Global well-posedness of a three-dimensional Brinkman-Forchheimer-Bénard convection model in porous media

Edriss S. Titi* Saber Trabelsi†

April 7, 2022

This work is dedicated to Professor Jerome A. Goldstein on the occasion of his 80th birthday

Abstract

We consider three-dimensional (3D) Boussinesq convection system of an incompressible fluid in a closed sample of a porous medium. Specifically, we introduce and analyze a 3D Brinkman-Forchheimer-Bénard convection problem describing the behavior of an incompressible fluid in a porous medium between two plates heated from the bottom and cooled from the top. We show the existence and uniqueness of global in-time solutions, and the existence of absorbing balls in $L^2$ and $H^1$. Eventually, we comment on the applicability of a data assimilation algorithm to our system.

MSC class: 35Q30, 35Q35, 76B03, 86A10, 93C20, 37C50, 76B75, 34D06.

Keywords: Porous media, Brinkman-Forchheimer-extended Darcy model, 3D Navier-Stokes equations, Bénard problem, Data assimilation.

Contents

1 Introduction and main results
   1.1 The physical model .................................................. 2
   1.2 The Main results ................................................... 3

2 Proof of well-posedness
   2.1 Galerkin approximation system ..................................... 5
   2.2 A priori estimates and existence of weak solutions ............... 6
   2.3 Energy and gradient estimates ....................................... 8
   2.4 Continuous dependence on the initial data and uniqueness of solutions ........................................... 12

3 Data Assimilation ....................................................... 14

References ................................................................. 15

1 Introduction and main results

In its original description, the Bénard convection problem is concerned with the motion of incompressible flow confined between two horizontal plates (or walls) heated at the bottom and cooled at the top. Density differences occur due to the temperature difference across the fluid as regions of the fluid near the bottom...
boundary are heated and thus expand. In turn, the density differences result in a buoyancy force pushing the lighter fluid to the top and the heavier to the bottom and the governing equation of motion is modeled with the Boussinesq approximation. Applications of Bénard convection range from weather forecasting to nuclear magnetic resonance pulsed-field-gradient diffusion measurements [20] and the security of liquefied natural gas packets [43] etc. In this paper we analyze the Bénard problem in a porous medium modeled by the Brinkman-Forchheimer extended Darcy system for the momentum equation coupled with the heat convection.

1.1 The physical model

In this paper, we consider the Bénard convection problem of an incompressible fluid saturating an infinite horizontal layer of porous medium confined between two horizontal solid walls located at $z = 0$ and $z = 1$. The fluid is heated from below and cooled from the top with temperatures normalized to 1 and 0, respectively. Using the Boussinesq approximation, the non-dimensional 3D equations governing the motion of the convected fluid through the porous medium are given by the Brinkman-Forchheimer-Bénard system

\[
\mathcal{S}_o : \begin{cases} 
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + a |u|^{2\alpha} u + \nabla q = T \mathbf{e}_3, \\
\partial_t T - \kappa \Delta T + (u \cdot \nabla) T = 0, \\
\nabla \cdot u = 0, \quad u|_{t=0} = u_0, \quad T|_{t=0} = T_0, 
\end{cases}
\]

where $\mathbf{e}_3 = (0, 0, 1)^T$. We consider this problem in an horizontal periodic domain $\Omega_r := [0, L] \times [0, L] \times [0, 1]^1$, and supplement it with the following boundary conditions (BC in short)

\[
\mathcal{B}_C : \begin{cases} 
T(t, x, y, 0) = 1, \quad T(t, x, y, 1) = 0, \quad u_3(t, x, y, 0) = u_3(t, x, y, 1) = 0, \\
\partial_3 u_3(t, x, y, 0) = \partial_3 u_3(t, x, y, 1) = \partial_3 u_2(t, x, y, 0) = \partial_3 u_2(t, x, y, 1) = 0, \\
q, u, T, \text{periodic in the } x \text{ and } y \text{ variables with period } L. 
\end{cases}
\]

In system $(\mathcal{S}_o - \mathcal{B}_C)$, the fluid velocity $u(t, x, y, z)$, the pressure $p = p(t, x, y, z)$, and the normalized temperature $T = T(t, x, y, z)$ are the unknowns. $\nu$ and $\kappa$ are positive constants representing the kinematic viscosity and the thermal diffusivity respectively, and $a$ is a positive coefficient that arises from the Darcy-Forchheimer law. In this model we use the Brinkman-Forchheimer-extended Darcy (BFeD in short) model for flow in porous media, instead of the simple Darcy law. The velocity BC in $\mathcal{B}_C$ are no-normal flow and stress-free at the solid boundary.

Observe that $u = 0, T = 1 - z$ and $q = z (1 - \frac{z}{2})$ is the pure conduction steady state of $(\mathcal{S}_o - \mathcal{B}_C)$. Considering a fluctuation around this steady state

\[
\theta = T - (1 - z) \quad \text{and} \quad p = q - z \left(1 - \frac{z}{2}\right),
\]

then system $(\mathcal{S}_o - \mathcal{B}_C)$ is equivalent to

\[
\mathcal{S} : \begin{cases} 
\partial_t u - \nu \Delta u + (u \cdot \nabla) u + a |u|^{2\alpha} u + \nabla p = \theta \mathbf{e}_3, \\
\partial_t \theta - \kappa \Delta \theta + (u \cdot \nabla) \theta = u \cdot \mathbf{e}_3, \\
\nabla \cdot u = 0, \quad u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0, 
\end{cases}
\]

supplemented with the corresponding set of BC; obtained from $\mathcal{B}_C$ using (1.1). More precisely

\[
\mathcal{B} : \begin{cases} 
\theta(t, x, y, 0) = \theta(t, x, y, 1) = 0, \quad u_3(t, x, y, 0) = u_3(t, x, y, 1) = 0, \\
\partial_3 u_3(t, x, y, 0) = \partial_3 u_3(t, x, y, 1) = \partial_3 u_2(t, x, y, 0) = \partial_3 u_2(t, x, y, 1) = 0, \\
p, u, \theta, \text{periodic in the } x \text{ and } y \text{ variables with respective periods } L. 
\end{cases}
\]

1 Obviously the domain can be chosen as $[0, L_1] \times [0, L_2] \times [0, L_3]$ with $L_1, L_2, L_3 > 0$ and $L_1 \neq L_2 \neq L_3$. 

2
In the case of $a = 0$, the first set of equations of system $S$ ($S_1$ in short) is nothing but the classical 3D Navier-Stokes equations forced by buoyancy, so that $S$ corresponds to the classical 3D Boussinesq equations. In this case, the mathematical analysis of the Bénard system ($S - BC$) has been studied in [17] (see also [39] and references therein). The authors prove the existence and uniqueness of weak solutions in two-dimensional space (2D), and the existence of weak solutions in 3D. Also, they proved the existence of a finite-dimensional global attractor in 2D. Let us mention that the authors used the third line of $BC$ and Dirichlet for $u$ and $\theta$ at the top and the bottom boundaries as BC.

When $a > 0$, equations $S_1$ (with $\theta \equiv 0$) are the so-called the 3D BFeD model. This model was formally derived (cf., e.g., [21]) using Darcy-Forchheimer equation of porous media that states

$$\nabla p = -\frac{\mu}{k} \mathbf{v}_f - \gamma \rho_f |\mathbf{v}_f|^2 \mathbf{v}_f,$$

where $\gamma > 0$, $\mathbf{v}_f$ and $\rho_f$ stand for the the Forchheimer coefficient, the Forchheimer velocity and the density, respectively. This equation add a correction to the Darcy law to model the increase of the pressure drop.

There is a rich literature dedicated to the mathematical analysis of this model and its variants, and we refer to, e.g., [10, 27, 28, 29, 35, 37, 9, 34, 41, 42, 44, 8, 2]. Recently, in [24, 30], the authors shows the existence and uniqueness of weak and strong solutions with Dirichlet boundary condition starting from a regular enough initial data. In the periodic setting, the authors of [30] improve the results of [24] and prove the well posedness for initial data in $H^1(\mathbb{T})$. Their result can be extended to the case of Dirichlet boundary condition using the regularity estimates of the Stokes operator). An anisotropic viscous version of the BFeD system was studied in [6]. Eventually, a relatively closed (from the mathematical point of view) MHD model was investigated in [40] and a its Boussinesq-MHD version (without diffusion) in [26]. Let us mention that in the latter reference, the uniqueness was obtained only for regular solutions, and our argument in the present contribution combined with ideas from [40] can improve the result.

To overcome technicalities related to the boundary conditions, we extend the domain $\Omega_r$ to $\Omega = [0, L] \times [0, L] \times [-1, 1]$ and consider problem $S$, subjected to the following set of BC

$$BC_e : \begin{cases} 
\theta, u_3, \text{ periodic odd functions with respect to the } z \text{ variable with period } 2, \\
u_1, u_2, \text{ periodic even functions with respect to the } z \text{ variable with period } 2, \\
p, u, \theta, \text{ periodic in the } x \text{ and } y \text{ variables with period } L. 
\end{cases}$$

It is rather easy to see that this set of periodic-symmetric BC are equivariant under the solution operator of system $S$ supplemented with periodic conditions $BC$. Most importantly, solutions with this periodic-symmetric BC $BC_e$ clearly satisfy the physical BC $BC$. Indeed, the fact that $\theta, u_3$ are periodic odd functions with respect to the $z$ variable with period 2 implies that $\theta(t, x, y, 1) = 0, u_3(t, x, y, 1) = 0$. Equivalently, the fact that $u_1, u_2$ are periodic even functions with respect to the $z$ variable with period 2 implies that $\partial_3 u_1(t, x, y, 1) = \partial_3 u_2(t, x, y, 1) = 0$. As a matter of fact, we shall focus on the mathematical analysis of system ($S - BC_e$). Obviously all the results obtained in the periodic boundary conditions setting will be valid for the physical system ($S - BC$) in $\Omega_r$. In other words, the restriction to $\Omega_r$ of a solution $(u(x, y, z), \theta(x, y, z), p(x, y, z))$ of system ($S - BC_e$) on $\Omega$, is a solution of ($S - BC$).

