Crossed products of dual operator spaces by locally compact groups

by

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Abstract. If \( \alpha \) is an action of a locally compact group \( G \) on a dual operator space \( X \), then two generally different notions of crossed products are defined, namely the Fubini crossed product \( X \rtimes^F_{\alpha} G \) and the spatial crossed product \( X \rtimes_{\alpha} G \). It is shown that \( X \rtimes^F_{\alpha} G = X \rtimes_{\alpha} G \) if and only if the dual comodule action \( \hat{\alpha} \) of the group von Neumann algebra \( L(G) \) on \( X \rtimes^F_{\alpha} G \) is non-degenerate. As an application, this yields an alternative proof of the result of Crann and Neufang (2019) that the two notions coincide when \( G \) satisfies the approximation property (AP) of Haagerup and Kraus. Also, it is proved that the \( L(G) \)-bimodules \( \text{Bim}(J^\perp) \) and \( (\text{Ran} J)^\perp \) defined by Anoussis, Katavolos and Todorov (2019) for a left ideal \( J \) of \( L^1(G) \) are respectively isomorphic to \( J^\perp \rtimes G \) and \( J^\perp \rtimes^F G \). Therefore a necessary and sufficient condition for \( \text{Bim}(J^\perp) = (\text{Ran} J)^\perp \) is deduced from the main result.

1. Introduction. Let \( s \mapsto \alpha_s \), \( s \in G \), be an action of a locally compact group \( G \) on a von Neumann algebra \( M \) by unital normal \(*\)-automorphisms. Then \( \alpha \) defines a unital normal \(*\)-monomorphism \( \alpha \colon M \to M \otimes L^\infty(G) \) via

\[
\langle \alpha(x), \omega \otimes f \rangle = \int_G \langle \alpha_s^{-1}(x), \omega \rangle f(s) \, ds
\]

for \( x \in M \), \( \omega \in M_* \) and \( f \in L^1(G) \). The crossed product \( M \rtimes_{\alpha} G \) of \( M \) by \( \alpha \) is defined to be the von Neumann subalgebra of \( M \overline{\otimes} B(L^2(G)) \) generated by \( \alpha(M) \) and \( C1 \overline{\otimes} L(G) \), where \( L(G) = \lambda(G)^\prime \) is the left group von Neumann algebra.

There are two alternative ways to describe \( M \rtimes_{\alpha} G \):

1. On the one hand, from the covariance relations

\[
\alpha(\alpha_s(x)) = (1 \otimes \lambda_s)\alpha(x)(1 \otimes \lambda_s)^*, \quad x \in M, \ s \in G,
\]

it follows that \( M \rtimes_{\alpha} G \) is the \( w^*\)-closure of the linear span of the products
of the form \((1 \otimes \lambda_s)\alpha(x), x \in M, s \in G\). That is, \(M \rtimes_\alpha G\) is the normal \(L(G)\)-bimodule generated by \(\alpha(M)\).

(2) On the other hand, from the Digernes–Takesaki theorem (see e.g. [21, Chapter X, §1, Corollary 1.22]) we have

\[ M \rtimes_\alpha G = \{ x \in M \otimes B(L^2(G)) : (\alpha_s \otimes \text{Ad}\rho_s)(x) = x, \forall s \in G \}, \]

where \(\rho\) is the right regular representation of \(G\).

These two alternative descriptions of the crossed product \(M \rtimes_\alpha G\) can both be generalized for group actions on dual operator spaces. Therefore, for a group action \(\alpha\) on a dual operator space \(X \subseteq B(H)\) by \(w^*\)-continuous completely isometric isomorphisms, one can define again a \(w^*\)-continuous complete isometry \(\alpha: X \to X \overline{\otimes} L^\infty(G)\) and obtain the spatial crossed product

\[ X \backslash_\alpha G = \text{span}^{w^*}\{ (1_H \otimes \lambda_s)\alpha(x) : s \in G, x \in X \} \]

and the Fubini crossed product

\[ X \rtimes_\alpha G = \{ x \in X \otimes B(L^2(G)) : (\alpha_s \otimes \text{Ad}\rho_s)(x) = x, \forall s \in G \}. \]

The equality \(X \rtimes_\alpha G = X \backslash_\alpha G\) does not hold in general (see Remark 3.6). However, P. Salmi and A. Skalski [17] have proved a generalization of the Digernes–Takesaki theorem in the case of a group action on a \(W^*\)-TRO by TRO-morphisms. More precisely, they proved that if \(X\) is a \(W^*\)-TRO, \(G\) is an arbitrary locally compact group and \(\alpha\) is a \(W^*\)-TRO morphism such that \(\alpha: X \to X \overline{\otimes} L^\infty(G)\) is a non-degenerate TRO-morphism, then \(X \rtimes_\alpha G = X \backslash_\alpha G\) (for the definition of a non-degenerate TRO-morphism see e.g. Remark 3.11).

On the other hand, J. Crann and M. Neufang [6] proved that if \(G\) is a locally compact group with the approximation property of U. Haagerup and J. Kraus [9] (for more details see Section 4 below) and \(X\) is an arbitrary dual operator space, then \(X \rtimes_\alpha G = X \backslash_\alpha G\). Also, they obtained the analogous result for group actions on general operator spaces (not necessarily dual), generalizing a result of O. Uuye and J. Zacharias for discrete group actions on operator spaces (see [22]).

The analogue of the Fubini crossed product, in the setting of general operator spaces, had already been studied by M. Hamana [10] who proved a Takesaki-type duality theorem for Fubini crossed products.

The main purpose of the present paper is to characterize those group actions on dual operator spaces for which the Fubini crossed product coincides with the spatial crossed product. More precisely, it is shown that, for an action \(\alpha\) of a locally compact group \(G\) on a dual operator space \(X\), we have \(X \rtimes_\alpha G = X \backslash_\alpha G\) if and only if the dual comodule action \(\hat{\alpha}\) of the group von Neumann algebra \(L(G)\) on \(X \rtimes_\alpha G\) is non-degenerate (see Definitions 3.3 and 3.13). As applications this yields alternative proofs of the
results of Crann and Neufang (see Proposition 4.3) and Salmi and Skalski (see Remark 3.19) mentioned above. We also prove that the $L(G)$-bimodules $\text{Bim}(J^\perp)$ and $(\text{Ran} J)^\perp$ defined in [2] for a closed left ideal $J$ of $L^1(G)$ can both be realized as crossed products. This fact provides a perhaps more conceptual perspective for these bimodules; it also yields a necessary and sufficient condition for their equality, as an application of our theorem.

In Section 2 below we begin with preliminaries and notation on tensor products, comodules over Hopf-von Neumann algebras and crossed products of von Neumann algebras. In Section 3 we compare the two notions of crossed products under discussion and we prove our main result (see Theorem 3.17). In the same section we obtain an alternative proof of the result of Salmi and Skalski. In Section 4 we give an alternative proof of the result of Crann and Neufang. In Section 5 it is shown that $\text{Bim}(J^\perp)$ and $(\text{Ran} J)^\perp$ arise naturally as crossed products and a necessary and sufficient condition for their equality is obtained.

2. Preliminaries and notation. Let $X \subseteq B(H)$ and $Y \subseteq B(K)$ be dual operator spaces, i.e. $w^*$-closed subspaces of $B(H)$ and $B(K)$ respectively, where $H$, $K$ are Hilbert spaces. The spatial tensor product of $X$ and $Y$ is the subspace of $B(H \otimes K)$ defined by

$$X \otimes Y = \text{span}^{w^*}\{x \otimes y : x \in X, y \in Y\},$$

where $(x \otimes y)(h \otimes k) = (xh) \otimes (yk)$ for $h \in H$, $k \in K$.

The Fubini tensor product of $X$ and $Y$ is the space

$$X \otimes_F Y = \{x \in B(H \otimes K) : (\text{id}_{B(H)} \otimes \phi)(x) \in X, (\omega \otimes \text{id}_{B(K)})(x) \in Y, \forall \omega \in B(H)^*, \forall \phi \in B(K)^*\}.$$

Obviously, $X \otimes Y \subseteq X \otimes_F Y$.

If $M$ is an injective von Neumann algebra (in particular, of type I) then $M$ has the dual slice map property, i.e. $X \bar{\otimes} M = X \otimes_F M$ for every dual operator space $X$ (see [12 Theorem 1.9]) and this implies that

$$X \otimes_F Y = (X \bar{\otimes} B(K)) \cap (B(H) \bar{\otimes} Y).$$

Also, $M \bar{\otimes} N = M \otimes_F N$ for any von Neumann algebras $M$ and $N$ (see [7 Theorem 7.2.4]).

A Hopf-von Neumann algebra is a pair $(M, \Delta)$, where $M$ is a von Neumann algebra and $\Delta : M \to M \bar{\otimes} M$ is a normal unital $^*$-monomorphism called the comultiplication of $M$, such that the coassociativity rule holds:

$$(\Delta \otimes \text{id}_M) \circ \Delta = (\text{id}_M \otimes \Delta) \circ \Delta.$$
$X \overline{\otimes}_F M$ which satisfies
\[(\alpha \otimes \text{id}_M) \circ \alpha = (\text{id}_X \otimes \Delta) \circ \alpha.\]
In this case, we say that $\alpha$ is an action of $M$ on $X$ or an $M$-action on $X$.

A $w^*$-closed subspace $Y$ of $X$ is called an $M$-subcomodule of $X$ if $\alpha(Y) \subseteq Y \overline{\otimes}_F M$. In this case we write $Y \leq X$ and $Y$ is indeed an $M$-comodule for the action $\alpha|_Y$.

An $M$-comodule morphism between $M$-comodules $(X, \alpha)$ and $(Y, \beta)$ is a $w^*$-$w^*$-continuous complete contraction $\phi: X \to Y$ such that
\[\beta \circ \phi = (\phi \otimes \text{id}_M) \circ \alpha.\]

An $M$-comodule morphism is called an $M$-comodule monomorphism (resp. isomorphism) if it is a complete isometry (resp. surjective complete isometry), and we write $X \cong Y$ for isomorphic $M$-comodules.

An $M$-comodule isomorphism $\pi: N \to N \overline{\otimes} M$ such that
\[(\pi \otimes \text{id}_M) \circ \pi = (\text{id}_N \otimes \Delta) \circ \pi\]
will be called a $W^*$-$M$-action of $M$ on $N$, and $(N, \pi)$ will be called a $W^*$-$M$-comodule. The terms $W^*$-$M$-subcomodule, $W^*$-$M$-comodule morphism etc. are defined accordingly.

**Remark 2.1.** Let $(M, \Delta)$ be a Hopf–von Neumann algebra. Every $M$-comodule $(X, \alpha)$ is isomorphic to an $M$-subcomodule of a canonical $M$-comodule, which may be taken to be the $W^*$-$M$-comodule $(B(H) \overline{\otimes} M, \text{id}_{B(H)} \otimes \Delta)$ for some Hilbert space $H$. Indeed, the image $\alpha(X)$ under the action $\alpha$ is an $M$-subcomodule of the canonical $M$-comodule $X \overline{\otimes}_F M$, since
\[(\text{id}_X \otimes \Delta) \circ \alpha(X) = (\alpha \otimes \text{id}_M) \circ \alpha(X) \subseteq \alpha(X) \overline{\otimes}_F M\]
and $\alpha$ is an $M$-comodule isomorphism of $X$ onto $\alpha(X)$ and thus
\[X \cong \alpha(X) \leq X \overline{\otimes}_F M.\]
Furthermore, we may suppose that $X \subseteq B(H)$ as a $w^*$-closed subspace for some Hilbert space $H$, thus $X \overline{\otimes}_F M \subseteq B(H) \overline{\otimes} M$.

**Remark 2.2.** For any Hopf–von Neumann algebra $(M, \Delta)$, the predual $M_*$ of $M$ becomes naturally a Banach algebra with the product defined as
\[f \cdot g = (f \otimes g) \circ \Delta\]
for $f, g \in M_*$. Furthermore, an $M$-comodule $(X, \alpha)$ becomes an $M_*$-Banach module with the module operation defined as
\[f \cdot x = (\text{id}_X \otimes f) \circ \alpha(x), \quad f \in M_*, \ x \in X.\]
Also, it is easy to see that a \( \omega^* \)-continuous complete contraction \( \phi: X \to Y \) between two \( M \)-comodules \( X \) and \( Y \) is an \( M \)-comodule morphism if and only if \( \phi \) is an \( M_\omega \)-module homomorphism.

