TORIC IDEALS WHICH ARE DETERMINANTAL

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Abstract. Given any equigenerated monomial ideal $I$ with the property that the defining ideal $J$ of the fiber cone $F(I)$ of $I$ is generated by quadratic binomials, we introduce a matrix such that the set of its binomial 2-minors is a generating set of $J$. In this way, we characterize the fiber cone of sortable and Freiman ideals.

Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in variables $x_1, \ldots, x_n$ over $K$. For a graded ideal $I \subset S$ the fiber cone $F(I)$ of $I$ is the standard graded $K$-algebra $\bigoplus_{k \geq 0} J^k/m I^k$, where $m$ denotes the unique maximal graded ideal of $S$. Indeed, $F(I) = R/I(R) \cap I\mathcal{R}(I)$, where $\mathcal{R}(I) = \bigoplus_{k \geq 0} J^k/m I^k \subset S[t]$ is the Rees ring of $I$.

Let $I$ be a monomial ideal with $G(I) = \{u_1, \ldots, u_q\}$ and $T = K[t_{u_1}, \ldots, t_{u_q}]$ be the polynomial ring in variables $t_{u_1}, \ldots, t_{u_q}$ over $K$. The $K$-algebra homomorphism $T \to F(I), t_{u_i} \mapsto u_i + \mathfrak{m} I$ induces the isomorphism $F(I) \cong T/J$. The ideal $J$ is called the defining ideal of $F(I)$. Finding the minimal generators of $J$ and the algebraic properties of $F(I)$ is a difficult problem even in concrete cases. For example, for the symmetric ideals

$$I_1 = (x_1^{11}, x_1^9x_2, x_1^7x_2^4, x_1^5x_2^6, x_1^4x_2^7, x_1^2x_2^9, x_2^{11}),$$

$$I_2 = (x_1^{11}, x_1^{10}x_2, x_1^7x_2^4, x_1^5x_2^6, x_1^4x_2^7, x_1^2x_2^9, x_1x_2^{10}, x_2^{11})$$

generated in degree 11 in $K[x_1, x_2]$, one can check by CoCoA [2] that the defining ideal of $F(I_1)$ is an ideal generated by quadratic binomials and $F(I_1)$ is a Cohen-Macaulay algebra, while the minimal generating set of the defining ideal of $F(I_2)$ includes binomials in degrees 2 and 4, and $F(I_2)$ is not Cohen-Macaulay. Note that, $G(I_1)$ and $G(I_2)$ just differ in 2 monomials. We recall that a monomial ideal $I \subset S = K[x_1, x_2]$ with $G(I) = u_1, \ldots, u_q$ and $u_i = x_1^{a_i}x_2^{b_i}$ satisfying the properties $a_1 > a_2 > \ldots > a_q = 0$ and $0 = b_1 < b_2 < \ldots < b_q$ is called a symmetric ideal, if $b_i = a_{q-i+1}$ for $i = 1, \ldots, q$. The fiber cones of symmetric ideals with 4 generators are well studied in [8] and [10]. Moreover, [8] includes a characterization of the fiber cones of concave and convex monomial ideals and their algebraic properties.

In this paper, for an equigenerated monomial ideal $I$ with the property that the defining ideal $J$ of $F(I)$ is generated by quadratic binomials, we interpret $J$ as the set of binomial 2-minors of a special matrix. By an equigenerated monomial ideal we
mean an ideal generated by monomials in a single degree. In this paper we consider
that a binomial has exactly two non-zero monomials.

In Section 1 we associate to a sortable ideal \( I \) a matrix \( T_I \), such that the defining ideal \( J \) of \( F(I) \) is generated by the set of binomial 2-minors of \( T_I \) (Theorem 1.3). Moreover, we show that equigenerated \( c \)-bounded strongly stable monomial ideals, in particular Veronese type ideals, are sortable and hence, their fiber cones are Cohen-Macaulay normal domains and reduced Koszul algebras.

In Section 2 for any equigenerated monomial ideal \( I \in K[x_1, \ldots, x_n] \) with \( n \geq 3 \), we show that if the defining ideal \( J \) of the fiber cone \( F(I) \) is generated by quadratic binomials, then \( J \) is generated by the set of binomial 2-minors of \( T_I \) (Theorem 2.3). For the case \( I \subset K[x_1, x_2] \), we associate to \( I \) a matrix \( T_I \) in a different way, and show that the defining ideal of \( F(I) \) is generated by the set of binomial 2-minors of \( T_I \) (Theorem 2.2). In particular, we determine the fiber cone of Freiman ideals. A Freiman ideal \( I \) is an equigenerated monomial ideal such that \( \mu(I^2) = \ell(I)\mu(I) - \binom{\ell(I)}{2} \), where \( \mu(I) \) is the minimal number of generators of \( I \), and \( \ell(I) \) denotes the analytic spread of \( I \) which is by definition the Krull dimension of \( F(I) \). Freiman ideals are studied in [6] and [9].

1. The Fiber Cone of Sortable Ideals

Let \( S = K[x_1, \ldots, x_n] \) be the polynomial ring in the variables \( x_1, \ldots, x_n \) over a field \( K \). We denote by \( m \) the unique maximal graded ideal of \( S \). Let \( I \) be a monomial ideal of \( S \) and \( G(I) = \{ u_1, \ldots, u_q \} \) be the minimal set of monomial generators of \( I \). The fibre cone \( F(I) \) of \( I \) is defined as the standard graded \( K \)-algebra \( \bigoplus_{k \geq 0} I^k/mI^k \). Let \( T = K[t_{u_1}, \ldots, t_{u_q}] \), where \( t_{u_1}, \ldots, t_{u_q} \) are independent variables. Then \( F(I) \cong T/J \), where \( J \) is the kernel of the \( K \)-algebra homomorphism \( T \to F(I) \) with \( t_{u_i} \mapsto u_i+mI \). In this section we determine the fiber cone of sortable ideals and introduce several classes of ideals which are sortable.

