COMPUTING EIGENVALUES OF LARGE SCALE SPARSE TENSORS ARISING FROM A HYPERGRAPH

JINGYA CHANG*, YANNAN CHEN†, AND LIQUN QI§

Abstract. The spectral theory of higher-order symmetric tensors is an important tool to reveal some important properties of a hypergraph via its adjacency tensor, Laplacian tensor, and signless Laplacian tensor. Owing to the sparsity of these tensors, we propose an efficient approach to calculate products of these tensors and any vectors. Using the state-of-the-art L-BFGS approach, we develop a first-order optimization algorithm for computing H- and Z-eigenvalues of these large scale sparse tensors (CEST). With the aid of the Kurdyka-Lojasiewicz property, we prove that the sequence of iterates generated by CEST converges to an eigenvector of the tensor. When CEST is started from multiple randomly initial points, the resulting best eigenvalue could touch the extreme eigenvalue with a high probability. Finally, numerical experiments on small hypergraphs show that CEST is efficient and promising. Moreover, CEST is capable of computing eigenvalues of tensors corresponding to a hypergraph with millions of vertices.

Key words. Eigenvalue, hypergraph, Kurdyka-Lojasiewicz property, Laplacian tensor, large scale tensor, L-BFGS, sparse tensor, spherical optimization.

AMS subject classifications. 05C65, 15A18, 15A69, 65F15, 65K05, 90C35, 90C53

1. Introduction. Since 1736, Leonhard Euler posed a problem called “seven bridges of Königsberg”, graphs and hypergraphs have been used to model relations and connections of objects in science and engineering, such as molecular chemistry [35, 18], image processing [21, 50], networks [32, 22], scientific computing [19, 30], and very large scale integration (VLSI) design [29]. For large scale hypergraphs, spectral hypergraph theory provides a fundamental tool. For instance, hypergraph-based spectral clustering has been used in complex networks [42], date mining [37], and statistics [51, 36]. In computer-aided design [60] and machine learning [23], researchers employed the spectral hypergraph partitioning. Other applications include the multilinear pagerank [24] and estimations of the clique number of a graph [8, 54].

Recently, spectral hypergraph theory is proposed to explore connections between the geometry of a uniform hypergraph and H- and Z-eigenvalues of some related symmetric tensors. Cooper and Dutle [13] proposed in 2012 the concept of adjacency tensor for a uniform hypergraph. Two years later, Qi [49] gave definitions of Laplacian and signless Laplacian tensors associated with a hypergraph. When an even-uniform hypergraph is connected, the largest H-eigenvalues of the Laplacian and signless Laplacian tensors are equivalent if and only if the hypergraph is odd-bipartite [28]. This result gives a certification to check whether a connected even-uniform hypergraph is odd-bipartite or not.

* Version 1.0. Data: March 25, 2016.
† Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong; and School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China ([jychang@zzu.edu.cn]).
‡ School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China (ynchen@zzu.edu.cn). This author was supported by the National Natural Science Foundation of China (Grant No. 11401539, 11571178), the Development Foundation for Excellent Youth Scholars of Zhengzhou University (Grant No. 1421315070), and the Hong Kong Polytechnic University Postdoctoral Fellowship.
§ Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (maqilq@polyu.edu.hk). This author’s work was partially supported by the Hong Kong Research Grant Council (Grant No. PolyU 501212, 501913, 15302114 and 15300715).
We consider the problem of how to compute H- and Z-eigenvalues of the adjacency tensor, the Laplacian tensor, and the signless Laplacian tensor arising from a uniform hypergraph. Since the adjacency tensor and the signless Laplacian tensor are symmetric and nonnegative, an efficient numerical approach named the Ng-Qi-Zhou algorithm [43] could be applied for their largest H-eigenvalues and associated eigenvectors. Chang et al. [10] proved that the Ng-Qi-Zhou algorithm converges if the nonnegative symmetric tensor is primitive. Liu et al. [40] and Chang et al. [10] enhanced the Ng-Qi-Zhou algorithm and proved that the enhanced one converges if the nonnegative symmetric tensor is irreducible. Friedland et al. [20] studied weakly irreducible nonnegative symmetric tensors and showed that the Ng-Qi-Zhou algorithm converges with an R-linear convergence rate for the largest H-eigenvalue of a weakly irreducible nonnegative symmetric tensor. Zhou et al. [58, 59] argued that the Ng-Qi-Zhou algorithm is Q-linear convergence. They refined the Ng-Qi-Zhou algorithm and reported that they could obtain the largest H-eigenvalue for any nonnegative symmetric tensors. A Newton’s method with locally quadratic rate of convergence is established by Ni and Qi [44].

With respect to the eigenvalue problem of general symmetric tensors, there are two sorts of methods. The first one could obtain all (real) eigenvalues of a tensor with only several variables. Qi et al. [50] proposed a direct approach based on the resultant. An SDP relaxation method coming from polynomial optimization was established by Cui et al. [14]. Chen et al. [11] preferred to use homotopy methods. Additionally, mathematical softwares Mathematica and Maple provide respectively subroutines “NSolve” and “solve” which could solve polynomial eigen-systems exactly. However, if we apply these methods for eigenvalues of a symmetric tensor with dozens of variables, the computational time is prohibitively long.

The second sort of methods turn to compute an (extreme) eigenvalue of a symmetric tensor, since a general symmetric tensor has plenty of eigenvalues [48] and it is NP-hard to compute all of them [27]. Kolda and Mayo [33, 34] proposed a spherical optimization model and established shifted power methods. Using fixed point theory, they proved that shifted power methods converge to an eigenvalue and its associated eigenvector of a symmetric tensor. For the same spherical optimization model, Hao et al. [26] prefer to use a faster subspace projection method. Han [25] constructed an unconstrained merit function that is indeed a quadratic penalty function of the spherical optimization. Preliminary numerical tests showed that these methods could compute eigenvalues of symmetric tensors with dozens of variables.

How to compute the (extreme) eigenvalue of the Laplacian tensor coming from an even-uniform hypergraph with millions of vertices? It is expensive to store and process a huge Laplacian tensor directly.

In this paper, we propose to store a uniform hypergraph by a matrix, whose row corresponds to an edge of that hypergraph. Then, instead of generating the large scale Laplacian tensor of the hypergraph explicitly, we give a fast computational framework for products of the Laplacian tensor and any vectors. The computational cost is linear in the size of edges and quadratic in the number of vertices of an edge. So it is cheap. Other tensors arising from a uniform hypergraph, such as the adjacency tensor and the signless Laplacian tensor, could be processed in a similar way. These computational methods compose our main motivation.

Since products of any vectors and large scale tensors associated with a uniform hypergraph could be computed economically, we develop an efficient first-order optimization algorithm for computing H- and Z-eigenvalues of adjacency, Laplacian, and
signless Laplacian tensors corresponding to the even-uniform hypergraph. In order to obtain an eigenvalue of an even-order symmetric tensor, we minimize a smooth merit function in a spherical constraint, whose first-order stationary point is an eigenvector associated with a certain eigenvalue. To preserve the spherical constraint, we derive an explicit formula for iterates using the Cayley transform. Then, the algorithm for a spherical optimization looks like an unconstrained optimization. In order to deal with large scale problems, we explore the state-of-the-art L-BFGS approach to generate a gradient-related direction and the backtracking search to facilitate the convergence of iterates. Based on these techniques, we obtain the novel algorithm (CEST) for computing eigenvalues of even-order symmetric tensors. Due to the algebraic nature of tensor eigenvalue problems, the smooth merit function enjoys the Kurdyka-Lojasiewicz (KL) property. Using this property, we confirm that the sequence of iterates generated by CEST converges to an eigenvector corresponding to an eigenvalue. Moreover, if we start CEST from multiple initial points sampled uniformly from a unit sphere, it can be proved that the resulting best merit function value could touch the extreme eigenvalue with a high probability.

Numerical experiments show that the novel algorithm CEST is dozens times faster than the power method for eigenvalues of symmetric tensors related with small hypergraphs. Finally, we report that CEST could compute H- and Z-eigenvalues and associated eigenvectors of symmetric tensors involved in an even-uniform hypergraph with millions of vertices.

The outline of this paper is drawn as follows. We introduce some latest developments on spectral hypergraph theory in Section 2. Section 3 address the computational issues on products of a vector and large scale sparse tensors arising from a uniform hypergraph. In Section 4, we propose the new optimization algorithm based on L-BFGS and the Cayley transform. The convergence analysis of this algorithm is established in Section 5. Numerical experiments reported in Section 6 show that the new algorithm is efficient and promising. Finally, we conclude this paper in Section 7.

2. Preliminary on spectral hypergraph theory. We introduce the definitions of eigenvalues and spectral of a symmetric tensor and then discuss developments in spectral hypergraph theory.

The conceptions of eigenvalues and associated eigenvectors of a symmetric tensor are established by Qi [48] and Lim [38] independently. Suppose
\[ \mathcal{T} = (t_{i_1 \ldots i_k}) \in \mathbb{R}^{[k,n]}, \text{ for } i_j = 1, \ldots, n, j = 1, \ldots, k, \]
is a $k$th order $n$ dimensional symmetric tensor. Here, the symmetry means that the value of $t_{i_1 \ldots i_k}$ is invariable under any permutation of its indices. For $x \in \mathbb{R}^n$, we define a scalar
\[ \mathcal{T}x^k = \sum_{i_1 = 1}^{n} \cdots \sum_{i_k = 1}^{n} t_{i_1 \ldots i_k} x_{i_1} \cdots x_{i_k} \in \mathbb{R}. \]
Two column vectors $\mathcal{T}x^{k-1} \in \mathbb{R}^n$ and $x^{[k-1]} \in \mathbb{R}^n$ are defined with elements
\[ (\mathcal{T}x^{k-1})_i = \sum_{i_2 = 1}^{n} \cdots \sum_{i_k = 1}^{n} t_{i_2 \ldots i_k} x_{i_2} \cdots x_{i_k} \]
and $(x^{[k-1]})_i \equiv x_i^{k-1}$ for $i = 1, \ldots, n$ respectively. Obviously, $\mathcal{T}x^k = x^\top (\mathcal{T}x^{k-1})$. 
If there exist a real \( \lambda \) and a nonzero vector \( x \in \mathbb{R}^n \) satisfying
\[
T x^{k-1} = \lambda x^{[k-1]},
\]
we call \( \lambda \) an H-eigenvalue of \( T \) and \( x \) its associated H-eigenvector. If the following system
\[
\begin{align*}
T x^{k-1} &= \lambda x, \\
x^\top x &= 1,
\end{align*}
\]
has a real solution \( (\lambda, x) \), \( \lambda \) is named a Z-eigenvalue of \( T \) and \( x \) is its associated Z-eigenvector. All of the H- and Z-eigenvalues of \( T \) are called its H-spectrum \( \text{Hspec}(T) \) and Z-spectrum \( \text{Zspec}(T) \) respectively.

These definitions on eigenvalues of a symmetric tensor have important applications in spectral hypergraph theory.

**Definition 2.1 (Hypergraph).** We denote a hypergraph by \( G = (V, E) \), where \( V = \{1, 2, \ldots, n\} \) is the vertex set, \( E = \{e_1, e_2, \ldots, e_m\} \) is the edge set, \( e_p \subset V \) for \( p = 1, \ldots, m \). If \( |e_p| = k \geq 2 \) for \( p = 1, \ldots, m \) and \( e_p \neq e_q \) in case of \( p \neq q \), then \( G \) is called a uniform hypergraph or a k-graph. If \( k = 2 \), \( G \) is an ordinary graph.

The k-graph \( G = (V, E) \) is called odd-bipartite if \( k \) is even and there exists a proper subset \( U \) of \( V \) such that \( |e_p \cap U| \) is odd for \( p = 1, \ldots, m \).

Let \( G = (V, E) \) be a k-graph. For each \( i \in V \), its degree \( d(i) \) is defined as
\[
d(i) = |\{e_p : i \in e_p \in E\}|.
\]
We assume that every vertex has at least one edge. Thus, \( d(i) > 0 \) for all \( i \). Furthermore, we define \( \Delta \) as the maximum degree of \( G \), i.e., \( \Delta = \max_{1 \leq i \leq n} d(i) \).

