Acentralizers of Abelian groups of rank 2

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Abstract

Let $G$ be a group. The Acentralizer of an automorphism $\alpha$ of $G$, is the subgroup of fixed points of $\alpha$, i.e., $C_G(\alpha) = \{ g \in G \mid \alpha(g) = g \}$. We show that if $G$ is a finite Abelian $p$-group of rank 2, where $p$ is an odd prime, then the number of Acentralizers of $G$ is exactly the number of subgroups of $G$. More precisely, we show that for each subgroup $U$ of $G$, there exists an automorphism $\alpha$ of $G$ such that $C_G(\alpha) = U$. Also we find the Acentralizers of infinite two-generator Abelian groups.

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1. Introduction

Throughout the article, the usual notation will be used, for example $\mathbb{Z}_n$ denotes the cyclic group of integers modulo $n$, $\mathbb{Z}_n^*$ denotes the group of invertible elements of $\mathbb{Z}_n$. Let $G$ be a group. We denote $\text{cent}(G) = \{ C_G(g) \mid g \in G \}$, where $C_G(g)$ is the centralizer of the element $g$ in $G$. Then for any natural number $n$, a group is called $n$-centralizer if $|\text{cent}(G)| = n$. There are some results on finite $n$-centralizers groups (see [1–7,10,13,15]). The study of $n$-centralizer infinite groups was initiated in [9]. Let $\text{Aut}(G)$ be the group of automorphisms of $G$. If $\alpha \in \text{Aut}(G)$, then the Acentralizer of $\alpha$ in $G$ is defined as

$$C_G(\alpha) = \{ g \in G \mid \alpha(g) = g \}$$

which is a subgroup of $G$. In particular, if $\alpha = \tau_a$ is an inner automorphisms of $G$ induced by $a \in G$, then $C_G(\tau_a) = C_G(a)$ is the centralizer of $a$ in $G$. Let $\text{Acent}(G)$ be the set of Acentralizers of $G$, that is

$$\text{Acent}(G) = \{ C_G(\alpha) \mid \alpha \in \text{Aut}(G) \}.$$

The group $G$ is called $n$-Acentralizer, if $|\text{Acent}(G)| = n$.

It is obvious that $G$ is 1-Acentralizer group if and only if $G$ is a trivial group or $\mathbb{Z}_2$. Nasrabadi and Gholamian [12] proved that $G$ is 2-Acentralizer group if and only if $G \cong \mathbb{Z}_4, \mathbb{Z}_p$ or $\mathbb{Z}_{2p}$ for some odd prime $p$. Furthermore, they characterized 3, 4, 5-Acentralizer groups.

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Lemma 1.1 ([12]). Let $H$ and $T$ be finite groups. Then
\[ |\text{Acent}(H)| |\text{Acent}(T)| \leq |\text{Acent}(H \times T)|. \]
In addition if $|H|$ and $|T|$ are relatively prime, then
\[ |\text{Acent}(H)| |\text{Acent}(T)| = |\text{Acent}(H \times T)|. \]
Therefore, if $G$ is a finite nilpotent group of order $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_i$, $i = 1, \ldots, r$, are distinct primes and $k_i \geq 1$, then
\[ |\text{Acent}(G)| = \prod_{i=1}^{r} |\text{Acent}(G_{p_i})|, \]
where $G_{p_i}$'s are the Sylow $p_i$-subgroup of $G$.

Thus in order to find the number of Acentralizers of a finite nilpotent (in particular Abelian) group $G$, it is enough to find the number of Acentralizers of its Sylow subgroups.

In this paper we compute $|\text{Acent}(G)|$, considering $G$ to be a cyclic group of prime power order and of order $p_1^{k_1} \cdots p_r^{k_r}$, where $p_i$ for $i = 1, \ldots, r$ are distinct primes, an elementary Abelian group of prime power order, group of the form $\mathbb{Z}_{p^m} \times \mathbb{Z}_{n^r}$, where $m, n$ are positive integers and $p$ is a prime and finally a free Abelian group of rank 2.

2. Preliminaries

We begin with computing of Acentralizers of finite cyclic groups. We show that if $G$ is a cyclic group of odd order, then $|\text{Acent}(G)|$ is equal to the number of subgroups of $G$, while if $|G|$ is even, $|\text{Acent}(G)|$ is less than the number of subgroups of $G$.

