VALUES OF PAIRS INVOLVING ONE QUADRATIC FORM AND ONE LINEAR FORM AT S-INTEGRAL POINTS

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Abstract. We prove the existence of $S$-integral solutions of simultaneous diophantine inequalities for pairs $(Q, L)$ involving one quadratic form and one linear form satisfying some arithmetico-geometric conditions. The proof uses strong approximation in algebraic groups and Ratner’s topological rigidity of unipotent actions on homogeneous spaces.

1. Introduction

The theory of unipotent flows on homogeneous spaces is a powerful tool used to solve many difficult problems in number theory and more particularly in diophantine approximation. One of the great achievement of those so-called dynamical methods is the proof made by G.A. Margulis of the Oppenheim conjecture: Let $Q$ be a nondegenerate indefinite real quadratic form in $n \geq 3$ variables which is not proportional to a form with rational coefficients then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$. A similar Oppenheim type problem concerns the existence of integral solutions of simultaneous diophantine inequalities involving one quadratic form and one linear form. More precisely given a pair $(Q, L)$ and $(a, b) \in \mathbb{R}^2$ the problem is to find sufficient conditions which guarantees the existence of a nonzero integral vector in $x \in \mathbb{Z}^n$ such that

(A) For any $\varepsilon > 0$ one has simultaneously $|Q(x) - a| < \varepsilon$ and $|L(x) - b| < \varepsilon$

This condition is equivalent to ask the density of the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^n\}$ in $\mathbb{R}^2$. The first result in that direction is due to S.G Dani and G.A. Margulis [DM90] and concerns the dimension 3 for a pair $(Q, L)$ consisting of one nondegenerate indefinite quadratic form and a nonzero linear form in dimension 3 such that the cone $\{Q = 0\}$ intersects tangentially the plane $\{L = 0\}$ and no linear combinaison of $Q$ and $L^2$ is rational. Under those conditions they proved using the original method used to prove the Oppenheim conjecture that the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^3\}$ is dense in $\mathbb{R}^2$. In higher dimension, the density for pairs holds if one replaces the previous transversality condition by the assumption that $Q|_{L=0}$ is indefinite, this result is due to A.Gorodnik [Gor04]:

**Theorem 1.1** (Gorodnik). Let $F = (Q, L)$ be a pair consisting of a quadratic form $Q$ and $L$ a nonzero linear form in dimension $n \geq 4$ satisfying the the following conditions

1. $Q$ is nondegenerate.
2. $Q|_{L=0}$ is indefinite.
3. No linear combination of $Q$ and $L^2$ is rational.

Then the set $F(\mathbb{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}^2$ where $\mathbb{P}(\mathbb{Z}^n)$ is the set of primitive integer vectors.

The conclusion of the theorem implies immediately that the set $F(\mathbb{Z}^n)$ is dense in $\mathbb{R}^2$. The proof of this theorem reduced to the case of the dimension 4. The condition (1) is a sufficient condition to ensure that we have $F(\mathbb{R}^n) = \mathbb{R}^2$ and this is a conjecture that this

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condition can be weakened in order to make it necessary (see §7). The most important obstruction to prove density for pairs is that the identity component of the stabilizer of a pair \((Q, L)\) is no longer maximal among the connected Lie subgroups of \(\text{SL}(4, \mathbb{R})\) in contrast with the case of the isotropy groups \(\text{SO}(3, 1)^0\) or \(\text{SO}(2, 2)^0\).

Let \((Q, L)\) be a pair consisting of a nondegenerate quadratic form and a nonzero linear form, the stabilizer of the pair \((Q, L)\) is defined by the following subgroup of \(G\),

\[
\text{Stab}(Q, L) = \{ h \in G | (Q^h, L^h) = (Q, L) \}.
\]

It is not difficult to see that there exists \(g \in G\) such that \((Q, L) = (Q_0^g, L_0^g)\) for some canonical pairs \((Q_0, L_0)\) given explicitly (see [Gor04], Proposition 2). Clearly one has \(\text{Stab}(Q, L) = g\text{Stab}(Q_0, L_0)g^{-1}\) and we are reduced to study the stabilizer of canonical pairs \((Q_0, L_0)\). The pairs such that \(Q|_{L=0}\) is nondegenerate (resp. degenerate) are said to be of type (I) (resp. II). The proof of Theorem 1.1 is divided in two parts following each type and consists to apply Ratner’s orbit closure theorem, and to study the action of \(\text{Stab}(Q_0, L_0)\) on the dual space of \(\mathbb{C}^4\). A remarkable fact is that the density is proved without showing the density of the orbit closure of the stabilizer in the homogeneous space \(G/\Gamma\). Indeed the intermediate subgroups which possess non-trivial irreducible components have closed orbits in \(G/\Gamma\), in particular they are not maximal. However, one is able to classify all the complex semisimple Lie algebras in \(\mathfrak{sl}(4, \mathbb{C})\), and Gorodnik used this classification to check density case by case using the constrain on rationality given by the condition (3). The situation for pairs of type (II) is more complicated compared with the pairs of type (I) since the dual action of the stabilizer has three irreducible components for the pairs of type (II), instead of two for the pairs of type (I).

We are going to show an \(S\)-arithmetic generalisation of this result for pairs of type (I). Our proof is influenced by the work of Borel-Prasad on the generalised the Oppenheim conjecture for quadratic forms ([BP92]) and of course also by Gorodnik’s proof of theorem 1.1.

2. Main result

2.1. \(S\)-arithmetic setting.\footnote{Note that if \(s \notin S\), since \(S \supset S_\infty\) \(s\) is necessarily nonarchimedean!} Let us recall what we mean by \(S\)-arithmetic setting by fixing some notations. Let \(k\) be a number field, that is a finite extension of \(\mathbb{Q}\) and let \(\mathcal{O}\) be the ring of integers of \(k\). For every normalised absolute value \(|\cdot|_s\) on \(k\), let \(k_s\) be the completion of \(k\) at \(s\). We identify \(s\) with the specific absolute value \(|\cdot|_s\) on \(k_s\) defined by the formula \(\mu(a\Omega) = |a|_s\mu(\Omega)\), where \(\mu\) is any Haar measure on the additive group \(k_s\), \(a \in k_s\) and \(\Omega\) is a measurable subset of \(k_s\) of finite measure. We denote by \(\Sigma_k\) the set of places of \(k\).

In the sequel \(S\) is a finite set of \(\Sigma_k\) which contains the set \(S_\infty\) of archimedean places\footnote{Note that if \(s \notin S\), since \(S \supset S_\infty\) \(s\) is necessarily nonarchimedean!}, \(k_S\) the direct sum of the fields \(k_s(s \in S)\) and \(\mathcal{O}_S\) the ring of \(S\)-integers of \(k\) (i.e. the ring of elements \(x \in k\) such that \(|x|_s \leq 1\) for \(s \notin S\)). For \(s\) non-archimedean, the valuation ring of the local field \(k_s\) is defined to be \(\mathcal{O}_s = \{ x \in k | |x|_s \leq 1 \}\).

