BOUNDED MULTIPLIER ALGEBRAS ARISING FROM FOCK REPRESENTATIONS ASSOCIATED TO SEMIGROUPS

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Abstract. In this article, we introduce the “Bounded multiplier algebra” associated to the Fock representation that arising from the left-cancellative semigroup $S$ (denoted by $M_b(S)$). We establish two key results of the bounded multiplier algebras. We show that $M_b(S)$ is a unital operator algebra if $S$ is a left-cancellative semigroup and $M_b(G)$ is a $C^*$-algebra if $G$ is a group. We illustrate that the associated bounded multiplier algebras $M_b(\mathbb{Z}_+), M_b(\mathbb{Z}_+^2)$ are identified with respective Hardy algebras $H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{D}^2)$ for $S = \mathbb{Z}_+, \mathbb{Z}_+^2$. Next, we discuss that bounded multiplier algebra associated to the free semigroup $S = F_n^+$. We clearly show that the well-known non-commutative Hardy algebra $F_n^\infty$ and the bounded multiplier algebra $M_b(F_n^+)$ are isometrically isomorphic.

1. Introduction

Multiplier algebras are widely used in the study of operator algebras, multivariate operator theory, and abstract Harmonic analysis for variety of purposes. Undoubtedly, the multiplier algebras of the reproducing kernel Hilbert spaces are among the most well-known and important objects in the study of analytic function theory and multivariate operator theory [1].

Multiplier algebra of the full Fock space over $\mathbb{C}^n$, (denoted $\mathcal{F}^2(\mathbb{C}^n)$) has been utilized to explore non-commutative analytic function theory (see for example [8, 9, 10, 11] and references therein). As the full Fock space $\mathcal{F}^2(\mathbb{C}^n)$ is unitarily equivalent to the Hilbert space $\ell_2(\mathbb{F}_n^+)$ where $\mathbb{F}_n^+$ denotes the free semigroup generated by $n$ symbols (see Section 3 for more details). In [3] the authors have referred to $\ell_2(\mathbb{F}_n^+)$ as the Fock representation associated to the free semigroup $\mathbb{F}_n^+$. Borrowing this terminology, for any left-cancellative semigroup $S$, we refer to $\ell_2(S)$ as the Fock representation associated to $S$. Here, it is worth mentioning that $\mathcal{F}^2(\mathbb{C}^n)$ is not a reproducing kernel Hilbert space and thus the notion of multiplier algebra of $\mathcal{F}^2(\mathbb{C}^n)$ (in the usual sense of reproducing kernel Hilbert spaces) does not make sense in this context. However, G. Popescu defined the non-commutative Hardy algebra $F_n^\infty$ that plays the role of multiplier algebra of the Fock space.

In this article, we address the following question. We show that Popescu non-commutative Hardy algebra can be generalize over semigroups if we consider the Definition 3.1. Below we summarize the content of Section 3 (see for the precise formulations).

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Question 1. Given a Fock representation of a left cancellative $S$ whether it is possible to define a notion of multiplier algebra?

We define (see Definition 3.1) the notion of bounded multiplier algebra of the Fock representation associated to a left-cancellable semigroup.

**Definition A.** The bounded multiplier algebra associated to $S$ is defined by:

$$M_b(S) = \{(L, R) : L, R \in B(\ell_2(S)), fL(g) = R(f)g \text{ for all } f, g \in \mathcal{P}[S]\}$$

where $\mathcal{P}[S]$ denotes the algebra polynomials over $S$.

**Theorem B.** Let $S$ be a left-cancellative semigroup. Then:

(a) $M_b(S)$ is always a unital Banach algebra with respect to the norm $||(\cdot, \cdot)||$ given by

$$||(L, R)|| = \max\{||L||, ||R||\} \text{ for } (L, R) \in M_b(S).$$

(b) If $S = G$, a group, then $M_b(G)$ is a $C^*$-algebra equipped with the $\ast$-operation given by

$$(L, R)^\ast = (R^\#, L^\#) \text{ for all } (L, R) \in M_b(G),$$

where $L^\#$ and $R^\#$ are defined by

$$L^\#(f) = UL(Uf) \text{ and } R^\#(g) = UR(Ug) \text{ for all } f, g \in \ell_2(G)$$

and $U : \ell_2(G) \to \ell_2(G)$ is the anti-unitary operator defined by

$$U(\sum c_g \delta_g) = \sum \overline{c_g} \delta_{g^{-1}}$$

where, for $g \in G$, $\delta_g$ denotes the Dirac delta function at $g$.

We observe that if $S$ is not a group, then $M_b(S)$ is not necessarily a $C^*$-algebra (see in Section 4). This raises the following natural question.

**Question 2.** Is $M_b(S)$ an operator algebra?

In Section 4 we answer to this question in the affirmative. Applying “Bleacher-Ruan-Sinclair” theorem, which provides the sufficient criteria for a unital Banach algebra to be a unital operator algebra, we prove that: (see Theorem 4.2)

**Theorem C.** $M_b(S)$ is always a unital operator algebra.

In addition, we show that following result (see Theorem 3.8 and Theorem 4.3).

**Theorem D.** Let $G$ and $H$ be two isomorphic groups. Then the associated $C^*$-algebras $M_b(G)$ and $M_b(H)$ are isomorphic.

**Theorem E.** Let $S$ and $S'$ be two isomorphic semigroups. Then the operator algebra $M_b(S)$ is completely isomorphic to the operator algebra $M_b(S').$

Finally, it seems quite interesting to compute concretely multiplier algebras corresponding to ‘nice’ semigroups. Thus, we have the following question. In Section 4, we have been able to concretely realize multiplier algebras of certain semigroups.
Proposition F. We explicitly compute the bounded multiplier algebras for the following semigroups.

