Exploring Subexponential Parameterized Complexity of Completion Problems

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Abstract

Let \( F \) be a family of graphs. In the \( F \)-Completion problem, we are given an \( n \)-vertex graph \( G \) and an integer \( k \) as input, and asked whether at most \( k \) edges can be added to \( G \) so that the resulting graph does not contain a graph from \( F \) as an induced subgraph. It appeared recently that special cases of \( F \)-Completion, the problem of completing into a chordal graph known as Minimum Fill-in, corresponding to the case of \( F = \{ C_4, C_5, C_6, \ldots \} \), and the problem of completing into a split graph, i.e., the case of \( F = \{ C_4, 2K_2, P_4 \} \), are solvable in parameterized subexponential time \( 2^{O(\sqrt{n \log k})} n^{O(1)} \). The exploration of this phenomenon is the main motivation for our research on \( F \)-Completion.

In this paper we prove that completions into several well studied classes of graphs without long induced cycles also admit parameterized subexponential time algorithms by showing that:

- The problem Trivially Perfect Completion is solvable in parameterized subexponential time \( 2^{O(\sqrt{n \log k})} n^{O(1)} \), that is \( F \)-Completion for \( F = \{ C_4, P_4 \} \), a cycle and a path on four vertices.

- The problems known in the literature as Pseudosplit Completion, the case where \( F = \{ 2K_2, C_4 \} \), and Threshold Completion, where \( F = \{ 2K_2, P_4, C_4 \} \), are also solvable in time \( 2^{O(\sqrt{n \log k})} n^{O(1)} \).

We complement our algorithms for \( F \)-Completion with the following lower bounds:

- For \( F = \{ 2K_2 \} \), \( F = \{ C_4 \} \), \( F = \{ P_4 \} \), and \( F = \{ 2K_2, P_4 \} \), \( F \)-Completion cannot be solved in time \( 2^{o(k)} n^{O(1)} \) unless the Exponential Time Hypothesis (ETH) fails.

Our upper and lower bounds provide a complete picture of the subexponential parameterized complexity of \( F \)-Completion problems for \( F \subseteq \{ 2K_2, C_4, P_4 \} \).

1 Introduction

Let \( F \) be a family of graphs. In this paper we study the following \( F \)-Completion problem.

\[
\begin{array}{ll}
\text{\( F \)-Completion} & \\
\text{Input:} & \text{A graph } G = (V, E) \text{ and a non-negative integer } k. \\
\text{Parameter:} & k \\
\text{Question:} & \text{Does there exist a supergraph } H = (V, E \cup S) \text{ of } G, \text{ such that } |S| \leq k \text{ and } H \text{ contains no graph from } F \text{ as an induced subgraph?}
\end{array}
\]

The \( F \)-Completion problems form a subclass of graph modification problems where one is asked to apply a bounded number of changes to an input graph to obtain a graph with some property. Graph modification problems arise naturally in many branches of science and have been studied extensively during the past 40 years. Interestingly enough, despite the long study of the problem, there is no known dichotomy classification of \( F \)-Completion explaining for which classes \( F \) the problem is solvable in polynomial time and for which the problem is NP-complete \( [29, 24, 6] \).

One of the motivations to study completion problems in graph algorithms comes from their intimate connections to different width parameters. For example, the treewidth of a graph, one of the most

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fundamental graph parameters, is the minimum over all possible completions into a chordal graph of the maximum clique size minus one [4]. The treedepth of a graph, also known as the vertex ranking number, the ordered chromatic number, and the minimum elimination tree height, plays a crucial role in the theory of sparse graphs developed by Nešetřil and Ossona de Mendez [26]. Mirroring the connection between treewidth and chordal graphs, the treedepth of a graph can be defined as the largest clique size in a completion to a trivially perfect graph. Similarly, the vertex cover number of a graph is equal to the minimum of the largest clique size taken over all completions to a threshold graph, minus one.

Recent developments have also led to subexponential parameterized algorithms for the problems INTERVAL COMPLETION [2] and PROPER INTERVAL COMPLETION [3]. Both these problems have strong connection to width parameters just like the ones mentioned above: The pathwidth of a graph is the minimum over the maximum clique size in an interval completion of the graph, minus one, whereas the bandwidth mirrors this relation for proper interval completions of the graph.

Parameterized algorithms for completion problems  For a long time in parameterized complexity, the main focus of studies in -COMPLETION was for the case when was an infinite family of graphs, e.g., MINIMUM FILL-IN or INTERVAL COMPLETION [19, 25, 27]. This was mainly due to the fact that when is a finite family, -COMPLETION is solvable on an -vertex graph in time for some function by a simple branching argument; This was first observed by Cai [7]. More precisely, if the maximum number of non-edges in a graph from is , then the corresponding -COMPLETION is solvable in time .

The interest in -COMPLETION problems started to increase with the advance of kernelization. It appeared that from the perspective of kernelization, even for the case of finite families the problem is far from trivial. Guo [16] initiated the study of kernelization algorithms for -COMPLETION in the case when the forbidden set contains the graph , see Figure 1. (In fact, Guo considered edge deletion problems, but they are polynomial time equivalent to completion problems to the complements of the forbidden induced subgraphs.) In the literature, the most studied graph classes containing no induced are the split graphs, i.e., graphs, threshold graphs, i.e., , trivially perfect graphs [5]. Guo obtained polynomial kernels for the completion problems for chain graphs, split graphs, threshold graphs and trivially perfect graphs and concluded that, as a consequence of his polynomial kernelization, the corresponding -COMPLETION problems: CHAIN COMPLETION, SPLIT COMPLETION, THRESHOLD COMPLETION and TRIVIALLY PERFECT COMPLETION are solvable in times , , and , respectively.

The work on kernelization of -COMPLETION problems continued by Kratsch and Wahlström [21] who showed that there exists a set consisting of one graph on seven vertices for which -COMPLETION does not admit a polynomial kernel. Guillet et al. [15] showed that COGRAPH COMPLETION, i.e., the case , admits a polynomial kernel, while for the complement of a path on 13 vertices, -COMPLETION has no polynomial kernel. These results were significantly improved by Cai and Cai [8]: For or , the problems -COMPLETION and -EDGE DELETION admit a polynomial kernel if and only if the forbidden graph has at most three edges.

It appeared recently that for some choices of , -COMPLETION is solvable in subexponential time. The exploration of this phenomenon is the main motivation for our research on this problem. The last chapter of Flum and Grohe’s textbook on parameterized complexity theory [10, Chapter 16] concerns subexponential fixed parameter tractability, the complexity class , which, loosely speaking—we skip here some technical conditions—is the class of problems solvable in time , where is the input length and is the parameter. Until recently, the only notable examples of problems in were problems on planar graphs, and more generally, on graphs excluding some fixed graph as a minor [9]. In 2009, Alon et al. [1] used a novel application of color coding, dubbed chromatic coding, to

Figure 1: Forbidden induced subgraphs. Trivially perfect graphs are \{C₄, P₄\}-free, threshold graphs are \{2K₂, P₄, C₄\}-free, and cographs are \(P₃\)-free.
Figure 2: Known subexponential complexity of \( \mathcal{F} \)-Completion for different sets \( \mathcal{F} \). All problems in this table are NP-hard and in FPT. The entry SUBEPT means the problem is solvable in subexponential time \( 2^{o(k)n^{O(1)}} \) whereas \( \mathcal{E} \) means that the problem is not solvable in subexponential time unless ETH fails.

Our results  In this work we extend the class of \( \mathcal{F} \)-Completion problems admitting subexponential time algorithms, see Figure 2. Our main algorithmic result is the following:

Trivially Perfect Completion is solvable in time \( 2^{O(\sqrt{\log k})}n^{O(1)} \) and is thus in \( \text{SUBEPT} \).

This problem is the \( \mathcal{F} \)-Completion problem for \( \mathcal{F} = \{C_4, P_4\} \).

On a very high level, our algorithm is based on the same strategy as the algorithm for completion into chordal graphs [12]. Just like in that algorithm, we enumerate subexponentially many special objects, here called trivially perfect potential maximal cliques which are the maximal cliques in some minimal completion into a trivially perfect graph that uses at most \( k \) edges. As far as we succeed in enumerating these objects, we apply dynamic programming in order to find an optimal completion. But here the similarities end. To enumerate trivially perfect potential maximal cliques (henceforth referred to as only potential maximal cliques) for trivially perfect graphs, we have to use completely different structural properties from those used for the case of chordal graphs.

We also show that within the same running time, the \( \mathcal{F} \)-Completion problem is solvable for \( \mathcal{F} = \{2K_2, C_4\} \), and \( \mathcal{F} = \{2K_2, P_3, C_4\} \). This corresponds to completion into threshold and pseudosplit graphs, respectively. Let us note that combined with the results of Fomin and Villanger [12] and Ghosh
et al. [13], this implies that all four problems considered by Guo in [16] are in $\text{SUBEPT}$, in addition to admitting a polynomial kernel. We finally complement our algorithmic findings by showing the following:

For $F = \{2K_2\}$, $F = \{C_4\}$, $F = \{P_4\}$ and $F = \{2K_2, P_4\}$, the $F$-COMPLETION problem cannot be solved in time $2^{O(k)} n^{O(1)}$ unless ETH fails.

Thus, we obtain a complete classification for all $F \subseteq \{2K_2, P_4, C_4\}$.

Organization of the paper In Section 2 we give some structural results about trivially perfect graphs and their completions, and give the main result of the paper: an algorithm solving TRIVIALLY PERFECT COMPLETION in subexponential time. This section also contains some structural results on trivially perfect graphs that might be interesting on its own. In Sections 3 and 4 we give subexponential time algorithms for THRESHOLD COMPLETION and PSEUDOSPLIT COMPLETION.

In Section 5, we give the lower bounds on $F$-COMPLETION when $F$ is $\{2K_2\}$, $\{C_4\}$, $\{P_4\}$, and $\{2K_2, P_4\}$. Finally, in Section 6 we give some concluding remarks and state some interesting remaining questions and future directions.

Notation and preliminaries on parameterized complexity We consider only finite simple undirected graphs. We use $n_G$ to denote the number of vertices and $m_G$ the number of edges in a graph $G$. If $G = (V, E)$ is a graph, and $A, B \subseteq V$, we write $E(A, B)$ for the edges with one endpoint in $A$ and the other in $B$, and we write $E(A) = m_A = m_{G[A]}$ for the edges inside $A$.

Given a graph $G = (V, E)$, recall that $N_G(v)$ for a vertex $v \in V$ denotes the set of neighbors of $v$ in $G$. We write $N_G[v]$ to mean the set $N_G(v) \cup \{v\}$. For sets of vertices $U \subseteq V$, we write $N_G(U)$ to denote the open neighborhood $\bigcup_{v \in U} (N_G(v)) \setminus U$, and $N_G[U] = N_G(U) \cup U$ to denote the closed neighborhood. For a set of pairs of vertices $S$, we write $G + S = (V, E \cup S)$ and if $U \subseteq V$ is a set of vertices, then $G - U = G[V \setminus U]$. We will skip the subscripts when this will not cause any confusion.

A universal vertex in a graph $G$ is a vertex $v$ such that $N[v] = V(G)$. Let uni($G$) denote the set of universal vertices of $G$. Observe that uni($G$), when non-empty, is always a clique, and we will refer to it as the (maximal) universal clique. The maximal universal cliques play an important role in the trivially perfect graphs; They are the main building blocks we will use to achieve the algorithm.

We here provide a simplified definition of parameterized problems, kernels and the class of parameterized subexponential time algorithms. A parameterized problem $\Pi$ is a problem whose input is a pair $(x, k)$, where $k \in \mathbb{N}$. The problem $\Pi$ is fixed-parameter tractable, and thus belongs to the class $\text{FPT}$, if there is an algorithm solving this problem in time $f(k) \cdot x^{O(1)}$ for some function $f$, depending only on $k$. A kernelization algorithm for $\Pi$ is a polynomial time algorithm which on input $(x, k)$ gives an output $(x', k')$ such that $|x'| \leq g(k)$ and $k' \leq g(k)$ for some function $g$ depending only on $k$, and such that $(x, k)$ is a yes instance for $\Pi$ if and only if $(x', k')$ is a yes instance for $\Pi$. We call the output the kernel. We say a problem admits a polynomial kernel if the function $g$ is polynomial.

The complexity class $\text{SUBEPT}$ is contained in $\text{FPT}$; It is the class of problems $\Pi$ for which there exists an algorithm with running time $2^{o(k)} \cdot n^{O(1)}$. That is, the parameter function $f$ is subexponential. Note that if the exponential time hypothesis is true, then $\text{SUBEPT} \subseteq \text{FPT}$.

2 Completion to trivially perfect graphs

In this section we study the TRIVIALLY PERFECT COMPLETION problem which is $F$-COMPLETION for $F = \{C_4, P_4\}$. The decision version of the problem was shown to be $\text{NP}$-complete by Yamakakis [28]. As already stated in the introduction, trivially perfect graphs are characterized by a finite set of forbidden induced subgraphs, and thus it follows from Cai [7] that the problem also is fixed parameter tractable, i.e., it belongs to the class $\text{FPT}$.

The main result of this section is the following theorem:

**Theorem 2.1.** For an input $(G, k)$, TRIVIALLY PERFECT COMPLETION is solvable in time $2^{O(\sqrt{k}\log k)} + O(kv^4)$.

Throughout this section, an edge set $S$ is called a completion for $G$ if $G + S$ is trivially perfect. Furthermore, a completion $S$ is called a minimal completion for $G$ if no proper subset of $S$ is a completion for $G$. The main outline of the algorithm is as follows:
Step A: On input \((G, k)\), we first apply the algorithm by Guo [16] to obtain a kernel \(O(k^3)\) vertices. The running time of this algorithm is \(O(kn^4)\). The kernelization algorithm of Guo can only reduce the parameter, i.e., \(k' \leq k\) where \(k'\) is the new parameter. Moreover, the output kernel is in fact of size \(O(k^3)\). Therefore, due to this preprocessing step we may assume without loss of generality that we work on an instance \((G, k)\) with \(|V(G)| \leq O(k^3)\).

Step B: Assuming our input instance has \(O(k^3)\) vertices, we show how to generate all special vertex subsets of the kernel which we call vital potential maximal cliques in time \(2^{O(\sqrt{k} \log k)}\). A vital potential maximal clique \(\Omega \subseteq V(G)\) is a vertex subset which is a maximal clique in some minimal completion of size at most \(k\).

Step C: Using dynamic programming, we show how to compute an optimal solution or to conclude that \((G, k)\) is a no instance, in time polynomial in the number of vital potential maximal cliques.

2.1 Structure of trivially perfect graphs

Apart from the aforementioned characterization by forbidden induced subgraphs, an inherently local characterization, several other equivalent definitions of trivially perfect graphs are known. These definitions reveal more structural properties of this graph class which will be essential in our algorithm. Therefore, before proceeding with the proof of Theorem 2.1, we establish a number of results on the global structure of trivially perfect graphs and minimal completions which will be useful.

The trivially perfect graphs have a rooted decomposition tree, which we call a universal clique decomposition, in which each node corresponds to a maximal set of vertices that all are universal for the graph induced by the vertices in the subtree rooted at this node. This decomposition is similar to that of a treedepth decomposition. We refer to Figure 3 for an example of the concepts that we introduce next. The following recursive definition is often used as an alternative definition of trivially perfect graphs.

**Proposition 2.2 ([18]).** The class of trivially perfect graphs can be defined recursively as follows:

- \(K_1\) is a trivially perfect graph.
- Adding a universal vertex to a trivially perfect graph results in a trivially perfect graph.
- The disjoint union of two trivially perfect graphs is a trivially perfect graph.

Let \(T\) be a rooted tree and \(t\) be a node of \(T\). We denote by \(T_t\) the maximal subtree of \(T\) rooted in \(t\). We can now use the universal clique uni\((G)\) of a trivially perfect graph \(G = (V, E)\) to make a decomposition structure.

