MULTI-PERIOD OPTIMAL INVESTMENT CHOICE
POST-RETIREMENT WITH INTER-TEMPORAL RESTRICTIONS
IN A DEFINED CONTRIBUTION PENSION PLAN

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ABSTRACT. This paper studies a multi-period portfolio selection problem during the post-retirement phase of a defined contribution pension plan. The retiree is allowed to defer the purchase of the annuity until the time of compulsory annuitization. A series of investment targets over time are set, and restrictions on the inter-temporal expected values of the portfolio are considered. We aim to minimize the accumulated variances from the time of retirement to the time of compulsory annuitization. Using the Lagrange multiplier technique and dynamic programming, we study in detail the existence of the optimal strategy and derive its closed-form expression. For comparison purposes, the explicit solution of the classical target-based model is also provided. The properties of the optimal investment strategy, the probabilities of achieving a worse or better pension at the time of compulsory annuitization and the bankruptcy probability are compared in detail under two models. The comparison shows that our model can greatly decrease the probability of achieving a worse pension at the compulsory time and can significantly increase the probability of achieving a better pension.

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1. **Introduction.** If we take the retirement time as a demarcation point, a defined contribution (DC for short hereafter) pension scheme can be split into two phases: the accumulation phase (or pre-retirement) and the distribution (or post-retirement) phase. In the accumulation phase, the pensioners periodically contribute their salary into the pension fund according to a given proportion. Pensioners have to face the volatility of the assets’ random return, i.e., the investment risk, which can greatly affect the accumulated wealth at retirement. Until now, the literature on how to address the investment risk during the accumulation phase is quite rich; see, for example, [11], [21], [27], [28] and [29], most of which aim to maximize the CRRA, CARA or mean-variance utility of the wealth at retirement. In addition to the investment risk during the pre-retirement phase, the risk borne by the pensioners in a DC pension plan also appears at the time of retirement. When the pensioners retire at a time of high annuity price (low bond yield), they have to accept a lower-than-expected pension income if purchasing a non-refundable life annuity at retirement is the only option. This is the so-called annuity risk. To reduce the annuity risk, in many countries, including Canada, Denmark, Japan, the UK and the USA, pensioners are permitted to defer the purchase of the life annuity at retirement; meanwhile, they can withdraw an amount from the fund for consumption and invest the remaining wealth in the financial market until the date of death or the compulsory time determined by the government. This flexibility is usually called an “income drawdown option”. In addition to the risk of low bond yield, according to some existing findings, such as [7], [13], [18] and [20], other reasons for the levels of the voluntary annuitization exhibited among retirees being extremely low might lie in the strong bequest motive or poor health situations. Therefore, the income drawdown option has some advantages over the life annuity in terms of the possibility to aim for a better annuity than that purchasable at retirement or to bequeath some of the funds to heirs in the case of an early death. As a result, the optimal control during the post-retirement phase has received substantial attention.

In the income drawdown option, the retirees have three principal degrees of freedom and three main strands of research objectives. The first freedom is to decide the investment strategy to optimize their objective, the second freedom is to decide when to buy the life annuity (if ever), and the third freedom is to decide the withdrawal amount to consume during the post-retirement phase (if necessary). Current optimal control papers on this topic will definitely choose the first freedom and only choose the other two freedoms if needed. In terms of the three main strands of research objectives, the first strand is to minimize the probability of shortfall, the second strand is to maximize the CRRA or CARA utility of the intertemporal consumption rate and the terminal wealth, and the third strand attempts to minimize the accumulated expected deviations of the consumption amount or the wealth from the desired levels by adopting the quadratic loss function. We first recall some papers on the shortfall probability. It is true that the income drawdown option has some advantages over immediate annuitization in terms of the possibility to buy a better life annuity or to bequeath some of the funds to heirs in the case of an early death. However, the retirees face a risk of outliving their wealth before the date of death or the time of compulsory annuitization. Therefore, some existing literature focuses on minimizing the shortfall probability. Among others, [23] compute the lifetime and eventual probability of ruin for retirees who have a fixed withdrawal amount for consumption, where the lifetime probability of ruin is the probability that the net wealth hits zero prior to a stochastic time.
of death, and the eventual probability of ruin is the probability that net wealth will ever hit zero for an infinitely lived individual. [2] assume that the retirees periodically withdraw a fixed amount and then minimize the probability of running out of money before the uncertain date of death with respect to German insurance and capital market conditions. [24] seek to minimize, over the admissible strategy set of the investment amount and the net consumption, the probability that the wealth of the retiree drops to zero before she dies. [4] find the minimum probability of lifetime ruin for a retiree who can purchase a deferred life annuity under the assumption that the admissible set of strategies of the annuity purchasing process is the set of increasing adapted processes.

In addition to the shortfall probability minimization, some papers focus on the optimal decision with the CARA or CRRA utility maximization. In some well-cited papers on this topic, [25] and [26] aim to maximize the expected sum of the power utility of consumption and the power utility of terminal wealth over the triples of annuitization time, consumption amount and investment amount. [12] tend to find the optimal investment strategy to minimize the expected loss of the retirees as measured by the performance of the fund against a benchmark by adopting exponential or power loss functions. [30] consider the effects of inflation and maximize the CRRA utility from the terminal real wealth by investing the fund in inflation-indexed bonds. In the third stand of research on this topic, some literature assumes that one important reason to defer annuitization is the desire to buy a better annuity at the time of compulsory annuitization and then resorts to the target-based model. In the target-based model, the consumption target and the wealth target that a retiree wishes to achieve are given in advance. Specifically, the terminal wealth target that aims to guarantee a better annuity than the one purchased at retirement is determined by the price of the annuity and the desired consumption amount at the time of compulsory annuitization. Then, the retirees aim to minimize the deviations of the obtained consumption and wealth from the desired levels. In this way, the classical target-based model attempts to increase the probability to buy a better annuity and decrease the probability to buy a worse annuity at the time of compulsory annuitization. In this strand of research, [15] investigate a continuous-time optimal investment choice for a DC pension scheme, assuming that the withdrawal amount is constant over time and that the pensioner has no bequest motive. Based on their work, there are many generations. Among others, [16] find the optimal investment and consumption choices when the annuitization time is fixed and the time of death of the retiree is random. [11] incorporates the idea of relative choice into the income drawdown option and finds the optimal investment strategy and withdrawal rate. The author also studies three classes of target: endogenous deterministic targets, exogenous deterministic targets and stochastic targets similar to the annuity products. [17] solve a problem of finding the optimal time of annuitization, and the investment and consumption strategy. [10] deal with a constrained portfolio selection problem where a no-short-selling constraint on the investment strategy and a final capital requirement constraint on the wealth level are presented. [9] studies an optimal investment-consumption problem when the annuitization time is fixed. The consumption rate is restricted to an interval to achieve a higher final annuity, and a minimum guarantee for the final annuity is given to eliminate the ruin probabilities. The approximations of the solution are obtained by the policy iteration method. The above-mentioned papers are continuous-time models. To fill the gap, [19] study the discrete-time counterpart of
They give a series of targets greater than the expected wealth and analyze the properties of the investment strategies, the accumulated withdrawal amounts, the targets and the bankruptcy probabilities under three withdrawal mechanisms. For more information on the income drawdown option, interested readers are referred to [3], [6] and [22].

This paper considers a discrete-time multi-period portfolio selection problem in an income drawdown option similar to [19]. Different from the existing literature, we assume that the inter-temporal expected value of the portfolio at each time is equal to the corresponding desired target at that time. In other words, inter-temporal restrictions are imposed in our model. For convenience, the term “target-based model” hereafter refers specifically to the model without inter-temporal restrictions, while the term “our model” refers to the model with inter-temporal restrictions in this present paper. This restriction can increase the essential difficulty in solving the problem and analyzing the related results. The mathematical analysis shows that the properties of the investment strategy in our model are essentially different from those in the target-based model. In addition, stated by [15], “In fact, retiring members of a DC scheme take the income drawdown option in hope of doing better than buying an annuity at retirement. Therefore, it makes sense for them to have the wish of being able to buy a better annuity at a certain point of time after retirement than the annuity they would have purchased had they bought it at retirement.”. However, the classical target-based model actually has a very low probability of buying a better annuity at the compulsory time. The numerical analysis either in [15] when the targets are always greater than the wealth levels or in this present paper indicates that the target-based model has an almost zero probability of buying a better annuity at the annuitization time, which is against the initial intent of the retirees. However, our numerical analysis shows that when the inter-temporal restrictions are imposed in the classical target-based model, the probability of buying a better annuity at the annuitization time can be significantly improved by at least 100 times as much as that in the target-based model. Moreover, the probability that the retirees obtain a worse income after the compulsory time is approximately half of that in the target-based model. Furthermore, these two probabilities in our model are more robust with respect to fluctuations of the financial market due to the intermediate investment restrictions.

The remainder of this paper is organized as follows. The problem formulation and the notations are described in Section 2. The optimal solution to the portfolio selection problem with intertemporal restrictions is obtained in Section 3. By mathematical or numerical analysis, we compare our model and the target-based model without intertemporal restrictions in Section 4 in terms of the optimal strategy, the expected wealth at each time, the probabilities of achieving a worse or better pension at the time of compulsory annuitization and the bankruptcy probability. Section 5 concludes this paper. Proofs of the theorems and lemmas are provided in Appendices A-E.

2. Problem formulation and notations. In this paper, we assume that a pensioner retires at time 0 with initial wealth $x_0$ and withdraws a certain income from the pension fund until the compulsory time $T$ for annuitization. Further assume that the pension fund is invested in one risk-free asset and one risky asset, and denote their gross returns over period $[n, n+1)$ by $r^n_f$ and $R^n$, respectively, where $r^n_f$ is a positive constant and $R^n$ is a non-negative random variable. Moreover,
$R_n$ is assumed to be independent of $R_m$ for all $n \neq m$, and it is also reasonable to assume that $E[R_n] > r_n > 1$ for all $n = 0, 1, \ldots, T - 1$. Over period $[n, n + 1)(n = 0, 1, \ldots, T - 1)$, the pensioner withdraws a deterministic amount $\zeta_n$ for daily-life consumption. Let $\pi_n$ be the amount invested in the risky asset; then, the wealth under the strategy $\pi$ evolves over time according to

$$X_{n+1}^\pi = \pi_n R_n + (X_n^\pi - \zeta_n - \pi_n) r_n^f$$

$$= \pi_n R_n^\pi + (X_n^\pi - \zeta_n) r_n^f, \quad n = 0, 1, \ldots, T - 1,$$  \hspace{0.5cm} (1)

where $R_n^\pi = R_n - r_n^f$ is the excess return of the risky asset. Moreover, let $r_n^e = E(R_n^e)$.

In contrast to the literature as mentioned in Section 1, this paper considers the inter-temporal investment performance during the whole decumulation phase. Specifically, we mathematically formalize the problem as follows.

$$P(\omega) \min_{\pi_0, \ldots, \pi_T} \sum_{k=1}^{T} \omega_k E_{0,x_0} (X_k^\pi - \eta_k)^2$$

$$s.t. \ E_{0,x_0} (X_k^\pi) = \eta_k, \ k = 1, 2, \ldots, T.$$ \hspace{0.5cm} (2)

This means that an acceptable value $\eta_k$ is given in advance at each time $k = 1, 2, \ldots, T$; then, the retirees hope that the expected wealth at each time is exactly equal to the corresponding value that they would accept when they are seeking to minimize the weighted sum of the risk at each time. Similar to [2], [12] and [15], we do not consider the mortality risk of the retirees. Furthermore, it should be noted that [8] and [31] also investigate the optimal portfolio selection problem with inter-temporal constraints. However, there exist essential differences between our paper and these two papers. First, the research topics are different. Our paper considers an optimal investment choice after retirement, leading to a non-self-financing model, whereas these two investigated papers do not. Second, the model settings are also different. [8] consider the following model:

$$\min_{\pi \in U} \sum_{t \in I_\alpha} \alpha(t) \text{Var}(X^\pi(t))$$

$$s.t. \ E(X^\pi(t)) \geq v(t), \ t \in I_v,$$ \hspace{0.5cm} (3)

where $\alpha(t) > 0$ is a set of numbers for $t \in I_\alpha = \{\tau_1, \tau_2, \ldots, \tau_n\}$ with $\tau_n = T$ and $v(t) > 0$ is also a set of numbers for $t \in I_v = \{\tau_1, \tau_2, \ldots, \tau_n\}$ with $\tau_n = T$. Furthermore, they assume that $I_v \subset I_\alpha$. [31] study the following model:

$$\max_{\pi} E(X_T^\pi) - \omega T \text{Var}(X_T^\pi)$$

$$s.t. \ P \{X_T^\pi \leq b_t\} \leq \varepsilon_t, \ t = 1, 2, \ldots, T - 1.$$ \hspace{0.5cm} (4)

These two models do not give the closed-form expressions for the investment strategy, but they give the steps to solve the problems.

