A novel method to test atomic quantum correlations

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Abstract. The use of continuous feedback control of the center of mass of a 1D trapped bosonic gas for probing atom-atom correlations is studied. Based on the Schwarz inequality an experimentally observable criterium for distinguishing between classical and truly quantum correlations is introduced. It is shown that thermal equilibrium states, including the case $T = 0$, will exhibit purely classical correlations.

1. Introduction
After the famous Hanburry-Brown Twiss experiment \cite{1} the issue of two-photon correlations attracted considerable interest. Measuring this quantity allows for distinguishing a truly coherent light from a filtered chaotic one \cite{2}. Moreover, the study of such correlations has allowed for the observation of specific non-classical states of optical fields, exhibiting so-called photon antibunching \cite{3–8}.

An analogous property can be defined for an atomic gas. Using the second quantization formalism such correlations can be written as

$$C(r, r') = \langle \hat{\phi}^\dagger(r) \hat{\phi}^\dagger(r') \hat{\phi}(r') \hat{\phi}(r) \rangle, \quad (1)$$

where the field (bosonic or fermionic) operators $\hat{\phi}^\dagger(r)$ and $\hat{\phi}(r)$ describe the creation and annihilation of an atom at the position $r$, respectively. Similar to photonic fields, certain types of peculiar many-atom quantum states can be detected via the measurement of such correlations (see for example Refs \cite{9, 10}). In particular, it has been proposed and experimentally verified, that the Mott-insulator state of atom trapped in an optical lattice can be tested via the measurement of correlations of this type.

To the best of our knowledge, there are two basic approaches used for detecting atomic correlations of this type: The first is based on the idea of the Hanburry-Brown Twiss experiment \cite{10–12}, whereas the second employs the fact that the correlations influence the rate of three-body recombinations and consequently can be measured via trap-loss detection \cite{13–15}.

However, as will be shown in this paper there is an alternative way to detect atomic correlations. The main idea is to drive the system in such a way as to map atom-atom correlations into an easily accessible single-atom property of the system. We propose to realize this mapping applying a linear negative feedback to the center-of-mass (CM) coordinate of the atomic gas. Such a feedback control stabilizes the CM of the gas and, as will be shown below, simultaneously alters the dynamics in such a way as to map the atom-atom correlations onto the spatial extension of the atomic cloud.
2. Continuous quantum feedback

2.1. The model

To demonstrate the feasibility of the method we consider a gas of $N$ non-interacting bosons of mass $m$, being trapped in a one dimensional harmonic potential of frequency $\omega$. The CM of these atoms, $\hat{X}$, is subject to continuous feedback control. It is measured with resolution $\sigma_0$ and with measurement outcomes being described by the trajectory $X(t)$. Given a measurement outcome $X(t)$ at time $t$, the CM is then shifted towards a target value $X_0$:

$$\hat{X}(t) \to \hat{X}(t) - \zeta_0 [X(t) - X_0],$$

where $\zeta_0$ characterizes the relative shift of the control action. Without loss of generality the target position may be chosen in the origin of the coordinate frame, so that $X_0 = 0$.

Mathematically to have a non-singular evolution of the system in the continuous limit, i.e. for the feedback rate $\gamma \to \infty$, the measurement resolution $\sigma_0$ and the feedback relative shift $\zeta_0$ should obey

$$\frac{\sigma_0^2}{\gamma} = \sigma^2 = \text{const}$$

$$\zeta_0 \gamma = \zeta = \text{const.}$$

This defines the physical parameters $\sigma$ and $\zeta$ which are the rms time-integrated measurement resolution and the relative feedback shift rate, respectively.

