Non-commutative stochastic processes with independent increments

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This article is on the research of Wilhelm von Waldenfels in the mathematical field of quantum (or non-commutative) probability theory. Wilhelm von Waldenfels certainly was one of the pioneers of this field. His idea was to work with moments and to replace polynomials in commuting variables by free algebras which play the role of algebras of polynomials in non-commuting quantities. Before he contributed to quantum probability he already worked with free algebras and free Lie algebras. One can imagine that this helped to create his own special algebraic method which proved to be so very fruitful. He came from physics. His PhD thesis, supervised by Heinz König, was in probability theory, in the more modern and more algebraic branch of probability theory on groups. Maybe the three, physics, abstract algebra and probability, must have been the best prerequisites to become a pioneer, even one of the founders, of quantum probability.

We concentrate on a small part of the scientific work of Wilhelm von Waldenfels. The aspects of physics are practically not mentioned at all. There is nothing on his results in classical probability on groups (Waldenfels operators). This is an attempt to show how the concepts of non-commutative notions of independence and of Lévy processes on structures like Hopf algebras developed from the ideas of Wilhelm von Waldenfels.

1 Algebraic central limit theorem

If nothing else is said, algebras are understood to be associative and over the field of complex numbers.

Gaussian (or normal) distributions arise as central limits. A gaussian distribution is given by its covariance matrix $Q = (Q_{ij})_{i,j \in [d]}$ with $d \in \mathbb{N}$, $Q_{ij} \in \mathbb{R}$, and where $[d]$ denotes the set $\{1, \ldots, d\}$. Here $d$ is the degree of freedom of the distribution $\gamma_Q$, and $Q_{ij} = \int_{\mathbb{R}^d} x_i x_j \, d\gamma_Q$. The (real) $d \times d$-matrix $Q$ is positive semi-definite. In particular, $Q$ is symmetric and we have $Q_{ij} = Q_{ji}$. By definition, a $d$-dimensional
real random variable $X : \Omega \to \mathbb{R}^d$ over a probability space $(\Omega, \mathcal{F}, P)$ is $\gamma_Q$-distributed if $P(X \in B) = \gamma_Q(B)$ for all Borel subsets $B$ of $\mathbb{R}^d$ or equivalently, if the expectation values $E(f \circ X)$ equal $\int_{\mathbb{R}^d} f d\gamma_Q =: \gamma_Q(f)$ for all bounded continuous functions $f$ on $\mathbb{R}^d$. The **moments** $\gamma_Q(x_1^{l_1} \cdots x_d^{l_d})$, $l_1, \ldots, l_d \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of the gaussian $\gamma_Q$ are given by putting $x_1^{l_1} \cdots x_d^{l_d} = x_1, \ldots, x_n$, $n = l_1 + \ldots + l_d$, and

$$\gamma_Q(x_{i_1} \cdots x_{i_n}) = \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ \sum Q_{S_1} \cdots Q_{S_k} & \text{if } n = 2k \text{ is even.} \end{array} \right.$$  \hspace{1cm} \text{(1)}$$

where the sum is taken over all pair partitions $\{S_1, \ldots, S_k\}$ of $[2k]$ (that is all partitions of $[2k]$ into sets with exactly two elements) and where $Q_S = Q_{i_k, i_l}$ for $S = \{k, l\}$, $k < l$. In the case of the normal distribution all moments exist and they determine the distribution. We can think of $\gamma_Q$ as a linear functional on the (real, unital) algebra $\mathbb{R}[x_1, \ldots, x_d]$ of polynomials in the **commuting** indeterminates $x_1, \ldots, x_d$. Then $\gamma_Q(p) = \int_{\mathbb{R}^d} p \, d\gamma_Q$ for $p \in \mathbb{R}[x_1, \ldots, x_d]$.

The fundamental idea of the paper [GvW78] of N. Giri and W. von Waldenfels is to replace $\mathbb{R}[x_1, \ldots, x_d]$ by the algebra $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ of polynomials with complex coefficients in **non-commuting** indeterminates $x_1, \ldots, x_d$. Then for a (complex) $d \times d$-matrix $Q$ the linear functional $\gamma_Q$ on $\mathbb{C}\langle x_1, \ldots, x_d \rangle$, formally again defined by equation (1), is called **gaussian** with covariance matrix $Q$. Notice that now $Q$ is **hermitian** and not automatically symmetric. In particular, we now have $\gamma_Q(x_i x_j) = Q_{ij} = \overline{Q_{ji}} = \gamma_Q(x_j x_i)$ which means that it can happen that $\gamma_Q$ does not vanish on the ideal in $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ generated by the element $x_i x_j - x_j x_i$ and that $\gamma_Q$ cannot be identified with a gaussian on either $\mathbb{C}[x_1, \ldots, x_d]$ or $\mathbb{R}[x_1, \ldots, x_d]$. For example, if $d = 2$ and $Q = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, then $Q$ is positive semi-definite. Here $\gamma_Q(x_1 x_2) = \frac{1}{2} i \neq -\frac{1}{2} i = \gamma_Q(x_2 x_1)$, and $\gamma_Q(x_1 x_2 - x_2 x_1) = i$. It can be shown that $\gamma_Q$ vanishes on the ideal in $\mathbb{C}\langle x_1, \ldots, x_d \rangle$ generated by the elements $x_i x_j - x_j x_i$ (see Theorem 2 of [GvW78]; cf. Theorem [1] below). Thus $\gamma_Q$ is a linear functional on the non-commutative *-algebra generated by two self-adjoint elements $q = x_1$ and $p = x_2$ with the canonical commutation relation between the canonical pair $(p, q)$ of momentum and position.

We state the classical Central Limit Theorem (CLT) in its elementary form. Let $X_1, X_2, \ldots$ be a sequence of, in the classical sense, independent identically distributed $d$-dimensional random variables such that the second moments exist. Moreover, let $X_n$ be centralized and put $Q_{ij} = E(X_i X_j)$, $i, j \in [d]$. Then $Q$ is real positive semi-definite. The CLT says that

$$\frac{X_1 + \ldots + X_n}{\sqrt{n}} \xrightarrow{n \to \infty} \gamma_Q \quad \text{in law}$$

which means that

$$\int_{\mathbb{R}^d} f \left( \frac{X_1 + \ldots + X_n}{\sqrt{n}} \right) \, dP \xrightarrow{n \to \infty} \int_{\mathbb{R}^d} f \, d\gamma_Q$$  \hspace{1cm} \text{(2)}$$

for all bounded continuous functions on $\mathbb{R}^d$. Non-constant polynomials are not bounded, so a polynomial cannot serve as an $f$ in (2). The left hand side of (2)
even makes no sense for a polynomial $f$, unless all moments of $X_n$ exist. If all moments exist, an algebraic version of the CLT says that (2) holds for all polynomials $f \in \mathbb{C}[x_1, \ldots, x_d]$. We may, for the moment, forget about positivity. Given a normalized, centralized (i.e. $\varphi(1) = 1$, $\varphi(x_i) = 0$) linear functional $\varphi$ on $\mathbb{C}[x_1, \ldots, x_d]$ the algebraic CLT

$$\varphi^n(p(\frac{x_1}{\sqrt{n}}, \ldots, \frac{x_d}{\sqrt{n}})) \xrightarrow{n \to \infty} \gamma_Q(p)$$

holds for all $p \in \mathbb{C}[x_1, \ldots, x_d]$. Here $p(\frac{x_1}{\sqrt{n}}, \ldots, \frac{x_d}{\sqrt{n}})$ denotes the polynomial obtained from $p$ by replacement of $x_1, \ldots, x_d$ by $\frac{x_1}{\sqrt{n}}, \ldots, \frac{x_d}{\sqrt{n}}$. The statement (3) matches (2) for $f \in \mathbb{C}[x_1, \ldots, x_d]$, because the distribution of the sum of $n$ independent identically distributed random variables is the $n$-fold convolution of the distribution of $X_1$. Of course, we must define the convolution of (arbitrary) linear functionals on polynomial algebras which we will do now.

Define the algebra homomorphism

$$\Delta : \mathbb{C}[x_1, \ldots, x_d] \to \mathbb{C}[x_1, \ldots, x_d] \otimes \mathbb{C}[x_1, \ldots, x_d]$$

by $\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$ on the generators. The convolution product of two linear functionals $\varphi$ and $\psi$ is then defined by $\varphi \ast \psi = (\varphi \otimes \psi) \circ \Delta$.

