Forbidden subgraphs for constant domination number

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In this paper, we characterize the sets $\mathcal{H}$ of connected graphs such that there exists a constant $c = c(\mathcal{H})$ satisfying $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph $G$, where $\gamma(G)$ is the domination number of $G$.

Keywords: Domination number, Forbidden induced subgraph, Ramsey number

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. Let $G$ be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. For a vertex $x \in V(G)$, let $N_G(x)$ and $N_G[x]$ denote the open neighborhood and the closed neighborhood, respectively; thus $N_G(x) = \{ y \in V(G) : xy \in E(G) \}$ and $N_G[x] = N_G(x) \cup \{ x \}$. For a set $X \subseteq V(G)$, let $N_G[X] = \bigcup_{x \in X} N_G(x)$. For a vertex $x \in V(G)$ and a non-negative integer $i$, let $N_G^i(x) = \{ y \in V(G) : \text{the distance between } x \text{ and } y \text{ in } G \text{ is } i \}$. Note that $N_G^0(x) = \{ x \}$ and $N_G^1(x) = N_G(x)$. Let $K_n$ and $P_n$ denote the complete graph and the path of order $n$, respectively. For terms and symbols not defined in this paper, we refer the reader to [3].

Let $G$ be a graph. For two sets $X, Y \subseteq V(G)$, we say that $X$ dominates $Y$ if $Y \subseteq N_G[X]$. A subset of $V(G)$ which dominates $V(G)$ is called a dominating set of $G$. The minimum cardinality of a dominating set of $G$, denoted by $\gamma(G)$, is called the domination number of $G$. Since the determining problem of the value $\gamma(G)$ is NP-complete (see [7]), many researchers have tried to find good bounds for the domination number (see [9]). One of the most famous results is due to Ore [11] who proved that every connected graph $G$ of order at least two satisfies $\gamma(G) \leq |V(G)|/2$. Here one problem naturally arises: What additional conditions allow better upper bounds on the domination number? In this paper, we focus on forbidden induced subgraph conditions.

For a graph $G$ and a set $\mathcal{H}$ of connected graphs, $G$ is said to be $\mathcal{H}$-free if $G$ contains no graph in $\mathcal{H}$ as an induced subgraph. In this context, members of $\mathcal{H}$ are called forbidden subgraphs. If $G$ is $\{H\}$-free, then $G$ is simply said to be $H$-free. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ such that $H_1$ is an induced subgraph of $H_2$. The relation

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“≤” between two sets of forbidden subgraphs was introduced in [6]. Note that if \( \mathcal{H}_1 \leq \mathcal{H}_2 \), then every \( \mathcal{H}_1 \)-free graph is also \( \mathcal{H}_2 \)-free.

Let \( K_{1,3} \) and \( K_{3} \) denote the two unique graphs having degree sequence \( (3, 1, 1, 1) \) and \( (3, 3, 1, 1, 1) \), respectively. Cockayne, Ko and Shepherd [1] (see also Theorem 2.9 in [9]) proved that every connected \( \{ K_{1,3}, K_{3} \} \)-free graph \( G \) satisfies \( \gamma(G) \leq \lceil |V(G)|/3 \rceil \). Indeed, Duffus, Gould and Jacobson [5] proved that every connected \( \{ K_{1,3}, K_{3} \} \)-free graph has a Hamiltonian path. Since \( \gamma(P_n) = \lceil n/3 \rceil \) for every integer \( n \), the above inequality is a consequence of this result. Furthermore, forbidden induced subgraph conditions for domination-like invariants were widely studied (see, for example, [2, 4, 8, 10]).

In this paper, we will characterize the sets \( \mathcal{H} \) of connected graphs satisfying the condition that

(A1) there exists a constant \( c = c(\mathcal{H}) \) such that \( \gamma(G) \leq c \) for every connected \( \mathcal{H} \)-free graph \( G \).

Let \( n \geq 1 \) be an integer. Let \( K_{n}^* \) denote the graph with \( V(K_{n}^*) = \{ x_i : 1 \leq i \leq n \} \cup \{ y_i : 1 \leq i \leq n \} \) and \( E(K_{n}^*) = \{ x_ix_j : 1 \leq i < j \leq n \} \cup \{ x_iy_i : 1 \leq i \leq n \} \), and let \( S_{n}^* \) denote the graph with \( V(S_{n}^*) = \{ x \} \cup \{ y_i : 1 \leq i \leq n \} \cup \{ z_i : 1 \leq i \leq n \} \) and \( E(S_{n}^*) = \{ xy_i : 1 \leq i \leq n \} \cup \{ y_iz_i : 1 \leq i \leq n \} \) (see Figure 1). Our main result is the following.

