Simultaneous Confidence Band for Partially Linear Panel Data Models with Fixed Effects

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Abstract

In this paper, we construct the simultaneous confidence band (SCB) for the nonparametric component in partially linear panel data models with fixed effects. We remove the fixed effects, and further obtain the estimators of parametric and nonparametric components, which do not depend on the fixed effects. We establish the asymptotic distribution of their maximum absolute deviation between the estimated nonparametric component and the true nonparametric component under some suitable conditions, and hence the result can be used to construct the simultaneous confidence band of the nonparametric component. Based on the asymptotic distribution, it becomes difficult for the construction of the simultaneous confidence band. The reason is that the asymptotic distribution involves the estimators of the asymptotic bias and conditional variance, and the choice of the bandwidth for estimating the second derivative of nonparametric function. Clearly, these will cause computational burden and accumulative errors. To overcome these problems, we propose a Bootstrap method to construct simultaneous confidence band. Simulation studies indicate that the proposed Bootstrap method exhibits better performance under the limited samples.

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1 Introduction

In the literature, there were a large amount of studies about parametric linear and non-linear panel data models, and Arellano (2003), Baltigi (2005), and Hsiao (2003) had provided excellent overview of parametric panel data model analysis. To relax the strong restrictions assumed in the parametric panel data models, nonparametric and semiparametric panel data models have received a lot of attention in recent years. Compared to traditional parametric panel data model, nonparametric or semiparametric panel data models are better and more flexible to fit the actual data. Thus, this kind of models have become the hot research topic for the econometricians and statisticians. For example, Henderson, Carroll and Li (2008), and Li, Peng and Tong (2013) considered the fixed effects nonparametric panel data model. Henderson and Ullah (2005), Lin and Ying (2001), and Wu and Zhang (2002) considered the random effects nonparametric panel data models. Li and Stengos (1996) considered a partially linear panel data model with some regressors being endogenous via IV approach. Su and Ullah (2006) investigated the fixed effects partially linear panel data model with exogenous regressors. Zhang et al. (2011) considered the empirical likelihood inference for the fixed effects partially linear panel data model. Sun, Carroll and Li (2009) considered the problem of estimating a varying coefficient panel data model with fixed effects using a local linear regression approach. Chen, Gao and Li (2013a, 2013b) and Lai, Li and Lian (2013) studied the semiparametric estimation for a single-index panel data model, and among others.

Recently, the fixed effects models are frequently used in econometrics and biometrics. In this paper, we consider the following partially linear panel data models with fixed effects:

\[ Y_{it} = X_{it}' \beta + g(Z_{it}) + \alpha_i + V_{it}, \quad i = 1, \cdots, n, \quad t = 1, \cdots, T, \quad (1.1) \]

where \( \{X_{it}\} \) are \( p \times 1 \) vector of observable regressors, \( \{Z_{it}\} \) are explanatory variables in \([0,1]\), \( \beta \) is a \( p \times 1 \) vector of unknown coefficients, \( g(\cdot) \) is an unknown smooth function in \([0,1]\), \( \{V_{it}\} \) are random errors with zero mean, and \( \{\alpha_i\} \) are fixed effects. In addition, \( T \) is the time series length, \( n \) is the cross section size.
For model (1.1), we assume that \( \{\alpha_i\} \) are unobserved time-invariant individual effects. Model (1.1) is called as a partially linear fixed effects model if \( \{\alpha_i\} \) are correlated with \( \{X_{it}, Z_{it}\} \) with an unknown correlation structure. For identification purpose, we impose \( \sum_{i=1}^{n} \alpha_i = 0 \). An application of fixed effects models is the study of individual wage rate, \( \alpha_i \) represents different unobserved abilities of individual \( i \), such as the unmeasured skills or unobservable characteristics of individual \( i \), which maybe correlate with some observed covariates: age, educational level, job grade, gender, work experience and et al.. As a special case, when \( \{\alpha_i\} \) are uncorrelated with \( \{X_{it}, Z_{it}\} \), model (1.1) becomes a partially linear random effects model.

Baltagi and Li (2002) applied the first-order difference to eliminate the fixed effects and used the series method to estimate the parametric and nonparametric components, and they further established the asymptotic properties. Su and Ullah (2006) considered the estimation of partially linear panel data models with fixed effects. Zhang et al. (2011) applied the empirical likelihood method to model (1.1).

For the partially linear panel data models, the existing literatures considered the point-wise asymptotic normality of the estimator for the nonparametric component, and the result can be used to construct the pointwise confidence bands. In practice, we need to construct the simultaneous confidence band of the nonparametric function in the model. The simultaneous confidence band is a powerful tool to check the graphical representation of the nonparametric function during the practical applications. Therefore, there are extensive literatures on the construction of the simultaneous confidence band. For example, Fan and Zhang (2000), and Zhang and Peng (2010) considered the simultaneous confidence bands for the coefficient functions in varying-coefficient models; Li, Peng and Tong (2013) considered the simultaneous confidence band for nonparametric fixed effects panel data model; Li et al. (2014) and Yang et al. (2014) studied the simultaneous confidence band and hypothesis testing for the link function in single-index models, and more literatures see Yothers and Sampson (2011), Brabanter et al. (2012), Cao et al. (2012), Liu et al. (2013), and Li and Yang (2015).

In this paper, combining the idea of least-squares dummy-variable approach in parametric panel data models with the local linear regression technique in nonparametric models, we use the profile least-squares dummy-variable method proposed in Su and Ullah (2006) to remove the fixed effects, and further obtain the estimators of parametric and nonparametric components, which do not depend on the fixed effects. Under some
suitable conditions, we establish the asymptotic distribution of their maximum absolute deviation between the estimated nonparametric component and the true nonparametric component, and hence the result can be used to construct the simultaneous confidence band of the nonparametric component. In order to construct the simultaneous confidence band based on the asymptotic distribution, we first need to estimate the asymptotic bias and conditional variance, and choose the bandwidth for estimating the second derivative of nonparametric function. These will cause computational burden and accumulative errors, and it becomes difficult for the construction of the simultaneous confidence band. To overcome these problems, we further propose a Bootstrap method to construct the simultaneous confidence band of the nonparametric component in model (1.1).

The rest of the paper is organized as follows. In Section 2, we use the profile least-squares dummy-variable approach to obtain the estimators of the parametric and nonparametric components, and present the asymptotic properties. In Section 3, we propose the Bootstrap method to construct the simultaneous confidence band. In Section 4, simulation studies are used to illustrate the proposed method under the limited samples. The technical proofs of the main theorems are presented in the Appendix.

2 Estimation procedure and asymptotic properties

2.1 Estimation procedure

Let \( \{(Y_{it}; X_{it}^T, Z_{it}), i = 1, \cdots, n, t = 1, \cdots, T\} \) be an independent identically distributed (i.i.d.) random sample which comes from model (1.1). In this paper, we consider the asymptotic theories by letting \( n \) approach infinity and holding \( T \) fixed. In this section, we consider the estimation procedure to first remove the fixed effects, and further obtain the efficient estimators of parametric and nonparametric components.