![Fig. 2.1. A 4-uniform hypergraph: sunflower.](image)

The first hypergraph is illustrated in Figure 2.1. There are ten vertices \( V = \{1, 2, \ldots, 10\} \) and three edges \( E = \{e_1 = \{1, 2, 3, 4\}, e_2 = \{1, 5, 6, 7\}, e_3 = \{1, 8, 9, 10\}\} \). Hence, it is a 4-graph, and its degrees are \( d(1) = 3 \) and \( d(i) = 1 \) for \( i = 2, \ldots, 10 \). So we have \( \Delta = 3 \). Moreover, this hypergraph is odd-bipartite since we could take \( U = \{1\} \subset V \).

**Definition 2.2 (Adjacency tensor [13]).** Let \( G = (V, E) \) be a k-graph with \( n \) vertices. The adjacency tensor \( A = (a_{i_1 \ldots i_k}) \) of \( G \) is a kth order n-dimensional

\footnote{Qi [18] pointed out that the tensor \( T \) should be regular, i.e., zero is the unique solution of \( T x^{k-1} = 0 \).}
symmetric tensor, whose elements are

\[ a_{i_1 \ldots i_k} = \begin{cases} 
\frac{1}{(k-1)!} & \text{if } \{i_1, \ldots, i_k\} \in E, \\
0 & \text{otherwise.} 
\end{cases} \]

**Definition 2.3** (Laplacian tensor and signless Laplacian tensor [49]). Let \( G \) be a \( k \)-graph with \( n \) vertices. We denote its degree tensor \( D \) as a \( k \)th order \( n \)-dimensional diagonal tensor whose \( i \)th diagonal element is \( d(i) \). Then, the Laplacian tensor \( L \) and the signless Laplacian tensor \( Q \) of \( G \) is defined respectively as

\[ L = D - A \quad \text{and} \quad Q = D + A. \]

Obviously, the adjacency tensor \( A \) and the signless Laplacian tensor \( Q \) of a hypergraph \( G \) are nonnegative. Moreover, they are weakly irreducible if and only if \( G \) is connected [47]. Hence, we could apply the Ng-Qi-Zhou algorithms [43, 10, 59] for computing their largest H-eigenvalues and associated H-eigenvectors. On the other hand, the Laplacian tensor \( L \) of a uniform hypergraph \( G \) is a \( M \)-tensor [57, 16]. Qi [49, Theorem 3.2] proved that zero is the smallest H-eigenvalue of \( L \). However, the following problems are still open.

- How to compute the largest H-eigenvalue of \( L \)?
- How to calculate the smallest H-eigenvalues of \( Q \) and \( A \)?
- How to obtain extreme Z-eigenvalues of \( A \), \( L \), and \( Q \)?

Many theorems in spectral hypergraph theory are proved to address H- and Z-eigenvalues of \( A \), \( L \), and \( Q \) when the involved hypergraph has well geometric structures. For convenience, we denote the largest H-eigenvalue and the smallest H-eigenvalue of a tensor \( T \) related to a hypergraph \( G \) as \( \lambda_{H \max}(T(G)) \) and \( \lambda_{H \min}(T(G)) \) respectively. We also define similar notations for Z-eigenvalues of that tensor.

**Theorem 2.4.** Let \( G \) be a connected \( k \)-graph. Then the following assertions are equivalent.

(i) \( k \) is even and \( G \) is odd-bipartite.
(ii) \( \lambda_{H \max}(L(G)) = \lambda_{H \max}(Q(G)) \) (from Theorem 5.9 of [28]).
(iii) \( \text{Hspec}(L(G)) = \text{Hspec}(Q(G)) \) (from Theorem 2.2 of [52]).
(iv) \( \text{Hspec}(A(G)) = -\text{Hspec}(A(G)) \) (from Theorem 2.3 of [52]).
(v) \( \text{Zspec}(L(G)) = \text{Zspec}(Q(G)) \) (from Theorem 8 of [7]).

Khan and Fan [31] studied a sort of non-odd-bipartite hypergraph and gave the following result.

**Theorem 2.5.** (Corollary 3.6 of [31]) Let \( G \) be a simple graph. For any positive integer \( k \), we blow up each vertex of \( G \) into a set that includes \( k \) vertices and get a \( 2k \)-graph \( G^{2k,k} \). Then, \( G^{2k,k} \) is not odd-bipartite if and only if \( G \) is non-bipartite. Furthermore,

\[ \lambda_{\min}(A(G)) = \lambda_{H \min}(A(G^{2k,k})) \quad \text{and} \quad \lambda_{\min}(Q(G)) = \lambda_{H \min}(Q(G^{2k,k})). \]

3. Computational methods on sparse tensors arising from a hypergraph. The adjacency tensor \( A \), the Laplacian tensor \( L \), and the signless Laplacian tensor \( Q \) of a uniform hypergraph are usually sparse. For instance, \( A \), \( L \) and \( Q \) of the 4-uniform sunflower illustrated in Figure 2.4 only contain 0.72%, 0.76%, and 0.76%
nonzero elements respectively. Hence, it is an important issue to explore the sparsity in tensors $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{Q}$ involved in a hypergraph $G$. Now, we introduce a fast numerical approach based on MATLAB.

**How to store a uniform hypergraph?** Let $G = (V, E)$ be a $k$-graph with $|V| = n$ vertices and $|E| = m$ edges. We store $G$ as an $m$-by-$k$ matrix $G_r$ whose rows are composed of the indices of vertices from corresponding edges of $G$. Here, the ordering of elements in each row of $G_r$ is unimportant in the sense that we could permute them.

For instance, we consider the 4-uniform sunflower shown in Figure 2.1. The edge-vertex incidence matrix of this sunflower is a 3-by-10 sparse matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
$$

From the viewpoint of scientific computing, we prefer to store the incidence matrix of the sunflower in a compact form

$$
G_r = \begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & 5 & 6 & 7 \\
1 & 8 & 9 & 10 \\
\end{bmatrix} \in \mathbb{R}^{3 \times 4}.
$$

Obviously, the number of columns of the matrix $G_r$ is less than the original incidence matrix, since usually $k \ll n$. We can benefit from this compact matrix in the process of computing. In MATLAB, this matrix $G_r$ is written in Line 2 of Figure 3.1.

**How to compute products $\mathcal{I}x^k$ and $\mathcal{I}x^{k-1}$ when $\mathcal{I} = \mathcal{A}, \mathcal{L}$, and $\mathcal{Q}$?** Suppose that the matrix $G_r$ representing a uniform hypergraph and a vector $x \in \mathbb{R}^n$ are available. Since $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$, it is sufficient to study the degree tensor $\mathcal{D}$ and the adjacency tensor $\mathcal{A}$.

We first consider the degree tensor $\mathcal{D}$. It is a diagonal tensor and its $i$th diagonal element is the degree $d(i)$ of a vertex $i \in V$. Once the hypergraph $G$ is given, the degree vector $d \equiv [d(i)] \in \mathbb{R}^n$ is fixed. So we could save $d$ from the start. Let $\delta(\cdot, \cdot)$ be the Kronecker delta, i.e., $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ if $i \neq j$. Using this notation, we could rewrite the degree as

$$
d(i) = \sum_{\ell=1}^{m} \sum_{j=1}^{k} \delta(i, (G_r)_{\ell j}), \quad \text{for } i = 1, \ldots, n.
$$

To calculate the degree vector $d$ efficiently, we construct an $n$-by-$mk$ sparse matrix $M_{sp} = [\delta(i, (G_r)_{\ell j})]$. By summarizing each row of $M_{sp}$, we obtain the degree vector $d$. For any vector $x \in \mathbb{R}^n$, the computation of

$$
\mathcal{D}x^{k-1} = d \ast (x^{k-1}) \quad \text{and} \quad \mathcal{D}x^k = d^\top (x^k)
$$

are straightforward, where “$\ast$” denotes the component-wise Hadamard product. In Figure 3.1, we show these codes in Lines 4-15.

Second, we focus on the adjacency tensor $\mathcal{A}$. We construct a matrix $X_{mat} = [v_{(G_r)_{\ell j}}]$ which has the same size as $G_r$. Assume that the $(\ell, j)$-th element of $G_r$ is $i$. 
% Store a 4-uniform sunflower
Gr = [1, 2, 3, 4; 1, 5, 6, 7; 1, 8, 9, 10];

% Calculate the degree vector
[m, k] = size(Gr); n = max(Gr(:));
Msp = sparse(Gr(:, (1:m*k)'), ones(m*k, 1), n, m*k);
degree = full(sum(Msp, 2));

% Suppose that x is an n-dimensional vector.
% Compute the vector Dxˆ(k−1)
Dx_ = degree .* (x.ˆ(k−1));

% Compute the scalar Dxˆk
Dxk = dot(degree, x.ˆk); % direct approach, or
Dxk2 = dot(x, Dx_); % if Dx_ is available.

% Compute the vector Axˆ(k−1)
Xmat = reshape(x(Gr(:)), [m, k]);
Ax_ = zeros(n, 1);
for j = 1:k
    Mj = sparse(Gr(:, j), (1:m)', ones(m, 1), n, m);
yj = prod(Xmat(:, [1:j−1, j+1:k]), 2);
    Ax_ = Ax_ + Mj*yj;
end

% Compute the scalar Axˆk
Axk = k*sum(prod(Xmat, 2)); % direct approach, or
Axk2 = dot(x, Ax_); % if Ax_ is available.

% Calculate Lxˆ(k−1) and Qxˆ(k−1)
Lx_ = Dx_ − Ax_;
Qx_ = Dx_ + Ax_;

% Calculate Lxˆk and Qxˆk
Lxk = Dxk − Axk;
Qxk = Dxk + Axk;

Fig. 3.1. Matlab codes for the sunflower illustrated in Figure 2.1.

Then, the (ℓ, j)-th element of Xmat is defined as x_. From this matrix, we rewrite the product Ax^k as

Ax^k = k \sum_{\ell=1}^{m} \prod_{j=1}^{k}(X_{mat})_{\ell j}.

See Lines 18 and 27 of Figure 3.1 To compute the vector Ax^k−1, we use the following
representation

\[(Ax^{k-1})_i = \sum_{j=1}^{k} \sum_{t=1}^{m} \left( \delta(i, (G_r)_{\ell j}) \prod_{s=1, s \neq j}^{k} (X_{mat})_{\ell s} \right), \quad \text{for } i = 1, \ldots, n.\]

For each \(j = 1, \ldots, k\), we construct a sparse matrix \(M_j = [\delta(i, (G_r)_{\ell j})] \in \mathbb{R}^{n \times m}\) and a column vector \(y_j = [\prod_{s \neq j} (X_{mat})_{\ell s}] \in \mathbb{R}^{m}\) respectively. Then, the vector

\[Ax^{k-1} = \sum_{j=1}^{k} M_j y_j\]

could be computed by using a simple loop. See Lines 17-24 of Figure 3.1.

The computational costs for computing products of tensors \(A, L,\) and \(Q\) with any vector \(x\) are about \(mk^2, mk^2 + nk\), and \(mk^2 + nk\) multiplications, respectively. Since \(mk^2 < mk^2 + nk \leq 2mk^2\), the computational cost of the product of a vector and a large scale sparse tensor related with a hypergraph is cheap. Additionally, the codes listed in Figure 3.1 could easily be extended to parallel computing.

4. The CEST algorithm. The design of the novel CEST algorithm is based on a unified formula for the H- and Z-eigenvalue of a symmetric tensor [9, 17]. Let \(I \in \mathbb{R}^{[k,n]}\) be an identity tensor whose diagonal elements are all one and off-diagonal elements are zero. Hence, \(I x^{k-1} = x^{[k-1]}\). If \(k\) is even, we define \(E \in \mathbb{R}^{[k,n]}\) such that \(E x^{k-1} = (x^T x)^{\frac{k-1}{2}} x\). Using tensors \(I\) and \(E\), we could rewrite systems (2.1) and (2.2) as

\[(4.1) \quad T x^{k-1} = \lambda B x^{k-1},\]

where \(B = I\) and \(B = E\) respectively. In the remainder of this paper, we call a real \(\lambda\) and a nonzero vector \(x \in \mathbb{R}^n\) an eigenvalue and its associated eigenvector respectively if they satisfies (4.1). Now, we devote to compute such \(\lambda\) and \(x\) for large scale sparse tensors.

