A BIJECTIVE PROOF FOR RECIPROCITY THEOREM

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Abstract. In this paper, we study the graph polynomial that records spanning rooted forests \( f_G \) of a given graph. This polynomial has a remarkable reciprocity property. We give a new bijective proof for this theorem which has Prüfer coding as a special case.

1. Introduction

A spanning tree \( T \) in some graph \( G \) is a connected acyclic subgraph of \( G \) that includes all vertices in \( V(G) \). Calculating the number \( t(G) \) of spanning trees for some graph \( G \) is one of the typical questions we will ask. For example, when \( G \) is a complete graph \( K_n \), \( t(K_n) = n^{n-2} \). There are several methods to calculate \( t(G) \), such as the matrix-tree theorem and Prüfer coding.

In this paper, we study some graph polynomial \( f_G \) that records the spanning trees of the extended graph \( \tilde{G} \) of graph \( G \). This polynomial can be used to compute the spanning tree of some complex graphs easily. For example, let \( \Gamma = \Gamma (G; G_1, \ldots, G_k) \) be the graph that is obtained by substitution of graphs \( G_1, \ldots, G_k \) instead of a vertices of a graph \( G \). Then we can easily obtain \( f_{\Gamma} \) by \( f_G \) and \( f_{G_i} \), for \( 1 \leq i \leq k \).

In fact, the polynomial \( f_G \) possess the remarkable property of reciprocity. A. Renyi [9] gives an inductive proof for this reciprocity theorem. I. Pak and A. Postnikov [1] also give an inductive proof. Throughout this paper, we present a new bijective proof for the reciprocity theorem. One interesting fact is that the map we used in the bijection is Prüfer coding when \( G \) is a complete graph.

This paper is organized as follows: In section 2, we define the graph polynomial \( f_G \) to enumerate spanning trees in \( \tilde{G} \). In section 3, we show the reciprocity theorem for \( f_G \) and defined some tools for the future bijective proof. In section 4, we define two maps \( \phi \) and \( \psi \) to show the bijection between \( A \) and \( B \). Finally, in section 5, we use this bijective correspondence to prove the reciprocity theorem of \( f_G \).

2. Graph Polynomials for Spanning Trees

Suppose that \( G = (V, E) \) is a graph with vertices \( 1, \ldots, n \), where \( |V| = n \). Let \( 0 \notin V \) and \( \tilde{V} := V \cup \{0\} \). We say the extended graph \( \tilde{G} \) of \( G \) is a graph on the set \( \tilde{V} \) obtained by adding edges \( \{0, v\} \) to \( G \) for all vertices \( v \in V \). Clearly, if \( G \) is a complete graph \( K_n \) with \( n \) vertices, then \( \tilde{G} \) is a complete graph \( K_{n+1} \) with \( n + 1 \) vertices. We denote the set of all spanning trees in \( G \) as \( T_G \), i.e. all acyclic connected subgraphs in \( G \) which contain all the vertices of \( G \).

First of all, we assign variables \( x_i \) to \( i \), for all \( 1 \leq i \leq n \). For any spanning tree \( T \) in \( T_G \), define a function \( m(T) \) associated to \( T \):

\[
m(T) = \prod_{v \in \tilde{V}} x_v^{\rho_T(v)-1},
\]

(2.1)
where $\rho_T(v)$ denotes degree of the vertex $v$ in the tree $T$, i.e. the number of edges adjacent to the vertex $v$.

Now, we set the graph polynomial $t_G$ to be,

$$t_G := \sum_{T \in \mathcal{T}(G)} m(T).$$

Let us associate the variable $x$ to vertex 0. Then, the graph polynomial $f_G$ of variables $x$ and $x_v$ for all $v \in V$ is defined as follows:

$$f_G := t_G = \sum_{T \in \mathcal{T}_G} m(T). \quad (2.2)$$

We denote $V = \{1, \ldots, n\}$ and $f_G = f_G(x; x_1, \ldots, x_n)$.

It is easy to see that the spanning trees in $\mathcal{T}_G$ correspond to spanning rooted forests in $G$, i.e. acyclic subgraphs in $G$ containing all vertices in $V$, with a root chosen in each component. In particular, the two polynomials $t_G$ and $f_G$ possess the following identity:

$$t_G(x_1, \ldots, x_n) \cdot (x_1 + \cdots + x_n) = f_G(0; x_1, \ldots, x_n). \quad (2.3)$$

An short proof for Eq. (2.3) is provided in Igor Pak and A. Postnikov [1].

