Optimal Status Update for Age of Information Minimization with an Energy Harvesting Source

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Abstract

In this paper, we consider a scenario where an energy harvesting sensor continuously monitors a system and sends time-stamped status updates to a destination. The destination keeps track of the system status through the received updates. We use the metric Age of Information (AoI), the time that has elapsed since the last received update was generated, to measure the “freshness” of the status information available at the destination. We assume energy arrives randomly at the sensor according to a Poisson process, and each status update consumes one unit of energy. Our objective is to design optimal online status update policies to minimize the long-term average AoI, subject to the energy causality constraint at the sensor. We consider three scenarios, i.e., the battery size is infinite, finite, and one unit only, respectively.

For the infinite battery scenario, we adopt a best-effort uniform status update policy and show that it minimizes the long-term average AoI. For the finite battery scenario, we adopt an energy-aware adaptive status update policy, and prove that it is asymptotically optimal when the battery size goes to infinity. For the last scenario where the battery size is one, we first show that within a broadly defined class of online policies, the optimal policy should have a renewal structure, i.e., the status update epochs form a renewal process, and the length of each renewal interval depends on the first energy arrival over that interval only. We then focus on a renewal interval, and prove that if the AoI in the system is below a threshold when the first energy arrives, the sensor should store the energy and hold status update until the AoI reaches the threshold; otherwise, it updates the status immediately. We analytically characterize the long-term average AoI under such a threshold-based policy, and explicitly identify the optimal threshold. Simulation results corroborate the theoretical bounds.

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I. INTRODUCTION

Enabled by the widespread wireless communications and the proliferation of ultra-low power sensors, ubiquitous sensing has profoundly changed almost every aspect of our daily lives. In many applications, such as environment monitoring [2], vehicle tracking [3], sensors are deployed to monitor the status of sensing objects, and communicate the status information to a fusion center (FC). To keep track of the status, it is desirable to keep the status information at the FC as fresh as possible. However, this is often constrained by limited physical resources, such as energy and bandwidth. In order to measure the freshness of the status updates at the FC, a metric called “Age of Information” (AoI) has been introduced in recent literature [4]. Specifically, AoI is defined as the time that has elapsed since the last received update was generated.

With this definition, AoI in various queueing systems has been analyzed, such as single-source single-server queues [4], the $M/M/1$ Last-Come First-Served (LCFS) queue with preemption in service [5], and the $M/M/1$ First-Come First-Served (FCFS) system with multiple sources [6], [7]. AoI with out-of-order packet delivery has been evaluated in [8]–[10]. A related metric, Peak Age of Information (PAoI), is introduced in [11], and has been studied in [12], [13]. In [14], Bedewy et al have identified the optimality properties of a Last Generated First Served (LGFS) service discipline when updates arrive out of order. Kam et al have shown that packet deadlines can improve AoI in [15]. AoI in the presence of errors has been evaluated in [16], and LCFS with non-memoryless gamma-distributed service times has been investigated in [17]. With knowledge of the server state, Sun et al have studied the optimal status update policy in [18].

One the other hand, optimal scheduling policies of energy harvesting (EH) sensors for various communication and sensing objectives have been extensively studied in the recent literature. Optimal transmission scheduling for throughput and delay optimization has been studied under both infinite battery setting [19]–[21] and finite battery setting [22]–[26]. With signal processing related performance metrics, such as detection delay and estimation error, optimal sensing scheduling policies have been developed to optimize the sensing performances of EH sensor networks [27]–[30]. As the newly introduced metric AoI comes into the picture, several works have begun to investigate AoI optimization in EH status updating systems. Under an EH setting, [31], [32] investigate several status update polices assuming the battery at the energy harvesting sensor is sufficiently large. It has been shown in [31] that with knowledge of the system state, updates should be submitted only when the server is free to avoid queueing delay. Moreover, a
greedy policy that submits a fresh update as the system becomes idle is shown to be inefficient; a lazy update policy that introduces inter-update delays is better. The optimal update policy remains open in this setting. In [32], under the assumption that a status update packet can be generated and served (transmitted) instantly, the authors investigate optimal offline and online policies. The optimal offline policy is to equalize the inter-update delays as much as possible, subject to the energy constraint imposed by the energy harvesting source. The online problem is cast as a Markov Decision Process in a discrete-time setting, and solved through dynamic programming. Although it is analytically intractable, the optimal policy is shown to have a threshold structure. I.e., with real-time knowledge of the energy arrival profile and its statistics, the source sends a status update if the expected AoI is above certain threshold, given it has sufficient energy. Other threshold type status update policies have been studied in [33], [34] and shown to be optimal under certain conditions recently. An offline policy to minimize AoI in a two-hop relay channel is studied in [35].

In this paper, we investigate optimal online status update policies for an energy harvesting source with various battery sizes in a continuous-time setting. Similar to [32], we assume a status update packet can be generated by the source at any time and transmitted to a FC instantly, given sufficient energy is available at the source. We assume that the energy unit is normalized so that each status update requires one unit of energy. This energy unit represents the cost of both measuring and transmitting a status packet. We assume energy arrives at the sensor according to a Poisson process, and the sensor only has causal information of the energy arrival profile in addition to the parameter of the Poisson process. Our objective is then to determine the sequence of update instants so that the long-term average AoI at the FC is minimized, subject to the energy causality constraint at the source.

We first study the properties of the time-average AoI as a function of inter-update delays, and establish a connection between this problem and the optimal sensing problem studied in [30]. This motivates us to adopt the (asymptotically) optimal sensing policies in [30] for AoI minimization, namely, a best-effort uniform status update policy for the infinite battery case, and an energy-aware adaptive status update policy for the finite battery case. Since the AoI function does not have all the properties required to establish the optimality of those policies in [30], we revise the proofs accordingly to re-establish their (asymptotic) optimality. We then study a special case where the battery size is one unit, and propose a threshold based status update policy, i.e., if the AoI in the system is below a threshold when an energy enters an empty battery, the sensor
should store the energy and hold status update until the AoI reaches the threshold; otherwise, it consumes the energy to update the status immediately. Through rigorous stochastic analysis, we show that within a broadly defined class of online policies, this threshold based status update policy is optimal.

II. System Model and Problem Formulation

Consider a scenario where an energy harvesting sensor continuously monitors a system and sends time-stamped status updates to a destination. The destination keeps track of the system status through the received updates. We use the metric Age of Information (AoI) to measure the “freshness” of the status information available at the destination.

We assume that the time used to collect and transmit a status update is negligible compared with the time scale of inter-update delays. Therefore, given sufficient energy is available at the source, a status update can be generated by the source at any time and transmitted to a FC instantly. In this case, a status update is transmitted immediately after it is generated to avoid unnecessary queueing delay.