In all the statements of the article, without loss of generality one can replace \(k\) by \(\mathbb{Q}\) but for sake of completeness we work with number fields.

Let \((Q, L)\) be a pair consisting of one quadratic form and one nonzero linear form on \(k^n_S\). Equivalently, \((Q, L)\) can be viewed as a family \((Q_s, L_s)(s \in S)\), where \(Q_s\) is a quadratic form on \(k^n_s\) and \(L_s\) a nonzero linear form on \(k^n_s\). The form \(Q\) is nondegenerate if and only each \(Q_s\) is nondegenerate. We say that \(Q\) is isotropic if each \(Q_s\) is so, i.e. if there exists
for every $s \in S$ an element $x_s \in k^n_s - \{0\}$ such that $Q_s(x_s) = 0$, in particular if $s$ is a real place an isotropic form is also said to be indefinite. For any quadratic form $Q$, we denote by $\text{rad}(Q)$ (resp. $c(Q)$) the radical (resp. the isotropy cone) of $Q$, by definition $Q$ is nondegenerate (resp. isotropic) if and only if $\text{rad}(Q) \neq 0$ (resp. $c(Q) \neq 0$). The form $Q$ is said to be rational (over $k$) if there exists a quadratic form $Q_0$ on $k^n$ and a unit $c$ of $k$ such that $Q = c.Q_0$, irrational otherwise. For any $s \in S$ let $k_s$ denote an algebraic closure of $k_s$. If $G$ is a locally compact group, $G^o$ denotes the connected component of the identity in $G$.

2.2. Main result. Let be given a pair $F = (Q_s, L_s)_{s \in S}$ on $k^n_S$ and let $(a, b) \in k^n_S$. We are interested in finding sufficient conditions which guarantees the existence of nontrivial $S$-integral solutions $x \in \mathcal{O}_S^n$ of the following simultaneous diophantine problem

\[(A_S) \text{ For any } \varepsilon > 0, |Q_s(x) - a_s|_s < \varepsilon \text{ and } |L_s(x) - b_s|_s < \varepsilon \text{ for each } s \in S.\]

Obviously as in the real case, we need to find sufficient conditions on $F$ so that the set $F(\mathcal{O}_S^n)$ would be dense in $k^n_S$. One have to be careful since the condition $(A_S)$ is not equivalent to density contrarily to real pairs (see [BP92], §6 and our §7).

Our main result gives the required conditions for assertion $(A_S)$ to hold when $(a, b) = (0, 0)$. In other words, we give sufficient conditions which implies that $F(\mathcal{O}_S^n)$ is not discrete around the origin in $k^n_S$. It may be seen as a weak $S$-arithmetic version of Theorem 1.1,

**Theorem 2.1.** Let $Q = (Q_s)_{s \in S}$ be a quadratic form on $k^n_S$ and $L = (L_s)_{s \in S}$ be a linear form on $k^n_S$ with $n \geq 4$. Suppose that the pair $F = (Q, L)$ satisfies the following conditions,

1. $Q$ is nondegenerate.
2. $Q_{|L=0}$ is nondegenerate and isotropic.
3. The quadratic form $\alpha Q + \beta L^2$ is irrational for any units $\alpha, \beta$ in $k_S$ such that $(\alpha, \beta) \neq (0, 0)$.

Then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

\[|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S.\]

2.3. Remarks. (a) This theorem reduces to dimension 4, (see §3). The key is the use of the weak approximation in $k_S$ follows in the same lines ([BP92], Proposition 1.3).

(b) Even if we assume that $\alpha Q + \beta L^2$ is irrational, it can be possible that the pencil form $\alpha_s Q_s + \beta_s L_s^2$ is rational for some place $s$. To carry out this situation we adapt the strategy of Borel and Prasad ([BP92]) to complete the picture. In particular we are forced to assume that $Q_{|L=0}$ is nondegenerate. Therefore the stabiliser of $(Q, L)$ is semisimple and this condition guarantees the existence of the universal covering for which strong approximation is known to hold provided that the stabiliser is isotropic.

(c) Unfortunately we are not able to show the density of $F(\mathcal{O}_S^n)$ under the conditions of theorem 2.1 with our method. We are also even unable to show that $|Q_s(x)|_s$ and $|L_s(x)|_s$ are both nonzero for any $s \in S$ and $x \in \mathcal{O}_S^n$ as in the conclusion of Theorem 2.1. We discuss those issues in §7.
(d) One can hope to relax condition (2) by only asking $\alpha Q + \beta L^2$ to be isotropic as it is conjectured by Gorodnik (see § 7, Conjecture 7.1). The major issue is that reduction to lower dimension fails to hold.

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3. Weak approximation and reduction to the dimension 4

Number fields satisfy a nice local-global principle called the weak approximation which can be seen as a refinement of the Chinese remainder theorem.

**Theorem 3.1.** Let $S$ be a finite set of $\Sigma_k$. Let given $\alpha_s \in k_s$ for each $s \in S$. Then there exists an $\alpha \in k$ which is arbitrarily close to $\alpha_s$ for all $s \in S$ with respect to the $s$-adic topology.

**Proof.** (See e.g. [L], Theorem1, p.35)

One can reformulate this theorem as follows: the diagonal embedding $k \hookrightarrow \prod_{s \in S} k_s$ is dense, the product being equipped with the product of the $s$-adic topologies.

**Proposition 3.2.** Let $F = (Q, L)$ be a pair consisting of a quadratic form $Q$ and a nonzero linear form $L$ in $k^n_S$ ($n \geq 5$) such that
1. $Q$ is nondegenerate
2. $Q|_{L=0}$ is isotropic
3. The quadratic form $\alpha Q + \beta L^2$ is irrational for any units $\alpha, \beta$ in $k_S$ such that $(\alpha, \beta) \neq (0, 0)$.

Then there exists a $k$-rational subspace $V$ of $k^n$ of codimension 1 such that $F|_V$ satisfies the conditions (1)(2)(3), moreover $V$ can be chosen such that $Q|_{\{L=0\} \cap V}$ is nondegenerate.

**Proof.** When $s$ is an archimedean real place, it is proved in ([Gor04], Proposition 4) that there exists a subspace $V_s$ of $k^n$ of codimension 1 such that $F|_{V_s}$ verifies conditions (1)(2)(3). In the case of archimedean complex places and nonarchimedean places, one may replace the condition $Q_s|_{L_s=0}$ of type (I) which only valid for real places by equivalent condition that $Q_s|_{L_s=0}$ is nondegenerate which is valid for all $s \in S$. Therefore there exists a subspace $V_s$ of $k^n$ of codimension 1 such that $F_s|_{V_s}$ verifies conditions (1)(2)(3), the proof of the latter existence of $V_s$ for non-archimedean places in $S$ is identical to the real places (see [Gor04], Proposition 4). Hence for any $s \in S$ we may find $V_s$ a subspace of $k^n$ of codimension 1 so that the conditions (1)(2)(3) are satisfied by $F_s|_{V_s}$ and one can choose $V_s$ to be such that $Q_s|_{\{L_s=0\} \cap V_s}$ is nondegenerate.

