QCD AND STRINGS IN 2D\textsuperscript{1}

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Abstract

In two dimensions large N QCD with quarks, defined on the plane, is equivalent to a modified string theory with quarks at the ends and taken in the zero fold sector. The equivalence that was established in 1975 was expressed in the form of an interacting string action that reproduces the spectrum and the 1/N interactions of 2D QCD. This action may be a starting point for an analytic continuation to a four dimensional string version of QCD. After reviewing the old work I discuss relations to recent developments in the pure QCD-string equivalence on more complicated background geometries.

1. Introduction

During the past year, starting with the work of Gross \textsuperscript{1}, there has been new progress in establishing a definite relationship between large-N QCD and strings in 2-dimensions \textsuperscript{2, 3, 4, 5, 6}. Pure QCD (without quarks) is trivial in 2 dimensions if it is defined on the plane. However, when defined on more complicated geometries, such as the cylinder, torus, pretzel, etc. there remains non-contractable Wilson lines (color glue strings) that survive as degrees of freedom. They have shown, to all orders in the 1/N expansion, that the path integral for large N QCD can be reorganized as a sum over surfaces analogous to the Polyakov path integral over

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all Riemann surfaces in string theory (all string loops). The parameter $1/N$ plays the role of the string coupling constant. This shows that there must be a string action from which this string path integral may be derived. However, so far it has not been possible to identify this QCD-string action.

On the other hand, back in 1975 we established other definite quantitative relationships between large-N QCD and strings [7, 8, 9, 10]. In that case quarks were included, but QCD was defined on the plane. Therefore, the new results are complementary to the old ones. Furthermore, in contrast to the recent progress, the old correspondence established a many-body formulation of an interacting string action [9, 10] that reproduced all the features of QCD on the plane, including the large-N propagators, and the $1/N$ interaction vertices. In explicit computations to first order in $1/N$ this string action gave the same results as QCD for the spectrum, the 3-meson form factors and the electroweak form factors. For agreement with QCD the string had to be limited to one topological sector, namely to the no-fold sector. This sector was self contained and consistent within the string theory. The interacting Hamiltonian that was derived from the string action implemented the restriction to the no fold sector consistently [10]. The recent developments have also emphasized the no fold sector.

The string theory version allowed additional topological sectors in which the string folded on itself. The folded string was analysed both classically and quantum mechanically [7]. The semi-classical spectrum of the folded string which was derived in [7] was later reproduced by the semi-classical spectrum of the Liouville mode discovered by Polyakov [11]. Thus, there is a close correspondence between the degrees of freedom provided by the folds on the one hand and the degrees of freedom of the Liouville field on the other hand. One may therefore associate the folds with the dynamics that relate to 2D gravity. The fold degrees of freedom were obviously absent in 2D QCD on the plane. On the other hand, from another point of view the folds could also be interpreted as points carrying color charge (i.e. dynamical degree of freedom, such as gluons or other adjoint matter), while the strings in between could be interpreted as the glue on a Wilson line [9]. These were also obviously absent in 2D QCD on the plane. But in fact,
In this interpretation of folds is realized in a hybrid lattice formulation of 4D QCD later developed by Bardeen and Pierson [14]. In trying to make a connection to the more recent developments that involve more complicated geometries or more complicated versions of QCD, there seems to be room for the folds that fit either kind of interpretation.

In this lecture I will first briefly review the old QCD-string correspondence on the plane, which includes quarks. In particular I will emphasize the action which may be analytically continued to four dimensions. Other lecturers in this conference will review the very interesting recent progress for pure QCD on more complicated fixed (non-dynamical) geometries, but without quarks or an action. In the second part of my lecture I will discuss the folds, and in the last part of my lecture I will speculate on possible relations between the new results and the old ones in a way that considers the folds.