**Theorem 1.1** Let \( \mathcal{H} \) be a set of connected graphs. Then \( \mathcal{H} \) satisfies (A1) if and only if \( \mathcal{H} \leq \{ K_{k}^*, S_{\ell}^*, P_m \} \) for some positive integers \( k, \ell \) and \( m \).

We conclude this section by considering the case where a set \( \mathcal{H} \) can contain disconnected graphs. Then the following proposition holds.

**Proposition 1.2** Let \( \mathcal{H} \) be a set of graphs. Then \( \mathcal{H} \) satisfies (A1) if and only if \( \mathcal{H} \leq \{ K_{k} \} \) for some positive integer \( k \).

**Proof:** Suppose that \( \mathcal{H} \) satisfies (A1). Then there exists a constant \( c = c(\mathcal{H}) \) such that \( \gamma(G) \leq c \) for every connected \( \mathcal{H} \)-free graph \( G \). Since \( \gamma(K_{c+1}) = c + 1 \), \( K_{c+1} \) is not \( \mathcal{H} \)-free, and so \( \mathcal{H} \leq \{ K_{c+1} \} \).

On the other hand, if \( \mathcal{H} \leq \{ K_{k} \} \), then every \( \mathcal{H} \)-free graph \( G \) satisfies \( \gamma(G) \leq k - 1 \) because every maximal independent set of \( G \) is a dominating set. \( \square \)
2 Proof of Theorem 1.1

For positive integers $s$ and $t$, let $R(s, t)$ denote the Ramsey number with respect to $s$ and $t$. For positive integers $k, \ell$ and $i$, we recursively define $g_{k, \ell}(i)$ as follows:

\[
\begin{align*}
g_{k, \ell}(1) &= 1 \\
g_{k, \ell}(i) &= R(k, (\ell - 1)g_{k, \ell}(i - 1) + 1) - 1 \quad (i \geq 2).
\end{align*}
\]

Lemma 2.1 Let $k, \ell$ and $i$ be positive integers. Let $G$ be a $\{K^*_k, S^*_\ell\}$-free graph, and let $a$ be a vertex of $G$. Then for an independent set $X \subseteq N^*_G(a)$, there exists $U \subseteq N_{G}^{i-1}(a)$ with $|U| \leq g_{k, \ell}(i)$ that dominates $X$.

Proof: We proceed by induction on $i$. If $i = 1$, then $U = \{a\}$ is a desired subset of $N_{G}^{i-1}(a) = \{a\}$. Thus we may assume that $i \geq 2$. Note that $N_{G}^{i-1}(a)$ dominates $X$. Let $U$ be a minimal subset of $N_{G}^{i-1}(a)$ that dominates $X$. It suffices to show that $|U| \leq R(k, (\ell - 1)g_{k, \ell}(i - 1) + 1) - 1 = g_{k, \ell}(i)$.

By way of contradiction, suppose that $|U| \geq R(k, (\ell - 1)g_{k, \ell}(i - 1) + 1)$. For each $u \in U$, since $U - \{u\}$ does not dominate $X$ by the minimality of $U$, there exists a vertex $x_u \in X$ such that $N_G(x_u) \cap U = \{u\}$. Recall that $X$ is an independent set. If there exists a clique $U_1 \subseteq U$ with $|U_1| = k$, then the subgraph of $G$ induced by $U_1 \cup \{x_u : u \in U_1\}$ is isomorphic to $K^*_k$, which contradicts the $K^*_k$-freeness of $G$. Since $|U| \geq R(k, (\ell - 1)g_{k, \ell}(i - 1) + 1)$, this implies that there exists an independent set $U_2 \subseteq U$ with $|U_2| = (\ell - 1)g_{k, \ell}(i - 1) + 1$. By the induction hypothesis, there exists $U' \subseteq N_{G}^{i-2}(a)$ with $|U'| = g_{k, \ell}(i - 1)$ that dominates $U_2$. By the pigeon-hole principle, there exists a vertex $u' \in U'$ such that $|N_G(u') \cap U_2| \geq \ell$. Let $U_2 \subseteq N_G(u') \cap U_2$ be a set with $|U_2| = \ell$. Then the subgraph of $G$ induced by $\{u'\} \cup U_2 \cup \{x_u : u \in U_2\}$ is isomorphic to $S^*_\ell$, which is a contradiction. \hfill \Box

For positive integers $k, \ell$ and $i$ with $i \geq 2$, let $f_{k, \ell}(i) = R(k, \ell)g_{k, \ell}(i)$.

