Motion of an active particle with dynamical disorder

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We propose a model for the motion of a single active particle in a heterogeneous environment where the heterogeneity may arise due to the crowding, conformational fluctuations and/or slow rearrangement of the surroundings. Describing the active particle in terms of the Ornstein-Uhlenbeck process (OUP) and incorporating the heterogeneity in the thermal bath using the two separate models, namely “diffusing diffusivity” and “switching diffusion”, we explore the essential dynamical properties of the particle for its one-dimensional motion. In addition, we show how the dynamical behavior is controlled by two timescales, namely diffusive time and persistence time. Our model is relevant in the context of single particle dynamics in crowded environment, driven by activity.
I. INTRODUCTION

Recent advances in the single-particle tracking technique have promoted the studies on the motion of single particles which are of great importance as they provide deep insights about the nonequilibrium physics, in addition to their immense potential applicability in the field of biophysics [1]. In biological systems, from cargo transport by a motor protein to the motion of a microorganism such as bacteria, a particle moves on its own by harnessing energy from an active process like ATP hydrolysis. Such system is termed as active and it operates out of equilibrium [2, 3]. One of the most common ways of modelling an active particle is by describing its propulsion velocity as the Ornstein-Uhlenbeck process (OUP), and it is termed as the “active Ornstein-Uhlenbeck particle” (AOUP) model [4–6]. In this model, the velocities at different times are exponentially correlated. Therefore, the motion has persistence. By construction, in the AOUP framework, the particle has a Gaussian distribution in space, and it ensures enhanced diffusion in the long-time limit. It is worth mentioning here that the OUP model can also be adapted to describe the dynamics of passive particles immersed in a bath containing active particles, e.g., see Refs. [7–11]. The above description is well suited for the diffusion in an environment where the diffusivity of the particle remains unchanged over the time, or in other words, the particle moves in a homogeneous environment.

Over the past decade, experiments on the dynamics of passive tracers in a crowded environment as well as computer simulations have been performed by several research groups [12–18]. The examples include diffusion of lipid vesicles in a solution of entangled filaments [13], tracer diffusion in an environment consisting of large particles or polymer melts or suspension of polymer chains [14, 19, 20], mobile biomolecules on a surface [16, 17], to name but a few. The most notable observation of these studies is the fact that particle has non-Gaussian position distribution. In many cases, the distribution has an exponential tail at intermediate times, which vanishes in the long-time limit, exhibiting a Gaussian behavior, as predicted by the central limit theorem [20, 21]. Interestingly, some studies have shown that though the distribution is non-Gaussian, the mean square displacement (MSD) all times grows linearly in time, or in other words, the dynamics is Fickian [12–14, 18]. Such observations are not consistent with the description of the normal diffusion, and thus the processes associated with similar features are, in general, termed as “anomalous, yet Brownian diffusion” (AYBD) [22]. A plausible explanation for AYBD comes from a theoretical formalism where the tagged particle possesses multiple diffusion coefficients, i.e., the diffusivity is a stochastic variable and it follows some distribution [22]. The idea of invoking fluctuations
into intensive properties such as temperature (or diffusivity) can be traced back to the work of Beck and Cohen, who called it “superstatistics” [23]. However, it was Robert Zwanzig who, in a different context, introduced essentially the same concept while describing the rate process with a rate constant that fluctuates in time due to fluctuations of the barrier height [20, 24, 25]. It can take discrete values or it can be a continuous function. Zwanzig called this “dynamical disorder”. In the context of diffusing particles, a similar formalism named “diffusing diffusivity” model was put forward by Chubynsky and Slater [22], and it was further extended by Sebastian’s group and Metzler’s group independently [26–31]. The model of Chubynsky and Slater is a case of static disorder as the diffusivity does not fluctuate with time as it has a stationary distribution. But for the models developed by Sebastian and Metzler, the diffusivity of a particle in a crowded, heterogeneous surroundings is a time-dependent random variable which is considered as the square of a random process such as OU process, since the diffusivity can take only positive values. But when the motion is sampled over all trajectories, it results diffusive motion with the ensemble-averaged diffusivity. This model is applied to describe the motion in a heterogeneous environment rearranging with a timescale comparable to the one corresponding to the diffusing particle. There are situations where the particle can have few conformations specified by their diffusivities, or there are spatial heterogeneity or specific kinds of interaction between the particle and its surroundings which leads to a motion associated with discrete diffusivities [32–35]. For instance, a freely diffusing particle intermittently trapped due to presence of random binding zones or traps exhibits a non-Gaussian spatial distribution [36]. Another example includes conformational fluctuations of a protein called transcription factor which switches between two conformations while searching for a promoter site to initiate the transcription; in one conformation it moves fast on the DNA track, but in another conformation it undergoes a slow motion due to strong interaction with the track [33]. The above dynamics is usually described by the two-state model or the switching diffusion [25, 37]. Apart from the Brownian dynamics, applications of the switching model in describing several biochemical processes such as cellular signalling, synaptic dynamics, growth of cell population, pattern formation are noteworthy in relation to the present topic [38–41].

In all the examples discussed above on the heterogeneous diffusion, the particle’s motion is mainly influenced by the thermal fluctuations, or more specifically, no intrinsic energy source or no external stochastic driving is considered, and therefore, the dynamics is in equilibrium. However, as mentioned earlier, the diffusive motion coupled to an active process is very common in biological systems, though its theoretical study in heterogeneous environment is largely lacking. Inclusion
of heterogeneity is important to study the activity-driven systems such as self-propelled particles in the presence of obstacles \cite{42}, active soft colloids with switching active interactions \cite{43}, run-and-tumble disks displaying avalanche dynamics \cite{44}, etc. For a detailed account on this topic, the reader is referred to Ref. \cite{45}. Along these lines, there have been theoretical investigations of the impact of heterogeneity on the transport properties of active particles \cite{46,47}. These studies have mostly considered the active Brownian particle (ABP) model to describe the dynamics, and the fluctuations are incorporated either in self-propulsion velocity or both in thermal and active forces. In another study \cite{43}, Bley \textit{et. al.} have investigated the dynamical properties of a suspension of active colloids where each particle switches randomly between two states differing in their size as well as mobility. It has been shown there that this switching induces non-Gaussian behavior at intermediate times.

Drawing motivations from these experimental and simulation based investigations, here we propose an analytically solvable model to capture the dynamics of a simple one-dimensional active particle with dynamical disorder. We consider two cases, namely the switching diffusion where the particle switches between two conformations with two different thermal diffusivities, and the diffusing diffusivity model in which the particle takes instantaneous thermal diffusivity which changes randomly over time due to the rearrangement of the environment. In our model, the activity does not get affected by the heterogeneity, and it is modeled as Ornstein-Uhlenbeck process. These models along with the dynamics of the particle is discussed in Sec. \ref{sec:II}. In Sec. \ref{sec:III} we showcase the main results. A summary is given in Sec. \ref{sec:IV}.

