SIGNS OF SELF-DUAL DEPTH-ZERO SUPERCUSPIDAL REPRESENTATIONS

MANISH MISHRA

Abstract. Let $G$ be a quasi-split tamely ramified connected reductive group defined over a $p$-adic field $F$. We show that if $-1$ is in the $F$-points of the absolute Weyl group of $G$, then self-dual supercuspidal representations of $G(F)$ exist. Now assume further that $G$ is unramified and that the center of $G$ is connected. Let $\pi$ be a generic self-dual depth-zero regular supercuspidal representation of $G(F)$. We show that the Frobenius–Schur indicator of $\pi$ is given by the sign by which a certain distinguished element of the center of $G(F)$ of order two acts on $\pi$.

1. Introduction

Let $G$ be a group and let $(\tau, V)$ be an irreducible representation of $G$. If $\tau$ is self-dual, i.e., it is isomorphic to its contragradient $\hat{\tau}$, then there exists a non-degenerate $G$-invariant bilinear form $B : V \times V \to \mathbb{C}$ which is unique up to scalars. It is thus either symmetric or skew symmetric. The sign or the Frobenius–Schur indicator $\text{sgn}(\tau)$ of $\tau$ is defined to be $+1$ (resp. $-1$) according as $B$ is symmetric (resp. skew-symmetric). When $G$ is a finite group,

$$\text{sgn}(\tau) = \frac{1}{|G|} \sum_{g \in G} \omega_\tau(g^2),$$

where $\omega_\tau$ denotes the character of $\tau$. The expression on the right hand side of the above equality is zero when $\tau$ is not self-dual.

Now let $G$ be a connected reductive group defined over a local or finite field $F$ and let $\pi$ be a smooth irreducible representation of $G(F)$. When $\pi$ is self-dual and also generic, i.e., it admits a Whittaker model, D. Prasad introduced the idea of studying the sign by the action of certain order two element of the center $Z(F)$ of $G(F)$ [8, 9]. See [4, Conjecture 8.3] for a possible connection of this element with the Deligne-Langlands local root number.

Now let $G$ be unramified, $F$ be $p$-adic and let $\pi$ be an irreducible regular depth-zero supercuspidal representation. Regular depth-zero supercuspidal representations are the ones which arise from certain Deligne-Lusztig cuspidal representations of finite reductive groups. These representations were studied by DeBacker
Assuming further that $G$ has connected center and that $\pi$ is self-dual generic, we show in Theorem 4 that the sign of $\pi$ is given by the central character $\omega_\pi$ evaluated at a certain order two element $\epsilon$ of the center of $G(F)$. The main idea in the proof of Theorem 4 is to reduce the problem to a question about finite reductive groups and use Prasad’s result in that setting. In Prasad’s result, we first observe that the central element $\epsilon$ has an explicit description in terms of the root data. The assumption of the genericity of $\pi$ is used to ensure - by a result of DeBacker and Reeder - that the finite reductive group in the inducing data of $\pi$ has the same root system as $G$. We finally use Kaletha’s description of regular depth-zero representations to relate $\omega_\pi(\epsilon)$ to the analogous element of the finite reductive group.

When $G$ is quasi split and tamely ramified over $F$, we give in Theorem 5 a sufficient condition for self-dual supercuspidal representations of $G(F)$ to exist. We show that if $-1$ is in the $F$-points of the absolute Weyl group of $G$, then self-dual supercuspidal representations do exist. The proof uses Kaletha’s description of regular supercuspidal representations [7] and Hakim-Murnaghan’s result about dual Yu-datum [5].

2. Notations

Let $G$ be a reductive group over a local or finite field $F$. The central character of a representation $\pi$ of $G(F)$ will be denoted by $\omega_\pi$. The contragradient of $\pi$ will be denoted by $\hat{\pi}$. If $\pi$ is irreducible self-dual, then its Frobenius–Schur indicator will be denoted by $\text{sgn}(\pi)$. When $F$ is non-archimedean local, we write $B(G, F)$ (resp. $B_{\text{red}}(G, F)$) to denote the Bruhat-Tits building (resp. reduced Bruhat-Tits building) of $G(F)$. We follow the standard notations (as in [7, Sec. 2] for instance) for parahoric subgroups of $G(F)$ and their Moy-Prasad filtrations.

3. Finite reductive group

Let $G$ be a connected reductive group defined over a finite field $\mathbb{F}_q$. We assume that center $Z$ of $G$ is connected. Let $B = TU$ be an $\mathbb{F}_q$-Borel subgroup of $G$, where $U$ is the unipotent radical of $B$ and $T$ is an $\mathbb{F}_q$-maximal torus of $G$ contained in $B$. We denote the adjoint torus by $T_{\text{ad}}$. The character lattice of $T$ (resp. $T_{\text{ad}}$) will be denoted by $X^*(T)$ (resp. $X^*(T_{\text{ad}})$).

