Metric Solutions in Torsionless Gauge for Vacuum Conformal Gravity

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Abstract

In a recent paper we have established the form of the metric-torsional conformal gravitational field equations; in the present paper we study their vacuum configurations: in this case we will consider torsion to have a specific form, and we will find exact solutions. A discussion will follow.

Introduction

In defining a relativistic theory of gravity in which to implement conformal transformations accounting for length scaling, the main goal is to find a possible model for gravity that is also renormalizable, so to eventually be able to treat the problem of gravitational quantization, as it has been discussed by Stelle in reference [1]: according to this procedure, one has to find solutions in the form of gravitational waves expanded in terms of monochromatic plane waves on which to transfer the commutation relations set by the rules of quantization, and in the past solutions in the form of gravitational waves have been discussed extensively, mainly by Mannheim, as for instance in [2], and also by Paranjape and collaborators, as for example in [3] and [4]. On the other hand however, if one wants to consider the graviton after a proper quantization, then according to the Wigner classification for which any particle must be described in terms of momentum and spin, one must consider a background in which not only metric but also torsional degrees of freedom are present: in this background, the energy and spin of the gravitational wave will be described in terms of the metric and torsion of the spacetime, and the problem of having a graviton that undergoes quantization is transferred to the problem of having metric and torsion obeying the standard rules of quantization. Thus, the problem of having a theory of gravitation suitable of being quantized relies upon the problem of having a theory of gravitation that is renormalizable in both translational and rotational degrees of freedom, and hence a theory of gravitation in which conformal invariance is implemented for both metric and torsion [5].

The problem of such an approach was that there was no known way to obtain for the Weyl conformal curvature in the purely metric case any torsional extension, and thus no way in which the Weyl conformal gravity could be obtained as a torsionless limit: nevertheless, such an extension has been found in a recent paper [6]. Then, the metric-torsional conformal gravitational field equations have been given, and their consistency has been checked in terms of the conservation laws that have to be satisfied by the spin and energy conformal quantities of any matter fields, once the conformally invariant matter field
equations are given. The next step is to look for special situations like those we have in absence of any matter field for the vacuum configurations.

Now, it is well-known that Schwarzschild-like vacuum solutions for Weyl conformal gravity are in fact extensions of the Schwarzschild vacuum solution of the Einsteinian gravitation [8]; on the other hand, it is essential for the present torsional extension of the metric Weyl gravity to find vacuum solutions in order to see how they may further extend the purely metric Weyl gravity, and so the Einsteinian gravitation. In this paper, we shall consider all such solutions and their relationships in cases of special symmetries.

1 The Metric-Torsional Conformal Gravitation

In this paper the Riemann-Cartan geometry is defined in terms of a Riemannian metric $g_{\alpha\beta}$ and a metric-compatible connection $\Gamma_{\alpha\sigma}^\mu$ so to be independent, and the Cartan torsion tensor $Q_{\rho\alpha\sigma} = \Gamma_{\rho\alpha\sigma}^{\mu} - \Gamma_{\rho\sigma\alpha}^{\mu}$ has contraction $Q_{\rho\mu\nu} = Q_{\nu}$ as usual; it is further possible to define the Riemann-Cartan curvature tensor in the form

$$G_{\rho\xi\mu\nu} = \partial_{\mu}\Gamma_{\rho\xi\nu}^{\alpha} - \partial_{\nu}\Gamma_{\rho\xi\mu}^{\alpha} + \Gamma_{\rho\pi\mu}^{\alpha}\Gamma_{\pi\xi\nu}^{\alpha} - \Gamma_{\rho\pi\nu}^{\alpha}\Gamma_{\pi\xi\mu}^{\alpha}$$

with contraction chosen to be given in the form $G_{\rho\mu\nu} = G_{\mu\nu}$ with contraction $G_{\eta\nu}g^{\eta\nu} = G$ setting our convention: from torsion and curvature it is possible to define the metric-torsional curvature tensor

$$M_{\alpha\beta\mu\nu} = G_{\alpha\beta\mu\nu} + (1 - q^3) (Q_{\beta}^{\alpha\mu\nu} - Q_{\alpha}^{\beta\mu\nu})$$