For ease of notation, let

\[
Y = (Y_{11}, \cdots, Y_{1T}, Y_{21}, \cdots, Y_{2T}, \cdots, Y_{n1}, \cdots, Y_{nT})^T, \\
g = \left(g(Z_{11}), \cdots, g(Z_{1T}), g(Z_{21}), \cdots, g(Z_{2T}), \cdots, g(Z_{n1}), \cdots, g(Z_{nT})\right)^T, \\
V = (V_{11}, \cdots, V_{1T}, V_{21}, \cdots, V_{2T}, \cdots, V_{n1}, \cdots, V_{nT})^T, \\
\alpha_0 = (\alpha_1, \cdots, \alpha_n)^T
\]

and \( X = (X_{11}, \cdots, X_{1T}, X_{21}, \cdots, X_{2T}, \cdots, X_{n1}, \cdots, X_{nT})^T \) is an \( nT \times p \) matrix, where
$X_{it} = (X_{it1}, \cdots, X_{itp})^\tau$. Then model (1.1) can be written as the following matrix form,

$$Y = X\beta + g + (I_n \otimes e_T)\alpha_0 + V,$$

(2.1)

where $I_n$ is an $n \times n$ identity matrix, $e_T$ is a $T$-dimensional column vector with all elements being 1, and $\otimes$ denotes the Kronecker product. Furthermore, by the identification assumption $\sum_{i=1}^{n} \alpha_i = 0$, we have $\alpha_1 = -\sum_{i=2}^{n} \alpha_i$. Define the $(nT) \times (n-1)$ matrix $D = [-e_{n-1}, I_{n-1}]^\tau \otimes e_T$, and $\alpha = (\alpha_2, \cdots, \alpha_n)^\tau$, model (2.1) can be rewritten as

$$Y = X\beta + g + D\alpha + V.$$

(2.2)

Given $\alpha$ and $\beta$, model (2.2) is a version of the usual nonparametric fixed effects panel data model

$$Y - X\beta - D\alpha = g + V.$$

(2.3)

We first apply the local polynomial method (see the details in Fan and Gijbels, 1996) to estimate the nonparametric function $g(\cdot)$. For $Z_{it}$ in a small neighborhood of $z \in [0, 1]$, approximate $g(Z_{it})$ by

$$g(Z_{it}) \approx g(z) + g'(z)(Z_{it} - z).$$

(2.4)

Let $K(\cdot)$ is a kernel function in $\mathbb{R}$, $K_h(z) = K(z/h)/h$, where $h$ is a bandwidth, and let

$$Z_z = \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ Z_{11} - z & \cdots & Z_{1T} - z & \cdots & Z_{n1} - z & \cdots & Z_{nT} - z \end{pmatrix}^\tau,$$

$$W_z = \text{diag}(K_h(Z_{11} - z), \cdots, K_h(Z_{1T} - z), K_h(Z_{21} - z), \cdots, K_h(Z_{2T} - z), \cdots, K_h(Z_{n1} - z), \cdots, K_h(Z_{nT} - z))$$

is an $(nT) \times (nT)$ diagonal matrix. Let $G(z) = (g(z), (g'(z)))^\tau$, $\eta = (\alpha^\tau, \beta^\tau)^\tau$.

In what follows, we outline the estimation procedure for $\beta$ and $g(\cdot)$.

Given $\eta = (\alpha^\tau, \beta^\tau)^\tau$, we define the following weighted least-squares objective function

$$(Y - X\beta - Z_zG(z) - D\alpha)^\tau W_z (Y - X\beta - Z_zG(z) - D\alpha).$$

(2.5)

Minimizing the above objective function (2.5) with respect to $G(z)$, we can obtain the solution of $G(z)$ as follows

$$\tilde{G}(z, \eta) = (Z_z^\tau W_z Z_z)^{-1} Z_z^\tau W_z (Y - X\beta - D\alpha).$$

(2.6)
Define the smoothing operator by
\[
M(z) = (Z_z^r W_z Z_z)^{-1} Z_z^r W_z.
\]

Then, we can define the estimator of \( g(z) \) by
\[
\tilde{g}(z, \eta) = m^r(z)(Y - X\beta - D\alpha),
\]
where \( m^r(z) = e^r M(z) \), \( e = (1, 0)^T \) is a 2 \times 1 vector.

Since the fixed effects is an \( n \)-dimensional unobserved variable, it is difficult to obtain the consistent estimator for the fixed effects. Therefore, we first need to remove the fixed effects from the model, and further obtain the estimators of parametric and nonparametric components. By (2.7), we define the following objective function
\[
(Y - X\beta - \tilde{g}_\eta(z) - D\alpha)^T (Y - X\beta - \tilde{g}_\eta(z) - D\alpha) = (Y - \tilde{X}\beta - \tilde{D}\alpha)^T (Y - \tilde{X}\beta - \tilde{D}\alpha)
\]
where \( \tilde{g}_\eta(z) = (\tilde{g}(Z_{11}, \eta), \cdots, \tilde{g}(Z_{1T}, \eta), \cdots, \tilde{g}(Z_{n1}, \eta), \cdots, \tilde{g}(Z_{nT}, \eta), \tilde{Y} = (I_{nT} - M)Y, \tilde{X} = (I_{nT} - M)X, \tilde{D} = (I_{nT} - M)D, \tilde{Q} = I_{nT} - \tilde{D}(\tilde{D}^r \tilde{D})^{-1} \tilde{D}^r, \) and \( M \) is an \((nT) \times (nT)\) smoothing matrix, that is
\[
M = \begin{pmatrix}
(1, 0)(Z_{Z_{11}}^r W_{Z_{11}} Z_{Z_{11}})^{-1} Z_{Z_{11}}^r W_{Z_{11}} \\
\vdots \\
(1, 0)(Z_{Z_{1T}}^r W_{Z_{1T}} Z_{Z_{1T}})^{-1} Z_{Z_{1T}}^r W_{Z_{1T}} \\
\vdots \\
(1, 0)(Z_{Z_{nT}}^r W_{Z_{nT}} Z_{Z_{nT}})^{-1} Z_{Z_{nT}}^r W_{Z_{nT}}
\end{pmatrix}.
\]

In addition, let \( P = (I_{nT} - M)^r (I_{nT} - M) \) be an \((nT) \times (nT)\) matrix.

Taking derivative of (2.8) with respect to \( \alpha \) and setting it equal to zero, we have
\[
\tilde{\alpha}(\beta) = (\tilde{D}^r \tilde{D})^{-1} \tilde{D}^r (\tilde{Y} - \tilde{X}\beta).
\]

Obviously, the estimator of the fixed effects depends on \( \beta \). Based on the idea of least-squares dummy-variable approach in panel data parametric models and the nonparametric local linear regression technique, we then apply the profile least-squares dummy variable method to estimate parameter vector \( \beta \).
Plugging (2.9) into (2.8), we then minimize the profile least-squares objective function with respect to \( \beta \). Thus, we obtain the profile least-squares estimator of \( \beta \) as

\[
\hat{\beta} = (\hat{X}^\top \hat{Q} \hat{X})^{-1} \hat{X}^\top \hat{Q} \hat{Y}.
\]

(2.10)

By (2.10) and (2.9), we have

\[
\hat{\alpha} = (\hat{\alpha}_2, \cdots, \hat{\alpha}_n) = (\hat{D}^\top \hat{D})^{-1} \hat{D}^\top (\hat{Y} - \hat{X} \hat{\beta}).
\]

(2.11)

By \( \sum \alpha_i = 0 \) and (2.11), the estimator of \( \alpha_1 \) is \( \hat{\alpha}_1 = -\sum_{i=2}^n \hat{\alpha}_i \).

