Robustness of multiparticle entanglement: specific entanglement classes and random states

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Abstract
We investigate the robustness of genuine multiparticle entanglement under decoherence. We consider different kinds of entangled three- and four-qubit states as well as random pure states. For amplitude damping noise, we find that the W-type states are most robust, while other states are not more robust than generic states. For phase damping noise the GHZ state is the most robust state, and for depolarizing noise several states are significantly more robust than random states.

Keywords: robustness, multiparticle entanglement, decoherence, random states

1. Introduction
Entanglement between different particles is one of the peculiar features of quantum physics and its characterization is an active field of research [1, 2]. In theory, pure state entanglement can be used for tasks like quantum teleportation or cryptography, but in real implementations one cannot avoid interactions with the environment, leading to noise and decoherence. Therefore it is important to study the robustness of entangled states and many research efforts have been undertaken in this direction [3–8].

Several problems concerning the robustness of entanglement under decoherence have been discussed so far. Many works considered the lifetime of entanglement under decoherence [3], but it should be noted that the lifetime may not characterize the decoherence process well, since the lifetime of entanglement can be large, but the actual amount of entanglement is already small after a short time, making the decohered state practically useless for information processing tasks [4]. Here, the lifetime of entanglement denotes the time with a nonzero value of the chosen measure of entanglement.

Further works considered bipartite aspects of the entanglement of several particles [5], but this can only give a partial characterization, since multiparticle entanglement is known to be different from entanglement between all bipartitions [2]. A further central problem behind existing studies is that the theory of multiparticle entanglement is still not fully developed, so for many cases one can only make statements about lower bounds on entanglement, but not the actual value [6]. The exact calculation of a multiparticle entanglement measure was, so far, only possible for special states and decoherence models [7]. Finally, in order to make a fair statement about the robustness of a specific state one has to compare it with random states. Due to the difficulty in evaluating multiparticle entanglement measures, this has not been investigated so far.

In this paper we study the robustness of various prominent multi-qubit states as well as random states under local decoherence. Our work is enabled by recent progress in the theory of multiparticle entanglement, especially the computable entanglement monotone for genuine multiparticle entanglement from [10]. To calculate the robustness of this monotone, we have chosen all those time scales for which the values of monotone are not vanishingly small. We study different models of decoherence and identify the most robust...
states for this scenarios. For amplitude damping noise, we find that the W-type states are most robust, for phase damping noise the GHZ state is the most robust state, and for depolarizing noise several states are significantly more robust than random states.

The reader should be aware of the fact that the term ‘robustness of entanglement’ was used in [9] as a kind of quantification of entanglement, but our approach is different from this work, since we consider the behaviour under decoherence. Also in this paper we speak of multipartite systems and multipartite entanglement, but we essentially discuss two-level quantum systems (or qubits). This means that our results may also be applied to multi-qubit systems, where the qubits are not separated as particles.

This paper is organized as follows. In section 2.1, we describe our physical model to obtain the dynamics of an arbitrary density matrix. In section 2.2, we briefly review the concept of multipartite entanglement and also review the derivation of entanglement monotone. In sections 2.3–2.5, we define the investigated states including the generation method of random pure states and weighted-graph states. We present the main results in section 3. Finally, we offer some conclusions in section 4.

2. Preliminaries

2.1. Local decoherence models for multipartite states

We consider N qubits (e.g., N two-level atoms) which are coupled to their own local reservoirs. The reservoirs are assumed to be independent from each other. We assume weak coupling between each qubit and the corresponding reservoir and no back action effect of the qubits on the reservoirs. We also assume that the correlation time between the qubits and the reservoirs is much shorter than the characteristic time of the evolution so that the Markovian approximation is valid. The interactions of the physical system with environment can be studied via various techniques, for example, solving a master equation, the Kraus operator formalism, or quantum trajectories, etc. We work in the Kraus operator formalism. The time evolution of an initial density matrix can be written as

\[ \rho(t) = \sum K_i(t) \rho(0) K_i^\dagger(t), \]

where \( K_i(t) \) are the Kraus operators, satisfying the normalization condition \( \sum_i K_i^\dagger(t) K_i(t) = 1 \). The precise form of these Kraus operators is given as \( K_i(t) = \omega_i^A \otimes \omega_i^B \otimes \cdots \otimes \omega_i^N \), where \( \omega_i^j \) are the single-qubit Kraus operators acting on the \( j \)th qubit.

