Erdős-Rényi laws for exponentially and polynomially mixing dynamical systems.

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Abstract

Erdős-Rényi limit laws give the length scale of a time-window over which time-averages in Birkhoff sums have a non-trivial almost-sure limit. We establish Erdős-Rényi type limit laws for Hölder observables on dynamical systems modeled by Young Towers with exponential and polynomial tails. This extends earlier results on Erdős-Rényi limit laws to a broad class of dynamical systems with some degree of hyperbolicity.

1 Introduction

The Erdős-Rényi fluctuation law gives the length scale of a time-window over which time-averages in Birkhoff sums have a non-trivial almost-sure limit. It was first proved in the independent and identically distributed (i.i.d.) case [9] in the following form:

Proposition 1.1. Let $(X_n)_{n \geq 1}$ be an i.i.d. sequence of non-degenerate random variables, $E[X_1] = 0$, and let $S_n = X_1 + \cdots + X_n$. Assume that the moment generating function $\phi(t) = E(e^{tX_1})$ exists in some open interval $U \subset \mathbb{R}$ containing $t = 0$. For each $\alpha > 0$, define $\psi_\alpha(t) = \phi(t)e^{-\alpha t}$. For those $\alpha$ for which $\psi_\alpha(t)$ attains its minimum at a point $t_\alpha \in U$, let $c_\alpha = \alpha t_\alpha - \ln \phi(t_\alpha)$. Then

$$\lim_{n \to \infty} \max_{1 \leq j \leq n} \left\{ \frac{(S_j + \lfloor \ln n/c_\alpha \rfloor - S_j)/\lfloor \ln n/c_\alpha \rfloor}{\ln n/c_\alpha} : 1 \leq j \leq n - \lfloor \ln n/c_\alpha \rfloor \right\} = \alpha$$

The existence of $\psi_\alpha(t)$ for all $t \in U$ implies exponential large deviations with a rate function (in fact $c_\alpha = I(\alpha)$ where $I$ is the rate function, defined later) and this implies that sampling over a window length $k(n)$ of larger than logarithmic length scale (in the sense that $k(n)/\ln n \to \infty$), allows the ergodic theorem to kick in and

$$\lim_{n \to \infty} \max_{1 \leq j \leq n - k(n)} \left\{ \frac{(S_j + k(n) - S_j)/k(n) : 1 \leq j \leq n - k(n)}{k(n)} \} = 0$$

while sampling over too small a window, for example $k(n) = 1$, gives similarly a trivial limit

$$\lim_{n \to \infty} \max_{1 \leq j \leq n - k(n)} \left\{ \frac{(S_j + k(n) - S_j)/k(n) : 1 \leq j \leq n - k(n)}{k(n)} \} = \|X_1\|_\infty$$

Define the function

$$\theta(n, k(n)) := \max_{0 \leq j \leq n - k(n)} \frac{S_{j+k(n)} - S_j}{k(n)}$$

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which may be interpreted as the maximal average gain over a time window of length \( k(n) \) up to time \( n \). In the setting of coin tosses the Erdős-Rényi law gives precise information on the maximal average gain of a player in a fair game in the case where the length of the time window ensures \( \lim_{n \to \infty} \theta(n, k(n)) \) has a non-degenerate almost sure limit.

In 1986 Deheuvels, Devroye and Lynch [6] in the i.i.d. setting of Proposition 1.1 gave a precise rate of convergence and showed that if \( k(n) = [\ln n/c_\alpha] \) then \( P \) a.s:

\[
\limsup \frac{\theta(n, k(n)) - \alpha k(n)}{\ln k(n)} = \frac{1}{2t_\alpha}
\]

and

\[
\liminf \frac{\theta(n, k(n)) - \alpha k(n)}{\ln k(n)} = -\frac{1}{2t_\alpha}
\]

In this paper we establish Erdős-Rényi limit laws for Hölder observables on dynamical systems modeled by Young Towers [21, 22] with exponential and polynomial tails. Tails refer to the measure \( \mu(R > n) \) of the return time \( R \) function to the base of the tower. Our exposition is based upon [13] Section 2.3 and [15] who present a framework more general than that of the original Tower construction of Young [21] in that uniform contraction of local stable manifolds is not assumed for polynomially mixing systems in dimensions greater than 1. We will give more details on Young Towers below but here note that Hölder observables on Young Towers with exponential (polynomial) tails have exponential (polynomial) decay of correlations, the precise rate is encoded in the return time function.

Our results extends the work of [16] from the class of non-uniformly expanding maps with exponential decay of correlations to all systems modeled by a Young Tower, including Sinai dispersing billiard maps; diffeomorphisms of Henon type; polynomially mixing billiards as in [4] (as long as the correlation decay rate is greater than \( n^{-\beta}, \beta > 1 \)); smooth unimodal and multimodal maps satisfying the Collet-Eckmann conditions [13] Example 4.10; certain Viana maps [13] Example 4.11; and Lorenz-like maps. Other examples to which our results apply are listed in [15].

In the setting of hyperbolic dynamical systems there are many earlier results. Grigull [10] established the Erdős-Renyi law for hyperbolic rational maps, Chazottes and Collet [5] proved Erdős-Renyi theorems with rates for uniformly expanding maps of the interval, while Denker and Kabluchko [7] proved Erdős-Renyi results for Gibbs-Markov dynamics. In [8] Erdős-Rényi limit laws for Lipschitz observations on a class of non-uniformly expanding dynamical systems, including logistic-like maps, were given as well as related results on maximal averages of a time series arising from Hölder observations on intermittent-type maps over a time window of polynomial length. Kifer [11, 12] has established Erdős-Rényi laws for non-conventional ergodic sums and in the setting of averaging or homogenization of chaotic dynamical systems. We mention also recent related work of [2, 3] on applications of Erdős-Rényi limit laws to multifractal analysis.

