BLACK HOLE CONFIGURATIONS
WITH TOTAL ENTROPY LESS THAN
A/4 *

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Abstract

If one surrounds a black hole with a perfectly reflecting shell and adiabatically squeezes the shell inward, one can increase the black hole area $A$ to exceed four times the total entropy $S$, which stays fixed during the process. $A$ can be made to exceed $4S$ by a factor of order unity before the one enters the Planck regime where the semiclassical approximation breaks down. One interpretation is that the black hole entropy resides in its thermal atmosphere, and the shell restricts the atmosphere so that its entropy is less than $A/4$. 

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The Generalized Second Law (GSL) [1] of black hole thermodynamics states that the total entropy $S$ does not decrease, and it further states that for Einsteinian gravity (to which this paper will be restricted, though the generalization to various other theories should be straightforward), $S$ is the GSL entropy

$$S_{\text{GSL}} \equiv \frac{1}{4}A + S_m, \quad (1)$$

where $A$ is the total event horizon area of all black holes and $S_m$ is the entropy of matter outside the black holes. (I am using Planck units in which $\hbar$, $c$, $4\pi \epsilon_0$, Boltzmann’s constant $k$, and the renormalized Newtonian gravitational constant $G$ are all set equal to unity.) Although the Generalized Second Law has only been proved under restricted conditions, such as for quasistationary semiclassical black holes [2], it is believed to have greater generality, such as to rapidly evolving black holes.

On the other hand, it is obvious that the GSL would not apply in the form above if the matter entropy $S_m$ were taken to be its von Neumann fine-grained entropy

$$S_{\text{vN}} = -\text{tr}\rho \ln \rho, \quad (2)$$

and if information is not fundamentally lost when a black hole forms and evaporates, since the initial and final fine-grained entropy of the matter could be zero when $A = 0$, whereas the GSL entropy $S_{\text{GSL}} = A/4 + S_m$, which ignores quantum correlations or entanglements between the black hole and the matter outside, would be positive when (and only when) the black hole exists and has positive area.

Nevertheless, under some suitable assumptions of coarse graining (such as ignoring entanglements between black holes and matter outside, and ignoring complex entanglements between different parts of the matter emitted from a black hole if the emission process is indeed described by a quantum unitary process), it has generally been assumed that it is a good approximation to take the total entropy $S$ to be $S_{\text{GSL}}$, the sum of $A/4$ and $S_m$.

An implicit further assumption that is often made is that the matter entropy $S_m$ cannot be negative (as indeed it could not, for example, if it were given by $S_{\text{vN}} = -\text{tr}\rho \ln \rho$). This assumption, plus the GSL, leads to the conclusion that the total entropy is bounded below by one-quarter the total event horizon area:

$$\frac{1}{4}A \leq S. \quad (3)$$

Here I shall show that the inequality (3) can be violated. This violation can be interpreted as either a violation of the Generalized Second Law (if $S_m$ is assumed to be restricted to nonnegative values) or as an indication that the matter entropy $S_m$ must be allowed to take negative values in order to conform to the GSL.
Briefly, a violation of the inequality (3) can be produced as follows: Take a Schwarzschild black hole of initial mass $M_i$ and radius $2M_i \gg 1$ (in the Planck units used herein) with negligible matter outside (e.g., before the black hole has had time to radiate significantly). Assuming the GSL for this initial state, the initial entropy $S_i$ is roughly $A_i/4 = 4\pi M_i^2 \gg 1$, one-quarter the initial area of the hole, since the initial matter entropy $S_m$ is negligible in comparison. (If one considers as matter the thermal atmosphere that forms when the horizon forms in the near-horizon region $r - 2M \ll 2M$, either the entropy of this atmosphere should be considered negligible if it is considered to be part of $S_m$, or it should be considered to be part of the black hole entropy $A/4$; one gets too large a value for the total entropy if one counts both $A/4$ and a large entropy associated with the near-horizon thermal atmosphere.)

Now surround the black hole by a spherical perfectly reflecting shell at a radius $r_i$ that is a few times the Schwarzschild radius $2M_i$ of the black hole. This region will soon fill up with thermal Hawking radiation to reach an equilibrium state of fixed energy $M_i$ inside the shell, but for $M_i \gg 1$, all but a negligible fraction ($\sim 1/M_i^2$ in Planck units) of the energy will remain in the hole, which can thus be taken still to have mass $M_i$. Outside the shell, one will have essentially the Boulware vacuum state with zero entropy (plus whatever apparatus that one will use to squeeze the shell in the next step, but this will all be assumed to be in a pure state with zero entropy).

Next, squeeze the shell inward. If this is done sufficiently slowly, this should be an adiabatic process, keeping the total entropy fixed. Also, the outside itself should remain in a zero-entropy pure state, since the perfectly reflecting shell isolates the region outside from the region inside with its black hole and thermal radiation, except for the effects of the gravitational field, which will be assumed to produce negligible quantum correlations between the inside and the outside of the shell (as one would indeed get in a semiclassical approximation in which the geometry is given by a spherically symmetric classical metric). Some of the thermal Hawking radiation will thus be forced into the black hole, increasing its area.

So long as the shell is not taken into the near-horizon region $r - 2M \ll 2M$, the radiation forced into the black hole will have negligible energy and so will not increase the black hole area significantly above its initial value $A_i$. (Indeed, some of this tiny increase in the area just compensates for the tiny decrease in the black hole area when it filled the region $r < r_i$ with thermal radiation.)

However, nothing in principle prevents one from squeezing the shell into the near-horizon region, where a significant amount of the near-horizon thermal radiation can be forced into the hole, increasing its mass $M$ and area $A = 4\pi M^2$ significantly. Since the entropy $S$ should not change by this adiabatic process, it remains very nearly at $A_i/4$. Therefore, one ends up with a squeezed black hole configuration.
with \( A > 4S \approx A_i \), or total entropy significantly less than \( A/4 \).

If the squeezing of the shell is accomplished purely by tensile forces within the shell that reduce its area, there is a minimum to the area of the shell (where the tensile forces needed to hold the shell in place diverge) and a maximum to the area of the squeezed black hole, such that \( A - 4S \) cannot rise above some constant (depending on the radiation constant and hence on the number of matter fields present) times \( M_i \) (or some related constant times \( \sqrt{S} \)). However, if the squeezing is accomplished by applying radial forces to the shell, it can be squeezed past its minimum area to a sequence of configurations in which both its area and the area of the black hole inside increase by a fraction of order unity, with now the limitation being the onset of Planck-scale curvatures. To borrow the language from another field, when it is time to push, the black hole dilates.

Perhaps the simplest way to incorporate these \( S < A/4 \) configurations into black hole thermodynamics is modify the Generalized Second Law to state that

\[
\tilde{S}_{\text{GSL}} \equiv S_{\text{bh}} + S_m
\]  

(4)

does not decrease for a suitably coarse-grained nonnegative \( S_m \) and for a suitable definition of \( S_{\text{bh}} \) that reduces to \( A/4 \) (in Einstein gravity) when there are no constraints on the near-horizon thermal atmosphere but which is less than \( A/4 \) when the atmosphere is constrained. One might interpret \( S_{\text{bh}} \) as arising entirely from the near-horizon thermal atmosphere, so that if the atmosphere is unconstrained in the vertical direction, its entropy is at least approximately \( A/4 \). (There is no fundamental difficulty in allowing that in this unconstrained case, \( S_{\text{bh}} \) might also have other smaller correction terms, such as a logarithm of the number of fields or a logarithm of \( A \) or of some other black hole parameter. It is just that in the unconstrained case, the leading term of \( S_{\text{bh}} \) should be proportional to \( A \), and the coefficient should be \( 1/4 \), at least in Einstein gravity.) But if the near-horizon thermal atmosphere is constrained, it has less entropy.

An alternative way to incorporate these \( S < A/4 \) configurations is to retain the Generalized Second Law in the original form of Eq. (1), which is the special case of Eq. (3) in which \( S_{\text{bh}} = A/4 \), but now to allow \( S_m \) to become negative when one squeezes the black hole. For example, one might use Eq. (1) not to define \( S_{\text{GSL}} \) in terms of \( A/4 \) and \( S_m \), but instead to define \( S_m \) as the total entropy \( S_{\text{GSL}} \) minus the black hole entropy \( A/4 \). (Of course, this procedure would make the GSL useless for telling what the total entropy is, so then \( S_{\text{GSL}} \) would have to be found by some other procedure.)

An analogue in which such a definition would give negative entropies for some subsystems would be the case in which one used the von Neumann fine-grained entropy (2) for the entropy of a total system (analogous to \( S_{\text{GSL}} \)) and for the entropy
of one subsystem (analogous to $S_{bh} = A/4$) and then simply defined the entropy of the second subsystem (analogous to $S_m$) to be the total system entropy minus the first subsystem entropy. If the two subsystems making up the total system have sufficient quantum correlations or entanglements, the entropy of the second subsystem thus defined can be negative. For example, if the complete system is in a pure state that entangles the two subsystems, the von Neumann fine-grained entropy of the total system would be zero, whereas the first subsystem would be in a mixed state with positive von Neumann entropy, so by the definition above, the entropy of the second subsystem would be the negative of that positive quantity.

In pursuing this analogue, one could certainly take $S_{GSL}$ to be the von Neumann fine-grained entropy of the total universe, though then under unitary evolution (e.g., no information loss in black hole formation and evaporation), this entropy would remain constant, so the Generalized Second Law would be rather trivial. (Or one could say the nontriviality is all in the fact that the evolution is unitary.) But the analogue would almost certainly break down for the black hole entropy $S_{bh}$ if it is assumed to be $A/4$, since the von Neumann fine-grained entropy of a black hole would not in general equal $A/4$. One would expect it to be approximately $A/4$ when the black hole is maximally mixed for its area, but, at least under the assumption of unitary evolution, the actual von Neumann entropy of a black hole subsystem formed from some system of significantly smaller entropy (e.g., from the collapse of a star) would be expected to be much smaller than this maximum value of approximately $A/4$ until the black hole has emitted radiation (with which it thus becomes entangled) with entropy at least as great as $A/4$ of the remaining black hole [3].