In exactly the same manner the comultiplication

$$\Delta : \mathbb{C}(x_1, \ldots, x_d) \to \mathbb{C}(x_1, \ldots, x_d) \otimes \mathbb{C}(x_1, \ldots, x_d)$$

can be defined in the non-commutative case. Again, for linear functionals $\varphi$ and $\psi$, this time on $\mathbb{C}(x_1, \ldots, x_d)$, the convolution is given by $\varphi \ast \psi = (\varphi \otimes \psi) \circ \Delta$. The algebraic CLT of Wilhelm von Waldenfels reads as follows.

Theorem 1.1 (N. Giri, W. von Waldenfels 1978 [GvW78])

(a) Let $\varphi$ be a normalized, centralized linear functional on $\mathbb{C}(x_1, \ldots, x_d)$. Then (3) holds for all $p \in \mathbb{C}(x_1, \ldots, x_d)$.

(b) The linear functional $\gamma_Q$ vanishes on the ideal in $\mathbb{C}(x_1, \ldots, x_d)$ generated by the elements

$$x_ix_j - x_jx_i - (Q_{ij} - Q_{ji}), \quad i, j \in [d].$$

Proof: In ([GvW78]) a proof based on direct combinatorial calculations is given. A proof using a convolution logarithm will be sketched in the next section.

In fact, we will see that $\gamma_Q$ is the convolution exponential $\exp_\ast g_Q$ of the cumulant functional $g_Q$ of $\gamma_Q$ which is the linear functional on $\mathbb{C}(x_1, \ldots, x_d)$ with

$$g_Q(x_{i_1} \ldots x_{i_n}) = \begin{cases} 0 & \text{if } n \neq 2 \\ Q_{i_1,i_2} & \text{if } n = 2 \end{cases}$$

2 Schoenberg correspondence, tensor case

Let us begin with a few remarks on tensor products of vector spaces. The tensor product of two vector spaces $V$ and $W$ in terms of bases is the vector space with basis
$B \times C$ if $B$ and $C$ are bases of $V$ and $W$. This definition has the disadvantage that one has to use bases. Alternatively and basis-free, the tensor product is the vector space $V \otimes W$ with basis $V \times W$ divided by the sub-vector space spanned by elements of the form $(\alpha v_1 + v_2, \beta w_1 + w_2) - \alpha \beta(v_1, w_1) - (v_2, w_2) - \alpha(v_1, w_2) - \beta(v_2, w_1)$, $\alpha, \beta \in \mathbb{C}, v_1, v_2 \in V, w_1, w_2 \in W$. For $v \in V, w \in W$ one writes $v \otimes w$ for the equivalence class of $(v, w)$. Then every element of $V \otimes W$ can be written in the form

$$\sum_{i=1}^{n} v_i \otimes w_i \quad (7)$$

with suitable $n \in \mathbb{N}, v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$. The representation (7) of an element $V \otimes W$ is not unique. A representation (7) of an element in $V \otimes W$ is called minimal if $n$ is the minimum of all representations (7) of this element.

**Lemma 2.1** (Paul-André Meyer)

A representation (7) of a non-zero element of $V \otimes W$ is minimal if and only if $v_1, \ldots, v_n$ are linearly independent in $V$ and $w_1, \ldots, w_n$ are linearly independent in $W$.

**Proof:** Each element $F \neq 0$ of $V \otimes W$ has a representation (7). If $n = 1$ we must have $v_1 \neq 0$ and $w_1 \neq 0$ because $F = v_1 \otimes w_1 \neq 0$. Suppose that $F \neq 0$ has a minimal representation of form (7) with $n \geq 2$. If $v_1, \ldots, v_n$ are not linearly independent, one of the $v_1, \ldots, v_n$ must be a linear combination of the others. By inserting this linear combination, a short computation shows that we obtain a representation (7) of our element $F$ with $n - 1$ summands which is a contradiction to the minimality of $n$. Similarly, $w_1, \ldots, w_n$ have to be linearly independent. \( \Box \)

**Lemma 2.2** Let $(\mathcal{V}_i)_{i \in I}$ and $(\mathcal{W}_i)_{i \in I}$ be two families of linear subspaces, $\mathcal{V}_i$ of the vector space $V$ and $\mathcal{W}_i$ of the vector space $W$, indexed by the same index set $I$. Then

$$\bigcap_{i \in I} \mathcal{V}_i \otimes \mathcal{W}_i = (\bigcap_{i \in I} \mathcal{V}_i) \otimes (\bigcap_{i \in I} \mathcal{W}_i) \quad (8)$$

**Proof:** The inclusion ‘$\supset$’ is clear. – For ‘$\subset$’ we observe that $F \in V \otimes W$ is in $\mathcal{V}_i \otimes \mathcal{W}_i$ if and only if $(\varphi \otimes \text{id})(F) \in \mathcal{W}_i$ for all $\varphi \in \mathcal{V}'$ and $(\text{id} \otimes \psi)(F) \in \mathcal{V}_i$ for all $\psi \in \mathcal{W}'$. Now for $F \in \bigcap_{i \in I} \mathcal{V}_i \otimes \mathcal{W}_i$ we have $(\varphi \otimes \text{id})(F) \in \mathcal{W}_i$ and $(\text{id} \otimes \psi)(F) \in \mathcal{V}_i$ for all $i \in I$ which means $(\varphi \otimes \text{id})(F) \in \bigcap_{i \in I} \mathcal{W}_i$ and $(\text{id} \otimes \psi)(F) \in \bigcap_{i \in I} \mathcal{V}_i$. \( \Box \)

**Lemma 2.3** Let $\mathcal{U}, V$ and $W$ be vector spaces and let $x \in \mathcal{U} \otimes V \otimes W$.

Then there are $n, m \in \mathbb{N}$, linearly independent vectors $u_1, \ldots, u_n \in \mathcal{U}$, vectors $v_{i,j} \in V, i = 1, \ldots, n, j = 1, \ldots, m$, and linearly independent vectors $w_1, \ldots, w_m \in W$ such that

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i \otimes v_{ij} \otimes w_j \quad (9)$$
Proof: It is easy to see that each element of $U \otimes V \otimes W$ can be written in the form (9) with $u_i \in U$, $v_{ij} \in V$ and $w_j \in W$. We must show that the $u_i$ and $w_j$ can be chosen in such a way that they are linearly independent. Assume that in (9) $u_1$ is a linear combination of $u_2, \ldots, u_n$ so that $u_1 = \sum_{i=2}^{n} \alpha_i u_i$ for some complex numbers $\alpha_2, \ldots, \alpha_n$. Then

$$x = \sum_{i=2}^{n} \sum_{j=1}^{m} u_i \otimes (\alpha_i v_{1j} + v_{ij}) \otimes w_j.$$  

By eliminating the linearly dependent vectors $u_i$, we arrive at a representation (9) of $x$ with linearly independent $u_1, \ldots, u_n$. In a similar manner the linearly dependent $w_j$ can be eliminated by changing the vectors $v_{ij}$. □

A coalgebra is a triplet $(C, \delta, \delta)$ consisting of a vector space $C$, a linear mapping $\Delta : C \to C \otimes C$ and a linear functional $\delta$ on $C$ such that the condition $(\Delta \otimes \id) \circ \Delta = (\id \otimes \Delta) \circ \Delta$ of co-associativity and the counit property $(\delta \otimes \id) \circ \Delta = \id = (\id \otimes \delta) \circ \Delta$ are fulfilled. (We use the natural identification $C \otimes C = C = C \otimes C$.) In other words, a coalgebra is a co-monoid in the monoidal category of vector spaces with the above described tensor product of vector spaces. Sometimes we use the Sweedler notation $\Delta c = \sum c(1) \otimes c(2)$.

**Corollary 2.1** If $C_i$, $i \in I$, is a family of sub-coalgebras of the coalgebra $C$ then $\bigcap_{i \in I} C_i$ is again a sub-coalgebra of $C$.

**Proof:** By Lemma 2.2 $\Delta C_i \subset C_i \otimes C_i$ implies that $\Delta C_i \subset (\bigcap_{i \in I} C_i) \otimes (\bigcap_{i \in I} C_i)$. □

For a subset $\mathcal{M}$ of a coalgebra $C$ we define the sub-coalgebra $\langle \mathcal{M} \rangle$ generated by $\mathcal{M}$ to be the intersection of all sub-coalgebras of $C$ which contain $\mathcal{M}$.

**Theorem 2.1** Let $\mathcal{M}$ be a subset of a coalgebra $(C, \Delta, \delta)$. Then

$$\# \mathcal{M} < \infty \implies \dim \langle \mathcal{M} \rangle < \infty$$  \hspace{1cm} (10)

**Proof:** Since it is clear that the sum of two sub-coalgebras is again a sub-coalgebra, it suffices to proof that $\langle \{c\} \rangle$ is finite-dimensional for each element $c \in C$.

