RECOLLEMENT FOR DIFFERENTIAL GRADED ALGEBRAS

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Abstract. A recollement of triangulated categories describes one such category as being “glued together” from two others.

This paper gives a precise criterion for the existence of a recollement of the derived category of a Differential Graded Algebra in terms of two other such categories.

0. Introduction

A recollement of triangulated categories is a diagram

\[
\begin{array}{c}
\mathcal{T}' \\
\downarrow i_* \\
\mathcal{T} \\
\uparrow i^! \\
\mathcal{T}'' \\
\end{array}
\begin{array}{c}
\mathcal{T} \\
\downarrow j^* \\
\mathcal{T}'' \\
\uparrow j_* \\
\mathcal{T}' \\
\end{array}
\]

of triangulated categories and functors satisfying various conditions, most importantly that \((i^*, i_*)\), \((i_!, i^!)\), \((j^!, j_*)\), and \((j^*, j_*)\) are adjoint pairs (see definition 3.1 for precise details).

This notion was introduced in \cite{1} with the idea that \(\mathcal{T}\) can be viewed as being “glued together” from \(\mathcal{T}'\) and \(\mathcal{T}''\). The canonical example of a recollement has \(\mathcal{T}, \mathcal{T}', \) and \(\mathcal{T}''\) equal to suitable derived categories of sheaves on spaces \(X, Z,\) and \(U,\) where \(X\) is the union of the closed subspace \(Z\) and its open complement \(U.\)

In a more algebraic vein, an important class of triangulated categories are the derived categories of Differential Graded modules over Differential Graded Algebras (abbreviated below to DG modules over DGAs). If \(R, S,\) and \(T\) are DGAs with derived categories of left DG modules \(\mathcal{D}(R), \mathcal{D}(S),\) and \(\mathcal{D}(T),\) it is therefore natural to ask: When is

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there a recollement

\[
\begin{array}{ccc}
D(S) & \xrightarrow{i^*} & D(R) \\
\downarrow{i_*} & & \downarrow{j_*} \\
D(T) & \xleftarrow{j^*} & D(T)
\end{array}
\]

The main result of this paper, theorem 3.4, provides a precise criterion. In informal terms, the criterion says that

\[
B = i_*(S) \quad \text{and} \quad C = j_!(T)
\]

must be suitably finite objects which together generate \(D(R)\) in a minimal way.

This sheds light on several earlier results:

First, theorem 3.4 can be viewed as a generalization of the main result of König’s [5] which dealt with the situation where \(R, S,\) and \(T\) are rings.

Secondly, when \(B\) and \(C\) are given then \(S\) and \(T\) will be constructed as endomorphism DGAs of K-projective resolutions of \(B\) and \(C\). The construction of \(T\) as an endomorphism DGA was originally considered in the Morita theory developed by Dwyer and Greenlees in [2]. I shall draw on their work, and improve one of their results in proposition 3.3.

Thirdly, the construction of \(T\) as an endomorphism DGA means that in the special case \(B = 0\), theorem 3.4 reduces to Keller’s theorem for DGAs, see [3, thm. 4.3], which states that if \(C\) is a compact K-projective generator for \(D(R)\), then \(D(R)\) is equivalent to \(D(\mathcal{E}^{\text{op}})\) where \(\mathcal{E}\) is the endomorphism DGA of \(C\). In the general case \(B \neq 0\), theorem 3.4 can therefore be viewed as a two object generalization of Keller’s theorem.

Finally, there is another connection to work by Keller who already in [4, rmk. 3.2] made some remarks on recollements of derived categories of Differential Graded Categories.

The paper is organized as follows: Section 1 gives an embedding result for derived categories of DGAs. Section 2 recalls the Morita theory of Dwyer and Greenlees. And section 3 combines and develops these themes to prove the main result.

1. AN EMBEDDING RESULT

**Definition 1.1.** Let \(T\) be a triangulated category with set indexed coproducts and let \(B\) be an object of \(T\).

Then \(\langle B \rangle\) denotes the triangulated subcategory of \(T\) consisting of objects built from \(B\) using distinguished triangles, retracts, and set indexed coproducts (cf. [8, def. 3.2.9]).
Definition 1.2. Let $\mathcal{T}$ be a triangulated category with set indexed coproducts. An object $B$ of $\mathcal{T}$ will be called self-compact if the restricted functor

$$\Hom_{\mathcal{T}}(B, -)_{(B)}$$

respects set indexed coproducts.

