Some Remarks on Tachyon Action in 2d String Theory

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We discuss the effective action of tachyon in the two dimensional string theory at tree level. We show that already starting from the cubic terms the action is nonlocal and the usually assumed simplest cubic term does not give the correct amplitude. Four point 1PI terms are also discussed.
Two dimensional string theory is a very simple string model, compared to the critical bosonic string and any critical superstring theory. The only dynamic mode is the tachyon field, besides some topological modes. Yet it is a nontrivial toy model, for its structure is rich and much of it is still to be unravelled. In a flat background with a linear dilaton, perturbation theory is ill-defined without a tachyon background, namely the cosmological term, since there is a large strong coupling region. Some even speculates that a tachyon background could be dynamically switched on, albeit via some nonperturbative effects, in the end of a process such as Hawking radiation of a blackhole. Once a tachyon background is present, any incident tachyon mode will be prevented from penetrating into the strong coupling region.

One of remarkable features of the 2d string theory is the absence of bulk tachyon amplitudes. There is only one delta function in the scattering amplitude due to the conservation of energy. This is not surprising, since there is no conservation of momentum because of the existence of a linear dilaton. The scattering of tachyon is solely due to the wall effects. There is only a finite interaction region for tachyon modes with a given energy, where the coupling constant is strong enough yet not too strong so that the tachyon modes still can penetrate in the tachyon barrier.

It is conventional to write the following action without a tachyon background

\[ S = \int \sqrt{-g} e^{-2\Phi} (R + 4(\nabla \Phi)^2 + (\nabla T)^2 + 2T^2 + \frac{a}{6}T^3 + \ldots - 8), \]

where \( a \) is a constant and dots denote higher order terms of the tachyon potential. In the 2d string theory, it is questionable to use such a seemingly background independent action. There are several points to argue against use of the above action. First, perturbation theory does not exist for small fluctuations of \( T \). This can be easily seen by rescaling \( T \) to absorb the dilaton factor. There will be extra factor \( \exp(\Phi) \) in the cubic term. This is precisely the string coupling constant and there is no bound of it with a linear dilaton field. Second, equation of motion for the tachyon derived from (1) does not allow an exact solution \( \exp(2\phi) \) whenever \( a \neq 0 \), where \( \phi \) is the Liouville dimension. We are convinced by some calculations of tachyon amplitudes on the world sheet that such a tachyon background is exactly marginal, at least in that framework of regularization. Another perhaps unpleasant fact is that no matter what the exact tachyon background is, when it

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2 It should be noted that the discrete states in the 2d string occur at imaginary energies, therefore are nonpropagating. Their major role may be to provide time-dependent backgrounds.
is plugged into the equation of motion for the metric derived from (1), there is backreaction to the metric. The metric is no longer flat [4].

So we are forced to introduce a tachyon background into the action. As will be seen, to reproduce even three point amplitude calculated by different methods [3], we have to introduce perhaps infinitely many cubic terms. This is a nonlocal action, similar to the nonlocal quadratic action introduced in [6]. It is appropriate to point out that we are considering the effective action for the tachyon. Although other modes such as the graviton do not propagate, integrating out these modes in principle generates nonlocal terms in the tachyon effective action. A similar situation occurs in the 2d Schwinger model, where integrating out the gauge field induces nonlocal terms. It is not clear whether the nonlocal terms we need to introduce can be generated in this way.

Let the dilaton field be $\Phi = 2\phi$. The simple generalization of the kinetic action is

$$S_0 = \int d\phi dt \left( \frac{1}{2} (\partial_t T)^2 - \frac{1}{2} (\partial_\phi T)^2 - 2\mu e^{2\phi} T^2 \right),$$  \hspace{1cm} (2)$$

where $T$ is the re-scaled tachyon field. The effect of the tachyon background is encoded in the last term. $T = 0$ is a solution to the equation of motion, this means that the tachyon background should be an exact solution to some unknown background independent equation of motion. We should emphasize at this point that this background independent action is highly nontrivial in view of the above effective action with the tachyon background. The Das-Jevicki collective field theory [7] is not such a background independent theory yet, as the parameter $\mu$ enters explicitly as the lagrangian multiplier. With (2) as our quadratic action, we will see that perturbation is now well-defined. Seiberg and Shenker argue that a tachyon background represents a superselection sector, so should not be minimized over [8]. This may be tied up with the difficulty of writing down a background independent action.