**Definition 2.3** (Universal clique decomposition). A universal clique decomposition of a connected trivially perfect graph \(G = (V, E)\) is a pair \((T = (V_T, E_T), B = \{B_t\}_{t \in V_T})\), where \(T\) is a rooted tree and \(B\) is a partition of the vertex set \(V\) into disjoint non-empty subsets, such that

- if \(vw \in E(G)\) and \(v \in B_t\) and \(w \in B_s\), then \(s\) and \(t\) are on a path from a leaf to the root, with possibly \(s = t\), and
- for every node \(t \in V_T\), the set of vertices \(B_t\) is the maximal universal clique in the subgraph \(G[\bigcup_{s \in V(T_t)} B_s]\).

We call the vertices of \(T\) nodes and the sets in \(B\) bags of the universal clique decomposition \((T, B)\). By slightly abusing the notation, we often do not distinguish between nodes and bags. Note that by the definition, in a universal clique decomposition every non-leaf node has at least two children, since otherwise the universal clique contained in the corresponding bag would not be maximal.

**Lemma 2.4.** A connected graph \(G\) admits a universal clique decomposition if and only if it is trivially perfect. Moreover, such a decomposition is unique up to isomorphisms.

**Proof.** From right to left, we proceed by induction on the number of vertices using Proposition 2.2. The base case is when we have one vertex, \(K_1\) which is a trivially perfect graph and also admits a unique universal clique decomposition. The induction step is when we add a vertex \(v\), and by the definition of
trivially perfect graphs, \( v \) is a universal vertex. Either we add a universal vertex to a connected trivially perfect graph, in which case we simply add the vertex to the root bag, or we add a universal vertex to the disjoint union of two or more trivially perfect graphs. In this case, we create a new tree, with \( r_1 \) being the root connected to the root of each of the trees for the disjoint union. Since \( v \) is the only universal vertex in the graph, the constructed structure is a universal clique decomposition. Observe that the constructed decompositions are unique (up to isomorphisms).

From left to right, we proceed by induction on the height of the universal clique decomposition. Suppose \((T, B)\) is a universal clique decomposition of a graph \( G \). Consider the case when \( T \) has height 1, i.e., we have only one single tree node (and one bag). Then this bag, by Proposition 2.2, is a clique (every vertex in the bag is universal), and since a complete graph is trivially perfect, the base case holds. Consider now the case when \( T \) has height at least 2. Let \( r \) be the root of \( T \), and let \( x_1, x_2, \ldots, x_p \) be children of \( r \) in \( T \). Observe that the tree \( T_{x_i} \), is a universal clique decomposition for the graph \( G[\bigcup_{i \in V(T_{x_i})} B_i] \) for each \( i = 1, 2, \ldots, p \). Hence, by the induction hypothesis we have that \( G[\bigcup_{i \in V(T_{x_i})} B_i] \) is trivially perfect. To see that \( G \) is trivially perfect as well, observe that \( G \) can be obtained by taking the disjoint union of graphs \( G[\bigcup_{i \in V(T_{x_i})} B_i] \) for \( i = 1, 2, \ldots, p \), and adding \( |B_r| \) universal vertices.

For the purposes of the dynamic programming procedure, we define the following notion.

**Definition 2.5** (Block). Let \((T = (V_T, E_T), B = \{B_t\}_{t \in V_T})\) be the universal clique decomposition of a connected trivially perfect graph \( G = (V, E) \). For each node \( t \in V_T \), we associate a block \( L_t = (B_t, D_t) \), where

- \( B_t \) is the subset of \( V \) contained in the bag corresponding to \( t \), and
- \( D_t \) is the set of vertices of \( V \) contained in the bags corresponding to the nodes of the subtree \( T_t \).
- The tail of a block \( L_t \) is the set of vertices \( Q_t \) contained in the bags corresponding to the nodes of the path from \( t \) to \( r \) in \( T \), where \( r \) is the root of \( T \), including \( B_t \) and \( B_r \).

When \( t \) is a leaf of \( T \), we have that \( B_t = D_t \) and we call the block \( L_t = (B_t, D_t) \) a leaf block. If \( t \) is the root, we have that \( D_t = V(G) \) and we call \( L_t \) the root block. Otherwise, we call \( L_t \) an internal block. Observe that for every block \( L_t = (B_t, D_t) \) with tail \( Q_t \) we have that \( B_t \subseteq Q_t \), \( B_t \subseteq D_t \), and \( D_t \cap Q_t = B_t \), see Figure 3. Note also that \( Q_t \) is a clique and the vertices of \( Q_t \) are universal to \( D_t \setminus B_t \).
The following lemma summarizes the properties of universal clique decompositions, maximal cliques, and blocks used in our proof.

**Lemma 2.6.** Let \((T, B)\) be the universal clique decomposition of a connected trivially perfect graph \(G\) and let \(L = (B, D)\) be a block with \(Q\) as its tail.

(i) If \(L\) is a leaf block, then \(Q = N_G[v]\) for every \(v \in B\).

(ii) The following are equivalent:

1. \(L\) is a leaf block,
2. \(D = B\), and
3. \(Q\) is a maximal clique of \(G\).

(iii) If \(L\) is a non-leaf block, then for every two vertices \(u, v\) from different connected components of \(G[D \setminus B]\), we have that \(Q = N_G(u) \cap N_G(v)\).

**Proof.** (i) Since \(Q\) is a clique, we have that \(Q \subseteq N_G[v]\). On the other hand, since \(v \in B\) and \(L\) is a leaf block, we have that \(Q \supseteq N_G[v]\) by the definition of universal clique decomposition.

(ii) We prove the chain \((1) \rightarrow (2) \rightarrow (3) \rightarrow (1)\). Suppose that \(L\) is a leaf block, and \(D\) is the set of vertices in the bags in the subtree rooted at \(L\), then \(B = D\). Then by (i) we have that \(N_G[v] = Q\) for any \(v \in B\), hence \(Q\) is maximal. Finally, if \(Q\) is a maximal clique in the graph, i.e., it cannot be extended, by definition \(L\) cannot have any children so \(L\) must be a leaf block.

(iii) Suppose \(L = (B, D)\) is a non-leaf block and \(D_1\) and \(D_2\) are two connected components of \(G' = G[D \setminus B]\). Let \(v \in D_1\) and \(u \in D_2\) and observe that since they are in different connected components of \(G[D \setminus B]\), \(N_G(v) \cap N_G(u) = \emptyset\). By the universality of \(Q\), the result follows: \(Q = N_G(v) \cap N_G(u)\). \(\square\)

### 2.2 Structure of minimal completions

Before we proceed with the algorithm, we provide some properties of minimal completions. The following lemma gives insight to the structure of a **yes** instance.

**Lemma 2.7.** Let \(G = (V, E)\) be a connected graph, \(S\) a minimal completion and \(H = G + S\). Suppose \(L = (B, D)\) is a block in some universal clique decomposition of \(H\) and denote by \(D_1, D_2, \ldots, D_\ell\) the connected components of \(H[D] - B\).

(i) If \(L\) is not a leaf block, then \(\ell > 1\);

(ii) If \(\ell > 1\), then in \(G\) every vertex \(v \in B\) has at least one neighbor in each set \(D_1, D_2, \ldots, D_\ell\);

(iii) The graph \(G[D_1]\) is connected for every \(i \in \{1, \ldots, \ell\}\);

(iv) For every \(i \in \{1, \ldots, \ell\}\), \(B \subseteq N_G(D \setminus (B \cup D_i))\).

**Proof.** We prove this case by case. (i) Let \((B, D)\) be a non-leaf block. Since \(B\) is maximal, \(D\) is not a clique, so by the recursive definition of trivially perfect graphs, \(H[D] - B\) is the disjoint union of two or more trivially perfect graphs, hence \(\ell > 1\).

(ii) Suppose, without loss of generality, that there exists a vertex \(v \in B\) that has no neighbor in \(D_i\). Let \(S' = S \setminus \{v\} \times V(D_1)\). Note that since \(v\) is universal to \(V(D_1)\) in \(H\) and completely non-adjacent to \(V(D_1)\) in \(G\), then \(\{v\} \times V(D_1) \subseteq S\) and \(S'\) is a proper subset of \(S\). We claim that \(H' = G + S'\) is also a trivially perfect graph, which contradicts the minimality of \(S\). Indeed, consider a universal clique decomposition obtained from the universal clique decomposition of \(H\) by (a), in case \(\ell = 2\), moving \(v\) from \(B\) to the root bag of \(D_2\), or (b), in case \(\ell > 2\), moving \(v\) from \(B\) to a new bag \(B' = \{v\}\) attached below \(B\), with all the root bags of \(D_2, D_3, \ldots, D_\ell\) re-attached from below \(B\) to below \(B'\). It can be easily seen that this new universal clique decomposition is indeed a universal clique decomposition of \(H'\), which proves that \(H'\) is trivially perfect.

(iii) For the sake of a contradiction, suppose \(G[D_a]\) was disconnected. Let \((D_{a_1}, D_{a_2})\) be a partition of \(D_a\) such that there is no edge between \(D_{a_1}\) and \(D_{a_2}\) in \(G\). Clearly, \(H[D_{a_1}]\) and \(H[D_{a_2}]\) are trivially perfect graphs as induced subgraphs of \(H\), hence they admit some universal clique decompositions. Since \(H[D_a]\) is connected, we infer that \(S\) contains some edges between \(D_{a_1}\) and \(D_{a_2}\). Let now \(S' = S \setminus \{uv\}_
As has been already mentioned, the following concept is crucial for our algorithm. Recall that when $\Omega$ is a maximal clique contains at most $k$ (Fill number). Step A. Kernelization

By the previous argument we have that $S' \subseteq S$. Modify now the given universal clique decomposition of $H$ by removing the subtree below $B$ that corresponds to $D_G$, and attaching instead two subtrees below $B$ that are universal clique decompositions of $H[D_G]$ and $H[D_G \setminus B]$. Observe that thus we obtain a universal clique decomposition of $G + S'$, which shows that $G + S'$ is trivially perfect. This is a contradiction with the minimality of $S$.

(iii) Follows directly from (i) and (ii): if $\ell > 0$, then $\ell > 1$ and every vertex of $B$ has edges in $G$ to all different connected components of $D \setminus B$.

2.3 The algorithm

As has been already mentioned, the following concept is crucial for our algorithm. Recall that when $\Omega$ is a set of vertices in a graph $G$, by $m_\Omega$ we mean the number of edges in $G[\Omega]$.

Definition 2.8 (Vital potential maximal clique). Let $(G, k)$ be an input instance to TRIVIALLY PERFECT COMPLETION. A vertex set $\Omega \subseteq V(G)$ is a trivially perfect potential maximal clique or simply potential maximal clique, if $\Omega$ is a maximal clique in some minimal trivially perfect completion of $G$. If moreover this trivially perfect completion contains at most $k$ edges, then the potential maximal clique is called vital.

Observe that given a yes instance $(G, k)$ and a minimal completion $S$ of size at most $k$, every maximal clique in $G + S$ is a potential maximal clique in $G$. Note also that in particular, any vital potential maximal clique contains at most $k$ non-edges. The following definition will be useful:

Definition 2.9 (Fill number). Let $G = (V, E)$ be a graph, $S$ a completion and $H = G + S$. We define the fill of a vertex $v$, denoted by $fn_H^G(v)$ as the number of edges incident to $v$ in $S$.

Observation 2.10. There are at most $2\sqrt{k}$ vertices $v$ such that $fn_H^G(v) > \sqrt{k}$.

It follows that for every set $U \subseteq V$ such that $|U| > 2\sqrt{k}$, there is a vertex $u \in U$ with $fn_H^G(u) \leq \sqrt{k}$. Any vertex $u$ such that $fn_H^G(u) \leq \sqrt{k}$ will be referred to as a cheap vertex.

Equivalently is settled to start the proof of Theorem 2.1. Our algorithm consists of three steps. We first compress the instance to an instance of size $O(k^2)$, then we enumerate all (subexponentially many) vital potential maximal cliques in this new instance, and finally we do a dynamic programming procedure on these objects.

Step A. Kernelization For a given input $(G, k)$, we start by applying the kernelization algorithm by Guo [16] to construct in time $O(kn^2)$ an equivalent instance $(G', k')$, where $G'$ has $O(k^2)$ vertices and $k' \leq k$. Thus, from now on we can assume that the input graph $G$ has $O(k^2)$ vertices. Without loss of generality, we will also assume that $G$ is connected, since we can treat each connected component of $G$ separately.

u ∈ D_{a_1}, v ∈ D_{a_2}, uv ∈ S \}; By the previous argument we have that $S' \subseteq S$. Modify now the given universal clique decomposition of $H$ by removing the subtree below $B$ that corresponds to $D_G$, and attaching instead two subtrees below $B$ that are universal clique decompositions of $H[D_G]$ and $H[D_G \setminus B]$. Observe that thus we obtain a universal clique decomposition of $G + S'$, which shows that $G + S'$ is trivially perfect. This is a contradiction with the minimality of $S$.

(iv) Follows directly from (i) and (ii): if $\ell > 0$, then $\ell > 1$ and every vertex of $B$ has edges in $G$ to all different connected components of $D \setminus B$.

2.3 The algorithm

As has been already mentioned, the following concept is crucial for our algorithm. Recall that when $\Omega$ is a set of vertices in a graph $G$, by $m_\Omega$ we mean the number of edges in $G[\Omega]$.

Definition 2.8 (Vital potential maximal clique). Let $(G, k)$ be an input instance to TRIVIALLY PERFECT COMPLETION. A vertex set $\Omega \subseteq V(G)$ is a trivially perfect potential maximal clique or simply potential maximal clique, if $\Omega$ is a maximal clique in some minimal trivially perfect completion of $G$. If moreover this trivially perfect completion contains at most $k$ edges, then the potential maximal clique is called vital.

Observe that given a yes instance $(G, k)$ and a minimal completion $S$ of size at most $k$, every maximal clique in $G + S$ is a potential maximal clique in $G$. Note also that in particular, any vital potential maximal clique contains at most $k$ non-edges. The following definition will be useful:

Definition 2.9 (Fill number). Let $G = (V, E)$ be a graph, $S$ a completion and $H = G + S$. We define the fill of a vertex $v$, denoted by $fn_H^G(v)$ as the number of edges incident to $v$ in $S$.

Observation 2.10. There are at most $2\sqrt{k}$ vertices $v$ such that $fn_H^G(v) > \sqrt{k}$.

It follows that for every set $U \subseteq V$ such that $|U| > 2\sqrt{k}$, there is a vertex $u \in U$ with $fn_H^G(u) \leq \sqrt{k}$. Any vertex $u$ such that $fn_H^G(u) \leq \sqrt{k}$ will be referred to as a cheap vertex.

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Step B. Enumeration In this step, we give an algorithm that in time $2^{O(\sqrt{k}\log k)}$ outputs a family $\mathcal{C}$ of vertex subsets of $G$ such that

- the size of $\mathcal{C}$ is $2^{O(\sqrt{k}\log k)}$, and
- every vital potential maximal clique belongs to $\mathcal{C}$.

We identify four different types of vital potential maximal cliques. For each type $i$, $1 \leq i \leq 4$, we list a family $\mathcal{C}_i$ of $2^{O(\sqrt{k}\log k)}$ subsets containing all vital potential maximal cliques of this type. Finally, $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_4$. We show that every vital potential maximal clique of $(G, k)$ is of at least one of these types and that all objects of each type can be enumerated in $2^{O(\sqrt{k}\log k)}$ time.

Let $\Omega$ be a vital potential maximal clique. By the definition of $\Omega$, there exists a minimal completion with at most $k$ edges into a trivially perfect graph $H$ such that $\Omega$ is a maximal clique in $H$. Let $(T = (V_T, E_T), B = \{B_t\}_{t \in V_T})$ be the universal clique decomposition of $H$. Recall that by Lemma 2.6, $\Omega$ corresponds to a path $P_t = B_{t_0}B_{t_1}\cdots B_{t_q}$ in $T$ from the root $r = t_0$ to a leaf $t = t_q$. Then for the corresponding leaf block $(B_t, D_t)$ with tail $Q_t$, we have that $\Omega = Q_t$. To simplify the notation, we use $B_t$ for $B_t$.