Assume that $n \geq 5$. For each $s \in S$, it is well known that the orbit of $V_s$ under the orthogonal group $SO(Q_s)$ is open in the Grassmanian variety $G_{n-1,n}(k_s)$ of the hyperplanes in $k^n_s$ for the analytic topology. This can be seen using the fact that the map

$$SO(Q_s) \ni g \mapsto g(V_s)$$

in particular the image contains a neighbourhood of the image of any $g$ after the implicit function theorem (see e.g. [EV08], Lemma 2). Moreover by weak approximation we can find a rational subspace in $V'$ of codimension 1 in $k^n$ such that $V' \otimes_k k_s$ is arbitrarily close

\footnote{i.e. the induced map on tangent space is surjective}
to $V_s$ for all $s \in S$, in particular they belong to the same open orbit. We have established that $F_{s|V_s}$ satisfies conditions (1) and (2), it is equivalent to say that

$$\text{rad}(Q_s) \cap V_s = \{0\} \quad \text{and} \quad c(Q_{s|L_s=0}) \cap V_s \neq \{0\} \quad \text{(*)}$$

The condition (2) remains true if we replace $V_s$ by any subspace sufficiently close to $V_s$. Since the subspace $\text{rad}(Q_s)$ is invariant under the action of the orthogonal group $SO(Q_s)$, the condition (1) above is verified by any element of $\mathbb{G}_{n-1,n}(k_s)$ which lies in the orbit of $V_s$ under $SO(Q_s)$. In particular, $V' \otimes_k k_s$ satisfies (*) for each $s \in S$. Hence we obtain a $k$-rational subspace $V'$ of $k^n$ such that $F_{|V'_s}$ satisfies the conditions (1)(2). It remains to prove that $F_{|V'_s}$ satisfies the condition (3). By weak approximation again, there exists $e \in V'(k) \cap \{L = 0\}$ such that $Q_{s|L_s=0}(e_s) \neq 0$ for all $s \in S$. Let $\bar{Q} = \alpha Q + \beta L^2$ for an arbitrarily choice of $\alpha, \beta \in k_S$ with $(\alpha, \beta) \neq (0,0)$. Applying $e$ to it, we have that

$$\bar{Q}(e_s) = \alpha_s Q_s(e_s) + \beta_s L_s(e_s)^2 = \alpha_s Q_s(e_s) \neq 0 \quad \text{for any} \quad s \in S.$$

Hence multiplying by some unit in $k_S$ we may assume that $\bar{Q}(e_s) = 1$ for all $s \in S$. Now let

$$V = \{V \in \mathbb{G}_{n-1,n}(k) | e \in V(k) \text{ and } F_{|V'_s} \text{ satisfies conditions (1)(2)}\}.$$

It is nonempty because it contains $V'$. Suppose that there is no $V \in V$ such that $F_{|V'_s}$ satisfies the condition (3). Then $\bar{Q}(x) \in k$ for any $x \in V(k), V \in V$. For each $s \in S$, the map $x \mapsto \bar{Q}(x)$ is a regular function on $\mathbb{F}_s^n$ and it takes values in $k$ on the union of $V(k), V \in V$. The latter union is clearly Zariski-dense in $\mathbb{F}_s^n$ and it is defined over $k$. Therefore $\bar{Q}(x) \in k$ for all $x \in k^n$ and this implies that $\bar{Q}$ is rational over $k$. Since $e \in V$ for any $V \in V$ and $\bar{Q}(e) = 1$ for all $s \in S$, $\bar{Q}(x)$ is independent of $s$ for any $x \in V(k), V \in V$. This implies that $\bar{Q}$ is rational, this is a contradiction. Hence there exists some $V \in V$ such that $F_{|V'_s}$ satisfies the conditions (1)(2)(3). Moreover such $V \in V$ is chosen so that $Q_{s|L=0} \cap V_s$ is nondegenerate since it is at each place of $S$.

**Corollary 3.3.** It suffices to prove Theorem 2.1 for $n = 4$.

**Proof.** It follows from the proposition by descending induction on $n$.

## 4. Strong approximation

### 4.1. Adeles and strong approximation for number fields

The set of adeles $\mathbb{A}$ of $k$ is the subset of the direct product $\prod_{s \in \Sigma} k_s$ consisting of those $x = (x_s)$ such that $x \in \mathcal{O}_s$ for almost all $s \in \Sigma_f$. The set of adeles $\mathbb{A}$ is a locally topological ring with respect to the adele topology given by the base of open sets of the form $\prod_{s \in S} U_s \times \prod_{s \notin S} \mathcal{O}_s$ where $S \subset \Sigma$ is finite with $S \supset S_\infty$ and $U_s$ are open subsets of $k_s$ for each $s \in S$. For any subset $S \subset \Sigma$ finite with $S \supset S_\infty$, the ring of $S$-integral adeles is defined by:

$$\mathbb{A}(S) = \prod_{s \in S} k_s \times \prod_{s \notin S} \mathcal{O}_s,$$

thus we can see that $\mathbb{A} = \bigcup_{S \supset S_\infty} \mathbb{A}(S)$.

We define also $\mathbb{A}_S$ to be the image of $\mathbb{A}$ onto $\prod_{s \notin S} k_s$, clearly $\mathbb{A} = k_S \times \mathbb{A}_S$.

**Theorem 4.1** (Strong approximation). If $S \neq \emptyset$ the image of $k$ under the diagonal embedding is dense in $\mathbb{A}_S$.

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Note that the fact that a quadratic form $Q = (Q_s)_{s \in S}$ is irrational over $k_S$ does not prevent some $Q_s$ to be rational!
4.2. **Strong approximation in algebraic groups.** Let $G$ be an algebraic group defined over $k$, we defined the group $G(\mathbb{A}_S)$ of $S$-adeles of $G$ to be the image of $G(\mathbb{A})$ under the natural projection of $\prod_{s \in \Sigma} G(k_s)$ onto $\prod_{s \not\in S} G(k_s)$.

**Definition 4.2.** A $k$-algebraic group $G$ is said to satisfy the strong approximation property relative to $S$ if the image under the diagonal embedding $G_k \hookrightarrow G(\mathbb{A}_S)$ is dense (equivalently in term of full adelic group $G(k)G(k_S)$ is dense in $G(\mathbb{A})$).

**Definition 4.3.** An isogeny between two algebraic groups $G$ and $H$ defined over $k$, is a surjective morphism $\mu : G \rightarrow H$ which has finite kernel. If $\mu$ is a $k$-morphism, we say that $\mu$ is a $k$-isogeny.

**Definition 4.4.** An algebraic group $G$ is said to be simply connected, if for any connected $H$, any isogeny $\mu : H \rightarrow G$ is an isomorphism.

An important class of algebraic groups which satisfies the strong approximation property is given by the unipotent subgroups (see e.g. [PR], §7.1). Thus, for a connected $k$-group $G$ the Levi decomposition $G = LR_u(G)$ implies that it suffices to check the strong approximation property for reductive groups.

