NOTE ON AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. This note is based on my talk at ICCM 2013, Taipei. We give an exposition of the group-theoretic method and recent results on the questions of non-emptiness and dimension of affine Deligne-Lusztig varieties in affine flag varieties.

Introduction

Affine Deligne-Lusztig varieties was first introduced by Rapoport in [18], generalizing Deligne and Lusztig's classical construction. Understanding the emptiness/non-emptiness pattern and dimension of affine Deligne-Lusztig varieties is fundamental in the study of reduction of Shimura varieties with parahoric level structures.

In this note, we will discuss the new group-theoretic method to study affine Deligne-Lusztig varieties in affine flag varieties and answer the above questions in terms of class polynomials of affine Hecke algebras. The two key ingredients are the Deligne-Lusztig reduction [2] and the combinatorial properties of affine Weyl groups [15].

1. Affine Deligne-Lusztig varieties

1.1. Let $\mathbb{F}_q$ be a finite field with $q$ elements and $k$ be an algebraic closure of $\mathbb{F}_q$. We consider the field $L = k((\epsilon))$ and its subfield $F = \mathbb{F}_q((\epsilon))$. The Frobenius automorphism $\sigma$ of $k/\mathbb{F}_q$ can be extended in the usual way to an automorphism of $L/F$ such that $\sigma(\sum a_n\epsilon^n) = \sum \sigma(a_n)\epsilon^n$.

Let $G$ be a split connected reductive group over $\mathbb{F}_q$ and let $I$ be a $\sigma$-stable Iwahori subgroup of $G(L)$. The affine Deligne-Lusztig variety associated with $w$ in the extended affine Weyl group $\tilde{W} \cong I\backslash G(L)/I$ and $b \in G(L)$ is defined to be

$$X_w(b) = \{gI \in G(L)/I; g^{-1}b\sigma(g) \in IwI\}.$$ 

Here we embed $\tilde{W}$ set-theoretically into $G(L)$.

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This is a locally closed sub-ind scheme of the affine flag variety $G(L)/I$. It is a finite dimensional $k$-scheme, locally of finite type over $k$.

1.2. To illustrate some difficulties in the study of affine Deligne-Lusztig varieties, let us begin with (classical) Deligne-Lusztig varieties. Let $B$ be a Borel subgroup of $G$ defined over $\mathbb{F}_q$, and $W$ be the associate finite Weyl group. For any $w \in W$ and $b \in G(k)$, the classical Deligne-Lusztig variety $X_{w,b}$ is defined to be

$$X_{w,b} = \{ gB \in G(k)/B(k); g^{-1}b \sigma(g) \in B(k)wB(k) \}.$$  

Note that in both the classical case and the affine cases, if $b$ and $b'$ are in the same $\sigma$-conjugacy class, then the associated classical/affine Deligne-Lusztig varieties are isomorphic. In other words, the classical/affine Deligne-Lusztig varieties depends on the finite/affine Weyl group element $w$ and the $\sigma$-conjugacy class $[b]$.

However, in the classical case, $G(k)$ is a single $\sigma$-conjugacy class. Therefore, we may omit $b$ in the definition and simply write $X_w = \{ gB \in G(k)/B(k); g^{-1} \sigma(g) \in B(k)wB(k) \}$. This is what appears in the literature. The classical Deligne-Lusztig variety $X_w$ is always nonempty and is a smooth variety of dimension $\ell(w)$.

The loop group $G(L)$, on the other hand, contains infinitely many $\sigma$-conjugacy classes. The extra parameter $[b]$ makes the study of affine Deligne-Lusztig varieties much harder than that of the classical ones.

From a different point of view, the two parameters $w$ and $[b]$ in the definition of affine Deligne-Lusztig varieties appear naturally. There are two important stratifications on the special fiber of a Shimura variety with Iwahori level structure: one is the Kottwitz-Rapoport stratification whose strata are indexed by specific elements $w \in \tilde{W}$; the other is the Newton stratification whose strata are indexed by specific $\sigma$-conjugacy classes $[b] \subseteq G(L)$. There is a close relation between the affine Deligne-Lusztig variety $X_w(b)$ and the intersection of the Newton stratum associated with $[b]$ with the Kottwitz-Rapoport stratum associated with $w$ (see [9] and [23]).