Note that the algorithm does not know neither the clique $\Omega$ nor the completed trivially perfect graph $H$. However, in the analysis we may partition all the vital potential maximal cliques $\Omega$ with respect to structural properties of $\Omega$ and $H$, and then provide simple enumeration rules that ensure that all vital potential maximal cliques of each type are indeed enumerated. We now proceed to the description of the types and enumeration rules and refer to Figure 4 for a visualization of the concepts. In the sequel, whenever we are referring to cheap or expensive vertices, we mean being cheap/expensive with respect to the fixed completion to $H$.

Type 1. Potential maximal cliques of the first type are such that $|V \setminus \Omega| \leq 2\sqrt{k}$. The family $\mathcal{C}_1$ consists of all sets $W \subseteq V$ such that $|V \setminus W| \leq 2\sqrt{k}$.

There are at most $(2\sqrt{k} + 1) \cdot \binom{|V|}{2\sqrt{k}}$ such sets and we enumerate all of them in time $2^{O(\sqrt{k}\log k)}$ by trying all vertex subsets of size at least $|V| - 2\sqrt{k}$. Thus every Type 1 vital potential maximal clique is in $\mathcal{C}_1$.

Type 2. By Lemma 2.6 (i), we have that $\Omega = Q_t = N_H[v]$ for each vertex $v \in D_t = B_t$. Vital potential maximal cliques of the second type are such that $|B_t| > 2\sqrt{k}$. Observe that then at least one vertex $v \in B_t$ should be cheap, i.e., $f_{\mathcal{O}}(v) \leq \sqrt{k}$. We generate the family $\mathcal{C}_2$ as follows. Every set in $\mathcal{C}_2$ is of the form $W_1 \cup W_2$, where $W_1 = N_G[v]$ for some $v \in V$, and $|W_2| \leq \sqrt{k}$. There are at most $\binom{|V|}{\sqrt{k}}$ such sets and they can be enumerated by computing for every vertex $v$ the set $W_1 = N_G[v]$ and adding to each such set all possible subsets of size at most $\sqrt{k}$. Hence every Type 2 vital potential maximal clique is in $\mathcal{C}_2$.

Thus if $\Omega$ is not of Types 1 or 2, then $|V \setminus \Omega| > 2\sqrt{k}$ and for the corresponding leaf block we have $|B_t| \leq 2\sqrt{k}$. Since $|V \setminus \Omega| > 2\sqrt{k}$ it follows that $V \setminus \Omega$ contains at least one cheap vertex, i.e., a vertex with fill number at most $\sqrt{k}$.

We partition the nodes of $T$ that are not on the path $B_0, B_1, \ldots, B_p$ into $q$ disjoint sets $Z_0, Z_1, \ldots, Z_{q-1}$ according to the nodes of the path $P_t$. Node $x \notin V(P_{t_i})$ belongs to $Z_i$, $i \in \{0, \ldots, q-1\}$ if $i$ is the largest integer such that $t_i$ is an ancestor of $x$ in $T$. In other words, $Z_i$ consists of bags of subtrees outside $P_{t_i}$ attached below $t_i$, see Figure 4a.

For integers $p_1, p_2$, we shall denote $B_{p_1,p_2} = \bigcup_{j=p_1}^{p_2} B_j$.

For the remaining two types of vital potential maximal cliques we distinguish cases depending on whether all cheap vertices in $V \setminus \Omega$ are located in exactly one set $Z_i$, or not. Recall that all vital potential maximal cliques for which $V \setminus \Omega$ does not contain any cheap vertex are already contained in Type 1.

Type 3. Vital potential maximal cliques $\Omega$ of the third type are the ones that do not belong to Type 1 or 2, but there exists an index $i \in \{0, 1, \ldots, q-1\}$ such that all cheap vertices of $V \setminus \Omega$ belong to $Z_i$. Since $\Omega$ is not of Type 1, $Z_i$ is non-empty. Also, since $\Omega$ is not of Type 2, we have that $|B_q| \leq 2\sqrt{k}$. Let us denote $Z_{<i} = \bigcup_{j=0}^{i-1} Z_j$ and $Z_{>i} = \bigcup_{j=i+1}^{q-1} Z_j$ (see Figure 4a). By our assumption, we have that $Z_{<i}$ and $Z_{>i}$ contain only expensive vertices, and hence $|Z_{<i}|, |Z_{>i}| \leq 2\sqrt{k}$. Let $u$ be any cheap vertex belonging to $Z_i$, and observe that the following equalities and inclusions are implied by Lemma 2.7 (ii):

- $B_{0,i-1} = N_G(Z_{<i})$;
• $B_{i+1,q-1} \subseteq N_G(Z_{>i}) \subseteq \Omega$;

• $B_i \subseteq N_G(B_q \cup (N_G[Z_{>i}] \setminus N_H(u))) \subseteq \Omega$.

It follows that

$$\Omega = N_G(Z_{<i}) \cup N_G(Z_{>i}) \cup N_G(B_q \cup (N_G[Z_{>i}] \setminus N_H(u))) \cup B_q.$$  \hspace{1cm} (1)

Given (1), we may define family $C_3$. Family $C_3$ comprises all the sets that can be constructed as follows:

• Pick three disjoint sets $W_1, W_2, W_3 \subseteq V$ of size at most $2\sqrt{k}$ each. This corresponds to the choice of $Z_{<i}, Z_{>i}$ and $B_q$, respectively.

• Pick a vertex $v \in V$ and a set $A \subseteq V$ of size at most $\sqrt{k}$. This corresponds to the choice of $u$ and fill-in edges adjacent to $u$. Let $N_v = N_G(v) \cup A$.

• Put the set $N_G(W_1) \cup N_G(W_2) \cup N_G(W_3 \cup (N_G(W_2) \setminus N_v)) \cup W_3$ into the family $C_3$.

Observe that since $|V| = \mathcal{O}(k^3)$, the number of sets included in $C_3$ is at most $2^{\mathcal{O}(\sqrt{k} \log k)}$, and that this family can be enumerated within the same asymptotic running time. From (1) it follows immediately that each vital potential maximal clique of Type 3 is contained in $C_3$.

**Type 4.** Vital potential maximal cliques $\Omega$ of the fourth type are the ones that do not belong to Type 1 or 2, but there exist at least two indices $i_1$ and $i_2$ such that $Z_{i_1}$ and $Z_{i_2}$ both contain a cheap vertex. Let $i_1, i_2$ be the two largest such indices, where $i_1 > i_2$. Let $Z_{<i_1,i_2} = \bigcup_{j=i_2+1}^{i_1-1} Z_j$ and $Z_{>i_1} = \bigcup_{j=i_1+1}^{\sqrt{k}} Z_j$. See Figure 4b for an illustration. By the maximality of $i_1, i_2$ we have that $Z_{<i_1,i_2}$ and $Z_{>i_1}$ contain only expensive vertices, and hence $|Z_{<i_1,i_2}|, |Z_{>i_1}| \leq 2\sqrt{k}$. Again, since $\Omega$ is not of Type 2, we have that $|B_q| \leq 2\sqrt{k}$. Let $u_1 \in Z_{i_1}$ and $u_2 \in Z_{i_2}$ be two cheap vertices. Observe that the following equalities and inclusions are implied by Lemma 2.7 (ii):

• $B_{0,i_2} = N_H(u_1) \cap N_H(u_2)$;

• $B_{i_2+1,i_1-1} \subseteq N_G(Z_{<i_1,i_2}) \subseteq \Omega$;

• $B_{i_1+1,q-1} \subseteq N_G(Z_{>i_1}) \subseteq \Omega$;

• $B_{i_1} \subseteq N_G(B_q \cup (N_G[Z_{>i_1}] \setminus N_H(u_1))) \subseteq \Omega$.

It follows that

$$\Omega = (N_H(u_1) \cap N_H(u_2)) \cup N_G(Z_{<i_1,i_2}) \cup N_G(Z_{>i_1}) \cup N_G(B_q \cup (N_G[Z_{>i_1}] \setminus N_H(u_1))) \cup B_q.$$  \hspace{1cm} (2)

Given (2), we may define the family $C_4$. This family comprises all the sets that can be constructed as follows:

• Pick three disjoint sets $W_1, W_2, W_3 \subseteq V$ of size at most $2\sqrt{k}$ each. This corresponds to the choice of $Z_{<i_1,i_2}, Z_{>i_1}$ and $B_q$, respectively.

• Pick two vertices $v_1, v_2 \in V$ and two sets $A_1, A_2 \subseteq V$, each of size at most $\sqrt{k}$. This corresponds to the choice of $u_1$ and $u_2$, and of the neighbors in $H$ adjacent to $u_1$ and $u_2$. Let $N_{v_i} = N_G(v_i) \cup A_i$, for $i = 1, 2$.

• Put the set $(N_{v_1} \cap N_{v_2}) \cup N_G(W_1) \cup N_G(W_2) \cup N_G(W_3 \cup (N_G(W_2) \setminus N_{v_1})) \cup W_3$ into the family $C_4$.

Observe that since $|V| = \mathcal{O}(k^3)$, the number of sets included in $C_4$ is at most $2^{\mathcal{O}(\sqrt{k} \log k)}$, and that this family can be enumerated within the same asymptotic running time. From (2) it follows immediately that each vital potential maximal clique of Type 4 is contained in $C_4$.

Summarizing, every vital potential maximal clique of Type 1, 2, 3, and 4 is included in the family $C_1, C_2, C_3$, and $C_4$, respectively. Since every vital potential maximal clique is of Type 1, 2, 3, or 4, by taking $\mathcal{C} = C_1 \cup C_2 \cup C_3 \cup C_4$ we can infer the following lemma that formalizes the result of Step B.

**Lemma 2.11 (Enumeration Lemma).** Let $(G, k)$ be an instance of Trivially Perfect Completion such that $|V(G)| = \mathcal{O}(k^3)$. Then in time $2^{\mathcal{O}(\sqrt{k} \log k)}$, we can construct a family $\mathcal{C}$ consisting of $2^{\mathcal{O}(\sqrt{k} \log k)}$ subsets of $V(G)$ such that every vital potential maximal clique of $(G, k)$ is in $\mathcal{C}$.
Step C. Dynamic programming  We first give an intuitive idea of the dynamic procedure: We start off by assuming that we have the family $C$ containing all vital potential maximal cliques of $(G, k)$. We start by generating in time $2^{O(\sqrt{k \log k})}$ a family $S$ of pairs $(X, Y)$, where $X, Y \subseteq V(G)$, such that for every minimal completion $S$ of size at most $k$, and the corresponding universal clique decomposition $(T, B)$ of $H = G + S$, it holds that every block $(B, D)$ is in $S$, and the size of $S$ is $2^{O(\sqrt{k \log k})}$. (See Definition 2.5 for the definition of a block.)

The construction of $S$ is based on the following observations about blocks and vital potential maximal cliques: Let $G$ be a graph, $S$ a minimal completion and $L = (B, D)$ a block of the universal clique decomposition of $H = G + S$, where $H$ is not a complete graph, with $Q$ being its tail. Then the following holds:

- If $L$ is a leaf block, then $B = \Omega_1 \setminus \Omega_2$ for some vital potential maximal cliques $\Omega_1$ and $\Omega_2$, and $D = B$.
- If $L$ is the root block, then the tail of $L$ is $B$, $B = \Omega_1 \cap \Omega_2$ for some vital potential maximal cliques $\Omega_1$ and $\Omega_2$, and $D = V$.
- If $L$ is an internal block, then $Q$ is the intersection of two vital potential maximal cliques $\Omega_1$ and $\Omega_2$ of $G$, $B = Q \setminus \Omega_3$ for some vital potential maximal clique $\Omega_3$, and $D$ is the connected component of $G - (Q \setminus B)$ containing $B$.

From this observation, we can conclude that by going through all triples $\Omega_1, \Omega_2, \Omega_3$, we can compute the set $S$ consisting of all blocks $(B, D)$ of minimal completions. We now define the value $\text{dp}(B, D)$ as follows: $\text{dp}(B, D)$ is equal to the minimum number of edges needed to be added to $G[D]$ to make it a trivially perfect graph with $B$ being the universal clique contained in the root of the universal clique decomposition, unless this minimum number is larger than $k$; In this case we put $\text{dp}(B, D) = +\infty$. We later derive recurrence equations that enable us to compute all the relevant values of $\text{dp}(\cdot, \cdot)$ using dynamic programming. Finally, the minimum cost of completing $G$ to a trivially perfect graph is equal to $\min_{(B, V(G)) \in S} \text{dp}(B, V(G))$. If this minimum is equal to $+\infty$, then no completion of size at most $k$ exists and we can conclude that $G, k$ is a no-instance.

We now proceed to a formal proof of the correctness of the dynamic programming procedure. Suppose that we have the family $C$ containing all vital potential maximal cliques of $(G, k)$. We start by generating in time $2^{O(\sqrt{k \log k})}$ a family $S$ of pairs $(X, Y)$, where $X, Y \subseteq V$, where $V = V(G)$, such that

- for every minimal completion $H$ that adds at most $k$ edges, every block $(B, D)$ of the universal clique decomposition of $H$ belongs to $S$, and
- the size of $S$ is $2^{O(\sqrt{k \log k})}$.

The construction of $S$ is based on the following lemmata.

Lemma 2.12. Let $G$ be a graph, $S$ a minimal completion of size at most $k$, and $(B, D)$ a non-leaf and non-root block of the universal clique decomposition of $H = G + S$, with $Q$ being its tail. Then

(i) $Q$ is the intersection of two vital potential maximal cliques $\Omega_1$ and $\Omega_2$ of $G$,

(ii) $B = Q \setminus \Omega_3$ for some vital potential maximal clique $\Omega_3$, and

(iii) $D$ is the connected component of $G - (Q \setminus B)$ containing $B$.

Proof. (i) Consider two connected components $D_1$ and $D_2$ of $H[D \setminus B]$ and let $\Omega'_1$ and $\Omega'_2$ be maximal cliques in $D_1$ and $D_2$. Observe that $\Omega_1 = \Omega'_1 \cup Q$ and $\Omega_2 = \Omega'_2 \cup Q$ are maximal cliques in $H$. By definition, $\Omega_1$ and $\Omega_2$ are vital potential maximal cliques in $G$ and $\Omega_1 \cap \Omega_2 = Q$.

(ii) Let $L = (B, D)$ be the parent block of $(B, D)$. Since $L$ is not a leaf-block, $L$ has at least two children and thus there is a block $(B', D')$ which is also a child of $L$. By the previous point, $Q$, the tail of $L$ is exactly $Q = \Omega_1 \cap \Omega_3$ for some vital potential maximal clique $\Omega_3$. It follows that $B = Q \setminus \Omega_3$.

(iii) It follows from Lemma 2.7 that $G[D]$ is connected. Then it follows immediately that $D$ is the unique connected component of $G - (Q \setminus B)$ containing $B$.  □
Lemma 2.13. Let \( G \) be a graph, \( S \) a minimal completion of size at most \( k \), and \( L = (B, D) \) a leaf block of the universal clique decomposition of \( H = G + S \). If \( H \) is not a complete graph, then

(i) \( B = \Omega_1 \setminus \Omega_2 \) for some vital potential maximal cliques \( \Omega_1 \) and \( \Omega_2 \), and

(ii) \( D = B \).

