Abstract. In this paper, we study the Newton polytopes of $F$-polynomials in a totally sign-skew-symmetric cluster algebra $\mathcal{A}$ and generalize them to a larger set consisting of polytopes $N_h$ associated to vectors $h \in \mathbb{Z}^n$ as well as $\tilde{P}$ consisting of polytope functions $\rho_h$ corresponding to $N_h$.

The main contribution contains that (i) obtaining a recurrence construction of the Laurent expression of a cluster variable in a cluster from its $g$-vector; (ii) proving the subset $P$ of $\tilde{P}$ is a strongly positive $\mathbb{Z}_{Trop}(Y)$-basis for $\mathcal{U}(\mathcal{A})$ consisting of certain universally indecomposable Laurent polynomials when $\mathcal{A}$ is a cluster algebra with principal coefficients. For a cluster algebra $\mathcal{A}$ over arbitrary semifield $\mathbb{F}$ in general, $P$ is a strongly positive $\mathbb{Z}_{Trop}(Y)$-basis for a subalgebra $\mathcal{I}_{P}(\mathcal{A})$ (called the intermediate cluster algebra of $\mathcal{A}$) of $\mathcal{U}(\mathcal{A})$. We call $P$ the polytope basis; (iii) constructing some explicit maps among corresponding $F$-polynomials, $g$-vectors and $d$-vectors to characterize their relationship.

As an application of (i), we give an affirmation to the positivity conjecture of cluster variables in a totally sign-skew-symmetric cluster algebra, which in particular provides a new method different from that given in [11] to present the positivity of cluster variables in the skew-symmetrizable case. As another application, a conjecture on Newton polytopes posed by Fei is answered affirmatively.

For (ii), we know that in rank 2 case, $P$ coincides with the greedy basis introduced by Lee, Li and Zelevinsky. Hence, we can regard $P$ as a natural generalization of the greedy basis in general rank.

As an application of (iii), the positivity of denominator vectors associated to non-initial cluster variables, which came up as a conjecture in [9], is proved in a totally sign-skew-symmetric cluster algebra.

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1. Introduction

Cluster algebras are first constructed by Fomin and Zelevinsky in [7]. Generally speaking, it is a commutative algebra with so-called exchange relations given by an extra combinatorial structure. Later, researchers found many relationships from the theory of cluster algebras to other topics, such as Lie theory, quantum groups, representation theory, Riemann surfaces with triangulation, number theory, tropical geometry and Grassmanian theory as well as many interesting properties. The most significant properties among them are the Laurent phenomenon and positivity of varieties which claim that each cluster variables can be expressed as a Laurent polynomial in any cluster over \( \mathbb{P} \). However, the calculation of the Laurent expression of a cluster variable in a given cluster is in general difficult. One of our aims in this paper is to provide recurrence formulas as a program to make the above calculation easier.

Two bases related to an upper cluster algebra \( \mathcal{U}(A) \), called the greedy basis and the theta basis respectively, are constructed in [17] and [11], which both contain coefficient free cluster monomials. It is known that each element in the above two bases satisfies the Laurent phenomenon and positivity, that is, its expression in every cluster is a Laurent polynomial over \( \mathbb{P} \). So in some sense such element can be seen as a generalization of cluster monomials. Moreover, the constant coefficients of the Laurent expression in the initial cluster are related to counting of some combinatorial objects. However, the greedy basis is only constructed for rank 2 case, while the theta basis relies on the cluster scattering diagram. Another goal of this paper is to directly construct a basis of \( \mathcal{U}(A) \) consisting of some universally indecomposable Laurent polynomials as a generalization of cluster monomials in general case. In order to achieve it, one useful tool we will apply is the Newton polytopes of \( F \)-polynomials associated to cluster variables.

In [5], Jiarui Fei defined the Newton polytope of an \( F \)-polynomial associated to representations of a finite-dimensional basic algebra, as well as showed some interesting combinatorial properties of such Newton polytopes. On the other hand, the authors of [17] and [16] focused on Newton polytopes of cluster variables in cluster algebras of rank 2 and rank 3 respectively. By definitions, the Newton polytope of a cluster variable can be obtained from that of the related \( F \)-polynomial by a transformation induced by its exchange matrix \( B \) since \( \bar{y}_{l,t} = \prod_{i=1}^{m} x_{i,l,t}^{b_{ij}} \) in the case of geometric type.

In this paper, we use the Newton polytopes of \( F \)-polynomials associated to cluster variables, because it seems more suitable to our construction since it keeps the information about \( Y \)-variables.

Based on the study of the Newton polytope \( N_{l,t} \) of \( F_{l,t} \), we introduce the polytope \( N_{h} \) associated to a vector \( h \in \mathbb{Z}^n \) and the polytope functions \( \rho_{h} \) as a generalization of \( N_{l,t} \) and \( x_{l,t} \) respectively. Then the properties of \( N_{h} \) and \( \rho_{h} \) naturally induce those of \( N_{l,t} \) and \( x_{l,t} \). Moreover, it will also be proved that the polytope functions compose a strongly positive basis of \( \mathcal{U}(A) \) for a cluster algebras with principal coefficients as well as certain cluster algebra over arbitrary semifield.

We would like to introduce the following notations for convenience: for any \( n \in \mathbb{N} \), \( x \in \mathbb{Z} \),

\[
[1, n] = \{1, 2, \ldots, n\}, \quad \text{sgn}(x) = \begin{cases} 
0 & x = 0 \\
\frac{x}{|x|} & \text{otherwise}
\end{cases}, \quad [x]_+ = \max\{x, 0\}.
\]

And for a vector \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^{r} \), \( [\alpha]_+ = ([\alpha_1]_+, \ldots, [\alpha_r]_+) \).

The paper is organized as follows.
In Section 2, we introduce preliminaries about cluster algebras, including the definitions of F-polynomials, g-vectors and d-vectors, as well as some results about them which we will use later.

In Section 3, we first construct \( N_h \) and \( \rho_h \) for every \( h \in \mathbb{Z}^2 \) and show that \( \{ \rho_h | h \in \mathbb{Z}^2 \} \) coincides with the greedy basis (Proposition 3.4). Then we furthermore define \( N_h \) as well as \( \rho_h \) for any integer vector \( h \) and obtain several properties of them. In particular, we have the following theorem.

\[ \star \text{(Theorem 3.9)} \]

Let \( \mathcal{A} \) be a cluster algebra having principal coefficients and \( h \in \mathbb{Z}^n \). Then,

(i) There is at most one indecomposable Laurent polynomial \( \rho_h^0 \) in \( U_{\geq 0}^+(\Sigma_{t_0}) \) having \( X^h \) as a summand.

(ii) For \( h \in \mathbb{Z}^n \) such that \( \rho_h^0 \in U_{\geq 0}^+(\Sigma) \) and any \( k \in [1, n] \), there is

\[ h^t_k = h - 2h_{k+1} \quad \text{such that} \quad L^t_k(p_{h^t_k}) = \rho^t_{h^t_k}, \]

where \( t_k \in T_n \) is the vertex connected to \( t_0 \) by an edge labeled \( k \).

(iii) Denote by \( N_{h^t}^0 \) the polytope corresponding to \( \rho^0_{h^t} \), where \( t_k \in T_n \) is the vertex connected to \( t_0 \) by an edge labeled \( k \).

Moreover, \( \rho^0_{h^t} \) with the greedy basis \( \mathcal{P} \) (Proposition 3.4). Then we furthermore define \( N_h \) as well as \( \rho_h \) for any integer vector \( h \) and obtain several properties of them. In particular, we have the following theorem.

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where \( t_k \in T_n \) is the vertex connected to \( t_0 \) by an edge labeled \( k \).

(iii) Denote by \( N_{h^t}^0 \) the polytope corresponding to \( \rho^0_{h^t} \), where \( t_k \in T_n \) is the vertex connected to \( t_0 \) by an edge labeled \( k \).

Moreover, \( \rho^0_{h^t} \) is parallel to the \( k \)-th coordinate axis for some \( k \in [1, n] \) and \( m_k(p), m_k(p') > 0 \), then \( m_k(p'') > 0 \) for any point \( p'' \in l \).

(iv) For any \( h \) such that \( \rho_h^0 \in U_{\geq 0}^+(\Sigma) \) and any \( t \in T_n \), \( \rho_h^0 \) is indecomposable in \( U_{\geq 0}^+(\Sigma) \).

Moreover, \( \rho_h^0 \) is universally indecomposable. It follows that the set \( \mathcal{P} = \{ L^t(p_{h^t}) \in \mathbb{N}[X_{\Sigma}] | h \in \mathbb{Z}^n \} \) is independent of the choice of \( t \). And, the set consisting of coefficient free cluster monomials \( \{ X_{\alpha}^\ell | \alpha \in \mathbb{N}^r, t \in T_n \} \) is a subset of \( \mathcal{P} \).

(v) Let \( S \) be an \( r \)-dimensional face of \( N_{h^t}^0 \) for \( h \in \mathbb{Z}^n \) such that \( \rho_{h^t}^0 \in U_{\geq 0}^+(\Sigma) \). Then there are a seed \( \Sigma \) in \( \mathcal{A} \), a vector \( h' \in \mathbb{Z}^r \) and a cluster algebra \( \mathcal{A} ' \) with principal coefficients of rank \( r \) which corresponds to a pure sub-cluster algebra \( \mathcal{A}(\Sigma_{\Phi}, \Sigma_{\Psi}) \) of \( \mathcal{A} \) with \( \Sigma \) as the initial seed such that the polytope \( N_{h^t}^0 \) is isomorphic to \( S \) via an isomorphism \( \tau \) whose induced linear map \( \tilde{\tau} \) (see the definition in [3]) satisfies

\[ \tilde{\tau}(e_i) \in \mathbb{N}^r \quad \text{for any} \quad i \in [1, r]. \]

In Section 4, we explain that since each cluster variable \( x_{t,i} \) equals \( \rho_{g_{t,i}} \), cluster variables and the Newton polytopes inherit all properties shown in the last section. In particular, the definitions of \( N_h \) and \( \rho_h \) present a recursive way to calculate the Laurent expression of any cluster variable in a given cluster from its g-vector.

\[ \star \text{(Theorem 4.1) \ (Recurrence formula)} \]

Let \( \mathcal{A} \) be a TSSS cluster algebra having principal coefficients, then \( x_{t,i} = \rho_{g_{t,i}} \), and \( N_{t,i} = N_{g_{t,i}} \). Following this view, we have that

\[ \text{co}_{p}(N_{t,i}) = \text{co}_{p}(N_{g_{t,i}}) = \sum_{N_{g_{t,i}}[w_j] \in U_{_{g_{t,i}}}^r} \text{co}_{p}(N_{\alpha_j}[w_j]) \]

and

\[ x_{t,i} = X_{g_{t,i}} \left( \sum_{p \in N_{g_{t,i}}} \text{co}_{p}(N_{g_{t,i}})[Y_p] \right), \]

where \( U_{g_{t,i}}^r \) running over all \( r \)-th strata of the polytope \( N_h \) for \( x_i \) along direction \( k \), and hence all \( N_{\alpha_j}[w_j] \) are defined in the construction 3.12 (2) with \( h = g_{t,i} \).

The \textbf{Laurent phenomenon}, given in [4, 5], is the most fundamental result in cluster theory, which says that for a cluster algebra \( \mathcal{A} \) and its fixed seed \( (X, Y, B) \), every cluster variable of \( \mathcal{A} \) is a Laurent polynomial over \( \mathbb{N}[P] \) in cluster variables in \( X \).
Following this fact, in \cite{Fei}, the positivity conjecture for cluster variables is suggested, that is,

**Conjecture 1.1** (\cite{Fei}). Every cluster variable of a cluster algebra $\mathcal{A}$ is a Laurent polynomial in cluster variables from an initial cluster $X$ with positive coefficients.

So far, the recent advance on the positivity conjecture is a proof in skew-symmetrizable case given in \cite{FP}. For totally sign-skew-symmetric cluster algebras, it was only proved in acyclic case in \cite{FangP}.

As a harvest of this polytope method, a natural conclusion of Theorem 4.1 is the following corollary, which actually completely confirms Conjecture 1.1 in the most general case:

\begin{center}
\begin{itemize}
  \item [(\*)] (Corollary 4.2) (Positivity for TSSS cluster algebras) Let $\mathcal{A}$ be a TSSS cluster algebra with principal coefficients and $(X, Y, B)$ be its initial seed. Then every cluster variable in $\mathcal{A}$ is a Laurent polynomial over $\mathbb{N}[Y]$ in $X$. In particular, the positivity of TSSS cluster algebras holds.

  Moreover, as a class of special elements in $\mathcal{P}$, they admit extra properties as the following theorem claims.

  \begin{itemize}
    \item [(\*)] (Theorem 4.3) Let $\mathcal{A}$ be a TSSS cluster algebra having principal coefficients, $l \in [1, n], t \in \mathbb{T}_n$. Then, for the $F$-polynomial $F_{lt}$ associated to $x_{lt}$ and its corresponding Newton polytope $N_{lt}$, the following statements hold:

    (i) The support of $F$-polynomial $F_{lt}$ is saturated.

    (ii) For any $p \in N_{lt}$, $c_p(N_{lt}) = 1$ if and only if $p \in V(N_{lt})$.

    (iii) Let $S$ be a $r$-dimensional face of $N_{lt}$. Then there is a cluster algebra $\mathcal{A}'$ with principal coefficients of rank $r$, a Newton polytope $S'$ corresponding to some coefficient free cluster monomial in $\mathcal{A}'$ and an isomorphism $\tau$ from $S'$ to $S$ with its induced linear map $\tilde{\tau}$ satisfying

    $$\tilde{\tau}(e_i) \in \mathbb{N}^n \text{ for any } i \in [1, r].$$

Then as a conclusion, in Corollary 4.3 we provide a positive answer to Conjecture 4.4 posed in \cite{ACP} by Fei.

On the other hand, when $\mathcal{A}$ is in particular skew-symmetrizable, we can calculate the cluster algebra associated to each face $S$ by the following result:

\begin{itemize}
    \item [(\*)] (Theorem 4.8) In Theorem 3.9 (v), if $\mathcal{A}$ is a skew-symmetrizable cluster algebra with principal coefficients, and denote by $B'$ the initial exchange matrix of the cluster algebra $\mathcal{A}'$, then $B' = W^t BW$, where $W = (\tilde{\tau}(e_1)^t, \ldots, \tilde{\tau}(e_r)^t)$, $W = (\overline{\tau(e_1)}, \ldots, \overline{\tau(e_r)})$ are $n \times r$ integer matrices, $\overline{\tau(e_i)} = \sum_{j=1}^r d_{ij} w_{ji} e_j$ for $\tilde{\tau}(e_i) = \sum_{j=1}^r w_{ji} e_j$, with $s \neq 0$ being the label of the edge in $S$ parallel to $\tilde{\tau}(e_i)$ while $\overline{\tau(e_i)} = \sum_{j=1}^r d_{ij} w_{ji} e_j$ when the label is 0.

    Then in Section 5 we use the results of above two sections to construct maps from non-initial $F$-polynomials to the corresponding $g$-vectors or $d$-vectors as well as maps from $g$-vectors to the corresponding $F$-polynomials or $d$-vectors.

\begin{itemize}
    \item [(\*)] (Theorem 5.6 and Theorem 5.13) Let $\mathcal{A}$ be a cluster algebra with principal coefficients. Then, for any $l = 1, \ldots, n$ and $t \in \mathbb{T}_n$,

    (i) There are two surjective maps

    $$\varphi : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial } d\text{-vectors of } \mathcal{A}\},$$

    and

    $$\eta : \{g\text{-vectors of } \mathcal{A}\} \rightarrow \{d\text{-vectors of } \mathcal{A}\}$$

    via $\eta(g_{lt}) = \text{denominator vector of } \rho_{g_{lt}}$.
(ii) There are two bijective maps
\[ \theta_1 : \{ \text{non-initial } F\text{-polynomials of } A \} \to \{ \text{non-initial } g\text{-vectors of } A \} \]
and
\[ \theta_2 : \{ g\text{-vectors of } A \} \to \{ F\text{-polynomials of } A \} \]
via \( \theta_2(g_{l,t}) = \rho_{g_{l,t}} \big|_{x_i \to 1, \forall i} \). Moreover, we have \( \theta_2 = \theta_1^{-1} \).

(iii) There are two bijective maps
\[ \chi_1 : \{ \text{non-initial } F\text{-polynomials of } A \} \to \{ \text{non-initial cluster variables of } A \} \]
and
\[ \chi_2 : \{ g\text{-vectors of } A \} \to \{ \text{cluster variables of } A \} \]
such that \( x_{l,t} = \chi_1(F_{l,t}) = \frac{\deg_k(P_{l,t})}{\deg_k(\phi_k(P_{l,t}))} \) and \( x_{l,t} = \chi_2(g_{l,t}) = \rho_{g_{l,t}} \) for any \( l, t \) with \( l \in N^n \).

Then we obtain that all non-initial \( d\)-vectors are non-negative vectors and that a non-initial cluster variable is uniquely determined by its corresponding \( F\)-polynomial, that is,

\[ \blackstar \text{(Theorem 5.5)} \]

Let \( A \) be a TSSS cluster algebra and \( x_{l,t} \) be a non-initial cluster variable in \( A \) with \( l \in [1,n] \), \( t \in T_n \). Then \( d_{l,t} \in N^n \). More precisely, for any \( k \in [1,n] \),
\[ d_k(x_{l,t}) = \frac{\deg_k(P_{l,t})}{\deg_k(\phi_k(P_{l,t}))}. \]

\[ \blackstar \text{(Corollary 5.7)} \]

Let \( A \) be a TSSS cluster algebra with two non-initial cluster variables \( x_{l,t}, x_{l',t'} \) and \( F_{l,t}, F_{l',t'} \) be the \( F\)-polynomials associated to \( x_{l,t}, x_{l',t'} \) respectively. If \( F_{l,t} = F_{l',t'} \), then \( x_{l,t} = x_{l',t'} \).

In Section 6, we show that \( \mathcal{P} \) is a strongly positive basis of the upper cluster algebra \( \mathcal{U}(A) \).

\[ \blackstar \text{(Theorem 6.3)} \]

Let \( A \) be a TSSS cluster algebra with principal coefficients. Then \( \mathcal{P} \) is a strongly positive \( Z\text{Trop}(Y)\)-basis for the upper cluster algebra \( \mathcal{U}(A) \).

Hence we call \( \mathcal{P} \) the polytope basis of \( \mathcal{U}(A) \).

2. Preliminaries and some lemmas

In this section, we recall some preliminaries of cluster algebras, \( F\)-polynomials and \( d\)-vectors mainly based on [9] as well as some important conclusions about them.

An \( n \times n \) integer matrix \( B = (b_{ij}) \) is called \textbf{sign-skew-symmetric} if either \( b_{ij} = b_{ji} = 0 \) or \( b_{ij}b_{ji} < 0 \) for any \( i, j \in [1,n] \). A \textbf{skew-symmetric} matrix is a sign-skew-symmetric matrix with \( b_{ij} = -b_{ji} \) for any \( i, j \in [1,n] \). Moreover, a \textbf{skew-symmetrizable} matrix is a sign-skew-symmetric matrix such that there is a positive diagonal integer matrix \( D \) satisfying that \( DB \) is skew-symmetric.

For a sign-skew-symmetric matrix \( B \), we define another \( n \times n \) matrix \( B' = (b'_{ij}) \) satisfying that for any \( k, i, j \in [1,n] \),
\[ b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \text{sgn}(b_{ik})|b_{ik}b_{kj}| & \text{otherwise}. \end{cases} \]

We call the formula (4) the exchange relation for sign-skew-symmetric matrices. Denote by \( B' = \mu_k(B) \) the mutation of \( B \) in direction \( k \).

For \( k_1, k_2 \in [1,n] \), if \( B' = \mu_{k_1}(B) \) is also sign-skew-symmetric, then we can mutate \( B' \) in direction \( k_2 \) to obtain \( B'' = \mu_{k_2} \mu_{k_1}(B) \).

**Definition 2.1.** For a sign-skew-symmetric matrix \( B \), if \( B^{(i)} = \mu_{k_1} \cdots \mu_{k_1}(B) \) are always sign-skew-symmetric for all \( i \in [1,s] \) and any sequences of mutations \( \mu_{k_1}, \cdots, \mu_{k_s} \), then \( B \) is called a \textbf{totally sign-skew-symmetric} matrix.
The notion of totally sign-skew-symmetric matrices was introduced in [1]. It is well-known that skew-symmetric and skew-symmetrizable matrices are totally sign-skew-symmetric matrices. An example of a $3 \times 3$ sign-skew-symmetric matrix which is not skew-symmetrizable was given in [1]. In that paper, Berenstein etc. conjectured that any acyclic sign-skew-symmetric matrices are total. In [12], Ming Huang and Fang Li proved this conjecture.

Hence, on sign-skew-symmetric matrices, one of the most important remaining problems is the condition under which sign-skew-symmetric matrices are total. In this paper, we always assume the involved sign-skew-symmetric matrices are total.

For convenience, we will denote a totally sign-skew-symmetric matrix (respectively, cluster algebra defined subsequently) briefly as a TSSS matrix (respectively, TSSS cluster algebra).

Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield, i.e. a free abelian multiplicative group endowed with a binary operation (auxiliary) addition $\oplus$ which is commutative, associative and distributive with respect to the multiplication in $\mathbb{P}$. And $\mathcal{F}$ is the field of rational functions in $n$ independent variables with coefficients in $\mathbb{Q}\mathbb{P}$.

**Definition 2.2.** A seed in $\mathcal{F}$ is a triple $\Sigma = (X, Y, B)$ such that

- $X = (x_1, x_2, \cdots, x_n)$ is an $n$-tuple whose components form a free generating set of $\mathcal{F}$;
- $Y = (y_1, y_2, \cdots, y_n)$ is an $n$-tuple of elements in $\mathbb{P}$;
- $B$ is an $n \times n$ totally sign-skew-symmetric integer matrix.

$X$ defined above is called a cluster with cluster variables $x_i$, $y_i$ is called a $Y$-variable and $B$ is called an exchange matrix.

**Definition 2.3.** For any seed $\Sigma = (X, Y, B)$ in $\mathcal{F}$ and $k \in [1, n]$, $\Sigma' = (X', Y', B')$ is obtained from $\Sigma$ by mutation in direction $k$ if

\begin{align}
    x_j' &= \begin{cases} 
        \frac{\prod_{i=1}^{n} x_i^{b_{ik}^+} + \prod_{i=1}^{n} x_i^{-b_{ik}^-}}{(y_k \oplus 1)x_k} & j = k; \\
        x_j & \text{otherwise.}
    \end{cases} \\
    y_j' &= \begin{cases} 
        y_k^{-1} y_j b_{jk}^+ (y_k \oplus 1)^{-b_{jk}} & j = k; \\
        y_j & \text{otherwise.}
    \end{cases}
\end{align}

and $B' = \mu_k(B)$. In this case, we write $\Sigma' = \mu_k(\Sigma)$.

It can be easily checked that $\Sigma'$ is a seed and the seed mutation $\mu_k$ in an involution.

**Definition 2.4.** Let $\mathbb{T}_n$ be the $n$-regular tree whose $n$ edges emanating from the same vertex are labeled bijectively by $[1, n]$. We assign a seed to each vertex of $\mathbb{T}_n$ such that if two vertices are connected by an edge labeled $k$, then the seeds assigned to them are obtained from each other by the mutation at direction $k$. This assignment is called a cluster pattern.

In this paper, the seed assigned to a vertex $t$ is denoted by $\Sigma_t = (X_t, Y_t, B_t)$ with

$X_t = (x_{1:t}, x_{2:t}, \cdots, x_{n:t})$, $Y_t = (y_{1:t}, y_{2:t}, \cdots, y_{n:t})$ and $B_t = (b_{ij})_{i,j \in [1, n]}$, where $B_t$ is totally sign-skew-symmetric.

Now we are ready to introduce the definition of cluster algebras.

**Definition 2.5.** Given a cluster pattern, let $\mathcal{S} = \{x_{i:t} \in \mathcal{F} \mid i \in [1, n], t \in \mathbb{T}_n\}$. The (totally sign-skew-symmetric) cluster algebra $\mathcal{A}$ associated with the cluster pattern is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{S}$. 
If there is a skew-symmetrizable (respectively, skew-symmetric) exchange matrix in a cluster algebra \( A \), then all exchange matrices of \( A \) are skew-symmetrizable (respectively, skew-symmetric). So, in this case we call \( A \) a \textit{skew-symmetrizable} (respectively, \textit{skew-symmetric}) \textbf{cluster algebra}.

In this paper, when saying a cluster algebra, we always mean a TSSS cluster algebra. And we always assume \( A \) is a cluster algebra with cluster variables \( x_{il} \) for any \( l \in [1, n], t \in \mathbb{T}_n \).

It can be seen from above that the cluster algebra \( A \) is related to the choice of semifield \( \mathbb{P} \). There are two special semifields which play important roles.

**Definition 2.6.** (i) The universal semifield \( \mathbb{Q}_{sf}(u_1, u_2, \ldots, u_l) \) is the semifield of all rational functions which have subtraction-free rational expressions in independent variables \( u_1, u_2, \ldots, u_l \), with usual multiplication and addition.

(ii) The tropical semifield \( Trop(u_1, u_2, \ldots, u_l) \) is the free abelian multiplicative group generated by \( u_1, u_2, \ldots, u_l \) with addition defined by \( \prod_{j=1}^l u_j^{a_j} \oplus \prod_{j=1}^l u_j^{b_j} = \prod_{j=1}^l u_j^{\min(a_j, b_j)} \).

In particular, we say a cluster algebra \( A \) is of \textit{geometric type} if \( \mathbb{P} \) is a tropical semifield. In this case, we can also denote it as \( Trop(x_{n+1}, x_{n+2}, \ldots, x_m) \). Then according to the definition, \( y_{jt} \) is a Laurent monomial of \( x_{n+1}, x_{n+2}, \ldots, x_m \) for any \( j \in [1, n], t \in \mathbb{T}_n \). Hence we can define \( b_{ij}^t \) for \( i \in [n, m], j \in [1, n] \) as

\[
y_{jt} = \prod_{r=1}^m x_i^{b_{ij}^t}.
\]

Let \( \tilde{B}_t \) be the \( m \times n \) matrix \( \tilde{B}_t = (b_{ij}^t)_{i \in [m], j \in [1, n]} \) and \( \tilde{X}_t = (x_{1; t}, \ldots, x_{n; t}, x_{n+1; t}, \ldots, x_m) \).

Then the seed assigned to \( t \) can be represented as \((\tilde{X}_t, \tilde{B}_t)\). The mutation formulas are the same for \( \tilde{B} \) while those of \( \tilde{X} \) at direction \( k \) become

\[
x_j' = \begin{cases} 
\prod_{k=1}^m x_i^{b_{ik}'} + \prod_{k=1}^m x_i^{\{b_{ik}'\}} + x_k & j = k; \\
x_j & \text{otherwise}.
\end{cases}
\]

**Definition 2.7.** A \textbf{cluster algebra} is said to have \textbf{principal coefficients} at a vertex \( t_0 \) if \( \mathbb{P} = Trop(y_1, y_2, \ldots, y_n) \) and \( Y_{t_0} = (y_1, y_2, \ldots, y_n) \).

Hence a cluster algebra having principal coefficients at some vertex is of geometric type. Then if we use \((\tilde{X}, \tilde{B})\) to represent a seed, the definition is equivalent to that there is a seed \((\tilde{X}_{t_0}, \tilde{B}_{t_0})\) at vertex \( t_0 \) satisfying \( \tilde{B}_{t_0} = \left( \begin{array}{c} B_{t_0} \\ I \end{array} \right) \), where \( I \) is a \( n \times n \) identity matrix.

Given a cluster algebra \( A \) with initial seed \( \Sigma_{t_0} = (X_{t_0}, Y_{t_0}, B_{t_0}) \), we denote by \( A_{\text{prin}} \) the cluster algebra with principal coefficients associated to \( B_{t_0} \), which is called the \textbf{principal coefficients cluster algebra corresponding to} \( A \) since it is unique up to cluster isomorphisms.

When \( A \) is a cluster algebra of geometric type with initial seed \( \Sigma_{t_0} = (\tilde{X}_{t_0}, \tilde{B}_{t_0}) \), we also denote its seed as \( \Sigma_t = (X_t, X_t^{fr}, \tilde{B}_t) \) for any \( t \in \mathbb{T}_n \), where \( X_t = \{x_{1; t}, \ldots, x_{n; t}\} \) and \( X_t^{fr} = \{x_{n+1; t}, \ldots, x_m\} \) to distinguish two kinds of variables.

**Definition 2.8.** (i) Let \( A \) be a cluster algebra of geometric type with a seed \( \Sigma = (X, X^{fr}, \tilde{B}) \). Assume \( X_0 \subseteq X \) and \( X_1 \subseteq \tilde{X} \) satisfy \( X_0 \cap X_1 = \emptyset \). Denote

\[
x'_1 = X \cap X_1, \quad X''_1 = X^{fr} \cap X_1, \quad X' = X \setminus (X_0 \cup X'_1), \quad X'' = \tilde{X} \setminus X_1
\]

and \( \tilde{B}' \) as a \([X'] \times |X'| \) matrix obtained from \( \tilde{B} \) by deleting the \( i \)-th row and column for \( x_i \in X_1 \) and deleting the \( i \)-th column for \( x_i \in X_0 \). The seed \( \Sigma_{X_0, X_1} = (X', (X^{fr} \cup X_0) \setminus X''_1, \tilde{B}') \) is called a \textbf{mixing-type sub-seed} or \((X_0, X_1)\)-\textbf{type sub-seed} of \( \Sigma \).
(ii) A cluster algebra $\mathcal{A}'$ with initial seed $\Sigma'$ is a (mixing-type) sub-rooted cluster algebra of type $(X_0, X_1)$ of a cluster algebra $\mathcal{A}$ with initial seed $\Sigma$ if $\mathcal{A}'$ is cluster isomorphic to the cluster algebra associated to $\Sigma_{X_0, X_1}$. In particular, $\mathcal{A}'$ is a pure sub-cluster algebra of $\mathcal{A}$ if $X_0 = \emptyset$.

**Definition 2.9.** For any seed $\Sigma_t$ associated to $t \in \mathbb{T}_n$, we denote by $\mathcal{U}(\Sigma_t)$ the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ given by

$$\mathcal{U}(\Sigma_t) = \mathbb{ZP}[X_{t_{1}}^{\pm 1}] \cap \mathbb{ZP}[X_{t_{2}}^{\pm 1}] \cap \cdots \cap \mathbb{ZP}[X_{t_{n}}^{\pm 1}],$$

where $t_i$ is the vertex connected to $t$ by an edge labeled $i$ in $\mathbb{T}_n$ for any $i \in [1, n]$. $\mathcal{U}(\Sigma_t)$ is called the upper bound associated with the seed $\Sigma_t$. And $\mathcal{U}(\mathcal{A}) = \bigcap_{t \in \mathbb{T}_n} \mathcal{U}(\Sigma_t)$ is called the upper cluster algebra associated to $\mathcal{A}$.