II. DYNAMICS

Here we consider an active particle moving in a heterogeneous media as pictorially depicted in Fig. \ref{fig:1}. For simplicity, the motion is restricted to one dimension although it can be in principle extended to higher dimensions. In the following section, more details about the dynamics is provided.

A. Model of active noise

Apart from thermal kicks experienced by the particle, it is also subjected to an extra noise termed as the active noise, and as a result, it executes directed motion. Therefore, the dynamics of the
particle is governed by the following stochastic equation:

\[ \dot{x}(t) = \eta_T(t) + v_A(t), \]  

(1)

where \( \eta_T(t) \) is the thermal noise modeled as a Gaussian white noise with zero mean. Thus, \( \langle \eta_T(t) \rangle = 0 \), and the autocorrelation function is

\[ \langle \eta_T(t)\eta_T(t') \rangle = 2D_T(t)\delta(t-t'). \]  

(2)

Here it is assumed that either the environment is crowded and stochastically rearranging, that is to say, the particle explores the heterogeneous environment, or the particle itself alters between many states, which results only the fluctuations in the thermal diffusivity \( D_T \). Following Zwanzig, it is referred here as the dynamically disorder system. To account for the disorder, two different models are discussed in Sec. II B.

On the other hand, \( v_A(t) \) corresponds to the self-propulsion velocity of the particle, and it can be conceived as the active noise which drives the particle out of equilibrium. As a standard model, the noise taken here is a Gaussian colored noise, which follows the OUP of the form

\[ \dot{v}_A(t) = -\frac{1}{\tau_A}v_A(t) + \frac{1}{\tau_A}\eta_A(t), \]  

(3)

with \( \eta_A(t) \) being the Gaussian white noise having correlation \( \langle \eta_A(t)\eta_A(t') \rangle = 2D_A\delta(t-t') \). So the auto-correlation function of \( v_A(t) \) can be expressed as

\[ \langle v_A(t)v_A(t') \rangle = \frac{D_A}{\tau_A}e^{-\frac{|t-t'|}{\tau_A}}, \]  

(4)

where \( D_A \) denotes the active diffusivity and \( \tau_A \) is the persistence time. Other than the strength \( D_A \), the characteristic timescale \( \tau_A \) characterizes the active noise, and higher value of \( \tau_A \) means a long-lived directed motion. In the limit \( \tau_A \to 0 \), the noise becomes delta-correlated and behaves like the thermal noise [48].

**B. Model of thermal diffusivity**

As discussed earlier, the thermal diffusivity of the particle fluctuates in time due to disorder. In the following sections, two models are considered to describe the time-dependent diffusivity \( D_T(t) \).
1. Switching diffusion

In the switching diffusion model, the particle switches between two states with two different thermal diffusivities $D_1$ and $D_2$. Let us call the states as state 1 and state 2, respectively. The transition happens from the state 2 to the state 1 with a rate $r_{21}$, and vice versa with a rate $r_{12}$. This is a simple version of the “Markov additive” model commonly used to capture the protein movement on a DNA track where it encounters different states of chromatin with variable affinities, and as a result, its motion accounts for the heterogeneous environment \(^{[49]}\). Here we assume that the particle has two discrete states and it starts its journey either from the state 1 or state 2 with their respective steady-state probabilities which are given by $p_{1,s} = \frac{r_{21}}{r_{12} + r_{21}}$ and $p_{2,s} = \frac{r_{12}}{r_{12} + r_{21}}$, where $\rho_{i,s}$ being the steady-state probability at state $i$. For such a case, the characteristic functional can be expressed as \(^{[50]}\)

$$
\langle e^{-\int_{0}^{t'} dt' D(t')} \rangle_{D(t')} = \frac{1}{2} \left[ 1 - \frac{\phi_3}{\phi_2} \right] e^{-(\phi_1 + \phi_2)t} + \frac{1}{2} \left[ 1 + \frac{\phi_3}{\phi_2} \right] e^{-(\phi_1 - \phi_2)t},
$$

(5)

where $\phi_1 = \frac{r_2}{2}(D_1 + D_2) + \frac{1}{2}(r_{12} + r_{21})$, $\phi_2 = \frac{1}{2} \sqrt{[(D_2 - D_1)p^2 + (r_{21} - r_{12})]^2 + 4r_{12}r_{21}}$, and $\phi_3 = \frac{1}{2}(r_{12} + r_{21}) - \frac{\sqrt{r_{12}r_{21}}}{2} \frac{r_{12}}{r_{12} + r_{21}} (D_2 - D_1)$. The equilibrium diffusivity is given by $D_{eq} = \langle D(t) \rangle = p_{2,s}D_2 + p_{1,s}D_1$. Here, $D_2 \geq D_1 \geq 0$. 
2. Diffusing diffusivity model

In the diffusing diffusivity model, $D_T(t)$ is considered as the position of $n-$dimensional harmonic oscillator, i.e., $D_T(\tau) = \sum_{i=1}^{n} \xi_i^2(\tau)$, where the evolution of $\xi_i(t)$ is governed by the Ornstein-Uhlenbeck process (OUP), $\dot{\xi}_i = -\omega \xi_i(t) + \eta_i(t)$. Here $\eta_i(t)$ is the white Gaussian noise with correlation $\langle \eta_i(t) \eta_j(t') \rangle = D_{eq} \delta_{ij} \delta(t-t')$, and $D_{eq}$ is the effective or equilibrium diffusivity of the particle in the medium. Here, we consider a two-dimensional model, i.e., $D_T(\tau) = \sum_{i=1}^{2} \xi_i^2(\tau)$. The characteristic functional for such model can be found easily by the path integral technique, and it reads

$$\langle e^{-p^2 \int_0^t dt_1 D(t_1)} \rangle_D(t_1) = \frac{4\omega \beta}{(\beta + \omega)^2 - (\beta - \omega)^2} e^{-2\beta t},$$ (6)

with $\beta = \sqrt{\omega^2 + 2\omega D_{eq} p^2}$. The relaxation timescale for the environment rearrangement is roughly related to the inverse of $\omega$, which implies that for higher values of $\omega$, the environment relaxes faster and as a result, the particle visits all possible configurations of the environment comparatively at a shorter timescale.

