NP-completeness Results for Graph Burning on Geometric Graphs

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Abstract. Graph burning runs on discrete time steps. The aim of the graph burning problem is to burn all the vertices in a given graph in the least amount of time steps. The least number of required time steps is known to be the burning number of the graph. The spread of social influence, an alarm, or a social contagion can be modeled using graph burning. The less the burning number, the quick the spread. Computationally, graph burning is hard. It has already been proved that burning of path forests, spider graphs, and trees with maximum degree three are NP-Complete. In this work we study graph burning on geometric graphs and show NP-completeness results on several sub classes. More precisely, we show burning problem to be NP-complete on interval graph, permutation graph and disk graph.

Keywords: Graph Burning, NP-complete, Interval Graph, Permutation Graph, Disk Graph

1 Introduction

The spread of social influence in order to analyze a social network is an important topic of study \([4,13,24,25,30,34]\). Kramer et al. (2014) \([26]\) have highlighted that the underlying network plays an essential role in the spread of an emotional contagion; they have nullified the necessity of in person interaction and non-verbal cues. With the aim to model such problems, graph burning was introduced in \([8]\). It was also inspired by other contact processes like firefighting \([5,11,14,15]\), graph cleaning \([1]\), and graph bootstrap percolation \([3]\). Burning a graph can be used to model the spread of a meme, social gossip, or a social contagion. It can also be used to model the viral infections: the exposure to infections and proliferation of virus.

Graph burning runs on discrete time steps; at each step \(t\), first an unburned vertex is burned, and then fire spreads to all vertices adjacent to the vertices which are already burned till step \(t - 1\). This process continues till each vertex of \(G\) has been burned. The aim of the graph burning problem is to burn all the vertices in a given graph \(G\) in least amount of time steps. The minimum time steps required to burn a graph is called the burning length or the burning number of the graph \(G\), and is denoted by \(b(G)\).
Physically, the less the value of $b(G)$, the quicker it is to reach, and therefore burn, all the vertices in $G$. Bessy et.al. [6] showed that graph burning is in NP. They also showed that burning spider graph, tree with maximum degree 3 and path forest are NP-Complete.

In this article, we study burning problem on geometric graphs. A geometric graph is formed from a general arrangement of objects on a plane. Those objects are connected under specific constraints. In the corresponding geometric graph, these objects become vertices and the corresponding connections become edges between those vertices [36]. For example, a graph $G$ is a disk graph if it represents the circles (or disks) drawn on a plane. The centers of each distinct disk can be replaced by a distinct vertex of $G$ and there is an edge $(v_a, v_b)$ between a pair of vertices $v_a$ and $v_b$ iff their respective disks overlap. These disks can generally be of arbitrary radius. If radius of all the disks are equal and one unit each, the graph is called a unit disk graph. In this article, we discuss graph burning on three sub classes of geometric graphs, namely interval graph, permutation graph and disk graph.

Our Contribution: One of the main contribution of our work is the NP-completeness proof for burning interval graph. Besides this, we show that burning permutation graph and disk graph is also NP-complete using existing results Bessy et.al. [6] that burning path-forest and spider-graph are NP-complete.

Organization of the article: Preliminaries are in section 2, followed by related works in section 3. In section 4, we prove that burning interval graph optimally is an NP-Complete problem. Section 5 involves in proving that burning general permutation graph is NP-Complete. In section 6, we prove that burning disk graph is NP-Complete followed by conclusion in Section 7.

2 Preliminary definitions and symbols

First we recall a few well known definitions. A graph is an interval graph if there exists a family of intervals such that the interval graph contains a vertex corresponding to each interval, and for any pair of distinct intervals, there is an edge between the corresponding vertices iff they overlap on the real line. On the other hand, if a set of intervals is given, we can form the corresponding interval graph. A permutation graph is constructed from an original sequence of numbers $O = (1, 2, 3, ..., k)$ and its permutation $P = (p_1, p_2, p_3, ..., p_k)$ such that there is an edge between vertices $v_i$ and $v_j$, corresponding to the numbers $i$ and $j$ in $O$, if $i < j$ in $O$, but $j$ occurs before $i$ in $P$. In a spider graph, only one vertex $c$, the head vertex, is of degree $d \geq 3$; the degree of all other vertices is less than 3. A spider graph with head degree $s$, and length $r$ of each arm is denoted as $SP(s, r)$. A path-forest is a graph which contains simple paths only, disconnected from each other.

Now we list down some of the notations used in this article. We denote the set of vertices in a graph $G$ by $G.V$. The set of vertices that are at most at a distance $i$ from $S$, including $S$ is denoted by $G.N_i[S]$. We use $(u, v)$ to represent an edge between two vertices $u$ and $v$ in a graph. The shortest distance between
the vertices $u$ and $v$ in a given graph is denoted by $\text{dist}(u, v)$. We define $\cup_{\delta}$ to be the left sequential union. If $P = \{a, b\}$, then after executing the statement $P = P \cup_{\delta} \{c\}$, $P$ becomes $\{c, a, b\}$. This operation has been used to add a single element to a sequence. Similarly $\cup_{\delta}$ is defined as the right sequential union. We denote the sum of all numbers in a set $A$ as $\sum(A)$. The largest element in the set $A$ is denoted by $\max(A)$. It is easy to see that, if a burning sequence is able to burn an arbitrary graph $G$ completely, then equation 1 must hold true [6].

$$G.N_{k'-1}[y_1] \cup G.N_{k'-2}[y_2] \cup G.N_{k'-3}[y_3] \cup ... \cup G.N_{k'-1}[y_{k'}] = G.V \quad (1)$$

3 Related Works

We divide the related work section in parts. We start with existing results on burning problem. And as we discuss burning problem on several sub classes of geometric graphs, namely interval graph, permutation graph and disk graph, we look on to some of the other results on these graph classes.