Proof. (i) Let \( \hat{L} = (\hat{B}, \hat{D}) \) be the parent block of \( L \), which exists since \( L \) is not the root block. Let \( L' = (B', D') \) be a child of \( \hat{L} \) which is not \( L \). If \( L' = (B', D') \) is a leaf, then set \( L'' = L \), and if not, then let \( L'' = (B'', D'') \) be a leaf having \( L' \) as an ancestor. The blocks \( L' \) and \( L'' \) exist since \( \hat{L} \) is not a leaf. Furthermore, let \( \hat{Q} \) be the tail of \( \hat{L} \), and let \( \Omega_1 = N_H[B] \) and \( \Omega_2 = N_H[B'] \) be two maximal cliques in \( H \). We know from above that \( \hat{Q} = \Omega_1 \cap \Omega_2 \) and hence \( B = \Omega_1 \setminus \Omega_2 \).

(ii) This follows immediately from Lemma 2.6.

Lemma 2.14. Let \( G \) be a connected graph, \( S \) a minimal completion of size at most \( k \), and \( L = (B, D) \) the root block of the universal clique decomposition of \( H = G + S \). If \( H \) is not a complete graph, then

(i) \( \text{the tail of } L \text{ is } B \),

(ii) \( B = \Omega_1 \cap \Omega_2 \) for some vital potential maximal cliques \( \Omega_1 \) and \( \Omega_2 \), and

(iii) \( D = V \).

Proof. (i) By definition, the tail is the collection of vertices from \( B \) to the root. Since \( L \) is a root block, the tail is \( B \) itself.

(ii) This follows in the same manner as in the proof of Lemma 2.12 (i), since \( B \) is the tail of block \( L \).

(iii) From the definition of universal clique decompositions we have that \( D \) is the connected component of \( H[V \setminus (Q \setminus B)] \) containing \( B \), but \( Q \setminus B = \emptyset \), hence \( D \) is the connected component of \( H \) containing \( B \) and since \( H \) is connected, the result follows.

By making use of Lemmata 2.12–2.14, one can construct the required family \( S \) by going through all possible triples of elements of \( C \). The size of \( S \) is at most \( |C|^3 = 2^{O(\sqrt{k} \log k)} \) and the running time of the construction of \( S \) is \( 2^{O(\sqrt{k} \log k)} \). Note here that by Lemma 2.7 (iii) and the fact that \( G \) is connected, we may discard from \( S \) every pair \((B, D)\) where \( G[D] \) is not connected.

For every pair \((X, Y)\) \(\in S\), with \( X \subseteq Y \subseteq V \), we define \( dp(X, Y) \) to be the minimum number of edges required to add to \( G[Y] \) to obtain a trivially perfect graph where \( X \) is the maximal universal clique; if this minimum value exceeds \( k \), we define \( dp(X, Y) = +\infty \). Thus, to compute an optimal solution, it is sufficient to go through the values \( dp(X, Y) \), where \((X, Y) \in S \) with \( Y = V \). In other words, to compute the size of a minimum completion we can find

\[
\min_{(X,Y) \in S} dp(X,Y),
\]

and if this value is \( +\infty \), then the size of a minimum completion exceeds \( k \).

In the following, for a subset of vertices \( A \) we write \( m_A \) to denote the number of edges inside \( A \), i.e., \( m_A = |E(A)| \). We compute (3) by making use of dynamic programming over sets of \( S \). For every pair \((X, Y) \in S \) which can be a leaf block for some completion, i.e., for all pairs with \( X = Y \), we put

\[
dp(X, X) = \binom{|X|}{2} - m_X.
\]

Of course, if the computed value exceeds \( k \), then we put \( dp(X, X) = +\infty \).

For \((X, Y) \in S \) with \( X \subseteq Y \), if \((X, Y) \) is a block of some minimal completion \( H \), then in \( H \), we have that \( X \) is a universal clique in \( H[Y] \), every vertex of \( X \) is adjacent to all vertices of \( Y \setminus X \) and the number of edges in \( H[Y \setminus X] \) is the sum of edges in the connected components of \( H[Y \setminus X] \). By Lemma 2.7, the vertices of every connected component \( Y' \) of \( H[Y \setminus X] \) induce a connected component in \( G[Y \setminus X] \). We can notice that for each connected component \( Y' \) of \( H[Y \setminus X] \) the decomposition of \( H \) contains a new block \((X', Y')\) and since \( S \) contains all blocks of minimal trivially perfect completions it follows that \((X', Y') \in S \).
Now for \((X, Y) \in \mathcal{S}\) in increasing size of \(Y\), we use \(m_{X,Y\setminus X} = |E(X, Y \setminus X)|\) to denote the number of edges between \(X\) and \(Y \setminus X\) in \(G\). Let \(C\) be the set of connected components of \(G[Y \setminus X]\). Then we have

\[
dp(X, Y) = \left(\frac{|X|}{2}\right) - m_X + |Y| \cdot |Y \setminus X| - m_{X,Y\setminus X} + \min_{G[Y|e \in C(X', Y') \in \mathcal{S}]} \dp(X', Y').
\]

Again, if the value on the right hand side exceeds \(k\), then we have \(dp(X, Y) = +\infty\).

The cardinality of \(Y'\) is less than \(|Y|\) since \(X \neq \emptyset\) and as blocks are processed in increasing cardinality of \(Y\), the value for \(dp(X', Y')\) has been calculated when it is needed for \(dp(X, Y)\).

The running time required to compute \(dp(X, Y)\) is up to a polynomial factor in \(k\) proportional to the number of sets \((X', Y') \in \mathcal{S}\), which is \(O(|\mathcal{S}|)\). Thus the total running time of the dynamic programming procedure is up to a polynomial factor in \(k\) proportional to \(O(|\mathcal{S}|^2)\), and hence (3) can be computed in time \(2^{O(\sqrt{k} \log k)}\). This concludes Step C and the proof of Theorem 2.1.

3 Completion to threshold graphs

In this section we give an algorithm which solves Threshold Completion, which is \(\mathcal{F}\)-Completion for the case when \(\mathcal{F} = \{2K_2, C_4, P_4\}\), in subexponential parameterized time. More specifically, we show the following theorem:

**Theorem 3.1.** **Threshold Completion** is solvable in time \(2^{O(\sqrt{k} \log k)} + O(kn^4)\).

The proof of Theorem 3.1 is a combination of the following known techniques: the kernelization algorithm by Guo [16], the chromatic coding technique of Alon et al. [1], also used in the subexponential algorithm of Ghosh et al. [13] for split graphs, and the algorithm of Fomin and Villanger for chain completion [12].

For the kernelization part we use the following result from Guo [16]. Guo stated and proved it for the complement problem Threshold Edge Deletion, but since the set of forbidden subgraphs \(\mathcal{F} = \{2K_2, C_4, P_4\}\) is self-complementary, the deletion and completion problems are equivalent.

**Proposition 3.2 ([16]).** **Threshold Completion** admits a kernel with \(O(k^3)\) vertices. The running time of the kernelization algorithm is \(O(kn^4)\).

**Universal sets** We start with describing the chromatic coding technique by Alon et al. [1]. Let \(f\) be a coloring (not necessarily proper) of the vertex set of a graph \(G = (V, E)\) into \(t\) colors. We call an edge \(e \in E\) monochromatic if its endpoints have the same color, and call a set of edges \(F \subseteq E\) colorful if no edge in \(F\) is monochromatic.

**Definition 3.3.** A universal \((n, k, t)\)-coloring family is a family \(\mathcal{F}\) of functions from \([n]\) to \([t]\) such that for any graph \(G\) with vertex set \([n]\), and \(k\) edges, there is an \(f \in \mathcal{F}\) such that \(f\) is a proper coloring of \(G\), i.e., \(E(G)\) is colorful.

**Proposition 3.4 ([1]).** For any \(n > 10k^2\), there exists an explicit universal \((n, k, O(\sqrt{k}))\)-coloring family \(\mathcal{F}\) of size \(|\mathcal{F}| \leq 2^{O(\sqrt{k} \log k)} \log n\).

Note that by explicit we mean here that the family \(\mathcal{F}\) not only exists, but can be constructed in \(2^{O(\sqrt{k} \log k)}n^{O(1)}\) time.

3.1 Split, threshold and chain graphs.

Here we give some known facts about split graphs, threshold graphs and chain graphs which we will use to obtain the main result.

**Definition 3.5.** Given a graph \(G = (V, E)\), a partition of the vertex set into sets \(C\) and \(I\) is called a **split partition** of \(G\) if \(C\) is a clique and \(I\) is an independent set.

We denote by \((C, I)\) a split partition of a graph.

**Definition 3.6** (Split graph). A graph is a split graph if it admits a split partitioning.
We now proceed to the details of the algorithm which solves Threshold Completion in the time stated in the theorem. Fomin and Villanger [12] showed that the following problem is solvable in subexponential time:

**Proposition 3.7** (Theorem 6.2, [14]). A split graph on \( n \) vertices has at most \( n + 1 \) split partitions and these partitions can be enumerated in polynomial time.

**Definition 3.8.** A **chain graph** is a bipartite graph \( G = (A, B, E) \) where the neighborhoods of the vertices are nested, i.e., there is an ordering of the vertices in \( A, a_1, a_2, \ldots, a_{n_1} \), such that for each \( i < n_1 \) we have that \( N(a_i) \subseteq N(a_{i+1}) \), where \( n_1 = |A| \).

We will use the following result, which is often used as an alternative definition of threshold graphs.

**Proposition 3.9** ([23]). A graph \( G \) is a threshold graph if and only if \( G \) has a split partition \((C, I)\) and the neighborhoods of the vertices of \( I \) are nested.

Thus, the class of threshold graphs is a subclass of split graphs and by Proposition 3.7, threshold graphs on \( n \) vertices have at most \( n + 1 \) split partitions.

### 3.2 The algorithm.

We now proceed to the details of the algorithm which solves Threshold Completion in the time stated in the theorem. From now on we assume that the number of vertices \( n \) in \( G \) is \( O(k^3) \).

Note that in the Chain Completion problem, the resulting chain graph must have the same bipartition as the input graph. Thus, despite the fact that chain graphs are exactly the \( \{2K_2, C_3, C_4, P_4\}\)-free graphs, formally Chain Completion is not an \( F\)-Completion problem according to our definition.

**Proposition 3.10** ([12]). Chain Completion is solvable in \( 2^{O(\sqrt{k} \log k)} + O(k^3nm) \) time.

We now have the results needed to give an algorithm for Threshold Completion, thus proving Theorem 3.1.

**Proof.** We start by using Proposition 3.2 to obtain a polynomial kernel with \( O(k^3) \) vertices in time \( O(k^4n^4) \). We will therefore from now on assume that the input graph \( G \) has \( n = O(k^3) \) vertices.

Suppose that \( (G, k) \) is a yes instance of Threshold Completion. Then there is an edge set \( S \) of size at most \( k \) such that \( G + S \) is a threshold graph. Without loss of generality, we can assume that \( n > 10k^2 \). By Proposition 3.4, we can construct in \( 2^{O(\sqrt{k} \log k)} + O(1) \) time an explicit universal \( (n, k, O(\sqrt{k})) \)-coloring family \( \mathcal{F} \) of size \( |\mathcal{F}| \leq 2^{O(\sqrt{k} \log k)} \) log \( n \) = \( 2^{O(\sqrt{k} \log k)} \). Since \( |S| \leq k \), there is a vertex coloring \( f \in \mathcal{F} \) such that \( S \) is colorful.

We iterate through all the colorings \( f \in \mathcal{F} \). Let us examine one coloring \( f \in \mathcal{F} \), and let \( V_1, V_2, \ldots, V_t \) be the partitioning of \( V(G) \) according \( f \), where \( t = O(\sqrt{k}) \). Then, since threshold graphs are hereditary and we assume \( S \) to be colorful, each \( V_i \) must induce a threshold graph—we cannot add edges within a color class.

By Proposition 3.7, \( G + S \) has \( O(k^3) \) split partitions. Each such split partition of \( G + S \) induces a split partition of \( G[V_i] \), \( i \in \{1, \ldots, t\} \). Again by Proposition 3.7, each \( G[V_i] \) also has \( O(k^3) \) split partitions. We use brute-force to generate the set of \( O((k^3)^t) = 2^{O(\sqrt{k} \log k)} \) partitions of \( G + S \), and the set of generated partitions contains all split partitions of \( G + S \). By Proposition 3.9, if \( (G, k) \) is a yes instance and \( f \) is colorful, then for at least one of the split partitions \((C, I)\) of \( G + S \) the neighborhoods of \( I \) are nested. To check if a split partition can be turned into a nested partition, we use Proposition 3.10.

To summarize, we perform the following steps:

**Step A. Kernelization** Apply Proposition 3.2 to obtain in time \( O(k^4n^4) \) a kernel with \( O(k^3) \) vertices. From now on we assume that the number of vertices \( n \) in \( G \) is \( O(k^3) \).
Step B. Generating universal families

If necessary, we add a set of isolated vertices to $G$ to guarantee that $n > 10k^2$. We apply Proposition 3.4 to construct a universal $(n, k, \mathcal{O}(\sqrt{k}))$-coloring family $\mathcal{F}$ of size $2^{\mathcal{O}(\sqrt{k} \log k)}$. For each generated coloring $f$ and the corresponding vertex partition $V_1, V_2, \ldots, V_t$, $t = \mathcal{O}(\sqrt{k})$, we perform the steps that follow.

Step C. Generating split partitions

We generate a set of partitions $C$ of $V(G)$ as follows. Each partition $(C, I) \in C$ is of the following form. For $i \in \{1, \ldots, t\}$, let $C_i$, $|C_i| = \mathcal{O}(k^3)$, be the set of split partitions of $G[V_i]$. Then for each $i \in \{1, \ldots, t\}$, $(C \cap V_i, I \cap V_i) \in C_i$. In other words, every partition of $C$ induces a split partition of $G[V_i]$. The time required to generate all partitions from $C$ is $\mathcal{O}(k^3)$. We also perform a sanity check by excluding from $C$ all pairs $(C, I)$, where $I$ is not an independent set. We perform the next step with each pair $(C, I) \in C$.

Step D. Computing nested split partitions

For a pair $(C, I) \in C$, such that $I$ is an independent set in $G$, we first compute the number of edges $c$ needed to turn $C$ into a clique, i.e., $c = \binom{|C|}{2} - n c$. Finally, we use Proposition 3.10 to check if the neighborhood of $I$ in $C$ can be made nested by adding at most $k - c$ edges.

From the discussions above, if $(G, k)$ is a yes instance of the problem, the solution will be found after completing the algorithm. Otherwise, we conclude that $(G, k)$ is a no instance. The running time to perform Step A is $\mathcal{O}(kn^4)$ and Step B is done in $2^{\mathcal{O}(\sqrt{k} \log k)}$. For every $f \in \mathcal{F}$, in Step C we generate $2^{\mathcal{O}(\sqrt{k} \log k)}$ partitions. The total number of times Step C is called is $|\mathcal{F}|$, $2^{\mathcal{O}(\sqrt{k} \log k)} = 2^{\mathcal{O}(\sqrt{k} \log k)}$. In Step D, we run the algorithm with running time $2^{\mathcal{O}(\sqrt{k} \log k)}$ on each of the $2^{\mathcal{O}(\sqrt{k} \log k)}$ partitions, resulting in a total running time of $2^{\mathcal{O}(\sqrt{k} \log k)} + \mathcal{O}(kn^4)$.

4 Completion to pseudosplit graphs

In this section we show that Pseudosplit Completion, or $\mathcal{F}$-Completion for $\mathcal{F} = \{2K_2, C_4\}$, can be solved by first applying a polynomial-time and parameter-preserving preprocessing routine, and then using the subexponential time algorithm of Ghosh et al. [13] for Split Completion.

The crucial property of pseudosplit graphs that will be of use is the following characterization:

Proposition 4.1 ([22]). A graph $G = (V, E)$ is pseudosplit if and only if one of the following holds

- $G$ is a split graph, or
- $V$ can be partitioned into $C, I, X$ such that $G[C \cup I]$ is a split graph with $C$ being a clique and $I$ being an independent set, $G[X] \cong C_5$, and moreover, there is no edge between $X$ and $I$ and every edge is present between $X$ and $C$.