By convex optimization theory, $P(\omega)$ can be solved via the following optimal stochastic control problem with Lagrange multipliers $\alpha_k$:

$$\text{PLM}(\omega, \alpha) \min_{\pi_0, \ldots, \pi_T} \sum_{k=1}^{T} \left[ \omega_k E_{0,x_0} (X_k^\pi - \eta_k)^2 - \alpha_k (E_{0,x_0} (X_k^\pi) - \eta_k) \right].$$ \hspace{0.5cm} (5)

According to [5] and [14], the relationship between the optimal strategies of $P(\omega)$ and $\text{PLM}(\omega, \alpha)$ is summarized in Lemma 2.1 below.
Lemma 2.1. Denote by \( \hat{\pi}_n(x_n; \alpha_1, \ldots, \alpha_T) \) and \( H(x_0; \alpha_1, \ldots, \alpha_T) \) the optimal investment and the value function of the problem \( PLM(\omega, \alpha) \), respectively. If

\[
(\alpha^*_1, \alpha^*_2, \ldots, \alpha^*_T) = \arg \max_{\alpha_1, \alpha_2, \ldots, \alpha_T} H(x_0; \alpha_1, \alpha_2, \ldots, \alpha_T)
\]

exists, then the optimal strategy of the problem \( P(\omega) \) is \( \pi^*_n(x_n) = \hat{\pi}_n(x_n; \alpha^*_1, \alpha^*_2, \ldots, \alpha^*_T) \) and the corresponding value function is \( H(x_0; \alpha^*_1, \alpha^*_2, \ldots, \alpha^*_T) \).

Now we first solve the problem \( PLM(\omega, \alpha) \). Notice that

\[
\omega_n E_{0,x_0} (X^\pi_n - \eta_n)^2 - \alpha_n (E_{0,x_0}(X^\pi_n) - \eta_n)
\]

\[= E_{0,x_0} (\omega_n(X^\pi_n)^2 - (2\omega_n\eta_n + \alpha_n)X^\pi_n) + \omega_n(\eta_n)^2 + \alpha_n\eta_n, \tag{6}\]

then, the problem \( PLM(\omega, \alpha) \) is essentially equal to the following problem:

\[
\min_{\pi_0, \ldots, \pi_{T-1}} \sum_{n=1}^{T} E_{0,x_0} (\omega_n(X^\pi_n)^2 - (2\omega_n\eta_n + \alpha_n)X^\pi_n) \tag{7}\]

in the sense that both of them have the same optimal investment strategy. To solve \( PLM(\omega, \alpha) \) and \( PL(\omega, \alpha) \), we give a general result, which is summarized in Lemma 2.2.

Lemma 2.2. Let

\[
F_n(x_n) = \min_{\pi_n, \pi_{n+1}, \ldots, \pi_{T-1}} \sum_{k=n+1}^{T} E_{n,x_n} \left[ a_k(\pi^\pi_k)^2 - b_k \pi^\pi_k \right], n = 0, 1, \ldots, T - 1,
\]

where \( \pi^\pi_k \) satisfies the wealth dynamics (1) and \( a_k > 0, k = 1, 2, \ldots, T \). Then, the optimal control is

\[
\hat{\pi}_n(x_n) = \left( \frac{b_{n+1} + \hat{B}_{n+1}}{2(a_{n+1} + \hat{A}_{n+1})} - (x_n - \zeta_n) r^n_n \right) \frac{r^n_n}{E(R^n_n)^2}, \tag{8}\]

and the corresponding value function is \( F_n(x_n) = \hat{A}_n(x_n)^2 - \hat{B}_nx_n + \hat{C}_n, \) where

\[
\hat{A}_n = (a_{n+1} + \hat{A}_{n+1}) (r^n_n)^2 \frac{\text{Var}(R^n_n)}{E(R^n_n)^2}, \tag{9}\]

\[
\hat{B}_n = \left( 2(a_{n+1} + \hat{A}_{n+1}) \zeta_n r^n_n + (b_{n+1} + \hat{B}_{n+1}) \right) \frac{r^n_n \text{Var}(R^n_n)}{E(R^n_n)^2}, \tag{10}\]

\[
\hat{C}_n = (a_{n+1} + \hat{A}_{n+1}) (r^n_n)^2 \frac{\text{Var}(R^n_n)}{E(R^n_n)^2} (\zeta_n)^2 + (b_{n+1} + \hat{B}_{n+1}) \frac{r^n_n \text{Var}(R^n_n)}{E(R^n_n)^2} \hat{\pi}_n + \hat{C}_{n+1} - \frac{(b_{n+1} + \hat{B}_{n+1})^2 (r^n_n)^2}{4(a_{n+1} + \hat{A}_{n+1}) E(R^n_n)^2}, \tag{11}\]

Specifically, let \( \hat{A}_T = \hat{B}_T = \hat{C}_T = 0 \).

Proof. See Appendix A. \( \square \)

In what follows, we will first use Lemma 2.2 to solve the problem \( PLM(\omega, \alpha) \) and then use Lemma 2.1 to solve the original problem \( P(\omega) \). Denote by \( \hat{\pi}_n(x_n; \alpha_1, \alpha_2, \ldots, \)
\(\alpha_T\) the optimal strategy of the problem \(PL(\omega, \alpha)\) and by
\[
V_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_T) = \min_{x_n} \sum_{k=n+1}^{T} E_{n,x_n} \left[ \omega_k (X_k^n)^2 - (2\omega_k \eta_k + \alpha_k) X_k^n \right],
\]
the corresponding value function. Then, the expressions of \(\tilde{\pi}_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_T)\) and \(V_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_T)\) are summarized in the following theorem.

**Theorem 2.3.** The optimal strategy of the problem \(PL(\omega, \alpha)\) is given as
\[
\tilde{\pi}_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_T) = \frac{2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}}{2(\omega_{n+1} + A_{n+1})} + \frac{r_n^c}{E(R_n^c)^2}\]
\[
- (x_n - \zeta_n) r_n^f \frac{r_n^e}{E(R_n^e)^2}, n = 0, 1, \ldots, T - 1
\]
and the corresponding value function is given as
\[
V_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_{T-1}) = A_n (x_n)^2 - B_n x_n + C_n, n = 0, 1, \ldots, T - 1,
\]
where
\[
A_n = (\omega_{n+1} + A_{n+1}) (r_n^f)^2 \frac{\text{Var}(R_n^c)}{E(R_n^c)^2},
\]
\[
B_n = 2(\omega_{n+1} + A_{n+1}) \zeta_n (r_n^f)^2 \frac{\text{Var}(R_n^c)}{E(R_n^c)^2}
\]
\[
+ (2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}) r_n^f \frac{\text{Var}(R_n^c)}{E(R_n^c)^2},
\]
\[
C_n = C_{n+1} + (\omega_{n+1} + A_{n+1}) (r_n^f)^2 \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} (\zeta_n)^2
\]
\[
+ (2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}) r_n^f \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} \zeta_n
\]
\[
- (2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}) r_n^e \frac{(r_n^e)^2}{4(\omega_{n+1} + A_{n+1})} \frac{\text{Var}(R_n^c)}{E(R_n^c)^2}.
\]

Specifically, \(A_T = B_T = C_T = 0\).

**Proof.** Referring to Lemma 2.2, when \(a_k = \omega_k\) and \(b_k = 2\omega_k \eta_k + \alpha_k\), we have the results of Theorem 2.3.

Now according to the relationship between the problems \(PLM(\omega, \alpha)\) and \(PL(\omega, \alpha)\), the global minimum value of \(PLM(\omega, \alpha)\) is
\[
H(x_0; \alpha_1, \alpha_2, \ldots, \alpha_T) = A_0 (x_0)^2 - B_0 x_0 + C_0 + \sum_{n=1}^{T} \alpha_n \eta_n + \sum_{n=1}^{T} \omega_n (\eta_n)^2
\]
and the optimal strategy is \(\tilde{\pi}_n(x_n; \alpha_1, \alpha_2, \ldots, \alpha_T)\) in (12). In the next section, according to Lemma 2.1, we tend to solve the original problem \(P(\omega)\).
3. Solution of problem \( P(\omega) \). Referring to Lemma 2.1, we have to obtain

\[
(\alpha_1^*, \alpha_2^*, \ldots, \alpha_T^*) = \arg \max_{\alpha_1, \alpha_2, \ldots, \alpha_T} H(x_0; \alpha_1, \alpha_2, \ldots, \alpha_T).
\]

However, in view of (16), we need to derive the explicit expressions of \( A_0, B_0 \) and \( C_0 \) and their partial derivatives to \( \alpha_n, n = 1, 2, \ldots, T. \) For convenience, denote by

\[
\psi_n = r_n^f \frac{\text{Var}(R_n^c)}{E(R_n^c)^2}, \quad n = 0, 1, \ldots, T - 1,
\]

\[
\theta_n = \frac{1}{2(\omega_n + A_n)} \frac{(r_{n-1})^2}{E(R_n^c)^2}, \quad n = 1, 2, \ldots, T,
\]

\[
\delta_n = 2 \sum_{k=n}^{T-1} \left( (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f + \omega_{k+1} \eta_{k+1} \right) \prod_{m=n}^k \psi_m, \quad n = 0, 1, \ldots, T - 1.
\]

By the formula

\[ Z_n = \lambda_n Z_{n+1} + \varrho_n = Z_T \prod_{m=n}^{T-1} \lambda_m + \sum_{k=n}^{T-1} \theta_k \prod_{m=n}^{k-1} \lambda_m, \]

we have the explicit expressions for \( A_n, B_n \) and \( C_n \) as follows.

\[ A_n = \sum_{k=n}^{T-1} \omega_{k+1} \prod_{m=n}^k \left( r_m^f \frac{\text{Var}(R_m^c)}{E(R_m^c)^2} \right), \quad (20) \]

\[ B_n = \left( 2(\omega_{n+1} + A_{n+1}) \zeta_n r_n^f + 2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1} \right) \psi_n = \psi_n B_{n+1} + \left( 2(\omega_{n+1} + A_{n+1}) \zeta_n r_n^f + 2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} \right) \psi_n = \sum_{k=n}^{T-1} \alpha_{k+1} \prod_{m=n}^k \psi_m + 2 \sum_{k=n}^{T-1} \left( (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f + \omega_{k+1} \eta_{k+1} \right) \prod_{m=n}^k \psi_m = \sum_{k=n}^{T-1} \alpha_{k+1} \prod_{m=n}^k \psi_m + \delta_n, \quad (21) \]

\[ C_n = C_{n+1} + (\omega_{n+1} + A_{n+1}) (r_{n}^f) \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} \left( \zeta_n \right)^2 + (2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}) r_n^f \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} \zeta_n - (2\omega_{n+1} \eta_{n+1} + \alpha_{n+1} + B_{n+1}) \frac{(r_n^c)^2}{4(\omega_{n+1} + A_{n+1})^2} \frac{\text{Var}(R_n^c)}{E(R_n^c)^2} \zeta_n \]

\[ = \sum_{k=n}^{T-1} \left[ \left( (\omega_{k+1} + A_{k+1}) (r_k^f) \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \left( \zeta_k \right)^2 \right) + (2\omega_{k+1} \eta_{k+1} + \alpha_{k+1} + B_{k+1}) r_k^f \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k - (2\omega_{k+1} \eta_{k+1} + \alpha_{k+1} + B_{k+1}) \frac{(r_k^c)^2}{4(\omega_{k+1} + A_{k+1})} \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k \right] \]
According to (20)-(21), the partial derivatives of \( A \) with respect to \( \alpha_n \) are given as

\[
\frac{\partial A_0}{\partial \alpha_n} = 0, \quad n = 1, 2, \ldots, T, \quad (24)
\]

\[
\frac{\partial B_0}{\partial \alpha_n} = \prod_{m=0}^{n-1} \psi_m, \quad n = 1, 2, \ldots, T. \quad (25)
\]

As for the partial derivative of \( C_0 \) with respect to \( \alpha_n \), according to (23), we first have

\[
\frac{\partial}{\partial \alpha_n} \sum_{k=0}^{T-1} \left( \sum_{l=k+1}^{T} \alpha_l \prod_{m=k+1}^{l-1} \psi_m \right)^2 \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \frac{(r_k^c)^2}{E(R_k^c)^2}.
\]