For example, suppose we take a star of ten solar masses, or $M \sim 10^{39}$ in Planck units. Its initial entropy will be of the order of the number of nucleons (of mass $m_n \sim 10^{-19}$) that it has, $S_i \sim M/m_n \sim 10^{58}$. If this star collapses into a black hole of the same mass (ignoring the mass ejection that would realistically take place), $A/4 = 4\pi M^2 \sim 10^{70}$, but the von Neumann entropy would remain near $10^{58}$, 21 orders of magnitude smaller, for a very long time, $\sim 10^3 M S_i \sim 10^{100} \sim 10^{49}$ yr, before the black hole increases its von Neumann entropy significantly above $S_i$ by emitting, and become entangled with, radiation of significantly more entropy than $S_i$. If one squeezed the black hole so that $A - 4S \sim S$ and assumed that the original GSL Eq. [1] were valid, one would need $S_m \sim -10^{79}$, which is not only negative but also is about 21 orders of magnitude in size larger than the relevant von Neumann entropies of the complete system and of the black hole that are of the order of $S_i$. Therefore, it is rather hard to interpret such an enormous negative $S_m$ as arising from differences of these two von Neumann entropies, though it still might be possible in terms of field correlations across a region of the height of the
constrained near-horizon thermal atmosphere.

Let us now try to estimate what the total entropy is of an uncharged, nonrotating black hole configuration of mass $M$ and area $A = 16\pi M^2$, in equilibrium with Hawking thermal radiation inside a perfectly reflecting pure-state shell of radius $R$ and local mass $\mu$, outside of which one has vacuum. We shall take a semiclassical approximation with a certain set of matter fields, which for simplicity will all be assumed to be massless free conformally coupled fields. Given the field content of the theory, the three parameters $(M, R, \mu)$ determine the configuration, though the entropy should depend only on $M$ and $R$, since the shell and the vacuum outside have zero entropy.

It is convenient to replace the shell radius $R$ with the classically dimensionless parameter

$$W = \frac{2M}{R},$$

which would be 0 if the shell were at infinite radius (though before one reached this limit the black hole inside the shell would become unstable to evaporating away) and 1 if the shell were at the black hole horizon (though in this limit the forces on the shell would have to be infinite). Then we would like to find $S(M, W)$.

If $W$ is neither too close to 0 nor to 1, the entropy will be dominated by $A/4 = 4\pi M^2$. The dominant relative correction to this will come from effects of the thermal radiation and vacuum polarization around the hole and so would have a factor of $\hbar$ if I were using gravitational units ($c = G = 1$) instead of Planck units ($\hbar = c = G = 1$). In gravitational units, $\hbar$ is the square of the Planck mass, so to get a dimensionless quantity from that, one must divide by $M^2$ (or by $R^2$, which is just $4M^2/W^2$ with $W$ being of order unity); for the free massless fields under consideration, there are no other mass scales in the problem other than the Planck mass. Therefore, in Planck units, the first relative correction to $A/4$ will have a factor of $1/M^2$ and hence give an additive correction term to $4\pi M^2$ that is of the zeroth power of $M$, a function purely of $W$. One might expect that if one proceeded further in this way, one would find that the entropy $S$ is given by $4\pi M^2$ times a whole power series in $1/M^2$, with each term but the zeroth-order one having a coefficient that is a function of $W$. If we had been considering the possibility of massive fields, then these coefficients of the various powers of $1/M^2$ would not be purely functions of $W$ but would also be functions of the masses of the fields. However, for simplicity we shall consider only the free massless field case here.

In fact, I shall consider only the first two terms in this power series,

$$S(M, W) = 4\pi M^2 + f_1(W) + f_2(W)M^{-2} + f_3(W)M^{-4} + \cdots \approx 4\pi M^2 + f_1(W).$$

(6)
The function $f_1(W)$ will depend on the massless matter fields present in the theory, most predominantly through the radiation constant

$$a_r = \frac{\pi^2}{30}(n_b + \frac{7}{8}n_f),$$

where $n_b$ is the number of bosonic helicity states and $n_f$ is the number of fermionic helicity states for each momentum. It also proves convenient to define

$$\alpha \equiv \frac{a_r}{384\pi^3} = \frac{n_b + \frac{7}{8}n_f}{11520\pi},$$

which makes the entropy density of the thermal Hawking radiation far from the hole (when $R \gg M$ or $W \ll 1$) simply $\alpha/M^3$, and also to set

$$f_1(W) = -32\pi\alpha s(W),$$

where $s(W)$ depends (weakly) only on the ratios of the numbers of particles of different spins and so stays fixed if one doubles the number of each kind of species. Then my truncated power series expression for the entropy of an uncharged spherical black hole of mass $M$ at the horizon (and hence horizon radius $2M$ and horizon area $4\pi M^2$) surrounded by a perfectly reflecting shell of radius $R = 2M/W$ is

$$S(M, W) \approx 4\pi M^2 - 32\pi\alpha s(W) = \frac{1}{4}A - 32\pi\alpha s(W),$$

Now I shall evaluate an approximate expression for $s(W)$ when the perturbation to the Schwarzschild geometry is small from the thermal radiation inside the shell and from the vacuum polarization inside and outside the shell. There will be an additive constant to $s(W)$, giving an additive constant to the entropy, that I shall not be able to evaluate, but for simplicity and concreteness I shall assume that $s(1/2) = 0$, so that the entropy is $A/4$ when the shell is at $W = 1/2$ or $R = 4M$.

First, I shall ignore the Casimir energy and related effects of the shell itself on the fields. I would expect that these effects would give additive corrections to $s(W)$ that are of order $W$ or smaller (and so never large compared with unity), whereas the leading term in the perturbative approximation for $s(W)$ will go as $1/W^3$ (proportional to the volume inside the shell) for $W \ll 1$ (shell radius $R \gg 2M$) and as $1/(1 - W)$ (inversely proportional to the redshift factor to infinity) for $1 - W \ll 1$ (shell radius relatively near the horizon), so one or other of these leading terms will dominate when $W$ is near 0 or 1. Therefore, I shall take the stress-energy tensor inside the shell to be approximately that of the Hartle-Hawking state in the Schwarzschild geometry, and that outside the shell to be approximately that of the Boulware vacuum.
The first part of the analysis will be done in a coordinate system \((x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)\) in which the spherically symmetric classical metric has, at each stage of the process, the approximately static form

\[
d s^2 = -e^{2\phi} dt^2 + U^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]  

(11)

with

\[
e^{2\phi} = e^{2\psi} U
\]  

(12)

and

\[
U = 1 - \frac{2m}{r} = 1 - w
\]  

(13)

with

\[
w \equiv \frac{2m}{r} = 1 - U = 1 - (\nabla r)^2.
\]  

(14)

Here \(\phi, U, \psi, m,\) and \(w\) are all functions of the \(x^1 = r\) coordinate alone, although they also have a global dependence on the black hole mass \(M\) (the value of \(r/2\) where \(e^{2\phi} = 0\)), and (for \(r > R\)) on the radius \(r = R = 2M/W\) of the shell and on the total rest mass energy \(\mu\) of the shell. The Einstein equations then give

\[
\frac{d\psi}{dr} = 4\pi r (\rho + P)U^{-1}
\]  

(15)

and

\[
\frac{dm}{dr} = 4\pi r^2 \rho,
\]  

(16)

where

\[
\rho = -\langle T^0_0(r) \rangle
\]  

(17)

is the expectation value of the energy density in the appropriate quantum state, and

\[
P = \langle T^1_1(r) \rangle
\]  

(18)

is the corresponding expectation value of the radial pressure, both functions of \(r\). The functional form of the expectation value of the tangential pressure \(\langle T^2_2(r) \rangle = \langle T^3_3(r) \rangle\) would then follow from the conservation of \(\langle T^\mu_\nu \rangle\) but will not be explicitly needed in this paper.

Since we are assuming that the state of the quantum fields inside the shell \((r < R)\) is the Hartle-Hawking [4] thermal state, for \(r < R\) we have

\[
\rho = \rho_H(M, r) \equiv \frac{3\alpha}{32\pi M^4} \varepsilon_H(w)
\]  

(19)

and

\[
P = P_H(M, r) \equiv \frac{\alpha}{32\pi M^4} p_H(w),
\]  

(20)
where on the extreme right hand side of each of these two equations I have factored out the dependence on the black hole mass $M$ from that on the classically dimensionless radial function $w \equiv 2m/r$, thereby defining two classically dimensionless functions of $w$, $\varepsilon_H(w)$ and $p_H(w)$.

Similarly, we are assuming that the state of the quantum fields outside the shell ($r > R$) is the Boulware [5] vacuum state, so for $r > R$ we have

$$\rho = \rho_B(M_\infty, r) \equiv \frac{3\alpha}{32\pi M_\infty^4} \varepsilon_B(w)$$

(21)

and

$$P = P_B(M_\infty, r) \equiv \frac{\alpha}{32\pi M_\infty^4} p_B(w),$$

(22)

thereby defining two new classically dimensionless functions of $w$, $\varepsilon_B(w)$ and $p_B(w)$. Here

$$M_\infty \equiv m(r = \infty)$$

(23)

is the ADM mass at radial infinity.

In some cases below we shall assume that there is some extra apparatus in the region $r > R$ holding the shell in. If so, its energy density and radial pressure can simply be included in $\rho_B$ and $P_B$. In any case, we shall assume that whatever is outside the shell is in a pure state with zero entropy.

