QUADRATIC GRÖBNER BASES ARISING FROM PARTIALLY ORDERED SETS

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Abstract
The order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ associated to a partially ordered set $P$ are studied. In this paper, we introduce the convex polytope $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ which is the convex hull of $\mathcal{O}(P) \cup (-\mathcal{C}(Q))$, where both $P$ and $Q$ are partially ordered sets with $|P| = |Q| = d$. It will be shown that $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is a normal and Gorenstein Fano polytope by using the theory of reverse lexicographic squarefree initial ideals of toric ideals.

introduction
A convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is integral if all vertices belong to $\mathbb{Z}^d$. An integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is normal if, for each integer $N > 0$ and for each $\mathbf{a} \in N\mathcal{P} \cap \mathbb{Z}^d$, there exist $\mathbf{a}_1, \ldots, \mathbf{a}_N \in \mathcal{P} \cap \mathbb{Z}^d$ such that $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_N$, where $N\mathcal{P} = \{N\alpha \mid \alpha \in \mathcal{P}\}$. Furthermore, an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is Fano if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$. A Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ is Gorenstein if its dual polytope $\mathcal{P}^\vee := \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{y} \in \mathcal{P}\}$ is integral as well. A Gorenstein Fano polytope is also said to be a reflexive polytope.

In recent years, the study of Gorenstein Fano polytopes has been more vigorous. It is known that Gorenstein Fano polytopes correspond to Gorenstein Fano varieties, and they are related with mirror symmetry (see, e.g., [1], [2]). On the other hand, to find new classes of Gorenstein Fano polytopes is one of the most important problem.

As a way to construct normal Gorenstein Fano polytopes, taking the centrally symmetric configuration [11] of an integer matrix is a powerful tool. In [11], it is shown that, for any matrix $A$ with $\text{rank}(A) = d$ such that all nonzero maximal minors of $A$ are $\pm 1$, the integral convex polytope arising from the centrally symmetric configuration of $A$ is normal Gorenstein Fano. Moreover, in [12], a way to construct non-symmetric normal Gorenstein Fano polytopes is introduced. In this paper, we treat the integral convex polytopes arising from combining the order polytopes and chain polytopes associated with two partially ordered sets.

Let $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets (posets, for short) with $|P| = |Q| = d$. A subset $I$ of $P$ is called a poset ideal of $P$ if $p_i \in I$ and $p_j \in P$ together with $p_j \leq p_i$ guarantee $p_j \in I$. Note that the empty set $\emptyset$ as well as $P$ itself is a poset ideal of $P$. Let $\mathcal{J}(P)$ denote the set of poset ideals of $P$. 
A subset $A$ of $Q$ is called an antichain of $Q$ if $q_i$ and $q_j$ belonging to $A$ with $i \neq j$ are incomparable. In particular, the empty set $\emptyset$ and each 1-element subsets $\{q_j\}$ are antichains of $Q$. Let $\mathcal{A}(Q)$ denote the set of antichains of $Q$.

Let $e_1, \ldots, e_d$ the canonical unit coordinate vectors of $\mathbb{R}^d$. Then, for each subset $I \subset P$ and for each subset $J$ of $Q$, we define the $(0,1)$-vectors $\rho(I) = \sum_{i \in I} e_i$ and $\rho(J) = \sum_{q_j \in J} e_j$, respectively. In particular $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. Recall that the order polytope $\mathcal{O}(P)$ [13, Definition 1.1] is the convex hull of $\{\rho(I) \mid I \in \mathcal{J}(P)\}$ and the chain polytope $\mathcal{C}(Q)$ ([13, Definition 2.1]) is the convex hull of $\{\rho(A) \mid A \in \mathcal{J}(Q)\}$.

Now, we define the convex polytope $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ as the convex hull of $\mathcal{O}(P) \cup (-\mathcal{C}(Q))$, where $-\mathcal{C}(Q) = \{-\beta \mid \beta \in \mathcal{C}(Q)\}$. This is a kind of $(-1,0,1)$-polytope, that is, each of its vertices belongs to $\{-1,0,1\}^d$. Note that $\dim \Gamma(\mathcal{O}(P), -\mathcal{C}(Q)) = d$. Moreover, since $\rho(P) = e_1 + \cdots + e_d \in \mathcal{O}(P)$ and $\rho(\{q_j\}) = e_j \in \mathcal{C}(Q)$ for $1 \leq j \leq d$, we have that the origin $0$ of $\mathbb{R}^d$ belongs to the interior of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$.

