Some Results on Dominating
Induced Matchings

S. Akbari\textsuperscript{1}, H. Baktash\textsuperscript{2}, A. Behjati\textsuperscript{2}, A. Behmaram\textsuperscript{3}, M. Roghani\textsuperscript{2}

\textsuperscript{1}Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
\textsuperscript{2}Department of Computer Engineering, Sharif University of Technology, Tehran, Iran.
\textsuperscript{3}Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Abstract

Let $G$ be a graph, a dominating induced matching (DIM) of $G$ is an induced matching that dominates every edge of $G$. In this paper we show that if a graph $G$ has a DIM, then $\chi(G) \leq 3$. Also, it is shown that if $G$ is a connected graph whose all edges can be partitioned into DIM, then $G$ is either a regular graph or a biregular graph and indeed we characterize all graphs whose edge set can be partitioned into DIM. Also, we prove that if $G$ is an $r$-regular graph of order $n$ whose edges can be partitioned into DIM, then $n$ is divisible by $\binom{2r-1}{r-1}$ and $n = \binom{2r-1}{r-1}$ if and only if $G$ is the Kneser graph with parameters $r - 1, 2r - 1$.

2010 Mathematics Subject Classification. 05C69, 05C70.

Keywords. Induced matching, dominating induced matching, Kneser graph

1 Introduction

Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. As usual $|V(G)| = n$ and $|E(G)| = m$ denote the number of vertices and edges of $G$. The minimum and the maximum degree of a graph $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a vertex $v$, $N(v)$ denotes the set of neighbors of $v$ in $G$. A cycle of order $r$ is denoted by $C_r$. Also, a vertex $k$-coloring of a simple graph $G$ is defined as an assignment of one element from a set $C$ of $k$ colors to each vertex in $V(G)$ such that no two adjacent vertices have the same color, and the minimum $k$ for which $k$-coloring is possible, is $\chi(G)$. A Kneser graph $KG_{n,k}$ is the graph whose vertices correspond to the $k$-subsets of a set of $n$ elements, and two vertices are adjacent if and only if their two corresponding sets are disjoint. The Petersen graph is an example of Kneser graph $KG_{5,2}$ given in Figure 1. A $BG_{m,n}$ is a bipartite graph, whose vertices in one part correspond to the $m$-subsets of a set of $m + n - 1$ elements and vertices in the other part correspond to $n$-subsets of the same set and two vertices in different parts are adjacent if and only if they are disjoint. A biregular graph is a bipartite graph $G = (X, Y)$ in which any two vertices in $X$ have the same degree and any two vertices in $Y$ have the same degree.

\footnote{E-mail addresses: s.\textsuperscript{2}akbari@sharif.edu (S. Akbari), hossein.bktsh@gmail.com (H. Baktash), amin.bahjati@gmail.com (A. Behjati), behmaram@tabrizu.ac.ir (A. Behmaram), mohammadroghani43@gmail.com (M. Roghani)}
A matching of $G$ is a set of mutually non-adjacent edges of $G$. An induced matching (IM) is a matching having no two edges joined by an edge. In other words, $M$ is an induced matching (also called strong matching) of $G$ if the subgraph of $G$ induced by $V(M)$ is 1-regular. A maximum induced matching is an induced matching of maximum cardinality. Finding maximum IM is NP-hard [2]. The concept of domination in graphs appears as a natural model for facility location problems and has many applications in design and analysis of communication networks, network routing, and coding theory, see [5] and [7].

An edge of $G$ dominates itself and every edge adjacent to it. A dominating induced matching (DIM) (or efficient edge dominating set (EEDS) in some papers) of $G$ is an induced matching that dominates every edge of $G$. We denote $\text{dim}(G)$, the size of the smallest DIM in $G$. Clearly, not every graph has DIM, for instance consider $C_4$. Denote $E(u)$ as the set of all edges incident with $u$. For every edge $e$, let $D_e$ be the set of edges that are dominated by $e$. Note that for a DIM, all edges are being dominated by exactly one of the edges in DIM. The cycle $C_n$ has an DIM if and only if $n = 3k$. Dominating induced matchings have been extensively studied by several authors, for instance see [1], [3], [6], and [4].