Let \( d \) be a positive integer and \( S_d \) be the \( \mathbb{K} \)-vector space generated by monomials of degree \( d \) in \( S \). For monomials \( u, v \in S_d \) we write \( uv = x_{i_1}x_{i_2} \ldots x_{i_{2d}} \) with \( 1 \leq i_1 \leq i_2 \leq \ldots \leq i_{2d} \leq n \). The pair \( (u', v') \) with \( u' = x_{i_1}x_{i_3} \ldots x_{i_{2d - 1}} \) and \( v' = x_{i_2}x_{i_4} \ldots x_{i_{2d}} \) is called the sorting of \( (u, v) \). So we get the map

\[
\text{sort} : S_d \times S_d \to S_d \times S_d, (u, v) \mapsto (u', v'),
\]
called sorting operator. A pair \( (u, v) \) is called sorted if \( \text{sort}(u, v) = (u, v) \), otherwise it is called unsorted. It is shown in [11] Section 6.2] that the pair \( (u, v) \) with \( u = x_{i_1}x_{i_2} \ldots x_{i_d} \) and \( v = x_{j_1}x_{j_2} \ldots x_{j_d} \) is sorted if and only if

\[
(1) \quad i_1 \leq j_1 \leq i_2 \leq j_2 \leq \ldots \leq i_d \leq j_d.
\]

Note that if \( (u, v) \) is sorted, then \( u \geq_{\text{lex}} v \), where \( \geq_{\text{lex}} \) denotes the lexicographic order on Mon(\( S \)), the set of monomials of \( S \).

**Definition 1.1.** (a) A set of monomials \( A \subset S_d \) is called sortable if \( \text{sort}(A \times A) \subset A \times A \).

(b) An equigenerated monomial ideal \( I \) is called a sortable ideal, if \( G(I) \) is a sortable set.
Let $A \subset S_d$ be a sortable set of monomials and $I$ be the ideal generated by $A$. We denote by $K[A]$ the semigroup ring generated over $K$ by $A$. Let $T = K[t_u : u \in A]$ be the polynomial ring with the order on variables given by $t_u > t_v$ if $u \geq_{\text{lex}} v$. Also, let $\varphi : T \to K[A]$ be the $K$-algebra homomorphism defined by $t_u \mapsto u$ for all $u \in A$ and $P_A$ be the kernel of $\varphi$. Since the ideal $I$ is equigenerated, $F(I) \cong K[A]$ (see [7, the proof of Corollary 1.2]) and so the toric ideal $P_A$ is the defining ideal $J$ of $F(I)$ in the representation $F(I) = T/J$ of the fiber cone of $I$.

The following well known theorem plays an important role in the proof of the main theorem of this section (See [4, Theorem 6.16]).

**Theorem 1.2.** Let $K[A]$ be a $K$-algebra generated by a sortable set of monomials $A \subset S_d$ and $P_A \subset R$ its toric ideal. Then $$G = \{t_u t_v - t_u' t_v' : u, v \in A, (u, v) \text{ unsorted}, (u', v') = \text{sort}(u, v)\}$$ is the reduced Gröbner basis of $P_A$ with respect to the sorting order.

It follows from [4, Theorem 6.15] that the ideal $\text{in}_{\prec}(G)$ is generated by the monomials $t_u t_v$ where $(u, v)$ is unsorted.

Before stating the main theorem of this section, we recall that an affine semigroup $H$ generated by the set $H = \{h_1, \ldots, h_k\} \subset \mathbb{Z}^n$ is called normal if it satisfies the following condition: if $mg \in H$ for some $g \in \mathbb{Z}H$ and $m > 0$, then $g \in H$, where $\mathbb{Z}H$ is the subgroup of $\mathbb{Z}^n$ generated by $H$. Also, a domain $R$ is called normal if it is integrally closed, that is $R = \overline{R}$ where $\overline{R}$ is the integral closure of $R$.

**Notation 1.3.** Let $I \subset S$ be an ideal generated minimally by a set of monomials of degree $d$. One can consider $G(I) = \{u_1, \ldots, u_q\}$ as a subset of $G(m^d)$. Let $M$ be the matrix which the entries of its $i$-th row are the monomials of $G(m^d)$ containing $x_i$, ordered lexicographically from left to right, for $i = 1, \ldots, n$. This matrix has $n$ rows and $\binom{n+d-2}{d-1}$ columns. We replace the entries of $M$ belonging to $G(m^d) \setminus G(I)$ by 0, remove its zero columns and denote the obtained matrix by $M_I$. Then we replace any non-zero element $u$ of $M_I$ by indeterminate $t_u$ of $R = K[t_{u_1}, \ldots, t_{u_q}]$. Denote this matrix by $T_I$ and call it the matrix associated to $I$.

