On the Existence of the Logarithmic Correction Term in Black Hole Entropy-Area Relation

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Abstract

In this paper we consider a model universe with large extra dimensions to obtain a modified black hole entropy-area relation. We use the generalized uncertainty principle to find a relation between the number of spacetime dimensions and the presence or vanishing of logarithmic prefactor in the black hole entropy-area relation. Our calculations are restricted to the microcanonical ensembles and we show that in the modified entropy-area relation, the microcanonical logarithmic prefactor appears only when spacetime has an even number of dimensions.

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1 Motivation

In Bekenstein-Hawking formalism of the black hole thermodynamics, entropy of the black hole is given by the following famous formula

$$S_{BH} = \frac{A}{4L_p^2}$$  \hspace{1cm} (1)

where $A$ is the cross-sectional area of the black hole horizon. Here spacetime is restricted to be a four dimensional manifold and the black hole is considered as a macroscopically large object. Originally, black hole entropy-area relation (1), was followed by some thermodynamical considerations. However, a statistical interpretation of the issue should be taken into account. A thorough statistical framework of the problem is provided by quantum gravity considerations. Several alternative approaches such as string theory, loop quantum gravity and noncommutative geometry have successfully been applied to find quantum gravitational corrections of black hole entropy-area relation. According to these approaches, a quantum corrected entropy of the black hole which may be observer independent, can be written as follows[1]

$$S_{BH} = \frac{A}{4L_p^2} + c_0 \ln(\frac{A}{4L_p^2}) + \sum_{n=1}^{\infty} c_n(\frac{A}{4L_p^2})^{-n} + \text{const.}$$  \hspace{1cm} (2)

The values of $c_0$ and $c_n$ are quantum gravity model dependent. $c_0$, which is called logarithmic prefactor, contains some information about details of the underlying quantum gravity proposal. A knowledge of its exact values in different approaches may reflect possible links between alternative quantum gravity scenarios[2]. Nevertheless, there are some controversies in existing literatures regarding the values of this prefactor even in a given scenario. Many authors have tried to determine the possible values of $c_0$. The matter which is obvious from the beginning is the fact that $c_0$ is an ensemble dependent quantity. Hod has employed statistical argument that constrain this prefactor to be a non-negative integer[3]. Medved has studied the quantum correction of microcanonical entropy of a fixed energy black hole[4]. He also has studied quantum corrections to the canonical entropy. He has argued that microcanonical entropy calculation leads to a logarithmic prefactor with nonzero value of $c_{0mc} = -\frac{3}{2}$ and the canonical entropy gives the value of $c_{0c} = \frac{3}{2}$. As a result, the total logarithmic prefactor will be zero due to mutual cancelation of microcanonical and canonical contributions. This conclusion is consistent with Hod’s result since it gives a non-negative integer. However, Medved in another paper
by considering some general considerations of ensemble theory, has argued that canonical and microcanonical corrections could not cancel out each other to result in vanishing logarithmic prefactor in entropy[5]. Recently, Medved and Vagenas have shown that a tunneling framework of the Hawking radiation, effectively constrains the coefficient of logarithmic term to be non-negative[6]. They have argued that this observation implies the necessity for including the canonical corrections in the quantum formulation of the black hole entropy.

There are other literatures considering logarithmic corrections to the black hole entropy-area relation (see for example[6-8]), but there is no explicit statement about the exact value of this prefactor and specially its dependence to spacetime dimensionality. There are several questions about the value of this prefactor. Some of these questions are: what is the role of the grand canonical ensemble? What is the exact role of spin, angular momentum, etc. in the black hole thermodynamics? Some of these questions have been discussed in literatures[4-6]. Here we won’t to answer these questions. What we want to do is just to show that from microcanonical point of view, there is a relation between spacetime dimensionality and the existence or vanishing of logarithmic prefactor. To elaborate our proposal, we calculate the quantum correction of entropy-area relation using the generalized uncertainty principle in a model universe with large extra dimensions. We use Arkani-Hamed, Dimpolous and Dvali(ADD) model of large extra dimensions to perform our calculations. In this model, $n$ extra spacelike dimensions with the same radius $L$ and without any curvature are suggested. All standard-model particles are confined to the observable 4-dimensional brane, whereas gravitons can access the whole $d$-dimensional bulk spacetime, being localized at the brane at low energies[9].

