Nonlinear distributional geometry and general relativity

Roland Steinbauer∗

Department of Mathematics, University of Vienna
Strudlhofg. 4, A-1090 Wien, Austria

Abstract

This work reports on the construction of a nonlinear distributional geometry (in the sense of Colombeau’s special setting) and its applications to general relativity with a special focus on the distributional description of impulsive gravitational waves.

Key words. Algebras of generalized functions, Colombeau algebras, generalized tensor fields, generalized pseudo-Riemannian geometry, general relativity, impulsive gravitational waves.

Mathematics Subject Classification (2000). Primary 46F30; Secondary 46T30, 46F10, 83C05, 83C35.

1 Introduction

Idealizations play an overall role in modeling physical situations; a particularly useful one is to replace smooth extended densities by “concentrated sources” whenever the density is confined to a “small region” in space and its internal structure is negligible (e.g. point charges in electrodynamics). On trying to describe this idealization mathematically one is led to distributions in a natural way. In the case of, e.g., electrodynamics distribution theory in fact furnishes a consistent framework, i.e., provides the following two features: first since Maxwell equations are linear with respect to sources and fields they make sense within distributions and second it is guaranteed that (say smooth) charge densities close—in the sense of $\mathcal{D}'$-convergence—to, e.g., a point charge produce fields that are close to the Coulomb field. While the first property allows for a mathematically sound formulation it is precisely the latter one which renders the idealization physically sensible.

∗Electronic mail: roland.steinbauer@univie.ac.at; supported by research grant P12023-MAT of the Austrian Science Foundation
One would wish for a similar mathematical description of concentrated sources in the theory of general relativity. However, its field equations, i.e., Einstein’s equations form a (complicated) system of nonlinear PDEs. More precisely, since the spacetime metric and its first derivatives enter nonlinearly, the field equations simply cannot be formulated for distributional metrics. For a more detailed discussion of the geometrical aspects and, in particular, weak singularities in general relativity see [Vic01]. Despite this conceptual obstacle spacetimes involving an energy-momentum tensor supported on a hypersurface of spacetime (so-called thin shells) have long since been used in general relativity (see [sr67] for the final formulation of this widely applied approach). The description of gravitational sources supported lower dimensional submanifolds of spacetime (e.g. cosmic strings and point particles), however, is more delicate. In fact by a result of Geroch and Traschen [Ger87] a mathematically sound and at the same time physically resonable description (in the sense of a “limit consistency” as discussed in the context of Maxwell fields above) explicitly excludes the treatment of sources of the gravitational field concentrated on a submanifold of codimension greater than one.

Recently nonlinear generalized function methods have been used to overcome this conceptual obstacle in the context of such different topics in general relativity as cosmic strings (e.g. [Cla96, Vic01]), (ultrarelativistic) black holes ([Bal97a, Ste97, Hei01]), impulsive gravitational waves (e.g. [Bal97b, Kun99a]) and signature change (e.g. [Man00]). For an overview see [Vic99].

In this work we are going to discuss the recently developed global approach to nonlinear distributional (in the sense of the special version of Colombeau’s construction) geometry ([Ste00, Kun01a, Kun01b]) and its applications to general relativity. While the following section is devoted to a review of the former and, in particular, to generalized pseudo-Riemannian geometry, applications to the distributional description of impulsive gravitational pp-waves will be presented in Section 3. We shall see that despite the absence of a canonical embedding of distributions Colombeau’s special setting, due to the fact that the basic building blocks automatically are diffeomorphism invariant provides a particularly flexible tool to model singular metrics in the nonlinear context of general relativity. For an introduction into the diffeomorphism invariant full algebras of generalized functions of [Gro01a] and in particular its global formulation ([Gro01b]), however, we refer to [Ste01, Gro01b, Kun01c] in this volume; its applications to general relativity are discussed in [Vic01].

2 Generalized pseudo-Riemannian geometry

In the following we use the notational conventions of [Kun01a, Kun01b]. The (special) algebra of generalized functions on the (separable, smooth Hausdorff) manifold X is defined as the quotient $G(X) := \mathcal{E}_M(X)/\mathcal{N}(X)$ of the space of moderately growing nets of smooth functions $(u_\varepsilon)_{\varepsilon \in (0,1]} \in C^\infty(X)^{(0,1]} =: \mathcal{E}(X)$ modulo negligible nets, where the respective notions of moderateness and negligibility are defined (denoting by $\mathcal{P}(X)$ the space of linear differential operators...
on $X$) by

\[
\mathcal{E}_M(X) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(X) : \forall K \subset X, \forall \sigma P \in \mathcal{P}(X) \exists N \in \mathbb{N} : \\
\sup_{p \in K} |P u_\varepsilon(p)| = O(\varepsilon^{-N}) \}
\]

\[
\mathcal{N}(X) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(X) : \forall K \subset X, \forall n \in \mathbb{N}_0 : \sup_{p \in K} |u_\varepsilon(p)| = O(\varepsilon^n) \}.
\]