Let \(k\) be even. We consider the spherical optimization problem

\[(4.2) \quad \min f(x) = \frac{T x^k}{B x^k} \quad \text{s.t. } x \in S^{n-1},\]

where the symmetric tensor \(T\) arises from a \(k\)-uniform hypergraph, so \(T\) is sparse and may be large scale. \(B\) is a symmetric positive definite tensor with a simple structure such as \(I\) and \(E\). Without loss of generality, we restrict \(x\) on a compact unit sphere \(S^{n-1} \equiv \{x \in \mathbb{R}^n : x^T x = 1\}\) because \(f(x)\) is zero-order homogeneous.

The gradient of \(f(x)\) \(\frac{\partial f(x)}{\partial x}\) is

\[(4.3) \quad g(x) = \frac{k}{B x^k} \left( T x^{k-1} - \frac{T x^k}{B x^k} B x^{k-1} \right).\]

Obviously, for all \(x \in S^{n-1}\), we have

\[(4.4) \quad x^T g(x) = \frac{k}{B x^k} \left( x^T T x^{k-1} - \frac{T x^k}{B x^k} x^T B x^{k-1} \right) = 0.\]
This equality implies that the vector \( x \in S^{n-1} \) is perpendicular to its (negative) gradient direction. The following theorem reveals the relationship between the spherical optimization (4.2) and the eigenvalue problem (4.1).

**Theorem 4.1.** Suppose that the order \( k \) is even and the symmetric tensor \( B \) is positive definite. Let \( x_\ast \in S^{n-1} \). Then, \( x_\ast \) is a first-order stationary point, i.e., \( g(x_\ast) = 0 \), and only if \( x_\ast \) is an eigenvector corresponding to a certain eigenvalue. In fact, the eigenvalue is \( f(x_\ast) \).

**Proof.** Since \( B \) is positive definite, \( Bx^k > 0 \) for all \( x \in S^{n-1} \). Hence, by (4.3), if \( x_\ast \in S^{n-1} \) satisfies \( g(x_\ast) = 0 \), \( f(x_\ast) \) is an eigenvalue and \( x_\ast \) is its associated eigenvector.

On the other hand, suppose that \( x_\ast \in S^{n-1} \) is an eigenvector corresponding to an eigenvalue \( \lambda_\ast \), i.e.,

\[
Tx_\ast^{k-1} = \lambda_\ast Bx_\ast^{k-1}.
\]

By taking inner products on both sides with \( x_\ast \), we get \( Tx_\ast^k = \lambda_\ast Bx_\ast^k \). Because \( Bx_\ast^k > 0 \), it yields that \( \lambda_\ast = \frac{Tx_\ast^k}{Bx_\ast^k} = f(x_\ast) \). Hence, by (4.3), we obtain \( g(x_\ast) = 0 \). \( \square \)

Next, we focus on numerical approaches for computing a first-order stationary point of the spherical optimization (4.2). First, we apply the limited memory BFGS (L-BFGS) approach for generating a search direction. Then, a curvilinear search technique is explored to preserve iterates in a spherical constraint.

**4.1. L-BFGS produces a search direction.** The limited memory BFGS method is powerful for large scale nonlinear unconstrained optimization. In the current iteration \( c \), it constructs an implicit matrix \( H_c \) to approximate the inverse of a Hessian of \( f(x) \). At the beginning, we introduce the basic BFGS update.

BFGS is a quasi-Newton method which updates the approximation of the inverse of a Hessian iteratively. Let \( H_c \) be the current approximation,

\[
y_c = g(x_{c+1}) - g(x_c), \quad s_c = x_{c+1} - x_c, \quad \text{and} \quad V_c = I - \rho_c y_c s_c^\top,
\]

where \( I \) is an identity matrix,

\[
\rho_c = \begin{cases} 
\frac{1}{y_c^\top s_c} & \text{if } y_c^\top s_c \geq \kappa_c, \\
0 & \text{otherwise},
\end{cases}
\]

and \( \kappa_c \in (0, 1) \) is a small positive constant. We generate the new approximation \( H_c^+ \) by the BFGS formula \( 56 \ 57 \)

\[
H_c^+ = V_c^\top H_c V_c + \rho_c s_c s_c^\top.
\]

For the purpose of solving large scale optimization problems, Nocedal \( 45 \) proposed the L-BFGS approach which implements the BFGS update in an economic way. Given any vector \( g \in \mathbb{R}^n \), the matrix-vector product \( -H_c g \) could be computed using only \( O(n) \) multiplications.

In each iteration \( c \), L-BFGS starts from a simple matrix

\[
H_c^{(0)} = \gamma_c I,
\]

where \( \gamma_c > 0 \) is usually determined by the Barzilai-Borwein method \( 39 \ 41 \). Then, we use BFGS formula \( 16 \) to update \( H_c^{(\ell)} \) recursively

\[
H_c^{(L-\ell+1)} = V_{c-\ell}^\top H_c^{(L-\ell)} V_{c-\ell} + \rho_{c-\ell} s_{c-\ell}^\top s_{c-\ell}^\top, \quad \text{for } \ell = L, L-1, \ldots, 1.
\]
Algorithm L-BFGS The two-loop recursion for L-BFGS [45, 46, 53].

1: \( q \leftarrow -g(x_c) \),
2: for \( i = c - 1, c - 2, \ldots, c - L \) do
3: \( \alpha_i \leftarrow \rho_i s_i^\top q_i \),
4: \( q \leftarrow q - \alpha_i y_i \),
5: end for
6: \( p \leftarrow \gamma c q \),
7: for \( i = c - L, c - L + 1, \ldots, c - 1 \) do
8: \( \beta \leftarrow \rho_i y_i^\top p \),
9: \( p \leftarrow p + s_i(\alpha_i - \beta) \),
10: end for
11: Stop with result \( p = -H_c g(x_c) \).

and obtain

\[
H_c = H^{(L)}_c.
\]

If \( \ell \geq c \), we define \( \rho_{c-\ell} = 0 \) and L-BFGS does nothing for that \( \ell \). In a practical implementation, L-BFGS enjoys a cheap two-loop recursion. The computational cost is about \( 4Ln \) multiplications.

For the parameter \( \gamma_c \), we have three candidates. The first two are suggested by Barzilai and Borwein [4]:

\[
\gamma^{BB1}_c = \frac{y_c^\top s_c}{\|y_c\|^2} \quad \text{and} \quad \gamma^{BB2}_c = \frac{\|s_c\|^2}{y_c^\top s_c}.
\]

The third one are their geometric mean [15]

\[
\gamma^{Dai}_c = \frac{\|s_c\|}{\|y_c\|}.
\]

Furthermore, we set \( \gamma_c = 1 \) if \( y_c^\top s_c < \kappa_c \).

4.2. Cayley transform preserves the spherical constraint. Suppose that \( x_c \in \mathbb{S}^{n-1} \) is the current iterate, \( p_c \in \mathbb{R}^n \) is a good search direction generated by Algorithm L-BFGS and \( \alpha \) is a damped factor. First, we construct a skew-symmetric matrix

\[
W = \alpha(x_c p_c^\top - p_c x_c^\top) \in \mathbb{R}^{n \times n}.
\]

Obviously, \( I + W \) is invertible. Using the Cayley transform, we obtain an orthogonal matrix

\[
Q = (I - W)(I + W)^{-1}.
\]

Hence, the new iterate \( x_{c+1} \) is still locating on the unit sphere \( \mathbb{S}^{n-1} \) if we define

\[
x_{c+1} = Q x_c.
\]

Indeed, matrices \( W \) and \( Q \) are not needed to be formed explicitly. The new iterate \( x_{c+1} \) could be generated from \( x_c \) and \( p_c \) directly with only about \( 4n \) multiplications.

Lemma 4.2. Suppose that the new iterate \( x_{c+1} \) is generated by (4.13), (4.14), and (4.15). Then, we have

\[
x_{c+1}(\alpha) = \frac{((1 - \alpha x_c^\top p_c)^2 - \|\alpha p_c\|^2) x_c + 2\alpha p_c}{1 + \|\alpha p_c\|^2 - (\alpha x_c^\top p_c)^2}.
\]
and

\[
\|x_{c+1}(\alpha) - x_c\| = 2\left(\frac{\|\alpha p_c\|^2 - (\alpha x_c^top p_c)^2}{1 + \|\alpha p_c\|^2 - (\alpha x_c^top p_c)^2}\right)^{\frac{1}{2}}.
\]

**Proof.** We employ the Sherman-Morrison-Woodbury formula: if \(A\) is invertible,

\[(A + UV^\top)^{-1} = A^{-1} - A^{-1}U(I + V^\top A^{-1}U)^{-1}V^\top A^{-1}.
\]

It yields that

\[
(I + W)^{-1}x_c = \left(I + \left[\begin{array}{c} x_c \\ -\alpha p_c \end{array}\right] \left[\begin{array}{c} \alpha p_c^top \\ x_c \end{array}\right]\right)^{-1}x_c
\]

\[
= \left(I - \left[\begin{array}{c} x_c \\ -\alpha p_c \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] + \left[\begin{array}{c} \alpha p_c^top \\ x_c \end{array}\right] \left[\begin{array}{c} \alpha p_c^top \\ x_c \end{array}\right] \right)^{-1} \left[\begin{array}{c} \alpha p_c^top \\ x_c \end{array}\right]x_c
\]

\[
= x_c - \left[\begin{array}{c} x_c \\ -\alpha p_c \end{array}\right] \left[\begin{array}{cc} 1 + \alpha x_c^top p_c & -\|\alpha p_c\|^2 \\ 1 & 1 - \alpha x_c^top p_c \end{array}\right]^{-1} \left[\begin{array}{c} \alpha x_c^top p_c \\ 1 \end{array}\right]
\]

\[
= x_c - \left[\begin{array}{c} x_c \\ -\alpha p_c \end{array}\right] \frac{1}{1 + \|\alpha p_c\|^2 - (\alpha x_c^top p_c)^2} \left[\begin{array}{c} \alpha x_c^top p_c (1 - \alpha x_c^top p_c) + \|\alpha p_c\|^2 \\ 1 \end{array}\right]
\]

where \(1 + \|\alpha p_c\|^2 - (\alpha x_c^top p_c)^2 \geq 1\) since \(x_c \in \mathcal{S}^{n-1}\) and \(|\alpha x_c^top p_c| \leq \|\alpha p_c\|\). Then, the calculation of \(x_{c+1}\) is straightforward

\[
x_{c+1} = (I - W) \left(1 - \alpha x_c^top p_c\right)x_c + \alpha p_c = \left[(1 - \alpha x_c^top p_c)^2 - \|\alpha p_c\|^2\right]x_c + 2\alpha p_c.
\]

Hence, the iterate formula (4.16) is valid.

Then, by some calculations, we have

\[
\|x_{c+1}(\alpha) - x_c\|^2
\]

\[
= \left\| \left[\begin{array}{c} 2\alpha x_c^top p_c (\alpha x_c^top p_c - 1) - 2\|\alpha p_c\|^2 x_c + 2\alpha p_c \end{array}\right] \right\|^2
\]

\[
= \left[\begin{array}{c} 2\alpha x_c^top p_c (\alpha x_c^top p_c - 1) - 2\|\alpha p_c\|^2(2\alpha x_c^top p_c)^2 + 4\|\alpha p_c\|^4 + 4\|\alpha p_c\|^2 \end{array}\right]^2
\]

\[
= \left[\begin{array}{c} (2\alpha x_c^top p_c)^2(\alpha x_c^top p_c)^2 - 1 - 4\|\alpha p_c\|^2\right]^2
\]

\[
= \left[\begin{array}{c} 4\|\alpha p_c\|^2 - 4(\alpha x_c^top p_c)^2 \end{array}\right]^2
\]

Therefore, the equality (4.17) holds. \(\square\)

Whereafter, the damped factor \(\alpha\) could be determined by an inexact line search owing to the following theorem.

