Well-posedness and robust stability of a nonlinear ODE-PDE system

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Abstract

This work studies stability and robustness of a nonlinear system given as an interconnection of an ODE and a parabolic PDE subjected to external disturbances entering through the boundary conditions of the parabolic equation. To this end we develop an approach for a construction of a suitable coercive Lyapunov function as one of the main results. Based on this Lyapunov function we establish the well-posedness of the considered system and establish conditions that guarantee the ISS property. ISS estimates are derived explicitly for the particular case of globally Lipschitz nonlinearities.

Keywords: Infinite-dimensional systems, coupled ODE-PDE equations, stability and robustness

1 Introduction

Studying stability of infinite dimensional systems has a long history of several decades [8, 5, 11]. During the last decade a lot of attention was devoted to the investigation of robust stability of such systems, especially in the input-to-state stability (ISS) framework [15, 25, 23, 12]. This framework is known to be suitable in studying interconnected systems. Coupled systems appear in many modern practical problems. A special class of interconnections are couplings of a PDE and ODE systems motivated by different applications from mechanics [7], control [10, 26, 27], biology [9], etc. Since we are interested in stability

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properties, we recall several recent works where this property was studied for the case of ODE-PDE couplings.

The work [27] deals with the problem of axial and rotational vibrations in the drilling process, where the rock-bit interaction leads to a nonlinear coupling function between an ODE and the wave equation. A feedback stabilization controller was developed there to guarantee the ultimate boundedness of solutions so that the undesired vibrations are suppressed successfully. To this end the authors proposed a Lyapunov-Krasovskii functional and developed stability conditions in form of linear and bilinear matrix inequalities.

Theorems of the small-gain type were used in [1] to establish exponential stability of a hybrid system given as an interconnection of an autonomous ODE and a parabolic PDE. Based on the ISS-type estimates the authors establish exponential stability for the case when a discrete time controller is applied. Their stability conditions restrict the maximal discretization of the time interval. By means of a coordinate transformations from [13] it was demonstrated that these conditions can be used for the design of suitable observers with discrete time output, for linear ODE-PDE cascades.

A feedback controller for local stabilization of a reaction-diffusion system was proposed in [24]. The Lyapunov stability of the uncontrolled system was not assumed there. By means of the direct Lyapunov method the authors derive estimates for the domain of attraction for the closed loop system, in the form of linear and bilinear inequalities. These results can be applied to the cases of distributed as well as scalar boundary controllers.

Boundary stabilization of a cascade of an ODE with a heat equation was considered in [13]. By means of the backstepping techniques the authors have designed a suitable stabilizing controller and derived estimates for the domain of attraction. These estimates were obtained by the direct Lyapunov method and with help of the Halanay inequality. These results were then extended in [14] to the case of coupled linear ODE-PDE system with time varying delay.

Stabilization of a linear ODE-PDE system with boundary control was considered in [30]. Using the backstepping approach the system was transformed into a cascade form for which a stabilizing controller was designed. By the direct Lyapunov method exponential estimates for the norm of solutions of the original system were derived.

The authors of [2] consider a system of linear hyperbolic equations, where the state of one boundary point is controlled by the state measured at another boundary point. Since the measurements are assumed to be perturbed the problem is to design controllers robust with respect to the measurement disturbances. For locally essentially bounded disturbances the conditions to guarantee the ISS or ISpS property are derived. The well-posedness and stability analysis are based on the $C_0$-semigroup theory and Lyapunov methods.

Adaptive stabilization problem for a cascade of an ODE with a hyperbolic PDE subjected to unknown harmonic disturbances was considered in [29]. Stabilizing controllers were developed there. By means of the linear semigroup theory and La Salle invariance principle conditions of well-posedness and for the asymptotic stability for the closed loop system were established.
Some older related works are \cite{4, 16, 17}, where stability of coupled parabolic PDE with an ODE was considered. These works use vector Lyapunov functions combined with theory of monotone dynamical systems in Banach spaces \cite{20}. Matrix valued Lyapunov functions were used in \cite{21} for stability investigation of ODE-PDE systems. The ISS framework for such and other infinite dimensional systems was used in \cite{15}.

Let us note that in the most of the above literature it is required that decoupled systems are asymptotically stable. This excludes a class of interconnections where one of the subsystems can be unstable, but the overall system is stabilized by the other one. Our result aims to fill this gap.

In this work we consider a coupled ODE with a parabolic PDE so that the decoupled ODE subsystem is not necessarily ISS. The disturbances enter to the system at the boundary, which is more difficult to handle than the distributed ones. We will derive conditions guaranteeing the ISS property for the coupled system under the assumption that the linearized PDE is globally asymptotically stable.

In contrary to well-known approaches such as vector Lyapunov functions or small-gain theory a construction of a Lyapunov function for the whole system in our case cannot be derived from the Lyapunov functions of subsystems, due to the presence of an unstable subsystem. Also the admissibility approach as used in \cite{12}, cannot be applied directly to our case because of the presence of essentially nonlinear functions in the subsystems. Even if we exclude such nonlinearities in our system this approach is hardly possible to extend as even in this case we will have difficulties to derive an explicit expression for the linear semigroup generated by the linear part of the system. Expansion of solution in series of eigenfunctions (of the linear part) as used for example in \cite{15} cannot be applied in our case due to the essential nonlinearities.

A new method is needed that takes nonlinearities of the subsystems, possible instability of one of them and the presence of disturbances at the boundary of the parabolic PDE into account.

To solve this problem we provide an approach to construct a Lyapunov function, which leads to the resolving of a boundary value problem for a second order PDE. The Green function will be derived for the latter problem explicitly, so that the needed Lyapunov function is obtained also explicitly. Based on this Lyapunov function we derive conditions for the ISS property of the coupled system. For the case of globally Lipschitz nonlinearities we provide an ISS-type estimation of solutions.

This paper is organized as follows. The next section introduces notation and several known basic facts that will be used in the main part of the paper. The problem statement is given in Section 3 together with related definitions. Several steps needed for the construction of a suitable Lyapunov function are explained in Section 4. The well-posedness and the main result establishing the ISS property is given in Section 5. Concluding remarks are collected in Section 6. Technical proofs are provided in the Appendix.
2 Notation and known facts

By $C[0, l]$ we denote the space of continuous functions defined on $[0, l]$ with values in $\mathbb{R}$ normed by $\|f\|_{C[0, l]} = \max_{x \in [0, l]} |f(x)|$, and $C^k[0, l]$ for $k \in \mathbb{N}$ denotes the space of continuously differentiable up to the order $k$ functions with the norm $\|f\|_{C^k[0, l]} = \max_{p=0, \ldots, k} \max_{x \in [0, l]} |f^{(p)}(x)|$. Let $L^p[0, l]$, $1 \leq p < \infty$ be the space of Lebesgue measurable functions with finite norm given by $\|f\|_{L^p[0, l]} = \left(\int_0^l |f(z)|^p \, dz\right)^{1/p}$. The Hilbert space $H^k[0, l] \subset L^2[0, l]$ is a subset of $L^2[0, l]$ of functions $f$ with (generalized) derivatives $f^{(p)} \in L^2[0, l]$ and scalar product is defined by

$$
(f, g)_{H^k[0, l]} = \sum_{p=0}^{k} \int_0^l f^{(p)}(z)g^{(p)}(z) \, dz.
$$

Recall that $H^k[0, l]$ is a completion of $C^k[0, l]$ with respect to the norm $\|f\|_{H^k[0, l]} = \sqrt{(f, f)_{H^k[0, l]}}$. $C^\infty_0[0, l]$ denotes the set of infinitely differentiable functions $[0, l]$ vanishing in a vicinity of the points $x = 0$ and $x = l$. $C^\infty_0([0, T], C^\infty_0[0, l])$ denotes the set infinitely smooth mappings $f : [0, T] \to C^\infty_0[0, l]$ with zero value in a vicinity of the points $t = 0$ and $t = T$. The completion of $C^\infty_0[0, l]$ with respect to the norm $H^k[0, l]$ is denoted by $H^k_0[0, l]$. $H^{-k}[0, l]$ denotes the dual space to $H^k_0[0, l]$ with the standard norm. For $\alpha, \beta \in \mathbb{Z}_+$ by $H^{\alpha, \beta}([0, l] \times [0, T])$ we denote the space of such functions $f : [0, l] \times [0, T] \to \mathbb{R}$ such that $\partial^\alpha_x f \in L^2([0, l] \times [0, T])$ for $k = 0, 1, \ldots$, $\alpha$ and $\partial^\beta_t f \in L^2([0, l] \times [0, T])$ for $k = 0, 1, \ldots, \beta$, see, e.g., [22].

For a Banach space $X$ its dual space is denoted by $X^*$ and $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$ is the usual duality pairing.

Let $X$ be a Banach space, then $L^p([0, T], X)$, $1 \leq p < \infty$ denotes the space of strongly measurable mappings $[0, T] \to X$ such that the norm $\|f\|_{L^p([0, T], X)} = \left(\int_0^T \|f(s)\|^p_X \, ds\right)^{1/p}$ is finite. For $p = \infty$ this norm is defined by $\|f\|_{L^\infty([0, T], X)} = \text{ess sup}_{t \in [0, T]} \|f(t)\|_X$.

$L^\infty(\mathbb{R}_+)$ is the Banach space of measurable functions essentially bounded on $\mathbb{R}_+ := [0, \infty)$. For $x \in \mathbb{R}^n$ we use the norm $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$. $C^1(\mathbb{U}, \mathbb{R}^n)$, $\mathbb{U} \subset \mathbb{R}$ is the set of continuously differential mappings $U \to \mathbb{R}^n$.

For $f : [0, l] \to (L^p([0, l]))^n$, $1 \leq p < \infty$ the norm is defined by $\|f\|_{L^p([0, l])} = \left(\sum_{k=1}^n \|f_k\|_{L^p([0, l])}^p\right)^{1/p}$ and for $p = \infty$ it is $\|f\|_{L^\infty([0, l])} = \max_{k=1, \ldots, n} \|f_k\|_{L^\infty([0, l])}$. We will use the following sets of comparison functions:

$$
\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ : \text{continuous, strictly increasing and } \gamma(0) = 0\},
$$

$$
\mathcal{L} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ : \text{continuous, strictly decreasing and } \lim_{t \to \infty} \gamma(t) = 0\},
$$

$$
\mathcal{K} \times \mathcal{L} = \{\beta : \mathbb{R}_+^2 \times \mathbb{R}_+ \to \mathbb{R}_+ : \text{cont., } \beta(r, t), \beta(r, t) \in \mathcal{K}, \beta(r, s, t) \in \mathcal{L}\}. 
$$
\( \mathbb{R}^{n \times m} \) is the linear space of \( n \times m \)-matrices, and if \( m = n \), then \( \mathbb{R}^{n \times n} \) is a Banach algebra. By \( \mathbb{S}^n \) we denote the set of the symmetric matrices of the size \( n \). For \( P, Q \in \mathbb{S}^n \), we write \( P \succ Q \) if and only if \( P - Q \) is positive definite. For \( A \in \mathbb{S}^n \) its minimal and maximal eigenvalues are denoted by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) respectively. The norm on \( \mathbb{R}^{n \times n} \) is induced by the euclidean norm in \( \mathbb{R}^n \): \( \|A\| = \sup_{\|x\|=1} \|Ax\| = \lambda_{\max}(AT A) \).

We will use the following well-known inequalities: Young’s inequality
\[
xy \leq \frac{x^{p_1}}{p_1} + \frac{y^{p_2}}{p_2}, \quad x \geq 0, \quad y \geq 0, \quad p_1 \in (1, \infty), \quad \frac{1}{p_1} + \frac{1}{p_2} = 1,
\]
Hölder’s integral inequality for functions \( f \in L^{p_1}[0, l] \) and \( g \in L^{p_2}[0, l] \)
\[
\left| \int_0^l f(t)g(t) \, dt \right| \leq \left( \int_0^l |f(t)|^{p_1} \, dt \right)^{1/p_1} \left( \int_0^l |g(t)|^{p_2} \, dt \right)^{1/p_2},
\]
\( p_1 \in (1, \infty), \quad \frac{1}{p_1} + \frac{1}{p_2} = 1, \)
in the particular case \( p_1 = p_2 = 1/2 \) it is also called the Cauchy-Bunyakovsky inequality. We will also use the following elementary inequality
\[
-az^2 + bz \leq -\frac{a}{2}z^2 + \frac{b^2}{2a}, \quad a > 0.
\]

For the proof of the existence of solutions to our problem we will use the following

**Theorem 2.1** (Theorem 1.4 in [3]). Let \( \mathcal{X}_0, \mathcal{X}, \mathcal{X}_1 \) be Banach spaces with \( \mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_0 \), where \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \) are reflexive and the embedding \( \mathcal{X}_1 \subset \mathcal{X} \) is compact. Let \( \{v_n\}_{n=1}^\infty \subset L^{p_0}([0, T], \mathcal{X}_1) \), \( 1 \leq p_0 < \infty \) be a bounded sequence in \( L^{p_0}([0, T], \mathcal{X}_1) \). Let the sequence of the generalized derivatives be such that \( \partial_t v_n \in L^{p_1}([0, T], \mathcal{X}_0), \ p_1 > 1 \) and bounded in \( L^{p_1}([0, T], \mathcal{X}_0) \). Then there is a subsequence of \( \{v_n\}_{n=1}^\infty \) which converges in \( L^{p_0}([0, T], \mathcal{X}) \).

We will also use the integration by parts formula, see (33) on page 43 in [3], as follows. Let \( u, v \in L^p([0, T], L^p[0, l]), \ p \geq 2, \ \partial_t u, \partial_t v \in L^{p'}([0, T], L^{p'}[0, l]), \ 1/p + 1/p' = 1 \), \( L^p[0, l] \subset L^2[0, l] \subset L^p[0, l], \ u, v \in L^\infty([0, T], L^2[0, l]), \ 0 \leq \tau \leq T \), then
\[
\int_0^\tau \langle \partial_t u, v \rangle \, dt + \int_0^\tau \langle u, \partial_t v \rangle = \langle u(\tau), v(\tau) \rangle - \langle u(0), v(0) \rangle.
\]

### 3 Problem statement

Consider a coupled system of the following differential equations
\[
u_t(z, t) = a^2 u_{zz}(z, t) + f(u(z, t)) + B^T(z)x(t),
\]
\[
x(t) = Cx(t) + X(x(t)) + \int_0^t D(z)u(z, t) \, dz,
\]
with initial conditions
\[ x(0) = x_0 \in \mathbb{R}^n, \quad u(z, 0) = \varphi(z), \quad \varphi \in L^2[0, l], \quad z \in (0, l), \quad t \in (0, +\infty) \tag{7} \]
and boundary conditions
\[ u(0, t) = d_1(t), \quad u(l, t) = d_2(t), \tag{8} \]
where \( a > 0, \ C \in \mathbb{R}^{n\times n}, \ B, D \in L^2([0, l], \mathbb{R}^n). \) For the nonlinear functions \( f: \mathbb{R} \rightarrow \mathbb{R} \) and \( X: \mathbb{R}^n \rightarrow \mathbb{R}^n \) we assume:

1) \( f(s) \) can be written as \( f(s) = f_0(s) + f_1(s), \) \( f_0(0) = f_1(0) = 0, \) where \( f_0(\cdot) \) is globally Lipschitz with Lipschitz constant \( L > 0, \) that is
\[ |f_0(s_2) - f_0(s_1)| \leq L|s_2 - s_1|, \quad s_1, s_2 \in \mathbb{R} \tag{9} \]
and \( f_1 \in C^1(\mathbb{R}, \mathbb{R}). \)

2) \( \exists \sigma \in \mathbb{R}, \ \alpha > 0 \) and \( q \geq 3/2 \) such that
\[ sf(s) \leq \sigma s^2 - \alpha|s|^{2q} \quad \text{for all} \quad s \neq 0; \tag{10} \]

3) \( \exists \zeta > 0, \ c_0 > 0 \) such that for all \( s \in \mathbb{R} \) it holds that:
\[ |f_1(s)| \leq c_0(1 + |s|^{2q-2}), \quad |f_1(s)| \leq \zeta|s|^{2q-1} \tag{11} \]

4) \( X \in C^1(\mathbb{R}^n, \mathbb{R}^n) \) and there exists \( \delta_2 > 0 \) such that
\[ \|X(x)\| \leq \delta_2\|x\|^{2q-1}, \quad x \in \mathbb{R}^n. \tag{12} \]

The disturbances \( d_i(t), \ i = 1, 2 \) are assumed to be of class \( L^\infty(\mathbb{R}_+) \) with \( d_i(0) = 0 \) and such that \( \frac{d}{dt}d_i \in L^\infty(\mathbb{R}_+). \)

**Remark 3.1.** A very special case of the problem \((6)-(8)\) was considered in \([21]\), namely with \( f = 0, \ X = 0 \) and \( d_1 = d_2 = 0, \) which leads to a linear system without disturbances. The construction of a Lyapunov function from \([21]\) is not suitable in the case of \((6)-(8)\).

