UNIQUENESS FOR INVERSE PROBLEM OF DETERMINING FRACTIONAL ORDERS FOR TIME-FRACTIONAL ADVECTION-DIFFUSION EQUATIONS

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Abstract. We consider initial boundary value problems of time-fractional advection-diffusion equations with the zero Dirichlet boundary value $\partial^\alpha_t u(x, t) = -Au(x, t)$, where $-A = \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j) + \sum_{j=1}^d b_j(x) \partial_j + c(x)$. We establish the uniqueness for an inverse problem of determining an order $\alpha$ of fractional derivatives by data $u(x_0, t)$ for $0 < t < T$ at one point $x_0$ in a spatial domain $\Omega$. The uniqueness holds even under assumption that $\Omega$ and $A$ are unknown, provided that the initial value does not change signs and is not identically zero. The proof is based on the eigenfunction expansions of finitely dimensional approximating solutions, a decay estimate and the asymptotic expansions of the Mittag-Leffler functions for large time.

Key words. fractional advection-diffusion equation, uniqueness, fractional order

AMS subject classifications. 35R30, 35R11

1. Introduction

Throughout this article, we assume that the spatial dimensions $d = 1, 2, 3$. We can similarly argue for higher dimensions $d \geq 4$, but we need more regularity for initial values in (1.1) and (1.5) later described. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \Omega$ and let

$$0 < \alpha, \beta < 1.$$ 

By $\partial^\alpha_t$ we denote the Caputo derivative:

$$\partial^\alpha_t g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} g(s) ds$$

for $\alpha \in (0, 1)$ (e.g., Podlubny [25]).

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Throughout this article, we set

\[-Av(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x)) + \sum_{j=1}^{d} b_j(x)\partial_j v(x) + c(x)v(x), \quad x \in \Omega, \quad (1.1)\]

where \(a_{ij} = a_{ji} \in C^1(\overline{\Omega}), b_j \in C^1(\Omega), 1 \leq i, j \leq d, c \in C(\overline{\Omega})\) are all real-valued. Moreover we assume that \(c(x) \leq 0\) for \(x \in \overline{\Omega}\), and there exists a constant \(\sigma = \sigma(a_{ij}) > 0\) such that

\[\sum_{i,j=1}^{d} a_{ij}(x)\zeta_i\zeta_j \geq \sigma(a_{ij}) \sum_{i=1}^{d} \zeta_i^2 \quad \text{for all} \quad x \in \overline{\Omega} \quad \text{and} \quad \zeta_1, \ldots, \zeta_d \in \mathbb{R}. \quad (1.2)\]

We consider an initial boundary value problem for a time-fractional advection-diffusion equation:

\[
\begin{aligned}
\partial_{\alpha}^t u(x,t) &= -Au(x,t), \quad x \in \Omega, \quad 0 < t < T, \\
 u \big|_{\partial \Omega \times (0,T)} &= 0, \\
 u(x,0) &= a(x), \quad x \in \Omega. 
\end{aligned}
(1.3)
\]

For \(\alpha \in (0,1)\), the first equation in (1.3) is called a fractional advection-diffusion equation with the first-order term \(\sum_{j=1}^{d} b_j \partial_j u\), and is a macroscopic model for anomalous diffusion in heterogeneous media. Fractional diffusion equations are studied related also to diffusions in fractals and we refer for example to Mainardi \([22]\), Metzler, Glöckle and Nonnenmacher \([23]\), Metzler and Klafter \([24]\), Roman and Alemany \([27]\).

Let \(x_0 \in \Omega\) and \(0 < t < T\) be arbitrarily chosen. The main subject of this article is **Inverse problem of determining the order \(\alpha\).**

*Determine \(\alpha\) by data \(u(x_0,t)\) for \(0 < t < T\) for (1.3).*

Several properties such as asymptotic behavior as \(t \rightarrow \infty\) of solution \(u\) to (1.1) depend on the fractional order \(\alpha\) of the derivative. It is known that \(\alpha\) is an essential physical parameter characterizing the anomaly of diffusion. Thus our inverse problem is important not only from the theoretical viewpoint but also for modelling actual anomalous advection-diffusion of substances such as contaminants by a relevant fractional diffusion equation.

As for inverse problems of determining orders and other parameters, we refer to Alimov and Ashurov \([3]\), Ashurov and Umarov \([4]\), Cheng, Nakagawa, Yamamoto and Yamazaki \([5]\), Hatano, Nakagawa, Wang and Yamamoto \([9]\), Janno \([10]\), Janno and Kinash \([11]\), Jin and Kian \([12]\), Krasnoschok, Pereverzyev, Siryk and Vasylyeva \([15]\), Li, Zhang, Jia and Yamamoto \([17]\), Li and Yamamoto \([20]\), Tatar, Tinaztepe and Ulusoy \([29]\), Tatar and Ulusoy \([30]\), Yamamoto \([32], [33]\), Yu, Jiang and Qi \([34]\), for example. See Li, Liu and Yamamoto \([19]\) as a survey.
In [15], the authors proved that for smooth function \( \varphi(t) \) on \([0,T]\), the order \( \alpha \) can be uniquely determined only by \( \partial^\alpha_0 \varphi \) and \( \varphi \) near \( t = 0 \) only, whether or not \( \varphi(t) \) is related to a solution to a fractional equation equation, and applied such uniqueness to other inverse problem of determining an order as well as a solution to semilinear subdiffusion equations. In [15], the continuity of \( u(x_0,t) \) and \( \partial_0^\alpha u(x_0,t) \) at \( t = 0 \) is essential and so initial value \( a \) must be smoother.

We can prove the uniqueness by specified data of solution of even if the coefficients of \( A \) are unknown, and we refer to [5] as early work, and see [12], [33] as related articles.

However, these works do not consider advection terms, that is, assume that \( b_1 = b_2 = \cdots = b_d = 0 \) in \( \Omega \) to study the uniqueness in determining the order \( \alpha \). For such a symmetric \( A \), relying on a well-known eigenfunction expansion of solution \( u \) to (1.3), we can establish the uniqueness for the inverse problem directly. However, for non-symmetric \( A \), such a method does not work, and to the best knowledge of the author, there are no works on the uniqueness in determining the order for non-symmetric \( A \) given by (1.1).

On the other hand, since the advection term \( \sum_{j=1}^d b_j(x) \partial_j u \) is of lower-order and does not drastically change the structure of the equation, for non-symmetric \( A \), we can naturally expect a similar uniqueness result to [4], [12], [32], [33]. The main purpose of this article is to prove that such a conjecture is correct for \( A \) given by (1.1). Moreover, suggested by [5], [12] and [33], the order \( \alpha \) can be uniquely determined independently of the operator \( A \) and the domain \( \Omega \).

For the formulation of our result, we introduce operators and domains. Let \( L^2(\Omega) \), \( H^2(\Omega) \), \( H^1_0(\Omega) \) denote usual Lebesgue space and Sobolev spaces (e.g., Adams [1]). By \( \| \cdot \|_{L^2(\Omega)} \) and \( (\cdot,\cdot)_{L^2(\Omega)} \), we denote the norm and the scalar product in \( L^2(\Omega) \) respectively.