Lemma 2.2 Let $k, \ell$ and $i$ be positive integers with $i \geq 2$. Let $G$ be a $\{K^*_k, S^*_\ell\}$-free graph, and let $a$ be a vertex of $G$. Then there exists $U \subseteq V(G)$ with $|U| \leq f_{k, \ell}(i)$ that dominates $N^*_G(a)$.

Proof: Let $X$ be a maximal independent subset of $N^*_G(a)$. By Lemma 2.1, there exists $U \subseteq N_{G}^{i-1}(a)$ with $|U| \leq g_{k, \ell}(i)$ that dominates $X$. By the maximality of $X$, $X$ dominates $N_{G}^{i-1}(a)$, and so $X$ dominates $N_{G}^{i-1}(a) - N_{G}[U]$. Let $X_0$ be a minimal subset of $X$ that dominates $N_{G}^{i-1}(a) - N_{G}[U]$.

Claim 2.1 We have $|X_0| \leq (R(k, \ell) - 1)g_{k, \ell}(i)$.

Proof: Suppose that $|X_0| \geq (R(k, \ell) - 1)g_{k, \ell}(i) + 1$. Since $U$ dominates $X_0$ and $|U| \leq g_{k, \ell}(i)$, there exists a vertex $u' \in U$ such that $|N_G(u') \cap X_0| \geq R(k, \ell)$. For each $x \in X_0$, since $X_0 - \{x\}$ does not dominate $N_{G}^{i-1}(a) - N_{G}[U]$ by the minimality of $X_0$, there exists a vertex $y_x \in N_{G}^{i-1}(a) - N_{G}[U]$ such that $N_G(y_x) \cap X_0 = \{x\}$. Set $Y = \{y_x : x \in N_G(u') \cap X_0\}$, and for each $y \in Y$, write $N_G(y) \cap X_0 = \{x_y\}$. Note that $y_x : y \in Y \subseteq N_G(u') \cap X_0$ and $y_{x_y} = y$ for each $y \in Y$. Since $|Y| = |N_G(u') \cap X_0| \geq R(k, \ell)$, there exists a clique $Y_1 \subseteq Y$ with $|Y_1| = k$ or an independent set $Y_2 \subseteq Y$ with $|Y_2| = \ell$. Recall that $Y \subseteq N_{G}(a) - N_{G}[U]$, and so $N_{G}(u') \cap Y = \emptyset$, and so $N_{G}(u') \cap Y = \emptyset$. If there exists a clique $Y_1 \subseteq Y$ with $|Y_1| = k$, then the subgraph of $G$ induced by $Y_1 \cup \{y_y : y \in Y_1\}$ is isomorphic to $K^*_k$; if there exists
an independent set $Y_2 \subseteq Y$ with $|Y_2| = \ell$, then the subgraph of $G$ induced by $\{u\} \cup \{x_y : y \in Y_2\} \cup Y_2$ is isomorphic to $S_\ell^c$. In either case, we obtain a contradiction.

Recall that $X_0$ dominates $N^c_G(a) - N_G[U]$. Hence $U \cup X_0$ dominates $N^c_G(a)$. Furthermore, by the definition of $U$ and Claim 2.1,

$$|U \cup X_0| = |U| + |X_0| \leq g_{k,\ell}(i) + (R(k, \ell) - 1)g_{k,\ell}(i) = f_{k,\ell}(i).$$

Thus $\hat{U} = U \cup X_0$ is a desired set.

\textbf{Proof of Theorem 1.1}: We first prove the “only if” part. Let $\mathcal{H}$ be a set of connected graphs satisfying (A1). Then there exists a constant $c = c(\mathcal{H})$ such that $\gamma(G) \leq c$ for every connected $\mathcal{H}$-free graph $G$. Since we can easily verify that $\gamma(K^*_c+1) = \gamma(S^*_c+1) = \gamma(P^*_m) = c+1$, none of $K^*_c+1, S^*_c+1$ and $P^*_m$ is $\mathcal{H}$-free. This implies that $\mathcal{H} \subseteq \{K^*_c+1, S^*_c+1, P^*_m\}$, as desired.

