LOG-CANONICAL THRESHOLD FOR CURVES ON A SMOOTH SURFACE

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Abstract. It is shown that the log-canonical threshold of a curve with an isolated singularity is computed by the term ideal of the curve in a suitable system of local parameters at the singularity. The proof uses the Enriques diagram of the singularity and shows that the log-canonical threshold depends only on a non-degenerate path of that diagram.

1. Introduction

Let \( X \) be an affine variety with coordinate ring \( A = \mathbb{C}[X] \). Any ideal \( a \subset A \) determines a sheaf on \( X \) that will also be denoted by \( a \subset \mathcal{O}_X \). Let \( f \in A \) with an isolated singularity at \( P \) and let \( a_f \) be its term ideal in a system of local parameters \( \mathcal{P} \) at \( P \). It is known that for a rational number \( \xi > 0 \) one can attach to the divisor \(( f )\) a collection of multiplier ideals \( \mathcal{J}(\xi \cdot (f)) \) that starts at \( \mathcal{O}_X \), diminishes exactly when \( \xi \) equals a jumping number—they represent an increasing discrete sequence of rationals—and finally ends at \( \mathcal{I}(f) \). The multiplier ideals reflect the singularity of the rational divisor \( \xi (f) \). Multiplier ideals and jumping numbers are also attached to ideal sheaves in a similar way. For \( 0 < \xi < 1 \), one has \( \mathcal{J}(\xi \cdot (f)) \subset \mathcal{J}(\xi \cdot a_f) \) with equality if the coefficients of \( f \) are 'sufficiently general'. The definition and theorem hereafter will make precise what 'sufficiently general' means and state the equality.

The aim of this paper is to show that the log-canonical threshold, i.e. the first jumping number, of a curve \( C \) with an isolated singularity is always computed by the term ideal in a suitable system of local parameters, regardless the genericity of the curve’s equation.

Definition 1.1 (see [1]). Let \( \mathcal{P} \) be a system of local parameters and let \( f \in \mathbb{C}[X] \) with term ideal \( a_f \). Let \( \text{Newt}(a_f) \) be the Newton polygon associated to \( a_f \). Given any face \( \sigma \) of \( \text{Newt}(a_f) \) denote by \( f_\sigma \) the sum of those terms of \( f \) corresponding to points lying on \( \sigma \). The function \( f \) is said to be non-degenerate along \( \sigma \) if and only if the 1-form \( df_\sigma \) is nowhere vanishing on the torus \((\mathbb{C}^*)^n \). The function \( f \) is said to have non-degenerate principal part if this condition holds for every compact face \( \sigma \) of \( \text{Newt}(a_f) \).

Theorem 1.2 (see [6] Proposition 9.2.28)). If \( f \) has non-degenerate principal part, then for every \( 0 < \xi < 1 \)

\[ \mathcal{J}(\xi \cdot (f)) = \mathcal{J}(\xi \cdot a_f) \]
in a neighbourhood of $P$.

To state the main result we need the following:

**Definition 1.3.** Let $C$ be a curve with an isolated singularity at $P$ and let $K$ be the cluster associated to the minimal log resolution of the singularity of $C$. A system of local parameters at $P$ is said to be an *adapted system of local parameters* at $P$ for $C$ if the divisor $(y)$ contains a free point of $K$ and $(x)$ does not.

**Theorem (5.1).** If $C$ is a germ of curve with an isolated singularity at $P$, then

$$\text{lct}(C; P) = \min \{ \text{lct}(a_{C, P}) \mid P \text{ adapted system of local parameters at } P \text{ for } C \}$$

where $a_{C, P}$ is the term ideal of $C$ with respect to $P$.

The proof is given in §5 and is based on the relation between monomial ideals and Enriques diagrams and on the study of the log-canonical threshold of unibranch Enriques diagrams. The former is established in §3 and the latter in §4 showing that for a unibranch Enriques diagram the log-canonical threshold is computed at a point belonging to its non-degenerate part; see Section 3 for definitions.

## 2. Preliminaries and notation

### 2.1. Log-canonical threshold.

Let $X$ be an affine variety and let $a \subset O_X$ be a non-zero ideal sheaf. A *log resolution* of $a$ is a projective birational mapping $\mu : Y \to X$ with $Y$ non-singular such that $\mu^{-1}a := a \cdot O_Y = O_Y(-F)$ where $F$ is an effective divisor on $Y$ such that $F + \text{except}(\mu)$ has simple normal crossing support.

**Definition.** Let $a \subset O_X$ be a non-zero ideal sheaf and $\xi > 0$ a rational number. If $\mu : Y \to X$ is a log resolution of $a$ with $a \cdot O_Y = O_Y(-F)$, then the multiplier ideal of $\xi \cdot a$ is

$$J(\xi \cdot a) = \mu_* O_Y(K_{Y/X} - \lfloor \xi F \rfloor).$$

**Definition.** The *log-canonical threshold* at $P \in X$ of a non-zero ideal sheaf is

$$\text{lct}(a; P) = \inf \{ \xi \in \mathbb{Q} \mid J(\xi \cdot a)_P \subset m_P \}.$$

The above definitions work for $D \subset X$ a rational effective divisor. In case $C$ is a curve and $X$ an affine smooth surface, if $\mu^* C = \sum_{\alpha} e_\alpha E_\alpha + \tilde{C}$ and $K_{Y/X} = \sum_{\alpha} k_\alpha E_\alpha$, then

$$\text{lct}(C; P) = \min_{\alpha} \frac{k_\alpha + 1}{e_\alpha},$$

i.e. $\text{lct}(C; P)$ is the first $\xi > 0$ for which each exceptional divisor appears in $K_{Y/X} - [\mu^*(\xi C)]$ with coefficient $\geq -1$ and at least one has coefficient $= -1$.

**Remark.** Let $a$ or $D$, $\mu : Y \to X$ and $F$ as above. The multiplier ideals of $D$ or $a$ do not depend on the log resolution, see [7 II, Thm.9.2.11]. On the contrary, if $\xi$ is non-integral, the ideal $\mu_* O_Y(- \lfloor \xi F \rfloor)$ may depend on the log resolution. The simplest example is given by the ideal $\langle x^2, y^4 \rangle$. The general element of $a$ defines a curve with a tacnode at the origin in the affine plane. The minimal log-resolution of the ideal is given by $\mu$, the composition of two blow-ups. If $\mu'$ is $\mu$ composed with the blowing up of the intersection point of the two exceptional
divisors, then \( \mu_\ast \mathcal{O}_Y(-\lceil 1/6F \rceil) = \mathcal{O}_{\mathbb{A}^2} \) and \( \mu_\ast' \mathcal{O}_Y(-\lceil 1/6F' \rceil) = \mathcal{I}_O \). Now, if \( \xi = 1 \), then \( \overline{a} = \mu_\ast \mathcal{O}_Y(-F) \) (see [7] II, p. 216-219).

**Remark.** If \( X \) is a smooth affine surface then, for an ideal sheaf or a curve \( C \) with a singularity at \( P \), there exists a unique minimal log resolution.