(a) $S = \mathbb{Z}^n_+: M_b(\mathbb{Z}^n_+) \text{ is isometrically isomorphic to the Hardy algebra } H^\infty(\mathbb{D}^n)$. 

(b) $S = \mathbb{F}^n_+: M_b(\mathbb{F}^n_+) \text{ is isometrically isomorphic to the non-commutative Hardy algebra } F^\infty$. 

2. Preliminaries

Let $F(X, \mathbb{C})$ denote the space of all complex valued functions on a set $X$ with respect to pointwise addition and scalar multiplication. Then a vector subspace $H \subseteq F(X, \mathbb{C})$ is said to be reproducing kernel Hilbert space on the set $X$ if $H$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ in which $H$ is a Hilbert space and the following condition holds: For each $x \in X$, the evolution map $E_x : H \to \mathbb{C}$, defined by $E_x(f) = f(x)$ for all $f \in H$, is bounded. The Multiplier algebra of a reproducing kernel Hilbert space is defined by

$$\text{Mult}(H) = \{ \varphi : X \to \mathbb{C} : \varphi f \in H \text{ for all } f \in H \}.$$ 

If $H$ is a reproducing kernel Hilbert space, then $\text{Mult}(H)$ is a closed subspace of $B(H)$, thanks to closed graph theorem [6]. There are many important classes of Hilbert spaces which are reproducing kernel Hilbert spaces. An important example is the Hardy Hilbert space $H^2(\mathbb{D})$ [2]. Next, we describe the Hardy Hilbert space and multiplier algebra of the Hardy Hilbert space.

Let $\mathbb{D}$ denote open unit disc in the complex plane $\mathbb{C}$ and let $\text{Hol}(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$. The Hardy space $H^2(\mathbb{D})$ (see [2]) is defined by

$$H^2(\mathbb{D}) := \left\{ f \in \text{Hol}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$ 

Moreover, Hardy space over unit circle is identified as

$$H^2(\mathbb{T}) := \left\{ f = \sum_{n=0}^{\infty} a_n e^{in\theta} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$ 

The Hilbert spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{T})$ are unitary equivalent via obvious correspondence

$$H^2(\mathbb{D}) \ni f(z) = \sum_{n=0}^{\infty} a_n z^n \to \tilde{f}(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta} \in H^2(\mathbb{T}).$$

Throughout, we assume that $G$ is a discrete group and $S$ is a left-cancellative semigroup. It is worth to record that every semigroup $S$ embedded in a group $G$ is automatically a left-cancellative semigroup. For example, the set of all isometries of a $C^*$-algebra forms a left-cancellative semigroup. Next, we discuss the Fock representation associated to the left-cancellative semigroup $S$. The associated Fock representation of $S$ is the Hilbert space $\ell_2(S) := \{ f : S \to \mathbb{C} | \sum_{s \in S} |f(s)|^2 < \infty \}$ with respect to the orthogonal basis given by
\{\delta_s : s \in \mathcal{S}\}, where the Dirac function \(\delta_s : \mathcal{S} \to \mathbb{C}\) is defined by \(\delta_s(r) = \begin{cases} 1 & \text{if } s = r \\ 0 & \text{if } s \neq r \end{cases}\). For each \(s \in \mathcal{S}\), the associated isometry \(V_s\) acting on \(\ell_2(\mathcal{S})\) is defined by \(V_s \delta_r = \delta_{sr}\) for all \(r \in \mathcal{S}\).

3. Bounded Multiplier algebras

Before coming to the definition of multiplier algebra, let us fix some notations. We denote the polynomial over \(\mathcal{S}\) by \(\mathcal{P}[\mathcal{S}]\). Moreover, it is worth to recall that \(\mathcal{P}[\mathcal{S}] \subset \ell_2(\mathcal{S})\) and the element of \(\mathcal{P}[\mathcal{S}]\) is of the form \(\sum c_g \delta_g\), where the coefficients are at most nonzero at finite points. Also, we denote \(\mathcal{P}_1[\mathcal{S}]\) the set of all polynomials in \(\mathcal{S}\) whose norms are less than or equal to one in \(\ell_2(\mathcal{S})\). When there are no confusions arise, we simply write \(\mathcal{P}\) for \(\mathcal{P}[\mathcal{S}]\) and \(\mathcal{P}_1\) for \(\mathcal{P}_1[\mathcal{S}]\). In general, we cannot define multiplication operation between two elements of Hilbert space \(\ell_2(\mathcal{S})\). However, it is standard practice to define a multiplication on the polynomial space \(\mathcal{P}\) by declaring the following convolution operation \(\delta_s * \delta_r = \delta_{sr}\) for all \(s, r \in \mathcal{S}\). Throughout the paper, we write \(\delta s \delta r\) to indicate \(\delta_s * \delta_r\) for the sake of simplicity. Moreover, we can extend \(f \varphi\) and \(\varphi f\) whenever \(f \in \mathcal{P}\) and \(\varphi \in \ell_2(\mathcal{S})\).

**Definition 3.1 (Bounded Multiplier Algebra).** The bounded multiplier algebra associated to the semigroup \(\mathcal{S}\) is denoted and defined by

\[
(1) \quad M_b(\mathcal{S}) = \{(L, R) : L, R \in B(\ell_2(\mathcal{S})), f L(g) = R(f)g \text{ for all } f, g \in \mathcal{P}[\mathcal{S}]\}.
\]

**Remark 3.2.** In the above definition, we have to assume the \(L, R\) are bounded. We have to add such assumption as \(f L(g) = R(f)g\) may not imply that \(L, R\) are bounded. This assumption has been used frequently throughout the paper. Due to such assumption, we called such algebra as bounded multiplier algebra.