2. CONJUGACY CLASSES OF $\tilde{W}$

2.1. To understand the affine Deligne-Lusztig varieties $X_w(b)$, one needs to understand the $\sigma$-conjugacy classes of $G(L)$ and its relation with the extended affine Weyl group $\tilde{W}$. We will relate the conjugacy classes of $\tilde{W}$ with the $\sigma$-conjugacy classes of $G(L)$. Based on the decomposition $G(L) = \sqcup_{w \in \tilde{W}} IwI$, this sounds plausible. However, the

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1One may consider any connected reductive group over $F$ that splits over a tamely ramified extension of $L$. As discussed for example in [12, Section 6 & 7], the questions we’ll discuss in this note can be reduced to quasi-split unramified groups. For simplicity, we only consider split groups in this note.
naive map sending $IwI$ to the $\sigma$-conjugacy class of $w$ does not work. The reasons are as follows:

(1) Given two elements $w$ and $w'$ in the same conjugacy class of $\tilde{W}$, the set $IwI$ and $Iw'I$ may not be $\sigma$-conjugated to each other.

(2) One $\sigma$-conjugacy class of $G(L)$ may contain elements of different conjugacy classes in $\tilde{W}$. For example, in the classical case, $G(k)$ is a single $\sigma$-conjugacy class and $W$ contains several conjugacy classes.

The ideas to overcome the difficulties is to use the minimal length elements in the conjugacy classes of $\tilde{W}$. We will recall some properties of the minimal length elements in this section and discuss the applications to $\sigma$-conjugacy classes of $G(L)$ in the next section.

2.2. We have the semidirect product

$$\tilde{W} = P \rtimes W = \{t^\lambda w; \lambda \in P, w \in W\};$$

where $P$ is the coweight lattice of a split maximal torus $T$ of $G$. Let $Q$ be the coroot lattice of $T$. Then $W_a = Q \rtimes W \subseteq \tilde{W}$ is an affine Weyl group. The length function and Bruhat order on $W_a$ extends in a natural way to $\tilde{W}$ and $\tilde{W} = W_a \rtimes \Omega$, where

$$\Omega = \{w \in \tilde{W}; \ell(w) = 0\} \cong P/Q.$$

Let $\kappa: \tilde{W} \to \tilde{W}/W_a$ be the natural projection. We call $\kappa$ the Kottwitz map.

Let $P_Q = P \otimes \mathbb{Z} Q$ and $P_Q/W$ be the quotient of $P_Q$ by the natural action of $W$. We may identify $P_Q/W$ with $P_Q^{+, +}$, the set of dominant rational coweights. For any $w \in \tilde{W}$, $w^{n_0} = t^\lambda$ for some $\lambda \in P$, where $n_0 = \sharp(W)$. Let $\nu_w$ be the unique element in $P_Q^{+, +}$ that lies in the $W$-orbit of $\lambda/n_0$. We call the map

$$\tilde{W} \to P_Q^{+, +}, \quad w \mapsto \nu_w$$

the Newton map.

Define $f: \tilde{W} \to \tilde{W}/W_a \times P_Q^{+, +}$ by $w \mapsto (\kappa(w), \nu_w)$. It is constant on each conjugacy class of $\tilde{W}$. We denote the image of $f$ by $B(\tilde{W})$. Note that in general, a fiber of $f$ contains more than one conjugacy class of $\tilde{W}$.

2.3. We call an element $w \in \tilde{W}$ straight if $\ell(w) = \langle \nu_w, 2\rho \rangle$, where $\rho$ is the half sum of all the positive roots. It is easy to see that $w$ is straight if and only if $\ell(w^n) = n\ell(w)$ for all $n \in \mathbb{N}$. We call a conjugacy class of $\tilde{W}$ straight if it contains some straight element. Note that a straight element is automatically of minimal length in its conjugacy class.

