CONSTRUCTION AND RESEARCH OF ADEQUATE
COMPUTATIONAL MODELS FOR QUASILINEAR
HYPERBOLIC SYSTEMS

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ABSTRACT. In the paper, we study a class of three-dimensional quasilinear hyperbolic systems. For such system, we set the initial boundary value problem and construct the energy integral. We construct the difference scheme and obtain an a priori estimate for its solution.

1. Introduction. Finite-difference methods are often used to find the numerical solutions of quasilinear hyperbolic systems. There are many different ways to form difference schemes for quasilinear hyperbolic systems, some of approaches to do it are given in [3, 1, 2]. In [3], an interesting approach is described, based on the possibility of writing the system of gas dynamics equations in two variants. First we explain the meaning of this approach for a differential problem.

2. One class of quasilinear hyperbolic systems. We consider the initial value problem for the symmetric system of quasilinear equations in the domain $\Pi = \{(t, x, y, z) : 0 \leq t \leq T; 0 \leq x \leq l; |y| < \infty; |z| < \infty\}$:

$$
A \frac{\partial U}{\partial t} + B \frac{\partial U}{\partial x} + C \frac{\partial U}{\partial y} + D \frac{\partial U}{\partial z} + QU = F,
$$

$$
0 \leq t \leq T; 0 \leq x \leq l; |y| < \infty; |z| < \infty;
$$

(1)

with periodic boundary conditions:

$$
U(t, 0, y, z) = U(t, l, y, z),
$$

(2)

and

$$
\|U\| = (U, U)^{1/2} \xrightarrow{|y|, |z| \to +\infty} 0 \text{ for any fixed } t, x,
$$

(3)

and with initial data for $t = 0$:

$$
U(0, x, y, z) = U_0(x, y, z), 0 \leq x \leq l; |y| < \infty; |z| < \infty.
$$

(4)

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Here \( A = A(U, t, x, y, z) = A^*(U, x, y, z); \) \( B = B(U, t, x, y, z) = B^*(U, x, y, z); \)
\( C = C(U, t, x, y, z) = C^*(U, x, y, z); \) \( D = D(U, t, x, y, z) = D^*(U, x, y, z); \) \( Q = Q(U, t, x, y, z) \) are real quadratic matrices of order \( N, \) elements of which are bounded functions; \( A^*, \ B^*, \ C^*, \ D^* \) are the corresponding transposed matrices; \( F = F(t, x, y, z), U_0(x, y, z) \) are given vector-functions, vanishing at infinity; \( U(t, x, y, z) \) is an unknown vector-function of dimension \( N; \) \( T, l \) are positive real numbers.

Furthermore, \( A > 0 \) is a positive defined matrix (see \([3]\)). We note that in \([3]\), systems of the form (1) defined in such a way are called the symmetric \( t-\)hyperbolic systems. Assume that there is a nonsingular transformation \( W = W(U, t, x, y, z), \) that reduces the initial boundary problem (1)-(4) to the form:

\[
A \frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(A^T W) + B \frac{\partial W}{\partial x} + \frac{\partial}{\partial x}(B^T W) + C \frac{\partial W}{\partial y} + \frac{\partial}{\partial y}(C^T W)
\]
\[
+D \frac{\partial W}{\partial z} + \frac{\partial}{\partial z}(D^T W) + Q W = F
\]

\( 0 < t \leq T; 0 < x < l; |y| < \infty; |z| < \infty \)

with periodic boundary conditions:

\[
W(t, 0, y, z) = W(t, l, y, z),
\]

(6)

and

\[
\|W\| = \langle W, W \rangle^{1/2} \rightarrow 0 \text{ for any fixed } t, x,
\]

(7)

with the initial data for \( t = 0: \)

\[
W(0, x, y, z) = W(U_0(x, y, z), 0, x, y, z), \ 0 \leq x \leq l; |y| < \infty; |z| < \infty.
\]

(8)

Here \( A = A(W, t, x, y, z) > 0; \) \( B = B(W, t, x, y, z); \) \( C = C(W, t, x, y, z); \) \( D = D(W, t, x, y, z); \) \( Q = Q(W, t, x, y, z) \) are quadratic matrices of order \( N; \) \( A^T, \ B^T, \ C^T, \ D^T \) are the corresponding transposed matrices. Multiplying (on the right) the both sides of the system (5) by the vector \( W \) we have

\[
(A \frac{\partial W}{\partial t}, W) + \left( \frac{\partial}{\partial t}[A^T W], W \right) + \left( B \frac{\partial W}{\partial x}, W \right) + \left( \frac{\partial}{\partial x}[B^T W], W \right)
\]
\[
+ \left( C \frac{\partial W}{\partial y}, W \right) + \left( \frac{\partial}{\partial y}[C^T W], W \right) + \left( D \frac{\partial W}{\partial z}, W \right) + \left( \frac{\partial}{\partial z}[D^T W], W \right) + (Q W, W) = (F, W).
\]