Elements of $\mathcal{G}(X)$ are denoted by capital letters, i.e., $U = \text{cl}[(u_\varepsilon)_\varepsilon] = (u_\varepsilon)_\varepsilon + \mathcal{N}(X)$. $\mathcal{G}(\_)$ is a fine sheaf of differential algebras with respect to the Lie derivative (w.r.t. smooth vector fields) defined by $L_\xi U := \text{cl}[(L_\xi u_\varepsilon)_\varepsilon]$. The spaces of moderate resp. negligible sequences and hence the algebra itself may be characterized locally, i.e., $U \in \mathcal{G}(X)$ iff $U \circ \psi_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))$ for all charts $(V_\alpha, \psi_\alpha)$. Smooth functions are embedded into $\mathcal{G}$ simply by the “constant” embedding $\sigma$, i.e., $\sigma(f) := \text{cl}[(f)_\varepsilon]$, hence $C^\infty(X)$ is a faithful subalgebra of $\mathcal{G}(X)$. In the absence of a canonical embedding compatibility with respect to the distributional setting is established via the notion of association, defined as follows: a generalized function $U$ is called associated to 0, $U \approx 0$, if $\int_X u_\varepsilon \mu \rightarrow 0 \ (\varepsilon \rightarrow 0)$ for all compactly supported one-densities and one (hence every) representative $(u_\varepsilon)_\varepsilon$ of $U$. The equivalence relation induced by this notion gives rise to a linear quotient space of $\mathcal{G}(X)$. If $\int_X u_\varepsilon \mu \rightarrow w(\mu)$ for some $w \in \mathcal{D}'(X)$ then $w$ is called the distributional shadow (or macroscopic aspect) of $U$ and we write $U \approx w$. Similarly we call a generalized function $U$ $k$-associated to 0 ($0 \leq k \leq \infty$), $U \approx_k 0$, if for all $l \leq k$, all $\xi_1, \ldots, \xi_l \in \mathbb{X}(X)$ and one (hence any) representative $L_{\xi_1} \ldots L_{\xi_l} u_\varepsilon \rightarrow 0$ uniformly on compact sets. Also we say that $U$ admits $f$ as $C^k$-associated function, $U \approx_k f$, if for all $l \leq k$, all $\xi_1, \ldots, \xi_l \in \mathbb{X}(X)$ and one (hence any) representative $L_{\xi_1} \ldots L_{\xi_l} (u_\varepsilon - f) \rightarrow 0$ uniformly on compact sets. Finally, inserting $x \in X_\varepsilon \subset X$ into $U \in \mathcal{G}(X)$ yields a well defined element of the ring of constants $\mathcal{K}$ (corresponding to $\mathbb{R} = \mathbb{R}$ resp. $\mathbb{C}$), defined as the set of moderate nets of numbers $|(r_\varepsilon)_\varepsilon| \in \mathbb{K}^{[0,1]}$ with $|r_\varepsilon| = O(\varepsilon^{-N})$ for some $N$ modulo negligible nets $|(r_\varepsilon)_\varepsilon| = O(\varepsilon^n)$ for each $n$.

The $\mathcal{G}(X)$-module of generalized sections $\mathcal{G}(X,E)$ of a vector bundle $E \rightarrow X$—and in particular the space of generalized tensor fields $\mathcal{G}_\bullet^*(X)$—is defined along the same lines using analogous asymptotic estimates with respect to the norm induced by any Riemannian metric on the respective fibers. We denote generalized sections by $S = \text{cl}[(s_\varepsilon)_\varepsilon] = (s_\varepsilon)_\varepsilon + \mathcal{N}(X,E)$. Alternatively we may describe a section $S \in \mathcal{G}(X,E)$ by a family $(S_\varepsilon)_\varepsilon = (S_\varepsilon)_\alpha = (S_\varepsilon)_\alpha^{\alpha_1} \in \mathcal{G}_\alpha(V_\alpha)$ for all $\alpha_1, \ldots, \alpha_N$ with $N$ denoting the dimension of the fibers) satisfying $S_\varepsilon(x) = (\psi_\alpha_1(x) \cdot \psi_\alpha_1^{-1}(x)) S_\varepsilon \circ \psi_\alpha_1^{-1}(x))$ for all $x \in \psi_\alpha(V_\alpha \cap V_\beta)$, where $\psi_\alpha_1$ denotes the transition functions of the bundle. Smooth sections of $E \rightarrow X$ again may be embedded as constant nets, i.e., $\Sigma(s) = \text{cl}[(s_\varepsilon)_\varepsilon]$. Since $C^\infty(X)$ is a subring of $\mathcal{G}(X)$, $\mathcal{G}(X,E)$ also may be viewed as $C^\infty(X)$-module and the two respective module structures are compatible with respect to the embeddings. Moreover we have the following
algebraic characterization of the space of generalized sections

\[ G(X, E) = \mathcal{G}(X) \otimes \Gamma(X, E), \tag{1} \]

where \( \Gamma(X, E) \) denotes the space of smooth sections and the tensor product is taken over the module \( \mathcal{C}^\infty(X) \). Compatibility with respect to the classical resp. distributional setting again is accomplished using the concept of \((k-)\)association which carries over from the scalar case by \([\mathbb{I}]\).

