Zero sets of some classes of entire functions

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Abstract. – A method of constructing an entire function with given zeros and estimates of growth is suggested. It gives a possibility to describe zero sets of certain classes of entire functions of one and several variables in terms of growth of volume of these sets in certain polycylinders.

Résumé. – On propose une méthode pour la construction de fonctions entières dont l’ensemble des zéros est donné avec des majorations de croissance. Cette méthode permet de décrire avec précision pour certaines classes de fonctions entières (d’une ou de plusieurs variables) la croissance du volume des diviseurs de leurs zéros dans certaines polycylindres.

Let $K$ be some set of entire functions in $\mathbb{C}^n$, $n \geq 1$. Denote by $D_f$ the divisor of zeros of a function $f(z)$ and by $Z_K$ the set $\{D_f : f \in K\}$. We are interested in descriptions of the sets $Z_K$ for certain classes $K$.

In this note we make a survey of a couple of results which appeared in [Ru1], [Ru2], [RoRu]. They are devoted mainly to descriptions of zeros of entire functions $f(z)$ with the property

$$M_f(h) \overset{\text{def}}{=} \sup\{|f(z)| : |Imz| \leq h\} < \infty, \ \forall h > 0$$

We denote the class of all entire functions satisfying (1) by $B$. This class is important, because it contains, for example, entire characteristic functions of probability distributions in $\mathbb{R}^n$ and Dirichlet series with imaginary frequencies.
Note, that even in one variable the usual canonical products seem to be a non-
adequate tool for constructing functions with given zeros satisfying (1), and one 
needs to develop special technique and use appropriate characteristics of zero sets 
in this case.

Let $\Pi(r, h) \subset \mathbb{C}^n$ be a polycylinder (rectangular box for $n = 1$) \{z : |Rez| \leq r, |Imz| \leq h\}, and let $D$ be a divisor in $\mathbb{C}^n$. Denote by $n_D(r, h)$ the volume of $D$
 in $\Pi(r, h)$, i.e.

$$n_D(r, h) = \int_{D \cap \Pi(r, h)} (dd^c|z|^2)^{n-1}.$$

For the case $n = 1$, the quantity $n_D(r, h)$ is just the number of points of $D$
 (counted with multiplicities) in $\Pi(r, h)$.

A complete description of zeros (divisors) for the whole class $B$ was obtained 
by I.P.Kamynin and I.V.Ostrovskii ([KO]): a divisor $D$ in $\mathbb{C}$ belongs to $Z_B$ if and only if

$$\forall h > 0 : \log n_D(r, h) = o(r), \ r \to \infty \quad (2)$$

Using this result, Kamynin and Ostrovskii give a complete description of zero 
divisors of Hermitian - positive entire functions (Fourier transforms of probability 
distributions in $\mathbb{R}$). We denote this class by $H$ and remind that an entire 
function $f(z), \ z \in \mathbb{C}^n$, is called Hermitian - positive if $f(0) = 1$ and for all 
$x^{(1)} \in \mathbb{R}^n, \ldots, x^{(n)} \in \mathbb{R}^n, \ w_1 \in \mathbb{C}, \ldots, w_n \in \mathbb{C},$

$$\sum_{k,j=1}^{n} f(x^{(k)} - x^{(j)}) w_k \overline{w_j} \geq 0.$$
According to the result in [KO], a divisor $D$ in $\mathbb{C}$ belongs to $Z_H$ if and only if the following holds:

(i) $D \cap \{\text{Re}z = 0\} = \emptyset$;
(ii) $D = -\overline{D}$, where $-\overline{D} = \{z : -\overline{z} \in D\}$;
(iii) $D$ satisfies (2).

Our theorem below extends this result to the case of several variables and hence both results together give an answer to a question posed by Yu.V.Linnik and I.V.Ostrovskii in [LO].

**Theorem 1** ([RoRu]). A divisor $D$ in $\mathbb{C}^n$ belongs to $Z_H$ if and only if conditions (i) – (iii) above hold.

If we put restrictions on the growth of $M_f(h)$, the problem of description of zero sets becomes more complicated. Let $\varphi(t)$ be a nonnegative increasing convex $C^2$-function on $\mathbb{R}_+$. By $B_\varphi$ we denote the class of functions $f(z)$ belonging to $B$ and satisfying the estimate

$$\log M_f(h) \leq C e^{C\varphi(h)}$$

for some $C = C(f)$.