Below we shall also need the vacuum polarization part of the stress-energy tensor inside the shell, whose components I shall denote by

$$\rho_V(M, r) \equiv \rho_H(M, r) - \rho_T(M, r)$$

$$\equiv \frac{3\alpha}{32\pi M^4} \varepsilon_V(w) \equiv \frac{3\alpha}{32\pi M^4} (\varepsilon_H(w) - \varepsilon_T(w))$$

(24)

and

$$P_V(M, r) \equiv P_H(M, r) - P_T(M, r)$$

$$\equiv \frac{\alpha}{32\pi M^4} p_V(w) \equiv \frac{\alpha}{32\pi M^4} (p_H(w) - p_T(w)),$$

(25)

where $\rho_T$ and $P_T$ denote the components of the thermal parts.

I shall assume that the vacuum polarization part is what the Boulware state would give if one had it inside the shell, so that $\rho_V$ and $P_V$ have the same dependence on the local mass $m(r)$ and radius $r$ as $\rho_B$ and $P_B$ do outside the shell (when there is no extra apparatus there). In the first-order (in $\alpha/M^2$) perturbative calculation being done here, the expectation value of the stress tensor is already first order (except possibly for that of the shell), so its functional dependence on $m$ can be replaced by its dependence on its zeroth approximation, which is the black hole
mass $M$ for $r < R$ and the ADM mass $M_\infty$ for $r > R$. Therefore, to sufficient accuracy for our purposes, $\rho_V$ and $P_V$ can be evaluated by using Eqs. (21) and (22) for $\rho_B$ and $P_B$ with the ADM mass $M_\infty$, which is approximately the value of the local mass $m(r)$ anywhere outside the massive shell, replaced by the black hole mass $M$, which is approximately the value of $m(r)$ anywhere inside the shell. In particular, this implies that we can use

$$\varepsilon_V(w) = \varepsilon_B(w)$$

(26)

and

$$p_V(w) = p_B(w)$$

(27)

For explicit approximate calculations, it is useful to have explicit approximate formulas for these various components of the stress-energy tensor, given by the equations above from the six functions $\varepsilon_H(w)$, $\varepsilon_B(w)$, $\varepsilon_T(w) = \varepsilon_H(w) - \varepsilon_B(w)$, $p_H(w)$, $p_B(w)$, and $p_T(w) = p_H(w) - p_B(w)$. For simplicity and concreteness, I shall use those obtained for a conformally invariant massless scalar field in the gaussian approximation \[6\], which gives

$$\varepsilon_H(w) \equiv \frac{32\pi M^4}{3\alpha} \rho_H \equiv \frac{(8\pi M)^4}{3a_r} \rho_H \approx \frac{1 - (4 - 3w)^2w^6}{(1 - w)^2} - 24w^6$$

$$= 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 - 33w^6,$$

(28)

$$\varepsilon_B(w) \equiv \frac{32\pi M_\infty^4}{3\alpha} \rho_B \equiv \frac{(8\pi M_\infty)^4}{3a_r} \rho_B \approx \frac{-(4 - 3w)^2w^6}{(1 - w)^2} - 24w^6$$

$$= -\frac{1}{(1 - w)^2} + 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 - 33w^6,$$

(29)

$$\varepsilon_T(w) = \varepsilon_H(w) - \varepsilon_B(w) \approx \frac{1}{(1 - w)^2} = \frac{1}{U^2},$$

(30)

$$p_H(w) \equiv \frac{32\pi M^4}{\alpha} P_H \equiv \frac{(8\pi M)^4}{3a_r} P_H \approx \frac{1 - (4 - 3w)^2w^6}{(1 - w)^2} + 24w^6$$

$$= 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 15w^6,$$

(31)

$$p_B(w) \equiv \frac{32\pi M_\infty^4}{\alpha} P_B \equiv \frac{(8\pi M_\infty)^4}{3a_r} P_B \approx \frac{-(4 - 3w)^2w^6}{(1 - w)^2} + 24w^6$$

$$= -\frac{1}{(1 - w)^2} + 1 + 2w + 3w^2 + 4w^3 + 5w^4 + 6w^5 + 15w^6,$$

(32)

and

$$p_T(w) = p_H(w) - p_B(w) \approx \frac{1}{(1 - w)^2} = \frac{1}{U^2}.$$
Note that this approximation gives

$$\rho T \approx 3P T \approx a_r T_{\text{local}}^4,$$  \hspace{1cm} (34)

just like thermal radiation in flat spacetime, where $T_{\text{local}}$ is the local value of the Hawking temperature,

$$T_{\text{local}} \approx \frac{1}{8\pi m}(1 - \frac{2m}{r})^{-1/2}. \hspace{1cm} (35)$$

Now we use the Einstein equations (15) and (16) with the appropriate $\rho$ and $P$ on the right hand side, and with the metric function $U$ there taking on its approximate Schwarzschild form, $1 - 2M/r$ for $r < R$ and $1 - 2M_{\infty}/r$ for $r > R$.

We also need to consider the effect of the shell, which has a surface stress-energy tensor with components

$$S^0_0 = -\frac{\mu}{4\pi R^2} \hspace{1cm} (36)$$

and

$$S^2_2 = S^3_3 = -\frac{F}{2\pi R} \hspace{1cm} (37)$$

where $\mu$ is the total local mass of the shell, the shell area $4\pi R^2$ multiplied by the local mass-energy per area $-S^0_0$ as seen by a local observer fixed on the shell, and $F$ is the local total tensile force pulling together the two hemispheres of the shell, the circumference $2\pi R$ multiplied by the local surface tension (tensile force per length) $-S^2_2 = -S^3_3$.

If one integrates the Einstein equations (15) and (16) through the shell and uses the conservation law for the stress-energy tensor, one get the junction conditions [7] in the static case that

$$\mu = R(U_-^{1/2} - U_+^{1/2}) \hspace{1cm} (38)$$

and

$$8F = \frac{\mu}{R} + (1 + 8\pi R^2 P_-)U_-^{-1/2} - (1 + 8\pi R^2 P_+)U_+^{-1/2}, \hspace{1cm} (39)$$

where

$$U_- = 1 - \frac{2M_-}{R} \hspace{1cm} (40)$$

is the value of $U$ just inside the shell $(r = R-)$, where the local mass function $m$ takes on the value $M_-$, and

$$U_+ = 1 - \frac{2M_+}{R} \hspace{1cm} (41)$$

is the value of $U$ just outside the shell $(r = R+)$, where the local mass function $m$ takes on the value $M_+$. Similarly, $P_-$ and $P_+$ are the expectation values of the radial pressure of the respective quantum states just inside and just outside the shell.

Thus we have at least five relevant masses for the configuration: the black hole mass $M = m(r = 2M)$, the mass $M_- = m(r = R-)$ just inside the shell, the
local mass (or local energy) $\mu$ of the shell itself at radius $r = R$, the mass $M_+ = m(r = R+)$ just outside the shell, and the ADM mass $M_{\infty} = m(r = \infty)$ at radial infinity. Since the stress-energy tensor inside the shell is that of the Hartle-Hawking state determined by $M$ and $r$, $M_-$ is a function of $M$ and $R$. Similarly, since the stress-energy tensor outside the shell is that of the Boulware state determined by $M_{\infty}$ and $R$ (at least when we do not have an extra apparatus there to hold the shell in place), $M_+$ is a function of $M_{\infty}$ and $R$. The junction condition (28) then gives $\mu$ as a function of $M_-, M_+$, and $R$, and hence as a function of $M, M_{\infty}$, and $R$. One can in principle invert this to get $M_{\infty}$ (and hence all the other masses as well) as a function of $M, R$, and $\mu$, or to get $M$ and all other masses as a function of $M_{\infty}, R$, and $\mu$. The main point is that if we just have a black hole with the Hartle-Hawking thermal state inside a shell, and the Boulware vacuum state outside the shell, the semiclassical configuration (for fixed field content of the quantum field theory) is determined by three parameters, though only two of them (say $M$ and either $R$ or $W = 2M/R$) are relevant for the entropy which resides purely inside the shell.

To evaluate the function $s(W)$ in the truncated entropy formula (10), I shall consider an adiabatic process of slowly squeezing the shell, keeping the total entropy constant and thereby getting

$$\frac{ds}{dW} = \frac{M}{4\alpha} \frac{dM}{dW}$$

(42)
during this process. Since this process is not strictly static, one cannot use precisely the static metric (1) with $\phi$ and $U$ (or $\psi$ and $m$) that are purely functions of $r$ and obey the static Einstein equations (15) and (16). However, one can consider a quasi-static metric in which $\phi$ and $U$ (or $\psi$ and $m$) have a very slow dependence on the time coordinate $t$ and the Einstein equations are only slightly different from Eqs. (13) and (10).

The specific calculation which I shall do will be to have the shell squeeze itself inward by using its own internal energy, so that no apparatus is used outside the shell to push it inward, and that region has only the Boulware vacuum polarization. The contraction of the shell is assumed to be so slow that it does not excite the vacuum outside it but rather leaves it in the Boulware vacuum state with constant $M_{\infty}$. However, as the shell moves in, it is enlarging the Boulware state region, so effectively the shell must be creating a larger volume of vacuum with its vacuum polarization. This means that in the slowly inmoving frame of the shell, there is a flux of energy from the shell into the Boulware region, needed to enlarge the Boulware region while keeping it static where it already exists. [For the stress-energy tensor components of the Boulware vacuum given by Eqs. (29) and (32), this energy influx into the Boulware region is actually negative, so it increases the energy of the shell as it moves inward.]
Similarly, if the inside of the shell were also vacuum that did not get excited by the adiabatic contraction of the shell, there would be a swallowing up of part of the vacuum region by the shell as it moves inward. This would give a flux of (negative) energy from the vacuum inside into the shell, decreasing its energy. Surely this flux into the shell also exists even if the inside is not vacuum, and I assume that it is given by the vacuum polarization part of the actual stress-tensor there, which I take to be approximately that of a Boulware state with the same \( m \) and \( r \). The remaining part of the total stress-energy tensor there, which I am calling the thermal part, and which is given approximately by Eq. (34), should simply be reflected by the shell and not give an energy flux into it (in the frame of the slowly contracting shell), though it will contribute to the force that needs to be counterbalanced very nearly precisely to obey the static junction equation (39) to high accuracy in order that the shell not have any significant acceleration relative to a static frame.

