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Estimates of the physical meson masses by an effective light-cone Hamiltonian

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Abstract. A recent renormalization group analysis of the effective light-cone Hamiltonian yields 5 flavor masses and 2 mass shifts as free parameters. These are determined in the present note and used to calculate the masses of all 30 physical mesons (The topped mesons have been omitted). The agreement between experiment and theoretical estimate is quite satisfactory over the whole dynamical range, from the lightest pseudo-scalar mesons like the pion up to the heaviest vector mesons like the upsilon. Evidence for an additive quark model is given and for how the concept of isospin is realized in a gauge field theory such as QCD.

PACS. 11.10.Ef Lagrangian and Hamiltonian approach – 12.38.Aw General properties of QCD – 11.10.St Bound and unstable states

1 The mass formula

A recent renormalization group analysis of the effective light-cone Hamiltonian [1] culminates in the result that the invariant mass-square eigenvalues $M^2$ of the physical mesons are given by the quasi-static mass formula

$$M^2 = (m_{\bar{q}} + m_q)^2 + 2(m_{\bar{q}} + m_q) \tilde{s}_\pm.$$  (1)

In deriving this equation the dynamical effects of the binding energy proper have been omitted with an estimated error of about 10%. Some more background is given in Sections 2 and 5. The physical quark masses $m_{\bar{q}}$ and the mass shifts $\tilde{s}_\pm$ are renormalization group invariants and have to be fixed by experiment. The mass shifts can have different values for the pseudo-scalar ($\tilde{s}_-$) and the vector mesons ($\tilde{s}_+$). Their determination is one of the objects of this note.

The familiar nomenclature of the physical mesons is given in Table 1. The empirical masses are taken from the data of the particle data group [2]. These data do not yet include the topped mesons and are compiled in Table 2. The following procedure was applied. First, the up and the down mass were chosen equal, $m_u = m_d = 350$ MeV. Then the empirical masses of $\pi^+$ and $\rho^+$ were used to

| $\pi$ | $\overline{\pi}$ | $\eta$ | $\overline{\eta}$ |
|-------|-----------------|--------|-----------------|
| $u$   | 768             | 1002   | 2301            |
| $d$   | *140            | 1002   | 2301            |
| $s$   | *494            | 494    | 2535            |
| $c$   | *1865           | 1865   | 2102            |
| $b$   | *5278           | 5278   | 5512            |

Table 1. The nomenclature of the ‘flavor-off-diagonal’ physical mesons. The vector mesons are given in the upper, the pseudo-scalar mesons in the lower triangle.

| $\pi$ | $\overline{\pi}$ | $\eta$ | $\overline{\eta}$ |
|-------|-----------------|--------|-----------------|
| $u$   | 768             | 1002   | 2301            |
| $d$   | *140            | 1002   | 2301            |
| $s$   | *494            | 494    | 2535            |
| $c$   | *1865           | 1865   | 2102            |
| $b$   | *5278           | 5278   | 5512            |

Table 3. Same as Table 2 but for the calculated masses. The *upperscript marks where shift and mass are determined.
determine the mass shifts for the singlet and the triplet,
\[
\bar{\pi}_- = -336 \text{ MeV}, \quad \text{and} \quad \bar{\pi}_+ = 71 \text{ MeV},
\]
respectively. The remaining quark masses are obtained from the pseudo-scalar mesons with an up quark, which exhausts all freedom in determining physical parameters. The resulting quark masses are given in Table 4. The remaining 13 off-diagonal pseudo-scalar and vector meson masses are calculated straightforwardly from Eq. (1), and compiled in Table 3. Equivalently, but with slightly different results, the quark masses could have been determined from the off-diagonal vector mesons.

2 The general problem

Since the off-diagonal mesons are so well described, one wonders about the flavor-diagonal ones.

Let us review in short the general aspects. The full light-cone Hamiltonian for gauge theory with its complicated many-body aspects is reduced in [3] by the method of iterated resolvents to the effective Hamiltonian \( H_{\text{eff}} \), which by definition acts only in the Fock space of a single quark \( q \) and a single anti-quark \( \bar{q} \). It has essentially the same as \( E_2 = \langle \bar{s}u|H_{\text{eff}}|\bar{s}u \rangle \) since charge exchange does not change the eigenvalues. Most blocks correspond to a zero matrix. The way these zero matrices arrange themselves in the table is peculiar to QCD. This demonstrates that most of the Hamiltonian is reducible and that one can diagonalize blockwise. The blockwise diagonalization of the off-diagonal mesons in Tables 3 and 4 survives thus even if the annihilation interaction is included.

