PARABOLIC SUBGROUPS OF COXETER GROUPS
ACTING BY REFLECTIONS ON CAT(0) SPACES

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Abstract. We consider a cocompact discrete reflection group $W$ of a CAT(0) space $X$. Then $W$ becomes a Coxeter group. In this paper, we study an analogy between the Davis-Moussong complex $\Sigma(W,S)$ and the CAT(0) space $X$, and show several analogous results about the limit set of a parabolic subgroup of the Coxeter group $W$.

1. Introduction and preliminaries

The purpose of this paper is to study the limit set of a parabolic subgroup of a reflection group of a CAT(0) space. A metric space $(X, d)$ is called a geodesic space if for each $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such a $\xi$ is called a geodesic). We say that an isometry $r$ of a geodesic space $X$ is a reflection of $X$, if

1. $r^2$ is the identity of $X$,
2. $\text{Int} F_r = \emptyset$ for the fixed-point set $F_r$ of $r$,
3. $X \setminus F_r$ has exactly two convex components $X^+_r$ and $X^-_r$, and
4. $rX^+_r = X^-_r$ and $rX^-_r = X^+_r$,

where the fixed-point set $F_r$ of $r$ is called the wall of $r$. Let $X^+_r$ and $X^-_r$ be the two convex connected components of $X \setminus F_r$, where $X^+_r$ contains a basepoint of $X$. An isometry group $\Gamma$ of a geodesic space $X$ is called a reflection group, if some set of reflections of $X$ generates $\Gamma$.

Let $\Gamma$ be a reflection group of a geodesic space $X$ and let $R$ be the set of all reflections of $X$ in $\Gamma$. Now we suppose that the action of $\Gamma$ on $X$ is proper, that is, $\{ \gamma \in \Gamma \mid \gamma x \in B(x, N) \}$ is finite for any $x \in X$ and $N > 0$ (cf. [2, p.131]). Then the set $\{ F_r \mid r \in R \}$ is locally finite. Let $C$ be a component of $X \setminus \bigcup_{r \in R} F_r$, which is called a chamber. Then $\Gamma C = X \setminus \bigcup_{r \in R} F_r$, $\overline{\Gamma C} = X$ and for each $\gamma \in \Gamma$, either $C \cap \gamma C = \emptyset$ or $C = \gamma C$. We say that $\Gamma$ is a cocompact discrete reflection group of $X$.

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if $\overline{C}$ is compact and $\{ \gamma \in \Gamma \mid C = \gamma C \} = \{1\}$. Every Coxeter group is a cocompact discrete reflection group of some CAT(0) space.

A Coxeter group is a group $W$ having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \rangle,$$

where $S$ is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

1. $m(s,t) = m(t,s)$ for any $s,t \in S$,
2. $m(s,s) = 1$ for any $s \in S$, and
3. $m(s,t) \geq 2$ for any $s,t \in S$ such that $s \neq t$.

The pair $(W,S)$ is called a Coxeter system. H.S.M. Coxeter showed that a group $\Gamma$ is a finite reflection group of some Euclidean space if and only if $\Gamma$ is a finite Coxeter group. Every Coxeter system $(W,S)$ induces the Davis-Moussong complex $\Sigma(W,S)$ which is a CAT(0) space ([5], [6], [14]). Then the Coxeter group $W$ is a cocompact discrete reflection group of the CAT(0) space $\Sigma(W,S)$. It is known that a group $\Gamma$ is a cocompact discrete reflection group of some geodesic space if and only if $\Gamma$ is a Coxeter group ([12]).

Let $W$ be a cocompact discrete reflection group of a CAT(0) space $X$, let $R$ be the set of reflections in $W$, let $S$ be a minimal subset of $R$ such that $C = \bigcap_{s \in S} X_s^+$ (i.e. $C \neq \bigcap_{s \in S \backslash \{s_0\}} X_s^+$ for any $s_0 \in S$). Then $\langle S \rangle C = X \setminus \bigcup_{r \in R} F_r = WC$, $S$ generates $W$ and the pair $(W,S)$ is a Coxeter system ([12]). For a subset $T$ of $S$, $W_T$ is defined as the subgroup of $W$ generated by $T$, and called a parabolic subgroup. It is known that the pair $(W_T,T)$ is also a Coxeter system.