Let \( (X, \alpha) \) be an \( M \)-comodule over a Hopf–von Neumann algebra \( (M, \Delta) \). The fixed point subspace of \( X \) is the operator space

\[
X^\alpha = \{ x \in X : \alpha(x) = x \otimes 1_M \}.
\]

Note that \( X^\alpha \) is obviously an \( M \)-subcomodule of \( X \).

Another important notion concerning actions of Hopf–von Neumann algebras is commutativity of actions:

**Definition 2.3.** Let \( (M_1, \Delta_1) \) and \( (M_2, \Delta_2) \) be two Hopf–von Neumann algebras and \( \alpha_1, \alpha_2 \) be actions of \( M_1 \) and \( M_2 \) respectively on the same operator space \( X \). We say that \( \alpha_1 \) and \( \alpha_2 \) commute if

\[
(\alpha_1 \otimes \text{id}_{M_2}) \circ \alpha_2 = (\text{id}_X \otimes \sigma) \circ (\alpha_2 \otimes \text{id}_{M_1}) \circ \alpha_1,
\]

where \( \sigma: M_2 \overline{\otimes} M_1 \to M_1 \overline{\otimes} M_2, x \otimes y \mapsto y \otimes x \), is the flip isomorphism.

**Lemma 2.4 (\cite{10} Lemma 5.2).** With the notation and assumptions of Definition 2.3, the fixed point subspace \( X^{\alpha_1} \) is an \( M_2 \)-subcomodule of \( (X, \alpha_2) \), i.e. the restriction \( \alpha_2|_{X^{\alpha_1}} \) is an action of \( M_2 \) on \( X^{\alpha_1} \).

For the rest of this paper, \( G \) will denote a locally compact (Hausdorff) group with left Haar measure \( ds \) and modular function \( \Delta_G \). We identify \( L^\infty(G) \) with the multiplicative operators acting on \( L^2(G) \). Also, the fundamental unitary operator \( V_G \in B(L^2(G)) \overline{\otimes} B(L^2(G)) \simeq B(L^2(G \times G)) \) defined by

\[
V_G f(s, t) = f(t^{-1} s, t), \quad f \in L^2(G \times G), \quad s, t \in G,
\]
gives rise to a comultiplication \( \alpha_G: L^\infty(G) \to L^\infty(G) \overline{\otimes} L^\infty(G) \) via

\[
\alpha_G(f) = V_G^* (f \otimes 1)V_G
\]

and it is easy to see that

\[
\alpha_G(f)(s, t) = f(ts), \quad s, t \in G,
\]

(identifying \( L^\infty(G \times G) \simeq L^\infty(G) \overline{\otimes} L^\infty(G) \)). Thus, \( (L^\infty(G), \alpha_G) \) is a Hopf–von Neumann algebra.

On the other hand, the unitary \( U_G \in B(L^2(G)) \overline{\otimes} B(L^2(G)) \) defined by

\[
U_G \xi(s, t) = \Delta_G(t)^{1/2} \xi(st, t), \quad s, t \in G, \xi \in L^2(G \times G),
\]

induces another comultiplication on \( L^\infty(G) \), namely

\[
\alpha'_G(f) = U_G (f \otimes 1) U_G^*; \quad f \in L^\infty(G),
\]

or equivalently

\[
\alpha'_G(f)(s, t) = f(st), \quad s, t \in G.
\]
Obviously,

\[ \alpha_G = \sigma \circ \alpha'_G, \]

where \( \sigma: B(L^2(G)) \overline{\otimes} B(L^2(G)) \to B(L^2(G)) \overline{\otimes} B(L^2(G)), x \otimes y \mapsto y \otimes x, \) is the flip mapping.

Another basic example of a Hopf–von Neumann algebra is the left von Neumann algebra of \( G, \) i.e. the algebra \( L(G) := (\lambda(G))'' \subseteq B(L^2(G)) \) generated by the left regular representation \( \lambda: G \ni s \mapsto \lambda_s \in B(L^2(G)), \)

\[ \lambda_s\xi(t) = \xi(s^{-1}t), \quad \xi \in L^2(G). \]

The unitary operator \( W_G \in B(L^2(G \times G)) \) given by the formula

\[ W_G f(s,t) = f(s,st), \quad f \in L^2(G \times G), \quad s,t \in G, \]

induces a comultiplication \( \delta_G: L(G) \to L(G) \overline{\otimes} L(G) \) via

\[ \delta_G(x) = W_G^*(x \otimes 1)W_G. \]

It is easy to verify that

\[ \delta_G(\lambda_s) = \lambda_s \otimes \lambda_s, \quad s \in G. \]

Recall the Fourier algebra \( A(G) \) of \( G \) (see e.g. [8]):

\[ A(G) = \{ u: G \to \mathbb{C} : \exists \xi, \eta \in L^2(G), \forall s \in G, u(s) = \langle \lambda_s \xi, \eta \rangle \}. \]

With the pointwise product, \( A(G) \) is a Banach algebra with respect to the norm

\[ \| u \|_{A(G)} = \inf \{ \| \xi \|, \| \eta \| : \xi, \eta \in L^2(G) \text{ are such that } u(s) = \langle \lambda_s \xi, \eta \rangle \} \]

and it is isometrically isomorphic to the predual \( L(G)_* \) of the group von Neumann algebra, where the duality is given by

\[ \langle \lambda_s, u \rangle = u(s), \quad s \in G, \quad u \in A(G). \]

The pointwise product on \( A(G) \) coincides with that induced on the predual \( L(G)_* \) by the comultiplication \( \delta_G \) of \( L(G), \) because

\[ \langle \lambda_s, uv \rangle = u(s)v(s) = \langle \lambda_s, u \rangle \langle \lambda_s, v \rangle = \langle \lambda_s \otimes \lambda_s, u \otimes v \rangle = \langle \delta_G(\lambda_s), u \otimes v \rangle. \]

In the following, \( A(G) \) will be identified with \( L(G)_*. \)

Note that \( V_G \in L(G) \overline{\otimes} L^\infty(G) \) and \( W_G \in L^\infty(G) \overline{\otimes} L(G). \) Furthermore, \( \alpha_G \) and \( \delta_G \) extend to actions of \( L^\infty(G) \) and \( L(G) \) on \( B(L^2(G)) \) respectively, via the formulas

\[ \alpha_G(x) = V_G^*(x \otimes 1)V_G, \quad \delta_G(x) = W_G^*(x \otimes 1)W_G, \quad x \in B(L^2(G)). \]

Also, recall the right regular representation of \( G \) on \( L^2(G) \):

\[ \rho_s f(t) = \Delta_G(s)^{1/2}f(ts), \quad s,t \in G, \quad f \in L^2(G). \]

We denote by \( R(G) = \rho(G)'' \) the right group von Neumann algebra and it is easy to verify that \( U_G \in R(G) \overline{\otimes} L^\infty(G). \)
In the following, we always assume that $L^\infty(G)$ and $L(G)$ are considered as Hopf–von Neumann algebras with respect to $\alpha_G$ and $\delta_G$ respectively.

Now, let us collect some well known facts about crossed products of von Neumann algebras (see for example [15], [18], [19], [20] and [21]), which will be used subsequently.

**Theorem 2.5 ([18] §18.6).** Given a (pointwise) group action on a von Neumann algebra $M$, i.e. a $w^*$-continuous representation $\alpha: G \to \text{Aut}(M)$ by $^*$-automorphisms of the von Neumann algebra $M$, for any $x \in M$ the $w^*$-continuous function $s \mapsto \alpha_{s}^{-1}(x)$ defines a unique element $\pi_\alpha(x) \in M \overline{\otimes} L^\infty(G)$ such that

$$\langle \pi_\alpha(x), \omega \otimes f \rangle = \int_G \langle \alpha_{s}^{-1}(x), \omega \rangle f(s) \, ds$$

for any $\omega \in M_*$ and $f \in L^1(G)$, and the map $\pi_\alpha: M \to M \overline{\otimes} L^\infty(G)$ is a $W^*$-action of $(L^\infty(G), \alpha_G)$ on $M$, i.e. a unital normal $^*$-monomorphism satisfying

$$(\pi_\alpha \otimes \text{id}_{L^\infty(G)}) \circ \pi_\alpha = (\text{id}_M \otimes \alpha_G) \circ \pi_\alpha.$$ 

Conversely, for any $W^*$-$L^\infty(G)$-action $\pi: M \to M \overline{\otimes} L^\infty(G)$ of $(L^\infty(G), \alpha_G)$ on $M$, there exists a unique $w^*$-continuous representation $\alpha: G \to \text{Aut}(M)$ by $^*$-automorphisms of $M$ such that $\pi_\alpha = \pi$.

The crossed product of $M$ by the action $\alpha$ is defined to be the von Neumann subalgebra of $M \overline{\otimes} B(L^2(G))$ generated by $\pi_\alpha(M)$ and $\mathbb{C} \overline{\otimes} L(G)$, that is,

$$M \rtimes_\alpha G = (\pi_\alpha(M) \cup \mathbb{C} \overline{\otimes} L(G))''.$$ 

Because of the covariance relations

$$\pi_\alpha(\alpha_s(x)) = (1 \otimes \lambda_s)\pi_\alpha(x)(1 \otimes \lambda_s)^*, \quad x \in M, s \in G,$$

we get

$$M \rtimes_\alpha G = \overline{\text{span}}^{w^*}\{(1 \otimes \lambda_s)\pi_\alpha(x) : s \in G, x \in M\},$$

where span denotes the linear span.

For a group action $\alpha$ on $M$ we set $M^\alpha := \{x \in M : \alpha_s(x) = x, \forall s \in G\}$. Then $M^{\pi_\alpha} = M^\alpha$, i.e.

$$\pi_\alpha(x) = x \otimes 1 \iff \alpha_s(x) = x, \forall s \in G.$$ 

Fix a group action $\alpha: G \to \text{Aut}(M)$ on a von Neumann algebra $M$ and consider the action $\beta = \alpha \otimes \text{Ad} \rho$ on $M \overline{\otimes} B(L^2(G))$, that is, $\beta_s = \alpha_s \otimes \text{Ad} \rho_s$ for all $s \in G$.

**Proposition 2.6 (Dual action, [15] I, Proposition 2.4).** Set

$$\widehat{\alpha} = (\text{id}_M \otimes \delta_G)|_{M \rtimes_\alpha G},$$
where $\delta_G$ is considered as an $L(G)$-action on $B(L^2(G))$ and $\text{id}_M \otimes \delta_G$ is an $L(G)$-action on $M \otimes B(L^2(G))$. Then $\hat{\alpha}$ is a $W^*-L(G)$-action on $M \rtimes_\alpha G$, called the dual of $\alpha$, and

$$\hat{\alpha}(x) = \text{Ad}_{(1 \otimes W_G^*)}(x \otimes 1), \quad x \in M \rtimes_\alpha G.$$ 

**Proposition 2.7 ([15, p. 9]).** The corresponding $W^*-L^\infty(G)$-actions $\pi_\alpha$ and $\pi_\beta$ satisfy

$$\pi_\beta = (\text{id}_M \otimes \text{Ad} U_G^*) \circ (\text{id}_M \otimes \sigma) \circ (\pi_\alpha \otimes \text{id}_{B(L^2(G))}),$$

where $U_G$ is the unitary defined above and were $\sigma$ is the flip mapping on $B(L^2(G)) \boxtimes B(L^2(G))$. Furthermore, $\pi_\beta$ commutes with the $L(G)$-action $\text{id}_M \otimes \delta_G$ (see Definition 2.3).

**Lemma 2.8 ([15, Proposition I.1.2]).** The action $\beta$ of $G$ on $M \otimes B(L^2(G))$ satisfies

$$\pi_\beta \big( (M \otimes L^\infty(G)) \big) = \pi_\beta \big( (M \otimes L^\infty(G)) \big) = \pi_\alpha(M).$$

**Proposition 2.9 ([15, II, Theorem 1.1]).** For the dual action $\hat{\alpha}$ of $\alpha$ we have

$$(M \rtimes_\alpha G)^{\hat{\alpha}} = \pi_\alpha(M).$$

**Theorem 2.10 (Digernes– Takesaki, [15, II, Theorem 1.2]).** The crossed product $M \rtimes_\alpha G$ is the fixed point space of $\beta$, i.e.

$$M \rtimes_\alpha G = \big( (M \boxtimes B(L^2(G))) \big)^{\beta} = \big( (M \boxtimes B(L^2(G))) \big)^{\pi_\beta}.$$ 

### 3. Crossed products of dual operator spaces and main results.

In this section, we prove some basic facts about crossed products of dual operator spaces which are generalizations of known results for crossed products of von Neumann algebras, and we characterize those $L^\infty(G)$-comodules for which the Fubini crossed product coincides with the spatial crossed product (see Theorem 3.17).