Remark 1.3. Recall that an object $C$ of $\mathcal{T}$ is called compact if the functor

$$\Hom_{\mathcal{T}}(C, -)$$

respects set indexed coproducts.

A compact object is self-compact, but there are self-compact objects which are not compact. For instance, if $\mathbb{Z}_2$, the integers with 2 inverted, are viewed as a complex of $\mathbb{Z}$-modules, then they are self-compact but not compact in $\mathcal{D}(\mathbb{Z})$, the derived category of the integers; see example 1.4.

Remark 1.4. Let $R$ be a DGA with derived category $\mathcal{D}(R)$. It is not hard to see that if $B$ is self-compact and $C$ compact in $\mathcal{D}(R)$, then the functors

$$\text{RHom}_R(B, -)_{(B)}$$

and

$$\text{RHom}_R(C, -)$$

respect set indexed coproducts.

For the following results, recall that if $\mathcal{E}$ is a DGA then $\mathcal{E}^{\text{op}}$ denotes the opposite DGA with product $\cdot$ defined in terms of the product of $\mathcal{E}$ by $e \cdot f = (-1)^{|e||f|} fe$. Left DG modules over $\mathcal{E}^{\text{op}}$ can be identified canonically with right DG modules over $\mathcal{E}$, so $\mathcal{D}(\mathcal{E}^{\text{op}})$, the derived category of left DG modules over $\mathcal{E}^{\text{op}}$, can be identified with the derived category of right DG modules over $\mathcal{E}$. Note that subscripts indicate left and right DG module structures.

Lemma 1.5. If $\mathcal{E}$ is a DGA then $\mathcal{D}(\mathcal{E}^{\text{op}}) = \langle \mathcal{E}_\mathcal{E} \rangle$.

Proof. This is a consequence of Neeman-Thomason localization, [7, thm. 2.1.2].

Theorem 1.6. Let $R$ be a DGA with a $K$-projective left DG module $B$ which is self-compact in $\mathcal{D}(R)$, and let $\mathcal{E}$ be the endomorphism DGA of $R B$. Then $B$ acquires the structure $\mathcal{E}^R B$, and there is an adjoint pair of functors

$$\mathcal{D}(\mathcal{E}^{\text{op}}) \overset{i^*(-) = - \otimes_{\mathcal{E}} B}{\longrightarrow} \mathcal{D}(R) \overset{i^*(-) = \text{RHom}_R(B, -)}{\longleftarrow}$$
where $i_*$ is a full embedding with essential image

$$\text{Ess.}\text{Im } i_* = \langle R B \rangle.$$

**Proof.** It is clear that $B$ acquires the structure $R, E B$ since $E$ is the endomorphism DGA of $R B$, and hence by definition acts on $B$ in a way compatible with the action of $R$.

Lemma 1.5 says $D(E^{\text{op}}) = \langle E \rangle$, that is, each object in $D(E^{\text{op}})$ is built from $E$ using the operations of distinguished triangles, retracts, and set indexed coproducts. The functor

$$i_*(-) = - \otimes_{E} R, E B$$

respects these operations, so each object in the essential image of $i_*$ is built from $i_*(E) = E \otimes_{E} R, E B \cong R B$ using distinguished triangles, retracts, and set indexed coproducts; that is,

$$\text{Ess.}\text{Im } i_* \subseteq \langle R B \rangle. \quad (1)$$

Since $i_*$ respects set indexed coproducts, it follows that $i_*$ sends set indexed coproducts in $D(E^{\text{op}})$ to set indexed coproducts in $\langle R B \rangle$.

Moreover, $R B$ is self-compact so the restriction of

$$i^!(-) = \text{RHom}_{R}(R, E B, -)$$

to $\langle R B \rangle$ respects set indexed coproducts by remark 1.4. Together, this shows that the functor

$$i^!i_*(-)$$

respects set indexed coproducts.

