Anti-fuzzy SA-ideals with degree $(\lambda, \kappa)$ of SA-algebra

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Abstract

In this paper, we will give some information about the anti-fuzzy SA-ideals with degree $(\lambda, \kappa)$ of SA-algebra, there are many examples with be provide in order to make the SA-ideal clear. We will define the image and the inverse image of anti-fuzzy SA-ideals with degree $(\lambda, \kappa)$ of SA-algebra, and show how the image and the inverse image of anti-fuzzy SA-ideals with degree $(\lambda, \kappa)$ of SA-algebra are studied. At the end, the Cartesian product of anti-fuzzy SA-ideals with degree $(\lambda, \kappa)$ of SA-algebra are given.

Keywords: SA-ideals, anti-fuzzy SA-ideals, image and pre-image of anti-fuzzy SA-ideals, cartesian product of anti-fuzzy SA-ideals.

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1. Introduction

The concept of fuzzy subset was firstly being introduced by L.A. Zadeh in [7] after that many writers in different branches of mathematics has was used this concept. Especially in the area of fuzzy topology. From that time many research has been talked about it. "In 1967 "two classes of abstract algebras: BCK-algebras and BCI-algebras has been introduced by K.Is'eki and S.Tanaka [4]. "It is familiar that the class of BCK-algebras is asuitable subclass of the class of BCI-algebras. O.G. Xi [6] used the concept of fuzzy subset to BCK-algebras" and"also showed some of it is importance and it is properties. After that. Y.S. Hwang and S.S. Ahn [3] gave us an idea of an enlarged p-ideal in BCI-algebras with degree."

"In 2015 [2], Areej Tawfeeq Hameed gave new idea named SA-algebras and discovered many basic properties that are related to SA-ideal and"showed the idea of fuzzy"SA-ideals with degree $(\lambda,\kappa)$ of SA-algebras. "How to deal with homomorphism of image and inverse image of fuzzy SA-ideals with degree $(\lambda,\kappa)$ of SA-algebras was also being described by her.

In this paper, if Allah willing, we will show you an idea about fuzzy SA-ideals with degree $(\lambda,\kappa)$ of SA-algebras . "After that we"will study the homomorphism image and inverse image of anti-fuzzy SA-ideals with degree" $(\lambda,\kappa)$ of SA-algebra. "At the end we will demonstrate that the certain product of anti-fuzzy SA-ideals with degree $(\lambda,\kappa)$ of SA-algebras are anti-fuzzy SA-ideals with $(\lambda,\kappa)$ of SA-algebras.

2. Preliminaries
Now, we give some definitions and preliminary results needed in the later parts

2.1. Definition
Let \((Ψ, +, −, 0)\) be an algebra with two binary operations \((+)\) and \((−)\) and constant \((0)\). \(Ψ\) is named a \(SA\)-algebra \((SA − g)\), if it satisfies the following identities: \(∀\pi, ε, ω ∈ Ψ\),
\((SA_1) \pi − π = 0\),
\((SA_2) \pi − 0 = π\),
\((SA_3) (π − ε) − ω = π − (ω + ε)\),
\((SA_4) (π + ε) − (π + ω) = ε − ω\).

In \(X\) we can define a binary relation \((≤)\) by: \(\pi ≤ ε\) if and only if \(π − ε = 0\).

2.2. Example
Let \(Ψ = \{0, 1, 2, 3\}\) be a set with the following tables:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| − | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 3 | 2 | 1 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 1 | 0 | 3 |
| 3 | 3 | 2 | 1 | 0 |

**TABLE (1)**

**TABLE (2)**

Then \((Ψ, +, −, 0)\) is a \(SA − g\).

Lemma

Let \((Ψ, +, −, 0)\) be a \(SA − g\). Then for any \(π, ε ∈ Ψ\),
\((L_1) \pi + ε = π − (−ε)\),
\((L_2) \pi − ε = π + (−ε)\),
\((L_3) \pi − ε = −ε + π\).

2.3. Proposition
Let \((Ψ, +, −, 0)\) be a \(SA − g\). Then the following holds:
\(∀\pi, ε, ω ∈ Ψ\),
\((a_1) (π − ε) − ω = (π − ω) − ε\),
\((a_2) 0 − (π − ε) = (ε − π)\),
\((a_3) π − ε ≤ ω ⇒ π − ω ≤ ε\),
\((a_4) π ≤ ε ⇒ ω + ε ≤ ω + π\),
\((a_5) (π − ε) − (ω − ω) ≤ π − ω\) and \((π − ε) − (π − ω) ≤ ω − ε\),
\((a_6) π ≤ ε & ε ≤ ω ⇒ π ≤ ω\).

2.4. Definition
Let \((Ψ, +, −, 0)\) be a \(SA − g\) and let \(δ\) be a non-empty set of \(Ψ\). \(δ\) is named a \(SA − sg\) of \(Ψ\), if \(π + ε ∈ δ\) whenever \(π, ε ∈ δ\).
2.5. Definition
A non-empty subset $\rho$ of $\mathcal{A} - g (\Psi, +, -, 0)$ is named a SA-ideal of $\Psi$, $(\mathcal{A} - i)$, if it satisfies: for $\forall \pi, \varepsilon, \omega \in \Psi$
(1) $0 \in \rho$, 
(2) $(\pi + \omega) \in \rho \& (\varepsilon - \omega) \in \rho \Rightarrow (\pi + \varepsilon) \in \rho$.

2.6. Proposition
Every $\mathcal{A} - i$ of $\mathcal{A} - g \Psi$ is a $\mathcal{A} - sg$ of $\Psi$ and the converse is not true.

2.7. Lemma
A $\mathcal{A} - i$, $\rho$ of $\mathcal{A} - g (\Psi, +, -, 0)$ has the following property:
$\forall \pi \in \Psi, \forall \varepsilon \in \rho, \pi \leq \varepsilon \Rightarrow \pi \in \rho$.
If $\forall \pi \in \rho \Rightarrow - \pi \in \rho$.

2.8. Definition
Let $\mathcal{A}$ be a non-empty set, a fuzzy subset $\eta$ in $\mathcal{A}$ is a function $\eta: \mathcal{A} \to [0,1]$.

2.9. Definition
Let $\mathcal{A}$ be a non-empty set and $\eta$ be a fuzzy subset in $\mathcal{A}$, for $t \in [0,1]$, the set $\eta_t = \{ \pi \in \mathcal{A} | \eta(\pi) \geq t \}$ is named a level subset of $\eta$.

2.10. Remark
Let $\lambda$ and $\kappa$ be members of $(0, 1]$, and let $n$ and $r$ denote a natural number and a real number, respectively, such that $r < n$ unless otherwise specified.

2.11. Definition
Let $(\Psi, +, -, 0)$ be a $\mathcal{A} - g$, a fuzzy subset $\eta$ of $\Psi$ is named a fuzzy SA-ideal with degree $(\lambda, \kappa)$ of $\mathcal{A}$, if it satisfies:
(FL1) $\eta(0) \geq \lambda \eta(\varepsilon)$,
(FL2) $\eta(\pi + \varepsilon) \geq \kappa \min \{ \eta(\pi + \omega), \eta(\varepsilon - \pi) \}$.

