Isometric, Symmetric and Isosymmetric Commuting $d$-Tuples of Banach Space Operators

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Abstract. Generalising the definition to commuting $d$-tuples of operators, a number of authors have considered structural properties of $m$-isometric, $n$-symmetric and $(m, n)$-isosymmetric commuting $d$-tuples in the recent past. This note is an attempt to take the mystique out of this extension and show how a large number of these properties follow from the more familiar arguments used to prove the single operator version of these properties.

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1. Introduction

Let $B(\mathcal{H})$ (resp., $B(\mathcal{X})$) denote the algebra of operators, i.e. bounded linear transformations, on an infinite dimensional complex Hilbert space $\mathcal{H}$ into itself (resp., on an infinite dimensional complex Banach space $\mathcal{X}$ into itself), $\mathbb{C}$ denote the complex plane, $B(\mathcal{H})^d$ (resp., $B(\mathcal{X})^d$ and $\mathbb{C}^d$) the product of $d$ copies of $B(\mathcal{H})$ (resp., $B(\mathcal{X})$ and $\mathbb{C}$) for some integer $d \geq 1$, $\overline{z}$ the conjugate of $z \in \mathbb{C}$ and $z = (z_1, z_2, ..., z_d) \in \mathbb{C}^d$. For a given polynomial $P$ in $\mathbb{C}^d$ and

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a $d$-tuple $A$ of commuting operators in $B(H)^d$, $A$ is a hereditary root of $P$ if $P(A) = 0$. Two particular operator classes of hereditary roots which have been studied extensively are those of $m$-symmetric (also called $m$-selfadjoint in the literature) and $m$-isometric operators, where $A \in B(H)$ is $m$-symmetric (for some integer $m \geq 1$) if

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} A^*(m-j)A^j = 0
$$

and $A \in B(H)$ is $m$-isometric if

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} A^*j A^j = 0
$$

Combining these two classes, we say $A \in B(H)$ is an $(m, n)$-isosymmetry (equivalently, the pair $(A^*, A)$ is $(m, n)$-isosymmetric) for some integers $m, n \geq 1$ if

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} A^*j \left( \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^*(n-k)A^k \right) A^j = 0.
$$

It is clear that $m$-symmetric operators arise as solutions of $P(z) = (z-z)^m = 0$, $m$-isometric operators arise as solutions of $P(z) = (z^2 - 1)^m = 0$ and $(m, n)$-isosymmetric operators arise as solutions of $(z^2 - 1)^m (z-z)^n = 0$. The class of $m$-symmetric operators was introduced by Helton [24] (albeit not as operator solutions of the polynomial equation $(z-z)^m = 0$), and the class of $m$-isometric operators was introduced by Agler [1]. These classes of operators, and their variants, have since been studied by a multitude of authors, amongst them Agler and Stankus [2–4], Sid Ahmed [5], Bayart [7], Bermudez et al [9–11], Botelho and Jamison [8], Duggal [14,15], Gu [21,22] and Gu and Stankus [23], Stankus [27] and Trieu Le [28].

A generalisation of the $m$-isometric property of operators $A \in B(H)$ to commuting $d$-tuples $A = (A_1, \cdots, A_d) \in B(H)^d$, $[A_i, A_j] = A_iA_j - A_jA_i = 0$ for all $1 \leq i, j \leq d$, is obtained as follows [20]: $A$ is $m$-isometric if

$$
\sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{|\beta|=j} j! \beta^* \beta A^\beta = 0,
$$

where

$$
\beta = (\beta_1, \cdots, \beta_d), \quad |\beta| = \sum_{i=1}^{d} \beta_i, \quad \beta! = \Pi_{i=1}^{d} \beta_i!,
$$

$$
A^\beta = \Pi_{i=1}^{d} A_i^{\beta_i}, \quad A^*\beta = \Pi_{i=1}^{d} A_i^{*\beta_i};
$$
A is $n$-symmetric if
$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} (A_1^* + \cdots + A_d^*)^{n-j} (A_1 + \cdots + A_d)^j = 0.$$ 
These generalisations, and certain of their variants (including $(m, n)$-isosymmetric operators), have recently been the subject matter of a number of studies, see [6,12,13,16,17,19] for further references.

This paper studies $(X, m)$-isometric, $(X, n)$-symmetric and $(X, (m, n))$-isosymmetric commuting Banach space $d$-tuples from the point of view of operators defined by elementary operators (of left and right multiplication) and shows how the arguments from the single operator case [16,17] work just as well in proving a number of the structural properties of these classes of operators. The plan of the paper is as follows. In Section 2, we introduce our generalised definition of $(X, m)$-isometric, $(X, n)$-symmetric and $(X, (m, n))$-isosymmetric commuting $d$-tuples in $B(X)^d$, and prove some well known and some not so well known (possibly new) results on the structure of these operators. Section 3 considers perturbation by commuting nilpotent $d$-tuples, and Sect. 4 considers commuting products.

## 2. Definitions and Introductory Properties

For $A, B \in B(\mathcal{X})$, let $L_A$ and $R_B \in B(B(\mathcal{X}))$ denote respectively the operators
$$L_A(\mathcal{X}) = AX \text{ and } R_B(\mathcal{X}) = XB$$
of left multiplication by $A$ and right multiplication by $B$. A $d$-tuple $A = (A_1, \cdots, A_d) \in B(\mathcal{X})^d$ is a commuting $d$-tuple if
$$[A_i, A_j] = A_i A_j - A_j A_i = 0, \text{ all } 1 \leq i, j \leq d.$$Given commuting $d$-tuples $A = (A_1, \cdots, A_d)$ and $B = (B_1, \cdots, B_d)$, define operators $L_A$ and $R_B$ by
$$L_A^\alpha = \Pi_{i=1}^d L_{A_i}^{\alpha_i}, \quad R_B^\alpha = \Pi_{i=1}^d R_{B_i}^{\alpha_i}$$
where
$$\alpha = (\alpha_1, \cdots, \alpha_d), \quad |\alpha| = \sum_{i=1}^d \alpha_i, \quad \alpha_i \geq 0 \text{ for all } 1 \leq i \leq d.$$For $d$-tuples $A$ and $B$, and an operator $X \in B(\mathcal{X})$, let "$*$" and "$\times$" denote, respectively, the multiplication operations
$$(L_A * R_B)^j(X) = \left( \sum_{|\alpha| = j} \frac{j!}{\alpha!} L_A^\alpha R_B^\alpha \right) (X) = \left( \sum_{i=1}^d L_{A_i} R_{B_i} \right)^j (X)$$
(all integers $j \geq 0$, $\alpha! = \alpha_1! \cdots \alpha_d!$) and
\((L_A \times R_B)(X) = \left( \sum_{i=1}^{d} L_{A_i} \right) \left( \sum_{i=1}^{d} R_{B_i} \right)(X).\)

We say that the \(d\)-tuples \(A\) and \(B\) commute, \([A, B] = 0\), if \([A_i, B_j] = 0\) for all \(1 \leq i, j \leq d\).

Evidently,
\([L_A, R_B] = 0\)
and if \([A, B] = 0\), then
\([L_A, L_B] = [R_A, R_B] = 0\).

A pair \((A, B)\) of commuting \(d\)-tuples \(A\) and \(B\) is said to be \((X, m)\)-isometric, \((A, B) \in (X, m)\)-isometric, for some positive integer \(m\) and operator \(X \in B(\mathcal{X})\), if
\[\Delta_{A,B}^m(X) = (I - L_A \times R_B)^m(X)\]
\[= \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (L_A \times R_B)^j \right)(X)\]
\[= \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{i=1}^{d} L_{A_i} R_{B_i} \right)^j \right)(X)\]
\[= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{|\alpha|=j} A^\alpha X B^\alpha \frac{j!}{\alpha!} \right)\]
\[= 0;\]

\((A, B)\) is \((X, n)\)-symmetric, for some positive integer \(n\) and operator \(X \in B(\mathcal{X})\), if
\[\delta_{A,B}^n(X) = (L_A - R_B)^n(X)\]
\[= \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} L_A^{n-j} \times R_B^j \right)(X)\]
\[= \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \sum_{i=1}^{d} L_{A_i} \right)^{n-j} \left( \sum_{i=1}^{d} R_{B_i} \right)^j \right)(X)\]
\[= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left( \sum_{i=1}^{d} A_i \right)^{n-j} X \left( \sum_{i=1}^{d} B_i \right)^j\]
[0].