**Example 1.4.** Let $n = 3$ and $d = 3$. The matrix $M$ is the following:

$$M = \begin{pmatrix} x_1^3 & x_1^2 x_2 & x_1 x_2^2 & x_1 x_3 & x_2 x_3 & x_3^2 \\ x_2^3 & x_2 x_1^2 & x_1 x_2 x_3 & x_1 x_3^2 & x_1 x_2^2 & x_2 x_3 \\ x_3^3 & x_3 x_1 x_2 & x_1 x_2 x_3 & x_1 x_3^2 & x_2 x_3 & x_3 x_2 \\ x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_2 x_3 \end{pmatrix}.$$

Let $I$ be the Veronese type ideal $I = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_1 x_2 x_3, x_1 x_2 x_3, x_2 x_3) \subset K[x_1, x_2, x_3]$. We set $u_1 = x_1^3$, $u_2 = x_1^2 x_2$, $u_3 = x_1 x_2^2$, $u_4 = x_1 x_2 x_3$, $u_5 = x_1 x_2 x_3$ and $u_6 = x_2 x_3$. Let $F(I) = K[t_{u_1}, \ldots, t_{u_6}]/J$. We have

$$T_I = \begin{pmatrix} t_{u_1} & t_{u_2} & t_{u_3} & t_{u_4} & t_{u_5} \\ t_{u_2} & t_{u_3} & t_{u_4} & t_{u_5} & 0 \\ t_{u_3} & t_{u_4} & t_{u_5} & 0 & t_{u_6} \\ t_{u_4} & t_{u_5} & t_{u_6} & 0 & 0 \end{pmatrix}.$$

**Theorem 1.5.** Let $I \subset S$ be a sortable ideal with $G(I) = \{u_1, \ldots, u_q\}$ and the fiber cone $F(I) = K[t_{u_1}, \ldots, t_{u_q}]/J$. Also, let $T_I$ be the matrix associated to $I$, introduced in Notation 1.3.
(a) The toric ideal $J$ is generated by the set of binomial 2-minors of $T_I$. Indeed,

$$J = \{ t_u t_v - t_{u'} t_{v'} : u, v, u', v' \in G(I), \begin{pmatrix} t_u & t_{u'} \\ t_v & t_{v'} \end{pmatrix} \text{ is a submatrix of } T_I \}.$$

(b) $F(I)$ is a reduced Koszul algebra.

(c) $F(I)$ is a Cohen-Macaulay normal domain.

Proof. (a) For $i = 1, \ldots, n$, dividing the entries of the $i$-th row of the matrix $M$ by $x_i$, we get a matrix whose entries of all rows are the monomials of $G(\mathfrak{m}^{d-1})$ ordered lexicographically from left to right. This implies that the set of binomial 2-minors of $T_I$ includes in $J$.

Now, we show that for the monomials $u, v, u', v' \in G(I)$ if the binomial $f = t_u t_v - t_{u'} t_{v'}$ belongs to $J$, then $f$ is a 2-minor of $T_I$. By theorem 1.2

$$G = \{ t_u t_v - t_{u'} t_{v'} : u, v \in G(I), (u, v) \text{ unsorted}, (u', v') = \text{sort}(u, v) \}$$

is the reduced Gröbner basis of $J$ with respect to the sorting order. We show that if monomials $u, v \in G(I)$ are unsorted, then $u$ and $v$ are the $i$-th and the $kl$-th entries of the matrix $M_I$ respectively, such that $i \neq k$ and $j \neq l$ and that sort$(u, v) = (u', v')$, where $u'$ and $v'$ are the $il$-th and the $kj$-th entries of the matrix $M_I$ respectively. This implies that the determinants of $2 \times 2$ submatrices of $T_I$ which have no zero entries, form a Gröbner basis of $J$. Let $(u, v)$ with $u = x_{i_1} x_{i_2} \ldots x_{i_q}$ and $v = x_{j_1} x_{j_2} \ldots x_{j_q}$ be an unsorted pair in $G(I) \times G(I)$. Notice that $u$ and $v$ belong to different columns of $M$. Indeed, if two different monomials $w, w'$ belong to the same column of $M$, we have $w = x_{p} w_1$ and $w' = x_{q} w_1$ for $1 \leq p \neq q \leq n$ and a monomial $w_1 \in G(\mathfrak{m}^{d-1})$. So, by 1 the pair $(w, w')$ is sorted. On the other hand, since $(u, v)$ is unsorted, $u$ is divisible by $x_i$ and $v$ is divisible by $x_j$ for some $1 \leq i \neq j \leq n$, because otherwise $u = v = x_r^q$ for a variable $x_r$, a contradiction. Hence, we can find $u$ and $v$ in different rows of the matrix $M$ (although they may appear in the same row as well). So we assume that $u = u_{ij}$ and $v = v_{kl}$ such that $i \neq k$ and $j \neq l$. Since $G(I)$ is a sortable set, it follows that sort$(u, v) \in G(I) \times G(I)$. Assume that sort$(u, v) = (u', v')$. So $t_u t_v - t_{u'} t_{v'}$ belongs to the reduced Gröbner basis of $J$ by Theorem 1.2. Note that $u_{ij} = x_{i} u_1$ and $v_{kj} = x_{k} u_1$ for a monomial $u_1 \in G(\mathfrak{m}^{d-1})$. Similarly, $u_{id} = x_{i} u_2$ and $v_{kl} = x_{k} u_2$ for a monomial $u_2 \in G(\mathfrak{m}^{d-1})$. Therefore, $u_{ij} v_{kl} = u_{id} v_{kj} = x_i x_k u_1 u_2$ and hence sort$(u_{ij}, v_{kl}) = \text{sort}(u_{id}, v_{kj})$. Suppose that $(u_{id}, v_{kj})$ is unsorted. It follows from Theorem 1.2 that $t_{u_{id}} t_{v_{kj}} - t_{u'} t_{v'}$ belongs to the reduced Gröbner basis of $J$ which is a contradiction, because $(t_u t_v - t_{u'} t_{v'}) - (t_{u_{id}} t_{v_{kj}} - t_{u'} t_{v'}) = t_u t_v - t_{u_{id}} t_{v_{kj}}$ belongs to the reduced Gröbner basis of $J$. Therefore, sort$(u_{ij}, v_{kl}) = (u_{id}, v_{kj})$. So, the assertion follows from Theorem 1.2. Notice that this Gröbner basis is not necessary reduced.

