EVALUATIONS OF SOME SERIES OF THE TYPE
\[ \sum_{k=0}^{\infty} \frac{(ak + b)x^k}{(mk)(nk)} \]

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Abstract. In this paper, via the beta function we evaluate some series of the type \( \sum_{k=0}^{\infty} \frac{(ak + b)x^k}{(mk)(nk)} \). We completely determine the values of
\[ \sum_{k=0}^{\infty} \left( \frac{49k + 1}{8k} \right) \left( \frac{-27}{4} < x < \frac{27}{4} \right) \]
and
\[ \sum_{k=0}^{\infty} \left( \frac{10k - 1}{4k} \right) \left( -16 < x < 16 \right) \]
for \( r = 0, \pm 1 \). For example, we prove that
\[ \sum_{k=0}^{\infty} \frac{10k - 1}{4k} = \frac{4\sqrt{3}}{27} \pi. \]

We also establish the following efficient formula for computing \( \log n \) with \( 1 < n \leq 85/4 \):
\[ \frac{1}{4} \sum_{k=0}^{\infty} \frac{x^{2k}}{\left( \frac{2k}{k} \right)} = \frac{\sqrt{4 - x^2} + x \arcsin(x/2)}{(4 - x^2)\sqrt{4 - x^2}} \quad (1.1) \]
and
\[ \frac{1}{4} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{\left( \frac{2k}{k} \right)} = \frac{1}{4 + x^2} - \frac{x \arcsinh(x/2)}{(4 + x^2)\sqrt{4 + x^2}} \quad (1.2) \]
where
\[ \arcsinh t = \sum_{n=0}^{\infty} \frac{(2n)!}{(2n+1)(-4)^n} = \log(t + \sqrt{t^2 + 1}) \]

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is the inverse hyperbolic sine function. It is also known that

$$
\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2 \binom{2k}{k}} = 2 \arcsin^2 \frac{x}{2}
$$

for any \( x \in \mathbb{R} \) with \( |x| < 2 \) (see, e.g., [3]).

Using (1.1) and (1.2) and their derivatives, we can easily deduce that

$$
\sum_{k=0}^{\infty} \frac{2(2n+1)^2k + 3}{(-n(n+1))^k \binom{2k}{k}} = -\frac{2n(n+1)}{2n+1} \log \left( 1 + \frac{1}{n} \right)
$$

(1.3)

if \( n < -\left(1 + \sqrt{2}\right)/2 \) or \( n > (\sqrt{2} - 1)/2 \). When \( n = 1/4 \) this yields the identity

$$
\sum_{k=0}^{\infty} \frac{(3k + 2)16^k}{(-5)^k \binom{2k}{k}} = -\frac{5}{18} \log 5.
$$

In contrast, if \( n > 1 \) or \( n < -1 \) then

$$
\log \left( 1 + \frac{1}{n} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{kn^k}
$$

by the Taylor series.

In 1974 R. W. Gosper announced the new identity

$$
\sum_{k=0}^{\infty} \frac{25k - 3}{2^k \binom{4k}{2k}} = \frac{\pi}{2},
$$

(1.4)

which was later used by F. Bellard [2] to find an algorithm for computing the \( n \)th decimal of \( \pi \) without computing the earlier ones. Inspired by Gosper’s identity, in 2003 Bellard [2] discovered the identity

$$
\pi = \frac{1}{740025} \left( \sum_{k=1}^{\infty} \frac{3P(k)}{2^{2k-1} \binom{2k}{2k}} - 20379280 \right),
$$

where

$$
P(n) = -885673181k^5 + 3125347237k^4 - 2942969225k^3
$$

$$
+ 1031962795k^2 - 196882274k + 10996648;
$$

he used this identity to set his world record of computing the \( 10^{11} \) binary digit of \( \pi \). Bellard [2]. Moreover, G. Almkvist, C. Krattenthaler and J. Petersson [1] gave a proof of Gosper’s identity and found 12 new identities of the type

$$
\pi = \sum_{k=0}^{\infty} \frac{P(k)}{a^k \binom{mk}{nk}},
$$

where \( P(x) \in \mathbb{Q}[x] \), and \((m, n, a, \deg P)\) is among the ordered quadruples

(8, 4, −4, 4), (10, 4, 4, 8), (12, 4, −4, 8), (16, 8, 16, 8),

(24, 12, −64, 12), (32, 16, 256, 16), (40, 20, −2^{10}, 20), (48, 24, 2^{12}, 24),

(56, 28, −2^{14}, 28), (64, 32, 2^{16}, 32), (72, 36, −2^{18}, 36), (80, 40, 2^{20}, 40).
For example, [1, Example 2] gives the identity
\[ \pi = \frac{1}{105^2} \sum_{k=0}^{\infty} \frac{P(k)}{(sk)(4k)}(-4)^k, \]
where
\[ P(k) = -89286 + 3875948k - 34970134k^2 + 11020472k^3 - 115193600k^4. \]

By Stirling’s formula,
\[ k! \sim \sqrt{2\pi k} \left( \frac{k}{e} \right)^k \text{ as } k \to +\infty. \]

If \( m > n > 0 \) are integers, then
\[ \binom{mk}{nk} = \frac{(mk)!}{(nk)!((m-n)k)!} \sim \frac{\sqrt{m}}{\sqrt{2\pi}(m-n)k} \left( \frac{m^m}{n^n(m-n)^{m-n}} \right)^k \]
as \( k \to +\infty. \)

In this paper we evaluate some series of the type
\[ \sum_{k=0}^{\infty} (ak + b) \frac{x^k}{\binom{mk}{nk}}, \]
where \( m > n > 0 \) are integers and \( a, b, x \) are real numbers with
\[ |x| < \frac{m^m}{n^n(m-n)^{m-n}}. \]

Note that
\[ \binom{3k}{k} \sim \frac{\sqrt{3}}{2\sqrt{k\pi}} \left( \frac{27}{4} \right)^k \text{ as } k \to +\infty. \]
Thus, for any real number \( x_0 \) with \(-27/4 < x_0 < 27/4\) the series
\[ \sum_{k=1}^{\infty} \frac{k^r x_0^k}{\binom{3k}{k}} \] (\( r = 0, \pm 1 \))
converge absolutely. How to evaluate them? Let
\[ c = \frac{3}{2} \left( (1 + \sqrt{2})^{1/3} - (1 + \sqrt{2})^{-1/3} \right) = 0.8941 \cdots. \]

For \( f(x) = x^3 - x_0(x-1) \), clearly \( f(-3) = -27 + 4x_0 < 0 \) and
\[ f(c) = c^3 + (1-c)x_0 = \frac{27}{4}(1-c) + (1-c)x_0 = (1-c) \left( x_0 + \frac{27}{4} \right) > 0. \]

So there is a real number \(-3 < x < c\) such that \( f(x) = 0 \) and hence \( x_0 = x^3/(x-1) \). Moreover, such \( x \in (-3,c) \) can be found by solving the cubic equation \( x^3 = x_0(x-1) \).

Now we state our first theorem.
Theorem 1.1. Let $-3 < x < c$ with $c$ given by (1.5).

(i) We have

$$\sum_{k=1}^{\infty} \frac{((2x - 3)^2 + 2x^2 + 2x - 3)x^{3k}}{(x - 1)^k \binom{3k}{k}} = -2x^3 \frac{x + 7}{(x + 3)^2} + \frac{8x^2(x - 1)q(x)}{(x + 3)^2 \sqrt{(1 - x)(3 + x)}},$$

where

$$q(x) = \begin{cases} \arctan \frac{x}{x + 2} \sqrt{\frac{3 + x}{1 - x}} & \text{if } -2 < x < 1, \\ -\frac{\pi}{2} & \text{if } x = -2, \\ \arctan \frac{x}{x + 2} \sqrt{\frac{3 + x}{1 - x}} - \pi & \text{if } -3 < x < -2. \end{cases}$$

