ON THE FUKAYA CATEGORY OF A FANO HYPERSURFACE IN PROJECTIVE SPACE

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Abstract. This paper computes part of the Fukaya category of a Fano hypersurface in projective space. The Fukaya category decomposes into components indexed by eigenvalues of the quantum cup-product with the first Chern class, and we compute one of these components (the ‘most complicated’ in a certain sense). This allows us to prove Kontsevich’s homological mirror symmetry conjecture for this component. We also consider the relationship between the quantum cohomology and the Hochschild cohomology of the Fukaya category, and use it to show it is possible to extract non-trivial information about Gromov-Witten invariants from the Fukaya category.

1. Introduction

1.1. The monotone Fukaya category and homological mirror symmetry. The central object of study in this paper is the Fukaya category of a Fano hypersurface $X \subset \mathbb{CP}^n$. Because these symplectic manifolds are monotone, one can define a particularly simple version of the Fukaya category (following [25, 26]), without appealing to the heavy machinery required for the fully general definition [12]. We call this the monotone Fukaya category, and denote it by $\mathcal{F}(X)$. Let us recall the important points here.

Firstly, we define the monotone Fukaya category to be $\mathbb{C}$-linear (no Novikov coefficient ring is necessary). Its objects are monotone Lagrangian submanifolds $L \subset X$ equipped with a brane structure (grading, spin structure, $\mathbb{C}^*$-local system), such that the image of $\pi_1(L)$ is torsion in $\pi_1(X)$. We observe that, because such Lagrangians are orientable, their minimal Maslov number is $\geq 2$. For each such $L$, the signed count of Maslov index 2 disks passing through a generic point on $L$, weighted by the monodromy of the local system around the boundary, defines a number $w(L) \in \mathbb{C}$.

For any two objects $L_0, L_1$, one defines the morphism space $\text{CF}^*(L_0, L_1)$ to be the $\mathbb{C}$-vector space generated by intersection points between $L_0$ and $L_1$ (perturbing by a Hamiltonian flow to make the intersections transverse). One defines $A_\infty$ structure maps

$$\mu^s : \text{CF}^*(L_{s-1}, L_s) \otimes \ldots \otimes \text{CF}^*(L_0, L_1) \rightarrow \text{CF}^*(L_0, L_s)$$

for $s \geq 1$ by counting pseudoholomorphic disks with boundary conditions on the $L_i$, weighted by the holonomy of the local systems around the boundary.

The $A_\infty$ relations are satisfied, with the sole exception that the differential

$$\mu^1 : \text{CF}^*(L_0, L_1) \rightarrow \text{CF}^*(L_0, L_1)$$

does not square to zero:

$$\mu^1(\mu^1(x)) = (w(L_0) - w(L_1))x.$$

To extract an honest $A_\infty$ category, we define $\mathcal{F}(X)_w$ to be the full subcategory whose objects are those $L$ with $w(L) = w$. Then each $\mathcal{F}(X)_w$ is individually an $A_\infty$ category.
There is a unital $\mathbb{C}$-algebra homomorphism from the quantum cohomology of $X$ to the endomorphism algebra of any object of $\mathcal{F}(X)_w$, which we denote by

$$\mathcal{C}O^0 : QH^*(X) \to HF^*(L,L)$$

(we define $QH^*(X)$ to be a $\mathbb{C}$-algebra, parallel with our convention for the monotone Fukaya category). Furthermore, it was proven in [4] that

$$\mathcal{C}O^0(c_1) = w \cdot e,$$

where $c_1$ is the first Chern class and $e$ is the unit. It follows that $\mathcal{F}(X)_w$ is trivial unless $w$ is an eigenvalue of quantum cup product with $c_1$ (see [4]).

Following [19], one expects the mirror to the monotone symplectic manifold $X$ to be a Landau-Ginzburg model $(X^\vee, W)$, where $X^\vee$ is a variety over $\text{Spec}(\mathbb{C})$ and $W : X^\vee \to \mathbb{C}$ is a regular function. The $A$-model on $X$ (i.e., the monotone Fukaya category) should be mirror to the $B$-model on $(X^\vee, W)$, which is Orlov’s triangulated category of singularities of the fibres of $W$, $D^b \text{Sing}(W^{-1}(w))$ (see [28]). The triangulated category of singularities is trivial if the fibre $W^{-1}(w)$ is non-singular. If $X^\vee = \text{Spec}(S)$ is affine, with $W \in S$, then the triangulated category of singularities is quasi-equivalent to the category of matrix factorizations:

$$D^b \text{Sing}(W^{-1}(w)) \cong MF(S,W - w)$$

for all $w$.

The eigenvalues of $c_1 \ast$, which index non-trivial components of the Fukaya category, should correspond to singular values of the superpotential $W$, which index fibres with non-trivial triangulated category of singularities. Homological mirror symmetry predicts quasi-equivalences of $\mathbb{Z}_2$-graded, split-closed, triangulated $A_\infty$ categories

$$D^\pi(\mathcal{F}(X)_w) \cong D^\pi \text{Sing}(W^{-1}(w))$$

for all $w$ (the superscript ‘$\pi$’ denotes the idempotent or Karoubi closure).

1.2. **Main result: HMS for a Fano hypersurface.** Let $X^n_a$ be a smooth degree-$a$ hypersurface in $\mathbb{C}P^{n-1}$ (we apologize for the awkward notation, which makes $X^n_a$ an $(n-2)$-dimensional manifold, but it really makes the formulae less complicated). It is monotone if $a \leq n-1$. Let $P \in QH^*(X^n_a)$ be the class Poincaré dual to a hyperplane. The relation satisfied by $P$ in $QH^*(X^n_a)$ is computed in [16]:

**Proposition 1.1.** [16] Corollaries 9.3 and 10.9] Define

$$p^n_a(x) := x^{a-1} - a^a x^{a-1}.$$  

Then the subalgebra of $QH^*(X^n_a)$ generated by $P$ is isomorphic to

$$\mathbb{C}[P]/p^n_a(P)$$

if $a \leq n - 2$, and

$$\mathbb{C}[P]/p^n_a(P + a!)$$

if $a = n - 1$.

**Corollary 1.2.** The eigenvalues of $P^* : QH^*(X^n_a) \to QH^*(X^n_a)$ are:

- 0 and the $(n-a)$th roots of $a^a$, if $a \leq n - 2$;
- $-a!$ and $a^n - a!$, if $a = n - 1$.

In particular, because $c_1 = (n-a)P$, the eigenvalues of $c_1 \ast$ are the same as those of $P^*$, multiplied by $(n-a)$.
This tells us where the non-trivial Fukaya categories are. Now we consider the mirror.

Let \( X^\vee := \mathbb{C}^n \), and let

\[
W_a^n := u_1 \ldots u_n + \sum_{j=1}^n u_j^a \in \mathbb{C}[u_1, \ldots, u_n].
\]

We define

\[
\Gamma_a^n := (\mathbb{Z}_a)^n / \mathbb{Z}_a,
\]

the quotient of \((\mathbb{Z}_a)^n\) by the diagonal subgroup. Its character group is

\[
(\Gamma_a^n)^* \cong \{(\zeta_1, \ldots, \zeta_n) \in (\mathbb{Z}_a)^n : \zeta_1 + \ldots + \zeta_n = 0\}.
\]

There is an obvious action of \((\Gamma_a^n)^*\) on \(\mathbb{C}^n\) by multiplying coordinates by \(a\)th roots of unity, and \(W_a^n\) is \((\Gamma_a^n)^*\)-equivariant. Therefore, we can define the \((\Gamma_a^n)^*\)-equivariant triangulated categories of singularities of the fibres of \(W_a^n\); they are quasi-equivalent to the corresponding \((\Gamma_a^n)^*\)-equivariant categories of matrix factorizations:

\[
D^b \text{Sing}_{(\Gamma_a^n)^*}((W_a^n)^{-1}(w)) \cong MF((\Gamma_a^n)^*)(W_a^n - w).
\]

An easy computation (related to computations in Proposition 2.11) shows that the critical values of \(W_a^n\) lie at roots of \(p_a^n\), up to an irrelevant constant factor. So there is a correspondence between non-singular fibres \((W_a^n)^{-1}(w)\) and non-trivial components of the Fukaya category \(\mathcal{F}(X_a^n)_{w}\).

Our main result is:

**Theorem 1.3.** If \(3 \leq a \leq n - 2\), then there is a quasi-equivalence of \(\mathbb{C}\)-linear, \(\mathbb{Z}_2\)-graded, triangulated, split-closed \(A_\infty\) categories

\[
D^a\mathcal{F}(X_a^n)_0 \cong D^a \text{Sing}_{(\Gamma_a^n)^*}((W_a^n)^{-1}(0)).
\]

If \(3 \leq a = n - 1\), then there is a quasi-equivalence

\[
D^a\mathcal{F}(X_a^n)_{-a!} \cong D^a \text{Sing}_{(\Gamma_a^n)^*}((W_a^n)^{-1}(0)).
\]

To prove Theorem 1.3, we construct a set of \(a^{n-1}\) Lagrangian spheres \(L \subset X_a^n\) (their construction is described in Section 5.1). These satisfy \(w(L) = 0\) when \(a \leq n - 2\), and \(w(L) = -a\) if \(a = n - 1\). We prove that these spheres split-generate the corresponding component of the monotone Fukaya category. This allows us to prove the following:

**Corollary 1.4.** If \(3 \leq a \leq n - 1\) and \(L \subset X_a^n\) is a monotone Lagrangian submanifold equipped with a grading, spin structure and \(\mathbb{C}^*\)-local system such that \(w(L) = 0\) (respectively, \(w(L) = -a\) in the case \(a = n - 1\)) and \(HF^*(L, L) \neq 0\), then \(L\) intersects at least one of the \(a^{n-1}\) Lagrangian spheres described above.

**Remark 1.1.** Theorem 1.3 proves that homological mirror symmetry holds for one of the components of the Fukaya category (namely, \(w = 0\) if \(a \leq n - 2\), and \(w = -a\) if \(a = n - 1\)), but does not address the other components. We will make some comments about the other components, at least in the case \(a = n - 1\), in Section 7.3. In future work, we will give a complete proof of a somewhat different version of homological mirror symmetry for these other components; as a corollary, it will be possible to remove the hypothesis on \(w(L)\) from Corollary 1.4.

**Remark 1.2.** The Fukaya category \(\mathcal{F}(X_a^n)\) can be equipped with a \(\mathbb{Z}_2(\mathbb{n-a})\)-grading (if appropriate restrictions are placed on the objects), because \(n - a\) is the minimal Chern number of spheres in \(X_a^n\). In Theorem 1.3, we simply work with the \(\mathbb{Z}_2\)-graded version of the Fukaya category, as the triangulated category of singularities is naturally \(\mathbb{Z}_2\)-graded. One could also define a category of matrix factorizations (again with certain restrictions on the objects) with a \(\mathbb{Z}_2(\mathbb{n-a})\)-grading, and prove a \(\mathbb{Z}_2(\mathbb{n-a})\)-graded version of Theorem 1.3.
1.3. Quantum cohomology of Fano hypersurfaces from the Fukaya category. An interesting feature of our calculations in the Fukaya category is that they allow us to compute non-trivial information about Gromov-Witten invariants. Namely, we give an alternative proof of Proposition 1.1, with the exception that in the case $a = n - 1$, we are only able to prove that the subalgebra is $\mathbb{C}[P]/p_a^n(P + w)$ where $w \in \mathbb{Z}$ is equal to a count of $J$-holomorphic disks with boundary on a certain Lagrangian sphere in $X^n_a$ (i.e., an open Gromov-Witten invariant). The proof uses a version of the closed-open string map from quantum cohomology to Hochschild cohomology (in fact, the $G$-graded Hochschild cohomology of the relative Fukaya category). These relations are well-known and have been computed multiple times by other methods, but nevertheless it is interesting that this information can be extracted from the Fukaya category.

Remark 1.3. We confess that there are two undetermined signs in our computations; these are irrelevant for the proof of Theorem 1.3, but become necessary when computing relations in quantum cohomology, so our re-proof of Proposition 1.1 is not complete. Nevertheless, we feel it is interesting as a proof of the concept that non-trivial information about Gromov-Witten invariants can be extracted from the Fukaya category.

We remark that, in Givental’s work on closed string mirror symmetry for Fano hypersurfaces in toric varieties (see [16, Section 10]), the extra factor of $a!$ that appears in Proposition 1.1 in the case $a = n - 1$ corresponds to an additional term $e^{-aQ}$ which has to multiply the correlators in this case, which did not appear for $a \leq n - 2$. So in the course of our alternative proof of Proposition 1.1, we equate this additional term with the open Gromov-Witten invariant $w$ mentioned above.

It is well-known, but interesting to remark, that information about genus-zero Gromov-Witten invariants can be extracted from these relations in quantum cohomology. For example, one can show that the number of genus-zero, degree-one curves on a cubic hypersurface $X^3_{n-3} \subset \mathbb{CP}^{n-1}$ (with $n \geq 5$), which send $n + 1$ fixed marked points to $n$ hyperplanes and one 2-dimensional linear subspace, is equal to 81. When $n = 4$, $X^3_4$ is the cubic surface, and one can show that the number of lines on the cubic surface is equal to 27 if and only if the open Gromov-Witten invariant $w$ mentioned above is equal to $-6$. We collect some other interesting facts about the case of the cubic surface in Section 7.

1.4. Outline of the paper. In Section 2 we collect all of the algebraic results we will need: classifications of $A_\infty$ structures and computations of various versions of Hochschild cohomology. In Section 3, we give a self-contained account of the construction of the monotone Fukaya category, as well as the closed-open and open-closed string maps, the \(A\)-inner product (a weak version of cyclicity), and the split-generation criterion of [2]. None of the material of Section 3 is due to the author, although the proof that the closed-open and open-closed string maps are dual (Proposition 3.4) is original (it avoids the need to construct a strictly cyclic structure on the Fukaya category). Furthermore, the results are far from being the most general possible, for which see [12]. Nevertheless we have included it, partly because [2] is still in preparation at the time of writing, and partly because it seems useful to collect these results together in the simplest (monotone) case, where the algebraic and analytic complications of the construction are minimal.

In Section 4 we establish a relationship between the relative Fukaya category as defined in [36] and the monotone Fukaya category as defined in Section 3, as well as the closed-open string map. In Section 5, we apply the techniques of [36] to make computations in the monotone Fukaya category of a Fano hypersurface $X^n_{a}$ via the relative Fukaya category. We compute a full subcategory $\mathcal{A}$ of (a component of) the monotone Fukaya category, and apply the split-generation criterion to prove that it split-generates.
In Section 6, we switch to making computations in the $B$-model. Again we follow [36] closely: we define a category of matrix factorizations mirror to the relative Fukaya category, compute a full subcategory $\tilde{B}$, apply the Homological Perturbation Lemma and the classification results from Section 2 to prove it is quasi-equivalent to $\tilde{A}$, and prove that it split-generates. This allows us to complete the proof of Theorem 1.3.

Finally, in Section 7, we collect a few results which are specific to the interesting special case of the cubic surface in $\mathbb{CP}^3$.

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2. Deformation theory

2.1. A classification result for $A_\infty$ algebras. Let $G$ be a grading datum, and $A = (A, \mu_0)$ a $G$-graded minimal $A_\infty$ algebra over $\mathbb{C}$. We recall [36, Definition 2.4] the definition of a $G$-graded minimal deformation of $A$ over a $G$-graded ring $R$ which is the completion of some $G$-graded polynomial ring $\tilde{R} \cong \mathbb{C}[r_1, \ldots, r_n]$ with respect to the order filtration, in the category of $G$-graded algebras. We denote by $R^j$ the part of $R$ of order $j \in \mathbb{Z}_{\geq 0}$. We recall [36, Definition 2.4] the notion of the first-order deformation class $[\mu_1] \in \text{THH}^2_G(A, A \otimes R^1)$ of a $G$-graded minimal deformation of $A$ over $R$.

Proposition 2.1. Suppose that $\text{THH}^2_G(A, A \otimes R^1) \cong 0$ for $j \geq 2$. Any two $G$-graded minimal deformations of $A$ over $R$, whose first-order deformation classes are equal, are related by a formal diffeomorphism.

Proof. Let $(A, \mu)$ and $(A, \eta)$ be two such deformations. We construct, order-by-order, a $G$-graded formal diffeomorphism $F \in TCC^1_G(A, A \otimes R)$ so that $\mu = F_0 \eta$ (we call this the $A_\infty$ relation for the purposes of the proof). We set

$$F = F_0 + F_1 + \ldots,$$

where $F_j \in TCC^1_G(A, A \otimes R^j)$.

We start with $F_0 = \text{id}$. The order-0 $A_\infty$ relation holds by definition. Now suppose, inductively, that we have constructed $F_j$ for all $j \leq k - 1$, so that

$$(\mu - F_j \eta)_j = 0 \text{ for all } j \leq k - 1.$$

We show it is possible to construct $F_k$ so that $(\mu - F_k \eta)_k = 0$.

First, note that

$$[\mu - F_k \eta, \mu + F_k \eta] = 0$$
by expanding brackets: cross-terms vanish by symmetry and the other terms vanish because \( \mu \) and \( F \eta \) are \( A_\infty \) structures. Now note that \( (\mu + F \eta)_0 = 2\mu_0 \) by definition, so the order-\( k \) part of this equation says that \( (\mu - F \eta)_k \) is a Hochschild cochain.

Now we observe that
\[
(\mu - F \eta)_k = \delta(F_k) + D_k,
\]
where \( D_k \) are the terms not involving \( F_k \). Our previous argument says that \( D_k \) is a Hochschild cochain; in fact the corresponding class
\[
[D_k] \in \text{THH}_2^G(A, A \otimes R^k)
\]
vanishes. When \( k = 1 \), this is true by assumption (the first-order deformation classes of \( \mu \) and \( \eta \) coincide); when \( k \geq 2 \), it is true because the Hochschild cohomology group vanishes by assumption.

Therefore, \( D_k \) is a Hochschild coboundary, so we can choose \( F_k \) to make the order-\( k \) \( A_\infty \) relation hold. This completes the proof, by induction. \( \square \)

2.2. \( A_\infty \) algebras of type \( A_n \). Let \( G \) be the grading datum \( G_1^\bullet \) of \cite{example} Example 2.1.13, given by
\[
Z \to Y := (\mathbb{Z} \oplus \mathbb{Z}(y_1, \ldots, y_n))/\langle 2(1 - n), y_1 + \ldots + y_n \rangle,
\]
\[
j \mapsto j \oplus 0.
\]
Let
\[
U \cong \mathbb{C}\langle u_1, \ldots, u_n \rangle
\]
be the \( G \)-graded vector space of \cite{example} Example 2.2.9, where
\[
\text{deg}(u_j) = (-1, y_j).
\]
Let \( A \) be the \( G \)-graded algebra of \cite{example} Definition 2.2.12,
\[
A \cong \Lambda^* U.
\]
Let \( V_a \) be the \( G \)-graded vector space of \cite{example} Example 2.2.10:
\[
V_a := \mathbb{C}\langle r_1, \ldots, r_n \rangle,
\]
where
\[
\text{deg}(r_j) = (2 - 2a, ay_j).
\]
By analogy with \cite{example} Example 2.2.13, we define the \( G \)-graded power series ring \( R_a \) to be the completion of
\[
\widetilde{R_a} := \mathbb{C}[V_a]
\]
in the category of \( G \)-graded algebras. In this case, we have the following

**Lemma 2.2.** If \( a \neq n \), and \( r^{e_1}, r^{e_2} \) are generators of \( R_a \), and
\[
\text{deg}(r^{e_1}) = \text{deg}(r^{e_2}) \in Y,
\]
then \( c_1 = c_2 \).

**Proof.** We have, by definition,
\[
ac_1 = ac_2 + qy[n],
\]
\[
(2 - 2a)|c_1| = (2 - 2a)|c_2| + 2(1 - n)q.
\]
Taking the dot product of the first equation with \( y[n] \) and substituting yields
\[
(2 - 2a)aq = 2a(1 - n)q
\]
\[
\Rightarrow (n - a)q = 0,
\]
So \( q = 0 \). \( \square \)
This means that
\[ R_a \cong \mathbb{C}[V_a] \]
(this actually follows from Lemma 4.1).

We recall the existence of the Hochschild-Kostant-Rosenberg map
\[ \Phi : \mathbb{C}^* (A \otimes R_a) \to R_a \otimes A \]
of [36, Definition 2.5.1].

**Definition 2.1.** We say that a (possibly curved) \( G \)-graded \( A_\infty \) algebra \( A \) over \( R_a \) has **type \( A_n^a \)** if it satisfies the following properties:

- Its underlying \( R_a \)-module and order-0 cohomology algebra is
  \[ (A, \mu^0) \cong A \otimes R_a; \]
- It satisfies
  \[ \Phi(\mu^3) = W_a^n + \mathcal{O}(r^2), \]
  where
  \[ W_a^n := -u_1 \ldots u_n + \sum_{j=1}^n r_j u_j^a \in R_a[U_a]; \]
- \( \mu^0_0 = 0 \).

Now we work out when an \( A_\infty \) algebra of type \( A_n^a \) can have a non-zero curvature \( \mu^0 \).

**Lemma 2.3.** Suppose \( n - 1 \geq a \geq 2 \) and \( n \geq 4 \), and \( A \) is an \( A_\infty \) algebra of type \( A_n^a \). Then we have:

- If \( a = n - 1 \), then \( \mu^0 \propto r^{y[n]} \);
- Otherwise, \( \mu^0 = 0 \).