with contractions given as for the Riemann-Cartan curvature tensor. Finally from this tensor we construct the modified metric-torsional curvature tensor as given by the Weyl procedure

$$T_{\alpha\beta\mu\nu} = M_{\alpha\beta\mu\nu} - \frac{1}{2} (M_{\alpha[\mu}g_{\nu]\beta} - M_{\beta[\mu}g_{\nu]\alpha}) + \frac{1}{12} M (g_{\alpha[\mu}g_{\nu]\beta} - g_{\beta[\mu}g_{\nu]\alpha})$$

for any value of the parameter $q$ and which has the same symmetry properties of the curvature tensors previously defined but it is irreducible. The conformal transformation for the metric is

$$g_{\alpha\beta} \rightarrow \sigma^2 g_{\alpha\beta}$$

and by defining $\ln \sigma = \phi$ we have that the simplest conformal transformation for the torsion tensor is given by the following

$$Q_{\rho\alpha} \rightarrow Q_{\rho\alpha}^{\sigma} + q (\delta_{\rho}^{\sigma}\partial_{\alpha}\phi - \delta_{\alpha}^{\sigma}\partial_{\rho}\phi)$$

in terms of the parameter $q$ which is completely generic, and in this way the conformal transformation is induced on the curvature tensors and as a simple calculation shows $T_{\rho\xi\mu\nu}$ is a conformally covariant curvature tensor.

We notice that in building the coordinate-conformal invariants, because the presence of torsion spoils some symmetry properties of indices transposition, we will not have only one but three possible contractions so that it is in terms of three parameters $A$, $B$, $C$ that the most general invariant is given by $AT_{\alpha\beta\mu\nu}a_{\alpha}^{\beta} + BT_{\mu\nu\alpha\beta} + CT_{\alpha\beta\mu\nu}T_{\alpha\beta\mu\nu}$ as a straightforward analysis shows; thus it is useful to define the parametric quantity

$$P_{\alpha\beta\mu\nu} = AT_{\alpha\beta\mu\nu} + BT_{\mu\nu\alpha\beta} + \frac{C}{T} (T_{\alpha\mu\beta\nu} - T_{\beta\mu\alpha\nu} + T_{\beta\nu\alpha\mu} - T_{\alpha\nu\beta\mu})$$
in terms of the parameters $A$, $B$, $C$, with the same symmetry properties of all curvature defined before but irreducible and still conformally covariant: in terms of this parametric conformal tensor $P_{\alpha \beta \mu \nu}$ the most general invariant further reduces to the form given by $T^{\alpha \beta \mu \nu} P_{\alpha \beta \mu \nu}$ and so we eventually have that the action is given by

$$S = \int [kT^{\alpha \beta \mu \nu} P_{\alpha \beta \mu \nu} + L_{\text{matter}}] \sqrt{|g|} d V$$

(6)

with constant $k$ complemented by the material Lagrangian and where it is over the volume of the spacetime that the integral is taken. By varying this action with respect to metric and connection one obtains the field equations

$$2k[P^{\rho \sigma \nu \alpha} T_{\theta \sigma \rho} - \frac{1}{4} g^{\alpha \mu} P^{\theta \sigma \rho \beta} T_{\theta \sigma \rho \beta} + P_{\mu \sigma \alpha \rho} M_{\sigma \rho} +$$

$$+ \left(\frac{1 - q}{3q}\right) (P_{\nu} (2 P^{\mu \rho \alpha \nu} Q_{\rho} - g^{\mu \alpha} P^{\nu \rho \sigma} Q_{\rho \sigma} + g^{\mu \nu} P^{\alpha \beta \rho \sigma} Q_{\theta \rho \sigma}) +$$

$$+ Q_{\nu} (2 P^{\mu \rho \alpha \nu} Q_{\rho} - g^{\mu \alpha} P^{\nu \rho \sigma} Q_{\rho \sigma} - P^{\mu \nu \rho \sigma} Q_{\theta \rho \sigma})) = \frac{k}{2} T_{\alpha \mu}$$

(7)

and

$$4k[D_{\rho} P^{\rho \alpha \beta \mu} + Q_{\rho} P^{\alpha \beta \mu} - \frac{1}{4} Q_{\rho}^{\mu} P^{\alpha \beta \rho} -$$

$$- \left(\frac{1 - q}{3q}\right) (Q_{\rho}^{\alpha \beta \mu} - \frac{1}{4} Q_{\rho \sigma \theta g^{\mu \rho \sigma \theta}} )] = T_{\mu \alpha \beta}$$

(8)

in terms of the parameter $q$ and the constant $k$ and where $T^{\mu \nu}$ and $S^{\rho \mu \nu}$ are the energy and spin densities of the conformal matter field one wants to study.