For any $k \in [1, n]$ and $t \in \mathbb{T}_n$, denote by $M_{k; t} = x_{k; t} \mu_k(x_{k; t})$ the exchange binomial in direction $k$ at $t$. Trivially, $M_{k; t}$ is a polynomial in $\mathbb{ZP}[x_{1; t}, \cdots, x_{k-1; t}, x_{k+1; t}, \cdots, x_{n; t}]$. In order to prove Laurent phenomenon of a cluster algebra, it is first proved in [1] that

**Theorem 2.10.** [1] For any vertices $t, t' \in \mathbb{T}_n$ connected by an edge labeled $k \in [1, n]$, assume $M_{i; s}$ and $M_{j; t}$ are coprime for any $i \neq j \in [1, n], s = t$ or $t'$. Then their corresponding upper bounds coincide, that is, $\mathcal{U}(\Sigma_t) = \mathcal{U}(\Sigma_{t'})$.

In particular, when $\mathcal{A}$ is a cluster algebra having principal coefficients, $M_{i; s}$ and $M_{j; t}$ are coprime for any $i \neq j \in [1, n], t \in \mathbb{T}_n$. So we can get from Theorem 2.10 that $\mathcal{U}(\Sigma_t) = \mathcal{U}(\Sigma_{t'})$ for any $t, t' \in \mathbb{T}_n$.

In this paper, we will use $A \mid B$ to imply that a Laurent polynomial $A$ can divide another Laurent polynomial $B$. $P|_{a \to b}$ means all $a$ in a Laurent polynomial $P$ is replaced by $b$.

**Lemma 2.11.** In a cluster algebra $\mathcal{A}$ with principal coefficients, let $P^t$ be a polynomial over $\mathbb{ZP}$ in $X_t$ and $\alpha = \frac{\mu^t}{X_t^d}$ be a Laurent polynomial in $X_t$ with $d^t \in \mathbb{N}^n$, where $t \in \mathbb{T}_n$. Then the following statements are equivalent:

(i) $\mathcal{U}(\Sigma_t) = \mathcal{U}(\Sigma_{t'})$, where $t'$ is connected to $t$ by an edge in $\mathbb{T}_n$;

(ii) $\alpha$ is a Laurent polynomial in expression of any $X_{t'}, t' \in \mathbb{T}_n$ if and only if $M_{k; t}^{d_{k; t}} \mid (P^t|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}})$ for any $k \in [1, n]$.

**Proof** (i) $\Rightarrow$ (ii): Firstly, we prove the necessity. Since $\alpha$ is a Laurent polynomial in expression of any $X_{t'}$ for any $t' \in \mathbb{T}_n$, in particular this holds when $t'$ is the vertex connected to $t$ by an edge labelled $k$ in $\mathbb{T}_n$. By the definition of mutations, we have that $\alpha = \frac{\mu^t}{X_t^d}|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}}$ and it is a Laurent polynomial. So, $M_{k; t}^{d_{k; t}} \mid (P^t|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}})$ for any $k \in [1, n]$.

Secondly, we prove the sufficiency. $M_{k; t}^{d_{k; t}} \mid (P^t|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}})$ for any $k \in [1, n]$ ensures that $\alpha = \frac{\mu^t}{X_t^d}|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}}$ is a Laurent polynomial for any $k$, i.e., $\alpha \in \mathcal{U}(\Sigma_t)$. Then by statement (i), we have that $\alpha \in \mathcal{U}(\Sigma_{t'}) \subseteq \mathbb{ZP}[X_{t'}^{\pm 1}]$ for any $t' \in \mathbb{T}_n$.

(iii) $\Rightarrow$ (i): If $M_{k; t}^{d_{k; t}} \mid (P^t|_{x_{k; t} \to x_{k; t}^{-1} M_{k; t}})$ for any $k \in [1, n]$ can lead to that $\alpha$ is a Laurent polynomial in expression of any $X_{t'}$, $t' \in \mathbb{T}_n$, then $\mathcal{U}(\Sigma_t) \subseteq \mathbb{ZP}[X_{t'}^{\pm 1}]$ for any $t' \in \mathbb{T}_n$. Hence $\mathcal{U}(\Sigma_t) \subseteq \mathcal{U}(\Sigma_{t'})$ for any $t' \in \mathbb{T}_n$. Therefore $\mathcal{U}(\Sigma_t) = \mathcal{U}(\Sigma_{t'})$ because of the arbitrary choice of $t$.

\[\square\]

Hence Lemma 2.11 (ii) gives an equivalent statement of Theorem 2.10.
According to the Laurent phenomenon, let $\mathcal{A}$ be a cluster algebra and $(X_{t_0}, Y_{t_0}, B_{t_0})$ be a seed of it, then for any cluster variable $x_{t,t}$, we can express it as a Laurent polynomial in the cluster $X_{t_0}$:

$$x_{t,t} = \frac{P_{t,t}^{t_0}}{\prod_{i=1}^{n} x_{i,t_0}^{d_{i,t}^{t_0}(x_{i,t})}}.$$ 

Here and in the following, we use $P_{t,t}^{t_0}$ to denote the numerator of the above Laurent expression of $x_{t,t}$ in $X_{t_0}$. The denominator vector $d_{i,t}^{t_0} = (d_{1,t}^{t_0}(x_{1,t}), d_{2,t}^{t_0}(x_{2,t}), \ldots, d_{n,t}^{t_0}(x_{n,t}))^T$ is called the d-vector of $x_{t,t}$ with respect to the cluster $X_{t_0}$. Moreover, if $\mathcal{A}$ has principal coefficients at $t_0$, then $P_{t,t}^{t_0}$ belongs to $\mathbb{Z}[x_{1,t_0}, \ldots, x_{n,t_0}; y_{1,t_0}, \ldots, y_{n,t_0}]$. $F_{t,t}^{t_0} = P_{t,t}^{t_0}|_{x_{i,t_0}=1, \forall i \in [1,n]}$ is a polynomial in $y_{1,t_0}, \ldots, y_{n,t_0}$ called the $F$-polynomial of $x_{t,t}$ with respect to $X_{t_0}$. Under the canonical $\mathbb{Z}^n$-grading given by $\text{deg}(x_i) = e_i$, $\text{deg}(y_i) = -b_i^{t_0}$ for any $i \in [1,n]$ where $e_1, \ldots, e_n$ are standard basis (column) vectors in $\mathbb{Z}^n$, and $b_i^{t_0}$ is the $i$-th column of $B_{t_0}$, the Laurent expression of $x_{t,t}$ in $X_{t_0}$ is homogeneous with degree $g_{t_0}^{t_0}$, which is called the g-vector of $x_{t,t}$ corresponding to $X_{t_0}$. Or we can also define g-vectors as follows: $g_{j,j'}^{t_0}$, and

$$g_{j,j'}^{t_0} = \begin{cases} -g_{k,k}^{t_0} + \sum_{i=1}^{n} [b_{k}^{t_0}g_{i,j}^{t_0}] + \sum_{i=1}^{n} [b_{k}^{t_0}g_{i,j'}^{t_0}] + b_{j}^{t_0} & \text{if } j = k; \\ g_{j,j'}^{t_0} & \text{otherwise}. \end{cases}$$

where $t$ and $t'$ are connected by an edge labeled $k$ in $T_n$.

Until now, there are so many researchers studying about them and many important properties are found. Here we would like to list some of them which are helpful in our research. Although we do not use these results directly in this paper, they help us to understand cluster algebras better and inspire our construction of $\rho_\text{t}$. 

**Theorem 2.12.** [10] For any skew-symmetrizable cluster algebra $\mathcal{A}$, $l \in [1,n]$ and $t, t' \in T_n$, $P_{t,t}^{t'}$ is irreducible as a polynomial in $\mathbb{Z}P[X_{t'}]$.

Theorem 2.10, Lemma 2.11 and Theorem 2.12 can lead to the result that non-initial cluster variable is uniquely determined by its corresponding $F$-polynomial (Corollary 5.7 for skew-symmetrizable case). But in the sequel, we will give the proof of this theorem in another way as an application of Newton polytope. In fact, we will provide a stronger result showing how a non-initial $F$-polynomial determines its corresponding d-vector specifically, which will lead to Corollary 5.7 directly.

**Theorem 2.13.** [9] For any cluster algebra $\mathcal{A}$ and any vertices $t$ and $t'$ in $T_n$, the cluster variable $x_{t,t}$ can be expressed as 

$$x_{t,t} = \frac{F_{t,t}^{t'}|_{\overline{x}(y_{1,t'}, \ldots, y_{n,t'})}}{F_{t,t}^{t'}|_{\overline{x}(y_{1,t'}, \ldots, y_{n,t'})}} \prod_{i=1}^{n} x_{i,t'}^{g_i^{t,t'}, t'}.$$

where $\hat{y}_{j,t'} = y_{j,t'} \prod_{i=1}^{n} x_{i,t}^{b_{j}^{t_0}}$, and $g_{j,t}^{t'} = (g_1, \ldots, g_n)^T$.

We denote $\hat{Y}_t = \{\hat{y}_{1,t}, \ldots, \hat{y}_{n,t}\}$ for any $t \in T_n$.

**Theorem 2.14.** [11] For any skew-symmetrizable cluster algebra $\mathcal{A}$, each $F$-polynomial $F_{t_0}^{t'}$ has constant term 1 and a unique monomial of maximal degree. Furthermore, this monomial has coefficient 1, and it is divisible by all the other occurring monomials.

A cluster monomial in $\mathcal{A}$ is a monomial in $X_t$ for some $t \in T_n$. In the following when we say a cluster monomial, it is of the form $aY_t^pX_t^q$ with $a, p \in \mathbb{Z}, q \in \mathbb{N}$ and $t \in T_n$. For a cluster monomial $f = aY_t^pX_t^q$ with $a, p \in \mathbb{Z}, q \in \mathbb{N}$, we call $a$ (respectively, $aY^p$) the constant coefficient...
(respectively, coefficient) of \( f \) and say \( f \) is constant coefficient (respectively, coefficient) free if \( a = 1 \) (respectively, \( aY^p = 1 \)). Similarly, we define a **cluster polynomial** to be a polynomial in a cluster \( X_t \).

In the sequel, for a cluster algebra \( \mathcal{A} \), we will always denote by \( t_0 \) the vertex of the initial seed unless otherwise specified. And when a vertex is not written explicitly, we always mean the initial vertex \( t_0 \). For example, we use \( X, x_t, P_{t:} \) to denote \( X_{t_0}, x_{t_0}, P_{t_0} \) respectively. For any cluster \( X_t \) and any vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \), we denote \( X_t^\alpha = \prod_{i=1}^n x_t^{\alpha_i} \).

As we introduced above, Laurent Phenomenon ensures that any cluster variable \( x_{t:t} \) can be expressed as a Laurent polynomial of any cluster \( X_{t'} \):

\[
x_{t:t} = \frac{P_{t:t}'}{\prod_{i=1}^n x_{t:t}^{d_i(t, t')}}
\]

where \( P_{t:t} \) is a polynomial in \( \mathbb{Z}[x_{t:t'}, x_{t', t'}, \ldots, x_{t', t'}] \) which is not divisible by \( x_{t:t'}, x_{t', t'}, \ldots, x_{t', t'} \).

**Definition 2.15.**

(i) For any \( l, k \in [1, n] \), \( t, t' \in \mathbb{T}_n \), we denote by \( \deg_k^{t'}(x_{t:t}) \) the \( k \)-**degree** of \( P_{t:t} \).

(ii) For any Laurent polynomial \( P \in \mathbb{Z}[X_\pm^1] \) and any Laurent monomial \( p = Y^\alpha X^\beta \) with \( \alpha, \beta \in \mathbb{Z}^n \), we denote by \( \text{cop}(P) \) the constant coefficient of \( p \) in \( P \).

(iii) For any Laurent polynomial \( P \), \( P' \) is called a **summand** of \( P \) if for any Laurent monomial \( p \) with constant coefficient 1, either \( 0 \leq \text{cop}(P') \leq \text{cop}(P) \) or \( \text{cop}(P) \leq \text{cop}(P') \leq 0 \). \( P' \) is called a **summand monomial** of \( P \) if it is moreover a Laurent monomial.

For any \( l, k \in [1, n] \), \( t, t' \in \mathbb{T}_n \), we can express \( P_{l:t} \) as

\[
P_{l:t} = \sum_{s=0}^{\deg_k^{t'}(x_{t:t})} x_{l:t}^s P_s(k),
\]

where \( P_s(k) \) is a polynomial in \( \mathbb{Z}[x_{t:t'}, x_{t', t'}, \ldots, x_{t', t'}] \) for any \( s \). And note that \( P_l(t) \neq 0 \). We say a polynomial \( P \) is \( x \)-homogeneous if \( \deg_x(p) \) is a constant for any summand monomial \( p \) of \( P \).

**Lemma 2.16.** For any \( l, k \in [1, n] \), \( t, t_1, t_2 \in \mathbb{T}_n \), if \( t_1 \) and \( t_2 \) are connected by an edge labeled \( j \), then

\[
d_k^2(x_{t:t}) = \begin{cases} 
  d_k^1(x_{t:t}) & \text{if } k \neq j; \\
  \deg_k^1(x_{t:t}) - d_k^1(x_{t:t}) & \text{if } k = j.
\end{cases}
\]

**Proof.** By mutation formula \( \bigcirc \), \( x_{t:t_1} = \begin{cases} 
  x_{t_1:t_2} & i = j; \\
  x_{t_1:t_2} & i \neq j.
\end{cases} \) Assume \( x_{t:t} \) can be expressed as a Laurent polynomial of \( X_{t_1} \) as

\[
x_{t:t} = \sum_{s=0}^{\deg_k^1(x_{t:t})} x_{t_1:t}^s P_s(j),
\]

then we can get the expression of \( x_{t:t} \) by \( X_{t_2} \) as

\[
x_{t:t} = \frac{\sum_{s=0}^{\deg_k^1(x_{t:t})} \left( \sum_{s=0}^\infty \left( M_{t:t_2} \right)^s P_s(j) \right) \prod_{i \neq j} \frac{d_i^1(x_{t:t})}{M_i^{t_2}} d_i^1(x_{t:t})}{\prod_{i \neq j} \frac{d_i^1(x_{t:t})}{M_i^{t_2}} M_i^{t_2} d_i^1(x_{t:t})} = \frac{M_{j:t_2}^{t_1} \sum_{s=0}^{\deg_k^1(x_{t:t})} \left( \sum_{s=0}^\infty \left( M_{j:t_2} \right)^s P_s(j) \right) \prod_{i \neq j} \frac{d_i^1(x_{t:t})}{M_i^{t_2}} d_i^1(x_{t:t})}{\prod_{i \neq j} \frac{d_i^1(x_{t:t})}{M_i^{t_2}} d_i^1(x_{t:t})}.}
\]
which completes the proof.

\[\square\]

**Definition 2.17.** For any \( t \in \mathbb{T}_n \), \( k \in [1,n] \) and \( x_{k,t} \)-homogeneous polynomial \( P \) in \( X_t \), denote by \( \widetilde{\deg}_k^t(P) := \deg_k^t(P) + \max\{s \in \mathbb{N} : M_{k,t}^s|P| \in \mathbb{Z}[X_t^{\pm 1}]\} \) the **general degree** of \( P \) in \( x_{k,t} \). Moreover, for any polynomial \( P = \sum_i P_i \), where \( P_i \) is a \( x_{k,t} \)-homogeneous polynomial in \( X_t \), define \( \widetilde{\deg}_k^t(P) := \min_i \{ \widetilde{\deg}_k^t(P_i) \} \).

According to the mutation formula \( 5 \), \( \widetilde{\deg}_k^t(P) \) is the maximal integer \( a \) such that \( \frac{P}{x_{k,t}} \) can be expressed as a Laurent polynomial in \( X_{t_k} \), where \( t_k \in \mathbb{T}_n \) is the vertex connected to \( t \) by an edge labeled \( k \).

**Lemma 2.18.** For any \( l, k \in [1,n] \), \( t, t' \in \mathbb{T}_n \) and cluster variable \( x_{l,t} \), \( \widetilde{\deg}_k^t(P_{l,t}^t) \geq d_k^t(x_{l,t}) \).

**Proof.** First, when \( d_k^t(x_{l,t}) \leq 0 \), this is true as \( \widetilde{\deg}_k^t(P_{l,t}^t) \geq 0 \).

When \( d_k^t(x_{l,t}) > 0 \), let \( X_t = \mu_k(x_{t'}) \). Then \( x_{l,t} = x_{l,t'} \) for \( i \neq k \) and \( x_{k,t} = \frac{M_k x_{t_k}}{x_{t_k}} \). \( x_{l,t} \) can be expressed as a Laurent polynomial of \( X_{t'} \) and \( X_{t_k} \) respectively as

\[
x_{l,t} = \frac{\sum_{s=0}^{\deg_k^t(x_{l,t})} x_{k,t'}^s P_s(k)}{\prod_{t' \in t, t' \neq k} d_{t'}^t(x_{l,t})},
\]

and

\[
x_{l,t} = \prod_{t \neq t_k} x_{l,t_k}^{-d_{l,t}^t(x_{l,t})} \left( \sum_{s=0}^{\deg_k^t(x_{l,t})} (M_{k,t_k} x_{k,t_k})^s d_{l,t}^t(x_{l,t}) P_s(k) + \frac{d_{l,t}^t(x_{l,t}) - s}{\prod_{t' \in t, t' \neq k})} \left( M_{k,t_k} x_{k,t_k} x_{k,t_k}^{-s} P_s(k) \right) \right).
\]

Therefore, \( M_{k,t_k}^{-d_{l,t}^t(x_{l,t})} \sum_{s=0}^{\deg_k^t(x_{l,t}) - 1} M_{k,t_k}^s P_s(k) x_{k,t_k}^{-s} \). Then for every \( s \in [0, d_k^t(x_{l,t}) - 1] \), because \( x_{k,t_k} \) does not appear in \( M_{k,t_k} \) or \( P_s(k) \), it follows that \( M_{k,t_k} \sum_{s=0}^{\deg_k^t(x_{l,t}) - 1} P_s(k) \).

Hence, by the definition of general degree,

\[
\widetilde{\deg}_k^t(x_{l,t}) = \deg_k^t(x_{l,t} P_s(k) + \max\{s \in \mathbb{N} : M_{k,t}^s|x_{k,t} P_s(k)| \geq s + d_k^t(x_{l,t}) - s = d_k^t(x_{l,t}) \}
\]

for any \( s \in [0, \deg_k^t(x_{l,t})] \). So \( \widetilde{\deg}_k^t(P_{l,t}^t) \geq d_k^t(x_{l,t}) \). \( \square \)

**Remark 2.19.** Due to above lemma, \( M_{k,t'}^{d_k^t(x_{l,t})} | (P_{l,t}^t)_{x_{k,t'} \rightarrow M_{k,t'}} \) for any \( k \in [1,n] \) and we can express \( P_{l,t}^t \) more explicitly by

\[
P_{l,t}^t = \sum_{s=0}^{\deg_k^t(x_{l,t})} x_{k,t'}^s P_s(k) + \sum_{s=0}^{d_k^t(x_{l,t}) - 1} x_{k,t'}^s M_{k,t'}^{d_k^t(x_{l,t}) - s} P_s(k).
\]

where \( P_s(k) \) is a polynomial in \( \mathbb{Z}[x_{l,t'}, \ldots, x_{k-1,t'}, x_{k+1,t'}, \ldots, x_{n,t}] \) for any \( s \in [0, \deg_k^t(x_{l,t})] \).

According to the definition of mutation, for \( t'' \in \mathbb{T}_n \) such that \( t'' \) and \( t' \) are adjacent and connected by an edge labeled \( k \), \( P_{l,t''} \) is obtained from \( P_{l,t}^t \) by the following way:

The \( x_{k,t'} \)-homogeneous term \( x_{k,t'}^{s} M_{k,t'}^{d_k^t(x_{l,t}) - s} P_s(k) \) with \( x_{k,t'} \)-degree \( s \) changes to \( x_{k,t''} \)-homogeneous term \( x_{k,t''}^{d_k^t(x_{l,t}) - s} M_{k,t''}^{s} P_s(k) \) with \( x_{k,t''} \)-degree \( \deg_k^t(x_{l,t}) - s \).
Here in this paper we would like to say any summand monomial of $x_{k;\ell}^{s}M_{k;\ell}^{[d_{1}^{i}(x_{1,i})-s]}p$ and that of $x_{k;\ell}^{s}M_{k;\ell}^{[s-d_{1}^{i}(x_{1,i})]}p$ correspond to each other (during mutation at direction $k$), where $p$ is a summand monomial of $P_{s}(k)$.

This together with Lemma 2.16 unveils how the Laurent expression of a cluster variable in one cluster changes under the mutations of the latter.

Next we briefly introduce some definitions and notations about polytopes mainly from [20], which will be used in this paper.

**Definition 2.20.**
(i) The **convex hull** of a finite set $V = \{\alpha_{1}, \cdots, \alpha_{r}\} \subseteq \mathbb{R}^{n}$ is
\[
\text{conv}(V) = \{ \sum_{i=1}^{r} a_{i}\alpha_{i} \mid a_{i} \geq 0, \sum_{i=1}^{r} a_{i} = 1 \},
\]
while the **affine hull** of $V$ is
\[
\text{aff}(V) = \{ \sum_{i=1}^{r} a_{i}\alpha_{i} \mid a_{i} \in \mathbb{R}, \sum_{i=1}^{r} a_{i} = 1 \}.
\]
(ii) A **polytope** is the convex hull of a certain finite set of points in $\mathbb{R}^{n}$ for some $n \in \mathbb{N}$, or equivalently, a polytope is the intersection of finitely many closed halfspaces in $\mathbb{R}^{n}$ for $n \in \mathbb{N}$. The **dimension of a polytope** is the dimension of its affine hull.

(iii) Let $N \subseteq \mathbb{R}^{n}$ be a polytope. For some chosen $w \in \mathbb{R}^{n}$ and $c \in \mathbb{R}$, a linear inequality $wp^{\top} \leq c$ is called **valid** for $N$ if it is satisfied for all points $p \in N$. A **face** of $N$ is a set of the form
\[
S = N \cap \{ p \in \mathbb{R}^{n} \mid wp^{\top} = c \},
\]
where $wp^{\top} \leq c$ is a valid inequality for $N$. The **dimension of a face** is the dimension of its affine hull.

(iv) The **vertices**, **edges** and **facets** of a polytope $N$ are its faces with dimension 0, 1, and $(\dim N) - 1$ respectively.

(v) The **sum** $N + N'$ of two polytopes $N$ and $N'$ is the convex hull of $N \cup N'$.

(vi) The **Minkowski sum** $N \oplus N'$ of two polytopes $N$ and $N'$ is the polytope consisting of all points $p + q$ for points $p \in N$ and $q \in N'$.

3. **The polytope associated to an integer vector**

We will construct a collection of polytopes associated to vectors and show that they admit some interesting properties. In this section, assume $\mathcal{A}$ is a totally sign-skew-symmetric cluster algebra with principal coefficients.

3.1. **Preliminaries.**

Before introducing the construction, we would like to explain some notations first.

In this paper, we denote by $z_{1}, \cdots, z_{n}$ the coordinates of $\mathbb{R}^{n}$ and represent the elements in $\mathbb{R}^{n}$ as row vectors unless otherwise specified.

To any Laurent monomial $a_{v}Y^{v}$ in $y_{1}, \cdots, y_{n}$, where $a_{v} \neq 0 \in \mathbb{Z}, v \in \mathbb{Z}^{n}$, we can associate a vector $v$. Hence a Laurent polynomial $f(Y) = \sum_{v \in \mathbb{Z}^{n}} a_{v}Y^{v}$ with $a_{v} \neq 0 \in \mathbb{Z}$ corresponds to a set of vectors $v$ called the **support** of $f(Y)$ together with an integer $a_{v}$ placed at each vector $v$. Moreover, each integer vector of dimension $n$ corresponds to a lattice point in $\mathbb{R}^{n}$. Denote by $N$ the convex hull of lattice points corresponding to the above vectors $v$ with integers $a_{v}$ placed at lattice points. Then,
Laurent polynomials \( f(Y) \) correspond one-by-one to polytopes \( N \) together with a family of integers \( a_v \) placed at lattice points. In particular, for a principal coefficients cluster algebra of rank \( n \) with initial cluster \( X \), the above bijection induces a bijection from homogeneous Laurent polynomials \( f(Y)X^h \) of degree \( h \) to the polytopes \( N \) in the non-negative quadrant together with a family of integers \( a_v \) placed at lattice points which corresponds to \( f(Y)X^{h\mid z_{-1,\cdots,v}} \) when a vector \( h \in \mathbb{Z}^n \) is given. In the sequel, we will call \( N \) together with a family of integers placed at lattice points the (Newton) polytope corresponding to the Laurent polynomial \( f(Y) \) (or to the Laurent polynomial \( f(\hat{Y})X^h \)). The support of a Laurent polynomial \( f(Y) \) is called saturated if any lattice point in the Newton polytope \( N \) corresponds to a nonzero summand monomial of \( f(Y) \), i.e., if the integer placed at any lattice point in \( N \) is nonzero.

Note that in the sequel, unless otherwise specified, points always imply lattice points in \( \mathbb{Z}^n \) and polytopes are those whose vertices are lattice points.

**Definition 3.1.** For a polytope \( N \), the weight of a point \( p \in N \) is the integer placed on this point, denoted as \( \text{co}_p(N) \), or simply \( \text{co}_p \) when the polytope \( N \) is known clearly.

The weight \( \text{co}_p \) of \( p \in N \) is indeed the coefficient of the correspondent Laurent monomial \( Y^p \) in the Laurent polynomial \( f(Y) \). Following this point of view, we assume that for a polytope \( N \),

\[
(7) \quad \text{co}_p(N) = 0 \quad \text{if} \quad p \notin N.
\]

In the sequel, when we say \( N \) is a polytope, it actually means that \( N \) is a polytope together with weight \( \text{co}_p(N) \) of each point \( p \in N \).

**Definition 3.2.** For two polytopes \( N \) and \( N' \), \( N' \) is a sub-polytope of \( N \) if there is \( w \in \mathbb{N}^n \) such that 0 ≤ \( \text{co}_p(N'[w]) \) ≤ \( \text{co}_p(N) \) or \( 0 \geq \text{co}_p(N'[w]) \geq \text{co}_p(N) \) for any point \( p \in N \), which is equivalent to that the \( Y \)-polynomial corresponding to \( N' \) is a summand of that corresponding to \( N \) up to multiplying a monomial in \( Y \). In this case, denote \( N' \leq N \). This relation \( \leq \) defines a partial order in the set of polytopes.

Two summations introduced in Definition 3.2 (v) and (vi) can be extended to polytopes with weights. For two polytopes \( N \) and \( N' \), we define the weights of \( N + N' \) as follows:

\[
\text{co}_p(N + N') = \text{co}_p(N) + \text{co}_p(N') \begin{cases} 
\text{co}_p(N) + \text{co}_p(N') & \text{if} \ p \in N \cap N'; \\
\text{co}_p(N) & \text{if} \ p \in N \setminus N'; \\
\text{co}_p(N') & \text{if} \ p \in N' \setminus N; \\
0 & \text{if} \ p \notin N \cup N'. 
\end{cases}
\]

This summation is induced from that of Laurent polynomials with respect to the correspondence between polytopes and Laurent polynomials. While for the Minkowski sum \( \oplus \), let

\[
\text{co}_q(N \oplus N') = \sum_{p + p' = q} \text{co}_p(N)\text{co}_{p'}(N')
\]

for any point \( q \in N \oplus N' \). This is induced from the multiplication of Laurent polynomials. It can easily verified that both summations are commutative and associative.

It is easy to see when the weights are all non-negative for polytopes \( N \) and \( N' \), they are both sub-polytopes of \( N + N' \) and \( N \oplus N' \).

For any polytope \( N \) in \( \mathbb{R}^n \) and \( w \in \mathbb{Z}^n \), we denote by \( N[w] \) the polytope obtained from \( N \) by translation along \( w \).

In this paper, we always use the following partial order in \( \mathbb{Z}^n \) unless otherwise specified:

For any \( a = (a_1, \cdots, a_n), b = (b_1, \cdots, b_n) \in \mathbb{Z}^n \), \( a \preceq b \) if \( a_i \leq b_i \) for all \( i \in [1, n] \).
Given a polytope $N$ with a point $v$ in it and a sequence $i_1, \cdots, i_r \in [1, n]$, define $\{i_1, \cdots, i_r\}$-section at $v$ of $N$ to be the convex hull of lattice points in $N$ whose $i_j$-th coordinates are equal to that of $v$ respectively for $j \in [1, r]$. Denote by $V(N)$ and $E(N)$ the set consisting of vertices and edges of $N$ respectively.

In this paper, for any $i \in \mathbb{Z}$, $j \in \mathbb{N}$, we denote binomial coefficients

$$\binom{i}{j} \triangleq \begin{cases} \binom{i(1-1) \cdots (i-j+1)}{j} & \text{if } j > 0; \\ 1 & \text{if } j = 0. \end{cases}$$

and denote

$$\tilde{C}^j_i \triangleq \begin{cases} \binom{i}{j} & \text{if } i \geq 0; \\ 0 & \text{if } i < 0. \end{cases}$$

as modified binomial coefficients.

For any two points $p, q$, denote $l(pq)$ to be the length of the segment $pq$.

An **isomorphism** $\tau$ of two polytopes $N$ and $N'$ is a bijection of two sets:

$$\tau: \{\text{(not necessary lattice) points in } N\} \rightarrow \{\text{(not necessary lattice) points in } N'\}$$

satisfying that $\tau(ap + bq) = a\tau(p) + b\tau(q)$ for any (not necessary lattice) points $p, q \in N$ and any $a, b \in \mathbb{R}_{\geq 0}$ with $a + b = 1$, and the weights associated to $p$ and $\tau(p)$ respectively are the same, where the weights of non-lattice points are zero.