The following theorem is due to V. Platonov, the proof consists to reduce the problem of strong approximation to the Kesner-Tits conjecture which was also proved by itself in [P69].

**Theorem 4.5 (Strong approximation).** Let $G$ be a reductive $k$-group and $S$ a set of places of $k$. Then $G$ has the strong approximation property with respect to $S$ if and only if it satisfies the following conditions

1. $G$ is simply connected.
2. Every $k$-simple factor of $G$ is isotropic.

The proof of this theorem (see e.g. [PR], §7.4) when $S \supset S_\infty$ is finite relies on the important observation that the closure of $G(\mathcal{O}_S)$ is open in $G(\mathbb{A}_S)$. This observation can be generalised to any $S$-arithmetic subgroup in $G(\mathcal{O}_S)$. Recall that an $S$-arithmetic subgroup $\Gamma$ is a subgroup of $G_S$ which is commensurable with $G(\mathcal{O}_S)$, i.e. if $\Gamma \cap G(\mathcal{O}_S)$ has finite index both in $G(\mathcal{O}_S)$ and $\Gamma_S$ (see e.g. [PR], §7.5).

**Proposition 4.6.** Assume $G$ satisfies conditions of the previous theorem. Given any $S$-arithmetic subgroup $\Gamma$ in $G(\mathcal{O}_S)$, its closure is an open subgroup in $G(\mathbb{A}_S)$.

For any reductive algebraic group $G$ which is isotropic, the strong approximation property should hold for the universal covering of $G$ which by definition is simply connected. In general the existence of such universal cover is not always guaranteed, however for semisimple algebraic groups the existence is always satisfied \(^4\).

**Proposition 4.7 ([PR], Thm 2.6).** For any semisimple group $G$, there exists a simply connected group $\tilde{G}$ and an isogeny $\sigma : \tilde{G} \rightarrow G$.

**Definition 4.8.** The isogeny $\sigma : \tilde{G} \rightarrow G$ is called the universal covering and the $\pi(G) = \ker(\sigma)$ is the fundamental group of $G$. Moreover if $G$ is $k$-split, the universal cover is defined over $k$.

\(^4\)The situation is quite different in the category of topological groups, this is due to the definition of simply connectedness which is more restrictive in the category of algebraic groups than in the topological context.
Theorem 4.9 ([PR], Thm 4.1). Let \( \varphi : G_S \to H_S \) be a surjective \( k \)-morphism of algebraic groups. If \( \Gamma \) is an \( S \)-arithmetic subgroup of \( G_S \), then \( \varphi(\Gamma_S) \) is an \( S \)-arithmetic subgroup of \( H_S \).

Then every \( k \)-isogeny sends any arithmetic subgroup to another one. In particular universal coverings do so providing the group is semisimple.

Corollary 4.10. Suppose \( G_S \) is semisimple. If \( \Gamma_S \) is an \( S \)-arithmetic subgroup of \( \tilde{G}_S \), then \( \sigma(\Gamma_S) \) is an \( S \)-arithmetic subgroup of \( G_S \).

5. Stabilizers and \( S \)-adic Ratner’s Theorem

For each \( s \in S \) define \( G_s = \text{SL}_{4}(k_s) \), \( G_S = \prod_{s \in S} \text{SL}_{4}(k_s) = \text{SL}_{4}(k_S) \). Let \( F = (Q, L) \) be a pair on \( k_S^{4} \) satisfying the conditions (1)(2)(3) of Theorem 2.1.

5.1. Stabilizer of a pair. For every \( s \in S \) we realize \( Q_s \) on a four-dimensional quadratic vector space \((W_s, Q_s)\) over \( k_s \) equipped with the standard basis \( B = \{ e_1, \ldots, e_4 \} \). For each \( s \in S \), let us define \( H_s \) the stabilizer of the pair \( F_s \) under the action of \( G_s \), in other words

\[
H_s = \{ g \in G_s \mid Q_s \circ g = Q_s, L_s \circ g = L_s \}. 
\]

Equivalently one can write \( H_s = \{ g \in \text{SO}(Q_s) \mid L_s \circ g = L_s \} \), clearly it is a linear algebraic group defined over \( k_s \). Also let us define \( V_s = \{ L_s = 0 \} \), it is an hyperplane of \( W_s \) which induces a quadratic isotropic subspace \((V_s, Q_s|_{V_s})\) of dimension 3 in \( W_s \). We have two cases following \((V_s, Q_s|_{V_s})\) is nondegenerate or not. If \( s \) is a real place the first case corresponds to pairs of type (I) in the terminology of [Gor04].

Lemma 5.1. Let be given a pair \((Q, L)\) satisfying the conditions of Theorem 2.1 in dimension 4. Then the stabilizer of \((Q, L)\) under the action of \( G \) is of the form

\[
H = \{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in \text{SO}(Q|_{L=0}) \} \subseteq \text{SL}_{4}(\overline{k_s}).
\]

In particular, \( H \) is semisimple.

Proof. Since \((V_s, Q_s|_{V_s})\) is nondegenerate, one can write the the following decomposition \( W_s = V_s \oplus V_s^\perp \) where the orthogonal complement is a one-dimensional subspace of \( W_s \). Let \( v \) a nonzero vector of \( W_s \) such that \( V_s^\perp = \langle v \rangle \). Since \( V_s^\perp = \{ L_s \neq 0 \} \) it is clearly \( H_s \)-invariant, therefore \( H_s \) acts by \( x \mapsto \lambda x \) on the line \( \langle v \rangle \). The linear form \( L_s|_{V_s^\perp} \) is nonzero and is \( H_s \)-invariant then \( H_s \) acts trivially on the line \( V_s^\perp \). Let define \( w_4 = v \) and complete to obtain a basis \( B' = \{ w_1, \ldots, w_4 \} \) of \( W_s \) where \( \langle w_1, w_2 \rangle \) is an hyperbolic plane in \( V_s \). Hence we obtain for any \( x \in W_s \), with respect to the basis \( B' = \{ w_1, \ldots, w_4 \} \) that

\[
Q_s(x) = x_1x_2 + a_3x_3^2 + a_4x_4^2 \quad \text{and} \quad L_s(x) = x_4 \quad \text{with} \quad a_3, a_4 \in k_s^*, \]

Therefore for every \( s \in S \), the stabiliser of \( F_s = (Q_s, L_s) \) is given by

\[
H_s = \{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid M \in \text{SO}(Q_s|_{L_s=0}) \} \subseteq \text{SL}_{4}(\overline{k_s}).
\]

It is well-known that the orthogonal group of a nondegenerate quadratic form is a semisimple group.
Let $F = (Q_s, L_s)_{s \in S}$ be a pair satisfying the conditions of the main theorem. Let $\mathcal{H}_s$ be the algebraic group defined over $k_s$ such that $\mathcal{H}_s(k_s) = H_s$. Define $H_s^+$ to be the subgroup of $H_s$ generated by its one-dimensional unipotent subgroups.

