ON THE MOTIVE OF THE NESTED QUOT SCHEME OF POINTS ON A CURVE

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Abstract. Let \( C \) be a smooth curve over an algebraically closed field \( k \), and let \( E \) be a locally free sheaf of rank \( r \). We compute, for every \( d > 0 \), the generating function of the motives \([\text{Quot}_C(E, n)]\) \( \in \text{K}_0(\text{Var}_k) \), varying \( n = (0 \leq n_1 \leq \cdots \leq n_d) \), where \( \text{Quot}_C(E, n) \) is the nested Quot scheme of points, parametrising 0-dimensional subsequent quotients \( E \to T_d \to \cdots \to T_1 \) of fixed length \( n_i = \chi(T_i) \). The resulting series, obtained by exploiting the Białyńcki-Birula decomposition, factors into a product of shifted motivic zeta functions of \( C \). In particular, it is a rational function.

0. Introduction

Let \( K_0(\text{Var}_k) \) be the Grothendieck ring of varieties over an algebraically closed field \( k \). If \( Y \) is a \( k \)-variety, its motivic zeta function

\[
\zeta_Y(q) = 1 + \sum_{n \geq 0} [\text{Sym}^n Y]q^n \in K_0(\text{Var}_k)[[q]]
\]

is a generating series introduced by Kapranov in [23], where he proved that for smooth curves it is a rational function in \( q \).

In this paper we compute the motive of the nested Quot scheme of points \( \text{Quot}_C(E, n) \) on a smooth curve \( C \), entirely in terms of \( \zeta_C(q) \). Here, \( E \) is a locally free sheaf on \( C \), and \( n = (0 \leq n_1 \leq \cdots \leq n_d) \) is a non-decreasing tuple of integers, for some fixed \( d > 0 \). The scheme \( \text{Quot}_C(E, n) \) generalises the classical Quot scheme of Grothendieck (recovered when \( d = 1 \)): it parametrises flags of quotients \( E \to T_d \to \cdots \to T_1 \) where \( T_i \) is a 0-dimensional sheaf of rank \( n_i \).

Our main result, proved in Theorem 4.2 in the main body, is the following.

**Theorem A.** Let \( C \) be a smooth curve over \( k \), let \( E \) be a locally free sheaf of rank \( r \) on \( C \). Then

\[
\sum_{0 \leq n_1 \leq \cdots \leq n_d} [\text{Quot}_C(E, n)]q_1^{n_1} \cdots q_d^{n_d} = \prod_{i=1}^{d} \zeta_C(\mathbb{L}^{r-1} q_i q_{i+1} \cdots q_d) \in K_0(\text{Var}_k)[q_1, \ldots, q_d],
\]

where \( \mathbb{L} = [A_k^1] \) is the Lefschetz motive. In particular, this generating function is rational in \( q_1, \ldots, q_d \).

The statement taken with \( d = 1 \), thus regarding the motive \([\text{Quot}_C(E, n)]\) of the usual Quot scheme of points, was proved in [1]. Our result is a natural generalisation, which was inspired by Mochizuki’s paper on “Filt schemes” [24].

Our formula fits nicely in the philosophical path according to which

“rank \( r \) theories factorise in \( r \) rank 1 theories”.

There are to date a number of examples of this phenomenon in Donaldson–Thomas theory, exhibiting a generating series of rank \( r \) invariants as a product of \( r \) (suitably shifted) generating series of rank 1 invariants: see for instance [2, 28] for enumerative DT invariants, [15] for K-theoretic DT invariants, [6, 7] for motivic DT invariants and [26, 14] for the parallel pictures in string theory.
The paper is organised as follows. In Section 1 we introduce the *nested Quot scheme* and prove its connectedness. In Section 2 we describe its tangent space and prove that, for a smooth quasiprojective curve, the nested Quot scheme is smooth. Under the assumption that the locally free sheaf is split, in Section 3 we describe a torus framing action and its associated Białynicki-Birula decomposition. In Section 4 we prove that the motive of the nested Quot scheme is independent of the locally free sheaf, and exploit the Białynicki-Birula decomposition to prove Theorem A. Our result readily implies closed formulae for the generating series of Hodge–Deligne polynomials, $\chi_y$-genera, Poincaré polynomials, Euler characteristics, since these are all motivic measures; we provide some explicit formulae in Section 4.4.

After our paper was written, we were informed that our formula for the motive of the nested Quot scheme on a projective curve can be alternatively obtained, after some manipulations, from general results on the stack of iterated Hecke correspondences [17, Corollary 4.10] (see also [20, Section 3] for a related computation of the Voevodsky motive with rational coefficients). Our paper provides a direct and self-contained argument for this formula, exploiting the geometry of the nested Quot scheme.

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**Conventions.** All schemes are of finite type over an algebraically closed field $k$. A variety is a reduced separated $k$-scheme. If $Y$ is a scheme and $Y_1, \ldots, Y_r$ are locally closed subschemes of $Y$, we say that they form a (locally closed) stratification, denoted '$Y = Y_1 \amalg \cdots \amalg Y_r$' with a slight abuse of notation, if the natural morphism of schemes $Y_1 \amalg \cdots \amalg Y_r \to Y$ is bijective. This is crucial in our calculations since this condition implies the identity $[Y] = [Y_1] + \cdots + [Y_r]$ in $K_0(\text{Var}_k)$.

1. **Nested Quot schemes of points**

1.1. **The moduli space.** Let $X$ be a quasiprojective $k$-variety and $E$ a coherent sheaf on $X$. Fix an integer $d > 0$ and a non-decreasing $d$-tuple $n = (n_1, \leq \cdots \leq n_d)$ of non-negative integers $n_i \in \mathbb{Z}_{\geq 0}$. We define the *nested Quot functor* associated to $(X, E, n)$ to be the functor $\text{Quot}_X(E, n): \text{Sch}^\text{op}_{k} \to \text{Sets}$ sending a $k$-scheme $B$ to the set of isomorphism classes of subsequent quotients

$$E_B \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1,$$

where $E_B$ is the pullback of $E$ along $X \times_k B \to X$ and $T_i \in \text{Coh}(X \times_k B)$ is a $B$-flat family of 0-dimensional sheaves of length $n_i$ over $X$ for all $i = 1, \ldots, d$. Two 'nested quotients'

$$E_B \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1, \quad E_B \twoheadrightarrow T_d' \twoheadrightarrow \cdots \twoheadrightarrow T_1'$$

are considered isomorphic when $\ker(E_B \twoheadrightarrow T_1) = \ker(E_B \twoheadrightarrow T_1')$ for all $i = 1, \ldots, d$.