2 Existence and Some Properties of Dominating Induced Matching

In this section, we prove that if a graph $G$ has a DIM, then its chromatic number does not exceed 3. Furthermore, we show that if $G$ has a DIM, then the size of all DIM of $G$ are the same. First, we start with the following result.

Theorem 1 If a graph $G$ has a DIM, then $\chi(G) \leq 3$.

Proof. Let $M = \{v_1w_1, \ldots, v_kw_k\}$ be a DIM for the graph $G$. For $i = 1, \ldots, k$ we color $v_i$ and $w_i$ by colors 1, 2, respectively and color the remaining vertices by color 3. By the definition of DIM, this coloring is a proper vertex coloring by 3 colors and so $\chi(G) \leq 3$. \hfill \Box

In the following result we provide an upper bound for the size of a graph which has a DIM.
Theorem 2 If a graph $G$ of order $n$ has a DIM, then $|E(G)| \leq \frac{n^2+n}{4}$.

Proof. Let $D$ be a DIM of $G$. For every vertex $v \in V(D)$, obviously, there are exactly $\dim(G)$ edges whose two ends are in $V(D)$. Moreover, since $V(G) \setminus V(D)$ is an independent set, there are no edges whose ends are in $V(G) \setminus V(D)$. Also, the number of edges between $V(D)$ and $V(G) \setminus V(D)$ is at most $\dim(G)(n-\dim(G))$. Therefore, we have the following inequality:

$$|E(G)| \leq 2\dim(G)(n-\dim(G)) + \dim(G).$$

This inequality will be maximized whenever $\dim(G) = \frac{n}{4}$. Therefore the following holds:

$$|E(G)| \leq \frac{n}{2}(n - \frac{n}{2}) + \frac{n}{4} = \frac{n^2 + n}{4},$$

as desired. \hfill $\square$

The next result shows that the size of all DIM of $G$ are the same.

Theorem 3 Let $G$ be a graph. If $D_1$ and $D_2$ are two DIM of $G$, then $|D_1| = |D_2|$.

Proof. Obviously, for every $e \in E(D_1)$, $|D_e \cap D_2| \leq 1$, because $e$ cannot be dominated by more than one edge in $D_2$. Since $E(G) = \bigcup_{e \in E(D_1)} D_e$, one can see that $|D_2| \leq |D_1|$. Similarly, we have $|D_1| \leq |D_2|$ and the proof is complete. \hfill $\square$

Theorem 4 Let $G$ be a graph of order $n$ with at least one DIM such that $\delta(G) \geq 2$. Then the following inequalities hold:

$$\frac{\delta(G)}{\Delta(G) - 1} \leq \frac{2\dim(G)}{n - 2\dim(G)} \leq \frac{\Delta(G)}{\delta(G) - 1}$$

In particular, if $G$ is a $k$-regular graph, then $\dim(G) = \frac{nk}{4k-2}$.

Proof. Let $D$ be a DIM of $G$. For every vertex $v \in V(D)$, since $v$ is adjacent to exactly one edge of $D$, $v$ has at least $\delta(G) - 1$ and at most $\Delta(G) - 1$ neighbors out of $V(D)$. Also for every vertex $v \notin V(D)$, since $V(G) \setminus V(D)$ is an independent set, $v$ has at least $\delta(G)$ and at most $\Delta(G)$ neighbors in $V(D)$. Then by double counting of the number of edges between $V(D)$ and $V(G) \setminus V(D)$, we find,

$$\delta(G)(n - 2\dim(G)) \leq 2\dim(G)(\Delta(G) - 1)$$

$$\Delta(G)(n - 2\dim(G)) \geq 2\dim(G)(\delta(G) - 1).$$

From these two inequalities, the proof of the first part is complete. Now, if $G$ is a $k$-regular graph, then the following hold:

$$\frac{k}{k-1} \leq \frac{2\dim(G)}{n - 2\dim(G)} \leq \frac{k}{k-1}.$$
This implies that,
\[ \frac{k}{k-1} = \frac{2\dim(G)}{n-2\dim(G)}, \]
and so we have,
\[ \dim(G) = \frac{nk}{4k-2}. \]

Now, we have an immediate corollary.