Let us put
\[ H_S = \prod_{s \in S} H_s \text{ and } H_S^+ = \prod_{s \in S} H_s^+ \text{.} \]

Therefore $H_S$ is an algebraic subgroup of $\text{SL}_4(k_S)$ which leaves invariant the pair $F = (Q, L)$ with respect to the basis $B'$ introduced in the previous lemma.

5.2. Unipotent actions on $S$-products. Let $G_S = \text{SL}_4(k_S)$ and let $\Gamma_S$ be the $S$-arithmetic subgroup of $G_S$ given by $\Gamma_S = \text{SL}_4(0_S)$. The ring $0_S$ is a lattice in $k_S$. Let us define $\Omega_S$ to be the quotient space given by $G_S/\Gamma_S$. It is the space of free of $0_S$-submodules of $k_S^4$ of maximal rank and determinant one. Then $\Omega_S$ is the homogeneous space of unimodular lattices of $0_S^4$, by lattice we mean a discrete subgroup of $G_S$ of finite covolume. For every $s \in S$, let $U_s$ be a unipotent $k_s$-algebraic subgroup of $\text{SL}_4(k_s)$ and denote by $U = \prod_{s \in S} U_s(k_s)$ the associated unipotent subgroup of $G_S$.

We are interested in the left action of $U$ on the homogeneous space $\Omega_S$ and more particularly with the closure of such orbits. If $x \in \Omega_S$ it turns out that the closure of the orbit $Ux$ is also an orbit of $x$. The following result is the generalisation of Ratner’s orbit closure theorem for $S$-products proven independently by Margulis-Tomanov and Ratner (see [MT94], [R93]).

Theorem 5.2 (Ratner’s Theorem for $S$-adic groups). Assume that $U$ is generated by its one-dimensional unipotent subgroups. Then for any $x \in \Omega_S$, there exists a closed subgroup $M = M(x) \subset G_S$ containing $U$ such that the closure of the orbit $Ux$ coincides with $Mx$ and $Mx$ admits $M$-invariant probability measure.

6. Proof of the Theorem 2.1

Let $F = (Q, L)$ be a pair in $k_S^3$ which satisfies the conditions of Theorem 2.1. After § 3, we know that it suffices to show it for $n = 4$. We divide the proof into two cases following the pencils of $Q$ and $L^2$ are all irrational for each place or not.

Case 1. Let us assume first that all the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational for each $\alpha_s, \beta_s$ in $k_s$ such that $(\alpha_s, \beta_s) \neq (0, 0)$ for any $s \in S$. Let $g \in G_S$ be the matrix of the basis $B'$ in the standard basis of $k_S^4$. By definition $g^{-1}H_S g$ leaves invariant the pair $F = (Q_s, L_s)_{s \in S}$, and $H_S^+$ is generated by one-dimensional unipotent subgroups. We consider $\Gamma_S$ as an element of the homogeneous space $\Omega_S$. By applying Ratner’s Theorem 5.2, one obtains
\[ g^{-1}H_S^+ g \Gamma_S = P \Gamma_S \]  
(1)

where $P$ is a closed subgroup of $G_S$ which contains $gH_S^+ g^{-1}$.

Assume first that $S = S_\infty$, thus $0_S^4 = 0^4$ and we simply write $k_\infty$, $H_\infty$ and $G_\infty$ respectively for $k_{S_\infty}$, $H_{S_\infty}$ and $G_{S_\infty}$. One notes also that $H_\infty^+$ is nothing else than the component of the identity $H_\infty^\circ$. Using equality (1) one deduces that the set $F(0^4)$ is dense in $k_\infty^4$. Indeed, we are going to adapt the proof of ([Gor04], Proposition 10) to the $S_\infty$-products, as follows. We first reduce the ground field from $k$ to the rational numbers. To achieve this we realise $G_\infty$ as the group of real points of an algebraic group $\mathfrak{g}$ defined over $\mathbb{Q}$. This is given explicitly by taking $\mathfrak{g} = R_{k/\mathbb{Q}} \text{SL}_4$ where $R_{k/\mathbb{Q}}$ is the functor restriction of

\[ \text{For more precisions, one is invited to read the original proof which is similar.} \]
scalars of the field extension $k/\mathbb{Q}$ and where $\text{SL}_4$ is regarded as the usual algebraic group over $k$ (see e.g. §2.1.2, [PR] for a definition of the functor restriction of scalars). In other words, $G_\infty = G(\mathbb{R})$ with $G$ an algebraic group defined over $\mathbb{Q}$. Now let us precise the structure of $P$. From Proposition 7.2 in [BP92], we infer that there exists an algebraic group $\tilde{M}$ defined over $\mathbb{Q}$ which is the smallest $\mathbb{Q}$-algebraic group whose group of real points $\tilde{P}$ contains $g^{-1}H_\infty g$. In the other hand, Proposition 3.2 in [Shah] implies that $P = \tilde{P}(\mathbb{R})^\circ$ and the unipotent radical $U$ of $\tilde{P}$ is also defined over $\mathbb{Q}$. Thus equality (1) may be read as

$$g^{-1}H_\infty g \Gamma = \tilde{P}(\mathbb{R})^\circ \Gamma.$$  

(2)

**Lemma 6.1.** For each $s \in S_\infty$, let $P_s$ be the intersection of $M$ with $G_s$. If $P_s$ acts irreducibly on $\mathbb{C}^4$, then $P_s = G_s$. Otherwise, $P_s = M_s U$ where

$$M_s = u g_s^{-1} \begin{pmatrix} \text{SL}_3 & 0 \\ 0 & 1 \end{pmatrix} g_s u^{-1}$$

for some $u \in U_s$.

**Proof the Lemma.** This result is the core of the proof of Proposition 10 in [Gor04] for which we recall the outlines. If $P_s$ acts irreducibly on $\mathbb{C}^4$, then $P_s$ is semisimple and the classification of irreducible semisimple Lie groups in $\text{SL}_4$ implies that $P_s$ is equal either to $G_s$ or $\text{SO}(B_s)$ for some nondegenerate form $B_s$ (Proposition 7 and Lemma 8, [Gor04]). Such form $B_s$ being $H_s$-invariant is necessarily of the form $\alpha_s Q_s + \beta_s L_s^2$ for some $(\alpha_s, \beta_s) \neq (0,0)$ (Lemma 9, [Gor04]). As seen before $P_s$ is defined over $\mathbb{Q}$, so that $B_s$ is forced to be rational which is a contradiction. Hence $P_s = G_s$. For the second assertion, we consider the induced action of $P_s$ on the space $L$ of linear forms in $\mathbb{C}^4$, it is reducible by hypothesis. There are only two $P_s$-invariant subspaces in $L$, namely $L_1 = \langle L_1, L_2, L_3 \rangle$ and $L_2 = \langle L_4 \rangle$ where $L_i(x) = (gx)_i$ for $i = 1, \ldots, 4$, note that $L_4 = L$. Since $P_s$ is defined over $\mathbb{Q}$, one infers that $M$ is semisimple thus admitting a Levi decomposition

$$P_s = M_s U_s$$

where $M_s$ and $U_s$ are respectively a Levi subgroup and the unipotent radical of $P_s$. The Levi subgroup $M_s$ is defined over $\mathbb{Q}$ since $P_s$ is. Also as seen above, $U$ is defined over $\mathbb{Q}$ and Malcev’s theorem ensures that the Levi subgroups are unique up to conjugacy (e.g. see §4.3 [OV]), in particular

$$g_s^{-1}H_\infty^s g_s \subseteq u^{-1}M_s u$$

for some $u \in U_s$.  