Substituting (21) into (22) and simplifying the terms, we have

\[
C_n = \sum_{k=n}^{T-1} (\omega_{k+1} + A_{k+1})(r_k^f)^2 \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} (\zeta_k)^2 + 2 \sum_{k=n}^{T-1} \omega_{k+1} \eta_{k+1} r_k^f \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k
\]

\[
+ \sum_{k=n}^{T-1} \frac{\sum_{l=k}^{T-1} \alpha_{l+1} \prod_{m=k+1}^{l} \psi_m + \delta_k}{4(\omega_{k+1} + A_{k+1})} \left( \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k \right)^2
\]

\[
- \sum_{k=n}^{T-1} \frac{2(2\omega_{k+1} \eta_{k+1} + \alpha_{k+1} + B_{k+1})^2}{4(\omega_{k+1} + A_{k+1})} \left( \frac{r_k^c)^2}{E(R_k^c)^2} \right)^2.
\]

Substituting (21) into (22) and simplifying the terms, we have

\[
C_n = \sum_{k=n}^{T-1} (\omega_{k+1} + A_{k+1})(r_k^f)^2 \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} (\zeta_k)^2 + 2 \sum_{k=n}^{T-1} \omega_{k+1} \eta_{k+1} r_k^f \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k
\]

\[
+ \sum_{k=n}^{T-1} \frac{\sum_{l=k}^{T-1} \alpha_{l+1} \prod_{m=k+1}^{l} \psi_m + \delta_k}{4(\omega_{k+1} + A_{k+1})} \left( \frac{\text{Var}(R_k^c)}{E(R_k^c)^2} \zeta_k \right)^2
\]

\[
- \sum_{k=n}^{T-1} \frac{2(2\omega_{k+1} \eta_{k+1} + \alpha_{k+1} + B_{k+1})^2}{4(\omega_{k+1} + A_{k+1})} \left( \frac{r_k^c)^2}{E(R_k^c)^2} \right)^2.
\]
To obtain the maximum value of \( \eta \), we solve the system of equations with respect to \( \alpha \) and (27), we have

\[
\frac{\partial}{\partial \alpha} \sum_{k=1}^{T} \left( \sum_{l=k}^{T} \alpha_i \prod_{m=k}^{l-1} \psi_m \right) \frac{(r_{k-1})^2}{4(\omega_k + A_k)} = \frac{\partial}{\partial \alpha} \sum_{k=1}^{n} \left( \sum_{l=k}^{n} \alpha_i \prod_{m=k}^{l-1} \psi_m \right) \frac{(r_{k-1})^2}{4(\omega_k + A_k)}
\]

According to (23) and (26), we have

\[
\frac{\partial}{\partial \alpha} C_0 = \sum_{k=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right) \sum_{m=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right) \sum_{m=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right)
\]

To obtain the maximum value of \( H(x; \alpha_1, \alpha_2, \ldots, \alpha_T) \), according to (16), (24)-(25) and (27), we solve the system of equations with respect to \( \alpha_1, \alpha_2, \ldots, \alpha_T \) as follows:

\[
\eta_n - x_0 \prod_{m=0}^{n-1} \psi_m + \sum_{k=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right) \sum_{m=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right) \sum_{m=0}^{n-1} \left( r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \right)
\]

Denote by \( \Gamma \) a \( T \times 1 \) vector whose \( n \)th component is given as

\[
\eta_n - x_0 \prod_{m=0}^{n-1} \psi_m + \sum_{k=1}^{n} \sum_{m=0}^{n-1} (2\omega_k \eta_k + \delta_k) \prod_{m=k}^{n-1} \psi_m + \varepsilon_n,
\]

where

\[
\varepsilon_n = \sum_{k=0}^{n-1} r_k \frac{\text{Var}(R_k)}{E(R_k)^2} \sum_{m=0}^{n-1} \prod_{m=k+1}^{n-1} \psi_m - x_0 \prod_{m=0}^{n-1} \psi_n.
\]
In addition, denote by $\Lambda$ a $T \times T$ matrix whose components are listed as follows:

$$
\Lambda_{nl} = \begin{cases} \\
\sum_{k=1}^{l} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m, & l \leq n, \\
\sum_{k=1}^{n} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m, & l \geq n + 1.
\end{cases}
$$

(30)

Thus, the system of equations (28) can be written as $\Lambda(\alpha_1, \ldots, \alpha_T)' = \Gamma$. In the following, we will prove the positive definiteness of $\Lambda$ to show that $H(x_0; \alpha_1, \ldots, \alpha_T)$ in (16) is a strictly concave function in $\alpha_1, \ldots, \alpha_T$.

**Theorem 3.1.** $\Lambda$ is a symmetric positive definite matrix and its inverse is given as

$$
\Lambda^{-1}_{11} = \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2}, \quad \Lambda^{-1}_{12} = -\frac{\psi_1}{\theta_2},
$$

$$
\Lambda^{-1}_{nn-1} = -\frac{\psi_{n-1}}{\theta_n}, \quad \Lambda^{-1}_{nn} = \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}}, \quad \Lambda^{-1}_{nn+1} = -\frac{\psi_n}{\theta_{n+1}}, \quad n = 2, 3, \ldots, T - 1,
$$

$$
\Lambda_{TT-1}^{-1} = -\frac{\psi_{T-1}}{\theta_T}, \quad \Lambda_{TT}^{-1} = \frac{1}{\theta_T}.
$$

Other elements of $\Lambda^{-1}$ are 0.

**Proof.** See Appendix B.

According to Theorem 3.1, the Hessian matrix $-\Lambda$ of $H$ in (16) is negative definite, leading to only one optimal solution to $\max_{\alpha_1, \ldots, \alpha_T} H(x_0; \alpha_1, \ldots, \alpha_T)$. Using the notations above, (28) can be written as $\Lambda(\alpha_1, \ldots, \alpha_T)' = \Gamma$ whose only optimal solution is

$$(\alpha_1^*, \ldots, \alpha_T^*)' = \Lambda^{-1}\Gamma. \tag{31}$$

Now, according to Lemma 2.1, the optimal strategy and the value function of the problem $P(\omega)$ can be summarized as in Theorem 3.2.

**Theorem 3.2.** The optimal strategy of the original problem $P(\omega)$ is

$$
\pi_n^*(x_n) = \left(2\omega_{n+1}\eta_{n+1} + \alpha_n^* + B_{n+1}(\alpha_1^*, \alpha_2^*, \ldots, \alpha_T^*)\right) \frac{\theta_{n+1}}{r_n} - \frac{r_n^2\varepsilon_n}{E(R_n^2)} (x_n - \zeta_n). \tag{32}
$$

The corresponding value function is

$$
H(x_0; \alpha_1^*, \ldots, \alpha_T^*) = A_0(x_0)^2 - B_0(\alpha_1^*, \ldots, \alpha_T^*)x_0
$$

$$
+ C_0(\alpha_1^*, \ldots, \alpha_T^*) + \sum_{n=1}^{T} \alpha_n^* \eta_n + \sum_{n=1}^{T} \omega_n(\eta_n)^2. \tag{33}
$$

In what follows, to perform some mathematical analysis of the optimal strategy, we first derive the explicit expressions for $\alpha_1^*, \ldots, \alpha_T^*$ and some formulas with respect to $\alpha_n^*$s.

**Theorem 3.3.** The expressions for $\alpha_1^*, \ldots, \alpha_T^*$ are given as

$$
\alpha_1^* = \left(1 - \frac{(\psi_1)^2}{\theta_2} \right) (\eta_1 + \varepsilon_1) - 2\omega_1\eta_1 - \frac{\psi_1}{\theta_2} (\eta_2 + \varepsilon_2)
$$

$$
-2(\omega_2 + A_2)r_1^2 \psi_1 \zeta_1, \tag{34}
$$
target results in 1 at the beginning of the period. In addition, one unit increment of the investment at the beginning of the period. This is a reasonable result. If one wants a higher target at the end of the period has a positive effect on the risky investment amount.

\[ \alpha_n^* = -\frac{\psi_n}{\theta_n} (\eta_n - \varepsilon_n) + \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_n+1} \right) (\eta_n + \varepsilon_n) - 2\omega_n \eta_n \]

\[ \alpha_T^* = -\frac{\psi_T}{\theta_T} (\eta_T - \varepsilon_T) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) - 2\omega_T \eta_T. \]

Theorem 3.4. (i) In terms of the effects of the targets \( \eta_1, \eta_2, \ldots, \eta_T \) on the investment amount, we have

\[ \frac{\partial \pi_n^*(x_0)}{\partial \eta_1} = \frac{1}{r_n^k}, \quad \frac{\partial \pi_n^*(x_0)}{\partial \eta_k} = 0, \quad k = 2, 3, \ldots, T. \]

For \( n = 1, 2, \ldots, T - 1, \)

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_k} = 0, \quad 1 \leq k \leq n - 1, \quad n + 2 \leq k \leq T, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_n} = -\frac{\psi_n}{r_n^k}, \quad \frac{\partial \pi_n^*(x_n)}{\partial \eta_{n+1}} = \frac{1}{r_n^k}. \]

(ii) In terms of the effects of \( \zeta_0, \zeta_1, \ldots, \zeta_{T-1} \), we have

\[ \frac{\partial \pi_n^*(x_n)}{\partial \zeta_n} = \frac{r_n^k}{r_n^k}, \quad n = 0, 1, \ldots, T - 1, \quad \frac{\partial \pi_n^*(x_n)}{\partial \zeta_k} = 0, \quad n \neq k. \]

Proof. See Appendix C. \( \square \)

We first study the properties of the optimal strategy \( \pi_n^*(x_n) \), which is summarized in the following theorem.

Theorem 3.4. (i) In terms of the effects of the targets \( \eta_1, \eta_2, \ldots, \eta_T \) on the investment amount, we have

\[ \frac{\partial \pi_n^*(x_0)}{\partial \eta_1} = \frac{1}{r_n^k}, \quad \frac{\partial \pi_n^*(x_0)}{\partial \eta_k} = 0, \quad k = 2, 3, \ldots, T. \]

For \( n = 1, 2, \ldots, T - 1, \)

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_k} = 0, \quad 1 \leq k \leq n - 1, \quad n + 2 \leq k \leq T, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_n} = -\frac{\psi_n}{r_n^k}, \quad \frac{\partial \pi_n^*(x_n)}{\partial \eta_{n+1}} = \frac{1}{r_n^k}. \]

(ii) In terms of the effects of \( \zeta_0, \zeta_1, \ldots, \zeta_{T-1} \), we have

\[ \frac{\partial \pi_n^*(x_n)}{\partial \zeta_n} = \frac{r_n^k}{r_n^k}, \quad n = 0, 1, \ldots, T - 1, \quad \frac{\partial \pi_n^*(x_n)}{\partial \zeta_k} = 0, \quad n \neq k. \]

Proof. See Appendix D. \( \square \)

Theorem 3.4 indicates the following: (a) In each period \( [n, n+1) \), the investment target at the end of the period has a positive effect on the risky investment amount at the beginning of the period. This is a reasonable result. If one wants a higher investment target at the end of the period, she will invest more wealth in risky assets at the beginning of the period. In addition, one unit increment of the investment target results in \( 1/r_n^k \) unit increment of the risky investment amount. Specifically, the incremental risky investment amount with respect to one unit increment of the future target decreases along with the excess return of the risky asset. We find this phenomenon as follows. As mentioned above, a larger investment target at time \( n \)
yields a larger risky investment amount at time $n - 1$, with a higher probability to obtain a larger wealth level. Consequently, when the target at time $n + 1$ is not changed, according to (32), a larger wealth level at time $n$ leads to a smaller investment amount to obtain the future target at time $n + 1$. As for the margin of decrease, one unit decrease in the investment target at time $n$ results in $\psi_n/r_n^x$ unit decrease in the risky investment amount. (c) Expect for the times $n$ and $n + 1$, changes in the targets at other times have no effect on the investment amount at time $n$. The influence of fluctuations of the target at time $k$ is absorbed by the strategy $\pi_{k+1}^*(x_{k+1})$ and will not transfer to the strategy at a more distant point in time. (d) The larger the withdrawal amount, the larger the risky investment amount in order to reach the investment target.