**Proof.** Suppose that \( \mu^0 \) has a term proportional to \( r^c \theta^K \). We apply [36, Lemma 2.5.3, Equation (2.5.2)]. It says that
\[ y_K + ac = g y[n]. \]
Thus, for each \( k \in [n] \), we have
\[ (0 or 1) + ac_k = q. \]
Because \( a \geq 2 \), this means all \( c_k \) must be equal. Suppose they are equal to \( c \), so we have \( c = c y[n] \) and \( j = cn \). From the above equation, we have \( K = \phi \) or \( [n] \).

**Case 1:** \( K = [n] \). In this case, we have \( 1 + ac = q \). Now [36, Lemma 2.5.3, Equation (2.5.3)] yields
\[ 2 = (n - 2)(1 + ac) + (2 - a)cn \]
\[ \Rightarrow 4 = n + 2c(n - a). \]
But now the RHS is strictly larger than 4, because we have \( n \geq 4 \), \( n - a > 0 \) by assumption, and \( c > 0 \) (this last follows from the assumption that \( \mu^0_0 = 0 \) in Definition 2.1). Therefore there are no terms in \( \mu^0 \) of this form.

**Case 2:** \( K = \phi \). In this case, we have \( ac = q \). Plugging into [36, Lemma 2.5.3, Equation (2.5.3)] yields
\[ 2 = (n - 2)ac + (2 - a)cn \]
\[ \Rightarrow 2 = 2c(n - a). \]
It follows that \( c = n - a = 1 \). Therefore \( a = n - 1 \) and \( \mu^0 \propto r^{y[n]} \), as claimed. \( \square \)
Lemma 2.4. Suppose that \( n - 1 \geq a \geq 2 \) and \( n \geq 4 \), and \( A = (A \otimes R, \mu^*) \) is an \( A_\infty \) algebra of type \( A^n_a \). Then for any \( \alpha \in CC^2_{c,G}(A \otimes R)^1 \), either \( \alpha \) sends \( \theta[n] \mapsto \gamma \cdot r^c y[n] \theta[\phi] \) for some constant \( \gamma \in \mathbb{C} \), and some integer \( c \) such that \( n = 1 + 2c(n - a) \) (and \( \alpha \) vanishes on all other generators), or \( \alpha = 0 \).

Proof. Suppose that \( \alpha \) sends \( \theta_K \mapsto r^c \theta_0 \).

By \cite[Lemma 2.5.3]{36}, we have
\[
1 = t = (n - 2)q + (2 - a)j.
\]
If \( a = 2 \) then we have \( 1 = (n - 2)q \), which is impossible as \( n \geq 4 \). So we may assume \( a \geq 3 \).

Applying \cite[Lemma 2.5.3]{36} again yields
\[
ac + y_K_0 = y_K_1 + qy[n].
\]
Thus, for each \( k \in [n] \), we have
\[
a c_k = q + ( -1 \text{ or } 0 \text{ or } 1 ).
\]
Because \( a \geq 3 \), this implies that all \( c_k \) are equal. Suppose they are all equal to \( c \), so we have \( c = c y[n] \) and \( j = nc \). Therefore, we have
\[
y_K_0 - y_K_1 = (q - ac)y[n].
\]
Now observe that we have
\[
1 = t = (n - 2)q + (2 - a)cn = (n - 2)(q - ac) + 2c(n - a).
\]
We saw earlier that \( q - ac \) is equal to \(-1, 0 \text{ or } 1 \). If \( q - ac = 1 \), the right-hand side is \( \geq 2 \), so we have a contradiction. If \( q - ac = 0 \), the right-hand side is even, so we have a contradiction.

If \( q - ac = -1 \), then we have
\[
n = 1 + 2c(n - a),
\]
and \( \alpha \) must send
\[
\theta[n] \mapsto r^c y[n] \theta[\phi].
\]
\[\square\]

Next we show that the differential \( \mu^1 \) vanishes, for any \( A_\infty \) algebra of type \( A^n_a \).

Corollary 2.5. Suppose that \( n - 1 \geq a \geq 2 \) and \( n \geq 4 \), and \( A \) is a non-curved \( A_\infty \) algebra of type \( A^n_a \). Then the differential \( \mu^1 \) vanishes.

Proof. Note that
\[
\mu^1 \in CC^2_{c,G}(A \otimes R)^1.
\]
So, by Lemma 2.4, \( \mu^1(\theta^K) = 0 \) for all \( K \neq [n] \). Now choose some \( K \subset [n] \) with \( K \neq \emptyset, [n] \), and apply the \( A_\infty \) equation:

\[
\mu^1\left( \mu^2\left( \theta^K, \theta^\bar{K} \right) \right) = \mu^2(\mu^1(\theta^K), \theta^\bar{K}) + \mu^2(\theta^K, \mu^1(\theta^\bar{K}))
\]

\[
\Rightarrow \mu^1\left( \theta^K + O(r) \right) = 0
\]

Therefore, \( \mu^1 \) vanishes.

\[\square\]

**Theorem 2.6.** Suppose that \( A_1 = (A \otimes R, \mu) \) and \( A_2 = (A \otimes R, \eta) \) are two non-curved \( A_\infty \) algebras of type \( A_n^a \), where \( n - 1 \geq a \geq 2 \) and \( n \geq 4 \). Then there exists a \( G \)-graded formal diffeomorphism \( F \in \mathfrak{G}_R(A) \), such that

\[ A_1 = F \ast A_2. \]

We will prove Theorem 2.6 by applying Proposition 2.1.

**Lemma 2.7.** The 0th-order parts \( A_j \otimes_R \mathbb{C} \) (where the map \( R \to \mathbb{C} \) sends each \( r_j \) to 0) are necessarily minimal and related by a formal diffeomorphism.

**Proof.** See [36, Corollary 2.5.7].

The next step is to determine the first-order deformation space.

**Lemma 2.8.** If \( n \geq 4 \), then the vector space

\[ HH^2_G(A, A \otimes R^1_a) \]

is generated by the elements

\[ r_j u^a_j, \]

for \( j = 1, \ldots, n \).

**Proof.** See [36, Lemma 2.5.11].

The next step is to show that the higher-order deformation spaces vanish.

**Lemma 2.9.** Suppose that \( n - 1 \geq a \geq 2 \) and \( j \geq 2 \). Then we have

\[ THH^2_G(A, A \otimes R^j) \cong 0. \]

**Proof.** If \( r^c u^b \theta^K \) is a generator of \( THH^2_G(A, A \otimes R^j) \), and \( j = |c| \geq 2 \), then we have \( c \geq y_{[n]} \) (see [36, Lemma 2.5.12]), and hence \( j \geq |y_{[n]}| = n \).

Now, applying [36, Lemma 2.5.4, Equation (2.5.8)], we have \( 2(1 + q - j) = |K| \geq 0 \), so \( q - j \geq -1 \). We also have \( |b| = s = 2 - t \geq 2 \), because \( t \leq 0 \) by definition for a truncated Hochschild cochain. Applying [36, Lemma 2.5.4, Equation (2.5.6)], we have

\[
y_K + ac = qy_{[n]} + b
\]

\[
\Rightarrow |K| + aj \geq nq + 2
\]

\[
\Rightarrow |K| \geq (n - a)j + n(q - j) + 2.
\]
We split into two cases: if \( q - j = -1 \), then \(|K| = 0\), so we obtain
\[
0 \geq (n - a)j - n + 2 \\
\Rightarrow n \geq (n - a)j + 2.
\]
If, on the other hand, \( q - j \geq 0 \), then because \(|K| \leq n\), we obtain
\[
n \geq |K| \geq (n - a)j + n(q - j) + 2 \geq (n - a)j + 2.
\]
In either case, we have
\[
n \geq (n - a)j + 2 \geq n + 2,
\]
where the second inequality follows because \( n - a \geq 1 \) by assumption, and we have shown that \( j = |c| \geq n \). This is a contradiction, so the result follows.

We remark that \( HH^2_G(A, A \otimes R^j) \) has an extra generator \( r^{a_j} \) in the case \( a = n - 1 \), which we found in Lemma 2.3, but this does not correspond to a generator of truncated Hochschild cohomology because it has degree \( t = 2 > 0 \).

**Proof.** (of Theorem 2.6). Suppose we are given \( A_1 \) and \( A_2 \) of type \( A^n \), both non-curved. First, by Lemma 2.7, we can apply a formal diffeomorphism \( F \) to \( A_2 \) so that \( F^* A_2 \cong A_1 \) to order 0. Thus, we may assume without loss of generality that \( A_1 \) and \( A_2 \) are both deformations of the same \( A_\infty \) algebra \( A := A_1 \otimes_R \mathbb{C} \), over the ring \( R \).

By Corollary 2.5 these deformations are minimal. Furthermore, by Lemma 2.9 we have
\[
THH^2_G(A, A \otimes R^j) \cong 0 \quad \text{for } j \geq 2.
\]
We recall that the spectral sequence induced by the length filtration converges:
\[
THH^2_G(A, A \otimes R) \Rightarrow THH^2_G(A, A \otimes R)
\]
(see [36, Lemma 2.5.8]). Therefore,
\[
THH^2_G(A, A \otimes R^j) \cong 0 \quad \text{for } j \geq 2.
\]
It follows by Proposition 2.1 that, if the first-order deformation classes of \( A_1 \) and \( A_2 \) coincide, then there is a formal diffeomorphism \( F \) such that \( A_1 \cong F_* A_2 \).

By Lemma 2.8
\[
THH^2_G(A, A \otimes R^1) \cong \mathbb{C} \langle r_1 u_1^a, \ldots, r_n u_n^a \rangle.
\]
It follows from the convergence of the length spectral sequence as above, that \( THH^2_G(A, A \otimes R^1) \) is generated by elements of the form
\[
r_j u_j^a + \text{(lower-order in length filtration)},
\]
for \( j = 1, \ldots, n \). Because \( A_1 \) and \( A_2 \) are both of type \( A^n \), by definition their first-order deformation classes are both of the form
\[
\sum_{j=1}^n r_j u_j^a + \text{(lower-order in length filtration)}.
\]
Therefore, their first-order deformation classes coincide. The claim now follows from Proposition 2.1. \( \square \)
2.3. Computation of Hochschild cohomology. For the purposes of this section, let \( A \) be a non-curved \( A_n \) algebra of type \( A_n^0 \), where \( n - 1 \geq a \geq 2 \) and \( n \geq 4 \). We will compute two versions of the Hochschild cohomology \( HH^*(A) \). Our strategy is suggested by [35, Section 3.6] (compare also [35]). It relies on a slightly modified version of [9, Proposition 1].

Let us start by stating this modified version explicitly. Let \( G = \{ f : Z \to Y \} \) be a grading datum. A \( G \)-graded \( L_\infty \) algebra is a \( G \)-graded vector space \( g \) together with \( L_\infty \) structure maps

\[ \ell^k : g^\otimes k \to g \]

for \( k \geq 1 \), of degree \( f(2 - k) \), satisfying a system of relations (see [21]). In fact, we will only consider differential graded Lie algebras, with \( \ell^{2+3} = 0 \); nevertheless, working with \( L_\infty \) algebras helps one see why the techniques we employ make sense.

If \( g \) and \( h \) are \( G \)-graded \( L_\infty \) algebras, then a \( G \)-graded \( L_\infty \) morphism \( \Phi \) from \( g \) to \( h \) consists of maps

\[ \Phi^k : g^\otimes k \to h \]

for \( k \geq 1 \), of degree \( f(1 - k) \in Y \), satisfying a system of relations (see [20, Section 4]).

Now let \( G \oplus \mathbb{Z} \) denote the grading datum \( \{ f \oplus \text{id} : Z \to Y \oplus \mathbb{Z} \} \). Let \( g \) be a \( G \oplus \mathbb{Z} \)-graded \( L_\infty \) algebra, where the \( \mathbb{Z} \)-grading is bounded below. The \( \mathbb{Z} \)-grading on \( g \) then induces a bounded-above, decreasing filtration:

\[ F_r g := \bigoplus_{y \oplus s \in Y \oplus \mathbb{Z} \geq r} g_{y \oplus s}. \]

Let \( \hat{g} \) denote the completion of \( g \) with respect to this filtration, in the category of \( G \)-graded modules. It is a \( G \)-graded \( L_\infty \) algebra. It has a filtration \( F_r \hat{g} \), but this is not a filtration by \( L_\infty \) subalgebras: the \( L_\infty \) products send

\[ \ell^k : (F_r \hat{g})^\otimes k \to F_{2+k(r-1)} \hat{g}. \]

This means that the Maurer-Cartan equation for \( \alpha \in \hat{g}_{f(1)} \):

\[ \sum_{j \geq 1} \frac{1}{j!} \ell^j (\alpha, \ldots, \alpha) = 0, \]

does not make sense because the infinite sum may not converge. However, if we assume that \( \alpha \in F_2 \hat{g}_{f(1)} \) then the sum does make sense, because the \( k \)th term in the Maurer-Cartan equation lies in \( F_{2+k} \hat{g} \), so the terms in the sum are of successively higher and higher orders in the filtration.

If \( \alpha \in F_2 \hat{g}_{f(1)} \) is a Maurer-Cartan element, we define the \( L_\infty \) structure on \( \hat{g} \) twisted by \( \alpha \):

\[ \ell^k : \hat{g}^\otimes k \to \hat{g}, \]

\[ \ell^k_\alpha (x_1 \otimes \ldots \otimes x_k) := \sum_{j \geq 0} \frac{1}{j!} \ell^j (\alpha \otimes^j x_1 \otimes \ldots \otimes x_k) . \]

This converges, and defines a new \( G \)-graded \( L_\infty \) structure on \( \hat{g} \) (see [13, Proposition 4.4]).

Now suppose that \( g \) and \( h \) are \( G \oplus \mathbb{Z} \)-graded \( L_\infty \) algebras, and \( \Phi \) is a \( G \oplus \mathbb{Z} \)-graded \( L_\infty \) morphism from \( g \) to \( h \). Then \( \Phi \) induces a \( G \)-graded \( L_\infty \) morphism \( \hat{\Phi} \) from \( \hat{g} \) to \( \hat{h} \). If \( \alpha \in F_2 \hat{g}_{f(1)} \) is a Maurer-Cartan element, then

\[ \hat{\Phi}_* \alpha := \sum_{j \geq 1} \frac{1}{j!} \hat{\Phi}^j (\alpha, \ldots, \alpha) \in F_2 \hat{h}_{f(1)} \]
is also a Maurer-Cartan element. Furthermore, there is a $G$-graded $L_\infty$ morphism $\hat{\Phi}_\alpha$ from $(\hat{g}, \ell_\alpha)$ to $(\hat{h}, \ell_{\hat{\Phi}_\alpha})$, defined by

$$\hat{\Phi}_\alpha^k : \hat{g}^\otimes k \rightarrow \hat{h}$$

$$\hat{\Phi}_\alpha^k(x_1 \otimes \ldots \otimes x_k) := \sum_{j \geq 0} \frac{1}{j!} \hat{\Phi}_\alpha^j(\alpha^\otimes j \otimes x_1 \otimes \ldots \otimes x_k).$$

These last two claims are proven by the same argument as [9, Proposition 1]. There are two differences in our case. Firstly, we are dealing with $L_\infty$ algebras rather than dg Lie algebras: however the necessary alterations to the proofs are obvious. Secondly, we have made different assumptions on the filtrations from those in [9] (we do not have a filtration by dg Lie subalgebras). Nevertheless, because we restrict ourselves to Maurer-Cartan elements in $F_2\hat{g}$, it is easy to check that all of the infinite sums we have written down converge.

Finally, suppose that $\Phi$ is a quasi-isomorphism, i.e., the chain map

$$\Phi^1 : (g, \ell^1) \rightarrow (h, \ell^1)$$

induces an isomorphism on cohomology. Then we claim that $\hat{\Phi}_\alpha$ is also a quasi-isomorphism, i.e., the chain map

$$\hat{\Phi}_\alpha^1 : (\hat{g}, \ell^1_{\hat{\Phi}_\alpha}) \rightarrow (\hat{h}, \ell^1_{\hat{\Phi}_\alpha})$$

induces an isomorphism on cohomology. The proof uses the Eilenberg-Moore comparison theorem for spectral sequences ([10, Theorem 5.5.11]), for the spectral sequences induced by the filtrations on the complexes $(\hat{g}, \ell^1_{\hat{\Phi}_\alpha})$ and $(\hat{h}, \ell^1_{\hat{\Phi}_\alpha})$, and the morphism between them induced by $\hat{\Phi}_\alpha^1$.

The $E^1$ pages of the induced spectral sequences are $g$ and $h$ respectively, with the differentials given by $\ell^1$ on both sides, and the chain map $\Phi^1$ between them. Therefore, the $E^2$ pages are the cohomologies $H^*(g, \ell^1)$ and $H^*(h, \ell^1)$, with the map between them induced by $\Phi^1$. By assumption, this map is an isomorphism. Because these filtrations are complete and bounded above, hence exhaustive, the Eilenberg-Moore comparison theorem shows that $\hat{\Phi}_\alpha^1$ is a quasi-isomorphism.

Now let us apply this to compute the Hochschild cohomology of a non-curved $A_\infty$ algebra $A$ of type $A_n$. Let $g$ be the $G \oplus \mathbb{Z}$-graded vector space $CC^*_{G}(A \otimes R)[1]$, the compactly-supported $G$-graded Hochschild cohomology of $A$. The Gerstenhaber bracket satisfies the graded Jacobi relation, and makes $g$ into a $G \oplus \mathbb{Z}$-graded Lie algebra. The completion $\hat{g}$ is the $G$-graded Hochschild cohomology $CC^*_{G}(A \otimes R)[1]$; the filtration $F_\hat{g}$ is the length filtration (shifted by 1).

The $A_\infty$ structure maps define a Maurer-Cartan element $\mu^* \in F_0\hat{g}(f(1))$. For a general $A_\infty$ structure, we only have $\mu^* \in F_0\hat{g}(f(1))$, so this does not fit into the setup outlined above; however, because we are only dealing with a Lie algebra at the moment, there are no higher products and the Maurer-Cartan equation makes sense.

In our case, by Corollary 2.5, $A$ is minimal, so $\mu^2$ defines an associative product. In particular, $\mu^2$ is itself a Maurer-Cartan element. Twisting our Lie algebra structure by this element defines a new $L_\infty$ (in fact dg Lie) algebra with the differential $\ell^2 = [\mu^2, -]$ and $\ell^2 \hat{\Phi} \hat{\Phi}$ given by the Gerstenhaber bracket. The cohomology of $\ell^2$ is the Hochschild cohomology of the associative algebra $A \otimes R$. The remainder of the $A_\infty$ products define a Maurer-Cartan element

$$\alpha := \mu^{\geq 3} \in F_2\hat{g}(f(1)).$$
which fits into the previously-described setup. Twisting by this Maurer-Cartan element defines a new \( L_{\infty} \) (in fact dg Lie) algebra structure, with differential equal to the Hochschild differential:
\[
\ell_2^1(x) = \ell^1(x) + \ell^2(\alpha, x) = [\mu^*, x].
\]
Therefore, the cohomology of \( \ell^1_\alpha \) is the Hochschild cohomology of the \( A_\infty \) algebra \( A \).

Now observe that, by Lemmata 2.8 and 2.9, \( THH_{\infty}^2(A \otimes R)^2 \cong 0 \). It follows that one can construct an isomorphism of algebras
\[
F : (A \otimes R, \mu^2) \cong (A \otimes R, \mu_0^2).
\]
Pushing forward \( \mu^* \) by \( F \) gives us a quasi-isomorphic \( A_\infty \) algebra of type \( A^n \), where the product \( \mu^2 \) is simply the exterior product \( \mu_0^2 \). So we may assume without loss of generality that \( \mu^2 \) is the exterior product.

Now let \( h \) be the \( G \oplus \mathbb{Z} \)-graded dg Lie algebra \( R[u_1, \ldots, u_n] \otimes A \), with vanishing differential and the Schouten-Nijenhuis bracket. The HKR isomorphism \([18]\) defines a \( G \oplus \mathbb{Z} \)-graded quasi-isomorphism
\[
\Phi^1 : g \to h.
\]
By the \( R \)-linear extension of Kontsevich’s formality theorem \([20]\), this extends to a \( G \oplus \mathbb{Z} \)-graded \( L_{\infty} \) quasi-isomorphism from \( g \) to \( h \). Kontsevich’s original paper actually considered the case of a polynomial algebra, but the case of an exterior algebra is parallel. It also considered an \( L_{\infty} \) morphism in the other direction, but this implies the existence of such a \( \Phi \) because \( L_{\infty} \) quasi-isomorphisms can be inverted.

We should explain why Kontsevich’s \( L_{\infty} \) morphism is \( G \oplus \mathbb{Z} \)-graded. The morphism is \( \mathbb{Z} \)-graded by construction, where the grading on \( g \) is by length (shifted by 1) and the grading on \( h \) is by the degree of the polynomial in \( R[u_1, \ldots, u_n] \) (shifted by 1, where \( R \) and \( A \) are equipped with the zero grading). So the Taylor coefficient \( \Phi^k \) of Kontsevich’s \( L_{\infty} \) morphism \( \Phi \) has degree 1 – \( k \). Furthermore, the morphism is \( GL(\mathbb{C}^n) \)-equivariant, and in particular \( (\mathbb{C}^*)^n \)-equivariant, where \( (\mathbb{C}^*)^n \) is the subgroup of invertible diagonal \( n \times n \) matrices. Equivalently, if we equip \( \mathbb{C}^n \) with the natural \( \mathbb{Z}^n \) grading, then the maps \( \Phi^k \) have degree \( 0 \in \mathbb{Z}^n \). We define the \( G \)-grading of \( A \cong \Lambda(\mathbb{C}^n) \) by pushing forward the grading coming from the obvious \( \mathbb{Z}^n \)-grading along a morphism from \( \mathbb{Z}^n \) to \( G \). This defines a \( G \)-grading on \( CC_{\infty,\mathcal{C}}(A) \) and hence on \( g \) := \( CC_{\infty,\mathcal{C}}(A \otimes R) \) (where we recall that \( R \) has its own \( G \)-grading), and similarly on \( h \). The formality morphisms \( \Phi^k \) have degree 0 with respect to this grading.