The representability of the functor $\text{Quot}_X(E, n)$ can be proved adapting the proof of [29, Theorem 4.5.1] or by an explicit induction on $d$ as in [21, Section 2.A.1]. We define $\text{Quot}_X(E, n)$ to be the moduli scheme representing the above functor. Its closed points are then in bijection with the set of isomorphism classes of nested quotients

$$E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1.$$
where each $T_i \in \text{Coh}(X)$ is a 0-dimensional quotient of $E$ of length $n_i$. The nested Quot scheme comes with a closed immersion

$$\text{Quot}_X(E, n) \hookrightarrow \prod_{i=1}^{d} \text{Quot}_X(E, n_i)$$

cut out by the nesting condition $\ker(E \to T_d) \hookrightarrow \ker(E \to T_{d-1}) \hookrightarrow \cdots \hookrightarrow \ker(E \to T_1)$. In particular, it is projective as soon as $X$ is projective. If $C$ is a smooth proper curve over $\mathbb{C}$ and $E \in \text{Coh}(C)$ is a locally free sheaf, the cohomology of $\text{Quot}_C(E, n)$ was studied by Mochizuki [24].

**Example 1.1.** The classical Quot scheme $\text{Quot}_X(E, n)$ of length $n$ quotients of $E$ is obtained by setting $n = (n)$, i.e. taking $d = 1$ and $n_d = n$. If we set $n = (1 \leq 2 \leq \cdots \leq d)$, we obtain Mochizuki’s complete Filt scheme $\text{Filt}(E, d)$, which for $d = 1$ reduces to $\text{Filt}(E, 1) = \mathbb{P}(E)$ [24]. When $E = \mathcal{O}_X$, we use the notation $\text{Hilb}^n(X)$ to denote $\text{Quot}_X(\mathcal{O}_X, n)$. This space is the nested Hilbert scheme of points, studied extensively by Cheah [9, 8, 10].

1.2. **Support map and nested punctual Quot scheme.** Fix a variety $X$, a coherent sheaf $E$ and a $d$-tuple of non-negative integers $n = (n_1 \leq \cdots \leq n_d)$ for some $d > 0$. Composing the embedding (1.1) with the usual Quot-to-Chow morphisms yields the **support map**

$$h_{E,n}: \text{Quot}_X(E, n) \to \prod_{i=1}^{d} \text{Sym}^{n_i}(X)$$

recording the 0-cycles $[\text{Supp } T_i \in \text{Sym}^{n}(X)]_{i \leq i \leq d}$ attached to a $d$-tuple $(E \to T_i)_{1 \leq i \leq d}$. Here, $\text{Sym}^n X = X^n / \mathfrak{S}_m$ is the $m$-th symmetric power of $X$.

We make the following definition.

**Definition 1.2** (Nested punctual Quot scheme). Let $X$ be a variety, $x \in X$ a point, $E \in \text{Coh}(X)$ a coherent sheaf, $n = (n_1 \leq \cdots \leq n_d)$ a tuple of non-negative integers. The **nested punctual Quot scheme** attached to $(X, E, n, x)$ is the closed subscheme

$$\text{Quot}_X(E, n)_x \subset \text{Quot}_X(E, n),$$

defined as the preimage of the cycle $(n_1 x, \ldots, n_d x)$ along the support map $h_{E,n}$.

The name ‘punctual’ refers, as for the classical Quot schemes, to the fact that all quotients are entirely supported at a single point. We will not need the following result.

**Lemma 1.3.** Let $X$ be a smooth quasi-projective variety of dimension $m$, and let $E$ be a locally free sheaf of rank $r$ on $X$. For every $d$-tuple $n = (n_1 \leq \cdots \leq n_d)$, and for every $x \in X$, one has a non-canonical isomorphism

$$\text{Quot}_X(E, n)_x \cong \text{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\otimes r}, n)_0.$$**

**Proof.** The result follows from the isomorphism $\text{Quot}_X(E, k)_x \cong \text{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\otimes r}, k)_0$ relating the classical punctual Quot schemes, which is proved in full detail in [27, Section 2.1] exploiting a choice of étale coordinates around $x$ (which exist by the smoothness assumption, and which explain the non-canonical nature of the isomorphism). It remains to observe that the induced isomorphism

$$\prod_{i=1}^{d} \text{Quot}_X(E, n_i)_x \cong \prod_{i=1}^{d} \text{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\otimes r}, n_i)_0$$

maps the subscheme $\text{Quot}_X(E, n)_x$ isomorphically onto $\text{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\otimes r}, n)_0$. □
1.3. Connectedness. We prove the following connectedness result for the nested Quot scheme. A proof in the case \((r, d, n) = (1, 1, n)\) of the classical Hilbert scheme was first given by Hartshorne [19], and by Fogarty in the surface case [16]. We shall also exploit Cheah’s connectedness result for \(\text{Hilb}^m(X)\), see [9, Sec. 0.4].

**Theorem 1.4.** If \(X\) is an irreducible quasiprojective \(k\)-variety and \(E\) is a locally free sheaf on \(X\), then \(\text{Quot}_X(E, n)\) is connected for every \(n = (n_1 \leq \cdots \leq n_d)\). In particular, the classical Quot scheme \(\text{Quot}_X(E, n)\) is connected for every \(n \geq 0\).

**Proof.** The proof consists of several steps.

**Step 1:** We reduce to proving the statement when \(E = \mathcal{O}_X^{\oplus r}\) is trivial. Let \(x = [E \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1] \in \text{Quot}_X(E, n)\) be a point, where \(E\) is arbitrary. Since \(T_d\) is 0-dimensional we can find an open neighbourhood \(U \subset X\) of the set-theoretic support of \(T_d\) such that \(E|_U = \mathcal{O}_U^{\oplus r}\) is trivial. The point \(x\) then lies in the image of the open immersion \(\text{Quot}_U(\mathcal{O}_U^{\oplus r}, n) \hookrightarrow \text{Quot}_X(E, n)\). By assumption, the space \(\text{Quot}_U(\mathcal{O}_U^{\oplus r}, n)\) is connected. Now if \(x' = [E \twoheadrightarrow T_d' \twoheadrightarrow \cdots \twoheadrightarrow T_1'] \in \text{Quot}_X(E, n)\) is another point, we can find another open subset \(U' \subset X\) surrounding the support of \(T_d'\) and trivialising \(E\). Since \(X\) is irreducible, \(U \cap U' \neq \emptyset\), which implies \(\text{Quot}_U(\mathcal{O}_U^{\oplus r}, n) \cap \text{Quot}_U(\mathcal{O}_U^{\oplus r}, n) \neq \emptyset\), so \(x\) and \(x'\) are connected in \(\text{Quot}_X(E, n)\) by any point in this intersection.