In other words, I am assuming that if a shell moves inward through a static geometry, the vacuum polarization part of the stress-energy tensor will stay static, with \( T^0_0 = -\rho_v(M, r) \), \( T^1_1 = P_v(M, r) \), and \( T^1_0 = T^0_1 = 0 \) inside the shell, and with \( T^0_0 = -\rho_B(M_{\infty}, r) \), \( T^1_1 = P_B(M_{\infty}, r) \), and \( T^1_0 = T^0_1 = 0 \) outside the shell. Then as the shell moves through this static stress-energy tensor, in the frame of the shell, there will be fluxes of energy into or out from the shell on its two sides. In constrast, I am assuming that the thermal radiation part of the stress-energy tensor will be perfectly reflected by the shell, so that in the frame of the shell it will give no energy fluxes into or out from the shell.

There is a modification of this picture that occurs when the inward motion of the shell squeezes thermal radiation into the black hole so that its mass goes up. While the hole mass is increasing, the vacuum polarization inside the shell is not quite static but instead has small \( T^0_1 \) and \( T^1_0 \) terms that, for sufficiently slow adiabatic processes, are proportional to \( \dot{M} \), the coordinate time derivative of the black hole mass \( M \). In the present calculation, in which the shell is squeezing itself inward by using its own internal energy, the ADM mass \( M_{\infty} \) stays fixed, and so the vacuum stress-energy tensor outside the shell stays static during the process, under my approximation of neglecting Casimir-type boundary effects of the shell itself on the quantum field. For a sufficiently slow inward squeezing of the shell, the \( T^0_1 \) and \( T^1_0 \) terms inside are small, but over the correspondingly long time of the squeezing they contribute an effect on the energy balance of the shell that is not completely negligible when one contemplates squeezing the shell to a final position very near the black hole horizon. (My original neglect of these terms caused me considerable confusion during early stages of this work and lectures I gave about it.)

My present procedure for calculating the small \( T^0_1 \) and \( T^1_0 \) terms inside the shell is to assume that the shell squeezing, and all consequent processes, occur so slowly
that $T^0_0$ and $T^1_1$ are given to high accuracy by the same functions of $M$ and $r$ as they are when the geometry is static, namely $-\rho_V(M,r)$ and $P_V(M,r)$. Then I assume that the vacuum polarization part of the stress-energy tensor is itself conserved away from the shell, so one can use the conservation of its energy to deduce the radial derivative of $e^\psi r^2 T^0_0$.

In particular, if we let the vacuum polarization part of the stress-energy tensor have the component

$$T^1_0 = \frac{\alpha \dot{M} w^2}{4\pi M^4} e^{-\psi} f,$$

with the factors chosen so that $f$ is a function purely of $w$, then $T^\mu_0;_\mu = 0$ becomes

$$\pi \frac{\partial f}{\partial r} = \frac{\pi M^2 e^\psi r^2}{\alpha \dot{M}} \left[ \dot{\rho}_V + \frac{\dot{m}}{r} (\rho_V + P_V) \right].$$

For the region inside the shell with $r$ not too much larger than $2M$, one has $m \approx M$ and $e^\psi \approx 1$ (possibly after suitably normalizing the time coordinate $t$). Then if one uses Eqs. (24)-(27), one can rewrite Eq. (44) as

$$\frac{df}{dr} = -\frac{3w}{4} \frac{d}{dw} \left( \frac{\varepsilon_B}{w^4} \right) - \frac{3\varepsilon_B + p_B}{8w^3(1 - w)}.$$  

Given the functions $\varepsilon_B(w)$ and $p_B(w)$, e.g., as given by Eqs. (24) and (32) from the gaussian approximation for a conformally invariant massless scalar field, one can integrate Eq. (15) to obtain $f(w)$ up to a constant of integration. Although the constant of integration is not important, it can also be fixed by assuming that an observer that remains at fixed $w = 2m/r$ as $m$ changes sees in its frame no energy flux in the limit that $w$ is taken to unity, which implies that the flux of vacuum polarization energy through the horizon is taken to be zero.

After one calculates the vacuum polarization part of the stress-energy tensor, which gives $T^1_1 - T^0_0 = \rho_B + P_B$ and $T^1_0 = 0$ outside the shell and $T^1_1 - T^0_0 = \rho_V + P_V$ and $T^1_0$ as given by Eq. (13) inside the shell, one can then calculate the fluxes of energy out from and into the shell and insert these into the conservation equations for the surface stress-energy tensor of the shell. For a very slowly expanding or contracting shell, one finds that

$$d\mu = 4F dR + 4\pi R^2 dR \left[ (\rho_B + P_B) U^{-1/2}_+ - (\rho_V + P_V) U^{-1/2}_- \right] - 4\pi R^2 T^0_1 U^{-1/2}_- dt.$$  

The first term on the right hand side is the work done by the tensile force within the shell, and the remaining terms are the energy input from the vacuum stress-tensor components $\rho_B$ and $P_B$ just outside the shell and the vacuum stress-tensor components $\rho_V$, $P_V$, and $T^1_0$ just inside the shell.
One now combines this local energy conservation equation for the shell with the static junction equations (38) and (39) that should still apply to high accuracy in this slowly evolving situation to keep the shell radius from accelerating too rapidly. When one also combines this with the integrals of Eq. (16),

\[ M_- = M + \int_{2M}^{R} 4\pi r^2 dr \rho_H, \]  

\[ M_+ = M_\infty - \int_{R}^{\infty} 4\pi r^2 dr \rho_B, \]

one finds

\[ \left(1 - \frac{4\alpha f}{M^2}\right) dM \approx -4\pi R^2 dR (\rho_T + P_T) \]  

during the adiabatic contraction of the shell, which, up to the small correction factor involving \( f \), is precisely what one would get in flat spacetime from adiabatically contracting a ball of thermal radiation.

Next, we can use the fact that \( R = 2M/W \) to derive that

\[ \frac{dW}{dM} = \frac{2}{R} \left(1 - \frac{M}{R} \frac{dR}{dM}\right) \approx \frac{2}{R} \left[1 + \frac{M(1 - 4\alpha f/M^2)}{4\pi R^3 (\rho_T + P_T)}\right]. \]

where \( f \) and \( \rho_T + P_T \) are to be evaluated at \( r = R \) or \( w \approx W \). Inserting this back into Eq. (12) then gives

\[ \frac{ds}{dW} \approx \frac{3\varepsilon_B + p_B}{4W^4} \left\{1 + \frac{4\alpha}{M^2} \left[\frac{3\varepsilon_B + p_B}{4W^3} - f\right]\right\}^{-1}. \]

For massless particles of any spin, it should be a good approximation to take \( \rho_T + P_T \approx (4/3)\alpha_T T_{\text{local}}^4 \) in terms of the local temperature \( T_{\text{local}} \), and this implies that \( 3\varepsilon_B + p_B \approx 4/(1 - W)^2 \), so

\[ \frac{ds}{dW} \approx \frac{1}{W^4(1 - W)^2} \left\{1 + \frac{4\alpha}{M^2} \left[\frac{1}{W^3(1 - W)^2} - f\right]\right\}^{-1}. \]

If we omitted the \( f \) term from the radial flux of vacuum polarization energy when \( M \) changes, as I indeed first erroneously did, then the factor inside the curly brackets above would diverge as one approached the horizon, where \( W = 1 \). This implies that the reciprocal of this factor would cancel the divergence in the factor before it, so \( ds/dW \) would stay finite all the way down to \( W = 1 \), and one would find that the increase of one-quarter the area over the entropy would be limited to an amount of the order of \( \alpha M \). For large \( M \) this is large in absolute units, but it is always much smaller than the entropy itself, which is of the order of \( 4\pi M^2 \).

However, one can use the fact that the regularity of the Hartle-Hawking stress-energy tensor at the horizon implies that \( \rho_H + P_H \), and hence \( 3\varepsilon_H + p_H \), must go
to zero at least as fast as $1 - w$ as one approaches the horizon. (This is easiest to see in the Euclidean section with imaginary time $t$, on which for fixed coordinates $\theta$ and $\varphi$, the horizon is at the center of a regular rotationally symmetric two-surface with angular coordinate $i\kappa t$ with $\kappa \approx 1/(4M)$ being the black hole surface gravity and with the radial distance being roughly $4M\sqrt{1 - w}$ when $1 - w \ll 1$. Then $P_H = T_1^1$ is the pressure in the radial direction, and $-\rho_H = T_0^0$ is the Euclidean pressure in the Euclidean angular direction, and regularity at the origin demands that the difference go to zero at least as fast as the square of the radial distance from the origin.) Then one can show that $f$ cancels the divergence in $W^{-3}(1 - W)^{-2} - f$ stays finite as one approaches the horizon. In fact, if one chooses the constant of integration of $f$ so that the flux of vacuum polarization energy through the horizon is zero as $M$ is slowly changed, then $W^{-3}(1 - W)^{-2} - f$ actually goes to zero linearly with $1 - W$ as one approaches the horizon. For example, using this constant of integration and the gaussian approximation for $3\varepsilon_B$ and $p_B$ leads to

$$\frac{ds}{dW} \approx \frac{1}{W^4(1 - W)^2} \left\{ 1 + \frac{4\alpha}{M^2W^3}(1 - W)(1 + 3W + 6W^2 + 2W^3 + 7W^4 + 13W^5) \right\}^{-1}. \quad (53)$$

Therefore, we see that the correction term that is first order in $\alpha/M^2$ inside the curly brackets of Eqs. (51)-(53) does not diverge as one takes $1 - W$ to zero but instead always remains small. Therefore, we can drop it (as we have also neglected other finite corrections that are linear in $\alpha/M^2$) and integrate the zeroth-order part of Eq. (53) to get an explicit formula for $s(W)$:

$$s(W) \approx \int_{1/2}^{W} \frac{dw}{w^4(1 - w)^2} = \frac{1}{1 - W} + 4\ln \frac{W}{1 - W} - \frac{1}{3W^3} - \frac{1}{W^2} + \frac{3}{W} + \frac{32}{3}. \quad (54)$$

As discussed above, I arbitrarily chose the constant of integration of this integral to make $s(W) = 0$ at $W = 1/2$ or $R = 4M$, but this is not likely to be valid, and there are also Casimir energy effects from the shell and corrections to Eqs. (30) and (33) that would give correction terms at least of order $W$ and likely also of the order of a constant and of order $1/W$.