Commuting tuples of \((X, m)\)-isometric, similarly \((X, n)\)-symmetric operators, share a large number of properties with their single operator counterparts.
However, there are instances where a property holds for the single operator
version but fails for the $d$-tuple version. For example, whereas
\[
\Delta^m_{A,B}(X) = 0 \iff \Delta^m_{A^t,B^t}(X) \quad \text{for all integers } t \geq 1
\]
and
\[
\Delta^m_{A,B}(X) = 0 \iff \Delta^m_{A^{-1},B^{-1}}(X) = 0 \quad \text{for all invertible } A \text{ and } B,
\]
these properties fail for $d$-tuples, as the following example shows.

**Example 2.1.** If we define operators $A_i, B_i \ (i = 1, 2)$ by $A = \mathbb{B} = \left( \frac{1}{\sqrt{2}} I, \frac{1}{\sqrt{2}} I \right)$,
then $A = (A_1, A_2)$ and $\mathbb{B} = (B_1, B_2)$ are commuting, invertible $2$-tuples such that $(A, \mathbb{B})$ is $1$-isometric, i.e. $(I, I)$-isometric, but neither of $(A^2, \mathbb{B}^2)$, $A^2 = \mathbb{B}^2 = (A_1^2, A_1 A_2, A_2 A_1, A_2^2)$, and $(A^{-1}, \mathbb{B}^{-1})$, $A^{-1} = \mathbb{B}^{-1} = (\sqrt{2} I, \sqrt{2} I)$, is $m$-isometric for any $m$.

In the following we show that where a property is shared by the single oper-
ator and the $d$-tuple versions, a proof of the $d$-tuple version of the result is
obtained from the argument of the single operator version of the result (if not
by a transliteration of the argument, then by a simple additional argument).
We remark here that a number of authors have considered $(X, m)$-isometric
and $(X, n)$-symmetric Hilbert space tuples with the operator $X$ replaced by a
positive operator $P$. The consideration of a general operator $X$, rather than
$P \geq 0$, does not involve extra argument and does not, in general, result in loss
of information. Any additional information that may result from a consider-
ation with $P \geq 0$ is usually a result of additional hypotheses on $P$, such as
injectivity, which lead to additional structure on the underlying Hilbert space.

We start in the following with a couple of basic observations. The definitions
imply
\[
\Delta^t_{A,B}(X) = \Delta^{t-m}_{A,B} \left( \Delta^m_{A,B}(X) \right), \quad \delta^t_{A,B}(X) = \delta^{t-n}_{A,B} \left( \delta^n_{A,B}(X) \right),
\]
\[
\Delta^{t_1}_{A,B} \left( \delta^{t_2}_{A,B}(X) \right) = \Delta^{t_1-m}_{A,B} \left[ \Delta^{m}_{A,B} \left( \delta^{t_2}_{A,B}(X) \right) \right]
\]
\[
= \Delta^{t_1-m}_{A,B} \left[ \delta^{t_2}_{A,B} \left( \Delta^{m}_{A,B}(X) \right) \right]
\]
\[
= \Delta^{t_1}_{A,B} \left[ \delta^{t_2-n}_{A,B} \left( \delta^{m}_{A,B}(X) \right) \right]
\]
and
\[
\Delta^{t_1}_{A,B} \left( \delta^{t_2}_{A,B}(X) \right) = \Delta^{t_1-m}_{A,B} \left[ \Delta^{m}_{A,B} \left( \delta^{t_2-n}_{A,B} \left( \delta^{m}_{A,B}(X) \right) \right) \right]
\]
\[
= \Delta^{t_1-m}_{A,B} \left[ \delta^{t_2-n}_{A,B} \left( \Delta^{m}_{A,B}(X) \right) \right]
\]
for all integers $t_1 \geq m$ and $t_2 \geq n$. Thus:

**Proposition 2.2.** Given commuting $d$-tuples $A, \mathbb{B}$ in $B(X)^d$ and an operator
$X \in B(X)$,
\((A, B) \in (X, m)\) – isometric
\(\implies (A, B) \in (X, t_1)\) – isometric for all integers \(t_1 \geq m\);
\((A, B) \in (X, n)\) – symmetric
\(\implies (A, B) \in (X, t_2)\) – symmetric for all integers \(t_2 \geq n\);
\((A, B) \in (X, m)\) – isometric \(\land (A, B) \in (X, n)\) – symmetric
\(\implies (A, B) \in (X, (t_1, t_2))\) – isosymmetric for all integers \(t_1 \geq m, t_2 \geq n\).

If \((A, B) \in (X, m)\)-isometric, then
\[
\triangle_{A, B}^m(X) = 0 \iff (I - L_A \ast R_B) \left( \triangle_{A, B}^{m-1}(X) \right) = 0
\]
\[
\iff (L_A \ast R_B) \left( \triangle_{A, B}^{m-1}(X) \right) = \triangle_{A, B}^{m-1}(X)
\]
\[
\iff (L_A \ast R_B)^2 \left( \triangle_{A, B}^{m-1}(X) \right) = (L_A \ast R_B) \left( \triangle_{A, B}^{m-1}(X) \right) = \triangle_{A, B}^{m-1}(X)
\]
\[\ldots\]
\[
\iff (L_A \ast R_B)^t \left( \triangle_{A, B}^{m-1}(X) \right) = \triangle_{A, B}^{m-1}(X).
\]
for all integers \(t \geq 0\). Since \(L_A \ast R_B\) commutes with \(\triangle_{A, B}^{m-1}\), we also have
\[
\triangle_{A, B}^{m-1} \left( (L_A \ast R_B)^t(X) \right) = \triangle_{A, B}^{m-1}(X)
\]
for all integers \(t \geq 0\).

Again, if \((A, B) \in (X, n)\)-symmetric, then
\[
\delta_{A, B}^n(X) = 0 \iff (L_A - R_B) \left( \delta_{A, B}^{n-1}(X) \right) = 0
\]
\[
\iff L_A \left( \delta_{A, B}^{n-1}(X) \right) = R_B \left( \delta_{A, B}^{n-1}(X) \right)
\]
\[\ldots\]
\[
\iff L_A^t \left( \delta_{A, B}^{n-1}(X) \right) = R_B^t \left( \delta_{A, B}^{n-1}(X) \right)
\]
for all integers \(t \geq 0\). Here
\[
L_A \left( \delta_{A, B}^{n-1}(X) \right) = L_A \left( \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} L_A^{n-1-j} \times R_B^j \right)(X)
\]
\[
= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} L_A^{n-j} \times R_B^j \right)(X)
\]
and
\[
R_B \left( \delta_{A, B}^{n-1}(X) \right) = \left( \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} L_A^{n-1-j} \times R_B^{j+1} \right)(X).
\]
Proposition 2.3. Given commuting $d$-tuples $A, B \in B(X)^d$ and an operator $X \in B(X)$, if $(A, B) \in (X, m)$-isometric, then
\[
\lim_{t \to \infty} \frac{1}{t} (L_A \ast R_B)^t(X) = \Delta_{A,B}^{m-1}(X).
\]
In particular, if $(L_A \ast R_B)$ is invertible, then $\Delta_{A,B}^{m-1}(X) = 0$.