**Step 2:** The scheme \(\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)\) has a framing \(T\)-action with non-empty fixed locus, where \(T = G_m^r\) (see Proposition 3.1 for an explicit description of this fixed locus: we shall exploit it in the next step). Let \(x \in \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)\) be an arbitrary point. Then the closure of its orbit contains a \(T\)-fixed point — this will be explained in Section 3. Therefore it is enough to prove that any two \(T\)-fixed points \(x, x' \in \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)^T\) are connected in \(\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)\).

**Step 3:** In principle, we should check connectedness for an arbitrary pair \((x, x')\) of \(T\)-fixed points

\[
x = [\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1] \in \prod_{a=1}^d \text{Hilb}^{n_a}(X) \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)^T,
\]

\[
x' = [\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_d' \twoheadrightarrow \cdots \twoheadrightarrow T_1'] \in \prod_{a=1}^d \text{Hilb}^{n_a}(X) \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)^T,
\]

where \(\sum_{1 \leq a \leq r} n_a = n = \sum_{1 \leq a \leq r} n_a'.\) But since each nested Hilbert scheme \(\text{Hilb}^m(X)\) is connected (cf. [9, Sec. 0.4]), we can in fact choose a pair of convenient \(x\) and \(x'\). We fix them satisfying the condition that \(\text{Supp}(T_d), \text{Supp}(T_d')\) consist of \(n_d\) distinct points. When viewed in the full space \(\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)\), the points \(x\) and \(x'\) both belong to the open subset

\[
U \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n),
\]

defined by the cartesian diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\overset{\text{open}}{\square}} & \prod_{i=1}^d (\text{Sym}^{n_i} X \setminus \Delta_{\text{big}}) \\
\text{Quot}_X(\mathcal{O}_X^{\oplus r}, n) & \overset{\text{h}_{\mathcal{O}_{\mathcal{X}}}^r}{\longrightarrow} & \prod_{i=1}^d \text{Sym}^{n_i} X
\end{array}
\]

where \(\Delta_{\text{big}} \subset \text{Sym}^{n_i} X\) is the big diagonal and the bottom map is the support map (1.2). In other words, \(U \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n)\) is the open subscheme consisting of the flags of quotients \([\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_d \twoheadrightarrow \cdots \twoheadrightarrow T_1]\) where each \(T_i\) is supported on \(n_i\) distinct points. This yields an open immersion

\[
U \hookrightarrow \prod_{i=1}^d V_i,
\]

where \(V_i \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n_i - n_{i-1})\) is the open subscheme consisting of points \([\mathcal{O}_X^{\oplus r} \twoheadrightarrow T_i']\) where the quotients \(T_i'\) are supported on \(n_i - n_{i-1}\) distinct points (and we set \(n_0 = 0\)). The scheme \(V_i\) is the image
of the étale map (cf. [2, Proposition A.3])

$$A_i \hookrightarrow \text{Quot}_X(\mathcal{O}_X^{\oplus r}, n_i - n_{i-1})$$

defined on the open subscheme

$$A_i \subset \text{Quot}_X(\mathcal{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}}$$

parametrising quotients $(\mathcal{O}_X^{\oplus r} \rightarrow \mathcal{O}_x)_k$ with $x_k \neq x_i$ for every $k \neq l$. On the other hand,

$$\text{Quot}_X(\mathcal{O}_X^{\oplus r}, 1)^{n_i - n_{i-1}} \cong \mathbb{P}(\mathcal{O}_X^{\oplus r})^{n_i - n_{i-1}} \cong (X \times_k \mathbb{P}^{r-1})^{n_i - n_{i-1}}$$

is irreducible, hence $A_i$ is irreducible, and in particular $V_i$ is irreducible, being the image of an irreducible space along a continuous map. Therefore $U \hookrightarrow \prod_i V_i$ is also irreducible, in particular connected, which completes the proof.

2. TANGENT SPACE AND SMOOTHNESS IN THE CASE OF CURVES

Fix $(X, E, n)$ as in the previous section. For any point $x \in \text{Quot}_X(E, n)$ representing a $d$-tuple of nested quotients

$$E \longrightarrow T_d \xrightarrow{p_{d-1}} T_{d-1} \xrightarrow{p_{d-2}} \cdots \xrightarrow{p_1} T_2 \xrightarrow{p_0} T_1$$

we set $K_i = \ker(E \rightarrow T_i)$. We have a flag of subsheaves

$$K_d \hookrightarrow K_{d-1} \hookrightarrow \cdots \hookrightarrow K_2 \hookrightarrow K_1 \hookrightarrow E$$

and, for any $i = 1, \ldots, d - 1$, maps

$$\phi_i : \text{Hom}_X(K_i, T_i) \rightarrow \text{Hom}_X(K_{i+1}, T_i), \quad g \mapsto g \circ t_i$$

$$\psi_i : \text{Hom}_X(K_{i+1}, T_{i+1}) \rightarrow \text{Hom}_X(K_{i+1}, T_i), \quad h \mapsto p_i \circ h$$

which we assemble in a matrix

$$\Delta_x = \begin{pmatrix}
-\phi_1 & \psi_1 & 0 & 0 & \cdots & 0 \\
0 & -\phi_2 & \psi_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\phi_{d-1} & \psi_{d-1}
\end{pmatrix}$$

defining a map

$$\Delta_x : \bigoplus_{i=1}^d \text{Hom}_X(K_i, T_i) \longrightarrow \bigoplus_{i=1}^{d-1} \text{Hom}_X(K_{i+1}, T_i).$$