In other words, a pseudosplit graph is either a split graph, or a split graph containing one induced $C_5$ which is completely non-adjacent to the independent set of the split graph, and completely adjacent to the clique set of the split graph. We call a graph which falls into the latter category a proper pseudosplit graph.

In order to ease the argumentation regarding minimal completions, we call a split partition $(C, I)$ $I$-maximal if there is no vertex $v \in C$ such that $(C \setminus \{v\}, I \cup \{v\})$ is a split partition. Our algorithm uses the subexponential algorithm of Ghosh et al. [13] for Split Completion as a subroutine. We therefore need the following result:

Proposition 4.2 ([13]). Split Completion is solvable in time $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$.

Formally, in this section we prove the following theorem:

Theorem 4.3. Pseudosplit Completion is solvable in time $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$.

The algorithm whose existence is asserted in Theorem 4.3 is given as Algorithm 1. We now proceed to prove that this algorithm is correct, and that its running time on input $(G, k)$ is $2^{\mathcal{O}(\sqrt{k} \log k)} n^{\mathcal{O}(1)}$. In the following we adopt the notation from Algorithm 1.
1. Use the algorithm from Proposition 4.2 to check in time $2^\mathcal{O}(\sqrt{k} \log k) n^{\mathcal{O}(1)}$ if $(G,k)$ is a yes instance of Split Completion. If $(G,k)$ is a yes instance of Split Completion, then return that $(G,k)$ is a yes instance of Pseudosplit Completion. Otherwise we complete to a proper pseudosplit graph.

2. For each $X = \{x_1, x_2, \ldots, x_3\} \subseteq V(G)$ such that there is a supergraph $G_X \supseteq G[X]$ and $G_X \cong C_5$, we construct an instance $(G',k')$ to Split Completion from $(G,k)$ as follows:
   
   (a) Let $k' = k + |E(G[X])| - 5$.
   
   (b) Add all the possible edges between vertices of $X$, so that $X$ becomes a clique.
   
   (c) Add a set $A$ of $k + 2$ vertices to $G$.
   
   (d) Add every possible edge between $A$ and $N_G[X]$.

3. Use Proposition 4.2 to check if $(G',k')$ is a yes instance of Split Completion. If $(G',k')$ is a yes instance of Split Completion, then return that $(G,k)$ is a yes instance of Pseudosplit Completion.

4. If for no set $X$ the answer yes was returned, then return no.

Algorithm 1: Algorithm solving Pseudosplit Completion.

As in the algorithm, we denote by $X$ the set of five vertices which will be used as the set inducing a $C_5$ (we try all possible subsets; note that their number is bounded by $\mathcal{O}(n^5)$). Note here that since $G[X]$ admits a supergraph isomorphic to a $C_5$, it follows that $|E(G[X])| \leq 5$ and, consequently, $k' \leq k$.

Similarly, by $A$ we denote the set of $k + 2$ vertices we add that are adjacent only to $N_G[X]$. Intuitively, this set will be used to force that in any minimal split completion of size at most $k$ it holds that $N_G[X] \subseteq C$. From now on $G'$ is the graph as in the algorithm, that is, $G'$ is constructed from $G$ by making $X$ into a clique, adding vertices $A$ and all the possible edges between $A$ and $N_G[X]$.

The following lemma will be crucial in the proof of the correctness of the algorithm.

**Lemma 4.4.** Assume that $S$ is a minimal split completion of $G'$ of size at most $k'$, and let $(C,I)$ be an $I$-maximal split partition of $G' + S$. Then:

(i) $N_G[X] \subseteq C$,

(ii) $A \subseteq I$,

(iii) no edge of $S$ has an endpoint in $A$,

(iv) $C \setminus X$ is fully adjacent to $X$ in $G' + S$, and

(v) $I \setminus A$ is fully non-adjacent to $X$ in $G' + S$.

Proof. (i) Aiming towards a contradiction, suppose that some $v \in N_G[X]$ is in $I$. Since $A \subseteq N(v)$, we must then have that $A \subseteq C$. However, since $A$ is stable in $G$, this demands adding at least $\left(\frac{k+2}{2}\right) > k'$ edges.

(ii) Aiming towards a contradiction, assume that $A \cap C \neq \emptyset$. Since $N_G(A) \subseteq C$ and $A$ is stable in $G$, it follows that $G' + S'$, where $S'$ is $S$ with all the edges incident to $A$ removed, is also a split graph with partition $(C',I')$, where $C' = C \setminus A$ and $I' = I \cup (A \cap C)$. Since $S' \subseteq S$, we have that either $|S'| < |S|$ which is a contradiction with minimality of $S$, or that $S' = S$ and we obtain a contradiction with the assumption that partition $(C,I)$ was $I$-maximal.

(iii) Suppose that there is an edge $e \in S$ incident to a vertex of $A$. Since $A \subseteq I$, we infer that $S \setminus \{e\}$ is still a split completion, which contradicts the minimality of $S$.

(iv) $C$ is a clique in $G' + S$ and $X \subseteq C$, so this holds trivially.

(v) Suppose for a contradiction that some $v^+ \in I \setminus A$ is adjacent to some $v^- \in X$. Since $N_G[X] \subseteq C$, we have that $v^+v^- \in S$. But then $S \setminus \{v^+v^-\}$ is also a split completion, and we have a contradiction with the minimality of $S$. \qed
The correctness of the algorithm is implied by the following lemma:

**Lemma 4.5.** The instance \((G, k)\) is a **yes** instance of Pseudosplit Completion if and only if Algorithm 1 returns **yes** on input \((G, k)\).

**Proof.** From left to right, let \((G, k)\) be a **yes** instance for Pseudosplit Completion. We immediately observe that \((G, k)\) is a **yes** instance for Split Completion if and only if our algorithm returns **yes** in the first test. We therefore assume that \(G\) has to be completed to a proper pseudosplit graph.

Let \(S_0\) be a completion set with \(|S_0| \leq k\) such that \(G_0 = G + S_0\) is a proper pseudosplit graph. Let \((C, I, X)\) be the pseudosplit partition of \(G + S_0\); hence \(G_0[X]\) is isomorphic to \(C_5\). We claim that the algorithm will return **yes** when considering the set \(X\) in the second point; let then \(G'\) be the graph constructed in the algorithm for the set \(X\). Let \(S\) be equal to \(S_0\) with all the edges of \(G_0[X]\) that were not present in \(G[X]\) removed; note that \(|S| = |S_0| + |E(G[X])| - 5 \leq k'\). We claim that \(G' + S\) is a split graph with split partition \((C \cup X, I \cup A)\). Indeed, since \(G'[X]\) is a clique, \(X\) is fully adjacent to \(C\) in \(G_0 \subseteq G' + S\), and \(C\) is a clique in \(G_0 \subseteq G' + S\), then \(C \cup X\) is a clique in \(G' + S\). On the other hand, \(I \cup A\) is independent in \(G'\) and all the edges of \(S\) have at least one endpoint belonging to \(C \cup X\), so \(I \cup A\) remains independent in \(G' + S\). As a result \(G' + S\) is a split graph, and so the algorithm will return **yes** after the application of Proposition 4.2 in the third point.

From right to left, assume that Algorithm 1 returns **yes** on input \((G, k)\). If it returned **yes** already on the first test, then \(G\) may be completed into a split graph by adding at most \(k\) edges, so in particular \((G, k)\) is a **yes** instance of Pseudosplit Completion. From now on we assume that the algorithm returned **yes** in the third point. More precisely, for some set \(X\) the application of Proposition 4.2 has found a minimal completion set \(S\) of size at most \(k'\) such that \(G' + S\) is a split graph, with \(I\)-maximal split partition \((C, I)\).

By Lemma 4.4 we have that \((i)\) \(N_G[X] \subseteq C\), \((ii)\) \(A \subseteq I\), \((iii)\) no edge of \(S\) has an endpoint in \(A\), \((iv)\) \(C \setminus X\) is fully adjacent to \(X\) in \(G' + S\), and \((v)\) \(I \setminus A\) is fully non-adjacent to \(X\) in \(G' + S\). By the choice of \(X\), there exists a supergraph \(G_X\) of \(G[X]\) such that \(G_X \cong C_5\). Let now \(S_0\) be equal to \(S\) with all the edges of \(G_X\) that were not present in \(G[X]\) included. Observe that \(|S_0| \leq k\) and that by \((iii)\) \(S_0\) contains only edges incident to vertices of \(G\). Consider now the partition \((C \setminus X, I \setminus A, X)\) of \(V(G + S_0)\). Since \((C, I)\) was a split partition of \(G' + S\), it follows that \(C \setminus X\) is a clique in \(G + S_0\) and \(I \setminus A\) is an independent set in \(G + S_0\). Moreover, from \((iv)\) and \((v)\) it follows that \(X\) is fully adjacent to \(C \setminus X\) in \(G + S_0\) and fully non-adjacent to \(I \setminus A\) in \(G + S_0\). Finally, the graph induced by \(X\) in \(G + S_0\) is \(G_X \cong C_5\).

By Lemma 4.1 we infer that \(G + S_0\) is a pseudosplit graph, and so the instance \((G, k)\) is a **yes** instance of Pseudosplit Completion.

As for the time complexity of the algorithm, we try sets of five vertices for \(X\), which is \(O(n^5)\) tries. For each such guess, we construct the graph \(G'\), which has \(n + k + 2\) vertices. Since \(k' \leq k\), by Proposition 4.2 solving Split Completion requires time \(2^{O(\sqrt{k \log k})} n^{O(1)}\), both in the first and the third point of the algorithm. Thus the total running time of Algorithm 1 is \(2^{O(\sqrt{k \log k})} n^{O(1)}\).

## 5 Lower bounds

In this section we will give the promised lower bounds described in Figure 2, i.e., we will show that \(F\)-Completion is not solvable in subexponential time for \(F\) being one of \(\{2K_2\}, \{C_4\}, \{P_4\}\), and \(\{2K_2, P_4\}\) under ETH.

Throughout this section we will reduce to the above problems from 3Sat; We will assume that the input formula \(\varphi\) is in 3-CNF, that is, it is a conjunction of a number of clauses, where each clause is a disjunction of at most three literals. By applying standard regularizing preprocessing for \(\varphi\) (see for instance [11, Lemma 13]) we may also assume that each clause of \(\varphi\) contains exactly three literals, and the variables appearing in these literals are pairwise different.

If \(\varphi\) is a 3Sat instance, we denote by \(V(\varphi)\) the variables in \(\varphi\) and by \(C(\varphi)\) the clauses. We assume we have an ordering \(c_1, c_2, \ldots, c_m\) for the clauses in \(C(\varphi)\) and the same for the variables, \(x_1, x_2, \ldots, x_n\). For simplicity, we also assume the literals in each clause are ordered internally by the variable ordering.

We restate here the Exponential Time Hypothesis, as that will be the crucial assumption for proving that the problems mentioned above do not admit subexponential time algorithms.
Exponential Time Hypothesis (ETH). There exists a positive real number $s$ such that $3$-CNF-SAT with $n$ variables cannot be solved in time $2^{sn}$.

By the Sparsification Lemma of Impagliazzo, Paturi and Zane [17], unless ETH fails, $3$Sat cannot be solved in time $2^{2^{(n+m)(n+m)^O(1)}}$.

For each considered problem we present a linear reduction from $3$Sat, that is, a reduction which constructs an instance whose parameter is bounded linearly in the size of the input formula. Pipelining such a reduction with the assumed subexponential parameterized algorithm for the problem would give a subexponential algorithm for $3$Sat, contradicting ETH.

5.1 $2K_2$-free completion is not solvable in subexponential time

For $F = \{2K_2\}$, we refer to $F$-Completion as to $2K_2$-Free Completion. We show the following theorem.

**Theorem 5.1.** The problem $2K_2$-Free Completion is not solvable in $2^{o(k)n^{O(1)}}$ time unless the Exponential Time Hypothesis (ETH) fails.

For the proof, however, instead of working directly on this problem, we find it more convenient to show the hardness of the (polynomially) equivalent problem $C_4$-Free Edge Deletion. We will throughout this section write $G - S$ when $S \subseteq E(G)$ for the graph $(V(G), E \setminus S)$.

**Construction** We reduce from $3$Sat and the gadgets can be seen in Figure 5. Let $\varphi$ be an instance of $3$Sat. We construct the instance $(G_\varphi, k_\varphi)$ for $C_4$-Free Edge Deletion and we begin by defining the graph $G_\varphi$. For every variable $x \in V(\varphi)$, we construct a variable gadget graph $G^x$. The graph $G^x$ consists of six vertices $w^x_0, w^x_1, w^x_2, n^x$ (for negative), $p^x$ (for positive), and $t^x$. The three vertices $w^x_0, w^x_1$ and $w^x_2$ will induce a triangle whereas $n^x$ and $p^x$ are adjacent to the vertices in the triangle and to $t^x$. We can observe that the four vertices $n^x, t^x, p^x, w^x_i$ induce a $C_4$ for $i = 0, 1, 2$, and that no other induced $C_4$ occurs in the gadget (see Figure 5a). It can also be observed that by removing either one of the edges $n^x t^x$ and $p^x t^x$, the gadget becomes $C_4$-free. We will refer to the edge $t^x p^x$ as the true edge and to $t^x n^x$ as the false edge. These edges are the thick edges in Figure 5a. This concludes the variable gadget construction.

For every clause $c \in C(\varphi)$, we construct a clause gadget graph $G^c$ as follows. The graph $G^c$ consists of two triangles, $a^c_0, a^c_1, a^c_2$ and $b^c_0, b^c_1, b^c_2$. We also add the edges $a^c_0 b^c_0, a^c_1 b^c_1$, and $a^c_2 b^c_2$. These three latter edges will correspond to the variables contained in $c$ and we refer to them as variable-edges (the thick edges in Figure 5b). No more edges are added. The clause gadget can be seen in Figure 5b. Observe that there are exactly three induced $C_4$s in $G^c$, all of the form $a^c_i, a^c_{i+1}, b^c_i, b^c_{i+1}$ for $i = 0, 1, 2$, where the
indices behave cyclically modulo 3. Moreover, removing any two edges of the form $a_i^c b_i^c$ for $i = 0, 1, 2$ eliminates all the induced $C_4$s contained in $G^\phi$.

To conclude the construction, we give the connections between variable gadgets and clause gadgets that encode literals in the clauses (see Figure 6). If a variable $x$ appears in a non-negated form as the $i$th (for $i = 0, 1, 2$) variable in a clause $c$, we add the edges $t^x a_i^c$ and $p^x b_i^c$. If it appears in a negated form, we add the edges $t^x a_i^c$ and $n^x b_i^c$. The connections can be seen in Figure 6. Observe that we get exactly one extra induced $C_4$ in the connection, and that this can be eliminated by removing either one of the thick edges.

This concludes the construction. We have now obtained a graph $G_\phi$ constructed from an instance $\phi$ of 3Sat. We let $k_\phi = |V(\phi)| + 2|C(\phi)|$ be the allowed (and necessary) budget, and the instance of $C_4$-Free Edge Deletion is then $(G_\phi, k_\phi)$.

We now proceed to prove the following lemma, which will give the result.

**Lemma 5.2.** A given 3Sat instance $\phi$ has a satisfying assignment if and only if $(G_\phi, k_\phi)$ is a yes instance of $C_4$-Free Edge Deletion.

**Proof.** Let $\phi$ be satisfiable and $G_\phi$ and $k_\phi$ be as above. We show that $(G_\phi, k_\phi)$ is a yes instance for $C_4$-Free Edge Deletion. Let $\alpha : V(\phi) \rightarrow \{\text{true}, \text{false}\}$ be a satisfying assignment for $\phi$. For every variable $x \in V(\phi)$, if $\alpha(x) = \text{true}$, we remove the edge corresponding to true, i.e., the edge $t^x p^x$, otherwise we remove the edge corresponding to false, i.e., the edge $t^x n^x$. Every clause $c \in C$ is satisfied by $\alpha$; we pick an arbitrary variable $x$ whose literal satisfies $c$ and remove two edges corresponding to the two other literals. If a clause is satisfied by more than one literal, we pick any of the corresponding variables.