2. Interacting action of strings and quarks and equivalence to 2D QCD on the plane

In 1975 'tHooft [15] derived an integral equation that determines the spectrum of large-N QCD. Soon afterwards Bardeen, Bars, Hanson and Peccei [7], working in 2D string theory, showed that the Nambu string modified with dynamical points added to the ends, and taken in the zero fold sector has the same spectrum as large-N QCD. The zero fold sector is a self consistent sector of the string+points theory. This sector has a massless “pion” (not a tachyon) in the zero quark mass limit [13, 15]. Bars and Hanson [8] modified the action at the end points by adding quark degrees of freedom that carry spin and color/flavor charges and showed how the string+quark system can interact with electroweak gauge bosons. Furthermore, Bars [9, 10] formulated an interacting string action in a many body formalism, by adding a term to the action which fused the end point quarks on different strings, or created a pair of quarks by splitting a string. The fusing/splitting occurs when the end points on different strings touch each other, and involves the spins of the quarks in a non-trivial manner. Making all terms gauge invariant with respect to the electroweak interactions gave a further modification of the total action. This final form of the action is

\[ \text{folds has also been useful in the discussion of other aspects of hadrons in four dimensions [3].} \]

\[ ^{4}\text{It is important to emphasize that the 'tHooft equation for bosonic quarks (i.e. scalars in the fundamental representation) is different than the one for fermions, and it does not match any known string theory.} \]
\[ S_{\text{total}} = \sum_{\text{mesons}} S_{\text{meson}} + \sum_{I,J} S_{\text{fuse/split}}(I, J) + \sum_{I,J} S_{\text{electroweak}}(I, J) \quad (1) \]

The action for a single meson is

\[ S_{\text{meson}} = \int d\tau L_0(x_1(\tau)) + \int d\tau L_0(x_2(\tau)) + \gamma_M \int d\tau \int_0^\pi d\sigma \sqrt{-g} \quad (2) \]

where the string \( x^\mu(\tau, \sigma) \) has end points \( x_1^\mu(\tau) = x_1^\mu(\tau, \sigma = 0) \) and \( x_2^\mu(\tau) = x_2^\mu(\tau, \sigma = \pi) \). The last term is the Nambu action, while the first two terms are the world line actions for the quarks at the end points

\[ L_0(x_I(\tau)) = \overline{\psi}_I(\tau) i \frac{x_I^\mu(\tau)}{2\sqrt{-x_I^2(\tau)}} - \sqrt{-x_I^2(\tau)} \frac{\overline{\psi}_I(\tau) m_I(\tau)}{\psi_I(\tau)} \quad (3) \]

where \( x_I^\mu(\tau) = \partial x_I^\mu(\tau)/\partial \tau \) is the velocity vector of the end point labelled by \( I \), and \( \psi_{Ia}^I(\tau) \) is the Dirac wavefunction for the quark attached to the same end point, with its spin “\( \alpha \)”, color “\( i \)" and flavor “\( a \)” labels. Furthermore, \( m_{ab} = m_a \delta_{ab} \) is a mass matrix in flavor space. The ’t Hooft spectrum for mesons, in the leading term of the \( 1/N \) expansion of QCD, is identically reproduced by the string meson action \( S_{\text{meson}} \) in the zero fold sector (which is a self consistent sector even after interactions are included).

The index \( I \) runs over all end points on all the strings that represent the different mesons in the many body formalism. The interactions for joining and/or splitting of any two end points is

\[ \sum_{I,J} S_{\text{join/split}}(I, J) = -\left(\frac{f}{4}\right) \sum_{I \neq J} \int d\tau d\tau' \delta(x_I^\mu(\tau) - x_J^\mu(\tau')) \sqrt{-x_{I1}^2(\tau)} \sqrt{-x_{J1}^2(\tau')} \times \{ \overline{\psi}_I(\tau) \left( i \gamma_\tau \frac{x_{\mu I}}{2 x_{I1}^2} + m \right) \psi_J(\tau') + \overline{\psi}_J(\tau') \left( i \gamma_\tau \frac{x_{\mu J}}{2 x_{J1}^2} + m \right) \psi_I(\tau) \} \quad (4) \]