2 Thermodynamics of a d-Dimensional Schwarzschild Black Hole

The generalized uncertainty principle is a model independent aspect of quantum gravity proposal. It can be addressed in several approaches to quantum gravity and a nonzero minimal observable length naturally emerges from this proposal. Within string theory, investigation of string collisions at Planckian energies and through a renormalization group type analysis, the emergence of a minimal observable distance leads to the generalized
uncertainty principle[10]

\[\delta x \geq \frac{\hbar}{2} \left( \frac{1}{\delta p} + \alpha^2 \ell_p^2 \delta p \right). \tag{3}\]

Here \(\alpha\) is dimensionless, positive and independent of \(\delta x\) and \(\delta p\) but may in general depend on the expectation values of \(x\) and \(p\). It is on the order of unity and depends on the details of the quantum gravity proposal. At energies much below the Planckian energy, the extra term can be ignored and usual uncertainty principle of Heisenberg is recovered. But at the high energy regime, the extra term plays very important role: the appearance of this extra term leads to the finite resolution of spacetime points in high energy regime. In a model universe with large extra dimensions, our GUP can be written as

\[\delta x_i \delta p_i \geq \frac{\hbar}{2} \left( 1 + \frac{\alpha^2 L_p^2 (\delta p_i)^2}{\hbar^2} \right), \tag{4}\]

where \(L_p = (\frac{\hbar G_d}{c^3})^{\frac{1}{d-2}}\) is Planck length in extra dimensional scenario. \(d\) is the total number of spacetime dimensions, \(G_d = G_4 L^n\) where \(n = d - 4\) and \(L\) is the extension of \(n\) extra dimensions in ADD model[11]. Now, after a brief review of the preliminaries we calculate black hole temperature and entropy in which follows.

A d-dimensional spherically symmetric black hole of mass \(M\) (to which the collider black hole will settle into before radiating) is described by the following metric[12]

\[ds^2 = -\left( 1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} c^2 r^{d-3}} \right) c^2 dt^2 + \left( 1 - \frac{16\pi G_d M}{(d-2)\Omega_{d-2} c^2 r^{d-3}} \right)^{-1} r^2 dr^2 + r^2 d\Omega_{d-2}^2, \tag{5}\]

where \(\Omega_{d-2} = \frac{2\pi^{\frac{d-3}{2}}}{(d-2)\Gamma(\frac{d-1}{2})}\) is the metric of the unit \(S^{d-2}\) sphere. Since Hawking radiation is a quantum gravitational phenomenon, the emitted quanta from an evaporating black hole should be treated within the generalized uncertainty principle framework. If we simulate a black hole as a \((d-1)\) dimensional cube of size equal to its Schwarzschild radius \(r_s\), the uncertainty in the position of a Hawking particle at the emission is

\[\delta x_i \approx r_s = \omega_d L_p m^\frac{1}{d-3}, \tag{6}\]

where \(\omega_d = \left( \frac{16\pi}{(d-2)\Omega_{d-2}} \right)^\frac{1}{d-3}\), \(m = \frac{M}{M_p}\) and \(M_p = (\frac{\hbar}{c^{d-3} G_d})^{\frac{1}{d-2}}\). Here \(\omega_d\) is a dimensionless area factor. A simple calculation based on equation (4) leads to

\[\delta p_i = \frac{2\hbar \delta x_i \pm \sqrt{4\hbar^2 (\delta x_i)^2 - 4\hbar^2 \alpha^2 L_p^2}}{2\alpha^2 L_p^2}. \tag{7}\]
To find correct limiting result, we should consider the minus sign. The minimum position uncertainty can be obtained simply and the result is given by

$$(\delta x)_{\text{min}} = \alpha L_p.$$  \hfill (8)