4. Comparison to the target-based model. In this section, we aim to compare our model with the classical target-based model without inter-temporal restrictions as follows:

$$\tilde{P}(\omega) = \min_{\pi_0,\ldots,\pi_{T-1}} \sum_{n=1}^{T} \omega_n E_{0,x_0}(X_n^\pi - \eta_n)^2$$  \hspace{1cm} (38)

where $\eta_n$ is the investment target at time $n$. In what follows, by mathematical and numerical analysis, we will compare the optimal strategy, the expected wealth at each time, the probabilities of achieving a worse or better annuity at the time of compulsory annuitization and the bankruptcy probability.

4.1. Mathematical analysis. In this subsection, we aim to compare the strategy and the expected wealth at each time between these two models by mathematical analysis. First, we need to obtain the explicit expression of the investment strategy of the problem $\tilde{P}(\omega)$. Because

$$\sum_{n=1}^{T} \omega_n E_{0,x_0}(X_n^\pi - \eta_n)^2 = \sum_{n=1}^{T} E_{0,x_0}(\omega_n(X_n^\pi)^2) - 2\omega_n X_n^\pi \eta_n + \sum_{n=1}^{T} \omega_n(\eta_n)^2,$$

according to Lemma 2.2, we can obtain the optimal investment strategy of $\tilde{P}(\omega)$.

**Theorem 4.1.** The optimal strategy is

$$\pi_n(x_n) = \left(\frac{2\omega_{n+1} \eta_n + \tilde{B}_{n+1}}{2(\omega_{n+1} + \tilde{A}_{n+1})} - (x_n - \zeta_n) r_n^f\right) r_n^x E(R_n^x)^2$$  \hspace{1cm} (39)

and the corresponding value function is

$$\tilde{H}_n(x_n) = \tilde{A}_n(x_n)^2 - \tilde{B}_n x_n + \tilde{C}_n + \sum_{k=n+1}^{T} \omega_k(\eta_k)^2,$$  \hspace{1cm} (40)

where

$$\tilde{A}_n = A_n = (\omega_{n+1} + \tilde{A}_{n+1})(r_n^f)^2 \frac{\text{Var}(R_n^x)}{E(R_n^x)^2},$$  \hspace{1cm} (41)

$$\tilde{B}_n = \left(2(\omega_{n+1} + \tilde{A}_{n+1})\zeta_n r_n^f + (2\omega_{n+1} \eta_n + \tilde{B}_{n+1})\right) \psi_n,$$  \hspace{1cm} (42)

$$\tilde{C}_n = \tilde{C}_{n+1} + (\omega_{n+1} + \tilde{A}_{n+1})(r_n^f)^2 \frac{\text{Var}(R_n^x)}{E(R_n^x)^2}(\zeta_n)^2$$

$$+ (2\omega_{n+1} \eta_n + \tilde{B}_{n+1}) r_n^f \frac{\text{Var}(R_n^x)}{E(R_n^x)^2} \zeta_n,$$
\[- \frac{(2\omega_n + 1)\eta_n + 1 + \tilde{B}_n + 1)^2}{4(\omega_n + A_n + 1)} \left( \frac{r_n^2}{E(R_n^c)^2} \right)^2, \quad n = 0, 1, \ldots, T - 1. \tag{43}\]

and \( \hat{A}_T = \tilde{B}_T = \tilde{C}_T = 0. \)

By the formula
\[
Z_n = \lambda_n Z_{n+1} + \varrho_n = Z_T \prod_{m=n}^{T-1} \lambda_m + \sum_{k=n}^{T-1} \varrho_k \prod_{m=n}^{k-1} \lambda_m
\]
and \( \hat{A}_n = A_n, \) we derive
\[
\tilde{B}_n = 2 \sum_{k=n}^{T-1} \left( (\omega_{k+1} + A_{k+1})\zeta_k r_k + \omega_{k+1}\eta_{k+1} \right) \prod_{m=n}^{k} \psi_m,
\]
leading to
\[
2\omega_{n+1}\eta_{n+1} + \tilde{B}_{n+1} = 2 \sum_{k=n+1}^{T} \omega_k \eta_k \prod_{m=n+1}^{k-1} \psi_m
\]
\[
+ 2 \sum_{k=n+1}^{T-1} (\omega_{k+1} + A_{k+1})\zeta_k r_k \prod_{m=n+1}^{k} \psi_m.
\tag{44}\]

Combining (39) and (44), it is clear that without the inter-period constraints, the optimal strategy \( \tilde{\pi}_n \) is affected by the investment targets \( \eta_{n+1}, \ldots, \eta_T \) but it has no memory about the target \( \eta_n. \) In addition, \( \tilde{\pi}_n \) is increasing along with \( \zeta_n, \ldots, \zeta_{T-1}. \)

Referring to Theorem 3.4, we can find that the properties of the optimal strategy between our model and the traditional target-based model are quite different. Moreover, the expected wealth at each time \( n \) in the two models is given by Theorem 4.2.

**Theorem 4.2.** Under the optimal strategy \( \pi_n^* \) in (32) for the problem \( P(\omega) \), we have
\[
E_{0,x_0}(X_n^{\pi_n^*}) = \eta_n, \quad n = 1, 2, \ldots, T. \tag{45}\]

In contrast, under the optimal strategy \( \tilde{\pi}_n \) in (39) for the problem \( \tilde{P}(\omega) \), we have
\[
E_{0,x_0}(X_{n+1}^{\tilde{\pi}}) = x_0 \prod_{k=0}^{n} \psi_k + \sum_{m=0}^{n} \left( 2\omega_{m+1}\eta_{m+1} + \tilde{B}_{m+1} \right) \theta_{m+1} \prod_{k=m+1}^{n} \psi_k
\]
\[
- \sum_{m=0}^{n} \zeta_m r_m \frac{\text{Var}(R_m^c)}{E(R_m^c)^2} \prod_{k=m+1}^{n} \psi_k, \tag{46}\]
where \( 2\omega_{m+1}\eta_{m+1} + \tilde{B}_{m+1} \) satisfies (44).

**Proof.** See Appendix E. \( \square \)

Theorem 4.2 shows that in our model, regardless of the situation, the expected wealth at each time is exactly equal to the target set at the corresponding time. However, in the target-based model without inter-temporal constraints, there exists a gap between the expected wealth and the investment target. In the next subsection, we tend to analyze how this gap is changed according to fluctuations in the financial market and the final target via numerical analysis.
4.2. Numerical analysis. In this subsection, due to the difficulty of the mathematical technique, we resort to numerical analysis to compare the probabilities of achieving a worse or better annuity at the time of compulsory annuitization and the bankruptcy probability. Finally, the gaps of the expected wealths at each time under the two models are also studied.

To conduct the simulations, we assume that the pensioners retire at the age of 60 and that the compulsory age for annuitization is 75. Specifically, $T = 15$. In addition, assume that the initial wealth is $x_0 = 200000$, the pensioners withdraw an amount from the fund yearly, the gross yearly return of the risk-free asset is a constant 1.02 and the yearly expectation and variance of the risky asset $R_n$ are assumed to be 1.12 and 0.4, respectively. Let $\hat{a}_k$ be the price of the annuity calculated by the female life table of the USA in 2015, where $k$ is the age of the pensioners. Then, when we adopt the above parameters, $\zeta_0 = x_0/\hat{a}_{60} = 10230$ is assumed to be the yearly withdrawal amount at time 0. In this subsection, the withdrawal amount is set to be a constant over time. This means that the withdrawal amount, as the benefit of the retiree, is designed to be the same as the amount when the life annuity is bought at the initial time. Considering the fact that the retiree defers annuitization in hopes of being able to have a better quality of life in retirement, we assume that they expect a higher retirement income after the time of compulsory annuitization, which is set to be approximately $k(>1)$ times the initial yearly withdrawal amount $\zeta_0 = x_0/\hat{a}_{60}$, where $k$ might adopt the values 1.5, 1.7 and 2.0 in this subsection. Therefore, the final wealth target $\eta_T$ is equal to or slightly greater than $(k\zeta_0)\hat{a}_{75}$. Here, we raise one example to demonstrate how the targets over time are obtained when given the final target. To study the properties of some indices in the following experiments, this subsection gives two mechanisms to decide the targets over time.

♠ Mechanism One When $k = 1.5$, for instance, $(k\zeta_0)\hat{a}_{75} \approx 181193$. The gap $200000 - 181193 = 18807$ is distributed evenly among $T$ periods. Specifically, the difference between two consecutive targets is approximately 1253, leading to the targets at each period listed in Table 1.

| $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ | $\eta_5$ |
|---|---|---|---|---|
| 198747 | 197494 | 196241 | 194988 | 193735 |

♠ Mechanism Two The investment target at time $n$ is set to be $\eta_n = (1 + n \times (k-1)/T)\zeta_0\hat{a}_n$, $n = 1, 2, \ldots, T$, which means that if the retirees buy the life annuity at time $n$, $\eta_n$ can guarantee $1 + n \times (k-1)/T$ times retirement income of that of annuitization at time 0. When $k = 1.5$, for example, the targets over time are listed in Table 2.

| $\eta_{11}$ | $\eta_{12}$ | $\eta_{13}$ | $\eta_{14}$ | $\eta_{15}$ |
|---|---|---|---|---|
| 186217 | 184964 | 183711 | 182458 | 181205 |

Except for the value of $k$, the values of the other parameters are kept throughout this subsection unless otherwise stated. In what follows, we conduct some experiments to analyze the frequency of bankruptcy and the frequencies that the retiree has a worse or better retirement income after time $T$.

Experiment 1 The frequencies that $X_T/\hat{a}_{75} < \zeta_0$

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1 data from: [http://www.mortality.org/](http://www.mortality.org/)
Table 2. The targets over time

| η_1 | η_2 | η_3 | η_4 | η_5 | η_6 | η_7 | η_8 | η_9 | η_{10} |
|------|-----|-----|-----|-----|-----|-----|-----|-----|--------|
| 201356 | 202333 | 202942 | 203160 | 202960 | 202362 | 201386 | 200057 | 198330 | 196284 |
| 193886 | 191191 | 188231 | 184908 | 181193 | 201356 | 202333 | 202942 | 198330 | 188231 |

This experiment aims to compare the frequencies at which the final wealth amount cannot guarantee a higher retirement income between our model and the target-based model, i.e., the frequencies that $X_T/a_{\theta_{75}} < \zeta_0$. Substituting (32) and (39) into (1), we obtain

$$X_{n+1}^\pi = \frac{2\omega_{n+1}\eta_{n+1} + \alpha_{n+1}^\pi + B_{n+1}(\alpha_{1}^\pi, \alpha_{2}^\pi, \ldots, \alpha_{T}^\pi)}{2(\omega_{n+1} + A_{n+1})} - \left(X_n^\pi - \zeta_n\right) r_n^f \frac{R_n^e}{E(R_n^e)^2} + \left(X_n^\pi - \zeta_n\right) r_n^f,$$

where $X_{n+1}^\pi$ is the wealth process corresponding to the optimal strategies in our model.