2.2. **Term ideals.** Let \( x, y \) be regular functions on \( X \) that form a system of local parameters \( \mathcal{P} \) at \( P \). The functions \( x \) and \( y \) will be seen as coordinates around \( P \) through the monomorphism \( \mathcal{O}_{X,P} \cong \mathbb{C}[x,y] \) which associates to each \( f \in \mathcal{O}_{X,P} \) its Taylor power series—the injection of \( \mathcal{O}_{X,P} \) in its formal completion at \( \mathfrak{m}_P \). With \( \mathcal{P} \) a system of local parameters at \( P \) given, if \( f \in \mathcal{O}_X \), the term ideal of \( f \) is the ideal \( \mathfrak{a}_f,\mathcal{P} \) generated by the monomials of \( f \) with respect to \( \mathcal{P} \). If \( C = (f) \) the notation \( \mathfrak{a}_{C,P} \) will also be used for \( \mathfrak{a}_f,\mathcal{P} \).

2.3. **Monomial ideals, Newton polygons and Howald’s result.** The multiplier ideals and the jumping numbers are hard to compute in general. An exception is the class of monomial ideals. Howald describes in [5] their multiplier ideals by a combinatorial formula. To state the result let \( \mathfrak{a} \subset \mathbb{C}[X] \) be a monomial ideal, i.e. an ideal generated by monomials of the form \( x^m = x_1^{m_1} \cdots x_n^{m_n} \) in a system of local parameters \( (x_1,\ldots,x_n) \), with \( m \in \mathbb{Z}^n \). Such a monomial ideal \( \mathfrak{a} \) can be identified with the set of exponents in \( \mathbb{Z}^n \) of its monomials. The convex hull of this set in \( \mathbb{R}^n = \mathbb{Z}^n \otimes \mathbb{R} \) is called the **Newton polyhedron** of \( \mathfrak{a} \) and it is denoted by \( \text{Newt}(\mathfrak{a}) \). If \( 1 = (1,\ldots,1) \in \mathbb{Z}^n \), then:

**Theorem 2.1 (Howald).** Let \( \mathfrak{a} \subset \mathbb{C}[X] \) be a monomial ideal in a system of local parameters. Then for every \( \xi > 0 \), the multiplier ideal \( \mathcal{J}(\xi \cdot \mathfrak{a}) \) is the monomial ideal generated by all monomials \( x^m \) with the vector \( m \) satisfying

\[
\mathbf{m} + 1 \in (\xi \cdot \text{Newt}(\mathfrak{a}))^\circ.
\]

For example if the polyhedron \( \text{Newt}(\mathfrak{a}) \) is cut out in the first orthant by the inequalities \( g_j(\mathbf{v}) \geq 1 \) with non-negative rational coefficients, then

\[
\text{lc}(\mathfrak{a}) = \min_j g_j(1).
\]

The picture of the Newton polygon of the monomial ideal \( \mathfrak{a} = (x^8,x^3y^2,y^4) \) in the figure below shows, using Howald’s result, that \( \mathcal{J}(5/12 \cdot \mathfrak{a}) = (x,y) \). Note that even though \( (0,0) + 1 \) lies in \( \text{Newt}(\mathfrak{a}) \) it does not lie in its interior. Therefore, the monomial 1 corresponding to \( (0,0) \) does **not** belong to \( \mathcal{J}(5/12 \cdot \mathfrak{a}) \). But for all \( c < 5/12, 1 \in \mathcal{J}(\xi \cdot \mathfrak{a}) \). Hence \( \text{lc}(\mathfrak{a}) = 5/12 \).

**Example 2.2.** Let \( f(x,y) = (x^3 - y^2)^2 - x^5y \). Its term ideal is generated by \( x^6, x^5y, x^3y^2 \) and \( y^4 \) and its Newton polygon has a unique compact face \( \sigma \) given by

\[
\begin{align*}
\text{deg } y &\quad \text{deg } x
\end{align*}
\]

[Diagram showing the Newton polygon with shaded area representing the unique compact face \( \sigma \)]
successive blowing ups with YE strict transforms of the exceptional divisors

The proof of Theorem 5.1 will mainly deal with the configuration formed by the minimal log resolution of C at P.

\[ \mu^*C = \tilde{C} + D = \tilde{C} + \sum_{\alpha} c_\alpha E_\alpha. \]

The proof of Theorem 5.1 will mainly deal with the configuration formed by the strict transforms of the exceptional divisors E_\alpha. To prepare the way for the proof we need to formalize the setup and recall some results from the theory of clusters.

Let Y = Y_{r+1} \rightarrow Y_r \rightarrow \cdots \rightarrow Y_1 = X be the decomposition of \mu : Y \rightarrow X into successive blowing ups with Y_{\alpha+1} = \text{Bl}_{P_\alpha} Y_\alpha. Each point P_\alpha is infinitely near to P = P_1 and has an associated exceptional divisor on Y_{\alpha+1}. The exceptional divisor and all its strict transforms will be denoted by E_\alpha. Its total transforms will be denoted by W_\alpha. When needed, E_\alpha^{(\beta)} will design the strict transform of E_\alpha on Y_\beta and similarly for the total transform. For example W_\alpha^{(\alpha+1)} = E_\alpha^{(\alpha+1)}. The strict transforms E_\alpha and the the total transforms W_\alpha form two bases of the \( \mathbb{Z} \)-module \( \Lambda_C = \bigoplus_\alpha \mathbb{Z}E_\alpha \subset \text{Pic} Y \). The divisor D becomes \( D = \sum_\alpha w_\alpha W_\alpha \) in the basis of the W's.

**Definition.** A *cluster* in X centered at a smooth point P is a finite set of weighted infinitely near points to P, \( K = \{ P_1^{w_1}, \ldots, P_r^{w_r} \} \), with \( P_1 = P \) and such that the ordering of the points is compatible with the partial order of the infinitely near points. The point \( P_1 \) is called the *proper* point of the cluster.

The combinatorics of the configuration of the strict transforms on Y is encoded in the notion of *proximity* for the points of the cluster: a point P_\beta is said to be proximate to P_\alpha, \( P_\beta \prec P_\alpha \), if P_\beta lies on \( E_\alpha^{(\beta)} \subset Y_\beta \). Besides, a point that is infinitely near, i.e., that is not proper, is always proximate to at most two other points of the cluster. It is said to be *free* if it is proximate to exactly one point and *satellite* if it is proximate to exactly two points of the cluster.