For the problem \((6)-(8)\) we are interested in weak solutions defined as follows

**Definition 3.2.** A pair of functions \( u \in L^2([0, T], H^1[0, l]) \cap L^2([0, T], L^2[0, l]) \cap C([0, T], L^2[0, l]) \) and \( x \in C([0, T], \mathbb{R}^n) \) satisfying
\[ -\int_0^T \langle u(\cdot, t), \phi_t(\cdot, t) \rangle \, dt = \int_0^T (-a^2(u_z(\cdot, t), \phi_z(\cdot, t)) \]
\[ + \langle f(u(\cdot, t)) + B^T(\cdot)x(t), \phi(\cdot, t) \rangle \, dt \tag{13} \]
\[ \dot{x}(t) = Cx(t) + X(x) + \int_0^l D(z)u(z, t) \, dz, \]
for any \( \phi \in C_0^\infty([0, T], C_0^\infty[0, l]) \) and conditions \((7)-(8)\) is called weak solution of \((6)-(8)\).
Remark 3.3. In this definition we mean that $u$ satisfies (7) in the sense that 
\[ \lim_{t \to 0^+} \| u(z, t) - \varphi(z) \|_{L^2[0, l]} = 0. \]
As well we mean that (8) holds for almost all $t \in (0, \infty)$, where the values $u(0, t)$ and $u(l, t)$ are well defined due to the compact embedding $H^1(0, l) \subset C[0, l]$.

Definition 3.4. We say that system (8)–(12) is input-to-state stable (ISS) from $(d_1, d_2)$ to $(u, x)$, if there exist $\beta_i \in KKL$, $\gamma_i \in K$, $i = 1, 2$ such that for any initial state $(\varphi, x_0)$ and any disturbance $(d_1, d_2)$ the solution satisfies
\[
\begin{align*}
\| u(\cdot, t) \|_{L^2[0, l]} &\leq \beta_1(\| \varphi \|_{L^2[0, l]}, \| x_0 \|, t) + \gamma_1(d_\infty), \quad t \geq 0, \\
\| x(t) \| &\leq \beta_2(\| \varphi \|_{L^2[0, l]}, \| x_0 \|, t) + \gamma_2(d_\infty), \quad t \geq 0
\end{align*}
\]
where $d_\infty := \max(\| d_1 \|_{L^\infty[0, \infty)}, \| d_2 \|_{L^\infty[0, \infty)}).

The aim of this paper is to establish the ISS property of the interconnected system (8)–(12) in the sense of this definition.

4 Preliminary results

To construct a suitable Lyapunov function we first prove the following:

Lemma 4.1. Let $(u, x) \in (L^2([0, T], H^1[0, l]) \cap L^{2q}([0, T], L^2[0, l])) \cap C([0, T], L^2[0, l]) \times C([0, T], \mathbb{R}^n)$ be a solution of the system (6), then
\[ u(z, t) = w(z, t) + v(z, t), \quad \text{(15)} \]
where $(v, x) \in (L^2([0, T], H^1[0, l]) \cap L^{2q}([0, T], L^2[0, l])) \cap C([0, T], L^2[0, l]) \times C([0, T], \mathbb{R}^n)$ is the solution of
\[
\begin{align*}
v_t(z, t) &= a^2 v_{zz}(z, t) + f(v(z, t)) + B^T(z)x(t) + g(z, t), \\
\dot{x}(t) &= Cx(t) + X(x(t)) + \int_0^t D(z)v(z, t) \, dz + p(t)
\end{align*}
\]
with initial condition
\[ x(0) = x_0, \quad v(z, 0) = \varphi(z) \quad \text{(17)} \]
and boundary condition
\[ v(0, t) = 0, \quad v(l, t) = 0, \quad \text{(18)} \]
for some functions $w(z, t)$, $g(z, t)$ and $p(t)$ (provided explicitly in the proof) satisfying for almost all $(z, t) \in [0, l] \times \mathbb{R}_+$ the following estimates: $|w(z, t)| \leq d_\infty$ and
\[ |g(z, t)| \leq \psi_1(\varepsilon)|v(z, t)|^{2q - 1} + \psi_0(\varepsilon, d_\infty), \quad \| p(t) \| \leq \sqrt{t}\| D \|_{L^2[0, l]}d_\infty, \quad \text{(19)} \]
where $\varepsilon > 0$ and
\[ \psi_0(\varepsilon, d_\infty) = 2^{2q - 3}c_0\varepsilon^{1 - 2q} \frac{1}{2q - 1}d_\infty^{2q - 1} + (L + c_0)d_\infty + 2^{2q - 3}c_0d_\infty^{2q - 1}, \]
\[ \psi_1(\varepsilon) = 2^{2q - 3}c_0 \frac{2q - 2}{2q - 1} \varepsilon^{(2q - 1)/(2q - 2)}. \]
Proof. Substituting \( u(z,t) = w(z,t) + v(z,t) \) into (10) we obtain
\[
v_t(z,t) = a^2 v_{zz}(z,t) + B^T(z)x(t) + a^2 w_{zz}(z,t) - w_t(z,t) + f(v(z,t) + w(z,t)),
\]
\[
\dot{x}(t) = Cx(t) + X(x(t)) + \int_0^t D(z)v(z,t) \, dz + \int_0^t D(z)w(z,t) \, dz.
\]
(20)

Let us denote \( g(z,t) := f(v(z,t) + w(z,t)) - f(v(z,t)), \) \( p(t) := \int_0^t D(z)w(z,t) \, dz. \)

Let \( w \in H^{1,1}([0,l] \times [0,T]) \) be the weak solution to
\[
w_t(z,t) - a^2 w_{zz}(z,t) = 0, \quad w(0,t) = d_1(t), \quad w(l,t) = d_2(t), \quad w(z,0) = 0,
\]
by definition this means that for any \( \phi \in C_0^\infty([0,l] \times [0,T]) \) the following equality holds
\[
\int_0^T \langle w_t(z,t), \phi(z,t) \rangle \, dt + a^2 \int_0^T \langle w_z(z,t), \phi_z(z,t) \rangle \, dt = 0.
\]
(22)

In this case (22) holds also for all \( \phi \in H_0^{1,1}([0,l] \times [0,T]). \)

Let us note that conditions \( d_i(0) = 0, \) \( d_i \in L^\infty(\mathbb{R}_+) \) guarantee the existence of a weak solution \( w \in L^2([0,T], H^1[0,l]) \cap L^{2q}([0,T], L^{2q}[0,l]) \) to the problem (21). Indeed, the function \( \tilde{w}(z,t) = w(z,t) - \left( \frac{z}{l} \dot{d}_2(t) + \frac{l-z}{l} \dot{d}_1(t) \right) \) is the weak solution to the problem
\[
\tilde{w}_t(z,t) - a^2 \tilde{w}_{zz}(z,t) = \Delta(z,t), \quad \Delta(z,t) = -\frac{z}{l} \dot{d}_2(t) - \frac{l-z}{l} \dot{d}_1(t),
\]
\[
\tilde{w}(0,t) = 0, \quad \tilde{w}(l,t) = 0, \quad \tilde{w}(z,0) = 0.
\]
(23)

The problems (21) and (22) are equivalent. Hence it is enough to show the existence of a weak solution to the problem (22). Since \( \Delta \in L^2([0,l] \times [0,T]), \) then by Theorem 4 from (22) (Chapter 6) the existence of a unique solution \( \tilde{w} \in H^{2,1}([0,l] \times [0,T]) \) to the problem (22) follows.

By the (weak) maximum principle, see e.g., §30 in (28) applied to \( w \) it follows that
\[
|w(z,t)| \leq d_\infty \text{ for almost all } (z,t) \in [0,l] \times [0,\infty).
\]

From the assumption 1)–2) for \( f \) we obtain the estimation of \( g(z,t) \) for almost all \( (z,t) \in [0,l] \times [0,\infty): \)
\[
|g(z,t)| \leq |f_0(v(z,t) + w(z,t))| + |f_1(v(z,t) + w(z,t))| - f_1(v(z,t))|f_0(v(z,t) + w(z,t))| \leq L[w(z,t)] + |f'(v(z,t) + \eta w(z,t))|w(z,t)| \leq Ld_\infty + c_0(1 + |v(z,t)| + \eta|w(z,t)|^{2q-2})d_\infty,
\]

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where \( \eta \in (0, 1) \). Since the function \( s \mapsto 1 + s^{2q-2} \), \( s \geq 0 \), is convex, we can estimate further
\[
|g(z, t)| \leq Ld_\infty + c_0(1 + (|v(z, t)| + \eta |w(z, t)|)^{2q-2})d_\infty
\]
\[
\leq Ld_\infty + c_0(1 + 2^{2q-3}(|v(z, t)|^{2q-2} + |w(z, t)|^{2q-2}))d_\infty
\]
\[
\leq 2^{2q-3}c_0d_\infty |v(z, t)|^{2q-2} + (L + c_0)d_\infty + 2^{2q-3}c_0d_\infty^{2q-1}.
\]
By the Young’s inequality with \( p_1 = (2q-1)/(2q-2) \), \( p_2 = 2q-1 \) we get
\[
d_\infty |v(z, t)|^{2q-2} \leq \frac{2q-2}{2q-1} \varepsilon^{(2q-1)/(2q-2)} |v(z, t)|^{2q-1} + \varepsilon^{1-2q} \frac{1}{2q-1} d_\infty^{2q-1},
\]
where \( \varepsilon > 0 \). Finally, we obtain that for almost all \( (z, t) \in [0, l] \times [0, \infty) \) it holds that
\[
|g(z, t)| \leq 2^{2q-3}c_0 \frac{2q-2}{2q-1} \varepsilon^{(2q-1)/(2q-2)} |v(z, t)|^{2q-1} + c_0 \varepsilon^{1-2q} \frac{2^{2q-3}}{2q-1} d_\infty^{2q-1}
\]
\[
+ (L + c_0)d_\infty + 2^{2q-3}c_0d_\infty^{2q-1} = \psi_1(z)|v(z, t)|^{2q-1} + \psi_0(\varepsilon, d_\infty)
\]
To estimate \( p(t) \) we use the Cauchy-Bunyakovsky inequality:
\[
\|p(t)\| \leq \int_0^t \|D(z)\| |w(z, t)| \, dz \leq \|D\|_{L^2[0, t]} \|w(\cdot, t)\|_{L^2[0, t]} \leq d_\infty \sqrt{7}\|D\|_{L^2[0, t]}.
\]

Taking into account that \( w \in H^{2,1}([0, l] \times [0, T]) \) from (20) follows that \( v \in L^2([0, T], H^1_0[0, l]) \cap L^{2q}([0, T], L^{2q}[0, l]) \) is a weak solution to (16) satisfying (17) and (18). This finishes the proof of the lemma.

To derive estimates of solutions to (16) we use the direct method of Lyapunov. For this we define
\[
V(v(\cdot, t), x) := \langle v(\cdot, t), v(\cdot, t) \rangle + 2 \langle x^T P_{12}, v(\cdot, t) \rangle + x^T P x,
\]
where \( P_{12} \in (H^2_0([0, l]))^n \) the solution to the boundary value problem
\[
a^2 P_{12}''(z) + C^T P_{12}(z) = -B(z) - PD(z), \quad P_{12}(0) = P_{12}(l) = 0,
\]
and \( P \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix. Also we denote
\[
\Pi_1 = \left( \begin{array}{ccc} 1 & \|P_{12}\|_{L^2[0, t]} & \lambda_{\min}(P) \\ -\|P_{12}\|_{L^2[0, t]} & -\lambda_{\min}(P) & 0 \\ \lambda_{\min}(P) & 0 & \lambda_{\min}(P) \end{array} \right), \quad \Pi_2 = \left( \begin{array}{ccc} 1 & \|P_{12}\|_{L^2[0, t]} & \lambda_{\max}(P) \\ -\|P_{12}\|_{L^2[0, t]} & -\lambda_{\max}(P) & 0 \\ \lambda_{\max}(P) & 0 & \lambda_{\max}(P) \end{array} \right)
\]
then \( V(v(\cdot, t), x) \) can be estimated as
\[
\lambda_{\min}(\Pi_1)(\|v(\cdot, t)\|^{2}_{L^2[0, t]} + \|x\|^2) \leq V(v(\cdot, t), x) \leq \lambda_{\max}(\Pi_2)(\|v(\cdot, t)\|^{2}_{L^2[0, t]} + \|x\|^2)
\]
(26)
Now let us consider the derivation of $P_{12}(z)$ as a solution to (25). The equation for $P_{12}(z)$ can be written as

$$P''_{12}(z) + \frac{1}{a^2}C^T P_{12}(z) = F(z), \quad P_{12}(0) = P_{12}(l) = 0,$$

(27)

where $F(z) := -\frac{1}{a^2}(B(z) + PD(z))$.

Let us first consider the case $C = C^T > 0$. Using the functional calculus of matrices we can state the following

**Lemma 4.2.** Let $C = C^T > 0$, $\det \sin(\frac{1}{a}(C^T)^{1/2}l) \neq 0$, then the solution of

$$P''_{12}(z) + \frac{1}{a^2}C^T P_{12}(z) = F(z), \quad P_{12}(0) = P_{12}(l) = 0$$

(28)

can be written as

$$P_{12}(z) = \int_0^l G(z, \xi)F(\xi)d\xi,$$

(29)

with the Green’s function $G(z, \xi)$ given by

$$G(z, \xi) = \begin{cases} 
\frac{\sin(\frac{1}{a}(C^T)^{1/2}z) \sin(\frac{1}{a}(C^T)^{1/2}(z-\xi))((C^T)^{1/2} \sin(\frac{1}{a}(C^T)^{1/2}l))^{-1}}{\cos(\frac{1}{a}(C^T)^{1/2}z)\sin(\frac{1}{a}(C^T)^{1/2}(\xi-l))((C^T)^{1/2} \sin(\frac{1}{a}(C^T)^{1/2}l))^{-1}} & \text{for } 0 \leq \xi \leq z \leq l; \\
\frac{\sin(\frac{1}{a}(C^T)^{1/2}z) \sin(\frac{1}{a}(C^T)^{1/2}(\xi-l))((C^T)^{1/2} \sin(\frac{1}{a}(C^T)^{1/2}l))^{-1}}{\cos(\frac{1}{a}(C^T)^{1/2}z)\sin(\frac{1}{a}(C^T)^{1/2}(\xi-l))((C^T)^{1/2} \sin(\frac{1}{a}(C^T)^{1/2}l))^{-1}} & \text{for } 0 \leq z \leq \xi \leq l.
\end{cases}$$

**Proof.** By the variation of constants method we look for a solution to (28) in the form

$$P_{12}(z) = \sin(\frac{1}{a}(C^T)^{1/2}z)C_1(z) + \cos(\frac{1}{a}(C^T)^{1/2}z)C_2(z)$$

where vectors $C_1$ and $C_2$ satisfy

$$\sin(\frac{1}{a}(C^T)^{1/2}z)C'_1(z) + \cos(\frac{1}{a}(C^T)^{1/2}z)C'_2(z) = 0.$$

Substituting this $P_{12}$ into (28) we obtain that $C_1$ and $C_2$ must satisfy

$$\frac{1}{a}(C^T)^{1/2}\left(\cos(\frac{1}{a}(C^T)^{1/2}z)C'_1(z) - \sin(\frac{1}{a}(C^T)^{1/2}z)C'_2(z)\right) = F(z).$$

This implies that

$$C'_1(z) = a(C^T)^{-1/2}\cos(\frac{1}{a}(C^T)^{1/2}z)F(z),$$

$$C'_2(z) = -a(C^T)^{-1/2}\sin(\frac{1}{a}(C^T)^{1/2}z)F(z).$$
After integration we get

\[ C_1(z) = C_1(0) + a(C^T)^{-1/2} \int_0^z \cos\left( \frac{1}{a} (C^T)^{1/2} \xi \right) F(\xi) d\xi, \]

\[ C_2(z) = C_2(0) - a(C^T)^{-1/2} \int_0^z \sin\left( \frac{1}{a} (C^T)^{1/2} \xi \right) F(\xi) d\xi. \]

Consequently,

\[ P_{12}(z) = \sin\left( \frac{1}{a} (C^T)^{1/2} z \right) C_1(0) + \cos\left( \frac{1}{a} (C^T)^{1/2} z \right) C_2(0) + a(C^T)^{-1/2} \int_0^z \sin\left( \frac{1}{a} (C^T)^{1/2} (z - \xi) \right) F(\xi) d\xi. \]  

(30)

From the boundary conditions \( P_{12}(0) = 0, P_{12}(l) = 0 \) follows \( C_2(0) = 0 \) and

\[ C_1(0) = -a(C^T)^{-1/2} \int_0^l \left( \sin\left( \frac{1}{a} (C^T)^{1/2} l \right) \right)^{-1} \sin\left( \frac{1}{a} (C^T)^{1/2} (l - \xi) \right) F(\xi) d\xi \]

\[ = -a(C^T)^{-1/2} \int_0^l \left( \cos\left( \frac{1}{a} (C^T)^{1/2} \xi \right) - \cot\left( \frac{1}{a} (C^T)^{1/2} l \right) \sin\left( \frac{1}{a} (C^T)^{1/2} \xi \right) \right) F(\xi) d\xi. \]

Substituting these \( C_1(0) \) and \( C_2(0) \) into (30) we obtain (29). By properties of the Green’s function it follows that this \( P_{12} \) defined by (29) satisfies (28).