(3)

Moreover this inclusion is strict because the subspace $\langle Q, L \rangle$ is not a $\mathbb{Q}$-subspace of $L$. The latter fact and the maximality of $\text{SO}(Q_{\mid L = 0})$ in $\text{SL}_3$ ($Q_{\mid L = 0}$ is isotropic) gives the equality

$$M_s = u g_s^{-1} \begin{pmatrix} \text{SL}_3 & 0 \\ 0 & 1 \end{pmatrix} g_s u^{-1}.$$  

This achieves the proof of the Lemma.

Let us define the subgroup

$$M'_s := u^{-1}M_s u = g_s^{-1} \begin{pmatrix} \text{SL}_3 & 0 \\ 0 & 1 \end{pmatrix} g_s u^{-1}.$$  

By the previous Lemma 6.1, one can rephrase equality (2) in the following way

$$g^{-1}H_\infty^s g \Gamma = M'(\mathbb{R})^\circ U(\mathbb{R}) \Gamma.$$

(4)

Now let be given $(a, b) \in k_\infty^2$ and let us choose $x \in \mathcal{O}^4 - \langle g^{-1} e_4 \rangle$. It is not difficult to see that there exists $m \in M'(\mathbb{R})^\circ$ and $u \in U(\mathbb{R})$ such that

$$F(mux) = (Q(mux), L(mux)) = (a, b).$$
Using density in (4), we infer that there exists $h_n \in g^{-1}H_{\infty}^s g$ and $\gamma_n \in \Gamma$ such that 

$$h_n \gamma_n \to mu \text{ as } n \to \infty.$$ 

We conclude that 

$$F(\gamma_n x) = F(h_n \gamma_n x) \to F(um x) = (a, b) \text{ as } n \to \infty.$$ 

In other words, $F(\mathcal{O}^4)$ is dense in $k_{\infty}^2$ and in particular this proves Case 1 when $S = S_{\infty}$.

Now let us assume $S \neq S_{\infty}$ and let be given $s \in S_f$. The set $\mathcal{O}^4$ is bounded in $k_{s}^4$, thus for any neighbourhood $U$ of the origin in $k_{s}^4$ one can find an integer $a_s \in \mathcal{O}_s$ such that $a_s.\mathcal{O}^4 \subset U$. In other words, given any $\varepsilon > 0$ one can find $a_s \in \mathcal{O}_s$ such that:

$$|Q_s(a_s x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s \leq \varepsilon \text{ for all } x \in \mathcal{O}^4.$$ 

Thus for each $s \in S_f$, we can associate an integer $a_s \in \mathcal{O}_s$ satisfying the previous inequalities. By strong approximation one can find $a \in \mathcal{O}$ such that $|a|_s = |a_s|_s$ for all $s \in S_f$.

Put $\|a\|_{\infty} = \max_{s \in S_{\infty}} |a|_s$, by the previous case we can find $x \in \mathcal{O}^4$ such that:

$$|Q_s(x)|_s \leq \varepsilon/\|a\|_{\infty}^2 \text{ and } |L_s(x)|_s \leq \varepsilon/\|a\|_{\infty} \text{ for all } s \in S_{\infty}.$$ 

We immediately obtain for all $s \in S_{\infty}$

$$|Q_s(a_s x)|_s = |a_s|^2 |Q_s(x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s = |a_s| |L_s(x)|_s \leq \varepsilon.$$ 

Hence given any $\varepsilon > 0$, we get a nonzero element $y = a.x \in \mathcal{O}_s^4$ satisfying the conclusion of the theorem, i.e.

$$|Q_s(y)|_s \leq \varepsilon \text{ and } |L_s(y)|_s \leq \varepsilon \text{ for all } s \in S.$$ 

**Case 2.** Now it remains to prove the theorem when some linear combination over $k_s$ of $Q_s$ and $L_s^2$ is rational for some $s \in S$. The proof is a slight modification of the argument used in ([BP92], §4). The only difference is that we deal with pairs instead of quadratic forms, but the conditions on $Q_{|L=0}$ force the stabiliser of such pairs to be isomorphic to a semisimple orthogonal group in $\text{SL}(n-1)$.

To do so, we assume that there exists $v \in S$ and a rational form $Q_0$ on $k^n$ such that,

$$\alpha_v Q_v + \beta_v L_v^2 = \lambda_v Q_0$$

for some $\lambda_v \in k_v^{\times}$ and $\alpha_v, \beta_v$ in $k_v$ such that $(\alpha_v, \beta_v) \neq (0, 0)$. Let us choose $\alpha, \beta$ in $k_S$ such that $(\alpha)_v = \alpha_v$ and $(\beta)_v = \beta_v$ and set $\tilde{Q} = \alpha Q + \beta L^2$, then $\tilde{Q}_v = \lambda_v Q_0$. Now, we define the set $S'$ to be the set of places $s \in S$ such that $\tilde{Q}_s = \lambda_s Q_0$ for some $\lambda_s \in k_s^{\times}$. Obviously $v \in S'$ and $S \neq S'$ since $\tilde{Q}$ is irrational, and one can put $T = S - S'$.

Let $Z_0 = \{Q_0 = L = 0\}$ be the $k_S$-algebraic variety defined by $Q_0$ and $L$ and for each $s \in S$ put $Z_{0,s} = Z_0 \times k_s$. Let us define also for each $s \in S$ the $k_s$-algebraic variety defined by $Z_s = \{Q_s = L_s = 0\}$. Obviously $Z_s$ characterises the pair $F_s = (Q_s, L_s)$ up to a multiple in $k_{s}^{\times}$ and $Z_{0,s} = Z_s$ for all $s \in S'$ whereas $Z_{0,t} \neq Z_t$ for any $t \in T$. Since $Z_{0} \neq \{0\}$ ($Q_{|L_{t}=0}$ is isotropic) and defined over $k_t$, $Z_t(k_t)$ is Zariski dense in $Z_t$ and $Z_{0,t}(k_t) \neq Z_t(k_t)$ for any $t \in T$. 

Let us define $H_0 = \text{SO}(Q_0) \cap \text{Stab}_G(L)$ and let $V$ be the subspace $\text{Ker} L = \{L = 0\}$. By surjectivity of $L$ we have immediately $\text{Stab}_G(L) = V^G$ i.e. $H_0 = \text{SO}(Q_0) \cap V^G$. Moreover by construction, for each $s \in S'$ one has $\lambda_s Q_{|V_s} = \tilde{Q}_{|V_s} = \alpha_s Q_{|V_s}$ thus $H_0(k_s) = \text{SO}(Q_{s|V_s})$ is noncompact for each $s \in S'$ since $Q_{|V}$ is isotropic. 