By Lemma 2.3 we have

$$(\Delta \otimes \id) \Delta c = \sum_{i=1}^{n} \sum_{j=1}^{m} u_i \otimes v_{ij} \otimes w_j$$  \hspace{1cm} (11)

with $u_1, \ldots, u_n \in C$ linearly independent and $w_1, \ldots, w_m \in C$ linearly independent. Denote by $\mathcal{D}$ the linear span of the vectors $v_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ in $C$. Then $\mathcal{D}$ is a finite-dimensional linear subspace of $C$. We will show that $\mathcal{D}$ is a sub-coalgebra of $C$ containing $c$.

In fact, we have

$$c = (\delta \otimes \id \otimes \delta)(\Delta \otimes \id) \Delta = \sum_{i=1}^{n} \sum_{j=1}^{m} \delta(u_i) \delta(w_j) v_{ij} \in \mathcal{D}.$$  \hspace{1cm} 5
Moreover,

\[(\Delta \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})\Delta c = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} (\Delta u_i) \otimes v_{ij} \right) \otimes w_j \quad (12)\]

\[= (\text{id} \otimes \Delta \otimes \text{id})(\Delta \otimes \text{id})\Delta c \]

\[= \sum_{j=1}^{m} \left( \sum_{i=1}^{n} u_i \otimes (\Delta v_{ij}) \right) \otimes w_j\]

Since \(w_1, \ldots, w_m\) are linearly independent, an application of \(\text{id} \otimes \text{id} \otimes \text{id} \otimes \varphi\) with \(\varphi \in \mathcal{C}'\), \(\varphi(w_j) = \delta_{jl}\) to (12) gives, for all \(l = 1, \ldots, m\),

\[\sum_{i=1}^{n} (\Delta u_i) \otimes v_{il} = \sum_{i=1}^{n} u_i \otimes (\Delta v_{il})\]

which is an element of \(\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{D}\). Using \(\varphi \in \mathcal{C}'\), \(\varphi(u_i) = \delta_{ik}\), we have

\[(\varphi \otimes \text{id} \otimes \text{id}) \left( \sum_{i=1}^{n} (\Delta u_i) \otimes v_{il} \right) = \Delta v_{kl} = (\varphi \otimes \text{id} \otimes \text{id}) \left( \sum_{i=1}^{n} u_i \otimes (\Delta v_{il}) \right) \in \mathcal{C} \otimes \mathcal{D}\]

for all \(k = 1, \ldots, n, l = 1, \ldots, m\). The same argument, but now applying \(\text{id} \otimes \text{id} \otimes \Delta\) and \(\text{id} \otimes \Delta \otimes \text{id}\) to (11), shows that \(\Delta v_{ij} \in \mathcal{D} \otimes \mathcal{C}\). Using again Lemma 2.2, we finally obtain

\[\Delta v_{ij} \in (\mathcal{C} \otimes \mathcal{D}) \cap (\mathcal{D} \otimes \mathcal{C}) = (\mathcal{C} \cap \mathcal{D}) \otimes (\mathcal{D} \cap \mathcal{C}) = \mathcal{D} \otimes \mathcal{D}. \square\]

Denote by \(F(\mathcal{C})\) the system of finite dimensional sub-coalgebras of the coalgebra \(\mathcal{C}\). The system \(F(\mathcal{C})\) is ordered by inclusion. If \(\iota_{\mathcal{D}, \mathcal{E}}, \mathcal{D}, \mathcal{E} \in F(\mathcal{C}), \mathcal{D} \subset \mathcal{E}\) denote the inclusion maps, then \(\left(\iota_{\mathcal{D}, \mathcal{E}}\right)_{\mathcal{D}, \mathcal{E} \in F(\mathcal{C})}\) is an inductive system. Since for \(c \in \mathcal{C}\) we have \(c \in \{\{c\}\} \in F(\mathcal{C})\), the coalgebra is the inductive limit of the system \(\iota_{\mathcal{D}, \mathcal{E}}\). This fact can often be used to define a linear mapping on \(\mathcal{C}\) by defining it on finite-dimensional sub-coalgebras and by using compatibility with the \(\iota_{\mathcal{D}, \mathcal{E}}\).

For instance, if \(A : \mathcal{C} \to \mathcal{C}\) is linear and such that it leaves invariant all sub-coalgebras of \(\mathcal{C}\), we first define \(\exp A\) on finite-dimensional sub-coalgebras of \(\mathcal{C}\), in the usual manner as an exponential on a finite-dimensional space. Next we observe that the restriction of the exponential \(\exp A\) on \(\mathcal{E} \supset \mathcal{D}\) agrees with the exponential \(\exp A\) on \(\mathcal{D}\). We define \(\exp A\) on \(\mathcal{C}\) as the inductive limit. In particular, for \(\psi \in \mathcal{C}'\), the mapping \(T_\psi = (\text{id} \otimes \psi) \circ \Delta\) leaves invariant all sub-coalgebras of \(\mathcal{C}\). We define \(\exp T_\psi\) on the whole of \(\mathcal{C}\) as an inductive limit by defining it on \(\mathcal{D} \in F(\mathcal{C})\) to be the exponential of the restriction of \(T_\psi\) to \(\mathcal{D}\). Then

\[(\delta \circ \exp T_\psi)(c) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{\psi^k(c)}{k!} (13)\]
showing that the series \( \sum_{k=0}^{\infty} \frac{\varphi^k(c)}{k!} \) converges for all \( c \in \mathcal{C} \). We denote the limit functional by \( \exp_x \psi \) and call it the *convolution exponential* of \( \psi \). Again arguing with the inductive limit, we also have

\[
(\exp_x \psi)(c) = \lim_{n \to \infty} \left( \delta + \frac{\psi}{n} \right)^n(c).
\]  

(14)

for all \( c \in \mathcal{C} \).

**Remark 2.1** We can now sketch a proof of the algebraic CLT 1.1 of von Waldenfels. The polynomial algebra \( \mathbb{C}(x_1, \ldots, x_d) \) is a coalgebra with the comultiplication \( \Delta \) of \( \mathbb{C} \) and the counit \( \delta \) defined as the algebra homomorphism \( \delta : \mathbb{C}(x_1, \ldots, x_d) \to \mathbb{C} \) with \( \delta x_i = 0 \), i.e. \( \delta(p) = p(0) \). It is a nice calculation to see that \( \gamma_Q = \exp_x g_Q \). Next apply the ‘convolution logarithmic series’ to the left of (3). This works again by an application of the inductive limit argument. Some estimation, just like for complex numbers (see [Chu68, p. 184]), shows that the log, of the left of (3) converges to \( g_Q \) pointwise. From this it follows that \( \varphi^n(p(\frac{x_1}{\sqrt{n}}, \ldots, \frac{x_d}{\sqrt{n}})) \) converges to \( (\exp, g_Q)(p) = \gamma_Q(p) \).

Since \( \gamma_Q \) is a probability measure it is positive in the sense that \( \gamma_Q(p^*p) \geq 0 \) for all polynomials \( p \). The involution \( p \mapsto p^* \) is given if we think of the indeterminates \( x_i \) as *self-adjoint* elements that is \( (x_i)^* = x_i \). However, in the non-commutative case we did not prove that \( \gamma_Q \) is positive. It is easy to check that \( g_Q \) is *conditionally positive*, which means \( g_Q(p^*p) \geq 0 \) for all \( p \) with \( \delta(p) = p(0) = 0 \). *Schoenberg correspondence* tells us that \( \exp_x \psi \) is positive if \( \psi \) is conditionally positive and hermitian with \( \psi(1) = 0 \). We are going to prove the Schoenberg correspondence for sesquilinear forms on coalgebras which is also fundamental for the proof of the much more general Schoenberg correspondence of [Ger21]. It was Wilhelm von Waldenfels who first proved Schoenberg correspondence for sesquilinear forms in the special case of anti-cocommutative coalgebras; see [VW84b].