Let $X$ be a CAT(0) space and let $\Gamma$ be a group which acts properly by isometries on $X$. The limit set of $\Gamma$ (with respect to $X$) is defined as

$$L(\Gamma) = \overline{\Gamma x_0} \cap \partial X,$$

where $\overline{\Gamma x_0}$ is the closure of the orbit $\Gamma x_0$ in $X \cup \partial X$ and $x_0$ is a point in $X$. We note that the limit set $L(\Gamma)$ is independent of the point $x_0 \in X$.

Also we say that (the action of) $\Gamma$ is convex-cocompact, if there exists a compact subset $K$ of $X$ such that $\mathcal{R}_{x_0}(L(\Gamma)) \subset \Gamma K$ for some $x_0 \in \mathbb{R}$, where $\mathcal{R}_{x_0}(L(\Gamma))$ is the union of the images of all geodesic rays $\xi$ issuing from $x_0$ with $\xi(\infty) \in L(\Gamma)$. We note that for a group acting on a proper CAT(0) space, “convex-cocompactness” agrees with “geometrically finiteness” (cf. [9] and [10]).

We first prove the following theorem in Section 2.

**Theorem 1.** For each subset $T \subset S$,

1. $W_T \overline{C}$ is convex (hence CAT(0)).
(2) the limit set \( L(W_T) \) of \( W_T \) coincides with the boundary \( \partial(W_T \overline{C}) \), and

(3) the action of \( W_T \) on \( X \) is convex-cocompact.

This theorem implies the following corollary ([9] and [10]).

Corollary 2. For each subset \( T \subset S \), the following statements are equivalent:

(1) \( [W : W_T] < \infty \);
(2) \( L(W_T) = \partial X \);
(3) \( \text{Int}_{\partial X} L(W_T) \neq \emptyset \).

In Section 3, we show the following theorem which is an analogue of Lemma 4.2 in [8].

Theorem 3. Let \( x_0 \in C \) and let \( w \in W \). Then there exists a reduced representation \( w = s_1 \cdots s_l \) such that

\[
d_H([x_0, wx_0], P_{s_1, \ldots, s_l}) \leq \text{diam} \overline{C},
\]

where \( d_H \) is the Hausdorff distance and \( P_{s_1, \ldots, s_l} = [x_0, s_1 x_0] \cup [s_1 x_0, (s_1 s_2) x_0] \cup \cdots \cup [(s_1 \cdots s_{l-1}) x_0, wx_0] \).

Using this theorem, we can obtain the following corollaries by the same argument used in [8] and [11].

Corollary 4. For each subset \( T \subset S \), the limit set \( L(W_T) \) is \( W \)-invariant if and only if \( W = W_\bar{T} \times W_{S \setminus \bar{T}} \).

Here \( W_\bar{T} \) is the essential parabolic subgroup of \( W_T \) (cf. [8]), that is, \( W_\bar{T} \) is the minimum parabolic subgroup of finite index in \( (W_T, T) \).

We denote by \( o(g) \) the order of an element \( g \) in the Coxeter group \( W \). For \( s_0 \in S \), we define \( W^{\{s_0\}} = \{ w \in W \mid \ell(ws) > \ell(w) \text{ for each } s \in S \setminus \{s_0\} \} \setminus \{1\} \). A subset \( A \) of a space \( Y \) is said to be dense in \( Y \), if \( \overline{A} = Y \). A subset \( A \) of a metric space \( Y \) is said to be quasi-dense, if there exists \( N > 0 \) such that each point of \( Y \) is \( N \)-close to some point of \( A \).

Corollary 5. Suppose that \( W^{\{s_0\}} \) is quasi-dense in \( W \) with respect to the word metric and \( o(s_0 t_0) = \infty \) for some \( s_0, t_0 \in S \). Then there exists \( \alpha \in \partial X \) such that the orbit \( W \alpha \) is dense in \( \partial X \).

Corollary 6. If the set

\[
\bigcup \{W^{\{s\}} \mid s \in S \text{ such that } o(st) = \infty \text{ for some } t \in S\}
\]

is quasi-dense in \( W \), then \( \{w^\infty \mid w \in W \text{ such that } o(w) = \infty\} \) is dense in \( \partial X \).
A subset $T$ of $S$ is said to be spherical, if $W_T$ is finite.

**Corollary 7.** Suppose that there exist a maximal spherical subset $T$ of $S$ and an element $s_0 \in S$ such that $o(s_0 t) \geq 3$ for each $t \in T$ and $o(s_0 t_0) = \infty$ for some $t_0 \in T$. Then

1. $W\alpha$ is dense in $\partial X$ for some $\alpha \in \partial X$, and
2. $\{w^\infty \mid w \in W$ such that $o(w) = \infty\}$ is dense in $\partial X$.