(ii) We have

$$\sum_{k=0}^{\infty} \frac{(s(x)k + t(x))x^{3k}}{(x - 1)^k \binom{3k}{k}} = 12x^2(1-x) \log(1-x) - 27(1-x)(x^2 - 6x + 3),$$

where

$$s(x) = (x + 3)(2x - 3)^2(x^2 - 12x + 9) = 4x^5 - 48x^4 + 9x^3 + 351x^2 - 567x + 243$$

and

$$t(x) = 2x^5 - 48x^4 + 69x^3 - 189x^2 + 243x - 81$$

Remark 1.1. As

$$\begin{vmatrix} (2x - 3)^2 & 2x^2 + 2x - 3 \\ s(x) & t(x) \end{vmatrix} = -4x(2x - 3)^5 \neq 0$$

for all $x \in (-3, c)$ with $x \neq 0$, combining the two parts of Theorem 1.1 we have actually determined the values of

$$\sum_{k=0}^{\infty} \frac{x^{3k}}{(x - 1)^k \binom{3k}{k}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{kx^{3k}}{(x - 1)^k \binom{3k}{k}}$$

for all $x \in (-3, c)$. To evaluate

$$\sum_{k=1}^{\infty} \frac{x^{3k}}{k(x - 1)^k \binom{3k}{k}}$$

for $x \in (-3, c)$, it suffices to note that

$$\sum_{k=1}^{n} \frac{x^{3k}}{(x - 1)^k \binom{3k}{k}} \left( \frac{6}{k}(1 - x) + (2x^3 + 27x - 27) + (4x^3 - 27x + 27)k \right)$$

$$= 2x^3 \left( \frac{2n + 1}{\binom{3n}{n}} \cdot \frac{x^{3n}}{(x - 1)^n} - 1 \right)$$
which tends to \(-2x^3\) as \(n \to +\infty\). Therefore, in light of Theorem 1.1, we have determined the values of

\[
\sum_{k=0}^{\infty} \frac{k^r x_0^k}{(3k)^k} \left( \frac{27}{4} < x_0 < \frac{27}{4} \right)
\]

for all \(r \in \{0, \pm 1\}\). In particular,

\[
\sum_{k=1}^{\infty} \frac{8^k}{k^3 (3k)^k} = \frac{2}{7} \left( \pi \sqrt{3} - \log 3 \right).
\]

We are also able to show

\[
\sum_{k=1}^{\infty} \frac{g^k}{k^2 3^k (3k)^k} = \frac{\pi^2 - 3 \log^2 3}{6}
\]

via our method to prove Theorem 1.1.

Note that (1.6) in the case \(x = -1\) gives Gosper’s identity (1.4).

Putting \(x = -2\) in (1.6) and \(x = 1/n\) in (1.7), we obtain the following corollary.

**Corollary 1.1.** (i) We have

\[
\sum_{k=0}^{\infty} \frac{(49k + 1)8^k}{3^k (3k)^k} = 81 + 16\sqrt{3}\pi.
\]

(ii) If \(n < -1/3\) or \(n > 1/c = 1.11843\ldots\), then

\[
\sum_{k=0}^{\infty} \frac{a_n k^r - b_n}{((1 - n)n^2)^k (3k)^k} = 3n^2(n - 1) \left( 4 \log \left( 1 - \frac{1}{n} \right) - 9(3n^2 - 6n + 1) \right),
\]

where

\[
\begin{align*}
a_n &= (3n + 1)(3n - 2)^2(9n^2 - 12n + 1) \\
b_n &= 81n^5 - 243n^4 + 189n^3 - 69n^2 + 48n - 2.
\end{align*}
\]

In particular,

\[
\begin{align*}
\sum_{k=0}^{\infty} \frac{275k - 158}{2^k (3k)^k} &= 6 \log 2 - 135, \\
\sum_{k=0}^{\infty} \frac{728k - 17}{(-4)^k (3k)^k} &= -54 - 24 \log 2, \\
\sum_{k=0}^{\infty} \frac{(1813k - 2707)8^k}{3^k (3k)^k} &= 9(16 \log 3 - 171),
\end{align*}
\]
\[
\sum_{k=0}^{\infty} \frac{5635k - 1156}{(-18)^k \binom{3k}{k}} = 54 \log \frac{2}{3} - 1215, \quad (1.15)
\]
\[
\sum_{k=0}^{\infty} \frac{63050k - 15959}{(-48)^k \binom{3k}{k}} = 72 \left(4 \log \frac{3}{4} - 225\right), \quad (1.16)
\]
\[
\sum_{k=0}^{\infty} \frac{112216k - 30847}{(-100)^k \binom{3k}{k}} = 300 \log \frac{4}{5} - 31050, \quad (1.17)
\]
\[
\sum_{k=0}^{\infty} \frac{615296k - 176777}{(-180)^k \binom{3k}{k}} = 270 \left(4 \log \frac{5}{6} - 657\right), \quad (1.18)
\]
\[
\sum_{k=0}^{\infty} \frac{710809k - 209926}{(-294)^k \binom{3k}{k}} = 441 \left(2 \log \frac{6}{7} - 477\right), \quad (1.19)
\]
\[
\sum_{k=0}^{\infty} \frac{2910050k - 875807}{(-448)^k \binom{3k}{k}} = 672 \left(2 \log \frac{7}{8} - 855\right), \quad (1.20)
\]
\[
\sum_{k=0}^{\infty} \frac{9490712k - 2926289}{(-1210)^k \binom{3k}{k}} = 1350 \left(4 \log \frac{9}{10} - 2169\right), \quad (1.21)
\]
\[
\sum_{k=0}^{\infty} \frac{7825423k - 2432776}{(-1210)^k \binom{3k}{k}} = 1815 \left(2 \log \frac{10}{11} - 1341\right). \quad (1.22)
\]

Let \(x \in (-16, 16)\). By induction, we have
\[
\sum_{k=1}^{n} \frac{x^k}{(4k)!(2k)!} \left(-\frac{6}{k} + x + 32 + 2k(x - 16)\right) = -x + \frac{(2n + 1)x^{n+1}}{(2n)!(2n+1)}.
\]
which tends to \(-x\) as \(n \to +\infty\). Thus, if we know the values of
\[
\sum_{k=0}^{\infty} \frac{x^k}{(4k)!(2k)!} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{kx^k}{(4k)!(2k)!}
\]
then the value of \(\sum_{k=1}^{n} \frac{x^k}{(4k)!(2k)!} / (k(4k)!(2k)!))\) is also determined.

For any \(x \in \mathbb{R}\) with \(x > 1\) or \(x < 0\), we define
\[
R(x) := \sqrt{x} \arctanh \frac{1}{\sqrt{x}} = \begin{cases} \frac{\sqrt{x}}{2} \log \frac{\sqrt{x}+1}{\sqrt{x}-1} & \text{if } x > 1, \\ \sqrt{|x|} \arctan \frac{1}{|x|} & \text{if } x < 0, \end{cases}
\]
where \(\arctanh t\) is the inverse hyperbolic tangent function. Note that for \(x > 1\) or \(x < -1\), we have
\[
R(x) = \sqrt{x} \sum_{k=0}^{\infty} \frac{(1/\sqrt{x})^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} \frac{x^{-k}}{2k+1}.
\]
SERIES OF THE TYPE $\sum_{k=0}^{\infty} (ak + b)x^k / \binom{mk}{nk}$

Now we are ready to state our second theorem which involves the binomial coefficients $\binom{4k}{2k}$ with $k \in \mathbb{N}$.

**Theorem 1.2.** For any $x > 1/4$, we have

$$\sum_{k=0}^{\infty} \frac{2(4x + 1)k - 2x + 1}{x^{2k} \binom{4k}{2k}} = \frac{8x^2}{(4x - 1)^2} \left( \frac{3}{\sqrt{4x - 1}} \arccot \sqrt{4x - 1} - 4x + 4 \right)$$

(1.25)

and

$$\sum_{k=0}^{\infty} \frac{2(4x - 1)k - 2x - 1}{x^{2k} \binom{4k}{2k}} = \frac{8x^2}{(4x + 1)^2} \left( \frac{3R(4x + 1)}{4x + 1} - 4x - 4 \right).$$

(1.26)

Consequently,

$$\sum_{k=0}^{\infty} \frac{10k - 1}{\binom{4k}{2k}} = \frac{4\sqrt{3}}{27} \pi,$$

(1.27)

$$\sum_{k=0}^{\infty} \frac{k4k}{\binom{4k}{2k}} = \frac{3\pi + 8}{12},$$

(1.28)

$$\sum_{k=0}^{\infty} \frac{(14k + 1)9k}{\binom{4k}{2k}} = 24\pi \sqrt{3} + 64,$$

(1.29)

$$\sum_{k=0}^{\infty} \frac{(22k - 1)9k}{4^k \binom{4k}{2k}} = \frac{32}{25} \left( 4 + \frac{27}{\sqrt{15}} \arctan \sqrt{\frac{3}{5}} \right),$$

(1.30)

$$\sum_{k=0}^{\infty} \frac{14k - 5}{4^k \binom{4k}{2k}} = \frac{16}{81}(\log 2 - 24).$$

(1.31)

**Remark 1.2.** As

$$\left| \frac{2(4x + 1)}{2(4x - 1)} \right| = \frac{-2x + 1}{-2x - 1} = -24x \neq 0$$

for all $x > 1/4$, combining the two parts of Theorem 1.2 we have actually determined the values of

$$\sum_{k=0}^{\infty} \frac{x_0^k}{\binom{4k}{2k}} \text{ and } \sum_{k=0}^{\infty} \frac{kx_0^k}{\binom{4k}{2k}}$$

for all $x_0 \in (0, 16)$.