We assume that the energy unit is normalized so that each status update requires one unit of energy. This energy unit represents the cost of both measuring and transmitting a status update. Assume energy arrives at the sensor according to a Poisson process with parameter $\lambda$. Hence, energy units arrive at discrete time instants $t_1, t_2, \ldots$. We assume $\lambda = 1$ throughout this paper for ease of exposition. The sensor is equipped with a battery with capacity $B$, $B \geq 1$. When $B = \infty$, it corresponds to the infinite battery case.

A status update policy is denoted as $\pi := \{S_n\}_{n=1}^{\infty}$, where $S_n$ is the $n$-th update epoch. We assume $S_0 = 0$, i.e., the system updates its status information right before time zero. Denote the inter-update delays as $X_n \triangleq S_n - S_{n-1}$, for $n = 1, 2, \ldots$. Then, we have $S_n = \sum_{i=1}^{n} X_i$.

Define $A(X_n)$ as the total amount of energy harvested in $[S_{n-1}, S_n)$, and $E(S_n^-)$ as the energy level of the sensor right before the scheduled updating epoch $S_n$. For a clear exposition of the paper, we assume the system has one unit amount of energy before it updates at time zero, and after that, the battery becomes empty, i.e.,

$$E(S_0^-) = 1.$$  (1)
Then, under any feasible status update policy, the energy queue evolves as follows

\[ E(S_n^-) = \min\{E(S_{n-1}^-) - 1 + A(X_n), B\} \tag{2} \]
\[ E(S_n^-) \geq 1 \tag{3} \]

for \( n = 1, 2, \ldots \). Eqn. (3) corresponds to the energy causality constraint in the system. Based on the Poisson arrival process assumption, \( A(X_n) \) is an independent Poisson random variable with parameter \( X_n \).

Under any feasible status update policy, the AoI as a function of time is shown in Fig. 1. We use \( N(T) \) to denote the number of status updates generated over \((0, T]\). Define \( R(T) \) as the total “reward”, i.e., age of information experienced by the system over \([0, T]\). Then,

\[ R(T) = \sum_{i=1}^{N(T)} X_i^2 + (T - S_{N(T)})^2 \tag{4} \]

Then, the time average AoI over the duration \([0, T]\) can be expressed as \( R(T)/T \).

Our objective is to determine the sequence of update epochs \( S_1, S_2, \ldots \), so that the time average AoI at the FC is minimized, subject to the energy causality constraint. We focus on a set of.online.policies.Π in which the information available for determining the updating epoch \( S_n \) includes the updating history \( \{S_i\}_{i=0}^{n-1} \), the energy arrival profile over \([0, S_n]\), as well as the energy harvesting statistics (i.e., \( \lambda \) in this scenario). The optimization problem can formulated

![Fig. 1: AoI as a function of \( T \). Circles represent status update instants.](image-url)
\[
\min_{\pi \in \Pi} \limsup_{T \to +\infty} \mathbb{E} \left[ \frac{R(T)}{T} \right] 
\] (5)

where the expectation in the objective function is taken over all possible energy harvesting sample paths. This problem does not admit a MDP formulation in general, and it is extremely challenging to explicitly identify the optimal solution.

### III. Optimal Status Update Policies when \( B \) is Large

In [30], we studied an optimal sensing scheduling problem. Our objective was to strategically select the sensing epochs, so that the long-term average sensing performance can be optimized. We assumed that the sensing performance over \([0, T]\) can be expressed as \( \sum_{i=1}^{N(T)} f(X_i) + f(T - N(T)) \), where \( X_i \) is the \( i \)-th inter-sensing delay. Under the assumption that 1) \( f(x) \) is convex and monotonically increasing in \( x \); 2) \( f(x)/x \) is increasing in \( x \); and 3) \( f(x)/x \) is upper bounded by a positive constant, we proposed two sensing policies, for the infinite and finite battery cases, respectively, and proved their (asymptotic) optimality.

We note that the AoI minimization problem can be treated as a particularized case of the optimal sensing scheduling problem studied in [30], by replacing the general sensing performance metric with AoI. Thus, for this particular case, \( f(x) = x^2/2 \). We note that this function exhibits the first two properties required to establish the optimality of the proposed sensing scheduling policies in [30]. However, the last condition i.e., \( f(x)/x \) is upper bounded by a positive constant, does not hold, due to the fact that \( f(x)/x = x/2 \) and it is unbounded. Therefore, the optimality of the policies proposed in [30] need to be carefully examined. In the following, we will utilize the specific form of the AoI function to bypass the last condition and reaffirm the optimality of the policies.

For the completeness of this paper, in this section, we adapt the major results and policies in [30] for the AoI minimization setup. We leave out the proofs that do not reply on the third assumption, and provide necessary new proofs only.

#### A. Status Update with Infinite Battery

When the battery size is infinite, no energy overflow will happen. Thus, the maximum achievable long-term average status update rate is one update per unit time. If we drop the energy
causality constraint, and replace it with this long-term average status update rate constraint, we obtain a lower bound on the long-term average AoI as follows:

**Lemma 1** The long-term average AoI is lower bounded by 1/2.

This lower bound corresponds to a uniform status update policy which updates once per unit time. However, it may become infeasible when the energy causality constraint is imposed. Thus, we propose the following policy to ensure the status update policy is always feasible.

**Definition 1 (Best-effort Uniform Status Update Policy)** The sensor is scheduled to update the status at \( s_n = n, n = 1, 2, \ldots \). The sensor performs the task at \( s_n \) if \( E(s_n) \geq 1 \); Otherwise, the sensor keeps silent until the next scheduled status update epoch.

Here we use \( s_n \) to denote the \( n \)-th scheduled status update epoch, which is in general different from the \( n \)-th actual status update epoch \( S_n \) since some of the scheduled status update epochs may be infeasible.

**Theorem 1** The best-effort uniform status update policy is optimal when the battery size is infinite, i.e.,

\[
\limsup_{T \to +\infty} \frac{R(T)}{T} = \frac{1}{2} \quad \text{a.s.}
\]

The proof of Theorem 1 is provided in Appendix A. Intuitively, when the battery size is infinite, the fluctuation in the energy harvesting process can be averaged out when \( T \) is sufficiently large, thus the uniform status update policy can be achieved asymptotically.

**B. Status Update with Finite Battery**

In order to minimize the long-term average AoI when the battery size is finite, intuitively, the status update policy should try to prevent any battery overflow, as wasted energy leads to performance degradation. Meanwhile, the properties of AoI require the status update rate to be as uniform as possible in time. Those two objectives are not aligned with each other, thus, the optimal status update policy should strike a balance between them.