By (2.6), (2.10) and (2.11), and some simple calculations, we can obtain the estimator of \( G(z) \) as follows

\[
\hat{G}(z) = \hat{G}(z, \hat{\eta}) = M(z)(Y - X \hat{\beta} - D \hat{\alpha})
\]

\[
= M(z)[Y - X \hat{\beta} - D(\hat{D}^\top \hat{D})^{-1} \hat{D}^\top (\hat{Y} - \hat{X} \hat{\beta})]
\]

\[
= M(z)(I_{nT} - D(D^\top PD)^{-1}D^\top P)(Y - X \hat{\beta}).
\]

(2.12)

By (2.7) and (2.12), we get the estimator of \( g(z) \) as

\[
\hat{g}(z) = m^\top(z)(I_{nT} - D(D^\top PD)^{-1}D^\top P)(Y - X \hat{\beta}).
\]

(2.13)

**Remark 1.** From (2.10) and (2.13), it is easy to see that the estimators of \( \beta \) and \( g(\cdot) \) do not depend on the fixed effects.

### 2.2 Asymptotic properties

Let \( \mu_l = \int z^l K(z) dz \) and \( \nu_l = \int z^l K^2(z) dz \) for \( l = 0, 1, 2 \). Define the observed covariate set by \( D = \{X_{it}, Z_{it}, 1 \leq i \leq n, 1 \leq t \leq T \} \). In order to obtain the main results, we first present the following technical conditions.

(C1) \( (\alpha_i, V_i, X_i, Z_i)_i, i = 1, \cdots, n, \) are i.i.d., where \( V_i = (V_{i1}, V_{i2}, \cdots, V_{iT})^\top \), and \( X_i \) and \( Z_i \) can be defined similarly. \( E\|X_it\|^{2+\delta} < \infty \) and \( E\|V_{it}\|^{2+\delta} < \infty \) for some \( \delta > 0 \). Let \( \sigma^2(x, z) = \text{Var}(Y_{it} | X_{it} = x, Z_{it} = z), \sigma^2(z) = \text{Var}(Y_{it} | Z_{it} = z), \) and \( 0 < \sigma^2(x, z), \sigma^2(z) < \infty \).

(C2) \( E(Y_{it} | X_i, Z_{it}, \alpha_i) = E(Y_{it} | X_{it}, Z_{it}, \alpha_i) = X_{it}^\top \beta + g(Z_{it}) + \alpha_i, i = 1, \cdots, n, t = 1, \cdots, T \).

(C3) Let \( f(z) = \sum_{t=1}^T f_t(z) \), where \( f_t(z) \) is the continuous density function of \( Z_{it}, \) and \( f_t(z) \) is bounded away from zero and infinity on \([0, 1]\) for each \( t = 1, \cdots, T \). Let \( \tilde{V}_{it} = V_{it} - \frac{1}{T} \sum_{s=1}^T V_{is}, \sigma_t^2(z) = E[\tilde{V}_{it}^2 | Z_{it} = z] \) and \( \sigma^2(z) = \sum_{t=1}^T \sigma_t^2(z) f(z) \).
(C4) Let \( p(z) = E(X_{it} | Z_{it} = z) \). The functions \( g(\cdot) \) and \( p(\cdot) \) have the bounded and continuous second derivatives on \([0, 1] \).

(C5) The kernel function \( K(\cdot) \) is a symmetric density function, and is absolutely continuous on its support set \([-A, A] \).

(C5a) \( K(A) \neq 0 \) or

(C5b) \( K(A) = 0 \), \( K(t) \) is absolutely continuous and \( K^2(t), [K'(t)]^2 \) are integrable on \(( -\infty, +\infty ) \).

(C6) The bandwidth \( h \) satisfies that \( nh^3 \log n \to \infty \), \( nh^5 \log n \to 0 \), as \( n \to \infty \).

**Theorem 1.** Assume that conditions (C1)–(C6) hold. Let \( b(z) = h^2 \nu_2 g''(z)/2, \Sigma_g = \nu_0 \sigma^2(z) f^{-2}(z), \Sigma_{g'} = \nu_2 \sigma^2(z)/(f^2(z) \mu_2^2) \), Then uniformly for \( z \in [0, 1] \), we have

\[
\| \hat{\beta} - \beta \| = O_p(n^{-1/2})
\]

and

\[
\sqrt{nh} \{ \hat{g}(z) - g(z) - b(z) \} \overset{L}{\to} N(0, \Sigma_g),
\]

\[
\sqrt{nh^3} \{ \hat{g}'(z) - g'(z) \} \overset{L}{\to} N(0, \Sigma_{g'}),
\]

where \( \overset{L}{\to} \) denotes the convergence in distribution.

**Theorem 2.** Assume that conditions (C1)–(C6) hold and \( h = O(n^{-\rho}) \) for \( 1/5 \leq \rho < 1/3 \). Then for all \( z \in [0, 1] \), we have

\[
P \left\{ \sup_{z \in [0,1]} \left| \left( n h \Sigma_g^{-1} \right)^{1/2} (\hat{g}(z) - g(z) - b(z)) - d_n \right| < u \right\} \to \exp \left( -2 \exp(-u) \right), \text{ as } n \to \infty,
\]

where if \( K(A) \neq 0 \),

\[
d_n = (-2 \log h)^{1/2} + \frac{1}{(-2 \log h)^{1/2}} \left\{ \log \frac{K^2(A)}{\nu_0 \pi^{1/2}} + \frac{1}{2} \log \log h^{-1} \right\},
\]

and if \( K(A) = 0 \),

\[
d_n = (-2 \log h)^{1/2} + \frac{1}{(-2 \log h)^{1/2}} \log \left\{ \frac{1}{4 \nu_0 \pi} \int (K'(z))^2 \, dz \right\}.
\]

Theorem 2 gives the asymptotic distribution of the maximum absolute deviation between the estimated nonparametric component \( \hat{g}(\cdot) \) and the true nonparametric component \( g(\cdot) \) when the estimator of \( \beta \) is \( \sqrt{n} \)-consistent. It provides us the theoretical foundation for constructing the simultaneous confidence band of the nonparametric function in model (1.1).
Remark 2. If the supremum in Theorem 2 is taken on an interval of $[c, d]$ instead of $[0, 1]$, Theorem 2 still holds under certain conditions by transformation. The asymptotic distribution is represented as
\[
P \left\{ \frac{1}{2} \left( 2 \log \frac{h}{(d - c)} \right)^{1/2} \left( \sup_{z \in [c, d]} \left| (n h \Sigma_g^{-1})^{1/2} (\hat{g}(z) - g(z)) \right| - \tilde{d}_n \right) < u \right\} \rightarrow \exp \left( -2 \exp (-u) \right),
\]
where $\tilde{d}_n$ is the same as $d_n$ in the Theorem 2 except that $h$ is replaced by $h/(d - c)$.

