On the Certification of Positive and Convex Gotoh’s Fourth-Order Yield Function

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Abstract. Gotoh proposed in 1977 a fourth-order homogeneous polynomial of three plane stress components as a yield function to model anisotropic yielding and plastic flow of sheet metals. The yield function admits up to eight experimental inputs from uniaxial tension tests and one measurement from an equal biaxial tension test for calibrating its nine material constants and can model the formation of up to eight ears in deep cup drawing. However, the superior Gotoh’s yield function has not been widely adopted in sheet metal forming analyses in both academy and industry, especially outside Japan. One major concern is uncertain about the positivity and convexity of a calibrated Gotoh’s yield function. Here the problem of certifying a calibrated Gotoh’s yield function to be strictly positive and convex is first described and its resolutions are summarized based on some very recent research results. Algebraic necessary and sufficient conditions for a positive and convex Gotoh’s yield function with only two non-zero stress components are first presented. Sufficient conditions in terms of semi-algebraic and algebraic inequalities for establishing the positivity and convexity of tri-component plane stress Gotoh’s yield function are then summarized. Finally, the complete necessary and sufficient conditions for a positive and convex Gotoh’s yield function in plane stress are realized as a fully numerical minimization problem. Examples of successfully applying those conditions in certifying positive and convex Gotoh’s yield functions of various sheet metals are also given.

1. Introduction
Following the suggestion made by Hill in 1950 about the general polynomial formulation of a plane stress yield function [1], Gotoh in 1977 [2, 3] detailed his study of a complete fourth-order homogeneous polynomial yield function for orthotropic sheets in the form of

\[ \Phi_y(\sigma) = A_1\sigma_x^4 + A_2\sigma_x^2\sigma_y^2 + A_3\sigma_y^4 + A_4\sigma_x\sigma_y^3 + A_5\sigma_y^4 + (A_6\sigma_x^2 + A_7\sigma_y^2 + A_8\sigma_y^2 + A_9\tau_{xy}^2)\tau_{xy}^2, \] (1)

where \(A_1, ..., A_9\) are its nine independent material constants and \(\sigma = (\sigma_x, \sigma_y, \tau_{xy})\) is the applied Cauchy stress with its three plane-stress components defined with respect to the orthogonal symmetric axes \(xyz\) of a sheet (i.e., its rolling, transverse and normal directions).

In Part I of his study [2], Gotoh put \(A_1 = 1\) (i.e., assuming the equivalent true stress-strain curve is defined by the uniaxial tensile stress-strain curve along the rolling direction), showed that the non-equivalence between the fourth-order stress-based yield function and the fourth-order strain-rate-based plastic potential, examined types of ear-formation in axisymmetrical deep-drawing as predicted by the yield function, presented explicit relations for yield stress and plastic strain ratio (via the associated flow rule) under a given uniaxial tension direction,
and most importantly, derived a set of nine **linear** equations for uniquely identifying the nine polynomial coefficients or material constants of the yield function based on four pairs of uniaxial tension yield stresses and plastic strain ratios and one equal biaxial tensile yield stress. In Part II of his study [3], Gotoh gave two examples of his yield function calibrated for commercial Al-killed steel and Cu-(1/4)H sheets and showed that the new yield function can describe very well variations of both plastic strain ratios and yield stresses as a function of the uniaxial tensile loading angle and has the flexibility of accommodate both the plastic strain ratio and the ratio of equal-biaxial yield stress over uniaxial yield stress for in-plane isotropic sheets.

Subsequently, Gotoh and his co-workers have applied successfully the fourth-order yield function in finite element analyses of various sheet metal forming problems [4, 5, 6, 7]. Nevertheless, Gotoh’s yield function has only been considered occasionally in sheet metal modeling analyses since its first introduction more than forty years ago [8, 9, 10, 11, 12, 13], but rarely outside Japan [14, 15]. In comparison, associated non-quadratic yield functions such as Yld2000-2d and non-associated quadratic anisotropic sheet metal plasticity modeling that have only been proposed and developed since 2000s [16, 17] are much more widely appeared nowadays in the literature. As the total number of material constants is only eight in an associated model with the popular Yld2000-2d yield function and is only seven in a non-associated model with a quadratic yield function and a quadratic plastic potential respectively, their modeling capability or flexibility of anisotropic yielding and plastic flow behavior of a sheet metal is at most only comparable with Gotoh’s yield function with nine material constants. The reason behind this striking difference in their acceptance in both academy and industry is well-known: a) the positivity and convexity of Yld2000-2d (with real-valued material constants) is mathematically guaranteed, b) the positivity and convexity of any calibrated quadratic yield function can always be checked or verified via simple algebraic inequalities, but c) the positivity and convexity of Gotoh’s yield function is however uncertain and unknown in general.

