Phases of $\mathcal{N} = 1$ $USp(2N_c)$ Gauge Theories with Flavors

Changhyun Ahn$^1$, Bo Feng$^2$ and Yutaka Ookouchi$^3$

$^1$Department of Physics, Kyungpook National University, Taegu 702-701, Korea
$^2$Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA
$^3$Department of Physics, Tokyo Institute of Technology, Tokyo 152-8511, Japan

ahn@knu.ac.kr, fengb@ias.edu, ookouchi@th.phys.titech.ac.jp

Abstract

We studied the phase structures of $\mathcal{N} = 1$ supersymmetric $USp(2N_c)$ gauge theory with $N_f$ flavors in the fundamental representation as we deformed the $\mathcal{N} = 2$ supersymmetric QCD by adding the superpotential for adjoint chiral scalar field. We determined the most general factorization curves for various breaking patterns, for example, the two different breaking patterns of quartic superpotential. We observed all kinds of smooth transitions for quartic superpotential. Finally we discuss the intriguing role of $USp(0)$ in the phase structure and the possible connection with observations made recently in hep-th/0304271 (Aganagic, Intriligator, Vafa and Warner) and in hep-th/0307063 (Cachazo).
1 Introduction and Summary

The $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions have rich structures and the nonperturbative aspects can be characterized by the holomorphic effective superpotential which determines the quantum moduli space. A new recipe for the calculation of the effective superpotential was proposed in [1, 2, 3] through the free energies in the bosonic matrix model. These matrix model analyses could be interpreted within purely field theoretic point of view [4]. A new kind of duality where one can transit several vacua with different broken gauge groups continuously and holomorphically by changing the parameters of the superpotential was given by [5] (similar ideas have appeared in the earlier work [6, 7, 8] and in the more recent one [9]). The extension of [5] to the $\mathcal{N} = 1$ supersymmetric gauge theories with the gauge group $SO(N_c)/USp(2N_c)$ was found in [10] where the phase structures of these theories, the matrix model curve, and a generalized Konishi anomaly equation were obtained. In [11], by adding the flavors in the fundamental representation to the theory of [5], the vacuum structures, an addition map, and multiplication map were developed. Recently, the phase structures of $\mathcal{N} = 1$ supersymmetric $SO(N_c)$ gauge theory with $N_f$ flavors in the vector representation [12] were obtained.

In this paper, we continue to study the phase structures of $\mathcal{N} = 1$ supersymmetric $USp(2N_c)$ gauge theory with $N_f$ flavors in the fundamental representation by deforming the $\mathcal{N} = 2$ supersymmetric QCD with the superpotential of arbitrary polynomial for the adjoint chiral scalar field, by applying the methods in [5, 10, 11, 12]. These kinds of study were initiated in [5], in which a kind of new duality was found. This paper is a generalization of [10] to the $USp(2N_c)$ with flavors. We found that with flavors, the phase structure is richer and that more interesting dualities show up. We refer to [12] for some relevant papers on the recent works, along the line of [1, 2, 3].

In section 2, we describe the classical moduli space of $\mathcal{N} = 2$ SQCD deformed to $\mathcal{N} = 1$ theory by adding the superpotential $W(\Phi)$ (2.2). The gauge group $USp(2N_c)$ will break to $USp(2N_0) \times \prod_{j=1}^n U(N_j)$ with $2N_0 + \sum_{j=1}^n 2N_j = 2N_c$ by choosing the adjoint chiral field $\Phi$ to be the root of $W'(x)$ or $\pm im_i$, the mass parameters. To obtain a pure Coulomb branch where no any factor $U(N_j)$ is higgsed, we restrict ourselves to the case $W'(\pm im_i) = 0$. For each factor with some effective massless flavors, there exists a rich structure of Higgs branches, characterized by an integer $r_i$, that meets the Coulomb branch along the submanifold.

In section 3.1, we discuss the quantum moduli space of $USp(2N)$ by both the weak and strong coupling analyses. When the difference between the roots of $W'(x)$ is much larger than $\mathcal{N} = 2$ dynamical scale $\Lambda$ (in the weak coupling region), the adjoint scalar field $\Phi$ can be integrated out, giving a low energy effective $\mathcal{N} = 1$ superpotential. Under this condition, the higher order terms except for the quadratic piece in the superpotential (2.2) can be ignored.
Then the effective superpotential consists of the classical part plus nonperturbative part. We summarize the quantum theory with mass deformation for \( \Phi \) in the weak coupling analysis. There exist two groups of solutions, i.e., Chebyshev vacua and Special vacua, according to the unbroken flavor symmetry, a meson-like matrix \( M \), and various phases of vacua. At scales below the \( \mathcal{N} = 1 \) scale \( \Lambda_1 \) (when the roots of \( W'(x) \) are almost the same), strong coupling analysis is relevant. We need to determine the special points where some number of magnetic monopoles (mutually local or non local) become massless, on the submanifold of the Coulomb branch of \( \mathcal{N} = 2 \) USp\((2N)\) which is not lifted by the \( \mathcal{N} = 1 \) deformation. The conditions for these special points are translated into a particular factorization form of the corresponding Seiberg-Witten curve. We discuss some aspects of these curves at the Chebyshev branch or the Special branch, in particular, the power of factor \( t = x^2 \) and the number of single roots.

In section 3.2, combining the quantum moduli space of USp\((2N)\) group with the quantum moduli space of \( U(N) \) group studied in [11, 12], we give the most general factorization curves with the proper number of single roots and double roots for various symmetry breaking patterns, which generalize the results in [13, 14, 15, 16]. From the point of view of the geometry, these various breaking patterns correspond to the various distributions of wrapping D5-branes among the roots of \( W'(x) \). In this subsection, we also describe the mysterious role of USp\((0)\) in the phase structure.

In section 4, we study the quartic tree level superpotential with massive flavors for USp\((4)\) and USp\((6)\) gauge groups. There exist two breaking patterns USp\((2N_c)\) \( \rightarrow \) USp\((2N_0) \times U(N_1)\) where \( N_0 + N_1 = N_c \), \( N_0 \geq 0 \) and USp\((2N_c)\) \( \rightarrow \) USp\((2N_c)\). Depending on the properties of various factors, the factorization problems lead to interesting smooth transitions among these two breaking patterns: USp\((2N_c) \leftrightarrow USp(2M_0) \times U(M_1), USp(2N_0) \times U(N_1) \leftrightarrow USp(2M_0) \times U(M_1) \) and USp\((N_0) \times U(N_1) \leftrightarrow USp(2M_0) \times U(M_1) \leftrightarrow USp(2L_0) \times U(L_1)\). The phase structures for various product gauge groups are summarized in the Tables in section 4 and section 5. The addition map application helps us to derive the phases from known vacua, without computing the details.

In section 5, we move to the massless flavors with quartic deformed superpotential for USp\((2N_c)\) where \( N_c = 2 \) and 3. In this case, at the IR limit the USp\((2N_0)\) factor has massless flavors instead of \( U(N_i) \) factor. Because of this difference, new features arise. For the smooth transition USp\((2N_0) \times U(N_1) \leftrightarrow USp(2M_0) \times U(M_1)\) in the Special branch, we have \( M_0 = (N_f - N_0 - 2) (M_0 < N_0 \) and \( 2M_0 + 2 \leq N_f < 2N_0 + 2), which is the relationship between USp\((2N_0)\) and USp\((2M_0)\) to be the Seiberg dual pair. In fact, the smooth transition in this case may be rooted in the Seiberg duality.

In section 6, we use the Brane setup to understand the mysterious role of USp\((0)\) and recent observations made in [17, 18].

In Appendix A, by using the \( \mathcal{N} = 2 \) curve together with monopole constraints we are
interested in and applying the contour integral formula, we derive the matrix model curve (A.3) for deformed superpotential with an arbitrary degree and the relationship (A.2) (or the most general expression between the matrix model curve and the deformed superpotential $W'(x)$). Using these results, we have also checked the generalized Konishi anomaly equation for our gauge theory with flavors (A.8).

In Appendix B, in order to understand the vacua of different theories, we discuss both the addition map and multiplication map. The addition map with flavors relates the vacua of $USp(2N_c)$ gauge theory with $N_f$ flavors in the $r$-th branch to those of $USp(2N'_c)$ gauge theory with $N'_f$ flavors in the $r'$-branch. This phenomenon is the same as the one in the $U(N_c)$ gauge theory with flavors. For the multiplication map, we present the most general form. Through this multiplication map, one obtains the unknown factorization of the gauge group with higher rank from the known factorization of the gauge group with lower rank. One can construct a multiplication map from $USp(2N_c)$ with $2l$ massive flavors to $USp(K(2N_c + 2) - 2)$ with $2Kl$ massive flavors where $K$ is a positive integer. In this derivation, the properties of Chebyshev polynomials are used.

2 The classical moduli space of $USp(2N_c)$ supersymmetric QCD

Although the classical picture will be changed by quantum corrections, in certain situations, the classical analysis can be used to find out some information that is also valid in the full quantum theory.

Let us discuss $\mathcal{N} = 1$ supersymmetric $USp(2N_c)$ gauge theory with $N_f$ flavors of quarks $Q_a^i$ where $a = 1 \cdots, 2N_c$ and $i = 1, \cdots, 2N_f$ in the fundamental representation. The tree level superpotential of the theory can be obtained from $\mathcal{N} = 2$ SQCD by including the arbitrary polynomial of the adjoint scalar $\Phi$ belonging to the $\mathcal{N} = 2$ vector multiplet:

$$W_{tree}(\Phi, Q) = \sqrt{2} Q_a^i \Phi_b^a J^{bc} Q_c^i + \sqrt{2} m_{ij} Q_a^i Q_b^j J^{ab} + \sum_{s=1}^{k+1} g_{2s} u_{2s}$$

(2.1)

where the first two terms come from the $\mathcal{N} = 2$ theory and the third term, $W(\Phi)$, is a small perturbation of $\mathcal{N} = 2$ $USp(2N_c)$ gauge theory [19, 20, 21, 22, 23, 24, 25, 26, 15]

$$W(\Phi) = \sum_{s=1}^{k+1} \frac{g_{2s}}{2s} \text{Tr} \Phi^{2s} \equiv \sum_{s=1}^{k} g_{2s} u_{2s}, \quad u_{2s} \equiv \frac{1}{2s} \text{Tr} \Phi^{2s}$$

(2.2)

where $\Phi$ plays the role of a deformation breaking $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ supersymmetry. The $J_{ab}$ is the symplectic metric and $m_{ij}$ is a quark mass that together are represented
\[
J_{ab} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \otimes I_{Nc\times Nc}, \quad m = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \otimes \text{diag}(m_1, \ldots, m_{N_f})
\]

where the symplectic metric \( J_{ab} \) is used to raise or lower the \( USp(2N_c) \) color indices. The \( \mathcal{N} = 2 \) theory is asymptotically free for \( N_f < 2N_c + 2 \) (generating a strong coupling scale \( \Lambda \)), conformal for \( N_f = 2N_c + 2 \) and IR free for \( N_f > 2N_c + 2 \).

The classical vacuum structure, the zeroes of the scalar potential, can be obtained by solving D-terms and F-terms. We summarize the following results from the mechanism of adjoint vevs as follows:

1) The eigenvalues of \( \Phi \), \( \pm \phi_j \) can only be the roots of \( W'(x) \) or \( \pm im_j \); thus, the gauge group \( USp(2N_c) \) with \( N_f \) flavors is broken to the product of blocks with or without the effective massless flavors. Among these blocks, at most one block is \( USp(\mu_{n+1}) \) and others, \( U(\mu_i) \) where \( i = 1, 2, \ldots, n \).

2) If \( \phi_i = \pm im_i \) but \( W'(\pm im_i) \neq 0 \), the corresponding gauge symmetry of that block will be completely higgsed. Due to this fact, in the following discussions we will restrict ourselves to the case of \( W'(\pm im) = 0 \) where there exist richer structures.

3) For each block, if it has the “effective” massless flavors, the vacua are classified by an integer \( r \).

3 The quantum moduli space of \( USp(2N_c) \) supersymmetric QCD

According to the previous section, the gauge group \( USp(2N_c) \) is broken to the product of various gauge group \( U(N_i) \) with at most one single \( USp(2N) \). In order to understand the quantum moduli space of \( USp(2N_c) \) supersymmetric QCD, we need to study the quantum moduli for each factor gauge group. In this section, we focus on the quantum moduli space of \( USp(2N) \) gauge group with massless flavors discussed in \([20, 25]\) because the quantum moduli of \( U(N_i) \) with effective massless flavors were already studied in \([27, 25, 11]\).

3.1 The quantum theory of \( USp(2N) \) supersymmetric QCD

The quantum theory of \( USp(2N) \) with mass deformation \( \frac{1}{2}\mu\text{Tr}\Phi^2 \) has been described in \([20, 25]\) for both weak and strong coupling analyses. We will summarize the main results.

3.1.1 The weak coupling analysis

When the mass \( \mu \) for the adjoint scalar \( \Phi \) is larger than the \( \mathcal{N} = 2 \) dynamical scale \( \Lambda \) (that is, \( \mu \gg \Lambda \)), we integrate out \( \Phi \) first and by resubstituting the \( \Phi \) into the superpotential
we get \(-\frac{1}{8\pi} \text{Tr}(M J M J)\) where \(M^i{}_{ij} = Q_i^a J^{ab} Q_j^b = -M^{ji} \) are the meson-like composite superfields. However, depending on the number of flavors, various terms can be added to the superpotential by quantum effects. To find \(\mathcal{N} = 1\) vacua, the effective superpotential should be minimized. We can study the vacuum structure, the number of vacua, and global symmetry breakings according to the range of the number of flavors \(N_f\).

We summarize two groups of solutions as follows:

1) The first group exists for arbitrary number of flavors with broken flavor symmetry \(SO(2N_f) \to U(N_f)\) and the counting of vacua is \((2N + 2 - N_f)\). It corresponds to the Chebyshev point (which will be discussed soon) where its position in the \(\mathcal{N} = 2\) moduli space is located by the roots of a Chebyshev polynomial. The light degrees of freedom are mutually nonlocal and the theory flows to an interacting \(\mathcal{N} = 2\) superconformal theory. The dynamic flavor symmetry breaks into \(U(N_f)\).

2) The second group exists only when \(N_f \geq N + 2\) with unbroken flavor symmetry \(SO(2N_f)\). It corresponds to the “baryonic-like” root. In this Special point, the gauge symmetry is enhanced to \(USp(2\tilde{N}) \times U(1)^{N - \tilde{N}}\) where \(\tilde{N} = N_f - N - 2\) and the full \(SO(2N_f)\) global symmetry remains unchanged since there are no meson condensates and no dynamic symmetry breaking occurs. These vacua are in the free magnetic phase.

3.1.2 The strong coupling analysis

The strong coupling analysis has been done in [20, 25]. Let us recall the curve of \(USp(2N)\) [19] first

\[
y^2 = \left[ t \prod_{j=1}^{N} (t - \phi_j^2) + 2\Lambda^{2N+2-N_f} \prod_{k=1}^{N_f} m_k \right]^2 - (-1)^{N_f} 4\Lambda^{2(2N+2-N_f)} \prod_{k=1}^{N_f} (t - m_k^2)
\]

where we have used a relation \(t = x^2\). When \(2N + 2 - N_f = 0\) corresponding to the conformal theory, we should replace \(2\Lambda^{2N+2-N_f}\) by the moduli form

\[
g(\tau) = \frac{\theta_4^4}{\theta_3^4 + \theta_4^4}.
\]

For our analysis where all \(m_k = 0\), the curve is simplified to be

\[
y^2 = \left[ t \prod_{j=1}^{N} (t - \phi_j^2) \right]^2 - 4\Lambda^{2(2N+2-N_f)} t^{N_f}.
\]

If \(r \phi_j\)'s vanish and the remaining \((N - r)'s do not, we can factorize the curve (3.1) as

\[
y^2 = t^{2(r+1)} \prod_{j=1}^{N-r} (t - \phi_j^2)^2 - 4\Lambda^{2(2N+2-N_f)} t^{N_f - 2(r+1)}.
\]

\[1\) Notice that for \(USp(2N)\) gauge group, there is no baryonic-like gauge invariant operator. The invariant tensor breaks up into sums of products of the \(J^{ab}\) and baryons break up into mesons [28]. In this subsection, we consider only massless flavors (the quark mass \(m\) vanishes).
The analysis done in [20, 25] implies that only when
\[ r = \tilde{N} = N_f - N - 2 \]
and
\[ r = N_f/2 - 1, \quad \text{for } N_f \text{ even} \]
or
\[ r = (N_f - 1)/2, \quad \text{for } N_f \text{ odd}, \]
the curve (3.2)\(^2\) gives the vacua which are not lifted by the mass deformation \( \frac{1}{2} \mu \text{Tr} \Phi^2 \).

The basic idea of the above results is as follows. For general \( r \) the low energy theory is of the trivial conformal theory, i.e., the class 1 in the classification of [30]. For these theories, to have unlifted vacua, we must have a sufficient number of mutually local massless monopoles, i.e., enough double roots in the SW-curve. For the \( USp(2N) \) gauge group, it is necessary that \( ty^2 \) is a total square form which can happen if and only if \( r = \tilde{N} = (N_f - N - 2) \) by noticing that
\[ ty^2 = t^{2(N+1)} \left( P_{N-N-\tilde{N}}^2(t) - 4\Lambda^2(N-\tilde{N}) t^{N-\tilde{N}} \right) = t^{2(N+1)} \left( t^{N-\tilde{N}} - \Lambda^2(N-\tilde{N}) \right)^2 \]
corresponding to (9.137) of [25] when we choose \( P_{N-N-\tilde{N}}(t) = t^{N-\tilde{N}} + \Lambda^2(N-\tilde{N}) \)\(^3\). It looks like the baryonic root \( r = N_f - N \) of the \( U(N) \) gauge group, so we will call it the baryonic root of \( USp(2N) \). As shown in [25], after perturbation by mass term of flavors, \( r = \tilde{N} \) will give the second group (Special Point) of solutions discussed in the previous subsection.

Except for the above trivial conformal theory, there exists a special case in which the low energy effective theory is of non-trivial conformal theory. It is when \( r = N_f/2 - 1 \) for \( N_f \) even or \( r = (N_f - 1)/2 \) for \( N_f \) odd, which is the class 2 or 3 in the classification of [30] respectively, and is called the Chebyshev point. To see it, assuming that \( N_f \) is even, the curve becomes
\[ ty^2 = t^{N_f} \left[ P_{N+1-N/2}^2(t) - 4\Lambda^4(N+1-N/2) \right]. \]
When we take \(^4\)
\[ P_{N+1-N/2}^2(t) = 2(\eta \Lambda)^2(N+1-N/2) \mathcal{T}_{2(N+1-N/2)} \left( \frac{\sqrt{t}}{2\eta \Lambda} \right) \]
\(^2\)When \( N_f \) is even, since \( \phi_1 = \phi_2 = \cdots = \phi_{r=N_f/2-1}=0 \), the curve becomes \( ty^2 = t^{N_f} \left[ \prod_{j=1}^{N+1-N/2} (t - \phi_j^2) - 4\Lambda^2(2N+2-N_f) \right] \). When \( N_f \) is odd, since \( \phi_1 = \phi_2 = \cdots = \phi_{r=(N_f-1)/2}=0 \), the curve becomes \( ty^2 = t^{N_f} \left[ \prod_{j=1}^{N-(N_f-1)/2} (t - \phi_j^2) - 4\Lambda^2(2N+2-N_f) \right] \).

\(^3\)In this case, one takes \( (\phi_1, \phi_2, \cdots, \phi_{N-\tilde{N}}) = \Lambda^2(\omega, \omega^3, \cdots, \omega^{2(N-\tilde{N})}) \) where \( \omega = e^{\pi i/(N-\tilde{N})} \). Then the expression of \( P_{N-N-\tilde{N}}(t) = \prod_{j=1}^{N-\tilde{N}} (t - \phi_j) \) can be written as \( t^{N-\tilde{N}} + \Lambda^2(N-\tilde{N}) \). The locations of double zeros of the factor in the parenthesis are at \( t = \Lambda^2/\omega^2, \Lambda^2/\omega^4, \cdots, \Lambda^2/\omega^{2(N-\tilde{N})} = \Lambda^2 \).