Assume the dimension of $N$ is $r$, so for each $i \in [1, r]$, there are two (not necessary lattice) points $p, p' \in N$ such that the segment $pp'$ connecting $p$ and $p'$ parallels to $e_i$. Then a linear map $\tilde{\tau}$ is induced by the isomorphism $\tau$ satisfying that:

$$\tilde{\tau}: \mathbb{R}^r \rightarrow \mathbb{R}^r$$

$$e_i \mapsto \frac{\tau(p) - \tau(p')}{l(pq)}.$$

It is easy to check that $\tilde{\tau}$ is well-defined. In the later discussion, $N'$ is often a face of some polytope with higher dimension $n$, so we usually slightly abuse the notation to use $\tilde{\tau}$ as the linear map:

$$\tilde{\tau}: \mathbb{R}^r \rightarrow \mathbb{R}^r \hookrightarrow \mathbb{R}^n.$$

In this paper we denote the canonical projections and embeddings respectively as

$$\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$$

$$(\alpha_1, \cdots, \alpha_i, \cdots, \alpha_n) \mapsto (\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n),$$

and

$$\gamma_{i,j}: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$$

$$(\alpha_1, \cdots, \alpha_{n-1}) \mapsto (\alpha_1, \cdots, \alpha_{i-1}, j, \alpha_i, \cdots, \alpha_{n-1}).$$

We extend $\gamma_{i,j}$ to be a map from the set of $n-1$-dimensional polytopes to that of $n$-dimensional polytopes, which is also denoted as $\gamma_{i,j}$, that is, $\gamma_{i,j}(N) = \{\gamma_{i,j}(p) \mid \forall p \in N\} \subset \mathbb{R}^n$ for any polytope $N \subseteq \mathbb{R}^{n-1}$. It is easy to see $\gamma_{i,j}(N)$ is a polytope in $\mathbb{R}^n$.

Following the definitions in [17], a Laurent polynomial $p$ in $X$ is called **universally positive** if $p \in \mathbb{NP}[X_{\pm 1}^\infty]$ for any $t \in T_n$. And a universally positive Laurent polynomial is said to be **universally indecomposable** if it cannot be expressed as a sum of two nonzero universally positive Laurent polynomials. Universal indecomposability can be regarded as the "minimalism" in the set of universally positive Laurent polynomials. Since the above two definitions are given for all $t \in T_n$, they are naturally mutation invariants.
For any semifield $\mathbb{P}$, $t, t' \in T_n$ connected by an edge labeled $k$ and any homogeneous Laurent polynomial

$$ f \in \mathbb{Z}[Y_t^{\pm 1}]|X_t^{\pm 1}] \cap \mathbb{Z}[Y_{t'}^{\pm 1}]|X_{t'}^{\pm 1}] \subseteq \mathbb{Z}\text{Trop}(Y_{t'})|X_{t'}^{\pm 1}] $$

with grading $h$, we naturally have $f = F|_{f'}(Y_{t'})X_{t'}^{h}$, where $F = f|_{x_{t' \to 1}} \forall i \in [1,n]$. We modify it into the Laurent polynomial

$$ (9) \frac{F|_{f'}(Y_{t'})}{y_{k, t'}} X_{t'}^{h} = \frac{F|_{f'}(Y_{t'})}{F|_{Trop(Y_{t'})}(Y_{t'})} g_{k, t'} X_{t'}^{h} $$

where

$$ a = \text{min} \left\{ \sum_{j=1}^{n} |b_j^{t'}| + p_j + p_k \mid (p_1, \cdots, p_n) \in N_F \right\} - [-h_k]^+, $$

with $N_F$ the polytope corresponding to $F$, and then

$$ (10) \text{ denote by } L^t(f) \text{ the Laurent expression of this modified form (9) in } X_t $$

with coefficients in $Y_t \subseteq \text{Trop}(Y_t)$.

Then we can define $L^{t, \gamma}(f) = L^t \circ L^{t^{(1)}} \circ \cdots \circ L^{t^{(s)}}(f)$ for any path $\gamma = t - t^{(1)} - \cdots - t^{(s)} - t'$ in $T_n$ if $L^t \circ L^{t^{(1)}} \circ \cdots \circ L^{t^{(s)}}(f) \in \mathbb{Z}[Y_t^{\pm 1}]|X_t^{\pm 1}]$ for $j \in [0, s]$.

Later we will show that $L^{t, \gamma}(f)$ only depends on the endpoints $t$ and $t'$ in Remark 3.14 so we usually omit the path in the superscript.

For any $t \in T_n$ denote

$$ U_{\geq 0}(\Sigma_t) = \text{NP}[X_t^{\pm 1}] \cap \text{NP}[X_{t_1}^{\pm 1}] \cap \cdots \cap \text{NP}[X_{t_n}^{\pm 1}] \subseteq U(\Sigma_t), $$

$$ U^{+}(\Sigma_t) = \{ f \in U_{\geq 0}(\Sigma_t) \mid L^t(f) \in \mathbb{N}[Y_t]|X_t^{\pm 1}] \text{ and } L^{t_i}(f) \in \mathbb{N}[Y_{t_i}]|X_{t_i}^{\pm 1}] \forall i \in [1, n] \} $$

and

$$ U_{> 0}^+(\Sigma_t) = U^+(\Sigma_t) \cap U_{> 0}(\Sigma_{t_1}) \cap \cdots \cap U_{> 0}(\Sigma_{t_n}), $$

where $t_i \in T_n$ is the vertex connected to $t$ by an edge labeled $i$. We say an element in $U_{> 0}^+(\Sigma_t)$ to be indecomposable if it can not be written as a sum of two nonzero elements in $U_{> 0}(\Sigma_t)$.

3.2. For rank 2 case.

Our construction is mainly based on a simple idea that for $X = \{x_1, x_2\}$ and $h \in \mathbb{Z}^2$, in order to make $\rho_h$ a universally positive Laurent polynomial having $X^h$ as a summand, we need a proper Laurent polynomial $x_k^{-a}M_k^a q \in \mathbb{N}[Y]|X^\pm 1]$ as a summand of $\rho_h$ for any summand monomial $x_k^{-a}p$ of $\rho_h$ such that $x_k^{-a}p$ is a summand of $x_k^{-a}M_k^a q$, where $k = 1, 2$, $a \in \mathbb{Z}_{> 0}$ and $p, q$ are some Laurent monomials in $\mathbb{N}[Y]|X^\pm 1]$. This is because when we express $\rho_h$ in $X_{t_k}$ via the mutation in direction $k$, where $t_k$ is adjacent to $t_0$ by an edge labeled $k$, there will be $M_k^a$ in the denominator, therefore we need to let $x_k^{-a}p$ be in $x_k^{-a}M_k^a q$ to get rid of $M_k^a$ in the denominator so as to maintain a Laurent polynomial in $X_{t_k}$.

In the above process we call $x_k^{-a}M_k^a q$ a complement of $x_k^{-a}p$ in direction $k$.

In the above process, we also try to keep $\rho_h$ “minimal” to make it universally indecomposable by avoiding unnecessary summands. This idea induces an inductive construction of the Laurent polynomial $\rho_h$ we wanted. And later we will furthermore do the similar thing in a cluster algebra of general rank.

Calculation under the above idea leads us to the following definition of $N_h$ as well as $\rho_h$. 
When $A$ is a cluster algebra with principal coefficients of rank 2, without loss of generality, assume that the initial exchange matrix is

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix},$$

where $b, c \in \mathbb{Z}_{>0}$. For $h = (h_1, h_2) \in \mathbb{Z}^2$, let $V_h = \{v_1, v_2, v_3, v_4, v_5\}$ where

1. $v_1 = (0, 0)$,
2. $v_2 = (\lfloor -h_1 \rfloor +, 0)$,
3. $v_3 = (0, \lfloor -h_2 \rfloor +)$,
4. $v_4 = (\lfloor -h_1 \rfloor +, -h_2 + c\lfloor -h_1 \rfloor +)$,
5. $v_5 = (\lfloor -h_1 \rfloor + - \lfloor -h_1 \rfloor - b[c\lfloor -h_1 \rfloor + - h_2] +, -h_2 + c\lfloor -h_1 \rfloor +)$,

and let $E_h$ be the set consisting of edges connecting points in $V_h$ and parallel to $e_1$ or $e_2$. For any point $p_0 = (u_0, v_0)$ in an arbitrary edge $p_1 p_2$ in $E_h$ with $p_1, p_2 \in V_h$, define the weight $co_{p_0} = C_{(p_1 \mid p_2)}$, and denote

$$m_1(p_0) = \begin{cases} co_{p_0}, & \text{if } u_0 = -h_1; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad m_2(p_0) = \begin{cases} co_{p_0}, & \text{if } v_0 = 0; \\ 0, & \text{otherwise.} \end{cases}$$

For a point $p = (u, v)$ which is not in $E_h$, define $co_p$ inductively as follows:

$$co_p = \max \left\{ \sum_{i=1}^{\lfloor -h_1 \rfloor + u} m_1((u + i, v)) \tilde{C}^i_{-h_1 - bv}, \sum_{i=1}^{v} m_2((u, v - i)) \tilde{C}^i_{-h_2 + cu} \right\}$$

while

$$m_1(p) = co_p - \sum_{i=1}^{\lfloor -h_1 \rfloor + u} m_1((u + i, v)) \tilde{C}^i_{-h_1 - bv}, \quad m_2(p) = co_p - \sum_{i=1}^{v} m_2((u, v - i)) \tilde{C}^i_{-h_2 + cu}.$$
When $h_1 \geq 0$, $x_1 \rho_h = \rho(h_{1+1},h_2)$, i.e., $J^{(1)}_h = \{(0,0)\}$. Dually, there is a decomposition for $x_2 \rho_h$ as

$$x_2 \rho_h = \sum_{(u,v) \in J^{(2)}_h} c'_{u,v} y_1^u y_2^v \rho(h_{1+bu,h_2+cu})$$

where $c'_{u,v} \in \mathbb{N}$ and

$$J^{(2)}_h = \{(u,v) \in N_h \mid m_1((u + [-h_1 - bv]_+,v) > 0, m_2((u + [-h_1 - bv]_+,v - 1) > 0 \} \cup \{(0,0)\}.$$

We would like to list some facts, which are not hard to be verified:

(a) $\{p \in N_h | m_1(p) m_2(p) \neq 0\} = \{([-h_1]_+,0)\}, \{([-h_1]_+,v) \in N_h\} \subseteq \{p \in N_h | m_1(p) > 0\}$ and $\{(u,0) \in N_h\} \subseteq \{p \in N_h | m_2(p) > 0\}$.

(b) For any point $(u,v) \in N_h$, $m_1((u,v)) > 0$ if and only if there is $(u',v') \in J^{(2)}_h$ such that $m_1(u - u', v - v') > 0$ in $N_{(h_{1+bu',h_2+cu'})}; m_2((u,v)) > 0$ if and only if there is $(u''',v''') \in J^{(1)}_h$ such that $m_2(u - u'', v - v''') > 0$ in $N_{(h_{1+bu''',h_2+cu''})}$.

(c) For any points $(u,v) \in N_h$, $m_1((u,v)) > 0$ or $m_1((u+1,v)) > 0$ if and only if there is $(u',v') \in J^{(1)}_h$ such that $m_1(u - u', v - v') > 0$ in $N_{(h_{1+bu',h_2-cu'})}; m_2((u,v)) > 0$ or $m_2((u-1,v)) > 0$ if and only if there is $(u''',v''') \in J^{(2)}_h$ such that $m_2(u - u'', v - v''') > 0$ in $N_{(h_{1+bu''',h_2+cu''})}$.

**Lemma 3.3.** Let $A$ be a cluster algebra with principal coefficients of rank 2 with initial exchange matrix

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$ 

Assume $h \in \mathbb{Z}^2$, $\{j, \bar{j}\} = \{1,2\}$ and $p = (u, v) \in N_h$. Then,

(i) If $m_j(p) > 0$, $p' = p + (-1)^j e_j \in N_h$ and $p'' = p + (-1)\bar{j} \in N_h$, then $m_j(p'), m_j(p'') > 0$.

(ii) $N_h$ is in the rectangular having $(0,0), ([h_1]_+,0)$ and $([-h_1]_+, [-h_2 + c[-h_1]_+]_+)$ as vertices, which would degenerate to a segment or a point when some vertices are coincident.

(iii) $N_h \setminus \{([-h_1]_+,0)\}$ can be divided into three areas as the form in Figure 1, where

- any point $p$ in Area I satisfies $m_1(p) \neq 0$ and $m_2(p) = 0$;
- any point $p$ in Area II satisfies $m_1(p) = m_2(p) = 0$;
- any point $p$ in Area III satisfies $m_1(p) = 0$ and $m_2(p) \neq 0$;

and $m_1([-h_1]_+,0) = m_2([-h_1]_+,0) = 1$.

**Figure 1.** The polytope $N_h$ can be divided into three areas.

**Proof** (i) When $h_1 \geq 0$ or $h_2 + ch_1 \geq 0$, $N_h$ degenerates to a segment or a point, so it is easy to see this lemma holds in this case. Hence now we only need to consider the case where $h_1 < 0$ and $h_2 + ch_1 < 0$. We will use induction on $(-h_1,-h_2-ch_1)$ with respect to the partial order.

When $(-h_1,-h_2-ch_1) = (1,1)$, there are only three points in $N_h$, namely, $(0,0),(1,0),(1,1)$, and the lemma is true in this case via direct calculation.

Assume it is true for those with $(-h_1,-h_2-ch_1) < (l_1,l_2) \in \mathbb{Z}_{>0}^2$. Then we will show that the lemma holds when $(-h_1,-h_2-ch_1) = (l_1,l_2)$. 


Suppose there are \((u, v), (u+1, v) \in N_h\) satisfying \(u < -h_1\), \(m_1((u, v)) > 0\) and \(m_1((u+1, v)) = 0\). Then according to (14), inductive assumption and fact (b), there is \((u + h_1 + bv', v') \in J_h^{(2)}\) such that \(v' < v\) and \(m_1((-h_1 - bv', v - v')) > 0\) in \(N_{(h_1+ bv', h_2 + 1 - ca - ch_1 - bev')}\). And \((u + h_1 + bv', v') \in J_h^{(2)}\) means \(m_1((u, v')) > 0\) while \(m_2((u, v' - 1)) > 0\). Therefore, \(m_1((u + 1, v')) > 0\), since otherwise by the similar discussion as above we will get \((u + h_1 + bv'', v'') \in J_h^{(2)}\) such that \(v'' < v'\) and \(m_1((-h_1 - bv'', v - v'')) > 0\) hence \(m_1((-h_1 - bv''', v - v'''')) = 0\) in \(N_{(h_1+ bv', h_2 + 1 - ca - ch_1 - bev')}\), which leads to \(m_1((u, v' - 1)) > 0\), contradicting with \(m_2((u, v' - 1)) > 0\) as \((u, v' - 1)\) can not be \((-h_1, 0)\).

Now we have \(m_1((u + 1, v)) = 0\) and \(m_1((u + 1, v')) > 0\). Let \(w\) be the least integer such that \(v' < w \leq v, m_1((u + 1, w)) = 0\) and \(m_1((u + 1, w - 1)) > 0\). If \(m_2((u + 1, w)) = 0\), let \(w' < w\) be the largest integer satisfying \(m_2((u + 1, w')) > 0\), then according to the definition of \(N_h\), \(w - w' < c(u + 1 - h_2)\). In (14), we can continue to multiply the right side with \(x_2\) and do similar decomposition to every summands. Then, repeat these to the new summand. If we do them \(w - w' - 1\) times, there is a summand

\[
y_1^{u+1 - [-h_1 - b(w-1)], u'_2} - 1 \rho(h_1 + b(w-1), h_2 + w - w' - 1 - c(u + 1 - [-h_1 - b(w-1)]_+))
\]

in the final decomposition. Since

\[
h_2 + w - w' - 1 - c(u + 1 - [-h_1 - b(w-1)]_+) - c([-h_1 - b(w-1)]_+) < -1,
\]

according to the definition of \(E_w\) for any \(w \in \mathbb{Z}^2\), we have

\[
([-h_1 - b(w-1)]_+, 1) \in N_{(h_1+b(w-1), h_2 + w - w' - 1 - c(u + 1 - [-h_1 - b(w-1)]_+))}
\]

and so,

\[
m_1([[-h_1 - b(w-1)]_+, 1]) > 0
\]

in \(N_{(h_1+b(w-1), h_2+w-w' - 1 - c(u + 1 - [-h_1 - b(w-1)]_+))}\). Therefore by the fact (b), \(m_1((u + 1, w)) = 0\), which contradicts the choice of \(w\). Hence \(m_2((u + 1, w)) > 0\). Now we have \(m_2((u + 1, w)) > 0\) and \(m_2((u + 1, w - 1)) = 0\), which is dual to the initial case \(m_1((u, v)) > 0\) and \(m_1((u + 1, v)) = 0\), but \((u + 1, w)\) is closer to \((-h_1, 0)\) than \((u, v)\). So continue above discussion we will finally reach a contradiction due to fact (a), which completes the proof of (i), i.e., if \(m_j(p) > 0\), \(p' = p + (-1)^j e_j \in N_h\) and \(p'' = p + (-1)^{j+1} e_j \in N_h\), then \(m_j(p'), m_j(p'') > 0\).

(ii) Note that according to the definition of \(N_h\), if \((u, v), (-h_1, v) \in N_h\), then \(u \in [0, -h_1]\) and \(v \in [0, -h_2 - ch_1]\). If there is \((u, v) \in N_h\) with \(v > -h_2 - ch_1\), without loss of generality, we assume \(u = \max\{w \mid (w, v) \in N_h\}\). Then \(u < -h_1\), and \(m_1((u, v)) > 0\). Through the same discussion as in the proof of (i), we can see that \(m_1((u + 1, v)) > 0\), which leads to an impossible conclusion that \((u + 1, v) \in N_h\). So for any \((u, v) \in N_h\), \(v \leq -h_2 - ch_1\). Dually, it can also be proved similarly for any \((u, v) \in N_h\), \(u \geq 0\). Therefore \(N_h\) is in the unique rectangular determined by \((0, 0), ([-h_1]_+, 0)\) and \(([[-h_1]_+, [-h_2 + c[-h_1]_+])\) as vertices.

(iii) The shape of \(N_h\) is of the form in Figure [1] according to (ii) and the definition of \(N_h\). Moreover, because of (i), we can divide \(N_h \setminus \{([[-h_1]_+, 0)\} into three areas as showed in Figure [1] while

\[
m_1([[-h_1]_+, 0) = m_2([-h_1]_+, 0) = 1
\]

can be calculated directly from the definition of \(N_h\). □
When \( \mathcal{A} \) is a cluster algebra without coefficients of rank 2, we know in this case \( \mathcal{A} = \mathcal{U}(\mathcal{A}) \) since \( \mathcal{A} \) is acyclic. A \( \mathbb{Z} \)-basis for \( \mathcal{U}(\mathcal{A}) \) is defined in [16] called the greedy basis \( \{x[d] \mid d \in \mathbb{Z}^2\} \), where \( x[d] = X^{-d} \sum_{u,v \in \mathbb{N}} c(u,v) x_1^{bu} x_2^{cv} \) with \( c(0,0) = 1 \) and
\[
c(u,v) = \max\left\{ \sum_{k=1}^{u} (-1)^{k-1} c(u-k,v) \left( \frac{d_2 - cv + k - 1}{k} \right), \sum_{k=1}^{v} (-1)^{k-1} c(u,v-k) \left( \frac{d_1 - bu + k - 1}{k} \right) \right\}
\]
for each \( (u,v) \in \mathbb{N}^2 \setminus \{(0,0)\} \).

On the other hand, we have also a set of Laurent polynomials \( \{\rho_h|_{y_i \rightarrow 1, \forall i \in [1,2]} | h \in \mathbb{Z}^2\} \), where \( \rho_h \) is defined for the principal coefficients cluster algebra corresponding to \( \mathcal{A} \) as above. Here we modify \( \rho_h \) for \( \mathcal{A} \) by setting \( y_i \) to be 1 for \( i = 1,2 \).

The following result claims that the greedy basis is in fact the same as \( \{\rho_h|_{y_i \rightarrow 1, \forall i \in [1,2]} | h \in \mathbb{Z}^2\} \) in the above case for rank 2.

**Proposition 3.4.** Let \( \mathcal{A} \) be a cluster algebra without coefficients of rank 2. Then \( \{\rho_h|_{y_i \rightarrow 1, \forall i \in [1,2]} | h \in \mathbb{Z}^2\} \) and the greedy basis \( \{x[d] \mid d \in \mathbb{Z}^2\} \) are the same. More precisely, \( \rho_h|_{y_i \rightarrow 1, \forall i \in [1,2]} = x[d] \) for any \( h = (h_1,h_2) \in \mathbb{Z}^2 \), where \( d = (d_1,d_2) = (-h_1,-h_2 + c[-h_1]+) \).

**Proof** According to the definition of \( \rho_h \) for any \( h \in \mathbb{Z}^2 \), we first calculate \( E_h \). We have \( V_h = \{v_1,v_2,v_3,v_4\} \) or \( V_h = \{v_1,v_2,v_4,v_5\} \), where
\[
\begin{align*}
v_1 &= (0,0), \\
v_2 &= ([-h_1]+,0), \\
v_3 &= (0,[-h_2]+), \\
v_4 &= ([h_1]+,[-h_2+c[-h_1]+]), \\
v_5 &= ([-h_1]+-[-h_1]+-b[-h_1]+,[-h_2+c[-h_1]+]).
\end{align*}
\]
So \( E_h \) is as shown in Figure 2 where two red points connected by an edge may be coincident. Here we deal with the first one, the other case is similar.

![Figure 2](image_url)  
*Figure 2. The shape of \( E_h \).*

Then, for any \( u,v \in \mathbb{N} \), we can calculate to obtain that
\[
m_1([-h_1]+,v) = co([-h_1]+,v) = \tilde{C}^v_{[-h_2+c[-h_1]+]}, m_2(u,0) = co(u,0) = \tilde{C}^u_{[-h_1]+},
\]
\[
co(u,v) = \max\left\{ \sum_{k=0}^{v-1} m_2(k,v) \tilde{C}^{v-k}_{[-h_2+cu]+}, \sum_{k=u+1}^{[h_1]+} m_1(k,v) \tilde{C}^{k-u}_{[-h_1+bu]+} \right\}
\]
and
\[
m_1(u,v) = co_{u,v} - \sum_{k=u+1}^{[h_1]+} m_1(k,v) \tilde{C}^{k-u}_{[-h_1+bu]+}, m_2(u,v) = co_{u,v} - \sum_{k=0}^{v-1} m_2(k,v) \tilde{C}^{v-k}_{[-h_2+cu]+}.
\]

It can be proved by induction that \( co_{u,v} = c(v,[-h_1]+-u) \). First \( co_{[-h_1]+,0} = c(0,0) = 1 \). Assume \( co_{u,0} = c(0,[-h_1]+-u) = \tilde{C}^u_{[-h_1]+} \) when \( u > r \). Then when \( u = r \), we have \( co_{r,0} = \tilde{C}^r_{[-h_1]+} \).
Then we can repeat such induction recursively on rows $z$ and $20 FANG LI AND JIE PAN$

So universally indecomposable and there is $\alpha, h$

It is claimed in theorem 1.7 of [17] that the elements of the greedy basis are universally indecomposable and there is $\alpha, h$

Proof

We can check that such $(u, v)$ is independent of the choice of $t$. Moreover, $L^h(t_0) = \rho^t_{h|t_0}$, where

$$h^t = h - 2h_k e_k + h_k b^t_k + [-h_k]_+ b^t_k$$

for any $k \in [1, 2]$.

(ii) Up to interchanging $p_1$ and $p_2$, $p_1 > p_2$ and there is $s \in \mathbb{Z}_{>0}$ such that $l$ is isomorphic to $N(0, -s)$.

Proof (i) It is claimed in theorem 1.7 of [17] that the elements of the greedy basis are universally indecomposable and that the greedy basis is independent of the choice of the initial seed. Therefore by Proposition 3.4, the definition of $L^t$ and the fact that $B$ is of full rank in rank 2 case, $\rho^t_h$ is universally indecomposable and there is $\alpha, h' \in \mathbb{Z}^2$ such that $L^h(t_0) = Y^\alpha_{t_0} \rho^t_h$, where $\rho^t_h$ is defined associated to $h'$ with initial seed being $\Sigma_{t_0}$.

On the other hand, by lemma 3.3(ii), the definition of $L^t$ and the mutation formula of $Y$-variables, it can be checked that $X^h_{t_k}$ is a summand of $L^h(M_{t_k}^{[-h_k]_+} X^h)$, while the latter is a summand of $L^t(\rho_h)$. Therefore $\alpha = 0$ and $h' = h^t$. And thus $\{L^h(t_0) \mid h \in \mathbb{Z}^2\}$ is independent of the choice of $t$.

(ii) Denote by $h' \in \mathbb{Z}^2$ for any $t \in \mathbb{T}_2$ such that $L^t(\rho_h) = \rho^t_{h'}$ and by $(u, v)$ the unique maximal point in $N^t_{h'}$. Because $u, v \geq 0$, there is a vertex $t \in \mathbb{T}_2$ such that $(u, v) \leq (u', v')$ for any $t'$ connected to $t$. It can be check that such $(u, v)$ is unique.

We may assume $t_0$ satisfies the above condition. In the sequel of this proof, we always denote $u = u_{t_0}, v = v_{t_0}$.
If \((u, v) = (0, 0)\), then \(N_h\) is the origin. According to the mutation formula \((5)\) and \((6)\), two Laurent monomials corresponding to two points in a proper face (i.e., a face with dimension in \([0, \dim(N_{h,t}^t) - 1]\)) of \(N_{h,t}^t\) respectively must correspond to two Laurent monomials corresponding to two points in some proper face of \(N_{h,t'}^t\), for any two connected vertices \(t, t' \in \mathbb{T}_2\). So for any \(l\) which is a face of \(N_{h,t}^t\), with dimension 1 for \(t \in \mathbb{T}_2\), \(l\) is either in \(E_{h,t}^t\) or is isomorphic to an edge in \(E_{h,t'}^t\), for some \(t' \in \mathbb{T}_2\). Therefore the lemma holds when there is \(t \in \mathbb{T}_2\) such that \(N_{h,t}^t\) is a point.

If \((u, v) \neq (0, 0)\), then according to the mutation formula \((5)\) and \((6)\), we have \(u_{t'} = u\) and \(v_{t'} = cu - v\), where \(t'\) is connected to \(t_0\) by an edge labeled 2. So by our assumption that \(v_{t'} \geq v\), we have \(2v \leq cu\). Similarly, \(u_{t''} = bv - u\) and \(v_{t''} = v\), where \(t''\) is connected to \(t_0\) by an edge labeled 1, hence we have \(2u \leq bv\).

First assume \(2v = cu\), and the Newton polytope is of the following form as Figure 3:

![Figure 3](image)

**Figure 3.** The Newton polytope \(N_h\) when \(2v = cu\).

where Laurent monomials corresponding to points lying at the left (resp. right) side of the dashed line \(CD\) have positive (resp. negative) exponents of \(x_2\) while those corresponding to points lying above (below) the dashed line \(CE\) have positive (resp. negative) exponents of \(x_1\). Let \(AGCFO\) (resp. \(ABCIH\)) denotes the Area \(III\) (resp. Area \(I\)) in Lemma \(3.3\). Hence for any line \(z_2 = r\) below \(CE\), it intersects with \(OF\) (or \(FC\)) and \(AH\) (or \(HC\)) at two points (not necessarily lattice points) and the length of the segment connecting these points is \(u - br\).

On the other hand, \(Z_2 = r\) also intersects with \(AC\) and \(OC\) at two points (not necessarily lattice points) and the length of the segment connecting these points is \(u - br\). Moreover, since \(2v = u\), \(N_{h,t}^t = N_h\) up to reflection and translation, so \(OFC\) is symmetric with \(AGC\) over \(CD\). Therefore, \(OFC\) lies below \(OC\) and \(AHC\) lies above \(AC\).

And in general for \(2v \leq cu\), according to \((13)\), Area \(I\) for \(2v \leq cu\) case lies in \(ABICH\), which leads to that in this case the new bound \(OFC'\) lies below \(OC'\), where the letters with prime mean those in \(2v \leq cu\) case.

Dually, we also have that \(BIC\) lies below \(BC\).

Therefore, we get that \(N_h = \Delta_{OAB}, \) where \(O\) and \(B\) are the only two points in \(OB\) with nonzero weights, which equal to 1. Again according to mutation formula \((5)\) and \((6)\), we see that for any \(l\) which is a face of \(N_{h,t}^t\) with dimension 1 for \(t \in \mathbb{T}_2\), \(l\) is either in \(E_{h,t}^t\) or is isomorphic to \(OB\), which is isomorphic to \(N_{(0,-1)}\).

\[\square\]

**Remark 3.6.** According to the discussion in above proof, we know that for any \(h \in \mathbb{Z}^2\), \(N_h\) is one of the following forms in Figure 4 (up to reflections over the line \(z_1 = z_2\)):

**Figure 4(1):** in the case where \(h_1, h_2 \geq 0\).

**Figure 4(2):** in the case where \(h_2 \geq 0\), \(h_1 < 0\) and \(-h_2 - ch_1 \leq 0\).
Lemma 3.7. Let \( \mathcal{A} \) be a cluster algebra with principal coefficients of rank 2 in the above setting, \( h \in \mathbb{Z}^2 \). Then \( \rho_h \) is the unique indecomposable Laurent polynomial in \( \mathcal{U}_\geq_0(\Sigma_{i_0}) \) having \( X^h \) as a summand.

**Proof** According to Lemma 3.3(i) and the definition of \( \rho_h \) for any \( h \in \mathbb{Z}^2 \), \( \rho_h \) is an indecomposable Laurent polynomial in \( \mathcal{U}_\geq_0(\Sigma_{i_0}) \) having \( X^h \) as a summand. Then we need to show the uniqueness.

Assume \( f \) is an indecomposable Laurent polynomial in \( \mathcal{U}_\geq_0(\Sigma_{i_0}) \) having \( X^h \) as a summand. Then \( f_1 = X^h(y_1 + 1)^{-[h_1]} \) and \( f_2 = X^h(y_2 + 1)^{-[h_2]} \) must be two summands of \( f \) as complements of \( X^h \) in direction 1 and 2 respectively.

Moreover, \( f_3 = X^h y_1^{[-h_1]} (y_2 + 1)^{-h_2 + c[-h_1]} \) is a summand of \( f \) as a complement of \( X^h y_1^{[-h_1]} \) in direction 2 while

\[
f_4 = X^h y_1^{[-h_1]} (-[h_1] - b[-h_1] - h_2) y_2^{[-h_2] + c[-h_1]} \]

is a complement of \( X^h y_1^{[-h_1]} y_2^{[-h_2] + c[-h_1]} \) in direction 1.