In the following, we propose an energy-aware adaptive status update policy, which adaptively changes the update rate based on the instantaneous battery level. When the battery level is high, the sensor updates more frequently in order to prevent battery overflow; When the battery level
is low, the sensor updates less frequently to avoid infeasible status update epochs. Meanwhile, the update rate does not vary significantly in time in order to control the increase of time-average AoI caused by the jittering updating epochs.

Definition 2 (Energy-aware Adaptive Status Update Policy) Assume $B > 1$. The adaptive status update policy defines status update epochs $s_n$ recursively as follows

$$s_n = s_{n-1} + \begin{cases} \frac{1}{1-\beta}, & E(s_{n-1}^-) < \frac{B}{2} \\ 1, & E(s_{n-1}^-) = \frac{B}{2} \\ \frac{1}{1+\beta}, & E(s_{n-1}^-) > \frac{B}{2} \end{cases}$$

where $s_0 = 0$, $E(s_0^-) = 1$, and $\beta := \frac{k \log B}{B}$, with $k$ being a positive number such that $0 < \beta < 1$. The sensor samples and updates the status at $s_n$ if $E(s_n^-) \geq 1$; Otherwise, the sensor keeps silent until the next scheduled status update epoch.

As $B \to \infty$, we have $\beta \to 0$ for any fixed $k$, i.e., the adaptive status update policy converges to the best-effort uniform status update policy as battery size increases. Thus, it is reasonable to expect that the adaptive status update policy is asymptotically optimal as the battery size approaches infinity.

Theorem 2 Under the adaptive status update policy, the gap between the long-term average AoI and its lower bound $1/2$ scales in $O\left(\frac{2k+1k(\log B)^2}{B^{k+1}} + (\frac{\log B}{B})^2\right)$ almost surely.

Theorem 2 implies that as battery size $B$ increases, the long-term average AoI under the adaptive status update policy approaches $1/2$, which is the lower bound on the long-term average AoI in a system with infinite battery. Thus, it is asymptotically optimal. The proof of Theorem 2 is provided in Appendix B.

IV. A Special Case: $B = 1$

In the previous section, we investigate the optimal and asymptotically optimal status update policies when battery size $B$ is infinite, and finite but sufficiently large, respectively. However, when the battery size is so small that the asymptotics cannot kick in, those policies may not perform very well. This motivates us to investigate other status update policies when battery size $B$ is small. One extreme case for this scenario is when $B = 1$, i.e., the battery can only store the energy for one status update operation. In this case, the battery only has two states: empty,
or full. When it is empty, obviously, any status update should not be scheduled. When one unit amount of energy arrives, the battery jumps to the other state, and it then need to decide when to spend the energy for status update. Denote $\Gamma_i$ as time duration between $S_{i-1}$ and the first energy arrival time after $S_{i-1}$. Then, we have the following observations

**Lemma 2** When $B = 1$, under any feasible online policy, we must have $X_i \geq \Gamma_i, \forall i$, and $\Gamma_i$s are independent and identically distributed (i.i.d.) random variables, with common distribution $\exp(1)$.

Lemma 2 is based on the energy causality constraint, and the memoryless property of the inter-arrival times of the Poisson energy arrivals.

As defined in Sec. II, the policy space $\Pi$ includes all of the online policies which make the status updating decision based on up-to-date updating history and energy arrival profile, as well as the energy harvesting statistics. In other words, $X_i$ is a function of $\Gamma_i$, among other variables.

In order to facilitate the analysis, in the following, we focus on a special class of online policies, termed as uniformly bounded policies.

**Definition 3 (Uniformly bounded policy)** For an online policy with $\{(X_i, \Gamma_i)\}_{i=1}^{\infty}$, if $\forall i, \Gamma_i \leq X_i$, and there exists a function $g(\Gamma_i)$ such that $X_i \leq g(\Gamma_i)$, and the second moment of $g(\Gamma_i)$ is finite, then this policy is a uniformly bounded policy.

**Theorem 3** Any uniformly bounded policy is sub-optimal to a renewal policy, i.e., a policy under which the updating epochs $\{S_i\}_{i=1}^{\infty}$ form a renewal process. Besides, under the renewal policy, $X_i$ only depends on $\Gamma_i$.

The proof is provided in Appendix C. Our approach involves two steps of averaging. The first step of averaging is in the space of status update sample paths under a given uniformly bounded policy. For each fixed $i$ and $\tau$, we group all of the sample paths with $\Gamma_i = \tau$, and obtain the corresponding average inter-update delay $X_i(\tau)$. This step essentially averages out all factors that may affect $X_i$ other than $\Gamma_i$. The second step is to do an averaging in the temporal domain. For each fixed $\tau$, we form a sophisticated linear combination of involved $X_i(\tau)$s, and use it as the inter-update delay under the new policy. Such a policy is a renewal policy, it is always feasible, and each renewal interval only depends on $\tau$. Through rigorous stochastic analysis, we prove that the new renewal policy always outperforms the original policy in terms of time-average AoI.
In the following, we will focus on renewal policies, and show that the optimal renewal policy has a threshold structure.

**Theorem 4** In the class of renewal policies, the optimal policy must have a threshold structure, i.e., $X_i$ equals constant $\tau_0$ if $\Gamma_i \leq \tau_0$; otherwise $X_i$ equals $\Gamma_i$.

The proof of Theorem 4 is provided in Appendix D. The proof is sketched as follows: We note that the long-term average AoI under a renewal policy can be expressed in the form of $\frac{E[X^2]}{2E[X]}$, where $X$ is a random variable with the same distribution of $X_i$. The objective is then to identify the optimal $X$ as a function of $\tau$, which follows the same as $\Gamma_i$. We first establish a necessary condition for $X(\tau)$ to be optimal, i.e., if we add a small perturbation on $X(\tau)$ without violating the energy causality condition, the objective function can only increase. Such a necessary condition indicates that $X(\tau)$ must have the threshold structure described in Theorem 4.

We propose the following threshold-based status update policy based on Theorem 4.

**Definition 4 (Threshold-based Status Update Policy)** When an energy unit enters an empty battery, the sensor performs a status update immediately if the AoI at the FC is greater than a threshold $\tau_0$; Otherwise, it holds its operation until the AoI is exactly equal to $\tau_0$.

The long-term average AoI under this policy can be analytically characterized based on the memoryless property of the exponentially distributed inter-arrival times of energy units.

**Theorem 5** Under the threshold-based status update policy, the long-term average AoI is $h(\tau_0) \triangleq \frac{2\tau_0 e^{-\tau_0} + 2e^{-\tau_0} + \tau_0^2}{2(e^{-\tau_0} + \tau_0)}$.

The proof of Theorem 5 can be found in Appendix E. Besides, by taking derivative of $h(\tau_0)$, we can show that $h(\tau_0)$ is first decreasing, then increasing in $\tau_0$. Therefore, the optimal $\tau_0$ corresponds to the point where $h'(\tau_0) = 0$. Solving the equation, we have $\tau_0^* = 0.901$, and the corresponding long-term average AoI is 0.9012.