\(^4\)The reason we define \( \tilde{t} = \frac{\sqrt{t}}{2\eta \Lambda} \) is because \( t \) has dimension two. We use a useful relation \( \mathcal{T}_{2K}^2(x) - 1 = (x^2 - 1)\mathcal{U}_{K-1}^2(x) \).
with $\eta^{4(N+1-N_f/2)} = 1$, the curve becomes

$$ty^2 = t^{N_f/4}(\eta \Lambda)^{4(N+1-N_f/2)} \left[ T_{2(N+1-N_f/2)}^2 \left( \frac{\sqrt{t}}{2\eta \Lambda} \right) - 1 \right]$$

$$= t^{N_f/4}(\eta \Lambda)^{4(N+1-N_f/2)} \left[ \left( \frac{\sqrt{t}}{2\eta \Lambda} \right)^2 - 1 \right] U_{2(N+1-N_f/2)-1}^2 \left( \frac{\sqrt{t}}{2\eta \Lambda} \right).$$

Notice that although $\eta^{4(N+1-N_f/2)} = 1$, the characteristic function $P_{N+1-N_f/2}(t)$ is a function of $t$ and $\eta^2$ because only the even power term appears in a first kind of Chebyshev polynomial $T_{2(N+1-N_f/2)} \left( \frac{\sqrt{t}}{2\eta \Lambda} \right)$ with even order $2(N + 1 - N_f/2)$; thus, we have $2N + 2 - N_f$ solutions as given in the weak coupling analysis. Additionally, note that except for a factor $t^{N_f}$, unlike the case of the total square form of $r = \tilde{N}$, there is a single root at $t = 0$ and at $t = 4\eta^2 \Lambda^2$ from the factor $\left[ \left( \frac{\sqrt{t}}{2\eta \Lambda} \right)^2 - 1 \right] \left( \frac{\sqrt{x}}{2\eta \Lambda} \right)$ since the second kind of Chebyshev polynomial $U_K(x)$ with an odd degree $K$ has a common factor $x$ and so there exists a factor $t^{N_f+1}$ which is odd for this case. Moreover, there are double zeros at $(N - N_f/2)$ different locations of $t$ from the structure of $U_{2(N+1-N_f/2)-1}^2 \left( \frac{\sqrt{x}}{2\eta \Lambda} \right)$. A similar discussion can be carried out for $N_f$ odd. It turns out that there are $N_f$ (which is odd) zeros at $t = 0$, a single zero at $t = 4\eta^2 \Lambda^2$ and double zeros at $N - (N_f - 1)/2$ different locations of $t$.

3.2 The factorized form of a hyperelliptic curve

Now combining the proper factorization form of $USp(2N)$ curve with the massless flavors in the previous subsection with the $U(N_i)$ curve with flavors in [11], the general form of the curve can be obtained. First we need to have the proper prefactor (like $t^0$ for $USp(2N)$ part and $(t - m^2)^{2r}$ for $U(N_i)$ part at the $r$-th branch). After factorizing out these prefactors, then we require the remaining curve to have a proper number of double roots and single roots, which will fix finally the form of factorization.

For every $U(N_i)$ factor at the non-baryonic branch, there exist two single roots while there is no single root at the baryonic branch. For the possible $USp(2N)$ factor, when it is at the Chebyshev branch, there are two single roots where one of them is at the origin, i.e., we have a factor $t$ for that single root. However, when it is at the Special branch, there exists no single root for the block $USp(2N)$. Adding the single roots together for all the blocks, the final number of the single roots can be found. Additionally notice that the Special branch (or the baryonic branch) will have one more double root than the Chebyshev branch (or the non-baryonic branch) in the factorization form.

However, there is one important point which differentiates the $U(N)$ and $SO(N)$ gauge groups; although the $USp(0)$ does not have the dynamical meaning, it does effect the form of factorization of the Seiberg-Witten curve. In other words, by including the $USp(0)$ factor group,
the breaking pattern $USp(2N_c) \rightarrow U(N_c)$ should be really written as,

$$USp(2N_c) \rightarrow U(N_c) \times USp(0).$$

The specialty of $USp(0)$ can be seen from the strong coupling analysis. Firstly, the number of vacua of $USp(2N)$ at the Chebyshev branch is $(2N + 2 - N_f)$ for $N_f \geq 1$ or $(N + 1)$ for $N_f = 0$. Setting $N = 0$ we find one for $N_f = 0, 1$ and zero for $N_f \geq 2$. Secondly, the power of a factor $t$ at the Special branch is $(N_f - N - 1)$, which is a positive number if $N_f \geq 2$. Combining these two observations, we get the following conclusion: $USp(0)$ has the Chebyshev branch with a factor $t$ for $N_f = 0, 1$ and the Special branch with a factor $t^2$ for $N_f \geq 2$.

Now we can discuss the general factorization form of the Seiberg-Witten curve by using the following simplest nontrivial examples in which $USp(2N)$ gauge group is broken to the following two cases:

$$USp(2N_c) \rightarrow USp(2N_0) \times U(N_1), \quad (N_0 + N_1 = N_c, \ N_0 \geq 0)$$

and

$$USp(2N_c) \rightarrow USp(2N_c).$$

(and the generalization to multiple blocks will be straightforward.)

For the broken pattern $USp(2N_c) \rightarrow USp(2N_0) \times U(N_1)$, by counting the number of the single roots, the following four possible curves are obtained:

$$ty^2 = tF_3(t)H_{N_c-2}^2(t), \quad (3.3)$$

$$ty^2 = F_2(t)H_{N_c-1}^2(t), \quad (3.4)$$

$$ty^2 = tF_1(t)H_{N_c-1}^2(t), \quad (3.5)$$

$$ty^2 = H_{N_c}^2(t). \quad (3.6)$$

The curve (3.3) is for $USp(2N_0)$ at the Chebyshev branch, and $U(N_1)$ at the non-baryonic branch. First we should have four single roots. Secondly, because $USp(2N_0)$ is at the Chebyshev branch, one of the four single roots must be at the origin $t = 0$ and finally one gets $F_4(t) = tF_3(t)$. The curve (3.4) is for $USp(2N_0)$ at the Special branch and $U(N_1)$ at the non-baryonic branch. The factor $F_2(t)$ with two single roots will record the information of $U(1) \subset U(N_1)$. The curve (3.5) is for $USp(2N_0)$ at the Chebyshev branch and $U(N_1)$ at the baryonic branch. Since in this case, $USp(2N_0)$ is at the Chebyshev branch one gets $F_2(t) = tF_1(t)$. Finally, the curve (3.6) is for $USp(2N_0)$ at the Special branch and $U(N_1)$ at the baryonic branch, where no single root is required. Notice that the function $H_p(t)$ will have a proper number of $(t - m^2)$ or $t$ to count the prefactor for various branches.
For the unbroken pattern $USp(2N_c) \rightarrow USp(2N_c)$, one gets

\[
ty^2 = t F_1(t) H_{N_c-1}^2(t),
\]

\[
ty^2 = H_{N_c}^2(t),
\]

where the curve (3.7) is for $USp(2N_c)$ at the Chebyshev branch, and the curve (3.8) is for $USp(2N_c)$ at the Special branch. The function $H_p(t)$ will have a proper number of factor $t$ to count the prefactor required by the Special or Chebyshev branch. It is noteworthy that although both curve (3.5) and curve (3.7) look the same, they can be distinguished by factor $H_p(t)$, where different powers of $t$ and $(t - m^2)$ can arise.

For $U(N_i)$ with $M_i$ flavors, the counting of vacua has been given in [11, 12] as

\[
\begin{cases}
2N_i - M_i & r < M_i/2, \\
N_i - M_i/2 & r = M_i/2, \\
2N_i - M_i & r \geq M_i - N_i, \\
N_i - r & r < M_i - N_i, \\
1 & r = N_i - 1 \text{(nonbaryonic)}, \\
1 & r = M_i - N_i, N_i \text{(baryonic)}. \\
\end{cases}
\]

The number of vacua

\[
\text{For pure } USp(2N) \text{ the counting of vacua is } (N + 1). \text{ For } USp(2N) \text{ with } M \geq 1 \text{ flavors, there are } (2N - M + 2) \text{ vacua from the Chebyshev branch (which means that there is no Chebyshev vacua if } M \geq 2N + 2) \text{ and one vacuum from the Special branch if } M \geq N + 2. \text{ The total number of vacua is the product of the number of vacua of various blocks.}
\]

4 Quartic superpotential with massive flavors

The general curve of $USp(2N)$ gauge theory with $M$ flavors [19, 20, 32] should be

\[
ty^2 = \left[ tP_N(t) + 2\Lambda^{2N+2-M} \prod_{j=1}^{M} m_j \right]^2 - (-1)^M 4\Lambda^{2(2N+2-M)} \prod_{j=1}^{M} (t - m_j^2).
\]

It is noteworthy that the factor $(-1)^M$ will be crucial for later analysis. The presence of the factor $(-1)^M$ makes sure that there is always a factor $t$ in the curve because the $t$-independent two terms on the right hand side exactly cancel each other.

There is a difference of the counting for the vacua between the gauge groups $U(N)$ and $USp(2N)$. For pure $U(N)$ gauge group, the number of vacua are $N$. However, with $N_f$ flavors it becomes $(2N - N_f)$, which shows very different behavior. For a pure $USp(2N)$ gauge group, the number of vacua is $(N + 1)$ while with $N_f$ flavors, it is given by $(2N + 2 - N_f)$. The new aspect for the phase on the breaking $USp(2N_c) \rightarrow U(N_c)$ is that it is, in fact, the breaking

\footnote{As we will see in the examples later, all the factorizations (3.3)-(3.8) except (3.6) appear in our study.}
pattern \( USp(2N_c) \to U(\overline{N_c}) \times USp(0) \), as we have described in section 3. For pure \( USp(0) \), the number of vacua is given by \((N + 1) = 1\) according to the above analysis, so it is not zero. \( ^6 \)

Let us recall that the baryonic branch for \( U(\overline{N_1}) \) factor can be characterized by

\[
\begin{align*}
    r &= N_1, \\
    \text{or} &
    r &= N_f - N_1
\end{align*}
\]

and that the index \( r \) satisfies

\[
0 \leq r \leq \lfloor N_f/2 \rfloor.
\]

The number of flavors \( N_f \) is restricted to \( N_f < 2N_c + 2 \) for asymptotically free theory. For a given number of colors \( N_c \), both the number of flavors \( N_f \) and index \( r \) are fixed. Now we are ready to consider explicit examples for \( USp(4) \) and \( USp(6) \) gauge theories with massive flavors.

### 4.1 \( USp(4) \) case

#### 4.1.1 \( N_f = 0 \)

The curve is given by

\[
    ty^2 = \left[ t(t^2 - s_1t + s_2) + 2\Lambda^6 \right]^2 - 4\Lambda^{12}
\]

by denoting the characteristic function as \( P_2(t) = t^2 - s_1t + s_2 \).

- **Non-baryonic** \( r = 0 \) branch

  We need to have the following factorization

  \[
  ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.
  \]

  The reason is that for \( N_f = 0 \), there is **no** Special branch for the \( USp(2N_0) \) part and the power of a factor \( t \) of the Chebyshev branch becomes \( N_f + 1 = 1 \) which coincides with the one on the right hand side of the above curve. There are two kinds of solutions. The first kind of solutions is given by

  \[
  s_1 = -2d, \quad s_2 = d^2, \quad a = 2d, \quad b = d^2, \quad c = 4\Lambda^6.
  \]

  The classical limit \( \Lambda \to 0 \) implies that the curve reduces to \( t^2(t + d)^4 \), so it gives the breaking pattern \( USp(4) \to USp(0) \times U(2) \) because the characteristic function becomes \( P_2(t) = (t + d)^2 \). The second kind of solution is summarized as follows:

  \[
  s_1 = -2d - \frac{4\Lambda^6}{d^2}, \quad s_2 = d^2 + \frac{6\Lambda^6}{d}, \quad a = 2d + \frac{8\Lambda^6}{d^2},
  \]

  \[
  b = d^2 + \frac{16\Lambda^6}{d} + \frac{16\Lambda^{12}}{d^2}, \quad c = 4 \left( \Lambda^6 + \frac{8\Lambda^{12}}{d^2} \right).
  \]

\( ^6 \)For \( USp(0) \) with \( N_f \) flavors, the number of vacua for the Chebyshev branch is \((2N + 2 - N_f) = 1\) only if \( N_f = 1 \). For the cases \( N_f \geq 2 \), there is **no** Chebyshev branch for the factor \( USp(0) \) due to the negativity of the vacua. However, by the relation \((N_f - 2 - N) = N_f - 2\), when \( N_f \geq 2 \), we have, instead, **Special branch** with a factor \( t^2 \) from the discussion of section 3.
There are two limits we can take: (1) If $\Lambda \to 0$, but $d \to \text{constant}$, the symmetry breaking patterns occur when $USp(4) \to U(2) \times USp(0)$; (2) If $\Lambda \to 0$, $d \to 0$, but $\Lambda^6/d^2 \to \text{constant}$, we have a symmetry breaking $USp(4) \to U(1) \times USp(2)$ where the characteristic function becomes $P_2(t) = t(t + 4\Lambda^6/d^2)$. Thus, we see there exists a smooth transition between $U(2) \times USp(0)$ and $U(1) \times USp(2)$.

Now let us count the number of vacua. For the non-degenerated case, we require 

$$F_3(t) = W'(x)^2 + O(t) = t(t - m^2)^2 + O(t) \Rightarrow -2m^2 = a$$

where the quadratic part of $t$ in $F_3(t)$ is identified with the one in $W'(x)^2$. The first kind of solution gives one vacuum. One special feature of this case is that the curve will have a factor $(t + d)^2 = (t - m^2)^2$ where we put $d = -m^2$. However, this does not mean it is at the $r = 1$ branch, because there is no $r = 1$ branch. The second kind of solution gives three vacua. One of them gives the breaking pattern $USp(4) \to U(2) \times USp(0)$ (therefore, there exist a total of two vacua which can also be seen from counting in the pure $U(2)$ gauge theory) and two others give $USp(4) \to U(1) \times USp(2)$ where there exist two roots for $d$: $-2m^2 = a = 8\Lambda^6/d^2$ where we know the $USp(2)$ has the number of vacua by $(N_0 + 1) = 2$.

We need to determine the $USp(4) \to USp(4)$. To achieve this, we require an extra double root in the curve where $F_3(t) = F_1(t)K_1^2(t)$. Putting these extra conditions into the three solutions we have found, all these cases give a pattern $USp(4) \to USp(4)$ at the classical limit. This matches the counting of vacua $(N + 1) = 3$ in this case.

### 4.1.2 $N_f = 1$

The curve can be written as 

$$ty^2 = [t(t^2 - s_1 t + s_2) + 2\Lambda^5 m]^2 + 4\Lambda^{10}(t - m^2)^2.$$ 

Note the sign of the second term in the right hand side.

- **Baryonic and Non-baryonic $r = 0$ branches**

  From (3.3), first we require that the factorization should be 

  $$ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.$$ 

  There are two kinds of solutions where 

  $$s_1 = -\frac{2(d^3 + \Lambda^5 m) \pm \frac{\Lambda^5(d + 2m^2)}{\sqrt{d^2 + m^2}}}{d^2},$$

  $$s_2 = \frac{d^3 + 4\Lambda^5 m \pm \frac{\Lambda^5(3d + 4m^2)}{\sqrt{d^2 + m^2}}}{d}. $$
There are the following limits: (1) as $\Lambda \to 0$, but $d \to \text{constant}$, we have $USp(4) \to \widehat{U}(2) \times USp(0)$ where the characteristic function behaves as $P_2(t) = (t + d)^2$; (2) when $\Lambda \to 0$, $d \to 0$, but $\Lambda^5/d^2 \to \text{constant}$, the breaking pattern is $USp(4) \to \widehat{U}(1) \times USp(2)$. The characteristic function in this case becomes $P_2(t) = t(t + \Lambda^5m/d^2)$. Thus, we have a smooth transition from $\widehat{U}(2) \times USp(0)$ to $\widehat{U}(1) \times USp(2)$.

To count the number of vacua, using the relation

$$F_3(t) = W'(x)^2 + \mathcal{O}(t) = t(t - m^2)^2 + \mathcal{O}(t) \Rightarrow -2m^2 = a.$$  

There exist five solutions. $^7$ Three of them belong to $\widehat{U}(2) \times USp(0)$ by $(2N - N_f) = 3$, and two of them to $\widehat{U}(1) \times USp(2)$ where $U(1)$ is at the non-baryonic $r = 0$ branch having one vacuum and $USp(2)$ contributes to $(N + 1) = 2$.

To get $USp(4) \to USp(4)$, we require another double root or the curve should be (3.6)

$$ty^2 = t(t + a)(t^2 + bt + c)^2.$$  

There are five equations for five variables so all the parameters can be fixed. Two of them belong to the breaking pattern $\widehat{U}(1) \times USp(2)$ where $\widehat{U}(1)$ is at the baryonic $r = 0$ branch and the $USp(2)$ gives two vacua $(N_0 + 1) = 2$. Therefore, we have the same counting both in the strong and weak coupling analyses. The other three solutions belong to the case $USp(4) \to USp(4)$, which matches the counting of vacua in the weak coupling analysis $(N + 1) = 3$.

However, we still need to discuss the smooth transition carefully as we did for $SO(5)$ with $N_f = 1$ at the baryonic $r = 0$ branch [12]. Carefully solving the problem, we find

$$s_1 = \frac{-a - 2b}{2}, \quad s_2 = \frac{-a(a - 4b)}{8} + c,$$

$$b = a - \frac{256\Lambda^{10}}{a^4}, \quad c = \frac{3a^5 - 2048\Lambda^{10}}{16a^3},$$

$$0 = m + \frac{16\Lambda^5}{a^2} - \frac{a^3}{64\Lambda^5}.$$  

From these results we see the two classical limits: (1) when $\Lambda^5/a^2 \to \text{constant}$, it gives the breaking pattern $\widehat{U}(1) \times USp(2)$, and (2) as $a^3/\Lambda^5 \to \text{constant}$, it provides $USp(4)$. Thus, both the case $USp(4)$ and the case $\widehat{U}(1) \times USp(2)$ where $\widehat{U}(1)$ is at the $r = 0$ baryonic branch are smoothly connected.

### 4.1.3 $N_f = 2$

The curve is given by

$$ty^2 = \left[t(t^2 - s_1 t + s_2) + 2\Lambda^4 m^2\right]^2 - 4\Lambda^8(t - m^2)^2.$$  

$^7$The mathematica calculation will give six solutions, but one of them is that $d = 0$ which is not the real solutions for our problem because $d$ is not equal to zero at the beginning. Another easy way to count the number of vacua correctly is to use the numerical method.
**Baryonic and Non-baryonic \( r = 1 \) branches**

First let us consider the \( r = 1 \) branch. There are two cases: one is \( \hat{U}(2) \) at \( r = 1 \) non-baryonic branch and the other, \( \hat{U}(1) \) at \( r = N_f - N = 1 \) baryonic branch. For \( r = 1 \) non-baryonic branch, we require the factor \((t - m^2)^2\) and the curve to be factorized as follows (3.3):

\[
ty^2 = tF_3(t)(t - m^2)^2 = t(t^3 + at^2 + bt + c)(t - m^2)^2.
\]

The solution is given by

\[
s_2 = -2\Lambda^4 - m^4 + m^2 s_1, \quad a = 2(m^2 - s_1), \quad b = -4\Lambda^4 + m^4 - 2m^2 s_1 + s_1^2, \quad c = -4(\Lambda^4 m^2 - \Lambda^4 s_1).
\]

To fully determine the solution, we need to use \( a = -2m^2 \) as we have observed previously which determines \( s_1 = 2m^2 \). There is only one vacuum which matches the counting of the \( r = 1 \) branch by \((2N - N_f)/2 = 1 \) for \( r = N_f/2 \) [11, 12]. To get the baryonic \( r = 1 \) branch, the curve should be factorized as, from (3.5),

\[
ty^2 = t(t + a)(t + b)^2(t - m^2)^2.
\]

It is easy to see by mathematica computation that there are two solutions giving a breaking pattern \( USp(4) \rightarrow \hat{U}(1) \times USp(2) \) with \( \hat{U}(1) \) at the \( r = 1 \) baryonic branch. The two vacua come from the \( USp(2) \) factor, which is also realized in the weak coupling analysis by \((N + 1) = 2\).