Generalized tensor fields may be viewed likewise as \( \mathcal{C}^\infty\)- resp. \( \mathcal{G}\)-multilinear mappings, i.e., as \( \mathcal{C}^\infty(X)\)- resp. \( \mathcal{G}(X)\)-modules we have

\[
\begin{align*}
\mathcal{G}_r^s(X) & \cong L_{\mathcal{C}_r^\infty(X)}(\mathfrak{X}(X)^r, \mathfrak{X}^*(X)^s; \mathcal{G}(X)) \\
\mathcal{G}_s^r(X) & \cong L_{\mathcal{G}(X)}(\mathcal{G}_r^0(X)^r, \mathcal{G}_1^0(X)^s; \mathcal{G}(X)),
\end{align*}
\]

where \( \mathfrak{X}(X) \) resp. \( \mathfrak{X}^*(X) \) denotes the space of smooth vector resp. covector fields on \( X \). In \([\mathbb{Kun01a}]\) many concepts of classical tensor analysis like e.g. Lie derivatives (with respect to both smooth and generalized vector fields), Lie brackets, tensor products and contraction have been generalized to the new setting and we shall use them freely in the sequel. Moreover several consistency results with respect to smooth resp. distributional geometry (cf. \([\mathbb{Mar68}]\)) have been established. We now begin to develop the basics of a generalized pseudo-Riemannian geometry.

2.1 Definition.

(i) A generalized \((0, 2)\) tensor field \( \hat{G} \in \mathcal{G}_2^0(X) \) is called a generalized Pseudo-Riemannian metric if it has a representative \((\hat{\gamma}_\varepsilon)\) satisfying

(a) \( \hat{\gamma}_\varepsilon \) is a smooth Pseudo-Riemannian metric for all \( \varepsilon \), and

(b) \( (\det \hat{\gamma}_\varepsilon)_\varepsilon \) is strictly nonzero on compact sets, i.e., \( \forall K \subset X \exists m \in \mathbb{N} : \inf_{p \in K} |\det \hat{\gamma}_\varepsilon(p)| \geq \varepsilon^m \).

(ii) We call a separable, smooth Hausdorff manifold \( M \) furnished with a generalized pseudo-Riemannian metric \( \hat{G} \) a generalized pseudo-Riemannian manifold or generalized spacetime and denote it by \( (M, \hat{G}) \) or merely by \( M \). The action of the metric on a pair of generalized vector fields will be denoted by \( \hat{G}(\Xi, H) \) and \( (\Xi, H) \), equivalently.

Note that condition (b) above is precisely equivalent to invertibility of \( \det \hat{G} \) in the generalized sense. The inverse metric \( \hat{G}^{-1} := \text{cl}[(\hat{\gamma}^{-1}_\varepsilon)]_\varepsilon \) is a well defined element of \( \mathcal{G}_0^0(M) \), depending exclusively on \( \hat{G} \) (i.e., independent of the particular representative \((\hat{\gamma}_\varepsilon)\)). Moreover if \( \hat{G} \approx_k g \), where \( g \) is a classical \( \mathcal{C}^k\)-pseudo-Riemannian metric then \( \hat{G}^{-1} \approx_k g^{-1} \). From now on we denote the inverse metric (using abstract index notation cf. \([\mathbb{Pen84}], \text{Ch. 2}\) by \( \hat{G}^{ab} \), its components by \( \hat{G}^{ij} \) and the components of a representative by \( \hat{g}^{ij}_\varepsilon \). Also we shall denote the generalized metric \( \hat{G}_{ab} \) by \( \hat{d}s^2 = \text{cl}[(\hat{d}s^2)_\varepsilon] \) and use summation convention.
2.2 Examples.

(i) A sequence \((\hat{g}_\varepsilon)\) of classical (smooth) metrics constitutes a representative of a generalized metric if it is moderate and zero-associated to a classical (then necessarily continuous) metric \(g\).

(ii) The metric of a two-dimensional cone was modeled in \cite{Cl96} by a generalized metric (in the full setting) obtained by using the embedding via convolution.

(iii) The metric of an impulsive pp-wave will be modeled by a generalized one in Section 3.

(iv) Further examples may be found e.g. in \cite{Bal97a, Ste97, Man00}.

