CONTRACTIBLE EDGES IN 3-CONNECTED GRAPHS THAT PRESERVE A MINOR

JOÃO PAULO COSTALONGA

Abstract. Let G be a 3-connected graph with a 3-connected (or sufficiently small) simple minor H. We establish that G has a forest F with at least \(|G| - |H| + 1)/2\) edges such that G/e is 3-connected with an H-minor for each e ∈ E(F). Moreover, we may pick F with \(|G| - |H|\) edges provided G is triangle-free. These results are sharp. Our result generalizes a previous one by Ando et. al., which establishes that a 3-connected graph G has at least \(|G|/2\) contractible edges. As another consequence, each triangle-free 3-connected graph has a spanning tree of contractible edges. Our results follow from a more general theorem on graph minors, a splitter theorem, which is also established here.

Key words: Graph, Contractible edges, 3-Connectedness, Splitter Theorem

1. Introduction

The graphs we consider are allowed to have loops and parallel edges. A graph G is said to be k-connected if the remotion of each set of vertices of G with less than k vertices leaves a connected graph (we do not consider the usual requirement that |G| > k). An edge e of a 3-connected graph G is said to be contractible if G/e is 3-connected. We refer the reader to [5] for more about contractible edges. The following result will be generalized here.

Theorem 1. (Ando, Enomoto and Saito [1]) Every 3-connected graph G has at least \(|G|/2\) contractible edges.

If G is a 3-connected graph with a simple H-minor (a minor isomorphic to H), we say that e is an H-contractible edge of G if G/e is 3-connected with an H-minor. We establish:

Theorem 2. Let G be a 3-connected graph with a 3-connected simple minor H. Then G has a forest with \(|G| - |H| + 1)/2\) H-contractible edges.

Theorem 2 for \(H \cong K_1\) implies Theorem 1 with the additional thesis that the \(|G|/2\) contractible edges are in a forest. An interesting consequence of Theorem 2 is:

Corollary 3. Let G be a 3-connected graph with a 3-connected simple minor H and a subgraph K. Then G has a forest F with \(|G| - |K| + 1)/2\) - |K| + 1 edges avoiding E(K), such that G/e is 3-connected with an H-minor and having K as subgraph for each e ∈ F (considering that the labels of V(K) are kept in G/e).

Whittle [9] established the particular case that \(|G| - |H| \leq 2\) in Theorem 2 (more generally for matroids). When \(|G| - |H| = 3\), we have the following strengthening:

Corollary 4. (Costalonga [2] Corollary 1.8) Suppose that G is a 3-connected graph with a 3-connected simple minor H and \(|G| - |H| \geq 3\). Then G has a forest with 3 H-contractible edges.

Corollary 4 also holds for matroids (Theorem 1.3 of [2]). When G has no triangles, we may improve Theorem 2.

Theorem 5. Suppose that G is a triangle-free 3-connected graph with a 3-connected simple minor H. Then G has a forest with \(|G| - |H|\) edges which are H-contractible.

Although Egawa et. al. [3] proved that a sufficiently large 3-connected graph G has \(|G| + 5\) contractible edges, the number of H-contractible edges in Theorem 5 is sharp. We conjecture that the analogue of Theorem 5 also holds for matroids, what is not true for Theorem 2, because \(M := M^{\ast}(K_{3,n})\) has only 3-elements e such that si(M/e) is 3-connected, see [10] Theorem 2.10. Theorem 5 also yields the following corollary:

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Corollary 6. If \( G \) is a triangle-free 3-connected graph, then \( G \) has a spanning forest whose edges are contractible.

In order to prove Theorems 2 and 5, we establish a more general results, but, first, we will need some definitions. We define an wye of \( G \) as a subgraph of \( G \) isomorphic to the star with 3 edges. We say that a simple subgraph \( H \) of \( G \) is a fan of \( G \) if:

(F1) \( F \) has at least 3 edges,

(F2) \( E(F) \) has an ordering \( a_0, a_1, \ldots, a_{m+1} \) of its distinct edges such that, for \( i = 1, \ldots, n \), \( \{a_{i-1}, a_i, a_{i+1}\} \) induces a wye or a triangle in \( G \) and,

(F3) for \( 0 < i < m \), \( \{a_{i-1}, a_i, a_{i+1}\} \) induces a wye in \( G \) if and only if \( \{a_i, a_{i+1}, a_{i+2}\} \) induces a triangle.

In this case, we say that \( a_0, a_1, \ldots, a_{m+1} \) is a fan ordering of \( F \). It is easy to check that a fan must be isomorphic to one of the graphs in Figures 1, 2 or 3, where \( u, v_0, \ldots, v_{n+1} \) are pairwise distinct with the possible exception that \( v_0 \) and \( v_n \) may be equal in figure 2 and \( v_0 \) and \( v_{n+1} \) may be equal in figure 3. Note that, if \( G \) is 3-connected and \( v_0 = v_n \) or \( v_0 \), then \( G \) is a wheel. To simplify our language, when there is no risk of confusion, we may identify a fan of \( G \) with its edge-set or with one of its fan orderings. We say that a fan is triangle-to-triangle, wye-to-triangle or wye-to-wye, according whether they begin or end with triangles or wyes, as described in Figures 1, 2 and 3. The edges \( y_1, \ldots, y_n \) like in the figures are the spokes of \( F \), the vertex \( u \) is the hub of \( F \) and the path induced by the edges other than the spokes is called the rim of \( F \).

Suppose that \( F^+ \) is a maximal wye-to-wye fan of \( G \) (this is, \( F^+ \) is not a proper subgraph of other wye-to-wye fan of \( G \)). Let \( x_0, y_1, x_1, \ldots, y_n, x_n \) be a fan ordering of \( F^+ \), we say that \( F := G[y_1, x_1, \ldots, y_n] \) is an H-inner fan of \( G \) provided \( G/F \) is 3-connected with an \( H \)-minor. An inner fan of \( G \) is an H-inner fan for \( H \equiv K_1 \). An H-inner fan \( F \) of \( G \) is non-degenerated if \( |E(F)| \geq 2 \). If \( |E(F)| = 1 \), then \( F \) is said to be degenerated.

The rank of \( X \subseteq E(G) \) in \( G \) is the number \( r_G(X) \) of edges in a spanning forest of \( G[X] \), or, equivalently, the number of vertices in \( G[X] \) minus the number of connected components of \( G[X] \). For a family \( \mathcal{F} := \{X_1, \ldots, X_n\} \) of subsets of \( E(G) \), we define the rank of \( \mathcal{F} \) in \( G \) by \( r_G(\mathcal{F}) := r_G(X_1 \cup \cdots \cup X_n) \).
and $G[\mathcal{F}] := G[X_1 \cup \cdots \cup X_n]$; moreover, the rank-sum of $\mathcal{F}$ is defined as $\text{rs}_G(\mathcal{F}) := r_G(X_1) + \cdots + r_G(X_n)$. We say that a family $\mathcal{F}$ of subsets of $E(G)$ is free if its members are pairwise disjoint and the edge-set of each circuit of $G[\mathcal{F}]$ is contained in a member of $\mathcal{F}$. Equivalently, $\mathcal{F} := \{X_1, \ldots, X_n\}$ is free when $r_G(\mathcal{F}) = r_{G}\mathcal{G}(\mathcal{F})$. A family $\mathcal{F}$ of subsets of $E(G)$ is an $H$-fan family if the members of $\mathcal{F}$ are pairwise disjoint and each member of $\mathcal{F}$ is an $H$-contractible edge. When we talk about an inner fan without mention to a minor $H$, it is the case that $H \cong K_1$. Now we are in conditions to state our main theorems:

**Theorem 7.** Let $G$, $H'$ and $H$ be 3-connected simple graphs such that $H$ is a minor of $H'$, $H'$ is a minor of $G$ and $|H| \geq 1$. Suppose that $H'$ has a free $H$-fan family with rank $r$. Then $G$ has a free $H$-fan family with rank at least $|G| - |H'| + r$.

For $H = H'$ in Theorem 7 we have:

**Theorem 8.** Let $G$ be a 3-connected simple graph with a 3-connected simple minor $H$ satisfying $|H| \geq 1$. Then, $G$ has a free $H$-fan family with rank at least $|G| - |H|$.

For $H \cong K_1$ in Theorem 8 we may derive the following structural result:

**Corollary 9.** Let $G$ be a 3-connected graph. Then $G$ has a subgraph $F$ such that $V(F) = V(G)$ and each block of $F$ is an inner fan of $G$ or is induced by a contractible edge of $G$.

It is clear that Theorem 5 is a corollary to Theorem 8. If $\mathcal{F}$ is a pairwise disjoint family of subsets of $E(G)$ and $T$ is a triangle of $G$, we say that $T$ is a crossing triangle of $F$ if $T$ is a triangle of $G[F]$ but $E(T)$ is not contained in any member of $\mathcal{F}$. If we weaken the freeness condition of Theorem 7 to the absence of crossing triangles, we have the following result:

**Theorem 10.** Let $G$, $H'$ and $H$ be 3-connected simple graphs such that $H$ is a minor of $H'$, $H'$ is a minor of $G$ and $|H| \geq 1$. Suppose that $H'$ has an $H$-fan family without crossing triangles with rank-sum $s$. Then $G$ has an $H$-fan family without crossing triangles with rank-sum at least $|G| - |H'| + s$.

For establishing Theorem 2 it is enough to combine Theorem 8 with:

**Theorem 11.** Let $G$ be a 3-connected simple graph with a 3-connected simple minor $H$. Suppose that $G$ has a free $H$-fan family with rank $r \geq 1$. Then $G$ has a forest with $\lceil (r + 1)/2 \rceil$ $H$-contractible edges.

These results may be used to improve the bounds for the number of contractible edges in classes of graphs with fixed minors. For instance, see the next corollary, obtained from Theorems 10 and 11 for $H := K_1$ and $H' = K_{n,n}$.

**Corollary 12.** If $G$ is a 3-connected simple graph with an $K_{n,n}$-minor, then $G$ has a fan-family with rank sum $|G| + n^2 - 2n$ and $G$ has $\lceil |G| + n^2 - 2n + 1)/2 \rceil$ contractible edges. Moreover $G$ has $|G| + n^2 - 2n$ contractible edges if $G$ is triangle-free.

All results we stated up to now follow from Theorems 7, 10 and 11. These theorems will be proved in Section 4.