Hence, taking into account the following equalities

\[
(A \frac{\partial W}{\partial t}, W) + \left( \frac{\partial}{\partial t}[A^T W], W \right) = \left( \frac{\partial W}{\partial t}, A^T W \right) + \left( \frac{\partial}{\partial t}[A^T W], W \right)
\]
\[
= \left( A^T W, \frac{\partial W}{\partial t} \right) + \left( \frac{\partial}{\partial t}[A^T W], W \right) = \left( \frac{\partial}{\partial t} \right) [A^T W, W];
\]
\[
(B \frac{\partial W}{\partial x}, W) + \left( \frac{\partial}{\partial x}[B^T W], W \right) = \left( \frac{\partial}{\partial x} \right) [B^T W, W];
\]
\[
(C \frac{\partial W}{\partial y}, W) + \left( \frac{\partial}{\partial y}[C^T W], W \right) = \left( \frac{\partial}{\partial y} \right) [C^T W, W];
\]
\[
(D \frac{\partial W}{\partial z}, W) + \left( \frac{\partial}{\partial z}[D^T W], W \right) = \left( \frac{\partial}{\partial z} \right) [D^T W, W].
\]
\[
(Q W, W) = \frac{1}{2} (Q W, W) + \frac{1}{2} (W, Q^* W) = \frac{1}{2} ([Q + Q^*] W, W),
\]
we get the following identity
\[
\frac{\partial}{\partial t}([A^T W], W) + \frac{\partial}{\partial x}([B^T W], W) + \frac{\partial}{\partial y}([C^T W], W) + \frac{\partial}{\partial z}([D^T W], W) = (GW, W) + (F, W).
\]
(9)

Here \( G = G(t, x, y, z) = \frac{1}{2} [Q + Q^T] \).

Integrating the both sides of the identity (9) over the domain \( \Gamma_{t_1, t_2} = \{(t,x,y,z) : t_1 \leq t \leq t_2; 0 \leq x \leq l; |y| < \infty; |z| < \infty \} \), we have
\[
I(t_2) - I(t_1) + \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ -(B^T W, W) \right]_{x=0} + (B^T W, W)_{x=l} \, dt \, dy \, dz
= \int_{\Gamma_{t_1, t_2}} \int \{(GW, W) + (F, W)\} \, dt \, dx \, dy \, dz,
\]
where \( I(t) = \int \int \int (AW, W) \, dx \, dy \, dz \).

The last identity was obtained in the class of functions \( W, \) for which the quadratic forms \( (CW, W) \) and \( (DW, W) \) tend to zero at infinity. Using periodic boundary conditions and the lemma on the integral inequality (see [3], page 152), we have the following inequality
\[
\sqrt{I(t)} \leq \sqrt{I(0)} e^{\frac{M t}{2}} + N e^{\frac{M t}{2}} - \frac{1}{M}.
\]
(10)

Here the constant \( M \) estimates the norm of the matrix \( \frac{1}{2} (Q + Q^T) \) and the constant \( N \) estimates the right side of equation (5).

3. **On symmetrization of the gas dynamics equations.** Consider the system of equations describing a three-dimensional motion of gas under the assumption that the gas is inviscid, non-heat-conducting and in the local thermodynamic equilibrium, i.e. there exists a state equation of gas. In the Cartesian coordinate system \( x = (x_1, x_2, x_3) \), the equations of gas dynamics can be written in the following conservative form
\[
\rho_t + \text{div} (\rho \cdot \mathbf{v}) = 0,
\]
\[
(\rho \cdot \mathbf{v}_i)_t + \sum_{k=1}^{3} (\Pi_{ik})_{x_k} = 0, \quad i = 1, 2, 3,
\]
\[
\left\{ \rho \left( E + \frac{|\mathbf{v}|^2}{2} \right)_t \right\} + \text{div} \left\{ \rho \cdot \mathbf{v} \left( E + \frac{|\mathbf{v}|^2}{2} + p \cdot V \right) \right\} = 0,
\]
where \( \rho \) is the density, \( \mathbf{v} = (v_1, v_2, v_3) \) is the velocity, \( \Pi_{ik} = \rho \cdot v_i \cdot v_k + p \cdot \delta_{ik} \) is the momentum flow density tensor, \( p \) is the pressure, \( E \) is the inner energy, and \( V = 1/\rho \) is the specific volume. Moreover, the following thermodynamical identity holds
\[
T \cdot dS = dE + p \cdot dV
\]
(12)
where \( T \) is the gas temperature and \( S \) is the entropy. It follows from (12) that
\[
p = - \frac{\partial E}{\partial V} \bigg|_S = p^2 \cdot \frac{\partial E}{\partial p} \bigg|_S, \quad T = \frac{\partial E}{\partial S} \bigg|_p.
\]
That is, if we add to the system (12) the state equation
\[ E = E(\rho, S), \]
then we get a closed system which can be now considered as the system for finding, for example, the vector of unknowns
\[ U = \begin{pmatrix} \rho \\ S \\ \nu \end{pmatrix}. \]

Finally, we add to (12) the following additional conservation law (the entropy conservation)
\[ (p \cdot S)_t + \text{div} (p \cdot S \cdot \mathbf{v}) = 0, \tag{13} \]
which is fulfilled on smooth solutions of the system (11). We also note that, taking into account (13), the system of equations (11) is equivalent on smooth solutions to the following non-conservative system
\[
\begin{align*}
\frac{1}{\rho \cdot c^2} \cdot \frac{dp}{dt} + \text{div} \mathbf{v} &= 0, \\
\frac{dS}{dt} &= 0, \\
\rho \cdot \frac{d\mathbf{v}}{dt} + \nabla p &= 0,
\end{align*}
\]
where \( \frac{d}{dt} = \frac{\partial}{\partial t} + (\nu, \nabla) \), \( c^2 = (\rho^2 \cdot E^\rho)_\rho \) is the square of the sound speed.