However, we recall from \([30]\) Definition 2.3.2 that, if a Hochschild cochain of length \( s \) changes \( G \)-degree by \( y \in G \), then we equip it with the grading \( y + f(s) \in G \). It follows that, after the shift by 1, the Taylor coefficient \( \Phi^k \) of the \( L_{\infty} \) morphism \( \Phi \) has degree \( f(1 - k) \oplus (1 - k) \in G \oplus \mathbb{Z} \), with respect to the standard grading. Therefore, \( \Phi \) is a \( G \oplus \mathbb{Z} \)-graded \( L_{\infty} \) quasi-isomorphism.

It follows from our preceding discussion that \( \Phi^1_\alpha \) induces an isomorphism
\[
HH^*_G(A) \cong H^*_g(\ell_\alpha^1)
\cong H^*_h(\ell_\Phi^1(\alpha, \ell_\Phi^1_\alpha)).
\]
We will now compute \( h \) and the differential \( \ell_\Phi^1_\alpha \). The first step is to show that \( h \cong h \).

**Lemma 2.10.** The filtration \( F, h \) is complete in the category of \( G \)-graded vector spaces; in particular, \( h \cong h \).

**Proof.** It suffices to show that each graded piece \( h_y \), for \( y = (y_1, y_2) \in Y \), is finite-dimensional. Recalling the proof of \([36]\) Lemma 2.5.3], if \( r^a u^b \theta^K \) is a generator of \( R[U] \otimes A \) of degree \( (y_1, y_2) \in Y \),
then we have
\[(0, y_K) - (0, b) + ((2 - a)|c|, ac) + (s, 0) = (y_1, y_2) + q((2 - n), y_{[n]})\]
for some \(q \in \mathbb{Z}\). Hence we have, setting \(|c| = j\) and \(|b| = s\),
\[(2 - a)j + (n - 2)q + s = \text{const} \quad \text{and} \quad |K| - s + aj - nq = \text{const}.
\]
Eliminating \(q\), and observing that \(|K| \geq 0\), gives an equation of the form
\[s + (n - a)j \leq \text{const}.
\]
Observing that \(n > a\) by assumption, and both \(s\) and \(j\) are non-negative, shows that \(\mathfrak{h}_y\) is finite-dimensional. Hence, the length filtration on each graded piece of \(\mathfrak{h}\) is complete, so the length filtration is complete in the category of \(G\)-graded vector spaces. \(\square\)

Now we compute the Maurer-Cartan element \(\Phi_x \alpha\). Because \(A\) is of type \(A^n\), we know the leading-order term
\[\Phi^1(\alpha) = W^n_a \in R[U] \subset R[U] \otimes A.
\]
We claim that the higher-order terms vanish for grading reasons. To see this, we introduce the grading
\[d := s + (n - a)j\]
on \(CC_G(A \otimes R)\) and \(R[U] \otimes A\). We denote by \(\mu^*_d\) the part of \(\mu^*\) with \((s + (n - a)j) = d\). Note that \(d \geq s\). It follows from [36, Lemma 2.6, Equation (2.5.3)], with \(s + t = 2\), that if \(\mu^*_d \neq 0\), then
\[d = s + (n - a)j = 2 + (n - 2)(q - j).
\]
In particular, \(d\) must be congruent to \(2\) modulo \(n - 2\). In particular, \(\mu^*_{d^3} = 0\) for \(d < n\).

Now we recall that \(\Phi^k\) has degree \(1 - k\), which after the shifts by \(1\) means it has degree \(2 - 2k\) with respect to the length grading \(s\), and it clearly preserves \(j\), the degree in \(R\), because \(\Phi^k\) is \(R\)-multilinear. It follows that the \(d\)-grading of the \(k\)th term in \(\Phi_x \alpha\),
\[\Phi^k(\alpha, \ldots, \alpha),\]
is \(2 + (n - 2)k\). In particular, if \(k \geq 2\) then the \(d\)-grading is \(> n\). But, by Lemmata [2.8 and 2.9] and [36, Lemma 2.5.6], \(HH^2_G(A \otimes R)\) is generated by monomials \(u_1 \ldots u_n\) (with \(d = s + (n - a)j = n\), \(r_j u_j\) (with \(d = s + (n - a)j = a + (n - a) = n\), and (if \(a = n - 1\), \(r_1 \ldots r_n\), which has length \(0\) and therefore does not lie in \(F_2\)). Therefore, the higher-order terms of \(\Phi_x \alpha\) necessarily vanish for grading reasons, and we have
\[\Phi_x \alpha = \Phi^1(\alpha) = W^n_a.
\]
The differential \(\partial_{\Phi_x \alpha}\) is therefore \([W^n_a, -]\), the Schouten-Nijenhuis bracket with \(W^n_a\). This is exactly the Koszul differential associated with the sequence
\[\frac{\partial W^n_a}{\partial u_j} = -u_1 \ldots \hat{u}_j \ldots u_n + ar_j u_j^{a - 1}\]
for \(j = 1, \ldots, n\), in the ring
\[R[U] \cong \mathbb{C}[r_1, \ldots, r_n, u_1, \ldots, u_n].
\]
Using an elimination order with respect to \(r_1, \ldots, r_n\), elementary Gröbner basis theory shows that this is a regular sequence in \(R[U]\), and therefore the cohomology of the Koszul complex is simply the Jacobian ring
\[R[U] \left(\frac{\partial W^n_a}{\partial u_1}, \ldots, \frac{\partial W^n_a}{\partial u_n}\right).\]
Proposition 2.11. Let $\mathcal{A}$ be a non-curved $A_\infty$ algebra of type $A_n^\alpha$. We introduce the element

$$\beta := \left[ r_j \frac{\partial}{\partial r_j} \alpha^* \right] \in HH^*_G(\mathcal{A}|R)$$

(note that it is indeed a Hochschild cochain, as can be seen by applying $r_j \partial/\partial r_j$ to the $A_\infty$ equation $\alpha^* \circ \mu = 0$). The $R$-subalgebra of $HH^*_G(\mathcal{A}|R)$ generated by $\beta$ is isomorphic to

$$R[\beta]/p_n^\alpha(\beta),$$

where we define

$$p_n^\alpha(\beta) := \beta^{n-1} - a^n T^a - 1,$$

where we recall

$$T := r_1 \ldots r_n$$

(note that this version of $p_n^\alpha$ coincides with that used in the introduction after a substitution $T \mapsto 1$).

Proof. We recall the quasi-isomorphism

$$\hat{\Phi}^1_\alpha : (CC^*_G(\mathcal{A}), [\alpha^*, -]) \to (R[U] \otimes A, [W^a_\alpha, -]),$$

and that

$$\hat{\Phi}^1_\alpha(\mu \geq 3) = W^n_\alpha.$$

It follows that

$$\hat{\Phi}^1_\alpha(\beta) = \hat{\Phi}^1_\alpha \left( r_j \frac{\partial \alpha^*}{\partial r_j} \right)$$

$$= r_j \frac{\partial W^n_\alpha}{\partial r_j}$$

$$= r_j u^a_j,$$

because we arranged that $\mu^2 = \mu_3^2$ is independent of $r_j$. Because $\hat{\Phi}^1_\alpha$ is a quasi-isomorphism, it suffices for us to compute the $R$-subalgebra of the Jacobian ring generated by

$$\tilde{\beta} := r_j u^a_j.$$

We observe that, in the Jacobian ring, we have relations

$$u_1 \ldots u_j \ldots u_n = a r_j u^a_j.$$

In particular, we have

$$\tilde{\beta} = \frac{1}{a} u_1 \ldots u_n.$$

Now if we take the product of the relations in the Jacobian ring, we obtain

$$(u_1 \ldots u_n)^{n-1} = a^n r_1 \ldots r_n (u_1 \ldots u_n)^{a-1}.$$

Plugging our expressing for $\tilde{\beta}$ into this gives the relation

$$(a\tilde{\beta})^{n-1} = a^n T(a\tilde{\beta})^{n-1}$$

$$\Rightarrow \tilde{\beta}^{n-1} = a^n T\tilde{\beta}^{a-1}.$$

One can use Gröbner bases to show that $\tilde{\beta}$ satisfies no further relations in the Jacobian ring, and the computation is complete. □
Now, let $G^n_a$ be the grading datum of [36 Example 2.1.13]. Let $p : G^n_a \to G^n_1 =: G$ be the morphism of grading data corresponding to the morphism of pseudo-grading data given by
\[
\tilde{p}_Y(y_j) = ay_j, \\
d(y_j) = 2(1 - a)
\]
(compare [36 Lemma 3.5.11]).

Let $\mathcal{A}$ be a non-curved $A_\infty$ algebra of type $A^n_a$, and $\overline{\mathcal{A}}$ be its extension to a $G^n_1$-graded $A_\infty$ category. Consider the $G^n_a$-graded $A_\infty$ category $\tilde{\mathcal{A}} := p^* \mathcal{A}$.

It is $R$-linear, where
\[
R \cong \mathbb{C}[r_1, \ldots, r_n]
\]
is now considered as a $G^n_a$-graded ring. In light of [36 Remark 2.12], we have

**Corollary 2.12.** Let
\[
\gamma := \left[ r_j \frac{\partial \mu^*}{\partial r_j} \right] \in HH^*_G(\tilde{\mathcal{A}}).
\]
Then the $R$-subalgebra generated by $\gamma$ is isomorphic to $R[\gamma]/p_a^n(\gamma)$.

Now we define
\[
\mathcal{A}_C := (\sigma_* \mathcal{A}) \otimes_R \mathbb{C},
\]
where $\mathbb{C}$ is an $R$-algebra via the map
\[
\mathbb{C}[r_1, \ldots, r_n] \to \mathbb{C}, \\
r_j \to 1 \text{ for all } j,
\]
and $\sigma : G^n_1 \to G_\sigma$ is the sign morphism (recall $G_\sigma := \{ \mathbb{Z} \to \mathbb{Z}_2 \}$, so $G_\sigma$-graded algebra is the same as $\mathbb{Z}_2$-graded algebra).

**Lemma 2.13.** There is an isomorphism of algebras,
\[
HH^*_G(\mathcal{A}_C) \cong \frac{\mathbb{C}[u_1, \ldots, u_n]}{\left( \frac{\partial W^n_a}{\partial u_1} \ldots \frac{\partial W^n_a}{\partial u_n} \right)},
\]
where
\[
\tilde{W}^n_a := -u_1 \ldots u_n + \sum_{j=1}^n a_j u_j \in \mathbb{C}[u_1, \ldots, u_n].
\]

**Proof.** We use the same strategy as we did to make computations in $HH^*_G(p^* \mathcal{A}|R)$. Let
\[
\mathfrak{g} := CC^*_{C, G_\sigma}(A)
\]
be the $G_\sigma \oplus \mathbb{Z}$-graded dg Lie algebra, where $\ell^1 = [\mu^2_{\mathbb{C}}, -]$, where $\mu^2_{\mathbb{C}}$ is the exterior product, and $\ell^2$ is the Gerstenhaber bracket. We then have, by definition,
\[
\tilde{\mathfrak{g}} := CC^*_{C,G_\sigma}(A),
\]
and the Maurer-Cartan element $\alpha := \mu^2_{\mathbb{C}} \in F_2 \tilde{\mathfrak{g}}$. The twisted differential is, as before, the Hochschild differential:
\[
HH^*_G(\mathcal{A}_C) \cong H^*(\tilde{\mathfrak{g}}, \ell^1_\alpha).
\]
Let $\mathfrak{h} := \mathbb{C}[U] \otimes A$ be the $G_\sigma \oplus \mathbb{Z}$-graded dg Lie algebra of polyvector fields, with zero differential and the Schouten-Nijenhuis bracket. Kontsevich’s formality theorem gives a $G_\sigma \oplus \mathbb{Z}$-graded $L_\infty$ quasi-isomorphism

$$\Phi : \mathfrak{g} \to \mathfrak{h}.$$ 

We now have

$$\hat{\mathfrak{h}} := \mathbb{C}[U] \otimes A,$$

and the corresponding $G_\sigma$-graded $L_\infty$ quasi-isomorphism $\hat{\Phi}$ from $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{h}}$. The pushed-forward Maurer-Cartan element is given by

$$\hat{\Phi}_* \alpha = W_\sigma^n \otimes 1 \in (R[U] \otimes A) \otimes_R \mathbb{C},$$

because the $A_\infty$ structure on $A_\mathbb{C}$ is given by $\mu^n := \mu^\alpha \otimes 1$ by definition. It follows that

$$\hat{\Phi}_* \alpha = \hat{W}_\sigma^n \in \mathbb{C}[U].$$

It follows as before that there is an isomorphism

$$\hat{\Phi}^1 : HH^*_{G_\sigma}(A_\mathbb{C}) \cong H^*(\hat{\mathfrak{h}}, \ell_{\hat{W}_\sigma}^1).$$

The right-hand side is given by the Koszul complex for the sequence $\partial \hat{W}_\sigma^n / \partial u_j$, for $j = 1, \ldots, n$. This sequence is regular, because the ideal it generates contains powers of $u_j$ for all $j$. To see this, observe that in the quotient by this ideal, we have relations

$$u_1 \cdots \hat{u}_j \cdots u_n = au_j^{a-1},$$

and hence (taking their product)

$$(u_1 \cdots u_n)^{n-1} = a^n(u_1 \cdots u_n)^{n-1}$$

$$\Rightarrow (u_1 \cdots u_n)^{a-1}((u_1 \cdots u_n)^{n-a} - a^a) = 0$$

$$\Rightarrow (u_1 \cdots u_n)^{a-1} = 0$$

where the last step follows because the expression in brackets is invertible in $\mathbb{C}[u_1, \ldots, u_n]$. Substituting in $u_1 \cdots u_n = au_j^a$, we obtain that

$$(u_j^a)^{a-1} = 0,$$

and hence $u_j^{a(a-1)}$ lies in the ideal generated by our sequence, for all $j$.

It follows that the sequence is regular, and hence that its cohomology is exactly the Jacobian ring, as required. \hfill \Box

We observe that the cokernel of the morphism $p$ of grading data is the quotient of $(\mathbb{Z}_a)^n$ by the diagonal subgroup $\mathbb{Z}_a$, and this is isomorphic to the character group $\Gamma^a_\mathbb{Z}$ of the group $(\Gamma_\mathbb{Z})^*$ introduced in Section 1.2.

The $G$-grading on $A$ induces a $\Gamma^a_\mathbb{Z}$-grading, with respect to which $R$ has degree 0; therefore, $A_\mathbb{C}$ comes with a $\Gamma^a_\mathbb{Z}$-grading. This equips $HH^*_{G_\sigma}(A_\mathbb{C})$ with a $\Gamma^a_\mathbb{Z}$-grading (because $\Gamma^a_\mathbb{Z}$ is a finite group, the completion in the category of $\Gamma^a_\mathbb{Z}$-graded modules coincides with the completion in the category of $G_\sigma$-graded modules). A $\Gamma^a_\mathbb{Z}$-grading induces a $(\Gamma^a_\mathbb{Z})^*$-action. We will be particularly interested in computing the $(\Gamma^a_\mathbb{Z})^*$-equivariant part of $HH^*_{G_\sigma}(A_\mathbb{C})$, or equivalently the part of degree 0 in $\Gamma^a_\mathbb{Z}$.

The isomorphism of Lemma 2.13 respects the $\Gamma^a_\mathbb{Z}$-grading, where the degree of the variable $u_j$ in the Jacobian ring is $y_j$, the image of the $j$th generator in $(\mathbb{Z}_a)^n / \mathbb{Z}_a$. In order to make computations in the Jacobian ring of the formal power series ring, we first need to understand the Jacobian ring of the polynomial ring:
Lemma 2.14. Let
\[ \beta := \beta \otimes 1 \in \mathbb{C}[u_1, \ldots, u_n] \]
denote the image of \( \beta \) under the identification
\[ R[U] \otimes_R \mathbb{C} \cong \mathbb{C}[U]. \]
Then we have an isomorphism of \( \mathbb{C} \)-algebras,
\[ \left( \frac{\mathbb{C}[u_1, \ldots, u_n]}{(\partial_{W_n} \partial_{u_1} \cdots, \partial_{W_n} \partial_{u_n})} \right)^{\langle \Gamma_n^a \rangle^*} \cong \mathbb{C}[\beta]/p_n^a(\beta), \]
after the substitution \( T \mapsto 1 \) in the definition of \( p_n^a \).

Proof. We compute a Gröbner basis for the Jacobian ideal, with respect to the homogeneous lexicographic order, such that \( u_i > u_j \) iff \( i > j \). If \( K \subset [n] \), we denote
\[ u_K := \prod_{k \in K} u_k. \]
It is convenient to do a change of variables so that the Jacobian ideal is generated by elements
\[ u_{\{j\}} - u_a^{a-1}. \]
Our Gröbner basis now consists of elements
\[ u_K u_n^{a(n-1-|K|)} - u_K^{a-1} \]
for all subsets \( K \subset [n] \) such that \( K \neq [n] \), and either \( n \in K \) or \( K = \{n\} \), together with the elements
\[ u_j^{a-1} - u_n^{a-1} \]
for all \( j \neq n \).

Now we use the Gröbner basis to compute the part of the Jacobian ring in degree 0 \( \mathbb{C}[U]^{\langle \Gamma_n^a \rangle^*} \). It is easy to check that the part of \( \mathbb{C}[U] \) of degree 0 \( \mathbb{C}[U]^{\langle \Gamma_n^a \rangle^*} \) is generated (as an algebra) by elements \( u_1 \ldots u_n \) and \( u_j^a \). Furthermore, these are all identified in the Jacobian ring; so the part of the Jacobian ring of degree 0 is generated by a single element \( \gamma \), hence must have the form
\[ \mathbb{C}[\gamma]/p(\gamma) \]
for some polynomial in \( \gamma \). Now observe that, using the computations from Proposition 2.11, \( \gamma \) satisfies the relation
\[ \gamma^{n-1} = \gamma^{a-1}. \]
Therefore, the polynomial \( p \) has degree \( \leq n-1 \). If \( p \) had degree \( < n-1 \), then we would be able to write \( \gamma^{n-2} = u_n^{a(n-2)} \) in terms of lower-order terms; in particular, because it is a monomial, it would lie in the initial ideal. One can easily check that it does not; therefore the degree-0 part of the Jacobian ring is isomorphic to
\[ \mathbb{C}[\gamma]/(\gamma^{n-1} - \gamma^{a-1}). \]
Reversing the change of variables now completes the proof. \( \square \)

Corollary 2.15. Let us denote by \( \hat{\beta} \) the image of \( \beta \otimes 1 \) in \( HH_{G_a}(A_C) \), where \( \beta \in HH_{G_a}(A|R) \) is the element defined in Proposition 2.11. We have
\[ HH_{G_a}(A_C)^{\langle \Gamma_n^a \rangle^*} \cong \mathbb{C}[\hat{\beta}]/\hat{\beta}^{a-1}. \]
Proof. By Lemma 2.13, we can make computations in the equivariant part of the power series Jacobian ring. As in Lemma 2.14, we can show that \( \hat{\beta} \) generates the equivariant part. We showed in the proof of Lemma 2.13 that \( \hat{\beta} a - 1 = 0 \). Suppose that a smaller power of \( \hat{\beta} \) vanished. Then we would have

\[
\hat{\beta} a - 2 = \sum_{j=1}^{n} f_j \frac{\partial W_n}{\partial u_j},
\]

where \( f_j \in \mathbb{C}[u_1, \ldots, u_n] \). By removing all of the terms in each \( f_j \) of length \( \geq N \), this shows that \( \beta a - 2 \) is equivalent to an element of length \( \geq N \) in the polynomial Jacobian ring, for arbitrarily large \( N \); this contradicts Lemma 2.14. Therefore \( \hat{\beta} a - 2 \neq 0 \), and the proof is complete. \( \square \)

We now define the \( G_\sigma \)-graded \( A_\infty \) category

\[
\tilde{A}_C := \sigma^* \mathcal{A} \otimes_R \mathbb{C}.
\]

Corollary 2.16. Let

\[
\hat{\beta} := \tilde{\beta} \otimes 1 \in HH_{G_\sigma}(\tilde{A}_C).
\]

We have an isomorphism of \( \mathbb{Z}_2 \)-graded \( \mathbb{C} \)-algebras

\[
HH_{G_\sigma}(\tilde{A}_C)^{\Gamma_a^n} \cong \mathbb{C}[\hat{\beta}]/\hat{\beta} a - 1,
\]

where \( \hat{\beta} \) has degree \( 0 \in \mathbb{Z}_2 \).

Proof. We recall from [36, Remark 2.11] that there is an action of \( \Gamma_a^n \) on \( CC_{G_\sigma}^*(p^* \mathcal{A}) \) (by shifts on the objects), and we have

\[
CC_{G_\sigma}^*(p^* \mathcal{A})^{\Gamma_a^n} \cong CC_{G_\sigma}^*(\mathcal{A})^{(\Gamma_a^n)^*}.
\]

Using the fact that taking invariants and taking direct products commute, it follows that

\[
CC_{G_\sigma}^*(\sigma^* \mathcal{A} \otimes_R \mathbb{C})^{\Gamma_a^n} \cong CC_{G_\sigma}^*(\sigma^* \mathcal{A} \otimes_R \mathbb{C})^{(\Gamma_a^n)^*}.
\]

It follows that

\[
HH_{G_\sigma}(\tilde{A}_C)^{\Gamma_a^n} \cong HH_{G_\sigma}(\tilde{A}_C)^{(\Gamma_a^n)^*} \cong \mathbb{C}[\hat{\beta}]/\hat{\beta} a - 1
\]

by Corollary 2.15. \( \square \)

We remark that the \( \mathbb{Z}_2 \)-grading in Corollary 2.16 can be enhanced to a \( \mathbb{Z}_2(n-a) \)-grading. Let \( G_{2(n-a)} \) denote the grading datum \( \{ \mathbb{Z} \to \mathbb{Z}_{2(n-a)} \} \). Let

\[
q : G_a^n \to G_{2(n-a)}
\]

be a morphism of grading data. Note that \( q \) is not unique; it corresponds to a choice of integers \( p_j \) for \( j = 1, \ldots, n \) whose sum is \( (n - a) \). The corresponding morphism of grading data sends \( y_j \) to \( 2p_j \); because it sends \( y_{[n]} \) to \( 2(n-a) \), it is well-defined.