For $x_0 \in (0, 16)$, how to evaluate

$$\sum_{k=1}^{\infty} \frac{(-x_0)^k}{k^{4k}} \text{ and } \sum_{k=1}^{\infty} \frac{k(-x_0)^k}{k^{4k}}?$$

If we take

$$x = \frac{1}{2} + \sqrt{\frac{4}{x_0} + \frac{1}{4}} > \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1 + \sqrt{2}}{2},$$

then...
then

\[-x_0 = \frac{4}{x(1-x)}.\]

Now we state our third theorem.

**Theorem 1.3.** If \(x > (1 + \sqrt{2})/2\) or \(x < (1 - \sqrt{2})/2\), then

\[
\sum_{k=0}^{\infty} \frac{(2(2x-1)^2(2x-3)k - (4x^3 - 16x^2 + 7x + 6))4^k}{(x(1-x))^k \binom{4k}{2k}} = (1-x)(3R(x) + 4x(x-3))
\] (1.32)

and

\[
\sum_{k=0}^{\infty} \frac{(2(2x-1)^2(2x+1)k - (4x^3 + 4x^2 - 13x - 1))4^k}{(x(1-x))^k \binom{4k}{2k}} = -x(3R(1-x) + 4(x-1)(x+2)).
\] (1.33)

**Remark 1.3.** As

\[
\left| \frac{2(2x-1)^2(2x-3)}{2(2x-1)^2(2x+1)} \right| = \left| \frac{-4x^3 + 16x^2 - 7x + 6}{-4x^3 - 4x^2 + 13x - 1} \right| = -6(2x-1)^5 \neq 0
\]

if \(x > (1 + \sqrt{2})/2\) or \(x < (1 - \sqrt{2})/2\), combining the two parts of Theorem 1.2 we have actually determined the values of

\[
\sum_{k=0}^{\infty} \frac{(-x_0)^k}{\binom{4k}{2k}} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{k(-x_0)^k}{\binom{4k}{2k}}
\]

for all \(x_0 \in (0, 16)\).

**Corollary 1.2.** We have

\[
\sum_{k=0}^{\infty} \frac{(30k - 7)(-2)^k}{\binom{4k}{2k}} = -\frac{3\pi + 64}{6}.
\] (1.34)

**Proof.** As

\[
\lim_{x \to -1} R(x) = \arctan 1 = \frac{\pi}{4}.
\]

Letting \(x \to -1\) in the identity (1.32), we immediately get (1.34). \(\square\)

**Corollary 1.3.** For

\[
1 < n < \frac{\sqrt{(1 + \sqrt{2})/2 + 1}}{\sqrt{(1 + \sqrt{2})/2 - 1}} = 21.2666866\ldots,
\]

we have the following formula for \(\log n:\)

\[
\sum_{k=0}^{\infty} \frac{(2(n^2 + 6n + 1)^2(n^2 - 10n + 1)k + P(n))(n-1)^{4k}}{(-n)^k(n+1)^{2k} \binom{4k}{2k}} = 6n(n+1)(n-1)^2 \log n - 32n(n+1)^2(n^2 - 4n + 1),
\] (1.35)

where

\[
P(n) := n^6 - 58n^5 + 159n^4 + 52n^3 + 159n^2 - 58n + 1.
\]
In particular,

\[
\sum_{k=0}^{\infty} \frac{2890k - 563}{(-18)^k (4k\choose 2k)} = -12(\log 2 + 48), 
\quad (1.36)
\]

\[
\sum_{k=0}^{\infty} \frac{245k - 17}{(-3)^k (4k\choose 2k)} = -24 - \frac{9}{2} \log 3, 
\quad (1.37)
\]

\[
\sum_{k=0}^{\infty} \frac{(77326k + 8951)81^k}{(-100)^k (4k\choose 2k)} = 40(80 - 81 \log 4), 
\quad (1.38)
\]

\[
\sum_{k=0}^{\infty} \frac{(196k + 73)64^k}{(-45)^k (4k\choose 2k)} = 15(3 - \log 5), 
\quad (1.39)
\]

\[
\sum_{k=0}^{\infty} \frac{(245134k + 181679)625^k}{(-294)^k (4k\choose 2k)} = 84(1456 - 375 \log 6), 
\quad (1.40)
\]

\[
\sum_{k=0}^{\infty} \frac{(127890k + 316933)2401^k}{(-28)^k (4k\choose 2k)} = 144(1584 - 343 \log 8), 
\quad (1.41)
\]

\[
\sum_{k=0}^{\infty} \frac{(1156k + 7031)1024^k}{(-225)^k (4k\choose 2k)} = 45(115 - 24 \log 9), 
\quad (1.42)
\]

\[
\sum_{k=0}^{\infty} \frac{(51842k - 3142679)6561^k}{(-1210)^k (4k\choose 2k)} = 220(2187 \log 10 - 10736), 
\quad (1.43)
\]

\[
\sum_{k=0}^{\infty} \frac{(2209k - 13421)625^k}{(-99)^k (4k\choose 2k)} = \frac{99}{2}(125 \log 11 - 624), 
\quad (1.44)
\]

\[
\sum_{k=0}^{\infty} \frac{(2354450k - 8037191)14641^k}{(-2028)^k (4k\choose 2k)} = 312(3993 \log 12 - 20176), 
\quad (1.45)
\]

\[
\sum_{k=0}^{\infty} \frac{(19220k - 46979)5184^k}{(-637)^k (4k\choose 2k)} = 91(81 \log 13 - 413), 
\quad (1.46)
\]

\[
\sum_{k=0}^{\infty} \frac{(3000515k - 5794357)28561^k}{(-3150)^k (4k\choose 2k)} = 420(2197 \log 14 - 11280), 
\quad (1.47)
\]

\[
\sum_{k=0}^{\infty} \frac{(118579k - 190573)2401^k}{(-240)^k (4k\choose 2k)} = 30(1029 \log 15 - 5312), 
\quad (1.48)
\]
\[
\sum_{k=0}^{\infty} \frac{(24174146k - 33367199)50625^k}{(-4624)^k \binom{4k}{2k}} = 544(10125 \log 16 - 52496), \quad (1.50)
\]
\[
\sum_{k=0}^{\infty} \frac{(48020k - 58117)16384^k}{(-1377)^k \binom{4k}{2k}} = 459(64 \log 17 - 333), \quad (1.51)
\]
\[
\sum_{k=0}^{\infty} \frac{(54371810k - 58537799)83521^k}{(-6498)^k \binom{4k}{2k}} = 684(14739 \log 18 - 76912), \quad (1.52)
\]
\[
\sum_{k=0}^{\infty} \frac{(608923k - 589327)6561^k}{(-475)^k \binom{4k}{2k}} = 95 \left(2187 \log 19 - 11440\right), \quad (1.53)
\]
\[
\sum_{k=0}^{\infty} \frac{(36377094k - 31893853)130321^k}{(-8820)^k \binom{4k}{2k}} = 840(6859 \log 20 - 35952), \quad (1.54)
\]
\[
\sum_{k=0}^{\infty} \frac{(584756k - 467339)40000^k}{(-2541)^k \binom{4k}{2k}} = 231 \left(375 \log 21 - 1969\right), \quad (1.55)
\]
\[
\sum_{k=0}^{\infty} \frac{(661704134402k - 517115569199)43046721^k}{2693140 \binom{4k}{2k}} = 60520 \left(1594323 \log 85 - 8374544\right). \quad (1.56)
\]

Proof. Putting \(x = \frac{(n + 1)^2}{(n - 1)^2}\) in (1.32), we get (1.35). Taking \(n = 2, \ldots, 21, 85/4\) in (1.35) we immediately obtain the remaining identities. \(\square\)