where the delta function insures that the points are at the same location, while the spins of the quarks \( (I, J) \) are also fused. The factors \( \sqrt{-x_{I1}^2(\tau)} \) insure \( \tau \) and \( \tau' \) reparametrization invariance. The operator sandwiched between the \( (I, J) \) quark wavefunctions is the Dirac operator in which the derivative is taken along the

\[^5\text{The color index is used only as a counting device in the 2D string theory. It is assumed that there are only color singlet states. Factors of } 1/N \text{ eventually appear because of the color counting.}\]
tangent to the worldline (or velocity of the end point). The equation of motion that follows from 3 also involves the same operator. Therefore, if the string coupling for mesons \( \gamma_M \) in 4 vanishes, the quarks can perform only free motion and do not interact. Thus, after accounting for the quark equations of motion, it is evident that the interaction above can take place only in the presence of strings. Indeed, the interaction Hamiltonian derived from 4 in [10] turns out to be proportional to \( \gamma_M \). Furthermore, the overall coefficient in 4 is fixed to \( f = \pi/2 \) by crossing symmetry. This interaction turns out to reproduce exactly the \( 1/N \) corrections in 2D QCD. This was shown by comparing the detailed QCD results [17] and string results [10] for the 3-meson vertex, which is a form factor describing the decay of one meson to two mesons.

Electroweak interactions of the quarks can be introduced by replacing ordinary derivatives by gauge covariant derivatives in the actions above, as follows

\[
\frac{x^\mu_I(\tau)}{x^2_I} i\partial_\tau \rightarrow \frac{x^\mu_I(\tau)}{x^2_I} i\partial_\tau - e A^\mu(x_I(\tau))
\]

where the gauge field \( A \) couples to the flavor indices. This produces the electroweak interactions

\[
\sum_{I,J} S_{\text{electroweak}} = \sum_I \int d\tau \sqrt{-x^2_I} \overline{\psi}_I(\tau) \gamma \cdot A(x_I(\tau)) \psi_I(\tau)
\]

\[
+ (f/4) \sum_{I \neq J} \int d\tau \int d\tau' \delta(x^\mu_I(\tau) - x^\mu_J(\tau')) \sqrt{-x^2_I} \sqrt{-x^2_J} \psi_I(\tau) \psi_J(\tau')
\]

\[
\times e \left\{ \overline{\psi}_I(\tau) A(x_I(\tau)) \psi_J(\tau') + \overline{\psi}_J(\tau') A(x_J(\tau)) \psi_I(\tau) \right\}
\]

where the first term comes from covariantizing 3 and the second term comes from covariantizing 4. A smooth parametrization in \( \tau \) or \( \tau' \) require that each worldline be timelike. Then, the first term in this expression describes the interaction of the electroweak field with a timelike quark line, while the second term corresponds to the creation or annihilation of a pair, where each member of the pair has a timelike propagation. Therefore these two terms must be crossing symmetric to each other. After computing some diagrams in momentum space, it was found that crossing symmetry requires \( f = \pi/2 \); so gauge invariance plus crossing symmetry fixes the overall coefficient of the interaction term in 4. Computations in both QCD [17, 18] and string theory [3, 10] have shown that the electroweak form factors of mesons agree in detail in both theories.

Thus the string action [4], which is a modification of the Nambu action, taken in the zero fold sector, agrees exactly with QCD on the plane, including the \( 1/N \)
corrections. The zero fold condition is implemented in the derivation of the Hamiltonian \[10\]. The spectrum is given by \[2\] and the $1/N$ interaction vertices come from \[4,6\]. Furthermore, the perturbative expansion of the total Hamiltonian has diagrams that are in one to one correspondence with the quark loops in QCD. Since the propagator (i.e. spectrum), the vertex, and the set of diagrams are the same in both theories, one expects agreement to all orders of $1/N$ for all quark loops, without introducing gluon loops; however, this latter point has not been checked in detail since the computations are not available in either QCD or string theory.