The main novelty of our technique is the use of the symbolic metric on the axiomatic Young Tower construction of [15] to control the norm of the indicator function of sets of the form \( (S_n > na) \) on the quotiented tower. This eliminates many difficulties involved with considering the Lipschitz norm of such sets with respect to the Riemannian metric on the phase space of the system. The structure allows us to consider, with small error, averaged Birkhoff sums as being constant on stable manifolds, and thence use the decay of correlations for observables on the quotiented tower in terms of their Lipschitz and \( L^\infty \) norms.

Our results in the case of Young Towers with exponential decay of correlations, Theorem 5.1, are optimal and replicate the i.i.d case, while in the case of Young Towers with polynomial tails
we investigate windows of polynomial length and give close to optimal upper and lower bounds, Theorem 7.1 and Theorem 7.2.

2 Young Towers.

We now describe more precisely what we mean by a non-uniformly hyperbolic dynamical system modeled by a Young Tower. Our exposition is based upon [13, Section 2.3] and [15] who present a framework more general than that of the original Tower of Young [21] in that uniform contraction of local stable manifolds is not assumed for polynomially mixing systems in dimensions greater than 1. This set-up is very useful for the study of almost sure fluctuations of Birkhoff sums of bounded variables.

We suppose $T$ is a diffeomorphism of a Riemannian manifold $(M, d)$, possibly with singularities. Fix a subset $\Lambda \subset M$ with a ‘product structure’. Product structure means there exists a family of disjoint stable disks (sometimes called local stable manifolds) $\{W^s\}$ that cover $\Lambda$ as well as a family of disjoint unstable disks (sometimes called local unstable manifolds) $\{W^u\}$ that cover $\Lambda$. The stable and unstable disks containing $x \in \Lambda$ are denoted $W^s(x)$ and $W^u(x)$. Each stable disk intersects each unstable disk in precisely one point.

Suppose there is a partition $\{\Lambda_j\}$ of $\Lambda$ such that each stable disk $W^s(x)$ lies in $\Lambda_j$ if $x \in \Lambda_j$. Suppose there exists a ‘return time’ integer-valued function $R : \Lambda \to \mathbb{N}$, constant with value $R(j)$ on each partition element $\Lambda_j$, such that $T^{R(j)}(W^s(x)) \subset W^s(T^{R(j)}x)$ for all $x \in \Lambda_j$. We assume that the greatest common denominator of the integers $\{R(j)\}$ is 1, which ensures that the Tower is mixing. We define the induced return map $f : \Lambda \to \Lambda$ by $f(x) = T^{R(x)}(x)$.

For $x, y \in \Lambda$ let $s(x, y)$ be the least integer $n \geq 0$ such that $f^n(x)$ and $f^n(y)$ lie in different partition elements of $\Lambda$. We call $s$ the separation time with respect to the map $f : \Lambda \to \Lambda$.

**Assumptions:** there exist constants $K \geq 1$ and $0 < \beta_1 < 1$ such that

- (a) if $z \in W^s(x)$ then $d(f^n z, f^n x) \leq K \beta_1^n$;
- (b) if $z \in W^u(x)$ then $d(f^n z, f^n x) \leq K \beta_1^{s(x,z)}$;
- (c) if $z, x \in \Lambda$ then $d(T^j z, T^j x) \leq K (d(z, x) + d(f z, f x))$ for all $0 \leq j \leq \min\{R(z), R(x)\}$.

Define an equivalence relation on $\Lambda$ by $z \sim x$ if $z \in W^s(x)$ and form the quotient space $\overline{\Lambda} = \Lambda / \sim$ with corresponding partition $\{\Lambda_j\}$. The return time function $R : \overline{\Lambda} \to \mathbb{N}$ is well-defined as each stable disk $W^s(x)$ lies in $\Lambda_j$ if $x \in \Lambda_j$ and $T^{R(j)}(W^s(x)) \subset W^s(T^{R(j)}x)$ for all $x \in \Lambda_j$. So we have a well-defined induced map $\overline{f} : \overline{\Lambda} \to \overline{\Lambda}$. Suppose that $\overline{f}$ and the partition $\{\overline{\Lambda}_j\}$ separates points in $\overline{\Lambda}$. Define $d_{\beta_1}(z, x) = \beta_1^{s(z, x)}$, then $d_{\beta_1}$ is a metric on $\overline{\Lambda}$.

Let $m$ be a reference probability measure on $\Lambda$ (in most applications this will be normalized Lebesgue measure). Assume that $\overline{f} : \overline{\Lambda} \to \overline{\Lambda}$ is a Gibbs-Markov uniformly expanding on $(\overline{\Lambda}, d_{\beta_1})$. By this we mean that $\overline{f}$ is a measure-theoretic bijection from each $\overline{\Lambda}_j$ onto $\overline{\Lambda}$.

We assume that $\overline{f} : \overline{\Lambda} \to \overline{\Lambda}$ has an invariant probability measure $\overline{\nu}$ and $0 < a < \frac{d\nu}{dm} < b$ for some constants $a, b$. We assume that $R$ is $\overline{\nu}$-integrable and there is an $f$ invariant probability measure on $\Lambda$ such that $\overline{\nu} \circ f = \overline{\nu}$ where $\overline{\nu}$ is the quotient map taking $\Lambda$ onto $\Lambda / \sim$. Now we define the Young Tower

$$\Delta = \{x, j\} \in \Lambda \times \mathbb{N} : 0 \leq j \leq R(x) - 1\}$$

and the tower map $F$ by

$$F(x, j) = \begin{cases} (x, j + 1) & \text{if } j < R(x) - 1; \\ (f x, 0) & \text{if } j = R(x) - 1. \end{cases}$$

and lift $\nu$ in a standard way to an invariant probability measure $\nu_\Delta$ for $F : \Delta \to \Delta$. In fact $\nu_\Delta = \nu \times \text{counting measure}$. 
Define the semi-conjugacy $\pi : \Delta \to M$, $\pi(x,j) = T^j(x)$. The measure $\mu = \pi^* \nu_\Delta$ is a $T$-invariant mixing probability measure on $M$. Given an observable $\varphi : M \to \mathbb{R}$ we may lift to an observable $\varphi : \Delta \to \mathbb{R}$ by defining $\varphi(x,j) = \varphi(T^jx)$ (we keep the same notation for the observable). The semi-conjugacy $\pi^*$ allows us to transfer statistical properties from lifted observables $\varphi$ on $(\Delta, F, \nu_\Delta)$ to the original observables $\varphi$ on $(T, M, \mu)$.

