \quad (10) \]

The considered metric takes the form
\[ ds^2_{aux} = r^4\{-f(r, t)dv^2 + 2dvdr\}. \quad (11) \]
Finding the geodesics of the above metric is equivalent to find Σv. Also, eq. (9) denotes the proper length of the geodesic in the above auxiliary metric (11) (times 4π) which is exactly the volume of Σ. Σ acts as space-like from the condition (8). This indicates that \( L > 0 \), since r ≥ 0. At r = 0, the Lagrangian is identical to zero, which is the last point for the geodesic. In \( M_{aux} \), γ is a space-like geodesic. A well-fitted parametrization is chosen as \( \lambda \), as the proper length in \( M_{aux} \). After the maximization, we set
\[ L(r, v, \dot{v}, \dot{r}) = 1 \Rightarrow r^4(-f(r, t)\dot{v}^2 + 2\dot{v}\dot{r}) = 1 \quad (12) \]
and from (9) we instantly have
\[ V = 4\pi \lambda_f. \quad (13) \]

Also, \( \xi^\mu = (\partial_\nu)\lambda^\mu \propto (1, 0) \) is the Killing vector of the metric \( g_{\alpha\beta} \). Again the inner product of ξ and its tangent \( \dot{x}^\alpha(\dot{v}, r) \) are conserved. As γ is an affinal parametrized geodesic in \( M_{aux} \), we get
\[ r^4(-f(r, t)\dot{v}^2 + 2\dot{v}\dot{r}) = A. \quad (14) \]
However, we can see from eq. (14) that \( \dot{r} \) turns to be infinite. Hence from eqs. (12) and (14), we can recast it in the following form:

\[
\dot{r} = -r^{-4}\sqrt{A^2 + r^4 f(r, t)}
\]  
(15)

(the plus sign choice in (15) would correspond to space-like geodesics outside the horizon)

\[and \quad \dot{v} = \frac{1}{A + r^4 f}. \quad (16)\]

We can see that the value of \( A \) is less than zero for the space-like geodesic. Then \( \dot{r} \) and \( \dot{v} \) are both less than zero and there exist only positive terms in (10). From eq. (15), we get

\[
\frac{V_\Sigma}{4\pi} = \lambda_f = \int_0^{2ma(t)} dr \frac{r^4}{\sqrt{A^2 + r^4 f(r, t)}}.
\]  
(17)

Equation (17) indicates some restriction on \( A \) as

\[
A^2 + r^4 f(r, t) > 0 \rightarrow A^2 > -r^4 f(r, t) = \frac{27}{16} m^4 a^4(t) = A^2_m.
\]  
(18)

By analysing the polynomial, we get the last condition. It gets a maximum value at \( r_V = \frac{3}{2} ma(t) \). Also \( r = 0 \) and \( r = 2ma(t) \) are the roots. Here, it is positive in this said range.

If the radial value is constant, eqs. (15) and (16) can be written as

\[
A^2 = -r^4 f(r, t)
\]  
(19)

and

\[\dot{v} = \frac{1}{A}. \quad (20)\]

For every constant value of \( r \), we have a solution in the range \( 0 < r < 2ma(t) \), since \(-r^4 f(r, t) > 0 \). Surfaces for the constant value of \( r \) are identically stationary (maximal) points of the volume \( (V_\Sigma r) \). By integrating eq. (20), we get

\[
\lambda_f = A(v_f - v).
\]  
(21)

Between two given \( v \) in the \( r = \) const surface, we get the largest volume when \( A \) is large, i.e., we get \( A = A_m \) for \( r = r_V \). These assumptions give the basic derivation of the asymptotic volume. The derivation is done in the remaining part of this letter.

Now, we are constructing the volume for large \( v \). \( \mathcal{S} \) is located at the point \((v, 2ma(t))\) up to \((v_f, 0)\). In that range, the proper length of the space-like geodesic should increase monotonically. The \( v \) coordinates can be simply estimated, where \( \Sigma_v \) reaches \( r = 0 \): it must be onwards the construction of the singularity, whereas the accessible volume increases for this, and we can choose \( v_f = 0 \) except for any significative error for the large-\( v \) limit. So, we choose the end point of \( \gamma \) at the coordinate \((0, 0)\).