V. Simulation Results

The performances of the proposed status update policies are evaluated in this section through simulations.

First, we fix the battery size $B = \infty$. We generate sample paths for the Poisson energy harvesting process, and perform status updating according to the best-effort uniform status update
The time average AoI as a function of $T$ is shown in Fig. 2. We plot one sample path and the sample average over 1,000 sample paths in the figure. We observe that both curves gradually approach the lower bound $1/2$ as $T$ increases. When $T = 500$, there is only a very small difference between the simulation results and the analytical lower bound. The results indicate that the proposed best-effort uniform status update policy is optimal.

Next, we study the time average AoI under the adaptive status update policy with finite battery sizes. We fix $T = 100,000$ and plot the average AoI over 1,000 sample paths in Fig. 3. We note that for each fixed $k$, the gap between the time average AoI and the lower bound $1/2$ monotonically decreases as $B$ increases, which is consistent with Theorem 2.

Last, we compare the performances of the three policies for $B = 1$. For a fair comparison, we optimize the parameters for the best-effort uniform status update policy and the adaptive status update policy numerically before we perform the comparison. We note that the optimal update rate for the best-effort uniform policy is once every $0.43$ unit of time. We also modify the adaptive status update policy to make it applicable for the case $B = 1$. Specifically, we schedule the next update $\frac{1}{1+\beta}$ away if the battery level is full right before the current update; otherwise, we schedule it in time $\frac{1}{1-\beta}$. We numerically search for the optimal value of $\beta$, and it turns out that when $\beta = -0.145$, the time-average AoI is minimized. This is opposite to the case when $B$ is large but finite. Although it is a bit counter intuitive, it is due to the memoryless property of the inter-arrival times of a Poisson process, i.e., the expected waiting time for the next energy arrival keeps unchanged after current scheduled update epoch, regardless of its feasibility. If $B = 1$ at
current scheduled update epoch, the battery will become empty immediately after it updates the status, and the AoI will then linearly grow from zero; If $B = 0$, the AoI has a positive value already, and will grow with the same rate. Thus, in order to balance the inter-update delays to minimize the time average AoI, the system should be more aggressive to update if the current scheduled update is infeasible. We then generate a sample path and plot the time average AoI as a function of time $T$ under each policy, as shown in Fig. 4 The corresponding sample-path average over 1,000 sample paths is plotted in Fig. 5 As we expect, for both scenarios, the threshold based updating policy outperforms the other two policies, and approaches its limit as
$T$ gets sufficiently large.

VI. CONCLUSIONS

In this paper, we investigated optimal status update policies for an energy harvesting source equipped with a battery. We considered three different cases, namely, the battery size is infinite, finite but large, and one unit. We showed that the best-effort uniform status update policy minimizes the long-term average AoI when the battery size is infinite. We then showed that the energy-aware adaptive status update policy is asymptotically optimal when the battery size is finite but sufficiently large. At last, we considered an extreme case when the battery size is one unit. We introduced the definition of uniformly bounded online policies, and explicitly identified the optimal policy within this class. Under the optimal policy, the status update epochs form a renewal process, where the length of each renewal interval relies on the first energy arrival in that interval, and has a threshold structure. We also evaluated the performances of the proposed policies through simulation.

We point out that the (asymptotically) optimal status update policies for the infinite battery and finite battery cases are closely related to our earlier work [30]. In [30], our objective was to optimize the long-term average sensing performance, which was measured by a general function of the inter-sensing delays. The average AoI as a function of the inter-update delays can be treated as a particularized case of that general function, thus the optimal policies and the corresponding analysis in both works are quite similar. Such inherent connection between time average AoI minimization and more general sensing performance optimization also implies
that AoI as a metric of information freshness does have deep implications on other performance metrics in sensing/status updating systems. Unveiling the intricate relationship between AoI and other performance metrics (i.e., estimation error) from an information theoretic perspective is one of our future directions.

APPENDIX

A. Proof of Theorem 1

To prove Theorem 1, it suffices to show that \( \limsup_{T \to +\infty} \frac{R(T)}{T} \leq 1/2 \) almost surely.

The uniform best-effort status update policy partitions the time axis into slots, each with length 1. Let \( E(n^-) \) be the energy level of the sensor right before the scheduled status update at time \( n \). Based on \( E(n^-) \), we can group the time slots into intervals labeled as \( v_1, u_1, \ldots, v_k, u_k, \ldots \), where \( u_i \) corresponds to the \( i \)-th interval that begins with \( E(n^-) = 0 \) for some \( n \) and ends when \( E(n^-) \) becomes positive as \( n \) increases; \( v_i \) corresponds to the \( i \)-th interval that begins with \( E(n^-) > 0 \) for some \( n \) and ends when \( E(n^-) \) becomes zero as \( n \) increases. Note that we assume one unit energy is available at time 0, i.e., \( E(0^-) = 1 \).

We note that \( E(n^-) \) jumps from zero to some positive value at the end of \( u_i \), due to random energy arrivals over the last time slot in \( u_i \). Based on assumption that the energy arrivals follow a Poisson process, the length of \( u_i \) follows an independent geometric distribution where

\[
P[u_i = k] = e^{-(k-1)}(1 - e^{-1}), \quad k = 1, 2, \ldots
\]  

With a bit abuse of notation, in (7), and in the following proofs, we use \( u_i \) to denote the length of the interval labeled as \( u_i \); similarly for \( v_i \).

Over the interval labeled as \( v_i \), all of the scheduled status update epochs are feasible, except for the last one bounding \( v_i \). Considering the duration bounded by the first and last feasible status update epochs over \( v_i \), the aggregated AoI equals \( (v_i - 1)f(1) \), where \( f(x) = x^2/2 \). Since all of the scheduled status updating epochs over \( u_i \) are infeasible except for the last one (which is also the first feasible status update epoch over \( v_{i+1} \)), the aggregated AoI over the duration bounded by the last feasible status update epoch over \( v_i \) and the first feasible status update epoch over \( v_{i+1} \) is \( f(u_i + 1) \).
Let $K(T)$ be the number of $u_i$s over $[0, T]$. Then the number of $v_i$s over $[0, T]$ is either $K(T)$ or $K(T) + 1$, depending on whether time $T - 1$ is a feasible update epoch or not. Therefore,

$$\limsup_{T \to +\infty} \frac{R(T)}{T} = \limsup_{T \to +\infty} \frac{\sum_{i=1}^{K(T)} f(u_i + 1)}{T} + \frac{T - \sum_{i=0}^{K(T)} (u_i + 1) f(1)}{T}$$

$$= \limsup_{T \to +\infty} \frac{\sum_{i=1}^{K(T)} (u_i + 1)^2}{2T} + \frac{1}{2} - \frac{\sum_{i=1}^{K(T)} u_i}{2T} - \frac{K(T)}{2T}$$

$$= \limsup_{T \to +\infty} \left( \frac{\sum_{i=1}^{K(T)} u_i^2}{2K(T)} + \frac{\sum_{i=1}^{K(T)} u_i}{K(T)} + 1 \right) \frac{K(T)}{T} + \frac{1}{2}$$

