Holographic phase transitions of p-wave superconductors in Gauss-Bonnet gravity with back-reaction

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We investigate the phase transitions of holographic p-wave superconductors in (4 + 1)-dimensional Einstein-Yang-Mills-Gauss-Bonnet theories, in a grand canonical ensemble. Turning on the back-reaction of the Yang-Mills field, it is found that the condensations of vector order parameter become harder if the Gauss-Bonnet coefficient grows up or the back-reaction becomes stronger. In particular, the vector order parameter exhibits the features of first order and second order phase transitions, while only the second order phase transition is observed in the probe limit. We discuss the roles that the Gauss-Bonnet term and the back-reaction play in changing the order of phase transition.

I. INTRODUCTION

The AdS/CFT correspondence [1–4] provides a novel approach to study the strongly coupling systems at finite density. Therefore, it may have some useful applications in condensed matter physics. It has been applied to study the holographic shear viscosity [5–8], holographic superconductors [9, 10] and holographic (non)fermi-liquids [11–13]. In this paper, we will focus on the holographic p-wave superconductors [14–16].

The phenomena of superconducting can be explained by the spontaneously breaking of U(1) gauge symmetry [15]. In the p-wave superconductor, the rotational symmetry is also broken by a special direction of some vector field in addition. This could be achieved by the condensation of a charged vector field. And the holographic modeling of this picture could be simply realized by adding an SU(2) Yang-Mills field to an AdS black hole background [14]. In this case, after making some ansatz of the SU(2) field, one U(1) subgroup of SU(2) is considered as the electromagnetic gauge group, in addition, a gauge boson generated by another SU(2) generator is charged under this U(1) subgroup through the nonlinear coupling of the non-Abelian gauge fields. In this setup, the superconductor phase transition is studied, and conductivities show some anisotropic behavior. Furthermore, the holographic p-wave superconductor with back-reaction was investigated in Ref. [16]. The crucial point is that

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the back-reaction will dramatically change the order of the phase transition. More specifically, when the matter field coupling goes beyond a critical value, the former second order phase transition with a $1/2$ mean-field theory critical exponent near the critical temperature \[14, 16, 19–22\] will be changed to a first order phase transition.

In a previous paper \[19\], we studied the holographic Gauss-Bonnet p-wave superconductors in the probe limit. The holographic Gauss-Bonnet superconductors are also discussed in 23–28. In this paper we will investigate the holographic p-wave superconductors with back-reaction in order to find out how the matter couplings and the Gauss-Bonnet coefficient affect the phase transition of the p-wave superconductor. We find that the bigger the Gauss-Bonnet coupling is, the bigger the condensation value of the order parameter is, and the lower the critical temperature is. This reflects that the big Gauss-Bonnet coefficient will make the superconducting phase transition hard, which is consistent with our previous conclusions \[19\]. Besides, we also find that the stronger the matter field couples to the background, the harder the condensation to be formed. In addition, we find that the phase transition will change from second order to first order when the back-reaction is strong, which is similar to the discussions of \[16\]. In grand canonical ensemble, we study the free energy and entropy of the p-wave superconductor which also support our claim of the change of phase transitions.

This paper is organized as follows: We will set up our model of the holographic superconductors in Sec. II and study the condensation behavior of the vector order parameter for different Gauss-Bonnet coefficients and different matter field couplings in Sec. III. In Sec. IV, we study the thermodynamics of the p-wave superconductor by exploring the free energy and entropy. We draw our conclusions in Sec. V.

II. HOLOGRAPHIC SET UP OF P-WAVE SUPERCONDUCTORS

We consider the Einstein-Gauss-Bonnet gravity with an SU(2) Yang-Mills field in (4+1)-dimensional asymptotically AdS space-time. The action is

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} \left( R + \frac{12}{L^2} + \frac{\alpha}{2} (R^2 - 4R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}) \right) - \frac{1}{4\hat{g}^2} \left( F_{\mu\nu}^a F^{a\mu\nu} \right) \right] + S_{bdy}. \tag{1}$$

where $\kappa_5$ is the five dimensional gravitational constant with $2\kappa_5^2 = 16\pi G_5$, and $G_5$ a (4+1)-dimensional Newton gravitational constant, $\hat{g}$ is the Yang-Mills coupling constant and $L$ is the AdS radius. The SU(2) Yang-Mills field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c \tag{2}$$