Proof. The identity
\[
(a-1)^m = a^m - \sum_{j=0}^{m-1} \binom{m}{j} (a-1)^j
\]
applied to $\Delta_{A,B}^{m}(X) = (I - L_A \ast R_B)^m(X) = 0$ implies
\[
(L_A \ast R_B)^m(X) = \sum_{j=0}^{m-1} \binom{m}{j} \Delta_{A,B}^{j}(X).
\]
Observing
\[
(L_A \ast R_B)^{m+1}(X) = \sum_{j=0}^{m-1} \binom{m+1}{j} \Delta_{A,B}^{j+1}(X) + \sum_{j=0}^{m-1} \binom{m}{j} \Delta_{A,B}^{m-j}(X)
\]
\[
= \binom{m}{m-1} \Delta_{A,B}^{m}(X) + \sum_{j=0}^{m-1} \binom{m+1}{j} \Delta_{A,B}^{j}(X)
\]
\[
= \sum_{j=0}^{m-1} \binom{m+1}{j} \Delta_{A,B}^{j}(X),
\]
an induction argument shows that
\[
(L_A \ast R_B)^t(X) = \sum_{j=0}^{m-1} \binom{t}{j} \Delta_{A,B}^{j}(X)
\]
\[
= \binom{t}{m-1} \Delta_{A,B}^{m-1}(X) + \sum_{j=0}^{m-2} \binom{t}{j} \Delta_{A,B}^{j}(X)
\]
for all integers $t \geq m$. Since $\binom{t}{m-1}$ is of the order of $tm^{-1}$ and $\binom{t}{j}$, $0 \leq j \leq m - 2$, is of the order of $tm^{-2}$ as $t \to \infty$, we have
\[
\Delta_{A,B}^{m-1}(X) = \lim_{t \to \infty} \frac{1}{t} \left( \binom{t}{m-1} \Delta_{A,B}^{m-1}(X) + \sum_{j=0}^{m-2} \binom{t}{j} \Delta_{A,B}^{j}(X) \right)
\]
\[
= \lim_{t \to \infty} \frac{1}{t} (L_A \ast R_B)^t(X).
\]
As seen above \((L_A \ast R_B)^t(\triangle_{A,B}^{m-1}(X)) = \triangle_{A,B}^{m-1}(X)\) for all integers \(t \geq 0\). Hence, if \(L_A \ast R_B\) is invertible, then \(\triangle_{A,B}^{m-1}(X) = (L_A \ast R_B)^{-t}(\triangle_{A,B}^{m-1}(X))\) for all integers \(t \geq 0\). Consequently, if \(L_A \ast R_B\) is invertible, then
\[
X = \left(\begin{array}{c} t \\ m-1 \end{array}\right) \triangle_{A,B}^{m-1}(X) + \sum_{j=0}^{m-2} \left(\begin{array}{c} t \\ j \end{array}\right) (L_A \ast R_B)^{-t}(\triangle_{A,B}^{j}(X)).
\]
Since
\[
\lim_{t \to \infty} \frac{1}{t} \left[\sum_{j=0}^{m-2} \left(\begin{array}{c} t \\ j \end{array}\right) (L_A \ast R_B)^{-t}(\triangle_{A,B}^{j}(X)) - X\right] = 0,
\]
we have \(\triangle_{A,B}^{m-1}(X) = 0\). □

The case \(B = (B_1, \ldots, B_d) \in B(\mathcal{H})^d, A = B^*, X = I\) and \(m = 2\) of Proposition 2.3 is of some interest: if \((A^*, A)\) is \((I, 2)\)-isometric and \(0 \notin \sigma\left(\sum_{i=1}^d |A_i|^2\right)\), then \(A\) is a spherical isometry (i.e., \((A^*, A)\) is \((I, 1)\)-isometric).

For tuples \(A \in B(\mathcal{H})^d\) such that \((A^*, A)\) is \((I, 2)\)-symmetric, we have the following analogue of the well known result that \((I, 2)\)-symmetric operators \(A \in B(\mathcal{H})\) are self-adjoint [16,25].

Recall that an operator \(A \in B(\mathcal{H})\) is hyponormal if \(AA^* \leq A^*A\). Hyponormal pairs \((A, B^*)\) satisfy the Putnam-Fuglede commutativity property, namely that \(\delta_{A^*,B}(I) = 0\) implies \(\delta_{A^*,B^*}(I) = 0\). Indeed, more is true [26]: if \(\delta_{A,B}(I) = 0\) for hyponormal \(A, B^* \in B(\mathcal{H})\) and some positive integer \(n\), then \(\delta_{A,B}(I) = \delta_{A^*,B^*}(I) = 0\).

**Proposition 2.4.** If \((A^*, A)\) is \((I, 2)\)-symmetric, then \(\sum_{i=1}^d A_i\) is self-adjoint.

**Proof.** For convenience, let
\[
\sum_{i=1}^d A_i = \sum A_i \quad \text{and} \quad \sum_{i=1}^d A_i^* = \sum A_i^*.
\]
The hypothesis \(\delta_{A^*,A}^2(I) = 0\) then implies
\[
\sum A_i^2 + 2\sum A_i^* + \sum A_i^2 = 0.
\]
Since already
\[
0 \leq \delta_{A^*,A}(I)^* \delta_{A^*,A}(I) = \sum A_i^2 - \sum A_i^* \sum A_i - \sum A_i^* \sum A_i + \sum A_i^2,
\]
we have (upon combining)
\[
\sum A_i^* \leq \sum A_i^2.
\]
Proposition 2.5. \( \sum \) is hyponormal. Evidently,

\[
\delta_{\lambda^*,\lambda}(I) = \sum_{j=0}^{2} (-1)^{2-j} \left( \begin{array}{c} 2 \\ j \end{array} \right) \sum_{i} s^j \sum_{i} 2^{-j} = 0 = \delta_{\sum,\sum^*}(I),
\]

the Putnam-Fuglede commutativity theorem applies and we conclude \( \delta_{\sum,\sum^*}(I) = \delta_{\sum,\sum}(I) = 0 \). Thus \( \sum = \sum^* \).

Proposition 2.4 is a particular case of the following more general result, which for the case of the single operator says that an \( (I, m) \)-symmetric operator \( T \in B(\mathcal{H}) \), \( m \) an even positive integer, is \( (I, m-1) \)-symmetric \([25]\). Let \( T - \lambda I = T - \lambda \).

**Proposition 2.5.** If \( \mathbb{A} = (A_1, \cdots, A_d) \in B(\mathcal{H})^d \) satisfies \( \delta_{\lambda^*,\lambda}(I) = 0 \) for some positive even integer \( m \), then \( \delta_{\lambda^*,\lambda}(I) = 0 \).

**Proof.** The idea of the proof below is to reduce the problem to that of a single operator. For this, we start by determining the approximate point spectrum for \( \sum = \sum^* \) hence \( \sum = \sum^* \).

Suppose \( \lambda \in \sigma_a(\mathbb{A}) \) and \( \lim_{n \to \infty} \| (A_i - \lambda) x_n \| = 0 \) for all \( 1 \leq i \leq d \). Then

\[
\delta_{\lambda^*,\lambda}(I) = \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \left( \sum_{i=1}^{d} A_i^* \right)^{m-j} \left( \sum_{i=1}^{d} A_i \right)^{j} x_n, x_n = 0
\]

implies

\[
\lim_{n \to \infty} \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \\ j \end{array} \right) \left( \sum_{i=1}^{d} A_i^* \right)^{m-j} \left( \sum_{i=1}^{d} A_i \right)^{j} x_n, x_n = 0
\]

Hence \( \sum_{i=1}^{d} \lambda_i \) is real for all \( \lambda \in \sigma_a(\mathbb{A}) \). The spectrum \( \sigma(\mathbb{A}) = \sigma_a(\mathbb{A}) \cup \sigma_a(\mathbb{A}^*) \) being a compact subset of \( \mathbb{C} \), there exists a real \( \lambda = (\lambda, \cdots, \lambda) \notin \sigma(\mathbb{A}) \) such that \( \sum_{i=1}^{d} (A_i - \lambda) = \sum_{i=1}^{d} A_i - d\lambda \) is invertible.
Let $\sum_{i=1}^{d} A_i - d\lambda = A_{\lambda}$. Then
\[
\delta_{A^*,A}^m(I) = (\mathbb{L}_{A^*} - \mathbb{R}_{A})^m(I) \\
= (\mathbb{L}_{(A^*-\lambda)} - \mathbb{R}_{(A^*-\lambda)})^m(I) \\
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{i=1}^{d} A_i^* - d\lambda \right) ^{m-j} \left( \sum_{i=1}^{d} A_i - d\lambda \right)^{j} \\
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} A_{\lambda}^{*(m-j)} A_{\lambda}^j \\
= \delta_{A_{\lambda}^*,A_{\lambda}}^m(I) \\
= 0.
\]

The operator $A_{\lambda}$ being invertible,
\[
\delta_{A^*,A}^m(I) = 0 \iff \delta_{A_{\lambda}^*,A_{\lambda}}^m(I) = 0 \\
\iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} A_{\lambda}^{*(m-j)} A_{\lambda}^j = 0 \\
\iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} A_{\lambda}^{*(-j)} A_{\lambda}^j = 0 \\
\quad (\text{multiply by } A_{\lambda}^{*m} \text{ on the right}) \\
\iff \triangle_{A_{\lambda}^{*m-1},A_{\lambda}}^m(I) = 0.
\]

Arguing as in the proof of [16, Theorem 3], see also Proposition 2.3, this implies (recall: $m$ is even)
\[
\triangle_{A_{\lambda}^{*m-1},A_{\lambda}}^m(I) = 0 \iff \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} A_{\lambda}^{*(m-1-j)} A_{\lambda}^j = 0 \\
\iff \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} A_{\lambda}^{*(m-1-j)} A_{\lambda}^j = 0 \\
\quad (\text{multiply on the left by } A_{\lambda}^{*(m-1)}) \\
\iff \delta_{A_{\lambda}^*,A_{\lambda}}^{m-1}(I) = 0 \iff \delta_{A_{\lambda}^*,A_{\lambda}}^{m-1}(I) = 0.
\]

This completes the proof. \[\square\]

Given a sequence of operators $\{A_n\} \in B(\mathcal{X})$, we write
\[
A_n \xrightarrow{s} A, \quad A_n \text{ converges strongly to } A,
\]
if
\[
\lim_{n \to \infty} \|A_n - A\| = 0.
\]
The $d$-tuple $A_n = (A_{1n}, \cdots, A_{dn})$ converges strongly to $A = (A_1, \cdots, A_d)$, $A_n \xrightarrow{s} A$, if $A_{in} \xrightarrow{s} A_i$ for all $1 \leq i \leq d$. The following proposition is an analogue of a result on the norm closure of the class of $m$-isometric, similarly, $m$-symmetric, operators.