**Corollary 5** Let $G$ be a $k$-regular graph of order $n$ with at least one DIM. Then, 
$4k - 2|nk$.

In the following, we study the size of intersection of a DIM with cycles in a graph.

**Theorem 6** Let $G$ be a graph. Then for every cycle $C_r$ of $G$, and each DIM, $D$ of $G$ the following hold:

\[ |E(C_r) \cap E(D)| \leq \frac{r}{3}, \quad |E(C_r) \cap E(D)| \equiv r \pmod{2}. \]

**Proof.** Let $e \in E(D)$. If $e \in E(C_r)$, then $e$ dominates exactly 3 edges of $C_r$, and if $e \notin E(C_r)$, then $e$ dominates an even number of edges in $C_r$. Since every edge of $C_r$ is dominated by exactly one edge of $D$, then $|E(C_r) \cap E(D)| \leq \frac{r}{3}$ and the parity of $r$ equals to the parity of $|E(C_r) \cap E(D)|$. \qed

Now, we have the following corollary.

**Corollary 7** Let $G$ be a graph with a DIM, $D$. Then the following hold:

(i) Each $C_3$, $C_5$, or $C_7$ in $G$, meets $D$ in exactly one edge.

(ii) $D$ contains no edge of $C_4$ in $G$.

3 Edge Partitioning of Graphs into Dominating Induced Matching

In this section, we show that the edge set of some families of Kneser graphs can be partitioned into dominating induced matchings. Also, we show that if $G$ is a connected graph whose all edges can be partitioned into DIM, then $G$ is either a regular graph or a biregular graph. For example, all edges of the Petersen graph can be partitioned into five DIM, as shown in Figure 2.

**Theorem 8** Let $G$ be a connected graph whose all edges can be partitioned into DIM. Then $G$ is either a regular graph or a biregular graph. Moreover, the number of DIM is $d(u) + d(v) - 1$, for each $uv \in E(G)$. 

4
Proof. Let $u$ and $v$ be two adjacent vertices of $G$. Note that if one of the edges of $E(u) \cup E(v)$ is contained in a DIM, then no other edge of $E(u) \cup E(v)$ is in the DIM. Also, every DIM has at least one edge in $E(u) \cup E(v)$. Thus, every DIM has exactly one edge in $E(u) \cup E(v)$. Hence, if $uv \in E(G)$, then the number of DIM that partition the edges of $G$ is $d(u) + d(v) - 1$. Choose a vertex $w$. By earlier statement, we see that the degree of vertices in $N(w)$ are the same. With the same argument, the degree of all vertices of distance 2 from $w$ is $d(w)$, and so on. Hence $G$ is either a regular graph, when the degree of $w$ and its neighbors are the same, or a biregular graph, when the degree of $w$ and its neighbors are not the same. \hfill\Box

Before characterization of all graphs whose edges can be partitioned into DIM, we need two following results.

Theorem 9 Let $r$ be a positive integer. Then all edges of $KG_{2r-1,r-1}$ can be partitioned into $2r - 1$ DIM.

Proof. First we define an edge coloring for $KG_{2r-1,r-1}$ with the colors $1, \ldots, 2r - 1$. Let $XY$ be an edge of $KG_{2r-1,r-1}$, where $X$ and $Y$ are $(r - 1)$-subsets of $S = \{1, \ldots, 2r - 1\}$. We color the edge $XY$ by $S \setminus (X \cup Y)$ (Note that $S \setminus (X \cup Y)$ has one element). Assume that $C_1, \ldots, C_{2r-1}$ are all color classes. We claim that for $i = 1, \ldots, 2r - 1$, $C_i$ is a DIM. Let $XY, X'Y' \in C_i$. Thus we have $Y = Y' = S \setminus (X \cup \{i\})$, a contradiction. Hence all edges of $C_i$ form a matching. Now, assume that $XY, X'Y' \in C_i$ and there exists the edge $XY$ between $XY$ and $X'Y'$, where $X'Y' \in C_j$. Clearly, $i, j \notin X' \cup Y$. On the other hand since $X' \cap Y = \emptyset$, we have $|X' \cup Y| = 2r - 2$, contradiction. Let $Z$ and $Z'$ be two vertices of $KG_{2r-1,r-1}$ and no edge in $C_i$ is incident with $Z$ and $Z'$. Obviously, $i \in Z \cap Z'$ and so $Z$ and $Z'$ are not adjacent. Therefore for $i = 1, \ldots, 2r - 1$, $C_i$ is a DIM. The proof is complete. \hfill\Box