(b) It follows from [4, Theorem 6.15]) that $\text{in}_{<}(J)$ is a square-free monomial ideal. This yields that $J$ is a radical ideal (see [5, Theorem 3.3.7]) and hence $F(I)$ is a reduced algebra. Moreover, since $J$ has a quadratic Gröbner basis, $F(I)$ is Koszul by a well known result of Fröberg (see [4, Theorem 6.7]).

(c) Since $\text{in}_{<}(J)$ is a squarefree monomial ideal, it follows from a result by Sturmfels ([13, Proposition 13.15]) that $F(I)$ is normal. Moreover, by a result of Hochster ([11, Theorem 1]) $F(I)$ is Cohen-Macaulay. \qed
In the rest of this section we show that any equigenerated \(c\)-bounded strongly stable monomial ideal is sortable.

Let \(c = (c_1, \ldots, c_n)\) be an integer vector with \(c_i \geq 0\). The monomial \(u = x_1^{a_1} \cdots x_n^{a_n}\) is called \(c\)-bounded, if \(a \leq c\), that is, \(a_i \leq c_i\) for all \(i\). Let \(I = (u_1, \ldots, u_m)\) be a monomial ideal. We set

\[I^{c} = (u_i: \text{ } u_i \text{ is } c\text{-bounded}).\]

We also set \(m(u) = \max\{i: \text{ } a_i \neq 0\}\). The following definition is obtained from [2].

**Definition 1.6.** Let \(I \subset S\) be a \(c\)-bounded monomial ideal.

1. \(I\) is called **\(c\)-bounded strongly stable** if for all \(u \in G(I)\) and all \(i < j\) with \(x_j|u\) and \(x_iu/x_j \in I\), it follows that \(x_iu/x_j \in I\).
2. \(I\) is called **\(c\)-bounded stable** if for all \(u \in G(I)\) and all \(i < m(u)\) for which \(x_iu/x_m(u)\) is \(c\)-bounded, it follows that \(x_iu/x_m(u) \in I\).

It is clear that a \(c\)-bounded strongly stable monomial ideal is \(c\)-bounded stable.

The smallest \(c\)-bounded strongly stable ideal containing \(c\)-bounded monomials \(u_1, \ldots, u_m\) is denoted by \(B^c(u_1, \ldots, u_m)\). A monomial ideal \(I\) is called a **\(c\)-bounded strongly stable principal ideal**, if there exists a \(c\)-bounded monomial \(u\) such that \(I = B^c(u)\). The smallest strongly stable ideal containing \(u_1, \ldots, u_m\) (with no restrictions on the exponents) is denoted \(B(u_1, \ldots, u_m)\). The monomials \(u_1, \ldots, u_m\) are called **Borel generators** of \(I = B(u_1, \ldots, u_m)\).

**Proposition 1.7.** Let \(I = B^c(u_1, \ldots, u_m)\) be an equigenerated \(c\)-bounded strongly stable monomial ideal. Then \(I\) is a sortable ideal.

**Proof.** First we prove the assertion for the case \(c\)-bounded strongly stable principal ideal \(I = B^c(u_k)\) where \(1 \leq k \leq m\). Assume that \(v, w \in G(B^c(u_k))\) and \(\text{sort}(v, w) = (v', w')\). For this purpose we first show that \(v', w'\) are \(c\)-bounded monomials. So, we must check that for all \(i \in \{1, \ldots, n\}\), the degrees of \(x_i\) in \(v'\) and \(w'\) are not greater than \(c_i\). Let \(\deg_{x_i}(v) = a_i\) and \(\deg_{x_i}(w) = b_i\). Note that \(a_i, b_i \leq c_i\). If \(a_i + b_i\) is even, \(\deg_{x_i}(v') = \deg_{x_i}(w') = (a_i + b_i)/2\) by the definition of the sorting operator. Therefore, \(\deg_{x_i}(v')\), \(\deg_{x_i}(w') \leq c_i\). Now let \(a_i + b_i\) be an odd integer. Then \(\deg_{x_i}(v') = (a_i + b_i + 1)/2\) and \(\deg_{x_i}(w') = (a_i + b_i - 1)/2\). Hence, \(\deg_{x_i}(v'), \deg_{x_i}(w') \leq c_i\). This means that \(v', w'\) are \(c\)-bounded monomials.

Now, since \(vw = v'w'\) and \(v, w \in G(B^c(u_k))\), it follows from [3, Lemma 2.7] that \(v', w' \in B^c(u_k)\), and since \(v, w, v'\) and \(w'\) are in the same degree, we get \(v', w' \in G(B^c(u_k))\).