Theorem 3. Assume that conditions (C1)-(C6) hold and $\Sigma_g = \nu_2 \delta^2(z)/(f^2(z) \mu_{2}^{2})$. Then for all $z \in [0, 1]$, we have
\[
P \left\{ \frac{1}{2} \left( 2 \log \frac{h}{(d - c)} \right)^{1/2} \left( \sup_{z \in [0, 1]} \left| (n h^3 \Sigma_g^{-1})^{1/2} (\hat{g}'(z) - g'(z)) \right| - d_{n_{1}} \right) < u \right\} \rightarrow \exp \left( -2 \exp (-u) \right), \quad \text{as } n \rightarrow \infty,
\]
where $d_{n_{1}} = (2 \log h)^{1/2} + \frac{1}{(2 \log h)^{1/2}} \log \left\{ \frac{1}{2 \pi \sqrt{2}} \left( \int z^2 (K'(z))^2 dz \right)^{1/2} \right\}$. If $K(0) = 0$, $K(z)$ is absolutely continuous and $K^2(z)$, $(K'(z))^{2}$ are integrable on $(-\infty, +\infty)$.

Theorem 3 presents the asymptotic distribution of the maximum absolute deviation for $\hat{g}'(\cdot)$

2.3 Simultaneous confidence band for the nonparametric function

Since the asymptotic bias and variance of $\hat{g}(\cdot)$ in Theorem 2 involve some unknown quantities, we cannot apply Theorem 2 to construct simultaneous confidence band of $g(\cdot)$ directly. In order to construct the simultaneous confidence band of $g(\cdot)$, we first need to get the consistent estimators of the asymptotic bias and variance of $\hat{g}(\cdot)$. By Theorem 1, the asymptotic bias of $\hat{g}(z)$ is
\[
(h^2 \mu_2/2) g''(z) (1 + o_p(1)).
\]
Thus, the consistent estimator of the asymptotic bias is $\hat{\text{bias}}(\hat{g}(z)) = h^2 \mu_2 \hat{g}''(z)/2$, where the estimator $\hat{g}''(z)$ of $g''(z)$ is obtained by using local cubic fit with an appropriate pilot bandwidth $h_\ast = O(n^{-1/7})$, which is optimal for estimating $g''(z)$ and can be chosen by the residual squares criterion proposed in Fan and Gijbels (1996).

Next we will estimate the asymptotic variance of $\hat{g}(z)$. For simplicity, suppose that the random errors $V_{it}$ are i.i.d. for all $i$ and $t$. By the proofs of theorem, we have
\[
\text{Var}\{\hat{g}(z)|D\} = (1, 0)(Z_z^\top W_z Z_z)^{-1}(Z_z^\top W_z Q_1 Q_1^\top W_z Z_z)(Z_z^\top W_z Z_z)^{-1}(1, 0)^\top,
\]
where $Q_1 = (I_{nT} - D(D^\top PD)^{-1}D^\top P)$ and $\Phi_1 = \text{diag}(\sigma^2(Z_{11}), \ldots, \sigma^2(Z_{1T}), \sigma^2(Z_{21}), \ldots, 
abla^2(Z_{2T}), \ldots, \sigma^2(Z_{n1}), \ldots, \sigma^2(Z_{nT}))$. Using the similar approximate local homoscedasticity in Li, Peng and Tong (2013), the asymptotic variance of $\hat{g}(z)$ is defined by

$$
\text{Var}\{\hat{g}(z)|D\} = (1, 0)\left(Z_z^\top W_z Z_z\right)^{-1}(Z_z^\top W_z Q_1 W_z Z_z)\left(Z_z^\top W_z Z_z\right)^{-1}(1, 0)^\top \sigma^2(z).
$$

Let $\hat{V} = Y - \hat{Y}$ be the residual, where $\hat{Y} = \hat{g} + X\hat{\beta} + D\hat{\alpha}$. By (2.10), (2.11) and (2.13), we have

$$
\hat{V} = Y - \hat{g} - X\hat{\beta} - D\hat{\alpha}
= Y - X\hat{\beta} - D\hat{\alpha} - M(Y - X\hat{\beta} - D\hat{\alpha})
= (I_{nT} - M)(Y - X\hat{\beta} - D\hat{\alpha})
= (I_{nT} - M)(I_{nT} - D(D^\top PD)^{-1}D^\top P)(Y - X\hat{\beta})
= (I_{nT} - M)Q_1(I_{nT} - X(X^\top PQ_1 X)^{-1}X^\top PQ_1)Y
= (I_{nT} - M)Q_1 Q_2 Y,
$$

(2.14)

where $Q_2 = I_{nT} - X(X^\top PQ_1 X)^{-1}X^\top PQ_1$. Obviously, the residual $\hat{V}$ does not depend on the fixed effects, and is a linear function of $Y$. By the normalized weighted residual sum of squares, $\sigma^2(z)$ can be estimated by

$$
\hat{\sigma}^2(z) = \frac{\hat{V}^\top \hat{V}}{\text{tr}(Q_2^\top Q_1^\top PQ_1 Q_2)} = \frac{Y^\top (Q_2^\top Q_1^\top PQ_1 Q_2) Y}{\text{tr}(Q_2^\top Q_1^\top PQ_1 Q_2)}.
$$

**Theorem 4.** Under the conditions in Theorem 2, and assume that $\hat{g}^{(3)}(\cdot)$ is continuous on $[0, 1]$ and the pilot bandwidth $h_\star$ satisfies that $h_\star = O(n^{-1/7})$. Then for all $z \in [0, 1]$, we have

$$
P\left\{(-2 \log h)^{1/2}\left(\sup_{z \in [0, 1]} \left| \frac{\hat{g}(z) - g(z) - \text{bias}(\hat{g}(z)|D)}{\text{Var}(\hat{g}(z)|D)} \right| - d_n \right) < u \right\} \longrightarrow \exp\left(-2 \exp(-u)\right),
$$

where $d_n$ is defined in Theorem 2.

By Theorem 4, we construct the $(1 - \alpha) \times 100\%$ simultaneous confidence band of the nonparametric function $g(z)$ as

$$
\left(\hat{g}(z) - \text{bias}(\hat{g}(z)|D) \pm \Delta_{1,\alpha}(z)\right),
$$

(2.15)

where $\Delta_{1,\alpha}(z) = (d_n + [\log 2 - \log\{-\log(1 - \alpha)\}](-2 \log h)^{-1/2})\left[\text{Var}(\hat{g}(z)|D)\right]^{1/2}$. 