Let \( \Pi = \| [p_{\alpha \beta}] \| \) be the decomposition matrix of the strict transforms in terms of the total transforms on Y; it is also called the proximity matrix of the cluster. Since

\[ E_\alpha = W_\alpha - \sum_{P_\beta \prec P_\alpha} W_\beta, \]

\( p_{\alpha \alpha} = 1 \) for any \( \alpha \) and \( p_{\alpha \beta} \) equals \(-1\) if \( P_\beta \) is proximate to \( P_\alpha \) and 0 if not. Along the \( \alpha \) column of \( \Pi \) the non-zero elements not on the diagonal correspond to the points to which \( P_\alpha \) is a satellite. The matrix \(-\Pi^{-1}\Pi \) is the intersection matrix of the curves \( E_\alpha \) on the surface Y. Since the intersection matrix of the curves \( W_\alpha \) is minus the identity, there exists effective divisors \( B_\alpha \) that form the dual basis for the divisors \(-E_\alpha\) with respect to the intersection form. In the sequel this basis will be referred to as the *branch basis*. The decomposition matrix of the basis of strict transforms in terms of the branch basis is \( \Pi^{-1}\Pi \).

\[ ^1 \text{If Q and R are infinitely near to P, then the point Q precedes the point R if and only if R is infinitely near to Q.} \]
2.5. **Unloaded clusters.** Let $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ be a cluster centered at $P$. As we have noticed, it defines a divisor $D_K = \sum w_\alpha W_\alpha$ on $Y$ and an ideal sheaf $\mu_*O_Y(-D_K)$ on $X$, or equivalently a subscheme $Z_K$ of $X$. A 0-dimensional subscheme defined by a cluster will be called a **complete subscheme.** The lemma below clarifies the relation between the divisor $D_K$ and the ideal sheaf $\mu_*O_Y(-D_K)$ or equivalently, the complete subscheme $Z_K$.

**Lemma 2.3.** Let $D_K = \sum b_\alpha B_\alpha$. If $b_\beta < 0$ for a certain $\beta$, then
\[ \mu_*O_Y(-D_K) = \mu_*O_Y(-D_K - E_\beta). \]

**Proof.** We take $\mu_*$ on the exact sequence
\[ 0 \to O_Y(-D_K - E_\beta) \to O_Y(-D_K) \to O_{E_\beta}(-D_K |_{E_\beta}) \to 0. \]
Since $\deg(-D_K |_{E_\beta}) = -\sum b_\alpha B_\alpha \cdot E_\beta = b_\beta < 0$, we have that
\[ \mu_*O_{E_\beta}(-D_K |_{E_\beta}) = 0. \]

**Definition.** A cluster $K$ is said to satisfy the **proximity relations** if for every $P_\alpha$ in $K$,
\[ \overline{w}_\alpha = \sum_{P_\beta \sim P_\alpha} w_\beta \leq w_\alpha. \]

**Corollary 2.4** (see [2], Theorem 4.2). Let $K = \{P_1^{w_1}, \ldots, P_r^{w_r}\}$ be a cluster that contains a point $P_\alpha$ at which the proximity relation is not satisfied. If $K' = \{\tilde{P}_1^{w_1'}, \ldots, P_r^{w_r'}\}$ is the cluster defined by $w_\alpha' = w_\alpha + 1$, $w_\beta' = w_\beta - 1$ for every $\beta$ with $P_\beta$ proximate to $P_\alpha$, and $w_\alpha' = w_\alpha$ otherwise, then $K$ and $K'$ define the same subscheme in $X$, i.e., $\mu_*O_Y(-D_K) = \mu_*O_Y(-D_{K'})$.

**Proof.** Let $D_K = \sum w_\alpha W_\alpha = \sum e_\alpha E_\alpha = \sum b_\alpha B_\alpha$ and $D_{K'} = \sum b_\alpha' B_\alpha$. The coefficients $b_\alpha$ are given by $b = e \cdot \Pi \cdot \Pi = w \cdot \Pi = w - \overline{w}$, with $b = (b_1, \ldots, b_r)$ and so on. Then
\[ b' = w' - \overline{w} = w - \overline{w} + (\Pi \cdot \Pi)_\alpha = b + (\Pi \cdot \Pi)_\alpha \]
and
\[ e' = b' \cdot (\Pi \cdot \Pi)^{-1} = b \cdot (\Pi \cdot \Pi)^{-1} + (\Pi \cdot \Pi)_\alpha \cdot (\Pi \cdot \Pi)^{-1} = e + (0, \ldots, 1, \ldots, 0), \]
hence $D_{K'} = D_K + E_\alpha$. But $b_\alpha = w_\alpha - \overline{w}_\alpha < 0$ and the result follows form the previous lemma.

The cluster $K'$ in the above corollary is said to be obtained from $K$ by the unloading procedure. Starting from $K$, iterated applications of this procedure lead to a cluster $\overline{K}$ that satisfies the proximity relations and defines the same subscheme in $X$. The cluster $\overline{K}$ is called the unloaded cluster associated to $K$. So a cluster and the unloaded cluster associated to it defines the same complete subscheme of $X$.

**Corollary 2.5.** A cluster is unloaded if and only if the coefficients of its divisor in the branch basis are non-negative.
Example. Let \( \{P_1^5, P_2^2, P_3^3, P_4^1, P_5^1\} \) and \( \{P_1^4, P_2^2, P_3^0, P_4^2, P_5^1\} \) be two clusters with the proximity matrix

\[
\begin{pmatrix}
1 & -1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1
\end{pmatrix}.
\]

The former is unloaded; it encodes the log-resolution of the singularity \( x^5 - y^7 = 0 \), i.e., its associated ideal is the integral closure of \((x^5, y^7)\). The latter does not satisfy the proximity relation at \( P_3 \). The unloaded associated cluster is \( \{P_1^4, P_2^2, P_3^1, P_4^1, P_5^0\} \) and the associated divisor \( B_2 + B_4 \).

2.6. Clusters and Enriques diagrams. For our purposes, it will be convenient to work with Enriques diagrams instead of clusters. The points of a cluster \( K \), their weights and proximity relations were encoded by Enriques in an appropriate tree diagram now called the Enriques diagram of the cluster (see \[2, 3, 4\]). If the weights are omitted, the tree reflects the combinatorics of the configuration of the strict transforms \( E_\alpha \subset Y \).

**Definition** (see \[3, 2, 4\]). An Enriques tree is a couple \((T, \varepsilon_T)\), where \( T = T(\mathcal{U}, \mathcal{E}) \) is an oriented tree (a graph without loops) with a single root, with \( \mathcal{U} \) the set of vertices and \( \mathcal{E} \) the set of edges, and where \( \varepsilon_T \) is a map

\[ \varepsilon_T : \mathcal{E} \to \{\text{slant}, \text{horizontal}, \text{vertical}\}. \]

fixing the graphical representation of the edges. An Enriques diagram is an weighted or labeled Enriques tree.

**Definition.** Let \( T \) be an Enriques tree. A horizontal (respectively vertical) \( L \)-shaped branch of \( T \) is a path of length \( \geq 1 \) such that all edges, but the first, are horizontal (respectively vertical) through \( \varepsilon_T \). An \( L \)-shaped branch is proper if it contains at least two edges. A maximal \( L \)-shaped branch is an \( L \)-shaped branch that cannot be continued to a longer one.