Note, that since $\varphi$ is supposed to be convex, we are dealing with functions of infinite order. We assume additionally, that $\varphi(2t) = O(\varphi(t))$.

Denote by $h(t)$ the solution of the equation

$$h \varphi(h) = t, \quad t \geq 0.$$

It is easy to see, that $h$ is an increasing function of $t$ with the properties
\[
\lim_{t \to \infty} \frac{h(t)}{t} = \lim_{t \to \infty} \frac{\varphi(h(t))}{t} = 0.
\]

A description of divisors in \(Z_{B^\varphi}\) is given by the following

**Theorem 2.** A divisor \(D\) in \(\mathbb{C}^n\) belongs to \(Z_{B^\varphi}\) if and only if

\[
\log n_D(r, h) \leq C(1 + \varphi(h) + \varphi(h(r)))
\]

for some \(C > 0\).

If we compare (3) to (2), we see that \(o(r)\) is replaced in (3) by a more precise expression \(\varphi(h(r))\).

A typical example of \(\varphi\) satisfying our conditions is \(t^\rho,\ \rho \geq 1\). In this case \(\varphi(h(t)) = t^{\rho^\frac{\varphi}{\rho+1}}\). We formulate the corresponding result below.

**Corollary.** A divisor \(D\) in \(\mathbb{C}^n\) is a divisor of an entire function \(f(z)\) with

\[
\limsup_{h \to \infty} \frac{\log \log M(f)(h)}{h^\rho} < \infty \quad (\rho \geq 1)
\]

if and only if for some \(C > 0\)

\[
\log n_D(r, h) \leq C(1 + h^\rho + r^{\varphi})
\]

For \(n = 1\) one can obtain more precise results by using a more precise characteristic of the divisors. Namely, we allow our boxes \(\Pi(r, h)\) to move along the real axis and add one more parameter setting

\[
n_D(x; r, h) = n_{D-x}(r, h),
\]

where \(D - x\) means a translation of \(D\).
For simplicity we formulate one result for a subclass of $B_\nu$. We denote by $[\rho, \sigma]$ the class of entire functions $f(z)$ of one variable satisfying the condition

$$
\limsup_{h \to \infty} \frac{\log \log M_f(h) - \rho}{h^\rho} \leq \sigma,
$$

and by $[\rho, \sigma]^*$ a subclass in $[\rho, \sigma]$ consisting of functions possessing the property

$$
\exists \varepsilon \in (0, \rho), \exists a, b > 0 : \forall x \in \mathbb{R},
$$

$$
\sup_{|x-t| \leq a} \log |f(t)| \geq -\exp\left(b|x|^{\frac{\rho - \varepsilon}{1+\rho - \varepsilon}}\right).
$$

**Theorem 3.** Let $D$ be a divisor in $\mathbb{C}$ and let $\rho > 1, \sigma \geq 0$ be given. Then $D$ belongs to $Z[\rho, \sigma]^*$ if and only if there exist $\varepsilon \in (0, \rho), C > 0$ and a function $\sigma(t) \to \sigma$ $(t \to \infty)$, such that

$$
\log n_D(x; r, h) \leq C + r + \sigma(h)h^\rho + C|x|^{\frac{\rho - \varepsilon}{1+\rho - \varepsilon}}.
$$

**Remark.** The condition $\varphi(2t) = O(\varphi(t))$ is not necessary and is used only to simplify the formulations. Results for more general scales of growth may be found in [Ru1].

The main tool for the above results is provided by a theorem on construction of entire function with the given divisor of zeros and estimates of its growth in terms of $n_D(r, h)$ (or $n_D(x; r, h)$). Similar results for exhaustion of $\mathbb{C}^n$ by balls or polydiscs are well-known (see, for example, [Le], [Lev], [LG], [Ro], [Stl]). A remarkable result of H.Skoda (see [Sk]) may be used in the proofs of some (not
all) of the theorems above instead of our construction. In any case, it seems that
the method described below might be of interest itself. The general idea of the
method was expressed by L.I.Ronkin. Actually, one goes along the lines of the
classical solution of Cousin’s second problem, but with estimates. We remind
briefly Cousin’s scheme.