III. RESULTS

Here our aim is to find the dynamical properties of the particle. To do so, we first outline the technique to obtain the probability distribution function (PDF) in space. Using Eq. (1), one can write

$$x(t) = x_0 + \int_0^t dt_1 \eta_T(t_1) + \int_0^t dt_1 v_A(t_1),$$ (7)

with $x_0$ being the initial position of the particle. Now the probability distribution function (PDF) of finding the particle at position $x$ at time $t$ considering $x(0) = x_0$, can be expressed as

$$P(x,t|x_0,0) = \langle \delta(x-x(t)) \rangle,$$ (8)

where $\langle \cdots \rangle$ represents the ensemble average over all trajectories of $x(t)$. Writing the definition of delta functional explicitly in the above and with the aid of Eq. (7), the PDF can be rewritten as

$$P(x,t|x_0,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} \langle e^{ipx(t)} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ip(x-x_0)} \langle e^{ip\int_0^t dt' \eta_T(t')} \eta_T(t') \langle e^{i\int_0^t dt' \eta_T(t') \eta_T(t')} \rangle \eta_T(t') \rangle \langle e^{i\int_0^t dt' v_A(t')} \rangle \eta_T(t') \rangle v_A(t').$$ (9)
Now the averages are taken over the noises. As the evolution of the two noises are decoupled to each other, the ensemble averages now can be computed separately as shown in the above equation. It is useful to note here that the fourth-order autocorrelation function of any Gaussian noise $\sigma_G$ can be expressed as \[ G_4(\tau) = \langle \sigma_G(t_1)\sigma_G(t_2)\sigma_G(t_3)\sigma_G(t_4) \rangle = \langle \sigma_G(t_1)\sigma_G(t_2) \rangle \langle \sigma_G(t_3)\sigma_G(t_4) \rangle + \langle \sigma_G(t_1)\sigma_G(t_3) \rangle \langle \sigma_G(t_2)\sigma_G(t_4) \rangle + \langle \sigma_G(t_1)\sigma_G(t_4) \rangle \langle \sigma_G(t_2)\sigma_G(t_3) \rangle, \] and the characteristic functional of the noise $\sigma_G$ can be evaluated using the relation \[ e^{i\int^0_t dt' p(t')\sigma_G(t')} \frac{d\langle e^\sigma_G \rangle}{d\sigma_G} = e^{-\frac{1}{2}\int^0_t dt_1 \int^0_t dt_2 p(t_1)\sigma_G(t_2) p(t_2)}. \] By the virtue of the above relation and Eq. (4), the average over the active noise can be easily obtained, and it reads \[ \langle e^{i\int^0_t dt' v_A(t')} \rangle_{v_A(t')} = e^{-\frac{1}{2}\int^0_t dt_1 \int^0_t dt_2 \langle v_A(t_1)v_A(t_2) \rangle} = e^{-D_A p^2 \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right)}. \] Similarly for the thermal noise, the average can be calculated using Eq. (2) as \[ \langle e^{i\int^0_t dt' \eta_T(t')} \rangle_{\eta_T,D_T} = \langle e^{-p^2 \int^0_t dt' D_T(t')} \rangle_{D_T(t')}. \] Since the diffusion is also a stochastic variable, one also needs to do the averaging over all realizations of $D_T(t)$, as shown in the above equation. Now we can take different models of $D_T(t)$ as discussed in Sec. II B to do the further analysis. To get a preliminary idea about the dynamics, we find the mean square displacement (MSD). For both the models of $D_T(t)$, the MSD can be computed using Eq. (7), and it yields \[ \langle (x(t) - x_0)^2 \rangle = \int^t_0 \int^t_0 dt_1 dt_2 \langle \eta_T(t_1)\eta_T(t_2) \rangle + \int^t_0 \int^t_0 dt_1 dt_2 \langle v_A(t_1)v_A(t_2) \rangle = 2D_{eq} t + 2D_A \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right). \] Notice that the first term of the right-hand side (RHS) is a linear function of time and is solely coming from the thermal contribution, which we denote here as $\langle (x(t) - x_0)^2 \rangle_T$. Now we can distinguish different time limits for which the MSD can be approximated as follows: \[ \langle (x(t) - x_0)^2 \rangle \approx \begin{cases} 2D_{eq} t & \text{if } t < 2D_{eq} / D_A \tau_A \\ D_A \tau_A^2 & \text{if } t > 2D_{eq} / D_A \tau_A \\ 2(D_{eq} + D_A) t & \text{if } t \gg \tau_A \end{cases}. \]
Figure 2. Log-Log plot of the mean square displacement (MSD) [Eq. (14)] scaled by the MSD in a purely thermal bath as a function of time \( t \) for different values of \( \tau_A \). The curves with symbols are the approximate results for the ballistic regime in the limit \( t \ll \tau_A \) [see Eq. (15)], and two black lines correspond to the very short-time (lower line) and long-time (upper line) diffusive limits. The other parameters used in the plots are \( D_{eq} = 5.0 \), \( D_A = 25 \), and \( x_0 = 0 \).

In the short-time limit, i.e., for \( t \ll \tau_A \), the motion approaches a ballistic behavior when \( t > \frac{D_{eq}}{D_A} \tau_A \), though at \( t < \frac{D_{eq}}{D_A} \tau_A \), the dynamics is Fickian with the thermal diffusivity \( D_{eq} \). On the other hand, the MSD for the long time limit \( t \gg \tau_A \) is linear in time, clearly suggesting a diffusive regime with an elevated diffusivity. The results are graphically displayed in Fig. 2 and are equivalent to the one observed for the case in a homogeneous environment [53].

To know more about the dynamics, particularly the tail behavior, we compute the non-Gaussian parameter (NGP) defined as

\[
\gamma_G = \frac{\langle x^4(t) \rangle}{3\langle x^2(t) \rangle^2} - 1,
\]

considering \( x_0 = 0 \), without any loss of information. For the switching diffusion, the NGP is given by Eq. (A7). In the limit \( t \ll \tau_A \), it approximates to

\[
\gamma_G(t) \approx \frac{2(D_2-D_1)^2 r_{12} r_{21} \left((r_{12}+r_{21})t-1+e^{-t(r_{12}+r_{21})}\right)}{(r_{12}+r_{21})^4 \left(D_{eq} t + \frac{1}{2} D_A \frac{r_{12}^2}{\tau_A^2}\right)^2},
\]

which implies \( \gamma_G(t) \propto \tau_A^2 \) at short times. This can be understood from Fig. 3 (a). In the extremely short-time limit \( t \to 0 \), or equivalently for \( r \ll r_{12}, r \ll r_{21}, r \ll \tau_A \), the NGP converges to a fixed value \( \frac{(D_2-D_1)^2 r_{12} r_{21}}{(r_{12}+r_{21})^4 D_{eq}^2} \), which is independent of \( \tau_A \) and \( D_A \). This indicates that the particle was initially at a nonequilibrium state solely due to disorder. A quick check for the accuracy of our result is to take \( D_2 = D_1 \), and notice that the NGP vanishes to zero at short times, which correctly reproduces the result for the case without disorder. As time passes, \( \gamma_G(t) \) decreases monotonically, and it decays faster for small values of \( \tau_A \), which suggests that the active noise mostly controls the dynamics at short and intermediate times. This
can also be inferred from Fig. 3(c) where one can see that the NGP is a monotonically increasing function of $\tau_A$, the slope in the short $\tau_A$ limit gradually decreases as $t$ increases, though its effect at intermediate $\tau_A$ prevails only at a relevant timescale. In the long-time limit, the NGP can be approximated to $\gamma_G(t) \approx \frac{2(D_2 - D_1)^2 r_{12} r_{21}}{(r_{12} + r_{21})^2 (D_{eq} + D_A)^2}$, and it becomes zero at $t \to \infty$, implying the Gaussian behavior, as anticipated.