Finally, we can insert this form for $s(W)$ into Eq. (10) to get

$$S(M, W) = \frac{1}{4} A[1 - \frac{8\alpha}{M^2} s(W)] = 4\pi M^2 - 32\pi \alpha s(W)$$

$$\approx 4\pi M^2 - 32\pi \alpha \left[ \frac{1}{1 - W} + 4\ln \frac{W}{1 - W} - \frac{1}{3W^3} - \frac{1}{W^2} + O\left( \frac{1}{W} \right) \right]$$

$$\approx 4\pi M^2 - 32\pi \alpha \left[ \frac{R}{R - 2M} + 4\ln \frac{2M}{R - 2M} - \frac{R^3}{24M^3} - \frac{R^2}{4M^2} \right]. \quad (55)$$

where after the last approximate equality I have dropped the terms in Eq. (54) that I suspect are always dominated by corrections to my approximations that I have not
included. Although I have retained four terms from $s(W)$, only the first two terms should be kept when $1 - W = 1 - 2M/R \ll 1$ (shell very near the horizon), and only the next two terms should be retained when $W = 2M/R \ll 1$ (shell very large compared with the black hole).

The result indicated by Eq. (54) is precisely the same that one would obtain by taking the geometry to be Schwarzschild with a thermal bath of radiation with local Hawking temperature

$$T_{\text{local}} = \frac{1}{8\pi M} \left(1 - \frac{2M}{r}\right)^{-1/2}. \quad (56)$$

and entropy density $(4/3)a_r T_{\text{local}}^3$, and then taking the total entropy to be $4\pi M^2$ plus the entropy difference between that inside the shell at radius $R$ and that inside the radius $4M$. If one naively integrates this assumed entropy density all the way down to the horizon, one would get a divergence, but one can take the attitude that this divergence is regulated so that the entropy in this thermal atmosphere below some radius like $4M$ (the precise value of which doesn’t matter much, since the assumed entropy density is this region is of the order of $\alpha/M^3$) is the black hole entropy $S_{\text{bh}} \approx A/4 = 4\pi M^2$. Then one can say that if the shell is put at a much larger radius, the entropy of the thermal Hawking radiation outside $4M$ or so would be matter entropy $S_m$ that would add to $S_{\text{bh}}$, which is certainly an uncontroversial assumption.

What I have found from my consideration of having the shell squeezed in adiabatically is that if the shell is put much nearer the horizon than a radius of $4M$ or so is, then the entropy is correspondingly less than the usual black hole entropy $S_{\text{bh}} \approx A/4 = 4\pi M^2$. Because the thermal atmosphere is restricted from filling up the region to $4M$ or so, it does not have the entropy needed to make the total entropy as large as $A/4$.

The next question is the range of $W$ over which one would expect that Eq. (55) is approximately valid. For very small $W$ or very large $R$, one essentially has a black hole of mass $M$ surrounded by a much bigger volume, $V \sim 4\pi R^3/3$, of radiation in nearly flat spacetime with Hawking temperature $1/(8\pi M)^{-1}$, energy density roughly $3\alpha/(32\pi M^4)$, and entropy density roughly $\alpha/M^3$. The dominant term for the total energy of the radiation is $E_r \sim \alpha R^3/(8M^4)$, and from Eq. (55), the dominant term for the total entropy of the radiation is $S_r \sim 4\pi \alpha R^3/(3M^3)$. This agrees with the standard expression for the entropy of thermal radiation of energy $E_r$ in a volume $V$,

$$S_r = \frac{4}{3}(a_r V)^{1/4}E_r^{3/4} \sim \frac{4\pi}{3} \alpha^{1/4} \left(8RE_r\right)^{3/4}. \quad (57)$$

For fixed total energy $M_\infty = M + E_r \ll R$, the total entropy

$$S \approx 4\pi M^2 + S_r \sim 4\pi(M_\infty - E_r)^2 + \frac{4\pi}{3} \alpha^{1/4} \left(8RE_r\right)^{3/4} \quad (58)$$
is indeed extremized for
\[ E_r \sim \frac{\alpha R^3}{8M^4} = \frac{\alpha R^3}{8(M_\infty - E_r)^4}, \tag{59} \]
but the extremum is a local entropy maximum if and only if \( 5E_r \leq M_\infty \) or \( 4E_r \leq M \mathbb{R} \), which implies that one needs \( R \leq (2M^5/\alpha)^{1/3} \) or
\[ W \geq \left( \frac{4\alpha}{M^2} \right)^{1/3} \tag{60} \]
for thermodynamic stability.

For smaller values of \( W \) (larger values of \( R \)), the radiation energy \( E_r \) is more than 20% of the total available energy \( M_\infty \) (assumed to be held fixed), and then if the black hole emits some extra radiation and shrinks, it heats up more than the radiation does, leading to an instability in which the black hole radiates away completely. On the other hand, if the black hole absorbs some extra radiation, it will grow and cool down more than the surrounding radiation, therefore cooling down more and absorbing more radiation, until the radiation energy drops to the lower positive root of Eq. (59), which is less than \( 0.2M_\infty \) and hence is at least locally stable.

At the opposite extreme, the question is how small \( 1 - W \) can be. Here the fundamental limit is the Planck regime, which is the boundary of the semiclassical approximation being used in this paper. The Boulware vacuum energy density \( \rho_B \) just outside a massless shell (so that the mass just outside, \( M_+ \), is very nearly the same as the black hole mass \( M \); for positive shell mass \( \mu \), \( \rho_B \) would have an even greater magnitude) is, for very small \( U = 1 - W \), \( \rho_B \sim -3\alpha/(32\pi M^4 U^2) \). Suppose the semiclassical theory is valid until the orthonormal Einstein tensor component \( G_0^0 = -8\pi \rho_B \sim 3\alpha/(4M^4 U^2) \) reaches a maximum value of, say, \( C_M \), which would be expected to be of order unity (orthonormal curvature component of the order of the Planck value). This gives the restriction
\[ U = 1 - W \geq \left( \frac{3\alpha}{4C_M M^4} \right)^{1/2}. \tag{61} \]

For \( U = 1 - W \ll 1 \), the spatial distance from the shell to the horizon is \( D \sim 4MU^{1/2} \), so this restriction on \( U \) gives a minimum distance the shell can be from the horizon:
\[ D \geq \left( \frac{192\alpha}{C_M} \right)^{1/4}, \tag{62} \]
in Planck units, as all quantities are in this paper unless otherwise specified.

If we combine the restriction (61) with the lower bound on \( W \) from Eq. (60) and re-express the combined restriction as a restriction on the radius \( R \) of the shell, we
get

\[ 2M + \frac{1}{M} \sqrt{\frac{3\alpha}{C_M}} \leq R \leq \left( \frac{2M^5}{\alpha} \right)^{1/3}. \]  

(63)

Alternatively, in terms of the distance \( D \) of the shell to the horizon (which is \( D \sim R \) for \( R \gg 2M \)), we get

\[ \left( \frac{192\alpha}{C_M} \right)^{1/4} \leq D \leq \left( \frac{2M^5}{\alpha} \right)^{1/3}. \]  

(64)

If we now insert the restriction (61) or (62) into the asymptotic form of the total entropy (55) for \( U = 1 - W \ll 1 \), which is

\begin{align*}
S(M, W) &\sim 4\pi M^2 - \frac{32\pi\alpha}{U} \sim 4\pi M^2 \left( 1 - \frac{128\alpha}{D^2} \right) \\
&\sim 4\pi M^2 - \frac{8\pi\alpha}{U} \sim 4\pi M^2 \left( 1 - \frac{32\alpha}{D^2} \right),
\end{align*}

(65)

we get the limitation

\[ S(M, W) \geq 4\pi M^2 \left( 1 - 16\sqrt{\frac{\alpha C_M}{3}} \right) = 4\pi M^2 \left( 1 - \sqrt{\frac{C_M}{135\pi}} \left( n_b + \frac{7}{8} n_f \right) \right). \]  

(66)

This can be re-expressed as a limitation on how much the area \( A \) of a black hole can exceed four time the entropy, \( 4S \):

\[ A - 4S \leq A \sqrt{\frac{C_M}{135\pi}} \left( n_b + \frac{7}{8} n_f \right). \]  

(67)

Therefore, unless we have \( N \equiv n_b + 7n_f/8 \), the effective number of one-helicity particles, comparable to or greater than \( 135\pi/C_M \approx 424/C_M \), the fractional increase of the black hole area \( A \) above \( 4S \) is restricted to be rather small, though even just \( N = 4 \) from two-helicity gravitons and photons would give a fractional increase of about 10% if the curvature limitation \( C_M \) is one in Planck units.