Proposition 2.1. Let $G$ be a cyclic group of order $m = p_1^{k_1} \cdots p_r^{k_r}$, where $p_1 < p_2 < \cdots < p_r$ are distinct primes and $k_1, \ldots, k_r$ are positive integers. Then
\[ |\text{Acent}(G)| = \begin{cases} (k_1 + 1) \cdots (k_r + 1) & \text{if } p_1 \neq 2 \\ k_1 (k_2 + 1) \cdots (k_r + 1) & \text{if } p_1 = 2 \end{cases} \]

Proof. First let $G = \langle a \rangle$ be a cyclic group of order $p^n$, where $p$ is an odd prime and $n$ a positive integer. For every $0 \leq k \leq n$, let $G_k = \langle a^{p^{n-k}} \rangle$ be the unique subgroup of $G$ of order $p^k$. If $\alpha$ is defined as $\alpha(a) = a^{1+p^k}$, then $\alpha$ is an automorphism of $G$ and
\[ \alpha(a^{p^{n-k}}) = (a^{p^{n-k}})^{(1+p^k)} = a^{p^{n-k}} \]
and so $C_G(\alpha) = G_k$. Hence every subgroup of $G$ is an Acentralizer of $G$ and $|\text{Acent}(G)| = n + 1$. Similarly we can see that if $p = 2$, then every non-identity subgroup of $G$ is an Acentralizer of $G$ and $|\text{Acent}(G)| = n$.

Now suppose that $G$ is a cyclic group of order $m = p_1^{k_1} \cdots p_r^{k_r}$, where $p_i$, $i = 1, \ldots, r$, are distinct odd primes. Then, by Lemma 1.1, $|\text{Acent}(G)| = (k_1 + 1) \cdots (k_r + 1)$, which is the number of subgroups of $G$. Also if $G$ is a cyclic group of order $m = 2^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_i$, $i = 1, \ldots, r$, are distinct odd primes, then $|\text{Acent}(G)| = k_1 (k_2 + 1) \cdots (k_r + 1)$. Note that in this case the number of subgroups of $G$ is $(k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$. □

The following question arises naturally.

What is $|\text{Acent}(G)|$, where $G$ is a finite Abelian group?

We show that an elementary Abelian $p$-group $G$ is a $m$-Acentralizer group, where $m$ is the number of subgroups of $G$. The proof of the following result is well-known which is brought for completeness.
Proposition 2.2. Let $G$ be an elementary Abelian group of order $p^n$. Then $|\text{Acent}(G)|$ is the number of subgroups of $G$, that is

$$|\text{Acent}(G)| = c_0 + c_1 + \cdots + c_{n-1} + c_n,$$

where $c_0 = 1$ and $c_k = \frac{(p^n-1)(p^n-p)\cdots(p^n-p^{k-1})}{(p^k-1)(p^k-p)\cdots(p^k-p^{k-1})}$ for $k = 1, \ldots, n-1$.

**Proof.** We note that $G$ is a vector space over $\mathbb{Z}_p$ and for each $1 \leq k \leq n$, there are $c_k$ subspaces of dimension $k$ in $V$. To see this, first we count the number of $k$-element linearly independent subsets in $G$. Every such set generates a $k$-dimensional subspace of $G$. Let $\{v_1, \ldots, v_k\}$ be linearly independent. The vector $v_1$ (which is a non-zero vector) could be selected in $p^n - 1$ ways, the vector $v_2$ (which is not a multiple of $v_1$) in $p^n - p$ ways, $\ldots$, and $v_k$ (which is not a linear combination of $v_1, v_2, \ldots, v_{k-1}$) in $p^n - p^{k-1}$ ways. So there are $t = (p^n - 1)(p^n - p)\cdots(p^n - p^{k-1})$ linearly independent $k$-element subsets of $G$.

Every basis of $W := \text{span}\{v_1, \ldots, v_k\}$ generate the same subspace and, as shown above, there are $s = (p^k - 1)(p^k - p)\cdots(p^k - p^{k-1})$ basis of $W$. Therefore there are $t/s$ distinct $k$-dimensional subspaces of $G$.