A generalized metric \(\hat{G}\) is non-degenerate in the following sense: \(\exists \in \mathcal{G}^1_0(X)\), 

\[
\hat{G}(\Xi, H) = 0 \forall H \in \mathcal{G}^1_0(M) \Rightarrow \Xi = 0.
\]

Moreover \(\hat{G}\) induces a \(\mathcal{G}(X)\)-linear isomorphism \(\mathcal{G}^1_0(M) \rightarrow \mathcal{G}^0_1(M)\) by

\[
\Xi \mapsto \hat{G}(\Xi, .),
\]

which—as in the classical context—extends naturally to generalized tensor fields of all types. Hence from now on we shall use the common conventions on upper and lower indices also in the context of generalized tensor fields. In particular, identifying a vector field \(\Xi^a \in \mathcal{G}^1_0(M)\) with its metrically equivalent one-form \(\Xi_a\) we denote its contravariant respectively covariant components by \(\Xi^i\) and \(\Xi_i\). A similar convention will apply to representatives.

2.3 Definition. A generalized connection \(\hat{D}\) on a manifold \(X\) is a map \(\mathcal{G}^1_0(X) \times \mathcal{G}^1_0(X) \rightarrow \mathcal{G}^1_0(X)\) satisfying

\((D1)\) \(\hat{D}_\Xi H\) is \(\mathcal{R}\)-linear in \(H\).

\((D2)\) \(\hat{D}_\Xi H\) is \(\mathcal{G}(X)\)-linear in \(\Xi\).

\((D3)\) \(\hat{D}_\Xi(UH) = U \hat{D}_\Xi H + \Xi(U)H\) for all \(U \in \mathcal{G}(X)\).

Let \((V_\alpha, \psi_\alpha)\) be a chart on \(X\) with coordinates \(x^i\). The generalized Christoffel symbols for this chart are given by the \((\dim X)^3\) functions \(\hat{\Gamma}^k_{ij}\) defined by

\[
\hat{D}_\partial_i \partial_j = \sum_k \hat{\Gamma}^k_{ij} \partial_k.
\]

We are already in the position to state the “Fundamental Lemma of (pseudo)-Riemannian Geometry” in our setting.

2.4 Theorem. Let \((M, \hat{G})\) be a generalized pseudo-Riemannian manifold. Then there exists a unique generalized connection \(\hat{D}\) such that

\((D4)\) \([\Xi, H] = \hat{D}_\Xi H - \hat{D}_H \Xi\) and
\( (D5) \quad \Xi(H, Z) = \langle \hat{D}_\Xi H, Z \rangle + \langle H, \hat{D}_\Xi Z \rangle \)

hold for all \( \Xi, H, Z \) in \( G_0^1(M) \). \( \hat{D} \) is called generalized Levi-Civita connection of \( M \) and characterized by the so-called Koszul formula

\[
2 \langle \hat{D}_\Xi H, Z \rangle = \Xi \langle H, Z \rangle + H \langle \Xi, Z \rangle - Z \langle \Xi, H \rangle - \langle \Xi, [H, Z] \rangle + [H, \langle Z, \Xi \rangle] + [Z, \langle \Xi, H \rangle].
\]

As in the classical case from the torsion-free condition \( i.e., (D4) \) we immediately infer the symmetry of the Christoffel symbols of the Levi-Civita connection in its lower pair of indices. Moreover, from \( (D3) \) and the Koszul formula \( (2) \) we derive \( \text{(analogously to the classical case)} \) the following

2.5 Proposition. On every chart \( (V_\alpha, \psi_\alpha) \) we have for the generalized Levi-Civita connection \( \hat{D} \) of \( (M, \hat{G}) \) and any vector field \( \Xi \in G_0^1(X) \)

\[
\hat{D}_\delta(\Xi^i \partial_j) = \left( \frac{\partial \Xi^k}{\partial x^i} + \hat{\Gamma}^k_{ij} \Xi^j \right) \partial_k.
\]

Moreover, the generalized Christoffel symbols are given by

\[
\hat{\Gamma}^k_{ij} = \frac{1}{2} \hat{G}^{kl} \left( \frac{\partial \hat{G}_{jm}}{\partial x^i} + \frac{\partial \hat{G}_{im}}{\partial x^j} - \frac{\partial \hat{G}_{ij}}{\partial x^m} \right).
\]

To be able to state the appropriate consistency results with respect to classical resp. distributional geometry we need to define the action of a classical (smooth) connection \( D \) on generalized vector fields \( \Xi, H \) by \( D_\Xi H := \text{cl}[D_\xi, \eta] \). Now we have

2.6 Proposition. Let \( (M, \hat{G}) \) be a generalized pseudo-Riemannian manifold.

(i) If \( \hat{G}_{ab} = \Sigma(g_{ab}) \) where \( g_{ab} \) is a classical smooth metric then we have, in any chart, \( \Gamma^i_{jk} = \Sigma(\Gamma^i_{jk}) \) \( \text{with} \ \Gamma^i_{jk} \ \text{denoting the Christoffel Symbols of} \ g_{ab}. \) Hence for all \( H \in G_0^1(M) \)

\[
\hat{D}_\Xi H = D_\Xi H.
\]