For the diffusing diffusivity model, the NGP is given by Eq. (A13). The NGP becomes unity and zero in the very short and large time limits, respectively. Like the previous case, the value of $\gamma_G(t)$ strongly depends on $\tau_A$, as can be seen in panels (b) and (d) of Fig. 3. Also, similar dependence of $\omega$ on the NGP can be found, as shown in Fig. 3(b). As the environment rearranges fast when $\omega$ is large, the motion is sampled over all realizations at a shorter timescale and consequently, the particle achieves a Gaussian distribution, which is reflected in the plot of $\gamma_G(t)$ shown in Fig. 3(b).

**Switching diffusion:** Here we further investigate the dynamical properties by finding the complete PDF for the model of switching diffusion. By virtue of Eqs. (5) and (12), the PDF given in Eq. (9) can be written explicitly as

$$P(x, t | x_0 = 0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} \left[ \frac{1}{\phi_2} \left( 1 - \frac{\phi_3}{\phi_2} \right) e^{-\left(\phi_1 + \phi_2\right)t} e^{-D_A p^2 t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right)} \right]$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} \left[ \frac{1}{\phi_2} \left( 1 + \frac{\phi_3}{\phi_2} \right) e^{-\left(\phi_1 - \phi_2\right)t} e^{-D_A p^2 t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right)} \right].$$

(16)

The integration in the above equation cannot be evaluated exactly. However, we can consider different limiting cases as demonstrated in Appendix B. At a timescale shorter than the inverse of switching rates, the distribution is the weighted average of two Gaussian functions centered at $x_0 = 0$, as given by Eq. (B1). So it shows a non-Gaussian behavior with a central peak region which becomes pronounced at a higher value of $\tau_A$, as can be seen in Fig. 5. With the progress of time, the height of the peak decreases and the distribution flattens and eventually, it converges to a single Gaussian, as illustrated in Fig. 4. Unlike the passive-particle case, here the width of the Gaussian distribution given in Eq. (B2) is enhanced, and it is expressed in terms of the equilibrium thermal diffusivity and active diffusivity. All these results are in consistent with the previous analysis on the non-Gaussian parameter. Note that here the pattern of convergence to a Gaussian is strikingly different from the one observed in a system with strong static disorder where the peak narrows down to a single point in the long time [54]. Our model is related to the one in Ref. [43] where a soft colloid diffusing in response to the gradients in chemical potential
Figure 3. The non-Gaussian parameter (NGP) is plotted (i) as a function of time $t$ for different values of $\tau_A$ in panels (a)-(b), and (ii) the plots of NGP scaled by its value at $\tau_A = 0$ [see Eqs. (A8) and (A14)] are shown in panels (c)-(d) as a function of the correlation time of the active noise $\tau_A$ at different times. In panels (a) and (c), the thermal diffusivity is modelled by the switching diffusion, and the curves correspond to Eq. (A7). The parameters used here are given by the set $\{D_1 = 5, D_2 = 15, D_A = 25, r_{12} = r_{21} = 1/2\}$. In panels (b) and (d), the diffusing diffusivity model is used to describe the fluctuating diffusivity, and the curves are obtained using Eq. (A13). Here, the values of other parameters are $D_{eq} = 5, D_A = 25$. For panel (d), we take $\omega = 1.0$.

and/or concentrations switches between two states with different masses due to active interactions. Unlike [43], here we consider the motion of a soft active colloid, which means that the dynamics is not only driven by the chemical or/and concentration gradient, but it also moves due to self-propulsion mechanism fueled by ATP hydrolysis. However, it is interesting to note that both studies capture the non-Gaussian behavior at a shorter timescale, which further emphasizes the fact that the non-Gaussianity is a very common feature associated with a disordered system.
Figure 4. Probability distribution function versus displacement at different times for the case where the thermal diffusivity switches between two values $D_1$ (state 1) and $D_2$ (state 2) at Poisson rates (from state 2 to 1) $r_{21}$ and (vice versa) $r_{12}$. The curves with symbols are obtained numerically using Eq. (16) by taking the set of parameters as follows: $\{D_1 = 1, D_2 = 14, r_{12} = 3/4, r_{21} = 1/4, \tau_A = 50, D_A = 25\}$. In panel (a) the distribution is for small times and is approximately similar to Eq. (B1) as depicted by solid lines. In panel (b) the solid lines correspond to Eq. (B2) and match well with the numerical results in the long-time limit.

Figure 5. Active particle with switching diffusion: spatial distribution computed numerically from Eq. (16) is plotted with symbols as a function of displacement $x$ for different values of $\tau_A$ at a short time $t = 0.10$. The values of other parameters are as follows: $D_1 = 1, D_2 = 14, r_{12} = 1/2, r_{21} = 1/2, D_A = 25$. The solid lines which correspond to the approximate result given in Eq. (B1) are in good agreement with the numerical ones.
Figure 6. Plot of probability distribution function at different times as a function of displacement considering that the thermal diffusivity $D(t)$ is a stochastic variable following the OU process. The characteristic timescales for thermal diffusivity and active noise are $1/\omega = 0.4$, $\tau_A = 5.0$, respectively. The numerical results obtained from Eq. (17) are plotted using symbols taking $D_A = 25$. In panel (a) the plots are for small times, and the solid lines correspond to the analytical result given by Eq. (B3). In panel (b) the solid lines correspond to Eq. (B4).

Diffusing diffusivity model: From Eq. (9), with aid of Eqs. (12) and (6), the spatial distribution can be expressed as

$$P(x|t|x_0 = 0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} \frac{4\omega \beta e^{-(\beta - \omega)t}}{(\beta + \omega)^2 - (\beta - \omega)^2 e^{-2\beta t} e^{-D_A p^2}} e^{-D_A p^2 \left(t + \tau_A \left(e^{-\tau_A} - 1\right)\right)}.$$  

(17)

The above integration is evaluated numerically and plotted in Figs. 6 and 7. Unlike the switching diffusion case, the distribution at short times is expressed by double exponential functions of the form given in Eq. (B3), which is also non-Gaussian in nature. Clearly, for large values of $\tau_A$, the distribution has a significantly long exponential tail, which indirectly suggests that the long persistence strongly enhances the effect of disorder on the particle’s motion, as has been found in the switching diffusion model too. This is also complemented by the previous NGP analysis. As time passes, the height of the distribution gets reduced with flattening of the curve, and after a long time, the PDF becomes a Gaussian distribution given by Eq. (B4).

IV. CONCLUSION

In this paper, we have investigated the dynamics of a single active Ornstein-Uhlenbeck particle (OUP) subjected to the dynamical disorder. Invoking the idea of fluctuating thermal diffusivity,
Figure 7. Active particle in a heterogeneous medium characterized by the diffusing diffusivity model: plots of the spatial distribution numerically obtained from Eq. (17) and marked by symbols are compared with the analytical solution [Eq. (B3)] represented by the solid curves for different values of $\tau_A$ at a short time $t = 0.10$. The values of other parameters are same as in Fig. 6.

we have shown that the distribution becomes non-Gaussian at intermediate times, which cannot be found in the usual OUP model, but the behavior is not very surprising as this is the signature of a disordered system. However, unlike the passive disordered case, the dynamics is not always Fickian; we have observed a ballistic behavior at a timescale shorter than its persistence time, signifying the directed motion of the active particle. Also it has been found that the non-Gaussianity in the distribution is strongly enhanced in a system with longer persistence time, which clearly suggests that the stronger correlation arising due to activity facilitates the effect of disorder on the dynamics.

