On the Tate-Shafarevich group of Abelian schemes over higher dimensional bases over finite fields

Timo Keller

October 21, 2014

Abstract

We study analogues for the Tate-Shafarevich group for Abelian schemes with everywhere good reduction over higher dimensional bases over finite fields.

Keywords: Étale cohomology, higher regulators, zeta and $L$-functions, Brauer groups of schemes

MSC 2010: 19F27, 14F22

1 Introduction

The Tate-Shafarevich group $\text{III}(\mathcal{A}/X)$ of an Abelian scheme $\mathcal{A}$ over a scheme $X$ is of great importance for the arithmetic of $\mathcal{A}$. It classifies everywhere locally trivial $\mathcal{A}$-torsors. If $X$ is the spectrum of the ring of integers of a number field or a smooth projective geometrically connected curve over a finite field, so that the function field $K = K(X)$ of $X$ is a global field, one has

$$\text{III}(\mathcal{A}/X) = \ker \left( H^1(K, A) \to \prod_v H^1(K_v, A) \right),$$

where $v$ runs over all places of $K$ and $K_v$ is the completion of $K$ with respect to $v$. The aim of this article is to generalise this definition to the case of a higher dimensional basis $X/F_q$ and prove some properties for this group.

In section 3, we show that an Abelian scheme $\mathcal{A}/X$ over $X$ regular, Noetherian, integral and separated satisfies the Néron mapping property, namely that $\mathcal{A} = g_* g^* \mathcal{A}$ on the smooth site of $X$, where $g : \{\eta\} \to X$ denotes the inclusion of the generic point. In section 4, we define the Tate-Shafarevich group for Abelian schemes $\mathcal{A}$ over higher dimensional bases $X$ as $\text{III}(\mathcal{A}/X) := H^1_{\text{ét}}(X, \mathcal{A})$ and show:

$$H^1(X, \mathcal{A}) = \ker \left( H^1(K, A) \to \prod_{x \in S} H^1(K_{nr}^x, \mathcal{A}) \right),$$

where $S$ is (a) the set of all points $X$, (b) the set of all closed points $|X|$, or (c) the set of all codimension-1 points $X^{(1)}$ and $\mathcal{A} = \text{Pic}_{\mathcal{O}/X}$, and $K_{nr}^x = \text{Quot}(\mathcal{O}_{X,x}^{\text{nr}})$. The obvious conjecture is that the Tate-Shafarevich group $\text{III}(\mathcal{A}/X)$ is finite.

In section 2 and 4.1, for a (split) relative curve $\mathcal{C}/X$ we relate the Brauer groups of $X$ and $\mathcal{C}$ to the Tate-Shafarevich group of $\text{Pic}_{\mathcal{C}/X}$: There is an exact sequence

$$0 \to \text{Br}(X) \xrightarrow{\pi_*} \text{Br}(\mathcal{C}) \to \text{III}(\text{Pic}_{\mathcal{C}/X}^0/X) \to 0.$$

This generalises results of Artin and Tate [Tat66].

In section 4.2, we show that finiteness of an $\ell$-primary component of the Tate-Shafarevich group descents under generically étale $\ell'$-alterations. This is used in [Kel14], Theorem 4.18 and Remark 4.19 to prove the finiteness of the Tate-Shafarevich group for isotrivial Abelian schemes under mild conditions. In section 4.3, we show that finiteness of an $\ell$-primary component of the Tate-Shafarevich group is invariant under étale isogenies. In section 4.4, we construct a Cassels-Tate pairing $\text{III}(\mathcal{A}/X) \times \text{III}(\mathcal{A}'/X) \to \mathbb{Q}/\mathbb{Z}$ in some cases. In another article [Kel14], we use the results of this article for studying the Birch-Swinnerton-Dyer conjecture for Abelian schemes over higher dimensional bases over finite fields.
Notation. For an Abelian group $A$, let $\text{Tor} A$ be the torsion subgroup of $A$, and $A_{\text{tors}} = A/\text{Tor} A$. Let $\text{Div} A$ be the maximal divisible subgroup of $A$ and $A_{\text{Div}} = A/\text{Div} A$. Denote the cokernel of $A \to A/n$ by $A[n]$, and the $p$-primary subgroup $\lim_{\rightarrow} A[p^n]$ by $A[p^{\infty}]$ for any prime $p$. Canonical isomorphisms are often denoted by “$\sim$”. If not stated otherwise, all cohomology groups are taken with respect to the étale topology. We denote Pontryagin duals, duals of $R$-modules and Abelian schemes by $(\cdot)^\vee$. It should be clear from the context which one is meant. The Henselisation of a (local) ring $A$ is denoted by $A^h$ and the strict Henselisation by $A^s$.

2 Brauer groups, Picard groups and cohomology of $G_m$

We collect some results on the cohomology of $G_m$.

Lemma 2.1. Let $X$ be a scheme and $\ell$ a prime invertible on $X$. Then there are exact sequences

$$0 \to H^{i-1}(X, G_m) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \to H^i(X, \mu_{\ell^{\infty}}) \to H^i(X, G_m)[\ell^{\infty}] \to 0$$

for each $i \geq 1$.

Proof. This follows from the long exact sequence induced by the Kummer sequence. $\square$

Definition 2.2. A variety over a field $k$ is a separated scheme of finite type over $k$.

Recall the definition [Mi80], IV.2, p. 140ff. of the Brauer group $Br(X)$ of a scheme $X$ as the group of equivalence classes of Azumaya algebras on $X$.

Definition 2.3. $Br'(X) := \text{Tor} H^2(X, G_m)$ is called the cohomological Brauer group.

Theorem 2.4. (a) $Br'(X) = H^2(X, G_m)$ if $X$ is a regular integral quasi-compact scheme.

(b) There is an injection $Br(X) \hookrightarrow Br'(X)$, where $Br(X)$ is the Brauer group of $X$.

(c) Let $X$ be a scheme endowed with an ample invertible sheaf. Then $Br(X) = Br'(X)$.

Proof. (a) See [Mi80], p. 106f., Example 2.22. (b) See [Mi80], p. 142, Theorem 2.5. (c) See [dJ]. $\square$

Corollary 2.5. Let $X/k$ be a regular quasi-projective geometrically connected variety. Then $Br(X) = Br'(X) = H^2(X, G_m)$.

Theorem 2.6. Let $X$ be a smooth projective geometrically connected variety over a finite field $k = F_q$, $q = p^n$.

(a) $H^i(X, G_m)$ is torsion for $i \neq 1$, finite for $i \neq 1, 2, 3$ and trivial for $i > 2 \dim(X) + 1$.

(b) For $\ell \neq p$ and $i = 2, 3$, one has $H^i(X, G_m)[\ell^{\infty}] = (\mathbb{Q}/\mathbb{Z})^{\psi_{i,\ell}} \oplus C_{i,\ell}$, where $C_{i,\ell}$ is finite and trivial for all but finitely many $\ell$, and $\psi_{i,\ell}$ a non-negative integer.

(c) Let $\ell \neq p$ be prime. Then one has

$$H^i(X, \mu_{\ell^{\infty}}) \cong H^i(X, G_m)[\ell^{\infty}]$$

for $i \neq 2$.

Proof. (a) and (b): See [Li83], p. 180, Proposition 2.1a–c), f). (c) follows from Lemma 2.1 and (a). $\square$

Lemma 2.7. Let $X$ be a quasi-compact scheme, quasi-projective over an affine scheme. Assume $A \in H^1(X, \text{PGL}_n)$ is an Azumaya algebra trivialised by $A \cong \text{End}(V)$ with $V \in H^1(X, \text{GL}_n)$ a locally free sheaf of rank $n$. Then every other such $V'$ differs from $V$ by tensoring with an invertible sheaf.

Proof. Consider for $n \in \mathbb{N}$ the central extension of étale sheaves on $X$ (see [Mi80], p. 146)

$$1 \to G_m \to \text{GL}_n \to \text{PGL}_n \to 1.$$ 

By [Mi80], p. 143, Step 3, this induces a long exact sequence in (Čech) cohomology of pointed sets

$$\text{Pic}(X) = \check{H}^1(X, G_m) \xrightarrow{g} \check{H}^1(X, \text{GL}_n) \xrightarrow{h} \check{H}^1(X, \text{PGL}_n) \xrightarrow{i} \check{H}^2(X, G_m).$$
Note that by assumption and \([\text{Mil80}], p. 104\), Theorem III.2.17, \(H^1(X, G_m) = H^1(X, G_m) = \text{Pic}(X)\) and \(H^2(X, G_m) = H^2(X, G_m)\). Further, \(\text{Br}(X) = \text{Br}(X)\) since a scheme quasi-compact and quasi-projective over an affine scheme has an ample line bundle (\([\text{Lin96}], p. 171\), Corollary 5.1.36), so Theorem 2.4 applies and \(\text{Br}'(X) \rightarrow H^2(X, G_m)\). Since \(A\) is an Azumaya algebra, \(f(A) = \{A\} \in \text{Br}(X) \rightarrow H^2(X, G_m)\). Therefore \(f\) factors through \(\text{Br}(X) \rightarrow H^2(X, G_m)\).

Assume the Azumaya algebra \(A \in H^1(X, \text{PGL}_n)\) lies in the kernel of \(f\), i.e. there is a \(V\) such that \(A \cong \text{End}(V)\). Then it comes from \(V \in H^1(X, \text{GL}_n)\) by \([\text{Mil80}], p. 143\), Step 2 (\(h\) is the morphism \(V \rightarrow \text{End}(V)\)). So, since \(G_m\) is central in \(\text{GL}_n\), by the analogue of \([\text{Scr02}], p. 54\), Proposition 42 for étale Čech cohomology, if \(V' \in H^1(X, \text{GL}_n)\) also satisfies \(A \cong \text{End}(V')\), they differ by an invertible sheaf.

**Lemma 2.8.** Let \(f : \mathcal{C} \rightarrow X\) be a projective smooth morphism with \(X\) the spectrum of a Henselisation of a variety at a regular prime ideal. Suppose the transition maps of \((\text{Pic}(\mathcal{C}_n))_{n \in \mathbb{N}}\) are surjective (in fact, the Mittag-Leffler condition would suffice). Then the canonical homomorphism

\[
\text{Br}(\mathcal{C}) \rightarrow \lim_{\leftarrow n} \text{Br}(\mathcal{C}_n)
\]

is injective.

**Proof.** Let \(A\) be an Azumaya algebra over \(\mathcal{C}\) which lies in the kernel of the map in this lemma, i.e. such that for every \(n \in \mathbb{N}\) there is an isomorphism

\[
u_n : A_n \cong \text{End}(V_n)\tag{2.1}
\]

with \(V_n\) a locally free \(\mathcal{O}_{\mathcal{C}_n}\)-module. Such a \(V_n\) is uniquely determined by \(A_n\) modulo tensoring with an invertible sheaf \(\mathcal{L}_n\) by Lemma 2.7.

Because of surjectivity of the transition maps of \((\text{Pic}(\mathcal{C}_n))_{n \in \mathbb{N}}\), one can choose the \(V_n, \nu_n\) such that the \(V_n\) and \(\nu_n\) form a projective system:

\[
V_n = V_{n+1} \otimes_{\mathcal{O}_{\mathcal{C}_{n+1}}} \mathcal{O}_{\mathcal{C}_n}\tag{2.2}
\]

and the isomorphisms \((2.1)\) also form a projective system: Construct the \(V_n, \nu_n\) inductively. Take \(V_0\) such that

\[
A \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\mathcal{C}_0} \cong A_0 \cong \text{End}(V_0).
\]

One has

\[
A_n = A \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{O}_{\mathcal{C}_n}
\]

and by Lemma 2.7 there is an invertible sheaf \(\mathcal{L}_n \in \text{Pic}(\mathcal{C}_n)\) such that

\[
V_{n+1} \otimes_{\mathcal{O}_{\mathcal{C}_{n+1}}} \mathcal{O}_{\mathcal{C}_n} \xrightarrow{\cong} V_n \otimes_{\mathcal{O}_{\mathcal{C}_n}} \mathcal{L}_n.
\]

By assumption, there is an invertible sheaf \(\mathcal{L}_{n+1} \in \text{Pic}(\mathcal{C}_{n+1})\) such that \(\mathcal{L}_{n+1} \otimes_{\mathcal{O}_{\mathcal{C}_{n+1}}} \mathcal{O}_{\mathcal{C}_n} \cong \mathcal{L}_n\), so redefine \(V_{n+1}\) as \(V_{n+1} \otimes_{\mathcal{O}_{\mathcal{C}_{n+1}}} \mathcal{L}_{n+1}^{-1}\). Then \((2.2)\) is satisfied.