2.12. Example
Let $\Psi = \{0, 1, 2, 3\}$ be a set with the following tables:

\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}

\begin{array}{c|cccc}
- & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 3 & 2 & 1 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 1 & 0 & 3 \\
3 & 3 & 2 & 1 & 0 \\
\end{array}

TABLE (3) \hspace{1cm} TABLE (4)

Then $(\Psi, +, -, 0)$ is $\mathcal{A} - g$. Define a fuzzy subset by: $\eta(\pi) = \begin{cases} 0.7 & \text{if } \pi \in \{0,1\} \\ 0.3 & \text{otherwise} \end{cases}$
$\rho_1 = \{0, 1\}$ is a $SA - i$ of $\Psi$. Routine calculation gives that $\eta$ is a $FSA - i$ with degree $\left(\frac{4}{7}, \frac{4}{7}\right)$ of $SA - g$ in $\Psi$.

2.13. Proposition
Let $A$ be a $SA - i$ of $SA - g$ in $\Psi$. Then for any fixed number $t$ in an open interval $(0,1)$, there exists $\eta$ of $SA - i$ of $\Psi$ such that $\eta$ is a $FSA - i$ of $\Psi$ with degree $rac{7}{4}$.

2.14. Theorem
Let $A$ be a non-empty subset of a $-g$, $\Psi$ and $\eta$ be a fuzzy subset of $\Psi \ni \eta$ into $\{0, 1\}$, so that $\eta$ is the characteristic function of $A$. Then $\eta$ is a $FSA - i$ of $\Psi$ $\Leftrightarrow \Lambda$ is a $SA - i$ of $\Psi$.

2.15. Proposition
Let $\eta$ be a $FSA - i$ of $\lambda, \kappa$ of $\Psi, +, -, 0$, then the following hold: $\forall \pi, \epsilon, \omega \in \Psi,\pi \leq \epsilon \Rightarrow \eta(\pi) \geq \lambda \kappa \eta(\epsilon)$, $\eta(\pi + \epsilon) \geq \lambda \kappa \min\{\eta(\pi + \epsilon), \eta(\epsilon)\}$.

2.16. Proposition
Let $\eta$ be a $FSA - i$ of $\lambda, \kappa$ of $\Psi$, $\omega \in \Psi$, and let $\eta_{t_1}, \eta_{t_2}$ be two level $SA - i$ of $\eta$, where $t_1 < t_2$, then the following are equivalent.

$E_1)$ $\eta_{t_1} = \eta_{t_2}$.

$E_2)$ There is no $\pi \in \Psi$ such that $t_1 \leq \eta(\pi) < t_2$.

2.17. Definition
Let $\eta$ be a fuzzy subset of $SA - g$ in $\Psi$. $\eta$ is a $FSA - i$ of $\Psi$, if $\forall \pi, \epsilon \in \Psi,\eta(\pi + \epsilon) \geq \min\{\eta(\pi), \eta(\epsilon)\}$.

Theorem
Let $\eta$ be a fuzzy subset of $SA - g$ in $\Psi$. If $\eta$ is a $FSA - i$ of $\Psi$ $\Leftrightarrow$ for every $t \in [0,1]$, $\eta_t$ is a $SA - i$ of $\Psi$, when $\eta_t \neq \emptyset$.

2.18. Theorem
Let $\eta$ be a fuzzy subset of $SA - g$ in $\Psi$. $\eta$ is a $FSA - i$ of $\lambda, \kappa$ of $\Psi$ $\Leftrightarrow$ for every $t \in [0,1]$, $\eta_t$ is a $SA - i$ of $\Psi$, when $\eta_t \neq \emptyset$.

2.19. Proposition
Every $FSA - i$ of $\lambda, \kappa$ of $SA - g$ in $\Psi$ is a $FSA - i$ of $\Psi$.

The converse of Proposition (2.21) is not true as the following example:

2.20. Example
Let $\Psi = \{0, 1, 2, 3\}$ in which $+$ and $-$ is defined by the following table:
It is easy to show that \((\leq g \pi_8, +, -, 0)\) is a \(SA - g\). \(I = \{0, 2\}\) is a \(SA - sg\) of \(\Psi\), but \(I\) is not a \(SA - i\) of \(\Psi\).

\(3 + 2 = 1 \in I\) and \(2/3 = 0 \in I\), but \(3 + 2 = 1 \notin I\).

Define \(\eta: (\leq g \pi_8, +, -) \rightarrow [0, 1]\) by \(\eta(\pi) = 0.5 \text{ if } \pi \in I\) and \(0 \text{ otherwise}\). \(\eta\) is a \(\leq g \pi_8 \Psi_2\) of \(\leq g \pi_8\), but \(\eta\) is not \(\leq g \pi_8 \Psi_2\) of \(\leq g \pi_8\).

2.21. Definition
Let \((\psi_1; +, -, 0)\) and \((\psi'_2; +, -, 0')\) be two \(\psi_1 - g\), the mapping \(f: (\psi_1; +, -, 0) \rightarrow (\psi'_2; +, -, 0')\) is named a homomorphism \((\psi_1 - homo)\), if it satisfies:
\[f(\pi + \epsilon) = f(\pi) + f(\epsilon), \quad f(\pi - \epsilon) = f(\pi) - f(\epsilon), \quad \forall \pi, \epsilon \in \psi_1.\]

2.22. Definition
Let \(f: (\psi_1; +, -, 0) \rightarrow (\psi'_2; +, -, 0')\) be a mapping non-empty sets \(\psi_1\) and \(\psi'_2\) respectively. If \(\mu\) is a fuzzy subset of \(\psi_1\), then the fuzzy subset \(\beta\) of \(\psi'_2\) defined by:
\[f(\eta)(\epsilon) = \sup \{ \eta(\pi): \pi \in f^{-1}(\epsilon) \}, \quad \text{if } f^{-1}(\epsilon) = \{ \epsilon \in \psi_1, f(\pi) = \epsilon \} \neq \emptyset, \quad \text{if } f^{-1}(\epsilon) = \emptyset, \text{ otherwise}\]
is said to be the image of \(\eta\) under \(f\).

Similarly, if \(\beta\) is a fuzzy subset of \(\psi'_2\), then the fuzzy subset \(\eta = (\beta \circ f)\) in \(\psi\) (i.e the fuzzy subset defined by \(\eta(\pi) = \beta(f(\pi))\), \(\forall \pi \in \psi_1\), is named the pre-image of \(\beta\) under \(f\).

2.23. Theorem
An into homomorphic pre-image of a \(\psi_1 - i\) of \((\lambda, \kappa)\) is also a \(\psi_2 - i\) of \((\lambda, \kappa)\).

2.24. Definition
A fuzzy subset \(\eta\) of \((\psi_1, +, -, 0)\) has sup property, if for any subset \(T\) of \(\psi_1\), there exist \(t_0 \in T\) such that \(\eta(t_0) = \sup \{ \mu(t) \mid t \in T\}\).