Now we consider the case when \( C^T = C \succ 0 \) is not true. In this case the (29) is the solution to (28), if in the expression for the Green-function we adopt that

\[ (C^T)^{1/2} \sin\left( \frac{l}{a} (C^T)^{1/2} \xi \right) = \sum_{k=0}^{\infty} \frac{(-1)^k (l/a)^{2k+1}}{(2k+1)!} \left( C^T \right)^{k+1}, \]

\[ \sin\left( \frac{(C^T)^{1/2} \xi}{a} \sin\left( \frac{(C^T)^{1/2} (z - l)}{a} \right) \right) \]

\[ = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^{k+p}}{(2k+1)! (2p+1)!} \left( \frac{\xi}{a} \right)^{2k+1} \left( \frac{z - l}{a} \right)^{2p+1} \left( C^T \right)^{k+p+1}. \]

These series can be calculated as follows. Let \( T \) be nonsingular matrix such that \( T^{-1} C^T T = \text{diag}\{ J_{n_1}(\lambda_1), \ldots, J_{n_r}(\lambda_r) \} \) is the Jordan normal form of \( C^T \). Then these series can be calculated separately for each Jordan block \( J_{n_k}(\lambda_k) \) explicitly.
5 Main results

Our first result provides conditions to guarantee that our system is well-posed. The second result establishes the ISS property.

**Theorem 5.1.** Assume that
1) \( f \) and \( X \) satisfy conditions 1)—4) from section 2 and \( f'_1(s) < 0 \) \( \forall \ s \in \mathbb{R}, \ s \neq 0; \)
2) \( \Pi_1 \) is positive definite and for some \( \delta_1 > 0 \)
   \[
   x^T PX(x) \leq -\delta_1 \|x\|^2 q, \quad x \in \mathbb{R}^n,
   \]
3) the following conditions are satisfied
   \[
   \omega := 2 \left( \frac{\pi^2 a^2}{l^2} - \|D\|_{L^2[0,l]}^2 \right) > 0,
   \]
   \[
   \Omega := - \left( C^T P + PC - \frac{1}{a^2} \int_0^l \int_0^l (G(z, \zeta)(B(\zeta) + PD(\zeta))B^T(z) + B(z)(B^T(\zeta) + D^T(\zeta)P)G^T(z, \zeta))d\zeta dz \right) > 0,
   \]
4) the matrix
   \[
   \Xi = \left( \begin{array}{cc}
   \omega & -L \|P_{12}\|_{L^2[0,l]} \\
   -L \|P_{12}\|_{L^2[0,l]} & \lambda_{\min}(\Omega)
   \end{array} \right)
   \]
   is positive definite.
5) Let (the unique pair) \( \tau_1 > 0 \) and \( \tau_2 > 0 \) satisfying
   \[
   \frac{\zeta \|P_{12}\|_{L^1[0,l]}^{2q-1}}{q} + \frac{\delta_2 \|P_{12}\|_{L^2[0,l]}(2q - 1)}{q} \tau_1^{2q/(2q-1)} = 2\delta_1
   \]
   \[
   \frac{\delta_2 \|P_{12}\|_{L^2[0,l]}^{2q-1}}{q} \tau_2^{-2q} + \frac{\zeta(2q - 1)}{q} \|P_{12}\|_{L^\infty[0,l]} \tau_2^{-2q/(2q-1)} = 2\alpha
   \]
be such that \( \tau_2 < \tau_1 \).

Then there exist \( u \in L^2([0, T], H^1[0, l]) \cap L^{2q}([0, T], L^{2q}[0, l]) \) and \( x \in C([0, T], \mathbb{R}^n) \) solving the problem \( \text{5} \) — \( \text{8} \) for any \( T > 0 \).

**Remark 5.2.** Note that \( u \in L^2([0, T], H^1[0, l]) \cap L^{2q}([0, T], L^{2q}[0, l]) \) and \( \partial_t u \in L^2([0, T], H^{-1}[0, l]) \cap L^{2q}([0, T], L^{2q}/2q[0, l]) \) and hence by \( \text{5} \) \( u \) is absolutely continuous.

The proof follows the ideas of the proof of Theorem 3.1 in \( \text{3} \) on page 38 (see also \( \text{14} \)) and is based on the Lyapunov function defined in Section 4. We postpone the detailed proof to the Appendix A and state our second main result as follows,
Theorem 5.3. Let \((6) - (8)\) be such that \(f\) and \(X(x)\) satisfy assumptions 1)–4) of Section 2 and let conditions 2)–5) from Theorem 5.1 be satisfied. Assume that for any initial condition \((7)\) and disturbances \((8)\) there exist a solution \(u \in L^2([0, T], H^1[0, l])\cap L^2q([0, T], L^2q[0, l])\) and \(x \in C([0, T], \mathbb{R}^n)\) for any \(T > 0\). Then \((6) - (8)\) is ISS from \((d_1, d_2)\) to \((u, x)\).

Remark 5.4. Let us note that conditions 2)–5) from Theorem 5.1 can be verified by means of standard numerical tools available in Matlab or Maple.

Proof. Using Lemma 2.2 from \([6]\) we calculate the time derivative of \(V(v(\cdot, t), x)\) along solutions to (16):

\[
\dot{V}(v(\cdot, t), x) = W_1 + W_2 + W_3 + W_4,
\]

where we have denoted (see detailed calculations in Appendix B)

\[
W_1 = -2a^2 \langle v_z(\cdot, t), v_z(\cdot, t) \rangle + 2 \int_0^t v(z, t) f(v(z, t)) \, dz 
\]

\[
+ 2 \int_0^t v(z, t) D^T(z) \, dz \int_0^t P_{12}(z) v(z, t) \, dz,
\]

\[
W_2 = x^T \left( C^T P + PC + \int_0^t (P_{12}(z) B^T(z) + B(z) P_{12}^T(z)) \, dz \right) x 
\]

\[
+ 2 x^T PX(x),
\]

\[
W_3 = 2 \int_0^t v(z, t) g(z, t) \, dz + 2p^T(t) \int_0^t P_{12}(z) v(z, t) \, dz 
\]

\[
+ 2x^T \int_0^t P_{12}(z) g(z, t) \, dz + 2x^T Pp(t),
\]

\[
W_4 = 2X^T(x) \int_0^t P_{12}(z) v(z, t) \, dz + 2x^T \int_0^t P_{12}(z) f(v(z, t)) \, dz.
\]

Using the Friedrich’s inequality

\[
- \langle v_z(\cdot, t), v_z(\cdot, t) \rangle = - \int_0^t v_z^2(z, t) \, dz \leq \frac{\pi^2}{l^2} \|v(\cdot, t)\|_{L^2[0, l]}^2
\]

and the Cauchy-Bunyakovskyi inequality

\[
\int_0^t v(z, t) D^T(z) \, dz \int_0^t P_{12}(z) v(z, t) \, dz \leq \|D\|_{L^2[0, l]} \|P_{12}\|_{L^2[0, l]} \|v(\cdot, t)\|_{L^2[0, l]}^2
\]

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we can estimate $W_1$ with help of integration by parts as follows

$$W_1 \leq \left( -\frac{2\pi^2 a^2}{l^2} + 2\|D\|_{L^2[0,1]}\|P_{12}\|_{L^2[0,1]} \right)\|v(\cdot, t)\|_{L^2[0,1]}^2 + 2 \int_0^l v(z, t)f(v(z, t)) \, dz.$$  

By the assumption 2) of Section 3 we obtain that

$$W_1 \leq \left( -\frac{2\pi^2 a^2}{l^2} + 2\|D\|_{L^2[0,1]}\|P_{12}\|_{L^2[0,1]} + 2\alpha\right)\|v(\cdot, t)\|_{L^{2a}(0,1)}^{2a}.$$  

Let us estimate $W_1$ in (32). Applying the first inequality from (19) we obtain

$$\int_0^l v(z, t)g(z, t) \, dz \leq \psi_0(\varepsilon, d_\infty) \int_0^l |v(z, t)| \, dz + \psi_1(\varepsilon) \int_0^l |v(z, t)|^2q \, dz$$

$$\leq \psi_0(\varepsilon, d_\infty)\sqrt{t}\|v(\cdot, t)\|_{L^2[0,1]} + \psi_1(\varepsilon)\|v(\cdot, t)\|_{L^{2q}(0,1)}^{2q}.$$  

By the triangle inequality and the Cauchy-Bunyakovskyi inequality we have

$$\|\int_0^l P_{12}(z)g(z, t) \, dz\| \leq \int_0^l \|P_{12}(z)\| (\psi_0(\varepsilon, d_\infty) + \psi_1(\varepsilon)|v(z, t)|^{2q-1}) \, dz$$

$$\leq \sqrt{t}\|P_{12}\|_{L^2[0,1]}\psi_0(\varepsilon, d_\infty)|x| + \psi_1(\varepsilon) \int_0^l \|P_{12}(z)\| |x| |v(z, t)|^{2q-1} \, dz,$$

and by the Cauchy-Bunyakovskyi inequality we estimate

$$x^T \int_0^l P_{12}(z)g(z, t) \, dz$$

$$\leq \sqrt{t}\|P_{12}\|_{L^2[0,1]}\psi_0(\varepsilon, d_\infty)|x| + \psi_1(\varepsilon) \int_0^l \|P_{12}(z)\| |x| |v(z, t)|^{2q-1} \, dz.$$  

By the Young’s inequality (2) with $p_1 = 2q, p_2 = 2q/(2q - 1)$, we have

$$\|x\| |v(z, t)|^{2q-1} \leq \frac{1}{2q} \|x\|^{2q} + \left(1 - \frac{1}{2q}\right) |v(z, t)|^{2q}.$$  

Hence we obtain

$$x^T \int_0^l P_{12}(z)g(z, t) \, dz$$

$$\leq \sqrt{t}\|P_{12}\|_{L^2[0,1]}\psi_0(\varepsilon, d_\infty)|x| + \frac{1}{2q} \psi_1(\varepsilon)\|P_{12}\|_{L^2[0,1]}|x|^{2q}$$

$$+ \left(1 - \frac{1}{2q}\right) \psi_1(\varepsilon)\|P_{12}\|_{L^\infty[0,1]}\|v(\cdot, t)\|_{L^{2q}(0,1)}^{2q}.$$  

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By the Cauchy-Bunyakovskiy inequality we have

\[
\left| p^T(t) \int_0^t P_{12}(z)v(z,t)\,dz \right| \leq d_\infty \sqrt{t} \|D\|_{L^2[0,t]} \|P_{12}\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]},
\]

Hence,

\[
x^T P p(t) \leq \sqrt{t} \|D\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]}.
\]

Thus, we have

\[
W_3 \leq 2\sqrt{t} \|p \|_{L^2[0,t]} \|P_{12}\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]}
+ 2\sqrt{t} \|P_{12}\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]}
+ 2 \psi_1(\epsilon) \left( 1 + \frac{2q - 1}{2} \|P_{12}\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]} \right)
+ \frac{1}{q} \psi_1(\epsilon) \|P_{12}\|_{L^2[0,t]} \|v(\cdot,t)\|_{L^2[0,t]}^2.
\]

To estimate \( W_4 \) we apply the Cauchy-Bunyakovskiy inequality to (36) and using (3)–(12) obtain

\[
W_4 \leq 2 \|X(\cdot)\| \int_0^t \|P_{12}(z)\| \|v(z,t)\| \,dz
+ 2 \|x\| \int_0^t \|P_{12}(z)\| \|f(v(z,t))\| \,dz
\leq 2 \delta_2 \|P_{12}\|_{L^2[0,t]} \|x\|^{2q-1} \|v(\cdot,t)\|_{L^2[0,t]}
+ 2 \sqrt{t} \|P_{12}\|_{L^2[0,t]} \|x\| \|v(\cdot,t)\|_{L^2[0,t]} + 2 \|x\| \int_0^t \|P_{12}(z)\| \|v(z,t)\|^{2q-1} \,dz.
\]

By the Young’s inequality with \( p_1 = \frac{2q}{2q-1}, \ p_2 = 2q, \ \tau > 0 \) we have

\[
\|x\|^{2q-1} \|v(\cdot,t)\|_{L^2[0,t]} \leq \frac{2q - 1}{2q} \tau^{2q/(2q-1)} \|x\|^{2q} + \frac{1}{2q} \tau^{-2q} \|v(\cdot,t)\|^{2q}_{L^2[0,t]}.
\]

By the Hölder’s inequality we have

\[
\|v(\cdot,t)\|^{2q}_{L^2[0,t]} = \left( \int_0^t |v(z,t)|^{2q} \,dz \right)^q \leq t^{q-1} \int_0^t |v(z,t)|^{2q} \,dz = t^{q-1} \|v(\cdot,t)\|^{2q}_{L^{2q}[0,t]},
\]

which implies that

\[
\|x\|^{2q-1} \|v(\cdot,t)\|_{L^2[0,t]} \leq \frac{2q - 1}{2q} \tau^{2q/(2q-1)} \|x\|^{2q} + \frac{t^{q-1}}{2q} \tau^{-2q} \|v(\cdot,t)\|^{2q}_{L^2[0,t]}.
\]

By the Young’s inequality with \( p_2 = \frac{2q}{2q-1}, \ p_1 = 2q, \ \tau > 0 \) we have

\[
\|x\| \|v(z,t)\|^{2q-1} \leq \frac{1}{2q} \tau^{2q} \|x\|^{2q} + \frac{2q - 1}{2q} \tau^{-2q} \|v(z,t)\|^{2q}.
\]
Finally $W_4$ can be estimated as follows

$$W_4 \leq 2L\|P_{12}\|_{L^2[0,1]}\|x\|_{L^2[0,1]}\|v(\cdot, t)\|_{L^2[0,1]} \left(\zeta\|P_{12}\|_{L^1[0,1]}\tau^{-2q} + \delta_2\|P_{12}\|_{L^2[0,1]}\frac{2q-1}{q} \tau^{-2q/(2q-1)}\right)\|x\|^{2q}$$

$$+ \left(\delta_2\|P_{12}\|_{L^2[0,1]}\frac{1}{q} \tau^{-2q} + \zeta\frac{(2q-1)}{q} \|P_{12}\|_{L^\infty[0,1]} \tau^{-2q/(2q-1)}\right)\|v(\cdot, t)\|_{L^2[0,1]}}^{2q}.$$  