For every clause we deleted exactly two edges and for every variable exactly one edge. Thus the total number of edges removed is $2|C(\phi)| + |V(\phi)| = k_\phi$. We argue now that the remaining graph $G'_\phi$ is $C_4$-free. Since variables appearing in clauses are pairwise different, it can be easily observed that every induced cycle of length four in $G'_\phi$ is either
The case of a 4-cycle of the form $t^x \gamma b_i^c a_i^c$, where $x$ is the $i$th variable of clause $c$, and $\gamma \in \{n, p\}$ denotes whether the literal in $c$ that corresponds to $x$ is negated or non-negated.

By the construction of $G_\varphi$, we destroyed all induced 4-cycles of the first two types. Consider a 4-cycle $t^x p^x b_i^c a_i^c$ of the third type, where $x$ appears positively in clause $c$. In the case when the literal of variable $x$ was not chosen to satisfy $c$, we have deleted the edge $a_i^c b_i^c$ and so this 4-cycle is removed. Otherwise we have that $\alpha(x) = \text{true}$, and we have deleted the edge $t^x p^x$, thus also removing the considered 4-cycle.

The case of a 4-cycle of the form $t^x n^x b_i^c a_i^c$ is symmetric.

Concluding, all the induced 4-cycles that were contained in $G_\varphi$ are removed in $G_\varphi'$. Since vertices in pairs $(a_i^c, b_i^c)$ and $(\gamma^x, t^x)$ for $\gamma \in \{n, p\}$ do not have common neighbors, it follows that no new $C_4$ could be created when obtaining $G_\varphi'$ from $G_\varphi$ by removing the edges. We infer that $G_\varphi'$ is indeed $C_4$-free.

We proceed with the opposite direction. Let $S$ be an edge set of $G_\varphi$ of size at most $k_\varphi$ such that $G - S$ is $C_4$-free. By the definition of the budget $k_\varphi$ and the observation that every variable gadget needs at least one edge to be in $S$ and every clause gadget needs at least two edges to be in $S$ (note here that the edge sets of clause and variable gadgets are pairwise disjoint), we have that $S$ contains exactly one edge from each variable gadget, exactly two edges from each clause gadget, and no other edges.

We construct an assignment $\alpha : \mathcal{V}(\varphi) \to \{\text{true}, \text{false}\}$ for the formula $\varphi$ as follows. For a variable $x \in \mathcal{V}(\varphi)$, put $\alpha(x) = \text{false}$ if the false edge $t^x n^x$ of $G^c$ is in $S$, put $\alpha(x) = \text{true}$ if the true edge $t^x p^x$ of $G^c$ is in $S$, and put an arbitrary value for $\alpha(x)$ otherwise. We claim that the assignment $\alpha$ satisfies $\varphi$.

Suppose for a contradiction that a clause $c \in C$ is not satisfied. Since exactly two edges in the clause gadget $G^c$ are in $S$, there is a variable $x$ appearing in $c$ such that the corresponding variable-edge of $G^c$ is not in $S$. If $\alpha(x) = \text{true}$, then because $c$ is not satisfied, we have that $\neg x \in c$. By the definition of $\alpha$ we have that the false edge of $G^c$ does not belong to $S$. Then in $G_\varphi$, the false edge of $G^c$ and the variable-edge of $G^c$ corresponding to $x$ form an induced $C_4$ that is not destroyed by $S$, a contradiction. The case $\alpha(x) = \text{false}$ is symmetric. This concludes the proof of the lemma.

Finally, the proof of Theorem 5.1 follows from Lemma 5.2: Combining the presented reduction with an algorithm for $C_4$-FREE EDGE DELETION working in $2^{o(k)} n^{O(1)}$ time would yield an algorithm for 3SAT with time complexity $2^{o(n+m)}(n + m)^{O(1)}$, which contradicts ETH by the results of Impagliazzo, Paturi and Zane [17].

### 5.2 $C_4$-free completion is not solvable in subexponential time

For every $\mathcal{F}$-COMPLETION problem that so far turned out to be solvable in subexponential time, we had the graph $C_4$ in $\mathcal{F}$ together with some other graphs: trivially perfect graphs are the class excluding $C_4$ and $P_4$, threshold graphs are the class excluding $2K_2$, $P_4$ and $C_4$, and pseudosplit graphs are the class excluding $2K_2$ and $C_4$. Previous known subexponentiality results in the area of graph modifications are completing to chordal graphs and chain graphs [12], completing to split graphs [13] and recently, completing to interval graphs [2] and proper interval graphs [3]. All these graph classes have $C_4$ as a forbidden induced subgraph.

It is therefore natural to ask whether the $C_4$ is the “reason” for the existence of subexponential algorithms. However, in this section we show that excluding $C_4$ alone is not sufficient for achieving a subexponential time algorithm. For $\mathcal{F} = \{C_4\}$, we refer to $\mathcal{F}$-COMPLETION as $C_4$-FREE COMPLETION.

**Theorem 5.3.** The problem $C_4$-FREE COMPLETION is not solvable in $2^{o(k)} n^{O(1)}$ time unless the Exponential Time Hypothesis (ETH) fails.

To prove the theorem, we reduce from 3SAT, and similarly as before we start with a formula where each clause contains exactly three literals corresponding to pairwise different variables. By duplicating clauses if necessary, we also assume that each variable appears in at least two clauses.

We again need two types of gadgets, one gadget to emulate variables in the formula and one type to emulate clauses. Let $\varphi$ be the 3SAT instance and denote by $\mathcal{V}(\varphi)$ the variables in $\varphi$ and by $\mathcal{C}(\varphi)$ the clauses. We construct the graph $G_\varphi$ as follows:
For each variable $x \in V(\varphi)$ we construct a variable gadget graph $G^x$ as depicted in Figure 8. Let $p_x$ be the number of clauses $x$ occurs in; by our assumption we have that $p_x \geq 2$. The graph $G^x$ consists of a “tape” of $4p_x$ squares arranged in a cycle, with additional vertices attached to the sides of the tape. The intuition is that every fourth square in $G^x$ is reserved for a clause $x$ occurs in. Formally, the vertex set of $G^x$ consists of

$$V(G^x) = \bigcup_{0 \leq i < 4p_x} \{ u^x_i, t^x_i, b^x_i, d^x_i \},$$

and the edge set of

$$E(G^x) = \bigcup_{0 \leq i < 4p_x} \{ u^x_i t^x_i, u^x_i t^x_{i+1}, t^x_i u^x_{i+1}, t^x_i t^x_{i+1}, \}
\quad t^x_i b^x_i, b^x_i b^x_{i+1}, b^x_i d^x_{i+1}, b^x_i d^x_i, d^x_i d^x_{i+1} \},$$

where the indices behave cyclically modulo $4p_x$. The letters for the vertices are chosen to correspond with top and bottom ($t^x$ and $b^x$) of tape, and up and down ($u^x$ and $d^x$). The construction is visualized in Figures 7a and 8.

Claim 5.4. The minimum number of edges required to add to $G^x$ to make it $C_4$-free is $4p_x$. Moreover, there are exactly two ways of eliminating all $C_4$ s with $4p_x$ edges, namely adding an edge on the diagonal for each square. Furthermore, if we add one edge to eliminate some cycle, all the rest must have the same orientation, i.e., either all added edges are of the form $t^x_i b^x_{i+1}$ or of the form $t^x_{i+1} b^x_i$. See Figure 7.

of claim. A gadget $G^x$ contains $4p_x$ induced $C_4$, and no two of them can be eliminated by adding one edge. Hence, to eliminate all $C_4$ s in $G^x$, we need at least $4p_x$ edges. On the other hand, it is easy to verify that after adding $4p_x$ diagonals to $C_4$s of the same orientation the resulting graph does not contain any induced $C_4$, see Figure 7 for examples. Whenever we have two consecutive cycles with completion edges of different orientation, we create a new $C_4$ consisting of the two completion edges, and (depending on their orientation) either two edges incident to vertex $u^x_i$ above their common vertex, or two edges incident to vertex $d^x_i$ below. See Figure 7d.

Corollary 5.5. The minimum number of edges required to eliminate all $C_4$ s appearing inside all the variable gadgets is $12|C(\varphi)|$.

Proof. Since each clause of $C(\varphi)$ contains exactly three occurrences of variables, it follows that $\sum_{x \in V(\varphi)} p_x = 3|C(\varphi)|$. The constructed variable gadgets are pairwise disjoint, so by Claim 5.4 we infer that the minimum number of edges required in all the variable gadgets is equal to $\sum_{x \in V(\varphi)} 4p_x = 3 \cdot 4|C(\varphi)| = 12|C(\varphi)|$. \qed
We now proceed to create the clause gadgets. For each clause \( c \in C(\varphi) \), we create the graph \( G^c \) as depicted in Figure 10. It consists of an induced 4-cycle \( v_1^c v_2^c v_3^c v_4^c \) and induced paths \( v_1^c u_1^c u_2^c v_1^c \) and \( v_3^c u_3^c u_4^c v_3^c \). We also attach a gadget consisting of \( k_{\varphi} \), internally disjoint induced paths of four vertices with endpoints in \( v_1^c \) and \( u_1^c \), where \( k_{\varphi} \) is the budget to be specified later. That makes it impossible to add an edge between \( v_1^c \) and \( u_1^c \) in any \( C_4 \)-free completion with at most \( k_{\varphi} \) edges.

By the \( i \)-th square we mean a quadruple \( (t_i^x, b_i^x, t_{i+1}^x, b_{i+1}^x) \). If a clause \( c \) is the \( \ell \)-th clause the variable \( x \) appears in, we will use the vertices of the \( 4(\ell - 1) \)-st square for connections to the gadget corresponding to \( c \). For ease of notation let \( j = 4(\ell - 1) \). We also use pairs \( \{v_i^c, u_j^c\} \), \( \{v_2^c, u_3^c\} \), and \( \{v_3^c, u_4^c\} \) of \( G^c \) for connecting to the corresponding variable gadgets. If a variable \( x \) appears in a non-negated form as the \( i \)th (for \( i = 1, 2, 3 \)) literal of a clause \( c \), then we add the edges \( t_i^x v_i^c \) and \( b_{i+1}^x u_i^c \). If it appears in a negated form, we add the edges \( t_i^x v_i^c \) and \( b_{i+1}^x u_i^c \). See Figure 11. This concludes the construction of \( G_{\varphi} \). Finally, we set the budget for the instance equal to \( k_{\varphi} = 14|C(\varphi)| \).

**Claim 5.6.** For each clause gadget \( G^c \) for a clause \( c \in C(\varphi) \), we need to add at least two edges between vertices of \( G^c \) to eliminate all induced \( C_4 \)s in \( G^c \). Moreover, there are exactly three ways of adding exactly two edges to \( G^c \) so that the resulting graph does not contain any induced \( C_4 \): by adding \( \{v_1^c v_2^c, v_2^c u_1^c\} \), \( \{v_3^c v_2^c, v_2^c u_3^c\} \), or \( \{v_2^c v_3^c, v_3^c u_2^c\} \).

**Claim.** There is a four-cycle \( v_1^c v_2^c v_3^c v_4^c \) which needs to be eliminated, either by adding the edge \( v_1^c v_2^c \) (Figure 9b) or \( v_3^c v_4^c \) (Figure 9c). In any case we create a new \( C_4 \), either \( v_1^c u_1^c v_2^c u_2^c \) in the former case, and \( v_3^c u_3^c v_4^c u_4^c \) in the latter case. In the former case we can eliminate the created \( C_4 \) by adding \( v_1^c u_1^c \) or \( v_2^c u_2^c \), and in the latter case we can eliminate it by adding \( v_3^c u_3^c \). Note that in the latter case we cannot add \( v_3^c u_3^c \), since then we would create \( k_{\varphi} \) new induced four-cycles. A direct check shows that all the three aforementioned completion sets lead to a \( C_4 \)-free graph. \( \square \)

**Lemma 5.7.** Given a 3Sat instance \( \varphi \), we have that \( (G_{\varphi}, k_{\varphi}) \) is a yes instance for \( C_4 \)-Free Comple-
Figure 11: The connections for a clause $c = x \lor \neg y \lor z$. In this example, $c$ is the first clause of appearance for $x$ thus $x$ is connected to $G^c$ via the 0th square. For $y$ and $z$, we assume that $c$ is the third clause they appear, thus $y$ and $z$ use the 8th square.

**Proof.** From right to left, suppose $\varphi$ is satisfiable. Let $\alpha: \mathcal{V}(\varphi) \rightarrow \{\text{true, false}\}$ be a satisfying assignment for $\varphi$. For every variable $x \in \mathcal{V}(\varphi)$, if $\alpha(x) = \text{true}$, we add edges $t_i^x b_i^r$ to $S$ for $i \in \{0, \ldots, 4p_x - 1\}$ and if $\alpha(x) = \text{false}$, we add edges $t_i^x b_i^l$ to $S$ for $i \in \{0, \ldots, 4p_x - 1\}$.

For a clause $c$ in $\mathcal{C}(\varphi)$, if the first literal satisfies the clause, we add the edges $v^c_1v^c_3$ and $v^c_2u^c_1$ to $S$. If the second literal satisfies the clause, we add $v^c_1v^c_4$ and $v^c_2u^c_2$ to $S$ and if it is the third literal, we add $v^c_1v^c_5$ and $v^c_2u^c_3$ to $S$. If more than one literal satisfies the clause, we pick any. In total this makes $12|\mathcal{C}(\varphi)|$ edges added to the variable gadgets and $2|\mathcal{C}(\varphi)|$ edges added to the clause gadgets.

Suppose now for a contradiction that $G_\varphi + S$ contains a cycle $L$ of length four. In Claims 5.4 and 5.6 it is already verified that $L$ is not completely contained in a variable or clause gadget. Each vertex has at most one incident edge ending outside the gadget of the vertex and there are only edges between variable and clause gadgets. Thus $L$ consist of one edge from a variable gadget and one from a clause gadget and two edges between. We can observe that $L$ then must contain either $v^c_1u^c_1$, $v^c_2u^c_2$, or $v^c_3u^c_3$ of the clause gadget, see Figure 11. Let us assume without loss of generality that $L$ contains the edge $v^c_1u^c_1$. By the construction of the set $S$ this implies that the literal of the first variable $x$ of $c$ satisfies $c$. If $x$ is non-negated in $c$, then we have that $\alpha(x) = \text{true}$ and that $v^c_1t^x_{j+1}$ and $u^c_1b^x_j$ are edges of $L$. To complete the cycle $t^x_{j+1}b^x_j$ must be an edge of $L$; however, by the definition of $S$ we have added the edge $t^x_{j+1}b^x_{j+1}$ to $S$ instead of $t^x_{j+1}b^x_j$ and we obtain a contradiction. The case where $x$ is negated is symmetric.

From left to right, suppose $(G_\varphi, k_\varphi)$ is a yes instance for $k_\varphi = 14|\mathcal{C}(\varphi)|$ and let $S$ be such that $G_\varphi + S$ is $C_4$-free with $|S| \leq k_\varphi$. By Corollary 5.5 and Claim 5.6 we know that we need to use at least $12|\mathcal{C}(\varphi)|$ edges to fix the variable gadgets and we need to use at least $2|\mathcal{C}(\varphi)|$ edges for the clause gadgets. Since $|S| \leq k_\varphi$, we infer that $|S| = k_\varphi$, that we use at exactly $4p_x$ edges to fix each variable gadgets $G^r$ (and
that the orientation of the added edges must be the same within the gadget), that we use exactly two edges for each clause gadget $G^c$, and that $S$ contains no edges other than the mentioned above.