The embedding (1.1) induces a $\mathbf{k}$-linear inclusion of tangent spaces

$$T_x \text{Quot}_X(E, n) \hookrightarrow \bigoplus_{i=1}^d \text{Hom}_X(K_i, T_i),$$

which can be described as follows: a $d$-tuple of maps $(\delta_1, \ldots, \delta_d) \in \bigoplus_{i=1}^d \text{Hom}_X(K_i, T_i)$ belongs to the tangent space of $\text{Quot}_X(E, n)$ at $x$ precisely when the diagram

$$\begin{array}{ccccccccc}
K_d & \xrightarrow{\delta_d} & K_{d-1} & \xrightarrow{\delta_{d-1}} & \cdots & \xrightarrow{\delta_2} & K_2 & \xrightarrow{\delta_1} & K_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
T_d & \xrightarrow{p_{d-1}} & T_{d-1} & \xrightarrow{p_{d-2}} & \cdots & \xrightarrow{p_1} & T_2 & \xrightarrow{p_0} & T_1
\end{array}$$

(2.1)

commutes. This is formalised in terms of the map $\Delta_x$ in the next proposition.
Proposition 2.1. Set \( n = (n_1 \leq \cdots \leq n_d) \). The tangent space of \( \text{Quot}_X(E, n) \) at a point \( x = [E \dashrightarrow T_d \dashrightarrow \cdots \dashrightarrow T_1] \) is

\[
T_x \text{Quot}_X(E, n) = \ker \left( \bigoplus_{i=1}^{d} \text{Hom}(K_i, T_i) \xrightarrow{\Delta} \bigoplus_{i=1}^{d-1} \text{Hom}(K_{i+1}, T_i) \right).
\]

In particular, if \( E \) is locally free of rank \( r \) on a smooth curve \( C \), then we have that \( \text{Quot}_C(E, n) \) is smooth of dimension \( r \cdot n_d \).

Proof. Along the same lines of [29, Prop. 4.5.3(i)] it is easy to see that the tangent space is given by the maps making Diagram (2.1) commute, which is equivalent to belonging to the kernel of \( \Delta_x \).

Let \( Q_i \) be the 0-dimensional sheaf fitting in the exact sequences

\[
0 \to K_i \to K_{i-1} \to Q_i \to 0
\]

for every \( i = 1, \ldots, d \). If \( X = C \) is a smooth curve, we have that each \( K_i \) is a locally free sheaf of rank \( r \) (because torsion free is equivalent to locally free on smooth curves); since \( Q_i \) is a 0-dimensional sheaf, we obtain the vanishing

\[
\text{Ext}^j_C(K_i, T_i) = \text{Ext}^j_C(K_{i+1}, T_i) = \text{Ext}^j_C(K_i, Q_i) = 0, \quad j > 0.
\]

Therefore each \( \psi_i \) is a surjective map, which implies that \( \Delta_x \) is surjective and that the dimension of the tangent space is computed as

\[
\text{dim}_k T_x \text{Quot}_C(E, n) = \text{dim}_k \left( \bigoplus_{i=1}^{d} \text{Hom}(K_i, T_i) \right) - \text{dim}_k \left( \bigoplus_{i=1}^{d-1} \text{Hom}(K_{i+1}, T_i) \right)
= \sum_{i=1}^{d} r n_i - \sum_{i=1}^{d-1} r n_i
= r n_d.
\]

Since the tangent space dimension is constant and \( \text{Quot}_C(E, n) \) is connected by Theorem 1.4, it is enough to find a smooth open subset \( U \subset \text{Quot}_C(E, n) \) of dimension \( r n_d \). We shall exploit the fact that the classical Quot scheme \( \text{Quot}_C(E, m) \) is smooth of dimension \( r m \), which follows from standard deformation theory and the vanishing \( \text{Ext}^1_C(K, T) = H^1(C, K^\vee \otimes T) = 0 \) for an arbitrary point \([K \dashrightarrow E \dashrightarrow T] \in \text{Quot}_C(E, m)\).

Let \( U \subset \text{Quot}_C(E, n) \) be the open subscheme as in Diagram (1.3) (which of course exists for arbitrary \( E \)), and write \( U \cong \prod_{i=1}^{d} V_i \) as in the proof of Theorem 1.4. We know that each \( V_i \subset \text{Quot}_C(E, n_i - n_{i-1}) \) is smooth of dimension \( r \cdot (n_i - n_{i-1}) \), therefore \( U \) is smooth of dimension \( r n_d \) as required.

Remark 2.2. The smoothness of \( \text{Quot}_C(E, n) \) was already proved by Mochizuki [24, Prop. 2.1], via a tangent-obstruction theory argument. See also [25] for the classification of smoothness of \( \text{Quot}_X(E, n) \) when \( X \) has arbitrary dimension.

3. Bialynicki-Birula decomposition

Let \( E \) be a locally free sheaf of rank \( r \) on a variety \( X \). Assume that \( E = \bigoplus_{a=1}^{r} L_a \) splits into a sum of line bundles on \( X \). Then \( \text{Quot}_X(E, n) \) admits the action of the algebraic torus \( T \cong G_m^r \) as in [4]. Indeed, \( T \) acts diagonally on the product \( \prod_{i=1}^{d} \text{Quot}_X(E, n_i) \) and the closed subscheme \( \text{Quot}_X(E, n) \) is \( T \)-invariant. Its fixed locus is determined by a straightforward generalisation of the main result of [4].

Proposition 3.1. If \( E = \bigoplus_{a=1}^{r} L_a \), there is a scheme-theoretic identity

\[
\text{Quot}_X(E, n)_T = \bigsqcup_{n_1 + \cdots + n_r = n} \prod_{a=1}^{r} \text{Quot}_X(L_a, n_a).
\]
Proof. We construct a bijection on $k$-valued points, which is straightforward to verify in families.

Fix tuples $n_a = (n_{a,i})$ such that $n_i = \sum_{a \in A} n_{a,i}$ for every $i = 1, \ldots, d$. An element of the connected component corresponding to $(n_1, \ldots, n_d)$ in the right hand side is a tuple of nested quotients

$$\left( [L_a \rightarrow T_d^{(a)} \rightarrow \cdots \rightarrow T_1^{(a)}] \right)_{a \in A},$$

where each $T_i^{(a)}$ is the structure sheaf of a finite subscheme of $X$ of length $n_{a,i}$. By Białynicki’s theorem on the $T$-fixed locus of ordinary Quot schemes, we have that

$$(3.1) \quad \bigoplus_{a \in A} \left( L_a \rightarrow T_i^{(a)} \right) \in \text{Quot}_X(E, n_i)^T$$

for each $i = 1, \ldots, d$, and since each of the original tuples of quotients was nested according to $n$, it follows that also the tuples (3.1) are nested according to $n$, and this proves that (3.1) defines a point in $\text{Quot}_X(E, n)^T$.