From (31) and the assumptions of the theorem we can estimate $W_2$ as

$$W_2 \leq x^T(t)\Omega x(t) - 2\delta_1\|x(t)\|^{2q}.$$

Now from (32), (31), (41), (38) and (40) we obtain

$$V(v(\cdot, t), x(t)) \leq -\omega\|v(\cdot, t)\|^2_{L^2[0,1]} - \lambda_{\min}(\Omega)\|x(t)\|^2$$

$$+ 2L\|P_{12}\|_{L^2[0,1]}\|x(t)\|_{L^2[0,1]}\|v(\cdot, t)\|_{L^2[0,1]} + H_1(\varepsilon)\|v(\cdot, t)\|_{L^2[0,1]}$$

$$+ H_2(\varepsilon)\|x(t)\| + H_3(\varepsilon, \tau)\|v(\cdot, t)\|^2_{L^2[0,1]} + H_4(\varepsilon, \tau)\|x(t)\|^2,$$

where we have denoted

$$H_1(\varepsilon) = 2\sqrt{T}(\psi_0(\varepsilon, d_{\infty}) + d_{\infty}\|D\|_{L^2[0,1]}\|P_{12}\|_{L^2[0,1]}),$$

$$H_2(\varepsilon) = 2\sqrt{T}(\|P_{12}\|_{L^2[0,1]}\|\psi_0(\varepsilon, d_{\infty}) + \|P\|\|D\|_{L^2[0,1]}d_{\infty}),$$

$$H_3(\varepsilon, \tau) = 2\psi_1(\varepsilon)\left(1 + \frac{2q-1}{2q}\|P_{12}\|_{L^\infty[0,1]}\right)$$

$$+ \zeta\frac{(2q-1)}{q} \|P_{12}\|_{L^\infty[0,1]} \tau^{-2q/(2q-1)} - 2\alpha,$$

$$H_4(\varepsilon, \tau) = \frac{1}{q} \psi_1(\varepsilon)\|P_{12}\|_{L^1[0,1]}$$

$$+ \zeta\frac{(2q-1)}{q} \|P_{12}\|_{L^1[0,1]} \tau^{-2q} + \delta_2\|P_{12}\|_{L^2[0,1]}\frac{2q-1}{q} \tau^{-2q/(2q-1)} - 2\delta_1.$$

By the assumption 5) of the theorem there exists $\tau \in (\tau_2, \tau_1)$ such that

$$\frac{\delta_2\|P_{12}\|_{L^2[0,1]}\frac{1}{q} \tau^{-2q}}{q} - \zeta\frac{(2q-1)}{q} \|P_{12}\|_{L^\infty[0,1]} \tau^{-2q/(2q-1)} - 2\alpha < 0,$$

$$\frac{\zeta\|P_{12}\|_{L^1[0,1]} \tau^{-2q}}{q} + \frac{\delta_2\|P_{12}\|_{L^2[0,1]}\frac{2q-1}{q} \tau^{-2q/(2q-1)}}{q} - 2\delta_1 < 0.$$  

(43)

Since $\psi_1(\varepsilon) \to 0$ for $\varepsilon \to 0+$, there is some small enough $\varepsilon > 0$ such that $H_3(\varepsilon, \tau) < 0$, $H_4(\varepsilon, \tau) < 0$. Hence for this choice of $(\varepsilon, \tau)$ we obtain from assumption 4) of Theorem [6, 1] that

$$V(v(\cdot, t), x(t)) \leq -\lambda_{\min}(\Xi)\|v(\cdot, t)\|^2_{L^2[0,1]} + \|x(t)\|^2$$

$$+ H_3(\varepsilon, \tau)\|v(\cdot, t)\|_{L^2[0,1]} + H_2(\varepsilon, \tau)\|x(t)\|.$$

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By means of the Gronwall’s inequality and using (5) we obtain the estimates
\[ \tilde{V}(v(\cdot), x(t)) \leq -\frac{\lambda_{\min}(\Xi)}{2} (\|v(\cdot), t\|_{L^2[0,1]}^2 + \|x(t)\|^2) + \frac{H_1^2(\epsilon, \tau)}{2\lambda_{\min}(\Xi)} + \frac{H_2^2(\epsilon, \tau)}{2\lambda_{\min}(\Xi)}. \]
We denote
\[ \theta_0 := \frac{\lambda_{\min}(\Xi)}{2\lambda_{\max}(\Pi_2)}, \quad \vartheta(d_\infty) := \frac{H_1^2(\epsilon, \tau)}{2\lambda_{\min}(\Xi)} + \frac{H_2^2(\epsilon, \tau)}{2\lambda_{\min}(\Xi)}, \]
and write
\[ \tilde{V}(v(\cdot), x(t)) \leq -\theta_0 V(v(\cdot), x(t)) + \vartheta(d_\infty). \]
By means of the Gronwall’s inequality and using (6) we obtain the estimates
\[ \|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)}} \varrho(x_0, \varphi) e^{-\theta_0 t/2} + \sqrt{\frac{\vartheta(d_\infty)}{\lambda_{\min}(\Pi_1)}} \theta_0, \]
\[ \|v(\cdot, t)\|_{L^2[0,1]} \leq \sqrt{\frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)}} \varrho(x_0, \varphi) e^{-\theta_0 t/2} + \sqrt{\frac{\vartheta(d_\infty)}{\lambda_{\min}(\Pi_1)}} \theta_0, \]
where \( \varrho(x_0, \varphi) := (\|x_0\|^2 + \|\varphi\|_{L^2[0,1]}^2)^{1/2} \).
Recall that \( u(z, t) = v(z, t) + w(z, t) \), hence finally the following ISS estimate holds
\[ \|u(\cdot, t)\|_{L^2[0,1]} \leq \sqrt{\frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)}} \varrho(x_0, \varphi) e^{-\theta_0 t/2} + \sqrt{\frac{\vartheta(d_\infty)}{\lambda_{\min}(\Pi_1)}} \theta_0. \]
\[ \Box \]

**Corollary 5.5.** Let \( f \) be globally Lipschitz so that for some \( L > 0 \) we have
\[ |f(s_1) - f(s_2)| \leq L|s_1 - s_2|, \quad s_1, s_2 \in \mathbb{R} \]
and such that for all \( s \in \mathbb{R}, s \neq 0 \) it hold that \( sf(s) \leq \sigma s^2, f(0) = 0 \) for some \( \sigma \in \mathbb{R} \). Further, we assume that the conditions 2) – 4) of Theorem 5.1 are satisfied. Then system (5) – (8) is ISS from \( (d_1, d_2) \) to \( (u, x) \) so that
\[ \|x(t)\| \leq \sqrt{\frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)}} \varrho(x_0, \varphi) e^{-\theta_0 t/2} + \sqrt{\frac{\beta}{\theta_0 \lambda_{\min}(\Pi_1)}} d_\infty, \quad t \geq 0, \]
\[ \|u(\cdot, t)\|_{L^2[0,1]} \leq \sqrt{\frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)}} \varrho(x_0, \varphi) e^{-\theta_0 t/2} + \left( \sqrt{\frac{\beta}{\theta_0 \lambda_{\min}(\Pi_1)}} + \sqrt{\frac{\beta}{\theta_0 \lambda_{\min}(\Pi_1)}} \right) d_\infty, \quad t \geq 0, \]
where
\[ \theta := \frac{\lambda_{\min}(\Xi)}{2\lambda_{\max}(\Pi_2)}, \quad \beta := \frac{2(L^2(K_1^2 + K_2^2) \lambda_{\min}(\Xi)}{\lambda_{\min}(\Xi)}, \quad \varrho(x_0, \varphi) := (\|x_0\|^2 + \|\varphi\|_{L^2[0,1]}^2)^{1/2} \]
\[ K_1 := L + \|D\|_{L^2[0,1]} \|P_0\|_{L^2[0,1]}, \quad K_2 := L\|P_1\|_{L^2[0,1]} + \|D\|_{L^2[0,1]} \|P\|. \]
Proof. In this case we have \( \psi_0(\varepsilon, d_\infty) = L d_\infty \), \( \psi_1(\varepsilon) \equiv 0 \), \( H_3 \equiv 0 \), \( H_4 \equiv 0 \).

\[
H_1(\varepsilon) = 2 \sqrt{L d_\infty} (L + \|D\|_{L^2[0,l]} \|P_1\|_{L^2[0,l]}) = 2 \sqrt{L d_\infty} K_1,
\]

\[
H_2(\varepsilon) = 2 \sqrt{\|P_1\|_{L^2[0,l]} L d_\infty + \|P_2\|_{L^2[0,l]} d_\infty}) = 2 \sqrt{L d_\infty} K_2,
\]

so that from (42) we have the estimate

\[
\dot{V}(v(\cdot, t), x(t)) \leq -\lambda_{\min}(\Xi)(\|v(\cdot, t)\|_{L^2[0,l]}^2 + \|x(t)\|^2) + 2 \sqrt{L d_\infty} (K_1 \|v(\cdot, t)\|_{L^2[0,l]} + K_2 \|x(t)\|).
\]

(45)

and by (4) we can write

\[
\dot{V}(v(\cdot, t), x(t)) \leq -\lambda_{\min}(\Xi) \|v(\cdot, t)\|_{L^2[0,l]}^2 + \|x(t)\|^2 + \beta d_\infty^2.
\]

by means of (26) for \( V(v(\cdot, t), x(t)) \) we obtain the differential inequality

\[
\dot{V}(v(\cdot, t), x(t)) \leq -\theta V(v(\cdot, t), x(t)) + \beta d_\infty^2,
\]

which implies after the integration that

\[
V(v(\cdot, t), x(t)) \leq e^{-\theta t} V(v, x_0) + \frac{2 \beta \lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Xi)} d_\infty^2.
\]

Using (26), the elementary inequality \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \), \( a, b \geq 0 \) and \( u(t, z) = v(t, z) + w(t, z) \) we arrive at \( \Box \).

Corollary 5.6. Under the conditions of Theorem 5.1 a solution \( (u, x) \) to the problem (6) – (8) is unique in the class \( L^2(\mathbb{R}^+, L^\infty[0,l]) \times C(\mathbb{R}^+, \mathbb{R}^n) \).

See Appendix B for the proof.

6 Example

To illustrate our results we consider an interconnection of a parabolic equation and an ordinary differential equation as follows

\[
u_t(z, t) = a^2 u_{zz}(z, t) + \sigma \sin u(z, t) + bx(t),
\]

\[
\dot{x}(t) = cx(t) + d \int_0^l u(z, t) \, dz
\]

(46)

with initial conditions

\[
x(0) = x_0 \in \mathbb{R}, \quad u(z, 0) = \varphi(z), \quad \varphi \in L^2([0,l]), \quad z \in (0,l), \quad t \in (0, +\infty)
\]

(47)
and boundary conditions
\[ u(0, t) = d_1(t), \quad u(l, t) = d_2(t). \] (48)

where \(a, c, l, \sigma\) are positive numbers and \(b, d \in \mathbb{R}\).

The solution to the (scalar) problem (25) with (scalar) \(P > 0\) can be calculated as follows. The problem (25) reads as
\[ a^2 P''_{12}(z) + c P_{12}(z) = -b - P, \quad P_{12}(0) = P_{12}(l) = 0. \]

Since \(c > 0\), the general solution to the differential equation is
\[ P_{12}(z) = C_1 \sin(\lambda z) + C_2 \cos(\lambda z) - \frac{b}{c} \frac{P}{a}, \quad \lambda = \sqrt{\frac{c}{a}}. \]

From the boundary conditions \(P_{12}(0) = P_{12}(l) = 0\) we get
\[ C_2 - \frac{b}{c} \frac{P}{a} = 0, \]
\[ C_1 \sin(\lambda l) + C_2 \cos(\lambda l) - \frac{b}{c} \frac{P}{a} = 0. \]

From this we derive
\[ C_1 = \frac{b}{c} \frac{P}{a} \left( \frac{1}{\sin(\lambda l)} - \cot(\lambda l) \right) = \tan \left( \frac{\lambda l}{2} \right) \frac{b}{c} \frac{P}{a}. \]

So that
\[ P_{12}(z) = \frac{b}{c} \frac{P}{a} \left( \cos(\lambda z) - 1 + \tan \left( \frac{\lambda l}{2} \right) \sin(\lambda z) \right). \]

Let us calculate \(\|P_{12}\|_{L^2([0,l])} = \left| \frac{b + Pd}{c} \right| \chi\), where
\[ \chi = \left( \int_0^l (\cos(\lambda z) - 1 + \tan \left( \frac{\lambda l}{2} \right) \sin(\lambda z))^2 \, dz \right)^{1/2}. \]

Under the integral we have
\[ \cos(\lambda z) - 1 + \tan \left( \frac{\lambda l}{2} \right) \sin(\lambda z) = -2 \sin^2 \left( \frac{\lambda z}{2} \right) + 2 \tan \left( \frac{\lambda l}{2} \right) \sin \left( \frac{\lambda z}{2} \right) \cos \left( \frac{\lambda z}{2} \right) \]
\[ = \frac{2 \sin \left( \frac{\lambda l}{2} \right)}{\cos \left( \frac{\lambda l}{2} \right)} \left( \sin \left( \frac{\lambda l}{2} \right) \cos \left( \frac{\lambda z}{2} \right) - \sin \left( \frac{\lambda z}{2} \right) \cos \left( \frac{\lambda l}{2} \right) \right) \]
\[ = \frac{2 \sin \left( \frac{\lambda l}{2} \right) \sin \left( \frac{\lambda (l-z)}{2} \right)}{\cos \left( \frac{\lambda l}{2} \right)} = \frac{\cos(\lambda(z - \frac{l}{2})) - \cos \left( \frac{\lambda l}{2} \right)}{\cos \left( \frac{\lambda l}{2} \right)} \]
\[ = \frac{2 \sin \left( \frac{\lambda l}{2} \right) (\cos(\lambda(z - \frac{l}{2})) - \cos \left( \frac{\lambda l}{2} \right))}{\sin(l)} = \frac{\sin(\lambda z) + \sin(\lambda(l-z)) - \sin(\lambda l)}{\sin(l)}. \]
Hence for the integral we can write

\[\chi = \frac{1}{|\sin(\lambda l)|} \left( \int_0^l (\sin(\lambda z) + \sin(\lambda (l - z)) - \sin(\lambda l))^2 \, dz \right)^{1/2}.\]

Obviously, \(|B|_{L^2([0,1])} = \sqrt{l}|b|\) and \(|D|_{L^2([0,1])} = \sqrt{l}|d|\), so that we can calculate the constants

\[
\omega = 2 \left( \frac{\pi^2 a^2}{l^2} - \frac{||D||_{L^2([0,1])} ||P_{12}||_{L^2([0,1])} - \sigma}{l} \right) = 2 \left( \frac{\pi^2 a^2}{l^2} - \frac{(b + Pd)\sqrt{l}d}{c} |\chi - \sigma| \right)
\]

\[
\Omega = - \left( C^T P + PC + \int_0^l (P_{12}(z)B^T(z) + B(z)P_{12}(z)) \, dz \right)
\]

\[
= -2 \left( cP + \frac{b(b + Pd)}{c} \int_0^l (\cos(\lambda z) - 1 + \tan \left( \frac{\lambda l}{2} \right) \sin(\lambda z)) \, dz \right).
\]

For the last integral we calculate

\[
\int_0^l (\cos(\lambda z) - 1 + \tan \left( \frac{\lambda l}{2} \right) \sin(\lambda z)) \, dz = \frac{\sin(\lambda z)}{\lambda} - z - \tan \left( \frac{\lambda l}{2} \right) \cos(\lambda l) \bigg|_{z=0}
\]

\[
= \frac{1}{\lambda} \left( \sin(\lambda l) + \tan \left( \frac{\lambda l}{2} \right) (1 - \cos(\lambda l)) \right) - l = \frac{2 \tan \left( \frac{\lambda l}{2} \right)}{\lambda} - l := \kappa,
\]

which implies

\[
\Omega = -2 \left( cP + \frac{b(b + Pd)\kappa}{c} \right).
\]