Graph Burning: The burning number was introduced in [8]. This work showed that the burning number of a path or cycle of length $p$ shall be $\sqrt{p}$. They have presented some other properties and results also related to the graph burning problem. Bessy et al. (2017) [6] showed that general graph burning is in NP. They also showed burning spider graphs, trees, and path-forests are NP-Complete. A 3-approximation algorithm for burning general graphs was described in [6]. Bonato et al (2019) [9] proposed a 2-approximation algorithm for burning trees. A 2-approximation algorithm for graphs bounded by a diameter of constant length was described in [21,22]. A 1.5 approximation algorithm for burning path forests was described in [10]. This article also showed that burning number of spider graphs of order $n$ is atmost $\sqrt{n}$. Simon et al. (2019) [37] presented systems that utilize burning in the spread of an alarm through a network. Bessy et al. (2018) [7] proved that the burning number of a connected graph of order $n$ is at most $\sqrt{\frac{n^2}{2} + 3} \approx 1.309\sqrt{n} + 3$. They also showed that the burning number of trees with $n_2$ vertices of degree 2, and $n(\geq 3)$ vertices of degree at least 3 is at most $\sqrt{(n + n_2) + \frac{1}{4} + \frac{1}{2}}$. [35] have provided heuristics to minimize the time steps in burning a graph. They have studied that which vertices should be selected to be burnt from ”outside” and in which time steps. In this article, we study the burning problem of interval graph, permutation graph, and disk graph.

Interval Graph: There are several linear time algorithms available to solve different problems on interval graph. Olariu (1991) [31] discovered linear time algorithm for coloring. Marathe et al. (1992) [29] gave a linear time algorithm to compute minimum vertex cover. Similarly, a linear time algorithm to compute interval graph isomorphism was given in [27]. Ibarra (2017) [20] proposed a linear time algorithm to compute the clique separator graph of a given interval graph. Fomin et al. (2016) [16] described an algorithm which solves the
firefighter problem on connected interval graph in $O(n^7)$ time. Interval graph ordering was introduced in [32]. As per interval graph ordering, the vertices are ordered according to the increasing order of the ending time of their corresponding intervals. Ravi et al. (1992) [33] proposed an algorithm to solve all pairs shortest path in $(O(n^2))$ time, where $n$ is the number of vertices. Authors in [21,22], along with [23] discussed bounds on the burning number of interval graph. Although they have not provided any algorithm to find an optimal burning sequence. On the contrary of all those linear time results, in this article, we prove that burning interval graph optimally is an NP-Complete problem despite having known bounds on optimal burning number. This solves one open question posed in [23].

**Permutation Graph:** Polynomial time algorithms exist for various properties in permutation graph also. If $r$ is the size of longest decreasing subsequence in the corresponding permutation $P$, then the chromatic number and the size of largest clique in the corresponding permutation graph $G$, both are equal to $r$ [18]. Atallah et al. (1998) [2] proposed an algorithm that finds the minimum dominating set in an arbitrary permutation graph with $n$ vertices in $O(n \log^2 n)$ time. In this article, we prove that computing an optimal burning sequence for permutation graph is NP-complete. This answers another open question of [23].

**Disk Graph:** Unlike interval and permutation graphs, computation of various properties on disk graph is NP-Hard. Clark et al. (1990) [12] showed that finding the chromatic number of disk graph is NP-Complete. They also showed that 3-coloring problem is NP-Complete on unit disk graph, even if their underlying disk representation is given. Although, in contrast, they also gave a polynomial time algorithm to find maximal cliques in unit disk graph when the geometrical representation of the underlying disks is given. Here in this article, we study burning problem on disk graph and show that finding optimal burning sequence is NP-complete even if the underlying disk representation is given.

4 Burning Interval Graph

4.1 Burning a path and an interval graph: almost similar

**Definition 1.** Burning cluster of a fire source. The cluster burnt by a fire source $y_i$ in a burning sequence $S = (y_1, y_2, ..., y_k)$ is the set $G.N_{k'-i}[y_i]$.

First we recall some of the existing results as observations. Bonato et al. (2016) [8] proved that a path or a cycle of $n$ vertices can be burned in $\lceil \sqrt{n} \rceil$ steps. In their proof, they have also provided an algorithm which is able to construct an optimal burning sequence for a given path or cycle. Note the following observation from the above fact.

**Observation 1** The burning clusters of each of the $n$ fire sources of any optimal burning sequence of a path of order $n^2$ are pairwise disjoint.
It is easy to see that the $i$th fire source burn at most $2(n - i) + 1$ vertices including itself. If we do summation over $i = 1$ to $n$, we get at most $n^2$ vertices can be burnt by those $n$ fire sources. As we have exactly $n^2$ vertices on the path of length $n^2$, the observation follows. We would like to recall another result from [21,22,23] on the bounds on burning number of interval graph in the following observation.