The time evolution of the density operator of the $N$ atoms, $\hat{\rho}_N(t)$, is then governed by the following master equation [16–18]

$$\partial_t \hat{\rho}_N = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}_N] + \frac{i \zeta}{2\hbar} [\hat{P}, \{\hat{X}, \hat{\rho}_N\}] - \frac{(\zeta \sigma)^2}{2\hbar^2} [\hat{P}, [\hat{P}, \hat{\rho}_N]],$$

where the operators $\hat{X}$ and $\hat{P}$ are the CM coordinate and momentum, respectively. They are defined as

$$\hat{X} = \frac{1}{N} \int dx \hat{\phi}^\dagger(x) x \hat{\phi}(x), \quad \hat{P} = -i\hbar \int dx \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x).$$

The Hamiltonian operator $\hat{H}$ of the ideal Bose gas in the trap is given by

$$\hat{H} = \int dx \hat{\phi}^\dagger(x) \left[ \frac{\hbar^2}{2m} \partial_x^2 + \frac{m\omega^2}{2} x^2 \right] \hat{\phi}(x).$$

An experimental implementation of the measurement of the CM position may be based on a non-resonant interaction of the trapped atoms with spatially varying optical fields, such as a cavity field as in Ref. [19] or optical lattices as in Ref. [20]. The shift of the coordinate is equivalent to the shift of the trapping potential, which may be realized changing magnetic or optical fields used in a trap. Since the oscillation of atoms in the trap is typically very much slower than the response time of modern electronic circuits, the feedback loop may be considered as acting instantaneously, as has been asserted in the derivation of Eq. (2).

2.2. Single-atom dynamics

Under the action of the feedback, the system’s atom-atom correlations will be mapped into single-atom properties. To show this, let us consider the single-atom density matrix, which describes all possible single-atom properties. It is defined as

$$\rho(x, x', t) = \langle \hat{\phi}^\dagger(x') \hat{\phi}(x) \rangle_t,$$
where \( \langle \ldots \rangle_t = \text{Tr} [ \ldots \hat{\varrho}_N(t) ] \). The evolution of this quantity is governed by atomic correlations of higher order. Formally this happens due to the presence of products of collective observables \( \hat{X} \) and \( \hat{P} \) in Eq. (2). These products, being composed of four matter-field operators, represent the influence of the many-atom correlations on the system’s dynamics in a way similar to the case of interacting atoms. An analytic solution of Eq. (2) for the total density operator is generally not possible due to the large number of degrees of freedom associated with the atomic gas. Nevertheless, as has been recently shown [21,22], analytic solutions for the single-atom density matrix still can be found.

For this purpose, instead of the single-atom density matrix the joint Wigner function of single atom (variables \( x, p \)) and center of mass of the other \( N - 1 \) atoms (variables \( X, P \)) should be considered:

\[
W(x, p; X, P, t) = (2\pi\hbar)^{-3} \int dx' e^{-ix'p/\hbar} \int dX' \int dP' \times \left\langle \hat{\phi}^\dagger(x - x'/2) e^{i[(\hat{P} - P)X' + (\hat{X} - X)P']/\hbar} \hat{\phi}(x + x'/2) \right\rangle_t.
\]

From Eq. (2) a closed Fokker-Planck equation for this distribution function can be obtained. It is of linear type with positive semi-definite diffusion matrix and thus allows for analytic solutions with a bound analytic Green function of Gaussian shape [23]. Then the analytic solution for the single-atom density matrix is obtained by integrating the analytic solution for (6) over the macroscopic phase-space variables \( X, P \):

\[
\rho(x + x', x - x', t) = \int dX \int dP \int dp W(x, p; X, P, t) e^{2ipx'}.
\]

2.3. Size of the atomic cloud

For simplicity we consider here the rms variance of the atomic density, which describes the quantum-statistically averaged temporal evolution of the extension of the atomic cloud. It is defined as

\[
\Delta x(t) = \left\{ \int \frac{dx}{N} x^2 \rho(x, x, t) - \left[ \int \frac{dx}{N} x \rho(x, x, t) \right]^2 \right\}^{1/2},
\]

and can be obtained from the analytic solution that has been outlined above. One finds that this rms variance exponentially converges at rate \( \zeta/2 \) to the asymptotic behavior

\[
\Delta x_a(t) = \sqrt{\Delta X_s^2 + \sigma_q^2(t)},
\]

where the first term in the square root is given by the stationary rms variance of the center of mass

\[
\Delta X_s = \delta X_0 \sqrt{(\eta + \eta^{-1})}/2.
\]