By means of the Taylor expansion

\[
\tan x = x + \frac{x^3}{3} + \cdots
\]

we can write

\[
\kappa \approx \frac{2}{\lambda} \left( \frac{\lambda l}{2} + \frac{(\lambda l)^3}{3} \right) - l = \frac{\lambda^2 l^3}{12}
\]

and similarly

\[
\chi \approx \frac{1}{|\sin(\lambda l)|} \left( \int_0^l \left( \lambda - \frac{(\lambda l)^3}{6} + \lambda(z - l) - \frac{(\lambda(z - l))^3}{6} - \lambda z + \frac{(\lambda z)^3}{6} \right)^2 \, dz \right)^{1/2}
\]

\[
\approx \frac{1}{\lambda} \left( \frac{\lambda^6 l^6}{4} \int_0^l z^2 (l - z)^2 \, dz \right)^{1/2} = \frac{l^2 \sqrt{\lambda}^2}{2 \sqrt{30}}.
\]
By Corollary 5.5 the system (46) is ISS if the conditions 2)-5) of Theorem 5.1 are satisfied, that is if there exists $P$ such that

$$ P > 0, \quad \omega(P) > 0, \quad \Omega(P) > 0, \quad \chi \left| \frac{b + dP}{c} \right| < \sqrt{P}.$$  \hspace{1cm} (49)

Let $P = p^*$ satisfy the inequalities (49), then

$$ K_1 = \sigma + \sqrt{l|d(b + p^*d)|}, \quad K_2 = \sigma \frac{|d(b + p^*d)|}{c} \chi + \sqrt{l|d|},$$

In case $\lambda l \ll 1$ these conditions can be simplified essentially, as in this case

$$ \kappa \approx \frac{\lambda^2 l^3}{12}, \quad \chi \approx \frac{\lambda^2 l^2 \sqrt{l}}{2 \sqrt{30}}$$

and

$$ \|P_{12}\|_{L^2[0,1]} \approx \left| \frac{b + Pd}{c} \right| \frac{\lambda^2 l^2 \sqrt{l}}{2 \sqrt{30}}, \quad \|B\|_{L^2[0,1]} = \sqrt{l|b|}, \quad \|D\|_{L^2[0,1]} = \sqrt{l|d|},$$

$$ \omega = \omega(P) \approx 2 \left( \frac{\pi^2 a^2}{l^2} - \frac{l^3|d(b + dP)|}{2 \sqrt{30} a^2} - \sigma \right),$$

$$ \Omega = \Omega(P) \approx -2(Pa^2 \lambda^2 + \frac{l^3 b(b + Pd)}{12 a^2}).$$

Let the parameters of our system be given by $c = 0.25, a = 1, l = 1, b = 1, d = -5, L = \sigma = 1$ then we can take $P = p^* = 1$, so that

$$ \Xi = \begin{pmatrix} 13.992949 & -0.374626 \\ -0.374626 & 0.183765 \end{pmatrix},$$

$$ \kappa = 0.021367, \quad \chi = 0.023414, \quad \omega = 13.992949, \quad \Omega = 0.183766,$$

$$ \|P_{12}\|_{L^2[0,1]} = 0.374626, \quad \lambda_{\min}(\Pi_1) = \lambda_{\min}(\Pi_2) = 0.625374,$$

$$ \lambda_{\max}(\Pi_1) = \lambda_{\max}(\Pi_2) = 1.374626, \quad K_1 = 2.873130, \quad K_2 = 8.7462728,$$

$$ \lambda_{\min}(\Xi) = 0.1736102, \quad \theta = 0.06315, \quad \beta = 785.0749.$$

By Corollary 5.5 we obtain the following ISS estimates for the solutions of (46)

$$ \|x(t)\| \leq 1.482594 e^{-0.0315 t} + 95.26 d_{\infty}, \quad t \geq 0,$$

$$ \|u(t)\|_{L^2[0,1]} \leq 1.482594 e^{-0.0315 t} + 96.26 d_{\infty}, \quad t \geq 0.$$

**Remark 6.1.** Let us not that the ODE subsystem is not ISS, because it is not globally asymptotically stable already in the disconnected case ($u = 0$), hence there is no way to establish the ISS property for the interconnections by means of the small-gain theory or with help of vector Lyapunov functions.
7 Conclusion and future work

In this work we have developed an approach for a construction of a Lyapunov function. With help of this function we have proved the ISS property of a nonlinear coupled systems of an ODE and a PDE with disturbances at the boundary. ISS-type estimation for solutions is also derived. Recall that in contrary to the most related works we allow the situation, where the decoupled ODE can be unstable. Also our problem is not self-adjoint, which is different from many of exiting works dealing with the ISS-like properties.

An interesting direction for future research is to extend the developed approach to the multidimensional case and to the case of coupled PDE systems with time varying coefficients, where the decoupled PDEs are not necessarily stable. Furthermore, it is of interest to consider other types of boundary conditions, e.g., of Neumann or Robin type.

A Proof of Theorem 5.1

Here we will prove Theorem 5.1 on the existence of solutions to the problem (6) – (8). We will use the Lyapunov function $V$ from Section 3 and the ideas from [3, 11]. Hence we begin with the following change of variables

$$\tilde{u}(z,t) = u(z,t) - H(z,t), \quad H(z,t) = \frac{z}{l}d_2(t) + \frac{l-z}{l}d_1(t)$$

(50)

for which the problem (4) – (8) transforms to the equivalent problem

$$\tilde{u}_t(z,t) = a^2\tilde{u}_{zz}(z,t) + f(\tilde{u}(z,t) + H(z,t)) + B^T(z)x(t) - H_t(z,t)$$

$$\dot{x}(t) = Cx(t) + X(x(t)) + \int_0^l D(z)\tilde{u}(z,t) dz + \int_0^l D(z)H(z,t) dz$$

(51)

with initial states $x(0) = x_0 \in \mathbb{R}^n$, $\tilde{u}(z,0) = \varphi(z)$, $\varphi \in L^2[0,l]$, $z \in (0,l)$, $t \in (0,\infty)$ and boundary conditions $\tilde{u}(0,t) = 0$, $\tilde{u}(l,t) = 0$.

In the sequel we drop the symbol $\tilde{}$ over $u$ to simplify notation. Consider the orthonormal basis $e_j(z) = \sqrt{\frac{2}{l}} \sin \frac{\pi j z}{l}$, $j \in \mathbb{N}$ in $L^2[0,l]$.

We define the following projection operator $\Pi_N f := \sum_{p=0}^N f_p e_p(z)$, $f_p = (f,e_p)_{L^2[0,l]}$ that maps the Hilbert space $L^2[0,l]$ onto the finite dimensional subspace $E_N = \text{span}\{e_1(z),\ldots,e_N(z)\}$.

The Galerkin system corresponding to the problem (51) is as follows

$$\partial_t u_N(z,t) = a^2\partial_{zz} u_N(z,t) + \Pi_N f(u_N(z,t) + H(z,t))$$

$$+ \Pi_N B^T(z)x_N(t) - \Pi_N H_t(z,t)$$

$$\dot{x}_N(t) = Cx_N(t) + X(x_N(t)) + \int_0^l D(z)u_N(z,t) dz + \int_0^l D(z)H(z,t) dz$$

(52)
with initial conditions $u_N(z,0) = \Pi_N \varphi(z)$, $x_N(0) = x_0$. The boundary conditions $u_N(0,t) = u_N(l,t) = 0$ are satisfied since $u_N(z,t)$ is an element of $E_N$. Due to the assumptions 1)–4) introduced in Section 3 it follows that the ODE-system (52) satisfies the local Lipschitz condition and hence possesses a solution defined for $t \in [0,t_N]$, $t_N > 0$.

To derive a priori estimates for the solutions $(u_N(z,t), x_N(t))$ of the ODE system (52) we use the function $V$ from Section 4

$$V(u_N(\cdot,t), x_N) = \int_0^l u_N^2(z,t) \, dz + 2x_N^T \int_0^l P_{12}(z)u_N(z,t) \, dz + x_N^T P x_N,$$  \hspace{1cm} (53)

for which along solutions of (52) we calculate

$$\dot{V}(u_N(\cdot,t), x_N(t)) = R_1(z,t) + R_2(z,t) + R_3(z,t) + R_4(z,t),$$  \hspace{1cm} (54)

where

$$R_1(z,t) = -2a^2 \int_0^l (\partial_z u_N(z,t))^2 \, dz + 2 \int_0^l u_N(z,t)\Pi_N f(u_N(z,t)) \, dz$$

$$+ 2 \int_0^l u_N(z,t)D^T(z) \, dz \int_0^l P_{12}(z)u_N(z,t) \, dz$$

$$R_2(z,t) = 2 \int_0^l u_N(z,t)(\Pi_N - I)B^T(z) \, dz x_N(t)$$

$$+ 2x_N^T(x_N(t)) \int_0^l P_{12}(z)u_N(z,t) \, dz + 2x_N^T(t) \int_0^l P_{12}(z)\Pi_N f(u_N(z,t)) \, dz$$

$$R_3(z,t) = x_N^T(t)(C^T P + PC + \int_0^l (P_{12}(z)B^T(z) + B(z)P_{12}^T(z)) \, dz) x_N(t)$$

$$+ 2x_N^T(t)P X(x_N(t))$$

$$+ x_N^T(t) \int_0^l (P_{12}(z)(\Pi_N - I)B^T(z) + (\Pi_N - I)B(z)P_{12}^T(z)) \, dz x_N(t).$$

(57)
Lemma A.1. There exists $N^*$ such that for any $N \geq N^*$ and $t \in \mathbb{R}_+$ the solution $(u_N, x_N)$ to (52) satisfies

$$
\|x_N(t)\|^2 + \|u_N(\cdot, t)\|^2_{L^2[0, l]} \leq e^{-\gamma_0 t} \frac{\lambda_{\max}(\Pi_2)}{\lambda_{\min}(\Pi_1)} (\|x_0\|^2 + \|\varphi\|^2_{L^2[0, l]}) + \frac{\gamma_2}{\lambda_{\min}(\Pi_1)^2 \gamma_0},
$$

where $\gamma_0$ and $\gamma_2$ are some positive constants. In particular the solution $(u_N, x_N)$ exists for all $t \geq 0$. Moreover for all $N \geq N^*$ and $T > 0$ the next a priori estimate is true

$$
\int_0^T (\|u_N(\cdot, s)\|^2_{L^2[0, l]} + \|x_N(s)\|^2_{L^2[0, l]}) ds \leq \frac{\gamma_2 T}{\gamma_1} \frac{\lambda_{\max}(\Pi_2)}{\gamma_1} (\|x_0\|^2 + \|\varphi\|^2_{L^2[0, l]})
$$

for some constant $\gamma_1 > 0$.

Proof. Step by step we estimate parts of the expressions for $R_i(z, t), i = 1, 2, 3, 4$.

By the Friedrich’s inequality, taking the boundary values of $u_N$ we have

$$
-\int_0^l (\partial_z u_N(z, t))^2 dz \leq -\frac{\pi^2}{l^2} \int_0^l u_N^2(z, t) dz.
$$

(61)

Since the orthogonal projector $\Pi_N$ is self-adjoint and by the property (10) we get

$$
\int_0^l u_N(z, t) \Pi_N f(u_N(z, t)) dz = (u_N, \Pi_N f(u_N))_{L^2[0, l]} = (\Pi_N u_N, f(u_N))_{L^2[0, l]}
$$

$$
= (u_N, f(u_N))_{L^2[0, l]} \leq \|u_N(\cdot, t)\|^2_{L^2[0, l]} - \alpha \|u_N(\cdot, t)\|^2_{L^2[0, l]}.
$$

(62)
By the Cauchy-Bunyakovskiy inequality we can estimate
\[
\int_0^t u_N(z,t)D^T(z)dz \int_0^t P_{12}(z)u_N(z,t)dz \leq \|D\|_{L^2[0,l]}\|P_{12}\|_{L^2[0,l]}\|u_N(\cdot,t)\|_{L^2[0,l]}^2.
\]
(63)

Since \(B \in (L^2[0,l])^n\), for any \(\eta > 0\) there is \(N_1 = N_1(\eta) \in \mathbb{N}\) such that for any \(N \geq N_1\) we have \(\|(I - \Pi_N)B^T\|_{L^2[0,l]} < \eta\), hence
\[
\int_0^t u_N(z,t)(\Pi N - I)B^T(z)dzx_N(t) \leq \eta\|x_N(t)\|\|u_N(\cdot,t)\|_{L^2[0,l]}.
\]
(64)

By the Cauchy-Bunyakovskiy inequality, taking (12) into account we have
\[
x_N^T(t)\int_0^t (P_{12}(z)(\Pi N - I)B^T(z))dzx_N(t) \leq 2\eta\|P_{12}\|_{L^2[0,l]}\|x_N(t)\|^2.
\]
(65)

By the Cauchy-Bunyakovskiy inequality, taking (12) into account we have
\[
X^T(x_N(t))\int_0^t P_{12}(z)u_N(z,t)dz \leq \delta_2\|x_N(t)\|^{2q-1}\int_0^t \|P_{12}(z)\|\|u_N(z,t)\|dz
\]
\[
\leq \delta_2\|P_{12}\|_{L^2[0,l]}\|x_N(t)\|^{2q-1}\|u_N(\cdot,t)\|_{L^2[0,l]}.
\]

By the Young’s inequality with \(p_1 = \frac{2q}{2q-1}\), \(p_2 = 2q\) and \(\tau > 0\) we obtain
\[
\|x_N(t)\|^{2q-1}\|u_N(\cdot,t)\|_{L^2[0,l]} \leq \frac{2q-1}{2q}\tau^{2q/(2q-1)}\|x_N(t)\|^{2q} + \frac{\tau^{-2q}}{2q}\|u_N(\cdot,t)\|_{L^2[0,l]}^{2q},
\]
and taking (39) into account we further estimate
\[
X^T(x_N(t))\int_0^t P_{12}(z)u_N(z,t)dz \leq \frac{\delta_2\|P_{12}\|_{L^2[0,l]}(2q-1)}{2q}\tau^{2q/(2q-1)}\|x_N(t)\|^{2q}
\]
\[
+ \frac{\delta_2\|P_{12}\|_{L^2[0,l]}\tau^{-2q}}{2q}\|u_N(\cdot,t)\|_{L^2[0,l]}^{2q}.
\]
(66)

By the self-adjointness of \(\Pi_N\) we derive that
\[
x_N^T(t)\int_0^t P_{12}(z)\Pi_Nf(u_N(z,t))dz = (x_N^T(t)P_{12}, \Pi_Nf(u_N))_{L^2[0,l]}
\]
\[
= (\Pi_Nx_N^T(t)P_{12}, f(u_N))_{L^2[0,l]}
\]
\[
= (x_N^T(t)\Pi_NP_{12}, f(u_N))_{L^2[0,l]} = x_N^T(t)\int_0^t \Pi_NP_{12}(z)f(u_N(z,t))dz.
\]
Hence, taking (9) and (11) into account and applying the Cauchy-Bunyakovsky inequality it follows that

\[
\left| \frac{T}{N} \int_0^l \Pi_N P_{12}(z) f(u_N(z, t)) \, dz \right| \leq \|x_N(t)\| \int_0^l \|\Pi_N P_{12}(z)\| |f(u_N(z, t))| \, dz \\
\leq \|x_N(t)\| \int_0^l \|\Pi_N P_{12}(z)\| |f_0(u_N(z, t))| \, dz + \|x_N(t)\| \int_0^l \|\Pi_N P_{12}(z)\| |f_1(u_N(z, t))| \, dz \\
\leq L \|x_N(t)\| \|\Pi_N P_{12}\|_{L^2[0, l]} \|u_N(\cdot, t)\|_{L^2[0, l]} + \zeta \|x_N(t)\| \int_0^l \|\Pi_N P_{12}(z)\| \|u_N(z, t)\|^{2q-1} \, dz \\
\leq L \|x_N(t)\| \|P_{12}\|_{L^2[0, l]} \|u_N(\cdot, t)\|_{L^2[0, l]} + \zeta \int_0^l \|\Pi_N P_{12}(z)\| \|x_N(t)\| \|u_N(z, t)\|^{2q-1} \, dz.
\]

To estimate \( \|x_N(t)\| \|u_N(z, t)\|^{2q-1} \) we apply the Young’s inequality (2) with 
\( p_1 = 2q, \, p_2 = \frac{2q}{2q-1}, \, \tau > 0 \)

\[
\|x_N(t)\| \|u_N(z, t)\|^{2q-1} \leq \frac{\tau^{2q}}{2q} \|x_N(t)\|^2 + \frac{2q-1}{2q} \tau^{-2q/(2q-1)} \|u_N(z, t)\|^{2q}. \tag{67}
\]

