On 021-Avoiding Ascent Sequences

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Abstract

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev in their study of $(2+2)$-free posets. An ascent sequence of length $n$ is a nonnegative integer sequence $x = x_1x_2\ldots x_n$ such that $x_1 = 0$ and $x_i \leq \text{asc}(x_1x_2\ldots x_{i-1}) + 1$ for all $1 < i \leq n$, where $\text{asc}(x_1x_2\ldots x_{i-1})$ is the number of ascents in the sequence $x_1x_2\ldots x_{i-1}$. We let $A_n$ stand for the set of such sequences and use $A_n(p)$ for the subset of sequences avoiding a pattern $p$. Similarly, we let $S_n(\tau)$ be the set of $\tau$-avoiding permutations in the symmetric group $S_n$. Duncan and Steingrimsson have shown that the ascent statistic has the same distribution over $A_n(021)$ as over $S_n(132)$. Furthermore, they conjectured that the pair $(\text{asc}, \text{rlm})$ is equidistributed over $A_n(021)$ and $S_n(132)$ where rlm is the right-to-left minima statistic. We prove this conjecture by constructing a bistatistic-preserving bijection.

Keywords: 021-avoiding ascent sequence, 132-avoiding permutation, right-to-left minimum, number of ascents, bijection

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1 Introduction

The objective of this note is to establish a bijection which leads to the equidistribution of the pair of statistics $(\text{asc}, \text{rlm})$ over 021-avoiding ascent sequences and over 132-avoiding...
permutations. This confirms a conjecture posed by Duncan and Steingrímsson [5].

Let us give an overview of the notation and terminology. Let \( S_n \) denote the set of permutations of \([n]\), where \([n] = \{1, 2, \ldots, n\}\). Given a permutation \( \pi \in S_n \) and a permutation \( \tau \in S_k \), we say that a subsequence \( \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} \), \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), of \( \pi \) is of pattern \( \tau \) if it is order isomorphic to \( \tau \), that is, this subsequence has the same relative order as \( \tau \). If \( \pi \) does not contain any subsequence of pattern \( \tau \), then we say that \( \pi \) avoids \( \tau \), or \( \pi \) is \( \tau \)-avoiding.

We denote by \( S_n(\tau) \) the set of \( \tau \)-avoiding permutations in \( S_n \). For example, the permutation 763894512 contains the subsequence 3952 of pattern 2431, but it is 1234-avoiding. Pattern avoiding permutations have been intensively studied in recent years from many points of view, see [1, 6, 7].

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev [2]. For a sequence \( x = x_1 x_2 \cdots x_n \) of nonnegative integers, we say that an index \( i \) (\( 1 \leq i < n \)) is an ascent if \( x_i < x_{i+1} \). We denote by asc\((x)\) or merely asc \( x \) the number of ascents of \( x \). A sequence \( x = x_1 x_2 \cdots x_n \) is called an ascent sequence if \( x_1 = 0 \) and

\[
x_i \leq \text{asc}(x_1 x_2 \cdots x_{i-1}) + 1
\]

for all \( 1 < i \leq n \). For example, \( x = 010122 \) is an ascent sequence while \( x = 010142 \) is not since \( x_5 = 4 \) > asc\((0101)\) + 1 = 3. We let \( A_n \) denote the set of ascent sequences of length \( n \). For an ascent sequence, a pattern is a word on a nonnegative integers \( \{0, 1, \ldots, k\} \), where each element \( i \) appears at least once. Containment and avoidance of patterns for ascent sequences are defined in the same way as for permutations. For example, the ascent sequence 01231234 has five occurrences of the pattern 001, namely, the subsequences 112, 113, 114, 223, 224, and the ascent sequence 01012203 is 021-avoiding. We denote by \( A_n(p) \) the set of ascent sequences of length \( n \) avoiding pattern \( p \).

In addition to the ascent statistic, we will be interested in the number of right-to-left minima. A right-to-left minimum of any sequence \( x \) of nonnegative integers is an index \( i \) such that \( x_i < x_j \) for all \( j > i \). The number of right-to-left minima of \( x \) is denoted by rlm\((x)\) = rlm \( x \). For example, \( \text{rlm}(010122) = 3 \).

Ascent sequences are closely connected to \((2+2)\)-free posets [2], upper-triangular matrices [4], Stoimenow’s matchings [3], and the Catalan numbers \( C_n \). In particular, a poset is called \((2+2)\)-free if it does not contain an induced subposet which is isomorphic to the disjoint union of two 2-element chains. Bousquet-Mélou, Claesson, Dukes and Kitaev [2] found a bijection from \((2+2)\)-free posets to ascent sequences which maps the number of levels of the poset to the number of ascents of the sequence. Dukes and Parviainen [4] established a bijection between ascent sequences and nonnegative upper-triangular matrices. Duncan and Steingrímsson [5] have shown that \( \#A_n(p) = C_n \) for any of the patterns \( p = 101, 0101, \) or \( 021 \). It is well known that \( \#S_n(132) = C_n \) and Duncan-Steingrímsson also showed that the ascent statistic is equidistributed over \( A_n(021) \) and \( S_n(132) \). Furthermore, they proposed the following conjecture.
Conjecture 1.1. The bistatistic \((\text{asc}, \text{rlm})\) has the same distribution over \(A_n(021)\) and \(S_n(132)\).