**Lemma 2.6** (see \[4, Proposition 1.2\]). There exists a unique map from the set of clusters in \( X \) centered at a smooth point \( P \) to the set of Enriques diagrams such that:

- For every cluster \( K = \{P_1^{w_1}, \ldots, P_r^{w_r}\} \) the set of vertices of the image tree is \( \mathcal{U} = \{P_1, \ldots, P_r\} \) with the weights given by the integers \( w_1, w_2, \ldots, w_r \). The root of the tree is the proper point.
- At every point ends at most one edge.
- A point \( P_\alpha \) is satellite if and only if there is either a horizontal or a vertical edge that ends at the vertex \( P_\alpha \).
- If there is an edge that begins at the vertex \( P_\alpha \) and ends at the vertex \( P_\beta \) then \( P_\beta \in E^{(3)}_\alpha \), and the converse is true if \( P_\beta \) is free.
- The point \( P_\beta \) is proximate to \( P_\alpha \) if and only if there is an \( L \)-shaped branch that starts at \( P_\alpha \) and ends at \( P_\beta \).
- The strict transforms \( E_\alpha \) and \( E_\beta \) intersect on \( Y \) if and only if the Enriques diagram contains a maximal \( L \)-shaped branch that has \( P_\alpha \) and \( P_\beta \) as its extremities.
- An edge that begins at a vertex of a free point and ends at a vertex of a satellite point is horizontal.
Due to this correspondence, in the sequel, we shall freely talk about the vertices of an Enriques tree as being free or satellite. Now, if $T$ is an Enriques diagram and $K$ a corresponding cluster, the divisor $D_K$ will also be denoted by $D_T$ and will be considered as associated divisor to $T$. Moreover, if the cluster is unloaded, the log-canonical threshold of the cluster is defined using (1) and the notation $\text{lct}(T)$ and $\text{lct}(D_K)$ will be used.

Examples 2.7. Let $p < q$ be relatively prime positive integers. The Enriques tree $T_{p,q}$ and diagram $T_{p,q}$ encode the minimal log resolution of the curve $x^p - y^q = 0$. More precisely, if $r_0 = a_1 r_1 + r_2, \ldots, r_{m-2} = a_{m-1} r_{m-1} + r_m$ and $r_{m-1} = a_m r_m$, with $r_0 = q$ and $r_1 = p$, the oriented tree has $\mathcal{V} = \{P_{\alpha} | 1 \leq \alpha \leq a_1 + \cdots + a_m\}$ and $\mathcal{E} = \{[P_{\alpha} P_{\alpha+1}] | 1 \leq \alpha \leq a_1 + \cdots + a_m - 1\}$. The map $\varepsilon$ is locally constant on the $a_j$ edges $[P_{\alpha} P_{\alpha+1}]$ with $a_1 + \cdots + a_{j-1} + 1 \leq \alpha < a_1 + \cdots + a_j$. The first constant value of $\varepsilon$—on the first $a_1$ edges—is ‘slant’. The other constant values are alternatively either ‘horizontal’ or ‘vertical’, starting with ‘horizontal’. The Enriques trees represent $T_{1,3}, T_{2,3}$ and respectively $T_{5,7}$. The tree $T_{5,7}$ together with the weights $5, 2, 2, 1, 1$ and become the Enriques diagram $T_{5,7}$ of the log resolution for $x^5 - y^7 = 0$. In general, the weights of $T_{p,q}$ are the corresponding remainders of the Euclidean algorithm.

Example (2.2 once more). The minimal log resolution of $(x^3 - y^2)^2 - x^5 y = 0$ needs five blowing ups with the following Enriques diagram.

Using the notation that will be introduced in the 5th section, the Enriques tree is the connected sum $T_{2,3} \# T_{2,3}$.

3. NEWTON POLYGONS AND ENRIQUES DIAGRAMS

We have seen in (2) that to any germ of plane singularity one can associate an Enriques diagram. The goal of this Section is to answer the following question: What are the Enriques diagrams who correspond to monomial ideals of the polynomial ring in two variables? Monomial ideals are in some sense the ‘simplest’ ideals. The main result is probably well-known to the experts, but due to the lack of suitable reference and for the coherence of the exposition we shall state and prove it in this section.

Definitions 3.1. An Enriques tree $(T, \varepsilon_T)$ is said to be non-degenerate if for every free point $P_{\alpha}$, the path going from the root to the vertex $P_{\alpha}$ contains only free points. An Enriques tree is called a binary Enriques tree if it is non-degenerate, the
outdegree of each vertex is \( \leq 2 \) and any vertex different from the root cannot have two proximate free vertices. An Enriques diagram is said to be non-degenerate or binary if its tree is.

For Enriques trees with the degree of the root equal to 1 the notion of union is defined as follows. Let \( T \) and \( T' \) be rooted at \( P \) and \( P' \) and let \( (S, S') \) be the maximal couple of subtrees of \( T \) and \( T' \) for which there exists an isomorphism of Enriques trees \( \varphi : S \to S' \). The Enriques tree \( T \cup T' \) is the disjoint union of \( T \) and \( T' \) modulo the isomorphism \( \varphi \). The definition extends to Enriques diagrams; a vertex belonging neither to \( S \) nor to \( S' \) keeps its weight, whereas a vertex corresponding to two vertices identified through \( \varphi \) inherits the sum of the respective weights. It is easy to see that \( T \cup T' \) is again a binary Enriques tree with the degree of the root equal to 1. Moreover, if \( T \) and \( T' \) are unloaded, then \( T \cup T' \) is unloaded.

**Example 3.2.** The Enriques diagram \( T_{5,7} \cup T_{4,7} \cup T_{3,4} \) is shown below.

![Enriques Diagram](image)

**Theorem 3.3.** Let \( a \subset \mathcal{O}_X \) be an ideal whose associated subscheme is 0-dimensional and centered at \( P \). There exists a system of local parameters at \( P \) such that \( a \) is monomial if and only if the Enriques diagram associated to \( a \) is a binary Enriques diagram. In this case the followings hold:

1. The boundary lines of \( \text{Newt}(a) \) have slopes \( -q_j/p_j \) and lengths \( d_j \), with \( j \in J \), if and only if the two subdiagrams with the degree of the root \( = 1 \) into which the Enriques diagram decomposes are

\[
T' = \bigcup_{j, -\frac{q_j}{p_j} \leq -1} T_{p_j, q_j}^{d_j} \quad \text{and} \quad T'' = \bigcup_{j, -\frac{q_j}{p_j} > -1} T_{p_j, q_j}^{d_j},
\]

2. If \( P' \) and \( P'' \) are the extremities of the longest paths of free points of these two subdiagrams, then \( P' \) and \( P'' \) lie on the coordinate curves.