**Observation 2** Let $P$ be a shortest path of maximum length among shortest paths between all pairs of vertices in an interval graph $G$. Then $b(P) \leq b(G) \leq b(P) + 1$.

This follows from the fact that each vertex of an interval graph is either lies on $P$ or at a distance at most 1 from at least one vertex of $P$ [21,22,23]. Since it is possible to burn $P$ optimally, so at the worst case, after choosing $b(P)$ fire sources according to [8], any unburnt vertex is at most at a distance one from at least one already burnt vertex. So any choice of $(b(P) + 1)^{th}$ fire source will do. Here we can see a similarity between burning a path and an interval graph.

Also note that, finding such $P$ is easy to do in polynomial time. We can simply compute all pair shortest path and choose the maximum length path among all. Then we can see from the above discussion that burning an interval graph in $(b(P) + 1)$ is trivial. But if $b(G) = b(P)$, is it always possible to find a burning sequence of length $b(P)$ that burns the entire interval graph, not only $P$? We show that this is an NP-Complete problem. Before going to the NP-completeness proof, we recall the distinct 3 partition problem.

**Definition 2. Distinct 3-partition problem.** In a distinct 3-partition problem, given input is a set of natural numbers, $X = \{a_1, a_2, ..., a_{3n}\}$, and $B$ such that $\sum_{i=1}^{3n} a_i = nB$, $\frac{B}{4} > a_i > \frac{B}{2}$; the task is to find if $X$ can be partitioned into $n$ sets, each containing 3 elements such that each set sums to $B$.

The distinct 3-partition problem has been proved to be NP-Complete in strong sense (see [17,19,6]).

### 4.2 Interval graph construction for NP-completeness

Let $X$ be an input set to the distinct 3-partition problem; let $n = \lfloor \frac{|X|}{3} \rfloor$, $m = \max(X)$, $B = \frac{s(X)}{m}$, and $k = m - 3n$. Let $F_m$ be the set of first $m$ natural numbers, $F_m = \{1, 2, 3, ..., m\}$; and $F'_m$ be the set of first $m$ odd numbers, $F'_m = \{2 f_i - 1 : f_i \in F_m\} = \{1, 3, 5, ..., 2m - 1\}$. Let $X' = \{2 a_i - 1 : a_i \in X\}$.

Observe that $s(X') = \sum_{i=1}^{3n} 2 a_i - 1 = 2nB - 3n$, so $B' = 2B - 3$.

Let $Y = F'_m \setminus X'$.

Let there be $n$ paths $Q_1, Q_2, ..., Q_n$, each of order $B'$; $k$ paths $Q'_1, Q'_2, ..., Q'_k$ such that each $Q'_j (1 \leq j \leq k)$ is of order of $j^{th}$ largest number in $Y$, and $m + 1$ paths $T_1, T'_2, ..., T_{m+1}$ such that each $T_j (1 \leq j \leq m + 1)$ is of order of $2(2m + 1 - j) + 1$. We join these paths in the following order to form a larger path:
We call this path $P_I$. Number of vertices in $P_I$ is $(2m + 1)^2$. Hence $b(P_I) = (2m + 1)$.

Now we add a few more vertices to $P_I$. We add a distinct vertex connected to each vertex from 2nd to 2nd-last vertices of $T_j \forall 1 \leq j \leq m + 1$. Let this graph be called $IG(X)$. Total number of vertices in $IG(X)$ is $(2m + 1)^2 + ((2m)^2 - (m - 1)^2) = 4m^2 + 1 + 4m + m^2 - m^2 - 1 + 2m = 7m^2 + 6m.$ Observe that, in $IG(X)$, $P_I$ is a shortest path of maximum length among shortest paths between all pairs of vertices, and all other vertices not in $P_I$ are at most at a distance one from some vertex of $P_I$. Also, $IG(X)$ is a tree: it contains no cycles. So $IG(X)$ is a valid interval graph (an example illustrated in section 4.3, and figure 2). We prove in lemma 2 that burning number of $IG(X)$ is still $2m + 1$, i.e., $b(IG(X)) = b(P_I)$. Then we show that burning an interval graph is NP-Complete because to burn $IG(X)$, we must solve the distinct 3-partition problem on $X$.

Figure 1 illustrates a particular $T_j$ along with the added vertices and edges (vertically upwards w.r.t. $T_j$). This forms a kind of comb structure, we call it $T^c_j$.

![Fig. 1. Structure of a $T_j$ with 33 vertices, along with the extra vertices connected to it. The dashed red line represents the fact that other subpaths may be connected to a $T_j$ on either or both ends.](image)

Let $u^j_1$ be the vertex connected to the 2nd vertex of each $T_j$ and $u^{\lceil T_j \rceil - 2}_{j}$ be the vertex connected to its 2nd-last vertex of $T_j$, where $\lceil T_j \rceil$ stands for the number of vertices in the sub path $T_j$. Let $A^T_j = \{u^{j}_1, u^{j}_2, ..., u^{\lceil T_j \rceil - 2}_{j}\}$ be the set of all $\lceil T_j \rceil - 2$ additional vertices corresponding to $T_j$. Here we mention one important observation regarding burning $T^c_j$.