In the following, we consider three models. For amplitude damping, there are two Kraus operators for a single qubit,

\[ \omega_1^A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_1 \end{pmatrix}, \quad \omega_2^A = \begin{pmatrix} 0 & \sqrt{1 - \gamma_1^2} \\ 0 & 0 \end{pmatrix}, \]

where \( \gamma_j = e^{-T_j/2} \). For phase damping, the corresponding single-qubit Kraus operators are given as

\[ \omega_1^A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_j \end{pmatrix}, \quad \omega_2^A = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - \gamma_j^2} \end{pmatrix}. \]

Finally, for the depolarizing channel there are four single-qubit Kraus operators,

\[ \omega_1^A = \sqrt{1 - q} \mathbb{I}, \quad \omega_2^A = \frac{q}{\sqrt{3}} \sigma_x, \]

\[ \omega_3^A = \frac{q}{\sqrt{3}} \sigma_y, \quad \omega_4^A = \frac{q}{\sqrt{3}} \sigma_z, \]

where \( q = 3p/4 \) with \( p = 1 - \gamma_j \), and \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices. Note that in ion-trap experiments, amplitude damping is the typical noise, while for photonic experiments phase damping is more relevant. For the case of N-qubit states, there are \( 2^N \) global Kraus operators \( K_i(t) \) for the amplitude and phase damping channels and \( 4^N \) Kraus operators for the depolarizing channel. For the sake of simplicity we assume onwards that \( \gamma_1 = \gamma_2 = \cdots = \gamma_N = \gamma \).

The time evolved density matrix for a single qubit can directly be computed. Under amplitude damping it is given as

\[ \rho(t) = \begin{pmatrix} \rho_{11} + e^{2\gamma t} (1 - e^{-\gamma t}) & e^{\gamma t} e^{-\gamma t/2} \\ e^{-\gamma t/2} & e^{-\gamma t} \end{pmatrix}, \]

whereas the time evolved density matrix for a single qubit state under phase damping is given as

\[ \rho(t) = \begin{pmatrix} \rho_{11} + p(1 - \rho_{11}) & e^{\gamma t} e^{-\gamma t/2} \\ e^{-\gamma t/2} & e^{-\gamma t} - p \rho_{22} + p(1 - \rho_{22}) \end{pmatrix}. \]

Finally, the density matrix for a single qubit state under depolarizing noise is

\[ \rho(t) = \begin{pmatrix} \rho_{11} + p(1 - \rho_{11}) & e^{\gamma t} e^{-\gamma t/2} \\ e^{-\gamma t/2} & e^{-\gamma t} - p \rho_{22} + p(1 - \rho_{22}) \end{pmatrix} = (1 - p) \rho + p \frac{\mathbb{I}}{2}. \]

For more qubits, the calculation of density matrices is straightforward.

2.2. Genuine multipartite entanglement and the multiparticle negativity

Let us first recall the basic definitions for genuine multipartite entanglement. We explain the main ideas by considering three particles \( A, B \) and \( C \), the generalization to more parties is straightforward. A state is separable with respect to some bipartition, say, \( A|BC \), if it is a mixture of product states with respect to this partition, that is, \( \rho = \sum q_k \phi_A^k \phi_B^k \otimes \phi_C^k \), where the \( q_k \) form a probability distribution. We denote these states as \( \rho_{A|BC}^{sep} \). Similarly, we can define separable states for the two other bipartitions \( \rho_{AB|C}^{sep} \) and \( \rho_{ABC}^{sep} \). Then a state is called biseparable if it can be written as a mixture of states which are separable with respect to different bipartitions, that is

\[ \rho^{bs} = p_1 \rho_{A|BC}^{sep} + p_2 \rho_{AB|C}^{sep} + p_3 \rho_{ABC}^{sep}. \]

Finally, a state is called genuinely multipartite entangled if it is not biseparable. In the remainder of this paper, we always mean genuine multipartite entanglement when we talk about entanglement.

Recently, a powerful technique has been worked out to detect and characterize multipartite entanglement [10]. The method is to use positive partial transpose mixes.
(PPT mixtures). Recall that a two-party state \( \varrho = \sum_{ijkl} \varrho_{ijkl} |i⟩⟨j| \otimes |k⟩⟨l| \) is PPT if its partially transposed matrix \( \varrho^T_k = \sum_{ijkl} \varrho_{ijkl} |i⟩⟨j| \otimes |k⟩⟨l| \) has no negative eigenvalues. A well-known fact is that separable states are always PPT [11]. The set of separable states with respect to some partition is therefore contained in a larger set of states which has a PPT for that bipartition.