Obviously \( \Pi_N P_{12} \in (L^1[0, l])^n \) and \( \|(I - \Pi_N)P_{12}\|_{L^1[0, l]} \leq \sqrt{1 - \|(I - \Pi_N)P_{12}\|_{L^2[0, l]}} \to 0 \) for \( N \to \infty \). Since \( P_{12} \in (L^2[0, l])^n \), for any \( \eta > 0 \) there is \( N_2 = N_2(\eta) \in \mathbb{N} \), such that \( \|\Pi_N P_{12}\|_{L^1[0, l]} \leq \|P_{12}\|_{L^1[0, l]} + \eta \). Also we have \( \Pi_N P_{12} \in (L^\infty[0, l])^n \) and we can show that \( \|(I - \Pi_N)P_{12}\|_{L^\infty[0, l]} \to 0 \) for \( N \to \infty \). Indeed, let 
\( P_{12}(z) = (P_{12}^{(1)}(z), \ldots, P_{12}^{(n)})^T \), then for \( k = 1, \ldots, n \) holds:

\[
(I - \Pi_N)P_{12}^{(k)}(z) = \sum_{p=N+1}^{\infty} (P_{12}^{(k)}, e_p)_{L^2[0, l]} e_p(z),
\]

\[
(P_{12}^{(k)}, e_p)_{L^2[0, l]} = \sqrt{\frac{2}{l}} \int_0^l P_{12}^{(k)}(z) \sin \frac{\pi p z}{l} \, dz.
\]

Taking into account that \( P_{12} \in (H_0^2[0, l])^n \) and integrating two times by parts we get

\[
(P_{12}^{(k)}, e_p)_{L^2[0, l]} = \sqrt{\frac{2}{l}} \frac{l^2}{\pi^2 p^2} \int_0^l \partial_z P_{12}^{(k)}(z) \cos \frac{\pi p z}{l} \, dz \\
= -\sqrt{\frac{2}{l}} \frac{l^2}{\pi^2 p^2} \int_0^l \partial_{zz} P_{12}^{(k)}(z) \sin \frac{\pi p z}{l} \, dz.
\]

From which follows

\[
\|(P_{12}^{(k)}, e_p)_{L^2[0, l]}\| \leq \frac{l^2}{\pi^2 p^2} \|\partial_{zz} P_{12}^{(k)}\|_{L^2[0, l]}.
\]
Hence
\[ ||(I - \Pi_N) P_{12}^{(k)}(z)|| \leq \frac{\sqrt{2}^{3/2} \|\partial_{zz} P_{12}^{(k)}\|_{L^2[0,t]}}{\pi^2} \sum_{p=N+1}^{\infty} \frac{1}{p^2}. \]
so that
\[ ||(I - \Pi_N) P_{12}||_{L^\infty[0,t]} \leq \frac{\sqrt{2}^{3/2} \|\partial_{zz} P_{12}\|_{L^2[0,t]}}{\pi^2} \sum_{p=N+1}^{\infty} \frac{1}{p^2} \to 0 \quad \text{for} \quad N \to \infty. \]

We conclude that for any \( \eta > 0 \) there exists \( N_3 = N_3(\eta) \in \mathbb{N} \) such that for all \( N \geq N_3 \) we have \( ||\Pi_N P_{12}\|_{L^\infty[0,t]} \leq ||P_{12}\|_{L^\infty[0,t]} + \eta. \)

With help of (67) we can estimate the integral
\[ \int_0^l \|\Pi_N P_{12}(z)||x_N(t)|| u_N(z,t) ||^2 q-1 \, dz \]
\[ \leq \frac{\tau^{2q}}{2q} ||\Pi_N P_{12}\|_{L^1[0,t]} ||x_N(t)||^2 q + \frac{2q - 1}{2q} \tau^{-2q/(2q-1)} ||\Pi_N P_{12}\|_{L^\infty[0,t]} ||u_N(\cdot,t)||^2 q \]
\[ \leq \frac{\tau^{2q}}{2q} (||P_{12}\|_{L^1[0,t]} + \eta) ||x_N(t)||^2 q \]
\[ + \frac{2q - 1}{2q} \tau^{-2q/(2q-1)} (||P_{12}\|_{L^\infty[0,t]} + \eta) ||u_N(\cdot,t)||^2 q. \]

Finally we obtain
\[ \left| x_N^2(t) \int_0^l \Pi_N P_{12}(z)f(u_N(z,t)) \, dz \right| \leq L ||x_N(t)|| ||P_{12}||_{L^2[0,t]} ||u_N(\cdot,t)||_{L^2[0,t]} \]
\[ + \frac{\tau^{2q}}{2q} (||P_{12}\|_{L^1[0,t]} + \eta) ||x_N(t)||^2 q \]
\[ + \frac{2q - 1}{2q} \tau^{-2q/(2q-1)} (||P_{12}\|_{L^\infty[0,t]} + \eta) ||u_N(\cdot,t)||^2 q. \]  
(68)

By the condition 2) of Theorem 5.1 it follows that
\[ x_N^2(t) PX(x_N(t)) \leq -\delta_1 ||x_N(t)||^2 q. \]  
(69)

Applying the Cauchy-Bunyakovskiy inequality we have
\[ \int_0^l u_N(z,t) \Pi_N H_t(z,t) \, dz \leq ||H_t||_{L^2[0,t]} ||u_N(\cdot,t)||_{L^2[0,t]}, \]  
(70)

\[ \left| \int_0^l H(z,t) D^T(z) \, dz \int_0^l P_{12}(z) u_N(z,t) \, dz \right| \leq ||D||_{L^2[0,t]} ||H||_{L^2[0,t]} ||P_{12}||_{L^2[0,t]} ||u_N(\cdot,t)||_{L^2[0,t]}, \]  
(71)
From the self-adjointness of $\Pi_N$ follows

$$\int_0^l u_N(z,t) \Pi_N (f(u_N(z,t) + H(z,t)) - f(u_N(z,t))) \, dz
= \int_0^l u_N(z,t) (f_0(u_N(z,t) + H(z,t)) - f_0(u_N(z,t))) \, dz
= \int_0^l u_N(z,t) (f_1(u_N(z,t) + H(z,t)) - f_1(u_N(z,t))) \, dz
+ \int_0^l u_N(z,t) (f_1(u_N(z,t) + H(z,t)) - f_1(u_N(z,t))) \, dz
\leq L \int_0^l |u_N(z,t)||H(z,t)| \, dz
+ \int_0^l |u_N(z,t)||f_1(u_N(z,t) + H(z,t)) - f_1(u_N(z,t))| \, dz$$

Using (11) and the mean value theorem there is some $\vartheta \in (0, 1)$ such that

$$|f_1(u_N(z,t) + H(z,t)) - f_1(u_N(z,t))| \leq |f'_1((1 - \vartheta)u_N(z,t) + \vartheta H(z,t))||H(z,t)|
\leq c_0(1 + ((1 - \vartheta)|u_N(z,t)| + \vartheta|H(z,t)|)^{2q-2})|H(z,t)|
\leq c_0(|H(z,t)| + |u_N(z,t)|^{2q-2}|H(z,t)| + |H(z,t)|^{2q-1}).$$

(74)
By the inequalities of Cauchy-Bunyakovsky and Young we derive that

\[
\int_0^t |u_N(z,t)||f_1(u_N(z,t) + H(z,t)) - f_1(u_N(z,t))|\,dz 
\]

\[
\leq c_0 \int_0^t |u_N(z,t)||H(z,t)| + |H(z,t)|^{2q-1}\,dz + c_0 \int_0^t |u_N(z,t)|^{2q-1}|H(z,t)|\,dz 
\]

\[
\leq \frac{c_0 \epsilon}{2} \|u_N(\cdot,t)\|_{L^2[0,t]}^2 + \frac{c_0 \epsilon^{-1}}{2} \|H(\cdot,t)\|_{L^2[0,t]} + \frac{c_0 \epsilon^{-2q}}{2q} \|H(\cdot,t)\|_{L^{2q}[0,t]}^{2q-1} 
\]

and finally we conclude

\[
\int_0^t u_N(z,t) \Pi_N (f(u_N(z,t) + H(z,t)) - f(u_N(z,t)))\,dz 
\]

\[
\leq \frac{L \epsilon}{2} \|u_N(\cdot,t)\|_{L^2[0,t]}^2 + \frac{L \epsilon^{-1}}{2} \|H(\cdot,t)\|_{L^2[0,t]} \quad \text{(75)} 
\]

\[
+ \frac{c_0 \epsilon}{2} \|u_N(\cdot,t)\|_{L^2[0,t]}^2 + \frac{c_0 \epsilon^{-1}}{2} \|H(\cdot,t)\|_{L^2[0,t]} + \frac{c_0 \epsilon^{-2q}}{2q} \|H(\cdot,t)\|_{L^{2q}[0,t]}^{2q-1} 
\]

Again we use that \( \Pi_N \) is self-adjoint to derive

\[
x_N^T(t) \int_0^t P_{12}(z) \Pi_N (f(u_N(z,t) + H(z,t)) - f(u_N(z,t)))\,dz 
\]

\[
= x_N^T(t) \int_0^t \Pi_N P_{12}(z)(f(u_N(z,t) + H(z,t)) - f(u_N(z,t)))\,dz 
\]

\[
\leq \|x_N(t)\| \int_0^t \|\Pi_N P_{12}(z)\| |f(u_N(z,t) + H(z,t)) - f(u_N(z,t))|\,dz 
\]
and taking (74) into account

\[ \|x(t)\| \int_0^t \|\Pi_t P_{12}(z)\| |f_1(u_N(z, t) + H(z, t)) - f_1(u_N(z, t))| \, dz \]

\[ \leq c_0 \|x(t)\| \int_0^t \|\Pi_t P_{12}(z)\| (|H(z, t)| + |u_N(z, t)|^{2q-2}|H(z, t)| + |H(z, t)|^{2q-1}) \, dz \]

\[ \leq c_0 \|P_{12}\|_{L^2[0, t]} \|H(\cdot, t)| + |H(\cdot, t)|^{2q-1}\|_{L^2[0, t]} \|x(t)\| + c_0 \|x(t)\| \int_0^t \|\Pi_t P_{12}(z)\| |u_N(z, t)|^{2q-2}|H(z, t)| \, dz \]

By the Young’s inequality (2) with \( p_1 = (2q - 1)/(2q - 2), p_2 = 2q - 1 \) we see that

\[ \|x(t)\| |u_N(z, t)|^{2q-2} \leq \frac{2q - 2}{2q - 1} |u_N(z, t)|^{2q-1} + \frac{1}{2q - 1} \|x(t)\|^{2q-1} \]

Hence

\[ \|x(t)\| \int_0^t \|\Pi_t P_{12}(z)\| |u_N(z, t)|^{2q-2}|H(z, t)| \, dz \]

\[ \leq \frac{2q - 2}{2q - 1} \int_0^t \|\Pi_t P_{12}(z)\| |H(z, t)| |u_N(z, t)|^{2q-1} \, dz \]

\[ + \frac{1}{2q - 1} \|x(t)\|^{2q-1} \int_0^t \|\Pi_t P_{12}(z)\| |H(z, t)| \, dz \]

Applying the Young’s inequality two times we obtain

\[ \|x(t)\|^{2q-1} \int_0^t \|\Pi_t P_{12}(z)\| |H(z, t)| \, dz \]

\[ \leq \frac{2q - 1}{2q} \|x(t)\|^{2q} \frac{1}{2q} e^{-2q} \left( \int_0^t \|\Pi_t P_{12}(z)\| |H(z, t)| \, dz \right)^{2q} \]

\[ \leq \frac{2q - 1}{2q} \|x(t)\|^{2q} \frac{1}{2q} e^{-2q} \|P_{12}\|_{L^2[0, t]}^2 \|H(\cdot, t)| + |H(\cdot, t)|^{2q-1}\|_{L^2[0, t]}^2. \]

\[ \|\Pi_t P_{12}(z)\| |H(z, t)| |u_N(z, t)|^{2q-1} \]

\[ \leq \|\Pi_t P_{12}\|_{L^2[0, t]} \left( \frac{2q - 1}{2q} e^{-2q} |u_N(z, t)|^{2q} + \frac{1}{2q} e^{-2q} |H(z, t)|^{2q} \right) \]

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and finally we get an estimate for the integral
\[
\int_0^t \| \Pi_N P_{12}(z) \| H(z, t) \| u_N(z, t) \|^{2q-1} dz
\leq \| \Pi_N P_{12} \|_{L^\infty[0, t]} \frac{2q - 1}{2q} \epsilon^{2q/(2q - 1)} \| u_N(\cdot, t) \|^{2q}_{L^2[0, t]}
+ \| \Pi_N P_{12} \|_{L^\infty[0, t]} \frac{1}{2q} \epsilon^{-2q} \| H(\cdot, t) \|^{2q}_{L^2[0, t]}.
\]
In the same way we can derive that
\[
x_N^T(t) \int_0^t P_{12}(z) \Pi_N (f_0(u_N(z, t) + H(z, t)) - f_0(u_N(z, t))) dz
\leq \frac{L}{2} \| P_{12} \|_{L^2[0, t]} \| x_N(t) \|^2 + \frac{L}{2} \| P_{12} \|_{L^2[0, t]}.
\]
Let \( N \geq N_3(\eta) \), then \( \| \Pi_N P_{12} \|_{L^\infty[0, t]} \leq \| P_{12} \|_{L^\infty[0, t]} + \eta \). Hence
\[
x_N^T(t) \int_0^t P_{12}(z) \Pi_N (f(u_N(z, t) + H(z, t)) - f(u_N(z, t))) dz
\leq \left( \frac{c_0 \| P_{12} \|_{L^2[0, t]} \| H(\cdot, t) \| + \| H(\cdot, t) \|^{2q-1}_{L^2[0, t]} \epsilon + \frac{L}{2} \| P_{12} \|_{L^2[0, t]} \right) \| x_N(t) \|^2
+ \frac{c_0 \| P_{12} \|_{L^2[0, t]} \| H(\cdot, t) \| + \| H(\cdot, t) \|^{2q-1}_{L^2[0, t]} \epsilon^{-1}}{2}
+ \frac{c_0 (q - 1)}{q} \left( \| P_{12} \|_{L^\infty[0, t]} + \eta \right) \epsilon^{2q/(2q - 1)} \| u_N(\cdot, t) \|^{2q}_{L^2[0, t]} + \frac{c_0}{2q} \epsilon^{2q/(2q - 1)} \| x_N(t) \|^{2q}
+ \| P_{12} \|_{L^2[0, t]}^{2q} \| H(\cdot, t) \|^{2q}_{L^2[0, t]} + \frac{L}{2} \| P_{12} \|_{L^2[0, t]}.
\]
Collecting the estimations \((61)\) to \((77)\) to the estimate for the time derivative of \( V(u_N(\cdot, t), x_N(t)) \) along solutions to the Galerkin system \((52)\), which holds for all \( N \geq N^*(\eta) = \max(N_1(\eta), N_2(\eta), N_3(\eta)) \):
\[
\dot{V}(u_N(\cdot, t), x_N) \leq -x_N^T(\cdot) \Omega(\eta, \epsilon) x_N(t) - \omega(\eta, \epsilon) \| u_N(\cdot, t) \|^{2q}_{L^2[0, t]}
+ (2L \| P_{12} \|_{L^2[0, t]} + \xi(\eta, \epsilon)) \| x_N(t) \| \| u_N(\cdot, t) \|_{L^2[0, t]}
- \dot{H}_1(\eta, \epsilon, \tau) \| u_N(\cdot, t) \|^{2q}_{L^2[0, t]} - \dot{H}_2(\eta, \epsilon, \tau) \| x_N(t) \|^{2q} + \dot{H}_3(\eta, \epsilon, \tau)
\]
where \( \Omega(\eta, \epsilon) \to \Omega, \omega(\eta, \epsilon) \to \omega, \xi(\eta, \epsilon) \to 0 \) for \( (\eta, \epsilon) \to (0, 0) \),
\[
\dot{H}_1(\eta, \epsilon, \tau) \to 2\alpha - \left( \frac{\delta_3 \| P_{12} \|_{L^2[0, t]}^{2q-1}}{q} \tau^{-2q} + \frac{\zeta(2q - 1)}{q} \| P_{12} \|_{L^\infty[0, t]}^{2q/(2q - 1)} \right),
\]
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\[ \tilde{H}_2(\eta, \epsilon, \tau) \rightarrow 2\delta_1 - \left( \frac{\zeta}{q} \| P_{12} \|_{L^1[0,l]} \tau^2 q + \frac{(2q - 1)\delta_2 \| P_{12} \|_{L^2[0,l]} \tau^{2q/(2q-1)}}{q} \right), \]

for \((\eta, \epsilon) \rightarrow (0, 0)\).