**Proof.** The proof proceeds by induction on the height of the root—the length of the path from the root to the furthest child. Suppose the theorem true when the corresponding Enriques diagram has height of the root equal to \( n \). Let \( a \) be an ideal such that the associated Enriques diagram is binary and the height of the root equals \( n + 1 \). Let \( \sigma : \tilde{X} = \text{Bl}_P(X) \to X \) be the blowing up at \( P \) and let \( E \) be the exceptional curve. Then

\[
\sigma^*a = \mathcal{I}_{\tilde{Z}_1 \cup \tilde{Z}_2} \otimes \mathcal{O}_{\tilde{X}}(-cE)
\]

with \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) 0-dimensional subschemes supported at \( \tilde{P}_1 \) and \( \tilde{P}_2 \), and \( c \geq \deg \tilde{Z}_{1|E} + \deg \tilde{Z}_{2|E} \). The Enriques diagrams associated to \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) are binary and the heights of the root are \( \leq n \). By the induction hypothesis \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) are monomial in suitable systems of local parameters \( \mathcal{P}_j \) at the points \( \tilde{P}_j \), \( j = 1, 2, \ldots \).
and satisfy the claims in Theorem 3.3. Let $\mathcal{P}$ be a local system of local parameters at $P$ such that $\sigma$ is given locally by
\begin{align*}
x = \tilde{x}_1 \tilde{y}_1 \\
y = \tilde{y}_1.
\end{align*}
Using the notion of staircase below and the lemma hereafter with $Z_1$ and $Z_2$ the complete subschemes associated to $\tilde{Z}_1$ and $\tilde{Z}_2$, we conclude that $a$ is monomial and that $\mu_*\mathcal{O}(-D_T) = \mathfrak{a}$ (the integral closure $\mathfrak{a}$ of a monomial ideal $a$ is the monomial ideal spanned by all monomial whose exponents lie in $\text{Newt}(a)$). The other claims of the theorem follow from the identity on the staircases and the way the Enriques diagram $T$ is constructed from the diagrams $T_j$, $j = 1, 2$:
\[\mathcal{D}(T) = \mathcal{D}(T_1) \cup \mathcal{D}(T_2) \cup \{P\}, \quad \mathcal{E}(T) = \mathcal{E}(T_1) \cup \mathcal{E}(T_2) \cup \{[P\tilde{P}_1], [P\tilde{P}_2]\}.\]
The map $\varepsilon_T$ inherits the values from the corresponding maps $\varepsilon_{T_j}$ except for the slant values of a possible path of free points in each $T_j$. More precisely, the possible path in $T_1$ of free points coming from a horizontal staircase in the previous step, becomes a path of horizontal edges in $T$. The possible path in $T_2$ of free points coming from a vertical staircase in the previous step, becomes a path of vertical edges in $T$.

In the next lemma it will be easier to work with staircases instead of Newton polygons. A staircase is a subset of $\Sigma \subset \mathbb{N}^2$ such that its complement verifies $\Sigma^c + \mathbb{N}^2 \subset \Sigma^c$. If a system of local parameters at $P \in X$ is given, to a staircase $\Sigma$, a monomial ideal is associated $I^\Sigma$. Conversely, if $a$ is a monomial ideal in a certain system of local parameters then $\Sigma(a)$, or $\Sigma(Z)$ if $Z$ is the associated 0-dimensional subscheme, will denote the corresponding staircase.

Clearly $\text{Newt}(a)$ is the convex hull of $\Sigma(a)^c$, but $\text{Newt}(a)$ and $\Sigma(a)$ encode the same information only for integrally closed monomial ideals.

A finite staircase may be seen as a non increasing finite support sequence of integers $(n_j)_j$, its horizontal slices. If the finite staircases $\Sigma'$ and $\Sigma''$ are defined by the sequences $(n'_j)_j$ and $(n''_j)_j$, then the sequence $(n'_j + n''_j)_j$ defines the horizontal sum $\Sigma' + h \Sigma''$. The vertical sum is similarly defined. We shall denote by $\Sigma_c$ the staircase $\{(m, n) \mid m + n < c\}$.

**Lemma 3.4.** Let $X$, $\tilde{X}$, $P$, $\mathcal{P}$, $P_j$ and $\mathcal{P}_j$, $j = 1, 2$ as in the above proof. Let $Z_j$ be a 0-dimensional complete subscheme of $\tilde{X}$ supported at $\tilde{P}_j$ and monomial in the system of local parameters $\mathcal{P}_j$, for each $j = 1, 2$. If the integer $c$ satisfies $c \geq \text{deg} Z_j|_E + \text{deg} Z_2|_E$, then there exists a unique 0-dimensional complete subscheme $Z$ of $X$ supported at $P$ and monomial in the system of local parameters $\mathcal{P}$ such that
\[I_Z = \sigma_*(I_{Z_1\cup Z_2} \otimes \mathcal{O}_{\tilde{X}}(-cE)) \quad \text{and} \quad \sigma^*I_Z = I_{Z_1\cup Z_2} \otimes \mathcal{O}_{\tilde{X}}(-cE).\]
Moreover
\[\Sigma(Z) = \Sigma_c + h \Sigma(Z_1) + h \Sigma(Z_2).\]

**Proof.** It is easy to see that if $c \geq \text{deg} Z_1|_E$, then $I_{Z'} = \sigma_*(I_{Z_1} \otimes \mathcal{O}_{\tilde{X}}(-cE))$ is a monomial ideal with the staircase $\Sigma_c + h \Sigma(Z_1)$—if the Newton polygon of $Z_1$ is defined by $m, n \geq 0$ and $g_i(m, n) \geq 0$, a finite number of linear inequalities with positive integer coefficients for $m$ and $n$, then the Newton polygon of $Z'$ is defined by $m, n \geq 0$, $g_i(m, m + n - c) \geq 0$ and $m + n - c \geq 0$—and similarly for
$\sigma_{\ast}(I_{Z_1} \otimes O_{\tilde{X}}(-cE))$. It follows that $\sigma_{\ast}(I_{Z_1 \cup Z_2} \otimes O_{\tilde{X}}(-cE))$ is the monomial ideal

$$\sigma_{\ast}(I_{Z_1} \otimes O_{\tilde{X}}(-cE)) \cap \sigma_{\ast}(I_{Z_2} \otimes O_{\tilde{X}}(-cE))$$

with the corresponding subscheme $(\Sigma_c + v \Sigma(Z_1)) \cup (\Sigma_c + v \Sigma(Z_2))$ and

$$\sigma_{\ast}(I_{Z_1} \otimes O_{\tilde{X}}(-cE)) \cap \sigma_{\ast}(I_{Z_2} \otimes O_{\tilde{X}}(-cE))$$

and

$$(\Sigma_c + v \Sigma(Z_1)) \cup (\Sigma_c + h \Sigma(Z_2)) = \Sigma_c + v \Sigma(Z_1) + h \Sigma(Z_2)$$

if and only if $c \geq \deg Z_1|_E + \deg Z_2|_E$. To finish the proof, let $Z$ be the 0-dimensional subscheme defined by $\sigma_{\ast}(I_{Z_1 \cup Z_2} \otimes O_{\tilde{X}}(-cE))$. It is sufficient to notice that $\sigma_{\ast}I_Z \otimes O_{\tilde{X}}(cE)$ defines a 0-dimensional subscheme if and only if the identity (4) holds. □

4. Unibranch Enriques trees and log-canonical threshold

The next step is to study the log-canonical threshold for unibranch Enriques trees. The proof of the main result will be reduced to this case.