The reverse inclusion follows by an analogous reasoning relying once more on Białynicki’s result [4].

\[ \square \]

Remark 3.2. For a locally free sheaf $L$ of rank 1, we naturally have the isomorphism

$$\text{Quot}_X(L, n) \cong \text{Hilb}^n(X),$$

where Hilb$^n(X)$ is the nested Hilbert scheme of points, see for example [9]. Moreover, if $X = C$ is a smooth quasiprojective curve, we have (see [9, Sec. 0.2])

$$\text{Hilb}^n(C) \cong \text{Sym}^{n_1}(C) \times \text{Sym}^{n_2-n_1}(C) \times \cdots \times \text{Sym}^{n_{d-1}}(C).$$

Assume now $X = C$ is a smooth quasiprojective curve and let $x \in \text{Quot}_C(E, n)^T$ be a $T$-fixed point, corresponding to the tuple

$$(3.3) \quad \left( [L_a \rightarrow T_d^{(a)} \rightarrow \cdots \rightarrow T_1^{(a)}] \right)_a \in \bigoplus_{a \in A} \text{Quot}_C(L_a, n_a).$$

Set $K_i^{(a)} = \ker(L_a \rightarrow T_i^{(a)})$. The tangent space at $x$ can be written as

$$T_x \text{Quot}_C(E, n) = \ker \left( \bigoplus_{1 \leq a \leq r \leq s} d \Hom_C(K_i^{(a)} \otimes w_\alpha, T_i^{(b)} \otimes w_\beta) \sim \bigoplus_{1 \leq a \leq r \leq s} d-1 \Hom_C(K_i^{(a)} \otimes w_\alpha, T_i^{(b)} \otimes w_\beta) \right).$$

Denote by $w_1, \ldots, w_s$ the coordinates of the algebraic torus $T$, which we see as irreducible $T$-characters. As a $T$-representation, the tangent space admits the following weight decomposition

$$T_x \text{Quot}_C(E, n) = \ker \left( \bigoplus_{1 \leq a \leq r \leq s} d \Hom_C(K_i^{(a)} \otimes w_\alpha, T_i^{(b)} \otimes w_\beta) \sim \bigoplus_{1 \leq a \leq r \leq s} d-1 \Hom_C(K_i^{(a)} \otimes w_\alpha, T_i^{(b)} \otimes w_\beta) \right).$$

We recall the classical result of Białynicki-Birula (see [3, Section 4]), by which we obtain a decomposition of $\text{Quot}_X(E, n)$ in the case when $E$ is completely decomposable.

Theorem 3.3 (Białynicki-Birula). Let $X$ be a smooth projective scheme with a $G_m$-action and let $\{ X_i \}$ be the connected components of the $G_m$-fixed locus $X^{G_m} \subset X$. Then there exists a locally closed stratification $X = \bigsqcup X_i^+$, such that each $X_i^+ \rightarrow X_i$ is an affine fibre bundle. Moreover, for every closed point $x \in X_i$, the tangent space is given by $T_x(X_i^+) = T_x(X_i)^{\text{fix}} \oplus T_x(X_i)^+$, where $T_x(X_i)^{\text{fix}}$ (resp. $T_x(X_i)^+$) denotes the $G_m$-fixed (resp. positive) part of $T_x(X)$. In particular, the relative dimension of $X_i^+ \rightarrow X_i$ is equal to $\dim T_x(X_i)^+$ for $x \in X_i$.

The Białynicki-Birula “strata” are constructed as follows. If $t$ denotes the coordinate of $G_m$, we have

$$X_i^+ = \left\{ x \in X \mid \lim_{t \rightarrow 0} t \cdot x \in X_i \right\}.$$

In particular, the properness assumption assures that the closure of each $G_m$-orbit in $X$ contains the $G_m$-fixed point $\lim_{t \rightarrow 0} t \cdot x$. Recently Jelisiejew–Sienkiewicz [22] generalised Theorem 3.3, proving the
the $X^+_C$ always exists even when $X$ is not projective (or even not smooth). However, in the smooth case they cover $X$ as long as the closure of every $G_m$-orbit contains a fixed point.

We now determine a Białynicki-Birula decomposition for $\text{Quot}_C(E, n)$, where $C$ is a smooth quasiprojective curve. See Mochizuki’s paper [24, Section 2.3.4] for an equivalent construction and tangent space calculation (in the projective case), using a slightly different, but technically equivalent, tangent complex.\footnote{We thank Takuro Mochizuki for kindly sharing with us a note proving that the tangent complex used in [24] is quasi-isomorphic to the one encoded by the map $\Delta_x$.}

Let $G_m \hookrightarrow T$ be the generic 1-parameter subtorus given by $w \mapsto (w, w^2, \ldots, w^r)$; it is clear that $\text{Quot}_C(E, n)^T = \text{Quot}_C(E, n)_{G_m}^\times$. Let

$$Q_n = \bigcap_{a=1}^r \text{Quot}_C(L_a, n_a) \subseteq \text{Quot}_C(E, n)_{G_m}^\times$$

be the connected component of the fixed locus corresponding to the $r$-tuple $n = (n_a)_{1 \leq a \leq r}$, decomposing $n_1 + \cdots + n_r = n$.

**Proposition 3.4.** Let $C$ be a smooth quasiprojective curve and $E = \bigoplus_{a=1}^r L_a$. Then the nested Quot scheme admits a locally closed stratification

$$\text{Quot}_C(E, n) = \bigcup_n Q_n^+$$

where $n = (n_a)_{1 \leq a \leq r}$ are such that $n_1 + \cdots + n_r = n$ and $Q_n^+ \hookrightarrow Q_n$ is an affine fibre bundle of relative dimension $\sum_{1 \leq a \leq r} (a-1)n_{a,d}$.