We denote the states which are PPT with respect to fixed bipartition by \( \varrho_{\text{A}B}^{\text{PPT}}, \varrho_{\text{ABC}}^{\text{PPT}}, \) and \( \varrho_{\text{ABC}}^{\text{PPT}} \) and ask the question whether a state can be written as

\[
\varrho^{\text{PPTmix}} = p_1 \varrho_{\text{A}B}^{\text{PPT}} + p_2 \varrho_{\text{B}C}^{\text{PPT}} + p_3 \varrho_{\text{C}A}^{\text{PPT}}.
\]

Such a mixing of PPT states is called a PPT mixture. The genuine multiparticle entanglement of four or more particles can be detected and quantified in an analogous manner by considering all bipartitions (like one particle versus \( N - 1 \) particles, two particles versus \( N - 2 \) particles, etc).

Obviously, any biseparable state is a PPT mixture, therefore any state which is not a PPT mixture is guaranteed to be genuinely multiparticle entangled. The major advantage of considering PPT mixtures instead of biseparable states comes from the fact that PPT mixtures can be fully characterized with the method of semidefinite programming (SDP), a standard method in convex optimization [12]. In general, the set of PPT mixtures is a very good approximation to the set of biseparable states and the W states for \( N \) qubits.

Let us briefly describe the SDP. As shown in [10], a state \( \varrho \) of entanglement is called the weighted-graph state, and it can be expressed as

\[
(\begin{array}{llllll}
1 & 0 & 0 & . & . & 0 \\
0 & 0 & 0 & . & . & 0 \\
0 & 0 & 0 & . & . & 0 \\
. & . & . & . & . & . \\
0 & 0 & 0 & . & . & 0 \\
0 & 0 & 0 & . & . & 0 \\
\end{array})^{\otimes N}.
\]

In the past, the structure of weighted-graph states made it possible to deal with thousands of spins (spin gases) to study their entanglement features [17].

To define weighted-graph states, consider a graph as a simple undirected graph with \( N \) vertices

\[
G = \{V, E\}, \quad V = \{0, 1, \ldots, N-1\}, \quad E = \{(i,j) : i \neq j \}.
\]

For us, it is important that this approach can be used to quantify genuine entanglement. In fact the absolute value of the above minimization was shown to be an entanglement monotone for genuine multiparticle entanglement [10]. In the following, we will denote this measure by \( E(\varrho) \). For bipartite systems, this monotone is equivalent to the so-called negativity [15]. A system of qubits, this measure is bounded by \( E(\varrho) \leq 1/2 \) [16].

2.3. Investigated quantum states

Let us introduce the multiparticle entangled states which we study in this paper. Two important types of states are the GHZ states and the W states for \( N \) qubits,

\[
|\text{GHZ}_N⟩ = \frac{1}{\sqrt{2}} (|00\ldots0⟩ + |11\ldots1⟩),
\]

\[
|\text{W}_N⟩ = \frac{1}{\sqrt{N}} (|00\ldots01⟩ + |00\ldots10⟩ + |10\ldots01⟩).
\]

For the GHZ state, the entanglement monotone has a value of \( E(|\text{GHZ}_N⟩⟨\text{GHZ}_N|) = 1/2 \), while for the W state, its value is \( E(|\text{W}_N⟩⟨\text{W}_N|) \approx 0.443 \) and \( E(|\text{W}_3⟩⟨\text{W}_3|) \approx 0.366 \).

For the case of four qubits, several other states are interesting and have been discussed in the literature. These states are the Dicke state \( |\text{D}_{2,4}⟩ \), the four-qubit singlet state \( |\text{S}_{4}⟩ \), the cluster state \( |\text{CL}⟩ \) and the so-called \( \chi \)-state \( |\chi⟩ \).

They are explicitly given by

\[
|\text{D}_{2,4}⟩ = |0011⟩ + |1100⟩ + |0101⟩ + |1010⟩,
\]

\[
|\text{S}_{4}⟩ = \frac{1}{2}(|0011⟩ + |1100⟩ - |0101⟩ + |1010⟩),
\]

\[
|\text{CL}⟩ = \frac{1}{2}(|0000⟩ + |0011⟩ + |1100⟩ - |1111⟩),
\]

\[
|\chi⟩ = \frac{1}{\sqrt{3}} (|0011⟩ + |0001⟩ + |0100⟩ + |0100⟩ + |1000⟩).
\]

Note that all of these states have the maximum value of entanglement \( E(|\text{D}_{2,4}⟩⟨\text{D}_{2,4}|) = E(|\text{S}_{4}⟩⟨\text{S}_{4}|) = E(|\text{CL}⟩⟨\text{CL}|) = E(|\chi⟩⟨\chi|) = 1/2 \). Further entanglement properties of these states are reviewed in [2].