The extensions of this work may be manyfold: In one direction, one can explore the effect of heterogeneity on several properties of a non-interacting active system which are at certain extent studied in the context of passive particle. In other direction, it can be extended to higher dimensions which is relevant while to be dealing with real biological systems. In addition, one may incorporate heterogeneity and interactions together to look into the many-body problem such as biopolymers and active gels [55, 56].
V. CONFLICTS OF INTEREST

There are no conflicts to declare.

VI. ACKNOWLEDGEMENTS

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Appendix A: Computation of non-Gaussian parameter

The fourth moment of the distribution given in Eq. (10) can be expressed using Eq. (7), as follows:

\[
\langle (x(t) - x_0)^4 \rangle = \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_3) \eta_T(t_4) \rangle \\
+ 2 \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \rangle \langle v_A(t_3) v_A(t_4) \rangle \\
+ \int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau dt_1 dt_2 dt_3 dt_4 \langle v_A(t_1) v_A(t_2) v_A(t_3) v_A(t_4) \rangle. \tag{A1}
\]

Now, with the aid of Eq. (10) and the autocorrelation function of \( v_A(t) \) as given in Eq. (4), the last term of the RHS in the above equation can be computed, and it reads

\[
\begin{align*}
&\int_0^\tau \int_0^\tau \int_0^\tau \int_0^\tau dt_1 dt_2 dt_3 dt_4 \langle v_A(t_1) v_A(t_2) v_A(t_3) v_A(t_4) \rangle \\
&= \int_0^\tau \int_0^\tau dt_1 dt_2 \langle v_A(t_1) v_A(t_2) \rangle \times \\
&\left[ D_A \tau_A \int_0^\tau dt_1 dt_2 e^{-\frac{|t_1-t_2|}{\tau_A}} \frac{D_A}{\tau_A} \int_0^\tau dt_3 dt_4 e^{-\frac{|t_3-t_4|}{\tau_A}} \right] \\
&= 12 \left[ D_A \left( \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) + t \right) \right]^2. \tag{A2}
\end{align*}
\]
Using the properties of white noise $\eta_T(t)$, the first term of the RHS in Eq. (A1) can be written as

$$
\int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_3) \eta_T(t_4) \rangle = \int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4
$$

$$
\langle \langle \eta_T(t_1) \eta_T(t_2) \rangle \eta_T(t_3) \eta_T(t_4) \rangle_{\eta_T} + \int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_3) \eta_T(t_4) \rangle_{\eta_T}
$$

$$
+ \int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_4) \rangle_{\eta_T} \eta_T(t_3)_{\eta_T}
$$

$$
= 24 \int_0^t dt_1 \int_0^t dt_2 \langle D_T(t_1) D_T(t_2) \rangle.
$$

(A3)

The middle term of the RHS in Eq. (A1) is given by

$$
\int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \rangle \langle \eta_T(t_3) v_A(t_4) \rangle = 12t \int_0^t dt_1 \int_0^t dt_2 \langle D(t_1) \rangle \langle v_A(t_1) v_A(t_2) \rangle.
$$

(A4)

For the two models of $D_T(t)$, we solve Eqs. (A3) and (A4). The calculations are shown below.

1. Switching diffusion

To perform the integration in Eqs. (A3) and (A4), it is required to know the average and auto-correlation function of $D(t)$ which are given by \[57\] $\langle D(t) \rangle = \frac{r_{12} D_2 + r_{21} D_1}{r_{12} + r_{21}}$, and $\langle D(t_1) D(t_2) \rangle = \langle D(t_1) \rangle^2 + \frac{r_{12} r_{21}}{(r_{12} + r_{21})^2} (D_2 - D_1)^2 e^{-(r_{12} + r_{21})|t_2 - t_1|}$. Using these, one can compute Eq. (A3) as

$$
\int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_3) \eta_T(t_4) \rangle
$$

$$
= 24 \int_0^t dt_1 \int_0^t dt_2 \left[ \left( \frac{r_{12} D_2 + r_{21} D_1}{r_{12} + r_{21}} \right)^2 + \frac{r_{12} r_{21}}{(r_{12} + r_{21})^2} (D_2 - D_1)^2 e^{-(r_{12} + r_{21})(t_1 - t_2)} \right]
$$

$$
= 12 \left( \frac{(r_{12} + r_{21})^2 (D_2 r_{12} + D_1 r_{21})^2 + 2 (D_1 - D_2)^2 r_{12} r_{21} \left( \frac{e^{-(r_{12} + r_{21})t}}{r_{12} + r_{21}} + t \right)}{(r_{12} + r_{21})^3} \right).
$$

(A5)

Similarly, one has

$$
2 \int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \rangle \langle v_A(t_3) v_A(t_4) \rangle
$$

$$
= 24 D_A t (D_2 r_{12} + D_1 r_{21}) e^{\frac{t}{\tau_A}} \left( \frac{e^{t/\tau_A} (t - \tau_A) + \tau_A}{r_{12} + r_{21}} \right).
$$

(A6)
By virtue of Eqs. (A2), (A5), (A6) and (14), the NGP defined by $\gamma_G = \frac{(x^4(t))}{3(x^2(t))^2} - 1$ can be calculated, and it yields

$$
\gamma_G(t) = \frac{2(D_2 - D_1)^2 r_{12} r_{21} \left( (r_{12} + r_{21}) t - 1 + e^{-t(r_{12} + r_{21})} \right)}{(r_{12} + r_{21})^2 \left( (r_{12} + r_{21}) D_A (t - \tau_A) + (D_2 r_{12} + D_1 r_{21}) t + e^{-\frac{t}{\tau_A}} (r_{12} + r_{21}) D_A \tau_A \right)^2}.
$$

(A7)

For $\tau_A = 0$, the NGP denoted here as $\gamma^0_G(t)$ can be expressed as

$$
\gamma^0_G(t) = \frac{2(D_2 - D_1)^2 r_{12} r_{21} \left( (r_{12} + r_{21}) t - 1 + e^{-t(r_{12} + r_{21})} \right)}{(r_{12} + r_{21})^4 \left( D_A + D_{eq} \right)^2}.
$$

(A8)

2. Diffusing diffusivity model

Using the dynamical equation of $\xi(t)$ given in Sec. [11B2] one can write

$$
\langle (\xi(t_1) - \xi(0)e^{-\omega t_1}) (\xi(t_2) - \xi(0)e^{-\omega t_2}) \rangle = \langle \xi(t_1)\xi(t_2) \rangle - \langle \xi(t_1)\xi(0) \rangle e^{-\omega t_2} - \langle \xi(t_2)\xi(0) \rangle e^{-\omega t_1} + \langle \xi(0)\xi(0) \rangle e^{-\omega(t_1 + t_2)}
$$

$$
= \int_0^{t_1} dt_1 e^{-\omega|t_1-t_2|} \langle \eta_A(t_1) \eta_A(t_2) \rangle,
$$