## 2. Preliminaries

When, in a graph $G$, an edge $e$ with endvertices $u$ and $v$ is not parallel to any other edge of $G$, we say that $e = uv$ in $G$. When there is no risk of confusion, we may refer to a vertex $v$ of $G$ in a minor $H$ of $G$ as the vertex obtained from the contraction of some subgraph of $G$ containing $v$. We define the operation of vertex splitting as the opposite of edge-contraction. We use the notation $[n] := \{1, \ldots, n\}$. We denote by $N_G(v)$ the set of neighbors of $v$ in $G$ and by $E_G(v)$ the set of edges of $G$ incident to $v$. Although some of the following lemmas are presented as corollaries to their more general versions for matroids, the reader shall have no problem to prove their graphic versions straightforwardly.
Lemma 13. (Corollary to [7, Proposition 8.2.7]) Let $G$ be a 2-connected graph with an edge $x$ such that $G/x$ is 3-connected but $G$ is not. Then, one of the endvertices of $x$ has exactly two neighbors in $G$.

Corollary 14. Suppose that $T$ is a triangle in a 3-connected simple graph $G$ such that $G/T$ is 3-connected. Let $v \in V(T)$ and $y \in E(T) - E_G(v)$. Then $G/y$ is 3-connected or $\deg_G(v) = 3$.

Proof. Use Lemma 13 on $G/y$ for some $x \in T - y$. □

We denote by $\text{si}(G)$ the simplification of $G$, a graph obtained from $G$ by removing all loops and deleting all but one edges in each class of parallel edges. The cosimplification of $G$, $\text{co}(G)$, is defined by a graph obtained from $G$ by removing all vertices with degree less than two and, in each path of $G$ maximal in respect to having all internal vertices with degree 2, contracting all but one edges. Note that $\text{co}(G)$ and $\text{si}(G)$ are uniquely determined up to choosing what labels of $G$ will remain. If the reader is familiar with matroids, it is important to note that our definition of cosimplification is slightly different from that one for matroids, since we keep pairs of non-adjacent edges in a 2-edge cut. But these definitions are coincident when $\text{co}(G)$ is 3-connected, which is the case we are going to use it.

Lemma 15. (Corollary to [9, Lemma 3.7]) Suppose that $G$ is a 3-connected graph, $T$ is a triangle and $Y$ is a wye of $G$. If $E(T) - E(Y) = \{y\}$, then $\text{si}(G/x_1, y) \equiv \text{si}(G/x_2, y)$ for all $x_1, x_2 \in E(Y)$.

Lemma 16. (Corollary to [9, Lemma 3.8]) Suppose that $G$ is a 3-connected graph, $T$ is a triangle and $Y$ is a wye of $G$. If $E(T) - E(Y) = \{y\}$ and $E(Y) - E(T) = \{x\}$ then $G/x$ and $\text{co}(G/y)$ are 3-connected.

From Lemma 16 we have the following corollaries:

Corollary 17. Suppose that $x_0, y_1, x_1, y_2, x_2$ is a fan ordering of a wye-to-wye fan in a 3-connected graph $G$. Then $G/x_1$ is 3-connected or $G$ has a wye containing $y_1$ and $y_2$.

Corollary 18. If $G$ is a 3-connected graph with a triangle $T$ containing 3 degree-3 vertices, then $G/T$ is 3-connected. Moreover if $G$ is simple and $G \not\cong K_4$, then $G/T$ is simple.

Corollary 19. Suppose that $x_0, y_1, x_1, \ldots, y_n, x_n$ is a fan ordering of a wye-to-wye fan of a 3-connected simple graph $G$ with $n \geq 3$. If $1 \leq i \leq n - 1$, then $G/x_i$ is 3-connected and simple and has $x_0, y_1, x_1, \ldots, y_{i-1}, x_{i-1}, y_{i+1}, x_{i+1}, \ldots, x_n$ as the fan ordering of a wye-to-wye fan.

Lemma 20. Suppose that $G$ is a simple 3-connected graph and that $x$ and $y$ are edges of $G$ such that $G/x, y$ is 3-connected but $G/y$ is not. Then $|G| \geq 5$ and $G$ has a wye $Y$ and a triangle $T$ such that $E(T) - E(Y) = \{x\}$ and $x \in E(Y)$.

Proof. Suppose the contrary. If $|G| \leq 4$, then it is clear that $G/y$ is 3-connected. Thus $|G| \geq 5$. By Lemma 13 on $G/y$, it follows that $x$ is incident to a vertex $u$ with exactly two neighbors $v$ and $w$ in $G/y$. Since there are no degree-2 vertices in $G$ and neither in $G/y$, then $u$ is incident to at least one pair $P$ of parallel edges of $G/y$. Since $G$ is simple, $G$ is obtained from $G/y$ by splitting one of the vertices incident to $P$. If $y$ is obtained by splitting $u$, then $G\{v, w\}$ is disconnected, a contradiction. So, $G$ is obtained by splitting one of $v$ or $w$. As $G$ is simple, $P \cup y$ is the edge set of a triangle $T$ of $G$ and $Y := G[E_G(u)]$ is a wye of $G$ meeting $P$ and containing $x$ but not $y$. This proves the lemma. □

Lemma 21. Suppose that $G$ is a 3-connected graph with $|G| \geq 4$, $e$ is an edge of $G$ other than a loop and $v$ is a vertex of $G$ not incident to $e$. Let $G'$ be the graph constructed from $G$ by putting a vertex $u$ in the middle of $e$ and adding an edge $f$ linking $u$ and $v$. Then $G'$ is 3-connected.

Proof. Let $w$ be an endvertex of $e$ in $G$. Note that $G'/uw \equiv G + vw$ is 3-connected. If $G'$ is not 3-connected, then, by Lemma 13, we have a vertex in $G'$ with only two neighbors. By construction, this implies that $G$ has a vertex with at most two neighbors. A contradiction. □

Lemma 22. Let $G$ be a 3-connected graph. Suppose that $F$ is a singleton set with an edge in a wye of $G$ or $F$ is a triangle-to-triangle fan of a wye-to-wye fan of $G$. If $G/F$ is 3-connected, then $F$ is an inner fan of $G$. 


Proof. The result is clear if \( G \) is a wheel. Suppose for a contradiction that \( G \) is not a wheel and there is a wye-to-wye fan \( F^{+} \) containing \( F \) with \( |E(F^{+})| > |E(F)| + 2 \). In particular, we may pick \( F^{+} \) such that \( |E(F^{+})| = |E(F)| + 4 \). Consider the labels of \( F^{+} \) as in Figure 3. Then \( y_{1} \) and \( y_{n-1} \) are the extreme spokes of \( F \). Note that \( v_{n} \) is a degree-2 vertex of \( s(G/F) \). This implies that \( |G/F| \leq 3 \) because \( G/F \) is 3-connected. If \( v_{0} = v_{n+1} \), it is clear that \( G \) is a wheel. So \( v_{0}, v_{n}, v_{n+1} \) and \( u \) are distinct vertices of \( G/F \). A contradiction. \( \square \)

Lemma 23. Let \( G \) be a graph, suppose that \( Y, X \subseteq E(G) \) are sets such that \( Y \) induces a wye in \( G \) and \( X \) is an union of edge-sets of circuits of \( G \). Then \( |Y \cap X| \neq 1 \).

3. Lemmas

In this section we prove some lemmas towards the proof of the theorems. We will use the symbol “\( \Diamond \)” to point the end of a nested proof. We denote by \( \Pi_{3} \) the prism with triangular bases.

Lemma 24. Let \( G \) be a 3-connected simple graph with an edge \( x \) such that \( G \setminus x \) is 3-connected with a simple minor \( H \). Suppose that \( F \) is a non-degenerated \( H \)-inner fan of \( G \setminus x \). Then, \( F \) contains the members of a free \( H \)-fan family of \( G \) with rank \( r_{G}(F) \).

Proof. Assume the contrary. Consider the labels for a maximal wye-to-wye fan \( F^{+} \) of \( G \setminus x \) containing \( F \) as in Figure 3. If possible choose \( F^{+} \) with hub having degree at least 4 in \( G \).

If \( F \) is a fan of \( G \), then, as \( G/F \setminus x \) is 3-connected, so is \( G/F \). By Lemma 22, \( F \) is an \( H \)-inner fan of \( G \) and the lemma holds. Thus, \( F \) is not a fan of \( G \). Hence, \( x \) is incident to \( v_{i} \) in \( G \) for some \( s \in [n] \).

This implies that \( \deg_{G}(u) = \deg_{G \setminus x}(u) \). If \( \deg_{G}(u) = 3 \) then \( F \) has all vertices with degree 3 in \( G \setminus x \) and we should have chosen \( F^{+} \) with \( v_{s} \) as hub instead of \( u \). Thus \( \deg_{G}(u) \geq 4 \). By Corollary 17, each element in the rim of \( F \) is \( H \)-contractible in \( G \setminus x \) and, therefore, in \( G \). So, if we find an \( H \)-inner fan \( F' \) of \( G \) contained in \( F \), then the family

\[
\{ \{ x_{i} \} : i \in [n-1] \text{ and } x_{i} \in E(F') \} \cup \{ F' \}.
\]

satisfies the lemma. To find \( F' \) we will consider two cases.

Case 1. \( x \) is incident to \( v_{t} \) for some \( t \in \{ 0, \ldots, n + 1 \} - s \): We may assume that \( t > s \). As \( G \) is simple, \( t \geq s + 2 \). Define \( G_{1} := G/\{ x_{s+2}, \ldots, x_{t-1}, y_{s+2}, \ldots, y_{t-1} \} \) (see Figure 4). By Corollary 19, \( G_{1} \setminus x \) is 3-connected and, as a consequence, so is \( G_{1} \). Now, note that \( \{ x, x_{s}, x_{s+1} \} \) induces a triangle and
[\{y_{s+1}, x_5, x_{s+1}\}] induces a wye in G_1. By Lemma \[16\] G_1/y_{s+1} is 3-connected. Now, let F' be the fan of G with fan ordering \(y_{s+1}, x_{s+1}, \ldots, y_{t-1}\). So, G/F' = G_1/y_{s+1} is 3-connected. By Lemma \[22\] F' is an H-inner fan of G and the lemma holds in Case 1.