Since

$$T_H^{\text{GUP}} = \frac{(d - 3)}{4\pi} c\delta p_i,$$  \hfill (9)

we find the following expression for the black hole temperature in a model universe with large extra dimensions

$$T_H^{\text{GUP}} = \frac{(d - 3)}{4\pi} \frac{hc\delta x_i}{\alpha^2 L_p^2} \left[ 1 - \sqrt{1 - \frac{\alpha^2 L_p^2}{\delta x_i^2}} \right],$$  \hfill (10)

where $\delta x_i$ is given by (6). This relation shows implicitly that black hole temperature increases with spacetime dimensions, $d$. The higher temperature yields to faster decay and less classical properties of the black hole. As a result, in models with large extra dimensions, black holes are hotter and shorter-lived[13].

Now we look at black hole entropy. When a quantum particle with energy $E$ and size $l$ is absorbed by a black hole, the minimum increase in the horizon area of the black hole can be expressed as follows

$$(\Delta A)_{\text{min}} \geq \frac{8 \ln 2L_p^{d-2}El}{(d-3)hc},$$  \hfill (11)

where by $E \sim c\delta p_i$ and $l \sim \delta x_i$, can be re-written as

$$(\Delta A)_{\text{min}} \geq \frac{8 \ln 2L_p^{d-2}c\delta p_i\delta x_i}{(d-3)hc}.$$  \hfill (12)

Using the value of $\delta p_i$ from (7), we obtain

$$(\Delta A)_{\text{min}} \geq \frac{8 \ln 2L_p^{d-4}\delta x_i^2}{(d-3)\alpha^2} \left[ 1 - \sqrt{1 - \frac{\alpha^2 L_p^2}{\delta x_i^2}} \right].$$  \hfill (13)

To calculate the microcanonical entropy of a large black hole, we put $\delta x_i \approx r_s$ and $r_s^2 = \Omega_{d-2} \frac{2^{d-1}}{\Gamma(\frac{d-1}{2})} A \frac{2^{d-2}}{\Gamma(\frac{d-2}{2})}$. If we set $(\Delta S)_{\text{min}} = b$, then we find

$$\frac{dS_d}{dA} \approx \frac{(\Delta S)_{\text{min}}}{(\Delta A)_{\text{min}}} \approx \frac{\frac{(d-3)b\alpha^2 \Omega_{d-2}^2}{8 \ln 2L_p^{d-4} A \frac{2^{d-2}}{\Gamma(\frac{d-2}{2})}}}{A \frac{2^{d-2}}{\Gamma(\frac{d-2}{2})} \left[ 1 - \sqrt{1 + \frac{-\alpha^2 L_p^2 \Omega_{d-2}^2}{A \frac{2^{d-2}}{\Gamma(\frac{d-2}{2})}} \right]}.$$  \hfill (14)
For simplicity, we define \( \lambda = \frac{(d-3)b\alpha^2\Omega_\frac{d}{d-2}}{8\ln 2\ell_p} \) and \( \eta = -\alpha^2\ell_p^2\Omega_\frac{d}{d-2} \). Thus we find

\[
\frac{dS_d}{dA} \simeq \frac{\lambda}{A^{\frac{d}{d-2}}} \left[ 1 - \sqrt{1 + \frac{\eta}{A^{\frac{d}{d-2}}}} \right],
\]

or in integral form

\[
S_d \simeq \lambda \int_{A_p}^A \frac{dA}{A^{\frac{d}{d-2}}} \left[ 1 - \sqrt{1 + \frac{\eta}{A^{\frac{d}{d-2}}}} \right],
\]

where \( A_p = \Omega_{d-2}(\alpha\ell_p)^{d-2} \) is the area of event horizon of black hole remnant. Existence of these remnants is a consequence of minimal length conjecture[14,15]. We apply a Taylor series expansion to obtain the following expression

\[
S_d \simeq -\frac{2\lambda}{\eta} \int_{A_p}^A dA \left[ 1 + \frac{\eta}{4} A^{-\frac{d}{d-2}} - \frac{\eta^2}{16} A^{-\frac{d}{d-2}} + \frac{\eta^3}{32} A^{-\frac{d}{d-2}} - \frac{5\eta^4}{256} A^{-\frac{d}{d-2}} + \frac{7\eta^5}{256} A^{-\frac{d}{d-2}} + \ldots \right].
\]

In which follows, we calculate \( S_d \) for some values of \( d \). Note that we choose \( b = \ln 2 \) (as one bit of information).