(9)

where (28) follows from the definition of $v_i$ and $u_i$, (9) follows from the results that $K(T)/T \to 0$ and $\sum_{i=1}^{K(T)} u_i/T \to 0$ almost surely, as proved in the proof of Theorem 1 in [30]. Since $u_i$’s are i.i.d. geometric random variables, $\sum_{i=1}^{K(T)} u_i$ and $\sum_{i=1}^{K(T)} u_i^2$ converges to the first and second moments of the geometric distribution specified in (7), which are finite constants. Therefore, we have (9) converges to $1/2$ almost surely.

**B. Proof of Theorem 2**

Consider the first $n$ scheduled status update epochs under the proposed adaptive status update policy for a sample path of the energy harvesting process. Let $n_+$ denote the number of intervals between two scheduled status updating epochs with duration $\frac{1}{1-\beta}$, $n_-$ be that with duration $\frac{1}{1+\beta}$, and $n_0$ be that with duration 1. Let $\bar{n}$ be the number of status updating epochs the battery overflows, and $\underline{n}$ be the number of infeasible status update epochs. Then, the $n$-th scheduled status update epoch happens at time $T_n := \frac{n_+}{1-\beta} + n_0 + \frac{n_-}{1+\beta}$. Let $A_n^+$ be the total amount of energy wasted. Then,

$$E(T_n^-) = (A(T_n) - A_n^+) - (n - \underline{n})$$

(10)

where $A(T_n)$ is a Poisson random variable with parameter $T_n$. Dividing both sides by $n$ and taking the limit as $n$ goes to $+\infty$, we have

$$\lim_{n \to \infty} \frac{E(T_n^-)}{n} = \lim_{n \to \infty} \frac{A(T_n)}{T_n} \cdot \frac{T_n}{n} - \lim_{n \to \infty} \frac{A_n^+}{n} - \left(1 - \lim_{n \to \infty} \frac{n}{n}\right)$$
According to Theorem 3 in [30], for almost every sample path,

\[
\lim_{n \to \infty} \frac{A^+_n}{n} = O \left( \frac{(2^{k+1}k(\log B)^2)}{B^{k+1}} \right) \quad (11)
\]

\[
\lim_{n \to \infty} \frac{\bar{n}}{n} = O \left( \frac{(2^{k+1}k(\log B)^2)}{B^{k+1}} \right) \quad (12)
\]

Combining with the fact that \( \lim_{n \to \infty} \frac{E(T_n)}{n} = 0 \) and \( \lim_{n \to \infty} \frac{A(T_n)}{T_n} = 1 \), we have

\[
\lim_{n \to \infty} \frac{T_n}{n} = 1 + O \left( \frac{(2^{k+1}k(\log B)^2)}{B^{k+1}} \right) \quad (13)
\]

Based on Taylor expansion and (13), we have

\[
\lim_{n \to \infty} \frac{n}{T_n} f \left( \frac{1}{1-\beta} \right) + n_0 f (1) + n_- f \left( \frac{1}{1+\beta} \right) = f(1) + O \left( \frac{2^{k+1}k(\log B)^2}{B^{k+1}} + \left( \frac{\log B}{B} \right)^2 \right)
\]

On the other hand, due to the existence of infeasible status update epochs, we have

\[
\lim_{n \to \infty} \frac{\sum_{n: X_n > \frac{1}{1-\beta}} f(X_n)}{T_n} \leq \lim_{n \to \infty} \frac{\sum_{n: X_n > \frac{1}{1-\beta}} X_n}{2\bar{n}} \frac{\bar{n}}{T_n} = \frac{\mathbb{E} \left[ \bar{X}_n^2 \right]}{(1-\beta)^2} = \frac{2-p}{2p^2(1-\beta)^2} \quad (14)
\]

where the inequality in (14) follows from the fact that \( X_n \) differs from the delay between two consecutive scheduled status update epochs only when battery outage happens, and \( \bar{n} \) denote the number of \( X_n \)'s with \( X_n > \frac{1}{1-\beta} \).

We note that for all \( X_n \geq \frac{1}{1-\beta} \), \( \bar{X}_n \triangleq X_n(1-\beta) \) follows a geometric distribution with parameter \( p \triangleq 1 - e^{-\frac{1}{1-\beta}} \), and its second moment is \( \frac{2-p}{p^2} \). Then,

\[
\lim_{n \to \infty} \frac{\sum_{n: X_n > \frac{1}{1-\beta}} X_n^2}{2\bar{n}} = \lim_{n \to \infty} \frac{\sum_{n: X_n > \frac{1}{1-\beta}} (\bar{X}_n)^2}{2(1-\beta)^2\bar{n}} = \frac{\mathbb{E} \left[ \bar{X}_n^2 \right]}{(1-\beta)^2} = \frac{2-p}{2p^2(1-\beta)^2} \quad (15)
\]

Meanwhile, we have \( \lim_{n \to \infty} \frac{\bar{n}}{T_n} \leq \lim_{n \to \infty} \frac{\bar{n}}{T_n} \). Combining with (12)(14), we have

\[
\lim_{n \to \infty} \frac{\sum_{n} f(X_n)}{T_n} = \frac{1}{2} + O \left( \frac{(2^{k+1}k(\log B)^2)}{B^{k+1}} + \left( \frac{\log B}{B} \right)^2 \right)
\]
C. Proof of Theorem

Let \( \{S_i\}_{i=1}^{\infty} \) be the status update epochs under a uniformly bounded policy, and \( \{X_i\}_{i=1}^{\infty} \) be the corresponding inter-update delays. Based on the definition of \( R(T) \) in (4), we have

\[
\frac{R(S_{N(T)})}{T} \leq \frac{R(T)}{T} \leq \frac{R(S_{N(T)+1})}{T}
\]

Thus,

\[
\mathbb{E}\left[\frac{R(T)}{T}\right] \geq \mathbb{E}\left[\frac{R(S_{N(T)})}{T}\right] = \mathbb{E}\left[\frac{R(S_{N(T)+1})}{T}\right] - \mathbb{E}\left[\frac{X^2_{N(T)+1}}{2T}\right]
\]

We aim to show that 1) \( \mathbb{E}\left[\frac{X^2_{N(T)+1}}{2T}\right] \to 0 \), and 2) \( \mathbb{E}\left[\frac{R(S_{N(T)+1})}{T}\right] \) is suboptimal to a renewal policy. In the following, we will show them separately.