Case 2. \(v_s\) is the unique vertex of \(F^+\) incident to \(x\): We may assume that \(s \geq 2\). Define:

\[G_2 := G/x_1, \ldots, x_{s-2}, y_1, \ldots, y_{s-2}\quad \text{and} \quad G_3 := G/x, \ldots, x_{n-1}, y_3, \ldots, y_n.\]

We represent \(G_2[F \cup x]\) and \(G_3[F \cup x]\) in Figures 5 and 6. We keep the labels of \(v_s\) and \(v_{s-1}\) in \(G_2\) and \(G_3\). Let \(v_s\) and \(v_b\) be the endvertices of \(x\). By Corollary \[19\] \(G_2/x\) and \(G_3/x\) are 3-connected and, therefore, so are \(G_2\) and \(G_3\).

By the description of Case 2, \(v_s, v_{s-1}, v'_{n+1}\) and \(u\) are distinct neighbors of \(v_s\) in \(G_3\). Thus \(\deg_{G_3}(v_s) \geq 4\). Note that \(G_3/[x_{s-1}, y_{s-1}, y_n] = G/F\) is 3-connected. As \(v_s\) is opposite to \(y_{s-1}\) in \(G_3/[x_{s-1}, y_{s-1}, y_n]\), thus, by Corollary \[14\] \(G_3/y_{s-1}\) is 3-connected.

As \(G_2/y_{s-1}\) can be obtained from \(G_3/y_{s-1}\) by successively applying Lemma \[21\] (see Figures 5 and 6), then \(G_2/y_{s-1}\) is 3-connected because so is \(G_3/y_{s-1}\). Let \(F'\) be the fan of \(G\) with fan ordering \(y_1, x_1, \ldots, y_{s-1}\). Note that \(G_2/y_{s-1} = G/F'\), which is 3-connected. By Lemma \[22\], \(F'\) is an H-inner fan of \(G\) and the lemma holds.

\[\square\]

Lemma 25. Suppose that \(F\) is an inner fan of a 3-connected graph \(G\) and \(|G/F| \leq 3\). Then \(G\) is a wheel.

\[\text{Proof.}\] We may assume that \(|G| \geq 5\). Consider a wye-to-wye fan \(F^+\) of \(G\) containing \(F\) labeled as in Figure 3. If \(v_{n+1} = v_0\), the result is clear, so, assume that \(v_0 \neq v_{n+1}\). Therefore, \(V(G/F) = \{u, v_0, v_{n+1}\}\). Hence, \(V(G) = V(F^+)\). As \(G\) has no vertices with degree less than 3, then \(X := \{uv_0, uv_{n+1}, v_0v_{n+1}\} \subseteq E(G)\). If \(G\) has some edge out of \(E(F)\) \(\cup X\), we have a contradiction to the fact that \(F^+\) is a wye-to-wye fan of \(G\). This proves the lemma.

\[\square\]

Lemma 26. Let \(G\) be a 3-connected graph with an edge \(x\) such that \(G/x\) is 3-connected and simple with a simple minor \(H\). If \(F\) is a non-degenerated H-inner fan of \(G/x\) such that \(G[E(F)]\) has no triangles, then one of the spokes of \(G\) is H-contractible in \(G\).

\[\text{Proof.}\] Consider, in \(G/x\), a maximal wye-to-wye fan \(F^+\) containing \(F\), labeled as in Figure 3. Since \(G[E(F)]\) has no triangles, \(G\) is obtained from \(G/x\) by splitting \(u\) into two vertices \(u_1\) and \(u_2\) in such a way that \(v_i\) is adjacent to \(u_i\) in \(G\) if \(i\) is odd and to \(u_2\) if \(i\) is even. If \(G/x\) is a wheel, then \(G\) is isomorphic to the graph in Figure 7 and, in this case, the result may be verified directly. So, assume that \(G/x\) is not a wheel. By Lemma \[25\] \(|G/F \cup x| \geq 4\). Hence, \(|G| \geq 7\). Moreover, \(v_0 \neq v_n\).

As \(G/x\) is not a wheel, then there is \(u' \in N_{G/x}(u) - V(F^+)\). If \(n\) is even, by symmetry, we choose the labels in such a way that \(u' \in N_G(u_1)\).

We may assume that \(G/u_2u_2\) is not 3-connected. As \(|G/u_2v_2| \geq 4\), thus \(G/u_2v_2\) has a 2-vertex cut. Since \(G\) is 3-connected, \(G\) has a 3-vertex cut in the form \(S := \{v_2, u_2, w\}\). Note that \(w \neq u_1\) because \(G/x = G/u_1u_2\) is 3-connected. So, \(G/S\) has a vertex \(s\) in a different connected component than \(u_1\). Denote by \(v_F\) the vertex of \(G/F \cup x\) obtained by the contraction of \(F \cup x\) in \(G\) and denote \(G' := G/F \cup x\).

If \(s \in V(F)\), as \(v_1u_1 \in E(G)\), \(s \neq v_1\). Thus \(n \geq 3\). As \(u_1v_3 \in E(G)\), then \(u\) is in the \((v_3, s)\)-path contained in the rim of \(F\). Let \(v_k := s\). As \(G'\) is 3-connected, \(|G'| \geq 4\) and \(v_0, v_n \in V(G') - v_F\), then \(G'\backslash v_F\) has a \((v_0, v_n)\)-path \(\gamma\). Note that \(v_k, v_k+1, \ldots, v_n, \gamma, v_0, v_1, u_1\) is an \((s, u_1)\)-path of \(G'S\). A contradiction. Therefore, \(s \notin V(F)\) and \(s\) is a vertex of \(G'\) distinct from \(v_F\).

If \(w \notin \{v_0, v_1\}\), define \(s := v_0, v_1, u_1\). Otherwise, if \(n \geq 3\), define \(s := v_n, \ldots, v_3, u_1\) and, if \(n = 2\), define \(s = u', u_1\). Denote by \(t\) the first vertex of \(s\). So, \(s\) is a \((t, u_1)\)-path of \(G'S\). As, \(s \in V(G') - \{w, v_F\}\), \(G'\backslash \{w, v_F\}\) has an \((s, t)\)-path \(\varphi\). Now, \(s, \varphi, t, s, u_1\) is an \((s, u_1)\)-path of \(G'S\). A contradiction.

\[\square\]

Lemma 27. Let \(G\) be a simple 3-connected graph with an edge \(x\) such that \(G/x\) is 3-connected and simple with a simple minor \(H\). Suppose that \(F\) is an H-inner fan of \(G/x\). Consider the labels for a maximal wye-to-wye fan \(F^+\) of \(G\) containing \(F\) as in Figure 3. If \(F\) is a fan of \(G\), then one of the following alternatives holds:
(a) $F$ is an $H$-inner fan of $G$ or
(b) $G$ contains an edge $y$ such that one of $x$, $y$, $x_0, y_0, x_1, y_1, \ldots, x_n, y_n$ or $x_0, y_1, x_1, \ldots, y_n, x_n, y, x$ is the fan ordering of a maximal wye-to-wye fan of $G$ containing an $H$-inner fan of $G$.

**Proof.** Since $F$ is a fan of $G$, then $F^+$ is a wye-to-wye fan of $G$. By Lemma 22, (a) holds if $G/F$ is 3-connected. So, assume that $G/F$ is not 3-connected. By Corollary 13 used iteratively, $G_1 := G/x_2, \ldots, x_{n-1}/y_2, \ldots, y_{n-1}$ is simple and 3-connected. If $\deg_{G_1}(u) = 3$, then $G_1/y_1, x_1, y_n = G/F$ is 3-connected by Corollary 13 a contradiction. Thus, $\deg_{G_1}(u) \geq 4$ and, by Corollary 17 $G_2 := G_1/x_1 \setminus y_1$ is 3-connected and simple.

Note that $G_2/x, y_n = G/F \cup x$ is 3 connected, but $G_2/y_n = G/F$ is not 3-connected. By Lemma 20 $G_2$ has a wye $Y$ and a triangle $T$ such that $T(E(T) - \{y_n\} = \{y_n\}$ and $x \in E(Y)$. But $y_n$ is in the wye induced by $\{x_0, x_n, y_n\}$ in $G_2$. So, we may assume without losing generality that $x_n \in T$ and, therefore, $T(E(T) = \{x_n, y_n, y\}$, where $y = uv_0$ in $G_2$. Note that $E(T)$ also induces a triangle in $G$. So $y = uv_{n+1}$ in $G$. Moreover, $x \neq y$ and, therefore, $x \in (E(Y) - E(T))$. This implies that $x_0, y_1, \ldots, x_n, y, x$ is a wye-to-wye fan of $G$. To conclude (b) we have to check that $G/(E(F) \cup \{x_n, y\}) = G_2/x, y_n$ is 3-connected. By Lemma 15 $G_2/x, y_n \cong \text{si}(G_2/y, y_n) \equiv \text{si}(G_2/x_n, y_n, y)$. As $G_2/x, y_n = G/F \cup x$, then $G_2/x_n, y_n, y$ is 3-connected and the lemma is valid. □

**Lemma 28.** Let $G$ be a 3-connected simple graph with $|G| \geq 4$ and with an edge $x$ such that $G/x$ and $G$ is 3-connected and simple. Suppose that $F$ is an inner fan of $G$ and $G$ is obtained from $G/x$ by splitting the hub of $F$. Consider the labels of a wye-to-wye fan $F^+$ of $G/x$ containing $F$ as in figure 3. If for $k \in [n-1]$ $G/x_k$ is not 3-connected, then $x_{k-1}, y_k, x_k, y_{k+1}, x_{k+1}$ is the fan ordering of a maximal wye-to-wye fan of $G$ with a degree-3 hub.

**Proof.** Suppose the contrary. Since $G$ is obtained for $G/x$ by splitting the hub $u$ of $F$, then $\deg_{G/x}(u) \geq 4$. This implies that $|G/x| \geq 5$. By Corollary 17 $G/x, x_k$ is 3-connected. By Lemma 20 there is a wye $Y$ of $G$ meeting a triangle $T$ such that $x \in E(Y)$ and $E(T) - E(Y) = \{x_k\}$. As $G/x$ is 3-connected and simple, $x$ is in no triangle of $G$, and, therefore, $T$ is a triangle of $G/x$. As $G/x$ is 3-connected with $|G/x| \geq 5$ and $x_k$ is in the rim of a fan of $G$, then it is straightforward to verify that $T$ is the unique triangle of $G/x$ containing $x_k$. Then, $E(T) = \{x_k, y_k, y_{k+1}\}$. As a consequence, $Y = \{x, y_k, y_{k+1}\}$ and the lemma holds. □

**Lemma 29.** Let $G$ be a 3-connected simple graph, other than a wheel, not isomorphic to $\Pi_3$ and with an edge $x$ such that $G/x$ is 3-connected and simple with a simple minor $H$. Suppose that $F$ is a non-degenerative $H$-inner fan of $G/x$ but $G[E(F)]$ is not a fan of $G$. Then $E(F)$ contains a free $H$-fan family $\mathcal{X}$ of $G$ such that:

(a) $r_\mathcal{X}(F) = r_G(x,F),$
(b) $\mathcal{X} \cup \{\{x\}\}$ is a free-family of $G$ and
(c) one of the members of $\mathcal{X}$ contains an edge incident to the hub of $F$ in $G/x$ and the other members are singleton sets in the rim of $F$.