In [6], the Cauchy problem was studied for the system (11), and the local-in-time theorem on existence of the smooth (classical) solution was proved for this problem. S.K. Godunov [4] (see also [4, 5]) proposed a special class of systems of conservation laws in the following form
\[
\frac{\partial}{\partial t} L_{q_i} + \text{div} (M_{q_i}) = 0, \quad i = 1, n, \tag{15}
\]
where \( M = (M^1, M^2, M^3) \), \( Q = (q_1, ..., q_n)^* \); \( L = L(Q) \), \( M^k = M^k(Q) \), \( k = 1, 2, 3 \) are nonlinear functions of the dependent variables \( q_i \), \( i = 1, n \) (so-called productive functions). Why systems (15) are interesting? Firstly, the system (15) enables one to get an additional \((n + 1)\)-th conservation laws. We multiply the \( i \)-th equation of (15) by \( q_i \) and sum up the results:
\[
\frac{\partial}{\partial t} L_{q_i} + \text{div} (M_{q_i}) = 0, \tag{16}
\]
where \( L_{q_i} = (L_{q_1}, ..., L_{q_n})^* \), \( M_{q_i} = (M^1_{q_i}, M^2_{q_i}, M^3_{q_i}) \) is a rectangle matrix of order \( n \times 3 \); \( M^k_{q_i} = (M^k_{q_1}, ..., M^k_{q_n})^* \), \( k = 1, 2, 3 \).

Secondly, the system (11) can be rewritten in the symmetric form
\[
A^0(Q) \cdot Q_t + \sum_{k=1}^3 A^k(Q) \cdot Q_{xx} = 0
\]
where \( A_0 = \|L_{q_i,q_j}\| \), \( A^k = \|M^k_{q_i,q_j}\| \), \( i, j = 1, n \), \( k = 1, 2, 3 \) are symmetric matrices. If, in addition \( A > 0 \) or \( A < 0 \), then (16) is a symmetric \( t \)-hyperbolic system (according to Friedrich). As a rule, we have in practice the opposite situation, when the additional conservation law is known for some equations of mathematical
physics. It is the law (13), that was used in [5] (see also [6]) for constructing the symmetric system, where [5] s for the functions $L, M^k, k = 1, 2, 3$, and the variables $q_i, i = 1, 5$ ($n = 5$) the following expressions were obtained:

$$q_1 = -\frac{E + p \cdot V - \frac{|v|^2}{2}}{T} + S,$$

$$q_{1+k} = -\frac{v_k}{T}, \quad k = 1, 2, 3;$$

$$q_5 = \frac{1}{T}, \quad L = -\frac{p}{T}, \quad M^k = v_k L, \quad k = 1, 2, 3.$$

Here, we consider only the case of polytrophic gas when the state equation

$$E = E(\rho S)$$

is defined as follows (see [8]):

$$E = \frac{p \cdot V}{\gamma - 1}, \quad p = c_1 \cdot \rho^\gamma \cdot \exp \left( \frac{S}{c_V} \right),$$

where $c_1 > 0$ is a constant, $c_V > 0$ is the gas specific heat capacity, $\gamma = (1 + R/c_V) > 1$ is the adiabatic index, $R > 1$ is the gas constant. Then, the matrices $A^0, A^k, \quad k = 1, 2, 3$ are the following:

$$A^0 = -B^0, \quad A^k = -B^k, \quad k = 1, 2, 3, \quad \varrho_1 = \frac{\rho}{c_V \cdot (\gamma - 1)}$$

$$B^0 = \varrho_1 \cdot \left( \begin{array}{cccccc}
1 & v_1 & v_2 & v_3 & \bar{E} \\
v_1 & v_1^2 + pV & v_1 v_2 & v_1 v_3 & v_1 \Omega \\
v_2 & v_1 v_2 & v_2^2 + pV & v_2 v_3 & v_2 \Omega \\
v_3 & v_1 v_3 & v_2 v_3 & v_3^2 + pV & v_3 \Omega \\
\bar{E} & v_1 \Omega & v_2 \Omega & v_3 \Omega & \bar{E} + p |v|^2 / 2 \\
\end{array} \right),$$

$$\bar{E} = E + \frac{|v|^2}{2}, \quad \Omega = \bar{E} + p \cdot V,$$

$$B^k = v_k \cdot B^0 + \frac{p}{c_V (\gamma - 1)} \cdot \bar{B}^k, \quad k = 1, 2, 3;$$

