Statistical properties of periodic points for infinitely renormalizable unimodal maps

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Abstract
For an infinitely renormalizable negative Schwarzian unimodal map $f$ with a non-flat critical point, we analyze statistical properties of periodic points as their periods tend to infinity. Since the standard sequence of probability measures constructed from periodic points weighted with Birkhoff sums of a given potential does not always converge to an equilibrium state, we consider another sequence of probability measures obtained by averaging over certain time windows. For a weight $\phi$ which is a continuous function or a geometric potential $-\beta \log |f'|$, we obtain level-2 large deviation bounds. From the upper bound, we deduce that weighted periodic points asymptotically distribute with respect to equilibrium states for the potential $\phi$. It follows that periodic points asymptotically distribute with respect to measures of maximal entropy, and periodic points weighted with their Lyapunov exponents asymptotically distribute with respect to the post-critical measure supported on the attracting Cantor set. In the case the pressure of $\phi$ is non-positive, we obtain the level-2 large deviation principle.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Hyperbolic periodic points are skeletons of smooth dynamical systems. Naively, one attempts to structure the dynamics using these points as a spine. This strategy is supported by Bowen [3] who proved that periodic points of topologically mixing Axiom A systems equidistribute with respect to the measures of maximal entropy.

Two key properties in Bowen’s proof are expansivity and specification. It is therefore natural to consider distributions of periodic points for dynamical systems lacking these two properties. For $C^\infty$ surface diffeomorphisms or interval maps with positive topological entropy, Burguet [6] proved that periodic points whose Lyapunov exponents are uniformly bounded away from zero equidistribute with respect to measures of maximal entropy. In this paper we establish a version of equidistribution of weighted periodic points for infinitely renormalizable unimodal maps. Such maps lack both expansivity and specification, and have periodic points with arbitrarily small Lyapunov exponents. We develop a large deviations approach that takes all periodic points into consideration.

The equidistribution of preimages and periodic points with respect to measures of maximal entropy has been shown in a variety of settings in complex dynamical systems (see e.g., [1, 4, 20]), although weighted periodic points were rarely considered (see [2]). For arbitrary rational maps of the Riemannian sphere, Pollicott and Sharp [23, corollary 2] used a large deviations approach to the equidistribution of preimages weighted with Hölder continuous potentials, thereby generalizing the result of Lyubich [20]. For topologically exact interval maps with non-degenerate critical points or rational maps satisfying weak hyperbolicity assumptions, large deviations results for weighted periodic points were obtained in [8, 9, 15], which imply the equidistribution of those periodic points with respect to equilibrium states. We aim to recover this type of result as much as possible for infinitely renormalizable unimodal maps.

1.1. Statements of results

Let $X$ be a compact interval. A $C^1$ map $f : X \to X$ is unimodal if it has a unique critical point $c$, that is contained in $\text{int}(X)$ and is an extremum, and satisfies $f(\partial X) \subset \partial X$. An $S$-unimodal map $f$ is a unimodal map of class $C^3$ on $X \setminus \{c\}$ with negative Schwarzian derivative $f''' / f' - (3/2)(f'' / f')^2 < 0$ such that if $x \in \partial X$ is a fixed point of $f$ then $|f'(x)| > 1$. The critical point $c$ is non-flat if there exist $\ell > 1$ and $C^1$ diffeomorphisms $\varphi$ and $\psi$ defined on a neighborhood of $c$ and $f(c)$ respectively such that $\varphi(c) = 0 = \psi(f(c))$ and $|\varphi(x)|^\ell = |\psi(f(x))|$ for all $x$ near $c$.

Let $f : X \to X$ be a unimodal map with a non-flat critical point $c$. A proper closed subinterval $J$ of $X$ is restrictive with period $p \geq 2$ if the following hold (cf [10, p 139]):

- the interiors of $J, \ldots, f^{p-1}(J)$ are pairwise disjoint;
- $f^p(J) \subset J$ and $f^p(\partial J) \subset \partial J$;
- one of the intervals $J, \ldots, f^{p-1}(J)$ contains $c$ in its interior;
- $J$ is maximal with respect to these properties: if $J' \supset J$ is a closed interval which is strictly contained in $X$ and satisfies the previous three properties with the same integer $p$, then $J' = J$.

We say $f$ is: (i) non-renormalizable if there is no restrictive interval; (ii) renormalizable if there is a restrictive interval; (iii) only finitely renormalizable if renormalizable and there are only finitely many restrictive intervals; (iv) infinitely renormalizable if there are restrictive intervals of arbitrarily high periods.
In what follows, let \( f \) be an \( S \)-unimodal map with a non-flat critical point. For all \( n \geq 1 \), the set \( \text{Fix}(f^n) = \{ x \in X : f^n(x) = x \} \) of periodic points of period \( n \) is a finite set \([25]\), and is non-empty as it contains the fixed point in \( \partial X \). Let \( \mathcal{M}(X) \) denote the space of Borel probability measures on \( X \) endowed with the weak* topology. For \( x \in X \) and \( n \geq 1 \) let \( \delta^n_x \) denote the empirical measure \( (1/n) \sum_{k=1}^{n-1} \delta_{f^k(x)} \in \mathcal{M}(X) \), where \( \delta_{f^k(x)} \in \mathcal{M}(X) \) denotes the unit point mass at \( f^k(x) \). For a function \( \phi : X \to \mathbb{R} \) and \( n \geq 1 \), we write \( S_n \phi \) for the Birkhoff sum \( \sum_{k=0}^{n-1} \phi \circ f^k \), and introduce a standard sequence \( \{ \nu_{n,\phi} \}_{n=1}^\infty \) in \( \mathcal{M}(X) \) by

\[
\nu_{n,\phi} = \frac{1}{Z_n(\phi)} \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)} \delta^n_x,
\]

with the normalizing constant

\[
Z_n(\phi) = \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)}.
\]

If \( f \) is non-renormalizable, the standard sequence \( \{ \nu_{n,\phi} \}_{n=1}^\infty \) converges in the weak* topology under additional assumptions on \( f \) and \( \phi \). If \( f \) satisfies the Collet–Eckmann condition \([7]\), the large deviations result of Keller and Nowicki \([15\text{, theorem 1.3}]\) implies that \( \{ \nu_{n,\log |f|^\alpha} \}_{n=1}^\infty \) converges to the uniquely absolutely continuous \( f \)-invariant probability measure. If \( f \) satisfies a certain assumption on the growth of derivatives of the critical orbit that is weaker than the Collet–Eckmann condition and \( \phi \) is H"older continuous, the large deviation results of Comman and Rivera-Letelier \([9]\) and Li \([19]\) imply that \( \{ \nu_{n,\phi} \}_{n=1}^\infty \) converges to the unique equilibrium state for the potential \( \phi \). If \( f \) is only finitely renormalizable, the convergence of \( \{ \nu_{n,\phi} \}_{n=1}^\infty \) may still hold: the results \([9, 15, 19]\) can be applied to periodic points contained in the deepest renormalization cycle, and the remaining periodic points are contained in the union of other finitely many renormalization cycles, which can be treated with the classical argument \([3]\).

We stress that if \( f \) is infinitely renormalizable, the standard sequence \( \{ \nu_{n,\phi} \}_{n=1}^\infty \) may not converge to an equilibrium state for the potential \( \phi \). This is essentially due to the sparseness of periodic points in the following sense: let \( m \geq 1 \) and let \( K_m \) be the \( m \)-th renormalization cycle of period \( p_m \), to be defined after corollary 1.1. If \( p_m \uparrow n \), there is no periodic point of period \( n \) in \( K_m \). Therefore, the standard sequence is not useful in the analysis of distribution of periodic points of infinitely renormalizable maps.

In order to compensate the sparseness and recover the convergence to equilibrium states, we take into consideration more periodic points of different periods. We introduce a scale \( \Delta \geq 0 \), and define a probability measure \( \mu_{n,\phi,\Delta} \) by taking an average over the time window

\[
\Delta_n = [n, (1 + \Delta)n] \cap \mathbb{N}.
\]

This means that we consider a sequence \( \{ \mu_{n,\phi,\Delta} \}_{n=1}^\infty \) in \( \mathcal{M}(X) \) obtained by normalizing the measures \( \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)} \delta^n_x \), namely

\[
\mu_{n,\phi,\Delta} = \frac{1}{\sum_{q \in \Delta_n} Z_q(\phi)} \sum_{q \in \Delta_n} Z_q(\phi) \nu_{q,\phi}.
\]

For example, in the case \( \phi \equiv 0 \) we have

\[
\mu_{n,\phi,\Delta} = \frac{1}{\sum_{q \in \Delta_n} \# \text{Fix}(f^q)} \sum_{x \in \bigcup_{q \in \Delta_n} \text{Fix}(f^q)} \# \{ q \in \Delta_n : x \in \text{Fix}(f^q) \} \delta_x.
\]
where the symbol \( \# \) denotes the cardinality of a finite set.

Since the elements of the standard sequence are all \( f \)-invariant, \( \mu_{n, \varphi, \Delta} \) is \( f \)-invariant too. However, \( \{ \mu_{n, \varphi, \Delta} \}_{n=1}^{\infty} \) should not be interpreted as an analogue of the standard sequence. For example, \( \mu_{n, \varphi, \Delta} \) and \( \mu_{n, \varphi + \alpha \Delta} \), \( \alpha \in \mathbb{R} \setminus \{0\} \) may not coincide, while the standard sequence is invariant under the shift of the potential.

For the analysis of \( \{ \mu_{n, \varphi, \Delta} \}_{n=1}^{\infty} \), it is convenient to consider a sequence \( \{ \tilde{\mu}_{n, \varphi, \Delta} \}_{n=1}^{\infty} \) of Borel probability measures on \( \mathcal{M}(X) \) given by

\[
\tilde{\mu}_{n, \varphi, \Delta} = \frac{1}{Z_n(\varphi)} \sum_{q \in \Delta_n} Z_q(\varphi) \tilde{\nu}_{q, \varphi},
\]

where \( \{ \tilde{\nu}_{n, \varphi} \}_{n=1}^{\infty} \) is a sequence of Borel probability measures on \( \mathcal{M}(X) \) given by

\[
\tilde{\nu}_{n, \varphi} = \frac{1}{Z_n(\varphi)} \sum_{\chi \in \text{Fam}(P_n)} e^{\chi_n(X)} \delta_{\chi},
\]

and \( \delta_{\chi} \) denotes the unit point mass at \( \chi \).

Let \( \mathcal{M}(X, f) \) denote the set of elements of \( \mathcal{M}(X) \) which are \( f \)-invariant. For \( \mu \in \mathcal{M}(X, f) \), let \( h(\mu) \) denote the measure-theoretic entropy of \( \mu \) relative to \( f \), and define its Lyapunov exponent by \( \chi(\mu) = \int \log |f'| \, d\mu \). Let \( C(X) \) denote the set of real-valued continuous functions on \( X \). For a function \( \varphi \in C(X) \cup \{ -\beta \log |f'| \}_{\beta \in \mathbb{R}} \) we introduce a free energy \( F_{\varphi} : \mathcal{M}(X, f) \to \mathbb{R} \) by

\[
F_{\varphi}(\mu) = h(\mu) + \int \varphi \, d\mu,
\]

and an associated pressure

\[
P(\varphi) = \sup \{ F_{\varphi}(\mu) : \mu \in \mathcal{M}(X, f) \}.
\]

Set \( \varphi_{\Delta} = \Delta \max \{ P(\varphi), 0 \} \). Our first result is stated as follows.