The tensor product of two coalgebras \( (\mathcal{C}_i, \Delta_i, \delta_i), i = 1, 2 \), is the coalgebra \( (\mathcal{C}_1 \otimes \mathcal{C}_2, (\id \otimes \tau \otimes \id) \circ (\Delta_1 \otimes \Delta_2), \delta_1 \otimes \delta_2) \) where \( \tau : \mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{C}_2 \otimes \mathcal{C}_1 \) denotes the flip operator. The complex conjugate coalgebra \( \overline{\mathcal{C}} = (\overline{\mathcal{C}}, \overline{\delta}) \) of a coalgebra \( (\mathcal{C}, \Delta, \delta) \) is the complex conjugate vector space of the vector space \( \mathcal{C} \), i.e. \( \overline{\mathcal{C}} = \{ \overline{\tau} \mid c \in \mathcal{C} \} \) is a copy of \( \mathcal{C} \) with \( \overline{\tau + \overline{\tau}} := \overline{\tau + \overline{\tau}}, \lambda \overline{\tau} = \overline{\lambda \tau} \). The comultiplication is given by \( \overline{\Delta(\tau)} = \overline{\tau(1)} \otimes \overline{\tau(2)} \) and the counit by \( \overline{\delta(\tau)} = \delta(c) \). A sesquilinear form \( K \) on \( \mathcal{C} \) can be identified with a linear functional on \( \overline{\mathcal{C}} \otimes \mathcal{C} \) via \( K(\overline{\tau} \otimes d) = K(c, d) \).

We define the convolution product \( K\ast L \) of two sesquilinear forms on a coalgebra to be the convolution of \( K \) and \( L \) with respect to the coalgebra structure of \( \overline{\mathcal{C}} \otimes \mathcal{C} \). A sesquilinear form \( K \) is called positive semi-definite (or simply positive) if \( K(c, c) \geq 0 \) for all \( c \in \mathcal{C} \), and it is called hermitian if \( K(c, d) = \overline{K(d, c)} \) for all \( c, d \in \mathcal{C} \). A sesquilinear form \( K \) on a coalgebra \( \mathcal{C} \) is called *conditionally positive* if \( K(c, c) \geq 0 \) for all \( c \in \ker \delta \) (that is for all \( c \in \mathcal{C} \) with \( \delta(c) = 0 \)). A graded version of the following is a generalization of the result [VW84b] of von Waldenfels to arbitrary graded coalgebras. We state and prove the result only in the non-graded setting.
because in our applications to *-bialgebras a ‘symmetrization procedure’ always allows a reduction to the non-graded case.

**Theorem 2.2** (Schoenberg correspondence for sesquilinear forms)

*For a sesquilinear form* $L$ on a coalgebra $C$ the following are equivalent:

(i) $L$ is conditionally positive and hermitian.

(ii) $\exp_*(tL)$ is positive for all $t \in \mathbb{R}_+$.

**Proof:** We show that $\exp_*(L)$ is positive if $L$ is conditionally positive and hermitian. In view of Theorem 2.1 we may assume that $C$ is finite-dimensional. Let $L$ be conditionally positive, hermitian. We fix a scalar product $\langle \cdot, \cdot \rangle : C \otimes C \to \mathbb{C}$ on $C$. Put $L_\epsilon := L + \epsilon \langle \cdot, \cdot \rangle$. We show that $\exp_*(L_\epsilon)$ is positive for all $\epsilon > 0$. Then we finished, because

$$\left(\exp_*(L)\right)(c, c) = \lim_{\epsilon \downarrow 0} \left(\exp_*(L_\epsilon)\right)(c, c) \geq 0$$

for all $c \in C$.

By compactness of the unit ball $B = \{ c \in C \mid \| c \| \leq 1 \}$ and because $L$ is conditionally positive, hermitian we have: For all $\eta > 0$ there is a $\gamma > 0$ such that

$$c \in B, \quad |\delta(c)|^2 \leq \gamma \implies L(c, c) > -\eta. \quad (15)$$

Fix $\epsilon > 0$. We claim that there is an $n_0 \in \mathbb{N}$ such that $\left(\exp_*(L_\epsilon)\right)(c, c) \geq 0$ for all $c \in B$. Proof of this claim: let $\gamma > 0$ such that (15) holds for $\eta = \epsilon$. For $c \in B$

$$\left(\exp_*(L_\epsilon)\right)(c, c) = |\delta(c)|^2 + \frac{L(c, c) + \epsilon}{n^2} + \frac{1}{n^2}R_n(c) \quad (16)$$

where $|R_n(c)| \leq C$ for a suitable constant $C$. For $c \in B$ with $|\delta(c)|^2 \leq \gamma$ we have $L(c, c) + \epsilon > 0$. A short calculation shows that the convolution product of two hermitian sesquilinear forms is again hermitian. Thus $R_n(c)$ is real, and (16) will be $\geq 0$ for $n$ large.

Using the fact that the Schur product of two positive semi-definite matrices is again positive semi-definite, one shows that the convolution of two positive sesquilinear forms is again positive. We have proved that $\exp_*(L_\epsilon)$ is positive. Therefore, $\exp_*(L_\epsilon) = (\exp_*(L_\epsilon))^{*n_0}$ is positive. □

We have seen that the polynomial algebras are examples of coalgebras. They are also equipped with an involution by assuming the indeterminates to be self-adjoint elements. Then $(x_{i_1} \ldots x_{i_n})^* = x_{i_n} \ldots x_{i_1}$. Thus $\mathbb{C}(x_1, \ldots, x_d)$ (and also the commutative polynomial algebra) becomes a *-algebra. Since $\Delta$ and $\delta$ are *-algebra homomorphisms (where the tensor product carries the usual involution), the polynomial algebras are Hopf *-algebras with the antipode given by extension of $x_i \mapsto -x_i$. The general definitions read as follows: A coalgebra $(\mathcal{B}, \Delta, \delta)$ which is also an (associative, unital) algebra such that $\Delta$ and $\delta$ are algebra homomorphisms is called a bialgebra. An antipode $S$ on a bialgebra is a linear mapping $S : \mathcal{B} \to \mathcal{B}$ which satisfies the equations $M \circ (\text{id} \otimes S) \circ \Delta = \delta 1 = M \circ (S \otimes \text{id}) \circ \Delta$. Here $M : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$
is the multiplication map of $B$. A bialgebra which has an antipode is called a Hopf algebra. A bialgebra which is also a $\ast$-algebra such that $\Delta$ and $\delta$ are $\ast$-mappings is called a $\ast$-bialgebra. A $\ast$-bialgebra with antipode is called a Hopf $\ast$-algebra. A linear functional $\varphi$ on a $\ast$-algebra $A$ is called positive if $\varphi(a^*a) \geq 0$ for all $a \in A$. If $A$ has a unit, a positive normalized linear functional is called a state. A linear functional $\psi$ on a $\ast$-bialgebra $B$ is called conditionally positive if $\psi(b^*b) \geq 0$ for all $b \in \text{ker} \delta$.

A main application of Theorem 2.2 is

**Corollary 2.2** For a linear functional $\psi$ on a $\ast$-bialgebra $B$ the following are equivalent:

(i) $\psi$ is conditionally positive and hermitian with $\psi(1) = 0$.

(ii) $\exp_s(t\psi)$ are states for all $t \in \mathbb{R}_+$.

**Proof:** Apply Theorem 2.2 to $L(b, c) := \psi(b^*c)$. □

In this article we are mainly interested in two classes of examples. The first are the polynomial Hopf $\ast$-algebras which we will rewrite a bit: Let $V$ be a vector space which carries an anti-linear mapping $v \mapsto v^*$ such that $(v^*)^* = v$. We form the tensor algebra

$$T(V) := \bigoplus_{n=0}^{\infty} V^\otimes n$$

with the multiplication given by $(v_1 \otimes \ldots \otimes v_m)(w_1 \otimes \ldots \otimes w_m) := v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots \otimes w_m$. (If $V$ is $d$-dimensional and $x_1, \ldots, x_d$ forms a basis of self-adjoint elements of $V$, we may identify $T(V)$ with $\mathbb{C}[x_1, \ldots, x_d]$.) By extension of $\ast$ as an involution from $V \subset T(V)$ to the algebra $T(V)$ in the only possible way we define an involution on $T(V)$. Moreover, $T(V)$ becomes a Hopf $\ast$-algebra by putting $\Delta v = v \otimes 1 + 1 \otimes v$, $\delta v = 0$ and $Sv = -v$.