We assume that all \( p_j \) are divisible by \( 2(n - a) \). Then one can check that \( q^* p^* R \) is concentrated in degree 0; therefore it makes sense to define the \( G_{2(n-a)} \)-graded \( \mathbb{C} \)-linear category

\[
\tilde{A}_C' := q^* \mathcal{A} \otimes_R \mathbb{C}.
\]

Then the result of Corollary 2.16 can be upgraded to:
Corollary 2.17. There is an isomorphism of $\mathbb{Z}_{2(n-a)}$-graded $\mathbb{C}$-algebras

$$\text{HH}_{\mathbb{Z}_{2(n-a)}}\left(\mathbb{A}_C^{\ast}\right)^{\Gamma_{\ast}} \cong \mathbb{C}[\hat{\beta}] / \hat{\beta}^{a-1},$$

where $\hat{\beta}$ has degree 2.

3. The monotone Fukaya category and split-generation

This section is not original. It gives a self-contained account of the construction of the monotone Fukaya category (following [25, 26] and [34]), and the split-generation result of Abouzaid, Fukaya, Oh, Ohta and Ono [2]. By restricting ourselves to the monotone setting, the algebraic and analytic details simplify significantly (the fully general results are in [12, 2]).

3.1. Algebraic preliminaries. We recall some basic facts about $A_\infty$ categories and modules.

A $G$-graded $A_\infty$ category $\mathcal{F}$ has a set of objects $L$, with a shift operation on objects. For each pair of objects there is a $G$-graded hom-space, which we will denote $\text{CF}^{\ast}(L_0, L_1)$. We introduce the convenient notation

$$\mathcal{F}(L_s, \ldots, L_0) := \text{CF}^{\ast}(L_{s-1}, L_s) \otimes \ldots \otimes \text{CF}^{\ast}(L_0, L_1).$$

We define the Hochschild cochain complex of $\mathcal{F}$,

$$\text{CC}^\ast(\mathcal{F}) := \prod_{L_0, \ldots, L_s} \text{Hom}(\mathcal{F}(L_s, \ldots, L_0), \mathcal{F}(L_s, L_0)),$$

with the $G$-grading given by adding the length $s$ to the natural degree as a homomorphism of $G$-graded vector spaces. It has the Gerstenhaber product

$$\phi \circ \psi(a_s, \ldots, a_1) := \sum_{i+j+k=s} (-1)^{\sigma'(\psi)} \Phi^{i+k+1}(a_i+k, \ldots, a_{i+j+1}, \psi^{i}(a_{i+j}, \ldots, a_{i+1}), a_i, \ldots, a_1),$$

where we establish running notation that, for any sign $\sigma$, $\sigma' := \sigma + 1$ denotes the opposite sign, and

$$\Phi^i_j := \sum_{k=j}^i \sigma'(a_k).$$

An $A_\infty$ structure on $\mathcal{F}$ is an element $\mu^\ast \in \text{CC}^2(\mathcal{F})$ satisfying $\mu^\ast \circ \mu^\ast = 0$.

We now recall basic notions about $A_\infty$ $\mathcal{F} - \mathcal{F}$ bimodules from [33] (our conventions are identical, except that the order of inputs is reversed in all operations).

An $A_\infty$ bimodule $\mathcal{M}$ over the $A_\infty$ category $\mathcal{F}$ associates to each pair $K, L$ of objects of $\mathcal{F}$ a $G$-graded vector space $\mathcal{M}(K, L)$, together with maps

$$\mu^{k|l} : \mathcal{F}(K_k, \ldots, K_0) \otimes \mathcal{M}(L_0, K_0) \otimes \mathcal{F}(L_0, \ldots, L_l) \rightarrow \mathcal{M}(L_l, K_k)$$
For any bimodule \( M \) with structure maps \( \mu \) satisfying the associativity equation

\[
\sum_{i \leq k, j \leq l} (-1)^{\Phi_{ij}[i]} \mu^{k-1}[i-j] (a_k, \ldots, a_{i+1}, \mu^{i}[i] (a_i, \ldots, a_j, a_{i+1}, \ldots, a_l)) \\
+ \sum_{i+j \leq l} (-1)^{\Phi_{ij}[i+j]} \mu^{k}[i-j] (a_k, \ldots, \mu^{i} (a_{i+j}, \ldots, a_{i+1}, \ldots, a_l)) \\
+ \sum_{i+j \leq k} (-1)^{\Phi_{ij}[k]} \mu^{i}[k-j] (a_k, \ldots, a_i, \mu^{j} (a_{i+j}, \ldots, a_l)) = 0
\]

where

\[
\Phi_{ij}[i] := \sum_{i=j}^{l} \sigma (a_i)
\]

\[
\Phi_{ij}[i] := \sigma (m) + \sum_{i=1}^{j} \sigma (a_i) + \sum_{i=1}^{l} \sigma (a_i).
\]

One obvious \( A_\infty \) bimodule is the diagonal bimodule, \( \mathcal{F}_\Delta \):

\[
\mathcal{F}_\Delta(K, L) := CF^*(K, L), \\
\mu^{k}[1][i] := (-1)^{\Phi_{ij}[i]} \mu^{k+1}[i+1].
\]

For any bimodule \( \mathcal{M} \), we can define the shifted bimodule \( \mathcal{M}[-1] \), with

\[
\mathcal{M}[-1](K, L) := \mathcal{M}(K, L)[-1] \\
\mu^{k}[1][i] := (-1)^{\Phi_{ij}[i]} \mu^{k+1}[i+1].
\]

For any bimodule \( \mathcal{M} \), we can define the linear dual bimodule \( \mathcal{M}^\vee \), where

\[
\mathcal{M}^\vee(K, L) := \mathcal{M}(L, K)^\vee,
\]

with structure maps \( \tilde{\mu}^{k}[1][i] \) defined by

\[
\tilde{\mu}^{k}[1][i](a_k, \ldots, a_1, \alpha, a_1, \ldots, a_l)(m) := (-1)^{\tilde{\sigma} (\alpha)} (\mu^{i}[i](a_k, \ldots, a_l, m, a_k, \ldots, a_l)).
\]

where

\[
\tilde{\sigma} := \sigma (\alpha) + \Phi_{ij}[i] \cdot \sigma (\alpha) + \Phi_{ij}[i] + \sigma (m).
\]

To any \( \mathcal{F} - \mathcal{F} \) bimodule \( \mathcal{M} \), there is associated the **Hochschild cochain complex**

\[
CC^*(\mathcal{F}, \mathcal{M}) := \prod_{L_0, \ldots, L_s} \text{Hom} (\mathcal{F}(L_s, \ldots, L_0), \mathcal{M}(L_0, L_s)),
\]

with differential

\[
\delta (\alpha)(a_s, \ldots, a_l) := \sum_{i+j \leq s} (-1)^{\tilde{\sigma} (\alpha)} \Phi_{ij}[i] \mu(a_s, \ldots, \alpha (a_{s+j}, \ldots, a_i), a_i, \ldots, a_1) \\
+ \sum_{i+j \leq s} (-1)^{\tilde{\sigma} (\alpha)} \Phi_{ij}[i] \alpha(a_s, \ldots, \mu^{i}(a_{i+j}, \ldots, a_i), a_i, \ldots, a_1).
\]

Its cohomology is the Hochschild cohomology group \( HH^*(\mathcal{F}, \mathcal{M}) \). We define

\[
HH^*(\mathcal{F}) := HH^*(\mathcal{F}, \mathcal{F}_\Delta).
\]
We define the **Yoneda product**, which is a map

\[
CC^*(\mathcal{F}) \otimes CC^*(\mathcal{F}) \to CC^*(\mathcal{F}),
\]

which we denote by

\[
\phi \otimes \psi \mapsto \phi \bullet \psi,
\]

and which is defined by

\[
\sum_{0 \leq j \leq k \leq l \leq m \leq s} (-1)^j \mu^\sigma(a_s, \ldots, \phi^\beta(a_m, \ldots), a_l, \ldots, \psi^\gamma(a_k, \ldots), a_j, \ldots, a_1),
\]

where

\[
\alpha = s + 2 + l + j - m - k, \quad \beta = m - l, \quad \gamma = k - j,
\]

and

\[
\dagger = \sigma'(\phi) \cdot \mathcal{K}_1 + \sigma'\psi \cdot \mathcal{K}_1.
\]

It descends to a \(G\)-graded associative product on \(HH^*(\mathcal{F})\).

We also define the Hochschild chain complex,

\[
CC_*(\mathcal{F}, \mathcal{M}) := \bigoplus_{L_0, \ldots, L_s} \mathcal{M}(L_s, L_0) \otimes \mathcal{F}(L_s, \ldots, L_0),
\]

with the differential

\[
b(m \otimes x_s \otimes \ldots \otimes x_1) := \sum_{i+j \leq s} (-1)^j \mu^{i|s-i-j}(x_i, \ldots, x_1, m, x_s, \ldots, x_{i+j+1}) \otimes x_{i+j} \otimes \ldots \otimes x_{i+1}
\]

\[
+ \sum_{i+j \leq s} (-1)^j m \otimes x_s \otimes \ldots \otimes x_{i+j+1} \otimes \mu^j(x_{i+j}, \ldots, x_{i+1}) \otimes x_i \otimes \ldots \otimes x_1,
\]

where \(\mathcal{K}_1 := \mathcal{K}_1[\sigma(m) + \mathcal{K}_{s+1}] + \mathcal{K}_{s+1}^{i+j} \).

Its cohomology is the Hochschild homology group \(HH_*(\mathcal{F}, \mathcal{M})\). We define

\[
HH_*(\mathcal{F}) := HH_*(\mathcal{F}, \mathcal{F}_\Delta).
\]

There is an isomorphism of chain complexes

\[
CC^*(\mathcal{F}, \mathcal{M}^\vee) \cong CC_*(\mathcal{F}, \mathcal{M})^\vee.
\]

Because these are complexes of \(\mathbb{C}\)-vector spaces, we have

\[
HH^*(\mathcal{F}, \mathcal{M}^\vee) \cong HH_*(\mathcal{F}, \mathcal{M})^\vee.
\]

We now recall the notion of an \(A_\infty\) bimodule homomorphism from \(\mathcal{M}\) to \(\mathcal{N}\). It consists of maps

\[
F^{k|l} : \mathcal{F}(K_k, \ldots, K_0) \otimes \mathcal{M}(L_0, K_0) \otimes \mathcal{F}(L_0, \ldots, L_l) \to \mathcal{N}(L_l, K_k)
\]

satisfying a set of \(A_\infty\) equations. The lowest-order \(A_\infty\) equation says that

\[
F^{0|0} : (\mathcal{M}(K, L), \mu^{0|0}) \to (\mathcal{N}(K, L), \mu^{0|0})
\]

is a chain map; if it is a quasi-isomorphism, then we say that \(F\) is a quasi-isomorphism.

Given an \(A_\infty\) bimodule homomorphism \(F\) from \(\mathcal{M}\) to \(\mathcal{N}\), there is an induced map

\[
HH^*(F) : HH^*(\mathcal{F}, \mathcal{M}) \to HH^*(\mathcal{F}, \mathcal{N}),
\]

given on the cochain level by

\[
CC^*(F)(\alpha)(a_s, \ldots, a_1) := \sum_{k+l \leq s} (-1)^{\sigma(\alpha)} F^{k|l}\alpha(a_s, \ldots, a_{s-l+1}, \alpha(a_{s-l}, \ldots, a_{k+1}), a_k, \ldots, a_1).
\]

If \(F\) is a quasi-isomorphism, then \(HH^*(F)\) is an isomorphism.
Similarly, there is an induced map
\[ HH_{\ast}(F) : HH_{\ast}(\mathcal{F}, \mathcal{M}) \to HH_{\ast}(\mathcal{F}, \mathcal{N}), \]
given on the cochain level by
\[ CC_{\ast}(F)(m \otimes a_s \otimes \ldots \otimes a_1) := \sum_{k+l \leq s} (-1)^{\$} F^{k\mid 1\mid}(a_k, \ldots \otimes a_1, m, a_s, \ldots, a_s-l \otimes \ldots a_{k+1}, \]
where
\[ \$ := \mathcal{K}_k^l \cdot (\sigma(m) + \mathcal{K}_{k+1}^s) + \sigma(F) \cdot \mathcal{K}_{k+1}^{s-l} \cdot \]
If \( F \) is a quasi-isomorphism, then \( HH_{\ast}(F) \) is an isomorphism.

We recall (from [39, Definition 5.3]) that an \( \infty \)-inner-product on the \( A_{\infty} \) category \( \mathcal{F} \) is an \( A_{\infty} \) bimodule homomorphism
\[ \phi : \mathcal{F}_\Delta \to \mathcal{F}_\Delta^\vee. \]
We define an \( n \)-dimensional \( \infty \)-inner-product to be an \( A_{\infty} \) bimodule homomorphism
\[ \phi : \mathcal{F}_\Delta \to \mathcal{F}_\Delta^\vee[-n]. \]
We say that the \( \infty \)-inner-product is non-degenerate if \( \phi \) is a quasi-isomorphism. Note that a non-degenerate \( n \)-dimensional \( \infty \)-inner-product induces an isomorphism
\[ HH^\ast(\mathcal{F}) \cong HH_{\ast}(\mathcal{F})^\vee[-n]. \]
The existence of a non-degenerate \( n \)-dimensional \( \infty \)-inner-product is a weaker version of being an \( n \)-Calabi-Yau \( A_{\infty} \) category.

Now we recall that, for any \( A_{\infty} \) category \( \mathcal{F} \), we can form the \( A_{\infty} \) category of twisted complexes, \( \mathcal{F}^{tw} \), which is triangulated, and its split-closure, \( \mathcal{F}^{perf} \), which is triangulated and split-closed (see [34, Chapters 3 and 4]). For any full subcategory \( \mathcal{G} \subset \mathcal{F} \), we can consider the smallest full subcategory of \( \mathcal{F}^{perf} \) which contains \( \mathcal{G} \), is closed under quasi-isomorphism, and is triangulated and split-closed. We say that the objects of this category are split-generated by \( \mathcal{G} \). If this category is all of \( \mathcal{F}^{perf} \), we say that \( \mathcal{G} \) split-generates \( \mathcal{F} \).

We recall that an \( A_{\infty} \) left \( \mathcal{F} \)-module \( \mathcal{M} \) associates to each object \( L \) of \( \mathcal{F} \) a graded vector space \( \mathcal{M}^\ast(L) \), together with maps
\[ \mu^{k\mid 1\mid} : \mathcal{F}(L_k, \ldots, L_0) \otimes \mathcal{M}(L_0) \to \mathcal{M}(L_k) \]
satisfying an \( A_{\infty} \) relation. \( A_{\infty} \) left \( \mathcal{F} \)-modules are the objects of a differential graded category, which we denote by \( \mathcal{F}^{\text{mod}} \). Given an object \( K \) of \( \mathcal{F} \), we define a left module \( \mathcal{Y}_{K}^l \), with
\[ \mathcal{Y}_{K}^l(L) := \mathcal{F}(K, L) \]
\[ \mu^{k\mid 1\mid} := \mu^{k+1}. \]
This extends to a cohomologically full and faithful \( A_{\infty} \) embedding
\[ \mathcal{Y}^l : \mathcal{F} \to \mathcal{F}^{\text{mod}}, \]
which is the \( A_{\infty} \) version of the Yoneda embedding (see [34, Section 2g]).

Similarly, one can define the dg category of \( A_{\infty} \) right \( \mathcal{F} \)-modules, and we denote it by \( \text{mod}^{\ast} \mathcal{F} \). For each object \( K \) of \( \mathcal{F} \), there is a right-module \( \mathcal{Y}_{K}^r \), with \( \mathcal{Y}_{K}^r(L) := \mathcal{F}(L, K) \) and structure maps given by \( \mu^r \).
We can then form the $\mathcal{F} - \mathcal{F}$ bimodule $\mathcal{Y}_{K}^d \otimes \mathcal{Y}_{\tilde{K}}^d$, where

$$
\mathcal{Y}_{K}^d \otimes \mathcal{Y}_{\tilde{K}}^d(L_0, L_1) := \mathcal{F}(K, L_0) \otimes \mathcal{F}(L_1, K)
$$

and

$$
\mu^{01|0}(p \otimes q) := (-1)^{\sigma(r)(q)} \mu^1(p) \otimes q - p \otimes \mu^1(q)
$$

$$
\mu^{k|0}(a_k, \ldots, a_1, p \otimes q) := (-1)^{\sigma(r)(q)} \mu^{k+1}(a_k, \ldots, a_1, p) \otimes q
$$

$$
\mu^{0|l}(p \otimes q, a_1, \ldots, a_l) := (-1)^{\mu^{l+1}(q, a_1, \ldots, a_l)}
$$

$$
\mu^{k|l} = 0 \text{ if both } k \text{ and } l \text{ are non-zero.}
$$

There is a map $H^*(\mu)$:

$$
H^*(\mu) : HH_*(\mathcal{F}, \mathcal{Y}_{K}^d \otimes \mathcal{Y}_{\tilde{K}}^d) \to HF^*(K, K),
$$

defined on the cochain level by contracting the chain of morphisms with the $A_\infty$ structure maps:

$$
\mu((p \otimes q) \otimes a_s \otimes \ldots \otimes a_1) := (-1)^{\sigma^r(p)(q + c^{*})} \mu^{s+2}(q, a_s, \ldots, a_1, p).
$$

Now let $\mathcal{G} \subset \mathcal{F}$ be a full subcategory. Denote by $\mathcal{Y}\mathcal{G}_{K}^d$ the restriction of $\mathcal{Y}_{K}^d$ to an object of $\mathcal{G}$-mod, and similarly define $\mathcal{Y}\mathcal{G}_{\tilde{K}}^d$.

**Lemma 3.1.** ([1] Lemma 1.4) If the identity of $HF^*(K, K)$ lies in the image of the map

$$
H^*(\mu) : HH_*(\mathcal{G}, \mathcal{Y}\mathcal{G}_{K}^d \otimes \mathcal{Y}\mathcal{G}_{\tilde{K}}^d) \to HF^*(K, K),
$$
	hen $\mathcal{G}$ split-generates $K$.

### 3.2. The monotone Fukaya category

It is well-known that defining Lagrangian Floer cohomology for monotone Lagrangian submanifolds of monotone symplectic manifolds, with minimal Maslov number $\geq 2$, requires significantly less machinery than the general case [12]. The original construction is due to Oh [23, 26]. In this section, we outline the definition of the **monotone Fukaya category**, $\mathcal{F}(X)$.

Let $X$ be a monotone symplectic manifold, i.e.,

$$[\omega] = \tau c_1$$

for some $\tau > 0$. Let $\mathcal{G}X$ be the Lagrangian Grassmannian of $X$, and let $\mathcal{G}'X$ be its universal abelian cover, with covering group $H_1(\mathcal{G}X)$. There is an associated grading datum $G(X)$, given by the map

$$Z \cong H_1(\mathcal{G}X) \to H_1(\mathcal{G}X).$$

To each $w \in \mathbb{C}$, we will associate a $\mathbb{C}$-linear, $G$-graded (non-curved) $A_\infty$ category $\mathcal{F}(X)_w$.

Objects of the categories $\mathcal{F}(X)_w$ are closed monotone Lagrangian submanifolds $L \subset X$ such that the image of $\pi_1(L)$ in $\pi_1(X)$ is torsion, together with a spin structure, a lift of $L$ to $\mathcal{G}'X$ (called a grading, see [21]) and a flat $\mathbb{C}^*$-local system. Note that, because our Lagrangians are spin, they are orientable, so they have minimal Maslov number $\geq 2$. For simplicity, we will choose a finite set $\mathcal{L}$ of such Lagrangians, and define the subcategory of the monotone Fukaya category with those objects.

**Remark 3.1.** For any abelian cover $\mathcal{G}'X$ of $\mathcal{G}X$, we can define a $G'(X)$-graded monotone Fukaya category, where $G'X$ is given by the composition

$$H_1(\mathcal{G}X) \to H_1(\mathcal{G}X) \to Y,$$

where $Y$ is the covering group. Objects of this category are Lagrangian submanifolds of $X$ equipped with a lift to $\mathcal{G}'X$. For the purposes of this paper, we will always consider the universal abelian cover.
We define $\mathcal{H} := C^\infty(X, \mathbb{R})$, the space of Hamiltonian functions on $X$. For each pair of objects $L_0, L_1 \in \mathcal{L}$, we choose a one-parameter family of Hamiltonians:

$$H \in C^\infty([0; 1], \mathcal{H})$$

such that the time-1 flow of the Hamiltonian vector field associated to $H$ makes $L_0$ transverse to $L_1$ (this is one half of a Floer datum, in the terminology of [24, Section 3]). We then define the morphism space $CF^*(L_0, L_1)$ to be the $\mathbb{C}$-vector space generated by length-1 Hamiltonian chords from $L_0$ to $L_1$.