(ii) If \( \hat{G}_{ab} \approx_k g_{ab} \), \( g_{ab} \) a classical smooth metric, \( \Xi, H \in G_0^1(M) \) and \( \Xi \approx_k \xi \in \mathcal{D}_0^1(M) \) \( \text{or vice versa, i.e.,} \ \Xi \approx \xi \in \mathcal{D}_0^1(M), \ H \approx_k \eta \in \mathcal{X}(M) \) \( \text{then} \)

\[
\hat{D}_\Xi H \approx_k D_\xi \eta.
\]

(iii) Let \( \hat{G}_{ab} \approx_k g_{ab} \), \( g_{ab} \) a classical \( C^k \)-metric, then, in any chart, \( \hat{\Gamma}^i_{jk} \approx_{k-1} \Gamma^i_{jk}. \) If in addition \( \Xi, H \in G_0^1(M), \ \Xi \approx_{k-1} \xi \in \Gamma^{k-1}(M, TM) \) \( \text{and} \ H \approx_k \eta \in \Gamma^k(M, TM) \) \( \text{then} \)

\[
\hat{D}_\Xi H \approx_{k-1} D_\xi \eta.
\]
Next we define the generalized Riemann, Ricci, scalar and Einstein curvature from an invariant point of view. However, all the classical formulae will hold on the level of representatives, i.e., all the symmetry properties of the respective classical tensor fields carry over to our setting. Moreover, the Bianchi identities hold in the generalized sense.

2.7 Definition. Let \((M, \hat{G})\) be a generalized pseudo-Riemannian manifold with Levi-Civita connection \(\hat{D}\).

(i) The generalized Riemannian curvature tensor \(\hat{R}^{d}{}_{abc} \in \mathcal{G}^{1}(M)\) is defined by
\[
\hat{R}_{\Xi, H} Z := \hat{D}_{[\Xi, H]} Z - [\hat{D}_{\Xi}, \hat{D}_{H}] Z.
\]

(ii) The generalized Ricci curvature tensor is defined by \(\hat{R}_{ab} := \hat{R}^{c}{}_{cab}\).

(iii) The generalized curvature (or Ricci) scalar is defined by \(\hat{R} := \hat{R}^{a}{}_{a}\).

(iv) Finally we define the generalized Einstein tensor by \(\hat{G}_{ab} := \hat{R}_{ab} - \frac{1}{2} \hat{R} \hat{G}_{ab}\).

The framework developed above opens a gate to a wide range of applications in general relativity. Definition 2.1 is capable of modeling a large class of singular spacetimes while at the same time its (generalized) curvature quantities simply may be calculated by the usual coordinate formulae. Hence we are in a position to mathematically rigorously formulate Einsteins equations for generalized metrics. Moreover we have at our disposal several theorems (which essentially are rooted in \cite{Kum01}, Prop. 3) guaranteeing consistency with respect to linear distributional geometry resp. the smooth setting.

2.8 Theorem. Let \((M, \hat{G})\) a generalized pseudo-Riemannian manifold with \(\hat{G}_{ab} \approx k g_{ab}\). Then all the generalized curvature quantities defined above are \(C^{k-2}\)-associated to their classical counterparts.

In particular, if a generalized metric \(\hat{G}_{ab}\) is \(C^{2}\)-associated to a vacuum solution of Einstein equations then we have for the generalized Ricci tensor
\[
\hat{R}_{ab} \approx_0 R_{ab} = 0.
\]

Hence \(\hat{R}_{ab}\) satisfies the vacuum Einstein equations in the sense of 0-association (cf. the remarks in \cite{Vic98}).

Generally speaking when dealing with singular spacetime metrics in general relativity we may apply the steps of the following scheme: first we transfer the classically singular metric to the generalized setting. This may be done by some “canonical” smoothing or by some other physically motivated regularization. Of course diffeomorphism invariance of the procedure employed has to be carefully investigated. Once the generalized setting has been entered, the relevant curvature quantities may be calculated componentwise according to the classical formulae. All classical concepts carry over to the new framework and one may
treat e.g. the Ricci tensor, geodesics, geodesic deviation, etc. within this nonlinear distributional geometry. Finally one may use the concept of \((k)\)-association to return to the distributional or \(\mathcal{C}^k\)-level for the purpose of interpretation.

This program has been carried out for a conical metric (representing a cosmic string) by Clarke, Vickers and Wilson [Cla96, Vic01], however, in the full setting of Colombeau’s construction) rigorously assigning to it a distributional curvature and (via the field equations) the heuristically expected energy-momentum tensor. In Section 3 we are going to review the distributional description of impulsive pp-wave spacetimes of [Ste98, Kun99a, Kun99b]. Further applications following the procedure described above may be found e.g. in [Bal97a, Hei01].