**Proof.** The strata $Q_n^+$ are induced by Theorem 3.3.3; we just need to show that the closure of every orbit contains a fixed point. Choose a compactification $C \hookrightarrow \overline{C}$, an extension $\overline{T}_a$ of each line bundle $L_a$ and consider the induced open immersion

$$\text{Quot}_C\left(\bigoplus_{a=1}^r L_a, n\right) \hookrightarrow \text{Quot}_{\overline{C}}\left(\bigoplus_{a=1}^r \overline{T}_a, n\right).$$

The closure of every orbit must contain a fixed point in $\text{Quot}_C\left(\bigoplus_{a=1}^r \overline{T}_a, n\right)$, but the $G_m$-action does not move the support of a nested quotient, by which we conclude that such a fixed point had to be already contained in $\text{Quot}_C\left(\bigoplus_{a=1}^r L_a, n\right)$.

Let $x \in Q_n$ be a fixed point as in (3.3). The positive part of the tangent space (3.4) is

$$T^+_x \text{Quot}_C(E, n) = \ker \left( \bigoplus_{a<\beta} \bigoplus_{i=1}^d \text{Hom}_C(K^a_i, T^\beta_i) \xrightarrow{\Delta^+_x} \bigoplus_{a<\beta} \bigoplus_{i=1}^{d-1} \text{Hom}_C(K^a_{i+1}, T^\beta_i) \right),$$

where $\Delta^+_x$ is the restriction of the map $\Delta_x$. Thanks to the vanishings (2.2), $\Delta^+_x$ is surjective, therefore the relative dimension is computed as

$$\dim_k T^+_x \text{Quot}_C(E, n) = \dim_k \left( \bigoplus_{a<\beta} \bigoplus_{i=1}^d \text{Hom}_C(K^a_i, T^\beta_i) \right) - \dim_k \left( \bigoplus_{a<\beta} \bigoplus_{i=1}^{d-1} \text{Hom}_C(K^a_{i+1}, T^\beta_i) \right)$$

$$= \sum_{a<\beta} \left( \sum_{i=1}^d n_{a,i} - \sum_{i=1}^{d-1} n_{a,i} \right)$$

$$= \sum_{a<\beta} \left( \sum_{i=1}^d n_{a,i} - \sum_{i=1}^{d-1} n_{a,i} \right)$$

$$= \sum_{\beta=1}^r (\beta - 1)n_{a,d}$$

where we used $n_{a,i} = \dim_k \text{Hom}_C(K^a_i, T^\beta_i)$ since $K^a_i = \ker(L_a \twoheadrightarrow T^\beta_i)$ has rank 1. The proof is complete. $\square$
4. **The motive of the nested Quot scheme on a curve**

**4.1. Grothendieck ring of varieties.** Let $B$ be a scheme locally of finite type over $k$. The *Grothendieck group of $B$-varieties*, denoted $K_0(\operatorname{Var}_B)$, is defined to be the free abelian group generated by isomorphism classes $[X \to B]$ of finite type $B$-varieties, modulo the scissor relations, namely the identities $[h \colon X \to B] = [h_2 \colon Z \to B] + [h_1 \colon X \setminus Z \to B]$ whenever $Z \hookrightarrow X$ is a closed $B$-subvariety of $X$. The neutral element for the addition operation is the class of the empty variety. The operation

$$[X \to B] \cdot [X' \to B] = [X \times_B X' \to B]$$

defines a ring structure on $K_0(\operatorname{Var}_B)$, with identity $1_B = [\text{id} : B \to B]$. Therefore $K_0(\operatorname{Var}_B)$ is called the *Grothendieck ring of $B$-varieties*. If $B = \operatorname{Spec} k$, we write $K_0(\operatorname{Var}_k)$ instead of $K_0(\operatorname{Var}_{\operatorname{Spec} k})$, and we shorten $[X] = [X \to \operatorname{Spec} k]$ for every $k$-variety $X$.

The main rules for calculations in $K_0(\operatorname{Var}_k)$ are the following:

1. If $X \to Y$ is a geometric bijection, i.e. a bijective morphism, then $[X] = [Y]$.
2. If $X \to Y$ is Zariski locally trivial with fibre $F$, then $[X] = [Y] \cdot [F]$.

These are, indeed, the only properties that we will use.

The *Lefschetz motive* is the class $L = [\mathbb{A}_k^1] \in K_0(\operatorname{Var}_k)$. It can be used to express, for instance, the class of the projective space, namely $[\mathbb{P}^n_k] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n \in K_0(\operatorname{Var}_k)$.

**4.2. Independence of the vector bundle.** The following result generalises [27, Corollary 2.5], which in turn generalises the main theorem of [1] extending it from proper smooth curves to arbitrary smooth varieties.

**Proposition 4.1.** Let $E$ be a locally free sheaf of rank $r$ on a $k$-variety $X$. For every $n$, the motivic class of $\operatorname{Quot}_X(E, n)$ is independent of $E$, that is

$$[\operatorname{Quot}_X(E, n)] = [\operatorname{Quot}_X(\mathcal{O}_X^{\oplus n}, n)] \in K_0(\operatorname{Var}_k).$$

**Proof.** Let $(U_k)_{1 \leq k \leq e}$ be a Zariski open cover trivialising $E$. We can refine it to a locally closed stratification $X = W_1 \cup W_2 \cdots \cup W_e$, so that in particular $E|_{W_k} = \mathcal{O}_W^{\oplus n}$ for every $k$. Each $W_k$ is taken with the reduced induced scheme structure.

Let $\operatorname{Quot}_{X,W_k}(E, n) \subset \operatorname{Quot}_X(E, n)$ be the preimage of $\operatorname{Sym}^{\oplus n}(W_k) \subset \operatorname{Sym}^{\oplus n}(X)$ along the projection

$$\operatorname{pr}_d \circ h_{E,n} : \operatorname{Quot}_X(E, n) \to \prod_{i=1}^d \operatorname{Sym}^{\oplus n}(X) \to \operatorname{Sym}^{\oplus n}(X),$$

where $h_{E,n}$ is the support map (1.2). We endow $\operatorname{Quot}_{X,W_k}(E, n)$ with the reduced scheme structure. We have a geometric bijection

$$\bigcap_{i=1}^e \operatorname{Quot}_{X,W_k}(E, n) \to \operatorname{Quot}_X(E, n),$$

therefore the motive $[\operatorname{Quot}_X(E, n)]$ is computed entirely in terms of the motives $[\operatorname{Quot}_{X,W_k}(E, n)]$. It is enough to prove that these are independent of $E$. In the cartesian diagram