**Remark 1.4.** Note that our identities (1.36)-(1.55) provide series for

\[
\log 2, \ldots, \log 21
\]

which converge rapidly. The identity (1.35) with \(n = 5/3, 7/5, 9/7\) yields the following examples:

\[
\sum_{k=0}^{\infty} \frac{27869k - 6203}{(-60)^k \binom{4k}{2k}} = -15 \left(416 + 3 \log \frac{5}{3}\right), \quad (1.57)
\]
\[
\sum_{k=0}^{\infty} \frac{115943k - 27691}{(-315)^k \binom{4k}{2k}} = -\frac{105}{2} \left(528 + 3 \log \frac{7}{5}\right), \quad (1.58)
\]
\[
\sum_{k=0}^{\infty} \frac{2016125k - 491747}{(-1008)^k \binom{4k}{2k}} = -126 \left(3904 + 3 \log \frac{9}{7}\right). \quad (1.59)
\]

In the spirit of [7], by combining previous theorems with some related finite identities, we obtain the following result.
Theorem 1.4. We have

\[ \sum_{k=1}^{\infty} \frac{5k - 2}{k(2k - 1)2^k \binom{3k}{k}} = \frac{\pi}{6}, \]  
(1.60)

\[ \sum_{k=1}^{\infty} \frac{(7k - 3)8^k}{k(2k - 1)3^k \binom{4k}{k}} = \frac{8\sqrt{3}}{9}, \]  
(1.61)

\[ \sum_{k=1}^{\infty} \frac{28k - 11}{k(2k - 1)(-4)^k \binom{3k}{2k}} = -2\log 2, \]  
(1.62)

\[ \sum_{k=1}^{\infty} \frac{7k - 2}{k(2k - 1)(-3)^k \binom{4k}{2k}} = -\log 3 \]  
(1.63)

\[ \sum_{k=1}^{\infty} \frac{10k - 3}{k(2k - 1) \binom{4k}{2k}} = \frac{2\sqrt{3}}{9}, \]  
(1.64)

\[ \sum_{k=1}^{\infty} \frac{(3k - 1)4^k}{k(2k - 1) \binom{4k}{2k}} = \frac{\pi}{2}, \]  
(1.65)

\[ \sum_{k=1}^{\infty} \frac{(6k - 1)(-2)^{k-1}}{k(2k - 1) \binom{4k}{2k}} = \frac{\pi}{4}, \]  
(1.66)

\[ \sum_{k=1}^{\infty} \frac{14k - 3}{k(2k - 1)4^k \binom{4k}{2k}} = \frac{2}{3}\log 2, \]  
(1.67)

\[ \sum_{k=1}^{\infty} \frac{(12k^2 + 1)4^k}{\binom{4k}{2k}} = \frac{11}{2}\pi + \frac{50}{3}, \]  
(1.68)

\[ \sum_{k=1}^{\infty} \frac{k(126k + 29)(-2)^k}{\binom{4k}{2k}} = -2\pi - \frac{65}{3}, \]  
(1.69)

\[ \sum_{k=1}^{\infty} \frac{k(70k - 37)}{4^k \binom{4k}{2k}} = \frac{8}{729}(46\log 2 + 111). \]  
(1.70)

Remark 1.5. Actually we have many other identities similar to those in Theorem 1.4.

In the next section we shall give an auxiliary proposition whose proof involves the beta function

\[ B(a, b) := \int_{0}^{1} x^{a-1}(1 - x)^{b-1}dx \quad \text{for } a > 0 \text{ and } b > 0. \]

Our proofs of Theorems 1.1-1.4 will be given in Sections 3-6 respectively.

In view of [4, 6, 8], we also have related conjectures on p-adic congruences motivated by our theorems.
2. An auxiliary proposition

**Lemma 2.1.** For any complex number \( z \) with \(|z| < 1\), we have

\[
\sum_{k=1}^{\infty} k z^k = \frac{z}{(1-z)^2} \quad \text{and} \quad \sum_{k=1}^{\infty} k^2 z^k = \frac{z(z+1)}{(1-z)^3}.
\] (2.1)

**Proof.** This is easy. Recall the well-known identity

\[
\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad (|z| < 1)
\]

Taking derivatives of both sides, we get

\[
\sum_{k=1}^{\infty} k z^{k-1} = \frac{1}{(1-z)^2}
\] (2.2)

and this implies the first identity in (2.1). Taking derivatives of both sides of (2.2), we obtain

\[
\sum_{k=1}^{\infty} k(k-1) z^k = \frac{2z^2}{(1-z)^3}.
\] (2.3)

Adding this and the first identity in (2.1), we immediately get the second identity in (2.1). \(\square\)

The beta function is connected with the Gamma function

\[
\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-t} dt \quad (x > 0)
\]

as first pointed out by Euler.

**Lemma 2.2 (Euler).** For any \( a > 0 \) and \( b > 0 \), we have

\[
B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.
\] (2.4)

Now we present an auxiliary proposition.

**Proposition 2.1.** Let \( m > n > 0 \) be integers, and let \( a, b, x \) be real numbers with \(|x| < m^m/(n^n(m-n)^{m-n})\), and set

\[
S_{m,n}(a, b, x) := \sum_{k=1}^{\infty} (ak + b) \frac{x^k}{\binom{mk}{nk}}.
\]

Then

\[
S_{m,n}(a, b, x) = n \int_0^1 T_{m,n}(a, b, x; t) dt,
\]

where

\[
T_{m,n}(a, b, x; t) := t^{n-1} (1-t)^{m-n} x \frac{(a-b)t^n(1-t)^{m-n}x + a + b}{(1-t^n(1-t)^{m-n}x)^3}.
\]
Proof. Clearly,

\[
S_{m,n}(a,b,x) = \sum_{k=1}^{\infty} (ak + b)x^k \frac{(nk)!((m - n)k)!}{(mk)!}
= \sum_{k=1}^{\infty} (ak + b)x^k \frac{n k \Gamma(nk) \Gamma((m - n)k + 1)}{\Gamma(mk + 1)}
= n \sum_{k=1}^{\infty} (ak^2 + bk)x^k B(nk, (m - n)k + 1)
= n \sum_{k=1}^{\infty} (ak^2 + bk)x^k \int_0^1 t^{nk-1}(1 - t)^{(m-n)k} dt
= n \int_0^1 \frac{1}{t} \sum_{k=1}^{\infty} (ak^2 + bk)(t^n(1 - t)^{(m-n)x})^k dt.
\]

Note that for \(0 \leq t \leq 1\) we have

\[
\sqrt{\left(\frac{t}{n}\right)^{n} \left(\frac{1-t}{m-n}\right)^{m-n}} \leq n \times \frac{t}{n} \times \frac{1-t}{m-n} = 1
\]
and hence

\[
|t^n(1 - t)^{m-n}x| \leq \frac{n^n(m - n)^{m-n}}{m^m} |x| < 1.
\]

Combining the above with Lemma 2.1, we get

\[
\frac{S_{m,n}(a,b,x)}{n} = \int_0^1 \left( a \frac{t^n-1(1-t)^{m-n}x(t^n(1-t)^{m-n}x + 1)}{(1-t^n(1-t)^{m-n}x)^3} + b \frac{t^n-1(1-t)^{m-n}x}{(1-t^n(1-t)^{m-n}x)^2} \right) dt
= \int_0^1 T_{m,n}(a,b;x;t) dt.
\]

This concludes the proof. \(\square\)

3. Proof of Theorem 1.1

Lemma 3.1. For \(-3 < x < 1\), we have

\[
\arctan \frac{x - 1}{\sqrt{(1 - x)(3 + x)}} + \arctan \frac{x + 1}{\sqrt{(1 - x)(3 + x)}} = q(x),
\]

where \(q(x)\) is as in Theorem 1.1.

Proof. Let

\[
\alpha = \arctan \frac{x - 1}{\sqrt{(1 - x)(3 + x)}} \text{ and } \beta = \arctan \frac{x + 1}{\sqrt{(1 - x)(3 + x)}}.
\]
If $x \neq -2$, then
\[
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - (\tan \alpha) \tan \beta}
\]
\[
= \frac{2x}{\sqrt{(1-x)(3+x)}} \left(1 - \frac{x^2 - 1}{(1-x)(3+x)}\right)^{-1} = \frac{x}{x+2} \sqrt{\frac{3+x}{1-x}}
\]
and hence $\alpha + \beta - \gamma \in \pi \mathbb{Z}$, where
\[
\gamma = \arctan \frac{x}{x+2} \sqrt{\frac{3+x}{1-x}} \in (\frac{-\pi}{2}, \frac{\pi}{2}).
\]
As $x < 1$ we have $\alpha + \beta < \beta < \pi/2$ and hence $\alpha + \beta \leq \gamma$. If $-1 \leq x < 1$, then $\alpha + \beta \geq \alpha > -\pi/2$ and hence $\alpha + \beta = \gamma$. If $x \in (-2,-1)$, then $\gamma - \pi < -\pi < \alpha + \beta$ and hence $\alpha + \beta = \gamma$.