Finally by taking into account the Jacobi-Bianchi identities we have that the field equations (7-8) are converted into conservation laws that are given in the following form

$$D_\mu T^{\mu \rho} + Q_\rho T^{\mu \rho} - T_{\mu \sigma} Q^{\sigma \rho \mu} + S_{\beta \mu \sigma} G^{\sigma \beta \rho \sigma} = 0$$

(9)

and

$$D_\rho S^{\rho \mu \nu} + Q_\rho S^{\rho \mu \nu} + \frac{1}{2} T^{[\mu \nu]} = 0$$

(10)

with trace condition as another conservation law

$$(1 - q)(D_\mu S^{\mu \nu} + Q_\mu S^{\mu \nu}) + \frac{1}{2} T_{\mu} = 0$$

(11)

which are satisfied once conformal matter field equations are given: we notice that the general conservation laws (9-10) are now accompanied by an additional conservation law for the trace (11) because there is not only general coordinate covariance but also conformal invariance. In the following we shall consider matter fields to be absent in order to study the vacuum configuration.

2 Metric Solutions in Torsionless Gauge for the Vacuum in Conformal Gravitation

This is the the gravitational model presented in [6], and in what follows we are going to apply it to the situation of vacuum configurations.

As explained in [7], a first special simplification we may perform consists in assuming that of all torsional decompositions some may vanish; of course,
because the conformal transformation for torsion $Q^{\mu}_{\alpha\sigma}$ is entirely loaded on the trace vector decomposition $Q_\nu$, then the axial vector and the remaining decompositions are conformally invariant: this means that it is meaningless to set the the trace vector $Q_\nu$ to zero because it can always be produced by a conformal transformation, and therefore the other two decompositions are the only ones of which we can meaningfully require the vanishing. Thus said, in the following we shall assume that the torsion tensor is entirely given by its trace vector part as

$$Q^{\mu}_{\alpha\sigma} \equiv \frac{1}{3}(\delta^\mu_\alpha Q_\sigma - \delta^\mu_\sigma Q_\alpha) \quad (12)$$

that is that containing the conformal transformation properties, setting to zero all other parts as they are conformally invariant. With this requirement, it is straightforward to see that $T_{\rho\sigma\mu\nu} \equiv C_{\rho\sigma\mu\nu}$ that is the conformal curvature defined here identically reduces to the conformal curvature of the purely metric case given by the Weyl curvature defined as usual; so the metric and torsional degrees of freedom decouple at least within the conformal curvature.

Also, as explained in [8], further simplifications comes by choosing special fine-tunings, in this case for the parameters $A, B, C$; however, because the conformal curvature is now the Weyl curvature, with all its symmetry properties of indices transposition, then there is no loss of generality in choosing the parameters in such a way that $P_{\rho\sigma\mu\nu} \equiv T_{\rho\sigma\mu\nu} = C_{\rho\sigma\mu\nu}$ so to have the parametric conformal curvature reduce to the conformal curvature equal to the Weyl curvature. With these requirements, the entire system of conformal field equations thus reduces to the following form

$$C_{\theta\sigma\rho\alpha} C_{\theta\sigma\rho} - \frac{1}{3}g^{\theta\alpha} C_{\theta\sigma\rho\beta} C_{\theta\sigma\rho\beta} + C^{\mu\sigma\alpha\rho} R_{\sigma\rho} +$$

$$+ \frac{2}{9q} C^{\mu\sigma\alpha\rho} Q_{\rho} Q_\sigma + \frac{2}{3q} C^{\mu\sigma\alpha\rho} \nabla_\rho Q_\sigma = 0 \quad (13)$$

and

$$\nabla_\rho C^{\alpha\beta\mu\rho} - \frac{1}{3q} Q_\rho C^{\alpha\beta\mu\rho} = 0 \quad (14)$$

in which $R_{\rho\sigma\mu\nu}$ and $\nabla_\mu$ are the curvature and the covariant derivative of the purely metric case given by the Riemann curvature and in terms of the Levi-Civita connection as usual; however we see that metric and torsional degrees of freedom are not decoupled within the conformal gravitational field equations.