Moreover, $G[E(F)]$ contains at most one triangle $T$ with three degree-3 vertices and when such triangle exists it is a member of $\mathcal{X}$.

**Proof.** Consider a maximal wye-to-wye fan $F^+$ of $G/x$ containing $F$, labeled as in Figure 3. Since $G[E(F)]$ is not a fan of $G$, then $G$ is obtained from $G/x$ by splitting $u$ into vertices $u_1$ and $u_2$. Let $F_1, \ldots, F_m$ be the maximal subsets of $E(F)$ such that each $G[F_k]$ is a triangle-to-triangle fan of $G$ or $F_k$ is a singleton set with a spoke of $F$. Let $y_{s_k}$ and $y_{t_k}$ be the extreme spokes of $F_k$ with $s_k \leq t_k$, which are incident to $v_{s_k}$ and $v_{t_k}$, respectively. Choose the labels in such a way that $k > l$ implies $s_k > s_l$ (this labeling is illustrated in Figure 8). First we check:

1. There is at most one index $k \in [m]$ such that $F_k$ is a triangle of $G$ with $3$ degree-3 vertices.

Suppose the contrary. Let $1 \leq i < j \leq m$ be such indices. Say that $u_1$ is a vertex of $F_i$. So $E_G(u_1) = \{y_{s_1}, y_{t_1}, x\}$. Thus $u_2 \in V(F_j)$. Analogously, $E_G(u_2) = \{y_{s_1}, y_{t_1}, x\}$. Thus $y_{s_2}, y_{t_2}, y_{s_3}$ and $y_{t_3}$ are the unique spokes of $F$ and $n = 3$. Define $W := \{u_1, u_2, v_1, v_2, v_3, v_4\}$. If $G$ has a vertex $v \in V(G) - W$,
then it is clear that \( v \) and \( u_1 \) are in different connected components of \( G \backslash \{v_1, v_4\} \). Thus \( V(G) = W \). Now it is straightforward to check that \( G \cong \Pi_3 \) or to \( \mathcal{W}_4 \). A contradiction to the hypothesis. \( \diamond \)

**II.** For some \( \alpha \in [m] \), \( G/F_\alpha \) is 3-connected and each edge \( z \) in the rim of \( F \) and out of \( E(F_\alpha) \) is \( H \)-contractible in \( G \).

We consider two cases for this.

**Case 1:** For some \( \alpha \in [m] \), \( F_\alpha \) is a triangle with 3 degree-3 vertices. By Lemma 18, \( G/F_\alpha \) is 3-connected. By (II), each edge \( z \) in the rim of \( F \) and out of \( F_k \) is not in a triangle with 3 degree-3 vertices and, therefore, by Lemma 28, \( z \) is \( H \)-contractible in \( G \). Note that the second part of the lemma is proved.

**Case 2:** Otherwise: By Lemma 28, each edge in the rim of \( G \) is \( H \)-contractible in \( G \). We just have to find \( \alpha \in [m] \) such that \( G/F_\alpha \) is 3-connected. For \( k \in [m] \), define \( Y_k := \{y_{k+1}, \ldots, y_{k-1}\} \) and \( X_k := \{x_{k+1}, \ldots, x_{k-1}\} \). Moreover, let:

\[
G_1 := G/X_1 \cup \cdots \cup X_m \cap \{Y_1 \cup \cdots \cup Y_m\}.
\]

Note that the unique edge of \( G_1 \) remaining from each \( F_i \) is \( y_{t+1} \) (see Figures 8 and 9). Since \( G[E(F)] \) is not a fan of \( G \), then \( m \geq 2 \). This implies that \( G_1/x \) is obtained from \( G/x \) by repeatedly performing the operation of Corollary 19. Hence, \( G_1/x \) is 3-connected and simple. Now we split this case two into two subcases:

**Case 2.1:** \( G_1 \) is not 3-connected: By Lemma 13, we may assume that \( \deg_{G_1}(u_2) = 2 \) because \( G_1/x \) is 3-connected. Say that \( y_{t+1} \) is incident to \( u_2 \). Therefore \( G_1/X_{t+1} \cap Y_{t+1} \) has \( u_2 \) as a degree-2 vertex incident to \( y_{t+1} \) and \( x \). Thus, \( G/F_\alpha = G/X_\alpha \cap Y_\alpha \cap y_{t+1} \cong (G/x)/(Y_\alpha \cap y_{t+1}) \), which is 3-connected by Corollary 19. So, we have the desired \( \alpha \) in this case.

**Case 2.2:** \( G_1 \) is 3-connected: Now, \( F'_+ := x_0, y_{t+1}, x_{t+1}, y_{t+2}, \ldots, x_{t+m}, y_{t+m}, x_n \) is a maximal wye-to-wye fan of \( G_1/x \). Since \( m \geq 2 \), then \( F' = y_{t+1}, x_{t+1}, y_{t+2}, \ldots, x_{t+m}, y_{t+m} \) is a maximal triangle-to-triangle fan of \( G_1/x \) contained in \( F'_+ \) (see Figure 9). Since \( G_1/F' \cup x = G/F \cup x \) is 3-connected, then \( F' \) is an
H-inner fan of $G_1/x$ by Lemma\textsuperscript{22}. By construction, none of the edge-sets of triangles of $F'$ is the edge-set of a triangle of $G_1$, thus, by Lemma\textsuperscript{25} for some $\alpha \in [m]$, $y_{t_{\alpha}}$ is $H$-contractible in $G_1$. Say that $y_{t_{\alpha}}$ is incident to $u_1$. Consider a graph $G_2$ obtained from $G_1/y_{t_{\alpha}}$ by changing the label of $y_{t_{\alpha}}$ by $y_i$ for $i > \alpha$ (see Figure\textsuperscript{10}). Now $G/F_{t_{\alpha}}$ can be rebuilt from $G_2$ using the operation described in Lemma\textsuperscript{21} (see Figures\textsuperscript{10} and\textsuperscript{11}). Therefore, $G/F_{t_{\alpha}}$ is 3-connected and\textsuperscript{11} holds.

Now, by Lemma\textsuperscript{22}, $F_{t}$ is an $H$-inner fan of $G$. Let
\[
\mathcal{X} := \{x_i : i \in [n-1] \text{ and } x_i \notin F_{t}\} \cup \{F_{t}\}.
\]
Recall that, for $k \in [m]$, $x_k$ is $H$-contractible in $G$ if $x_k \notin F_{t}$. Now, items (a), (b) and (c) are easy to verify. $\Box$

4. PROOFS FOR THE THEOREMS

In this section we prove theorems \textsuperscript{7} \textsuperscript{10} and \textsuperscript{11}. We define the vertex cleaving operation as the inverse of the identification of non-adjacent vertices. The next lemma is the key for proving Theorems \textsuperscript{10} and \textsuperscript{11}.

**Lemma 30.** Let $G$ and $H$ be 3-connected simple graphs such that $H$ is a minor of $G$. Suppose that $G$ is not isomorphic to $\Pi_3$ and neither to a wheel. Suppose that $x$ is an edge of $G$ such that some $G' \in \{G/x, G \setminus x\}$ is 3-connected with an $H$-minor and $\mathcal{F}$ is an $H$-fan family of $G'$ without crossing triangles. Then $G$ has an $H$-fan family $\mathcal{X}$ such that:

(a) $\mathcal{X}$ has no crossing triangles,

(b) $\mathcal{X}$ is free if $\mathcal{F}$ is free and

(c) $r_{S_G}(\mathcal{X}) \geq r_{S_G}(\mathcal{F}) + 1$ if $G' = G/x$ and $r_{S_G}(\mathcal{X}) \geq r_{S_G}(\mathcal{F})$ if $G' = G \setminus x$.

**Proof.** Write $\mathcal{F} := \{F_1, \ldots, F_m\}$. We first make the proof in the simple case, when $G' = G/x$. If $F_k$ is an $H$-inner fan of $G/x$, then, by Lemma\textsuperscript{24}, $F_k$ contains a free $H$-fan family $\mathcal{F}_k$ with rank $r_{S_G}(F_k)$. Otherwise, $F_k$ is singleton and contains an $H$-contractible element of $G/x$ and, therefore, of $G$. In this case we define $\mathcal{F}_k := \{F_k\}$. It is straightforward to check that $\mathcal{X} := \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m$ is the family we are looking for in this case.

Now, assume that $G' = G/x$. Let $F$ be the union of the members of $\mathcal{F}$. Next, we define a partition $\{I_1, I_2, I_3, J_1, J_2, J_3, K, L\}$ of $[m]$ and families $\mathcal{X}_k$, for $k \in [m] - L$ as follows. First we will define the sets $I_1$, $I_2$ and $I_3$. For $i \in [m]$, we let:

- $i \in I_1$ if $G/F_i$ is not 3-connected and $F_i$ is a fan of $G$,
- $i \in I_2$ if $G/F_i$ is not 3-connected, $|F_i| = 1$ and $F_i$ is in a wye of $G$ and
- $i \in I_3$ if $G/F_i$ is not 3-connected, $|F_i| = 1$ and $F_i$ is not in a wye of $G$.

For $i \in I_1$, let $F_i^+$ be a wye-to-wye fan of $G$ containing $F_i$ with $|E(F_i^+)| = |E(F_i)| = 2$. By Lemma\textsuperscript{27}, $G$ has an edge $\psi(i)$ such that, for some ordering $x_{0_i}^i, y_{1_i}^i, x_{1_i}^i, \ldots, y_{n_i}^i, x_{n_i}^i$ of $F_i^+$ and for $\chi(i) := x_{0_i}^i$, we have that $x, \psi(i), \chi(i), y_{1_i}^i, x_{1_i}^i, \ldots, y_{n_i}^i, x_{n_i}^i$ is a wye-to-wye fan of $G$ and $F_i' := -\psi(i), \chi(i), y_{1_i}^i, x_{1_i}^i, \ldots, y_{n_i}^i$ is an $H$-inner fan of $G$. In this case, we define $\mathcal{X}_i := \{F_i'\}$. Note that $G/\psi(i)$ is not 3-connected for $i \in I_1$.