$$\bar{B}^1 = \left( \begin{array}{cccccc}
0 & 1 & 0 & 0 & v_1 \\
1 & 2 & v_1 & v_2 & v_1 \Omega + \Omega \\
0 & v_2 & 0 & 0 & v_1 v_2 \\
0 & v_3 & 0 & 0 & v_1 v_3 \\
v_1 & v_1^2 + \Omega & v_1 v_2 & v_1 v_3 & 2v_1 \Omega \\
\end{array} \right),$$

$$\bar{B}^2 = \left( \begin{array}{cccccc}
0 & 0 & 1 & 0 & v_2 \\
0 & 0 & v_1 & 0 & v_1 v_2 \\
1 & v_1 & 2v_2 & v_3 & v_3^2 + \Omega \\
0 & v_3 & 0 & 0 & v_2 v_3 \\
v_2 & v_1 v_2 & v_2^2 + \Omega & v_2 v_3 & 2v_2 \Omega \\
\end{array} \right),$$

$$\bar{B}^3 = \left( \begin{array}{cccccc}
0 & 0 & 0 & 1 & v_3 \\
0 & 0 & 0 & v_1 & v_1 v_3 \\
0 & 0 & 0 & v_2 & v_2 v_3 \\
1 & v_1 & v_2 & v_2 v_3 & v_3^2 + \Omega \\
v_3 & v_1 v_3 & v_2 v_3 & v_3^2 + \Omega & 2v_3 \Omega \\
\end{array} \right).$$
We note that \( B^0 > 0 \) (if \( \rho > 0 \)). With the help of simple calculations we easily find

\[
\mathbf{L}_q = H(S)(A^0 \cdot \mathbf{Q} + \mathbf{N}^0),
\]
\[
\mathbf{M}_q^k = H(S) \cdot (A^k \cdot \mathbf{Q} + \mathcal{M}^k), \quad k = 1, 2, 3.
\] (17)

Here

\[
\mathbf{N}^0 = \frac{p}{\gamma - 1} \cdot (0, 0, 0, 0, 1)^*,
\]
\[
\mathcal{M}^1 = p \cdot (0, 1, 0, \frac{\gamma \cdot v_1}{\gamma - 1})^*,
\]
\[
\mathcal{M}^2 = p \cdot (0, 0, 1, 0, \frac{\gamma \cdot v_2}{\gamma - 1})^*,
\]
\[
\mathcal{M}^3 = p \cdot (0, 0, 1, \frac{\gamma \cdot v_3}{\gamma - 1})^*, \quad H(S) = \frac{1}{1 - \frac{s}{c_r(\tau - 1)}}.
\]

By virtue of (16), the system of gas dynamics equations can be rewritten as

\[
H(S) \cdot B^0 \cdot \mathbf{Q}_t + 3 \sum_{k=1}^3 H(S) \cdot B^k \cdot \mathbf{Q}_{x_k} = 0
\] (18)

and, in view of (17), in the form

\[
(H(S) \cdot B^0 \cdot \mathbf{Q})_t + \sum_{k=1}^3 \left\{ (H(S) \cdot B^k \cdot \mathbf{Q})_{x_k} - (p \cdot H(S))_{x_k} \cdot \mathbf{N}^k \right\} = 0
\] (19)

where

\[
\mathbf{N}^1 = (0, 1, 0, 0, v_1)^*, \quad \mathbf{N}^2 = (0, 0, 1, 0, v_2)^*, \quad \mathbf{N}^3 = (0, 0, 0, 1, v_3)^*.
\]

It should be noted that when we obtain the system (19), we use the nonconservative form (14) of the gas dynamics equations.

We assume that (14) has a smooth solution \( \mathbf{U} = \mathbf{U}(t, \mathbf{x}) \), \( (t, \mathbf{x}) \in R^+ = \{ t > 0, \mathbf{x} \in R^3 \} \) satisfying, for example, the periodicity conditions

\[
\mathbf{U}(t, \mathbf{x} + \mathbf{m} \cdot \mathbf{L}) = \mathbf{U}(t, \mathbf{x}),
\] (20)

where

\[
\mathbf{L} = \text{diag}(l_1, l_2, l_3), \quad \mathbf{m} = (m_1, m_2, m_3),
\]
is a vector with integer constants: \( m_k = 0, \pm 1, \ldots, k = 1, 2, 3; l_k > 0, k = 1, 2, 3 \) are some real constants. Using system (18), (19), we now have an a priori estimate which can be a basis for the proof of local-in-time existence of smooth solutions.

Multiplying systems (18), (19) by the vector \( \mathbf{Q} \) and summing up the results, we obtain

\[
(\mathbf{Q}, H(S) \cdot B^0 \cdot \mathbf{Q})_t + \sum_{k=1}^3 (\mathbf{Q}, H(S) \cdot B^k \cdot \mathbf{Q})_{x_k} = 0
\] (21)

When we obtain the equality (21), we used the relations

\[
(\mathbf{Q}, \mathbf{N}^k) = 0, \quad k = 1, 2, 3.
\]

Integrating the equality (21) over the domain \( \Pi = \{ x; 0 < x_k < l_k, k = 1, 2, 3 \} \) and taking into account (20), we get the desired a priori estimate

\[
J(t) = J(0), 0 < t < T < \infty,
\] (22)
where
\[ J(t) = \int \int \int (Q, H(S) \cdot B^0 \cdot Q) \, dx. \]