**Definition 3.1 (M. Hamana, [10]).** For an $L^\infty(G)$-comodule $(X, \alpha)$, we define the map

$$\tilde{\alpha} : X \boxtimes B(L^2(G)) \to X \boxtimes B(L^2(G)) \boxtimes L^\infty(G)$$

by

$$\tilde{\alpha} = (\text{id}_X \otimes \text{Ad} U_G^*) \circ (\text{id}_X \otimes \sigma) \circ (\alpha \otimes \text{id}_{B(L^2(G))})$$

where $\sigma$ is the flip mapping on $B(L^2(G)) \boxtimes B(L^2(G))$.

The proof of the next result is essentially the same as that of [10, Lemma 5.3(i)] and so we omit it.

**Proposition 3.2.** Let $(X, \alpha)$ be an $L^\infty(G)$-comodule. Then $\tilde{\alpha}$ is an $L^\infty(G)$-action on $X \boxtimes B(L^2(G))$, which commutes with the $L(G)$-action $\text{id}_X \otimes \delta_G$. 
Definition 3.3 (M. Hamana, [10]). Let \( (X, \alpha) \) be an \( L^\infty(G) \)-comodule. The Fubini crossed product of \( X \) by \( \alpha \) is the \( L(G) \)-comodule \( (X \rtimes_\alpha^F G, \hat{\alpha}) \), where

\[
X \rtimes_\alpha^F G := (X \boxtimes B(L^2(G)))^\hat{\alpha} \quad \text{and} \quad \hat{\alpha} := (\text{id}_X \otimes \delta_G)|_{X \rtimes_\alpha G}.
\]

The \( L(G) \)-action \( \hat{\alpha} : X \rtimes_\alpha^F G \to (X \rtimes_\alpha^F G) \boxtimes_F L(G) \) is called the dual of \( \alpha \).

Remark 3.4. It follows immediately from Proposition 3.2 and Lemma 2.1 that the dual action \( \hat{\alpha} \) is indeed an \( L(G) \)-action on the Fubini crossed product \( X \rtimes_\alpha^F G \), since \( \text{id}_X \otimes \delta_G \) commutes with \( \hat{\alpha} \).

Let \( \alpha \) be a pointwise \( w^* \)-action of \( G \) on a von Neumann algebra \( M \) and let \( X \) be a \( G \)-invariant \( w^* \)-closed subspace of \( M \). In this setting, J. Crann and M. Neufang [6] use an alternative definition of the Fubini crossed product of \( X \) by \( \alpha \) (see [6] Definition 3.1) and they prove that it coincides with

\[
\{ y \in M \rtimes_\alpha G : (\text{id} \otimes \omega)(y) \in X, \forall \omega \in B(L^2(G))_+ \}
\]

(see [6] Proposition 3.2), that is,

\[
(X \boxtimes_F B(L^2(G))) \cap (M \rtimes_\alpha G),
\]

which is exactly

\[
(X \boxtimes B(L^2(G)))^\hat{\alpha} = X \rtimes_\alpha^F G,
\]

since \( X \boxtimes B(L^2(G)) = X \boxtimes_F B(L^2(G)) \) (because \( B(L^2(G)) \) has the dual slice map property by [12]) and \( M \rtimes_\alpha G = (M \boxtimes B(L^2(G)))^\hat{\alpha} \) (by the Digernes–Takesaki theorem).

Therefore, the definition of Crann–Neufang coincides with Definition 3.3.

Definition 3.5. Let \( (X, \alpha) \) be an \( L^\infty(G) \)-comodule and suppose that \( X \) is \( w^* \)-closed in \( B(H) \) for some Hilbert space \( H \). The spatial crossed product of \( X \) by \( \alpha \) is defined to be the space

\[
X \vartriangleright_\alpha G := \overline{\text{span}}^{w^*} \{(1_H \otimes \lambda_s)\alpha(x) : s \in G, x \in X \} \subseteq B(H) \boxtimes B(L^2(G)).
\]

Remark 3.6. It is clear, from the Digernes–Takesaki theorem (see Theorem 2.10 above), Theorem 2.5 and Proposition 2.7 that \( X \rtimes_\alpha^F G = X \vartriangleright_\alpha G \) when \( X \) is a von Neumann algebra and \( \alpha \) is in addition a unital normal \( * \)-homomorphism.

However, this is not true for general \( L^\infty(G) \)-comodules. For example, consider any discrete group \( G \) failing the approximation property (see Section 1), such as \( G = \text{SL}(3, \mathbb{Z}) \) (see [14]). Then \( L(G) \) does not have the dual slice map property (see [9] Theorem 2.1), which means that there exists a dual operator space \( X \) such that \( X \boxtimes L(G) \not\subseteq X \boxtimes_F L(G) \). Consider the trivial \( L^\infty(G) \)-action \( \alpha : X \to X \boxtimes L^\infty(G) \), \( \alpha(x) = x \otimes 1 \), for any \( x \in X \). Then, obviously,

\[
X \vartriangleright_\alpha G = X \boxtimes L(G).
\]
On the other hand, it is not hard to see that if $\alpha$ is trivial, then

$$X \rtimes F_\alpha G = X \overline{\otimes}_F L(G)$$

and therefore $X \rtimes F_\alpha G \neq X \overline{\chi}_\alpha G$.

**Remark 3.7.** Let $H$, $K$ be Hilbert spaces, $X \subseteq B(H)$ a $w^*$-closed subspace and $b, c \in B(K)$. Then

$$(1_H \otimes b)(X \overline{\otimes} B(K))(1_H \otimes c) \subseteq X \overline{\otimes} B(K).$$

Indeed, for all $a, b, c \in B(K)$ and $v \in X$, we have $(1_H \otimes b)(v \otimes a)(1_H \otimes c) = v \otimes (bac) \in X \overline{\otimes} B(K)$, thus the above inclusion follows from the definition of the spatial tensor product and the fact that the multiplication in $B(H) \overline{\otimes} B(K)$ is separately $w^*$-continuous. As a consequence, if $(X, \alpha)$ is an $L^\infty(G)$-comodule, then $X \overline{\chi}_\alpha G \subseteq X \overline{\otimes} B(L^2(G))$, because $\alpha(X) \subseteq X \overline{\otimes} L^\infty(G) \subseteq X \overline{\otimes} B(L^2(G))$.

Also, if in addition $Y$ is a $w^*$-closed subspace of $B(L)$ for some Hilbert space $L$ and $\phi : X \rightarrow Y$ is a $w^*$-continuous completely bounded map, then $\phi \otimes \text{id}_{B(K)} : X \overline{\otimes} B(K) \rightarrow Y \overline{\otimes} B(K)$ is a $w^*$-continuous $B(K)$-bimodule map in the sense that

$$(\phi \otimes \text{id}_{B(K)})((1_H \otimes a)x(1_H \otimes b)) = (1_L \otimes a)(\phi \otimes \text{id}_{B(K)})(x)(1_L \otimes b)$$

for all $a, b \in B(K)$ and $x \in X \overline{\otimes} B(K)$.

**Proposition 3.8.** Let $(X, \alpha)$ be an $L^\infty(G)$-comodule and suppose that $X$ is $w^*$-closed in $B(H)$ for some Hilbert space $H$. Then $X \rtimes F_\alpha G$ is an $L(G)$-bimodule, i.e.

$$(1_H \otimes \lambda_s)y(1_H \otimes \lambda_t) \in X \rtimes F_\alpha G, \quad s, t \in G, \quad y \in X \rtimes F_\alpha G,$$

and $\alpha(X) \subseteq X \rtimes F_\alpha G$. Therefore

$$X \overline{\chi}_\alpha G \subseteq X \rtimes F_\alpha G.$$

Furthermore, $\tilde{\alpha}(X \overline{\chi}_\alpha G) \subseteq (X \overline{\chi}_\alpha G) \overline{\otimes}_F L(G)$, that is, $X \overline{\chi}_\alpha G$ is an $L(G)$-subcomodule of $(X \rtimes F_G, \tilde{\alpha})$.

**Proof.** Let $s \in G$ and $y \in X \rtimes F_\alpha G$. Then by Remark 3.7 we deduce that $(1_H \otimes \lambda_s)y \in X \overline{\otimes} B(L^2(G))$ and $\tilde{\alpha}(y) = y \otimes 1$, by Definition 3.3. Also, by Remark 3.7

$$(\alpha \otimes \text{id}_{B(L^2(G))})(1_H \otimes \lambda_s)y = (1_H \otimes 1_{L^2(G)} \otimes \lambda_s)(\alpha \otimes \text{id}_{B(L^2(G))})(y).$$

Thus,

$$\tilde{\alpha}((1_H \otimes \lambda_s)y) = (\text{id}_X \otimes \text{Ad} U^*_G) \circ (\text{id}_X \otimes \sigma) \circ (\alpha \otimes \text{id}_{B(L^2(G))})(1_H \otimes \lambda_s)y$$

$$= (\text{id}_X \otimes \text{Ad} U^*_G) \circ (\text{id}_X \otimes \sigma)(1_H \otimes 1_{L^2(G)} \otimes \lambda_s)(\alpha \otimes \text{id}_{B(L^2(G))})(y)$$

$$= [(\text{id}_{B(H)} \otimes \text{Ad} U^*_G) \circ (\text{id}_{B(H)} \otimes \sigma)(1_H \otimes 1_{L^2(G)} \otimes \lambda_s)]\tilde{\alpha}(y)$$

$$= [(1_H \otimes U^*_G)(1_H \otimes \lambda_s \otimes 1_{L^2(G)})(1_H \otimes U_G)](y \otimes 1_{L^2(G)})$$

$$= (1_H \otimes \lambda_s)y \otimes 1_{L^2(G)},$$
where the third equality above follows from the fact that \( (\text{id}_{B(H)} \otimes \text{Ad} U^*_G) \circ (\text{id}_{B(H)} \otimes \sigma) \) is a \(*\)-homomorphism and thus multiplicative, while the last equality is because \( U_G \in R(G) \overline{\otimes} L^\infty(G) \) and \( R(G) = L(G)' \). Therefore, \( (1_H \otimes \lambda_s)y \in X \rtimes^\mathcal{F}_\alpha G \). Similarly, \( y(1_H \otimes \lambda_t) \in X \rtimes^\mathcal{F}_\alpha G \) for all \( t \in G \) and \( y \in X \rtimes^\mathcal{F}_\alpha G \).

On the other hand, if \( x \in X \), then

\[
\tilde{\alpha}(\alpha(x)) = (\text{id}_X \otimes \text{Ad} U^*_G) \circ (\text{id}_X \otimes \sigma) \circ (\alpha \otimes \text{id}_{B(L^2(G))})(\alpha(x))
= (\text{id}_X \otimes \text{Ad} U^*_G) \circ (\text{id}_X \otimes \sigma) \circ (\text{id}_X \otimes \alpha_G)(\alpha(x))
= (\text{id}_X \otimes \text{Ad} U^*_G) \circ (\text{id}_X \otimes \alpha'_G)(\alpha(x))
= (1_H \otimes U^*_G)(1_H \otimes U_G)(\alpha(x) \otimes 1_{L^2(G)})(1_H \otimes U^*_G)(1_H \otimes U_G)
= (\alpha(x) \otimes 1_{L^2(G)},
\]

because \( \alpha'_G = \sigma \circ \alpha_G \) and \( \alpha'_G(f) = U_G(f \otimes 1)U^*_G \) for all \( f \in L^\infty(G) \). Hence, \( \alpha(X) \subseteq X \rtimes^\mathcal{F}_\alpha G \).

