Wormholes Immersed in Rotating Matter

Christian Hoffmann\textsuperscript{1}, Theodora Ioannidou\textsuperscript{2}, Sarah Kahlen\textsuperscript{1}, Burkhard Kleihaus\textsuperscript{1}, Jutta Kunz\textsuperscript{1}

\textsuperscript{1} Institut für Physik, Universität Oldenburg, Postfach 2503
D-26111 Oldenburg, Germany
E-mail: christian.hoffmann@uni-oldenburg.de; sarah.kahlen@uni-oldenburg.de; b.kleihaus@uni-oldenburg.de; jutta.kunz@uni-oldenburg.de

\textsuperscript{2} Faculty of Civil Engineering, School of Engineering
Aristotle University of Thessaloniki, 54249, Thessaloniki, Greece
E-mail: ti3@auth.gr

We consider Ellis wormholes immersed in rotating matter in the form of an ordinary complex boson field. The resulting wormholes may possess full reflection symmetry with respect to the two asymptotically flat spacetime regions. However, there arise also wormhole solutions where the reflection symmetry is broken. The latter always appear in pairs. We analyse the properties of these rotating wormholes and show that their geometry may feature single throats or double throats. We also discuss the ergoregions and the lightring structure of these wormholes.

Keywords: wormholes, boson stars, lightrings

1. Introduction

The non-trivial topology of wormholes requires the presence of exotic matter in Einstein’s General Relativity (see e.g. the recent review\textsuperscript{1} of the field). Choosing a massless phantom (scalar) field for the exotic matter, Ellis\textsuperscript{2,3} and Bronnikov\textsuperscript{4} found static spherically symmetric wormhole solutions, which connect two asymptotically flat regions of space-time.

Their rotating generalizations were first constructed perturbatively for slow rotation\textsuperscript{5,6} and later numerically for rapid rotation\textsuperscript{7}. In these wormhole solutions the rotation of the throat and thus the spacetime is achieved by an appropriate choice of the boundary conditions. However, this results in the fact that the two asymptotic regions are rotating with respect to one another. Thus while both asymptotic regions are asymptotically flat, the spacetime is not symmetric with respect to reflection at the throat.

In order to obtain rotating wormholes that exhibit a reflection symmetry at the throat, one can immerse the throat inside rotating matter\textsuperscript{8,9}. Then
the rotation of the matter will drag the spacetime and thus the throat of
the wormhole.

2. Wormholes Immersed in Rotating Matter

A nice model to study such wormholes immersed in rotating matter is
obtained by adding an ordinary massive complex scalar field to the real
phantom scalar field into the action, coupling both to gravity. Without
the phantom scalar field the model would yield non-rotating and rotating
boson stars. The combination of both scalar fields then allows for rotating
wormholes immersed in scalar matter, that are reflection symmetric. More-
over, a new type of non-symmetric wormholes emerges. In the following we
will briefly discuss the model and present its rotating wormhole solutions
and analyze their properties\textsuperscript{8–11}.

2.1. Theoretical Setting

We consider the action
\[ S = \int \left[ \frac{1}{2\kappa} R + \mathcal{L}_{bs} + \mathcal{L}_{ph} \right] \sqrt{-g} \, d^4x \] (1)

with the Einstein-Hilbert term, the Lagrangian $\mathcal{L}_{bs}$ of the complex scalar
field $\Phi$
\[ \mathcal{L}_{bs} = \frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi^* \partial_\nu \Phi + \partial_\mu \Phi \partial_\nu \Phi^*) - m_{bs}^2 |\Phi|^2 , \] (2)

and the Lagrangian $\mathcal{L}_{ph}$ of the phantom field $\Psi$,
\[ \mathcal{L}_{ph} = \frac{1}{2} \partial_\mu \Psi \partial_\mu \Psi . \] (3)

The field equations then consist of the Einstein equations
\[ G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu} \] (4)

and the matter field equations
\[ \nabla^\mu \nabla_\mu \Psi = 0 \] (5)

and
\[ \nabla^\mu \nabla_\mu \Phi = m_{bs}^2 \Phi . \] (6)

An appropriate Ansatz for the metric is given by
\[ ds^2 = -e^f dt^2 + e^{g-F} \left[ e^b (dr^2 + h d\theta^2) + h \sin^2 \theta(d\varphi - \omega dt)^2 \right] , \] (7)
where \( f, q, b \) and \( \omega \) are functions of \( \eta \) and \( \theta \), \( h = \eta^2 + \eta_0^2 \) with throat parameter \( \eta_0 \), and \( \eta \) takes positive and negative values, \(-\infty < \eta < \infty\). The ansatz for the rotating bosonic matter is taken as for boson stars

\[
\Phi(t, \eta, \theta, \varphi) = \phi(\eta, \theta) e^{i\omega_s t + in\varphi},
\]

where \( \phi(\eta, \theta) \) is a real function, \( \omega_s \) is the boson frequency, \( n \) is a rotational quantum number, and the ansatz for the phantom field \( \Psi \) is simply

\[
\Psi(t, \eta, \theta, \varphi) = \psi(\eta, \theta).
\]

(8)

The resulting set of six coupled non-linear partial differential equations is then solved numerically subject to an appropriate set of boundary conditions in the two asymptotic regions \( \eta \rightarrow \pm \infty \), on the axis of rotation \( \theta = 0 \), and in the equatorial plane \( \theta = \pi/2 \).

2.2. **Symmetric Wormholes**

Let us now discuss the properties of the symmetric wormholes immersed in bosonic matter. In Fig.1a and b we show their mass \( M \) and particle number \( Q \) versus the boson frequency \( \omega_s \) for a typical set of such wormholes \( (\eta_0 = 1, n = 0, 1, 2) \). Their angular momentum \( J \) is given by \( J = nQ \).