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3 The Bootstrap method

Despite the fact that Theorem 4 provides the asymptotic distribution to construct the simultaneous confidence band (2.15) for the nonparametric component, we need to estimate the asymptotic bias and the asymptotic conditional variance. First, the estimator of the asymptotic bias involves the estimator the second derivative \( g''(\cdot) \) and the choice of the pilot bandwidth \( h_* \) for estimating the second derivative \( g''(\cdot) \). The estimator of the second derivative \( g''(\cdot) \) has a slow convergence rate, and is very sensitive with the pilot bandwidth \( h_* \). This will influence the estimator of the asymptotic bias. Second, the asymptotic variance estimation is very complicated, especially for panel data semiparametric fixed effects model. Finally, the asymptotic critical value \( c_\alpha \) depends on the double exponential distribution, the estimators of asymptotic bias and the asymptotic conditional variance. These will not only cause computational burden and accumulative errors, but also lead to the difficulty to construct simultaneous confidence band. To overcome these problems, we extend the Bootstrap method used in Li, Peng and Tong (2013) to partially linear panel data fixed effects model (1.1).

Now we discuss how to use the Bootstrap procedure to construct simultaneous confidence band for \( g(\cdot) \). Let

\[
T = \sup_{z \in [0,1]} \frac{|\hat{g}(z) - g(z)|}{\text{Var}(\hat{g}(z|D))^{1/2}}.
\]

Suppose that the upper \( \alpha \) quantile of \( T \) is \( c_\alpha \). If \( c_\alpha \) and \( \text{Var}(\hat{g}(z|D)) \) are known, the simultaneous confidence band of \( g(\cdot) \) with \( (1 - \alpha) \times 100\% \) on the interval \([0, 1]\) should be

\[
\hat{g}(z) \pm \{\text{Var}(\hat{g}(z|D))\}^{1/2} c_\alpha.
\]

However, \( c_\alpha \) and \( \text{Var}(\hat{g}(z|D)) \) are unknown. We will get their estimators using the bootstrap method. Suppose that we have the estimators \( \hat{c}_\alpha \) and \( \text{Var}^*(\hat{g}(z|D)) \) of \( c_\alpha \) and \( \text{Var}(\hat{g}(z|D)) \), respectively. Then we can obtain the \( (1 - \alpha) \times 100\% \) simultaneous confidence band of \( g(\cdot) \) as follows

\[
\hat{g}(z) \pm \{\text{Var}^*(\hat{g}(z|D))\}^{1/2} \hat{c}_\alpha.
\]

The Bootstrap procedure is given as follows:

1. By (2.14), obtain the residuals \( \hat{V} = (I_nT - M)Q_1Q_2Y \), where \( \hat{V} = (\hat{V}_{11}, \ldots, \hat{V}_{1T}, \hat{V}_{21}, \ldots, \hat{V}_{2T}, \ldots, \hat{V}_{n1}, \ldots, \hat{V}_{nT})^r \).

2. For each \( i = 1, \ldots, n, \ t = 1, \cdots, T \), obtain the bootstrap error \( V^*_{it} = \hat{V}_{it}\varepsilon_{it} \), where
\(\epsilon_{it}\) are i.i.d. \(\sim N(0,1)\) across \(i\) and \(t\). Generate the bootstrap sample member \(Y_{it}^*\) by
\[Y_{it}^* = \hat{Y}_{it} + V_{it}^*, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T.\]

(3) Given the bootstrap resample \(\{(Y_{it}^*, X_{it}, Z_{it}), i = 1, \ldots, n, \quad t = 1, \ldots, T\}\), obtain the estimators of \(\beta\) and \(g(\cdot)\), and denote the resulting estimate by \(\hat{\beta}^*\) and \(\hat{g}^*(\cdot)\), as the bootstrap estimators of \(\beta\) and \(g(\cdot)\), respectively.

(4) Repeat (2)–(3) \(N\) times to get a size \(N\) bootstrap sample of \(g(\cdot)\), \(\hat{g}^*_k(\cdot), k = 1, \ldots, N\). The estimator \(\text{Var}^*(\hat{g}(z))\) of \(\text{Var}(\hat{g}(\cdot))\) is taken as the sample variance of \(\hat{g}^*_k(\cdot)\).

(5) Compute the bootstrap sample of \(T\) by
\[T_k^* = \sup_{z \in [0,1]} \frac{|\hat{g}^*_k(z) - \hat{g}(z)|}{\left\{\text{Var}^*(\hat{g}(z|D))\right\}^{1/2}}, \quad k = 1, \ldots, N.\]
Use the upper \(\alpha\) percentile \(\hat{c}_\alpha\) of \(T_k^*, k = 1, \ldots, N\), to estimate the upper \(\alpha\) quantile \(c_\alpha\) of \(T\).

We can construct the \((1 - \alpha) \times 100\%\) simultaneous confidence band of \(g(\cdot)\) by (3.1) when we obtain the estimators of \(c_\alpha\) and \(\text{Var}(\hat{g}(z|D))\).

4 Simulation studies

We conduct simulation studies to assess the performance of our proposed method. Our simulated data are generated from the following model:
\[Y_{it} = X_{it}'\beta + 0.8 \cos(\pi Z_{it}) + \alpha_i + V_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \quad (4.1)\]
where \(\beta = (-1, 3, 5)'\), \(X_{it}\) are three dimensional i.i.d. random variables from uniform \([-1,1]\), \(Z_{it}\) are i.i.d. from uniform \([-1,1]\), and the random errors \(V_{it}\) are i.i.d. from \(N(0,1)\).

In this simulation, we only consider \(\alpha_i\) are correlated with the covariate \(Z_{it}\), and generate \(\alpha_i = \epsilon_i + cZ_{it}, i = 2, \ldots, n\), where \(\epsilon_i \sim N(0,1)\), \(Z_{it} = \frac{1}{T} \sum_{t=1}^{T} Z_{it}\) and \(\alpha_1 = - \frac{n}{n} \sum_{i=2}^{n} \alpha_i, i = 1, \ldots, n\). We consider three cases for \(c = 0, 0.5, 1\). When \(c \neq 0\), \(Z_{it}\) and \(\alpha_i\) are correlated, model (4.1) is the partially linear fixed effects model. When \(c = 0\), model (4.1) leads to the usual partially linear random effects model.

In our simulation studies, we apply the Epanechnikov kernel \(K(z) = 0.75(1 - z^2)_+\) for estimating the nonparametric function. Finding an appropriate bandwidth can be of both theoretical and practical interest. To implement the estimation procedure described in Section 2, we need to choose the bandwidth \(h\). One can select \(h\) by minimizing the generalized cross validation criterion. Here we use the following cross validation method
to automatically select the optimal bandwidth \( h_{CV} \).

\[
CV(h) = \sum_{i=1}^{n} \sum_{t=1}^{T} (Y_{it} - \hat{Y}_{it})^2 = \sum_{i=1}^{n} \sum_{t=1}^{T} \left( \frac{Y_{it} - \hat{Y}_{it}}{1 - l_{kk}} \right)^2 = \sum_{i=1}^{n} \sum_{t=1}^{T} \left( \frac{\hat{V}_{it}}{1 - l_{kk}} \right)^2, \quad (4.2)
\]

where \( Y_{it} \) denote the fitted values that are computed from data with measurements of the \( \{Y_{it}, X_{it}\} \) observation deleted. \( k = (i - 1)T + t \), \( \hat{V}_{it} = Y_{it} - \hat{Y}_{it} \) and \( l_{kk} \) is the \((k,k)\) element of matrix \([I_{nT} - (I_{nT} - M)Q_1Q_2]\). The cross validation bandwidth \( h_{CV} \) is then defined to be the minimizer of \( CV(h) \).