The objective of this note is to give a bijective proof of the above conjecture.

2 Proof of the conjecture

In order to construct our bijection, we will need the concept of the special maximum value of an ascent sequence \(x = x_1 x_2 \ldots x_n\). The special maximum value of \(x\) is the largest integer \(M\) such that there is an index \(i\) with \(x_i = M\) and

\[
x_i = \text{asc}(x_1 x_2 \ldots x_{i-1}) + 1
\]

(2.1)
giving us equality in the defining relation for an ascent sequence. To illustrate the notion, if \(x = 01013312434\) then \(M = 3\). Note that, except for the zero sequence, \(M\) will always exist since the first 1 in any nonzero sequence satisfies (2.1). So we define \(M = 0\) for a zero sequence.

Also define a special maximum index as an index \(i\) where \(x_i\) satisfies \(x_i = M\) as well as condition (2.1). The first index \(i\) with \(x_i = M\) is always a special maximum index. In fact, if \([i, j]\) is the largest interval of indices starting with the first special maximum index and satisfying \(x_i = x_{i+1} = \cdots = x_j = M\) then we claim that these are exactly the special maximum indices. To see this, first note that if \(k \in [i, j]\) then \(k\) is a special maximum index because

\[
x_k = x_i = \text{asc}(x_1 x_2 \ldots x_{i-1}) + 1 = \text{asc}(x_1 x_2 \ldots x_{k-1}) + 1
\]

since there are no ascents between \(x_i\) and \(x_k\). To see that no other index can be special maximum, suppose \(x_k = M\) with \(k \geq j + 2\). Now \(x_j > x_{j+1}\) because if \(x_j < x_{j+1}\) then the special maximum value would be at least \(M + 1\). Thus there must be an ascent between \(x_{j+1}\) and \(x_k\) so that (2.1) is no longer an equality when \(i = k\). Call the special maximum value unique if there is only one special maximum index and repeated otherwise.

Theorem 2.1. The bistatistic \((\text{asc}, \text{rlm})\) has the same distribution over \(A_n(021)\) and \(S_n(132)\).

Proof. We will inductively build a bijection \(\phi_n : A_n(021) \rightarrow S_n(132)\) preserving the bistatistic. To do so, we will need decompositions of \(A_n(021)\) and \(S_n(132)\) into pieces indexed by smaller subscripts. We will start on the ascent side.

A simple but important observation for what follows is that \(p \in A_n(021)\) if and only if the nonzero entries of \(p\) are weakly increasing. We will use this fact to construct a bijection \(f = f_n\) between \(A_n(021)\) and the set of pairs

\[
\bigcup_{i=1}^{n} A_{i-1}(021) \times A_{n-i}(021).
\]
Consider $x \in \mathcal{A}_n(021)$ and suppose first that $x$ has a repeated special maximum value $M$. Let $k$ be any of the special maximum indices and define

$$f(x) = (\epsilon, z)$$

where $\epsilon$ is the empty sequence and $z$ is $x$ with $x_k$ removed. For example, if $x = 01013300304$ then $z = 0101300304$. Clearly $z$ still avoids 021 since its nonzero entries still increase and, since $x$’s special maximum value was repeated, $z$ still has $M$ as its special maximum value. Since the special maximum value does not change, one can construct an inverse map from $\mathcal{A}_0(021) \times \mathcal{A}_{n-1}(021)$ back to the elements of $\mathcal{A}_n$ which have a repeated special maximum in the obvious way. Finally note that in this case

$$\text{asc} x = \text{asc} z \quad \text{and} \quad \text{rlm} x = \text{rlm} z.$$ 

Now suppose that $x$ has a unique special maximum value $x_i = M$. Here we let

$$f(x) = (y, z)$$

where $y = x_1 \ldots x_{i-1}$ and $z$ is obtained from $x' = x_{i+1} \ldots x_n$ by subtracting $M - 1$ from all the nonzero entries. To illustrate, if $x = 0101300304$ then $y = 0101$ and $z = 00102$. It is clear that $y \in \mathcal{A}_{i-1}(021)$. To show that $z \in \mathcal{A}_{n-i}(021)$, we first note that $z$ still has weakly increasing nonzero elements and so avoids 012. We must also demonstrate that $z$ is an ascent sequence. Since the defining condition for an ascent sequence is trivial for zero elements, we need only consider $z_r \neq 0$. But since we have subtracted the same amount from all nonzero entries of $x'$, the index $r$ is an ascent of $z$ if and only if the index $r+i$ is an ascent of $x'$. Also, since $M$ was the special maximum value, we have $\text{asc}(x_1 \ldots x_i) = M$, $x_i > x_{i+1}$ and $x_{r+i} \leq \text{asc}(x_1 x_2 \ldots x_{r+i-1})$ for any $r \geq 1$. It follows that for any $z_r \neq 0$ we have

$$z_r = x_{r+i} - M + 1$$

$$\leq \text{asc}(x_1 \ldots x_{r+i-1}) - M + 1$$

$$= \text{asc}(x_1 \ldots x_i) + \text{asc}(x_{i+1} \ldots x_{r+i-1}) - M + 1$$

$$= \text{asc}(z_1 \ldots z_{r-1}) + 1$$

which is what we wished to prove. Constructing the inverse of this part of the map is similar to what was done in the first case.