Finally, since \(I = B^c(u_1, \ldots, u_m) = B^c(u_1) + \cdots + B^c(u_m)\), the assertion follows. \(\square\)

**Corollary 1.8.** The statements of Theorem [1,3] hold for equigenerated \(c\)-bounded strongly stable monomial ideals.

**Remark 1.9.** (a) An equigenerated \(c\)-bounded stable ideal is not necessarily sortable. For example, the ideal \(I = (x_1^3, x_2^3x_2, x_1x_2^2, x_1x_2x_3) \subset K[x_1, x_2, x_3]\) is a \(c\)-bounded stable ideal of degree 3, where \(c = (3, 2, 1)\). Note that \(\text{sort}(x_1^3, x_1x_2x_3) = (x_1^2x_2, x_1^2x_3) \notin G(I) \times G(I)\).
(b) Let \( v \) be a monomial in \( S \) and \( I = B(v) \). It follows from \([3]\) Lemma 2.7 that \( G(I) \) is a sortable set. So, since for equigenerated monomials \( u_1, \ldots, u_m \) we have \( B(u_1, \ldots, u_m) = B(u_1) + \cdots + B(u_m) \), the statements of the Theorem 1.5 hold for equigenerated strongly stable monomial ideals.

Now we come with an important class of equigenerated \( c \)-bounded strongly stable monomial ideals, called Veronese type ideals. Let \( n \) be a positive integer, \( d \) be an integer, and \( \mathbf{a} = (a_1, \ldots, a_n) \) be an integer vector with \( a_1 \geq a_2 \geq \cdots \geq a_n \). The monomial ideal \( I_{\mathbf{a},n,d} \subset S = K[x_1, \ldots, x_n] \) with the minimal generating set

\[
G(I_{\mathbf{a},n,d}) = \{ x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \mid \sum_{i=1}^{n} b_i = d \text{ and } b_i \leq a_i \text{ for } i = 1, \ldots, n \}
\]

is called a Veronese type ideal. It is obvious that \( I_{\mathbf{a},n,d} \) is \( \mathbf{a} \)-bounded strongly stable.

Corollary 1.10. The statements of Theorem 1.5 hold for Veronese type ideals.

Example 1.11. In the Example 1.4, the ideal \( I \) is Veronese type \( I_{\mathbf{a},3,3} \) with \( \mathbf{a} = (3, 2, 1) \). Therefore,

\[
J = (t_{u_1} t_{u_5} - t_{u_2}^2, t_{u_1} t_{u_5} - t_{u_2} t_{u_3}, t_{u_2} t_{u_3} - t_{u_3} t_{u_4}, t_{u_2} t_{u_4} - t_{u_4} t_{u_5}, t_{u_3} t_{u_5} - t_{u_2}^2),
\]

which is confirmed by CoCoA.

2. Toric ideals generated by quadratic binomials

Let \( I \subset K[x_1, \ldots, x_n] \) be an equigenerated ideal, such that the toric defining ideal \( J \) of \( F(I) \) is generated by quadratic binomials. We associate to \( I \) a matrix, and show that \( J \) is generated by the set of binomial \( 2 \)-minors of this matrix. Indeed, \( J \) is generated by the set of the determinants of \( 2 \times 2 \) submatrices of this matrix which have no zero entries. The construction of the associated matrix when \( n = 2 \) is different from the cases \( n \geq 3 \). For this purpose we introduce the following notation.

Notation 2.1. Let \( I \subset K[x_1, x_2] \) be an ideal generated in degree \( d \) with the minimal set of monomial generators \( G(I) = \{ u_1, \ldots, u_q \} \) which can be considered as a subset of \( G(\mathbb{m}^d) \). We assume that \( I \) contains \( x_1^d \), because otherwise there exist a positive integer \( d' \) and an ideal \( J \) such that \( I = x_2^{d'} J \) and \( G(J) \) contains \( x_1^{d - d'} \), for which we have \( F(I) = F(J) \). Also, we assume that

\[
u_1 = x_1^d >_{\text{lex}} u_2 = x_1^{d-a} x_2^a >_{\text{lex}} u_3 \ldots >_{\text{lex}} u_{q-1} >_{\text{lex}} u_q = x_1^{d-a-b} x_2^{a+b},\]

where \( 1 \leq a, b \leq d - 1 \) and \( 2 \leq a + b \leq d \).

We arrange the columns of the matrix \( \mathcal{M} \) in the following way

\[
\mathcal{M} = \begin{pmatrix}
x_1^d & u_1 & x_1^{d-1} x_2 & x_1^{d-2} x_2^2 & \ldots & x_1^{d-b} x_2^b \\
x_1^{d-1} x_2 & x_1^{d-2} x_2 & x_1^{d-3} x_2^3 & \ldots & x_1^{d-b-1} x_2^{b+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_1^{d-a} x_2^a = u_2 & x_1^{d-a-1} x_2^{a+1} & x_1^{d-a-2} x_2^{a+2} & \ldots & x_1^{d-a-b} x_2^{a+b} = u_q
\end{pmatrix}.
\]

We replace the entries of \( \mathcal{M} \) belonging to \( G(\mathbb{m}^d) \setminus G(I) \) by 0 and denote the obtained matrix by \( \mathcal{M}_I \). We also replace any nonzero element \( u \) of \( \mathcal{M}_I \) by the indeterminate \( t_u \) of \( R = K[t_{u_1}, \ldots, t_{u_q}] \) and denote this matrix by \( \mathcal{T}_I \).
Theorem 2.2. Let \( I \subset K[x_1, x_2] \) be a monomial ideal generated in degree \( d \) with the unique minimal set of monomial generators \( G(I) = \{u_1, \ldots, u_q \} \) and let \( F(I) = K[t_{u_1}, \ldots, t_{u_q}]/J \). If the toric ideal \( J \) is generated by quadratic binomials, then \( J \) is the ideal generated by the set of binomial 2-minors of \( T_I \).