The graph polynomial $f_G$ has two important properties that allow us to compute the number of spanning rooted forests for certain graph. The first property is the composition of graphs. Let $G_1$ and $G_2$ be two graphs on disjoint sets of vertices, and $G_1 + G_2$ be the disjoint union of the graphs. We associate variable $x$ to the root 0, variables $y_1, \ldots, y_{r_1}$ to the vertices of $G_1$, and variables $z_1, \ldots, z_{r_2}$ to the vertices of $G_2$. Then the following formula holds:

$$f_{G_1 + G_2}(x; y_1, \ldots, y_{r_1}, z_1, \ldots, z_{r_2}) = x \cdot f_{G_1}(x; y_1, \ldots, y_{r_1}) \cdot f_{G_2}(x; z_1, \ldots, z_{r_2}).$$

One can prove the above equation by some simple arguments.

### 3. Reciprocity Theorem For Polynomials $f_G$

A graph $\overline{G} = (V, E)$ is called the complement of some graph $G = (V, E)$ if $\overline{E} = (V \choose 2) \setminus E$. That is to say, $e \in \overline{E}$ iff $e \notin E$. The graph polynomials $f_G$ possess the following reciprocity property:

$$f_G(x; x_1, \ldots, x_n) = (-1)^{n-1} f_{\overline{G}}(-x - x_1 - \cdots - x_n; x_1, \ldots, x_n). \quad (3.1)$$

The case that $x_1 = \cdots = x_n = 1$ for (3.1) was found by S. D. Bedrosian [2] and A. Kelmans.

Before we give the bijective proof for Eq. (3.1), we first introduce some notation.

First of all, let $\hat{F}_G$ be a spanning tree of some extended graph $\overline{G}$ with root 0 and vertices $1, \ldots, n$ so that $F_G$ is a spanning rooted forest of $G$. It is easy to show that for any vertex $u$ of $G$, there is a unique path from $u$ to root 0. Therefore, we can assign a direction to every edge in $\hat{F}_G$ such that each arrow points toward the root 0. This implies that every vertex $u \neq 0$ has outdegree 1. For convention, in this paper, when we say graphs $\hat{F}_G \in \mathcal{T}_G$ or $F_G$, we always consider it as a directed graph, and thus for every $u \neq 0$, there is a unique directed edge $(u, v) \in E(\hat{F}_G)$. In addition, a vertex $u$ is the child of vertex $u_1$ if there is a directed path from $u$ to $u_1$ in $\mathcal{T}_G$. 
Secondly, we say that a valid pair of some tree $\mathcal{F}_K$ is a pair $(u, v) \in E(K)$, and $Z_{G, \mathcal{F}_K}$ is a subset of valid pairs of $\mathcal{F}_K$ such that

$$Z_{G, \mathcal{F}_K} = \{(u, v) : (u, v) \notin E(G), (u, v) \in E(F_K)\}. \tag{3.2}$$

Now, given a subset $C$ of all valid pairs not in $Z_{G, \mathcal{F}_K}$, we define an operational set $O_{G, \mathcal{F}_K, C}$ as follows:

$$O_{G, \mathcal{F}_K, C} = C \cup Z_{G, \mathcal{F}_K}. \tag{3.3}$$

One can see that for a spanning tree $\mathcal{F}_G$ and graph $G \in K_n$, there could be many possible operational sets. An example is in figure 1.

![Figure 1](image)

**Figure 1.** For $\mathcal{F}_K$ and $G$ as above, we have two possible operational sets for $\mathcal{F}_K$. (The green marks are the graph after we apply all the pair in the operation sets to $\mathcal{F}_K$.)

Now, for any $\mathcal{F}_G$, suppose its induced subgraph $F_G$ in $K_n$ has $k$ connected components. We say a weight sequence $W_{\mathcal{F}_G}$ of $\mathcal{F}_G$ is

$$W_{\mathcal{F}_G} = (w_1, \ldots, w_{k-1}), \tag{3.4}$$

where $w_j \in \{0, 1, \ldots, n\}$, for $1 \leq j \leq k-1$. By convention, if $k = 1$, we set $W_{\mathcal{F}_G}$ to be empty. Therefore, there are $(n+1)^{k-1}$ possible weight sequences for spanning tree $\mathcal{F}_G$ that has $k$ connected components in $F_G$.

