CHARACTERISATIONS OF $V$-SUFFICIENCY AND $C^0$-SUFFICIENCY OF RELATIVE JETS

KARIM BEKKA AND SATOSHI KOIKE

Dedicated to Tzee-Char Kuo for his 80th birthday

Abstract. The Kuiper-Kuo theorem ([14], [15]) is well-known in Singularity Theory, as a result giving sufficient conditions for $r$-jets to be $V$-sufficient and $C^0$-sufficient in $C^r$ functions or $C^{r+1}$ functions. The converse is also proved by J. Bochnak and S. Lojasiewicz [3]. Generalisations of the criteria for $V$-sufficiency and $C^0$-sufficiency to the mapping case are established by T.-C. Kuo [17] and J. Bochnak and W. Kucharz [4], respectively. In this paper we consider the problems of sufficiency of jets relative to a given closed set. We give several equivalent conditions; they generalise the Kuo condition to the relative case. We also discuss characterisations of $V$-sufficiency and $C^0$-sufficiency in the relative case, corresponding to the above non-relative results.

Applying the results obtained in the relative case, we construct examples of polynomial functions whose relative $r$-jets are $V$-sufficient in $C^r$ functions and $C^{r+1}$ functions but not $C^0$-sufficient in $C^r$ functions and $C^{r+1}$ functions, respectively. In addition, we give characterisations of relative finite $V$-determinacy and also relative finite $C^r$ contact determinacy.

1. Introduction

Sufficiency of jets is one of the most important notions introduced by René Thom for the structural stability theory. The property of sufficiency of $r$-jet is a kind of local stability at degree higher than $r$. Implicit Function Theorem and Morse Lemma may be regarded as results on sufficiency. The notion of sufficiency of jets also has applications to the bifurcation problems in Differential Equation. Hence this notion has been explored by many researchers in the 1970s and 1980s (see C. T. C. Wall [33] for the survey of this field).

Any mapping realisation of a $C^0$-sufficient jet has an isolated singularity, and the zero-set of any mapping realisation of a $V$-sufficient jet also has an isolated singularity. Therefore the above works on sufficiency of jets only deal with the isolated singularity case. On the other hand, the works on characterisations of
sufficiency of jets relative to a given closed set have been also started, e.g. V. Grandjean [8], S. Izumiya and S. Matsuoka [9], L. Kushner and B. Terra Leme [20], V. Thilliez [27], X. Xu [35], P. Migus, T. Rodak and S. Spodzieja [24] and so on. This relative case includes the non-isolated case.

The goal of this paper is to carry on the study of sufficiency of jets of differentiable map-germs \((\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0), n \geq p\), possibly with non-isolated singularities. We consider the following situation; for a given closed set-germ \(\Sigma\) in \((\mathbb{R}^n, 0)\), we define the notion of map jets relative to \(\Sigma\), and using the group of homeomorphisms which fixes \(\Sigma\) pointwise, we define some topological sufficiencies of jets relative to \(\Sigma\).

In this paper we mainly treat the problems of \(V\)-sufficiency and \(C^0\)-sufficiency of relative jets.

Before we describe our main results, we recall the conditions characterising the aforementioned sufficiencies and their related results in the non-relative case, in order to expose the difference between the two cases. For simplicity, we mention them for \(r\)-jets sufficient in \(C^r\) functions.

Let \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)\) be a \(C^r\) function germ. The \(r\)-jet of \(f\) at \(0 \in \mathbb{R}^n\), \(j^r f(0)\), has a unique polynomial representative \(z\) of degree not exceeding \(r\). We do not distinguish the \(r\)-jet \(j^r f(0)\) and the polynomial representative \(z\) here.

**Kuiper-Kuo condition.** There is a strictly positive number \(C\) such that

\[
\| \text{grad } z(x) \| \geq C \| x \|^{r-1}
\]

holds in some neighbourhood of \(0 \in \mathbb{R}^n\).

The Kuiper-Kuo condition is equivalent to the \(C^0\)-sufficiency of \(z\) in \(C^r\) functions (N. Kuiper [14], T.-C. Kuo [15], J. Bochnak and S. Lojasiewicz [3]).

**Kuo condition.** There are strictly positive numbers \(C, \alpha\) and \(\bar{w}\) such that

\[
\| \text{grad } z(x) \| \geq C \| x \|^{r-1}
\]

in \(\mathcal{H}_r^{C^r}(f; \bar{w}) \cap \{\| x \| < \alpha\}\).

Here \(\mathcal{H}_r^{C^r}(f; \bar{w})\) denotes the horn-neighbourhood of \(f^{-1}(0)\) of degree \(r\) and width \(\bar{w}\) (see §2.2 for the definition). The Kuo condition is equivalent to the \(V\)-sufficiency of \(z\) in \(C^r\) functions (T.-C. Kuo [17]).

**Condition \((\tilde{K}_\Sigma)\).** There is a strictly positive number \(C\) such that

\[
\| x \| \| \text{grad } z(x) \| + |f(x)| \geq C \| x \|^r
\]

holds in some neighbourhood of \(0 \in \mathbb{R}^n\).

This condition is the Kuo condition in a different way.

**Thom type inequality.** There are strictly positive numbers \(K\) and \(\beta\) such that

\[
\sum_{i<j} \left| x_i \frac{\partial z}{\partial x_j} - x_j \frac{\partial z}{\partial x_i} \right|^2 + |f(x)|^2 \geq K \| x \|^{2\beta}
\]
for $\|x\| < \beta$.

At almost the same time as Kuiper and Kuo, R. Thom proved in [28] that the Thom type inequality implies the $C^0$-sufficiency of $z$ in $C^r$-functions. Using the curve selection lemma, we can show the equivalence between the Thom type inequality and condition $\langle \tilde{K}_\Sigma \rangle$ ([1]).

Now, we recall the Bochnak-Lojasiewicz inequality ([3]).

**Bochnak-Lojasiewicz inequality.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a $C^\omega$ function germ, and let $0 < \theta < 1$. Then

$$\|x\| \|\text{grad} \ f(x)\| \geq \theta |f(x)|$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

This inequality plays a very important role in proving that the Kuo condition, or in fact condition $\langle \tilde{K}_\Sigma \rangle$ is equivalent to the Kuiper-Kuo condition in the analytic case. It follows that $V$-sufficiency in $C^r$ functions is equivalent to $C^0$-sufficiency in $C^r$ functions, and we can see that Thom had proved an equivalent result to the Kuiper-Kuo theorem.

The Kuiper-Kuo condition, the Kuo condition, condition $\langle \tilde{K}_\Sigma \rangle$ and the Thom type inequality are $r$-compatible in the sense of [1] (see also Definition 3.3). Therefore we can replace $z$ with $f$ in those conditions.

We can consider the conditions for $r$-jets sufficient in $C^{r+1}$ functions, corresponding to the Kuiper-Kuo condition, the Kuo condition and condition $\langle \tilde{K}_\Sigma \rangle$. We call them the second Kuiper-Kuo condition, the second Kuo condition and condition $\langle \tilde{K}^\delta \rangle$, respectively. The first 4 conditions in the $C^r$ function case and the last 3 conditions in the $C^{r+1}$ function case can be generalised to the mapping case.

Now we describe the plan of the rest of the paper. The main results in this paper are the characterisations of $V$-sufficiency and $C^0$-sufficiency of jets relative to a given closed set. Therefore let $\Sigma$ be a germ of a closed set at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$, as above.

In §2 we first introduce the notion of jet relative to $\Sigma$. In the non-relative case, namely in the case where $\Sigma = \{0\}$ any $r$-jet, $r \in \mathbb{N}$, has a unique polynomial representative of degree not exceeding $r$ as mentioned above. But in the relative case some jets do not have even an analytic realisation (cf. Remark 2.1(1)). Therefore, when we consider the problem of sufficiency of relative jets in the general setting, we cannot use the analytic method like the curve selection lemma. Note that the non-relative case is a special case of the analytic setting, namely $\Sigma$ is a subanalytic closed set-germ and any relative $r$-jet has a subanalytic $C^r$ or $C^{r+1}$ realisation. Subsequently we define the notions of $\Sigma$-$C^0$-sufficiency, $\Sigma$-$SV$-sufficiency and $\Sigma$-$V$-sufficiency of relative jets, and give the formulations of the above 7 conditions in the relative case. Then we give various characterisations of the relative Kuo condition and condition $\langle \tilde{K}_\Sigma^\delta \rangle$ in the general setting (Propositions 2.21, 2.22), and of the relative second Kuo condition in the analytic setting (Propositions 2.23, 2.26).
In [1] we proved the equivalence between the Kuo condition and the Thom type inequality. We call the left side $K_m$ of the inequality in the Kuo condition and the left side $T_m$ of the Thom type inequality the Kuo quantity and the Thom quantity, respectively (see Definition 2.3 for $K_m$ and $T_m$). The main result of §3 is the equivalence for analytic map-germs of the Kuo quantity and the Thom quantity (Theorem 3.1). This result includes a generalisation of the previous result to the relative case as a special case.

The criteria for $C^0$-sufficiency of $r$-jets in $C^r$ functions and in $C^{r+1}$ functions are the Kuiper-Kuo condition and the second Kuiper-Kuo condition, respectively, in the non-relative case. J. Bochnak and W. Kucharz proved in [4] the corresponding results in the mapping case to the above ones. The main results of §4 are the generalisations of these results on $\Sigma$-$C^0$-sufficiency of relative jets in $C^r$ mappings and $C^{r+1}$ mappings (Theorems 4.1, 4.3).

One of the main results of §5 is a characterisation of $\Sigma$-$V$-sufficiency of relative $r$-jets in $C^r$ mappings : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n \geq p$, using the relative Kuo condition. We give such a characterisation in the general setting (Theorem 5.2) when $n > p$, and give it in the analytic setting when $n = p$ (Theorem 5.3). Using this result and the criterion of $\Sigma$-$C^0$-sufficiency in $C^r$ mappings in §4, we construct an example of a polynomial function to show that $\Sigma$-$V$-sufficiency in $C^r$ functions does not always imply $\Sigma$-$C^0$-sufficiency in $C^r$ functions (Example 5.6). From this example, we can see that the Bochnak-Lojasiewicz inequality does not always hold in the relative case even if $f$ is a polynomial function and $\Sigma$ is a line (see Remark 5.3 also). Another main result of §5 is a sufficient condition for the relative $r$-jets to be $\Sigma$-$V$-sufficient in $C^{r+1}$ mappings (Theorem 5.10). Using this result and the criterion of $\Sigma$-$C^0$-sufficiency in $C^{r+1}$ mappings in §4, we construct an example of a polynomial function to show that $\Sigma$-$V$-sufficiency in $C^{r+1}$ functions does not always imply $\Sigma$-$C^0$-sufficiency in $C^{r+1}$ functions (Example 5.14).

In [2] J. Bochnak and T.-C. Kuo gave characterisations of finite $V$-determinacy for $C^\infty$ map-germs using $C^r$-rigidity and ellipticity of some ideal. Under the assumption that $\Sigma$ is coherent (see Definition 6.6), we generalise the Bochnak-Kuo theorem to the relative case (Theorems 6.7, 6.8) in §6.

In §7 we deal with the contact equivalence in the relative case, generalising the main result of H. Brodersen [5]. We study the relative contact equivalence, and show the equivalence for $C^\infty$ map-germs of infinite $\Sigma$-contact determinacy and finite $\Sigma$-contact determinacy (Theorem 7.3).

Throughout this paper, let us denote by $\mathbb{N}$ the set of natural numbers in the sense of positive integers.

2. Preliminaries

2.1. Definitions. Let $\mathcal{E}_{[s]}(n, p)$ denote the set of $C^s$ map-germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let $j^r f(0)$ denote the r-jet of $f$ at $0 \in \mathbb{R}^n$ for $f \in \mathcal{E}_{[s]}(n, p)$ ($s \geq r$), and let $J^r(n, p)$ denote the set of r-jets in $\mathcal{E}_{[s]}(n, p)$. 
Throughout this paper, let Σ be a germ of a closed subset of \( \mathbb{R}^n \) at 0 \( \in \mathbb{R}^n \) such that 0 \( \in \Sigma \). Then we denote by \( \mathcal{R}_\Sigma^{\text{fix}} \) the group of germs of homeomorphisms \( \varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) at 0 \( \in \mathbb{R}^n \) which fixes \( \Sigma \), namely \( \varphi(x) = x \) for all \( x \in \Sigma \). Finally we denote by \( d(x, \Sigma) \) the distance from a point \( x \in \mathbb{R}^n \) to the subset \( \Sigma \).

We consider on \( E_{[r]}(n, p) \) the following equivalence relation:

Two map-germs \( f, g \in E_{[r]}(n, p) \) are \( r\Sigma\)-equivalent, denoted by \( f \sim g \), if there exists a neighbourhood \( U \) of 0 in \( \mathbb{R}^n \) such that the \( r \)-jet extensions of \( f \) and \( g \) satisfy \( j^r f(\Sigma \cap U) = j^r g(\Sigma \cap U) \).

We denote by \( j^r f(\Sigma; 0) \) (or simply \( j^r f(\Sigma) \)) the equivalence class of \( f \), and by \( J^{\Sigma}_{[r]}(n, p) \) the quotient set \( E_{[r]}(n, p)/\sim \).

**Remark 2.1.** (1) In the case where \( \Sigma = \{0\} \), an \( r \)-jet \( j^r f(0) \) has a polynomial realisation for any \( f \in E_{[r]}(n, p) \). But this property does not always hold in the relative case. In fact, let \( f : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \) be a \( C^\infty \) function defined by

\[
f(x) := \begin{cases} e^{-\frac{1}{x^2}} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}
\]

Let \( \Sigma = \{ \frac{2}{m\pi} \mid m \in \mathbb{N} \} \cup \{0\} \). Then \( f(\frac{2}{m\pi}) = 0 \) for even \( m \), but \( f(\frac{2}{m\pi}) \neq 0 \) for odd \( m \). Therefore, for any \( r \in \mathbb{N} \), \( j^r f(\Sigma; 0) \) does not have even a subanalytic \( C^r \)-realisation.

(2) Let \( f \in E_{[r]}(n, p) \), and let \( \Sigma \) be a germ of a closed subset of \( \mathbb{R}^n \) at 0 \( \in \mathbb{R}^n \) such that 0 \( \in \Sigma \). Then \( j^r f(\Sigma; 0) \) has a \( C^r \)-realisation \( \tilde{f} \) whose restriction to \( \mathbb{R}^n \setminus \Sigma \) is smooth, namely of class \( C^\infty \) (Theorem 2.2, page 73 in J.-C. Tougeron [29]).

Let us introduce some equivalences for elements of \( J^{\Sigma}_{[r]}(n, p) \).

**Definition 2.2.** (1) We say that \( f, g \in E_{[r]}(n, p) \) are \( \Sigma\)-\( C^0 \)-equivalent, if there is \( \varphi \in \mathcal{R}_\Sigma^{\text{fix}} \) such that \( f = g \circ \varphi \).

(2) We say that \( f, g \in E_{[r]}(n, p) \) are \( \Sigma\)-V-equivalent, if \( f^{-1}(0) \) is homeomorphic to \( g^{-1}(0) \) as germs at 0 \( \in \mathbb{R}^n \) by a homeomorphism which fixes \( f^{-1}(0) \cap \Sigma \).

(3) We say that \( f, g \in E_{[r]}(n, p) \) are \( \Sigma\)-SV-equivalent, if there is a local homeomorphism \( \varphi \in \mathcal{R}_\Sigma^{\text{fix}} \) such that \( \varphi(f^{-1}(0)) = g^{-1}(0) \).

Let \( w \in J^{\Sigma}_{[r]}(n, p) \). We call the relative jet \( w \) \( \Sigma\)-\( C^0 \)-sufficient, \( \Sigma\)-V-sufficient, and \( \Sigma\)-SV-sufficient in \( E_{[r]}(n, p) \) \((s \geq r)\), if any two realisations \( f, g \in E_{[r]}(n, p) \) of \( w \), namely \( j^r f(\Sigma; 0) = j^r g(\Sigma; 0) = w \), are \( \Sigma\)-\( C^0 \)-equivalent, \( \Sigma\)-V-equivalent, and \( \Sigma\)-SV-equivalent, respectively.

**Definition 2.3.** Let \( f \in E_{[r]}(n, p) \) \((n \geq p)\), and let \( m \geq 1 \) be an integer. Let us define two functions of the variable \( x \):

\[
K_m(f, x) := \|x\|^m \sum_{1 \leq i_1 < \cdots < i_p \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{i_1}, \ldots, x_{i_p})} \right)(x) \right|^m + \|f(x)\|^m
\]

\[
T_m(f, x) := \sum_{1 \leq i_1 < \cdots < i_{p+1} \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p, \rho)}{D(x_{i_1}, \ldots, x_{i_{p+1}})} \right)(x) \right|^m + \|f(x)\|^m
\]
where $\rho(x) = ||x||^2$. Note that $T_m(f,x) = ||f(x)||^m$ in the case where $n = p$.

We prepare some notations.

**Definition 2.4.** Let $f, g : U \to \mathbb{R}$ be non-negative functions, where $U \subset \mathbb{R}^N$ is an open neighbourhood of $0 \in \mathbb{R}^N$. If there are real numbers $K > 0$, $\delta > 0$ with $B_\delta(0) \subset U$ such that

$$f(x) \leq Kg(x) \text{ for any } x \in B_\delta(0),$$

where $B_\delta(0)$ is a closed ball in $\mathbb{R}^N$ of radius $\delta$ centred at $0 \in \mathbb{R}^N$, then we write $f \preceq g$ (or $g \succeq f$). If $f \preceq g$ and $f \succeq g$, we write $f \approx g$.

The following lemma is useful in establishing many of the results in this paper.

**Lemma 2.5.** Let $\Sigma$ be a germ at $0 \in \mathbb{R}^n$ of a closed subset, and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^k$ map-germ, $k \geq 1$, such that $j^k f(\Sigma; 0) = \{0\}$. Then

$$\|f(x)\| = o(d(x, \Sigma)^k).$$

If moreover $f$ is of classe $C^{k+1}$, then

$$\|f(x)\| \preceq d(x, \Sigma)^{k+1}.$$

**Proof.** It is a consequence of the Taylor formula for $C^k$ mapping and the assumption on $f$. Let $\delta > 0$ and $y \in B(0, \delta) \cap \Sigma$. By the $k$th order Taylor formula we have

$$f(x) = \sum_{j=0}^{k} \frac{(D^j f)(y)}{j!}((x-y)^{(j)}) + R_{k,y}(x-y)$$

for $x \in B(0, r)$, where

$$R_{k,y}(x-y) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!}((D^k f)(y + t(x-y)) - (D^k f)(y))((x-y)^{(k)}) \, dt$$

satisfies

$$\|R_{k,y}(x-y)\| \leq C_{k,h,y} \|x-y\|^k; \lim_{x-y \to 0} C_{k,x-y,y} = 0$$

with

$$C_{k,x-y,y} = \sup_{t \in [0,1]} \frac{\|(D^k f)(y + t(x-y)) - (D^k f)(y)\|}{k!}.$$ 

The convergence $C_{k,h,y} \to 0$ as $h \to 0$ is uniform for $y$ supported in a compact subset of $B(0, \delta)$. Where $h^{(j)} = (h, \ldots, h) \in (\mathbb{R}^n)^j$ and

$$(D^j f)(y)(h, \ldots, h) = \sum_{(i_1, \ldots, i_j)} h_{i_1} \cdots h_{i_j} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_j}}(\alpha).$$

Now, if $j^k f(\Sigma; 0) = \{0\}$, we have $\|f(x)\| \leq C_{k,h,y} \|x-y\|^k$ and taking the infimum on $y \in \Sigma$, we obtain $\|f(x)\| = o(d(x, \Sigma)^k)$.

Moreover, if $f$ is of classe $C^{k+1}$, taking the $(k+1)$th order Taylor formula, and the infimum on $y \in \Sigma$, we get easily $\|f(x)\| \preceq d(x, \Sigma)^{k+1}$. $\square$
2.2. Relative Kuo condition and relative Thom’s type inequality. We suppose now on the germ $\Sigma$ fixed, and introduce the relative notions to $\Sigma$ of the Kuo condition and the Thom type inequality. We first give the notion of the relative Kuiper-Kuo condition. The original condition was introduced by N. Kuiper [14] and T.-C. Kuo [15] as a sufficient condition of $C^0$-sufficiency of jets in the function case. 

Let $v_1, \ldots, v_p$ be $p$ vectors in $\mathbb{R}^n$ where $n \geq p$. The Kuo distance $\kappa$ is defined by

$$\kappa(v_1, \ldots, v_p) = \min_i \text{distance of } v_i \text{ to } V_i,$$

where $V_i$ is the span of the $v_j$'s, $j \neq i$. In the case where $p = 1$, $\kappa(v) = \|v\|$.

Definition 2.6 (The relative Kuiper-Kuo condition). A map germ $f \in \mathcal{E}[r](n,p)$, $n \geq p$, satisfies the relative Kuiper-Kuo condition $(K-K_{\Sigma})$ if

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

Definition 2.7 (The second relative Kuiper-Kuo condition). A map germ $f \in \mathcal{E}[r](n,p)$, $n \geq p$, satisfies the second relative Kuiper-Kuo condition $(K-K_{\delta\Sigma})$ if there is a strictly positive number $\delta$ such that

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

For a map germ $f \in \mathcal{E}[r](n,p)$, we denote by Sing$(f)$ the singular points set of $f$.