**Proposition 2.6.** If $A_n = (A_{1n}, \cdots, A_{dn})$ and $B = (B_{1n}, \cdots, B_{dn})$ are sequences of $d$-tuples in $B(X)^d$ such that $A_{in} \xrightarrow{s} A_i$ and $B_{in} \xrightarrow{s} B_n$ for all $1 \leq i \leq d$ and if either of $\Delta_{A_n,B_n}^{m_1}(X)$ and $\delta_{A_n,B_n}^{m_2}(X)$ equals 0 for all $n$, then $\Delta_{A,B}^{m_1} \left( \delta_{A,B}^{m_2}(X) \right) = 0$.

**Proof.** We start by proving that $\Delta_{A_n,B_n}^{m_1}(X) = 0$ for all $n$ implies $\Delta_{A,B}^{m_1}(X) = 0$ and $\delta_{A_n,B_n}^{m_2}(X) = 0$ for all $n$ implies $\delta_{A,B}^{m_2}(X) = 0$. The hypotheses $A_n \xrightarrow{s} A$ and $B_n \xrightarrow{s} B$ implies

$$
\lim_{n \to \infty} \|A_{in} - A_i\| = \lim_{n \to \infty} \|B_{in} - B_i\| = 0
$$

for all $1 \leq i \leq d$. Since

$$
\left\| \Delta_{A_n,B_n}^{m_1}(X) - \Delta_{A,B}^{m_1}(X) \right\| \\
\leq \left\| \Delta_{A_n,B_n}^{m_1}(X) - \Delta_{A_n,B_n}^{m_1}(X) \right\| + \left\| \Delta_{A_n,B_n}^{m_1}(X) - \Delta_{A_n,B_n}^{m_1}(X) \right\|
$$

$$
\leq \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ j \end{array} \right) \left( \left\| \sum_{i=1}^{d} A_{in} X B_{in}^j - B_i^j \right\| + \left\| \sum_{i=1}^{d} (A_{in} - A_i) X B_{in}^j \right\| \right)
$$

$$
\leq \sum_{j=0}^{m_1} \left( \begin{array}{c} m_1 \\ j \end{array} \right) \left( \sum_{i=1}^{d} \|A_{in}\| \|X\| \|B_{in}^j - B_i^j\| + \|A_{in} - A_i\| \|X\| \|B_{in}^j\| \right) \to 0
$$

as $n \to \infty$ for all integers $j \geq 1$,

$\Delta_{A_n,B_n}^{m_1}(X) = 0$ implies $\Delta_{A,B}^{m_1}(X) = 0$. Considering next $\delta_{A_n,B_n}^{m_2}(X)$, we have

$$
\left\| \delta_{A_n,B_n}^{m_2}(X) - \delta_{A,B}^{m_2}(X) \right\| \\
\leq \left\| \delta_{A_n,B_n}^{m_2}(X) - \delta_{A_n,B_n}^{m_2}(X) \right\| + \left\| \delta_{A_n,B_n}^{m_2}(X) - \delta_{A_n,B_n}^{m_2}(X) \right\|
$$

$$
\leq \sum_{j=0}^{m_2} \left( \begin{array}{c} m_2 \\ j \end{array} \right) \left( \left\| \sum_{i=1}^{d} A_{in} \right\| \left( \sum_{i=1}^{d} B_{in}^j \right) - \left( \sum_{i=1}^{d} B_i \right)^j \right)
$$

$$
\left. + \left\| \left( \sum_{i=1}^{d} A_{in} \right) - \left( \sum_{i=1}^{d} A_i \right) \right\| \left( \sum_{i=1}^{d} B_{in}^j \right) \right).$$
Since
\[ \left\| \left( \sum_{i=1}^{d} B_{in} \right)^{j} - \left( \sum_{i=1}^{d} B_{i} \right)^{j} \right\| \leq \left\| \sum_{i=1}^{d} (B_{in} - B_{i})^{j} \right\| P \left( \sum_{i=1}^{d} B_{in}, \sum_{i=1}^{d} B_{i} \right) \]
for some polynomial \((P, ., .)\),
\[ \lim_{n \to \infty} \left\| \left( \sum_{i=1}^{d} B_{in} \right)^{j} - \left( \sum_{i=1}^{d} B_{i} \right)^{j} \right\| = 0. \]
Similarly,
\[ \lim_{n \to \infty} \left\| \left( \sum_{i=1}^{d} A_{in} \right)^{m_{2} - j} - \left( \sum_{i=1}^{d} A_{i} \right)^{m_{2} - j} \right\| = 0. \]
Hence
\[ \delta_{A_{n}, B_{n}}^{m_{2}} (X) = 0 \text{ for all } n \implies \lim_{n \to \infty} \left\| \delta_{A_{n}, B_{n}}^{m_{2}} (X) - \delta_{A_{n}, B_{n}}^{m_{2}} (X) \right\| = 0 \]
\[ \implies \delta_{A_{n}, B_{n}}^{m_{2}} (X) = 0. \]
Finally, since
\[ \lim_{n \to \infty} \Delta_{A_{n}, B_{n}}^{m_{1}} \left( \delta_{A_{n}, B_{n}}^{m_{2}} (X) \right) = \lim_{n \to \infty} \Delta_{A_{n}, B_{n}}^{m_{1}} \left( \lim_{n \to \infty} \delta_{A_{n}, B_{n}}^{m_{2}} (X) \right) \]
\[ = \lim_{n \to \infty} \delta_{A_{n}, B_{n}}^{m_{2}} \left( \lim_{n \to \infty} \Delta_{A_{n}, B_{n}}^{m_{1}} (X) \right), \]
the proof is complete. \(\Box\)

Proposition 2.6 is a generalisation of a number of extant results, amongst them [6, Theorem 2].

**Remark 2.7.** Let \( U = [U_{ij}]_{1 \leq i, j \leq d} \) be a unitary operator in \( B(\mathcal{H})^{d} \). Given a \( d \)-tuple \( T = (T_{1}, \ldots, T_{d}) \in B(\mathcal{H})^{d} \), define the \( d \)-tuple \( S = (S_{1}, \ldots, S_{d}) \) by \( S_{j} = \sum_{i=1}^{d} U_{ij} T_{i} \); \( 1 \leq j \leq d \). Then \( \sum_{i=1}^{d} U_{ij} U_{ik} = 1 \) if \( 1 \leq j = k \leq d \) and 0 otherwise. [19, Proposition 2.2] claims that if \( (T^{*}, T) \) is \( (A, m) \)-isometric for some positive operator \( A \in B(\mathcal{H}) \), then \( (S^{*}, S) \) is \( (A, m) \)-isometric. This is false, even for single operators, as the following example shows.