3 Large deviations and rate functions.

Before stating precisely our main result we recall the definition of rate function and some other notions of large deviations theory. Suppose $(T, M, \mu)$ is a probability preserving transformation and $\varphi : M \to \mathbb{R}$ is a mean-zero integrable function i.e. $\int_M \varphi \, d\mu = 0$. Throughout this paper we will write $S_n(\varphi) := \varphi + \varphi \circ T + \ldots + \varphi \circ T^{n-1}$ for the $n$th ergodic sum of $\varphi$. Sometimes we will write $S_n$ instead of $S_n(\varphi)$ for simplicity of notation or when $\varphi$ is clear from context.

Definition 3.1. A mean-zero integrable function $\varphi : M \to \mathbb{R}$ is said to satisfy a large deviation principle with rate function $I(\alpha)$, if there exists a non-empty neighborhood $U$ of 0 and a strictly convex function $I : U \to \mathbb{R}$, non-negative and vanishing only at $\alpha = 0$, such that

$$\lim_{n \to \infty} \frac{1}{n} \log \mu(S_n(\varphi) \geq n\alpha) = -I(\alpha)$$

for all $\alpha > 0$ in $U$ and

$$\lim_{n \to \infty} \frac{1}{n} \log \mu(S_n(\varphi) \leq n\alpha) = -I(\alpha)$$

for all $\alpha < 0$ in $U$.

In the literature this is referred to as a first level or local (near the average) large deviations principle.

For H"older observables on Young Towers with exponential tails (which are not $L^1$ coboundaries in the sense that $\varphi \neq \psi \circ T - \psi$ for any $\psi \in L^1(\mu)$) such an exponential large deviations result holds with rate function $I_\varphi(\alpha)$ [16, 19, 14, 18]. A formula for the width of $U$ is given in [19] following a standard approach but it is not useful in concrete estimates.

4 Erdős-Rényi laws: background.

Proposition 4.1 given below is found in a proof from Erdős and Rényi [9] (see [?, Theorem 2.4.3], Grigull [10] Denker and Kabluchko [7] or [8] where this method has been used). The Gauss bracket $[.]$ denotes the integer part of a number. Throughout the proofs of this paper we will concentrate on the case $\alpha > 0$ as the case $\alpha < 0$ is identical with the obvious modifications of statements.

Proposition 4.1. Let $(T, M, \mu)$ be an ergodic dynamical system and $\varphi : M \to \mathbb{R}$ is an observable.

(a) Suppose that $\varphi$ satisfies a large deviation principle with rate function $I$ defined on the open set $U$ and assume $\mu(\varphi) = 0$ Let $\alpha > 0$, $\alpha \in U$ and set

$$L_n = L_n(\alpha) = \left\lfloor \frac{\ln n}{I(\alpha)} \right\rfloor \quad n \in \mathbb{N}.$$ 

Then the upper Erdős-Rényi law holds, that is, for $\mu$ a.e. $x \in X$

$$\limsup_{n \to \infty} \max_{0 \leq j \leq n - L_n} \frac{1}{L_n} S_{L_n}(\varphi) \circ T^j(x) \leq \alpha.$$
(b) If for some constant $C > 0$ and integer $\kappa \geq 0$ for each interval $A$

$$\mu \left( \bigcap_{m=0}^{n-L_n} \{ S_{L_n}(\varphi) \circ T^m \in A \} \right) \leq C[\mu(S_{L_n} \in A)]^{n/(L_n)^\kappa} \quad (3)$$

then the lower Erdős-Rényi law holds as well, that is, for $\mu$ a.e. $x \in X$

$$\liminf_{n \to \infty} \max_{0 \leq j \leq n-L_n} \frac{1}{L_n} S_{L_n}(\varphi) \circ T^j \geq \alpha.$$ 

Remark 4.2. If both Assumptions (a) and (b) of Proposition 4.1 hold then

$$\lim_{n \to \infty} \max_{0 \leq m \leq n-L_n} \frac{S_{L_n} \circ T^m}{L_n} = \alpha.$$ 

Remark 4.3. The proof of the lemma shows that the upper Erdős-Rényi law follows from the existence of exponential large deviations given by a rate function, while for the lower Erdős-Rényi law it suffices to show that for every $\epsilon > 0$ the series $\sum_{n>0} \mu(B_n(\epsilon))$, where $B_n(\epsilon) = \left\{ \max_{0 \leq m \leq n-L_n} S_{L_n} \circ T^m \leq L_n(\alpha - \epsilon) \right\}$ is summable. This is usually the harder part to prove in the deterministic case.

5 Erdős-Rényi limit laws for Young Towers with exponential tails.

We now state our main theorem in the case of exponential tails.

Theorem 5.1. Suppose $(T, M, \mu)$ is a dynamical system modeled by a Young Tower with $\nu_\Delta(R > j) \leq C\delta_2^j$ for some $\delta_2 \in (0, 1)$ and some constant $C_2$. Let $\varphi : M \to \mathbb{R}$ be Hölder with $\int \varphi \, d\mu = 0$. Assume $\varphi \neq \psi \circ T - \psi$ for any $\psi \in L^1(\mu)$. Let $I(\alpha)$ denote the non-degenerate rate function defined on an open set $U \subset \mathbb{R}$ containing 0. Define $S_n(x) = \sum_{j=0}^{n-1} \varphi(T^j x)$.