While obtaining the a priori estimate (21), we also assumed that the smooth solution of system (11) has the property
\[ H(S) > 0. \]

We note that
\[ \langle Q, H(S) \cdot B^0 \cdot Q \rangle = c_V \cdot \rho \cdot \left\{ H(S) + \frac{\gamma - 1}{H(S)} \right\}. \]

We now discuss the symmetrization of the gas dynamics equations (11) when the additional conservation law is
\[ \frac{\partial}{\partial t} (\rho \cdot h(S)) + \text{div} (\rho \cdot h(S) \cdot v) = 0 \quad (23) \]
which is satisfied on smooth solutions of system (11) (see, for example, [7]). Here \( h(S) \) is a smooth function of \( S \). Simply calculating we get for the functions \( L, M_k, k = 1, 2, 3 \), of the variables \( q_i, i = 1, 5 \) the following expressions
\[ q_1 = -h' E + p \cdot V - \frac{|v|^2}{2} + h, \]
\[ q_{1+k} = -h' v_k T, k = 1, 2, 3, \]
\[ q_5 = \frac{h'}{T}, L = -\frac{h'}{T}, M_k = v_k L, k = 1, 2, 3. \]
The matrices \( A^0, A^k, k = 1, 2, 3 \) take the form
\[ A^0 = -B^0, A^k = -B^k, k = 1, 2, 3, \quad \rho = \frac{\rho \cdot \hat{h}}{c_V \cdot (\gamma - 1) h' \hat{h}}, \]
\[ B^0 = \rho \cdot \left( \begin{array}{cccc}
1 & v_1 & v_2 & v_3 & E \\
v_1 & v_1^2 + \frac{1}{h} \cdot pV & v_1 v_2 & v_1 v_3 & v_1 \hat{E} \\
v_2 & v_1 v_2 & v_2^2 + \frac{h}{5} \cdot pV & v_2 v_3 & v_2 \hat{E} \\
v_3 & v_1 v_3 & v_2 v_3 & v_3^2 + \frac{h}{5} \cdot pV & v_3 \hat{E} \\
\hat{E} & v_1 \hat{E} & v_2 \hat{E} & v_3 \hat{E} & \hat{E} \Omega + \Omega_1
\end{array} \right), \]
\[ \hat{E} = \Omega - \frac{pV}{h}, \quad \Omega = E + \frac{|v|^2}{2} + pV; \]
\[ \hat{E} = \hat{E} + \frac{h' \hat{V}}{h}, \quad \Omega_1 = \frac{pV|v|^2(2\hat{h} - 1)}{2h}; \]
\[ \hat{h} = 1 - \frac{\hat{h}''}{h'}, \quad \hat{h} = 1 - \gamma c_V \frac{h''}{h'}; \]
\[ B^k = v_k \cdot B^0 + \frac{\rho}{c_V (\gamma - 1) h'} \cdot \hat{B}^k, \quad k = 1, 2, 3 \]
(the matrices \( \hat{B}^k \) are described above). The condition of positive definiteness of the matrix \( \hat{B}^0 \) leads us to the following restrictions on the function \( h(S) \)
\[ h'(S) > 0, \quad \frac{h''(S)}{H'(S)} < \frac{1}{c_V \gamma}. \quad (24) \]
Further, simply calculating, we find
\[
\mathbf{L}_q = \bar{H}(S)(\mathbf{A}^0 \cdot \mathbf{Q} + \mathbf{N}^0), \\
\mathbf{M}_q = \bar{H}(S) \cdot (\mathbf{A}^k \cdot \mathbf{Q} + \mathbf{M}^k), \quad k = 1, 2, 3.
\]
(25)