Given a graph $G \in K_n$, let $A$ be the set of all possible pairs $\left(\mathcal{F}_K, O_{G, \mathcal{F}_K, C}\right)$ and $B$ be the set of all possible pairs $\left(\mathcal{F}_G, W_{\mathcal{F}_G}\right)$. In the following section, we show a bijection between $A$ and $B$.

### 4. Bijection Between $A$ to $B$

Suppose that $G$ is a graph with $n$ vertices labeled $1, \ldots, n$ where each vertex $i$ is associated to a variable $x_i$, for $1 \leq i \leq n$. For the root in the extended graph, we assign variable $x$ to root 0. We first construct a map $\phi$ from $A$ to $B$.

**Definition 4.1.** Given a pair $\left(\mathcal{F}_K, O_{G, \mathcal{F}_K, C}\right) \in A$, the map $\phi$ outputs a pair $(\mathcal{F}, W)$ and is defined as follows:
Let $S$ be the set of vertices $u$ in $\tilde{F}_{Kn}$, where the directed edge $(u, v) \in E(\tilde{F}_{Kn})$ is a pair in $\mathcal{O}_{G,\tilde{F}_{Kn},C}$ or $v = 0$. Construct an empty sequence $W$ and a graph $\tilde{F}$ which is a duplicate of $\tilde{F}_{Kn}$.

WHILE $|S| > 1$,

1: Suppose there is a leaf $u' \neq 0$ in $\tilde{F}_{Kn}$ such that the edge $(u', v') \in E(\tilde{F}_{Kn})$ is not in $S$. We remove $u'$ and $(u', v')$ from $\tilde{F}_{Kn}$.

2: Repeat step 1 until every leaf $u \neq 0$ in $\tilde{F}_{Kn}$ is also in $S$. Let $M$ to be the set of all these vertices.

3: Delete the largest vertex $u^*$ in $M$ and the directed edge $(u^*, v^*)$ in $\tilde{F}_{Kn}$. We set $S$ to be $S\backslash\{u^*\}$, and add $v^*$ to the end of the sequence $W$.

4: Remove edge $(u^*, v^*)$ and add edge $(u^*, 0)$ to $\tilde{F}$.

RETURN $(\tilde{F}, W)$.

An example of this algorithm is in figure 2. In the following proposition, we prove that $\phi$ is well-defined.

**Proposition 4.2.** The map $\phi$ is a well-defined map from $A$ to $B$.

**Proof.** It is easy to see that all the steps in WHILE loop work. Now, we show that $\tilde{F}$ is a spanning tree of $\tilde{K}_n$ after each step 4. We proceed this by induction.

![Figure 2](image-url)
Initially, $\hat{F} = \hat{F}_{K_n}$ is a tree. Suppose that at some step 4, we delete edge $(u^*, v^*)$ and add edge $(u^*, 0)$ to the spanning tree $\hat{F} \in T_{\tilde{K}_n}$. Furthermore, since for any vertex $u \neq 0$, $u$ and root 0 is connected in graph $\hat{F}$, it remains connected after we change some edge $(u^*, v^*)$ to edge $(u^*, 0)$. Since $|E(\hat{F})| = n$, $\hat{F}$ is always a spanning tree of $\tilde{K}_n$ after any step 4.

Now, from (3.3), we know that $Z_{G, \hat{F}_{K_n}} \in O_{G, \hat{F}_{K_n}, C}$ and all the edges $(u, v)$ in the operational set $O_{G, \hat{F}_{K_n}, C}$ became $(u, 0)$ in the output graph $\hat{F}$. Thus, every edge in $E(F)$ is also in $E(\overline{G})$, and $\hat{F}$ is a spanning tree of $\overline{G}$.

Finally, we show that $W$ is a weight sequence of $\hat{F}$. Clearly, $S$ is the set of all roots in the spanning rooted forest $F$. Since the WHILE loop ends when $|S| = 1$, there are totally $|S| - 1$ elements added to the sequence $W$. Consequently, $W$ satisfies the length requirement in Eq. (3.4).

The above arguments tell us that $(F, W) \in B$ as desired.