Let $\alpha > 0$, $\alpha \in U$ and define

$$L_n = L_n(\alpha) = \left\lceil \frac{\ln n}{I(\alpha)} \right\rceil \quad n \in \mathbb{N}.$$ 

Then

$$\lim_{n \to \infty} \max_{0 \leq j \leq n-L_n} \frac{S_{L_n} \circ T^j(x)}{L_n} = \alpha,$$

for $\mu$ a.e. $x \in \Omega$.

6 Proof of Theorem 5.1

We now give the proof of Theorem 5.1, beginning with some preliminary lemmas. Throughout this proof we will assume that $\varphi$ is Lipschitz, as the modification for Hölder $\varphi$ is straightforward.

The next lemma is not optimal but is useful in allowing us to go from uniform contraction along stable manifolds upon returns to the base of the Young Tower (Property (P3) of [21]) to estimates of the contraction along stable leaves in the whole manifold.

Lemma 6.1. Let $\beta_1$ be defined as in Section (2.1) Assumption (a) and $\beta_2$ be as in Theorem 2.2. Let $D(m) = \{(x, j) \in \Delta : |T^k W^s(x, j)| < (\sqrt{m})^k$ for all $k \geq m\}$. Then for any $\delta > 0$ there exists $K(\delta) > 0$ such that for all $m \geq K$, $\nu_\Delta(D(m)^c) \leq C(\beta_2 + \delta)^{m/2}$ for some constant $C > 0$. 5
Proof. Let \( \tau_r(x,j) := \#\{k : 1 < k \leq r : F^k(x,j) \in \Lambda \} \), so that \( \tau_r(x,j) \) denotes the number of times \( k \in [1,r] \) that \( F^k(x,j) \) lies in the base of the Young Tower. Let \( B_r = \{ (x,j) \in \Delta : \tau_r(x,j) \leq \sqrt{r} \} \). If \( \tau_r(x,j) \leq \sqrt{r} \) then there is at least one \( k \in [0,r] \), such that \( R(F^k(x,j)) > \sqrt{r} \) and hence \( B_r \subset \bigcup_{k=1}^{r} F^{-k}(R > \sqrt{r}) \). Thus \( \nu_{\Delta}(B_r) \leq r\nu(R > \sqrt{r}) < C_{2r}\beta_2^{\sqrt{r}} \).

Suppose now that \((x,j) \in B_r^c \). Then \( |T^rW_r((x,j))| \leq 2K\beta_1^{\sqrt{r}} \) by (a) and (c) and moreover \( \nu_{\Delta}(\bigcup_{r \geq m}(B_r)) \leq \sum_{r \geq m} C_{2r}\beta_2^{\sqrt{r}} \). Now the lemma follows from a straightforward calculation.

\[ \square \]

Corollary 6.2. Lift \( \varphi : M \to \mathbb{R} \) to \( \varphi : \Delta \to \mathbb{R} \) by defining \( \varphi(x,j) = \varphi(T^jx) \). Let \( \beta_1 \) be defined as in Section (2.1) (a). Suppose \( p \in D(m) = \{ (x,j) \in \Delta : |T^kW^s(x,j)| < (\sqrt{\beta_1})^k \text{ for all } k \geq m \} \) and let \( L_n = [\frac{\ln n}{\ln(t\gamma)}] \). Then if \( q \in W^s(p) \), \( |S_{L_n}\varphi \circ F^m(p) - S_{L_n}\varphi \circ F^m(q)| \leq C\|\varphi\|_\infty L_n\beta_1^{m/2} \).

Proof of Theorem 5.1. The main idea of the proof of Theorem 5.1 is to approximate functions on \( \Delta \) by functions constant on stable manifolds, so that correlation decay estimates on the quotiented tower from [13 Corollary 2.9] can be used.

We define an equivalence relation on \( \Lambda \) by \( z \sim x \) if \( z \in W^s(x) \) and form the quotient space \( \overline{\Lambda} = \Lambda/\sim \) with corresponding partition \( \{ \overline{\Lambda}_j \} \). The return time function \( R : \overline{\Lambda} \to \mathbb{N} \) is well-defined (and the same in the quotiented and unquotiented tower) as each stable disk \( W^s(x) \) lies in \( \Lambda_j \) if \( x \in \Lambda_j \) and \( T^{R(j)}(W^s(x)) \subset W^s(T^{R(j)}x) \) for all \( x \in \Lambda_j \). So we have a well-defined induced map \( \overline{f} : \overline{\Lambda} \to \overline{\Lambda} \). We similarly define the quotient space of \( \Delta \), denoted \( \overline{\Delta} \). The separation time for \( f : \Delta \to \overline{\Delta} \) extends to a separation time on \( \overline{\Delta} \) by defining

\[
s((x,l),(y,l')) = \begin{cases} 
  s(x,y) & \text{if } l = l' \\
  1 & \text{if } l \neq l'.
\end{cases}
\]

We fix \( \beta_1 \) from Section 2.1 Assumption (a) and define the metric \( d_{\beta_1} \) on \( \overline{\Delta} \) by \( d_{\beta_1}(p,q) = \beta_1^{s(p,q)} \). Here we write \( p = (x,l) \in \overline{\Delta} \), \( q = (y,l') \). We define the \( \| \cdot \|_{\beta_1} \)-norm by \( \| \phi \|_{\beta_1} := \| \phi \|_\infty + \sup_{p,q \in \Delta} \frac{|\phi(p)-\phi(q)|}{d_{\beta_1}(p,q)} \). Functions \( \phi \) and \( \psi \) which are constant on stable manifolds in \( \Delta \) naturally project to functions \( \overline{\phi} \) and \( \overline{\psi} \) (we use the same notation) on \( \overline{\Delta} \) with the same \( d_{\beta_1} \) Lipschitz constant and \( L^\infty \) norm. If \( \phi : \Delta \to \mathbb{R} \) is constant on stable manifolds we define the \( \| \cdot \|_{\beta_1} \)-norm by \( \| \phi \|_{\beta_1} := \| \phi \|_\infty + \sup_{p,q \in \Delta} \frac{|\phi(p)-\phi(q)|}{d_{\beta_1}(p,q)} \).