Here
\[
\bar{H}(S) = \frac{\hat{h}}{1 - \frac{hh''}{hV^2}}, \\
\mathbf{N}^0 = \frac{\rho}{\gamma - 1} \cdot \frac{1}{\hat{h}} \cdot (1 - \frac{hh''}{hV^2}) \cdot (0, 0, 0, 1)^*, \\
\mathbf{M}^k = \frac{1}{\hat{h}} \cdot (1 - \frac{hh''}{hV^2}) \cdot (0, 0, 0, 0, 1)^*, \\
\mathbf{M}^k = (0, 0, 0, 0, 1)^*, \\
\bar{H}(S) = \frac{\gamma - 1}{\hat{h}} + (\gamma - 1)\hat{h} \bar{h}, \quad k = 1, 2, 3
\]
(the vectors \(\mathbf{M}^k\) are described above). As above the gas dynamics system (11) can be rewritten either as
\[
(\bar{H}(S) \cdot \mathbf{B}^0 \cdot \mathbf{Q}_t + 3 \sum_{k=1}^{3} \bar{H}(S) \cdot \mathbf{B}^k \cdot \mathbf{Q}_{x_k} = 0)
\]
(26)
or, by virtue of (25), as follows:
\[
(\bar{H}(S) \cdot \mathbf{B}^0 \cdot \mathbf{Q}_t + \\
\sum_{k=1}^{3} (\bar{H}(S) \cdot \mathbf{B}^k \cdot \mathbf{Q})_{x_k} - (p \cdot \bar{H}(S) \cdot \frac{1}{\hat{h}} \cdot (1 - \frac{hh''}{hV^2}))_{x_k} \cdot \mathbf{N}^k = 0)
\]
(27)
where the vectors \(\mathbf{N}^k, k = 1, 2, 3\) are described above. We can also (see [5], [6]) easily obtain the a priori estimate
\[
\bar{J}(t) = \bar{J}(0), \quad 0 < t < T < \infty,
\]
(28)
where
\[
\bar{J}(t) = \int \int \int (\mathbf{Q}, \bar{H}(S) \cdot \mathbf{B}^0 \cdot \mathbf{Q})dx.
\]
(28)
While obtaining the a priori estimation (28) we assumed that the smooth solution of (11) has the property
\[
\bar{H}(S) > 0.
\]
We note that
\[
(\mathbf{Q}, \bar{H}(S) \cdot \mathbf{B}^0 \cdot \mathbf{Q}) = c_V \cdot \rho \cdot \frac{h'}{\hat{h}} \left\{ \frac{\bar{H}(S)}{h} \cdot (\gamma \hat{h} - (\gamma - 1)) + \frac{(\gamma - 1)\hat{h}}{\bar{H}(S)} \right\}.
\]
Remark. Let
\[
h(S) = K \cdot \exp \left\{ \frac{S}{c_V(\alpha + \gamma)} \right\},
\]
(29)
where \(K, \alpha > 0\) are constants. Then \(\mathbf{N}^0 = \mathbf{M}^k = 0, k = 1, 2, 3\), \(\bar{H}(S) = -\frac{\gamma - 1}{\alpha + \gamma}\), \(\hat{h} = \frac{\alpha + \gamma - 1}{\alpha + \gamma}, \hat{h} = \frac{\alpha}{\alpha + \gamma}\).
At the same time systems (26), (27) can be rewritten as
\[
\mathbf{B}^0 \cdot \mathbf{Q}_t + \sum_{k=1}^{3} \mathbf{B}^k \cdot \mathbf{Q}_{x_k} = 0,
\]
(30)
\[(B^0 \cdot \mathbf{Q})_t + \sum_{k=1}^{3} (B^k \cdot \mathbf{Q})_{x_k} = 0.\] (31)

4. Difference schemes for the quasilinear hyperbolic systems. In the area of

If we assume that \(A = E\) is the unit matrix and the matrix \(B\) satisfies the condition of periodicity on \(x\) with the period \(l\). The difference model of the initial boundary

problem (5)-(8) is formulated as follows:

\[\begin{align*}
[V + V^{m+1}]_T W + r_x V_{i+1} B_{i+1}^+ \xi W &+ r_x V \xi [(B^T)^T W] + r_x V_{i+1} B_{i+1}^- \xi W \\
&+ r_y V_{j+1} C_{j+1}^+ \eta W + r_y V \eta [(C^T)^T W] + r_y V_{j+1} C_{j+1}^- \eta W + r_y V \eta [(C^T)^T W] \\
&+ r_z V_{k+1} D_{k+1}^+ \zeta W + r_z V \zeta [(D^T)^T W] + \Delta V Q W = \Delta V F; \\
n = 0, M - 1; i = 0, n - 1; j, k = 0, 1, \ldots
\end{align*}\] (33)

\[W_{0,j,k} = W_{n,j,k}^m, W_{i,0,k}^m = W_{i,n,k}^m, W_{i,j,0}^m = W_{i,j,n}^m; m = 0, M; |j|, |k| = 0, 1, \ldots; (34)\]

\[W_{i,j,k}^0 = W(U_0(ihx, jhy, khz), 0, ihx, jhy, khz); i = 0, n; |j|, |k| = 0, 1, \ldots; (35)\]

\[V = \text{diag}(W_1, W_2, \ldots, W_N).\]

Here

\[\begin{align*}
B(u) &= B^+(u) + B^-(u), B^+(u) \geq 0, B^-(u) \leq 0, \forall u \in \mathbb{R}^N, \\
C(u) &= C^+(u) + C^-(u), C^+(u) \geq 0, C^-(u) \leq 0, \forall u \in \mathbb{R}^N, \\
D(u) &= D^+(u) + D^-(u), D^+(u) \geq 0, D^-(u) \leq 0, \forall u \in \mathbb{R}^N,
\end{align*}\]

\[r_x = \frac{\Delta}{hx}, r_y = \frac{\Delta}{hy}, r_z = \frac{\Delta}{hz}, B = B(W_{i,j,k}^m, m \Delta, ihx, jhy, khz)\]

and so on.