**Example 2.8.** Consider operators \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( U = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix} \). Then \( A \geq 0 \), \( U \) is unitary and \( \Delta_{T^{*}, T}^{2} (A) = 0 \). However,
\[ S^{*} AS = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad S^{*} AS^{2} = \begin{pmatrix} 1 & 1 - i \\ 1 + i & 2 \end{pmatrix} \]
and \( \Delta_{S^{*}, S}^{2} (A) \neq 0 \).
We observe here that if $\Delta_{m,T,T}^{m}(I) = 0$, and the $d$-tuple $S$ and the unitary $U$ are as in the remark above, then $\Delta_{S^*S}^{m}(I) = 0$, as the following argument shows. We have:

$$
\begin{align*}
\Delta_{S^*S}^{m}(I) &= \sum_{j=0}^{m} (-1)^{j} \left( \sum_{i=1}^{d} S^*_i S_i \right)^j \\
&= \sum_{j=0}^{m} (-1)^{j} \left( \sum_{i=1}^{d} \left( \sum_{1 \leq s,t \leq d} T^*_s U^*_is U_it T_t \right) \right)^j \\
&= \sum_{j=0}^{m} (-1)^{j} \left( \sum_{i=1}^{d} \left( \sum_{1 \leq s \leq d} T^*_s U^*_is U_it \right) \right)^j \\
&= \sum_{j=0}^{m} (-1)^{j} \left( \sum_{i=1}^{d} T^*_i T_i \right)^j \\
&= \Delta_{T,T,T}^{m}(I),
\end{align*}
$$

since

$$
\sum_{i=1}^{d} \left( \sum_{1 \leq s,t \leq d} T^*_s U^*_is U_it T_t \right) = \sum_{1 \leq s,t \leq d} T^*_s \left( \sum_{i=1}^{d} U^*_is U_it \right) T_t = \sum_{1 \leq s \leq d} T^*_s \left( \sum_{i=1}^{d} U^*_is U_is \right) T_s = \sum_{1 \leq s \leq d} T^*_s T_s.
$$

$(A_i, B_i)$ is $(X, m)$-isometric, even $(X, 1)$-isometric, for all $1 \leq i \leq d$ does not imply $(A, B)$ is $(X, m)$-isometric. Consider $A_i = B_i = I$ for all $1 \leq i \leq d$, when it is seen that $(A_i, B_i)$ is $(X, 1)$-isometric for all $X \in B(X)$ and $\Delta_{A,B}(X) = (d - 1)X \neq 0$. The following proposition goes some way towards explaining this phenomenon.

**Proposition 2.9.** (a) If $(A_i, B_i)$ is $(X, 1)$-isometric for all $1 \leq i \leq d - 1$, then $(A, B)$ is $(X, m)$-isometric if and only if $((d - 2)I + L_{A_i} R_{B_i})^m(X) = 0$.
(b) If $(A_i, B_i)$ is $(X, 1)$-symmetric for all $1 \leq i \leq d - 1$, then $(A, B)$ is $(X, m)$-symmetric if and only if $(A_i, B_i)$ is $(X, m)$-symmetric.

**Proof.** (a) If $\Delta_{A_i,B_i}(X) = 0$, then

$$
\begin{align*}
\Delta_{A,B}^{m}(X) &= (I - L_{A}* R_{B})^m(X) = \left( I - \sum_{i=1}^{d} L_{A_i} R_{B_i} \right)^m(X) \\
&= \left[ (I - L_{A_i} R_{B_i}) - \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right) \right]^m(X)
\end{align*}
$$
\[= \sum_{j=0}^{m} (-1)^j \binom{m-j}{j} \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^j (X)\]
\[= \sum_{j=0}^{m} (-1)^j \binom{m-j}{j} \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^j \left( \Delta_{A_1, B_1}^{m-j} (X) \right)\]
\[= 0\]

for all \( m - j \neq 0 \), and if \( j = m \), then \( \Delta_{A, B}^{m} (X) = 0 \) if and only if
\[\Delta_{A, B}^{m} (X) = \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0.\]

Assume next that (also) \( \Delta_{A_2, B_2}^{m} (X) = 0 \). Then \( \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0 \) if and only if
\[(-1)^m \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^m (X) = \left( I + \sum_{i=3}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0.\]

Thus \( \left( \sum_{i=2}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0 \) if and only if \( \left( I + \sum_{i=3}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0.\)

Repeating the argument, we have eventually that \( \left( (d-3)I + \sum_{i=d-1}^{d} L_{A_i} R_{B_i} \right)^m (X) = 0 \) if and only if \( \left( (d-2)I + L_{A_d} R_{B_d} \right)^m (X) = 0.\)

Conclusion: \( \Delta_{A, B}^{m} (X) = 0 \iff \left( (d-2)I + L_{A_d} R_{B_d} \right)^m (X) = 0.\)

(b) If \( \delta_{A_1, B_1}^{m} (X) = 0 \), then
\[\delta_{A, B}^{m} (X) = (L_{A} - R_{B})^{m} (X) = \left( \sum_{i=1}^{d} (L_{A_i} - R_{B_i}) \right)^m (X)\]
\[= \left[ \sum_{i=2}^{d} \delta_{A_i, B_i} + \delta_{A_1, B_1} \right]^m (X)\]
\[= \sum_{j=0}^{m} \binom{m}{j} \left( \sum_{i=2}^{d} \delta_{A_i, B_i} \right)^j \left( \delta_{A_1, B_1}^{m-j} (X) \right)\]
\[= 0\]

for all \( m - j \neq 0 \), and if \( j = m \), then \( \delta_{A, B}^{m} (X) = 0 \) if and only if \( \left( \sum_{i=2}^{d} \delta_{A_i, B_i} \right)^m (X) = 0.\)

Repeating the argument, we have eventually that \( \delta_{A, B}^{m} (X) = 0 \) if and only if \( \delta_{A_d, B_d}^{m} (X) = 0.\)
Proposition 2.9 subsumes [6, Proposition 3], and proves that an analogous result holds for \((X, M) \to (X, m)\)-symmetric tuples.

### 3. Perturbation by Commuting Nilpotents

The single operator techniques extend to proving results on perturbation by commuting nilpotents of commuting tuples of operators satisfying an isometric or symmetric property. A commuting \(d\)-tuple \(N = (N_1, \cdots , N_d) \in B(X)^d\) is an \(n\)-nilpotent for some positive integer \(n\) if

\[
N^\alpha = \Pi_{i=1}^d N_i^{\alpha_i} = 0
\]

for all \(d\)-tuples \(\alpha = (\alpha_1, \cdots , \alpha_d)\) of non-negative integers \(\alpha_i\) such that \(|\alpha| = \sum_{i=1}^d \alpha_i = n\) and \(N^\alpha \neq 0\) for at least one \(\alpha\) with \(|\alpha| \leq n - 1\). As usual, given \(d\)-tuples \(\mathbb{A} = (A_1, \cdots , A_d)\) and \(\mathbb{N} = (N_1, \cdots , N_d)\), we define

\[
\mathbb{A} + \mathbb{N} = (A_1 + N_1, \cdots , A_d + N_d).
\]

Recall that \([\mathbb{A}, \mathbb{N}] = 0\) if and only if \([A_i, N_j] = 0\) for all \(1 \leq i, j \leq d\).

**Theorem 3.1.** Given commuting \(d\)-tuples \(\mathbb{A} = (A_1, \cdots , A_d)\) and \(\mathbb{B} = (B_1, \cdots , B_d)\) in \(B(X)^d\) such that \(\Delta^{m_1}_{A,B} (\delta^{m_2}_{A,B}(X)) = 0\) for some positive integers \(m_1\) and \(m_2\), let

\[
\mathbb{N}_i = (N_{i1}, \cdots , N_{id}), \ i = 1, 2,
\]

be two commuting \(n_i\)-nilpotent \(d\)-tuples such that

\[
[A, \mathbb{N}_1] = [\mathbb{B}, \mathbb{N}_2] = 0.
\]

Then

\[
\Delta^{t_1}_{A+N_1,B+N_2} ( \delta^{t_2}_{A+N_1,B+N_2}(X) ) = 0; \ t_i = m_i + n_1 + n_2 - 2, \ i = 1, 2.
\]

**Proof.** The commutativity hypotheses on \(\mathbb{A}, \mathbb{B}, \mathbb{N}_1\) and \(\mathbb{N}_2\), taken alongside the commutativity of the left and the right multiplication operators, imply

\[
\Delta^{s_1}_{A+N_1,B+N_2} ( \delta^{s_2}_{A+N_1,B+N_2}(X) ) = \delta^{s_2}_{A+N_1,B+N_2} ( \Delta^{s_1}_{A+N_1,B+N_2}(X) )
\]

for all positive integers \(s_1\) and \(s_2\). We prove the theorem in two steps. In the first step we let \(\delta^{m_2}_{A,B}(X) = Y\); then \(\Delta^{m_1}_{A,B}(Y) = 0\), and we prove that \(\Delta^{t_1}_{A+N_1,B+N_2}(Y) = 0\). In the second step, we let \(\Delta^{t_1}_{A+N_1,B+N_2}(Y) = Z\). Then \(\Delta^{t_1}_{A+N_1,B+N_2}(Y) = \delta^{m_2}_{A,B}(Z) = 0\), and we prove that \(\delta^{t_2}_{A+N_1,B+N_2}(Z) = 0\).

