RESONANT LASER COOLING OF CIRCULAR ACCELERATOR BEAMS

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April 30, 2019

Abstract

The resonant laser cooling of circular accelerator beams of relativistic charged particle is studied. It is shown that in the approximation of the given external electromagnetic wave amplitude (small gain free electron laser) the emittance of a beam of charged particles decreases. In the range of particle energy about 100 in the mass energy units the beam energy losses are negligible. The discovered effect can be used for cooling of charged particle beams in various accelerators. The significant cooling rates are possible to achieve by placing of the cooling device in the appropriate points of the accelerator orbit. Resonant laser cooling for various cases are considered. This method of charged particle beams cooling can be applied to circular accelerators of electron and various relativistic heavy particle beams and has significantly, about 3-4 orders of magnitude, shorter cooling time in comparison to any other cooling method.
1 Introduction

The increasing of the accelerator beam lifetime and luminosity is very important. But the beam lifetime and emittance is restricted by various effects. For example, in the electron circular accelerators the beam lifetime is restricted by synchrotron radiation, especially in the machines with combined function magnets as ARUS (Yerevan) and old DESY (Germany). In the proton and ion circular accelerators the beam lifetime and emittance is restricted by the different scattering effects. The influence of these effects on the beam lifetime and emittance can be decreased by applying of any effective cooling method. So, the cooling of charged particle beams is a very important issue for attaining of higher lifetime and luminosity beams. To date, various methods of cooling are used in various accelerators: there are radiative [1], electron [2], stochastic [3] and Doppler [4] cooling. The first method is mainly used to cool electron or light particle beams in circular accelerators and storage rings. Electron cooling proposed by Budker [2] is advantageous to cool proton and heavy particle beams. Excellent reviews on cooling of charged particle beams are given in [3]-[7]. Stochastic cooling is applicable to various particle beams, but it is very slow. Doppler cooling is used for ion beams. In [8] laser cooling of relativistic ion beam is observed. The lowest temperature of a laser cooled stored ion beam is reported in [9]. However, above methods do not meet the modern requirements. For instance, these methods do not provide the high luminosity beams in linear, circular accelerators and storage rings. Also they have small cooling rate resulting in long cooling time. Moreover, there are no effective cooling methods for high energy, relativistic particle beams used in linacs and colliders. In order to improve the cooling efficiency new methods have been proposed, among which we note a resonant laser [10], a undulator [12], ionization cooling [13], the stimulated radiation or stimulated Doppler cooling [16], and the 3-dimensional laser cooling [14].

Transverse laser cooling of ions is demonstrated in [15]. Cooling by backward Compton scattering of free electrons on photons is of a special interest [17]-[19]. It is known, however, that the stimulated scattering under the same conditions has a greater cross-section [16], [20], than the cross-section of free particle scattering. Therefore, in contrast to cooling methods depicted above, it is advantageous to consider stimulated or resonant scattering. In this paper we discuss the applicability of stimulated or resonant interaction of particles with a flat electromagnetic wave (photons) for fast transverse cooling of a beam of relativistic charged particles. The resonant bunch - wave interaction is mediated by applying the homogeneous magnetic field, directed along of a beam movement direction. The magnetic field creates beam current density correlations proportional to laser wavelength. This increases bunch - wave interaction and makes it selective. The usual and known distribution functions, for example Gaussian or Boltzman, are useful for ensembles of particles with irregular,
stochastic trajectories, when the exact trajectories are not known and the ensemble is described by the density of probability of particle states. When particles are moved at certain trajectories, i.e. we know the particle state, the distribution is not purely probabilistic, rather it becomes correlated. This can be seen from the distribution function of particles with certain trajectories, which is equal to the usual distribution function for random variables. Since the trajectories depend on time, the distribution function depends on time too. This shows that temporary evolution of the ensemble is not probabilistic, and is determined by the certainty of trajectories. All average macroscopic values of the system depend on time and we know exactly their temporal evolution, which means that correlations of states of particles exist in such system.