2. **Convex-cocompactness of parabolic subgroups**

Let $W$ be a cocompact discrete reflection group of a CAT(0) space $X$, let $C$ be a chamber containing a basepoint of $X$, let $R$ be the set of reflections in $W$, and let $S$ be a minimal subset of $R$ such that $C = \bigcap_{s \in S} X_s^+$. (Then the pair $(W, S)$ is a Coxeter system [12].) For each reflection $r$ in $W$, $F_r$ is the wall of $r$ and $X_r^+$ and $X_r^-$ are the two convex components of $X \setminus F_r$ such that $C \subset X_r^+$ and $C \cap X_r^- = \emptyset$. We note that $F_r$, $X_r^+ \cup F_r$ and $X_r^- \cup F_r$ are convex.

The following lemmas are known.

**Lemma 2.1** ([12, Lemma 3.4]). Let $w \in W$ and let $s \in S$. Then $\ell(w) < \ell(sw)$ if and only if $wC \subset X_s^+$.

**Lemma 2.2** ([7, Lemma 1.3]). Let $w \in W$ and let $T \subset S$. Then there exists a unique element of shortest length in the coset $W_T w$. Moreover, the following statements are equivalent:

1. $w$ is the element of shortest length in the coset $W_T w$;
2. $\ell(sw) > \ell(w)$ for any $s \in T$;
3. $\ell(vw) = \ell(v) + \ell(w)$ for any $v \in W_T$.

We first show the following lemma.

**Lemma 2.3.** Let $T \subset S$. Then $wX_s^+ = X_{ws^{-1}}^+$, for any $w \in W_T$ and $s \in S \setminus T$.

**Proof.** Let $T \subset S$, $w \in W_T$ and $s \in S \setminus T$. Then $\ell(sw^{-1}) > \ell(w^{-1})$. Hence $w^{-1}C \subset X_s^+$ by Lemma 2.1. Thus $C \subset wX_s^+$, i.e., $wX_s^+ = X_{ws^{-1}}^+$. \hfill \square

Using lemmas above, we prove the following theorem.

**Theorem 2.4.** For each subset $T \subset S$,

1. $W_T \overline{C}$ is convex (hence CAT(0)),
2. the limit set $L(W_T)$ of $W_T$ coincides with the boundary $\partial(W_T \overline{C})$, and
3. the action of $W_T$ on $X$ is convex-cocompact.
Proof. Let $T \subset S$. Then we show that

$$W_T \overline{C} = \bigcap \{X_{wsw^{-1}}^+ \mid w \in W_T, \ s \in S \setminus T\}.$$ 

For each $v, w \in W_T$ and $s \in S \setminus T$, $C \subset v^{-1}wX^+_s$ by Lemma 2.3. Hence $vC \subset wX^+_s = X^+_{wsw^{-1}}$ by Lemma 2.3. Thus $vC \subset \overline{X^+_{wsw^{-1}}}$ for any $v, w \in W_T$ and $s \in S \setminus T$, that is,

$$W_T \overline{C} \subset \bigcap \{X_{wsw^{-1}}^+ \mid w \in W_T, \ s \in S \setminus T\}.$$ 

To prove

$$W_T \overline{C} \supset \bigcap \{X_{wsw^{-1}}^+ \mid w \in W_T, \ s \in S \setminus T\},$$ 

we show that for each $v \in W \setminus W_T$, there exist $w \in W_T$ and $s \in S \setminus T$ such that $vC \subset X^-_{wsw^{-1}}$. Let $v \in W \setminus W_T$. By Lemma 2.2, there exists a unique element $x \in W_Tv$ of shortest length. Let $w = vx^{-1}$. Here we note that $w \in W_T$ and $\ell(v) = \ell(w) + \ell(x)$. Let $s \in S$ such that $\ell(sx) < \ell(x)$. By Lemma 2.2 (2), $s \in S \setminus T$. Then

$$\ell(sw^{-1}v) = \ell(sx) < \ell(x) = \ell(w^{-1}v).$$ 

Hence $w^{-1}vC \subset X^-_s$ by Lemma 2.1. By Lemma 2.3, $vC \subset wX^-_s = X^-_{wsw^{-1}}$. Therefore

$$W_T \overline{C} \supset \bigcap \{X_{wsw^{-1}}^+ \mid w \in W_T, \ s \in S \setminus T\}.$$ 

Since $X^+_{wsw^{-1}} = X^+_{wsw^{-1}} \cup F_{wsw^{-1}}$ is convex for any $w \in W_T$ and $s \in S \setminus T$, $W_T \overline{C}$ is convex. Hence $L(W_T) = \partial(W_T \overline{C})$ and the action of $W_T$ on $X$ is convex-cocompact. \hfill \Box

3. ON GEODESICS AND REDUCED REPRESENTATIONS

We give the following lemma which is an analogue of a result about Davis-Moussong complexes.