### 1.2 The Main results

Let us introduce the functional setting that we shall use along this paper. Let $\mathcal{X}_e$ be the set of trigonometric polynomials with period $L$ in the $x$ and $y$ variables, and are even with period 2 in the $z$ variable. Let $\mathcal{X}_o$ be the set of trigonometric polynomials with period $L$ in the $x$ and $y$ variables, and are odd with period 2 in the $z$ variable. Eventually, let $\mathcal{Y}$ be the set of divergence-free vector fields belonging to $\mathcal{X}_e \times \mathcal{X}_e \times \mathcal{X}_o$. In the sequel, we will not make a difference in the notation of scalar and vector Lebesgue and Sobolev spaces, which shouldn’t confuse the reader.

Now, we define $H_0$ and $H_1$ as the closure of $\mathcal{Y}$ and $\mathcal{X}_o$ in $L^2(\Omega)$, respectively. We endow $H_0$ and $H_1$ with the following scalar products
\[(u, v)_{H_0} = \sum_{i=1}^{3} \int_{\Omega} u_i(x) v_i(x) \, dx, \quad \text{and} \quad (\varphi, \phi)_{H_1} = \int_{\Omega} \varphi(x) \phi(x) \, dx.\]

The associated norms are given by \(\|u\|_{H_0} = [(u, u)_{H_0}]^{\frac{1}{2}}\) and \(\|\varphi\|_{H_1} = [(\varphi, \varphi)_{H_1}]^{\frac{1}{2}}\), respectively. Equivalently, we define \(V_0\) and \(V_1\) as the closure of \(Y\) and \(\mathcal{X}_\alpha\) in \(H^1(\Omega)\), respectively. \(V_0\) and \(V_1\) are Hilbert spaces endowed with the following scalar products

\[(u, v)_{V_0} = (u, v)_{H_0} + (u, v)_{V_0} := (u, v)_{H_0} + \sum_{i,j=1}^{3} \int_{\Omega} \partial_j u_i(x) \partial_j v_i(x) \, dx,\]

and

\[(\varphi, \phi)_{V_1} = \sum_{j=1}^{3} \int_{\Omega} \partial_j \varphi(x) \partial_j \phi(x) \, dx,\]

where \(\partial_j\) denotes the partial derivative with respect to the variable \(x\) if \(j = 1, y\) if \(j = 2\), and \(z\) if \(j = 3\). The associated norms are given by \(\|u\|_{V_0} = [(u, u)_{V_0}]^{\frac{1}{2}}\), and \(\|\varphi\|_{V_1} = [(\varphi, \varphi)_{V_1}]^{\frac{1}{2}}\), respectively. It is worth noticing that since \(\theta, u, v, \varphi \in V_1\) are odd in the \(z\) variable and periodic in the \(x\) and \(y\) variables, they have average zero over \(\Omega\), thus by the Poincaré inequality, \(\|\cdot\|_{V_1}\) defines a norm on \(V_1\).

In the sequel, we shall use the notation \(\|\cdot\|_p\) for the \(L^p(\Omega)\) norms, and \(\|\cdot\|_{H^1}\) and \(\|\cdot\|_{H^2}\) for \(H^1(\Omega)\) and \(H^2(\Omega)\) norms respectively. Now, let \(\mathcal{A}_i\), for \(i = 0, 1\), be the unbounded nonnegative self-adjoint linear operators with domains \(D(\mathcal{A}_i) = V_i \cap H^2(\Omega)\) satisfying \((\mathcal{A}_i \varphi, \phi)_{H_1} = ((\varphi, \phi))_{V_1}\), for all \(\varphi, \phi \in D(\mathcal{A}_i)\) and \(i = 0, 1\). The operator \(\mathcal{A}_1\) is positive definite with compact inverse \(\mathcal{A}_1^{-1}\). Observe that with periodic BC, we have \(\mathcal{A}_0 = -\Delta\), which is not invertible whose kernel consists of constant vector fields corresponding to the eigenvalue 0. Thanks to the elliptic regularity of the operator \(\mathcal{A}_0 + I\) and Cauchy-Schwarz inequality, it is rather easy to see that \(\|u\|_{H^2} \simeq \|u\|_{L^2} + \|\mathcal{A}_0 u\|_{L^2}\). Consequently, there exists a basis of orthonormal eigenfunctions \(w_{i,j}\) of \(\mathcal{A}_i\) for \(i = 0, 1\) and \(j = 1, 2, \ldots\) such that \(\mathcal{A}_i w_{i,j} = \lambda_{i,j} \ w_{i,j}\) where \(w_{i,j} \in H_1\) denotes the \(j^{th}\) eigenfunction of \(\mathcal{A}_i\) and \(\lambda_{i,j}\) the associated positive eigenvalue satisfying \(0 < \lambda_{i,j} \leq \lambda_{i,j+1}\) for all \(i = 0, 1\) and \(j = 1, 2, \ldots\). Let us mention that, by abuse of notation, we denote \(\lambda_{0,1}\) the second eigenvalue of \(\mathcal{A}_0\) since the first eigenvalue is 0 as it was stated above, and we add the associated eigenvector to the basis spanning the kernel of \(\mathcal{A}_0\). Thus, introducing \(\lambda := \inf_{i=0; j=1; 2, \ldots} \lambda_{i,j} > 0\), the Poincaré and Sobolev inequalities read \(\lambda^{1/2} |\varphi|_2 \leq |\nabla \varphi|_2\) and \(\|\varphi\|_{\mathcal{X}_\alpha} \leq \gamma |\nabla \varphi|_2\), respectively, where \(\varphi = \theta, u, v, \partial_3 u_1, \partial_3 u_2, \partial_1 u_3, \partial_2 u_3\) thanks to the set of symmetric periodic BC, \(BC_e\) (with \(\gamma > 0\) being a constant depending only on the size of the domain).

Now, we are able to state our first result about the existence of solutions to system \((S - BC_e)\)

**Theorem 1.1.** Let \((u_0, \theta_0) \in H_0 \times H_1, \alpha \geq 0\) and \(a, \nu, \kappa > 0\), then system \((S - BC_e)\) has global weak solutions \((u, \theta)\) satisfying

\[u(x, t) \in C^0_b(\mathbb{R}^+; H_0) \cap L^2_{\text{loc}}(\mathbb{R}^+; V_0) \cap L^{2n+2}_{\text{loc}}(\mathbb{R}^+; L^{2n+2}(\Omega)),\]

and

\[\theta(x, t) \in C^0_b(\mathbb{R}^+; H_1) \cap L^2_{\text{loc}}(\mathbb{R}^+; V_1).\]

In particular

\[
\limsup_{t \to +\infty} \|\theta(t)\|_{H_1}, \quad \limsup_{t \to +\infty} \|u(t)\|_{H_0} \leq \frac{4aL^2}{\min(\alpha, \kappa \lambda)} \left(\frac{2a\kappa \lambda}{a\kappa \lambda + 32}\right)^{\frac{a-1}{2}}. \tag{1.2}
\]

In addition, if \(\alpha > 1\), then the weak solutions depend continuously on the initial data in the \(H_0 \times H^{-1}(\Omega)\) topology, in particular they are unique.

This Theorem ensures the existence of weak solutions to system \((S - BC_e)\) and their uniqueness for a damping parameter’s range \(\alpha > 1\). Moreover, it shows the existence of absorbing ball in \(H_0 \times H_1\) for the solutions of \((S - BC_e)\). This property plays a crucial role in the design and analysis of a data assimilation (DA) algorithm for system \((S - BC_e)\) (see section 3). Also, we have the following
**Theorem 1.2.** Let \((u_0, \theta_0) \in V_0 \times H_1, \alpha > 1\) and \(a, \nu, \kappa > 0\), then system \((S - BC_e)\) has global solutions \((u, \theta)\) satisfying

\[
\begin{align*}
    u(x, t) &\in C^0_0(\mathbb{R}^+; V_0) \cap \mathcal{L}^2_{\text{loc}}(\mathbb{R}^+; V_0 \cap H^2(\Omega)) \cap \mathcal{L}^{2\alpha+2}_{\text{loc}}(\mathbb{R}^+; \mathcal{L}^{2\alpha+2}(\Omega)), \\
    \theta(x, t) &\in C^0_0(\mathbb{R}^+; H_1) \cap \mathcal{L}^2_{\text{loc}}(\mathbb{R}^+; V_1),
\end{align*}
\]

In particular, in addition to (1.2), we have

\[
\limsup_{t \to +\infty} \|u(t)\|_{V_0} \leq \frac{\Gamma_2}{\Gamma_1} \left[ (a + 1)\Gamma_1 + 4aL^2 + (3 + a)\Gamma_1 + 4aL^2 \right],
\]

where

\[
\Gamma_1 := \left( \frac{2a\kappa \lambda}{a\kappa \lambda + 32} \right)^{\frac{\alpha+1}{\alpha}} \frac{4aL^2}{\min(a, \kappa \lambda)} \quad \text{and} \quad \Gamma_2 := \left( \frac{a\nu^\alpha}{2^{2-\alpha}} \right)^{1/2-\alpha}.
\]

Moreover, if \(\theta_0 \in L^0(\Omega)\), then the solutions are unique. Furthermore, if \(u_0 \in V_0 \cap \mathcal{L}^{2\alpha+2}(\Omega)\), then \(u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}^+; \mathcal{L}^{2\alpha+2}(\Omega))\) and \(\partial_t u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}^+; H_0)\). Also, the solutions depend continuously on the initial data in the \(H_0 \times H^{-1}(\Omega)\) topology, in particular they are unique.