With this set-up the correlation estimate of [13 Corollary 2.9] can be stated:

Proposition 6.3. [13 Corollary 2.9]

Suppose that \( \phi, \psi : \Delta \to \mathbb{R} \) are constant on stable manifolds then for some constants \( C, \beta_3 \in (0,1) \),

\[
|\int_{\Delta} \phi(\psi \circ F^j) \, d\nu_{\Delta} - \int_{\Delta} \phi \, d\nu_{\Delta} \int_{\Delta} \psi \, d\nu_{\Delta}| \leq C\|\phi\|_{\beta_1} \|\psi\|_\infty \beta_3^j
\]

for all \( j \geq 0 \).

In the case that \( \varphi \) is not an \( L^1 \) coboundary i.e. there exists no \( \psi \) such that \( \varphi = \psi \circ T - \psi \), \( \psi \in L^1(m) \) it has been shown [16, 19] under the assumptions of Theorem 5.1 that \( \varphi \) has exponential large deviations with a rate function \( I(\alpha) \). Thus assumption (a) of Proposition 4.1 holds and we therefore only need to prove \( \mu(\{ \max_{0 \leq m \leq n \leq N} S_{L_n} \circ T^m \leq L_n(\alpha-\varepsilon) \}) \) is summable in order to get the lower bound by an application of the Borel-Cantelli lemma. This direction is more difficult and uses differential and dynamical information on the system.

For the reader’s convenience we recall our assumptions:

Assumptions: there exist constants \( K \geq 1 \) and \( 0 < \beta_1 < 1 \) such that

(a) if \( z \in W^s(x) \) then \( d(f^n z, f^n x) \leq K\beta_1^n \);
(b) if \( z \in W^u(x) \) then \( d(f^n z, f^n x) \leq K\beta_1^{s(x,z)-n} \);
(c) if $z,x \in \Lambda$ then $d(T^jz,T^jx) \leq K(d(z,x) + d(fz,fx))$ for all $0 \leq j \leq \min\{R(z),R(x)\}$. We lift $\varphi$ from $M$ to $\Delta$ by defining $\varphi(x,j) = \varphi(T^jx)$. We will use the same notation for $\varphi$ on $\Delta$ as we use for $\varphi$ on $M$.

To simplify notation we will sometimes write $p = (x,j)$ for a point $p \in \Delta$.

For $0 < \epsilon \ll \alpha$ put

$$A_n(\epsilon) := \{(x,j) \in \Delta : S_{\Delta} \leq L_n(\alpha - \epsilon)\},$$

where

$$S_{\Delta}(x,j) = \sum_{k=0}^{n-1} \varphi \circ F^k(x,j)$$
is the $\epsilon$th ergodic sum of $\varphi$. Define

$$B_n(\epsilon) = \bigcap_{m=0}^{n-L_n} F^{-m}A_n(\epsilon) = \left\{(x,j) \in \Delta : \max_{0 \leq m \leq n-L_n} S_{\Delta} \circ F^m \leq L_n(\alpha - \epsilon)\right\}.$$

The theorem follows by the Borel-Cantelli lemma once we show that $\sum_{n=1}^{\infty} \nu_{\Delta}(B_n(\epsilon)) < \infty$.

To do this we will use a blocking argument to take advantage of decay of correlations and intercalate by blocks of length $\kappa_n := \ln^n(n)$, where $\kappa$ will be specified later.

For $1 \leq j < r_n := \lceil \frac{2}{\kappa_n} \rceil$ put

$$E_n^j(\epsilon) := \bigcap_{m=1}^{j} F^{-m[k\kappa]}A_n(\epsilon)$$

which is a nested sequence of sets. Note that $\nu_{\Delta}(B_n(\epsilon)) \leq \nu_{\Delta}(E_{n}^{r_n}(\epsilon))$.

We also have the recursion

$$E_n^j(\epsilon) = A_n(\epsilon) \cap F^{-k\kappa}E_n^{j-1}(\epsilon)$$

$j = 1, \ldots, r_n$, which implies

$$\nu_{\Delta}(E_n^j(\epsilon)) = \nu_{\Delta}(A_n(\epsilon) \cap F^{-k\kappa}E_n^{j-1}(\epsilon))$$

Recall $D(m) = \{(x,j) \in \Delta : |T^kW^s(x)| < (\sqrt{\beta_2})^k \text{ for all } k \geq m\}$. Hence given $\delta > 0$ such that $\beta_\delta := \beta_2 + \delta < 1$ by Lemma 6.1 we may estimate $\nu_{\Delta}(D(\kappa)) \leq (\beta_\delta)^{\kappa n/2}$ for sufficiently large $n$.

Furthermore if $m \geq \kappa_n$, $p \in D(m)$ and $q \in W^s(p)$ then $|S_{\Delta} \circ F^m(p) - S_{\Delta} \circ F^m(q)| \leq C\|\varphi\|_{\infty}L_n\beta_\delta^{\kappa n/2}$ by the corollary to Lemma 6.1. We will take $\kappa$ and $n$ large enough that $C\|\varphi\|_{\infty}L_n\beta_\delta^{\kappa n/2} < \frac{\epsilon}{2}$.

Accordingly for large $n$ if $m \geq \kappa_n$, $p \in D(m) \cap F^{-m}A_n(\epsilon)$ and $q \in W^s(p)$ then $F^m q \in A_n(\frac{\epsilon}{2})$.