(A9)

which implies $\langle \xi(t_1)\xi(t_2) \rangle = \langle \xi^2(0) \rangle e^{-\omega|t_1-t_2|}$. With these, one can get

$$
\sum_{i,j=1}^{2} \langle \xi^2_i(t_1)\xi^2_j(t_2) \rangle = \sum_{i,j=1, i \neq j} \langle \xi^2_i(t_1) \rangle \langle \xi^2_j(t_2) \rangle + \sum_{i=1}^{2} \langle \xi^2_i(t_1)\xi^2_i(t_2) \rangle
$$

$$
= 2\langle \xi^2(0) \rangle^2 + \sum_{i=1}^{2} \langle (\xi^2_i(t_1))^2 + 2\langle \xi_i(t_1)\xi_i(t_2) \rangle \rangle
$$

$$
= 4\langle \xi^2(0) \rangle^2 + 4\langle \xi^2(0) \rangle^2 e^{-2\omega|t_1-t_2|}.
$$

(A10)

Here, $\langle D(t) \rangle = \sum_{n=1}^{2} \xi^2_n(t)$, and $\langle \xi^2_i(t) \rangle = \langle \xi^2_i(0) \rangle = \frac{1}{2}D_{eq}$. Using Eq. (A10), Eq. (A3) can be calculated as follows:

$$
\int_0^{t} dt_1 \int_0^{t} dt_2 \int_0^{t} dt_3 \int_0^{t} dt_4 \langle \eta_T(t_1) \eta_T(t_2) \eta_T(t_3) \eta_T(t_4) \rangle
$$

$$
= 24 \int_0^{t} dt_1 \int_0^{t} dt_2 \langle D_T(t_1) D_T(t_2) \rangle = 24 \int_0^{t} dt_1 \int_0^{t} dt_2 \sum_{i,j=1}^{2} \langle \xi^2_i(t_1)\xi^2_j(t_2) \rangle
$$

$$
= 24 \int_0^{t} dt_1 \int_0^{t} dt_2 \frac{1}{4} \langle \xi^2(t_1) \rangle^2 + 24 \int_0^{t} dt_1 \int_0^{t} dt_2 \frac{1}{4} \langle \xi^2(0) \rangle^2 e^{-2\omega|t_1-t_2|}
$$

$$
= 6D_{eq}^2 \left[ 2t\omega(\omega + 1) + e^{-2t\omega} \right].
$$

(A11)
Similarly, from Eq. (A4) one can obtain
\[
2 \int_0^t \int_0^t \int_0^t \int_0^t dt_1 dt_2 dt_3 dt_4 \langle \eta_T(t_1) \eta_T(t_2) \rangle \langle v_A(t_3) v_A(t_4) \rangle = 24 t \int_0^t dt_1 \int_0^{t_1} dt_2 \langle D(t_1) \rangle \langle v_A(t_1) v_A(t_2) \rangle = 24 D_{eq} D_A t \left( \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) + t \right). \tag{A12}
\]

Using Eqs. (A2), (A11), (A12) and (A4), the NGP can be calculated, and it reads
\[
\gamma_G(t) = \frac{\langle x^4(t) \rangle}{3 \langle x^2(t) \rangle^2} - 1 = \frac{D_{eq}^2 \left( 2t \omega - 1 + e^{-2t\omega} \right)}{2 \left( \omega t (D_A + D_{eq}) - D_A \omega \tau_A + \omega D_A \tau_A e^{-\frac{t}{\tau_A}} \right)^2}. \tag{A13}
\]

From the above equation one can find the NGP at \( \tau_A = 0 \), which reads
\[
\gamma_G^0(t) = \frac{\langle x^4(t) \rangle}{3 \langle x^2(t) \rangle^2} - 1 = \frac{D_{eq}^2 \left( 2t \omega - 1 + e^{-2t\omega} \right)}{2 \omega^2 \tau^2 (D_A + D_{eq})^2}. \tag{A14}
\]

**Appendix B: Limiting values of PDF**

Here we show the calculation of approximate PDFs at different limits. Two models of the fluctuating thermal diffusivity are considered for the analysis.

1. **Switching diffusion**

Here one can assign two diffusive timescales \( D_2 p^2 \) and \( D_1 p^2 \), to compare them with the switching rates \( r_{12} \) and \( r_{21} \). For \( \frac{D_2 p^2}{r_{12}} \gg 1 \) and \( \frac{D_1 p^2}{r_{21}} \gg 1 \), one gets \( \phi_2 \approx \frac{1}{2} \left[ (D_2 - D_1)p^2 + (r_{21} - r_{12}) \right] + \frac{r_{12} r_{21}}{(D_2 - D_1)p^2 + (r_{21} - r_{12})} \). Clearly, it corresponds to the limiting case at shorter times. With the above approximation, Eq. (16) modifies to
\[
P(x,t|x_0,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ip(x-x_0)} \left( \frac{r_{12}}{r_{12} + r_{21}} e^{-D_2 p^2 t} e^{-D_A p^2 \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right)} \right. \\
+ \left. \frac{r_{21}}{r_{12} + r_{21}} e^{-D_1 p^2 t} e^{-D_A p^2 \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right)} \right) = \frac{r_{12}}{r_{12} + r_{21}} \sqrt{4\pi \left[ D_2 t + D_A \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right) \right]} + \frac{r_{21}}{r_{12} + r_{21}} \sqrt{4\pi \left[ D_1 t + D_A \left( t + \tau_A \left( e^{-\frac{t}{\tau_A}} - 1 \right) \right) \right].} \tag{B1}
\]
In the short-time limit, \( \frac{D_t}{r_{21}} p^2 \ll 1 \) and \( \frac{D_p}{r_{12}} p^2 \ll 1 \), the following approximations can be done, \( \phi_2 \approx \frac{1}{2} (r_{12} + r_{21}) - \frac{p^2}{2} \frac{r_{12} - r_{21}}{r_{12} + r_{21}} (D_2 - D_1) \), and \( \phi_1 - \phi_2 \approx p^2 \left( \frac{r_{12}}{r_{12} + r_{21}} D_2 + \frac{r_{21}}{r_{12} + r_{21}} D_1 \right) \approx p^2 D_{eq} \). Therefore, Eq. (16) approximates to

\[
P(x, t|x_0, 0) = \frac{-\frac{(x-x_0)^2}{4D_{eq}t+DA_t}}{\sqrt{4\pi[D_{eq}t+DA_t(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))]}}.
\] (B2)