Finally, for \( x \in X \) and \( s \in G \), we have

\[
\tilde{\alpha}((1_H \otimes \lambda_s)\alpha(x)) = (\text{id}_{B(H)} \otimes \delta_G)((1_H \otimes \lambda_s)\alpha(x))
= (\text{id}_{B(H)} \otimes \delta_G)(1_H \otimes \lambda_s)(\text{id}_{B(H)} \otimes \delta_G)(\alpha(x))
= (1_H \otimes \delta_G(\lambda_s))(1_H \otimes W^*_G)(\alpha(x) \otimes 1_{L^2(G)})(1_H \otimes W_G)
= (1_H \otimes \lambda_s \otimes \lambda_s)(\alpha(x) \otimes 1_{L^2(G)}),
\]

because \( 1_H \otimes W_G \in C1_H \overline{\otimes} L^\infty(G) \overline{\otimes} L(G) \) commutes with \( \alpha(x) \otimes 1_{L^2(G)} \in B(H) \overline{\otimes} L^\infty(G) \overline{\otimes} C1_{L^2(G)}. \) Therefore,

\[
\tilde{\alpha}((1_H \otimes \lambda_s)\alpha(x)) = ((1_H \otimes \lambda_s)\alpha(x)) \otimes \lambda_s
\]

and it follows that \( \tilde{\alpha}(X \overline{\rtimes}_\alpha G) \subseteq (X \overline{\rtimes}_\alpha G) \overline{\otimes}\mathcal{F} L(G) \). \( \blacksquare \)

The next result proves that, for any \( L^\infty(G) \)-comodule \( X \), both the Fubini crossed product and the spatial crossed product are independent of the Hilbert space on which \( X \) is represented.

**Proposition 3.9 (Uniqueness of the crossed product).** Let \( (X, \alpha) \) and \( (Y, \beta) \) be two \( L^\infty(G) \)-comodules and suppose that \( X \) and \( Y \) are \( w^* \)-closed subspaces of \( B(H) \) and \( B(K) \) respectively. If there exists an \( L^\infty(G) \)-comodule isomorphism \( \Phi: X \to Y \), then the isomorphism \( \Psi := \Phi \otimes \text{id}_{B(L^2(G))}: X \overline{\otimes} B(L^2(G)) \to Y \overline{\otimes} B(L^2(G)) \) is an \( L^\infty(G) \)-comodule isomorphism from \( (X \overline{\otimes} B(L^2(G)), \tilde{\alpha}) \) onto \( (Y \overline{\otimes} B(L^2(G)), \tilde{\beta}) \), which maps \( X \rtimes^\mathcal{F}_\alpha G \) onto \( Y \rtimes^\mathcal{F}_\beta G \) and \( X \overline{\rtimes}_\alpha G \) onto \( Y \overline{\rtimes}_\beta G \). Also, \( \Psi|_{X \rtimes^\mathcal{F}_\alpha G} \) is an \( L(G) \)-comodule isomorphism from \( (X \rtimes^\mathcal{F}_\alpha G, \tilde{\alpha}) \) onto \( (Y \rtimes^\mathcal{F}_\beta G, \tilde{\beta}) \) and \( \Psi|_{X \overline{\rtimes}_\alpha G} \) is an \( L(G) \)-comodule isomorphism from \( (X \overline{\rtimes}_\alpha G, \tilde{\alpha}) \) onto \( (Y \overline{\rtimes}_\beta G, \tilde{\beta}) \). Furthermore, \( \Psi \) is an \( L(G) \)-bimodule map, i.e.

\[
\Psi((1_H \otimes \lambda_s)x(1_H \otimes \lambda_t)) = (1_K \otimes \lambda_s)\Psi(x)(1_K \otimes \lambda_t)
\]

for all \( s, t \in G \) and \( x \in X \overline{\otimes} B(L^2(G)) \).
Therefore \( \Psi \) is an \( L^\infty(G) \)-comodule isomorphism from \( (X \bar{\otimes} B(L^2(G)), \tilde{\alpha}) \) onto \( (Y \bar{\otimes} B(L^2(G)), \tilde{\beta}) \). This implies that \( \Psi \) maps the fixed point subspace \( X \rtimes_\alpha^F G \) of \( \tilde{\alpha} \) onto the fixed point subspace \( Y \rtimes_\beta^F G \) of \( \tilde{\beta} \). On the other hand, the relation \( \beta \circ \Phi = (\Phi \otimes \text{id}) \circ \alpha \) yields
\[
\Psi(\alpha(X)) = (\Phi \otimes \text{id})(\alpha(X)) = \beta(\Phi(X)) = \beta(Y)
\]
and since \( \Psi \) is an \( L(G) \)-bimodule map (see Remark 3.7) it follows that \( \Psi \) maps \( X \bar{\rtimes}_\alpha G \) onto \( Y \bar{\rtimes}_\beta G \). It remains to show that
\[
\tilde{\beta} \circ \Psi = (\Psi \otimes \text{id}) \circ \tilde{\alpha}.
\]
Indeed,
\[
\tilde{\beta} \circ \Psi = (\text{id}_Y \otimes \delta_G) \circ (\Phi \otimes \text{id}_{B(L^2(G))}) = (\Phi \otimes \text{id}_{B(L^2(G))} \otimes \text{id}_{L(G)}) \circ (\text{id}_X \otimes \delta_G) = (\Psi \otimes \text{id}_{L(G)}) \circ \tilde{\alpha}.
\]

The equality of the spatial and Fubini crossed products does not pass to subcomodules. Indeed, any \( L^\infty(G) \)-comodule is isomorphic to a subcomodule of a von Neumann algebra (see Remark 2.1), and the latter will always satisfy the equality condition. However, if a subcomodule is covariantly \( w^* \)-complemented, i.e., it is the range of a normal projection of norm 1 commuting with the action, the situation is better. In a more general setting, the following proposition gives a sufficient condition for \( X \rtimes_\alpha^F G = X \bar{\rtimes}_\alpha G \) to hold for an \( L^\infty(G) \)-comodule \( (X, \alpha) \).

**Proposition 3.10.** Let \( (X, \alpha) \) and \( (Y, \beta) \) be two \( L^\infty(G) \)-comodules and suppose that \( X \) and \( Y \) are \( w^* \)-closed subspaces of \( B(H) \) and \( B(K) \) respectively. Suppose that \( \zeta : X \rightarrow Y \) is an \( L^\infty(G) \)-comodule monomorphism and there exists an \( L^\infty(G) \)-comodule morphism \( \phi : Y \rightarrow X \) onto \( X \) such that \( \phi \circ \zeta = \text{id}_X \). If \( Y \rtimes_\beta^F G = Y \bar{\rtimes}_\beta G \), then \( X \rtimes_\alpha^F G = X \bar{\rtimes}_\alpha G \).

**Proof.** Let \( \psi := \phi \otimes \text{id}_{B(L^2(G))} : Y \bar{\otimes} B(L^2(G)) \rightarrow X \bar{\otimes} B(L^2(G)) \). Then, with the same argument as in the proof of Proposition 3.9, we show that \( \psi \) is an \( L^\infty(G) \)-comodule morphism from \( (Y \bar{\otimes} B(L^2(G)), \beta) \) to \( (X \bar{\otimes} B(L^2(G)), \tilde{\alpha}) \), and since \( \phi \) is onto \( X \), it follows that \( \psi \) is onto \( X \bar{\otimes} B(L^2(G)) \). Similarly, the map \( \theta := \zeta \otimes \text{id}_{B(L^2(G))} : X \bar{\otimes} B(L^2(G)) \rightarrow Y \bar{\otimes} B(L^2(G)) \) is an \( L^\infty(G) \)-comodule monomorphism such that \( \psi \circ \theta = \text{id}_{X \bar{\otimes} B(L^2(G))} \).
We will show that $\psi$ maps $Y \rtimes^F_\beta G$ onto $X \rtimes^F_\alpha G$. Indeed, on the one hand, since $\psi$ is an $L^\infty(G)$-comodule morphism from $(Y \otimes B(L^2(G)), \tilde{\beta})$ to $(X \otimes B(L^2(G)), \tilde{\alpha})$, it maps the fixed point subspace of $(Y \otimes B(L^2(G)), \tilde{\beta})$ into the fixed point subspace of $(X \otimes B(L^2(G)), \tilde{\alpha})$, that is, $\psi(Y \rtimes^F_\beta G) \subseteq X \rtimes^F_\alpha G$. On the other hand, for any $x \in X \rtimes^F_\alpha G$, we have $x = \psi(y)$, where $y := \theta(x) \in Y \otimes B(L^2(G))$. Thus

$$
\tilde{\beta}(y) = \tilde{\beta}(\theta(x)) = (\theta \otimes \text{id}_{L^\infty(G)})(\tilde{\alpha}(x)) = (\theta \otimes \text{id}_{L^\infty(G)})(x \otimes 1) = \theta(x) \otimes 1 = y \otimes 1
$$

and so $y \in Y \rtimes^F_\beta G$. This yields $X \rtimes^F_\alpha G \subseteq \psi(Y \rtimes^F_\beta G)$ and therefore we have the equality $X \rtimes^F_\alpha G = \psi(Y \rtimes^F_\beta G)$.

Now if we assume that $Y \rtimes^F_\beta G = Y \tilde{\alpha} G$, then

$$
X \rtimes^F_\alpha G = \psi(Y \rtimes^F_\beta G) = \psi(Y \tilde{\alpha} G) \subseteq \text{span}^w \{\psi((1_K \otimes \lambda_s)\beta(y)) : s \in G, y \in Y\}
$$

$$
= \text{span}^w \{(\phi \otimes \text{id})((1_K \otimes \lambda_s)\beta(y)) : s \in G, y \in Y\}
$$

$$
= \text{span}^w \{(1_H \otimes \lambda_s)\phi(y) : s \in G, y \in Y\}
$$

$$
= \text{span}^w \{(1_H \otimes \lambda_s)\alpha(\phi(y)) : s \in G, y \in Y\}
$$

and therefore $X \rtimes^F_\alpha G = X \tilde{\alpha} G$, because the inclusion $X \tilde{\alpha} G \subseteq X \rtimes^F_\alpha G$ holds in general (see Proposition 3.8).

**Remark 3.11.** Let $Y$ and $Z$ be two W*-TRO’s. A W*-TRO-morphism $\beta : Y \to Z$ is called non-degenerate if the linear spans of $\beta(Y)Z^*Z$ and $\beta(Y)^*ZZ^*$ are $w^*$-dense respectively in $Z$ and $Z^*$.

In [17], P. Salmi and A. Skalski proved essentially that $X \rtimes^F_\alpha G = X \tilde{\alpha} G$ when $X$ is a W*-TRO and the action $\alpha$ is in addition a non-degenerate W*-TRO morphism. An alternative way to prove this result is the following. Consider the linking von Neumann algebra $R_X$ and the canonical $w^*$-continuous projection $P : R_X \to X$. Then the action $\alpha$ extends uniquely to a $W^*-L^\infty(G)$-action $\beta : R_X \to R_X \otimes L^\infty(G)$ since $\alpha$ is a non-degenerate TRO-morphism (see [17, Proposition 1.2 and Theorem 2.3]), such that $\alpha \circ P = (P \otimes \text{id}) \circ \beta$. Therefore, $X \rtimes^F_\alpha G = X \tilde{\alpha} G$ follows from Proposition 3.10.