For each $t \in T$, suppose $R_t$ is an open subgroup of $H_0$ and let

$$X_t = \{x \in k_t^{\times} - Z_0(k_t) \mid r_t.x \in Z_t(k_t) \text{ for some } r_t \in R_t\}.$$
Clearly $X_t$ is an nonempty subset of $k_t^4$ defined up to multiple in $k_t^4$. The key of the proof of the theorem lies on the following result of geometry of numbers.

**Lemma 6.2.** Given a polydisc $\Delta = \prod_{s \in S'} \Delta_s$ centred in the origin in $k_{S'}^4$, then there exists an integral vector $x \in \mathcal{O}_S^4$ such that $x_s \in \Delta_s$ for all $s \in S'$ and $x_t \in X_t$ for all $t \in T$.

In other words we can find an integral vector $x$ with $T$-components lying in $X_T$ and arbitrarily small $S$-components. For sake of clarity, we give a detailed proof of this lemma (see [BP92], §4(*)).

**Proof of the Lemma.** Let be given an arbitrary $t \in T$, since $X_t$ is nonempty and defined up to a multiple in $k_t - \{0\}$, we can find a nonzero vector $e_{t,1} \in X_t$ such that the pinched line $k_t^* e_{t,1}$ is still contained in $X_t$. Let us complete $e_{t,1}$ into a basis $\{e_{t,1}, \ldots, e_{t,n}\}$ of $k_t^n$, and let us define for any positive real $r$

$$D_{t,r} = \{ x \in k_t : |x|_t \leq r \}.$$  

$$B_{t,r} = \oplus_{2 \leq j \leq n} D_{t,r} e_{t,j}.$$  

$$C_{t,r} = D_{t,r} e_{t,1} \oplus B_{t,r} = \oplus_{1 \leq j \leq n} D_{t,r} e_{t,j}.$$  

The facts that the line $k_t^* e_{t,1}$ is contained in $X_t$ and that $R_t$ is an open subgroup of $H_0$ provide two reals $a \geq b > 0$ so small that

(i) $e_{t,1} + C_{t,a}$ is contained in $X_t$ and $(D_{t,m} - D_{t,1}) e_{t,1} \oplus B_{t,a} \subset X_t$ for any real $m > 1$

(ii) The sum of any $\lvert T \rvert - 1$ elements of $C_{t,b}$ is contained in $C_{t,a}$.

Now let us consider $\Theta = \prod_{s \in S'} \Theta_s$ be a bounded polydisc centred on the origin such that the sum of any $2 \lvert T \rvert$ elements of $\Theta$ is contained in $\Delta$. If we are able to prove the claim $(\ast)_t$ below the proof of the lemma will follows.

$(\ast)_t$ There exists $x(t) \in \mathcal{O}_S^4$ and a real $m \geq 2$ such that $x(t)_s \in \Theta_s + \Theta_s$, if $s \in S'$, $x(t)_t \in (D_{t,m} - D_{t,1}) e_{t,1} \oplus B_{t,b}$ and $x(t)' \in C_{t,b}$ if $t' \in T - \{t\}$.

Indeed, assume $(\ast)_t$ is fullfilled and let us define $x = \sum_{t \in T} x(t) \in \mathcal{O}_S^4$.

For any $s \in S'$, $x_s = \sum_{t \in T} x(t)_s$ is the sum of $2 \lvert T \rvert$ elements of $\Theta_s$ hence it is contained in $\Delta_s$. On the other hand, by condition $(\ast)_t$, for any $t' \in T$

$$x_{t'} = \sum_{t \in T} x(t)_{t'} = x(t')_{t'} + \sum_{t \in T - t'} x(t)_{t'} \in (D_{t,m} - D_{t,1}) e_{t',1} + B_{t',b} + C_{t',a}.$$  

The fact that $B_{t',b} \subset B_{t',a}$ and the condition $(\ast)_t$ above imply together that $x_{t'} \in X_{t'}$. Hence $x \in \mathcal{O}_S^4$ satisfies all the condition of the lemma.

It remains to show that $(\ast)_t$ is satisfied for any given $t \in T$ which we fix from now. For a positive real number $r$, let us put

$$\Omega_r = \Theta \times (D_{t,r} e_{t,1} \oplus B_{t,b/2}). \prod_{t' \in T - \{t\}} C_{t',b/2} \subset k_{S'}^4.$$  

Recall that each completion $k_s (s \in S)$ is equipped with an additive Haar measure $\nu_s$, we denote by $\text{vol}$ the product of the $\nu_s$ ($s \in S$) which gives a Haar measure on $\mathcal{O}_S^4$. Let $\mu$ be the pullback of the Haar measure on $k_{S'}^4$ with respect to the natural projection $\pi : k_{S'}^4 \rightarrow k_{S}^4 / \mathcal{O}_S^4$. Also let us denote by $\text{vol}$ the covolume of $\mathcal{O}_S^4$ in $k_S^4$, that is, $\text{vol} = \mu(\mathcal{O}_S^4 / \mathcal{O}_S^4)$. Since $\mathcal{O}_S^4$ is discrete in $k_{S'}^4$, the set $(\Omega_1 + \Omega_1) \cap \mathcal{O}_S^4$ is finite with cardinality, say $q \geq 1$. Now the map
be the central
Proof of the Theorem 2.1
and the lemma is proved.

This contradicts inequality (5), hence it must exist at least one fiber, say
Suppose that all the fibers \( \pi^{-1}_{\Omega_{m/2}}(y) \) have less or equal than \( q + 1 \) elements, then
so that we get \( q + 1 \) distincts elements in \( \Theta^4_S \) by taking \( x_i := y_0 - y_i \) for every \( i = 1, \ldots, q + 1 \). Obviously, since \( (\Theta_1 + \Theta_1) \cap \Theta^4_S \) has \( q \) elements, one of the \( x_i \)'s, say \( x_1 \), must lie outside \( \Theta_1 + \Theta_1 \).
Let us define \( x(t) := x_1 \), then immediately \( x(t) \in \Theta^4_S \). It is easy to check that \( x(t) \) satisfies \((*)_t\). Indeed, let be given \( s \in S' \), then
At the place \( t \), we have
\[
x(t)_t \in (D_t,m/2 - D_{t,1})e_{t,1} \oplus B_{t,b/2} + (D_t,m/2 - D_{t,1})e_{t,1} \oplus B_{t,b/2} = (D_t,m - D_{t,1})e_{t,1} \oplus B_{t,b}.
\]
on the other hand it is clear that
\[
x(t)_{t'} \in C_{t',b/2} + C_{t',b/2} = C_{t',b} \text{ if } t' \in T - \{t\}
\]
and the lemma is proved.