Now let $\mathcal{J}$ denote the space of almost-complex structures on $X$ compatible with $\omega$. For every object $L$, we choose an almost-complex structure $J_L \in \mathcal{J}$, and consider the moduli space $\mathcal{M}(L)$ of Maslov index 2 $J_L$-holomorphic disks with boundary on $L$, with a single marked boundary point. For generic $J_L$, the moduli space of somewhere-injective $J_L$-holomorphic disks of Maslov index 2 is regular, by standard transversality results à la [24, Section 3]. It follows from [27, 22] that any $J_L$-holomorphic disk $u$ with boundary on $L$ contains a somewhere-injective $J_L$-holomorphic disk $v$ with boundary on $L$ in its image. In particular, if $u$ has Maslov index 2, then $v$ has Maslov index $\leq 2$ by monotonicity. Because $L$ has minimal Maslov number $\geq 2$, this means we must have $u = v$, and $u$ is somewhere-injective. Therefore, $\mathcal{M}(L)$ is regular for generic $J_L$.

Standard index theory shows that the moduli space $\mathcal{M}(L)$ is a $d$-dimensional manifold, where $d$ is half the dimension of $X$, and it is compact by Gromov compactness (because the homology class of a Maslov index 2 disk cannot be expressed as a non-trivial sum of two homology classes with positive energy in $H_2(X, L)$). There is an evaluation map at the boundary marked point:

$$ev : \mathcal{M}(L) \to L.$$ 

If the $\mathbb{C}^*$-local system on $L$ is trivial, then we define $w(L) \in \mathbb{Z}$ by

$$ev_*[\mathcal{M}(L)] := w(L)[L] \in H_0(L; \mathbb{Z}).$$

If it is non-trivial, we weight the count by the monodromy of the local system around the boundary of the disk, so in general $w(L) \in \mathbb{C}$. The complex number $w(L)$ is independent of the choice of $J_L$. We furthermore require that the evaluation map is transverse to the finite set of start-points and end-points of time-1 Hamiltonian chords between $L$ and the other Lagrangians $L' \in \mathcal{L}$.

Now for each pair of objects $L_0, L_1 \in \mathcal{L}$, we choose a one-parameter family of almost-complex structures:

$$J \in C^\infty([0; 1], \mathcal{J})$$

such that $J(i) = J_{L_i}$ for $i = 0, 1$. We consider the moduli space $\mathcal{M}_0(L_0, L_1)$ of spheres of Chern number 1 which are $J_t$-holomorphic for some $t \in [0; 1]$, and are equipped with a marked point. Any $J_t$-holomorphic sphere is necessarily a branched cover of a somewhere-injective one by [24, Section 2.5], and hence any $J_t$-holomorphic sphere of Chern number 1 must be somewhere-injective, so this moduli space is regular for generic choice of $J_t$. Standard index theory shows it has dimension $2d - 1$.

Note that the parameter $t$ can be thought of as a map

$$t : \mathcal{M}_0(L_0, L_1) \to [0; 1].$$

We also have the evaluation map at the marked point,

$$ev : \mathcal{M}_0(L_0, L_1) \to X.$$ 

For any time-1 Hamiltonian chord $\gamma : [0; 1] \to X$ which starts on $L_0$ and ends on $L_1$, we consider the map

$$(\gamma \circ t, ev) : \mathcal{M}_0(L_0, L_1) \to X \times X.$$ 

We require that, for any such $\gamma$ (there are finitely many, by assumption), the image of this map avoids the diagonal in $X \times X$. This is true for generic $J_t$, because the domain has dimension $2d - 1$ and the diagonal has codimension $2d$. 


Now, the pairs \((H, J)\) constitute Floer data for the pairs of objects in our category. For any generators \(x, y\) of \(CF^*(L_0, L_1)\), we can define the corresponding moduli space \(\mathcal{M}(x, y)\) of pseudoholomorphic strips (following [34] Section 8f), with translation-invariant perturbation data given by the Floer data. For a generic choice of \(J_t\), these moduli spaces are regular by [11]. In particular, all moduli spaces of negative virtual dimension are empty. Furthermore, the moduli spaces admit an \(\mathbb{R}\)-action, by translation of the domain. It follows that the \(\mathbb{R}\)-action on a moduli space of virtual dimension 0 must be trivial. That means the only moduli spaces of virtual dimension 0 which are non-empty, are those in \(\mathcal{M}(x, x)\) which are constant along their length.

We now define the differential \(\mu^1 : CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_1)\), by counting elements of the moduli spaces \(\mathcal{M}(x, y) / \mathbb{R}\), where \(\mathcal{M}(x, y)\) has dimension 1 (we weight the count by the monodromy of the local systems around the boundary). A simple argument using monotonicity together with the fact that the image of each \(\pi_1(L_j)\) in \(\pi_1(X)\) is torsion shows that these moduli spaces have fixed energy (since they have fixed index), hence are compact. By considering the Gromov compactification of the moduli spaces \(\mathcal{M}(x, z) / \mathbb{R}\), where \(\mathcal{M}(x, z)\) has dimension 2, the standard argument shows that

\[
\mu^1(\mu^1(x)) = (w(L_0) - w(L_1))x.
\]

To see why, suppose we have a nodal strip \(u_1 \neq u_2\) in the compactification of this moduli space. If \(u_1\) and \(u_2\) are Maslov-index 1 strips, we have a broken strip contributing to the left-hand side. If \(u_1\) is a holomorphic disk of Maslov index > 2 or a holomorphic sphere of Chern number > 1, then \(u_2\) is a strip of virtual dimension < 0, hence can’t exist by regularity. If \(u_1\) is a holomorphic sphere of Chern number 1, then \(u_2\) is a strip of virtual dimension 0, hence must be a strip which is constant along its length. Our regularity assumptions for \(J_t\)-holomorphic spheres of Chern number 1 ensure that they can not bubble off a constant holomorphic strip. If \(u_1\) is a holomorphic disk of Maslov index 2, then \(u_2\) is a strip of virtual dimension 0, hence is constant along its length. The signed count of such configurations with \(u_1\) a disk on \(L_0\) is \(w(L_0)\), and the signed count with \(u_1\) a disk on \(L_1\) is \(-w(L_1)\). These are regular boundary strata, by our regularity assumptions on the almost-complex structures \(J_L\). The signed count of boundary components of this compact 1-manifold gives the result.

In particular, if \(w(L_0) = w(L_1)\), then \((\mu^1)^2 = 0\) on \(CF^*(L_0, L_1)\), so we can define \(HF^*(L_0, L_1)\) to be the cohomology of the differential \(\mu^1\).

Now let us make a consistent choice of perturbation data for all moduli spaces of holomorphic disks with \(s \geq 2\) incoming and 1 outgoing boundary punctures, following [34] Section 9i]. For a generic choice of perturbation data these moduli spaces are regular. In particular, moduli spaces of negative virtual dimension are empty. As before, these moduli spaces have fixed energy; hence are compact, and we can define the \(A_\infty\) structure maps \(\mu^s \in CC^2(\mathcal{F})\) by signed counts of the zero-dimensional components (weighted by monodromy of the local systems around the boundary). More explicitly, they are maps

\[
\mu^s : \mathcal{F}(L_s, \ldots, L_0) \rightarrow \mathcal{F}(L_s, L_0)
\]

degree \(2 - s\) (note that \(\mathcal{F}(L_s, L_0) := CF^*(L_0, L_s)\)).

By considering the boundary of one-dimensional moduli spaces as usual, we find that the \(A_\infty\) relations \((\mu^s \circ \mu^s)^s = 0\) are satisfied for all \(s \geq 2\). In particular, if a nodal pseudoholomorphic disk \(u_1 \neq u_2\) appears, and \(u_1\) is a disk of Maslov index \(\geq 2\) or a sphere of Chern number \(\geq 1\), then \(u_2\) is a disk of negative virtual dimension, hence does not exist.

The result is that, for each \(w \in \mathbb{C}\), we have a \(G\)-graded, \(\mathbb{C}\)-linear, non-curved \(A_\infty\) category \(\mathcal{F}(X)_w\), whose objects are exactly those \(L\) such that \(w(L) = w \in \mathbb{C}\).
3.3. The unit. We recall the Lagrangian version [3] of the Plunikhin-Salamon-Schwarz morphism [29]. It is a morphism of chain complexes between $C^*(L)$ and $CF^*(L,L)$, defined in degrees up to the minimal Maslov number of $L$ minus two. We will only be interested in the degree-zero part of this map, and in particular the element $e \in CF^*(L,L)$ which is the image of the identity. To define it, we consider the moduli space of pseudoholomorphic disks illustrated in Figure 1(a). There is a single outgoing boundary puncture, a single internal marked point which is unconstrained, and a single boundary marked point which is unconstrained (unconstrained marked points serve to stabilize the moduli space). Counting the zero-dimensional component of this moduli space defines $e$. Counting the boundary points of the one-dimensional component shows that $\mu^1(e) = 0$.

To see that $e$ is a unit, consider the one-parameter family of holomorphic disks in Figure 1(b) parametrized by $t \in [0,1]$. We make a consistent choice of perturbation data on this family, so that at $t = 0$ the perturbation data are invariant under $\mathbb{R}$-translation, just given by the Floer data on the strip. Counting the zero-dimensional component of the corresponding moduli space of pseudoholomorphic disks defines a map

$$H : CF^*(L,L) \to CF^*(L,L).$$

Counting the boundary points of the one-dimensional component of the moduli space shows that

$$\mu^2(x,e) = x + \mu^1(H(x)) + H(\mu^1(x)).$$

The boundary points at $t = 1$ contribute the left-hand side, the boundary points at $t = 0$ contribute the first term on the right-hand side (because the perturbation data are invariant under translation, the moduli space admits an $\mathbb{R}$-action, but the moduli space is zero-dimensional so the $\mathbb{R}$-action must be trivial, hence the strip must be constant along its length), and the remaining terms correspond to strip breaking. Unstable disk and sphere bubbling are ruled out exactly as in the definition of the Fukaya category.

It follows that right-multiplication with $e$ is homotopic to the identity. It follows similarly that left multiplication with $e$ is homotopic to the identity, and therefore that $e$ is a cohomological unit in $\mathcal{F}(X)_w$.

3.4. Quantum cohomology, $QH^*(X)$. If $X$ is a monotone symplectic manifold, then we can define the (small) quantum cohomology ring $QH^*(X)$ as in [24, Chapter 11]. It is convenient for us to use an equivalent Morse-theoretic model for quantum cohomology, so we will briefly outline the idea.
Choose a Morse function on $X$, and define

$$(QC^*(X), d) := (C^*_M(X; \mathbb{C}), d_M)$$

to be the Morse cohomology complex, and $QH^*(X)$ to be its cohomology. $QC^*(X)$ is $\mathbb{Z}$-graded by the Morse index, but we regard it as a $G$-graded cochain complex by equipping a critical point of index $j$ with degree $f(j) \in Y$, where $G = \{ f : \mathbb{Z} \to Y \}$ is the grading datum associated to $X$.

To define the quantum cup product on $QC^*(X)$, we consider a moduli space of pseudoholomorphic spheres with three marked points, at which two incoming and one outgoing semi-infinite Morse flowlines are attached. For a generic choice of perturbation data, this moduli space is regular. Counting the zero-dimensional component of this moduli space defines the quantum cup product

$$
* : QC^*(X) \otimes QC^*(X) \to QC^*(X).
\alpha \otimes \beta \mapsto \alpha \ast \beta.
$$

Counting the boundary points of the one-dimensional component of this moduli space shows that the cup product satisfies the Leibniz rule:

$$
d(\alpha \ast \beta) = d\alpha \ast \beta + (-1)^{\sigma(\alpha)} \alpha \ast d\beta.
$$

Sphere bubbling can be ruled out using monotonicity. Standard index theory shows that this is a differential $G$-graded $\mathbb{C}$-algebra. It follows that the quantum cup product descends to a $G$-graded product on $QH^*(X)$.

To prove the product is associative, we consider the moduli space illustrated in Figure 2. It consists of three one-dimensional moduli spaces glued together. On the first, a broken Morse flowline decreases in length until it shrinks to a node; on the second, we have a moduli space of holomorphic spheres with four marked points, interpolating between two nodal holomorphic spheres; and on the third, a Morse flowline increases in length from a node to a broken flowline. Counting the zero-dimensional component of this moduli space defines an operator

$$
H : QC^*(X)^{\otimes 3} \to QC^*(X).
$$

Counting the boundary points of the one-dimensional component shows that

$$
\alpha \ast (\beta \ast \gamma) - (\alpha \ast \beta) \ast \gamma = dH(\alpha, \beta, \gamma) + H(d\alpha, \beta, \gamma) + (-1)^{\sigma(\alpha)} H(\alpha, d\beta, \gamma) + (-1)^{\sigma(\alpha) + \sigma(\beta)} H(\alpha, \beta, d\gamma).
$$

It follows that the quantum cup product is associative on the level of cohomology.

3.5. The closed-open string map, $\mathcal{CO}$. In this section, we consider the closed-open string map, which relates quantum cohomology of $X$ to Hochschild cohomology of the Fukaya category of $X$ (see e.g. [12, Section 3.8.4], [13, Section 6]).

Let us fix $w \in \mathbb{C}$, and denote $\mathcal{F} := \mathcal{F}(X)_w$ to avoid notational clutter.

The closed-open string map is a $G$-graded $\mathbb{C}$-algebra homomorphism

$$
\mathcal{CO} : QH^*(X) \to HH^*(\mathcal{F}).
$$

To define $\mathcal{CO}$, we consider moduli spaces of holomorphic disks with $s \geq 0$ incoming strip-like ends, 1 outgoing strip-like end, and a single internal marked point. We choose perturbation data for these moduli spaces, and require them to be consistent with the Deligne-Mumford compactification.

We consider the corresponding moduli space of pseudoholomorphic disks, with an incoming semi-infinite Morse flowline attached at the internal marked point. These moduli spaces are regular for generic choice of perturbation data. Counting the zero-dimensional component of this moduli space defines a map

$$
\mathcal{CO} : QC^*(X) \to CC^*(\mathcal{F}).
$$
Counting the boundary points of the one-dimensional component shows that $\mathcal{CO}$ is a chain map, and hence defines a map on the level of cohomology. Unstable sphere and disk bubbling are ruled out as in the definition of the Fukaya category. Standard index theory shows that $\mathcal{CO}$ respects the $G$-grading.

To show that $\mathcal{CO}$ is a homomorphism of $\mathbb{C}$-algebras, we consider a certain subset of the moduli space of disks with two internal marked points, $s \geq 0$ incoming strip-like ends, and one outgoing strip-like end. Namely, parametrizing our disk by the unit disk in $\mathbb{C}$, we require that the outgoing boundary puncture lies at $-i$, and the internal marked points lie at $\pm t$, where $t \in [0,1]$ (see Figure 3(a)). At $t = 0$, where a node develops in the moduli space, we glue on another moduli space where the node is replaced by a Morse flowline, which increases in length from 0 (the node) to $\infty$ (a broken Morse flowline), analogously to the moduli space used to prove associativity of the quantum cup product. We choose consistent perturbation data for these moduli spaces.

We consider the corresponding moduli space of pseudoholomorphic disks with incoming semi-infinite Morse flowlines attached at the incoming marked points. Counting the zero-dimensional component of the moduli space defines a map

$$H : QC^*(X) \otimes QC^*(X) \to CC^*(\mathcal{F})$$

of degree $-1$. Counting the boundary points of the one-dimensional component shows that

$$\mathcal{CO}(\alpha \ast \beta) = \mathcal{CO}(\alpha) \bullet \mathcal{CO}(\beta) + \delta(H(\alpha, \beta)) - H(d\alpha, \beta) - (-1)^{s(\alpha)}H(\alpha, d\beta),$$

where $\ast$ is the quantum cup product, $\bullet$ is the Yoneda product, and $\delta$ is the Hochschild differential. The boundary components where the Morse flowline connecting the sphere to the disk (glued on past $t = 0$) contribute the left-hand side. The boundary components at $t = 1$ are illustrated in Figure 3(c) and contribute the first term on the right-hand side. Disk bubbling contributes the second term on the right-hand side, and breaking of the incoming Morse flowlines contributes the final two terms. It follows that $\mathcal{CO}$ is an algebra homomorphism on the level of cohomology.
Now we consider the length-zero part of the closed-open string map,

\[ \text{CO}^0 : QH^*(X) \to HF^*(L,L) \]

for each object \( L \) of \( \mathcal{F}(X)_w \). By the above argument, \( \text{CO}^0 \) is a homomorphism of \( \mathbb{C} \)-algebras. The homomorphism is obviously unital (the moduli spaces defining \( \text{CO}^0(1) \) and \( e \) count the same objects).

**Lemma 3.2.** (due to Auroux, Kontsevich and Seidel, see [4, Section 6]) If \( c_1 \in QH^*(X) \) is the first Chern class of \( TX \), then we have

\[ \text{CO}^0(c_1) = w(L) \cdot e \in HF^*(L,L). \]

**Proof.** Consider a smooth cycle \( A \subset X \setminus L \) which is Poincaré dual to the Maslov class \( \mu \in H^2(X,L) \). Because the diagram

\[ 
\begin{array}{ccc}
H^2(X,L) & \xrightarrow{\cong} & H_{2n-2}(X \setminus L) \\
\downarrow & & \downarrow \\
H^2(X) & \xrightarrow{\cong} & H_{2n-2}(X)
\end{array}
\]

commutes, and the left vertical arrow sends \( \mu \) to \( 2c_1 \), the cycle \( A \) is Poincaré dual to \( 2c_1 \) in \( X \).
Choose a Morse function on $X$ so that $A$ is the ascending manifold of a critical point $\alpha$. Then $\alpha$ is a generator of $QC^*(X)$ corresponding to $2c_1$ (it is easy to check that $QH^*(X)$ and $CO$ do not depend on the choice of Morse function, on the level of cohomology). Therefore, $CO^0(2c_1)$ is obtained by counting pseudoholomorphic disks as in Figure 4(a), with the internal marked point constrained to lie on $A$. We now consider a one-parameter family of holomorphic disks as in Figure 4(b), parametrized by $t \in [0, 1]$. We choose perturbation data on this family which coincide with those used to define $CO^0$ at $t = 0$, and which coincide with those used to define $e$ (with the constant almost-complex structure $J = J_L$ on the disk bubble) at $t = 1$. We consider the corresponding moduli space of pseudoholomorphic disks.

Counting the zero-dimensional component defines an element

$$H \in CF^*(L, L).$$

Counting the boundary points of the one-dimensional component shows that

$$CO^0(2c_1) = 2w(L) \cdot e + \mu^1(H).$$

The boundary points at $t = 0$ contribute the left-hand side (Figure 4(a)). The boundary points at $t = 1$ contribute the first term on the right-hand side (Figure 4(c)). To see why, observe that these boundary points consist of a $J_L$-holomorphic disk bubble $u_1$, together with an internal marked point of $u_1$ constrained to lie on $A$, together with a holomorphic disk $u_2$ which is an element of the moduli space used to define $e$.

Now $u_1$ can not be a constant bubble, because $A$ does not intersect $L$. Therefore it must have Maslov index $\geq 2$. If it has Maslov index $> 2$ then $u_2$ would generically not exist, so $u_1$ must have Maslov

![Figure 4. Proving Lemma 3.2](image-url)
index 2. It follows that \( u_2 \) is rigid, with output \( e \), and we must count the number of \( J_L \)-holomorphic Maslov index 2 disks \( u_1 \) with an internal marked point lying on \( A \), whose boundary marked point coincides with the boundary marked point of the disk \( u_2 \). By definition, there are \( w(L) \) such disks \( u_1 \), and for each we have a signed count of \( u_1 \cdot A = \mu(u_1) = 2 \) choices of internal marked point lying on \( A \). So the contribution of the boundary points at \( t = 1 \) is exactly \( 2w(L) \cdot e \).

The remaining boundary points correspond to breaking off a strip, and contribute the remaining term. The equation implies the result. □

Now let us consider the map

\[ c_1^* : QH^* (X) \to QH^* (X) \]

of quantum cup product with \( c_1 \). Denote the set of eigenvalues of \( c_1^* \) by \( \Lambda \). Let

\[ QH^* (X) \cong \bigoplus_{w \in \Lambda} QH^*(X)_w \]

be the decomposition of \( QH^* (X) \) into generalized eigenspaces of \( c_1^* \), and let

\[ e = \sum_{w \in \Lambda} e_w \]

be the corresponding decomposition of the identity. Then

\[ e_w^* : QH^* (X) \to QH^* (X) \]

is the projection onto the generalized eigenspace \( QH^* (X)_w \).

**Lemma 3.3.** (see [3] Proposition 6.8) The restricted map

\[ CO^0 : QH^* (X)_w \to HF^* (L, L) \]

vanishes if \( w(L) \neq w \), and is a unital homomorphism of \( \mathbb{C} \)-algebras if \( w(L) = w \).

**Proof.** If \( \alpha \in QH^* (X)_w \), then we have \( (c_1 - w)^k \star \alpha = 0 \) for some \( k \). By Lemma 3.2 this means that

\( (w(L) - w)^k CO^0(\alpha) = 0 \).

Hence, if \( w(L) \neq w \), we must have \( CO^0(\alpha) = 0 \). Because \( CO^0 \) is unital, and kills all \( e_w \) for \( w \neq w(L) \), it follows that the restriction to \( QH^* (X)_{w(L)} \) is unital. □

It follows, in particular, that \( F(X)_w \) is trivial unless \( w \) is an eigenvalue of \( c_1^* \).

3.6. **The open-closed string map, **\( OC \). In this section, we consider the open-closed string map, which relates quantum cohomology to Hochschild homology (see e.g. [12] Section 3.8.1, [1] Section 5.3).