**Baryonic and Non-baryonic \( r = 0 \) branches**

Next we will discuss the \( r = 0 \) branch. In this case, except for the \( r = 0 \) non-baryonic branch for both \( \hat{U}(2) \) and \( \hat{U}(1) \), there is also the \( r = 0 \) baryonic branch for \( \hat{U}(2) \). For the \( r = 0 \) non-baryonic branch, we require the factorization to be

\[
ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.
\]

The first kind of solution is given by

\[
s_1 = -2d, \quad s_2 = d^2 - 2\Lambda^4, \quad a = 2d, \quad b = d^2 - 4\Lambda^4, \quad c = 4\Lambda^4 m^2.
\]

This gives \( USp(4) \rightarrow \hat{U}(2) \) at the classical limit. To count the number of vacua, using the relationship

\[
-2m^2 = a
\]

we get one vacuum. Some remarks are in order. Although the curve has a factor of \((t + d)^2 = (t - m^2)^2\) and it seems to belong to the \( r = 1 \) branch, it is a kind of an illusion and we should take it to be the \( r = 0 \) branch. Similar phenomena have been observed previously in the \( N_f = 0 \) case.
The second kind of solution is given by

\[ s_1 = -2d - \frac{4\Lambda^4 m^2}{d^2}, \quad s_2 = d^2 + 2\Lambda^4 + \frac{8\Lambda^4 m^2}{d}, \]
\[ a = 2d + \frac{8\Lambda^4 m^2}{d^2}, \quad b = d^2 + 4\Lambda^4 + \frac{16\Lambda^4 m^2}{d} + \frac{16\Lambda^8 m^4}{d^4}, \]
\[ c = \frac{4\Lambda^4 m^2 (d^3 + 4d\Lambda^4 + 8\Lambda^4 m^2)}{d^3}. \]

There are the following limits: (1) when \( \Lambda \to 0 \), but \( d \to \) constant, it produces \( USp(4) \to USp(0) \times \tilde{U}(2) \); (2) when \( \Lambda \to 0 \), \( d \to 0 \) and \( \Lambda^4/d^2 \to \) constant, there exists \( USp(4) \to \tilde{U}(1) \times USp(2) \). Thus, we have the smooth transition from \( USp(0) \times \tilde{U}(2) \) to \( \tilde{U}(1) \times USp(2) \).

To count the number of vacua, we use the fact \( a = -2m^2 \) which gives three solutions for \( d \). Two of them give the symmetry breaking \( USp(4) \to USp(0) \times \tilde{U}(2) \) where \( \tilde{U}(1) \) is at the \( r = 0 \) non-baryonic branch. One of them gives \( USp(4) \to USp(0) \times \tilde{U}(2) \). Therefore, by combining with the one vacuum from the first kind of solution, we get two vacua for \( \tilde{U}(2) \) at the \( r = 0 \) non-baryonic branch which is exactly that in the weak coupling analysis obtained through the relation \((2N - N_f) = 2\).

Finally we require

\[ ty^2 = t(t + a)(t + b)^2(t + c)^2 \]

for the \( USp(4) \to USp(4) \). Since it is at the \( r = 0 \) branch, it is equal to

\[ t(t^2 - s_1 t + s_2) + 2\Lambda^4 m^2 + 2\Lambda^4(t - m^2) = t(t + b)^2, \]
\[ t(t^2 - s_1 t + s_2) + 2\Lambda^4 m^2 - 2\Lambda^4(t - m^2) = (t + a)(t + c)^2. \]

Here the first relation can be understood that the left hand side does not have a \( t \)-independent term, it does have a factor \( t \) and therefore it is a consistent equation. The solution is given by

\[ s_1 = -2c - \frac{4\Lambda^4 m^2}{c^2}, \quad s_2 = -2\Lambda^4 + \frac{(c^3 + 2\Lambda^4 m^2)^2}{c^4}, \]
\[ a = \frac{4\Lambda^4 m^2}{c^2}, \quad b = c + \frac{2\Lambda^4 m^2}{c^2}, \]
\[ 0 = c^4 + c^3 m^2 - \Lambda^4 m^4. \]

There are four solutions. Keeping the first and second terms of the last equation, we get one solution \( c, b \sim -m^2 \) which gives a breaking pattern \( USp(4) \to USp(0) \times \tilde{U}(2) \) where \( \tilde{U}(2) \) is at the \( r = 0 \) baryonic branch. The vacuum number one is also realized in the weak coupling analysis \([11, 12]\). Keeping the second and third terms of the last equation, we get three solutions \( c \sim \Lambda^4/3 \) which give a pattern \( USp(4) \to USp(4) \). Finally, there is a smooth transition from \( USp(4) \to USp(0) \times \tilde{U}(2) \) (where \( \tilde{U}(2) \) is at the \( r = 0 \) baryonic branch) to \( USp(4) \to USp(4) \), which matches the counting of vacua in the weak coupling analysis \((N + 1) = 3\).
4.1.4 \( N_f = 3 \)

The curve can be written as

\[ ty^2 = \left[ t(t^2 - s_1 t + s_2) + 2\Lambda^3 m^3 \right]^2 + 4\Lambda^6 (t - m^2)^3. \]

**Baryonic and Non-baryonic \( r = 1 \) branches**

First let us consider the \( r = 1 \) branch. There will be an \( r = 1 \) non-baryonic branch for \( \widehat{U}(2) \) and an \( r = 1 \) baryonic branch for both \( \widehat{U}(1) \) and \( \widehat{U}(2) \). For the \( r = 1 \) non-baryonic branch, the curve should be, from (3.3),

\[ ty^2 = tF_3(t)(t - m^2)^2 = t(t^3 + at^2 + bt + c)(t - m^2)^2. \]

The solution is given by

\[
\begin{align*}
    s_2 &= -m(2\Lambda^3 + M^3 - ms_1), \quad c = 4\Lambda^3(\Lambda^3 - m^3 + ms_1), \\
    a &= 2(m^2 - s_1), \quad b = -4\Lambda^3 m + (m^2 - s_1)^2. 
\end{align*}
\]

Using the relation \( a = -2m^2 \), we find that \( s_1 = 2m^2 \) and the curve gives one vacuum for \( USp(4) \to USp(0) \times \widehat{U}(2) \) at the \( r = 1 \) branch, which is also observed in the weak coupling analysis by counting \( (2N - N_f) = 1 \).

To get the \( r = 1 \) baryonic branch for \( \widehat{U}(1) \times USp(2) \) we need to have the following factorized curve (3.5)

\[ ty^2 = t(t + a)(t + b)^2(t - m^2)^2. \]

There are three solutions given by

\[
\begin{align*}
    a^3 - 64\Lambda^6 + 16a\Lambda^3 m &= 0, \quad b = \frac{a^3 + 2a^2 m^2 - 32\Lambda^3(\Lambda^3 + m^3)}{2a(a + 4m^2)}. 
\end{align*}
\]

Two of them give a breaking pattern \( USp(4) \to \widehat{U}(1) \times USp(2) \) at the \( r = 1 \) baryonic branch (the weak coupling analysis provides the number of vacua as two from the \( USp(2) \) factor by \( (N + 1) = 2 \)) and the remaining one, \( USp(4) \to USp(0) \times \widehat{U}(2) \) where \( \widehat{U}(2) \) is at the \( r = 1 \) baryonic branch, which is coincident with the number of vacua in the weak coupling approach. Thus, we see the smooth transition of the \( r = 1 \) baryonic branch between \( USp(0) \times \widehat{U}(2) \) and \( \widehat{U}(1) \times USp(2) \).

**Non-baryonic \( r = 0 \) branch**

Now let us discuss the \( r = 0 \) branch. There is no baryonic \( r = 0 \) branch. The curve is given by

\[ ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2. \]
The solutions are

\[ s_1 = \frac{\pm \Lambda^3 (d - 2m^2) \sqrt{d + m^2} - 2 (d^3 + \Lambda^3 m^3)}{d^2}, \]
\[ s_2 = \frac{d^3 + 4 \Lambda^3 m^3 \pm \Lambda^3 \sqrt{d + m^2} (d + 4m^2)}{d}. \]

There are the following limits: (1) when \( \Lambda \to 0 \), but \( d \to \) constant, there exists a breaking pattern \( USp(4) \to USp(0) \times \hat{U}(2) \); (2) when \( \Lambda \to 0 \), \( d \to 0 \), but \( \Lambda^3/d^2 \to \) constant, so that \( USp(4) \to \hat{U}(1) \times USp(2) \). Thus, we have a smooth transition from \( USp(0) \times \hat{U}(2) \) to \( \hat{U}(1) \times USp(2) \). To count the number of vacua, we set \( a = -2m^2 \) as before and solve \( s_1, s_2, b, c \) in terms of \( d \) with the following constraint (we have neglected the case where \( d = -m^2 \))

\[ 0 = d^4 - d \Lambda^6 + 2d^2 m^2 + 3 \Lambda^6 m^2 + 4d \Lambda^3 m^3 + d^2 m^4 + 4 \Lambda^3 m^5. \]

There are four solutions. Two of them give \( USp(4) \to \hat{U}(1) \times USp(2) \) (in the weak coupling approach the number two comes from the \( USp(2) \) part while the number of vacua for the \( \hat{U}(1) \) part gives one vacuum from the relation \( (N - r) = 1 \)) and another two, \( USp(4) \to USp(0) \times \hat{U}(2) \). It is noteworthy that for the \( r = 0 \) non-baryonic branch of \( \hat{U}(2) \) with \( N_f = 3 \), the counting is given by \( (N - r) = 2 \) from [11, 12] in the weak coupling approach instead of in the \( (2N - N_f) = 1 \).

Finally we require the following factorization curve

\[ ty^2 = t(t + a)(t^2 + bt + c)^2. \]

It is hard to solve analytically, but numerically there exist twelve solutions. However, six of them have \( s_1 \to m^2 \), three of them have \( s_1 \to 2m^2 \), and three of them have \( s_1 \to 0 \), which is the one we are looking for and give \( USp(4) \to USp(4) \). The counting of vacua in the weak coupling analysis gives \( (N + 1) = 3 \), which is the same as the one in the above strong coupling analysis.

4.1.5 \( N_f = 4 \)

The curve is

\[ ty^2 = \left[ t(t^2 - s_1 t + s_2) + 2 \Lambda^2 m^4 \right]^2 - 4 \Lambda^4 (t - m^2)^4. \]

- **Baryonic \( r = 2 \) branch**

There is only one baryonic \( r = 2 \) branch for \( \hat{U}(2) \). The curve is given by (3.5)

\[ ty^2 = t(t + a)(t - m^2)^4. \]

16
There is one solution:

\[ a = 4\Lambda^2, \quad s_1 = 2m^2 - 2\Lambda^2, \quad s_2 = m^4 - 4\Lambda^2m^2 \]

which gives a breaking pattern \( USp(4) \to USp(0) \times \hat{U}(2) \) at the baryonic \( r = 2 \) branch. In the weak coupling analysis [11, 12] the number of vacua is also one because it is a baryonic branch.

- **Non-baryonic \( r = 1 \) branch**

  For the \( r = 1 \) branch, it is non-baryonic and the curve should be (3.3)

\[ ty^2 = t(t^3 + at^2 + bt + c)(t - m^2)^2. \]

The solution is given by:

\[
\begin{align*}
  s_2 &= -m^2(2\Lambda^2 + m^2 - s_1), \quad c = 4\Lambda^2m^2(2\Lambda^2 - m^2 + s_1), \\
  a &= 2(m^2 - s_1), \quad b = -4\Lambda^4 - 4\Lambda^2m^2 + (m^2 - s_1)^2.
\end{align*}
\]

Using \( a = -2m^2 \), we see that there is one vacuum for \( USp(4) \to USp(0) \times \hat{U}(2) \), which can be checked also by the counting \( (N - r) = (2 - 1) = 1 \) in the weak coupling analysis. We did not see the breaking pattern \( USp(4) \to \hat{U}(1) \times USp(2) \) because for the \( \hat{U}(1) \) factor, we have \( (N - r) = 0 \).

- **Non-baryonic \( r = 0 \) branch**

  For the \( r = 0 \) branch, it is also non-baryonic and the curve should be

\[ ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2. \]

There are two kinds of solutions. The first kind of solution is given by

\[
\begin{align*}
  s_1 &= -2(d + \Lambda^2), \quad s_2 = d^2 - 4\Lambda^2m^2, \quad c = 4\Lambda^2m^4, \\
  a &= 2(d + 2\Lambda^2), \quad b = d^2 - 8\Lambda^2m^2.
\end{align*}
\]

It gives one vacuum for the breaking \( USp(4) \to USp(0) \times \hat{U}(2) \). The second kind of solution is given by

\[
\begin{align*}
  s_1 &= -2d + 2\Lambda^2 - \frac{4\Lambda^2m^4}{d^2}, \quad s_2 = d^2 + 4\Lambda^2m^2 + \frac{8\Lambda^2m^4}{d}, \\
  a &= 2(d - 2\Lambda^2 + \frac{4\Lambda^2m^4}{d^2}), \quad b = d^2 + 8\Lambda^2m^2 + \frac{16\Lambda^2(d - \Lambda^2)m^4}{d^2} + \frac{16\Lambda^4m^8}{d^4}, \\
  c &= \frac{4\Lambda^2m^4(d^3 + 8d\Lambda^2m^2 + 8\Lambda^2m^4)}{d^3}.
\end{align*}
\]

There exist the following limits: (1) when \( \Lambda \to 0 \), but \( d \to \) constant, there is a breaking pattern \( USp(4) \to USp(0) \times \hat{U}(2) \); (2) when \( \Lambda \to 0 \), \( d \to 0 \), but \( \Lambda^2/d^2 \to \) constant, we
have $USp(4) \rightarrow \hat{U}(1) \times USp(2)$. Thus, we have the smooth transition from $USp(0) \times \hat{U}(2)$ to $\hat{U}(1) \times USp(2)$. To count the number of vacua, we set $a = -2m^2$ as we did before and find three solutions. Two of them give a breaking pattern $\hat{U}(1) \times USp(2)$ where the $\hat{U}(1)$ factor is with the counting $(N - r) = 1$ and $USp(2)$ has two vacua. The remaining one gives $USp(0) \times \hat{U}(2)$ and by combining the one vacuum from the first kind of solutions we get a total of two vacua, which is also computed by the counting $(N - r) = 2$ in the weak coupling analysis.

Finally we require the following factorization

$$t(t^2 - s_1 t + s_2) + 2\Lambda^2 m^4 - 2\Lambda^2 (t - m^2)^2 = t(t + b)^2,$$
$$t(t^2 - s_1 t + s_2) + 2\Lambda^2 m^4 + 2\Lambda^2 (t - m^2)^2 = (t + a)(t + c)^2.$$  

The solution is given by some function of $c$ while the parameter $c$ satisfies the following constraint

$$0 = -c^3 + c^2 \Lambda^2 - 2c\Lambda^2 m^2 + \Lambda^2 m^4.$$  

Obviously there are three solutions which provide the three vacua of $USp(4) \rightarrow USp(4)$.

4.1.6 $N_f = 5$

The curve can be described by:

$$ty^2 = \left[t(t^2 - s_1 t + s_2) + 2\Lambda m^5\right]^2 + 4\Lambda^2 (t - m^2)^5.$$  

- **Baryonic $r = 2$ branch**

  The $r = 2$ branch is baryonic and the curve is from (3.5)

  $$ty^2 = t(t + a)(t - m^2)^4.$$  

  The solution is given by:

  $$s_1 = 2m(-\Lambda + m), \quad s_2 = m^3(-4\Lambda + m), \quad a = 4\Lambda(\Lambda + m)$$  

  which gives the one vacuum of $USp(4) \rightarrow USp(0) \times \hat{U}(2)$ at the baryonic $r = 2$ branch, which is also consistent with the count in the weak coupling analysis.

- **Non-baryonic $r = 1$ branch**

  For the $r = 1$ branch, it is non-baryonic and the curve should be (3.3)

  $$ty^2 = t(t^3 + at^2 + bt + c)(t - m^2)^2.$$  

  The solution is given by

  $$s_2 = m^2(-m(2\Lambda + m) + s_1), \quad a = 2(2\Lambda^2 + m^2 - s_1),$$
  $$b = -12\Lambda^2 m^2 - 4\Lambda m^3 + (m^2 - s_1)^2, \quad c = 4\Lambda m^3(3\Lambda m - m^2 + s_1).$$
Using $a = -2m^2 + 4\Lambda^2$, we get one vacuum for $USp(4) \to USp(0) \times \widehat{U}(2)$ that can be interpreted in the weak coupling analysis by counting $(N - r) = 1$. We did not see the $USp(4) \to \widehat{U}(1) \times USp(2)$ because for the $\widehat{U}(1)$ factor we have no vacua $(N - r) = 0$.

- **Non-baryonic $r = 0$ branch**
  For $r = 0$ branch, it is also non-baryonic and the curve should be
  \[ ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2. \]
  The solutions are given by
  \[
  s_1 = \frac{\pm \Lambda (3d - 2m^2) (d + m^2)\frac{a}{2} - 2(d^3 + \Lambda m^5)}{d^2}, \\
  s_2 = \frac{d^3 + 4\Lambda m^5 \mp \Lambda (d - 4m^2) (d + m^2)\frac{3}{2}}{d}.
  \]
  There are the following limits: (1) when $\Lambda \to 0$, but $d \to$ constant, we see $USp(4) \to USp(0) \times \widehat{U}(2)$; (2) as $\Lambda \to 0$, $d \to 0$, but $\Lambda/d^2 \to$ constant, so that the following breaking pattern appears $USp(4) \to \widehat{U}(1) \times USp(2)$. Thus, we have the smooth transition from $USp(0) \times \widehat{U}(2)$ to $\widehat{U}(1) \times USp(2)$. To count the number of vacua, setting $a = -2m^2 + 4\Lambda^2$, we solve other parameters in terms of $d$ with the constraint
  \[ 0 = d^4 + d\Delta m^4 (5\Lambda + 4m) + \Lambda m^6 (15\Lambda + 4m) + d^3 \left(-9\Lambda^2 + 2m^2\right) + d^2 \left(-15\Lambda^2 m^2 + m^4\right). \]
  Among these four solutions, two of them give a breaking pattern $USp(4) \to USp(0) \times \widehat{U}(2)$ which can be realized in the weak coupling analysis by counting $(N - r) = 2$, and in the two others, $USp(4) \to \widehat{U}(1) \times USp(2)$ where two vacua are also obtained from the analysis of weak coupling $(N_1 - r) \times (N_0 + 1) = 2$.
  Finally we set
  \[ ty^2 = t(t + a)(t^2 + bt + c)^2. \]
  It should give three vacua, but we are unable to show it numerically.

  We summarize the results in Table 1 by writing the flavors $N_f$, symmetry breaking patterns, various branches, the power of $t$ in the curve, $U(1)$ at the IR, the number of vacua, and the possible smooth connection.