However, this does raise the interesting theoretical question of what would happen in a theory in which the effective number \( N \) of particles is so large that \( NC_M/(135\pi) \) is bigger than unity. Naïvely it would then appear that, without running into excessive curvatures (i.e., \( G_0^0 > C_M \)), one could put the shell sufficiently close to the black hole so that the total entropy, given approximately by Eq. (63), would be negative, which is surely nonsense. Of course, one could never get to such a configuration by adiabatically compressing the shell, since one would then have started out with positive entropy that would not decrease (though it does appear that one could in principle push hard and far enough on the shell so that the black hole could dilate to an arbitrarily large radius for the fixed initial entropy, which by
itself would be rather remarkable). However, one could imagine constructing such a shell in place and then evacuating the thermal radiation from above it, which would seem to leave behind a black hole configuration of negative entropy. In our world this possibility might be excluded by the limited number of particle species that exist and contribute to the thermal Hawking radiation, but one would like to see a direct argument of why negative total entropies of black hole configurations cannot be achieved even in a model universe in which one had a huge number of fields.

The first guess that came to me off the top of my head is that of course no physical shell can be perfectly reflecting. A partially reflecting shell should be able to squeeze enough of the Hawking radiation into a black hole to raise its area above four times the entropy, but the transmission will put a limit on how effective this process can be. If there are more species of particles that can be partially transmitted, the shell may become less effective in increasing the black hole area, and conceivably this could offset the increase in the otherwise theoretically allowed fractional area increase from the increase of the number of species. However, this is just a wild guess, and so the problem will be left as an exercise for the reader.

One refinement of the results above that should be explained is that although I have used a Schwarzschildian coordinate system \( \Sigma \) in the analysis described so far, this system breaks down when one follows the Boulware vacuum sufficiently far inward. In particular, the Schwarzschildian radial coordinate \( r \), which is \( 1/(2\pi) \) times the circumference of the two-sphere, is only a good coordinate when it decreases monotonically inward. This indeed occurs when \( U = 1 - 2m/r = (dr/dD)^2 \) remains positive, where \( D \) is the proper radial distance. But because the Boulware vacuum energy density \( \rho_B \) is negative for sufficiently small \( U \), as one moves inward with \( r \) initially decreasing, the mass function \( m(r) \) increases rather than decreases, and eventually one reaches a radius where \( 2m = r \) and hence \( U = 0 \). In this case, \( -g_{00} = e^{2\phi} \) remains positive (which implies that \( e^{2\psi} = e^{2\phi} U^{-1} \) diverges, so \( \psi \) is no longer a good metric function either), which means that one has not reached the horizon, but rather just a location where the gradient of the Schwarzschildian radial coordinate vanishes. Inside this point, the gradient of \( r \) reverses sign, so now \( r \) increases as one moves inward. The Einstein equation (16) implies, since the energy density remains negative, that the mass function \( m \) now decreases as one moves inward.

Since at small radii the Boulware vacuum polarization gives a stress-energy tensor that is dominated by contributions that look like thermal radiation but with the opposite sign (e.g., an isotropic pressure that is one third the energy density, both of which are negative), one can incorporate an approximation for its back reaction on the metric simply by solving the Tolman-Oppenheimer-Volkoff equations for such
a thermal fluid with
\[ P_B \approx \frac{1}{3} \rho_B \approx -P_T \approx -\frac{1}{3} a_r T_{\text{local}}^4 \approx -\frac{\alpha}{32 \pi M_\infty^4} e^{-4\phi}. \]  
(68)

If one defines the function
\[ W \equiv 1 + 8\pi r^2 P_B \]  
(69)
(not to be confused with the previous use of \( W \) as \( 2M/R \)), then one gets the two differential equations
\[ \begin{align*}
U (4 - U - 3W) dW &= 2(W - 2U)(1 - W) dU \\
&= 2(W - 2U)(1 - W)(4 - U - 3W) dr/r.
\end{align*} \]  
(70)

One can match with the Schwarzschild metric, slightly perturbed by the Boulware vacuum polarization, at \( \sqrt{\alpha}/M_\infty \ll U \ll 1 \), where \( 1 - W \approx \alpha M_\infty^2 (1 - U)^{-2} U^{-2} \) and \( r \approx 2M_\infty (1 - U)^{-1} \). Then as one integrates Eq. (70) inward, initially \( r, U, \) and \( W \) all decrease, until \( U \) and \( W \) simultaneously go to zero. This is a singular point of Eq. (70), but one can easily see that \( U \) goes to zero quadratically with \( W \), so that as \( W \) crosses zero and becomes negative, \( U \) becomes positive again. If we define the new variable
\[ Y \equiv \frac{U W}{1 + 8\pi r^2 P}, \]  
(71)
we find that it decreases monotonically as we go inward, starting at the small positive value \( Y = Y_0 \ll 1 \) in order that one be in the regime where Eq. (68) is valid, but then with \( Y \) going negative where \( U \) goes to zero quadratically with \( W \).

In terms of \( Y \) and \( W \), the first differential equation of (70) becomes
\[ \frac{dW}{dY} = \frac{2(1 - 2Y)W(1 - W)}{Y[(1 + 2Y)(2 - W) - 3YW]} \]  
(72)
Although this equation is also singular at \( Y = 0 \) and \( W = 0 \), \( Y dW/W dY = 1 \) there, so \( W \) just passes through zero linearly with \( Y \), with a positive coefficient that one can calculate is roughly \( M_\infty/\sqrt{\alpha} \). Then one can easily see that the right hand side of Eq. (72) is always positive, since \( 1 - W > 0 \) and \( 1 - 2Y > 0 \) everywhere where Eq. (68) is valid, so \( Y \) and \( W \) are both monotonically decreasing variables as one moves inward through the negative pressure (and negative energy-density) vacuum Boulware stress-energy tensor.

Eq. (72) is not separable, but by pulling out the separable parts, one can get
\[ \frac{W}{Y \sqrt{1 - W}} \propto \exp \left[ - \int \frac{(8 - 7W) dY}{(2 - W) + (4 - 5W) Y} \right]. \]  
(73)
For the initial conditions above with \( \sqrt{\alpha}/M_\infty \ll 1 \) one can now integrate Eq. (73) approximately in the various regimes for \( W \) and match them:
Initially, \( W \) is near 1, and so

\[
\frac{W}{Y \sqrt{1 - W}} \approx \frac{M_\infty}{\sqrt{\alpha}} (1 - Y). \tag{74}
\]

In particular, the right hand side is correct to first order in \( Y \) when \( \sqrt{\alpha}/M_\infty \ll Y \). Then as \( W \) goes from near 1 to much less than \(-1\), \( Y \) is so small and changes so little that the integral in Eq. (73) has only a negligible contribution. Therefore, Eq. (74) remains approximately valid while \( W \) drops from being near 1 until it becomes sufficiently negative that \( Y \) no longer has a small magnitude, though the right hand side is no longer correct to first order in the small quantity \( Y \); an expression correct to first order in \( Y \) for \(-\sqrt{\alpha}/M_\infty \ll Y \ll \sqrt{\alpha}/M_\infty \), which implies \(|W| \ll 1\), is

\[
\frac{W}{Y \sqrt{1 - W}} \approx \frac{M_\infty}{\sqrt{\alpha}} (1 + 2Y)^{-2}. \tag{75}
\]

Finally, when \( W \) is very large and negative, \( Y \) can grow from its tiny value and gives, for \( 0.2 < Y \ll -\sqrt{\alpha}/M_\infty \),

\[
\frac{W}{Y \sqrt{1 - W}} \approx \frac{M_\infty}{\sqrt{\alpha}} (1 + 5Y)^{-7/5}. \tag{76}
\]

If one only needs an expression for \( W/(Y \sqrt{1 - W}) \) that is approximately correct but not necessarily correct to first order in \( Y \), then Eq. (76) can be taken to apply over the whole allowed range where Eq. (68) is valid, namely \(-0.2 < Y \leq Y_0 \ll 1\). One can then algebraically solve Eq. (76) for \( W \equiv 1 + 8\pi r^2 P_B \) to get the explicit approximate formulas for \( W \equiv 1 + 8\pi r^2 P_B \) and \( U \equiv 1 - 2m/r \equiv WY \) in terms of \( Y \equiv U/W \):

\[
W \approx \frac{M_\infty}{\sqrt{\alpha}} Y(1 + 5Y)^{-7/5} \left[ 1 + \frac{M_\infty^2}{4\alpha} Y^2(1 + 5Y)^{-14/5} \right]^{1/2} - \frac{M_\infty^2}{2\alpha} Y^2(1 + 5Y)^{-14/5}, \tag{77}
\]

\[
U \approx \frac{M_\infty}{\sqrt{\alpha}} Y^2(1 + 5Y)^{-7/5} \left[ 1 + \frac{M_\infty^2}{4\alpha} Y^2(1 + 5Y)^{-14/5} \right]^{1/2} - \frac{M_\infty^2}{2\alpha} Y^3(1 + 5Y)^{-14/5}. \tag{78}
\]

One can go on to use Eqs. (68)-(70) to solve for the behavior of \( r, m, \) and \( \phi \). To avoid expressions that are too cumbersome, it is useful to divide up the region where Eq. (68) is valid into three overlapping regions and give explicit approximate results for each region separately.