This Theorem shows the existence of global solutions with regular initial velocity. For a given temperature \(\theta \in H_1\), the velocity solution \(u \in V_0\) is a classical solution of the BFeD equations with the forcing term \(\theta e_1\). Equivalently, for a given velocity \(u \in V_0\), the solution \(\theta\) of the thermal diffusion equation is a weak solution. Of course, if the initial temperature is considered in \(V_1\), then we can extend the Theorem and reach strong solutions for both velocity and temperature (see Remark 2.6). Most importantly, this Theorem shows the existence of an absorbing ball in \(V_0 \times H_1\) for the solutions of \((S - BC_e)\). Obviously, the Theorem still holds if the initial temperature is in \(L^\infty(\Omega)\), which corresponds to the physical case. Also, if one consider initial temperature in \(V_1\), then system \((S - BC_e)\) have an absorbing ball in \(V_0 \times V_1\). Eventually, let us mention that in Theorems 1.1 and 1.2, we do not discuss the regularity of the pressure which can be recovered from the velocity in a classical way using the divergence free property and standard elliptic regularity. We refer to any textbook for details about this point (i.e., [36, 38])

This work is dedicated to Professor Jerome A. Goldstein on the occasion of his 80th birthday as a token of admiration for his contribution to the mathematical analysis of Partial Differential Equations and their applications.

## 2 Proof of well-posedness

First, let us recall the following version of Young’s inequality,

\[
    ab \leq \epsilon a^p + \epsilon^{-q/p} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \text{for all} \quad \epsilon > 0 \quad \text{and} \quad a, b \geq 0.
\]

### 2.1 Galerkin approximation system

The well posedness of \((S - BC_e)\) can be shown using a standard approximation argument. First, one uses the Faedo-Galerkin approximation method based on an orthonormal basis of eigenfunctions of the operators \(A_0\) and \(A_1\) (see, e.g. [38, 39]) to show the existence and uniqueness of approximate solutions. Next, one obtains uniform \(a \text{ priori}\) estimates using the approximate system, and eventually pass to the limit using compactness arguments, e.g., Aubin-Lions Lemma, [25] (Lemma I-6.5).

First, we define the bilinear forms \(B_0(\cdot, \cdot) : V_0 \times \mathcal{D}(A_1) \to H_0\) and \(B_1(\cdot, \cdot) : V_1 \times \mathcal{D}(A_1) \to H_1\) such that

\[
    B_0(u, v) := \mathbb{P}(u \cdot \nabla) v, \quad \text{and} \quad B_1(u, \theta) := (u \cdot \nabla) \theta,
\]

where \(\mathbb{P}\) denotes the Leray projector on divergence-free vector fields. Let \(\mathcal{W}_k(x) \subset \mathcal{D}(A_0)\) and \(\tilde{\mathcal{W}}_k(x) \subset \mathcal{D}(A_1)\) be orthonormal basis of \(H_0\) and \(H_1\), respectively. Denote by...
$V_0^m = \text{span}\{W_1, W_2, \ldots, W_m\}$ and $V_1^m = \text{span}\{\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_m\}$, and $P_m : H_0 \to V_0^m$ and $\hat{P}_m : H_1 \to V_1^m$ be the corresponding the projections. Now, set

$$u_m(x,t) = \sum_{i=1}^{m} g^i_m(t) \hat{W}_i(x) \in V_0^m \quad \text{and} \quad \theta_m(x,t) = \sum_{i=1}^{m} \hat{g}^i_m(t) \hat{W}_i(x) \in V_1^m.$$ 

by the solution of the Galerkin approximation system associated with $(S - BC_e)$, namely, the unknown coefficients $g^i_m = (u_m, W_i)_{H_0}$ and $\hat{g}^i_m = (\theta_m, \hat{W}_i)_{H_1}$, for $i = 1, 2, \ldots, m$, solve the following system of ordinary differential equations

$$\begin{aligned}
\frac{d}{dt} u_m + \nu A_0 u_m + \mathcal{P}_m B_0(u_m, u_m) + a \mathcal{P}_m (\mathbb{P} (|u_m|^{2\alpha} u_m)) - \mathcal{P}_m (\mathbb{P} (\theta_m e_3)) &= 0, \\
\frac{d}{dt} \theta_m + \kappa A_1 \theta_m + \hat{P}_m B_1(u_m, \theta_m) - \hat{P}_m (u_m \cdot e_3) &= 0,
\end{aligned}$$

(2.1a, 2.1b)

$$u_m(0) = \mathcal{P}_m u_0, \quad \theta_m(0) = \hat{P}_m \theta_0,$$

(2.1c)

where $(u_0, \theta_0) \in H_0 \times H_1$. Observe that for $(u_0, \theta_0) \in H_0 \times H_1$ one has

$$(\mathcal{P}_m u_0, \hat{P}_m \theta_0) \to (u_0, \theta_0) \quad \text{strongly in} \ H_0 \times H_1 \quad \text{as} \ n \to +\infty.$$

(2.2)

Since the vector field in system (2.1) is locally Lipschitz in $V_0^m \times V_1^m$, the system admits a unique solution $(u_m, \theta_m) \in C^1([0, \tau^*_m], V_0^m) \times C^1([0, \tau^*_m], V_1^m)$, for some $\tau^*_m > 0$.

2.2 A priori estimates and existence of weak solutions

Let $T \in (0, \infty)$ be arbitrary. Our goal is to show that the unique solution of the Galerkin approximation system (2.1) exists on the interval $[0, T]$. Let $[0, \tau^*_m)$ be the maximal interval existence of solutions to (2.1), and assume by contradiction that $\tau^*_m \leq T < \infty$. This in turn implies that

$$\limsup_{t \to (\tau^*_m)^-} (\|u_m(t)\|_{H_0}^2 + \|\theta_m(t)\|_{H_1}^2) = \infty.$$ 

(2.3)

Next, we focus on the interval $[0, \tau^*_m)$ and establish a priori estimates for the solution of (2.1). Taking the $H_0$–inner product of equation (2.1) with $u_m(t)$ and thanks to Cauchy-Schwarz and Young inequalities one obtains

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_{H_0}^2 + \nu \|u_m\|_{V_0}^2 + a \|u_m\|_{2\alpha + 2}^2 \leq \frac{1}{2} \|\theta_m\|_{H_1}^2 + \frac{1}{2} \|u_m\|_{H_0}^2.$$ 

(2.4)

Similarly, we take the $H_1$–inner product of equation (2.1) with $\theta_m(t)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\theta_m\|_{H_1}^2 + \kappa \|\theta_m\|_{V_1}^2 \leq \frac{1}{2} \|\theta_m\|_{H_1}^2 + \frac{1}{2} \|u_m\|_{H_0}^2.$$ 

(2.5)

Summing (2.4) and (2.5), we get for all $t \in [0, \tau^*_m)$

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_{H_0}^2 + \frac{1}{2} \frac{d}{dt} \|\theta_m\|_{H_1}^2 + \nu \|u_m\|_{V_0}^2 + \kappa \|\theta_m\|_{V_1}^2 + a \|u_m\|_{2\alpha + 2}^2 \leq \|\theta_m\|_{H_1}^2 + \|u_m\|_{H_0}^2.$$ 

(2.6)

Now, using Gronwall’s inequality, we obtain

$$\|u_m(t)\|_{H_0}^2 + \|\theta_m(t)\|_{H_1}^2 \leq (\|P_m u_0\|_{H_0}^2 + \|\hat{P}_m \theta_0\|_{H_1}^2) e^{2t} \leq (\|u_0\|_{H_0}^2 + \|\theta_0\|_{H_1}^2) e^{2t}.$$ 

Therefore, integrating the inequality (2.6), we get

$$\|u_m(t)\|_{H_0}^2 + \|\theta_m(t)\|_{H_1}^2 + 2\nu \int_0^t \|u_m(s)\|_{V_0}^2 ds + 2\kappa \int_0^t \|\theta_m(s)\|_{V_1}^2 ds + 2a \int_\Omega \|u_m(s)\|_{2\alpha + 2}^2 ds \leq (\|P_m u_0\|_{H_0}^2 + \|\hat{P}_m \theta_0\|_{H_1}^2) (1 + e^{2t}) \leq (\|u_0\|_{H_0}^2 + \|\theta_0\|_{H_1}^2) (1 + e^{2t}),$$