The second class of examples is inspired by the coefficient Hopf algebra $K[d]$ of the group $U_d$ of unitary $d \times d$-matrices. $K[d]$ is the sub-algebra of the algebra of continuous complex-valued functions on $U_d$ which is formed by the coefficients of the continuous finite-dimensional representations of $U_d$. By a result of Hermann Weyl $K[d]$ is $\ast$-isomorphic to the algebra $\mathbb{C}[x_{ij}, x_{ij}^*; i, j \in [d]]/J$ where $J$ denotes the ideal generated by the elements

$$\sum_{n=1}^{d} x_{in}x_{jn}^* - \delta_{ij}1, \quad \sum_{n=1}^{d} x_{ni}^*x_{nj} - \delta_{ij}1, \quad i, j \in [d]. \quad (17)$$

The $\ast$-isomorphism is given by $x_{ij} \mapsto u_{ij}$ with $u_{ij} : U_d \to \mathbb{C}$, $u_{ij}(U) = U_{ij}$. By extension of $\Delta_{ij} = \sum_{n=1}^{d} x_{in} \otimes x_{nj}$, $\delta(x_{ij}) = \delta_{ij}$ and $Sx_{ij} = x_{ji}$ the $\ast$-algebra $K[d]$ becomes a Hopf $\ast$-algebra.

Wilhelm von Waldenfels introduced a non-commutative version of $K[d]$. He was studying a mathematical model for the light emission and absorption in a laser; see [vW84a]. In this model the time evolution is described by unitary operators on $\mathbb{C}^d \otimes \mathcal{H}$, $\mathcal{H}$ an infinite-dimensional Hilbert space. Here the physical system $\mathbb{C}^d$ is
coupled to a ‘heat bath’ which is represented by $\mathcal{H}$. The mappings $\xi_{ij} : U(\mathbb{C}^d \otimes \mathcal{H}) \to \mathcal{B}(\mathcal{H})$ from the group $U(\mathbb{C}^d \otimes \mathcal{H})$ of unitary operators on $\mathbb{C}^d \otimes \mathcal{H}$ to the $C^*$-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$ are defined by $\xi_{ij}(U) = U_{ij}$ for $U \in U(\mathbb{C}^d \otimes \mathcal{H})$, $U$ considered as a $d \times d$-matrix with entries in $\mathcal{B}(\mathcal{H})$. Denote by $\mathcal{K}(d)$ the sub-$*$-algebra generated by the $\xi_{ij}$ of the $*$-algebra of mappings from $U(\mathbb{C}^d \otimes \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ (which is a $*$-algebra with the pointwise structure coming from $\mathcal{B}(\mathcal{H})$). On the other hand, von Waldenfels considered the algebra $\mathbb{C}\langle x_{ij}, x_{ij}^*; i, j \in [d] \rangle / \bar{J}$ where $\bar{J}$ is the ideal, now in the non-commutative polynomial algebra, generated by formally the same elements $[17]$. The following analogue of Hermann Weyl’s classical result was found by von Waldenfels in the 1980’s and published in [GvW89] as a cooperation with Peter Glockner.

**Theorem 2.3** There is a unique $*$-algebra homomorphism

$$\Phi : \mathbb{C}\langle x_{ij}, x_{ij}^*; i, j \in [d] \rangle / \bar{J} \to \mathcal{K}(d)$$

such that $\Phi(x_{ij} + \bar{J}) = \xi_{ij}$.

See [GvW89] for a remarkable proof which uses Hermann Weyl’s result for all $U_m$, $m \in \mathbb{N}$.

In fact, it is not difficult to see that $\Delta x_{ij} = \sum_{n=0}^{d} x_{in} \otimes x_{nj}$ and $\delta x_{ij} = \delta_{ij}$ defines a $*$-bialgebra structure on $\mathbb{C}\langle x_{ij}, x_{ij}^*; i, j \in [d] \rangle$. Moreover, $\bar{J}$ is a $*$-bi-ideal of this $*$-bialgebra so that $\mathcal{K}(d)$ itself is turned into a $*$-bialgebra. We sometimes write $x_{ij}$ for the equivalence class $x_{ij} + \bar{J}$.

The $*$-algebra $\mathcal{K}(d)$ has been introduced by L.G. Brown in [Bro81] and is now called the Brown-Glockner-Waldenfels algebra. One shows that $\mathcal{K}(d)$ is not a Hopf $*$-algebra, i.e. it does not allow for an antipode!

3 Lévy processes, tensor case

A good source for this section is [Fra06] or [Lac15] or [Ger15].

A stochastic process $X_t : \Omega \to G$, $t \in \mathbb{R}_+$, with values in a topological group $G$ is called a Lévy process on $G$ (over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$) if the increment process $(X_{st})_{0 \leq s \leq t}$, $X_{st}(g) := X_s(g)^{-1} X_t(g)$ fulfills the following conditions. The distribution of the increment $X_{st}$ only depends on the difference $t - s$ (that is the distribution of $X_{st}$ equals the distribution of $X_{0,t-s}$), the increments $X_{t_1, t_2}, \ldots, X_{t_n, t_{n+1}}$ for consecutive times $0 \leq t_1 \leq t_2 \leq \ldots, t_n \leq t_{n+1}$ are stochastically independent, and $X_{st}$ converges weakly to the constant random variable $e = X_{ss}$ for $t \downarrow s$.

In order to avoid inverses one starts right away from an ‘increment process’ $(X_{st})_{0 \leq s \leq t}$, with the extra increment conditions $X_{rs} X_{st} = X_{rt}$, $X_{tt} = e$. Thus we may define a Lévy process on a topological monoid $G$ to be a stochastic process $X_{st} : \Omega \to G$, $0 \leq s \leq t$, which satisfies the increment condition and with stationary, independent increments in the above sense. In the non-commutative case, where Hopf algebras will replace groups, this will mean that we will have Lévy processes
of \( \ast \)-independence for all \( \ast \)-algebras; see the example of the \( \ast \)-bialgebra \( K[d] \) (Brown-Glockner-Waldenfels algebra) which is not a Hopf algebra. Since we wish to enter the non-commutative world we will have to ‘reverse all the arrows’ and replace the measurable spaces by \( \ast \)-algebras of functions and later by not necessarily commutative \( \ast \)-algebras.

The underlying probability space is, for example, replaced by the pair \((L^\infty(\Omega), E)\) with \(L^\infty(\Omega)\) the \( \ast \)-algebra of complex-valued bounded measurable function on \( \Omega \) and \( E(F) \) the expectation value of \( F \in L^\infty(\Omega) \). For example, if \( G = U_d \) we could replace \( U_d \) by its coefficient algebra \( K[d] \). The process becomes the family \( j_{st} : K[d] \to L^\infty(\Omega) \) of \( \ast \)-algebra homomorphisms defined by \( j_{st}(f) = f \circ X_{st}, f \in K[d] \). The increment condition now reads \( j_{rs} \ast j_{st} := M \circ (j_{rs} \otimes j_{st}) \circ \Delta = j_{rt} \) and \( j_{tt} = \delta 1 \). Stationarity becomes \( E \circ j_{st} = E \circ j_{0,t-s} \), weak continuity now reads \( E \circ j_{st} \xrightarrow{t \downarrow s} \delta \), and independence of increments means that the state \( E \) factorizes on the image algebras \( j_{st}(K[d]) \), i.e.

\[
E(j_{t_1,t_2}(f_1) \ldots j_{t_n,t_{n+1}}(f_n)) = E(j_{t_1,t_2}(f_1)) \ldots E(j_{t_n,t_{n+1}}(f_n))
\]

for all \( t_1 \leq \ldots \leq t_{n+1}, f_1, \ldots, f_n \in K[d] \). In his work on light emission and absorption [vW84a] Wilhelm von Waldenfels replaced \( K[d] \) by \( K(d) \); see \( \S 2 \) of [vW84a] where von Waldenfels uses the notions \( K(U(d)) \) for \( K[d] \) and (handwritten \( K) U(d) \) for \( K(d) \). We will return to this work of von Waldenfels later.

What is a Lévy process on a \( \ast \)-bialgebra over a ‘quantum probability space’? The latter is a pair \((A, \Phi)\) where \( A \) is a (unital) \( \ast \)-algebra and \( \Phi : A \to \mathbb{C} \) is a state on \( A \). In the classical case one may choose \( A = L^\infty(\Omega) \) and \( \Phi = E \). A tensor Lévy process on a \( \ast \)-bialgebra \( B \) over \((A, \Phi)\) is a family \( j_{st} : B \to A, 0 \leq s \leq t, \) of \( \ast \)-algebra homomorphisms such that \( j_{rs} \ast j_{st} = j_{rt} \) and \( j_{tt} = \delta 1_A \), and such that \( \Phi \circ j_{st} = \Phi \circ j_{0,t-s} \) and \( \Phi \circ j_{st} \xrightarrow{t \downarrow s} \delta \) and the sub-algebras \( j_{t_1,t_2}(B), \ldots, j_{t_n,t_{n+1}}(B) \) are ‘independent’ for all \( t_1 \leq \ldots \leq t_{n+1} \).