Then according to the definition of \( E_h \), for any \( p \) corresponding to a point in \( E_h \), \( c_{p}(\rho_h)X^h \) is a summand of one of the Laurent polynomials \( f_1, f_2, f_3, f_4 \) discussed above, so it is also a summand of \( f \). Therefore, \( c_{p}(f) \geq c_{p}(\rho_h) \) when \( p \) corresponds to a point in \( E_h \).

When \([[-h_1] - h_2 + c[-h_1]] \neq 0\), we already have \( \rho_h \) is a summand of \( f \), so \( f = \rho_h \) due to the indecomposability.

Otherwise, when \([[-h_1] - h_2 + c[-h_1]] = 0\), similar to the definition of \( \rho_h \), by induction on the partial order of \( (\sum_1 - u, v) \) for a point \( p = (u, v) \) we can prove \( c_{p}(f) \geq c_{p}(\rho_h) \) and \( m_i(p)|f|_{\rho_h} \) for \( i = 1, 2 \) because \( f \) is an indecomposable Laurent polynomial in \( \mathcal{U}_\geq_0(\Sigma_{i_0}) \), i.e., \( \rho_h \) is a summand of \( f \). Hence \( f = \rho_h \) due to the indecomposability.

\[\Box\]

3.3. For any rank case.

The idea for the construction of \( \rho_h \) when \( h \in \mathbb{Z}^n \) is similar to that in rank 2 case, i.e., we try to find a "minimal" Laurent polynomial \( \rho_h \) having \( X^h \) as a summand satisfies that there is a summand \( x_k^{-a}M_k^q \) of \( \rho_h \) for any summand monomial \( x_k^{-a}p \) of \( \rho_h \) such that \( x_k^{-a}p \) is a summand of \( x_k^{-a}M_k^q \), where \( k \in [1, n] \), \( a \in \mathbb{Z}_{\geq 0} \) and \( p, q \) are some Laurent monomials in \( \mathbb{N}[Y][X^{\pm 1}] \).

However in general case, the process of this construction is pretty complex. So our strategy is using sections as well as formulas similar to (13) and (14) in general rank case to provide an inductive construction. It may not be able to calculate concrete coefficients conveniently. But, this inductive algorithm already contains much information about cluster algebras, especially about cluster monomials.
In the definition of \( \rho_h \), the combinatorial objects of a polytope, such as sections and faces, help us to understand the structure of \( \rho_h \) better by the fundamental properties in the theory of polytopes. It is why we introduce polytopes corresponding to homogeneous Laurent polynomials in our study.

Let \( \mathcal{A} \) be a cluster algebra of rank \( n \) with principal coefficients. For any Laurent monomial \( p \) in \( X \) and a subset \( I \subseteq [1, n] \), define \( E(p, I) \) consisting of segments \( \frac{\text{sgn}(\epsilon_i)}{\epsilon_i-1}q_i \) for any sequence of Laurent monomials \( q_0, \ldots, q_r = p' \) in \( X \) and any sequence of indices \( i_1, \ldots, i_r \) with \( i_j \in I \) for any \( j \in [1, r] \) and \( i_j \neq i_{j-1} \) if \( j > 1 \) satisfying:

(i) \( q_0 = p, \epsilon_0 = 1 \) and \( \epsilon_j = \epsilon_{j-1} \text{sgn}(b_{i_k,i_j}) \), where \( k < j \) is the maximal satisfying \( b_{i_k,i_j} \neq 0 \) and \( b_{i_0,i_j} > 0 \) for any \( j \in [1, r] \);

(ii) \( \deg_{x_i}(q_{j-1}) < 0 \) for any \( j \in [1, r] \);

(iii) For any \( j \in [1, r] \) and \( k \in [1, n] \), \( \epsilon_{j-1}b_{k,i_j} \geq 0 \) if \( \deg_{x_k}(q_{j-1}) < 0 \);

(iv) \( q_j = q_{j-1} - \epsilon_{j-1} \deg_{x_i}(q_{j-1}) \) for any \( j \in [1, r] \).

\( \frac{\text{sgn}(\epsilon_i)}{\epsilon_i-1}q_i \) in the above sequence is parallel to the \( i_j \)-th coordinate axis for any \( j \in [1, r-1] \), so we label it by \( i_j \). Note that there may be some parallel segments in the same sequence labeled by the same integer.

Then we define the essential skeleton \( E_h = E(X^h, [1, n]) \). Denote by \( V_h \) the set consisting of vertices of the segments in \( E_h \).

It can be verified that for any subset \( I \subseteq [1, n] \), if we delete all segments in \( E_h \) with labels not in \( I \), then there are unique maximal and minimal points in every connected component respectively because of the sign-skew-symmetry of the exchange matrix by induction on \( |[1, n] \setminus I| \).

**Definition 3.8.** Fix \( \epsilon_i \in \{ \pm 1 \} \) for any \( i \in I \), define an order with respect to \( \epsilon_i \) in a polytope \( N \) as \( p \preceq r \), \( p' \) for any point \( p, p' \in N \) if \( p \) is in \( i \)-section at \( p' \) and \( \epsilon_i z_i(p) \leq \epsilon_i z_i(p') \), where \( z_i(p) \) is the \( i \)-th coordinate of \( p \). Then denote

\[
(15) \quad m_i(p) = c_\rho(p)N - \sum_{v < i, p} m_i(v)\hat{c}(\hat{v}(\rho))_{-\deg_{x_i}(p)} \in \mathbb{N}.
\]

**Theorem 3.9.** Let \( \mathcal{A} \) be a cluster algebra having principal coefficients and \( h \in \mathbb{Z}^n \). Then,

(i) There is at most one indecomposable Laurent polynomial \( \rho_h^0 \) in \( U_{0}^+(\Sigma_{t_0}) \) having \( X^h \) as a summand.

(ii) For \( h \in \mathbb{Z}^n \) such that \( \rho_h^0 \in U_{0}^+(\Sigma) \) and any \( k \in [1, n] \), there is

\[
(16) \quad h^{t_k} = h - 2h_k e_k + h_k[b_k] + [-h_k] + b_k
\]

such that \( L^{t_k}(\rho_h^0) = \rho_h^{0,t_k} \), where \( t_k \in \mathbb{T}_n \) is the vertex connected to \( t_0 \) by an edge labeled \( k \).

(iii) Denote by \( N_h^0 \) the polytope corresponding to \( \rho_h^0 \vert_{x_i = 1 \forall i \in [1, n]} \), which we call the polytope from \( \mathcal{A} \) associated to \( h \) when \( \rho_h^0 \in U_{0}^+(\Sigma) \). For any \( p, p' \in N_h^0 \), if the segment \( l \) connecting \( p \) and \( p' \) is parallel to the \( k \)-th coordinate axis for some \( k \in [1, n] \) and \( m_k(p), m_k(p') > 0 \), then \( m_k(p'') > 0 \) for any point \( p'' \in l \).

(iv) For any \( h \) such that \( \rho_h^0 \in U_{0}^+(\Sigma) \) and any \( t \in \mathbb{T}_n \), \( \rho_h^0 \) is indecomposable in \( U_{0}^+(\Sigma) \). Moreover, \( \rho_h^0 \) is universally indecomposable. It follows that the set \( \mathcal{P} = \{ L^{t_k}(\rho_h^0) \in \mathbb{NP}[X^{\pm 1}] \vert h \in \mathbb{Z}^n \} \) is independent of the choice of \( t \). And, the set consisting of coefficient free cluster monomials \( \{ X^\alpha \vert \alpha \in \mathbb{N}^n, t \in \mathbb{T}_n \} \) is a subset of \( \mathcal{P} \).

(v) Let \( S \) be an \( r \)-dimensional face of \( N_h^0 \) for \( h \in \mathbb{Z}^n \) such that \( \rho_h^0 \in U_{0}^+(\Sigma) \). Then there are a seed \( \Sigma \) in \( \mathcal{A} \), a vector \( h' \in \mathbb{Z}^r \) and a cluster algebra \( \mathcal{A}' \) with principal coefficients of rank \( r \) which corresponds to a pure sub-cluster algebra \( \mathcal{A}(\Sigma_{\emptyset}, X_{A}) \) of \( A \) with \( \Sigma \) as the initial seed such that the polytope \( N_h \vert_{\mathcal{A}'} \) is isomorphic to \( S \) via an isomorphism \( \tau \) whose induced linear map \( \hat{\tau} \) (see the
\[ \hat{\tau}(e_i) \in \mathbb{N}^n \text{ for any } i \in [1, r]. \]

**Proof** Here and in the sequel, we will use \( \rho_h \) and \( N_h \) to represent \( \rho_h^{t_0} \) and \( N_h^{t_0} \) respectively.

We prove (i), (ii), (iii), (v) and the following statement using double inductions:

For any \( h \) such that \( \rho_h^{t_0} \in \mathcal{U}_{\geq 0}^+(\Sigma) \) and any \( i \in [1, n] \), there is a finite decomposition

\[ x_i \rho_h^{t_0} = \sum_{w, \alpha} c_{w, \alpha} Y_w \rho_\alpha, \]

where \( w \in \mathbb{N}^n, \alpha \in \mathbb{Z}^n \) and \( c_{w, \alpha} \in \mathbb{N} \).

First, when \( h \in \mathbb{Z}^2 \), (i) follows from Lemma 3.5 (iii) follows from Lemma 3.3 (ii) is ensured by Lemma 3.6 (i), (v) is exactly Lemma 3.5 (ii) and (17) is equivalent to (13) and (14) for \( i = 1 \) and 2 respectively. Then we assume they hold when \( h \in \mathbb{Z} < n \), so we need to show it is true when \( h \in \mathbb{Z}^n \). In order to achieve this we apply another induction. When \( N_h \) is the origin, (i), (iii), (v) and (17) are trivial while (ii) is not hard to verify. Assume they hold for all \( h' \in \mathbb{Z}^n \) satisfying \( \rho_{h'} \), exists, and \( N_{h'} < N_h \). Then the proof is reduced to \( h \) case.

Suppose there is \( k \in [1, n] \) such that \( \sum_{p \in N} Y^p X^h \) is a Laurent polynomial satisfying conditions in (i), where \( N = \gamma_{k, 0}(N_{\pi_k(h)}|A') \) and \( A' \) is a cluster algebra with the initial exchange matrix being obtained from \( B \) by deleting its \( k \)-th row and column. Then following inductive assumption, \( \rho_{\pi_k(h)}|A' \) is a summand of \( \rho_h|_{x_k \to 1} \) since \( \rho_h \in \mathcal{U}_{\geq 0}^+(\Sigma_{t_0}) \). Because of the indecomposability of \( \rho_h \) and the fact that \( \text{deg}(\check{y}_i) = 0 \) for any \( i \in [1, n] \), \( \rho_h \) is homogeneous. Thus \( \sum_{p \in N} Y^p X^h \) is a summand of \( \rho_h \).

Therefore, \( \rho_h = \sum_{p \in N} \hat{Y}^p X^h \) is the unique element satisfying conditions in (i) and the theorem holds following inductive assumption for smaller rank case. So in the following we always assume there is no \( k \in [1, n] \) such that \( \sum_{p \in N} \hat{Y}^p X^h \) is a Laurent polynomial satisfying conditions in (i), that is, we can find \( p \in N \) satisfying \( \text{deg}_{x_k}(\hat{Y}^p X^h) < 0 \).

For any \( t \in T_n \) denote

\[ \hat{\mathcal{U}}_{\geq 0}(\Sigma_i) = \text{NF}([X_i^{\pm 1}]) \cap \text{NF}([X_i^{\pm 1}]) \cap \cdots \cap \text{NF}([X_i^{\pm 1}]], \]

\[ \hat{\mathcal{U}}^+(\Sigma_i) = \{ f \in \hat{\mathcal{U}}_{\geq 0}(\Sigma_i)|L^i(f) \in \mathbb{N}[Y_i][[X_i^{\pm 1}]] \text{ and } L^i(f) \in \mathbb{N}[Y_i][[X_i^{\pm 1}]], \forall i \in [1, n] \} \]

and

\[ \hat{\mathcal{U}}_{\geq 0}(\Sigma_i) = \hat{\mathcal{U}}^+(\Sigma_i) \cap \hat{\mathcal{U}}_{\geq 0}(\Sigma_{t_1}) \cap \cdots \cap \hat{\mathcal{U}}_{\geq 0}(\Sigma_{t_n}), \]

where \( t_i \in T_n \) is the vertex connected to \( t \) by an edge labeled \( i \).

Next we provide a way to inductively construct a formal Laurent polynomial \( \rho_h \in \hat{\mathcal{U}}_{\geq 0}^+(\Sigma) \) for each \( h \in \mathbb{Z}^n \) such that \( \rho_h \) is indecomposable in \( \hat{\mathcal{U}}_{\geq 0}^+(\Sigma) \) and \( X^h \) is a summand of \( \rho_h \). We choose \( r \in [1, n] \) such that \( N_{\pi_r(h)}|A_r \) has dimension \( n - 1 \) and \( i \in \{ [1, n] \setminus \{ r \} \} \), where \( A_r \) is the cluster algebra with principal coefficients whose initial exchange matrix is obtained from \( B \) by deleting its \( r \)-th row and column. Such \( i, r \) exist because due to our assumption, \( N_{\pi_r(h)}|A_r \) has larger dimension than \( N_{\pi_r(h)}|A_r \) if \( N_{\pi_r(h)}|A_r \) lies in the hyperplane \( z_j = 0 \). Then for the cluster algebra \( A' = A_r \), we have a decomposition

\[ \rho_{\pi_r(h)}x_i = \rho_{\pi_r(h)} + e_i + \sum_j c_{w_j, \alpha_j} Y^{w_j} \rho_{\alpha_j} \]

for some \( w_j \in \mathbb{N}^{n-1}, \alpha_j \in \mathbb{Z}^{n-1} \) and \( c_{w_j, \alpha_j} \in \mathbb{Z}_{> 0} \) by inductive assumption.
Denote the multiset
\[ U_h^0 = \bigcup_j \{ (N_{w_j} [\gamma_{r,0}(w_j)])_{c_{w_j,\alpha_j}} \mid \nu_j = \gamma_{r,0}(w_j) b^T \} \bigcup \{ N_{h^+} \}, \]
where \((N_{w_j} [\gamma_{r,0}(w_j)])_{c_{w_j,\alpha_j}}\) represents \(N_{w_j} [\gamma_{r,0}(w_j)]; \ldots, N_{w_j} [\gamma_{r,0}(w_j)]\), that is, the \(c_{w_j,\alpha_j}\) copies of \(N_{w_j} [\gamma_{r,0}(w_j)]\), and \(h_r\) is the \(r\)-th element of \(h\) and \(b^T_r\) is the \(r\)-th column of \(B^T\). According to inductive assumption, the Laurent polynomial \(f\) corresponding to the intersection of \(z_t = 0\) and \( \sum_{N \in U_h^0} N \) with degree \(h\) is the unique minimal Laurent polynomial in \( \bigcap_{l \in [1,n] \setminus \{r\}} \mathcal{U}^0_{\mathcal{Y}_l}(\Sigma_{\mathcal{U}_l}) \) having \(X^h\) as a summand and \(L^h(f) \in \mathbb{N}[Y_{\mathcal{U}^1}][X_{\mathcal{U}^1}]\) for any \(l \in [0,n] \setminus \{r\}\), where \( \mathcal{U}^0_{\mathcal{Y}_l}(\Sigma) = \bigcap_{l \in [0,n] \setminus \{r\}} \mathbb{N}[X_{\mathcal{U}^1}]\).

After \(U_h^0\) being constructed for \(j < s\), we define \(U_h^s\) inductively for any \(s \in \mathbb{Z}_{>0}\) to be a minimal multiset of polytopes \(N_{\alpha_j}[w_j]\) with \(\alpha_j \in \mathbb{Z}^n\) and \(w_j \in \mathbb{N}^n\), which have been constructed by induction, satisfying
\[ f_s \subseteq \bigcap_{l \in [1,n] \setminus \{r\}} \mathcal{U}^0_{\mathcal{Y}_l}(\Sigma_{\mathcal{U}_l}) \text{ and } L^l_i(f_s) \in \mathbb{N}[Y_{\mathcal{U}^1}][X_{\mathcal{U}^1}] \text{ for any } l \in [0,n] \setminus \{r\}, \]
where \(f_s\) is the Laurent polynomial corresponding to the intersection of \(z_r = s\) and \( \sum_{N \in U_h^0} N_{\alpha_j}[w_j] \) with degree \(h\).

Now, we present the concrete way to construct \(U_h^s\).

First, the intersection of \(z_r = s\) and \( \sum_{N \in U_h^0} N_{\alpha_j}[w_j] \) equals \( \sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_j]) \) for some \(\alpha'_j \in \mathbb{Z}^{n-1}\) and \(w'_j \in \mathbb{N}^{n-1}\).

Let \(\gamma_{r,s}(N_{\alpha_j}[w_{j_0}])\) be a summand of \( \sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_j]) \) containing a maximal (respectively, minimal) point in \( \sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_j]) \) when \(b_{tr} \geq 0\) (respectively, \(b_{tr} \leq 0\)). Then denote \(N_1 = \sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_j])\), where \(J'\) is the maximal subset of \(J\) making \(N_1\) a summand of \(\gamma_{r,s}(N_{\alpha_j}[w_j])\) (respectively, \(\gamma_{r,s}(N_{\alpha_j}[w_j])\) when \(b_{ir} \geq 0\) (respectively, \(b_{ir} \leq 0\)), such that for \(\alpha'\) and \(w'\), the Laurent polynomial \(Y_{\alpha', w'}\) is a summand of \(x_i \rho_{\alpha'}\) for \(\mathcal{A}'\) and the maximal \(\mathcal{Y}\)-degree term of \(x_i \rho_{\alpha'}\) is contained in \(Y_{\alpha', w'}\) as a summand. According to the inductive construction and Corollary proved below for \(n-1\) case, such \(\alpha'\) and \(w'\) uniquely exist. Similar to \(\mathcal{Y}\), we have
\[ Y_{\alpha', w'} \rho_{\alpha'} x_i = \sum_{j \in J'} Y_{\alpha', w'} \rho_{\alpha'} x_i \] when \(b_{ir} \geq 0\) (respectively, \(b_{ir} \leq 0\)) for \(\mathcal{A}'\). Now, we let
\[ U_1 = \{ \{N_{w_j} [\gamma_{r,s}(w_j)] \mid \nu_j = \gamma_{r,s}(w_j) b^T \} \bigcup \{ N_{h^+} \}, \]
where \(\alpha'_{r,s}\) is the \(r\)-th element of \(\alpha'\).

Replacing \(\sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_{j_0}])\) by \(\sum_{j \in J} \gamma_{r,s}(N_{\alpha_j}[w_{j_0}]) - N_1\), we repeat the above process to get \(N_2\) and \(U_2\). This goes on until \( \sum_{n \in J} \gamma_{r,s}(N_{\alpha_j}[w_{j_0}]) - N_1 = 0 \). Then finally, let \(U_h^s = \bigcup_i U_i\).

We call such \(U_h^s\) for any \(s \geq 0\) the \(s\)-th stratum of \(N_h\) for \(x_i\) along direction \(r\).

Let \(N_h = \sum_{N \in U_h^0} N_{\alpha_j}[w_j]\) and the corresponding Laurent polynomial \(\rho_h = \sum_{p \in N_h} c_{p}(N_h) \mathcal{Y}^p X^h\), where \(U_h^s\) runs over all strata of \(N_h\) for \(x_i\) along direction \(r\).
It can be seen from the above construction that \( \rho_h \) is indecomposable in \( \bar{U}_{>0}^+(\Sigma) \) having \( X^h \) as a summand and the following decomposition holds:

\[
    x_i \rho_h = \sum_{N_\alpha[w] \in \bigcup_{i} U^i_h} Y^w \rho_\alpha
\]

for any \( i \in [1, n] \), which is in fact the Laurent polynomial version of \( N_h = \sum_{N_\alpha[w] \in \bigcup_{i} U^i_h} N_\alpha[w] \).

Because \( \rho_h = X^h \) when \( h \in \mathbb{N}^n \) and \( \rho_h \) is a Laurent polynomial for any \( h \in \mathbb{Z}^n \), all two dimensional sections of \( N_h \) for \( h \in \mathbb{Z}^n \) defined above are finite, hence the above “polytopes” or formal Laurent polynomials can be defined formally.

Denote by \( I_h \) the subset of \([1, n]\) such that \( i \in I_h \) if and only if there is an edge \( l \in E(N_h) \) parallel to \( e_i \).

Then we claim this is the only possible construction for \( \rho_h \) by showing that \( \rho_h = f \) if \( f \) is indecomposable in \( \bar{U}_{>0}^+(\Sigma) \) having \( X^h \) as a summand. Because \( X^h \) is a summand of \( f \) and \( f \in \mathcal{U}_{>0}^+(\Sigma_{t_k}) \), we can see that \( x_i^{-1}( \sum_{N_\alpha[w] \in U_h^i} Y^w \rho_\alpha|_{A'} ) = \rho_{\pi_i(h)}|_{A'} X^{h_r} \) must be a summand of \( f \).

Hence \( x_i^{-1}( \sum_{N_\alpha[w] \in U_h^i} Y^{\gamma_{r,n}(w_i)} \rho_{\gamma_{r,n}(w_i) \pi_i(\alpha_j)}|_{A'} ) \) is a summand of \( f \). So similar to the above discussion, by using induction on \( s \) we can see that \( x_i^{-1}( \sum_{N_\alpha[w] \in U_h^i} Y^{\gamma_{r,n}(w_i)} \rho_{\gamma_{r,n}(w_i) \pi_i(\alpha_j)}|_{A'} ) \) is a summand of \( f \). Therefore, there is a summand \( \rho_h \) of \( f \). So due to the indecomposability of \( f \), \( f = \rho_h \). In summary, we get that the Laurent polynomial satisfying conditions in (i) exists if and only if \( \rho_h \) is a Laurent polynomial, and it is unique when it exists. Thus we get (i) and (ii).

Due to the uniqueness, we can see that in the above construction, \( \rho_h \) is independent of the choice of \( r \) and \( i \) while the union of strata of \( N_h \) for \( x_i \) along direction \( r \) is independent of the choice of \( r \).

In order to show (ii), we check how the corresponding polytope changes under the change of initial seed. Denote by \( L^{t_k}(N_h) \) the corresponding polytope after the initial seed changing to \( \Sigma_{t_k} \), i.e., the polytope corresponding to \( L^{t_k}(\rho_h) \). According to (19), we have

\[
    L^{t_k}(\rho_h) = \sum_{N_\alpha[w] \in \bigcup_{s} U_s^t} L^{t_k}(Y^w \rho_\alpha X^{-e_i}).
\]

Then by inductive assumption, \( L^{t_k}(\rho_h) \in \mathcal{U}_{>0}^+(\Sigma) \). In particular, \( N_{h+e_i} \in U_h^0 \), so \( L^{t_k}(\rho_{h+e_i} X^{-e_i}) = \rho_{(h+e_i)}^{t_k} x_{i=t_k}^{-1} \) is a summand of \( L^{t_k}(\rho_h) \). Since \( (h+e_i)^{t_k} = h^k + e_i \), \( X_h^{t_k} \) is a summand of \( \rho_{(h+e_i)}^{t_k} x_{i=t_k}^{-1} \), hence a summand of \( L^{t_k}(\rho_h) \).

For any \( N_\alpha[w] \in \bigcup_{s} U_s^t \), \( Y^w F_\alpha|_{Trop(Y_{t_k})} = \bigoplus_{p \in V(N_\alpha[w])} Y^p \) in \( Trop(Y_{t_k}) \), where \( F_\alpha = \rho_\alpha|_{x_i=1, \forall i \in [1,n]} \).

We divide \( V(N_\alpha[w]) \) into several classes \( R_1, \ldots, R_s \) such that \( p, p' \) are in the same class if and only if there is a path connecting them which consists of edges of \( N_\alpha[w] \) each parallel to some coordinate axis. Then due to (v) for \( N_\alpha[w] \) by inductive assumption, we can find \((p_1, I_1), \cdots, (p_s, I_s)\) satisfying \( R_i \) consists of vertices in \( E(p_i, I_i) \) for any \( i \in [1, s] \). According to (3), the definition of \( E(p_i, I_i) \) and the construction of \( \rho_h \) in (i), it can be checked that in \( Trop(Y_{t_k}) \), \( y_k^{[-h_k]+} = \bigoplus_{i=1}^{s} Y_i^{[-h_k]+} \bigoplus_{i=1}^{s} Y_i^{[-h_k]+} = y_k^{[-h_k]+} \), where \( h_k \) is the \( k \)-th element of \( h \). Therefore by the definition of \( L^{t_k} \) and (20), \( L^{t_k}(\rho_h) = F_{\rho_h} (Y) y_k^{[-h_k]+} X^h = F_{\rho_h}(Y) X^h \in \mathbb{N}[Y_j] [X_j^\pm] \). Similarly, \( L^{t_j'} \circ L^{t_k}(\rho_h) \in \mathbb{N}[Y_j] [X_j^\pm] \) for any \( t_j' \) connected to \( t_k \) by an edge labeled \( j \). So \( L^{t_k}(\rho_h) \in \mathcal{U}_{>0}^+(\Sigma_{t_k}) \). Then by (i) for \( \rho_h^{t_k} \), \( \rho_h^{t_k} \) is a summand of \( L^{t_k}(\rho_h) \). So due to the symmetry of \( t_0 \) and \( t_k \), \( L^{t_k}(\rho_h) = \rho_h^{t_k} \).
Then we consider (iii). Note that this result does not depend on the choice of $\epsilon_k$. In the definition of $N_h$, if there is $r \neq k$ satisfying that $N_{\pi, (h)}|_{A_r}$ has dimension $n - 1$, then we can choose $i \neq r, k$ in the definition of $N_h$ and get $x_i \rho_h = \sum_{N, [w] \in \cup U_k} Y^w \rho$. By inductive assumption, (iii) holds for each $N$, i.e., for any $p \in N$, points $q$ in the $k$-section of $N$ at $p$ with $m_k(q) > 0$ consist an interval. Therefore for any $p \in N_h$, points $q$ in the $k$-section of $N_h$ at $p$ with $m_k(q) > 0$ consist a union of several intervals. Moreover, due to the definition of $\rho_h$, particularly $U_k$ and $N_h$, we can see that the union of these intervals is an interval. Hence (iii) holds.

If $k$ is the only index satisfying that $N_{\pi, (h)}|_{A_k}$ has dimension $n - 1$, then we consider $N_{h, k}^{\text{ap}}$ instead. According to (iii) and the definition of $m_h$, (iii) holds for $N_h$ if and only if it holds for $N_{h, k}^{\text{ap}}$. By (ii), either $N_{h, k}^{\text{ap}}$ lies in $z_k = 0$, or there is $r \neq k$ satisfying that $N_{\pi, (h, k)}|_{A_r}$ has dimension $n - 1$, where $A_r$ is the cluster algebra with principal coefficients whose initial exchange matrix is obtained from $\mu_k$ by deleting its $r$-th row and column. In the former case, $m_{h, k}(p) > 0$ for any point $p \in N_{h, k}^{\text{ap}}$, while the latter case is the one we discussed above, so (iii) holds.

(iv) It is a direct corollary of (i) and (ii).

(v) Denote $E^0 = E_h$ and $E^i = \{ l \in E(N_h) \mid l \notin \bigcup_{j=0}^{i-1} E^j \}$, and it is an edge of a 2-dimensional face $N$ of $N_h$ with two non-parallel edges in $\bigcup_{j=0}^{i-1} E^j$.

For any face $S$ of $N_h$, with dimension 2 which has two non-parallel edges in $E^0$, there is $i \neq j \in [1, n]$ satisfying that $S$ is parallel to the 2-dimensional plane determined by the $i$-th and $j$-th coordinate axes (in fact, $i$ and $j$ are the labels of the above two edges in $S$ assigned in the definition of $E_h$, respectively). Then due to the uniqueness and the construction of $\rho_h$, as we explain before, there is $h' \in \mathbb{Z}^2, w \in \mathbb{N}^n$ such that $S = N_{h'}|_{A'}[w]$, where the initial exchange matrix is obtained from $B$ by deleting all but the $i$-th and $j$-th rows and columns. Hence (v) holds for such face.

While $E^i$ consists of edges $l$ of faces $S$ in the above case, which in the mean time are not included in $E^0$. In the proof of Lemma 3.5 (ii), we divide such faces $S$ into two kinds, those with $(u, v) = 0$ and with $(u, v) > 0$ respectively. When $(u, v) = 0$, there is a mutation sequence $\mu$ such that $N_{h'}^{\mu(t_0)}$ is of dimension 1. Therefore, the face $S$ corresponds to an edge in $E^0$ of $N_{h'}^{\mu(t_0)}$, which means that $l$ corresponds to an edge of $N_{h'}^{\mu(t_0)}$ parallel to some coordinate axis. When $(u, v) > 0$, $l$ has no point other than two vertices.

For a face $S$ of $N_h$ with dimension 2, let $s$ be the minimal integer such that there are two non-parallel edges of $S$ in $\bigcup_{j=0}^{s-1} E^j$. We take induction on $s$ to prove $S$ satisfies one of the following conditions:

(a) There is a mutation sequence $\mu = \mu_{i_1} \circ \cdots \circ \mu_{i_s}$ such that there is a face $S_p$ in $N_{h'}^{\mu(t_0)}$ corresponding to $S$ equals to $N_{f}^{\mu(t_0)}$ up to translation for some $f \in \mathbb{Z}^n$, where $S_{j}$ corresponds to $S_{j-1}$ under mutation at direction $i_j$ for any $j \in [1, p]$. So $S$ is isomorphic to $N_{f'}^{\mu'(t_0)}$, where $\mu'$ is obtained from $\mu$ by deleting $\mu_{i_1}$ such that there is no edge of $S_{s-1}$ parallel to the $i_s$-th coordinate axis.

(b) $S$ is isomorphic to a polytope $N_{f}|_{A'}$ whose support is one of the following forms (up to reflection) in Figure 5 for some $f \in \mathbb{Z}^2$ (some vertices may coincide), as there is at least one edge in its essential skeleton with length 1. And $\tilde{\tau}(e_i) \in \mathbb{N}^n$ for $i = j_1, j_2$, where $\tau_S$ is the isomorphism from $N_{f}|_{A'}$ to $S$ and $j_1, j_2$ are labels of edges in its essential skeleton.
The $s = 0$ case has been showed above. Assume the claim is true for less than $s$ case, and now let us deal with the $s$ case.