The open-closed string map is a map of \( G \)-graded \( \mathbb{C} \)-vector spaces

\[ OC : HH_*(F) \to QH^{n+*}(X) \]

(where \( X \) has real dimension \( 2n \)). It is defined by considering moduli spaces of pseudoholomorphic disks with \( s \geq 1 \) incoming strip-like ends, and an internal marked point. We choose consistent perturbation data for these moduli spaces.

We consider the corresponding moduli space of pseudoholomorphic disks, with an outgoing semi-infinite Morse flowline attached at the internal marked point. Counting the zero-dimensional component of these moduli spaces defines the map

\[ OC : CC_*(F) \to QC^{n+*}(X). \]
3.7. The $\infty$-inner product on the Fukaya category. We now explain how the monotone Fukaya category of $X$ can be equipped with a non-degenerate $n$-dimensional $\infty$-inner-product $\phi$, where $n$ is half of the real dimension of $X$ (following [31, Section 12]), where this structure is called an ‘$n$-dimensional weak Calabi-Yau structure’). To define $\phi$, we consider a subset of the moduli space of disks with $k + l + 2$ incoming boundary punctures, and an internal marked point. We label the boundary punctures $p_{\text{out}}, q_1, \ldots, q_k, p_{\text{in}}, q_{k+1}, \ldots, q_{k+l}$ in order around the boundary, and consider the moduli space of disks such that, if we parametrize the disk as the unit disk in $\mathbb{C}$, then $p_{\text{in}}$ lies at $-1$, $p_{\text{out}}$ lies at $+1$, and the internal marked point lies at $0$ (see Figure 5(a)). We make a consistent choice of perturbation data for this moduli space, and consider the corresponding moduli space of pseudoholomorphic disks. Note that there is no constraint placed on the marked point.

Counting the zero-dimensional component of this moduli space defines a map

$$\phi^{k | 1 | l} : \mathcal{F}(K_k, \ldots, K_0) \otimes \mathcal{F}(K_0, L_0) \otimes \mathcal{F}(L_0, \ldots, L_l) \to \mathcal{F}(K_k, L_l)^{\vee}$$

Standard index theory shows it is $G$-graded. Counting the boundary points of the one-dimensional component shows that $\mathcal{O}$ is a chain map. It follows that $\mathcal{O}$ gives a well-defined map on the level of homology.

Remark 3.2. In fact, $HH_*(\mathcal{F})$ is naturally a $HH^*(\mathcal{F})$-module, and hence, via $\mathcal{O}$, a $QH^*(X)$-module, and the map $\mathcal{O}$ is a homomorphism of $QH^*(X)$-modules (the analogous result for the wrapped Fukaya category of an exact symplectic manifold was proven by Ganatra [14], see also [30]).
of degree $n - k - l$. Explicitly, the puncture $p_{\text{out}}$ corresponds to the output in
\[ CF^*(L_l, K_k)^\vee \cong \mathcal{F}(K_k, L_l)^\vee, \]
the marked points $q_1, \ldots, q_k$ correspond to generators of $\mathcal{F}(K_k, \ldots, K_0)$, the marked point $p_{\text{in}}$ corresponds to a generator of $\mathcal{F}(K_0, L_0)$, and the marked points $q_{k+1}, \ldots, q_{k+l}$ correspond to generators of $\mathcal{F}(L_0, \ldots, L_l)$. Counting the boundary points of the one-dimensional component of this moduli space shows that $\phi^{k|1|l}$ satisfies the $A_\infty$ bimodule homomorphism equations, and therefore defines an $A_\infty$ bimodule homomorphism
\[ \phi : \mathcal{F} \to \mathcal{F}^\vee[-n]. \]
To show that $\phi$ is a quasi-isomorphism, we must show that
\[ \phi^{0|1|0} : CF^*(K, L) \to CF^*(K, L)^\vee \]
is a quasi-isomorphism for all $K, L$. In fact, we can regard $\phi^{0|1|0}$ as a continuation map from $CF^*(K, L)$ (defined using Floer datum $(H, J)$) to $CF^*(K, L)$ (defined using Floer datum $(-H, J)$), so we can apply the usual argument showing that continuation maps are quasi-isomorphisms. This quasi-isomorphism can be thought of as a version of Poincaré duality for the Fukaya category.

**Proposition 3.4.** The bimodule quasi-isomorphism $\phi$ defines an isomorphism
\[ HH^*(\phi) : HH^*(\mathcal{F}) \to HH^*(\mathcal{F}, \mathcal{F}^\vee[-n]) \cong HH_*(\mathcal{F})^\vee[-n], \]
We have
\[ HH^*(\phi) \circ CO \circ PD = OC^\vee, \]
where
\[ PD : QH^*(X)^\vee \to QH^*(X) \]
denotes Poincaré duality.

**Proof.** We consider a moduli space of holomorphic disks with $s + 1 \geq 1$ incoming boundary punctures $p_0, \ldots, p_s$ and two internal marked points, so that if we parametrize the disk as the unit disk in $\mathbb{C}$, the first internal marked point lies at 0, the second lies at $t \in [0, 1]$, and $p_0$ lies at +1 (see Figure 5(b)). We make a consistent choice of perturbation data for these moduli spaces, and consider the corresponding moduli space of pseudoholomorphic disks, with an incoming Morse flowline attached at the first internal marked point (and no constraint on the second internal marked point). Counting the one-dimensional component of these moduli spaces defines a map
\[ H : QC^*(X) \to CC_*(\mathcal{F})^\vee. \]
Counting the one-dimensional component of these moduli spaces shows that
\[ OC^\vee = CC^*(\phi) \circ CO \circ PD + \delta^\vee \circ H - H \circ d, \]
as maps from $QC^*(X) \to CC_*(\mathcal{F})^\vee \cong CC^*(\mathcal{F}, \mathcal{F}^\vee)$, Here $\delta^\vee$ is the dual of the Hochschild differential. The boundary points at $t = 0$ correspond to the left-hand side, the boundary points at $t = 1$ correspond to the first term on the right-hand side (illustrated in Figure 5(c)), disk bubbling for $0 < t < 1$ contributes the second term on the right-hand side, and breaking of the Morse flowline contributes the final term. This proves the result. \(\square\)

**Remark 3.3.** Proposition 3.4 is closely related to the result that $OC$ is a homomorphism of $QH^*(X)$-modules (see [14, Proposition 5.4] and [30, Theorem 6.2]). Namely, the $\infty$-inner product
\[ \phi \in \text{Hom}_{F-F}(\mathcal{F}, \mathcal{F}^\vee[-n]) \cong HH_*(\mathcal{F})^\vee \]
can be identified on the chain level with the class $2OC^\vee(1)$, in the notation of [14, Section 5.6]. Then, because $OC$ is a homomorphism of $QH^*$-modules, so is $OC^\vee$, so
\[ OC^\vee(\alpha) = CO(\alpha) \cap OC^\vee(1) \]
which amounts to a proof of Proposition 3.4.

3.8. The split-generation criterion. In this section, we summarize a criterion for split-generating the Fukaya category, which is due to Abouzaid, Fukaya, Oh, Ohta and Ono [2]. We follow [1] closely, which proved the analogous result for the wrapped Fukaya category. We remark that the proof is particularly simple in the case of a closed monotone symplectic manifold, because one does not have to deal with weights on the strip-like ends (as in the wrapped Fukaya category), and it is easier to ensure transversality of our moduli spaces in the monotone setting.

For any object $K$ of $\mathcal{F}$, we define a map of $\mathcal{F}$ bimodules

$$\Delta : \mathcal{F}_\Delta \to \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r$$

of degree $n$. To define $\Delta$, we consider moduli spaces of holomorphic disks with $k+l+3$ boundary punctures, labelled $p_{\text{in}}, q_1, \ldots, q_k, p_{\text{out}}^l, p_{\text{out}}^r, q_{k+1}, \ldots, q_{k+l}$ as in Figure 6(a). We make a consistent choice of perturbation data and consider the corresponding moduli spaces of pseudoholomorphic disks (the boundary component between $p_{\text{out}}^l$ and $p_{\text{out}}^r$ is labelled $K$).

Counting the zero-dimensional component of the moduli space defines the map $\Delta^{k|l|} : \mathcal{F}(K_k, \ldots, K_0) \otimes \mathcal{F}(L_0, \ldots, L_l) \to \mathcal{F}(K, K_l) \otimes \mathcal{F}(K, K)$. Counting the boundary points of the one-dimensional component of the moduli space shows that $\Delta$ is an $A_\infty$ bimodule homomorphism.

Hence it defines a map

$$HH_* (\Delta) : HH_* (\mathcal{F}, \mathcal{F}_\Delta) \to HH_* (\mathcal{F}, \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r).$$

Lemma 3.5. The following diagram commutes up to a sign $(-1)^{\frac{n(n+1)}{2}}$:

$$\begin{array}{ccc}
HH_* (\mathcal{F}) & \xrightarrow{\mathcal{O}C} & QH^*(X) \\
\downarrow HH_* (\Delta) & & \downarrow CC^0 \\
HH_* (\mathcal{F}, \mathcal{Y}_K^l \otimes \mathcal{Y}_K^r) & \xrightarrow{H^* (\mu)} & HF^*(K, K).
\end{array}$$

Proof. We consider a moduli space of holomorphic annuli which can be parametrized as the region between the unit circle and the circle of radius $r \in [1, \infty)$ in $\mathbb{C}$, with $s+1 \geq 1$ incoming boundary punctures $p_0, \ldots, p_s$ on the outer boundary component, with $p_0$ sitting at $-r$, and one outgoing boundary puncture on the inner boundary component, sitting at 1 (see Figure 6(b)). We make a consistent choice of perturbation data, and consider the corresponding moduli spaces of pseudoholomorphic annuli. As $r \to \infty$, the annulus degenerates to two disks connected at a node. At these boundary points, we glue in a constant Morse flowline connecting the two disks, then allow its length to grow to $\infty$.

Counting the zero-dimensional component of this moduli space defines a map $H : CC_* (\mathcal{F}) \to \mathcal{F}(K, K)$.

Counting boundary points of the one-dimensional component of the moduli space shows that

$$\mathcal{C}C^0 \circ \mathcal{O}C = \left(-1\right)^{\frac{n(n+1)}{2}} \mu \circ CC_* (\Delta) + H \circ \delta + \left(-1\right)^n \mu^1 \circ H.$$
remaining terms on the right-hand side correspond to internal breaking of the moduli space. This equation shows that the diagram commutes (up to the sign) on the level of cohomology.

\[ \text{□} \]

**Corollary 3.6.** \([2]\) If \( G \subset F(X) \) is a full subcategory, \( K \) is another object of \( F(X) \), and if the map

\[ CO^0 \circ OC : HH_*(G) \to HF^*(K, K) \]

contains the identity in its image, then \( K \) is split-generated by \( G \).

**Proof.** Follows immediately from Lemmata 3.1, 3.5. \( \text{□} \)

**Corollary 3.7.** \([2]\) If \( G \subset F(X) \) is a full subcategory, and if the map

\[ OC : HH_*(G) \to QH^*(X) \]

contains the idempotent \( e_w \) in its image, then \( G \) split-generates \( F(X) \).
Proof. Follows from Corollary 3.6 and Lemma 3.3.

Corollary 3.8. (2) If \( G \subset F(X)_w \) is a full subcategory, and if the map
\[
\mathcal{C}O : QH^*(X)_w \to HH^*(G)
\]
is injective, then \( G \) split-generates \( F(X)_w \).

Proof. By Corollary 3.7, it suffices to prove that the map
\[
OC : HH_*(G) \to QH^*(X) \overset{e_w \ast}{\to} QH^*(X)_w
\]
is surjective, where we recall that the final map is projection onto the \( w \)-generalized eigenspace. This is equivalent to injectivity of the dual map, which is equal to the composition
\[
QH^*(X)_w^\vee \overset{(e_w \ast)^\vee}{\to} QH^*(X)^\vee \overset{PD}{\to} QH^*(X) \overset{\mathcal{C}O}{\to} HH^*(F(X)_w) \overset{HH^*(\phi)}{\to} HH_*(F(X)_w)^\vee
\]
by Proposition 3.4.

Because \( QH^*(X) \) is a Frobenius algebra,
\[
\langle e_w \ast a, b \rangle = \langle a, e_w \ast b \rangle
\]
for each \( w \), so \( e_w \ast \) is symmetric. It follows that we have a decomposition
\[
QH^*(X)^\vee \cong \bigoplus_{w \in \Lambda} QH^*(X)_w^\vee,
\]
and a decomposition of the Poincaré duality isomorphism into components
\[
PD_w : QH^*(X)_w^\vee \xrightarrow{\sim} QH^*(X)_w,
\]
such that
\[
PD = \bigoplus_{w \in \Lambda} PD_w.
\]
In particular, the map
\[
QH^*(X)_w^\vee \overset{(e_w \ast)^\vee}{\to} QH^*(X)^\vee \to QH^*(X)
\]
factors as
\[
QH^*(X)_w^\vee \overset{PD_w}{\to} QH^*(X)_w \to QH^*(X).
\]
So it suffices for us to prove injectivity of the composition
\[
QH^*(X)_w^\vee \overset{PD_w}{\to} QH^*(X)_w \overset{\mathcal{C}O}{\to} HH^*(F(X)_w) \overset{HH^*(\phi)}{\to} HH_*(F(X)_w)^\vee.
\]
We now observe that the maps on either end of this composition are isomorphisms (the one on the right-hand end because \( \phi \) is a quasi-isomorphism), so it suffices to prove that the middle map is injective. This completes the proof.

4. The relative Fukaya category

We recall (from [36 Section 5]) the definition of \( F(X, D) \), the Fukaya category of a Kähler manifold \( X \) relative to a smooth normal-crossings divisor \( D \), where \( D \) has \( \geq n + 1 \) components (\( n \) is the complex dimension of \( D \)), each of which is Poincaré dual to some multiple of the symplectic form. In this section, we explain the relationship between the monotone Fukaya category and the relative Fukaya category. We use the relationship to prove results about the closed-open string map, analogous to the divisor axiom for Gromov-Witten invariants. Finally, we use these results to make computations in the Fukaya category of a Fano hypersurface in projective space.
4.1. Relating the monotone and relative Fukaya categories. Suppose that \((X,D)\) is a Kähler pair, and \(X\) is monotone. We recall that the coefficient ring \(R\) of \(\mathcal{F}(X,D)\) is the completion of a polynomial ring
\[
\tilde{R} := \mathbb{C}[r_1, \ldots, r_k]
\]
in the category of \(G(X,D)\)-graded algebras.

**Lemma 4.1.** If \(X\) is monotone, then the completion is unnecessary: \(R \cong \tilde{R}\).

**Proof.** It suffices to prove that no two generators \(r^{c_1}, r^{c_2}\) of \(\tilde{R}\) have the same degree. Observe that there is an isomorphism
\[
H_2(X, X \setminus D) \to \mathbb{Z}^k
\]
(where \(k\) is the number of divisors), given by taking intersection numbers with the divisors. Thus we can regard \(c_1\) and \(c_2\) as living in \(H_2(X, X \setminus D)\).

Recall (from [36, Definition 5.1]) that the grading of the generator \(r_j\) corresponding to divisor \(D_j\) is defined by choosing a disk
\[
u : (D^2, \partial D^2) \to (X, X \setminus D)
\]
with intersection number +1 with \(D_j\) and 0 with all other divisors, then choosing a lift \(\tilde{u}\) of \(u\) to \(G X\); the grading of \(r_j\) is then \(\tilde{u}|_{\partial D^2} \in H_1(G(X \setminus D))\).

It follows that the class \(c_1 - c_2\) lifts to a class \(\tilde{c} \in H_2(G X, G(X \setminus D))\), whose image under the boundary map to \(H_1(G(X \setminus D))\) vanishes. By the homology long exact sequence for the pair \((G X, G(X \setminus D))\), the class \(\tilde{c}\) must lie in the image of \(H_2(G X)\). Such a class corresponds to a surface in \(X\) with a lift to the Lagrangian Grassmannian \(G X\); such surfaces have vanishing Chern class, so by monotonicity the symplectic area vanishes, and hence the intersection numbers with the divisors all vanish because, by the definition of a Kähler pair, each divisor is Poincaré dual to a multiple of the class of the symplectic form.

Therefore \(c_1 = c_2\), and the proof is complete. \(\square\)

Now we examine the relationship between the gradings of the relative Fukaya category and the monotone Fukaya category. The relative Fukaya category \(\mathcal{F}(X,D)\) is \(G(X,D)\)-graded, where we recall that \(G(X,D)\) is the grading datum
\[
\mathbb{Z} \cong H_1(G_z(X \setminus D)) \to H_1(G(X \setminus D)).
\]
The monotone Fukaya category is \(G(X)\)-graded, where \(G(X)\) is the grading datum
\[
\mathbb{Z} \cong H_1(G_z X) \to H_1(G X).
\]

**Definition 4.1.** Let \((X,D)\) be a Kähler pair. There is an obvious inclusion of fibrations
\[
G(X \setminus D) \hookrightarrow G X.
\]
We denote the resulting morphism of grading data by
\[
q_{X,D} : G(X,D) \to G(X).
\]
We will write \(q\) when no confusion is possible.

**Lemma 4.2.** If \(R\) is the coefficient ring of the relative Fukaya category, then \(q_* R\) is concentrated in degree 0.

**Proof.** As we saw in the proof of Lemma [4.1], the grading of class \(r_j\) is a boundary in \(G X\), by its definition. \(\square\)
We would now like to relate the relative Fukaya category, $\mathcal{F}(X,D)$, to the monotone Fukaya category $\mathcal{F}(X)_w$. First we relate the objects.

**Lemma 4.3.** (See [4, Lemma 3.1] and comments immediately after). Let $(X,D)$ be a monotone Kähler pair, where $D$ has smooth irreducible components $D_j$ Poincaré dual to $d_j[\omega]$, and

$[\omega] = \tau c_1.$

If $L \subset X \setminus D$ is an exact, anchored Lagrangian brane, then any disk with boundary on $L$, $[u] \in H_2(X,L)$, satisfies

$\tau \mu(u) = 2d_j u \cdot D_j = 2\omega(u),$

where $\mu$ denotes the Maslov class.

**Proof.** Recall that an anchored Lagrangian brane $L$ is equipped with a lift to the universal abelian cover of the Lagrangian Grassmannian of $X \setminus D$. Hence, the boundary of $u$ is nullhomologous in the Lagrangian Grassmannian $\mathcal{G}(X \setminus D)$, hence there is a surface $u' \subset X \setminus D$ with the same boundary as $u$, which lifts to the Lagrangian Grassmannian. Then the closed surface $\tilde{u} = u \cup u'$ satisfies

$2c_1(\tilde{u}) = \mu(u),$

$\omega(\tilde{u}) = \omega(u),$ and

$\tilde{u} \cdot D_j = u \cdot D_j.$

The first line follows because $u'$ lifts to the Lagrangian Grassmannian, hence has vanishing Maslov class. The second line follows by exactness of $L$ in $X \setminus D$, which implies that $\omega(u') = 0$ by Stokes' theorem. The third line follows because $u'$ lies in the complement of $D$. The result follows, because

$\tau c_1(\tilde{u}) = \omega(\tilde{u}) = d_j \tilde{u} \cdot D_j$

by assumption. 

It follows, in particular, that objects of $\mathcal{F}(X,D)$ are monotone Lagrangians. Let $\mathcal{F}(X,D)_w$ denote the full subcategory of $\mathcal{F}(X,D)$ whose objects are Lagrangians $L$ with $w(L) = w$.

**Corollary 4.4.** If $w \neq 0$, then $\mathcal{F}(X,D)_w$ has an empty set of objects unless $\tau$ is an integer divisible by $d_j$ for all $j$.

**Proof.** If $L$ is an object of $\mathcal{F}(X,D)_w$ and $w \neq 0$, then $L$ must bound a Maslov index 2 disk $u$. It follows by Lemma 4.3 that

$d_j(u \cdot D_j) = \tau$

for all $j$. 

**Corollary 4.5.** After choosing a map $\tilde{i}$ that makes the diagram

$\begin{align*}
\tilde{G}(X \setminus D) & \xrightarrow{\tilde{i}} \tilde{G}X \\
\downarrow & \\
G(X \setminus D) & \xrightarrow{i} GX
\end{align*}$

commute, there is a natural map from objects of $\mathcal{F}(X,D)_w$ to objects of $\mathcal{F}(X)_w$.

We would like to extend this to an $A_\infty$ functor. We do this by first introducing a new category.
Definition 4.2. For each \( w \in \mathbb{C} \), we define the monotone relative Fukaya category \( \mathcal{F}_m(X, D)_w \). It is a \( G(X, D) \)-graded, \( \mathbb{R} \)-linear \( A_\infty \) category. Its objects are the anchored Lagrangian branes \( L \subset X \setminus D \) such that \( w(L) = w \). The morphism spaces and \( A_\infty \) structure maps are defined as for \( \mathcal{F}(X)_w \), except that:

- All Floer and perturbation data are chosen so that the almost-complex structure makes each divisor \( D_j \) into an almost-complex submanifold, and the Hamiltonian part vanishes with its first derivative along each divisor; then intersection numbers of pseudoholomorphic disks with divisors are non-negative.
- Each pseudoholomorphic disk is counted with a coefficient \( r^u D \in \mathbb{R} \).

This category is \( G(X, D) \)-graded, for the same reason that the relative Fukaya category is.

Remark 4.1. It is still possible to achieve transversality with this restricted set of Floer and perturbation data: moduli spaces of disks and spheres transverse to the divisors are still generically regular, but moduli spaces of holomorphic spheres inside the divisor may not be. However, the only place where we needed regularity of a moduli space of holomorphic spheres in the construction of the monotone Fukaya category was when we ruled out spheres bubbling off a pseudoholomorphic strip that is constant along its length, its image coinciding with a Hamiltonian chord between two Lagrangians. The Hamiltonian chords lie in the complement of the divisors, so this type of sphere bubbling is still ruled out, without the need for regularity of moduli spaces of holomorphic spheres inside the divisors.