3 Impulsive gravitational waves

Plane fronted gravitational waves with parallel rays (pp-waves) are spacetimes characterized by the existence of a covariantly constant null vector field, which can be used to write the metric tensor in the form

\[ ds^2 = h(u,x,y)du^2 - du\, dv + dx^2 + dy^2 \]

where \(u, v\) is a pair of null coordinates \((u = t - z, \ v = t + z)\) and \(x, y\) are transverse (Cartesian) coordinates. We are especially interested in impulsive pp-waves as introduced by R. Penrose (see e.g. [Pen72]) where the profile function \(h\) is proportional to a \(\delta\)-distribution, i.e., takes the form \(h(u,x,y) = f(x,y)\delta(u)\), where \(f\) is a smooth function of the transverse coordinates. This metric is flat everywhere except on the null hypersurface \(u = 0\), where it has a \(\delta\)-shaped “shock” and—due to the appearance of a distribution in one component—clearly lies beyond the scope of linear distributional geometry.

Physically this form of the metric arises as the impulsive limit of a sequence of sandwich waves, i.e., \(h_\varepsilon\) taking the form \(h_\varepsilon(u,x,y) = \delta_\varepsilon(u,f(x,y)\delta(u)\), where \(\delta_\varepsilon \to \delta\) weakly. This is our motivation to model the impulsive pp-wave metric by a generalized metric of the form

\[ ds^2 = f(x,y)D(u)du^2 - du\, dv + dx^2 + dy^2, \]

where \(D\) denotes a generalized delta function which allows for a strict delta net \((\rho_\varepsilon)_\varepsilon\) as a representative, i.e.,

\(\text{(a)}\) \quad \text{supp}(\rho_\varepsilon) \to \{0\} \quad (\varepsilon \to 0),

\(\text{(b)}\) \quad \int \rho_\varepsilon(x) \, dx \to 1 \quad (\varepsilon \to 0) \quad \text{and}

\(\text{(c)}\) \quad \exists \eta > 0 \ \exists \varepsilon \geq 0 : \int |\rho_\varepsilon(x)| \, dx \leq C \quad \forall \varepsilon \in (0, \eta].

In [Kun99a] it has been shown that the geodesic as well as the geodesic deviation equation for the metric \((3)\) may be solved uniquely within our present setting. Moreover these unique generalized solutions possess physically reasonable distributional shadows which shows that we have achieved a physically
sensible distributional description of impulsive pp-waves. Diffeomorphism invariance of these results is assured by diffeomorphism invariance of the class of strict delta nets.

Here, however, we shall be interested in modeling the heuristically motivated singular transformation of the distributional pp-wave metric first given by R. Penrose ([Pen72]) within our framework (cf. [Kun99b]). For later use we introduce the notation $X^i(x_0^i, u), V(v_0, x_0^j, u) \ (i, j = 1, 2)$ for the unique generalized geodesics of (3) with vanishing initial speeds. Here the fourth coordinate $u$—due to the special geometry—may be used as an affine parameter along the geodesics and the real constants $x_0^i, v_0$ denote the initial positions, i.e., $X^i(x_0^i, -1) = x_0^i$ and analogously for $V$.

In the literature impulsive pp-waves have frequently been described in different coordinates where the metric tensor is actually continuous, i.e., (in the special case of a plane wave $(f(x, y) = x^2 - y^2$ and $u_+$ denoting the kink function),

$$ds^2 = (1 + u^+)^2dX^2 + (1 - u^+)^2dY^2 - dudV.$$  \quad (4)

Clearly a transformation relating these two metrics cannot even be continuous, hence in addition to involving ill-defined products of distributions it changes the topological structure of the manifold. In the special case envisaged above this discontinuous change of variables was given in [Pen72] (denoting by $H$ the Heaviside function)

$${x} = (1 + u^+)X, \quad {y} = + (1 - u^+)Y$$

$$v = V + \frac{1}{2}X^2(u^+ + H(u)) + \frac{1}{2}Y^2(u^+ - H(u)).$$  \quad (5)

However, the two mathematically distinct spacetimes are equivalent from a physical point of view, i.e., the geodesics and the particle motion agree on a heuristic level (see [Ste99]). We are now going to model this transformation by a generalized coordinate transformation, that is

3.1 Definition. Let $\Omega$ be an open subset of $\mathbb{R}^n$. We call $T \in \mathcal{G}(\Omega, \mathbb{R}^n)$ a generalized diffeomorphism if there exists $\eta > 0$ such that

(i) There exists a representative $(t_\varepsilon)_\varepsilon$ such that $t_\varepsilon : \Omega \to t_\varepsilon(\Omega)$ is a diffeomorphism for all $\varepsilon \leq \eta$ and there exists $\tilde{\Omega} \subseteq \mathbb{R}^n$ open, $\Omega \subseteq \bigcap_{\varepsilon \leq \eta} t_\varepsilon(\tilde{\Omega})$.