The frequencies that $p_1 = P(X_T^\pi/a_{\theta_{75}} < \zeta_0)$ and $p_2 = P(X_T^\pi/a_{\theta_{75}} < \zeta_0)$ are compared in Table 3. We note the following features: (a) In both two mechanisms, $p_1$ and $p_2$

Table 3. The frequencies that $X_T/a_{\theta_{75}} < \zeta_0$

| Final target | 1.5\zeta_0a_{\theta_{75}} | 1.7\zeta_0a_{\theta_{75}} | 2\zeta_0a_{\theta_{75}} |
|--------------|--------------------------|--------------------------|------------------------|
| Var(R_n) = 0.4 | \begin{tabular}{l|l} p_1: Our model & 0.2774 \\ p_2: Target-based model & 0.5394 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.2761 \\ p_2: Target-based model & 0.5497 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.2535 \\ p_2: Target-based model & 0.3975 \\ \end{tabular} |
| Var(R_n) = 0.6 | \begin{tabular}{l|l} p_1: Our model & 0.3300 \\ p_2: Target-based model & 0.6624 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.3151 \\ p_2: Target-based model & 0.5736 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.3024 \\ p_2: Target-based model & 0.4957 \\ \end{tabular} |
| Var(R_n) = 0.8 | \begin{tabular}{l|l} p_1: Our model & 0.3661 \\ p_2: Target-based model & 0.7428 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.3507 \\ p_2: Target-based model & 0.6532 \\ \end{tabular} | \begin{tabular}{l|l} p_1: Our model & 0.3360 \\ p_2: Target-based model & 0.5672 \\ \end{tabular} |

are increased with respect to the variance Var(R_n). Specifically, when the financial market has a greater volatility, the retiree has more risk to have a worse retirement.
life after time $T$. Meanwhile, $p_1$ and $p_2$ are decreasing along with the value of the final target regardless of the decision mechanisms and the value of $\text{Var}(R_n)$. The readers can refer to the third row of Table 3 when $\text{Var}(R_n) = 0.4$, for example, to find that $p_1$ is decreased from 0.2774 to 0.2535 in response to an increased final target. This is an expected result. When the final target is increased, the targets from time 1 to time $T - 1$ are increased accordingly. Because the two models want to minimize the distance between the wealth and the target, when the target is increased, the wealth will be increased with a certain probability, leading to a lower value of $p_1$ or $p_2$; (b) $p_1$ in our model is more robust to fluctuations in the financial market and changes in the final targets. For example, when $k = 1.5$, in the first mechanism of target decision, when $\text{Var}(R_n)$ is increased by 0.2, $p_1$ is increased by 0.0526 and 0.0361, while $p_2$ is increased by 0.1230 and 0.0804; in the second mechanism of target decision, $p_1$ is increased by 0.0543 and 0.0339, while $p_2$ is increased by 0.1270 and 0.0783. When $\text{Var}(R_n)$ is fixed to be 0.4, in the first decision mechanism, $p_1$ is changed by approximately 0.01 while $p_2$ is changed by approximately 0.06 and 0.08 when the final target is changed. (c) Regardless of the values of $\text{Var}(R_n)$, the final target and the decision mechanism, the frequency $p_1$ in our model is lower than $p_2$ in the target-based model. Specially, when the final target is equal to $1.5\bar{a}_{75}$, $p_2$ is approximately twice as much as $p_1$. The gap between $p_1$ and $p_2$ in both mechanisms is listed in Table 4, which indicates that the gap between $p_1$ and $p_2$ is increasing along with the fluctuations in the financial market while decreasing along with the final target. This is an expected result because $p_1$ in our model is more robust than $p_2$ with respect to the financial market and changes in the final target. From Table 4, we also know that the frequency that the retirees would buy a worse life annuity in the target-based model without intertemporal restrictions is at least 14.40% greater than that in our model.

In this experiment, the probability that the retirees receive a lower retirement income after time $T$ is considered. In the next experiment, we shall study the opposite case, that is, the retirees would have a better quality of life in retirement if they deferred annuitization.

**Experiment 2** The frequencies that $X_T/\bar{a}_{75} \geq k\bar{z}_0$

This experiment considers the frequency that the retirees can obtain a better retirement life when they defer annuitization. To this end, we assume that the retirees expect the retirement income after time $T$ is 1.5, 1.7 and 2 times the retirement income.
income when a life annuity is bought at time 0. Denote by \( g_1 = P(X_T^T / \bar{a}_{75} \geq k\zeta_0) \) and \( g_2 = P(X_T^T / \bar{a}_{75} \geq k\zeta_0) \) where \( k > 1 \) and then we obtain Table 5. Table 5 indicates the following: (a) The target-based model has an extremely small probability of guaranteeing a better quality of life after time \( T \), especially when the variance of the risky return is high or the final target is low. In contrast, this probability in our model is substantially higher, more than 100 times that in the target-based model. This conclusion holds regardless of the decision mechanism of the target, the value of \( \text{Var}(R_n) \) and the final target. (b) In our model, if the retirees adopt the first decision mechanism, the frequency at which they obtain a better quality of life after time \( T \) is more stable with respect to changes in the value of the final target. Because one important reason why the retirees defer annuitization is that they want to be able to buy a better annuity at time \( T \) than the one they could buy at retirement, it follows from Table 5 that, compared with the classical target-based model, our model with intertemporal restrictions can greatly improve the probability of ending up with a higher annuity than the one that could be purchased at retirement.

**Table 5.** The frequencies that \( X_T / \bar{a}_{75} \geq k\zeta_0 \)

| Final target | \( 1.5\zeta_0\bar{a}_{75} \) | \( 1.7\zeta_0\bar{a}_{75} \) | \( 2\zeta_0\bar{a}_{75} \) |
|--------------|-----------------|-----------------|-----------------|
| \( \text{Var}(R_n) = 0.4 \) | \( g_1 \) Our model | 0.5717 | 0.5725 | 0.5762 |
| | \( g_2 \) Target-based model | 0.0038 | 0.0045 | 0.0040 |
| \( \text{Var}(R_n) = 0.6 \) | \( g_1 \) Our model | 0.5458 | 0.5466 | 0.5493 |
| | \( g_2 \) Target-based model | 0.0036 | 0.0028 | 0.0027 |
| \( \text{Var}(R_n) = 0.8 \) | \( g_1 \) Our model | 0.5198 | 0.5167 | 0.5250 |
| | \( g_2 \) Target-based model | 0.0037 | 0.0029 | 0.0027 |

Experiment 3 The probability of bankruptcy

The retirees also want to know whether the investment can guarantee the withdrawal. Thus, the following numerical experiments aim to analyze the probability that the wealth level is below the consumption amount for the two models, i.e., the probability of bankruptcy defined in this paper. Let \( S \) be the total number of bankruptcies in \( T = 15 \) decision points. For example, \( S = 2 \) means that bankruptcy occurs two times in \( T = 15 \) decision points. Under the assumption that \( \text{Var}(R_n) = 0.4 \), we obtain Table 6. This table shows the following: (a) In both models, the probability of bankruptcy is not high. In the 20000 simulations, bankruptcy does not occur with a probability of greater than 0.7059. The reason for
Table 6. The frequencies of bankruptcies–20000 simulations

| Events       | Our model        | Target-based model |
|--------------|------------------|--------------------|
|              | Number of simulations (Frequencies) | Number of simulations (Frequencies) |
| $S = 0$      | 16355(0.8177)    | 19774(0.9887)      |
| $S \neq 0$   | 3645(0.1823)     | 226(0.0113)        |
| Mean of $S$  | 0.6833           | 0.0181             |

| Events       | Our model        | Target-based model |
|--------------|------------------|--------------------|
|              | Number of simulations (Frequencies) | Number of simulations (Frequencies) |
| $S = 0$      | 14424(0.7212)    | 18909(0.9455)      |
| $S \neq 0$   | 5576(0.2788)     | 1091(0.0546)       |
| Mean of $S$  | 1.3019           | 0.1135             |

| Events       | Our model        | Target-based model |
|--------------|------------------|--------------------|
|              | Number of simulations (Frequencies) | Number of simulations (Frequencies) |
| $S = 0$      | 16270(0.8135)    | 19667(0.9833)      |
| $S \neq 0$   | 3730(0.1865)     | 333(0.0167)        |
| Mean of $S$  | 0.8719           | 0.0297             |

| Events       | Our model        | Target-based model |
|--------------|------------------|--------------------|
|              | Number of simulations (Frequencies) | Number of simulations (Frequencies) |
| $S = 0$      | 14118(0.7059)    | 18726(0.9363)      |
| $S \neq 0$   | 5882(0.2941)     | 1274(0.0637)       |
| Mean of $S$  | 1.6798           | 0.1457             |

This is that both models aim to minimize the distance between the wealth level and the investment target, leading the wealth to fluctuate around the target. Because the value of the target at each time is set to be substantially higher than the withdrawal amount, we have a high non-bankruptcy probability. (b) The non-bankruptcy probability in our model is slightly less than that in the target-based model. The explanation is given below. The intertemporal constraints in our model can increase the value function, i.e., the accumulated distances between the wealth and the target from time 1 to time $T$, leading to greater volatility of the wealth. This greater volatility, on the one hand, leads the wealth to have a high probability of being a larger value. This is why the retirees in our model can obtain a substantially higher probability to receive a better retirement income after time $T$. On the other hand, the greater volatility can increase the probability that the wealth levels over time have small values, leading to a higher bankruptcy frequency in our model.

In light of the three experiments above, we briefly summarize the comparison between our model and the target-based model in the following table under the same assumption that $\text{Var}(R_n) = 0.4$. It follows from Table 7 that the positive effects of our model dominate the negative effects. By Table 7, the probability that the retiree has a better income after time $T$ in our model is more than 144 times
Table 7. Comparison summary

| Comparison items                                      | Our model | Target-based model | Times |
|------------------------------------------------------|-----------|--------------------|-------|
| Frequencies that $X_T / \hat{a}_{T75} < \zeta_0$    | 0.2774    | 0.5394             | 0.5143|
| $\eta_T = 1.5 \zeta_0 \hat{a}_{T75}$                |           |                    |       |
| Frequencies that $X_T / \hat{a}_{T75} \geq k \zeta_0$| 0.5717    | 0.0038             | 150.4 |
| Bankruptcy probabilities $\eta_T = 1.5 \zeta_0 \hat{a}_{T75}$ | 0.1823    | 0.0113             | 16.63 |
| Frequencies that $X_T / \hat{a}_{T75} < \zeta_0$    | 0.2535    | 0.3975             | 0.6377|
| $\eta_T = 2.0 \zeta_0 \hat{a}_{T75}$                |           |                    |       |
| Frequencies that $X_T / \hat{a}_{T75} \geq k \zeta_0$| 0.5762    | 0.0040             | 144.05|
| Bankruptcy probabilities $\eta_T = 2.0 \zeta_0 \hat{a}_{T75}$ | 0.2788    | 0.0546             | 5.106 |

| Comparison items                                      | Our model | Target-based model | Times |
|------------------------------------------------------|-----------|--------------------|-------|
| Frequencies that $X_T / \hat{a}_{T75} < \zeta_0$    | 0.2759    | 0.5125             | 0.5383|
| $\eta_T = 1.5 \zeta_0 \hat{a}_{T75}$                |           |                    |       |
| Frequencies that $X_T / \hat{a}_{T75} \geq k \zeta_0$| 0.3042    | 0.0001             | 3042  |
| Bankruptcy probabilities $\eta_T = 1.5 \zeta_0 \hat{a}_{T75}$ | 0.1865    | 0.0167             | 11.17 |
| Frequencies that $X_T / \hat{a}_{T75} < \zeta_0$    | 0.2523    | 0.3760             | 0.6710|
| $\eta_T = 2.0 \zeta_0 \hat{a}_{T75}$                |           |                    |       |
| Frequencies that $X_T / \hat{a}_{T75} \geq k \zeta_0$| 0.5427    | 0.0018             | 301.5 |
| Bankruptcy probabilities $\eta_T = 2.0 \zeta_0 \hat{a}_{T75}$ | 0.2941    | 0.0637             | 4.617 |

that of the target-based model, while the bankruptcy probability in our model is no more than 16.63 times that in the target-based model. In addition, the probability that the retiree has a worse income after time $T$ is approximately half of that in the target-based model.

**Experiment 4** The gap between $E_{0,x_0}(X^\pi_n)$ and $\eta_n$, $n = 1, 2, \ldots, T$

It follows from Theorem 4.2 that the expected wealth at each time in our model is equal to the investment target; therefore, in this experiment, we simply need to analyze the gap between the distance between $E_{0,x_0}(X^\pi_n)$ and $\eta_n$ for $n = 1, 2, \ldots, T$. The comparison is summarized in Figure 1, where Figure 1(a) and Figure 1(b) represent the comparison result corresponding to the first and second decision mechanism of the targets, respectively. In Figures 1(a)-1(b), the variance $\text{Var}(R_n)$ of the risky asset takes on values of 0.4, 0.6 and 0.8 and the $k$ related to the final investment target takes on values of 1.5, 1.7 and 2.0. Finally, in Figure 1(a), for example, in three cases corresponding to $k = 1.5, 1.7$ and 2.0, the lines “—” and “-” correspond to the gaps when $\text{Var}(R_n) = 0.4, 0.6$ and 0.8, respectively. Some comments on Figure 1 are given as follows: (a) In both decision mechanisms and regardless of the values of $k$ and $\text{Var}(R_n)$, the expected wealth at each time in the target-based model is lower than the investment target. (b) In both decision mechanisms, when $k$ is fixed, a higher variance results in a larger value of $\eta_n - E_{0,x_0}(X^\pi_n)$. This means that the expected wealth in the classical target-based model more easily
fluctuates with the volatility of the financial market. (c) When \( \text{Var}(R_n) \) is fixed, a higher final investment target (a larger \( k \)) leads to a higher deviation between \( \eta_n \) and \( E_{0,x_0}(X_n^{\bar{x}}) \), which indicates that the value of \( E_{0,x_0}(X_n^{\bar{x}}) \) is also volatile and easily changed by the final target.

\[
\eta = E_{0,x_0}(X_n^{\bar{x}}), \quad n = 1, 2, \ldots, T
\]

