Constant Curvature Solutions of Minimal Massive Gravity

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Abstract

We study the constant curvature solutions of the minimal massive gravity (MMGR). After introducing a condition on the physical and the fiducial metrics as well as the Skückelberg scalars which truncates the action to the Einstein-Hilbert one in the presence of an effective cosmological constant we focus on the solutions of the constraint equations written for the constant curvature physical metrics. We discuss two distinct formal solution methods for these constraint equations then present an explicit class of solutions for the Skückelberg scalars and the fiducial metrics giving rise to constant curvature solutions of MMGR.

Keywords: Non-linear theories of gravity, massive gravity, solutions of constant curvature

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1 Introduction

A Boulware-Deser (BD) [1, 2] ghost-free non-linear extension of the Fierz-Pauli [3] massive gravity has been constructed in a series of works in recent years. The entire programme has originated from an approach which was constructed in [4, 5] that enables the cancellation of the BD-ghost order-by-order in the expansion of the mass term in terms of the Goldstone bosons which is a consequence of the fact that the mass term can be linked to the spontaneous symmetry breaking of the general covariance. This method motivated de Rham, Gabadadze, and Tolley to construct a two parameter family of massive non-linear gravity actions free of ghost up to an order at the decoupling limit [6, 7]. This construction was based on a fixed flat background (fiducial or reference) metric which couples to the physical metric in the graviton mass terms of massive gravities via coordinate transformations. Later in [8, 9, 10] a generalized formulation of the dRGT theory including a general background or fiducial metric has been presented. In these later works it has also been shown that both the dRGT as well as the so-called newly constructed minimal and the general non-linear massive gravity (which was shown to contain the dRGT action) theories are ghost-free at all orders.

In this work starting from the minimal massive gravity (MMGR) action of [8] written for the general background metric by proposing a solution scheme which truncates the action to the Einstein gravity on-shell we focus on the constant curvature solutions of the theory. Following the introduction of such a constraint on the solutions which we show to be also compatible with the field equations, as the total action is truncated to the vacuum Einstein form with a cosmological constant the constant curvature Einsteinian metrics become the solutions of the MMGR. In this context instead of solving the coupled Skückelberg scalars, the constant curvature Einstein metric and the fiducial metric directly from the highly non-linear field equations one has to consider solving a matrix constraint equation for any choice of constant curvature Einstein metric. After formulating the integral form of these constraint equations we will inspect two distinct solution methodologies for a set of fiducial metrics admitting a special form. Finally we will present a class of exact solutions of the Skückelberg scalars and the fiducial metric when they couple to any constant curvature Einstein metrics.

Section two is devoted to the construction of the solution scheme of the MMGR field equations for constant curvature physical metrics. In Section three we will obtain the integral form of the constraint equations for diago-
nal form of fiducial metrics. Section four contains the discussion about the two distinct solution methodologies of these constraint equations. Finally in Section five by using one of these methods we will exactly integrate out the constraint equations for an explicitly chosen form of the fiducial metric so that the resulting class of solutions of Skückelberg scalars and the fiducial reference metric lead to the constant curvature solutions of the MMGR.

2 The set-up

The minimal non-linear modification to the massive Fierz-Pauli action [3] is constructed in [8] which is the minimal ghost free massive gravity action. It reads

\[ S_{\text{MMGR}} = -M_p^2 \int \left[ R \ast 1 + 2m^2 \text{tr}(\sqrt{\Sigma}) \ast 1 + \Lambda' \ast 1 \right], \tag{2.1} \]

where \( M_p \) is the planck mass, \( m \) is the graviton mass, \( R \) is the Ricci scalar, and the Hodge star operator is defined as

\[ \ast 1 = \sqrt{-g} dx^4, \tag{2.2} \]

with \( g = \text{det}(g_{\mu\nu}) \). The effective cosmological constant is

\[ \Lambda' = \Lambda - 6m^2. \tag{2.3} \]