Let \(\hat{X}\) be the completion of \(X\), and denote by \(\hat{\mathcal{C}}, \hat{A}, \ldots\) the base change of \(\mathcal{C}, \mathcal{A}, \ldots\) by \(\hat{X} \rightarrow X\).

Recall that an adic Noetherian ring \(A\) with defining ideal \(\mathcal{J}\) is a Noetherian ring with a basis of neighbourhoods of zero of the form \(\mathcal{J}^n, n > 0\) such that \(A\) is complete and Hausdorff in this topology. For such a ring \(A\), there is the formal spectrum \(\text{Spf}(A)\) with underlying space \(\text{Spec}(A/\mathcal{J})\).

According to \([\text{EGAIII}], p. 150, Théorème (5.1.4)\), to give a projective system \((V_n, \nu_n)_{n \in \mathbb{N}}\) on \((\mathcal{C}_n)_{n \in \mathbb{N}}\) as in \((2.1)\) and \((2.2)\) is equivalent to giving a locally free module \(\hat{V}\) on \(\hat{\mathcal{C}}\) and an isomorphism

\[
\hat{u} : \hat{A} \xrightarrow{\cong} \text{End}(\hat{V}).\tag{2.3}
\]

If \(X = \hat{X}\), we are done: \(\hat{A} = \hat{A}\) is trivial.

In the general case, one has to pay attention to the fact that one does not know if, with the preceding construction, \(\hat{V}\) comes from a locally free module \(\hat{V}\) on \(\hat{\mathcal{C}}\). However, there is a locally free module \(\mathcal{E}\) on \(\mathcal{C}\) such that there exists an epimorphism

\[
\mathcal{E} \rightarrow \hat{V}.
\]

Indeed, choosing a projective immersion for \(\mathcal{C}\) (by projectivity of \(\mathcal{C}/\text{Spec}(\hat{A})\)) with an ample invertible sheaf \(\mathcal{O}_{\mathcal{E}}(1)\), it suffices to take a direct sum of sheaves of the form \(\mathcal{O}_{\mathcal{E}}(-N), N \gg 0\). Now, for \(N \gg 0\), there is an epimorphism \(\mathcal{O}_{\mathcal{E}} \rightarrow \hat{V}(N)\) for a suitable \(k \in \mathbb{N}\), so twisting with \(\mathcal{O}_{\mathcal{E}}(-N)\) gives

\[
\mathcal{O}_{\mathcal{E}}(-N)^{\oplus k} \rightarrow \hat{V}.
(“there are enough vector bundles”). Set \( \mathcal{O}_E(-N)^{\oplus k} \)

Now consider, for schemes \( X' \) over \( X \), the contravariant functor \( F: (\text{Sch}/X) \to \text{Set} \) given by \( F(X') = \{ \text{pairs } (V', \varphi') \text{ where } V' \text{ is a quotient of a locally free module } \mathcal{O}_X \otimes \mathcal{O}_X X' \text{ and } \varphi': A' = \mathcal{O}_X X' \to \text{End}(V') \}. \)

Since \( f \) is projective and flat, by [SGA4.3], p. 133 ff., Lemme XIII 1.3, one sees that the functor \( F \) is representable by a scheme, also denoted \( F \), locally of finite type over \( X \), hence locally of finite presentation since our schemes are Noetherian (what matters is the functor being locally of finite presentation, not its representability).

By assumption of Lemma 2.8 and (2.3), \((\hat{V}, \hat{u})\) is an element from \( F(X) \). By Artin approximation [Art69], p. 26, Theorem (1.10) resp. Theorem (1.12), \( F(X) \neq \emptyset \) implies \( F(X) \neq \emptyset \).

This proves that \( A \) is isomorphic to an algebra of the form \( \text{End}(V) \) with \( V \) locally free over \( \mathcal{O} \), so it is trivial as an element of \( \text{Br}(\mathcal{O}) \).

The following is a generalisation of [Gro65], pp. 98–104, Théorème (3.1) from the case of \( X/Y \) with \( \dim X = 2 \), \( \dim Y = 1 \) to \( X/Y \) with relative dimension 1. One can remove the assumption \( \dim X = 1 \) if one uses Artin’s approximation theorem [Art69], p. 26, Theorem (1.10) resp. Theorem (1.12) instead of Greenberg’s theorem on p. 104, 1. 4 and 1. -2, and replaces “proper” by “projective” and does some other minor modifications; also note that in our situation the Brauer group coincides with the cohomological Brauer group by Theorem 2.6 and Theorem 2.3.

**Theorem 2.9.** Let \( f: \mathcal{O} \to X \) be a smooth projective morphism with fibres of dimension \( \leq 1 \), \( \mathcal{O} \) and \( X \) regular and \( X \) the spectrum of a Henselisation of a variety at a prime ideal with closed point \( x \), and \( \mathcal{O}_0 \to \mathcal{O} \) the subscheme \( f^{-1}(x) \). Then the canonical homomorphism

\[
H^2(\mathcal{O}, \mathbb{G}_m) \to H^2(\mathcal{O}_0, \mathbb{G}_m)
\]

induced by the closed immersion \( \mathcal{O}_0 \to \mathcal{O} \) is bijective.

*Proof.* Let \( X = \text{Spec}(A), X_n = \text{Spec}(A/m^{n+1}), \mathcal{O}_n = \mathcal{O} \times_X X_n. \)

Note that for \( \mathcal{O} \) and \( X \), \( \text{Br} \) and \( H^2(\mathcal{O}, \mathbb{G}_m) \) are equal since there is an ample sheaf (Theorem 2.4) and by regularity (Theorem 2.4).

There are exact sequences for every \( n \)

\[
0 \to \mathcal{F} \to \mathbb{G}_m, \mathcal{O}_{n+1} \to \mathbb{G}_m, \mathcal{O}_n \to 1
\]

(2.4)

with \( \mathcal{F} \) a coherent sheaf on \( \mathcal{O}_0 \); Zariski-locally on the source, \( \mathcal{O} \to X \) is of the form \( \text{Spec}(B) \to \text{Spec}(A) \) and hence \( \mathcal{O}_n \to X_n \) of the form \( \text{Spec}(B/m^{n+1}) \to \text{Spec}(A/m^{n+1}) \). There is an exact sequence

\[
1 \to (1 + m^n/m^{n+1}) \to (B/m^{n+1})^\times \to (B/m^n)^\times \to 1.
\]

The latter map is surjective since \( m^n/m^{n+1} \subset B/m^{n+1} \) is nilpotent (deformation of units: Let \( f: B \to A \) be a surjective ring homomorphism with nilpotent kernel. If \( f(b) \) is a unit, so is \( b \): this is because a unit plus a nilpotent element is a unit: Let \( f(b)c = 1_A \). Then there is a \( \tilde{c} \in B \) such that \( bc - 1_B \in \ker(f) \), so \( bc \) is a unit, so \( b \) is invertible in \( B \). By the logarithm, \((1 + m^n/m^{n+1}) \to m^n/m^{n+1} \) is a coherent sheaf on \( \text{Spec}(B) \). The sequences for a Zariski-covering of \( \mathcal{O}_0 \) glue to an exact sequence of sheaves on \( \mathcal{O}_0 \) [2.4], equivalently, on \( \mathcal{O}_n \) for any \( n \) since these have the same underlying topological space.

Therefore, the associated long exact sequence to (2.4) yields

\[
H^2(\mathcal{O}_0, \mathcal{F}) \to H^2(\mathcal{O}_0, \mathbb{G}_m, \mathcal{O}_{n+1}) \to H^2(\mathcal{O}_0, \mathbb{G}_m, \mathcal{O}_n) \to H^3(\mathcal{O}_0, \mathcal{F}).
\]

Now, \( H^2_{\mathcal{O}_0}(\mathcal{O}_0, \mathcal{F}) = H^2_{\mathcal{O}_0}(\mathcal{O}_0, \mathcal{F}) \) since \( \mathcal{F} \) is coherent by [SGA4.2a], VII 4.3. Thus, since \( \dim \mathcal{O}_0 \leq 1 \), \( H^2(\mathcal{O}_0, \mathcal{F}) = H^3(\mathcal{O}_0, \mathcal{F}) = 0 \). Thus we get an isomorphism

\[
H^2(\mathcal{O}_0, \mathbb{G}_m, \mathcal{O}_{n+1}) \cong H^2(\mathcal{O}_0, \mathbb{G}_m, \mathcal{O}_n)
\]

Next note that \( \mathcal{O}_0 \to \mathcal{O}_n \) is a closed immersion defined by a nilpotent ideal sheaf, so there is an equivalence of categories of \( \mathcal{O}_0 \)-sheaves and \( \mathcal{O}_n \)-sheaves by [Mil80], p. 30, Theorem I.3.23, so we get

\[
H^2(\mathcal{O}_{n+1}, \mathbb{G}_m) \cong H^2(\mathcal{O}_n, \mathbb{G}_m).
\]

Taking torsion, it follows that \( \text{Br}^t(\mathcal{O}_{n+1}) \cong \text{Br}^t(\mathcal{O}_n) \), and then Theorem 2.4 yields that the \( \text{Br}(\mathcal{O}_{n+1}) \to \text{Br}(\mathcal{O}_n) \) are isomorphisms (in fact, injectivity suffices for the following). Therefore the injectivity of \( \text{Br}(\mathcal{O}) \to \text{Br}(\mathcal{O}_0) \).
follows from Lemma 2.8. One can apply this in our situation since the transition maps \( \text{Pic}(\mathscr{C}_{n+1}) \to \text{Pic}(\mathscr{C}_n) \) are surjective by [EGAIV], p. 288, Corollaire (21.9.12).

The surjectivity in Theorem 2.12 is shown analogously: Take an element of \( \text{Br}(\mathscr{C}_0) \), represented by an Azumaya algebra \( A_0 \). As \( \text{Br}(\mathscr{C}_n) \twoheadrightarrow \text{Br}(\mathscr{C}_0) \), see above, there is a compatible system of Azumaya algebras \( A_n \) on \( \mathscr{C}_n \). Therefore, as above, there is an Azumaya algebra \( A \) on \( \mathscr{C} \). Choose a locally free module \( E \) on \( \mathscr{C} \) such that there is an epimorphism \( \mathscr{E} \twoheadrightarrow A \) and consider the functor \( F : (\text{Sch}/X)^{\circ} \to \text{Set} \) defined by \( F(X') = \{ \text{the set of pairs } (E', p') \} \) where \( E' \) is a locally free module of \( \mathscr{E} \otimes X' \) and \( p' \) a multiplication law on \( E' \) which makes it into an Azumaya algebra. Then \( F \) is representable by a scheme locally of finite type over \( X \) (loc. cit.), and the point in \( F(X) \) (recall that \( x \) is the closed point of \( X \)) defined by \( A_0 \) gives us a point in \( F(X) \), so by Artin approximation [Art69], p. 26, Theorem (1.10) resp. Theorem (1.12) it comes from a point in \( F(X) \), which proves surjectivity.

\[ \square \]

**Lemma 2.10.** Let \( X \) be a scheme, \( \ell \) a prime number and \( n \) an integer such that \( \text{cd}_\ell(X) \leq n \). Then \( H^i(X, G_m)[\ell^\infty] = 0 \) if \( i > n + 1 \), resp. \( i > n \) if \( \ell \) is invertible on \( X \).

**Proof.** If \( \ell \) is invertible on \( X \), the Kummer sequence

\[
1 \to \mu_{\ell^r} \to G_m \xrightarrow{\ell^r} G_m \to 1
\]

induces a long exact sequence in cohomology, part of which is

\[
0 = H^{i+1}(X, \mu_{\ell^r}) \to H^{i+1}(X, G_m) \xrightarrow{\ell^r} H^{i+1}(X, G_m) \to H^{i+2}(X, \mu_{\ell^r}) = 0.
\]

for \( i > n \), i.e. multiplication by \( \ell^r \) induces an isomorphism on \( H^{i+1}(X, G_m) \) for \( i > n \). If \( c \in H^{i+1}(X, G_m)[\ell^r] \subseteq H^{i+1}(X, G_m)[\ell^\infty] \), then \( 0 = \ell^rc \), therefore \( c = 0 \) by the injectivity of \( \ell^r \), so \( H^i(X, G_m)[\ell^\infty] = 0 \) for \( i > n \).