Remark 2.8. For a map $f \in \mathcal{E}[r](n,p)$ satisfying the relative Kuiper-Kuo condition or the second relative Kuiper-Kuo condition, we have Sing$(f) \subset \Sigma$ in a neighbourhood of $0 \in \mathbb{R}^n$. Therefore these conditions include the case where $\Sigma = \text{Sing}(f)$, as a special case.

We next give the notion of the relative Kuo condition. The original condition was introduced by T.-C. Kuo [17] as a criterion of $V$-sufficiency of jets in the mapping case.

Definition 2.9 (The relative Kuo condition). A map germ $f \in \mathcal{E}[r](n,p)$, $n \geq p$, satisfies the relative Kuo condition $(K_{\Sigma})$ if there are strictly positive numbers $C, \alpha$ and $\bar{w}$ such that

$$\kappa(df(x)) \gtrsim C d(x, \Sigma)^{r-1}$$

in $H_{\Sigma}(f; \bar{w}) \cap \{\|x\| < \alpha\}$, namely

$$\kappa(df(.)) \gtrsim d(., \Sigma)^{r-1}$$

on a set of points where $\|f\| \gtrsim d(., \Sigma)^r$.

In the definition 2.9, $H_{\Sigma}(f; \bar{w})$ denotes the horn-neighbourhood of $f^{-1}(0)$ relative to $\Sigma$ of degree $r$ and width $\bar{w}$,

$$H_{\Sigma}^v(f; \bar{w}) = \{x \in \mathbb{R}^n : \|f(x)\| \leq \bar{w} d(x, \Sigma)^r\}.$$

The original notion of this horn-neighbourhood was introduced in [16].

We have also a variant of the previous condition:
Definition 2.10 (The second relative Kuo condition). A map germ $f \in \mathcal{E}_{[r+1]}(n, p)$, $n \geq p$, satisfies the second relative Kuo condition ($K_{2}^{\Sigma}$) if for any map $g \in \mathcal{E}_{[r+1]}(n, p)$ satisfying $j^{*}g(\Sigma; 0) = j^{*}f(\Sigma; 0)$ there are numbers $C, \alpha, \delta$ and $\bar{w}$ (depending on $g$), such that
\[ \kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta} \]
in $\mathcal{H}_{r+1}^{\Sigma}(g; \bar{w}) \cap \{\|x\| < \alpha\}$, namely
\[ \kappa(df(.)) \gtrsim d(., \Sigma)^{r-\delta} \]
on a set of points where $\|g(.)\| \gtrsim d(., \Sigma)^{r+1}$.

Remark 2.11. 1) For a map $f \in \mathcal{E}_{[r]}(n, p)$ satisfying the relative Kuo condition or the second relative Kuo condition, in a neighbourhood of $0 \in \mathbb{R}^{n}$, the intersection of the singular set of $f$, $\text{Sing}(f)$, and the horn neighbourhood $\mathcal{H}_{r+1}^{\Sigma}(f; \bar{w})$ is contained in $\Sigma$, namely
\[ \text{Sing}(f) \cap \mathcal{H}_{r+1}^{\Sigma}(f; \bar{w}) \subset \Sigma. \]
In particular, in a neighbourhood of $0 \in \mathbb{R}^{n}$, $\text{grad}f_{1}(x), \ldots, \text{grad}f_{p}(x)$ are linearly independent on $f^{-1}(0) \setminus \Sigma$.

2) For a map $f \in \mathcal{E}_{[r]}(n, p)$ satisfying the second relative Kuo, we have for any map $g \in \mathcal{E}_{[r+1]}(n, p)$ satisfying $j^{*}g(\Sigma; 0) = j^{*}f(\Sigma; 0)$, in a neighbourhood of $0 \in \mathbb{R}^{n}$, the intersection of the singular set of $f$, $\text{Sing}(f)$, and the horn neighbourhood $\mathcal{H}_{r+1}^{\Sigma}(g; \bar{w})$ is contained in $\Sigma$, namely
\[ \text{Sing}(f) \cap \mathcal{H}_{r+1}^{\Sigma}(g; \bar{w}) \subset \Sigma. \]
Since $\|(f - g)(x)\| \gtrsim d(x, \Sigma)^{r+1}$, we have $f^{-1}(0) \subset \mathcal{H}_{r+1}^{\Sigma}(g; \bar{w})$, then, in a neighbourhood of $0 \in \mathbb{R}^{n}$, $\text{grad}f_{1}(x), \ldots, \text{grad}f_{p}(x)$ are linearly independent on $f^{-1}(0) \setminus \Sigma$.

Definition 2.12 (Condition ($\bar{K}_{\Sigma}$)). A map germ $f \in \mathcal{E}_{[r]}(n, p)$, $n \geq p$, satisfies condition ($\bar{K}_{\Sigma}$) if
\[ d(x, \Sigma)\kappa(df(x)) + \|f(x)\| \gtrsim d(x, \Sigma)^{r} \]
holds in some neighbourhood of $0 \in \mathbb{R}^{n}$.

Remark 2.13. 1) Condition ($\bar{K}_{\Sigma}$) was introduced in [1], in the case $\Sigma = \{0\}$, in the proof of the equivalence between V-sufficiency and SV-sufficiency.

2) It is easy to see that condition ($\bar{K}_{\Sigma}$) and the relative Kuo condition ($K_{\Sigma}$) are equivalent.

3) The relative Kuiper-Kuo condition ($\bar{K}$-$K_{\Sigma}$), the relative Kuo condition ($K_{\Sigma}$), and condition ($\bar{K}_{\Sigma}$) are invariant under rotation.

Definition 2.14 (Condition ($\tilde{K}_{\Sigma}^{\delta}$)). A map germ $f \in \mathcal{E}_{[r+1]}(n, p)$, $n \geq p$, satisfies condition ($\tilde{K}_{\Sigma}^{\delta}$) if for any map $g \in \mathcal{E}_{[r+1]}(n, p)$ satisfying $j^{*}g(\Sigma; 0) = j^{*}f(\Sigma; 0)$ there exists $\delta > 0$ (depending on $g$), such that
\[ d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta} \]
holds in some neighbourhood of $0 \in \mathbb{R}^{n}$.
Remark 2.15. (1) The second relative Kuiper-Kuo condition \((K_2^2)\), the second relative Kuo condition \((K_5^2)\), and condition \((K_7^2)\) are invariant under rotation.

(2) This condition can be equivalently written as: for any map \(g \in E_{r+1}(n, p)\) satisfying \(\delta > 0\) (depending on \(g\)), such that
\[
d(x, \Sigma)\kappa(dg(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}
\]
holds in some neighbourhood of 0 \(\in \mathbb{R}^n\).

Definition 2.16 (The relative Thom type inequality). A map germ \(f \in E_{r}(n, p)\) \((n \geq p)\), satisfies a relative Thom type inequality \((T_S)\) if there are strictly positive numbers \(K, \beta\) and \(a\) such that
\[
T_2(f, x) \geq Kd(x, \Sigma)^a \text{ for } \|x\| < \beta,
\]
namely
\[
T_2(f, .) \gtrsim d(, \Sigma)^a.
\]

2.3. Equivalent conditions to the relative Kuo conditions. Let \(\mathcal{L}(E, F)\) denote the space of linear mappings from \(\mathbb{R}^n\) to \(\mathbb{R}^p\). For \(T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\), we denote by \(T^{\ast}\) the adjoint map in \(\mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)\).

Definition 2.17. Let \(T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\). Set
\[
\nu(T) = \inf\{|T^\ast(v)| : v \in \mathbb{R}^p, \|v\| = 1\},
\]
where \(T^\ast\) is the dual operator. This function is called the Rabier’s function \([25]\).

We have the following facts on the Rabier’s function \(\nu(T)\) (see \([18, 25]\) and \([10]\) for instance):

(1) Let \(S\) be the set of non surjective linear map in \(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\). Then for all \(T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\), we have:

(a) \(\nu(T) = d(T, S) := \inf_{T' \in S} \|T - T'\|\).

(b) \(\nu(T) = \sup\{r > 0 : B(0, r) \subseteq T(B(0, r))\}\).

(c) If \(T \in GL_n(\mathbb{R})\), then \(\nu(T) = \frac{1}{\|T^{-1}\|}\).

(2) For \(T, T' \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\) we have \(\nu(T + T') \geq \nu(T) - \|T'\|\).

(3) The relationship \(\nu \approx \kappa\) holds between the Rabier’s function and the Kuo distance. More precisely, for \(T = (T_1, \ldots, T_p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\), we have
\[
\nu(T) \leq \kappa(T) \leq \sqrt{p} \nu(T).
\]

Definition 2.18. Let \(A = [a_{ij}]\) be the matrix in \(\mathcal{M}_{n,p}(\mathbb{R})\), \(n \geq p\). By \(M_I(A)\), we denote a \(p \times p\) minor of \(A\) indexed by \(I\), where \(I = (i_1, \ldots, i_p)\) is any subsequence of \((1, \ldots, n)\). Moreover, if \(J = (j_1, \ldots, j_{p-1})\) is any subsequence of \((1, \ldots, n)\) and \(j \in \{1, \ldots, p\}\), then by \(M_J(j)(A)\) we denote an \((p-1) \times (p-1)\) minor of a
matrix given by columns indexed by $J$ and with deleted $j$-th row (if $p = 1$ we put $M_J(j)(A) = 1$). We define $\eta : M_{n,p}(\mathbb{R}) \to \mathbb{R}_+$ by

$$\eta(A) := \left( \frac{\sum_I |M_I(A)|^2}{\sum_{J,j} |M_J(j)(A)|^2} \right)^{\frac{1}{2}}.$$

It is easy to see that the above $\eta$ is equivalent to the following non-negative function $\tilde{\eta}$ in the sense that $\eta \approx \tilde{\eta}$:

$$\tilde{\eta}(A) := \max_I \frac{|M_I(A)|}{h_I(A)},$$

where $h_I(A) = \max\{|M_J(j)(A)| : J \subset I, j = 1, \ldots, p\}$, with the convention that $0_0 = 0$.

**Lemma 2.19.** The relationship $\eta \approx \kappa$ holds. More precisely, for $T = (T_1, \ldots, T_p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, we have

$$\eta(T) \leq \kappa(T) \leq \sqrt{p} \eta(T).$$

**Remark 2.20.** The functions $\nu, \kappa, \eta$ and $\tilde{\eta}$ are continuous. In order to see these facts, it suffices to show that $\eta$ is continuous at $A \in M_{p,p}(\mathbb{R})$. It is obvious if the denominator is bigger than 0. Let us assume that the denominator is equal to 0. Then $\eta(A) = 0$ and thus $A \in \mathcal{S}$. Let $\{A_k\}$ be a sequence of elements of $M_{p,p}(\mathbb{R})$ which tend to $A$. Then there exists $C > 0$ such that as $k \to \infty$, we have

$$\eta(A_k) \leq C\nu(A_k) = C d(A_k, \mathcal{S}) \to 0.$$

We have the following characterisations of the relative Kuo condition.

**Proposition 2.21.** Let $\Sigma$ be a (non empty) germ at 0 of a closed subset of $\mathbb{R}^n$. For $f \in \mathcal{E}[r](n, p)$, $n \geq p$, the following conditions are equivalent:

1. $f$ satisfies condition $(K_\Sigma)$.
2. $f$ satisfies condition $(\widetilde{K}_\Sigma)$
3. The inequality

$$d(x, \Sigma) \left( \frac{\Gamma((\text{grad } f_i(x))_{1 \leq i \leq p})}{\sum_{j=1}^p \Gamma((\text{grad } f_i(x))_{i \neq j})} \right)^{\frac{1}{2}} + \|f(x)\| \gtrsim d(x, \Sigma)^r$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$, where

$$\Gamma(v_1, \ldots, v_k) := \det(<v_i, v_j>_{i,j \in \{1, \ldots, k\}})$$

is the Gram determinant.
4. The inequality

$$d(x, \Sigma)\|df^*(x)y\| + \|f(x)\| \gtrsim d(x, \Sigma)^r$$

holds for $x$ in some neighbourhood of $0 \in \mathbb{R}^n$, uniformly for all $y \in \mathbb{S}^{p-1}$, where $df^*(x)$ is the dual map of $df(x)$, and $\mathbb{S}^{p-1}$ denotes the unit sphere in $\mathbb{R}^p$ centred at $0 \in \mathbb{R}^p$. 

Proof. Concerning the equivalence (1) \(\iff\) (2), see Remark 2.13.

The equivalence (2) \(\iff\) (3) follows from Lemma 2.19 and

\[ \Delta(f, x) := \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left( \det \frac{D(f_{i_1}, \ldots, f_{i_p})}{D(x_{i_1}, \ldots, x_{i_p})} \right)^2 = \Gamma((\text{grad } f_i(x))_{1 \leq i \leq p}). \]

Lastly we show the equivalence (2) \(\iff\) (4). For a given set of vectors \(v_1, v_2, \ldots, v_p \in \mathbb{R}^n\), let \(\tilde{\kappa}(v_1, v_2, \ldots, v_p)\) be defined as follows:

\[ \tilde{\kappa}(v_1, v_2, \ldots, v_p) = \min \left\{ \sum_{i=1}^p \lambda_i v_i : \lambda_i \in \mathbb{R}, \sum_{i=1}^p \lambda_i^2 = 1 \right\}, \]

Then the equivalence between conditions (2) and (4) follows from the equivalence between \(\kappa\) and \(\tilde{\kappa}\) (see Lemma 2.19) and the fact that

\[ \|d f^* (x) y\| = \| \sum_{i=1}^p y_i \text{grad } f_i(x) \| \gtrsim d(x, \Sigma)^{r-1} \text{ for all } y \in S^{p-1} \]

is equivalent to

\[ \tilde{\kappa}(\text{grad } f_1(x), \ldots, \text{grad } f_p(x)) \gtrsim d(x, \Sigma)^{r-1}. \]

\(\square\)

We have also the following characterisations of the \((\tilde{K}_\Sigma^p)\) condition.

**Proposition 2.22.** Let \(\Sigma\) be a (non empty) germ at 0 of a closed subset of \(\mathbb{R}^n\). For a map \(f \in \mathcal{E}_{[r+1]}(n, p)\), \(n \geq p\), the following conditions are equivalent:

1. \(f\) satisfies condition \((\tilde{K}_\Sigma^p)\),
2. For any map \(g \in \mathcal{E}_{[r+1]}(n, p)\) satisfying \(j^r g(\Sigma; 0) = j^r f(\Sigma; 0)\), there exists \(\delta > 0\) (depending on \(g\)), such that the inequality

\[ d(x, \Sigma) \left( \frac{\Gamma((\text{grad } g_i(x))_{1 \leq i \leq p})}{\sum_{j=1}^p \Gamma((\text{grad } g_j(x))_{i \neq j})} \right)^{\frac{1}{2}} + \| g(x) \| \gtrsim d(x, \Sigma)^{r+1-\delta} \]

holds in some neighbourhood of \(0 \in \mathbb{R}^n\).
3. For any map \(g \in \mathcal{E}_{[r+1]}(n, p)\) satisfying \(j^r g(\Sigma; 0) = j^r f(\Sigma; 0)\), there exists \(\delta > 0\) (depending on \(g\)), such that the inequality

\[ d(x, \Sigma)\|dg^* (x) y\| + \| g(x) \| \gtrsim d(x, \Sigma)^{r+1-\delta} \]

holds for some \(\delta > 0\), for \(x\) in some neighbourhood of \(0 \in \mathbb{R}^n\) and uniformly for all \(y \in S^{p-1}\).
4. For any map \(g \in \mathcal{E}_{[r+1]}(n, p)\) satisfying \(j^r g(\Sigma; 0) = j^r f(\Sigma; 0)\), there exists \(\delta > 0\) (depending on \(g\)), such that the inequality

\[ d(x, \Sigma)\|df^* (x) y\| + \| g(x) \| \gtrsim d(x, \Sigma)^{r+1-\delta} \]

in a neighbourhood of the origin, (uniformly) for all \(y \in S^{p-1}\).
Proof. The proofs of the equivalences between (1) (2), and (3) are identical to the corresponding ones in Proposition 2.21.

The equivalence between (3) and (4) is an application of Lemma 2.5. □

In the next proposition, we establish the equivalence between the second relative Kuo condition ($K_\Sigma^g$) and condition ($K_\Sigma^f$) for subanalytic maps and $\Sigma$.

**Proposition 2.23.** Let $\Sigma$ be a germ of a closed subanalytic set at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. For a subanalytic map $f \in \mathcal{E}_{[r+1]}(n, p)$, $n \geq p$, the following conditions are equivalent:

1. $f$ satisfies the second relative Kuo condition ($K_\Sigma^f$) : for any subanalytic map $g \in \mathcal{E}_{[r+1]}(n, p)$ with $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$, there are numbers $C, \alpha, \delta$ and $\bar{w}$ (depending on $g$) such that

\[
\kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta} \quad \text{in } \mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \cap \{\|x\| < \alpha\}.
\]

2. For any subanalytic map $g \in \mathcal{E}_{[r+1]}(n, p)$ with $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$,

\[
\frac{d(x, \Sigma)\kappa(df(x)) + \|g(x)\|}{d(x, \Sigma)^{r+1}} \to \infty \quad \text{as } x \to \Sigma.
\]

3. $f$ satisfies condition ($K_\Sigma^g$) : for any subanalytic map $g \in \mathcal{E}_{[r+1]}(n, p)$ with $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$, there exists $\delta > 0$ (depending on $g$) such that

\[
d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \gtrsim d(x, \Sigma)^{r+1-\delta}
\]

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

**Proof.** We first show the implication (1) $\implies$ (2). If condition (2) doesn’t hold, then there exist a realisation $g$ of $j^r f(\Sigma; 0)$, and an analytic arc $\gamma : I \to \mathbb{R}^n$ where $I = [0, \beta)$, $\beta > 0$, such that $\gamma(0) = 0 \in \mathbb{R}^n$ and for $t \in I$

\[
\|g(\gamma(t))\| \lesssim d(\gamma(t), \Sigma)^{r+1}, \quad \kappa(df(\gamma(t))) \lesssim d(\gamma(t), \Sigma)^r.
\]

In particular, for any sequence $\{x_i\}$ where $x_i = \gamma(t_i)$, $t_i \to 0$, $t_i \neq 0$, and for any $\delta > 0$ we have

\[
\|g(x_i)\| = o(d(x_i, \Sigma)^{r+1-\delta}), \quad \kappa(df(x_i)) = o(d(x_i, \Sigma)^{r-\delta}).
\]

Then (2.8) implies that for any choice of the positive numbers $C, \alpha, \delta$ and $\bar{w}$, the inequality

\[
\kappa(df(x)) \geq Cd(x, \Sigma)^{r-\delta}
\]

cannot hold in $\mathcal{H}_{r+1}^\Sigma(g; \bar{w}) \cap \{\|x\| < \alpha\}$, namely condition ($K_\Sigma^g$) is not satisfied. Therefore the implication (1) $\implies$ (2) is shown.

We next show the implication (2) $\implies$ (3). By condition (2), the continuous subanalytic function germ on $(\mathbb{R}^n \setminus \Sigma, 0)$, defined by

\[
h(x) := \frac{d(x, \Sigma)^{r+1}}{d(x, \Sigma)\kappa(df(x)) + \|g(x)\|},
\]
can be extended continuously by 0 on Σ. Since Σ = \( h^{-1}(0) \), by the Lojasiewicz inequality (\([21] \S 18\)), there is some \( \delta > 0 \) such that
\[
0 \leq h(x) \lesssim d(x, \Sigma)^\delta
\]
in some neighbourhood of 0 ∈ \( \mathbb{R}^n \). Thus \( (\tilde{K}_\Sigma^\delta) \) is satisfied.

We lastly show the implication (3) \( \implies \) (1). Suppose that \( f \) satisfies condition \( (\tilde{K}_\Sigma^\delta) \). Let \( g \in \mathcal{E}_{[r+1]}(n,p) \) with \( j^rg(\Sigma; 0) = j^rf(\Sigma; 0) \) which satisfies the condition that there are positive constants \( \delta, C \) and \( \alpha \) such that
\[
(2.9) \quad d(x, \Sigma)\kappa(df(x)) + \|g(x)\| \geq Cd(x, \Sigma)^{r+1-\delta}
\]
for \( x \in \mathbb{R}^n, \|x\| < \alpha \). If \( x \) is in the horn-neighbourhood \( \mathcal{H}_{r+1}(g; C_2^2) \cap \{\|x\| < \alpha\} \), then
\[
\|g(x)\| \leq C_2^2d(x, \Sigma)^{r+1} \leq C_2^2d(x, \Sigma)^{r+1-\delta}
\]
and by \((2.9)\)
\[
\kappa(df(x)) \geq C_2^2d(x, \Sigma)^{r-\delta}.
\]
Therefore condition \( (K\Sigma^\delta) \) is satisfied; which shows the implication (3) \( \implies \) (1).

Remark 2.24. From the proof, we can see that without the assumption of subanalyticity, the implication (3) \( \implies \) (1) in Proposition 2.23 holds, namely condition \((\tilde{K}_\Sigma^\delta)\) implies the second relative Kuo condition \((K\Sigma^\delta)\).

Let us now introduce a (ostensibly) weaker condition in terms of \( f \) only (namely, not using all the realisations of the jet \( j^r f(\Sigma; 0) \)), to be compared to \([12]\).

Definition 2.25. A map germ \( f \in \mathcal{E}_{[r+1]}(n,p), n \geq p, \) satisfies condition \((K\Sigma)\) if
\[
(2.10) \quad \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} \to \infty \quad \text{as } x \to \Sigma
\]
in a neighbourhood of the origin, (uniformly) for all \( y \in \mathbb{S}^{p-1} \).