Note that the unit morphism

$$E \longrightarrow i^!i_*(E)$$

is just the canonical morphism

$$E \longrightarrow \text{RHom}_{R}(R, E B, E \otimes_{E} R, E B) \cong \text{RHom}_{R}(R, E B, R B)$$

which is an isomorphism since $E$ is the endomorphism DGA of the K-projective DG module $R B$. Since $i^!i_*$ respects set indexed coproducts, it follows that the unit morphism

$$Y \longrightarrow i^!i_* Y$$

is an isomorphism for each object $Y$ which can be built from $E$, that is, for each object in $\langle E \rangle = D(E^{\text{op}})$.

By adjoint functor theory this implies that $i_*$ is a full embedding of $D(E^{\text{op}})$ into $D(R)$.
To conclude the proof, I must show \( \text{Ess} \text{Im } i_* = \langle R B \rangle \). The inclusion \( \subseteq \) was proved in equation (1), so I must show

\[ \text{Ess} \text{Im } i_* \supseteq \langle R B \rangle. \]

For this, note that \( i_*(\mathcal{E}_\mathcal{E}) \cong_R R B \) is in the essential image of \( i_* \). Since \( i_* \) is a full embedding respecting set indexed coproducts, it follows that each object built from \( R B \) is in the essential image of \( i_* \), as desired. □

**Example 1.7.** The purpose of this example is to prove that \( \mathbb{Z}_2 \), the integers with 2 inverted, is self-compact but not compact in \( D(\mathbb{Z}) \), the derived category of the integers.

There is an adjoint pair of functors

\[
\begin{array}{ccc}
D(\mathbb{Z}_2) & \xrightarrow{i^*} & D(\mathbb{Z}) \\
\xleftarrow{i_*} & & \xleftarrow{i_*}
\end{array}
\]

where \( i^*(-) = \mathbb{Z}_2 \otimes_{\mathbb{Z}} - \) while \( i_* \) is the forgetful functor which takes a complex of \( \mathbb{Z}_2 \)-modules and views it as a complex of \( \mathbb{Z} \)-modules.

By adjoint functor theory, the unit morphism

\[
i_*\mathbb{Z}_2 \longrightarrow i_*i^*(i_*\mathbb{Z}_2)
\]

is an isomorphism. The functors \( i_* \) and \( i^* \) clearly respect set indexed coproducts, so \( i_*i^* \) respects set indexed coproducts, and it follows that the unit morphism

\[
X \longrightarrow i_*i^*X
\]

is an isomorphism for each \( X \) in \( \langle i_*\mathbb{Z}_2 \rangle \).

This permits the computation

\[
\text{Hom}_{D(\mathbb{Z})}(i_*\mathbb{Z}_2, -)|_{\langle i_*\mathbb{Z}_2 \rangle} \cong \text{Hom}_{D(\mathbb{Z})}(i_*i_*\mathbb{Z}_2, i_*i^*(-))|_{\langle i_*\mathbb{Z}_2 \rangle} \cong \text{Hom}_{D(\mathbb{Z})}(i^*i_*\mathbb{Z}_2, i^*(-))|_{\langle i_*\mathbb{Z}_2 \rangle} \cong \text{Hom}_{D(\mathbb{Z})}(\mathbb{Z}_2, i^*(-))|_{\langle i_*\mathbb{Z}_2 \rangle},
\]

where \( (a) \) is because \( i^*i_*\mathbb{Z}_2 = \mathbb{Z}_2 \otimes_{\mathbb{Z}} i_*i_*\mathbb{Z}_2 = \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong \mathbb{Z}_2 \). But \( i^* \) respects set indexed coproducts, so the same holds for the right-hand side of the computation and in consequence for the left-hand side,

\[
\text{Hom}_{D(\mathbb{Z})}(i_*\mathbb{Z}_2, -)|_{\langle i_*\mathbb{Z}_2 \rangle}.
\]

This shows that \( i_*\mathbb{Z}_2 \) is a self-compact object of \( D(\mathbb{Z}) \), and \( i_*\mathbb{Z}_2 \) is just \( \mathbb{Z}_2 \) viewed as a complex of \( \mathbb{Z} \)-modules, so \( \mathbb{Z}_2 \) is self-compact in \( D(\mathbb{Z}) \).