Since
\[
\lim_{x \to -2} \tan(\alpha + \beta) = \lim_{x \to -2} \frac{x}{x+2} \sqrt{\frac{3+x}{1-x}} = -\infty,
\]
we have $\alpha + \beta = -\pi/2$ in the case $x = -2$.

Now we consider the case $x \in (-3,-2)$. Note that $\alpha < 0$ and $\beta < 0$, but $\gamma > 0$. So $\alpha + \beta = \gamma - \pi$.

Combining the above, we obtain the desired identity (3.1). \hfill \Box

**Proof of Theorem 1.1.** (i) Let
\[
f(t) = \frac{(x+3)(2t-1)(3t-2)x^3 - 9x + 9}{(1-x + t(1-t)^2x^3)^2}
\]
\[
+ \frac{2((t-1)(2t-1)x^4 + (t-1)x^3 - 9x^2 - 9x + 18)}{(x-1)(1-x + t(1-t)^2x^3)}
\]
and
\[
g(t) = \frac{8x^2}{(x-1)\sqrt{(1-x)(3+x)}} \arctan \frac{(2t-1)x-1}{\sqrt{(1-x)(3+x)}}.
\]
It is easy to verify that
\[
\frac{d}{dt} \left(\frac{(x-1)^2}{(x+3)^2}(f(t) + g(t))\right) = T_{3,1} \left( (2x-3)^2, 2x^2 + 2x - 3, \frac{x^3}{x-1}, t \right).
\]
Thus, with the aid of Proposition 2.1, we get
\[
S_{3,1} \left( (2x-3)^2, 2x^2 + 2x - 3, \frac{x^3}{x-1} \right)
\]
\[
= \frac{(x-1)^2}{(x+3)^2} (f(t) + g(t)) \bigg|_{t=0} = \frac{(x-1)^2}{(x+3)^2} (f(1) - f(0) + g(1) - g(0)).
\]
Note that
\[
f(1) - f(0) = \frac{x+3}{(1-x)^2} \left(2x^3 + 2 - 2x^3\right) + \frac{2(-x^4 + x^3)}{(x-1)(1-x)}
\]
\[
= -4x^3 \frac{x+3}{(1-x)^2} + \frac{2x^3}{x-1} = -2x^3 \frac{x+7}{(1-x)^2}
\]
and
\[
g(1) - g(0) = \frac{8x^2}{(x - 1)\sqrt{(1 - x)(3 + x)}} \times \left( \arctan \frac{x - 1}{\sqrt{(1 - x)(3 + x)}} + \arctan \frac{x + 1}{\sqrt{(1 - x)(3 + x)}} \right) = \frac{8x^2q(x)}{(x - 1)\sqrt{(1 - x)(3 + x)}}
\]
with the help of Lemma 3.1. Therefore
\[
\sum_{k=1}^{\infty} \frac{(2x - 3)^2k + 2x^2 + 2x - 3)x^{3k}}{(x - 1)^k(3k)} = \frac{(x - 1)^2}{(x + 3)^2} \left( -2x^3 \frac{x + 7}{(x - 1)^2} + \frac{8x^2q(x)}{(x - 1)\sqrt{(1 - x)(3 + x)}} \right)
\]
\[
= -2x^3 \frac{x + 7}{(x + 3)^2} + \frac{8x^2(x - 1)q(x)}{(x + 3)^2\sqrt{(1 - x)(3 + x)}}.
\]
This proves Theorem 1.1(i).

(ii) Set
\[
f_1(t) = \frac{(x^3 - 13x^2 + 21x - 9)(2x^3(3t^2 - 5t + 2) - 9x + 9)}{(1 - x + t(1 - t)^2x^3)^2},
\]
\[
f_2(t) = \frac{2((2t^2 - 3t + 1)x^5 + 6t(1 - t)x^4 + 3(t - 4)x^3 + 90x^2 - 135x + 54)}{1 - x + t(1 - t)^2x^3}
\]
and
\[
f_3(t) = 4x^2 \left( 2\log(1 + (t - 1)x) - \log(1 + t^2x^2 - tx(1 + x)) \right).
\]
It is easy to verify that
\[
\frac{d}{dt}(x - 1)(f_1(t) + f_2(t) + f_3(t)) = T_{3,1} \left( s(x), t(x), \frac{x^3}{x - 1}, t \right).
\]
Thus, by applying Proposition 2.1, we obtain
\[
S_{3,1} \left( s(x), t(x), \frac{x^3}{x - 1} \right) = (x - 1)(f_1(1) - f_1(0) + f_2(1) - f_2(0) + f_3(1) - f_3(0))
\]
\[
= (x - 1) \left( \frac{(x^3 - 13x^2 + 21x - 9)(0 - 4x^3)}{(1 - x)^2} + \frac{2}{1 - x}(-9x^3 - (x^5 - 12x^3)) \right) + 4x^2(x - 1)(-\log(1 - x) - 2\log(1 - x))
\]
\[
= -2x^3(x^2 - 24x + 21) + 12x^2(1 - x)\log(1 - x).
\]
Therefore
\[
\sum_{k=0}^{\infty} \frac{(s(x)k + t(x))x^{3k}}{(x - 1)^k(\frac{3k}{2})^k} = t(x) + S_{3,1} \left( s(x), t(x), \frac{x^3}{x - 1} \right)
\]
\[
= 2x^5 - 48x^4 + 69x^3 - 189x^2 + 243x - 81 - 2x^3(x^2 - 24x + 21) + 12x^2(1 - x) \log(1 - x).
\]
\[
= 27(x - 1)(x^2 - 6x + 3) + 12x^2(1 - x) \log(1 - x).
\]
This proves the identity (1.7).

In view of the above, we have completed the proof of Theorem 1.1. □

4. PROOF OF THEOREM 1.2

Lemma 4.1. For \( x > 1/4 \) we have
\[
\sum_{k=0}^{\infty} \frac{1}{x^{2k}(\frac{4k}{2})^k} = \frac{16x^2}{16x^2 - 1} + 2x \left( \frac{\arccot \sqrt{4x - 1}}{(4x - 1)^{1/2}} - \frac{\arccoth \sqrt{4x + 1}}{(4x + 1)^{1/2}} \right).
\]
(4.1)

Proof. By Proposition 2.1,
\[
\sum_{k=1}^{\infty} \frac{1}{x^{2k}(\frac{4k}{2})^k} = 2 \int_0^1 T_{1,2} \left( 0, 1, \frac{1}{x^2}; t \right) \, dt
\]
\[
= \int_0^1 \frac{t(1-t)(1-2t) + t(1-t)}{x^2(1-t^2)(1/t^2)^2/x^2} \, dt
\]
\[
= \frac{1}{2x^2} \left[ \frac{t(1-t)}{(1-t^2)(1/t^2)^2/x^2} \right]_{t=0}^{1/2} + \frac{1}{x^2} \int_0^1 \frac{t(1-t)}{(1-t^2)(1/t^2)^2/x^2} \, dt
\]
\[
= \frac{1}{x^2} \int_0^{1/2} \frac{t(1-t)}{(1-t^2)(1/t^2)^2/x^2} \, dt + \frac{1}{x^2} \int_{1/2}^1 \frac{t(1-t)}{(1-t^2)(1/t^2)^2/x^2} \, dt
\]
\[
= \frac{2}{x^2} \int_0^{1/2} \frac{t(1-t)}{(1-t^2)(1/t^2)^2/x^2} \, dt.
\]
For \( t \in [0, 1/2] \), if we set \( u = t(1-t) \) then
\[
t = \frac{1 - \sqrt{1 - 4u}}{2} \quad \text{and} \quad dt = \frac{du}{\sqrt{1 - 4u}}.
\]
Thus
\[
\sum_{k=1}^{\infty} \frac{1}{x^{2k}(\frac{4k}{2})^k} = \frac{2}{x^2} \int_0^{1/4} \frac{u}{(1-u^2/x^2)^2 \sqrt{1 - 4u}} \, du
\]
(4.2)