where $a, b, c = (1, 2, 3)$ are the indices of the generators of SU(2) algebra. $\mu, \nu = (t, r, x, y, z)$ are the labels of space-time with $r$ being the radial coordinate of AdS. The $A_\mu^a$ are the components of the mixed-valued gauge fields $A = A^a_\mu \tau^a dx^\mu$, where $\tau^a$ are the SU(2) generators with commutation relation $[\tau^a, \tau^b] = \epsilon^{abc} \tau^c$. $\epsilon^{abc}$ is the totally antisymmetric tensor with $\epsilon^{123} = +1$. The quadratic curvature term is the Gauss-Bonnet term with $\alpha$ the Gauss-Bonnet coefficient and $R_{\mu\nu\rho\sigma} = \partial_\rho \Gamma^\rho_{\mu\sigma} - \cdots$. $S_{bdy}$ includes boundary terms that do not affect the
equations of motion, namely the Gibbons-Hawking surface term, as well as counter-terms required for the on-shell action to be finite. We will write $S_{\text{bdy}}$ term explicitly in Sec. IV. The Einstein field equations can be derived from the above action as
\begin{equation}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + \frac{12}{L^2}) + \frac{\alpha}{2}[H_{\mu\nu} - \frac{1}{2}g_{\mu\nu}H] = \kappa_5^2 T_{\mu\nu}
\end{equation}
with
\begin{align}
T_{\mu\nu} &= \frac{1}{g^2}\text{tr}(F^a_{\mu\rho}F^{a\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F^a_{\rho\sigma}F^{a\rho\sigma}), \\
H_{\mu\nu} &= 2RR_{\mu\nu} + 2R_{\mu\alpha\tau\rho}R_{\nu}^{\alpha\tau\rho} - 4R_{\mu\rho}R^\rho_{\nu} + 4R^\sigma_{\rho}R_{\nu\mu\sigma}, \\
H &= H^a_{\mu}.
\end{align}
where “tr” takes the trace over the indices of SU(2) generators. The Yang-Mills equations of motion are:
\begin{equation}
\nabla_{\mu}F^{a\mu\nu} = -\epsilon^{abc}A^b_{\mu}F^{c\mu\nu}.
\end{equation}
Following Refs. [14–16], we choose the ansatz of the gauge fields as
\begin{equation}
A(r) = \phi(r)\tau^3 dt + w(r)\tau^1 dx.
\end{equation}
In this ansatz we regard the U(1) symmetry generated by $\tau^3$ as the U(1) subgroup of SU(2). We call this U(1) subgroup as U(1)$_3$. The gauge boson with nonzero component $w(r)$ along $x$ direction is charged under $A^3_\mu = \phi(r)$. According to AdS/CFT dictionary, $\phi(r)$ is dual to the chemical potential in the boundary field theory while $w(r)$ is dual to the $x$ component of some charged vector operator $J$. The condensation of $w(r)$ will spontaneously break the U(1)$_3$ gauge symmetry and induce the phenomena of superconducting on the boundary field theory.

Our metric ansatz following Ref. [16, 17] is
\begin{equation}
ds^2 = -N(r)\sigma(r)^2 dt^2 + \frac{1}{N(r)}dr^2 + r^2 f(r)^{-1}dx^2 + r^2 f(r)^2(dy^2 + dz^2).
\end{equation}
with
\begin{equation}
N(r) = \frac{r^2}{2\alpha}\left(1 - \sqrt{1 - \frac{4\alpha}{L^2} + \frac{4\alpha m(r)}{r^4}}\right).
\end{equation}
where $m(r)$ is a function related to the mass and charge of the black hole. The reason for this metric ansatz is that: the back-reaction of nonzero $w(r)$ will change the background of space-time. The condensation of $w(r)$ will preserves only SO(2) symmetry of the spatial direction, i.e., $(y, z)$-direction. The horizon of the black hole is located at $r_h$ while the boundary of the bulk is at $r_{\text{bdy}} \to \infty$. Note that when $r \to \infty$,
\begin{equation}
N(r) \sim \frac{r^2}{2\alpha}\left(1 - \sqrt{1 - \frac{4\alpha}{L^2}}\right).
\end{equation}
So we can define an effective radius $L_c$ of AdS space-time as

$$L_c \equiv L \sqrt{1 + \frac{U}{2}}, \quad U = \sqrt{1 - \frac{4\alpha}{L^2}}. \quad (12)$$

From this relation we can see that in order to have a well-defined vacuum for the gravity theory, there is an upper bound for $\alpha \leq L^2/4$. The saturation $\alpha = L^2/4$ is called Chern-Simons limit. If we further consider the causality constraint of the boundary CFT, there is an additional constraint on the Gauss-Bonnet coefficient with $-7L^2/36 \leq \alpha \leq 9L^2/100$ in five dimensions [29–35].