2 Physical principles of resonant laser cooling

2.1 Introduction

Interaction of electrons, which move by periodic trajectories, with a flat monochromatic electromagnetic wave is one way of amplification of an electromagnetic wave in the free electron laser. The autoresonant free electron laser [21] based on the interaction of the charged particle with the electromagnetic wave extending along a homogeneous magnetic field $B_0 \parallel z$, is characterized by the fact that detuning from an exact resonance stays independent on charged particle energy changes. In [22] it was shown, that detuning is constant due to the existence of the integral of motion $I$

$$I = \gamma - P_z,$$  \hspace{1cm}(1)  

which is applicable only in a constant external wave approximation. Here $\gamma$ is the full energy, and $P_z$ is the momentum of a particle along magnetic field (in the system of units with $m=c=1$). This formula can be obtained easily from the usual hamiltonian of a charged particle in the presence of a plane laser wave proportional to $exp(\omega t - kz)$. The dependence on time means that energy is not conserved and hamiltonian is not a motion integral. The new motion integral can be found from the Hamiltonian by using the canonical transformation $\phi = t - z$. After such transformation of $z$, the new hamiltonian equals to $I$ and do not depend on time. The time independence of the new Hamiltonian means that new energy $I$ is conserved and $I$ is a motion integral. In this article we examine the influence of autoresonant interaction of a particle with a wave on a transverse beam emittance. Taking into account the expression $\gamma = \sqrt{P_z^2 + P_\perp^2 + 1}$, it is easy to receive from (1) for a transverse kinematical momentum of a particle

$$P_\perp^2 = 2I\gamma - 1 - I^2$$

After differentiating of this expression with respect to time and taking an average of all particles in a beam, we shall receive the equation for changing
of the mean square of a transverse momentum of a beam

\[ \frac{d}{dt}(P_{\perp}^2) = 2\langle \dot{\gamma} \rangle \quad (2) \]

As shown in works [21]-[23], the trajectories of electrons are spirals, which radii varies along the way. As the drift of the center of a Larmor circle is absent and \( I \geq 0 \), interaction in the mode of the laser (\( \dot{\gamma} < 0 \)) means reduction of a beam transverse emittance, because of the negativity of the right hand side of (2). Here and below the dot means the differentiating on time. In the laser mode the amplified wave carries out a part of beam energy.

It is very important to find which part of the energy is taken out from full beam energy and which part is taken from beam transverse emittance, i.e. what constitutes beam cooling (emittance reduction). As it is not difficult to receive from [7], relative speed \( P_z/P_z \) of a longitudinal momentum change is \( 2P_{\perp}^2/(1 + P_{\perp}^2) \) times less than relative speed \( P_L/P_L \) of a transverse momentum change. Therefore emittance reduction will be effective for beams with \( \langle P_{\perp}^2 \rangle \ll 1 \), when full energy losses are small. But in practical applications it is also possible to compensate beam energy losses by external accelerating elements.

### 2.2 Equations of motion

In the limit of infinitesimal changes of a transverse momentum during the beam interaction with a wave, i.e. if motion in a transverse plane is determined by a magnetic field \( B_0 \), it is easy to receive for \( \dot{\gamma} \) in the field of the circular-polarized wave with frequency \( \omega \) and amplitude

\[ \dot{\gamma} = \frac{\xi \omega}{\gamma} P_{\perp} \cos \varphi \]

where \( \xi = eE_0/\omega \) is the dimensionless amplitude of an electric field of a wave, and \( \varphi \) is the phase of the Larmor rotation, counted from a wave phase at the location of a particle. In the approximation used it is easy to derive the equation for the change of an average longitudinal momentum or energy of a beam

\[ \langle \dot{p} \rangle = \langle \dot{\gamma} \rangle = \left\langle \frac{\xi \omega}{\gamma} P_{\perp} \cos \varphi \right\rangle \quad (3) \]

Averaging on all particles of a beam in the equation (2) allows to investigate the average beam size, using the kinetic approach instead of particle-by-particle detailed trajectory analysis. If a weak (in comparison to the magnetic field \( \vec{B}_0 \)) electromagnetic wave with \( \vec{E}, \vec{B} \sim \exp(\omega t - k z) \) is present, there are correlations of density which result in modulation of a beam on \( \varphi \) and accordingly, give a nonzero right hand side in (2). To quantify this effect we shall present distribution function of a beam as [24]

\[ f = f_0 + \delta f \]

where \( f_0 \) is the basic equilibrium function of distribution, and \( \delta f \ll f_0 \) is the small addition to \( f_0 \) caused by a wave. Close to a resonance it is convenient to write the addition as Fourier series

\[ \varphi : \delta f = \sum g_s(P_{\perp}, P_z) \exp(is\varphi). \]