**2. Diffusing diffusivity model**

In the short-time limit, \( \text{i.e.} \), for \( \beta t \ll 1 \), and for small and intermediate displacements (\( \text{i.e.} \), \( D_{eq} p^2 / \omega \gg 1 \)), the above can be approximated as

\[
P(x, t|x_0 = 0, 0) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} dpe^{-ipx} \frac{e^{-ipx}}{1 + D_{eq}p^2} e^{-D_{eq}p^2 \left( t + \tau_A \left( e^{-\frac{x}{\tau_A}} - 1 \right) \right)}
\]

\[
\approx \frac{1}{2\sqrt{D_{eq}t}} \frac{1}{4\pi[D_{A}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))]} \int_{-\infty}^{\infty} dx' e^{-\frac{|x-x'|}{\sqrt{D_{eq}t}}} e^{-\frac{x^2}{4D_{eq}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))}}
\]

\[
\approx e^{-\frac{DA_t}{D_{eq}}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))} \cosh \left( \frac{x}{\sqrt{D_{eq}t}} \right) - e^{-\frac{DA_t}{D_{eq}}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))} \text{erf} \left( \frac{x}{\sqrt{4[D_{A}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))]} \right) \sinh \left( \frac{x}{\sqrt{D_{eq}t}} \right).
\] (B3)

In the long \( t \) limit, \( \text{i.e.} \), for \( \beta t \gg 1 \), the PDF [Eq. (17)] can be approximated as

\[
P(x, t|x_0 = 0, 0) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} dpe^{-ipx} e^{-D_{eq}p^2 i} e^{-DA_t(p^2 \left( t + \tau_A \left( e^{-\frac{x}{\tau_A}} - 1 \right) \right))}
\]

\[
\approx \frac{1}{\sqrt{4\pi[D_{eq}t+DA_t(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))]} e^{-\frac{x^2}{4D_{eq}(t+\tau_A(e^{-\frac{x}{\tau_A}}-1))}}}
\] (B4)

[1] H. Shen, L. J. Tazin, R. Baiyasi, W. Wang, N. Moringo, B. Shuang, C. F. Landes, Single particle tracking: from theory to biophysical applications, Chem. Rev. 117 (11) (2017) 7331–7376.
[2] S. Ramaswamy, The mechanics and statistics of active matter, Annu. Rev. Condens. Matter Phys. 1 (1) (2010) 323–345.

[3] É. Fodor, M. C. Marchetti, The statistical physics of active matter: From self-catalytic colloids to living cells, Physica A 504 (2018) 106–120.

[4] C. Maggi, M. Paoluzzi, N. Pellicciotta, A. Lepore, L. Angelani, R. Di Leonardo, Generalized energy equipartition in harmonic oscillators driven by active baths, Phys. Rev. Lett. 113 (23) (2014) 238303.

[5] L. L. Bonilla, Active orstein-uhlenbeck particles Phys. Rev. E 100 (2019) 022601. doi:10.1103/PhysRevE.100.022601
URL https://link.aps.org/doi/10.1103/PhysRevE.100.022601

[6] D. Martin, J. O’Byrne, M. E. Cates, É. Fodor, C. Nardini, J. Tailleur, F. van Wijland, Statistical mechanics of active orstein-uhlenbeck particles, Phys. Rev. E 103 (3) (2021) 032607.

[7] S. Chaki, R. Chakrabarti, Entropy production and work fluctuation relations for a single particle in active bath, Physica A 511 (2018) 302–315.

[8] K. Goswami, Heat fluctuation of a harmonically trapped particle in an active bath, Phys. Rev. E 99 (1) (2019) 012112.

[9] K. Goswami, Work fluctuation relations for a dragged brownian particle in active bath, Physica A 525 (2019) 223–233.

[10] S. Chaki, R. Chakrabarti, Effects of active fluctuations on energetics of a colloidal particle: Superdiffusion, dissipation and entropy production, Physica A 530 (2019) 121574.

[11] K. Goswami, Work fluctuations in a generalized gaussian active bath, Physica A 566 (2021) 125609.

[12] B. Wang, S. M. Anthony, S. C. Bae, S. Granick, Anomalous yet brownian, Proc. Natl. Acad. Sci. U.S.A. 106 (36) (2009) 15160–15164.

[13] J. Guan, B. Wang, S. Granick, Even hard-sphere colloidal suspensions display fickian yet non-gaussian diffusion, ACS nano 8 (4) (2014) 3331–3336.

[14] G. Kwon, B. J. Sung, A. Yethiraj, Dynamics in crowded environments: is non-gaussian brownian diffusion normal?, J. Phys. Chem. B 118 (28) (2014) 8128–8134.

[15] P. Kumar, L. Theeyancheri, S. Chaki, R. Chakrabarti, Transport of probe particles in a polymer network: effects of probe size, network rigidity and probe–polymer interaction, Soft Matter 15 (44) (2019) 8992–9002.

[16] I. Chakraborty, G. Rahamim, R. Avinery, Y. Roichman, R. Beck, Nanoparticle mobility over a surface as a probe for weak transient disordered peptide–peptide interactions, Nano Lett. 19 (9) (2019) 6524–
[17] M. J. Skaug, J. Mabry, D. K. Schwartz, Intermittent molecular hopping at the solid-liquid interface, Phys. Rev. Lett. 110 (25) (2013) 256101.

[18] I. Chakraborty, Y. Roichman, Disorder-induced fickian, yet non-gaussian diffusion in heterogeneous media, Phys. Rev. Research 2 (2020) 022020. doi:10.1103/PhysRevResearch.2.022020 URL https://link.aps.org/doi/10.1103/PhysRevResearch.2.022020

[19] C. Xue, X. Zheng, K. Chen, Y. Tian, G. Hu, Probing non-gaussianity in confined diffusion of nanoparticles, J. Phys. Chem. Lett. 7 (3) (2016) 514–519.

[20] N. Samanta, R. Chakrabarti, Tracer diffusion in a sea of polymers with binding zones: mobile vs. frozen traps, Soft Matter 12 (41) (2016) 8554–8563.

[21] B. Wang, J. Kuo, S. C. Bae, S. Granick, When brownian diffusion is not gaussian, Nat. Mater. 11 (6) (2012) 481–485.

[22] M. V. Chubynsky, G. W. Slater, Diffusing diffusivity: A model for anomalous, yet brownian, diffusion, Phys. Rev. Lett. 113 (2014) 098302. doi:10.1103/PhysRevLett.113.098302 URL https://link.aps.org/doi/10.1103/PhysRevLett.113.098302

[23] C. Beck, E. G. Cohen, Superstatistics, Physica A 322 (2003) 267–275.

[24] R. Zwanzig, Rate processes with dynamical disorder, Acc. Chem. Res. 23 (5) (1990) 148–152.

[25] N. Tyagi, B. J. Cherayil, Non-gaussian brownian diffusion in dynamically disordered thermal environments, J. Phys. Chem. B 121 (29) (2017) 7204–7209.