(2.7)
for all \( t \in [0, T^*] \). The above contradicts (2.3). Therefore, the solution exists on \([0, T]\). Moreover, we also conclude from the above that \((u_m, \theta_m)\) remains bounded uniformly in \(m\) in
\[
L^\infty([0, T], H_0) \times L^\infty([0, T], H_1) \cap L^2([0, T], V_0) \times L^2([0, T], V_1).\tag{2.7}
\]
Furthermore, \(u_m\) remains bounded in \(L^{2\alpha+2}([0, T], L^{2\alpha+2}(\Omega))\) so that \(|u_m|^{2\alpha}u_m\) remains bounded, uniformly in \(m\), in
\[
L^{\frac{2\alpha+2}{\alpha+1}}([0, T], L^{\frac{2\alpha+2}{\alpha+1}}(\Omega)), \quad \text{since} \quad \|u_m|^{2\alpha}u_m\|^{\frac{2\alpha+2}{\alpha+1}} = \|u_m|^{2\alpha+2}.\tag{2.8}
\]
Now, from system (2.1) and by virtue of (2.8) it is rather standard to see that \((\frac{d}{dt}u_m, \frac{d}{dt}\theta_m)\) remains bounded, uniformly in \(m\), in
\[
L^2([0, T], V_0') + L^{\frac{2\alpha+2}{\alpha+1}}([0, T], L^{\frac{2\alpha+2}{\alpha+1}}(\Omega)) \times L^2([0, T], V_1').
\]
Now, thanks to (2.7), we infer that there exist \(u\) and \(\theta\), and there exist two subsequences, that we still denote \((u_m, \theta_m)\), such that
\[
(u_m, \theta_m) \to (u, \theta) \quad \text{weakly-* in} \quad L^\infty([0, T], H_0) \times L^\infty([0, T], H_1) \quad \text{as} \quad m \to +\infty, \tag{2.9}
\]
and
\[
(u_m, \theta_m) \to (u, \theta) \quad \text{weakly in} \quad L^2([0, T], V_0) \times L^2([0, T], V_1) \quad \text{as} \quad m \to +\infty, \tag{2.10}
\]
Furthermore, thanks to (2.8), it holds
\[
u_{m} \to u \quad \text{weakly in} \quad L^{2\alpha+2}([0, T], L^{2\alpha+2}(\Omega)) \quad \text{as} \quad m \to +\infty, \tag{2.11}
\]
and there exist \(v\) such that
\[
|u_m|^{2\alpha}u_m \to v \quad \text{weakly in} \quad L^{\frac{2\alpha+4}{\alpha+1}}([0, T], L^{\frac{2\alpha+4}{\alpha+1}}(\Omega)) \quad \text{as} \quad m \to +\infty. \tag{2.12}
\]
Eventually, it holds
\[
\frac{d}{dt}u_m \to \frac{d}{dt}u \quad \text{weakly in} \quad L^2([0, T], V_0') + L^{\frac{2\alpha+2}{\alpha+1}}([0, T], L^{\frac{2\alpha+2}{\alpha+1}}(\Omega)) \quad \text{as} \quad m \to +\infty. \tag{2.13}
\]
Thus, thanks to Aubin-Lions compactness Lemma \([25, 38]\), we have
\[
(u_m, \theta_m) \to (u, \theta) \quad \text{strongly in} \quad L^2([0, T], H_0) \times L^2([0, T], H_1) \quad \text{as} \quad m \to +\infty. \tag{2.14}
\]
Now, let us fix \(i\), such that \(i < m\). We take the \(H_0\)-inner product of (2.1) with \(W_i\), and the \(H_1\)-inner product of (2.1) with \(\tilde{W}_i\), and integrate with respect to time over \([0, t] \subset [0, T]\) to obtain
\[
(u_m(t), W_i)_{H_0} - (u_0, W_i)_{H_0} + \nu \int_0^t (u_m(s), W_i)_{V_0} \, ds + \int_0^t (B_0(u_m(s), u_m(s)) + a|u_m(s)|^{2\alpha}u_m(s) - \theta_m(s) e_3, W_i)_{H_0} \, ds = 0,
\]
\[
(\theta_m(t), \tilde{W}_i)_{H_1} - (\theta_0, \tilde{W}_i)_{H_1} + \kappa \int_0^t (\theta_m(s), \tilde{W}_i)_{V_1} \, ds + \int_0^t (B_1(u_m(s), \theta_m(s)) - u_m \cdot e_3, \tilde{W}_i)_{H_1} \, ds = 0.
\]
In the above we used the facts that \(\mathcal{P}_m, \tilde{\mathcal{P}}_m\) and \(P\) are orthogonal projections in the corresponding \(L^2\) spaces, and the fact that \(\mathcal{P}_m W_i = W_i\) and \(\mathcal{P}_m \tilde{W}_i = \tilde{W}_i\), since \(i < m\).
Now, we can pass to the limit in these equalities. The linear terms and the convective nonlinear terms are handled using (2.10-2.14), and we refer to \([38, 39]\) for details. For the velocity power term, up to extracting a
subsequence, we can show that \( v = |u|^{2\alpha}u \) thanks to (2.14) using Aubin-Lions compactness Lemma, [25, 38]. As a result we have shown that for every \( i = 1, 2, \ldots \) the limit pair \((u, \theta)\) satisfies

\[
(u(t), \mathcal{W}_i)_{H_0} - (u_0, \mathcal{W}_i)_{H_0} + \nu \int_0^t (u(s), \mathcal{W}_i)_{V_0} ds \\
+ \int_0^t (B_0(u(s), u(s)) + a |u(s)|^{2\alpha} u(s) - \theta(s) e_3, \mathcal{W}_i)_{H_0} ds = 0,
\]

\[
(\theta(t), \mathcal{W}_i)_{H_1} - (\theta_0, \mathcal{W}_i)_{H_1} + \kappa \int_0^t (\theta(s), \mathcal{W}_i)_{V_1} ds \\
+ \int_0^t (B_1(u(s), \theta(s)) - u \cdot e_3, \mathcal{W}_i)_{H_1} ds = 0.
\]

Consequently, the pair \((u, \theta)\) is a weak solution to \((S - BC_e)\) (see [38, 39] for details).

### 2.3 Energy and gradient estimates

In this section we prove the regularity of the solutions. For this purpose, we shall perform formal calculation using system \((S - BC_e)\). We remark that these calculation can be justified rigorously by performing them first for the Galerkin approximate system and then pass to the limit as in the previous section. We start by revisiting the estimates we established in the previous section dedicated to the existence of weak solutions. Specifically, we show the following

**Proposition 2.1.** Let \((u_0, \theta_0) \in H_0 \times H_1, \alpha \geq 0\) and \(a, \nu, \kappa > 0\). If \((u(t), \theta(t))\) is a weak solution of \((S - BC_e)\), then

\[
u [u]_{H_0}^2 + a [u]_{2\alpha+2}^2 + \kappa [\theta]_{V_1}^2 \leq \epsilon [\theta]_{H_1}^2 + \epsilon^{-1} [u]_{H_0}^2,
\]

(2.15)

and

\[
u [\theta]_{H_1}^2 + \kappa [\theta]_{V_1}^2 \leq \epsilon [\theta]_{H_1}^2 + \epsilon^{-1} [u]_{H_0}^2,
\]

(2.16)

respectively. Observe that for all \(\epsilon > 0\), we have

\[
[u]_{H_0}^2 \leq \epsilon [u]_{2\alpha+2}^2 + 2 \epsilon^{-\frac{a\alpha}{3}} L^2,
\]

(2.17)

Setting \(\epsilon = \frac{2a\alpha\lambda}{a\alpha + 32}\) in (2.17), and summing-up (2.15) and (2.16) with \(\epsilon = \frac{a\lambda}{4}\), we obtain

\[
\frac{d}{dt} \left( [u]_{H_0}^2 + [\theta]_{H_1}^2 \right) + 2 \nu [u]_{V_0}^2 \left( \frac{k}{2} [\theta]_{V_1}^2 + \frac{a}{2} [u]_{H_0}^2 + \frac{k\lambda}{2} [\theta]_{H_1}^2 + a [u]_{2\alpha+2}^2 \right) \\
\leq 2 \left( \frac{2a\alpha\lambda}{a\alpha + 32} \right)^{\frac{a+1}{a}} a L^2.
\]

(2.18)

Thanks to Gronwall’s inequality, we get

\[
[u(t)]_{H_0}^2 + [\theta(t)]_{H_1}^2 \leq \left( [u_0]_{H_0}^2 + [\theta_0]_{H_1}^2 \right) e^{-\frac{\min(a, \kappa)}{2} t} \\
+ \left( \frac{2a\alpha\lambda}{a\alpha + 32} \right)^{\frac{a+1}{a}} a L^2 \left( 1 - e^{-\frac{\min(a, \kappa)}{a} t} \right),
\]

(2.19)
Consequently, we have \( u(t) \in L^\infty(\mathbb{R}^+; H_0) \) and \( \theta(t) \in L^\infty(\mathbb{R}^+; H_0) \). In particular, we have
\[
\limsup_{t \to +\infty} \left( |u(t)|^2_{H_0} + |\theta(t)|^2_{H_1} \right) \leq \Gamma_1. \tag{2.20}
\]

Eventually, integrating (2.18) with respect to time, and using (2.19) we get
\[
\int_0^t \|u(s)\|^2_{V_0} \, ds \leq \frac{\|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1}}{2\nu} + \frac{\Gamma_0}{2\nu} t, \tag{2.21}
\]
and
\[
\int_0^t \|\theta(s)\|^2_{V_1} \, ds \leq \frac{2\left(\|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1}\right)}{\kappa} + \frac{2\Gamma_0}{\kappa}. \tag{2.22}
\]

In addition, we have
\[
\int_0^t \|u(s)\|_{2\alpha+2}^2 \, ds \leq \frac{\|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1}}{a} + \frac{\Gamma_0}{a} t.
\]

Therefore, we infer that \( u(t) \in L^2_{loc}(\mathbb{R}^+; V_0) \), \( \theta(t)L^2_{loc}(\mathbb{R}^+; V_1) \), and \( L^{2\alpha+2}_{loc}(\mathbb{R}^+; L^{2\alpha+2}(\Omega)) \).