**Definition 4.1.** Let $T$ and $T'$ be unibranch Enriques trees with $\mathfrak{B}(T) = \{P_1, \ldots, P_r\}$ and $\mathfrak{B}(T') = \{P'_1, \ldots, P'_{r'}\}$. The connected sum of $T$ and $T'$ is the Enriques tree $T \# T'$ with the set of vertices $\mathfrak{B}(T \# T') = \mathfrak{B}(T) \cup \mathfrak{B}(T')/\{P_r = P'_1\}$, the set of edges $\mathcal{E}(T \# T') = \mathcal{E}(T) \cup \mathcal{E}(T')$ and the map $\varepsilon_{T \# T'}$ defined by $\varepsilon_T$ and $\varepsilon_{T'}$ through the natural restrictions.

In what follows, if $T$ is an Enriques tree, then $\Lambda_T \subset \text{Pic} Y$ denotes the $\mathbb{Z}$-module $\bigoplus_\alpha E^T_{\alpha}$ with $E^T_{\alpha}$'s the strict transforms, and $(e^T_{\alpha})$ denotes the basis for $\Lambda_T^r$, dual of $(E^T_{\alpha})$. Similarly, $(w^T_{\alpha})$ is the dual basis of the basis of total transforms $(W^T_{\alpha})$ and $(b^T_{\alpha})$ the dual basis of the branch basis $(B^T_{\alpha})$.

**Proposition 4.2.** Let $T$ be a unibranch Enriques tree that contains at least one proper L-shaped branch and has $r$ vertices. Let $T'$ be the unibranch tree $T'_{p', q'}$ that contains $r'$ vertices with $p' \geq 1$ and $q' \geq 2$ relatively prime integers. Then the branch basis of the Enriques tree $S = T \# T'$ satisfies

$$e^S_r(B_{\alpha}) \geq e^S_{r+r'-1}(B_{\beta})$$

for any $1 \leq \alpha \leq r + r' - 1$, where $k^S_\alpha = e^S_s(\omega^S)$ are the coefficients of the ‘relative canonical divisor’, $\omega^S = \sum_\alpha W^S_{\alpha}$, in the strict transform basis.

As a consequence we get the second technical ingredient needed for Theorem [5.1]

**Corollary 4.3.** Let $T$ be an unloaded Enriques diagram such that its associated tree $T$ is unibranch and degenerate. If $T'$ is the Enriques diagram obtained by taking away the last node from $T$, then

$$\text{lct}(T') = \text{lct}(T).$$

**Proof.** Let $D_T = \sum_\alpha b_{\alpha} B^T_{\alpha}$. Since $T$ is unloaded, by Corollary [2.5] $b^T_{\alpha}(D_T) \geq 0$ for any $\alpha$. Then

$$\frac{1}{\text{lct}(T')} = \frac{1}{\text{lct}(D_T)} = \frac{1}{\max_{\beta=1}^{r} \frac{e^T_\beta(D_T)}{k^T_\beta + 1}} = \frac{1}{\max_{\beta=1}^{r} \frac{\sum_\alpha b_{\alpha} e^T_\beta(B^T_{\alpha})}{k^T_\beta + 1}} = \frac{1}{\max_{\beta=1}^{r} \frac{\sum_\alpha b_{\alpha} e^T_\beta(B^T_{\alpha})}{k^T_\beta + 1}}.$$

the last equality being given by Proposition [4.2] since $T$ is degenerate. Hence

$$\frac{1}{\text{lct}(T')} = \frac{1}{\max_{\beta=1}^{r} \frac{e^T_\beta(D_T)}{k^T_\beta + 1}}.$$
Now, let $T'$ be the Enriques diagram obtained by taking away the last node from $T$. Then

$$D_{T'} = \sum_{\alpha=1}^{r-1} w_\alpha W'_\alpha$$

and

\begin{equation}
\frac{1}{\lct(T')} = \max_{\beta=1}^{r-1} \frac{e^T_\beta(D_{T'})}{k^T_\beta + 1}
\end{equation}

Claim. $e^T_\beta(D_{T'}) = e^T_\beta(D_T)$ for any $1 \leq \beta \leq r - 1$.

Indeed, let $\sigma^* : \Lambda_{T'} \to \Lambda_T$ be the monomorphism given by $W^T_{T'} \mapsto W^T_\alpha$ and $\sigma_*$ be the dual epimorphism. Then $D_T = \sigma^* D_{T'} + w_r W^T_r$ and, for any $1 \leq \beta \leq r - 1$,

$$e^T_\beta(D_T) = e^T_\beta(\sigma^* D_{T'}) + w_r e^T_\beta(W^T_r) = \sigma_* e^T_\beta(D_{T'}).$$

Using the identity (3) and denoting by $\Pi_T$ the decomposition matrix for the strict and total transforms associated to $T$, we have

$$\Pi_T = \begin{bmatrix} \Pi_{T'}^T & * \\ 0 & 1 \end{bmatrix}.$$ 

Hence

\begin{equation}
\sigma_* e^T_{\beta} = \sigma_*(w^T \cdot \Pi_{T'}^T)^\beta = \sigma_* \left( w^T \cdot \begin{bmatrix} \Pi_{T'}^T & * \\ 0 & 1 \end{bmatrix} \right)_{\beta}
\end{equation}

$$= \left( [w^{T'}_1 \ldots w^{T'}_{r-1} 0] \cdot \begin{bmatrix} \Pi_{T'}^T & * \\ 0 & 1 \end{bmatrix} \right)_{\beta} = e^T_{\beta}$$

for any $1 \leq \beta \leq r - 1$.

Finally, since $k^T_{\beta} = k^T_{\beta'}$ for any $1 \leq \beta \leq r - 1$, using the claim and the formulae (6) and (5) the corollary follows.

For the proof of the proposition we need four preliminary lemmas. The first three deal with the combinatorics of $T_{p,q}$, see [9].

**Lemma 4.4.** If $(f_j)_{-1 \leq j \leq m}$ and $(\delta_j)_{1 \leq j \leq m+1}$ are two finite sequences defined by

$$f_j = f_{j-2} + a_j \delta_j, \text{ for any } 1 \leq j \leq m,$$

$$\delta_j = \delta_{j-2} + a_{j-1} f_{j-2}, \text{ for any } 2 \leq j \leq m + 1$$

and such that $f_{-1} = f_0 = 0$ and $\delta_0 = \delta_1 = 1$, then the remainder $r_j$ in the Euclid algorithm is given by $-f_{j-1}q + \delta_j p$ if $j$ is odd and $\delta_j q - f_{j-1}p$ if $j$ is even.

**Proof.** Left to the reader.

**Remark 4.5.** If $m$ is odd, then $f_m = q$ and $\delta_{m+1} = p$, and if $m$ is even, then $f_m = p$ and $\delta_{m+1} = q$. Indeed, let us suppose that $m$ is odd. Then the equalities follow since for any $1 \leq j \leq m$ the integers $f_j$ and $\delta_{j+1}$ are relatively prime and $0 = r_{m+1} = \delta_{m+1}q - f_mp$.