**Proposition 3.12.** For any $L^\infty(G)$-comodule $(X, \alpha)$, we have

$$
(X \rtimes^F_\alpha G)\hat{\alpha} = (X \tilde{\alpha} G)\hat{\alpha} = \alpha(X).
$$

**Proof.** By Proposition 3.9 and Remark 2.1 we may assume that $(X, \alpha)$ is a subcomodule of a $W^*-L^\infty(G)$-comodule $(M, \alpha)$, i.e. $M$ is a von Neumann algebra that contains $X$ and the action $\alpha$ extends to a $W^*-L^\infty(G)$-action
on $M$, which we denote again by $\alpha$. Then, since $\tilde{\alpha}$ commutes with $\text{id}_M \otimes \delta_G$ (see Proposition 3.2), we get
\[
(X \rtimes^F_\alpha G)\tilde{\alpha} = ((X \bar{\otimes} B(L^2(G)))\tilde{\alpha}) = ((X \bar{\otimes} B(L^2(G)))^{\text{id}_X \otimes \delta_G})\tilde{\alpha} = (X \bar{\otimes} L^\infty(G))\tilde{\alpha},
\]
because $B(L^2(G))^{\delta_G} = L^\infty(G)$ (see e.g. [19, §0.3.9]). By Theorem 2.5, there exists a $w^*$-group action $\gamma: G \to \text{Aut}(M)$ such that $\pi_\gamma = \alpha$. Therefore, by Proposition 2.7 and Lemma 2.8 we get $(M \bar{\otimes} L^\infty(G))\tilde{\alpha} = \alpha(M)$, and so
\[
(X \bar{\otimes} L^\infty(G))\tilde{\alpha} \subseteq (M \bar{\otimes} L^\infty(G))\tilde{\alpha} \cap (X \bar{\otimes} L^\infty(G)) = \alpha(M) \cap (X \bar{\otimes} L^\infty(G)).
\]
Thus, it suffices to show that $\alpha(M) \cap (X \bar{\otimes} L^\infty(G)) = \alpha(X)$. The inclusion $\alpha(M) \cap (X \bar{\otimes} L^\infty(G)) \subseteq \alpha(X)$ is obvious. To prove the inclusion $\alpha(M) \cap (X \bar{\otimes} L^\infty(G)) \subseteq \alpha(X)$ consider an $x \in M$ such that $\alpha(x) \in X \bar{\otimes} L^\infty(G)$. Then $(\text{id}_M \otimes f)(\alpha(x)) \in X$ for all $f \in L^1(G)$ and hence, for any $\omega \in M_*$ with $\omega|_X = 0$, we have
\[
\langle (\text{id}_M \otimes f)(\alpha(x)), \omega \rangle = 0, \forall f \in L^1(G)
\]
\[
\implies \langle \alpha(x), \omega \otimes f \rangle = 0, \forall f \in L^1(G)
\]
\[
\implies \int_G f(s)\langle \gamma_{s^{-1}}(x), \omega \rangle ds = 0, \forall f \in L^1(G),
\]
because $\alpha(x)(s) = \pi_\gamma(x)(s) = \gamma_{s^{-1}}(x)$ for all $s \in G$. Since the function $G \ni s \mapsto \langle \gamma_{s^{-1}}(x), \omega \rangle$ is bounded and continuous, the last equality implies that it is identically zero. Therefore, by taking $s = e$ (where $e$ is the unit of $G$), we get $\langle x, \omega \rangle = 0$ for any $\omega \in M_*$ with $\omega|_X = 0$. Thus, $x \in X$. So, we have proved that $(X \rtimes^F_\alpha G)\tilde{\alpha} = \alpha(X)$.

On the other hand, we have $(X \bar{\rtimes}_\alpha G)\tilde{\alpha} = (X \rtimes^F_\alpha G)\tilde{\alpha} \cap (X \bar{\rtimes}_\alpha G) = \alpha(X) \cap (X \bar{\rtimes}_\alpha G) = \alpha(X)$. 

**Definition 3.13.** Let $(M, \Delta)$ be a Hopf–von Neumann algebra acting on a Hilbert space $K$ and $(X, \alpha)$ be an $M$-comodule with $X$ being a $w^*$-closed subspace of some $B(H)$. We say that $(X, \alpha)$ is *non-degenerate* if
\[
X \bar{\otimes} B(K) = \text{span}^{w^*} \{ (1_H \otimes y)\alpha(x) : y \in B(K), x \in X \}.
\]

One can easily verify that if $X$ and $Y$ are two $M$-comodules with $X$ non-degenerate and $\phi: X \to Y$ is an $M$-comodule isomorphism, then $Y$ is non-degenerate too. Therefore, Definition 3.13 does not depend on the representation $X \subseteq B(H)$. On the other hand, the author does not know whether it depends on the representation $M \subseteq B(K)$. However, this is not an issue since all Hopf–von Neumann algebras under discussion in this article are concrete.

Also, it is known that if $(M, \Delta, \phi, \psi)$ is a locally compact quantum group in the sense of [13] and $\alpha: N \to N \bar{\otimes} M$ is a $W^*$-action of $M$ on a von
Neumann algebra $N$, then $N \overline{\otimes} B(H) = (\mathbb{C}1 \overline{\otimes} B(H) \cup \alpha(N))''$, where $H$ is the GNS space corresponding to the left Haar weight $\phi$ (see [23] Theorem 2.6]). Observe that our notion of non-degeneracy for $M$-comodules, which are not necessarily von Neumann algebras, is a priori stronger.

Note that it follows from Theorem 3.17 below and Remark 3.6 that there exist $L(G)$-comodules which do not satisfy non-degeneracy, at least when $G$ is a discrete group failing the approximation property in the sense of [9]. On the other hand, for any locally compact group $G$, every $L^\infty(G)$-comodule is non-degenerate. This result and related applications appear in the preprint [1] of the author.

**Lemma 3.14.** Let $(X, \alpha)$ be an $M$-comodule for a Hopf–von Neumann algebra $(M, \Delta)$. If $(X, \alpha)$ is non-degenerate, then

\[
X = \text{span}^w \{ \omega \cdot x : \omega \in M_*, x \in X \},
\]

where $\omega \cdot x = (id_X \otimes \omega)(\alpha(x))$ is the corresponding $M_*$-module action on $X$ (see Remark 2.2).

**Proof.** Suppose that $(X, \alpha)$ is non-degenerate and that the von Neumann algebra $M$ acts on a Hilbert space $K$ and $X$ is $w^*$-closed in $B(H)$ for some Hilbert space $H$. Let $\phi \in X_*$ be such that $\phi(\omega \cdot x) = 0$ for all $\omega \in M_*$ and $x \in X$. Then

\[
\phi \circ (id_X \otimes \omega) \circ \alpha(x) = 0, \forall \omega \in M_*, x \in X
\]

\[
\implies \omega \circ (\phi \otimes id_{B(K)}) \circ \alpha(x) = 0, \forall \omega \in M_*, x \in X
\]

\[
\implies (\phi \otimes id_{B(K)}) \circ \alpha(x) = 0, \forall x \in X
\]

\[
b(\phi \otimes id_{B(K)}) \circ \alpha(x) = 0, \forall b \in B(K), x \in X
\]

\[
\implies (\phi \otimes id_{B(K)}) (1_H \otimes b) \alpha(x) = 0, \forall b \in B(K), x \in X.
\]

Since $(X, \alpha)$ is non-degenerate, the last condition implies that $(\phi \otimes id_{B(K)})(y) = 0$ for any $y \in X \overline{\otimes} B(K)$; thus $\phi(x)1 = (\phi \otimes id_{B(K)})(x \otimes 1) = 0$ for any $x \in X$ and hence $\phi = 0$. So the desired conclusion follows from the Hahn–Banach theorem.

For the proof of the next result see [20] Lemma II.1.4 and Corollary II.1.5.

**Proposition 3.15 ([20 Corollary II.1.5]).** Let $\delta : N \rightarrow N \overline{\otimes} L(G)$ be a $W^*$-$L(G)$-action on a von Neumann algebra $N$. For any $x \in N$ and any $k \in A(G)$, we have

\[
(k \cdot x) \otimes 1_{L^2(G)} \in \text{span}^w \{ (1_N \otimes \lambda_s) \delta(h \cdot k \cdot x) : s \in G, h \in A(G) \},
\]

where $k \cdot x = (id_N \otimes k)(\delta(x))$ for $k \in A(G)$ and $x \in N$.

**Corollary 3.16.** Let $\delta : N \rightarrow N \overline{\otimes} L(G)$ be a $W^*$-$L(G)$-action on a von Neumann algebra $N \subseteq B(K)$ and let $Y \subseteq N$ be an $L(G)$-subcomodule
of $N$, i.e. $Y$ is a $w^*$-closed subspace of $N$ and $\delta(Y) \subseteq Y \overline{\otimes}_F L(G)$. Then the following are equivalent:

(i) $Y = \overline{\text{span}}^{w^*}\{h \cdot y : h \in A(G), y \in Y\}$,
(ii) $(Y, \delta)$ is non-degenerate,

where $h \cdot y = (\text{id}_Y \otimes h)(\delta(y))$ for $h \in A(G)$ and $y \in Y$.

Proof. The implication (ii)$\Rightarrow$(i) is immediate by Lemma 3.14.

(i)$\Rightarrow$(ii): Since $Y = \overline{\text{span}}^{w^*}\{h \cdot y : h \in A(G), y \in Y\}$, from Proposition 3.15 it follows that

$$z \otimes 1_{L^2(G)} \in \overline{\text{span}}^{w^*}\{(1_K \otimes b)\delta(y) : b \in B(L^2(G)), y \in Y\}$$

for any $z \in Y$. Therefore, for any $z \in Y$ and $c \in B(L^2(G))$, we have

$$z \otimes c = (1_K \otimes c)(z \otimes 1_{L^2(G)}) \in \overline{\text{span}}^{w^*}\{(1_K \otimes b)\delta(y) : b \in B(L^2(G)), y \in Y\},$$

because the multiplication in $B(K) \overline{\otimes} B(L^2(G))$ is separately $w^*$-continuous. Thus,

$$Y \overline{\otimes} B(L^2(G)) \subseteq \overline{\text{span}}^{w^*}\{(1_K \otimes b)\delta(y) : b \in B(L^2(G)), y \in Y\}.$$ 

The converse inclusion follows from $\delta(Y) \subseteq Y \overline{\otimes}_F L(G) \subseteq Y \overline{\otimes} B(L^2(G))$ and the fact that $Y \overline{\otimes} B(L^2(G))$ is a $\mathbb{C}1_K \overline{\otimes} B(L^2(G))$-bimodule (see Remark 3.7).

In what follows, for a Hilbert space $H$ and any subsets $A, B \subseteq B(H)$, we will denote by $AB$ the set of all products $ab$, where $a \in A$ and $b \in B$.

The next result gives a characterization of those $L^\infty(G)$-comodules for which the corresponding Fubini crossed product is equal to the spatial crossed product.

**Theorem 3.17.** Let $(X, \alpha)$ be an $L^\infty(G)$-comodule. The following are equivalent:

(i) $X \rtimes^F_{\alpha} G = X \bar{\rtimes}_{\alpha} G$,
(ii) $(X \rtimes^F_{\alpha} G, \hat{\alpha})$ is a non-degenerate $L(G)$-comodule,
(iii) $X \rtimes^F_{\alpha} G = \overline{\text{span}}^{w^*}\{h \cdot y : h \in A(G), y \in X \rtimes^F_{\alpha} G\},$

where $h \cdot y = (\text{id}_X \rtimes^F_{\alpha} G \otimes h)(\hat{\alpha}(y))$ for $h \in A(G), y \in X \rtimes^F_{\alpha} G$.

Proof. By Proposition 3.9 and Remark 2.1 we may assume that $(X, \alpha)$ is a submodule of a $W^*$-$L^\infty(G)$-comodule $(M, \alpha)$, with $M$ acting on a Hilbert space $H$. Also, let us fix some notation. Set $K := H \otimes L^2(G), N := B(K), Y := X \rtimes^F_{\alpha} G, Y_0 := X \bar{\rtimes}_{\alpha} G$ and $\delta := \text{id}_{B(H)} \otimes \delta_G : N \rightarrow N \overline{\otimes} L(G)$.

The equivalence (ii)$\Leftrightarrow$(iii) is immediate from Corollary 3.16.