The Hawking temperature of this black hole is

$$T = \frac{\sigma N'}{4\pi} \bigg|_{r = r_h} = \left( \frac{\sigma}{\pi L^2} - \frac{\kappa_g^2 \phi'^2}{12\pi \sigma} \right) \bigg|_{r = r_h} \quad (13)$$

where, $\kappa_g \equiv \kappa_5/\hat{g}$ is regarded as the effective matter field coupling, and “$'$” denotes the derivative with respect to $r$. The Bekenstein-Hawking entropy of the black hole is:

$$S = \frac{A}{4G_5} = \frac{2\pi A}{\kappa_5^2} = \frac{2\pi}{\kappa_5^2} V r_h^3, \quad (14)$$

where $A$ denotes the area of the horizon and $V = \int d^3x$.

The Einstein and Yang-Mills equations of motion with the ansatz [89] can be explicitly written as

$$N' = -\frac{\kappa_g^2 r f'^w f^2 \phi^2}{3r^2 N \sigma^2 - 6\alpha N^2 \sigma^2 (r f'^2)(r f)^2} - \frac{\kappa_g^2 \phi'^2}{3r^2 \sigma^2 - 6\alpha N \sigma^2 (r f'^2)(r f)^2}$$

$$\phi'' = \frac{f'^w \phi}{r^2 N} + \left( -\frac{3}{r} + \frac{\sigma'}{\sigma} \right) \phi', \quad (18)$$

$$w'' = -\frac{w \phi'^2}{N^2 \sigma^2} - w' \left( \frac{1}{r} + 4 \frac{f'}{f} + \frac{N'}{N} + \frac{\sigma'}{\sigma} \right), \quad (19)$$

$$f'' = f_1 + f_2 + f_3, \quad (17)$$

$$\sigma' = \frac{\kappa_g^2 r f'^w \phi^2}{6r^2 N \sigma^2 - 12\alpha N^3 \sigma (r f'^2)(r f)^2} - \frac{\kappa_g^2 \phi'^2}{6r^2 \sigma^2 - 12\alpha N^2 \sigma (r f'^2)(r f)^2}$$

$$+ \left( \frac{12r^3 \sigma f^2 + L^2 \sigma \left( \kappa_g^2 r N w'^2 f^6 - 3f^2 (N'^2 r^2 + 2N (r - \alpha (r f'^2)(r f)^2 N'))) + 6r^3 N f'^2 \right)}{6L^2 f N (r^2 - 2\alpha N (r f'^2)(r f)^2)}, \quad (15)$$

where $e$ denotes the energy density of the black hole.
where

\[ f_1 = \left( -\kappa_g^2 r^2 f^7 w^2 \phi^2 + \kappa_g^2 \alpha f^5 N w^2 \phi^2 (2f - rf') (rf') \right) / \left( 3 r^4 f^2 N^2 \sigma^2 
+ 3 \alpha^2 N^2 \sigma (2r^2 N \sigma f^2 - 2r^2 f' (\sigma N' + 2N \sigma') - f^2 (r \sigma N' + 2N (r \sigma')) 
+ 6 \alpha^2 r N^3 \sigma (rf')^2 (\sigma N' + 2N \sigma') \right) \] (20)

\[ f_2 = \left( \kappa_g^2 r^2 f^7 \sigma w^2 - \kappa_g^2 \alpha f^5 N \sigma (2f - rf') (rf') w^2 \right) / \left( 3 r^4 f^2 \sigma 
+ 3 \alpha^2 (2r^2 N \sigma f^2 - 2r^2 f' (\sigma N' + 2N \sigma') - f^2 (r \sigma N' + 2N (r \sigma')) 
+ 6 \alpha^2 r N (r f')^2 (\sigma N' + 2N \sigma') \right) \] (21)