In the sequel, we need the following uniform Gronwall type lemma

**Lemma 2.2** ([39]). Let \( \varphi, \phi \) and \( \psi \) be three non-negative locally integrable functions on \([t_0, +\infty[\) such that \( \psi \) is absolutely continuous with \( \psi' \) being local integrable on \([t_0, +\infty[\), and which satisfy
\[
\psi' \leq \varphi \psi + \phi, \quad \text{for all} \quad t \geq t_0,
\]
and
\[
\int_t^{t+s} \varphi(\tau) \, d\tau \leq a_1, \quad \int_t^{t+s} \phi(\tau) \, d\tau \leq a_2, \quad \text{and} \quad \int_t^{t+s} \psi(\tau) \, d\tau \leq a_3, \quad \text{for all} \quad t \geq t_0,
\]
where \( s, a_1, a_2 \) and \( a_3 \) denote positive constants. Then
\[
\psi(t+s) \leq \left(\frac{a_3}{s} + a_2\right) e^{a_1}, \quad \text{for all} \quad t \geq t_0.
\]

Now, we improve these weak solutions by considering more regular initial data for the velocity

**Proposition 2.3.** Let \( (u_0, \theta_0) \in V_0 \times H_1 \), \( \alpha > 1 \) and \( a, \nu, \kappa > 0 \). If \((u(t), \theta(t))\) is a weak solution of \((\mathcal{S} - BC_c)\), then in addition to the conclusion of Proposition 2.1, it holds that \( u(t) \in L^\infty(\mathbb{R}^+; V_0) \cap L^2_{loc}(\mathbb{R}^+, H^2(\Omega)) \).

**Proof.** First, we show that if \((u_0, \theta_0) \in V_0 \times H_1 \) and \( \alpha > 1 \), then \( u(t) \in L^\infty(\mathbb{R}^+; V_0) \). We take the \( H_0 \)-inner product of \((\mathcal{S} - BC_c)_1 \) with \(-\Delta u\) to obtain
\[
\frac{1}{2} \frac{d}{dt} |u|^2_{V_0} + \nu |\Delta u|_2^2 + \int_{\Omega} (u \cdot \nabla) u \cdot (-\Delta u) \, dx + a |u|^\alpha \nabla u|_2^2 \leq \frac{1}{\nu} |\theta|_H^2 + \frac{\nu}{4} |\Delta u|_2^2,
\]
since
\[
-a \int_{\Omega} \Delta u \cdot |u|^2 u \, dx = a \int_{\Omega} |u|^{2\alpha} |\nabla u|_2^2 \, dx + \frac{2a\alpha}{(\alpha+1)^2} \int_{\Omega} |\nabla u|^{(\alpha+1)} \, dx.
\]

Now, using Cauchy-Schwarz, Hölder and Young inequalities and assuming \( \alpha > 1 \), we can write for all \( \epsilon, \epsilon_0 > 0 \)
\[
\left| \int_{\Omega} (u \cdot \nabla) u \cdot (-\Delta u) \, dx \right| \leq \int_{\Omega} |u| |\nabla u|^\frac{\alpha}{2} |\nabla u|^{1-\frac{\alpha}{2}} |\Delta u| \, dx
\leq \|u\| |\nabla u|_{2\alpha} \|\nabla u|^{1-\frac{\alpha}{2}} \|\nabla u|_{\epsilon_0} \|\Delta u|_2
\leq \frac{1}{4\epsilon_0} |u|^\alpha |\nabla u|_2^2 \|u|_{V_0}^{2(1-\frac{\alpha}{2})} + \epsilon_0 |\Delta u|_2^2
\leq \frac{\epsilon}{4\epsilon_0} |u|^\alpha |\nabla u|_2^2 + \frac{\epsilon^{1-\alpha}}{4\epsilon_0} |u|_{V_0}^2 + \epsilon_0 |\Delta u|_2^2. \tag{2.23}
\]
Optimizing in the $\epsilon$’s, we obtain
\[
\frac{d}{dt} \|u\|_{V_0}^2 + \nu \| \Delta u \|^2_2 + a \|u\| | \nabla u|_2^2 \leq \frac{1}{\nu} \|\theta(t)\|_{H_1}^2 + \left( \frac{a\nu^\alpha}{2^2-\alpha} \right)^{1/\alpha} \|u\|_{V_0}^2. \tag{2.24}
\]
Integrating this inequality with respect to time and using the fact that $\theta \in L^\infty(\mathbb{R}^+, H_1)$ and $u(t) \in L_{loc}^2(\mathbb{R}^+; V_0)$, we obtain that $\Delta u \in L_{loc}^2(\mathbb{R}^+; V_0)$, hence $u \in L_{loc}^2(\mathbb{R}^+, H^2(\Omega))$. Also, we have
\[
\|u\|^\alpha | \nabla u| \in L_{loc}^2(\mathbb{R}^+, L^2(\Omega)). \tag{2.25}
\]
Furthermore, using the fact that $\theta \in L^\infty(\mathbb{R}^+, H_1)$ then by means of the Gronwall’s inequality we obtain that $u(t) \in L_{loc}^\infty(\mathbb{R}^+, V_0)$.

Now, we show that actually $u(t) \in L^\infty(\mathbb{R}^+, V_0)$. We proceed in two steps, first for all $t \in [0, 1]$ and then for all $t \geq 0$.

On the one hand, let $t \in [0, 1]$, integrating (2.24), using (2.19) along with (2.21), we obtain
\[
\|u(t)\|_{V_0}^2 \leq \|u_0\|_{V_0}^2 + \|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1} + \frac{\Gamma_1}{\nu} + \frac{\Gamma_2}{2\nu} \left( \|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1} + \Gamma_0 \right).
\]

On the other hand, let $t \geq 0$ and we set
\[
\psi = \|u(t)\|_{V_0}^2, \quad \varphi = \Gamma_2, \quad \text{and} \quad \phi = \frac{1}{\nu} \|\theta(t)\|_{H_1}^2.
\]

Now, integrating (2.18) over $[t, t+1]$ we obtain for all $t \geq 0$
\[
\int_t^{t+1} |u(\tau)|_{V_0}^2 d\tau \leq \frac{1}{2\nu} \left( \|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1} + \Gamma_0 + \Gamma_1 \right),
\]
and
\[
\int_t^{t+1} |\theta(\tau)|_{H_1}^2 d\tau \leq \frac{2}{\kappa\lambda} \left( \|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1} + \Gamma_0 + \Gamma_1 \right).
\]

Eventually, thanks to Lemma 2.2 with $s=1$, we obtain for all $t \geq 1$
\[
|u(t)|_{V_0}^2 \leq \frac{\kappa\lambda + 4}{2\kappa\lambda\nu} \left( \|u_0\|^2_{H_0} + \|\theta_0\|^2_{H_1} + \Gamma_0 + \Gamma_1 \right) \exp(\Gamma_2).
\]

All in all, we get $u(t) \in L^\infty(\mathbb{R}^+, V_0)$. \qed

Now, we show the existence of absorbing ball in $V_0$ for the velocity. For this purpose, we let $t \geq 1$ and
\[
\|u(t)\|_{V_0} \leq \|u(s)\|_{V_0} + \frac{1}{\nu} \int_{t-1}^t |\theta(\tau)|_{H_1}^2 d\tau + \Gamma_2 \int_{t-1}^t |u(\tau)|_{V_0}^2 d\tau. \tag{2.26}
\]

Now, on the one hand, integrating (2.19), we obtain
\[
\frac{1}{\nu} \int_{t-1}^t |\theta(\tau)|_{H_1}^2 d\tau \leq \frac{2}{\nu \min \{a, \kappa\lambda\}} \left( \frac{e^{-\min \{a, \kappa\lambda\}}}{2} - 1 \right) e^{-\frac{\min \{a, \kappa\lambda\}}{2} \tau} + \frac{\Gamma_1}{\nu}. \tag{2.27}
\]

On the other hand, going back to (2.15), and using (2.17), we get
\[
\frac{d}{dt} \|u\|^2_{H_0} + 2\nu \|u\|_{V_0}^2 + a \|u\|_{H_0}^2 \leq a \|\theta\|_{H_1}^2 + 4aL^2.
\]
Integrating this inequality over \([t - 1, t]\), and use (2.27) to obtain
\[
\int_{t-1}^{t} \|u(\tau)\|_{V_0}^2 d\tau \leq \frac{1}{2\nu} \|u(t-1)\|_{H_0}^2 + \frac{a}{2\nu} \int_{t-1}^{t} \|\theta(\tau)\|_{H_1}^2 d\tau + \frac{2aL^2}{\nu} \\
\leq \frac{1}{2\nu} \|u(t-1)\|_{H_0}^2 + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} + \frac{\Gamma_1 + 4aL^2}{2\nu}. \tag{2.28}
\]
Substituting (2.27) and (2.28) in (2.26), we get for \(s \in [t - 1, t]\)
\[
\|u(t)\|_{V_0}^2 \leq \|u(s)\|_{V_0}^2 + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} \\
+ \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}.
\]
Thanks to (2.19), we have, for \(s \in [t - 1, t]\)
\[
\|u(t)\|_{V_0}^2 \leq \|u(s)\|_{V_0}^2 + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} \\
+ \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}.
\]
Integrating this inequality with respect to \(s\) over \([t - 1, t]\), we get
\[
\|u(t)\|_{V_0}^2 \leq \|u(t-1)\|_{H_0}^2 + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} \\
+ \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}.
\]
Now, using again (2.28), we get
\[
\|u(t)\|_{V_0}^2 \leq \frac{1}{2\nu} \|u(t-1)\|_{H_0}^2 + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} \\
+ \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}.
\]
Eventually, using (2.19), the latter inequality gives
\[
\|u(t)\|_{V_0}^2 \leq + \frac{\nu}{\min(a, \kappa)} \left( e^{\frac{\nu}{2} \min(a, \kappa) - 1} - 1 \right) e^{-\frac{\nu}{2} t} \\
+ \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}.
\]
Therefore, one has
\[
\limsup_{t \to +\infty} \|u(t)\|_{V_0}^2 \leq \frac{\Gamma_2 \left( (a + 1)\Gamma_1 + 4aL^2 \right)}{2\nu}. \tag{2.29}
\]
Next, we have

**Proposition 2.4.** Let \((u_0, \theta_0) \in V_0 \cap L^{2\alpha+2}(\Omega) \times H_1, \alpha > 1\) and \(a, \nu, \kappa > 0\). If \((u(t), \theta(t))\) is a weak solution of \((S - BC_e)\), then in addition to the conclusion of Proposition 2.3, it holds that \(u \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^{2\alpha+2}(\Omega))\) and \(\partial_t u \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\Omega))\).