In the above action we have defined the four by four matrix \( \Sigma \) as

\[ (\Sigma)_\mu^\nu = g^{\mu\rho} f_{\rho\nu}, \tag{2.4} \]

with \( g^{\mu\nu} \) being the inverse metric and the four by four matrix functional \( f \) is

\[ f_{\rho\nu} = \partial_\rho \phi^a \partial_\nu \phi^b f_{ab}(\phi^c). \tag{2.5} \]

Here\(^1\) \( \phi^a \) with \( a, b, c = 0, 1, 2, 3 \) are the generalized Stückelberg scalar fields which correspond to the coordinate transformations \(^2\) and \( f_{ab}(\phi^c) \) is an auxiliary metric to be determined (or chosen) depending on the physical

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\(^1\)Here we slightly change notation and consider a global form of the action rather than a coordinate basis one.

\(^2\)For the entire formulation below we will assume that \( \mu, \nu \cdots = 0, 1, 2, 3 \) as well as \( a, b, c \cdots = 0, 1, 2, 3 \) where as \( i, j, k \cdots = 1, 2, 3 \).
problem at hand and it is a function of the scalars. Following the original definition in [8] we have the square root matrix
\[ \sqrt{\Sigma} \sqrt{\Sigma} = \Sigma. \] (2.6)

In the absence of matter the metric equation [8] is
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \Lambda g_{\mu\nu} + \frac{1}{2} m^2 [g \sqrt{\Sigma} + \sqrt{\Sigma}^T g]_{\mu\nu} + m^2 g_{\mu\nu} (3 - tr [\sqrt{\Sigma}]) = 0. \] (2.7)

Also in [8] the Stückelberg scalar field equations are equivalently formulated as
\[ \nabla_\mu \left( [\sqrt{\Sigma}]^\mu_{\nu} + [g^{-1} \sqrt{\Sigma}^T g]^\mu_{\nu} - 2 tr [\sqrt{\Sigma}] \delta^\mu_{\nu} \right) = 0, \] (2.8)
where \( \nabla_\mu \) is the covariant derivative of the Levi-Civita connection acting on \( \sqrt{\Sigma} \) as a (1,1) tensor. Now consider solutions satisfying the constraint
\[ \frac{1}{2} [g \sqrt{\Sigma} + \sqrt{\Sigma}^T g] - tr [\sqrt{\Sigma}] g = C g, \] (2.9)
where \( C \) is an arbitrary constant and we prefer the matrix notation. When this solution constraint is substituted in (2.7) one obtains
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \tilde{\Lambda} g_{\mu\nu} = 0, \] (2.10)
where
\[ \tilde{\Lambda} = -\frac{1}{2} \Lambda + 3 m^2 + C m^2. \] (2.11)

Therefore upon applying the constraint (2.9) on the Stückelberg scalars, the physical and the fiducial metrics the metric sector of MMGR is truncated to the cosmological constant type vacuum-Einstein theory. Now (2.9) implies
\[ \sqrt{\Sigma} + g^{-1} \sqrt{\Sigma}^T g - 2 tr [\sqrt{\Sigma}] \mathbf{1}_4 = 2 C \mathbf{1}_4, \] (2.12)
where \( \mathbf{1}_4 \) is the \( 4 \times 4 \) identity matrix. If we take the covariant derivative of (2.12) and contract it with the upper indices of the matrices we see that solutions satisfying this constraint automatically and trivially satisfy the Stückelberg scalar field equations (2.8). On the other hand we also infer

\[ \text{At this stage there is no guarantee whether such solutions are compatible with the field equations and exist but below we will explicitly prove their existence.} \]
from the on-shell-redundant metric equation (2.10) that such solutions have the constant Ricci scalar

$$R = 4\tilde{\Lambda}. \quad (2.13)$$

So via (2.10) we have the $\Lambda$-type Einstein metric vacuum solutions whose Ricci curvature 2-forms are constant and can be written as

$$R^{\alpha\beta} = K e^{\alpha\beta}, \quad (2.14)$$

for some moving co-frame $e^\alpha$. Here $R = 12K$ and for our solutions we have

$$K = \frac{1}{3}\tilde{\Lambda}. \quad (2.15)$$

We observe that in this scheme one obtains the flat solutions if $C = \Lambda/(2m^2) - 3$. Now if we take the trace of (2.12) we find that