The equation of motion for $T$ is simply

$$(-\partial_t^2 + \partial_\phi^2 - 4\mu e^{2\phi})T = 0.$$ 

This is just the Wheeler-de Wit equation which is satisfied exactly by the macroscopic loop [9]. The solution which drops to zero when $\phi \to \infty$ and satisfies the asymptotic condition

$$T(\phi, t) \to e^{i\omega(\phi-t)} + S(\omega)e^{-i\omega(\phi+t)}$$
when $\phi \to -\infty$ is
\[
T_\omega(\phi, t) = \frac{1}{\pi} \sqrt{\sinh \pi \omega} K_{i\omega}(2\sqrt{\mu} e^{i\phi}) e^{-i\omega t}. \tag{3}
\]
The first term in the asymptotic form of $T$ is the incoming plane wave, the second is the reflected outgoing plane wave. $S(\omega)$ is just the reflection coefficient. It is easily found from the definition of function $K$: 
\[
S(\omega) = -\mu^{-i\omega} \frac{\Gamma(1 + i\omega)}{\Gamma(1 - i\omega)}, \tag{4}
\]
this is the exact tree level two point amplitude, it is a pure phase and can be absorbed by a redefinition. It is straightforward to establish the following orthogonal relation
\[
\int_{-\infty}^{\infty} \, d\phi T_{\omega'}(\phi) T_\omega(\phi) = \frac{1}{2\omega} \delta(\omega - \omega'),
\]
where $T_\omega(\phi)$ is the function in (3) without the time factor. It is a real function of $\phi$.

Before setting off to discuss scattering amplitudes, we perform the standard second quantization. First we should emphasize that for a given $\omega$, there is only one single particle state represented by (3) which contains the incident right moving mode and the reflected left moving mode. In the usual quantum field theory with one spatial dimension, there are two modes associated with a given energy $\omega$, namely the right moving and left moving modes. In our case, if we like, we can view the incident right moving mode as an in-state, and the reflected mode as an out-state, then the reflection coefficient (4) represents the time delay. Now we expand the quantum field $T$ in terms of $T_\omega$:
\[
T(\phi, t) = \int_{0}^{\infty} \, d\omega (a_\omega T_\omega(\phi, t) + a_\omega^+ T_\omega^*(\phi, t)), \tag{5}
\]
where $a_\omega$ and $a_\omega^+$ are destruction and creation operators respectively. With the help of the orthogonal relation, we obtain the commutation relation
\[
[a_{\omega'}, a_{\omega}^+] = \delta(\omega' - \omega).
\]
The hamiltonian corresponding to the free action (2) is then diagonalized:
\[
H = \int_{0}^{\infty} \, d\omega \omega a_{\omega}^+ a_{\omega}.
\]
There are only half modes as can be seen obviously from the above equation. We comment at this point that the authors of [6] introduced a nonlocal quadratic action which is significantly different from (2) at high energies, as they believe this is the signal of soft high
energy behavior of a string model. This nonlocality is certainly the most common feature in the model under discussion, as it arises also in the cubic interaction as will be seen soon. If we use the nonlocal quadratic action in [6], the three point amplitude calculated from the simplest cubic action is even more different from the correct answer. We will not use that quadratic action here.

The most natural cubic term in the lagrangian is proportional to \( \exp(2\phi)T^3 \), here the exponential factor represents the string coupling constant. It is easy to discuss three point amplitudes in the Hamiltonian framework. We then write the contribution of this term to the interaction part of hamiltonian as

\[
H_I^{(3)} = -\frac{a}{3!} \int d\phi e^{2\phi} : T^3 :, \tag{6}
\]

where : : denotes the normal ordering, \( a \) is a constant to be determined.

Three point amplitude \( T(\omega_1 \rightarrow \omega'_1 + \omega'_2) \) will be calculated only, since the result for another amplitude is the same. The correct answer given in [5] is

\[
T(\omega_1 \rightarrow \omega'_1 + \omega'_2) = \frac{2\pi i}{8\pi \mu} \delta(\omega_1 - \omega'_1 - \omega'_2) \sqrt{\omega_1 \omega'_1 \omega'_2}, \tag{7}
\]

a fairly simple result.