According to the inductive assumption, an edge $l$ of $S$ in $E^j$ with $j < s$ is in a 2-dimensional face $S'$ of $N_h$ with two non-parallel edges in $\bigcup_{i=0}^{j-1} E^i$ satisfying (a) or (b). $S'$ satisfies (b), $l$ has no point other than its vertices. Then considering the construction of $N_h$ as well, $S$ is isomorphic to a polytope whose support is one of the forms in Figure 5. Otherwise $S'$ satisfies (a), so there is a mutation sequence $\mu$ such that $\mu(S')$ equals to a polytope $N$ associated to a vector up to translation. For $N$, if $(u, v) > 0$ (here we use the same notation with that in the proof of Lemma 3.5), then due to the above discussion, again $l$ has no point other than its vertices and $S$ satisfies (b). If $(u, v) = 0$, there is a mutation sequence such that $N$ corresponds to an edge parallel to some coordinate axis under this sequence.

Recall that for a polytope whose corresponding Laurent polynomial is in $\mathcal{U}(A)$, due to the mutation formula \( \delta \), \( \mu \) and $y_j = y_j \prod_{i=1}^{n} x_i^{b_{ij}}$, its face $R$ either corresponds to a face isomorphic to itself if this is an edge parallel to the $k$-th coordinate axis or corresponds to $\mu_k(R)$ under the mutation in direction $k$ otherwise, where the mutation here as usual means change of the initial seed. Hence in the above case where $(u, v) = 0$, we can find a sequence $\mu^1$ of mutations such that $l$ corresponds to an edge parallel to some coordinate axis. Assume the other edge of $S$ in $E^{j'}$ with $j' < s$, which is not parallel to $l$, corresponds to $l'$. Similarly, either $l'$ has no point other than its vertices and so $S$ satisfies (b) or we can furthermore find a mutation sequence $\mu^2$ such that $l'$ corresponds to an edge parallel to some coordinate axis and thus $\mu^2 \circ \mu^1(S)$ corresponds to a face parallel to a plane determined by two coordinate axes, i.e., $S$ satisfies (a). Hence by induction any face of $N_h$ with dimension 2 satisfies (a) or (b), which leads to (v).

Next we take induction on $r$, the dimension of a face $S$. Assume (v) holds for faces with dimension $r - 1$. Then for a face $S$ with dimension $r$, there is an isomorphism from the polytope associated to certain vector to any proper face of $S$. Then according to inductive assumption for $r - 1$ case, we can choose $r$ vectors $l_1, \ldots, l_r$ of $S$ such that vertices are the only points in $l_i$, there is an edge of $S$ parallel to $l_i$ for any $i \in [1, r]$ and $l' = \sum_{i=1}^{r} a_i l_i$ with all $a_i \in \mathbb{Z}_{\geq 0}$ or all $a_i \in \mathbb{Z}_{\leq 0}$ for any edge $l'$ of $S$. If there is some $l_k$ such that we can not find a mutation sequence under which $l_k$ corresponds to an edge parallel to a coordinate axis, then as we discussed above, there is no interior point in $S$ and (v) for $S$ can degenerate to $r - 1$ case.

Otherwise, choose a face $R$ of $S$ with dimension $r - 1$ having a minimal point of $S$ satisfying that we can exactly find one index $i \in [1, r]$ such that there is no edge in $R$ parallel to $l_i$. Then there must be another $r - 1$-dimensional face $R'$ of $S$ satisfying that $R \cap R' \neq \emptyset$, there is an edge in $R'$ parallel to $l_i$ and $j \in [1, r]$ be the only one such that there is no edge in $R'$ parallel to $l_j$. The above setting is ensured by inductive assumption for $r - 1$ case and the construction of $N_h'$ for any $h'$. According to inductive assumption, we have two polytopes $N_{\alpha|A_1}$ and $N_{\alpha'|A_2}$ isomorphic to $R$ and $R'$ respectively for some cluster algebras of rank $r - 1$ with principal coefficients whose initial exchange matrices are
Definition 3.11. From Theorem 3.9 (ii), we have the definition as follows:

Let \( B' \) be an \( r \times r \) matrix such that \( B_1 \) (respectively, \( B_2 \)) is obtained from \( B' \) by deleting the \( i \)-th (respectively, \( j \)-th) row and column (because there is a mutation sequence \( \mu^k \) under which \( l_k \) corresponds to an edge parallel to a coordinate axis for any \( k \in [1, r] \), the matrices obtained from \( B_1 \) by deleting the \( j \)-th row and column and from \( B_2 \) by deleting the \( i \)-th row and column are the same, so the above condition can be satisfied). While \( b_{ij}' \) is determined by that \( ((\mu^t)'(B'))_{ij} \) equals \( \deg_{x_i}(p) - \deg_{x_j}(q) \), where \( (\mu^t)' \) is obtained from \( \mu^t \) in the same way as that in \( \alpha \), \( p, q \) are two points in \( N^\mu(t_{ho}) \), such that \( \mu^t \) parallel to the edge corresponding to \( l_j \) and \( p > q \). We can determine \( b_{ij}' \) similarly. Then \( B' \) is totally determined. Then let \( f \in \mathbb{Z}^n \) such that \( \pi_i(f) = \alpha \) and \( \pi_j(f + \gamma_{i0}(p)(B')^\top) = \alpha' + qB_2^\top \), where \( p = v_{\alpha}|_{A_1} \) and \( q = v_{\alpha'}|_{A_2} \) corresponding to the same point in \( R \cap R' \) under the isomorphism. It can be checked that \( f \) is unique as it does not depend on the choice of \( p, q \).

Then we show \( S \) is isomorphic to \( \rho_f|_{A'} \), where \( A' \) is the cluster algebra with principal coefficients associated to \( B' \). First following the discussion for 2-dimensional faces, it can be verified inductively on \( s \) for \( E^s \) that there is a bijection from the set of 2-dimensional faces of \( S \) to that of \( N_f|_{A'} \) such that the corresponding two faces under the bijection are isomorphic to the same \( N_{\alpha}|_{A'} \) for some \( \alpha \in \mathbb{Z}^2 \) and some cluster algebra \( A'' \) of rank 2, so they are isomorphic. Given such a bijection from the set of \( s \)-dimensional faces of \( S \) to that of \( N_f|_{A'} \), due to the construction of \( \rho_{h'} \) for any \( h' \) and (i), we can further calculate with the help of above \( N_{\alpha}|_{A''} \) to see that the above bijection induces a bijection from the set of \( s+1 \)-dimensional faces of \( S \) to that of \( N_f|_{A'} \) such that the corresponding two faces under the bijection are isomorphic when forgetting weights and \( c_{pq}(S) \geq c_{pq}(N_f|_{A'}) \) for any corresponding points \( p, q \). Since the \( s \)-dimensional faces of two corresponding \( s+1 \)-dimensional faces under the bijection are isomorphic accordingly and \( \rho_f|_{A'} \) is indecomposable by (i) of Theorem 3.10, \( \rho_f|_{A'} \) is indecomposable by (ii) of Theorem 3.10.

Remark 3.10. The analogous conclusions of (ii) and (iv) in Theorem 3.7 were proved in [17] and [10] for Newton polytopes of cluster variables in rank 2 and rank 3 cases respectively. Note that this is not a coincidence. According to Theorem 2.13 up to multiplying a Laurent monomial in \( X_{t_{ho}}, F_{t_{ho}} \) determines the Laurent expression of \( x_{1,t} \) in \( X_{t_{ho}} \). So the Newton Polytope of \( F_{t_{ho}} \) determines that of \( x_{1,t} \) up to a translation along the exponential vector of above Laurent monomial.

Theorem 4.11 in the sequel can be regarded as an enhanced version of Theorem 3.10 (v) for skew-symmetrizable case. Both of them are inspired by Theorem 6.8 of [1] for Newton polytopes associated to modules, which presents a specific relation between such a polytope and its facets.

As a direct corollary of Theorem 3.9 we can see that \( E_h \in E(N_h) \) when \( \rho_h \in U(A) \).

Starting from the proof of Theorem 3.9 we use \( \rho_h \) and \( N_h \) to represent \( \rho_h^\kappa \) and \( N_h^{\kappa_0} \) respectively. From Theorem 3.10 (ii), we have the definition as follows:

Definition 3.11. The polytope \( N_{h^k}^{\kappa_0} \) of \( \rho_k^{\kappa_0} \) is called the mutation of the polytope \( N_k \) of \( \rho \) in direction \( k \), and denote it as \( \mu_k(N_h) \).

The relation among \( N_h, \mu_k(N_h) \) and their corresponding Laurent polynomials is shown in Figure 3.10.
The construction of \(N_h\) and \(\rho_h\) in the proof of Theorem 3.9 can be restated as follows.

**Construction 3.12.** Given a TSSS cluster algebra \(A\) of rank \(n\) with principal coefficients and a vector \(h \in \mathbb{Z}^n\), we can construct \(N_h\) as well as \(\rho_h\) as follows:

When \(h \in \mathbb{N}^n\), then \(N_h\) is the origin and \(\rho_h = X^h\).

When \(h \in \mathbb{Z}^n \setminus \mathbb{N}^n\), \(N_h\) as well as \(\rho_h\) can be constructed inductively by step by step:

(i) For \(n = 1\), then \(N_h\) equals \(\mathbb{N}\) and \(\rho_h = X^h(\hat{y}_1 + 1)^{-|h|_+}\), where \(p, q\) are the origin and \([-h]_+\), respectively.

(ii) For \(n > 1\) and \(h \in \mathbb{Z}^n \setminus \mathbb{N}^n\), we have the following two cases:

(a) In the case there is \(k \in [1, n]\) such that \(\text{deg}_{z_k} (Y^p X^h) \geq 0\) for any point \(p \in \gamma_{k,0}(N_{\pi_k(h)}|A_k)\). Then \(N_h = \gamma_{k,0}(N_{\pi_k(h)}|A_k)\) with dimension not larger than \(n - 1\), where \(N_{\pi_k(h)}|A_k\) is constructed for \(A_k\) by inductive assumption and \(A_k\) is a cluster algebra of rank \(n - 1\) with principal coefficients whose initial exchange matrix is obtained from \(B\) by deleting its \(k\)-th row and column.

(b) In the case for any \(k \in [1, n]\), \(\text{deg}_{z_k} (Y^p X^h) < 0\) holds for some point \(p \in \gamma_{k,0}(N_{\pi_k(h)}|A_k)\). Then the dimension of \(N_h\) is \(n\). Choose any \(k\) satisfying \(N_{\pi_k(h)}|A_k\) has dimension \(n - 1\) and \(i \neq k \in [1, n]\). Let \(A'\) be the cluster algebra \(A_k\). Then we can construct \(N_h\) inductively in the following steps (1) to (3):

1. Let \(U_k^0 = \bigcup_j \{(N_{\nu_j}[\gamma_{k,0}(w_j)])^{a_{w_j}, \alpha_j} | \nu_j = \gamma_{k; h_k + \gamma_k;0(w_j)b_k^{-1}}(\alpha_j)\}\), where \(a_{w_j}, \alpha_j, \alpha_j\) and \(w_j\) are appearing in the right hand side of the equation \(\rho_{\pi_k(h)}x_i = \sum_j c_{w_j, \alpha_j} Y^{w_j} \rho_{\alpha_j}\) in the cluster algebra \(A'\).

2. Let \(\gamma_{k,0}(N_{\alpha'_{i_0}}[w'_{j_0}]\) be a summand of \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\) containing a maximal (minimal, respectively) point in \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\) when \(b_{ik} \geq 0\) (\(b_{ik} \leq 0\), respectively), where \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\) equals the intersection of \(z_k = s\) and \(\bigcup_{i \in \alpha'_j} N_{\alpha_j}[w_j] \in \bigcup U_k^0\). Then denote \(N_1 = \sum_{j \in J'} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\),

where \(J'\) is the maximal subset of \(J\) making \(N_1\) a summand of \(\gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\) (\(\gamma_{k,0}(N_{\alpha'_j} - e_{i}[w'_{j_0}])\), respectively) when \(b_{ik} \geq 0\) (\(b_{ik} \leq 0\), respectively), such that for \(\alpha'\) and \(w'\), the Laurent polynomial \(Y^{w'_{i_0} - w'} \rho_{\alpha'_{i_0}}\) is a summand of \(x_i \rho_{\alpha'}\) for \(A'\) and the maximal \(Y\)-degree term of \(x_i \rho_{\alpha'}\) is contained in \(Y^{w'_{i_0} - w'} \rho_{\alpha'_{i_0}}\) as a summand. Calculate the decomposition \(Y^{w'_{i_0} - w'} \rho_{\alpha'_{i_0}} x_i = \sum_{j \in J''} Y^{w'_{j_0} - w'} \rho_{\alpha'_{j}}\) (\(Y^{w'_{i_0} - w'} \rho_{\alpha'_{i_0}} x_i = \sum_{j \in J''} Y^{w'_{j_0} - w'} \rho_{\alpha'_{j}}\), respectively) when \(b_{ik} \geq 0\) (\(b_{ik} \leq 0\), respectively) for \(A_k\). Now, we let

\[U_1 = \{N_{\nu_j}[\gamma_{k,0}(w'_{j})] | \nu_j = \gamma_{k; \alpha'_j, h + \gamma_k;0(w'_j)b_k^{-1}}(\alpha'_j)\} \text{ for } j \in J'' \setminus J'\],

where \(\alpha'_{j, k}\) is the \(k\)-th element of \(\alpha'_{j}\).

Replacing \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}])\) by \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}]) - N_1\) and repeating the above process to get \(N_2\) and \(U_2\). This goes on until \(\sum_{j \in J} \gamma_{k,0}(N_{\alpha'_j}[w'_{j}]) - \sum_i N_i = 0\). Then finally, let \(U_h^k = U_{i_1} \cup \ldots \cup U_{i_n}\).
(3) \[ N_h = \sum_{N_{\gamma|\{w_i\}} \in \hat{U}_h \cap T} N_{\gamma|\{w_i\}} \] and \( \rho_h = \sum_{p \in N_h} c_{\gamma}(N_h)\hat{y}^p X^h, \) where \( U_h \) runs over all strata of \( N_h \) for \( x_i \) along direction \( k. \)

**Remark 3.13.** It is ensured by the proof of Theorem 3.9 (i) that (a) of the construction is well-defined. Also we can consider it in polytope way as follows. In (a), if there are two different indices \( j, k \in [1, n] \) with \( j \neq k \) such that

\[
\deg_{x_k}(\hat{y}^p X^h) \geq 0 \quad \text{for any point } p \in \gamma_{k;0}(N_{\gamma|\{w_i\}} | A_k);
\]

\[
\deg_{x_j}(\hat{y}^p X^h) \geq 0 \quad \text{for any point } p \in \gamma_{j;0}(N_{\gamma|\{w_i\}} | A_j),
\]

then \( \deg_{x_j}(\hat{y}^p X^h) \geq 0 \) for any point \( p \in \gamma_{k;0}(N_{\gamma|\{w_i\}} | A_k), \) where \( A_{k,j} \) is a cluster algebra with principal coefficients whose initial exchange matrix is obtained from \( B \) by deleting its \( k \)-th and \( j \)-th rows and columns. Therefore we have

\[
\gamma_{k;0}(N_{\gamma|\{w_i\}} | A_k) = \gamma_{k;0}(\gamma_{j;0}(N_{\gamma|\{w_i\}} | A_k)) = \gamma_{j;0}(\gamma_{k;0}(N_{\gamma|\{w_i\}} | A_k)).
\]

So (a) of the construction is well-defined.

**Remark 3.14.** As in the proof of Theorem 3.9. We define \( F_h = \rho_h|_{x_i \to 1, \forall i \in [1, n]} \). When \( h = g_{1:t} \), the above \( F_h \) is exactly the \( F \)-polynomial of \( x_{1:t} \). So, the polynomial \( F_h \) defined here is a generalization of the \( F \)-polynomial associated to a cluster variable. So we call \( F_h \) the \( F \)-polynomial associated to the vector \( h \). This definition of \( F \)-polynomials is inspired by that of \( F \)-polynomials defined in [4] for representations of a quiver with potential associated to \( g \)-vectors.

For any circle \( \gamma \) in \( T \), with endpoint \( t, L^{t;\gamma}(f) = qf \) for some \( q \in \text{Trop}(Y_t) \) and any homogeneous Laurent polynomial \( f \in \mathbb{N}[Y_t][X_{t+1}] \) by the definition of \( L^{t;\gamma} \); on the other hand \( L^{t;\gamma}(\rho_h) = \rho_h \) following Theorem 3.9 (ii). So \( q = 1 \) and \( L^{t;\gamma} \) depends only on the endpoints of \( \gamma \) for any \( t \in T \) and any path \( \gamma \) in \( T \).

According to Theorem 3.9 we get that \( L^{t;\gamma}(\rho_h) = \frac{F_h|_{x(\hat{Y})}}{F_h|_{x(\hat{Y})}} X^h \). So it is natural to generalize the definition of \( \rho_h \) for a cluster algebra over an arbitrary semifield \( \mathbb{P} \) as

\[
\rho_h|_{\mathbb{P}} := \frac{F_h|_{x(\hat{Y})}}{F_h|_{x(\hat{Y})}} X^h \in \mathbb{N}[X^{\pm 1}],
\]

It can be verified that this does not depend on the choice of the initial vertex \( t_0 \).

Obviously, \( \rho_h|_{\mathbb{P}} \) is related to the choice of the semifield \( \mathbb{P} \). However we in general omit the subscript of semifield if there is no risk of confusion. For example when we talk about \( \rho_0^i \) and \( \rho_0^i \), the semifields are \( \text{Trop}(Y_t) \) and \( \text{Trop}(Y_t') \) respectively.

For a cluster algebra \( A \) of rank \( n \), denote by \( \hat{P}|_{\mathbb{P}} \) the set consisting of all such formal Laurent polynomials \( \rho_h|_{\mathbb{P}} \), i.e.,

\[
\hat{P}|_{\mathbb{P}} = \{ \rho_h|_{\mathbb{P}} \in \mathbb{N}[X^{\pm 1}] \mid h \in \mathbb{Z}^n \}.
\]

Then \( \mathcal{P}|_{\mathbb{P}} \) is a subset of \( \hat{P}|_{\mathbb{P}} \) such that

\[
\mathcal{P}|_{\mathbb{P}} = \hat{P}|_{\mathbb{P}} \cap \mathbb{N}[X^{\pm 1}].
\]

It follows that both \( \hat{P}|_{\mathbb{P}} \) and \( \mathcal{P}|_{\mathbb{P}} \) do not depend on the choice of the initial vertex \( t_0 \).

It is easy to check that according to the definition of \( \rho_h \), in principal coefficients case \( \rho_h \) is homogeneous with degree \( h \) under the canonical \( \mathbb{Z}^n \)-grading since

\[
\deg(\hat{y}_i) = \deg(y_i X^h) = -b_i + \sum_{j=1}^n b_{ij} e_j = 0
\]

for any \( i \in [1, n], \) so \( \deg(\rho_h) = \deg(F_h|_{x(\hat{Y})}) + \deg(X^h) = h. \)
Define $H$ to be the index set of $\mathcal{P}|_{\mathcal{P}}$, i.e., $H := \{ h \in \mathbb{Z}^n \mid \rho_h|_{\mathcal{P}} \in \mathcal{P} \}$. According to Theorem 3.9 (iv), this set is independent of the choice of semifield $\mathbb{P}$ or initial seed.

**Corollary 3.15.** Let $A$ be a cluster algebra having principal coefficients, $h \in H$. Then there is a unique maximal point $p$ in $N_h$. Hence the $F$-polynomial $F_h$ has a unique term $Y^p$ with maximal $Y$-degree as well as a constant term and $\text{co}_{Y^p}(F_h) = \text{co}_1(F_h) = 1$.

**Proof** According to Theorem 3.9 (v), for two vertices $p$ and $q$ in an edge of $N_h$, we always have $p > q$ or $p < q$. And according to Remark 3.13 it is easy to verified when $N_h$ has dimension 2.

In this proof, when we say a path in $N_h$ we always mean segments $l_1, \cdots, l_s$ in $N_h$ with endpoints of each segment in $V(N_h)$ satisfying the target of $l_i$ equals the source of $l_{i+1}$ for any $i \in [1, s-1]$. And a path is an edge path if all segments are edges of $N_h$. We say a path is reduced if there is no cycle and reducing a path is deleting all cycles. For two paths $l_1,l_2,\cdots,l_s$ and $l'_1,\cdots,l'_{s+1}$, we say the latter is an up (respectively, down) move of the former if there is $i \in [1,s]$ such that $l_j = l'_j$ when $j < i$, $l_j = l'_{j+1}$ when $j > i$ and $p + q < 2r$ (respectively, $p + q > 2r$), where $p,q$ are the endpoints of $l_i$ while $r$ is the target of $l'_i$. For two distinguished reduced paths $\zeta$ and $\zeta'$, we say the former lies above the latter if there is a sequence of paths $\zeta_1 = \zeta, \cdots, \zeta_s$ such that $\zeta_{i+1}$ is an up move of $\zeta_i$ or $\zeta_i$ is an down move of $\zeta_{i+1}$ for $i \in [1,s-1]$ and $\zeta'$ is obtained from $\zeta_s$ by reducing.

Assume there are two different maximal point $p$ and $q$ in $N_h$. Choose a reduced edge path in $N_h$ from $p$ to $q$ such that there is no reduced edge path lying above it. Denote it by $l_1,l_2,\cdots,l_r$. Then there is $s \in [1,r-1]$ such that $p_1 > p_2$ and $p_3 > p_2$, where $p_1$ and $p_2$ are the vertices of $l_s$ while $p_2$ and $p_3$ are those of $l_{s+1}$. Then due to the convexity of $N_h$, $p_1 P p_3$ is in $N_h$ and it lies above $l_s,l_{s+1}$. Therefore, following the convexity of $N_h$, Theorem 3.9 (v) and Remark 3.16 we can find an edge path $l'_1,\cdots,l'_j$ lying above $l_s,l_{s+1}$. So we obtain a reduced edge path lying above $l_1,l_2,\cdots,l_r$ by reducing the path $l_1,\cdots,l_{s-1},l'_1,\cdots,l'_j,l_{s+2},\cdots,l_r$, which contradicts the choice of $l_1,l_2,\cdots,l_r$.

Hence there is unique maximal point $p$ in $N_h$, i.e., the $F$-polynomial $F_h$ has unique maximal term $Y^p$.

And the definition of $\rho_h$ ensures that the origin is the unique minimal point in $N_h$. This induces that the origin and $p$ are both vertices of $N_h$. By Theorem 3.9 (v), if $q$ is a vertex of $N_h$, then $\text{co}_q(N_h) = 1$. Thus $\text{co}_q(N_h) = \text{co}_p(N_h) = 1$.

It can be seen that when restricted to $F$-polynomials associated to cluster variables, this corollary is a generalization of the corresponding result in [14] from skew-symmetrizable case to TSSS case.

4. Polytopes in specific cases

4.1. Newton polytopes of $F$-polynomials and recurrence formula on cluster variables.

We have known in [5] that the Newton polytope of an $F$-polynomial is defined associated to representations of a finite-dimensional basic algebra, as well as some interesting combinatorial properties of these Newton polytopes. But those cluster algebras, whose categorification have not been found so far, are not suitable for the theory in [5]. In this sense, it is necessary for us to establish the theory of these Newton polytopes for general TSSSS cluster algebras. This is one of the motivations for our construction in the last section.

In this subsection we will take a look at cluster variables, which turn out to be contained in $\mathcal{P}$. Hence the results in this section hold for cluster variables. In particular, we have a recurrence formula and universally positivity for cluster variables as a special case of the above $\rho_h$ (Theorem 3.11).
First we introduce some definitions from [5], then we will obtain some results analogous to those in [4], but in the context of totally sign-skew-symmetric cluster algebras.

As we said in the last section, any polynomial in \( Y \) corresponds to a polytope. Denote by \( N_{lt,t} \) the Newton polytope of an \( F \)-polynomial \( F_{lt,t} \). The \( F \)-polynomial of a cluster variable can be generalized to any cluster monomial \( X_t^\alpha \) as \( F_{X_t^\alpha} = \prod_{i} F_{lt,i}^{\alpha_i} \). We denote the Newton polytope associated to \( F_{X_t^\alpha} \) as \( N(X_t^\alpha) \).

According to Theorem 3.9 (iv), any cluster monomial \( X_t^\alpha = \rho^{\prime}_t \) is in \( P \) for any \( t \in T_n, \alpha \in \mathbb{N}^n \), and \( h^{+k} = h - 2h_k + h_k[b_k^+] + [-h_k]b_k^+ \) for any \( t_k \) connected to \( t \) by an edge labeled \( k \), which coincides with the mutation formula of \( g \)-vectors. Hence we can see that \( X_t^\alpha = \rho_g(X_t^\alpha) \) as well as \( N(X_t^\alpha) = N_g(X_t^\alpha) \). In particular, this is true for any cluster variable. Thus \( x_{l,t} \) and \( N_{lt,t} \) naturally inherit the properties about \( \rho_h \) and \( N_h \) claimed in Theorem 3.9. And the definitions of \( \rho_h \) and \( N_h \) provide a recurrence formulas for \( N_{lt,t} \) and \( x_{l,t} \).

**Theorem 4.1. (Recurrence formula)** Let \( A \) be a TSSS cluster algebra having principal coefficients, then \( x_{l,t} = \rho_{g_{lt}} \) and \( N_{lt,t} = N_{g_{lt}} \). Following this view, we have that

\[
(21) \quad c_{p}(N_{lt,t}) = c_{p}(N_{g_{lt}}) = \sum_{\alpha_j(w_j) \in \bigcup_{i} U_{g_{lt}}} c_{p}(N_{\alpha_j}[w_j])
\]

and

\[
(22) \quad x_{l,t} = X^{g_{lt}} \left( \sum_{p \in N_{g_{lt}}} c_{p}(N_{lt}) \hat{Y}^p \right),
\]

where \( U_{g_{lt}} \), running over all \( r \)-th strata of the polytope \( N_{h} \) for \( x_{l} \) along direction \( k \), and hence all \( N_{\alpha_j}[w_j] \) are defined in the construction 3.12 (2) with \( h = g_{lt} \).

Repeating the Recurrence formula in the above theorem, we can express each weight of \( N_{lt,t} \) as a sum of weights of some polytopes of the form \( N_{h^i}[w] \) with \( h^i \in \mathbb{Z}^2 \) and \( w \in \mathbb{N}^n \), thus by (12) it is in fact a sum of certain binomial coefficients, which are naturally non-negative. So this provides a proof of the positivity of cluster variables in a totally sign-skew-symmetric cluster algebra. Then we obtain directly the following:

**Corollary 4.2. (Positivity for TSSS cluster algebras)** Let \( A \) be a TSSS cluster algebra with principal coefficients and \((X,Y,B)\) be its initial seed. Then every cluster variable in \( A \) is a Laurent polynomial over \( \mathbb{N}[Y] \) in \( X \). In particular, the positivity of TSSS cluster algebra holds.

As a special case, the proof of this result provides a new method different from that in [11] to present the positivity of cluster variables in a skew-symmetrizable cluster algebra.

Now we show some properties of Newton polytopes associated to cluster variables, which may usually not be true for Newton polytopes associated to general vectors in \( H \).

**Theorem 4.3.** Let \( A \) be a TSSS cluster algebra having principal coefficients, \( l \in [1,n], t \in T_n \). Then, for the \( F \)-polynomial \( F_{lt,t} \) associated to \( x_{l,t} \) and its corresponding Newton polytope \( N_{lt,t} \), the following statements hold:

(i) The support of \( F \)-polynomial \( F_{lt,t} \) is saturated.

(ii) For any \( p \in N_{lt,t}, c_{p}(N_{lt}) = 1 \) if and only if \( p \in V(N_{lt,t}) \).

(iii) Let \( S \) be a \( r \)-dimensional face of \( N_{lt,t} \). Then there is a cluster algebra \( A' \) with principal coefficients of rank \( r \), a Newton polytope \( S' \) corresponding to some coefficient free cluster monomial in \( A' \) and an isomorphism \( \tau \) from \( S' \) to \( S \) with its induced linear map \( \hat{\tau} \) satisfying

\[
\hat{\tau}(e_i) \in \mathbb{N}^{n} \quad \text{for any } i \in [1,r].
\]
**Proof** We prove this theorem by induction on the length of the path connecting $t$ and $t_0$. It is trivial when the length is $0$, i.e., $t_0 = t$. Assume it is true for $t' \in \mathbb{T}_n$. We claim that it is also true for $t_0 \in \mathbb{T}_n$ connected to $t'$ by an edge labeled $k \in [1, n]$.

(i) Assume there is a lattice point $p \in N_{t,t}$ such that $\text{co}_p(P_{t,t}) = 0$. If there is no $p_1 \in N_{t,t}$ such that $\frac{p_1}{p_1} = \tilde{y}_k^a_1$ and $\text{co}_{p_1}(P_{t,t}) \neq 0$ for some $a \in \mathbb{Z}_{>0}$, then according to the mutation formulas, we know that there is $p'$ corresponding to $p$ (with respect to the mutation at direction $k$) satisfying that $\text{co}_{p'}(P_{t,t}') = 0$. Moreover, as $p \in N_{t,t}$, we have $p' \in N_{t,t}'$. Thus $N_{t,t}'$ is not saturated, which contradicts our inductive assumption. Similarly, we obtain a contradiction if there is no $p_1 \in N_{t,t}$ such that $\frac{p_1}{p_1} = \tilde{y}_k^a_2$ and $\text{co}_{p_1}(P_{t,t}) \neq 0$ for some $a \in \mathbb{Z}_{<0}$.

Therefore, there must be $p_1, p_2 \in N_{t,t}$ such that $\frac{p_1}{p_1} = \tilde{y}_k^a_1$, $\frac{p_2}{p_2} = \tilde{y}_k^a_2$, where $a_1 \in \mathbb{Z}_{>0}$, $a_2 \in \mathbb{Z}_{<0}$ and $\text{co}_{p_1}(P_{t,t}) \text{co}_{p_2}(P_{t,t}) \neq 0$. If $\deg_{z_k}(p) \geq 0$, then $\deg_{z_k}(p_1) \geq 0$ and $\deg_{z_k}(p_2) \geq 0$. Hence we always have $m_k(p) = \text{co}_p(P_{t,t}) = 0$ while $m_k(p_1) = \text{co}_{p_1}(P_{t,t}) \neq 0$ and $m_k(p_2) = \text{co}_{p_1}(P_{t,t}) \neq 0$, which contradicts Theorem 4.3 (ii). Otherwise if $\deg_{z_k}(p) < 0$, then due to the definition of $m_k$, we can find $p_1$ and $p_2$ satisfying $m_k(p_1)m_k(p_2) \neq 0$, which contradicts Theorem 3.9 (iii). So in conclusion we get that $N_{t,t}$ is saturated.