**Lemma 4.6.** If $(f_j)_{-1 \leq j \leq m}$ and $(\delta_j)_{1 \leq j \leq m+1}$ are the finite sequences defined in Lemma [4.4], then for any $1 \leq j \leq m$ and any $1 \leq k \leq a_j$

$$e^T_r (B^r_{a_1 + \ldots + a_{j-1} + k}) = \begin{cases} (f_j - 2 + k \delta_j)p & \text{if } j \text{ is odd} \\ (f_j - 2 + k \delta_j)q & \text{if } j \text{ is even} \end{cases}$$
Proof. The proof proceeds by induction on \( j \) and \( k \). It is clear for \( j = 1 \) and any \( k \). Suppose that \( j \) is even, \( k < a_j \) and that \( e_r^T(B_{a_1 + \cdots + a_{j-1}+k}) = (f_{j-2} + k \delta_j)q \). We recall that \( B^T_\alpha \) is given by the Enriques diagram for which the weight of the point \( P_\alpha \) is \( w_\alpha = 1 \), the weights of all the points that do not precede \( P_\alpha \) are 0, and all the others are computed by imposing equalities in the proximity relations. Then

\[
B^T_{a_1 + \cdots + a_{j-1}+k+1} = B^T_{a_1 + \cdots + a_{j-1}+k} + W^T_{a_1 + \cdots + a_{j-1}+k+1} + B^T_{a_1 + \cdots + a_{j-1}}
\]

and

\[
e_r^T(B^T_{a_1 + \cdots + a_{j-1}+k+1}) = (f_{j-2} + k \delta_j)q + r_j + (f_{j-3} + a_{j-1} \delta_{j-1})p
\]

\[
= (f_{j-2} + k \delta_j)q + \delta_j q - f_{j-1} p + f_{j-1}p
\]

\[
= (f_{j-2} + (k+1) \delta_j)q.
\]

The argument is similar in all the other cases, i.e. when either \( k = a_j \) or \( j \) odd. \( \square \)

**Lemma 4.7.** For any \( 1 \leq j \leq m \) and any \( 2 \leq k \leq \) either \( a_j + 1 \) if \( j < m \) or \( a_m \) if not,

\[
w^T_1(B^T_{a_1 + \cdots + a_{j-1}+k}) = \begin{cases} \delta_{j-1} + k f_{j-1} & \text{if } j \text{ is odd} \\ f_{j-2} + k \delta_{j} & \text{if } j \text{ is even} \end{cases}
\]

**Proof.** We know the result for the last branch divisor \( B^T_\alpha \), where \( r = a_1 + \cdots + a_m \); we have \( w^T_1(B^T_\alpha) = p \). For an arbitrary \( B^T_\alpha \) we reduce the computation to this known situation.

Let us suppose that \( j \) is even, the argument being similar in the other case. Let \( T \) be the Enriques tree given by the first \( a_1 + \cdots + a_{j-1} + k \) vertices of \( T \) and let \( \tilde{p} < \tilde{q} \) be the relatively prime positive integers such that \( T = T_{\tilde{p}, \tilde{q}} \). By what we have just noticed,

\[
w^T_1(B^T_{a_1 + \cdots + a_{j-1}+k}) = w^T_1(B^T_{a_1 + \cdots + a_{j-1}+k}) = \tilde{p}.
\]

The sequences of Lemma 4.4 associated to \( \tilde{p} \) and \( \tilde{q} \) are given by \( f_{-1}, \ldots, f_{j-1} \) and \( \tilde{f}_j = f_{j-2} + k \delta_j \), and by \( \delta_{1}, \ldots, \delta_{j} \) and \( \tilde{\delta}_{j+1} = \delta_{j-1} + k f_{j-1} \). Here the \( f_i \)'s and the \( \delta_i \)'s are the elements of the sequences associated to \( p \) and \( q \). We end the proof using the remark 4.5. \( \square \)

**Lemma 4.8.** Let \( T \) and \( T' \) be two Enriques unibranch trees with \( r \) and respectively \( r' \) vertices. For every \( 1 \leq \beta \leq r' \) the following formula holds for the connected sum \( S = T \# T' \):

\[
B^S_{r+\beta-1} = w^T_1(B^T_\beta) (B^S_r - W^S_r) + \sum_{\alpha'=1}^{r'} w^T_\alpha (B^T_\beta) W^S_{r+\alpha'-1}.
\]
Proof. Using that \( w_{r+\alpha'}^{T'}(B_{r+\beta-1}^S) = w_{r'}^{T'}(B_{\beta}^S) \) for any \( 1 \leq \alpha' \leq r' - 1 \) and discarding the exponent \( S = T\#T' \), we have

\[
B_{r+\beta-1} = w_r(B_{r+\beta-1}) B_r + \sum_{\alpha' = 1}^{r'-1} w_{r+\alpha'}(B_{r+\beta-1}) W_{r+\alpha'}
\]

\[
= w_1^{T'}(B_{\beta}^S) B_r + \sum_{\alpha' = 1}^{r'-1} w_{\alpha'+1}^{T'}(B_{\beta}^S) W_{r+\alpha'}
\]

\[
= w_1^{T'}(B_{\beta}^S)(B_r - W_r) + \sum_{\alpha' = 1}^{r'-1} w_{\alpha'+1}^{T'}(B_{\beta}^S) W_{r+\alpha'-1}.
\]

Proof of Proposition 4.2. Let \( s = r + r' - 1 \). Clearly \( k_s^S = e_r(\omega^S) = e_r^{T'}(\omega^{T'}) \). As for \( k_s^S \), we have

\[
k_s^S = e_s(\omega^S)
\]

\[
= e_s(\sum_{\alpha = 1}^r W_{\alpha}) - e_s(W_r) + e_s(\sum_{\beta = r}^s W_{\beta})
\]

\[
= e_r^{T'}(e_r^{T'}(\omega^{T'})W_1^{T'}) - e_r^{T'}(W_1^{T'}) + e_r^{T'}(\omega^{T'})
\]

\[
= (e_r^{T'}(\omega^{T'}) - 1)p' + (p' + q' - 1).
\]

Now, if \( 1 \leq \alpha \leq r \), then

\[
e_r(r+\alpha-1)(B_{\alpha}) = e_r(r+\alpha-1)(e_r^{T'}(B_{\alpha}) W_r) = e_r^{T'}(B_{\alpha}) e_r(W_1^{T'}) = e_r^{T'}(B_{\alpha}) p'.
\]

The desired inequality is equivalent to \((q' - p')e_r^{T'}(B_{\alpha}) > 0\) which is satisfied. If \( \alpha = r + \beta - 1 \) with \( \beta \geq 2 \), then by Lemma 4.8

\[
e_r(r+\beta-1) = w_1^{T'}(B_{\beta}^S) e_r(B_r) = e_r^{T'}(B_{\beta}^S) w_1^{T'}(B_{\beta}^S)
\]

and

\[
e_r(r+\beta-1) = w_1^{T'}(B_{\beta}^S)(e_r^{T'}(B_{\beta}^S) - 1)e_r^{T'}(W_1^{T'}) + \sum_{\alpha' = 1}^{r'-1} w_{\alpha'+1}^{T'}(B_{\beta}^S) e_r^{T'}(W_{\alpha'+1}^{T'})
\]

\[
= (e_r^{T'}(B_{\beta}^S) - 1)p' w_1^{T'}(B_{\beta}^S) + e_r^{T'}(B_{\beta}^S).
\]