On the other hand, \( \mathbb{Z}_2 \) is not finitely generated over \( \mathbb{Z} \) so cannot be compact in \( D(\mathbb{Z}) \) since the compact objects in \( D(\mathbb{Z}) \) have finitely generated cohomology as follows from [3, thm. 5.3]. □
2. DWYER AND GREENLEES’S MORITA THEORY

Setup 2.1. The following is taken from [2], up to the trivial change of $R$ being a DGA and not a ring.

Let $R$ be a DGA with a $K$-projective left DG module $C$ which is compact in $D(R)$, and let $\mathcal{F}$ be the endomorphism DGA of $R C$. Then $C$ acquires the structure $R, \mathcal{F} C$.

Observe that

$$\text{RHom}_R(R, \mathcal{F} C, -) \simeq \text{RHom}_R(R, \mathcal{F} C, R R \otimes_R -)$$

where (a) is because $R C$ is compact. Setting

$$C^*_{R, \mathcal{F}} = \text{RHom}_R(R, \mathcal{F} C, R R)$$

hence gives

$$\text{RHom}_R(R, \mathcal{F} C, -) \simeq C^*_{R, \mathcal{F}} \otimes_R -.$$ 

There are therefore functors

$$\text{D}(R) \xrightarrow{j^*} \text{D}(\mathcal{F}^{\text{op}}) \xleftarrow{j_*} \text{D}(R)$$

given by

$$j_*(-) = - \otimes_{\mathcal{F}, R} C,$$

$$j^*(-) = \text{RHom}_R(R, \mathcal{F} C, -) \simeq C^*_{R, \mathcal{F}} \otimes_R -,$$

$$j_*(-) = \text{RHom}_{\mathcal{F}^{\text{op}}}(C^*_{R, \mathcal{F}}, -),$$

where $(j_!, j^*)$ and $(j^*, j_*)$ are adjoint pairs. □

The following result was established in [2, sec. 2], up to the change of $R$ being a DGA and not a ring.

Proposition 2.2. In the situation of setup 2.1 the functors $j_!$ and $j_*$ are full embeddings.

3. RECOLLEMENT

Let me first recall the definition of recollement from [1, sec. 1.4].
**Definition 3.1.** A recollement of triangulated categories is a diagram of triangulated categories and triangulated functors

\[ \xymatrix{T' & T 
\ar@/^/[r]^i & \ar@/_/[r]_j \ar@/^/[l]^i \ar@/_/[l]_j} \square \]

satisfying

(i) \((i^*, i_*)\), \((i_*, i^!\))\), \((j_!, j^*\))\), and \((j^*, j_*\)) are adjoint pairs.

(ii) \(j^*i_* = 0\).

(iii) \(i_*, j_!, j^*\) are full embeddings.

(iv) Each object \(X\) in \(T\) determines distinguished triangles

(a) \(i_*i^!X \rightarrow X \rightarrow j_*j^*X \rightarrow\)

(b) \(j_!j^*X \rightarrow X \rightarrow i_*i^*X \rightarrow\)

where the arrows to and from \(X\) are counit and unit morphisms.

\[\square\]

**Remark 3.2.** The following are easy formal consequences of definition 3.1.

(i) \(i^*j_! = 0\) and \(i^!j_* = 0\).

(ii) The restriction of \(i_*i^*\) to the essential image of \(i_*\) is naturally equivalent to the identity functor.

\[\square\]

For the following results, note that if \(X\) is a full subcategory of a triangulated category \(T\), then there are full subcategories

\[X^\perp = \{ Y \in T | \text{Hom}_T(\Sigma^\ell X, Y) = 0 \text{ for each } \ell \}\]

and

\[^\perp X = \{ Y \in T | \text{Hom}_T(Y, \Sigma^\ell X) = 0 \text{ for each } \ell \}\]

It turns out that Dwyer and Greenlees’s Morita theory can be improved in terms of recollement. Specifically, equation (2) from setup 2.1 is the right-hand part of a recollement as follows.

