On the descent algebra of type $D$  
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1 Introduction

Given a Coxeter group, $W$, we can construct an algebra - the descent algebra - which is a sub-algebra of the group algebra $Q[W]$. These were introduced in 1976 by Louis Solomon [Sol76], however, his description, in terms of double coset representatives and parabolic subgroups of $W$, although ingenious, made them, in practice, difficult to handle.

Consequently, little advance was made in this area until a matrix interpretation for multiplying together basis elements of the descent algebra of the symmetric groups was found, for example [GR89]. This led to a revival of interest, and much development in the subject, [Atk92], [BGR92], [BBHT92], including a matrix interpretation by François and Nantel Bergeron for multiplication in the descent algebra of the hyperoctahedral groups, [BB92].

Until now, there has been little success in developing such an interpretation for the Coxeter groups of type $D$. However, in this paper we shall reveal the rule for this remaining Coxeter family.

The Coxeter group, $D_n$, is the group whose involutory generators are the set $S = \{1',1,2,\ldots,n-1\}$, and whose relations are given by the following diagram:

$$
\begin{array}{c}
1 \\
\vdots \\
n-1
\end{array}
\quad
1' \quad 2 \quad 3 \\
\vdots
$$

where an edge between nodes $i$ and $j$ gives us the relation $(ij)^3 = 1$, and no edge gives $(ij)^2 = 1$.

Solomon proved that if $J$ is a subset of $S$, and $W_J$ is the subgroup generated by $J$, then if $X_J (X_J^{-1})$ is the set of unique left (right) coset representatives of minimal length of $W_J$, and $x_J$ is the formal sum of the elements in $X_J$ then for $J,K,L \subseteq S$,

$$
x_J x_K = \sum a_{JKL} x_L
$$

where $a_{JKL}$ is the number of elements $x \in X_J^{-1} \cap X_K$ such that $x^{-1} J x \cap K = L$. Moreover, the set of all $x_J$'s form a basis for an algebra - the descent algebra of $D_n$, $\Sigma D_n$. Our interpretation of this multiplication rule uses this basis, but for ease of computation, we use a different notation.

We define a composition, $q$, of an integer, $n$, to be an ordered list $[q_1, q_2, \ldots, q_k]$ of positive integers whose sum is $n$, and shall write $q \equiv n$ to denote this. We shall call the integers $\{q_i\}_{i=1}^k$ the components of $q$.

There exists a natural bijection between the subsets of $\{1',1,2,3,\ldots,n-1\}$ and the union, $C(n)$, of the sets $C_{<n} = \{q | q \equiv m, m \leq n-2\}$, $C_n = \{q | q \equiv n\}$ and $C'_n = \{q' | q \equiv n, q_1 \geq 2\}$ (the $'$ here is purely notational, and so in practice we write $q' \equiv n$). The subset corresponding to such a composition, $q$, in $C(n)$ is

1. $\{q_1, q_0 + q_1, \ldots + q_k, q_0 + \ldots + q(k-1)\}$ if $q \in C_{<n}$,
2. $\{1, q_1, \ldots q_1 + \ldots + q(k-1)\}$ if $q \in C_n'$,
3. $\{1', q_1, \ldots q_1 + \ldots + q(k-1)\}$ if $q \in C_n$. 


where \( q_0 = n - m \)

We can also define a partial order relation on the set of compositions in \( C(n) \). Let \( q, r \in C(n) \), with \( q \vdash m_1 \) and \( r \vdash m_2 \), where \( m_2 \leq m_1 \). Then we say \( q \preceq r \) if the components of \( r \) can be obtained from the components of \( q \) by deleting components of \( q \) to give \( q \vdash m_2 \) and replacing adjacent components of \( q \) by their sum.

2 The multiplication rule and further results

If \( J' \) is the complement of \( J \) in \( S \), then we let \( B_q = x_{J'} \) where \( q \) is the composition in \( C(n) \) that corresponds to \( J \) by the above bijection. Solomon’s rule can now be described as follows.

Consider “templates” with the following form

\[
\begin{pmatrix}
\begin{array}{cccc}
  a_{00} & a_{01} & a_{02} & \cdots & a_{0l} \\
  b_{11} & b_{12} & \cdots & b_{1l} \\
  a_{10} & a_{11} & a_{12} & \cdots & a_{1l} \\
  & & & & \\
  & & & & \\
  a_{s0} & a_{s1} & a_{s2} & \cdots & a_{sl}
\end{array}
\end{pmatrix}
\]

where

1. \( a_{00} = n - N \), where \( N \) is the sum of all other entries in the template,
2. All entries in a template are non-negative integers,
3. The \( b \)-lines do not have entries in column 0.

**Definition 1.** We define the periphery-sum, \( P \), of the template to be the sum

\[
a_{00} + \sum_{j=1}^{l} a_{0j} + \sum_{i=1}^{s} a_{i0}
\]

and the \( b \)-sum, \( B \), to be \( \sum_{i,j} b_{ij} \). The reading word, \( r(t) \), of a given template \( t \) is given by

\[
[a_{01}, a_{02}, \ldots, a_{0l}, b_{11}, b_{12}, b_{1l}, a_{10}, a_{11}, \ldots, a_{1l}, \ldots, a_{s0}, a_{s1}, a_{s2}, \ldots, a_{sl}]
\]

unless \( a_{00} = 1 \), in which case \( r(t) \) is given by

\[
[1, a_{01}, a_{02}, \ldots, a_{0l}, b_{11}, b_{12}, b_{1l}, a_{10}, a_{11}, \ldots, a_{1l}, \ldots, a_{s0}, a_{s1}, a_{s2}, \ldots, a_{sl}]
\]

If \( q \) and \( r \) are compositions in \( C(n) \), such that \( q \) has components \( q_1, q_2, \ldots, q_l \), and \( r \) has components \( r_1, r_2, \ldots, r_s \), then we define \( S(q,r) \) to be the set of templates, \( T \), such that

1. the row sum, \( a_{00} + \sum_{j=1}^{l} (b_{ij} + a_{ij}) = r_i \),
2. the column sum, \( a_{0j} + \sum_{i=1}^{s} (b_{ij} + a_{ij}) = q_j \),
3. If \( P = 0 \), \( B \) is odd if

   (a) \( q \in C \) and \( r \in C' \), or
   (b) \( q \in C' \) and \( r \in C \).