[26] R. Jain, K. L. Sebastian, Diffusion in a crowded, rearranging environment, J. Phys. Chem. B 120 (16) (2016) 3988–3992. doi:10.1021/acscpcb.6b01527 URL https://doi.org/10.1021/acscpcb.6b01527

[27] R. Jain, K. L. Sebastian, Diffusing diffusivity: survival in a crowded rearranging and bounded domain, J. Phys. Chem. B 120 (34) (2016) 9215–9222.

[28] R. Jain, K. L. Sebastian, Lévy flight with absorption: A model for diffusing diffusivity with long tails, Phys. Rev. E 95 (2017) 032135. doi:10.1103/PhysRevE.95.032135 URL https://link.aps.org/doi/10.1103/PhysRevE.95.032135

[29] A. V. Chechkin, F. Seno, R. Metzler, I. M. Sokolov, Brownian yet non-gaussian diffusion: from superstatistics to subordination of diffusing diffusivities, Phys. Rev. X 7 (2) (2017) 021002.

[30] R. Metzler, Gaussianity fair: the riddle of anomalous yet non-gaussian diffusion, Biophys. J. 112 (3) (2017) 413.
[31] V. Sposini, A. V. Chechkin, F. Seno, G. Pagnini, R. Metzler, Random diffusivity from stochastic equations: comparison of two models for brownian yet non-gaussian diffusion, New J. Phys. 20 (4) (2018) 043044.

[32] R. Yamamoto, A. Onuki, Heterogeneous diffusion in highly supercooled liquids, Phys. Rev. Lett. 81 (1998) 4915–4918. doi:10.1103/PhysRevLett.81.4915
URL https://link.aps.org/doi/10.1103/PhysRevLett.81.4915

[33] J. S. Leith, A. Tafvizi, F. Huang, W. E. Uspal, P. S. Doyle, A. R. Fersht, L. A. Mirny, A. M. van Oijen, Sequence-dependent sliding kinetics of p53, Proc. Natl. Acad. Sci. U.S.A. 109 (41) (2012) 16552–16557.

[34] B. R. Parry, I. V. Surovtsev, M. T. Cabeen, C. S. O’Hern, E. R. Dufresne, C. Jacobs-Wagner, The bacterial cytoplasm has glass-like properties and is fluidized by metabolic activity, Cell 156 (1-2) (2014) 183–194.

[35] L. Cuculis, Z. Abil, H. Zhao, C. M. Schroeder, Direct observation of tale protein dynamics reveals a two-state search mechanism, Nat. Commun. 6 (1) (2015) 1–11.

[36] S. Mora, Y. Pomeau, Brownian diffusion in a dilute field of traps is fickean but non-gaussian, Phys. Rev. E 98 (2018) 040101. doi:10.1103/PhysRevE.98.040101
URL https://link.aps.org/doi/10.1103/PhysRevE.98.040101

[37] T. Miyaguchi, T. Akimoto, E. Yamamoto, Langevin equation with fluctuating diffusivity: A two-state model, Phys. Rev. E 94 (2016) 012109. doi:10.1103/PhysRevE.94.012109
URL https://link.aps.org/doi/10.1103/PhysRevE.94.012109

[38] P. C. Bressloff, Stochastic switching in biology: from genotype to phenotype, J. Phys. A: Math. Theor. 50 (13) (2017) 133001.

[39] G. Mongillo, D. Hansel, C. van Vreeswijk, Bistability and spatiotemporal irregularity in neuronal networks with nonlinear synaptic transmission, Phys. Rev. Lett. 108 (2012) 158101. doi:10.1103/PhysRevLett.108.158101
URL https://link.aps.org/doi/10.1103/PhysRevLett.108.158101

[40] A. Godec, R. Metzler, First passage time statistics for two-channel diffusion, J. Phys. A: Math. Theor. 50 (8) (2017) 084001.

[41] S. Paul, S. Ghosh, D. S. Ray, Reaction–diffusion systems with fluctuating diffusivity; spatio-temporal chaos and phase separation, J. Stat. Mech.: Theory Exp. 2018 (3) (2018) 033205.

[42] O. Chepizhko, E. G. Altmann, F. Peruani, Optimal noise maximizes collective motion in heterogeneous media, Phys. Rev. Lett. 110 (2013) 238101. doi:10.1103/PhysRevLett.110.238101
[43] M. Bley, P. I. Hurtado, J. Dzubiella, A. M. Jorda, Active interaction switching controls the dynamic heterogeneity of soft colloidal dispersions, Soft Matter (2022). doi:10.1039/D1SM01507A.
[44] C. O. Reichhardt, C. Reichhardt, Avalanche dynamics for active matter in heterogeneous media, New J. Phys. 20 (2) (2018) 025002.
[45] C. Bechinger, R. Di Leonardo, H. Löwen, C. Reichhardt, G. Volpe, G. Volpe, Active particles in complex and crowded environments, Rev. Mod. Phys. 88 (2016) 045006. doi:10.1103/RevModPhys.88.045006.
[46] J.-L. Thiffeault, J. Guo, Shake your hips: An anisotropic active brownian particle with a fluctuating propulsion force, arXiv preprint arXiv:2102.11758 (2021).
[47] S. M. J. Khadem, N. H. Siboni, S. H. L. Klapp, Transport and phase separation of active brownian particles in fluctuating environments (2021). arXiv:2103.16582.
[48] S. Chaki, R. Chakrabarti, Escape of a passive particle from an activity-induced energy landscape: emergence of slow and fast effective diffusion, Soft Matter 16 (30) (2020) 7103–7115.
[49] D. A. Garcia, G. Fettweis, D. M. Presman, V. Paakinaho, C. Jarzynski, A. Upadhyaya, G. L. Hager, Nar breakthrough article power-law behavior of transcription factor dynamics, Nucleic Acids Res. (2021) 1.
[50] K. Goswami, K. L. Sebastian, Exact solution to the first-passage problem for a particle with a dichotomous diffusion coefficient, Phys. Rev. E 102 (2020) 042103. doi:10.1103/PhysRevE.102.042103.
[51] H. Risken, Fokker-planck equation, in: The Fokker-Planck Equation, Springer, 1996, pp. 63–95.
[52] K. Goswami, Nonequilibrium dynamics and thermodynamics of some single-particle activity-induced diffusive systems, Ph.D. thesis (2021).
[53] K. Goswami, K. L. Sebastian, Diffusion caused by two noises-active and thermal, J. Stat. Mech.: Theory Exp. 2019 (8) (2019) 083501.
[54] A. Pacheco-Pozo, I. M. Sokolov, Convergence to a gaussian by narrowing of central peak in brownian yet non-gaussian diffusion, Phys. Rev. Lett. 127 (2021) 120601. doi:10.1103/PhysRevLett.127.120601.
[55] T. Uneyama, T. Miyaguchi, T. Akimoto, Fluctuation analysis of time-averaged mean-square displacement for the langevin
[56] N. Ahamad, P. Debnath, Rouse model in crowded environment modeled by “diffusing diffusivity”, Physica A 549 (2020) 124335.

[57] V. Balakrishnan, On a simple derivation of master equations for diffusion processes driven by white noise and dichotomic markov noise, Pramana 40 (4) (1993) 259–265.