\[ f_3 = \left( -L^2 r^3 \sigma f' (f (r \sigma N' + N (3 \sigma + r \sigma')) - r \sigma N f') f^3 
+ r \alpha f f' \left[ (L^2 r^2 N^2 r^2 + N (\kappa_g^2 \phi^2 L^2 + 2 \sigma N' \sigma' L^2 + 12 \sigma^2)) \right) r^2 + 2L^2 N^2 \sigma (2 \sigma + r \sigma') \right) f^3 
+ r f' \left( -4L^2 N^2 \sigma^2 + L^2 r^2 N^2 \sigma^2 + r N \left( \kappa_g^2 \phi^2 L^2 + 2 \sigma N' \sigma' L^2 + 2 \sigma^2 (6r - L^2 N') \right) \right) f^2 
- 2L^2 r^2 N \sigma f^2 (2 \sigma N' + 3N f (2 \sigma + r \sigma')) - 4L^2 r^3 N^2 \sigma f^3 \right] 
+ 4L^2 \alpha^2 N^2 \sigma f' (f + rf')^2 (-f^2 + rf' f + r^2 f'^2) (\sigma N' + 2N \sigma') \right) / \left( L^2 r^4 N \sigma^2 f^4 + 2L^2 r^2 \alpha N^2 \sigma (f + rf')^2 (\sigma N' + 2N \sigma') \right) f^2 
+ L^2 r^2 \alpha N \sigma (2 \sigma N f^2 r^2 - 2f f' (\sigma N' + 2N \sigma') r^2 - f^2 (r \sigma N' + 2N (\sigma + r \sigma'))) \right) f^2 \] (22)

There are four useful scaling symmetries in the above equations:

\[ (I) \quad f \rightarrow \lambda f, \quad w \rightarrow \lambda^{-2} w, \] (23)
\[ (II) \quad \sigma \rightarrow \lambda \sigma, \quad \phi \rightarrow \lambda \phi, \] (24)
\[ (III) \quad r \rightarrow \lambda r, \quad m \rightarrow \lambda^4 m, \quad \omega \rightarrow \lambda \omega, \quad \phi \rightarrow \lambda \phi, \quad N \rightarrow \lambda^2 N \] (25)
\[ (IV) \quad r \rightarrow \lambda r, \quad m \rightarrow \lambda^2 m, \quad L \rightarrow \lambda L, \quad \phi \rightarrow \lambda^{-1} \phi, \quad \kappa_g \rightarrow \lambda \kappa_g, \quad \alpha \rightarrow \lambda^2 \alpha. \] (26)

We can use the symmetries (25\ 26) to set \( r_h = 1 \) and \( L = 1 \), and use symmetries (23\ 24) to set \( \sigma (r \rightarrow \infty) = f (r \rightarrow \infty) = 1 \) in order to make the solution asymptotically approach to an AdS.

### III. TWO BRANCHES OF SOLUTIONS

#### A. Analytic charged GB-AdS solution with \( w(r) = 0 \)

In the ansatz (8) with \( w(r) = 0 \), there is an straightforward analytic black hole solution which is a charged generalization of GB-AdS black holes [36, 37]. In this case,

\[ f(r) = \sigma (r) = 1, \quad \phi = \mu - \frac{Q}{2 r^2}. \] (27)
and
\[ N(r) = \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 + 4\alpha\left(\frac{\tilde{M}}{r^4} - \frac{1}{L^2} - \frac{\kappa^2}{6} \frac{Q^2}{r^6}\right)} \right], \]  
where \( \tilde{M} \) is related to the mass of the black hole, \( Q \) is the charge of the black hole and \( Q = 2\mu^2_h \). Using the formula (13), the temperature for this charged GB-AdS black hole is
\[ T = \frac{r_h}{\pi L^2} \left( 1 - \frac{\mu^2}{3\pi r_h^3} \right). \]  

**B. Superconducting solutions with \( w(r) \neq 0 \)**

In order to investigate the superconducting solutions with \( w(r) \neq 0 \), we have to solve numerically the set of equations (15) (16) (17) (18) and (19).

First of all, we should impose boundary conditions on the fields. The horizon is located at \( r = r_h \) with \( N(r_h) = 0 \). On the horizon, we should impose \( \phi(r_h) = 0 \) for the U(1) gauge field to have a finite norm, and \( \sigma(r), f(r), w(r) \) should be finite. We can expand the fields in the powers of \( (1 - r_h/r) \) near the horizon, they are
\[
\begin{aligned}
N &= 0 \Rightarrow m = \frac{r^4_h}{L^2} + m_H^{(1)} (1 - r_h/r) + \cdots \\
\sigma &= \sigma_H^{(0)} + \sigma_H^{(1)} (1 - r_h/r) + \cdots \\
f &= f_H^{(0)} + f_H^{(1)} (1 - r_h/r) + \cdots \\
\phi &= \phi_H^{(1)} (1 - r_h/r) + \phi_H^{(2)} (1 - r_h/r)^2 + \cdots \\
w &= w_H^{(0)} + w_H^{(1)} (1 - r_h/r) + \cdots,
\end{aligned}
\]
where all the coefficients of the expansions are constants.