**Observation 3** If $T^c_j$ is burnt by $m \geq 2$ fire sources put on $T_j$, then the burning clusters of at least two of these fire sources will overlap (contain common vertices) in $T_j$.

**Proof.** We first show this observation true for a pair of fire sources and then it will follow similarly for more than two fire sources.

Let us assume that there is not a single vertex which commonly comes in the burning cluster of both the fire sources and we can still burn $T^c_j$ completely. This implies, there is some vertex $v$ on $T_j$ such that all the vertices in the left side of it, including it, are burnt by the first fire source and all the vertices in the right by the second fire source.

Let the vertex that is just right to $v$ is $v_r$. By pigeonhole principle, we have that at least one of $v$ or $v_r$ having a neighbor $v_t$ in $T^c_j$ which is not in $T_j$. 


Lemma 1. If at least one $T_j$ is burnt using more than one fire sources, then $P_I$ can not be burnt optimally, i.e., $b(P_I) = (2m + 1)$ steps.

Proof. Since $P_I$ is a simple path of length $(2m+1)^2$, according to Observation 1, each fire source in an optimal burning sequence must burn disjoint set of vertices of $P_I$. Let $x_1, x_2, ..., x_{2m+1}$ be an optimal burning sequence of $P_I$ such that some $T_j$ is burnt using more than one fire sources, then according to Observation 3, at least two fire sources burn a common vertex of $P_I$ and hence $x_1, x_2, ..., x_{2m+1}$ can not be an optimal burning sequence.

Now we state the following lemma.

Lemma 1. If at least one $T_j$ is burnt using more than one fire sources, then $P_I$ can not be burnt optimally, i.e., $b(P_I) = (2m + 1)$ steps.

Proof. Since $P_I$ is a simple path of length $(2m+1)^2$, according to Observation 1, each fire source in an optimal burning sequence must burn disjoint set of vertices of $P_I$. Let $x_1, x_2, ..., x_{2m+1}$ be an optimal burning sequence of $P_I$ such that some $T_j$ is burnt using more than one fire sources, then according to Observation 3, at least two fire sources burn a common vertex of $P_I$ and hence $x_1, x_2, ..., x_{2m+1}$ can not be an optimal burning sequence.

Before going to the Np-completeness proof, we construct a specific example of $IG(X)$ below.

4.3 Example construction

Let $X = \{10, 11, 12, 14, 15, 16\}$. Then $n = 2$, $m = 16$, $B = 39$, and $k = 10$. Also $F_m = \{1, 2, ..., 16\}$ and $F'_m = \{1, 3, ..., 31\}$. Further, $X' = \{19, 21, 23, 27, 29, 31\}$, $B' = 75 = 2B - 3$ and $Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 25\}$. Observe that $Q_1$ and $Q_2$ are paths of size 75, and each $Q'_1, Q'_2, ..., Q'_k$ are paths of order of 25, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively. $T_1, T_2, T_3, ..., T_{m+1}$ are of order of 65, 63, 61, ..., 33 respectively. We add a vertex connected to each vertex from 2nd to 2nd-last vertices of $T_j (1 \leq j \leq m + 1)$. Observe that this is a valid interval graph. The constructed example $IG(X)$ is shown in Figure 2.

Next we show that this interval graph can be burned optimally only if 3-partition problem can be solved for $X = \{10, 11, 12, 14, 15, 16\}$.

4.4 NP-Completeness

Lemma 2. Given that $Q_1, ..., Q_n$ can be partitioned into $Q'_1, ..., Q'_m$ of orders in $X'$, then burning number of $IG(X)$ is $2m + 1$.

Proof. Let $P' = \{Q'_1, ..., Q'_m, Q_1, ..., Q'_k, T_1, ..., T_{m+1}\}$. Let $r_i$ be the $(2m + 1) - i + 1)^{th}$ vertex on the $i^{th}$ largest sub path in $P'$. Then, we can burn $P_I$ and subsequently $IG(X)$ if we put $2m + 1$ fire sources such that the fire source $y_i$ is put on $r_i$. So, $S' = (y_1, y_2, ..., y_{2m+1})$ is the burning sequence
in this case. This implies that \( b(IG(X)) \leq 2m + 1 \). Since the union of all sub paths in \( P' \) produces entire \( P_I \) which is a path of length \((2m + 1)^2\), we have \( b(IG(X)) \geq 2m + 1 \). Hence, \( b(IG(X)) = 2m + 1 \).

Fig. 2. Construction of example \( IG(X) \).

Here after, by \( P' \), we denote the set of all sub paths of \( P_I \) such that \( P' = \{Q_1'', ..., Q_{3n}'', Q_1', ..., Q_k', T_1, ..., T_{m+1}\} \).

**Lemma 3.** Each fire source must be on \( P_I \).

**Proof.** If the all the fire sources are on \( P_I \), then we have that \( G.N_{2m+1-i}[y_i] \cap P_I \) is a path of order at most \( 2(2m + 1 - i) + 1 \).