We show that for every subspace $W$ of $V$, there exists $\alpha \in \text{Aut}(V)$ such that $\alpha$ induces identity just on $W$, that is $C_V(\alpha) = W$. Let $W$ be a $k$-dimensional subspace of $V$. Then there exists a subspace $U$ of $V$ such that $V = W \oplus U$. Let $\mathcal{S} = \{w_1, \ldots, w_k, u_1, \ldots, u_t\}$ be a basis of $V$, where $\{w_1, \ldots, w_k\}$ is a basis of $W$ and $\{u_1, \ldots, u_t\}$ be a basis of $U$. If $t \geq 2$, then we can define $\alpha$ on $V$ as follows: $\alpha(w_i) = w_i$, $i = 1, \ldots, k$, $\alpha(u_i) = u_1 + u_2$, $\alpha(u_i) = u_{i+1}$, $i = 2, \ldots, t - 1$, $\alpha(u_t) = u_1$. Thus $\alpha$ is an automorphism of $V$ inducing identity just on $W$. If $t = 1$ then $\alpha$ can defined as $\alpha(w_i) = w_i$, $i = 1, \ldots, k$, $\alpha(u_1) = u_1 + w_1$. If $t = 0$, that is $W = V$, then $\alpha$ is the identity automorphism.

We can generalize the above result. In fact we can show that if $V$ is a vector space over any field, then every subspace of $V$ is a centralizer of an automorphism of $V$.

We need to know the structure of subgroups of direct products. We briefly recall the discussions on pages 34-36 of [14] about subgroups of direct products. A subgroup $D$ of $G = H \times K$ such that $DH = G = DK$ and $D \cap H = \{1\} = D \cap K$ is called a diagonal in $G$ (with respect to $H$ and $K$). If $H \cong K$ and $\delta : H \rightarrow K$ is an isomorphism, then

$$D(\delta) = D(H, \delta) = \{x\delta(x) : x \in H\}$$

is a diagonal in $G$ (with respect to $H$ and $K$). Conversely, if $D$ is a diagonal in $G$ (with respect to $H$ and $K$), then there exists a unique isomorphism $\delta : H \rightarrow K$ such that $D = D(\delta)$. Thus there is a bijection between diagonals (with respect to $H$ and $K$) and isomorphisms of $H$ to $K$.

Every subgroup $U$ of a direct product $G = H \times K$ is a diagonal in a certain section of $G$. More precisely, there is natural isomorphism

$$\frac{UK \cap H}{U \cap H} \cong \frac{UK \cap K}{U \cap K}.$$  
Conversely, let $W_1 \leq U_H \leq H$ and $W_2 \leq U_K \leq K$ be subgroups of direct factors. For every isomorphism $\delta : \frac{U_H}{W_1} \rightarrow \frac{U_K}{W_2}$ there exists a subgroup $U \leq G$ such that $U_H = UK \cap H, U_K = UH \cap K, W_1 = U \cap H$ and $W_2 = U \cap K$, namely

$$U = D(U_H, \delta) = \{xy : x \in U_H, y \in \delta(xW_1)\}.$$  

Thus in order to recover the subgroups of $G = H \times K$ we need the isomorphisms between the sections (i.e., intervals in the subgroup lattice) in $\{[1], H\}$ respectively $\{[1], K\}$. Also every subgroup of a direct product $G = H \times K$ is a direct of the form $H_1 \times K_1$, where $H_1 \leq H$ and $K_1 \leq K$ or is a diagonal.

Set $G \cong \mathbb{Z}_{pm} \times \mathbb{Z}_{pn}$, where $1 \leq m \leq n$. First of all, we have the direct product of chains of length $m$ respectively $n$, that is, $(m+1)(n+1)$ subgroups. Second, we have $m$
sections of order \( p \) from the first direct factor and \( n \) sections of order \( p \) from the second direct factor. Thus for each pair of 1-segments correspond to the isomorphisms \( \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) of these sections we have \( p - 1 \) diagonals i.e., \( mn(p - 1) \).

Third, we have \( m - 1 \) sections of order \( p^2 \) from the first direct factor and \( n - 1 \) sections of order \( p^2 \) from the second direct factor. Thus for each pair of 2-segments correspond to the isomorphisms \( \mathbb{Z}_{p^2} \rightarrow \mathbb{Z}_{p^2} \) of these sections we have \( p^2 - p \) diagonals i.e., \( (m - 1)(n - 1)(p^2 - p) \).

In general, for every \( k \in \{0, 1, \ldots, n - m\} \), we have \( m - (k - 1) \) sections of order \( p^k \) from the first direct factor and \( n - (k - 1) \) sections of order \( p^k \) from the second direct factor. Thus for each pair of \( k \)-segments correspond to the isomorphisms \( \mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^k} \) of these sections we have \( p^k - p^{k-1} \) diagonals i.e., \( (m - k + 1)(n - k + 1)(p^k - p^{k-1}) \). Thus we have the following result.