We recall the Poincaré inequality: there exists a constant \( C(\Omega) > 0 \), depending on a bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \) such that

\[
C(\Omega) \int_\Omega |w|^2 dx \leq \int_\Omega |\nabla w|^2 dx \quad \text{for} \ w \in H^1_0(\Omega). \tag{1.4}
\]

For a bounded domain \( \Omega \subset \mathbb{R}^d \) with smooth boundary \( \partial \Omega \), we define \( -A \) by (1.1) and assume (1.2), and for another bounded domain \( \widetilde{\Omega} \) with smooth boundary \( \partial \widetilde{\Omega} \), we additionally set

\[
(-\widetilde{A}v)(x) = \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j v(x)) + \sum_{j=1}^d \tilde{b}_j(x) \partial_j v(x) + \tilde{c}(x)v(x), \quad x \in \Omega, \tag{1.5}
\]
where \( \tilde{a}_{ij} = \tilde{a}_{ji} \in C^1(\bar{\Omega}) \), \( \tilde{b}_j \in C^1(\Omega) \), \( 1 \leq i, j \leq d \), \( \tilde{c} \in C(\bar{\Omega}) \) are all real-valued. Moreover we assume that \( \tilde{c}(x) \leq 0 \) for \( x \in \bar{\Omega} \), and there exists a constant \( \sigma = \sigma(\tilde{a}_{ij}) > 0 \) such that

\[
\sum_{i,j=1}^{d} \tilde{a}_{ij}(x)\zeta_i\zeta_j \geq \sigma(\tilde{a}_{ij})\sum_{i=1}^{d} \zeta_i^2 \quad \text{for all } x \in \bar{\Omega} \text{ and } \zeta_1, ..., \zeta_d \in \mathbb{R}.
\]  

(1.6)

We consider an initial boundary value problem for a time-fractional advection-diffusion equation:

\[
\begin{cases}
\partial_t^\alpha \tilde{u}(x,t) = -\tilde{A}\tilde{u}(x,t), & x \in \bar{\Omega}, 0 < t < T, \\
\tilde{u}|_{\partial \Omega \times [0,T]} = 0, \\
\tilde{u}(x,0) = \tilde{a}(x), & x \in \bar{\Omega}.
\end{cases}
\]

(1.7)

We define an operator \( \tilde{A} \) in \( L^2(\Omega) \) attaching the zero Dirichlet boundary condition, that is, the domain \( \mathcal{D}(\tilde{A}) \) of \( \tilde{A} \) is \( H^2(\Omega) \cap H^1_0(\Omega) \), and the same for \( \tilde{\Lambda} : \mathcal{D}(\tilde{\Lambda}) := H^2(\Omega) \cap H^1_0(\Omega) \). Then it is known that the spectrum \( \sigma(\tilde{A}) \) of \( \tilde{A} \) consists entirely of eigenvalues with finite multiplicities (e.g., Agmon [2]). With some numbering, we can set

\[
\sigma(\tilde{A}) = \{ \lambda_n \}_{n \in \mathbb{N}} \subset \mathbb{C}, \quad \lim_{n \to \infty} \Re \lambda_n = \infty.
\]

We note \( \lambda_n \notin \mathbb{R} \) in general. Similarly we set \( \sigma(\tilde{\Lambda}) = \{ \tilde{\lambda}_n \}_{n \in \mathbb{N}} \).

We further assume

\[
\begin{cases}
\inf_{x \in \Omega} \left( \frac{1}{2} \sum_{j=1}^{d} \partial_j b_j(x) - c(x) \right) + C(\Omega)\sigma(\tilde{a}_{ij}) > 0, & c(x) \leq 0, \quad x \in \bar{\Omega}, \\
\inf_{x \in \Omega} \left( \frac{1}{2} \sum_{j=1}^{d} \partial_j \tilde{b}_j(x) - \tilde{c}(x) \right) + C(\bar{\Omega})\sigma(\tilde{a}_{ij}) > 0, & \tilde{c}(x) \leq 0, \quad x \in \bar{\Omega}.
\end{cases}
\]

(1.8)

Condition (1.8) yields

\[
\{ \Re \lambda_n; \lambda_n \in \sigma(\tilde{A}) \} > 0, \quad \{ \Re \tilde{\lambda}_n; \tilde{\lambda}_n \in \sigma(\tilde{\Lambda}) \} > 0.
\]

(1.9)

For completeness, we verify (1.9) in Appendix.

In this article, for simplicity, we do not discuss the case where there exist eigenvalues \( \lambda_n \) such that \( \Re \lambda_n < 0 \).

Now we are reay to state the main result in this article.

**Theorem.**

Let \( 0 < \alpha, \beta < 1 \) and \( \Omega \cap \tilde{\Omega} \neq \emptyset \), and \( x_0 \in \Omega \cap \tilde{\Omega} \), \( T > 0 \) be arbitrarily chosen. We assume that each of initial values \( a \in \mathcal{D}(\tilde{A}) \) and \( \tilde{a} \in \mathcal{D}(\tilde{\Lambda}) \) does not change signs in \( \Omega \) and \( \tilde{\Omega} \) respectively, and

\[
either \tilde{a} \neq 0 \text{ in } \Omega \text{ or } \tilde{a} \neq 0 \text{ in } \tilde{\Omega}.\]

(1.10)
If
\[ u(x_0, t) = \tilde{u}(x_0, t), \quad 0 < t < T, \]
then \( \alpha = \beta \).

By Lemma 1 in Section 2, we see that \( u(x_0, \cdot) \in C(0, \infty) \) and so \( u(x_0, t) \) makes sense for \( t > 0 \).

Remark.

From the proof in Section 3, we can see the following:

Let each of \( a \) and \( \tilde{a} \) does not change signs in \( \Omega \) and \( \tilde{\Omega} \) respectively.

\begin{equation}
\text{If } u(x_0, t) = \tilde{u}(x_0, t) \text{ for } 0 < t < T \text{ and } \alpha \neq \beta, \text{ then } u(x, t) = 0 \text{ in } \Omega \times (0, T) \text{ and } \tilde{u}(x, t) = 0 \text{ in } \tilde{\Omega} \times (0, T).
\end{equation}

In other words, under assumption (1.11), we can conclude that \( u(x_0, t) = \tilde{u}(x_0, t) \) for \( 0 < t < T \) implies either

\( \alpha = \beta \),

or

\[ u = 0 \text{ in } \Omega \times (0, T) \text{ and } \tilde{u} = 0 \text{ in } \tilde{\Omega} \times (0, T). \]

The proof relies on the asymptotic behavior of \( u \) and \( \tilde{u} \) as \( t \to \infty \), which is an idea similar to Sakamoto and Yamamoto [28], Yamamoto [33], but by the non-symmetry of \( A \) and \( \tilde{A} \), we cannot make use of the eigenfunction expansions of the solutions \( u \) and \( \tilde{u} \) themselves. Alternatively we derive asymptotic expansions of eigenprojections of \( u \) and \( \tilde{u} \) in view of the completeness in \( L^2(\Omega) \) of generalized eigenfunctions for each of \( A \) and \( \tilde{A} \).

The article is composed of three sections and one appendix. In Section 2, we prepare fundamental qualitative properties of the solutions and a representation formula of eigenprojections of the solution. In Section 3, on the basis of the results in Section 2, we complete the proof of the main result.

2. Preliminaries

2.1. Qualitative properties of the solution \( u \) to (1.3).

We recall that the spatial dimensions \( d \) is either 1, 2 or 3.

Lemma 1.
Let $a \in L^2(\Omega)$. Then $u \in C(\overline{\Omega} \times (0, \infty))$ for any $t > 0$.

**Proof.**

Similarly to Gorenflo, Luchko and Yamamoto [8], we can prove $u \in C((0, \infty); H^2(\Omega))$. Since $d \leq 3$, the Sobolev embedding yields $H^2(\Omega) \subset C(\overline{\Omega})$, and so we see the lemma. ■

**Lemma 2.**

Let $a \in L^2(\Omega)$ and let $x_0 \in \Omega$ be fixed. Then $u(x_0, t)$ is analytic in $t > 0$.

**Proof.**

We can repeat the proof of e.g., Theorem 2.2 in Li, Imanuvilov and Yamamoto [18], and we omit the details.

Next we show decay estimates of the solution $u$.

**Proposition 1.**

(i) There exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq \frac{C}{t^\alpha} \|a\|_{L^2(\Omega)}, \quad t > 0.$$ 

(ii) Let $d = 1, 2, 3$ and let $x_0 \in \Omega$. Then there exists a constant $C > 0$ such that

$$|u(x_0, t)| \leq \frac{C}{t^\alpha} \|Aa\|_{L^2(\Omega)}, \quad t > 0$$

for each $a \in \mathcal{D}(A)$.

The part (i) of the proposition is well-known for the case of symmetric $A$ and can be proved directly by the eigenfunction expansions (e.g., [28]). Moreover, Vergara and Zacher [31] proved the same decay estimate for symmetric $A$ with time dependent coefficients which does not admit eigenfunction expansions, and so the proof requires more technicality. See also Chapter 5 in the book Kubica, Ryszewska and Yamamoto [16]. For non-symmetric $A$, to the best knowledge of the author, there are no published works and so Proposition 1 (i) can be an independent interest.