Let \( \psi(u) \) denote the expression
\[
\frac{x(1 + 4u) \sqrt{1 + 4u}}{u^2 - x^2} \frac{2x + 1 \sqrt{4x - 1}}{\sqrt{4x - 1}} + \frac{2x + 1 \sqrt{4x + 1}}{\sqrt{4x + 1}} \arctan \frac{\sqrt{1 - 4u}}{\sqrt{4x - 1}} + \frac{2x + 1 \sqrt{4x + 1}}{\sqrt{4x + 1}} \arctanh \frac{\sqrt{1 - 4u}}{\sqrt{4x + 1}}.
\]
SERIES OF THE TYPE $\sum_{k=0}^{\infty} (a_k + b)x^k / \binom{m_k}{n_k}$

It is easy to verify that
\[
\frac{d}{du} \left( \frac{x}{16x^2 - 1} \psi(u) \right) = \frac{2}{x^2} \cdot \frac{u}{(1 - u^2/x^2)^2 \sqrt{1 - 4u}}.
\]

Combining this with (4.2), we get
\[
\sum_{k=1}^{\infty} \frac{1}{x^{2k} \binom{4k}{2k}} = \frac{x}{16x^2 - 1} \left( \psi \left( \frac{1}{4} \right) - \psi(0) \right) = -\frac{x}{16x^2 - 1} \psi(0)
\]
\[
= -\frac{x}{16x^2 - 1} \left( \frac{x}{-x^2} - \frac{2(4x + 1)}{\sqrt{4x - 1}} \arctan \frac{1}{\sqrt{4x - 1}} \right)
\]
\[
- \frac{x}{16x^2 - 1} \cdot \frac{2(4x - 1)}{\sqrt{4x + 1}} \arctanh \frac{1}{\sqrt{4x + 1}}
\]
\[
= \frac{1}{16x^2 - 1} + \frac{2x \arccot \sqrt{4x - 1}}{\left( \sqrt{4x - 1} \right)^2} - \frac{2x \arccoth \sqrt{4x + 1}}{\left( \sqrt{4x + 1} \right)^2}
\]
and hence (4.1) follows immediately. □

**Remark 4.1.** We can prove Lemma 4.1 in another way by noting that
\[
\sum_{k=0}^{\infty} \frac{1}{x^{2k} \binom{4k}{2k}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1 + (-1)^k}{x^k \binom{2k}{k}} \text{ for } |x| > \frac{1}{4},
\]
and using the identities (1.1) and (1.2) for $|x| \leq 2$, which can be proved via Proposition 2.1.

By Lemma 4.1,
\[
\sum_{k=1}^{\infty} \frac{1}{x^{2k} \binom{4k}{2k}} = \frac{45 + 25\pi \sqrt{3} - 54\sqrt{5} \arctanh(1/\sqrt{5})}{675},
\]
\[
\sum_{k=1}^{\infty} 4^k \frac{1}{x^{2k} \binom{4k}{2k}} = \frac{12 + 9\pi - 4\sqrt{3} \arctanh(1/\sqrt{3})}{36},
\]
\[
\sum_{k=1}^{\infty} 9^k \frac{1}{x^{2k} \binom{4k}{2k}} = \frac{189 + 98\pi \sqrt{3} - 6\sqrt{21} \arctanh \sqrt{3/7}}{147},
\]
\[
\sum_{k=1}^{\infty} \frac{(9/4)^k}{x^{2k} \binom{4k}{2k}} = \frac{9}{55} + \frac{12}{55} \left( \frac{11}{\sqrt{15}} \arctan \sqrt{3/5} - \frac{5}{\sqrt{33}} \arctanh \sqrt{3/11} \right).
\]

We are also able to prove that
\[
\sum_{k=1}^{\infty} \frac{1}{k \binom{4k}{2k}} = \frac{\sqrt{3}}{9} \pi - \frac{2}{5} \sqrt{5} \log \frac{1 + \sqrt{5}}{2}
\]
and
\[
\sum_{k=1}^{\infty} \frac{1}{k4^k \binom{4k}{2k}} = \frac{2}{\sqrt{7}} \arctan \frac{1}{\sqrt{7}} - \log \frac{2}{3}
\]
via the beta function.
Proof of Theorem 1.2. Taking derivatives of both sides of (4.1), we deduce that

\[
\sum_{k=0}^{\infty} \frac{k}{x^{2k}(4k)_{2k}} - \frac{24x^2}{(16x^2 - 1)^2} = \frac{x(1 + 2x)}{(4x - 1)^2\sqrt{4x - 1}} \arccot\sqrt{4x - 1} + \frac{x(1 - 2x)}{(4x + 1)^2\sqrt{4x + 1}} \arccoth\sqrt{4x + 1}.
\]

(4.3)

Via \(2(4x + 1) \times (4.3) + (1 - 2x) \times (4.1)\) we see that (1.25) holds. Similarly, via \(2(4x - 1) \times (4.3) - (2x + 1) \times (4.1)\) we obtain

\[
\sum_{k=0}^{\infty} \frac{2(4x - 1)k - 2x - 1}{x^{2k}(4k)_{2k}} = \frac{8x^2}{(4x + 1)^2} \left(\frac{3}{\sqrt{4x + 1}} \arccot\sqrt{4x + 1} - 4x - 4 \right),
\]

which is equivalent to (1.26) since

\[
\frac{\arccoth\sqrt{4x + 1}}{\sqrt{4x + 1}} = \sum_{k=0}^{\infty} \frac{1}{(2k + 1)(\sqrt{4x + 1})^{2k+2}} = \frac{1}{4x + 1} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)(4x + 1)^k} = \frac{R(4x + 1)}{4x + 1}.
\]

Putting \(x = 1, 1/2, 1/3, 2/3\) in (1.25) we immediately get (1.27)-(1.30). In light of (1.24), the identity (1.31) follows from (1.26) with \(x = 2\).

In view of the above, we have completed the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

Lemma 5.1. For any \(u < 1\) with \(u \neq 0\), we have

\[
\sum_{k=0}^{\infty} \frac{u^k}{2k + 1} \left( \left(1 - i\sqrt{1 - u}\right)^{-2k-1} + \left(1 + i\sqrt{1 - u}\right)^{-2k-1} \right) = \frac{\arctanh\sqrt{u}}{\sqrt{u}}.
\]

(5.1)

Proof. It suffices to prove that

\[
\sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k + 1} \left( \left(1 - i\sqrt{1 - t^2}\right)^{-2k-1} + \left(1 + i\sqrt{1 - t^2}\right)^{-2k-1} \right) = \arctanh t
\]

(5.2)

for each \(t \in \mathbb{C}\) with \(t^2 < 1\). Note that

\[
\left| \frac{t}{1 \pm i\sqrt{1 - t^2}} \right|^2 = \frac{|(1 - t^2) - 1|}{1 + (1 - t^2)} < 1.
\]
Let $f(t)$ and $g(t)$ denote the left-hand side and the right-hand side of (5.2). Then
\[
f'(t) = \sum_{k=0}^{\infty} t^{2k} \left( (1 - i \sqrt{1 - t^2})^{-2k-1} + (1 + i \sqrt{1 - t^2})^{-2k-1} \right)
- \sum_{k=0}^{\infty} t^{2k+1} \left( (1 - i \sqrt{1 - t^2})^{-2k-2} - (1 + i \sqrt{1 - t^2})^{-2k-2} \right) \frac{it}{\sqrt{1 - t^2}}
= \frac{1}{1 - i \sqrt{1 - t^2}} \sum_{k=0}^{\infty} \left( \frac{t^2}{(1 - i \sqrt{1 - t^2})^2} \right)^k
+ \frac{1}{1 + i \sqrt{1 - t^2}} \sum_{k=0}^{\infty} \left( \frac{t^2}{(1 + i \sqrt{1 - t^2})^2} \right)^k
- \frac{i}{\sqrt{1 - t^2}} \frac{t^2}{(1 - i \sqrt{1 - t^2})^2} \sum_{k=0}^{\infty} \left( \frac{t^2}{(1 + i \sqrt{1 - t^2})^2} \right)^k
+ \frac{i}{\sqrt{1 - t^2}} \frac{t^2}{(1 + i \sqrt{1 - t^2})^2} \sum_{k=0}^{\infty} \left( \frac{t^2}{(1 - i \sqrt{1 - t^2})^2} \right)^k
= \frac{1}{1 - i \sqrt{1 - t^2}} - \frac{2i \sqrt{1 - t^2}}{2i \sqrt{1 - t^2}}
+ \frac{1}{1 + i \sqrt{1 - t^2}} - \frac{i \sqrt{1 - t^2}}{i \sqrt{1 - t^2}}
- \frac{i t^2}{\sqrt{1 - t^2}} \frac{1}{\sqrt{1 - t^2}} + \frac{i t^2}{\sqrt{1 - t^2}} \frac{1}{\sqrt{1 - t^2}}
= 1 + \frac{1}{1 - t^2} = \frac{1}{1 - t^2} = g'(t).
\]
Thus $f(t) - g(t)$ is a constant. Since $f(0) = 0 = g(0)$, we have $f(t) = g(t)$. This concludes our proof of Lemma 5.1. \qed