We now give a map $\psi$ from $B$ to $A$.

**Definition 4.3.** Given a pair $\left(\hat{F}_{\overline{G}}, W_{\hat{F}_{\overline{G}}}\right) \in B$, the map $\psi$ outputs $\left(\hat{F}^*, O\right)$ and is defined as follows:

Assume that the forest $\hat{F}_{\overline{G}}$ has $k$ connected components and the associated weight sequence $W_{\hat{F}_{\overline{G}}} = (w_1, \ldots, w_{k-1})$. Create a tree $\hat{F}^* = \hat{F}_{\overline{G}}$, sequence $W_{\hat{F}^*} = W_{\hat{F}_{\overline{G}}}$, and an empty set $O$. Let $R$ be the set of roots in $\hat{F}_{\overline{G}}$.

**WHILE** the length of $W_{\hat{F}^*}$ is larger than 0.

1: We choose the first element $w$ in the sequence $W_{\hat{F}^*}$. Let $u$ be the largest vertex in $R$ such that $w_i$ is not $u$ nor a child of $u$ in $\hat{F}^*$, for any $w_i$ in $W_{\hat{F}^*}$. Delete the element $w$ from the sequence $W_{\hat{F}^*}$ and $u$ from the set $R$.

2: Remove the edge $(u, 0)$ and add the edge $(u, w)$ to the graph $\hat{F}^*$. If $w \neq 0$, we add pair $(u, w)$ to the set $O$, i.e. $O = O \cup \{(u, w)\}$.

**RETURN** $(\hat{F}^*, O)$.

An example of this mapping $\psi$ is in figure [3]. In the following lemma, we prove that $\psi$ is well-defined.

**Proposition 4.4.** The map $\psi$ is a well-defined map from $B$ to $A$.

**Proof.** We first show that at any stage, the set $R$ and graph $\hat{F}^*$ satisfy the following properties:

1. $\hat{F}^*$ is a spanning tree of $\tilde{K}_n$, i.e. $F^*$ is a spanning rooted forest of $K_n$.
2. $R$ is the sets of roots of forest $F^*$.

We proceed by induction on the number of loops. Initially, $R$ is the set of all the roots in forest $\hat{F}_{\overline{G}}$ and $W_{\hat{F}^*}$ is a sequence of length $k - 1 = |R| - 1$. Moreover, at each step 1, we remove an element in $W_{\hat{F}^*}$ and an element in $R$. Thus, the length of sequence $W_{\hat{F}^*}$ is always $|R| - 1$.

Now, suppose at some stage, we have that properties (1) and (2) hold and sequence $W_{\hat{F}^*} = \{w'_{k_1}, \ldots, w'_{k_1-1}\}$, where $k_1 = |R|$. During step 1, since there are $k_1$ connected components in $\hat{F}^*$, there exists at least one connected component that contains no elements in $W_{\hat{F}^*}$. Consider the component with the largest root $u$.
that meets this condition. It is not hard to see that for any $1 \leq i \leq k_1 - 1$, $w'_i$ is not $u$ nor a child of $u$. Consequently, step 1 works.

For step 2, by the choice of vertex $u$, we have $w'_i$ and $u$ are not connected in $F^*$. Suppose $\tilde{F}^*$ becomes cyclic after we delete edge $(u, 0)$ and add edge $(u, w'_1)$ to this graph. This implies that there is a cycle containing edge $(u, w'_1)$. It is not possible since vertices $u$ and $w'_1$ would be connected in $F^*$ before we add edge $(u, w'_1)$.

The above arguments show that after step 1 and 2, $\tilde{F}^*$ remains acyclic, and is a spanning tree of $\tilde{K}_n$. Furthermore, after step 2, since $u$ is no longer a root, $R$ remains as the set of all roots in $F^*$. As a result, properties (1) and (2) always hold.

Finally, we need to show that $(v, v') \in \mathcal{O}$, for every directed edge $(v, v') \notin E(\mathcal{G})$ and $(v, v') \in E(F^*)$. Clearly, $\tilde{F}^*$ is obtained from $\tilde{F}_G$ by a series of removing and adding edges in step 2. If edge $(v, v') \notin E(\mathcal{G})$, then $(v, v') \notin E(\tilde{F}^*)$. Therefore, edge $(v, v')$ is added to graph $\tilde{F}^*$ in some step 2, and $(v, v') \in \mathcal{O}$. This implies that $(\tilde{F}^*, \mathcal{O}) \in A$ as desired. \qed
Theorem 4.5. The two maps $\phi$ and $\psi$ define a bijective correspondence between sets $A$ and $B$.