Therefore, according to the old results, we have an action formulation for the correspondence between string theory with quarks at the ends on the plane, and QCD with quarks on the plane. Note that the string action is written in any number of dimensions in flat spacetime, although it was worked out in detail in only two dimensions. The question naturally arises whether this string action may be valid for the correspondence between strings and QCD in higher dimensions? Of course, the whole point of this kind of exercise in two dimensions is based on the hope that a QCD-string formulation that is valid in two dimensions might be “analytically continued” to four dimensions and be used as a good approximation to understand hadronic physics. We know that the form of string theory given above is incomplete for all applications in four dimensions since closed strings that represent glueballs and their interactions are not yet included. However, one can still attempt a comparison of 4D QCD to the above string action, by restricting oneself to only planar graphs of QCD which do not include glue balls. With this in mind, I believe it is a good exercise for someone to attempt a comparison in four dimensions of the string theory defined above to planar QCD (but with $1/N$ corrections due to quark loops, which make holes on the sheet, included in the comparison).

If such a limited comparison in 4D is successful to begin with, then the following additional suggested modifications of the string action may reproduce the gluonic aspects of QCD: (1) Closed strings representing glueballs should be included along with the meson action \[2\]. (2) The interaction \[4\] allows open strings to make transitions to closed string, (3) An interaction that allows the string to interact at interior points should be included. Such a many body type action formulation, to be used at low orders of $1/N$, could be used for phenomenological applications to low energy hadronic physics since it would account for confinement and Regge behaviour, etc.. A more ambitious step would be to find a field theoretic string action instead of a many body type string action, but that formalism
would be harder to work with for computations of standard processes.

3. Folds in 2D string theory and longitudinal modes

In [7] folded 2D strings were discussed using several approaches. It was shown that the classical solutions for folded strings emerged in different gauges and the gauge invariant motions were displayed graphically. The quantum theory of folded open massless strings was solved semi-classically in a formalism of action-angle variables (closed strings could be done just easily, see below). The normal modes were identified and shown to be equivalent to those of an additional longitudinal string degree of freedom, and the Lorentz invariance of the system was proven in the second paper in [7].

The semi-classical mass operator was expressed as

$$\alpha'(M^2 - m_0^2) = \sum_{n=1}^{\infty} \alpha_n \alpha_n$$

with $m_0^2 = \text{const.}$. Notice that the $n = 0$ term is excluded [7]. It has the spectrum $\alpha'(M^2 - m_0^2) = \sum_{l=0}^{\infty} n l_n$ with $l_n = 0, 1, 2, \cdots$. The classical solutions showed that each normal mode corresponds to an open string that has $n - 1$ folds, wraps around by folding upon itself in equal lengths, and oscillates. Later, with Polyakov’s discovery of the Liouville mode, one could associate semi-classically the Liouville degrees of freedom with those of folded strings.

The classical solutions for folded strings are most easily given in the conformal gauge (for the same physics in other gauges see [7]). The classical equations of motions and constraints for a free string in 2D are

$$\partial_+ \partial_- x^\mu = 0, \quad (\partial_\pm x^\mu)^2 = 0 = (\partial_- x^\mu)^2. \quad (7)$$

The solutions are $x^\mu(\tau, \sigma) = x^\mu_L(\sigma^+) + x^\mu_R(\sigma^-)$, where the left and right movers depend on $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$ respectively, and they are constrained by $\partial x^0_L = \pm \partial x^1_L$, $\partial x^0_R = \pm \partial x^1_R$. The $\pm$ can generally switch signs discontinuously in various regions of $\sigma^\pm$, but the $x^\mu$’s have to be continuous to obtain a continuous string. For a physical description, the time coordinate $x^0(\tau, \sigma) = x^0_L(\sigma^+) + x^0_R(\sigma^-)$ must be monotonically increasing with $\tau$, while both $x^\mu(\tau, \sigma)$ must be periodic in $\sigma$ (this periodicity is obvious for closed strings, but is also required for open strings to satisfy boundary conditions). Using the remaining gauge invariance of conformal transformations one can fix the gauge as $x^0_L = \frac{\alpha' E}{2\pi}(\tau + \sigma)/2$, $x^0_R = \frac{\alpha' E}{2\pi}(\tau - \sigma)/2$, so that $x^0 = \frac{\alpha' E}{\pi} \tau$ satisfies the periodicity and monotonicity conditions. The other string coordinate may be written as