The condition 5) of Theorem 5.1 implies the existence of \(\tau > 0\) such that the inequalities \((13)\) hold. We fix such \(\tau > 0\) and note that by the positive definiteness of the matrix \(\Xi\) it follows that one can choose \(\eta\) and \(\epsilon\) small enough, so that the quadratic form \((78)\) becomes negative definite and \(\tilde{H}_1(\eta, \epsilon, \tau) > 0\), \(\tilde{H}_2(\eta, \epsilon, \tau) > 0\). In this case the inequality \((78)\) for \(N \geq N^*\) can be written as

\[ \dot{V}(u_N(\cdot, t), x_N(t)) \leq -\gamma_0 V(u_N(\cdot, t), x_N(t)) - \gamma_1 (\|u_N(\cdot, t)\|_{L^2}^2 + \|x_N(t)\|_{L^2}^2) + \gamma_2 \]

where \(\gamma_i > 0, i = 1, 2, 3\). From \((79)\) follows

\[ \lambda_{\text{min}}(\Pi_1)(\|x_N(t)\|^2 + \|u_N(\cdot, t)\|_{L^2}^2) \leq V(u_N(\cdot, t), x_N(t)) \]

\[ \leq e^{-\gamma_0 t} V(\varphi_N, x(0)) + \gamma_2 \gamma_0^{-1} \leq e^{-\gamma_0 t} \lambda_{\text{max}}(\Pi_2)(\|x_0\|^2 + \|\varphi\|_{L^2}^2) + \gamma_2 \gamma_0^{-1}. \]

From which the a priori estimate \((69)\) follows.

Integrating \((79)\) from 0 to \(T\) we obtain the a priori estimate \((60)\). \(\square\)

**Lemma A.2.** There exists \(N^*\) such that for all \(N \geq N^*\) and \(t \in \mathbb{R}_+\) for the solutions \((u_N, x_N)\) to the ODE system \((52)\) the following is true

(i) The mapping \(N_f : L^{2q}[0,l] \rightarrow L^{2q/(2q-1)}[0,l]\) defined by \(N_f : u \mapsto f(u(x))\) is bounded.

(ii) The linear operator \(\partial_{zz} : H_0^1[0,l] \rightarrow H^{-1}[0,l]\) is bounded.

(iii) For any \(T > 0\) the mapping \(\partial_{zz} u_N \in L^2([0,T], L^2[0,l])\) and

\[ \|u_N(\cdot, T)\|_{L^2[0,l]} + a^2 \|\partial_{zz} u_N\|_{L^2([0,T], L^2[0,l])} \leq \|\varphi\|_{L^2[0,l]} + c_2 T. \]

(iv) For any \(s \geq 1\) the sequence \(\{\partial_{zz} u_N\}_N^\infty\) is bounded in \(L^{2q/(2q-1)}([0,T], (H^s_0)^*)\).

**Proof.** (i) Note that if \(u \in L^{2q}[0,l]\), then \(f(u) \in L^{2q/(2q-1)}[0,l]\) and the mapping \(f : L^{2q}[0,l] \rightarrow L^{2q/(2q-1)}[0,l]\) is bounded, because by the Hölder’s inequality there is some constant \(c^* > 0\) such that

\[ \|f(u)\|_{L^{2q/(2q-1)}[0,l]} \leq c^* \|u\|_{L^{2q}[0,l]}. \]

(ii) The linear operator \(\partial_{zz}\) is bounded as a mapping from \(H^1[0,l]\) to \(H^{-1}[0,l]\). Indeed, for any \(\varphi \in C_0^\infty[0,l]\) the following inequality holds

\[ \sup_{\varphi \in C_0^\infty[0,l], \varphi \neq 0} \frac{\|\partial_{zz} u, \varphi\|_{H^{-1}[0,l]}}{\|\varphi\|_{H^1[0,l]}} = \sup_{\varphi \in C_0^\infty[0,l], \varphi \neq 0} \frac{\|\partial_{zz} u\|_{L^2[0,l]} \|\partial_{zz} \varphi\|_{L^2[0,l]}}{\|\varphi\|_{H^1[0,l]}}, \]

\[ \leq \sup_{\varphi \in C_0^\infty[0,l], \varphi \neq 0} \frac{\|\partial_{zz} u\|_{L^2[0,l]} \|\partial_{zz} \varphi\|_{L^2[0,l]}}{\|\varphi\|_{H^1[0,l]}}, \]
Taking (75) into account we get

\[ L \text{ weakly in } L^2(0, l). \]

Hence there is a subsequence in the latter one that converges weakly in

\[ L^2(0, l). \]

The space \( L^2(0, l) \) is proved in Lemma A.1 we obtain for some

\[ c > 0 \]

for some \( c_2 > 0, c_3 > 0 \).

Choosing \( \epsilon \) small enough and using the boundedness of \( \{ u_N \}_N \) in \( L^2[0, l] \) proved in Lemma A.1 we obtain for some \( c_2 > 0 \)

\[ \dot{V}_1(u_N(\cdot, t)) \leq -a^2 \| \partial_z u_N(\cdot, t) \|^2_{L^2[0, l]} + (\sigma + L) \| u_N(\cdot, t) \|^2_{L^2[0, l]} + \epsilon_3 \| u_N(\cdot, t) \|_{L^2[0, l]}, \]

for some \( \epsilon_2 > 0, \epsilon_3 > 0 \).

Integrating this inequality from 0 to \( T \) we obtain (iii).

(iv) Note that the sequence \( \{ u_N \}_N \) is bounded in \( L^2(0, T), H^1_0[0, l] \) \( \cap \) \( L^2/(2q-1)(0, T), L^2/(2q-1)(0, l) \), the mappings \( \partial_z u_N \) and \( f \) are bounded in \( L^2(0, T), H^{-1}[0, l] \) and \( L^2/(2q-1)(0, T), L^2/(2q-1)(0, l) \) respectively. Hence \( \partial_z u_N \) and \( f \) are bounded in \( L^2/(2q-1)(0, T), (H^s_0)^* \) for any \( s > 1 \), this means that \( \{ \partial_t u_N \}_N \) is bounded in \( L^2/(2q/2q-1)(0, T), (H^s_0)^* \).

Proof of Theorem 5.1 Since the space \( L^2(0, T), L^2[0, l] \) is a Hilbert space, its bounded subsets are compact in weak topology. By the Lemma A.1 we have the boundedness of the sequence \( \{ u_N \}_N \) in \( L^2(0, T), H^2[0, l] \), hence there is a subsequence which converges weakly to the function \( u \in L^2(0, T), L^2[0, l] \). The space \( L^2q(0, T), L^2q[0, l] \) is reflexive as well, hence its bounded subsets are weakly compact. Hence there is a subsequence in the latter one that converges weakly in \( L^2q(0, T), L^2q[0, l] \) and its limit is again \( u \).

The time derivatives of the elements \( t \) of this subsequence is bounded in \( L^2q/(2q-1)(0, T), (H^s_0)^* \) and by its reflexivity there is a subsequence \( \partial_t u_N \) that converges weakly in \( L^2q/(2q-1)(0, T), H^{-s} \) to some function \( v \) and we have \( v = \partial_t u \).
The mapping \( u \mapsto f(u + H(\cdot, t)) \) is bounded in \( L^{2q/(2q-1)}([0, T], L^{2q/(2q-1)}[0, l]) \), hence we can choose a subsequence \( u_N \) from the latter one, such that \( f(u_N(\cdot, t) + H(\cdot, t)) \rightarrow \mu \) weakly in \( L^{2q/(2q-1)}([0, T], L^{2q/(2q-1)}[0, l]) \). By Theorem 2.1 applied to \( X_0 = (H_N^0)^*, X = L^2[0, l], X_1 = L^2[0, l], p_0 = 2, p_1 = 2q/(2q - 1) \) and by Lemmas A.1,A.2, it follows that the chosen subsequence from \( \{u_N\}_{N=N_0}^\infty \) converges to \( u \) in \( L^2([0, T], L^2[0, l]) \). Taking a further subsequence (if necessary) by the Riesz theorem follows that there is a subsequence of functions \( \{u_N(\cdot, t)\}_{N=N_0}^\infty \) that converges almost everywhere on \([0, T]\) to \( u(\cdot, t) \) in \( L^2[0, l] \).

From the sequence of functions \( x_N \in C([0, T], \mathbb{R}^n) \) we select the subsequence \( \{x_N\}_{N=N_0}^\infty \) corresponding to the last chosen subsequence of \( \{u_N\}_{N=N_0}^\infty \). From (59) and the second equation of (62) it follows that there exists \( c_1 > 0 \) such that for all \( N \geq N^* \) the time derivatives of \( x_N \) are uniformly bounded: \( \sup_{t \in \mathbb{R}^+} |\dot{x}_N(t)| \leq c_1 \). By the theorem of Arzela-Ascoli follows that there is a subsequence of \( \{x_N\}_{N=N_0}^\infty \) which converges in \( C([0, T], \mathbb{R}^n) \). The final subsequence obtained from \( \{(u_N, x_N)\}_{N=N_0}^\infty \), in this way we denote by \((u_m, x_m)\). Let \( x(t) = \lim_{m \rightarrow \infty} x_m(t) \) in \( C([0, T], \mathbb{R}^n) \).

The sequence \( \{u_m\}_{m=m_0}^\infty, m_0 \geq N^* \) is a subsequence of \( \{u_N\}_{N=N_0}^\infty \), and possess all the properties mentioned above. Since \( \partial_t u_m \rightarrow \partial_t u \) weakly in \( L^{2q/(2q-1)}([0, T], H^{-s}) \) and \( u_m(\cdot, 0) \rightarrow \varphi \) in \( L^2[0, l] \), then

\[
\int_0^t \partial_t u_m(\cdot, \tau) \, d\tau + u_m(\cdot, 0) \rightarrow \int_0^t \partial_t u(\cdot, \tau) \, d\tau + u(\cdot, 0)
\]

for each \( t \in [0, T] \) weakly in \( H^{-s} \) for \( m \rightarrow \infty \). Then it follows that \( u(\cdot, t) \rightarrow u_0 \) weakly in \( H^{-s} \) fro \( t \rightarrow 0^+ \). This proves that the initial condition is satisfied.

From (62) follows

\[
\langle \partial_t u_m(\cdot, t), e_j \rangle = a^2 \langle \partial_{zz} u_m(\cdot, t), e_j \rangle + \langle f_0(u_m(\cdot, t) + H(\cdot, t)), e_j \rangle
\]

\[
+ \langle f_1(u_m(\cdot, t) + H(\cdot, t)), e_j \rangle + \langle B^T x_m(t), e_j \rangle - \langle H_t(\cdot, t), e_j \rangle. \tag{80}
\]

By the weak convergence \( u_m \rightarrow u \) for \( m \rightarrow \infty \) and definition of the generalized derivative it follows that

\[
\langle \partial_{zz} u_m(\cdot, t), e_j \rangle \rightarrow \langle u(\cdot, t), \partial_{zz} e_j \rangle = \langle \partial_{zz} u(\cdot, t), e_j \rangle
\]

for \( m \rightarrow \infty \).

Recall that \( f_0 \) as a part of \( f \) is globally Lipschitz, so that

\[
\langle f_0(u_m(\cdot, t) + H(\cdot, t)) - f_0(u(\cdot, t) + H(\cdot, t)), e_j \rangle
\]

\[
= \int_0^t (f_0(u_m(z, t) + H(z, t)) - f_0(u(z, t) + H(z, t))e_j(z) 
\]

\[
\leq L \int_0^t |u_m(z, t) - u(z, t)||e_j(z)| \, dz \leq L \|u_m(\cdot, t) - u(\cdot, t)\|_{L^2[0, l]} \rightarrow 0, \quad \text{for} \quad m \rightarrow \infty. \tag{81}
\]
Applying the integration by parts as in (5) we get
\[ \langle \partial_t u(\cdot, t), e_j \rangle = a^2 \langle \partial_{zz} u(\cdot, t), e_j \rangle + (f_0(u(\cdot, t) + H(\cdot, t)), e_j) + (\mu_1, e_j) + \langle B^T(\cdot)x(t), e_j \rangle - \langle H_t(\cdot, t), e_j \rangle \] (82)

By the completeness of the set of eigenfunctions \( \{e_j\}_{j=1}^\infty \) it follows that for almost all \( t \in [0, T] \) we have
\[ \partial_t u(z, t) = a^2 \partial_{zz} u(z, t) + f_0(u(z, t) + H(z, t)) + \mu_1 + B^T(z)x(t) - H_t(z, t). \] (83)

Hence \( \partial_t u(z, t) \) is a sum of functions from \( L^{2q/(2q-1)}([0, T], L^{2q/(2q-1)}[0, l]) \) and \( L^2([0, T], H^{-1}[0, l]) \).

We show that \( \mu_1 = f_1(u(\cdot, t) + H(t)) \). Let
\[ \varphi_m := \int_0^t \langle f_1(u_m(\cdot, s) + H(\cdot, s)) - f_1(v(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) - v(\cdot, s) \rangle \, ds \] (84)

By the above considerations \( f_1(u_m(\cdot, s) + H(\cdot, s)), f_1(v(\cdot, s) + H(\cdot, s)) \in L^{2q/(2q-1)}[0, l] \).