**Theorem A (Level-2 large deviations).** Let \( f : X \to X \) be an infinitely renormalizable S-unimodal map with a non-flat critical point, and let \( \varphi \in C(X) \cup \{ -\beta \log |f'| \}_{\beta \in \mathbb{R}} \). There exists a non-convex lower semi-continuous function \( I_{\varphi} : \mathcal{M}(X) \to [0, \infty] \) such that for any \( \Delta > 0 \) the following hold:

(a) (Lower bound) for any open set \( \mathcal{G} \subset \mathcal{M}(X) \),

\[
\liminf_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_{n, \varphi, \Delta}(\mathcal{G}) \geq -\inf_{\mathcal{G}} I_{\varphi} - \varphi_{\Delta};
\]

(b) (Upper bound) for any closed set \( \mathcal{C} \subset \mathcal{M}(X) \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_{n, \varphi, \Delta}(\mathcal{C}) \leq -\inf_{\mathcal{C}} I_{\varphi}.
\]

For the meanings of level-2 or level-1 we refer the reader to the book of Ellis [13, chapter 1]. We call \( I_{\varphi} \) the level-2 rate function, to be defined later in (1.1). The level-2 rate function is finite at invariant measures which are approximated by periodic points. Since convex combinations of such measures may not have this property due to the structure of renormalization, the level-2 rate function becomes a non-convex function.
The term $\varphi_\Delta$ in the lower bound in theorem A(a) arises as a result of the averaging over the time window $\Delta_n$. For details, see the proof of proposition 4.1(b). If the pressure is non-positive, then $\varphi_\Delta = 0$ and so theorem A yields the following the level-2 large deviation principle (LDP).

**Corollary 1.1 (The level-2 LDP).** Let $f : X \to X$ and $\varphi : X \to \mathbb{R}$ be as in theorem A. If $P(\varphi) \leq 0$, then for any $\Delta > 0$ the following hold:

(a) (Lower bound) for any open set $G \subset \mathcal{M}(X)$,

$$
\liminf_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_{n,\varphi,\Delta}(G) \geq -\inf_G I_{\varphi};
$$

(b) (Upper bound) for any closed set $C \subset \mathcal{M}(X)$,

$$
\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_{n,\varphi,\Delta}(C) \leq -\inf_C I_{\varphi}.
$$

Large deviations for measures constructed from weighted periodic points have been considered for various dynamical systems satisfying some hyperbolicity assumptions, plus some sort of specification which leads to the convexity of rate functions. The level-2 LDP for topologically mixing Axiom A systems with Hölder continuous weight functions was shown by Kifer [17]. For arbitrary rational maps of the Riemannian sphere, Pollicott and Sharp [23] obtained upper bounds for closed sets, for Hölder continuous weight functions whose supremum is strictly less than the topological pressure. For maps with neutral fixed points, Pollicott et al [24] obtained level-2 upper bounds. For non-renormalizable $S$-unimodal maps satisfying the Collet–Eckmann condition, Keller and Nowicki [15, theorem 1.3] obtained a level-1 large deviations result for periodic points weighted with their Lyapunov exponents or functions of bounded variation. For topologically exact interval maps with non-degenerate critical points or rational maps satisfying certain weak hyperbolicity assumptions, Coman and Rivera-Letelier [9], Li [19] established the level-2 LDP for periodic points weighted with Hölder continuous potentials. The weight functions treated in [9, 19] are required to be Hölder continuous, in order to apply the method in [16, theorem 3.4] that is based on the uniqueness of equilibrium states. The results in [9] have been generalized and strengthened in [8]. For large deviations on hyperbolic flows, see [18, 22].

Let $f$ and $\varphi$ be as in theorem A. In order to define the level-2 rate function $I_{\varphi}$ we need a few ingredients in the renormalization of unimodal maps. We begin by defining a strictly decreasing sequence of closed intervals

$$
X = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots
$$

which contain the critical point $c$, and a strictly increasing sequence of integers

$$
1 = p_0 < p_1 < p_2 < \cdots
$$

so that $J_m$ is restrictive with period $p_m$ for $m \geq 1$, inductively as follows. Given $J_m$ and $p_m$ for some $m \geq 0$, the restriction $f^{p_m}|_{J_m} : J_m \to J_m$ is an $S$-unimodal map. Define $J_{m+1}$ to be the restrictive interval of $f^{p_m}|_{J_m}$ with the smallest period $r_m$ and containing $c$, and set $p_{m+1} = p_m r_m$. For $m \geq 0$, the $m$th renormalization cycle is defined by

$$
K_m = \bigcup_{k=0}^{p_m-1} f^k(J_m).
$$
The omega-limit set of Lebesgue almost every initial point in $X$ coincides with the invariant attracting Cantor set

$$\Lambda = \bigcap_{m=0}^{\infty} K_m,$$

on which $f$ acts as an adding machine [10, chapter III, section 4]. The unique element of $\mathcal{M}(X, f)$ that is $f|_{\Lambda}$-invariant is denoted by $\mu_{\infty}$ and called the post-critical measure. It is obtained as the weak$^*$ limit of the sequence $\{\delta_c^n\}_{n=1}^{\infty}$ of empirical measures along the orbit of the critical point $c$, and satisfies $h(\mu_{\infty}) = \chi(\mu_{\infty}) = 0$ [5, theorem 3.4(b)].

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The level-2 rate function $I_\varphi$ is given by

$$I_\varphi(\mu) = \begin{cases} P(\varphi) - F_\varphi(\mu) & \text{if } \mu \in \mathcal{M}(X, f) \text{ and } \text{supp}(\mu) \subset \text{cl}(K_m \setminus K_{m+1}) \\
0 & \text{if } \mu = \mu_{\infty}; \\
\infty & \text{otherwise}, \end{cases}$$

(1.1)

where supp($\mu$) denotes the support of $\mu$, the smallest closed set with full $\mu$-measure. By definition, $I_\varphi$ is non-negative. We say $\mu \in \mathcal{M}(X)$ is a minimizer if $I_\varphi(\mu) = 0$. The level-2 LDP in corollary 1.1 asserts that in the limit $n \to \infty$ the measure $\tilde{\mu}_n(\varphi, \Delta)$ on $\mathcal{M}(X)$ assigns all but exponentially small mass to the set of minimizers.

The contraction principle [11, theorem 4.2.1] applied to the level-2 LDP in corollary 1.1 yields the level-1 LDP. We say a Borel function $\psi : X \to \mathbb{R}$ is permissible if the function $\mu \in \mathcal{M}(X) \mapsto \int \psi \, d\mu$ is well-defined and continuous. For a permissible $\psi$, define a level-1 rate function $J_{\varphi, \psi} : \mathbb{R} \to [0, \infty]$ given by

$$J_{\varphi, \psi}(\alpha) = \inf \left\{ I_\varphi(\mu) : \mu \in \mathcal{M}(X), \int \psi \, d\mu = \alpha \right\}.$$  

(1.2)

Since $I_\varphi$ is lower semi-continuous by theorem A, $J_{\varphi, \psi}$ is also lower semi-continuous.

**Corollary 1.2 (The level-1 LDP).** Let $f : X \to X$ and $\varphi : X \to \mathbb{R}$ be as in theorem A. If $P(\varphi) \leq 0$, then for any $\Delta > 0$ and any permissible function $\psi : X \to \mathbb{R}$ the following hold:

(a) (Lower bound) for any open set $G \subset \mathbb{R}$;

$$\liminf_{n \to \infty} \frac{1}{n} \log \left( \frac{\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f_q)} e^{\psi(x)} \psi(x)}{\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f_q)} e^{\psi(x)}} \right) \geq -\inf_{G} J_{\varphi, \psi};$$

(b) (Upper bound) for any closed set $C \subset \mathbb{R}$.

$$\limsup_{n \to \infty} \frac{1}{n} \log \left( \frac{\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f_q)} e^{\psi(x)} \psi(x)}{\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f_q)} e^{\psi(x)}} \right) \leq -\inf_{C} J_{\varphi, \psi}.$$
Figure 1. (i) a typical graph of the level-1 rate function \( \alpha \in \mathbb{R} \mapsto J_{\varphi}(\alpha) \in [0, \infty ] \) for \( \varphi = -\log |f'| \); \( J_{\varphi}(\alpha) = 0 \) if and only if \( \alpha = 0 \), and there can be infinitely many discontinuities (ii) typical graphs of \( \beta \in \mathbb{R} \mapsto \sup \{ F_{\beta \varphi}(\mu) : \mu \in \mathcal{M}_m(X, f) \} \) for \( \varphi = -\log |f'| \) and \( m = 0, 1, 2 \). For those \( \beta \) at which the curves intersect, equilibrium states are not unique. For the definition of \( \mathcal{M}_m(X, f) \), see section 2.1 and (2.1).

As expected, the profile of the level-1 rate function for weighted periodic points in corollary 1.2 is rather different from that for non-renormalizable \( S \)-unimodal maps satisfying the Collet–Eckmann condition considered in [15]. For such a map \( f \) and \( \varphi = -\log |f'| \), Keller and Nowicki [15, theorem 1.3] obtained a local level-1 rate function, which is defined on a small neighborhood of the mean and vanishes only at the mean. By [5, theorem 3.4(b)] we have \( P(\varphi) = 0 \), and so the level-2 LDP as in corollary 1.2 holds, and the post-critical measure \( \mu_\infty \) is the unique minimizer of the level-2 rate function \( I_{\varphi} \). From Hahn–Banach’s theorem applied to the topological vector space of signed measures on \( X \), the mapping \( \mu \in \mathcal{M}(f) \mapsto \chi(\mu) \), which is continuous by lemma 2.6 admits a continuous extension to \( \mathcal{M}(X) \). Hence, \( \varphi \) is permissible. The level-1 rate function \( J_{\varphi, \psi} \) given by (1.2) vanishes only at the mean 0, possibly with infinitely many discontinuities accumulating at 0, see figure 1(i). This is indeed the case for the logistic map \( f(x) = a_\infty x(1 - x) \) at the first accumulation parameter \( a_\infty = 3.5699456 \ldots \) of period doubling cascade.

We also remark that minimizers of the level-2 rate function in theorem A are not always unique, as inferred from the definition (1.1). Figure 1(ii) gives the idea on a typical shape of the graph of the function \( \beta \in \mathbb{R} \mapsto P(\beta \varphi) = \sup \{ F_{\beta \varphi}(\mu) : \mu \in \mathcal{M}(X, f) \} \) for \( \varphi = -\log |f'| \). It can happen that for some \( \beta < 1 \), measures maximizing \( F_{\beta \varphi} \) are not unique. For those \( \beta \), minimizers of the level-2 rate function \( I_{\beta \varphi} \) are not unique. As a result, the level-1 rate function \( J_{\beta \varphi, \psi} \) can vanish at multiple points.