This presentation aims to address this widely perceived shortcoming about Gotoh’s yield function by concisely summarizing recent research progresses on its convexity certification. As shown in Fig.1, it is proposed that a cascaded sequence of necessary or/and sufficient conditions should be used to certify the convexity of any calibrated Gotoh’s yield function and to adjust if necessary an as-calibrated but non-convex Gotoh’s yield function to be convex. Details of those conditions are described first and followed by some application examples and a brief discussion.

### 2. Algebraic necessary conditions for positive and convex Gotoh’s yield function

A simple algebraic necessary condition is often useful for screening out quickly any non-convex yield function. A set of some semi-algebraic and algebraic necessary conditions for ensuring a calibrated Gotoh’s yield function to be positive and convex have explicitly been given perhaps for the first time by Soare et al. [15] by setting one of three plane stress components to be zero. For example, they showed that the following inequalities should have to be met for a convex...
Gotoh’s yield function $\Phi_g(\sigma_x, 0, \tau_{xy})$ or $\Phi_g(0, \sigma_y, \tau_{xy})$ as

$$0 \leq A_6 \leq 6\sqrt{A_1 A_5}, \quad \text{or} \quad 0 \leq A_8 \leq 6\sqrt{A_3 A_9}.$$  \hspace{1cm} (2)

These and some additional inequalities per a convex Gotoh’s yield function in one or two stress components are listed in Appendix B of [18]. More recently, the necessary and sufficient conditions for a convex Gotoh’s function $\Phi_g(\sigma_x, \sigma_y, 0)$ has been reported as [19]

$$3A_1 + A_3 + 3A_5 \geq 0, \quad 24A_1A_3 + 72A_1A_5 + 24A_3A_5 - 9A_2^2 - 2A_3^2 - 9A_1^2 \geq 0, \quad (3)$$

$$72A_4A_3A_5 + 9A_2A_3A_4 - 2A_3^3 - 27A_1A_3^2 - 27A_2^2A_5 \geq 0. \quad (4)$$

All these algebraic inequalities serve as necessary conditions for a general tri-component Gotoh’s yield function $\Phi_g(\sigma_x, \sigma_y, \tau_{xy})$ to be positive and convex.

3. Sufficient conditions for positive and convex Gotoh’s yield function

Once a calibrated Gotoh’s yield function passes the necessary conditions given above, one can then check if it can pass one of sufficient convexity conditions as well. If it does, then the task of convexity certification is competed for this particular as-calibrated Gotoh’s yield function. In the following, two different sufficient conditions are described, one is semi-algebraic and another is fully algebraic.

As shown in [20], Gotoh’s yield function can always be rewritten in terms of intrinsic variables of a plane stress as a generalized 1979 Hill’s yield function of four degrees. Two examples are

$$\phi_{h1}(\sigma_1, \sigma_2, \theta) = f(\theta)\sigma_1^4 + g(\theta)\sigma_2^4 + h(\theta)(\sigma_1 - \sigma_2)^4 + a(\theta)(2\sigma_1 - \sigma_2)^4 + b(\theta)(2\sigma_2 - \sigma_1)^4,$$

$$\phi_{h2}(\sigma_1, \sigma_2, \theta) = f(\theta)\sigma_1^4 + g(\theta)\sigma_2^4 + a(\theta)(2\sigma_1 - \sigma_2)^4 + b(\theta)(2\sigma_2 - \sigma_1)^4 + c(\theta)(\sigma_1 + \sigma_2)^4,$$

where $(\sigma_1, \sigma_2, \theta)$ are two principal stresses and the loading orientation angle of the Cauchy plane stress, and $f(\theta), g(\theta), h(\theta), a(\theta), b(\theta), c(\theta)$ are Fourier cosine series function in $\theta$ with their Fourier coefficients being linearly dependent on the nine polynomial coefficients $A_1, A_2, ..., A_9$.