The next observation is useful when we consider the bounded multiplier algebra of an abelian semigroup.

**Proposition 3.3.** Let \(\mathcal{S}\) be an abelian semigroup. If \((L, R) \in M_b(\mathcal{S})\), then \(L = R\).

**Proof.** Let \((L, R) \in M_b(\mathcal{S})\). Then for all \(f, g \in \mathcal{P}\), we have \(f L(g) = R(f)g\). Putting \(f = g = 1\), we have \(L(1) = R(1)\). We assume \(\varphi = L(1)\). Putting \(f = 1\), we obtain \(L(g) = R(1)g\) and therefore \(L(g) = \varphi g\) for all \(g \in \mathcal{P}\). Similarly putting \(g = 1\), we have \(f L(1) = R(f)\) and therefore \(R(f) = f \varphi\) for all \(f \in \mathcal{P}\). Since \(\mathcal{S}\) is abelian, we have \(\varphi f = f \varphi\). It implies that \(L(f) = R(f)\) for all \(f \in \mathcal{P}\). Thus by boundedness property of \(L, R\), and density property of \(\mathcal{P}\) in \(\ell_2(\mathcal{S})\), we conclude that \(L = R\). \(\square\)

**Theorem 3.4.** Let \(\mathcal{S}\) be a semigroup. Then \(M_b(\mathcal{S})\) is a unital Banach algebras with respect to the norm \(\|(\ldots)\|\) given by

\[
\|(L, R)\| = \max\{\|L\|, \|R\|\} \text{ for all } (L, R) \in M_b(\mathcal{S}).
\]

**Proof.** Clearly, \(M_b(\mathcal{S})\) is an algebra with respect to addition, scalar multiplication, and multiplication given by

\[
(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2),
\]
\[ \lambda(L, R) = (\lambda L, \lambda R), \]
\[ (L_1, R_1)(L_2, R_2) = (L_1 L_2, R_2 R_1). \]

Moreover \((1, 1) \in M_b(S)\) is the multiplicative identity as \((1, 1)(L, R) = (L, R)(1, 1) = (L, R)\) for all \((L, R) \in M_b(S)\). Next to see \(\|(. .)\|\) is a norm, it is enough to notice the following triangle inequality:

\[
\| (L_1, R_1) + (L_2, R_2) \| = \max \{ \| L_1 + L_2 \|, \| R_1 + R_2 \| \} \\
\leq \max \{ \| L_1 \| + \| L_2 \|, \| R_1 \| + \| R_2 \| \} \\
\leq \max \{ \| L_1 \|, \| R_1 \| \} + \max \{ \| L_2 \|, \| R_2 \| \}.
\]

Let \(\{(L_n, R_n)\}\) be a Cauchy sequence in \(M_b(S)\). Then \(\{L_n\}, \{R_n\}\) are Cauchy sequences in \(B(\ell_2(S))\). Therefore, the limits \(\lim_{n \to \infty} L_n = L\) and \(\lim_{n \to \infty} R_n = R\) exist in \(B(\ell_2(S))\). To see \((L, R) \in M_b(S)\), we notice that \(f L(g) = \lim_{n \to \infty} f L_n(g) = \lim_{n \to \infty} R_n(f)g = R(f)g\). It implies \((L, R) \in M_b(S)\) and consequently \(M_b(S)\) forms a Banach space. Finally, to show \(M_b(S)\) is a Banach algebra, we need to see the following inequality:

\[
\| (L_1, R_1)(L_2, R_2) \| \leq \max \{ \| L_1 \| \| L_2 \|, \| R_1 \| \| R_2 \| \} \\
\leq \max \{ \| L_1 \|, \| R_1 \| \} \max \{ \| L_2 \|, \| R_2 \| \} \\
= \| (L_1, R_1) \| \| (L_2, R_2) \|.
\]

This completes the proof. \(\Box\)

Next, we investigate bounded multiplier algebra \(M_b(G)\) when \(G\) is a group. As we understand in Theorem 3.4, \(M_b(G)\) is a Banach algebra. In order to prove \(M_b(G)\) is a \(C^*\)-algebra, we must find an appropriate * operation on \(M_b(G)\). In next Lemma 3.5, we able to construct anti linear unitary as \(G\) is a group. This anti-unitary operator will play the main role to define \(*\)-operation.

**Lemma 3.5.** Let \(G\) be a group. Then the anti-linear operator \(U : \ell_2(G) \to \ell_2(G)\) is defined by

\[
(2) \quad U(\sum_g c_g \delta_g) = \sum_g \overline{c_g} \delta_g^{-1}.
\]

satisfies the following property:

1. \(U^* = U\), and \(U^2 = 1\).
2. \(U(f \varphi) = U(\varphi) U(f)\) and \(U(\varphi f) = U(f) U(\varphi)\) for all \(f \in \mathcal{P}, \varphi \in \ell_2(G)\).

**Proof.** It is clear from the definition that \(U^2 = 1\). Since \(U\) is an anti-linear map, thus the anti-linear adjoint \(U^*\) should satisfy

\[
\langle U^* f_1, f_2 \rangle = \overline{\langle f_1, U f_2 \rangle} \text{ for all } f_1, f_2 \in \ell_2(G).
\]

Let \(f_1 = \sum_g c_g \delta_g, f_2 = \sum_h a_h \delta_h\). The following computation reveals:

\[
\langle U^* (\sum_g c_g \delta_g), \sum_h a_h \delta_h \rangle = \overline{\langle \sum_g c_g \delta_g, U(\sum_h a_h \delta_h) \rangle}.
\]
\[
\begin{align*}
&= \left\langle \sum_{g} c_g \delta_g, \left( \sum_{h} a_h \delta_{h^{-1}} \right) \right\rangle \\
&= \sum_{h} c_{h^{-1}} a_h \\
&= \left\langle \sum_{g} c_g \delta_{g^{-1}}, \sum_{h} a_h \delta_h \right\rangle.
\end{align*}
\]