2.25. Theorem
Let \(f: (\psi_1; +, -, 0) \rightarrow (\psi'_2; +, -, 0')\) be a homo. Between \(\psi_1 - g, \psi'_2\) and \(\psi'_2 - g, \psi'_2\), respectively. For every \(\eta\) is \(\psi_1 - i\) of \((\lambda, \kappa)\) of \(\psi_1\) and with sup property, \(f(\eta)\) is a \(\psi_2 - i\) of \((\lambda, \kappa)\) of \(\psi'_2\).

3. The Structure of anti-fuzzy \(\psi_1\)-ideals with degree \((\lambda, \kappa)\) of \(\psi_2\)-algebra.
In this part, we introduce a new notion named an anti-fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of $SA$-algebra and study some of its basic properties.

### 3.1. Definition

Let $(\Psi, +, -, 0)$ be a $SA - g$, a fuzzy subset $\eta$ of $\Psi$ is named an anti-fuzzy $SA$-ideal with degree $(\lambda, \kappa)$ of $SA$ $(\text{AFSA} - i$ of $(\lambda, \kappa)$), if it satisfies: $\forall \pi, \varepsilon, \omega \in \Psi$, and $\lambda, \kappa$ is members of $[0, 1]$, 

\begin{align*}
(A\text{SA}_1) \quad & \eta(0) \leq \lambda \eta(\pi), \\
(A\text{SA}_2) \quad & \eta(\pi + \varepsilon) \leq \kappa \max \{\eta(\pi + \omega), \eta(\varepsilon - \omega)\}.
\end{align*}

### 3.2. Example

Let $\Psi = \{0, 1, 2, 3\}$ be a set with the following tables:

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 3 | 0 |
| 3 | 3 | 3 | 0 | 1 |

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 3 | 2 |
| 1 | 1 | 1 | 0 | 3 |
| 2 | 2 | 2 | 1 | 0 |
| 3 | 3 | 3 | 2 | 1 |

Then $(\Psi, +, -, 0)$ is a $SA - g$. Define a fuzzy subset $\eta: \Psi \rightarrow [0, 1]$ by: $\eta(0) = t_1, \eta(1) = \eta(2) = \eta(3) = t_2$, where $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Routine calculation gives that $\eta$ is an $\text{AFSA} - i$ of $(\lambda, \kappa)$.

### 3.3. Lemma

Let $\eta$ be an $\text{AFSA} - i$ of $(\lambda, \kappa)$ of $SA - g$, $(\Psi, +, -, 0)$. If $\pi \leq \varepsilon$, then $\eta(\pi) \leq \kappa \eta(\varepsilon)$, $\forall \pi, \varepsilon \in \Psi$.

**Proof:**

Assume that $\pi \leq \varepsilon$, then $\varepsilon - \pi = 0$, this together with $\pi - 0 = \pi$ and $\eta(\pi) \leq \kappa \eta(\varepsilon)$, we get

\begin{align*}
\eta(\pi - 0) &= \eta(\pi) \\
&= \max \{\eta(\varepsilon), \eta(0)\} \\
&\leq \kappa \max \{\eta(\varepsilon), \eta(0)\}.
\end{align*}

Subsequently $\eta(\pi) \leq \kappa \eta(\varepsilon)$.

### 3.4. Proposition

Let $\eta$ be an $\text{AFSA} - i$ of $(\lambda, \kappa)$ of $SA - g$, $(\Psi, +, -, 0)$. If $\forall \pi, \varepsilon, \omega \in \Psi$, the inequality $\pi + \varepsilon \leq \omega$ implies $\eta(\pi) \leq \kappa \max \{\eta(\varepsilon), \eta(\omega)\}$.

**Proof:**

Assume that the inequality $\pi + \varepsilon \leq \omega$ holds on $\Psi$, by Lemma (3.3), $\eta(\pi + \varepsilon) \leq \eta(\omega)$ --- (1).

By $(A\text{SA}_2)$, $\eta(\pi + \omega) \leq \kappa \max \{\eta(\pi + \varepsilon), \eta(\varepsilon - \omega)\}$. Put $\omega = 0$, then $\eta(\pi + 0) = \eta(\pi) \leq \kappa \max \{\eta(\pi + \varepsilon), \eta(\varepsilon - 0)\} = \max \{\eta(\pi + \varepsilon), \eta(\varepsilon)\}$ --- (2).
From (1) and (2), we get \( \eta(\pi) \leq \kappa \max \{ \eta(\varepsilon), \eta(\omega) \} \), \( \forall \, \pi, \varepsilon, \omega \in \Psi \). \( \triangle \)

3.5. Theorem

Let \( \eta \) be an anti-fuzzy subset of \( SA - g, (\Psi, +, -, 0) \). \( \eta \) is an AFSA \(- \) of \( \lambda, \kappa \) of \( \Psi \) \( \Leftrightarrow \) it satisfies: \( \forall \, \alpha \in [0, 1], \, U'(\eta; \alpha) \neq \phi \) imply \( U'(\eta; \alpha) \) is a \( SA - i \) of \( \Psi \), where \( U'(\eta; \alpha) \) = \{ \pi \in \Psi : \eta(\pi) \leq \alpha \} \) --- (A).

Proof: Assume that \( \eta \) is an AFSA \(- i \) of \( \lambda, \kappa \) of \( \Psi \), let \( \alpha \in [0, 1] \) be \( \exists \ U'(\eta; \alpha) \neq \phi \) and let \( \forall \, \pi, \varepsilon, \omega \in \Psi \) be \( \exists \, \pi \in U'(\eta; \alpha) \), then \( \eta(\pi) \leq \alpha \) and so by \( (ASA_1) \), \( \eta(0) \leq \kappa \eta(\pi) \leq \alpha \). Thus \( 0 \in U'(\eta; \alpha) \).

Now, let \( (\pi + \varepsilon), (\varepsilon - \omega) \in U'(\eta; \alpha) \). It follows from \( (ASA_2) \) that \( \eta(\pi + \varepsilon) \leq \kappa \max \{ \eta(\pi + \varepsilon), \varepsilon(\pi - \omega) \} \leq \alpha \), so that \( (\pi + \varepsilon) \in U'(\eta; \alpha) \). Subsequently \( U'(\eta; \alpha) \) is a \( SA - i \) of \( \Psi \).

On the contrary, Assume that \( \eta \) satisfies (A), then we only need to show that \( (ASA_1) \) and \( (ASA_2) \) are true. If \( (ASA_1) \) is false, then there exists \( \pi \in \Psi \) such that \( \eta(0) > \eta(\pi) \). If we take

\[ t = \frac{1}{n}[\eta(0) + \eta(\pi)], \] where \( n \in \mathbb{N}, n>1 \), then \( \eta(0)>t \) and \( 0 \leq \lambda \eta(\pi) < t \leq 1 \), thus \( \pi \in U'(\eta; \alpha) \) and \( U'(\eta; t) \neq \phi \).

As \( U'(\eta; \alpha) \) is a \( SA - i \) of \( \Psi \), we have \( 0 \in U'(\eta; \alpha) \), and so \( \eta(0) \leq t \). This is a contradiction. Subsequently \( \eta(0) \leq \lambda \eta(\pi), \, \forall \, \pi \in \Psi \).