We now move on to distributions of periodic points. Elements of \( \mathcal{M}(X, f) \) which maximize the free energy \( F_{\beta} \) are called equilibrium states (for the potential \( \varphi \)), and in the particular case \( \varphi \equiv 0 \) measures of maximal entropy. If the topological entropy of \( f \) is zero, like the logistic map \( f(x) = a_\infty x(1 - x) \), any element of \( \mathcal{M}(X, f) \) is a measure of maximal entropy. If the topological entropy of \( f \) is positive, there are only finitely many measures of maximal entropy [14], which may not be unique since \( f \) is non-transitive. Our second result is stated as follows.

**Theorem B (Accumulation to equilibrium states).** Let \( f : X \to X \) and \( \varphi : X \to \mathbb{R} \) be as in theorem A. For any \( \Delta > 0 \), any accumulation point of the sequence \( \{ \mu_{n, \varphi, \Delta} \}_{n=1}^{\infty} \) in the weak* topology is an equilibrium state for the potential \( \varphi \).
Taking \( \varphi \) in theorem B to be a constant function yields the following.

**Corollary 1.3 (Accumulation to measures of maximal entropy).** Let \( f : X \to X \) be as in theorem A and let \( \varphi : X \to \mathbb{R} \) be a constant function. For any \( \Delta > 0 \), any accumulation point of the sequence \( \{ \mu_{n,\varphi,\Delta} \}_{n=1}^\infty \) in the weak* topology is a measure of maximal entropy.

For the one-parameter family \( \{ -\beta \log |f'| \}_{\beta \in \mathbb{R}} \) of geometric potentials, qualitative changes occur at \( \beta = 1 \). For \( \beta < 1 \) the pressure is positive [5, example 3.13], and each ergodic equilibrium state has positive entropy, is supported on a certain maximal invariant set associated with the renormalization. There are only finitely many ergodic equilibrium states, and they need not be unique [12]. For \( \beta \geq 1 \) the pressure is 0, and \( \mu_\infty \) is the unique equilibrium state [5, theorem 3.4(b)].

**Corollary 1.4 (Convergence to the post-critical measure).** Let \( f : X \to X \) be as in theorem A. For any \( \beta \geq 1 \) and any \( \Delta > 0 \), the sequence \( \{ \mu_{n,-\beta \log |f'|,\Delta} \}_{n=1}^\infty \) converges in the weak* topology to the post-critical measure as \( n \to \infty \).

### 12. Necessity of \( \Delta > 0 \)

Taking \( \Delta = 0 \) we recover the standard sequence, namely

\[
\mu_{n,\varphi,0} = \nu_{n,\varphi} \quad \text{and} \quad \tilde{\mu}_{n,\varphi,0} = \tilde{\nu}_{n,\varphi}.
\]

We explain the necessity of \( \Delta > 0 \) with elementary observations. The next proposition implies that the lower bounds as in theorem A(a) and corollary 1.1(a) for the sequence \( \{ \tilde{\nu}_{n,\varphi} \}_{n=1}^\infty \) do not hold.

**Proposition 1.5.** Let \( f \) and \( \varphi \) be as in theorem A. There exists an open set \( G \subset \mathcal{M}(X) \) such that \( \inf_G I_\varphi < \infty \) and \( \liminf_{n \to \infty} (1/n) \log \tilde{\nu}_{n,\varphi}(G) = -\infty \).

**Proof.** If \( n > p_1 \) and \( p_1 \nmid n \), then \( \text{Fix}(f^{n}) \subset K_0 \setminus K_1 \). Let \( x \in \text{int}(K_1) \) be a periodic point of \( f \) of period \( p \), and let \( G \) be a open subset of \( \mathcal{M}(X) \) containing \( \delta^p_x \) such that \( \{ x \in K_0 \setminus K_1 : \delta^p_x \in G \} = \emptyset \) holds for all \( n \geq 1 \).

The next proposition implies that the assumption \( \Delta > 0 \) is necessary in theorem B. Indeed, choose a potential for which there is no equilibrium state whose support is contained in \( \text{cl}(K_0 \setminus K_1) \).

**Proposition 1.6.** Let \( f \) and \( \varphi \) be as in theorem B. There exists a subsequence of the standard sequence which converges in the weak* topology to a measure whose support is contained in \( \text{cl}(K_0 \setminus K_1) \).

**Proof.** Let \( \{ \mu_{n,\varphi,\Delta} \}_{n=1}^\infty \) be a subsequence of the standard sequence such that \( n_j > p_1 \) and \( p_1 \nmid n_j \) for all \( j \geq 1 \). Since \( \text{Fix}(f^{n_j}) \subset K_0 \setminus K_1 \), any accumulation point of this subsequence is supported on \( \text{cl}(K_0 \setminus K_1) \).

Theorem B indicates that this problem does not occur in the case \( \Delta > 0 \). If \( \varphi \) is a potential for which there is no equilibrium state whose support is contained in \( \text{cl}(K_0 \setminus K_1) \), any equilibrium state must be supported on a subset of \( K_1 \). By theorem B, for any \( \Delta > 0 \) the support of any accumulation point of \( \{ \mu_{n,\varphi,\Delta} \}_{n=1}^\infty \) must be contained in \( K_1 \). Contrary to proposition 1.6, the situation here is that for any \( n \), the number of periodic points which are contained in \( K_1 \) and comprise the measure \( \mu_{n,\varphi,\Delta} \) has a positive definite fraction. Indeed, the number of periods \( q \in \Delta_x \) for which \( \text{Fix}(f^{q}) \subset K_1 \) is of order at least \( \Delta n / p_1 \). These periodic points contribute to the asserted convergence to equilibrium states in theorem B.
1.3. Structure of the paper

The rest of this paper consists of three sections. Throughout we assume $f : X \to X$ is an infinitely renormalizable $S$-unimodal map with a non-flat critical point and $\varphi \in C(X) \cup \{-\beta \log |f'|\}_{\beta \in \mathbb{R}}$. Section 2 provides a few preliminary results on such maps. In section 3 we establish key lower and upper bounds, and use them in section 4 to complete proofs of the theorems.

Proofs of the two bounds in theorem A have different sorts of difficulties. In the proof of the upper bound, a set of periodic points is already given, while for the lower bound we need to construct suitable periodic points. Section 2 contains ingredients needed for the construction of periodic points.

A main difficulty in the proof of theorem B is that minimizers of the level-2 rate function are not always unique, as already pointed out. We use a continuous projection of the sequence $\{\tilde{\mu}_{n,\varphi,\Delta}\}_{n=1}^{\infty}$ to $\{\mu_{n,\varphi,\Delta}\}_{n=1}^{\infty}$, to show that any accumulation point of the sequence $\{\mu_{n,\varphi,\Delta}\}_{n=1}^{\infty}$ is contained in the closure of the set of convex combinations of minimizers. We then show that this accumulation point is an equilibrium state.

2. Preliminaries

This section provides a few preliminary results on the dynamics of an infinitely renormalizable $S$-unimodal map $f$ with a non-flat critical point. We introduce in sections 2.1 and 2.2 main ingredients for the construction of periodic points in the proof of theorem A(a). In section 2.3 we show the continuity of Lyapunov exponents. In section 2.4 we introduce a projection of measures which will be used to relate minimizers of the level-2 rate function and equilibrium states in the proof of theorem B.

2.1. Construction of periodic points

For the proof of theorem A(a), in principle, we will construct periodic points by pulling back intervals on each renormalization cycle that is equipped with a Markov structure. This Markov structure will also be used in the proof of theorem A(b).

Let $W$ be a subinterval of $X$ and $n \geq 1$ an integer. Any connected component of $f^{-n}(W)$ is called a pullback of $W$ by $f^n$. If $U$ is a pullback of $W$ by $f^n$ and $f^n|_U$ is a diffeomorphism, $U$ is called a diffeomorphic pullback of $W$ by $f^n$.

For each integer $m \geq 0$, let $L_m$ denote the closed subinterval of the restrictive interval $J_m$ bordered by $f^{p_m}(c)$ and $f^{2p_m}(c)$. Note that $f^{p_m}(L_m) = L_m$. Let $\mathcal{P}_m$ denote the collection of connected components of $K_m \setminus \bigcup_{k=0}^{p_m-1} \text{int}(f^k(J_{m+1}))$. It has the Markov property: if $P, Q \in \mathcal{P}_m$ and $f(P) \cap Q \neq \emptyset$ then $f(P) \supset Q$. Hence, the restriction of $f$ to the set $\bigcap_{m=0}^{\infty} f^{-m} \left( \bigcup_{P \in \mathcal{P}_m} P \right)$ is topologically conjugate to a topological Markov chain $\sigma_T : \Sigma_T \to \Sigma_T$ over the finite alphabet $\mathcal{P}_m$ whose transition matrix is given by

$$T = (T_{P,Q})_{P,Q \in \mathcal{P}_m}, \quad T_{P,Q} = \begin{cases} 1 & \text{if } f(P) \supset Q; \\ 0 & \text{otherwise}. \end{cases}$$

Since no power of $\sigma_T$ is topologically mixing (see the caption of figure 2), $\sigma_T$ itself cannot be used for our construction of periodic points. In order to obtain a topologically mixing Markov chain, we restrict to the subset...
Figure 2. The structure of renormalization: (i) the case $p_{m+1}/p_m = 3$. The four thick segments are the elements of $\mathcal{P}_m$ that are contained in $J_m$. The two red ones are the elements of $\mathcal{P}_m$ that are contained in $L_m$; (ii) the case $p_{m+1}/p_m = 2$. The two thick segments are the elements of $\mathcal{P}_m$ that are contained in $J_m$. The red dot indicates the element of $P_m$ that is the singleton (a periodic point of prime period $p_m$) contained in $L_m$. In both (i) and (ii), no forward iteration of the left blue interval covers the right blue one. Hence, the topological Markov chain $\sigma_T$ is not mixing.

$P_m = \{ P \in \tilde{P}_m : P \subset \bigcup_{k=0}^{p_m-1} f^k(L_m) \}$.

In the case $p_{m+1}/p_m \neq 2$, $\mathcal{P}_m$ consists of non-trivial intervals. Otherwise, $\mathcal{P}_m$ consists of the singletons $f^k(J_{m+1}) \cap f^{pm+k}(J_{m+1})$ for $0 \leq k < p_m$, which form a hyperbolic repelling periodic orbit of prime period $p_m$. The topological Markov chain over the finite alphabet $\mathcal{P}_m$ is topologically mixing by the next lemma that is an immediate consequence of [26, lemma 2.9].

**Lemma 2.1 (cf [26, lemma 2.9]).** Let $f : X \to X$ be an infinitely renormalizable $S$-unimodal map with a non-flat critical point. If $m \geq 0$ and $p_{m+1}/p_m \neq 2$, there exists a positive integer multiple $M_m$ of $p_m$ such that any element of $\mathcal{P}_m$ that is contained in $L_m$ contains a diffeomorphic pullback of $L_m$ by $f^{M_m}$.