### 4.2 $USp(6)$ case

The calculation becomes more complicated as we increase the rank of the gauge group for $USp(2N_c)$. For the $USp(6)$ gauge group, let us deal with only the even number of flavors because there are no new and interesting phenomena for odd number of flavors.
| $N_f$ | Group                  | Branch            | Power of $t(=x^2)$ | $U(1)$ | Number of vacua | Connection |
|-------|------------------------|-------------------|--------------------|--------|-----------------|------------|
| 0     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               |            |
|       | $USp(2) \times U(1)$  | $(C,0_{NB})$      | $t^1$              | 1      | 2               | A          |
|       | $USp(0) \times U(2)$  | $(C,0_{NB})$      | $t^1$              | 1      | 1               | A          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
| 1     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               | B          |
|       | $USp(2) \times U(1)$  | $(C,0_B)$         | $t^1$              | 0      | 2               | B          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | C          |
|       | $USp(0) \times U(2)$  | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{B})$       | $t^1$              | 0      | 1               | D          |
| 2     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               |            |
|       | $USp(2) \times U(1)$  | $(C,1_B)$         | $t^1$              | 0      | 2               | F          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | G          |
|       | $USp(0) \times U(2)$  | $(C,1_{NB})$      | $t^1$              | 1      | 1               | E          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{B})$       | $t^1$              | 0      | 1               | D          |
|       |                        | $(C,1_{B})$       | $t^1$              | 0      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | G          |
| 3     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               |            |
|       | $USp(2) \times U(1)$  | $(C,1_B)$         | $t^1$              | 0      | 2               | F          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | G          |
|       | $USp(0) \times U(2)$  | $(C,1_{NB})$      | $t^1$              | 1      | 1               | E          |
|       |                        | $(C,1_{B})$       | $t^1$              | 0      | 1               | F          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | G          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
| 4     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               |            |
|       | $USp(2) \times U(1)$  | $(C,0_{NB})$      | $t^1$              | 1      | 2               | H          |
|       | $USp(0) \times U(2)$  | $(C,2_B)$         | $t^1$              | 0      | 1               |            |
|       |                        | $(C,1_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,2_{B})$       | $t^1$              | 0      | 1               | H          |
|       |                        | $(C,1_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | I          |
| 5     | $USp(4)$               | $(C)$             | $t^1$              | 0      | 3               |            |
|       | $USp(2) \times U(1)$  | $(C,0_{NB})$      | $t^1$              | 1      | 2               | I          |
|       | $USp(0) \times U(2)$  | $(C,2_B)$         | $t^1$              | 0      | 1               |            |
|       |                        | $(C,1_{NB})$      | $t^1$              | 1      | 1               |            |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               | I          |
|       |                        | $(C,0_{NB})$      | $t^1$              | 1      | 2               |            |

Table 1: The summary of the phase structure of the $USp(4)$ gauge group with massive flavors. The flavors are charged under the $U(N_i)$ group. Here we use $C$ for the Chebyshev branch for the $USp(2N_i)$ factor and $r_{NB}/r_B$ for the $r$-th non-baryonic or baryonic branch. In this table, we list the power of $t$ that is $t^1$ and the $U(1)$ which is present in the nonbaryonic branch, at the IR. They are indices to see whether the two phases could have smooth transition. Note the explicit presence of the $USp(0)$ group factor in the second column. The same capital letters in the last column denote the smooth connected phases between them. We could not find three vacua denoted by $3^*$ above.
4.2.1 \( N_f = 0 \)

The curve is

\[
y^2 = \left[ t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^8 \right]^2 - 4\Lambda^{16}.
\]

- **Non-baryonic** \( r = 0 \) branch

First we require that

\[
y^2 = tF_3(t)H_2^2(t).
\]

There are two kinds of solutions. The first kind of solution is given by, from the simple further factorization,

\[
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^8 - 2\Lambda^8 = tF_3(t),
\]

\[
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^8 + 2\Lambda^8 = H_2(t)^2 = (t^2 + at + b)^2.
\]

The solution is given by:

\[
s_1 = -2a, \quad s_2 = a^2 \mp 4\Lambda^4, \quad s_3 = \pm 4a\Lambda^4, \quad b = \mp 2\Lambda^4
\]

so that the function \( F_3(t) \) can be written as

\[
F_3(t) = (t + a)(t(t + a)^2 \mp \Lambda^4).
\]

Using the relationship of \( F_3(t) \) and \( W'(x) \), we get \( a = -m^2 \). There are two vacua for \( USp(6) \rightarrow U(2) \times USp(2) \). The second kind of solution is given by:

\[
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^8 - 2\Lambda^8 = t(t + a)(t + b)^2,
\]

\[
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^8 + 2\Lambda^8 = (t^2 + ct + d)(t + e)^2.
\]

The solution is given by

\[
s_1 = -3 \left( \frac{b + e}{2} \right) - \frac{2\Lambda^8}{(b - e)e^2}, \quad s_2 = b \left( \frac{3e + 4\Lambda^8}{(b - e)e^2} \right), \quad s_3 = \frac{b^2 \left( b - 3e + \frac{4\Lambda^8}{e^2(b+e)} \right)}{2},
\]

\[
a = \frac{-b}{2} + \frac{3e}{2} + \frac{2\Lambda^8}{(b - e)e^2}, \quad c = \frac{3b}{2} - \frac{e}{2} + \frac{2\Lambda^8}{(b - e)e^2}, \quad d = \frac{4\Lambda^8}{e^2},
\]

\[
0 = \frac{(-b + e)^3}{2} + \frac{2(b - 3e)\Lambda^8}{e^2}.
\]

There are the following limits: (1) as \( \Lambda \rightarrow 0 \), \( b \sim e \neq 0 \), so there is a relation \( (b - e) \sim \Lambda^{8/3} \). It gives \( USp(6) \rightarrow USp(0) \times U(3) \); (2) as \( \Lambda \rightarrow 0 \), \( b \neq 0 \), we have \( \Lambda^8/e^2 \sim b^2/4 \). It gives
$USp(6) \to U(2) \times USp(2)$; (3) when $\Lambda \to 0$, $b, e \to 0$, but $\Lambda^8/(e^2(b - e)) \neq 0$, it gives $USp(6) \to U(1) \times USp(4)$. Thus, we see the smooth transition among all of these three phases.

To count the number of vacua, we find $F_3(t)$ and get a relation

$$b + e + \frac{4\Lambda^8}{(b - e)e^2} = -2m^2.$$  

Combining with this constraint, we get the following equation for $b$

$$0 = -4\Lambda^{16} + \left(b + m^2\right)^2 \left(b^6 + 28b^2\Lambda^8 + 3b\left(b^4 + 18\Lambda^8\right)m^2 + 3\left(b^4 + 9\Lambda^8\right)m^4 + b^3m^6\right).$$

There are eight solutions. Three of them give $USp(6) \to USp(0) \times U(3)$, which is consistent with the counting of weak coupling analysis. Another three of them give $USp(6) \to U(1) \times USp(4)$ where the number of vacua three comes from the $USp(4)$ part in the weak coupling approach. The remaining two give $USp(6) \to U(2) \times USp(2)$. Combining the two vacua from the first kind of factorization we have found before with this gives four vacua of this phase (in the weak coupling analysis two comes from $USp(2)$ and two does from $U(2)$).

Finally we consider the case which is relevant to the pattern $USp(6) \to USp(6)$

$$t(t^3 - s_1t^2 + s_2t - s_3) + 2\Lambda^8 - 2\Lambda^8 = t(t + a)(t + b)^2,$$

$$t(t^3 - s_1t^2 + s_2t - s_3) + 2\Lambda^8 + 2\Lambda^8 = (t + c)^2(t + d)^2.$$  

Solving the factorization will give us eight solutions, but putting back to curve we find only four different curves. Thus, the four vacua match the prediction by $(N + 1) = 4$ for $USp(6)$ factor from the analysis of weak coupling.

### 4.2.2 $N_f = 2$

The curve is given by

$$ty^2 = \left[t(t^3 - s_1t^2 + s_2t - s_3) + 2\Lambda^6m^2\right]^2 - 4\Lambda^{12}(t - m^2)^2.$$  

**Baryonic and Non-baryonic $r = 1$ branches**

The $r = 1$ branch is as follows: $\hat{U}(3)$ and $\hat{U}(2)$ have only non-baryonic branch while $\hat{U}(1)$ is at the baryonic branch. To do this, first we require

$$t(t^3 - s_1t^2 + s_2t - s_3) + 2\Lambda^6m^2 = (t - m^2)(t^3 - u_1t^2 + u_2t - u_3).$$

This can be solved as

$$s_1 = m^2 + u_1, \quad s_2 = m^2u_1 + u_2, \quad s_3 = 2\Lambda^6 + u^2u_2, \quad u_3 = 2\Lambda^6.$$
Now we can have the following factorization

\[(t^3 - u_1 t^2 + u_2 t - 2\Lambda^6)^2 - 4\Lambda^{12} = \bigl(t(t^2 - u_1 t + u_2) - 2\Lambda^6\bigr)^2 - 4\Lambda^{12} = t(t^3 + at^2 + bt + c)(t + d)^2\]

which remarkably is the \(r = 0\) nonbaryonic branch of \(USp(4)\) without flavors (4.1). From these results, we conclude immediately that: (1) there is smooth transition between \(USp(6) \rightarrow USp(0) \times \hat{U}(3)\) and \(USp(6) \rightarrow \hat{U}(2) \times USp(2)\); (2) there are two vacua for \(USp(6) \rightarrow USp(0) \times \hat{U}(3)\), which match the counting \((2N_c - N_f)/2 = 2\) in the weak coupling analysis; and (3) there are two vacua for \(USp(6) \rightarrow \hat{U}(2) \times USp(2)\), which match the counting \(2 \times (2N - N_f)/2 = 2\); (4) If we require the extra double root, we get three vacua for \(USp(6) \rightarrow \hat{U}(1) \times USp(4)\) where \(\hat{U}(1)\) is at the \(r = 1\) baryonic branch and the number three in the weak coupling analysis comes from the \(USp(4)\) factor.

- **Baryonic and Non-baryonic \(r = 0\) branches**

Now we will discuss the \(r = 0\) branch. There are non-baryonic branches for \(\hat{U}(3), \hat{U}(2)\) and \(\hat{U}(1)\). For \(\hat{U}(2)\), there is also the baryonic branch. First let us discuss the non-baryonic branch. For the factorization (2, 0) where the numbers \((X, Y)\) are indicating the numbers of double roots, it is given by

\[t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 + 2\Lambda^6 (t - m^2) = tF_3(t),\]
\[t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 - 2\Lambda^6 (t - m^2) = (t^2 + at + b)^2.\]

The solutions are given by

\[s_1 = -2a, \quad s_2 = a^2 \mp 4\Lambda^3 m, \quad s_3 = -2\Lambda^6 \pm 4\Lambda^3 am, \quad b = \mp 2\Lambda^3 m\]

which give the breaking \(USp(6) \rightarrow \hat{U}(2) \times USp(2)\). Calculating the \(F_3(t)\), we get a relation \(a = -m^2\) and there are two vacua.

For factorization (1, 1) the curve is given by

\[t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 + 2\Lambda^6 (t - m^2) = t(t + a)(t + b)^2,\]
\[t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 - 2\Lambda^6 (t - m^2) = (t^2 + ct + d)(t + e)^2.\]

The solution is given by

\[s_1 = \frac{-3(b + e)}{2} - \frac{2\Lambda^6 m^2}{(b - e) e^2}, \quad s_2 = b \left(3e + \frac{4\Lambda^6 m^2}{(b - e) e^2}\right), \quad d = \frac{4\Lambda^6 m^2}{e^2},\]
\[s_3 = \frac{4\Lambda^6 + b^2 \left(b - 3e + \frac{4\Lambda^6 m^2}{e^2 (b - e)}\right)}{2}, \quad a = -\frac{b}{2} + \frac{3e}{2} + \frac{2\Lambda^6 m^2}{(b - e) e^2},\]
\[c = \frac{3b - e}{2} + \frac{2\Lambda^6 m^2}{(b - e) e^2}, \quad 0 = \frac{(-b + e)^3 - 8\Lambda^6}{2} + \frac{(b - 3e)\Lambda^6 m^2}{e^2}.\]
There are the following limits: (1) when $\Lambda \to 0$, $b \sim e \neq 0$, so it gives $(b - e) \sim \Lambda^{8/3}$ which implies a breaking pattern $USp(6) \to USp(0) \times \widetilde{U}(3)$; (2) when $\Lambda \to 0$, $b \neq 0$, we have a relation $\Lambda^6 m^2 / e^2 \sim b^2 / 4$. This gives a breaking pattern $USp(6) \to \widetilde{U}(2) \times USp(2)$; (3) when $\Lambda \to 0$, $b, e \to 0$, but $\Lambda^6 m^2 / (e^2 (b - e)) \neq 0$, it gives a symmetry breaking pattern $USp(6) \to \widetilde{U}(1) \times USp(4)$. Thus we see the smooth transition among all of these three phases.

To count the number of vacua, we find the $F_3(t)$ and get the following relationship

$$-2m^2 = b + e + \frac{4\Lambda^6 m^2}{(b - e)e^2}.$$

Combined with the above constraint and eliminating $e$, we get an equation for $b$ of degree nine. Among these nine solutions, four give a breaking pattern $USp(6) \to USp(0) \times \widetilde{U}(3)$ which is the same as the number $(2N - N_f) = 4$; three give $USp(6) \to \widetilde{U}(1) \times USp(4)$, which is the same as the counting $(N_1 - r) \times (N_0 + 1) = 1 \times 3 = 3$; and the remainder two, plus the other two from the previous $(2, 0)$ factorization, for $USp(6) \to \widetilde{U}(2) \times USp(2)$ which is also obtained from $(2N_1 - N_f) \times (N_0 + 1) = 2 \times 2 = 4$ in the weak coupling analysis.

Finally we discuss the following factorization

$$t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 + 2\Lambda^6(t - m^2) = t(t + a)(t + b)^2,$$

$$t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^6 m^2 - 2\Lambda^6(t - m^2) = (t + c)^2(t + d)^2.$$

The solution is given by

$$s_1 = -a - 2b, \quad s_2 = b(2a + b), \quad s_3 = -ab^2 + 2\Lambda^6,$$

$$u_1 \equiv c + d = \frac{a}{2} + b, \quad u_2 \equiv cd = \mp 2\Lambda^3 m,$$

$$b = \frac{a}{2} - \frac{16\Lambda^6}{a^2}, \quad \frac{a^2}{\Lambda^3} = \mp 64\Lambda^3 a \mp 16m = 0.$$

There are six solutions. Four of them have $\frac{a^2}{\Lambda^3} \neq 0$ and give the four vacua of $USp(6) \to USp(6)$ which is the same as the counting $(N + 1) = 4$. The remainder two have $\frac{64\Lambda^3}{a} \neq 0$ and give the breaking pattern $USp(6) \to \widetilde{U}(2) \times USp(2)$ where $\widetilde{U}(2)$ is at the $r = 0$ baryonic branch (in the weak coupling analysis the number two comes from the factor $USp(2)$ with $(N + 1) = 2$). These two phases are smoothly connected.

### 4.2.3 $N_f = 4$

The curve is

$$ty^2 = \left[t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^4 m^4\right]^2 - 4\Lambda^8(t - m^2)^4.$$

- **Non-baryonic $r = 0$ branch**
Now we discuss the \( r = 0 \) branch. There are only non-baryonic branches for \( \tilde{U}(3) \), \( \tilde{U}(2) \), and \( \tilde{U}(1) \). For the factorization \((2, 0)\) it is given by
\[
\begin{align*}
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^4 m^4 - 2\Lambda^4(t - m^2)^2 &= tF_3(t), \\
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^4 m^4 + 2\Lambda^4(t - m^2)^2 &= (t^2 + at + b)^2.
\end{align*}
\]
The solutions are given by
\[
s_1 = -2a, \quad s_2 = a^2 - 2\Lambda^4 - 4\eta\Lambda^2 m^2, \quad s_3 = 4\eta\Lambda^2 m^2(a - \eta\Lambda^2), \quad b = -2\eta\Lambda^2 m^2
\]
which give the breaking \( USp(6) \to \tilde{U}(2) \times USp(2) \). Calculating the \( F_3(t) \), we get a relation \( a = -m^2 \) with two vacua.

For the factorization \((1, 1)\) type, the curve is given by
\[
\begin{align*}
tP_3(t) + 2\Lambda^4 m^4 - 2\Lambda^4(t - m^2)^2 &= t(t + a)(t + b)^2, \\
tP_3(t) + 2\Lambda^4 m^4 + 2\Lambda^4(t - m^2)^2 &= (t^2 + ct + d)(t + e)^2.
\end{align*}
\]
The solution is given by
\[
\begin{align*}
d &= \frac{4\Lambda^4 m^4}{e^2}, \quad a = \frac{e(b + e)}{2b} + \frac{\Lambda^4(e + m^2)(e + 3m^2)}{be(b - e)}, \\
c &= \frac{b(b + e)}{2e} + \frac{2\Lambda^4(e + m^2)(be - bm^2 + 4em^2)}{e^3(b - e)}, \\
0 &= \frac{(b - e)^3}{2} + \frac{2\Lambda^4}{e^2}(e + m^2)(be + e^2 - bm^2 + 3em^2).
\end{align*}
\]
There are the following limits: (1) when \( \Lambda \to 0, b \sim e \neq 0 \), so it gives \( b - e \sim \Lambda^{4/3} \) which implies a symmetry breaking pattern \( USp(6) \to USp(0) \times \tilde{U}(3) \); (2) when \( \Lambda \to 0, b \neq 0 \), we have the following relation \( \Lambda^4/e^2 \sim b^2/(4m^4) \). It gives a breaking pattern \( USp(6) \to \tilde{U}(2) \times USp(2) \); (3) as \( \Lambda \to 0, b, e \to 0 \), but \( a \neq 0 \), it gives other breaking pattern \( USp(6) \to \tilde{U}(1) \times USp(4) \). Thus, we see the smooth transition among all of these three phases.

To count the number of vacua, we find the \( F_3(t) \) and get the following relationship
\[
-2m^2 = \frac{b^2 + 3e^2}{2e} + \frac{2\Lambda^4 m^2(e + m^2)(5e - b)}{e^3(b - e)}.
\]
Note that we use the above conditions \( d \) and \( e \). Combining with the constraint in the above and eliminating \( e \), we get an equation for \( b \) of degree eight. Among these eight solutions, three give a breaking pattern \( USp(6) \to USp(0) \times \tilde{U}(3) \) which is the same as the counting \( (N_1 - r) = 3 \), three for \( USp(6) \to \tilde{U}(1) \times USp(4) \) that can be also obtained from the counting \( (N_0 + 1) = 3 \) in the \( USp(4) \) part and the remainder two, plus the other two from the \((2, 0)\) factorization, for \( USp(6) \to \tilde{U}(2) \times USp(2) \) which is equivalent to the counting \((N_1 - r) \times (N_0 + 1) = 4 \).
Finally we will discuss the following factorization

\[ tP_3(t) + 2\Lambda^4 m^4 - 2\Lambda^4(t - m^2)^2 = t(t + a)(t + b)^2, \]
\[ tP_3(t) + 2\Lambda^4 m^4 + 2\Lambda^4(t - m^2)^2 = (t + c)^2(t + d)^2. \]

The solution is given by

\[ u_1 \equiv c + d = \frac{3a}{4} + \frac{2(\lambda^4 + \eta\sqrt{2}\lambda^2 m^2)}{a}, \quad b = \frac{a^2 + 8\lambda^4 + 8\eta\sqrt{2}\lambda^2 m^2}{4a}, \]
\[ u_2 \equiv cd = \eta\sqrt{2}\lambda^2 m^2, \quad \frac{a^3}{16} - 4\lambda^4 m^2 - \frac{4\lambda^6(\lambda^2 + \eta\sqrt{2}m^2)}{a} + a \left( -\lambda^4 + \frac{\eta\lambda^2 m^2}{\sqrt{2}} \right) = 0. \]

There are eight solutions. However putting back to curve, we find only four different curves. These solutions give only one semiclassical limit \( \Lambda \to \) with \( a/\Lambda \to \) const. It gives \( USp(6) \to USp(6) \). Thus, the four vacua match the prediction by \( (N + 1) = 4 \) in the weak coupling analysis.

**Baryonic and Non-baryonic \( r = 1 \) branches**

Next we discuss the \( r = 1 \) branch. There are non-baryonic branches for \( \hat{U}(3) \) and \( \hat{U}(2) \). For \( \hat{U}(1) \) and \( \hat{U}(3) \), there are baryonic branches.

By using the addition map, we can reduce the discussion to the one for \( USp(4) \) gauge theory with \( N_f = 2, r = 0 \). To better understand the phase structure without explicit calculations. The first kind of solution gives rise to the breaking pattern \( USp(6) \to USp(0) \times \hat{U}(3) \). There is one vacuum. The second kind of solution gives rise to the two vacua with \( USp(6) \to \hat{U}(2) \times USp(2) \) and one vacuum with \( USp(0) \times \hat{U}(3) \). These vacua are smoothly interpolated to each other. The number of vacua matches the counting from the weak coupling analysis. Note that we do not obtain the breaking pattern \( USp(6) \to \hat{U}(1) \times USp(4) \) because the number of vacua is zero, which matches the counting from the weak coupling analysis because of \( (N_i - r) = 0 \).

Finally from the factorization for the baryonic branch, we obtain one vacuum with \( USp(6) \to USp(0) \times \hat{U}(3) \) and three vacua with \( USp(6) \to \hat{U}(1) \times USp(4) \). These numbers of vacua match those in the weak coupling analysis. Due to the baryonic branch \( \hat{U}(1) \) and \( \hat{U}(3) \), we have only one vacuum respectively and for the \( USp(2N) \) factor there exist \( (N + 1) \) vacua.