At the boundary \( r \to \infty \), the asymptotical behavior of these fields are
\[
\begin{aligned}
m &= m_B^{(0)} + m_B^{(2)} / r^2 + \cdots \\
\sigma &= \sigma_B^{(0)} + \sigma_B^{(4)} / r^4 + \cdots \\
f &= f_B^{(0)} + f_B^{(4)} / r^4 + \cdots \\
\phi &= \phi_B^{(0)} + \phi_B^{(2)} / r^2 + \cdots \\
w &= w_B^{(0)} + w_B^{(2)} / r^2 + \cdots.
\end{aligned}
\]

From the AdS/CFT dictionary we know that \( \phi_B^{(0)} = \mu, \phi_B^{(2)} = \rho \), where \( \mu \) and \( \rho \) are respectively the chemical potential and density of the charge at the boundary; \( w_B^{(0)} \) is the source of the boundary operator \( J \) while \( w_B^{(2)} \) is the expectation value of \( J \) and the nonzero \( w_B^{(2)} \) will induce superconducting phase as we have mentioned.

Near horizon (30), \( m_H^{(1)}, \sigma_H^{(1)}, f_H^{(1)} \), and \( \phi_H^{(1)} \) can be evaluated as functions of \( \sigma_H^{(0)}, f_H^{(0)}, \phi_H^{(1)} \) and \( w_H^{(0)} \), by substituting the expansions into the equations of motion. Therefore, there are four independent initial values left to be specified, i.e., \( \sigma_H^{(0)}, f_H^{(0)}, \phi_H^{(1)}, w_H^{(0)} \).
FIG. 1: The condensation values of vector operator $J$ versus temperature for different $\alpha$ and different $\kappa_g$. The black, red, blue and pink curves correspond to $\kappa_g = 0.0001, 0.2, 0.4$ and $0.45$, respectively.

For the boundary conditions at infinity, we impose $\sigma(r) = f(r) = 1$ to have an asymptotic AdS boundary, this could be reached by using the scaling symmetries (23) and (24). We also impose $w_B^{(0)} = 0$ to turn off the the source of $J$. In the numerical calculations, we will set $r_h = L = 1$ by using the scaling symmetries (25) and (26).

Armed with these equations of motion and boundary conditions we can numerically solve the set of equations by the shooting method. Because we will work in the grand canonical ensemble, we can fix the chemical potential value $\mu$ and then vary the four independent near-horizon coefficients $(\sigma_H^{(0)}, f_H^{(0)}, \phi_H^{(1)}, w_H^{(0)})$ until we find a solution which produces the desired value of $\mu$ and $\sigma_B^{(0)} = f_B^{(0)} = 1, w_B^{(0)} = 0$.

In the following, we will present our numerical results of the condensation value of the order parameter $J$. We will scan through value of $\alpha$ from $\alpha = -0.19$ to $\alpha = 0.09$ as well as $\kappa_g$ from $\kappa_g = 0.0001$ to $\kappa_g = 0.45$. From the AdS/CFT dictionary [4], we know that the conformal dimension of vector field in five dimension is $\lambda = 3$, therefore, $J^{1/3}/T_c$ is the right dimensionless quantity.

Fig. 1 shows the condensation value of vector operator $J$ for different Gauss-Bonnet
TABLE I: The condensation values of $J$ for different $\alpha$ and $\kappa_g$, numbers in boldface represent first order phase transition

| $\kappa_g$ | $\alpha$ | -0.19 | 0.0001 | 0.01 | 0.09 |
|-----------|---------|-------|--------|------|------|
| 0.0001    | 5.89103 | 6.01942 | 6.02764 | 6.10352 |
| 0.2       | 6.86288 | 7.35303 | 7.38871 | 7.74425 |
| 0.4       | 12.28375 | **16.42425** | **16.77913** | **20.71994** |
| 0.45      | **16.17033** | **24.76784** | **25.65623** | **35.20014** |

coefficients and different matter field couplings. We can see that the condensation value grows if the Gauss-Bonnet coefficient grows or the matter field coupling grows. Table I lists all the explicit condensation data of Fig. 1.