Set $\mathcal{P}_{-1} = \mathcal{P}_0 \setminus \mathcal{P}_1$. For each integer $m \geq -1$ we define

$$\Gamma_m = \bigcap_{n=0}^{\infty} f^{-n} \left( \bigcup_{P \in \mathcal{P}_m} P \right).$$

The sets $\Gamma_m$ are non-empty, pairwise disjoint closed sets. Since they are $f$-invariant and do not contain the critical point of $f$, they are hyperbolic sets [21, theorem A]. We also define

$$\mathcal{M}_m(X, f) = \{ \mu \in \mathcal{M}(X, f) : \text{supp}(\mu) \subset \Gamma_m \}.$$ (2.1)

Then for all $m \geq 0$,

$$\{ \mu \in \mathcal{M}(X, f) : \text{supp}(\mu) \subset \text{cl}(K_m \setminus K_{m+1}) \} \setminus \mathcal{M}_m(X, f) \subset \mathcal{M}_{m-1}(X, f).$$ (2.2)

The left set in (2.2) is a singleton that is supported on the periodic orbit containing the boundary point of the restrictive interval $J_m$. 

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The following notation will be used along the way. For an integer \( l \geq 1 \) define

\[
C(X)^l = \{ \vec{\phi} = (\phi_1, \ldots, \phi_l) : \phi_j \in C(X) \text{ for all } 1 \leq j \leq l \}.
\]

For \( \vec{\phi} \in C(X)^l \), \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}^l \) and \( \mu \in M(X) \), the expression \( \int \vec{\phi} \, d\mu > \vec{\alpha} \) indicates that \( \int \phi_j \, d\mu > \alpha_j \) holds for \( 1 \leq j \leq l \). For \( n \geq 1 \) define

\[
A_n(\vec{\phi}, \vec{\alpha}) = \left\{ x \in X : \int \vec{\phi} \, d\mu_x^* > \vec{\alpha} \right\}
\]

and

\[
\overline{A}_n(\vec{\phi}, \vec{\alpha}) = \left\{ x \in X : \int \vec{\phi} \, d\mu_x^* \geq \vec{\alpha} \right\},
\]

and \( S_n \vec{\phi} = (S_n \phi_1, \ldots, S_n \phi_l) \). For \( a \in \mathbb{R} \) and \( v = (v_1, \ldots, v_l) \in \mathbb{R}^l \), write \( \vec{\alpha} = (a, \ldots, a) \in \mathbb{R}^l \) and \( \| \vec{v} \| = \max_{1 \leq j \leq l} |v_j| \). The next lemma is an immediate consequence of [26, lemma 3.3].

**Lemma 2.2 (cf [26, lemma 3.3]).** Let \( f : X \to X \) be an infinitely renormalizable \( S \)-unimodal map with a non-flat critical point. Let \( m \geq 0 \) be such that \( p_{n+1}/p_n \neq 2 \). Let \( \mu \in M_m(X, f) \) and let \( P \in \mathcal{P}_n \) be such that \( P \subset \mathcal{L}_m \) and \( \mu(P) > 0 \). For any \( \epsilon > 0 \), there exists \( N \geq 1 \) such that for each integer \( k \geq N \), there exists a finite collection \( \mathcal{B}_{p_nk} \) of diffeomorphic pullbacks of \( P \) by \( f^{p_nk} \) which are contained in \( P \) such that the following hold:

\[
\begin{align*}
(a) & \quad \frac{1}{p_nk} \log \# \mathcal{B}_{p_nk} - h(\mu) < \epsilon; \\
(b) & \quad \sup_{\vec{\phi} \in \mathcal{B}_{p_nk}} \sup_{\vec{\beta}} \left\| \frac{1}{p_nk} \mathcal{S}_{p_nk} \vec{\phi} - \int \vec{\phi} \, d\mu \right\| < \epsilon; \\
(c) & \quad \sup_{\vec{\phi} \in \mathcal{B}_{p_nk}} \sup_{\vec{\beta}} \left\| \frac{1}{p_nk} \mathcal{S}_{p_nk} \vec{\phi} - \int \vec{\phi} \, d\mu \right\| < \epsilon.
\end{align*}
\]

We will combine lemmas 2.1 and 2.2 to construct periodic points in elements of \( \mathcal{B}_{p_nk} \).

### 2.2. Construction of periodic points related to the post-critical measure

The next lemma indicates that empirical measures along orbits contained in sufficiently deep renormalization cycles are approximated in the weak* topology by the post-critical measure \( \mu_\infty \).

**Lemma 2.3.** Let \( l \geq 1 \), \( \vec{\phi} \in C(X)^l \), \( \vec{\alpha} \in \mathbb{R}^l \). For any \( \epsilon > 0 \) there exist integers \( m \geq 1 \) and \( N \geq 1 \) such that the following hold:

\[(a) \text{ If } n \geq N \text{ and } A_n(\vec{\phi}, \vec{\alpha}) \cap K_m \neq \emptyset \text{ then } \int \vec{\phi} \, d\mu_\infty > \vec{\alpha} - \vec{\epsilon}; \]

\[(b) \text{ If } n \geq N \text{ then for all } x \in K_m, \]

\[
\left\| \frac{1}{n} \mathcal{S}_n \vec{\phi}(x) - \int \vec{\phi} \, d\mu_\infty \right\| < \epsilon.
\]

**Proof.** Let \( \epsilon > 0 \). Take \( \delta > 0 \) such that \( |x - y| < \delta \) implies \( \| \vec{\phi}(x) - \vec{\phi}(y) \| < \epsilon/2 \). Let \( m > 1 \) be such that \( |f^k(J_m)| < \delta \) holds for \( 0 \leq k < p_m \). If \( A_n(\vec{\phi}, \vec{\alpha}) \cap K_m \neq \emptyset \) for \( n \geq 1 \), there exists \( 0 \leq k < p_m \) such that \( (1/n)\mathcal{S}_n \vec{\phi}(y) > \vec{\alpha} - (1/2)\vec{\epsilon} \) for all \( y \in f^kJ_m \). Since the
support of \( \mu_\infty \) is contained in \( K_{m_\infty} \), item (a) holds if \( N_\tau \) is sufficiently large. Since \( f|_X \) is uniquely ergodic, \( \sup_{X}(1/n)S_n \phi - \int \phi \, d\mu_\infty \to 0 \) as \( n \to \infty \). Take \( N_\tau \geq 1 \) such that
\[
\sup_{X}(1/n)S_n \phi - \int \phi \, d\mu_\infty < \epsilon/2
\]
for every \( n \geq N_\tau \). Then (b) follows. \( \square \)

For ease of notation, for a periodic point \( x \in \text{Fix}(f^\alpha) \) we also define its Lyapunov exponent by \( \chi(x) = (1/n)\log|f^\alpha(x)| \). Clearly, \( \chi(x) \) equals the Lyapunov exponent of the measure supported on the periodic orbit containing \( x \). Since all periodic points are hyperbolic repelling, their Lyapunov exponents are positive. By the next lemma, periodic points contained in deep renormalization cycles have small Lyapunov exponents.

\textbf{Lemma 2.4.} \textit{We have}
\[
\lim_{m \to \infty} \sup \{ \chi(x) : x \in \text{cl}(K_m \setminus K_{m+1}) \cap \bigcup_{n=1}^{\infty} \text{Fix}(f^n) \} = 0.
\]

\textbf{Proof.} Suppose the statement is false. Then there are an increasing sequence \( \{m_j\}_{j=1}^{\infty} \) in \( \mathbb{N} \) and a sequence \( \{x_j\}_{j=1}^{\infty} \) of periodic points such that \( x_j \in \text{cl}(K_{m_j} \setminus K_{m_j+1}) \) and\( \lim_{j \to \infty} \chi(x_j) > 0 \). From lemma 2.3(b), the sequence of empirical measures on the orbits of \( x_j \) converges in the weak* topology to \( \mu_\infty \) as \( j \to \infty \). Since \( \mu \in \mathcal{M}(X, f) \mapsto \int \log|f^\alpha| \, d\mu \) is upper semi-continuous and \( \int \log|f^\alpha| \, d\mu_\infty = 0 \) [5, theorem 3.4(b)], we obtain \( \lim \sup \chi(x_j) \leq 0 \), a contradiction. \( \square \)

From lemmas 2.3 and 2.4 we obtain the following corollary, which asserts that the free energy of the post-critical measure for the potential \( \varphi \in C(X) \cup \{-\beta \log|f^\alpha|\} \), \( \beta \in \mathbb{R} \), is approximated by the Birkhoff sum of \( \varphi \) along a periodic orbit contained in a sufficiently deep renormalization cycle.

\textbf{Corollary 2.5.} \textit{For any} \( \epsilon > 0 \), \textit{there exist integers} \( m_\tau \geq 1 \) \textit{and} \( N_\tau \geq 1 \) \textit{such that if} \( n \geq N_\tau \) \textit{and} \( x \in K_{m_\tau} \cap \text{Fix}(f^n) \) \textit{then}
\[
\left| \frac{1}{n}S_n \varphi(x) - F_\varphi(\mu_\infty) \right| < \epsilon.
\]

\textbf{Proof.} Let \( \epsilon > 0 \). If \( \varphi \in C(X) \), then let \( m_\tau \geq 1 \), \( N_\tau \geq 1 \) be the integers for which the conclusion of lemma 2.3(b) holds with \( l = 1 \) and \( \phi = \varphi \). Since \( h(\mu_\infty) = 0 \) [5, theorem 3.4(b)], if \( n \geq N_\tau \) and \( x \in K_{m_\tau} \cap \text{Fix}(f^n) \) then \( |(1/n)S_n \varphi(x) - F_\varphi(\mu_\infty)| = |(1/n)S_n \varphi(x) - \int \phi \, d\mu_\infty| < \epsilon \) as required.

If \( \varphi = -\beta \log|f^\alpha| \) (\( \beta \in \mathbb{R} \setminus \{0\} \)), then by lemma 2.4 there exists \( m_\tau \geq 1 \) such that \( \sup \{ \chi(x) : x \in \text{cl}(K_{m_\tau} \setminus K_{m_\tau+1}) \cap \bigcup_{n=1}^{\infty} \text{Fix}(f^n) \} < |\beta| \) for all \( m \geq m_\tau \). Let \( N_\tau = 1 \). Since \( h(\mu_\infty) = \chi(\mu_\infty) = 0 \) [5, theorem 3.4(b)], if \( n \geq N_\tau \) and \( x \in K_{m_\tau} \cap \text{Fix}(f^n) \) then we obtain \( |(1/n)S_n \varphi(x) - F_\varphi(\mu_\infty)| = | -\beta \chi(x) - h(\mu_\infty) + \beta \chi(\mu_\infty)| < \epsilon \) as required. \( \square \)

\subsection*{2.3. Continuity of Lyapunov exponents}

We prove the next lemma.