We have another characterisation of the second relative Kuo condition.

Proposition 2.26. Let \( \Sigma \) be a germ at 0 of closed subanalytic subset of \( \mathbb{R}^n \), and let \( f \in \mathcal{E}_{[r+1]}(n,p), n \geq p, \) be a subanalytic map. Then \( f \) satisfies the second relative Kuo condition, equivalently for any subanalytic map \( g \in \mathcal{E}_{[r+1]}(n,p) \) such that \( j^rg(\Sigma; 0) = j^rf(\Sigma; 0) \), there are positive constants \( \delta, C \) and \( \alpha \) such that
\[
(2.11) \quad d(x, \Sigma)\kappa(dg(x)) + \|g(x)\| \geq Cd(x, \Sigma)^{r+1-\delta},
\]
for \( \|x\| < \alpha \) if and only if \( f \) satisfies condition \((K\Sigma)\).

Proof. By Proposition 2.22 condition \((2.11)\) is equivalent to condition (4), which implies \((K\Sigma)\) for any \( g \in \mathcal{E}_{[r+1]}(n,p) \) such that \( j^rg(\Sigma; 0) = j^rf(\Sigma; 0) \), in particular, for \( f \).

To prove the converse, we will use the subanalyticity. Suppose that condition \((K\Sigma)\) is satisfied. Let \( g \in \mathcal{E}_{[r+1]}(n,p) \) such that \( j^rg(\Sigma; 0) = j^rf(\Sigma; 0) \), and set
$h(x) := g(x) - f(x)$. By Lemma 2.5, \( \|h(x)\| \lesssim d(x, \Sigma)^r \) for sufficiently small values of \( \|x\| \). Then, for all \( y \in \mathbb{S}^{p-1} \)

\[
d(x, \Sigma)\|df^*(x)y\| + \|g(x)\| = d(x, \Sigma)\|df^*(x)y\| + \|f(x) + h(x)\|
\]

and

\[
\frac{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|}{d(x, \Sigma)^{r+1}} \geq \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} - \frac{\|h(x)\|}{d(x, \Sigma)^{r+1}}.
\]

Since \( \frac{\|h(x)\|}{d(x, \Sigma)^{r+1}} \) is bounded,

\[
\lim_{d(x, \Sigma) \to 0} \frac{d(x, \Sigma)\|df^*(x)y\| + \|f(x)\|}{d(x, \Sigma)^{r+1}} = \infty
\]

which implies

\[
\lim_{d(x, \Sigma) \to 0} \frac{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|}{d(x, \Sigma)^{r+1}} = \infty.
\]

Therefore condition (2.3) is satisfied.

Now the continuous subanalytic function germ on \((\mathbb{R}^n \setminus \Sigma, 0) \times \mathbb{S}^{p-1}\), defined by

\[
q(x, y) := \frac{d(x, \Sigma)^{r+1}}{d(x, \Sigma)\|df^*(x)y\| + \|g(x)\|},
\]

can be extended continuously by 0 to \( \Sigma \times \mathbb{S}^{p-1} \). Since \( \Sigma \times \mathbb{S}^{p-1} = h^{-1}(0) \), by the Lojasiewicz inequality, there is some \( \delta > 0 \) such that

\[
|q(x, y)| \lesssim d((x, y), \Sigma \times \mathbb{S}^{p-1}) \delta = d(x, \Sigma)^\delta
\]

in some neighbourhood of \( 0 \in \mathbb{R}^n \). This is exactly condition (4) in Proposition 2.22. \(\square\)

2.4. Integrability. The standard condition for proving local integrability of vector fields is the Lipschitz condition. We shall use a more general controllability condition which yields the Lipschitz equivalence or even \(C^k\)-equivalence as a special case (see for instance T.-C. Kuo [15, 17], N. Kuiper [14], F. Takens [26], for the isolated singular case and E. Looijenga [22], J. Damon [6] for family of isolated singularities).

Let \( \Sigma \) be a germ of closed set of \( \mathbb{R}^n \) such that \( 0 \in \Sigma \). Let \( G \) be a germ of \( C^1 \) vector field on \( \mathbb{R}^n \times \mathbb{R}^m \setminus \Sigma \times \mathbb{R}^m \) which satisfies the relative Lipschitz condition:

\[
\|G(x, t)\| \leq C d(x, \Sigma)
\]

where \( d(x, \Sigma) \) denotes the distance of the point \( x \) to \( \Sigma \).

For a fixed vector \( v \in \{0\} \times \mathbb{R}^m \), we define

\[
X(x, t) := \begin{cases} G(x, t) + v & \text{if } x \notin \Sigma \\ v & \text{if } x \in \Sigma. \end{cases}
\]

Then we have the following proposition.
Proposition 2.27. For $G(x, t)$ satisfying the preceding conditions, $X(x, t)$ is locally integrable in the sense that there are a neighbourhood $W$ of $(x_0, t_0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, $\delta > 0$, and a family of homeomorphisms $\phi_s(x, t)$ defined on $W$ for $|s| < \delta$ so that $\phi_0 = id$ and for $(x, t, s) \in W \times (-\delta, \delta)$,

$$\frac{\partial \phi_s}{\partial s} = X \circ \phi_s.$$ 

Lemma 2.28. Let $U$ be an open subset of $\mathbb{R}^n \setminus \Sigma$, let $0 \in (a, b)$ and let $G : U \times (a, b) \to \mathbb{R}^n$ be a continuous mapping which satisfies

$$\|G(x, t)\| \leq Cd(x, \Sigma)$$

for some $C > 0$ and $(x, t) \in U \times (a, b)$. Let $\varphi : (\alpha, \beta) \to U$ be an integral solution of the system of differential equations $y' = G(y, t)$ with the initial condition $\varphi(0) = x_0$ where $x_0 \in U$ and $0 \in (\alpha, \beta) \subset (a, b)$. Then we have

$$d(x_0, \Sigma)e^{-C|t|} \leq d(\varphi(t), \Sigma) \leq d(x_0, \Sigma)e^{C|t|}$$

for $t \in (\alpha, \beta)$.

Proof. Since $\|\varphi(t)\| > 0$ for $t \in (\alpha, \beta)$, we can define the function $\rho : (\alpha, \beta) \to \mathbb{R}$ by $\rho(t) = \frac{1}{2} \ln \|\varphi(t)\|^2$ for $t \in (\alpha, \beta)$. This function is differentiable and

$$\rho'(t) = \frac{\langle \varphi(t), \varphi'(t) \rangle}{\|\varphi(t)\|^2} = \frac{\langle \varphi(t), G(\varphi(t), t) \rangle}{\|\varphi(t)\|^2}$$

for $t \in (\alpha, \beta)$. From the mean value theorem, for every $t \in (0, \beta)$ there exists $\theta \in (0, t)$ such that $\rho(t) - \rho(0) = \rho'(\theta)t$. Then we have

$$|\rho(t) - \rho(0)| \leq |\rho'(\theta)|t \leq \frac{\|\varphi(0)\||G(\varphi(0), \theta)\|}{\|\varphi(\theta)\|^2}t \leq Ct.$$ 

Therefore for every $t \in (0, \beta)$,

$$\rho(0) - Ct \leq \rho(t) \leq \rho(0) + Ct.$$ 

The above inequalities hold also for $t = 0$. This ends the proof of the lemma. 

Proof of Proposition 2.27. Let $\gamma(s)$, $|s| < \epsilon$, be an integral curve of $X$ in $(\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^m$. Then, by Lemma 2.28 $\gamma(s)$ stays within a compact subset of $(U \setminus \Sigma) \times \mathbb{R}^m$ when $\gamma(0)$ does. Thus, together with $\gamma(s) = \gamma(0) + sv$ for $\gamma(0) \in \Sigma \times \mathbb{R}^m$, we obtain a continuous flow $\phi_s$, $|s| < \epsilon$ for $\gamma(0)$ in a sufficiently small compact neighbourhood of $x_0$. Thus there is a compact neighbourhood $W$ of $(x_0, t_0)$ in $\mathbb{R}^{n+m}$ and a positive number $\delta$ so that the integral curves $\gamma(s)$ of $X$ with $\gamma(0) \in W \setminus (\Sigma \times \mathbb{R}^m)$ are defined for $|s| \leq \delta$ and belong to $W \setminus (\Sigma \times \mathbb{R}^m)$. Then, we define $\phi(x, t, s) : W \times [-\delta, \delta] \to W$ by $\phi(x, t, s) = \gamma(s)$ where $\gamma$ is the integral curve of $X$ with $\gamma(0) = (x, t)$ (if $(x, t) \in \Sigma \times \mathbb{R}^m$, then $\gamma(s) = (x, t) + sv$). This flow has a continuous inverse (in a smaller neighbourhood) by the same argument applied to $-X$ and uniqueness. Thus, it is a parametrised family of local homeomorphisms. 

□
2.5. Bochnak-Kuo Lemma. J. Bochnak and T.-C. Kuo proved a lemma in [2] in order to show a characterisation of finite $V$-determinacy of map-germs. Using a similar argument to the lemma, we can show the following lemma which will be used to show characterisations of relative $C^0$-sufficiency of jets and relative $V$-sufficiency of jets.

**Lemma 2.29.** Let $\{u_\nu^{(1)}, \ldots, u_\nu^{(p)}\}_{\nu \in \mathbb{N}}$ be a sequence of $p$-tuples of vectors in $\mathbb{R}^n$, and let $s \in \mathbb{N}$. Suppose that there is a sequence of positive numbers $\alpha_\nu, \alpha_\nu \to 0$ such that

$$d \left( u_\nu^{(1)}, \sum_{k=2}^p \mathbb{R} u_\nu^{(k)} \right) = o(\alpha_\nu^s).$$

Then we can find a sequence $\{\lambda_\nu^{(2)}, \ldots, \lambda_\nu^{(p)}\}_{\nu \in \mathbb{N}}$ of $(p-1)$-tuples of vectors in $\mathbb{R}^n$, satisfying the following three conditions:

(i) $|\lambda_\nu^{(k)}| = o(\alpha_\nu^s), \quad 2 \leq k \leq p$;

(ii) For each $\nu \in \mathbb{N}$, $u_\nu^{(2)} + \lambda_\nu^{(2)}, \ldots, u_\nu^{(p)} + \lambda_\nu^{(p)}$ are linearly independent;

(iii) For each $\nu \in \mathbb{N}$, $u_\nu^{(1)}$ belongs to the subspace spanned by $u_\nu^{(k)} + \lambda_\nu^{(k)}, \quad 2 \leq k \leq p$.

**Remark 2.30.** In the case of the original Bochnak-Kuo Lemma, we suppose that

$$d \left( u_\nu^{(1)}, \sum_{k=2}^p \mathbb{R} u_\nu^{(k)} \right) = o(\alpha_\nu^s)$$

for all $s \in \mathbb{N}$ not a given $s \in \mathbb{N}$. Then statement (i) holds for all $s \in \mathbb{N}$.

The original Bochnak-Kuo Lemma will be used to show a characterisation of finite $SV$-determinacy of map-germs in the relative case.

3. Equivalence of relative Kuo condition and Thom type inequality

In this section we show the equivalence between the Kuo quantity $K_m$ and the Thom quantity $T_m$.

Let $\text{ord}(\gamma(t))$ denotes the order of $\gamma$ in $t$ for a $C^\omega$ function $\gamma : [0, \delta) \to \mathbb{R}$.

**Theorem 3.1.** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0), \ n \geq p$, be a $C^\omega$ map-germ. Then for any $m \in \mathbb{N}$,

$$K_m(f,.) \approx T_m(f,.)$$

**Proof.** It is obvious that $K_m(f,.) \gtrsim T_m(f,.)$. Therefore we have to show the converse.

We first remark that if $x$ and $y$ are bigger than or equal to 0, we have

$$(x + y)^m \geq x^m + y^m \geq \frac{(x + y)^m}{2^m}.$$ 

It follows that

$$K_m(f, x) \approx v^m(x) + u^m(x) \approx (h(x))^m$$

$$T_m(f, x) \approx w^m(x) + u^m(x) \approx (g(x))^m,$$

where $u(x) = \|f(x)\|$. 
It follows from (3.3) that
\[ h(x) = v(x) + u(x) \] and \[ g(x) = w(x) + u(x). \]

Suppose now that \( K_m(f, .) \not\subset T_m(f, .) \) does not hold. Then by the curve selection lemma, there is a \( C^{\omega} \) curve \( \hat{\lambda} = (\lambda, C) : [0, \delta) \to \mathbb{R}^n \times \mathbb{R} \) with \( \hat{\lambda}(0) = (0, 0) \) and \( \hat{\lambda}(t) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^*, \) for \( t \neq 0, \) such that
\[
(C(t))^m K_m(f, \lambda(t)) > T_m(f, \lambda(t)).
\]
We may write (3.1) as:
\[
(C(t)(h \circ \lambda(t)))^m > (g \circ \lambda(t))^m.
\]
Here we remark that the functions \( g \circ \lambda, h \circ \lambda, u \circ \lambda, v \circ \lambda \) and \( w \circ \lambda \) are real analytic on \([0, \delta)\) and satisfying the conditions
\[
g \circ \lambda(0) = h \circ \lambda(0) = u \circ \lambda(0) = v \circ \lambda(0) = w \circ \lambda(0) = 0
\]
and
\[
\lambda(t) \neq 0, \quad C(t) > 0, \quad h \circ \lambda(t) > 0, \quad g \circ \lambda(t) \geq 0 \quad \text{for} \quad 0 < t < \delta
\]
By (3.2), \( C(t)(h \circ \lambda(t)) > u \circ \lambda(t), \) \( C(t)(h \circ \lambda(t)) > w \circ \lambda(t) \) and \( v \circ \lambda(t) = h \circ \lambda(t) - u \circ \lambda(t) \geq h \circ \lambda(t)(1 - C(t)). \) Then we have
\[
\left\{
\begin{array}{l}
\text{ord}(C) + \text{ord}(h \circ \lambda) \leq \text{ord}(u \circ \lambda) \\
\text{ord}(C) + \text{ord}(h \circ \lambda) \leq \text{ord}(w \circ \lambda) \\
\text{ord}(v \circ \lambda) \leq \text{ord}(h \circ \lambda)
\end{array}
\right.
(3.3)
\]
Note that we are not considering the second inequality in the case where \( n = p. \)

Let \( \hat{\lambda} \) be written as follows \( \lambda_i(t) = a_1^{(i)} t^{\varepsilon_1(i)} + a_2^{(i)} t^{\varepsilon_2(i)} + \ldots \)
where \( 1 \leq \varepsilon_1(i) < \varepsilon_2(i) < \ldots \) and \( a_1^{(i)} \neq 0 \) \( \text{if} \ \lambda_i(t) \neq 0 \) \( \varepsilon_1(i) = \infty \) \( \text{if} \ \lambda_i(t) \equiv 0 \) \( 1 \leq i \leq n, \)
\[ C(t) = u_1 t^{b_1} + u_2 t^{b_2} + \ldots \text{where} \ 1 \leq b_1 < b_2 < \ldots \text{and} \ u_1 \neq 0. \]
Since condition (3.1) is invariant under rotation, we can assume that \( \varepsilon_1(1) < \varepsilon_1(i) \) for \( i \neq 1. \)

Let \( f_j(\lambda(t)) = d_1^{(j)} t^{q_1^{(j)}} + d_2^{(j)} t^{q_2^{(j)}} + \ldots, \) where \( 1 \leq q_1^{(j)} < q_2^{(j)} < \ldots (1 \leq j \leq p). \)
Then
\[
\frac{df_j \circ \lambda}{dt}(t) = q_1^{(j)} d_1^{(j)} t^{q_1^{(j)} - 1} + q_2^{(j)} d_2^{(j)} t^{q_2^{(j)} - 1} + \ldots \quad (1 \leq j \leq p).
\]
It follows from (3.3) that
\[
q_1^{(j)} \geq \text{ord}(C) + \text{ord}(h \circ \lambda) \quad \text{for all} \ j \in \{1, \ldots, p\}.
(3.4)
\]
By (3.3) again, we have
(3.5) \[ \varepsilon_1(1) + \text{ord} \left( \sum_{1 \leq i_1 < \cdots < i_p \leq n} \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_1, \ldots, x_p)}(\lambda(t)) \right) \right| \right) \leq \text{ord}(h \circ \lambda) \]

Therefore there is a \( p \)-tuple of integers \((k_1, \ldots, k_p)\) with \(1 \leq k_1 < \cdots < k_p \leq n\) such that

\[
\text{ord}(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_1, \ldots, x_p)}(\lambda(t)) \right)|) \leq \text{ord}(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_1, \ldots, x_p)}(\lambda(t)) \right)|)
\]

for any \((i_1, \ldots, i_p)\), and

\[
\text{ord}(|\det \left( \frac{D(f_1, \ldots, f_p)}{D(x_1, \ldots, x_p)}(\lambda(t)) \right)|) \leq \text{ord}(h \circ \lambda) - \varepsilon_1(1).
\]

We continue the proof of the converse, dividing it into two cases. We first consider the case where \( n > p \). Then we have the following.

**Claim:** \( k_1 > 1 \).

**Proof:** Since

\[
\frac{df_j \circ \lambda}{dt}(t) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\lambda(t)) \frac{d\lambda_i}{dt}(t) \quad (1 \leq j \leq p)
\]

we have

\[
\begin{pmatrix}
\frac{df_1 \circ \lambda}{dt}(t) \\
\vdots \\
\frac{df_p \circ \lambda}{dt}(t)
\end{pmatrix} = \frac{d\lambda_1}{dt}(t) \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\lambda(t)) \\
\vdots \\
\frac{\partial f_1}{\partial x_1}(\lambda(t))
\end{pmatrix} + \cdots + \frac{d\lambda_n}{dt}(t) \begin{pmatrix}
\frac{\partial f_p}{\partial x_1}(\lambda(t)) \\
\vdots \\
\frac{\partial f_p}{\partial x_1}(\lambda(t))
\end{pmatrix}
\]

Here we remark that, by (3.4)

\[
\text{ord} \left( \frac{1}{\lambda_1(t)} \frac{df_j \circ \lambda}{dt}(t) \right) = q_1^{(j)} - \varepsilon_1(1) \geq \text{ord}(C) + \text{ord}(h \circ \lambda) - \varepsilon_1(1) \quad (1 \leq j \leq p),
\]

and

\[
\text{ord} \left( \frac{\lambda_i(t)}{\lambda_1(t)} \right) \geq 1 \quad (2 \leq i \leq n).
\]

Assume, by contradiction, that \( k_1 = 1 \) in (3.6). For simplicity, set

\[
A(t) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_p}(\lambda(t)) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_p}{\partial x_1}(\lambda(t)) & \frac{\partial f_p}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_p}{\partial x_p}(\lambda(t))
\end{pmatrix}
\]

Then the determinant of the matrix \( A(t) \) is the summation of determinants of the following matrices:
Therefore we have

\[ \frac{1}{\lambda'(t)} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_p}(\lambda(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(\lambda(t)) & \frac{\partial f_p}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_p}{\partial x_p}(\lambda(t)) \end{pmatrix} \]

\[ \frac{1}{\lambda'(t)} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_p}(\lambda(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1}(\lambda(t)) & \frac{\partial f_p}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_p}{\partial x_p}(\lambda(t)) \end{pmatrix} \]

By (3.8) the order of the determinant of the matrix (3.10) is bigger than or equal to \( ord(C) + ord(h \circ \lambda) - \varepsilon_1(1) \), and by (3.5) the order of the determinant of the matrix (3.11) is bigger than the order of the determinant of the matrix (3.10). Therefore we have

\[ ord(|\det A(t)|) \geq ord(C) + ord(h \circ \lambda) - \varepsilon_1(1) > ord(h \circ \lambda) - \varepsilon_1(1) \]

which contradicts (3.6). This completes the proof of the claim.

It follows from the Claim that there is a \( p \)-tuple \((k_1, \ldots, k_p)\) with \( 1 < k_1 < \cdots < k_p \leq n \) such that condition (3.6) holds. Then

\[ ord \left( \left| \det \left( \frac{D(f_1, \ldots, f_p, \rho)}{D(x_1, x_{k_1}, \ldots, x_{k_p})} (\lambda(t)) \right) \right| \right) \leq ord(\lambda) + ord \left( \left| \det \left( \frac{D(f_1, \ldots, f_p)}{D(x_{k_1}, \ldots, x_{k_p})} (\lambda(t)) \right) \right| \right) \]

\[ \leq \varepsilon_1(1) + ord(h \circ \lambda) - \varepsilon_1(1) = ord(h \circ \lambda) \]

This contradicts (3.3), and it follows that \( K_m(f, \cdot) \lesssim T_m(f, \cdot) \).

We next consider the case where \( n = p \). Using a similar argument to the proof of the above Claim, we get the same contradiction for

\[ A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\lambda(t)) & \frac{\partial f_1}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_1}{\partial x_n}(\lambda(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\lambda(t)) & \frac{\partial f_n}{\partial x_2}(\lambda(t)) & \cdots & \frac{\partial f_n}{\partial x_n}(\lambda(t)) \end{pmatrix} \]

Therefore it follows that \( K_m(f, \cdot) \lesssim T_m(f, \cdot) \), and this completes the proof.

**Example 3.2.** Let \( f = (f_1, f_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \) be a polynomial mapping defined by

\[ f_1(x, y) = x - y^2, \quad f_2(x, y) = x^2. \]

Then we have

\[ f_1(x, y)^2 + f_2(x, y)^2 = (x - y^2)^2 + x^4, \]

\[ \det \left( \frac{D(f_1, f_2)}{D(x, y)}((x, y)) \right) = 4xy. \]

Therefore we have

\[ T_2(f, (x, y)) = (x - y^2)^2 + x^4, \quad K_2(f, (x, y)) = 16(x^2 + y^2)x^2y^2 + (x - y^2)^2 + x^4. \]

To show that \( T_2(f, (x, y)) \approx K_2(f, (x, y)) \), we consider two cases.