In Region 1, where \( 1 - W \ll 1 \) or \( Y \gg \sqrt{\alpha}/M_\infty \), one gets

\[
r \approx \frac{2M_\infty}{1 - Y}, \tag{79}
\]

\[
m \approx M_\infty + \frac{3\alpha}{M_\infty Y}, \tag{80}
\]

22
\[ \phi \approx \frac{1}{2} \ln Y. \]  
(81)

In Region 2, where \( U \ll 1 \), which is always true for positive \( Y \leq Y_0 \ll 1 \) and is true also for negative \( Y \) if \(-Y \ll (M_\infty^2/\alpha)^{1/3} \), one gets
\[ r \approx 2M_\infty(1 + \sqrt{4\alpha/M_\infty^2 + Y^2}), \]  
(82)
\[ m \approx M_\infty \left[ 1 + \frac{M_\infty}{2\sqrt{\alpha}} Y^2 + \left( 1 - \frac{M_\infty^2}{4\alpha} Y - \frac{M_\infty^2}{4\alpha} Y^2 \right) \sqrt{\frac{4\alpha}{M_\infty^2}} + Y^2 \right], \]  
(83)
\[ \phi \approx \frac{1}{4} \ln \frac{\alpha}{M_\infty^2} - \frac{1}{2} \sinh^{-1} \frac{M_\infty}{2\sqrt{\alpha} Y}. \]  
(84)

One can see that \( r \) reaches a minimum value of roughly \( r_m \approx 2M_\infty + 4\sqrt{\alpha} \) at \( Y = 0 \), where the mass function \( m \) attains its maximum value \( m_m = r_m/2 \approx M_\infty + 2\sqrt{\alpha} \). Then as one moves further inward, to negative values of \( Y \) and \( W \), \( r \) increases again, and \( m \) decreases.

In Region 3, where \(-W \gg 1 \) or \(-Y \gg \sqrt{\alpha/M_\infty^2} \), one gets
\[ r \approx 2M_\infty(1 + 5Y)^{-1/5}, \]  
(85)
\[ m \approx M_\infty - \frac{1}{\alpha} \left( \frac{-M_\infty Y}{1 + 5Y} \right)^3, \]  
(86)
\[ \phi \approx -\frac{1}{2} \ln \left( \frac{-M_\infty Y}{\alpha} \right) + \frac{3}{5} \ln (1 + 5Y). \]  
(87)

One can see that the mass function \( m \) crosses zero at \( Y \approx (\alpha/M_\infty^2)^{1/3} \) and thereafter becomes negative, approaching negative infinity as \( Y \) approaches its lower limit of \(-0.2 \).

If we solve Eq. (83) for \( Y \) in terms of \( r \), then we can write the approximate metric in Region 3 in terms of explicit functions of \( r \):
\[ ds^2 \approx -\frac{5\alpha}{M_\infty^2} \left( \frac{2M_\infty}{r} \right)^6 \left[ 1 - \left( \frac{2M_\infty}{r} \right)^5 \right]^{-1} dt^2 \]
\[ + \frac{125\alpha}{M_\infty^2} \left( \frac{2M_\infty}{r} \right)^{14} \left[ 1 - \left( \frac{2M_\infty}{r} \right)^5 \right]^{-3} dr^2 \]
\[ + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \]  
(88)

One can see that there is a naked singularity, where \( r \) goes to \(+\infty \) and \(-g_{00} \) goes to zero, at a finite proper radial distance. In fact, for \( r \gg M_\infty \), the proper distance to the singularity along a geodesic with constant \( t \), \( \theta \), and \( \varphi \) is, if one were to continue using this metric in a region where the curvature it indicates is comparable to the Planck values,
\[ \ell \approx \sqrt{\frac{125\alpha}{3}} \left( \frac{2M_\infty}{r} \right)^6. \]  
(89)
Therefore, for $\ell \ll \sqrt{\alpha}$, the metric (88) has the approximate form

$$ds^2 \approx -\sqrt{\frac{9\alpha}{5}} \frac{\ell dt^2}{M_\infty^2} + d\ell^2 + \left(\frac{125}{9\alpha}\right)^{1/6} (2M_\infty)^2 \ell^{-1/3} (d\theta^2 + \sin^2 \theta d\varphi^2).$$  \hspace{1cm} (90)

A slightly cruder form for the approximate metric of the Boulware vacuum region with backreaction, but one which applies over the whole spacetime, uses (in this section only; elsewhere $R$ is the circumference of the shell divided by $2\pi$)

$$R \equiv \frac{2M_\infty}{1 - e^{2\phi}} \equiv \frac{2M_\infty}{1 + g_{00}}$$  \hspace{1cm} (91)

as the independent radial variable, since this variable, like $e^{2\phi} \equiv -g_{00}$, varies monotonically with radial distance, from $R = 2M_\infty$ at the naked singularity to $R = \infty$ at radial infinity. Then the metric takes the approximate form

$$ds^2 \approx - \left(1 - \frac{2M_\infty}{R}\right) dt^2$$

$$+ \left(1 + \frac{8\alpha}{M_\infty^2} - \frac{2M_\infty}{R}\right)^{-1} dR^2$$

$$+ R^2 \left[1 + \frac{6\alpha}{M_\infty^2 (1 - 2M_\infty/R)}\right]^{1/3} (d\theta^2 + \sin^2 \theta d\varphi^2).$$  \hspace{1cm} (92)

For $\alpha/M_\infty^2 \ll 1$ as we have always been assuming, this metric reduces to very nearly the Schwarzschild metric for $R \ll 2M_\infty \gg \alpha/M_\infty$, which includes the region with $Y \gg Y_0$ for which Eq. (38) is not valid, and it also gives a reasonably good approximation to the metric in Regions 1 and 2 where Eq. (38) is likely to be valid. In the part of Region 3 where $r \gg M_\infty$, the metric (92) gives a proper distance to the singularity that is roughly $\sqrt{0.9}$ times the distance in the metric (90) at the same value of $-g_{00}$, and the circumference of the spheres in the metric (92) is roughly $(1.2)^{1/6}$ the amount given by the metric (90), but at least the qualitative behavior agrees. Furthermore, the metric (90) is supposed to apply at a distance $\ell$ that is less than one Planck length from the naked singularity, and there one would expect quantum gravity effects to change the form of the metric or invalidate the use of a semiclassical metric altogether. Therefore, I propose that the metric (92) may be taken as a reasonably good approximation to the metric of an asymptotically flat static spherically symmetric vacuum region when the backreaction of the Boulware stress-tensor is self-consistently taken into account in a semiclassical approximation, and when one avoids the high curvature region where the semiclassical approximation is expected to break down.

We found in Eq. (53) that for a neutral spherical black hole of area $A = 4\pi M^2$ surrounded by a perfectly reflecting shell at $R - 2M \ll M$, the entropy is roughly

$$S \approx \frac{1}{4} A - \frac{32\pi \alpha R}{R - 2M} = \frac{1}{4} A - \frac{n_n + \frac{7}{8} n_f}{360(1 - 2M/R)}.$$  \hspace{1cm} (93)
The last term represents the leading term for the reduction of the entropy below one-quarter the area. Let us ask how large this term can be for various assumptions about the shell.

First, consider the case that the shell is held up entirely by its own stresses, with no external forces (other than gravity) on it. In particular, we shall consider the static shell junction conditions (38) and (39), applying the strong energy condition to the shell so that its surface stress obeys the inequality $S_2^2 = S_3^3 \leq -S_0^0$. As we shall soon see, it then turns out that $U^- = 1 - 2M/R \approx 1 - W = 1 - 2M/R$ cannot be very small, so the terms involving the pressures inside and outside the shell are then negligible. Then the strong energy condition applied to the junction conditions (38) and (39) imply that $U^+ + U^- \geq 1/25$, and since Eq. (38) implies that a shell with positive local mass has $U^+ < U^-$, we see that $1 - W \approx U^- > 1/5$, or $R > 2.5M$. If Eq. (93) applied for such a large value of $1 - W$, it would then give

$$\frac{1}{4} A - S \approx \frac{n_b + \frac{7}{8} n_f}{360(1 - 2M/R)} < \frac{n_b + \frac{7}{8} n_f}{72},$$

(94)

a quite negligible decrease in the entropy, unless somehow $n_b + (7/8)n_f$ is very large.

Next, consider the case that the shell has charge $Q$, so that its electrostatic repulsion holds it up. Since we found above that the stresses within the surface of the shell are quite ineffectual in holding up the shell at $R - 2M \ll M$, let us drop them from the junctions equations but add the tension of the electromagnetic field outside the shell and assume that that tension is much greater than the radial pressures (or tensions) of the quantum fields. Then the junction conditions (38) and (39) become

$$\frac{\mu}{R} = U^{1/2} - U^{1/2}_+$$

(95)

and

$$0 = 8F = \frac{\mu}{R} + U^{-1/2} - (1 - \frac{Q^2}{R^2})U^{-1/2}_+,$$

(96)

Now for fixed charge-to-mass ratio $Q/\mu$, if we let $\gamma = (\mu/R)/U^{1/2}_+ < 1$, Eq. (95) implies that $U^{1/2}_+ = (1 - \gamma)(\mu/R)$, which when inserted into Eq. (96) gives

$$\frac{1}{1 - 2M/R} \approx \frac{1}{U^-} = 1 - \gamma + \gamma (Q/\mu)^2 < (Q/\mu)^2.$$

(97)

If we take the charge-to-mass ratio of an electron, we get $(Q/\mu)^2 \approx 4.17 \times 10^{42}$. If we then suppose that somehow a shell of electrons reflects electromagnetic (but not other) radiation and thereby manages to keep the electromagnetic field in its Boulware vacuum state outside the shell (rather than in the Hartle-Hawking thermal
state that exists within the shell), then \( n_b = 2 \) (from the two helicities of photons) and \( n_f = 0 \), so one gets
\[
\frac{1}{4} A - S \approx \frac{1}{180(1 - 2M/R)} < 2.31 \times 10^{40}.
\] (98)

Of course, there are severe problems in attaining anything near this limit. First, electrons in a shell around a black hole, even if in static equilibrium as I have calculated they can be, will not be in stable equilibrium, and some unknown mechanism would have to be invoked to keep the shell in place. Second, without specifying how the electrons are to be kept in place, it is hard to say how they will respond to the black hole thermal radiation impinging upon them from below. However, it is interesting that the upper limit given by Eq. (98) for the reduction in the entropy below one-quarter the (neutral) black-hole area, from a shell held up by electrostatic forces, is so large (because the charge-to-mass ratio of an electron is so large).