Proof. We have shown that $\phi$ and $\psi$ are well-defined. The remaining task is to prove that $\phi$ is the inverse map of $\psi$.

Given a pair $(\hat{F}_{K_n}, O_{G,\hat{F}_{K_n},C}) \in A$, we apply the map $\phi$ and obtain an output $(\hat{F}, W) \in B$. Suppose that during the map $\phi$, we record the largest vertex $u^*$ in every step 3 into a sequence $U$ in order. It is easy to see that $|U| = |W| = |S| - 1$, where $S$ is the original set before the WHILE loop in map $\phi$. Let $|S| = k$, and we set $U = \{u_1, \ldots, u_{k-1}\}$ and $W = \{w_1, \ldots, w_{k-1}\}$. Thus, for any $1 \leq j \leq k - 1$, $(u_j, w_j)$ is the directed edge removed from $\hat{F}_G$ in step 3 in the $j$-th WHILE loop.

Now, let us apply the map $\psi$ on pair $(\hat{F}, W) \in B$, and denote the output pair by $(\hat{F}'_{K_n}, O_{G,\hat{F}'_{K_n},C'}) \in A$. Therefore, initially, $R = S$ is the set of roots of forest $F$. Our goal is to prove that

$$\hat{F}_{K_n} = \hat{F}'_{K_n} \text{ and } O_{G,\hat{F}_{K_n},C} = O_{G,\hat{F}'_{K_n},C'}.$$ (4.1)

We record the vertex $u$ we picked in every step 1 in the map $\psi$ and get a sequence $U' = \{u'_1, \ldots, u'_k\}$ in order. Clearly, if $U$ and $U'$ are the same sequence, Eq. (4.1) holds since every move in step 2 in $\psi$ will be the reverse move in step 4 in $\phi$.

Before we show that $U = U'$, we first prove the following property:

1. In the $i$-th WHILE loop of the map $\phi$, where $1 \leq i \leq k - 1$, consider the graph $\hat{F}_{K_n}$ after step 2. Then for any $u$ in that current set $S$, it is not a leaf in $\hat{F}_{K_n}$ if there exists some $w_i$, where $i \leq i \leq k - 1$, such that $w_i$ is $u$ or a child of $u$.

If $u$ is not a leaf in $\hat{F}_{K_n}$, then there must be a vertex $u'$ in current set $S$ that is child of $u$. Consider the vertex $w'$ which edge $(u', w')$ is in $E(\hat{F}_{K_n})$. Consequently, $u' \notin \{w_1, \ldots, w_{k-1}\}$ is vertex $u$ or child of $u$. By some easy arguments, one can see that the reverse statement is true, and thus prove property (1).

We now show $U = U'$ by induction on the index $i$, where $1 \leq i \leq k - 1$. When $i = 1$, clearly, from (1), we know that $u'_1$ is a leaf in $\hat{F}_{K_n}$. By the choice of $u_1$, we have $u'_1 \leq u_1$. On the other hand, since $u'_1$ is the largest element in $S$ that no element in $W$ is $u'_1$ or child of $u'_1$, we have $u_1 \leq u'_1$. As a result, $u_1 = u'_1$.

Secondly, suppose for $i$ from 1 to $r - 1$, where $r \leq k - 1$, we have $u_i = u'_i$. That is to say, the set $S$ and $R$ in the $r$-th WHILE loop of map $\phi$ and $\psi$ are the same. When $i = r$, from (1) and the choice of $u'_r$, we have that both $u_r \in S$ and $u'_r \in R = S$ are the largest vertex $z$ such that no element $w \in \{w_r, \ldots, w_{k-1}\}$ is $z$ or child of $z$. Consequently, $u_r = u'_r$.