By [15, Proposition 3.2], a conjugacy class is straight if and only if it contains a length-zero element in the extended affine Weyl group $\tilde{W}_M$ for some standard Levi subgroup $M$ of $G$. 
By [15, Theorem 3.3], the map \( f : \tilde{W} \to \tilde{W}/W_0 \times P_{Q_+} \) induces a bijection from the set of straight conjugacy classes to \( B(\tilde{W}) \).

2.4. Now we discuss the minimal length elements in a conjugacy class \( \mathcal{O} \) of \( \tilde{W} \). The following result is obtained in [15], motivated by previous results of Geck and Pfeiffer [8] for finite Coxeter groups.

**Theorem 2.1.** Let \( \mathcal{O} \) be a conjugacy class of \( \tilde{W} \). Then

1. Each element of \( \mathcal{O} \) can be brought to a minimal length element by conjugation by simple reflections which reduce or keep constant the length.

2. Any two minimal length elements in \( \mathcal{O} \) are conjugate in the associated Braid group.

3. If moreover, \( \mathcal{O} \) is straight, then any two minimal length elements in \( \mathcal{O} \) are conjugate by cyclic shifts.

Let \( \mathcal{O} \) be a straight conjugacy class and \( \mathcal{O}' \) be another conjugacy class such that \( f(\mathcal{O}) = f(\mathcal{O}') \). Then the minimal length elements in \( \mathcal{O}' \) are related to the straight elements in \( \mathcal{O} \) in the sense of [15, Theorem 3.4]. This property is used to overcome the difficulty \( \S 2.1 \) (2).

3. \( \sigma \)-CONJUGACY CLASSES OF \( G(L) \)

3.1. Recall that \( G(L) = \bigsqcup_{w \in \tilde{W}} IwI \). Let \( w \in \tilde{W} \) and \( s \in \tilde{W} \) be a simple reflection. Then

\[
IswI = \begin{cases} 
IswI, & \text{if } sw > w; \\
IswI \sqcup IwI, & \text{if } sw < w.
\end{cases}
\]

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\end{cases}
\]

Therefore,

\[
G(L) \cdot_\sigma IwI = \begin{cases} 
G(L) \cdot_\sigma IswI, & \text{if } \ell(sws) = \ell(w); \\
G(L) \cdot_\sigma IswI \sqcup G(L) \cdot_\sigma IswI, & \text{if } \ell(sws) < \ell(w).
\end{cases}
\]

Here \( \cdot_\sigma \) denotes the \( \sigma \)-conjugation action.

This equality, together with Theorem 2.1, gives a reduction method in the study of \( G(L) \cdot_\sigma IwI \) and allows us to reduce the general case to the case where \( w \) of minimal length in its conjugacy class. This is used to overcome the difficulty \( \S 2.1 \) (1).

We have the following parameterisation of \( \sigma \)-conjugacy classes of \( G(L) \).

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In [15, Section 3], we assume that \( G \) is adjoint. However, [15, Theorem 3.3] holds for any reductive group and the proof is the same as in loc. cit.
Theorem 3.1. There is a canonical bijection between
(a) The set of $\sigma$-conjugacy classes of $G(L)$;
(b) The set of straight conjugacy classes of $\tilde{W}$;
(c) The image of $f: \tilde{W} \rightarrow \tilde{W}/W_a \times P_{Q,+}$.

Here the bijection between (a) and (c) follows from Kottwitz’s classification of $\sigma$-conjugacy classes [17] together with the fact that any $\sigma$-conjugacy class is represented by an element in $\tilde{W}$ [7, Corollary 7.2.2]. The bijection between (b) and (c) is discussed in §2.3. The bijection between (a) and (b) (and a new proof of the classification of $\sigma$-conjugacy classes) is obtained in [12, Section 3] using the strategy in §3.1.

3.2. Now we introduce partial orders on the three sets in Theorem 3.1.

Let $C, C'$ be $\sigma$-conjugacy classes of $G(L)$. We write $C \leq C'$ if $C$ is contained in the closure of $C'$. Let $O, O'$ be straight conjugacy classes of $\tilde{W}$. We write $O \leq O'$ if for some $w'$ of minimal length in $O'$, there exists $w$ of minimal length in $O$ such that $w \leq w'$ with respect to the Bruhat order in $\tilde{W}$. It is proved in [10, §4.7] and [11, Corollary 7.5] that $\leq$ is a partial order on the set of straight conjugacy classes.