Proof. Let \( m_j \) be the least common multiple of all entries of the \( j \)-th column of \( M \) for \( j = 1, \ldots, b + 1 \). Dividing the \( j \)-th column of \( M \) by \( m_j \) for all \( j \), we get a matrix whose entries of all columns are the monomials of \( G(m^d) \) ordered lexicographically up to down. So, every 2-minor of this matrix is zero. This implies that, every binomial 2-minor of \( T_I \) belongs to \( J \).

Conversely, we show that any quadratic binomial of \( G(J) \) stands as the determinant of a \( 2 \times 2 \) submatrix of \( T_I \) which has no zero entries. Let \( f = t_u t_v - t_w t_{v'} \in J \), where \( u, v, u', v' \in G(J) \). It is clear from the arrangement of the columns of \( M \) that \( u \) and \( v \) appear on the main diagonal of a \( 2 \times 2 \) submatrix of \( M \) (note that \( uv \neq x_1^{2d-1}x_2 \) and also \( u'v' \neq x_1x_2^{2d-1} \)). We need to show that \( u' \) and \( v' \) appear on the secondary diagonal of the same submatrix. Let \( u = x_1^{d-q}x_2^p, v = x_1^{d-r}x_2^r, u' = x_1^{d-q}x_2^q, v' = x_1^{d-s}x_2^s \). Since \( f \in J \), it follows that \( uv = u'v' \) and hence \( p + q = r + s \). So, without loss of generality, we may assume that \( s > p \) and \( q > r \). In addition, we let \( u \) and \( v \) be the \( ij \)-th and the \( kl \)-th entries of \( M \) respectively. We show that \( u' \) and \( v' \) are the \( il \)-th and the \( kj \)-th entries of \( M \) respectively. Since \( p + q = r + s \), therefore \( s - p = q - r \). So, it follows from the arrangement of the columns of \( M \) that \( u' \) and \( v' \) appear as the \( il \)-th and the \( kj \)-th entries of \( M \) respectively, and the proof is complete. \( \square \)

In the next theorem we let \( I \) be an equigenerated monomial ideal in the polynomial ring \( S = K[x_1, \ldots, x_n] \) with \( n \geq 3 \), and \( T_I \) be its associated matrix introduced in Notation 1.3.

Theorem 2.3. Let \( I \subset S = K[x_1, \ldots, x_n] \) with \( n \geq 3 \) be a monomial ideal generated in degree \( d \) with the minimal set of monomial generators \( G(I) = \{u_1, \ldots, u_q \} \), and let \( F(I) = K[t_{u_1}, \ldots, t_{u_q}]/J \). If the toric ideal \( J \) is generated by quadratic binomials, then \( J \) is the ideal generated by the set of binomial 2-minors of \( T_I \).

Proof. As we stated in the proof of Theorem 1.3, if we divide the entries of the \( i \)-th row of the matrix \( M \) by \( x_i \) for \( i = 1, \ldots, n \), we get a matrix whose entries of all rows are the monomials of \( G(m^d) \) ordered lexicographically from left to right. This implies that all 2-minors of \( M \) are zero and therefore, any binomial 2-minors of \( T_I \) is contained in \( J \).

Conversely, for the nonzero monomials \( u, v, u', v' \in G(I) \), let the nonzero binomial \( f = t_u t_v - t_w t_{v'} \) belongs to \( J \). We show that \( f \) is a 2-minor of \( T_I \). Set \( u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, v = x_1^{\beta_1} \cdots x_n^{\beta_n}, u' = x_1^{\alpha'_1} \cdots x_n^{\alpha'_n} \) and \( v' = x_1^{\beta'_1} \cdots x_n^{\beta'_n} \). It is clear that \( t_u t_v - t_u t_{v'} \in J \) if and only if \( uv = u'v' \). Therefore, \( \alpha_i + \beta_i = \alpha'_i + \beta'_i \) for \( i = 1, \ldots, n \). Since \( f \) is a nonzero binomial, then \( uv \) is not pure power of a variable and so there are indices \( k \neq l \) such that \( x_k | u \) and \( x_l | v \). Thus, \( x_k | u'v' \) and \( x_l | u'v' \). We distinguish the following cases:

i) If \( x_k | u' \) and \( x_l | v' \), then \( u, u' \) appear in the \( k \)-th row of \( M \), and \( v, v' \) appear in \( l \)-th row of \( M \). We need to show that \( v' \) appears in the same column of \( u \), and \( v \)
appears in the same column of \( u' \). Since \( \alpha_i + \beta_i = \alpha'_i + \beta'_i \), we get \( \alpha_i - \beta'_i = \alpha'_i - \beta_i \) for \( i = 1, \ldots, n \). Set

\[
\frac{u}{\text{lcm}(u, u')} = \hat{u} = x_1^{\hat{\alpha}_1} \ldots x_n^{\hat{\alpha}_n},
\]
\[
\frac{u'}{\text{lcm}(u, u')} = \hat{u}' = x_1^{\hat{\alpha}'_1} \ldots x_n^{\hat{\alpha}'_n},
\]
\[
\frac{v}{\text{lcm}(v', v)} = \hat{v} = x_1^{\hat{\beta}_1} \ldots x_n^{\hat{\beta}_n},
\]

and

\[
\frac{v'}{\text{lcm}(v', v)} = \hat{v}' = x_1^{\hat{\beta}'_1} \ldots x_n^{\hat{\beta}'_n}.
\]