**Baryonic and Non-baryonic \( r = 2 \) branches**

Next we will discuss the \( r = 2 \) branch. As in the \( r = 1 \) branch, we can use the addition map and thus reduce to the discussion for \( USp(4) \) theory with \( r = 1, N_f = 2 \). From this discussion, we can obtain two kinds of solutions. The first kind of solution, which comes from \( r = 1 \) branch for \( USp(4) \) theory, gives rise to the two vacua with the breaking pattern \( USp(6) \to USp(0) \times \hat{U}(3) \). The other kind of solution gives rise to the two vacua with \( USp(6) \to \hat{U}(2) \times USp(2) \). These numbers of vacua match those from the weak coupling analysis. On the baryonic branch, \( \hat{U}(2) \) has one vacuum on the non-baryonic branch and \( \hat{U}(3) \) has two vacua because of \( (N_i - r) = 2 \).
4.2.4 $N_f = 6$

The curve is given by
\[ ty^2 = \left[ t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^2 m^6 \right]^2 - 4\Lambda^4 (t - m^2)^6. \]

- **Non-baryonic $r = 0$ branch**

Now let us discuss the $r = 0$ branch. There are only non-baryonic branches for $\widehat{U(3)}$, $\widehat{U(2)}$, and $\widehat{U(1)}$. For the factorization (2, 0) it is given by
\[
\begin{align*}
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^2 m^6 + 2\Lambda^2 (t - m^2)^3 &= tF_3(t), \\
t(t^3 - s_1 t^2 + s_2 t - s_3) + 2\Lambda^2 m^6 - 2\Lambda^2 (t - m^2)^3 &= (t^2 + at + b)^2.
\end{align*}
\]

The solutions are given by
\[ s_1 = -2(a + \Lambda^2), \; s_2 = a^2 - 2\eta\Lambda m^2(3\eta\Lambda + 2m), \; s_3 = 2\eta\Lambda m^3(2a - 3\eta\Lambda m), \; b = -2\eta\Lambda m^3 \]
which give the breaking $USp(6) \rightarrow \widehat{U(2)} \times USp(2)$. Calculating the $F_3(t)$, we get $a = m^2$ with two vacua.

For the factorization (1, 1), the curve is given by
\[
\begin{align*}
tP_3(t) + 2\Lambda^2 m^6 + 2\Lambda^2 (t - m^2)^3 &= (t + a)(t + b)^2, \\
tP_3(t) + 2\Lambda^2 m^6 - 2\Lambda^2 (t - m^2)^3 &= (t^2 + ct + d)(t + e)^2.
\end{align*}
\]

The solution is given by
\[
\begin{align*}
c &= \frac{b (b + e)}{2e} - \frac{2\Lambda^2 m^2 (3be^2 + 6e^2 m^2 - bm^4 + 4em^4)}{e^3 (-b + e)}, \\
a &= \frac{e (b + e)}{2b} + \frac{6\Lambda^2 m^2 (e + m^2)^2}{b (b - e) e}, \quad b = \frac{4\Lambda^2 m^6}{e^2}, \\
0 &= \frac{-(b - e)^3}{2} - \frac{2\Lambda^2 (e + m^2)^2 (2be - bm^2 + 3em^2)}{e^2}.
\end{align*}
\]

There are the following limits: (1) when $\Lambda \rightarrow 0, b \sim c \neq 0, \; (b - e) \sim \Lambda^{2/3}$. It gives $\widehat{U(3)}$; (2) when $\Lambda \rightarrow 0, b \neq 0$, there is $\Lambda^2/e^2 \sim b^2/4m^6$ which gives $USp(6) \rightarrow \widehat{U(2)} \times USp(2)$; (3) when $\Lambda \rightarrow 0, b, e \rightarrow 0$ with $a \neq 0$, it gives a breaking pattern $USp(6) \rightarrow \widehat{U(1)} \times USp(4)$. Thus, we see a smooth transition among all of these three phases.

To count the number of vacua, we find the $F_3(t)$ and get the following relationship
\[ -2m^2 = \frac{(b + e) (b^2 + e^2)}{2be} + \frac{2\Lambda^2 m^2 \left( 3e^2 (e + m^2)^2 + 2be^2 (3e + 2m^2) + b^2 (3e^2 - m^4) \right)}{b (b - e) e^3}. \]

Combining with the constraint in the above and eliminating $e$, we get an equation for $b$ of degree eight. Among these eight solutions, three of them give $USp(6) \rightarrow USp(0) \times \widehat{U(3)}$, three
give \( USp(6) \to \hat{U}(1) \times USp(4) \) and the remainder two, plus the other two from the \((2,0)\) factorization, for \( USp(6) \to \hat{U}(2) \times USp(2) \). The consistency with the weak coupling analysis is similar to the \( r = 0, N_f = 4 \) case.

Finally let us discuss the following factorization
\[
tP_3(t) + 2\Lambda^2 m^6 + 2\Lambda^2(t - m^2)^3 = t(t + d - 4\Lambda^2)(t + c)^2,
\]
\[
tP_3(t) + 2\Lambda^2 m^6 - 2\Lambda^2(t - m^2)^3 = (t^2 + at + b)^2.
\]
The solution gives
\[
a = \frac{3d^2 - 32d\Lambda^2 - 8\Lambda(-8\Lambda^3 + 6\Lambda m^2 + 2\eta m^3)}{4d},
\]
\[
c = \frac{d^2 - 16d\Lambda^2 - 8\Lambda(-8\Lambda^3 + 6\Lambda m^2 + 2\eta m^3)}{4d}.
\]
It is easy to solve numerically. There exist ten solutions. However, six of them have \( s_1 \to m \) and the remaining of them have \( s_1 \to 0 \), which is the one we are looking for and which gives \( USp(6) \to USp(6) \). The counting of vacua in the weak coupling analysis gives \((N + 1) = 4\), which is the same as in the above strong coupling analysis.

- **Baryonic and Non-baryonic \( r = 1 \) branches**

Next we will discuss the \( r = 1 \) branch. There are non-baryonic branches for \( \hat{U}(3) \) and \( \hat{U}(2) \). For \( \hat{U}(1) \), there is also the baryonic branch.

By using the addition map, we can reduce the discussion to the one for \( USp(4) \) theory with \( N_f = 4, r = 0 \). We can understand the phase structure without explicit calculations. The first kind of solution for \( USp(4) \) case gives rise to one vacuum with a breaking pattern \( USp(6) \to USp(0) \times \hat{U}(3) \). The second kind of solution gives rise to the two vacua with \( USp(6) \to \hat{U}(2) \times USp(2) \) and one vacuum with \( USp(6) \to USp(0) \times \hat{U}(3) \). These vacua are smoothly interpolated to each other. The number of vacua matches the counting from the weak coupling analysis. As in \( N_f = 4 \) case, we do not obtain the breaking pattern \( USp(6) \to \hat{U}(1) \times USp(4) \), which matches the one from the weak coupling analysis because \((N_i - r) = 0\).

Finally from the factorization for baryonic branch, we obtain three vacua with \( USp(6) \to \hat{U}(1) \times USp(4) \). Thus, the number of vacua matches with the prediction by \((N + 1) = 3\).

- **Non-baryonic \( r = 2 \) branch**

Next let us discuss the \( r = 2 \) branch. As in the \( r = 1 \) branch, we can use the addition map and reduce to the discussion for \( USp(4) \) with \( r = 1, N_f = 4 \). We obtain only one vacuum with \( USp(6) \to \hat{U}(3) \). This number of vacuum matches the prediction from the weak coupling analysis that \((N - r) = 1\).

### 5 Quartic superpotential with massless flavors

In this section, we will consider \( USp(2N_c) \) gauge theories with \( N_f \) massless flavors. As
| $N_f$ | Group       | Branch  | Power of $t = x^2$ | $U(1)$ | Number of vacua | Connection |
|-------|-------------|---------|--------------------|--------|-----------------|------------|
| 0     | $USp(6)$    | $(C)$   | $t^1$              | 0      | 4               |            |
|       | $USp(2) \times U(2)$ | $(C,0_{NB})$ | $t^1$            | 1      | 2               | A          |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 2               | A          |
|       | $USp(4) \times U(1)$ | $(C,0_{NB})$ | $t^1$            | 1      | 3               | A          |
|       | $USp(0) \times U(3)$ | $(C,0_{NB})$ | $t^1$            | 1      | 3               | A          |
| 2     | $USp(6)$    | $(C)$   | $t^1$              | 0      | 4               | D          |
|       | $USp(4) \times U(1)$ | $(C,1_B)$      | $t^1$            | 0      | 3               |            |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 3               | C          |
|       | $USp(2) \times U(2)$ | $(C,1_{NB})$ | $t^1$            | 1      | 2               | B          |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 2               | C          |
|       |             | $(C,0_{B})$  | $t^1$            | 0      | 2               | D          |
|       | $USp(0) \times U(3)$ | $(C,1_{NB})$ | $t^1$            | 1      | 2               | B          |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 4               | C          |
| 4     | $USp(6)$    | $(C)$   | $t^1$              | 0      | 4               |            |
|       | $USp(4) \times U(1)$ | $(C,0_{NB})$ | $t^1$            | 1      | 3               | F          |
|       |             | $(C,1_B)$  | $t^1$            | 0      | 3               | G          |
|       | $USp(2) \times U(2)$ | $(C,0_{NB})$ | $t^1$            | 1      | 2               |            |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 2               | F          |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 1               | E          |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 1               | E          |
|       |             | $(C,2_B)$  | $t^1$            | 0      | 1               | G          |
|       | $USp(0) \times U(3)$ | $(C,0_{NB})$ | $t^1$            | 1      | 2               |            |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 1               |            |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 1               |            |
|       |             | $(C,2_{NB})$ | $t^1$            | 1      | 2               |            |
| 6     | $USp(6)$    | $(C)$   | $t^1$              | 0      | 4               |            |
|       | $USp(4) \times U(1)$ | $(C,0_{NB})$ | $t^1$            | 1      | 3               | H          |
|       |             | $(C,1_B)$  | $t^1$            | 0      | 3               |            |
|       | $USp(2) \times U(2)$ | $(C,0_{NB})$ | $t^1$            | 1      | 2               |            |
|       |             | $(C,0_{NB})$ | $t^1$            | 1      | 2               | H          |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 2               | J          |
|       | $USp(0) \times U(3)$ | $(C,0_{NB})$ | $t^1$            | 1      | 3               | H          |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 1               | J          |
|       |             | $(C,1_{NB})$ | $t^1$            | 1      | 1               | J          |
|       |             | $(C,2_{NB})$ | $t^1$            | 1      | 1               |            |

Table 2: The summary of the phase structure of the $USp(6)$ gauge group with massive flavors. In particular, the phases having the smooth transitions $USp(2N_0) \times U(N_1) \leftrightarrow USp(2M_0) \times U(M_1) \leftrightarrow USp(2L_0) \times U(L_1)$ give the dualities between three different gauge groups (not two) which were not present in the $SO(6)$ gauge theory with massive flavors. As we increase the rank of the gauge group $USp(2N_c)$, there will be more dualities connecting more groups.
previously discussed, we have studied the massive flavor case and the tree level superpotential had an extremization at \( x = \pm m \). However, for \( USp(2N) \) case in this section, since we assume that all the flavors are massless, the tree level superpotential is extremized at \( x = 0 \). Thus, we have one free parameter which we denote by \( \alpha \) and as in the \( SO(N_c) \) case \([12]\) the quartic tree level superpotential is given by

\[
W'(x) = x(x^2 - \alpha^2). \tag{5.1}
\]

In these examples, the gauge group breaks into the two factors, \( USp(2N_c) \rightarrow USp(2N_0) \times U(N_1) \) where \( 2N_c = 2N_0 + 2N_1 \) and \( N_0 \geq 0, \ N_1 \geq 0 \), under the semiclassical limit, \( \Lambda \rightarrow 0 \) (Of course, it includes the degenerated case \( USp(2N_c) \rightarrow USp(2N_c) \) and \( USp(2N_c) \rightarrow U(N_c) \times USp(0) \)). We consider the examples that all the flavors are massless and charged under the \( USp(2N_0) \). \(^8\) The Seiberg-Witten curves for these gauge theories are given by

\[
x^2y^2 = \left( x^2 P_{2N_c}(x) \right)^2 - 4(-1)^{N_f} \Lambda^{4N-2N_f+4}x^{2N_f}.
\]

(or sometimes we use \( t = x^2 \)).

Notice that since the \( U(N_1) \) factor does not have any flavors, it can only be the non-baryonic branch while the \( USp(2N_0) \) factor can be either Chebyshev branch with \( t^2(N_f/2+1) \) or the Special branch with \( t^2(N_f-N_0-1) \) when \( N_f \leq 2N_0 + 2 \) or \( t^2(N_0+1) \) when \( N_f \geq 2N_0 + 2 \). It is noteworthy that the power of \( t \) is odd for the Chebyshev branch and even for the Special branch.

Now we are ready to deal with the explicit examples for \( USp(4) \) and \( USp(6) \) gauge theories.

5.1 \( USp(4) \) case

5.1.1 \( N_f = 1 \)

The curve is given by

\[
ty^2 = \left[ t(t^2 - s_1 t + s_2) \right]^2 + 4\Lambda^{10} t.
\]

**Non-degenerated case**

First the curve takes the following factorization form

\[
ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.
\]

The solutions are

\[
\begin{align*}
s_1 &= -2d \pm \frac{\Lambda^5}{d^{3/2}}, & s_2 &= d^2 \mp \frac{3\Lambda^5}{\sqrt{d}}, \\
a &= 2d \mp \frac{2\Lambda^5}{d^{3/2}}, & b &= d^2 \mp \frac{6\Lambda^5}{\sqrt{d}}, & c &= \frac{4\Lambda^{10}}{d^2}.
\end{align*}
\]

\(^8\) As in \([11, 12]\), we use the notation for hat in \( USp(2N_0) \) to denote a gauge theory with flavors charged under the \( USp(2N_0) \) group.
There are two limits: (1) when $\Lambda \to 0$, but $d \to \text{constant}$, we have $USp(4) \to U(2) \times USp(0)$; (2) when $\Lambda \to 0$, $d \to 0$, but $\Lambda^d/d^2 \to \text{constant}$, it gives $USp(4) \to U(1) \times USp(2)$. Thus, we see a smooth transition from $U(2) \times USp(0)$ to $U(1) \times USp(2)$. Notice that for $USp(2)$ and $USp(0)$ with $N_f = 1$, there is no Special branch because the number $(N_f - N - 2)$ becomes negative. For the Chebyshev branch, there is factor $t^{N_f} = t$, as we see above. Because of the same factor $t$, this explains why we observe a smooth transition in this case.

To count the number of vacua, using the relationship

$$-2\alpha^2 = a$$

we get five solutions totally. Two of them give $U(2) \times USp(0)$ where the number of vacua can be obtained also from the weak coupling analysis $2 \times (2N_0 + 2 - N_f) = 2$ and three others, $U(1) \times USp(2)$ which match with the counting $(2N_0 + 2 - N_f) = 3$ in the weak coupling analysis.

- **Degenerated case**

  To get the $USp(4) \to USp(4)$, we require

  $$ty^2 = t(t + a)(t^2 + bt + c)^2.$$ 

  We found five vacua which match with the counting $(2N + 2 - N_f) = 5$.

5.1.2 $N_f = 2$

The curve is

$$ty^2 = \left[t(t^2 - s_1 t + s_2)\right]^2 - 4\Lambda^8 t^2.$$ 

- **Non-degenerated case**

  The curve takes the following factorization form

  $$ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.$$ 

  If $d \neq 0$, the solutions are given by

  $$s_1 = -2d, \quad s_2 = d^2 \mp 2\Lambda^4, \quad a = 2d, \quad b = d^2 \mp 4\Lambda^4, \quad c = 0.$$ 

  These solutions give the breaking $USp(4) \to U(2) \times USp(0)$. By using $-2\alpha^2 = a$, we get two vacua. Additionally since $c = 0$, the curve has a factor $t^2$, which is exactly the character of $USp(0)$ with $N_f = 2$ at the Special branch. $^9$ Note also (3.5).

  If $d = 0$, the solution is

  $$s_2 = \mp 2\Lambda^4, \quad a = -2s_1, \quad b = \mp 4\Lambda^4 + s_1^2, \quad c = \pm 4\Lambda^4 s_1$$

  $^9$Recall the condition that $\tilde{N} = N_f - 2 - N$ with a factor $t^{2(\tilde{N} + 1)}$.  

31
which gives $USp(4) \rightarrow U(1) \times USp(2)$ with a factor $t^3$. To count the number of vacua, we need to use $-2\alpha^2 = a$ and find that $s_1 = \alpha^2$. This gives two vacua of $USp(4) \rightarrow U(1) \times USp(2)$, which match with the counting $(2N_0 + 2 - N_f) = 2$ in the weak coupling approach.

- **Degenerated case**
  
  To get $USp(4) \rightarrow \hat{USp}(4)$, we require
  
  $$ty^2 = t(t + a)(t(t + b))^2.$$  

  Solving this equation we get $b = -4\eta\Lambda^2$, $a = 2\eta\Lambda^2$ where $\eta$ is 4-th roots of unity. These four vacua match the counting $(2N + 2 - N_f) = 4$ in the weak coupling analysis.

5.1.3 $N_f = 3$

The curve is given by

$$ty^2 = \left[t(t^2 - s_1t + s_2)\right]^2 + 4\Lambda^6 t^3.$$  

- **Non-degenerated case**
  
  First, the curve takes the following factorization form

  $$ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.$$  

  If $d \neq 0$, we have the following solutions

  $$s_1 = -2d \pm \frac{\Lambda^3}{\sqrt{d}}, \quad s_2 = d^2 \pm \sqrt{d}\Lambda^3, \quad a = 2d \mp \frac{2\Lambda^3}{\sqrt{d}}, \quad b = \frac{(\sqrt{d^3} + \Lambda^3)^2}{d}, \quad c = 0.$$  

  There are two classical limits: (1) when $\Lambda \rightarrow 0$, but $d \rightarrow$ constant, there is $USp(4) \rightarrow U(2) \times USp(0)$; (2) when $\Lambda \rightarrow 0$, $d \rightarrow 0$, but $\Lambda^3/\sqrt{d} \rightarrow$ constant, it gives $USp(4) \rightarrow U(1) \times USp(2)$. Thus, we see a smooth transition from $U(2) \times USp(0)$ to $U(1) \times USp(2)$. Notice since $c = 0$, we have a factor $t^2$ in the curve. This indicates that both $USp(0)$ and $USp(2)$ are at the Special branch. To count the number of vacua, using $-2\alpha^2 = a$, we find three solutions. Two of them give $USp(4) \rightarrow U(2) \times USp(0)$ which can be obtained also from the weak coupling analysis, and the remaining one gives $USp(4) \rightarrow U(1) \times USp(2)$ for the Special branch.

  If $d = 0$, the solution is given by

  $$s_2 = 0, \quad a = -2s_1, \quad b = s_1^2, \quad c = 4\Lambda^6.$$  

  Using $-2\alpha^2 = a$, we find one vacuum for $USp(4) \rightarrow U(1) \times USp(2)$ where $USp(2)$ is at the Chebyshev branch by the factor $t^{N_f} = t^3$ and matches with the counting from the weak coupling analysis $(2N + 2 - N_f) = 1$.

- **Degenerated case**
Finally we require the curve to be
\[ ty^2 = t(t + a)(t^2 + bt + c)^2. \]

There are three vacua with \( c = 0 \) of \( USp(4) \to \hat{USp}(4) \) which match with the counting \( (2N + 2 - N_f) = 3 \). Notice that \( \hat{USp}(4) \) is at the Chebyshev branch by the factor \( t^3 \).

5.1.4 \( N_f = 4 \)

The curve is
\[ ty^2 = [t(t^2 - s_1 t + s_2)]^2 - 4\Lambda^4 t^4. \]

- Non-degenerated case
The curve takes the following factorization form
\[ ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2. \]

For \( d \neq 0 \), the solution is given by
\[ s_1 = -2d \pm 2\Lambda^2, \quad s_2 = d^2, \quad a = 2d \mp 4\Lambda^2, \quad b = d^2, \quad c = 0. \]

These are two vacua of \( USp(4) \to U(2) \times \hat{USp}(0) \) with a factor \( t^2 \). Notice that to have a smooth transition to \( USp(4) \to U(1) \times \hat{USp}(2) \), since \( \hat{USp}(0) \) is at the Special branch, we require that \( \hat{USp}(2) \) be at the Special branch. However, the \( \hat{USp}(2) \) at the Special branch has the factor \( t^4 \) which explains why we do not find the smooth transition.

To have \( USp(4) \to U(1) \times \hat{USp}(2) \) where \( \hat{USp}(2) \) is at the Special branch, the curve should have a factor \( t^4 \). Setting \( d = 0 \), we find the solution
\[ s_2 = 0, \quad a = -2s_1, \quad b = -4\Lambda^4 + s_1^2, \quad c = 0. \]

Using the \(-2\alpha^2 = a\) we get one vacuum for \( USp(4) \to U(1) \times \hat{USp}(2) \) where \( \hat{USp}(2) \) is \( r = \tilde{N} = 1 \) branch at the Special branch. Notice that there is no Chebyshev branch for \( \hat{USp}(2) \) with \( N_f = 4 \) just by the simple counting \( (2N + 2 - N_f) = 0 \).