(ii) If there is a point $p \in N_{t,t}$ satisfying that $\text{co}_p(N_{t,t}) = 1$ but $p \notin V(N_{t,t})$, then according to the mutation formula 4.3 and 4.5, there is a point $p' \in N_{t,t}'$ corresponding to $p$ with $\text{co}_{p'}(N_{t,t}') = 1$. Hence $p' \in V(N_{t,t}')$, then due to the mutation formula 4.3 and 4.5, Theorem 3.9 (iii) and (i) of this theorem, either $p \notin V(N_{t,t})$ or $\text{co}_{p}(N_{t,t}) > 1$, which contradicts our assumption.

(iii) Because of Theorem 3.9 (v), $S$ is isomorphic to $N_{h,h'}$ for some $h' \in \mathbb{Z}^+$. So we only need to show that $N_{h,h'}$ equals the Newton polytope associated to some coefficient free cluster monomial in $\mathcal{A}'$.

According to the mutation formula 4.3 and 4.5, any face $S$ of $N_{t,t}$ corresponds to a face $S''$ of $N_{t,t}'$, which is isomorphic to the Newton polytope $N$ of some coefficient free cluster monomial in $\mathcal{A}'$ as the inductive assumption. Denote this isomorphism by $\tau'$. Since $S$ corresponds to $S''$ with respect to the mutation in direction $k$, it is either isomorphic to $N$ when $\tilde{\tau}'(e_i) \neq e_k$ for any $i$ or to $\mu_{k}(N)$ when there is $i$ such that $\tilde{\tau}'(e_i) = e_k$. In both cases, $S$ is isomorphic to the Newton polytope associated to some coefficient free cluster monomial.

The following statements hold.

Conjecture 4.4. Let $\mathcal{A}$ be a TSSS cluster algebra with principal coefficients. Then,

(i) A point $p$ in the Newton polytope associated to any coefficient free cluster monomial is a vertex if and only if $\text{co}_p = 1$.

(ii) The support of the $F$-polynomial of any cluster monomial is saturated.

In 4.3, this conjecture was proved to be true when the initial exchange matrix $B_{t_0}$ is acyclic and skew-symmetric. Here, as a direct corollary of Theorem 4.3 we give a positive answer to the conjecture.

Corollary 4.5. Let $\mathcal{A}$ be a TSSS cluster algebra with principal coefficients and $t \in \mathbb{T}_n$. Then,

(i) A point $p$ in the Newton polytope $N(X_{t,t}^\alpha)$ associated to any cluster monomial $X_{t,t}^\alpha$ is a vertex if and only if $\text{co}_p(N_{t,t}(X_{t,t}^\alpha)) = 1$ for any $\alpha \in \mathbb{N}^n$.

(ii) The support of the $F$-polynomial $F_{X_{t,t}^\alpha}$ of any cluster monomial $X_{t,t}^\alpha$ is saturated.
Proof (i) According to the definition of Minkowski sum, a point \( q \in N \oplus N' \) is a vertex if and only if there is unique vertices \( p \in N \) and \( p' \in N' \) such that \( q = p + p' \). Then by Theorem 4.3 (ii), \( p \in V(N(X^\circ)) \) is equivalent to \( \c_\rho = 1 \) since \( N(X^\circ) = \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{\alpha_i} N_{i:t}) \).

(ii) For any point \( q \in N \oplus N' \), there are points \( p \in N \) and \( p' \in N' \) such that \( q = p + p' \). If the supports of \( N \) and \( N' \) are both saturated, then \( \c_\rho(N), \c_\rho'(N') \neq 0 \). So (ii) is induced by Theorem 4.3 (i) and the universally positivity of any cluster variable since \( N(X^\circ) = \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{\alpha_i} N_{i:t}) \).

\( \square \)

Example 4.6. Let \( \mathcal{A} \) be a cluster algebra having principal coefficients with the initial seed \((X, Y, B)\), where \( X = (x_1, x_2, x_3) \), \( Y = (y_1, y_2, y_3) \) and

\[
B = \begin{pmatrix} 0 & -3 & 16 \\ 3 & 0 & -6 \\ -16 & 6 & 0 \end{pmatrix}.
\]

Then in the seed \((X_t, Y_t, \hat{B}_t) = \mu_3 \circ \mu_1 \circ \mu_2((X, \hat{B}))\), the Laurent expression of \( x_{3; t} \) in \( X \) is

\[
x_{3; t} = \frac{y_1 y_2 y_3 x_1 x_2 x_3 (y_2^2 y_3 x_1^2 x_2 + 2y_1 y_2^2 y_3 x_2^2 x_3 + 6y_1 y_2 y_3^2 x_3^2 x_1 + 6y_1 y_2 y_3 x_1 x_2^2 x_3 + 2y_2 y_3 x_1 x_2^3 + y_2 y_3 x_2 x_3^2 + 6y_2 y_3^2 x_3^2 x_1 + 15y_2^2 y_3 x_3^3 + 20y_2 y_3 x_1 x_2 y_3^2 + 15y_2 y_3^2 x_2 x_3^2 + 6y_2 x_1^5 y_3 + 6y_2 x_1^4 y_3 + 6y_2 x_1^3 y_3 + 6y_2 x_1^2 y_3 + 6y_2 x_1 y_3 + 6y_2 y_3^2 x_2 x_3)}{x_1 x_2 x_3}.
\]

Hence the corresponding Newton polytope \( N_{3; t} \) is as follows

\[\text{Figure 7. The Newton polytope } N_{3; t}\]

where the set of points is the support of \( F_{3; t} \) and we mark points in \( V_{3; t} \) and edges in \( E_{3; t} \) in red. It can be seen that the support of \( F_{3; t} \) is saturated, \( V_{3; t} = V(N_{3; t}) \) and \( E_{3; t} \subseteq E(N_{3; t}) \). Moreover, it can be checked that the constant coefficients of these Laurent monomials satisfy Theorem 2.1. We calculate the coefficients of blue points in \( \rho_{3; t} \) for example, where \( g_{3; t} = (0, 0, -1) \).

\[
\c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(\rho_{3; t}) = C_6^{10} = 6 = \c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(P_{3; t}),
\]

\[
\c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(\rho_{3; t}) = C_2^{10} = 2 = \c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(P_{3; t}),
\]

and

\[
\c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(\rho_{3; t}) = \max\{\c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(\rho_{3; t})C_4^{10} - \deg_{x_1}(y_2 y_3 x_1 x_2^{-6} x_3^{-6}), \c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(\rho_{3; t})C_4^{10} - \deg_{x_2}(y_2 y_3 x_1 x_2^{-6} x_3^{-6})\}
\]

\[
= \max\{0, 6\}
\]

\[
= 6
\]

\[
= \c_{g_2 y_1 y_2 y_3 x_1 x_2^{-6} x_3^{-6}}(P_{3; t}).
\]
We can calculate to see that \( N_{3,1} = N_{g_{3,1}} \) and \( x_{3,1} = \rho_{g_{3,1}} \).

4.2. Newton polytopes in a skew-symmetrizable cluster algebra.

In this subsection we assume \( \mathcal{A} \) is a skew-symmetrizable cluster algebra. Recall that in this paper we denote the skew-symmetrizer \( D = \text{diag}(d_1, \cdots, d_n) \). Then for the exchange matrix \( B_t \) of \( \mathcal{A} \) at any vertex \( t, DB_t \) is skew-symmetric, which will help us to construct the proper faces of \( N_h \) more explicitly.

When \( \mathcal{A} \) is of rank 2, we have listed the possible shape of all \( N_h \) of dimension 2 in Figure 4.4(3)-(7). Any polytope \( N_h \) satisfies one of the two following conditions (a) and (b) from the proof of Theorem 4.9.

(a) There is \( t \in T_n \) such that \( N^t_{h,t} \) is the origin, i.e. the polytope has dimension 0.
(b) For any \( t \in T_n \), \( N^t_{h,t} \) has dimension 2.

A polytope \( N_h \) whose shape is one of (3)-(6) satisfies the condition (a) while that with shape (7) may satisfy (a) or (b). In order to determine which condition such a polytope satisfies, we need to analyze the lengths of two legs. Assume the initial exchange matrix is

\[
B' = \begin{pmatrix} 0 & \epsilon b \\ -\epsilon c & 0 \end{pmatrix}
\]

with \( b, c \in \mathbb{N}, \epsilon \in \{ \pm 1 \} \) and the lengths of two legs in \( N_h \) are \( w_1 \) and \( w_2 \) respectively.

A polytope with shape (7) satisfying (a) is mutation equivalent to \( N^\alpha_h \) for \( \alpha \in \mathbb{N}^2 \). So by calculating the polytope \( N^\alpha_h \) for \( i = 1, 2 \) and \( t \in T_n \), a polytope with shape (7) satisfies (a) if and only if \( w_1 \) and \( w_2 \) satisfy the following Condition 4.7 for five cases, where all of the sets are multi-sets:

**Condition 4.7.** Case 1: when \( bc = 1 \), it holds that \( \{w_1, w_2\} = \{1, 1\} \) and let \( s = 1 \) or 2;

Case 2: when \( bc = 2 \), it holds that \( \{w_1, w_2\} \) equals \( \{1, 1\} \) or \( \{1, 2\} \) and let \( s \) satisfy \( |b'_{rs}| = \max \{w_1, w_2\} \) for some \( r \);

Case 3: when \( bc = 3 \), it holds that \( \{w_1, w_2\} \) equals \( \{1, 1\}, \{1, 2\}, \{1, 3\} \) or \( \{2, 3\} \) and let \( s \) satisfy \( |b'_{rs}| = 3 \) if \( 3 \in \{w_1, w_2\} \) while \( |b'_{rs}| = 1 \) if \( 3 \notin \{w_1, w_2\} \) for some \( r \);

Case 4: when \( bc = 4 \), there is \( i \in \mathbb{Z}_>0 \) such that \( \{w_1, w_2\} \) equals \( \{2i-1, 4i\}, \{2i+1, 4i\}, \{i, i+1\}, \{2i-1, i\} \) or \( \{2i+1, i\} \) and let \( s \in \{1, 2\} \) satisfy that either both \( w_1 \) and \( w_2 \) are odd and \( |b'_{rs}| = 1 \) for some \( r \) or \( w_s \) is the only odd in \( \{w_1, w_2\} \);

Case 5: when \( bc > 4 \), there is \( j, i \in \mathbb{N} \) such that \( j - i \) equals 0 or 1,

\[
\{w_1, w_2\} = \left\{ \frac{f^{j}+f^{j-2}+(-k)^{-j}+(-k)^{-j+1}}{f+k}, \frac{f^{j+1}+f^{j+2}+f^{j+1}+(-k)^{j+1}+(-k)^{j+2}}{|b'_{sr}|(f+k)} \right\}
\]

and \( s \) satisfies that \( w_s = \frac{f^{j}+f^{j-1}+(-k)^{-j}+(-k)^{-j}}{f+k} \), where \( s \neq r \), \( f = \sqrt{bc^{2}+4bc+bc^{2}} \) and \( k = \frac{\sqrt{bc^{2}+4bc+bc^{2}}^{2}}{2} \).

It can be checked that \( \frac{f^{j}+f^{j-1}+(-k)^{-j}+(-k)^{-j}}{f+k} \) and \( \frac{f^{j+1}+f^{j+2}+f^{j+1}+(-k)^{j+1}+(-k)^{j+2}}{|b'_{sr}|(f+k)} \) are increasing along the increase of \( j \) and \( i \) respectively when \( bc > 4 \). So in practice Condition 4.7 is not hard to verify.

We assign a label to each edge in \( E(N_h) \). Denote \( E^0 = E_h \). And to each edge in \( E^0 \) assign a label by the way introduced in the definition of \( E_h \). Then inductively let \( E^{r+1} \) consist of the edges not in each of which is an edge of a face \( S \) of \( N_h \) with dimension 2 having at least two non-parallel edges in \( \bigcup_{i=0}^{r} E^i \). According to Theorem 3.9(v), there are \( hS \in \mathbb{Z}^2 \), a cluster algebra \( \mathcal{A}' \) and an isomorphism \( \tau \) from \( N_{hS} \mathcal{A}' \) to \( S \) satisfying \( \tau(e_i) \in \mathbb{N}^2 \) for \( i = 1, 2 \). Denote the initial exchange matrix of \( \mathcal{A}' \) by

\[
B''_{ww} = \begin{pmatrix} 0 & \epsilon b \\ -\epsilon c & 0 \end{pmatrix}
\]

with \( b, c \in \mathbb{N}, \epsilon \in \{ \pm 1 \} \).
To each edge \( l \in S \cap E^{r+1} \) we assign the label \( i(l) \) by labels of edges in \( \bigcup_{i=0}^{r} E^i \) as

\[
\begin{array}{ll}
\text{the label assigned to the edge} & \text{if } N_{h_i} | \mathcal{A}' \text{ is of the form (3), (4) or (5) in Figure 4} \\
\text{of } S \text{ not intersecting with } l & \\
\text{the label assigned to the edge} & \text{if } N_{h_i} | \mathcal{A}' \text{ is of the form (6) in Figure 4} \\
\text{of } S \text{ intersecting with } l & \\
\text{the label assigned to the edge} & \text{if } N_{h_i} | \mathcal{A}' \text{ is of the form (7) in Figure 4} \\
\text{of } S \text{ parallel to } \tilde{\tau}(e_i) & \text{and } w_1, w_2 \text{ satisfy the following Condition 4.7} \\
0 & \text{if } N_{h_i} | \mathcal{A}' \text{ is of the form (7) in Figure 4} \\
\end{array}
\]

where \( w_1, w_2 \) is calculated by \( \tilde{\tau}^{-1}(p - q) = w_1 e_1 + w_2 e_2 \) for arbitrary two points \( p > q \) in \( S \) corresponding to two points in the hypotemuse of \( N_{h_i} | \mathcal{A}' \) such that there is no other point in \( \overline{W} \). Note that the edges in the right-hand side of (23) are those in \( \bigcup_{i=0}^{r} E^i \), so the inductive definition is well-defined.

In the skew-symmetrizable case, the following result help us to find the cluster algebra \( \mathcal{A}' \) so as to construct the proper face of \( N_h \) more conveniently.

**Theorem 4.8.** In Theorem 3.9 (v), if \( \mathcal{A} \) is a skew-symmetrizable cluster algebra with principal coefficients, and denote by \( B' \) the initial exchange matrix of the cluster algebra \( \mathcal{A}' \), then \( B' = \overline{W}^T \overline{B} W \), where \( W = (\overline{\tau}(e_1)^T, \ldots, \overline{\tau}(e_r)^T), \overline{B} = (\overline{\tau}(e_1)^T, \ldots, \overline{\tau}(e_r)^T) \) are \( n \times r \) integer matrices,

\[
\overline{\tau}(e_i) = \sum_{j=1}^{r} d_{ji} w_{ji} e_j \quad \text{for } \overline{\tau}(e_i) = \sum_{j=1}^{r} w_{ji} e_j, \text{ with } s \neq 0 \text{ being the label of the edge in } S \text{ parallel to } \tilde{\tau}(e_i)
\]

while \( \overline{\tau}(e_i) = \sum_{j=1}^{r} d_{ji} w_{ji} e_j \) when the label is 0.

**Proof** We denote \( W = (w_{ji})_{n \times r} \) and \( w_i \) the \( i \)-th column of \( W \) for \( i \in [1, r] \). If there is \( i \in [1, r] \) such that the label of the edge in \( S \) parallel to \( \tilde{\tau}(e_i) \) is 0, then there is no interior point in \( N_{h_i} | \mathcal{A}' \) and in the construction of \( N_{h_i} | \mathcal{A}' \), the \( i \)-th column of \( B' \) is not used. Hence the \( i \)-th column of \( B' \) can be arbitrary so long as \( B' \) is skew-symmetrizable. So we can let \( \overline{\tau}(e_i) = \sum_{j=1}^{r} d_{ji} w_{ji} e_j \) and then focus on proper faces of \( S \). We deal with all \( i \in [1, n] \) such that the label of the edge in \( S \) parallel to \( \tilde{\tau}(e_i) \) is 0 in the above way. Hence in the following we assume the label of the edge in \( S \) parallel to \( \tilde{\tau}(e_i) \) is not 0 for any \( i \in [1, r] \).

According to the discussion in the proof of Theorem 3.9, there is some \( t \in \mathbb{T}_n \), a subset \( J \subset [1, n] \) and \( h_i' \in \mathbb{Z}^{\mid J \mid} \) such that there is a face \( S^t \) of \( N_{h_i'} \) which corresponds to \( S \) under certain mutation sequence satisfying that \( S^t \) equals \( N_{h_i'} | \mathcal{A}' \), up to translation, where \( \mathcal{A}' \) is a cluster algebra associated to \( B'_t \) obtained from \( B_t \) by deleting rows and columns with labels not in \( J \). This means Theorem 3.9 (iv) holds for \( S^t \) with an isomorphism \( \tau_t \) such that \( \tilde{\tau}^t(e_j) = e_j \) for any \( j \in J \). Then we use induction on the length of the path from \( t_0 \) to \( t \) to complete the proof. Assume it holds for \( t_1 \) and \( t_2 \) is connected to \( t_1 \) by an edge labeled \( k \). For any face \( S^{t_2} \) of \( N_{h_i'} \), there is a face \( S^{t_1} \) of \( N_{h_i'} \) corresponding to it according to Theorem 3.9 (ii).

If the dimension of \( S^{t_1} \) equals that of \( S^{t_2} \) and there is no edge \( e \in E(N_{h_i'} | \mathcal{A}'_{t_1}) \) satisfying \( \tilde{\tau}_{t_1}^t(e) = e_k \), then according to the mutation formula (5) and (6), it can be seen that \( S^{t_2} \) is isomorphic to \( S^{t_1} \). Hence we can find an isomorphism \( \tau_{t_2} \) from \( N_{h_i'} | \mathcal{A}'_{t_1} \) to \( S^{t_2} \), i.e., in this case \( h_i' = h_i' \) and \( B'_t = B'_t \). Denote \( B'_t = (b_{ij})_{i,j \in I_{h_i'}}. \)
On the other hand, again due to the mutation formula (5) and (9), we can calculate that \( w_{j_i}^{t_2} = w_{j_i}^{t_1} \) if \( j \neq k \), while \( w_{k_i}^{t_1} = \sum_l w_{l_i}^{t_1} [\epsilon_k b_{l_k}^{t_1}]_+ - w_{k_i}^{t_1} \) for some \( \epsilon_k \in \{ \pm 1 \} \) (here the choice of \( \epsilon \) is according to that \( S^{t_2} \) is an upper face or a bottom face with respective to the \( k \)-th coordinate). Hence we also have \( \overline{w}_{j_i}^{t_2} = \overline{w}_{j_i}^{t_1} \) if \( j \neq k \), while

\[
\overline{w}_{k_i}^{t_1} = \sum_l \frac{d_k}{d_l} \overline{w}_{l_i}^{t_1} [\epsilon_k b_{l_k}^{t_1}]_+ - \overline{w}_{k_i}^{t_1} = \sum_l \overline{w}_{l_i}^{t_1} [-\epsilon_k b_{l_k}^{t_1}]_+ - \overline{w}_{k_i}^{t_1}.
\]

Therefore, it can be checked that

\[
\sum_{l,s} \overline{w}_{l_i}^{t_1} b_{l_s}^{t_2} u_{s_j}^{t_2} = \sum_{l,s \neq k} \overline{w}_{l_i}^{t_1} (b_{l_s}^{t_2} + [-\epsilon_k b_{l_k}^{t_1}]_+ b_{k_s}^{t_1} + b_{l_k}^{t_1} [\epsilon_k b_{k_s}^{t_1}]_+) u_{s_j}^{t_2}
\]

\[- \sum_{s \neq k} \overline{w}_{l_i}^{t_1} [-\epsilon_k b_{l_k}^{t_1}]_+ b_{k_s}^{t_1} u_{s_j}^{t_2} - \sum_{l \neq k} \overline{w}_{l_i}^{t_1} b_{l_k}^{t_1} (\sum_{v} w_{v_j}^{t_1} [\epsilon_k b_{v_k}^{t_1}]_+ - w_{k_j})
\]

\[
= \sum_{l,s} \overline{w}_{l_i}^{t_1} b_{l_s}^{t_2} u_{s_j}^{t_2} = b_{j_j}^{t_2}
\]

So \( B_{t_2} \) is hyperplane \( z_k = i \) for some \( i \in \mathbb{N} \). We can embed \( N_{h_{t_2}} \) to a higher space with an extra coordinate \( z_k \) and extend \( \ell_{t_1} \) as well as \( B'_{t_1} \) by setting \( \ell_{t_1} (e_k) = e_k \) and adding the \( k \)-th row and column to \( B'_{t_1} \) according to the \( k \)-th row and column of \( B_{t_1} \) respectively.

Otherwise there is an edge \( e \in E(N_{h_{t_2}}) \sim A_{t_2} \), satisfying \( \ell_{t_1} (e) = e_k \) or there is an edge in \( S^{t_2} \) parallel to \( e_k \) (in the former case we also use \( k' \) to denote the label of \( e \)), then \( S^{t_2} \) is isomorphic to \( \mu_k' (N_{h_{t_1}}) \sim A' \). Hence \( B'_{t_2} = \mu_k' (B'_{t_1}) \). On the other hand, since \( \ell_{t_1} (e) = e_k \), we have \( w_{j_k}^{t_1} = \delta_{j_k} \), where is. And similar to the first case, in this case we can also calculate to see that

\[
w_{j_i}^{t_2} = \begin{cases} \sum_{l} w_{l_i}^{t_1} [\text{sgn}(b_{ik'}) b_{l_k}^{t_1}]_+ - w_{k_i}^{t_1} & \text{if } j = k; \\ \overline{w}_{j_i}^{t_1} & \text{otherwise.} \end{cases}
\]

and

\[
\overline{w}_{j_i}^{t_2} = \begin{cases} \sum_{l} w_{l_i}^{t_1} [-\text{sgn}(b_{ik'}) b_{l_k}^{t_1}]_+ - \overline{w}_{k_i}^{t_1} & \text{if } j = k; \\ \overline{w}_{j_i}^{t_1} & \text{otherwise.} \end{cases}
\]

Therefore, for any \( i, j \in I_{h_{t_2}} \setminus \{ k' \} \),

\[
\sum_{l,s} \overline{w}_{l_i}^{t_2} b_{l_s}^{t_2} u_{s_j}^{t_2} = \sum_{l,s \neq k} \overline{w}_{l_i}^{t_1} [b_{l_s}^{t_2} + [-\epsilon_k b_{l_k}^{t_1}]_+ b_{k_s}^{t_1} + b_{l_k}^{t_1} [\epsilon_k b_{k_s}^{t_1}]_+] u_{s_j}^{t_2}
\]

\[- \sum_{s \neq k} \overline{w}_{l_i}^{t_1} [-\epsilon_k b_{l_k}^{t_1}]_+ b_{k_s}^{t_1} u_{s_j}^{t_2} - \sum_{l \neq k} \overline{w}_{l_i}^{t_1} b_{l_k}^{t_1} (\sum_{v} w_{v_j}^{t_1} [\epsilon_k b_{v_k}^{t_1}]_+ - w_{k_j})
\]

\[
= \sum_{l,s} \overline{w}_{l_i}^{t_1} b_{l_s}^{t_2} u_{s_j}^{t_2} = b_{j_j}^{t_2} + [b_{ik}'] + b_{k'} + [b_{jk}'] + ,
\]

while

\[
\sum_{l,s} \overline{w}_{l_i}^{t_2} b_{l_s}^{t_2} u_{s_j}^{t_2} = \sum_s b_{ik}^{t_2} u_{s_j}^{t_2} = \sum_s b_{ik}^{t_2} u_{s_j}^{t_2} = \sum_{l,s} \overline{w}_{l_i}^{t_1} b_{l_s}^{t_2} u_{s_j}^{t_2} = -b_{jk}.'
and similarly $\sum_{i,s} \sum_{t_1} b_{t_1}^{i,s} w_{sk}^{t_1} = -b_{ik}$. So we get $B'_{t_2} = \mu_k(B'_{t_1}) = \sum_{t_2} B_{t_2} W_{t_2}$, which completes the proof.

According to Theorem 3.9 Theorem 4.8 and the construction of $\rho_h$ in the proof of Theorem 3.9 when $A$ is a skew-symmetric cluster algebra with principal coefficients, we can calculate the proper faces of $N_h$ in the following way.

The key point is to determine $E(N_h) = \bigcup_i E^i$. Denote $E^0 = E_h$ and $N^0$ the convex hull of $E^0$. Assign a label set to each edge in $E$. Then we get $E^r$ with dimension 2 having two non-parallel edges $l_1$ and $l_2$ in $\bigcup_i E^i$ and at least one edge not in $\bigcup_i E^i$, we construct a polytope $S$ with dimension 2 such that $S' \subseteq S$ and $l_1, l_2$ are edges of $S$ by the following properties:

(a) There is a vector $h_S \in \mathbb{Z}^2$ and an isomorphism $\tau : N_{h_S} \rightarrow S$.

(b) $\hat{\tau}(e_i) = \sum_{j=1}^N w_{ji} e_j \in \mathbb{N}^n$ for $i = 1, 2$.

(c) The above $N_{h_S}$ means the polytope is defined in the cluster algebra $A'$, whose initial exchange matrix $B' = BW^\top = \tau(e_1) = (\tau(e_1), \tau(e_2))$, $W = (\tau(e_1), \tau(e_2))$ and $\tau(e_i) = \sum_{j=1}^N d_{ji} e_j$ with $s \neq 0$ the label of an edge in $S$ parallel to $\hat{\tau}(e_i)$ and $\tau(e_i) = \sum_{j=1}^N d_{ji} e_j$ when the label is 0.

Then $S$ is in fact a face of $N_h$ according to Theorem 4.8. Let an edge $l$ of $S$ be in $E^{r+1}$ if $l$ is not in $\bigcup_{i=0}^r E^i$ and $N^{r+1}$ be the convex hull of $E^{r+1}$. Then $E^{r+1} = \bigcup_{p} E(p, I(p))$ and $N^{r+1}$ is the convex hull of $E^{r+1}$, where $p$ runs over all vertices of edges in $E^{r+1}$ and $I(p) = \{i \in [1, n] | p - \deg x_p \cdot (\hat{Y}^p X^h)_{+} e_i \notin N^{r+1}\}$. The label $i(l)$ to each edge $l \in E^{r+1}$ is assigned by (23).

When $N_h$ is a finite set, calculation along the above way ends in finitely many times, and we would get $E^0, \cdots, E^r$. Note that according to our construction, $\bigcup_i E^i$ includes all edges of $N_h$. Then the support of $N_h$ is determined. And similarly, we can also calculate any proper face of $N_h$ with higher dimension by Theorem 4.8. If $N_h$ is not finite, the procedure has no end, however it is finite locally since any face of dimension 2 is finite. So we can still have such definition formally.

The later process is just as that in Construction 3.12 but it is slightly easier in calculation since we have known all proper faces now.

Example 4.9. Let $A$ be a cluster algebra having principal coefficients with the initial seed $(X, Y, B)$, where $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ and

$$B = \begin{pmatrix} 0 & 2 & -1 \\ -1 & 0 & 2 \\ 1 & -4 & 0 \end{pmatrix}.$$ 

Let $h = (-5, 2, 4)$, then we can calculate $N_h$ as well as $\rho_h$ according to their definitions. We can first obtain $V_h$ as the set consisting of red points with coordinates listed in (1) of the following figure. Then we get $E_h$ consisting of red edges and the proper faces of $N_h$ can be calculated recursively.
Then, following the definitions, we can calculate that

\[ U_0 = \{ \gamma_{3;0}(N_{(−5, 2)}) \}, \quad U_1 = \{ \gamma_{3;1}(N_{(−4, 2)}) \}^2, \gamma_{3;1}(N_{(−2, 1)} \}^3 \}, \]
\[ U_2 = \{ \gamma_{3;2}(N_{(−3, 2)}) \}, \gamma_{3;2}(N_{(−2, 1)} \}^2, \gamma_{3;2}(N_{−1, 1)} \}^3 \} \] and \[ U_3 = \{ \gamma_{3;3}(N_{(−2, 3)}) \}. \]

Thus combining them together we get the polytope \( N_h \) and the associated Laurent polynomial

\[ \rho_h = \sum_{\gamma \in U_0 \cup U_1 \cup U_2 \cup U_3} \frac{x_1^{-5} x_2^2 x_3^3 + 5y_1 x_1^{-5} x_2 x_3^3 + 10y_1^2 x_1^{-5} x_2^{-1} x_3^7 + 5y_1^3 x_1^{-5} x_2^2 x_3^3 \}}{\gamma} \]

associated to points in \( U_0 \)

\[ + y_1 x_1^{-5} x_2^{-3} x_3^9 + y_1 y_2 x_1^{-3} x_2 x_3 + 9y_1^2 y_2 x_1^{-3} x_2 x_3^3 + 12y_1^3 y_2^2 x_1^{-3} x_2 x_3^3 \]

associated to points in \( U_1 \)

\[ + 10y_1 y_2 x_1^{-3} x_2^3 x_3^3 + 3y_2^3 y_3 x_1^{-3} x_2 x_3^3 + 2y_1 y_2^2 x_1^{-3} x_2^{-1} x_3^3 + 5y_1^3 y_2^2 x_1^{-3} x_2 x_3^3 \]

associated to points in \( U_2 \)

\[ + 3y_1^2 y_2^2 x_1^{-3} x_2 x_3^3 + y_1^2 y_2^2 x_1^{-3} x_2^2 x_3 + 8y_1^2 y_2^2 x_1^{-3} x_2^2 x_3^3 + 12y_1^2 y_2^2 x_1^{-3} x_2 x_3^3 \]

associated to points in \( U_3 \)

\[ + 2y_1 y_2 y_3 x_1^{-3} x_2 x_3^3 + 3y_2^3 y_3 x_1^{-3} x_2 x_3^3 + 2y_1 y_2^2 x_1^{-3} x_2 x_3^3 + 11y_1^2 y_2^2 x_1^{-3} x_2 x_3^3 \]

associated to points in \( S \)

Then for example, for the proper face \( S \) containing \((1, 1, 0), (5, 3, 0) \) and \((5, 3, 3) \), we find a linear map:

\[ \tilde{\tau} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

such that \( \tilde{\tau}(1, 0) = (2, 1, 0) \) and \( \tilde{\tau}(0, 1) = (0, 0, 1) \). It can be checked that there is an isomorphism

\[ \tau : N_{(−2, 1)}|A' \rightarrow S \]

which can induce \( \tilde{\tau} \), where the initial exchange matrix \( B' \) of the cluster algebra \( A' \) is

\[ B' = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} = \overline{W}^T BW, \]

as

\[ W = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{W} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \]

5. Application to relations among \( F \)-polynomials, \( d \)-vectors and \( g \)-vectors

In this section, we will use the results in the last two sections to investigate some concrete relations among \( F \)-polynomials, \( d \)-vectors and \( g \)-vectors in a totally sign-skew-symmetric cluster algebra.
5.1. From F-polynomials to d-vectors and g-vectors.