Clearly, the domain of existence is limited by a maximal value \( \omega_{\text{max}} = m_b \), where a vacuum configuration with \( M = 0 = Q \) is reached, analogous to boson stars. For large values of \( \omega_s \) the global charges of the wormholes follow those of boson stars (see the thin black lines in the figures). However, for small \( \omega_s \) the spiralling behaviour present in boson stars is basically lost. In fact, the would-be spirals unwind with respect to the frequency and continue to lower frequencies (possibly reaching a singular configuration \(^{10}\)).

We emphasize that these solutions satisfy the same boundary conditions in both asymptotic regions. Thus in the case of rotation, it is the complex scalar field with its finite rotational quantum number \( n \) that imposes the rotation on the configuration. Then the rotation of the scalar field is reflected in the rotation of the spacetime, leading to a rotating throat and frame dragging. Not too surprisingly therefore a sufficiently fast rotation will lead to ergoregions in the wormhole spacetimes. The circumferential radii of the ergoregions are exhibited by the black lines in Fig. 1c and d.

When we consider the geometry of the wormhole solutions we realize that there arises a transition from ordinary single throat wormholes to double throat wormholes with an equator in between, as the frequency \( \omega_s \) is decreased. At the transition value the throat degenerates to an inflection
point, i.e., the circumferential radius at the center $\eta = 0$ has vanishing first and second derivative.

Of interest are also the lightrings of these spacetimes, as exhibited in Fig. 1c and d for corotating and counterrotating photon orbits, respectively. In particular, we show their circumferential radii $R_e(\eta_0^+)$ and $R_e(\eta_0^-)$. We note that a single lightring exists for large $\omega_s$. For smaller $\omega_s$ two more lightrings emerge. One lightring is always located at $\eta = 0$ (i.e., at the throat or equator), and the additional ones are located symmetrically w.r.t. $\eta = 0$. In the $n = 2$ case up to five lightrings of counterrotating massless particles exist.

2.3. **Asymmetric Wormholes**

Let us now focus on another new aspect, namely the presence of symmetric and asymmetric wormhole solutions immersed in bosonic matter. Whereas

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Fig. 1. Properties of symmetric wormhole solutions (throat parameter $\eta_0 = 1$, rotational quantum numbers $n = 0, 1, 2$) versus the boson frequency $\omega_s$: (a) the mass $M$; (b) the particle number $Q$; the dashed lines indicate the respective boson star solutions; (c) the corotation lightring circumferential radius $R_e(\eta_0^+)$; (d) the counterrotation lightring circumferential radius $R_e(\eta_0^-)$; also shown are the circumferential radii of the ergosurfaces (black lines).
the field equations are symmetric with respect to reflection of the radial coordinate at the center, and the same boundary conditions are employed in both asymptotically flat regions, the solutions may, however, still be either symmetric or asymmetric with respect to such a reflection. It is the non-trivial topology, which allows for asymmetric solutions, as well.

Starting again the discussion with the global charges, we exhibit the mass and the particle number of these asymmetric solutions in Fig. 2, and compare to the corresponding symmetric solutions. Because of the asymmetry the boson field is different in both regions of the spacetime, resulting in different global charges, read off asymptotically. Thus for a given \( n \) there is one curve for the symmetric wormholes, but there are two curves for the asymmetric wormholes. Note, that for these asymmetric solutions the angular momentum no longer satisfies \( J = nQ \).

As in the case of the symmetric wormholes, there arise double throat wormholes, but now the equator will not reside at \( \eta = 0 \), but instead arise somewhere in one of the regions. Of course, for each asymmetric solution there exists a second solution, obtained for \( \eta \to -\eta \). The asymmetric wormholes may also possess ergoregions and multiple lightrings.

3. Conclusions and Outlook

We have obtained a new type of rotating wormhole by immersing the throat in rotating bosonic matter, adopting some features from boson stars. We have studied various physical properties of these solutions, like their global charges, their ergoregions, and their lightrings. To conclude, let us illustrate these new wormhole solutions immersed in rotating matter, via embeddings
of a symmetric and an asymmetric double throat wormhole in Fig. 3.

All these wormholes are based on General Relativity and therefore need exotic matter for their existence. It will be interesting to consider this new type of wormhole solutions in generalized theories of gravity, which allow for wormholes without the need for exotic matter. Another point of interest will be the study of the stability of these solutions.

References
1. F. S. N. Lobo, Fundam. Theor. Phys. 189, pp. (2017).
2. H. G. Ellis, J. Math. Phys. 14, 104 (1973).
3. H. G. Ellis, Gen. Rel. Grav. 10, 105 (1979).
4. K. A. Bronnikov, Acta Phys. Polon. B4, 251 (1973).
5. P. E. Kashargin and S. V. Sushkov, Grav. Cosmol. 14, 80 (2008).
6. P. E. Kashargin and S. V. Sushkov, Phys. Rev. D 78, 064071 (2008).
7. B. Kleihaus and J. Kunz, Phys. Rev. D 90, 121503 (2014)
8. C. Hoffmann, T. Ioannidou, S. Kahlen, B. Kleihaus and J. Kunz, Phys. Lett. B 778, 161 (2018)
9. C. Hoffmann, T. Ioannidou, S. Kahlen, B. Kleihaus and J. Kunz, Phys. Rev. D 97, no. 12, 124019 (2018)
10. V. Dzhunushaliev, V. Folomeev, C. Hoffmann, B. Kleihaus and J. Kunz, Phys. Rev. D 90, no. 12, 124038 (2014)
11. C. Hoffmann, T. Ioannidou, S. Kahlen, B. Kleihaus and J. Kunz, Phys. Rev. D 95, no. 8, 084010 (2017)