**Theorem 4.3.** Suppose that \(p_c\) is a gradient-related direction satisfying (5.1) and \(x_{c+1}(\alpha)\) is generated by (4.10). Let \(\eta \in (0, 1)\) and \(g(x_c) \neq 0\). Then, there is an
Owing to $\eta < H$, hence, we have

$$g_p$$

where the last equality holds for (4.4). Since $g_p$ and $c$ are iteration touch the extreme eigenvalue of a tensor with a high probability. When we start CEST from plenty of randomly initial points, resulting eigenvalues may condition holds at the limiting point, CEST enjoys a linear convergence rate. Finally, we present the new Algorithm CEST formally. Roughly speaking, CEST is a modified version of the state-of-the-art L-BFGS method for unconstrained optimization. Due to the spherical constraint imposed here, we use the Cayley transform explicitly to preserve iterates on a unit sphere. An inexact line search is employed to determine a suitable damped factor. Theorem 4.3 indicates that the inexact line search is well-defined.

Algorithm CEST Computing eigenvalues of sparse tensors.

1. For a given uniform hypergraph $G_c$, we compute the degree vector $d$.
2. Choose an initial unit iterate $x_1$, a positive integer $L$, parameters $\eta \in (0, 1)$, $\beta \in (0, 1)$, and $c \leftarrow 1$.
3. while the sequence of iterates does not converge do
4. Compute $Tx_c^{k-1}$ and $Tx_c^k$ using the codes in Figure 3.1 where $T \in \{A, L, Q\}$.
5. Calculate $\lambda_c = f(x_c)$ and $g(x_c)$ by (4.2) and (4.3) respectively.
6. Generate $p_c = -H_c g(x_c)$ by Algorithm L-BFGS.
7. Choose the smallest nonnegative integer $\ell$ and calculate $\alpha = \beta^\ell$ such that (4.18) holds.
8. Let $\alpha_c = \beta^\ell$ and update the new iterate $x_{c+1} = x_c + (\alpha_c) \beta^\ell$ by (4.18).
9. Compute $s_c, y_c$ and update the new iterate $x_{c+1} = x_c + (\alpha_c) \beta^\ell$ by (4.18).
10. $c \leftarrow c + 1$.
11. end while

$$\alpha_c > 0 \text{ such that for all } \alpha \in (0, \alpha_c],$$

(4.18) \[ f(x_{c+1}(\alpha)) \leq f(x_c) + \eta \alpha p_c \top g(x_c). \]

Proof. From (4.16), we obtain $x_{c+1}(0) = x_c$ and $x'_{c+1}(0) = -2x_c \top p_c x_c + 2p_c$. Hence, we have

$$\frac{d f(x_{c+1}(\alpha))}{d\alpha} \bigg|_{\alpha=0} = g(x_{c+1}(0)) \top x'_{c+1}(0) = g(x_c) \top (-2x_c \top p_c x_c + 2p_c) = 2p_c \top g(x_c),$$

where the last equality holds for (4.4). Since $g(x_c) \neq 0$ and $p_c$ satisfies (5.1), we have $p_c \top g(x_c) < 0$. Then, by Taylor's theorem, for a sufficiently small $\alpha$, we obtain

$$f(x_{c+1}(\alpha)) = f(x_c) + 2 \alpha p_c \top g(x_c) + o(\alpha^2).$$

Owing to $\eta < 2$, there exists a positive $\alpha_c$ such that (4.18) is valid.

5. Convergence analysis. First, we prove that the sequence of merit function values $\{f(x_c)\}$ converges and every accumulation point of iterates $\{x_c\}$ is a first-order stationary point. Second, using the Kurdyka-Lojasiewicz property, we show that the sequence of iterates $\{x_c\}$ is also convergent. When the second-order sufficient condition holds at the limiting point, CEST enjoys a linear convergence rate. Finally, when we start CEST from plenty of randomly initial points, resulting eigenvalues may touch the extreme eigenvalue of a tensor with a high probability.

5.1 Basic convergence theory. If CEST terminates finitely, i.e., there exists an iteration $c$ such that $g(x_c) = 0$, we immediately know that $f(x_c)$ is an eigenvalue and $x_c$ is the corresponding eigenvector by Theorem 4.1. So, in the remainder of this section, we assume that CEST generates an infinite sequence of iterates $\{x_c\}$. 
Since the symmetric tensor $B$ is positive definite, the merit function $f(x)$ is twice continuously differentiable. Owing to the compactness of the spherical domain of $f(x)$, we obtain the following bounds $[12]$.

**Lemma 5.1.** There exists a constant $M > 1$ such that

\[ |f(x)| \leq M, \quad \|g(x)\| \leq M, \quad \text{and} \quad \|\nabla^2 f(x)\| \leq M, \quad \forall \ x \in S^{n-1}. \]

Because the bounded sequence $\{f(x_c)\}$ decreases monotonously, it converges.

**Theorem 5.2.** Assume that CEST generates an infinite sequence of merit functions $\{f(x_c)\}$. Then, there exists a $\lambda_*$ such that

\[ \lim_{c \to \infty} f(x_c) = \lambda_. \]

The next theorem shows that $p_c = -H_c g(x_c)$ generated by L-BFGS is a gradient-related direction.

**Theorem 5.3.** Suppose that $p_c = -H_c g(x_c)$ is generated by L-BFGS. Then, there exist constants $0 < C_L \leq 1 \leq C_U$ such that

\[ p_c^\top g(x_c) \leq -C_L \|g(x_c)\|^2 \quad \text{and} \quad \|p_c\| \leq C_U \|g(x_c)\|. \]

**Proof.** See Appendix A. 

Using the gradient-related direction, we establish bounds for damped factors generated by the inexact line search.

**Lemma 5.4.** There exists a constant $\alpha_{\min} > 0$ such that

\[ \alpha_{\min} \leq \alpha_c \leq 1, \quad \forall c. \]

**Proof.** Let $0 < \alpha \leq \hat{\alpha} = \frac{(2-\eta)C_L}{(2+\eta)MC_U^2}$. Hence, $\alpha C_U M \leq \frac{(2-\eta)C_L}{(2+\eta)MC_U^2} < 1$. From $\text{(5.1)}$ and Lemma $5.1$, we obtain

\[ -\alpha p_c^\top g(x_c) \leq \alpha \|p_c\| \|g(x_c)\| \leq \alpha C_U \|g(x_c)\|^2 \leq \alpha C_U M^2 < M \]

and

\[ \|\alpha p_c\|^2 - (\alpha x_c^\top p_c)^2 \leq \alpha^2 \|p_c\|^2 \leq \alpha^2 C_U^2 \|g(x_c)\|^2. \]

The above two inequalities and $\alpha \in (0, \hat{\alpha}]$ yield that

\[
\begin{align*}
2\alpha p_c^\top g(x_c) + 2M(\|\alpha p_c\|^2 - (\alpha x_c^\top p_c)^2) - \eta \alpha p_c^\top g(x_c)(1 + \|\alpha p_c\|^2 - (\alpha x_c^\top p_c)^2) \\
= (2 - \eta)\alpha p_c^\top g(x_c) + (2M - \eta \alpha p_c^\top g(x_c))(\|\alpha p_c\|^2 - (\alpha x_c^\top p_c)^2) \\
< (2 - \eta)\alpha p_c^\top g(x_c) + (2 + \eta)M\alpha^2 C_U^2 \|g(x_c)\|^2 \\
\leq (2 - \eta)\alpha p_c^\top g(x_c) + (2 - \eta)C_L \alpha \|g(x_c)\|^2 \\
\leq 0, \tag{5.2}
\end{align*}
\]

where the last inequality holds for $\text{(5.1)}$. 

\[ \]
From the mean value theorem, Lemmas 5.1 and 4.2 and the equality (4.4), we have
\[
f(x_{c+1}(\alpha)) - f(x_c) \leq g(x_c)^\top (x_{c+1}(\alpha) - x_c) + \frac{1}{2} M \|x_{c+1}(\alpha) - x_c\|^2
\]
\[
= 2\alpha p_c^\top g(x_c) + 2M(||\alpha p_c||^2 - (\alpha x_c^\top p_c)^2)
\]
\[
< \eta \alpha p_c^\top g(x_c) \left(1 + ||\alpha p_c||^2 - (\alpha x_c^\top p_c)^2\right)
\]
\[
= \eta \alpha p_c^\top g(x_c),
\]
where the last inequality is valid owing to (5.2). Due to the rule of the inexact search, the damped factor \(\alpha_c\) satisfies \(1 \geq \alpha_c \geq \beta \hat{\alpha} \equiv \alpha_{\text{min}}\).

The next theorem proves that every accumulation point of iterates \(\{x_c\}\) is a first-order stationary point.

**Theorem 5.5.** Suppose that CEST generates an infinite sequence of iterates \(\{x_c\}\). Then,
\[
\lim_{c \to \infty} ||g(x_c)|| = 0.
\]

**Proof.** From (4.18) and (5.1), we get
\[
(5.3) \quad f(x_c) - f(x_{c+1}) \geq -\eta \alpha_c p_c^\top g(x_c) \geq \eta \alpha_c C_L ||g(x_c)||^2.
\]
Since Lemmas 5.1 and 5.4 we have
\[
2M \geq f(x_1) - \lambda = \sum_{c=1}^{\infty} [f(x_c) - f(x_{c+1})] \geq \sum_{c=1}^{\infty} \eta \alpha_c C_L ||g(x_c)||^2 \geq \sum_{c=1}^{\infty} \eta \alpha_{\text{min}} C_L ||g(x_c)||^2.
\]
That is to say,
\[
\sum_{c=1}^{\infty} ||g(x_c)||^2 \leq \frac{2M}{\eta \alpha_{\text{min}} C_L} < +\infty.
\]
Hence, this theorem is valid.

**5.2. Convergence of the sequence of iterates.** The Kurdyka-Łojasiewicz property was discovered by S. Łojasiewicz [11] for real-analytic functions in 1963. Bolte et al. [5] extended this property to nonsmooth functions. Whereafter, KL property was widely applied in analyzing proximal algorithms for nonconvex and nonsmooth optimization [2, 3, 6, 55].

We remark that the merit function \(f(x) = T x^k - \lambda B x^k\) is a semialgebraic function since its graph
\[
\text{Graph} f = \{(x, \lambda) : T x^k - \lambda B x^k = 0\}
\]
is a semialgebraic set. Therefore, \(f(x)\) satisfies the following KL property [5, 11].

**Theorem 5.6 (KL property).** Suppose that \(x_*\) is a stationary point of \(f(x)\). Then, there is a neighborhood \(U\) of \(x_*\), an exponent \(\theta \in [0, 1)\), and a positive constant \(C_K\) such that for all \(x \in U\), the following inequality holds
\[
(5.4) \quad |f(x) - f(x_*)|^\theta \leq C_K ||g(x)||.
\]
Here, we define $0^0 \equiv 0$.

Using KL property, we will prove that the infinite sequence of iterates $\{x_c\}$ converges to a unique accumulation point.