**Lemma 5.1.** Let $A$ be a cluster algebra having principal coefficients. Denote by $p$ a vertex of $N_h$, and $p'$ another vertex of the $k$-section at $p$ for some $k \in [1, n]$. If $l(pp') > [-\deg x_k(p)]_+$, then $-d_k(\rho_h) < \deg x_k(p) < \deg x_k(\rho_h)$, where $d_k(\rho_h)$ denotes the $k$-th element of the denominator vector of $\rho_h$.

**Proof** There is nothing to say when the dimension is 0 or 1 since the assumption never holds. So in the we assume the dimension is at least 2.

Let $K$ be the maximal section of $N_h$ at $p$ satisfying that $\deg x_k(q) = \deg x_k(p)$ for any point $q \in K$. Obviously, the segment connecting $p$ and $p'$ lies in $K$. If $K$ is a face of $N_h$, then by Theorem 3.9 (v), $K$ equals a polytope $N_{h'}$ for some $h' \in \mathbb{Z}^r$ with some $r \leq n$ up to translation. So according to the definition of $N_{h'}$, we always have $l(pp') = [-\deg x_k(p)]_+$, which contradicts our assumption. Hence $K$ cannot be a face of $N_h$.

On the other hand, by Theorem 2.13 the $x_k$-degree of a point in $N_h$ is linearly dependent on its coordinates, which leads to that the convex hull of the set of points with maximal or minimal $x_k$-degree is a face of $N_h$. Therefore, combining above facts we get $-d_k(\rho_h) < \deg x_k(p) < \deg x_k(\rho_h)$.

According to the definitions of $N_h$ and $\rho_h$, for any $k \in [1, n]$, we have the following $x_k$-degree decomposition:

$$\rho_h = \sum_{i=-d_k(\rho_h)}^{\deg x_k(\rho_h)} x_i^i \rho_h^{(i)} = \sum_{i=-d_k(\rho_h)}^{\deg x_k(\rho_h)} x_i^i M_k^{[-i]} \rho_h^{(i;k)}.$$ 

where $\rho_h^{(i)} = M_k^{[-i]} \rho_h^{(i;k)}$ and $\rho_h^{(i;k)}$ is a Laurent polynomial in $\mathbb{N}[x_1^\pm, x_2^\pm, \ldots, x_n^\pm]$.

**Lemma 5.2.** Let $A$ be a cluster algebra with principal coefficients, $h \in H$. Then for any $k \in [1, n]$, we have

(i) $M_k \upharpoonright \rho_h^{(-d_k(\rho_h);k)}$;  
(ii) $M_k \upharpoonright \rho_h^{(\deg x_k(\rho_h);k)}$.

**Proof** Note that the $x_k$-degree of a summand Laurent monomial of $\rho_h$ is linearly dependent on the coordinates of its corresponding point in $N_h$. Hence there must be a vertex $p$ in $N_h$ having maximal or minimal $x_k$-degree. Denote by $p'$ the other vertex of $k$-section at $p$. Then by Lemma 5.1 $l(pp') \leq [-\deg x_k(p)]_+$. Hence there is no summand $P$ of $\rho_h$ such that $p$ is a summand of $P$ and $M_k^{[-\deg x_k(p)]_+} \upharpoonright P$.

In particular, $M_k^{[d_k(\rho_h)]_+} \upharpoonright x_k^{-d_k(\rho_h)} M_k^{[d_k(\rho_h)]_+} \rho_h^{(-d_k(\rho_h);k)}$ as well as $M_k^{[-\deg x_k(\rho_h)]_+} \upharpoonright x_k^{\deg x_k(\rho_h)} M_k^{[-\deg x_k(\rho_h)]_+} \rho_h^{(\deg x_k(\rho_h);k)}$,

i.e., $M_k \upharpoonright \rho_h^{(-d_k(\rho_h);k)}$ as well as $M_k \upharpoonright \rho_h^{(\deg x_k(\rho_h);k)}$.

**Example 5.3.** (i) When $-d_k(\rho_h) < s < \deg x_k(\rho_h)$, there can be $M_k \upharpoonright \rho_h^{(s;k)}$. Let the initial seed be as that in Example 4.9, then in the seed $\mu_3 \circ \mu_1 \circ \mu_2((X, \tilde{B}))$ there is a cluster variable

$$\rho_{(0,0,-1)} = \frac{y_3(y_2y_1x_2^3 + (y_2x_3^6 + x_1^3)^3 + x_2^2x_3^6)}{x_2^2x_3^6x_3}.$$ 

Hence $-d_1(\rho_{(0,0,-1)}) = 0$ and $\deg x_1(\rho_{(0,0,-1)}) = 16$. The summands with all $x_1$-degree 1 terms is $x_1 \cdot 6(y_1x_2^3 + x_3^{16})y_2y_3x_2^6x_3^{14}$. So $p_{(1;1)}^{(0,0,-1)} = 6(y_1x_2^3 + x_3^{16})y_2y_3x_2^6x_3^{14}$. Since $M_1 = y_1x_2^3 + x_3^{16}$, we have $M_1 \upharpoonright \rho_{(0,0,-1)}^{(s)}$ in this case.
(ii) Lemma 5.2 does not hold for any semifield \( \mathbb{P} \). One counterexample is given in \( \mathbb{P} \). Let \( \mathbb{P} = \{1\} \) and let the initial seed be \((X, B)\), where

\[
B = \begin{pmatrix}
0 & 2 & -1 \\
-2 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}.
\]

Then in seed \( \mu_1 \circ \mu_2 \circ \mu_3((X, B)) \) there is a cluster variable \( \rho_{(0,0,-1)} \mid \mathbb{P} = \frac{x_1^2 + 2x_1x_2 + x_2^2 + x_3}{x_1x_2x_3} \).

We can see that \( -d_3(\rho_{(0,0,-1)}) = -1, M_3 = x_1 + x_2 \) and \( \rho_{(0,0,-1)}^{(-1:3)} = M_3 \). So \( M_3 \mid \rho_{(0,0,-1)}^{(-1:3)} \).

For any \( i \in [1, n] \) and \( t \in \mathbb{T}_n \), define a map

\[
\phi_i^t : \mathbb{Z}P[x_{i:t}, \ldots, x_{n:t}] \rightarrow \mathbb{Z}P[x_{i-1:t}, x_{i+1:t}, \ldots, x_{n:t}]
\]

such that \( \phi_i^t(P) = P \mid_{x_i \to 0} \).

**Theorem 5.4.** Let \( \mathcal{A} \) be a TSSS cluster algebra having principal coefficients, \( h \in H \). Then \([d_k(\rho_h)]_+\) equals the maximal length of edges of \( N_h \) parallel to the \( k \)-th coordinate axis for any \( k \in [1, n] \).

**Proof** This is a direct corollary of Theorem 3.9 (v), Lemma 5.2 and the fact that there must be a vertex of \( N_h \) having minimal \( x_k \)-degree for any \( k \in [1, n] \) we mentioned above.

The above relation between \( d \)-vectors and polytopes induces the positivity of \( d \)-vectors associated to non-initial cluster variables which was first come up as a conjecture in \( \mathbb{P} \) and then proved in \( \mathbb{P} \) for skew-symmetrizable case.

**Theorem 5.5.** Let \( \mathcal{A} \) be a TSSS cluster algebra and \( x_{l:t} \) be a non-initial cluster variable in \( \mathcal{A} \) with \( l \in [1, n] \), \( t \in \mathbb{T}_n \). Then \( d_{l:t} \in \mathbb{N}^n \). More precisely, for any \( k \in [1, n] \),

\[
d_k(x_{l:t}) = \deg_k(P_{l:t}) = \deg_k(\phi_k(P_{l:t})).
\]

**Proof** Since this result is independent of the choice of the semifield \( \mathbb{P} \), we can assume \( \mathcal{A} \) has principal coefficients. If \( d_{l:t} \not\in \mathbb{N}^n \), then there is \( l \in [1, n] \) such that \( d_i(x_{l:t}) < 0 \).

Thus, by Theorem 5.4 \( N_{g_{l:i}} \) lies in the hyperplane \( z_i = 0 \) and \( \deg_{x_i}(Y_{pX^{g_{l:i}}}) \geq -d_i(x_{l:t}) \) for any point \( p \in N_{g_{l:i}} \). So according to the construction of \( N_{g_{l:i}} \) and \( N_{g_{l:i} + d_i(x_{l:t})e_i} \), we have

\[
N_{g_{l:i}} = \gamma_t;0(N_{x_i (g_{l:i})}) = \gamma_t;0(N_{x_i (g_{l:i} + d_i(x_{l:t})e_i)}) = N_{g_{l:i} + d_i(x_{l:t})e_i}.
\]

Hence \( x_{i:t} = \rho_{g_{l:i}} = x_i^{-d_i(x_{l:t})} \rho_{g_{l:i} + d_i(x_{l:t})e_i} \).

\[
(24) \quad x_{i:t} = L^t(x_i^{-d_i(x_{l:t})} \rho_{g_{l:i} + d_i(x_{l:t})e_i}) = L^t(x_i^{-d_i(x_{l:t})}) L^t(\rho_{g_{l:i} + d_i(x_{l:t})e_i}) = \rho_{g_{l:i} + d_i(x_{l:t})} h,
\]

where \( h = (g_{l:i} + d_i(x_{l:t})e_i)^t \).

According to Theorem 3.9 \( \rho_{g_{l:i} + d_i(x_{l:t})} \rho_{h} \in \mathbb{NTrop}(Y_{t})[X^{\pm}] \). So \( 24 \) induces that both \( \rho_{g_{l:i} + d_i(x_{l:t})} \) and \( \rho_{h} \) are Laurent monomials, which means \( g_{l:i}^t h \in \mathbb{N}^n \) and thus \( x_{i:t} = X_t^{-d_i(x_{l:t})} g_{l:i}^t + h \).

Hence \( e_i = -d_i(x_{l:t}) g_{l:i}^t + h \). It holds only when \( g_{l:i}^t = e_i, d_i(x_{l:t}) = -1 \) and \( h = 0 \), which contradicts the non-initial assumption.

Therefore \( d_{l:t} \in \mathbb{N}^n \).

Thus by Theorem 5.4 we have \( d_k'(x_{l:t}) = [-d_k'(x_{l:t})]_+ = \deg_k'(\phi_k(P_{l:t})) \). Moreover, by Lemma 218 and the definition of general degree we have

\[
d_k'(x_{l:t}) \leq \deg_k'(P_{l:t}) \leq \deg_k(\phi_k(P_{l:t})),
\]

so they are all equal.
For a cluster algebra $\mathcal{A}$ having principal coefficients, define a map
\[
\psi : \mathbb{ZP}[x_1^\pm 1, \ldots, x_n^\pm 1] \rightarrow \mathbb{ZP}[x_1, \ldots, x_n]
\]
such that $\psi(x) = P$ when $x = PX^\alpha$, where $P \in \mathbb{ZP}[x_1, \ldots, x_n]$, $\alpha \in \mathbb{Z}^n$ and $P$ is coprime with $x_i, i \in [1, n]$. Theorem 5.5. means that
\[
x_{l;t} = \frac{F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)}{\tilde{F}_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)} X^{g_{l;t}} = \frac{F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)}{\tilde{F}_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)} X^{\tilde{g}_{l;t}}.
\]
Due to Theorem 5.5.
\[
P_{l;t} = \psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)).
\]
We obtain a map from above theorem
\[
\varphi : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial } d\text{-vectors of } \mathcal{A}\}
\]
such that
\[
\varphi(F_{l,t}) = (\deg_1 \circ \phi_1 \circ \psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)), \ldots, \deg_n \circ \phi_n \circ \psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)))^\top,
\]
which maps a $F$-polynomial associated to a non-initial cluster variable to the $d$-vector associated to the same cluster variable.

Again by Theorem 2.13 we can see that $X^{g_{l;t}} = \frac{p_0}{X^{\tilde{g}_{l;t}}}$ for any $l \in [1, n], t \in T_n$, where $p_0$ is the unique summand monomial of $P_{l;t}$ with coefficient 1. So we can also define a map from a $F$-polynomial associated to a non-initial cluster variable to the $g$-vector associated to the same cluster variable:
\[
\theta_1 : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial } g\text{-vectors of } \mathcal{A}\}
\]
such that
\[
\theta_1(F_{l,t}) = (\deg_{x_1}(\psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)) |_{y_1 = \ldots = y_n = 0}), \ldots, \deg_{x_n}(\psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)) |_{y_1 = \ldots = y_n = 0}))^\top.
\]
In conclusion, we have the following theorem.

**Theorem 5.6.** Let $\mathcal{A}$ be a TSSS cluster algebra with principal coefficients. Then, for any $l = 1, \ldots, n$ and $t \in T_n$,

(i) there is a surjective map
\[
\varphi : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial } d\text{-vectors of } \mathcal{A}\}
\]
such that
\[
\varphi(F_{l,t}) = (\deg_1 \circ \phi_1 \circ \psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)), \ldots, \deg_n \circ \phi_n \circ \psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)))^\top;
\]

(ii) there is a bijective map
\[
\theta_1 : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial } g\text{-vectors of } \mathcal{A}\}
\]
such that
\[
\theta_1(F_{l,t}) = (\deg_{x_1}(\psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)) |_{y_1 = \ldots = y_n = 0}), \ldots, \deg_{x_n}(\psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)) |_{y_1 = \ldots = y_n = 0}))^\top.
\]

(iii) there is a bijective map
\[
\chi_1 : \{\text{non-initial } F\text{-polynomials of } \mathcal{A}\} \rightarrow \{\text{non-initial cluster variables of } \mathcal{A}\}
\]
such that
\[
x_{l;t} = \chi_1(F_{l,t}) = \frac{\psi(F_{l,t} | x(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n))}{X^{\psi(F_{l,t})}}.
\]
Corollary 5.7. Let \( A \) be a TSSSS cluster algebra having principal coefficients with two non-initial cluster variables \( x_{i; t}, x_{i; t}' \) and \( F_{i; t}, F_{i; t}' \) be the F-polynomials associated to \( x_{i; t}, x_{i; t}' \) respectively. If \( F_{i; t} = F_{i; t}' \), then \( x_{i; t} = x_{i; t}' \).

Proof. (i) It is directly induced by Theorem 5.5.

(ii) The surjectivity naturally holds by the definition of \( \theta_1 \) and that of g-vectors.

On the other hand, if there are \( l, l' \in [1, n] \) and \( t, t' \in \mathbb{T}_n \) satisfying \( \theta_1(F_{l; t}) = \theta_1(F_{l'; t'}) \), then \( F_{l; t} = \rho_{\theta_1(F_{l; t})}|_{x_i \rightarrow 1, \forall i \in [1, n]} = \rho_{\theta_1(F_{l'; t'})}|_{x_i \rightarrow 1, \forall i \in [1, n]} = F_{l'; t'} \). Hence \( \theta_1 \) is also injective.

(iii) Following (i), we have \( x_{i; t} = \frac{P_{i; t}}{X^{\theta_1(F_{i; t})} \psi_{i; t}} = \frac{\phi(F_{i; t}) x_{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_l}}{X^{\psi(F_{i; t})}} \), which means the surjectivity.

On the other hand, since \( F_{i; t} = P_{i; t}|_{x_i \rightarrow 1, \forall i \in [1, n]} \), the injectivity follows.

And the map \( \chi_1 \) directly leads to the following results:

Corollary 5.7. Let \( A \) be a TSSSS cluster algebra having principal coefficients with two non-initial cluster variables \( x_{i; t}, x_{i; t}' \) and \( F_{i; t}, F_{i; t}' \) be the F-polynomials associated to \( x_{i; t}, x_{i; t}' \) respectively. If \( F_{i; t} = F_{i; t}' \), then \( x_{i; t} = x_{i; t}' \).

Proposition 5.8. Let \( A \) be a skew-symmetrizable cluster algebra having principal coefficients at some vertex and \( t, t' \in \mathbb{T}_n \). If \( \prod_{i=1}^{n} x_{i; t}^a_i = \prod_{i=1}^{n} x_{i; t'}^a_i \) and they are cluster monomials in \( X_t \) and \( X_{t'} \) respectively, then for every \( a_i \neq 0 \), there is \( j \in [1, n] \) such that \( x_{i; t} = x_{j; t'} \) and \( a_i = a_j \).

The following Corollary is a generalization of Proposition 5.8.

Corollary 5.9. Let \( A \) be a TSSSS cluster algebra having principal coefficients, \( t, t' \in \mathbb{T}_n \) and \( \alpha, \beta \in \mathbb{Z}^n \) are non-zero. If \( X_\alpha^\beta = X_\beta^\alpha \), then this is a permutation \( \sigma \) of \([1, n]\) such that \( x_{i; t} = x_{\sigma(i); t'} \) and \( a_i = \beta_{\sigma(i)} \) for any \( \alpha_i \neq 0 \).

Proof

\[
X_\beta^\alpha = X_\alpha^\beta = \prod_{i=1}^{n} \left( \frac{P_{i; t}'}{X_{\beta}^{\gamma_{i; t}}} \right)^{a_i}
\]

is a Laurent monomial in \( X_{t'} \). So \( \prod_{i=1}^{n} (P_{i; t}')^{a_i} \) is a Laurent monomial. Let \( I_1 = \{ i \in [1, n] | x_{i; t} \in X_{t'} \} \), \( I_2 = [1, n] \setminus I_1 \). By Corollary 5.7, \( F_{i; t}' = F_{i; t} \) for \( i = j \) and \( i \in I_2 \). So \( \alpha_i = 0 \) for \( i \in I_2 \). Then there is a permutation \( \sigma \) of \([1, n]\) such that \( x_{i; t} = x_{\sigma(i); t'} \) for \( i \in I_1 \) and

\[
\prod_{i \in I_1} x_{\sigma(i); t'} = \prod_{i=1}^{n} x_{i; t}'.
\]

Therefore \( \alpha_i = \beta_{\sigma(i)} \) for \( i \in I_1 \) and \( \alpha_i = 0 \) for \( i \in I_2 \).

Theorem 5.10. For any skew-symmetrizable cluster algebra \( A \) and any collection \( U \) of cluster variables in \( A \), if each pair in \( U \) is contained in some cluster of \( A \), then there is a cluster of \( A \) containing \( U \) as a subset.

Lemma 5.11. For any \( l \in [1, n] \) and \( t \in \mathbb{T}_n \), \( x_{l; t} \) is an initial cluster variable if and only if \( F_{l; t} = 1 \).

Proof. The necessity is directly from the definition of F-polynomials.

By Theorem 5.10, we have

\[
x_{l; t} = \frac{F_{l; t} \psi(y_1, y_2, \ldots, y_l)}{F_{l; t} \psi(y_1, y_2, \ldots, y_n)} X_{l; t}.
\]

So when \( F_{l; t} = 1 \), \( x_{l; t} \) is a Laurent monomial of \( X_{l_0} \) and of \( X_l \) respectively. According to Corollary 5.8 \( x_{l; t} = x_j \) for some \( j \), which shows the sufficiency.

Then by Lemma 5.11 Theorem 5.10 and Corollary 5.7 we have the following corollary:
Corollary 5.12. Let \(\mathcal{A}\) be a skew-symmetrizable cluster algebra with \(X_1, X_{t'}\) two clusters. If \(\{F_{i.t}\}_{i \in [1,n]} = \{F_{t'i}\}_{i \in [1,n]}\), then \(X_1 = X_{t'}\).

Proof For convenience we order two sets of \(F\)-polynomials such that \(F_{i.t} = F_{t'i}\) for any \(i \in [1,n]\). If \(F_{i.t} \neq 1\), by the last Lemma, \(x_{i.t}\) is non-initial and so following Corollary 5.7 we have \(x_{i.t} = x_{t'i}\). If \(F_{i.t} = 1\), then \(x_{i.t}\) is an initial cluster variable. Without loss of generality, we can order \(F\)-polynomials such that \(F_{i.t} = 1\) when \(i \in [1,k]\) and \(F_{i.t} \neq 1\) otherwise. Then it is enough to prove that given a set of cluster variables \(x_{k+1:t'}, \ldots, x_{n.t}\), there is at most one \(k\)-set of initial cluster variables \(x_{i_1}, \ldots, x_{i_k}\) such that \((x_{i_1}, \ldots, x_{i_k}, x_{k+1:t'}, \ldots, x_{n.t})\) is a cluster.

Assume there are two different \(k\)-set satisfying above condition, then by Theorem 5.10 there is a cluster containing all of them. However there are totally at least \(n + 1\) variables. The contradiction completes the proof.

\[\square\]

5.2. From \(g\)-vectors to \(F\)-polynomials and \(d\)-vectors.

The fact that a \(g\)-vector uniquely determines its corresponding cluster variable is already proved in \[11\] for skew-symmetrizable case, here in this subsection we would like to express the maps from a \(g\)-vector associated to a cluster variable to the \(F\)-polynomial and the \(d\)-vector associated to the same cluster variable respectively.

Theorem 5.13. Let \(\mathcal{A}\) be a TSSS cluster algebra with principal coefficients. Then, for any \(l = 1, \ldots, n\) and \(t \in \mathbb{T}_n\),

(i) there is a bijective map

\[
\theta_2 : \{g\text{-vectors of } \mathcal{A}\} \longrightarrow \{F\text{-polynomials of } \mathcal{A}\} \quad \text{via} \quad \theta_2(g_{l.t}) = \rho_{g_{l.t}} |_{x_{\rightarrow 1}, y_{t_i}}.
\]

In fact, we have \(\theta_2 = \theta_1^{-1}\) for the map \(\theta_1\) in Theorem 5.6.

(ii) there is a surjective map

\[
\eta : \{g\text{-vectors of } \mathcal{A}\} \longrightarrow \{d\text{-vectors of } \mathcal{A}\} \quad \text{via} \quad \eta(g_{l.t}) = \text{denominator vector of } \rho_{g_{l.t}}.
\]

(iii) there is a bijective map

\[
\chi_2 : \{g\text{-vectors of } \mathcal{A}\} \longrightarrow \{\text{cluster variables of } \mathcal{A}\}
\]

via

\[
x_{l.t} = \chi_2(g_{l.t}) = \rho_{g_{l.t}}.
\]

Proof According to Theorem 5.9 we have \(x_{l.t} = \rho_{g_{l.t}}\) and \(\deg(\rho_{g_{l.t}}) = g_{l.t}\), which lead to (iii). Moreover, (i) and (ii) follow from (iii), due to Theorem 5.10 (ii) and the definitions of \(F\)-polynomials and \(d\)-vectors.

\[\square\]

Remark 5.14. When \(\mathcal{A}\) is a cluster algebra over a semifield \(\mathbb{P}\), by Theorem 5.6 (iii) and Theorem 5.13 (iii), for a cluster variable \(x_{l.t}\) in \(\mathcal{A}\), we have

\[
x_{l.t} = \frac{\psi(F_{l.t}) x(y_1, y_2, \ldots, y_n)}{F_{l.t}|x(y_1, y_2, \ldots, y_n)} \quad \text{and} \quad x_{l.t} = \frac{\rho_{g_{l.t}}}{\theta_2(g_{l.t})|p(y_1, y_2, \ldots, y_n)},
\]

where \(F_{l.t}\) is the \(F\)-polynomial associated to \(x_{l.t}\) in the corresponding principal coefficients cluster algebra \(\mathcal{A}_{\text{prin}}\) of \(\mathcal{A}\).
Thus we obtain the relations among cluster variables, $g$-vectors, $F$-polynomials and $d$-vectors as the following diagram. Note that the maps from \{F-polynomials\} are restricted in the subset consisting of non-initial $F$-polynomials since all initial $F$-polynomials equal to 1.

![Relation Diagram](image)

**Figure 9. Relation Diagram**

A natural question is that whether we can construct a map from the set of $d$-vectors to one of the other three sets in Figure 9.

**Problem 5.15.** In a cluster algebra $A$, is an element in $P$ uniquely determined by its denominator vector in $\mathbb{Z}^n$? In particular, is the $g$-vector $g_{i,t}$ associated to a cluster variable $x_{i,t}$ determined uniquely by its denominator vector $d_{i,t}$?

The answer is positive in some special cases. For example, when $A$ is of rank 2, according to Proposition 3.4, $P$ is the greedy basis whose elements are parameterized by denominator vectors. However, in general this is still an open question. We think that the Newton polytope might be helpful in considering this problem.

6. **Polytope basis for an upper cluster algebra**

Recall that in Theorem 3.9 we associate a (formal) Laurent polynomial $\rho_h^p$ with each $h \in \mathbb{Z}^n$ in a cluster algebra with principal coefficients. While in a cluster algebra $A$ over an arbitrary semifield $\mathbb{Z}P$, let $F_h = \rho_h^p |_{x_i \rightarrow 1, \forall i \in [1,n]}$ and define a formal Laurent polynomial

$$\rho_h|_P := \frac{F_h|_P(Y)}{F_h|_P(\hat{Y})} X^h \in \mathbb{NP}[[X^{\pm 1}]].$$

We denote by $\hat{P}|_P$ the set consisting of all such formal Laurent polynomials $\rho_h|_P$, i.e.,

$$\hat{P}|_P = \{ \rho_h|_P \in \mathbb{NP}[[X^{\pm 1}]] | h \in \mathbb{Z}^n \},$$

and $P|_P = \{ \rho_h|_P \in \mathbb{NP}[[X^{\pm 1}]] | h \in H \}$.

In this section, the subscript of semifield is always compatible with the cluster algebra we are talking about, so we omit the subscript for convenience. We want to take further discussion about $\rho_h$ to construct a basis of $\mathcal{U}(A)$ for a TSSS cluster algebra $A$.

**Lemma 6.1.** For a TSSS cluster algebra $A$ with principal coefficients, a universally positive elements $f$ in $\mathcal{U}(A)$ can be expressed as a $\mathbb{Z}Trop(Y)$-linear combination of $P$, that is, $f = \sum_{h \in H} a_h \rho_h$ with finitely many nonzero $a_h \in \mathbb{Z}Trop(Y)$.
Proof Without loss of generality we can assume $f$ is universally indecomposable and it is written as a Laurent polynomial in $X_{t_0}$.

The universal indecomposability of $f$ leads to the fact that for any constant coefficient free summand monomial $p$ of $f$, there is another constant coefficient free summand monomial $p'$ of $f$ satisfying $\frac{p}{p'} = \hat{y}_j^a$ for some $j \in [1, n]$ and $a \in \mathbb{Z}$. Therefore, $f$ should be homogeneous under canonical $\mathbb{Z}^n$-grading, since $\hat{y}_j$ is homogeneous with degree 0 for any $j \in [1, n]$. Denote this degree by $h(f)$.

Let $N(f)$ be the corresponding polytope of $f$. Choose a minimal vector $w$ in $N(f)$. It corresponds to a summand monomial $\hat{Y}^w X^{h(f)}$ of $f$. Denote $h = h(f) + w B^\top$. Let $t_k$ be the vertex connected to $t_0$ by an edge labelled $k$ in $T_n$ for any $k \in [1, n]$. Since $f$ is univerally positive, $L^{t_k}(f)$ is a positive Laurent polynomial in $X_{t_k}$. Then for any summand monomial $p$ of $f$, the sum of all summand monomials of $f$ having $x_k$-degree $\deg_{x_k}(p)$ must be of the form
\begin{equation}
\sum_j m_j,
\end{equation}
where $m_j$ is a Laurent monomial in $X \setminus \{x_k\}$.

Let $w' \in V_h$. Then according to the definition of $V_h$, in $N_h$ there is a sequence $p_0 = X^h, p_1, \ldots, p_r = \hat{Y}^w X^h$ and a sequence $i_1, \ldots, i_r$ satisfying the conditions listed in the definition of $V_h$.

Assume $p = \hat{Y}^w + w' X^{h(f)}$ and $r = k$ (thus $\deg_{x_k}(p) < 0$). We then use induction on $r > 0$ to prove that there is some $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} \sum_j m_j = \hat{Y}^{w' + w} X^{h(f)}$.

When $r = 1$, there is no $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} m_j = \hat{Y}^{w' + w} X^{h(f)}$, which is equivalent to there is no $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} m_j = \hat{Y}^{w' + w} X^{h(f)}$, then there is some $m_j$ and $s \in [1, \deg_{x_k}(p)]$ such that $\hat{Y}^w X^{h(f)} = x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} m_j$, which means there is a vector $w - se_k$ in $N_f$ less than $w$. This contradicts our choice of $w$. Hence there is some $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} m_j = \hat{Y}^{w' + w} X^{h(f)}$.

Suppose when $r < l$, there is some $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} m_j = \hat{Y}^{w' + w} X^{h(f)}$.

When $r = l$, if there is no $m_j$ in (25) such that $x_k^{\deg_{x_k}(p)} \frac{1}{\hat{y}_k} m_j = \hat{Y}^{w' + w} X^{h(f)}$, then according to the mutation formula, the Laurent monomial corresponding to the cross is not a summand of $L^{t_k}(f)$ as showed in Figure 10. Here the red line in the left side corresponds to the red point in the right side.

![Figure 10. The shape of a part of $N(f)$ under mutation.](image-url)

Let $t'$ be the vertex connected to $t_k$ by an edge labelled $i_{r-1}$ in $T_n$. Since $L^{t'}(f)$ is a positive Laurent polynomial in $X_{t'}$, the Laurent monomial $\hat{Y}^w p_{r-2}$ is a summand of
\begin{equation}
x_{i_{r-1}}^{\deg_{x_{i_{r-1}}}(p_{r-2})} \frac{1}{\hat{y}_{i_{r-1}}} \sum_j m_j.
\end{equation}
for some $j$, where $s \in [1, -\deg x_{i+1}(p_{r-2})]$ rather than $s = 0$. This contradicts the inductive assumption.

Hence there is some $m_j$ in \([23]\) such that
\[
x_k^{\deg x_k(p)} y_k^{[-(r-1) + \deg x_k(p)]} m_j = \hat{y}^{w+w'} X^h(f).
\]

Moreover, the similar discussion as above at \(\hat{y}^{w'} X^h\) in \(E_h\) claims that for any \(w' \in E_h, k \in [1, n]\) satisfying \(m_k(w') = \co w'(\rho_k) > 0\), there is unique \(\epsilon_{w', k} \in \{\pm 1\}\) such that
\[
m_k(w') \hat{y}^{w'} X^h(1 + \hat{y}_{k}^{[\epsilon_{w', k}] - \deg x_k}(\hat{y}^{w'} X^h))
\]
is a summand of \(\rho_h\) as a complement of \(m_k(w') \hat{y}^{w'} X^h\) in direction \(k\), and
\[
m_k(w') \hat{y}^{w+w'} X^h(1 + \hat{y}_{k}^{[\epsilon_{w', k}] - \deg x_k}(\hat{y}^{w'} X^h))
\]
is a summand of \(f\).