**Proposition 3.3.** In the situation of setup 2.1 there is a recollement

\[ \xymatrix{(R) \ar@/^/[r]^j & D(R) 
\ar@/^/[r]^j & \ar@/_/[r]_j \ar@/^/[l]^i \ar@/_/[l]_i} \]

where $i_*$ is the inclusion of the full subcategory $(RC)^\perp$ and $i^*$ and $i^!$ are its left- and right-adjoint functors, while the functors $j_!, j^*$, and $j_*$ are given as in setup 2.1,

$$
\begin{align*}
  j_!(-) &= - \otimes_{\mathcal{F}} R_{\mathcal{F}C}, \\
  j^*(-) &= R\text{Hom}_R(R, \mathcal{F}C, -), \\
  j_*(-) &= R\text{Hom}_{\mathcal{F}\text{op}}(C^{\ast}_{\mathcal{F}}, -).
\end{align*}
$$

Proof. The functors $j_!$ and $j^*$ are left- and right-adjoint to $j^*$, and by proposition 2.2 both $j_!$ and $j^*$ are full embeddings.

This situation is exactly the one considered in [6, prop. 2.7] which now gives existence of a recollement where the left-hand category is the kernel of $j^*$ and $i^*$ is the inclusion. But the kernel of $j^*(-) = R\text{Hom}_R(R, \mathcal{F}C, -)$ is clearly $(RC)^\perp$, so the present proposition follows. □

The preceding material allows me to prove the following main result.

**Theorem 3.4.** Let $R$ be a DGA with left DG modules $B$ and $C$. Then the following are equivalent.

(i) There is a recollement

$$
\begin{array}{c}
\xymatrix{ D(S) \ar@/^/[rr]^{i^*} & \ar@/_/[rr]^{j^*} & D(R) \ar@/^/[rr]^{j_!} & \ar@/_/[rr]^{i^!} & D(T) }
\end{array}
$$

where $S$ and $T$ are DGAs, for which

$$
i_*(S S) \cong B, \quad j_!(T T) \cong C.
$$

(ii) In the derived category $D(R)$, the DG module $B$ is self-compact, $C$ is compact, $B^\perp \cap C^\perp = 0$, and $B \in C^\perp$.

Proof. (i) $\Rightarrow$ (ii) The functor $i_*$ is triangulated and a full embedding by definition 3.1(iii); hence the essential image of $i_*$ is a triangulated subcategory of $D(R)$. And $i_*$ is a left-adjoint so respects set indexed coproducts; hence the essential image of $i_*$ is closed under set indexed coproducts in $D(R)$.

Since $B \cong i_*(S S)$ is in the essential image of $i_*$, it follows that

$$
\langle B \rangle \subseteq \text{Ess.Im} \ i_*,
$$

(3)
and remark 3.2(ii) then implies that the restriction of \(i_*i^*\) to \(\langle B \rangle\) is naturally equivalent to the identity functor. This permits the computation

\[
\text{Hom}_{\mathcal{D}(R)}(B, -)|_{\langle B \rangle} \cong \text{Hom}_{\mathcal{D}(R)}(i_*(SS), i_*i^*(-))|_{\langle B \rangle}
\]

\[
\cong \text{Hom}_{\mathcal{D}(S)}(SS, i^*(-))|_{\langle B \rangle},
\]

where (a) is because \(i_*\) is a full embedding. But \(i^*\) is a left-adjoint so respects set indexed coproducts, so the same holds for the right-hand side of the computation and in consequence for the left-hand side,

\[
\text{Hom}_{\mathcal{D}(R)}(B, -)|_{\langle B \rangle}.
\]

This shows that \(B\) is self-compact.

Similarly, there is the computation

\[
\text{Hom}_{\mathcal{D}(R)}(C, -) \cong \text{Hom}_{\mathcal{D}(R)}(j_!(TT), -) \cong \text{Hom}_{\mathcal{D}(T)}(TT, j^*(-)),
\]

and \(j^*\) is a left-adjoint so respects set indexed coproducts, so the same holds for the right-hand side of the computation and in consequence for the left-hand side,

\[
\text{Hom}_{\mathcal{D}(R)}(C, -).
\]

This shows that \(C\) is compact.

Let \(X\) be in \(B^\perp \cap C^\perp\). Then

\[
0 = \text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell B, X)
\]

\[
\cong \text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell i_*(SS), X)
\]

\[
\cong \text{Hom}_{\mathcal{D}(S)}(\Sigma^\ell (SS), i^*X)
\]

and

\[
0 = \text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell C, X)
\]

\[
\cong \text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell j_!(TT), X)
\]

\[
\cong \text{Hom}_{\mathcal{D}(T)}(\Sigma^\ell (TT), j^*X)
\]

for each \(\ell\), proving \(i^*X = 0 = j^*X\). But then the distinguished triangle in definition 3.1(iv)(a) shows \(X = 0\), and \(B^\perp \cap C^\perp = 0\) follows.