**Lemma 5.7.** Suppose that $x_*$ is a stationary point of $f(x)$, and $\mathcal{B}(x_*, \rho) = \{x \in \mathbb{R}^n : \|x - x_*\| \leq \rho\} \subseteq \mathcal{W}$ is a neighborhood of $x_*$. Let $x_1$ be an initial point satisfying

$$\rho > \rho(x_1) \equiv \frac{2C_U C_K}{\eta C_L(1 - \theta)} |f(x_1) - f(x_*)_1^{-\theta} + \|x_1 - x_*\|.$$

Then, the following assertions hold:

$$\mathcal{B}(x_*, \rho), \quad c = 1, 2, \ldots,$$

and

$$\sum_{c=1}^{\infty} \|x_{c+1} - x_c\| \leq \frac{2C_U C_K}{\eta C_L(1 - \theta)} |f(x_1) - f(x_*)_1^{-\theta}.$$

**Proof.** We proceed by induction. Obviously, $x_1 \in \mathcal{B}(x_*, \rho)$.

Now, we assume that $x_i \in \mathcal{B}(x_*, \rho)$ for all $i = 1, \ldots, c$. Hence, KL property holds in these points. Let

$$\phi(t) \equiv \frac{C_K}{1 - \theta} |t - f(x_*)|_1^{-\theta}.$$

It is easy to prove that $\phi(t)$ is a concave function for $t > f(x_*)$. Therefore, for $i = 1, \ldots, c$, we have

$$\phi(f(x_i)) - \phi(f(x_{i+1})) \geq \phi'(f(x_i))(f(x_i) - f(x_{i+1}))$$

$$= C_K |f(x_i) - f(x_*)|_1^{-\theta}(f(x_i) - f(x_{i+1}))$$

[KL property] $\geq \|g(x_i)\|^{-1}(f(x_i) - f(x_{i+1}))$

[for (5.3)] $\geq \frac{\eta C_L}{2C_U} \|x_{i+1} - x_i\|$

where the last inequality is valid because

$$\|x_{c+1} - x_c\| \leq 2 \left(\|\alpha_c p_c\|^2 - (\alpha_c x_c^T p_c)^2\right)^{\frac{1}{2}} \leq 2\alpha_c\|p_c\| \leq 2\alpha_c C_U \|g(x_c)\|$$

by (4.17) and (5.1). Then,

$$\|x_{c+1} - x_*\| \leq \sum_{i=1}^{c} \|x_{i+1} - x_i\| + \|x_1 - x_*\|$$

$$\leq \frac{2C_U}{\eta C_L} \sum_{i=1}^{c} \phi(f(x_i)) - \phi(f(x_{i+1})) + \|x_1 - x_*\|$$

$$\leq \frac{2C_U}{\eta C_L} \phi(f(x_1)) + \|x_1 - x_*\|$$

$$< \rho.$$
Hence, we get \( x_{c+1} \in \mathcal{S}(x_*, \rho) \) and (5.8) holds. Moreover,
\[
\sum_{c=1}^{\infty} \|x_{c+1} - x_c\| \leq \frac{2CU}{\eta C_L} \sum_{c=1}^{\infty} \phi(f(x_c)) - \phi(f(x_{c+1})) \leq \frac{2CU}{\eta C_L} \phi(f(x_1)).
\]

The inequality (5.7) is valid. \( \square \)

**Theorem 5.8.** Suppose that CEST generates an infinite sequence of iterates \( \{x_c\} \). Then,
\[
\sum_{c=1}^{\infty} \|x_{c+1} - x_c\| < +\infty.
\]

Hence, the total sequence \( \{x_c\} \) has a finite length and converges to a unique stationary point.

**Proof.** Owing to the compactness of \( \mathbb{S}^{n-1} \), there exists an accumulate point \( x_* \) of iterates \( \{x_c\} \). By Theorem 5.5, \( x_* \) is also a stationary point. Then, there exists an iteration \( K \) such that \( \rho(x_K) < \rho \). Hence, by Lemma 5.7, we have \( \sum_{c=K}^{\infty} \|x_{c+1} - x_c\| < \infty \). The proof is complete. \( \square \)

Next, we estimate the convergence rate of CEST. The following lemma is useful.

**Lemma 5.9.** There exists a positive constant \( C_m \) such that
\[
\|x_{c+1} - x_c\| \geq C_m \|g(x_c)\|.
\]

**Proof.** Let \( \langle a, b \rangle \) be the angle between nonzero vectors \( a \) and \( b \), i.e.,
\[
\langle a, b \rangle \equiv \arccos \frac{a^\top b}{\|a\| \|b\|} \in [0, \pi].
\]
In fact, \( \langle \cdot, \cdot \rangle \) is a metric in a unit sphere and satisfies the triangle inequality
\[
\langle a, b \rangle \leq \langle a, c \rangle + \langle c, b \rangle
\]
for all nonzero vectors \( a, b, \) and \( c \).

From the triangle inequality, we get
\[
\langle x_c, -g(x_c) \rangle - \langle -g(x_c), p_c \rangle \leq \langle x_c, p_c \rangle \leq \langle x_c, -g(x_c) \rangle + \langle -g(x_c), p_c \rangle.
\]
Owing to (4.4), we know \( \langle x_c, -g(x_c) \rangle = \frac{\pi}{2} \). It yields that
\[
\frac{\pi}{2} - \langle -g(x_c), p_c \rangle \leq \langle x_c, p_c \rangle \leq \frac{\pi}{2} + \langle -g(x_c), p_c \rangle.
\]
Hence, we have
\[
\sin \langle x_c, p_c \rangle \geq \sin \left( \frac{\pi}{2} - \langle -g(x_c), p_c \rangle \right) = \cos \langle -g(x_c), p_c \rangle = \frac{-p_c^\top g(x_c)}{\|p_c\| \|g(x_c)\|} \geq \frac{C_L}{CU},
\]
where the last inequality holds because of (5.1). Recalling (4.17) and \( x_c \in \mathbb{S}^{n-1} \), we obtain
\[
\|x_{c+1} - x_c\| = 2 \left( \frac{\|p_c\|^2 (1 - \cos^2 \langle x_c, \alpha_c p_c \rangle)}{1 + \|p_c\|^2 (1 - \cos^2 \langle x_c, \alpha_c p_c \rangle)} \right)^{\frac{1}{2}} = \frac{2 \alpha}{\sqrt{1 + \alpha^2} \|p_c\|^2 \sin^2 \langle x_c, \alpha_c p_c \rangle}.
\]
Since $\alpha_{\min} \leq \alpha_c \leq 1$ and $\|p_c\| \leq C_U\|g(x_c)\| \leq C_U M$, it yields that

$$\|x_{c+1} - x_c\| \geq \frac{2\alpha_{\min}C_LC_c^{-1}}{\sqrt{1+C_U^2 M^2}}\|p_c\| \geq \frac{2\alpha_{\min}C_L}{C_U(1+C_U M)}\|p_c\|.\]$$

From (5.1), we have $\|p_c\|\|g(x_c)\| \geq -p_c^\top g(x_c) \geq C_L\|g(x_c)\|^2$. Hence, $\|p_c\| \geq C_L\|g(x_c)\|$. Therefore, this lemma is valid by taking $C_m \equiv \frac{2\alpha_{\min}C_c^2}{C_U(1+C_U M)}$. \(\blacksquare\)

**Theorem 5.10.** Suppose that $x_*$ is the stationary point of an infinite sequence of iterates $\{x_n\}$ generated by CEST. Then, we have the following estimations.

- If $\theta \in (0, \frac{1}{2}]$, there exists a $\gamma > 0$ and $\phi \in (0, 1)$ such that
  $$\|x_n - x_*\| \leq \gamma \phi^n.$$  
- If $\theta \in (\frac{1}{2}, 1)$, there exists a $\gamma > 0$ such that
  $$\|x_n - x_*\| \leq \gamma c^{-\frac{n}{m-1}}.$$  

**Proof.** Because of the validation of Lemma 5.1, the proof of this theorem is similar to [1] Theorem 2 and [12] Theorem 7. \(\blacksquare\)

Liu and Nocedal [39] Theorem 7.1 proved that L-BFGS converges linearly if the level set of $f(x)$ is convex and the second-order sufficient condition at $x_*$ holds. We remark here that, if the second-order sufficient condition holds, the exponent $\theta = \frac{1}{2}$ in KL property (5.4). According to Theorem 5.10 the infinite sequence of iterates $\{x_n\}$ has a linear convergence rate. Hence, to obtain the same local linear convergence rate in theory, we assume $\theta = \frac{1}{2}$ in KL property is weaker than the second-order sufficient condition.

5.3. **On the extreme eigenvalue.** For the target of computing the smallest eigenvalue of a large scale sparse tensor arising from a uniform hypergraph, we start CEST from plenty of randomly initial points. Then, we regard the resulting smallest merit function value as the smallest eigenvalue of this tensor. The following theorem reveals the successful probability of this strategy.

**Theorem 5.11.** Suppose that we start CEST from $N$ initial points which are sampled from $S^{n-1}$ uniformly and regard the resulting smallest merit function value as the smallest eigenvalue. Then, there exists a constant $\varsigma \in (0, 1]$ such that the probability of obtaining the smallest eigenvalue is at least

$$1 - (1 - \varsigma)^N.$$  

Therefore, if the number of samples $N$ is large enough, we obtain the smallest eigenvalue with a high probability.

**Proof.** Suppose that $x^*$ is an eigenvector corresponding to the smallest eigenvalue and $\mathcal{B}(x^*, \rho) \subseteq \mathcal{W}$ is a neighborhood as defined in Lemma 5.7. Since the function $\rho(\cdot)$ in (5.5) is continuous and satisfies $\rho(x^*) = 0 < \rho$, there exists a neighborhood $\mathcal{V}(x^*) \equiv \{x \in S^{n-1} : \rho(x) < \rho \} \subseteq \mathcal{W}$. That is to say, if an initial point $x_1$ happens to be sampled from $\mathcal{V}(x^*)$, the total sequence of iterates $\{x_n\}$ converges to $x^*$ by Lemma 5.7 and Theorem 5.8. Next, we consider the probability of this random event.

Let $S$ and $A$ be hypervolumes of $(n-1)$ dimensional solids $S^{n-1}$ and $\mathcal{V}(x^*)$ respectively.\(^2\) (That is to say, the “area” of the surface of $S^{n-1}$ in $\mathbb{R}^n$ is $S$ and the

\(^2\)The hypervolume of the $(n-1)$ dimensional unit sphere is $S = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, where $\Gamma(\cdot)$ is the Gamma function.
“area” of the surface of $\mathcal{Y}(x^*) \subseteq \mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is $A$. Hence, $A \leq S$. Then, $S$ and $A$ are positive. By the geometric probability model, the probability of one randomly initial point $x_1 \in \mathcal{Y}(x^*)$ is

$$\zeta \equiv \frac{A}{S} > 0.$$  

In fact, once $\{x_c\} \cap \mathcal{Y}(x^*) \neq \emptyset$, we could obtain the smallest eigenvalue. When starting from $N$ initial points generated by a uniform sample on $\mathbb{S}^{n-1}$, we obtain the probability as (5.9).

If we want to calculate the largest eigenvalue of a tensor $\mathcal{J}$, we only need to replace the merit function $f(x)$ in (4.2) with

$$\tilde{f}(x) = -\frac{\mathcal{J}x^k}{2x^k}.$$  

All of the theorems for the largest eigenvalue of a tensor could be proved in a similar way.

6. Numerical experiments. The novel CEST algorithm is implemented in Matlab and uses the following parameters

$$L = 5, \quad \eta = 0.01, \quad \text{and} \quad \beta = 0.5.$$  

We terminate CEST if

$$\|g(x_c)\|_{\infty} < 10^{-6} \quad (6.1)$$

or

$$\|x_{c+1} - x_c\|_{\infty} < 10^{-8} \quad \text{and} \quad \frac{|f(x_{c+1}) - f(x_c)|}{1 + |f(x_c)|} < 10^{-16} \quad (6.2)$$

If the number of iterations reaches 5000, we also stop. All of the codes are written in Matlab 2012a and run in a ThinkPad T450 laptop with Intel i7-5500U CPU and 8GB RAM.

We compare the following four algorithms in this section.

- An adaptive shifted power method \cite{33,34} (Power M.). In Tensor Toolbox 2.6\footnote{See http://www.sandia.gov/tgkolda/TensorToolbox/index-2.6.html.}, it was implemented as eig_schopm and eig_geap for Z- and H-eigenvalues of symmetric tensors respectively.
- Han’s unconstrained optimization approach (Han’s UOA) \cite{25}. We solve the optimization model by fminunc in Matlab with settings: GradObj:on, LargeScale:off, TolX:1.e-8, TolFun:1.e-16, MaxIter:5000, Display:off. Since iterates generated by Han’s UOA are not restricted on the unit sphere $\mathbb{S}^{n-1}$, the tolerance parameters are different from other algorithms.
- CESTde: we implement CEST for a dense symmetric tensor, i.e., the skills addressed in Section 3 are not applied.
- CEST: the novel method is proposed and analyzed in this paper.