2.4. Weighted-graph states

Weighted-graph states form a family of multi-qubit states that includes states with a large variety of entanglement features (such as GHZ and cluster states) [17, 18]. In the past, the structure of weighted-graph states made it possible to deal with thousands of spins (spin gases) to study their entanglement features [17].

To define weighted-graph states, consider a graph as a set of vertices and edges. Physically, the vertices denote the physical systems (qubits), whereas the edges represent the interactions among physical systems. In the beginning, one prepares all of the qubits in the state \( |+⟩ = (|0⟩ + |1⟩)/\sqrt{2} \). Then, if two qubits \( k, l \) are connected with an edge, one applies an interaction according to the Hamiltonian

\[
H_{kl} = \frac{1}{2} (|i⟩⟨i| - |j⟩⟨j|) \otimes (|i⟩⟨i| - |j⟩⟨j|).
\]

This leads to a unitary transformation of the type \( U_{kl} = e^{i\phi_{kl} H_{kl}} \), where \( \phi_{kl} \) is the interaction time. The resulting state is then called the weighted-graph state, and it can be expressed as

\[
|G⟩ = \bigotimes_{k,l} U_{kl}(\phi_{kl}) |+⟩^{\otimes N}.
\]

In this way, the weighted-graph state is uniquely determined by the \( N(N - 1)/2 \) parameters \( \phi_{kl} \). Clearly, weighted-graph states form only a small subset of all pure states (which are described by \( 2^N - 1 \) parameters), but many interesting states fall in this class. For generating random weighted-graph states, we have chosen the interaction times \( \phi_{kl} \in [0, 2\pi] \) uniformly distributed in the interval.
Three more remarks are in order. First, if one considers only the possibilities $\phi_{ij} = \pi$ or $\phi_{ij} = 0$, the usual graph states (to which the GHZ and cluster state belong) emerge. Second, it should be noted that the unitaries $U_{ij}$ commute, so the temporal order of the interaction does not matter. Finally, some generalizations of weighted-graph states have been proposed and investigated recently [19].

### 2.5. Random pure states

Finally, let us describe how we have generated random pure states. A state vector randomly distributed according to the Haar measure can be generated in the following way [20]: first one generates a vector such that both the real and the imaginary parts of the vector elements are Gaussian distributed random numbers with a zero mean and unit variance. Second we normalize the vector. It is easy to prove that the random vectors obtained in this way are equally distributed on the unit sphere [20]. Note that this generates random pure states in the global Hilbert space of three and four qubits, so the unit sphere is not the Bloch ball.

### 3. Results

In this section we study the robustness of entanglement of three and four qubits under local decoherence. Before studying the different states and models, let us define how one can quantify the robustness. First, as already mentioned, the lifetime of entanglement may lead to inconclusive results, since the state under decoherence may be entangled for a long time, but the amount of entanglement may be nearly zero and therefore of little use for quantum information processing tasks. Second, the lifetime of entanglement is clearly not a reasonable figure of merit, if the initial states that should be compared already have a different amount of entanglement. For the same reason, also the comparison of the values of $E(\rho_i(t))$ for different $t$ and initial states $i$ is not useful.

In our approach, we consider the logarithmic derivative

$$\eta(t) = \frac{d}{dt} \left( \ln\left[ E(\rho_i(t)) \right] \right) = \frac{d}{dt} \left[ \frac{E(t)}{E(\rho_i(t))} \right],$$

where $E(t)$ is the entanglement monotone [7]. This describes the relative decay of entanglement, and allows states to be compared with a different initial entanglement. In this way, the introduction of the logarithmic derivative makes sure that comparative change in monotone is meaningful for several different quantum states. Note that for a state where the entanglement just decays exponentially, $\eta(t)$ is constant and the inverse of the half-life.

#### 3.1. Amplitude damping

##### 3.1.1. Three qubits

Let us first consider the effects of amplitude damping on quantum states of three qubits. Figure 1 shows the logarithmic derivative $\eta(t)$ of the entanglement monotone, plotted against the parameter $\gamma t$ for different types of states. In this figure we show this parameter for the usual GHZ and W state (denoted by GHZ$_1$ and W$_1$). In addition, we computed the value for two different forms of the GHZ and W state, namely $|GHZ_2\rangle = 1/\sqrt{2} (|001\rangle + |110\rangle)$ and $|W_2\rangle = 1/\sqrt{3} (|110\rangle + |101\rangle + |011\rangle)$. Finally, we also give the mean value for random pure states (MRS) and random weighted-graph states (MWGS). For computing these mean values, we considered 100 realizations of the respective states, an explicit figure is given in the appendix. From these data we also obtain an error estimate to indicate the reliability of measure. This can, for instance, be defined as a confidence interval

$$CI = \mu \pm \sqrt{\delta},$$

where $\mu$ stands for mean value and $\delta$ for the variance of the quantity being measured. Note, however, that this is not a confidence interval in the mathematical sense.