We consider the following reconstruction:

\[\begin{align*}
U_{i+1/2}^L &= W_{i+1}^m - \frac{1}{2} \psi(R_i^m)(W_{i+1}^m - W_i^m), \\
U_{i+1/2}^R &= W_i^m - \frac{1}{2} \psi(R_i^m)(W_{i+1}^m - W_i^m), \\
U_{j+1/2}^L &= W_{j+1}^m - \frac{1}{2} \psi(R_j^m)(W_{j+1}^m - W_j^m), \\
U_{j+1/2}^R &= W_j^m - \frac{1}{2} \psi(R_j^m)(W_{j+1}^m - W_j^m), \\
U_{k+1/2}^L &= W_{k+1}^m - \frac{1}{2} \psi(R_k^m)(W_{k+1}^m - W_k^m), \\
U_{k+1/2}^R &= W_k^m - \frac{1}{2} \psi(R_k^m)(W_{k+1}^m - W_k^m),
\end{align*}\]
we obtain the following inequality
\[ \psi(R) = \text{diag}(\psi(R_1), \psi(R_2), \ldots, \psi(R_N)), \psi(R) = \text{diag}(\psi(\frac{1}{R_1}), \psi(\frac{1}{R_2}), \ldots, \psi(\frac{1}{R_N})), \]
\[
(R_i)^m = \frac{(w_i)_{i+1} - (w_i)_i}{(w_i)_{i+1} - (w_i)_{i-1}}, (R_i)^m = \frac{m}{w_i} + (w_i)_{j+1} - (w_i)_j, (R_i)_{k}^m = \frac{w_i}{(w_i)_k - (w_i)_{k-1}}.
\]

Here \( \psi : \mathbb{R} \to \mathbb{R} \) is a continuous function called \textit{limiter}. \( \psi = 1 \) corresponds to one sided scheme of the second order with upwind differences. Now we prove that the difference model (33)-(35) admits presence of difference analog of the dissipative energy integral.

This gives us possibility to get energetic estimation (the difference analog a priori estimation (10)), from which it follows stability of the difference scheme (33)-(35). For this, we multiply both sides of the obtained inequality by \( \tau \in (0, 1) \) and sum up over \( i \) from 1 to \( n - 1 \); over \( j \) from \( -\infty \) to \( +\infty \) and over \( k \) from \( -\infty \) to \( +\infty \). Then, taking into account the following relations
\[
\tau(W, W) + r_x \xi((B^T(U_{i+1/2}^L)W_{i+1/2}), W) + r_x \xi((B^T(U_{i-1/2}^R)W_{i-1/2}), W) + r_y \eta((C^T(U_{j+1/2}^L)W_{j+1/2}), W) + r_y \eta((C^T(U_{j-1/2}^R)W_{j-1/2}), W) + r_z \gamma((D^T(U_{k+1/2}^L)W_{k+1/2}), W) + \Delta V W = \Delta V F.
\]

Using the following formula of difference differentiation (for any grid functions \( u = u_{i,j,k} \) and \( v = v_{i,j,k} \) of the type
\[
\xi(u) = (u_{i-1} - \xi v) + (\xi u)
\]
and the following relations
\[
(\tau W, W) = \tau(W, W); (QW, W) = -(GW, W); (W, F) \leq \frac{1}{2}(W, W) + \frac{1}{2}(F, F),
\]
we have
\[
\tau(W, W) + r_x \xi((B^T W)W) + r_y \eta((C^T W)W) + r_z \gamma((D^T W)W) \leq \Delta \{GW, W\} + \frac{1}{2}(W, W) + \frac{1}{2}(F, F),
\]
\[
(37)
\]

We multiply both sides of the obtained inequality by \( h_x h_y h_z \) and sum up over \( i \) from 1 to \( n - 1 \); over \( j \) from \( -\infty \) to \( +\infty \) and over \( k \) from \( -\infty \) to \( +\infty \). Then, taking into account the following relations
\[
\sum_{i=1}^{n-1} \{\xi(\xi(U_{i+1/2}^L U_{i+1/2}^L), W) + \xi((B^T(U_{i-1/2}^R)U_{i-1/2}^R), W)\} = 0;
\]
\[
\sum_{j=-\infty}^{\infty} \{\eta((B^T(U_{j+1/2}^L)U_{j+1/2}^L), W) + \eta((B^T(U_{j-1/2}^R)U_{j-1/2}^R), W)\} = 0;
\]
\[
\sum_{k=1}^{n-1} \{\gamma((B^T(U_{k+1/2}^L)U_{k+1/2}^L), W) + \gamma((B^T(U_{k-1/2}^R)U_{k-1/2}^R), W)\} = 0
\]
we obtain the following inequality
\[
J_m \leq \frac{e^{2T(6\mu + 3)} J_0}{2 \mu + 1} - \Phi, \quad m = \frac{1}{M},
\]
\[
(39)
\]
which means the stability of the difference scheme (33) - (35) in the energetic norm \( \sqrt{J_m} \).
As an example, we consider the hyperbolic equation (the Burgers equation)

$$u_t + uu_x = 0.$$ 

We introduce the following notations $f(u) = \frac{u^2}{2}$, $f'(u) = f^+(u) + f^-(u)$, $\frac{df^+}{du} \geq 0$, $\forall u \in \mathbb{R}$, and we construct the following difference scheme for the last equation

$$U_{i}^{k+1} = U_{i}^{k} - \frac{\Delta t}{\Delta x}[F^+(U_{i+\frac{1}{2}}^L) - F^+(U_{i-\frac{1}{2}}^L) + F^-(U_{i+\frac{1}{2}}^R) - F^-(U_{i-\frac{1}{2}}^R)].$$