For tensors arising from an even-uniform hypergraph, each algorithm starts from one hundred random initial points sampled from a unit sphere $\mathbb{S}^{n-1}$ uniformly. Then,
we obtain one hundred estimated eigenvalues \( \lambda_1, \ldots, \lambda_{100} \). If the extreme eigenvalue \( \lambda^* \) of that tensor is available, we count the accuracy rate of this algorithm as

\[
\text{Accu.} = \left| \left\{ i : \frac{|\lambda_i - \lambda^*|}{1 + |\lambda^*|} \leq 10^{-8} \right\} \right| \times 1\%.
\]

After using the global strategy in Section 5.3, we regard the best one as the estimated extreme eigenvalue.

### 6.1. Small hypergraphs

First, we investigate some extreme eigenvalues of symmetric tensors corresponding to small uniform hypergraphs.

**Squid.** A squid \( G_k^S = (V, E) \) is a \( k \)-uniform hypergraph which has \( (k^2 - k + 1) \) vertices and \( k \) edges: legs \( \{i_{1,1}, \ldots, i_{1,k}\}, \ldots, \{i_{k-1,1}, \ldots, i_{k-1,k}\} \) and a head \( \{i_1, \ldots, i_{k-1,1}, i_k\} \).

When \( k \) is even, \( G_k^S \) is obviously connected and odd-bipartite. Hence, we have \( \lambda_{\text{min}}^H(A(G_k^S)) = -\lambda_{\text{max}}^H(A(G_k^S)) \) because of Theorem 2.4(iv). Since the adjacency tensor \( A(G_k^S) \) is nonnegative and weakly irreducible, its largest H-eigenvalue \( \lambda_{\text{max}}^H(A(G_k^S)) \) could be computed by the Ng-Qi-Zhou algorithm \[43\]. For the smallest H-eigenvalue of \( A(G_k^S) \), we perform the following tests.

With regards to the parameter \( L \) for L-BFGS, Nocedal suggested that L-BFGS performs well when \( 3 \leq L \leq 7 \). Hence, we compare L-BFGS with \( L = 3, 5, 7 \) and the Barzilai-Borwein method (\( L = 0 \)). The parameter \( \gamma_c \) is chosen from \( \gamma_{BB1}^c, \gamma_{BB2}^c, \) and \( \gamma_{\text{Dai}}^c \) randomly. For \( k \)-uniform squids with \( k = 4, 6, 8 \), we compute the smallest H-eigenvalues of their adjacency tensors. The total CPU times for one hundred runs are illustrated in Figure 6.1. Obviously, L-BFGS is about five times faster than the Barzilai-Borwein method. Following Nocedal’s setting\[4\], we prefer to set \( L = 5 \) in CEST.

Next, we consider the 4-uniform squid \( G_4^S \) illustrated in Figure 6.2. For reference, we remind \( \lambda_{\text{max}}^H(A(G_4^S)) = 1.3320 \) by the Ng-Qi-Zhou algorithm. Then, we compare four kinds of algorithms: Power M., Han’s UOA, CESTde, and CEST. Results are shown in Table 6.1. Obviously, all algorithms find the smallest H-eigenvalue of the adjacency tensor \( A(G_4^S) \) with probability 1. Compared with Power M., Han’s UOA and CESTde save 78% and 63% CPU times, respectively. When the sparse structure

---

4See http://users.iems.northwestern.edu/~nocedal/lbfgs.html.
Fig. 6.2. A 4-uniform squid $G^4_S$.

Table 6.1
Comparison results for the smallest $H$-eigenvalue of the adjacency tensor $A(G^4_S)$.

| Algorithms     | $\lambda_{\min}^H(A(G^4_S))$ | Time (s) | Accu. |
|----------------|-------------------------------|----------|-------|
| Power M.       | $-1.3320$                     | 97.20    | 100%  |
| Han’s UOA      | $-1.3320$                     | 21.20    | 100%  |
| CESTde         | $-1.3320$                     | 35.72    | 100%  |
| CEST           | $-1.3320$                     | 2.43     | 100%  |

Fig. 6.3. The Petersen graph $G_P$.

of the adjacency tensor $A(G^4_S)$ is explored, CEST is forty times faster than the power method.

Blowing up the Petersen graph. Figure 6.3 illustrates an ordinary graph $G_P$: the Petersen graph. It is non-bipartite and the smallest eigenvalue of its signless Laplacian matrix is one. We consider the $2k$-uniform hypergraph $G^{2k,k}_P$ which is generated by blowing up each vertex of $G_P$ to a $k$-set. Hence, $G^{2k,k}_P$ contains $10k$ vertices and 15 edges. From Theorem 2.5, we know that the smallest $H$-eigenvalue of the signless Laplacian tensor $Q(G^{2k,k}_P)$ is exactly one.

Table 6.2 reports comparison results of four sorts of algorithms for the 4-uniform hypergraph $G^4_P$. Here, Han’s UOA missed the smallest $H$-eigenvalue of the signless Laplacian tensor. Power M., CESTde, and CEST find the true solution with a high
Table 6.2

Comparison results for the smallest $H$-eigenvalue of the signless Laplacian tensor $Q(G_P^{4,2})$. (*) means a failure.

| Algorithms      | $\lambda_{H\min}(Q(G_P^{4,2}))$ | CPU time(s) | Accu. |
|-----------------|---------------------------------|-------------|-------|
| Power M.        | 1.0000                          | 657.44      | 95%   |
| Han’s UOA       | 1.1877(*)                       | 93.09       |       |
| CESTde          | 1.0000                          | 70.43       | 100%  |
| CEST            | 1.0000                          | 3.82        | 100%  |

Table 6.3

CEST computes the smallest $H$-eigenvalues of signless Laplacian tensors $Q(G_P^{2k,k})$.

| $2k$ | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | 18 | 20 |
|------|----|----|----|----|----|----|----|----|----|----|
| Accu. (%) | 100 | 100 | 100 | 100 | 99 | 98 | 86 | 57 | 20 | 4  |

Fig. 6.4. Some 4-uniform grid hypergraphs.

Grid hypergraphs. The grid $G_{G}^{s}$ is a 4-uniform hypergraph generated by subdividing a square. If the subdividing order $s = 0$, the grid $G_{G}^{0}$ is the square with 4 vertices and only one edge. When the subdividing order $s \geq 1$, we subdivide each edge of $G_{G}^{s-1}$ into four edges. Hence, a grid $G_{G}^{s}$ has $(2^s + 1)^2$ vertices and $4^s$ edges. The 4-uniform grid $G_{G}^{s}$ with $s = 1, 2, 3, 4$ are illustrated in Figure 6.4.

We study the largest $H$-eigenvalue of the Laplacian tensor of a 4-uniform grid $G_{G}^{2}$ as shown in Figure 6.4(b). Obviously, the grid $G_{G}^{2}$ is connected and odd-bipartite. Hence, we have $\lambda^H_{\max}(\mathcal{L}(G_{G}^{2})) = \lambda^H_{\max}(Q(G_{G}^{2}))$ by Theorem 2.4(ii). Using the Ng-Qi-Zhou algorithm, we calculate $\lambda^H_{\max}(Q(G_{G}^{2})) = 6.5754$ for reference. Table 6.4 shows the performance of four kinds of algorithms: Power M., Han’s UOA, CESTde, and CEST. All of them find the largest $H$-eigenvalue of $\mathcal{L}(G_{G}^{2})$ with probability one. Compared with Power M., Han’s UOA and CESTde saves about 75% and 70% CPU probability. When compared with power M., CESTde saves more than 88% CPU times. Moreover, the approach exploiting the sparsity improves CESTde greatly, since CEST saves about 99% CPU times.

For $2k$-uniform hypergraph $G_{P}^{2k,k}$ with $k = 1, \ldots, 10$, we apply CEST for computing the smallest $H$-eigenvalues of their signless Laplacian tensors. Detailed results are shown in Table 6.3. For each case, CEST finds the smallest $H$-eigenvalue of the signless Laplacian tensor in at most one minute. With the increment of $k$, the percentage of accurate estimations decreases.
Table 6.4
Comparison results for the largest H-eigenvalue of the Laplacian tensor $L(G^2_G)$.

| Algorithms   | $\lambda_{max}^H(L(G^2_G))$ | Time(s) | Accu. |
|--------------|------------------------------|---------|-------|
| Power M.     | 6.5754                       | 142.51  | 100%  |
| Han’s UOA    | 6.5754                       | 35.07   | 100%  |
| CESTde       | 6.5754                       | 43.35   | 100%  |
| CEST         | 6.5754                       | 2.43    | 100%  |

Table 6.5
CEST computes the largest H-eigenvalue of Laplacian tensors $L(G^s_G)$.

| $s$ | $n$ | $m$ | $\lambda_{max}^H(L(G^s_G))$ | Iter. | Time(s) | Accu. |
|-----|-----|-----|------------------------------|-------|---------|-------|
| 1   | 9   | 4   | 4.6344                       | 2444  | 1.39    | 100%  |
| 2   | 25  | 16  | 6.5754                       | 4738  | 2.43    | 100%  |
| 3   | 81  | 64  | 7.5293                       | 12624 | 6.44    | 98%   |
| 4   | 289 | 256 | 7.8648                       | 34558 | 26.08   | 65%   |

times respectively. CEST is about fifty times faster than the Power Method.

In Table 6.5, we show the performance of CEST for computing the largest H-eigenvalues of the Laplacian tensors of grids illustrated in Figure 6.4.

6.2. Large hypergraphs. Finally, we consider two large scale even-uniform hypergraphs.

**Sunflower.** A $k$-uniform sunflower $G_S = (V, E)$ with a maximum degree $\Delta$ has $n = (k - 1)\Delta + 1$ vertices and $\Delta$ edges, where $V = V_0 \cup V_1 \cup \cdots \cup V_\Delta$, $|V_0| = 1$, $|V_i| = k - 1$ for $i = 1, \ldots, \Delta$, and $E = \{V_0 \cup V_i | i = 1, \ldots, \Delta\}$. Figure 2.1 is a 4-uniform sunflower with $\Delta = 3$.

Hu et al. [28] argued in the following theorem that the largest H-eigenvalue of the Laplacian tensor of an even-uniform sunflower has a closed form solution.

**Theorem 6.1.** (Theorems 3.2 and 3.4 of [28]) Let $G$ be a $k$-graph with $k \geq 4$ being even. Then

$$\lambda_{max}^H(L) \geq \lambda_H^*,$$

where $\lambda_H^*$ is the unique real root of the equation $(1 - \lambda)^k - (\lambda - \Delta) + \Delta = 0$ in the interval $(\Delta, \Delta + 1)$. The equality holds if and only if $G$ is a sunflower.

We aim to apply CEST for computing the largest H-eigenvalues of Laplacian tensors of even-uniform sunflowers. For $k = 4$ and 6, we consider sunflowers with maximum degrees from ten to one million. Since we deal with large scale tensors, we slightly enlarge tolerance parameters in (6.1) and (6.2) by multiplying $\sqrt{n}$. To show the accuracy of the estimated H-eigenvalue $\lambda_{max}^H(L)$, we calculate the relative error

$$RE = \frac{|\lambda_{max}^H(L) - \lambda_H^*|}{\lambda_H^*},$$

where $\lambda_H^*$ is defined in Theorem 6.1. Table 6.6 reports detailed numerical results. Obviously, the largest H-eigenvalues of Laplacian tensors returned by CEST have a high accuracy. Relative errors are in the magnitude $O(10^{-10})$. The CPU time costed by CEST does not exceed 80 minutes.