It should be mentioned that if the entanglement vanishes, the quantity $\eta(t)$ diverges. In our investigations we always consider timescales which are much shorter than the lifetime of entanglement, so this effect does not occur in our figures. Only for random states, we found singular cases (with a probability less than 1%) where the entanglement vanishes fast, these are then not included in our sample of random states.

From figure 1 and figure A1 in the appendix we can conclude two things: first, the W state is for this type of decoherence clearly the most robust state. The other states do not deviate significantly from random states. Second, for nearly all states the decay of entanglement is roughly exponential, while for the GHZ state it is super-exponential.

#### 3.1.2. Four qubits

For the case of four qubits, the main results are given in figure 2. Here, we consider all the four-qubit states as introduced in section 2.3: the W state, the GHZ state, the Dicke state $D_{2,4}$, the singlet state $S_4$, the $\chi_4$-state $X_4$ and the cluster state CL. Again, we compare them with random pure states and random weighted-graph states. The conclusions are similar as for the three-qubit case: the entanglement present in the W state is most robust against decoherence, the other states are, in this respect, not significantly different from random states.
3.2. Phase damping

For three qubits, we study the robustness of genuine entanglement under phase damping for GHZ type states, W-type states, random pure states and random weighted-graph states, the results are given in figure 3(a). In this figure, one may be surprised about the fact that the logarithmic derivative for the W state seems to be non-analytic at some point. This, however, has also been observed for other measures [7]. Such points can occur, if the entanglement measure is a so-called convex roof measure [1] and the optimal decomposition in this convex roof construction changes qualitatively. In our case, we defined the measure $E(\rho)$ via an SDP, but the same measure can also be viewed as a convex roof measure [21].

For four qubits, we investigate again the various entangled states mentioned above, see figure 3(b). The variances for the random states show the same behaviour as before, so we have not displayed them in the diagram.

Summarizing, we can state that for dephasing noise the GHZ state turns out to be the most robust state. Other states show a similar behaviour as random states, whereas the $|W\rangle$ state is a very fragile state.

3.3. Depolarizing noise

Finally, we study the effects of depolarizing noise on multiparticle entanglement of three- and four-qubit quantum states. For three qubits, the results are given in figure 4(a). Clearly, the GHZ state is the most robust state and the W state does not significantly deviate from random states.

The four-qubit results are given in figure 4(b). Here, an interesting feature emerges: apart from the W state, all states
4. Discussion and summary

We studied the effects of local decoherence on genuine multiparticle entanglement for various quantum states, including random states. We found that for amplitude damping noise the W-type states are most robust, for phase damping noise the GHZ state is the most robust state, and for depolarizing noise several states are significantly more robust than random states. It is worth mentioning that the robustness of GHZ and W states has also been discussed in [6]. Our results are in line with these findings for amplitude damping noise and depolarizing noise. For dephasing noise, however, the results of [6] are opposite to ours. A reason for this discrepancy may be that the lower bound on entanglement used in [6] works better for W class states, suggesting that they are more robust to noise.

There are several directions in which our work can be extended. First, it would be very interesting to investigate the scaling of the robustness with the number of qubits $N$. For that, one needs analytical formulas for entanglement measure $E(\varrho)$. First results indicate that such formulas may be derived [21]. Second, it would be interesting to find out why certain states are more robust to noise than others. This may be investigated by modelling the interaction between the particles and the environment. Third, it is of interest to compare for a given multiparticle state the robustness of entanglement with the
usefulness for some task in quantum information processing. For instance, the usefulness for quantum metrology can be determined via calculating the Fisher information [22], so the states which are most robust for metrology can be identified. Finally, the robustness of entangled states is also of relevance for the characterization of quantum channels [23]. Therefore, the algorithms used in our paper may be helpful for this task.

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Appendix

In order to show the typical behaviour of random states, figure A1(a) shows the logarithmic derivative $\eta(t)$ of the entanglement monotone against parameter $\gamma t$ for 100 random pure states under amplitude damping, while A1(b) shows the same quantity for random weighted-graph states.

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