It ensures that \( U^* \left( \sum_{g} c_g \delta_g \right) = \sum_{g} c_g \delta_{g^{-1}} \) for all \( \sum_{g} c_g \delta_g \in \ell_2(G) \). Next let \( f = \sum_{g} c_g \delta_g \in \mathcal{P}, \varphi = \sum_{h} a_h \delta_h \in \ell_2(G) \). Next, two computations are as follows:

\[
U(f \varphi) = U \left( \sum_{g} c_g \delta_g \sum_{h} a_h \delta_h \right) \\
= U \left( \sum_{s} \left( \sum_{g} c_g a_{g^{-1} s} \right) \delta_s \right) \quad \text{[put} \ s = gh \text{]} \\
= \sum_{s} \left( \sum_{g} c_g a_{g^{-1} s} \right) \delta_{s^{-1}},
\]

and \( U(\varphi) U(f) = U \left( \sum_{h} a_h \delta_h \right) U \left( \sum_{g} c_g \delta_g \right) \)

\[
= \sum_{h} a_h \delta_{h^{-1}} \sum_{g} c_g \delta_{g^{-1}} \\
= \sum_{s,g} c_g a_{g^{-1} s} \delta_{s^{-1}} \quad \text{[put} \ s = gh \text{]} \\
= \sum_{s} \left( \sum_{g} c_g a_{g^{-1} s} \right) \delta_{s^{-1}}.
\]

As a consequence, we obtain \( U(f \varphi) = U(\varphi) U(f) \). Using a similar calculation, we can prove that \( U(\varphi f) = U(f) U(\varphi) \) for all \( f \in \mathcal{P}, \varphi \in \ell_2(G) \). □

Now we are in a position to claim \( M_b(G) \) is a \( C^* \)-algebra.

**Theorem 3.6.** Let \( G \) be a group. Then \( M_b(G) \) is a \( C^* \)-algebra equipped with the \( \ast \)-operation given by

\[
(L, R)^\ast = (R^\#, L^\#) \quad \text{for all} \ (L, R) \in M_b(G),
\]

where \( L^\# \) and \( R^\# \) are defined by

\[
L^\#(f) = U L(U f) \quad \text{and} \quad R^\#(g) = U R(U g) \quad \text{for all} \ f, g \in \ell_2(G).
\]

**Proof.** By Theorem 3.4, \( M_b(G) \) is a Banach algebra. Let \( (L, R) \in M_b(G) \). Then for all \( f, g \in \mathcal{P}, \) we have \( f L(g) = R(f) g \). Putting \( f = g = 1 \), we notice that \( L(1) = R(1) \in \ell_2(G) \) and we denote \( L(1) = \varphi \). Putting \( f = 1 \), we obtain \( L(g) = R(1) f = \varphi f \). Similarly, when \( g = 1 \), we have \( R(f) = f L(1) = f \varphi \). Let us define an unitary operator \( U : \ell_2(G) \to \ell_2(G) \) by

\[
U \left( \sum_{g} c_g \delta_g \right) = \sum_{g} c_g \delta_{g^{-1}}.
\]
Then by Lemma 3.5, we have \( U^* = U \) and \( U \) possesses the following property:

\[
U(f \varphi) = U(\varphi)U(f) \quad \text{and} \quad U(\varphi f) = U(f)U(\varphi)
\]

for all \( f \in \mathcal{P}, \varphi \in \ell_2(G) \).

Now we define the \(*\)-operation as follows: Let \((L, R) \in M_b(G)\). Then we consider \( L^#, R^# \) defined by \( L^#(f) = UL(Uf) \) and \( R^#(g) = UR(Ug) \) for all \( f, g \in \ell_2(G) \). Since \((L, R) \in M_b(G)\), there exists \( \varphi \in \ell_2(G) \) such that \( L(f) = \varphi f \) and \( R(f) = f \varphi \) for all \( f \in \ell_2(G) \). Then \( L^#, R^# \) can be written explicitly as

\[
R^#(g) = UR(Ug) = U((Ug)\varphi) = U\varphi g,
\]

\[
L^#(f) = UL(Uf) = U(\varphi(Uf)) = fU\varphi \quad \text{for all} \quad f \in \mathcal{P}.
\]

To see \((R^#, L^#) \in M_b(G)\), we observe that \( L^#(f)g = (fU\varphi)g = f(U\varphi g) = fR^#(g) \) for all \( f, g \in \mathcal{P} \). Finally, we define a \(*\)-operation on \( M_b(G) \) by

\[
(L, R)^* = (R^#, L^#) \quad \text{for all} \quad (L, R) \in M_b(G).
\]

Then we see that \((L, R)^{**} = (R^#, L^#)^* = (L^#, R^#)^* = (L, R)\). Now, it is remain to show that \( \|(.,.)\| \) satisfies \( C^*\)-norm condition. Let \((L, R) \in M_b(S)\). Then let \( L(1) = R(1) = \varphi \in \ell_2(G) \).