For \( d = 4 \) (our four-dimensional brane)

\[
S_4 \simeq \frac{A}{4\ell_p^2} - \frac{\pi\alpha^2}{4} \ln \frac{A}{4\ell_p^2} + \frac{\pi^2\alpha^4}{16} \left( \frac{4\ell_p^2}{A} \right) + \frac{\pi^3\alpha^6}{16} \left( \frac{4\ell_p^2}{A} \right)^2 + \frac{5\pi^4\alpha^8}{32} \left( \frac{4\ell_p^2}{A} \right)^3 - \frac{7\pi^5\alpha^{10}}{256} \left( \frac{4\ell_p^2}{A} \right)^4 + \ldots + C,
\]

where \( C \) is a constant. This relation has the standard form of entropy-area relation as given by string theory. The matter which should be stressed here is the fact that our calculation rules out the possibility of a power-law correction, that is, there are no correction terms proportional to \( \left( \frac{A}{4\ell_p^2} \right)^n \). On the other hand, our approaches give the value of \( c_0 = -\frac{\pi\alpha^2}{3} \) for logarithmic prefactor. Since \( \alpha \) is a string theory parameter related to the minimal observable length, \( (\Delta x)_{\text{min}} = \alpha\ell_p \), it is a positive quantity and therefore microcanonical prefactor is negative. If we accept the Hod’s prescription[3], we should add the positive contribution of the canonical ensemble to find non-negative prefactor. This result is consistent with the result of Medved and Vagenas[6] regarding the necessity for including the canonical corrections in the quantum formulation of the black hole entropy to find non-negative prefactor.

Now we consider the effect of the extra dimensions.
For $d = 5$, we obtain

$$
S_5 \simeq \frac{1}{2L_p^3} \left[ A - 1.19\pi^{4/3}\alpha^2 L_p^2 A^{1/3} + 0.47\pi^{8/3}\alpha^4 L_p^4 \frac{1}{A^{4/3}} + 0.13\pi^4\alpha^6 L_p^6 \frac{1}{A} + 0.07\pi^{16/3}\alpha^8 L_p^8 \frac{1}{A^{5/3}} 
- 0.12\pi^{20/3}\alpha^{10} L_p^{10} \frac{1}{A^{7/3}} + ... + C \right],
$$

which has no terms consisting $\ln A$ but contains some extraordinary powers of area. For $d = 6$, we find

$$
S_6 \simeq \frac{3}{4L_p^3} \left[ A - 0.81\pi\alpha^2 L_p^2 A^{1/2} - 0.17\pi^2\alpha^4 L_p^4 \ln \left( \frac{A}{4L_p^2} \right) + 0.27\pi^3\alpha^6 L_p^6 \frac{1}{A^{1/2}} + 0.035\pi^4\alpha^8 L_p^8 \left( \frac{4L_p^2}{A} \right) 
- 0.21\pi^5\alpha^{10} L_p^{10} \frac{1}{A^{3/2}} + ... + C \right],
$$

with an explicit logarithmic correction term and some extraordinary area dependent terms. For $d = 7$, we find

$$
S_7 \simeq \frac{1}{L_p^3} \left[ A - 0.42\pi^{6/5}\alpha^2 L_p^2 A^{3/5} - 0.31\pi^{12/5}\alpha^4 L_p^4 A^{1/5} + 0.16\pi^{18/5}\alpha^6 L_p^6 \frac{1}{A^{1/5}} + 0.03\pi^{24/5}\alpha^8 L_p^8 \frac{1}{A^{3/5}} 
- 0.03\pi^6\alpha^{10} L_p^{10} \frac{1}{A} + ... + C \right],
$$