**Theorem 2.3** ([8]). Let \( G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \) with \( m \leq n \). Then, the number of subgroups of \( G \) is

\[
(m + 1)(n + 1) + \sum_{k=0}^{m-1} (m - k)(n - k)(p^{k+1} - p^k).
\]

Fix an isomorphism

\[
G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}
\]

with \( 1 \leq m \leq n \) and let \( \mathbb{Z}_{p^m} \cong \langle a \rangle \), \( \mathbb{Z}_{p^n} \cong \langle b \rangle \). Given an endomorphism \( \alpha : G \rightarrow G \) we get \( \alpha(a) = a^i b^j \) and \( \alpha(b) = a^r b^s \), for some integers \( 0 \leq i, r < p^m \) and \( 0 \leq j, s < p^n \).

We indicate this situation by a matrix \( \begin{bmatrix} i & j \\ r & s \end{bmatrix} \). Observe that the relations \( a^{p^m} = 1 \) and \( b^{p^n} = 1 \) yield \( j \equiv 0 \pmod{p^{n-m}} \). Note that if \( n = m \), then certainly \( \text{Aut}(G) = \text{GL}_2(p^m) \), the group of invertible 2 \times 2 matrices over the ring \( \mathbb{Z}_{p^m} \) of integers \( \pmod{p^m} \).

**Theorem 2.4** ([11, Corollary 3]). Let \( G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \) with \( m < n \). Then, the matrix

\[
\begin{bmatrix} i & r \\ j & s \end{bmatrix}
\]

represents

1. an endomorphism of \( G \) if and only if \( i \in \mathbb{Z}_{p^m} \), \( j \equiv 0 \pmod{p^{n-m}} \), \( r \in \mathbb{Z}_{p^n} \) and \( s \in \mathbb{Z}_{p^n} \);
2. an automorphism of \( G \) if and only if \( i \in \mathbb{Z}_{p^m} \), \( j \equiv 0 \pmod{p^{n-m}} \), \( r \in \mathbb{Z}_{p^n} \) and \( s \in \mathbb{Z}_{p^n} \).

3. Main results

In this section we compute \( |\text{Acent}(G)| \), where \( G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \). First we show that if \( p \)

is odd, then \( |\text{Acent}(G)| \) is equal to total number of subgroups of \( G \).

**Theorem 3.1.** Let \( G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \), where \( m \leq n \) and \( p \) is odd prime, then \( |\text{Acent}(G)| \) is equal to the number of subgroups of \( G \), that is

\[
|\text{Acent}(G)| = (m + 1)(n + 1) + \sum_{k=0}^{m-1} (m - k)(n - k)(p^{k+1} - p^k).
\]

**Proof.** Let \( G = A \times B, A = \langle a \rangle \cong \mathbb{Z}_{p^m}, B = \langle b \rangle \cong \mathbb{Z}_{p^n}, m \leq n \). Let \( \alpha \) be an automorphism of \( G \) such that \( \alpha(a) = a^i b^j \) and \( \alpha(b) = a^r b^s \), where \( 0 \leq i, r < p^m \) and \( 0 \leq j, s < p^n \). By Theorem 2.4, \( \gcd(i, p^m) = 1 \), \( \gcd(s, p^n) = 1 \), and \( j \equiv 0 \pmod{p^{n-m}} \). Since

\[
\alpha(a^x b^y) = (\alpha(a))^x (\alpha(b))^y = (a^i b^j)^x (a^r b^s)^y = a^{ix+ry} b^{jx+sy}
\]


we have
$$C_G(\alpha) = \{a^xb^y \mid \alpha(a^xb^y) = a^xb^y\}$$
$$= \{a^xb^y \mid a^{ix+ry}b^{ix+sy} = a^xb^y\}.$$  
Hence the elements of $C_G(\alpha)$ is of the form $a^xb^y$, where $(x, y)$ is a solution of the following equation
$$\begin{cases} ix + ry = x \pmod{p^m}, \\ jx + sy = y \pmod{p^n} \end{cases}$$
that is
$$\begin{cases} (i - 1)x + ry = 0 \pmod{p^m}, \\ jx + (s - 1)y = 0 \pmod{p^n}. \end{cases}$$
(1)
Let $A_u = \langle a^{p^{m-u}} \rangle$ be the unique subgroup of $A$ of order $p^u$, $u = 0, 1, \ldots, m$; and let $B_v = \langle b^{p^{n-v}} \rangle$ be the unique subgroup of $B$ of order $p^v$, $v = 0, 1, \ldots, n$. Then $G$ has $(m + 1)(n + 1)$ subgroups of the form $A_u \times B_v$, $u = 0, \ldots, m$, $v = 0, \ldots, n$. For every $u = 0, \ldots, m$ and $v = 0, \ldots, n$ we find an automorphism $\alpha$ of $G$ inducing identity just on $A_u \times B_v$. If we choose $i = 1 + p^u$, $j = 0$, $r = 0$, and $s = 1 + p^v$ then $\alpha$ defined by $\alpha(a) = a^{1+p^u}$ and $\alpha(b) = b^{1+p^v}$ is an automorphism of $G$, such that $C_G(\alpha) = A_u \times B_v$.