First, we need to see the more explicit form of \( L^# \) and \( R^# \). Let \( \varphi = \sum c_g \delta_g \in \ell_2(G) \). Then for all \( f = \sum a_h \delta_h \in \mathcal{P} \), we have

\[
L^#(f) = fU\varphi
\]

\[
= \sum_h a_h \delta_h \sum_g c_g \delta_g^{-1}
\]

\[
= \sum_{s,g} a_{sg} c_g \delta_s \quad \text{[put} \quad s = hg^{-1}].
\]

Similarly, we can show that \( R^#(f) = \sum_{g,s} a_{gs} c_g \delta_s \). Now, let \( g = \sum b_t \delta_t \in \mathcal{P} \). Then we compute the following:

\[
\langle L^#(f), g \rangle = \left\langle \sum_{s,g} a_{sg} \overline{c_g} \delta_s, \sum_t b_t \delta_t \right\rangle
\]

\[
= \left\langle \sum_s \left( \sum_g a_{sg} \overline{c_g} \right) \delta_s, \sum_t b_t \delta_t \right\rangle
\]

\[
= \sum_{s,g} a_{sg} \overline{c_g} b_s.
\]

Similarly, it is easy to see that \( \langle R^#(f), g \rangle = \sum_{s,g} a_{gs} \overline{c_g} b_s \). Finally, to get a relation between \( L^* \) and \( R^# \), we need to see the following:

\[
\langle L^* f, g \rangle = \left\langle L^* \left( \sum_h a_h \delta_h \right), \sum_t b_t \delta_t \right\rangle
\]

\[
= \left\langle \sum_h a_h \delta_h, \sum_{g,t} c_g b_t \delta_{gt} \right\rangle
\]
The last two computations ensure that $L^*(f) = R^#(f)$ for all $f \in \mathcal{P}$. Hence by continuity properties of $L^#$ and $R^#$, we conclude that $R^# = L^*$. Similarly, we can show that $L^# = R^*$. Finally, we observe that norm \[ \|(L, R)^*(L, R)\| = \|(R^# L, R L^#)\| = \|(L^* L, R R^*)\| = \max\{\|L\|^2, \|R\|^2\} = \max\{\|L\|, \|R\|\}^2 = \|(L, R)\|^2. \]

Next, we observe an important remark before coming to some application of the Theorem 3.6.

**Remark 3.7.** Let $G$ be an abelian group and let $(L, R) \in M_b(G)$. Then $L = R$ and consequently $L^# = L^*$.

**Theorem 3.8.** Let $G$ and $H$ be two isomorphic groups. Then the associated $C^*$-algebras $M_b(G)$ and $M_b(H)$ are isomorphic.

**Proof.** Let $\psi : G \to H$ be a group isomorphism. We define a unitary map $\psi : \ell_2(G) \to \ell_2(H)$ by $U_\psi(\sum_{g \in G} c_g \delta_g) = \sum_{g \in G} c_g \delta_{\psi(g)}$. Here our plan is to use unitary operator (see Equation (8)) to determine isomorphism between two $C^*$-algebras $M_b(G)$ and $M_b(H)$. In what follows: let $(L, R) \in M_b(H)$. Then we define a map $\Pi : M_b(H) \to M_b(G)$ by

$$\Pi((L, R)) = (U_\psi^* L U_\psi, U_\psi^* R U_\psi) \text{ for all } (L, R) \in M_b(H).$$

We claim that $\Pi$ is well defined. Notice that $L(\delta_h) = R(\delta_h) = \varphi \in \ell_2(H)$. So $\varphi$ can be expressed as $\varphi = \sum_{h \in H} c_h \delta_h$. Now choose $g, g' \in G$, then we can notice the following observation:

$$U_\psi^* L U_\psi(\delta_g) = U_\psi^* L \delta_{\psi(g)} = U_\psi^* \left( \sum_{h \in H} c_h \delta_{h \psi(g)} \right) = \sum_{h \in H} c_h \delta_{\psi^{-1}(h)g} = (\sum_{h \in H} c_h \delta_{\psi^{-1}(h)}) \delta_g.$$ 

Similarly, we can notice that $U_\psi^* R U_\psi(\delta_{g'}) = \delta_{g'}(\sum_{h \in H} c_h \delta_{\psi^{-1}(h)})$. This follows that $\delta_g'(U_\psi^* L U_\psi) \delta_{g'} = ((U_\psi^* R U_\psi) \delta_{g'}) \delta_g$ for all $g, g' \in G$, and consequently the map $\Pi$ is a well defined map. It is immediate to see that $\Pi$ is a bijective and unital homomorphism. It remains
to show that $\Pi((L, R)^*) = \Pi((L, R))^*$. Let us invoke two anti-unitaries $U_G : \ell_2(G) \to \ell_2(G)$ and $U_H : \ell_2(H) \to \ell_2(H)$ from Equation (2) as follows:

$$U_G \left( \sum c_g \delta_g \right) = \sum \overline{c}_g \delta_g^{-1} \text{ for all } \sum c_g \delta_g \in \ell_2(G),$$

$$U_H \left( \sum c_h \delta_h \right) = \sum \overline{c}_h \delta_h^{-1} \text{ for all } \sum c_h \delta_h \in \ell_2(H).$$

Then by Theorem 3.6, we observe that $(L, R)^* = (U_H L U_H, U_H R U_H)$ for all $(L, R) \in M_b(H)$ and $(P, Q)^* = (U_G P U_G, U_G Q U_G)$ for all $(P, Q) \in M_b(H)$. We notice that

$$U_H U_\psi(\delta_g) = U_H(\delta_{\psi(g)})$$

$$= \delta_{\psi(g)^{-1}}$$

$$= \delta_{\psi(g^{-1})}$$

$$= U_\psi \delta_{g^{-1}}$$

$$= U_\psi U_G(\delta_g) \text{ for all } g \in G.$$ 

Therefore, we have $U_H U_\psi = U_\psi U_G$. Finally, we notice that

$$\Pi((L, R)^*) = (U_\psi^* L U_\psi, U_\psi^* R U_\psi)^*$$

$$= (U_G U_\psi^* L U_\psi U_G, U_G U_\psi^* R U_\psi U_G)$$

$$= (U_\psi^* U_H L U_H U_\psi, U_\psi^* U_H R U_H U_\psi)$$

$$= \Pi((U_H L U_H, U_H R U_H))$$

$$= \Pi((L, R)^*) \text{ for all } (L, R) \in M_b(H).$$

This completes the proof. □

4. BOUNDED MULTIPLIER ALGEBRAS OF CERTAIN SEMIGROUPS

In this section, we discuss the important connection between bounded multiplier algebra and non-commutative function space associated to the semigroup $S$. Before defining non-commutative function space, we recall that Popescu’s non-commutative Hardy algebra (see [10, Equation 3.2]) associated to the free groups generated by $n$ symbols:

$$F_n^\infty = \{ f \in \mathcal{F}^2(\mathbb{C}^n) : \sup_{p \in \mathcal{P}_1} \| f \otimes p \| < \infty \}. $$