$$\begin{array}{ccc}
\operatorname{Quot}_{U_k,W_k}(E|_{U_k}, n_k) & \to & \operatorname{Quot}_{X,W_k}(E, n_k) \\
\downarrow & & \downarrow \\
\operatorname{Quot}_{U_k}(E|_{U_k}, n_k) & \to & \operatorname{Quot}_X(E, n_k)
\end{array}$$

the open immersion $j$ is in fact surjective, hence an isomorphism. But we can repeat this process with $\mathcal{O}_X^{\oplus n}$ in the place of $E$. It follows that

$$\operatorname{Quot}_{X,W_k}(E, n) \cong \operatorname{Quot}_{U_k,W_k}(\mathcal{O}_X^{\oplus n}, n_k) \cong \operatorname{Quot}_X(\mathcal{O}_X^{\oplus n}, n_k),$$

which yields the result. \[\square\]
4.3. Proof of the main theorem. Let $X$ be a smooth quasi-projective variety and $E$ a locally free sheaf of rank $r$. Define
\[ Z_{X,r,d}(q) = \sum_{n} \left[ \text{Quot}_{X}(E, n) \right] q^n \in K_{\text{Gr}}[\text{Var}_{\mathbb{k}}][q_1, \ldots, q_d], \]
where $n = (n_1, \ldots, n_d)$ and we use the multivariable notation $q = (q_1, \ldots, q_d)$ and $q^n = \prod_{i=1}^{d} q^n_i$. The notation $Z_{X,r,d}$ reflects the independence on $E$ that we proved in Proposition 4.1. If $X = C$ is a smooth quasi-projective curve and $r = d = 1$, then $Z_{C,1,1}(q)$ is simply the Kapranov motivic zeta function
\[ Z_{C,1,1}(q) = \sum_{n \geq 0} \left[ \text{Sym}^n(C) \right] q^n. \]

We can now prove our main theorem, first stated in Theorem A in the Introduction.

**Theorem 4.2.** Let $C$ be a smooth quasi-projective curve. The generating series $Z_{C,r,d}(q)$ is a product of shifted motivic zeta functions: there is an identity
\[ Z_{C,r,d}(q) = \prod_{a=1}^{r} \prod_{i=1}^{d} \zeta_{C}(L^{a-1} q_i q_{i+1} \cdots q_d). \]
In particular, $Z_{C,r,d}(q)$ is a rational function in $q_1, \ldots, q_d$.

**Proof.** By Proposition 4.1 the motive $[\text{Quot}_{C}(E, n)]$ is independent on the vector bundle $E$, so we may assume $E = \mathcal{O}_C^\oplus r$. In this case, we may compute the motive exploiting the decomposition of $\text{Quot}_{C}(\mathcal{O}_C^\oplus r, n)$ given by Proposition 3.4. Every stratum is a Zariski locally trivial fibration over a connected component of the fixed locus, with fibre an affine space whose dimension we computed in Proposition 3.4.

In what follows, we denote by $n_a = (n_{a,1}, \ldots, n_{a,d})$ a nested tuple of non-negative integers and by $l_a = (l_{a,1}, \ldots, l_{a,d})$ a tuple of non-negative integers. Clearly the two collections of tuples are in bijection, by means of the correspondence
\[ (n_{a,1} \leq \cdots \leq n_{a,d}) \leftrightarrow (n_{a,1} - n_{a,2}, \ldots, n_{a,d} - n_{a,d-1}). \]
We compute
\[ \sum_{n} [\text{Quot}_{C}(\mathcal{O}_C^\oplus r, n)] q^n = \sum_{n} q^n \sum_{n_1 + \cdots + n_d = n} \prod_{a=1}^{r} [\text{Quot}_{C}(\mathcal{O}_C, n_a)] \cdot L^{(a-1)n_{a,d}} \quad \text{by Proposition 3.4} \]
\[ = \sum_{n_1, \ldots, n_d} \prod_{a=1}^{r} q^{n_a} \left[ \text{Hilb}^{n_a}(C) \right] \cdot L^{(a-1)n_{a,d}} \]
\[ = \sum_{l_1, \ldots, l_d} \prod_{a=1}^{r} \prod_{i=1}^{d} q^{l_i} \cdot \left[ \text{Hilb}^{l_i}(C) \right] \cdot L^{(a-1)\sum_{i=1}^{d} l_{i,a}} \quad \text{by (4.2)} \]
\[ = \sum_{l_1, \ldots, l_d} \prod_{a=1}^{r} \prod_{i=1}^{d} q^{l_i} \cdot \left[ \text{Sym}^{l_i}(C) \right] \cdot L^{(a-1)l_{i,a}} \quad \text{by (3.2)} \]
\[ = \sum_{l_1, \ldots, l_d} \prod_{a=1}^{r} \prod_{i=1}^{d} q^{l_i} \cdot \left[ \text{Sym}^{l_i}(C) \right] \cdot L^{(a-1)l_{i,a}} \]
\[ = \prod_{a=1}^{r} \prod_{i=1}^{d} \zeta_{C}(L^{a-1} q_i q_{i+1} \cdots q_d) \quad \text{by (4.1)} \]
The rationality follows by the rationality of the Kapranov zeta function, proved in [23, Theorem 1.1.9].

**Remark 4.3.** We can reformulate our main theorem in terms of the motivic exponential, for which a minimal background is provided in Appendix A. The case $r = d = 1$ yields the classical expression
\[ \zeta_{C}(q) = \text{Exp}_{\mathbb{k}}(\mathbb{C}[q]). \]
The general case becomes

\[ Z_{C,r,d}(q) = \text{Exp}_+ \left( \left[ C \times_k \mathbb{P}^{r-1}_k \sum_{i=1}^d \frac{1}{q_i q_{i+1} \cdots q_d} \right] \right) \]

Setting \( d = 1 \) we recover the calculations of [1, 27].