Can we estimate the path for which the volume is maximized? The final inspection is that for maximization of \( \lambda_f \), when \( v \) is very large, the geodesic must allocate the maximum probable time for the radius \( r \). Also the line element is longer and occurs to have a maximum. So we may relate the geodesic with a final and an initial transient and an intermediate long phase (where \( \dot{r} \sim 0 \)). Hence the auxiliary line element (11) is given by

\[ds_{M_{aux}} \sim \sqrt{-r^4 f(r, t)} dv,
\]  
(22)

where the increment of \( v \) is improving the approximation. The minus sign is taken because \( dv < 0 \). To get the maximum length, we have to maximize \( ds/dv \), where the steady phase of the geodesic runs for the value of \( r \). Then we get

\[
\frac{d}{dv} \sqrt{-r^4 f(r, t)} = 0.
\]  
(23)

The solution of the above equation gives the maximum value of the polynomial \(-r^4 f(r, t)\), which already is mentioned as \( r_V \),

\[r_V = \frac{3}{2} ma(t).
\]  
(24)

So for large \( v \), we get the maximized spherically symmetric space-like surface which is constructed by a long stretch near to the constant radius \( r_V = \frac{3}{2} ma(t) \), joined from \( r = 0 \) on one side to the horizon given by \( r = 2ma(t) \) to the opposite extremity by transients. Hence,

\[
V \approx -4\pi r_V^4 f(r_V, t)v = 3\sqrt{3}\pi m^4 a^4(t)v,
\]  
(25)

it is the composite form of eqs. (13) and (21) for \( v_f = 0 \).

From fig. 1, we are trying to show that if we are considering \( A \) to tend to \( A_m \) then the total volume increases. The surface \( \Sigma \) has some regions depending on the radius of the black hole. It has two transient regions for the lower and larger value of the radius. But when the radius is equal to \( \frac{3}{2} ma(t) \), it has a long steady region. So from a small region neighboring \( r_V = \frac{3}{2} ma(t) \), we get a major contribution on the volume increasing as \( A \rightarrow A_m \).

Here, we get the volume which is modified form of CR volume which might exist inside the black hole. It is necessary to determine the number of modes of quantum field which can be combined in this volume. For the standard quantum statistical method [15], one can determine the number of quantum states of some volume for one specific phase space which are labeled by \( \lambda, \theta, \phi, p_x, p_y \) and \( p_\phi \). The uncertainty relation of quantum mechanics is stated as \( \Delta x \Delta p \sim 2\pi \), one quantum state being similar to a “cell” of volume equal to \((2\pi)^3\) in the phase space. Then the number of states is given by

\[
\frac{d\lambda d\theta d\phi dp_x dp_y dp_\phi}{(2\pi)^3}.
\]  
(26)

We have to calculate the integral of the density of states in the phase space. Therefore, we assume the massless scalar field \( \Phi \) in the spacetime with the coordinates \([13]\)

\[
ds^2 = -dT^2 + (f(r, t))c^2 + 2c^2 \dot{r}d\lambda^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\]  
(27)
This metric is equivalent to the form
\[
ds^2 = -dT^2 + H(T) d\lambda^2 + r(T)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{28}\]
which implies that the hypersurface is dynamical for the defined time \( T \) in the interior of the BH. The interior of the BH is sometimes interpreted as cosmological model for this reason [16] and it evolves to the singularity of BH. Due to the physical quantities, statistical mechanics cannot be defined clearly in the dynamical background. We must take a static background to gain the statistical property of the scalar field in the interior of the BH. According to [9] the hypersurface at constant \( r \) can be identified as static with \( v \gg m \) and \( \dot{r} = 0 \). So the hypersurface is maximized as \( r = r_{\text{max}} = \frac{2}{3} m a(t) \), where \( t \) is large enough [17]. Near the maximal hypersurface, the proper time between neighbouring hypersurfaces tends to 0. As \( t \) increases, there is no evolution found. Thus, our calculation is free from the effect of the non-static character of the metric, since it is determined with approximately \( T = \text{const} \). So we will use usual method in the curved spacetime to know the motion of the scalar field in the interior of the BH.