If \( \ell = p \), one has an exact sequence

\[
1 \to \ker(\ell^r) \to G_m \xrightarrow{\ell^r} G_m \to \text{coker}(\ell^r) \to 1
\]

of \( \text{étale} \) sheaves which splits up into

\[
1 \longrightarrow \ker(\ell^r) \longrightarrow G_m \xrightarrow{\ell^r} G_m \longrightarrow \text{coker}(\ell^r) \longrightarrow 1
\]

\( \text{im}(\ell^r) \)

By the same argument as in the case \( \ell \) invertible on \( X \), one finds \( H^i(X, G_m) \xrightarrow{\sim} H^i(X, \text{im}(\ell^r)) \) for \( i > n \), and, since \( \ell^r \text{coker}(\ell^r) = 0 \), \( H^i(X, \text{im}(\ell^r)) \xrightarrow{\sim} H^i(X, G_m) \) for \( i > n + 1 \). So, altogether

\[
\ell^r : H^i(X, G_m) \xrightarrow{\sim} H^i(X, \text{im}(\ell^r)) \xrightarrow{\sim} H^i(X, G_m)
\]

is injective for \( i > n + 1 \), and therefore \( H^i(X, G_m)[\ell^\infty] = 0 \) for \( i > n + 1 \).

\[ \square \]

**Lemma 2.11.** Let \( C/K \) be a projective regular curve over a separably closed field. Then \( \text{Br}(C) = \text{Br}^r(C) = H^2(C, G_m) = 0 \).

**Proof.** One has \( \text{Br}(C) = \text{Br}^r(C) = 0 \) by [Gro68], p. 132, Corollaire (5.8) since \( C \) is a proper curve over a separably closed field. Moreover, Theorem 2.4 implies \( \text{Br}^r(C) = H^2(C, G_m) \).

\[ \square \]

**Theorem 2.12.** Let \( \pi : \mathscr{C} \to X \) be projective and smooth with \( \mathscr{C} \) and \( \mathscr{X} \) regular, all fibres of dimension 1 and \( X \) be a variety. Then

\[
R^q\pi_* G_m = 0 \quad \text{for } q > 1.
\]

**Proof.** By [SGA4.2a], VIII 5.2, resp. [Mil80], p. 88, III.1.15 one can assume \( X \) strictly local and we must prove \( H^i(\mathscr{C}, G_m) = 0 \) for \( i > 1 \). By the proper base change theorem [Mil80], p. 224, Corollary VI.2.7, one has for torsion sheaves \( \mathscr{F} \) on \( \mathscr{C} \) with restriction \( \mathscr{F}_0 \) to the closed fibre \( \mathscr{C}_0 \) restriction isomorphisms

\[
H^i(\mathscr{C}, \mathscr{F}) \to H^i(\mathscr{C}_0, \mathscr{F}_0).
\]
Since \( \dim \mathfrak{C}_0 = 1 \), the latter term vanishes for \( i > 2 \) and for \( i > 1 \) if \( \mathfrak{C} \) is \( p \)-torsion, where \( p \) is the residue field characteristic. Therefore
\[
\text{cd}(\mathfrak{C}) \leq 2, \quad \text{and} \quad \text{cd}_p(\mathfrak{C}) \leq 1.
\]
The relation \( H^i(\mathfrak{C}, G_m) = 0 \) for \( i > 2 \) follows from the fact that these groups are torsion by Theorem 2.6(a) and from Lemma 2.10.

It remains to treat the case \( i = 2 \), i.e. to prove
\[
H^2(\mathfrak{C}, G_m) = 0.
\]
If \( \ell \) is invertible on \( X \), then
\[
H^2(\mathfrak{C}, G_m)[\ell^\infty] = 0
\]
follows as in the case \( i > 2 \). From the Kummer sequence Lemma 2.1 and the inclusion \( \mathfrak{C}_0 \hookrightarrow \mathfrak{C} \), one gets a commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Pic}(\mathfrak{C}) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H^2(\mathfrak{C}, \mu_{\ell^\infty}) & \longrightarrow & H^2(\mathfrak{C}, G_m)[\ell^\infty] & \longrightarrow & 0 \\
0 & \longrightarrow & \text{Pic}(\mathfrak{C}_0) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & H^2(\mathfrak{C}_0, \mu_{\ell^\infty}) & \longrightarrow & H^2(\mathfrak{C}_0, G_m)[\ell^\infty] & \longrightarrow & 0,
\end{array}
\]
and the middle vertical arrow is bijective by proper base change \([\text{EGAIV}, \text{p. 224, Corollary VI.2.7}]\), and the first vertical arrow is surjective by \([\text{EGAIV}, \text{p. 288, Corollaire (21.9.12)}]\) and the right exactness of the tensor product. Hence, by the five lemma, the right vertical morphism is bijective.

Thus, by Lemma 2.11, the diagram gives us \( H^2(\mathfrak{C}, G_m)[\ell^\infty] = 0 \) and the middle vertical arrow is bijective by proper base change \([\text{EGAIV}, \text{p. 224, Corollary VI.2.7}]\), and the first vertical arrow is surjective by \([\text{EGAIV}, \text{p. 288, Corollaire (21.9.12)}]\) and the right exactness of the tensor product.

For \( \ell = p \), one uses Theorem 2.9 which gives us
\[
H^2(\mathfrak{C}, G_m) \xrightarrow{\sim} H^2(\mathfrak{C}_0, G_m),
\]
and \( H^2(\mathfrak{C}_0, G_m) = 0 \) by Lemma 2.11.

**Corollary 2.13.** In the situation of Theorem 2.13 assume we have locally Noetherian separated schemes with geometrically reduced and connected fibres. Then one has the long exact sequence
\[
0 \rightarrow H^3(X, G_m) \xrightarrow{\pi^*} H^1(\mathfrak{C}, G_m) \rightarrow H^0(X, R^1\pi_*G_m) \rightarrow \ldots
\]
\[
H^8(X, G_m) \xrightarrow{\pi^*} H^4(\mathfrak{C}, G_m) \rightarrow H^{n-1}(X, R^1\pi_*G_m) \rightarrow \ldots
\]

**Lemma 2.14.** Let \( \pi : \mathfrak{C} \rightarrow X \) be a proper relative curve with \( X \) integral and let \( D \) be an irreducible Weil divisor in the generic fibre \( C \) of \( \pi \). Then \( \pi_D : D \rightarrow X \) is a finite morphism.

**Corollary 2.15.** If in the situation of Corollary 2.13 \( \pi \) has a section \( s : X \rightarrow \mathfrak{C} \), one has split short exact sequences
\[
0 \rightarrow H^i(X, G_m) \xrightarrow{s^*} H^i(\mathfrak{C}, G_m) \rightarrow H^{i-1}(X, R^1\pi_*G_m) \rightarrow 0
\]
for \( i \geq 1 \).

In the general case, denote by \( C/K \) the generic fibre of \( \mathfrak{C}/X \), and assume that for every Weil divisor \( D \) in \( C \), \( \overline{D} \subseteq \mathfrak{C} \) has everywhere the same dimension as \( X \) with no embedded components, and denote by \( \delta \) the greatest common divisor of the degrees of Weil divisors on \( C/K \), i.e. the index of \( C/K \). Then one has an exact sequence
\[
0 \rightarrow K_2 \rightarrow H^2(X, G_m) \xrightarrow{\pi^*} H^2(\mathfrak{C}, G_m) \rightarrow H^1(X, R^1\pi_*G_m) \rightarrow K_3 \rightarrow 0,
\]
where \( K_1 = \ker(H^1(X, G_m) \xrightarrow{\pi^*} H^1(\mathfrak{C}, G_m)) \) are Abelian groups annihilated by \( \delta \) whose prime-to-\( p \) torsion is finite.
Proof. The first assertion is obvious from the previous Corollary 2.13 and the existence of a section.

For the second claim, take an irreducible Weil divisor \( D \) in \( C \). Then \( \pi_D : \bar{D} \to X \) is a finite morphism by Lemma 2.14. We have the commutative diagram

\[
\begin{array}{ccc}
\bar{D} & \xrightarrow{i} & \mathfrak{C} \\
\downarrow{\pi_D} & & \downarrow{\pi} \\
X & & \\
\end{array}
\]

By the Leray spectral sequence, we have that \( H^i(\bar{D}, G_m) = H^i(X, \pi|_{\bar{D}}*, G_m) \) as \( \pi|_{\bar{D}} \) is finite, hence exact for the étale topology, see [Mil80], p. 72, Corollary II.3.6. If \( \pi|_{\bar{D}} \) is also flat, by finite locally freeness we have a norm map \( \pi|_{\bar{D}}*G_m \to G_m \) whose composite with the inclusion \( G_m \to \pi|_{\bar{D}}*G_m \) is the \( \delta \)-th power map. If not, there is still a norm map since \( f \) is flat in codimension 1 since \( \bar{D} \) has everywhere the same dimension as \( X \) with no embedded components, so one can take the norm there, which then will land in \( G_m \) as \( X \) is normal.

We have \( \pi|_{\bar{D}}*i* \circ \pi^* = \pi|_{\bar{D}}*i* \circ \pi|_{\bar{D}}^* = \deg(D) \), so \( \ker(\pi^*: H^*(X, G_m) \to H^*(\mathfrak{C}, G_m)) \) is annihilated by \( \deg(D) \) for all \( D \), hence by the index \( \delta \). Now the finiteness of the prime-to-\( p \) part of the \( K_i \) follows from Theorem 2.6.

In the following, assume \( \pi \) is smooth (automatically projective since \( \mathfrak{C}, X \) are projective over \( F_q \)), with all geometric fibres integral and of dimension 1, and that it has a section \( s : X \to \mathfrak{C} \). Assume further that \( \pi_*\mathcal{O}_{\mathfrak{C}} = \mathcal{O}_X \) holds universally and \( \pi \) is cohomologically flat in dimension 0, e. g. if \( \pi \) is a flat proper morphism of locally Noetherian separated schemes with geometrically connected fibres.

We recall some definitions from [FGI+05], p. 252, Definition 9.2.2.

Definition 2.16. The relative Picard functor \( \text{Pic}_{\mathfrak{C}/X} \) on the category of (locally Noetherian) \( S \)-schemes is defined by \( \text{Pic}_{X/S}(T) := \text{Pic}(X \times S T)/\text{pr}_2^* \text{Pic}(T) \). Its associated sheaves in the Zariski, étale and fppf topology are denoted by \( \text{Pic}_{X/S,Zar} \), \( \text{Pic}_{X/S,\text{ét}} \) and \( \text{Pic}_{X/S,\text{fppf}} \).

Now we come to the representability of the relative Picard functor by a group scheme, whose connected component of unity is an Abelian scheme.

Theorem 2.17. \( \text{Pic}_{\mathfrak{C}/X} \) is represented by a separated smooth \( X \)-scheme \( \text{Pic}_{\mathfrak{C}/X} \) locally of finite type. \( \text{Pic}_{\mathfrak{C}/X}^0 \) is represented by an Abelian \( X \)-scheme \( \text{Pic}_{\mathfrak{C}/X}^0 \). For every \( T/X \),

\[
0 \to \text{Pic}(T) \to \text{Pic}(\mathfrak{C} \times_X T) \to \text{Pic}_{\mathfrak{C}/X}(T) \to 0
\]

is exact.

3 The Néron model

The following result is easy.

Lemma 3.1. Let \( A \) be a regular local ring. Then \( A^h \otimes_A \text{Quot}(A) = \text{Quot}(A^h) \) and \( A^{sh} \otimes_A \text{Quot}(A) = \text{Quot}(A^{sh}) \).

Theorem 3.2 (The Néron model). Let \( S \) be a regular, Noetherian, integral, separated scheme, and \( g : \{\eta\} \to S \) the inclusion of the generic point. Let \( X/S \) be a smooth projective variety with geometrically integral fibres that admits a section such that its Picard functor is representable (e. g., \( X/S \) a smooth projective curve admitting a section or an Abelian scheme). Then \( \text{Pic}_{X/S} \xrightarrow{\sim} g_*g^*\text{Pic}_{X/S} \) as sheaves on \( S_{\text{sm}} \), the smooth site. Let \( \mathcal{A} / S \) be an Abelian scheme. Then

\[
\mathcal{A} \xrightarrow{\sim} g_*g^*\mathcal{A}
\]

as sheaves on \( S_{\text{sm}} \).