By induction, we can prove that $U$ and $U'$ are the same sequence. Therefore, Eq. (4.1) holds and $\psi$ is the inverse map of $\phi$. Finally, this shows us that the two maps $\phi$ and $\psi$ define a correspondence relation between sets $A$ and $B$. \hfill \Box

In particular, consider the case that $G = K_n$. Since $\overline{G}$ is empty, we have that every valid pair $(u, v)$ in $\hat{F}_{K_n}$ is not in $\overline{G}$. Therefore, for every spanning tree $\hat{F}_{K_n}$ in $\overline{K_n}$, there is only one possible operational set $O_{K_n,\hat{F}_{K_n},C} = Z_{K_n,\hat{F}_{K_n}}$. In addition, there is only one spanning tree $\hat{F}_G$ which is the graph with every vertex connected to root 0. Consequently, for every pair $(\hat{F}_G, W_{R_{K_n}}) \in B$, we have that $|W_{R_{K_n}}| = n - 1$. That is to say, every element in $B$ is associated to a sequence of length $n - 1$. One
can easily see that the map $\phi$ now is a prufer coding for spanning trees in $K_{n+1}$ and therefore, prufer coding is a special case for this bijection.

5. A New Proof of The Reciprocity Theorem

In this section, we show how to use this bijection to prove the reciprocity theorem.

**Theorem 5.1.** Let $G$ be a graph on the set of vertices $\{1, \ldots, n\}$. Then

$$f_G(x; x_1, \ldots, x_n) = (-1)^{n-1} f_G(-x; -x_1 - \cdots - x_n; x_1, \ldots, x_n). \quad (5.1)$$

**Proof.** First of all, we show that

$$(-1)^{n-1} f_G(x; x_1, \ldots, x_n) = f_G(-x; -x_1, \ldots, -x_n). \quad (5.2)$$

If we can show that the degree of every monomial in $f_G(x; x_1, \ldots, x_n)$ is $n-1$, then Eq. (5.2) will be true. Note that each monomial in $f_G(x; x_1, \ldots, x_n)$ corresponds to some spanning tree $\tilde{F}_{K_n}$ of $K_n$, and we have

$$\deg \left( m \left( \tilde{F}_{K_n} \right) \right) = \sum_{v \in \{0, \ldots, n\}} (\deg(v) - 1) = \sum_{v \in \{0, \ldots, n\}} \deg(v) - (n + 1) \quad (5.3)$$

$$= 2|E| - (n + 1) = n - 1. \quad (5.4)$$

This implies that Eq. (5.2) is true.

Now, we show that

$$f_G(x; x_1, \ldots, x_n) = f_G(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n). \quad (5.5)$$

Consider some spanning tree $\tilde{F}_{K_n}$ of $K_n$ associated to a monomial $x^{d_1} x_1^{d_1} \cdots x_n^{d_n}$ in polynomial $f_G$ and an operational set $O_{G, \tilde{F}_{K_n}, C}$ for $\tilde{F}_{K_n}$. Let us apply the map $\phi$ on $\left( \tilde{F}_{K_n}, O_{G, \tilde{F}_{K_n}, C} \right)$. Denote the output pair by $\left( \tilde{F}_{\tilde{G}}, W_{\tilde{F}_{\tilde{G}}} \right) \in B$, where sequence $W_{\tilde{F}_{\tilde{G}}} = (w_1, \ldots, w_k-1)$, and $k$ is the number of connected components in $\tilde{F}_{\tilde{G}}$. Moreover, the contribution of graph $\tilde{F}_{\tilde{G}}$ in the polynomial $f_{\tilde{G}}$ is

$$(x + x_1 + \cdots + x_n)^{k-1} (-x_1)^{\deg(v_1)-1} \cdots (-x_n)^{\deg(v_n)-1}, \quad (5.6)$$

where $\deg(v_i)$ is the degree of vertex $i \neq 0$ in $\tilde{F}_{\tilde{G}}$. We associate the pair $\left( \tilde{F}_{\tilde{G}}, W_{\tilde{F}_{\tilde{G}}} \right)$ to the monomial

$$x_{w_1} \cdots x_{w_{k-1}} (-x_1)^{\deg(v_1)-1} \cdots (-x_n)^{\deg(v_n)-1}$$

in $f_{\tilde{G}}$, where $x_0 = x$ and $x_{w_j}$ is the variable corresponding to vertex $w_j$, for $1 \leq j \leq k - 1$. Clearly, $x_{w_1} \cdots x_{w_{k-1}}$ is a monomial in $(x + x_1 + \cdots + x_n)^{k-1}$. By the choice of $W_{\tilde{F}_{\tilde{G}}}$ shown in section 3, we have that the set $B$ and set of all monomials in $f_{\tilde{G}}(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n)$ have a bijective correspondence.