**Proof.** We multiply \((S - BC_e)\) by \(\partial_t u\) and integrate over \(\Omega\) to get thanks to the Cauchy-Schwarz and Young inequalities
\[
\left| \partial_t u \right|_2^2 + \int_{\Omega} (u \cdot \nabla) u \cdot \partial_t u \, dx + \nu \frac{d}{dt} \|\nabla u\|_2^2 + \frac{a}{2\alpha + 2} \frac{d}{dt} \|u\|_{2\alpha+2}^2 \leq \|\theta\|_{H_1}^2 + \frac{1}{4} \|\partial_t u\|_2^2.
\]
In order to show the uniqueness, first we take the
Remark 2.5.
Thus, we have
\[
\int_0^t \| \partial_t u(s) \|_{L^2}^2 \, ds + \frac{a}{2\alpha + 2} \| u(t) \|_{L^{2\alpha + 2}}^{2\alpha + 2} \leq \frac{\nu}{2} \| u_0 \|_{H^\alpha}^2 + \frac{a}{\alpha + 1} \| u_0 \|_{L^{2\alpha + 2}}^{2\alpha + 2} + 2 \int_0^t \left( \| \theta(s) \|_{H^\alpha}^2 + \| u(s) \|_{L^{\alpha}}^\alpha \| \nabla u(s) \|_{L^2}^2 + \| u(s) \|_{H^\alpha}^2 \right) \, ds.
\]
Using (2.19),(2.21), (2.25), and Proposition 2.3, we obtain for all \((u_0, \theta_0) \in V_0 \cap L^{2\alpha + 2}(\Omega) \times H_1 \)
\[
u \in L^\infty(\mathbb{R}^+; L^{2\alpha + 2}(\Omega)), \quad \text{and} \quad \partial_t u \in L^2(\mathbb{R}^+; L^2(\Omega)).
\]
\[
\square
\]

### 2.4 Continuous dependence on the initial data and uniqueness of solutions

In this section, we prove the continuous dependence of the weak solutions on the initial data with respect to the \(H_0 \times H^{-1}(\Omega)\) topology, in particular their uniqueness. For this purpose, let \((u, \theta)\) and \((v, \eta)\) be two weak solutions of system \((S - \mathcal{B}_C)\), and let \(w = u - v\) and \(\xi = \theta - \eta\). It is rather clear that \((w, \xi)\) enjoys
\[
\mathcal{S}_d : \begin{cases}
\partial_t w + \nu \mathcal{A}_0 w + (v \cdot \nabla)w + (w \cdot \nabla)u + a (u^{2\alpha}u - a |v|^{2\alpha}v + \nabla \xi = \xi \, e_3, \\
\partial_t \xi + \kappa \mathcal{A}_1 \xi + (v \cdot \nabla)\xi + (w \cdot \nabla)\theta = w \cdot e_3, \\
\nabla \cdot w = 0, \quad w_{|t=0} = w_0 = u_0 - v_0, \quad \xi_{|t=0} = \xi_0 = \theta_0 - \eta_0,
\end{cases}
\]
with the associated periodic BC obtained from \(\mathcal{B}_C\) satisfied by \((u, \theta)\) and \((v, \eta)\). In the sequel, we will need the following strong monotonicity form (see, e.g., [5]): There exists a positive constant \(\delta(\alpha)\) such that
\[
\delta |u - v|^2 (|u| + |v|)^{2\alpha} \leq (|u|^{2\alpha}u - |v|^{2\alpha}v) \cdot (u - v).
\]
In order to show the uniqueness, first we take the \(H_0\)-inner product of \((S - \mathcal{B}_c)\) with \(w\) and integrate over \(\Omega\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2_{H_0} + \nu \| w \|^2_{V_0} + \int_\Omega (v \cdot \nabla) w \cdot u \, dx + a \int_\Omega |u|^{2\alpha} u \cdot w \, dx - a \int_\Omega |v|^{2\alpha} v \cdot w \, dx = \int_\Omega \xi (w \cdot e_3) \, dx.
\]
Next, let \(\psi_1\) and \(\psi_2\) be the unique periodic functions solutions of the elliptic equations \(\theta = \Delta \psi_1\) and \(\eta = \Delta \psi_2\) satisfying \(\int_\Omega \psi_1 dx = \int_\Omega \psi_2 dx = 0\). Since \(\theta\) and \(\eta\) are odd functions with respect of the \(z\)-variable then \(\psi_1\) and \(\psi_2\) enjoy the same property. Now, let \(\psi = \psi_1 - \psi_2\), therefore \(\psi\) satisfies the following equation
\[
\partial_t \Delta \psi + \kappa \mathcal{A}_1 \Delta \psi + (v \cdot \nabla) \Delta \psi + (w \cdot \nabla) \Delta \psi_1 = w \cdot e_3.
\]
Taking the duality action of the latter equation with \(\psi\) gives
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \psi \|^2_{L^2} + \kappa \| \Delta \psi \|^2_{L^2} + \int_\Omega (v \cdot \nabla) \Delta \psi \psi \, dx + \int_\Omega (w \cdot \nabla) \Delta \psi_1 \psi \, dx = \int_\Omega \psi (w \cdot e_3) \, dx.
\]
Now, using Cauchy-Schwarz and Young inequalities, we can write
\[
\int_{\Omega} \xi (w \cdot e_3) \, dx = \int_{\Omega} \Delta \psi (w \cdot e_3) \, dx \leq |\Delta \psi|_{L_2} |w|_{H^0} \leq \frac{\kappa}{3} |\Delta \psi|_{H^1}^2 + \frac{3}{\kappa} |w|_{H^0}^2. \tag{2.34}
\]

Next, since \( \nabla \cdot w = 0 \) then by integrating by parts and using the generalized Hölder inequality, we have for all \( \alpha > 1 \)
\[
| \int_{\Omega} (w \cdot \nabla) u \cdot w \, dx | = | - \int_{\Omega} (w \cdot \nabla) w \cdot u \, dx | \leq \int_{\Omega} |u| |\nabla w| \, dx = \int_{\Omega} |u| \frac{1}{2} |w|^{1 - \frac{1}{2}} |\nabla w| \, dx
\leq \frac{\kappa}{2} |u|_{2\alpha} |w|^{1 - \frac{1}{2}} |\nabla w|_{V_0}
\leq \frac{3}{2} \kappa |u| |\nabla w|^2_{L^2} |w|_{H^0}^{2(1 - \frac{1}{2})} + \nu \frac{1}{2} |w|_{V_0}^2
\leq a \delta |u|^{\alpha} |\nabla w|^2_{L^2} + (a \delta \nu^\alpha)^{1/\alpha} |w|^2_{H^0} + \frac{\nu}{2} |w|_{V_0}^2.
\]