4.4. Hodge–Deligne Polynomial. In this subsection we work over \( k = \mathbb{C} \). Ring homomorphisms \( K_0(\text{Var}_C) \to R \) are called motivic measures. A typical example of a motivic measure is the Hodge–Deligne polynomial

\[ E: K_0(\text{Var}_C) \to \mathbb{Z}[u,v] \]

defined by sending the class \([Y]\) of a smooth projective variety to

\[ E(Y; u, v) = \sum_{p,q \geq 0} \dim_C H^p(Y, \Omega^q_Y)(-u)^p(-v)^q. \]

**Notation 4.4.** If \( f(u, v) = \sum_{i,j} p_{ij} u^i v^j \in \mathbb{Z}[u, v] \), we set

\[ (1-q)^{-f(u,v)} = \prod_{i,j} (1-u^i v^j q)^{-p_{ij}}. \]

This is actually the formula defining the power structure on \( \mathbb{Z}[u,v] \). The motivic measure \( E \) can be proved to be a morphism of rings with power structure, see [18] for full details.

Let \( C \) be a smooth projective curve of genus \( g \). We have

\[ E(\xi_C(q)) = \sum_{n \geq 0} E(\text{Sym}^n(C); u, v)q^n = (1-q)^{-E(C; u,v)} \]

\[ = (1-q)^{-g+g+g} \]

\[ = (1-uq)^g(1-vq)^g \]

\[ = \frac{(1-q)^g}{(1-q)(1-u v q)}. \]

For \( E \) a locally free sheaf of rank \( r \) over \( C \), define

\[ E_{C,r,d}(q) = \sum_n E(\text{Quot}_C(E; \mathbb{n}); u, v)q^n. \]

As a direct consequence of Theorem 4.2, we obtain the following corollary.

**Corollary 4.5.** There is an identity

\[ E_{C,r,d}(q) = \prod_{a=1}^r \prod_{i=1}^d \left( 1 - u^a v^{a-1} q_i q_{i+1} \cdots q_d \right)^g \left( 1 - v^{a-1} u^a q_i q_{i+1} \cdots q_d \right)^g \left( 1 - u^a v^{a-1} q_i q_{i+1} \cdots q_d \right)^g. \]

**Proof.** This follows by combining Theorem 4.2 and Equation (4.3) with one another, after observing that \( E \) is multiplicative (being a ring homomorphism) and sends \( L \) to \( u v \).

The generating function of the signed Poincaré polynomials is obtained from \( E_{C,r,d}(q) \) by setting \( u = v = 1 \). The result confirms a result of L. Chen [11] obtained in the case \( C = \mathbb{P}^1 \). The generating series of topological Euler characteristics is obtained from \( E_{C,r,d}(q) \) by setting \( u = v = 1 \), also in the quasiprojective case. So we obtain

\[ \sum_n e_{\text{top}}(\text{Quot}_C(E; \mathbb{n)})q^n = \prod_{i=1}^d \left( 1 - q_i q_{i+1} \cdots q_d \right)^{-r a_{\text{top}}(C)}. \]

By a beautiful result of Bittner [5], the classes of smooth projective varieties generate \( K_0(\text{Var}_k) \) as soon as \( \text{char} k = 0 \). But of course \( E \) can be defined on arbitrary varieties via mixed Hodge structures.
APPENDIX A. MOTIVIC EXPONENTIALS

If $\Lambda = \Lambda \times \Lambda \to \Lambda$ is a commutative monoid in the category of $k$-schemes, where $\mu : \Lambda \times \Lambda \to \Lambda$ is the multiplication map and $e : \text{Spec} k \to \Lambda$ is the identity element, then by [12, Example 3.5 (4)], one has a $\lambda$-ring structure $\sigma_{\mu}$ on the Grothendieck ring $K_0(\text{Var}_\lambda)$, determined by the operations

$$\sigma_{\mu}^{\Lambda}[Y \xrightarrow{f} \Lambda] = \left[\text{Sym}^{\sigma} Y \xrightarrow{\text{Sym}^{\sigma} f} \text{Sym}^{\sigma} \Lambda \xrightarrow{\mu} \Lambda\right].$$

Assume $\Lambda_+ \subset \Lambda$ is a sub-monoid such that $\bigsqcup_{n \geq 0} \Lambda_+^n \to \Lambda$ is of finite type. Then we can define the *motivic exponential*

$$\text{Exp}_{\mu} : K_0(\text{Var}_{\Lambda_+}) \to K_0(\text{Var}_{\Lambda})^\times$$

by setting

$$\text{Exp}_{\mu}(A) = \sum_{n \geq 0} \sigma_{\mu}^{\Lambda_+}(A)$$

for an effective class $A$, and extending via

$$\text{Exp}_{\mu}(A - B) = \text{Exp}_{\mu}(A) \cdot \text{Exp}_{\mu}(B)^{-1}$$

whenever $A$ and $B$ are effective. The map $\text{Exp}_{\mu}$ is injective. See [13, Section 1] for more details.

We use this construction in the case $(\Lambda, \mu, e) = (\mathbb{N}^d, +, 0)$, and setting $\Lambda_+ = \mathbb{N}^d \setminus 0$. Of course here we are seeing $\mathbb{N}^d$ as the $k$-scheme $\bigsqcup_{n \in \mathbb{N}^d} \text{Spec} k$. There is an isomorphism

$$K_0(\text{Var}_k)[[q_1, \ldots, q_d]] \xrightarrow{\sim} K_0(\text{Var}_{\mathbb{N}^d})$$

defined by sending

$$\sum_{n \in \mathbb{N}^d} Y_n \cdot q_1^{n_1} \cdots q_d^{n_d} \to \bigsqcup_{n \in \mathbb{N}^d} Y_n \to \text{Spec} k[n]$$

for varieties $Y_n$, and extending by linearity. Under this identification, if we let $m$ be the ideal generated by $(q_1, \ldots, q_d)$ in $K_0(\text{Var}_k)[[q_1, \ldots, q_d]]$, we can see $\text{Exp}_{+}$ as a group isomorphism

$$\text{Exp}_{+} : m \cdot K_0(\text{Var}_k)[[q_1, \ldots, q_d]] \sim 1 + m \cdot K_0(\text{Var}_k)[[q_1, \ldots, q_d]] \subset (K_0(\text{Var}_k)[[q_1, \ldots, q_d]])^\times$$

between an additive group (on the left) and a multiplicative group (on the right). In particular, one has the identity

$$\text{Exp}_{+}\left(\sum_{i=1}^s f_i(q_1, \ldots, q_d)\right) = \prod_{i=1}^s \text{Exp}_{+}(f_i(q_1, \ldots, q_d))$$

for $f_i(q_1, \ldots, q_d) \in m \cdot K_0(\text{Var}_k)[[q_1, \ldots, q_d]]$.

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