Now expanding \( H_I^{(3)} \) in terms of modes, the relevant part for this amplitude is

\[
-\frac{a}{2} \int d\phi e^{2\phi} (\int d\omega_1 d\omega_2 d\omega_3 T^{*}_{\omega_1} T^{*}_{\omega_2} a_{\omega_1} a_{\omega_2} a_{\omega_3} + a^+_{\omega_1} a^+_{\omega_2} a_{\omega_3}).
\]

The three point amplitude is read off straightforwardly from the above equation

\[
T_0(\omega_1 \rightarrow \omega'_1 + \omega'_2) = 2\pi ai\delta(\omega_1 - \omega'_1 - \omega'_2) f_{\omega'_1} f_{\omega'_2} f_{\omega_1}, \tag{8}
\]

\[
\int_{-\infty}^{\infty} d\phi e^{2\phi} K_{i\omega_1}(2\sqrt{\mu e^\phi}) K_{i\omega'_1}(2\sqrt{\mu e^\phi}) K_{i\omega'_2}(2\sqrt{\mu e^\phi}),
\]

where

\[
f_{\omega} = \frac{1}{\pi} \sqrt{\sinh \pi \omega}.
\]

The reason for us to denote this amplitude by \( T_0 \) is that this does not give us the correct one and we reserve the name \( T \) for the correct one. The low energy behavior of the integral involving Bessel functions in (8) is discussed in [8] and agrees with calculations done in the matrix model approach. We shall perform the integral exactly and show that the amplitude differs significantly from the standard result (7) at high energies. This will force
us to introduce more cubic terms in order to reproduce the correct amplitude. Using new
variable \( x = \exp(\phi) \), the integral reads

\[
\frac{1}{(2\sqrt{\mu})^2} \int_0^\infty dx x K_{i\omega_1}(x) K_{i\omega'_1}(x) K_{i\omega'_2}(x).
\]

This integral is well defined, as the strong coupling region is suppressed by the wave
functions. We also see explicitly how the effective coupling constant \( 1/\mu \) emerges. We are
interested in the case when \( \omega_1 = \omega'_1 + \omega'_2 \), the integral is performed in the appendix in this
case. The result is

\[
\frac{\pi}{8\mu \sinh(\pi \omega_1)} \text{Im} \left( \sum_{n=0}^\infty B(1 - i\omega'_1 + n, 1 - i\omega'_2 + n) \right).
\]

The sum of beta functions can be expressed in terms of a generalized hypergeometric
function. It seems unlikely that this can be reduced to a combination of elementary
functions. To estimate the high energy behavior, we write the sum of beta functions as an
integral

\[
\int_0^1 dt t^{-i\omega'_1} (1 - t)^{-i\omega'_2} \frac{1}{1 - t + t^2}.
\]

When both \( \omega'_1 \) and \( \omega'_2 \) are small, the imaginary part of this integral is about

\[
-\omega_1 \int_0^1 dt \frac{\ln t}{1 - t + t^2} = C \omega_1,
\]

where the constant \( C = 1/3(\psi'(1/3) - 2\pi^2/3) = 0.781302 \ldots \). Substituting this back to (8)
we find that the low energy behavior of the scattering amplitude agrees with the correct
one, provided the constant in the cubic action is

\[
a = \frac{\pi^{3/2}}{C}
\]

When \( \omega'_1 \) and \( \omega'_2 \) are both large, the integral can be estimated by the stationary phase
approximation. Although the amplitude of this integral is about the same as one should
expect from the correct scattering amplitude, it oscillates fast. The high energy behavior
of the scattering amplitude calculated from the cubic action is then

\[
T_0(\omega_1 \to \omega'_1 + \omega'_2) = 2\pi a \delta(\omega_1 - \omega'_1 - \omega'_2) f_{\omega_1} f_{\omega'_1} f_{\omega'_2} \sqrt{2\pi \omega_1 \omega'_1 \omega'_2} \sin \left( \ln \frac{\omega_1}{(\omega'_1)^{\omega_1} (\omega'_2)^{\omega'_2}} + \frac{\pi}{4} \right). \tag{9}
\]
We have seen that the simplest cubic term does not generate the correct three point amplitude, although it does at low energy. Apparently there are many ways to introduce additional cubic terms in order to generate the three point amplitude. We give one simple choice, maybe not the minimal one, in the following. Let $f(\omega_1', \omega_2') = T/T_0$, the ratio of the correct amplitude and the amplitude generated by the simple cubic term. This function has an expansion

$$f(\omega_1', \omega_2') = \sum_{m,n=0} f_{mn}(\omega_1')^m(\omega_2')^n.$$  

The leading term is 1. Consider a function $g(\omega_1', \omega_2')$ satisfy

$$\frac{1}{3}(g(\omega_1', \omega_2') + g(\omega_1' + \omega_2', \omega_1') + g(\omega_1' + \omega_2', \omega_2')) = f(\omega_1', \omega_2').$$