The main idea for injectivity is to use the separatedness of our schemes, and the main idea for surjectivity is that Weil divisors spread out and that the Picard group equals the Weil divisor class group by regularity.

Proof. We first prove everything for sheaves on the étale site on \( S \).

Let \( f : \text{Pic}_{X/S} \to g_*g^*\text{Pic}_{X/S} \) be the natural map of étale sheaves induced by adjointness. Let \( s \to S \) be a geometric point. We have to show that \( (\text{coker}(f))_s = 0 = (\text{ker}(f))_s \).
Taking stalks and using [EGAIV$_3$], p. 52, Proposition 8.14.2, we get the following commutative diagram:

\[
\begin{array}{c}
\text{Spec } \mathcal{O}_{S,s}^{\text{sh}} \ar[d] \ar[r]^f & \text{Pic}_{X/S}(\mathcal{O}_{S,s}^{\text{sh}}) \ar[d] \\
\text{Pic}(X \times_S \mathcal{O}_{S,s}^{\text{sh}}) \ar[r] & \text{Pic}(X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}})) \\
\text{Div}(X \times_S \mathcal{O}_{S,s}^{\text{sh}}) \ar[r] & \text{Div}(X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}}))
\end{array}
\]

Note that $\mathcal{O}_{S,s}^{\text{sh}}$ is a domain since it is regular as a strict Henselisation of a regular local ring by [Fil11], p. 111, Proposition 2.8.18. By [EGAIV$_3$], p. 52, Proposition 8.14.2, one has $(\text{Pic}_{X/S})_s = \text{Pic}_{X/S}(\mathcal{O}_{S,s}^{\text{sh}})$ and $(g_\sharp g^* \text{Pic}_{X/S})_s = \text{Pic}_{X/S}(\mathcal{O}_{S,s}^{\text{sh}} \otimes_{\mathcal{O}_S} K(S))$. But for a regular local ring $A$, one has $A^{\text{sh}} \otimes_A \text{Quot}(A) = \text{Quot}(A^{\text{sh}})$ by Lemma 3.1.

By [Har83], p. 145, Corollary II.6.16, one has a surjection from $\text{Div}$ to $\text{Pic}$ since $S$ is Noetherian, integral, separated and locally factorial. By Theorem 2.17, the upper vertical arrows are surjective (under the assumption that $X/S$ has a section).

But here, the lower horizontal map is surjective: A preimage under $\iota : X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}}) \to X \times_S \mathcal{O}_{S,s}^{\text{sh}}$ of $D \in \text{Div}(X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}}))$ is $\tilde{D} \in \text{Div}(X \times_S \mathcal{O}_{S,s}^{\text{sh}})$, the closure taken in $X \times_S \mathcal{O}_{S,s}^{\text{sh}}$. In fact, note that $D$ is closed in $X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}})$ since it is a divisor; the closure of an irreducible subset is irreducible again, and the codimension is also 1 since the codimension is the dimension of the local ring at the generic point $\eta_D$ of $D$, and the local ring of $\eta_D$ in $X \times_S \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}})$ is the same as the local ring of $\eta_D$ in $X \times_S \mathcal{O}_{S,s}^{\text{sh}}$ as it is the colimit of the global sections taken for all open neighbourhoods of $\eta_D$. Hence $(\text{coker}(f))_s = 0$.

For $(\ker(f))_s = 0$, consider the diagram:

\[
\begin{array}{ccc}
\text{Spec } \mathcal{O}_{S,s}^{\text{sh}} & \text{Spec Quot}(\mathcal{O}_{S,s}^{\text{sh}}) & \text{Pic}_{X/S} \\
\text{Spec } \mathcal{O}_{S,s}^{\text{sh}} & \text{Spec } \mathcal{O}_{S,s}^{\text{sh}} & S \\
\end{array}
\]

We want to show that a lift $\text{Spec } \mathcal{O}_{S,s}^{\text{sh}} \to \text{Pic}_{X/S}$ of $\text{Spec Quot}(\mathcal{O}_{S,s}^{\text{sh}})$ to $\text{Pic}_{X/S}$ is unique. As $\text{Pic}_{X/S}/S$ is separated, this is true for all valuation rings $\mathcal{O} \subset \text{Quot}(\mathcal{O}_{S,s}^{\text{sh}})$ by the valuative criterion of separatedness [EGAII], p. 142, Proposition (7.2.3). But by [Mat86], p. 72, Theorem 10.2, every local ring $(\mathcal{O}_{S,s}^{\text{sh}})_s$ is dominated by a valuation ring $\mathcal{O}$ of $\text{Quot}(\mathcal{O}_{S,s}^{\text{sh}})$. It follows from the valuative criterion for separatedness that the lift is topologically unique. Assume $\varphi, \varphi'$ are two lifts. Now cover $\text{Pic}_{X/S}/S$ by open affines $U_i = \text{Spec } A_i$ and their preimages $\varphi^{-1}(U_i) = \varphi'^{-1}(U_i)$ by standard open affines $\{D(f_{ij})\}_j$.

It follows that $\varphi = \varphi'$.

Now we prove the last statement of the theorem for Abelian schemes, so one can deduce the statement for Abelian varieties by noting that $\mathcal{A} = (\mathcal{A}^\vee)\vee = \text{Pic}_{\mathcal{A}^\vee/S}^0$. 

We want to show that
\[
\text{Pic}^0_{X/S}(S') \to \text{Pic}^0_{X/S}(S'_n)
\] (3.1)
is bijective for any étale $S$-scheme $S'$.

First note that such an $S'$ is regular, so its connected components are integral, and $S'_n$ is the disjoint union of the generic points of the connected components of $S'$, so we can replace $X \to S$ with the restrictions of the base change $X' \to S'$ over each connected component of $S'$ separately to reduce to checking for $S' = S$.

For any section $S \to \text{Pic}_{X/S}$, since $S$ is connected and $\text{Pic}^0_{X/S}$ is open and closed in $\text{Pic}_{X/S}$ by [SGA6], p. 647f., exp. XIII, Théорème 4.7, the preimage of $\text{Pic}^0_{X/S}$ under the section is open and closed in $S$, hence empty or $S$. Thus, if even a single point of $S$ is carried into $\text{Pic}^0_{X/S}$ under the section, then the whole of $S$ is.

More generally, when using $S'$-valued points of $\text{Pic}_{X/S}$ for any étale $S$-scheme $S'$, such a point lands in $\text{Pic}^0_{X/S}$ if and only if some point in each connected component of $S'$ does, such as the generic point of each connected component $S'_n$.

This proves the statement in (3.1) for $\text{Pic}^0_{X/S}$ replaced by $\text{Pic}^0_{X/S}$, and hence for $\text{Pic}^0_{X/S}$ in cases where it coincides with $\text{Pic}^0_{X/S}$ (i. e., when the geometric fibers have component group for $\text{Pic}_{X/S}$—i. e., Néron-Severi group—that is torsion-free, e. g. for Abelian schemes or curves).

The Néron mapping property for smooth $S$-schemes $S'$ follows formally. Since $S' \to S$ is smooth, $S'$ is regular as well and its generic points lie over $\eta$. Since an $S'$-point of $\text{Pic}_{X/S}$ is the same as an $S$'-point of $\text{Pic}_{X'/S'}$, an $S'_n$-point extends uniquely to an $S'$-point.

4 The Tate-Shafarevich group

**Proposition 4.1.** Let $X$ be integral and $\mathcal{A}/X$ be an Abelian scheme over $X$ regular, Noetherian, integral and separated. Then $H^i(X, \mathcal{A})$ is torsion for $i > 0$.

**Proof.** Consider the Leray spectral sequence for the inclusion $g : \{\eta\} \hookrightarrow X$ of the generic point
\[
H^p(X, R^qg_*\mathcal{A}) \Rightarrow H^{p+q}(\eta, g^*\mathcal{A}).
\]
Calculation modulo the Serre subcategory of torsion sheaves, and exploiting the fact that Galois cohomology groups are torsion in dimension $> 0$, and therefore also the higher direct images $R^qg_*\mathcal{A}$ are torsion sheaves by [Mil80], p. 88, Proposition III.1.13, the spectral sequence degenerates giving
\[
H^p(X, g_*\mathcal{A}) = E^0_{p,0} = E^p = H^p(\eta, g^*\mathcal{A}) = 0
\]
for $p > 0$. Because of the Néron mapping property we have $H^p(X, \mathcal{A}) \overset{\sim}{\longrightarrow} H^p(X, g_*\mathcal{A})$, which finishes the proof. \qed

**Definition 4.2.** Define the **Tate-Shafarevich group** of an Abelian scheme $\mathcal{A}/X$ by
\[
\text{III}(\mathcal{A}/X) = H^1_{\text{ét}}(X, \mathcal{A}).
\]

**Theorem 4.3.** Let $\mathcal{A}/X$ be an Abelian scheme over $X$ regular, Noetherian, integral and separated. For $x \in X$, denote the function field of $X$ by $K$, the quotient field of the strict Henselisation of $\mathcal{O}_{X,x}$ by $K^\text{nr}_x$, the inclusion of the generic point by $j : \{\eta\} \hookrightarrow X$ and let $j_x : \text{Spec}(K^\text{nr}_x) \hookrightarrow \text{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$ be the composition. Then we have
\[
H^1(X, \mathcal{A}) \overset{\sim}{\longrightarrow} \ker \left( H^1(K, j^*\mathcal{A}) \to \prod_{x \in X} H^1(K^\text{nr}_x, j^*_x\mathcal{A}) \right).
\]

**Proof.** The Leray spectral sequence $H^p(X, R^qj_*(j^*\mathcal{A})) \Rightarrow H^{p+q}(K, j^*\mathcal{A})$ yields the exactness of $0 \to E^1_{2,0} \to E^1 \to E^0_{1,1}$, i. e.
\[
H^1(X, j_*j^*\mathcal{A}) = \ker \left( H^1(K, j^*\mathcal{A}) \to H^0(X, R^1j_*(j^*\mathcal{A})) \right).
\]
Since
\[
H^0(X, R^1j_*(j^*\mathcal{A})) \to \prod_{x \in X} R^1j_*(j^*\mathcal{A})_x
\]
is injective ([Mil80], p. 60, Proposition II.2.10: If a section of an étale sheaf is non-zero, there is a geometric point for which the stalk of the section is non-zero) and
\[ R^1 j_*(j^* \mathcal{A})_x = H^1(K_{x, \mathcal{A}}^nr, j_x^* \mathcal{A}) \]
by Lemma 3.1, the theorem follows from the kernel-cokernel exact sequence and the Néron mapping property
\[ H^1(X, \mathcal{A}) \to H^1(X, j_*j^* \mathcal{A}). \]

**Theorem 4.4.** In the situation of Theorem 4.3, one can replace the product over all points by

(a) the codimension-1 points if one disregards the p-torsion (\( p = \text{char } k \)) (for \( \dim X \leq 2 \), this also holds for the p-torsion), if \( X/k \) is smooth projective over \( k \) finitely generated and \( \mathcal{A}/X \) is an Abelian scheme such that the vanishing theorem Lemma 4.11 below is satisfied (e.g. \( \mathcal{A} = \text{Pic}_{\mathcal{E}/X} \)):
\[ H^1(X, \mathcal{A}) = \ker \left( H^1(K, j^* \mathcal{A}) \to \bigoplus_{x \in X^{(1)}} H^1(K_{x, \mathcal{A}}^nr, j_x^* \mathcal{A}) \right), \]
or (b) the closed points
\[ H^1(X, \mathcal{A}) = \ker \left( H^1(K, j^* \mathcal{A}) \to \prod_{x \in X} H^1(K_{x, \mathcal{A}}^nr, j_x^* \mathcal{A}) \right) \]
One can also replace \( K_{x, \mathcal{A}}^nr \) by the quotient field of the completion \( \hat{O}_{\mathcal{E}, X, x}^{sh} \), in the case of \( x \in X^{(1)} \).

We first establish some vanishing results for étale cohomology with supports.