Obviously, the sufficient condition for them to be positive and convex is

$$f(\theta) \geq 0, \quad g(\theta) \geq 0, \quad h(\theta) \geq 0, \quad a(\theta) \geq 0, \quad b(\theta) \geq 0, \quad c(\theta) \geq 0.$$  \hspace{1cm} (7)

These inequalities have to be checked however numerically over $\theta \in [0, \pi/2]$ [20].

The second sufficient condition is given very recently in [19] as the following 9-by-9 matrix being positive semidefinite

$$G_{9 \times 9} = \begin{pmatrix}
12A_1 & 3A_2 & 0 & 3A_2 & 2A_3 & 0 & 0 & 0 & 2A_6 \\
3A_2 & 2A_3 & 0 & 2A_3 & 3A_4 & 0 & 0 & 0 & A_7 \\
0 & 0 & 2A_6 & 0 & 0 & A_7 & 2A_6 & A_7 & 0 \\
3A_2 & 2A_3 & 0 & 2A_3 & 3A_4 & 0 & 0 & 0 & A_7 \\
2A_3 & 3A_4 & 0 & 3A_4 & 12A_5 & 0 & 0 & 0 & 2A_8 \\
0 & 0 & A_7 & 0 & 0 & 2A_8 & A_7 & 2A_8 & 0 \\
0 & 0 & 2A_6 & 0 & 0 & A_7 & 2A_6 & A_7 & 0 \\
0 & 0 & A_7 & 0 & 0 & 2A_8 & A_7 & 2A_8 & 0 \\
2A_6 & A_7 & 0 & A_7 & 2A_8 & 0 & 0 & 0 & 12A_9
\end{pmatrix} \geq 0. \quad (9)$$

The coefficients of its characteristic polynomial $|\lambda I + G_{9 \times 9}|$ can be used for testing positive semi-definiteness as $|\lambda I + G_{9 \times 9}|$ has the following form

$$\lambda^9 + d_8 \lambda^8 + d_7 \lambda^7 + d_6 \lambda^6 + d_5 \lambda^5 + d_4 \lambda^4 + d_3 \lambda^3.$$  \hspace{1cm} (10)
The necessary and sufficient conditions for the 9-by-9 matrix $G_{9 \times 9}$ to be positive semidefinite are that the six coefficients $(d_8, d_7, d_6, d_5, d_4, d_3)$ of its characteristic polynomial should be nonnegative. The lengthy algebraic expressions for these six coefficients in terms of $(A_1, \ldots, A_9)$ are listed in Appendix A of [19] but are omitted here for brevity.

4. The necessary and sufficient conditions for convex Gotoh's yield function

On the occasion when a calibrated Gotoh’s yield function fails the sufficient conditions above, one then has to further check if it satisfies the necessary and sufficient convexity condition given below. As a convex Gotoh’s yield function is always positive [18], one can skip checking on its positivity. Gotoh’s yield function is twice differentiable so the most easily usable convexity condition is the so-called second order or local condition in terms of its Hessian matrix

$$\nabla^2 \Phi_g(\sigma_x, \sigma_y, \tau_{xy}) = \begin{pmatrix} \frac{\partial^2 \Phi_g}{\partial \sigma_x^2} & \frac{\partial^2 \Phi_g}{\partial \sigma_x \partial \sigma_y} & \frac{\partial^2 \Phi_g}{\partial \sigma_x \partial \tau_{xy}} \\ \frac{\partial^2 \Phi_g}{\partial \sigma_x \partial \sigma_y} & \frac{\partial^2 \Phi_g}{\partial \sigma_y^2} & \frac{\partial^2 \Phi_g}{\partial \sigma_y \partial \tau_{xy}} \\ \frac{\partial^2 \Phi_g}{\partial \sigma_x \partial \tau_{xy}} & \frac{\partial^2 \Phi_g}{\partial \sigma_y \partial \tau_{xy}} & \frac{\partial^2 \Phi_g}{\partial \tau_{xy}^2} \end{pmatrix}.$$  \hspace{1cm} (11)

That is, Gotoh’s yield function $\Phi_g(\sigma_x, \sigma_y, \tau_{xy})$ is convex if and only if its Hessian matrix is positive semidefinite per the yield condition $\Phi_g(\sigma_x, \sigma_y, \tau_{xy}) = \sigma_f^4$ (where $\sigma_f$ is the current yield strength of the sheet metal under consideration). As the strict convexity is preferred in practice, one only needs to ensure that three leading principal minors $\Psi_{4A}, \Psi_{4B}$ and $\Psi_{4C}$ of the Hessian matrix are strictly positive on the yield surface [21, 18].