without any logarithmic correction term. For $d = 8$, our calculations gives

$$
S_8 \simeq \frac{5}{4L_p^3} \left[ A - 0.38\pi\alpha^2 L_p^2 A^{2/3} - 0.19\pi^2\alpha^4 L_p^4 A^{1/3} - 0.03\pi^3\alpha^6 L_p^6 \ln \frac{A}{4L_p^2} + 0.06\pi^4\alpha^8 L_p^8 \frac{1}{A^{1/3}} 
- 0.05\pi^5\alpha^{10} L_p^{10} \frac{1}{A^{2/3}} + ... + C \right],
$$

with an explicit logarithmic correction term. For $d = 9$, we find

$$
S_9 \simeq \frac{3}{2L_p^3} \left[ A - 0.26\pi^{8/7}\alpha^2 L_p^2 A^{5/7} - 0.08\pi^{16/7}\alpha^4 L_p^4 A^{3/7} - 0.08\pi^{24/7}\alpha^6 L_p^6 A^{1/7} + 0.04\pi^{32/7}\alpha^8 L_p^8 \frac{1}{A^{1/7}} 
- 0.01\pi^{40/7}\alpha^{10} L_p^{10} \frac{1}{A^{4/7}} + ... + C \right],
$$

without any logarithmic correction term. And finally, for $d = 10$, we find

$$
S_{10} \simeq \frac{7}{4L_p^3} \left[ A - 0.25\pi\alpha^2 L_p^2 A^{3/4} - 0.07\pi^2\alpha^4 L_p^4 A^{1/2} - 0.05\pi^3\alpha^6 L_p^6 A^{1/4} - 0.01\pi^4\alpha^8 L_p^8 \ln \frac{A}{4L_p^2} 
- 0.02\pi^5\alpha^{10} L_p^{10} \frac{1}{A^{1/4}} + ... + C \right],
$$

(24)
which has a logarithmic correction term. Note that $L_p$ and $C$ have different values in different spacetime dimensions. From these relations, one can deduce the following conclusions

- In four dimensional spacetime (our brane), our approach gives the standard prescription of string theory as given by the relation (2) or
  \[
  S_4 = \frac{A}{4L_p^2} + c_0 \ln\left(\frac{A}{4L_p^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4L_p^2}\right)^{-n} + C. \tag{25}
  \]
  There is a logarithmic correction term with exact value of prefactor, $c_0 = -\frac{\pi \alpha'^2}{4}$. Our approach rules out the possibility of power-law expansion of correction terms in four dimensions.

- Comparison between the functional form of entropy in different spacetime dimensionalities, shows that the microcanonical logarithmic prefactor appears only in spacetimes with even number of dimensions. For spacetimes with odd dimensionality, there is no microcanonical logarithmic prefactor.

- For $d > 4$, we observe that some unusual powers of area have been appeared in entropy formula. These unusual terms are not consistent with general prescription as is described by relation (2). One may argue that entropy-area relation in extra dimensional scenarios do not obeys the prescription provided by (2). Even it is possible to have a power-law correction of entropy-area relation in extra dimensions. If we insist on the validity of the prescription (2), we should omit unusual terms in entropy-area relations for $d > 4$. This procedure leads to severe constraints on the functional form of modified dispersion relations (see [2] and references therein). But there is no obvious reason for omitting these terms in extra dimensional scenarios. The only statement that one can say is that the higher dimensional scenarios lead to entropy-area relations which have considerable departure from their four-dimensional counterpart.

- The existence or vanishing of logarithmic prefactor not only depends on the spacetime dimensionality but also depends on the statistical ensemble used in the course of the calculations. We have shown that this prefactor in the case of microcanonical ensemble, exists only when spacetime has an even number of dimensions and is negative. Comparison between our finding and the results of Hod and Medved, shows
that to have a non-negative prefactor, the contribution of the canonical ensemble should be positive. But this is not the complete argument of the issue since contributions of grand-canonical ensemble and spin effects should be taken into account. In this paper we have shown the explicit role of the microcanonical ensemble.

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