**Icosahedron.** An icosahedron has twelve vertices and twenty faces. The subdivision of an icosahedron could be used to approximate a unit sphere. The $s$-order
The largest $H$-eigenvalues of Laplacian tensors corresponding to $k$-uniform sunflowers.

| $k$ | $n$ | $\lambda_{max}^H(\mathcal{L})$ | RE | Iter. | Time(s) | Accu. |
|-----|-----|-------------------------------|-----|-------|---------|-------|
| 4   | 31  | 10.0137 x 10^{-16}           | 5.3218 x 10^{-16} | 4284  | 2.39    | 100%  |
|     | 301 | 100.0001                     | 7.3186 x 10^{-14} | 4413  | 3.73    | 42%   |
|     | 3,001 | 1,000.0000                  | 1.2917 x 10^{-10} | 1291  | 4.84    | 100%  |
|     | 30,001 | 10,000.0000              | 5.9652 x 10^{-12} | 1280  | 38.14   | 100%  |
|     | 300,001 | 100,000.0000            | 9.6043 x 10^{-15} | 1254  | 512.04  | 100%  |

Table 6.7

The largest $Z$-eigenvalues of Laplacian tensors and signless Laplacian tensors of hypergraphs $G^s_I$. Then, the 4-graph $G^s_I$ must be connected and odd-bipartite. See Figure 6.5.

According to Theorem 2.4(v), we have $\lambda_{max}^Z(\mathcal{L}(G^s_I)) = \lambda_{max}^Z(Q(G^s_I))$, although they are unknown. Experiment results are reported in Table 6.7. It is easy to see that CEST could compute the largest $Z$-eigenvalues of both Laplacian tensors and signless Laplacian tensors of hypergraphs $G^s_I$ with dimensions up to almost two millions. In each case of our experiment, CEST costs at most twenty-one minutes.

Additionally, for 4-graphs $G^s_I$ generated by subdividing an icosahedron, the fol-
Table 6.7

| s | n  | m  | $\lambda_{max}^Z(L(G_s^I))$ | Iter. | time(s) | $\lambda_{max}^Z(Q(G_s^I))$ | Iter. | time(s) |
|---|----|----|-----------------|------|--------|-----------------|------|--------|
| 0 | 32 | 20 | 5               | 1102 | 0.89   | 5               | 1092 | 0.75   |
| 1 | 122| 80 | 6               | 1090 | 1.09   | 6               | 1050 | 0.75   |
| 2 | 482| 320| 6               | 1130 | 1.39   | 6               | 1170 | 1.23   |
| 3 | 1,922 | 1,280 | 6 | 1226 | 3.15 | 6 | 1194 | 2.95 |
| 4 | 7,682 | 5,120 | 6 | 1270 | 10.11 | 6 | 1244 | 10.06 |
| 5 | 30,722 | 20,480 | 6 | 1249 | 36.89 | 6 | 1282 | 35.93 |
| 6 | 122,882 | 81,920 | 6 | 1273 | 166.05 | 6 | 1289 | 161.02 |
| 7 | 491,522 | 327,680 | 6 | 1300 | 744.08 | 6 | 1327 | 739.01 |
| 8 | 1,966,082 | 1,310,720 | 6 | 574 | 1251.36 | 6 | 558 | 1225.87 |

The following equality seems to hold

$$\lambda_{max}^Z(L(G_s^I)) = \lambda_{max}^Z(Q(G_s^I)) = \Delta.$$  

Bu et al. [7] proved that (6.4) holds for a $k$-uniform sunflower with $3 \leq k \leq 2\Delta$. However, it is an open problem whether the equality (6.4) hold for a general connected odd-bipartite uniform hypergraph.

7. Conclusion. Motivated by recent advances in spectral hypergraph theory, we proposed an efficient first-order optimization algorithm CEST for computing extreme $H$- and $Z$-eigenvalues of sparse tensors arising form large scale uniform hypergraphs.

Due to the algebraic nature of tensors, we could apply the Kurdyka-Lojasiewicz property in analyzing the convergence of the sequence of iterates generated by CEST. By using a simple global strategy, we prove that the extreme eigenvalue of a symmetric tensor could be touched with a high probability.

We establish a fast computational framework for products of a vector and large scale sparse tensors arising from a uniform hypergraph. By using this technique, the storage of a hypergraph is economic and the computational cost of CEST in each iteration is cheap. Numerical experiments show that the novel algorithm CEST could deal with uniform hypergraphs with millions of vertices.

Appendix A. In this appendix, we will prove that L-BFGS produces a gradient-related direction, i.e., Theorem 5.3 is valid. First, we consider the classical BFGS update (4.5)–(4.7) and establish the following two lemmas.

**Lemma A.1.** Suppose that $H_c^+$ is generated by BFGS (4.5)–(4.7). Then, we have

$$\|H_c^+\| \leq \|H_c\| \left(1 + \frac{4M}{\kappa_c}\right)^2 \left(\frac{4}{\kappa_c}\right).$$  

*Proof.* If $y_c^Ts_c < \kappa_c$, we get $\rho_c = 0$ and $H_c^+ = H_c$. Hence, the inequality (A.1) holds.

Next, we consider the case $y_c^T s_c \geq \kappa_c$. Obviously, $\rho_c \leq \frac{1}{\kappa_c}$. From Lemma 5.1 and all iterates $x_c \in S^{n-1}$, we get

$$\|s_c\| \leq 2 \quad \text{and} \quad \|y_c\| \leq 2M.$$
Since
\[ \|V_c\| \leq 1 + \rho_c\|y_c\|s_c\| \leq 1 + \frac{4M}{\kappa_c} \quad \text{and} \quad \|\rho_c s_c s_c^\top\| \leq \rho_c\|s_c\|^2 \leq \frac{4}{\kappa_c}, \]
we have
\[ \|H_c^+\| \leq \|H_c\|\|V_c\|^2 + \|\rho_c s_c s_c^\top\| \leq \|H_c\| \left(1 + \frac{4M}{\kappa_c}\right)^2 + \frac{4}{\kappa_c}. \]
Hence, the inequality (A.1) is valid. \( \square \)

**Lemma A.2.** Suppose that \( H_c \) is positive definite and \( H_c^+ \) is generated by BFGS (4.5)–(4.7). Let \( \mu_{\text{min}}(H) \) be the smallest eigenvalue of a symmetric matrix \( H \). Then, we get \( H_c^+ \) is positive definite and
\[ \mu_{\text{min}}(H_c^+) \geq \frac{\kappa_c}{\kappa_c + 4M^2\|H_c\|\mu_{\text{min}}(H_c)}. \]

**Proof.** For any unit vector \( z \), we have
\[ z^\top H_c^+ z = (z - \rho_c s_c^\top y_c)\top H_c (z - \rho_c s_c^\top y_c) + \rho_c (s_c^\top z)^2. \]
Let \( t \equiv s_c^\top z \) and \( \phi(t) \equiv (z - t\rho_c y_c)\top H_c (z - t\rho_c y_c) + \rho_c t^2. \)
Because \( H_c \) is positive definite, \( \phi(t) \) is convex and attaches its minimum at \( t_* = \frac{\rho_c y_c^\top H_c z}{\rho_c + \rho_c y_c^\top H_c y_c}. \) Hence,
\[ z^\top H_c^+ z \geq \phi(t_*) = z^\top H_c z - t_* \rho_c y_c^\top H_c y_c = \rho_c z^\top H_c z + \rho_c^2 (y_c^\top H_c y_c) (z^\top H_c z - (y_c^\top H_c z)^2) \rho_c + \rho_c^2 y_c^\top H_c y_c \geq \frac{z^\top H_c z}{1 + \rho_c y_c^\top H_c y_c} \geq 0, \]
where the penultimate inequality holds because the Cauchy-Schwarz inequality is valid for the positive definite matrix norm \( \|\cdot\|_{H_c}, \) i.e., \( \|y_c\|_{H_c}\|z\|_{H_c} \geq y_c^\top H_c z. \) Therefore, \( H_c^+ \) is also positive definite. From (A.2), it is easy to verify that
\[ 1 + \rho_c y_c^\top H_c y_c \leq 1 + \frac{4M^2\|H_c\|}{\kappa_c}. \]
Therefore, we have
\[ z^\top H_c^+ z \geq \frac{\kappa_c}{\kappa_c + 4M^2\|H_c\|\mu_{\text{min}}(H_c)}. \]
Hence, we get the validation of (A.3). \( \square \)

Second, we turn to L-BFGS. Regardless of the selection of \( \gamma_c \) in (4.8), we get the following lemma.

**Lemma A.3.** Suppose that the parameter \( \gamma_c \) takes Barzilai-Borwein steps (4.11) or its geometric mean (4.12). Then, we have
\[ \frac{\kappa_c}{4M^2} \leq \gamma_c \leq \frac{1}{\kappa_c}. \]
Proof. If \( y_c^T s_c < \kappa_e \), we get \( \gamma_c = 1 \) which satisfies the bounds in (A.4) obviously. Otherwise, we have \( \kappa_e \leq y_c^T s_c \leq \|y_c\|\|s_c\| \). Recalling (A.2), we get

\[
\frac{\kappa_e}{2} \leq \|y_c\| \leq 2M \quad \text{and} \quad \frac{\kappa_e}{2M} \leq \|s_c\| \leq 2.
\]

Hence, we have

\[
\kappa_e \frac{4M^2}{\|y_c\|^2} \leq \frac{\|s_c\|\|y_c\|}{\|y_c\|^2} = \frac{\|s_c\|^2}{\|y_c\|^2} \leq \frac{\|s_c\|^2}{\|y_c\|^2} \leq \frac{4}{\kappa_e},
\]

which means that three candidates \( \gamma_c^\text{BB1} \), \( \gamma_c^\text{BB2} \), and \( \gamma_c^\text{Dai} \) satisfy the inequality (A.4). \( \square \)

Third, based on Lemmas A.1, A.2, and A.3, we obtain two lemmas as follows.

**Lemma A.4.** Suppose that the approximation of a Hessian’s inverse \( H_c \) is generated by L-BFGS (4.8)–(4.10). Then, there exists a positive constant \( C_U \geq 1 \) such that

\[ \|H_c\| \leq C_U. \]

**Proof.** From Lemma A.3 and (4.8), we have \( \|H_c^{(0)}\| \leq \frac{4}{\kappa_e} \). Then, by (4.10), (4.9) and Lemma A.1, we get

\[
\|H_c\| = \|H_c^{(L)}\| \leq \|H_c^{(L-1)}\| \left(1 + \frac{4M}{\kappa_e}\right)^2 + \frac{4}{\kappa_e} \\
\leq \|H_c^{(L-1)}\| \left(1 + \frac{4M}{\kappa_e}\right)^2 + \frac{4}{\kappa_e} \sum_{\ell=0}^{L-1} \left(1 + \frac{4M}{\kappa_e}\right)^{2\ell} \\
\leq \frac{4}{\kappa_e} \sum_{\ell=0}^{L} \left(1 + \frac{4M}{\kappa_e}\right)^{2\ell} = C_U.
\]

The proof is complete. \( \square \)

**Lemma A.5.** Suppose that the approximation of a Hessian’s inverse \( H_c \) is generated by L-BFGS (4.8)–(4.10). Then, there exists a constant \( 0 < C_L < 1 \) such that

\[ \mu_{\min}(H_c) \geq C_L. \]

**Proof.** From Lemma A.3 and (4.3), we have \( \mu_{\min}(H_c^{(0)}) \geq \frac{\kappa_e}{4M^2} \). Moreover, Lemma A.4 means that \( \|H_c^{(\ell)}\| \leq C_U \) for all \( \ell = 1, \ldots, L \). Hence, Lemma A.2 implies

\[ \mu_{\min}(H_c^{(\ell+1)}) \geq \frac{\kappa_e}{\kappa_e + 4M^2C_U} \mu_{\min}(H_c^{(\ell)}). \]
Then, from (4.10) and (4.9), we obtain
\[
\mu_{\min}(H_c) = \mu_{\min}(H_c^{(L)}) \\
\geq \frac{\kappa_c}{\kappa_c + 4M^2C_U}\mu_{\min}(H_c^{(L-1)}) \\
\geq \cdots \\
\geq \left( \frac{\kappa_c}{\kappa_c + 4M^2C_U} \right)^L \mu_{\min}(H_c^{(0)}) \\
\geq \frac{\kappa_c}{4M^2C_U} \left( \frac{\kappa_c}{\kappa_c + 4M^2C_U} \right)^L \equiv C_L.
\]

We complete the proof. \(\square\)

Finally, the proof of Theorem 5.3 is straightforward from Lemmas A.4 and A.5.