Substituting this in the kinetic equation and neglecting terms of the second order, gives the following expressions for factors

\[ g_s = \frac{Q_s}{i(s + \alpha)}; Q_s = \frac{1}{2\pi} \int_0^{2\pi} d\tau \exp(-is\tau) Q(P_z, P_{\perp}, \tau); Q = \frac{e}{\omega B} \frac{\partial f_0}{\partial P_z}(\vec{E} + \vec{B}) \]
where \( \alpha = (kv - \omega)/\omega_B \), \( \omega_B = eB_0/\gamma \) is the Larmor frequency, \( \tau \) is the phase of integration and the relation \( \omega B = |kE| \) for the field of the wave is used. Close to a simple cyclotron resonance the leading term in the Fourier series will be the term with \( s=1 \). For Gaussian distribution function on transverse momentum

\[
f_0 = \frac{1}{\pi (P_{\perp}^2)} \exp\left(-\frac{P_{\perp}^2}{(P_{\perp}^2)}\right)\delta(P_{\perp} - P_0).
\]

(4)

In a relativistic case \( P_0 \gg 1 + P_{\perp}^2 \) we can neglect the \( I^2 \) in (1) and taking into account \( I \approx (1 + P_{\perp}^2)/2P_0 \) we find a small addition to (4)

\[
\delta f = -\xi \frac{P_{\perp}(1 + P_{\perp}^2)}{(P_{\perp}^2)} i(\Delta_{\parallel} - P_{\perp}^2) \exp(i\varphi) f_0
\]

(5)

where \( \Delta_{\parallel} = 2\Omega P_0 - 1; \Omega = \omega_B \gamma \). Approximation of a longitudinal - monochromatic beam \( f_0 \) is fair, if the relative spread on a longitudinal momentum \( (P_{\perp} - P_0)/P_0 \) is much less than \( \langle P_{\perp}^2 \rangle \). This approximation is quite useful as it allows us to estimate \( f_0 \) by a resonant term \( (1 + \alpha)^{-1} \).

Then we average on \( f \) we note rather usual in such cases pole located at \( p_{\perp}^2 = \Delta_{\parallel} \), which is caused by a resonant multiplier. By a rule of detour of Landau poles (substitution \( \omega \rightarrow \omega + i\delta \)) we find the vale of the integral of interest [24]:

\[
\int_0^\infty \frac{f(z)}{\Delta_{\parallel} - z} dz = \int \frac{f(z)}{\Delta_{\parallel} - z} dz + i\pi \delta(\Delta_{\parallel} - z)
\]

where \( z = p_{\perp}^2 \), and the integral in the right hand side is treated in sense of a principal value. As only a real part of [24] is physical meaningful, if \( \Delta_{\parallel} \leq 0 \), when the right hand side is pure imaginary, the speed of change of a transverse momentum is equal to zero. And when \( \Delta_{\parallel} > 0 \) we find:

\[
\frac{d}{dt} \langle p_{\perp}^2 \rangle = -\xi \frac{\pi \omega}{2P_{\perp}^2} \Delta_{\parallel} (1 + \Delta_{\parallel}) \exp(-\Delta_{\parallel} / \langle p_{\perp}^2 \rangle)
\]

(6)

2.3 Particular solutions

The maximum of the right hand side in case with \( \langle p_{\perp}^2 \rangle \ll 1 \) (\( p_0 \approx const \)) is achieved at \( \Delta_{\parallel} = \langle p_{\perp}^2 \rangle \), and its value is

\[
\langle p_{\perp}^2 \rangle = (\langle p_{\perp}^2 / \langle p_{\perp}^2 \rangle \rangle^2 - \frac{2\pi^2}{\xi^2} \frac{z}{2\lambda p_0})^{1/2}
\]

(7)

where \( P_{\perp,0} \) is the initial value of an average transverse momentum of a beam squared, \( \lambda \) is a wavelength, and \( z = ct \) is a length of a beam way. When \( \Delta_{\parallel} = const \) and much less \( \langle p_{\perp}^2 \rangle \), we shall receive:

\[
\langle p_{\perp}^2 \rangle = (\langle p_{\perp,0}^2 \rangle^2 - \frac{3\pi^2}{\xi^2} \frac{2\lambda p_0}{\Delta_{\parallel}})^{1/3}
\]

(8)

The decreasing of a beam transverse emittance by autoresonant bunch - wave interaction must be significant as well for a bunch modulated \( \varphi \). In this case we choose the distribution function of a different kind:

\[
f(p_{\perp}, p_{\perp}, \varphi) = f_0(1 - \varepsilon \cos \varphi)
\]

where \( 0 < \varepsilon < 1 \) is the depth of modulation, and \( f_0 \) is defined in (4).