$\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is an integral convex polytope arising from combining the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(Q)$ associated with two posets $P$ and $Q$. We consider the question when such polytope is normal Gorenstein Fano. This kind of question has been studied. It is known that $\Gamma(\mathcal{O}(P), -\mathcal{O}(P))$ is normal Gorenstein Fano for any poset $P$ [8] and $\Gamma(\mathcal{O}(P), -\mathcal{O}(Q))$ is normal Gorenstein Fano if $P$ and $Q$ possess a common linear extension [7]. Moreover, it is shown that $\Gamma(\mathcal{C}(P), -\mathcal{C}(Q))$ is normal Gorenstein Fano for any posets $P$ and $Q$ [12].

In this paper, we prove that $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ is normal Gorenstein Fano for any posets $P$ and $Q$ by using the theory of reverse lexicographic squarefree initial ideals of toric ideals. For fundamental materials on Gröbner bases and toric ideals, see [3].

1. Squarefree Quadratic Gröbner basis

Let, as before, $P = \{p_1, \ldots, p_d\}$ and $Q = \{q_1, \ldots, q_d\}$ be finite partially ordered sets with the same cardinality. For a poset ideal $I \in \mathcal{J}(Q)$, we write $\max(I)$ for the set of maximal elements of $I$. In particular, $\max(I)$ is an antichain of $Q$. Note that for each antichain $A$ of $Q$, there exists a poset ideal $I$ of $Q$ such that $A = \max(I)$.

Let $K[t, t^{-1}, s] = K[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}, s]$ denote the Laurent polynomial ring in $2d + 1$ variables over a field $K$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, then $t^{\alpha}$ is the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d}$ in $K[t, t^{-1}, s]$. In particular $t^0 = s$. We define the toric ring of $\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))$ as the subring $K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ of $K[t, t^{-1}, s]$ which is generated by those Laurent monomials $t^\alpha$ with $\alpha \in \Gamma(\mathcal{O}(P), -\mathcal{C}(Q)) \cap \mathbb{Z}^d$. Let

$$K[x, y, z] = K[\{x_I\}_{\emptyset \neq I \in \mathcal{J}(P)} \cup \{y_{\max(J)}\}_{\emptyset \neq J \in \mathcal{J}(Q)} \cup \{z\}]$$

denote the polynomial ring over $K$ and define the surjective ring homomorphism $\pi : K[x, y, z] \to K[\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))]$ by the following:

- $\pi(x_I) = t^{\rho(I)}s$, where $\emptyset \neq I \in \mathcal{J}(P)$;
- $\pi(y_{\max(J)}) = t^{-\rho(\max(J))}s$, where $\emptyset \neq J \in \mathcal{J}(Q)$;
\[ \pi(z) = s. \]

Then the toric ideal \( I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))} \) of \( \Gamma(\mathcal{O}(P), -\mathcal{C}(Q)) \) is the kernel of \( \pi \).

Let \( < \) denote a reverse lexicographic order on \( K[x, y, z] \) satisfying

- \( z < y_{\max(J)} < x_I; \)
- \( x_{I'} < x_I \) if \( I' \subset I; \)
- \( y_{\max(J')} < y_{\max(J)} \) if \( J' \subset J; \)

and \( \mathcal{G} \) the set of the following binomials:

- (i) \( x_I x_{I'} - x_{I \cup I'} x_{I \cap I'}; \)
- (ii) \( y_{\max(J)} y_{\max(J')} - y_{\max(J \cup J')} y_{\max(J \cup J')}; \)
- (iii) \( x_I y_{\max(J)} - x_{I \setminus \{p_i\}} y_{\max(J) \setminus \{q_i\}}; \)

where

- \( x_\emptyset = y_\emptyset = z; \)
- \( I \) and \( I' \) are poset ideals of \( P \) which are incomparable in \( \mathcal{J}(P); \)
- \( J \) and \( J' \) are poset ideals of \( Q \) which are incomparable in \( \mathcal{J}(Q); \)
- \( J * J' \) is the poset ideal of \( Q \) generated by \( \max(J \cap J') \cap (\max(J) \cup \max(J')); \)
- \( p_i \) is a maximal element of \( I \) and \( q_i \) is a maximal element of \( J. \)

First, we have

**Theorem 1.1.** Work with the same situation as above. Then \( \mathcal{G} \) is a Gröbner basis of \( I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))} \) with respect to \( <. \)