In Appendix we provide a sketch of the proof of Proposition 1.

**2.2. Spectral decomposition and representation of solution.**

It is sufficient to argue for the operator $A$ in $\Omega$, because for $\tilde{A}$ in $\tilde{\Omega}$, we can do similarly to obtain the same results.

We recall that $A$ is defined by (1.1) with $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and satisfies (1.2), and that the spectrum $\sigma(A)$ consists only of isolated eigenvalues $\lambda_n \in \mathbb{C}$ with $n \in \mathbb{N}$ and that $\infty$ is a unique accumulation point (e.g., Agmon [2]). For each $n \in \mathbb{N}$, we take a circle
\( \gamma_n \) centered at \( \lambda_n \) with sufficiently small radius such that \( \gamma_n \) does not enclose \( \lambda_m \) with any \( m \neq n \). We define

\[
P_n := \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_n} (z - A)^{-1} dz
\]  

(2.1)

and then we have

\[
P_n P_m = \begin{cases} 
0, & n \neq m, \\
P_n, & n = m
\end{cases}
\]  

(2.2)

(e.g., Kato [13]). We call \( P_n \) the eigenprojection for \( \lambda_n \) and \( \varphi \in P_n L^2(\Omega), \neq 0 \) a generalized eigenfunction.

We see that \( \dim P_n L^2(\Omega) < \infty \), and we set

\[
i_n := \dim P_n L^2(\Omega) < \infty.
\]  

(2.3)

Then

\[
(A - \lambda_n)^{i_n} P_n = 0
\]  

(2.4)

(e.g., [13]). We note that if \( A \) is symmetric, then \( \lambda_n \in \mathbb{R} P_n L^2(\Omega) = \text{Ker} (\lambda_n - A) \) and \( i_n = \dim \text{Ker} (\lambda_n - A) \) for all \( n \in \mathbb{N} \).

More we set

\[
D_n := (\lambda_n - A)P_n, \quad n \in \mathbb{N}.
\]  

(2.5)

By (2.1) and (2.2), we prove \( P_n \varphi = \varphi \) for \( \varphi \in P_n L^2(\Omega) \), and

\[
P_n L^2(\Omega) \subset D(A), \quad AP_n L^2(\Omega) \subset P_n L^2(\Omega), \quad D_n P_n L^2(\Omega) \subset P_n L^2(\Omega), \quad D_n^{i_n} = 0.
\]  

(2.6)

Moreover since \( 0 \notin \sigma(A) \), we see that \( A^{-1} \) exists and

\[
A^{-1} \varphi = \sum_{k=0}^{i_n-1} \frac{D_n^k}{\lambda_k^{k+1}} \varphi \in P_n L^2(\Omega) \quad \text{for all } \varphi \in P_n L^2(\Omega).
\]  

(2.7)

Indeed (2.5) yields

\[
A \left( \sum_{k=0}^{i_n-1} \frac{D_n^k}{\lambda_k^{k+1}} \varphi \right) = (\lambda_n - D_n) \left( \sum_{k=0}^{i_n-1} \frac{D_n^k}{\lambda_k^{k+1}} \varphi \right)
\]

\[
= \sum_{k=0}^{i_n-1} \frac{D_n^k}{\lambda_k^{k+1}} \varphi - \sum_{k=0}^{i_n-1} \frac{D_n^{k+1}}{\lambda_k^{k+1}} \varphi = \sum_{k=0}^{i_n-1} \frac{D_n^k}{\lambda_k^{k+1}} \varphi - \sum_{k=1}^{i_n} \frac{D_n^k}{\lambda_k^{k+1}} \varphi = \varphi - \frac{D_n^{i_n}}{\lambda_{i_n}} \varphi = \varphi,
\]

which proves (2.7).
We state the completeness of the generalized eigenfunctions of $A$.

**Lemma 3.**

(i) For any $a \in L^2(\Omega)$, there exists a sequence $a_N$, $N \in \mathbb{N}$ such that

$$a_N \in \sum_{k=1}^{N} P_k L^2(\Omega), \quad \lim_{N \to \infty} \|a_N - a\|_{L^2(\Omega)} = 0.$$ 

(ii) For any $a \in \mathcal{D}(A)$, there exists a sequence $a_N$, $N \in \mathbb{N}$ such that

$$a_N \in \sum_{k=1}^{N} P_k L^2(\Omega) \subset \mathcal{D}(A), \quad \lim_{N \to \infty} \|Aa_N - Aa\|_{L^2(\Omega)} = 0.$$ 

In particular,

$$\lim_{N \to \infty} \|a_N - a\|_{C(\overline{\Omega})} = 0. \tag{2.8}$$

**Proof.**

Since the linear subspace spanned by all the generalized eigenfunctions of $A$ is dense in $L^2(\Omega)$ (e.g., [2]), we see part (i). Next let $a \in \mathcal{D}(A)$. Then $Aa \in L^2(\Omega)$. By (i) we can choose a sequence $b_N$, $N \in \mathbb{N}$ such that

$$b_N \in \sum_{k=1}^{N} P_k L^2(\Omega), \quad \lim_{N \to \infty} \|b_N - Aa\|_{L^2(\Omega)} = 0.$$ 

Since $0 \notin \sigma(A)$, we see that $A^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ exists and is bounded, and

$$\lim_{N \to \infty} \|A(A^{-1}b_N) - Aa\|_{L^2(\Omega)} = 0.$$ 

By (2.7), we see that $a_N := A^{-1}b_N \in \sum_{k=1}^{N} P_k L^2(\Omega) \subset \mathcal{D}(A)$ and

$$\lim_{N \to \infty} \|Aa_N - Aa\|_{L^2(\Omega)} = 0.$$ 

Therefore, in view of $d \leq 3$, the Sobolev embedding $\mathcal{D}(A) \subset H^2(\Omega) \subset C(\overline{\Omega})$ implies (2.8). Thus the proof of Lemma 3 is complete. ■

Henceforth, we set

$$\binom{k}{j} := \frac{k!}{j!(k-j)!}, \quad 0! := 1.$$ 

Now we establish a representation formula of solution to (1.3) with $a \in P_n L^2(\Omega)$.

**Lemma 4.**
Let \( n \in \mathbb{N} \) be arbitrarily fixed. Then the solution \( u_n \) to (1.3) with \( a \in P_n L^2(\Omega) \) is given by

\[
u_n(x, t) = \sum_{k=0}^{\infty} \frac{t^\alpha}{\Gamma(\alpha k + 1)}(-\lambda_n + D_n)^k a(x) = \sum_{k=0}^{\infty} \frac{t^\alpha}{\Gamma(\alpha k + 1)} \sum_{j=0}^{k} \binom{k}{j} (-\lambda_n)^{k-j} D_n^j a(x),
\]

where the series is convergent in \( C(\overline{\Omega} \times [\delta, \infty)) \) with any \( \delta > 0 \).

**Proof.**

We can interpret (2.9) as

\[
E_{\alpha,1}(- (AP_n)t^\alpha) a = E_{\alpha,1}((-\lambda_n + D_n)t^\alpha) a = \sum_{k=0}^{\infty} \frac{t^\alpha}{\Gamma(\alpha k + 1)} (-\lambda_n + D_n)^k a, \quad t > 0,
\]

and we can verify as follows.