Lemma 4.6. The map on objects defined in Corollary 4.3 extends to a strict full embedding of \( G(X) \)-graded, \( \mathbb{C} \)-linear \( A_\infty \) categories \( q_* \mathcal{F}_m(X, D)_w \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathcal{F}(X)_w \), where we regard \( \mathbb{C} \) as an \( \mathbb{R} \)-algebra via the map \( \mathbb{R} \to \mathbb{C} \) sending each \( r_j \mapsto 1 \): this map is well-defined by Lemma 4.1 and respects the \( G(X) \)-grading by Lemma 4.2.

Proof. The embedding is tautologous: the \( A_\infty \) structure maps on both sides count the same moduli spaces of pseudoholomorphic disks. The coefficient \( r^u D \) with which disks contribute to an \( A_\infty \) structure map in \( \mathcal{F}_m(X, D)_w \) reduces to 1 after tensoring with \( \mathbb{C} \). \( \square \)

Now we would like to compare \( \mathcal{F}_m(X, D)_w \) with \( \mathcal{F}(X, D)_w \). The fact that \( \mathcal{F}(X, D) \) may be curved makes it a bit complicated to prove the categories are quasi-equivalent, because we would first need to construct a non-curved model for \( \mathcal{F}(X, D)_w \). Anyway this turns out to be unnecessary for our purposes. Namely, it turns out to be sufficient to prove this result to first order, so we need only a \( \mathbb{R}/m^2 \)-linear quasi-equivalence (where \( m \subset \mathbb{R} \) denotes the maximal ideal generated by \( r_1, \ldots, r_n \)). For this purpose, we need only consider moduli spaces of pseudoholomorphic disks with \( \leq 1 \) intersection points with the divisors. The problem of curvature does not arise, because any non-constant pseudoholomorphic disk with boundary on a single Lagrangian \( L \) must necessarily intersect all of the divisors \( D_j \) (by Lemma 4.3).

Proposition 4.7. There is a quasi-equivalence of \( G(X, D) \)-graded, \( \mathbb{R}/m^2 \)-linear \( A_\infty \) categories \( \mathcal{F}_m(X, D)_w / m^2 \cong \mathcal{F}(X, D)_w / m^2 \).

Proof. The two categories clearly have the same set of objects. We use the usual trick of ‘doubling’ the category: consider a category \( \mathcal{F}^{\text{tot}} \) which contains two copies of each object, and so that there are strict \( G(X, D) \)-graded, \( \mathbb{R}/m^2 \)-linear embeddings \( \mathcal{F}_m(X, D)_w / m^2 \hookrightarrow \mathcal{F}^{\text{tot}} \hookrightarrow \mathcal{F}(X, D)_w / m^2 \).
The $A_\infty$ structure maps on the left are defined by choosing perturbation data on moduli spaces of disks with at most 1 marked point which are pulled back via the forgetful map forgetting the marked point. The $A_\infty$ structure maps on the right are defined by choosing perturbation data which are consistent with respect to the Deligne-Mumford compactification, including when the internal marked point bubbles off at the boundary. We extend these to choices of perturbation data for all other moduli spaces of pseudoholomorphic disks with at most 1 marked points and boundary conditions on the objects of our category. We require consistency of these perturbation data with respect to the Deligne-Mumford compactification, except when the internal marked point bubbles off in a disk on its own. This type of degeneration cannot happen in the corresponding moduli spaces of pseudoholomorphic disks, because any pseudoholomorphic disk with boundary on a single Lagrangian has to intersect each divisor $D_j$ at least once (by Lemma 4.3), and in particular has at most 2 internal marked points.

We use the corresponding moduli spaces of pseudoholomorphic disks to define an $A_\infty$ structure on $\mathcal{F}^{\text{tot}}$. It follows as in [34, Section 10a] that the order-zero components of these categories are quasi-equivalent, and as in [36, Section 2.6] that the $R/m^2$-linear categories are quasi-equivalent.

4.2. The relative Fukaya category and the closed-open string map. If $(X, D)$ is a Kähler pair, then we can define a differential $G(X, D)$-graded $R$-algebra $QC^*(X, D)$, by analogy with $QC^*(X)$. The underlying $G(X, D)$-graded $R$-module is the Morse cohomology $C^*_M(X; \mathbb{C}) \otimes R$. The differential is the Morse differential $d$, and its cohomology is denoted $QH^*(X, D)$. Each holomorphic sphere $u$ (with attached Morse flowlines) contributes to the quantum cup product with a coefficient $\tau/d_j \in R$.

Furthermore, there is a $G(X, D)$-graded homomorphism of $R$-algebras,

$$CO_{X,D} : QH^*(X, D) \to HH^*_G(X, D)(\mathcal{F}_m(X, D)_w),$$

for every $w \in \mathbb{C}$. It is defined by analogy with $CO$, where every pseudoholomorphic disk $u$ is counted with a coefficient $\tau/d_j \in R$. It is $G(X, D)$-graded by standard index theory.

**Proposition 4.8.** Let $(X, D)$ be a Kähler pair. For a given $w \in \mathbb{C}$, we have

$$CO_{X,D}(d_j[w]) = \left[ r_j \frac{\partial \mu^*}{\partial r_j} + r_j \frac{\partial (w \cdot T)}{\partial r_j} \cdot \hat{e} \right] \in HH^*_G(X, D)(\mathcal{F}_m(X, D)_w),$$

where $\hat{e} \in HH^0_G(X, D)(\mathcal{F}_m(X, D)_w)$ is a class whose length-zero component is given by the identity elements $e \in CT^0(L, L)$ for all $L$, and

$$T = r_1^{\cdot \tau_1} \ldots r_k^{\cdot \tau_k} \in R.$$

We remark that the exponents $\tau/d_j$ are not always integral, so $T$ does not always lie in $R$; however, when that happens, $\mathcal{F}(X, D)_w$ has an empty set of objects unless $w = 0$ by Corollary 4.4, in which case the term involving $T$ vanishes.

**Remark 4.2.** The length-0 component of this equation was proven in the course of the proof of Lemma 3.2.

**Remark 4.3.** If we had instead followed [12] and defined the Fukaya category as a curved $A_\infty$ category, then the term involving $\hat{e}$ could be absorbed into the first term, where it would correspond to the curvature term $\mu^0$. However we have chosen to use a different definition of the Fukaya category (taking advantage of the monotonicity of our manifolds), in which case $\mu^0$ is set equal to zero.

**Proof.** Because $D_j$ is Poincaré dual to $d_j[w]$, the left-hand side is given by the count of pseudoholomorphic disks with a single internal marked point constrained to lie on $D_j$ (for appropriate choice of Morse function, as in the proof of Lemma 3.2). On the other hand, the first term of the right-hand side is given by the count of pseudoholomorphic disks defining the $A_\infty$ structure map, multiplied by...
their intersection number with $D_j$. Equivalently, it is given by the count of pseudoholomorphic disks with a choice of internal marked point which lies on divisor $D_j$.

So we have two moduli spaces of pseudoholomorphic disks with an internal marked point: on the first, the perturbation data are chosen to be consistent with the Deligne-Mumford compactification, whereas on the second, the perturbation data are pulled back from the perturbation data used to define the $A_\infty$ structure maps, via the map forgetting the internal marked point. We observe that the latter choice is not consistent with the Deligne-Mumford compactification: consistency would require that, when the marked point approaches the boundary, a ‘thin’ region modeled on a strip would develop separating the marked point from the rest of the disk, and the perturbation data along the thin region should coincide with the Floer data. In the pulled-back perturbation data, when the marked point approaches the boundary, a holomorphic disk bubbles off on which the perturbation data has vanishing Hamiltonian part and constant almost-complex structure part $J = J_L$.

To compare the two choices, we consider the same moduli space of holomorphic disks used in the proof that $\mathcal{CO}$ is a homomorphism of $\mathbb{C}$-algebras (see Section 3.5, in particular Figure 3). We define perturbation data on this moduli space, so that at $t = 1$, the perturbation data on the right-hand marked disk coincides with that used to define the closed-open string map $\mathcal{CO}$, while that on the left-hand marked disk is given by the perturbation data used to define the $A_\infty$ structure maps $\mu^*$, pulled back via the forgetful map forgetting the marked point.

Now we consider the moduli space of pseudoholomorphic disks with these perturbation data, where the left-hand marked point is constrained to lie on $D_j$ and the right-hand marked point is unconstrained. Counting the boundary points of the one-dimensional component of this moduli space shows that

$$\mathcal{CO}_{X,D}(d_j \omega) = \mathcal{CO}(1) \bullet \left( r_j \frac{\partial \mu^*}{\partial r_j} \right) + r_j \frac{\partial (w \cdot T)}{\partial r_j} \cdot \tilde{e} + [d, H],$$

where $H$ is defined by counting the zero-dimensional components of the moduli space.

The left-hand side corresponds to the boundary component at $t = 0$ (the holomorphic sphere that bubbles off is necessarily constant, so it is equivalent to constraining the centre marked point to lie on $D_j$).

The first term on the right-hand side corresponds to the boundary component at $t = 1$, when the disk that bubbles off on the left has $\geq 1$ incoming marked boundary points. By the argument given at the start of the proof, the count of such disks contributes exactly $r_j \partial \mu^* / \partial r_j$. The right-hand disk corresponds to $\mathcal{CO}(1)$, and the middle disk gives the Yoneda product.

The second term on the right-hand side corresponds to the boundary component at $t = 1$, when the disk that bubbles off on the left has no incoming marked boundary points. Because the perturbation data on this disk are pulled back via the forgetful map, the disk must have constant perturbation data given by the almost-complex structure $J = J_L$. The count of such disks is $w$ by definition. We must also choose an internal marked point lying on $D_j$; the disk has intersection number $\tau / d_j$ with divisor $D_j$ by Lemma 4.3, and hence there are $\tau / d_j$ possible choices for the internal marked point lying on divisor $D_j$. Each disk contributes with a coefficient $r^w d = T$, so the total count of these disks is $r_j \partial (w \cdot T) / \partial r_j$. We define the class $\tilde{e}$ by counting the pseudoholomorphic disks in the remainder of the configuration: namely, the right-hand disk gives $\mathcal{CO}(1)$ as before, but the central disk now has an unconstrained boundary marked point where the left-hand disk bubbled off. It is easy to check that the count of these configurations defines a Hochschild cohomology class $\tilde{e}$ with properties as stated.

Finally, disk bubbling for $t \in (0, 1)$ contributes the remaining term. The result now follows (using the fact that $\mathcal{CO}$ is a unital algebra homomorphism).
5. Fermat hypersurfaces

5.1. Computations in the Fukaya category. We consider the hyperplane

\[ X_1^n := \mathbb{CP}^{n-1} \cong \left\{ \sum_{j=1}^n z_j = 0 \right\} \subset \mathbb{CP}^n, \]

with the smooth normal-crossings divisor

\[ D := \bigcup_{j=1}^n D_j, \text{ where } \]

\[ D_j := \{ z_j = 0 \}. \]

We recall from [36, Section 6] the construction of an immersed Lagrangian sphere \( L : S^{n-2} \to \mathbb{CP}^{n-2} \setminus D \). There is an \((a, \ldots, a)\)-branched cover

\[ \phi : (X^n_a, D) \to (X^n_1, D), \]

given by

\[ \begin{align*}
\left\{ \sum_{j=1}^n z_j^n = 0 \right\} & \to \left\{ \sum_{j=1}^n z_j = 0 \right\}, \\
[z_1 : \ldots : z_n] & \mapsto [z_1^n : \ldots : z_n^n].
\end{align*} \]

We recall that the associated grading data are

\[ G(X^n_1, D) \cong G^n_1, \quad \text{and} \]

\[ G(X^n_a, D) \cong G^n_a. \]

The branched cover determines a morphism of grading data,

\[ p : G^n_a \to G^n_1, \]

which can be determined by [36, Lemma 3.5.11]. The cokernel of \( p \) is the covering group, and is naturally isomorphic to \( \Gamma^n_a \).

**Proposition 5.1.** Let \( \tilde{A} \) denote the full subcategory of the \( G^n_a \)-graded monotone relative Fukaya category \( \mathcal{F}_m(X^n_a, D)_w \) whose objects are lifts of the immersed Lagrangian sphere \( L \subset X^n_1 \) under the branched cover \( \phi \). Then there exists a non-curved \( A_\infty \)-algebra \( \tilde{A} \) of type \( A^n_a \), such that

\[ \tilde{A} \cong p^* \tilde{A} \]

(where, as usual, \( \tilde{A} \) denotes the unique extension of \( A \) to a \( G^n_1 \)-graded \( A_\infty \) category).

**Proof.** Because \( \Gamma^n_a \) acts freely on the set of lifts of \( L \), we can choose \( \Gamma^n_a \)-equivariant perturbation data for \( \tilde{A} \). It follows that

\[ \tilde{A} \cong p^* \tilde{A} \]

for some \( G^n_1 \)-graded, \( R \)-linear \( A_\infty \) algebra \( A \).

We recall that, by [36, Proposition 5.1], there is a \( G^n_1 \)-graded \( R_a \)-linear \( A_\infty \) category \( \mathcal{F}(\phi) \), such that there are is a strict \( R/m^2 \)-linear isomorphism

\[ \mathcal{F}(\phi)/m^2 \cong \mathcal{F}(X^n_1, D, a)/m^2, \]

where

\[ m := (r_1, \ldots, r_n) \subset R_a \]
and
\[ a = (a_1, \ldots, a), \]
and there is also a full strict \( p^* R_n \)-linear embedding
\[ p^* F(\phi) \hookrightarrow F(X^n_\alpha, D). \]

Altering the proof of Proposition 4.7 to take into account the \( \Gamma^n_\alpha \)-equivariance, we obtain an \( R/m^2 \)-linear quasi-isomorphism
\[ A/m^2 \cong CF_{F(\phi)}(L, L), \]
and hence an \( R/m^2 \)-linear quasi-isomorphism
\[ A/m^2 \cong A'/m^2, \]
where
\[ A' := CF_{F(X^n_1, D, a)}(L, L)/m^2. \]

Because being of type \( A^n_\alpha \) only depends on the zeroth- and first-order coefficients of the \( A_\infty \) structure maps, it suffices to prove that \( A' \) is of type \( A^n_\alpha \).

\( A' \) is \( G_i \)-graded and \( R_n \)-linear by definition. By [36] Proposition 6.1], its order-0 cohomology algebra is an exterior algebra and
\[ \Phi(\mu^{\geq 3}) = u_1 \ldots u_n + m. \]

[36] Proposition 6.1 also shows that the first-order deformation class of the endomorphism algebra of \( L \) in the non-orbifold relative Fukaya category \( F(X^n_1, D) \) is
\[ \pm \sum_{j=1}^n r_j u_j. \]

It follows by [36] Theorem 5.1] that the first-order deformation class in the orbifold relative Fukaya category \( F(X^n_\alpha, D, a) \) is
\[ \pm \sum_{j=1}^n r_j u^a_j. \]

It follows that (after an appropriate change of variables to fix the signs)
\[ \Phi(\mu^{\geq 3}) = W^n_\alpha + m^2, \]
and hence that \( A' \) (and hence \( A \)) is of type \( A^n_\alpha \), as required.

\[ \square \]

**Lemma 5.2.** If \( \pi_1(X) \cong \{e\} \), then
\[ G(X) \cong G_{2N}, \]
where \( N \in \mathbb{Z} \) is the minimal Chern number on spherical classes, i.e., the generator of the image of the map
\[ c_1(TX) : \pi_2(X) \to \mathbb{Z}, \]
and we recall that \( G_{2N} \) denotes the grading datum \( \{\mathbb{Z} \to \mathbb{Z}_{2N}\} \).

**Proof.** Follows from the long exact sequence for the fibration \( G X \):
\[ \pi_2(X) \to \pi_1(G_x X) \to \pi_1(G X) \to \pi_1(X), \]
together with the observation that \( \pi_1(G_x X) \cong \mathbb{Z} \) (given by the Maslov class), and the first map in the exact sequence coincides with evaluation of \( 2c_1(TX) \). \[ \square \]
The map

\[ q : G(X_a^n, D) \to G(X_a^n) \]

coincides with the map

\[ G_a^n \to G_{2(n-a)} \]

introduced immediately before Corollary 2.16.

**Corollary 5.3.** If \( 3 \leq a \leq n - 1 \), then there is a fully faithful embedding of \( G_{2(n-a)} \)-graded, \( \mathbb{C} \)-linear, non-curved \( A_\infty \) categories

\[ q_* \mathcal{A} \otimes_R \mathbb{C} \hookrightarrow \mathcal{F}(X_a^n)_w, \]

where \( \mathcal{A} \) is an \( A_\infty \) algebra of type \( A_n^a \). If \( a \leq n - 2 \) then \( w = 0 \), but if \( a = n - 1 \) then \( w \) may be non-zero.

**Proof.** Recall that the minimal Maslov number of a lift of \( L \) is \( 2(n-a) \), so if \( a \leq n - 2 \) then all lifts of \( L \) do not bound Maslov index 2 disks, and therefore have \( w(L) = 0 \). If \( a = n - 1 \), then \( L \) may bound Maslov index 2 disks. However, by symmetry, all lifts of \( L \) have the same value \( w \) of \( w(L) \).

The result then follows from Proposition 5.1 and Lemma 4.6.

**Remark 5.1.** We will prove (in Corollary 5.8) that, when \( a = n - 1 \), we have \( w = -a! \).

### 5.2. The closed-open string map and split-generation.

In this section, we will study the closed-open string map for the Fermat hypersurfaces \( X_a^n \). First we would like to understand \( H^*(X_a^n) \) better. We recall the Lefschetz decomposition of \( H^*(X_a^n) \):

\[ H^*(X_a^n) \cong H^H(X_a^n) \oplus H^p(X_a^n), \]

where ‘\( H \)’ stands for ‘Hodge’ and ‘\( P \)’ stands for ‘primitive’. By the Lefschetz hyperplane theorem, the primitive cohomology is concentrated in the middle degree \( d = n - 2 \). So the Hodge part is generated by the hyperplane class \( P \):

\[ H^H(X_a^n) \cong \mathbb{C}[P]/P^{n-1}, \]

and the primitive part is

\[ H^p(X_a^n) \cong \ker(\wedge P : H^d(X_a^n) \to H^{d+2}(X_a^n)). \]

**Lemma 5.4.** The image of the map

\[ H^c_*(X_a^n \setminus D) \to H^*(X_a^n) \]

contains the primitive cohomology \( H^p_*(X_a^n) \).

**Proof.** We observe that

\[ H^c_*(X_a^n \setminus D) \cong H^*(X_a^n, D), \]

and apply the long exact sequence in cohomology for the pair \( (X_a^n, D) \):

\[ \ldots \to H^d(X_a^n, D) \to H^d(X_a^n) \to H^d(D) \to \ldots \]

It suffices to prove that the image of the primitive cohomology in \( H^d(D) \) vanishes. To do this, it is sufficient to show that the map

\[ \wedge P : H^d(D) \to H^{d+2}(D) \]

is an isomorphism (where \( P \) is the restriction of the hyperplane class to \( D \)). In other words, \( D \) has no ‘primitive cohomology’ in degree \( d \).
To understand the cohomology of $D$, we apply the generalized Mayer-Vietoris principle [6, Proposition 8.8] to the open cover by neighbourhoods of the components $D_i$. This yields a bounded double complex, hence a spectral sequence converging to $H^\ast(D)$, with $E_1$ page

$$E_1^{p,q} \cong \bigoplus_{K \subset [k], |K| = p-1} H^q(D_K),$$

where $D_K$ denotes the intersection of all divisors indexed by $i \in K$. The map $\wedge P$ defines a morphism from this spectral sequence to itself, of degree 2.

Now observe that $D_K$ is a hypersurface in a projective space $\mathbb{P}^{d+1-|K|}$, so only has primitive cohomology in degree $d-|K|$. It follows that the map $\wedge P : H^q(D_K) \to H^{q+2}(D_K)$ is an isomorphism for all $q + |K| = d + 1$. In particular, the map

$$\wedge P : E_1^{p,q} \to E_1^{p,q+2}$$

is an isomorphism for all $p + q = d$. Since the spectral sequence converges, it follows that

$$\wedge P : E_1^d \to H^d(D)$$

is an isomorphism. This completes the proof. \qed

Now let $\Gamma^n_a$ denote the covering group of the $(a, \ldots, a)$-branched cover

$$\pi : (X^n_a, D) \to (X^n_1, D).$$

It clearly acts on $X^n_a$, and hence on the cohomology $H^\ast(X^n_a)$.

**Lemma 5.5.** The $\Gamma^n_a$-fixed part of $H^\ast(X^n_a)$ is exactly the Hodge part of cohomology:

$$H^\ast(X^n_a)^{\Gamma^n_a} \cong H^\ast_H(X^n_a).$$

**Proof.** The action of $\Gamma^n_a$ obviously fixes the Hodge part of the cohomology, and preserves the primitive cohomology, so it suffices to prove that the $\Gamma^n_a$-fixed part of $H^p(X^n_a)$ is trivial. By Lemma 5.4, the map

$$H^\ast_c(X^n_a \setminus D) \to H^\ast_p(X^n_a)$$

is surjective. This map is clearly $\Gamma^n_a$-equivariant, and it follows that the induced map

$$H^\ast_c(X^n_a \setminus D)^{\Gamma^n_a} \to H^\ast_p(X^n_a)^{\Gamma^n_a}$$

is surjective.

The restriction $\pi : X^n_a \setminus D \to X^n_1 \setminus D$ is an unbranched cover, so we have

$$H^\ast_c(X^n_a \setminus D)^{\Gamma^n_a} \cong H^\ast_c(X^n_1 \setminus D)$$

(using De Rham cohomology, this can be realized by averaging differential forms).