The scalar field \( \Phi \) can be written as
\[
\Phi = \exp \{-iET\} \exp \{i(\lambda, \theta, \phi)\} \tag{29}\]
by using the WKB approximation.

Substituting this into the Klein-Gordon equation in curved spacetime,
\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) = 0,
\]
we get
\[
E^2 - \frac{1}{f(r, t)\dot{r}^2 + 2\dot{v}r^2 \rho_\lambda^2} \frac{1}{r^2 \rho_\theta^2} - \frac{1}{r^2 \sin^2 \theta \rho_\phi^2} \rho_\phi^2 = 0, \tag{30}\]
where \( \rho_\lambda = \frac{\partial}{\partial \lambda} \), \( \rho_\theta = \frac{\partial}{\partial \theta} \), \( \rho_\phi = \frac{\partial}{\partial \phi} \). From eq. (26), the number of states with energy \((< E)\) is obtained as
\[
g(E) = \frac{1}{2(2\pi)^4} \int d\lambda d\phi d\rho_\lambda d\rho_\phi = \frac{1}{2(2\pi)^4} \int \sqrt{-f(r, t)\dot{r}^2 + 2\dot{v}r^2 \rho_\lambda^2} \frac{1}{r^2 \rho_\theta^2} \rho_\phi^2 \rho_\phi^2 dr d\theta d\phi \]
\[
\times \left( \frac{2\pi}{3} E^3 r^2 \sin \theta \right) = \frac{E^3}{12\pi^2} \left[ 4\pi \int d\lambda \sqrt{f(r, t)\dot{r}^2 + 2\dot{v}r} \right] = \frac{\sqrt{3} E^3}{4\pi} m^2 a^2(t) v. \tag{31}\]

Thus, the number of quantum states is proportional to the volume which we have obtained as \( V \). Ignoring the exotic
feature of $V$ we can continue to compute the free energy at inverse temperature $\beta$,

$$F(\beta) = \frac{1}{\beta} \int \frac{g(E)}{e^\beta E - 1} dE = -\frac{V}{12\pi^2} \int \frac{E^2 dE}{e^\beta E - 1} = -\frac{\pi^3}{20\sqrt{3}\beta^4} m^2 a^2(t)v.$$  

(32)

Also, the entropy is

$$S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{\pi^3}{5\sqrt{3}\beta^3} m^2 a^2(t)v.$$  

(33)

This entropy is calculated normally but it is inevitable considering the Hawking radiation into our calculation. For the consideration of the Hawking radiation we need the mass loss rate from the black hole which we get from the Stefan-Boltzmann law for a Schwarzschild BH,

$$\frac{dm}{dv} = -\frac{1}{\gamma m^2}, \quad \gamma > 0,$$  

(34)

where $\gamma$ is a constant which has no effect directly in the calculation. From this law we have for a black hole with mass $m$

$$v \sim \gamma m^3.$$  

(35)

It satisfies our requirement that is $v \gg m$. Therefore, we get the entropy $S$ as

$$S \sim \frac{\gamma m^2 a^2(t)}{(5\sqrt{3} \times 8^3)} = \frac{\gamma}{(10\sqrt{3} \times 8^4)\pi} A a^2(t),$$  

(36)

where we are considering the inverse temperature for a Schwarzschild black hole, $\beta = T^{-1} = 8\pi m$ and $A = 16\pi m^2$ is the surface area of a Schwarzschild black hole. It is an intriguing and surprising result that the entropy is proportional to the square of $a(t)$ and the surface area of the BH horizon that covers the volume. Also this result depends on the relation (34), which was demonstrated in [18] to hold as long as the mass of the Schwarzschild BH is greater than the Planck mass.