- Degenerated case
Finally to have \( USp(4) \to \hat{USp}(4) \), we require
\[ ty^2 = t(t + a)(t^2 + bt + c)^2. \]

There are three solutions. Two of them with \( c = b = 0 \) give a factor \( t^5 \). They are the vacua where \( \hat{USp}(4) \) is at the Chebyshev branch with \( (2N + 2 - N_f) = 2 \). The third one has \( a = b = 0 \) (so with a factor \( t^2 \)) and gives \( \hat{USp}(4) \) at the Special branch (the dual rank is \( \tilde{N} = 0 \)). Note also (3.8).
5.1.5 $N_f = 5$

The curve is given by

$$ty^2 = \left[ t(t^2 - s_1t + s_2) \right]^2 + 4\Lambda^2 t^5.$$ 

- **Non-degenerated case**
  First, the curve takes the following factorization form

$$ty^2 = tF_3(t)H_1^2(t) = t(t^3 + at^2 + bt + c)(t + d)^2.$$ 

If $d \neq 0$, there are two solutions

$$s_1 = -2d \pm 3\sqrt{d}\Lambda, \quad s_2 = d^2 \mp d^{3/2}\Lambda, \quad a = 2(s + 2\Lambda^2) \mp 6\sqrt{d}\Lambda, \quad b = d(\sqrt{d} \mp \Lambda)^2, \quad c = 0.$$ 

These give the breaking $USp(4) \to U(2) \times USp(0)$. To count the number of vacua, we need to use the relationship between $F_3(t)$ and $W'(x)$. However, now it is modified to

$$F_3(t) - 4\Lambda^2 t^2 = t(t - \alpha^2)^2 + O(t) \Rightarrow -2\alpha^2 + 4\Lambda^2 = a.$$ 

From this relationship, we find two vacua for $USp(4) \to U(2) \times USp(0)$ with a factor $t^2$ which is also consistent with the counting from the weak coupling analysis.

If $d = 0$, we get solution

$$s_2 = 0, \quad a = 4\Lambda^2 - 2s_1, \quad b = s_1^2, \quad c = 0.$$ 

The curve has a factor $t^4$ which gives one vacuum of $USp(4) \to U(1) \times USp(2)$ where $USp(2)$ is at the Special branch. This number of vacua can be computed in the weak coupling analysis as follows: the $USp(2)$ gauge theory with $N_f = 5$ on the $r = 2$ branch has the number $(\tilde{N} - r + 1) = 1$. Let us stress that although the weak coupling analysis seems to tell us that there are $r$ different branches characterized by $0 \leq r \leq N_f - 2 - N_c$, at the strong coupling analysis, (i.e., for the Seiberg-Witten curve), they collapse to the same point, the Special point, in the curve.

- **Degenerated case**
  Finally to have $USp(4) \to USp(4)$, we require

$$ty^2 = t(t + a)(t^2 + bt + c)^2.$$ 

There are two solutions. One is with $a = c = 0$ so having the factor $t^4$ and gives the Special branch of $USp(4)$. The other is with $c = b = 0$ so having a factor $t^5$ and gives the Chebyshev branch of $USp(4)$ by counting $(2N_0 + 2 - N_f) = 1$. We have seen that for the $N_f > N_c + 2$ case, there are two kinds of vacua. The Chebyshev branch has $U(N_f)$ flavor symmetry and the Special branch has $SO(2N_f)$ flavor symmetry. These two kinds of solutions each have one vacuum.
| $N_f$ | Group | Branch | Power of $t(=x^2)$ | $U(1)$ | Number of vacua | Connection |
|-------|-------|--------|-------------------|-------|----------------|------------|
| 1     | $USp(4)$ | $(C)$ | $t^1$ | 0 | 5 |  |
|       | $USp(2) \times U(1)$ | $(C,0_{NB})$ | $t^1$ | 1 | 3 | A |
|       | $USp(0) \times U(2)$ | $(C,0_{NB})$ | $t^1$ | 1 | 2 | A |
| 2     | $USp(4)$ | $(C)$ | $t^3$ | 0 | 4 |  |
|       | $USp(2) \times U(1)$ | $(C,0_{NB})$ | $t^3$ | 1 | 2 |  |
|       | $USp(0) \times U(2)$ | $(S,0_{NB})$ | $t^2$ | 1 | 2 |  |
| 3     | $USp(4)$ | $(C)$ | $t^3$ | 0 | 3 | B |
|       | $USp(2) \times U(1)$ | $(S,0_{NB})$ | $t^2$ | 1 | 1 | B |
|       | $(C,0_{NB})$ | $t^3$ | 1 | 1 |  |
|       | $USp(0) \times U(2)$ | $(S,0_{NB})$ | $t^2$ | 1 | 2 | B |
| 4     | $USp(4)$ | $(S)$ | $t^2$ | 0 | 1 |  |
|       | $(C)$ | $t^3$ | 0 | 2 |  |
|       | $USp(2) \times U(1)$ | $(S,0_{NB})$ | $t^4$ | 1 | 1 |  |
|       | $(C,0_{NB})$ | $t^2$ | 1 | 2 |  |
| 5     | $USp(4)$ | $(S)$ | $t^4$ | 0 | 1 |  |
|       | $(C)$ | $t^3$ | 0 | 1 |  |
|       | $USp(2) \times U(1)$ | $(S,0_{NB})$ | $t^4$ | 1 | 1 |  |
|       | $(C,0_{NB})$ | $t^2$ | 1 | 2 |  |

Table 3: The summary of the phase structure of $USp(4)$ gauge group with massless flavors. The flavors are charged under the $USp(2N_i)$ factor. We use $S$ for the Special branch for $USp(2N_i)$ factor. All the conventions are the same as in previous Tables. It is also worth comparing with the results for $USp(4)$ gauge theory with massive flavors in the previous section. For the smooth transition in the Special branch, the condition $(N_f - N_0 - 2) = M_0$ is exactly the condition for Seiberg dual pair between $USp(2N_0)$ and $USp(2M_0)$ where $N_f = 3, N_0 = 1$ and $M_0 = 0$. For this reason (the number of flavors are fixed by Seiberg dual condition above), one cannot see any smooth transitions when one increases the number of flavors, $N_f = 4, 5$. This kind of observation for $SO(6)$ with one massless flavor was found also in [12].
5.2 \textit{USp}(6) case

We will do for even number of flavor cases as we did for massive case.

5.2.1 $N_f = 2$

The curve is given by

$$ty^2 = \left[ t(t^3 - s_1t^2 + s_2t - s_3) \right]^2 - 4\Lambda^{12}t^2.$$

- **Non-degenerated case**

First the curve takes the following factorization form

$$ty^2 = tF_3(t)H_2^2(t) = t[t(t + d)]^2(t^3 - at^2 + bt + c).$$

We find two kinds of solutions. The one kind of solution is given by

$$s_1 = -2d + \frac{4\eta\Lambda^6}{d^2}, \quad s_2 = d^2 - \frac{8\eta\Lambda^6}{d}, \quad s_3 = 2\eta\Lambda^6,$$

$$a = -2d + \frac{8\eta\Lambda^6}{d^2}, \quad b = d^2 - \frac{16\eta\Lambda^6}{d} + \frac{16\Lambda^{12}}{d^4}, \quad c = -4\Lambda^6 + \frac{32\Lambda^{12}}{d^3}.$$  

There are the two limits: (1) as $\Lambda \to 0$, $d \to$ constant, we have $USp(6) \to U(2) \times USp(2)$; (2) when $\Lambda \to 0$, $d \to 0$, $\Lambda^6/d^2 \to$ constant, it gives $USp(6) \to U(1) \times USp(4)$. Thus, we see a smooth transition between these phases.

To count the number of vacua, using the relationship between $F_3(t)$ and $W'(x)$, there is a relation

$$2\alpha^2 = -2d + \frac{8\eta\Lambda^6}{d^2}.$$  

Taking into account $\eta$ from this equation, we find six solutions. Four of them give $U(1) \times USp(4)$ which can be seen by the counting $(2N_0 + 2 - N_f) = 4$ in the weak coupling analysis and two others, $U(2) \times USp(2)$.

The other kind of solutions is given by

$$s_1 = -2d, \quad s_2 = d^2, \quad s_3 = 2\eta\Lambda^6,$$

$$a = -2d, \quad b = d^2, \quad c = -4\eta\Lambda^6,$$

where $\eta$ is 2-nd roots of unity. These two solutions give a breaking pattern $USp(6) \to U(2) \times USp(2)$. Therefore, there exist four vacua in the strong coupling analysis, which is consistent with the counting of $N_1 \times (2N_0 + 2 - N_f) = 4$ in the weak coupling analysis.

- **Degenerated case**

36
Next we shall consider the degenerate case:

$$ty^2 = H_3^2F_2(t) = (t^3 - at^2 + bt + c)^2(t^2 + dt + f).$$

We find two kinds of solutions. One is given by

$$d = -a, \ c = 0, \ f = \frac{a^2}{4} - 4\epsilon\Lambda^4, \ b = \frac{a^2}{4} - \epsilon\Lambda^4$$

where $\epsilon$ is 3-rd roots of unity. These solutions give $USp(6) \to U(3)$. In this case also the condition $\widetilde{N} = 0$ with a factor $t^{2(\widetilde{N}+1)} = t^2$ is applied and as we have seen before the correct symmetry breaking pattern here is $USp(6) \to USp(0) \times U(3)$. The number of vacua in the weak coupling analysis for pure $U(3)$ theory has three.

The other solutions are

$$a = 4\epsilon\Lambda^2, \ b = 3\epsilon^2\Lambda^4, \ c = 0, \ d = -4\epsilon\Lambda^2, f = 0$$

where $\epsilon$ is 6-th roots of unity. These solutions give $USp(6) \to USp(6)$ with a factor $t^3$. The number of solution matches with the prediction $(2N + 2 - N_f) = 6$ in the weak coupling analysis.

5.2.2 $N_f = 4$

The curve is given by

$$ty^2 = \left[\left(\frac{t(t^3 - s_1t^2 + s_2t - s_3)}{t^3 - a t^2 + b t + c}\right)^2 - 4\Lambda^8 t^4\right].$$

- Non-degenerated case

First the curve takes following form

$$ty^2 = tF_3(t)H_2^2(t) = t[(t + e)(t + d)]^2(t^3 - at^2 + bt + c).$$

The solution is given by

$$a = -2d, \ b = d^2 - 4\eta\Lambda^4, \ c = 0, \ e = 0,$$

where $\eta$ is 2-nd roots of unity. These solutions give $USp(6) \to U(2) \times USp(2)$. These solutions come from the Special branch of $USp(2)$. The number of vacua is also seen from the weak coupling analysis: Pure $U(2)$ gauge theory sets the number of vacua at two while the $USp(2)$ has the $r = \widetilde{N} = 1$ branch at the Special branch. This is consistent with the fact that there is an overall factor $t^{2(\widetilde{N}+1)} = t^4$ in the curve. This breaking pattern does not appear in the Chebyshev branch because $(2N + 2 - N_f) = 0$.

The other solution is given by

$$ty^2 = tF_3(t)H_2^2(t) = t^2[(t + a - b)(t + b)]^2(t + c)(t + d)$$
and
\[ d = \frac{2(a - b)^2}{a}, \quad c = \frac{2b^2}{a}, \quad 0 = (a - 2b)^3 + 4a\Lambda^4. \]

If \( a = b \), then \( d \) becomes zero. These lead to the unbroken case we will discuss below. For the case \( a \neq b \), there are two limits. (1) \( \Lambda \to 0, d \neq 0, b \to 0 \). It gives \( U(1) \times \hat{U}Sp(4) \). (2) \( \Lambda \to 0, a, b \neq 0 \). It gives \( U\hat{S}p(0) \times U(3) \).

To count the number of vacua, using the relationship \( F_3(t) \) and \( W'(x) \) there is
\[ -2\alpha^2 = \frac{2b^2}{a} + \frac{2(a - b)^2}{a}. \]

Under the limit of (2), we obtain \( 2b = a - (a\Lambda^4)^{1/3}, a = -2\alpha^2 \), so there are three vacua with \( U(3) \) which is also consistent with the counting in the weak coupling analysis. On the other hand, for the limit (1), we obtain \( a = 0, \pm 2\Lambda^2 \) and \( b = 0 \), as we have three vacua. This number of vacua matches with the results of the weak coupling analysis. Notice that one of three vacua with \( U(1) \times \hat{U}Sp(4) \) comes from the Special branch of \( U\hat{S}p(4) \) factor. The remaining two vacua come from Chebyshev branch with a factor \( t^5 \).

- **Degenerated case**

  Next we will consider degenerate case:

  \[ ty^2 = H_3^2F_2(t) = (t^3 - at^2 + bt + c)^2(t^2 + dt + f). \]

  We find two solutions. One is given by

  \[ d = 4\eta\Lambda^2, \quad c = 0, \quad f = 0, \quad b = 0 \]

  where \( \eta \) is 4-th roots of unity. These solutions give \( USp(6) \to U\hat{S}p(6) \) with a factor \( t^5 \). The number of vacua matches with the prediction \( (2N + 2 - N_f) = 4 \) in the weak coupling analysis.

5.2.3 \( N_f = 6 \)

The curve is
\[ ty^2 = \left[ t(t^3 - s_1t^2 + s_2t - s_3) \right]^2 - 4\Lambda^4 t^6. \]

- **Non-degenerated case**

  The curve takes the following factorization form

  \[ ty^2 = tF_3(t)H_2^2(t) = t[(t + e)(t + d)]^2(t^3 - at^2 + bt + c). \]

  The solution is given by

  \[ s_1 = -2d + 2\eta\Lambda^2, \quad s_2 = d^2, \quad s_3 = 0, \quad a = -2d + 4\eta\Lambda^2, \quad b = d^2, \quad c = 0, \quad e = 0, \]
where \( \eta \) is 2-nd roots of unity. These solutions give \( USp(6) \to U(2) \times USp(2) \). These solutions have a \( t^4 \) factor, and so come from the Special branch with \( r = \tilde{N} = 3 \) branch of \( USp(2) \) factor where \( \tilde{N} = 3 \). The number of vacua matches the prediction of \( (\tilde{N} - r + 1) = 1 \) in the weak coupling analysis.

To get \( USp(4) \times U(1) \) with \( USp(4) \) at the Special branch, we need a factor \( t^6 \) so \( P_3(t) = t^2(t - s_1) \). There is only one solution. In this case, since \( \tilde{N} = 2 \), the \( USp(4) \) has the \( r = \tilde{N} = 2 \) branch at the Special branch.

To get \( USp(0) \times U(3) \), the curve is factorized as
\[
\begin{align*}
t^3 - s_1 t^2 + s_2 t - s_3 - 2\Lambda^2 t^2 &= (t + a)(t + b)^2, \\
t^3 - s_1 t^2 + s_2 t - s_3 + 2\Lambda^2 t^2 &= (t + c)(t + d)^2.
\end{align*}
\]

The solution is given by
\[
\begin{align*}
s_1 &= 2 \left( \frac{b^3 - d^3 + (b^2 + d^2) \Lambda^2}{-b^2 + d^2} \right), \\
s_2 &= b \left( b + \frac{4d^2 (-b + d - 2\Lambda^2)}{-b^2 + d^2} \right), \\
s_3 &= \frac{-2b^2 d^2 (b - d + 2\Lambda^2)}{b^2 - d^2}, \\
a &= \frac{2d^2 (-b + d - 2\Lambda^2)}{-b^2 + d^2}, \\
c &= \frac{2b^2 (b - d + 2\Lambda^2)}{b^2 - d^2}, \\
0 &= (b - d)^3 - 8bd\Lambda^2.
\end{align*}
\]

There is only one limit we can take: \( b, d \neq 0 \), but \( (b - d) \sim \Lambda^{2/3} \). To count the vacua, using \( a + c = -2\alpha^2 \) we have three solutions.

- **Degenerated case**

Next we consider degenerate case:
\[
ty^2 = H_3^2 F_2(t) = t^3(t^2 + at^1 + b)^2(t + d).
\]

We find three solutions.
\[
d = a = 0, \quad b = -\Lambda^4, \\
d = 4\eta\Lambda^2, \quad a = b = 0,
\]

where \( \eta \) is 2-nd roots of unity. These solutions give \( USp(6) \to USp(6) \). The number of vacua matches the prediction \( (2N + 2 - N_f) = 2 \) in the weak coupling analysis which is at the Chebyshev branch with a factor \( t^7 \) and the one comes from the Special \( r = 1 \) branch with a factor \( t^4 \) where the counting \( (\tilde{N} - r + 1) = (1 - r + 1) = 1 \) provides with the \( r = 1 \).

### 6 Discussion

In this section, we will discuss about some observations made recently by Aganagic et al in [17], which show that the ‘trivial’ group \( USp(0) \) plays a role in the calculation of ‘nontrivial’
| $N_f$ | Group         | Branch | Power of $t = x^2$ | $U(1)$ | Number of vacua | Connection |
|-------|---------------|--------|-------------------|--------|-----------------|------------|
| 2     | $USp(6)$      | (C)    | $t^3$             | 0      | 6               |            |
|       | $USp(2) \times U(2)$ | (C, $0_{NB}$) | $t^3$     | 1      | 2               | A          |
|       |               | (C, $0_{NB}$) | $t^3$     | 1      | 2               |            |
|       | $USp(4) \times U(1)$ | (C, $0_{NB}$) | $t^3$     | 1      | 4               | A          |
|       | $USp(0) \times U(3)$ | (S, $0_{NB}$) | $t^2$     | 1      | 3               |            |
| 4     | $USp(6)$      | (C)    | $t^5$             | 0      | 4               |            |
|       | $USp(2) \times U(2)$ | (S, $0_{NB}$) | $t^4$     | 1      | 2               |            |
|       | $USp(4) \times U(1)$ | (S, $0_{NB}$) | $t^2$     | 1      | 1               | B          |
|       |               | (C, $0_{NB}$) | $t^5$     | 1      | 2               |            |
|       | $USp(0) \times U(3)$ | (S, $0_{NB}$) | $t^2$     | 1      | 3               | B          |
| 6     | $USp(6)$      | (C)    | $t^7$             | 0      | 2               |            |
|       |               | (S)    | $t^4$             | 0      | 1               |            |
|       | $USp(2) \times U(2)$ | (S, $0_{NB}$) | $t^4$     | 1      | 2               |            |
|       | $USp(4) \times U(1)$ | (S, $0_{NB}$) | $t^4$     | 1      | 1               |            |
|       | $USp(0) \times U(3)$ | (S, $0_{NB}$) | $t^2$     | 1      | 3               |            |

Table 4: The summary of the phase structure of $USp(6)$ gauge group with massless flavors. All the conventions are the same as in previous Tables. The phases having smooth transition look similar to those in the $SO(6)$ gauge group with massless flavors where $SO(\hat{N}_0) \times U(N_1) \leftrightarrow SO(\hat{M}_0) \times U(M_1)$, in the context of breaking patterns. For the smooth transition in the Special branch, the condition $(N_f - N_0 - 2) = M_0$ is exactly the condition for the Seiberg dual pair between $USp(2N_0)$ and $USp(2M_0)$ where $N_f = 4, N_0 = 2$ and $M_0 = 0$. One can also understand the reason why one does not see any smooth transition for $N_f = 6$ in Special branch by checking the Seiberg dual condition: The number of flavors are constrained to this.
superpotential and makes an agreement between the matrix model results and standard gauge theory results for $USp(2N)$ gauge theory with an adjoint for all Higgs breaking patterns and for any $N > 0$ (the F-completion of $USp(2N)$ with an adjoint that differs from the standard gauge theory UV completion for only $USp(0)$ case), and by Cachazo in [18] which states that the dynamics of $USp(2N)$ is related to the one of $U(2N + 2)$ with the confining index $t = 2$ in the notation of [5]. This observation is related to the mysterious behavior of $USp(0)$ in the factorization form described in the current paper. Here we will show that such a relationship can also be observed with the Brane setup [33, 34].

Recall that the standard Brane setup to get $USp(2N)$ gauge theory in four dimensions (in type IIA string theory) is to put $2N$ D4-branes with parallel orientifold plane, $O4$-plane, between the two NS5-branes. The orientifold plane is $O4^+$ (with D4-brane charge $+1$) between two NS5-branes and $O4^−$ (with D4-brane charge $−1$) outside these two NS5-branes. If we add one single D4-brane top on the orientifold plane, the total D4-brane charge outside the two NS5-branes will be zero due to the cancellation of the effect of both single D4-brane and $O4^−$, just like the construction of the $U(N)$ gauge group. However, the total D4-brane charge inside the two NS5-branes becomes $(2N + 2)$. This explains the mapping $USp(2N) \rightarrow U(2N + 2)$. Furthermore, the orientifold action hints the $U(2N + 2)$ gauge theory side to be the confining index $t = 2$.