(ii) $(t_\varepsilon)^{-1}_\varepsilon \in \mathcal{E}_M(\tilde{\Omega}, \mathbb{R}^n)$ and there exists $\Omega_1 \subseteq \mathbb{R}^n$ open such that $\Omega_1 \subseteq \bigcap_{\varepsilon \leq \eta} (t_\varepsilon)^{-1}(\tilde{\Omega})$.

(iii) Writing $T^{-1} := \text{cl}[(t_\varepsilon)^{-1}_\varepsilon],\ T \circ T^{-1}$ as well as $T^{-1} \circ T|_{\Omega_1}$ are well-defined elements of $\mathcal{G}(\tilde{\Omega}, \mathbb{R}^n)$ resp. $\mathcal{G}(\Omega_1, \mathbb{R}^n)$.

It is then clear that $T \circ T^{-1} = \text{id}_\tilde{\Omega}$ resp. $T^{-1} \circ T|_{\Omega_1} = \text{id}_{\Omega_1}$. Let us now consider the transformation $T = \text{cl}[(t_\varepsilon)_\varepsilon] : (u, v, x^i) \mapsto (u, V, X^i)$ depending on the regularization parameter $\varepsilon$ according to

$${t_\varepsilon} : \quad x^i = x^i_\varepsilon(X^j, u)$$

$$v = v_\varepsilon(V, X^j, u),$$  \quad (6)
where \( x_i^\varepsilon(x_0^j, u) \) and \( v_\varepsilon(v_0, x_0^j, u) \) are representatives of the generalized geodesics \( X^i(x_0^j, u) \) resp. \( V(v_0, x_0^j, u) \) and are given by

\[
x_i^\varepsilon(x_0^j, u) = x_i^0 + \frac{1}{2} \int_{-\varepsilon}^{\varepsilon} \partial_i f(x_i^\varepsilon(x_0^j, r)) \rho_\varepsilon(r) \, dr \, ds
\]

\[
v_\varepsilon(v_0, x_0^j, u) = v_0 + \int_{-\varepsilon}^{\varepsilon} f(x_i^\varepsilon(x_0^j, s)) \rho_\varepsilon(s) \, ds + \int_{-\varepsilon}^{\varepsilon} \partial_i f(x_i^\varepsilon(x_0^j, r)) \dot{x}_i^\varepsilon(x_0^j, r) \rho_\varepsilon(r) \, dr \, ds.
\]

(7)

Now one may prove the following

3.2 Theorem. The generalized function \( T \) defined above is a generalized coordinate transformation on a suitable open subset \( \Omega \) of \( \mathbb{R}^4 \) containing the shock hyperplane at \( u = 0 \). The new coordinates are constant along the geodesics given by (7). Moreover the macroscopic aspect of \( T \) coincides with the discontinuous transformation used in the literature (hence in the special case of a plane wave is given by (5)).

Applying this generalized transformation to the metric (3) we find that in the new coordinates it is given by the class of

\[
\hat{ds}_\varepsilon^2 = -dudV + \left( 2 \sum_{i=1}^{2} (\dot{x}_i^\varepsilon \partial_j x_j^\varepsilon - \partial_j v_\varepsilon) duX^j + \sum_{i=1}^{2} (\partial_j x_i^\varepsilon dX^j)^2 \right),
\]

(8)

where \( \cdot \) and \( \partial_i \) denote derivatives with respect to \( u \) and \( X^i \), respectively. Moreover we find the following distributional shadow of the metric (8)

\[
\text{cl}[(\hat{ds}_\varepsilon^2)_\varepsilon] \approx -dudV + (1 + \frac{1}{2} \partial_{11} f(X^j) u_+)^2 dX^2 + (1 + \frac{1}{2} \partial_{22} f(X^j) u_+)^2 dY^2 + \frac{1}{2} \partial_{12} f(X^j) \Delta f(X^j) u_+^2 dX dY + 2u_+ \partial_{12} f(X^j) dX dY + \frac{1}{4} (\partial_{12} f(X^j))^2 u_+^2 (dX^2 + dY^2)
\]

(9)

which is precisely the continuous (or so-called Rosen-) form of the metric of an impulsive pp-wave (cf. [Aic97, Pod98]).

Summing up we have shown the following: after modeling the (distributional form of the) impulsive pp-wave metric in a diffeomorphism invariant way by the generalized metric (3) we have subjected the latter to the generalized change of coordinates \( T \). In either coordinates the distributional shadow is computed giving the distributional resp. the continuous form of the pp-wave metric. (Note that although the action of a smooth diffeomorphism is compatible with the notion of association, generalized coordinate transformations clearly are not.) Physically speaking the two forms of the impulsive metric arise as the (distributional) limits of a sandwich wave in different coordinate systems. Hence impulsive pp-waves indeed are sensibly modeled by the generalized spacetime metric (3): in different coordinate systems related by generalized coordinate transformations, different distributional pictures arise.
References

[Aic97] Aichelburg, P. C., Balasin, H. Generalized symmetries of impulsive gravitational waves. *Class. Quant. Grav.*, pages A31–A41, 1997.