Finally,

\[
\text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell C, B) \cong \text{Hom}_{\mathcal{D}(R)}(\Sigma^\ell j_!(TT), i_*(SS))
\]

\[
\cong \text{Hom}_{\mathcal{D}(T)}(\Sigma^\ell (TT), j^*i_*(SS)),
\]

and this is 0 for each \(\ell\) because \(j^*i_* = 0\) by definition 3.1(ii), so \(B\) is in \(C^\perp\).
(ii) $\Rightarrow$ (i) It is enough to construct a recollement

$$
\begin{array}{c}
\text{D}(\mathcal{E}^{\text{op}}) \\
\text{i}^* \\
\text{i}^! \\
\text{j}^* \\
\text{j}^! \\
\end{array}
\begin{array}{c}
\text{D}(\mathcal{R}) \\
\text{i}^* \\
\text{i}^! \\
\text{j}^* \\
\text{j}^! \\
\end{array}
\begin{array}{c}
\text{D}(\mathcal{F}^{\text{op}}) \\
\text{i}^* \\
\text{i}^! \\
\text{j}^* \\
\text{j}^! \\
\end{array}

(4)
$$

for which

$$i_*(\mathcal{E}) \cong B, \quad j_!(\mathcal{F}) \cong C,$$

(5)

because the recollement in part (i) of the theorem can be obtained from this by setting $S = \mathcal{E}^{\text{op}}$ and $T = \mathcal{F}^{\text{op}}$.

I can clearly replace $B$ and $C$ with K-projective resolutions. Let $\mathcal{E}$ and $\mathcal{F}$ be the endomorphism DGAs of $\mathcal{R}B$ and $\mathcal{R}C$ so I have the full embedding of theorem 1.6 because $\mathcal{R}B$ is self-compact, and the recollement of proposition 3.3 because $\mathcal{R}C$ is compact.

The recollement of proposition 3.3 goes some way towards giving (4), except that the left-hand category is $(\mathcal{R}C)^{\perp}$ and not $\text{D}(\mathcal{E}^{\text{op}})$. But if I could prove

$$(\mathcal{R}C)^{\perp} = \langle \mathcal{R}B \rangle,$$

(6)

then I could replace $(\mathcal{R}C)^{\perp}$ by $\langle \mathcal{R}B \rangle$ which could again be replaced by $\text{D}(\mathcal{E}^{\text{op}})$ using the full embedding of theorem 1.6 and this would give (4). In this case, (6) would be clear because theorem 1.6 would imply

$$i_*(\mathcal{E}) = \mathcal{E} \overset{1}{\otimes} \mathcal{R} \mathcal{E} B \cong \mathcal{R} B$$

while proposition 3.3 would imply

$$j_!(\mathcal{F}) = \mathcal{F} \overset{1}{\otimes} \mathcal{R} \mathcal{F} C \cong \mathcal{R} C.$$

To show (6), note that $\supseteq$ is clear since $\mathcal{R}B$ is in $(\mathcal{R}C)^{\perp}$ by assumption while $\mathcal{R}C$ is compact. To prove $\subseteq$, let $X$ be in $(\mathcal{R}C)^{\perp}$. The adjunction in theorem 1.6 gives a counit morphism $i_*i^!X \overset{\epsilon}{\to} X$ (where $i_*$ and $i^!$ are now used in the sense of theorem 1.6) which can be extended to a distinguished triangle

$$i_*i^!X \overset{\epsilon}{\to} X \to Y \to .$$

By adjoint functor theory, $i^!(\epsilon)$ is an isomorphism, so $i^!Y = 0$, that is, $\text{RHom}_{\mathcal{R}(\mathcal{R}B,Y)} = 0$, so $Y$ is in $(\mathcal{R}B)^{\perp}$.