Applying the integration by parts as in (5) we get
\[
\begin{align*}
\int_0^t &\langle f_1(u_m(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) \rangle \, ds \\
= &\int_0^t \langle \partial_t u_m(\cdot, s) - f_0(u_m(\cdot, s) + H(\cdot, s)) - a^2 \partial_{zz} u_m(\cdot, s) \\
&\quad - \Pi_m B^T(z)x_m(s) + \Pi_m H_t(\cdot, s), u_m(\cdot, s) \rangle \, ds \\
= & \frac{1}{2} \| u_m(\cdot, t) \|^2_{L^2[0,l]} - \frac{1}{2} \| u_m(\cdot, 0) \|^2_{L^2[0,l]} + a^2 \int_0^t \| \partial_z u_m(\cdot, s) \|^2_{L^2[0,l]} \, ds \\
&\quad - \int_0^t \langle f_0(u_m(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) \rangle \, ds \quad - \int_0^t \langle B^T(z)x_m(s), u_m(\cdot, s) \rangle \, ds \\
&\quad + \int_0^t \langle H_t(\cdot, s), u_m(\cdot, s) \rangle \, ds
\end{align*}
\] (85)
Taking (86) into account we have

\[ \varpi_m = \frac{1}{2} \| u_m(\cdot,t) \|_{L^2[0,l]}^2 - \frac{1}{2} \| u_m(\cdot,0) \|_{L^2[0,l]}^2 + a^2 \int_0^t \| \partial_z u_m(\cdot,s) \|_{L^2[0,l]}^2 \, ds \]

\[ - \int_0^t \langle B^T(z)x_m(s), u_m(\cdot,s) \rangle \, ds + \int_0^t \langle H_1(\cdot,s), u_m(\cdot,s) \rangle \, ds \]

\[ - \int_0^t \langle f_1(u_m(\cdot,s) + H(\cdot,s)), v(\cdot,s) \rangle \, ds \]

\[ - \int_0^t \langle f_1(v(\cdot,s) + H(\cdot,s)), u_m(\cdot,s) - v(\cdot,s) \rangle \, ds \]

\[ - \int_0^t \langle f_0(u_m(\cdot,s) + H(\cdot,s)), u_m(\cdot,s) \rangle \, ds \]

(86)

Due to the weak convergence \( u_m(\cdot,s) \to u(\cdot,s) \) in \( L^2[0,l] \) it follows that

\[ \lim_{m \to \infty} \inf \| u_m(\cdot,t) \|_{L^2[0,l]} \geq \| u(\cdot,t) \|_{L^2[0,l]} . \]  

(87)

Similarly follows

\[ \lim_{m \to \infty} \inf \| \partial_z u_m(\cdot,t) \|_{L^2[0,l]} \geq \| \partial_z u(\cdot,t) \|_{L^2[0,l]} . \]  

(88)

Note that \( \| u_m(\cdot,0) \|_{L^2[0,l]}^2 \to \| u(\cdot,0) \|_{L^2[0,l]}^2 \). From the weak convergence \( u_m \to u \), uniform boundedness in \( L^2([0,T], L^2[0,l]) \) and by the convergence \( x_m \to x \)
in $C([0,T],\mathbb{R}^n)$ for $m \to \infty$ we get the convergence of

\[
\int_0^t \langle B^T(z)x_m(s), u_m(\cdot, s) \rangle \, ds \to \int_0^t \langle B^T(z)x(s), u(\cdot, s) \rangle \, ds,
\]

\[
\int_0^t \langle H_t(\cdot, s), u_m(\cdot, s) \rangle \, ds \to \int_0^t \langle H_t(\cdot, s), u(\cdot, s) \rangle \, ds,
\]

\[
\int_0^t \langle f_0(u_m(\cdot, s) + H(\cdot, s)), v(\cdot, s) \rangle \, ds \to \int_0^t \langle \mu, v(\cdot, s) \rangle \, ds,
\]

\[
\int_0^t \langle f_1(u_m(\cdot, s) + H(\cdot, s)), v(\cdot, s) \rangle \, ds \to \int_0^t \langle \mu, v(\cdot, s) \rangle \, ds,
\]

\[
\int_0^t \langle f_0(v(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) - v(\cdot, s) \rangle \, ds
\]

\[
\to \int_0^t \langle f_0(v(\cdot, s) + H(\cdot, s)), u(\cdot, s) - v(\cdot, s) \rangle \, ds,
\]

\[
\int_0^t \langle f_1(v(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) - v(\cdot, s) \rangle \, ds
\]

\[
\to \int_0^t \langle f_1(v(\cdot, s) + H(\cdot, s)), u(\cdot, s) - v(\cdot, s) \rangle \, ds
\]

for $m \to \infty$. Hence for $m \to \infty$ we have

\[
\int_0^t \langle f_0(u_m(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) \rangle \, ds \to \int_0^t \langle f_0(u(\cdot, s) + H(\cdot, s)), u(\cdot, s) \rangle \, ds.
\]

From the other side we have

\[
\varpi_m = \int_0^t \langle f_1(u_m(\cdot, s) + H(\cdot, s)) - f_1(v(\cdot, s) + H(\cdot, s)), u_m(\cdot, s) - v(\cdot, s) \rangle \, ds \quad (89)
\]

by the assumptions for $f_1$, in particular that $f_1'(s) < 0$ for all $s \in \mathbb{R}$, $s \neq 0$ we
Using (5) with \( u \)
the following estimation holds

\[
0 \geq \varpi_m \geq \frac{1}{2} \left\| u(\cdot, t) \right\|_{L^2[0, l]}^2 - \frac{1}{2} \left\| u(\cdot, 0) \right\|_{L^2[0, l]}^2 - a^2 \int_0^t \left\| \partial_x u(\cdot, s) \right\|_{L^2[0, l]}^2 ds
\]

\[
- \int_0^t \langle B^T(z)x(s), u(\cdot, s) \rangle ds + \int_0^t \langle H_t(\cdot, s), u(\cdot, s) \rangle ds
\]

\[
- \int_0^t \langle \mu_1, v(\cdot, s) \rangle ds - \int_0^t \langle f_1(v(\cdot, s) + H(\cdot, s)), u(\cdot, s) - v(\cdot, s) \rangle ds
\]

\[
- \int_0^t \langle f_0(u(\cdot, s) + H(\cdot, s)), u(\cdot, s) \rangle ds
\]

(90)

Using (9) with \( u = v \) from (83)
 follows

\[
\frac{1}{2} \left\| u(\cdot, t) \right\|_{L^2[0, l]}^2 - \frac{1}{2} \left\| u(\cdot, 0) \right\|_{L^2[0, l]}^2 = -a^2 \int_0^t \left\| \partial_x u(\cdot, s) \right\|_{L^2[0, l]}^2 ds
\]

\[
+ \int_0^t \langle f_0(u(\cdot, s) + H(\cdot, s)), u(\cdot, s) \rangle ds + \int_0^t \langle \mu_1, u(\cdot, s) \rangle ds
\]

\[
+ \int_0^t \langle B^T(z)x(t), u(\cdot, s) \rangle ds - \int_0^t \langle H_t(z, t), u(\cdot, s) \rangle ds
\]

Comparing this equality with (91)
we conclude that

\[
\int_0^t \langle \mu_1 - f_1(v(\cdot, s) + H(\cdot, s)), u(\cdot, s) - v(\cdot, s) \rangle ds \leq 0
\]

(91)

for all \( v \in L^{2q}([0, T], L^{2q}[0, l]) \). Let \( v = u - \nu w, \nu \in [0, 1], \) for some \( \nu \in (0, 1) \)
the following estimation holds

\[
|f_1(u + \nu w + H) - f_1(u + H)| \leq |f_1^\prime(u + \nu \vartheta w + H)|\nu|w|
\leq c_0(1 + |u + \nu \vartheta w + H|^{2q-2})\nu|w|
\leq c_0(1 + 3^{2q-3}(|u|^{2q-2} + |w|^{2q-2} + |H|^{2q-2}))\nu|w|
\leq \nu(c_0 + 3^{2q-2}c_0 \sup_{(z,s)\in [0,l] \times [0,T]} |H(z,s)|^{2q-2})|w|
\]

\[
+ 3^{2q-3}c_0 \nu|u|^{2q-2}|w| + 3^{2q-3}c_0 \nu|w|^{2q-1}
\]

\[
:= \nu(\kappa_1|w| + \kappa_2|u|^{2q-2}|w| + \kappa_3|w|^{2q-1}).
\]

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Let \( w \in L^{2q}([0, T], L^{2q}[0, l]) \). Taking (59) and (60) into account we obtain
\[
\int_0^t \langle f_1(u(\cdot, s) + \nu w(\cdot, s) + H(\cdot, s) - f_1(u(\cdot, s) + H(\cdot, s), w(\cdot, s)) \rangle \, ds
\]
\[
= \nu \int_0^t \left( \kappa_1 |w(z, s)|^2 + \kappa_2 |u(z, s)|^{2q-2} |w(z, s)|^2 + \kappa_3 |w(z, s)|^{2q} \right) dz \, ds
\]
\[
\leq \nu(\kappa_1 \|w\|_{L^2([0, T], L^2[0, l])}^2 + (\kappa_3 + \frac{\kappa_2}{q}) \|w\|_{L^{2q}([0, T], L^{2q}[0, l])}^{2q}
+ \frac{\kappa_2(q-1)}{q} \|u\|_{L^{2q}([0, T], L^{2q}[0, l])}^{2q}) \to 0 \quad \text{for} \quad \nu \to 0.
\]
We substitute \( v = u - \nu w, \nu \in (0, 1] \) into (91) and get
\[
\int_0^t \langle \mu_1 - f_1(u(\cdot, s) + H(\cdot, s) - \nu w(\cdot, s), w(\cdot, s) \rangle \, ds \leq 0. \quad (92)
\]
Taking the limit for \( \nu \to 0 \) it follows that
\[
\int_0^t \langle \mu_1 - f_1(u(\cdot, s) + H(\cdot, s), w(\cdot, s) \rangle \, ds \leq 0.
\]
Since \( w \) is arbitrary it follows that
\[
\int_0^t \langle \mu_1 - f_1(u(\cdot, s) + H(\cdot, s), w(\cdot, s) \rangle \, ds = 0.
\]
and \( \mu_1 = f_1(u(\cdot, t) + H(\cdot, t). \) From Lemma \[A.1\] and the equality
\[
\dot{x}_m(t) = Cx_m(t) + X(x_m(t)) + \int_0^t D(z)u_m(z, t) \, dz + \int_0^t D(z)H(z, t) \, dz
\]
we conclude the uniform convergence of \( \{\dot{x}_m(t)\}_{m=0}^\infty \) on \([0, T]\). Hence in the last equality we can take the limit for \( m \to \infty \) which finishes the proof of Theorem 5.1. \hfill \Box

B Calculation of \( \dot{V} \) in the proof of Theorem 5.3

Here we calculate the full time derivative of the Lyapunov functional
\[
V(v(\cdot, t), x) = \|v(\cdot, t)\|_{L^2([0, T])}^2 + 2x^T \int_0^T P_{12}(z)v(z, t) \, dz + x^TPx
\]
as follows. We note that by Lemma 2.2 from [6] we have
\[
\frac{d}{dt}\|v(\cdot,t)\|_{L^2(0,l)}^2 = 2\langle \partial_t v, v \rangle,
\]
and similarly
\[
\frac{d}{dt}\int_0^l P_{12}(z)v(z,t)\,dz = 2\langle \partial_t v, P_{12} \rangle,
\]
so that
\[
\dot{V}(v(\cdot,t), x) = 2\langle \partial_t v(\cdot,t), v(\cdot,t) \rangle + 2\int_0^l (C x + X(x) + \int_0^l D(\xi)v(\xi,t)\,d\xi + p(t))^T P_{12}(z)v(z,t)\,dz
\]
\[
+ 2\int_0^l (\partial_z v)\,\partial_z P_{12} + 2x^T P(C x + X(x) + \int_0^l D(z)v(z,t)\,dz + p(t))
\]
\[
= 2a^2 \langle \partial_{zz} v, v(\cdot,t) \rangle + 2\int_0^l f(v(z,t))v(z,t)\,dz + 2\int_0^l B^T(z)xv(z,t)\,dz
\]
\[
+ 2\int_0^l g(z,t)v(z,t)\,dz + 2x^T \int_0^l C^T P_{12}(z)v(z,t)\,dz + 2X^T(x) \int_0^l P_{12}(z)v(z,t)\,dz
\]
\[
+ 2\int_0^l \left( \int_0^l v(\xi,t)D^T(\xi)\,d\xi \right) P_{12}(z)v(z,t)\,dz
\]
\[
+ 2p^T(t) \int_0^l P_{12}(z)v(z,t)\,dz + 2x^T(a^2\partial_{zz} v, P_{12})
\]
\[
+ 2x^T \int_0^l P_{12}(z)(f(v(z,t)) + B^T(z)x + g(z,t))\,dz
\]
\[
+ x^T(C^T P + PC)x + 2x^T PX(x) + 2x^T P \int_0^l D(z)v(z,t)\,dz + 2x^T P p(t)
\]
By the definition of the generalized function derivative and taking the boundary conditions \(P_{12}(0) = P_{12}(l) = 0\) into account we have
\[
\langle \partial_{zz} v, v(\cdot,t) \rangle = -\langle \partial_z v, \partial_z v \rangle = -\|\partial_z v(\cdot,t)\|_{L^2(0,l)}^2,
\]
\[
\langle \partial_{zz} v, P_{12} \rangle = \langle v, \partial_z P_{12} \rangle = \int_0^l v(z,t)\partial_z P_{12}(z)\,dz.
\]
Hence
\[
\dot{V}(v(t), x) = -2\|\partial_z v\|_{L\infty[0,l]}^2 + 2 \int_0^l f(v(z,t))v(z,t) dz \\
+ 2 \int_0^l P_{12}(z)v(z,t) dz \int_0^l v(\xi,t)D^T(\xi) \, d\xi \\
+ x^T(C^T P + PC)x + 2x^T \int_0^l P_{12}(z)B^T(z) \, dz x + 2x^T PX(x) \\
+ 2 \int_0^l g(z,t)v(z,t) \, dz \\
+ 2x^T Pp(t) + 2x^T \int_0^l P_{12}(z)g(z,t) \, dz + 2p^T(t) \int_0^l P_{12}(z)v(z,t) \, dz \\
+ 2x^T \int_0^l P_{12}(z)v(z,t) \, dz + 2x^T \int_0^l P_{12}(z)f(v(z,t)) \, dz + 2 \int_0^l B^T(z)xv(z,t) \, dz \\
+ 2x^T \int_0^l C^T P_{12}(z)v(z,t) \, dz \\
+ 2x^T \int_0^l v(z,t)a^2 \partial_{zz}P_{12}(z) \, dz + 2x^T P \int_0^l D(z)v(z,t) \, dz \\
= W_1 + W_2 + W_3 + W_4 \\
+ 2x^T \int_0^l (a^2 P''_{12}(z) + C^T P_{12}(z) + B(z) + PD(z))v(z,t) \, dz
\]
By the choice of \( P_{12}(z) \) the last integral vanishes (see (25)).

C \hspace{1em} \textbf{Proof of Corollary 5.6}

\textit{Proof.} Assume that \((u_i, x_i) \in L^2(\mathbb{R}^+, L^\infty[0,l]) \cap L^{2q}(\mathbb{R}^+, L^{2q}[0,l]) \in C(\mathbb{R}^+, \mathbb{R}^n)\), \(i = 1, 2\) satisfy (10) – (13). Then \( w := u_2 - u_1, x := x_2 - x_1 \) solve the following problem
\[
w_1(z,t) = a^2 w_{zz}(z,t) + f(w(z,t) + u_1(z,t)) - f(u_1(z,t)) + B^T(z)x(t), \\
\dot{x}(t) = Cx(t) + X(x(t) + x_1(t)) - X(x_1(t)) + \int_0^t D(z)w(z,t) \, dz,
\]
subject to initial conditions

\[ x(0) = 0 \in \mathbb{R}^n, \quad w(z, 0) = 0, \quad z \in (0, l), \quad t \in (0, +\infty) \]

and boundary conditions

\[ w(0, t) = 0, \quad w(l, t) = 0. \]

From Theorem follows \( \sup_{t \in \mathbb{R}_+} |x_1(t)| < \infty, \sup_{t \in \mathbb{R}_+} \|u_1(\cdot, t)\|_{L^\infty[0, l]} < \infty, \)
hence for some constant \( L_1 > 0 \) we have

\[ \| X(x(t) + x_1(t)) - X(x_1(t)) \| \leq L_1 \| x(t) \|. \]

We introduce a vector Lyapunov function with components defined by

\[ V_1(w(\cdot, t)) = \frac{1}{2}\|w(\cdot, t)\|_{L^2[0, l]}^2 \quad V_2(x(t)) = \frac{1}{2}\|x(t)\|^2 \]

so that

\[
\dot{V}_1(w(\cdot, t)) \leq -\frac{2\pi^2a^2}{l^2}V_1(w(\cdot, t)) + \int_0^l \int f'(u_1(z, t) + \vartheta w(z, t))w^2(z, t) \, dz \\
+ \|B\|_{L^2[0, l]}\|x(t)\|\|w(\cdot, t)\|_{L^2[0, l]} \\
\leq (-\frac{2\pi^2a^2}{l^2} + \|B\|_{L^2[0, l]}V_1(w(\cdot, t)) + \|B\|_{L^2[0, l]}V_2(x(t)),
\]

\[
\dot{V}_2(x(t)) \leq \|C\|V_2(x(t)) + 2L_2V_2(x(t)) + \|D\|_{L^2[0, l]}\|x(t)\|\|w(\cdot, t)\|_{L^2[0, l]} \\
\leq (2\|C\| + 2L_2 + \|D\|_{L^2[0, l]}V_2(x(t)) + \|D\|_{L^2[0, l]}V_1(w(\cdot, t)),
\]

for some \( \vartheta \in (0, 1) \), which implies that for all \( t \geq 0 \)

\[
0 \leq 0 \leq e^{Mt} \begin{pmatrix} V_1(w(\cdot, t)) \\ V_2(x(t)) \end{pmatrix} \leq e^{Mt} \begin{pmatrix} V_1(w(\cdot, 0)) \\ V_2(x(0)) \end{pmatrix} = 0,
\]

with some constant matrix \( M \). Hence \( w(\cdot, t) = 0, x(t) = 0 \) for a.a. \( t \), which proves the lemma.

\[ \square \]

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