\textbf{Lemma 2.6.} \textit{The map} \( \mu \in \mathcal{M}(X, f) \mapsto \chi(\mu) \) \textit{is non-negative and continuous.}

\textbf{Proof.} Any measure \( \mu \in \mathcal{M}(X, f) \) is represented in a unique way as a convex combination
\[
\mu = a_\infty \mu_\infty + \sum_{m=-1}^{\infty} a_m \mu_m,
\]
where \( \mu_m \in \mathcal{M}(X, f) = \{ \mu \in \mathcal{M}(X, f) : \text{supp}(\mu) \subset \Gamma_m \} \) and all coefficients are non-negative reals adding up to one. Elements of \( \mathcal{M}(X, f) \) have a positive Lyapunov exponent since \( \Gamma_m \) is a hyperbolic set, and \( \chi(\mu_\infty) = 0 \) [5, theorem 3.4(b)]. Therefore, \( \chi(\mu) \geq 0 \) holds.

\[\text{Nonlinearity 35 (2022) 6399}\]
Let \( \{\mu^{(k)}\}_{k=1}^{\infty} \) be a sequence of measures in \( \mathcal{M}(X, f) \) converging to \( \mu \in \mathcal{M}(X, f) \) as \( k \to \infty \) in the weak* topology. For each \( L > 0 \) consider the continuous function \( \varphi_L : x \in X \mapsto \max\{\log|Df(x)|, -L\} \). Then \( \int \varphi_L \, d\mu^{(k)} \to \int \varphi_L \, d\mu \) as \( k \to \infty \), and the monotone convergence theorem gives \( \lim_{L \to \infty} \int \varphi_L \, d\mu = \chi(\mu) \). Since \( \chi(\mu^{(k)}) \leq \int \varphi_L \, d\mu^{(k)} \), it follows that \( \limsup_{k \to \infty} \chi(\mu^{(k)}) \leq \chi(\mu) \), proving the upper semi-continuity.

To show the lower semi-continuity, write \( \mu^{(k)} \) and \( \mu \) in \( \mathcal{M}(X, f) \) as in (2.3):

\[
\mu^{(k)} = \alpha^{(k)} \mu_{\infty} + \sum_{m=1}^{\infty} \alpha^{(k)}_m \mu^{(k)}_m, \quad \mu = \alpha \mu_{\infty} + \sum_{m=1}^{\infty} \alpha_m \mu_m.
\]

Let \( \epsilon > 0 \). Let \( M \geq 0 \) be an integer such that \( \chi(\mu) < \sum_{m=1}^{M} \alpha_m \chi(\mu_m) + \epsilon \). Since each \( \Gamma_m \) does not contain the critical point of \( f \), we have

\[
\sum_{m=1}^{M} \alpha^{(k)}_m \chi(\mu^{(k)}_m) > \sum_{m=1}^{M} \alpha_m \chi(\mu_m) - \epsilon
\]

for sufficiently large \( k \). Therefore

\[
\liminf_{k \to \infty} \chi(\mu^{(k)}) \geq \liminf_{k \to \infty} \sum_{m=1}^{M} \alpha^{(k)}_m \chi(\mu^{(k)}_m) > \sum_{m=1}^{M} \alpha_m \chi(\mu_m) - \epsilon > \chi(\mu) - 2\epsilon.
\]

Letting \( \epsilon \to 0 \) proves the lower semi-continuity. \( \square \)

### 2.4. Minimizers and equilibrium states, projection of measures

By the definition (1.1), minimizers of the level-2 rate function \( I_2 \) are precisely equilibrium states supported on \( cl(K_m \setminus K_{m+1}) \) for some \( m \geq 0 \), plus the post-critical measure \( \mu_{\infty} \). Let \( \mathcal{K} \) denote the closure of the set of convex combinations of minimizers in the weak* topology. Later in the proof of theorem B we will show that \( \mathcal{K} \) contains any accumulation point of \( \{\mu_{a_0, \Delta}\}_{m=1}^{\infty} \). Let \( \mathcal{M}(\mathcal{M}(X)) \) denote the space of Borel probability measures on \( \mathcal{M}(X) \) endowed with the weak* topology. Let \( \mathcal{M}(\mathcal{K}) \) denote the set of elements of \( \mathcal{M}(\mathcal{M}(X)) \) whose supports are contained in \( \mathcal{K} \). A key ingredient in the proof of theorem B is a projection \( \Pi : \tilde{\mu} \in \mathcal{M}(\mathcal{M}(X)) \to \Pi(\tilde{\mu}) \in \mathcal{M}(X) \) defined by

\[
\int_{\mathcal{M}(X)} \left( \int \phi \, d\mu \right) d\tilde{\mu}(\mu) = \int \phi \, d\Pi(\tilde{\mu}) \quad \text{for all } \phi \in C(X).
\]

The left-hand side is a normalized non-negative linear functional on \( C(X) \), and so \( \Pi \) is well-defined by Riesz’s representation theorem. It is clear that \( \Pi \) is continuous. For \( \mu \in \mathcal{M}(X) \), let \( \delta_{\mu} \in \mathcal{M}(\mathcal{M}(X)) \) denote the unit point mass at \( \mu \). For \( \mu, \nu \in \mathcal{M}(X) \) and \( a \in [0, 1] \) we have \( \Pi((1-a)\delta_{\mu} + a\delta_{\nu}) = (1-a)\mu + a\nu \). Hence \( \Pi(\tilde{\mu}_{a_0, \Delta}) = \mu_{a_0, \Delta} \) holds.

**Lemma 2.7.** We have \( \Pi(\mathcal{M}(\mathcal{K})) \subset \mathcal{K} \).

**Proof.** Let \( \mathcal{M}_0(\mathcal{K}) \) denote the set of elements of \( \mathcal{M}(\mathcal{K}) \) which are finite convex combinations of unit point masses at points in \( \mathcal{K} \). By the above observation on \( \Pi \), \( \Pi(\mathcal{M}_0(\mathcal{K})) \subset \mathcal{K} \) holds.

Since \( \Pi \) is continuous and \( \mathcal{K} \) is closed, it is enough to show that \( \mathcal{M}_0(\mathcal{K}) \) is dense in \( \mathcal{M}(\mathcal{K}) \).

Let \( \tilde{\mu} \in \mathcal{M}(\mathcal{K}) \). Let \( I \geq 1 \) and let \( \psi_1, \ldots, \psi_I : \mathcal{M}(X) \to \mathbb{R} \) be continuous functions. By the definition of Lebesgue integration, for any \( \epsilon > 0 \) there exist a finite partition \( \{P_i\}_{i=1}^{k} \) of the support of \( \tilde{\mu} \) into Borel sets and a finite subset \( \{\mu_i\}_{i=1}^{k} \) of \( \mathcal{K} \) such that \( \mu_i \in P_i \) for \( 1 \leq i \leq k \).
and $|\int \psi_j d\tilde{\mu} - \sum_{i=1}^k \psi_j(\mu_i) \tilde{\mu}(P_i)| < \epsilon$ for $1 \leq j \leq l$. Set $\tilde{\nu} = \sum_{i=1}^k \tilde{\mu}(P_i) \delta_{\mu_i}$. Then $\tilde{\nu} \in \mathcal{M}_0(\mathcal{K})$ and $|\int \psi_j d\tilde{\nu} - \int \psi_j d\tilde{\mu}| < \epsilon$ for $1 \leq j \leq l$. Hence $\mathcal{M}_0(\mathcal{K})$ is dense in $\mathcal{M}(\mathcal{K})$. □

The next lemma characterizes the set $\mathcal{K}$.

**Lemma 2.8.** The set $\mathcal{K}$ consists of convex combinations of equilibrium states for the potential $\varphi$ and the post-critical measure.

**Proof.** Let $\mu \in \mathcal{K}$. Let $\{\mu^{(k)}\}_{k=1}^{\infty}$ be a sequence of convex combinations of minimizers converging to $\mu$ as $k \to \infty$ in the weak* topology. From (1.1) we can write $\mu^{(k)}$ and $\mu$ as

$$
\mu^{(k)} = \rho^{(k)} \mu_\infty + \sum_{m=0}^{\infty} \rho^{(k)} m, \quad \mu = \rho \mu_\infty + \sum_{m=0}^{\infty} \rho m, \mu_m,
$$

where $\varphi^{(k)}$ and $\mu_m$ are minimizers with $\text{supp}(\varphi^{(k)}(m)) \cup \text{supp}(\mu_m) \subset \text{cl}(\mathcal{K}_m \setminus \mathcal{K}_{m+1})$ and all the coefficients in the equations are non-negative reals adding up to one. We assume $\sum_{m=0}^{\infty} \rho m > 0$ for otherwise there is nothing to prove. Using the relation (2.2) we can re-write the equalities as in (2.3):

$$
\mu^{(k)} = \rho^{(k)} \mu_\infty + \sum_{m \in R_k} \rho_m^{(k)} \mu_m^{(k)}, \quad \mu = \rho \mu_\infty + \sum_{m \in R_\infty} \rho_m \mu_m,
$$

where $R_k, R_\infty \subset \{-1, 0, 1, \ldots\}$, $\mu_m^{(k)} \in \mathcal{M}_m(X, f)$ and $\mu_m \in \mathcal{M}_m(X, f)$ are minimizers, and all the coefficients in the series are positive reals adding up to $1 - \rho^{(k)}_\infty$ and $1 - \rho \infty$ respectively, with $\rho_m^{(k)} \in \{\rho_m^{(k)} \rho_m^{(k)} - \rho^{(k)} + \rho_m^{(k)} + \rho_m^{(k+1)}\}$ and $\rho_m \in \{\rho_m, \rho_m + 1, \rho_m + \rho_m^{(k)}\}$. Since $\mu^{(k)}$ is a minimizer, it is an equilibrium state for the potential $\varphi$. Since $\mu^{(k)} \to \mu$ in the weak* topology as $k \to \infty$, for each $m \in R_k$ we have $m \in R_k$ for all sufficiently large $k$, and $\rho_m^{(k)} \to \rho_m$ and therefore $\mu_m^{(k)} \to \mu_m$. Since the free energy $F_\mu$ is upper semi-continuous and $\mathcal{M}_m(X, f)$ is compact, $\mu_m$ is an equilibrium state. Since $m \in R_\infty$ is arbitrary, it follows that $\mu$ is a convex combination of equilibrium states and the post-critical measure $\mu_\infty$.

Conversely, let $\mu$ be a convex combination of equilibrium states and the post-critical measure $\mu_\infty$. From the ergodic decomposition theorem and the affinity of the free energy, $\mu$ is in the closure of convex combinations of ergodic equilibrium states and $\mu_\infty$. Since ergodic equilibrium states are supported on subsets of $\Gamma_m$ for some $m \geq 0$ or of $\Lambda$, they are minimizers, which implies $\mu \in \mathcal{K}$. □

3. Key bounds

This section establishes three key bounds. Sections 3.1 and 3.2 contain proofs of lower and upper bounds in a flavor of large deviations. In section 3.3 we obtain an upper bound on the number of periodic points with small exponents.

3.1. Large deviations lower bound

The following bound is a main step in the proof of theorem A(a), which we will use with $L = 1$ and $L = 1 + \Delta$ in section 4.

**Proposition 3.1.** Let $l \geq 1$, $\tilde{\varphi} \in C(X)^l$, $\tilde{\alpha} \in \mathbb{R}^l$. Let $\mu \in \mathcal{M}(X, f)$ be such that $\int \tilde{\phi} d\mu > \tilde{\alpha}$, and $\text{supp}(\mu) \subset \text{cl}(\mathcal{K}_m \setminus \mathcal{K}_{m+1})$ for some $m \geq 0$ or $\mu = \mu_\infty$. For any $\Delta > 0$ and $1 \leq L \leq 1 + \Delta$ we have
\[
\liminf_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^q) \setminus A_{q}(\vec{\alpha}, \vec{\alpha})} e^{\hat{q} \varphi(x)} \geq F_{\varphi}(\mu)L.
\]

**Proof.** Our strategy is to construct for each \( n \in \mathbb{N} \) a set of periodic points with periods in \( \Delta_n \) that contribute to the series. Let \( \epsilon > 0 \) be such that
\[
\int \hat{q} \, d\mu - \epsilon > \vec{\alpha}.
\]
(3.1)

In what follows we treat four cases separately.