Let $(k, \nu), (k', \nu') \in \tilde{W}/W_a \times P_{Q,+}$. We write $(k, \nu) \leq (k', \nu')$ if $k = k'$ and $\nu' - \nu \in \sum_{\alpha} \mathbb{R}_{\geq 0} \alpha$, where $\alpha$ runs over all the simple roots. This partial order is studied in detail by Chai in [1].

Theorem 3.2. Let $C, C'$ be $\sigma$-conjugacy classes of $G(L)$ and $O, O'$ the corresponding straight conjugacy classes in $\tilde{W}$. Then the following conditions are equivalent:

(1) $C \leq C'$;
(2) $O \leq O'$;
(3) $f(O) \leq f(O').$

Here (1) $\Rightarrow$ (3) is proved by Rapoport and Richartz in [19, Theorem 3.6], (3) $\Rightarrow$ (1) is proved by Viehmann in [21, Theorem 2] and the equivalence between (1) and (2) is proved in [11, Section 11] and [13]. It is easy to show that (2) $\Rightarrow$ (3). It is interesting to give a direct (combinatorial proof) that (3) $\Rightarrow$ (2). 3

4. “Dimension=degree” Theorem

4.1. We first recall the reduction method of Deligne and Lusztig [2, Theorem 1.6].

Theorem 4.1. Let $w \in \tilde{W}$, and let $s$ be a simple affine reflection.

(1) If $\ell(ws) = \ell(w)$, then there exists a universal homeomorphism $X_w(b) \rightarrow X_{ws}(b)$.

3In fact, (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2) holds for any tamely ramified group. We expect that (2) $\Rightarrow$ (1) also holds for any tamely ramified group. As explained in the beginning of this note, it suffices to prove it for quasi-split unramified groups.
(2) If $\ell(sws) = \ell(w) - 2$, then $X_w(b)$ can be written as a disjoint union $X_w(b) = X_1 \sqcup X_2$ where $X_1$ is closed and $X_2$ is open, and such that there exist morphisms $X_1 \to X_{sws}(b)$ and $X_2 \to X_{sw}(b)$ which are compositions of a Zariski-locally trivial fiber bundle with one-dimensional fibers and a universal homeomorphism.

The reduction method, together with Theorem 2.1, in principle, reduce the study of $X_w(b)$ to the case where $w$ is of minimal length in its conjugacy class. The latter one, is studied in detail in [14] and [12]. In particular,

**Theorem 4.2.** Let $w \in \tilde{W}$ be an element of minimal length in its conjugacy class and $b \in G(L)$.

1. If $b$ and $w$ are not in the same $\sigma$-conjugacy class of $G(L)$, then $X_w(b) = \emptyset$.
2. If $b$ and $w$ are in the same $\sigma$-conjugacy class of $G(L)$, then $\dim X_w(b) = \ell(w) - \langle \nu_w, 2\rho \rangle$.

The emptiness/nonemptiness pattern and dimension formula for affine Deligne-Lusztig varieties is obtained if we can keep track of the reduction step from an arbitrary element to a minimal length element. This is accomplished via the class polynomials of affine Hecke algebras.

4.2. Let $\tilde{H}$ be the Hecke algebra associated with $\tilde{W}$, i.e., $\tilde{H}$ is the associated $A = \mathbb{Z}[v, v^{-1}]$-algebra with basis $T_w$ for $w \in \tilde{W}$ and multiplication is given by

$$T_w T_w' = T_{ww'}, \quad \text{if } \ell(w) + \ell(w') = \ell(ww');$$

$$(T_s - v)(T_s + v^{-1}) = 0, \quad \text{for any simple reflection } s.$$  

It is proved in [15] that for any $w \in \tilde{W}$ and conjugacy class $\mathcal{O}$ of $\tilde{W}$, there exists a unique polynomial $f_{w, \mathcal{O}} \in \mathbb{N}[v - v^{-1}]$ such that for any finite dimensional representation $V$ of $\tilde{H}$,

$$\text{Tr}(T_w, V) = \sum_{\mathcal{O}} f_{w, \mathcal{O}} \text{Tr}(T_{w_{\mathcal{O}}}, V),$$

where $w_{\mathcal{O}}$ is a minimal length element in $\mathcal{O}$.