Let for contradiction that we put a fire source \( y_i \) on any vertex adjacent to some vertex on \( P_I \) and not on \( P_I \), and \( IG(X) \) can still be burnt within \( 2m + 1 \) steps (lemma 2). Then we have that the subgraph induced by \( G.N_{2m+1-i}[y_i] \cap P_I \) is a path of order less that \( 2(2m + 1 - i) + 1 \). This implies that (from equation 1) \( |\bigcup_{i=1}^{2m+1} G.N_{2m+1-i}[y_i] \cap P_I| < (2m + 1)^2 \) which is a contradiction. Therefore each \( y_i \) must be a put on some vertex on \( P_I \).

Let \( S' = (y_1, y_2, ..., y_{2m+1}) \) be an optimal burning sequence. Let \( r_i \) be the \((2m - i + 2)^{th}\) vertex on the \( i^{th} \) largest sub path in \( P' \). Observe that \( T_j \)'s are the largest \( m + 1 \) sub paths in \( P' \).

**Lemma 4.** We must have \( y_i = r_i \forall 1 \leq i \leq m + 1 \).
Proof. We are going to prove this lemma using the strong induction hypothesis. We have that each \( u_i \) must receive fire from a \( y_i \) in \( P_t \) (lemma 3). For \( i = 1 \), the only vertex connected to both \( u_1 \) and \( u_{i-1} \) within a distance \( 2m+1-i = 2m \) is \( r_1 \). Now we must have \( y_1 = r_1 \), else, if we put \( y_1 \) somewhere else, then no other fire source can burn \( T_1^r \) alone. If we utilize more than one fire sources to burn \( T_1^r \), then then at least one vertex of \( T_1 \) would be burnt by both of those two fire sources (observation 3), following that \( P_t \) cannot be burnt completely (lemma 1) which is a contradiction.

So, we must have that \( y_1 = r_1 \). Let we need to have \( y_k \) on \( r_k \) for all \( 1 \leq k \leq m \). We need to show for \( y_{k+1} \). After burning \( T_k \), we must burn \( T_{k+1} \) first of all, otherwise we will again obtain overlaps in the burning clusters of the fire sources and we will not be able to burn \( P_t \) completely with is a contradiction.

So, using strong induction hypothesis, we must have that \( y_{k+1} = r_{k+1} \) to burn entire \( T_{k+1} \) (\( 1 \leq k \leq m \)) since the only vertex connected to both \( u_{k+1} \) and \( u_{m+1} \) within distance \( 2m+1-(k+1) \) is \( r_{k+1} \).

Let us define \( P'' \) by \( P'' = IG(X) \setminus (T_1^c \cup T_2^c \cup ... \cup T_m^c) \). Now we present the following lemma on burning this remaining graph \( P'' \).

Lemma 5. There is a partition of \( P'' \), induced by the fire sources \( y_i(m+1 \leq i \leq 2m+1) \), into paths of orders in \( F_m' \).

Proof. From lemma 4, we have that \( \forall 1 \leq i \leq m+1 \), all the vertices in \( T_i \), along with all the vertices connected to it, shall be burnt by \( y_i \). Therefore, we have to burn the vertices in \( Q_1, ..., Q_{m+1} \) by the fire sources \( y_{m+2}, x_{m+3}, ..., y_{2m+1} \) (the last \( m \) sources of fire). Since \( P'' \) is a disjoint union of paths, so we have that \( \forall m+2 \leq i \leq 2m+1 \), the subgraph induced by the vertices in \( G.N_{2m+1-i}[y_i] \) is a path of length at most \( 2(2m+1-i)+1 \). Moreover, we have that the path forest \( P'' \) is of order \( \sum_{i=1}^{m}(2i-1) = m^2 \). This implies that \( \forall m+2 \leq i \leq 2m+1 \), the subgraph induced by the vertices in \( G.N_{2m+1-i}[y_i] \) is a path of order equal to \( 2(2m+1-i)+1 \), otherwise we cannot burn all the vertices of \( P'' \) which is a contradiction. Therefore there must be a partition of \( P'' \), induced by the burning sequence \( y_{m+2}, y_{m+3}, ..., y_{2m+1} \), into sub paths of order as per each element in \( F_m' \).

Theorem 1. Optimal burning of an interval graph is NP-Complete.

Proof. Considering the partition provided in lemma 5, we claim that there is a partition of \( P'' \) into sub paths of orders in \( F_m' \). Now since each \( Q_j \) is of order of a distinct element in \( Y \), and \( |Y| = k \), we have that there must be a partition of \( Q_i(1 \leq i \leq n) \) in sub paths of orders in \( X' \).

Let \( l = |Q_j| \). Assume that a \( Q_j \) is partitioned by a union of paths of orders in \( F_m' \), instead of being burnt by a single fire source. Then \( P_i \) must be used in partitioning some other sub path in \( P'' \setminus Q_j \). Hence we can easily modify the partitioning by switching the place of \( P_i \) and those paths that have covered
\[ \sum_{i=1}^{s} h \cdot \text{number of switching in a partition for } P_i \]

Hence, total number of elements in \( \bigcup_{j=1}^{m} P_j \) only.

Since each \( Q_i \) is of order \( B' = 2B - 3 \), there must be a partition of \( Y \) into \( n \) sets such that sum of elements in each set is \( B' \). Equivalently, there must be a partition of \( X \) into \( n \) sets such that sum of elements in each set is \( B \). Since \( IG(X) \) is an interval graph, then we have a polynomial time reduction from the distinct 3-partition problem to burning interval graph.