For every $k = 1, \ldots, m$ we have the diagonals corresponding to the automorphisms $\mathbb{Z}_{p^k} \rightarrow \mathbb{Z}_{p^k}$, which give $p^k - p^{k-1}$ diagonals for each pair of $k$-segments. So there are $(m - k + 1)(n - k + 1)(p^k - p^{k-1})$ diagonal subgroups corresponding to the automorphisms between sections of order $p^k$. We find these subgroups explicitly and automorphisms of $G$ inducing identity just on these subgroups.

For every $u = k, \ldots, m$, $v = k, \ldots, n$ and for every $t$ with $\gcd(p^k, t) = 1$, the isomorphism
$$\delta_k : A_u/A_{u-k} \rightarrow B_v/B_{v-k}$$
$$a^{p^{m-u}}A_{u-k} \rightarrow b^{p^{n-v}}B_{v-k}$$
gives a diagonal subgroup
$$D_{u,v,t} = \{xy \mid x \in A_u, y \in \delta_k(xA_{u-k})\}$$
$$= \{a^{p^{m-u}\ell_1}y \mid 1 \leq \ell_1 \leq p^u, y \in b^{p^{n-v}\ell_1}B_{v-k}\}$$
$$= \{a^{p^{m-u}\ell_1}b^{p^{n-v}\ell_1}b^{p^{n-(v-k)}}\ell_2 \mid 1 \leq \ell_1 \leq p^u, 1 \leq \ell_2 \leq p^{v-k}\}$$
$$= \{a^{p^{m-u}\ell_1}b^{p^{n-v}(\ell_1 t + \ell_2 p^k)} \mid 1 \leq \ell_1 \leq p^u, 1 \leq \ell_2 \leq p^{v-k}\}.$$  
We find an automorphism of $G$ inducing identity just on $D_{u,v,t}$. We must choose $i, j, r, s$ such that $(p^{m-u}\ell_1, p^{n-v}(\ell_1 t + \ell_2 p^k))$ is a solution of (1) that is
$$\begin{cases} p^{m-u}(i - 1)\ell_1 + p^{n-v}(\ell_1 t + \ell_2 p^k) = 0 \pmod{p^m}, \\ p^{m-u}j\ell_1 + p^{n-v}(s - 1)(\ell_1 t + \ell_2 p^k) = 0 \pmod{p^n}. \end{cases}$$
If we choose $i = 1 + p^u$, $s = 1 + p^v$, $j = p^{n-m+u}$, and $r = p^{n-n+v}$, then $\alpha$, defined by $\alpha(a) = a^{1+p^u}b^{p^{m-u}}$ and $\alpha(b) = b^{1+p^v}$, is an automorphism of $G$ such that $C_G(\alpha) = D_{u,v,t}$.

Thus we have shown for every subgroup $M$ of $G$ there exists $\alpha \in \text{Aut}(G)$ such that $C_G(\alpha) = M$. Hence $|\text{Acent}(G)|$ is equal to total number of subgroups of $G$ and the proof is completed.

To compute $|\text{Acent}(\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n})|$, we need to find the subgroups of $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, which are not Acentralizers.
**Lemma 3.2.** Let $G = A \times B$, $A = \langle a \rangle \cong \mathbb{Z}_{2^m}$, $B = \langle b \rangle \cong \mathbb{Z}_{2^n}$, $m \leq n$. The following subgroups are not centralizers of $G$.

1. $A_u = \langle a^{2^m-u} \rangle$, where $u = 0, 1, \ldots, m$.
2. $B_v = \langle b^{2^n-v} \rangle$, where $v = 1, 2, \ldots, n - m - 1$, and
3. $D_{u,v,t}$, where $k = v \leq u, u = k, \ldots, m, v = k, \ldots, n$ for every $t$ with gcd$(2^k, t) = 1$ and $k = 1, \ldots, m$.