In the case where \(|x - y|^2| \leq \frac{1}{2} y^2\), we have \(x \geq \frac{1}{2} y^2\). Therefore \(64x^4 \geq 16x^2 y^4\) and since for any constant \(C > 65\), \(16x^4 y^2 = o((C - 65)x^4)\) we get
\[
CT_2(f, (x, y)) \geq K_2(f, (x, y))
\]
in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\).

In the case where \(|x - y|^2| \geq \frac{1}{2} y^2\) we can see that
\[
(x - y)^2 + x^4 \geq \frac{1}{4} y^4 + x^4 \geq 16x^2 y^4 + 16x^4 y^4 = 16(x^2 + y^2)x^2 y^2
\]
in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\).

Thus, for any constant \(C > 65\), we have
\[
T_2(f, (x, y)) \leq K_2(f, (x, y)) \leq CT_2(f, (x, y))
\]
in a small neighbourhood of \((0, 0) \in \mathbb{R}^2\), it follows that \(T_2(f, (x, y)) \approx K_2(f, (x, y))\).

We now introduce some notion for a \(C^r\)-map germ \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) in order to extend to the relative case the previous equivalence defined in the non-relative case.

Let \(\Sigma\) be a germ at \(0 \in \mathbb{R}^n\) of closed set such that \(0 \in \Sigma\). Given a map \(g \in \mathcal{E}_{\Sigma}(n, p)\) with \(j^r g(\Sigma; 0) = j^r f(\Sigma; 0)\). Let \(f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) denote the \(C^r\) mapping defined by
\[
f_t(x) = f(x) + t(g(x) - f(x)) \quad \text{for} \quad t \in [0, 1].
\]

**Definition 3.3.** A condition (\(*\)) on a \(C^r\) map \(f\) is called \(\Sigma\)-\(r\)-compatible in the direction \(g\), if \(f_t\) satisfies condition (\(*\)) for any \(t \in [0, 1]\). If condition (\(*\)) is \(\Sigma\)-\(r\)-compatible in any direction \(g \in \mathcal{E}_{\Sigma}(n, p)\) with \(j^r g(\Sigma; 0) = j^r f(\Sigma; 0)\), we simply say condition (\(*\)) is \(\Sigma\)-\(r\)-compatible.

Let \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) be a \(C^1\) map-germ, \(\Sigma \subset \mathbb{R}^n\) be a germ of a closed set such that \(0 \in \Sigma\) and \(r \in \mathbb{N}\). For \(m \in \mathbb{N}\), we introduce the following conditions:
\[
I^r_T(m) : \exists c, \delta > 0 \text{ such that } T_m(f, x) \geq c(d(x, \Sigma))^m \text{ for } ||x|| < \delta,
\]
\[
I^r_K(m) : \exists c, \delta > 0 \text{ such that } K_m(f, x) \geq c(d(x, \Sigma))^m \text{ for } ||x|| < \delta.
\]

**Remark 3.4.** If \(f\) is \(C^\omega\), we have from Theorem 3.1, for any \(m \in \mathbb{N}\),
\[
I^r_T(m) \text{ holds if and only if } I^r_K(m) \text{ holds.}
\]

**Theorem 3.5.** Let \(\Sigma\) be a germ at \(0 \in \mathbb{R}^n\) of a closed set such that \(0 \in \Sigma\). Let \(r\) be a positive integer, and let \(f \in \mathcal{E}_{\Sigma}(n, p)\), \(n \geq p\), such that \(j^r f(\Sigma, 0)\) has a \(C^\omega\) realisation. Then for any \(m \in \mathbb{N}\),
\[
I^r_T(m) \text{ holds if and only if } I^r_K(m) \text{ holds.}
\]

**Proof.** Let \(g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) be a \(C^\omega\) realisation of \(j^r f(\Sigma, 0)\). From Theorem 3.1 conditions \(I^r_T(m)\) and \(I^r_K(m)\) are equivalent for \(g\). Therefore it suffices to show that conditions \(I^r_T(m)\) and \(I^r_K(m)\) are \(r\)-compatible. Let \(f_t = f + th\) with \(h = g - f\).

Then \(\|h\| = o(d(., \Sigma)^r), \|f_t\| \geq \|f\| - \|h\|\) and the expansion of the determinants give
\[
T_m(f_t, x) = T_m(f, x) + o(d(x, \Sigma)^m),
\]
Thus the r-compatibilities of $I_T^r(m)$ and $I^K_r(m)$ follow. □

Corollary 3.6. Let $\Sigma$ be a germ at 0 of a closed set. Let $r$ be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, such that $j^r f(\Sigma,0)$ has a $C^\omega$ realisation. Then the following conditions are equivalent:

1) There exists $m \in \mathbb{N}$ such that $I_T^r(m)$ holds
2) For all $m \in \mathbb{N}$, $I_T^r(m)$ holds
3) There exists $m \in \mathbb{N}$ such that $I^K_r(m)$ holds
4) For all $m \in \mathbb{N}$, $I^K_r(m)$ holds

Remark 3.7. 1) It follows from the proof of Theorem 3.1 that the equivalence between conditions $T_m$ and $K_m$ holds for any $C^1$ map $f$ in a category where the analytic curve selection lemma is valid. 2) Since for $X_1, \ldots, X_l \geq 0$ and a positive integer $m \in \mathbb{N}$, we have

$$(X_1 + \ldots + X_l)^m \approx X_1^m + \ldots + X_l^m,$$

then $K_1 \approx T_1$ if and only if for any $m \in \mathbb{N}$, $K_m \approx T_m$.

4. Relative $C^0$-sufficiency of jets

Let us recall that $\Sigma$ is a germ of a non-empty, closed subset at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. In this section we give criteria for $\Sigma$-$C^0$-sufficiency of relative $r$-jets in $C^r$ mappings and in $C^{r+1}$ mappings, and compute some examples on relative $C^0$-sufficiency of jets using the criteria.

4.1. Relative $C^0$ sufficiency of $r$-jets in $C^r$ mappings. In this subsection we give a criterion of $\Sigma$-$C^0$-sufficiency of $r$-jets in $C^r$ mappings, using the relative Kuiper-Kuo condition.

In the following theorem, the relative Kuiper-Kuo condition implies $\Sigma$-$C^0$-sufficiency, is also proved, with slightly different method in [24].

Theorem 4.1. Let $r$ be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,p)$ where $n \geq p$. Then the following conditions are equivalent.

1) $f$ satisfies the relative Kuiper-Kuo condition ($K-K_\Sigma$), namely

$$\kappa(df(x)) \gtrsim d(x,\Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.
2) The relative $r$-jet $j^r f(\Sigma,0)$ is $\Sigma$-$C^0$-sufficient in $\mathcal{E}_{[r]}(n,p)$.

Proof. We first show the implication (1) $\implies$ (2). In the case where $r = 1$, $0 \in \mathbb{R}^n$ is a regular point of $f$. Therefore the theorem follows from the Implicit Function Theorem. We may assume that $r \geq 2$ after this.

Let $g \in \mathcal{E}_{[r]}(n,p)$ be an arbitrary mapping such that $j^r g(\Sigma;0) = j^r f(\Sigma;0)$. We define a $C^r$ mapping $h : (\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$ by $h(x) := g(x) - f(x)$. Then $j^r h(x) = 0$ for any $x \in \Sigma$. 

Let \( t_0 \) be an arbitrary element of \( I := [0, 1] \). Define \( F(x, t) := f(x) + th(x) \) for \( t \in I \). Since \( j^*h = 0 \) on \( \Sigma \) near \( 0 \in \mathbb{R}^n \), by Lemma 2.25 \( \|h(x)\| = o(d(x, \Sigma)^r) \). Then there exists a small neighbourhood \( T \) of \( t_0 \) in \( I \) such that
\[
\|F(x, t) - F(x, t_0)\| = o(d(x, \Sigma)^r)
\]
for any \( t \in T \). Therefore there are \( \bar{w}, \alpha > 0 \) such that
\[
\nu(d_x F(x, t_0)) = \nu(df(x) + t_0dh(x)) \geq \nu(df(x)) - t_0\|dh(x)\| \geq \frac{C}{2} d(x, \Sigma)^{r-1}
\]
in \( \{\|x\| < \alpha\} \). Then there exists \( C' > 0 \) such that
\[
(4.1) \quad \kappa(d_x F(x, t)) \geq C'd(x, \Sigma)^{r-1}
\]
for \( (x, t) \in W := \{\|x\| < \alpha\} \times T \). Thus, for \( (x, t) \in W \setminus \Sigma \times T \) the vectors \( \text{grad}_x F_j(x, t) (1 \leq j \leq p) \) are linearly independent. Let for \( (x, t) \in W \setminus \Sigma \times T \), \( V_{x,t} \) be the subspace spanned by the \( \{\text{grad}_x F_1(x, t), \ldots, \text{grad}_x F_p(x, t)\} \).

Let us consider now \( \{N_1(x, t), \ldots, N_p(x, t)\} \) the basis of \( V_{x,t} \) constructed as follows:
\[
N_j(x, t) = \text{grad}_x F_j(x, t) - \tilde{N}_j(x, t) \quad (1 \leq j \leq p)
\]
where \( \tilde{N}_j(x, t) \) is the projection of \( \text{grad}_x f_j(x, t) \) to the subspace \( V^j_{x,t} \) spanned by the \( \text{grad}_x F_k(x, t) \), \( k \neq j \). Hence for \( j \in \{1, \ldots, p\} \), \( \|N_j(x, t)\| \) is the distance of \( \text{grad}_x F_j(x, t) \) to \( V^j_{x,t} \). From the above we get, for any \( j \in \{1, \ldots, p\} \) and \( (x, t) \in W \),
\[
\|N_j(x, t)\| \geq \kappa(d_x F(x, t)) \geq C'd(x, \Sigma)^{r-1}.
\]

To trivialise the family of level sets, we use a version of the Kuo vector field \([15]\),
\[
X(x, t) = \begin{cases} \frac{\partial}{\partial t} + \sum_{j=1}^{p} h_j(x) \frac{N_j(x, t)}{\|N_j(x, t)\|^2} & \text{if } (x, t) \in W \setminus \Sigma \times T \\ \frac{\partial}{\partial t} & \text{if } (x, t) \in W \cap \Sigma \times T. \end{cases}
\]
Since,
\[
\left| \sum_{j=1}^{p} h_j(x) \frac{N_j(x, t)}{\|N_j(x, t)\|^2} \right| \lesssim \frac{\|h(x)\|}{\|N_j(x, t)\|} \lesssim d(x, \Sigma)
\]
by Proposition 2.27, the following system of differential equations:
\[
(4.2) \quad y' = X(y, t).
\]
is integrable.

Now for \( (x, t) \in W \) define \( \gamma(x,t) \) to be the maximal solution of (4.2) such that \( \gamma(x,t)(t) = x \). Let \( H_0, \tilde{H}_0 : W \rightarrow \{\|x\| < \alpha\} \) be given by \( H_0(x, t) := \gamma(x,t_0)(t) \) and \( \tilde{H}_0(y, t) := \gamma(y,t_0)(t) \). By Proposition 2.27 the mappings \( H_0 \) and \( \tilde{H}_0 \) are continuous and by uniqueness of the solutions of (4.2) we have for any \( (x, t) \in W \)
\[
\tilde{H}_0(H_0(x, t), t) = x, \quad H_0(x, t_0) = x, \quad \text{and} \quad H_0(H_0(y, t), t) = y, \quad H_0(x, t) = x,
\]
and \( F(\gamma(x,t_0)(t), t) = F(x, t_0) \) for all \( t \in T \), namely we have
\[
f(H_0(x, t)) + th(H_0(x, t)) = F(x, t_0),
\]
in (4.2).
for \((x, t) \in W\) (since on \(\Sigma, h \equiv 0\)). In particular, for all \(t, t' \in T\), the germs of \(F(x, t)\) and \(F(x, t')\) at \(0 \in \mathbb{R}^n\) are \(\Sigma\)-homeomorphic. Finally, by compactness of \([0, 1]\), we obtain that the germs of maps \(f\) and \(g\) at \(0 \in \mathbb{R}^n\) are \(\Sigma\)-homeomorphic. It follows that \(j^r f(\Sigma; 0)\) is \(\Sigma\)-\(C^0\)-sufficient in \(E_{[\nu]}(n, p)\).

We next show the implication \((2) \implies (1)\) in the case where \(p \geq 2\). Let the relative \(r\)-jet \(j^r f(\Sigma; 0)\) be \(\Sigma\)-\(C^0\)-sufficient in \(E_{[\nu]}(n, p)\). Suppose by reductio ad absurdum, that

\[
\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-1}
\]

is not satisfied in a neighbourhood of the origin. One can then find a sequence \((x_\nu)_{\nu \geq 1}\) of points of \(\mathbb{R}^n \setminus \Sigma\) converging to \(0 \in \mathbb{R}^n\) such that

\[
(4.3) \quad \kappa(df(x_\nu)) = o(d(x_\nu, \Sigma)^{r-1}).
\]

Extracting a subsequence from \((x_\nu)_{\nu \geq 1}\) if necessary, one can assume that

\[
\|x_{\nu+1}\| < \frac{1}{3} d(x_\nu, \Sigma)
\]

(which implies, in particular, that \(d(x_\nu, \Sigma)\) decreases), and that condition \((4.3)\) implies:

\[
\delta_\nu = o(d(x_\nu, \Sigma)^{r-1})
\]

where

\[
\delta_\nu := \kappa(df(x_\nu)) = \text{dist} \left( \text{grad} f_j(x_\nu), \sum_{k \neq j} \mathbb{R} \text{grad} f_k(x_\nu) \right)
\]

for some \(j, 1 \leq j \leq p\). By Remark 2.13(3), we may assume \(j = 1\) after this.

Now we apply Lemma 2.29 with \(d_{\nu}^{(k)} = \text{grad} f_k(x_\nu), \alpha_\nu = d(x_\nu, \Sigma)\) and \(s = r - 1\), to find for each \(\nu \in \mathbb{N}, p - 1\) vectors, \(\lambda_\nu^{(2)}, \ldots, \lambda_\nu^{(p)} \in \mathbb{R}^n\) such that:

(a) \(\|\lambda_\nu^{(k)}\| = o(d(x_\nu, \Sigma)^{r-1}), k = 2, \ldots, p\);

(b) \(\text{grad} f_2(x_\nu) + \lambda_\nu^{(2)}, \ldots, \text{grad} f_p(x_\nu) + \lambda_\nu^{(p)}\) are linearly independant in \(\mathbb{R}^n\);

(c) \(\text{grad} f_1(x_\nu) \in \sum_{k=2}^{p} \mathbb{R} (\text{grad} f_k(x_\nu) + \lambda_\nu^{(k)}).\)

Let \(\psi : \mathbb{R}^n \to \mathbb{R}\) be a \(C^\infty\) function such that \(\psi(t) = 1\) in a neighbourhood of \(0 \in \mathbb{R}^n\) and \(\psi(t) = 0\) for \(|t| \geq \frac{1}{4}\). We define a map-germ \(\eta = (\eta_1, \ldots, \eta_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) by:

\[
\eta_1(x) = \epsilon_\nu \psi \left( \frac{x - x_\nu}{d(x_\nu, \Sigma)} \right) \|x - x_\nu\|^2,
\]

\[
\eta_k(x) = \psi \left( \frac{x - x_\nu}{d(x_\nu, \Sigma)} \right) \langle \lambda_\nu^{(k)}, (x - x_\nu) \rangle, k = 2, \ldots, p,
\]

for \(x \in B_\nu\) and \(\eta(x) = 0\) for \(x \not\in \bigcup_{\nu=1}^{\infty} B_\nu\), where for \(\nu \in \mathbb{N}\),

\[
B_\nu = \{ x \in \mathbb{R}^n : \|x - x_\nu\| \leq \frac{1}{4} d(x_\nu, \Sigma) \},
\]

and \((\epsilon_\nu)_{\nu \geq 1}\) is a sequence of real numbers.
Since $|\psi(t)|$ is bounded in $\mathbb{R}^n$, we have

\begin{equation}
|\eta_1(x)| \gtrsim \epsilon \nu d(x, \Sigma)^2,
\end{equation}

\begin{equation}
|\eta_k(x)| = o(d(x, \Sigma)^r), \quad k = 2, \ldots, p.
\end{equation}

Therefore, if we choose the sequence $(\epsilon_\nu)_{\nu \geq 1}$ so that $\epsilon_\nu = o(d(x, \Sigma)^r)$, we have

\[ \eta(x) = o(d(x, \Sigma)^r). \]

It follows that $g = f - \eta$ is a $C^r$-realisation of $j^*f(\Sigma; 0)$ and $g(x_\nu) = f(x_\nu)$, for any $\nu \in \mathbb{N}$.

By condition (b), there is a small neighbourhood $V_\nu$ of $x_\nu$ such that the set

\[ M_\nu = \{ x \in V_\nu : f_j(x) - \eta_j(x) = f_k(x_\nu), \quad k = 2, \ldots, p \} \]

is a differentiable manifold of codimension $p - 1$.

From condition (c), for each $\nu \in \mathbb{N}$, there are real numbers $a_{2,\nu}, \ldots, a_{p,\nu}$ such that,

\[ \text{grad} f_1(x_\nu) = \sum_{k=2}^{p} a_{k,\nu} (\text{grad} f_k(x_\nu) + \lambda^{(k)}). \]

Choose now $\epsilon_\nu = o(d(x_\nu, \Sigma)^r)$ more finely such that $x_\nu$ is a non-degenerate critical point of the restriction to $M_\nu$ of

\[ h_\nu(x) = f_1(x) - \eta_1(x) + \sum_{k=2}^{p} a_{k,\nu} (\eta_k(x) - f_k(x)). \]

Then

\[ g^{-1}(g(x_\nu)) \cap V_\nu = \{ x \in V_\nu : g_j(x) = f_j(x_\nu), \quad 1 \leq j \leq p \} \]
\[ = \{ x \in M_\nu : h_\nu(x) = h_\nu(x_\nu) \}. \]

By the choice of $\epsilon_\nu$, this set is the intersection of the locus of a non-degenerate quadratic form $h_\nu^{-1}(h_\nu(x_\nu))$ with a codimension $p - 1$ manifold $M_\nu$. Then if it is a topological manifold, necessarily it is reduced to a point.

Now if $n - p \geq 1$, $g^{-1}(x_\nu)$ cannot be a topological manifold of codimension $p$ and if $n = p$, for $x \in M_\nu$, $g(x) = (h_\nu(x), 0)$, thus $g$ is not injective in any neighbourhood of $x_\nu$, since (the quadratic form) $h_\nu$ restricted to the one dimensional manifold $M_\nu$ is not injective, but this contradicts the following lemma:

**Lemma 4.2.** Let $j^*f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E_{[r]}(n, p)$. Then for all maps $\theta \in E_{[r]}(n, p)$ such that $j^*\theta(\Sigma; 0) = j^*f(\Sigma; 0)$, and for all sequence $\{x_m\}_{m \in \mathbb{N}}$ in $\mathbb{R}^n \setminus \Sigma$, $x_m \to 0$, $m \to \infty$, there is a neighbourhood of $x_m$, for sufficiently large $m$, such that $g^{-1}(g(x_m))$ is a topological manifold of codimension $p$ if $n > p$, or $g$ is injective if $n = p$.

**Proof.** We need for this, the following fact, which is a consequence of Sard’s theorem: Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^p$ be open sets, let $F : U \times V \to \mathbb{R}^p$ be a smooth map and let $\{b_m\}_{m \in \mathbb{N}}$ be a sequence of points in the regular values of $F$. Then the set

\[ \mathcal{R} = \{ y \in V : \forall m \in \mathbb{N}, b_m \text{ is a regular value of the map } F_y : U \ni x \to F(x, y) \} \]
is residual in $V$.

By Remark 2.1(2), there exists a $C^r$-realisation $\tilde{f}$ of $j^rf(\Sigma; 0)$ such that the restriction of $\tilde{f}$ to $\mathbb{R}^n \setminus \Sigma$ is smooth.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth flat function such that $h^{-1}(0) = \Sigma$. We consider now, the map $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ defined by

$$F_j(x, y) = \tilde{f}_j(x) + y_j h(x), \quad j = 1, \ldots, p.$$  

The restriction of $F$ to $(\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^p$ is a submersion around $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^p$.

Let $(x_m)_{m \in \mathbb{N}}$ be a sequence of points of $\mathbb{R}^n \setminus \Sigma$ which tends to 0, then $(\tilde{f}(x_m))_{m \in \mathbb{N}}$ is a sequence of regular values of $F|_{(\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}^p}$, and by the quoted version of Sard theorem, there is $y_0 \in \mathbb{R}^p$ such that $(\tilde{f}(x_m))_{m \in \mathbb{N}}$ is a sequence of regular values of $F_{y_0}|_{\mathbb{R}^n \setminus \Sigma}$. Let $g_0 = F_{y_0}$. Since $h$ is flat on $\Sigma$, for all $r \in \mathbb{N}$, $j^r g_0(\Sigma; 0) = j^r f(\Sigma; 0)$.