Now, we would like to prove that the edge set of $BG_{r-1,s-1}$ can be partitioned into DIM, where $r$ and $s$ are two positive integers.

Theorem 10 For every positive integers $r, s$, $E(BG_{r-1,s-1})$ can be partitioned into $r + s - 1$ DIM.

Proof. First, we introduce an edge coloring for $BG_{r-1,s-1}$ using the colors $S = \{1, \ldots, r + s - 1\}$. For the edge $XY$, we color $XY'$ by the color $S \setminus (X \cup Y)$. For $i = 1, \ldots, r + s - 1$ denote the $i$-th color class by $C_i$. Let $XY$ and $XY'$ be two edges in $C_i$. Then, $Y = Y' = S \setminus (X \cup \{i\})$, a contradiction. Now, Let $XY, X'Y' \in C_i$ and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{petersen_graph_dim.png}
\caption{Partitioning of all edges of Petersen graph into five DIM.}
\end{figure}
Let $Z$ and $Z'$ be two vertices of $BG_{r-1,s-1}$ and no edge of $C_i$ is incident with $Z$ and $Z'$. Obviously, $i \in Z \cap Z'$ and so $Z$ and $Z'$ are not adjacent. Therefore, for $i = 1, \ldots, r + s - 1$, $C_i$ is a DIM.

Now, we are in a position to prove our main theorem.

**Theorem 11** Let $G$ be an $r$-regular graph and $E(G)$ can be partitioned into DIM. Then there exists a list of assignment $L : V(G) \rightarrow A$, where $A$ is the set of all $(r-1)$-subsets of $S = \{1, \ldots, 2r-1\}$, which has the following properties:

(i) For every $uv \in E(G)$, $L(u) \cap L(v) = \emptyset$,

(ii) $L$ is surjective,

(iii) For every $a, b \in A$, $|L^{-1}(a)| = |L^{-1}(b)|$.

**Proof.** By Theorem 8 let $M_1, M_2, \ldots, M_{2r-1}$ be all DIM which partition $E(G)$. Now, we define an edge coloring for $G$. For every $e \in E(G)$ define $c(e) = t$ if and only if $e \in M_t$. Now, we introduce the list assignment $L : V(G) \rightarrow A$ as follows. For every $v \in V(G)$, define $L(v) = S \setminus \{c(e) \mid e \text{ is incident with } v\}$:

(i) With no loss of generality, one can assume that $G$ is a connected graph. First we prove Part (i). By contradiction assume that $uv \in E(G)$ and $j \in L(u) \cap L(v)$. By the definition, no edge incident with $u$ or $v$ has color $j$. Hence $M_j$ does not dominate $uv$, a contradiction. So (i) is proved.

(ii) First we claim that for every $v \in V(G)$, if $a$ is an $(r-1)$-subset of $S$ such that $L(v) \cap a = \emptyset$, then there is a vertex $u \in N(v)$ such that $L(u) = a$. Suppose that $u, u' \in N(v)$. It is not hard to see that, $c(uv) \not\in L(u)$ and $c(uv) \in L(u')$. Thus, $L(u) \neq L(u')$. Since $d(v) = r$ and $S \setminus L(v)$ have exactly $r$, $(r-1)$-subsets, we conclude that every $(r-1)$-subset of $S \setminus L(v)$ appears as a list of a vertex in $N(v)$ once and the claim is proved. Let $v \in V(G)$ and $L(v) = a$. Assume that $b$ is an $(r-1)$-subset of $S$ and $k = |a \cap b|$. By induction on $k$ we show that there exists a vertex $u \in V(G)$ such that $L(u) = b$. For $k = 0$, there is nothing to prove. Let the assertion hold for $k-1$ and $k = |a \cap b|$. Assume that $x \in a \cap b$ and $y \in S \setminus (a \cup b)$. Let $c = (b \setminus \{x\}) \cup \{y\}$. Clearly, $|a \cap c| = k - 1$ and so by the induction hypothesis there exists $w \in V(G)$ such that $L(w) = c$. Suppose that $d = S \setminus (\{x\} \cup c)$. Since $L(w) \cap d = \emptyset$, By the claim, there exists $f \in V(G)$ which is adjacent to $w$ and $L(f) = d$. Since $b \cap d = \emptyset$, by the claim there exists $g \in V(G)$ such that $L(g) = b$.