Considering \(\Delta^{t_1}_{A+N_1,B+N_2}(Y)\), we have

\[
\Delta^{t_1}_{A+N_1,B+N_2} = (I - L_{A+N_1} * R_{B+N_2})^{t_1} = [(I - L_A * R_B) - ((L_{N_1} * R_{B+N_2}) + (L_A * R_{N_2}))]^{t_1}
\]
Theorem 3.1 translates to:
\[ \Delta_{A,B} \] and since \((\mathbb{L}_{N_1} \ast \mathbb{R}_{B+N_2})^k = 0\) for all \(k \geq n_1\) and \((\mathbb{L}_A \ast \mathbb{R}_{N_2})^{j-k} = 0\) for all \(j - k \geq n_2\), or \(k \leq j - n_2\). Hence

\[ \Delta_{A+N_1,B+N_2}^t(Y) = 0 \]

for all \(n_1 \leq k \leq j - n_2\). This leaves us with the case \(n_1 - 1 \geq k \geq j - n_2 + 1\). But then \(j \leq n_1 + n_2 - 2\) implies \(t_1 - j \geq m_1 + n_1 + n_2 - 2 - (n_1 + n_2 - 2) = m_1\), and this, since \(\Delta_{A,B}^t(X) = 0\), forces \(\Delta_{A,B}^{t_1-j}(Y) = 0\). Conclusion: \(\Delta_{A+N_1,B+N_2}^t(Y) = 0\).

Consider now \(\delta_{A+N_1,B+N_2}^{t_2}(Z)\). Since

\[ \delta_{A+N_1,B+N_2}^{t_2} = (\mathbb{L}_{A+N_1} - \mathbb{R}_{B+N_2})^{t_2} = ((\mathbb{L}_A - \mathbb{R}_B) + (\mathbb{L}_{N_1} - \mathbb{R}_{N_2}))^{t_2} = \sum_{j=0}^{t_2} \binom{t_2}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^k \mathbb{L}_{j-k}^k \mathbb{R}_{N_2}^{j-k} \delta_{A,B}^{t_2-j}, \]

and since \(\mathbb{L}_{N_1}^{j-k} \mathbb{R}_{N_2}^k = 0\) for all \(j - k \geq n_1\) and \(k \geq n_2\), \(\delta_{A+N_1,B+N_2}^{t_2}(Z) = 0\) for all \(n_2 \leq k \leq j - n_1\). If \(j - n_1 + 1 \leq k \leq n_2 - 1\) (implies \(j \leq n_1 + n_2 - 2\)), then \(t_2 - j = m_2 + n_1 + n_2 - 2 - j \geq m_2\) and \(\delta_{A,B}^{t_2-j}(Z) = 0\). Hence \(\delta_{A+N_1,B+N_2}^{t_1}(Z) = 0\), and the proof is complete.

Theorem 3.1 subsumes a number of extant results, amongst them [19, Theorem 3.1] and [6, Theorem 3]. The \(d\)-tuples \(A\) and \(B\) in the theorem, in the presence of suitable commutativity hypotheses, may be replaced by \(d\)-tuples \(A_i\) and \(B_i\); \(i = 1, 2\). The argument of the proof of the theorem implies the following corollary.

**Corollary 3.2.** Given commuting \(d\)-tuples \(A_i\), \(B_i\) and \(N_i\) in \(B(\mathcal{X})^d\), \(i = 1, 2\), such that

\[ [A_1, N_1] = [B_1, N_2] = [A_1, A_2] = [B_1, B_2] = 0, \]

if \(\Delta_{A_1,B_1}^{m_1}(X) = 0\), then \(\Delta_{A_1+N_1,B_1+N_2}^{m_1+m_1+n_2-2}(X) = 0\),

if \(\delta_{A_2,B_2}^{m_2}(X) = 0\), then \(\delta_{A_2+N_1,B_2+N_2}^{m_2+m_1+n_2-2}(X) = 0\)

and

if \(\Delta_{A_1,B_1}^{m_1} \left(\delta_{A_2,B_2}^{m_2}(X)\right) = 0\), then \(\Delta_{A_1+N_1,B_1+N_2}^{m_1+m_1+n_2-2} \left(\delta_{A_2+N_1,B_2+N_2}^{m_2+m_1+n_2-2}(X)\right) = 0\).
Corollary 3.3. \( \Delta_{T^*}^{m_1} \left( \delta_{T^*}^{m_2}(X) \right) = 0 \) implies \( \Delta_{T^*}^{m_1+2n-2} \left( \delta_{T^*}^{m_2+2n-2}(X) \right) = 0 \).

4. Isosymmetric Products

If \( A = (A_1, \ldots, A_{d_1}) \) and \( S = (S_1, \ldots, S_{d_2}) \) are two commuting \( d_i \) tuples in \( B(\mathcal{A})^{d_1}, i = 1, 2 \), then the product \( SA \) is the operator

\[
SA = (S_1A_1, \ldots, S_1A_{d_1}, S_2A_2, \ldots, S_2A_{d_1}, \ldots, S_{d_2}A_1, \ldots, S_{d_2}A_{d_1}).
\]

The tuples \( A \) and \( S \) commute, \( [A, S] = 0 \), if

\[
[A_i, S_j] = 0 \quad \text{for all } 1 \leq i \leq d_1 \text{ and } 1 \leq j \leq d_2.
\]

If \( A, B \in B(\mathcal{A})^{d_1} \) are commuting \( d_1 \)-tuples and \( S, T \in B(\mathcal{A})^{d_2} \) are commuting \( d_2 \)-tuples such that

\[
[A, S] = [B, T] = 0,
\]

then \( (L_{SA} = L_{AS}, R_{TB} = R_{BT} \text{ and}) \)

\[
L_{SA} * R_{TB}(X) = \sum_{j=1}^{d_2} S_j \left[ \sum_{i=1}^{d_1} A_i X B_i \right] T_j
\]

\[
= \sum_{j=1}^{d_2} S_j \left( (L_A * R_B)(X) \right) T_j
\]

\[
= (L_S * R_T) \left( (L_A * R_B)(X) \right),
\]

\[
\Delta_{SA, TB}(X) = (I - L_{SA} * R_{TB})(X)
\]

\[
= (I - L_S L_A * R_B R_T)(X)
\]

\[
= ((L_S * R_T)(I - L_A * R_B) + (I - L_S * R_T))(X)
\]

\[
= ((L_S * R_T) \Delta_{A,B} + \Delta_{S,T})(X)
\]

and

\[
((L_S * R_T) \Delta_{A,B})^n(X) = (L_S * R_T)^n(\Delta_{A,B}^n(X)).
\]

Again, if \( A, B, S, T \) are the tuples above, then

\[
\delta_{SA, TB}(X) = (L_{SA} - R_{TB})(X)
\]

\[
= \left[ \sum_{j=1}^{d_2} L_S j \left( \sum_{i=1}^{d_1} L_{A_i} \right) - \sum_{j=1}^{d_2} R_{T_j} \left( \sum_{i=1}^{d_1} R_{B_i} \right) \right](X)
\]

\[
= \left[ L_S \times L_A - R_T \times R_B \right](X)
\]

\[
= \left[ L_S \times (L_A - R_B) + R_B \times (L_S - R_T) \right](X)
\]

\[
= \left[ L_S \times \delta_{A,B} + R_B \times \delta_{S,T} \right](X),
\]
and

\[
(\mathbb{L}_S \times \delta_{A,B})^n(X) = \left[\sum_{j=1}^{d_2} L_{S_j} \left( \sum_{i=1}^{d_1} L_{A_i} - R_{B_i} \right) \right]^{n} (X)
\]

\[
= \left[\left( \sum_{j=1}^{d_2} L_{S_j} \right)^n \left( \sum_{i=1}^{d_1} (L_{A_i} - R_{B_i}) \right)^n \right] (X)
\]

\[
= (\mathbb{L}_S^n \times \delta_{A,B}^n)(X) = \mathbb{L}_S^n \times \delta_{A,B}^n(X)
\]

and similarly

\[
(\mathbb{R}_B \times \delta_{S,T})^n(X) = \mathbb{R}_B^n \times \delta_{S,T}^n(X).
\]

It is well known, see for example [9,16,18,21,28], that if \([A_1, A_2] = [B_1, B_2] = 0\) and \(\Delta_{A_1,B_1}^{m_1}(X) = 0\) (similarly, \(\delta_{A_1,B_1}(X) = 0\)) for \(i = 1, 2\), then \(\Delta_{A_1,A_2,B_1,B_2}^{m_1+m_2-1}(X) = 0\) (resp., \(\delta_{A_1,A_2,B_1,B_2}(X) = 0\)). Using an argument similar in spirit to the one used to prove Theorem 3.1 (see also [16]), we prove in the following an analogous result for products (\(\mathbb{S}_A,\mathbb{T}_B\)) of commuting \(d\)-tuples \(A, B, S\) and \(T\). We remark that the order \(d_i, i = 1, 2\) of the \(d\)-tuples plays no role in the workings of our argument: there is no loss of generality in assuming \(d_1 = d_2 = d\).