Now, by $\Sigma$-$C^d$-sufficiency of $j^rf(\Sigma; 0)$, there is a germ of homeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that the restriction of $\varphi$ to $\Sigma$ is the identity and $g_0 \circ \varphi = f$. Thus $f^{-1}(f(x_m)) = \varphi^{-1}(g_0^{-1}(f(x_m)))$ is a topological manifold of codimension $p$ in a neighbourhood of $x_m$ (for large $m$), because $\varphi^{-1}$ is a homeomorphism and $g_0^{-1}(f(x_i))$ is a smooth submanifold of $\mathbb{R}^n \setminus \Sigma$ of codimension $p$ and if $n = p$, $f$ is injective in a neighbourhood of $x_m$. Therefore, by $\Sigma$-$C^0$-sufficiency of $j^rf(\Sigma; 0)$, any mapping $\theta \in E_{r+1}(n, p)$ of classe $C^r$ such that $j^r \theta(\Sigma; 0) = j^r f(\Sigma; 0)$, shares these properties. $\square$

Therefore we can see that $f$ satisfies the relative Kuiper-Kuo condition ($K\Sigma$).

The implication $(2) \implies (1)$ in the function case, namely $p = 1$, follows similarly to the above mapping case, but more simply. In this case we have

$$\delta_r := \kappa(df(x)) = \|\text{grad} f(x)\| = o(d(x, \Sigma)^{r-1}).$$

We do not need to apply the Bochnak-Kuo Lemma. We take $\eta(x) = \eta_1(x)$ as the same function, and consider $M_\nu = V_\nu$. We do not define $h_\nu$. Instead, $g^{-1}(g(x_\nu)) \cap V_\nu$ takes the same role as $h_\nu^{-1}(h_\nu(x_\nu))$ in this case. Then the remainder follows in the same way.

This completes the proof of the theorem. $\square$

4.2. Relative $C^0$ sufficiency of $r$-jets in $C^{r+1}$ mappings. In this subsection we give a criterion of $\Sigma$-$C^0$-sufficiency of $r$-jets in $C^{r+1}$ mappings, using the second relative Kuiper-Kuo condition.

**Theorem 4.3.** Let $r$ be a positive integer, and let $f \in E_{[r+1]}(n, p)$ where $n \geq p$.

Then the following conditions are equivalent.

1. $f$ satisfies the second relative Kuiper-Kuo condition ($K\Sigma$), namely there is a strictly positive number $\delta$ such that

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

2. The relative $r$-jet $j^r f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E_{[r+1]}(n, p)$. 
Proof. This theorem is shown in the same way as Theorem 4.1.

In the case of $r - \delta$, the implication (1) $\implies$ (2) follows, after noticing that, by Lemma 2.5 if $F(x, t) := f(x) + th(x)$ for $t \in I = [0, 1]$ and $j^r h(\Sigma; 0) = 0$, then $\|h(x)\| \lesssim d(x, \Sigma)^{r+1}$ and there exists a small neighbourhood $T$ of $t_0$ in $I$ such that

$$\|F(x, t) - F(x, t_0)\| \lesssim d(x, \Sigma)^{r+1}$$

for any $t \in T$.

On the other hand, we can show the implication (2) $\implies$ (1) in the same way as above, by replacing everywhere $r - 1$ with $r - \delta$. \qed

4.3. $\Sigma$-$C^0$-sufficiency of jets in the function case. In this subsection we restate Theorems 4.1, 4.3 in the function case. Related to these results, we shall discuss in the next section if the Bochnak-Lojasiewicz inequality holds in the relative case, and the relationship between the relative $C^0$-sufficiency of jets and the relative $V$-sufficiency of jets through the relationship between the relative Kuiper-Kuo condition and condition $(\tilde{K}_\Sigma)$.

**Theorem 4.4.**

(1) Let $r$ be a positive integer, and let $f \in E[r](n, 1)$. Then the inequality

$$\|\grad f(x)\| \gtrsim d(x, \Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$ if and only if the relative $r$-jet $j^r f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E[r](n, 1)$.

(2) Let $r$ be a positive integer, and let $f \in E[r+1](n, 1)$. Then there is a strictly positive number $\delta$ such that the inequality

$$\|\grad f(x)\| \gtrsim d(x, \Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$ if and only if the relative $r$-jet $j^r f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E[r+1](n, 1)$.

**Remark 4.5.** X. Xu also has obtained in [35] a result that the inequality in Theorem 4.4 (1) implies $\Sigma$-$C^0$-sufficiency in $E[r](n, 1)$.

**Example 4.6.** Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a polynomial function defined by

$$f(x, y) := x^3,$$

and let $\Sigma := \{x = 0\}$. Then we can easily see that $d((x, y), \Sigma) = |x|$, and

$$\|\grad f(x, y)\| \gtrsim |x|^2$$

in a neighbourhood of $(0, 0) \in \mathbb{R}^2$. It follows from Theorem 4.4 (1) that $j^3 f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E[3](2, 1)$.

**Example 4.7.** Let $f_m : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $m \geq 3$, be a polynomial function defined by

$$f_m(x, y) := x^3 - 3xy^m.$$

Then we have $\grad f_m(x, y) = (3x^2 - y^m, -3mx^m)$. 

(1) Let $\Sigma := \{(0, 0)\}$. Then we have $d((x, y), \Sigma) = \|(x, y)\|$, and

$$\|\grad f_m(x, y)\| \gtrsim \|(x, y)\|^m - 1$$
in a neighbourhood of $(0,0) \in \mathbb{R}^2$. We can check the above inequality, dividing a neighbourhood of $0 \in \mathbb{R}^2$ into the following three regions:

$$A := \{3|x^2 - y^m| \leq |x|^2\}, \quad B := \{3|x|^2 \leq |y|^m\}, \quad \mathbb{R}^2 \setminus (A \cup B).$$

By the Kuiper-Kuo theorem\cite{[14],[15]}, $j^{\frac{3m}{m-1}}f(0)$ is $C^0$-sufficient in $E_{[\frac{3m+1}{m}]}(2,1)$ if $m$ is odd, and $j^{\frac{3m}{m-1}}f(0)$ is $C^0$-sufficient in $E_{[\frac{3m}{m-1}]}(2,1)$ if $m$ is even.

(2) Let $\Sigma := \{x = 0\}$. Then we can see that

$$\|\text{grad}f_m(x, y)\| \gtrsim |x|^{3-\frac{2}{m}}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$.

We can show (4.6) as follows. Let $\lambda : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be an arbitrary analytic arc on $\mathbb{R}^2$ passing through $(0,0) \in \mathbb{R}^2$, not identically zero, denoted by

$$\lambda(t) = (a_kt^k + \cdots, b_st^s + \cdots).$$

In the case where $\lambda(0) = (0,0), \lambda$ is contained in $\Sigma$, and $d((x, y), \Sigma) = |x| = 0$ on $\lambda$. Therefore we have

$$\|\text{grad}f_m(x, y)\| \gtrsim |x|$$

on $\lambda$. Thus we may assume after this that $a_k \neq 0$.

In the case where $2k < ms$, we have

$$\|\text{grad}f_m(x, y)\| \gtrsim |t|^{2k}, \quad |x| \approx |t|^k$$

on $\lambda$. Therefore we have

$$\|\text{grad}f_m(x, y)\| \gtrsim |x|^2$$

on $\lambda$ near $(0,0) \in \mathbb{R}^2$.

In the case where $2k \geq ms$, we have

$$\|\text{grad}f_m(x, y)\| \gtrsim |t|^k |t|^{(m-1)s} \gtrsim |t|^{k + \frac{2(m-1)s}{m}} = |t|^{(3-\frac{2}{m})k}, \quad |x| \approx |t|^k$$

on $\lambda$. Therefore we have

$$\|\text{grad}f_m(x, y)\| \gtrsim |x|^{3-\frac{2}{m}}$$

on $\lambda$ near $(0,0) \in \mathbb{R}^2$.

Thus we have (4.6) in a neighbourhood of $(0,0) \in \mathbb{R}^2$.

Note that

$$\frac{\partial f_m}{\partial x}(t^k, t^s) \equiv 0, \quad \frac{\partial f_m}{\partial y}(t^k, t^s) = 3m|t|^{(3-\frac{2}{m})k}$$

in the case where $2k = ms$. Therefore it follows from Theorem 1.4(1), (2) that $j^{3}f_m(\Sigma; 0)$ is not $\Sigma$-$C^0$-sufficient in $E_{[3]}(2,1)$ but $\Sigma$-$C^0$-sufficient in $E_{[4]}(2,1)$ for any $m \geq 3$.

(3) Let $\Sigma := \{y = 0\}$. Then, using a similar computation to the above one, we can see that

$$\|\text{grad}f_m(x, y)\| \gtrsim |y|^{\frac{m}{m-1}}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. It follows from Theorem 1.4(1), (2) that $j^{\frac{3m}{m-1}}f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E_{[\frac{3m+1}{m-1}]}(2,1)$ if $m$ is odd, and $j^{\frac{3m}{m-1}}f(\Sigma; 0)$ is $\Sigma$-$C^0$-sufficient in $E_{[\frac{3m}{m-1}]}(2,1)$ if $m$ is even.
5. **Relative V-sufficiency of jets**

5.1. **Relative V-sufficiency of r-jets in \( C^r \) mappings.** In this subsection we discuss the relationship between the Kuo condition and V-sufficiency of r-jets in \( C^r \) mappings which are relative to the closed set \( \Sigma \subset \mathbb{R}^n \) such that \( 0 \in \Sigma \).

**Theorem 5.1.** Let \( r \) be a positive integer, and let \( f \in \mathcal{E}_{[r]}(n, p) \), \( n \geq p \). If \( f \) satisfies condition \((K_\Sigma)\), then the relative r-jet, \( j^r f(\Sigma; 0) \), is \( \Sigma \)-V-sufficient in \( \mathcal{E}_{[r]}(n, p) \).

**Proof.** Because of the same reason as the theorem above, we may assume that \( r \geq 2 \).

Let \( g \in \mathcal{E}_{[r]}(n, p) \) be an arbitrary mapping such that \( j^r g(\Sigma; 0) = j^r f(\Sigma; 0) \). We define a \( C^r \) mapping \( h : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) by \( h(x) := g(x) - f(x) \). Then \( j^r h(x) = 0 \) for any \( x \in \Sigma \). By Lemma 2.5, \( \|h(x)\| = o(d(x, \Sigma)^r) \). Let \( F(x, t) := f(x) + th(x) \) for \( t \in I = [0, 1] \), and \( t_0 \in I \). Then there exists a small neighbourhood \( T \) of \( t_0 \) in \( I \) such that

\[
\|F(x, t) - F(x, t_0)\| \leq \tilde{w} \, d(x, \Sigma)^r
\]

for any \( t \in T \). Thus \( F(x, t) = 0 \) is contained in \((\mathcal{H}_x^\Sigma(F(x, t_0); \tilde{w}) \cap \{\|x\| < \alpha\}) \times T, \) hence we will concentrate our attention to this set.

Moreover there are \( \tilde{w} \), \( \alpha > 0 \) such that

\[
\nu(d_x F(x, t_0)) = \nu(df(x) + t_0 dh(x)) \geq \nu(df(x)) - t_0\|dh(x)\| \geq \frac{C}{2} d(x, \Sigma)^{r-1}
\]

for \( x \in \mathcal{H}_x^\Sigma(f(x); \tilde{w}) \cap \{\|x\| < \alpha\} \). Then there exists \( C' > 0 \) such that

\[
(5.1) \quad \kappa(d_x F(x, t)) \geq C'd(x, \Sigma)^{r-1}
\]

for \( (x, t) \in (\mathcal{H}_x^\Sigma(F(x, t_0); \tilde{w}) \cap \{\|x\| < \alpha\}) \times T \).

Set \( W := (\mathcal{H}_x^\Sigma(F(x, t_0); \tilde{w}) \cap \{\|x\| < \alpha\}) \times T \). Thus, for \( (x, t) \in W \setminus \Sigma \times T \) the vectors \( \text{grad}_x F_j(x, t) \) \((1 \leq j \leq p)\) are linearly independent. Let for \( (x, t) \in W \setminus \Sigma \times T \), \( V_{x,t} \) be the subspace spanned by the \( \{\text{grad}_x F_1(x, t), \ldots, \text{grad}_x F_p(x, t)\} \).

Let us consider now \( \{N_1(x, t), \ldots, N_p(x, t)\} \) the basis of \( V_{x,t} \) constructed in the case of relative \( C^0 \)-sufficiency of jets:

\[
N_j(x, t) = \text{grad}_x F_j(x, t) - \tilde{N}_j(x, t) \quad (1 \leq j \leq p)
\]

where \( \tilde{N}_j(x, t) \) is the projection of \( \text{grad}_x f_j(x, t) \) to the subspace \( V^j_{x,t} \) spanned by the \( \text{grad}_x F_k(x, t), k \neq j \). Then, for any \( j \in \{1, \ldots, p\} \) and \( (x, t) \in W \),

\[
\|N_j(x, t)\| \geq \kappa(d_x F(x, t)) \geq C'd(x, \Sigma)^{r-1}.
\]

To trivialise the family of zero sets, we use a version of Kuo vector field as in the proof of Theorem 4.1

\[
X(x, t) = \begin{cases}
\frac{\partial}{\partial t} + \sum_{j=1}^p h_j(x) \frac{N_j(x, t)}{\|N_j(x, t)\|^2} & \text{if } (x, t) \in W \setminus \Sigma \times T \\
\frac{\partial}{\partial t} & \text{if } (x, t) \in W \cap \Sigma \times T.
\end{cases}
\]
Since,
\[
\left\| \sum_{j=1}^{p} h_j(x) \frac{N_j(x, t)}{\|N_j(x, t)\|^2} \right\| \lesssim \frac{\|h(x)\|}{\|N_j(x, t)\|} \lesssim d(x, \Sigma)
\]
by Proposition 2.27, the following system of differential equations:
\[
y' = X(y, t).
\]
is integrable. Now for \((x, t) \in W\) define \(\gamma(x, t)\) to be the maximal solution of (5.2) such that \(\gamma(x, t)(t) = x\).

Let \(H_0, \tilde{H}_0 : W \to \mathcal{H}_r^C(F(x, t_0); \tilde{\nu}) \cap \{\|x\| < \alpha\}\) be given by \(H_0(x, t) = \gamma(x, t_0)(t)\) and \(\tilde{H}_0(y, t) = \gamma(y, t_0)(t)\). By Proposition 2.27, the mappings \(H_0\) and \(\tilde{H}_0\) are continuous and by uniqueness of the solutions of (5.2), we have for any \((x, t) \in W\)
\[
\tilde{H}_0(H_0(x, t), t) = x, H_0(x, t_0) = x, \text{ and } H_0(\tilde{H}_0(y, t), t) = y, H_0(x, t) = x
\]
and \(F(\gamma(x, t_0)(t), t) = F(x, t_0)\) for all \(t \in T\). Namely, we have
\[
f(H_0(x, t)) + th(H_0(x, t)) = F(x, t_0),
\]
for \((x, t) \in W\) since on \(\Sigma\), \(h \equiv 0\). In particular, for all \(t, t' \in T\), the germs of \(F(x, t) = 0\) and \(F(x, t') = 0\) at \(0 \in \mathbb{R}^n\) are \(\Sigma\)-homeomorphic.

Finally, using the same compactness argument as above, we obtain that the germs of zero-sets \(f(x) = 0\) and \(g(x) = 0\) at \(0 \in \mathbb{R}^n\) are \(\Sigma\)-homeomorphic.

\[\square\]

**Theorem 5.2.** Let \(r\) be a positive integer, and let \(f \in E_{[r]}(n, p)\) where \(n > p\). Then the following conditions are equivalent.

1. \(f\) satisfies the relative Kuo condition \((K_\Sigma)\).
2. \(f\) satisfies condition \((\tilde{K}_\Sigma)\).
3. The relative \(r\)-jet \(j^r f(\Sigma; 0)\) is \(\Sigma\)-\(V\)-sufficient in \(E_{[r]}(n, p)\).

**Theorem 5.3.** Let \(r\) be a positive integer, and let \(f \in E_{[r]}(n, n)\). Suppose that \(j^r f(\Sigma; 0)\) has a subanalytic \(C^r\)-realisation and that \(\Sigma\) is a subanalytic closed subset of \(\mathbb{R}^n\) such that \(0 \in \Sigma\). Then the following conditions are equivalent.

1. \(f\) satisfies the relative Kuo condition \((K_\Sigma)\).
2. \(f\) satisfies condition \((\tilde{K}_\Sigma)\).
3. The relative \(r\)-jet \(j^r f(\Sigma; 0)\) is \(\Sigma\)-\(V\)-sufficient in \(E_{[r]}(n, n)\).

**Proofs of Theorem 5.2 and Theorem 5.3.** We first assume that \(f \in E_{[r]}(n, p)\), \(n \geq p\), and we do not necessarily assume that \(j^r f(\Sigma; 0)\) has a subanalytic \(C^r\)-realisation or \(\Sigma\) is subanalytic in the case where \(n = p\).

As mentioned in Remark 2.13(2), conditions (1) and (2) are equivalent. The implication (1) \(\implies\) (3) follows from Theorem 5.1. Therefore we shall show that condition (3) implies condition (2). Namely, \(\Sigma\)-\(V\)-sufficiency of jets implies condition \((\tilde{K}_\Sigma)\).

Let the relative \(r\)-jet \(j^r f(\Sigma; 0)\) be \(\Sigma\)-\(V\)-sufficient in \(E_{[r]}(n, p)\). Suppose by reductio ad absurdum, that \((\tilde{K}_\Sigma)\) is not satisfied. One can then find a sequence \((x_\nu)_{\nu \geq 1}\) of points of \(\mathbb{R}^n \setminus \Sigma\) converging to \(0 \in \mathbb{R}^n\) such that
\[
d(x_\nu, \Sigma) \kappa(df(x_\nu)) + \|f(x_\nu)\| = o(d(x_\nu, (\Sigma)')).
\]
Extracting a subsequence from \((x_\nu)_{\nu \geq 1}\) if necessary, one can assume that
\[
\|x_{\nu+1}\| < \frac{1}{3}d(x_\nu, \Sigma)
\]
(which implies, in particular, that \(d(x_\nu, \Sigma)\) decreases), and that condition (5.3) implies:
1) \(|f_k(x_\nu)| = o(d(x_\nu, \Sigma)^r), \text{ for all } 1 \leq k \leq p;
2) \(\delta_\nu = o(d(x_\nu, \Sigma)^{r-1})\) where
\[
\delta_\nu := \kappa(df(x_\nu)) = \text{dist}(\text{grad} f_\nu, \sum_{k \neq j} \mathbb{R} \text{grad} f_k(x_\nu))
\]
for some \(j, 1 \leq j \leq p\). By Remark 2.13(3), we may assume \(j = 1\) after this.

Now we apply Lemma 2.29 with \(u_\nu^{(k)} = \text{grad} f_k(x_\nu), \alpha_\nu = d(x_\nu, \Sigma)\) and \(s = r - 1\), to find for each \(\nu \in \mathbb{N}, p - 1\) vectors, \(\lambda_\nu^{(2)}, \ldots, \lambda_\nu^{(p)} \in \mathbb{R}^n\) such that:
(a) \(\|\lambda_\nu^{(k)}\| = o(d(x_\nu, \Sigma)^{r-1}), \text{ for all } k = 2, \ldots, p;\)
(b) \(\text{grad} f_2(x_\nu) + \lambda_\nu^{(2)}, \ldots, \text{grad} f_p(x_\nu) + \lambda_\nu^{(p)}\) are linearly independant in \(\mathbb{R}^n;\)
(c) \(\text{grad} f_1(x_\nu) \in \sum_{k=2}^p \mathbb{R} (\text{grad} f_k(x_\nu) + \lambda_\nu^{(k)}).\)

Let \(\psi : \mathbb{R}^n \to \mathbb{R}\) be a \(C^\infty\) function such that \(\psi(t) = 1\) in a neighbourhood of \(0 \in \mathbb{R}^n\) and \(\psi(t) = 0\) for \(|t| \geq \frac{1}{4}\). We define a map-germ \(\eta = (\eta_1, \ldots, \eta_p) : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\) by:
\[
\eta_1(x) = \psi \left( \frac{x - x_\nu}{d(x_\nu, \Sigma)} \right) (f_1(x_\nu) + \epsilon_\nu \|x - x_\nu\|),
\eta_k(x) = \psi \left( \frac{x - x_\nu}{d(x_\nu, \Sigma)} \right) (f_k(x_\nu) - \langle \lambda_\nu^{(k)}, (x - x_\nu) \rangle), \quad k = 2, \ldots, p,
\]
for \(x \in B_\nu\) and \(\eta(x) = 0\) for \(x \notin \bigcup_{\nu=1}^\infty B_\nu\), where for \(\nu \in \mathbb{N}\),
\[
B_\nu = \{ x \in \mathbb{R}^n : \|x - x_\nu\| \leq \frac{1}{4}d(x_\nu, \Sigma) \},
\]
and \((\epsilon_\nu)_{\nu \geq 1}\) is a sequence of real numbers.

Let \(K > 0\) such that \(\|\psi(t)\| \leq K\) in \(\mathbb{R}^n\). Then we have
\[
|\eta_1(x)| \leq K (|f_1(x_\nu)| + \epsilon_\nu \|x - x_\nu\|^2) \lesssim o(d(x_\nu, \Sigma)^r) + \epsilon_\nu d(x_\nu, \Sigma)^2,
\]
\[
|\eta_k(x)| \leq K (|f_k(x_\nu)| + \|\lambda_\nu^{(k)}\| \|x - x_\nu\|) \lesssim o(d(x_\nu, \Sigma)^r), \quad k = 2, \ldots, p.
\]
Therefore, if we take the sequence \((\epsilon_\nu)_{\nu \geq 1}\) so that \(\epsilon_\nu = o(d(x_\nu, \Sigma)^r)\), we have
\[
\eta(x) = o(d(x, \Sigma)^r).
\]
It follows that \(g = f - \eta\) is a \(C^r\)-realisation of \(j^* f(\Sigma; 0)\).