**Lemma 3.1.** Let $w \in W$, let $w = s_1 \cdots s_l$ be a reduced representation and let $T = \{s_1, \ldots, s_l\}$. Then

$$\overline{C} \cap w \overline{C} = \bigcap_{t \in T} (F_t \cap \overline{C}) = \bigcap_{t \in T} (t \overline{C} \cap \overline{C}) = \bigcap_{v \in W_T} v \overline{C}.$$

**Proof.** Let $y \in \overline{C} \cap w \overline{C}$. Since $\ell(s_1 w) < \ell(w)$, $wC \subset X^-_{s_1}$ by Lemma 2.1. Then

$$y \in \overline{C} \cap w \overline{C} \subset X^+_{s_1} \cap X^-_{s_1} = F_{s_1}.$$

Hence $s_1 y = y$ and

$$y = s_1 y \in s_1 (\overline{C} \cap w \overline{C}) = s_1 \overline{C} \cap (s_2 \cdots s_l) \overline{C},$$ 

i.e., \( y \in \overline{C} \cap (s_2 \cdots s_l)\overline{C} \). By iterating the above argument, \( s_iy = y \) for any \( i \in \{1, \ldots, l\} \), that is, \( ty = y \) for any \( t \in T \). Hence \( y \in \bigcap_{t \in T}(F_t \cap \overline{C}) \). Thus \( \overline{C} \cap w\overline{C} \subset \bigcap_{t \in T}(F_t \cap \overline{C}) \).

Since \( F_t \cap \overline{C} = t\overline{C} \cap \overline{C} \) for any \( t \in T \), \( \bigcap_{t \in T}(F_t \cap \overline{C}) = \bigcap_{t \in T}(t\overline{C} \cap \overline{C}) \).

Let \( y \in \bigcap_{t \in T}(F_t \cap \overline{C}) \). Then \( ty = y \) for any \( t \in T \). Since \( T \) generates \( W_{T} \), \( vy = y \) for any \( v \in W_{T} \). Hence \( y = vy \in v\overline{C} \) for each \( v \in W_{T} \). Thus \( \bigcap_{t \in T}(F_t \cap \overline{C}) \subset \bigcap_{v \in W_{T}} v\overline{C} \).

It is obvious that \( \bigcap_{v \in W_{T}} v\overline{C} \subset \overline{C} \cap w\overline{C} \), since \( 1, w \in W_{T} \).

\( \square \)

**Lemma 3.2.** Let \( w \in W \), let \( w = s_1 \cdots s_l \) be a reduced representation and let \( T = \{s_1, \ldots, s_l\} \). Then \( \overline{C} \cap w\overline{C} \neq \emptyset \) if and only if \( W_{T} \) is finite.

**Proof.** Suppose that \( \overline{C} \cap w\overline{C} \neq \emptyset \). Then \( \bigcap_{v \in W_{T}} v\overline{C} \neq \emptyset \) by Lemma 3.1. Hence \( W_{T} \) is finite because the action of \( W \) on \( X \) is proper.

Suppose that \( W_{T} \) is finite. Then \( W_{T} \) acts on the CAT(0) space \( W_{T}\overline{C} \) by Theorem 2.4. By [2, Corollary II.2.8(1)], there exists a fixed-point \( y \in W_{T}\overline{C} \) such that \( vy = y \) for any \( v \in W_{T} \). Then \( y \in \overline{C} \cap w\overline{C} \) which is non-empty. \( \square \)

By the proof of [8, Lemma 4.2], we can obtain the following theorem from Lemmas 3.1, 3.1 and 3.2.

**Theorem 3.3.** Let \( x_0 \in C \) and let \( w \in W \). Then there exists a reduced representation \( w = s_1 \cdots s_l \) such that

\[
d_H([x_0, wx_0], P_{s_1, \ldots, s_l}) \leq \operatorname{diam} \overline{C},
\]

where \( d_H \) is the Hausdorff distance and \( P_{s_1, \ldots, s_l} = [x_0, s_1x_0] \cup [s_1x_0, (s_1s_2)x_0] \cup \cdots \cup [(s_1 \cdots s_{l-1})x_0, wx_0] \).

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