The last three rows and columns in Table 3 however show, that the annihilation mixes the flavors in the charge-neutral flavor-diagonal mesons. The QCD-flavor mixing for 3 flavors is thus the 3 \( \times 3 \) block matrix in Table 4. For \( N \) flavors one has correspondingly a \( N \times N \) matrix.

How can one include the annihilation graph \( U_{\text{TGA}} \), at least approximately? The numerators in the annihilation graph of Figure 2 must yield a Lorentz scalar like the current coupling \( \bar{q}q\gamma^\mu v(q)|\bar{q}q\gamma^\mu u(q') \rangle \). Because of our Dirac-spinor convention [3] one can thus anticipate that \( U_{\text{TGA}} \sim \bar{q}q|\bar{q}q \rangle \). The propagator in the two-gluon annihilation interaction \( U_{\text{TGA}} \) distroys a \( q\bar{q} \)-pair and creates another one with the same or an other flavor. When addressing to the problem of diagonalizing the full effective interaction \( H_{\text{eff}} \), one must provide the Fock space for that. For three flavors, the complete Fock-space with any quark and any anti-quark is given schematically in Table 3. The effective Hamiltonian is displayed as a block matrix to illustrate the flavor mixing in QCD. The abbreviation \( e_i = E_i + A_i \) is used by obvious reasons.

![Fig. 1. The three graphs of the effective interaction in the \( q\bar{q} \)-space. — The lower two graphs correspond to the one-gluon-exchange interaction \( U_{\text{OGE}} \), the upper corresponds to the two-gluon-annihilation interaction \( U_{\text{TGA}} \). The figure is taken from Ref. [1].](image)

| \( u \bar{d} \) | \( u \bar{s} \) | \( d \bar{s} \) | \( d \bar{u} \) | \( s \bar{u} \) | \( s \bar{d} \) | \( u \bar{u} \) | \( d \bar{d} \) | \( s \bar{s} \) |
|---|---|---|---|---|---|---|---|---|
| \( u \bar{d} \) | \( E_1 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( u \bar{s} \) | 0 | \( E_2 \) | 0 | 0 | 0 | 0 | 0 | 0 |
| \( d \bar{u} \) | 0 | 0 | \( E_3 \) | 0 | 0 | 0 | 0 | 0 |
| \( s \bar{d} \) | 0 | 0 | 0 | \( E_1' \) | 0 | 0 | 0 | 0 |
| \( u \bar{u} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( d \bar{d} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( s \bar{s} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4. The kernel of the effective Hamiltonian is displayed as a block matrix to illustrate the flavor mixing in QCD. The abbreviation \( e_i = E_i + A_i \) is used by obvious reasons.

In the present context, the coefficient \( b \) is subject to be determined by a fit to the experiment. It can have either sign, and can be different for pseudo-scalar and vector mesons.
The nature and physical interpretation of the parameter \(b\) is very different from for example \(\pi_\pm\). The latter is a physical (renormalization) constant, while Eq. (3) and the coefficients \(b_S\) and \(b_V\) are kind of empirical constraints on a future and more accurate analysis of the matrix elements of \(U_{TGA}\).

### 3 Flavor SU(2)

Consider first 2 flavors with equal masses \(\overline{m}_u = \overline{m}_d\) for the charge neutral pseudo-scalar mesons. The flavor-mixing matrix is

\[
H_{\text{eff}} = \frac{u\bar{u}}{dd} \left( \begin{array}{ccc}
 u\bar{u} & a + M_{u\bar{u}}^2 & a \\
 a & a + M_{d\bar{d}}^2 & a \\
 a & a & a + M_{s\bar{s}}^2 \\
\end{array} \right).
\]

(6)

Because of the equal quark masses, the Hamiltonian \(H_{OGE}\) is identical in the \(u\bar{u}\) and the \(d\bar{d}\) sector with identical eigenstates \(|q_i\rangle\) and eigenvalues \(M_{q_i}^2\). The matrix elements of \(U_{TGA}\) are therefore all equal, \(a = (u\bar{u})|U_{TGA}|u\bar{u} = (u\bar{u})|U_{TGA}|d\bar{d} = (d\bar{d})|U_{TGA}|d\bar{d}\). One can thus diagonalize state after state, and can interpret Eq. (4) as a simple 2 by 2 matrix (instead of a 2 by 2 block matrix), with \(M_{q_i}^2\) given by Eq. (5).