Now, Assume \( (ASA_2) \) is not true then there exists \( \forall \, \pi, \varepsilon, \omega \in \Psi \exists \eta \eta(\pi + \varepsilon) > \kappa \max \{ \eta(\pi + \varepsilon), \varepsilon(\pi - \omega) \} \).

Taking \( \beta_0 = \frac{1}{n}[\eta(\pi + \varepsilon) + \kappa \max \{ \eta(\pi + \varepsilon), \varepsilon(\pi - \omega) \}] \), where \( n \in \mathbb{N}, n>1 \), we have

\( \beta_0 \in [0,1] \) and \( \kappa \max \{ \eta(\pi + \varepsilon), \varepsilon(\pi - \omega) \} < \beta_0 < \eta(\pi + \varepsilon) \), it follows that

\( \kappa \max \{ \eta(\pi + \varepsilon), \varepsilon(\pi - \omega) \} \} \in U'(\eta, \beta_0) \) and \( \pi + \varepsilon \notin U'(\eta, \beta_0) \), this is a contradiction, then \( (ASA_2) \) is true. Subsequently \( \eta \) is an AFSA \(- i \) of \( \lambda, \kappa \) of \( \Psi \). \( \triangle \)

3.6. Corollary

A fuzzy subset \( \eta \) of \( SA - g, (\Psi, +, -, 0) \) is an AFSA \(- i \) of \( \lambda, \kappa \) of \( \Psi \) \( \Leftrightarrow \) for every \( t \in [0, 1], \eta'_t \) is a \( SA - i \) of \( \Psi \), where \( \eta'_t = \{ \pi \in \Psi : \eta(\pi) \leq t \} \).

Proof: \( \text{ Ago } U'(\eta; \alpha) = \eta'_t = \{ \pi \in \Psi : \eta(\pi) \leq t \}, \forall t \in [0, 1], \) then it is clear by Theorem (3.5). \( \triangle \)

3.7. Corollary

If a fuzzy subset \( \mu \) of \( SA - g, (\Psi, +, -, 0) \) is an AFSA \(- i \) of \( \lambda, \kappa \) of \( \Psi \), then for every \( t \in \text{Im}(\mu) \), \( \mu'_t \) is a \( SA - i \) of \( \Psi \).

Proof: It is clear by Corollary (3.6). \( \triangle \)

3.8. Corollary

Let \( I \) be a \( SA - i \) of \( SA - g, (\Psi, +, -, 0) \), then for any fixed number \( (t) \) in the open interval \( (0, 1) \), there exists \( \eta \) is AFSA \(- i \) of \( \lambda, \kappa \) of \( \Psi \) \( \exists \eta'_t = 1 \).

Proof:
Define $\eta: \Psi \rightarrow [0:1]$ by $\eta(\pi) = \left\{ \begin{array}{ll} 0, & \text{if } \pi \in I \\ t, & \text{if } \pi \not\in I \end{array} \right.$, where $t$ is a fixed number in $(0,1)$.

Clearly, $\eta(0) \leq \lambda \eta(\pi)$ and we have one of two level sets $\mu'_{\alpha} = 1$, $\mu'_{\alpha} = \Psi$, which are two $SA - i$ of $\Psi$, then from Corollary (3.6), $\mu$ is $AFSA - i$ of $(\lambda, \kappa)$ of $\Psi$. \( \triangle \)

### 3.9. Proposition

Let $\mu$ be an $AFSA - i$ of $(\lambda, \kappa)$ of $SA - g, (\Psi', +, - , 0)$, then the following hold:

1. $\epsilon \leq \omega \Rightarrow \eta(\epsilon) \leq \lambda \kappa \eta(\omega)$
2. $\eta(\pi + \epsilon) \leq \lambda \kappa \min \{\eta(\pi + \epsilon), \eta(\epsilon)\}$

**Proof:**

By Definition (3.1), we have $\eta(\pi + \epsilon) \leq \kappa \min \{\eta(\pi + \omega), \eta(\epsilon - \omega)\}$.

After $\epsilon \leq \omega$, then $\epsilon - \omega = 0$. Putting $\epsilon = 0$, we have

$$\eta(\epsilon) = \eta(0 + \epsilon) \leq \kappa \min \{\eta(0 + \omega), \eta(0)\}$$

$$\leq \kappa \min \{\eta(\omega), \lambda \eta(\omega)\}, \text{by } (\eta(0) \geq \lambda \eta(\omega))$$

$$= \lambda \kappa \eta(\omega).$$

We have $\eta(\pi + \epsilon) \leq \kappa \min \{\eta(\pi + \omega), \eta(\epsilon - \omega)\}$.

If $\pi = \epsilon$, then

$$\eta(\pi + \epsilon) \leq \kappa \min \{\eta(\pi + \epsilon), \eta(\epsilon - \epsilon)\}$$

$$= \kappa \min \{\eta(\pi + \epsilon), \eta(0)\}$$

$$\leq \kappa \min \{\eta(\pi + \epsilon), \lambda \eta(\epsilon)\}$$

$$= \lambda \kappa \min \{\eta(\pi + \epsilon), \eta(\epsilon)\}.$$

The proof is completed. \( \triangle \)

### 3.10. Proposition

The interpart of any set of $AFSA - i$ of $(\lambda, \kappa)$ of $SA - g, (\Psi', +, - , 0)$ is also an $AFSA - i$ of $(\lambda, \kappa)$ of $\Psi$.

**Proof:**

Let $\{\eta_i | i \in \Lambda\}$ be a family of $AFSA - i$ of $(\lambda, \kappa)$ of $SA - g, \Psi$, then $\forall \pi, \epsilon, \omega \in \Psi$,

$$\left( \bigcap_{i \in \Lambda} \eta_i \right)(0) = \inf(\eta_i(0)) \leq \lambda \inf(\eta_i(\pi)) = \lambda \bigcap_{i \in \Lambda} \eta_i(\pi)$$

and

$$\left( \bigcap_{i \in \Lambda} \eta_i \right)(\pi + \epsilon) = \inf(\eta_i(\pi + \epsilon)) \leq \kappa \inf \left( \max \{\eta_i(\pi + \omega), \eta_i(\epsilon - \omega)\} \right)$$

$$\leq \kappa \max \{\inf(\eta_i(\pi + \omega)), \inf(\eta_i(\epsilon - \omega))\}$$

$$= \kappa \max \left\{ \left( \bigcap_{i \in \Lambda} \eta_i \right)(\pi + \omega), \left( \bigcap_{i \in \Lambda} \eta_i \right)(\epsilon - \omega) \right\}. \triangle$$

### 3.11. Proposition

Let $\{\eta_i | i \in \Lambda\}$ be a family of $AFSA - i$ of $(\lambda, \kappa)$ of $SA - g, (\Psi', +, - , 0)$, then $\bigcup_{i \in \Lambda} \eta_i$ is an $AFSA - i$ of $(\lambda, \kappa)$ of $\Psi$, where $\eta_i \subseteq \eta_{i+1}$, $\forall i \in \Lambda$.