Moreover, $i_*i^!X$ is in the essential image of $i_*$ which equals $\langle \mathcal{R}B \rangle$ by theorem 1.6 so since $\langle \mathcal{R}B \rangle \subseteq (\mathcal{R}C)^{\perp}$ it follows that $i_*i^!X$ is in $(\mathcal{R}C)^{\perp}$. But $X$ is in $(\mathcal{R}C)^{\perp}$ by assumption, and it follows that also $Y$ is in $(\mathcal{R}C)^{\perp}$.
So $Y$ is in $(R B)^\perp \cap (R C)^\perp$ which is 0 by assumption, so $Y = 0$, so the distinguished triangle shows $X \cong i_* i^! X$ and this is in the essential image of $i_*$ which is equal to $(R B)^\perp$. \hfill \Box

**Remark 3.5.** Note that the proof of theorem 3.4 (ii) $\Rightarrow$ (i), gives a recipe for constructing $S$ and $T$ when $R$, $B$, and $C$ are known:

Replace $B$ and $C$ with K-projective resolutions, set $E$ and $F$ equal to $\text{End}(R B)$ and $\text{End}(R C)$, and set $S = E^\circ$ and $T = F^\circ$.

Similarly, there is a recipe for constructing the functors $i_*$, $i^!$, $j!$, $j^*$, and $j_*$:

After replacing $B$ and $C$ with K-projective resolutions, $B$ and $C$ acquire the structures $R \cdot E B$ and $R \cdot F C$, that is, $R B$ and $R C$, and the functors are then given by

$$j_!(\cdot) = R C_T \otimes_T \cdot$$

$$i_*(\cdot) = R B_S \otimes_S \cdot$$

$$i^!(\cdot) = \text{RHom}(R B_S, -)$$

$$j^*(\cdot) = \text{RHom}(R C_T, -)$$

$$j_*(\cdot) = \text{RHom}(R C^*_R, -).$$

\hfill \Box

**Example 3.6.** Let

$$R = \mathbb{Z}, \quad B = \mathbb{Z}_2, \quad C = \mathbb{Z}/(2)$$

where $\mathbb{Z}_2$ is $\mathbb{Z}$ with 2 inverted. The purpose of this example is to show that these data satisfy the conditions of theorem 3.4 (ii). Since example 1.7 proves that $\mathbb{Z}_2$ is self-compact but not compact in $\text{D}(\mathbb{Z})$, this shows that theorem 3.4 really needs the notion of self-compactness.

To check the conditions of theorem 3.4 (ii) apart from self-compactness of $B = \mathbb{Z}_2$ which is already known, note that there is a distinguished triangle

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(2) \to$$

in $\text{D}(\mathbb{Z})$. So $\mathbb{Z}/(2)$ is finitely built from $\mathbb{Z}$ and hence $\mathbb{Z}/(2)$ is compact in $\text{D}(\mathbb{Z})$; that is, $C$ is compact in $\text{D}(\mathbb{R})$.

Also, the distinguished triangle gives a distinguished triangle

$$\text{RHom}(\mathbb{Z}/(2), X) \to \text{RHom}(\mathbb{Z}, X) \to \text{RHom}(\mathbb{Z}, X) \to$$

for each $X$ in $\text{D}(\mathbb{Z})$, that is,

$$\text{RHom}(\mathbb{Z}/(2), X) \to X \to X \to,$$

and the long exact sequence of this implies that $\text{RHom}(\mathbb{Z}/(2), X) = 0$ if and only if 2 acts invertibly on each cohomology module of $X$.

So for example, $\text{RHom}(\mathbb{Z}/(2), \mathbb{Z}_2) = 0$, and this implies that $\mathbb{Z}_2$ is in $(\mathbb{Z}/(2))^\perp$, that is, $B$ is in $C^\perp$. 

Finally, let $X$ be in $(\mathbb{Z}/2)\perp \cap (\mathbb{Z}/(2))\perp$. It is well known that since $\mathbb{Z}$ has global dimension one, $X$ is isomorphic in $D(\mathbb{Z})$ to the complex $\tilde{X}$ having $H^i(X)$ in cohomological degree $i$ and having zero differential. It was shown above that since $X$ is in $(\mathbb{Z}/(2))\perp$, the integer $2$ acts invertibly on each $H^i(X)$. That is, each $H^i(X)$ is in fact a $\mathbb{Z}_2$-module, so $\tilde{X}$ can be viewed as a complex of $\mathbb{Z}_2$-modules which I will denote $Y$.