First Approximation.

We now approximate $1_{A_n(\epsilon)} \cap D(\kappa)$, by a function $g_n^\epsilon$ which is constant on stable manifolds by requiring that if $p \in A_n(\epsilon) \cap D(\kappa)$ then $g_n^\epsilon(p) = 1$ on $W^s(p)$ and $g_n^\epsilon = 0$ otherwise. Thus

$$\{g_n^\epsilon = 1\} \subset A_n(\frac{\epsilon}{2})$$

and

$$\nu_{\Delta}(g_n^\epsilon = 1) \leq \nu_{\Delta}(A_n(\frac{\epsilon}{2}))$$

Furthermore

$$A_n(\epsilon) \subset \{g_n^\epsilon = 1\} \cup D(\kappa)$$

hence

$$\nu_{\Delta}(A_n(\epsilon)) \leq \nu_{\Delta}(g_n^\epsilon = 1) + \nu_{\Delta}(D(\kappa)).$$
For $j = 1, \ldots, r_n$ let
\[
G_n^j(\epsilon) := \prod_{i=1}^j g_n^i \circ F_{i}[\kappa_n]
\]
and note $\nu(\beta^0_n(\epsilon)) \leq \nu(\beta^j_n(\epsilon)) + j\nu(\beta^j(\kappa_n \epsilon))$.

Second Approximation

We will approximate $g_n^i$ (considered as a function on $\Delta$) by a $d_{\beta_1}$ Lipschitz function $h_n^i$ which extends to a function on $\Delta$ by requiring $h_n^i$ to be constant on stable manifolds.

First define
\[
h_n^i(p) := \max\{0, 1 - d_{\beta_1}(p, \text{supp}(g_n^i))\beta_1^{-\sqrt{n\kappa}}\}
\]
on $\Delta$ and then extend so that it is constant on local stable manifolds and hence is a function on $\Delta$. In particular $h_n^i$ has support in points such that $d_{\beta_1}(p, \text{supp}(g_n^i)) \leq \beta_1^{-\sqrt{n\kappa}}$ and $\|h_n^i\|_{\beta_1} \leq \beta_1^\kappa$ by [20, Section 2.1].

By (b) and (c) if $z \in W^u(p)$ and $d_{\beta_1}(p, z) < \beta_1^{-\sqrt{n\kappa}}$ then $d(F^j p, F^j z) \leq 2K \beta_1^{-\sqrt{n\kappa} - L_n}$ for all $j \leq L_n$.

Hence if $d_{\beta_1}(z, \text{supp}(g_n^i)) \leq \beta_1^{-\sqrt{n\kappa}}$ then there exists $p \in \text{supp}(g_n^i)$ such that $d(F^j p, F^j z) \leq 2K \beta_1^{-\sqrt{n\kappa} - L_n}$ for all $j \leq L_n$ and hence
\[
\left| \sum_{j=0}^{L_n} [\varphi \circ F^j(z) - \varphi \circ F^j(p)] \right| \leq CL_n \beta_1^{-\sqrt{n\kappa} - L_n} \leq \epsilon/2
\]
for sufficiently large $n$. This implies that $\nu(\beta_n^i) \leq \nu(h_n^i) \leq \nu(\beta_n(A_n(\epsilon)))$.

As $h_n^i$ Lipschitz in the $d_{\beta_1}$ metric we obtain by Proposition 6.3
\[
\nu(\beta_n^i(\epsilon)) \leq \int_{\Delta} (G_n^i(\epsilon)) d\nu + j\nu(\beta(\kappa_n \epsilon))
\]
\[
\leq \int_{\Delta} (g_n^i \cdot G_n^{i-1} \circ F_{i} \cdot \kappa_n) d\nu + Cn(\beta_1)^{\kappa_n/2}
\]
\[
\leq \int_{\Delta} h_n^i d\nu + \int_{\Delta} G_n^{i-1}(\epsilon) d\nu + C_3 \beta^{\kappa_n} \|h_n^i\|_{\beta_1} \|G_n^{i-1}(\epsilon)\|_{\infty} + C_4(\beta_1)^{\kappa_n/2}
\]
\[
\leq \nu(\beta_n(A_n(\epsilon/2))) \nu(\beta_n^i(\epsilon)) + C_3 \beta^{\kappa_n} \beta_1^{-\sqrt{n\kappa}} + C_4(\beta_1)^{\kappa_n/2}.
\]
Iterating this estimate yields
\[
\nu(\beta_n^i(\epsilon)) \leq \nu(\beta_n(A_n(\epsilon/2)))^{[n/\kappa_n]} + nC_3 \beta_1^{-\sqrt{n\kappa}} + n^2 C \beta_1^{\kappa_n/2}.
\]
The terms $nC_3 \beta_1^{-\sqrt{n\kappa}}$ and $n^2 C \beta_1^{\kappa_n/2}$ are summable if we take $\kappa > 3$ in the definition of $\kappa_n$.

In order to verify summability of the $\nu(\beta_n(A_n(\epsilon/2)))^{[n/\kappa_n]}$ term we proceed as in the proof of Proposition 11 using large deviations. By the existence of a rate function we obtain $\nu(\beta_n(A_n(\epsilon/2))) \geq e^{-L_n(I(\alpha/2) + \delta_1)}$ for some $0 < \delta_1$ and hence $1 - \nu(\beta_n(A_n(\epsilon/2))) \geq e^{-L_n(I(\alpha/2) + \delta_1)}$ for some $0 < \delta_1$. Hence $\nu(\beta_n(A_n(\epsilon/2))) \leq 1 - n^{-\rho}$ where $\rho = I(\alpha/2) + \delta_1$ is less than 1 for $\delta_1 > 0$ small enough. The principal term can be bounded by
\[
\nu(\beta_n(A_n(\epsilon/2)))^{[n/\kappa_n]} \leq (1 - n^{-\rho})^{[n/\kappa_n]}
\]
which is also summable over $n$. Hence by Borel-Cantelli we conclude that the set $\{B_n(\epsilon) \text{ i.o.}\}$ has measure zero. This concludes the proof.
7 Erdös-Rényi laws for Young Towers with polynomial tails.