The inequality we want to prove becomes

\[
e_r^{T'}(B_{\beta}^S) w_1^{T'}(B_{\beta}^S) \geq \frac{(e_r^{T'}(B_{\beta}^S) - 1)p' w_1^{T'}(B_{\beta}^S) + e_r^{T'}(B_{\beta}^S)}{(e_r^{T'}(\omega^{T'}) + 1)p' + (q' - p')}
\]

and we shall use the lemmas 4.6 and 4.7 to establish it. Let \( \beta = \alpha' + \cdots + \alpha_{j-1} + k \) with \( 2 \leq k \leq \alpha_j' + 1 \). Let us suppose that \( j \) is odd—when \( j \) is even the argument is similar and simpler. The integer \( k \) is either \( \leq \alpha_j' \) or \( = \alpha_j' + 1 \). In the former case,

\[
w_1^{T'}(B_{\beta}^S) = \delta_j' - 1 + k f_{j-1}' \quad \text{and} \quad e_r^{T'}(B_{\beta}^S) = (f_{j-2}' + k \delta_j') p'.
\]
Inequality \[8\] is then equivalent to
\[9\]
\[e_r^T(B_r^T) (\delta_{j-1} + k f_{j-1}' (q' - p')) > (e_r^T(\omega^T) + 1) (- (\delta_{j-1} + k f_{j-1}' p' + (f_{j-2}' + k \delta_j) p')).\]

Since \(T\) contains at least one proper \(L\)-shaped branch \(e_r^T(B_r^T) > e_r^T(\omega^T) + 1\). So, to see that the inequality \[9\] is true, it is sufficient to show that
\[(\delta_{j-1} + k f_{j-1}') (q' - p') \geq - (\delta_{j-1} + k f_{j-1}') p' + (f_{j-2}' + k \delta_j) p',\]
or equivalently, that
\[\delta_{j-1} q' - f_{j-2}' p' \geq k(\delta_j p' - f_{j-1}' q').\]

By Lemma 4.4, this last inequality becomes \(r_{j-1}' \geq kr_{j}\) which is true since \(r_{j-1}' = a_j' r_j' + r_{j+1}'\). In the latter case, i.e. if \(k = a_j' + 1\), we have
\[\omega_r^T(B_{r_j}^T) = \delta_{j-1} + (a_j' + 1) f_{j-1}' = \delta_{j+1}' + f_{j-1}'\]
and
\[e_r^T(B_{r_j}^T) = (\delta_{j+1}' + f_{j-1}') q'.\]

The inequality \[8\] becomes equivalent to
\[\frac{e_r^T(B_r^T)}{e_r^T(\omega^T) + 1} > \frac{e_r^T(B_r^T) p' + (q' - p')}{(e_r^T(\omega^T) + 1) p' + (q' - p')},\]
or again to
\[(e_r^T(B_r^T) - e_r^T(\omega^T) - 1)(q' - p') > 0.\]

The first term is positive since \(T\) contains at least a proper \(L\)-shaped branch. \(\Box\)

5. The Result

The proof of the main result relies on the preceding investigations where we have established the relationship between the Enriques diagrams and the monomial ideals and computed the log-canonical threshold of unibranch Enriques diagrams.

**Theorem 5.1.** If \(C\) is a germ of curve with an isolated singularity at \(P\), then
\[
\lct(C; P) = \min \{\lct(a_{C, P}) \mid P \text{ adapted system of local parameters at } P \text{ for } C\}
\]
where \(a_{C, P}\) is the term ideal of \(C\) with respect to \(P\).

**Proof.** We start with a remark. Let \(T\) be the Enriques diagram corresponding to the minimal resolution of \(C\) and let \(P = (x, y)\) be an adapted system of parameters for \(C\). Let \(T(P)\) be the subdiagram of \(T\) determined as follows: If \(P_0\) is the highest free point on the coordinate curve \((y)\), then \(T(P)\) is the biggest binary Enriques subdiagram of \(T\) that contains as free points only the ones of the path from \(P_0\) to \(P_\rho\). Then
\[10\]
\[\mathfrak{a}_{C, P} = \mu_* \mathcal{O}_Y(-D_{T(P)}).\]

Indeed, \(\mathfrak{a}_{C, P}\) is the smallest integrally closed monomial ideal containing \(f\). It follows that \(\mathfrak{a}_{C, P} \subseteq \mu_* \mathcal{O}_Y(-D_{T(P)})\) since \(\mu_* \mathcal{O}_Y(-D_{T(P)})\) is monomial by Theorem 5.3. So \(T(P) \subseteq T_{\mathfrak{a}_{C, P}} \subseteq T\). By construction, the \(T(P)\) is the biggest binary Enriques subdiagram of \(T\) that contains as free points those between \(P_0\) and \(P_\rho\), hence \(T(P) = T_{\mathfrak{a}_{C, P}}\) and the assertion follows.
Second, let \( s(T) \) be a subdiagram of \( T \) whose Enriques tree is a path from the root to a leaf and that contains a vertex on which \( T \) realizes its log-canonical threshold. By Corollary 4.3 and descending induction

\[
lct(s(T)) = lct(s(T)^o),
\]

hence

\[
(11) \quad lct(T) = lct(s(T)) = lct(s(T)^o).\]

Here, the biggest non-degenerate subdiagram of \( T \) is denoted by \( T^o \).

To finish the proof, we consider for \( P \) a system of local parameters such that one coordinate contains the highest free point of \( s(T)^o \). By (10) and since \( s(T)^o \) is a subdiagram of \( T(P) \),

\[
lct(a_{C,P}) = lct(\alpha_{C,P}) = lct(\mu_* \mathcal{O}_Y(-DT(P))) = lct(T(P)) \leq lct(s(T)^o)\]

Using (11), \( lct(a_{C,P}) \leq lct(T) = lct(C; P) \) finishing the proof.

It has to be noticed that if \( f \) does not have non-degenerate principal part, then other jumping numbers may be different for \( f \) and for its term ideal as it will be seen below.

**Example (2.2 last time).** For \( f(x, y) = (x^3 - y^2)^2 - x^5 y \) we have seen that its Newton polygon is given by \( m/6 + n/4 - 1 \geq 0 \) and that its Enriques diagram is modeled on \( T_{2,3} # T_{2,3} \) with weights 4, 2, 2, 1, 1. We have the same log-canonical threshold for \( (f) \) and \( a_f \), but the next jumping number for \( a_f \) is 7/12 and the next jumping number for \( f \) is 15/26 < 7/12, showing that the singularity of \( (f) \) at the origin is worse than that defined by a general element of \( a_f \) as expected.

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