Now the forgetful functor $i_*$ from example 1.7 satisfies

$$i_*Y = \tilde{X} \cong X,$$

and using also the left-adjoint functor $i^*$ from example 1.7 I can compute for each $\ell$,

$$\text{Hom}_{\text{D}(\mathbb{Z}_2)}(\Sigma^\ell \mathbb{Z}_2, Y) \cong \text{Hom}_{\text{D}(\mathbb{Z}_2)}(\Sigma^\ell i^* \mathbb{Z}_2, Y) \cong \text{Hom}_{\text{D}(\mathbb{Z})}(\Sigma^\ell \mathbb{Z}_2, i_* Y) \cong \text{Hom}_{\text{D}(\mathbb{Z})}(\Sigma^\ell \mathbb{Z}_2, X) \cong 0,$$

where (a) is because $\mathbb{Z}_2 \cong \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_2 \cong i^* \mathbb{Z}_2$ while (b) is by equation (7) and (c) is because $X$ is in $(\mathbb{Z}_2)\perp$.

This proves $Y = 0$ whence $X = 0$ by equation (7), and altogether, I have shown $(\mathbb{Z}_2)\perp \cap (\mathbb{Z}/(2))\perp = 0$, that is, $B\perp \cap C\perp = 0$. \hfill \Box

The proof of the following theorem could have used the concrete construction of a recollement at the end of the proof of theorem 3.4 as a crutch. However, a direct proof is just as easy.

**Theorem 3.7.** In the situation of theorem 3.4 the following hold.

(i) $\text{Ess. Im } i_* = \ker j^* = \langle B \rangle = C\perp = \frac{1}{2}(B\perp)$.

(ii) $\text{Ess. Im } j_! = \ker i^* = \langle C \rangle = \frac{1}{2}(C\perp)$.

(iii) $\text{Ess. Im } j_* = \ker i^! = B\perp = (C\perp)\perp$.

**Proof.** To prove $\text{Ess. Im } i_* = \ker j^*$ in (i), note that the inclusion $\subseteq$ holds because $j^* i_* = 0$ by definition 3.4(ii), and that the inclusion $\supseteq$ follows from the distinguished triangle in definition 3.4(iv)(b).

The equalities $\text{Ess. Im } j_! = \ker i^*$ and $\text{Ess. Im } j_* = \ker i^!$ in (ii) and (iii) are proved by similar arguments.

To prove $\text{Ess. Im } i_* = \langle B \rangle$ in (i), note that the inclusion $\supseteq$ is already known from equation (3) in the proof of theorem 3.4. To see the inclusion $\subseteq$, it is enough to see $i_*(\text{D}(S)) \subseteq \langle B \rangle$. But this is easy,

$$i_*(\text{D}(S)) \cong i_*(\langle sS \rangle) \subseteq \langle i_*(sS) \rangle = \langle B \rangle,$$
where (a) is by lemma 1.5 and (b) is because \( i_* \) is a left-adjoint so respects set indexed coproducts.

The equality \( \text{Ess.} \text{Im} j_! = \langle C \rangle \) in (ii) is proved by a similar argument.

The equality \( \langle B \rangle = C^\perp \) in (i) is already known from equation (6) in the proof of theorem 3.4.

To show \( \text{Ker} i^* = (C^\perp) \) in (ii), note that \( X \) is in \( \text{Ker} i^* \) if and only if

\[
0 = \text{Hom}_{D(S)}(i^* X, Y) \cong \text{Hom}_{D(R)}(X, i_* Y)
\]

for each \( Y \) in \( D(S) \). But I have already proved \( \text{Ess.} \text{Im} i_* = C^\perp \), so up to isomorphism, the objects of the form \( i_* Y \) are exactly the objects in \( C^\perp \), so equation (8) is equivalent to \( X \) being in \( (C^\perp) \).

Similar arguments show first the equality \( \text{Ker} i^! = B^\perp \) in (iii) and then \( \text{Ker} j^* = (B^\perp) \) in (i).

Finally, \( B^\perp = (C^\perp)^\perp \) in (iii) follows from \( \langle B \rangle = C^\perp \) which I have already proved. \( \square \)

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