We now consider Young Towers with polynomial tails in the sense that \( \nu_\Delta(R > n) \leq C n^{-\beta} \).

7.1 Upper bounds.

We first prove a general result. We suppose that \( (T, M, \mu) \) is an ergodic dynamical system and \( \varphi : M \to \mathbb{R} \) is a bounded observable. We assume also

\[
\mu \left( \left| \frac{1}{n} S_n(\varphi) - \bar{\varphi} \right| > \epsilon \right) \leq C(\epsilon) n^{-\beta}.
\]

**Theorem 7.1.** Assume that \( \bar{\varphi} = \mu(\varphi) = 0 \), \( \varphi \) is bounded and for every \( \epsilon > 0 \) there exists a constant \( C(\epsilon) > 0 \) and \( \beta > 1 \) so that

\[
\mu \left( \left| \frac{1}{n} S_n(\varphi) \right| > \epsilon \right) \leq C(\epsilon) n^{-\beta}.
\]

Then if \( \tau > \frac{1}{\beta} \) for \( \mu \) a.e. \( x \in M \),

\[
\lim_{n \to \infty} \max_{0 \leq m \leq n - \tau} n^{-\tau} S_n \circ T^m(x) = 0.
\]

**Proof.** Choose \( \tau > \frac{1}{\beta} \) and put \( L_n = n^\tau \). Let \( \epsilon > 0 \) and define

\[
A_n := \{ x \in X : \max_{0 \leq m \leq n - L_n} |S_L \circ T^m| \geq L_n \epsilon \}.
\]

Then \( \mu(A_n) \leq n \mu(S_{L_n} \geq \epsilon L_n) \leq c_1(\delta)n^{1-\tau\beta} = c_1 n^{-\beta} \), for some \( c_1 > 0 \), where \( \delta = \tau \beta - 1 \).

Let \( p > \frac{1}{\beta} \) (i.e. \( \delta p > 1 \)) and consider the subsequence \( n = kp \). Since \( \sum_k \mu(A_{kp}) \leq c_1 \sum_k k^{-\beta p} < \infty \), we obtain via the Borel-Cantelli lemma that for \( \mu \) a.e. \( x \in X \)

\[
\limsup_{k \to \infty} \max_{0 \leq m \leq kp - L_{kp}} L_{kp}^{-1} |S_{kp} \circ T^m| \leq \epsilon.
\]

To fill the gaps use that \( kp - (k - 1)p = O(k^{p-1}) \) and we obtain (as \( \varphi \) is bounded) that

\[
\frac{S_{kp} \circ T^m}{L_{kp}} = \frac{S_{kp(k-1)p} \circ T^m}{L_{kp}} + O \left( \frac{1}{k} \right)
\]

where the implied constant is uniform in \( x \in X \) as \( \varphi \) is bounded. As \( \lim_{k \to \infty} \frac{k^p}{(k-1)^p} = 1 \) we conclude

\[
\lim_{k \to \infty} \left| \frac{S_{kp}}{L_{kp}} \right| = \lim_{k \to \infty} \left| \frac{S_{kp(k-1)p}}{L_{kp}} \right|.
\]

Since any \( n \in \mathbb{N} \) satisfies \( (k - 1)p \leq n \leq kp \) for some \( k \) and \( \varphi \) is bounded, it follows that

\[
\limsup_{n \to \infty} \max_{0 \leq m \leq n - L_n} \frac{|S_{L_n} \circ T^m|}{L_n} \leq \epsilon.
\]

As \( \epsilon \) was arbitrary this gives the upper bound. \( \square \)
7.2 Lower bounds.

Now we suppose there exists $\gamma \geq \beta$, an observable $\varphi$ and an $\alpha > 0$ such that for all $n$, 
$$\mu\left(\left|\frac{1}{n}S_n(\varphi) - \bar{\varphi}\right| > \alpha\right) \geq C(\alpha)n^{-\gamma}.$$ 
We show if we take a window of length $n^\tau$, $\tau < \frac{1}{\beta+1}$ then the time-averaged fluctuation persists almost surely. In the case that $\gamma$ limits to $\beta$ then we require $\tau < \frac{1}{2\beta+1}$. Comparing Theorem 7.1 and Theorem 7.2 there is a gap $\frac{1}{\beta+1} < \tau < \frac{1}{2\beta+1}$ for which we don’t know the almost sure limit of windows of length $n^\tau$. In Example 7.3 we show that $\tau < \frac{1}{\beta+1}$ is required to ensure that a time-averaged fluctuation persists almost surely.

Theorem 7.2. Suppose that $(T, M, \mu)$ is modeled by a Young Tower and $\nu_\Delta(R > n) \leq Cn^{-\beta}$. Suppose that $\gamma \geq \beta$ and there exists a function $C$ which is continuous on a neighborhood of $\alpha > 0$ such that

$$\mu\left(\frac{1}{n}S_n(\varphi) - \bar{\varphi} > \alpha\right) \geq C(\alpha)n^{-\gamma}$$

Then if $0 < \tau < \frac{1}{1+\gamma} - \frac{1}{\beta+1}$ for $\mu$ a.e. $x \in M$

$$\lim_{n \to \infty} \max_{0 \leq m \leq n^{-\tau}} n^{-\tau}S_n \circ T^m(x) \geq \alpha$$

Proof. Let $0 < \epsilon \ll \alpha$ and put

$$A_{n^\tau}(\epsilon) = \{(x, j) : \sum_{r=1}^{n^\tau} \varphi \circ F^r(x, j) \leq \alpha - \epsilon\}.$$ 