**Theorem 4.1.** Let \(A, B, S\) and \(T\) be commuting \(d\)-tuples in \(B(X)^d\) such that

\([A, S] = [B, T] = 0\).

If

\[\Delta_{A,B}^{m}(\delta_{A,B}^{n}(X)) = 0 = \Delta_{S,T}^{r}(\delta_{S,T}^{s}(X))\]

and

\[\Delta_{S,T}^{r}(\delta_{S,T}^{n}(X)) = 0 = \Delta_{A,B}^{m}(\delta_{A,B}^{s}(X))\]

for some positive integers \(m, n, r\) and \(s\), then

\[\Delta_{S,A,T,B}^{t_1}(\delta_{S,A,T,B}^{t_2}(X)) = 0,\]

where \(t_1 = m + r - 1\) and \(t_2 = n + s - 1\).

**Proof.** The commutativity hypothesis \([A, S] = [B, T] = 0\), taken alongwith the commutativity of the left and the right multiplication operators implies

\[
\Delta_{A,B}^{n_1}(\delta_{S,T}^{n_2}(X)) = \Delta_{A,B}^{n_1-m}(\delta_{S,T}^{n_2-n}(\delta_{S,T}^{n}(X)))
\]

\[
= \Delta_{A,B}^{n_1-m}(\delta_{S,T}^{n_2-n}(\Delta_{A,B}^{m}(\delta_{S,T}^{n}(X))))
\]

for all integers \(n_1 \geq m\) and \(n_2 \geq n\). Hence

\[
\Delta_{A,B}^{m}(\delta_{S,T}^{n}(X)) = 0 \Rightarrow \Delta_{A,B}^{n_1}(\delta_{S,T}^{n_2}(X)) = 0
\]
for all integers \( n_1 \geq m \) and \( n_2 \geq n \). Similarly,
\[
\Delta_{S,T}^r (\delta_{A,B}^n (X)) = 0 \implies \Delta_{S,T}^{n_1} (\delta_{A,B}^{n_2} (X)) = 0
\]
for all integers \( n_1 \geq r \) and \( n_2 \geq n \).

The proof below is divided into two parts. In the first part we prove
\[
\Delta_{SA,TB}^{t_1} (\delta_{A,B}^n (X)) = 0, \quad t_1 = m + r - 1,
\]
and in the second part we prove
\[
\delta_{SA,TB}^{t_2} \left( \Delta_{SA,TB}^{t_1} (X) \right) = 0, \quad t_2 = n + s - 1.
\]
Set \( \delta_{A,B}^n (X) = Y \). Then \( \Delta_{SA}^m (Y) = 0 \) and
\[
\Delta_{SA,TB}^{t_1} (Y) = (\Delta_{S,T} + (S \ast T) \Delta_{A,B})^{t_1} (Y)
\]
\[
= \left( \sum_{j=0}^{t_1} \binom{t_1}{j} (S \ast T)^{t_1-j} \Delta_{A,B}^{t_1-j} \Delta_{S,T}^j \right) (Y).
\]
The commutativity hypotheses ensure
\[
\left[ (S \ast T)^{t_1-j} \Delta_{A,B}^{t_1-j}, \Delta_{S,T}^j \right] = 0;
\]

hence
\[
(S \ast T)^{t_1-j} \Delta_{A,B}^{t_1-j} \Delta_{S,T}^j (Y) = (S \ast T)^{t_1-j} \left( \Delta_{A,B}^{t_1-j} \Delta_{S,T}^j (Y) \right)
\]
\[
= (S \ast T)^{t_1-j} \left( \Delta_{S,T}^{t_1-j} (Y) \right).
\]
Since
\[
\Delta_{S,T}^j (Y) = \Delta_{S,T}^j (\delta_{A,B}^n (X)) = 0
\]
for all \( j \geq r \) and
\[
\Delta_{A,B}^{t_1-j} (Y) = \Delta_{A,B}^{t_1-j} (\delta_{A,B}^n (X)) = 0
\]
for all \( t_1 - j \geq m \), equivalently \( j \leq t_1 - m = r - 1 \), we have
\[
\Delta_{SA,TB}^{t_2} (\delta_{A,B}^n (X)) = 0.
\]
Now set \( \delta_{S,T}^s (X) = M \). Then \( \Delta_{SA}^m (M) = 0 \). Arguing as above, it is seen that
\[
\Delta_{SA,TB}^{t_1} (M) = \left[ \sum_{j=0}^{t_1} \binom{t_1}{j} (S \ast T)^{t_1-j} \Delta_{A,B}^{t_1-j} \Delta_{S,T}^j \right] (M).
\]
Since \( \Delta_{S,T}^j (M) = 0 \) for all \( j \geq r \) and \( \Delta_{A,B}^{t_1-j} (M) = 0 \) for all \( t_1 - j \geq m \), equivalently \( j \leq r - 1 \), we have
\[
\Delta_{SA,TB}^{t_1} (\delta_{S,T}^s (X)) = 0.
\]
To conclude the proof, set \( \Delta_{SA,TB}^{t_1} (X) = Z \). Then \( \Delta_{SA,TB}^{t_2} (\delta_{SA,TB}^{t_2} (X)) = \delta_{SA,TB}^{t_2} (Z) \) and
\[ \delta_{SA,TB}^{t_2}(Z) = (L_{SA} - R_{TB})^{t_2}(Z) \]
\[ = (L_S \times \delta_{A,B} + R_T \times \delta_{S,T})^{t_2}(Z) \]
\[ = \sum_{j=0}^{t_2} \left( \begin{array}{c} t_2 \\ j \end{array} \right) \left( L_S^{t_2-j} \times \delta_{A,B}^{t_2-j} \right) \left( R_T^j \times \delta_{S,T}^j \right) (Z). \]

Evidently,
\[ [L_S \times \delta_{A,B}, R_S \times \delta_{S,T}] = 0. \]

Since
\[ R_S \times \delta_{S,T}^j(Z) = R_S \times \left( \delta_{S,T}^j \left( \Delta_{SA,TB}^{t_1}(X) \right) \right) = 0 \]
for all \( j \geq s \) and
\[ L_S^{t_2-j} \times \delta_{A,B}^{t_2-j}(Z) = L_S^{t_2-j} \times \left( \delta_{A,B}^{t_2-j} \left( \Delta_{SA,TB}^{t_1}(X) \right) \right) = 0 \]
for all \( t_2 - j \geq n \), equivalently for all \( j \leq t_2 - n = s - 1 \), we have
\[ \delta_{SA,TB}^{t_2}(Z) = \delta_{SA,TB}^{t_2} \left( \Delta_{SA,TB}^{t_1}(X) \right) \]
\[ = \Delta_{SA,TB}^{t_1} \left( \delta_{SA,TB}^{t_2}(X) \right) \]
\[ = 0. \]

\[ \square \]

**Corollary 4.2.** If \( A, B, S \) and \( T \) are commuting \( d \)-tuples in \( B(\mathcal{X})^d \) such that \([A, S] = [B, T] = 0\), then
\begin{align*}
(i) & \quad \Delta_{A,B}^m(X) = \Delta_{S,T}^n(X) = 0 \implies \Delta_{A,S,B,T}^{m+n-1}(X) = 0; \\
(ii) & \quad \delta_{A,B}^m(X) = \delta_{S,T}^n(X) = 0 \implies \delta_{A,S,B,T}^{m+n-1}(X) = 0.
\end{align*}

Let \( C \) be a conjugation of \( \mathcal{H} \). (Thus \( C : \mathcal{H} \rightarrow \mathcal{H} \) is a conjugate linear operator such that \( C^2 = I \) and \( \langle Cx, y \rangle = \langle Cy, x \rangle \) for all \( x, y \in \mathcal{H} \).) The first part of the following corollary has been proved in [6, Theorem 4], but with the additional hypothesis that \([S^*, CTC] = 0\).