5. **Conclusion.** This paper considers a multi-period portfolio optimization post retirement in a defined contribution pension plan. The retirees are allowed to defer purchasing the life annuity until the time \( T \) of compulsory annuitization. From the retirement time to the time \( T \), the retirees periodically withdraw a certain amount from the pension fund for daily-life consumption and invest the remaining money in the financial market. To monitor the investment performance and increase the probability of achieving a better pension after time \( T \), in contrast to the existing literature, we set a series of investment targets over time and restrict the expected wealth at each time to be equal to the corresponding investment target. The retirees aim to find the optimal investment strategy to minimize the accumulated variances from the initial time to time \( T \). By the Lagrange multiplier technique and dynamic programming, the existence of the optimal strategy is analyzed, and its explicit expression is derived. For comparison purposes, the classical target-based model in multi-period setting is also provided in our paper. By mathematical and numerical analysis, we find that the results are distinctly different in our model and the target-based model. (1) Our investment amount is influenced only by the investment targets in two adjacent periods. More specifically, the target at the end of the time period has a positive effect on the investment amount, while the target at the beginning of the time period has a negative effect. In addition, a larger deterministic withdrawal amount leads to more wealth invested in the risky asset. In contrast, in the traditional target-based model without inter-temporal constraints, the investment strategy is affected by all the future targets and withdrawal parameters. (2) The expected wealth in our model is proved to be equal to the investment target as required. However, in the target-based model, the deviation of the expected wealth from the target is increased along with the variance of the risky asset and the value of the final target. (3) Regardless of the values of \( \text{Var}(R_n) \), the final target and the decision mechanism of the targets over time, the frequency at which the retirees buy a worse life annuity at time \( T \) in our model is lower than that in the
classical target-based model. The former is at least 14.40% lower than the latter under our parameter settings. Furthermore, this frequency in our model is more robust to fluctuations in the financial market and changes in the final targets. \(4\) Regardless of the values of \(\text{Var}(R_n)\), the final target and the decision mechanism of the targets over time, the target-based model has an extremely small probability of guaranteeing a better retirement quality of life after time \(T\). In contrast, this probability is greatly increased in our model, being more than 100 times that in the target-based model. \(5\) In both models, bankruptcy is a rare event, but the bankruptcy probability in our model is slightly higher than that in the target-based model.

Our paper also has some limitations: \(1\) The mortality risk of the retirees and the motivations of bequest are neglected. \(2\) The time of annuitization is not a control variable. However, the retirees might also care about the optimal time to buy a life annuity before the compulsory time. In future works, we will relax these assumptions and consider more general models.

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**Appendix A. The proof of Lemma 2.2.**

**Proof.** According to dynamic programming and Bellman’s principle of optimality, we have the following recursive formula of \(F_n(x_n), n = 0, 1, \ldots, T - 1:\)

\[
F_n(x_n) = \min_{\pi_n} \mathbb{E}_{n,x_n} \left[ a_{n+1}(X_{n+1}^T)^2 - b_{n+1}X_{n+1}^T + F_{n+1}(X_{n+1}^T) \right], \quad (49)
\]

with the terminal condition \(F_T(x_T) = 0\). When \(n = T - 1\), we have

\[
F_{T-1}(x_{T-1}) = \min_{\pi_{T-1}} \mathbb{E}_{T-1,x_{T-1}} \left[ a_T(X_T^{\pi_{T-1}})^2 - b_TX_T^{\pi_{T-1}} \right] \\
= \min_{\pi_{T-1}} \mathbb{E}_{T-1,x_{T-1}} \left[ a_T \left( \pi_{T-1}R_{T-1}^f + (x_{T-1} - \zeta_{T-1}) r_{T-1}^f \right)^2 \right] \\
= \min_{\pi_{T-1}} \left[ a_T \pi_{T-1}^2 \mathbb{E}(R_{T-1}^f)^2 + \left( 2a_T x_{T-1} - \zeta_{T-1} \right) r_{T-1}^f \right] \pi_{T-1}^{r_{T-1}} \\
+ a_T(x_{T-1} - \zeta_{T-1})^2 r_{T-1}^f - b_T(x_{T-1} - \zeta_{T-1}) r_{T-1}^f, \]

The optimal solution of \(\min_{\pi_{T-1}} \mathbb{E}_{T-1,x_{T-1}} \left[ a_T(X_T^{\pi_{T-1}})^2 - b_TX_T^{\pi_{T-1}} \right]\) obviously exists and is given as

\[
\pi_{T-1}(x_{T-1}) = \left( \frac{b_T}{2a_T} - (x_{T-1} - \zeta_{T-1}) r_{T-1}^f \right) \frac{r_{T-1}^f}{\mathbb{E}(R_{T-1}^f)^2}. \quad (50)
\]
Substituting (50) into the formula for $F_{T-1}(x_{T-1})$ results in

$$F_{T-1}(x_{T-1})$$

$$= -a_T \left( (x_{T-1} - \zeta_{T-1}) r_{T-1}^f - b_T \right)^2 \frac{(r_{T-1}^c)^2}{E(R_{T-1}^c)^2}$$

$$+ a_T (x_{T-1} - \zeta_{T-1})^2 r_{T-1}^f (r_{T-1}^c)^2 - b_T (x_{T-1} - \zeta_{T-1}) r_{T-1}^f$$

$$= a_T (x_{T-1} - \zeta_{T-1})^2 (r_{T-1}^f)^2 \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2}$$

$$- b_T (x_{T-1} - \zeta_{T-1}) r_{T-1}^f \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2} - \frac{(b_T)^2}{4a_T} \frac{(r_{T-1}^c)^2}{E(R_{T-1}^c)^2}.$$

Further simplification to $F_{T-1}(x_{T-1})$ yields

$$F_{T-1}(x_{T-1})$$

$$= a_T (r_{T-1}^f)^2 \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2} (x_{T-1})^2$$

$$- \left( 2a_T \zeta_{T-1} r_{T-1}^f + b_T \right) r_{T-1}^f \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2} x_{T-1}$$

$$+ a_T (r_{T-1}^f)^2 \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2} (\zeta_{T-1})^2 + b_T r_{T-1}^f \frac{\text{Var}(R_{T-1}^c)}{E(R_{T-1}^c)^2} \zeta_{T-1}$$

$$- \frac{(b_T)^2}{4a_T} \frac{(r_{T-1}^c)^2}{E(R_{T-1}^c)^2}$$

$$:= \hat{A}_{T-1}(x_{T-1})^2 - \hat{B}_{T-1} x_{T-1} + \hat{C}_{T-1}.$$  \hspace{1cm} (51)

Formulas (50) and (51) indicate that Lemma 2.2 holds for $n = T - 1$. Now we assume that the results in Lemma 2.2 hold for $T - 1, T - 2, \ldots, n + 1$, then for $n$, according to (49), we first have

$$F_n(x_n) = \min_{\pi_n} E_{n,x_n} \left[ a_{n+1}(X_{n+1}^n)^2 - b_{n+1}X_{n+1}^n + F_{n+1}(X_{n+1}^n) \right]$$

$$= \min_{\pi_n} E_{n,x_n} \left[ (a_{n+1} + \hat{A}_{n+1})(X_{n+1}^n)^2 - (b_{n+1} + \hat{B}_{n+1})X_{n+1}^n + \hat{C}_{n+1} \right].$$

Substituting (1) into the formula above and rearranging terms yields

$$F_n(x_n)$$

$$= \min_{\pi_n} \left[ (a_{n+1} + \hat{A}_{n+1})E(R_n^c)^2(\pi_n)^2 \right.$$

$$- \left( b_{n+1} + \hat{B}_{n+1} - 2(a_{n+1} + \hat{A}_{n+1}) (x_n - \zeta_n) r_{n}^f \right) r_{n}^c \pi_n \right.$$

$$+ (a_{n+1} + \hat{A}_{n+1})(x_n - \zeta_n)^2 (r_{n}^f)^2$$

$$- (b_{n+1} + \hat{B}_{n+1}) (x_n - \zeta_n) r_{n}^f + \hat{C}_{n+1}. \hspace{1cm} (52)$$

Because $a_n > 0$, $n = 1, 2, \ldots, T$, we have a positive $\hat{A}_n > 0$ for $n = 0, 1, \ldots, T - 1$ according to the recursive formula (9) of $\hat{A}_n$. As a result, $a_{n+1} + \hat{A}_{n+1} > 0$, which yields the optimal strategy

$$\hat{\pi}_n(x_n) = \left( \frac{b_{n+1} + \hat{B}_{n+1}}{2(a_{n+1} + \hat{A}_{n+1})} - (x_n - \zeta_n) r_{n}^f \right) \frac{r_{n}^c}{E(R_n^c)^2}. \hspace{1cm} (53)$$
Substituting (53) into (52) results in
\[ F_n(x_n) = - (a_{n+1} + \hat{A}_{n+1}) (x_n - \zeta_n) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} \]
\[ + (a_{n+1} + \hat{A}_{n+1})(x_n - \zeta_n)^2 (r_n^f)^2 \]
\[ - (a_{n+1} + \hat{B}_{n+1}) (x_n - \zeta_n) r_n^f \hat{C}_{n+1} \]
\[ = (a_{n+1} + \hat{A}_{n+1})(x_n - \zeta_n)^2 (r_n^f)^2 \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} \]
\[ - (a_{n+1} + \hat{B}_{n+1}) (x_n - \zeta_n) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} \]
\[ - \frac{(b_{n+1} + \hat{B}_{n+1})^2 (r_n^c)^2}{4(a_{n+1} + \hat{A}_{n+1}) E(R^c_n)^2} \hat{C}_{n+1} \]
\[ = (a_{n+1} + \hat{A}_{n+1}) (r_n^f)^2 \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} (x_n)^2 \]
\[ - \left(2(a_{n+1} + \hat{A}_{n+1}) \zeta_n r_n^f + (b_{n+1} + \hat{B}_{n+1})\right) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} x_n \]
\[ + (a_{n+1} + \hat{A}_{n+1})(r_n^f)^2 \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} (\zeta_n)^2 + (b_{n+1} + \hat{B}_{n+1}) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} \zeta_n \]
\[ + \hat{C}_{n+1} - \frac{(b_{n+1} + \hat{B}_{n+1})^2 (r_n^c)^2}{4(a_{n+1} + \hat{A}_{n+1}) E(R^c_n)^2}. \]

Denote by
\[ \hat{A}_n = (a_{n+1} + \hat{A}_{n+1}) (r_n^f)^2 \frac{\text{Var}(R^c_n)}{E(R^c_n)^2}, \]
\[ \hat{B}_n = \left(2(a_{n+1} + \hat{A}_{n+1}) \zeta_n r_n^f + (b_{n+1} + \hat{B}_{n+1})\right) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2}, \]
\[ \hat{C}_n = (a_{n+1} + \hat{A}_{n+1})(r_n^f)^2 \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} (\zeta_n)^2 + (b_{n+1} + \hat{B}_{n+1}) r_n^f \frac{\text{Var}(R^c_n)}{E(R^c_n)^2} \zeta_n \]
\[ + \hat{C}_{n+1} - \frac{(b_{n+1} + \hat{B}_{n+1})^2 (r_n^c)^2}{4(a_{n+1} + \hat{A}_{n+1}) E(R^c_n)^2}. \]

Thus, \( F_n(x_n) \) can be written as \( \hat{A}_n(x_n)^2 - \hat{B}_n x_n + \hat{C}_n \), which means that Lemma 2.2 also holds for \( n \). By mathematical induction, we complete the proof of Lemma 2.2.

\[ \square \]

Appendix B. The proof of Theorem 3.1.