$$x^1(\tau, \sigma) = q + \frac{\alpha' E}{2\pi}(f(\tau + \sigma) + g(\tau - \sigma)), \quad f'(\tau + \sigma) = \pm 1, \quad g'(\tau - \sigma) = \pm 1. \quad (8)$$
Thus, the functions $f, g$ see-saw with slopes $\pm 1$ in the range $[-\pi, \pi]$ and are periodic with a period of $2\pi$. For the closed string the two functions are independent, but for the open string $f = g$. The simplest example of such a function is $f = |\tau + \sigma|_{\text{per}}$, i.e. the absolute value taken in the range $[-\pi, \pi]$ and then periodically repeated (see [7, 8] for pictures). For the open string one gets the solution

$$x^1(\tau, \sigma) = q + \frac{\alpha' E}{2\pi}(|\tau + \sigma|_{\text{per}} + |\tau - \sigma|_{\text{per}})$$

This function describes an open string stretched out without any folds, the two ends oscillate against each other while the intermediate points move in such a way as to allow the whole string to twist around when the two end points pass each other (remember that $\sigma = [0, \pi]$ for open strings). The parameter $E$ is the energy of the system. For a closed string with $\sigma = [-\pi, \pi]$, taking for illustration $f = g$, one obtains a closed string looped around once, each point moving such that the loop twists while the two end points (folds) oscillate against each other. At the end of one period $\tau \rightarrow \tau + 2\pi$ the string comes back to its original shape. There are an infinite number of such periodic see-saw functions which may be distinguished by the locations of the see-saw points in the range $[-\pi, \pi]$. The general folded string may fold and unfold during different time intervals in one period $\tau = [0, 2\pi]$. However, the normal modes correspond to strings that remain completely folded in equal lengths and oscillate in that form [7]. Although we have referred to such strings as “folded”, it is hard to distinguish between strings that fold or strings that wrap in the present formalism, since the string lies in one space dimension. The physical content of these two cases is quite different from the point of view of oriented color flux, and this point will be addressed below.

To quantize the theory one may identify physical degrees of freedom by fixing the gauge completely. This was done in [7] in a gauge that differs from the conformal gauge above. Starting with the generally covariant formalism, and using the $\tau, \sigma$ reparametrization invariance, one may choose arbitrarily two functions. Thus, one may choose (i) $x^0(\tau, \sigma) = \frac{\alpha' E}{2\pi} \tau$ and (ii) $\partial_\tau x^1(\tau, \sigma) = \text{independent of} \sigma$ in between $Z$ folds at $\sigma = \sigma_i$, $i = 1, 2, \cdots, Z$ (i.e. $x^1(\tau, \sigma)$ is linear in $\sigma$ in between the folds). This leaves only the end points ($x^1(\tau, 0) \equiv x_0(\tau)$, $x^1(\tau, \pi) \equiv x_{Z+1}(\tau)$) and the folds at $x^1(\tau, \sigma_i) \equiv x_i(\tau)$ and their momenta $p_i$ as canonical degrees of freedom (note that at a fold or end point $\partial_\sigma x^\mu = 0$ and therefore these points must move with the speed of light to satisfy the constraint $(\partial_\tau x^\mu)^2 = (\partial_\sigma x^\mu)^2$). It
was shown that, in the $Z$-fold sector, the open string Hamiltonian is given by

$$H = \sum_{i=0}^{Z+1} |p_i| + \frac{1}{\alpha} \sum_{i=0}^{Z} |x_i - x_{i+1}| .$$  \hspace{1cm} (10)

The classical solutions of this Hamiltonian coincide with those displayed above in the conformal gauge [7]. For a closed string with $2Z$ folds, one gets the same form, with a small modification that replaces the lower limit by $i = 1$, the upper limit on both sums by $2Z$ instead of $Z + 1$ and $Z$ respectively, and also requires $x_{2Z+1} = x_1$ (redefining $x^i(\tau, 0) \equiv x_1(\tau)$).