**Corollary 4.3.** Let \( S \) and \( T \) be commuting \( d \)-tuples in \( B(\mathcal{X})^d \) such that \([S, T] = 0\). Then
\begin{align*}
(i) & \quad \Delta_{S*,CSC}^m(X) = \Delta_{T*,CTC}^n(X) = 0 \implies \Delta_{S*,T*,CSTC}^{m+n-1}(X) = 0; \\
(ii) & \quad \delta_{S*,CSC}^m(X) = \delta_{T*,CTC}^n(X) = 0 \implies \delta_{S*,T*,CSTC}^{m+n-1}(X) = 0.
\end{align*}

If \( A = A \in B(\mathcal{X}) \) and \( B = B \in B(\mathcal{X}) \) are single operators, then the products \( AS \) and \( BT \) are the \( d \)-tuples
\[ AS = (AS_1, \cdots, AS_d) \text{ and } BT = (BT_1, \cdots, BT_d). \]
If also \([A, S] = [B, T] = 0\), then:
Corollary 4.4. (i) $\Delta_{A,B}^m (X) = \Delta_{S,T}^n (X) = 0 \implies \Delta_{A \otimes S, B \otimes T}^{m+n-1} (X) = 0$;
(ii) $\delta_{A,B}^m (X) = \delta_{S,T}^n (X) = 0 \implies \delta_{A \otimes S, B \otimes T}^{m+n-1} (X) = 0$.

Part (i) of the corollary is a generalisation of [6, Theorem 6] and part (ii) of the corollary, in so far as the authors can ascertain, is new.

Tensor products $\Delta_{A \otimes S, B \otimes T}^k (I)$ and $\delta_{A \otimes S, B \otimes T}^k (I)$

Let $\mathcal{X} \otimes \mathcal{X}$ denote the completion, endowed with a reasonable cross norm, of the algebraic tensor product of $\mathcal{X}$ with itself. Let $S \otimes T$ denote the tensor product of $S \in B(\mathcal{X})$ and $T \in B(\mathcal{X})$. The tensor product of the $d$-tuples $A = (A_1, \cdots, A_d)$ and $B = (B_1, \cdots, B_d)$ is the $d^2$-tuple $A \otimes B = (A_1 \otimes B_1, \cdots, A_1 \otimes B_d, A_2 \otimes B_1, \cdots, A_2 \otimes B_d, \cdots, A_d \otimes B_1, \cdots, A_d \otimes B_d)$.

Let $A$ and $B$ be commuting $d$-tuples such that $\Delta_{A,B}^m (I) = 0$ (i.e., the pair $(A, B)$ is $m$-isometric). Then

$\Delta_{A,B}^m (I) = 0 \iff \Delta_{A,B}^m (I) \otimes I = 0$

$\iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{i=1}^{d} A_i B_i \right)^j \otimes I$

$\iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{i=1}^{d} (A_i \otimes I)(B_i \otimes I) \right)^j = 0$

$\iff \Delta_{A \otimes I, B \otimes I}^m (\mathbb{I}) = 0$, $\mathbb{I} = I \otimes I$.

(Here, $\left( \sum_{i=1}^{d} A_i B_i \right)^j$ in the second two way implication is to be interpreted as meaning $\left( \sum_{i=1}^{d} A_i B_i \right)^j = \sum_{i=1}^{d} A_i \left( \sum_{i=1}^{d} A_i B_i \right)^{j-1} B_i$.) Similarly,

$\Delta_{A,B}^m (I) = 0 \iff \Delta_{I \otimes A, I \otimes B}^m (\mathbb{I}) = 0$.

Considering $m$-symmetric pairs $(A, B)$, we have

$\delta_{A,B}^m (I) = 0 \iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left[ \left( \sum_{i=1}^{d} A_i \right)^{m-j} \left( \sum_{i=1}^{d} B_i \right)^j \right] \otimes I = 0$

$\iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left[ \left( \sum_{i=1}^{d} A_i \otimes I \right)^{m-j} \left( \sum_{i=1}^{d} B_i \otimes I \right)^j \right] = 0$

$\iff \delta_{A \otimes I, B \otimes I}^m (\mathbb{I}) = 0$

$\iff \delta_{I \otimes A, I \otimes B}^m (\mathbb{I}) = 0$.

The extension of Theorem 4.1 to tensor products is now almost automatic.

Theorem 4.5. Given tuples $A, B, S$ and $T$ in $B(\mathcal{X})^d$, if
(i) $\Delta_{A,B}^m(I) = \Delta_{S,T}^n(I) = 0$ (resp., $\delta_{A,B}^m(I) = \delta_{S,T}^n(I) = 0$), then $\Delta_{A \otimes S, B \otimes T}^{m+n-1}(I) = 0$ (resp., $\delta_{A \otimes S, B \otimes T}^{m+n-1}(I) = 0$); 
(ii) $\Delta_{A,B}^m(\delta_{A,B}^n(I)) = \Delta_{S,T}^n(I) = \delta_{S,T}^n(I) = 0$ for some positive integers $m, n, r$ and $s$, then $\Delta_{A \otimes S, B \otimes T}^{m+r-1}(\delta_{A \otimes S, B \otimes T}^{n+s-1}(I)) = 0$.

Proof. Define operators $A, B, S$ and $T$ by

\[ A = A \otimes I, \quad B = B \otimes I, \quad S = I \otimes S, \quad T = I \otimes T. \]

Then $[A, S] = [B, S] = [B, T] = 0$, $\Delta_{A,B}^m(I) = \Delta_{S,T}^n(I) = 0 \iff \Delta_{A,B}^m(I) = \Delta_{S,T}^n(I) = 0$

and

\[ \delta_{A,B}^n(I) = \delta_{S,T}^n(I) = 0 \iff \delta_{A,B}^n(I) = \delta_{S,T}^n(I) = 0. \]

Applying Corollary 4.4,

\[ \triangle_{A \otimes B, S \otimes T}^{m+n-1}(I) = \triangle_{A \otimes S, B \otimes T}^{m+n-1}(I) = 0 \]

and

\[ \delta_{A \otimes B, S \otimes T}^{m+n-1}(I) = \delta_{A \otimes S, B \otimes T}^{m+n-1}(I) = 0. \]

This proves (i).

To prove (ii), we start by observing that

\[
\Delta_{A,B}^m(\delta_{A,B}^n(I)) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \left( \sum_{i=1}^{d} L_{A_i} R_{B_i} \right)^j \left( \sum_{i=1}^{d} L_{A_i} \right)^{n-k} \left( \sum_{i=1}^{d} R_{B_i} \right)^k \right)(I)
\]

\[
= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left( \sum_{i=1}^{d} (-1)^k \binom{n}{k} P(A, B) \right)(I)
\]

where $P(A, B)(I)$ is a polynomial with entries which are constant multiples of terms of type $A_1^{\alpha_1} \cdots A_d^{\alpha_d} B_1^{\beta_1} \cdots B_d^{\beta_d}$ for some non-negative integers $\alpha_i, \beta_i$ $(1 \leq i \leq d)$. Hence

\[
\Delta_{A,B}^m(\delta_{A,B}^n(I)) = 0 \iff \Delta_{A,B}^m(\delta_{A,B}^n(I)) \otimes I = 0
\]

\[
\iff \Delta_{A \otimes I, B \otimes I}^m(\delta_{A \otimes I, B \otimes I}^n(I)) = 0
\]

\[
\iff \Delta_{A,B}^m(\delta_{A,B}^n(I)) = 0.
\]

Again

\[
\delta_{S,T}^r(I) = 0 \iff \delta_{S,T}^r(I) = 0 \iff \delta_{A,B}^n(\Delta_{S,T}^r(I)) = 0,
\]

\[
\delta_{S,T}^r(I) = 0 \iff \delta_{S,T}^r(I) = 0 \iff \Delta_{A,B}^m(\delta_{S,T}^r(I)) = 0.
\]
and
\[ \Delta^r_{S,T} \left( \delta^s_{S,T}(I) \right) = 0. \]
Since \( A, B, S \) and \( T \) satisfy the hypotheses of Theorem 4.1, the proof of (ii) follows. \( \square \)

Declarations

Conflicts of interest The authors have not disclosed any competing interests.

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