Now, considering this system of field equations, and taking the first in its decomposition we have that its antisymmetric part is given by

$$\frac{1}{3q}(\nabla_\rho Q_\sigma - \nabla_\sigma Q_\rho) C^{\mu\sigma\alpha\rho} = 0 \quad (15)$$

identically, whose solutions in the case in which the spacetime is not conformally flat reduce to $\frac{1}{3q}(\nabla_\rho Q_\sigma - \nabla_\sigma Q_\rho) = 0$ then implying in a connected and simply connected spacetime that the trace vector is the gradient of a certain scalar field $Q_\sigma = 3q \nabla_\sigma \varphi$ removable by means of suitable conformal transformations; in fact by choosing $\varphi \equiv -\phi$ it is possible to see that a conformal transformation would map the trace vector into the vanishing one, thus forcing torsion to be equal to zero: it is important to remark that the torsionless theory has not been obtained by setting torsion to zero by hand, but by exploiting
conformal transformation. On the other hand however, we cannot perform further conformal transformations as we have already exhausted the only degree of freedom conceded by the conformal invariance of the theory; the conformal field equations in this conformal gauge are torsionless and so given in the purely metric case by

\[
C^\theta\sigma\rho\alpha C^\mu\sigma\rho\beta C^\theta\sigma\rho\beta + C^{\mu\sigma\alpha\rho} R_{\sigma\rho} = 0 \quad (16)
\]

and

\[
\nabla_\rho C^{\alpha\beta\mu\rho} = 0 \quad (17)
\]

for which we stress that they are a system of two field equations of which one is of the second-order derivative and quadratic and the other is linear and of the third-order derivative, differently from what we had in the purely metric case where with only one equation of fourth-order derivative. Finally, we write them in the explicit form

\[
R^{\theta\sigma\rho\alpha} R^\theta_{\sigma\rho\beta} R^\mu_{\theta\sigma\rho\beta} - R^{\alpha\beta\sigma\rho} R^\beta_{\alpha\beta\mu\rho} R^\alpha_{\sigma\rho\beta} + 1\frac{R^{\alpha\mu} R^{\rho\beta} R_{\rho\beta}}{4} + 1\frac{R^{\alpha\mu} R^{\beta\rho} R_{\alpha\beta}}{4} + 1\frac{R^{\alpha\mu} R^{\beta\rho} R_{\alpha\beta}}{4} + 1\frac{R^{\rho\beta} R_{\rho\beta}}{4} = 0 \quad (18)
\]

and

\[
\nabla_\mu (R_{\beta\nu} - \frac{1}{6} R g_{\beta\nu}) - \nabla_\nu (R_{\beta\mu} - \frac{1}{6} R g_{\beta\mu}) = 0 \quad (19)
\]

which are the final system of purely metric field equations defining our theory.

Next we notice the fundamental fact that by taking the quadratic field equation (18) and adding the divergence of the differential field equation (19) we get

\[
R^{\theta\sigma\rho\alpha} R^\theta_{\sigma\rho\beta} R^\mu_{\theta\sigma\rho\beta} - R^{\alpha\beta\sigma\rho} R^\beta_{\alpha\beta\mu\rho} R^\alpha_{\sigma\rho\beta} + 1\frac{R^{\alpha\mu} R^{\rho\beta} R_{\rho\beta}}{4} + 1\frac{R^{\alpha\mu} R^{\beta\rho} R_{\alpha\beta}}{4} + 1\frac{R^{\alpha\mu} R^{\beta\rho} R_{\alpha\beta}}{4} + 1\frac{R^{\rho\beta} R_{\rho\beta}}{4} + \nabla^2 R^{\alpha\mu} - \frac{1}{6} g^{\alpha\mu} \nabla^2 R + \frac{1}{6} g^{\alpha\mu} \nabla \nabla R - \nabla_\rho \nabla^\rho R^{\alpha\rho} = 0 \quad (20)
\]