Proof of the Theorem 2.1. In order to use strong approximation (see §4.2) let \( \sigma : \tilde{H}_0 \to H_0 \) be the central \( k \)-isogeny where \( \tilde{H}_0 \) is the universal covering of \( H_0 \), such isogeny exists because \( H_0 \) is semisimple, indeed this comes from the fact that \( Q_{|V} \) is nondegenerate:
\[
H_0 = \left\{ \left( \begin{array}{cc} -M & 0 \\ 0 & 1 \end{array} \right) \mid M \in SO(Q_{|V}) \right\} \simeq SO(Q_{|V}) \text{ is semisimple.}
\]
Let \( \Lambda_S \) be the stabilizer of \( \Theta^4_S \) in \( H_0(k) \), and via the diagonal embedding we realize \( \Lambda_S \) as a \( S \)-arithmetic subgroup of \( H_0(k_S) \). Let \( \Lambda_T \) be the projection of \( \Lambda_S \) on \( H_0(k_T) \). We know that \( H_0 \) and \( \tilde{H}_0 \) are both isotropic over \( k_s \) (\( s \in S' \)) and that it has no compact factors over \( k_s \) for \( s \in S' \). Hence the strong approximation property applied to \( \tilde{H}_0 \) with respect to \( S' \) yields a set of open subgroups of finite index \( R_t \) in \( H_0(k_t) \) for \( t \in T \) such that the product \( R_T = \prod_{t \in T} R_t \) is contained in the closure of \( \Lambda_T \).
Indeed, let \( \Lambda \) be an arbitrary \( S \)-arithmetic subgroup in \( \tilde{H}_0(k) \), which can be seen as a discrete subgroup in \( \tilde{H}_0(k_S) \). Let denote by \( \Lambda_T \) its projection on \( \tilde{H}_0(k_T) \). By strong approximation (Prop. 4.6) the closure of \( \Lambda \) is open in \( k_{S'} \), in particular its projection \( \Lambda_T \) is dense in a open subgroup of \( H_0(k_T) \). Therefore, using Corollary 4.10 the subgroup \( \sigma(\Lambda_T) \) is also an arithmetic subgroup which is dense in an open subgroup of \( H_0(k_T) \). Hence
the subgroup $R_T := \text{cl}(\sigma(\tilde{\Lambda}_T)) \cap \text{cl}(\Lambda_T)$ is the good candidate since $\sigma(\tilde{\Lambda}_T)$ and $\Lambda_T$ are commensurable as arithmetic subgroups in $H_0(k_T)$.

Now let $\varepsilon > 0$ be chosen arbitrarily and choose a polydisc $(\Delta_s)_{s \in S'}$ in $k_S$ centred at the origin and small enough so that the following simultaneous inequalities are satisfied:

$$\text{for each } s \in S', \ |Q_s(z)|_s \leq \varepsilon \text{ and } |L_s(z)|_s \leq \varepsilon \text{ for any } z \in \Delta_s.$$ 

As above we introduce the set $X_t$ for $t \in T$ associated to with $R_T$. By the Lemma 6.2 we get an $x \in \mathcal{O}_S^1$ such that $x_s \in \Delta_s$ for all $s \in S'$ and $x_t \in X_t$ for all $t \in T$. In one hand for each $t \in T$ there exists $r_t \in R_t$ such that $z_t = r_t x_t \in Z_t(k_t)$. Since $R_t$ is open we can find $y_t \in R_t x_t$ close enough to $z_t$ which satisfies the following inequalities:

$$0 < |Q_t(y_t)|_t \leq \varepsilon/2 \text{ and } 0 < |L_t(y_t)|_t \leq \varepsilon/2.$$

In the other hand since $R_T$ is contained in the closure of $\Lambda_T$, we can find also $\gamma \in \Lambda_S$ such that:

$$0 < |Q_t((\gamma x)_t)|_t \leq \varepsilon \text{ and } 0 < |L_t((\gamma x)_t)|_t \leq \varepsilon \text{ for each } t \in T$$

It suffices to show that the element $\gamma x \in \mathcal{O}_S^1$ satisfies the last inequalities for $s \in S'$.

Since $\gamma \in H_0$, it leaves invariant both $Q_s$ and $L_s$ for all $s \in S'$ thus it leaves also invariant $Q_s$ for all $s \in S'$. Therefore we get the following inequalities

$$|Q_s((\gamma x)_s)|_s \leq \varepsilon \text{ and } |L_s((\gamma x)_s)|_s \leq \varepsilon \text{ for each } s \in S'.$$

This finishes the proof of Theorem 2.1.

7. Comments and open problems

The problem of null values. The argument used in [BP92] (Theorem A, (iii)) to prove that for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^1$ such that $0 < |Q_s(x)|_s < \varepsilon$ does not easily generalise for pairs. In fact, we do not prove that the $x \in \mathcal{O}_S^1$ of Theorem 2.1 satisfies the stronger condition that $0 < |Q_s(x)|_s < \varepsilon$ and $0 < |L_s(x)|_s < \varepsilon$ for finite places $s \in S_f$.

Towards density. It should be possible to obtain the density of $F(\mathcal{O}_S^2)$ for a pair $F = (Q, L)$ over $k_S$ under the same assumptions generalising those of Theorem 1.1 i.e. without the condition $Q_{|L=0}$ is nondegenerate added here for our purpose. For this we need a analog of Lemma 6 of [Gor04] for nonarchimedean completions which has no clear reason to fail in characteristic zero. A significant difference with the classical Oppenheim conjecture is that the stabilizer of such pairs is no more maximal, and the classification of intermediate subgroups is much more involved. Unfortunately we are not able to prove Lemma 6.1 for non archimedean completions and to avoid the use of strong approximation.

An Open problem. We conclude by mentioning a conjecture of Gorodnik (see [Gor04], conjecture 15) which concerns the assumption (2) of Theorem 1.1 in the real case. It is conjectured that the condition $Q_{|L=0}$ is isotropic can be replaced by the weaker assumption that the pencil $\alpha Q + \beta L^2$ is isotropic for any real numbers $\alpha, \beta$ such that $(\alpha, \beta) \neq (0, 0)$.

Conjecture 7.1 (Gorodnik). Let $F = (Q, L)$ be a pair consisting of one nondegenerate quadratic and one nonzero linear form in dimension $n \geq 4$. Suppose that

1. For every $\beta \in \mathbb{R}$, $Q + \beta L^2$ is indefinite.
2. For every $(\alpha, \beta) \neq (0, 0)$, with $\alpha, \beta \in \mathbb{R}$, $\alpha Q + \beta L^2$ is irrational.

Then $F(\mathcal{P}(\mathbb{Z}^n))$ is dense in $\mathbb{R}^2$. 

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The first condition is necessary for density to hold. The main issue is that this condition (contrarily to the condition that $Q_{|L=0}$ is indefinite) does not allow us to reduce to the four dimensional case. Hence all the strategy of the proof of Theorem 1.1 becomes needless regarding the impossibility to classify all the intermediate subgroups in higher dimension.

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