For a somewhat more nearly realistic example of a shell around a black hole, consider a thin aluminum foil that is charged so that, like the shell of pure electrons, the electrostatic forces balance the gravitational forces. In this case there will be limitations from the mass density \( \rho \) of the foil, the minimum practical thickness \( \tau \) of the foil and the maximum charge per surface area, \( \sigma \), that it can hold. Again the possible tensile forces within the foil itself are negligible in comparison with the electrostatic forces and hence will be ignored. (It turns out that the strong energy condition also implies that they are not nearly sufficient to stabilize the shell against radial perturbations, which are unstable because the local gravitational forces go up rapidly as the shell is brought closer to the black hole horizon, whereas the electrostatic repulsion forces depend only on the circumference of the shell, which changes only slowly as the shell is moved inward or outward near the horizon. However, just as modern jet fighter planes fly under the control of computer servomechanisms while being deliberately constructed to be unstable, and thus rapidly maneuverable, I shall suppose that some unspecified servomechanism can be used to keep the shell in place. I shall leave to the reader the engineering problem of constructing such a servomechanism and just tell how to balance the forces in the unstable equilibrium.)

Let me now give some parameters associated with the aluminum foil, using both conventional units, atomic units, and Planck units (always the case in this paper when no units are explicitly given). For this discussion I shall use the charge of the positron as
\[
e \equiv e/\sqrt{4\pi \epsilon_0 \hbar c} \approx 0.0854245329,
\] (99)
the mass of the positron or electron as
\[
m \equiv m/\sqrt{\hbar c/G} \approx 4.185 \times 10^{-23},
\] (100)
the mass of the proton as
\[ m_p \equiv m_p/\sqrt{\hbar c/G} \approx 7.684 \times 10^{-20}, \tag{101} \]
the Rydberg energy as
\[ E_R \equiv \frac{1}{2} m e^4 \approx 13.6057 \text{ eV} \approx 1.114 \times 10^{-27}, \tag{102} \]
the Bohr radius as
\[ a_B \equiv \frac{1}{m e^2} \approx 5.2917721 \times 10^{-9} \text{ cm} \approx 3.2755 \times 10^{24}, \tag{103} \]
the tropical year as
\[ 1 \text{ yr} \approx 5.854 \times 10^{50}, \tag{104} \]
and the solar mass as
\[ M_\odot \approx 9.137 \times 10^{37} \approx 0.5395/m_p^2. \tag{105} \]

The density of solid aluminum is then
\[ \rho \equiv N_\rho m_p/a_B^3 \approx 2.70 \text{ g/cm}^2 \approx 5.233 \times 10^{-94}, \tag{106} \]
with
\[ N_\rho \approx 0.2391. \tag{107} \]
The aluminum will be so cold it will be superconducting, in which case it has a Meissner magnetic penetration depth of about 50 nm \[9\]. As we shall see, the local Hawking temperature will be far below the superconducting gap energy, so the foil will be almost completely reflecting to the thermal electromagnetic radiation if its thickness is several times the magnetic penetration depth. To be very conservative, I shall take the thickness of the foil to be about 100 times the magnetic penetration depth,
\[ \tau \equiv N_\tau a_B = 50 \text{ microns} = 0.0005 \text{ cm} \approx 3.094 \times 10^{29}, \tag{108} \]
with
\[ N_\tau \approx 94486. \tag{109} \]

I shall choose the electric surface charge density \[\sigma\] so that if it were an excess of electrons, the probability for one to tunnel off, say \[P\], is very small, say \[\exp(-100)\] in some suitable atomic time unit. Since \[P \sim \exp(-2I)\] with tunneling amplitude \[I\], I shall choose \[I = (1/2) \ln (1/P) = 50\]. Now the work function for polycrystalline aluminum is
\[ V_0 \equiv v_0 E_R \approx 4.28 \text{ eV}, \tag{110} \]
If one takes the potential energy of an electron, relative to that at the Fermi surface, to be \( V_0 - eEx \) at distance \( x \) from the surface, where \( E = 4\pi\sigma \) is the magnitude of the charge density (dropping the minus sign that \( \sigma \) would have in actuality, because of Ben Franklin’s inconvenient sign convention, if there were an excess of electrons on the surface), then the WKB amplitude for the electron to tunnel through the classically forbidden region \( 0 < x < x_0 = \frac{V_0}{eE} \) is

\[
I = \int_0^{x_0} \sqrt{2mV} \, dx = \frac{\sqrt{8mV_0^3}}{3eE}.
\]

This then gives

\[
\sigma \equiv \frac{m^2e^5}{N_\sigma} = \frac{m^2e^5v_0^{3/2}}{6\pi \ln (1/P)} \approx \frac{m^2e^5v_0^{3/2}}{600\pi}
\]

\[
\approx 3.34 \times 10^{12} \, \text{e/cm}^2 \approx 0.00893 \, \text{e}/a_B \approx 7.46 \times 10^{-55},
\]

(113)

with

\[
N_\sigma = 6\pi \ln (1/P)v_0^{-3/2} = 600\pi v_0^{-3/2} \approx 10700.
\]

(114)

Expressed in terms of the unit area \( n_\sigma^{-2/3} \) formed from the aluminum atomic number density \( n \), one needs about one excess electron per 460 of these unit areas to give this surface charge density \( \sigma \), so this does not seem excessive.

From these parameters, one gets that the charge-to-mass ratio of the aluminum foil is

\[
\frac{Q}{\mu} = \frac{\sigma}{\rho\tau} = \frac{e/m_p}{N_\sigma N_p N_{\tau}} \approx \frac{1.112 \times 10^{18}}{2.41 \times 10^8} \approx 4.61 \times 10^9.
\]

(115)

This is that of the pure electron shell, divided by \( N_\sigma N_p N_{\tau} (m_p/m) \approx 4.43 \times 10^{11} \).

Then, by the same analysis used above for the pure electron shell, one finds that if one takes \( \mu/R = U_{-1/2} = \mu/Q \), one gets

\[
\frac{1}{1 - 2M/R} = \left( \frac{Q}{\mu} \right)^2 \approx 2.12 \times 10^{19},
\]

(116)

and hence the reduction of the entropy from excluding the thermal photons from above the shell is

\[
\Delta S = \frac{1}{4} A - S \approx \frac{1}{180(1 - 2M/R)} \approx \frac{(e/m_p)^2}{180(N_\sigma N_p N_{\tau})^2}
\]

\[
\approx \frac{1.24 \times 10^{36}}{1.05 \times 10^{19}} \approx 1.18 \times 10^{17}.
\]

(117)
As we shall see, this is small in comparison with the total entropy of the black hole, but the reduction means that the number of states is fewer by a factor of about

\[ e^{\Delta S} \sim 10^{51,000,000,000,000,000}, \tag{118} \]

which is quite a large factor.

One can further derive properties of the black hole around which this aluminum foil is placed to minimize \( 1 - 2M/R \) and hence maximize \( \Delta S \). The radius of the shell (which is very nearly that of the hole) is

\[ R = \frac{1}{4\pi\sigma} \approx 1.07 \times 10^{53} \approx 182 \text{ light years}, \tag{119} \]

giving a circumference of about 1150 light years. If the redshift factor to infinity were \( \sqrt{1 - 2M/R} \), the time as seen at infinity for a photon to circumnavigate the shell would be about \( 5.3 \times 10^{12} \) years, hundreds of times longer than the current age of the universe. (In actuality, if one does take the limit of setting \( \mu/R = U^{1/2} = \mu/Q \), one finds that as seen from the outside, the shell is at what would be the horizon of an extreme Reissner-Nordstrom black hole, so there would be an infinite redshift factor to infinity. But if one set \( \mu/R \) to be, say, half as large, the previous quantities would be shifted by merely factors of two or so, and then the time as seen at infinity for a photon to circumnavigate the foil shell would be of the order of hundreds of times the present age of the universe.)

The mass of the black hole is then

\[ M \approx \frac{1}{2} R = \frac{1}{4\pi\sigma} \approx 5.34 \times 10^{52} \approx 5.84 \times 10^{14} M_\odot, \tag{120} \]

of the order of mass of a supercluster of galaxies. This mass then gives a total entropy for the black hole of

\[ S \approx 4\pi M^2 \approx 3.58 \times 10^{106}, \tag{121} \]

which is about \( 3.04 \times 10^{89} \) times the reduction \( \Delta S \) in the entropy calculated above. Therefore, as already mentioned, the relative reduction of the entropy is negligible in this case, but it does reduce the total number of states (far, far more than a googolplex in this example) by the huge factor given by Eq. (118).

The proper radial distance of the shell to the horizon is

\[ D \approx 2M\sqrt{1 - 2M/R} \approx 2.50 \text{ light seconds}. \tag{122} \]

The local acceleration of gravity as seen from just inside the shell is

\[ g \approx \frac{c^2}{D} \approx 1.20 \times 10^8 \text{ m/s}^2 \approx 1.22 \times 10^7 g_\oplus, \tag{123} \]
over twelve million times the acceleration of gravity $g_{\oplus}$ at the surface of the earth.

The local Hawking temperature as seen at the inner edge of the shell is

$$T_{\text{local}} \approx \frac{1}{2\pi D} \approx 3.44 \times 10^{-45} \approx 4.87 \times 10^{-13} \text{ K},$$

which is indeed far below the critical superconducting temperature of 1.175 K \cite{10}, so the shell should stay superconducting and indeed reflect almost all of the electromagnetic radiation emitted by the black hole.

Thus we have seen that by placing a reflecting shell around a black hole, we can make the entropy have a value that is below one-quarter its area. If we are allowed an idealized perfectly reflecting shell that can be placed within roughly one Planck length of the horizon, then this entropy reduction can be of the same order as the area of the hole. For a more realistic shell, such as a superconducting aluminum foil, the entropy reduction can only be a tiny fraction of the area, but it still can be huge in absolute units, markedly reducing the number of black hole states from what would be erroneously estimated by exponentiating one quarter the horizon area.

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