For \( i = 1, \ldots, n \), it is clear that \( \hat{\alpha}_i \neq 0 \) if and only if \( \alpha'_i = 0 \), and also \( \hat{\beta}'_i \neq 0 \) if and only if \( \beta_i = 0 \). Let \( \hat{\alpha}_i \neq 0 \). Then \( \hat{\alpha}'_i = 0 \). Now, \( \hat{\beta}'_i \neq 0 \), because otherwise the equality \( \alpha_i - \beta'_i = \alpha'_i - \beta_i \) gives a contradiction, since the left side is positive and the right side is negative. Therefore, \( \beta_i = 0 \) and so \( \alpha_i - \beta'_i = \alpha'_i - \beta_i = 0 \). It follows that \( \hat{u} = \hat{u}' \) and \( \hat{u}' = \hat{v} \). Now, since \( u = \text{lcm}(u, u')\hat{u}, \ u' = \text{lcm}(u, u')\hat{u}' \) and also \( v' = \text{lcm}(v', v)\hat{v}', v = \text{lcm}(v', v)\hat{v} \), it follows that \( \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \) is a submatrix of \( M \) and hence \( f \) is a 2-minor of \( T_I \).

ii) Let \( x_k, x_l \) do not divide \( v' \). It follows that \( x_k, x_l | u' \). So, there exists an index \( t = k, l \) such that \( x_t | v' \). Therefore, \( x_t | u \). If \( x_t | u \), then \( u, u' \) appear in the \( k \)-th row of \( M \) and \( v, v' \) appear in the \( t \)-th row of \( M \), and the conclusion is exactly the same as in the case (i). Now, assume that \( x_t \) does not divide \( v \). So \( x_t | u \). Therefore, \( u, v' \) appear in the \( t \)-th row of \( M \), and \( u', v \) appear in the \( l \)-th row of \( M \). We need to show that \( u' \) appears in the same column of \( u \), and \( v \) appears in the same column of \( v' \). Since \( \alpha_i + \beta_i = \alpha'_i + \beta'_i \), we get \( \alpha_i - \alpha'_i = \beta'_i - \beta_i \) for \( i = 1, \ldots, n \). Set

\[
\frac{u}{\text{lcm}(u, u')} = \hat{u} = x_1^{\hat{\alpha}_1} \ldots x_n^{\hat{\alpha}_n},
\]
\[
\frac{u'}{\text{lcm}(u, u')} = \hat{u}' = x_1^{\hat{\alpha}'_1} \ldots x_n^{\hat{\alpha}'_n},
\]
\[
\frac{v}{\text{lcm}(v', v)} = \hat{v} = x_1^{\hat{\beta}_1} \ldots x_n^{\hat{\beta}_n},
\]

and

\[
\frac{v'}{\text{lcm}(v', v)} = \hat{v}' = x_1^{\hat{\beta}'_1} \ldots x_n^{\hat{\beta}'_n}.
\]

For \( i = 1, \ldots, n \), it is clear that \( \hat{\alpha}_i \neq 0 \) if and only if \( \beta'_i = 0 \), and also \( \hat{\alpha}'_1 \neq 0 \) if and only if \( \beta_i = 0 \). Let \( \hat{\alpha}_i \neq 0 \). Then \( \hat{\alpha}'_1 = 0 \). Now, \( \hat{\beta}_1 \neq 0 \), because otherwise the equality \( \alpha_i - \alpha'_i = \beta'_i - \beta_i \) gives a contradiction, since the left side is positive and the right side is negative. Thus, \( \beta_i = 0 \) and hence \( \alpha_i - \alpha'_i = \beta'_i - \beta_i = 0 \). Therefore, \( \hat{u} = \hat{u}' \) and \( \hat{u}' = \hat{v} \). Now, since \( u = \text{lcm}(u, v')\hat{u}, \ v = \text{lcm}(u, v')\hat{v} \), it follows that \( \begin{pmatrix} u & u' \\ v & v' \end{pmatrix} \) is a submatrix of \( M \) and hence \( f \) is a 2-minor of \( T_I \). \( \square \)

An important consequence of Theorem 2.2 and Theorem 2.3 is a characterization of the fiber cone of Freiman ideals.

We recall that the analytic spread \( \ell(I) \) of an ideal \( I \) is by definition the Krull dimension of \( F(I) \). The following definition is obtained from [9].
**Definition 2.4.** An equigenerated monomial ideal $I$ is called a Freiman ideal, if $\mu(I^2) = \ell(I)\mu(I) - \binom{\ell(I)}{2}$.

**Corollary 2.5.** Assume that $T_I$ and $T_J$ are the matrices introduced in Notation 1.3 and Notation 2.1.

(a) Let $I = (u_1, \ldots, u_q) \subset K[x_1, x_2]$ be a Freiman ideal with the fiber cone $F(I) = K[t_{u_1}, \ldots, t_{u_q}]/J$. Then, the toric ideal $J$ is generated by the set of binomial 2-minors of $T_I$.

(b) Let $I = (u_1, \ldots, u_q) \subset K[x_1, \ldots, x_n]$ with $n \geq 3$ be a Freiman ideal and $F(I) = K[t_{u_1}, \ldots, t_{u_q}]/J$ be its fiber cone. Then, the toric ideal $J$ is generated by the set of binomial 2-minors of $T_I$.