$$tr[\sqrt{\Sigma}] = -\frac{4}{3}C. \quad (2.16)$$

Substituting this result back in (2.12) we get the matrix relation

$$g\sqrt{\Sigma} + \sqrt{\Sigma}^T g = -\frac{2C}{3}g, \quad (2.17)$$

which must be satisfied by the constant curvature metric solutions of (2.10), the St"uckelberg scalar fields $\phi^c$ and implicitly by the yet not specified fiducial metric $\bar{f}_{ab}$ so that they also satisfy (2.7) and (2.8) and become least action solutions of the minimal massive gravity action (2.1). (2.17) is the most general form of the constraint on such solutions to be solved. Next after refining this constraint matrix equation for a diagonal fiducial metric form we will discuss solution methods which lead to functionally parametric class of solutions composed of the physical metric, the St"uckelberg scalars and the fiducial metric. Thus we will show that there exists solutions of (2.7) and (2.8) which have constant curvature that satisfy the pre-assumed constraint (2.9). Explicit construction of a class of solutions in Section five will justify and finalize the validity of our formalism suggested above.

### 3 St"uckelberg scalar equations

Locally a space of constant curvature always has coordinates $x^\mu$ for which the corresponding metric can be written as

$$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{(1 + \frac{K}{4}x^2)^2}, \quad (3.1)$$
where \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) and \( x^2 = x^\mu x^\nu \eta_{\mu\nu} \). Now let us assume that \( f \) which must satisfy (2.17) is also a diagonal matrix. Below we will show that this is a consistent choice as for the coordinate system generating (3.1) there exist solutions of (2.17) for which \( f \) indeed becomes a diagonal matrix. Now by bearing in mind that the constant curvature metric and its inverse via (3.1) are diagonal and also by using the diagonal matrix assumption for \( f \) the constraint equation (2.17) becomes a more simplified matrix relation

\[
f = \frac{C^2}{9} g,
\]

(3.2)

here we have used the fact that when \( g^{-1} \) and \( f \) are diagonal \( \Sigma = g^{-1} f \) and thus \( \sqrt{\Sigma} \) also become diagonal matrices. Thus \( g \) commutes with \( \sqrt{\Sigma} \) and we have \( \sqrt{\Sigma}^T = \sqrt{\Sigma} \). Eq. (3.2) proves the consistency of our assumption of diagonal \( f \) as the RHS of (3.2) via (3.1) has become indeed diagonal. In component form this constraint matrix equation becomes

\[
\partial_\mu \phi^a \partial_\nu \phi^b \tilde{f}_{ab}(\phi^c) = \frac{C^2}{9} g_{\mu\nu}.
\]

(3.3)

We should state that in this equation only the diagonal components are non-zero. Another consequence of Eq. (3.3) is the fact that the constant curvature Einstein metric \( g_{\mu\nu} \) which minimizes the action (2.1) must be a multiple of the induced metric which is the pull-back of the fiducial metric \( \tilde{f}_{ab}(\phi^c) \) via the coordinate transformations generated by the St"ückelberg scalar fields \( \phi^c \). To solve (3.3) explicitly let us assume that the fiducial metric is of the form

\[
\tilde{f} = \begin{pmatrix}
f_{00}(\phi^a) & 0 & 0 & 0 \\
0 & f_{11}(\phi^a) & 0 & 0 \\
0 & 0 & f_{22}(\phi^a) & 0 \\
0 & 0 & 0 & f_{33}(\phi^a)
\end{pmatrix}.
\]

(3.4)

Following this choice and substituting (3.1) in (3.3) we obtain the set of equations when \( \mu = \nu \)

\[
\sum_{a=0}^{3} (\partial_\mu \phi^a)^2 f_{aa}(\phi^b) = C' \frac{\eta_{\mu\mu}}{(1+K x^2)^2},
\]

(3.5)

and when \( \mu \neq \nu \)

\[
\sum_{a=0}^{3} \partial_\mu \phi^a \partial_\nu \phi^a f_{aa}(\phi^b) = 0,
\]