**Proof.** First we show that the element $b^{2^n-1}$ is a unique element of order 2 in $G$, which is fixed by every automorphism of $G$. Let $\alpha$ be an automorphism of $G$. We know that $\alpha(a) = a^i b^j$ and $\alpha(b) = a^k b^s$, where $0 \leq i, j < 2^m$ and $0 \leq k, s < 2^n$. By Theorem 2.4, gcd$(i, 2^m) = 1$, gcd$(s, 2^n) = 1$, and $j \equiv 0 \pmod{2^{n-m}}$; so $i - 1$ and $s - 1$ are even. Therefore,

$$\alpha(b^{2^n-1}) = (a^i b^j)^{2^n-1} = a^{2^n-1} b^{2^n-1 s} = (a^2)^{2^{n-1} - 1} b^{2^n - 1(s-1)} b^{2^n - 1} = b^{2^n - 1}.$$ 

Since $b^{2^n-1} \notin A_u$, $u = 0, \ldots, m$, and $b^{2^n-1} \notin D_{u,v,t}$, $v \leq u$, it follows that $A_u$ and $D_{u,v,t}$ are not centralizers.

We show the centralizer of $\alpha$ is not equal to $B_v$, $v = 1, 2, \ldots, n - m - 1$. Suppose that $C_G(\alpha) = B_v$. Then $b_v^{2^n-1} = \alpha(b_v^{2^n-1}) = a^{2^n-1} b^{2^n-1 s}$. Since $m + 1 \leq n - v \leq n - 1$, $a^{2^n-1} = 1$. Therefore $b_v^{2^n-1} = b^{2^n-1 s}$ so $b^{2^n-1(s-1)} = 1$. Hence $2^{n-1}(s-1) \equiv 0 \pmod{2^n}$.

If $s = 1$, then $\alpha(b) = a^i b$ and so $\alpha(b^{2^n-1}) = a^{2^n-1} b^{2^n-1 s} = b^{2^n-1}$. But $b^{2^n-1} \notin B_v$. Thus $s \neq 1$.

If $j = 0$, then $\alpha(a) = a^i$ and so $\alpha(a^{2^n-1}) = a^{2^n-1 i} = a^{2^n-1(i-1)} a^{2^n-1} = a^{2^n-1}$. But $a^{2^n-1} \notin B_v$. Hence $j \neq 0$. Since gcd$(2^n, s) = 1$, there exists $t$ with $t = 1, \ldots, n - 1$, such that gcd$(2^n, s-1) = 2^t$. If $n - m - t \leq n - 1$, then $\alpha(b^{2^n-1}) = a^{2^n-1} b^{2^n-1 s} = b^{2^n(s-1)} b^{2^n} = b^{2^m}$. Thus $b^{2^n} \notin B_v$. Hence $1 \leq t \leq n - m - 1$. Hence $s - 1 = 2^t k'$ where $k'$ is odd.

If $j = 2^{n-t} h$, where $h$ is odd, then

$$\alpha(a^{2^n-1} b^{2^n-1}) = \alpha(a)^{2^n-1} \alpha(b)^{2^n-1} = a^{2^n-1} b^{2^n-1 j} = a^{2^n-1} b^{2^n-1}.$$ 

Since $2^{n-1} j + 2^{n-t}(s-1) = 2^n - 1 h + 2^{n-t} k' = 2^{n-1} (h + k')$ and $h + k'$ is even, $2^{n-1} j + 2^{n-t} (s-1) \equiv 0 \pmod{2^n}$, we have $\alpha(a^{2^n-1} b^{2^n-1}) = a^{2^n-1} b^{2^n-1}$. But $a^{2^n-1} b^{2^n-1} \notin B_v$.

If $j = 2^{n-m} h$, where $h$ is even, then

$$\alpha(a^{2^n-1} b^{2^n-1}) = \alpha(a)^{2^n-1} \alpha(b)^{2^n-1} = a^{2^n-1} b^{2^n-1 j} = a^{2^n-1} b^{2^n-1}.$$ 

Since $2^{n-1} j + 2^{n-t} (s-1) = 2^n - 1 h + 2^t k' = 2^{n-1} (h + 2k')$ and $h + 2k'$ is even, $2^{n-1} j + 2^{n-t} (s-1) \equiv 0 \pmod{2^n}$. Hence $\alpha(a^{2^n-1} b^{2^n-1}) = a^{2^n-1} b^{2^n-1}$. But $a^{2^n-1} b^{2^n-1} \notin B_v$. Thus $B_v$ is not an Automizer.