For $i \in I_2 \cup I_3$, we denote $F_i = \{y_{1_i}^i\}$. By Lemma\textsuperscript{20}, as $x$ is in no triangle of $G$, there are edges $\chi(i)$ and $\psi(i)$ such that $(x, \chi(i), \psi(i))$ induces a wye and $F_i' := (\chi(i), \psi(i), y_{1_i}^i)$ induces a triangle of $G$. By Lemma\textsuperscript{15}, $s_i(G/F_i') = s_i(G/F_i \cup x)$ is 3-connected with an $H$-minor.

If $i \in I_2$, then, by Lemma\textsuperscript{22}, $F_i'$ is an $H$-inner fan of $G$ and we define $\mathcal{X}_i := \{F_i'\}$. Moreover, we pick the labels of $\chi(i)$ and $\psi(i)$ in such a way that $\chi(i)$ is in a wye of $G$ with $y_{1_i}^i$. In particular, this implies that $G/\psi(i)$ is not 3-connected.

If $i \in I_3$, then by Corollary\textsuperscript{14}, $\chi(i)$ and $\psi(i)$ are $H$-contractible in $G$. In this case, we define $\mathcal{X}_i := \{\chi(k), \psi(k)\}$. We may pick the labels of $\chi(i)$ and $\psi(i)$ in such a way that $\psi(i) \notin F$ because of the following:

(I). If $i \in I$, then $|\{\chi(i), \psi(i)\} \cap F| \leq 1$

If $\chi(i), \psi(i) \notin F_i$, then $F_i'$ is a crossing triangle of $\mathcal{F}$. This proves\textsuperscript{11}.

Moreover, as we observed before:
(II). If \( i \in I_1 \cup I_2 \), then \( G \psi(i) \) is not 3-connected.

Define \( I = I_1 \cup I_2 \cup I_3 \). We defined functions \( \psi, \chi : I \to E(G) \). For each \( j \in |m| - I \) such that \( F_j \) do not intersect \( \psi(I) \cup \chi(I) \), we let:

- \( j \in J_1 \) if \(|F_j| = 1 \) and \( G/F_j \) is 3-connected,
- \( j \in J_2 \) if \(|F_j| > 1 \), \( F_j \) is a fan of \( G \) and \( G/F_j \) is 3-connected and
- \( j \in J_3 \) if \(|F_j| > 1 \) and \( F_j \) is not a fan of \( G \).

For \( j \in J_1 \), we simply define \( \mathcal{X}_j := \{F_j\} \).
If \( j \in J_2 \), then by Lemma \ref{lem:fan} \( F_j \) is an \( H \)-inner fan of \( G \) and we define \( \mathcal{X}_j := \{F_j\} \).
For \( j \in J_3 \), by Lemma \ref{lem:fan} there is a free \( H \)-fan family \( \mathcal{X}_j \) of \( G \) with the members contained in \( F_j \) satisfying items (a), (b) and (c) of such lemma.

We define \( J := J_1 \cup J_2 \cup J_3 \). For each \( k \in |m| - I \) with \( F_k \) meeting \( \psi(I) \cup \chi(I) \), we let:

- \( k \in K \) if \(|F_k| > 1 \) and
- \( k \in L \) if \(|F_k| = 1 \).

For \( k \in K \), we define \( \mathcal{X}_k \) as the partition of the edge-set of the rim of \( F_k \) in singleton sets. We will check on (IV)(viii) that \( \mathcal{X}_k \) is a free \( H \)-fan family of \( G \).

We will not define \( \mathcal{X}_k \) for \( k \in L \). Observe that \( \{I, I_2, I_3, J_1, J_2, J_3, K, L\} \) is indeed a partition of \(|m| \).
Moreover each \( \mathcal{X}_i \) is a free \( H \)-fan family of \( G \). Next we prove:

(III). If \( (i, j) \) is a 2-subset of \( I \), then \( \chi(i), \psi(i) \cap F'_{j} = \chi(i), \psi(i) \cap (\chi(j), \psi(j)) = \psi_j \).

First we check that \( \chi(i), \psi(i) \cap (\chi(j), \psi(j)) = \phi \). Suppose the contrary. Then, for \( k = i, j \), \( Y_k := \chi(k), \psi(k), x \) induces a wye of \( G \). But this implies that \( Y_i = Y_j \) since such wyes have a common pair of edges. Thus, \( \chi(i), \psi(i) \cap (\chi(j), \psi(j)) \). Moreover, for \( k = i, j \), \( T_k := G[\chi(k), \psi(k), y] \) is a triangle. As \( G \) is simple, \( y' \) is \( y \) and, therefore, \( F_i \) intersects \( F_j \). A contradiction.

Now it is left to show that \( \chi(i), \psi(i) \cap F_j = \phi \). Suppose for a contradiction that \( z \in \chi(i), \psi(i) \cap F_j \). As \( G(F_j) \) is a union of circuits of \( G \), by Lemma \ref{lem:intersection} \( Y_i \) meets \( F_j \) in at least two edges. As \( x \notin F_j \), then \( \chi(i), \psi(i) \in F_j' \). As \( \chi(i), \psi(i) \cap (\chi(j), \psi(j)) = \phi \), thus \( \chi(i), \psi(i) \subseteq F_j' \cap (\chi(j), \psi(j)) \subseteq F_j \subseteq F \). A contradiction to (I) \( \Box \).

By (III) for each \( l \in L \) there is an unique index \( \varphi(l) \in I \) such that \( F_j \) is either equal to \( \chi(\varphi(l)) \) or \( \psi(\varphi(l)) \). This defines a function \( \varphi : L \to I \). By (II), \( \varphi \) is injective. We will extend the domain of \( \varphi \) to \( L \cup K \) further. Next we prove:

(IV). If \( k \in K \), then there is an unique index \( i \in I \) such that \( \chi(i), \psi(i) \) meets \( F_k \). Moreover:

(i) \( F_k \) is not a fan of \( G \). In particular, \( G \) is obtained from \( G/x \) by splitting the hub of \( F_k \).
(ii) \( i \in I_2 \).
(iii) \( \chi(i) \) is a spoke of \( F_k \).
(iv) \( \psi(i) \notin F \).
(v) \( \psi(i) \cup \chi(i) \) meets no member of \( \mathcal{X}_k \),
(vi) \( i \notin \varphi(l) \) and \(|l| > |L| \).
(vii) \(|K| = 1 \).
(viii) \( \mathcal{X}_k \) is a free \( H \)-fan family of \( G \).

By the definition of \( K \), for some \( i \in I \) there is an element \( z \in \chi(i), \psi(i) \cap F_k \). To prove (ii), suppose for a contradiction that \( F_k \) is a fan of \( G \). Thus \( F_k \) is an union of circuits of \( G \). But \( Y := G[\chi(i), \psi(i), x] \) is a wye of \( G \) meeting \( F_k \) and, by Lemma \ref{lem:intersection} \( Y \) meets \( F_k \) in at least two edges. By (I) \( x \in F_k \), a contradiction. Thus, \( F_k \) is not a fan of \( G \). The second part of (ii) follows straightforwardly from this fact. So, (i) holds.

Say that \( G \) is obtained from \( G/x \) by splitting the hub of \( F_k \) into vertices \( u_1 \) and \( u_2 \) linked by \( x \). Since \( x \) is adjacent to \( z \), then \( z \) is a spoke of \( F_k \) and we may assume that \( z \) is incident to \( u_1 \). Let \( v_1 \) be the other endvertex of \( z \) in \( G \). Since \( v_1 \) is in the rim of \( F_k \), then \( E_G(v_1) \) induces a wye \( Y_1 \) of \( G \) meeting the triangle induced by \( F'_i := \chi(i), \psi(i), y' \). As \( x \) and \( v_1 \) are not incident, then \( x \notin E(Y_1) \) and \( Y_1 \neq Y \). So \( Y \) and \( Y_1 \) are distinct wyes of \( G \) meeting \( F'_i \) and, therefore, \( F'_i \subseteq E(Y) \cup E(Y_1) \). Since
$y^i$ is not adjacent to $x$, hence $y^i \in E(Y_1) = E_G(v_1)$. So, $v_1$ is incident to an edge out of $F_2$ and, consequently, $v_1$ is an extreme of the rim of $F_2$. Let $F^+_2$ be a wye-to-wye fan of $G/x$ containing $F_2$ labeled as in Figure 3. Note that $x_0 = y^i$ is in the wye $Y_1$ of $G$, which is also a wye of $G/x$. Then $i \notin I_3$. Moreover, as $Y_1$ meets a triangle of $F_2$, then by Lemma 16, $y^i$ is contractible in $G/x$. This implies that $i \notin I_1$. So, $i \in I_2$ and (ii) holds.

By Lemma 15, $\xi(G/x, y^i) \geq \xi(G/F^+_2)$ is 3-connected. If $\deg_G(v_0) = 3$, then $\deg_{G/x}(v_0) = 3$ and we have a contradiction to the maximality of $\deg_G(v_0)$ of $\psi$. This implies that $\deg_G(v_0) \neq 3$. By Lemma 14 $G/z$ is 3-connected. By (ii) and (III) $z = \chi(i)$ and (iii) holds.

Note that (iv) follows directly from (i). We checked that if $j \in I$ and $z' \in \{\psi(j), \chi(j)\} \cap F_2$, then $z' = \chi(j)$ is a spoke of $F_2$ and $\psi(j) \notin F$. This implies (v) because the members of $H_k$ are in the rim of $F_2$. By (iv), $\{\psi(i)\} \notin \Phi$. Moreover, $\{\chi(i)\} \notin \Phi$. Hence, there is no index $l \in L$ for which $i = \varphi(l)$. This implies that $\varphi$ is not surjective and (vi) holds.

Now we check that $\deg_G(u_2) \geq 4$. Since $E_G(u_1) = E(Y)$, $\deg_G(u_1) = 3$. Moreover, $G[\{v_0, u_1, v_1\}]$ is a triangle. Suppose for a contradiction that $\deg_G(u_2) = 3$. Hence, $N_G(u_2) = \{u_1, v_2, v_3\}$ and, for $X := \{u_1, u_2, v_0, v_1, v_3, v_4\}$, $G[X]$ has as subgraph the graph in figure 12 where $v_1, v_2, u_1$ and $u_2$ have degree 3 in $G$. As $G[u_0, v_3]$ is connected, then $V(G) = X$ and $G \equiv \Pi_3$ or $\mathcal{W}_4$, a contradiction the hypothesis. So, $\deg_G(u_2) \geq 4$.