Similarly, since \( \nabla \cdot w = 0 \) once again we integrate by parts and use the Hölder and Gagliardo-Nirenberg inequalities we deduce
\[
| \int_{\Omega} (w \cdot \nabla) \Delta \psi \psi \, dx | = | - \int_{\Omega} (w \cdot \nabla) \psi \Delta \psi \, dx |
\leq |w|_{L_6} |\nabla \psi|_{L_3} |\Delta \psi|_{L_2}
\leq |w|_{L_6} |\Delta \psi|_{L_2}^\frac{1}{2} |\nabla \psi|_{L_2}^\frac{1}{2} |\psi|_{H^1}
\leq \frac{\nu}{2} |w|_{L_6}^2 + \frac{3}{\kappa} |\Delta \psi|_{L_2}^2 |\nabla \psi|_{L_2}^2 |\psi|_{H^1}
\leq \frac{\nu}{2} |\nabla w|_{L_2}^2 + \frac{\kappa}{3} |\Delta \psi|_{L_2}^2 + \frac{12}{\kappa^2} |\nabla \psi|_{L_2}^4 |\psi|_{H^1}^4
\leq \frac{\nu}{2} |\nabla w|_{L_2}^2 + \frac{\kappa}{3} |\Delta \psi|_{L_2}^2 + \frac{3}{\kappa} |\nabla \psi|_{L_2}^2.
\]
where \( c \) is a constant depending on \( |u_0|_{H^0} \) and \( |\theta_0|_{H^1} \) (see (2.19)). Also, we have the following
\[
\int_{\Omega} (v \cdot \nabla) \Delta \psi \psi \, dx = - \int_{\Omega} (v \cdot \nabla) \psi \Delta \psi \, dx
\leq |v|_{L_6} |\nabla \psi|_{L_3} |\Delta \psi|_{L_2}
\leq |v|_{V_0} |\Delta \psi|_{L_2}^\frac{1}{2} |\nabla \psi|_{L_2}^\frac{1}{2}
\leq \frac{\kappa}{3} |\Delta \psi|_{L_2}^2 + \frac{3}{\kappa} |v|_{V_0}^2 |\nabla \psi|_{L_2}^2
\]
Thanks to (2.31), we have
\[
\delta |u|^{\alpha} |w|^2_{L^2} \leq \delta |(|u| + |v|)^\alpha |w|^2_{L^2} \leq \int_{\Omega} |u|^{2\alpha} - |v|^{2\alpha} \cdot w \, dx.
\]
Summing-up (2.32) and (2.33) and collecting the estimates above, there exist \( c_1 > 0 \) depending only on \( \alpha, \nu \) and \( \kappa \) and \( c_2 > 0 \) depending on \( \alpha, \nu, \kappa \) and the \( L^2 \) norm of the initial data such that
\[
\frac{d}{dt} \left( |w|^2_{H^0} + |\xi|^2_{H^{-1}} \right) \leq \max \left\{ c_1 + c_2 + |v|^2_{V_0} \right\} \left( |w|^2_{H^0} + |\xi|^2_{H^{-1}} \right).
\]
Eventually, Gronwall’s Lemma combined to the fact that \( v(t) \in L^2_{\text{loc}}(\mathbb{R}^+; V_0) \) (see (2.21)), shows the continuous dependence of the solutions on the initial data in \( L^\infty([0, T], H^0 \times H^{-1}(\Omega)) \), in particular their uniqueness.

**Remark 2.6.** Let us mention that Theorem 1.2 can be easily extended to the case where the initial data \((u_0, \theta_0) \in V_0 \times V_1\) where it can be shown that the solution satisfies
\[
u(x, t) \in C^0_b(\mathbb{R}^+; V_0) \cap L^2_{\text{loc}}(\mathbb{R}^+; V_0 \cap H^2(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}^+; L^{2\alpha+2}(\Omega)),
\]
and
\[
\theta(x, t) \in C^0_b(\mathbb{R}^+; V_1) \cap L^2_{\text{loc}}(\mathbb{R}^+; V_1 \cap H^2(\Omega)).
\]
The proof of the uniqueness given in section 2.4 remains unchanged. Indeed, the necessary estimate is obtained by taking the the $V_1$-inner product of $(S - BC)_2$ with $-\Delta \theta$ to obtain

$$\frac{1}{2} \frac{d}{dt} \| \theta \|^2_{V_1} + \kappa \| \Delta \theta \|^2_2 + \int_{\Omega} (u \cdot \nabla) \theta \cdot (-\Delta \theta) \, dx = \int_{\Omega} u_3 \Delta \theta \, dx$$

Now,

$$\int_{\Omega} u_3 \Delta \theta \, dx \leq \frac{\kappa}{4} \| \Delta \theta \|^2_2 + \frac{\kappa}{4} \| u \|^2_{H_0} \leq \frac{\kappa}{4} \| \Delta \theta \|^2_2 + \text{Const.}$$

and (for instance)

$$\int_{\Omega} (u \cdot \nabla) \theta \cdot (-\Delta \theta) \, dx \leq \| \nabla u \|^2_2 + \frac{\kappa}{4} \| \Delta \psi \|^2_2 + \frac{\kappa}{4} \| \nabla \psi \|^2_2 \leq \text{Const.} + \frac{\kappa}{4} \| \Delta \psi \|^2_2 + \frac{\kappa}{4} \| \nabla \psi \|^2_2.$$  

Collecting these estimates, and using (2.16), one can show immediately that $\theta \in L^\infty(\mathbb{R}^+, V_1)$. Furthermore, it can be also shown the existence of absorbing ball in $V_0 \times V_1$ following the same argument used for the velocity above.

### 3 Data Assimilation

Data assimilation belongs to the family of semi-empirical methods that aim to enhance the prediction’s quality of the physical phenomena at hand, by synchronizing information collected from measurements at coarse spatial scales with the theoretical models. This method was originally proposed for atmospheric predictions such as weather forecasting in [11]. A new approach was introduced in [4] based on ideas from control theory (see also [32, 33]) consisting in the introduction of a feedback control term that nudges the large spatial scales of the model towards those of the reference solution. In the context of data assimilation, the large spatial scales of the reference solution are constructed by an interpolation operator from the spatial coarse scale measurements. This new approach was applied for several models including the 3D Navier-Sotokes-α in [1], the 2D Bénard-convection where only measurement of velocity is needed in [13], the 2D Navier-Stokes where the authors show the convergence with measurements of a single velocity component in [14], the 3D Brinkman-Forchheimer-extended Darcy model in [30] etc. Other features of this new approach were developed and investigated in several recent publications such as [7, 12, 18, 19, 22, 23, 31] and references therein. In this paragraph, we present the data assimilation algorithm applied to our system $(S - BC_e)$ and state the result that can be readily obtained by combining Theorems 1.1 and 1.2 and techniques from [16]. Specifically, we state the convergence of the algorithm in the case of system $(S - BC)$ with measurement of only two components of the velocity. Before going further, let us present the idea of the algorithm. Briefly speaking, the observation are introduced into the system $(S - BC)$ by the mean of an interpolant operator $I_h(u(t))$ where $u(t)$ denotes the velocity part of the solution $(u(t), \theta(t))$. This operator interpolates the observations of the system $(S - BC)$ at coarse scale spatial resolution of size $h$ to be specified: We shall consider two types of interpolants on the space domain $\Omega_r := [0, L] \times [0, L] \times [0, 1]$:

$$I_h : H^1(\Omega_r) \to L^2(\Omega_r) \text{ satisfying } \| \psi - I_h(\psi) \|^2_{L^2} \leq c_0 \, h^2 \| \nabla \psi \|^2_{L^2}, \quad (3.1)$$

for all vectors $\psi = (\psi_1, \psi_2) \in (H^1(\Omega_r))^2$, and

$$I_h : H^2(\Omega_r) \to L^2(\Omega_r) \text{ satisfying } \| \psi - I_h(\psi) \|^2_{L^2} \leq c_0 \, h^2 \| \nabla \psi \|^2_{L^2} + c_1 \, h^4 \| \Delta \psi \|^2_{L^2}, \quad (3.2)$$

for all vectors $\psi = (\psi_1, \psi_2) \in (H^2(\Omega_r))^2$, where $c_0$ and $c_1$ denote dimensionless nonnegative constants. We refer the reader to [1, 3, 30] for physical examples of such interpolants. To simplify our notation, we will denote $v_\perp := (v_1, v_2)^T$ for all three-components vector $v = (v_1, v_2, v_3)^T$. Assume that the initial data $(u_0, \theta_0)$ of system $(S - BC)$ is missing. The idea is to recover this initial data as accurately as possible. This will be
 achieved by constructing a solution \((v(t), \eta(t))\) from the observations that satisfies

\[
s_{da} : \begin{cases}
\partial_t v_\perp - \nu \Delta v_\perp + B_0(v, v_\perp) + a |v|^{2\alpha} v_\perp + \nabla_\perp q = \mu I_h(u_\perp - v_\perp), \\
\partial_t v_3 - \nu \Delta v_3 + B_0(v, v_3) + a |v|^{2\alpha} v_3 + \partial_3 q = \eta e_3, \\
\partial_t \eta - \kappa \Delta \eta + B_1(v, \eta) = v \cdot e_3, \\
\nabla \cdot v = 0, \quad u_{|t=0} = v_0, \quad \eta_{|t=0} = \eta_0,
\end{cases}
\]

subjected to the set of boundary conditions \(BC\) (with \((u, \theta, p)\) replaced by \((v, \eta, q)\)). Observe that we do not use measurements of the temperature and the third component of the velocity. The parameter \(\mu > 0\) is the nudging parameter and \(h\) is the resolution parameter of the collected data. To avoid technical difficulties, one proceeds as for the original system \((S - BC)\) by extending the domain from \(\Omega_r\) to \(\Omega\) and therefore extending the BC to \(BC_e\). It is important to notice that the operator \(I_h\) should be as well extended to act on functions in \(H^1(\Omega)\) and \(H^1(\Omega)\) since the collected measurement are on the reference solution of \((S - BC)\). Specifically, its extension should be even since it acts only on the first two components of the velocity.

The extended data assimilation system \((S_{da} - BC_e)\) can be shown to admit unique weak and strong solutions equivalently to \((S - BC_e)\). Most importantly, it can be shown that for a space resolution \(h\) small enough and nudging parameter \(\mu\) large enough the weak solution of \((v(t), \eta(t))\) of \((S_{da} - BC_e)\) converges asymptotically in time at an exponential rate to the reference solution \((u(t), \theta(t))\) of system \((S - BC_e)\) in \(H^1_0(\Omega)\) for interpolants of type (3.1) for all \(\alpha > 1\). Equivalently, the strong solution \((v(t), \eta(t))\) of \((S_{da} - BC_e)\) converges asymptotically in time at an exponential rate to the reference solution \((u(t), \theta(t))\) of system \((S - BC_e)\) in \(V_\alpha \times H^{-1}(\Omega)\) for interpolants of type (3.2) but only for \(1 < \alpha < 2\).