We set \( A_n := AP_n \). Then \( A_n : P_n L^2(\Omega) \rightarrow P_n L^2(\Omega) \). Since \( P_n L^2(\Omega) < \infty \), we see that \( A_n \) is a bounded operator and

\[
\|A_n\|_{B(P_n L^2(\Omega))} := \rho < \infty,
\]

where \( \|A_n\|_{B(P_n L^2(\Omega))} \) denotes the operator norm of \( A_n : P_n L^2(\Omega) \rightarrow P_n L^2(\Omega) \). The Sobolev embedding yields

\[
\|a\|_{C(\overline{\Omega})} \leq C \|Aa\|_{L^2(\Omega)} = C \|A_n a\|_{L^2(\Omega)} \leq C \rho \|a\|_{L^2(\Omega)}
\]

for all \( a \in P_n L^2(\Omega) \). Moreover

\[
\|A_n^k a\|_{C(\overline{\Omega})} \leq C \|A_n^{k+1} a\|_{L^2(\Omega)} \leq C \|A_n\|_{B(P_n L^2(\Omega))} \|a\|_{L^2(\Omega)} \leq C \rho^{k+1} \|a\|_{L^2(\Omega)}.
\]

Since

\[
\partial_t^\alpha \left( \frac{t^\alpha}{\Gamma(\alpha k + 1)} \right) = \frac{t^{(k-1)\alpha}}{\Gamma(1 + (k-1)\alpha)}, \quad k \in \mathbb{N},
\]

we have

\[
\sum_{k=0}^{\infty} \left| \partial_t^\alpha \left( \frac{t^\alpha}{\Gamma(\alpha k + 1)} \right) \right| \|(A_n)^k a\|_{C(\overline{\Omega})} \leq \sum_{k=1}^{\infty} \frac{t^{(k-1)\alpha}}{\Gamma(1 + (k-1)\alpha)} \rho^{k+1} \|a\|_{L^2(\Omega)}
\]

\[
= C \|a\|_{L^2(\Omega)} \rho \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} < \infty,
\]

noting that

\[
E_{\alpha,1}(\rho t^\alpha) = \sum_{k=0}^{\infty} \frac{(\rho t^\alpha)^k}{\Gamma(\alpha k + 1)}
\]

and \( E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \) is an entire function in \( z \in \mathbb{C} \). Therefore, we can justify

\[
\partial_t^\alpha u_n(x, t) = \sum_{k=0}^{\infty} \partial_t^\alpha \left( \frac{t^\alpha}{\Gamma(\alpha k + 1)} \right) (-A_n)^k a = \sum_{k=0}^{\infty} \frac{t^{(k-1)\alpha}}{\Gamma(1 + (k-1)\alpha)} (-A_n)^k a
\]
for \( x \in \Omega \) and \( t > 0 \). We can directly verify that \( u_n(x,0) = a(x) \) and \( u_n(\cdot, t) \in D(A) \subset H^1_0(\Omega) \), so that \( u_n \) given by (2.9) satisfies (1.3) with \( a \in P_n L^2(\Omega) \). The proof of Lemma 4 is complete. ■

Next we calculate the right-hand side of (2.9). To this end, we prove

**Lemma 5.**

For \( j \in \mathbb{N} \) and \( 1 \leq \ell \leq j \), we have

\[
\left( \frac{d}{dt^\alpha} \right)^j = \sum_{\ell=1}^{j} \theta_{j\ell} t^{\ell-j\alpha} \left( \frac{d}{dt} \right)^\ell,
\]

where \( \theta_{j\ell} \in \mathbb{R} \) are defined by

\[
\theta_{11} = \frac{1}{\alpha}, \quad \theta_{j+1,\ell} = \begin{cases} 
\frac{\theta_{j1}(1-\alpha)}{\alpha}, & \ell = 1, \\
\frac{\theta_{j\ell}(\ell-\alpha)}{\alpha} + \frac{c_{j\ell-1}}{\alpha}, & 2 \leq \ell \leq j, \\
\frac{\theta_{jj}}{\alpha}, & \ell = j + 1
\end{cases}
\]

for \( j \geq 1 \).

Here in the case of \( j = 1 \) in (2.11), we neglect the possibility \( 2 \leq \ell \leq j \), that is,

\[
\theta_{2,\ell} = \begin{cases} 
\frac{\theta_{11}(1-\alpha)}{\alpha}, & \ell = 1, \\
\frac{\theta_{11}}{\alpha}, & \ell = 2.
\end{cases}
\]

**Proof.**

For \( j = 1 \), we have \( \frac{d}{dt^\alpha} = \frac{dt}{dt^\alpha} \frac{d}{dt} = \frac{1}{\alpha} t^{1-\alpha} \frac{d}{dt} \), and (2.10) holds with \( \theta_{11} = \frac{1}{\alpha} \). Let (2.10) hold for \( j \). Then

\[
\left( \frac{d}{dt^\alpha} \right)^{j+1} = \frac{d}{dt^\alpha} \left( \sum_{\ell=1}^{j} \theta_{j\ell} t^{\ell-j\alpha} \left( \frac{d}{dt} \right)^\ell \right) = \frac{1}{\alpha} t^{1-\alpha} \frac{d}{dt} \left( \sum_{\ell=1}^{j} \theta_{j\ell} t^{\ell-j\alpha} \left( \frac{d}{dt} \right)^\ell \right)
\]

\[
= \frac{1}{\alpha} t^{1-\alpha} \left( \sum_{\ell=1}^{j} \theta_{j\ell}(\ell-\alpha) t^{\ell-j\alpha-1} \left( \frac{d}{dt} \right)^\ell + \sum_{\ell=1}^{j} \theta_{j\ell} t^{1-\alpha} (\ell+1) \left( \frac{d}{dt} \right)^{\ell+1} \right)
\]

\[
= \sum_{\ell=1}^{j} \frac{\theta_{j\ell}(\ell-\alpha)}{\alpha} t^{\ell-(j+1)\alpha} \left( \frac{d}{dt} \right)^\ell + \sum_{\ell=2}^{j+1} \frac{\theta_{j\ell-1}}{\alpha} t^{\ell-(j+1)\alpha} \left( \frac{d}{dt} \right)^\ell,
\]

which proves (2.10) for \( j + 1 \), and (2.10) is seen for \( j \in \mathbb{N} \) by the induction. The recurrence formula (2.11) follows from (2.12). Thus the proof of Lemma 5 is complete. ■
Now we can show a representation formula for $u_n(x,t)$.

**Proposition 2.**

$$u_n(x,t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\lambda_n^j j!} \left( \sum_{\ell=1}^{j} \theta_{j\ell} E_{\alpha,1-\ell}(-\lambda_n t^{\alpha}) \right) D_n^j P_n a \quad \text{in } C(\Omega) \text{ for } t \geq 0. \quad (2.13)$$

**Proof.**

By (2.9) and $D_n^0 = 0$, the absolute convergence of the series in $j$ and $k$ allows us to exchange the order of the summation, and we have

$$u_n(x,t) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \sum_{j=0}^{k} \binom{k}{j} (-\lambda_n)^{k-j} D_n^j a$$

$$= \sum_{j=0}^{\infty} \frac{D_n^j a}{j!} \sum_{k=j}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} k(k-1) \cdots (k-j+1)(-\lambda_n)^{k-j}$$

$$= \sum_{j=0}^{\infty} \frac{t^{\alpha j}}{j!} \sum_{k=j}^{\infty} \frac{k(k-1) \cdots (k-j+1)}{\Gamma(\alpha k + 1)} (-\lambda_n t^\alpha)^{k-j} = \sum_{j=0}^{\infty} \frac{D_n^j a}{j!} t^{\alpha j} (-1)^j \frac{d^j}{dz^j} E_{\alpha,1}(-z) \bigg|_{z=\lambda_n t^\alpha}.$$

Now we calculate $\frac{d^j}{dz^j} E_{\alpha,1}(-z)$. Since the series $E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ is convergent uniformly in any compact set of $z \in \mathbb{C}$, the termwise differentiation yields

$$\frac{d^\ell}{dt^\ell} E_{\alpha,1}(-t^\alpha) = t^{-\ell} E_{\alpha,1-\ell}(-t^\alpha), \quad \ell \in \mathbb{N} \quad (2.15)$$

by

$$\Gamma(\alpha k + 1) = \alpha k(\alpha k - 1) \cdots (\alpha k - \ell + 1) \Gamma(\alpha k - \ell + 1).$$

Here we note that since $|\Gamma(1-\ell)| = \infty$ for $\ell \in \mathbb{N}$, we can have

$$E_{\alpha,1-\ell}(-t^\alpha) = \sum_{k=0}^{\infty} \frac{(-t^\alpha)^k}{\Gamma(\alpha k + 1 - \ell)} = \sum_{k=1}^{\infty} \frac{(-t^\alpha)^k}{\Gamma(\alpha k + 1 - \ell)}.$$