Take the bottom-left plot in Fig. 1 as an example (i.e., the plot with $\alpha = 0.01$), when we decrease the temperature, the condensation values for black ($\kappa_g = 0.0001$) and red ($\kappa_g = 0.2$) curve will emerge from zero at some critical temperature $T_c$. When we keep on cooling down the system, those condensation values will continuously tend to some constant values. Besides, when $T \sim T_c$ the condensation will take a mean-field theory critical exponent $1/2$ in the form of $J \propto (1-T/T_c)^{1/2}$. These are the second order phase transitions which have been explored in our previous paper [19]. However, for the blue ($\kappa_g = 0.4$) and pink ($\kappa_g = 0.45$) lines the behavior is much different from the above two curves. We see that these curves will bend to the right for $T > T_c$ and there will be two condensation values for $T > T_c$. However, when $T \ll T_c$ they will also tend to some constant values. The mean-field theory critical exponent $1/2$ never exists for these condensations. We will argue in Sec. IV that these peculiar condensation behaviors are features of first order phase transitions and the real critical temperature for these phase transitions is not exactly $T_c$.

In the holographic p-wave superconductors, the normal charge density $\rho_n$ is $\phi_H^{(1)}$ near the horizon, while the total charge density is $\rho_t = 2\phi_B^{(2)} = 2\rho$ where the factor 2 appears due to the scaling behavior of $\phi$ on the infinite boundary. The superconducting charge density is defined as $\rho_s = \rho_t - \rho_n$ [14, 19]. We plot the ratio $\rho_s/\rho_t$ in Fig. 2. For the black and red curves which are of second order phase transitions, the intersecting points where the curves meet the horizontal axis represent the critical temperatures $T_c/\mu$ at which the superconducting phase occurs. However, for the blue and pink curves which are of first order phase transitions (see Sec. IV), the intersecting points are not the critical temperature for the phase transitions. But the real critical temperature $T_c^{R}$ for the first order phase transition is a little above the intersecting points $T_c$. From Sec. IV we can find the values of $T_c^{R}/\mu$ for $\alpha = 0.01$ and varying $\kappa_g$s, which are listed in Table II. Despite of the first and second order phase transitions, we still see that the dimensionless critical temperature $T_c^{R}/\mu$ decreases when $\kappa_g$ grows, which reflects that the stronger the matter field couples to the gravity the harder the phase transition occurs. For sufficiently low temperature, all the curves will tend to one, which reflects the fact that the superconducting charge is dominated
FIG. 2: The ratio of superconducting charge density and the total charge density versus the dimensionless temperature $T/\mu$ at $\alpha = 0.01$. The black, red, blue and pink curves correspond to $\kappa_g = 0.0001, 0.2, 0.4$ and 0.45, respectively.

TABLE II: The quantity of $T_c^R/\mu$ for different $\kappa_g$ while fixing $\alpha = 0.01$. Boldfaces represent first order phase transition.

| $\kappa_g$ | $T_c^R/\mu$ |
|------------|-------------|
| 0.0001     | 0.07885     |
| 0.2        | 0.06437     |
| 0.4        | 0.02880     |
| 0.45       | 0.02063     |

in the total charge.

IV. THERMODYNAMICS

A. Euclidean action and counter-term methods

From the AdS/CFT correspondence, a non-extremal black hole corresponds to a thermal equilibrium state at the boundary. The Hawking temperature of black hole is the temperature of the boundary field. In the following we will work in the grand canonical ensemble with the fixed value of chemical potential $\mu$. The partition function of the bulk theory is

$$Z = e^{-I_E[g_*]}.$$  \hfill (32)

where $I_E[g_*]$ is the Euclidean action evaluated on the on-shell value of $g$. Because of the Euclidean action, the compactified time direction has a period $1/T$. The free energy now is

$$\Omega = -T \log Z = T \ I_E[g_*].$$  \hfill (33)
In the path integral, the Euclidean on-shell action should additionally include the Gibbons-Hawking surface term to give the correct Dirichlet variational problem and some other boundary counter-terms to render the action finite \[38\]. In the computation we introduce a hypersurface at large but finite \(r = r_{bdy}\) as the boundary, and then calculate the on-shell action by putting \(r_{bdy} \to \infty\).