**Proof:**
Ago \( \{ \eta_i \mid i \in \Lambda \} \) be a family of AFSA \(- i \) of \((\lambda, \kappa) \) of \( \Psi \) and \( \eta_i \subseteq \eta_i \), \( \forall \) \( i \in \Lambda \), then \( \forall \) \( \pi, \varepsilon, \omega \in \Psi \).

\[
\bigcup_{i \in \Lambda} \eta_i(0) = \sup(\mu_i(0)) \leq \lambda \sup(\eta_i(\pi)) = \lambda \bigcup_{i \in \Lambda} \eta_i(\pi)
\]

and

\[
\bigcup_{i \in \Lambda} \eta_i(\pi + \varepsilon) = \sup(\eta_i(\pi + \varepsilon))
\]

\[
\leq \kappa \sup(\eta_i(\pi + \omega), \eta_i(\varepsilon - \omega))
\]

\[
= \kappa \max\{\sup(\eta_i(\pi + \omega)), \sup(\eta_i(\varepsilon - \omega))\}
\]

\[
= \kappa \max\{\bigcup_{i \in \Lambda} \eta_i(\pi + \omega), \bigcup_{i \in \Lambda} \eta_i(\varepsilon - \omega)\}.
\]

This completes the proof. \( \triangle \)

4. The relation of anti-fuzzy SA-subalgebras and anti-fuzzy SA-ideals with degree \((\lambda, \kappa)\) of SA-algebra.

In this part, we give the relation between anti-fuzzy SA-subalgebras and anti-fuzzy SA-ideals with degree \((\lambda, \kappa)\) of SA-algebras and we gives some properties of its.

4.1. Definition

Let \((\Psi, +, - , 0)\) be a SA \(- g\), a fuzzy subset \( \eta \) of \( \Psi \) is named an anti-fuzzy SA-subalgebra of \( \Psi \), \((AFSA - sg)\), \( \forall \) \( \pi, \varepsilon \in \Psi \), \( \eta(\pi + \varepsilon) \leq \max\{\eta(\pi), \eta(\varepsilon)\} \).

4.2. Theorem

Let \( \eta \) be a fuzzy subset of \( SA - g, (\Psi, +, - , 0) \). Then \( \eta \) is an \( AFSA - sg \) of \( \Psi \) \( \Leftrightarrow \) for every \( t \in [0,1] \), \( \eta_t \) is a \( SA - sg \) of \( \Psi \).

Proof:

Assume that \( \eta \) is an \( AFSA - sg \) of \( \Psi \), let \( \pi, \varepsilon \in \Psi \) be \( \exists \) \( \pi \in \eta \), and \( \varepsilon \in \eta \), then \( \eta(\pi) \geq t \) and \( \eta(\varepsilon) \geq \delta \), \( t \). Ago \( \eta \) is an \( AFSA - sg \), it follows that \( \eta(\pi + \varepsilon) \leq \max\{\eta(\pi), \eta(\varepsilon)\} \geq t \) and that \( (\pi + \varepsilon) \in \eta \). Subsequently \( \eta \) is a \( SA - sg \) of \( \Psi \).

On the contrary, Assume \( \eta(\pi + \varepsilon) \leq \max\{\eta(\pi), \eta(\varepsilon)\} \) is not true, then \( \exists \) \( \pi' \) and \( \varepsilon' \in \Psi \exists \eta(\pi' + \varepsilon') > \max\{\eta(\pi'), \eta(\varepsilon')\} \). Putting \( t' = (\eta(\pi' + \varepsilon')) \) \( + \) \( \max\{\eta(\pi'), \eta(\varepsilon')\} \) \( /n \), where \( n \) \( \in \mathbb{N} \), \( n > 1 \), then \( \eta(\pi' + \varepsilon') < t' \) and \( 0 < t' < \max\{\eta(\pi'), \eta(\varepsilon')\} \leq 1 \), subsequently \( \eta(\pi') > t' \) and \( \eta(\varepsilon') > t' \), which imply that \( \pi' \in \eta \), and \( \varepsilon' \in \eta \), Ago \( \eta \) is a \( SA - sg \), it follows that \( \pi' + \varepsilon' \in \eta_t \), and that \( \eta(\pi' + \varepsilon') \geq t' \), this is also a contradiction. Therefore \( \eta(\pi + \varepsilon) \leq \max\{\eta(\pi), \eta(\varepsilon)\} \).

Subsequently \( \eta \) is an \( AFSA - sg \) of \( \Psi \). \( \triangle \)

4.3. Corollary

Let \( \eta \) be a fuzzy subset of \( - g, (\Psi, +, - , 0) \). If \( \eta \) is an \( AFSA - sg \), then \( \forall t \in \text{Im}(\eta) \), \( \eta_t \) is a \( SA - sg \) of \( \Psi \), when \( \eta_t \neq \phi \).

Proof:

It is clear by Theorem (4.2). \( \triangle \)

4.4. Theorem
Let \( \eta \) be a fuzzy subset of \( SA = g_\lambda (\Psi, +, -, 0) \). \( \eta \) is an \( AFSA - i \) of \( (\lambda, \kappa) \) of \( \Psi \) \( \Leftrightarrow \) for every \( t \in [0,1] \), \( \eta \) is a \( SA - i \) of \( \Psi \).

Proof:

Assume that \( \eta \) is an \( AFSA - i \) of \( (\lambda, \kappa) \) of \( \Psi \), by (ASA_1), we have \( \eta(0) \leq \lambda \eta(\pi) \), \( \forall \pi \in \Psi \), therefore, \( \eta(0) \leq \lambda \eta(\pi) \geq t \) for \( \pi \in \eta \), and so \( 0 \in \eta \).

Let \( \pi, \varepsilon, \omega \in \Psi \) be such that \( (\pi + \omega) \in \eta \) and \( (\varepsilon - \omega) \in \eta \), then \( \eta(\pi + \omega) \geq t \) and \( \eta(\varepsilon - \omega) \geq t \).

Putting \( t' = (\lambda \eta(\pi') + \eta(0))/2 \), then \( \eta(0) > t' \) and \( 0 \leq t' < \eta(\pi') \leq 1 \), then \( \pi' \in \eta \) and \( \eta \neq \phi \). As \( \eta \) is a \( SA - i \) of \( \Psi \), we have \( 0 \in \eta \), and so \( \eta(0) \geq t' \).

This is a contradiction. Subsequently \( (ASA_1) \) is true.

Now, Assume \( (ASA_2) \) is false, then \( \exists \pi', \varepsilon', \omega' \in \Psi \exists \eta(\pi' + \varepsilon') > \kappa \max \{\eta(\pi' + \omega'), \eta(\varepsilon' - \omega')\} \).

Putting \( t' = (\eta(\pi' + \varepsilon') + \kappa \max \{\eta(\pi' + \omega'), \eta(\varepsilon' - \omega')\})/n \), where \( n \in N, n>1 \), then \( \eta(\pi' + \varepsilon') > t' \) and \( 0 \leq t' < \kappa \max \{\eta(\pi' + \omega'), \eta(\varepsilon' - \omega')\} \leq 1 \), hence \( \eta(\pi' + \omega') > t' \) and \( \eta(\varepsilon' - \omega') > t' \), which imply that \( \pi' + \omega' \) \( \in \eta \), and \( \varepsilon' - \omega' \) \( \in \eta \). As \( \eta \) is a \( SA - i \), it follows that \( \pi' + \varepsilon' \in \eta \), and that \( \eta(\pi' + \varepsilon') \geq t' \), this is also a contradiction.