**Lemma 4.5.** In the situation of the previous lemma, one has
\[ H^i_Z(X, G_m) = 0 \quad \text{for } i \leq 2, \]
and for \( i = 3 \) at least away from \( p \).

**Proof.** See [Gro68], p. 133 ff.: Using the local-to-global spectral sequence ([Gro68], p. 133, (6.2))
\[ E_2^{p,q} = H^p(X, \mathcal{H}^q_Z(G_m)) \Rightarrow H^{p+q}_Z(X, G_m) \]
and [Gro68], p. 133–135
\[ \mathcal{H}^0_Z(G_m) = 0 \quad \text{[Gro68], p. 133, (6.3)} \]
\[ \mathcal{H}^1_Z(G_m) = 0 \quad \text{[Gro68], p. 133, (6.4)} \]
\[ \mathcal{H}^2_Z(G_m) = 0 \quad \text{[Gro68], p. 134, (6.5)} \]
\[ \mathcal{H}^3_Z(G_m)^{(n)} = 0 \quad \text{[Gro68], p. 134 f., Thm. (6.1),} \]
even \( \mathcal{H}^2_Z(G_m) = 0 \) for \( \dim X = 2 \); if not, we have to calculate modulo suitable Serre subcategories in the following), we have \( E_2^{p,q} = 0 \) for \( q \leq 3 \), and hence the result \( E^n = 0 \) for \( 0 \leq n \leq 3 \) follows from the exact sequences
\[ 0 \to E_2^{n,0} \to E^n \to E_2^{0,n} \]
for \( n = 1, 2, 3 \). \( \square \)

**Lemma 4.6.** If \( f : X \to Y \) is a flat morphism of locally Noetherian schemes and \( Z \hookrightarrow Y \) is a closed immersion of codimension \( \geq c \), then also the base change \( Z' := Z \times_Y X \hookrightarrow X \) is a closed immersion of codimension \( \geq c \).

**Lemma 4.7.** Let \( X \) be a normal scheme and \( C/X \) a smooth proper relative curve. Then there is an exact sequence
\[ 0 \to \text{Pic}^0_{C/X} \to \text{Pic}_{C/X} \to Z \to 0. \]

**Lemma 4.8.** Let \( X/k \) be a smooth variety and \( \pi : C \to X \) a smooth proper relative curve which admits a section \( s : X \to C \). Let \( Z \hookrightarrow X \) be a reduced closed subscheme of codimension \( \geq 2 \). Assume \( \dim X \leq 2 \). Then
\[ H^i_Z(X, \text{Pic}^0_{C/X}) = 0 \quad \text{for } i \leq 2. \]

For \( \dim X > 2 \), this holds at least up to p-torsion.
Proof. By Theorem 2.12 and the Leray spectral sequence with supports $H^q_\eta(Y, R^i\pi_* F) \Rightarrow H^{i+q}_{\overline{\eta}}(X, F)$, we get a long exact sequence

$$0 \to E^1_{2,0} \to E^1_{1,0} \to E^2_{2,0} \to E^2_{1,0} \to E^2_{2,1} \to E^3_{0,0} \to E^3_{1,0} \to E^4_{2,0} \to E^4_{3,0}.$$ (4.1)

But $E^i_{2,0} = H^i_\eta(X, G_m) = 0$ for $i \leq 3$ by Lemma 4.5 (for $i = 3$ at least away from $p$). Therefore the long exact sequence $\pi_\eta : H^2_\eta(\mathcal{E}, G_m) \to H^2_\eta(\mathcal{E}', G_m)$ yields isomorphisms $E^i \to E^i_{2,1}$ for $i \leq 2$, but $E^i = H^i_\eta(\mathcal{E}', G_m) = 0$ for $i \leq 3$, again by Lemma 4.5 and Lemma 4.6 hence $H^i_\eta(X, R^1\pi_* G_m) = E^i_{2,1} = 0$ for $i \leq 2$ from (4.1). For the vanishing of $E^2_{2,1}$ note that

$$0 \to E^2_{2,1} \to E^4_{2,0} \to E^4$$

is exact by (4.1), but the latter map is $\pi^*: H^2_\eta(X, G_m) \to H^2_\eta(\mathcal{E}', G_m)$, which is injective as $\pi$ admits a section. □

Lemma 4.9. Let $X$ be a connected normal Noetherian scheme with generic point $\eta$. Then $H^p(X, Q) = 0$ for all $p > 0$ and $H^1(X, Z) = 0$.

Proposition 4.10. Let $X$ be a connected Noetherian scheme and $\bar{x}$ a geometric point. Then

$$H^1(X, Q/Z) = \text{Hom}_{cont}(\pi^1_\eta(X, \bar{x}), Q/Z)$$

and

$$H^1(X, Z_\eta) = \text{Hom}_{cont}(\pi^1_\eta(X, \bar{x}), Z_\eta).$$

Lemma 4.11. Let $X/k$ be a smooth variety and $\mathcal{E}/X$ a smooth proper relative curve. Assume $\dim X \leq 2$. Let $Z \hookrightarrow X$ be a reduced closed subscheme of codimension $\geq 2$. Then

$$H^i_\eta(X, \text{Pic}^0_{\mathcal{E}/X}) = 0 \quad \text{for } i \leq 2.$$ If $\dim X > 2$, this holds at least up to $p$-torsion.

Proof. Taking the long exact sequence associated to the short exact sequence of Lemma 4.7 with respect to $H^2_\eta(X, -)$, by Lemma 4.8 it suffices to show that $H^i_\eta(X, Z) = 0$ for $i = 0, 1, 2$.

For this, consider the long exact sequence

$$\ldots \to H^2_\eta(X, Z) \to H^1_\eta(X, Z) \to H^1_\eta(X \setminus Z, Z) \to \ldots$$

It suffices to show that $H^1_\eta(X, Z) \to H^1_\eta(X \setminus Z, Z)$ is an isomorphism for $i = 0, 1$ and an injection for $i = 2$.

For $i = 0$, this map is $Z \rightarrow Z$.

For $i = 1$, both groups are equal to 0 by Lemma 4.9

For $i = 2$, this map is $H^2_\eta(X, Z) \to H^2_\eta(X \setminus Z, Z)$. Consider the long exact sequence associated to

$$0 \to Z \to Q \to Q/Z \to 0.$$
Lemma 4.12. Let $I$ be a filtered category and $(i \mapsto X_i)$ a contravariant functor from $I$ to schemes over $X$. Assume that all schemes are quasi-compact and that the transition maps $X_i \leftarrow X_j$ are affine. Let $X_{\infty} = \lim X_i$, and, for a sheaf $\mathcal{F}$ on $X_{et}$, let $\mathcal{F}_i$ and $\mathcal{F}_{\infty}$ be its inverse images on $X_i$ and $X_{\infty}$ respectively. Then

$$\lim \mathcal{H}^p((X_i)_{et}, \mathcal{F}_i) \sim \mathcal{H}^p((X_{\infty})_{et}, \mathcal{F}_{\infty}).$$

Assume the $X_i \subseteq X$ are open, the transition morphisms are affine and all schemes are quasi-compact. Let $Z \hookrightarrow X$ be a closed subscheme. Then

$$\lim \mathcal{H}^p_{Z \cap X, ((X_i)_{et}, \mathcal{F}_i)} \sim \mathcal{H}^p_{Z \cap X_{\infty}, ((X_{\infty})_{et}, \mathcal{F}_{\infty})}.$$

Corollary 4.13. We have

$$\lim \mathcal{H}^p(U, \mathcal{F}|_U) \sim \mathcal{H}^p(K, \mathcal{F}_K),$$

the colimit with respect to the restriction maps, where $U$ runs through the non-empty standard affine open subschemes $D(f_i)$ of an non-empty affine open subscheme $Spec(A) \subseteq X$.

Lemma 4.14 (Excision of codimension $\geq 2$ subschemes). One can excise subschemes $Z \hookrightarrow Y$ of codimension $\geq 2$ in $X$:

$$\ker (\mathcal{H}^1(U, \mathcal{O}) \to \mathcal{H}^2_{Z}(X, \mathcal{O})) = \ker \left(\mathcal{H}^1(U, \mathcal{O}) \to \mathcal{H}^2_{Z \setminus Z}(X \setminus Z, \mathcal{O}|_{X \setminus Z})\right). \quad (4.2)$$

Proof. From the long exact localisation sequence for cohomology with supports [Mil80], p. 92, Remark III.1.26, and from Lemma 4.11 one gets the injectivity

$$0 \to \mathcal{H}^2_{Z}(X, \mathcal{O}) \hookrightarrow \mathcal{H}^2_{Z \setminus Z}(X \setminus Z, \mathcal{O}|_{X \setminus Z}), \quad (4.3)$$

hence by the kernel-cokernel exact sequence the claim. □

Lemma 4.15. Let $Y \hookrightarrow X$ be a closed subscheme with open complement $U = X \setminus Y$ and with all of its irreducible components of codimension 1 in $X$. Denote the finitely many irreducible components of $Y$ by $(Y_i)_{i=1}^n$. Then

$$\ker (\mathcal{H}^1(U, \mathcal{O}) \to \mathcal{H}^2_{Z}(X, \mathcal{O})) = \ker \left(\mathcal{H}^1(U, \mathcal{O}) \to \bigoplus_{i=1}^n \mathcal{H}^2_{Z \setminus Z_i}(X \setminus Z_i, \mathcal{O})\right) \quad (4.4)$$

with certain closed subschemes $Z_i \hookrightarrow Y_i$.

Proof. Excise the intersections $Y_i \cap Y_j$ for $i \neq j$ (they are of codimension $\geq 2$ in $X$ since our schemes are catenary as they are varieties by [Liu06], p. 338, Corollary 8.2.16 and

$$2 = 1 + 1 \leq \operatorname{codim}(Y_i \cap Y_j \hookrightarrow Y) + \operatorname{codim}(Y \hookrightarrow X) = \operatorname{codim}(Y_i \cap Y_j \hookrightarrow X)).$$

Now, by a repeated application of the Mayer-Vietoris sequence with supports, one gets the claim. □

Lemma 4.16. Let $X$ be a regular variety over a finitely generated field and $x \in X^{(1)}$. Then there is an open affine subscheme $Spec A = X_0 \subseteq X$ containing $x$ such that $A$ is a unique factorisation domain.

Proof. The class group $Cl(X)$ is finitely generated by [Kah00], p. 396, Corollaire 2. □

Lemma 4.17. One has

$$\ker (\mathcal{H}^1(U, \mathcal{O}) \to \mathcal{H}^2_{Z}(X, \mathcal{O})) = \ker \left(\mathcal{H}^1(U, \mathcal{O}) \to \bigoplus_{i=1}^n \mathcal{H}^2_{Z_i}(X \setminus \tilde{Z}_i, \mathcal{O})\right) \quad (4.5)$$

with the $x_i \in X \setminus \tilde{Z}_i$ closed points and generic points of the $Y_i$ and $\tilde{Z}_i \hookrightarrow X$ certain subschemes.
Proof. This follows basically by excising (using \((4.2)\)) everything except the generic points of the \(Y_i\) in \((4.4)\). The only technical difficulty is that for applying Lemma \((4.12)\) one has to make sure that the transition maps are affine.

Fix an irreducible component \(Y_i\) of \(Y\) and call it \(Y\) with \(x\) its generic point. We have to construct an injection

\[
\text{H}^2_{\X \setminus Z}(X \setminus Z, \mathscr{A}) \hookrightarrow \text{H}^2_{U}(X \setminus Z, \mathscr{A}).
\] (4.6)

Using Lemma \((4.16)\) choose an affine open \(X_0 := \text{Spec}(A \subset X \setminus Z)\) containing the point \(x\) such that \(A\) is a unique factorisation domain. One has \(\text{H}^2_Y(X, \mathscr{A}) \rightarrow \text{H}^2_{Y \cap X_0}(X_0, \mathscr{A})\): \(X \setminus X_0\) is a closed subscheme \(V \hookrightarrow X\) such that \(V \cap Y \hookrightarrow Y\) is of codimension \(\geq 1\) and hence (since varieties are catenary by \([\text{Liu06}], \text{p. 338, Corollary 2.16}\) \(V \cap Y \hookrightarrow X\) of codimension \(\geq 2\), so we conclude by excision (\((4.2)\)). We construct a sequence \((X_i)_{i=0}^\infty\) of standard affine open subsets \(X_i = D(f_i) \subset X_0, X_{i+1} \subset X_i\), all of them containing the point \(x\) such that

\[
\text{H}^2_{Y \cap X_0}(X_i, \mathscr{A}) \hookrightarrow \text{H}^2_{Y \cap X_{i+1}}(X_{i+1}, \mathscr{A}), \quad \text{and thus by } (4.2)
\]

\[
\ker(\text{H}^1(U, \mathscr{A}) \rightarrow \text{H}^2(Y, \mathscr{A})) = \ker(\text{H}^1(U, \mathscr{A}) \rightarrow \text{H}^2_{Y \cap X_0}(X_i, \mathscr{A})),
\]

and such that \(\lim_{i \rightarrow i} X_i \cap Y = \{x\}\). Then \((4.6)\) follows from Lemma \((4.12)\).