Here we use the notion of what nowadays sometimes is called tensor independence to replace the classical stochastic independence. Tensor independence or, more generally, graded tensor independence, goes back to the early work of another pioneer of quantum probability, Robin L. Hudson, see [CH77] and [CH71]. Tensor independence would probably be the first answer of a physicist if he was asked for a reasonable notion of quantum or non-commutative independence: The sub-algebras \( A_1, \ldots, A_n \) are called tensor independent in the state \( \Phi \) if \( A_k \) and \( A_l \) commute for different values of \( k, l \leq \lfloor n \rfloor \) (that is \( a_k a_l = a_l a_k = 0 \) for all \( a_k \in A_k \) and \( a_l \in A_l \)) and if \( \Phi(a_1 \ldots a_n) = \Phi(a_1) \ldots \Phi(a_n) \) for all \( a_1 \in A_1, \ldots, a_n \in A_n \).

**Proposition 3.1** Let \( j_{st} \) be a tensor Lévy process on a \( \ast \)-bialgebra \( B \). Then we have with \( \varphi_t := \Phi \circ j_{0,t}, t \in \mathbb{R}_+ \),

\[
\varphi_{s+t} = \varphi_s \ast \varphi_t \quad \text{for } s, t \in \mathbb{R}_+ \tag{18}
\]

\[
\varphi_t(b) \xrightarrow{t \downarrow s} \delta(b) \quad \text{for all } b \in B \tag{19}
\]
Moreover, \( \frac{1}{t}(\varphi_t - \delta)(b) \) converges for \( t \downarrow 0 \) for all \( b \in \mathcal{B} \) to a limit which we denote by \( \psi(b) \). The linear functional \( \psi \) on \( \mathcal{B} \) is conditionally positive and hermitian with \( \psi(1) = 0 \). We have \( \varphi_t = \exp_t(t\psi) \).

**Proof:** It follows straightforward from the properties of a tensor Lévy process that \( \varphi_t \) is a weekly continuous convolution semi-groups of states on \( \mathcal{B} \), in particular the validity of (18) and (19).

For a \( d \)-dimensional, \( d \in \mathbb{N} \), sub-coalgebra \( \mathcal{D} \) of \( \mathcal{B} \) the linear operators \( A_t \) with \( A_t(b) = (\text{id} \otimes \varphi_t) \circ \Delta(b), \ b \in \mathcal{D} \), satisfy \( A_{s+t} = A_s \circ A_t \) and \( A_t \xrightarrow{t \downarrow 0} \text{id}_\mathcal{D} \). By a well-known result for matrices \( A_t \) must be of the form \( A_t = \exp(tG) \) with \( G \) a \( d \times d \)-matrix and \( \frac{1}{t}(A_t - \text{id}_\mathcal{D}) \xrightarrow{t \downarrow 0} G \). Now everything follows from the fact that the coalgebra \( \mathcal{B} \) is the inductive limit of its finite-dimensional sub-coalgebras; see Section 2. \( \blacklozenge \)

The conditionally positive linear functional of the proposition is called the **generator** of the tensor Lévy process \( j_{st} \). The generator determines the numbers \( \Phi(j_{s_1,t_1}(b_1) \ldots j_{s_n,t_n}(b_n)) \) where \( n \) runs through the natural numbers, \( s_i, t_i \in \mathbb{R}_+ \), \( b_1, \ldots, b_n \in \mathcal{B} \), which are the numbers relevant for the process. In this sense, \( \psi \) determines the process. The converse construction is now important. Start from a conditionally positive, hermitian linear functional \( \psi \) on \( \mathcal{B} \). Then by Schoenberg correspondence (Corollary 2.2) \( \varphi_t = \exp_t(t\psi) \) is a convolution semi-group of states on \( \mathcal{B} \). Inspired by the Daniel-Kolmogorov construction for stochastic processes of classical probability theory, in the late 1980’s Luigi Accardi achieved, in co-operation with Wilhelm von Waldenfels and M. Schirmann, the construction of a tensor Lévy process with a given convolution semi-group as an inductive limit; see [ASvW88]. This established a 1-1-correspondence between conditionally positive, hermitian linear functionals on a \( * \)-bialgebra \( \mathcal{B} \) (the generators) and (equivalence classes) of tensor Lévy processes on the \( * \)-bialgebra \( \mathcal{B} \).

Let us turn to a more detailed investigation of tensor Lévy processes on Hopf \( * \)-algebra of type \( T(\mathcal{V}) \) with \( \mathcal{V} \) a vector space with involution *\( \). Generators on \( T(\mathcal{V}) \) can be described as follows. Let \( H \) be a pre-Hilbert space, i.e. a complex vector space with a fixed scalar product \( \langle \cdot, \cdot \rangle \). Denote by \( L_a(H) \) the \( * \)-algebra of adjointable linear operators on \( H \). (A linear \( T : H \rightarrow H \) is by definition in \( L_a(H) \) if there is a linear \( T^* : H \rightarrow H \) such that \( \langle x, Ty \rangle = \langle T^*x, y \rangle \) for all \( x, y \in H \).) Let \( (\rho_0, \eta_0, \psi_0) \) be a triplet formed by a linear \( * \)-map \( \rho_0 : \mathcal{V} \rightarrow L_a(H) \), a linear map \( \eta_0 : \mathcal{V} \rightarrow H \) and a hermitian linear functional \( \psi_0 \) on \( \mathcal{V} \). Then an interesting calculation shows that there is a unique generator \( \psi \) on \( T(\mathcal{V}) \) such that \( \psi_0 \) is the restriction of \( \psi \) to \( \mathcal{V} \subset T(\mathcal{V}) \), for \( v_1, v_2 \in \mathcal{V} \) we have \( \psi(v_1 \otimes v_2) = \langle \eta_0(v_1^*), \rho_0(v_2) \rangle \), and

\[
\psi(v_1 \otimes \ldots \otimes v_n) = \langle \eta_0(v_1^*), \rho_0(v_2) \ldots \rho_0(v_{n-1}) \eta_0(v_n) \rangle
\]

for \( n \geq 3 \) and \( v_1, \ldots, v_n \in \mathcal{V} \). This construction yields a 1-1-correspondence between generators and triplets of this kind.
We form the pre-Hilbert space $\Gamma(H) := \bigoplus_{n=0}^{\infty} K^\otimes n$ with $K = L^2(\mathbb{R}_+) \otimes H$ where $K^\otimes n$ denotes the symmetric tensor product of $n$ copies of $K$. On this symmetric Fock space $\Gamma(H)$ we define the operators

$$A_t(x) = A(\chi_{[0,t]} \otimes x)$$
$$A_t^*(x) = A(\chi_{[0,t]} \otimes x)^*$$
$$\Lambda_t(T) = \Lambda(\chi_{[0,t]} \otimes T)$$

for $x \in H$, $T \in L_a(H)$. Here $A(\xi)$, $A(\xi)^*$, $\xi \in K$ and $\Lambda(T)$, $T \in L_a(K)$ denote the annihilation, creation and additive second quantization (= preservation) operators on symmetric Fock space respectively; cf. [Par92]. Let $\Phi$ be the state on annihilation, creation and additive second quantization (= preservation) operators $x$. We arrive at Robin Hudson’s quantum Wiener process. Of course, in general the $d$-algebra $B$ is associated with a pre-Hilbert space $H$, a $*$-algebra map $\rho : B \rightarrow L_a(H)$ and a linear surjective linear map $\eta : B \rightarrow H$ via the cohomological equations ($b, c \in \text{ker} \delta$)

$$\rho(b)\eta(c) = \eta(bc)$$
$$\langle \eta(b^*), \eta(c) \rangle = \psi(bc);$$

where $\psi$ is an extension of a conditional positive, hermitian linear functional $\eta$.

**Theorem 3.1** Let $\psi$ be a conditionally positive, hermitian linear functional on $T(V)$ with pre-Hilbert space $H$ and triplet $(\rho_0, \eta_0, \psi_0)$ as above. Then equation (23) defines a tensor Lévy process on $T(V)$ with generator $\psi$.