By condition (b), there is a small neighbourhood \(V_\nu\) of \(x_\nu\) such that the set
\[
M_\nu = \{ x \in V_\nu : f_k(x) - \eta_k(x) = 0, \quad k = 2, \ldots, p \}
\]
is a differentiable manifold of codimension \(p - 1\).
From condition (c), for each $\nu \in \mathbb{N}$, there are real numbers $a_{2,\nu}, \ldots, a_{p,\nu}$ such that,

$$\text{grad} f_\nu (x_\nu) = \sum_{k=2}^{p} a_{k,\nu} (\text{grad} f_k (x_\nu) + \lambda_k^{(k)}).$$

Choose now $\epsilon_\nu = o(d(x_\nu, \Sigma)^r)$ more finely such that $x_\nu$ is a non-degenerate critical point of

$$h_\nu (x) = f_1 (x) - \eta_1 (x) + \sum_{k=2}^{p} a_{k,\nu} (\eta_k (x) - f_k (x)).$$

Then

$$g^{-1}(0) \cap V_\nu = \{ x \in V_\nu : g_1 (x) = g_2 (x) = \ldots = g_p (x) = 0 \}$$

$$= \{ x \in M_\nu : h_\nu (x) = 0 \}.$$

By the choice of $\epsilon_\nu$, this set is the intersection of the locus of a non-degenerate quadratic form $h_k^{-1}(0)$ with a codimension $p - 1$ manifold $M_\nu$. Therefore, modifying the sequence $\epsilon_\nu, \epsilon_\nu = o(d(x_\nu, \Sigma)^r)$), if necessary, in the case where $n > p$, $g^{-1}(0)$ cannot be a topological manifold of codimension $p$, around $x_\nu$, $\nu \in \mathbb{N}$.

By construction, the map-germ $g$ has the same $r$-jet as $f$ as mentioned above, and its zero set $g^{-1}(0)$ contains the sequence $(x_\nu)_{\nu \in \mathbb{N}}$ which is not in $\Sigma$. Therefore the germ $g^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty. Since $j^r f(\Sigma; 0)$ is $\Sigma$-$V$-sufficient, the germ $f^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty, either.

It follows from the above arguments that we have the following properties for $f \in \mathcal{E}_{[r]}(n, p)$ under the assumption that $j^r f(\Sigma; 0)$ is $\Sigma$-$V$-sufficient in $\mathcal{E}_{[r]}(n, p)$, but $f$ does not satisfy condition $(K_{\Sigma})$:

(P1) The germ $f^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty.

(P2) In the case where $n > p$, $j^r f(\Sigma; 0)$ has a $C^r$-realisation $g \in \mathcal{E}_{[r]}(n, p)$ such that near each $x_\nu$, $g^{-1}(g(x_\nu)) = g^{-1}(0)$ is not a topological manifold of codimension $p$.

On the other hand, we have the following lemma.

**Lemma 5.4.** Let $j^r f(\Sigma; 0)$ is $\Sigma$-$V$-sufficient in $\mathcal{E}_{[r]}(n, p)$. Suppose that the germ $f^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty. Then there exists $g_0 \in \mathcal{E}_{[r]}(n, p)$ with $j^r g_0 (\Sigma; 0) = j^r f(\Sigma; 0)$ such that the germ of $g_0^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty and a smooth submanifold of $\mathbb{R}^n$ of codimension $p$.

**Proof.** We need for this, the following fact, which is a consequence of Sard’s theorem: Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^p$ be open sets, let $F : U \times V \to \mathbb{R}^p$ be a smooth map. and let $b$ be a regular values of $F$. Then for almost all $y \in V$, $b$ is a regular value of a map $F_y : U \ni x \mapsto F(x, y)$.

By Remark 2.1(2), there exists a $C^r$-realisation $\hat{f}$ of $j^r f(\Sigma; 0)$ such that the restriction of $\hat{f}$ to $\mathbb{R}^n \setminus \Sigma$ is smooth. Since the jet $j^r f(\Sigma; 0)$ is $\Sigma$-$V$-sufficient in $\mathcal{E}_{[r]}(n, p)$, the germ of $\hat{f}^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth flat function such that $h^{-1}(0) = \Sigma$. We consider now, the map $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ defined by

$$F_j (x, y) = \hat{f}_j (x) + y_j h(x), \quad j = 1, \ldots, p.$$
Theorem 4.1, 

inequality does not hold along an analytic arc $K$ that the relative Kuiper-Kuo condition (precisely, if we let $V$ to $K$ satisfies condition $(P^2)$ in a neighbourhood of $(0,0)$, then we ask whether the Bochnak-Lojasiewicz inequality holds also in the relative case. More important role in the proof of the equivalence. Therefore it may be natural to $g$ in a neighbourhood of $(0,0)$, $g$ is regular value of $F$. Therefore by the above fact, there is $y_0 \in \mathbb{R}^p$ close to $0 \in \mathbb{R}^p$ such that $0 \in \mathbb{R}^p$ is a regular value of $F_{y_0}|_{\mathbb{R}^n \setminus \Sigma}$.

Now we let $y_0 = F_{y_0}$. By construction, $g_0 \in E_{[\tau]}(n,p)$, and $j^*g_0(0;0) = j^*f(0;0)$. Since $j^*f(0;0)$ is $\Sigma$-V-sufficient in $E_{[\tau]}(n,p)$, $g_0$ is not empty and a germ of a smooth submanifold of $\mathbb{R}^n$ of codimension $p$.

We first consider the case where $n > p$. Since $j^*f(0;0)$ is $\Sigma$-V-sufficient in $E_{[\tau]}(n,p)$, property $(P^2)$ contradicts Lemma 5.4. Therefore $f$ satisfies condition ($K_{\Sigma}$). This completes the proof of Theorem 5.2.

We next consider the case where $n = p$. In this case we are assuming that $j^*f(0;0)$ has a subanalytic $C^\omega$-realisation $\tilde{f}$, and that $\Sigma$ is a subanalytic subset of $\mathbb{R}^n$. By property $(P^1)$, the germ of $f^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty. Since $j^*f(0;0)$ is $\Sigma$-V-sufficient in $E_{[\tau]}(n,p)$, the germ of $\tilde{f}^{-1}(0) \setminus \Sigma$ at $0 \in \mathbb{R}^n$ is not empty. Then, by the Curve Selection Lemma, there exists a $C^\omega$ arc $\lambda : [0, \delta) \to \mathbb{R}^n$, $\delta > 0$, such that $\lambda(0) = 0 \in \mathbb{R}^n$ and $\lambda(t) \in \tilde{f}^{-1}(0)$, $t \in [0, \delta)$. Therefore, because of $\Sigma$-V-sufficiency of $j^*f(0;0)$, this contradicts Lemma 5.4. Therefore $f$ satisfies condition ($\tilde{K}_{\Sigma}$). This completes the proof of Theorem 5.3.

Remark 5.5. In the non-relative case $C^0$-sufficiency of $r$-jets in $E_{[\tau]}(n,1)$ is equivalent to $V$-sufficiency of $r$-jets in $E_{[\tau]}(n,1)$. The Bochnak-Lojasiewicz inequality takes a very important role in the proof of the equivalence. Therefore it may be natural to ask whether the Bochnak-Lojasiewicz inequality holds also in the relative case. More precisely, if we let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ a $C^\omega$ function germ, then we ask whether the following inequality

$$d(x, \Sigma)\|\text{grad} f(x)\| \gtrsim |f(x)|$$

holds in a neighbourhood of $0 \in \mathbb{R}^n$.

If this Bochnak-Lojasiewicz inequality holds in the relative case, then it follows that the relative Kuiper-Kuo condition ($K-K_{\Sigma}$) and condition ($\tilde{K}_{\Sigma}$) are equivalent like in the non-relative case. But we give an example below to show that conditions ($K-K_{\Sigma}$) and ($\tilde{K}_{\Sigma}$) are not necessarily equivalent in the relative case. As a result, we can see that the Bochnak-Lojasiewicz inequality does not always hold in the relative case, and it follows from Theorems 4.1, 5.2 that $\Sigma$-V-sufficiency of $r$-jets in $E_{[\tau]}(n,1)$ does not always imply $\Sigma$-$C^0$-sufficiency of $r$-jets in $E_{[\tau]}(n,1)$.

Example 5.6. Let us recall the situation in Example 4.7. Namely, $f_m(x,y) = x^3 - 3xy^m$, $m \geq 3$, and $\Sigma = \{x = 0\}$. Let $r = 3$. In this setting, the relative Kuiper-Kuo condition is

$$\|\text{grad} f_m(x,y)\| \gtrsim |x|^{3-1}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. But as seen in Example 4.7, the above inequality does not hold along an analytic arc $\lambda(t) = (t^m, t^2)$ for $m \geq 3$. In other words, the relative Kuiper-Kuo condition ($K-K_{\Sigma}$) is not satisfied. Therefore, by Theorem 4.1, $j^3f_m(0;0)$ is not $\Sigma$-$C^0$-sufficiency in $E_{[\Sigma]}(2,1)$. 
On the other hand, condition \((\widetilde{K}_\Sigma)\) is
\[ |x||\text{grad} f_m(x, y)|| + |f_m(x, y)| \gtrsim |x|^3 \]
in a neighbourhood of \((0, 0) \in \mathbb{R}^2\). We show that \(f_m, m \geq 3\), satisfies this condition. Let
\[ \lambda(t) = (a_k t^k + \cdots, b_s t^s + \cdots) \]
be an analytic arc passing through \((0, 0) \in \mathbb{R}^2\) as in Example 4.7. Then we may assume \(a_k \neq 0\), and \(|x| \approx |t|^k\).

In the case where \(2k < ms\), we have
\[ |x||\text{grad} f_m(x, y)|| \geq 3|x|\cdot|x^2 - y^m| \geq |x||x| \approx |t|^k |t|^{2k} = |t|^{3k} \]
on \(\lambda\) near \((0, 0) \in \mathbb{R}^2\).

In the case where \(2k > ms\), we have
\[ |x||\text{grad} f_m(x, y)|| \geq 3|x|\cdot|x^2 - y^m| \gtrsim |x||x| \approx |t|^k |t|^{2k} = |t|^{3k} \]
on \(\lambda\) near \((0, 0) \in \mathbb{R}^2\).

In the case where \(2k = ms\) and \(a_k \neq b_s\), we have
\[ |x||\text{grad} f_m(x, y)|| \geq 3|x|\cdot|x^2 - y^m| \gtrsim |x||x| \approx |t|^k |t|^{2k} = |t|^{3k} \]
on \(\lambda\) near \((0, 0) \in \mathbb{R}^2\).

In the case where \(2k = ms\) and \(a_k = b_s\), we have
\[ |f_m(x, y)| = |x^3 - 3xy^m| = |x||x^2 - 3y^m| \gtrsim |x||x| \approx |t|^k |t|^{2k} = |t|^{3k} \]
on \(\lambda\) near \((0, 0) \in \mathbb{R}^2\).

On any analytic arc \(\lambda\), condition \((\widetilde{K}_\Sigma)\) is satisfied. Therefore we can see that \(f_m, m \geq 3\), satisfies condition \((\widetilde{K}_\Sigma)\). It follows that conditions \((K-K_\Sigma)\) and \((\widetilde{K}_\Sigma)\) are not necessarily equivalent in the relative case. In addition, by Theorem 5.2 we see that \(j^3 f_m(\Sigma; 0)\) is \(\Sigma\)-\(V\)-sufficient in \(E_{[3]}(2, 1)\) for any \(m \geq 3\).

Incidentally, the Bochnak-Łojasiewicz inequality does not hold along an analytic arc \(\lambda(t) = (t^m, t^2)\) for \(m \geq 3\).

As a corollary of the proofs of Theorems 5.2 and 5.3, we have the following.

**Corollary 5.7.** Let \(r\) be a positive integer, and let \(f \in E_{[r]}(n, p)\) such that \(j^r f(\Sigma, 0)\) is \(\Sigma\)-\(V\)-sufficient in \(E_{[r]}(n, p)\).

1) if \(n > p\), then for any \(C^r\) realisation \(g\) of \(j^r f(\Sigma, 0)\), \(g^{-1}(0) \setminus \Sigma\) is a germ of \(C^r\) submanifold of codimension \(p\) at \(0\) or empty.

2) if \(n = p\), \(j^r f(\Sigma; 0)\) has a subanalytic \(C^r\)-realisation and \(\Sigma\) is a germ at \(0 \in \mathbb{R}^n\) of a closed subanalytic subset of \(\mathbb{R}^n\), then for any \(C^r\) realisation \(g\) of \(j^r f(\Sigma, 0)\), the set-germs \((g^{-1}(0), 0)\) and \((f^{-1}(0), 0)\) are equal and are contained in \((\Sigma, 0)\).

**Remark 5.8.** It is well-known that the Kuiper-Kuo condition and \(V\)-sufficiency of jets are equivalent for function-germs. But, by Example 5.6 and Theorem 5.2 we can see that they are not always equivalent in the relative case.

**Remark 5.9.** It is worth to mention that if \(f \in E_{r}(n, p)\) and a subanalytic mapping then it has a subanalytic realisation in \(E_{q}(n, p)\) for any \(q \geq r\) (see [19]).
5.2. Relative $V$-sufficiency of $r$-jets in $C^{r+1}$ mappings. In this subsection we give some characterisations for the relative $r$-jets to be $\Sigma$-$V$-sufficient in $C^{r+1}$ mappings.

**Theorem 5.10.** Let $r$ be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$. If $f$ satisfies condition $(K^r_{\Sigma})$, then the relative $r$-jet, $j^r f(\Sigma; 0)$ is $\Sigma$-$V$-sufficient in $\mathcal{E}_{[r+1]}(n,p)$.

**Proof.** Because of the same reason as the theorem above, we may assume that $r \geq 2$.

Let $g \in \mathcal{E}_{[r+1]}(n,p)$ be an arbitrary mapping such that $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$. We define a $C^{r+1}$ mapping $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by $h(x) := g(x) - f(x)$. Then $j^r h(x) = 0$ for any $x \in \Sigma$.

Let $F(x,t) := f(x) + th(x)$ for $t \in I = [0,1]$. Since $j^r h = 0$ on $\Sigma$ near $0 \in \mathbb{R}^n$, by Lemma [2.5], $\|h(x)\| \lesssim d(x, \Sigma)^{r+1}$. Then there exists a small neighbourhood $T$ of $t_0$ in $I$ such that

$$
\|F(x,t) - F(x,t_0)\| \leq \bar{w} \, d(x, \Sigma)^{r+1}
$$

for any $t \in T$. Thus the zero-set $F(x,t) = 0$ is contained in

$$
(\mathcal{H}^\Sigma_{r+1}(F(x,t_0); \bar{w}) \cap \{\|x\| < \alpha\}) \times T,
$$

hence we will concentrate our attention to this set.

Moreover there are $\bar{w}$, $\alpha > 0$ such that

$$
\nu(d_x F(x,t_0)) = \nu(df(x) + t_0 dh(x)) \geq \nu(df(x)) - t_0 \|dh(x)\| \geq \frac{C}{2} d(x, \Sigma)^{r-\delta}
$$

in $\mathcal{H}^\Sigma_{r+1}(f; \bar{w}) \cap \{\|x\| < \alpha\}$. Then there exists $C' > 0$ such that

$$
(5.6) \quad \kappa(d_x F(x,t)) \geq C' d(x, \Sigma)^{r-\delta}
$$

for $(x,t) \in (\mathcal{H}^\Sigma_{r+1}(F(x,t_0); \bar{w}) \cap \{\|x\| < \alpha\}) \times T$. Set

$$
W := (\mathcal{H}^\Sigma_{r+1}(F(x,t_0); \bar{w}) \cap \{\|x\| < \alpha\}) \times T.
$$

Now we consider as in the proof of Theorem [5.1] the basis $\{N_1(x,t), \ldots N_p(x,t)\}$ of $V_{x,t}$ constructed as follows:

$$
N_j(x,t) = \text{grad}_x F_j(x,t) - \tilde{N}_j(x,t) \quad (1 \leq j \leq p),
$$

where $\tilde{N}_j(x,t)$ is the projection of $\text{grad}_x f_j(x,t)$ to the subspace $V_{x,t}^j$ spanned by the $\text{grad}_x F_k(x,t)$, $k \neq j$.

From the above we get, for any $j \in \{1, \ldots, p\}$ and $(x,t) \in W$;

$$
\|N_j(x,t)\| \geq \kappa(d_x F(x,t)) \geq C' d(x, \Sigma)^{r-\delta}.
$$

and then we use the same vector field of trivialisation as above

$$
X(x,t) = \begin{cases}
\frac{\partial}{\partial t} + \sum_{j=1}^p h_j(x) \frac{N_j(x,t)}{\|N_j(x,t)\|^2} & \text{if } (x,t) \in W \setminus \Sigma \times T \\
\frac{\partial}{\partial t} & \text{if } (x,t) \in W \cap \Sigma \times T.
\end{cases}
$$
Since
\[ \left\| \sum_{j=1}^{p} h_j(x) \frac{N_j(x, t)}{\|N_j(x, t)\|^2} \right\| \lesssim \frac{\|h(x)\|}{\|N_j(x, t)\|} \lesssim d(x, \Sigma)^{1+\delta}, \]
we use Proposition 2.27 to end the proof as in Theorem 5.5. \qed

**Example 5.14.** Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0), n \geq 3, \) be a polynomial function defined by
\[ f(x_1, x_2, \ldots, x_n) := x_1^3 - 3x_1x_2^5, \]
and \( \Sigma := \{ x_1 = x_2 = 0 \}. \) Then we have
\[ \text{grad} f_m(x, y) = (3(x_1^2 - x_2^3), -15x_1x_2^4), \]
and \( d(x, \Sigma) = \| (x_1, x_2) \|. \) From the computation in Example 4.7(1),
\[ \| \text{grad} f(x) \| \gtrsim d(x, \Sigma)^{7/2} \]
in a neighbourhood of \( 0 \in \mathbb{R}^n. \) Therefore, by Theorem 4.1(2), \( j^7 f(\Sigma; 0) \) is \( \Sigma \)-C\( ^0 \)-sufficient in \( \mathcal{E}_{[8]}(n, 1) \).

Now, since \( g(x) = x_1^3 - 3x_1x_2^5 + x_2^7 = (x_1 - x_2^7)^2(x_1 + 2x_2^7) \) is a realisation of the jet \( j^7f(\Sigma; 0) \) in \( \mathcal{E}_{[7]}(n, 1) \), which is not \( \Sigma \)-V-equivalent to \( f \); therefore \( j^7f(\Sigma; 0) \) is not \( \Sigma \)-V-sufficient in \( \mathcal{E}_{[7]}(n, 1) \). The proof can be carried out like in [11].

By Lemma 2.5 we have the following as a corollary of Theorem 5.10.

**Corollary 5.12.** Let \( r \) be a positive integer, and let \( f \in \mathcal{E}_{[r+1]}(n, p), n \geq p. \) If there exists \( \delta > 0 \) such that
\[ d(x, \Sigma)\kappa(df(x)) + \| f(x) \| \gtrsim d(x, \Sigma)^{r+1-\delta} \]
holds in some neighbourhood of \( 0 \in \mathbb{R}^n, \) then \( j^r f(\Sigma; 0) \) is \( \Sigma \)-V-sufficient in \( \mathcal{E}_{[r+1]}(n, p). \)

**Remark 5.13.** In the non-relative case \( C_0 \)-sufficiency of \( r \)-jets in \( \mathcal{E}_{[r+1]}(n, 1) \) is equivalent to \( V \)-sufficiency of \( r \)-jets in \( \mathcal{E}_{[r+1]}(n, 1), \) too. But this does not holds in the relative case, namely we give an example below to show that \( \Sigma \)-V-sufficiency of \( r \)-jets in \( C^{r+1} \) functions does not always imply \( \Sigma \)-C\( ^0 \)-sufficiency of \( r \)-jets in \( C^{r+1} \) functions, either.

**Example 5.14.** Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \) be a polynomial function defined by
\[ f(x, y) := (x - y^3)^2 + y^{10}, \]
and let \( \Sigma = \{ x = 0 \}. \) Then we have
\[ \text{grad} f(x, y) = (2(x - y^3), -6y^2(x - y^3) + 10y^9). \]
Let
\[ \lambda(t) = (a_k t^k + \cdots, b_k t^s + \cdots) \]
be an analytic arc passing through \( (0, 0) \in \mathbb{R}^2 \) as in Example 4.7. Then we may assume \( a_k \neq 0, \) and then \( |x| \approx |t|^k. \)

In the case where \( k < 3s, \) we have
\[ \| \text{grad} f(x, y) \| \gtrsim 2|x - y^3| \geq |x| \]
on λ near (0, 0) ∈ ℝ².

In the case where k > 3s, we have

\[\|\nabla f(x, y)\| \geq 2|x - y^3| \geq |y|^3 \geq |x|\]

on λ near (0, 0) ∈ ℝ².

In the case where k = 3s, |x| ≈ |y|^3. Therefore we have

\[|f(x)| = (x - y^3)^2 + y^{10} \approx |x|^{4 - \frac{2}{5}}\]

on λ near (0, 0) ∈ ℝ².