Assume for simplicity that we have a divisor \( D \) in a domain \( \Omega \in \mathfrak{C} \). Cover
\( \Omega \) by disks or squares \( G_j \). Take a polynomial \( P_j(z) = \prod_{w_k \in D \cap G_j} (z - w_k) \). On
\( G_{ij} \overset{\text{def}}{=} G_i \cap G_j \) the ratio \( \frac{P_i}{P_j} \) is a nonvanishing holomorphic function, hence having
the form \( \exp(g_{ij}) \). Suppose, we are able to find such holomorphic functions \( g_k \), that
\( g_{ij} = g_i - g_j \) on \( G_{ij} \) for all \( i, j \). Then we can put \( f(z) = P_j \exp(-g_j) \) on \( G_j \), thus
obtaining a function with given zeros in \( \Omega \). However, to be able to represent \( g_{ij} \) in
the form of the difference, we need to have a cocycle condition

\[
g_{ij} + g_{jk} + g_{ki} = 0.
\]

All we have is that

\[
\exp(g_{ij} + g_{jk} + g_{ki}) = 1,
\]
or

\[
g_{ij} + g_{jk} + g_{ki} = 2\pi i \cdot N
\]

with \( N = N_{ijk} \) integer.

The next step is to represent \( N_{ijk} \) in the form

\[
N_{ijk} = M_{ij} + M_{jk} + M_{ki}, \quad M_{\alpha \beta} - \text{integers}, \quad (4)
\]

\[6\]
which is possible exactly when the second group of cohomologies with integer coefficients in $\Omega$ is trivial, this being the necessary and sufficient condition for the solvability of Cousin’s second problem. This condition holds for every domain $\Omega$ in $\mathbb{C}$, which is not true for an arbitrary domain in $\mathbb{C}^n$; however it obviously holds for $\Omega = \mathbb{C}^n$.

Once we have passed this step, we put $h_{ij} = g_{ij} - 2\pi i \cdot M_{ij}$. For these functions the cocycle condition takes place and, representing them in the form $g_i - g_j$, we obtain our solution.

The idea is to follow the described scheme with bounds on each step. We can have good bounds for polynomials $P_k$ in terms of the number of points of the divisor in $G_k$. Thus we get estimates for $g_{ij}$ and $N_{ijk}$. The crucial point is to be able to solve the cohomological equation (4) with ”good” estimates of the solution. Once this is reached, it remains to give estimates for $g_i$, which is done in a standard way with the help of Hörmander’s $\bar{\partial}$– methods ([Hö]).

It appears to be possible to give solution with ”good” bounds for (4). Moreover, the solution may be chosen in such a way, that the bounds include the number of zeros in elements of the covering, situated along one direction (this is where the quantities $n_D(x; r, h)$ appear). This is verified by evaluating the number of ”free parameters” and assigning them certain appropriate values recurrently. The precise statement for one-dimensional situation may be formulated as follows.

**Theorem 4.** Let $D$ be a divisor in $\mathbb{C}$ and let

$$\log n_D(x; 1, |y|) \leq u(x + iy), \quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R},$$

where $u(z)$ is a subharmonic function.

Then there exists such an entire function $f(z)$ with $D_f = D$, that
\[
\log \log |f(z)| \leq C + 2 \log(1 + |z|^2) + \sup_{|w-z|\leq 1} u(w).
\]

In higher dimensions it is also possible to solve (4) with bounds. However, the number of ”free parameters” allows us to eliminate only one real direction, and we could obtain our estimates in terms of, say, \( n_D(x_1; r, h) \), which is enough for the above results.

Some details in several variables must be modified. We are not able to take polynomials any longer. However, appropriate ”local solutions” are M.Anderson’s solutions of \( \partial \overline{\partial} \)– problem in the unit ball of \( \mathbb{C}^n \) (see [An]). We only need to choose a center for the ball far enough from the divisor. This can be done so that the distance to the divisor is estimated from below in terms of its volume, and we obtain the ”right” estimates. The corresponding statement which we don’t formulate precisely looks similar to theorem 4.

We conclude with a remark, that in one dimension the outlined scheme may be used for constructing functions with given zeros in domains of \( \mathbb{C} \) with control of growth near the boundary of the domain.

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