Consequently, in view of (2.10), we have

$$\left( \frac{d}{dt^\alpha} \right)^j E_{\alpha,1}(-t^\alpha) = \sum_{\ell=1}^{j} \theta_{j\ell} t^{\ell-\alpha \ell} \frac{d^\ell}{dt^\ell} E_{\alpha,1}(-t^\alpha)$$

$$= \sum_{\ell=1}^{j} \theta_{j\ell} t^{\ell-\alpha \ell} E_{\alpha,1-\ell}(-t^\alpha) = t^{-\alpha j} \sum_{\ell=1}^{j} \theta_{j\ell} E_{\alpha,1-\ell}(-t^\alpha).$$

Setting $t^\alpha = z$, we obtain

$$\left( \frac{d}{dz} \right)^j E_{\alpha,1}(-z) = z^{-j} \sum_{\ell=1}^{j} \theta_{j\ell} E_{\alpha,1-\ell}(-z)$$
if Re \( z > 0 \). Therefore,

\[
\left( \frac{d}{dz} \right)^j E_{\alpha, 1}(-z) \big|_{z = \lambda_n t^\alpha} = (\lambda_n t^\alpha)^{-j} \sum_{\ell=1}^{j} \theta_{j \ell} E_{\alpha, 1-\ell}(-\lambda_n t^\alpha).
\]

Substituting this into (2.14), we obtain (2.13). The proof of Proposition 2 is complete. □

Finally we prove a key proposition for the proof of Theorem. Henceforth, we fix \( t_0 > 0 \) arbitrarily, and we omit the dependency of the constants on \( t_0 > 0 \) and \( a \).

**Proposition 3.**

There exists a constant \( C(n) > 0 \) such that

\[
\|R_n(\cdot, t)\|_{C(\mathbb{R})} \leq \frac{C(n)}{\eta^{2\alpha}}, \quad t \geq t_0.
\]

**Proof.**

For \( \ell \in \mathbb{N} \), we have asymptotics

\[
E_{\alpha, 1-\ell}(-\lambda_n t^\alpha) = \frac{1}{\Gamma(1 - \ell - \alpha)} \frac{1}{\lambda_n t^\alpha} + R_{\alpha, \ell}(-\lambda_n t^\alpha), \quad t \geq t_0;
\]

where there exists a constant \( C_{\alpha, \ell} > 0 \) such that

\[
|R_{\alpha, \ell}(-\eta)| \leq \frac{C_{\alpha, \ell}}{\eta^2}, \quad \eta > 0.
\]

Henceforth constants \( C_{\alpha, \ell}, C, C(n) > 0 \), etc. depend also on the initial value \( a \) and the order \( \alpha \), but we omit the dependency otherwise we need to specify it. As for (2.16), we refer to Section 4.5 of Chapter 4 in Gorenflo, Kilbas, Mainardi and Rogosin [7], formula (1.8.28) (p.43) in Kilbas, Srivastava and Trujillo [14] or Theorem 1.4 (pp.33-34) in Podlubny [25], and see also Popov and Sedletskii [26] as recent research article.

We substitute (2.16) into (2.13):

\[
u_n(x, t) = \sum_{j=0}^{\infty} \frac{1}{\lambda_n^{j+1}} \left( \frac{(-1)^j}{j!} \left( \theta_{j \ell} \frac{1}{\Gamma(1 - \ell - \alpha)} \sum_{\ell=1}^{j} \theta_{j \ell} R_{\alpha, \ell}(-\lambda_n t^\alpha) \right) \right) D_{\alpha}^j P_n a
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{\lambda_n^{j+1}} \left( \frac{(-1)^j}{j!} \left( \theta_{j \ell} \frac{1}{\Gamma(1 - \ell - \alpha)} \sum_{\ell=1}^{j} \theta_{j \ell} R_{\alpha, \ell}(-\lambda_n t^\alpha) \right) \right) \frac{1}{\eta^{\alpha}} D_{\alpha}^j P_n a + S_n(t^\alpha)(x).
\]
We note that by $D_n^i = 0$ the series are finite. Here, by (2.16), we can find a constant $C_1(n) > 0$ such that

$$
\|S_n(t^\alpha)(\cdot)\|_{C(\Omega)} \leq \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{\lambda_n^j j!} \sum_{\ell=1}^{j} \frac{|\theta_{j\ell}C_{\alpha,\ell}|}{\lambda_n^2 \ell^{2\alpha}}\right) \|D_n^j P_n a\|_{C(\Omega)} \leq \frac{C_1(n)}{t^{2\alpha}}
$$

(2.19)

for all $t \geq t_0$.

We calculate

$$
\Phi_j := \frac{(-1)^j}{j!} \sum_{\ell=1}^{j} \frac{\theta_{j\ell}}{\Gamma(1 - \ell - \alpha)}, \quad j \in \mathbb{N}.
$$

By (2.11) and $\Gamma(1 - \alpha) = -\alpha \Gamma(-\alpha)$, we see

$$
\Phi_1 = \frac{(-1)^1}{1!} \frac{\theta_{11}}{\Gamma(-\alpha)} = -1 \frac{1}{\alpha \Gamma(-\alpha)} = \frac{1}{\Gamma(1 - \alpha)}.
$$

(2.20)

Now we will verify $\Phi_j = \Phi_1$ for each $j \in \mathbb{N}$. By (2.11) we obtain

$$
\Phi_{j+1} = \frac{(-1)^{j+1}}{(j+1)!} \sum_{\ell=1}^{j+1} \frac{\theta_{j+1,\ell}}{\Gamma(1 - \ell - \alpha)}
$$

$$
= \frac{(-1)^{j+1}}{(j+1)!} \left\{ \frac{\theta_{j1}(1 - \alpha j)}{\alpha \Gamma(-\alpha)} + \sum_{\ell=2}^{j} \frac{1}{\Gamma(1 - \ell - \alpha)} \left( \frac{\theta_{j\ell}(\ell - \alpha j)}{\alpha} + \frac{\theta_{j,\ell-1}}{\alpha} \right) \right\}
$$

$$
+ \frac{1}{\Gamma(-j - \alpha)} \frac{\theta_{jj}}{\alpha \Gamma(-\alpha)}
$$

$$
= \frac{(-1)^{j+1}}{(j+1)!} \left\{ \frac{\theta_{j1}(1 - \alpha j)}{\alpha \Gamma(-\alpha)} + \sum_{\ell=2}^{j} \frac{\theta_{j\ell}(\ell - \alpha j)}{\alpha \Gamma(1 - \ell - \alpha)} \right\}
$$

$$
+ \sum_{\ell=1}^{j-1} \frac{\theta_{j\ell}}{\alpha \Gamma(-\ell - \alpha)} + \frac{\theta_{jj}}{\alpha \Gamma(-j - \alpha)} \frac{1}{\alpha \Gamma(-\alpha)}
$$

$$
= \frac{(-1)^{j+1}}{(j+1)!} \left\{ \frac{\theta_{j1}(1 - \alpha j)}{\alpha \Gamma(-\alpha)} + \sum_{\ell=2}^{j} \frac{\theta_{j\ell}(\ell - \alpha j)}{\alpha \Gamma(1 - \ell - \alpha)} \right\}
$$

Here we used

$$
\sum_{\ell=2}^{j} \frac{\theta_{j,\ell-1}}{\alpha \Gamma(1 - \ell - \alpha)} = \sum_{\ell=1}^{j-1} \frac{\theta_{j\ell}}{\alpha \Gamma(-\ell - \alpha)}
$$