**Proof.** It is clear that \( \mathcal{G} \subset I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))} \). For a binomial \( f = u - v, \) \( u \) is called the first monomial of \( f \) and \( v \) is called the second monomial of \( f. \) We note that the initial monomial of each of the binomials (i) – (iii) with respect to \( < \) is its first monomial. Let \( \text{in}_<(\mathcal{G}) \) denote the set of initial monomials of binomials belonging to \( \mathcal{G}. \) It follows from [10] (0.1)] that, in order to show that \( \mathcal{G} \) is a Gröbner basis of \( I_{\Gamma(\mathcal{O}(P), -\mathcal{C}(Q))} \) with respect to \( <, \) we must prove the following assertion: (♠) If \( u \) and \( v \) are monomials belonging to \( K[x, y, z] \) with \( u \neq v \) such that \( u \notin \langle \text{in}_<(\mathcal{G}) \rangle \) and \( v \notin \langle \text{in}_<(\mathcal{G}) \rangle \), then \( \pi(u) \neq \pi(v). \)

Let \( u, v \in K[x, y, z] \) be monomials with \( u \neq v. \) Write

\[ u = z^{\alpha} x_{I_1}^{\xi_1} \cdots x_{I_a}^{\xi_a} y_{\max(J_1)}^{\nu_1} \cdots y_{\max(J_b)}^{\nu_b}, \quad v = z^{\alpha'} x_{I_1'}^{\xi_1'} \cdots x_{I_a'}^{\xi_a'} y_{\max(J_1')}^{\nu_1'} \cdots y_{\max(J_b')}^{\nu_b'}, \]

where

- \( \alpha \geq 0, \alpha' \geq 0; \)
- \( I_1, \ldots, I_a, I_1', \ldots, I_a' \in \mathcal{J}(P) \setminus \{\emptyset\}; \)
- \( J_1, \ldots, J_b, J_1', \ldots, J_b' \in \mathcal{J}(Q) \setminus \{\emptyset\}; \)
- \( \xi_1, \ldots, \xi_a, \nu_1, \ldots, \nu_b, \xi_1', \ldots, \xi_a', \nu_1', \ldots, \nu_b' > 0, \)
and where \( u \) and \( v \) are relatively prime with \( u \not\in \langle \text{in}_<(G) \rangle \) and \( v \not\in \langle \text{in}_<(G) \rangle \). Note that either \( \alpha = 0 \) or \( \alpha' = 0 \). Hence we may assume that \( \alpha = 0 \). Thus

\[
\begin{align*}
u &= z^\alpha x_1^{\xi_1} \cdots x_n^{\xi_n} y_1^{\nu_1} \cdots y_m^{\nu_m}, \quad v = x_1^{\xi_1'} \cdots x_n^{\xi_n'} y_1^{\nu_1'} \cdots y_m^{\nu_m'}.
\end{align*}
\]

By using (i) and (ii), it follows that

- If \( I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_a \) and \( J_1 \subsetneq J_2 \subsetneq \cdots \subsetneq J_b \);
- If \( I_1' \subsetneq I_2' \subsetneq \cdots \subsetneq I_{a'}' \) and \( J_1' \subsetneq J_2' \subsetneq \cdots \subsetneq J_{b'}' \).

Furthermore, by virtue of \([3]\) and \([4]\), it suffices to discuss \( u \) and \( v \) with \( (a, a') \neq (0, 0) \) and \( (b, b') \neq (0, 0) \).

Since \( I_a \neq I_{a}' \), we may assume that \( I_a \setminus I_{a}' \neq \emptyset \). Then there exists a maximal element \( p_i \) of \( I_a \) with \( p_i \notin I_{a}' \).

Now, suppose that \( \pi(u) = \pi(v) \). Then we have

\[
\sum_{I \in (I_1, \ldots, I_a)} \xi_I - \sum_{J \in (J_1, \ldots, J_b)} \nu_J = \sum_{I' \in (I_1', \ldots, I_{a'})} \xi_{I'} - \sum_{J' \in (J_1', \ldots, J_{b'})} \nu_{J'},
\]

for all \( 1 \leq i \leq d \) by comparing the degree of \( t_i \). Since \( p_i \notin I_{a}' \), one has

\[
\sum_{J \in (J_1, \ldots, J_b)} \nu_J = - \sum_{J' \in (J_1', \ldots, J_{b'})} \nu_{J'} \leq 0.
\]

Moreover, since \( p_i \) is belonging to \( I_a \), we also have

\[
\sum_{I \in (I_1, \ldots, I_a)} \xi_I > 0.
\]

Hence there exists an integer \( c \) with \( 1 \leq c \leq b \) such that \( q_i \) is a maximal element of \( J_c \). Therefore we have \( x_{I_a} y_{\max(J_c)} \in \langle \text{in}_<(G) \rangle \), but this is a contradiction.