The bulk action evaluated on the on-shell values is

\[
I_{\text{on-shell}}^{(1)} = \frac{V}{T} \left[ \left. \int 2 f^2 \left( (r \sigma N')' + r f' (r \sigma N' + 2 N (r \sigma)' f - 4r^2 N \sigma f^2) \right) - \alpha N (r f)' \left( 3r^2 N \sigma + \frac{r^3 \sigma N'}{2} + r^3 N \sigma' \right) \right|_{r=r_{bdy}} \right],
\]

where \(V/T = \int dtd^3x\) is the volume of the \((3+1)\)-dimensional hypersurface. The usual Gibbons-Hawking term is

\[
I_{GH}^{(1)} = -\frac{1}{\kappa_5^2} \int d^4 x \sqrt{-\gamma} K = -\frac{V}{T \kappa_5^2} \left. (3r^2 N \sigma + \frac{r^3 \sigma N'}{2} + r^3 N \sigma') \right|_{r=r_{bdy}},
\]

where \(\gamma\) is the induced metric on the \(r = r_{bdy}\) hypersurface, \(n^\mu = \sqrt{N(r)} \delta^\mu_r\) is the outward-pointing normal vector to the hypersurface and \(K = K_{\mu}^\mu\) is the trace of the extrinsic curvature \(K_{\mu\nu} = \nabla_{(\mu} n_{\nu)}\).

For the GB term there is also a generalized Gibbons-Hawking term \[39, 40\]

\[
I_{GH}^{(2)} = -\frac{\alpha}{\kappa_5^2} \int d^4 x \sqrt{-\gamma} \left( J - 2G_{ij} K^{ij} \right) = \frac{\alpha V N}{T \kappa_5^2 f^3} \left. \left( (3r^2 N \sigma + 2 N (\sigma + 3 r \sigma')) f^2 - 3r^2 f^2 (r \sigma N' + 2 N (\sigma + r \sigma')) f - 4r^3 N \sigma f^3 \right) \right|_{r=r_{bdy}},
\]

where \(G_{ij}\) is the Einstein tensor of the metric \(\gamma_{ij}\) and \(J = \gamma^{ij} J_{ij}\) with

\[
J_{ij} = \frac{1}{3} \left( 2KK_{ik}K_{kj}^k + K_{kl}K^{kl}K_{ij} - 2K_{ik}K^{kl}K_{lj} - K^2 K_{ij} \right).
\]

No additional counter-terms for matter fields are necessary because the fields fall off sufficiently near the boundary \[41\].\(^1\) Besides, in our metric ansatz the scalar curvature \(R\) for the hypersurface is zero. So the simplest counter-term is \[43, 46\]

\[
I_{ct} = \frac{1}{\kappa_5^2} \int d^4 x \sqrt{-\gamma} \left. \frac{3}{L_c} \left( \frac{2 + U}{3} \right) \right|_{r=r_{bdy}}.
\]

Thus the total on-shell Euclidean action \(I_E[g_s]\) is

\[
I_E[g_s] = I_{\text{on-shell}}^{(1)} + I_{GH}^{(1)} + I_{GH}^{(2)} + I_{ct}.
\]

Then the free energy is

\[
\Omega = T \left( I_E[g_s] - T (I_{\text{on-shell}}^{(1)} + I_{GH}^{(1)} + I_{GH}^{(2)} + I_{ct}) \right).
\]

\(^1\) If one works in the canonical ensemble, the charge density \(\rho\) should be fixed and we should add an additional boundary term to the Euclidean action as \(\Delta I_E \propto \int d^4 x \sqrt{-\gamma} \text{tr}(n^\mu F_{\mu\nu}^a A^{a\nu})\). \[42\]
B. Phase diagrams of free energy and entropy

In this subsection, we will discuss the numerical results of the free energy $\Omega$ and the entropy $S$. In Fig. 3 we plot the free energy and entropy for some typical values of $\alpha$ and $\kappa_g$.

From the first two up-left plots in Fig. 3 we find that for $\alpha = -0.19, \kappa_g = 0.4$ and $\alpha = 0.01, \kappa_g = 0.3$, the charged GB-AdS solution (purple line) (28) exists for all the temperature $T > T_c$ and $T \leq T_c$. On the contrary, the superconducting solution (blue line) exists only for $T \leq T_c$. But for $T < T_c$ the free energy of superconducting solution is smaller than that of the charged GB-AdS solution, which means that the superconducting solution is thermodynamically favored, compared to the charged GB-AdS solution. This represents that when $T$ decreases across $T_c$, a phase transition occurs when the bulk solution going from the charged GB-AdS solution to a solution with nonzero vector hair which induces an order parameter of the p-wave holographic superconductor. In addition, we see that at $T = T_c$ the purple line and blue line are not only continuous but also $C^1$ differentiable which can be seen from the plots of entropy $S$ because $S = -\partial \Omega / \partial T$, see the first two up-right plots in Fig. 3. The entropy for these two solutions at $T = T_c$ is continuous but not differentiable. So according to Ehrenfest’s classification of phase transitions, these phase transitions are second order for $\alpha = -0.19, \kappa_g = 0.4$ and $\alpha = 0.01, \kappa_g = 0.3$. This is consistent with our remarks in the previous section.