After averaging of (2) with this function we shall find:

\[
\frac{d}{dt} \langle p_{\perp} \rangle = -\varepsilon \frac{\pi \omega}{2P_{\perp}} (1 + \frac{6}{\pi} \langle p_{\perp} \rangle^2),
\]

(9)

where \( \langle p_{\perp} \rangle = \sqrt{\frac{2\pi}{\xi^2}} \sqrt{\langle p_{\perp}^2 \rangle} \) relation is taken into account. From here at \( \langle p_{\perp}^2 \rangle \ll 1 \) (\( p_0 \approx const \)), we shall obtain
\[
\langle p_\perp \rangle = p_{\perp 0} - \varepsilon \xi \pi \frac{z}{2\lambda p_0^2} \quad (10)
\]

The condition of small change of the transverse momentum, which is used for derivation of all formulae actually means the limitation on the length of the way passed:

\[
1 \leq \frac{z}{2\lambda p_0^2} \ll A, \quad (11)
\]

where \(A = \frac{\varepsilon (p_{\perp 0}^2)^2}{\pi \lambda^2}; \frac{\varepsilon p_{\perp 0}^3}{\pi \lambda^3}; \frac{\varepsilon p_{\perp 0}}{\pi \lambda} \) for formulae (4), (8) and (10) respectively. The left inequality lets us to describe an adiabatic switching of a wave field. As in our case the field is switched instantly, the approximation of the adiabatic slow wave field change is fair at times greater than relaxation time. That is expression (6) for fair if the beam has passed a way greater than \(2p_{\perp 0}^2 \lambda \). We shall emphasize that the condition (11) comes from a method of calculation. The equation (2) is fair in the approximation of the given external field, when Langmuir (plasma) frequency of a beam is much less than frequency of the laser [22]. It specifies relation between a beam transverse momentum variation and an energy variation. For a beam of charged particles with \(P_0 = 100 \) and divergence \(\vartheta = 10^{-3} \) and the laser with \(\xi = 10^{-2} \) the length of a way on which transverse emittance changes by 100 percents is about 80 cm, while the length of the relaxation is about 20cm. This example shows that very short cooling time is achievable by using resonant laser cooling. Thus, the autoresonant cooling of beams of relativistic charges has some advantages. First, it is much faster and has shorter cooling time, than other cooling methods mentioned in the introduction. Second, this method is applicable as to circular accelerators and storage rings as well as in linear accelerators. Thirdly, though this method is most effective for beams with the Lorenz-factor about 100, is also can be used for cooling of GeV particles. The method allows to reach cooling rates comparable to that described in [17, 18] by the use of electromagnetic radiation sources with much lower intensities. We notice that formulae which describe cooling depend on parameter \(\xi \), which is easier be made of the value of about unity in long-wave radiation sources such as masers. Fourth, there is a possibility of application of a longitudinal or transversal non-uniform magnetic field that enables scanning in case of a beam with large divergence.

3 Dynamics of a circular accelerator beam in the presence of resonant laser cooling

3.1 Introduction

We consider in this section the influence of resonant laser cooling device on the beam dynamics of the circular accelerator. The beam dynamics in the presence of focusing forces of the accelerator is described by betatron oscillations around equilibrium orbit [25]. During these oscillations the divergence (transverse momentum) and position of a particle varying periodically and the points of
the maximum of one of them are the points of minimum of the other. So, the changing of the divergence of a particle by the resonant wave-beam interaction results in the appropriate changing of the particle position in respect to equilibrium orbit. This means, that the beam divergence reduction result in the accelerator beam size reduction. Therefore, the resonant beam-wave interaction result in decreasing of the betatron oscillation amplitude, i.e. in the accelerator beam cooling. This cooling is possible to apply in two regimes: fast cooling and slow cooling regimes. The first one is more suitable in case of electron beam due to a possibility to achieve very high cooling rates of light particles, such as electrons. This regime is necessary also for decreasing of the influence of the longitudinal magnetic field on the beam dynamics. This regime is considered in the first subsection. In the second subsection is considered the case of slow cooling more suitable for heavy particles such as protons and ions, but applicable to electrons too, under certain conditions.