5 Burning Permutation Graph

Bessy et. al. (2017) [6] have proved that burning a path-forest is NP-Complete. We use this fact and construct a suitable path-forests from permutations to show permutation graph is NP-Complete.

5.1 Permutation graph construction for NP-completeness

Let \( X \) be an input set to a distinct 3-partition problem; let \( n = \frac{3n}{X} \), \( m = \max(X) \), \( B = \frac{s(X)}{n} \), and \( k = m - 3n \). Let \( F_m \) be the set of first \( m \) numbers, \( F_m = \{1, 2, 3, ..., m\} \), and \( F'_m \) be the set of first \( m \) odd numbers, \( F'_m = \{2 f_i - 1 : f_i \in F_m \} = \{1, 3, 5, ..., 2m - 1\} \). Let \( X' = \{2 a_i - 1 : a_i \in X\} \), \( B' = \frac{s(X')}{n} \). Observe that \( s(X') = \sum_{i=1}^{3n} 2 a_i - 1 = 2nB - 3n \), so \( B' = 2B - 3 \). Let \( Y = F'_m \setminus X' \). Let \( O \) be the original sequence of numbers 1 to \( s(F'_m) \), \( O = \{1, 2, 3, ..., m^2\} \).

Now, we are going to construct \( n + k \) permutations \( P_1, P_2, ..., P_{n+k} \) in a specific manner such that these will produce path forests of \( (n + k) \) disjoint simple paths. Each \( P_j \) is a permutation of the numbers \( x_j \) to \( y_j \) belonging to \( O \).

Let \( t_j = y_j - x_j + 1 \); then \( P_j = \{p^1_j, p^2_j, ..., p^t_j\} \). We construct each \( P_j \) based on the subsequence \( (x_j, y_j) \in O \) and each of the \( p^h_j \) where \( h \in [1, t_j] \). Below we first provide a formula to calculate \( x_j, y_j \). We divide the range of \( j \) in two parts, \( 1 \leq j \leq n \) and \( n + 1 \leq j \leq n + k \).

We define \( y_0 = 0 \). Now \( \forall 1 \leq j \leq n, x_j = y_{j-1} + 1 \), and \( y_j = j \times 2B' \).

For the remaining values of \( j \), i.e., \( \forall 1 \leq j \leq k, x_{n+j} = y_{n+j-1} + 1 \) and \( y_{n+j} = y_{n+j-1} + L_j' \), where \( L_j' \) is the \( j^{th} \) largest element of \( Y \). See that, \( y_{n+k} = y_n + s(F'_m \setminus X') = nB' + s(F'_m \setminus X') = s(X') + s(F'_m \setminus X') = s(F'_m) = m^2 \). Hence, total number of elements in \( \bigcup_{j=1}^{n+k} P_j \) is \( m^2 \). Now we provide formula to find \( p^h_j \) for each \( j \) and all \( h \in (1, t_j) \).

If \( t_j \) is even, then \( \forall 1 \leq j \leq n + k \), we define as follows:

\( \forall \text{ odd } i, 1 \leq i \leq (t_j - 3), p^i_j = 2 + (x_j + i - 1) \). The only odd value of \( i = t_j - 1 \) remains and we define it as, \( p^{t_j-1}_j = y_j \).
Further, \( \forall \) even \( i, 4 \leq i \leq t_j, p^i_j = i - 2 \) and for the remaining value of even \( i = 2 \), we define \( p^2_j = x_j \).

Else, if \( t \) is odd, then \( \forall 1 \leq j \leq n + k \), we define as follows:

\[ \forall \text{ odd } i, 1 \leq i \leq t_j - 2, p^i_j = 2 + (x_j + i - 1). \]

The only odd value of \( i = t_j \) remains and we define it as, \( p^i_j = y_j - 1 \).

Further, \( \forall \) even \( i, 4 \leq i \leq t_j - 1, p^i_j = (x_j + i - 1) - 2 \) and for the remaining value of even \( i = 2 \), we define \( p^2_j = x_j \).

We follow the above construction where we have to compute the permutation of an a subsequence of \( O \) of length 5 or above, that is if \( y_j - x_j + 1 = t_j \geq 5 \). If otherwise \( t_j \leq 4 \), we construct the permutation \( P_j \) as follows. If \( t_j = 1 \), then \( P_j = (x_j) \). If \( t_j = 2 \), then \( P_j = (y_j, x_j) \). If \( t_j = 3 \), then \( P_j = (y_j, x_j, x_j + 1) \). If \( t_j = 4 \), then \( P_j = (x_j + 1, x_j, x_j + 2) \).

Now, \( P = P_1 \cup s_1 P_2 \cup s_2 \ldots \cup s_k P_{n+k} = (p^1_1, p^2_1, p^3_1, p^4_1, p^5_2, p^6_2, \ldots, p^1_{n+k}, p^2_{n+k}, \ldots, p^m_{n+k}) \) is the subject permutation of \( O \).