Using $\Gamma(1 - \eta) = -\eta \Gamma(-\eta)$ for $\eta \not\in \mathbb{N} \cup \{0\}$, we obtain $(-\ell - \alpha) \Gamma(-\ell - \alpha) = \Gamma(1 - \ell - \alpha)$ for $\ell = 1, 2, ..., j$, and so

$$
\frac{1}{\alpha \Gamma(-\ell - \alpha)} = -\frac{\ell + \alpha}{\alpha \Gamma(1 - \ell - \alpha)},
$$

which means

$$
\frac{\ell - \alpha j}{\alpha \Gamma(1 - \ell - \alpha)} + \frac{1}{\alpha \Gamma(-\ell - \alpha)} = \frac{1}{\alpha \Gamma(1 - \ell - \alpha)}(\ell - \alpha j - \ell - \alpha) = \frac{-(j+1)}{\Gamma(1 - \ell - \alpha)}.
$$
Therefore,

\[ \Phi_{j+1} = \frac{(-1)^{j+1}}{(j+1)!} \left\{ \left( \frac{\theta_{j1}(1 - \alpha_j)}{\alpha \Gamma(-\alpha)} + \frac{\theta_{j1}}{\alpha \Gamma(-1 - \alpha)} \right) \right. \\
\left. + \sum_{\ell=2}^{j-1} \left( \frac{\theta_{j\ell}(\ell - \alpha_j)}{\alpha \Gamma(1 - \ell - \alpha)} + \frac{\theta_{j\ell}}{\alpha \Gamma(-\ell - \alpha)} \right) + \left( \frac{\theta_{jj}(j - \alpha_j)}{\alpha \Gamma(1 - j - \alpha)} + \frac{\theta_{jj}}{\alpha \Gamma(-j - \alpha)} \right) \right\} \]

\[ = \frac{(-1)^{j+1}}{(j+1)!} \sum_{\ell=1}^{j} \left( \frac{\ell - \alpha_j}{\alpha \Gamma(1 - \ell - \alpha)} + \frac{1}{\alpha \Gamma(-\ell - \alpha)} \right) \theta_{j\ell} \]

\[ = \frac{(-1)^{j+1}}{(j+1)!} (-1)(j + 1) \sum_{\ell=1}^{j} \frac{\theta_{j\ell}}{\Gamma(1 - \ell - \alpha)} = \frac{(-1)^{j}}{j!} \sum_{\ell=1}^{j} \frac{\theta_{j\ell}}{\Gamma(1 - \ell - \alpha)} = \Phi_{j} \]

for each \( j \in \mathbb{N} \). Thus (2.20) yields \( \Phi_{j} = \Phi_{1} = \frac{1}{\Gamma(1-\alpha)} \) for all \( j \in \mathbb{N} \).

Hence, applying also (2.7), we obtain

\[ \sum_{j=0}^{\infty} \frac{1}{\lambda_{n}^{j+1}} \left( \frac{(-1)^{j}}{j!} \sum_{\ell=1}^{j} \frac{\theta_{j\ell}}{\Gamma(1 - \ell - \alpha)} \right) \frac{1}{t^{\alpha}} D_{n}^{j} P_{n}a = \sum_{j=0}^{\infty} \frac{1}{\lambda_{n}^{j+1}} \Phi_{j} \frac{1}{t^{\alpha}} D_{n}^{j} P_{n}a \]

\[ = \frac{1}{\Gamma(1 - \alpha)} \left( \sum_{j=0}^{\infty} \frac{1}{\lambda_{n}^{j+1}} D_{n}^{j} P_{n}a \right) \frac{1}{t^{\alpha}} = \frac{1}{\Gamma(1 - \alpha)} A^{-1}(P_{n}a)(x) \frac{1}{t^{\alpha}}. \]

Combining (2.19), we complete the proof of Proposition 3. ■

3. COMPLETION OF PROOF OF THEOREM

First Step.

We recall that \( u \) and \( \tilde{u} \) are the solutions to (1.3) and (1.7) with the initial values \( a \) and \( \tilde{a} \) respectively. For \( a \in \mathcal{D}(A) \), by Lemma 3, for each \( n \in \mathbb{N} \), we can find \( a_{N} \in \sum_{k=1}^{N} P_{k} L^{2}(\Omega) \) such that

\[ \lim_{N \to \infty} \rho_{N} = 0, \quad \text{where} \quad \rho_{N} := \| A(a - a_{N}) \|_{L^{2}(\Omega)}. \]

Let \( v_{N} \) and \( w_{N} \) satisfy

\[ \begin{cases} \partial_{t}^{\alpha} v_{N} = -A v_{N} \quad \text{in} \ \Omega \times (0, \infty), \\ v_{N}|_{\partial \Omega \times (0, \infty)} = 0, \\ v_{N}(x, 0) = a_{N}(x), \quad x \in \Omega \end{cases} \]

and

\[ \begin{cases} \partial_{t}^{\alpha} w_{N} = -A w_{N} \quad \text{in} \ \Omega \times (0, \infty), \\ w_{N}|_{\partial \Omega \times (0, \infty)} = 0, \\ w_{N}(x, 0) = a(x) - a_{N}(x), \quad x \in \Omega. \end{cases} \]
Then \( u = v_N + w_N \) in \( \Omega \times (0, \infty) \).

First applying Proposition 1 to (3.3) and setting \( W_N(t) = w_N(x_0, t) \), we obtain

\[
|W_N(t)| \leq \frac{C}{\rho_N}, \quad t \geq t_0. \tag{3.4}
\]

Since \( v_N = \sum_{n=1}^{N} u_n \) by the uniqueness of solution to (3.2), the application of Proposition 3 to \( u_n \) yields

\[
v_N(x_0, t) = \sum_{n=1}^{N} u_n(x_0, t) = \frac{1}{\Gamma(1 - \alpha)} \sum_{n=1}^{N} (A^{-1}(P_n a))(x_0) + \sum_{n=1}^{N} R_n(x_0, t)
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} (A^{-1}a_N)(x_0) + \sum_{n=1}^{N} R_n(x_0, t), \quad t \geq t_0.
\]

Therefore, we can find a constant \( C_0 = C_0(N) > 0 \) and a function \( S_N(t) \) such that

\[
v_N(x_0, t) = \frac{1}{\Gamma(1 - \alpha)} (A^{-1}a_N)(x_0) + S_N(t), \quad t \geq t_0, \tag{3.5}
\]

where

\[
|S_N(t)| \leq \frac{C_0(N)}{t^{2\alpha}}, \quad t \geq t_0. \tag{3.6}
\]

Consequently,

\[
u(x_0, t) = \frac{1}{\Gamma(1 - \alpha)} (A^{-1}a_N)(x_0) + S_N(t) + W_N(t), \quad t \geq t_0. \tag{3.7}
\]

We can similarly argue for \( \tilde{A} \) to see that there exist \( \tilde{a}_N \in D(\tilde{A}) \) and functions \( \tilde{S}_N \) and \( \tilde{W}_N \) for \( N \in \mathbb{N} \) such that

\[
\tilde{\rho}_N := \|\tilde{A}(\tilde{a} - \tilde{a}_N)\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty,
\]

\[
\tilde{u}(x_0, t) = \frac{1}{\Gamma(1 - \beta)} (\tilde{A}^{-1}\tilde{a}_N)(x_0) + \tilde{S}_N(t) + \tilde{W}_N(t), \quad t \geq t_0 \tag{3.8}
\]

and

\[
|\tilde{S}_N(t)| \leq \frac{C_0(N)}{t^{2\beta}}, \quad |\tilde{W}_N(t)| \leq \frac{C}{t^\beta} \tilde{\rho}_N, \quad t \geq t_0. \tag{3.9}
\]

**Second Step.**

By Lemma 2, from \( u(x_0, t) = \tilde{u}(x_0, t) \) for \( 0 < t < T \), we derive

\[
u(x_0, t) = \tilde{u}(x_0, t) \quad \text{for all} \quad t > 0.
\]

Therefore, (3.7) and (3.8) yield

\[
\frac{1}{\Gamma(1 - \alpha)} (A^{-1}a_N)(x_0) + S_N(t) + W_N(t)
\]

\[
= \frac{1}{\Gamma(1 - \beta)} (\tilde{A}^{-1}\tilde{a}_N)(x_0) + \tilde{S}_N(t) + \tilde{W}_N(t), \quad t \geq t_0.
\]
Assume that $\alpha < \beta$. Then we multiply by $t^\alpha$, we have

\[
\frac{1}{\Gamma(1-\alpha)}(A^{-1}a_N)(x_0) + t^\alpha S_N(t) + t^\alpha W_N(t)
\]

\[
= \frac{1}{\Gamma(1-\beta)t^{\beta-\alpha}}(\tilde{A}^{-1}\tilde{a}_N)(x_0) + t^\alpha \tilde{S}_N(t) + t^\alpha \tilde{W}_N(t), \quad t \geq t_0
\]  

for all $N \in \mathbb{N}$ and all $t \geq t_0$.