Subsequently \( \eta \) is an \( AFSA - i \) of \( (\lambda, \kappa) \) of \( \Psi \) \( \Rightarrow \).

4.5. Corollary

Let \( \eta \) be a fuzzy subset of \( SA = g_\lambda (\Psi, +, -, 0) \). If \( \eta \) is an \( AFSA - i \) of \( (\lambda, \kappa) \) of \( \Psi \), then for every \( t \in \text{Im}(\eta) \), \( \mu_i \) is a \( SA - i \) of \( \Psi \), when \( \eta \neq \phi \).

Proof:

It is clear by Theorem (4.4) \( \Rightarrow \).

4.6. Proposition

Every \( AFSA - i \) of \( (\lambda, \kappa) \) of \( SA = g_\lambda (\Psi, +, -, 0) \) is an \( AFSA - sg \) of \( \Psi \).

Proof:

Assume \( \eta \) is \( AFSA - i \) of \( (\lambda, \kappa) \) of \( \Psi \), then by Theorem (3.4), \( \forall t \in [0,1], \eta \) is a \( SA - i \) of \( \Psi \). By Proposition (3.7), \( \forall t \in [0,1], \eta \) is a \( SA - sg \) of \( \Psi \). Subsequently \( \eta \) is an \( AFSA - sg \) of \( \Psi \) by Theorem (4.2) \( \Rightarrow \).

Note that: The converse of Proposition (4.6) is not true as the following example:

4.7. Example

Let \( \Psi = \{0, 1, 2, 3\} \) in which \((+\) \) and \((-\) \) is defined by the following table:
It is easy to show that $(\Psi, +, - , 0)$ is a $SA - g$. $S = \{0, 2\}$ is a $SA - sg$ of $\Psi$, but $S$ is not a $SA - i$ of $i(\lambda, \kappa)$ of $\Psi$, Ago $3 + 3 = 2 \in S$ and $2 - 3 = 0 \in S$, but $3 + 2 = 1 \not\in S$.

Define $\eta$: $\Psi \rightarrow [0, 1]$ by $\eta(\pi) = \begin{cases} 0 & \text{if } \pi \in S \\ 0.5 & \text{otherwise} \end{cases}$.

$\eta$ is an $AFSA - sg$ of $\Psi$, but $\eta$ is not $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi$.

4.8. Theorem

A homomorphic pre-image of an $AFSA - i$ of $i(\lambda, \kappa)$ is also an $AFSA - i$ of $i(\lambda, \kappa)$.

Proof:

Let $f: (\Psi'; +', -', 0') \rightarrow (\Psi', +', -', 0')$ be a $SA - homo$, $\beta$ an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$ and $\eta$ the pre-image of $\beta$ under $f$, then $\beta(f(x)) = \mu(x)$, for all $x \in X$. Ago $f(x) \in Y$ and $\beta$ is an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$, it follows that $\beta(0') \geq \beta(f(\pi)) = \eta(\pi), \forall \pi \in \Psi$, where $0'$ is the zero element of $\Psi'$.

But $\beta(0') = \beta(f(0)) = \mu(0)$ and so $\eta(0) \leq \lambda \eta(x)$, for $x \in \Psi$.

Now let $\pi, \varepsilon, \omega \in \Psi$, then we get

$$\eta(\pi + \varepsilon) = \beta(f(\pi + \varepsilon)) = \beta(f(\pi) + f(\varepsilon))$$

$$\leq \kappa \max \{ \beta(f(\pi) + f(\omega)), \beta(f(\varepsilon) + f(\omega)) \}$$

$$= \kappa \max \{ \beta(f(\pi + \omega)), \beta(f(\varepsilon - \omega)) \}$$

$$= \kappa \max \{ \eta(\pi + \omega), \eta(\varepsilon - \omega) \}$$

i.e., $\eta(\pi + \varepsilon) \leq \kappa \max \{ \eta(\pi + \varepsilon), \eta(\varepsilon - \omega) \}$. Subsequently $\eta$ is an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi$. ⊢

4.9. Theorem

Let $f: (\Psi'; +', -', 0') \rightarrow (\Psi', +', -', 0')$ be a $SA - homo$. For every $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$ and with sup property, $f(\eta)$ is an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$.

Proof:

By definition $\beta(\varepsilon') = f(\mu(\varepsilon')) = \sup \{ \eta(\pi) : \pi \in f^{-1}(\varepsilon') \}, \forall \varepsilon' \in \Psi'$ (sup($\phi$) = 0). We have to prove that $\beta(\pi' + \varepsilon') \leq \kappa \max \{ \beta(\pi' + \omega'), \beta(\varepsilon' - \omega') \}, \forall \pi', \varepsilon', \omega' \in \Psi'$.

Let $f: (\Psi'; +', -', 0') \rightarrow (\Psi', +', -', 0')$ be a onto $SA - homo$, $\eta$ is an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$ and with sup property and $\beta$ the image of $\eta$ under $f$. Ago $\eta$ is an $AFSA - i$ of $i(\lambda, \kappa)$ of $\Psi'$, we have $\eta(0) \geq \lambda \eta(\pi), \forall \pi, \varepsilon \in \Psi$.

Note that $0 \in f^{-1}(0')$, where $0$ and $0'$ are the zero elements of $\Psi$ and $\Psi'$ respectively. Thus $\beta(0') = \sup_{\pi \in f^{-1}(0')} \eta(t) = \eta(0) \leq \lambda \eta(\pi), \forall \pi \in \Psi$ which implies that $\beta(0') \leq \sup_{\pi \in f^{-1}(\pi')} \eta(t) = \beta(\pi')$, for any $\pi' \in \Psi$.
Let \( \eta \) and \( \beta \) be fuzzy subsets of a set \( \Psi \). If \( \eta(\pi) \leq \max \{ \beta(\pi), \beta(\epsilon) \} \),

\[
\eta(\pi + \epsilon) = \max \{ \eta(\pi), \eta(\epsilon) \}
\]

\[
\eta(\pi - \epsilon) = \max \{ \eta(\pi), \eta(\epsilon) \}.
\]

Therefore, \( \eta(\pi + \epsilon) \leq \max \{ \beta(\pi + \epsilon), \beta(\epsilon) \} \).

Subsequently, \( \eta \) is an \( \text{AFSA} \) of \( \Psi \).

If \( f \) is not onto, \( \forall \pi' \in \Psi \), we define \( \Psi_{\pi'} := f^{-1}(\pi') \). Again, \( f \) is a \( \text{homo} \) of \( \Psi \), we get \( \forall \pi', \epsilon', \omega' \in \Psi_{\pi'} \),

\[
\Psi_{\pi'} \cup \Psi_{\epsilon'} \subset \Psi_{\pi' + \epsilon'} \quad \text{and} \quad \Psi_{\epsilon'} \cup \Psi_{\omega'} \subset \Psi_{\pi' + \omega'}.
\]

If \( (\pi' + \epsilon') \not\in \text{Im}(f) = f(\Psi) \), then by definition \( \beta(\pi' + \epsilon') = 0 \).