All these brane pictures can also be seen from the Seiberg-Witten curve:

$$ty^2 = \left[ tP_N(t) + 2\Lambda^{2N+2-M} \prod_{j=1}^{M} m_j \right]^2 - (-1)^M 4\Lambda^{2(2N+2-M)} \prod_{j=1}^{M} (t - m_j^2).$$

The addition of one single D4-brane top on the orientifold plane can be seen from the factors $t$ in front of both $y^2$ and the characteristic function $P_N(t)$. The orientifold action (or the confining index 2) can be seen by the factor $t = x^2$. Then the whole curve is exactly the same form as the one of $U(N)$ gauge group.

We can use the Brane setup to discuss the relationship between $U(N)$ and $SO(N)$ gauge theories. In this case, the orientifold is $O4^−$ (with D4-brane charge $−1$) for $SO(2N)$ or $O4^−$ (with D4-brane charge $−1$) for $SO(2N + 1)$ between the two NS5-branes, but $O4^+/\tilde{O4}^+$ (with D4-brane charge $+1$) outside the two NS5-branes for $SO(2N)/SO(2N + 1)$. To redeem the D4-brane chargelessness outside the two NS5-branes so that we can compare with the brane setup of $U(N)$ gauge group, we need to add some object with negative D4-brane charge, i.e., anti-D4-brane, to top on the orientifold in order to cancel the D4-brane charge for $O4^+/\tilde{O4}^+$. However, the addition of the anti-D4-brane will break the supersymmetry completely. Thus, we do not expect the similar relationship between $U(N)$ and $USp(2N)$ gauge theories we have described above holds between pure $SO(N)$ and $U(N)$ in the supersymmetric gauge theories.

However, if we allow the gauge group to carry the fundamental flavors, we may establish
the relationship between \(SO(N)\) and \(U(N)\) gauge theories. The reason is as follows. To give the flavors in the Brane setup, we just need to add D4-branes outside these two NS5-branes. Then we can trade the \(O4^+\) as one single D4-brane while bringing down one D4-brane inside these two NS5-branes to make the charge from \(O4^-\) be neutral. After that, we can guess that the \(SO(2N)\) gauge group with \(N_f\) flavors has some relationship with \(U(2N - 1)\) gauge group with \((2N_f + 1)\) flavors at the confining index 2. Obviously, it would be interesting to check this picture by Konishi anomaly equation as in [18].

**Acknowledgments**

This research of CA was supported by Korea Research Foundation Grant(KRF-2002-015-CS0006). CA thanks Dept. of Physics, Tokyo Institute of Technology where part of this work was undertaken. YO would like to thank Hiroaki Kanno for useful discussions. CA and YO thank Katsushi Ito and Kenichi Konishi for discussions. BF would like to thank Freddy Cachazo and Oleg Lunin for perfectly delightful discussions. The work of BF is supported by the NSF grant PHY-0070928.

**Appendix A Strong gauge coupling approach: superpotential and a generalized Konishi anomaly equation for \(USp(2N_c)\) case**

The superpotential considered as a small perturbation is given by (2.2) and the adjoint scalar chiral superfield \(\Phi\) has the form of

\[
\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \text{diag}(i\phi_1, i\phi_2, \ldots, i\phi_{N_c-r}, 0, 0, \ldots, 0).
\]

Let us consider a special \((N_c - r)\) dimensional submanifold of the Coulomb branch where some of the branching points of the moduli space collide. On the \(r\)-th branch, the effective theory is \(USp(2r) \times U(1)^{N_c-r}\) with \(N_f\) massless flavors. Following the previous work [12], the exact effective superpotential near a point with \((N_c - r - n)\) massless monopoles is given by

\[
W_{\text{eff}} = \sqrt{2} \sum_{l=1}^{N_c-r} M_l(u_{2s})q_l\bar{q}_l + \sum_{s=1}^{k+1} g_{2s}u_{2s}.
\]

By varying this with respect to \(u_{2s}\), we get an equation of motion similar to a pure Yang-Mills theory except that the extra terms on the left hand side since the \(u_{2s}\) with \(2s > 2(N_c - r)\) are dependent on \(u_{2s}\) with \(2s \leq 2(N_c - r)\). There exist \((N_c - n - r)\) equations for the \((k + 1)\) parameters \(g_{2s}\).

Let us consider a singular point in the moduli space where \((N_c - r - n)\) monopoles are massless. The \(\mathcal{N} = 2\) curve of genus \((2N_c - 2r - 1)\) degenerates to a curve of genus \(2n\) and it
is given by
\[ y^2 = \left( x^2 P_{Nc-r}(x) \right)^2 - 4 \Lambda^{2(Nc-r)-(Nf-2r)} x^{(Nf-2r)} \]
\[ = H_{Nc-n-r}^2(x) F_{2(2n+1)}(x), \]
where the double root part and single root part can be characterized by the following two even functions respectively
\[ H_{2(Nc-n-r)}(x) = \prod_{i=1}^{Nc-n-r} (x^2 - p_i^2), \quad F_{2(2n+1)}(x) = \prod_{i=1}^{2n+1} (x^2 - q_i^2) \]
which is similar to the one in $SO(N_c)$ case. Moreover, the characteristic function can be written as $P_{(Nc-r)}(x) = \prod_{i=1}^{Nc-r} (x^2 - \phi_i^2)$.

- **Field theory analysis for superpotential**

Now we extend the discussion in [11, 12] to $USp(2N_c)$ gauge theory and later generalize to the more general cases. The massless monopole constraint for $USp(2N_c)$ gauge theory with $N_f$ flavors is described as follows \(^{10}\):

\[ y^2 = B_{2Nc+2}^2(x) - 4 \Lambda^{4Nc-4-2Nf} A(x) = x^2 H_{2Nc-2n}^2(x) F_{2(2n+1)}(x), \]
\[ = x^2 \prod_{i=1}^{N_f} \left( x^2 - p_i^2 \right)^2 F_{2(2n+1)}(x), \quad A(x) \equiv \prod_{j=1}^{N_f} (x^2 - m_j^2) \quad (A.1) \]
where we used $l$ as the number of massless monopoles and
\[ B_{2Nc+2}(x) = x^2 P_{2Nc}(x) + 2 \Lambda^{2Nc+2-Nf} m^{N_f}. \]

We have an effective superpotential with $l$ massless monopole constraints (A.1),
\[ W_{low} = \sum_{t=1}^{k+1} g_{2t} u_{2t} + \sum_{i=1}^{l} \left[ L_i \left( B_{2Nc+2}(p_i) - 2 \epsilon_i \Lambda^{2Nc+2-Nf} \sqrt{A(p_i)} \right) \right. \]
\[ + \left. Q_i \frac{\partial}{\partial p_i} \left( B_{2Nc+2}(p_i) - 2 \epsilon_i \Lambda^{2Nc+2-Nf} \sqrt{A(p_i)} \right) \right] \]
where $L_i, Q_i$ are Lagrange multipliers and $\epsilon_i = \pm 1$. From the equation of motion for $p_i$ and $Q_i$ we obtain the following equations,
\[ Q_i = 0, \quad \frac{\partial}{\partial p_i} \left( B_{2Nc+2}(p_i) - 2 \epsilon_i \Lambda^{2Nc+2-Nf} \sqrt{A(p_i)} \right) = 0. \]
The variation of $W_{low}$ with respect to $u_{2t}$ leads to
\[ g_{2t} + \sum_{i=1}^{l} L_i \frac{\partial}{\partial u_{2t}} \left( B_{2Nc+2}(p_i) - 2 \epsilon_i \Lambda^{2Nc+2-Nf} \sqrt{A(p_i)} \right) = 0. \]
\(^{10}\)It is straightforward to apply it to other cases we have developed before where the structures of single and double roots are different from the one we are considering here.
Since $A(p_i)$ is independent of $u_{2l}$, the third term vanishes. By using $P_{2N_c}(p_i) = \sum_{j=0}^{N_c} s_{2j} p_i^{2N_c-2j}$, we can obtain

$$g_{2l} = \sum_{i=1}^{l} L_i \frac{\partial}{\partial u_{2l}} P_{2N_c}(p_i) = \sum_{i=1}^{l} \sum_{j=0}^{N_c} L_i s_{2j} p_i^{2N_c-2j} s_{2j-2l}.$$  

With this relation as in [16], the following relation is obtained:

$$W'(x) = \sum_{i=1}^{l} x^3 P_{2N_c}(x) L_i - x \sum_{i=1}^{l} 2c_i L_i A^{2N_c+2-N_f} \sqrt{A(p_i)} + \mathcal{O}(x^{-1}).$$

Defining a new polynomial $B_{2(l-1)}$ of order $2(l-1)$ as in [31],

$$\sum_{i=1}^{l} \frac{L_i}{x^2 - p_i^2} \equiv \frac{B_{2l}(x)}{x^2 H_{2l}(x)}.$$  

Then the deformed superpotential can be expressed as

$$W'(x) = \frac{x P_{2N_c}(x) B_{2l}(x)}{H_{2l}(x)} + \mathcal{O}(x^{-1}).$$

Now we can compare both sides. In particular, for the power behavior of $x$ it is easy to see that the left hand side behaves like $(2n+1)$ while the right hand side behaves like $(2N_c + 1 - 2l)$ except for the factor $B_{2l}(x)$, and therefore the condition $l = (N_c - n)$ will give rise to the consistency and the polynomial $B_{2l}(x)$ becomes a constant. By using this relation and substituting the characteristic polynomial $P_{2N_c}(x)$ from the monopole constraints (A.1), we can obtain the following relation together with the replacement of the polynomial $H_{2l}(x)$,

$$F_{4n+2} + \frac{4A^{4N_c-2N_f+4} \prod_{i=1}^{N_c} (x^2 - m_i^2)}{x^2 \prod_{j=1}^{N_f} (x^2 - p_j^2)^2} = \frac{1}{g_{2n+2}^2} \left( W'_{2n+1}(x)^2 + \mathcal{O}(x^{2n}) \right).$$

(A.2)

Thus, if $n > -N_f + 2N_c + 1$, the effect of flavor changes the geometry. This is a new feature compared with the pure gauge theory without any flavors and was used in the specific examples in the previous sections.

When the breaking pattern is $USp(2N_c) \rightarrow \prod_{i=1}^{n} U(N_i)$, the above analysis must be changed. The general form was given in [12]. Moreover, when the origin does not contain the wrapping D5-branes, as we did for $SO(N_c)$ gauge theory, the parts relevant to the single roots of SW curve can be expressed as the superpotential as follows:

$$F_{4n} + \frac{4A^{4N_c-2N_f+4} \prod_{i=1}^{N_f} (x^2 - m_i^2)}{\prod_{j=1}^{N_f-n+1} (x^2 - p_j^2)^2} = \frac{1}{g_{2n+2}^2} \left( \left( \frac{W'_{2n+1}(x)}{x} \right)^2 + \mathcal{O}(x^{2n-2}) \right).$$

Similarly, when $n > -N_f + 2N_c + 1$, the effect of flavor changes the geometry and this phenomenon appeared in the explicit examples.
Superpotential of degree $2(k + 1)$ less than $2N_c$

We would like to generalize (A.2) to $2n < 2k$. Let us describe the superpotential in the range $2n + 2 \leq 2k + 2 \leq 2N_c$ for $USp(2N_c)$ gauge theory with massless $N_f$ flavors in the $r$-th branch with the appropriate constraints by starting from a pure case and modifying the curve

$$W_{\text{eff}} = \sum_{s=1}^{k+1} 2g_{2s}u_{2s} + \sum_{i=0}^{2N_c-2r-2n} \left[ L_i \int \frac{B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}}{(x-p_i)^2} dx \right] + B_i \int \frac{B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}}{(x-p_i)^2} dx.$$ 

For an equal massive case, we simply replace $x^{N_f-2r}$ in the numerator with $(x^2 - m^2)^{N_f/2-r}$ and the function $B_{2(N_c-r)+2}(x)$ can be understood as $x^2 P_{2(N_c-r)}(x) + 2\Lambda^{2N_c-N_f+2} m^{N_f-2r}$. The $p_i$'s where $i = 0, 1, 2, \cdots, (2N_c - 2r - 2n)$ are the locations of the double roots of $y^2 = B_{2(N_c-r)+2}(x) - 4x^{2N_f-4r}\Lambda^{4N_c+1-2N_f}$. The massless monopole points appear in pair $(p_i, -p_i)$ and both the function $B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}$ at $x = \pm p_i$ and its derivative with respect to $x$ at $x = \pm p_i$ become zero where $i = 1, 2, \cdots, (N_c - r - n)$. The total number of constraints is $(N_c - r - n)$ due to the fact that the half of the Lagrange multipliers are not independent. For given constraints, there are only a Lagrange multiplier $L_i$ with $(x-p_i)^{-1}$ and a Lagrange multiplier $B_i$ with $(x-p_i)^{-2}$.

The variation of $W_{\text{eff}}$ with respect to the Lagrange multiplier $B_i$ will produce

$$0 = \int \frac{B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}}{(x-p_i)^2} dx$$

$$= \left( B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f} \right)' \big|_{x=p_i}$$

$$= \left( x^2 P'_{2(N_c-r)}(x) + 2x P_{2(N_c-r)}(x) - \frac{N_f-2r}{x} - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f} \right) \big|_{x=p_i}$$

$$= x^2 P'_{2(N_c-r)}(x) \left( \text{Tr} \left( \frac{1}{x - \Phi_{cl}} - \frac{N_f-2}{x} \right) \right) \big|_{x=p_i}.$$ 

Here we used the fact that there exist the equations of motion for the Lagrange multiplier $L_i$ when we replace $2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}$ with $x^2 P_{2(N_c-r)}(x)$ at $x = p_i$ and the last equality comes from $\text{Tr} \frac{1}{x - \Phi_{cl}} + \frac{2r}{x} = \sum_{i=1}^{N_c-r} \frac{2x}{x_i - \Phi_{cl}} = P'_{2(N_c-r)}(x)/P_{2(N_c-r)}(x)$. For the variation with respect to $p_j$ one obtains

$$0 = 2B_j \int \frac{B_{2(N_c-r)+2}(x) - 2\epsilon_i x^{N_f-2r} \Lambda^{2N_c+2-N_f}}{(x-p_i)^3} dx$$

where the equation of motion for the Lagrange multiplier $B_i$ is used and therefore there is no $L_i$ term. This expression does not lead to zero because the contour integral is not equal to zero. This implies that $B_i = 0$. Now we turn to the variation of $W_{\text{eff}}$ with respect to $u_{2s}$

$$0 = g_{2s} - \sum_{i=0}^{2N_c-2r-2n} L_i \int \frac{2x P_{2(N_c-r)}(x)}{x^{2s}} \frac{1}{x-p_i} dx.$$
By performing the multiplication with $z^{2s-1}$ and summing over $s$, the first derivative of superpotential can be written as

$$W'(z) = \sum_{s=1}^{k+1} g_{2s} z^{2s-1} = \sum_{i=0}^{2N_c-2r-2n} \int \sum_{s=1}^{k+1} z^{2s-1} \frac{x^2 P_{2(N_c-r)}(x)}{x^{2s}} \frac{L_i}{(x - p_i)} dx.$$

As before [10, 12], after we introduce the new polynomial $Q(x)$ together with the fact that there exist only half of the Lagrange multipliers and the locations of the double roots, one gets

$$W'(z) = \int \sum_{s=1}^{k+1} \frac{z^{2s-1} Q(x) x^2 P_{2(N_c-r)}(x)}{x H_{2N_c-2n-2r}(x)} dx.$$

By performing the summation to the infinite sum so as to get a closed form, we get

$$W'(z) = \int \sum_{s=1}^{\infty} \frac{z^{2s-1} Q_{2k-2n}(x) x^2 P_{2(N_c-r)}(x)}{x H_{2N_c-2n-2r}(x)} dx = \int z Q_{2k-2n}(x) x^2 P_{2(N_c-r)}(x) x H_{2N_c-2n-2r}(x) dx.$$

One can use

$$x^2 P_{2(N_c-r)}(x) = x \sqrt{F_{2(2n+1)}(x) H_{2N_c-2n-2r}(x) + O(x^{-2N_c+2N_f+2r-2})}$$

from the monopole constraints. Additionally, for $USp(2N_c)$ case, the second terms can not be ignored due to the $N_f$- and $r$-dependent parts.

By substituting this into the above, we get

$$W'(z) = \int z y_m^2(x) dx + \int O \left( x^{-4N_c+2N_f+2k-5} \right) dx, \quad y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}(x).$$

For the condition $-4N_c + 2N_f + 2k - 5 \geq -1$, the second term does contribute and one can classify the matrix model curve as follows:

$$y_m^2(x) = F_{2(2n+1)}(x) Q_{2k-2n}(x) = \begin{cases} W_{2k+1}^2(x) + O \left( x^{2k} \right) & k \geq 2N_c - N_f + 2 \\ W_{2k+1}^2(x) + O \left( x^{4N_c-2N_f+4} \right) & k < 2N_c - N_f + 2 \\ \end{cases} \equiv W_{2k+1}^2(x) + f_{2M}(x), \quad 2M = \max(2k, 4N_c - 2N_f + 4). \quad (A.3)$$

When $2k = 2n$, we reproduce (A.2) with $Q_0 = g_{2n+2}$. The second term on the left hand side of (A.2) behaves like as $x^{2N_f-4(N_c-n-1)} = x^{2N_f-4N_c+4n+2}$. Depending on whether the power of this is greater than or equal to $2n = 2k$, the role of flavor is effective or not. When $n = k > -N_f + 2N_c - 1$, the flavor-dependent part will contribute the $W'(x)$.

\footnote{When there are no wrapping D5-branes around the origin, then the matrix model curve in this case can be obtained similarly and summarized by

$$y_m^2(x) = F_{4n}(x) Q_{2k-2n}(x) = \begin{cases} \left( \frac{W_{2k+1}(x)}{x} \right)^2 + O \left( x^{2k-2} \right) & k \geq 2N_c - N_f + 2 \\ \left( \frac{W_{2k+1}(x)}{x} \right)^2 + O \left( x^{4N_c-2N_f+2} \right) & k < 2N_c - N_f + 2 \\ \end{cases} \equiv \left( \frac{W_{2k+1}(x)}{x} \right)^2 + f_{2M}(x), \quad 2M = \max(2k - 2, 4N_c - 2N_f + 2).$$}
• A generalized Konishi anomaly

Let us assume that the degree of superpotential is less than $2N_c$ and consider the following quantity

$$W' (\phi_I) = \sum_{i=0}^{2N_c-2r-2n} \int \phi_I \frac{x^2 P_{2N_c-r}(x)}{(x^2 - \phi_I^2)} L_i (x - p_i) dx.$$  

(A.4)

where we put $B_i = 0$. Therefore, by using the above expression one can write down the following relation

$$\text{Tr} \frac{W' (\Phi_{cl})}{z - \Phi_{cl}} = \sum_{i=0}^{2N_c-2r-2n} \int \frac{x^2 P_{2N_c-r}(x)}{(x^2 - z^2)(x - p_i)} \left( z \text{Tr} \frac{1}{z - \Phi} - x \text{Tr} \frac{1}{x - \Phi} \right) dx.$$  

(A.5)

According to the change of contour integration [10, 12]

$$\oint_{z_{out}} = \oint_{z_{in}} - \oint_{C_z + C_{-z}},$$

the first term of (A.5) can be written as

$$\left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \oint_{z_{out}} \frac{zQ_{2k-2n+2}(x)x^2P_{2N_c-r}(x)}{H_{2N_c-2n-2r}(x)(x^2 - z^2)} dx.$$

Then one obtains the first term of (A.5) in terms of two parts, after change of an integration,

$$\left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( \oint_{z_{in}} \frac{zQ_{2k-2n+2}(x)x^2P_{2N_c-r}(x)}{H_{2N_c-2n-2r}(x)(x^2 - z^2)} dx + \oint_{C_z + C_{-z}} \frac{zQ_{2k-2n+2}(x)x^2P_{2N_c-r}(x)}{H_{2N_c-2n-2r}(x)(x^2 - z^2)} dx \right)$$

$$= \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( W'(z) - \frac{y_m(z)x^2P_{2N_c-r}(x)}{\sqrt{B_{2N_c-2n-2r}(x)(x^2 - z^2)}} \right),$$

(A.6)

where we have used

$$H_{2N_c-2n-2r}(z) = \sqrt{B_{2N_c-2n-2r}(z)^2 - 4x^2N_f-4r A^{4N_c+4-2N_f}},$$

$$y_m(z) = F_{2N_c+1}(z)Q_{2k-2n+2}(z).$$

By simple manipulation, the second term of (A.5) can be reduced to

$$\sum_{i=0}^{2N_c-2r-2n} \int \frac{(N_f - 2r - 2)x^2P_{2N_c-r}(x)}{x^2 - z^2} L_i (x - p_i) dx = - \frac{W'(z)}{z}.$$  