One may also discuss the theory in the lightcone gauge, $x^+(\tau, \sigma) = \frac{\alpha' p^+_i}{\pi} \tau$, provided one slows down the motion of the folds or end-points by making them massive, and then taking the mass to zero at the end of the calculation (this is necessary because of the singular lightcone description of massless particles that move with the speed of light). The canonical lightcone degrees of freedom of the folds or end-points are $(x^-_i, p^+_i)$. The lightcone gauge open string Hamiltonian takes the form

$$P^- = \sum_{i=1}^{2Z} \frac{m_i^2}{2p_i^2} + \frac{1}{\alpha'} \sum_{i=1}^{2Z} |x^-_i - x^-_{i+1}| .$$  \hspace{1cm} (11)

Again, the classical motions coincide with those of the conformal gauge in the limit of zero masses $m_i \to 0$ [7]. The advantage of the lightcone formalism is its obvious Lorentz covariance, but it can be argued that in a certain sense the timelike gauge is Lorentz covariant as well [7]. The closed string Hamiltonian is obtained with the same modifications mentioned in the previous paragraph is

$$P^- = \sum_{i=1}^{2Z} \frac{m_i^2}{2p_i^2} + \frac{1}{\alpha'} \sum_{i=1}^{2Z} |x^-_i - x^-_{i+1}| .$$  \hspace{1cm} (12)

with $x^-_{2Z+1} = x^-_1$.

The semi-classical quantization was carried out by transforming to action-angle variables $(\vartheta_i, J_i)$, and showing that the mass operator takes the form $\alpha' M^2 = \sum J_i$. It was shown that the $i$'th mode describes a string that is wrapped around $i$ times, and that it covers a basic phase space $i$ times, thus its Bohr-Sommerfeld quantization was given as $J_i = i(l_i + \text{const})$ where $l_i = 0, 1, 2, \cdots$. This is the same spectrum that would be obtained from string oscillators, by identifying $J_i = \alpha_{-i} \alpha_i$. For a closed string the same arguments lead to the semi-classical mass spectrum

$$\alpha'(M^2 - m_0^2) = \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \tilde{\alpha}_n .$$  \hspace{1cm} (13)
The exact quantum spectrum of $\text{10}$ or $\text{11}$ is not known for any value of $Z$. But it is known that, for the zero fold sector $Z = 0$, $\text{11}$ coincides with the 'tHooft equation, and that in the limit of $m_0^2 = 0$ there is an exact zero mass eigenstate ($M^2 = 0$) that may be interpreted as a pion constructed from massless quarks in the chiral symmetry limit. This interpretation is more appropriate for the string theory version of the previous section (which gives the same equation) since it includes fermionic degrees of freedom, thus introducing the concept of chiral symmetry in the context of string theory. Also, numerical studies of the spectrum of the 'tHooft equation with massive quarks shows that its spectrum becomes linear after the first few excited states, thus coming into agreement with the semi-classical massless string spectrum described by oscillators that give linearly rising Regge trajectories.

One may also carry out a covariant quantization of the D=2 open or closed massless string by applying the Virasoro constraints on the Fock space constructed from two covariant oscillators $\alpha_{\mu}, \mu = 0, 1$. This was done a long time ago [20], and here we give the main results. By keeping $c = d = 2$ (rather than $c = 26$) and taking an intercept $\alpha_0 = \frac{d-2}{24} = 0$ (that is $L_0 = 0$), it is seen that the only states that satisfy the Virasoro constraints, $L_n|\psi> = 0$, have positive or zero norm. The positive norm states are in one-to-one correspondence with the states described by [11], and have the same spectrum and same degeneracy as the one longitudinal oscillator of [13], provided $m_0^2 = 0$. The vacuum state $|p^\mu>$ is a massless “tachyon” that may be interpreted as the “pion” of the zero-mass 'tHooft equation. Thus, in an interacting theory (i.e. the $1/N$ corrections) this “tachyon” or “pion” must decouple (suggestive of Adler zeroes for pions) since massless particles cannot exist in two dimensions due to their infrared behaviour. The absence of the “tachyon” in the interacting theory has also been deduced from a very different point of view in the recent work of Gross [11, 3].