But if \( (\pi' + \epsilon') \not\in f(\Psi) \), i.e., \( \Psi_{\pi' + \epsilon'} = \emptyset \), then by (*) at least one of \( \pi', \epsilon', \omega' \not\in f(\Psi) \) and consequently \( \beta(\pi' + \epsilon') \leq 0 = \max \{ \beta(\pi' + \epsilon'), \beta(\epsilon') \} \).

5. Cartesian product of anti-fuzzy SA-ideals with degree \( (\lambda, \kappa) \) of SA-algebra:

In this part, we give the Cartesian product of anti-fuzzy SA-ideals with degree of SA-algebras and we gives some properties of its.

5.1. Definition
A fuzzy relation \( \Omega \) on any set \( \gamma \) is a fuzzy subset \( \Omega \times \gamma \rightarrow [0,1] \).

5.2. Definition
If \( \Omega \) is a fuzzy relation on sets \( \gamma \) and \( \beta \) is a fuzzy subset of \( \gamma \), then \( \Omega \) is a fuzzy relation on \( \beta \), if \( \Omega(\pi, \epsilon) \geq \max \{ \beta(\pi), \beta(\epsilon) \} \), \( \forall \pi, \epsilon \in \gamma \).

5.3. Definition
Let \( \eta \) and \( \beta \) be fuzzy subsets of a set \( \gamma \). The Cartesian product of \( \eta \) and \( \beta \) is defined by \( \eta \times \beta \) is a fuzzy relation on \( \eta \),

\[
(\eta \times \beta)(\pi, \epsilon) = \max \{ \eta(\pi), \beta(\epsilon) \}, \forall \pi, \epsilon \in \gamma.
\]

5.4. Lemma
Let \( \gamma \) be a set and \( \eta \) and \( \beta \) be fuzzy subsets of \( \gamma \). Then

(1) \( \eta \times \beta \) is a fuzzy relation on \( \eta \),

(2) \( (\eta \times \beta)(\pi, t) = \eta(\pi) \times \beta(t), \forall t \in [0,1] \).

5.5. Definition
Let \( \gamma \) be a set and \( \beta \) be fuzzy subset of \( \gamma \). The strongest fuzzy relation on \( \gamma \), that is, a fuzzy relation on \( \beta \) is \( \Omega_{\beta} \) given by

\[
\Omega_{\beta}(\pi, \epsilon) = \max \{ \beta(\pi), \beta(\epsilon) \}, \forall \pi, \epsilon \in \gamma.
\]
5.6. Lemma
For a given fuzzy subset $\beta$ of a set $\gamma$, let $\Omega_\beta$ be the strongest fuzzy relation on $\gamma$. Then for $t \in [0,1]$, we have $(\Omega_\beta)_t = \beta_t \times \beta_t$.

5.7. Proposition
For a given fuzzy subset $\beta$ of $SA - g, (\Psi, +, -, 0)$, let $\Omega_\beta$ be the strongest fuzzy relation on $\Psi$.
If $\beta$ is an AFSA $- i o f (A, \kappa)$ of $\Psi \times \Psi$, then $\Omega_\beta (0,0) \leq \lambda \Omega_\beta (\pi, \pi), \forall \pi \in \Psi$.
Proof:
Ago $\Omega_\beta$ is a strongest fuzzy relation of $\Psi \times \Psi$, it follows from that, $\lambda \Omega_\beta (\pi, \pi) = \lambda \max\{\beta (\pi), \beta (\pi)\} \geq \max\{\beta (0), \beta (0)\} = \Omega_\beta (0,0)$, which implies that $\lambda \Omega_\beta (\pi, \pi) \geq \Omega_\beta (0,0)$. ◊

5.8. Proposition
For a given fuzzy subset $\beta$ of $SA - g, (\Psi, +, -, 0)$, let $\Omega_\beta$ be the strongest fuzzy relation on $\Psi$.
If $\beta$ is an $A \Phi 27 \Phi 527 \Phi 59 (A \Phi 9, A \Phi 8)$ of $\Psi \times \Psi$. Then $\beta (0) \leq \lambda \beta (\pi), \forall \pi \in \Psi$.
Proof:
Ago $\Omega_\beta$ is an $A \Phi 27 \Phi 527 \Phi 59 (A \Phi 9, A \Phi 8)$ of $\Psi \times \Psi$. But this means that, $\lambda \max\{\beta (\pi), \beta (\pi)\} \geq \max\{\beta (0), \beta (0)\}$ which implies that $\beta (0) \leq \lambda \beta (\pi)$. ◊

5.9. Remark
Let $\Psi$ and $\Psi'$ be two $SA - g$, we define $(\cdot)$ and $(-)$ on $\Psi \times \Psi'$ by $:(\pi, \varepsilon), (\tau, \upsilon) \in \Psi \times \Psi', (\pi, \varepsilon) + (\tau, \upsilon) = (\pi + \tau, \varepsilon + \upsilon)$ and $(\pi, \varepsilon) - (\tau, \upsilon) = (\pi - \tau, \varepsilon - \upsilon)$. Then clearly $(\Psi \times \Psi', +, -, (0,0))$ is a $SA - g$.

5.10. Theorem
Let $\mu$ and $\beta$ be an $A \Phi 27 \Phi 527 \Phi 59 (A \Phi 9, A \Phi 8)$ of $SA - g, (\Psi, +, -, 0)$. Then $\eta \times \beta$ is an AFSA $- i o f (A, \kappa)$ of $\Psi \times \Psi$.
Proof:
Note first that $\Psi (\pi, \varepsilon) \in \Psi \times \Psi$,
$(\eta \times \beta)(0,0) = \max \{\eta (0), \beta (0)\} \leq \lambda \max \{\eta (\pi), \beta (\varepsilon)\} = \lambda (\eta \times \beta)(\pi, \varepsilon)$. Now let $(\pi_1, \pi_2), (\varepsilon_1, \varepsilon_2) \subseteq (\omega_1, \omega_2) \subseteq \Psi \times \Psi$.
$(\eta \times \beta)(\pi_1 + \varepsilon_1, \pi_2 + \varepsilon_2) = \max \{\eta(\pi_1 + \varepsilon_1), \beta(\pi_2 + \varepsilon_2)\}$
$\leq \max \{\kappa \max \{\eta (\pi_1 + \omega_1), \eta (\varepsilon_1 - \omega_1)\}, \kappa \max \{\beta (\pi_2 + \omega_2), \beta (\varepsilon_2 - \omega_2)\}\}$
$= \kappa \max \{\eta (\pi_1 + \omega_1), \beta (\pi_2 + \omega_2)\} \leq \kappa \max \{\eta (\varepsilon_1 - \omega_1), \beta (\varepsilon_2 - \omega_2)\}$
$\leq \max \{\eta (\pi_1 + \omega_1), \beta (\pi_2 + \omega_2)\} \leq \max \{\eta (\varepsilon_1 - \omega_1), \beta (\varepsilon_2 - \omega_2)\}$
$\leq \kappa \max \{\eta (\pi_1 + \omega_1), \beta (\pi_2 + \omega_2)\} \leq \kappa \max \{\eta (\varepsilon_1 - \omega_1), \beta (\varepsilon_2 - \omega_2)\}$
Subsequently $\eta \times \beta$ is an AFSA $- i o f (A, \kappa)$ of $\Psi \times \Psi$. ◊

5.11. Theorem
Let $\eta$ and $\beta$ be anti-fuzzy subsets of $SA - g, (\Psi, +, -, 0)$ such that $\eta \times \beta$ is an AFSA $- i o f (A, \kappa)$ of $\Psi \times \Psi$. Then $\forall \pi \in \Psi$, (i) Either $\eta (0) \leq \lambda \eta (\pi)$ or $\beta (0) \leq \lambda \beta (\pi)$.
(ii) \( \eta(0) \leq \lambda \eta(\pi), \forall \pi \in \Psi \), then either \( \beta(0) \leq \lambda \beta(\pi) \) or \( \beta(0) \leq \lambda \eta(\pi) \).