For proving (vii), suppose for a contradiction that $j \in K - k$. Note that $u_1$ is a degree-3 endvertex of $x$ in $G$ incident to $\psi(i) \notin F$ and to $\chi(i) \in F_2$. Analogously, for $j$, one endvertex $u$ of $x$ has degree 3 and is incident to an edge of $F_j$ and an edge out of $F \cup x$. Clearly, $u = u_2$. But this contradicts the fact that $\deg_G(u_2) \geq 4$.

For proving (viii) it is enough to check that each edge in the rim of $F$ is $H$-contractible in $G$. As $\deg_G(u_2) \geq 4$, it follows from Lemma 28.

It is left to prove the uniqueness of $i$. Suppose for a contradiction that, for some $j \in I - i$. As $E_G(u_1) = \{x, \psi(i), \chi(i)\}$, analogously, for $j$, one endvertex $u$ of $x$ satisfies $N_G(u) = \{x, \psi(j), \chi(j)\}$. By (III) $u = u_2$. But this contradicts the fact that $\deg_G(u_2) \geq 4$.

By items (vi), (vii) and (viii) of (IV) we may extend the function $\varphi$ previously defined:

(V). There is an injective function $\Phi : K \cup L \rightarrow I$ such that:

- If $k \in K$, then $\Phi(k)$ is the unique index $i \in I$ such that $\chi(i) \in F_k$.
- If $l \in L$, $\Phi(l) := \varphi(l)$ is the index $i \in I$ such that $\{\chi(i), \psi(i)\}$ meets $F_l$.

By (V) $|I| \geq |K| + |L|$, then, in every possible case, we may define $\mathcal{X}$ as follows:

$$\mathcal{X} := \begin{cases} \{x\} \cup \left( \bigcup_{k \in [m] - L} \mathcal{X}_k \right) & \text{if } |I| = |K| + |L| \\ \bigcup_{k \in [m] - L} \mathcal{X}_k & \text{if } |I| > |K| + |L|. \end{cases}$$

We will prove that $\mathcal{X}$ is a family satisfying the lemma. Denote by $X$ the union of the members of $\mathcal{X}$. We shall prove now:

(VI). The members of $\mathcal{X}$ are pairwise disjoint.

Suppose for a contradiction that there are distinct members $A$ and $B$ in $\mathcal{X}$ with a common element $z$. By construction, each family $\mathcal{X}_k$ has pairwise disjoint members and does not contain $\{x\}$. So, there are distinct $i, j \in [m]$ such that $A \in \mathcal{X}_i$ and $B \in \mathcal{X}_j$. Note that each member of $\mathcal{X}_k$ is contained...
in $F_k \cup \chi(I) \cup \psi(I)$ for $k \in [m] - L$. Hence, if $z \in \chi(I) \cup \psi(I)$, then $z \in F_i \cap F_j$, contradicting the disjointness of $\mathcal{F}$. We may assume that $i \in I$ and $z \in \{\chi(i), \psi(i)\}$. In particular $z$ is in the wye $Y := G[\{\chi(i), \psi(i), x\}]$.

By $\[III\]$, $j \notin I$. If $j \in J$, then, by definition, $F_j$ does not meet $\chi(I) \cup \psi(I)$. So, $z \notin F_j$. Therefore, $z$ is in no member of $\mathcal{X}_j$ by construction. Hence, $j \notin J$. The remaining possibility is that $j \in K$. But this contradicts $\[IV\]$. \hfill \textcircled{v}

(VII). $r_{sG}(\mathcal{X}) \geq r_{sG/k}(\mathcal{F}) + 1$

By $\[V\]$ $|I| - |K| - |L| \geq 0$ and, by $\[I\]$, $|X \cap \{x\}| + |I| - |K| - |L| \geq 1$. Moreover, observe that $r_{sG}(\mathcal{X}_i) = r_{G/x}(F_i) + 1$ for each $i \in I$, $r_{sG}(\mathcal{X}_j) = r_{G/x}(F_j) = r_{G/x}(F_j) - 1$ for each $j \in J$, $r_{sG}(\mathcal{X}_k) = r_{G/x}(F_k) = r_{G/x}(F_k) - 1$ for each $k \in K$ and $r_{sG}(F_i) = r_{G/x}(F_i) = 1$ for each $i \in L$. Therefore, the rank-sum of $\mathcal{X}$ is given by:

$$r_{sG}(\mathcal{X}) = |X \cap \{x\}| + \sum_{k \in [m] - L} r_{sG}(\mathcal{X}_k)$$

$$= |X \cap \{x\}| + \sum_{i \in I} (r_{G/x}(F_i) + 1) + \sum_{j \in J} r_{G/x}(F_j) + \sum_{k \in K} (r_{G/x}(F_k) - 1)$$

$$= |X \cap \{x\}| + |I| - |K| - |L| + \sum_{k \in [m]} r_{G/x}(F_k)$$

$$= |X \cap \{x\}| + |I| - |K| - |L| + r_{sG/k}(\mathcal{F})$$

$$\geq r_{sG/k}(\mathcal{F}) + 1.$$ 

This proves $\[VII\]$. \hfill \textcircled{v}

(VIII). Suppose that $k \in [m]$ and $D$ is a circuit of $G$ with $E(D) \subseteq (F_k \cup x)$. Then either

- $k \notin K$ and $E(D)$ is contained in a member of $\mathcal{X}$ or
- $k \in K$ and $E(D) \notin \chi(I)$.

Since $|F_k| \geq |D| - 1 \geq 2$, then $k \in I_1 \cup I_2 \cup J_3 \cup K$.

If $k \in I_1 \cup I_2$, then $F_k$ is a fan of $G$. So, it is clear that $G[F_k \cup x]$ has no circuits containing $x$, and therefore $E(D) \subseteq F_k$, which is contained in a member of $\mathcal{X}_k$. So, we may assume that $k \in J_3 \cup K$.

If $k \in J_3$, then $\mathcal{X}_k$ satisfies item (c) of Lemma $29$ and $\[VIII\]$ holds.

So, assume that $k \in K$. Let $i$ be the index given by $\[IV\]$. Note that $D$ has at least two spokes of $F_k$. Let $s$ be a spoke of $F_k$ in $D$ other that $\chi(i)$. Suppose for a contradiction that $s \in X$. Then $s$ is in a member of $\mathcal{X}_j$ for some $j \in [m] - L$. By the definition of $\mathcal{X}_j$, $j \neq k$. By the uniqueness of $i$ and by $\[IV\]$, $s \notin \chi(I) \cup \psi(I)$. But, by construction, $E(G[\mathcal{X}_j]) \subseteq F_j \cup \chi(I) \cup \psi(I)$. So, $s \in F_k \cap F_j$, contradicting the disjointness of $\mathcal{F}$. \hfill \textcircled{v}

(IX). $\mathcal{X}$ has no crossing triangles.

Suppose for a contradiction that $T$ is a crossing triangle of $\mathcal{X}$. As $G/x$ is simple, $x \notin T$ and $T$ is a triangle of $G/x$.

If $E(T) \subseteq F$, then, as $\mathcal{F}$ has no crossing triangles, $T \subseteq F_k$ for some $k \in [m]$ and, by $\[VIII\]$, $E(T)$ is contained in a member of $\mathcal{X}$ or $E(T) \notin X$, a contradiction.

Thus $E(T) \notin F$ and there is an edge $z \in E(T) \cap \{\chi(i), \psi(i)\}$ for some $i \in I$. Recall that $\{x, \psi(i), \chi(i)\}$ induces a wye in $G$. As $x \notin T$, hence $\{\chi(i), \psi(i)\} \subseteq T$ and, therefore, $E(T) = \{\chi(i), \psi(i), y_1\}$. If $i \in I_1 \cup I_2$, then $T$ is contained in $F_i$, but $\mathcal{X}_i = \{F_i\}$ in this case, a contradiction. So, $i \in I_3$. But, now, $y_1 \in T - X$ by construction. A contradiction again. So, $\[IX\]$ holds.

Items (a) and (c) of the lemma follows from $\[VI\], \[VII\]$ and $\[IX\]$. It is left to prove item (b), this is, it is enough to prove that $\mathcal{X}$ is free provide $\mathcal{F}$ is free to finish the proof. Suppose the contrary. By $\[VI\]$, $G$ has a circuit $C$ such that $E(C) \subseteq X$ but $E(C)$ is contained in no member of $\mathcal{X}$. Choose such $C$ minimizing $|E(C)|$. We will prove some assertions next:

(X). If $e \in X$, then $e$ is not a chord of $C$.

Suppose the contrary. Then, there are circuits $C_1$ and $C_2$ of $G$ such that $E(C_1) \cup e = E(C_1) \cup E(C_2)$ and $E(C_1) \cap E(C_2) = \{e\}$. Let $A$ and $B$ be distinct members of $\mathcal{X}$ meeting $E(C)$ with $e \notin A$. For some $i \in [2]$, $C_i$ meets $A$ and the member of $\mathcal{X}$ containing $e$. Moreover, $E(C_i) \subseteq X$. Since $G$ is simple, $C_i$ contradicts the minimality of $C$. \hfill \textcircled{v}
For each $l \in L \cup K$, $\chi(\Phi(l)) \in F_l \subseteq F$.

By (IV) (vi), we may assume that $l \in L$. Let $i := \Phi(l)$. So, one of $\chi(i)$ or $\psi(i)$ is in $F_l \subseteq F$. If $i \in I_3$, the result follows from our choice of labels for $\chi(i)$ and $\psi(i)$. Assume that $i \in I_1 \cup I_2$. By (III), $G/\psi(i)$ is not 3-connected. If $G/x, \psi(i)$ is 3-connected, then by Lemma 20 $x$ and $\psi(i)$ are not adjacent. A contradiction. As $(G/x)/F_l$ is 3-connected, then $F_l = \{\chi(i)\}$. ◇

(E). $E(C) \subseteq F \cup x$.