and in (20) in the last line the last term can equivalently be rewritten by means of \( \nabla^\rho \nabla_\rho R_{\alpha\beta} \equiv \nabla_\mu \nabla^\mu R_{\alpha\beta} + [\nabla^\rho, \nabla_\mu] R_{\alpha\beta} \equiv \frac{1}{2} \nabla_\mu \nabla_\rho R - R^{\mu \rho} R_{\alpha \beta} + R_{\alpha \mu} R^{\mu \rho} \) in a form for which we can eventually simplify the write field equation in the form

\[
\nabla^2 R^{\alpha\mu} - \frac{1}{6} g^{\alpha\mu} \nabla^2 R - \frac{1}{3} \nabla^\alpha \nabla^\mu R + R^{\theta\sigma\rho\alpha} R^\theta_{\sigma\rho\beta} R^\mu_{\theta\sigma\rho\beta} - R^{\alpha\beta\sigma\rho} R^\beta_{\alpha\beta\mu\rho} R^\alpha_{\sigma\rho\beta} + 1\frac{R^{\alpha\mu} R^{\rho\beta} R_{\rho\beta}}{4} + 1\frac{R^{\alpha\mu} R^{\beta\rho} R_{\alpha\beta}}{4} + 1\frac{R^{\rho\beta} R_{\rho\beta}}{4} = 0 \quad (21)
\]

which is precisely the form of the field equations of the metric pure case in the vacuum of matter fields as considered in the standard theory of conformal gravitation; therefore we have proven that there are two possible schemes, the one in which there is an initial torsion later removed via a conformal transformation and the one in which torsion is never considered at all, and that in the vacuum they are such that the former has field equations that in the torsionless gauge reduce to be purely metric and such that they imply the latter field equations in which torsion is never present: so the theory discussed here is more restrictive that the standard one. This is true in general, but to see that in particular the
present theory is strictly contained in the standard one we will have to find specific counterexamples, and therefore in the following we will look for solutions in special cases given when particular symmetries are implemented.

In the next sub-sections we shall study two cases given when the spacetime has temporal and spatial projections are split and independent and for which the space is isotropic: then we discuss the case in which the space is isotropic solely and the case in which the space is isotropic and homogeneous, comparing the results with those of the usual treatment.

2.1 Isotropic Space

We shall now look for solutions in the case of isotropic space, that is with spherical symmetry; the case we will take into account is stationary: in the frame at rest with respect to the origin of the coordinates \((t, r, \theta, \phi)\), the metric given in terms of two functions of the radial coordinate \(A(r)\) and \(B(r)\) has the following form

\[
g_{tt} = A \\ g_{rr} = -B \\ g_{\theta\theta} = -r^2 \\ g_{\phi\phi} = -r^2(\sin \theta)^2
\]

(22)

with all other components equal to zero. The Levi-Civita connection is given as in the following

\[
\Lambda^t_r = \frac{\Lambda^r_t}{A} \\ \Lambda^t_\theta = \frac{\Lambda^\theta_t}{A} \\ \Lambda^r_r = \frac{B^r_t}{B} \\ \Lambda^r_\theta = -\frac{r}{B} \\ \Lambda^r_{\phi\phi} = -\frac{r}{B}(\sin \theta)^2 \\
\Lambda^\theta_r = \frac{1}{r} \\ \Lambda^\theta_{\phi\phi} = -\cot \theta(\sin \theta)^2 \\ \Lambda^\phi_r = \frac{1}{r} \\ \Lambda^\phi_{\phi\phi} = \cot \theta
\]

(23)

with all other components vanishing and the Riemann curvature is given as in the following

\[
R^t_{\ rtr} = -\frac{A''}{2A} + \frac{A^2}{A^2} + \frac{A'B'}{AB} \\ R^t_{\ r\theta} = -\frac{A'^r}{2AB} \\ R^t_{\ \phi\phi} = -(\frac{A'}{A})^2(\sin \theta)^2 \\
R^r_{\ r\theta} = \frac{B'}{2B} \\ R^r_{\ \phi\phi} = \frac{B'}{2B}(\sin \theta)^2 \\ R^\theta_{\ r\phi} = (1 - \frac{1}{r}) (\sin \theta)^2
\]