(3.6)
where we define $C' = C^2/9$. For the sake of generating solutions in the rest of our analysis instead of (3.6) we will consider the stronger conditions

$$\partial_\mu \phi^a \partial_\nu \phi^a = 0,$$

(3.7)

for each $a = 0, 1, 2, 3$ and again for $\mu \neq \nu$ by bearing in mind that (3.7) satisfy (3.6) trivially. Now by using (3.7) one can show easily via multiplying both sides of (3.5) for each $\mu$ by $\partial_\alpha \phi^a \partial_\beta \phi^b \partial_\gamma \phi^c$ where $\alpha, \beta, \gamma \neq \mu$ that (3.5) leads for each $a = 0, 1, 2, 3$ to the equations

$$(\partial_a \phi^a)^2 f_{aa}(\phi^b) = C' \frac{\eta_{aa}}{(1 + \frac{K}{4} x^2)^2},$$

(3.8)

where there is no sum on $a$. Inspecting (3.8) one realizes that $f_{00}$ must be negative. Keeping this fact in mind, we have

$$\partial_0 \phi^0 \sqrt{-f_{00}} = \pm \sqrt{C'} \frac{1 + \frac{K}{4} x^2}{},$$

(3.9)

$$\partial_i \phi^i \sqrt{f_{ii}} = \pm \sqrt{C'} \frac{1 + \frac{K}{4} x^2}{},$$

where $i = 1, 2, 3$ are the spatial indices on which there is no sum. These four equations must be solved together with (3.7) which define twenty four independent equations.

### 4 Solution methodology

One can consider two distinct methods to solve (3.7) and (3.9) for the Stückelberg scalar fields $\phi^c$. The first root one can follow depends on the observation that (3.9) enable one to choose any functional form for the fiducial metric components $f_{aa}(\phi^b)$ of (3.4) then do separation of variables and perform partial integration on both sides. This leads to the set of equations

$$\int \sqrt{-f_{00}(\phi^b)} \partial \phi^0 = F_0^0,$$

(4.1)

$$\int \sqrt{f_{ii}(\phi^b)} \partial \phi^i = F_i^i,$$

(4.2)
where \( \Delta \) defines the region of spacetime and it is \( \Delta = C''C''' \) with the definitions

for \( \mu = 0 \):

\[
C'' = -\frac{K}{4}, \quad C''' = \frac{K}{4}x^i x_i + 1,
\]

for \( \mu = i = 1, 2, 3 \):

\[
C'' = \frac{K}{4}, \quad C''' = \frac{K}{4}(-(x^0)^2 + \sum_{j\neq i} x^j x_j) + 1.
\]

The region dependent \( F^\mu_\Delta \) functions on the RHS of (4.1) result from the partial integration of the RHS of (3.9) with respect to \( x^\mu \). When \( \Delta > 0 \)

\[
F^\mu_\Delta = \pm \frac{2\sqrt{C'}}{\sqrt{4\Delta}} \arctan \left( \frac{2C''x^\mu}{\sqrt{4\Delta}} \right) + C^\mu_\Delta(x^{\nu\neq\mu}),
\]

when \( \Delta < 0 \)

\[
F^\mu_\Delta = \pm \frac{\sqrt{C'}}{\sqrt{-4\Delta}} \ln \left| \frac{2C''x^\mu - \sqrt{-4\Delta}}{2C''x^\mu + \sqrt{-4\Delta}} \right| + C^\mu_\Delta(x^{\nu\neq\mu}),
\]

and when \( \Delta = 0 \)

\[
F^\mu_\Delta = \pm \frac{\sqrt{C'}}{C''x^\mu} + C^\mu_\Delta(x^{\nu\neq\mu}),
\]