**Case 1:** \( \text{supp}(\mu) \subset \text{cl}(K_m \setminus K_{m+1}) \) for some \( m \geq 0, \mu \in M_m(X, f) \) and \( p_{m+1}/p_m \neq 2 \). In this case, there exists \( P \in \mathcal{P}_m \) such that \( P \subset L_m \) and \( \mu(P) > 0 \). For \( \epsilon > 0 \) in (3.1) and a large integer \( k \), let \( \mathcal{A}_{mk} \) be the collection of diffeomorphic pullbacks of \( P \) by \( f^m \) for which the conclusion of lemma 2.2 holds. From (3.1) and lemma 2.2(b), elements of \( \mathcal{A}_{mk} \) are contained in \( A_{mk}(\vec{\alpha}, \vec{\alpha}) \).

Let \( \Delta > 0 \) and \( 0 < L \leq 1 + \Delta \). For an integer \( n \geq N/L \), we fix \( k \in \mathbb{N} \) such that
\[
Ln \leq M_{mk} \leq \min\{Ln + M_m, [(1 + \Delta)n]\},
\]
(3.2)

where \( \lfloor \cdot \rfloor \) denotes the floor function. By lemma 2.1, Fix(\( f^{mk} \)) intersects any element of \( \mathcal{A}_{mk} \). For each \( B \in \mathcal{A}_{mk} \), we fix a point from Fix(\( f^{mk} \) \( \cap B \)) and denote it by \( x_B \). Using (a) and (c) in lemma 2.2 we have
\[
\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^q) \setminus A_q(\vec{\alpha}, \vec{\alpha})} e^{\hat{q} \varphi(x)} \geq \sum_{x \in \text{Fix}(f^{mk}) \setminus A_{mk}(\vec{\alpha}, \vec{\alpha})} e^{\hat{q} \varphi(x)}
\]
\[
\geq \sum_{B \in \mathcal{A}_{mk}} e^{\hat{q} \varphi(x_B)}
\]
\[
> e^{M_{mk}(\mu)-\epsilon} e^{M_{mk}(f \varphi d\mu - \epsilon)}
\]
\[
= e^{F_{\varphi}(\mu)-2\epsilon M_{mk}}.
\]

Taking logarithms of both sides and then dividing by \( n \), and then using (3.2),
\[
\frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^q) \setminus A_q(\vec{\alpha}, \vec{\alpha})} e^{\hat{q} \varphi(x)} \geq (F_{\varphi}(\mu) - 2\epsilon) \frac{M_{mk}}{n}
\]
\[
= F_{\varphi}(\mu)L + F_{\varphi}(\mu) \frac{M_{mk} - Ln}{n} - 2\epsilon \frac{M_{mk}}{n}
\]
\[
> F_{\varphi}(\mu)L + \min\left\{0, F_{\varphi}(\mu) \frac{M_m}{n}\right\} - 2\epsilon(1 + \Delta).
\]

Letting \( n \to \infty \) and then \( \epsilon \to 0 \) yields the desired inequality in proposition 3.1.

**Case 2:** \( \text{supp}(\mu) \subset \text{cl}(K_m \setminus K_{m+1}) \) for some \( m \geq 0, \mu \in M_m(X, f) \) and \( p_{m+1}/p_m = 2 \). In this case, \( \mu \) is the empirical measure on the periodic orbit of prime period \( p_m \). There is no need for lemma 2.1, and we simply redo the above argument replacing \( M_{mk} \) by \( p_m \) to obtain the desired inequality in proposition 3.1.

**Case 3:** \( \text{supp}(\mu) \subset \text{cl}(K_m \setminus K_{m+1}) \) for some \( m \geq 0 \) and \( \mu \notin M_m(X, f) \). In this case, \( \mu \) is the empirical measure on the periodic orbit containing the boundary point of the restrictive interval \( J_m \). There is no need for lemma 2.1 either, and we redo the above argument replacing \( M_{mk} \) by the period of this periodic orbit to obtain the desired inequality in proposition 3.1.
Case 4: $\mu = \mu_\infty$. Let $m_0 \geq 1$, $N \geq 1$ be integers for which the conclusion of lemma 2.3 and that of corollary 2.5 hold. For $n \geq N/k$, we fix an integer $k \geq 1$ such that

$$ Ln \leq p_m \leq \min \{ Ln + p_m, \lfloor (1 + \Delta)n \rfloor \}. \quad (3.3) $$

For $x \in \text{Fix}(f^{p_m k}) \cap K_m$, lemma 2.3 gives $x \in A_{p_m k}(\phi, \bar{\alpha})$, and corollary 2.5 gives $e^{\delta_{p_m k} + \varphi(x)} > e^{(F_x(\mu_\infty) - \epsilon)p_m k}$. Combining these with the obvious estimate $\#(\text{Fix}(f^{p_m k}) \cap K_m) \geq 1$ yields

$$ \sum_{q \in \Delta_k} \sum_{x \in \text{Fix}(f^q) \cap A_{q}(\phi, \bar{\alpha})} e^{S_q x} \geq \sum_{x \in \text{Fix}(f^{p_m k}) \cap A_{p_m k}(\phi, \bar{\alpha})} e^{\delta_{p_m k} + \varphi(x)} \geq \sum_{x \in \text{Fix}(f^{p_m k}) \cap K_m} e^{\delta_{p_m k} + \varphi(x)} > e^{(F_x(\mu_\infty) - \epsilon)p_m k}. $$

The rest of the argument is completely analogous to case 1. We use (3.3) as a substitute for (3.2). This completes the proof of proposition 3.1. \hfill \Box

3.2. Large deviations upper bound

For an integer $n \geq 1$ and $\theta > 0$, define

$$ \text{Fix}_\theta(f^n) = \{ x \in \text{Fix}(f^n) : \chi(x) \leq \theta \}. $$

The following upper bound is a main step in the proof of theorem A(b).

**Proposition 3.2.** Let $l \geq 1$, $\bar{\phi} \in C(X)$, $\bar{\alpha} \in \mathbb{R}^l$. Let $\Delta > 0$. For any $\theta > 0$ and any $\epsilon > 0$ there exists $n_0 \geq 1$ such that for all $n \geq n_0$ with $\bigcup_{q \in \Delta_n} \text{Fix}(f^q) \cap A_q(\bar{\phi}, \bar{\alpha})) \neq \emptyset$, there exists a measure $\mu \in \mathcal{M}(X, f)$ such that supp($\mu$) $\subset \text{cl}(K_m \setminus K_m+1)$ for some $m \geq 0$ or $\mu = \mu_\infty$, and we have

$$ \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^q) \cap A_q(\bar{\phi}, \bar{\alpha})} e^{S_q x} \leq \sum_{q \in \Delta_n} \#(\text{Fix}_\theta(f^q)) e^{(F_{q, \mu}(\mu) + 2\epsilon) q} \quad \text{and} \quad \int \bar{\phi} \, d\mu > \bar{\alpha} - \bar{\epsilon}. $$

A proof of proposition 3.2 is briefly outlined as follows. We will fix a large integer $m_0 \geq 1$, and for each sufficiently large $q > m_0$ decompose the set $\text{Fix}(f^q)$ into two groups, one in the complement of $K_m$, and the other in $K_m$. For each $0 \leq m \leq m_0 - 1$, we use the thermodynamic formalism for finite topological Markov chains to evaluate contributions from all suitable periodic points in $\text{cl}(K_m \setminus K_m+1)$ as in lemma 3.3 below. To deal with periodic points in $K_m$, we use the results in section 2.2.

**Lemma 3.3.** Under the assumption of proposition 3.2, for any $\epsilon > 0$ and $m \geq 0$, there exists an integer $q_0 \geq 1$ such that for $W \in \mathcal{P}_m$ and $q \geq q_0$ such that $\text{Fix}(f^q) \cap A_q(\bar{\phi}, \bar{\alpha}) \cap W \neq \emptyset$, there exists $\mu = \mu_W \in \mathcal{M}(X, f)$ such that supp($\mu$) $\subset \text{cl}(K_m \setminus K_m+1)$ and we have

$$ \sum_{x \in \text{Fix}(f^q) \cap A_q(\bar{\phi}, \bar{\alpha}) \cap W} e^{\delta_{q, x}} \leq e^{(F_{q, \mu}(\mu) + \epsilon) q} \quad \text{and} \quad \int \bar{\phi} \, d\mu > \bar{\alpha} - \bar{\epsilon}. $$

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Proof. Let $\epsilon > 0$ and $m \geq 0$. Since $\tilde{\phi}$ is a finite string of elements of $C(X)$ and $\text{cl}(K_{m+1})$ does not contain the critical point of $f$, there exists $q_0 \geq 1$ such that for $W \in \mathcal{D}_m$, $q \geq q_0$ and $y, z \in \bigcap_{i=0}^{q-1} f^{-i}(\text{cl}(K_{m+1}))$ which are contained in the same diffeomorphic pullback of $W$ by $f^q$ we have

$$\|S_q \tilde{\phi}(y) - S_q \tilde{\phi}(z)\| < \epsilon q \text{ and } |S_q \varphi(y) - S_q \varphi(z)| < \epsilon q. \quad (3.4)$$

Let $q \geq q_0$, and let $W \in \mathcal{P}_m$ be such that $\text{Fix}(f^q) \cap \overline{A}_q(\phi, \alpha) \cap W \neq \emptyset$. If $W$ is not contained in $L_m$, then $W$ contains only one point from $\text{Fix}(f^q)$, and this point is contained in $\text{cl}(K_{m+1})$. Taking the measure $\mu$ to be the empirical measure on the periodic orbit of this point, we have the desired inequalities in lemma 3.3.