REFERENCES

[1] Hédy Attouch and Jérôme Bolte, On the convergence of the proximal algorithm for nonsmooth functions involving analytic features, Mathematical Programming, 116 (2009), pp. 5–16.
[2] Hédy Attouch, Jérôme Bolte, Patrick Redont, and Antoine Soubeyran, Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the Kurdyka-Lojasiewicz inequality, Mathematics of Operations Research, 35 (2010), pp. 438–457.
[3] Hedy Attouch, Jérôme Bolte, and Benar Fux Svaiter, Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward-backward splitting, and regularized Gauss-Seidel methods, Mathematical Programming, Series A, 137 (2013), pp. 91–129.
[4] Jonathan Barzilai and Jonathan M. Borwein, Two-point step size gradient methods, IMA Journal of Numerical Analysis, 8 (1988), pp. 141–148.
[5] Jérôme Bolte, Aris Daniilidis, and Adrian Lewis, The Lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems, SIAM Journal on Optimization, 17 (2007), pp. 1205–1223.
[6] Jérôme Bolte, Shoham Sabach, and Marc Teboulle, Proximal alternating linearized minimization for nonconvex and nonsmooth problems, Mathematical Programming, Series A, 146 (2014), pp. 459–494.
[7] Changjiang Bu, Yamin Fan, and Jiang Zhou, Laplacian and signless laplacian z-eigenvalues of uniform hypergraphs, Frontiers of Mathematics in China, (2015), p. TBA.
[8] Samuel Rota Bulo and Marcello Pelillo, New bounds on the clique number of graphs based on spectral hypergraph theory, in Learning and Intelligent Optimization, Springer, 2009, pp. 45–58.
[9] Kung-Ching Chang, Kelly J. Pearson, and Tan Zhang, On eigenvalue problems of real symmetric tensors, Journal of Mathematical Analysis and Applications, 350 (2009), pp. 416–422.
[10] Liping Chen, Lixing Han, and Liangmin Zhou, Computing tensor eigenvalues via homotopy methods, (2015). http://arxiv.org/abs/1501.04201.
[11] Yannan Chen, Liqun Qi, and Qun Wang, Computing eigenvalues of large scale hankel tensors, Journal of Scientific Computing, (2015). DOI: 10.1007/s10915-015-0155-8.
[12] Joshua Cooper and Aaron Dutle, Spectra of uniform hypergraphs, Linear Algebra and its Applications, 436 (2012), pp. 3208–3292.
[13] Chun-Feng Cui, Yu-Hong Dai, and Jiawang Nie, All real eigenvalues of symmetric tensors, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 1582–1601.
[14] Yu-Hong Dai, A positive BB-like stepsize and an extension for symmetric linear systems, Workshop on Optimization for Modern Computation, Beijing, China, 2014. http://bicmr.pku.edu.cn/conference/opt-2014/slides/Yuhong-Dai.pdf.
[16] Wei Yang Ding, Liqun Qi, and Yimin Wei, $M$-tensors and nonsingular $M$-tensors, Linear Algebra and its Applications, 439 (2013), pp. 3264–3278.

[17] Wei Yang Ding and Yimin Wei, Generalized tensor eigenvalue problems, SIAM Journal on Matrix Analysis and Applications, 36 (2015), pp. 1073–1099.

[18] V. A. Skorobogatov E. V. Konstantinova, Molecular structures of organoelement compounds and their representation as labeled molecular hypergraphs, Journal of Structural Chemistry, 39 (1998), pp. 268–276.

[19] Eldar Fischer, Arie Matsliah, and Asaf Shapira, Approximate hypergraph partitioning and applications, SIAM Journal on Computing, 39 (2010), pp. 3155–3185.

[20] S. Friedland, S. Gaubert, and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra and its Applications, 438 (2013), pp. 738–749.

[21] Yue Gao, Meng Wang, Dacheng Tao, Rongrong Ji, and Qionghai Dai, 3d object retrieval and recognition with hypergraph analysis, IEEE Transactions on Image Processing, 21 (2012), pp. 4290–4303.

[22] Gourab Ghoshal, Vinko Zlatić, Guido Caldarelli, and M. E. J. Newman, Random hypergraphs and their applications, Physical Review E, 79 (2009), p. 066118.

[23] Debarghya Ghoshdastidar and Ambedkar Dukkipati, A provable generalized tensor spectral method for uniform hypergraph partitioning, in Proceedings of The 32nd International Conference on Machine Learning, 2015, pp. 400–409.

[24] David F. Gleich, Lek-Heng Lim, and Yong Yang Yu, Multilinear pagerank, SIAM Journal on Matrix Analysis and Applications, 36 (2015), pp. 1507–1541.

[25] Lixing Han, An unconstrained optimization approach for finding real eigenvalues of even order symmetric tensors, Numerical Algebra, Control and Optimization (NACO), 3 (2013), pp. 583–599.

[26] Chun-Lin Hao, Chun-Feng Cui, and Yu-Hong Dai, A sequential subspace projection method for extreme $Z$-eigenvalues of supersymmetric tensors, Numerical Linear Algebra with Applications, 22 (2015), pp. 283–298.

[27] Christopher J. Hillar and Lek-Heng Lim, Most tensor problems are NP-hard, Journal of the ACM, 60 (2013), pp. Article 45:1–39.

[28] Shenglong Hu, Liqun Qi, and Jinshan Xie, The largest Laplacian and signless Laplacian $H$-eigenvalues of a uniform hypergraph, Linear Algebra and its Applications, 469 (2015), pp. 1–27.

[29] George Karypis, Rajat Aggarwal, Vipin Kumar, and Shashi Shekhar, Multilevel hypergraph partitioning: applications in vlsi domain, IEEE Transactions on Very Large Scale Integration (VLSI) Systems, 7 (1999), pp. 69–79.

[30] Enver Kayaaslan, Ali Pinar, Ümit Catalyurek, and Cevdet Aykanat, Partitioning hypergraphs in scientific computing applications through vertex separators on graphs, SIAM Journal on Scientific Computing, 34 (2012), pp. A970–A992.

[31] Murad-ul-Islam Khan and Yi-Zheng Fan, The $H$-spectrum of a generalized power hypergraph, (2015). http://arxiv.org/abs/1504.03839.

[32] Steffen Klamti, Utz-Uwe Haus, and Fabian Theis, Hypergraphs and cellular networks, PLOS Computational Biology, 5 (2009), p. e1000385.

[33] Tamara G. Kolda and Jackson R. Mayo, Shifted power method for computing tensor eigenpairs, SIAM Journal on Matrix Analysis and Applications, 32 (2011), pp. 1095–1124.

[34] Tamara G. Kolda and Jackson R. Mayo, An adaptive shifted power method for computing generalized tensor eigenpairs, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 1563–1581.

[35] Elena V. Konstantinova and Vladimir A. Skorobogatov, Molecular hypergraphs: The new representation of nonclassical molecular structures with polycentric delocalized bonds, Journal of Chemical Information and Computer Sciences, 35 (1995), pp. 472–478.

[36] Jing Lei and Alessandro Rinaldo, Consistency of spectral clustering in stochastic block models, The Annals of Statistics, 43 (2015), pp. 215–237.

[37] Xi Li, Weiming Hu, Chunhua Shen, Anthony Dick, and Zhongfei Zhang, Context-aware hypergraph construction for robust spectral clustering, IEEE Transactions on Knowledge and Data Engineering, 26 (2014), pp. 2588–2597.

[38] Lek-Heng Lim, Singular values and eigenvalues of tensors: a variational approach, in Computational Advances in Multi-Sensor Adaptive Processing, 2005 1st IEEE International Workshop on, IEEE, 2005, pp. 129–132.

[39] Dong C. Liu and Jorge Nocedal, On the limited memory BFGS method for large scale optimization, Mathematical Programming, 45 (1989), pp. 503–528.

[40] Yongjun Liu, Guanlu Zhou, and Nur Fadhilah Ibrahim, An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor, Journal of Computational
and Applied Mathematics, 235 (2010), pp. 286–292.

[41] Stanislaw Lojasiewicz, Une propriétie topologique des sous-ensembles analytiques réels, Les Équations aux Dérivées Partielles, (1963), pp. 87–89.

[42] Tom Michel and Bruno Nachtergaele, Alignment and integration of complex networks by hypergraph-based spectral clustering, Physical Review E, 86 (2012), p. 056111.

[43] Michael Ng, Liqun Qi, and Guanglu Zhou, Finding the largest eigenvalue of a nonnegative tensor, SIAM Journal on Matrix Analysis and Applications, 31 (2009), pp. 1090–1099.

[44] Qin Ni and Liqun Qi, A quadratically convergent algorithm for finding the largest eigenvalue of a nonnegative homogeneous polynomial map, Journal of Global Optimization, 61 (2015), pp. 627–641.

[45] Jorge Nocedal, Updating quasi-newton matrices with limited storage, Mathematics of Computation, 35 (1980), pp. 773–782.

[46] Jorge Nocedal and Stephen J. Wright, Numerical Optimization, Springer, 2006.

[47] Kelly J. Pearson and Tan Zhang, On spectral hypergraph theory of the adjacency tensor, Graphs and Combinatorics, 30 (2014), pp. 1233–1248.

[48] Liqun Qi, Eigenvalues of a real supersymmetric tensor, Journal of Symbolic Computation, 40 (2005), pp. 1302–1324.

[49] Liqun Qi, H'-eigenvalues of Laplacian and signless Laplacian tensors, Communications in Mathematical Sciences, 12 (2014), pp. 1045–1064.

[50] Liqun Qi, Fei Wang, and Yuu Wang, Z-eigenvalue methods for a global polynomial optimization problem, Mathematical Programming, 118 (2009), pp. 301–316.

[51] Karl Rohe, Sourav Chatterjee, and Bin Yu, Spectral clustering and the high-dimensional stochastic blockmodel, The Annals of Statistics, (2011), pp. 1878–1915.

[52] Jia-Yu Shao, Hai-Ying Shan, and Bao Feng Wu, Some spectral properties and characterizations of connected odd-bipartite uniform hypergraphs, Linear and Multilinear Algebra, 63 (2015), pp. 2359–2372.

[53] Wenyu Sun and Ya Xiang Yuan, Optimization Theory and Methods: Nonlinear Programming, Springer, 2006.

[54] Jinshan Xie and Liqun Qi, The clique and coclique numbers bounds based on the h-eigenvalues of uniform hypergraphs, International Journal of Numerical Analysis and Modeling, 12 (2015), pp. 318–327.

[55] Yangyang Xu and Wotao Yin, A block coordinate descent method for regularized multiconvex optimization with applications to nonnegative tensor factorization and completion, SIAM Journal on Imaging Sciences, 21 (2012), pp. 3262–3272.

[56] Jun Yu, Daoyong Du, and Meng Wang, Adaptive hypergraph learning and its application in image classification, IEEE Transactions on Image Processing, 21 (1012), pp. 3262–3272.

[57] Liping Zhang, Liqun Qi, and Guanglu Zhou, M-tensors and some applications, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 437–452.

[58] Guanglu Zhou, Liqun Qi, and Soon-Yi Wu, Efficient algorithms for computing the largest eigenvalue of a nonnegative tensor, Frontiers of Mathematics in China, 8 (2013), pp. 155–168.

[59] Guanglu Zhou, Liqun Qi, and Soon-Yi Wu, On the largest eigenvalue of a symmetric nonnegative tensor, Numerical Linear Algebra with Applications, 20 (2013), pp. 913–928.

[60] Jason Y Zien, Martine DF Schlag, and Pak K Chan, Multilevel spectral hypergraph partitioning with arbitrary vertex sizes, IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 18 (1999), pp. 1389–1399.