(ii)$\Rightarrow$(i). Suppose that $(X \rtimes^F_{\alpha} G, \hat{\alpha})$ is non-degenerate and consider the following faithful $^*$-representation of $L(G)$ on $K$:

$$u : L(G) \rightarrow B(K), \quad u(x) = 1_H \otimes x.$$
Then observe that

\(1\) \(u(L(G))Y u(L(G)) \subseteq Y\), i.e. \(Y\) is a \(C1_H \otimes L(G)\)-subbimodule of \(N,\)

(2) \(\delta \circ u = (u \otimes \text{id}_{L(G)}) \circ \delta_{G}\), i.e. \(u\) is a comodule monomorphism from 

\((L(G), \delta),\) to \((N, \delta).\)

The representation \(u\) defines a unitary operator \(R:\ K \otimes L^2(G) \to K \otimes L^2(G)\)

by the formula

\[(R\xi)(s) = u(\lambda_{s^{-1}})(\xi(s)) = (1_H \otimes \lambda_{s^{-1}})(\xi(s))\]

for \(s \in G\) and \(\xi \in L^2(G, K) \simeq K \otimes L^2(G).\)

We claim that \(R \in C1_H \otimes L(G) \otimes L^\infty(G).\) Indeed, if we take \(T \in B(H),\)

\(r \in G\) and \(f \in L^\infty(G),\) then, for every \(\eta \in H, \phi, \psi \in L^2(G)\) and \(s \in G,\) we have

\[(R(T \otimes \rho_r \otimes f)(\eta \otimes \phi \otimes \psi))(s) = (R(T \eta \otimes \rho_r \phi \otimes f \psi))(s)\]

\[= (1_H \otimes \lambda_{s^{-1}})(f(s) \psi(s)(T \eta \otimes \rho_r \phi))\]

\[= f(s) \psi(s)(T \eta \otimes \lambda_{s^{-1}} \rho_r \phi)\]

\[= f(s) \psi(s)(T \eta \otimes \rho_r \lambda_{s^{-1}} \phi)\]

\[= (T \otimes \rho_r \otimes f)(\eta \otimes \lambda_{s^{-1}} \phi \otimes \psi)(s)\]

\[= ((T \otimes \rho_r \otimes f)R(\eta \otimes \phi \otimes \psi))(s),\]

thus \(R \in (B(H) \otimes R(G) \otimes L^\infty(G))' = C1_H \otimes L(G) \otimes L^\infty(G).\)

Combining \(R \in C1_H \otimes L(G) \otimes L^\infty(G)\) with (1) and Remark 3.7 shows that the normal \(*\)-isomorphism

\[\text{Ad} R:\ N \otimes B(L^2(G)) \to N \otimes B(L^2(G)), \quad T \mapsto RTR^*,\]

maps \(Y \otimes B(L^2(G))\) onto \(Y \otimes B(L^2(G)).\)

Now, we consider two \(W^*-L(G)\)-actions on the von Neumann algebra \(N \otimes B(L^2(G)),\) namely

\[\tilde{\delta} := (\text{id}_N \otimes \text{Ad} W_G) \circ (\text{id}_N \otimes \sigma) \circ (\delta \otimes \text{id}_{B(L^2(G)))},\]

\[\tilde{\delta} := (\text{id}_N \otimes \sigma) \circ (\delta \otimes \text{id}_{B(L^2(G))}).\]

Let \(\Sigma \in B(K \otimes L^2(G \times G))\) with \(\Sigma \xi(s, t) = \xi(t, s)\) for \(\xi \in K \otimes L^2(G \times G).\)

Then, for any \(\xi \in K \otimes L^2(G \times G),\) we have

\[\tilde{\delta}(R\xi)(s, t) = \Sigma(\delta \otimes \text{id}_{B(L^2(G))})(R)(\Sigma \xi)(s, t) = (\delta \otimes \text{id}_{B(L^2(G))})(R)(\Sigma \xi)(t, s)\]

\[= \delta(u(\lambda_{s^{-1}}))(\Sigma \xi)(t, s) = (u \otimes \text{id}_{L(G)})(\delta_{G}(\lambda_{s^{-1}}))(\Sigma \xi)(t, s)\]

\[= (u(\lambda_{s^{-1}}) \times \lambda_{s^{-1}})(\Sigma \xi)(t, s) = u(\lambda_{s^{-1}})(\Sigma \xi)(st, s)\]

\[= u(\lambda_{s^{-1}})(\xi(s, st) = (R \otimes 1)(1 \otimes W_G)\xi(s, t).\]

Thus, \(\tilde{\delta}(R) = (R \otimes 1)(1 \otimes W_G),\) which implies that \(\text{Ad} R\) is an \(L(G)\)-comodule isomorphism from \((N \otimes B(L^2(G)), \tilde{\delta})\) onto \((N \otimes B(L^2(G)), \tilde{\delta}).\) Indeed, for
any $T \in N \overline{\otimes} B(L^2(G))$, we have
\[
(\tilde{\delta} \circ \text{Ad } R)(T) = \tilde{\delta}(R)\tilde{\delta}(T)\tilde{\delta}(R)^* = (R \otimes 1)(1 \otimes W_G)\tilde{\delta}(T)(1 \otimes W_G^*)(R^* \otimes 1) \\
= (\text{Ad } R \otimes \text{id}_{B(L^2(G))}) \circ (\text{id}_N \otimes \text{Ad } W_G) \circ \tilde{\delta}(T) \\
= (\text{Ad } R \otimes \text{id}_{B(L^2(G))}) \circ \tilde{\delta}(T).
\]

Therefore, $\text{Ad } R$ preserves fixed points, i.e.
\[
\text{Ad } R((N \overline{\otimes} B(L^2(G)))\tilde{\delta}) = (N \overline{\otimes} B(L^2(G)))\tilde{\delta}.
\]

On the other hand, $N^\delta \overline{\otimes} B(L^2(G)) = (N \overline{\otimes} B(L^2(G)))\tilde{\delta}$. Indeed, for every $x \in N^\delta$ and $b \in B(L^2(G))$, we have
\[
\tilde{\delta}(x \otimes b) = (\text{id}_N \otimes \sigma)(\tilde{\delta}(x) \otimes b) = (\text{id}_N \otimes \sigma)(x \otimes 1 \otimes b) = x \otimes b \otimes 1,
\]
hence we have the inclusion $N^\delta \overline{\otimes} B(L^2(G)) \subseteq (N \overline{\otimes} B(L^2(G)))\tilde{\delta}$. For the converse inclusion, take $T \in N \overline{\otimes} B(L^2(G))$ such that $\tilde{\delta}(T) = T \otimes 1$. Then, for $\omega \in B(L^2(G))_+$, $\phi \in N_+$ and $h \in A(G)$, we have
\[
\langle \delta \circ (\text{id}_N \otimes \omega)(T), \phi \otimes h \rangle = \langle (\text{id}_N \otimes \omega)(T), (\phi \otimes h) \circ \delta \rangle \\
= \langle T, (\phi \otimes h \otimes \omega) \circ (\tilde{\delta} \circ \text{id}_{B(L^2(G))}) \rangle \\
= \langle T, (\phi \otimes h \otimes \omega) \circ (\text{id}_N \otimes \sigma) \circ (\tilde{\delta} \circ \text{id}_{B(L^2(G))}) \rangle \\
= \langle (\tilde{\delta}(T), \phi \otimes \omega \otimes h \rangle = \langle T \otimes 1, \phi \otimes \omega \otimes h \rangle \\
= \langle T, \phi \otimes \omega \rangle \langle 1, h \rangle = \langle (\text{id} \otimes \omega)(T) \otimes 1, \phi \otimes h \rangle,
\]
thus $(\text{id}_N \otimes \omega)(T) \in N^\delta$ for all $\omega \in B(L^2(G))_+$, which implies that $T \in N^\delta \overline{\otimes} B(L^2(G))$.

Furthermore, by Proposition 3.12, we have $\alpha(X) = Y^{\tilde{\alpha}}$. Also, recall that $B(L^2(G)) = \overline{\text{span}}^w \{ yf : f \in L^\infty(G), y \in L(G) \}$ and observe that
\[
(\mathcal{C}1_K \overline{\otimes} L^\infty(G))\delta(N) \subseteq (N \overline{\otimes} B(L^2(G)))\tilde{\delta}.
\]
Indeed, if $y \in N$, then
\[
\tilde{\delta}(\delta(y)) = (\text{id}_N \otimes \text{Ad } W_G) \circ (\text{id}_N \otimes \sigma) \circ (\delta \circ \text{id}_{B(L^2(G))})(\delta(y)) \\
= (\text{id}_N \otimes \text{Ad } W_G) \circ (\text{id}_N \otimes \sigma) \circ (\text{id}_N \otimes \delta_G)(\delta(y)) \\
= (\text{id}_N \otimes \text{Ad } W_G) \circ (\text{id}_N \otimes \delta_G)(\delta(y)) \quad \text{(because } \sigma \circ \delta_G = \delta_G) \\
= (\text{id}_N \otimes \text{Ad } W_G) \circ (\text{id}_N \otimes \text{Ad } W_G^*)(\delta(y) \otimes 1) = \delta(y) \otimes 1
\]
and for any $f \in L^\infty(G)$ we have
\[
\tilde{\delta}(1_K \otimes f) = (\text{id}_N \otimes \text{Ad } W_G) \circ (\text{id}_N \otimes \sigma) \circ (\delta \circ \text{id}_{B(L^2(G))})(1_K \otimes f) \\
= (\text{id}_N \otimes \text{Ad } W_G)(1_K \otimes f \otimes 1_{L^2(G)}) = 1_K \otimes W_G^*(f \otimes 1_{L^2(G)})W_G \\
= 1_K \otimes f \otimes 1_{L^2(G)} \quad (W_G \in L^\infty(G) \overline{\otimes} L(G)).
\]

By assumption, $(Y, \delta)$ is non-degenerate, i.e.
\[
Y \overline{\otimes} B(L^2(G)) = \overline{\text{span}}^w \{ (1_K \otimes b)\delta(x) : x \in Y, b \in B(L^2(G)) \}$
and \( B(L^2(G)) = \text{span}^w \{ L(G)L^\infty(G) \} \). Thus
\[
(*) \quad Y \overline{\otimes} B(L^2(G)) = \text{span}^w \{(1_K \otimes y)(1_K \otimes f)\delta(x) : x \in Y, y \in L(G), f \in L^\infty(G)\} \\
\subseteq \text{span}^w \{(1_K \otimes y)T : y \in L(G), T \in (Y \overline{\otimes} B(L^2(G)))^\tilde{Y}\}.
\]

The last inclusion follows from \((C1_K \overline{\otimes} L^\infty(G))\delta(N) \subseteq (N \overline{\otimes} B(L^2(G)))^\tilde{\delta}\) and from \((Y \overline{\otimes} B(L^2(G)), \tilde{\delta})\) being an \(L(G)\)-subcomodule of \((N \overline{\otimes} B(L^2(G)), \tilde{\delta})\). Therefore, since \(\text{Ad} R\) maps \(Y \overline{\otimes} B(L^2(G))\) onto \(Y \overline{\otimes} B(L^2(G))\), the above inclusion \((*)\) implies that
\[
Y \overline{\otimes} B(L^2(G)) = R(Y \overline{\otimes} B(L^2(G)))R^* \\
\subseteq \text{span}^w \{R(C1_K \overline{\otimes} L(G))R^* R(Y \overline{\otimes} B(L^2(G)))^\tilde{\delta}R^* \}.
\]

On the other hand, we have
\[
R(Y \overline{\otimes} B(L^2(G)))^\tilde{\delta}R^* = (R(N \overline{\otimes} B(L^2(G)))^\tilde{\delta}R^*) \cap (Y \overline{\otimes} B(L^2(G))) \\
= (N \overline{\otimes} B(L^2(G)))^\delta \cap (Y \overline{\otimes} B(L^2(G))) \\
= (N^\delta \overline{\otimes} B(L^2(G))) \cap (Y \overline{\otimes} B(L^2(G))) \\
= ((N^\delta \cap Y) \overline{\otimes} B(L^2(G))) = Y^\delta \overline{\otimes} B(L^2(G)) \\
= Y^\alpha \overline{\otimes} B(L^2(G)) = \alpha(X) \overline{\otimes} B(L^2(G))
\]

and
\[
R(C1_K \overline{\otimes} L(G))R^* \subseteq C1_H \overline{\otimes} L(G) \overline{\otimes} B(L^2(G)),
\]
because \(R \in C1_H \overline{\otimes} L(G) \overline{\otimes} B(L^2(G))\).

Hence, the inclusion
\[
Y \overline{\otimes} B(L^2(G)) \subseteq \text{span}^w \{R(C1_K \overline{\otimes} L(G))R^* R(Y \overline{\otimes} B(L^2(G)))^\tilde{\delta}R^* \}
\]
yields
\[
Y \overline{\otimes} B(L^2(G)) \subseteq \text{span}^w \{(C1_H \overline{\otimes} L(G) \overline{\otimes} B(L^2(G)))\alpha(X) \overline{\otimes} B(L^2(G))\} \\
\subseteq \text{span}^w \{(C1_H \overline{\otimes} L(G))\alpha(X) \overline{\otimes} B(L^2(G))\} \\
= Y_0 \overline{\otimes} B(L^2(G)).
\]

It follows that \(Y = Y_0\), that is, \(X \rtimes_\alpha^F G = X \rtimes_\alpha G\).