Here \( \delta X_0 \) is rms variance of the CM in the ground state of the trapping potential

\[
\delta X_0 = \sqrt{\hbar/(2Nm\omega)},
\]

and thus represents the standard quantum limit (SQL) for the CM coordinate. Furthermore, the parameter

\[
\eta = \delta X_0^2/(\zeta\sigma^2)
\]

denotes the ratio of spatial localization due to the potential over that due to the feedback.
The second term in Eq. (9) is, in general, explicitly time dependent and reads

\[ \sigma_q^2(t) = \frac{d}{dN} \left\langle \hat{\phi}(x) \left[ q(x, \partial_x, t) \right]^2 \hat{\phi}(x) \right\rangle_0 \]

\[ - \frac{d}{dN} \int \frac{dx}{N} \int \frac{dx'}{N} \left\langle \hat{\phi}(x) q(x, \partial_x, t) \hat{\phi}(x) \hat{\phi}(x') q(x', \partial_{x'}, t) \hat{\phi}(x') \right\rangle_0, \]

where the expectation value is performed over the initial \( N \)-atom density matrix \( \hat{\rho}_N(0) \). The time dependence enters via the single-atom quadrature, defined as

\[ q(x, \partial_x, t) = x \cos(\omega t) - i \frac{\hbar}{m\omega} \partial_x \sin(\omega t). \]

From Eq. (13) it is explicitly seen that atom-atom correlations appear in the single-atom quantity \( \Delta x_a(t) \).

Thus, the measurement of the width of the atomic density profile does yield certain information on the atom-atom correlations.

3. Classical vs quantum correlations

3.1. Schwarz inequality

Consider the classical limit of Eq. (13), which can be formulated classically as follows

\[ \sigma_q^2(t) = \frac{1}{N} \sum_{i=1}^{N} q_i^2(t) - \frac{1}{N^2} \sum_{i,j=1}^{N} q_i(t)q_j(t), \]

where \( q_i(t) \) is the classical counterpart of the quadrature of the \( i \)th atom and the averaging is performed over the corresponding classical phase space. A lower bound for Eq. (15) is obtained as

\[ \sigma_q^2(t) \geq \frac{1}{N} \sum_{i=1}^{N} q_i^2(t) - \frac{1}{N^2} \left| \sum_{i,j=1}^{N} q_i(t)q_j(t) \right| \]

\[ \geq \frac{1}{N} \sum_{i=1}^{N} q_i^2(t) - \frac{1}{N^2} \sum_{i,j=1}^{N} q_i(t)q_j(t) \]

Being classical correlations, they obey the Schwarz inequality

\[ \left| \sum_{i,j=1}^{N} q_i(t)q_j(t) \right|^2 \leq \sum_{i=1}^{N} q_i^2(t) \sum_{j=1}^{N} q_j^2(t), \]

which can be used to obtain the lower bound

\[ \sigma_q^2(t) \geq \frac{1}{N} \sum_{i=1}^{N} q_i^2(t) - \frac{1}{N^2} \sum_{i,j=1}^{N} \left[ q_i^2(t) q_j^2(t) \right]^{1/2} = 0. \]

Applying this result to Eq. (9) leads to a classical inequality for the size of the atomic cloud:

\[ \Delta x_a(t) \geq \Delta x_s. \]

A violation of the Schwarz inequality would indicate true quantum correlations between atoms as opposed to classical ones, since then the expectation value in Eq. (13) cannot be described by integration over a proper probability density. Given that \( \Delta x_s \) is a known quantity determined solely by the feedback and trap parameters [cf. Eq. (10)], a violation of (19) can be experimentally probed for by measuring the spatial extension of the atomic cloud over a half period of the trap oscillation. Measurements of this kind have been implemented with various techniques [24–26].
3.2. Thermal equilibrium state