(A.7)

---

12 One writes down

$$\text{Tr} \frac{W' (\Phi_{cl})}{z - \Phi_{cl}} = \text{Tr} \sum_{k=0}^{\infty} \frac{z^{-\theta} P_{2N_c-r}(x)}{(x^2 - \theta)(x^2 - \Phi_{cl}^2)} \sum_{i=1}^{N_c-r} \frac{2\delta_{i}^2}{(x^2 - \Phi_{cl}^2)} \sum_{i=0}^{2N_c-2r-2n} \int \frac{x^2P_{2N_c-r}(x)}{(x^2 - \Phi_{cl}^2)} L_i (x - p_i) dx.$$  

From this we change the expression into the linear combination of trace part

$$\frac{2\delta_{i}^2}{(x^2 - \Phi_{cl}^2)} = \frac{z \text{Tr} \frac{1}{z - \Phi} - x \text{Tr} \frac{1}{x - \Phi}}{x^2 - z^2}.$$  

13 The second term of (A.5) becomes

$$\oint \frac{L_i \frac{z^2P_{2N_c-r}(x)}{(x^2 - p_i)(x^2 - z^2)}}{x - \Phi_{cl}^2} dx = - \text{Tr} \frac{L_i \frac{z^2P_{2N_c-r}(x)}{(x^2 - p_i)(x^2 - z^2)}}{x - \Phi_{cl}^2} dx = - \oint \frac{(N_f - 2)x^2P_{2N_c-r}(x)}{x^2 - z^2} L_i (x - p_i) dx.$$
Therefore we obtain (A.5) by adding the two contributions (A.6) and (A.7)

\[
\text{Tr} \frac{W''(\Phi_{cl})}{z - \Phi_{cl}} = \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \left( W'(z) - \frac{y_m(z) z^2 P_{2(N_c-r)}(z)}{\sqrt{B_{2(N_c-r)+2}^2(z)} - 4z^{2N_f-4r} \Lambda^{4N_c+4-2N_f}} \right) - (N_f - 2) \frac{W'(z)}{z} + \left( \frac{y_m(z) z^2 P_{2(N_c-r)}(z)}{\sqrt{B_{2(N_c-r)+2}^2(z)} - 4z^{2N_f-4r} \Lambda^{4N_c+4-2N_f}} \right).
\]

Recalling that

\[
\text{Tr} \frac{1}{z - \Phi_{cl}} = \frac{\left( z^2 P_{2(N_c-r)}(z) \right)'}{z^2 P_{2(N_c-r)}(z)} - \frac{2}{z}
\]

and the quantum mechanical form for \( \langle \text{Tr} \frac{1}{z - \Phi} \rangle \) we have described before, we finally obtain the second term of the above expression:

\[
- \left( \text{Tr} \frac{1}{z - \Phi_{cl}} \right) \frac{y_m(z) z^2 P_{2(N_c-r)}(z)}{\sqrt{B_{2(N_c-r)+2}^2(z)} - 4z^{2N_f-4r} \Lambda^{4N_c+4-2N_f}} = - \left( \frac{\left( z^2 P_{2(N_c-r)}(z) \right)'}{z^2 P_{2(N_c-r)}(z)} - \frac{2}{z} \right) \frac{y_m(z) z^2 P_{2(N_c-r)}(z)}{\sqrt{B_{2(N_c-r)+2}^2(z)} - 4z^{2N_f-4r} \Lambda^{4N_c+4-2N_f}} = - \left( \langle \text{Tr} \frac{1}{z - \Phi} \rangle - \frac{(N_f - 2)}{z} \right) y_m(z) - \frac{(N_f - 2)}{z} \frac{y_m(z) z^2 P_{2(N_c-r)}(z)}{\sqrt{B_{2(N_c-r)+2}^2(z)} - 4z^{2N_f-4r} \Lambda^{4N_c+4-2N_f}}.
\]

Using the relation

\[
\text{Tr} \frac{W'(\Phi_{cl}) - W'(z)}{z - \Phi_{cl}} = \left( \text{Tr} \frac{W'(\Phi) - W'(z)}{z - \Phi} \right),
\]

our generalized Konishi anomaly equation can be summarized as follows:

\[
\langle \text{Tr} \frac{W'(\Phi)}{z - \Phi} \rangle = \left( \langle \text{Tr} \frac{1}{z - \Phi} \rangle - \frac{(N_f - 2)}{z} \right) \left[ W'(z) - y_m(z) \right]
\]

which is the generalized Konishi anomaly equation for \( USp(2N_c) \) case with flavors. This reproduces the one in a pure adjoint case [10] in which \( N_f = 0 \). The resolvent of the matrix model is related to \( W'(z) - y_m(z) \).

### Appendix B  Addition and multiplication maps

In [5] it was discussed that all the confining vacua were constructed from the Coulomb vacua with lower rank gauge groups by using the Chebyshev polynomial through the multiplication map. The vacua with a classical gauge group \( \prod_{i=1}^n U(N_i) \) are transformed into the vacua with
We have extended the $U(KN_i)$ for the same superpotential where $K$ is a multiplication index. This multiplication map was extended to the $SO(N_c)/USp(2N_c)$ gauge theories in [10], the $U(N_c)$ gauge theories with flavors in [11], and $SO(N_c)$ gauge theories with flavors [12]. In addition to this multiplication map, another map (called the addition map) was introduced in [11]. Thus, we will also study these two maps, the addition map and multiplication map for the $USp(2N_c)$ gauge theories with flavors.

- **Addition map**

Starting with $USp(2N)$ and $K$ flavors, the curve is characterized by

$$ty^2_\text{old} = \left[tP_N(t) + 2\Lambda_{\text{old}}^{2N+2-K}m^K\right]^2 - (-1)^K 4\Lambda_{\text{old}}^{2(2N+2-K)}(t - m^2)^K$$

where the characteristic function is given by

$$P_N(t) = \sum_{j=0}^{N} s_j t^{N-j}, \quad s_0 = 1.$$ 

Now we want to factorize out $(t - m^2)^{2r}$ with $2r \leq K$, so we should have

$$tP_N(t) + 2\Lambda_{\text{old}}^{2N+2-K}m^K = (t - m^2)^r \tilde{P}_{N+1-r}(t).$$

Parameterized by

$$\tilde{P}_{N+1-r}(t) = \sum_{j=0}^{N+1-r} \tilde{s}_j t^{N+1-r-j},$$

we get the following relationship, by writing $(t - m^2)^r$ in terms of binomial expansion,

$$\sum_{j=0}^{N} s_j t^{N+1-j} + 2\Lambda_{\text{old}}^{2N+2-K}m^K = \left(\sum_{l=0}^{r} t^{r-l}(-1)^l C_l^r m^{2l}\right) \left(\sum_{j=0}^{N+1-r} \tilde{s}_j t^{N+1-r-j}\right).$$

By reading off the $t$-independent part in both sides and recombinig the power of $\Lambda$, the key part is that

$$\tilde{s}_{N+1-r}(-1)^r m^{2r} = 2\Lambda_{\text{old}}^{2N+2-K}m^K \implies \tilde{s}_{N+1-r} = 2(-1)^r \Lambda_{\text{old}}^{2(N-r)+2-(K-2r)}m^{K-2r}.$$ 

Now after factorizing out the $(t - m^2)^{2r}$, by substituting the above value for $\tilde{s}_{N+1-r}$, the reduced curve becomes

$$\tilde{P}_{N+1-r}^2(t) - (-1)^K 4\Lambda_{\text{old}}^{2(2N+2-K)}(t - m^2)^{K-2r}$$

$$= \left[\sum_{j=0}^{N-r} \tilde{s}_j t^{N+1-r-j} + \tilde{s}_{N+1-r}\right]^2 - (-1)^K 4\Lambda_{\text{old}}^{2(2N+2-K)}(t - m^2)^{K-2r}$$

$$= \left[t \sum_{j=0}^{N-r} \tilde{s}_j t^{N-r-j} + 2\Lambda_{\text{new}}^{2(N-r)+2-(K-2r)}m^{K-2r}\right]^2 - (-1)^{K-2r} 4\Lambda_{\text{new}}^{2(2(N-r)+2-(K-2r))}(t - m^2)^{K-2r}$$
which is the curve of $USp(2N - 2r)$ theory with $(K - 2r)$ flavors while the new coupling scale has the following relationship

$$(-1)^r \Lambda_{\text{new}}^{2N+2-K} = \Lambda_{\text{old}}^{2N+2-K}.$$ 

Thus, we have the familiar addition map just like the cases of $U(N)$ and $SO(N)$ gauge theories. For example, the phase structure of $USp(6)$ with $N_f = 4$ massive flavors on the $r = 1$ branch can be read off the one of $USp(4)$ with $N_f = 2$ massive flavors on the $r = 0$ branch using the addition map.

- **Multiplication map**

  Let us assume that the SW curve for $USp(2N_c)$ gauge theory with $2l$ flavors and the scale $\Lambda_0$ has the following massless monopole constraint

  $$\left[ x^2 P_{2N_c}(x) + 2\Lambda_0^{(2N_c+2-2l)} m^{2l} \right]^2 - 4\Lambda_0^{(2N_c+2-2l)} (x^2 - m^2)^{2l} = f_{2p}(x) H_{2N_c+2-p}^2(x) \quad \text{(B.1)}$$

  where $p$ is some number and let us define

  $$\bar{x} = \frac{x^2 P_{2N_c}(x) + 2\eta \Lambda^{(2N_c+2-2l)} m^{2l}}{2\eta \Lambda^{(2N_c+2-2l)} (x^2 - m^2)^l}. \quad \text{(B.2)}$$

  We claim that the solution is given by

  $$x^2 P_{K(2N_c+2)-2}(x) + 2\Lambda^{K(2N_c+2-2l)} m^{2Kl} = 2 \left[ \eta \Lambda^{(2N_c+2-2l)} (x^2 - m^2)^l \right]^K T_K(\bar{x}) \quad \text{(B.3)}$$

  where the phase $\eta$ is determined later. Before we show it is true, let us see the consistency of the above expression. The key part is that when $x = 0$, the equation should also hold. For $x = 0$, the left hand side of (B.3) is $2\Lambda^{K(2N_c+2-2l)} m^{2Kl}$. For the right hand side of (B.3), one realizes that $\bar{x} = (-1)^l$ at $x = 0$, so by substituting this into the first kind of Chebyshev polynomial one gets $T_K((-1)^l) = (-1)^{Kl}$. To match both sides, we need to have

  $$\eta^K = 1.$$ 

  This is consistent with the case in pure adjoint. \(^\text{15}\)

  Now we can show that it is a really true solution. At first we rewrite the following curve for $USp(2(KN_c + K - 1))$ gauge theory with $2Kl$ flavors and the scale $\Lambda$:

  $$\left[ x^2 P_{K(2N_c+2)-2}(x) + 2\Lambda^{K(2N_c+2-2l)} m^{2Kl} \right]^2 - 4\Lambda^{2K(2N_c+2-2l)} (x^2 - m^2)^{2Kl}$$

  $$= 4\Lambda^{2K(2N_c+2-2l)} (x^2 - m^2)^{2Kl} \left[ \frac{x^2 P_{K(2N_c+2)-2}(x) + 2\Lambda^{K(2N_c+2-2l)} m^{2Kl}}{2(\eta \Lambda^{(2N_c+2-2l)} (x^2 - m^2)^l)^K} \right]^2 - 1 \quad \text{(B.4)}$$

\(^{14}\)When we put $l = 0$ (flavorless case) into (B.2), then it becomes $\bar{x} = \frac{x^2 P_{2N_c}(x)}{2(\eta \Lambda^{2N_c+2-2l})^{1/2}} + 1$ which appeared in the pure case [10].

\(^{15}\)Note that the $\eta$ here corresponds to $\eta^2$ in [10].
Then we substitute the solution (B.3) by squaring of it into the left hand side of the above

\[ 4\Lambda^{2K(2N_c+2-2l)}(x^2 - m^2)^{2Kl} \left[ T^2_K(\bar{x}) - 1 \right] \]

\[ = 4\Lambda^{2K(2N_c+2-2l)}(x^2 - m^2)^{2Kl}U_{K-1}^2(\bar{x}) \left[ \left( \frac{x^2P_{2N_c}(x) + 2\eta\Lambda^{(2N_c+2-2l)m^2l}}{2\eta^{\Lambda^{(2N_c+2-2l)}}(x^2 - m^2)^l} \right)^2 - 1 \right] \]

after applying the property of Chebyshev polynomial \(^{16}\) and plugging the expression of \(\bar{x}\) (B.2). Finally we obtain the factorization of a gauge group with higher rank, by using the information for the factorization of a gauge group with lower rank (B.1),

\[ \left[ (\eta\Lambda^{(2N_c+2-2l)}(x^2 - m^2)^l)^{K-1}U_{K-1}(\bar{x})H_{2N_c+2-p}(x) \right]^2 f_{2p}(x) \]  

(B.5)

with the identification \(\eta\Lambda^{(2N_c+2-2l)} = \Lambda_0^{(2N_c+2-2l)}\). The new solutions satisfy the relation by putting the left hand side of (B.4) to be equal to the equation (B.5)

\[ \left[ x^2P_{K(2N_c+2)-2l}(x) + 2\Lambda^{(2N_c+2-2l)}m^{2Kl} \right]^2 - 4\Lambda^{2K(2N_c+2-2l)}(x^2 - m^2)^{2Kl} \]

\[ = H_{K(2N_c+2)-p}^2(x)f_{2p}(x) \]

where \(H_{K(2N_c+2)-p}(x) \equiv (\eta\Lambda^{(2N_c+2-2l)}(x^2 - m^2)^l)^{K-1}U_{K-1}(\bar{x})H_{2N_c+2-p}(x)\).

It is easy to check the degree of this polynomial really gives \(K(2N_c+2) - p\) by adding the powers of \(x\). Again we see that \(USp(2N_c)\) with \(2l\) flavors is mapped to \(USp(2KN_c+2K-2)\) with \(2Kl\) flavors. The vacua constructed in this way for the \(USp(2KN_c+2K-2)\) with \(2Kl\) flavors have the same superpotential as the vacua of the \(USp(2N_c)\) with \(2l\) flavors because they have the common function \(f_{2p}(x)\). If we define \(2N'_c = 2KN_c+2K-2\), we find \(2N'_c + 2 = K(2N_c+2)\), implying that under the multiplication map, the combination \((2N_c+2)\) has simple multiplication by \(K\), as shown in the pure adjoint case \([10]\). For example, when \(N_c = 2\) and \(K = 2\), the information on the \(USp(4)\) gauge theory with flavors will give \(USp(2KN_c+2K-2 = 10)\) theory with flavors which we will not discuss in this paper. However, in future studies we will develop the explicit application of this multiplication map, as in \([10]\).

References

[1] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” Nucl. Phys. B 644, 3 (2002), hep-th/0206255.

[2] R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” Nucl. Phys. B 644, 21 (2002), hep-th/0207106.

\(^{16}\)We use a useful relation \(T^2_K(\bar{x}) - 1 = (\bar{x}^2 - 1)U_{K-1}(\bar{x})\).
[3] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," hep-th/0208048.

[4] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP 0212, 071 (2002), hep-th/0211170.

[5] F. Cachazo, N. Seiberg and E. Witten, "Phases of N = 1 supersymmetric gauge theories and matrices," JHEP 0302, 042 (2003), hep-th/0301006.

[6] T. Friedmann, "On the quantum moduli space of M theory compactifications," Nucl. Phys. B 635, 384 (2002), hep-th/0203256.

[7] F. Ferrari, "Quantum parameter space and double scaling limits in N=1 super Yang-Mills theory," Phys. Rev. D 67, 085013 (2003), hep-th/0211069.

[8] F. Ferrari, "Quantum parameter space in super Yang-Mills. II," Phys. Lett. B 557, 290 (2003), hep-th/0301157.

[9] R. Casero and E. Trincherini, "Phases and geometry of the N = 1 A(2) quiver gauge theory and matrix models," hep-th/0307054.

[10] C. Ahn and Y. Ookouchi, "Phases of N = 1 supersymmetric SO / Sp gauge theories via matrix model," JHEP 0303, 010 (2003), hep-th/0302150.

[11] V. Balasubramanian, B. Feng, M. x. Huang and A. Naqvi, "Phases of N=1 supersymmetric gauge theories with flavors," hep-th/0303065.

[12] C. Ahn, B. Feng and Y. Ookouchi, "Phases of N=1 SO(N(c)) gauge theories with flavors," hep-th/0306068.

[13] H. Fuji and Y. Ookouchi, "Confining phase superpotentials for SO/Sp gauge theories via geometric transition," JHEP 0302, 028 (2003), hep-th/0205301.

[14] F. Cachazo and C. Vafa, "N = 1 and N = 2 geometry from fluxes," hep-th/0206017.

[15] B. Feng, "Geometric dual and matrix theory for SO/Sp gauge theories," hep-th/0212010.

[16] Y. Ookouchi, "N = 1 gauge theory with flavor from fluxes," hep-th/0211287.

[17] M. Aganagic, K. Intriligator, C. Vafa and N. P. Warner, "The glueball superpotential," hep-th/0304271.

[18] F. Cachazo, "Notes on supersymmetric Sp(N) theories with an antisymmetric tensor," hep-th/0307063.
[19] P. C. Argyres and A. D. Shapere, “The vacuum structure of N=2 Super QCD with classical gauge groups,” Nucl. Phys. B 461, 437 (1996), hep-th/9509175.

[20] P. C. Argyres, M. Ronen Plesser and A. D. Shapere, “N = 2 moduli spaces and N = 1 dualities for SO(n(c)) and USp(2n(c)) super-QCD,” Nucl. Phys. B 483, 172 (1997), hep-th/9608129.

[21] T. Kitao, S. Terashima and S. K. Yang, “N = 2 curves and a Coulomb phase in N = 1 SUSY gauge theories with adjoint and fundamental matters,” Phys. Lett. B 399, 75 (1997), hep-th/9701009.

[22] S. Terashima and S. K. Yang, “Confining phase of N = 1 supersymmetric gauge theories and N = 2 massless solitons,” Phys. Lett. B 391, 107 (1997), hep-th/9607151.

[23] C. Ahn, “Confining phase of N = 1 Sp(N(c)) gauge theories via M theory fivebrane,” Phys. Lett. B 426, 306 (1998), hep-th/9712149.

[24] S. Terashima, “Supersymmetric gauge theories with classical groups via M theory fivebrane,” Nucl. Phys. B 526, 163 (1998), hep-th/9712172.

[25] G. Carlino, K. Konishi and H. Murayama, “Dynamical symmetry breaking in supersymmetric SU(n(c)) and USp(2n(c)) gauge theories,” Nucl. Phys. B 590, 37 (2000), hep-th/0005076.

[26] J. D. Edelstein, K. Oh and R. Tatar, “Orientifold, geometric transition and large N duality for SO/Sp gauge theories,” JHEP 0105, 009 (2001), hep-th/0104037.

[27] P.C. Argyres, M.R. Plesser and N. Seiberg, “The moduli space of vacua of N=2 SUSY QCD and duality in N=1 SUSY QCD,” Nucl. Phys. B 471, 159 (1996), hep-th/9603042.

[28] K. Intriligator and P. Pouliot, “Exact superpotentials, quantum vacua and duality in supersymmetric Sp(N(c)) gauge theories,” Phys. Lett. B 353, 471 (1995), hep-th/9505006.

[29] K. Hori, H. Ooguri and Y. Oz, “Strong coupling dynamics of four dimensional N=1 gauge theories from M theory five-brane,” Adv. Theor. Math. Phys. 1, 1 (1998), hep-th/9706082.

[30] T. Eguchi, K. Hori, K. Ito and S. K. Yang, “Study of N=2 superconformal field theories in 4 dimensions,” Nucl. Phys. B 471, 430 (1996), hep-th/9603002.

[31] F. Cachazo, K. A. Intriligator and C. Vafa, “A large N duality via a geometric transition,” Nucl. Phys. B 603, 3 (2001), hep-th/0103067.
[32] G. Bertoldi, B. Feng and A. Hanany, “The splitting of branes on orientifold planes,” JHEP 0204, 015 (2002), hep-th/0202090.

[33] A. Hanany and E. Witten, “Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,” Nucl. Phys. B 492, 152 (1997), hep-th/9611230.

[34] A. Giveon and D. Kutasov, “Brane dynamics and gauge theory,” Rev.Mod.Phys. 71, 983 (1999), hep-th/9802067.