From the two down-left plots of Fig. 3 i.e., plots with $\alpha = 0.01, \kappa_g = 0.4$ and $\alpha = 0.01, \kappa_g = 0.45$, the behavior of free energy is dramatically different from the one we mentioned above, there is a characteristic “swallowtail” shape of the free energy indicating a first order phase transition. Consider the $\alpha = 0.01, \kappa_g = 0.45$ case for example. When we decrease the temperature, entering the figure along the purple line from the right, we reach the temperature $T \approx 1.2 T_c$ where new solutions appear (the blue curves represent the new solutions). However, for now the charged GB-AdS solution is thermodynamically favored because it has a lower free energy. If we keep on cooling down the system we will still remain in the charged GB-AdS solution until $T \approx 1.12 T_c$. Then for $T < 1.12 T_c$, the superconducting solution will have a lower free energy than the charged GB-AdS solution. Therefore, superconducting solution will be thermodynamically favored, compared to the charged GB-AdS solution in the range $T < 1.12 T_c$. So the exact critical temperature for this kind of phase transition is $T^R_c = 1.12 T_c$. This is the reason why the quantities for $\kappa_g = 0.4$ and $\kappa_g = 0.45$ in Table. II are not the exact values which the intersecting points denote. This is a first order phase transition due to the non-differentiable free energy at $T = 1.12 T_c$. From the point view of entropy, we see that the entropy will jump from the purple curve to the lowest part of the blue curve at $T = 1.12 T_c$ (see the down-right plot in Fig. 3. The entropy is not continuous which also reveals the feature of the first order phase transition. Therefore, from the plots in Fig. 3 we can read the real critical temperature for the first order phase transition as $T = 1.02 T_c$ when $\alpha = 0.01, \kappa_g = 0.4$ and $T = 1.12 T_c$ when $\alpha = 0.01, \kappa_g = 0.45$, respectively.
FIG. 3: The free energy and entropy versus temperature for different $\alpha$ and $\kappa_g$. The purple line is for the charged GB-AdS black hole (i.e., $w(r) = 0$) while the blue line is for the superconducting solutions (i.e., $w(r) \neq 0$).
FIG. 4: Classification of phase transition versus $\alpha \ (-0.19 \leq \alpha \leq 0.09)$ and $\kappa_g \ (\kappa_g \geq 0)$. The order of phase transition depends on the coefficients $\alpha$ and $\kappa_g$. The yellow region below the dashed curve is of the second order phase transition while the orange region is of the first order phase transition.

V. CONCLUSIONS

In a previous paper [19], we studied holographic p-wave superconductors within Gauss-Bonnet gravity in the probe limit. In this paper, we continued this study by including back-reaction of Yang-Mills field. We found that both the Gauss-Bonnet coefficient and back-reaction will make the superconducting condensation difficult. This difficulty can be seen both from the growing condensation values and the decreasing critical temperatures. By studying the thermodynamics of the system in grand canonical ensemble, we found two kinds of phase transitions of the holographic p-wave superconductors. Note that in the probe limit, the superconducting phase transition is always second order. With back-reaction, we found that when fixing the Gauss-Bonnet coefficient, there was a critical value for the matter field couplings $\kappa_g(c)$ (see Fig. 4). If $\kappa_g < \kappa_g(c)$, the phase transition is second order (yellow region), however, if $\kappa_g > \kappa_g(c)$, the phase transition becomes first order (orange region). It was found that the stronger back-reaction will not only make the condensation value bigger but also will change the order of the phase transition. This is consistent with the conclusions of [16]. On the contrary, if we fixed the matter field couplings $\kappa_g$, the Gauss-Bonnet couplings would change the order of the phase transition just for a small range of $\kappa_g$, i.e., $0.366 \leq \kappa_g \leq 0.427$. However, out of this range, the Gauss-Bonnet term would not change the order of the phase transition. Therefore, we may conclude that although both the Gauss-Bonnet coefficient and back-reaction will make the p-wave condensation hard, the back-reaction plays a major role in changing the order of the phase transition.
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