**Construction of the \((X_i)_{i=0}^\infty\).** Since \(X\) is countable, one can choose an enumeration \((Z_i)_{i=1}^\infty\) of the closed integral subschemes of codimension \(1\) of \(X_0\) not equal to \(X_0 \cap Y\).

Given \(X_i = D(f_i)\), take \(f \in \text{Spec}(A_i)\) such that \(V(f) = Z_{i+1}\). (By the converse of Krull’s Hauptideal- satz \([\text{Ein95}], \text{p. 233, Corollary 10.5}\), since \(Z_{i+1}\) is of codimension \(1\) in \(X_0\), there is an \(f \in A\) such that \(Z_{i+1} \subset V(f)\) is minimal. Since \(X_0\) is a unique factorisation domain, one can assume \(f\) prime, hence \(Z_{i+1} = V(f)\) by codimension reasons.) Then \(V(f) \cap (X^{(1)} \cap Y) = \emptyset\) and \(V(f) \cap Y \subset Y\) has codimension \(\geq 1\) in \(Y\), hence (again, varieties being catenary) \(V(f) \cap Y\) has codimension \(\geq 2\) in \(X\). Therefore one can apply \((4.3)\) to yield an injection

\[
\text{H}^2_{Y \cap X_0}(X_i, \mathscr{A}) \hookrightarrow \text{H}^2_{Y \cap X_{i+1}}(X_{i+1}, \mathscr{A}),
\]

Now, by excision \([\text{Mil80}], \text{p. 92, Proposition III.1.27}\) of \(V(f)\), one has

\[
\text{H}^2_{Y \cap X_{i+1}}(X_{i+1}, \mathscr{A}) \hookrightarrow \text{H}^2_{Y \cap X_i}(X_i \setminus (Y \cap V(f)), \mathscr{A}).
\]

Set \(X_{i+1} = X_i \setminus V(f) = D(f_i f), f_{i+1} = f_i f\).

Apply this to the direct summands in \((4.4)\). \(\square\)

**Proof of Theorem \((4.3)\)(a).** Now for the proof of Theorem \((4.4)\) at least for prime-to-\(p\) torsion (conferring Lemma \((4.8)\):

First note that the map \(\text{H}^1(X, \mathscr{A}) \rightarrow \ker(\ldots)\) in Theorem \((4.4)\) is well-defined since if \(x \in \text{H}^1(X, \mathscr{A})\) is restricted to \(\text{H}^1(K, \mathscr{A})\) via \(g: \{y\} \hookrightarrow X\), its pullback to \(K_x^\text{ur} = \text{Spec}(\text{Quot}(\text{O}_{\X, x}^\text{sh}))\) factors as

\[
\text{H}^1(X, \mathscr{A}) \rightarrow \text{H}^1(\text{Spec}(\text{O}_{\X, x}^\text{sh}, \mathscr{A})), \rightarrow \text{H}^1(\text{Spec}(\text{Quot}(\text{O}_{\X, x}^\text{sh})), \mathscr{A}),
\]

but the étale site of \(\text{Spec}(\text{O}_{\X, x}^\text{sh})\) is trivial.

Let \(\emptyset \neq U \hookrightarrow X\) be open with reduced closed complement \(Y \hookrightarrow X\). Then one has the long exact localisation sequence \([\text{Mil80}], \text{p. 92, Proposition III.1.25}\)

\[
0 \rightarrow \text{H}^0_Y(X, \mathscr{A}) \rightarrow \text{H}^0(X, \mathscr{A}) \rightarrow \text{H}^0(U, \mathscr{A}) \rightarrow \text{H}^2_Y(X, \mathscr{A}) \rightarrow \text{H}^1(X, \mathscr{A}) \rightarrow \text{H}^1(U, \mathscr{A}) \rightarrow \ldots.
\]

Because of the injectivity of \(\text{H}^0(Y, \mathscr{A}) \hookrightarrow \text{H}^0(U, \mathscr{A})\) (If two sections of \(\mathscr{A}/X\) coincide on \(U\) open dense, they agree on \(X\) since \(\mathscr{A}\) is separated and \(X\) reduced), the exactness of the sequence yields \(\text{H}^0_Y(X, \mathscr{A}) = 0\), and hence a short exact sequence

\[
0 \rightarrow \text{H}^2_Y(X, \mathscr{A}) \rightarrow \text{H}^2(X, \mathscr{A}) \rightarrow \text{H}^2(U, \mathscr{A}) \rightarrow \text{H}^2_Y(X, \mathscr{A}) \rightarrow 0.
\] (4.7)
Surjectivity I. Applying excision in the form of [Mil80], p. 93, Corollary III.1.28 to (4.5) (using that the $x_i \in X \setminus \hat{Z}_i$ are closed points), one gets

$$\ker \left( H^1(U, \mathcal{O}) \to H^1_0(X, \mathcal{O}) \right) = \ker \left( H^1(U, \mathcal{O}) \xrightarrow{(r_i)} \bigoplus_{i=1}^n H^1_{(x_i)}(X^h_{x_i}, \mathcal{O}) \right).$$

Now $(r_i)$ factors as

$$r_i : H^1(U, \mathcal{O}) \xrightarrow{j_i} H^1(K^h_{x_i}, \mathcal{O}) \xrightarrow{\delta_i} H^2_{(x_i)}(X^h_{x_i}, \mathcal{O}),$$

where the latter map is the boundary map of the localisation sequence associated to the discrete valuation ring $\mathcal{O}^h_{X,x_i}$. (X is normal as it is smooth over a field, $x_i$ is a codimension-1 point, and the Henselisation of a normal ring is normal again by [Fu11], p. 106, Proposition 2.8.10, and normal rings are $(R, O)$)

$$\text{injectivity.}$$

So the cokernel of $r_i$ injects into $H^1(K^h_{x_i}, \mathcal{O})$ and in $H^2_{(x_i)}(X^h_{x_i}, \mathcal{O})$, so we get

$$\ker \left( H^1(U, \mathcal{O}) \to \bigoplus_{i=1}^n \text{coker}(j_{x_i}) \right) \xrightarrow{\sim} \ker \left( H^1(U, \mathcal{O}) \to \bigoplus_{i=1}^n H^2_{(x_i)}(X^h_{x_i}, \mathcal{O}) \right)$$

and hence

$$\ker \left( H^1(U, \mathcal{O}) \to H^2_0(X, \mathcal{O}) \right) = \ker \left( H^1(U, \mathcal{O}) \xrightarrow{(r_i)} \bigoplus_{i=1}^n H^1(K^nr_{x_i}, \mathcal{O}) \right).$$

Surjectivity II. Taking the limit over all $Y$ (choose an enumeration $(Y_i)_{i=1}^\infty$ of the integral closed subschemes of codimension 1 of $X$ [X is countable]) yields by Lemma [4.13] and the fact that (Ab) satisfies (AB5)

$$\ker \left( H^1(K, \mathcal{O}) \to \lim_{Y \to X} H^1_0(X, \mathcal{O}) \right) = \ker \left( H^1(K, \mathcal{O}) \to \bigoplus_{x \in X^{(1)}} H^1(K^nr_{x}, \mathcal{O}) \right).$$

Now (4.7) gives us

$$0 \to \lim_{Y \to X} H^1_0(X, \mathcal{O}) \to H^1(X, \mathcal{O}) \to \ker \left( H^1(K, \mathcal{O}) \to \bigoplus_{x \in X^{(1)}} H^1(K^nr_{x}, \mathcal{O}) \right) \to 0.$$
Proof of Theorem 4.19 (b). In the proof of Theorem 4.3 even
\[ H^0(X, \mathcal{R}^1 j_*(j^* \mathcal{A})) \rightarrow \prod_{x \in |X|} \mathcal{R}^1 j_*(j^* \mathcal{A})_x \]
is injective by [Mil80], p. 65, Remark II.2.17 (b): If a section of an étale sheaf is non-zero, there is a closed geometric point for which the stalk of the section is non-zero. This is because \(X/k\) is a variety and hence Jacobson.

One can replace the strict Henselisation \(\mathcal{O}_{X,x}^{\text{sh}}\) of \(\mathcal{O}_{X,x}\) by its completion \(\hat{\mathcal{O}}_{X,x}\) (respectively by their quotient fields) in the case of \(x \in X^{(1)}\).

Lemma 4.18. Let \((A, m)\) be a discrete valuation ring of a variety. Let \(Z\) be a smooth proper scheme of finite type over \(A^{sh}\). The following are equivalent:
1. \(Z\) has a point over \(\text{Quot}(A^{sh})\).
2. \(Z\) has a point over \(A^{sh}\).
3. \(Z\) has points over \(A^{sh}/m^n A^{sh} = \hat{A}^{sh}/m^n \hat{A}^{sh}\) for all \(n \gg 0\).
4. \(Z\) has a point over \(A^{sh}\).
5. \(Z\) has a point over \(\text{Quot}(\hat{A}^{sh})\).

4.1 Relation between the Brauer group and the Tate-Shafarevich group

In analogy with the conjectures of Birch and Swinnerton-Dyer, and of Artin and Tate, one could hope that the Tate-Shafarevich group \(\text{III}(\mathcal{A}/X)\) and the Brauer group \(\text{Br}(\mathcal{C})\) are finite. We can extend the results of Artin and Tate [Tat66] as follows:

Theorem 4.19 (The Artin-Tate and the Birch-Swinnerton-Dyer conjecture). Let \(\pi : \mathcal{C} \rightarrow X\) be projective and smooth with \(\mathcal{C}\) and \(X\) regular, all fibres of dimension 1 and \(X\) be a variety. Then one has an exact sequence
\[ 0 \rightarrow K_2 \rightarrow \text{Br}(X) \xrightarrow{\pi_*} \text{Br}(\mathcal{C}) \rightarrow \text{III}(\text{Pic}_{\mathcal{C}/X}/X) \rightarrow K_3 \rightarrow 0 \]
in which the groups \(K_i\) annihilated by \(\delta\), the index of the generic fibre \(C/K\), e.g. \(\delta = 1\) if \(\mathcal{C}/X\) has a section, and their prime-to-\(p\) parts are finite, and \(K_i = 0\) if \(\pi\) has a section. Here, \(\text{III}(\text{Pic}_{\mathcal{C}/X}/X)\) sits in a short exact sequence
\[ 0 \rightarrow \mathbb{Z}/d \rightarrow \text{III}(\text{Pic}^0_{\mathcal{C}/X}/X) \rightarrow \text{III}(\text{Pic}_{\mathcal{C}/X}/X) \rightarrow 0, \]
where \(d \mid \delta\).

Hence the finiteness of the \((\ell\text{-tor})\) of the Brauer group of \(\mathcal{C}\) is equivalent to the finiteness of the \((\ell\text{-tor})\) of the base \(X\) and the finiteness of the \((\ell\text{-tor})\) of the Tate-Shafarevich group of \(\text{Pic}_{\mathcal{C}/X}\).

Proof. Combining Corollary 2.15 (here the theory of the Picard functor and the exactness of the sequence still works if we do not have a section, but étale locally a section; and the latter is the case since a smooth morphism factors locally into an étale morphism followed by an affine projection) with \(\mathcal{R}^1 \pi_* \mathbb{G}_m = \text{Pic}_{X/S}\) by [BLR90], p. 202f. and Theorem 4.3 yields the exact sequence
\[ 0 \rightarrow K_2 \rightarrow \text{Br}(X) \xrightarrow{\pi_*} \text{Br}(\mathcal{C}) \rightarrow H^1(X, \text{Pic}_{\mathcal{C}/X}) \rightarrow K_3 \rightarrow 0 \]
with the prime-to-\(p\) part of the \(K_i\) finite, and \(K_i = 0\) if \(\pi\) has a section.