Therefore, because the diagram

$$\begin{array}{ccc}
H^\ast_c(X^n_a \setminus D)^{\Gamma^n_a} & \xrightarrow{\pi^*} & H^\ast(X^n_a)^{\Gamma^n_a} \\
\downarrow & & \downarrow \\
H^\ast_c(X^n_1 \setminus D) & \xrightarrow{\pi^*} & H^\ast(X^n_1)
\end{array}$$

commutes, the map

$$\pi^* : H^\ast(X^n_1) \to H^\ast(X^n_a)^{\Gamma^n_a}$$
contains the equivariant primitive cohomology classes in its image. However, \( X^n \cong \mathbb{CP}^{n-2} \), so its cohomology is generated by the hyperplane class \( P \); it follows that the only equivariant primitive cohomology class is 0. \( \square \)

We do not have an immediate use for the following Lemma (it will turn up in the proof of Proposition 7.4), but it gives an interesting description of the primitive homology of \( X^n \):

**Lemma 5.6.** The homology classes of the lifts of the Lagrangian sphere \( L \) to \( X^n \) generate the primitive homology of \( X^n \).

**Proof.** We regard \( X^n \setminus D \) as a submanifold of \( \mathbb{CP}^{n-1} \setminus D \cong (\mathbb{C}^*)^{n-1} \).

We observe that the argument map \( \text{Arg} : X^n \setminus D \rightarrow (S^1)^{n-1} \) is a homotopy equivalence onto its image (the ‘coamoeba’). This follows from \( [37, \text{Proposition 2.2}] \). The coamoeba of \( X^n \setminus D \) is a \( \Gamma^n \)-cover of the coamoeba of the pair of pants \( X^n \setminus D \), and in particular is homotopy-equivalent to an \((n-1)\)-torus with \( a^{n-1} \) points removed. The homology classes of the lifts of the Lagrangian \( L \) correspond to balls around each removed point by construction (see \( [37, \text{Proposition 2.6}] \)).

Therefore the lifts of \( L \), together with the \( n-1 \) coordinate \((n-2)\)-tori, generate the middle-dimensional homology of \( X^n \setminus D \). The coordinate \((n-2)\)-tori correspond to small tori near the zero-dimensional strata of the boundary divisor \( D \); when we compactify by adding the divisor back in, their homology classes disappear because they are bounded by polydisks near \( D \).

It follows that the homology classes of the Lagrangian spheres span the image of the map \( H_{n-2}(X^n \setminus D) \rightarrow H_{n-2}(X^n) \).

It follows by Lemma 5.4, using Poincaré duality, that the homology classes of the Lagrangian spheres generate the primitive homology. \( \square \)

Now we consider the quantum cohomology, \( QH^*(X^n, D) \). It also admits a \( \Gamma^n \)-action, so we may talk about the equivariant part. The closed-open string map intertwines this \( \Gamma^n \)-action with the \( \Gamma^n \)-action on the Hochschild cohomology of the relative Fukaya category, so we have a map

\[ \text{CO} : QH^*(X^n, D)^{\Gamma^n} \rightarrow HH^*_G(\mathcal{F}_m(X^n, D))^{\Gamma^n} \]

on the equivariant parts.

**Proposition 5.7.** Let \( \tilde{A} \) be the full subcategory of \( \mathcal{F}_m(X^n, D) \) generated by lifts of \( L \). If \( 3 \leq a \leq n-1 \), then the map

\[ \text{CO} : QH^*(X^n, D)^{\Gamma^n} \rightarrow HH^*_G(\tilde{A})^{\Gamma^n} \]

is an isomorphism and, in the notation of Corollary 2.12, sends \( P \mapsto \gamma + w \cdot T \), where \( w \in \mathbb{Z} \) is the integer appearing in Corollary 5.3. In particular, the subalgebra of \( QH^*(X^n, D) \) generated by \( P \) is isomorphic to

\[ \mathbb{C}[P]/p^n_a(P - w \cdot T). \]

**Proof.** By Corollary 5.3 we have \( \tilde{A} \cong p^* A \).
where \( \mathcal{A} \) is an \( A_\infty \) algebra of type \( A_n^a \). We therefore have

\[
HH_*^G(\tilde{\mathcal{A}})^\Gamma_n^a \cong R[\gamma]/\rho_n^a(\gamma)
\]

by Corollary 2.12. We have

\[
\mathcal{CO}(P) = \left[ r_j \frac{\partial \mu^*}{\partial r_j} \right] + w \cdot T = \gamma + w \cdot T
\]

by Proposition 4.8.

We know that \( \mathcal{CO} \) is a unital algebra homomorphism, and the equivariant part of the Hochschild cohomology is generated by \( \gamma \). It follows that \( \mathcal{CO} \) is surjective. By Lemma 5.5, the equivariant part of the quantum cohomology is a free \( R \)-module of rank \( n-2 \), and by Corollary 2.12, so is the equivariant part of the Hochschild cohomology. Since \( \mathcal{CO} \) is surjective, it follows that it is an isomorphism. □

The computations of quantum cohomology from Proposition 5.7 agree with [16, Corollaries 9.3 and 10.9]. In fact, comparison with [16, Corollary 10.9] immediately gives the following:

**Corollary 5.8.** If \( L \) is a lift of the Lagrangian sphere to \( X_{n-1}^n \), for \( n \geq 4 \), then

\[
w(L) = -(n-1)!
\]

Thus, we see that the additional parameter that shows up in the Fano index 1 case in [16], can be related to a count of holomorphic disks with boundary on a certain Lagrangian.

**Corollary 5.9.** If \( a \leq n-2 \), then the operator of quantum cup product with the divisor class:

\[
*P : QH^*(X_n^a) \to QH^*(X_n^a)
\]

has the following eigenvalues: 0, and \((n-a)\) times the \((n-a)\)th roots of \( a^a \). If \( a = n-1 \), then the eigenvalues are \(-a!\) and \(-a! + a^a\).

Now we examine the consequences for the closed-open string map to the Hochschild cohomology of the monotone Fukaya category.

**Corollary 5.10.** Let \( \tilde{\mathcal{A}}_C \) be the full subcategory of \( \mathcal{F}(X_n^a) \) whose objects are lifts of \( L \). The map

\[
\mathcal{CO} : QH^*(X_n^a)^\Gamma_n^a \to HH^*(\tilde{\mathcal{A}}_C)^\Gamma_n^a
\]

restricts to an isomorphism on the generalized eigenspace \( QH^*(X_n^a)_0 \) (respectively, the generalized eigenspace \( QH^*(X_n^a)_{a!} \) when \( a = n-1 \)), and vanishes on all other generalized eigenspaces.

**Proof.** Let us assume \( a \leq n-2 \); the case \( a = n-1 \) is analogous. By Proposition 5.7, the generalized eigenspace associated to the eigenvalue 0 has rank \( a-1 \), with basis

\[
(P^{n-a} - a^a), P(P^{n-a} - a^a), \ldots, P^{a-2}(P^{n-a} - a^a).
\]

The other generalized eigenspaces have rank 1, and are generated by elements of the form \( P^{a-1}f(P) \).

As in the proof of Proposition 5.7, we have

\[
\tilde{\mathcal{A}}_C \cong q_* p^* \mathcal{A} \otimes_R \mathbb{C},
\]

where \( \mathcal{A} \) is an \( A_\infty \) algebra of type \( A_n^a \). By Corollary 2.17, we have

\[
HH^*(\tilde{\mathcal{A}}_C)^\Gamma_n^a \cong \mathbb{C}[\hat{\beta}]/\hat{\beta}^{a-1},
\]
and as in the proof of Proposition 5.7 we have
\[ \mathcal{C}O(P) = \hat{\beta}. \]
It follows immediately that \( \mathcal{C}O \) vanishes on elements of the form \( P^{a-1} f(P) \), and hence on all of the generalized eigenspaces other than that associated to 0. Furthermore, on the generalized eigenspace associated to 0, the image is generated by the elements
\[ (\hat{\beta}^{n-a} - a^a), \ldots, \hat{\beta}^{a-2}(\hat{\beta}^{n-a} - a^a), \]
which span \( \mathbb{C}[\hat{\beta}]/\hat{\beta}^{a-1} \). So the restriction to this generalized eigenspace is surjective; because the eigenspace has rank \( a - 1 \), this means it is an isomorphism.

**Proof.** Follows immediately from (a \( \Gamma_n^a \)-equivariant version of) Corollary 3.8, together with Corollary 5.10. \( \square \)

6. **Matrix factorization computations**

Recall from Section 2.2 the grading datum \( \mathbf{G} := \mathbf{G}^1_n \), the \( \mathbf{G} \)-graded ring
\[ R_a := \mathbb{C}[r_1, \ldots, r_n] \]
and the \( \mathbf{G} \)-graded vector space \( U \); we define the \( \mathbf{G} \)-graded polynomial ring
\[ S_a := R_a[U] \cong R_a[u_1, \ldots, u_n] \]
together with the element
\[ W^n_a = -u_1 \ldots u_n + \sum_{j=1}^n r_j u_j^a \in S_a \]
of degree 2.

In accordance with [36, Definition 7.2], we consider the differential \( \mathbf{G} \)-graded category of matrix factorizations of \( S, MF^G(S_a, W^n_a) \). We introduce a matrix factorization \( \mathcal{O}_0 := (K, \delta_K) \), where
\[ K := S_a \otimes \Lambda^*(U^\vee) \cong S_a[\theta_1, \ldots, \theta_n], \]
where the \( \theta_i \) anti-commute. The differential is given by
\[ \delta_K := \sum_j u_j \frac{\partial}{\partial \theta_j} + w_j \theta_j, \]
where
\[ w_j := -\frac{u_1 \ldots u_n}{nu_j} + r_j u_j^{a-1}, \]
so that \( \sum u_j w_j = W^n_a \). It is easy to check that \( \delta_K^2 = W^n_a \cdot \text{id} \), so \( (K, \delta_K) \) is an object of \( MF^G(S_a, W^n_a) \). We define its endomorphism DG algebra,
\[ \mathcal{B} := \text{Hom}_{MF^G(S_a, W^n_a)}(\mathcal{O}_0, \mathcal{O}_0). \]

**Proposition 6.1.** If \( n - 1 \geq a \geq 3 \), then \( \mathcal{B} \) is quasi-isomorphic to an \( A_\infty \) algebra of type \( A_n^a \).

**Proof.** We use the homological perturbation lemma to construct a quasi-isomorphic minimal \( A_\infty \) algebra structure on the cohomology of \( \mathcal{B} \); it follows from [36, Proposition 7.1] that this \( A_\infty \) algebra is of type \( A_n^a \). \( \square \)
Now we consider the morphisms of grading data

$$p : G^n_a \to G^n_1$$

and the sign morphism

$$\sigma : G^n_a \to G_{\sigma}.$$ 

We obtain a $\mathbb{C}$-linear, differential $\mathbb{Z}_2$-graded category

$$\sigma^* p^* MF^{G}(S_a, W^n_a) \otimes_R \mathbb{C},$$

where $\mathbb{C}$ is an $R$-algebra via the map sending all $r_j \mapsto 1$.

**Proposition 6.2.** Let $\tilde{B}_C$ be the full subcategory of $D^b Sing^{(\Gamma^n_a)^*}(W^{-1}_C(0))$ whose objects are equivariant twists of the skyscraper sheaf of the origin. Then $\tilde{B}_C$ split-generates, and there is a quasi-isomorphism

$$\tilde{B}_C \cong \sigma^* p^* B \otimes_R \mathbb{C}$$

where $B$ is an $A_\infty$ algebra of type $A^n_a$.

**Proof.** The fact that $\tilde{B}_C$ split-generates follows from the fact that $W_C$ has an isolated singularity at the origin (see [35, Section 12] and [10]).

Now recall that there is a quasi-equivalence of DG categories

$$D^b Sing^{(\Gamma^n_a)^*}(W^{-1}_C(0)) \cong MF^{(\Gamma^n_a)^*}(S_C, W_C)$$

(equivalence of the homotopy categories is proven in [28], and quasi-equivalence as DG categories then follows from the results of [23]).

By [36, Remark 7.1], there is a fully faithful embedding of $\mathbb{Z}_2$-graded categories

$$\sigma^* p^* MF^{G}(S_a, W^n_a) \hookrightarrow MF^{(\Gamma^n_a)^*}(S_C, W_C),$$

where the right-hand side denotes the category of $(\Gamma^n_a)^*$-equivariant matrix factorizations of

$$W_C := -u_1 \ldots u_n + \sum_{j=1}^n u^a_j \in S_C := \mathbb{C}[u_1, \ldots, u_n],$$

and $(\Gamma^n_a)^*$ acts on $S_C$ by multiplying coordinate functions by $\text{th}$ roots of unity. Under this quasi-equivalence, the equivariant twists of the skyscraper sheaf of the origin correspond to the lifts of $(K, \delta_K)$; the result now follows by Proposition 6.1.

**Proof. of Theorem 1.3.** It follows from Corollary 5.3, Proposition 6.2 and Theorem 2.6 that the subcategory of $F(X^n_a)_w$ generated by lifts of $L$ is quasi-isomorphic to the subcategory of $D^b Sing^{(\Gamma^n_a)^*}(W^{-1}_C(0))$ generated by equivariant twists of skyscraper sheaf of the origin. It follows from Corollary 5.11 and Proposition 6.2 that these subcategories split-generate, so the proof is complete.

**7. The cubic surface**

For the purposes of this section, let $X = X^4_3$ be the cubic hypersurface in $\mathbb{C}P^3$. 
7.1. The 27 lines and an open Gromov-Witten invariant. Following a computation from [16] 'Control example 3' after Corollary 10.9, we can compare the Gromov-Witten invariant counting the number of lines on $X$ to the open Gromov-Witten invariant $w(L)$ which counts the number of Maslov index 2 disks with boundary on one of our Lagrangians $L$.

By Proposition 5.7 the class $P$ Poincaré dual to a hyperplane satisfies the relation 
\[(P - w)^3 = 3^3(P - w)^2\]

in $QH^*(X)$. Using the axioms of the Gromov-Witten invariants, we compute the number of rigid rational curves in $X$ (such a curve necessarily has degree 1): it is equal to 
\[
\langle P^2, P \rangle = 9(w + 9).
\]

Thus, the number of lines is 27, if and only if $w(L) = -6$.

7.2. The quantum cohomology of the cubic surface. Recall that the cubic surface can be expressed as $\mathbb{CP}^2$ blown up at six points. Therefore, $QH^*(X)$ has a basis $\{1, h, p, e_1, \ldots, e_6\}$, where $1, h, p$ are the standard basis for $H^*(\mathbb{CP}^2)$ and $e_j$ are the classes of the exceptional divisors. We introduce convenient auxiliary classes 
\[
M := e_1 + \ldots + e_6,
\]
and 
\[
A := 3h - M + 6.
\]

We remark that the first Chern class is $c_1 = A - 6$.

Proposition 7.1. (From [8]). We have a complete description of $QH^*(X)$:
\[
\begin{align*}
p * p &= 84A + 36 \\
h * p &= 42A - 6h \\
e_i * p &= 14A - 6e_i \\
h * h &= p + 25A - 12h - 30 \\
h * e_i &= 9A - 2h - 6e_i - 12 \\
e_i * e_i &= -p + 5A - 4e_i - 10 \\
e_i * e_j &= 3A - 2(e_i + e_j) - 4.
\end{align*}
\]

Proposition 7.2. We now consider the operation 
\[
c_1 * : QH^*(X) \to QH^*(X).
\]
It has eigenvalues $-6$ and 21. The generalized eigenspace corresponding to eigenvalue $-6$ has rank 8, and the generalized eigenspace corresponding to eigenvalue 21 has rank 1.

Proof. Because $c_1 = A - 6$, it suffices to compute the eigenvalues and generalized eigenspaces of $A *$. It follows from Proposition 7.1 that 
\[
A^{*3} = 27A^{*2},
\]
so the eigenvalues are 0 and 27. It also follows from Proposition 7.1 that 
\[
A * (3e_i - A - 6) = 0
\]
for all $i$. It follows that the 0-eigenspace of $A *$ contains $A - 27$, $A * (A - 27)$, and $3e_i - A - 6$ for $i = 1, \ldots, 6$, and hence has rank 8. The 27-eigenspace therefore has rank 1, and is generated by $A^{*2}$.

\[\square\]
7.3. The other eigenvalue. We recall that components of the monotone Fukaya category are indexed by eigenvalues of $c_1^*$. We have determined the category $\mathcal{F}(X)_{-6}$ indexed by $-6$, and we expect that there is another category $\mathcal{F}(X)_{21}$ indexed by the other eigenvalue $21$. Here we speculate about what the Lagrangian corresponding to this other component may look like.

The cubic surface can be degenerated to the union of coordinate lines \{ $z_1z_2z_3 = 0$ \}, giving a tropical manifold in the sense of Gross and Siebert (see [17, Chapter 1]). The resulting manifold is illustrated in op. cit., Figure 24. There is a monotone Lagrangian torus $L$ in the cubic surface, which corresponds to the centroid of the triangle which is the only compact cell of the tropical manifold.

This construction generalizes immediately to a construction of a monotone Lagrangian torus $L_n$ in $X^n_a$ for all $a = n-1$, lying over the centroid of the simplex which is the only compact cell of the tropical manifold corresponding to a degeneration of $X^n_a$ to a union of coordinate hyperplanes.

Conjecture 7.3. Let $a = n-1$ and $L_n \subset X^n_a$ be the monotone Lagrangian torus just constructed. Then we have $w(L_n) = a^a - a!$, and $L_n$ split-generates $\mathcal{F}(X^n_a)_{a^a-a!}$.

We can give a non-rigorous explanation of why this conjecture ought to be true in the case of the cubic surface. Firstly, there are exactly 21 tropical disks of Maslov index 2 with boundary on this Lagrangian torus fibre, so one may hope that the honest holomorphic disk count is also 21. Secondly, one can check that $HF^*(L,L) \neq 0$ when $L$ is equipped with the trivial $\mathbb{C}^*$-local system: that is because the obvious $\mathbb{Z}_3$ symmetry implies that the disk potential function (in the sense of, e.g., [7]) necessarily has a critical point at the origin, so the Floer cohomology does not vanish. This implies that $HH^*(\mathcal{G})$ contains a non-trivial identity class, where $\mathcal{G} \subset \mathcal{F}(X)_{21}$ is the full subcategory whose only object is $L$. Since the closed-open string map $\mathcal{CO}$ is unital, and $QH^*(X)_{21}$ is one-dimensional by Proposition 7.2 (generated by the corresponding idempotent), it follows that

$$\mathcal{CO} : QH^*(X)_{21} \to HH^*(\mathcal{G})$$

is injective, so $L$ split-generates $\mathcal{F}(X)_{21}$ by Corollary 3.8. Again, this argument is not rigorous by any means.

7.4. Non-semi-simplicity of $CF^*(L,L)$ and $w$. This section explains a computation made by Seidel (unpublished). We consider the endomorphism algebra $CF^*(L,L)$ of a single lift of our Lagrangian sphere $L$ to $X$. We know that

$$CF^*(L,L) \cong \mathbb{C}(e, \theta)$$

as a vector space. The differential vanishes, so we have a unital $\mathbb{C}$-algebra of rank 2. It must be of the form $\mathbb{C}[\theta]/p(\theta)$, where $p(\theta)$ is a quadratic polynomial. There are two possibilities up to isomorphism: either the algebra is semisimple ($p$ has distinct roots) or not ($p$ has a double root). It turns out not to be semi-simple, and the proof relies crucially on the value of $w(L) = -6$.

Proposition 7.4. (Seidel) The endomorphism algebra $CF^*(L,L)$ is not semi-simple.

Proof. We recall the algebra homomorphism

$$\mathcal{CO}^0 : QH^*(X) \to CF^*(L,L),$$

and that we have

$$\mathcal{CO}^0(c_1) = w \cdot e$$

(Lemma 3.2).

We derive the following relation in $QH^*(X)$ from Proposition 7.1

$$c_1^2 = 3p + 9c_1 + 108.$$
Applying $CO^0$, it follows that

$$CO^0(p) = \frac{w^2 - 9w - 108}{3}.$$ 

We also have relations from Proposition 7.1:

$$(h - 6)^* = p + 25c_1 + 156$$

and

$$(e_i + 2)^* = -p + 5c_1 + 20.$$ 

Applying $CO^0$, we obtain

$$(CO^0(h) - 6)^2 = \frac{1}{3}(w + 6)(w + 60)$$

and

$$(CO^0(e_i) + 2)^2 = -\frac{1}{3}(w + 6)(w - 30).$$

In particular, because $w = -6$, the right-hand sides vanish.

Now if either $CO^0(h)$ or $CO^0(e_i)$ had a non-trivial $\theta$ term, these equations would immediately imply that $CF^*(L, L)$ is non-semisimple. The $\theta$ term of $CO^0(\alpha)$, for $\alpha \in H^2(X)$, corresponds to the image of $\alpha$ under the restriction map

$$H^*(X) \rightarrow H^*(L),$$

because in this case the disk count defining $CO^0$ reduces to a count of constant holomorphic disks (see, for example, [32, Section 5a] for the argument in the exact case). In particular, because the classes $h$ and $e_i$ span $H^2(X)$, it suffices to show that the homology class of $L$ is non-trivial.

We argue by contradiction: if $L$ were homologically trivial, then all lifts of the Lagrangian sphere to $X$ would be homologically trivial, by symmetry. However we proved, in Lemma 5.6, that the homology classes of the lifts of the Lagrangian sphere span the primitive homology, and we know the primitive homology to be non-trivial; therefore, the homology class of $L$ is non-trivial. This completes the proof. □

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