The proof is based on a combination of the arguments presented in the previous sections and ideas from [15, 30].

Acknowledgments

This publication was made possible by NPRP grant# S-0207-200290 from the Qatar National Research Fund (a member of Qatar Foundation). The findings herein reflect the work, and are solely the responsibility, of the authors. E.S.T. would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme “Mathematical aspects of turbulence: where do we stand?” where part of the work on this paper was undertaken. This work was supported in part by EPSRC grant no EP/R014604/1.

References

[1] D. A. Albanez, H. J. Nussenzveig Lopes, and E. S. Titi, Continuous data assimilation for the three-dimensional Navier–Stokes–\(\alpha\) model, Asymptotic Analysis, 97 (2016), pp. 139–164.

[2] S. Antontsev and H. de Oliveira, The Navier–Stokes problem modified by an absorption term, Applicable Analysis, 89 (2010), pp. 1805–1825.

[3] A. Azouani, E. Olson, and E. S. Titi, Continuous data assimilation using general interpolant observables, Journal of Nonlinear Science, 24 (2014), pp. 277–304.

[4] A. Azouani and E. S. Titi, Feedback control of nonlinear dissipative systems by finite determining parameters - a reaction-diffusion paradigm, Evolution Equations and Control Theory, 3 (2014), pp. 579–594.

[5] J. W. Barrett and W. Liu, Finite element approximation of the parabolic p-laplacian, SIAM Journal on Numerical Analysis, 31 (1994), pp. 413–428.
H. Bessaih, S. Trabelsi, and H. Zorgati, *Existence and uniqueness of global solutions for the modified anisotropic 3d Navier-Stokes equations*, ESAIM: Mathematical Modelling and Numerical Analysis, 50 (2016), pp. 1817–1823.

A. Biswas, C. Foias, C. F. Mondaini, and E. S. Titi, *Downscaling data assimilation algorithm with applications to statistical solutions of the Navier-Stokes equations*, in Annales de l’Institut Henri Poincaré C, Analyse non linéaire, vol. 36, Elsevier, 2019, pp. 295–326.

X. Cai and Q. Jiu, *Weak and strong solutions for the incompressible Navier–Stokes equations with damping*, Journal of Mathematical Analysis and Applications, 343 (2008), pp. 799–809.

A. Celebi, V. Kalantarov, and D. Uğurlu, *Continuous dependence for the convective Brinkman–Forchheimer equations*, Applicable Analysis, 84 (2005), pp. 877–888.

A. O. Celebi, V. K. Kalantarov, and D. Uğurlu, *On continuous dependence on coefficients of the Brinkman–Forchheimer equations*, Applied mathematics letters, 19 (2006), pp. 801–807.

R. Daley, *Atmospheric Data Analysis*, no. 2, Cambridge University Press, 1993.

S. Desamsetti, H. P. Dasari, S. Langodan, E. S. Titi, O. Knio, and I. Hoteit, *Efficient dynamical downscaling of general circulation models using continuous data assimilation*, Quarterly Journal of the Royal Meteorological Society, 145 (2019), pp. 3175–3194.

A. Farhat, M. S. Jolly, and E. S. Titi, *Continuous data assimilation for the 2d Bénard convection through velocity measurements alone*, Physica D: Nonlinear Phenomena, 303 (2015), pp. 59–66.

A. Farhat, E. Lunasin, and E. S. Titi, *Abridged continuous data assimilation for the 2d Navier–Stokes equations utilizing measurements of only one component of the velocity field*, Journal of Mathematical Fluid Mechanics, 18 (2016), pp. 1–23.

A. Farhat, *Data assimilation algorithm for 3d Bénard convection in porous media employing only temperature measurements*, Journal of Mathematical Analysis and Applications, 438 (2016), pp. 492–506.

A. Farhat, *Continuous data assimilation for a 2d Bénard convection system through horizontal velocity measurements alone*, Journal of Nonlinear Science, 27 (2017), pp. 1065–1087.

C. Foias, O. Manley, and R. Temam, *Attractors for the Bénard problem: existence and physical bounds on their fractal dimension*, Nonlinear Analysis: Theory, Methods & Applications, 11 (1987), pp. 939–967.

B. García-Archilla, J. Novo, and E. S. Titi, *Uniform in time error estimates for a finite element method applied to a downscaling data assimilation algorithm for the Navier–Stokes equations*, SIAM Journal on Numerical Analysis, 58 (2020), pp. 410–429.

M. Geshou, E. Olson, and E. S. Titi, *A computational study of a data assimilation algorithm for the two-dimensional Navier-Stokes equations*, Communications in Computational Physics, 19 (2016), pp. 1094–1110.

W. J. Goux, L. A. Verkruyse, and S. J. Saltert, *The impact of Rayleigh–Bénard convection on nmr pulsed-field-gradient diffusion measurements*, Journal of Magnetic Resonance (1969), 88 (1990), pp. 609–614.

C. Hsu and P. Cheng, *Thermal dispersion in a porous medium*, International Journal of Heat and Mass Transfer, 33 (1990), pp. 1587–1597.

H. A. Ibdah, C. F. Mondaini, and E. S. Titi, *Fully discrete numerical schemes of a data assimilation algorithm: uniform-in-time error estimates*, IMA Journal of Numerical Analysis, 40 (2020), pp. 2584–2625.
[23] M. S. Jolly, V. R. Martinez, E. J. Olson, and E. S. Titi, Continuous data assimilation with blurred-in-time measurements of the surface quasi-geostrophic equation, Chinese Annals of Mathematics, Series B, 40 (2019), pp. 721–764.

[24] V. K. Kalantarov and S. Zelik, Smooth attractors for the Brinkman–Forchheimer equations with fast growing nonlinearities, Communications on Pure and Applied Analysis, 11 (2012), pp. 2037–2054.

[25] J. L. Lions, Quelques Méthodes De Résolution Des Problemes Aux Limites Non Linéaires, Paris, Dunod, (1969).

[26] H. Liu, D. Bian, and X. Pu, Global well-posedness of the 3d Boussinesq-MHD system without heat diffusion, Zeitschrift für angewandte Mathematik und Physik, 70 (2019), pp. 1–19.

[27] Y. Liu and C. Lin, Structural stability for Brinkman–Forchheimer equations., Electronic Journal of Differential Equations (EJDE)[electronic only], 2007 (2007), pp. Paper–No.

[28] M. Louaked, N. Seloula, S. Sun, and S. Trabelsi, A pseudocompressibility method for the incompressible Brinkman-Forchheimer equations., Differ. Integral Equ., 28 (2015), pp. 361–382.

[29] M. Louaked, N. Seloula, and S. Trabelsi, Approximation of the unsteady Brinkman-Forchheimer equations by the pressure stabilization method., Numer. Methods Partial Differ. Equations, 33 (2017), pp. 1949–1965.

[30] P. A. Markowich, E. S. Titi, and S. Trabelsi, Continuous data assimilation for the three-dimensional Brinkman–Forchheimer-extended Darcy model, Nonlinearity, 29 (2016), p. 1292.

[31] C. F. Mondaini and E. S. Titi, Uniform-in-time error estimates for the postprocessing Galerkin method applied to a data assimilation algorithm, SIAM Journal on Numerical Analysis, 56 (2018), pp. 78–110.

[32] E. Olson and E. S. Titi, Determining modes for continuous data assimilation in 2d turbulence, Journal of statistical physics, 113 (2003), pp. 799–840.

[33] ———, Determining modes and Grashof number in 2d turbulence: a numerical case study, Theoretical and Computational Fluid Dynamics, 22 (2008), pp. 327–339.

[34] Y. Ouyang and L.-e. Yang, A note on the existence of a global attractor for the Brinkman–Forchheimer equations, Nonlinear Analysis: Theory, Methods & Applications, 70 (2009), pp. 2054–2059.

[35] L. E. Payne and B. Straughan, Convergence and continuous dependence for the Brinkman–Forchheimer equations, Studies in Applied Mathematics, 102 (1999), pp. 419–439.

[36] H. Sohr, The Navier-Stokes equations: An Elementary Functional Analytic Approach, Springer Science & Business Media, 2012.

[37] B. Straughan, Stability And Wave Motion In Porous Media, vol. 165, Springer Science & Business Media, 2008.

[38] R. Temam, Navier-Stokes Equations: Theory And Numerical Analysis, vol. 343, American Mathematical Soc., 2001.

[39] ———, Infinite-Dimensional Dynamical Systems In Mechanics And Physics, vol. 68, Springer Science & Business Media, 2012.

[40] E. S. Titi and S. Trabelsi, Global well-posedness of a 3d mhd model in porous media, Journal of Geometric Mechanics, 11 (2019), pp. 621–637.

[41] D. Uğurlu, On the existence of a global attractor for the Brinkman–Forchheimer equations, Nonlinear Analysis: Theory, Methods & Applications, 68 (2008), pp. 1986–1992.
[42] B. Wang and S. Lin, Existence of global attractors for the three-dimensional Brinkman–Forchheimer equation, Mathematical Methods in the Applied Sciences, 31 (2008), pp. 1479–1495.

[43] J.-J. Wang and X.-q. Ma, A study on the chaotic behavior of LNG after stratification in main stream region of storage tank, Journal of Hydrodynamics (Ser. A), 1 (2007), p. 004.

[44] Y. You, C. Zhao, and S. Zhou, The existence of uniform attractors for 3d Brinkman–Forchheimer equations, Discrete & Continuous Dynamical Systems-A, 32 (2012), pp. 3787–3800.