Now the long exact sequence associated to the short exact sequence in Lemma 4.7 yields the exact sequence
\[ H^0(X, \text{Pic}_{\mathcal{C}/X}) \rightarrow H^0(X, \mathcal{Z}) \rightarrow H^1(X, \text{Pic}^0_{\mathcal{C}/X}) \rightarrow H^1(X, \text{Pic}_{\mathcal{C}/X}) \rightarrow H^1(X, \mathcal{Z}) = 0, \]
and \(H^1(X, \mathcal{Z}) = 0\) using Lemma 4.9. Now, choose a Weil divisor \(D\) on the generic fibre \(C/K\) of \(\mathcal{C}/X\) with degree \(\delta\) the index of \(C/K\). By Lemma 2.14 and [Con], p. 3, Proposition 4.1, \(D\) is a Weil divisor on \(\mathcal{C}\) of degree \(\delta\), and its image under \(H^0(X, \text{Pic}_{\mathcal{C}/X}) \rightarrow H^0(X, \mathcal{Z}) = \mathcal{Z}\) (here we use that \(X\) is connected) is \(\delta\). Hence \(\text{coker}(H^0(X, \text{Pic}_{\mathcal{C}/X}) \rightarrow H^0(X, \mathcal{Z}) = \mathcal{Z})\) is a quotient of \(\mathcal{Z}/\delta\).
4.2 Descent of finiteness of \( \text{III} \) under generically étale alterations

**Lemma 4.20.** Let \( \ell \) be invertible on \( X \). Then the \( \mathbb{Z}_\ell \)-corank of \( \text{III}(\mathcal{A}/X)[\ell^\infty] \) is finite.

**Proof.** The short exact sequence of étale sheaves \( 0 \to \mathcal{A}[\ell^n] \to \mathcal{A} \to \mathcal{A} \to 0 \) induces

\[
0 \to \mathcal{A}(X)/\ell^n \to H^1(X, \mathcal{A}[\ell^n]) \to H^1(X, \mathcal{A})[\ell^n] \to 0.
\]

From this, one sees that \( H^1(X, \mathcal{A})[\ell] \) is finite as it is a quotient of \( H^1(X, \mathcal{A})[\ell^n] \) and \( \mathcal{A}[\ell]/X \) is constructible. Hence \( \text{III}(\mathcal{A}/X)[\ell^\infty] \) is cofinitely generated.

**Theorem 4.21.** Let \( f : X' \to X \) be a proper, surjective, generically étale \( \ell \)-morphism of regular schemes (an \( \ell \)-alteration). Let \( \ell \) be some prime invertible on \( X \). If \( \mathcal{A} \) is an Abelian scheme on \( X \) such that the \( \ell \)-torsion of the Tate-Shafarevich group \( \text{III}(\mathcal{A}/X') \) of \( \mathcal{A}' := f^* \mathcal{A} = \mathcal{A} \times_X X' \) is finite, then the \( \ell \)-torsion of the Tate-Shafarevich group \( \text{III}(\mathcal{A}/X) \) is finite.

**Proof.** Without loss of generality, assume \( X \) integral with generic point \( \eta \). Define \( X'_\eta \) by the commutativity of the cartesian diagram

\[
\begin{array}{ccc}
X'_\eta & \xrightarrow{g'} & X' \\
\downarrow f & & \downarrow f \\
\{\eta\} & \xrightarrow{g} & X.
\end{array}
\]

**Step 1:** \( H^1(X, f_*\mathcal{A}')[\ell^\infty] \) is finite. This follows from the low terms exact sequence

\[
0 \to H^1(X, f_*\mathcal{A}') \to H^1(X', \mathcal{A}')
\]

associated to the Leray spectral sequence \( H^p(X, R^q f_*\mathcal{A}') \Rightarrow H^{p+q}(X', \mathcal{A}') \) and the finiteness of \( H^1(X', \mathcal{A}')[\ell^\infty] = \text{III}(\mathcal{A}'/X')[\ell^\infty] \).

**Step 2:** The theorem holds if there is a trace morphism. Assume there is a trace morphism \( f_*f^*\mathcal{A} \to \mathcal{A} \) such that the composition with the adjunction morphism

\[
\mathcal{A} \to f_*f^*\mathcal{A} \to \mathcal{A}
\]

is multiplication with \( \deg f \) and \( \deg f \) is invertible on \( X \). Let \( A = H^1(X, \mathcal{A})[\ell^\infty] \) and \( B = H^1(X, f_*\mathcal{A}')[\ell^\infty] \) and denote the induced morphisms on cohomology by \( g : A \to B \) und \( h : B \to A \). By Lemma 4.20, \( A \) is cofinitely generated. Since \( B \) is a finite \( \ell \)-group by Step 1, there is an \( N \in \mathbb{N} \) such that \( \ell^N \cdot B = 0 \). Then one has \( \ell^N g(A) = 0 \), thus \( \ell^N (h \circ g) = \ell^N [\deg f] = 0 \) as an endomorphism of \( A \). As \( A \) is cofinitely generated and \( \ell^N \cdot \deg f \neq 0 \), the finiteness of \( A \) follows.

**Step 3:** Construction of the trace morphism for \( X = \text{Spec} k \) a field. Since \( f \) is étale, by [Fu11], p. 205, Proposition 5.5.1 (i), \( f_1 \) is left adjoint to \( f^* \) and by loc. cit. (iv), one has

\[
(f_1 \mathcal{F})_x = \bigoplus_{x' \in X', x \sim x'} \mathcal{F}_{x'}.
\]

Since \( f \) is étale, proper and of finite type, by [Fu11], p. 207, Proposition 5.5.2, one has \( f_1 = f_* \). Hence, there is the adjunction morphism \( f_*f^*\mathcal{A} = f_1 f^*\mathcal{A} \to \mathcal{A} \), and we have to prove that

\[
\mathcal{A}_x \to (f_*f^*\mathcal{A})_x = (f_1 f^*\mathcal{A})_x \to \mathcal{A}_x
\]

is multiplication by \( \deg f \). Since one can assume \( \bar{x} = \text{Spec} k = X \) with \( k \) separably closed is strictly Henselian, \( X' = \amalg_{i=1}^{\deg f} \text{Spec} k \). Hence the morphisms are

\[
\mathcal{A}_x \to (f_*f^*\mathcal{A})_x, \quad s \mapsto (s)_{i=1}^{\deg f},
\]

\[
(f_1 f^*\mathcal{A})_x \to \mathcal{A}_x, \quad (s_i)_{i=1}^{\deg f} \mapsto \sum_{i=1}^{\deg f} s_i,
\]

and the claim is obvious.
Step 4: Construction of the trace morphism for \( X \) arbitrary regular. One has to construct a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\cong} & g_\ast g^\ast \mathcal{A} \\
\downarrow & & \downarrow \\
f_\ast f^\ast \mathcal{A} & \to & g_\ast g^\ast (f_\ast f^\ast \mathcal{A}) \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{\cong} & g_\ast g^\ast \mathcal{A}
\end{array}
\] (4.9)

such that the composition of the vertical maps equals multiplication by \( \deg f \).

The upper square comes from the functoriality of the adjunction morphism associated to the natural transformation of functors \( \text{id} \to g_\ast g^\ast \). The crucial point is to construct the lower right morphism \( g_\ast g^\ast (f_\ast f^\ast \mathcal{A}) \to \mathcal{A} \) corresponding to the trace morphism (it is not directly possible to apply \( g_\ast g^\ast \) to a map \( f_\ast f^\ast \mathcal{A} \to \mathcal{A} \) since it is not clear how to define the latter). This map arises as follows: Since the domain of \( g \) is a field, by Step 3, there is a trace morphism \( f_\ast f^\ast (g^\ast \mathcal{A}) = f_\ast g^\ast (f_\ast f^\ast \mathcal{A}) \to g^\ast \mathcal{A} \).

By the commutativity of the diagram (4.8), one has

\[
g_\ast f_\ast (f^\ast \mathcal{A}) = f_\ast g^\ast (f_\ast f^\ast \mathcal{A}) \to g^\ast \mathcal{A}.
\]

By proper base change [Den88], p. 231, Theorem 1.1 (The assumptions are: Noetherian, \( f \) proper, \( X' \) excellent, \( g' \) normal, i.e. flat with geometrically normal fibres: \( g' \) is flat as it is the base change of the flat morphism \( g : \{\eta\} \to X \); flat since \( g \) is the inclusion of \( \mathcal{O}_{X, \eta} \), \( g'^\ast f_\ast = f_\ast g'^\ast \), so

\[
g^\ast f_\ast (f^\ast \mathcal{A}) = g^\ast \mathcal{A}.
\]

By Step 3, it is obvious that the composition of the vertical arrows on the right hand side is multiplication by \( \deg f \).

Finally, applying \( g_\ast \), we get our trace morphism on the right hand side.

**Lemma 4.22.** In the situation of Corollary 2.15 if \( \text{Br}(\mathcal{C})[\ell^\infty] \) is finite for \( \ell \) invertible on \( X \), \( \text{Br}(X)[\ell^\infty] \) is finite.

**Proof.** Obvious from Corollary 2.15.

**4.3 Isogeny invariance**

**Theorem 4.23.** Let \( X/k \) be proper, \( \mathcal{A} \) and \( \mathcal{A}' \) Abelian schemes over \( X \) and \( f : \mathcal{A}' \to \mathcal{A} \) an étale isogeny. Let \( \ell \neq \text{char } k \) be a prime. Then \( \text{III}(\mathcal{A}/X)[\ell^\infty] \) is finite if and only if \( \text{III}(\mathcal{A}'/X)[\ell^\infty] \) is finite.

**Proof.** This follows from the long exact cohomology sequence associated to the short exact sequence of étale sheaves \( (f \text{ is étale and surjective}) 0 \to \ker(f) \to \mathcal{A}' \to \mathcal{A} \to 0 \) and [Mil80], p. 224f., Corollary VI.2.8.

**4.4 The Cassels-Tate pairing**

Let \( k = \mathbb{F}_q \) be a finite field with absolute Galois group \( \Gamma = G_k \) and \( X/k \) a smooth projective geometrically connected variety of dimension \( d \). Let \( \mathcal{A}/X \) be an Abelian scheme and \( \ell \neq p = \text{char } k \) be prime.
Reduction to the curve case. Let $Y \hookrightarrow X$ be an ample smooth geometrically connected hypersurface section (this exists by [Poo05], Proposition 2.7) with (necessarily) affine complement $U \hookrightarrow X$. Base changing to $\bar{k}$ and writing $\bar{X} = X \times_k \bar{k}$ etc., one has by [Mil80], p. 94, Remark III.1.30 a long exact sequence

$$\ldots \to H^i_c(\bar{U}, \mathcal{O}[\bar{\ell}^n]) \to H^i(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \to H^i(Y, \mathcal{O}[[\ell^n]]) \to H^{i+1}_c(\bar{U}, \mathcal{O}[\bar{\ell}^n]) \to \ldots$$

(Note that $H^2(\bar{X}, \mathcal{O}) = H^2(Y, \mathcal{O})$ since $\bar{X}$ is proper, and likewise for $Y$.)

Since $\mathcal{O}[[\ell^n]]/X$ is étale, Poincaré duality [Mil80], p. 276, Corollary VI.11.2 gives us

$$H^i_c(\bar{U}, \mathcal{O}[\bar{\ell}^n]) = H^{2d-i}(\bar{U}, (\mathcal{O}[[\ell^n]])^\vee(d)).$$

(Note that the varieties live over a separably closed field.) By the affine Lefschetz theorem [Mil80], p. 253, Theorem VI.7.2, one has $H^{2d-i}(\bar{U}, (\mathcal{O}[[\ell^n]])^\vee(d)) = 0$ for $2d-i > d$, i.e. for $i < d$. Analogously, $H^{i+1}_c(\bar{U}, \mathcal{O}[\bar{\ell}^n]) = 0$ for $i + 1 < d$. Plugging this into (4.10), one gets an isomorphism

$$H^i(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \sim H^i(\bar{Y}, \mathcal{O}[\bar{\ell}^n])$$

for $i + 1 < d$. Inductively, it follows that the cohomology groups in dimension 0 and 1 are isomorphic to the cohomology groups of a surface $S$.