The question arises of what type of many-atom quantum states will exhibit a violation of the Schwarz inequality in form of Eq. (19). Since the underlying correlations are similar in form to those of photon antibunching in the optical case, one may conclude that thermal states will not violate (19) but behave classically. To prove this let us calculate \( \sigma^2_q(t) \) for an initial thermal equilibrium state at temperature \( T \), given by the density operator

\[
\hat{\rho}_N = Z^{-1} \exp\left[-\beta(\hat{H} - \mu \hat{N})\right], \quad Z = \text{Tr} \exp\left[-\beta(\hat{H} - \mu \hat{N})\right],
\]

where \( \mu \) is the chemical potential and \( \beta = (k_B T)^{-1} \) with \( k_B \) being the Boltzmann constant.

Having the system in a thermal state one can apply the generalized Wick theorem (see for example Ref. [27]) to rewrite the four-field expectation value in the second integral of Eq. (13) as

\[
\langle \hat{\phi}^\dagger(x) q \hat{\phi}(x) \hat{\phi}^\dagger(x') q' \hat{\phi}(x') \rangle = \langle \hat{\phi}^\dagger(x) q \hat{\phi}(x) \rangle \langle \hat{\phi}^\dagger(x') q' \hat{\phi}(x') \rangle + \langle \hat{\phi}^\dagger(x) q' \hat{\phi}(x') \rangle \langle q \hat{\phi}(x) \hat{\phi}^\dagger(x') \rangle,
\]

where \( q = q(x, \partial_x, t) \) and \( q' = q(x', \partial_{x'}, t) \). After a partial integration the expression for \( \sigma^2_q(t) \) can then be written as

\[
\sigma^2_q(t) \big|_{\text{th}} = \frac{N-1}{N^2} \int dx \langle \hat{\phi}^\dagger(x) [q(x, \partial_x, t)]^2 \hat{\phi}(x) \rangle - \left[ \frac{1}{N} \int dx \langle \hat{\phi}^\dagger(x) q(x, \partial_x, t) \hat{\phi}(x) \rangle \right]^2 - \frac{1}{N^2} \int dx \int dx' \langle \hat{\phi}^\dagger(x) q(x', \partial_{x'}, t) \hat{\phi}(x') \rangle \langle \hat{\phi}^\dagger(x') q(x, \partial_x, t) \hat{\phi}(x) \rangle.
\]

Note that since for a thermal state the averaged CM coordinate and momentum are zero, the second term in this equation will vanish.

With the help of the normalized modes of the harmonic oscillator, \( \psi_n(x) \), the single-atom density matrix for the thermal equilibrium state can be written as

\[
\langle \hat{\phi}^\dagger(x) \hat{\phi}(x') \rangle = \sum_{k=0}^{\infty} \psi_k^*(x) \psi_k(x') N_k,
\]

where the Bose-Einstein distribution for the \( N \) atoms is given by

\[
N_k = \left(e^{\beta \epsilon_k - \mu} - 1\right)^{-1},
\]

Here \( \epsilon_k = \hbar \omega (k + \frac{1}{2}) \) are the harmonic-oscillator eigenenergies and the chemical potential \( \mu(N) \) is determined by the relation

\[
\sum_{k=0}^{\infty} N_k = N.
\]