**Theorem 1** (Prasad). There exists an element $s_0$ in $T(\mathbb{F}_q)$ such that it operates by $-1$ on all the simple root spaces of $U$. Further, $t_0 := s_0^2$ belongs to $Z(\mathbb{F}_q)$ and $t_0$ acts on an irreducible, generic, self-dual representation by 1 iff the representation is orthogonal.
From the short exact sequence

\[ 1 \rightarrow Z \rightarrow T \rightarrow T_{\text{ad}} \rightarrow 1, \]

we get the long exact sequence

\[ 1 \rightarrow \mathbb{Z}(F_q) \rightarrow T(F_q) \rightarrow T_{\text{ad}}(F_q) \rightarrow H^1(\Gamma, Z) \rightarrow \cdots. \]

Since $Z$ is connected, $H^1(\Gamma, Z)$ is trivial by Lang’s theorem. Therefore,

\[ 1 \rightarrow \mathbb{Z}(F_q) \rightarrow T(F_q) \rightarrow T_{\text{ad}}(F_q) \rightarrow 1 \]

is exact. Let $\bar{\rho}$ denote half the sum of positive co-roots. Let $s'$ be the element of $T_{\text{ad}}(F_q) = \text{Hom}(X^*(T_{\text{ad}}), \mathbb{G}_m)(F_q)$ given by

\[ \chi \in X^*(T_{\text{ad}}) \mapsto (-1)^{(\chi, \bar{\rho})} \in \mathbb{G}_m. \]

Let $s$ denote any pull back of $s'$ in $T(F_q)$. Then $s$ operates by $-1$ on all simple root spaces of $\mathfrak{u}$. The element $t := s^2 \in Z(F_q)$ has the description

\[ \chi \in X^*(T) \mapsto (-1)^{(\chi, 2\bar{\rho})} \in \mathbb{G}_m. \]

We can thus rewrite the above Theorem as

**Theorem 1’.** Let $\pi$ be an irreducible generic representation of $G(F_q)$. Then $\text{sgn}(\pi) = \omega_{\pi}(t)$.

**Remark 2.** The assumption in Theorem 1 that $Z$ is connected cannot be entirely dropped, as shown in the counter example in [3 Sec. 9].

4. Regular depth-zero representations

Let $G$ be an unramified connected reductive group defined over a $p$-adic field $F$. Assume that the center $Z$ of $G$ is connected. Let $\mathfrak{f}$ denote the residue field of $F$.

4.1. Construction of regular depth-zero supercuspidal. For the definition of regular depth-zero supercuspidal representations and the details of the construction in this section, see [7] Sec. 3.2.3. Let $S$ be an elliptic maximal torus of $G$ and let $\theta : S(F) \rightarrow \mathbb{C}^\times$ be a depth-zero character. Let $S(F)_0$ be the Iwahori subgroup of $S(F)$. Assume that $\theta$ is regular, i.e., the stabilizer of $\theta |_{S(F)_0}$ in $N(S(F), G(F))/S(F)$ is trivial, where $N(S(F), G(F))$ denotes the normalizer of $S(F)$ in $G(F)$. The restriction of $\theta |_{S(F)_0}$ factors through a character $\tilde{\theta}$ of $S(F)_{0;0+}$. Let $x \in \mathcal{B}^{\text{red}}(G, F)$ be the vertex associated to $S$. The group $G(F)_{0;0+}$ is the $\mathfrak{f}$-points of a connected reductive $\mathfrak{f}$-group $\mathcal{G}_x$ and $S(F)_{0;0+}$ is the $\mathfrak{f}$-points of an elliptic maximal $\mathfrak{f}$-torus $S'$ of $\mathcal{G}_x$. Let $\kappa(S, \tilde{\theta})$ denote the irreducible cuspidal Deligne-Lusztig representation of $\mathcal{G}_x(\mathfrak{f})$ associated to the pair $(S', \tilde{\theta})$. Denote again by $\kappa(S, \theta)$ its inflation to $G(F)_{x,0}$. This representation extends to a representation $\tilde{\kappa}(S, \theta)$ of $Z(F)G(F)_{x,0} = G(F)_x$. 

\[ \text{This representation extends to a representation} \]
Therefore the root system \( \Phi(\be) \) is irreducible (and hence supercuspidal) and every regular depth-zero supercuspidal representation is of this form.