**Proof.** First, we prove that \( A \) is symmetric. For \( n, l = 1, 2, \ldots, T \) and \( n < l \), according to (30), we have
\[ A_{nl} = \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m, \quad A_{ln} = \sum_{k=1}^{n} \theta_k \prod_{m=k}^{l-1} \psi_m \prod_{m=k}^{n-1} \psi_m. \]
As a result, the matrix $\Lambda$ is symmetric. Next, we will prove that $\Lambda$ is positive definite. To this end, we prove that all order principal minor determinants of $\Lambda$ are positive. When $l \leq n$, we have

$$\Lambda_{nl} = \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m, \quad \Lambda_{n+l} = \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{l-1} \psi_m.$$  

When $l \geq n + 1$, we have

$$\Lambda_{nl} = \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m, \quad \Lambda_{n+l} = \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{l-1} \psi_m + \theta_{n+1} \prod_{m=n+1}^{l-1} \psi_m.$$  

Therefore, starting from the last row, the $n + 1$th row of $\Lambda$ plus $-\psi_n$ times the $n$th row yields the following matrix $\tilde{\Lambda}$ with elements

$$\tilde{\Lambda}_{nl} = \begin{cases} 0, & l \leq n - 1, \\ \theta_n \prod_{m=n}^{l-1} \psi_m, & l \geq n. \end{cases}$$  

Clearly, $\Lambda$ and $\tilde{\Lambda}$ have the same order principal minor determinants by the elementary row transformation mentioned above. In addition, $\tilde{\Lambda}$ is an upper triangular matrix; therefore, all order principal minor determinants of $\tilde{\Lambda}$ are positive according to (54). Now, we know that all order principal minor determinants of $\Lambda$ are also positive, leading to the positive definiteness of $\Lambda$. Next, we attempt to prove the inverse of $\Lambda$. Let $Q$ be a $T \times T$ matrix, and let $Q(n, :)$ and $Q(:, n)$ be the $n$th row and $n$th column of $Q$, respectively. Therefore, we have

$$\Lambda^{-1}(1, :) \times \Lambda(:, 1) = \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \theta_1 - \frac{\psi_1}{\theta_2} \theta_1 \psi_1 = 1,$$

$$\Lambda^{-1}(1, :) \times \Lambda(:, 2) = \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \Lambda_{12} - \frac{\psi_1}{\theta_2} \Lambda_{2, 2}$$

$$= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \theta_1 \psi_1 - \frac{\psi_1}{\theta_2} \psi_1 \prod_{k=1}^{2} \theta_k \prod_{m=k}^{1} \psi_m \prod_{m=k}^{1} \psi_m = 0,$$

$$\Lambda^{-1}(1, :) \times \Lambda(:, l) = \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \Lambda_{1l} - \frac{\psi_1}{\theta_2} \Lambda_{2, l}$$

$$= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \theta_1 \prod_{m=1}^{l-1} \psi_m - \frac{\psi_1}{\theta_2} \prod_{k=1}^{2} \theta_k \prod_{m=k}^{1} \psi_m \prod_{m=k}^{1} \psi_m = 0,$$

$$l = 3, \ldots, T.$$
For $n = 2, \ldots, T - 1$,

$$\Lambda^{-1}(n, :) \times \Lambda(:, n)$$

$$= \Lambda_{nn}^{-1} \Lambda_{n-1n} + \Lambda_{nn}^{-1} \Lambda_{nn} + \Lambda_{nn+1}^{-1} \Lambda_{n+1n}$$

$$= -\frac{\psi_{n-1}}{\theta_n} \sum_{k=1}^{n-1} \theta_k \prod_{m=k}^{n-2} \psi_m \prod_{m=k}^{n-1} \psi_m$$

$$+ \left( \frac{1}{\theta_n} + \left( \frac{\psi_n}{\theta_n+1} \right)^2 \right) \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{n} \psi_m - \frac{\psi_n}{\theta_n+1} \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{n} \psi_m$$

$$= -\frac{1}{\theta_n} \sum_{k=1}^{n-1} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{n} \psi_m + \frac{1}{\theta_n} \sum_{k=1}^{n} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{n} \psi_m = 1.$$

For $l = 1, 2, \ldots, n - 1$,

$$\Lambda^{-1}(n, :) \times \Lambda(:, l)$$

$$= \Lambda_{nn}^{-1} \Lambda_{n-1l} + \Lambda_{nn}^{-1} \Lambda_{nl} + \Lambda_{nn+1}^{-1} \Lambda_{n+1l}$$

$$= -\frac{\psi_{n-1}}{\theta_n} \sum_{k=1}^{l-1} \theta_k \prod_{m=k}^{n-2} \psi_m \prod_{m=k}^{n-1} \psi_m$$

$$+ \left( \frac{1}{\theta_n} + \left( \frac{\psi_n}{\theta_n+1} \right)^2 \right) \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m - \frac{\psi_n}{\theta_n+1} \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{l-1} \psi_m$$

$$= 0.$$

For $l = n + 1, n + 2, \ldots, T$,

$$\Lambda^{-1}(n, :) \times \Lambda(:, l)$$

$$= \Lambda_{nn}^{-1} \Lambda_{n-1l} + \Lambda_{nn}^{-1} \Lambda_{nl} + \Lambda_{nn+1}^{-1} \Lambda_{n+1l}$$

$$= -\frac{\psi_{n-1}}{\theta_n} \sum_{k=1}^{l-1} \theta_k \prod_{m=k}^{n-2} \psi_m \prod_{m=k}^{n-1} \psi_m$$

$$+ \left( \frac{1}{\theta_n} + \left( \frac{\psi_n}{\theta_n+1} \right)^2 \right) \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n-1} \psi_m \prod_{m=k}^{l-1} \psi_m - \frac{\psi_n}{\theta_n+1} \sum_{k=1}^{l} \theta_k \prod_{m=k}^{n} \psi_m \prod_{m=k}^{l-1} \psi_m$$

$$= \prod_{m=n}^{l-1} \psi_m - \psi_n \prod_{m=n+1}^{l-1} \psi_m = 0.$$

Finally,

$$\Lambda^{-1}(T, :) \times \Lambda(:, T)$$

$$= \Lambda_{TT^{-1}}^{-1} \Lambda_{T-1T} + \Lambda_{TT^{-1}}^{-1} \Lambda_{TT}$$

$$= -\frac{\psi_{T-1}}{\theta_T} \sum_{k=1}^{T-1} \theta_k \prod_{m=k}^{T-2} \psi_m \prod_{m=k}^{T-1} \psi_m + \frac{1}{\theta_T} \sum_{k=1}^{T} \theta_k \prod_{m=k}^{T-1} \psi_m \prod_{m=k}^{T-1} \psi_m = 1.$$
For $l = 1, 2, \ldots, T - 1$
\[
\Lambda^{-1}(T, \cdot) \propto \Lambda(\cdot, l)
\]
\[
= \Lambda_{TT-l}^{-1} \Lambda_{T-l}^{-1} + \Lambda_{TT}^{-1} \Lambda_l
\]
\[
= -\frac{\psi_{T-1}}{\theta_T} \sum_{k=1}^{l} \frac{T-2}{\theta_k} \prod_{m=k}^{T-1} \psi_m \prod_{m=k}^{l-1} \psi_m + \frac{1}{\theta_T} \sum_{k=1}^{l} \frac{T-1}{\theta_k} \prod_{m=k}^{l-1} \psi_m \prod_{m=k}^{l-1} \psi_m = 0.
\]

Now we have proved that $\Lambda^{-1}$ in Theorem 3.1 is the inverse of $\Lambda$. \hfill \Box

Appendix C. The proof of Theorem 3.3.

Proof. According to Theorem 3.1, (21) and (31), we have
\[
\alpha_1^n = \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \left( \eta_1 - \frac{1}{\sum_{k=1}^{n-1}} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n-1} \psi_m + \varepsilon_1 \right)
\]
\[
- \frac{\psi_1}{\theta_2} \left( \eta_2 - \frac{2}{\sum_{k=1}^{n}} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n} \psi_m + \varepsilon_2 \right)
\]
\[
= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \left( \eta_1 + \varepsilon_1 \right) - \frac{\psi_1}{\theta_2} \left( \eta_2 + \varepsilon_2 \right) - (2\omega_1 \eta_1 + \delta_1) + \psi_1 (2\omega_2 \eta_2 + \delta_2)
\]
\[
= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \left( \eta_1 + \varepsilon_1 \right) - 2\omega_1 \eta_1 - \frac{\psi_1}{\theta_2} \left( \eta_2 + \varepsilon_2 \right) - 2(\omega_2 + A_2) r_1^l \psi_1 \zeta_1,
\]
which indicates that (34) holds. For $n = 2, 3, \ldots, T - 1$,
\[
\alpha_n = -\frac{\psi_{n-1}}{\theta_n} \left( \eta_{n-1} - \frac{n-1}{\sum_{k=1}^{n-1}} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n-1} \psi_m + \varepsilon_{n-1} \right)
\]
\[
+ \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}} \right) \left( \eta_n - \frac{n}{\sum_{k=1}^{n}} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n} \psi_m + \varepsilon_n \right)
\]
\[
- \frac{\psi_n}{\theta_{n+1}} \left( \eta_{n+1} - \frac{n+1}{\sum_{k=1}^{n+1}} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n+1} \psi_m + \varepsilon_{n+1} \right)
\]
\[
= -\frac{\psi_{n-1}}{\theta_n} \left( \eta_{n-1} + \varepsilon_{n-1} \right) + \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}} \right) \left( \eta_n + \varepsilon_n \right) - \frac{\psi_n}{\theta_{n+1}} \left( \eta_{n+1} + \varepsilon_{n+1} \right)
\]
\[
+ \frac{\psi_{n-1}}{\theta_n} \sum_{k=1}^{n-1} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n-2} \psi_m
\]
\[
- \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}} \right) \sum_{k=1}^{n} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n-1} \psi_m
\]
\[
+ \frac{\psi_n}{\theta_{n+1}} \sum_{k=1}^{n+1} (2\omega_k \eta_k + \delta_k) \sum_{m=k}^{n} \psi_m
\]
\[
= -\frac{\psi_{n-1}}{\theta_n} \left( \eta_{n-1} + \varepsilon_{n-1} \right) + \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}} \right) \left( \eta_n + \varepsilon_n \right) - \frac{\psi_n}{\theta_{n+1}} \left( \eta_{n+1} + \varepsilon_{n+1} \right)
\]
\[
- (2\omega_n \eta_n + \delta_n) + \psi_n (2\omega_{n+1} \eta_{n+1} + \delta_{n+1}).
\]
According to (21), we have

\[-(2\omega_n\eta_n + \delta_n) + \psi_n(2\omega_{n+1}\eta_{n+1} + \delta_{n+1})\]

\[= -2 \sum_{k=n}^{T} \omega_k \eta_k \prod_{m=n}^{k-1} \psi_m - 2 \sum_{k=n}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=n}^{k} \psi_m \]

\[+ 2 \sum_{k=n+1}^{T} \omega_k \eta_k \prod_{m=n}^{k-1} \psi_m + 2 \sum_{k=n+1}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=n}^{k} \psi_m \]

\[= -2\omega_n \eta_n - 2(\omega_{n+1} + A_{n+1})r_n^f \psi_n \zeta_n.\]

Consequently, the expression of \(\alpha_n^*\) is simplified as

\[\alpha_n^* = -\psi_n^{-1} \eta_{n-1} (\eta_{n-1} + \varepsilon_{n-1}) + \left( \frac{1}{\theta_n} + \frac{(\psi_n)^2}{\theta_{n+1}} \right) (\eta_n + \varepsilon_n) - 2\omega_n \eta_n\]

which indicates that (35) holds. For \(n = T\), (21) indicates that \(\delta_T = 0\), leading to

\[\alpha_T^* = -\psi_T^{-1} \eta_{T-1} + \sum_{k=1}^{T-1} (2\omega_k \eta_k + \delta_k) \theta_k \prod_{m=k}^{T-1} \psi_m + \varepsilon_{T-1}\]

\[+ \frac{1}{\theta_T} \left( \eta_T - \sum_{k=1}^{T} (2\omega_k \eta_k + \delta_k) \theta_k \prod_{m=k}^{T-1} \psi_m + \varepsilon_T \right)\]

\[= -\psi_T^{-1} (\eta_{T-1} + \varepsilon_{T-1}) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) - (2\omega_T \eta_T + \delta_T)\]

\[= -\psi_T^{-1} (\eta_{T-1} + \varepsilon_{T-1}) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) - 2\omega_T \eta_T.\]