Proceeding to the case $W \subset L_m$, let $\mathcal{D}_q$ denote the set of diffeomorphic pullbacks of $W$ by $f^m$ which are contained in $W$. Elements of $\mathcal{D}_q$ are pairwise disjoint. The Markov property of $\mathcal{D}_q$ over the finite alphabet $\Sigma$ implies that the restriction of $f^q$ to the set $\bigcup_{V \in \mathcal{D}_q} V$ defines the full topological Markov chain $\Sigma$ over the finite alphabet $\Sigma$. The restriction of $f^q$ to $\bigcup_{V \in \mathcal{D}_q} V$, denoted by $\hat{f}$, is topologically conjugate to this shift. The function $S_q \varphi : \bigcup_{V \in \mathcal{D}_q} V \to \mathbb{R}$ induces a potential on $\Sigma$ which is continuous with respect to the shift metric. By the variational principle [27, section 9.3], there exists a $\hat{f}$-invariant Borel probability measure $\hat{\mu}$ supported on the set $\bigcup_{V \in \mathcal{D}_q} \text{Fix}(f^q) \cap \overline{A}_q(\phi, \alpha)$ such that the measure $\mu = (1/q)\sum_{V \in \mathcal{D}_q} \sum_{k=0}^{q-1} \hat{\mu} \circ (f^k \circ \pi)^{-1}$ is in $\mathcal{M}(X, \hat{f})$ and satisfies

$$\sum_{V \in \mathcal{D}_q} \sup_{v \in V} e^{S_q \varphi(v)} \leq e^{(F_q(\mu) + \epsilon)q}.$$

Clearly, $\text{supp}(\mu) \subset \bigcup_{i=0}^{q-1} f^i(\bigcup_{V \in \mathcal{D}_q} V) \subset \text{cl}(K_{m+1})$ holds. From the first inequality in (3.4) and $\text{Fix}(f^q) \cap \overline{A}_q(\phi, \alpha) \cap W \neq \emptyset$ we have $\bigcup_{V \in \mathcal{D}_q} V \subset \mathcal{A}_q(\phi, \alpha - \epsilon)$, and therefore $\int \tilde{\phi} \, d\mu > \alpha - \epsilon$. Since any element of $\mathcal{D}_q$ contains exactly one element of $\text{Fix}(f^q) \cap \overline{A}_q(\phi, \alpha) \cap W$, we have

$$\sum_{x \in \text{Fix}(f^q) \cap \mathcal{A}_q(\phi, \alpha) \cap W} e^{S_q \varphi(x)} \leq \sum_{V \in \mathcal{D}_q} \sup_{v \in V} e^{S_q \varphi(v)} \leq e^{(F_q(\mu) + \epsilon)q},$$

as required. This completes the proof of lemma 3.3. \qed

Proof of proposition 3.2. Let $\theta > 0$ and let $\epsilon > 0$. Let $m_1 \geq 1$, $N_1 \geq 1$ be integers for which the conclusion of lemma 2.3 and that of corollary 2.5 hold. By lemma 2.4, enlarging $m_1$ if necessary we may assume that Lyapunov exponents of periodic points in $K_{m_1}$ do not exceed $\theta$. For $q$ large enough, for each $0 \leq m \leq m_1 - 1$ and $W \in \mathcal{P}_m$, let $\mu_W \in \mathcal{M}(X, f)$ be the measure for which the conclusion of lemma 3.3 holds. Pick a measure $\mu$ from the finite set $\{ \mu_{w_{m}} \} \cup \{ \mu_{w_{m-1}} \}$ which maximizes the free energy restricted to this finite set. By lemmas 2.3 and 3.3 we have $\int \tilde{\phi} \, d\mu > \alpha - \epsilon$. Then

$$\sum_{x \in \text{Fix}(f^q) \cap \mathcal{A}_q(\phi, \alpha) \cap \text{cl}(K_{m+1})} e^{S_q \varphi(x)} \leq \sum_{W \in \mathcal{P}_m} \sum_{x \in \text{Fix}(f^q) \cap \mathcal{A}_q(\phi, \alpha) \cap W} e^{S_q \varphi(x)} \leq e^{(F_q(\mu) + \epsilon)q} \leq \# \mathcal{P}_m e^{(F_q(\mu) + \epsilon)q}.$$
For periodic points in $K_m$, we have

$$\sum_{x \in \text{Fix}(f^q \mid \mathcal{A}_q)} e^{S_q \phi(x)} \leq \#\text{Fix}_\theta(f^q) e^{(F_{\psi}(\mu_\infty) + \epsilon) q}.$$  

Combining these two estimates yields

$$\sum_{x \in \text{Fix}(f^q \mid \mathcal{A}_q)} e^{S_q \phi(x)} \leq \sum_{m=0}^{m_\infty - 1} \sum_{x \in \text{Fix}(f^q \mid \mathcal{A}_q \cap \text{cl}(K_m \setminus K_{m+1}))} e^{S_q \phi(x)}$$

$$+ \sum_{x \in \text{Fix}(f^q \mid \mathcal{A}_q \cap \text{cl}(K_m))} e^{S_q \phi(x)} \leq \left( \sum_{m=0}^{m_\infty - 1} \# \tilde{\mathcal{E}}_m + \#\text{Fix}_\theta(f^q) \right) e^{(F_{\psi}(\mu_\infty) + \epsilon) q}$$

$$\leq \#\text{Fix}_\theta(f^q) e^{(F_{\psi}(\mu_\infty) + 2) q}.$$  

The last inequality holds for sufficiently large $q$. Summing this result over all $q \in \Delta_n$ yields the desired inequality in proposition 3.2.  

\[\square\]

### 3.3. Upper bound on the number of periodic points with small exponents

The next lemma asserts that the exponential growth rate of the number of periodic points with small Lyapunov exponent is small.

**Lemma 3.4.** We have $\lim_{\theta \to 0} \limsup_{n \to \infty} (1/n) \log \#\text{Fix}_\theta(f^n) = 0.$

**Proof.** Periodic points in $K_m$ are of period $p_m k$, $k \geq 1$ an integer. Since all periodic points of $f$ are hyperbolic repelling, each monotone branch of $f^{p_m k} \mid J_m$ contains at most one point from $\text{Fix}(f^{p_m k})$. Since $f^{p_m} \mid J_m$ is unimodal, the number of monotone branches of $f^{p_m k} \mid J_m$ is at most $2^k$. Therefore, the number of periodic points of period $p_m k$ which are contained in $K_m$ is bounded from above by $2^k p_m k$. Since $p_{m+1}/p_m \geq 2$ for all $m \geq 0$, $K_{\lfloor \log n \rfloor}$ does not intersect $\text{Fix}(f^n)$.

For $\theta > 0$, let $m(\theta) \geq 0$ denote the minimal integer for which there is a periodic point $x \in \text{cl}(K_m(\theta) \setminus K_{m(\theta)+1})$ such that $\chi(x) \leq \theta$. The above observations yield

$$\#\text{Fix}_\theta(f^n) \leq \sum_{k=0}^{\lfloor \log n \rfloor} 2^{p_m k} n \leq 2^{p_m \log n} n \log n.$$  

As $\theta \to 0$, $m(\theta) \to \infty$ and so $p_{m(\theta)} \to \infty$, which yields the assertion of the lemma.  

\[\square\]

### 4. Proofs of the theorems

In this section we put together the results in the previous sections to complete the main results. To handle various terms arising from the positive constant $\Delta$, we will use the following two inequalities. For $\mu \in \mathcal{M}(X, f)$, define

$$\Delta_{\psi}(\mu) = \Delta \max\{F_{\psi}(\mu), 0\}.$$
It is easy to see that for $n \geq 1$ and $q \in \Delta_n$, we have
\begin{equation}
F_q(\mu)q \leq F_q(\mu)n + \Delta_q(\mu)n. \tag{4.1}
\end{equation}
Moreover, the variational principle implies
\begin{equation}
\Delta_q(\mu) \leq \varphi_\Delta. \tag{4.2}
\end{equation}

4.1. Proof of theorem A

Let $f : X \rightarrow X$ be an infinitely renormalizable $\mathcal{S}$-unimodal map with a non-flat critical point, and let $\varphi \in \mathcal{C}(X) \cup \{ -\beta \log |f'| |_{\beta \in \mathbb{R}} \}$. The lower semi-continuity of the level-2 rate function $I_\varphi : \mathcal{M}(X) \rightarrow [0, \infty]$ defined by (1.1) is a consequence of the continuity of Lyapunov exponent in lemma 2.6 and the upper semi-continuity of entropy. To show the lack of its convexity, let $m_i \geq 0$ and let $\mu_i \in \mathcal{M}(X, f)$ (i.e., $m_0 - m_1 \geq 2$ and $\text{supp}(\mu_i) \subset \text{cl}(K_{m_i \setminus \mathcal{M}(m_i + 1)})$).

Then we have
\begin{align*}
\liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\geq \left( \mu \varphi(q) \right) L, \\
\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\leq \left( \mu \varphi(q) \right) L.
\end{align*}

Proposition 4.1. For any $\Delta > 0$ the following hold:

(a) For all $1 \leq L \leq 1 + \Delta$ and any open set $\mathcal{G} \subset \mathcal{M}(X)$,
\begin{align*}
\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\geq \left( \mu \varphi(q) \right) L, \\
\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\leq \left( \mu \varphi(q) \right) L.
\end{align*}

(b) For any closed set $\mathcal{C} \subset \mathcal{M}(X)$,
\begin{align*}
\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\geq \left( \mu \varphi(q) \right) L, \\
\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} &\leq \left( \mu \varphi(q) \right) L.
\end{align*}

Proof. Let $1 \leq L \leq 1 + \Delta$ and let $\mathcal{G} \subset \mathcal{M}(X)$ be a non-empty open set. Sets of the form $\{ \nu \in \mathcal{M}(X) : \int f_i \nu > \overline{\alpha} \}$ with $L \geq 1$, $\overline{\phi} \in \mathcal{C}(X)$, $\overline{\alpha} \in \mathbb{R}^l$ constitute a base of the topology of $\mathcal{M}(X)$. Hence, it is possible to write $\mathcal{G} = \bigcup \mathcal{G}_i$ with each $\mathcal{G}_i$ of this form. By the definition (1.1) and proposition 3.1 we have
\begin{displaymath}
\liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \mathcal{P}_n(q)} e^{\varphi(q)} \geq \left( \mu \varphi(q) \right) L.
\end{displaymath}

Since the subscript $\lambda$ is arbitrary, the lower limit is bounded from below by $\left( P(\varphi) + \sup_{\mathcal{G}_i} (\inf_{\mathcal{G}_i} I_{\varphi}) \right) L = \left( P(\varphi) - \inf_{\mathcal{G}_i} I_{\varphi} \right) L$ as required in proposition 4.1(a).

Moving on to a proof of proposition 4.1(b), let $\mathcal{C} \subset \mathcal{M}(X)$ be a non-empty closed set and let $\mathcal{G}$ be an arbitrary open set containing $\mathcal{C}$. Since $\mathcal{M}(X)$ is metrizable and $\mathcal{C}$ is compact, there exist finitely many closed sets $\mathcal{C}_i$, $1 \leq i \leq t$ of the form $\mathcal{C}_i = \{ \nu \in \mathcal{M}(X) : \int \overline{\phi} \nu > \overline{\alpha}_i \}$ with $L_i \geq 1$, $\overline{\phi}_i \in \mathcal{C}(X)$, $\overline{\alpha}_i \in \mathbb{R}^l$ and $c_0 > 0$ such that $\mathcal{C} \subset \bigcup_{i=1}^t \mathcal{C}_i \subset \bigcup_{i=1}^t \mathcal{C}_i(\epsilon) \subset \mathcal{G}$ for all $\epsilon \in (0, c_0)$, where $\mathcal{C}_i(\epsilon) = \{ \nu \in \mathcal{M}(X) : \int \overline{\phi}_i \nu > \overline{\alpha}_i - \epsilon \}$.