Given a solution, expanding it into power series

$$g(\omega_1', \omega_2') = \sum_{m,n=0} g_{mn}(\omega_1')^m(\omega_2')^n;$$

we introduce in the action the cubic terms

$$\frac{a}{3!} \sum_{m,n=0} g_{mn}e^{2\phi}T(H^{m/2}T)(H^{n/2}T), \quad (10)$$

where operator $H$ is

$$H = -\frac{\partial^2}{\partial\phi^2} + 4\mu e^{2\phi}.$$

(10) will reproduce the correct three point amplitude, obviously it is a nonlocal action.

Four point amplitudes are more difficult to calculate. Consider for example amplitude $T(\omega_1 + \omega_2 \rightarrow \omega_1' + \omega_2')$. There are two difficulties. First, to calculate the one point reducible part, we have to know the off-shell three point amplitude, namely when $\omega_1 \neq \omega_1' + \omega_2'$ in (8). Second, to calculate the 1PI part, we need to calculate an integral involving the product of four Bessel functions. Let us first discuss the one point reducible part. According to the standard formula, this amplitude is given by

$$2\pi i\delta(\omega_1 + \omega_2 - \omega_1' - \omega_2') \int_0^\infty d\omega_3 \frac{T(\omega_3 \rightarrow \omega_1 + \omega_2)T(\omega_3 \rightarrow \omega_1' + \omega_2')}{\omega_1 + \omega_2 - \omega_3 - i\epsilon}, \quad (11)$$

where $T(\omega_3 \rightarrow \omega_1 + \omega_2)$ is the three point amplitude containing no delta function corresponding to energy conservation, $\epsilon$ is a small positive constant. The off-shell three point amplitude depends crucially on what cubic action we choose, as we know there is no unique
way to choose one to reproduce the correct on-shell amplitude. According to our choice made before, the off-shell amplitude is

\[ T(\omega_3 \rightarrow \omega_1 + \omega_2) = T_0(\omega_3 \rightarrow \omega_1 + \omega_2) f(\omega_1, \omega_2), \]

where \( T_0 \) (without the delta function factor) is calculated by the simplest cubic action \( \exp(2\phi)T^3 \) with arbitrary \( \omega_i \), and \( f(\omega_1, \omega_2) \) is the function defined before. A formula for \( T_0 \) is given in the appendix.

Now we turn to calculating the 1PI amplitude, starting from

\[ H^{(4)}_I = -\frac{b}{4!} \int d\phi e^{4\phi} :T^4:, \]

where \( b \) is another constant. Denote the amplitude calculated from this hamiltonian by \( T_0 \) again,

\[ T_0(\omega_1 + \omega_2 \rightarrow \omega'_1 + \omega'_2) = 2\pi b i \delta(\omega_1 + \omega_2 - \omega'_1 - \omega'_2) f_{\omega_1} f_{\omega_2} f_{\omega'_1} f_{\omega'_2} \int_{-\infty}^{\infty} d\phi e^{4\phi} K_{i\omega_1}(2\sqrt{\mu} e^{\phi}) K_{i\omega_2}(2\sqrt{\mu} e^{\phi}) K_{i\omega'_1}(2\sqrt{\mu} e^{\phi}) K_{i\omega'_2}(2\sqrt{\mu} e^{\phi}). \] (12)

Again an integral with Bessel functions need be calculated. Using \( x = e^\phi \), the integral in (12) is just

\[ \frac{1}{(2\sqrt{\mu})^4} \int dx x^3 K_{i\omega_1}(x) K_{i\omega_2}(x) K_{i\omega'_1}(x) K_{i\omega'_2}(x), \] (13)

where we see how the coupling constant \( 1/\mu \) appears naturally. We have tried to reduce this integral to a simpler one. It turns out that this is possible, but the simpler integral can not be carried out. To estimate the low energy behavior of (13), use the following transform

\[ K_{i\omega}(x) = \int dt e^{-x \cosh t} \cos(\omega t). \]

From this formula we immediately see that when all \( \omega \)'s are small, the leading term in (13) is a constant. So (12) does give the right low energy behavior. The next order of the integral (13) is quadratic in \( \omega \)'s, therefore (12) alone does not give the correct next order term [5].