(24)

with contraction

\[
R^t_{\ t} = \frac{A''}{2AB} - \frac{A^2}{A^2} - \frac{A'B'}{AB^2} + \frac{A'}{AB} \\ R^r_r = \frac{A'^r}{2AB^2} - \frac{A^2}{AB^2} - \frac{A'B'}{AB^2} - \frac{B'}{B^2} \\
R^\theta_{\ \theta} = \frac{A'}{2ABr} - \frac{B'}{2Br} + \frac{1}{r} - \frac{1}{r^2} \\ R^\phi_{\ \phi} = \frac{A'}{2ABr} - \frac{B'}{2Br} + \frac{1}{r} - \frac{1}{r^2}
\]

(25)

and contraction

\[
R = \frac{A''}{AB} - \frac{A'^2}{2AB^2} - \frac{A'B'}{AB^2} + \frac{A'}{AB} - \frac{2B'}{B^2} + \frac{2}{r^2}
\]

(26)

all other components vanishing identically. From these quantities it is possible to calculate all covariant derivatives and square curvatures in the field equations.

With this symmetry, the field equations [13] are diagonal in which the component \((\phi\phi)\) is proportional to the component \((\theta\theta)\), and by employing the tracelessness of these field equations we see that also the component \((\theta\theta)\) can be written in terms of the component \((rr)\) and \((tt)\); the component \((rr)\) is solved in terms of the component \((tt)\) once the constraint \(AB = 1\) is imposed: what now is left to be the only independent component \((tt)\) can eventually be written after calling \(A = 1 + h\) in a form that is factorized in two linear field equation of the second-order derivative. With these symmetry and for the assumptions above, also the field equations [13] reduce to have the single independent component
given by \((rtt)\) in the form of an integral of the field equations. The general solution is then obtained by integrating this field equation \((19)\) and then constraining the result by means of the constraining equation \((15)\) as usually done.

However, we know that here the field equations \((19-18)\) are more restricted than the field equations of the standard theory \((21)\), and therefore any solution of the \((19-18)\) is a restricted solution of the \((21)\); now a possible strategy to find the most general solution of the field equations \((19-18)\) is to consider them as constraints with respect to the looser field equations \((21)\), and so using the constraints \((19-18)\) to constrain what is the most general solution of the field equations \((21)\); the most general solution of \((21)\) has been found by Mannheim and Kazanas in [9] to be given by

\[
h = -\frac{2m(1 - 3m\gamma)}{r} - 6m\gamma + 2\gamma r - kr^2
\]

and insisting that this solution is constrained by \((19-18)\) we get that here the most general solution is given by \((27)\) with constraint \(m\gamma = 0\) as the restriction we were seeking. In this case then we have two alternative solutions: we have either the solution given by

\[
h = -\frac{2m}{r} - kr^2
\]

which is asymptotically Schwarzschild for short distances and asymptotically de Sitter for large distances with constant spacetime curvature \(-12k\); or we have the solution

\[
h = 2\gamma r - kr^2
\]

which is conformal to the Friedmann-Lemaître-Robertson-Walker solution with constant spatial curvature \(-k\) for which the conformal tensor of curvature is equal to zero identically and so this solution is trivial.

We shall now compare the two approaches: we notice that in [9] Mannheim and Kazanas have a unique solution able to describe both the Schwarzschild limit for solar system scales and the Friedmann-Lemaître-Robertson-Walker limit for cosmological scales, and in addition as a by-product of the simultaneous presence of both terms it is also capable to fit the rotation curves for intermediate galactic scales; here instead we do not have such intermediate behaviour in either solution, and the short and large distance behaviours are described independently by two disconnected solutions of which one is only the Friedmann-Lemaître-Robertson-Walker solution without the Schwarzschild limit near the origin which is known to be correct. However, we do not regard this as a weak point of the approach followed here; to explain we first remark that any meaningful quantity in conformal relativity is to be constituted by conformal tensors, that is the Weyl curvature: if we calculate the Weyl tensor in these examples we see that for solution \((27)\) it is not zero even at infinity where a constant value proportional to the product \(m\gamma\) is reached, whereas for solution \((28)\) it drops to zero as the distance tends to infinity and solution \((29)\) it is always zero as expected. If we were to assume the reasonable hypothesis that all relevant quantities of a theory should vanish at infinity then \(m\gamma = 0\), which is precisely what we have in the present cases for both solutions \((28)\) or \((29)\); and if we neglect the trivial solution then there will be the solution \((28)\) alone. This solution
is such that it interpolates the Schwarzschild behaviour near the origin and it is able to interpolate the de Sitter behaviour for cosmological scales, therefore having the Einsteintian limit for the near universe and giving rise to an effective cosmological constant relevant for cosmological scales.