□
In the following theorem we show that $|\text{Acent}(G)|$, where $G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$ is less than the number of subgroups of $G$.

**Theorem 3.3.** Let $G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, where $m \leq n$, then

$$|\text{Acent}(G)| = (m + 1)(n + 1) + \sum_{k=0}^{m-1} (m - k)(n - k)(2^{k+1} - 2^k) - (n - m - 2 + 2^{m+1}).$$

**Proof.** Using the notation of the proof of Theorem 3.1, we have,

$$\begin{cases}
(i - 1)x + ry = 0 \pmod{2^m}, \\
(jx + (s - 1)y = 0 \pmod{2^n}.
\end{cases}$$

Let $A_u = \langle a^{2^{m-u}} \rangle$ be the unique subgroup of $A$ of order $2^u$, $u = 0, 1, \ldots, m$; and let $B_v = \langle b^{2^{n-v}} \rangle$ be the unique subgroup of $B$ of order $2^v$, $v = 0, 1, \ldots, n$. Then $G$ has $(m + 1)(n + 1)$ subgroups of the form $A_u \times B_v$, $u = 0, \ldots, m$, $v = 0, \ldots, n$. By Lemma 3.2, $A_u = \langle a^{2^{m-u}} \rangle$ for $u = 0, 1, \ldots, m$ and $B_v = \langle b^{2^{n-v}} \rangle$ for $v = 1, \ldots, n - m - 1$ are not Acentralizers. For every $u = 1, \ldots, m$ and $v = 1, \ldots, n$, we find an automorphism $\alpha$ of $G$ inducing identity just on $A_u \times B_v$. If we choose $i = 1 + 2^u$, $j = 0$, $r = 0$, and $s = 1 + 2^v$ then defined by $\alpha(a) = a^{1+2^u}$ and $\alpha(b) = b^{1+2^v}$ is an automorphism of $G$, such that $C_G(\alpha) = A_u \times B_v$. For $u = 0$ and $v = n - m, \ldots, n$, we find an automorphism $\alpha$ of $G$ inducing identity just on $A_u \times B_v$. If we choose $i = 1$, $j = 2^{m-r}$, $r = 2^{m-n+v}$, and $s = 1$ then defined by $\alpha(a) = ab^{2^{m-v}}$ and $\alpha(b) = a^{2^{m-n}+v}$ is an automorphism of $G$, such that $C_G(\alpha) = A_u \times B_v$. Also by Lemma 3.2, $D_{u,v,t}$, $v \leq u$ are not Acentralizers. For other $D_{u,v,t}$ the proof is similar to Theorem 3.1. Hence

$$|\text{Acent}(G)| = (m + 1)(n + 1) + \sum_{k=0}^{m-1} (m - k)(n - k)(2^{k+1} - 2^k) - (m + 1)(n + 1) + \sum_{k=0}^{m-1} (m - k)(n - k)(2^{k+1} - 2^k) - (n - m - 2 + 2^{m+1})$$

and the result follows. \hfill \Box

In the rest of the paper we find the Acentralizers of infinite two generator Abelian groups. We start with free Abelian groups. Let $G$ be a free Abelian group of rank 2. Note that $\text{Aut}(G) = \text{GL}_2(\mathbb{Z})$, the group of invertible 2 by 2 matrices over $\mathbb{Z}$. If $\{a, b\}$ is a basis of $G$ and $\alpha$ is an automorphism of $G$, then $\alpha(a) = ai^{ij}b^j$ and $\alpha(b) = a^r b^s$, where $i, j, r, s \in \mathbb{Z}$, and $i s - j r \neq 0$. Since

$$C_G(\alpha) = \{a^{x}b^{y} | a^{ix+ry}b^{ix+sy} = a^{x}b^{y}\},$$

the elements of $C_G(\alpha)$ is of the form $a^{x}b^{y}$, where $(x, y)$ is a solution of the following equation