4. Correspondence to recent developments

In the previous sections I described strings propagating in flat spacetime and exhibited an action that reproduces 2D QCD interacting with quarks in flat spacetime. How does this relate to the recent studies of QCD that are conducted without quarks and in curved spacetime with non-trivial topology? There are at least two important observations that have been rediscovered from a different

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6The spectrum is dramatically different if $c=26$. The positive norm states become then zero norm states.
point of view in the recent developments: (i) The string-QCD correspondence seems to require strings without folds, (ii) The interacting string, including the $1/N$ correction, has no massless “tachyon” or “pion” (i.e. even if it is there in the free spectrum, it must decouple in the interacting theory, or the theory must undergo a phase transition).

Is there a role that the folds may play to make some further connections? Note that the mathematical formalism of “folds” that we have outlined may describe different physical situations. A fold may occur by (1) the string folding back on itself in the opposite orientation, thus making a worldsheet that folds back on itself, or by (2) wrapping around, which makes a worldsheet with non-trivial topology. For a folded string lying in one dimension at a constant time one cannot tell the difference between the two cases. Therefore, the folded string formalism that we have discussed may apply to either possibility, however they will have different physical interpretations from the point of view of color flux tubes. The first possibility would correspond to having a color charge at the point of the fold, so that at constant time the color flux goes in and out along the same line in opposite orientations (one color in, and another color or the same color out). The second possibility would correspond to the same flux line (same color) continuing to wrap around until it finds a color charge or its own starting point. The second possibility can produce a big color flux passing through the same point, while the first possibility cannot do that. The QCD-string correspondence tested so far excludes the first kind of fold, while allowing the second kind to occur only with non-trivial geometrical backgrounds. This is understandable and expected since the theories considered do not have any color charges in the adjoint representation of the color group (no dynamical gluons).

Thus, to test for either kind of folds in the QCD-string correspondence we can consider enlarging our present 2D theories in two possible directions: (1) Consider 2D QCD coupled to adjoint matter in either flat or non-trivial geometries. One example is to consider a sector of 4D QCD in the hybrid formulation given in [14]; another example is 2D QCD coupled to adjoint fermions, with or without quarks in the fundamental representation. This will introduce color charges that can absorb and emit color flux, thus allowing a fold to occur. (2) Consider strings propagating in non-trivial topologies (like the recent work on QCD) and allow the string to wrap around the topology thus producing “folds” at the edges of the geometry. For example a cylinder which is squashed to lie on the table, that is representing the worldsheet of a 2D string, has the string wrapped around it at any fixed time, and the string is forced to fold at the edges of the squashed cylinder.
The cylinder may be multiple sheeted, which would lead to the interpretation that the string is wrapped around many times, thus giving strings with many folds. Let me say a bit more about tentative results for either case.

For the enlarged model of QCD coupled to adjoint matter let us consider bosonic versus fermionic matter. The string-like ’tHooft equation is easily derived in a Hamiltonian approach in the lightcone gauge $A_+ = 0$. Such equations were derived in [14] for the case of bosons, and in [21] for the case of fermions. For all cases, the Hamiltonian contains kinetic terms for the matter degrees of freedom and an interaction term that consists of a linear potential between the total color densities. In the large $N$ limit this Hamiltonian produces coupled ’tHooft-like equations for multi-particle states. Thus the two particle state mixes with the three particle states, etc. The coupling occurs because one can draw leading planar graphs that include the propagation of the matter fields in addition to the glue $\Omega$. One may split these equations into two parts: the first part may be considered “zeroth order” in the sense that it involves only the wavefunctions with a fixed number of matter particles, and the second part is the coupling. The zeroth order may be associated with strings binding the particles, and making worldsheets during their propagation. Then the coupling may be viewed as string-string interactions that are not suppressed by factors of $1/N$. The zeroth order term may be compared to a string theory before interactions. For bosonic adjoint matter the details of the ’tHooft-like equation do not match any known string theory. However, for fermions the ’tHooft-like equation for $Z$ fermions is identically reproduced by the folded string Hamiltonian given in [22]. This is a new observation that relates strings and large $N$ gauge theory. However, one must emphasize that the wavefunctions for the $Z$ fermions must be completely antisymmetric, so that only the antisymmetric solutions of these equations can be admitted. This additional input that comes from the QCD formulation is missing in the string formulation of [22]. So, there is a bit more to understand.