We can compare our metric given by eq. (2) with a standard non-commutative black hole [20] as follows:

$$\gamma \left( \frac{3}{2} r^2 \frac{d}{d\theta} \right) = \frac{r^6 t^3}{192 \theta^3} \left[ 1 - \frac{r^2 t}{2660} + \frac{r^4 t^2}{53330} - \frac{r^6 t^3}{768 \theta^3} + \cdots + \frac{r^2 n t^n}{n!4^n \pi^{2n} \theta^n} \right] = \frac{r^6 t^3}{192 \theta^3} [1 + a_1 t + a_2 t^2 + \cdots + a_n t^n],$$

where $a_1 = \frac{r^2}{2660}$, $a_2 = \frac{r^4}{53330}$, $\ldots$, $a_n = \frac{r^n}{n!4^n \pi^{2n} \theta^n}$. Then we write $f(r, t)$ of eq. (2) as

$$f(r, t) = 1 - \frac{768 \pi m \theta^3}{r^4 t^2 \sqrt{\pi}} \left( \frac{3}{2} \frac{r^2 t}{4d\theta} \right),$$

for a large black hole if we assume $\theta = r^2 t$ we get back a static symmetric trivial black-hole solution.

The entropy associated to the non-commutative volume remains insufficient for a statistical interpretation of the black-hole entropy if $v$ only is accounted for the first evaporation stage. To incorporate the second evaporation stage we will follow ref. [19]. This work deals with non-commutative spacetime from the point of view of the final explosion/evaporation stage with a diverging temperature (when $M_f \to M_0$). In this regime the approximation of the temperature can be taken as $T_h \simeq \alpha(M_f - M_0)$ with $\alpha = \frac{dM}{d\gamma}|_{r_h=r_0}$. Following the analysis of [20] we have the expression for large $V$ as

$$v \sim \frac{1}{(M_f - M_0)^3}.$$  

It is quite clear that the final evaporation stage will require an infinite time. This phenomenon will consist in the third law of thermodynamics, the statement of which claims that the zero temperature state cannot be reached with a countable number of steps or within a finite time. Following refs. [21–24] we find

$$V_{NC} \sim 3\sqrt{3}\pi M_f^2 \left( 1 - \frac{2x_m}{\sqrt{\pi}} \exp\left( \frac{-r^2}{\sqrt{\pi}} \right) \right)$$

for $v \to \infty$, when $M_f \to M_0$, the $V_{NC}$ is divergent. This is a black hole with an infinite volume wrapped by a finite horizon.

So we obtain from $S_{NC} = \frac{\pi^2 V_{NC}}{4\beta h}$ the expression of $S_{NC}$ as

$$S_{NC} \sim \eta V a^3 A_h$$

with

$$\eta = \frac{\sqrt{3}\pi^2(1 - \frac{2x_m}{\sqrt{\pi}} \exp\left( \frac{-r^2}{\sqrt{\pi}} \right))}{15 \times 16(1 - \frac{2M}{\sqrt{\pi}} \exp\left( \frac{-M^2}{\sqrt{\pi}} \right))^2} \simeq 0.05$$

when $\alpha \to \infty$; when $r_h \to r_0$, from $\frac{dM}{dr_h}|_{r_h=r_0} = 0$ we see that this expression approaches infinity as well.

So we can conclude that the non-commutative black hole can possess an infinite CR volume. This is a member of a class of black holes with finite surface, i.e., entropy, but an infinite interior. This conclusion supports the fact that the black-hole entropy may be independent of the interior of a black hole.

We have observed that the volume inside a sphere which is a spherically symmetric context is maximal for a space-like spherically symmetric 3D hypersurface covered by the sphere. Also we have calculated this proper volume for a black hole which is spherically symmetric created by a collapsed object. Also we have shown that the volume inside the black hole is defined by eq. (25). The attractive view of this result is that $V$ is large and it is proportional to the function $a(t)$ and $v$, i.e., it increases with a time function. But Christodoulou and Rovelli [9] have found that $V$ increases linearly with time since the black hole collapses.
Also we have computed the entropy in this volume with a standard statistical method. The entropy is proportional to the square of $a(t)$ and the entropy is increasing with time.

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