The three leading principal minors $\Psi_{4A}, \Psi_{4B}$ and $\Psi_{4C}$ of Gotoh’s yield function are homogeneous polynomials of two, four and six degrees respectively. There are no known algebraic inequalities put forth so far for positive non-quadratic homogeneous polynomials of three variables $(\sigma_x, \sigma_y, \tau_{xy})$, the positivity of both $\Psi_{4B}$ and $\Psi_{4C}$ has to be certified numerically instead (the positivity of the quadratic polynomial $\Psi_{4A}$ can however be checked algebraically). By using either polar coordinate [21] or spherical coordinate [18] parametrization of $\psi_4(\omega, \theta)$ and $\psi_{4C}(\omega, \theta)$ and by considering the homogeneity of Gotoh’s yield function, one actually needs only to find their minimum values over a two-dimensional orientation space of $\omega \in [0, \pi]$ and $\theta \in [0, \pi/2]$ and makes sure they are positive. This can be carried out nowadays using MATLAB or similar numerical calculation tools in seconds at most for a given calibrated Gotoh’s yield function.

5. Application Examples

Here we apply the necessary and sufficient conditions discussed so far to certify the convexity of some calibrated Gotoh’s yield functions. The first case is the fourteen in-plane isotropic sheets studied by Gotoh [3]. The corresponding Gotoh’s yield function $\phi_g(\sigma_1, \sigma_2)$ in terms of only two principal stresses $(\sigma_1, \sigma_2)$ will have the same form as the biaxial case $\Phi_g(\sigma_x, \sigma_y, 0)$ with $A_5 = A_1$ and $A_4 = A_2$. When the uniaxial tension yield stress $\sigma_0$ and plastic strain ratio $R_0$ along with the equal biaxial tensile yield stress $\sigma_b$ are given as experimental inputs (assuming $\sigma_f = \sigma_0$), the three independent material constants $(A_1, A_2, A_3)$ of $\phi_g(\sigma_1, \sigma_2)$ are given as [18, 22] $A_1 = 1$, $A_2 = -4R_0/(1 + R_0)$, $A_3 = \sigma_b^2/\sigma_0^2 + 6R_0 - 2/(1 + R_0)$. By introducing the plastic thinning ratio $\eta_b = 1/(1 + R_0) \in (0, 2)$ and the stress ratio $b = \sigma_0/\sigma_b \in (0, 2)$, the convexity conditions of Eq.(3) and Eq.(4) reduce to [19]

$$\psi_0(\eta_b, b) = -64\eta_b^3 + 144\eta_b b^4 - 120\eta_b^2 b^4 - 18b^8 + 24\eta_b b^8 - b^{12} \geq 0.$$  \hspace{1cm} (12)

As shown in Table 1, the convexity of each in-plane isotropic sheet can now be firmly established for the first time via the computed $\psi_0 > 0$ of Eq.(12). The results can also be visualized in Fig.2 in terms of the convex domain encompassing all sheet metals listed sequentially in the Table 1.
Table 1. Convexity of fourteen in-plane isotropic sheets list in Table 3 of [3].

| Metals       | $R_0$ | $\sigma_b/\sigma_0$ | $b$   | $\eta_0$ | $\psi_0$ | Metals       | $R_0$ | $\sigma_b/\sigma_0$ | $b$   | $\eta_0$ | $\psi_0$ |
|--------------|-------|---------------------|-------|-----------|----------|--------------|-------|---------------------|-------|-----------|----------|
| Rimmed Steel | 1.49  | 1.12                | 0.89  | 0.40      | +        | Titanium    | 4.41  | 1.35                | 0.74  | 0.19      | +        |
| AK steel     | 1.80  | 1.25                | 0.80  | 0.36      | +        | Pure Al     | 0.86  | 0.98                | 1.02  | 0.54      | +        |
| AK steel-2   | 1.54  | 1.23                | 0.81  | 0.39      | +        | Al-Mn alloy | 0.61  | 1.11                | 0.90  | 0.62      | +        |
| TK steel     | 2.00  | 1.25                | 0.80  | 0.33      | +        | Al-Cu       | 0.77  | 0.92                | 1.09  | 0.57      | +        |
| 18-8 SS steel| 0.84  | 1.19                | 0.84  | 0.54      | +        | Al-Mg       | 0.60  | 1.01                | 0.99  | 0.63      | +        |
| Cu-O         | 0.90  | 1.05                | 0.95  | 0.53      | +        | Al-Mg-Si    | 0.67  | 1.07                | 0.94  | 0.60      | +        |
| 7/3 Brass-O  | 0.81  | 1.03                | 0.97  | 0.55      | +        | Al-Mg 2     | 0.74  | 1.02                | 0.98  | 0.58      | +        |