(5)

Inspired by the Popescu’s non-commutative Hardy algebra, we introduce the non-commutative function space on the semigroup $S$ by

$$\mathcal{F}_n^\infty(S) := \{ \varphi \in \ell_2(S) : \sup_{p \in \mathcal{P}_1} \| \varphi p \| \leq \infty, \sup_{p \in \mathcal{P}_1} \| p \varphi \| \leq \infty \}. $$

We first recall that for $S = \mathcal{F}_n^+$, $\mathcal{F}_n^\infty(S)$ can be identified with Popescu’s non-commutative Hardy algebra (see Equation (9)). Next, we see the bounded multiplier algebra $M_b(S)$ has connection with $\mathcal{F}_n^\infty(S)$.

**Theorem 4.1.** Let $(L, R) \in M_b(S)$. Then there exists a unique $\varphi \in \mathcal{F}_n^\infty(S)$ such that

$$L(f) = \varphi f \text{ and } R(f) = f \varphi \text{ for all } f \in \ell_2(S).$$

(6)
Next, we define \( L(1) = \varphi = R(1) \). To see \( \varphi \in \mathbb{F}^\infty(S) \), notice that \( \sup_{p \in P_1} \| \varphi p \| = \sup_{p \in P_1} \| L(p) \| = \| L \| < \infty \), and \( \sup_{p \in P_1} \| p \varphi \| = \sup_{p \in P_1} \| R(p) \| = \| R \| < \infty \). Let \( f \in \ell_2(S) \). Then there exists a sequence \( \{ p_n \} \) with \( p_n \in P \) such that \( \lim p_n = f \). We claim that \( \lim \varphi_{p_n} \) exists in \( \ell_2(S) \). This is true because \( \| \varphi_{p_n} - \varphi_{p_m} \| = \| L(p_n - p_m) \| \leq \| L \| \| p_n - p_m \| \) and \( \{ p_n \} \) is a convergent sequence. Therefore, the \( \lim \varphi_{p_n} \) exists in \( \ell_2(S) \). We denote the limits by \( \varphi f \). Consequently, we have \( L(f) = \varphi f \) for all \( f \in \ell_2(S) \). □

Now, we show an important result.

**Theorem 4.2.** Let \( S \) be a semigroup. Then \( M_b(S) \) is a unital operator algebra.

**Proof.** We define a Banach space \( M_n(M_b(S)) := \{ ([L_{ij}], [R_{ij}]): (L_{ij}, R_{ij}) \in M_b(S) \text{ for all } 1 \leq i, j \leq n \} \) with respect to the norm \( \| ([..]) \|_n \) by
\[
\| ([L_{ij}], [R_{ij}]) \|_n := \max \{ \| [L_{ij}] \|, \| [R_{ij}] \| \},
\]
here \( [L_{ij}], [R_{ij}] \) are bounded linear operators acting on the Hilbert space \( \ell_2(S)^{\oplus n} \), so \( \| [L_{ij}] \|, \| [R_{ij}] \| \) denote the operator norms on the Hilbert space \( \ell_2(S)^{\oplus n} \). For \( \alpha = [\alpha_{ij}] \in M_{m,n}, \beta = [\beta_{ij}] \in M_{n,r} \), and \( ([L_{ij}], [R_{ij}]) \in M_n(M_b(S)) \), we define bimodule action by
\[
\alpha([L_{ij}], [R_{ij}]) = \left[ \sum_{k=1}^n \alpha_{ik} L_{kj} \right], \quad ([L_{ij}], [R_{ij}]) \beta = \left[ \sum_{k=1}^n L_{ik} \beta_{kj} \right].
\]
Next, we define \( \oplus \) operation by \( ([L_{ij}], [R_{ij}]) \oplus ([P_{ij}], [Q_{ij}]) = ([L_{ij}] \oplus [P_{ij}], [R_{ij}] \oplus [Q_{ij}]) \) for all \( ([L_{ij}], [R_{ij}]) \in M_n(M_b(S)), ([P_{ij}], [Q_{ij}]) \in M_m(M_b(S)) \). Next, we claim that \( M_b(S) \) is an operator space. Let \( (L, R) \in M_n(M_b(S)) \) and \( (P, Q) \in M_m(M_b(S)) \). We can notice that \( \| ([L, R] \oplus (P, Q)) \|_{m+n} = \max \{ \| L \|, \| R \|, \| P \|, \| Q \| \} = \max \{ \| (L, R) \|_m, \| (P, Q) \|_n \} \), and for all \( \alpha, \beta \in M \), we have
\[
\| \alpha(L, R) \|_m = \max \{ \| \alpha L \|, \| \alpha R \| \} \leq \max \{ \| \alpha \| \| L \|, \| \alpha \| \| R \| \} = \| \alpha \| \| (L, R) \|_m,
\]
\[
\| (L, R) \beta \|_m = \max \{ \| L \beta \|, \| R \beta \| \} \leq \max \{ \| L \| \| \beta \|, \| R \| \| \beta \| \} = \| \beta \| \| (L, R) \|_m.
\]
Therefore \( M_b(S) \) is an operator space equipped with matrix norm \( \{ \|([..]) \|_n \} \). For each \( n \in \mathbb{N} \), we define a product \( M_n(M_b(S)) \times M_n(M_b(S)) \to M_n(M_b(S)) \) by
\[
(L, R)(P, Q) = (LR, (Q^t P^t)^t)
\]
for all \( (L, R), (P, Q) \in M_n(S) \).