It will be useful to record what happens to our two statistics in the second case defining $f$. For the ascent statistic, we have everything in place from the previous paragraph and the fact that, by definition of $M$, $x_i > x_{i-1}$. Thus

$$\text{asc} x = \text{asc}(x_1 \ldots x_i) + \text{asc}(x_{i+1} \ldots x_n)$$

$$= \text{asc}(x_1 \ldots x_{i-1}) + 1 + \text{asc}(x_{i+1} \ldots x_n)$$

$$= \text{asc} y + \text{asc} z + 1.$$
In terms of right-to-left minima, we distinguish two subcases. If \( i \leq n - 1 \) then, since \( x_{i+1} = 0 \), the right-to-left minima of \( x \) must occur in the sequence \( x_{i+1} \ldots x_n \). Since the subtraction of \( M - 1 \) does not change the positions of these minima, we have

\[
\text{rlm } x = \text{rlm } z.
\]

On the other hand, if \( i = n \) then \( z = \epsilon \) and \( x_n \) is a right-to-left minimum, giving

\[
\text{rlm } x = \text{rlm } y + 1.
\]

We will now review the standard decomposition of \( S_n(132) \) which gives a bijection \( g \) from this set to

\[
\bigcup_{i=1}^{n} S_{i-1}(132) \times S_{n-i}(132).
\]

If \( \pi \in S_n(132) \) then we write \( \pi = \pi_L n \pi_R \) where \( \pi_L, \pi_R \) are the elements to the left and right of \( n \), respectively. Define the index \( i \) by \( \pi_i = n \). Then it is well known that \( \pi \in S_n(132) \) if and only if \( \pi_L, \pi_R \) avoid 132 and every element of \( \pi_L \) is bigger than every element of \( \pi_R \). So we let

\[
g(\pi) = (\rho, \sigma)
\]

where \( \rho \in S_{i-1}(132) \) and \( \sigma \in S_{n-i}(132) \) are order isomorphic to \( \pi_L \) and \( \pi_R \), respectively.

As with the ascent sequence decomposition, we have to consider what happens to our statistics in two separate cases. The first is when \( \rho = \epsilon \), equivalently, \( i = 1 \). So \( \pi = n \pi_R \) and so \( n \) makes no contribution either to the ascents or right-to-left maxima. It follows that

\[
\text{asc } \pi = \text{asc } \sigma \quad \text{and} \quad \text{rlm } \pi = \text{rlm } \sigma
\]

just as in the corresponding case for ascent sequences.

Now suppose \( 1 < i \leq n \). So there will be an ascent ending at \( n \) and all other ascents of \( \pi \) correspond to ascents of \( \rho \) or ascents of \( \sigma \). It follows that

\[
\text{asc } \pi = \text{asc } \rho + \text{asc } \sigma + 1.
\]

For the right-to-left minima we again break into two subcases depending on whether the second component of our bijection is \( \epsilon \) or not. If \( \sigma \neq \epsilon \) then \( i < n \) and the right-to-left minima of \( \pi \) are all in \( \pi_R \) because of the relative sizes of the elements of \( \pi_R \) and \( \pi_L \). This gives

\[
\text{rlm } \pi = \text{rlm } \sigma.
\]

Now consider \( \sigma = \epsilon \) so that \( \pi = \pi_L n \). Thus \( n \) is a right-to-left minimum of \( \pi \) as is every right-to-left minimum of \( \pi_L \). So in this subcase

\[
\text{rlm } \pi = \text{rlm } \rho + 1.
\]
Finally, we construct $\phi_n: A_n(021) \to S_n(132)$ as follows. Start with $\phi_0(\epsilon) = \epsilon$. Assuming that $\phi_i$ has been defined for all $i < n$, we define $\phi_n$ to be the composition

$$A_n(021) \xrightarrow{f} \bigcup_{i=1}^{n} A_{i-1}(021) \times A_{n-i}(021) \xrightarrow{h} \bigcup_{i=1}^{n} S_{i-1}(132) \times S_{n-i}(132) \xrightarrow{g^{-1}} S_n(132)$$

where the restriction of $h$ to $A_{i-1}(021) \times A_{n-i}(021)$ is $\phi_{i-1} \times \phi_{n-i}$. It should be clear from the equations derived for asc and rlm when defining $f$ and $g$ that this bijection preserves the bistatistic.

\[\Box\]

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