(iii) If \( \beta(0) \leq \lambda \beta(\pi), \forall \pi \in \Psi \), then either \( \eta(0) \leq \lambda \eta(\pi) \) or \( \eta(0) \leq \lambda \beta(\pi) \).

(iv) Either \( \eta \) or \( \beta \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \).

Proof:

Assume that \( \eta(0) > \lambda \eta(\pi) \) and \( \beta(0) > \lambda \beta(\varepsilon) \), for some \( \pi, \varepsilon \in \Psi \). Then
\[
\lambda (\eta \times \beta)(\pi, \varepsilon) = \lambda \max \{ \eta(\pi), \beta(\varepsilon) \} < \max \{ \eta(0), \beta(0) \} = (\eta \times \beta)(0,0).
\]
This is a contradiction and we obtain (i).

(ii) Assume that there exist \( \pi, \varepsilon \in \Psi \) such that \( \beta(0) > \lambda \eta(\pi) \) and \( \beta(0) > \lambda \beta(\varepsilon) \). Then
\[
(\eta \times \beta)(0,0) = \max \{ \eta(0), \beta(0) \} = \beta(0)
\]
follows that
\[
\lambda (\eta \times \beta)(\pi, \varepsilon) = \lambda \max \{ \eta(\pi), \beta(\varepsilon) \} < \beta(0) = (\eta \times \beta)(0,0)
\]
which is a contradiction. Subsequently (ii) holds.

(iii) It is true by similar method to part (ii).

(iv) Assume \( \beta(0) \leq \lambda \beta(\pi) \) by (i), then form (iii) either \( \eta(0) \leq \lambda \eta(\pi) \) or \( \eta(0) \leq \lambda \beta(\pi) \), \( \forall \pi \leq \pi_{06} \).

\[
\text{If } \eta(0) \leq \lambda \beta(\pi), \forall \pi \leq \pi_{06}, \text{ then}
\]
\[
\beta(x+y) \leq \max \{ \eta(0), \beta(x+y) \} \leq \max \{ \eta(x), \beta(0) \} = \lambda \eta(x)
\]
Which proves that \( \mu \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \). Subsequently either \( \eta \) or \( \beta \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \).

5.12. Theorem

Let \( \beta \) be a fuzzy subset of \( \Psi \times g, (\Psi, +, - , 0) \) and let \( \Omega_{\beta} \) be the strongest fuzzy relation on \( \Psi \), then \( \beta \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \) \( \iff \) \( \Omega_{\beta} \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \times \Psi \).

Proof:

Assume that \( \beta \) is an AFSA \(-i\) of \((\lambda, \kappa)\) of \( \Psi \). By Proposition (5.7), we get, \( \Omega_{\beta}(0,0) \leq \lambda \Omega_{\beta}(\pi, \varepsilon), \forall (\pi, \varepsilon) \in \Psi \times \Psi \).

\[
\text{Let } (\pi, \pi_{1}) \in \Psi \times \Psi, \text{ we have from (ASA}_{2}\ :
\]
\[\Omega_{\beta} (\pi_1 + \varepsilon_1, \pi_2 + \varepsilon_2) = \max \{\beta (\pi_1 + \varepsilon_1), \beta (\pi_2 + \varepsilon_2)\}\]
\[\leq \max \{\kappa \max \{\beta (\pi_1 + \omega_1), \beta (\varepsilon_1 - \omega_1)\}, \kappa \max \{\beta (\pi_2 + \omega_2), \beta (\varepsilon_2 - \omega_2)\}\}\]
\[= \max \{\kappa \max \{\beta (\pi_1 + \omega_1), \beta (\pi_2 + \omega_2)\}, \kappa \max \{\beta (\varepsilon_1 - \omega_1), \beta (\varepsilon_2 - \omega_2)\}\}\]
\[= \max \{\kappa \Omega_{\beta} ((\pi_1 + \omega_1), (\pi_2 + \omega_2)), \kappa \Omega_{\beta} ((\varepsilon_1 - \omega_1), (\varepsilon_2 - \omega_2))\}\]

Subsequently \(\Omega_{\beta}\) is an \(AFSA - i of (\lambda, \kappa) of \Psi \times \Psi\).

On the contrary, assume that \(\Omega_{\beta}\) is an \(AFSA - i of (\lambda, \kappa) of \Psi \times \Psi\), by Proposition (5.8) \(\beta(0)\)
\[\leq \lambda \beta (\pi), \forall \pi \in \Psi,\) which prove \((ASA_1)\).

Now, let \((\pi_1,\pi_2), (\varepsilon_1,\varepsilon_2), (\omega_1,\omega_2) \in \Psi \times \Psi\). Then,
\[\max \{\beta (\pi_1 + \varepsilon_1), \beta (\pi_2 + \varepsilon_2)\} = \Omega_{\beta} (\pi_1 + \varepsilon_1, \pi_2 + \varepsilon_2)\]
\[\leq \kappa \max \{\Omega_{\beta} ((\pi_1, \pi_2), (\omega_1, \omega_2)), \Omega_{\beta} ((\varepsilon_1, \varepsilon_2), (\omega_1, \omega_2))\}\]
\[= \kappa \max \{\Omega_{\beta} ((\pi_1 + \omega_1), (\pi_2 + \omega_2)), \Omega_{\beta} ((\varepsilon_1 - \omega_1), (\varepsilon_2 - \omega_1))\}\]
\[= \max \{\kappa \max \{\beta ((\pi_1 + \omega_1)), \beta ((\pi_2 + \omega_2))\}, \kappa \max \{\beta ((\varepsilon_1 - \omega_1)), \beta ((\varepsilon_2 - \omega_1))\}\}\]
In particular if we take \(\pi_2 = \varepsilon_2 = \omega_2 = 0\), then
\[\beta (\pi_1 + \varepsilon_1) \leq \kappa \max \{\beta ((\pi_1 + \omega_1)), \beta ((\varepsilon_1 - \omega_1))\}\]. This proves \((ASA_2)\) and \(\beta\) is an \(AFSA - i of (\lambda, \kappa) of \Psi\). \(\triangle\)

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