Here, \( P^t \) denote the transpose of \( P \), in other words \( P^t = [P_{ji}] \) whenever \( P = [P_{ij}] \in M_n(M_b(S)) \). It is not hard to see that the product \( M_n(M_b(S)) \times M_n(M_b(S)) \to M_n(M_b(S)) \) is a well defined map. Moreover we see that
\[
\| (L, R)(P, Q) \|_m = \| (LR, (Q^t P^t)^t) \|_m
\]
\[
\leq \max \{ \| LR \|, \| (Q^t P^t)^t \| \}
\]
\[
\leq \max \{ \| L \| \| R \|, \| Q \| \| P \| \}
\]
\[
= \| (L, R) \|_m \| (P, Q) \|_m.
\]
This ensures that \( M_b(S) \) is an operator algebra, thanks to “Blecher-Ruan-Sinclair” theorem (see [5]). □
Theorem 4.4. Let $S$ and $S'$ be two isomorphic semigroups. Then the operator algebras $M_b(S)$ and $M_b(S')$ are completely isomorphic.

Proof. Let $\psi : S \to S'$ be the semigroup isomorphism. We can invoke proof of Theorem 3.8 to define a unitary map $\psi : \ell_2(S) \to \ell_2(S')$ by

$$U_\psi(\sum_{s \in S} c_s \delta_s) = \sum_{g \in S} c_g \delta_{\psi(s)}.$$  

Then we define map $\Pi : M_b(S) \to M_b(S')$ by $\Pi((L, R)) = (U_\psi^*LU_\psi, U_\psi^*RU_\psi)$ for all $(L, R) \in M_b(S')$. Now, it will be immediate to see that the map $\Pi : M_b(S') \to M_b(S)$ is a completely isometry and unital homeomorphism. This completes the proof. \qed

4.1. Realization of bounded multiplier algebras. Before coming to the main result of the theorem, we recall some standard results of the Hardy space.

Proposition 4.4. The multiplier algebra $M_b(\mathbb{Z}_+)$ is isomorphic to the Hardy algebra $H^\infty(\mathbb{D})$.

Proof. Let $(L, R) \in M_b(\mathbb{Z}_+)$ and assume $\varphi = L(1)$. Then $L = R$ and $L(f) = \varphi f$ for all $f \in \mathcal{P}$. Then for all $\varphi \in \ell_2(\mathbb{Z}_+)$ we have $\sup_{f \in \mathcal{P}} \|\varphi f\| = \sup_{f \in \mathcal{P}_1} \|L(f)\| = \|L\| < \infty$. Notice that the Banach algebra $\{\varphi \in \ell_2(\mathbb{Z}_+) : \varphi f \in \ell_2(\mathbb{Z}_+)\}$ equipped with norm $\|f\| = \sup_{f \in \mathcal{P}_1} \|\varphi f\|$ is isometrically isomorphic to $M(H^2(\mathbb{D}))$ as the Hilbert space $H^2(\mathbb{D})$ is unitarily equivalent to $\ell_2(\mathbb{Z}_+)$. Also, note that $M(H^2(\mathbb{D}))$ is isometrically isomorphic to $H^\infty(\mathbb{D})$. Therefore, $\{\varphi \in \ell_2(\mathbb{Z}_+) : \varphi f \in \ell_2(\mathbb{Z}_+)\}$ is isometrically isomorphic to $H^\infty(\mathbb{D})$. Let us define a map $\pi : M_b(\mathbb{Z}_+) \to \{\varphi \in \ell_2(\mathbb{Z}_+) : \varphi f \in \ell_2(\mathbb{Z}_+)\}$ for all $f \in \ell_2(\mathbb{Z}_+)$ by $\pi((L, R)) = L(1)$ for all $(L, R) \in M_b(\mathbb{Z}_+)$. It is immediate to notice that $\pi$ is a unital algebra homomorphism. Moreover, $\pi$ is an isometry. This completes that proof. \qed

The same set of arguments can be used to demonstrate the next result.

Proposition 4.5. The bounded multiplier algebra $M_b(\mathbb{Z}_+^n)$ is isomorphic to the Hardy algebra $H^\infty(\mathbb{D}^n)$.

Let $(\mathbb{F}_n)^+$ be the free semigroup generated by $n$-symbols $g_1, g_2, \ldots, g_n$ with identity $g_0$. Any word $\alpha \in (\mathbb{F}_n)^+$ generated by the alphabets $g_1, g_2, \ldots, g_n$ has of the form $\alpha = g_{\mu_1}g_{\mu_2} \cdots g_{\mu_k}$, where $1 \leq \mu_1, \mu_2, \ldots, \mu_k \leq n$ and $k \in \mathbb{Z}_+$. Then length of the word $\alpha$ is a non-negative integer defined as follows:

$$|\alpha| = \begin{cases} k & \text{if } \alpha = g_{\mu_1}g_{\mu_2} \cdots g_{\mu_k} \\ 0 & \text{if } \alpha = g_0, \end{cases}$$