(i) \(\Rightarrow\) (ii). Suppose that condition (i) holds and keep the same notation as above. By Remark 3.7 we see that \(Y \overline{\otimes} B(L^2(G))\) is a \(C1_K \overline{\otimes} B(L^2(G))\)-bimodule. Thus, since \(\hat{\alpha}(Y) \subseteq Y \overline{\otimes} F L(G) \subseteq Y \overline{\otimes} B(L^2(G))\), we have the inclusion \(Y \overline{\otimes} B(L^2(G)) \subseteq \text{span}^w \{(1_K \otimes b)\hat{\alpha}(y) : b \in B(L^2(G)), y \in Y\}\).

For the converse inclusion, it suffices to show that \(Y_0 \overline{\otimes} B(L^2(G)) \subseteq \text{span}^w \{(1_K \otimes b)\hat{\alpha}(y) : b \in B(L^2(G)), y \in Y_0\}\), since \(Y = Y_0\) (by the assump-
tion that (i) holds). Indeed, for any \( s \in G, x \in X \) and \( b \in B(L^2(G)) \),
\[
((1_H \otimes \lambda_s)\alpha(x)) \otimes b = (1_H \otimes 1_{L^2(G)} \otimes b\lambda_s^{-1})((1_H \otimes \lambda_s)\alpha(x) \otimes \lambda_s) \\
= (1_H \otimes 1_{L^2(G)} \otimes b\lambda_s^{-1})\widehat{\alpha}((1_H \otimes \lambda_s)\alpha(x)).
\]
Thus, \((1_H \otimes \lambda_s)\alpha(x) \otimes b \in \text{span}^w\{(1_K \otimes c)\widehat{\alpha}(y) : c \in B(L^2(G)), y \in Y_0\}.\) Since \(Y_0 \otimes B(L^2(G))\) is the \(w^*\)-closed linear span of the elements of the form \((1_H \otimes \lambda_s)\alpha(x) \otimes b\), we get the desired inclusion. \(\blacksquare\)

**Remark 3.18.** Note that from the last part of the proof of Theorem 3.17 (that of the implication \((i) \Rightarrow (ii)\)) it follows that the spatial crossed product \((X \nabla_\alpha G, \widehat{\alpha})\) is always a non-degenerate \(L(G)\)-comodule.

**Remark 3.19.** The result of Salmi and Skalski (see Remark 3.11) can also be proved using Theorem 3.17 as follows.

Let \(X\) be a \(W^*\)-TRO, \(\alpha : X \to X \nabla \infty(G)\) be an \(L\infty(G)\)-action on \(X\) such that \(\alpha\) is a non-degenerate \(W^*\)-TRO-morphism (see Remark 3.11 for the definition) and consider the canonical \(w^*\)-continuous projection \(P : R_X \to X\), where \(R_X\) is the linking von Neumann algebra of \(X\). Also, let \(\beta : R_X \to R_X \nabla \infty(G)\) be the unique \(W^*-L\infty(G)\)-action that extends \(\alpha\). Then, by the proof of Proposition 3.9, \(P \otimes \text{id}_{B(L^2(G))} : R_X \nabla B(L^2(G)) \to X \nabla B(L^2(G))\) is also a \(w^*\)-continuous projection and an \(L\infty(G)\)-comodule morphism with respect to the actions \(\beta\) and \(\widehat{\alpha}\). Therefore, as in the proof of Proposition 3.10, \(P \otimes \text{id}_{B(L^2(G))}\) restricts to a \(w^*\)-projection \(Q : R_X \rtimes_\beta G \to X \rtimes_\beta G\). Furthermore, following the last part of the proof of Proposition 3.9 we get
\[
\widehat{\alpha} \circ Q = (Q \otimes \text{id}_{L(G)}) \circ \beta,
\]
which by Remark 2.2 means exactly that
\[
Q(u \cdot y) = u \cdot Q(y) \quad \text{for any } u \in A(G) \text{ and } y \in R_X \rtimes_\beta G.
\]
Thus \(Q : R_X \rtimes_\beta G \to X \rtimes_\alpha \infty(G)\) is a \(w^*\)-continuous \(A(G)\)-module morphism onto \(X \rtimes_\alpha \infty(G)\). On the other hand, \(R_X \rtimes_\beta G = \text{span}^w\{A(G) \cdot (R_X \rtimes_\beta \infty(G))\}\) by the Digernes–Takesaki theorem and Theorem 3.17. It follows that \(X \rtimes_\alpha \infty(G) = \text{span}^w\{A(G) \cdot (X \rtimes_\alpha \infty(G))\}\) and hence \(X \rtimes_\alpha \infty(G) = X \nabla_\alpha G\) by Theorem 3.17.

**4. An application to groups with the approximation property.**

Let \(G\) be a locally compact group. A complex-valued function \(u : G \to \mathbb{C}\) is called a multiplier for the Fourier algebra \(A(G)\) if the linear map \(m_u(v) = uv\) maps \(A(G)\) into \(A(G)\). The space of all multipliers of \(A(G)\) is denoted by \(MA(G)\), and \(MA(G) \subseteq C_b(G)\). For \(u \in MA(G)\), we denote by \(M_u\) the \(w^*\)-continuous linear map from \(L(G)\) to \(L(G)\) defined by \(M_u = m_u^*\). The function \(u\) is called a completely bounded multiplier if \(M_u\) is completely bounded. The
space of all completely bounded multipliers is denoted by $M_0A(G)$ and it is a Banach space with the norm $\|u\|_{M_0} = \|M_u\|_{cb}$. Moreover, $A(G) \subseteq M_0A(G)$.

It is known that $M_0A(G)$ is the dual Banach space of $Q(G)$, where $Q(G)$ denotes the completion of $L^1(G)$ in the norm

$$\|f\|_Q = \sup\left\{ \left| \int_G f(s)u(s)\,ds \right| : u \in M_0A(G), \|u\|_{M_0} \leq 1 \right\}.$$ 

We say that $G$ has the approximation property (AP) if there is a net $\{u_i\}_{i \in I}$ in $A(G)$ such that $u_i \to 1$ in the $\sigma(M_0A(G), Q(G))$-topology.

For more details on completely bounded multipliers and the approximation property see for example [4], [5] and [9].

We will need the following theorem due to U. Haagerup and J. Kraus (see [9], Proposition 1.7 and Theorem 1.9).

**Theorem 4.1.** For any locally compact group $G$, the following conditions are equivalent:

1. $G$ has the AP.
2. There is a net $\{u_i\}$ in $A(G)$ such that, for any von Neumann algebra $N$, $(\text{id}_N \otimes M_{u_i})(x) \to x$ in the $w^*$-topology for all $x \in N \overline{\otimes} L(G)$.

**Proposition 4.2.** Let $X \subseteq B(H)$ be a $w^*$-closed subspace of $B(H)$, $G$ be a locally compact group with the AP and $\alpha: X \to X \otimes L^\infty(G)$ be an $L^\infty(G)$-action. Then, for any $x \in X \rtimes^\alpha G$, we have

$$x \in \overline{A(G) \cdot x}_{w^*},$$

where $h \cdot x = (\text{id}_X \rtimes^\alpha G \otimes h) \circ \hat{\alpha}(x)$ for any $x \in X \rtimes^\alpha G$ and $h \in A(G)$.

**Proof.** Let $Y := X \rtimes^\alpha G$ and $K := H \otimes L^2(G)$. First, we prove that the dual action $\hat{\alpha}: Y \to Y \otimes_\alpha L(G)$ satisfies

(1) $$(\text{id}_Y \otimes M_u) \circ \hat{\alpha} = \hat{\alpha} \circ (\text{id}_Y \otimes u) \circ \hat{\alpha}$$

for any $u \in A(G)$. Indeed, for any $u, h \in A(G)$ and $y \in L(G)$ we have

$$\langle M_u(y), h \rangle = \langle y, hu \rangle = \langle \delta_G(y), h \otimes u \rangle = \langle (\text{id}_{L(G)} \otimes u) \circ \delta_G(y), h \rangle,$$

therefore $M_u = (\text{id}_{L(G)} \otimes u) \circ \delta_G$ for any $u \in A(G)$. So, for any $u \in A(G)$,

$$\hat{\alpha} \circ (\text{id}_Y \otimes u) \circ \hat{\alpha} = (\text{id}_Y \otimes \text{id}_{L(G)} \otimes u) \circ (\hat{\alpha} \otimes \text{id}_{L(G)}) \circ \hat{\alpha} = (\text{id}_Y \otimes \text{id}_{L(G)} \otimes u) \circ (\text{id}_Y \otimes \delta_G) \circ \hat{\alpha} = [\text{id}_Y \otimes ((\text{id}_{L(G)} \otimes u) \circ \delta_G)] \circ \hat{\alpha} = (\text{id}_Y \otimes M_u) \circ \hat{\alpha},$$

and (1) is proved.

Since $G$ has the AP, by Theorem 4.1 there exists a net $\{u_i\}$ in $A(G)$ such that $\text{id}_{B(K) \otimes M_{u_i}}(y) \to y$ ultraweakly for all $y \in B(K) \overline{\otimes} L(G)$.

Therefore, $(\text{id}_Y \otimes M_{u_i})(\hat{\alpha}(x)) \to \hat{\alpha}(x)$ ultraweakly for any $x \in Y$, because $\hat{\alpha}(Y) \subseteq Y \otimes_\alpha L(G) \subseteq B(K) \overline{\otimes} L(G)$. Thus (1) implies that $\hat{\alpha} \circ (\text{id}_Y \otimes u_i)$
\( \circ \hat{\alpha}(x) \to \hat{\alpha}(x) \) ultraweakly for any \( x \in Y \). On the other hand, \( \hat{\alpha} \) is a \( w^* \)-continuous isometry, therefore it is a \( w^*-w^* \)-homeomorphism from \( Y \) onto \( \hat{\alpha}(Y) \) (see e.g. [3] Theorem A.2.5) and thus \((\text{id}_Y \otimes u_i) \circ \hat{\alpha}(x) \to x \) ultraweakly, that is, \( u_i \cdot x \to x \) ultraweakly, for any \( x \in Y \). 

The next result is due to J. Crann and M. Neufang (see [6]). Here, we give an alternative proof as an application of Theorem 3.17 and Proposition 4.2.

**Proposition 4.3.** Let \( G \) be a locally compact group with the AP and let \((X, \alpha)\) be an \( L^\infty(G) \)-comodule. Then \( X \rtimes_{\alpha}^F G = X \varpi_{\alpha} G \).

**Proof.** As \( G \) has the AP, by Proposition 4.2 it follows that \( y \in A(G) \cdot y^w \) for any \( y \in X \rtimes_{\alpha}^F G \), which implies that

\[
X \rtimes_{\alpha}^F G = \overline{\text{span}^w} \{ h \cdot y : y \in X \rtimes_{\alpha}^F G, h \in A(G) \}.
\]

Therefore, Theorem 3.17 yields \( X \rtimes_{\alpha}^F G = X \varpi_{\alpha} G \). 

5. Crossed products and ideals of \( L^1(G) \). For this section, let \( J \) be a closed left ideal of \( L^1(G) \) and consider its annihilator \( J^\perp \subseteq L^\infty(G) \). In [2], M. Anoussis, A. Katavolos and I. Todorov defined two \( L(G) \)-submodules of \( B(L^2(G)) \), namely \( \text{Bim}(J^\perp) \) and \((\text{Ran} J)^\perp \), and asked whether these bimodules are equal. They proved that this is the case when \( G \) is either weakly amenable discrete or compact or abelian.

This result was later generalized by J. Crann and M. Neufang [6] who proved that if \( G \) has the AP, then \( \text{Bim}(J^\perp) = (\text{Ran} J)^\perp \). Their approach is based on their non-commutative Fejér-type theorem for crossed products of von Neumann algebras by groups with the AP.

Here, we prove that \( \text{Bim}(J^\perp) \) and \((\text{Ran} J)^\perp \) can be realized respectively as the spatial crossed product and the Fubini crossed product of a certain \( L^\infty(G) \)-action on \( J^\perp \). Thus, Theorem 3.17 provides a necessary and sufficient condition for the equality \( \text{Bim}(J^\perp) = (\text{Ran} J)^\perp \).