We fix \( T = 5 \) and examine the finite sample performance of the proposed method when the sample size is taken as \( n = 100, 150 \) and \( 200 \). For each case, 1000 replicates of simulated realizations are generated, and the nominal level is taken as \( 1 - \alpha = 0.95 \). The results are given in Tables 1–2 and Figure 1. Table 1 gives the bias, the standard deviation and the mean squared error of the estimator \( \hat{\beta} \) for \( c = 0 \) and \( c = 1 \). From Table 1, we can find that the bias, the standard deviation and the mean squared error are decreased as the sample size \( n \) increases for two cases. For the same sample size \( n \), the results of \( c = 1 \) are better than those of \( c = 0 \). Model (4.1) is reduced to partially linear random effects model when \( c = 0 \). From (2.10) and (2.13), it is easy to see that, in order to remove the fixed effects from the model, we loss some sample information to obtain the estimators of parametric and nonparametric components. So the profile least-squares dummy-variable method is not suitable for the partially linear random effects model, and the resulting estimators of parametric and nonparametric components are not efficient. Thus, we need develop the effective estimation procedure to estimate the random effects models, such as the generalized profile least squares method or the generalized estimating equation (GEE).

Based on the asymptotic distribution and the Bootstrap method, Table 2 gives the average probabilities of the simultaneous confidence band for the nonparametric function \( g(\cdot) \) when the nominal level is \( 1 - \alpha = 0.95 \), where “method one” denotes the method based on asymptotic distribution and “Bootstrap” denotes the method based on the Bootstrap procedure in Table 2. For the bootstrap procedure, we use \( M = 200 \) bootstrap replications to estimate \( c_\alpha \) and \( \text{Var}(\hat{g}(z|D)) \).

From Table 2, it is easy to see that the average coverage probabilities of the simultaneous confidence band for the nonparametric function obtained by the two methods tend to 0.95 as the sample size \( n \) increases for three cases. When \( c = 0 \), the average coverage probabilities are lower than those of \( c = 0.5 \) and 1. In addition, we also can find that the
Table 1: The bias, standard deviation (SD) and mean squared error (MSE) of $\hat{\beta}$

|       | $c = 0$ | $c = 1$ |
|-------|---------|---------|
| $\hat{\beta}$ | 100 | 150 | 200 | 100 | 150 | 200 |
| Bias | 0.0063 | 0.0059 | 0.0048 | 0.0045 | 0.0046 | 0.0023 |
| $\hat{\beta}_1$ SD | 0.0859 | 0.0720 | 0.0682 | 0.0841 | 0.0647 | 0.0635 |
| $\hat{\beta}_1$ MSE | 0.0074 | 0.0052 | 0.0046 | 0.0071 | 0.0042 | 0.0040 |
| Bias | 0.0057 | 0.0046 | 0.0031 | 0.0053 | 0.0027 | 0.0022 |
| $\hat{\beta}_2$ SD | 0.0901 | 0.0696 | 0.0620 | 0.0906 | 0.0687 | 0.0601 |
| $\hat{\beta}_2$ MSE | 0.0081 | 0.0049 | 0.0038 | 0.0082 | 0.0048 | 0.0036 |
| Bias | 0.0062 | 0.0049 | 0.0042 | 0.0041 | 0.0029 | 0.0026 |
| $\hat{\beta}_3$ SD | 0.0912 | 0.0770 | 0.0650 | 0.0857 | 0.0679 | 0.0545 |
| $\hat{\beta}_3$ MSE | 0.0083 | 0.0059 | 0.0042 | 0.0074 | 0.0046 | 0.0031 |

Table 2: Coverage probabilities of nonparametric component with the nominal level 95%

|       | $n$ | $c = 0$ | $c = 0.5$ | $c = 1$ |
|-------|-----|---------|-----------|---------|
|        | 100 | 0.926 | 0.933 | 0.941 |
| method one | 150 | 0.933 | 0.940 | 0.949 |
|         | 200 | 0.946 | 0.951 | 0.953 |
| Bootstrap | 100 | 0.928 | 0.934 | 0.942 |
|        | 150 | 0.937 | 0.946 | 0.950 |
|         | 200 | 0.948 | 0.952 | 0.954 |

average coverage probabilities based on the asymptotic distribution is lower than those of the Bootstrap method, which implies that the Bootstrap method performs better than the asymptotic distribution method. The reason is that the Bootstrap method avoids estimating the asymptotic bias and variance and reduces the computational burden and accumulative errors.

Based on the asymptotic distribution and the Bootstrap method, Figure 1 gives the 95% pointwise confidence bands of $g(\gamma)$ for $n = 100, 150, 200$ and $c = 0, 0.5, 1$, respectively. Figure 1 reveals that the performance of asymptotic confidence bands is not worse than that based on the bootstrap procedure. In addition, the confidence bands obtained by the two methods become narrow as the sample size $n$ increases for three cases. From Table
2 and Figure 1, it is easy to observe that, although the bootstrap method works better than the method based on asymptotic distribution, the proposed asymptotic distribution method is comparable with the bootstrap method.

5 Appendix: proofs of the main results

Let \( P = (I_{nT} - M)^\tau (I_{nT} - M) \) and \( \Phi = \sum_{t=1}^{T} \sum_{s=1}^{T} E\{ (\widetilde{X}_{it} - \overline{X}_{in} - \sum_{t=1}^{T} \widetilde{X}_{it} / T)^\tau V_{it} V_{is} \} \). The following Lemmas 1–5 play a very important role in proving the main results of Theorems 1–4, and the details of proofs can be found in Su and Ullah (2006) and Zhang et al. (2011), we omit the details here.

**Lemma 1.** Assume that conditions (C1)–(C6) hold. Let \( C \) be a positive constant and \( m(Z_{it}, z) = e^{\tau^\top (Z_{it} W_z Z_z)^{-1} Z_{it} K_h(Z_{it} - z)} \), where \( Z_{it} \) is a typical column of \( Z_z \), we have

(i) \( m(Z_{it}, z) = n^{-1} K_h(Z_{it} - z) f^{-1}(z) \{ 1 + o_p(1) \} \), where \( f(z) = \sum_{t=1}^{T} f_t(z) \);

(ii) \( \lim_{n \to \infty} P_n \left\{ \sup_{z \in [0,1]} \max_{1 \leq i \leq n, 1 \leq t \leq T} |m(Z_{it}, z)| \leq C(nh)^{-1} \right\} = 1. \)

**Lemma 2.** Assume that conditions (C1)–(C6) hold, we have

\[
(D^\top P D)^{-1} = (D^\top D)^{-1} + O_p(\zeta_n) = T^{-1} I_{n-1} + O_p(\zeta_n),
\]

where \( \zeta_n = (e_{n-1} e_{n-1}^\top) (nh)^{-1} \sqrt{\ln n} \).