[Bal97a] Balasin, H. Distributional energy-momentum tensor of the extended Kerr geometry. *Class. Quant. Grav.*, pages 3353–3362, 1997.

[Bal97b] Balasin, H. Geodesics for impulsive gravitational waves and the multiplication of distributions. *Class. Quant. Grav.*, **14**:455–462, 1997.

[Cla96] Clarke, C. J. S., Vickers, J. A., Wilson, J. P. Generalised functions and distributional curvature of cosmic strings. *Class. Quant. Grav.*, **13**, 1996.

[Ger87] Geroch, R., Traschen, J. Strings and other distributional sources in general relativity. *Phys. Rev. D*, **36**(4):1017–1031, 1987.

[Gro99] Grosser, M., Kunzinger, M., Steinbauer, R., Vickers, J. A global theory of algebras of generalized functions. *Preprint*, math.FA/9912216, 1999.

[Gro01a] Grosser, M., Farkas, E., Kunzinger, M., Steinbauer, R. On the foundations of nonlinear generalized functions I, II. *Mem. Am. Math. Soc.*, to appear (available electronically at http://arXiv.org/abs/math.FA/9912214, 9912215), 2001.

[Gro01b] Grosser, M. Diffeomorphism invariant Colombeau algebras. Part II: Classification. *This volume*.

[Hei01] Heinzle, J. M., Steinbauer, R. Remarks on the distributional Schwarzschild geometry. *Preprint*, 2001.

[Isr66] Israel, W. Singular hypersurfaces and thin shells in general relativity. *Nouv. Cim.*, **44B**(1):1–14, 1966.

[Kun99a] Kunzinger, M., Steinbauer, R. A rigorous solution concept for geodesic and geodesic deviation equations in impulsive gravitational waves. *J. Math. Phys.*, **40**:1479–1489, 1999.

[Kun99b] Kunzinger, M., Steinbauer, R. A note on the Penrose junction conditions. *Class. Quant. Grav.*, **16**:1255–1264, 1999.

[Kun01a] Kunzinger, M., Steinbauer, R. Nonlinear distributional geometry. *Preprint*, math.FA/0102019, 2001.

[Kun01b] Kunzinger, M., Steinbauer, R. Generalized pseudo-Riemannian geometry. *Preprint*, 2001.

[Kun01c] Kunzinger, M. Diffeomorphism invariant Colombeau algebras. Part III: Global theory. *This volume*.
[Man00] Mansouri, R., Nozari, K. A new distributional approach to signature change. Gen. Relativity Gravitation 32 (2000), no. 2, 253–269., 32(2):235–269, 2000.

[Mar68] Marsden, J. E. Generalized Hamiltonian mechanics. Arch. Rat. Mech. Anal., 4(28):323–361, 1968.

[Pen72] Penrose, R. The geometry of impulsive gravitational waves. In L. O’Raifeartaigh, editor, General Relativity, Papers in Honour of J. L. Synge, pages 101–115. Clarendon Press, Oxford, 1972.

[Pen84] Penrose, R., Rindler, W. Spinors and space-time I. Cambridge University Press, 1984.

[Pod98] Podolský, J., Vesely, K. Continuous coordinates for all impulsive pp-waves. Phys. Lett. A, pages 145–147, 1998.

[Ste97] Steinbauer, R. The ultrarelativistic Reissner-Nordstrøm field in the Colombeau algebra. J. Math. Phys., 38:1614–1622, 1997.

[Ste98] Steinbauer, R. Geodesics and geodesic deviation for impulsive gravitational waves. J. Math. Phys., 39:2201–2212, 1998.

[Ste99] Steinbauer, R. On the geometry of impulsive gravitational waves. In Vucanov, D., Cotaescu, I., editor, Proceedings of the 8th Romanian Conference on General Relativity and Gravitation. Mirton Publishing House (available electronically at http://arXiv.org/abs/gr-qc/9809054), 1999.

[Ste00] Steinbauer, R. Distributional Methods in General Relativity. Ph.D. thesis, University of Vienna (available electronically at http://www.mat.univie.ac.at/~stein/work/PhD/PhD.php3), 2000.

[Ste01] Steinbauer, R. Diffeomorphism invariant Colombeau algebras. Part I: Local theory. This volume.

[Vic98] Vickers, J., Wilson, J. A nonlinear theory of tensor distributions. ESI-Preprint (available electronically at http://www.esi.ac.at/ESI-Preprints.html), 566, 1998.

[Vic99] Vickers, J. A. Nonlinear generalized functions in general relativity. In Grosser, M., Hörmander, G., Kunzinger, M., Oberguggenberger, M., editor, Nonlinear Theory of Generalized Functions, volume 401 of CRC Research Notes, pages 275–290, Boca Raton, 1999. CRC Press.

[Vic01] Vickers, J. A. Nonlinear generalised functions and weak singularities in general relativity. This volume.