For any \( h \in L^1(G) \), the map \( \Theta(h) : B(L^2(G)) \to B(L^2(G)) \) is defined by

\[
\Theta(h)(T) = \int_T h(s) \text{Ad} \rho_s(T) \, ds,
\]

where the integral is understood in the \( w^* \)-topology. We define

\[
(\text{Ran} J)^\perp = \ker \Theta(J) := \{ T \in B(L^2(G)) : \Theta(h)(T) = 0, \forall h \in J \}.
\]

On the other hand, \( \text{Bim}(J^\perp) \) is the normal \( L(G) \)-bimodule generated by \( J^\perp \), that is,

\[
\text{Bim}(J^\perp) = \overline{\text{span}^w} \{ \lambda_s f \lambda_t : s, t \in G, f \in J^\perp \}.
\]

Then, \( \text{Bim}(J^\perp) \) and \((\text{Ran} J)^\perp \) are \( L(G) \)-bimodules and \( \text{Bim}(J^\perp) \subseteq (\text{Ran} J)^\perp \). For more details regarding \( \text{Bim}(J^\perp) \) and \((\text{Ran} J)^\perp \), as well as for the original definition of \((\text{Ran} J)^\perp \), see [2].
Consider the normal *-monomorphism
\[
\Phi: B(L^2(G)) \to B(L^2(G)) \bar{\otimes} B(L^2(G))
\]
given by
\[
\Phi(T) = V_G^\ast \delta_G(T)V_G = V_G^\ast W_G^*(T \otimes 1) W_G V_G.
\]
It is not hard to see that
\[
\begin{align*}
(2) & \quad \Phi(f) = \alpha_G(f), \quad f \in L^\infty(G), \\
(3) & \quad \Phi(\lambda_s) = 1 \otimes \lambda_s, \quad s \in G.
\end{align*}
\]
Also,
\[
\tilde{\alpha}_G \circ \Phi = (\Phi \otimes \text{id}_{L(G)}) \circ \delta_G
\]
and therefore \(\Phi\) is a \(W^\ast\)-\(L(G)\)-comodule isomorphism from \((B(L^2(G)), \delta_G)\) onto \((L^\infty(G) \bar{\otimes} \alpha_G G, \tilde{\alpha}_G)\).

Since \(J\) is a closed left ideal of \(L^1(G)\), it follows that
\[
\alpha_G(J^\perp) \subseteq J^\perp \bar{\otimes} L^\infty(G),
\]
i.e. \(J^\perp\) is an \(L^\infty(G)\)-subcomodule of \((L^\infty(G), \alpha_G)\).

**Proposition 5.1.** The \(L(G)\)-comodule isomorphism \(\Phi\) defined above maps \(\text{Bim}(J^\perp)\) onto \(J^\perp \bar{\otimes} \alpha_G G\) and \((\text{Ran} J)^\perp\) onto \(J^\perp \ltimes_{\alpha_G} G\). In particular, \(\text{Bim}(J^\perp)\) and \((\text{Ran} J)^\perp\) are \(L(G)\)-subcomodules of \((B(L^2(G)), \delta_G)\).

**Proof.** First, from the covariance relations
\[
\lambda_s f \lambda_{s^{-1}} = f_s, \quad s \in G,
\]
where \(f_s(t) = f(s^{-1}t)\), we get
\[
\text{Bim}(J^\perp) = \text{span}^w\{\lambda_s f : s \in G, f \in J^\perp\},
\]
thus \((2)\) and \((3)\) imply that \(\Phi(\text{Bim}(J^\perp)) = J^\perp \bar{\otimes} \alpha_G G\). It remains to show that \(\Phi((\text{Ran} J)^\perp) = J^\perp \ltimes_{\alpha_G} G\).

Note that
\[
J^\perp \ltimes_{\alpha_G} G = (J^\perp \bar{\otimes} B(L^2(G)))^\alpha_G
\]
\[
= (L^\infty(G) \bar{\otimes} B(L^2(G)))^\alpha_G \cap (J^\perp \bar{\otimes} B(L^2(G)))
\]
\[
= (L^\infty(G) \bar{\otimes} \alpha_G G) \cap (J^\perp \bar{\otimes} B(L^2(G))),
\]
since \((L^\infty(G) \bar{\otimes} B(L^2(G)))^\alpha_G = (L^\infty(G) \bar{\otimes} \alpha_G G)\) by the Digernes–Takesaki theorem.

Therefore, if \(y \in L^\infty(G) \bar{\otimes} \alpha_G G\), then
\[
y \in J^\perp \ltimes_{\alpha_G} G \iff (h \otimes \text{id}_{B(L^2(G))})(y) = 0, \forall h \in J.
\]
Since \((\text{Ran} J)^\perp\) is the intersection of the kernels of the maps \(\Theta(h)\) for \(h \in J\), and \(J^\perp \ltimes_{\alpha_G} G\) is the intersection of the kernels of the maps \((h \otimes \text{id}_{B(L^2(G))})\)
for $h \in J$, it suffices to prove that
\[ \Theta(h) = (h \otimes \text{id}_{B(L^2(G))}) \circ \Phi, \quad \forall h \in L^1(G). \]
Since both maps are linear and $w^*$-continuous, it suffices to prove the equality
for elements of the form $f \lambda_s$ for $f \in L^\infty(G)$ and $s \in G$, whose linear span is $w^*$-dense in $B(L^2(G))$. But $\Theta(h)$ and $(h \otimes \text{id}_{B(L^2(G))}) \circ \Phi$ are $L(G)$-module maps since clearly
\[ \Theta(h)(f \lambda_s) = \Theta(h)(f) \lambda_s \]
and
\[ (h \otimes \text{id})(\Phi(f \lambda_s)) = (h \otimes \text{id})(\Phi(f)(1 \otimes \lambda_s)) = (h \otimes \text{id})(\Phi(f)) \lambda_s, \]
therefore it suffices to prove that
\[ \Theta(h)(f) = (h \otimes \text{id})(\Phi(f)) \quad \text{for all } h \in L^1(G) \text{ and } f \in L^\infty(G). \]
Indeed, first observe that $\Theta(h)(f) = f_h$, where $f_h(t) = \int_G h(s)f(ts) \, ds$. Thus, for any $k \in L^1(G),$
\[ \langle \Theta(h)(f), k \rangle = \langle f_h, k \rangle = \int_G f_h(t)k(t) \, dt = \int_G h(s)f(ts)k(t) \, ds \, dt \]
\[ = \langle \alpha_G(f), h \otimes k \rangle = \langle (h \otimes \text{id})(\alpha_G(f)), k \rangle = \langle (h \otimes \text{id})(\Phi(f)), k \rangle \]
and so $\Theta(h)(f) = (h \otimes \text{id})(\Phi(f))$. \qed

The isomorphism $B(L^2(G)) \cong L^\infty(G) \rtimes G$ is usually viewed under the extended comultiplication $\Psi(T) := U_G(T \otimes 1)U^*_G$ for $T \in B(L^2(G))$ (this extends the comultiplication $\alpha'_G = \sigma \circ \alpha_G$ on $L^\infty(G)$; see Section 2) as opposed to the map $\Phi(T) = V^*_G W^*_G(T \otimes 1)W_G V_G$. The use of $\Phi$ instead of $\Psi$ in the proof of Proposition 5.1 is justified because we consider right comodule actions and $L^\infty(G)$ is viewed with the opposite comultiplication $\alpha_G$.

**Remark 5.2.** Following [16], let $CB^\sigma_{L^\infty(G)}(B(L^2(G)))$ be the algebra of all $w^*$-continuous completely bounded $L^\infty(G)$-bimodule maps on $B(L^2(G))$ which leave $L(G)$ invariant. Also, let $V^b_{\text{inv}}(G)$ and $V^\infty_{\text{inv}}(G)$ be the space of continuous right invariant Schur multipliers and the space of measurable right invariant Schur multipliers respectively (see [16] Section 4 for the precise definitions). It is shown in [16, Theorem 4.3] that there are completely isometric isomorphisms
\[ M_0A(G) \cong V^b_{\text{inv}}(G) \cong V^\infty_{\text{inv}}(G) \cong CB^\sigma_{L^\infty(G)}(B(L^2(G))). \]
In particular, every completely bounded multiplier $u \in M_0A(G)$ is identified with a map $S_u \in CB^\sigma_{L^\infty(G)}(B(L^2(G)))$ (sometimes called a Herz–Schur multiplier), which is the unique element in $CB^\sigma_{L^\infty(G)}(B(L^2(G)))$ that extends the completely bounded and $w^*$-continuous map $M_u = m^*_u : L(G) \to L(G)$ defined in Section 4 above.
In the case of an element \( u \in A(G) \), the map \( S_u \) can be described in terms of the \( L(G) \)-action \( \delta_G \) on \( B(L^2(G)) \), namely,

\[
S_u = (\text{id}_{B(L^2(G))} \otimes u) \circ \delta_G : B(L^2(G)) \to B(L^2(G))
\]

for any \( u \in A(G) \).

Indeed, since \( \delta_G(f) = f \otimes 1 \) and \( \delta_G(\lambda_s) = \lambda_s \otimes \lambda_s \) for any \( f \in L^\infty(G) \) and \( s \in G \), it follows easily that

\[
(id_{B(L^2(G))} \otimes u) \circ \delta_G(f \lambda_s) = u(s)f \lambda_s, \quad s \in G, \ f \in L^\infty(G), \ u \in A(G).
\]

On the other hand, \( M_u(\lambda_s) = u(s)\lambda_s \) for any \( s \in G \), and therefore if \( u \in A(G) \), then \( (id_{B(L^2(G))} \otimes u) \circ \delta_G \) is the unique completely bounded \( w^* \)-continuous \( L^\infty(G) \)-bimodule morphism on \( B(L^2(G)) \) that extends \( M_u \). That is, \( S_u = (id_{B(L^2(G))} \otimes u) \circ \delta_G \) for all \( u \in A(G) \).

Therefore, the \( A(G) \)-module structure of \( B(L^2(G)) \) induced by the \( L(G) \)-action \( \delta_G : B(L^2(G)) \to B(L^2(G)) \otimes L(G) \) (recall Remark 2.2) can be described in terms of Schur multipliers \( S_u \) with \( u \in A(G) \), that is,

\[
S_u(T) = (id_{B(L^2(G))} \otimes u) \circ \delta_G(T) = u \cdot T
\]

for all \( u \in A(G) \) and \( T \in B(L^2(G)) \).

It should be mentioned here that the fact that \( S_u = (id \otimes u) \circ \delta_G \) for all \( u \in A(G) \) as well as its dual version, i.e. \( \Theta(h) = (id \otimes h) \circ \Psi \) or equivalently \( \Theta(h) = (h \otimes id) \circ \Phi \) for \( h \in L^1(G) \), are known even at the level of locally compact quantum groups (see \([11]\)).

Thanks are due to the referee for pointing this out.

The next corollary, which is an immediate consequence of Theorem 3.17 and Proposition 5.1 provides a necessary and sufficient condition for \( \text{Bim}(J^\perp) = (\text{Ran} \ J)^\perp \) to hold.

**COROLLARY 5.3.** The following conditions are equivalent:

1. \( \text{Bim}(J^\perp) = (\text{Ran} \ J)^\perp \),
2. \((\text{Ran} \ J)^\perp, \delta_G) \) is a non-degenerate \( L(G) \)-comodule,
3. \((\text{Ran} \ J)^\perp = \text{span} \{ S_u(T) : u \in A(G), \ T \in (\text{Ran} \ J)^\perp \} \).

**Proof.** The equivalence between (a) and (b) is clear from Theorem 3.17 and Proposition 5.1. Also, the equivalence between (b) and (c) follows again from Theorem 3.17 because \( S_u(T) = (id_{B(L^2(G))} \otimes u) \circ \delta_G(T) = u \cdot T \) for any \( u \in A(G) \) and \( T \in B(L^2(G)) \).

**Remark 5.4.** Note that from Propositions 5.1 and 4.2 it follows that if \( G \) has the AP, then there exists a net \( \{u_i\}_{i \in I} \) in \( A(G) \) such that

\[
S_{u_i}(T) \to T \quad \text{ultraweakly for all } T \in B(L^2(G)).
\]

Therefore, Corollary 5.3 implies that if \( G \) has the AP, then \( \text{Bim}(J^\perp) = (\text{Ran} \ J)^\perp \).
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