The diagonalization of \(H_{\text{eff}}|\Phi_1\rangle = M_1^2|\Phi_1\rangle\) is easy. The two eigenstates are

\[
|\Phi_1\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
 |u\bar{u}\rangle \\
 |d\bar{d}\rangle \\
 |s\bar{s}\rangle \\
\end{array} \right), \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
 |u\bar{u}\rangle \\
 -|d\bar{d}\rangle \\
 0|s\bar{s}\rangle \\
\end{array} \right),
\]

(7)

and are associated with the eigenvalues

\[
M_1^2 = M_{u\bar{u}}^2 + 2a, \quad M_2^2 = M_{d\bar{d}}^2.
\]

(8)

The coherent state \(\Phi_1\) picks up all the strength. Since the eigenvalue of the incoherent state \(\Phi_2\) is identical with the unperturbed value \(M_{u\bar{u}}^2\), it is reasonable to interpret it as the \(\pi^0\), which combines with the \(\pi^+\) and the \(\pi^-\) into a mass degenerate triplet of isospin \(I = 1\). The coherent state can be fitted to the \(\eta\) which determines \(b \equiv b_S\). The outcome of this fit is reported in Table 3. The same procedure can be applied to the vector mesons, correspondingly, with the resulting fit values

\[
b_S = (1494 \text{ MeV})^2, \quad b_V = (348 \text{ MeV})^2.
\]

The results for the vector mesons are compiled in Table 4.

### 4 Flavor SU(3)

Next, consider 3 flavors with equal masses \(\overline{m}_u = \overline{m}_d = \overline{m}_s\). The flavor mixing matrix is then

\[
H_A = \frac{u\bar{u}}{dd} \left( \begin{array}{ccc}
 u\bar{u} & a + M_{u\bar{u}}^2 & a \\
 a & a + M_{d\bar{d}}^2 & a \\
 a & a & a + M_{s\bar{s}}^2 \\
\end{array} \right),
\]

(10)

with \(H_{\text{eff}} = H_A\). Up to an obvious normalization constant the three eigenstates are

\[
|\Phi_1\rangle = \left( \begin{array}{c}
 |u\bar{u}\rangle \\
 |d\bar{d}\rangle \\
 |s\bar{s}\rangle \\
\end{array} \right), \quad |\Phi_2\rangle = \left( \begin{array}{c}
 |u\bar{u}\rangle \\
 -|d\bar{d}\rangle \\
 0|s\bar{s}\rangle \\
\end{array} \right), \quad |\Phi_3\rangle = \left( \begin{array}{c}
 |u\bar{u}\rangle \\
 |d\bar{d}\rangle \\
 -2|s\bar{s}\rangle \\
\end{array} \right)
\]

and have the eigenvalues

\[
M_1^2 = M_{u\bar{u}}^2 + 3a, \quad M_2^2 = M_{u\bar{u}}^2, \quad M_3^2 = M_{d\bar{d}}^2.
\]

(12)

The coherent state picks up all the strength again. The eigenvalues of the remaining two states are again the unperturbed ones, which now are degenerate. State \(\Phi_2\) can again be interpreted as the eigenstate for the charge neutral \(\pi^0\) and the mass of the coherent state \(\Phi_1\) can be fitted with the \(\eta\). But then state \(\Phi_3\) is degenerate in mass with the \(\pi^0\).

Obviously, one cannot abstract from the appreciable mass difference of up and strange quark. Including the different masses yields \(H_{\text{eff}}\) which for convenience is split up like \(H_{\text{eff}} = H_A + H_C\), with \(H_A\) given in Eq. (10) and thus \(H_C\) by

\[
H_C = \frac{u\bar{u}}{dd} \left( \begin{array}{ccc}
 0 & 0 & C_1 \\
 0 & 0 & C_2 \\
 C_1 & C_2 & C_d \\
\end{array} \right).
\]

(13)
Because of $\overline{m}_u = \overline{m}_d$ one has $C_1 = C_2$ and
\begin{align}
C_1 &= \langle u\bar{u}|U_{\text{TGA}}|s\bar{s} \rangle - \langle u\bar{u}|U_{\text{TGA}}|u\bar{u} \rangle, \\
C_2 &= \langle s\bar{s}|U_{\text{TGA}}|s\bar{s} \rangle - \langle u\bar{u}|U_{\text{TGA}}|u\bar{u} \rangle + M_{ss}^2 - M_{us}^2.
\end{align}