1) \( \mathbb{E}\left[\frac{X^2_{N(T)+1}}{2T}\right] \to 0 \): First, we denote \( F_n(t) \) as the cumulative distribution function (cdf) of \( S_n \) under the uniformly bound policy, and \( N(t) \) be the total number of updates over \((0, t]\). Then, we have

\[
\mathbb{E}[N(t)] = \sum_{n=0}^{\infty} F_n(t)
\]

Next, we note that

\[
\mathbb{E}\left[X^2_{n+1} 1_{S_n+1 > T} \mid S_n = t\right]
\]

\[
= \mathbb{E}\left[X^2_{n+1} 1_{X_{n+1} > T-t} \mid S_n = t\right]
\]

\[
\leq \mathbb{E}_{\Gamma_{n+1}} \left[g^2(\Gamma_{n+1}) 1_{g(\Gamma_{n+1}) > T-t} \mid S_n = t\right]
\]

\[
= \mathbb{E}_{\Gamma_{n+1}} \left[g^2(\Gamma_{n+1}) 1_{g(\Gamma_{n+1}) > T-t} \right] \triangleq G(T - t)
\]

where (22) follows from the definition of uniformly bounded policy, and (23) follows from the memoryless property of the inter-arrival time of a Poisson process.
Therefore, by first conditioning on the time of the last renewal prior to (or at) time \( t \), we have

\[
\mathbb{E} \left[ X_{N(T)+1}^2 \right] = \sum_{n=0}^{\infty} \int_0^T \mathbb{E} \left[ X_{n+1}^2 1_{S_{n+1} > T} \mid S_n = t \right] dF_n(t) 
\]

\[
\leq \int_0^T G(T-t) d\left( \sum_{n=0}^{\infty} F_n(t) \right)  
\]

\[
= \int_0^T G(T-t) d\mathbb{E}[N(t)]  
\]

(24)

where (25) follows from (23), and (26) follows from (19).

For any fixed \( 0 \leq \Delta \leq T \), we have

\[
\frac{1}{T} \int_0^T G(T-t) d\mathbb{E}[N(t)] = \frac{1}{T} \int_0^{T-\Delta} G(T-t) d\mathbb{E}[N(t)] + \frac{1}{T} \int_{T-\Delta}^T G(T-t) d\mathbb{E}[N(t)] 
\]

\[
\leq G(\Delta) \frac{\mathbb{E}[N(T-\Delta)]}{T} + G(0) \frac{\mathbb{E}[N(T)] - \mathbb{E}[N(N-\Delta)]]}{T}  
\]

(27)

where (28) follows from the fact that \( G(t) \) is monotonically decreasing in \( t \).

Note that \( A(t) \) is defined as the total number of energy arrivals over \([0, t]\), which upper bounds the total number of status updates over \((0, t)\), i.e., \( N(t) \), due to energy causality constraint. Thus,

\[
N(T) - N(T-\Delta) = N(T) - (N(T-\Delta) + 1) + 1  
\]

\[
\leq A(S_{N(T)}) - A(S_{N(T-\Delta)} + 1)  
\]

\[
\leq A(T) - A(T - \Delta) + 1  
\]

(29)

(30)

(31)

under each status update sample path. Plugging in (28) and letting \( T \to \infty \), we have

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T G(T-t) d\mathbb{E}[N(t)] \leq \lim_{T \to \infty} G(\Delta) \frac{T - \Delta}{T} + G(0) \frac{\Delta + 1}{T}  
\]

(32)

\[
= G(\Delta)  
\]

(33)

where (33) holds for any \( \Delta \geq 0 \), due to the assumption in Definition 3 that \( \mathbb{E}[g^2(\Gamma_i)] \) is bounded. Since \( \lim_{\Delta \to 0} G(\Delta) = 0 \), we have \( \mathbb{E} \left[ \frac{X_{N(T)+1}^2}{2T} \right] \to 0 \) as \( T \to \infty \).

2) \( \mathbb{E} \left[ \frac{R(S_{N(T)+1})}{T} \right] \) is sub-optimal to a renewal policy: For any given uniformly bounded policy, we will construct a renewal policy as follows: For all of the status update sample paths under the given uniformly bounded policy, we will group those with \( S_i \geq T \) based on the value of \( \Gamma_i \),
and find the corresponding average inter-update delay $X_i$. Specifically, we define

$$\hat{X}_i^T(\tau) \triangleq \mathbb{E}[X_i|S_{i-1} \leq T, \Gamma_i = \tau]$$  \hspace{1cm} (34)$$

Since each $X_i \geq \Gamma_i$ under the given policy, we have $\hat{X}_i^T(\tau) \geq \tau$, and it depends only on $\tau$. Besides, we have the following observation:

**Lemma 3** For any fixed $T > 0$,

$$\mathbb{E}[X_i 1\{i \leq N(T) + 1\}] = \mathbb{E}_\tau[\hat{X}_i^T(\tau)] \cdot \mathbb{E}[1\{i \leq N(T) + 1\}]$$  \hspace{1cm} (35)$$

**Proof:** Based on the property of conditional expectation, we have

$$\hat{X}_i^T(\tau) = \frac{\mathbb{E}[X_i 1\{S_{i-1} \leq T\} | \Gamma_i = \tau]}{\mathbb{P}[S_{i-1} \leq T | \Gamma_i = \tau]}$$  \hspace{1cm} (36)$$
$$= \frac{\mathbb{E}[X_i 1\{i \leq N(T) + 1\} | \Gamma_i = \tau]}{\mathbb{E}[1\{i \leq N(T) + 1\}]}$$  \hspace{1cm} (37)$$

where $1\{\mathcal{E}\}$ in (36) is an indicator function, which takes value 1 if event $\mathcal{E}$ is true; otherwise, it equals 0. (37) follows from the fact that events $S_{i-1} \leq T$ and $i \leq N(T) + 1$ are equivalent, and

$$\mathbb{P}[S_{i-1} \leq T | \Gamma_i = \tau] = \mathbb{P}[S_{i-1} \leq T] = \mathbb{E}[1\{i \leq N(T) + 1\}]$$  \hspace{1cm} (38)$$

The Lemma is proved after taking expectation of both sides of (37) with respect to $\tau$. 

Next, we will construct a renewal policy based on the definition of $\hat{X}_i^T(\tau)$. Define

$$\rho_i^T \triangleq \frac{\mathbb{E}[1\{i \leq N(T) + 1\}]}{\mathbb{E}[N(T) + 1]}$$  \hspace{1cm} (39)$$
$$\tilde{X}_T(\tau) \triangleq \sum_{i=1}^{\infty} \rho_i^T \hat{X}_i^T(\tau)$$  \hspace{1cm} (40)$$

Then, we have the following observations.