where \( C^\mu_\Delta(x^{\nu\neq\mu}) \) are integration variable-independent functions of the partial integrations. The method one has to follow in finding solutions to (4.1) subject to the conditions (3.7) is first to choose functional forms for \( f_{\mu a}(\phi^b) \) for each \( a = 0, 1, 2, 3 \), perform the partial integration on the LHS of (4.1) and solve algebraically \( \phi^a \)’s. Then one can apply the constraints (3.7) to solve \( C^\mu_\Delta(x^{\nu\neq\mu}) \)’s. We must remark a couple of important points here. Firstly this analysis is done region by region depending on \( \Delta \). Secondly not all functional forms of \( f_{\mu a}(\phi^b) \) may lead solutions. One may have to inspect the conditions (3.7) carefully in order to choose sensible functional dependence of the fiducial metric components on the St"uckelberg scalars so that there exist solutions. In this direction for example one may assume that \( f_{\mu a} = f_{\mu a}(\phi^a) \) only. Finally one should observe that the conditions for each \( \mu = 0, 1, 2, 3 \)

\[
\partial_{\nu\neq\mu} \phi^\mu = 0,
\]
solve (3.7) and reduce the number of conditions to be satisfied by each $\phi^\mu$ from six to three. Therefore they are more preferable to construct the right form of the fiducial metric components out of the St"uckelberg fields that will lead to solutions. As an alternate to the above-mentioned method which is based on the explicit choice of the functional dependence of the fiducial metric components on the St"uckelberg scalars one can consider their implicit dependence as well namely one can propose how $f_{\alpha\alpha}$'s depend on the space-time coordinates directly suppressing the St"uckelberg scalars which are just coordinate transformations to an implicit level. In this direction first for each $\mu = 0, 1, 2, 3$ let us introduce three functions $G^\mu_j(x^\nu)$ for $j = 1, 2, 3$. Next we will define the fiducial metric components as any functionals of these functions

$$f_{\mu\nu} = f_{\mu\nu}[G^\mu_1(x^\nu), G^\mu_2(x^\nu), G^\mu_3(x^\nu)].$$ (4.7)

When one specifies the form of these functionals in terms of the yet undetermined functions and substitute them in (3.9) one can do the partial integration which yields the St"uckelberg scalars as

$$\phi^0(x^\nu) = \pm \int \frac{\sqrt{C'} \partial x^0}{\sqrt{-f_{00}[G^0_j(x^\nu)](1 + \frac{K}{4}x^2)}},$$ (4.8)

$$\phi^i(x^\nu) = \pm \int \frac{\sqrt{C'} \partial x^i}{\sqrt{f_{ii}[G^i_j(x^\nu)](1 + \frac{K}{4}x^2)}}.$$

We should not forget that these coordinate transformations are subject to the conditions (3.7) which we have discussed to be satisfied by the special choice (4.6). Therefore the unknown functions $G^\mu_j(x^\nu)$ can be determined by applying (4.6) for each St"uckelberg scalar field in (4.8). If one fixes the resultant integration variable-independent functions of the partial integration to be zero applying (4.6) in (4.8) yields

$$\partial_i \left( \frac{1}{\sqrt{-f_{00}[G^0_j(x^\nu)](1 + \frac{K}{4}x^2)}} \right) = 0,$$ (4.9)

$$\partial_{\nu \neq i} \left( \frac{1}{\sqrt{f_{ii}[G^i_j(x^\nu)](1 + \frac{K}{4}x^2)}} \right) = 0.$$
For each set of three functions $G_{\mu}^1(x^\nu), G_{\mu}^2(x^\nu), G_{\mu}^3(x^\nu)$ for $\mu = 0, 1, 2, 3$ one obtains three coupled non-linear first order pde’s. However we should state that these equations are de-coupled in differentiation variable sense. Therefore as a result when one chooses the functional forms in (4.7) one can substitute them in (4.9), then solve for $G_{\mu}^j(x^\nu)$’s from these pde’s thus the fiducial metric components are determined explicitly. Then one can substitute them in (4.8) and integrate out the St"uckelberg scalars.