It is easy to show that the monomial for the pair $\left( \tilde{F}_{K_n}, O_{G, \tilde{F}_{K_n}, C} \right)$ is the monomial associated to the pair $\left( \tilde{F}_{\tilde{G}}, W_{\tilde{F}_{\tilde{G}}} \right)$ with several sign changes, where the number of sign changes is $\sum_{i=1}^{n} (\deg(v_i) - 1)$. That is to say, we have

$$x^{d_1} x_1^{d_1} \cdots x_n^{d_n} = (-1)^{l} x_{w_1} \cdots x_{w_{k-1}} (-x_1)^{\deg(v_1)-1} \cdots (-x_n)^{\deg(v_n)-1}, \quad (5.7)$$
where \( l = \sum_{i=1}^{n} (\deg(v_i) - 1) = n - \deg(v_0) \).

Now, suppose that \( \hat{F}_{K_n} \in T(\tilde{G}) \). Since every valid pair in \( \hat{F}_{K_n} \) is not in graph \( \tilde{G} \), the only operational set for \( \hat{F}_{K_n} \) is \( \mathbf{Z}_{G,\hat{F}_{K_n}} \). In addition, the output spanning tree \( \hat{F}_{\tilde{G}} \) is the extended graph of empty graph. Therefore, the only pair \( \left( \hat{F}_{K_n}, \mathbf{Z}_{G,\hat{F}_{K_n}} \right) \in \mathbf{A} \) for \( \hat{F}_{K_n} \) is mapped to a monomial in \( (x + x_1 + \cdots + x_n)^n \). This implies that the coefficient of the monomial associated to \( \hat{F}_{K_n} \) is 1 in \( f_G(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n) \).

Secondly, if \( \hat{F}_{K_n} \notin T(\tilde{G}) \), then there is an edge \( (u,v) \in E(\hat{F}_{K_n}) \) such that \( (u,v) \in E(G) \). For every operational set \( \mathcal{O}_{G,\hat{F}_{K_n}} \) for \( \hat{F}_{K_n} \), we consider the two operational sets:

\[
\mathcal{O}_1 = \mathcal{O}_{G,\hat{F}_{K_n},G} \cup \{(u,v)\}, \quad \text{and} \quad \mathcal{O}_2 = \mathcal{O}_1 \setminus \{(u,v)\} \quad (5.8)
\]

Clearly, \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are both operational sets for \( \hat{F}_{K_n} \). Denote the output pair for \( (\hat{F}_{K_n}, \mathcal{O}_1) \) as \( (\hat{F}_1, W_1) \) and the output pair for \( (\hat{F}_{K_n}, \mathcal{O}_2) \) as \( (\hat{F}_2, W_2) \) in the map \( \phi \). From Eq. (5.7), one can see that the monomials associated to the two pairs \( (\hat{F}_1, W_1) \) and \( (\hat{F}_2, W_2) \) are the same. Moreover, the degrees of root 0 in \( \hat{F}_1 \) and \( \hat{F}_2 \) are differ by 1. Consequently, by (5.7), the summation of the coefficients of the monomial associated to \( (\hat{F}_1, W_1) \) and \( (\hat{F}_2, W_2) \) is 0. Finally, because we can pair up all the operational sets for \( \hat{F}_{K_n} \) by (5.8), the contribution of the monomial for \( \hat{F}_{K_n} \) in \( f_G(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n) \) is 0.

From the above argument, we conclude that the only monomials left in \( f_G \) after cancellation of coefficients are the monomials in \( f_G(x; x_1, \ldots, x_n) \). Moreover, each monomial in \( f_G \) has coefficient 1 in \( f_G(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n) \). As a result, we have that \( f_G(x; x_1, \ldots, x_n) = f_G(x + x_1 + \cdots + x_n; -x_1, \ldots, -x_n) \), and Eq. (5.4) holds as desired. \( \square \)

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CR = (2, 4, 5)
W = (3, 5)

CR = (4, 5)
W = (5)
σ = {(2, 3)}

CR = (5)
W = ()
σ = {(2, 3), (4, 5)}
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