On any analytic arc λ,

\[|x||\nabla f(x, y)| + |f(x)| \gtrsim |x|^{4 - \frac{2}{5}}\]

holds near (0, 0) ∈ ℝ². Therefore the above inequality holds in a neighbourhood of (0, 0) ∈ ℝ². It follows from Corollary 5.12 that \(j^3 f(\Sigma; 0)\) is \(\Sigma\)-V-sufficient in \(E[4](2, 1)\).

Let \(\lambda(t) := (t^3, t)\). Then \(|x| = |t|^3 = |y|\) on λ. Therefore we have

\[\|\nabla f(x, y)\| = \|(0, 10t^9)\| = 10|t|^9 = 10|x|^3\]

on λ near (0, 0) ∈ ℝ². By Theorem 4.4(2), \(j^3 f(\Sigma; 0)\) cannot be \(\Sigma\)-\(C^0\)-sufficient in \(E[4](2, 1)\).

We gave a sufficient condition for the relative r-jets to be \(\Sigma\)-V-sufficient in \(C^{r+1}\) mappings. We next give a necessary condition.

**Definition 5.15** (Σ-Regular Horn neighbourhood). Let \(f \in E[\nu](n, p)\) and \(d \in \mathbb{N}\). We say that the horn neighbourhood of f, \(\mathcal{H}_d^\Sigma(f)\) is \(\Sigma\)-regular if for some \(w > 0\),

\[\kappa(df(x)) \gtrsim d(x, \Sigma)^{d-1}\]

for \(x \in \mathcal{H}_d^\Sigma(f; w), \ x \) near 0.

**Remark 5.16.** For germ \(f \in E[\nu](n, p)\), \(n \geq p\), the following conditions are equivalent:
1) \(\mathcal{H}_d^\Sigma(f)\) is \(\Sigma\)-regular
2) \(f\) satisfies condition (\(\mathcal{K}_{\Sigma}\)).

**Proposition 5.17.** Let \(r\) be a positive integer, and let \(f \in E[\nu+1](n, p)\), \(n \geq p\), such that the relative r-jet \(j^r f(\Sigma; 0)\) is \(\Sigma\)-V-sufficient in \(E[\nu+1](n, p)\). Then for any realisation \(g\) of \(j^r f(\Sigma; 0)\) in \(E[\nu+1](n, p)\), the horn neighbourhood \(\mathcal{H}_d^{\Sigma}(g)\) is \(\Sigma\)-regular.

**Proof.** If not, then we can find a realisation \(\bar{g}\) of \(j^r f(\Sigma; 0)\) in \(E[\nu+1](n, p)\), a sequence \((x_{\nu})_{\nu \geq 1}\) of points of \(\mathbb{R}^n \setminus \Sigma\) converging to 0 ∈ \(\mathbb{R}^n\) such that

\[d(x_{\nu}, \Sigma)\kappa(d\bar{g}(x_{\nu})) + \|\bar{g}(x_{\nu})\| = o(d(x_{\nu}, \Sigma)^{r+1}).\]  

(5.7)

Extracting a subsequence from \((x_{\nu})_{\nu \geq 1}\) if necessary, one can assume that

\[\|x_{\nu+1}\| < \frac{1}{3}d(x_{\nu}, \Sigma)\]

which implies, in particular, that \(d(x_{\nu}, \Sigma)\) decreases, and that condition (5.7) implies:
1) \(|\bar{g}_k(x_{\nu})| = o(d(x_{\nu}, \Sigma)^{r+1})\), for all \(1 \leq k \leq p\);
2) $\delta_\nu = o(d(x_\nu, \Sigma)^r)$ where
$$\delta_\nu := \kappa(d(g(x_\nu))) = \text{dist} (\text{grad} g_j(x_\nu), \sum_{k \neq j} \mathbb{R} \text{grad} g_k(x_\nu)) \text{ for some } j, 1 \leq j \leq p.$$ 

Now adapting the proofs of Theorem 5.2 and Theorem 5.3, we first notice that the germ $\eta$, in this case satisfies the inequalities
\begin{align*}
|\eta_1(x)| & \leq K(|\tilde{g}_1(x_\nu)| + \epsilon_\nu \|x - x_\nu\|^2) \lesssim o(d(x_\nu, \Sigma)^{r+1}) + \epsilon_\nu d(x_\nu, \Sigma)^2, \\
|\eta_k(x)| & \leq K(|\tilde{g}_k(x_\nu)| + \|\lambda^{(k)}\|\|x - x_\nu\|) \lesssim o(d(x_\nu, \Sigma)^{r+1}), \quad k = 2, \ldots, p.
\end{align*}

Then for a suitable choice of the sequence $(\epsilon_\nu)_{\nu \geq 1}$, $(\epsilon_\nu = o(d(x_\nu, \Sigma)^{r+1}))$, we have $\eta(x) = o(d(x, \Sigma)^{r+1})$, and then $g = \tilde{g} - \eta$ is a $C^{r+1}$-realisation of $j^*f(\Sigma; 0)$. We carry on the same argument to contruct in each cases, $n > p$ and $n = p$, a realisation which contradict Lemma 5.3.

\[ \square \]

6. Rigidity and Relative SV-determinacy

Let $E(n)^p$, $n \geq p$, be the set of $C^\infty$ map-germs : $\mathbb{R}^n \to \mathbb{R}^p$ at 0 $\in \mathbb{R}^n$, and let $\Sigma$ be a germ of closed subset of $\mathbb{R}^n$ such that 0 $\in \Sigma$. We say that $f \in E(n)^p$ is finitely $\Sigma$-SV-determined (resp. finitely $\Sigma$-V-determined) if there is a positive integer $k$ such that for any $g \in E(n)^p$ with $j^k g(\Sigma; 0) = j^k f(\Sigma; 0)$, $g$ is $\Sigma$-SV-equivalent (resp. $\Sigma$-V-equivalent) to $f$. Concerning finite $SV$-determinacy or finite $V$-determinacy in the non-relative case, lots of characterisations have been obtained (see J. Bochnak - T.-C. Kuo \[2\]).

Let $\varphi = (\varphi_1, \ldots, \varphi_p) : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, be a $C^\infty$ map-germ at 0 $\in \mathbb{R}^n$. We denote by $I_K(\varphi)$ the ideal of $E(n)$ generated by $\varphi_1, \ldots, \varphi_p$ and the Jacobian determinants
$$\frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \quad (1 \leq i_1 < \ldots < i_p \leq n),$$

and we let
$$Z(\varphi, x) := \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \right|^2 + \sum_{j=1}^p \varphi_j(x)^2.$$

In the case where $n > p$, we define also the ideal of $E(n)$, denoted by $I_T(\varphi)$, generated by $\varphi_1, \ldots, \varphi_p$ and the Jacobian determinants
$$\frac{D(\varphi_1, \ldots, \varphi_p, \rho)}{D(x_{i_1}, \ldots, x_{i_p+1})}(x) \quad (1 \leq i_1 < \ldots < i_{p+1} \leq n).$$

In the case where $n = p$, we define the ideal $I_T(\varphi)$ of $E(n)$, as the ideal generated by only $\varphi_1, \ldots, \varphi_p$.

Recall that for $1 \leq s \leq \infty$, $m^s_\Sigma$ is the ideal of $E(n)$ of germs of $(s-1)$-flat functions at $\Sigma$, namely
$$m^s_\Sigma = \{ f \in E(n) : j^{s-1}f(\Sigma; 0) = 0 \}.$$

Therefore we have, $m^\infty_\Sigma = \cap_{s=1}^\infty m^s_\Sigma$. 

\[ \square \]
Let $r$ be a positive integer, and let \( \varphi = (\varphi_1, \ldots, \varphi_p) : \mathbb{R}^n \to \mathbb{R}^p \), \( n \geq p \), be a $C^r$ map-germ at \( 0 \in \mathbb{R}^n \). We denote by \( \mathcal{E}_{[r]}(n) \) be the ring of $C^r$ function-germs : \( \mathbb{R}^n \to \mathbb{R}^p \) at \( 0 \in \mathbb{R}^n \), and by \( \mathcal{E}_{[r]}(n)(\varphi) \) the ideal of \( \mathcal{E}_r(n) \) generated by \( \varphi_1, \ldots, \varphi_p \).

**Definition 6.1.** We call \( \varphi \in \mathcal{E}(n)^p \) $\Sigma$-C$^r$-rigid if there is a positive integer \( k \) for which the following holds:

for any \( \psi \in \mathcal{E}(n)^p \) such that \( j^k \varphi = j^k \psi \) on \( \Sigma \), there exists \( \tau \in \mathcal{R}_{\Sigma}^{\text{fix}} \) such that \( \mathcal{E}_{[r]}(n)(\varphi \circ \tau) = \mathcal{E}_{[r]}(n)(\psi) \).

**Definition 6.2.** Let \( I \) be an ideal of \( \mathcal{E}(n) \). We say that \( I \) is $\Sigma$-elliptic if there is \( f \in I \) such that

\[ |f(x)| \geq Cd(x, \Sigma)^\alpha \]

in a neighbourhood of \( 0 \), where \( C \) and \( \alpha \) are positive constants. We call such \( f \) an elliptic element of \( I \).

**Remark 6.3.** If the ideal \( I \) is $\Sigma$-elliptic and generated by \( f_1, \ldots, f_k \), then \( f_1^2 + \ldots + f_k^2 \) is an elliptic element of \( I \).

We have the following Lemma, which is a slight modification of a result of J.-C. Tougeron and J. Merrien \[29\]. We give the proof for completeness.

**Lemma 6.4.** Let \( I \) be a finitely generated ideal of \( \mathcal{E}(n) \). Then the following conditions are equivalent:

1. \( I \) is $\Sigma$-elliptic.
2. \( m_\Sigma^\infty \subset m_\Sigma^\infty I \)
3. \( \mathcal{R}_{\Sigma}^{\text{fix}} \subset I \)

**Proof.** Let \( f_1, \ldots, f_l \) be the generators of \( I \).

We first show the implication (1) $\Rightarrow$ (2). Let \( f = f_1^2 + \ldots + f_l^2 \). By Leibniz formula and the assumption on \( I \), in a neighbourhood of \( 0 \), we have: for each multi-index \( \beta \) there exists \( C_\beta > 0 \) such that

\[ \left| \frac{\partial^{\beta}(1/f)(x)}{\partial x^\beta} \right| \leq \frac{C_\beta}{|f(x)|(|\beta|+1)^r}, \]

and there exists \( C > 0 \) such that

\[ |f(x)| \geq Cd(x, \Sigma)^\alpha. \]

Now, by Proposition IV 4.2 of \[29\], for any \( \varphi \in m_\Sigma^\infty \), \( \varphi \neq f \in m_\Sigma^\infty \). It follows that \( \varphi \in m_\Sigma^\infty I \).

The implication (2) $\Rightarrow$ (3) is obvious.

We lastly show the implication (3) $\Rightarrow$ (1). Suppose that \( I \) is not $\Sigma$-elliptic. Then we can construct a sequence of points \( x_k \in \mathbb{R}^n \setminus \Sigma \), converging to \( 0 \in \mathbb{R}^n \) such that

\[ \sum_{i=1}^l |f_i(x_k)| \leq d(x_k, \Sigma)^{k+1}. \]

Taking a subsequence if necessary, we may assume that the balls \( B_k = B(x_k, \frac{1}{2}d(x_k, \Sigma)) \) are all disjoints.
Let \( g_k \in \mathcal{E}(n) \) such that \( g_k(x_k) = 1 \) and \( g_k = 0 \) on the complement of \( B_k \) and satisfying:

for each multi-index \( \beta \) there exists a positive constant \( C_\beta \) such that on \( B_k \)

\[
\left| \frac{\partial^{\beta} g(x)}{\partial x^\beta} \right| \leq \frac{C_\beta}{d(x_k, \Sigma)^{\|\beta\|}}.
\]

Then \( \sum_{k \in \mathbb{N}} g_k d(x_k, \Sigma)^k \) converges to a function \( g \in m^\infty_{\Sigma} \).

By the assumption (3) we have \( g \in I \). Then it follows that there exists \( C > 0 \) such that \( |g(x_k)| \leq C \sum_{i=1}^k |f_i(x_k)| \). Therefore we have \( d(x_k, \Sigma)^k \leq Cd(x_k, \Sigma)^{k+1} \), which is impossible. This is a contradiction. Thus \( I \) is \( \Sigma \)-elliptic \( \square \)

As a consequence we have the following proposition:

**Proposition 6.5.** For \( \varphi \in \mathcal{E}(n)^p \), the following conditions are equivalent:

1. There exist \( C, \alpha, \beta > 0 \) such that \( Z(\varphi, x) \geq Cd(x, \Sigma)^\alpha \) for \( |x| < \beta \).
2. \( m^\infty_{\Sigma} \subset I_K(\varphi) \).
   
   If moreover \( \Sigma \) is subanalytic and \( \varphi \) is analytic, they are also equivalent to:
3. \( m^\infty_{\Sigma} \subset I_T(\varphi) \).
4. The set germ at 0, \( \text{Sing}(\varphi) \cap \varphi^{-1}(0) \), is contained in \( \Sigma \).

**Proof.** The equivalence between (1), (2) and (3) follows from Theorem 3.1 and Lemma 6.4; (1) implies (4) trivially and the converse is the inequality of Lojasiewicz, since \( \Sigma \) is subanalytic, \( Z(\varphi, x) \) is analytic and \( \{Z(\varphi, x) = 0\} = \text{Sing}(\varphi) \cap \varphi^{-1}(0) \). \( \square \)

**Definition 6.6.** A germ of closed subset \( \Sigma \) of \( \mathbb{R}^n \) is called coherent if \( m_{\Sigma} \) is a finitely generated ideal of \( \mathcal{E}(n) \).

This definition is inspired by the following result of W. Kucharz proved in [13]: an analytic and semi-algebraic subset \( X \) in an open subset \( U \) of \( \mathbb{R}^n \) is coherent if and only if \( m_X \) is a finitely generated ideal of \( \mathcal{E}(n) \). In particular, \( \Sigma = \{0\} \) is coherent.

Let us give a generalisation of the Bochnak-Kuo theorem in [2] as follows.

**Theorem 6.7.** Let \( \Sigma \) be a coherent germ of closed subset of \( \mathbb{R}^n \) such that \( 0 \in \Sigma \). Then the following conditions are equivalent for \( \varphi \in \mathcal{E}(n)^p \) where \( n > p \):

1. For each \( r \in \mathbb{N} \), \( \varphi \) is \( \Sigma-C^r \)-rigid.
2. \( \varphi \) is finitely \( \Sigma \)-SV-determined.
3. \( \varphi \) is finitely \( \Sigma \)-V-determined.
4. \( I_K(\varphi) \) is \( \Sigma \)-elliptic.
5. \( m^\infty_{\Sigma} \subset I_K(\varphi) \).
   
   If moreover \( \varphi \) is analytic, they are also equivalent to:
6. \( m^\infty_{\Sigma} \subset I_T(\varphi) \).

**Proof.** The implications (1) \( \implies \) (2) \( \implies \) (3) are obvious by definition, and the equivalence (4) \( \iff \) (5) follows from Lemma 6.4. Concerning the equivalence (5) \( \iff \) (6) in the analytic case, see Proposition 6.5.
We first show the implication (5) \(\implies\) (1), namely we will show that \(m_\Sigma^2 \subseteq I_K(\varphi)\) implies that for any \(r \in \mathbb{N}\), there exists \(s \in \mathbb{N}\) such that \(m_\Sigma^2 \subseteq \mathcal{E}_{[r+1]}(n)(I_K(\varphi))\). Let \(\{f_1, \ldots, f_k\}\) be a system of generators of \(m_\Sigma\).

Since condition (5) is equivalent to condition (4), for \(s\) large enough, \(\frac{f_i^s}{Z(\varphi, x)}\) is of class \(C^{r+1}\) for any \(i \in \{1, \ldots, k\}\). Hence \(f_i^s \in \mathcal{E}_{[r+1]}(n)(I_K(\varphi))\) and then \(m_\Sigma^{(s-1)k+1} \subseteq \mathcal{E}_{[r+1]}(n)(I_K(\varphi))\). We set \(q := (s-1)k + 1\).

We now show that \(j^{2q}(\varphi)\) is \(\Sigma\)-\(C^{r+1}\)-rigid in \(\mathcal{E}(n)^p\). Let \(\psi \in \mathcal{E}^p\) be any element with \(j^{2q}(\psi) = j^{2q}(\varphi)\) on \(\Sigma\). Then \(\mathcal{E}(n)(\varphi - \psi) \subseteq m_\Sigma^{2q+1}\), hence
\[
(6.1) \quad \mathcal{E}_{[r+1]}(n)(\varphi - \psi) \subseteq \mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(I_K(\varphi)))^2.
\]

We first remark that
\[
\mathcal{E}_{[r+1]}(n)(I_K(\varphi)) = \mathcal{E}_{[r+1]}(n)(\varphi) + \mathcal{E}_{[r+1]}(n)(J(\varphi)),
\]
where \(J(\varphi)\) is the Jacobian ideal of \(\varphi\). From (6.1), we have
\[
\mathcal{E}_{[r+1]}(n)(I_K(\varphi)) = \mathcal{E}_{[r+1]}(n)(\psi) + \mathcal{E}_{[r+1]}(n)(J(\varphi)) + \mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(I_K(\varphi)))^2
\]
and by Nakayama’s Lemma we obtain:
\[
\mathcal{E}_{[r+1]}(n)(\psi) + \mathcal{E}_{[r+1]}(n)(J(\varphi)) = \mathcal{E}_{[r+1]}(n)(I_K(\varphi)).
\]

Thus there exist
\[
\varphi_1 \in \mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(J(\varphi)))^2 \text{ and } \psi_1 \in \mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(\psi))
\]
such that \(\varphi - \psi = \psi_1 - \varphi_1\).

For \(x\) and \(y\) in \((\mathbb{R}^n, 0)\), we define \(F\) by
\[
F(x, y) := \varphi(x + y) - \varphi(x) - \varphi_1(x).
\]
Since, for \(i = 1, \ldots, n\), \(F_i(x, 0)\) belongs to \(\mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(J(\varphi)))^2\), by Tougeron’s Implicit Function Theorem, there is a map \(g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\) with components in \(\mathcal{E}_{[r+1]}(n)(m_\Sigma)(\mathcal{E}_{[r+1]}(n)(J(\varphi)))\) such that \(F(x, g(x)) = 0\). Let \(\tau(x) = x + g(x)\). Clearly \(\tau\) is a germ of diffeomorphism at the origin, and it coincides with the identity on \(\Sigma\), namely \(\tau \in \mathcal{R}_\Sigma^{kx}\). Now, by construction, we have
\[
\varphi(\tau(x)) = \varphi(x) + \varphi_1(x) = \psi(x) + \psi_1(x).
\]
Thus \(\mathcal{E}_{[r+1]}(n)(\varphi \circ \tau) = \mathcal{E}_{[r+1]}(n)(\psi)\), namely \(\varphi\) is \(\Sigma\)-\(C^r\)-rigid.

It remains to prove the implication (3) \(\implies\) (4). Let \(\varphi\) be a finitely \(\Sigma\)-\(V\)-determined germ. We suppose that the germ at \(0 \in \mathbb{R}^n\) of \(\varphi^{-1}(0)\) \(\setminus \Sigma\) is not empty. Let \(r \in \mathbb{N}\) such that \(\varphi\) is \(\Sigma\)-\(V\)-determined at degree \(r\). Then, using a similar argument to the proof of Lemma 5.4, we can see that for all map \(g \in \mathcal{E}_{[\infty]}(n, p)\) with \(j^r g(\Sigma; 0) = j^r \varphi(\Sigma; 0)\) the germ of \(g^{-1}(0)\) \(\setminus \Sigma\) at \(0 \in \mathbb{R}^n\) is not empty and a topological manifold of \(\mathbb{R}^n\) of codimension \(p\).

We suppose that, on the contrary, condition (4) is not satisfied. One can then find a sequence \((x_\nu)_{\nu \geq 1}\) of points of \(\mathbb{R}^n\) converging to \(0 \in \mathbb{R}^n\) and such that
\[
(6.2) \quad Z(\varphi, x_\nu) = o(d(x_\nu, \Sigma)^s) \text{ for any } s \in \mathbb{N} \text{ and any } \nu \geq 1.
\]
Extracting a subsequence if necessary, one can assume that \( \|x_{\nu+1}\| < \frac{1}{3}d(x_{\nu}, \Sigma) \), which implies, in particular, that \( d(x_{\nu}, \Sigma) \) decreases.

As in the proofs of Theorems \[5.2\] and \[5.3\] we may assume that

\[
\kappa(df(x_{\nu})) = d(\text{grad}\varphi_1(x_{\nu}), \sum_{k=2}^{p} \text{grad}\varphi_k(x_{\nu})).
\]

Let

\[
\delta_{\nu} := d(\text{grad}\varphi_1(x_{\nu}), \sum_{k=2}^{p} \text{grad}\varphi_k(x_{\nu})),
\]

Since

\[
Z(\varphi, x_{\nu}) = \sum_{k=1}^{p} \varphi_k^2(x_{\nu}) + \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left| \frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})} \right|^2 \geq \sum_{k=1}^{p} \varphi_k^2(x_{\nu}) + \delta_{\nu}^2,
\]

condition \[6.2\] implies:
1) \( |\varphi_k(x_{\nu})| = o(d(x_{\nu}, \Sigma)^s) \), for all \( s \in \mathbb{N} \) and \( 1 \leq k \leq p \);
2) \( \delta_{\nu} = o(d(x_{\nu}, \Sigma)^s) \), for all \( s \in \mathbb{N} \).