We call \( P(X) \) to be the permutation graph corresponding to the original sequence \( O \), and its subject permutation \( P, \forall 1 \leq j \leq n + k \) let \( Q_j \) be the subgraph in \( P(X) \) induced by the permutation \( P_j = (p^1_j, p^2_j, \ldots, p^m_j) \) of the original sequence \( (x_j, \ldots, y_j) \). Observe that \( P(X) = Q_1 \cup Q_2 \cup \ldots \cup Q_{n+k} \) is a path forest where the paths \( Q_1, Q_2, \ldots, Q_{n+k} \) are disjoint from each other.

The burning number of \( P(X) \) is \( m \). It follows trivially from the arguments that we have used in the proofs of lemma 5 and theorem 1 to argue the burning procedure that should be followed to burn the path forest \( P'' \) because \( P'' \) is similar to \( P(X) \).

### 5.2 Example construction

Let \( X = \{10, 11, 12, 14, 15, 16\} \Rightarrow n = 2, m = 16, B = 39, \) and \( k = 10 \).

\( F_m = \{1, 2, \ldots, 16\}, \) and \( F'_m = \{1, 3, \ldots, 31\} \). \( X' = \{19, 21, 23, 27, 29, 31\}, B' = 75 = 2B - 3, Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 25\} \).

We finally form paths \( Q_1 \) and \( Q_2 \) each of order of 75. Also, we form paths \( Q_3, Q_4, \ldots, Q_{12} \) of order of 25, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively. \( P(X) \) is a path forest of the paths \( Q_1, \ldots, Q_{12} \), which are disjoint from each other. Burning number of \( P(X) \) in this case is \( m = 16 \).

The above example is followed from the general construct that we used to reduce burning permutation graph from a distinct 3-partition problem. In this example, we have constructed paths from subsequences (of \( O \)) of odd length only. For the sake of another example, let the original sequence be \((1, 2, 3, \ldots, 34) \). Let the \( x_1 = 1, y_1 = 9, x_2 = 10, y_2 = 17, x_3 = 18, y_3 = 26, x_4 = 27, y_4 = 34 \). Now the subject permutation of this sequence becomes \((3, 1, 5, 2, 7, 4, 9, 6, 8, 12, 10, 14, 11, 16, 13, 17, 15, 20, 18, 22, 19, 24, 21, 26, 23, 25, 29, 27, 31, 28, 33, 30, 34, 32) \). The resultant permutation graph is a path forest of four paths of order of 9, 8, 9 and 8 respectively. This shows that we can induce a path forest of any shape.
and size (containing paths of both even and odd lengths) from a permutation of an original sequence.

5.3 NP-Completeness

NP-Completeness of burning path forests had already been shown in [6]. We have also constructed \( P(X) \) on the basis of the distinct 3-partition problem and provided a general construction, hence according to [6], optimal burning of permutation graph is NP-complete. We state the following theorem. The proof is immediate from the above discussion.

**Theorem 2.** Burning of a general permutation graph is NP-Complete.

6 Burning Disk Graph

Bessy et al. (2017) [6] have proved that burning a spider graph is NP-Complete. We first construct a spider graph from disks which helps us show the NP-Completeness result.

6.1 Disk graph construction for NP-completeness

Let \( X \) be an input set to a distinct 3-partition problem; let \( n = |X| \), \( m = \max(X) \), \( B = s(X) \), and \( k = m - 3n \). Let \( p = m - 1 \).

Let \( C' \) be a circle of radius \( R' \). Let there be a set \( \text{Cir} \) of \( q \) disks \( \{c_1, c_2, ..., c_q\} \) of unit radius placed around \( C' \) such that their circumference touches circumference of \( C' \), but they do not overlap with each other, or with \( C' \). The maximum value that \( q \) can take is limited. As an example, we can put a maximum of 6 unit radius disks around a disk of unit radius. As the radius \( R' \) of the central disk tends to \( \infty \), the amount of unit radius disks that we can put tends to \( R' \times \pi \) [28].

In our construction, the value of \( R' \) is chosen such that we can put \( q \geq 2(p + 2) \) disks of unit radius around \( C' \).

We give the definition of disk-chain (of a certain size) below.

**Definition 3.** Disk-chain of size \( k \). A disk-chain of size \( k \) is a sequence of disks \( Ch_x = (c^1_x, c^2_x, c^3_x, ..., c^k_x) \) such that \( c^1_x \) overlaps only with \( c^2_x \), \( c^k_x \) overlaps only with \( c^{k-1}_x \), and \( \forall 2 \leq j \leq k - 1, c^j_x \) overlaps only with \( c^{j-1}_x \) and \( c^{j+1}_x \).

1 \( \leq i \leq q \), let that a disk chain of size \( p \), \( Ch_i = \{c^1_{i1}, c^2_{i1}, ..., c^p_{i1}\} \), is attached to each circle \( ci \) such that, apart from the overlaps that give it a chain structure, \( c^1_{i1} \) overlaps with \( c_{i1} \) and \( c^2_{i1} \) only. Let \( Ch \) be the set of all these \( q \) chains, \( Ch = \{Ch_i\}_{i=1}^q \).

Let there be a disk \( C \) of radius \( R : R' < R \leq R' + 0.5 \) is positioned with its center exactly at the center of \( C' \) defined above. Observe that all the disks in \( \text{Cir} \) now overlap with \( C \). Now consider the corresponding disk graph. Let the vertex corresponding to \( C \) be called head \( h \), vertices corresponding to \( c_i \) be
called \( v_1 \), and the vertices corresponding to \( c_i^j \) be called \( v_i^j \), \( \forall \ 1 \leq i \leq q \), and \( \forall \ 1 \leq j \leq p \).