Using the expression (23) the first integral in Eq. (22) can be rewritten as

\[
\frac{N-1}{N^2} \sum_k N_k \left\{ (x^2)_k \cos^2(\omega t) + \frac{\langle p^2 \rangle_k}{m \omega^2} \sin^2(\omega t) + \frac{\{x, p\}_k}{m \omega} \sin(\omega t) \cos(\omega t) \right\},
\]

where the coefficients read

\[
(x^2)_k = \int dx \psi_k^*(x) x^2 \psi_k(x) = \frac{\hbar}{m \omega} (k + \frac{1}{2}),
\]

\[
\langle p^2 \rangle_k = -\hbar^2 \int dx \psi_k^*(x) \partial_x^2 \psi_k(x) = \hbar m \omega (k + \frac{1}{2}),
\]

\[
\{x, p\}_k = -i \hbar \int dx \psi_k^*(x) (\partial_x x \partial_x + \partial_x x) \psi_k(x) = 0.
\]
Inserting Eqs (27) – (29) into Eq. (26) we thus obtain

$$\delta X_0^2 \frac{N-1}{N} \sum_k (2k+1) N_k,$$

(30)

with $\delta X_0$ being the CM SQL of Eq. (10). Consider now the last term in Eq. (22). With the help of Eq. (23) it can be rewritten after some straightforward calculation as

$$- \frac{\delta X_0^2}{N} \sum_{k,l} N_k N_l \left[ k \delta l,k - 1 + (k+1) \delta l,k+1 \right].$$

(31)

which results as

$$- \frac{\delta X_0^2}{N} \sum_k N_k \left[ k N_k - 1 + (k+1) N_{k+1} \right].$$

(32)

Combining Eqs (30) with (32) one obtains the following result for correlation part of the atomic-cloud size

$$\sigma_q^2(t) |_{th} = \delta X_0^2 \sum_k \frac{N_k}{N} \left[ (2k+1)(N-1) - k N_{k-1} - (k+1) N_{k+1} \right].$$

(33)

Let us analyze this result and find out whether it can become negative, i.e. whether it can violate the Schwarz inequality. The expression inside the square brackets can be written as

$$k \left[ 2(N-1) - N_{k-1} - N_{k+1} \right] + (N-1) - N_{k+1}.$$

(34)

The bracket in Eq. (34) is always non-negative for $N \geq 2$, i.e. for a many-atom system with at least two atoms. For the one-dimensional system considered here, the $N_k$ are monotonically decreasing with $k$. Thus $N_{k+1} \leq N/2 (k = 0, 1, \ldots)$ which means that also the combination of the remaining two terms in Eq. (34) is non-negative for $N \geq 2$.

Thus in thermal equilibrium $\sigma_q^2(t) |_{th} \geq 0$, which means that a thermal state of a 1D bosonic gas exhibits purely classical correlations with respect to the Schwarz inequality. Moreover, as can be seen from Eq. (33) the asymptotic behavior of the size of the atomic cloud does not depend on time in this case.

Note that this also holds for $T = 0$, i.e. for a condensed state. In this case the size of the atomic cloud after the application of feedback will reach the asymptotic (and stationary) value

$$\Delta x_a |_{T=0} = \sqrt{\Delta X_0^2 + \delta X_0^2 (N-1)}.$$

(35)

The minimum stationary noise in the CM is $\min(\Delta X_s) = \delta X_0$. In this case the stationary cloud size (35) also reaches its minimum value of $\min[\Delta x_a(t) |_{T=0}] = \delta X_0 \sqrt{N} = \sqrt{\hbar/(2m \omega)}$, which is exactly the rms variance of a single atom in the ground state of the trap potential. This minimum noise is obtained for $\eta = 1$, meaning that the CM localization due to the feedback must match that due to the trapping potential.

4. Summary

In conclusion, we have presented an alternative approach to detect atom-atom correlations in a 1D bosonic gas. The method is based on the application of continuous feedback control of the CM coordinate of the atoms. It has been shown, that this control results in a mapping of the correlations into the easily measurable spatial extension of the cloud. Moreover, based on the classical Schwarz inequality a criterium to distinguish between classical and quantum correlations in the many-atom quantum state has been introduced. For a thermal equilibrium state it has been shown that only classical correlations prevail, so that a violation of the Schwarz inequality can only be expected for non-equilibrium quantum states.
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