**Lemma 3.** Assume that conditions (C1)–(C6) hold, we have

(i) \( \frac{1}{n} X^\top P X \xrightarrow{P} \sum_{t=1}^{T} E[(X_{it} - p(Z_{it}))(X_{it} - p(Z_{it}))^\top] \),

(ii) \( \frac{1}{n} X^\top P D (D^\top D)^{-1} D^\top P X \xrightarrow{P} \frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{T} E[(X_{it} - p(Z_{it}))(X_{is} - p(Z_{is}))^\top] \),

(iii) \( \frac{1}{n} \widetilde{X}^\top \widetilde{Q} \widetilde{X} \xrightarrow{P} \Phi \).

**Lemma 4.** Assume that conditions (C1)–(C6) hold, we have

\[
\frac{1}{\sqrt{n}} \widetilde{X}^\top \widetilde{Q} (I_n - M) g(Z) = o_p(1).
\]

**Lemma 5.** Assume that conditions (C1)–(C6) hold, we have

\[
\frac{1}{\sqrt{n}} X^\top P V = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} (X_{it} - p(Z_{it})) V_{it} + o_p(1),
\]

\[
\frac{1}{\sqrt{n}} X^\top P D (D^\top D)^{-1} D^\top P V = \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} (X_{it} - p(Z_{it})) V_{is} + o_p(1).
\]

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Figure 1: The solid lines denote the true curve, the dotted lines denote the estimated curve, and the long-dashed lines denote the 95% simultaneous confidence bands based on the asymptotic distribution and the dash-dotted lines denote the 95% simultaneous confidence bands based on the Bootstrap procedure for $g(\cdot)$, where figures are displayed for $c = 0, 0.5, 1$ from top to bottom and for the sample sizes $n = 100, 150, 200$ from left to right, respectively.
Proof of Theorem 1. The proofs of Theorem 1 can immediately be obtained from Su and Ullah (2006) and Zhang et al. (2011) by Lemmas 1–5. So we omit the details here. □

Proof of Theorem 2. Note that \((I_nT - D(D^T PD)^{-1}D^T P)D\alpha = 0\). By (2.11), (2.13) and Lemma 2, we have

\[
\hat{g}(z) = m^T(z)(Y - D\hat{\alpha} - X\hat{\beta})
\]

\[
= m^T(z)(I_nT - D(D^T PD)^{-1}D^T P)(Y - X\hat{\beta})
\]

\[
= m^T(z)(I_nT - D(D^T PD)^{-1}D^T P)(g + V - X(\hat{\beta} - \beta))
\]

\[
= m^T(z)Q_1(g + V - X(\hat{\beta} - \beta)).
\]

(5.1)

Invoking the Taylor expansion, we have

\[
g(Z_{it}) \approx g(z) + g'(z)(Z_{it} - z) + \frac{1}{2}g''(z)(Z_{it} - z)^2,
\]

(5.2)

where \(Z_{it}\) is close to \(z \in [0, 1]\). By (5.1) and (5.2), we have

\[
\hat{g}(z) \approx m^T(z)I_nTg(z)e_{nT} + m^T(z)Q_1g'(z)Z_z
\]

\[
+ \frac{1}{2}m^T(z)Q_1g''(z)Z_z^2 + m^T(z)Q_1V - m^T(z)Q_1X(\hat{\beta} - \beta)
\]

\[
= m^T(z)I_nTg(z)e_{nT} - m^T(z)D(D^T PD)^{-1}D^T P g(z)e_{nT} + m^T(z)Q_1g'(z)Z_z
\]

\[
+ \frac{1}{2}m^T(z)Q_1g''(z)Z_z^2 + m^T(z)Q_1V - m^T(z)Q_1X(\hat{\beta} - \beta),
\]

(5.3)

where \(Z_z = (Z_{11} - z, \cdots, Z_{1T} - z, Z_{21} - z, \cdots, Z_{2T} - z, \cdots, Z_{n1} - z, \cdots, Z_{nT} - z)^T\). For ease of notation, let \(S_{nT,l}(z) = \sum_{i=1}^n \sum_{t=1}^T K_h(Z_{it} - z)(Z_{it} - z)^l, \ l = 0, 1, 2\). For the first term of (5.3), some simple calculations yield that

\[
m^T(z)I_nTg(z)e_{nT} = (1, 0)(Z_z^TW_zZ_z)^{-1}Z_z^TW_zI_nT e_{nT}g(z)
\]

\[
= (1, 0)
\left[
\begin{array}{cc}
S_{nT,0}(z) & S_{nT,1}(z) \\
S_{nT,1}(z) & S_{nT,2}(z)
\end{array}
\right]^{-1}
\left[
\begin{array}{c}
S_{nT,0}(z) \\
S_{nT,1}(z)
\end{array}
\right] g(z)
\]

\[
= (1, 0)
\left[
\begin{array}{cc}
S_{nT,2}(z) & -S_{nT,1}(z) \\
-S_{nT,1}(z) & S_{nT,0}(z)
\end{array}
\right]^{-1}
\left[
\begin{array}{c}
S_{nT,0}(z) \\
S_{nT,1}(z)
\end{array}
\right] g(z)
\]

\[
\times (S_{nT,0}(z)S_{nT,2}(z) - S_{nT,1}(z)^2)^{-1}
\]

\[
= (1, 0)
\left[
\begin{array}{cc}
S_{nT,0}(z)S_{nT,2}(z) - S_{nT,1}(z)^2 \\
0
\end{array}
\right] g(z)
\]

\[
\times (S_{nT,0}(z)S_{nT,2}(z) - S_{nT,1}(z)^2)^{-1}
\]

\[
= g(z).
\]

(5.4)
By (5.3), (5.4) and some calculations, we have
\[
\sqrt{nh}(g(z) - g(z)) \approx \sqrt{nh}m^\tau(z)Q_1g'(z)Z_z + \frac{\sqrt{nh}}{2}m^\tau(z)Q_1g''(z)Z_z^2 + \sqrt{nh}m^\tau(z)Q_1V \\
- \sqrt{nh}D(D^\tau PD)^{-1}D^\tau Pe_{nT} - \sqrt{nh}m^\tau(z)Q_1X(\hat{\beta} - \beta) \\
=: J_{11} + J_{12} + J_{13} - J_{14} - J_{15}. \tag{5.5}
\]

From the results of Lemmas 1–4, it is easy to show that $J_{11} = o_p(1)$ and $J_{14} = o_p(1)$. Again invoking the results of Lemmas 1–3 and $\|\hat{\beta} - \beta\| = O_p(n^{-1/2})$ in Theorem 1, we can prove that $J_{15} = o_p(1)$.

Now we consider $J_{12}$ and $J_{13}$. Let $M(Z_{it}, z)$ be a typical column of $M(z)$, where $M(z) = (M(Z_{11}, z), \cdots, M(Z_{1T}, z), M(Z_{21}, z), \cdots, M(Z_{2T}, z), \cdots, M(Z_{n1}, z), \cdots, M(Z_{nT}, z))$.