We now define an assignment $\alpha$ for $\mathcal{V}(\varphi)$ and prove that it is indeed a satisfying assignment. If $S$ contains the edge $t_0^v b_v^c$, we let $\alpha(x) = \text{true}$, and if $S$ contains the edge $t_1^v b_v^c$ we let $\alpha(x) = \text{false}$. Let $c \in C(\varphi)$ be a clause and suppose that $c$ is not satisfied. We know by Claim 5.6 that the gadget for $c$ contains $\{v_1^c, v_2^c, v_3^c\}$ or $\{v_1^c, v_2^c, v_3^c, v_4^c\}$.

Without loss of generality assume that $G^c$ contains $\{v_1^c, v_2^c, v_3^c\}$ and that $x$ is the first variable in $c$, and it appears non-negated. Since $x$ does not satisfy $c$, we infer that $\alpha(x) = \text{false}$. This means that $t_1^v b_v^c \in S$, and since the orientation of the added edges in the gadget $G^c$ is the same, then also $t_1^v b_0^c \in S$. As a result, both edges $t_1^{v_1} b_0^c$ and $v_1^{c} u_1^c$ are present in $G_\alpha + S$. But then we have an induced four-cycle $v_1^c u_1^c b_0^c t_1^{v_1} v_1^c$, contradicting the assumption that $G_\alpha + S$ was $C_4$-free. The cases for $y$, $z$ and negative literals are symmetric. This concludes the proof. \hfill $\square$

Similarly as before, the proof of Theorem 5.3 can be completed as follows: combining the presented reduction with an algorithm for $C_4$-FREE COMPLETION working in $2^{o(n)} n^{O(1)}$ time would give an algorithm for 3SAT working in $2^{o(n+m)(n+m)}$ time, which contradicts ETH by the results of Impagliazzo, Paturi and Zane [17].

### 5.3 $P_4$-free completion is not solvable in subexponential time

In this section we show that there is no subexponential algorithm for $F$-COMPLETION for $F = \{P_4\}$ unless the ETH fails. Let us recall that since $P_4 = P_4$, the problems $P_4$-FREE EDGE DELETION and $P_4$-FREE COMPLETION are polynomial time equivalent, and that this graph class more commonly goes under the name cographs. In other words, we aim to convince the reader of the following.

**Theorem 5.8.** The problem $P_4$-FREE COMPLETION is not solvable in $2^{o(n)} n^{O(1)}$ time unless ETH fails.

We reduce from 3SAT to the complement problem $P_4$-FREE EDGE DELETION. Let $\varphi$ be the input 3SAT formula, where we again assume that every clause of $\varphi$ contains exactly three literals corresponding to pairwise different variables. For a variable $x \in \mathcal{V}(\varphi)$ we denote by $p_x$ the number of clauses in $\varphi$ containing $x$. Note that since each clause contains exactly three variables, we have that $\sum_{x \in \mathcal{V}(\varphi)} p_x = 3|C(\varphi)|$. We construct a graph $G_\varphi$ such that for $k_\varphi = 4|C(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = 16|C(\varphi)|$, $\varphi$ is satisfiable if and only if ($G_\varphi, k_\varphi$) is a yes instance of $P_4$-FREE EDGE DELETION. Since the complement of $P_4$ is $P_4$, this will prove the theorem.

**Variable gadget** For each variable $x \in \mathcal{V}(\varphi)$, we create a gadget $G^x$ which looks like the one given in Figure 12. Before providing the construction formally, let us first describe it informally. We call a triangle with a pendant vertex a tower, where the triangle will be referred to as the base of the tower, and the pendant vertex the spike of the tower. The towers will always come in pairs, and they are joined in one of the vertices in the bases (two vertices are identified, see Figure 12). Pairs of towers will be separated by $k'$ (defined below) triangles sharing an edge. The vertices not shared between the $k'$ triangles will be called the stack, whereas the edge shared among the triangles will be called the shortcut.
The gadget $G^x$ for a variable $x$ consists of $p_x$ pairs of towers arranged in a cycle, one for each clause $x$ appears in, where every two consecutive pairs are separated by a shortcut edge and a stack of vertices. The stack is chosen to be big enough ($k' = k_x + 3$ vertices) so that we will never delete the edge that connects the two towers on each side of the stack, nor any edge incident to a vertex from the stack. We will refer to the two towers in the pairs as Tower 1 (the one with lower index) and Tower 2.

Formally, let $\phi$ be an instance of 3SAT. The budget for the output instance will be $k_x = 4|C(\phi)| + \sum_{x \in \mathcal{V}(\phi)} 4p_x = 16|C(\phi)|$. Let $k' = k_x + 3$. For a variable $x$ which appears in $p_x$ clauses, we create vertices $s_{i,j}^x$ for $i \in \{1, \ldots, p_x\}$ and $j \in \{1, \ldots, k'\}$. These are the vertices for the stacks. For the spikes of the towers, we add vertices $s_{i,1}^x$ and $s_{i,2}^x$ for $i \in \{1, \ldots, p_x\}$. For the base of the towers, we add vertices $b_{i,j}^x$ for $j \in \{1, \ldots, 5\}$ and $i \in \{1, \ldots, p_x\}$. These are all the vertices of the gadget $G^x$ for $x \in \mathcal{V}(\phi)$.

The vertices denoted by $t$ are the two spikes in the tower, i.e., $t_{i,1}^x$ is the spike of the Tower 1 of the $i$th pair for variable $x$. The vertices denoted by $b$ are for the bases (there are five vertices in the bases of the two towers).

Now we add the edges to $G^x$, see Figure 13:

- For the stack, we add edges $s_{i,j}^x \oplus b_{i,j}^x$ for all $i \in \{1, \ldots, p_x\}$ and $j \in \{1, \ldots, k'\}$ (right side of the stack) and edges $b_{i,j}^x \oplus s_{i+1,j}^x$ for $i \in \{1, \ldots, p_x\}$ and $j \in \{1, \ldots, k'\}$ (the left side of the next stack), where the indices behave cyclically modulo $p_x$.

- For the bases, we add the edges $b_{i,1}^x \oplus b_{i,2}^x$, $b_{i,2}^x \oplus b_{i,3}^x$, $b_{i,3}^x \oplus b_{i,4}^x$, $b_{i,4}^x \oplus b_{i,5}^x$ and $b_{i,5}^x \oplus b_{i+1,1}^x$ for $i \in \{1, \ldots, p_x\}$.

To attach the towers, we add the edges $t_{i,1}^x \oplus s_{i,1}^x$ and $t_{i,2}^x \oplus s_{i,2}^x$. The set of these eight edges will be denoted by $R_i^x$.

- The last edges to add are the shortcut edges $b_{i,5}^x \oplus b_{i+1,1}^x$ for $i \in \{1, \ldots, p_x\}$, where again the indices behave cyclically modulo $p_x$.

**Elimination from variable gadgets** We will now show that there are exactly two ways of eliminating all $P_4$s occurring in a variable gadget using at most $4p_x$ edges. To state this claim formally, we need to control how the variable gadget is situated in a larger construction of the whole output instance that will be defined later. We say that a variable gadget $G^x$ is *properly embedded* in the output instance $G_\phi$, if $G^x$ is an induced subgraph of $G_\phi$, and moreover the only vertices of $G^x$ that are incident to edges outside $G^x$ are the spikes of the towers, i.e., vertices $t_{i,1}^x$ and $t_{i,2}^x$ for $i \in \{1,2, \ldots, p_x\}$. This property will be satisfied for gadgets $G_x$ for all $x \in \mathcal{V}(\phi)$ in the next steps of the construction. Using this notion, we can infer properties of the variable gadget irrespective of what the whole output instance $G_\phi$ constructed later looks like.

We first show that an inclusion minimal deletion set $S$ that has size at most $k_\phi$ cannot touch the stacks nor the shortcut edges.

**Claim 5.9.** Assume gadget $G^x$ is embedded properly in the output graph $G_\phi$, and that $S$ is an inclusion minimal $P_4$-free edge deletion set in $G_\phi$ of size at most $k_\phi$. Then $S$ does not contain any edge of type $b_{i,5}^x \oplus b_{i+1,1}^x$ (a shortcut edge), nor any edge incident to a vertex of the form $s_{i,3}^x$. 

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Figure 13: Variable gadget $G^x$. The counter $i$ ranges from 1 to $p_x$, the number of clauses $x$ appears in. This figure does not illustrate that the gadget is a cycle, see Figure 12 for a zoomed-out version.
of claim. Suppose first that a shortcut edge $b_{i,5}^r b_{i+1,1}^r$ belongs to $S$. (See Figure 13 for indices.) Let $S' = S \setminus \{b_{i,5}^r b_{i+1,1}^r\}$. Since $S$ was inclusion minimal, the graph $G_\varphi - S'$ must contain an induced $P_4$ that contains the edge $b_{i,5}^r b_{i+1,1}^r$; denote this $P_4$ by $L$. By the assumption that $G^r$ is properly embedded in $G_\varphi$ we have that $L$ is entirely contained in $G^r$. Since the stack between pairs of towers $i$ and $i + 1$ has height $k' = k + 3$, we know that there are at least three vertices of the form $s_{t+1,j}^r$ for some $j \leq k$ which are not incident to an edge in $S$. Since $L$ passes through 2 vertices apart from $b_{i,5}^r$ and $b_{i+1,1}^r$, we infer that one of these vertices, say $s_{t+1,j_0}^r$, is not incident to any edge of $S$, nor it lies on $L$. Create $L'$ by replacing the edge $b_{i,5}^r b_{i+1,1}^r$ with the path $b_{i,5}^r - s_{t+1,j_0}^r - b_{i+1,1}^r$ on $L$. We infer that $L'$ is an induced $P_5$ in $G_\varphi - S$, which in particular contains an induced $P_4$. This is a contradiction to the definition of $S$.

Second, without loss of generality suppose now that the edge $b_{i,5}^r s_{t+1,j}^r$ belongs to $S$ for some $j \in \{1, 2, \ldots, k\}$. Let $S' = S \setminus \{b_{i,5}^r s_{t+1,j}^r, b_{i+1,1}^r s_{t+1,j}^r\}$; note here that the edge $b_{i+1,1}^r s_{t+1,j}^r$ might not have belonged to $S$, but if it had, then we remove it when constructing $S'$. Since $S$ was inclusion minimal, the graph $G_\varphi - S'$ must contain an induced $P_4$ that contain the vertex $s_{t+1,j}^r$, so also one of the vertices $b_{i,5}^r$ or $b_{i+1,1}^r$; denote this $P_4$ by $L$. Again, by the definition of $G^r$ we have that $L$ is entirely contained in $G^r$. By the same argumentation as before we infer that there exists a vertex $s_{t+1,j_0}^r$ such that $s_{t+1,j_0}^r$ is not traversed by $L$ and is not incident to an edge of $S$. Since vertices $s_{t+1,j_0}^r$ and $s_{t+1,j}^r$ are twins in $G_\varphi - S'$, it follows that the path $L'$ constructed from $L$ by substituting $s_{t+1,j}^r$ with $s_{t+1,j}^r$ is an induced $P_4$ in $G_\varphi - S$. This is a contradiction to the definition of $S$.

Now we show that every minimal deletion set $S$ must use at least 4 edges in each pair of towers, and if it uses exactly 4 edges then there are exactly 4 ways how the intersection of $S$ with this pair of towers can look like.

Claim 5.10. Assume that the gadget $G^r$ is embedded properly in the output graph $G_\varphi$, and that $S$ is an inclusion minimal $P_4$-free edge deletion set in $G_\varphi$ of size at most $k_\varphi$. Then for each $i \in \{1, 2, \ldots, p_r\}$ it holds that $|R_i^r \cap S| \geq 4$, and if $|R_i^r \cap S| = 4$ then either:

Elimination A: $R_i^r \cap S$ consists of the edges of the base of Tower 1 and the spike of Tower 2, or

Elimination B: $R_i^r \cap S$ consists of the edges of the base of Tower 2 and the spike of Tower 1, or

Elimination C: $R_i^r \cap S$ consists of the edges of both spikes and of the base of Tower 1 apart from the edge $b_{1,2}^r b_{1,2}^r$, or

Elimination D: $R_i^r \cap S$ consists of the edges of both spikes and of the base of Tower 2 apart from the edge $b_{1,2}^r b_{1,2}^r$.

We refer to Figure 14 for visualization of all the four types of eliminations. We will say that $R_i^r \cap S$ realizes Elimination $X$ for $X$ being $A$, $B$, $C$, or $D$, if $R_i^r \cap S$ is as described in the statement of Claim 5.10. Similarly, we say that the $i$th pair of towers realizes Elimination $X$ if $R_i^r \cap S$ does.

of Claim 5.10. By Claim 5.9 we infer that $S$ does not contain any edge incident to stacks $i$ and $i + 1$, nor any of the shortcut edges incident to the considered pair of towers. We consider four cases, depending on
how the set \( S \cap \{ t_{i_1}^x b_{i_2}^x, t_{i_2}^x b_{i_4}^x \} \) looks like. In each case we prove that \( |R_i^x \cap S| \geq 4 \), and that \( |R_i^x \cap S| = 4 \) implies that one of four listed elimination types is used.

First assume that \( S \cap \{ t_{i_1}^x b_{i_2}^x, t_{i_2}^x b_{i_4}^x \} = \emptyset \) and observe that \( t_{i_1+1}^x - b_{i_2}^x - b_{i_4}^x - t_{i_2}^x \) and \( s_{i_1+1}^x - b_{i_2}^x - b_{i_3}^x - b_{i_4}^x \) are induced \( P_3 \)'s in \( G_x \). Since on each of these \( P_3 \)'s there is only one edge that is not assumed to be not belonging to \( S \), it follows that both \( b_{i_1}^x b_{i_2}^x \) and \( b_{i_3}^x b_{i_4}^x \) must belong to \( S \). Suppose that \( b_{i_1}^x b_{i_2}^x \notin S \). Then we infer that \( b_{i_2}^x b_{i_3}^x b_{i_4}^x b_{i_5}^x \in S \), since otherwise any of these edges would form an induced \( P_4 \) in \( G_x - S \) together with edges \( b_{i_1}^x b_{i_3}^x \) and \( b_{i_1}^x s_{i_1}^x \). We infer that in this case \( |R_i^x \cap S| \geq 5 \), and a symmetric conclusion can be drawn when \( b_{i_3}^x b_{i_4}^x \notin S \). We are left with the case when \( b_{i_1}^x b_{i_2}^x b_{i_4}^x b_{i_5}^x \in S \). But then \( S \) must include also one of the edges \( b_{i_1}^x b_{i_2}^x \) or \( b_{i_3}^x b_{i_4}^x \) so that the induced \( P_3 t_{i_1}^x - b_{i_2}^x - b_{i_3}^x - b_{i_4}^x \) is destroyed. Hence, in all the considered cases we conclude that \( |R_i^x \cap S| \geq 5 \).

Second, assume that \( S \cap \{ t_{i_1}^x b_{i_2}^x, t_{i_2}^x b_{i_4}^x \} = \{ t_{i_2}^x b_{i_4}^x \} \). The same reasoning as in the previous paragraph shows that \( b_{i_1}^x b_{i_2}^x \) must belong to \( S \). Again, if \( b_{i_1}^x b_{i_2}^x \notin S \), then all the edges \( b_{i_2}^x b_{i_3}^x b_{i_4}^x b_{i_5}^x \) must belong to \( S \), and so \( |R_i^x \cap S| \geq 5 \). Assume then that \( b_{i_1}^x b_{i_2}^x \in S \). Note now that we have two induced \( P_3 \)'s: \( t_{i_1}^x - b_{i_2}^x - b_{i_3}^x - b_{i_4}^x \) and \( t_{i_2}^x - b_{i_3}^x - b_{i_3}^x - b_{i_5}^x \) that share the edge \( t_{i_2}^x b_{i_4}^x \) about which we assumed that it does not belong to \( S \), and the edge \( b_{i_2}^x b_{i_3}^x \). To remove both these \( P_3 \)'s we either remove at least two more edges, which results in conclusion that \( |R_i^x \cap S| \geq 5 \), or remove the edge \( b_{i_2}^x b_{i_4}^x \), which results in Elimination A.