2.2 Isotropic-Homogeneous Space

We shall next look for solutions in the case of homogeneous and isotropic space with spatial curvature $k$; this case has only temporal dependence: with coordinates $(t, r, \theta, \phi)$ the metric is given in terms of only one function of the cosmological time $A(t)$ as in the following

$$
g_{tt} = 1 \quad g_{rr} = -\frac{A^2}{(1-kr^2)} \quad g_{\theta\theta} = -A^2r^2 \quad g_{\phi\phi} = -A^2r^2(sin \theta)^2
$$

with all other components equal to zero. The Levi-Civita connection is

$$
\Lambda^t_{rr} = \frac{\dot{A}}{A} \quad \Lambda^t_{r\theta} = \frac{k}{A} \quad \Lambda^t_{r\phi} = \frac{kr}{A}
$$

$$
\Lambda^\theta_{rr} = \frac{\dot{A}}{A} \quad \Lambda^\theta_{r\theta} = \frac{1}{r} \quad \Lambda^\theta_{r\phi} = -\cot \theta(sin \theta)^2
$$

$$
\Lambda^\phi_{rr} = \frac{\dot{A}}{A} \quad \Lambda^\phi_{r\theta} = \frac{1}{r} \quad \Lambda^\phi_{r\phi} = \cot \theta
$$

all other components vanishing and the Riemann curvature is

$$
R^t_{rr} = -3 \left( \frac{\dot{A}}{A} \right) \quad R^t_{r\theta} = -2 \left( \frac{\dot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k}{A} \right)
$$

$$
R^\theta_{r\theta} = -2 \left( \frac{\dot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k}{A} \right) \quad R^\phi_{\phi\theta} = -2 \left( \frac{\dot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{k}{A} \right)
$$

all other components vanishing identically. And from these quantities we can calculate all covariant derivatives and square curvatures in the field equations.

In this case, the metric is conformal to the solution (28) and (29) with both coefficients $m = \gamma = 0$, whose conformal tensor of curvature vanishes; so this is always a solution although trivial. Thus we have that although the solution of the present theory is more restricted than the solution of the standard theory in general, in this specific case instead they happen to be equivalent.

Conclusion

In this paper, we have considered the metric-torsional conformal gravity in vacuum, and we have studied a specific case in which torsion was equivalent to its
trace vector, eventually removable via a conformal transformation by choosing what we have called the torsionless gauge, and we have studied two cases of particular symmetries: in the case of isotropy, we found that a solution could be given as a restriction of the solution found in \[9\], that is as solution of the type \((27)\) with constraint \(m = 0\) or \(\gamma = 0\), and so of the type \((28)\) having a Schwarzschild behaviour for short distances and a de Sitter behaviour for large distances or of an alternative type \((29)\) which is trivial; in the case of isotropy and homogeneity, we found that metric is that we had in the previous case having both \(m = 0\) and \(\gamma = 0\), and so of the type \((29)\) and trivial. Eventually in future it would be interesting to know what happens for gravitational waves.

The solutions obtained here are more restrictive than the solutions given in standard situation, although in some case it may happen that they are completely equivalent: that is the solutions given in models in which torsion is present and later removed in the torsionless gauge are more constrained than those in which torsion is never present; this is of course expected because there are third-order derivative equations in the former case and so restricted with respect to the fourth-order derivative equations of the latter case.

We stress once again that what we have studied here is a torsionless case in which torsion is chosen to vanish by exploiting conformal transformations, and not in a forced manner, so that here torsionless is obtained as an allowed choice of gauge, and not as an imposition; nevertheless further generalizations could then be obtained by letting the whole torsion to be present, and therefore studying complete metric-torsional vacuum solutions.

All these are open problems that will have to be addresses if we want to see whether metric-torsional conformal gravitation is a good candidate for a renormalizable model of quantum gravity.

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