3.2 Fast cooling regime

In this subsection we consider the case of significant changing, about unity or more times, of the beam angular divergence by resonant laser cooling on each turn on the accelerator orbit in the cooling device. This means that divergence decreased instantly and significantly at the cooling device. Such cooling regime is possible in case of cooling of electrons with \( \gamma \) about 100 by \( CO_2 \) laser with energy about 0.5 kJ and magnetic field about 5-6 T. For describing of such process we consider the instantenous decreasing of the beam divergence \( k \) times by passing through cooling device. It is well known, that the transverse dynamics of the beam with emittance \( E \) in the focusing lattice of the accelerators is described by the ellipse

\[
y^2 + (\alpha y + \beta y')^2 = \beta E
\]

(12)

where \( y \) and \( y' \) denotes the transverse coordinate and the angular divergence, respectively. After passing the cooling device the transverse divergence in two transverse directions is decreased in accordance to (11). For effective cooling it is necessary to place the cooling device in those points of the accelerator orbit, where \( y' \) is maximal. Usually, these points are the centers of free spaces and in these points the relation

\[
y'_{\text{max}} = \sqrt{E(1 + \alpha^2)/\beta}
\]

(13)

take place. So, the decreasing of the angular divergence \( k \) times means the decreasing of the transverse emittance \( k^2 \) times. By using the relation

\[
y_{\text{max}} = \sqrt{E/\beta}
\]

(14)

one can find, that the mean square transverse size of the accelerator beam is decreased by factor \( k \). The necessary cooling rate depend on laser intensity and pulse duration defined by period of one turn. Laser with intensity 5-6 MW decreases the transverse size of the electron beam with \( \gamma = 100 \) approximately 2 times on each turn. In case \( p_\perp << 1 \) the longitudinal momentum change is negligible. Notice, that such
decreasing of beam transverse size means decreasing of the beam emittance and increasing of the accelerator luminosity and of the beam lifetime in depend on factors which limited the beam lifetime.

3.3 Slow cooling regime

In this subsection we consider the case of small change of beam divergence by one pass through cooling device. This regime is more suitable in case of γ about 1000 and in case of heavy particles: protons and ions. It is clear, that in this case the cooling time is very long and we can consider the square route of the equation (6) as a component of a force in the transverse directions

\[ F_y = -a\zeta \frac{\pi \omega}{2p_0} \Delta && (1+\Delta ||)exp(-\Delta ||/\langle p^2 \parallel \rangle) \]

where

\[ a = L_m/l, \quad L_m \text{ is the length of the circumference of the closed orbit}, \]
\[ l \text{ is the length of the interaction way in the cooling device}. \]

Notice, that in this case the cooling time is longer than in case of fast cooling considered in the previous subsection about 100 times, but much shorter than the heating time, which is about or longer than msec for electrons with γ about 10000. So, even in this case and with laser of lower intensity the resonant laser cooling is very effective. The longitudinal cooling force can be found from the equation (6) by assuming, that the average longitudinal momentum \( p_0 \) equals to equilibrium momentum \( p_s \), which is defined, as well known, by the resonanticity of accelerating process.

\[ F_\parallel = -a\zeta \frac{\pi \omega}{2p_0} \Delta && (1+\Delta ||)exp(-\Delta ||/\langle p^2 \parallel \rangle) \]

It is necessary to note, that the \( p_0 \) in the right hand size is a given function of time determined by acceleration. This force give an additional term in the longitudinal cooling rate determined by synchrotron radiation. Cooling time equals to

\[ t_{cool} = \zeta^{-1} = (Fc/E_0)^{-1} \]

where the \( E_0 \) is the equilibrium energy of the particle. Calculations show, that by using of solenoid with magnetic field about 5-10 T and laser with power about 5-10 kW, the cooling time is about few seconds for heavy particles with \( \gamma = 100 \). The protons of this energy it is possible to cool by laser with wavelength 10 mcw (CO2 laser) but ions with maser with wavelength about or shorter 1 mm. For comparison, the electron cooling time of ions in the magnetic field about 5-10 T is about 10000 sec. So, the cooling time of resonant laser cooling of heavy particles is 10000 times shorter, than cooling time of electron cooling.

4 Conclusion

The resonant laser cooling, i.e. the laser cooling of particle beams in a longitudinal magnetic field has a number of advantages as compared with the newest laser [17, 18] and undulator cooling methods. This method provides much faster cooling given that all other conditions are the same. The cooling time of about few msec is achievable in case of cooling
of electrons in comparison to few msecs in other methods, for example, damper-magnets. In case of relativistic heavy particles the cooling time of about few seconds are achievable in comparison to $10^4$ seconds cooling time of electron cooling. More realistic intensities of lasers (masers) are necessary in comparison to other laser cooling methods. Resonant laser cooling, if applied, can increase significantly the luminosity and lifetime of synchrotron accelerator beams, due to beam emittance decrease. All this advantages come from the using of the stimulated radiation of the beam in the autoresonant laser for beam cooling. ACKNOWLEDGEMENTS. I am thankful to E. Laziev and M. Petrosian for interest to this work and useful discussions.

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