As an example, consider the linear functional $g_Q$ defined by equation (16). We assume that $Q$ is positive semi-definite. For simplicity let us also assume that $Q$ is regular, i.e. that $Q$ is positive definite. It is clear that $g_Q$ is hermitian and conditionally positive so that it is the generator of a tensor Lévy process on $\mathbb{C}(x_1, \ldots, x_d)$. An interesting calculation shows that indeed $\gamma_Q = \exp_g g_Q!$ The pre-Hilbert space $H$ in this case is $\mathbb{C}^d$ equipped with the scalar product $\langle x, Qy \rangle$. The map $\eta$ vanishes on all components $V^\otimes n$ with $n \neq 1$ where we put $V = \mathbb{C}^d = H$ equal to the linear span of $x_1, \ldots, x_d$ and use the identification of $\mathbb{C}(x_1, \ldots, x_d)$ with $T(V)$. Now for $v \in V$ we have $\eta(v) = v \in H$. Moreover, $\rho(v) = 0$ for all $v \in V$. The Lévy process is given by

$$F_t(v) = A_t(\eta_0(v^*)) + A_t^*(\eta_0(v)) + \psi_0(v) t$$

for $t \in \mathbb{R}_+$. We obtain $d$ realizations of Brownian motions as operator processes on the same symmetric Fock space. Of course, in general the $d$ Brownian motions do not commute. We arrive at Robin Hudson’s quantum Wiener process.

Let us have a look at the general tensor case. A generator $\psi$ of a tensor Lévy process $j$ on a $*$-bialgebra $B$ is associated with a pre-Hilbert space $H$, a $*$-algebra map $\rho : B \rightarrow L_a(H)$ and a linear surjective linear map $\eta : B \rightarrow H$ via the cohomological equations ($b, c \in \text{ker} \delta$)
see [Sch91] and [Fra06]. Put $B_0 = \ker \delta$. Then use the restrictions $\rho_0 := \rho \mid B_0$, $\eta_0 = \eta \mid B_0$ and $\psi_0 = \psi \mid B_0$ as a triplet to define the generator of a tensor Lévy process $k_{st}$ on $T(B_0)$ which we call the generator process of the tensor Lévy process $j_{st}$ on $B$. It will be given by $F_t$ of (23). Since $T(B_0)$ is a Hopf $*$-algebra, the process $k_{st}$ is determined by the process $k_t$ with $k_t := k_{0t}$.

Now we use quantum stochastic calculus as it was developed by Robin L. Hudson and K.R. Parthasarathy in the 1980’s; see [HPS84] and [Par92]. The tensor Lévy process $j_{st}$ with generator $\psi$ can always be realized on the symmetric Fock space $\Gamma(H)$ as the solution of the quantum stochastic differential equation (QDE)

$$
\frac{d}{dt} j_{st} = j_{st} \ast dk_t, \quad j_{ss} = \delta
$$

which is driven by the generator process $k_t$; see [Sch91] and [Lac15] for details. (Here we identify $B$ with $C_1 \oplus B_0 \subset T(B_0)$.) Again there is also pioneering work of Wilhelm von Waldenfels in this field of quantum probability. See again his paper [vW84a] where he constructed a tensor Lévy process on $\mathcal{K}(d)$ using a multiplicative quantum stochastic Itô integral.

The process which von Waldenfels constructed in [vW84a] is a tensor Lévy process on the $*$-bialgebra $\mathcal{K}(d)$.

Given a tensor Lévy process $j_{st}$ on $\mathcal{K}(d)$ we put $(U_t)_{ij} := j_{0t}(x_{ij})$ to obtain a unitary operator $U_t$ on $\mathbb{C}^d \otimes \mathcal{H}$. Here $\mathcal{H}$ is the completion of the pre-Hilbert space obtained through the Gelfand-Naimark-Segal construction applied to the pair $(\mathcal{A}, \Phi)$. Indeed, the unitarity relations (17) force the operators $j_{st}(x_{ij})$ to be bounded and $U_t$ is a unitary matrix of bounded operators on $\mathcal{H}$. Moreover, $j_{st}(x_{ij}) = (U_s^{-1} U_t)_{ij}$, so that the process $j_{st}$ is determined by the unitary process $U_t$.

One shows that all conditionally positive, hermitian linear functionals $\psi$ with $\psi(1) = 0$ (that is all generators) on $\mathcal{K}(d)$ arise in the following way. (For a vector space $\mathcal{V}$, denote by $M_d(\mathcal{V})$ the set of $d \times d$-matrices with entries in $\mathcal{V}$.) Fix a pre-Hilbert space $H$ and a unitary operator $W$ on $\mathbb{C}^d \otimes H$, a matrix $L$ in $M_d(H)$ and a hermitian matrix $D \in M_d(\mathbb{C})$. Then there is a unique generator $\psi$ such that $\rho(x_{ij}) = W_{ij}, \eta(x_{ij}) = L_{ij}$ and $2i D_{ij} = \psi(x_{ij} - x^*_j)$. The QSDE (24) for $U_t$ with generator triplet $(\rho, \eta, \psi)$ becomes

$$
dU_t = U_t \, dI_t, \quad U_0 = 1
$$

with $I_t$ the matrix

$$
(I_t)_{ij} = A_t(\eta(x^*_{ij})) + A_t(\rho(x_{ij}) - \delta_{ij}) + A_t^*(\eta(x_{ij})) + \psi(x_{ij}) \, t
$$

$$
= A_t^*(L_{ij}) + A_t((W - 1)_{ij}) - A_t((W^*L)_{ji}) + (D - \frac{1}{2} L^* L)_{ij} \, t.
$$

We have (cf. [Fra06])

**Theorem 3.2** All tensor Lévy processes on $\mathcal{K}(d)$ are of type (23).

For the process in [vW84a] we have $H = \mathbb{C}$ and $W = 1, D = 0$, so that (25) can be written

$$
dU_t = U_t (L \, dA^*_t - L^* \, dA_t - \frac{1}{2} L^* L \, dt).
$$
4 Generalisations

Again it was pioneering work of Wilhelm von Waldenfels that led to a new notion of independence, to ‘Boolean independence’. In the papers [vW73] and [vW75], which again are motivated by physics, von Waldenfels used cumulants of Boolean independence at a time when there was no discussion at all on notions of non-commutative independence. Boolean independence had not even received its name. When later asked why he chose Boolean cumulants, he used to say: ‘Because they were the simplest to calculate’.

If we neglect questions of positivity, the category of classical probability spaces is formed by pairs \((A, \varphi)\) as objects. Here \(A\) is a commutative unital algebra and \(\varphi\) is a normalized linear functional on \(A\), i.e. a linear map \(\varphi : A \to \mathbb{C}\). The morphisms \(j : (A_1, \varphi_1) \to (A_2, \varphi_2)\) are given by algebra homomorphisms \(j : A_1 \to A_2\) which satisfy \(\varphi_2 \circ j = \varphi_1\). The joint distribution \(\varphi_{12} = \varphi_1 \circ j\). The joint distribution of two random variables \(j_i : \mathcal{A}_i \to (A, \varphi), i = 1, 2\), is the linear functional \(\varphi \circ M_{\mathcal{A}} \circ (j_1 \otimes j_2)\) on \(A_1 \otimes A_2\). The random variables \(j_1, j_2\) are called independent if their joint distribution equals \(\varphi_1 \otimes \varphi_2\), i.e. \(\varphi = \varphi_1 \circ j_1\) the distribution of \(j_i, i = 1, 2\). This is the classical commutative situation.

One passes to non-commutativity by starting with the category \(\mathcal{F}\) with objects \((\mathcal{A}, \varphi)\) where \(\mathcal{A}\) is a (not necessarily unital) algebra and \(\varphi\) is a linear functional on \(\mathcal{A}\). (In order to include all known examples, for instance Boolean independence, we allow for non-unital algebras.) Morphisms (= random variables) are defined as before as algebra homomorphisms with \(\varphi_1 = \varphi_2 \circ j\). The joint distribution of \(j_i\), \(i = 1, 2\), is given as the linear functional \(\varphi \circ (j_1 \sqcup j_2)\) on the free product \(\mathcal{A} \sqcup \mathcal{A}_2\) of \(\mathcal{A}_1\) and \(\mathcal{A}_2\). The free product of algebras is the co-product in the category of algebras like the tensor product is the co-product in the category of commutative unital algebras. \(\mathcal{A}_1 \sqcup \mathcal{A}_2\) can be realized as the quotient of the tensor algebra \(T(\mathcal{A}_1 \oplus \mathcal{A}_2)\) by the ideal generated by the elements \(a_1 \otimes b_1 - a_1 b_1, a_1, b_1 \in \mathcal{A}_1\) and \(a_2 \otimes b_2 - a_2 b_2, a_2, b_2 \in \mathcal{A}_2\).