By proposition 3.2, for any
\( \theta > 0, \) any \( \epsilon \in (0, \epsilon_0) \) and for each \( 1 \leq i \leq t, \) there exists a measure \( \mu_i \in \mathcal{M}(X, f) \cap C_{i}(\epsilon) \) such that \( \text{supp}(\mu_i) \subset \text{cl}(K_m \backslash K_{m+1}) \) for some \( m \geq 0 \) or \( \mu_i = \mu_{i_\infty}, \) and

\[
\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^{(q)}) \cap C_{i}} e^{\mathcal{S}_{q}(\epsilon)} \leq \sum_{q \in \Delta_n} \# \text{Fix}_{\theta}(f^{q}) e^{(f_{\theta}(\mu_i)+2\epsilon)q}.
\]

Using (4.1) and (4.2) to bound the right-hand side, we have

\[
\sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^{(q)}) \cap C_{i}} e^{\mathcal{S}_{q}(\epsilon)} \leq \max_{q \in \Delta_n} \# \text{Fix}_{\theta}(f^{q}) e^{f_{\theta}(\mu_i)+2\epsilon(1+\Delta) n} e^{2(1+\Delta) n} \epsilon.
\]

Using lemma 3.4 and \( F_{\varphi}(\mu_i) \leq P(\varphi) - \inf_{C_{i}(\epsilon)} I_{\varphi}, \) we have

\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^{(q)}) \cap C_{i}} e^{\mathcal{S}_{q}(\epsilon)} \leq P(\varphi) - \inf_{C_{i}(\epsilon)} I_{\varphi} + \varphi + 2\epsilon(1+\Delta).
\]

Combining this estimate for all \( 1 \leq i \leq t \) gives

\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^{(q)}) \cap C_{i}} e^{\mathcal{S}_{q}(\epsilon)} \leq \max_{1 \leq i \leq t} \lim_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} \sum_{x \in \text{Fix}(f^{(q)}) \cap C_{i}} e^{\mathcal{S}_{q}(\epsilon)}
\]

\[
\leq P(\varphi) + \max_{1 \leq i \leq t} \left( -\inf_{C_{i}(\epsilon)} I_{\varphi} \right) + \varphi + 2\epsilon(1+\Delta)
\]

Since \( \epsilon > 0 \) is arbitrary, the upper limit is bounded from above by the number \( P(\varphi) - \inf_{C_{i}(\epsilon)} I_{\varphi} + \varphi. \) Since \( \mathcal{G} \) is an arbitrary open set containing \( C, \) this number is bounded from above \( P(\varphi) + \inf_{C_{i}(\epsilon)} (-\inf_{C_{i}(\epsilon)} I_{\varphi}) + \varphi = P(\varphi) - \inf_{C} I_{\varphi} + \varphi. \) The last equality is due to the lower semi-continuity of \( I_{\varphi}. \)

Returning to the proof of theorem A, let \( \Delta > 0. \) Substituting \( C = \mathcal{M}(X) \) in proposition 4.1(b) we get

\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} Z_{q}(\varphi) \leq P(\varphi) + \varphi.
\]

Combining proposition 4.1(a) with \( L = 1 \) and (4.3) we obtain the inequality in theorem A(a).

Substituting \( \mathcal{G} = \mathcal{M}(X) \) in proposition 4.1(a) we get

\[
\lim_{n \to \infty} \frac{1}{n} \log \sum_{q \in \Delta_n} Z_{q}(\varphi) \geq P(\varphi)L.
\]

If \( P(\varphi) \leq 0 \) then \( \varphi = 0. \) Combining proposition 4.1(b) and (4.4) with \( L = 1 \) we obtain the inequality in theorem A(b). If \( P(\varphi) > 0, \) then from proposition 4.1(b) and (4.4) with \( L = 1 + \Delta \) we obtain the inequality in theorem A(b). The proof of theorem A is complete. \( \square \)
4.2. Proof of theorem B

Let \( f : X \to X \) be an infinitely renormalizable S-unimodal map with a non-flat critical point, and let \( \varphi \in C(X) \cup \{-\beta \log |f'|\}_{\beta \geq 2} \). We proceed in two steps. First, using theorem A we show that any accumulation point of \( \{ \mu_{n,\varphi,\delta} \}_{n=1}^{\infty} \) is contained in the set \( K \) in section 2.4. We then show that this accumulation point is an equilibrium state for the potential \( \varphi \).

**Lemma 4.2.** Let \( \{ \tilde{\mu}_{n,\varphi,\delta} \}_{n=1}^{\infty} \) be a subsequence of \( \{ \mu_{n,\varphi,\delta} \}_{n=1}^{\infty} \) which converges in the weak* topology to \( \tilde{\mu}_0 \in \mathcal{M}(\mathcal{M}(X)) \). Then we have \( \tilde{\mu}_0 \in \mathcal{M}(K) \).

**Proof.** The support of \( \tilde{\mu}_0 \) is contained in the closed subset \( \mathcal{M}(X, f) \) of \( \mathcal{M}(X) \). In the case \( K = \mathcal{M}(X, f) \) like the logistic map at the first accumulation parameter \( a_\infty \) of period doubling cascade, there is nothing to prove since \( \tilde{\mu}_0 \) obviously belongs to \( \mathcal{M}(K) \). In what follows we assume \( K \) is a proper subset of \( \mathcal{M}(X, f) \).

We fix a metric on \( \mathcal{M}(X) \) which generates the weak* topology. For \( \epsilon > 0 \) let \( M_\epsilon = \{ \mu \in \mathcal{M}(X) : I_\varphi(\mu) \leq \epsilon \} \). Since \( I_\varphi \) is lower semi-continuous, \( M_\epsilon \) is a closed set. Since \( \mathcal{M}(X) \) is compact, \( M_\epsilon \) is compact. Let \( \nu \in \mathcal{M}(X) \) be such that \( I_\varphi(\nu) > 0 \). Clearly, \( \nu \notin M_{I_\varphi(\nu)/2} \) holds. Take \( \delta > 0 \) such that the closed ball \( B_\delta(\nu) \) of radius \( \delta \) about \( \nu \) does not intersect \( M_{I_\varphi(\nu)/2} \). By the weak* convergence \( \tilde{\mu}_{n,\varphi,\delta} \to \tilde{\mu}_0 \) as \( j \to \infty \) and the inequality in theorem A(b), we have

\[
\tilde{\mu}_0(\text{int}(B_\delta(\nu))) \leq \liminf_{j \to \infty} \tilde{\mu}_{n,\varphi,\delta}(\text{int}(B_\delta(\nu))) \\
\leq \limsup_{j \to \infty} \tilde{\mu}_{n,\varphi,\delta}(B_\delta(\nu)) \leq \limsup_{j \to \infty} e^{-I_\varphi(\nu)/2} = 0.
\]

Hence, the support of \( \tilde{\mu}_0 \) does not contain \( \nu \). Since \( \nu \) is an arbitrary measure in \( \mathcal{M}(X) \) satisfying \( I_\varphi(\nu) > 0 \), it follows that the support of \( \tilde{\mu}_0 \) is contained in the set of minimizers of \( I_\varphi \), and so \( \tilde{\mu}_0 \in \mathcal{M}(K) \) as required.

Let \( \{ \mu_{n,\varphi,\delta} \}_{n=1}^{\infty} \) be a convergent subsequence of \( \{ \mu_{n,\varphi,\delta} \}_{n=1}^{\infty} \). Taking a further subsequence if necessary, we may assume \( \{ \tilde{\mu}_{n,\varphi,\delta} \}_{n=1}^{\infty} \) converges in the weak* topology to a measure \( \tilde{\mu}_0 \). Lemma 4.2 gives \( \tilde{\mu}_0 \in \mathcal{M}(K) \), and lemma 2.7 gives \( \Pi(\tilde{\mu}_0) \in K \). By lemma 2.8 we can write \( \Pi(\tilde{\mu}_0) \) as a convex combination of an equilibrium state \( \mu_\varphi \) for the potential \( \varphi \) and the post-critical measure \( \mu_\infty \), namely

\[
\Pi(\tilde{\mu}_0) = (1 - a)\mu_\varphi + a\mu_\infty,
\]

where \( 0 \leq a \leq 1 \). If \( P(\varphi) = F_\varphi(\mu_\infty) = \int \varphi \, d\mu_\infty \), then by the affinity of the free energy on measures, \( \Pi(\tilde{\mu}_0) \) is an equilibrium state for the potential \( \varphi \) as required in theorem B.

In what follows we assume \( P(\varphi) > F_\varphi(\mu_\infty) \). To finish, it suffices to show \( a = 0 \). By lemma 3.4, for any \( \theta > 0 \) there exist \( \delta = \delta(\theta) > 0 \) and \( q_1 = q_1(\theta) \geq 1 \) such that \( \delta \to 0 \) as \( \theta \to 0 \), and

\[
\#\text{Fix}_0(f^j) \leq e^{q_1} \text{ for } q \geq q_1.
\]

Let \( m = m(\epsilon) \geq 1 \), \( N = N(\epsilon) \geq 1 \) be integers for which the conclusion of corollary 2.5 holds. By lemma 2.4, enlarging \( m \) if necessary we may assume that Lyapunov exponents of periodic points in \( K_m \) do not exceed \( \theta \). For \( j \geq 1 \) such that \( n_j \geq \max\{N, q_1\} \) and \( q \in \Delta n_j \), we have

\[
\sum_{x \in \text{int}(K_m), y^j \text{Fix}(f^j)} e^{S_{\lambda_j}(x)} \leq \#\text{Fix}_0(f^j) e^{(F_\varphi(\mu_\infty)+\epsilon)q} \leq e^{(F_\varphi(\mu_\infty)+\epsilon+\delta)q}.
\]
Summing this over all $q \in \Delta_{nj}$ and then using (4.1),
\[
\sum_{q \in \Delta_{nj}} \sum_{x \in \text{int}(K_m^*) \cap \text{Fix}(f^q)} e^{\psi(x)} \leq e^{F(\mu_\infty)} e^{\Delta_{nj}(\mu_\infty)} e^{\epsilon + \delta} = 1.
\] (4.6)

From (4.4) with $L = 1$ and $L = 1 + \Delta$, for all sufficiently large $j$ we have
\[
\sum_{q \in \Delta_{nj}} Z_q(\psi) \geq \max \{ e^{P(\psi)}_{nj}, e^{P(\psi)(1 + \Delta)}_{nj} \}.
\] (4.7)

We divide the rest of the proof into two cases. If $F(\mu_\infty) \leq 0$ then $\Delta(\mu_\infty) = 0$. Combining (4.6) with the first alternative in (4.7) we have
\[
\mu_{nj, \psi, \Delta}(\text{int}(K_m)) = \frac{1}{\sum_{q \in \Delta_{nj}} Z_q(\psi) \sum_{x \in \text{int}(K_m) \cap \text{Fix}(f^q)} e^{\psi(x)}} \leq \frac{e^{F(\mu_\infty)(1 + \Delta)}_{nj}}{e^{\epsilon + \delta} - 1}.
\] (4.8)

By (4.5), the upper bound in (4.8) decreases to 0 as $j$ increases. Hence we obtain
\[
0 = \liminf_{j \to \infty} \mu_{nj, \psi, \Delta}(\text{int}(K_m)) \geq H(\mu_{nj})(\text{int}(K_m)) \geq a \mu_{\infty}(\text{int}(K_m)) = a.
\]

The second inequality is because $\Lambda \subset \text{int}(K_m)$ and $\mu_{\infty}(\Lambda) = 1$.

If $F(\mu_\infty) > 0$ then $\Delta(\mu_\infty) \leq \Delta P(\psi)$. Combining (4.6) with the second alternative in (4.7) gives (4.8). Proceeding in the same way as the previous case we conclude that $a = 0$. The proof of theorem B is complete.

\[\square\]

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