One may also study strings in curved spacetime in 2D and compare to QCD. One possibility is to consider higher Riemann surfaces (e.g. cylinder) that are squashed to lie on the plane since we are in 2D. A Wilson line that is wrapped around such a cylinder must fold at the end of it. If this cylinder changes its radius as a function of time, then the folds appear to oscillate just like the fold solutions

For quarks in the fundamental representation alone there are only 2-body bound states, and thus no complicated coupling to many body bound states in the large $N$ limit.

However, for reasons that are not well understood this coupling seems to be numerically small [21]. I thank M. Douglas for making me aware of this paper.
of the string theory given in (8,9). A cylinder that changes as a function of time is possible in QCD provided it is coupled to dynamical Gravity in 2D. Therefore, one possibility for including the folds in the string-QCD correspondance is to consider the larger theory of QCD plus gravity in 2D and compare it to a string theory on the same curved manifold. So far 2D QCD has been analyzed on static 2D manifolds. As another example, it should be fun to couple QCD with or without quarks to the 2D SL(2,R)/R black hole, and study the relations to the corresponding string theory. I point out that the 2D black hole string theory has classical solutions that describe folded strings moving in the vicinity of the black hole.

It is evident that there is much to do even in two dimensions, but I would urge those interested in this subject to begin to seriously explore the possible correspondance between 4D QCD and the string action given in [1], as outlined at the end of section 2.

References

[1] D. Gross, Nucl. Phys. B400 (1993) 161.
[2] J. Minahan, Phys. Rev. D47 (1993) 3430.
[3] D. Gross and W. Taylor, Nucl. Phys. B400 (1993) 181; B403 (1993) 395.
[4] J. Minahan and Polychronachos, hepth 9303153, 9309044.
[5] M. Douglas, hepth 9303159.
[6] M. Douglas and V.A.Kazakov, hepth 9305047.
[7] W.A.Bardeen, I.Bars, A.Hanson and R.Peccei, Phys. Rev. D13 (1976) 2364.
   For the Poincare invariance of the quantum theory see also Phys. Rev. D14 (1976) 2193.
[8] I. Bars and A. Hanson, Phys. Rev. D13 (1976) 1744.
[9] I. Bars, Phys. Rev. Lett. 36 (1976) 1521.
[10] I. Bars, Nucl. Phys. B111 (1976) 413.
[11] A. Polyakov, Phys. Lett. 103B (1981) 207.
[12] B. Andersson et. al., Zeit. Phys. C1 (1979) 105; Zeit. Phys. C3 (1980) 223; J. Math. Phys. 26 (1985) 112; Nucl. Phys. B355 (1991) 82; J. Phys. G17 (1991) 1507.

[13] R. Pisarski and J.D. Stack, Nucl. Phys. B286 (1987) 657.

[14] W. A. Bardeen and R. Pierson, Phys. Rev. D14 (1976) 547. W.A. Bardeen, R. Pierson and E. Rabinovici, Phys. Rev. D21 (1980) 1037.

[15] G. ’tHooft, Nucl. Phys. B72 (1974) 461; B75 (1974) 461.

[16] M. Durgut and N. Pak, Phys. Lett. 117B (1982) 453.

[17] C. Callan, N. Coote and D. Gross, Phys. Rev. D13 (1976) 3451.

[18] M. Einhorn, Phys. Rev. D13 (1976) 3451.

[19] C. Thorn, Nucl. Phys. B248 (1974) 551.

[20] I. Bars, unpublished.

[21] S. Dalley and I. Klebanov, Phys. Rev. D47 (1993) 2517. D. Kutasov, hep-th 9306013.