A universal product is a bi-functor \(\circ\) on \(\mathcal{F}\) of the type \((\mathcal{A}_1, \varphi_1) \circ (\mathcal{A}_2, \varphi_2) = (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \circ \varphi_2), j_1 \circ j_2 = j_1 \sqcup j_2\) such that \(\circ\) turns \(\mathcal{F}\) into a monoidal category with unit object \(0 : \{0\} \to \mathbb{C}\); see [Lac15] and [Ger21]. The term universal in this connection might sound strange for a category theorist. It comes from the condition that the product ‘is the same’ for all algebras which, of course, means nothing else but that it is the tensor product of a category. Given \(\circ\) two random variables are called \(\circ\)-independent if their joint distribution equals \(\varphi_1 \circ \varphi_2\) that is the \(\circ\)-product of the marginal distributions. Examples of \(\circ\) are Boolean independence, tensor independence, free independence, monotone independence and anti-monotone independence.

All these products are positive in the following sense. If \(\mathcal{A}_i, i = 1, 2,\) are *-algebras and if \(\varphi_i : \mathcal{A}_i \to \mathbb{C}\) are such that the unitizations \(1 \varphi_i\) are states on the unitizations \(1 \mathcal{A}_i\), then the unitization \(1 (\varphi_1 \circ \varphi_2)\) is a state on \(1 (\mathcal{A}_1 \sqcup \mathcal{A}_2)\), see [Ger21]. The first classification result for products \(\circ\) was in the paper ‘On univer-
sal products’ [Spe97] by Roland Speicher. The complete classification of positive universal products was achieved by Naofumi Muraki in [Mur03] where it is shown that the above mentioned five examples are the only products which satisfy an additional property (which is satisfied by positive products). For a classification of general universal products see [GL15].

Still there are more examples. One arose by work of Marek Bożejko and Roland Speicher who investigated conditional freeness; see [BS91]. More examples of this kind were found by Takahiro Hasebe; see [Has11]. In these examples the linear functional $\varphi$ is generalized to a vector valued linear functional $\varphi : A \to \mathbb{C}^d$, $d$ a fixed natural number. This gives many more possibilities and a full classification seems to be out of reach.

Another direction of generalization is to work with $m$-faced algebras, $m \in \mathbb{N}$, i.e. with algebras $A$ which come with a free decomposition $A = A_1 \sqcup \ldots \sqcup A_m$. This means that the $A_i$ are sub-algebras of $A$ such that the natural mapping from $A_1 \sqcup \ldots \sqcup A_m$ to $A$ is an isomorphism of algebras. The $d$- and $m$-generalizations can be done in one step. Consider the category $\mathcal{F}_{d,m}$ with objects pairs $(A,\varphi)$ where $A$ is an $m$-faced algebra and $\varphi : A \to \mathbb{C}^d$ linear. The morphisms $j$ are algebra homomorphisms which respect the free decompositions and with $\varphi_1 = \varphi_2 \circ j$. A natural product $\otimes$, as before, is a bi-functor on $\mathcal{F}_{d,m}$ with $(A_1,\varphi_1) \otimes (A_2,\varphi_2) = (A_1 \sqcup A_2, \varphi_1 \circ \varphi_2)$, $j_1 \circ j_2 = j_1 \sqcup j_2$, and such that $\otimes$ turns $\mathcal{F}$ into a monoidal category with unit object $0 : \{0\} \to \mathbb{C}^d$; see again [Ger21]. $\otimes$-independence is defined as before. The first example of this kind was the bi-free independence which Dan V. Voiculescu introduced in 2014; see [Voi14].

Given such a product $\otimes$ we are in the situation to define $\otimes$-Lévy processes on $m$-faced dual semi-groups. The latter are co-monoids in the monoidal category of $m$-faced algebras with product $\sqcup$ and unit object $\{0\}$. More precisely, an $m$-faced dual semi-group is a triplet $(B,\Lambda,0)$ where $B$ is an $m$-faced $\ast$-algebra and $\Lambda : B \to B \sqcup B$ is an $m$-faced $\ast$-algebra homomorphism such that $(\Lambda \sqcup \text{id}) \circ \Lambda = (\text{id} \sqcup \Lambda) \circ \Lambda$ and $(\text{id} \sqcup 0) \circ \Lambda = \text{id} = (0 \sqcup \Lambda) \circ \Lambda$; see [Ger21]. A $\otimes$-Lévy process on the $m$-faced dual semi-group $B$ over the object $(A,\varphi)$ of $\mathcal{F}_{d,m}$, is, in analogy to the tensor case, a family $j_{st} : B \to A$ of $m$-faced $\ast$-algebra homomorphisms such that the increment property, the stationarity and independence of increments, and the week continuity (which now reads $\Phi \circ j_{st} \to \{0\}$ are satisfied.

As before the 1-dimensional distributions $\varphi_t = \Phi \circ j_{0,t}$ form a convolution semigroup where now the convolution is defined by

$$\varphi_1 \star \varphi_2 = (\varphi_1 \circ \varphi_2) \circ \Lambda.$$ 

We also require our product $\otimes$ to be positive. Using the Lachs functor, one can show (see [Lac15] and [Ger21]) that $\frac{1}{t} \varphi_t(b)$ converges to a limit which we denote by $\psi(b)$. Moreover, with the help of the Lachs functor $\varphi_t$ can be reconstructed from $\psi$ as an ‘exponential’ $\varphi_t = \exp_\varphi(t\psi)$. A very nice result of Malte Gerhold (Theorem 5.9 in [Ger21]) says that in this situation the Schoenberg correspondence again holds. The proof of this general Schoenberg correspondence uses the tensor result (Theorem 16 of [Ger21]) which shows that $\Phi \circ j_{st} \to \{0\}$ is satisfied.
which had been proved in the co-commutative case by Wilhelm von Waldenfels. With this result we have that $\odot$-Lévy processes are given by conditionally positive, hermitian linear functionals.

We close by mentioning that a realization of $\odot$-Lévy processes as solutions of QSDE is a nice problem. If $d = m = 1$ this, at least partially, has been done (free Fock space, Boolean Fock space). The general $d, m$-case seems to be an open problem. A classification of products $\odot$ in the case $d = 1, m = 2$ is in progress and there exist very interesting partial results that can be found in the PhD thesis of Philipp Varšo [Var].

References

[ASvW88] L. Accardi, M. Schürmann, and W. von Waldenfels. Quantum independent increment processes on superalgebras. *Math. Z.*, 198:451–477, 1988.

[Bro81] L.G. Brown. Ext of certain free product $C^*$-algebras. *J. Operator Theory*, 6:135–141, 1981.

[BS91] M. Bożejko and R. Speicher. $\psi$-independent and symmetrized white noises. In *Quantum probability & related topics*, QP-PQ, VI, pages 219–236. World Sci. Publ., River Edge, NJ, 1991.

[CH71] C.D. Cushen and R.L. Hudson. A quantum-mechanical central limit theorem. *Journal of Applied Probability*, 82:454–469, 1971.

[CH77] A.M. Cockroft and R.L. Hudson. Quantum mechanical Wiener processes. *J. Multivariate Anal.*, 7(1):107–124, 1977.

[Chu68] K.L. Chung. *A course in probability theory*. Harcourt, Brace and World, 1968.

[Fra06] U. Franz. Lévy processes on quantum groups and dual groups. In *Quantum independent increment processes II*, volume 1866 of *Lecture Notes in Math.*, pages 161–257. Springer, Berlin, 2006.

[Ger15] M. Gerhold. *On several problems in the theory of comonoidal systems and subproduct systems*. PhD thesis, Greifswald, 2015. [http://ub-ed.ub.uni-greifswald.de/opus/volltexte/2015/2244/](http://ub-ed.ub.uni-greifswald.de/opus/volltexte/2015/2244/)

[Ger21] M. Gerhold. Schoenberg correspondence for multifaced independences. *arXiv:2104.02985v2*, 2021.

[GL15] M. Gerhold and S. Lachs. Classification and GNS-construction for general universal products. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 18(1):1550004, 29, 2015.
[vW84a] W. von Waldenfels. Itô solution of the linear quantum stochastic differential equation describing light emission and absorption. In Quantum probability and applications to the quantum theory of irreversible processes (Villa Mondragone, 1982), volume 1055 of Lecture Notes in Math., pages 384–411. Springer, Berlin, 1984.

[vW84b] W. von Waldenfels. Positive and conditionally positive sesquilinear forms on anticommutative coalgebras. In Probability measures on groups, VII (Oberwolfach, 1983), volume 1064 of Lecture Notes in Math., pages 450–466. Springer, Berlin, 1984.