By (3.4), (3.6) and (3.9), for each fixed $N \in \mathbb{N}$, we have

\[
\begin{aligned}
&\lim_{t \to \infty} |t^\alpha S_N(t)| \leq \lim_{t \to \infty} \frac{C_0(N)}{t^\alpha} = 0, \\
&\lim_{t \to \infty} |t^\alpha \tilde{S}_N(t)| \leq \lim_{t \to \infty} \frac{C_0(N)}{t^{2\beta-\alpha}} = 0
\end{aligned}
\]

by $\beta > \alpha$. Moreover, whenever we fix $N \in \mathbb{N}$ arbitrarily, by (3.4) and (3.9) we see

\[
|t^\alpha W_N(t)| \leq C\rho_N, \quad |t^\alpha \tilde{W}_N(t)| \leq \frac{C\rho_N}{t^{\beta-\alpha}} \to 0 \quad \text{as} \quad t \to \infty
\]  

and

\[
\lim_{t \to \infty} \frac{1}{\Gamma(1-\beta)t^{\beta-\alpha}}(\tilde{A}^{-1}\tilde{a}_N)(x_0) = 0
\]  

by $\beta > \alpha$.

Therefore, applying (3.11) - (3.13) in (3.10) and letting $t \to \infty$, we obtain

\[
\left| \frac{1}{\Gamma(1-\alpha)}(A^{-1}a_N)(x_0) \right| \leq C\rho_N.
\]  

Since $a_N \to a$ in $L^2(\Omega)$ as $N \to \infty$, we see that $A^{-1}a_N \to A^{-1}a$ in $\mathcal{D}(A)$, and the Sobolev embedding implies

\[
\lim_{N \to \infty} \|A^{-1}a_N - A^{-1}a\|_{C(\Omega)} = 0,
\]

that is,

\[
\lim_{N \to \infty} A^{-1}a_N(x_0) = A^{-1}a(x_0).
\]

Hence, letting $N \to \infty$ in (3.14), we see

\[
A^{-1}a(x_0) = 0.
\]

We set $f := A^{-1}a$ in $\Omega$. Then $Af = a$ in $\Omega$. By (1.10), without loss of generality, we can assume that $a \leq 0$ on $\overline{\Omega}$. In view of $c \leq 0$ from (1.8), by noting that $f|_{\partial\Omega} = 0$, the weak maximum principle (e.g., Theorem 3.1 (p.32) in Gilbarg and Trudinger [6]) yields $f(x) \leq 0$ for $x \in \overline{\Omega}$.

Since $f \leq 0$ on $\overline{\Omega}$ and $f(x_0) = A^{-1}a(x_0) = 0$, we see that $f$ attains the maximum 0 at an interior point $x_0 \in \Omega$. In view of $(-A)f \geq 0$ in $\Omega$, we can apply the strong maximum principle (e.g., Theorem 3.5 (p.35) in [6]) to conclude that $f(x)$ is a constant function. Since
\( f \in D(A) \), we see \( f|_{\partial \Omega} = 0 \), so that \( f = 0 \) in \( \Omega \). Then \( u = 0 \) in \( \Omega \times (0, \infty) \). In terms of (3.8) and \( \overline{u}(x_0, t) = u(x_0, t) = 0 \) for \( t > 0 \), we have
\[
\frac{1}{t^{\beta}}(\mathcal{A}^{-1} \overline{a_N})(x_0) + \overline{S_N}(t) + \overline{W_N}(t) = 0, \quad t \geq t_0.
\]
Multiplying with \( t^\beta \) and letting \( t \to \infty \), we similarly obtain
\[
\left| \frac{1}{t^{1-\beta}}(\mathcal{A}^{-1} \overline{a_N})(x_0) \right| \leq C \overline{\rho_N} \quad \text{for all} \quad N \in \mathbb{N}.
\]
Letting \( N \to \infty \), we see \( \mathcal{A}^{-1} \overline{a}(x_0) = 0 \). By the same way as for \( a \), the weak and the strong maximum principles similarly yield \( \overline{a} = 0 \) in \( \Omega \). This means that \( a = 0 \) in \( \Omega \) and \( \overline{a} = 0 \) in \( \overline{\Omega} \), which is a contradiction for (1.10). Therefore, \( \alpha > \beta \) is impossible. Similarly we can prove that \( \alpha > \beta \) is impossible. Thus the proof of Theorem is complete. \( \blacksquare \)

4. Appendix

4.1. Proof of (1.9).

It sufficient to prove only for \( A \). Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( A \): \( A \varphi = \lambda \varphi \) and \( \varphi \neq 0 \) in \( \Omega \). That is,
\[
\begin{cases}
- \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j \varphi(x)) - \sum_{j=1}^{d} b_j(x) \partial_j \varphi - c(x) \varphi = \lambda \varphi & \text{in } \Omega, \\
\varphi|_{\partial \Omega} = 0.
\end{cases}
\]
(4.1)
Henceforth let \( \overline{\lambda} \) denote the complex conjugate of \( \lambda \in \mathbb{C} \). Since \( a_{ij}, b_j \) and \( c \) are real-valued, we have
\[
- \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j \overline{\varphi}(x)) - \sum_{j=1}^{d} b_j(x) \partial_j \overline{\varphi} - c(x) \overline{\varphi} = \overline{\lambda} \overline{\varphi} & \text{in } \Omega.
\]
(4.2)
Multiplying (4.1) and (4.2) with \( \overline{\varphi} \) and \( \varphi \) respectively, integrating by parts in \( \Omega \) and adding, we obtain
\[
\int_{\Omega} \sum_{i,j=1}^{d} a_{ij}((\partial_i \varphi) \partial_j \overline{\varphi} + (\partial_i \overline{\varphi}) \partial_j \varphi) \, dx - \int_{\Omega} b_j((\partial_j \varphi) \overline{\varphi} + (\partial_j \overline{\varphi}) \varphi) \, dx \\
- \int_{\Omega} 2c|\varphi|^2 \, dx = (\lambda + \overline{\lambda}) \int_{\Omega} |\varphi|^2 \, dx.
\]
On the other hand,
\[
((\partial_i \varphi) \partial_j \overline{\varphi} + \partial_i \overline{\varphi} \partial_j \varphi) = 2(\text{Re} (\partial_i \varphi) \text{Re} (\partial_j \varphi) + \text{Im} (\partial_i \varphi) \text{Im} (\partial_j \varphi))
\]
and so (1.2) yields
\[
\sum_{i,j=1}^{d} a_{ij}((\partial_i \varphi) \partial_j \overline{\varphi} + \partial_i \overline{\varphi} \partial_j \varphi)
\]
\[ 2 \left( \sum_{i,j=1}^{d} a_{ij} \text{Re} (\partial_i \varphi) \text{Re} (\partial_j \varphi) + \sum_{i,j=1}^{d} a_{ij} \text{Im} (\partial_i \varphi) \text{Im} (\partial_j \varphi) \right) \]
\[ \geq 2\sigma(a_{ij}) \sum_{i,j=1}^{d} \left( |\text{Re} (\partial_i \varphi)|^2 + |\text{Im} (\partial_i \varphi)|^2 \right) = 2\sigma |\nabla \varphi|^2. \]

Moreover,
\[ - \int_{\Omega} \sum_{j=1}^{d} b_j ((\partial_j \varphi) \varphi + (\overline{\partial_j \varphi}) \varphi) dx \]
\[ = - \int_{\Omega} \sum_{j=1}^{d} b_j (|\varphi|^2) dx = \int_{\Omega} \left( \sum_{j=1}^{d} \partial_j b_j \right) |\varphi|^2 dx. \]