4.3. Now we state the main result in [12], which relates the dimension of affine Deligne-Lusztig varieties with the degree of the class polynomials.

**Theorem 4.3.** Let $b \in G(L)$ and $w \in \tilde{W}$. Then

$$\dim(X_w(b)) = \max_{\mathcal{O}} \frac{1}{2}(\ell(w) + \ell(\mathcal{O}) + \deg(f_{w, \mathcal{O}})) - \langle \bar{\nu}_b, 2\rho \rangle,$$

here $\mathcal{O}$ runs over conjugacy classes of $\tilde{W}$ with $f(\mathcal{O}) = f(b)$ and $\ell(\mathcal{O})$ is the length of any minimal length element in $\mathcal{O}$.
Here we use the convention that the dimension of an empty variety and the degree of a zero polynomial are both $-\infty$. So in particular, $X_w(b) \neq \emptyset$ if and only if $f_{w,0} \neq 0$ for some conjugacy class $\mathcal{O}$ of $\tilde{W}$ with $f(\mathcal{O}) = f(b)$.

5. AFFINE DELIGNE-LUSZTIG VARIETIES FOR BASIC $b$

5.1. We say that an element $b \in G(L)$ is basic if $\langle \nu_b, \alpha \rangle = 0$ for any root $\alpha$ of $G$. For basic $b$, we are able to give a more explicit description of the emptiness/nonemptiness pattern and dimension formula, as conjectured by Görtz, Haines, Kottwitz and Reuman in [7, Conjecture 1.1.1 & Conjecture 1.1.3]. We first discuss the emptiness/nonemptiness pattern. It is given in terms of the $P$-alcoves introduced in [7, Definition 2.1.1].

Let $P = MN$ be a semistandard parabolic subgroup of $G$ and $w \in \tilde{W}$. We say that $w$ is a $P$-alcove element if $w \in \tilde{W}_N$ and $N(L) \cap wIw^{-1} \subseteq N(L) \cap I$.

**Theorem 5.1.** Let $w \in \tilde{W}$ and $b \in G(L)$ be a basic element. Then $X_w(b) \neq \emptyset$ if and only if for any semistandard parabolic subgroup $P = MN$ such that $w$ is a $P$-alcove element, $\kappa_M(b) \in \kappa_M([b] \cap M(L))$.

The “only if” part is proved in [7, Theorem 1.1.2] as a consequence of the Hodge-Newton decomposition for affine Deligne-Lusztig varieties [7, Theorem 1.1.4]. The “if” part is proved in [5, Theorem A] by showing that the notion of $P$-alcoves is compatible with the Deligne-Lusztig reduction. An algebraic proof of the analogy of Hodge-Newton decomposition for affine Hecke algebras and a new proof of the “only if” part of the Theorem 5.1 is obtained in [16].

5.2. Define

- $\eta_1 : \tilde{W} = P \rtimes W \to W$, the projection map.
- $\eta_2 : \tilde{W} \to W$ such that $\eta_2(w)^{-1}w$ lies in the dominant Weyl chamber.
- $\eta(w) = \eta_2(w)^{-1}\eta_1(w)\eta_2(w)$.

As a consequence of Theorem 5.1, there is a simpler criterion for emptiness/nonemptiness if $w$ lies in the shrunken Weyl chamber.

**Corollary 5.2.** Assume that the Dynkin diagram of $G$ is connected. Let $b \in G(L)$ be a basic element and $w \in \tilde{W}$ lies in the shrunken Weyl chamber. Then $X_w(b) \neq \emptyset$ if and only if $\kappa(b) = \kappa(w)$ and $\eta(w)$ is not in any proper parabolic subgroup of $W$.