Now, we have proved the expressions of \(\alpha_1^*, \ldots, \alpha_T^*\). In the following, we will prove the formula (37). According to (21), for \(n = 1, 2, \ldots, T\), we first have

\[2\omega_n \eta_n + \alpha_n + B_n\]

\[= \sum_{k=n-1}^{T-1} (\alpha_{k+1} + 2\omega_{k+1} \eta_{k+1}) \prod_{m=n}^{k} \psi_m + 2 \sum_{k=n}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=n}^{k} \psi_m\]

\[= \sum_{k=n}^{T} (\alpha_k + 2\omega_k \eta_k) \prod_{m=n}^{k-1} \psi_m + 2 \sum_{k=n}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=n}^{k} \psi_m.\]
For $n = 1, 2, \ldots, T - 1$, in view of the expressions of $\alpha^*_n$ and $\alpha^*_T$ above, we obtain

\[
\sum_{k=n+1}^{T} (\alpha^*_k + 2\omega_k \eta_k) \prod_{m=n+1}^{k-1} \psi_m + 2 \sum_{k=n+1}^{T} (\omega_{k+1} + A_{k+1}) \zeta_k r^f_k \prod_{m=n+1}^{k-1} \psi_m \\
= \sum_{k=n+1}^{T-1} \left( -\frac{\psi_{k-1}}{\theta_k} (\eta_{k-1} + \varepsilon_{k-1}) + \left( \frac{1}{\theta_k} + \frac{(\psi_k)^2}{\theta_{k+1}} \right) (\eta_k + \varepsilon_k) \right) \prod_{m=n+1}^{k-1} \psi_m \\
+ (\alpha^*_T + 2\omega_T \eta_T) \prod_{m=n+1}^{T-1} \psi_m \\
= \sum_{k=n+1}^{T-1} \left( -\frac{\psi_{k-1}}{\theta_k} (\eta_{k-1} + \varepsilon_{k-1}) + \left( \frac{1}{\theta_k} + \frac{(\psi_k)^2}{\theta_{k+1}} \right) (\eta_k + \varepsilon_k) \right) \prod_{m=n+1}^{k-1} \psi_m \\
+ (\alpha^*_T + 2\omega_T \eta_T) \prod_{m=n+1}^{T-1} \psi_m \\
= -\sum_{k=n+1}^{T-1} \frac{\psi_{k-1}}{\theta_k} (\eta_{k-1} + \varepsilon_{k-1}) \prod_{m=n+1}^{k-1} \psi_m + \sum_{k=n+1}^{T-1} \frac{1}{\theta_k} (\eta_k + \varepsilon_k) \prod_{m=n+1}^{k-1} \psi_m \\
+ \sum_{k=n+1}^{T-1} \left( \frac{(\psi_k)^2}{\theta_{k+1}} (\eta_k + \varepsilon_k) \prod_{m=n+1}^{k-1} \psi_m - \frac{\psi_k}{\theta_{k+1}} (\eta_{k+1} + \varepsilon_{k+1}) \prod_{m=n+1}^{k-1} \psi_m \right) \\
+ \left( -\frac{\psi_{T-1}}{\theta_T} (\eta_{T-1} + \varepsilon_{T-1}) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) \right) \prod_{m=n+1}^{T-1} \psi_m \\
= -\sum_{k=n}^{T-2} \frac{\psi_{k-1}}{\theta_k} (\eta_{k} + \varepsilon_{k}) \prod_{m=n+1}^{k} \psi_m + \sum_{k=n+1}^{T-1} \frac{1}{\theta_k} (\eta_k + \varepsilon_k) \prod_{m=n+1}^{k-1} \psi_m \\
+ \sum_{k=n+1}^{T-1} \frac{\psi_k}{\theta_{k+1}} (\eta_k + \varepsilon_k) \prod_{m=n+1}^{k} \psi_m - \sum_{k=n+2}^{T} \frac{1}{\theta_k} (\eta_k + \varepsilon_k) \prod_{m=n+1}^{k-1} \psi_m \\
+ \left( -\frac{\psi_{T-1}}{\theta_T} (\eta_{T-1} + \varepsilon_{T-1}) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) \right) \prod_{m=n+1}^{T-1} \psi_m \\
= \frac{1}{\theta_{n+1}} (\eta_{n+1} + \varepsilon_{n+1}) - \frac{\psi_n}{\theta_{n+1}} (\eta_n + \varepsilon_n).
\]
Therefore, according to (29), we derive

\[
\left( \sum_{k=n+1}^{T} (\alpha_k^* + 2\omega_k \eta_k) \prod_{m=n+1}^{k-1} \psi_m + 2 \sum_{k=n+1}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=n+1}^{k} \psi_m \right) \theta_{n+1}
\]

\[
= \eta_{n+1} - \psi_n \eta_n + \varepsilon_{n+1} - \psi_n \varepsilon_n
\]

\[
= \eta_{n+1} - \psi_n \eta_n + \sum_{k=0}^{n} \frac{r_k}{E(R_k^e)^2} \frac{\text{Var}(R_k^e)}{E(R_k^e)^2} \prod_{m=k+1}^{n} \psi_m - x_0 \prod_{m=0}^{n} \psi_m
\]

\[
= \eta_{n+1} - \psi_n \eta_n + r_n \frac{\text{Var}(R_n^e)}{E(R_n^e)^2} \zeta_n,
\]

which indicates that (37) holds for \( n \geq 1 \). For \( n = 0 \), we have

\[
\sum_{k=1}^{T} (\alpha_k^* + 2\omega_k \eta_k) \prod_{m=1}^{k-1} \psi_m + 2 \sum_{k=1}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=1}^{k} \psi_m
\]

\[
= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) \left( \eta_1 + \varepsilon_1 \right) - \frac{\psi_1}{\theta_2} \left( \eta_2 + \varepsilon_2 \right) - 2(\omega_2 + A_2) r_1^f \psi_1 \zeta_1
\]

\[
+ \sum_{k=2}^{T-1} (\alpha_k^* + 2\omega_k \eta_k) \prod_{m=1}^{k-1} \psi_m + (\alpha_T^* + 2\omega_T \eta_T) \prod_{m=1}^{T-1} \psi_m
\]

\[
+ 2 \sum_{k=1}^{T-1} (\omega_{k+1} + A_{k+1}) \zeta_k r_k^f \prod_{m=1}^{k} \psi_m
\]
\[ \begin{aligned}
&= \left( \frac{1}{\theta_1} + \frac{(\psi_1)^2}{\theta_2} \right) (\eta_1 + \varepsilon_1) - \frac{\psi_1}{\theta_2} (\eta_2 + \varepsilon_2) - 2(\omega_2 + A_2) \phi_1^T \psi_1 \zeta_1 \\
&\quad + \sum_{k=2}^{T-1} \left( \frac{\psi_{k-1}}{\theta_k} (\eta_{k-1} + \varepsilon_{k-1}) + \left( \frac{1}{\theta_k} + \frac{(\psi_k)^2}{\theta_{k+1}} \right) (\eta_k + \varepsilon_k) \right) \prod_{m=1}^{k-1} \psi_m \\
&\quad + \frac{\psi_{T-1}}{\theta_T} (\eta_{T-1} + \varepsilon_{T-1}) + \frac{1}{\theta_T} (\eta_T + \varepsilon_T) \prod_{m=1}^{T-1} \psi_m \\
&\quad + \frac{2}{\theta_1} (\eta_1 + \varepsilon_1) - \frac{T-2}{\theta_1} \psi_k \prod_{m=1}^{T-2} \psi_m - \sum_{k=3}^{T-2} \frac{1}{\theta_k} (\eta_k + \varepsilon_k) \prod_{m=1}^{k-1} \psi_m \\
&\quad + \frac{2}{\theta_1} \psi_k \prod_{m=1}^{T-2} \psi_m - \sum_{k=3}^{T-2} \frac{1}{\theta_k} (\eta_k + \varepsilon_k) \prod_{m=1}^{k-1} \psi_m \\
&\quad = \frac{1}{\theta_1} (\eta_1 + \varepsilon_1).
\end{aligned} \]

As a result,

\[ \left( \sum_{k=1}^{T} (\alpha_k^2 + 2\omega_k \eta_k) \prod_{m=1}^{k-1} \psi_m + 2 \sum_{k=1}^{T-1} (\omega_{k+1} + A_{k+1}) \phi_k^T \psi_k \prod_{m=1}^{k} \psi_m \right) \theta_1 = \eta_1 + \varepsilon_1 = \eta_1 + r_0 \frac{\text{Var}(R_0^t)}{E(R_0^t)^2} \varsigma_0 - x_0 \psi_0. \]
Appendix D. The proof of Theorem 3.4.

Proof. Referring to (32) and (37), we have

\[ \frac{\partial \pi_n^*(x_0)}{\partial \eta_1} = \frac{1}{r_0^n}, \quad \frac{\partial \pi_n^*(x_0)}{\partial \eta_k} = 0, \quad k = 2, 3, \ldots, T, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_k} = 0, \quad k = 1, 2, \ldots, n - 1, n + 2, \ldots, T, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \eta_n} = -\frac{\psi_n}{r_n^n}, \quad \frac{\partial \pi_n^*(x_n)}{\partial \eta_{n+1}} = \frac{1}{r_n^n}, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \zeta_n} = \frac{r_n^n}{r_n^n} \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} + r_n^n \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} = \frac{r_n^n}{r_n^n}, \quad n = 0, 1, \ldots, T - 1, \]

\[ \frac{\partial \pi_n^*(x_n)}{\partial \zeta_k} = 0, \quad n \neq k. \]

\[ \square \]

Appendix E. The proof of Theorem 4.2.

Proof. In view of (37), substituting \( \pi_n^0 \) in (32) when \( n = 0 \) into (1) and then taking the expectation yield

\[ E_{0,x_0}(X_1^n) = \pi_n^0 r_n^n + (x_0 - \zeta_0) r_0^n \]

\[ = \eta_1 + r_0^n \frac{\text{Var}(R_0^n)}{E(R_0^n)^2} \zeta_0 - x_0 \psi_0 - (x_0 - \zeta_0) r_0^n \frac{(r_0^n)^2}{E(R_0^n)^2} + (x_0 - \zeta_0) r_0^n \]

\[ = \eta_1 + r_0^n \frac{\text{Var}(R_0^n)}{E(R_0^n)^2} \zeta_0 - x_0 \psi_0 + (x_0 - \zeta_0) r_0^n \frac{\text{Var}(R_0^n)}{E(R_0^n)^2} \]

\[ = \eta_1 - x_0 \psi_0 + x_0 r_0^n \frac{\text{Var}(R_0^n)}{E(R_0^n)^2} = \eta_1. \]

Now we assume that \( E_{0,x_0}(X_1^n) = \eta_n \) holds, then for \( n + 1 \), similarly we have

\[ E_{0,x_0}(X_{n+1}^n) = \pi_n^r r_n^n + (E_{0,x_0}(X_{n+1}^n) - \zeta_n) r_n^n \]

\[ = \eta_{n+1} - \psi_n \eta_n + \eta_n \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} \zeta_n + \left( E_{0,x_0}(X_{n+1}^n) - \zeta_n \right) r_n^n \]

\[ - \left( E_{0,x_0}(X_{n+1}^n) - \zeta_n \right) r_n^n \frac{(r_n^n)^2}{E(R_n^n)^2} \]

\[ = \eta_{n+1} - \psi_n \eta_n + \eta_n \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} \zeta_n \]

\[ + \left( E_{0,x_0}(X_{n+1}^n) - \zeta_n \right) r_n^n \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} \]

\[ = \eta_{n+1} - \psi_n \eta_n + \eta_n r_n^n \frac{\text{Var}(R_n^n)}{E(R_n^n)^2} = \eta_{n+1}. \]

By mathematical induction, we prove that in our model, the expected wealth under the optimal strategy is exactly equal to the investment target. In the following, we
will prove (46) holds. Substituting (39) into (1) leads to

\[
X_{n+1}^π = \left( \frac{2ω_n+1η_n+1 + 2}{2(ω_n+1 + A_n+1)} \right) \left( X_n^π - ζ_n r^f \right) \frac{r_n^c R_n^c}{E(R_n^c)^2} \\
+ (X_n^π - ζ_n) r^f_n.
\]

Taking the expectation of both sides of the above formula yields

\[
E_{0,x_0} \left( X_{n+1}^π \right) = \frac{2ω_n+1η_n+1 + 2}{2(ω_n+1 + A_n+1)} \left( \frac{r_n^c}{E(R_n^c)^2} \right) - ζ_n r^f_n Var(R_n^c) \]

\[
\frac{2(ω_n+1 + A_n+1)}{E(R_n^c)^2} E_0,x_0 \left( X_{n+1}^π \right).
\]

(55) gives a recursive equation which can be solved to find out

\[
E_{0,x_0} \left( X_{n+1}^π \right) = x_0 \prod_{k=0}^{n} \psi_k + \sum_{m=0}^{n} \left( 2ω_m+1η_m+1 + 2 \right) θ_{m+1} - ζ_m r^f_m \frac{Var(R_m^c)}{E(R_m^c)^2} \prod_{k=m+1}^{n} \psi_k
\]

\[
= x_0 \prod_{k=0}^{n} \psi_k + \sum_{m=0}^{n} \left( 2ω_m+1η_m+1 + 2 \right) θ_{m+1} \prod_{k=m+1}^{n} \psi_k
\]

\[- \sum_{m=0}^{n} ζ_m r^f_m \frac{Var(R_m^c)}{E(R_m^c)^2} \prod_{k=m+1}^{n} \psi_k.
\]

Now we complete the proof of Theorem 4.2.

\[\square\]

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