5 A class of solutions

In this section we will present a class of solutions based on a simple observation. The conditions in (4.9) state that the expressions inside the parentheses must depend on only the spacetime variable of the corresponding St"uckelberg scalar field index. In other words in (4.9) the first parenthesis must only depend on $x^0$ and the following ones only on $x^1, x^2, x^3$ respectively. Therefore if we set

$$f_{\mu\mu} = G_{\mu}^1 G_{\mu}^2 G_{\mu}^3,$$ \hspace{1cm} (5.1)

then the following choice of functions

$$G_1^0 = \frac{1}{(1 + \frac{K}{4}x^2)^2}, \hspace{1cm} G_2^0 = -1, \hspace{1cm} G_3^0 = \frac{1}{(F_0(x^0))^2},$$ \hspace{1cm} (5.2)

$$G_1^i = \frac{1}{(1 + \frac{K}{4}x^2)^2}, \hspace{1cm} G_2^i = 1, \hspace{1cm} G_3^i = \frac{1}{(F_i(x^i))^2},$$ \hspace{1cm} (5.2)

where $F_{\mu}(x^\mu)$ are arbitrary functions of a single variable solve (4.9). In this case the fiducial metric components become

$$f_{00} = \frac{-1}{(1 + \frac{K}{4}x^2)^2(F_0(x^0))^2}, \hspace{1cm} f_{ii} = \frac{1}{(1 + \frac{K}{4}x^2)^2(F_i(x^i))^2},$$ \hspace{1cm} (5.3)

and when these are substituted in (4.8) we can solve the St"uckelberg scalars as

$$\phi^0(x^0) = \pm \int \sqrt{C'} F_0(x^0) dx^0, \hspace{1cm} \phi^i(x^i) = \pm \int \sqrt{C'} F_i(x^i) dx^i.$$ \hspace{1cm} (5.4)
If instead one chooses all $F_\mu(x^\mu) = 1$ then one obtains the linear coordinate transformations
\[
\phi^0(x^0) = \pm \sqrt{C'} x^0 + C^0, \quad \phi^i(x^i) = \pm \sqrt{C'} x^i + C^i, \tag{5.5}
\]
for which
\[
f_{00} = \frac{-1}{(1 + \frac{K}{4} x^2)^2}, \quad f_{ii} = \frac{1}{(1 + \frac{K}{4} x^2)^2}. \tag{5.6}
\]

As a summary we have found that for any constant curvature $\Lambda$–type vacuum Einstein solution metric which is parameterized by $K$ via (2.15) the St"uckelberg scalar fields, and the fiducial metrics which form solutions of (3.9) and (3.7) derived as a result of either method discussed in Section four or as a special class of solutions explicitly constructed in this section are also the constant curvature solutions of the minimal massive gravity namely the field equations (2.7) and (2.8).

6 Conclusion

We have introduced a straightforward scheme to study the constant curvature solutions of MMGR which replaces the method of directly solving the metric and the St"uckelberg scalar field equations upon fixing the fiducial metric by the method of solving a constraint matrix equation containing the physical, fiducial (background) metrics and the scalars of the mass term for a given constant curvature Einstein metric. After obtaining the general form of this constraint equation which has no a-priori guarantee to be compatible with the field equations we have simplified it by using the local form of the constant curvature metric which solves the on-shell remainder part of the metric equation upon the substitution of our constraint in the MMGR action. We then proceeded in discussing the solution methods of this constraint equation by assuming a diagonal form for the fiducial metric which later is justified by leading such form of solutions. Therefore under these conditions we were able to construct an integral form of the St"uckelberg scalars and thus the fiducial metric solutions locally which can be integrated when one chooses the functional form of the fiducial metric either in terms of the scalars or the spacetime coordinates. We have also derived a class of explicit solutions by integrating out these formal equations for the St"uckelberg fields in the later method.
Though depending on a somewhat simple observation and approach we should state that our construction enables one to find the particular fiducial metric needed to solve the MMGR field equations when one determines the desired physical metric solutions of the MMGR. Moreover as it must be clear from Sections three, four, and five one has the degree of freedom to arbitrarily generate fiducial metrics which correspond to the same constant curvature physical metrics by tuning the functional form of the diagonal elements of the fiducial metric either in Stückelberg fields or spacetime dependence sense. This methodology is quite different from the currently preferred one in the literature which fixes the reference metric from the beginning and then studies the solutions of the resulting MMGR action. This methodology may help to cure the current challenge of finding stable thus physically interesting solutions to the minimal and the general massive gravity theories constructed in [8] for a general fiducial metric.

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