The $3 \times 3$ matrix can not be diagonalized analytically, at least not in a simple way. Therefore, $H_{\text{cl}}$ is included by calculating the expectation values of $H_{\text{cl}}$ with the eigenstates of $H_A$, i.e.
\begin{align}
\langle \Phi_1|H_{\text{cl}}|\Phi_1 \rangle &= \frac{1}{3}(C_d + 2C_1 + 2C_2), \\
\langle \Phi_2|H_{\text{cl}}|\Phi_2 \rangle &= 0, \\
\langle \Phi_3|H_{\text{cl}}|\Phi_3 \rangle &= \frac{2}{3}(C_d - C_1 - C_2).
\end{align}

Adding these to the eigenvalues of $H_A$ gives
\begin{align}
M_1^2 &= M_{s\bar{s}}^2 + \langle \Phi_1|H_{\text{cl}}|\Phi_1 \rangle + 3a, \\
M_2^2 &= M_{d\bar{d}}^2 + \langle \Phi_2|H_{\text{cl}}|\Phi_2 \rangle, \\
M_3^2 &= M_{u\bar{u}}^2 + \langle \Phi_3|H_{\text{cl}}|\Phi_3 \rangle.
\end{align}

The coefficient $b = b_S$ is fixed for the pseudo-scalar mesons by requiring $M_1 = M_\pi$. For the vector mesons, the coefficient $b = b_V$ is fixed by requiring $M_1 = M_\omega$. The result is
\begin{align}
b_S &= (539 \text{ MeV})^2, \quad \text{and} \quad b_V = (652 \text{ MeV})^2.
\end{align}

Finally, the physical mass of the neutral charm pseudo-scalar meson is corrected according to Eq.\ref{eq:mass_correction}, like
\begin{align}
(M_{cc}^2)_{\text{physical}} = M_{cc}^2 + b_S \frac{\overline{m}_c^2}{M_{gg}^2 - M_{cc}^2},
\end{align}
in order to account for the annihilation channel approximately. The flavor-diagonal $b$-mesons and vector mesons are treated correspondingly. The results are compiled in Table\ref{tab:hadron_masses}.

### 5 Discussion and conclusions

In this necessarily short note all known hadron masses from the lightest ones like the pion up to the heaviest ones like the upsilon are described within the same model using the same parameters. Its ingredients are based on the effective Hamiltonian, down to which the genuine many-body aspects of gauge field theory and the full light-cone Hamiltonian for QCD are reduced by the method of iterated resolvents, and its subsequent analysis with the renormalization group.

The so obtained Eq.\ref{eq:mass_correction} is the quasi-static limit of the eigenvalues of the full effective Hamiltonian. For sufficiently large quark masses one can expand Eq.\ref{eq:mass_correction} as a power series in $\overline{m}_u/(\overline{m}_u + \overline{m}_d)$,
\begin{align}
M \simeq \overline{m}_q + \overline{m}_q + \pi_\pm + \cdots,
\end{align}
which to leading order yields an additive quark model.

Isospin is not a dynamic symmetry of QCD. In the present light-cone approach to gauge theory with an effective interaction isospin arises because the mass of the strange quark is similar to but different from the mass of up and down. Flavor-SU(3) is an approximate symmetry. The light-cone approach can even explain why the phenomenological flavor-SU(3) symmetry works better than flavor-SU(4) or SU(5): the large mass of the heavy mesons dominates the flavor-mixing matrix so strongly that the symmetry induced by the annihilation interaction is destroyed. The model predicts however that all flavors are mixed in the charge neutral mesons. The $\pi^+$ and the $\pi^-$ are obtained by fixing the quark mass and the mass shift. The third member of the isospin triplet, the $\pi^0$ is then obtained for free. The $\eta$ is obtained by fixing the parameter $b_S$ which accounts for the two-gluon annihilation interaction in an approximate way. The $\eta'$ is then obtained for free with an accuracy of about 5%. The present work contributes thus to the $\eta$-$\eta'$ puzzle \cite{Kilcup:1994} and exposes an accuracy comparable to state-of-art lattice gauge calculations \cite{Bazavov:2017}. The empirical and the calculated masses for the flavor-off-diagonal mesons show strong correlation. The largest discrepancy of roughly 15% is observed for the charmed vector mesons. On the average they agree with each other within 5% or less. It can be expected that the agreement gets even better in more accurate work where the dynamics is included.

To our recollection there is no other model which covers the whole range of flavored hadrons with the same set of parameters. This includes the phenomenological models \cite{Biefer:1999} where the light flavors are very successfully described by the Nambu-Jona-Lasinio model and its derivatives.

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