**Proposition 1** For any fixed $T \geq 0$, $\{\rho_i^T\}_{i=1}^{\infty}$ is a valid distribution.

This proposition can be proved based on the facts that $\rho_i^T \geq 0$, and $\sum_{i=1}^{\infty} \rho_i^T = 1$.

**Proposition 2** For any fixed $T \geq 0$, $\tilde{X}_T(\tau) \geq \tau$, and it depends on $\tau$ only.

This proposition is due to $\hat{X}_i^T(\tau) \geq \tau$, and it depends only on $\tau$, as well as Proposition 1. Proposition 2 indicates that if we define a status update policy such that the corresponding inter-
update delay is determined by the delay between the last status update epoch and the first energy arrival time after that according to $\bar{X}_T(\tau)$, then, the corresponding policy always satisfies the energy causality constraint, and the inter-update delays over $[0, T]$ are independent and identically distributed, thus it is a renewal policy over $[0, T]$.

With a little abuse of notation, in the following, we use $\tau$ to denote a random variable that has the same distribution as $\Gamma_i$.

**Lemma 4** For any fixed $T > 0$,

$$
\frac{\mathbb{E}[N(T) + 1]}{T} \geq \frac{1}{\mathbb{E}_\tau[\bar{X}_T(\tau)]} = \frac{1}{\sum_{i=1}^{\infty} \rho^T_i \mathbb{E}_\tau[\hat{X}_T^T(\tau)]}.
$$

**(Proof):** First, we note that $S_{N(T)+1} := \sum_{i=1}^{N(T)+1} X_i \geq T$. Thus, we have

$$
1 \leq \mathbb{E} \left[ \frac{S_{N(T)+1}}{T} \right]
$$

$$
= \frac{1}{T} \mathbb{E} \left[ \sum_{i=1}^{\infty} X_i \cdot 1\{i \leq N(T) + 1\} \right]
$$

$$
= \frac{1}{T} \sum_{i=1}^{\infty} \mathbb{E} [X_i \cdot 1\{i \leq N(T) + 1\}]
$$

$$
= \frac{1}{T} \sum_{i=1}^{\infty} \mathbb{E}_\tau[\hat{X}_T^i(\tau)] \cdot \mathbb{E} [1\{i \leq N(T) + 1\}]
$$

$$
= \left( \sum_{i=1}^{\infty} \rho_i^T \mathbb{E}_\tau[\hat{X}_T^i(\tau)] \right) \cdot \frac{\mathbb{E}[N(T) + 1]}{T}
$$

$$
= \mathbb{E}_\tau[\bar{X}_T(\tau)] \cdot \frac{\mathbb{E}[N(T) + 1]}{T}
$$

where we switch the order of summation and expectation in (44) since $X_i \geq 0$, (45) follows from Lemma 3, (46) follows from the definitions of $\rho_i^T$ in (39), and (47) follows from the definition of $\bar{X}_T$ in (40). Dividing $\mathbb{E}_\tau[\bar{X}_T(\tau)]$ on both sides of (47), we have (41) proved.

**Lemma 5** Under the uniformly bounded policy, we have

$$
(\mathbb{E}[X_i \cdot 1\{i \leq N(T) + 1\}|\Gamma_i = \tau])^2 \leq \mathbb{E}[X_i^2 \cdot 1\{i \leq N(T) + 1\}|\Gamma_i = \tau] \cdot \mathbb{E}[1\{i \leq N(T) + 1\}]
$$

(48)
Proof: Based on Cauchy-Schwarz inequality, we have

\[
\left( \mathbb{E}[X_i \cdot 1\{i \leq N(T) + 1\} | \Gamma_i = \tau] \right)^2 \\
\leq \mathbb{E}[X_i^2 \cdot (1\{i \leq N(T) + 1\})^2 | \Gamma_i = \tau] \cdot \mathbb{E}[(1\{i \leq N(T) + 1\})^2 | \Gamma_i = \tau]
\]  \tag{49}

Eqn. (48) then follows from the fact that 1\{i \leq N(T) + 1\} is independent with \Gamma_i. ■

Last, we will show that the corresponding renewal policy always outperforms the original uniformly bounded policy in terms of AoI.

\[
\mathbb{E} \left[ \frac{R(S_{N(T)+1})}{T} \right] = \frac{1}{2T} \mathbb{E} \left[ \sum_{i=1}^{\infty} X_i^2 \cdot 1\{i \leq N(T) + 1\} \right] \\
= \frac{1}{2T} \sum_{i=1}^{\infty} \mathbb{E} \mathbb{E}[X_i^2 \cdot 1\{i \leq N(T) + 1\} | \Gamma_i = \tau] \\
\geq \frac{1}{2T} \sum_{i=1}^{\infty} \mathbb{E} \left[ (\mathbb{E}[X_i \cdot 1\{i \leq N(T) + 1\} | \Gamma_i = \tau])^2 \right] \\
= \frac{1}{2T} \sum_{i=1}^{\infty} \mathbb{E} \left[ \left( \hat{X}_i^T(\tau) \right)^2 \right] \mathbb{E}[1\{i \leq N(T) + 1\}] \\
= \sum_{i=1}^{\infty} \mathbb{E} \left[ \left( \hat{X}_i^T(\tau) \right)^2 \right] \frac{\rho_i \cdot \mathbb{E}[N(T) + 1]}{2T} \tag{50}
\]

\[
\frac{\mathbb{E} \left[ \hat{X}_i^T(\tau) \right]}{2T} \geq \mathbb{E}_r \left[ \left( \sum_{i=1}^{\infty} \hat{X}_i^T(\tau) \right) \rho_i \cdot \mathbb{E}[N(T) + 1] \right] \geq \frac{\mathbb{E}_r[X_i^2(\tau)]}{2} \tag{51}
\]

where (52) follows from the Lemma 5, (53) follows from Lemma 3, (55) follows from Jensen’s inequality and Proposition 1. Combining with Lemma 4, we have (56), which is greater than or equal to (57), the minimum long-term average AoI of the optimal renewal policy. We use \Pi' to denote the set of feasible renewal policies under which \(X_i\) only depends on \(\Gamma_i\). Since the inequality holds for every \(T\), we have

\[
\limsup_{T \to \infty} \mathbb{E} \left[ \frac{R(T)}{T} \right] \geq \limsup_{T \to \infty} \mathbb{E} \left[ \frac{R(S_{N(T)+1})}{T} \right] \geq \min_{X(\tau) \in \Pi'} \frac{\mathbb{E}_r[X^2(\tau)]}{2} \tag{58}
\]
D. Proof of Theorem 4

Based on Theorem 3, we assume the inter-update delays under a renewal policy is a function of $\tau$, the duration between the last update epoch and the first energy arrival after it. Specifically, we denote the function as

$$X(\tau) = \tau + l(\tau)$$

(59)

where $l(\tau) \geq 0$. Since $X(\tau) \geq \tau$, such a policy is always feasible. Assume $X(\tau)$ is measurable. Then, to minimize the long-term average AoI is equivalent to

$$\min_{l(\tau)} \frac{\mathbb{E}_{\tau}[X^2(\tau)]}{2\mathbb{E}_{\tau}[X(\tau)]} \quad \text{s.t.} \quad l(\tau) \geq 0, \forall \tau$$

(60)

Based on the assumption that the energy arrival process is Poisson with $\lambda = 1$, $\tau$ is an exponential random variable with rate 1. Since continuous function is dense in the space of measurable functions, it suffices to show that a continuous $X(\tau)$ is suboptimal to a threshold policy.