where $1 \leq \mu_1, \mu_2, \ldots, \mu_k \leq n$. For fixed $n$, the full Fock space over $\mathbb{C}^n$ is defined as $\mathcal{F}^2(\mathbb{C}^n) = \mathbb{C}\Omega \bigoplus_{k=1}^\infty \mathbb{C}^n \otimes^k$, where $(\mathbb{C}^n)^{\otimes k}$ denotes the $k$-times tensor products of $\mathbb{C}^n$ and $\Omega$ is called the vacuum state. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. Then for a fixed word $\alpha = g_{\mu_1}g_{\mu_2}g_{\mu_3} \cdots g_{\mu_k}$, we denote

$$e_\alpha = \begin{cases} e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k} & \text{if } \alpha = g_{\mu_1}g_{\mu_2}g_{\mu_3} \cdots g_{\mu_k} \\ \Omega & \text{if } \alpha = g_0. \end{cases}$$
For $1 \leq i \leq n$, the left-creation operator $S_i$ and right creation operator $R_i$ are defined by

$$S_i(f) = e_i \otimes f, \quad R_i(f) = f \otimes e_i$$

for all $f \in \mathcal{F}^2(\mathbb{C}^n)$.

Non-commutative Hardy algebra $\mathbb{F}_n^\infty \subseteq \mathcal{F}^2(\mathbb{C}^n)$ (see [10, Equation 3.2]) introduced by G. Popescu is defined by

$$\mathbb{F}_n^\infty := \{ f \in \mathcal{F}^2(\mathbb{C}^n) : \sup_{p \in \mathcal{P}} \| f \otimes p \| < \infty \}$$

together with a norm $\| \cdot \|_\infty$ given by $\| f \| = \sup_{p \in \mathcal{P}} \| f \otimes p \|$ for all $f \in \mathbb{F}_n^\infty$.

**Lemma 4.6.** Let $(L, R) \in M_b(\mathbb{F}_n^+)$. Then there exists a flip operator $W$ acting on the full Fock space $\mathcal{F}^2(\mathbb{C}^n)$ such that

$$L(f) = W^* R(f) W$$

for all $f \in \mathcal{F}^2(\mathbb{C}^n)$.

**Proof.** Let $(L, R) \in M_b(\mathbb{F}_n^+)$. Then for all $f, g \in \mathcal{P}$, we have $f \otimes L(g) = R(f) \otimes g$. Let us choose $f = g = \Omega$ in the above equation, we obtain $L(\Omega) = R(\Omega)$. Now for $f = \Omega$, we obtain $L(g) = L(\Omega) \otimes g$ and similarly we can get $R(f) = f \otimes L(\Omega)$ for all polynomials $f, g \in \mathcal{P}$. Let us define a flip operator $W : \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ by $W(\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_\alpha) = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha e_{\tilde{\alpha}}$, where $\tilde{\alpha} = g_{i_1}g_{i_2} \ldots g_{i_1}$ whenever $\alpha = g_{i_1}g_{i_2} \ldots g_{i_1}$. Clearly $W$ is a unitary and $W^2 = 1$. Now it is straightforward to verify that $W^* (f \otimes L(\Omega))W = L(\Omega) \otimes f$ for all $f \in \mathcal{P}$. Therefore, we have $L(f) = W^* R(f) W$ for all polynomials $f \in \mathcal{P}$. Since $L, R$ are continuous, we have $L(f) = W^* R(f) W$ for all $f \in \mathcal{F}^2(\mathbb{C}^n)$. \hfill $\square$

**Remark 4.7.** Lemma 4.6 holds true as $L, R$ are bounded for each $(L, R) \in M_b(\mathbb{F}_n^+)$.  

**Proposition 4.8.** $M_b(\mathbb{F}_n^+)$ is isometrically isomorphic to non-commutative Hardy algebra $\mathbb{F}_n^\infty$.

**Proof.** Let $(L, R) \in M_b(\mathbb{F}_n^+)$. Then $L(f) = L(\Omega) \otimes f$ and $R(f) = f \otimes L(\Omega)$ for all $f \in \mathcal{P}$. By an appeal to Lemma 4.6, there exists a flip operator $W : \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ such that $L(f) = W^* R(f)W$ for all $f \in \mathcal{F}^2(\mathbb{C}^n)$. This implies that $\| L(f) \| = \| R(f) \|$ for all $f \in \mathcal{F}^2(\mathbb{C}^n)$. Now, we claim that $L(\Omega) \in \mathbb{F}_n^\infty$. To see this, we notice that $\| L(\Omega) \otimes f \| = \| L(\Omega) \| = \| L(f) \|$ for all $f \in \mathcal{F}^2(\mathbb{C}^n)$ and $\| L(\Omega) \|$ is finite. Also, notice that $\| L(\Omega) \| = \sup_{p \in \mathcal{P}} \| L(f) \|$ for all $f \in \mathcal{F}^2(\mathbb{C}^n)$. Therefore, we have $\sup_{p \in \mathcal{P}} \| L(\Omega) \otimes f \| = \| L(\Omega) \| < \infty$. This conclude that $L(\Omega) \in \mathbb{F}_n^\infty$. Hence we can define a linear map $\pi : M_b(\mathbb{F}_n^+) \to \mathbb{F}_n^\infty$ by $\pi((L, R)) = L(\Omega)$ for all $(L, R) \in M_b(\mathbb{F}_n^+)$. Moreover, $\pi$ is an isometry, as $\| (L, R) \| = \max\{ \| L \|, \| R \| \} = \sup_{p \in \mathcal{P}} \| L(\Omega) \otimes f \| = \| L(\Omega) \|_\infty$. Suppose $(L_1, R_1), (L_2, R_2) \in M_b(\mathbb{F}_n^+)$, then we have $L_1 L_2(\Omega) = L_1(\Omega) \otimes L_2(\Omega)$. This implies that $\pi((L_1, R_1)(L_2, R_2)) = \pi((L_1, R_1)) \pi((L_2, R_2))$. This completes the proof. \hfill $\square$

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