(iii) Since the restriction of $L$ on each connected component of $G$ is surjective, it suffices we prove that if $a \cap b = \emptyset$, then $|L^{-1}(a)| = |L^{-1}(b)|$. By the claim of Part (ii) we note that for every $v \in L^{-1}(a)$ there exists $u \in L^{-1}(b)$ which is adjacent to $v$, such that $c(uv) = S \setminus (a \cup b)$. Since by the claim of Part (ii) for every $v' \in N(u), c(uv) \neq c(uv')$, then every vertex in $L^{-1}(a)$ is adjacent to exactly one vertex in $L^{-1}(b)$. This implies that $|L^{-1}(a)| \leq |L^{-1}(b)|$. Similarly, $|L^{-1}(b)| \leq |L^{-1}(a)|$ and the proof is complete. \qed
Remark 12 By the same proof of Theorem 11 one can see that the three following statements hold for biregular graphs \( G(X, Y) \) in which \( E(G) \) can be partitioned into \( \text{DIM} \), where for every \( u \in X \) and \( v \in Y \), \( d(u) = x \) and \( d(v) = y \). If we replace \( L \) (define in Theorem 11) with \( L': V(G) \to A_1 \cup A_2 \), where \( A_1 \) and \( A_2 \) are respectively the \((x-1)\)-subsets and \((y-1)\)-subsets of \( S = \{1, 2, \ldots, x+y-1\} \) and \( L'(X) \subseteq A_1 \), \( L'(Y) \subseteq A_2 \):

(i) For every \( uv \in E(G) \), \( L'(u) \cap L'(v) = \emptyset \),

(ii) \( L' \) is surjective,

(iii) For every \( a, b \in A_1 \cup A_2 \), \(|L'^{-1}(a)| = |L'^{-1}(b)|\).

Now, we have the following corollary.

Corollary 13 Let \( G \) be an \( r \)-regular graph and \( E(G) \) can be partitioned into \( \text{DIM} \). Then \(|V(G)| \) is divisible by \( \binom{2r-1}{r-1} \) and \(|V(G)| = \binom{2r-1}{r-1} \) if and only if \( G \) is \( KG_{2r-1,r-1} \).

References

[1] A. Brandstädt, R. Mosca, Dominating induced matchings for P7-free graphs in linear time, Algorithmica 68(4) (2014): 998-1018.

[2] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97-102.

[3] D.M. Cardoso, E.A. Martins, L. Medina, O. Rojo, Spectral results for the dominating induced matching problem, Discrete Appl. Math. 234 (2018) 22-31.

[4] D.M. Cardoso, N. Korpelainen, V.V. Lozin, On the complexity of the dominating induced matching problem in hereditary classes of graphs, Discrete Appl. Math. 159 (2011) 521-531.

[5] D.L. Grinstead, P.J. Slater, N.A. Sherwani, N.D. Holmes, Efficient edge domination problems in graphs, Inform. Process. Lett. 48 (1993) 221-228.

[6] A. Hertz, V. Lozin, B. Ries, V. Zamaraev and D. de Werra, Dominating induced matchings in graphs containing no long claw, Journal of Graph Theory 88 (2018), 18-39.

[7] M. Livingston and Q.F. Stout, Distributing resources in hypercube computers, Proceedings of the Third Conference on Hypercube Concurrent Computers and Applications, (1988) 222-231.