**Proof.** (a), (b). Let $I$ be a Freiman ideal. Then, the toric defining ideal $J$ of $F(I)$ is generated by binomials, (e.g., see [4, Lemma 5.2]). On the other hand, $J$ has a 2-linear resolution by [6, Theorem 2.3]. Therefore, $J$ is generated by quadratic binomials. Now, (a) follows from Theorem 2.2 and (b) follows from Theorem 2.3.

**Remark 2.6.** In [12] there exists an example of a non-Koszul square-free semigroup ring whose toric ideal is generated by quadratic binomials but possesses no quadratic Gröbner basis (12, Example 2.1). Therefore, the set of binomial 2-minors of $T_I$ in Theorem 2.5 may not be a Gröbner basis of the toric ideal $J$.

**Remark 2.7.** Theorem 1.3 may fail when $I \subset K[x_1, x_2]$ is not generated by a sortable set of monomials, even the defining ideal of $F(I)$ is generated by quadratic binomials. For example, let $I = (x_1^5, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4)$. We set $u_1 = x_1^5, u_2 = x_1^3x_2, u_3 = x_1^2x_2^2$ and $u_4 = x_1x_2^3$. Note that $G(I)$ is not sortable, since sort($u_1, u_2) = (x_1^4x_2, x_1^3x_2) \notin G(I) \times G(I)$. One can check by CoCoA that $F(I) = K[t_{u_1}, \ldots, t_{u_4}]/J$ where $J = (t_{u_1}t_{u_4} - t_{u_2}^2, t_{u_2}t_{u_4} - t_{u_3}^2)$. While,

$$T_I = \begin{pmatrix} t_{u_1} & 0 & t_{u_2} & t_{u_3} & t_{u_4} \\ 0 & t_{u_2} & t_{u_3} & t_{u_4} & 0 \end{pmatrix}.$$ 

So, the ideal generated by the set of binomial 2-minors of $T_I$ is $(t_{u_2}t_{u_4} - t_{u_3}^2)$. Thus, in this case we use Theorem 2.2 to find $J$. We have

$$T_J = \begin{pmatrix} t_{u_1} & 0 & t_{u_2} \\ 0 & t_{u_2} & t_{u_3} \\ t_{u_2} & t_{u_3} & t_{u_4} \end{pmatrix}.$$ 

It is easy to see that $J$ is the ideal generated by the set of binomial 2-minors of $T_J$.

**Example 2.8.** (a) Let $I = (x_1^{12}, x_1^9x_2^3, x_1^6x_2^6, x_1^3x_2^9) \subset K[x_1, x_2]$. The ideal $I$ is a Freiman ideal, since $\ell(I) = 2$ and $\mu(I^2) = 7 = 2\mu(I) - \binom{2}{2}$. Set $u_1 = x_1^{12}, u_2 = x_1^9x_2^3, u_3 = x_1^6x_2^6$ and $u_4 = x_1^3x_2^9$. Checking by CoCoA we get $F(I) = K[t_{u_1}, \ldots, t_{u_4}]/J$ where

$$J = (t_{u_1}t_{u_3} - t_{u_2}^2, t_{u_1}t_{u_4} - t_{u_2}t_{u_3}, t_{u_2}t_{u_4} - t_{u_3}^2).$$
Note that $G(I)$ is not sortable, since $\text{sort}(u_1, u_2) = (x_1^{11}x_2, x_1^{10}x_2^2) \notin G(I) \times G(I)$. Using Theorem 2.2 to find $J$ we get

$$
T_I = \begin{pmatrix}
  t_{u_1} & 0 & 0 & t_{u_2} & 0 & 0 & t_{u_3} \\
  0 & t_{u_2} & 0 & 0 & t_{u_3} & 0 \\
  0 & t_{u_2} & 0 & 0 & t_{u_3} & 0 \\
  t_{u_2} & 0 & 0 & t_{u_3} & 0 & 0
\end{pmatrix}.
$$

We see that $J$ is the ideal generated by the set of binomial 2-minors of $T_I$.

(b) Let $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_3^2) \subset K[x_1, x_2, x_3]$ . Then $I$ is a Freiman ideal, because $\ell(I) = 3$ and $\mu(I^2) = 12 = 3\mu(I) - \binom{3}{2}$. Set $u_1 = x_1^3$, $u_2 = x_1^2x_3$, $u_3 = x_1x_3^2$, $u_4 = x_2^3$ and $u_5 = x_3^3$. Let $F(I) = K[u_1, \ldots, u_5]/J$. It follows from Theorem 2.3 that $J$ is the ideal generated by the set of binomial 2-minors of $T_I$, where

$$
T_I = \begin{pmatrix}
  t_{u_1} & 0 & t_{u_2} & 0 & 0 & t_{u_3} \\
  0 & 0 & 0 & t_{u_4} & 0 & 0 \\
  t_{u_2} & 0 & t_{u_3} & 0 & 0 & t_{u_5}
\end{pmatrix}.
$$

So $J = (t_{u_1}t_{u_3} - t_{u_2}^2, t_{u_1}t_{u_5} - t_{u_2}t_{u_3}, t_{u_2}t_{u_5} - t_{u_3}^2)$. The result is confirmed by CoCoA.

Note that $G(I)$ is not sortable, since $\text{sort}(u_1, u_4) = (x_1^2x_2, x_1x_2^2) \notin G(I) \times G(I)$.

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