The second case is the three other calibrated Gotoh’s yield functions as reported by Gotoh in [5]. As shown in Table 2, the three leading principal minors of the Hessian matrix of each of calibrated yield functions are found to be strictly positive and thus each yield function is shown to be strictly convex (and positive) as well. In fact, all of these three calibrated Gotoh’s yield functions meet the algebraic sufficient condition for convexity given by Eq.(9).

Table 2. Convexity of three as-calibrated Gotoh’s yield functions given in [5].

| Metals | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $\psi_{4A}$ | $\psi_{4B}$ | $\psi_{4C}$ |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------------|-------------|-------------|
| R.S.   | -2.19 | 3.18  | -2.34 | 0.98  | 6.73  | -6.19 | 5.64  | 8.69  | +           | +           | +           |
| K.S.2  | -2.52 | 3.75  | -2.91 | 1.05  | 6.24  | -8.38 | 7.28  | 11.20 | +           | +           | +           |
| T.S.   | -2.40 | 3.62  | -2.40 | 1.00  | 4.62  | -4.48 | 4.62  | 10.42 | +           | +           | +           |

6. Discussion
The first ever rigorous proof that certifies the strict convexity of some calibrated Gotoh’s yield functions including the very first two obtained by Gotoh in 1977 has only been reported in 2016 based on the semi-algebraic sufficient condition given in [20]. Numerous calibrated Gotoh’s yield functions have since been verified to be convex as well via a numerical minimization algorithm of necessary and sufficient convexity conditions [21, 18]. There exist the algebraic necessary and sufficient conditions Eq.(3) and Eq.(4) for convex $\Phi_g(\sigma_x, \sigma_y, 0)$ and the algebraic sufficient condition of Eq.(9) for $\Phi_g(\sigma_x, \sigma_y, \tau_{xy})$. These conditions are valuable additions to the fully numerical minimization algorithm of necessary and sufficient conditions for certifying and adjusting the convexity of a calibrated Gotoh’s yield function per the procedure shown in Fig.1.

Although the positive semi-definiteness of the 9-by-9 matrix $G_{9 \times 9}$ per Eq.(9) is only a sufficient condition for $\Phi_g(\sigma_x, \sigma_y, \tau_{xy})$ to be convex, this algebraic condition appears to be a fairly extensive one in practice. That is, almost all calibrated Gotoh’s yield functions that have been verified to be strictly convex via the numerical minimization algorithm so far meet this sufficient condition as well. It was however found that $d_3 < 0$ of Eq.(10) when applying the sufficient condition to sheet metal No.15 in Table 4 of [18] with $(A_1, A_2, ..., A_9) = (1, -1.7273, 2.3675, -1.9619, 1.2945, 6.3148, 0.0096, 7.1386, 5.0025)$. A typo was indeed found in the original code implementing the numerical minimization algorithm. Its third leading principal minor is
actually not all positive, see Fig.3. Using the procedure described in [18], a new set of material constants was found (when the adjustable parameter $\xi = 0.955$) as $(A_1, A_2, ..., A_9) = (-1.7273, 2.3675, -1.9619, 1.2945, 6.30162, 0.0289005, 7.03541, 5.15319)$ that makes the adjusted Gotoh’s yield function convex per its positive definite Hessian matrix. Nevertheless, this convex Gotoh’s yield function does still not satisfy the sufficient condition of Eq.(9).

Almost all Gotoh’s yield functions appeared in the literature have assumed isotropic hardening. That is, its nine polynomial coefficients are constant. One can extend Gotoh’s yield function to anisotropic hardening with its nine polynomial coefficients being functions of either plastic work or equivalent plastic strain. Convexity certification of Gotoh’s yield function with anisotropic hardening can also be readily carried out using the strategy described here.

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**Figure 2.** Graphic illustration on the convexity of Gotoh’s yield functions for 14 in-plane isotropic sheets in [3].

**Figure 3.** Graphic illustration of the negative third leading principal minor $\psi_{AC}(\omega, \theta)$ for sheet metal No.15 in [18].