How the mathematical understanding of the theory of dynamical systems can be improved with computer-assisted experimentation

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This article presents several examples of how the theory of dynamical systems can be studied by exploration (in particular, by using computer simulations), shows typical cases where the inclusion of computer tools is useful for learning and understanding these dynamical systems, and also points out typical difficulties and limitations of this approach.

The application of computer simulations to the study of the theory of dynamical systems is a particularly interesting case of seamless learning.

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1 The concept of seamless learning and its role for understanding mathematics

Seamless learning is a new approach to learning and teaching. It emphasized the use of electronic media to assist traditional learning methods. Computers, particularly portable ones such as laptop computers and smartphones, have become very widespread and easy to use.

The author of this article is interested in teaching and learning mathematical theories, particularly the theory of dynamical systems, and believes that it is particularly promising to use computer simulations in order to study some aspects of dynamical systems theory.

The learning-theoretical aspects of computer simulations in mathematics and in the natural sciences in general are discussed in more detail in [1].

2 Dynamical systems theory and exploratory ways to study it

Dynamical systems theory (see e.g. [2] for a more comprehensive overview) consists of two main parts: the study of discrete dynamical systems which are usually modeled as maps \( f : X \rightarrow X \) and the study of continuous dynamical systems which are usually modeled as ordinary differential equations \( \frac{d}{dt} x(t) = f(x) \). In both cases, the set \( X \) is the state space of the system. The map \( f \) describes the time evolution of the system. Time is either discrete (integers) or continuous (real numbers). Often there is additional structure on \( X \), e.g. a measure or the structure of a smooth manifold. There are several relevant generalizations of these concepts, such non-autonomous systems, non-deterministic systems, and partial differential equations such as the wave equation \( \frac{\partial^2}{\partial t^2} \phi = c^2 \cdot \Delta \phi \) which describes the propagation of waves.

The theory of dynamical systems places particular emphasis on the study of the long-term and asymptotic behavior of such systems. This includes attractors, periodic points, closed orbits, limit cycles, invariant sets (particularly invariant tori), growth rates of periodic orbits, ergodicity, and mixing behavior.

The focus on long-term behavior makes it very hard to make useful predictions via numerical simulations, harder than in most branches of applied mathematics and numerical analysis. Many problems of applied and numerical mathematics are static or on a short time scale. Thus, dynamical systems theory needs some additional methods for using computer simulations.

For example, it is possible to accurately predict the weather in the short term (shorter than a few days) with high spatial resolution, but predicting the weather more than a few days into the future is very error prone. This is a typical example of the butterfly effect, where small differences in initial conditions give rise to huge differences in the system state after large time.

In particular, climate modeling is more difficult than short-term weather forecasting. These are similar tasks, but climate modeling needs to make predictions on a longer time scale. Hence climate simulations use simplified models with fewer variables and make less detailed predictions.

3 Types of dynamical systems that can be studied well with computer simulations

Examples of dynamical systems that are especially easy to study via computer simulations include time-discrete systems having finite state space, and systems where the relevant parts of the evolution happen in a finite subspace. This applies to cellular automata and finite-state machines. Researchers and students who wish to study these systems often benefit from exploring them early on via computer simulations, and attempt only afterwards to make precise theoretical statements of their limit behavior. In many cases, it is possible to easily observe phenomena whose existence is hard to prove rigorously. (In some cases it is the other way around. E.g., any system with finitely many states is globally (pre-)periodic. This is very easy
to prove rigorously. But if the period is too large then this periodicity will not be seen in a simulation. This shows that computer simulations on their own are not enough. Theoretical arguments and mathematical proofs are also needed.)

The next simplest class of systems are those with finite-dimensional (preferably compact) state space, especially if the dynamics (both map and underlying space) are smooth. However, it may still be difficult to make meaningful observations in computer simulation which correspond to a long-term or asymptotic phenomenon in the real system.

Of particular interest are systems where the occurrence of a relevant abstract mathematical object is rigorously proven just by the approximate occurrence in a discretized version of the system. A well-known case is the shadowing theorem [2, ch. 10.2] in the theory of hyperbolic dynamical systems which shows (among other things) that if a computer simulation of the discretized system finds an $\epsilon$-pseudo-orbit which is $\epsilon$-closed (this is a sequence of points such that each point is mapped to the $\epsilon$-neighborhood of the next point in the sequence, and the last point is mapped to the $\epsilon$-neighborhood of the first one) near a hyperbolic set then the original system (not the discretized system in the computer simulation) must necessarily have a closed orbit (an actual closed orbit, not just an $\epsilon$-closed $\epsilon$-pseudo-orbit).

The existence of an attractor for a dynamical system can be difficult to prove theoretically, whereas a simulation may show it easily. (On the other hand, examples exist of systems whose numerical simulation seems to show very clearly what appears to be an attractor; however, after a much larger number of iterations, that “attractor” disappears.)

The attractor may have a fractal structure which may be easily understood visually (but which may appear unusual to someone trying to study it theoretically without seeing it). E.g., the chaos game starts with the points $a_1$, $a_2$, $a_3$ of an equilateral triangle in the plane. Each point $x$ is mapped to $(x + a_1)/2$, the midpoint between $x$ and $a_i$, where $i$ is chosen at random. (This is a random dynamical system.) Typical Orbits approximate the Sierpinski triangle, a self-similar fractal. This system is easily programmed on a computer (including appealing graphical output). There are many more systems that produce geometrically interesting fractals, including the Menger Sponge, fern-like and tree-like fractals, and other self-similar patterns.

A discretized version of a dynamical system may show behavior that has nothing to do with the behavior of the original system. One of the most obvious examples is the fact that, if the discretized system is finite, it must necessarily be (pre-)periodic, even if the original system is not periodic. Moreover, there may be recurrence in the discretized version that is not present in the original. E.g., the cat map on the 2-torus $f : [0, 1]^2 \to [0, 1]^2$, $(x, y) \mapsto (2x + y, x + y)$ is an Anosov map, hyperbolic, mixing and non-periodic. But its discretized version is periodic, and returns to the original state after a surprisingly small number of iterations. Moreover, for an even lower number of iterations, “ghosts” of the original state can be seen [3].

4 Types of dynamical systems that are difficult to study with computer simulations

Dynamical systems with infinite but finite-dimensional state space are more difficult to simulate in a computer than those with finite state space, since the computer simulation needs to model the state by some finite set. Discretized versions may not have the same long-term dynamical behavior as the original. Infinite-dimensional systems are even harder to simulate on a computer; they may not have a finite-dimensional approximation whose dynamic behavior is similar to the original system.

Systems of any dimension (even low-dimensional systems) can have sensitive dependence on initial conditions (the butterfly effect). This means that the result of numerical simulations may diverge exponentially from the true behavior of the system. This can make it very hard to study long-term or asymptotic behavior. As mentioned before, weather and climate simulations can exhibit this behavior.

5 Visual fact-finding and proof vs. convincing demonstrations in geometry

An interesting mathematical problem in geometry is the sphere eversion: turning a 2-sphere inside out in a continuous movement while avoiding ripping and several other types of singularities. One might re-state this problem as follows: does there exist a certain type of continuous dynamical system (a suitable flow) with prescribed initial and final state (which both happen to be the 2-sphere, just with different unit normal vector field)? Several concrete constructions have been created, and computer visualizations are helpful to understand and communicate them.

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