Define $I_{\psi} := \{k \in I : \psi(k) \in E(C)\}$. Recall that, for each $k \in I$, $Y_k := G[\{x, \psi(k), \chi(k)\}]$ is a wye and $T_k := G[\{x, \psi(k), \chi(k)\}]$ is a triangle. Hence, $|I| \leq 2$ and $|I_{\psi}| \leq 2$. We will use the symbol $\Delta$ for symmetric difference of sets. Denote $I_{\psi} := \{i_1, \ldots, i_n\}$ and define $Z := E(C) \setminus E(T_{i_1}) \setminus E(T_{i_k})$. Note that $G[Z]$ is a union of edge disjoint circuits of $G$. By (III) $T_1, \ldots, T_n$ are pairwise disjoint. We consider two cases:

Case (i). $x \in C$. In this case, for each $k \in I$, by Lemma 23 applied on $C$ and $Y_k$, we have that $(\psi(k), \chi(k))$ is contained in $E(C)$ or disjoint from $E(C)$. Thus $k \in I_{\psi}$ if and only if $(\psi(k), \chi(k)) \in E(C)$. This implies that $Z \subseteq F$. If $I_{\psi} = \emptyset$, then $E(C) = Z$ and $[\psi(x)]$ holds, so, assume that $i \in I_{\psi}$.

If $y_1 \in E(C)$, then $E(C) = (y_1, \psi(i), \chi(i))$. Moreover, $y_1 \in X$ and $i \notin I_3$. So, $i \in I_1 \cup I_2$. This implies that $E(C) \subseteq F_1$. But $(F_1, \chi(i)) \in \mathcal{X}$. A contradiction. Thus $y_1 \notin E(C)$.

Now, $y_1$ is a chord of $C$ and, by (V), $y_1 \notin X$. Therefore, $i \in I_2$ and $F_l := \{y_1\}$. Note that $y_1 \in Z$. Let $D$ be a circuit of $G[Z]/x$ containing $y_1$. As $E(D) \subseteq Z \subseteq F$ and $F_l = \{y_1\} \subseteq E(D) \not\subseteq F_1$, thus $D$ contradicts the freeness of $\mathcal{X}$. ◇

Case (ii). $x \in C$. Then $x \in X$. By the definition of $\mathcal{X}$, $|I| = |K| + |L|$. So, the function $\Phi$, defined in (VIII) is surjective. Hence, by (XII), $\chi(I) \subseteq F$. Note that $Z = \chi(I) \cup F \cup x$, and, therefore, $Z \subseteq F \cup x$. If $I_{\psi} = \emptyset$, then $E(C) = Z \subseteq F \cup x$ and we have (XII). So, assume that $i \in I_{\psi}$. As $\psi(i) \notin Z$, then $x$ and $\chi(i)$ are incident to a common degree 2 vertex of $G[Z]$ and are in a same circuit $D$ of $G[Z]$. Let $k \in L \cup K$ be the index such that $\chi(i) \in F_k$. As $D/x$ is a circuit of $(G/x)[F]$ and $\mathcal{X}$ is a free family of $G/x$, then $E(D/x) \subseteq F_k$. So, $|F_k| \geq 1$ and $k \in K$. By (VIII) $E(D) \not\subseteq X$. As $E(D) \subseteq Z \subseteq X \cup \chi(I) \cup \{y_1 : j \in I_{\psi}\} = X \cup \{y_1 : j \in I_{\psi}\}$, hence, for some $j \in I_{\psi}$, $y_1 \in E(D)$. Thus $y_1 \in F_k$, contradicting the disjointness of $\mathcal{X}$. ◇

(xiii). $x$ is a chord of $C$.

Suppose the contrary. Thus one of $C$ or $C/x$ is a circuit of $G/x$; call such circuit $B$. By (XII), $E(B) \subseteq F$ and, as $\mathcal{X}$ is free, $E(B) \subseteq F_k$ for some $k \in [m]$. So, by (VIII) for $D := C, C \not\subseteq X$ or $E(C)$ is in a member of $\mathcal{X}$. A contradiction. ◇

By (XIII) $x$ is a chord of $C$. By (X), $\{x\} \notin \mathcal{X}$. By (I), $|I| \geq 1$ and there is $i \in I$. By (XII), $E(C) \subseteq F$. Since $\chi(i), \psi(i), x$ is a wye of $G$ and $x$ is a chord of $C$, then $\chi(i), \psi(i) \subseteq E(C) \subseteq F$. But this contradicts (I). The lemma is proved.

From Seymour Splitter Theorem 8 (we refer the reader also to Corollary 12.1.3) we may conclude:

Corollary 31. Suppose that $G$ is a 3-connected simple graph with at least 4 vertices and a 3-connected simple minor $H$. If $G$ is not isomorphic to a wheel, then $G$ has an edge $x$ such that $G/x$ or $G\setminus x$ is 3-connected and simple with an $H$-minor.

Now we prove Theorems 7 and 10. The same argument prove both Theorems, differing only in the use of item (b) of Lemma 30 in the end.

Proof of Theorems 7 and 10. First, note that the theorem holds when $G$ is a wheel or $G \cong \Pi_3$ and assume the contrary. We proceed by induction on $k := |E(G)| - |E(H')|$. When $k = 0$, the result is trivial. Suppose that $k \geq 1$ and the theorem holds for smaller values of $k$. By Corollary 31 $G$ has an edge such that some $G' \in \{G/x, G\setminus x\}$ is 3-connected and simple with an $H'$-minor. By induction hypothesis we have a fan family $\mathcal{F}$ of $G'$ satisfying the theorem for $G'$. By Lemma 30 there is a family $\mathcal{X}$ satisfying the theorem for $G$. □
Proof of Theorem 11: Let $\mathcal{F}$ be a free $H$-fan family of $G$ with $r_G(\mathcal{F}) \geq r$. Consider a partition $\mathcal{F} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, where:

- $\mathcal{A}$ is the family of singleton sets of $\mathcal{F}$,
- $\mathcal{B}$ is the family of the edge sets of triangles of $G$ in $\mathcal{F}$ with 3 degree-3 vertices and
- $\mathcal{C}$ is the family of edge-sets of non-degenerated $H$-inner fans in $\mathcal{F} - \mathcal{B}$.

In particular, choose $\mathcal{F}$ maximizing $|\mathcal{A}|$. Let $U$ be the union of the members of $\mathcal{F}$. Let us check the following:

(I). If $x_0, y_1, x_1, \ldots, y_n, x_n$ is a wye-to-wye fan of $G$ containing a member $X$ of $\mathcal{C}$, then $\{x_0\} \not\in \mathcal{F}$ or $\{x_n\} \in \mathcal{F}$.

Suppose the contrary. Note that $x_0, x_n \not\in U$. By Lemma 16, $G/x_0$ and $G/x_n$ are 3-connected. By Lemma 15, $x_0$ and $x_n$ are $H$-contractible in $G$. Let $F_1$ be a spanning forest for $G[U]$. By the maximality of $|\mathcal{A}|$, $\mathcal{F} \cup \{x_n\}$ is not free. So, $E(F_1 \cup x_n)$ has a circuit $C$ containing $x_n$. Since $x_0 \not\in C$, then $C$ contains a spoke of $X$. Now, $F_2 := G[(E(F_1) - X) \cup \{x_1, \ldots, x_n\}]$ is a forest with the same number of edges as $F_1$. Note that $\mathcal{F}' := (\mathcal{F} - \{x\}) \cup \{x_1, x_2, x_3\}$ has rank $r_G(\mathcal{F})$ since it induces a subgraph of $G$ having $F_2$ as spanning forest. Moreover, each $x_i$ is $H$-contractible in $G$ by Corollary 17. So, $\mathcal{F}'$ contradicts the maximality of $|\mathcal{A}|$.

(II). If $T \in \mathcal{B}$ and $x_1, x_2, x_3$ are the edges of $G$ adjacent to $T$, then $|\{x_1, x_2, x_3\} \cap \mathcal{F}| \geq 2$.

Note that each circuit of $G$ meeting $\{x_1, x_2, x_3\}$ also meets $E(T)$. Thus, as $\mathcal{F}$ is free, then so is $\mathcal{F}' := (\mathcal{F} - \{T\}) \cup \{x_1, x_2, x_3\}$. If (II) fails, then $\mathcal{F}'$ has rank at least $r$, contradicting the maximality of $\mathcal{A}$. By (I) and (II), $\mathcal{A} \neq \emptyset$. Let $A$ and $B$ be the union of the members of $\mathcal{A}$ and $\mathcal{B}$ respectively. We define a vertex of $G$ to be green if it is incident to an edge of $B$ and to be red otherwise. We define the non-red vertices of $G/B$ to be blue. Next, we prove:

(III). $G/B$ is simple or $G$ is isomorphic to $K_4$ or $\Pi_3$.

Suppose the contrary. Let $\mathcal{B} := \{B_1, \ldots, B_n\}$ and let $k$ be the least index for which $G/B_1, \ldots, B_k$ is not simple. By the second part of Corollary 18, $G' := G/B_1, \ldots, B_{k-1}$ $\cong K_4$ and $B_k$ induces a triangle $T$ of $G'$. So, there is an unique vertex $w \in V(G') - T$. If $w$ is red, then $G = G' \cong K_4$, otherwise $w$ is blue and $G \cong \Pi_3$. If $G$ is isomorphic to $K_4$ or $\Pi_3$, the theorem may be verified directly. So, assume the contrary. Therefore $G/B$ is simple.

Define $R$ as the union of the edge-sets of the rims of the members of $\mathcal{C}$. Moreover, define $W := \{x \in E(G) - U : x$ is adjacent to and edge of $B\}$. Note that (III) implies:

(IV). Each blue vertex of $G/B$ is incident to at most one edge of $W$.

Let $F$ be the graph obtained from $G' := (G/B)[A \cup R \cup W]$ by cleaving each red vertex $v$ of $G'$ into $\deg_{G'}(v)$ degree-one so said pink vertices. Note that $F$ has two types of vertices: the blue ones, with degree three, and the pink ones, with degree one.