We are now ready to state our multiplication rule:
**Theorem 1.** Let \( q \in \mathcal{C}(n) \). Let \( T \) be a template, and \( u = [u_1, u_2, \ldots, u_k] \) be the composition obtained by omitting zero components of \( r(T) \). Then

1. If \( r \in C_n \),
   \[
   B_q B_r = \sum_{T \in S_{(q,r)}} B_{r(T)}
   \]
   where \( B_{r(T)} = B_u \).

2. If \( r \in C_n' \),
   \[
   B_q B_r = \sum_{T \in S_{(q,r)}} B_{r(T)}
   \]
   where \( B_{r(T)} = B_{u'} \).

3. If \( r \in C_{<n} \),
   \[
   B_q B_r = \sum_{T \in S_{(q,r)}} B_{r(T)}
   \]
   where
   
   (a) if \( q \in C_n \) and \( B \) is odd, or \( q \in C_n' \) and \( B \) is even, then \( B_{r(T)} = B_{u'} \),
   
   (b) if \( q \in C_{<n} \) and \( a_{00} = 0 \), then \( B_{r(T)} = B_u + B_{u'} \).
   
   (c) Otherwise \( B_{r(T)} = B_u \).

**Remark.** Note that in point 3b, if \( u_1 = 1 \), then \( u \equiv u' \), hence \( B_{r(T)} = 2B_u \).

A formal proof of this theorem can be found in [vW97]. However, since it can follow a similar argument to either [vW], or [GRS5], for the analogous theorem in the symmetric groups case, we feel it would be more beneficial to replace the proof with a collection of illuminating examples.

**Examples.** To illustrate our rule we shall work in \( \Sigma D_4 \). Each example, \( B_q B_r \), shall consist of \( S_{(q,r)} \), and the resulting summands it generates according to the rule.

1. \( B_{31} B_4 \)
   
   \[
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 3 & 1
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   
   B_{31} B_4 = B_{31} + B_{13} + B_{211} + B_{112}
   
2. \( B_{31'} B_4 \)
   
   \[
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 1
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 3 & 0
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   =
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   
   B_{31} B_4 = B_{31} + B_{13} + 2B_{121}
   
3
3. $B_{22'}B_{4'}$
\[
\begin{pmatrix}
0 & 0 & 0 \\
2 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
2 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 0 \\
0 & 2 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]
\[B_{22'}B_{4'} = 4B_{22'} + B_{13} + B_{1111}\]

4. $B_4B_2$
\[
\begin{pmatrix}
0 & 2 \\
2 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 2 \\
0 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
\[B_4B_2 = 2B_{22} + B_{211'}\]

5. $B_2B_2$
\[
\begin{pmatrix}
2 & 0 \\
2 & 2 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 2 \\
1 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix}
\]
\[B_2B_2 = 2B_2 + B_{11} + B_{22} + B_{22'} + 2B_{1111}\]

6. $B_{11}B_2$
\[
\begin{pmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 1 \\
2 & 0 & 0
\end{pmatrix}
\]
\[B_{11}B_2 = 4B_{11} + 2B_{112} + 4B_{1111}\]

Remark. Note, in particular, that examples 1 and 2 illustrate point 1, and moreover the influence of $P = 0$; 3 illustrates point 2; and 4, 5 and 6 illustrate point 3. More specifically, examples 4, 5 and 6 illustrate respectively points 3a, 3b, and the remark associated with 3b.

**Corollary 1.** Let $q, r, s \in C(n)$. If the coefficient of $B_s$ in $B_4B_r$ is non-zero, then $s \not\preceq r$.

**Proof.** Let $r_i$ be a component of $r$, and let $q$ contain $k$ parts. If $B_s$ occurs in $B_4B_r$ with non-zero multiplicity then, by Theorem 1, there exists a template whose reading word corresponds to $s$. However, since $a_{ik} + \sum_{j=1}^{k}(b_{ij} + a_{ij}) = r_i$, it follows that $s \not\preceq r$. \qed

By Corollary 1, it follows that $\mathcal{T} = \langle B_q | q \in C_n \cup C_n' \rangle$ is a left ideal. Moreover, we have the following,
Theorem 2. Let $B_n$ be the Coxeter group of type $B$, whose Dynkin diagram is on $n$ nodes, and let $\Sigma B_n$ be its associated descent algebra. Then

$$\Sigma B_n \cong \Sigma D_n/I$$

Proof. For clarity, for $q \in C_{<n}$, let $B_q^D$ be a basis element of $\Sigma D_n$, and let $B_q^B$ be a basis element of $\Sigma B_n$. Let $S(q,r)$ be the set of templates corresponding to templates in $\Sigma (q,r)$ with $a_{oo} \geq 2$. We denote this set of templates by $I(q,r)$. Note that if we subtract 2 from the $a_{oo}$ of any template $T$, the reading word, row sum, and column sum of $T$ are unaffected. Moreover, if this is performed on all $T \in I(q,r)$ the resulting templates are precisely those that arise if we calculate the product $B_q^B B_r^B$ in $\Sigma B_n$. Since this argument is reversible, the result follows.

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