For $J_{12}$, by Lemma 1 and some calculations, we can show that
\[
J_{12} \approx \frac{\sqrt{nh}}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} (1, 0)M(Z_{it}, z)g''(z)(Z_{it} - z)^2 \\
= \frac{\sqrt{nh}}{2} \frac{1}{nf(z)} \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)g''(z)(Z_{it} - z)^2 + o_p(1) \\
= \frac{\sqrt{nh}}{2} \frac{1}{nf(z)} g''(z) \int z^2K(z)dz + o_p(h^2) \\
= \frac{\sqrt{nh}}{2} b(z) + o_p(h^2). \tag{5.6}
\]

By Lemma 2 and Lemma 5, and using the same argument for $J_{13}$ and some simple calculations, we can show that
\[
J_{13} = \sqrt{nh}m^\tau(z)Q_1V \\
= \frac{\sqrt{nh}}{n} \frac{1}{f(z)} \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)\tilde{V}_{it} + o_p(1) \\
\xrightarrow{L} N(0, \Sigma_g), \tag{5.7}
\]

where $\tilde{V}_{it} = V_{it} - \frac{1}{T} \sum_{s=1}^{T} V_{is}$ and $\Sigma_g = \nu_0\tilde{\sigma}^2(z)f^{-2}(z)$.

By (5.5) and (5.6), it is easy to obtain that
\[
\hat{g}(z) - g(z) - b(z) = m^\tau(z)(I_{nT} - D(D^\tau PD)^{-1}D^\tau P)V + o_p(1) \\
\approx m^\tau(z)\tilde{V} + o_p(1) \\
= (1, 0)(Z_{i}^\tau W_z Z_z)^{-1}Z_{i}^\tau W_z \tilde{V} + o_p(1) \\
=: I_1(z) + o_p(1), \tag{5.8}
\]

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where \( \hat{\mathbf{V}} = (\hat{V}_{11}, \ldots, \hat{V}_{1T}, \hat{V}_{21}, \ldots, \hat{V}_{2T}, \ldots, \hat{V}_{n1}, \ldots, \hat{V}_{nT})^T \) and \( \hat{V}_{it} = V_{it} - \frac{1}{T} \sum_{s=1}^{T} V_{is} \).

Next, we approximate the process \( I_1(z) \) as follows. Note that

\[
\mathbf{Z}_z^T \mathbf{W}_z \mathbf{Z}_z = \begin{pmatrix}
\sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z) & \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z) \\
\sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z) & \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z)^2
\end{pmatrix}.
\]

By Lemma 1, we have

\[
\sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z) = \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z) = \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z)^2 = 0.
\]

Next, we approximate the process \( I_1(z) \) as follows. Note that

\[
\mathbf{Z}_z^T \mathbf{W}_z \mathbf{Z}_z = \begin{pmatrix}
\sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z) & \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z) \\
\sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z) & \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z)(Z_{it} - z)^2
\end{pmatrix}.
\]

By Lemma 1, we have

\[
n\mathbf{H}(\mathbf{Z}_z^T \mathbf{W}_z \mathbf{Z}_z)^{-1} \mathbf{H} = f^{-1}(z) \Omega^{-1} + O_p(h + (\log n/nh)^{1/2}),
\]
where \( \mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \) and \( \Omega = \begin{pmatrix} 1 & 0 \\ 0 & \mu_2 \end{pmatrix} \).

By Lemma 1, we further obtain that

\[
\left\| \frac{1}{n} \mathbf{H}^{-1} \mathbf{Z}_z^T \mathbf{W}_z \hat{\mathbf{V}} \right\|_\infty = O_p(h + (\log n/nh)^{1/2}). \tag{5.10}
\]

By (5.9) and (5.10), we have

\[
\left\| I_1(z) - \frac{1}{n f(z)} (1, 0) \Omega^{-1} \mathbf{H}^{-1} \mathbf{Z}_z^T \mathbf{W}_z \hat{\mathbf{V}} \right\|_\infty = O_p \left( h(\log n/nh)^{1/2} + (\log n/nh) \right). \tag{5.11}
\]

Let

\[
I_2(z) =: \frac{1}{n f(z)} (1, 0) \Omega^{-1} \mathbf{H}^{-1} \mathbf{Z}_z^T \mathbf{W}_z \hat{\mathbf{V}}
\]

\[
= \frac{1}{n f(z)} \sum_{i=1}^{n} \sum_{t=1}^{T} K_h(Z_{it} - z) \hat{V}_{it}.
\]

Invoking Theorem 1 and Lemma 1 in Fan and Zhang (2000), for \( h = n^{-\rho}, 1/5 \leq \rho \leq 1/3 \), we have

\[
P \left\{ (-2 \log h)^{1/2} \left( \left\| (nh \Sigma_g^{-1})^{1/2} I_2(z) \right\|_\infty - d_n \right) < u \right\} \rightarrow \exp \left( -2 \exp(-u) \right), \tag{5.12}
\]

where \( \Sigma_g = \nu_0 \sigma_2^2(z) f^{-2}(z) \) is defined in Theorem 1 and \( d_n \) is defined in Theorem 2. By (5.10), (5.11) and (5.12), we complete the proof of Theorem 2. \( \square \)

**Proof of Theorem 3.** Along the same lines as the proof of Theorem 2, it is easy to prove Theorem 3. Thus, we omit the details of proof. \( \square \)

**Proof of Theorem 4.** To prove Theorem 4, we need derive the rate of convergence for the bias and variance estimators. We first consider the difference between \( \text{bias}(\hat{g}(z)) \) and \( b(z) = \frac{1}{2} h^2 \mu_2 g''(z) \). By (5.9) and its similar arguments, we have

\[
\left\| \hat{\text{bias}}(\hat{g}(z)|D) - b(z) \right\|_\infty = O_p \left( h^2 \left( \sqrt{\log n/nh} \right) \right) = O_p \left( h^2 (n^{-1/7} \log^{1/2} n) \right), \tag{5.13}
\]

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where \( h_* = O(n^{-1/7}) \).

Furthermore, by Lemmas 1–2, and similar argument of (5.10), we have

\[
\left\| \frac{h}{n} H^{-1}(Z_z^TW_zQ_1W_zZ_z)H^{-1} - f(z)\Lambda \right\|_\infty = o_p(1),
\]

where \( \Lambda = \begin{pmatrix} \nu_0 & 0 \\ 0 & \nu_2 \end{pmatrix} \). By the similar argument, it is easy to check that \( \| \sigma^2(z) - \sigma^2(z) \|_\infty = o_p(1) \). These results, together with Theorem 2, we can show that, uniformly for \( z \in [0, 1] \),

\[
\left\| nh\text{Var}\{\hat{g}(z) | D\} - \Sigma_g \right\|_\infty = o_p(1). \tag{5.14}
\]

By (5.13) and (5.14), and invoking the result of Theorem 2, we finish the proof of Theorem 4.

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