This theorem guarantees that the toric ideal \( I_\Gamma(O(P), -C(Q)) \) possesses a squarefree initial ideal with respect to a reverse lexicographic order for which the variable corresponding to the column vector \([0, \ldots, 0, 1]^t\) is smallest. Therefore, we have the following corollary by using \([3]\) Lemma 1.1).

**Corollary 1.2.** For any partially ordered sets \( P \) and \( Q \) with \( |P| = |Q| = d \), \( \Gamma(O(P), -C(Q)) \) is a normal Gorenstein Fano polytope.

### 2. Example and remark

As the end of this paper, we give some examples.

Let \( \mathcal{P} \subset \mathbb{R}^d \) be an integral convex polytope of dimension \( d \). Given integers \( t = 1, 2, \ldots \), let \( i(\mathcal{P}, t) := \#(t\mathcal{P} \cap \mathbb{Z}^d) \). It is known that \( i(\mathcal{P}, t) \) is a polynomial in \( t \).
of degree $d$. This polynomial is called the Ehrhart polynomial of $P$. The generating function of $i(P, t)$ satisfies

$$1 + \sum_{t=1}^{\infty} i(P, t)\lambda^t = \frac{\delta_P(\lambda)}{(1 - \lambda)^d + 1}$$

where $\delta_P(\lambda) = \sum_{i=0}^{d} \delta_i \lambda^i$ is a polynomial in $t$ of degree $\leq d$. Then the vector $(\delta_0, \ldots, \delta_d)$ is called the $\delta$-vector of $P$. It is known that a Fano polytope $P \subset \mathbb{R}^d$ is Gorenstein if and only if $\delta_i = \delta_{d-i}$ for all $i = 0, 1, \ldots, d$ (See [3]).

**Example 2.1.** Let $P = \{p_1 < p_2 < \cdots < p_d\}$ and $Q = \{q_i_1 < q_i_2 < \cdots < q_i_d\}$ be chains $(1 \leq i_1, \ldots, i_d \leq d)$ and $(\delta_0, \ldots, \delta_d)$ be the $\delta$-vector of $\Gamma(O(P), -C(Q))$. By [3, Corollary 1.5], we have $i(\Gamma(O(P), -C(Q)), t) = i(\Gamma(C(Q), -C(P)), t)$, where $\Gamma(C(Q), -C(P))$ is the convex hull of $C(Q) \cup (-C(P))$. Since $\Gamma(C(Q), -C(P))$ is the same as the convex hull of the centrally symmetric configuration of $d \times d$ identity matrix, hence $\delta_P(\lambda) = (1 + \lambda)^d$. In particular, $\delta_i = \binom{d}{i}$ for all $i = 0, 1, \ldots, d$.

On the other hand, the convex polytope $\Gamma(O(P), -O(Q))$ which is the convex hull of $O(P) \cup (-O(Q))$ is not Gorenstein Fano [7, Lemma 1.1]. Indeed, the $\delta$-vector of $\Gamma(O(P), -O(Q))$ is $(1, d, 0, \ldots, 0)$.

**Example 2.2.** Let $P = \{p_1 < p_2 < p_3\}$ be a partially ordered set and $Q = \{q_1, q_2, q_3, q_4\}$ be a partially ordered set such that the shape of its Hasse diagram is the same as that of $P$. If $Q = \{q_1 < q_3, q_2 < q_4\}$ then the $\delta$-vector of $\Gamma(O(P), -C(Q))$ is $(1, 12, 28, 12, 1)$. On the other hand, if $Q = \{q_1 < q_2, q_3 < q_4\}$ or $\{q_1 < q_4, q_2 < q_3\}$, then the $\delta$-vector of $\Gamma(O(P), -C(Q))$ is $(1, 12, 26, 12, 1)$.

**Remark 2.3.** We proved that the convex polytope $\Gamma(C(P), -C(Q))$ is normal Gorenstein Fano for all partially ordered sets $P, Q$ with $|P| = |Q|$ ([3, Corollary 1.3]). Moreover, we proved that the Ehrhart polynomial of $\Gamma(O(P), -C(Q))$ is the same as that of $\Gamma(C(P), -C(Q))$ for all partially ordered sets $P, Q$ with $|P| = |Q|$. In addition, if $P$ and $Q$ possess a common linear extension, these polytopes $\Gamma(O(P), -O(Q))$, $\Gamma(C(Q), -C(P))$, and $\Gamma(C(P), -C(Q))$ have the same Ehrhart polynomial ([3, Theorem 1.1]).

One of the future problem is to determine the $\delta$-vectors of the above polytopes in terms of partially ordered sets $P$ and $Q$.

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