Consider the following modification of the obtained scheme

$$\psi \rightarrow \delta = 0$$

It easily follows from the last equality the energetic estimate

$$\|U\|_{\infty} \leq \frac{1}{\Delta x} \int_{x_{i-1}}^{x_i} \psi(u, k\Delta t) du, \psi : \mathbb{R} \rightarrow \mathbb{R}$$

is the continuous function called limiter. $\psi = 1$ is one sided scheme of the second order with upwind difference. We consider the following reconstruction:

$$U_{i+\frac{1}{2}}^L = U_{i}^{k} + \frac{1}{2} \psi(R_{i}^k)(U_{i+1}^k - U_{i}^k), U_{i-\frac{1}{2}}^R = U_{i}^{k} - \frac{1}{2} \psi(R_{i}^k)(U_{i}^k - U_{i-1}^k), \psi : \mathbb{R} \rightarrow \mathbb{R}$$

Now we prove that the difference model (40) admits availability of difference analogue of the energy integral. For this, we multiply the system (40) by $[\tilde{U} + U]$:

$$\tau U[\tilde{U} + U] + \frac{4}{3} \tau [F^+(U_{i-\frac{1}{2}}^L) \tilde{\xi}(U_i)] + \frac{\Delta t}{\Delta x} [(\tilde{\xi}[F^+(U_{i+\frac{1}{2}}^L)]U)]$$

$$+ \frac{\Delta t}{\Delta x} [(\tilde{\xi}[F^+(U_{i+\frac{1}{2}}^L)]U)] = 0; \quad (40)$$

$$U_{i}^{0} = \phi(i \cdot h); |i| = 0, 1, \ldots$$

Here $\tau = \Delta / h$.

Using formulas of difference differentiation, we have the following identity

$$(F^+(U_{i-\frac{1}{2}}^L) \cdot \tilde{\xi}(U)) + (\tilde{\xi}[F^+(U_{i+\frac{1}{2}}^L)] \cdot U) = \tilde{\xi}(F^+(U_{i+\frac{1}{2}}^L) \cdot U),$$

$$(F^-(U_{i+\frac{1}{2}}^R) \cdot \xi(U)) + (\xi[F^+(U_{i-\frac{1}{2}}^R)] \cdot U) = \xi(F^-(U_{i-\frac{1}{2}}^R) \cdot U).$$

Taking into account all these transformations, we have

$$\tau(U^2) + \frac{4}{3} \tau \tilde{\xi}(F^+(U_{i+\frac{1}{2}}^L) \cdot U) + \xi(F^-(U_{i-\frac{1}{2}}^R) \cdot U) = 0.$$

We multiply both sides of the obtained inequality by $h$ and sum up over $i$ from $-\infty$ to $+\infty$, taking into account that the function $u$ tends to zero at infinity and denoting the quantity $h \sum_{i=-\infty}^{\infty} U^2$ by $I_k$ we have the equality

$$I_{k+1} - I_k = 0.$$ 

It easily follows from the last equality the energetic estimate

$$I_k = I_0, k = \overline{1, M}$$

which means stability of the difference scheme in the norm $\sqrt{I_k}$

**Numerical results.** Consider the following problem

$$u_t + (u^2/2)_x = 0, -\infty < x < \infty, t > 0,$$
\[ u(x, 0) = \begin{cases} 
1, & x < 0, \\
0, & x > 0. 
\end{cases} \]

As parameters of numerical calculations we take \(-1 \leq x \leq 1, 0 \leq t \leq 1, \Delta = 0.001, h = 0.01\). We rewrite the difference scheme (40) in the following form

\[
U_{i}^{n+1} = U_{i}^{k} - \frac{4}{3(U_{i+1}^{k} + U_{i-1}^{k})} \times \]

\[ \times (F^{+}(U_{i-\frac{1}{2}}^{L})\xi(U_{i})) + (\xi[F^{+}(U_{i-\frac{1}{2}}^{L})]U_{i}) + (F^{-}(U_{i+\frac{1}{2}}^{R})\xi(U_{i})) + (\xi[F^{-}(U_{i+\frac{1}{2}}^{R})]U_{i}). \]

For the numerical solution of the difference scheme we apply an iterative method for nonlinear coefficients

\[
U_{i}^{n+1} = U_{i}^{k} - \frac{4}{3(U_{i}^{k} + U_{i}^{k})} \times \]

\[ \times (F^{+}(U_{i-\frac{1}{2}}^{L})\xi(U_{i})) + (\xi[F^{+}(U_{i-\frac{1}{2}}^{L})]U_{i}) + (F^{-}(U_{i+\frac{1}{2}}^{R})\xi(U_{i})) + (\xi[F^{-}(U_{i+\frac{1}{2}}^{R})]U_{i}). \]

In Figure 1, we give results of calculation of the numerical solution on Mathcad. Comparing the numerical solution with the exact one, it can be concluded that the modified scheme with limiter well modulates the jump.

For clearness we consider this solution at \( t = 1 \) (see Fig. 2)

5. **Conclusion.** In the present paper, we construct the class of difference schemes for systems of hyperbolic equations. It is proved stability of such schemes in the energetic norm \( \sqrt{I_{F}} \) (see (41)). Numerical experiments are done on model examples. The numerical results showed operability of the difference scheme.
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