It is not particularly illuminating to exactly calculate the one point reducible part in (11) and the irreducible part in (12), and to combine them to get a closed formula. Suffices it to say that the simplest quartic term together with the cubic terms we have introduced does not give the correct four point amplitude, as given in [5]. Once again infinitely many quartic terms are to be introduced.
In conclusion we have seen that to reproduce scattering amplitudes calculated by other means, infinitely many terms have to be introduced in the effective action of tachyon. This effective action is nonlocal, and we suspect that nonlocality persists to all orders. We have also noted that the effective action can not be fixed uniquely, upon comparing scattering amplitudes. It may be helpful to employ a spacetime realization of those discrete symmetries \cite{10} \cite{11} to fix the effective action. In view of the above discussion, it is remarkable that the Das-Jevicki collective field theory summarizes the physical content neatly in a cubic action, with derivative couplings. Discrete symmtries are also readily realized in that theory. There must be an interesting transformation between the usual effective action and the collective field theory. This transformation is most desirable to know in order to better understand spacetime physics of the 2d string, and eventually to uncover a background independent formulation. Finally, we comment that it is not clear whether the collective field theory is just the effective theory for tachyon in the flat background, or it indeed encodes more information about the 2d string. To go beyond the flat background such as to discuss blackhole physics in this framework therefore requires more justification.

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Appendix A.

In this short appendix we calculate integral

\[ \int dx x K_{i\omega_1}(x) K_{i\omega'_1}(x) K_{i\omega'_2}(x). \]  

(A.1)

Use the integral representation of the product of two Bessel functions \cite{12}

\[ K_{i\omega'_1}(x) K_{i\omega'_2}(x) = 2 \int_0^\infty dt \cos(\omega'_1 - \omega'_2) t K_{i(\omega_1 + \omega_2)}(2 \cosht) \]

the integral (A.1) is reduced to

\[ 2 \int_0^\infty dt \cos(\omega'_1 - \omega'_2) t \int_0^\infty dx x K_{i\omega_1}(x) K_{i(\omega'_1 + \omega'_2)}(2 \cosht) \]
The integral involving the product of two Bessel functions can be expressed in terms of the hypergeometric function \([12]\). When \(\omega_1 = \omega'_1 + \omega'_2\), this hypergeometric function reduces to an elementary function, so (A.1) reduces to

\[
\frac{\pi}{i \sinh \pi \omega_1} \int_0^{\infty} dt \frac{\cos(\omega'_1 - \omega'_2)t}{4 \cosh^2 t - 1} \left( (2 \cosh t)^{i \omega_1} - (2 \cosh t)^{-i \omega_1} \right)
= \frac{\pi}{2 \sinh \pi \omega_1} \text{Im} \left( \sum_{n=0}^{\infty} B(1 - i \omega'_1 + n, 1 - i \omega'_2 + n) \right).
\]

(A.2)

For \(\omega_1 \neq \omega'_1 + \omega'_2\), integral (A.1) can be calculated similarly, the result is

\[
\frac{\pi}{2 \sinh \pi \omega_1} \text{Im} \left[ \sum_{n=0}^{\infty} \frac{\Gamma(1 + \frac{i}{2} (\omega'_1 + \omega'_2 - \omega_1) + n) \Gamma(1 - \frac{i}{2} (\omega_1 + \omega'_1 + \omega'_2) + n)}{n! \Gamma(1 - i \omega_1 + n) \Gamma(2 - i \omega_1 + 2n)} \times \Gamma(1 - \frac{i}{2} (\omega_1 + \omega'_1 - \omega'_2) + n) \Gamma(1 - \frac{i}{2} (\omega_1 + \omega'_2 - \omega'_1) + n) \right].
\]

(A.3)

Substituting this back into (8) and dropping out the delta function, the off-shell three point amplitude is obtained.
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