Now we apply the Bochnak-Kuo Lemma in \[2\] (see Lemma \[2.29\] with Remark \[2.30\] with \( u_{\nu}^{(k)} = \text{grad}\varphi_k(x_{\nu}) \) and \( \alpha_{\nu} = d(x_{\nu}, \Sigma) \), to find for each \( \nu \in \mathbb{N} \), \( p - 1 \) vectors, \( \lambda_{\nu}^{(2)}, \ldots, \lambda_{\nu}^{(p)} \in \mathbb{R}^n \) such that:
(a) \( \|\lambda_{\nu}^{(k)}\| = o(d(x_{\nu}, \Sigma)^s) \), \( k = 2, \ldots, p \);
(b) \( \text{grad}\varphi_2(x_{\nu}) + \lambda_{\nu}^{(2)}, \ldots, \text{grad}\varphi_p(x_{\nu}) + \lambda_{\nu}^{(p)} \) are linearly independant in \( \mathbb{R}^n \);
(c) \( \text{grad}\varphi_1(x_{\nu}) \in \sum_{k=2}^{p} \mathbb{R} (\text{grad}\varphi_2(x_{\nu}) + \lambda_{\nu}^{(2)}). \)

Let \( \psi : \mathbb{R} \to [0, 1] \) be a \( C^\infty \) function such that \( \psi(t) = 1 \) in a neighbourhood of \( 0 \in \mathbb{R} \) and \( \psi(t) = 0 \) for \( |t| \geq \frac{1}{4} \). Let \( f : \mathbb{R}^n \to \mathbb{R} \) a smooth flat function such that \( f^{-1}(0) = \Sigma \). We define a germ \( \eta = (\eta_1, \ldots, \eta_p) \) by:

\[
\eta_1(x) = \sum_{\nu=1}^{\infty} \psi \circ f \left( \frac{x - x_{\nu}}{\|x_{\nu}\|} \right) (\varphi_1(x_{\nu}) + \epsilon_{\nu} \|x - x_{\nu}\|^2),
\]

\[
\eta_k(x) = \sum_{\nu=1}^{\infty} \psi \circ f \left( \frac{x - x_{\nu}}{\|x_{\nu}\|} \right) (\varphi_k(x_{\nu}) - \langle \lambda_{\nu}^{(k)}, x - x_{\nu} \rangle), \quad k = 2, \ldots, p.
\]

Then
i) If we choose \( \epsilon_{\nu} > 0 \) such that for each \( s \in \mathbb{N} \), \( \epsilon_{\nu} = o(d(x_{\nu}, \Sigma)^s) \), then \( \eta = (\eta_1, \ldots, \eta_p) \) is of class \( C^\infty \);
ii) \( \eta \) is flat on \( \Sigma \);
iii) For each \( \nu \in \mathbb{N} \), \( (\varphi - \eta)(x_{\nu}) = 0 \).

Let \( g = \varphi - \eta \). We shall show that we can choose \( \epsilon_{\nu}, \epsilon_{\nu} = o(d(x_{\nu}, \Sigma)^s) \), such that near each \( x_{\nu} \), \( g^{-1}(g(x_{\nu})) = g^{-1}(0) \) is not a topological manifold of codimension \( p \), if
By condition (b), there is a small neighbourhood $V_\nu$ of $x_\nu$, such that the set

$$M_\nu = \{ x \in V_\nu : \varphi_k(x) - \eta_k(x) = 0, \ k = 2, \ldots, p\}$$

is a smooth manifold of codimension $p-1$.

From condition (c), for each $\nu \in \mathbb{N}$, there are real numbers $a_{2,\nu}, \ldots, a_{p,\nu}$ such that,

$$\text{grad}\varphi_1(x_\nu) = \sum_{k=2}^{p} a_{k,\nu}(\text{grad}\varphi_k(x_\nu) + \lambda^{(k)}_\nu).$$

Choose now $\epsilon_\nu = o(d(x_\nu, \Sigma)^s)$ such that $x_\nu$ is a non-degenerate critical point of

$$h_\nu(x) = \varphi_1(x) - \eta_1(x) + \sum_{k=2}^{p} a_{k,\nu}(\eta_k(x) - \varphi_k(x)).$$

Then

$$g^{-1}(0) \cap V_\nu = \{ x \in V_\nu : g_1(x) = g_2(x) = \ldots = g_p(x) = 0\}$$

$$= \{ x \in M_\nu : h_\nu(x) = 0\}.$$

By the choice of $\epsilon_\nu$, this set is the intersection of the locus of a non-degenerate quadratic form $h_\nu^{-1}(0)$ with a codimension $p-1$ manifold $M_\nu$. Then if it is a topological manifold, necessarily it is a point. Now if $n - p \geq 1$, $g^{-1}(0)$ cannot be a topological manifold of codimension $p$. This is a contradiction. Thus the implication $(3) \implies (4)$ is shown. \hfill \Box

**Theorem 6.8.** Let $\Sigma$ be a coherent subanalytic germ of closed subset at $0 \in \mathbb{R}^n$. Then the following conditions are equivalent for analytic germ $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$:

1. For each $r \in \mathbb{N}$, $\varphi$ is $\Sigma$-$C^r$-rigid.
2. $\varphi$ is finitely $\Sigma$-SV-determined.
3. $\varphi$ is finitely $\Sigma$-V-determined.
4. $I_K(\varphi)$ is $\Sigma$-elliptic.
5. $\mathfrak{m}_0^\Sigma \subset I_K(\varphi)$.
6. $\mathfrak{m}_0^\Sigma \subset I_T(\varphi)$.

**Proof.** The proofs are the same as in the previous theorem except for the implication $(3) \implies (4)$, where we conclude by the following: since $n = p$ and the set-germ $f^{-1}(0) \setminus \Sigma$ is not empty, this contradicts the $\Sigma$-V-sufficiency (see Corollary 5.7). \hfill \Box

### 7. Relative $K$ equivalence

Let $E(n)$ be the local ring of germs of $C^\infty$ functions $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ with maximal ideal $m_n$. For a germ of closed subset $\Sigma$ of $\mathbb{R}^n$ such that $0 \in \Sigma$, we suppose moreover that $\Sigma$ is coherent (see 6.3). We now generalise Mather’s notion of contact equivalence (see [23]); we say that two map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are $K$-equivalent if there exists a germ of diffeomorphism $h \in R^{fix}_\Sigma$ and a $C^\infty$ germ $A : (\mathbb{R}^n, 0) \to GL(\mathbb{R}, p)$ such that $f = A \cdot g \circ h$; here $\cdot$ denotes matrix multiplication of $A$ by the vector-valued function $g \circ h$ in $\mathbb{R}^p$. Now let $f = (f_1, \ldots, f_p)$ be an
element in $m_n \mathcal{E}(n)^p$, let $J_{\Sigma}(f)$ be the ideal in $\mathcal{E}(n)$ generated by $f_1, \ldots, f_p$ and let
\[
\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle
\]
be the $\mathcal{E}(n)$ submodule of $\mathcal{E}(n)^p$ generated by the partial derivatives $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n$. In analogy with the $\mathcal{K}$ tangent space of a map germ (see [23]), we define the $\mathcal{E}(n)$ submodule of $\mathcal{E}(n)^p$:
\[
TK_{\Sigma}(f) := m_{\Sigma}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) + J_{\Sigma}(f)\mathcal{E}(n)^p.
\]

Proposition 7.1. A necessary and sufficient condition for $f$ to be $\mathcal{K}_{\Sigma}$ determined by a finite jet (resp. $\infty$-jet) is that for some $k < \infty$, $m_k^\infty \mathcal{E}(n)^p \subset TK_{\Sigma}f$ (resp. $m_k^\infty \mathcal{E}(n)^p \subset TK_{\Sigma}f$).

Proof. Assume that $f \in \mathcal{E}(n)^p$ satisfies the condition $m_k^\infty \mathcal{E}(n)^p \subset TK_{\Sigma}f$, denoted by (t) as in [23]. Let $h \in m_k^\infty \mathcal{E}(n)^p$. It is clear that we only have to prove that $f$ and $f + h$ are $\mathcal{K}_{\Sigma}$ equivalent. Define $F : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ by $F(t, x) = f(x) + th(x).

Lemma 7.2. If $f$ satisfies condition (t) then we can find a germ of a smooth vector field $X$ around $(0,0)$ in $\mathbb{R} \times \mathbb{R}^n$ of the following form:
\[
X = \frac{\partial}{\partial t} + \sum_{j=1}^n X_j(t, x) \frac{\partial}{\partial x_j}
\]
where $X_j(t, x) = 0$ for $x \in \Sigma$ and such that:
\[
DF(X) \in J_{\Sigma}(F)\mathcal{E}(n+1)^p.
\]

Proof. From the coherency condition, the following $\mathcal{E}(n)$ module
\[
m_{\Sigma}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) + J_{\Sigma}(f)\mathcal{E}(n)^p = TK_{\Sigma}f
\]
is finitely generated. Considering $\mathcal{E}(n)$ as a subset of $\mathcal{E}(n+1)$ it follows from condition (t) that:
\[
m_{n+1}m_k^\infty \mathcal{E}(n+1)^p \subset m_{n+1}\left(m_{\Sigma}(n+1)\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle + J_{\Sigma}(f)\mathcal{E}(n+1)^p \right).
\]
Since $F - f \in m_{n+1}m_k^\infty \mathcal{E}(n)^p$, this implies that
\[
m_{\Sigma}(n+1)\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \rangle + J_{\Sigma}(F)\mathcal{E}(n+1)^p \subseteq m_{\Sigma}(n+1)\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle + J_{\Sigma}(f)\mathcal{E}(n+1)^p.
\]
On the other hand we have
\[
m_{\Sigma}(n+1)\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle + J_{\Sigma}(f)\mathcal{E}(n+1)^p
\]
\[
\subseteq m_{\Sigma}\mathcal{E}(n+1)\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \rangle + J_{\Sigma}(F)\mathcal{E}(n+1)^p + m_{n+1}m_k^\infty \mathcal{E}(n+1)^p
\]
\[
\subseteq m_{\Sigma}\mathcal{E}(n+1)\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \rangle + J_{\Sigma}(F)\mathcal{E}(n+1)^p
\]
\[
+ m_{n+1}\left(m_{\Sigma}(n+1)\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle + J_{\Sigma}(f)\mathcal{E}(n+1)^p \right).
\]
Hence Nakayama’s Lemma gives that
\[ m_\Sigma \mathcal{E}(n+1) \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} + J_C(f) \mathcal{E}(n+1)^p = m_\Sigma \mathcal{E}(n+1) \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} + J_C(F) \mathcal{E}(n+1)^p. \]

From condition (t) it follows that
\[ h \in m_\Sigma^\infty \mathcal{E}(n+1)^p \subseteq m_\Sigma \mathcal{E}(n+1) \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} + J_C(F) \mathcal{E}(n+1)^p. \]

This shows that we can find germs \( X_j \in m_\Sigma \mathcal{E}(n+1) \) such that
\[ h + \sum_{j=1}^n X_j \frac{\partial F}{\partial x_j} \in J_C(F) \mathcal{E}(n+1)^p. \]

Then \( X(t, x) = \frac{\partial}{\partial t} + \sum_{j=1}^n X_j(t, x) \frac{\partial}{\partial x_j} \) satisfies the conditions of this lemma. \( \square \)

Now, integrate the vector field \( X \) in the lemma above around \((0, 0)\) in \( \mathbb{R} \times \mathbb{R}^n \). We get a family \( \{ h_t \} \) of diffeomorphisms \((\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\) such that \( h_t|_\Sigma = Id \). The condition \( DF(X) \in J_C(F) \mathcal{E}(n+1)^p \) gives that we can find a \( p \times p \) matrix \( A(t, x) \) with entries in \( \mathcal{E}(n+1) \) such that
\[ \frac{d}{dt} F(t, h_t(x)) = DF(t, h_t(x)) X(t, h_t(x)) = A(t, h_t(x)) \cdot F(t, h_t(x)) \]
which gives that \( F(t, h_t(x)) \) is a solution of the differential equation \( y' = A(t, h_t(x)) \cdot y \) with initial condition \( y(0) = f(x) \) for fixed \( x \). Since the solution of this differential equation is unique and smooth in \( x \) and \( t \) of form \( y(x, t) = A(t, x) \cdot y(x, 0) \) where \( A(t, x) \) is an invertible matrix, we can conclude that \( F(t, h_t(x)) = A(t, x) \cdot f(x) \).

Since this holds in a neighbourhood of \((0, 0)\) in \( \mathbb{R} \times \mathbb{R}^n \) we can conclude that \( f \) and \( f + th \) are \( K_\Sigma \) equivalent for small \( t \). Now fix an arbitrary \( t_0 \in [0, 1] \), and let \( f_{t_0} \in \mathcal{E}(n)^p \) denote \( f + t_0 h \). From condition (t) it follows easily that \( TK_\Sigma f_{t_0} \subseteq TK_\Sigma f \) and that \( TK_\Sigma f \subseteq TK_\Sigma f_{t_0} + m_n(TK_\Sigma f) \). Therefore Nakayama’s Lemma gives that \( TK_\Sigma f = TK_\Sigma f_{t_0} \). Thus \( TK_\Sigma f_{t_0} \) also satisfies condition (t), and it follows from above that \( f_{t_0} \) and \( f + th \) are \( K_\Sigma \) equivalent when \( |t - t_0| \) is small. Connectedness of \([0, 1] \) gives that \( f \) and \( f + h \) are \( K_\Sigma \) equivalent. \( \square \)

Let us also introduce the notion of \( C^r-K_\Sigma \) equivalence \( 0 \leq r \leq \infty \) which is the analogue of ordinary \( K_\Sigma \) equivalence just using \( C^k \) diffeomorphisms instead of \( C^\infty \) diffeomorphisms in the definition of \( K_\Sigma \) equivalence. Let \( J_K(f) \) be the ideal in \( \mathcal{E}(n) \) generated by the \( p \times p \) minors of \( DF \) and put \( J_K(f) = J_K(f) + J_C(f) \). Following \([5]\) we denote:

\( (a_r) \) for \( 0 \leq r \leq \infty \), \( f \) is infinitely \( C^r-K_\Sigma \) determined.
\( (b_r) \) for \( 0 \leq r \leq \infty \), \( f \) is finitely \( C^r-K_\Sigma \) determined.
\( (c) \) \( J_K(f) \) is \( \Sigma \)-elliptic.
\( (t) \) \( m_\Sigma^\infty \mathcal{E}(n)^p \subseteq TK_\Sigma f \).
\( (v_1) \) \( f \) is infinitely \( \Sigma \)-V determined.
\( (v'_1) \) \( f \) is infinitely \( \Sigma \)-SV-determined.
\( (v_2) \) \( f \) is finitely \( \Sigma \)-V determined.
(v'_2) f is finitely Σ-SV-determined.

Now we have:

**Theorem 7.3.** For \( f \in \mathcal{E}(n)^p, n > p \), the conditions

\[
(a_r) \ (0 \leq r \leq \infty), \ (b_r) \ (0 \leq r < \infty), \ (c), \ (t), \ (v_1), \ (v'_1), \ (v_2) \ and \ (v'_2)
\]

are all equivalent.

In Proposition 7.1 we have proved the equivalence \((t) \iff (a_\infty)\). By definition, the implications \((a_\infty) \implies (a_r) \implies (v_1), (v_2) \implies (v_1)\) and \((v'_2) \implies (v'_1)\) are obvious. The proof of \((3) \implies (4)\) of Theorem 6.7 can be easily adapted to prove the implication \((v_1) \implies (c)\), and Lemma 6.4 gives the implication \((c) \implies (t)\). In addition, the notion of \(C^r\) rigid is equivalent to \((b_r)\). Therefore the equivalences \((c) \iff (v_2) \iff (v'_2) \iff (b_r), (r < \infty)\), are proved in the same way as in Theorem 6.7.

Thus Theorem 7.3 is established.

**References**

[1] K. Bekka and S. Koike: *The Kuo condition, an inequality of Thom’s type and \((C)\)-regularity*, Topology 37 (1998), 45–62.
[2] J. Bochnak and T.-C. Kuo: *Rigid and finitely \(V\)-determined germs of \(C^\infty\)-mappings*, Canadian J. Math. 25 (1973), 727–732.
[3] J. Bochnak and S. Lojasiewicz: *A converse of the Kuiper-Kuo theorem*, Proc. of Liverpool Singularities Symposium I (C.T.C. Wall, ed.), Lectures Notes in Math. 192, pp. 254–261, (Springer, 1971).
[4] J. Bochnak and W. Kucharz: *Sur les germes d’applications différentiables à singularités isolées*, Trans. Amer. Math. Soc. 252, (1979) 115–131.
[5] H. Brodersen: *A note on infinite determinacy of smooth map germs*, Bull. London Math. Soc. 13 (1981), 397–402.
[6] J. Damon: *The unfolding and determinacy theorems for subgroups of \(A\) and \(K\)*, Memoirs Amer. Math. Soc. 306 (1984).
[7] T. Fukuda: *Topological triviality of real analytic singularities*, Analytic Varieties and Singularities (Kyoto, 1992), RIMS Kokyuroku 807 (1992), pp. 7–11.
[8] V. Grandjean: *Infinite relative determinacy of smooth function germs with transverse isolated singularities and relative Lojasiewicz conditions*, J. London Math. Soc. 69 (2004), 518–530.
[9] S. Izumiya and S. Matsuoka: *Notes on smooth function germs on varieties*, Proc. Amer. Math. Soc. 97 (1986), 146–150.
[10] Z. Jelonek: *On the generalized critical values of a polynomial mapping*, Manuscripta Math. 110 (2003), 145–157.
[11] S. Koike and W. Kucharz, *Sur les réalisations de jets non-suffisants, C. R. Acad. Sci. Paris* 288 (1979), 457–459.
[12] V. Kozyakin: *Polynomial reformulation of the Kuo criteria for \(V\)-sufficiency of map-germs*, Discret and Continuous Dynamical Systems (2) 14 (2010), 587–602.
[13] W. Kucharz: *Analytic and differentiable functions vanishing on an algebraic set*, Proc. American Math. Soc. 102 (1988), 514–516.
[14] N. Kuiper: *\(C^1\)-equivalence of functions near isolated critical points*, Symp. Infinite Dimensional Topology, Princeton Univ. Press, Baton Rouge, 1967, R. D. Anderson ed., Annales of Math. Studies 69 (1972), pp. 199–218.
[15] T.-C. Kuo: On $C^0$-sufficiency of jets of potential functions, Topology 8 (1969), 167–171.
[16] T.-C. Kuo: A complete determination of $C^0$-sufficiency in $J^r(2,1)$, Invent. math. 8 (1969), 226–235.
[17] T.-C. Kuo: Characterizations of $v$-sufficiency of jets, Topology 11 (1972), 115–131.
[18] K. Kurdyka, P. Orro and S. Simon, Semialgebraic Sard theorem for generalized critical values, J. Differential Geometry 56 (2000), 67–92.
[19] K. Kurdyka and W. Pawlucki, Subanalytic version of Whitney’s extension theorem, Studia mathematica 124 (1997), 269–280.
[20] L. Kushner and B. Terra Leme: Finite relative determination and relative stability, Pacific J. Math. 192 (2000), 315–328.
[21] S. Lojasiewicz: Ensembles semi-analytiques, I.H.E.S. (France 1965), Available on: http://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf
[22] E. Looijenga: Semi-universal deformation of a simple elliptic of hypersurface singularity I: unimodularity, Topology 6, (1977), 257–262.
[23] J. N. Mather: Stability of $C^\infty$ mappings III: Finitely determined map-germs, Inst. Hautes Études Sci. Publ. Math. 35 (1968), 279–308.
[24] P. Migus, T. Rodak and S. Spodzieja: Finite determinacy of non-isolated singularities, Ann. Polon. Math. 117 (2016), no. 3, 197–206.
[25] P. J. Rabier: Ehresmann fibrations and Palais-Smale conditions for morphisms of Finsler manifolds, Ann. of Math. (2) 146 (1997), 647–691.
[26] F. Takens: A note on sufficiency of jets, Invent. math. 13 (1971), 225–231.
[27] V. Thilliez: Infinite determinacy on a closed set for smooth germs with non-isolated singularities, Proc. Amer. Math. Soc. 134 (2006), (5) 1527–1536.
[28] R. Thom: Letter to T.-C. Kuo.
[29] J.-C. Tougeron: Idéaux de fonctions différentiables. Ergebnisse der Mathematik, Band 71, Springer-Verlag 1972.
[30] J.-C. Tougeron: Inégalités de Lojasiewicz globales, Ann. Inst. Fourier 41 (1991), 841–865.
[31] J.-C. Tougeron and J. Merrien: Idéaux de fonctions différentiables. II, Ann. Inst. Fourier 20 (1970), 179–233.
[32] D. J. A. Trotman and L. C. Wilson: Stratifications and finite determinacy, Proc. London Math. Soc. 78 (1999), 334–368.
[33] C. T. C. Wall: Finite determinacy of smooth map-germs, Bull. London Math. Soc. 13 (1981), 481–539.
[34] L. C. Wilson: Infinitely determined map-germs, Canadian J. Math. 33 (1981), 671–684.
[35] X. Xu: $C^0$-sufficiency, Kuiper-Kuo and Thom conditions for non-isolated singularity Acta Mathematica Sinica (English Series), 23, No. 7, (2007), 1251-1256.

Institut de recherche Mathematique de Rennes, Université de Rennes 1, Campus Beaulieu, 35042 Rennes cedex, France

Department of Mathematics, Hyogo University of Teacher Education, Kato, Hyogo 673-1494, Japan

E-mail address: karim.bekka@univ-rennes1.fr
E-mail address: koike@hyogo-u.ac.jp