Then in \(f - \co \hat{y}^{w'} X^h(f) Y^w \rho_h\), the vectors corresponding to Laurent monomials with negative constant coefficients are in \((N_h \setminus E_h)[w]\). Denote by \(W_h\) the set consisting of these vectors. Since the support of \(f - \co \hat{y}^{w'} X^h(f) Y^w \rho_h\) is finite, we can find a minimal Laurent polynomial \(\sum_{u \in W_h} a_u \rho_h(f) + uB\) for some \(a_u \in \mathbb{N}\) such that
\[
f - \co \hat{y}^{w'} X^h(f) Y^w \rho_h + \sum_{u \in W_h} a_u \rho_h(f) + uB^\top
\]
is universally positive. Denote it by \(f_1\) and its corresponding polytope by \(N(f_1)\). Note that
\[
\co \hat{y}^{w'} X^h(f)(f_1) = 0,
\]
and \(\co \hat{y}^{w'} X^h(f)(f_1) = 0\) for any \(u \in W_h\) with \(a_u \neq 0\) due to the minimality of \(\sum_{u \in W_h} a_u \rho_h(f) + uB^\top\).

Since the origin is the unique minimal point in \(N_h\), \(u \geq w\) for any \(u \in W_h\). So any minimal point \(p\) in \(N(f_1)\), is either a minimal point in \(N(f)\) or \(p \geq w\). Moreover, if there is \(u \in W_h\) such that the maximal point \(p\) in \(N_{h(f)+uB^\top}[u]\) is a maximal point in \(N(f_1)\), then similar to our above discussion (the maximal point case is dual to the minimal point case in the above), \(\co p(N(f_1)) \neq 0\) for any point \(p \in E_{h(f)+uB^\top}[w]\), in particular \(\co \hat{y}^{w'} X^h(f)(f_1) \neq 0\), which contradicts the fact. Hence there is no point \(p'\) in \(N(f_1)\) which is a maximal point in \(N(f_1)\) but not a maximal point in \(N(f)\). So the set consisting of minimal points in \(N(f_1)\) is a proper subset of that of minimal points in \(N(f)\) while the set consisting of maximal points in \(N(f_1)\) is a proper subset of that of maximal points in \(N(f)\).

Therefore, replacing \(f\) by \(f_1, \cdots\), we can continue the above way to produce new universally positive element \(f_2, \cdots\), and will finally get 0 in finitely many times as the minimal point set becomes strictly smaller and the maximal point set becomes smaller. Combining these together we have an equation
\[
f = \sum_{h \in H} a_h \rho_h\text{ for some } a_h \in \mathbb{N}^P.
\]

\[\square\]

In Lemma 6.1 \(a_h\) is not necessarily in \(\mathbb{N}Trop(Y)\). There is a characterization about when \(a_h\) is always in \(\mathbb{N}Trop(Y)\) for a cluster algebra without coefficients of rank 2 in [13]. So we may ask naturally how it is in general rank case.

**Problem 6.2.** In Lemma 6.1, for what kind of cluster algebras, is \(a_h\) always in \(\mathbb{N}Trop(Y)\)?

A basis \(\{a_s\}_{s \in I}\) over \(\mathbb{Z}\)P is called **strongly positive** if for any \(i, j \in I\), \(a_i a_j = \sum_{s \in I} a_{ij}^s a_s\), where \(a_{ij}^s \in \mathbb{N}\) for any \(s \in I\).
Theorem 6.3. Let $A$ be a TSSS cluster algebra with principal coefficients. Then $\mathcal{P}$ is a strongly positive $ZTrop(Y)$-basis for the upper cluster algebra $\mathcal{U}(A)$.

Proof First, we prove that $\mathcal{U}(A)$ is linearly generated by $\mathcal{P}$ over $ZTrop(Y)$.

Let $f$ be an element in $\mathcal{U}(A)$ as a Laurent polynomial in $X$. Then $f$ has a decomposition $f = \sum_{i=1}^{l} f_i$, where $f_i \neq 0$ is a summand of $f$ such that $f_i$ is a Laurent polynomial in $X_i$ for any $t \in T_n$, and there is $t \in T_n$ for any proper summand $p$ of $f_i$ such that $p$ is not a Laurent polynomial in $X_i$. Then as we said before, $f_i$ is homogeneous under canonical $\mathbb{Z}^n$-grading since the exchange binomials are all homogeneous with degree 0. Denote $h_i = \deg(f_i)$.

Denote by $N_i$ the Newton polytope of $f_i$. For a minimal lattice point $p$ in $N_i$, according to the proof of Lemma 6.1, the maximal point in $N_{h_i,-pB^+}[p]$ is not larger than a maximal point of $N_i$. Let $f_i^{(1)} = f_i - \text{co}_p(N_i)Y^p\rho_{h_i,-pB^+}$ and iteratively the above process on $f_i^{(1)}$ to get $f_i^{(l+1)}$. As explained in the proof of Lemma 6.1 it will stop in finitely many steps. Thus we decompose $f_i$ as a $ZTrop(Y)$-linear combination of $\mathcal{P}$.

Next, we need to show that $\mathcal{P}$ is linearly independent.

Assume $\sum_{i=1}^{l} a_i Y^{w_i} P_{h_i} = 0$ for some $a_i \neq 0 \in \mathbb{Z}$, $w_i \in \mathbb{Z}^n$ and $h_i \in H$ with $Y^{w_i} P_{h_i} \neq Y^{w_j} P_{h_j}$. Without loss of generality, we suppose all $a_i Y^{w_i} P_{h_i}$ have the same degree. Choose a minimal $w_j$ among all $w_i$. Then $\text{co}_{w_j} (\sum_{i=1}^{l} N_i[w_i]) = a_j \neq 0$, which contradicts our assumption. So $\mathcal{P}$ is linearly independent.

In summary, we get that $\mathcal{P}$ is a $ZTrop(Y)$-basis of $\mathcal{U}(A)$.

Finally we show that $\mathcal{P}$ is strongly positive.

Choose arbitrary $\rho_h, \rho_{h'} \in \mathcal{P}$. Then $\rho_h \rho_{h'} = \sum_f a_f \rho_f$ with $a_f \in \mathbb{Z}[Y^{\pm 1}]$. Therefore, we only need to prove that $a_f \in \mathbb{N}[Y]$.

If $h, h' \in \mathbb{N}^n$, then $\rho_h = X^h$ and $\rho_{h'} = X^{h'}$. So $\rho_h \rho_{h'} = X^{h+h'} = \rho_{h+h'}$, the claim holds.

Otherwise, at least one of $h, h'$ is in $\mathbb{Z}^n \setminus \mathbb{N}^n$. Without loss of generality, suppose $h \in \mathbb{Z}^n \setminus \mathbb{N}^n$. Now we take induction on the partial order $\preceq$ in the set of polytopes. Assume

$$\rho_r \rho_{r'} = \sum_{r'' \in \mathbb{Z}^n} c_{r''} \rho_{r''},$$

where $c_{r''} \in \mathbb{N}[Y]$ for any $\rho_r, \rho_{r'} \in \mathcal{P}$ with $N_r < N_h$.

According to (19), we have a decomposition

$$x_i \rho_h = \sum_{\alpha \in \mathbb{Z}^n, w \in \mathbb{Z}^n} Y^w \rho_{\alpha} = \sum_{r \in \mathbb{Z}^n} b_r \rho_r,$$

where $b_r \in \mathbb{N}[Y]$ and there are only finitely many such $b_r$ being nonzero. Therefore,

$$x_i \rho_h \rho_{h'} = \sum_{r \in \mathbb{Z}^n} b_r \rho_r \rho_{h'},$$

Because $h \in \mathbb{Z}^n \setminus \mathbb{N}^n$, there is $i \in [1, n]$ such that $N_h$ is not in the hyperpane $z_i = 0$. Then there must be more than one terms in the right-hand side of (27). Then, combining (26) and (27), we have

$$x_i \rho_h \rho_{h'} = \sum_{r \in \mathbb{Z}^n} \sum_{r' \in \mathbb{Z}^n} b_r c_{r'} \rho_{r'},$$

where $c_{r'} \in \mathbb{N}[Y]$ for any $\rho_r, \rho_{r'} \in \mathcal{P}$ with $N_r < N_h$. Therefore, $a_f \in \mathbb{N}[Y]$ for any $a_f \in \mathbb{Z}[Y^{\pm 1}]$. Thus $\mathcal{P}$ is strongly positive.

In summary, we get that $\mathcal{P}$ is a $ZTrop(Y)$-basis of $\mathcal{U}(A)$.

Finally we show that $\mathcal{P}$ is strongly positive.

Choose arbitrary $\rho_h, \rho_{h'} \in \mathcal{P}$. Then $\rho_h \rho_{h'} = \sum_f a_f \rho_f$ with $a_f \in \mathbb{Z}[Y^{\pm 1}]$. Therefore, we only need to prove that $a_f \in \mathbb{N}[Y]$.

If $h, h' \in \mathbb{N}^n$, then $\rho_h = X^h$ and $\rho_{h'} = X^{h'}$. So $\rho_h \rho_{h'} = X^{h+h'} = \rho_{h+h'}$, the claim holds.

Otherwise, at least one of $h, h'$ is in $\mathbb{Z}^n \setminus \mathbb{N}^n$. Without loss of generality, suppose $h \in \mathbb{Z}^n \setminus \mathbb{N}^n$. Now we take induction on the partial order $\preceq$ in the set of polytopes. Assume

$$\rho_r \rho_{r'} = \sum_{r'' \in \mathbb{Z}^n} c_{r''} \rho_{r''},$$

where $c_{r''} \in \mathbb{N}[Y]$ for any $\rho_r, \rho_{r'} \in \mathcal{P}$ with $N_r < N_h$.

According to (19), we have a decomposition

$$x_i \rho_h = \sum_{\alpha \in \mathbb{Z}^n, w \in \mathbb{Z}^n} Y^w \rho_{\alpha} = \sum_{r \in \mathbb{Z}^n} b_r \rho_r,$$

where $b_r \in \mathbb{N}[Y]$ and there are only finitely many such $b_r$ being nonzero. Therefore,

$$x_i \rho_h \rho_{h'} = \sum_{r \in \mathbb{Z}^n} b_r \rho_r \rho_{h'},$$

Because $h \in \mathbb{Z}^n \setminus \mathbb{N}^n$, there is $i \in [1, n]$ such that $N_h$ is not in the hyperpane $z_i = 0$. Then there must be more than one terms in the right-hand side of (27). Then, combining (26) and (27), we have

$$x_i \rho_h \rho_{h'} = \sum_{r \in \mathbb{Z}^n} \sum_{r' \in \mathbb{Z}^n} b_r c_{r'} \rho_{r'},$$

where $c_{r'} \in \mathbb{N}[Y]$ for any $\rho_r, \rho_{r'} \in \mathcal{P}$ with $N_r < N_h$. Therefore, $a_f \in \mathbb{N}[Y]$ for any $a_f \in \mathbb{Z}[Y^{\pm 1}]$. Thus $\mathcal{P}$ is strongly positive.
Denote \( J = \{(r, r') \in \mathbb{Z}^n \times \mathbb{Z}^n | b_{r, r'} \neq 0\} \). Again due to the construction of strata, we can find a partition \( J = \bigcup_{\lambda \in \Lambda} J_\lambda \) where for any \( \lambda \in \Lambda \), \( J_\lambda \) consists of \((r, r')\) satisfying

\[
(28) \quad \sum_{(r, r') \in J_\lambda} b_{r, r'} \rho_{r'} = x_i Y^{w_\lambda} \rho_{f_\lambda}
\]

for some \( w_\lambda \in \mathbb{N}^n \) and \( f_\lambda \in \mathbb{Z}^n \). The partition \( \bigcup_{\lambda \in \Lambda} J_\lambda \) of \( J \) can be calculated in the following way:

Choose a minimal point \( p \in N \), where \( N \) is the polytope corresponding to \( \rho_h \rho_{h'} \), then with the help of the strata of \( N_h \) for \( x_i \), it can be calculated that the strata of \( N_{h+h'-pB^T}[p] \) for \( x_i \) are in \( N \). Moreover it follows that \( N_{h+h'-pB^T}[p] \) is a sub-polytope of \( N \). So, we set \( J_{\lambda_0} \) to consist of the parameters \((r, r')\) of the Laurent polynomials corresponding to these strata by \((28)\) for \( w_{\lambda_0} = p \) and \( f_{\lambda_0} = h + h' - pB^T \) corresponding to one \( \lambda_0 \in \Lambda \). Then, we have

\[
\sum_{(r, r') \in J} b_{r, r'} \rho_{r'} + x_i Y^{w_{\lambda_0}} \rho_{f_{\lambda_0}}.
\]

Let \( N_1 \) denote the sub-polytope of \( N \) corresponding to \( \sum_{(r, r') \in J \setminus J_{\lambda_0}} b_{r, r'} \rho_{r'} \). Repeat the above process for \( N_1, \cdots, N_j \), then we can determine all \( J_\lambda \) and thus obtain \( J = \bigcup_{\lambda \in \Lambda} J_\lambda \) as well as \( \{(w_\lambda, f_\lambda)\}_{\lambda \in \Lambda} \).

So in conclusion, we have \( \rho_h \rho_{h'} = \sum_{\lambda \in \Lambda} \sum_{(r, r') \in J_\lambda} x_i^{-1} b_{r, r'} \rho_{r'} = \sum_{\lambda \in \Lambda} Y^{w_\lambda} \rho_{f_\lambda} \), thus \( a_f = \sum_{f_\lambda = f} Y^{w_\lambda} \in \mathbb{N}[Y] \) for \( \rho_h \rho_{h'} = \sum_{f} a_f \rho_f \).

\( \square \)

Due to Theorem \(6.3\) we call \( \mathcal{P} \) the \textbf{polytope basis} of \( \mathcal{U}(\mathcal{A}) \). Although this basis can be regarded as a generalization of greedy basis, we call it polytope basis instead of \textit{greedy basis} since the method of polytopes is used effectively in the construction of \( \mathcal{P} \).

However in general for a cluster algebra \( \mathcal{A} \) over an arbitrary semifield \( \mathbb{P} \), we find that \( \mathcal{P} \) is a \( \mathbb{Z}\mathbb{P} \)-basis for a subalgebra of \( \mathcal{U}(\mathcal{A}) \) rather than \( \mathcal{U}(\mathcal{A}) \) itself. For this reason, we call the \( \mathbb{Z}\mathbb{P} \)-subalgebra generated by \( \mathcal{P} \) the \textbf{intermediate cluster algebra} associated to \( \mathcal{A} \), and denote it by \( \mathcal{I}_\mathcal{P}(\mathcal{A}) \). In fact, we have:

\textbf{Proposition 6.4.} Let \( \mathcal{A} \) be a cluster algebra over a semifield \( \mathbb{P} \). Then \( \mathcal{P} \) is a strongly positive \( \mathbb{Z}\mathbb{P} \)-basis for the intermediate cluster algebra \( \mathcal{I}_\mathcal{P}(\mathcal{A}) \).

\textbf{Proof} First we prove that \( \mathcal{P} \) is \( \mathbb{Z}\mathbb{P} \)-linearly independent, which is more complicated than that in the case for principal coefficients given in Theorem \(6.3\).

Let \( P \in \mathcal{U}(\mathcal{A}) \) be a \( \mathbb{N}\mathbb{P} \)-linear combination of \( \mathcal{P} \) as \( P = \sum_{i=1}^{l} a_i P_i \), where \( a_i \in \mathbb{N}\mathbb{P} \) and \( P_i \in \mathcal{P} \) is a universally indecomposable summand of \( P \) for each \( i \in [1, l] \). We claim such decomposition is unique.

Assume there are two decompositions \( P = \sum_{i=1}^{l} a_i P_i = \sum_{j=1}^{l'} a'_j P'_j \), where \( a_i, a'_j \in \mathbb{N}\mathbb{P} \) and \( P_i = \rho_{h_i}, P'_j = \rho_{h'_j} \), for some \( h_i, h'_j \in H \) for each \( i \in [1, l], j \in [1, l'] \). For any \( k \in [1, n] \) and any summand monomial \( p \) of \( P \) with constant coefficient 1, let \( P^p \) be the sum of all summand monomials of \( P \) with the same \( x_k \)-degree as that of \( p \). So \( P^p \) is of the form \( P^p = x_k^{\deg x_k(p)} M_k^{\deg x_k(p)+1} L \), where \( L \) is a Laurent polynomial in \( \mathbb{N}\mathbb{P}[X^\pm] \), and there is a (not necessarily unique) summand monomial \( r_k \) of \( L \) such that \( p \) is a summand of \( x_k^{\deg x_k(p)} M_k^{\deg x_k(p)+1} r_k \). Denote

\[
J = \{ p \mid p \text{ is a summand monomial of } P \text{ with constant coefficient } 1 \}
\]
there is \( r_k \) of \( L \) such that \( p = x_k^{deg_{x_k}(p)} \prod_{s=1}^{n} \frac{a_{s-k}^{[a_{s-k}]}+}{y_k+1}x_k^{[a_{s-k}]+}r_k, \forall k \in [1, n] \).

Recall that in principal coefficients case, \( X^h \) is the only minimal summand monomial in \( \rho_h \) and \( \rho_h \) is universally indecomposable for any \( h \in \mathbb{Z} \). Hence because of the decomposition \( P = \sum_{i=1}^{l} a_i P_i \), we have \( J = \{ \frac{a_i}{c_{x_i h_i}(P)} X^{h_i} \mid i \in [1, l] \} \). On the other hand, \( J = \{ \frac{a_i'}{c_{x_i h_i'}(P)} X^{h_i'} \mid j \in [1, l'] \} \) due to the decomposition \( P = \sum_{j=1}^{l'} a_j' P_{j'} \). Therefore the above two decompositions must be the same.

So for any finite equation \( \sum_{h \in H} a_h \rho_h = 0 \), with \( a_h \in \mathbb{Z}_P \), we get \( \sum_{h \in H} a_h' \rho_h = \sum_{h \in H} a_h'' \rho_h \) with \( a_h', a_h'' \in \mathbb{NP} \). Following the above discussion, \( a_h' = a_h'' \), i.e, \( a_h = 0 \) for each \( h \in H \).

Due to \( \rho_h p = \frac{g}{h_{k_h}} X^h \), the strongly positivity in principal coefficients case leads to that in other semifield cases. Hence \( P \) is a strongly positive \( \mathbb{Z}_P \)-basis for \( \mathcal{I}_P(A) \).

\[ \square \]

**Remark 6.5.** When \( A = U_0(A) \), \( P \) is a strongly positive \( \mathbb{Z}_P \)-basis for it.

This is because \( A \subseteq \mathcal{I}_P(A) \subseteq U_0(A) \) by Theorem 53 and Proposition 54 so \( A = U_0(A) \) leads to \( A = \mathcal{I}_P(A) = U_0(A) \).

We wonder whether \( \mathcal{I}_P(A) \) coincides with \( U_0(A) \), and when does it happen if it is not the case in general. The following corollary of Theorem 6.3 and Proposition 6.4 provides an equivalent condition of \( \mathcal{I}_P(A) = U_0(A) \).

**Corollary 6.6.** Let \( A \) be a cluster algebra over a semifield \( \mathbb{P} \). Then \( \mathcal{I}_P(A) = U_0(A) \) if and only if for any universally indecomposable \( f \in U_0(A) \), there is \( \tilde{f} \in U_0(A_{prin}) \) satisfying \( a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = f \), where \( a \in \mathbb{NP} \), \( F = L(\tilde{f})|_{x_i \to 1, \forall i \in [1, n]} \) and \( g \) is the degree of \( \tilde{f} \).

In particular, \( \mathcal{I}_P(A) = U_0(A) \) if \( B \) (or \( \tilde{B} \) for geometric type) has full rank.

**Proof** ("only if"): Because \( \mathcal{I}_P(A) = U_0(A) \), \( P \) is a \( \mathbb{Z}_P \)-basis of \( U_0(A) \). So for any universally indecomposable element \( f \in U_0(A) \), there is a unique \( \mathbb{Z}_P \)-linear combination \( f = \sum_{i} a_i \rho_i \) with finitely many \( a_i \neq 0 \in \mathbb{Z}_P \). Then because \( f \) is universally indecomposable, we can find \( \tilde{a}_i \in \mathbb{Z} Trop(Y) \) such that \( \tilde{f} = \sum_{i} \tilde{a}_i \rho_i \in U_0(A_{prin}) \) is universally indecomposable and \( a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = f \) for some \( a \in \mathbb{NP} \), \( F = L(\tilde{f})|_{x_i \to 1, \forall i \in [1, n]} \) and \( g \) being the degree of \( \tilde{f} \).

("if"): Because for any universally indecomposable \( f \in U_0(A) \), there is \( \tilde{f} \in U_0(A_{prin}) \) satisfying \( a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = f \), where \( a \in \mathbb{NP} \), \( F = L(\tilde{f})|_{x_i \to 1, \forall i \in [1, n]} \) and \( g \) is the degree of \( \tilde{f} \), so for any \( f \) which is indecomposable in \( U_0(A) \), similar to the proof of Theorem 6.3 we can find a \( \mathbb{Z}_P \)-linear combination of \( f \) in universally indecomposable elements with the help of the set analogous to \( J \) in (i). So we assume \( f \) is universally indecomposable. Then there is \( \tilde{f} \in U_0(A_{prin}) \) satisfying \( a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = f \), where \( a \in \mathbb{NP} \), \( F = L(\tilde{f})|_{x_i \to 1, \forall i \in [1, n]} \) and \( g \) is the degree of \( \tilde{f} \).

Since \( f \) is universally indecomposable, so is \( \tilde{f} \). Then by Lemma 6.4 or Theorem 6.3 we have a \( \mathbb{Z} Trop(Y) \)-linear combination of \( \tilde{f} \) as \( \tilde{f} = \sum_{j} \tilde{c}_j \rho_j \in U_0(A_{prin}) \) for some \( \tilde{c}_j \in \mathbb{Z} Trop(Y) \). Hence \( f = a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = \sum_{j} c_j \rho_j \in U_0(A) \) for some \( c_j \in \mathbb{Z}_P \). So \( \mathcal{I}_P(A) = U_0(A) \).

In particular, when \( B \) or \( \tilde{B} \) has full rank, for any universally indecomposable \( f \in U_0(A) \), there is unique \( \tilde{f} \in U_0(A_{prin}) \) up to multiplying a Laurent monomial in \( Y \) satisfying \( a_{F_{|\tilde{f}|}(\tilde{Y})} X^g = f \), where \( a \in \mathbb{NP} \), \( F = L(\tilde{f})|_{x_i \to 1, \forall i \in [1, n]} \) and \( g \) is the degree of \( \tilde{f} \). So in this case \( \mathcal{I}_P(A) = U_0(A) \).
The following example given by Yan Zhou in [21] is a counterexample where the condition in Corollary 6.8 fails, thus \( \mathcal{I}_P(A) \not\subseteq U(A) \).

**Example 6.7 ([21]).** Let \( A \) be the cluster algebra without coefficients associated to the exchange matrix

\[
B = \begin{pmatrix}
0 & 2 & -2 \\
-2 & 0 & 2 \\
2 & -2 & 0
\end{pmatrix}.
\]

Then \( P := \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3} \in U(A) \) and it is universally indecomposable. But \( P \) can not be written as a \( \mathbb{Z} \)-linear combination of \( \mathcal{P} \). Otherwise by Corollary 6.6 there must be \( a_1, a_2, a_3 \in \mathbb{Z} \text{Tr}op(Y) \) satisfying

\[
\hat{P} = \frac{a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2}{x_1 x_2 x_3} \in U(A_{\text{prin}}) \text{ and } a_1|_{y_j \to 1, \forall j \in [1, n]} = 1 \text{ for } i \in \{1, 2, 3\}.
\]

However, by calculating the Laurent expression of \( \hat{P} \) in \( X_{t_i} \), where \( t_i \) is connected with \( t_0 \) by an edge labeled \( i \in \{1, 2, 3\} \), we get that either \( a_3 y_2 \) is a proper summand of \( a_1 \) and at the same time \( a_1 \) is a proper summand of \( a_3 y_2 \) or \( P \) has some monomial summand other than \( x_i^{-1} x_j^{-1} x_k \), where \( \{i, j, k\} = \{1, 2, 3\} \), which is impossible.

**Corollary 6.8.** Let \( A \) be a cluster algebra over a semifield \( \mathbb{P} \). Then for any \( h, h' \in \mathbb{Z}^n \), \( \rho_{h+h'} \) is a summand of \( \rho_h \rho_{h'} \). Hence \( H \) is an additive sub-monoid of \( \mathbb{Z}^n \).

**Proof** It is sufficient to deal with the principal coefficients case. Since \( X^h \) and \( X^{h'} \) are summand of \( \rho_h \) and \( \rho_{h'} \) respectively, \( X^{h+h'} \) is a summand of \( \rho_h \rho_{h'} \). Moreover, according to Theorem 6.8

\[
\rho_h \rho_{h'} = \sum_g a_g \rho_g \text{ with } a_g \in \mathbb{N}[\mathbb{Y}^\pm].
\]

Then because \( \rho_{h+h'} \) is the unique element in \( \hat{P} \) having \( X^{h+h'} \) as a summand, \( \rho_{h+h'} \) is a summand of \( \rho_h \rho_{h'} \).

We would like to end this section with the following result showing that in many “good” situations, \( \rho_h \) is a Laurent polynomial for any \( h \in \mathbb{Z}^n \) and hence \( \mathcal{P} = \hat{P} \).

**Proposition 6.9.** Let \( A \) be a cluster algebra over a semifield \( \mathbb{P} \). Then \( \mathcal{P} = \hat{P} \) if and only if \( \rho_{-e_i} \in \mathcal{P} \) for any \( i \in [1, n] \).

**Proof** The necessity is due to the definition of \( \hat{P} \). So, we only need to show the sufficiency. It is enough to deal with the case where \( A \) has principal coefficients. Assume \( \rho_{-e_i} \in \mathcal{P} \) for any \( i \in [1, n] \). \( \rho_{e_i} = x_i \in \mathcal{P} \) for any \( i \in [1, n] \). For any \( h \in \mathbb{Z}^n \), denote \( h = (h_1, \cdots, h_n) \). Hence

\[
h = \sum_{i=1}^n h_i e_i = \sum_{i=1}^n [h_i]_+ e_i + \sum_{i=1}^n [-h_i]_+ (-e_i).
\]

So according to Corollary 6.8 we have \( h \in H \). Hence, \( \rho_h \in \mathcal{P} \), and thus \( \mathcal{P} = \hat{P} \).

Motivated by applications to non-commutative Donaldson-Thomas theory, Keller introduced the idea of maximal green sequences and reddening sequences in [14, 15]. An equivalent definition for a skew-symmetric cluster algebra to admit reddening sequences is that it has \( -e_i \) as a \( g \)-vector associated to some cluster variable for any \( i \in [1, n] \) (see [19] for more details). Hence the following result is a direct corollary of the above proposition.

**Corollary 6.10.** Let \( A \) be a skew-symmetric cluster algebra over a semifield \( \mathbb{P} \). If \( A \) admits reddening sequences, then \( \mathcal{P} = \hat{P} \).
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References

[1] A. Berenstein, S. Fomin and A. Zelevinsky, Cluster algebras, III. Upper bounds and double Bruhat cells, Duke Math. J. 126 (2005), 1–52.
[2] A. Berenstein, A. Zelevinsky, Quantum cluster algebras. Adv. in Mathematics, 195 (2005): 405-455.
[3] Peigen Cao and Fang Li, The enough $g$-pairs property and denominator vectors of cluster algebras, Mathematische Annalen. 377 (2020), 1547-1572.
[4] H. Derksen, J. Weyman and A. Zelevinsky, Quivers with potentials and their representations II: Applications to cluster algebras, J. Amer. Math. Soc. 23 (2010), 749-790.
[5] Jiariui Fei, Combinatorics of $F$-polynomials, [arXiv:1909.10151] to appear in IMRN, 2022.
[6] Changjian Fu and B. Keller, On cluster algebras with coefficients and 2-Calbi-Yau categories, Trans. Amer. Math. Soc. 362 (2010) no. 2, 859-895.
[7] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497-529 (electronic).
[8] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification. Invent. Math., 154 (2003), no. 1, 63-121.
[9] S. Fomin and A. Zelevinsky, Cluster algebras, IV. Coefficients. Compos. Math., 143 (2007), 112-164.
[10] C. Geiss, B. Leclerc and J. Schröer: Factorial cluster algebras. Doc. Math., 18 (2013), 249-274.
[11] M. Gross, P. Hacking, S. Keel and M. Kontsevich, Canonical bases for cluster algebras, J. Amer. Math. Soc. 31 (2018), 497-608.
[12] Ming Huang and Fang Li, Unfolding of acyclic sign-skew-symmetric cluster algebras and applications to positivity and $F$-polynomials, Advances in Mathematics 340 (2018): 221-283.
[13] Ming Huang, Fang Li and Yichao Yang, On structure of cluster algebras of geometric type I: in view of sub-seeds and seed homomorphisms, Sci. China Math. 61 (5) (2018) 831-854.
[14] B. Keller, On cluster theory and quantum dilogarithm identities, In Representations of algebras and related topics, EMS Ser. Congr. Rep., (2011), 85-116.
[15] B. Keller, Cluster algebras and derived categories, in Derived Categories in Algebraic Geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, 123-183
[16] K. Lee, L. Li, R. Schiffler, Newton polytopes of rank 3 cluster variables, [arXiv:1910.14372]
[17] K. Lee, L. Li, A. Zelevinsky, Greedy elements in rank 2 cluster algebras, Selecta Mathematica. New Series, 20 (2012), no. 1, 57-82.
[18] K. Lee, L. Li, A. Zelevinsky, Positivity and tameness in rank 2 cluster algebras. J Algebr Comb 40, 823-840 (2014).
[19] G. Muller, The existence of a maximal green sequence is not invariant under quiver mutation, Electron. J. Combin. 23 (2016), no. 2, P2.47.
[20] G. Ziegler, Lectures on polytopes. In Graduate texts in mathematics, Vol. 152. Berlin: Springer, 1995.
[21] Yan Zhou, Cluster Structures and Subfans in Scattering Diagrams. Symmetry Integrability and Geometry-Methods and Applications 16 (2020): 013.

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