Note that each edge of $R$ has red endvertices in $G$ and, therefore, pink endvertices in $F$. So, each edge of $R$ induces a connected component of $F$. As $\mathcal{A} \neq \emptyset$, at least one of the connected components of $F$ is not induced by an edge of $R$. Let $k$ be the number of connected components of $F$. Hence:

$$|\mathcal{C}| \leq |R| \leq k - 1.$$  

As $G/B$ is simple, $F$ is simple. As $\mathcal{F}$ is free, $G[A \cup R]$ is a forest and so is $F[A \cup R]$. Hence, $F$ has a spanning forest $T$ containing $A \cup R$. Define $D := F - E(T)$. We say that a blue vertex is dark blue if it has degree 3 in $T$; otherwise, we say that such vertex is light blue. As each edge of $D$ is in a circuit of $F$, then it has light blue endvertices. Conversely, the light blue endvertices are exactly those incident to edges of $D$. As $D \subseteq W$, then, by (IV), each light blue vertex is incident to exactly one edge of $D$ and, therefore, have degree 2 in $T$. Moreover, each light blue vertex has degree one in $G[D]$, and,
therefore, the number \( l \) of light blue vertices satisfies \( l = 2|D| \). Let \( d \) be the number of dark blue vertices and let \( T' \) be a forest obtained from \( T \) by replacing by a single edge each maximal path in respect to having all inner vertices light blue. As \( T' \) is a cubic forest with \( \kappa \) connected components and \( d \) inner vertices, then \( |E(T')| = \kappa + 2d \). By construction, \( |E(T)| = |E(T')| + l = \kappa + 2d + l \). This implies that \( |E(F)| = |E(T)| + |D| = \kappa + 2d + l + |D| \). As \( G \), has \( |\mathcal{B}| = l + d \) blue vertices, then \( |E(F)| = \kappa + 2|\mathcal{B}| + l + |D| \). But \( l = 2|D| \), so:

\[
|\mathcal{B}| = |E(F)| - \kappa + |D|.
\]

Since \( D \subseteq W \) and \( E(F) = A \cup R \cup W \), then

\[
|A| + |R| \leq |E(F)| - |D|.
\]

Note that \( r \leq r_G(\mathcal{F}) = 2|\mathcal{B}| + (|A| + |R|) + |\mathcal{E}| \). So, by (3), (4) and (2):

\[
r \leq (|E(F)| - \kappa + |D|) + (|E(F)| - |D|) + (\kappa - 1) = 2|E(F)| - 1.
\]

This implies that \( |E(F)| \geq [(r + 1)/2] \). Recall that \( |E(F)| = A \cup R \cup W \). By Corollary\([17]\) the edges in \( R \) are \( H \)-contractible in \( G \). By Lemmas\([16]\) and \([15]\) so are the edges in \( W \). By definition, the elements of \( A \) are \( H \)-contractible in \( G \) and, therefore, so are the edges of \( F \). Now, it suffices to prove that \( G[E(F)] \) is a forest to establish the theorem. Indeed, recall that \( E(F) = A \cup R \cup W \). Since \( \mathcal{F} \) is free, then \( G[A \cup R] \) is a forest. So, every circuit of \( G[E(F)] \) meets an edge of \( W \). But each circuit meeting an edge of \( W \) also meets an edge of \( B \). As \( E(F) \cap B = \emptyset \), hence \( G[E(F)] \) is a forest and the theorem is valid. \( \Box \)

5. Sharpness

We denote by \( V_n(G) \) the set of vertices of \( G \) with degree \( n \). Consider the graphs \( J_1 \) and \( J_2 \) as in the figures below.

**Figure 13.** \( J_1 \)

**Figure 14.** \( J_2 \)

For \( i = 1, 2 \), let \( A_i \) be the set of edges in \( J_i \) with some endvertex of degree one and let \( B_i := E(J_i) - A_i \). Let \( 2n := |V_1(J_i)| \). For \( m \geq 2n + 1 \), let \( K_m \) be a copy of the complete graph with \( m \) vertices disjoint from \( J_i \). Consider the graph \( G_i \) obtained by identifying \( V_1(J_i) \) with \( 2n \) distinct vertices of \( K_m \). Note that \( |G_i[B_i]| = 4n - 6 \) and \( |G_i[B_i]| = 4n \).

Define \( H_i := G_i/B_i \). Note that \( H_i \) is \( 2n \)-connected. Let \( T_i \subseteq E(G_i) \) be a set such that \( G_i[T_i] \) is a forest, \( |G_i/T_i| = |H_i| \) and \( G_i/T_i \) has an \( H_i \)-minor. As \( G[T_i] \) is a forest, then \( |T_i| = r_G(B_i) \). Choose \( m \gg 2n \) in such a way that \( si(G_i/x) \) has less edges than \( H_i \) for each \( x \in E(K_m) \). So \( E(K_m) \cap T_i = \emptyset \) and, therefore, \( T_i \subseteq E(J_i) \).

Let us prove that \( T_i \subseteq B_i \). Suppose the contrary. Since \( T_i \cap E(K_m) = \emptyset \), then there is \( x_1 \in A_i \cap T_i \). Since \( m \geq 2n + 1 \), there is a vertex \( v \) in \( G_i \) incident to no edges of \( J_i \). Let \( T_i := \{x_1, \ldots, x_r\} \). For \( 0 \leq s \leq r \), define \( I_s := G_i/[x_1, \ldots, x_s] \) and \( W_s := V(I_s) - V(I_s[E(K_m)]) \). Consider the graph \( J'_i \) obtained by the identification of all degree-1 vertices of \( J_i \) into a single vertex \( w_1 \). Note that \( |J'_i| = |G[B_i]| + 1 \) and \( 0 \leq s \leq r = |G[B_i]| - 1 \). Thus \( J_{i,s} := J'_i/[x_1, \ldots, x_s] \) has at least two vertices. Keep the label of \( w_1 \in J_{i,s} \). Now note that \( \emptyset \neq V(J_{i,s}) - w_{(r+1)} \subseteq W_r \). Now, observe that \( I_r \) has a set with \( 2n - 1 \) vertices separating \( v \) from \( W_k \). By an inductive argument, we conclude that each \( G_i \) has an set with less than \( 2n \) edges separating \( v \) from \( W_k \). So, \( I_r \) is not \( 2n \)-connected. Since \( |H| = |I_r| \) and \( I_r \) has an \( H \)-minor, then \( H \) is not \( 2n \)-connected, a contradiction. Therefore, \( T_i \subseteq B_i \).

Since \( r_G(T_i) = r_G(B_i) \), then \( T_i \) induces a spanning tree of \( J_i[B_i] \) and \( si(G[T_i]) = G[B_i] \). So, all \( H_i \)-contractible edges of \( G_i \) are in \( B_i \). Hence, for \( i = 1 \), the largest subset of \( H_1 \)-contractible edges of \( G_1 \) has \( 2n - 3 \) edges, while \( |G_1| - |H_1| + 1 = 4n - 6 = 2(2n - 3) \). Similarly, the largest subset of
$H_2$-contractible edges of $G_2$ has $2n$ edges, while $|G_2| - |H_2| + 1 = 4n = 2(2n)$. This gives a sharp examples for Theorem\textsuperscript{2} for sufficiently large odd values of $|G| - |H|$. When $|G| - |H|$ is even, we consider for $i = 1, 2$, an edge $x$ in the graph $G_i$ previously defined such that $x \in E_i$ but $x$ is adjacent to an edge of $A_i$. Note that $x$ is adjacent to an unique triangle $T$, which has $3$ degree-$3$ vertices. The edge of $T$ not adjacent to $x$ is $H_i$-contractible in $G_i / x$ by Lemma\textsuperscript{17}. Moreover, the property of the other edges of $B_i$ in being $H$-contractible in $G_i$ or $G_i / x$ is the same. As $\left\lceil \frac{|G_i| - |H_i| + 1}{2} \right\rceil = \left\lceil \frac{|G_i / x| - |H_i| + 1}{2} \right\rceil$, we have a sharp example for theorem\textsuperscript{2} for $|G| - |H|$ even and sufficiently large.

For a sharp example for Theorems\textsuperscript{5} and \textsuperscript{8}, consider two disjoint copies $G_1$ and $G_2$ of a $(k + 1)$-connected triangle-free graph such that each $G_i$ has a stable $k$-set of vertices $X_i := \{x_1^i, \ldots, x_k^i\}$ and the vertices of $V(G_i) - X_i$ are not covered by less than $k + 1$ edges (we may choose, for instance, $G_1$ and $G_2$ as hypercubes of a suitable size). Now consider the graph $G := (G_1 \cup G_2) + \{x_1^j x_2^j : j \in [k]\}$. Define $Z := \{x_1^j x_2^j : j \in [k]\}$ and $H := G / Z$. Note that $H$ has an unique vertex cut $X$ with at most $k$ elements. Note that $X$ separates the edge sets of $G_1$ and $G_2$.

Now suppose that $Z'$ is a set of edges such that $H' := G / Z' \cong H$ but there is an edge $z \in Z' - Z$. Say that $z \in E(G_1)$. Now, in an inductive way, similarly as we did in the last class of examples, we may prove that there is a vertex cut $X'$ of $H'$ separating $F := E(G_1) \cap E(H')$ from $E(H') - F$. As $H \cong H'$, $X'$ is the unique vertex cut of $H'$ with up to $k$ elements. Moreover, both classes of edges separated by $X'$ induces copies of $G_1$ in $H'$. But one of them is induced by $E(G_1) \cap E(H') \subseteq E(G_1) - z$. A contradiction. Thus $Z$ is the unique $k$-subset of $G$ such that $G / Z \cong H$. As a consequence, each $H$-contractible edge of $G$ is in $Z$.

**References**

[1] K. Ando, H. Enomoto and A. Saito, *Contractible edges in 3-connected graphs*, J. Combin. Theory Ser. B 42 (1987) 87-93.

[2] J. P. Costalonga, *On 3-connected minors of 3-connected matroids on graphs*, European J. Combin. 33 (2012) 72-81.

[3] Y. Egawa, Y. Enomoto and A. Saito, *Contractible edges in triangle-free graphs*, Combinatorica 6 (1986) 269-274.

[4] W. Gu, X. Jia and H. Wu, *Chords in Graphs*, Aust. J. Comb 32 (2005) 117-124.

[5] M. Kriesell, *A Survey on Contractible Edges in Graphs of a Prescribed Vertex Connectivity*, Graphs and Combinatorics 18 (2002) 1-30.

[6] W. McCuaig, *Edge contractions in 3-connected graphs*, Ars Comb. 29 (1990) 299-308.

[7] J. G. Oxley, *Matroid Theory*, Second Edition, Oxford University Press, New York, 2011.

[8] R. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B 28 (1980) 305-359.

[9] G. Whittle, *Stabilizers of classes of representable matroids*, J. Combin. Theory Ser. B 77 (1999) 39-72.

[10] H. Wu, *On contractible and vertically contractible elements in 3-connected matroids and graphs*, Discrete Math. 179 (1998) 185-203.

joacostalonga@gmail.com, Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514; Campus de Goiabeiras, 29075-910 - Vitória - ES - Brazil.