**Definition 5 (Admissible Direction)** $\delta(\tau)$ is an admissible direction at $X(\tau)$ if 1) $\mathbb{E}_{\tau}[\delta(\tau)] = 0$; and 2) $\exists \epsilon > 0, X(\tau) + s\delta(\tau) \geq \tau$ for $s \in [0, \epsilon)$.

**Lemma 6** If $X(\tau)$ is optimal, then for any admissible direction $\delta(\tau)$,

$$\mathbb{E}[X(\tau)\delta(\tau)] \geq 0$$

(61)

**Proof:** For the optimal $X(\tau)$, construct a new policy as $X'(\tau) \triangleq X(\tau) + s\delta(\tau)$, where $\delta(\tau)$ is an admissible direction and $s \in [0, \epsilon]$. Then, the difference between the time average AoI under the new policy and that under the optimal policy is

$$\frac{\mathbb{E}_{\tau}[X'(\tau)]^2}{2\mathbb{E}_{\tau}[X'(\tau)]} - \frac{\mathbb{E}[X^2(\tau)]}{2\mathbb{E}[X(\tau)]}$$

(62)

$$= \frac{\mathbb{E}[X^2(\tau)] + 2s\mathbb{E}[X(\tau)\delta(\tau)] + s^2\mathbb{E}[\delta^2(t)]}{2\mathbb{E}[\alpha(\tau) + s\delta(\tau)]} - \frac{\mathbb{E}[X^2(\tau)]}{2\mathbb{E}[X(\tau)]}$$

(63)

$$= \frac{2s\mathbb{E}[X(\tau)\delta(\tau)] + s^2\mathbb{E}[\delta^2(t)]}{2\mathbb{E}[X(\tau)]}$$

(64)

which should be greater than or equal to zero for every $s \in [0, \epsilon]$. Thus

$$\lim_{s \to 0} \frac{2s\mathbb{E}[X(\tau)\delta(\tau)] + s^2\mathbb{E}[\delta^2(t)]}{2s\mathbb{E}[X(\tau)]} = \frac{\mathbb{E}[X(\tau)\delta(\tau)]}{\mathbb{E}[X(\tau)]} \geq 0$$

(65)
which implies that $\mathbb{E}[X(\tau)\delta(\tau)] \geq 0$. ■

Assume that under the optimal policy, we can find a point $a$ where $X(a) > a$. Since $X(\tau)$ is continuous, there must exist $\Delta > 0$, such that $X(\tau) > \tau$ for $\forall \tau \in [a, a + \Delta]$. Find another point $b \geq 0$ such that $[a, a + \Delta]$ and $[b, b + \Delta]$ do not overlap. Define

$$
\delta_\Delta(\tau) = \begin{cases} 
-\int_a^{a+\Delta} \frac{1}{p_r(t)dt} & \tau \in (a, a + \Delta) \\
\int_b^{b+\Delta} \frac{1}{p_r(t)dt} & \tau \in (b, b + \Delta)
\end{cases}
$$

(66)

where $p_r(t)$ is the probability density function of $\tau$. We can verify that $\delta_\Delta(\tau)$ is an admissible direction for every $\Delta > 0$.

Let $\Delta \to 0$. Then $\delta_{\Delta \to 0}(\tau) = \delta(\tau - b) - \delta(\tau - a)$ where $\delta(\tau)$ is the Dirac delta function with respect to $p_r$. Therefore (61) gives

$$
\mathbb{E}[X(\tau)\delta_{\Delta \to 0}(\tau)] = \mathbb{E}[X(\tau)(\delta(\tau - b) - \delta(\tau - a)))] = X(b) - X(a) \geq 0
$$

(67)

It implies the following necessary condition for $X(\tau)$ to be optimal: for every point $a$ where $X(a) \neq a$, we must have $X(a)$ no larger than any other $X(b)$. I.e., $X(a)$ either equals $a$, or equals $\inf_\tau \{X(\tau)\}$. Therefore, $X(\tau)$ must be a function in the following form

$$
X(\tau) = \begin{cases} 
\tau_0, & \tau \in (0, \tau_0) \\
\tau, & \tau \geq \tau_0
\end{cases}
$$

(68)

which corresponds to a threshold policy.

E. Proof of Theorem 5

Under the threshold-based status update policy, the status update epochs form a renewal process, and $X_n$s are i.i.d random variables. Based on the memoryless property of the inter-arrival
time for Poisson processes, $\Gamma_i$ is an exponential random variable with parameter 1. Therefore,

$$E[X_i] = E[X_i | \Gamma_i > \tau_0] P[\Gamma_i > \tau_0] + E[X_i | \Gamma_i \leq \tau_0] P[\Gamma_i \leq \tau_0]$$

$$= \int_{\tau_0}^{+\infty} \tau e^{-\tau} d\tau + \tau_0 (1 - e^{-\tau_0})$$

$$= (1 + \tau_0)e^{-\tau_0} + \tau(1 - e^{-\tau_0}) = e^{-\tau_0} + \tau_0$$

(69)

$$E[X_i^2] = E[X_i^2 | \Gamma_i > \tau_0] P[\Gamma_i > \tau_0] + E[X_i^2 | \Gamma_i \leq \tau_0] P[\Gamma_i \leq \tau_0]$$

$$= \int_{\tau_0}^{+\infty} \tau^2 e^{-\tau} d\tau + \tau_0^2 (1 - e^{-\tau_0})$$

$$= (\tau_0^2 + 2\tau_0 + 2)e^{-\tau_0} + \tau_0^2 (1 - e^{-\tau_0})$$

(70)

Thus, based on the property of renewal processes \cite{36}, we have

$$\lim_{T \to +\infty} \frac{R(T)}{T} = \frac{E[X_i^2]}{2E[X_i]} = \frac{(2\tau_0 + 2)e^{-\tau_0} + \tau_0^2}{2(e^{-\tau_0} + \tau_0)}$$

(71)

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