Hence, using also (1.4), we have
\[ 2\sigma C(\Omega) \int_{\Omega} |\varphi|^2 dx + \int_{\Omega} \left( \sum_{j=1}^{d} \partial_j b_j \right) - 2c(x) \right) |\varphi|^2 dx \]
\[ \leq 2\sigma \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\Omega} \left( \sum_{j=1}^{d} \partial_j b_j \right) |\varphi|^2 dx - 2 \int_{\Omega} c(x)|\varphi(x)|^2 dx \]
\[ \leq (\lambda + \overline{\lambda}) \int_{\Omega} |\varphi|^2 dx = 2(\text{Re} \lambda) \int_{\Omega} |\varphi|^2 dx. \]

In view of (1.8), we see
\[ 2\sigma C(\Omega) + \sum_{j=1}^{d} \partial_j b_j(x) - 2c(x) > 0 \quad \text{for all } x \in \overline{\Omega}. \]

Thus Re \( \lambda > 0 \) follows and we complete the proof of (1.9). \( \blacksquare \)

4.2. Proof of Proposition 1.

For the case of symmetric \( A \), in terms of the eigenfunction expansions by the Mittag-Leffler functions, we can directly prove the proposition (e.g., Sakamoto and Yamamoto [28]). However, for non-symmetric \( A \), we have to take other way. We make use of the argument by Vergara and Zacher [31] which is the first work proving Proposition 1 (i) for a case where the eigenfunction expansion does not work, that is, \( A \) is symmetric but the coefficients are time dependent. For the proof of Proposition 1, relying on Chapter 5 in [16] which modifies [31], we here give a sketch of the proof of Proposition 1. We can mostly repeat the argument in [16] and so we describe the main differences.

For the proof, we have to return to the construction of the solution \( u \) to (1.3) by the Galerkin method. More precisely, let \( \{\psi_k\}_{k \in \mathbb{N}} \) be an orthonormal basis in \( L^2(\Omega) \) which
consists of all the eigenfunctions of \(-\Delta\) with the zero Dirichlet boundary condition. We construct approximating solutions to \(u\) in the form of

\[
    u_N(x, t) := \sum_{k=1}^{N} p_k^N(t) \psi_k(x), \quad N \in \mathbb{N}
\]

such that

\[
    u_N(x, 0) = \sum_{k=1}^{N} (u, \psi_k)_{L^2(\Omega)} \psi_k(x)
\]

and

\[
    (\partial_t^\alpha u_N + Au_N, \psi_{\ell})_{L^2(\Omega)} = 0, \quad 1 \leq \ell \leq N, \quad t > 0,
\]

which is rewritten as

\[
    \int_{\Omega} \partial_t^\alpha u_N(x, t) \psi_{\ell}(x) dx + \sum_{i,j=1}^{d} a_{ij}(x)(\partial_j u_N)(x, t) \partial_i \psi_{\ell}(x) dx \\
    - \sum_{j=1}^{d} b_j(\partial_j u_N) \psi_{\ell}(x) dx - \int_{\Omega} c(x) u_N(x, t) \psi_{\ell}(x) dx = 0.
\]

Equation (4.4) is equivalent to a system of time-fractional ordinary differential equations in \(\{p_k^N(t)\}_{1 \leq k \leq N}\). Such \(p_k^N\) exist uniquely and \(u_N\) converges to the solution \(u\) weakly in \(L^2(0, T; H^2(\Omega))\). We can refer to e.g., Chapter 4 in [10], and we omit the details.

Multiplying (4.5) with \(p_{\ell}^N(t)\) and adding over \(\ell = 1, \ldots, N\), and integrating by parts the third term on the left-hand side, we obtain

\[
    \int_{\Omega} (\partial_t^\alpha u_N(x, t)) u_N(x, t) dx + \sum_{i,j=1}^{d} a_{ij}(x)(\partial_j u_N)(x, t) \partial_i u_N(x, t) dx \\
    + \frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^{d} \partial_j b_j \right) |u_N|^2 dx - \int_{\Omega} c(x) |u_N(x, t)|^2 dx = 0, \quad t > 0.
\]

By (1.2) and (1.4), we have

\[
    \int_{\Omega} (\partial_t^\alpha u_N(x, t)) u_N(x, t) dx + C(\Omega) \sigma(a_{ij}) \int_{\Omega} |u_N(x, t)|^2 dx \\
    + \frac{1}{2} \int_{\Omega} \left( \sum_{j=1}^{d} \partial_j b_j \right) |u_N|^2 dx - \int_{\Omega} c(x) |u_N(x, t)|^2 dx \leq 0, \quad t > 0.
\]

Applying (1.8), we can choose a constant \(\mu_0 > 0\) such that

\[
    \int_{\Omega} (\partial_t^\alpha u_N(x, t)) u_N(x, t) dx + \mu_0 \|u_N(\cdot, t)\|_{L^2(\Omega)}^2 \leq 0, \quad t > 0 \quad \text{for all } N \in \mathbb{N}.
\]
Hence Lemma 5.1 in [16] yields
\[ \|u_N(\cdot,t)\|_{L^2(\Omega)} \|u_N(\cdot,t)\|_{L^2(\Omega)} \leq -\mu_0 \|u_N(\cdot,t)\|_{L^2(\Omega)}^2 \leq 0, \quad t > 0. \] (4.6)

Then, we can formally argue as follows. By setting \(d_N(t) := \|u_N(\cdot,t)\|_{L^2(\Omega)}\) and assuming that \(d_N(t) \neq 0\) for all \(t > 0\), inequality (4.6) yields
\[
\begin{cases}
\partial_t^\alpha d_N(t) \leq -\mu_0 d_N(t), & t > 0, \\
d_N(0) = \left\| \sum_{k=1}^{N}(a,\psi_k)_{L^2(\Omega)}\psi_k \right\|_{L^2(\Omega)}.
\end{cases}
\] (4.7)

Noting that \(\partial_t^\alpha E_{\alpha,1}(-\mu_0 t^\alpha) = -\mu_0 E_{\alpha,1}(-\mu_0 t^\alpha)\), \(E_{\alpha,1}(0) = 1\), we apply the comparison principle (e.g., Luchko and Yamamoto [21]) to obtain
\[ d_N(t) \leq d_N(0) E_{\alpha,1}(-\mu_0 t^\alpha), \quad t > 0. \]

Therefore, for all \(N \in \mathbb{N}\), we have
\[ d_N(t) \leq \|a\|_{L^2(\Omega)} E_{\alpha,1}(-\mu_0 t^\alpha) \leq \frac{C \|a\|_{L^2(\Omega)}}{t^\alpha}, \quad t > 0, \]
where we use
\[ |E_{\alpha,1}(-\mu_0 t^\alpha)| \leq \frac{C}{1 + \mu_0 t^\alpha}, \quad t > 0 \]
(e.g., Theorem 1.6 (p.35) in [23]). This is an outline and for example, but we need to justify in deriving (4.7) by dividing with \(\|u_N(\cdot,t)\|_{L^2(\Omega)}\), because it may vanish at some \(t\). Such details are still the same as the proof of Theorem 5.1 of Chapter 5 in [16], and so we omit.

Finally we have to prove part (ii). Let \(d = 1, 2, 3\). By \(a \in \mathcal{D}(A)\), we consider
\[
\begin{cases}
\partial_t^\alpha U = -AU(x,t), & x \in \Omega, t > 0, \\
U|_{\partial \Omega \times (0,\infty)} = 0, \\
U(x,0) = -Aa(x), & x \in \Omega.
\end{cases}
\]

There exists a unique solution \(U \in C([0,\infty); L^2(\Omega))\) and applying part (i) to \(U\), we have
\[ \|U(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{t^{\alpha}} \|Aa\|_{L^2(\Omega)}, \quad t > 0. \]

Moreover the uniqueness of the solution yields \(U = \partial_t^\alpha u\) in \(\Omega \times (0,\infty)\). Therefore \(U = -Au\) in \(\Omega \times (0,\infty)\), and so
\[ \|Au(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{t^{\alpha}} \|Aa\|_{L^2(\Omega)}, \quad t > 0. \]

Since \(d = 1, 2, 3\), the Sobolev embedding implies
\[ \|u(\cdot,t)\|_{C(\overline{\Omega})} \leq \frac{C}{t^{\alpha}} \|Aa\|_{L^2(\Omega)}, \quad t > 0. \]
Thus the proof of Proposition 1 (ii) is complete. ■

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