5.3. The following dimension formula is conjectured in [7, Conjecture 1.1.3] and proved in [12, Theorem 12.1].
Theorem 5.3. Let $b \in G(L)$ be a basic element and $w \in \tilde{W}$ lie in the shrunken Weyl chamber. If $X_w(b) \neq \emptyset$, then
\[
\dim X_w(b) = \frac{1}{2}(\ell(w) + \ell(\eta(w)) - \text{def}(b)),
\]
where $\text{def}(b)$ is the defect of $b$.

Here the lower bound is obtained by constructing an explicit sequence from $w$ to a minimal length element $w'$ in some conjugacy class of $\tilde{W}$ (in most cases, $w$ and $w'$ are not in the same conjugacy class). The upper bound is obtained by combining the “partial conjugation” method in [10] and the dimension formula for affine Deligne-Lusztig varieties in affine Grassmannian in [6, Theorem 2.15.1] and [20, Theorem 1.1]. If $w$ lies in the shrunken Weyl chamber, then the lower bound and the upper bound coincide and the theorem is proved.\footnote{I learned from E. Viehmann [22] that her student Paul Hamacher recently proved the conjectural dimension formula for affine Deligne-Lusztig varieties in affine Grassmannian for unramified groups. Combining this result, with the proof in [12] as we outlined above, and [5, Proposition 2.5.1], we are able to generalize the above theorem to any tamely ramified groups.}

It is a challenging problem to give a close formula of $\dim X_w(b)$ for $w$ in the critical strips.

6. Affine Deligne-Lusztig varieties for nonbasic $b$

6.1. An element $w \in \tilde{W}$ can be written in a unique way as $xt^\mu y$, where $x, y \in W$ and $\mu \in P$ such that $t^\mu y$ sends the fundamental alcove to an alcove in the dominant Weyl chamber. In this case, $\eta(xt^\mu y) = yx$.

Let $w_0$ be the longest element in $W$. For any $\mu \in P_+$, $w_0 t^\mu$ is the unique maximal element in its $W \times W$-coset. For such element, there is a complete answer for the emptiness/nonemptiness pattern and dimension formula.

Theorem 6.1. Let $b \in G(L)$ and $\mu \in P_+$. Then $X_{w_0 t^\mu}(b) \neq \emptyset$ if and only if $f(b) \leq f(t^\mu)$. In this case,
\[
\dim X_{w_0 t^\mu}(b) = \langle \mu - \nu_b, \rho \rangle + \ell(w_0) - \frac{1}{2}\text{def}(b).
\]
6.2. For other $w$, not much is known. Görtz, Haines, Kottwitz and Reuman made the following conjecture [7, Conjecture 9.5.1 (b)] which relates the affine Deligne-Lusztig variety $X_w(b)$ with $X_w(b')$ for basic $b'$.

**Conjecture 6.2.** Let $b \in G(L)$ and $b'$ be a basic element in $G(L)$ such that $\kappa(b) = \kappa(b')$. Then for $w \in W$ with sufficiently large length, $X_w(b) \neq \emptyset$ if and only if $X_w(b') \neq \emptyset$. In this case,

$$\dim X_w(b) = \dim X_w(b') - \langle \nu_b, \rho \rangle + \frac{1}{2}(\text{def}(b') - \text{def}(b)).$$

Many numerical evidence for group of small rank is obtained by computer in support of this conjecture. Another evidence for split $b$ is obtained in [13].

**Theorem 6.3.** Assume that the Dynkin diagram of $G$ is connected. Let $\mu \in P_+$ and $\lambda \in Q$ be a dominant and regular coweight. Then for any $x, y \in W$, $X_{x\mu + \lambda y}(t^{\nu_y}) \neq \emptyset$ if and only if $yx$ is not in any proper parabolic subgroup of $W$. In this case,

$$\dim X_{x\mu + \lambda y}(t^{\nu_y}) = \langle \lambda, \rho \rangle + \frac{1}{2}(\ell(x) + \ell(y) + \ell(yx)).$$

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