The third case when \( S \cap \{ t_{i_1}^x b_{i_2}^x, t_{i_2}^x b_{i_4}^x \} = \{ t_{i_1}^x b_{i_2}^x \} \) is symmetric to the second case, and leads to a conclusion that either \( |R_i^x \cap S| \geq 5 \) or \( R_i^x \cap S \) realizes Elimination B.

Finally, assume that \( t_{i_1}^x b_{i_2}^x, t_{i_2}^x b_{i_4}^x \in S \). Observe that we have an induced \( P_3 s_{i_1}^x - b_{i_1}^x - b_{i_3}^x - b_{i_5}^x \) in \( G_x \), so one of the edges \( b_{i_1}^x b_{i_3}^x \) or \( b_{i_3}^x b_{i_5}^x \) must be included in \( S \). Assume first that \( b_{i_1}^x b_{i_3}^x \in S \). Consider now \( P_3 s_{i_1}^x - b_{i_1}^x - b_{i_2}^x - b_{i_3}^x \) and \( t_{i_2}^x - b_{i_3}^x - b_{i_5}^x - s_{i_3}^x \). Both these \( P_3 \)'s need to be destroyed by \( S \) since after removing \( b_{i_1}^x b_{i_3}^x \) the first \( P_3 \) becomes induced, while the second is induced already in \( G_x \). Moreover, these \( P_3 \)'s share only the edge \( b_{i_2}^x s_{i_3}^x \), which means that each \( |R_i^x \cap S| \geq 5 \) or \( b_{i_2}^x b_{i_4}^x \in S \) and \( R_i^x \cap S \) realizes Elimination C. The case when \( b_{i_2}^x b_{i_4}^x \in S \) is symmetric and leads to a conclusion that either \( |R_i^x \cap S| \geq 5 \) or \( R_i^x \cap S \) realizes Elimination D.

Finally, we are able to prove that the variable gadget \( G^x \) requires at least \( 4p_x \) edge deletions, and that there are only two ways of destroying all \( P_3 \)'s by using exactly \( 4p_x \) edge deletions: either by applying Elimination A or Elimination B to all the pairs of towers.

**Claim 5.11.** Suppose a gadget \( G^x \) is embedded properly in the output graph \( G_x \), and that \( S \) is an inclusion minimal \( P_3 \)-free edge deletion set in \( G_x \) of size at most \( k_x \). Then \( |E(G^x) \cap S| \geq 4p_x \), and if \( |E(G^x) \cap S| = 4p_x \), then either \( R_i^x \cap S \) realizes Elimination A for all \( i \in \{ 1, 2, \ldots, p_x \} \), or \( R_i^x \cap S \) realizes Elimination B for all \( i \in \{ 1, 2, \ldots, p_x \} \).

*Proof of claim.* By Claims 5.9 and 5.10 we have that \( S \) does not contain any shortcut edge or edge incident to a stack vertex, and moreover that \( |R_i^x \cap S| \geq 4 \) for all \( i \in \{ 1, 2, \ldots, p_x \} \). Since sets \( R_i^x \) are pairwise disjoint, it follows that \( |E(G^x) \cap S| \geq 4p_x \). Moreover, if \( |E(G^x) \cap S| = 4p_x \), then \( |R_i^x \cap S| = 4 \) for all \( i \in \{ 1, 2, \ldots, p_x \} \) and, by Claim 5.10, for all \( i \in \{ 1, 2, \ldots, p_x \} \) the set \( R_i^x \cap S \) must realize Elimination A, B, C, or D.

We say that one pair of towers is followed by another, if the former has index \( i \), and the latter has index \( i + 1 \) (of course, modulo \( p_x \)). To obtain the conclusion that either all the sets \( R_i^x \cap S \) realize Elimination A or all of them realize Elimination B, we observe that when some pair of towers realize Elimination A, C, or D, then the following pair must realize Elimination A. Indeed, otherwise the graph \( G_x - S \) would contain an induced \( P_3 \) of the form \( b_{i_4}^x - b_{i_5}^x - b_{i+1,1}^x - b_{i+1,3}^x \), where the 4th pair of towers is the considered pair that realizes Elimination A, C, or D. Now observe that since the pairs of towers are arranged on a cycle, then either all pairs of towers realize Elimination B, or at least one realizes Elimination A, C, or D, which means that the following pair realizes Elimination A, and so all the pairs must realize Elimination A.

**Clause gadget** We now move on to construct the clause gadget \( G^c \) for a clause \( c \in C(\varphi) \). Assume that \( c = \ell_y \lor \ell_y \lor \ell_z \), where \( \ell_y \) is a literal of variable \( r \) for \( r \in \{ x, y, z \} \). We create seven vertices: one vertex \( u^c \) and vertices \( u^x \) and \( u^y \) for \( r = x, y, z \). We also add the edges \( u^c u^x, u^c u^y, u^x w^y \) and \( u^y w^x \). Now, for non-negated \( r \in \{ x, y, z \} \) in \( c \), we construct \( c \) to be the spike of Tower 1 in tower pair \( i \). If \( r \) appears negated, we add the edges \( u^x w^y, u^z w^x \) instead, see Figure 16. Let \( M^c \) be the set comprising all the 15 created edges, including the ones incident to the
spikes of the towers. By $M^{c,r}$ for $r \in \{x, y, z\}$ we denote the subset of $M^c$ containing 5 edges that are incident to vertex $u^c_r$ or $u^c_t$. This construction closes the construction of the graph $G_\varphi$; note that all the variable gadgets are properly embedded in $G_\varphi$. Before showing the correctness of the reduction, we prove the following claims about the number of edges needed for the clause gadgets:

**Claim 5.12.** Assume that $S$ is a $P_4$-free deletion set of graph $G_\varphi$. Let $c$ be a clause of $\varphi$, and assume that $x, y, z$ are the variables appearing in $c$. Then $|S \cap M^c| \geq 4$, and if $|S \cap M^c| = 4$, then $S \cap M^{c,r} = \emptyset$ for some $r \in \{x, y, z\}$ (see Figure 15b for an example where $S \cap M^{c,z} = \emptyset$).

of claim. To simplify the notation, let $t^x, t^y, t^z$ be the corresponding vertices of the variable gadgets that are incident to edges of $M^c$.

If $|S \cap M^{c,r}| \geq 2$ for all $r \in \{x, y, z\}$, then $|S \cap M^c| \geq 6$ and we are done. Assume then without loss of generality that $|S \cap M^{c,z}| \leq 1$. Hence, at least one of the paths $t^x - u^c_2 - u^c_3$ and $t^x - u^c_2 - u^c_5$ does not contain an edge of $S$. Assume without loss of generality that it is $t^x - u^c_2 - u^c_5$. Now observe that in $G_\varphi$ we have 4 induced $P_4$ created by prolonging this $P_4$ by vertex $u^c_3$, $u^c_3$, $u^c_3$, or $u^c_3$. Since $t^x - u^c_2 - u^c_5$ is disjoint with $S$, it follows that all the four edges connecting these vertices with $u^c$ must belong to $S$. Hence $|S \cap M^c| \geq 4$, and if $|S \cap M^c| = 4$ then $M^{c,x}$ must be actually disjoint with $S$.

We are finally ready to prove the following lemma, which implies correctness of the reduction.

**Lemma 5.13.** Given an input instance $\varphi$ to 3SAT, $\varphi$ is satisfiable if and only if the constructed graph $G_\varphi$ has a $P_4$ deletion set of size $k_\varphi = 16|C(\varphi)|$.

**Proof.** From left to right, suppose $\varphi$ is satisfiable by an assignment $\alpha$, and let $G_\varphi$ and $k_\varphi$ be as above. If a variable $x$ is assigned false in $\alpha$, we delete as in Figure 14a, that is, we apply Elimination $A$ to all the pairs of towers in the variable gadget $G^x$. Otherwise we delete as in Figure 14b, that is, we apply Elimination $B$ to all the pairs of towers in the variable gadget $G^x$. In other words, if $x$ assigned to false (Elimination $A$), we delete the edges $t^x_2b^x_2b^x_4$, $b^x_2b^x_2b^x_4$, $b^x_2b^x_2b^x_4$, $b^x_2b^x_2b^x_4$, otherwise, when $x$ is assigned to true (Elimination $B$), we delete the edges $t^x_1b^x_1b^x_2$, $t^x_1b^x_1b^x_2$, $t^x_1b^x_1b^x_4$, $b^x_2b^x_2b^x_2$, for all $i \in \{1, \ldots, p_x\}$.

Furthermore, for every clause $c = \ell \lor \ell \lor \ell$ we choose an arbitrary variable whose literal satisfies $c$, say $r$. We remove the edges $u^c_r u^c_r$ and $u^c_r u^c_r$ for $r' \neq r$. We have thus used exactly four edge removals per clause, $4|C(\varphi)|$ in total, and for each $x \in \mathcal{V}(\varphi)$ we have removed $4p_x$ edges. This sums up exactly to $4|C(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = 4|C(\varphi)| + 4 \sum_{x \in \mathcal{V}(\varphi)} p_x = 4|C(\varphi)| + 4 \cdot 3|C(\varphi)| = 16|C(\varphi)| = k_\varphi$ edge removals.
follows: combining the reduction with an algorithm for
variable gadget, nor inside any clause gadget. Therefore, any induced
vertex of the form \(t^i_q\) for some \(x \in \mathcal{V}(\varphi)\), \(i \in \{1, 2, \ldots, p_x\}\), and \(q \in \{1, 2\}\), together with the edge of the
spike incident to this vertex and one of the edges of gadget \(G^c\) incident to this vertex, where \(c\) is the \(i\)th clause \(x\) appears in. Assume without loss of generality that \(q = 1\), so \(x\) appears in \(c\) positively. Since we
did not delete the spike edge \(t^i_{1,1}b^r_{i,2}\), we infer that \(\alpha(x) = \text{false}\). Therefore \(x\) does not satisfy \(c\), so we
must have deleted edges \(u^r_u^x\) and \(u^r_u^z\). Thus in the remaining graph \(G_\varphi - S\) the connected component of the
vertex \(t^i_{1,q}\) is a triangle with a pendant edge, which is \(P_4\)-free. We conclude that \(G_\varphi - S\) is indeed \(P_4\)-free.

From right to left, suppose now that \(G_\varphi\) is the constructed graph from a fixed \(\varphi\) and that for \(k_\varphi\) as
above, we have that \((G_\varphi, k_\varphi)\) is a \textbf{yes} instance of \(P_4\)-\textsc{Free Edge Deletion}. Let \(S\) be a \(P_4\) deletion set
of size at most \(k_\varphi\), and without loss of generality assume that \(S\) is inclusion minimal. By Claims 5.11
and 5.12, set \(S\) must contain at least \(4p_x\) edges in each set \(E(G^2)\), and at least four edges in each set
\(M^c\). Since \(4|\mathcal{C}(\varphi)| + \sum_{x \in \mathcal{V}(\varphi)} 4p_x = k_\varphi\), we infer that \(S\) contains exactly \(4p_x\) edges in each set \(E(G^2)\),
and exactly four edges in each set \(M^c\). By Claim 5.11 we infer that for each variable \(x\), all the pairs of
towers in \(G^x\) realize Elimination \(A\), or all of them realize Elimination \(B\). Let \(\alpha: \mathcal{V}(\varphi) \rightarrow \{\text{true, false}\}\)
be an assignment that assigns value \textbf{false} if Elimination \(A\) is used throughout the corresponding gadget,
and value \textbf{true} otherwise. We claim that \(\alpha\) satisfies \(\varphi\).

Consider a clause \(c \in \mathcal{C}(\varphi)\) and assume that \(x, y, z\) are variables appearing in \(c\). By Claim 5.12 we infer that there exists \(r \in \{x, y, z\}\) such that \(S \cap M^{c-r} = \emptyset\). Assume without loss of generality that \(r = x\), and that \(x\) appears positively in \(c\). Moreover, assume that \(c\) is the \(i_x\)th clause \(x\) appears in. We claim that \(\alpha(x) = \text{true}\), and thus \(c\) is satisfied by \(x\). Indeed, otherwise the edge \(t^i_{i_x,1}b^r_{i_x,2}\) would not be deleted,
and thus \(b^r_{i_x,2} - t^i_{i_x,1} - u^x - u^c\) would be an induced \(P_4\) in \(G_\varphi - S\); this is a contradiction to the definition of
\(S\).

Again, the proof of Theorem 5.8 follows: combining the presented reduction with an algorithm for
\(P_4\)-\textsc{Free Edge Deletion} working in \(2^{o(k)}n^{O(1)}\) time would give an algorithm for 3\textsc{Sat} working in
\(2^{o(n+m)(n+m)}\) time, which contradicts ETH by the results of Impagliazzo, Paturi and Zane [17].

It is easy to verify that in the presented reduction, both the graph \(G_\varphi\) and \(G_\varphi - S\) for \(S\) being the
deletion set constructed for a satisfying assignment for \(\varphi\) are actually \(C_4\)-free. Thus the same reduction
also shows that \(\{C_4, P_4\}\)-\textsc{Free Deletion} is not solvable in \(2^{o(k)}n^{O(1)}\) time unless ETH fails; Since
\(P_4 = P_4\) and \(C_4 = C_4\), it follows that \(\{2K_2, P_4\}\)-\textsc{Free Completion} is hard under ETH as well. In
other words we derive the following result: \textbf{Co-Trivially Perfect Completion} is not solvable in
Theorem 5.14. The problem \( \{2K_2, P_4\}\)-Free Completion is not solvable in \( 2^{o(k)}n^{O(1)} \) time unless ETH fails.

6 Conclusion and future work

In this paper, we provided several upper and lower subexponential parameterized bounds for \( F\)-Completion. The most natural open question would be to ask for a dichotomy characterizing for which sets \( F \), \( F\)-Completion problems are in \( P \), in \( SUBEPT \), and not in \( SUBEPT \) (under ETH). Keeping in mind the lack of such characterization concerning classes \( P \) and \( NP \), an answer to this question can be very non-trivial. Even a more modest task—deriving general arguments explaining what causes a completion problem to be in \( SUBEPT \)—is an important open question.

Similarly, from an algorithmic perspective obtaining generic subexponential algorithms for completion problems would be a big step forwards. With the current knowledge, for different cases of \( F \), the algorithms are built on different ideas like chromatic coding, potential maximal cliques, \( k \)-cuts, etc. and each new case requires special treatment.

Another interesting property is that all the graph classes for which subexponential algorithms for completion problems are known, are tightly connected to chordal graphs. Indeed, all the known algorithms exploit existence of a chordal-like decomposition of the target completed graph. Are there natural \( NP \)-hard graph modification problems admitting subexponential time algorithms where the graph class target is not related to chordal graphs?

Finally, in this paper we have presented \( SUBEPT \) lower bounds (under ETH) for \( F\)-Completion for several different cases of \( F \), but we lack a method for proving tight lower bounds on the running time for problems that actually are in \( SUBEPT \). For instance, it may be the case that Trivially Perfect Completion or Chordal Completion can be solved in time \( 2^{O(k^{1/4})}n^{O(1)} \). As Fomin and Villanger [12] observed, in the case of Chordal Completion known \( NP \)-hardness reductions provide lower bounds much weaker than the current upper bound of \( 2^{O(\sqrt{k}\log k)}n^{O(1)} \). However, we feel that a \( 2^{o(\sqrt{k})}n^{O(1)} \) running time should be impossible to achieve, since such an algorithm would immediately imply the existence of an exact algorithm with running time \( 2^{o(n)} \). Is it possible to prove \( 2^{o(\sqrt{k})}n^{O(1)} \) lower bounds under ETH for Trivially Perfect Completion, Chordal Completion, and other completion problems to subclasses of chordal graphs known to be contained in \( SUBEPT \)?

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