If $Y = C \hookrightarrow X$ is a curve, one gets at least

$$H^0(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \sim H^0(\bar{C}, \mathcal{O}[\bar{\ell}^n]),$$

$$H^1(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \sim H^1(\bar{C}, \mathcal{O}[\bar{\ell}^n]).$$

There are short exact sequences

$$0 \to H^{i-1}(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \to H^i(X, \mathcal{O}[[\ell^n]]) \to H^i(\bar{X}, \mathcal{O}[\bar{\ell}^n]) \to 0.$$

Using (4.11), one gets a commutative diagram

$$\begin{array}{ccc}
H^0(X, \mathcal{O}[\ell^n]) & \xrightarrow{\sim} & H^0(\bar{X}, \mathcal{O}[\bar{\ell}^n])^\Gamma \\
\downarrow & & \downarrow \\
H^0(C, \mathcal{O}[\ell^n]) & \xrightarrow{\sim} & H^0(\bar{C}, \mathcal{O}[\bar{\ell}^n])^\Gamma,
\end{array}$$

and hence an isomorphism $H^0(X, \mathcal{O}[\ell^n]) \sim H^0(C, \mathcal{O}[\ell^n])$.

This implies

$$\text{Tor}(\mathcal{O}(\ell^\infty)) = H^0(X, \mathcal{O}[\ell^\infty]) = H^0(C, \mathcal{O}[\ell^\infty]) = \text{Tor}(\mathcal{O}(\ell^\infty)) = H^0(C, \mathcal{O}[\ell^n]).$$

Using (4.11) and (4.12), one gets a commutative diagram

$$\begin{array}{cccccc}
0 & \to & H^0(\bar{X}, \mathcal{O}[\bar{\ell}^n]) & \xrightarrow{\sim} & H^1(\bar{X}, \mathcal{O}[\bar{\ell}^n])^\Gamma & \xrightarrow{\sim} & H^1(\bar{C}, \mathcal{O}[\bar{\ell}^n])^\Gamma & \to 0 \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \\
0 & \to & H^0(\bar{C}, \mathcal{O}[\bar{\ell}^n]) & \xrightarrow{\sim} & H^1(\bar{C}, \mathcal{O}[\bar{\ell}^n]) & \xrightarrow{\sim} & H^1(\bar{C}, \mathcal{O}[\bar{\ell}^n])^\Gamma & \to 0.
\end{array}$$

and hence, by the snake lemma, an injection $H^1(X, \mathcal{O}[\bar{\ell}^n]) \hookrightarrow H^1(C, \mathcal{O}[\bar{\ell}^n])$.

Remark 4.24. By the snake lemma, this injection is an isomorphism if $H^1(X, \mathcal{O}[\bar{\ell}^n])^\Gamma \hookrightarrow H^1(C, \mathcal{O}[\bar{\ell}^n])^\Gamma$ is surjective, e.g. dim $C \geq 2$. Under this condition, one gets (by passing to the inverse limit $\varprojlim$ and tensoring with $\mathbf{Q}_l$) the equality of the $L$-functions $L(\mathcal{O}/X, s) = L(\mathcal{O}/C, s)$, and hence, under assumption of the Birch-Swinnerton-Dyer conjecture (III($\mathcal{O}/C)[\ell^\infty]$ finite), the equality of the ranks $\text{rk}_Z(\mathcal{O}(\ell^\infty)) = \text{rk}_Z(\mathcal{O}(\ell^\infty))$.

We want the map $H^1(X, \mathcal{O}[\bar{\ell}^n]) \to H^1(C, \mathcal{O}[\bar{\ell}^n])$ of the $\ell^n$-torsion of the Tate-Shafarevich groups to be injective in order for the generalised Cassels-Tate pairing to be non-degenerate (at least on the non-divisible part).
By the snake lemma, the left vertical map is an injection and \( H^1(X, \mathcal{A})[\ell^n] \rightarrow H^1(C, \mathcal{A})[\ell^n] \) is injective if this injection \( \mathcal{A}(X)[\ell^n] \hookrightarrow \mathcal{A}(C)[\ell^n] \) is surjective. This is e.g. the case if \( \text{rk} \mathcal{A}(C) = 0 \) since \( \text{Tor}(\mathcal{A}(X))[\ell^n] = \text{Tor}(\mathcal{A}(C))[\ell^n] \) by (4.13) and \( \mathcal{A}(X)[\ell^n] \rightarrow \mathcal{A}(C)/[\ell^n] \) is injective. For example, one has \( \text{rk} \mathcal{A}(C) = 0 \) if \( \mathcal{A}/\mathcal{C} \) is constant and \( C \cong \mathbb{P}^1 \); The rank of the Mordell-Weil group of a constant Abelian variety over a projective space has rank 0, since there are no non-constant \( k \)-morphisms \( \mathbb{P}^1_k \rightarrow A \), see [MI86a], p. 107, Corollary 3.9.

**Remark 4.25.** Note that for \( \dim Y \geq 2 \) and \( Y \hookrightarrow X \) excised by a sequence of ample hypersurface sections, \( \text{rk} \mathcal{A}(X) = \text{rk} \mathcal{A}(Y) \) by Remark 4.24.

**The curve case.** By [MI86b], p. 176, (a), one has for \( C/k \) a smooth proper geometrically connected curve over the finite ground field \( k \):

\[
H^3(C, G_m) = \mathbb{Q}/\mathbb{Z}.
\]

By the local-to-global spectral sequence \( H^r(X, \mathcal{E}xt^1_X(\mathcal{A}, G_m)) \Rightarrow \text{Ext}^{r+1}_X(\mathcal{A}, G_m) \), and using \( \mathcal{H}om_X(\mathcal{A}, G_m) = 0 \) and the Barsotti-Weil formula \( \mathcal{E}xt^1_X(\mathcal{A}, G_m) = \mathcal{A}^{\vee} \), we get edge morphisms

\[
H^r(X, \mathcal{A}^{\vee}) \rightarrow \text{Ext}^{r+1}_X(\mathcal{A}, G_m),
\]

which are injective for \( r = 1 \). Composing this for \( r = 1 \) with the Yoneda pairing for \( r = 2 \)

\[
\text{Ext}^r_C(\mathcal{A}, G_m) \times H^{3-r}(C, \mathcal{A}) \rightarrow H^3(C, G_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},
\]

induces a pairing

\[
H^1(C, \mathcal{A}^{\vee}) \times H^1(C, \mathcal{A}) \xrightarrow{\cup} H^3(C, G_m) = \mathbb{Q}/\mathbb{Z}.
\]

This is the Cassels-Tate pairing, see [MI86b], p. 199–203.

**Acknowledgements.** I thank Uwe Jannsen, Jean-Louis Colliot-Thélène, Brian Conrad, Peter Jossen, Moritz Kerz, Niko Naumann, Jakob Stix, and, from mathoverflow, abx, Angelo, anon, Martin Bright, Kestutis Cesnavicius, Torsten Ekedahl, Laurent Moret-Bailly, ulrich and xuhan; for proofreading Patrick Forré, Peter Jossen, Niko Naumann and Antonella Perucca; finally the Studienstiftung des deutschen Volkes for financial and ideational support.

**References**

[Art69] ARTIN, Michael: Algebraic approximation of structures over complete local rings. In: Publ. Math. IHÉS, 36 (1969), 23–58.

[BLR90] BOSCH, Siegfried; LÜTKEBOHMERT, Werner and RAYNAUD, Michel: Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 21. Berlin etc.: Springer-Verlag. x, 325 p. 1990.

[Con] CONRAD, Brian: Math 248B. Picard functors for curves. URL http://math.stanford.edu/~conrad/248BPage/handouts/pic.pdf

[Den88] DENINGER, Christopher: A proper base change theorem for non-torsion sheaves in étale cohomology. In: J. Pure Appl. Algebra, 50(3) (1988), 231–235.

[dJ] DE JONG, Aise Johan: A result of Gabber. URL www.math.columbia.edu/~dejong/papers/2-gabber.pdf

[Eis95] EISENBUD, David: Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics. 150. Berlin: Springer-Verlag. xvi, 785 p. 1995.

[FGI+05] FANTECHI, Barbara; GÖTTSCHE, Lothar; ILLUSIE, Luc et al.: Fundamental algebraic geometry: Grothendieck’s FGA explained. Mathematical Surveys and Monographs 123. Providence, RI: American Mathematical Society (AMS). x, 339 p. 2005.
REFERENCES

[Fu11] Fu, Lei: Étale cohomology theory. Nankai Tracts in Mathematics 13. Hackensack, NJ: World Scientific. ix, 611 p. 2011.

[Gro68] Grothendieck, Alexandre: Le groupe de Brauer. III : Exemples et compléments. Dix Exposes Cohomologie Schémas, Advanced Studies Pure Math. 3, 88–188. 1968.

[Har83] Hartshorne, Robin: Algebraic geometry. Corr. 3rd printing. Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag. XVI, 496 p. 1983.

[Kah06] Kahn, Bruno: Sur le groupe des classes d’un schéma arithmétique. (Avec un appendice de Marc Hindry). In: Bull. Soc. Math. Fr., 134(3) (2006), 395–415.

[Kel14] Keller, Timo: On the conjecture of Birch and Swinnerton-Dyer for Abelian schemes over higher dimensional bases over finite fields. Preprint, 2014.

[Lic83] Lichtenbaum, Stephen: Zeta-functions of varieties over finite fields at $s = 1$. Arithmetic and geometry, Pap. dedic. I. R. Shafarevich, Vol. I: Arithmetic, Prog. Math. 35, 173–194. 1983.

[Liu06] Liu, Qing: Algebraic geometry and arithmetic curves. Transl. by Reinie Erné. In: Oxford University Press. xv, 577 p. 2006.

[Mi06] Milne, James S.: Étale cohomology. Princeton Mathematical Series. 33. Princeton, New Jersey: Princeton University Press. XIII, 323 p. 1980.

[Mii86a] Milne, James S.: Abelian varieties. Arithmetic geometry, Pap. Conf., Storrs/Conn. 1984, 103–150 (1986). 1986.

[Mii86b] Milne, James S.: Arithmetic duality theorems. Perspectives in Mathematics, Vol. 1. Boston etc.: Academic Press. Inc. Harcourt Brace Jovanovich, Publishers. X, 421 p. 1986.

[Ser02] Serre, Jean-Pierre: Galois cohomology. Transl. from the French by Patrick Ion. 2nd printing. Berlin: Springer 2002.

[Tat66] Tate, John T.: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Dix Exposés Cohomologie Schémas, Advanced Studies Pure Math. 3, 189–214 (1968); Sémin. Bourbaki 1965/66, Exp. No. 306, 415–440. 1966.

[EGAII] Grothendieck, Alexandre and Dieudonné, Jean: Éléments de géométrie algébrique : II. Étude globale élémentaire de quelques classes de morphismes. In: Publ. Math. IHÉS, 8 (1961), 5–222.

[EGAIII1] ——— Éléments de géométrie algébrique : III. Étude cohomologique des faisceaux cohérents, Première partie. In: Publ. Math. IHÉS, 11 (1961), 5–167.

[EGAIV3] ——— Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. In: Publ. Math. IHÉS, 28 (1966), 5–255.

[EGAIV4] ——— Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. In: Publ. Math. IHÉS, 32 (1967), 5–361.

[SGA4.2a] Artin, Michael; Grothendieck, Alexandre and Verdier, Jean-Louis: Séminaire de Géométrie Algébrique du Bois Marie – 1963–64 – Théorie des topos et cohomologie étale des schémas – (SGA 4) – vol. 2. Lecture notes in mathematics (in French) 270, Berlin; New York: Springer-Verlag. iv+418 1972.

[SGA4.3b] ——— Séminaire de Géométrie Algébrique du Bois Marie – 1963–64 – Théorie des topos et cohomologie étale des schémas – (SGA 4) – vol. 3. Lecture notes in mathematics (in French) 305, Berlin; New York: Springer-Verlag. vi+640 1972.

[SGA6] Berthelot, Pierre; Grothendieck, Alexandre and Illusie, Luc: Séminaire de Géométrie Algébrique du Bois Marie – 1966–67 – Théorie des intersections et théorème de Riemann-Roch – (SGA 6). Lecture notes in mathematics (in French) 225, Berlin; New York: Springer-Verlag. xii+700 1971.

Timo Keller, Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-Mail address: firstname.lastname@mathematik.uni-regensburg.de