4.2. Sign. Choose a system of positive roots \( \Phi^+(G, S) \) for the set of roots \( \Phi(G, S) \). Let \( \bar{t} \) denote half the sum of positive roots and let \( t \) denote the element \( 2\bar{t}(-1) \in \text{Z}(F) \) (see [4, Sec. 8.5]).

\[ \text{Theorem 4.} \quad \text{Let } \pi \text{ be a generic regular depth-zero self-dual supercuspidal representation of } G(F). \text{ Then } \text{sgn}(\pi) = \omega_x(t). \]

\[ \text{Proof.} \quad \text{By Lemma 3 the representation } \pi \text{ arises out of a pair } (S, \theta) \text{ as in Section 4.1. Let } x \in B^{\text{red}}(G, F) \text{ be the vertex associated to } S. \text{ The normalizer in } G(F) \text{ of } G(F)_{x,0} \text{ is } G(F)_x [10, Lemma 3.3]. \text{ Therefore } G(F)_x \text{ is self normalizer. Since } \hat{\pi} = c\text{-Ind}_{G(F)_x}^{G(F)} \hat{\kappa} \text{ if and only if } \exists g \in G(F) \text{ such that } (G(F)_x, \hat{\kappa}) \text{ is conjugate by an element } g \in G(F) \text{ to the pair } (G(F)_x, \hat{\kappa}) \text{ (by [3] Theorem 6.7 for instance without any hypothesis). But then } g \in G(F)_x \text{ since } G(F) \text{ is self normalizer. Therefore, } \pi(S, \theta) \text{ is self-dual if and only if } \kappa(S, \theta) \text{ is so. Thus, } \text{sgn}(\pi) = \text{sgn}(\kappa). \text{ Since } \pi(S, \theta) \text{ is generic, the vertex } x \text{ associated to } S \text{ is hyperspecial [3, Theorem 1.1] (also [2, Lemma 6.1.2]). Therefore the root system } \Phi(G, S) \text{ can be identified with } \Phi(G_x, S'). \text{ Let } \Phi^+(G_x, S') \text{ be the positive roots of } \Phi(G_x, S') \text{ under the identification. Let } \bar{t} \text{ be half the sum of positive roots of } \Phi(G_x, S') \text{ and } t = 2\bar{t}(-1) \in Z_F. \text{ Since } x \text{ is hyperspecial, } Z \text{ is connected implies } Z_x \text{ is connected. Also, } \kappa(S, \theta) \text{ is generic [2, Lemma 6.1.2]. We therefore have by Theorem 3 that } \text{sgn}(\kappa) = \omega_t(\tilde{\theta}). \text{ But } \omega_t(\tilde{\theta}) = \tilde{\theta}(t) = \theta(t). \text{ The Theorem now follows because } \omega_x = \theta |_{Z(F)} \text{ by [7] Fact 3.38}. \]

5. Existence of self-dual representations

Let \( G \) be a quasi-split tamely ramified connected reductive group over a \( p \)-adic field \( F \). Let \( \Omega(S, G) \) be the absolute weyl group. In [6, Sec. 3.2.1], a vertex \( x \in B^{\text{red}}(G, F) \) is called superspecial if it is special in \( B^{\text{red}}(G, E) \), where \( E \) is any finite Galois extension of \( F \) splitting \( G \).

\[ \text{Theorem 5.} \quad \text{If } -1 \in \Omega(S, G)(F), \text{ then self-dual supercuspidal representations of } G(F) \text{ exist.} \]

\[ \text{Proof.} \quad \text{Let } (S, \theta) \text{ be a tame regular elliptic pair [7, Def. 3.23] such that } S \text{ is relatively unramified [7, Sec. 3.2.1] and the point } x \in B^{\text{red}}(G, F) \text{ associated to } S \text{ is superspecial. Let } \pi(S, \theta) \text{ be the associated regular supercuspidal representation. By [3, Theorem 4.25], } \pi(S, \theta) \equiv \pi(S, \theta^{-1}). \text{ By [7, Lemma 3.37], } \pi(S, \theta) \equiv \pi(S, \theta^{-1}) \text{ if and only if } (S, \theta) \text{ is } G(F)\text{-conjugate to } (S, \theta^{-1}). \text{ By [7, Lemma 3.11] } \Omega(S, G)(F) \cong N(S, G)(F)/S(F), \text{ where } N(S, G) \text{ denotes the normalizer of } S \text{ in } G. \text{ Thus if } -1 \in \Omega(S, G)(F), \text{ then it follows that } (S, \theta) \text{ is } G(F)\text{-conjugate to } (S, \theta^{-1}). \]
Remark 6. If the root system of $G$ is of type $B_n$, $C_n$, $E_7$, $E_8$, $G_2$ or $D_n$ ($n$-even), then the longest Weyl group element of $G$ is $-1$.

Remark 7. When $G = \text{GL}_n$, Adler [1] showed that the necessary and sufficient condition for self-dual regular supercuspidal representations of $G(F)$ to exist is that either $n$ or the residue characteristic of $F$ be even.

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E-mail address: manish@iiserpune.ac.in

Department of Mathematics, Indian Institute of Science Education and Research (IISER), Dr. Homi Bhabha Road, Pashan, Pune 411 008 INDIA