**Lemma 5.2.** Let $x > 1$ or $x < 0$, and let
\[
F(s) = \frac{20x^2 - 56x + 35}{4s^2 + x(x - 1)^2} - \frac{4(x - 1)(2x - 3)(2x - 1)^2}{4s^2 + x(x - 1)^3}.
(5.3)
\]
Then
\[
8x^2(1 - x) \int_0^{1/4} \frac{sF(s)}{\sqrt{1 - 4s}} ds = 3(x - 1)R(x) + 5x - 6.
(5.4)
\]
Proof. Note that \( x(x-1) > 0 \). Let \( G(s) \) denote the expression

\[
(24s^3 + 4s^2x + 2sx(x-1)(8x-9) + x(x-1)(5x-6))\frac{\sqrt{1-4s\sqrt{x}}}{(4s^2 + x(1-x))^2} \\
+ \frac{3((1-x)i + \sqrt{x}\sqrt{x-1})}{\sqrt{x-1}\sqrt{1-2i\sqrt{x}\sqrt{x-1}}} \arctanh \left( \frac{\sqrt{1-4s}}{\sqrt{1-2i\sqrt{x}\sqrt{x-1}}} \right) \\
+ \frac{3((x-1)i + \sqrt{x}\sqrt{x-1})}{\sqrt{x-1}\sqrt{1+2i\sqrt{x}\sqrt{x-1}}} \arctanh \left( \frac{\sqrt{1-4s}}{\sqrt{1+2i\sqrt{x}\sqrt{x-1}}} \right)
\]

As

\[
\frac{d}{dz} (\arctanh z) = \frac{1}{1-z^2},
\]

it is routine to verify that

\[
\frac{d}{ds} \left( \frac{G(s)}{8x^{3/2}} \right) = \frac{sF(s)}{\sqrt{1-4s}}.
\]

Actually we find the expression of \( G(s) \) by Mathematica. Since \( x > 1 \) or \( x < 0 \), we have

\[
\left| \sqrt{1+2i\sqrt{x}\sqrt{x-1}} \right| = \left| \sqrt{x} \pm i\sqrt{x-1} \right| = \sqrt{|2x-1|} > 1.
\]

Thus

\[
8x^{3/2} \int_0^{1/4} \frac{sF(s)}{\sqrt{1-4s}} ds \\
= G\left( \frac{1}{4} \right) - G(0) = -G(0) \\
= -\frac{x(x-1)(5x-6)\sqrt{x}}{x^2(1-x)^2} \\
- \frac{3((1-x)i + \sqrt{x}\sqrt{x-1})}{\sqrt{x-1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)\left( \sqrt{1-2i\sqrt{x}\sqrt{x-1} \right)^{2k+2}} \\
- \frac{3((x-1)i + \sqrt{x}\sqrt{x-1})}{\sqrt{x-1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)\left( \sqrt{1+2i\sqrt{x}\sqrt{x-1} \right)^{2k+2}} \\
= \frac{5x-6}{x(1-x)\sqrt{x}} \\
- 3 \sum_{k=0}^{\infty} \left( \frac{\sqrt{x} - i\sqrt{x-1}}{(2k+1)(1-2i\sqrt{x}\sqrt{x-1})^{k+1}} + \frac{\sqrt{x} + i\sqrt{x-1}}{(2k+1)(1+2i\sqrt{x}\sqrt{x-1})^{k+1}} \right)
\]
which coincides with
\[
\frac{5x - 6}{x(1 - x)} \sqrt{x} - 3 \sum_{k=0}^{\infty} \frac{1}{2k + 1} \left( \frac{\sqrt{x} - i\sqrt{x - 1}}{(\sqrt{x} - i\sqrt{x - 1})^{2k+2}} + \frac{\sqrt{x} - i\sqrt{x - 1}}{(\sqrt{x} - i\sqrt{x - 1})^{2k+2}} \right)
\]
\[
= \frac{5x - 6}{x(1 - x)} \sqrt{x} - 6 \sum_{k=0}^{\infty} \frac{\sqrt{x}}{(2k + 1)x^{2k+1}} \Re \left( \left( 1 - i\sqrt{\frac{x - 1}{x}} \right)^{-2k-1} \right)
\]
This reduces the desired identity (5.4) to
\[
\sum_{k=0}^{\infty} \frac{x^{-k}}{2k + 1} \left( \left( 1 - i\sqrt{\frac{x - 1}{x}} \right)^{-2k-1} + \left( 1 + i\sqrt{\frac{x - 1}{x}} \right)^{-2k-1} \right) = R(x)
\]
which follows from the identity (5.1) with \( u = 1/x \). This ends our proof. □

**Lemma 5.3.** Let \( x > 1 \) or \( x < 0 \), and let \( F(s) \) be defined by (5.3). Then
\[
\int_{0}^{1} t(1 - t)^{2} F(t(1 - t)) dt = \int_{0}^{1/4} \frac{s F(s)}{\sqrt{1 - 4s}} ds.
\]

**Proof.** Note that
\[
\int_{0}^{1} \frac{8t(1 - t)(1 - 2t)}{(4t^{2}(1 - t)^{2} + x(x - 1))^{2}} dt = \int_{0}^{1} \frac{-1}{(4t^{2}(1 - t)^{2} + x(x - 1))} \prime dt = 0
\]
and
\[
\int_{0}^{1} \frac{8t(1 - t)(1 - 2t)}{(4t^{2}(1 - t)^{2} + x(x - 1))^{3}} dt = \int_{0}^{1} \frac{-1/2}{(4t^{2}(1 - t)^{2} + x(x - 1))^{2}} \prime dt = 0.
\]
Thus
\[
2 \int_{0}^{1} t(1 - t)^{2} F(t(1 - t)) dt
\]
\[
= \int_{0}^{1} t(1 - t)((1 - 2t) + 1) F(t(1 - t)) dt
\]
\[
= \int_{0}^{1/2} t(1 - t) F(t(1 - t)) dt + \int_{1/2}^{1} u(1 - u) F(u(1 - u)) du
\]
\[
= \int_{0}^{1/2} t(1 - t) F(t(1 - t)) dt + \int_{1/2}^{0} (1 - t) t F((1 - t)t) d(1 - t)
\]
\[
= 2 \int_{0}^{1/2} t(1 - t) F(t(1 - t)) dt.
\]
For \( t \in [0, 1/2] \), if we set \( s = t(1 - t) \), then \( t = (1 - \sqrt{1 - 4s})/2 \) and hence
\[
dt = \frac{-1}{4} \cdot \frac{-4}{\sqrt{1 - 4s}} ds = \frac{ds}{\sqrt{1 - 4s}}.
\]
Therefore
\[
\int_{0}^{1} t(1 - t)^{2} F(t(1 - t)) dt = \int_{0}^{1/2} t(1 - t) F(t(1 - t)) dt = \int_{0}^{1/4} \frac{s F(s)}{\sqrt{1 - 4s}} ds
\]
as desired. □
Proof of Theorem 1.3. Note that
\[
\binom{4k}{2k} \sim \frac{16^k}{\sqrt{2k\pi}} \quad \text{and} \quad 0 < \frac{4}{x(x-1)} < 16.
\]
So the series in (1.32) converges absolutely. By Proposition 2.1, we have
\[
\sum_{k=1}^\infty \frac{(2(2x-1)^2(2x-3)k - (4x^3 - 16x^2 + 7x + 6))4^k}{(x(1-x))^{k/2} k^{4k/2k}} = 2 \int_0^1 T_{4,2} \left( \frac{4(2x-1)^2(2x-3) - (4x^3 - 16x^2 + 7x + 6)}{x(1-x)}, t \right) dt.
\]
It is easy to verify that
\[
T_{4,2} \left( \frac{4(2x-1)^2(2x-3) - (4x^3 - 16x^2 + 7x + 6)}{x(1-x)}, t \right) = 4x^2(x-1)\left(2 - x^2 - 56x + 35\right)t(1-t)^2/(4t^2(1-t)^2 + x(1-x))^2 - 16x^2(x-1)^2(2x-3)t(1-t)^2/(4t^2(1-t)^2 + x(1-x))^3 = 4x^2(x-1)t(1-t)^2F(t(1-t))
\]
where the function $F$ is given by (5.3). Combining this with Lemmas 5.2 and 5.3, we get
\[
\sum_{k=1}^\infty \frac{(2(2x-1)^2(2x-3)k - (4x^3 - 16x^2 + 7x + 6))4^k}{(x(1-x))^{k/2} k^{4k/2k}} = 8x^2(x-1) \int_0^1 t(1-t)^2F(t(1-t))dt = 8x^2(x-1) \int_0^{1/4} \frac{sF(s)}{\sqrt{1-4s}} ds = 3(1-x)R(x) - 5x + 6
\]
and hence
\[
\sum_{k=0}^\infty \frac{(2(2x-1)^2(2x-3)k - (4x^3 - 16x^2 + 7x + 6))4^k}{(x(1-x))^{k/2} k^{4k/2k}} = -(4x^3 - 16x^2 + 7x + 6) + 3(1-x)R(x) - 5x + 6 = (1-x)(3R(x) + 4(x(x-3))).
\]
This proves (1.32).