Next we prove the “if” part. Let $\mathcal{H}$ be a set of connected graphs such that $\mathcal{H} \subseteq \{K^*_k, S^*_\ell, P^*_m\}$ for some positive integers $k, \ell$ and $m$. Choose $k, \ell$ and $m$ so that $k + \ell + m$ is as small as possible. Then $k, \ell$ and $m$ are uniquely determined. In particular, the value $1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i)$ only depends on $\mathcal{H}$. Furthermore, every $\mathcal{H}$-free graph is also $\{K^*_k, S^*_\ell, P^*_m\}$-free. Thus it suffices to show that every connected $\{K^*_k, S^*_\ell, P^*_m\}$-free graph $G$ satisfies $\gamma(G) \leq 1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i)$. Let $a \in V(G)$. Since $G$ is $P^*_m$-free, $N_G^0(a) = \emptyset$ for all $i \geq m - 1$. Since $G$ is connected, this implies that $V(G) = \bigcup_{0 \leq i \leq m-2} N_G^1(a)$. Since $G$ is $\{K^*_k, S^*_\ell\}$-free, it follows from Lemma 2.2 that for each $i$ with $2 \leq i \leq m - 2$, there exists a set $\hat{U}_i \subseteq V(G)$ with $|\hat{U}_i| \leq f_{k,\ell}(i)$ that dominates $N_G^1(a)$. Since $\{a\}$ dominates $N_G^0(a) \cup N_G^1(a)$, $\{a\} \cup (\bigcup_{2 \leq i \leq m-2} \hat{U}_i)$ is a dominating set of $G$, and so

$$\gamma(G) \leq |\{a\}| + \sum_{2 \leq i \leq m-2} |\hat{U}_i| \leq 1 + \sum_{2 \leq i \leq m-2} f_{k,\ell}(i),$$

as desired.

This completes the proof of Theorem 1.1. \hfill \Box

3 Concluding remark

In this paper, we characterized the sets $\mathcal{H}$ of connected graphs satisfying (A1). For similar problems concerning many domination-like invariants, we can use the sets appearing in Theorem 1.1.

Let $\mu$ be an invariant of graphs, and assume that

\begin{itemize}
  \item[(D1)] there exist two constants $c_1, c_2 \in \mathbb{R}^+$ such that $c_1 \gamma(G) \leq \mu(G) \leq c_2 \gamma(G)$ for all connected graphs $G$.
\end{itemize}

Note that many important domination-like invariants (for example, total domination number $\gamma_t$, paired domination number $\gamma_{pr}$, Roman domination number $\gamma_R$, rainbow domination number $\gamma_{rk}$, etc.) satisfy (D1). Furthermore, we focus on the condition that

\begin{itemize}
  \item[(A'1)] there exists a constant $c' = c'(\mu, \mathcal{H})$ such that $\mu(G) \leq c$ for every connected $\mathcal{H}$-free graph $G$.
\end{itemize}

We first suppose that a set $\mathcal{H}$ of connected graphs satisfies (A'1). Note that
For $\mu(K^*_1) \geq c_1 \gamma(K^*_1) = c_1 \cdot \lceil (\ell + 1)/c_1 \rceil \geq \ell + 1,$

\begin{itemize}
  \item $\mu(S^*_1) \geq c_1 \gamma(S^*_1) = c_1 \cdot \lceil (\ell + 1)/c_1 \rceil \geq \ell + 1,$ and
  \item $\mu(P^*_1) \geq c_1 \gamma(P^*_1) = c_1 \cdot \lceil (\ell + 1)/c_1 \rceil \geq \ell + 1.$
\end{itemize}

Thus, by similar argument to the proof of “only if” part of Theorem 1.1, we have $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for some positive integers $k, \ell$ and $m$.

On the contrary, suppose that a set $\mathcal{H}$ of connected graphs satisfies $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for some positive integers $k, \ell$ and $m$. Then by Theorem 1.1, (A1) holds, and hence for a connected $\mathcal{H}$-free graph $G$, we have

$$\mu(G) \leq c_2 \gamma(G) \leq c_2 \cdot c(\mathcal{H}).$$

Consequently (A1') holds (for $\ell = c_2 \cdot c(\mathcal{H})$). Therefore, we obtain the following theorem.

**Theorem 3.1** Let $\mu$ be an invariant for graphs satisfying (D1), and let $\mathcal{H}$ be a set of connected graphs. Then $\mathcal{H}$ satisfies (A1') if and only if $\mathcal{H} \leq \{K^*_k, S^*_\ell, P_m\}$ for positive integers $k, \ell$ and $m$.

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