$$\begin{cases}
(i - 1)x + ry = 0 \\
jx + (s - 1)y = 0.
\end{cases}$$

Let $H$ be a non-trivial subgroup of $G$. First suppose that $\text{rank}(H) = 1$. Then there exists a basis $\{a, b\}$ of $G$ such that $\{a^{u}\}$, where $u$ is a positive integer, is a basis of $H$. If $u = 1$, then $H = \langle a \rangle$ and so $H = C_G(\alpha)$, where $\alpha$ is an automorphism of $G$ defined by $\alpha(a) = a$ and $\alpha(b) = ab$. We claim that if $u > 1$, then there is no automorphism $\alpha$ with $C_G(\alpha) = H$. Suppose that $C_G(\alpha) = H$, for some $\alpha \in \text{Aut}(G)$. Since $a^{u} = \alpha(a^{u}) = a^{uj}b^{ju}$, $j = 0$, and $i = 1$ and so $\alpha(a) = a$. Thus $\langle a \rangle \leq C_G(\alpha) = H$, and so $u = 1$, which is contradiction.
Suppose that \( \text{rank}(H) = 2 \). Then there exists a basis \( \{a, b\} \) of \( G \) such that \( \{a^u, b^v\} \), where \( u \) and \( v \) are positive integers with \( u \parallel v \), is a basis of \( H \). We find an automorphism \( \alpha \) such that \( H = C_G(\alpha) \). If \( u + v + 1 \neq 0 \), then we define \( \alpha(a) = a^{1+v}b^{-v} \) and \( \alpha(b) = a^{-u}b^{1+v} \) (that is \( i = 1 = v, j = -v, r = -u, \) and \( s = 1 = u \)). If \( u + v + 1 = 0 \), then we define \( \alpha(a) = a^{u^2+uv+1}b^{u^2+uv-2} \) and \( \alpha(b) = a^{-u^2}b^{1-u^2} \) (that is \( i = 1 = u^2 + v, j = u^2 + u - 2, \) \( r = -u^2, \) and \( s = 1 = 1 - u^2 \)). In any case it is easy to see that \( H = C_G(\alpha) \).

Let \( G = A \times B \), where \( A = \langle a \rangle \cong \mathbb{Z} \) and \( B = \langle b \rangle \cong \mathbb{Z}_m \). If \( \alpha \) is an automorphism of \( G \), then \( \alpha(a) = a^tb \) and since \( \alpha(b) \) is of finite order, \( \alpha(b) = b^s \), where \( \gcd(n, s) = 1 \). Since \( B \) is a characteristic subgroup of \( G \), it follows that a subgroup of \( A \) is not an Acentralizer of \( \alpha \).

Suppose that \( C \) is a subgroup of \( G \) and \( \alpha \) is an automorphism of \( G \) such that \( C = C_G(\alpha) \).

Case I: If \( i \neq 1 \), then \( x = 0 \). So \( C = C_G(\alpha) \) is a subgroup of \( B \). For any divisor \( d \) of \( n \), let \( B_d = \langle b^{n/d} \rangle \) be the unique subgroup of order \( d \). It is easy to see that such automorphism exists. In fact if we define \( \alpha(a) = a^tb \) and \( \alpha(b) = b^{1+d} \), then \( \alpha \) is an automorphism of \( G \) and \( C_G(\alpha) = B_d \).

Case II: If \( x 
eq 0 \), then \( i = 1 \). Let \( t = \gcd(j, n) \). Then \( \alpha(a^{n/t}) = a^{n/t}b^{j/t} = a^{n/t}(b^{j/t})^n \) and so \( a^{n/t} \in C_G(\alpha) \). If \( a^{t} = a^{b^\ell} \), then \( a^\ell = a^{b^\ell} \) and \( n \mid j\ell \). Therefore \( \frac{n}{t} \mid \frac{j}{\ell} \). Hence \( x \) is a multiple of \( n/t \). It follows that \( b^{n/t} \in C_G(\alpha) \) and \( n \mid (s - 1)y \). Let \( v = \gcd(s - 1, n) \). Then \( \frac{n}{t} \mid \frac{s - 1}{v} \) and so \( \frac{n}{t} \mid y \). Hence \( b^{n/v} \in C_G(\alpha) \). It follows that \( C_G(\alpha) = \langle a^{n/t} \rangle \times \langle b^{n/v} \rangle \).

It is easy to see that such automorphism exists. In fact, if \( t \) and \( v \) are two arbitrary divisors of \( n \) then \( \alpha(a) = ab^t \) and \( \alpha(b) = b^{1+v} \) defines an automorphism of \( G \) and \( C_G(\alpha) = \langle a^{n/t} \rangle \times \langle b^{n/v} \rangle \).

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