As $x > (1 + \sqrt{2})/2$ or $x < (1 - \sqrt{2})/2$, we see that $1 - x < (1 - \sqrt{2})/2$ or $1 - x > (1 + \sqrt{2})/2$. Note that (1.32) with $x$ replaced by $1 - x$ yields (1.33). This concludes our proof of Theorem 1.3. \(\square\)
6. Proof of Theorem 1.4

Lemma 6.1. For any positive integer \( n \), we have

\[
\sum_{k=1}^{n} \frac{(4x - 27)k^2 + (2x + 27)k - 2x - 6)x^k}{(2k - 1)\binom{3k}{k}} = -2x + 2(n + 1)\frac{x^{n+1}}{(3n)_n}, \quad (6.1)
\]

\[
\sum_{k=1}^{n} \frac{(4x - 27)k^2 + (27 - 2x)k - 2x - 6)x^k}{k(2k - 1)\binom{3k}{k}} = -2x + 2\frac{x^{n+1}}{(3n)_n}, \quad (6.2)
\]

\[
\sum_{k=1}^{n} \frac{2(x - 16)k^2 + (x + 32)k - 2x - 6)x^k}{(2k - 1)\binom{4k}{2k}} = -x + (n + 1)\frac{x^{n+1}}{(2n)_n}, \quad (6.3)
\]

\[
\sum_{k=1}^{n} \frac{2(x - 16)k^2 + (32 - x)k - 2x - 6)x^k}{k(2k - 1)\binom{4k}{2k}} = -x + \frac{x^{n+1}}{(2n)_n}, \quad (6.4)
\]

\[
\sum_{k=1}^{n} \frac{(4x - 27)k^2 + (6x + 27)k - 2x - 6)x^k}{\binom{3k}{k}} = -2x + 2\frac{(n + 1)(2n + 1)x^{n+1}}{(3n)_n}, \quad (6.5)
\]

\[
\sum_{k=1}^{n} \frac{(2x - 16)k^2 + (3x + 32)k - 2x - 6)x^k}{\binom{4k}{2k}} = -x + \frac{(n + 1)(2n + 1)x^{n+1}}{(2n)_n}. \quad (6.6)
\]

Proof. Just use induction on \( n \).

Proof of Theorem 1.4. By Lemma 6.1,

\[
\sum_{k=1}^{\infty} \frac{(12k^2 - 18k + 5)4^k}{(2k - 1)\binom{4k}{2k}} = 2.
\]

Combining this with (1.28), we get

\[
2 = \sum_{k=1}^{\infty} \frac{6k(2k - 1) - (12k - 5)4^k}{(2k - 1)\binom{4k}{2k}} = 6 \times \frac{3\pi + 8}{12} - \sum_{k=1}^{\infty} \frac{(12k - 5)4^k}{(2k - 1)\binom{4k}{2k}}
\]

and hence

\[
\sum_{k=1}^{\infty} \frac{(12k - 5)4^k}{(2k - 1)\binom{4k}{2k}} = \frac{3\pi + 4}{2}. \quad (6.7)
\]

Combining this with the identity

\[
\sum_{k=1}^{\infty} \frac{(12k^2 - 14k + 3)4^k}{k(2k - 1)\binom{4k}{2k}} = 2
\]

obtained from Lemma 6.1, we find that

\[
2 = \sum_{k=1}^{\infty} \frac{k(12k - 5) - 3(3k - 1)4^k}{k(2k - 1)\binom{4k}{2k}} = \frac{3\pi + 4}{2} - 3\sum_{k=1}^{\infty} \frac{(3k - 1)4^k}{k(2k - 1)\binom{4k}{2k}}
\]

and hence (1.65) follows.
By Lemma 6.1,
\[
\sum_{k=1}^{\infty} \frac{(18k^2 - 15k + 2)(-2)^k}{(2k - 1) \binom{3k}{2k}} = -1.
\]
Combining this with (1.34), we get
\[
-10 = \sum_{k=1}^{\infty} \frac{3(2k-1)(30k - 7) - (18k + 1)(-2)^k}{(2k - 1) \binom{4k}{2k}}
\]
\[
= 3 \left( 7 - \frac{3\pi + 64}{6} \right) - \sum_{k=1}^{\infty} \frac{(18k + 1)(-2)^k}{(2k - 1) \binom{4k}{2k}}
\]
and hence
\[
\sum_{k=1}^{\infty} \frac{(18k + 1)(-2)^k}{(2k - 1) \binom{4k}{2k}} = -\frac{3\pi + 2}{2}.
\] (6.8)
Combining this with the identity
\[
\sum_{k=1}^{\infty} \frac{(18k^2 - 17k + 3)(-2)^k}{k(2k - 1) \binom{4k}{2k}} = -1
\]
obtained from Lemma 6.1, we find that
\[
-1 = \sum_{k=1}^{\infty} \frac{k(18k + 1) - 3(6k - 1)(-2)^k}{k(2k - 1) \binom{4k}{2k}} = -\frac{3\pi + 2}{2} - 3 \sum_{k=1}^{\infty} \frac{(6k - 1)(-2)^k}{k(2k - 1) \binom{4k}{2k}}
\]
and hence (1.66) follows.
By Lemma 6.1, we have
\[
\sum_{k=1}^{\infty} \frac{126k^2 - 129k + 25}{(2k - 1)4^k \binom{4k}{2k}} = 1.
\]
Combining this with (1.31), we get
\[
2 = \sum_{k=1}^{\infty} \frac{2(126k^2 - 129k + 25)}{(2k - 1)4^k \binom{4k}{2k}}
\]
\[
= \sum_{k=1}^{\infty} \frac{9(14k - 5)(2k - 1) - (42k - 5)}{(2k - 1)4^k \binom{4k}{2k}}
\]
\[
= 9 \left( 5 + \frac{16}{81} \log 2 - 24 \right) - \sum_{k=1}^{\infty} \frac{42k - 5}{(2k - 1)4^k \binom{4k}{2k}}
\]
and hence
\[
\sum_{k=1}^{\infty} \frac{42k - 5}{(2k - 1)4^k \binom{4k}{2k}} = \frac{16}{9} \log 2 + \frac{1}{3}.
\] (6.9)
Combining this with the identity
\[
\sum_{k=1}^{\infty} \frac{126k^2 - 127k + 24}{k(2k - 1)4^k \binom{4k}{2k}} = 1
\]
obtained from Lemma 6.1, we find that
\[ 1 = \sum_{k=1}^{\infty} \frac{3k(42k-5) - 8(14k - 3)}{k(2k - 1)4^k \binom{4k}{2k}} = 3 \left( \frac{16}{9} \log 2 + \frac{1}{3} \right) - 8 \sum_{k=1}^{\infty} \frac{14k - 3}{k(2k - 1)4^k \binom{4k}{2k}} \]
and hence (1.67) follows.

As other identities in Theorem 1.4 can be proved similarly, we omit the details. \(\square\)

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