We shall call this setting of disks \( DK(R, r, q, p, C, Cir, Ch) \). This setting of disks correspond to the graph \( SP(q, p + 1) \). Now we are going to extend \( DK(R, r, q, p, C, Cir, Ch) \) by adding more disks to it; in fact, we are going to add chains at the terminus of the chains that are already present, which we can do very easily. Before that, let us define what do we mean by attaching disk chain behind another disk chain.

**Definition 4. Attaching a disk-chain behind another.** If disk-chain \( C_1 = (c_1^1, c_1^2, c_1^3, \ldots, c_1^k) \) is attached behind another chain \( C_2 = (c_2^1, c_2^2, c_2^3, \ldots, c_2^k) \), a new chain is formed \( Ch_{FIN} = c_2^1, c_2^2, c_2^3, \ldots, c_2^k, c_1^1, c_1^2, c_1^3, \ldots, c_1^k \). Clearly, this attachment is done in such a manner that \( c_1^1 \) overlaps only with \( c_2^k \) and \( c_1^k \).

Let \( F_m \) be the set of first \( m \) numbers, \( F = \{1, 2, 3, \ldots, m\} \), and \( F_m' \) be the set of first \( m \) odd numbers, \( F_m' = \{2 f_i - 1 : f_i \in F_m\} = \{1, 3, 5, \ldots, 2m - 1\} \). Let \( X' = \{2 a_i - 1 : a_i \in X\} \). \( B' = \sum_{i=1}^{n} 2 a_i - 1 = 2nB - 3n \), so \( B' = 2B - 3 \). Let \( Y = F_m' \setminus X' \).

Let there be \( n \) disk-chains \( Q_1, Q_2, \ldots, Q_n \), each of size \( B' \), \( \forall \ 1 \leq j \leq k \), let \( Q_{n+j} \) be a disk chain of size \( L_i^Y \), where \( L_i^Y \) is the \( i \)-th largest element of \( Y \).

We now attach disk-chains \( Q_1, Q_2, \ldots, Q_{n+k} \) behind \( Ch_1, Ch_2, \ldots, Ch_{n+k} \) respectively. In the corresponding disk graph, let \( P_i' \) be the path induced by the disk chain \( Q_i \), \( \forall \ 1 \leq i \leq n + k \).

Let the corresponding disk graph of this updated setting of disks be called \( DK(X) \). We have to put the first fire source at the spider head \( h \) [6] to burn optimally. Bow we are left with a path forest similar to \( P'' \) of section 4. So the burning number of \( DK(X) \) is \( m + 1 \). It follows trivially from the arguments used in the proofs of lemma 5 and theorem 1 that we have used to argue the burning procedure that should be followed to burn the path forest \( P'' \).

### 6.2 Example construction

Let \( X = \{10, 11, 12, 14, 15, 16\} \implies n = 2, m = 16, B = 39, \) and \( k = 10, p = 15 \).

Let \( R' = 10 \), then we can attach \( q = 34 \) chains each of length \( p \). Take \( R = 10.5 \).

\[ F_m = \{1, 2, \ldots, 16\}, \text{ and } F_m' = \{1, 3, \ldots, 31\}. \]

\( X' = \{19, 21, 23, 27, 29, 31\}, B' = 75 = 2B - 3. \)

\( Y = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 25\}. \)

We finally obtain paths \( P_1' \) and \( P_2' \) each of order of 75. Also, we form paths \( P_3', P_4', \ldots, P_{12}' \) of order of 25, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively.

The central spider graph formed is \( SP(34, 16) \), and \( P_1', P_2', \ldots, P_{12}' \) are attached to \( v_1^{15}, v_2^{15}, \ldots, v_2^{12} \) respectively at vertices on one of their ends.

Construction of this example \( DK(X) \) is demonstrated in figure 3. Burning number of \( DK(X) \) in this case is \( m + 1 = 17 \).
6.3 NP-Completeness

Observe that $G$ can be burnt optimally only if $Q_1, Q_2, ..., Q_{n+k}$ can be broken into paths of length in $F'_m$.

Clearly, we can vary the size of the central disk and construct a connected spider graph of any shape and size: all spider graphs are valid disk graph. NP-Completeness of burning spider graphs has been shown in [6] (more specifically, Theorem 13). We have constructed $DK(X)$ on the basis of the distinct 3-partition problem, which strongly suggests the NP-Completeness of burning disk graph. So theorem 3 follows from here.

**Theorem 3.** Burning disk graph is NP-Complete even if the underlying disk representation is given.

7 Conclusion

In this paper we show graph burning problem is NP-complete on some of the sub classes of geometric graphs, namely interval graph, permutation graph and disk graph. Usually many other hard problems are solvable in polynomial time over interval graph. Surprisingly, even after having an almost tight bound on the optimal burning number of interval graph, burning turns out to be NP-complete on interval graph. We prove similar results on disk graph as well as permutation graph. One immediate future work can be studying the same on several other graph classes. Another very much related direction is to try and improve the 3-approximation algorithm provided in [6] for burning general graphs.
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