Since $\varphi$ is Lipschitz continuous with Lipschitz constant $L$, then if $y \in A_{n^\tau}(\epsilon)$ and $d(y, y') < \frac{\epsilon}{2Ln^\tau}$, then $y' \in A_{n^\tau}(\epsilon/2)$. Hence let us choose $n_1$ so that $K\beta^1_1 < \frac{\epsilon}{2Ln^\tau}$ and define

$$B_{n^\tau}(\epsilon) = \{(x, 0) \in \Lambda : \exists 0 \leq j < R(f^{n_1}x) \mbox{ with } (f^{n_1}x, j) \in A_{n^\tau}(\epsilon)\} = f^{-n_1}(\pi A_{n^\tau}(\epsilon)),$$

where $\pi : \Delta \to \Lambda$ is the projection given by $\pi((x, j)) = (x, 0)$ $(j < R(x))$. The choice of the integer achieves that if $(x, 0) \in B_{n^\tau}(\epsilon)$ and $(x', 0) \in W^s(x, 0)$ then $(f^{n_1}x', 0) \in \pi(A_{n^\tau}(\epsilon/2))$. This is a consequence of Assumption (a). By assumption

$$\nu_\Delta(A_{n^\tau}(\epsilon)) \geq C(\alpha - \epsilon)n^{-\gamma\tau}.$$ 

For $\delta > \frac{\gamma\tau}{\beta}$ we have

$$\nu_\Delta(R > n^\delta) = o(n^{-\delta})$$

as by assumption $\nu_\Delta(R > \ell) \leq C\ell^{-\beta}$.

Since $\nu_\Delta = \nu \times$ (counting measure) we get for $D \subset \Delta$

$$\bar{\nu}(\pi(D)) \geq \frac{\nu_\Delta(D) - \nu_\Delta(R > n^\delta)}{n^\delta}.$$ 

Consequently

$$\bar{\nu}(\pi(A_{n^\tau}(\epsilon))) \geq \left(C(\alpha - \epsilon)n^{-\gamma\tau} - o(n^{-\delta\beta})\right)n^{-\delta}$$

and since $\delta\beta > \gamma\tau$ the first term dominates and we obtain

$$\bar{\nu}(\pi(A_{n^\tau}(\epsilon))) \geq c_1n^{-\gamma\tau - \delta}$$

for some $c_1 > 0$ and since $f^{n_1}$ preserves $\bar{\nu}$,

$$\bar{\nu}(B_{n^\tau}(\epsilon)) \geq c_1n^{-\gamma\tau - \delta}.$$
We can now define
\[ \tilde{B}_{n^*}(\epsilon) = \bigcup_{x \in B_{n^*}(\epsilon)} W^*(x) \]
which by choice of \( n_1 \) implies that
\[ \tilde{B}_{n^*}(\epsilon) \subseteq B_{n^*}(\epsilon/2). \]
We now approximate \( 1_{B_{n^*}(\epsilon)} \) by a function \( h_{n^*}(\epsilon) \) which has Lipschitz constant \( \beta_1^{-n^*} \) in the \( d_{\beta_1} \)-norm, that is we define
\[ h_{n^*}(\epsilon)(p) = \max(0, 1 - d(p, B_{n^*}(\epsilon))\beta_1^{-n^*}) \]
where we write \( p \) for \((p, 0)\). We can choose \( n_1 \) to be much smaller than \( n^* \) and therefore, since by Assumption (b) and (c) if \( d(p, B_{n^*}(\epsilon)) < \beta_1^{-n_1} \) then \( d(f^{n_1} p, B_{n^*}(\epsilon)) < K \beta_1^{n_1-n_1} < \frac{K}{2\epsilon n^*} \) which implies that the support of \( h_{n^*}(\epsilon) \) is contained in \( B_{n^*}(\epsilon/2) \).

Now we let \( \tau_1 > \tau \) but \( \tau_1 - \tau < 1 - (\tau - \frac{\beta_1+1}{\beta}) \) and consider
\[ G_n(\epsilon) = \bigcap_{m=0}^{[n/\tau_1]} f^{-m\tau_1} B_{n^*}(\epsilon) \]
We will show that
\[ \sum_n \bar{\nu}(G_n(\epsilon)) < \infty \]
Now
\[
\bar{\nu}(G_n(\epsilon)) \leq \bar{\nu}\left(\prod_{m=0}^{n_1-\tau_1} h_{n^*}(\epsilon) \circ f^{m\tau_1}\right) \\
\leq \bar{\nu}(h_{n^*}(\epsilon))\bar{\nu}(G_{n-1}(\epsilon)) + c_3 \|h_{n^*}(\epsilon)\|_{\beta_1} \|G_{n-1}(\epsilon)\|_{\infty} \beta_3^{-\tau_1} \\
\leq \left(\bar{\nu}(h_{n^*}(\epsilon))\right)^{n_1-\tau_1} + nC_3 \beta_3^{-\tau_1} \beta_1^{-n^*} \]
The term \( nC_3 \beta_3^{-\tau_1} \beta_1^{-n^*} \) is summable in \( n \) as \( \tau_1 > \tau \). The principal term is estimated by
\[
[\bar{\nu}(h_{n^*}(\epsilon))]^{n_1-\tau_1} \leq \left(1 - C(\alpha - \frac{\epsilon}{2})n^{-\tau-\delta}\right)^{n_1-\tau_1} \leq \exp(-C(\alpha - \epsilon/2)n^{1-\tau_1-\tau-\delta}) \\
\]
Since \( \tau_1 > \tau \) can be chosen arbitrarily close to \( \tau \) and \( \delta > \frac{\tau_2}{\beta} \) can be chosen to achieve the power \( 1 - \tau - \tau\gamma - \delta \) is positive for any chosen \( \tau < (1 + \gamma \frac{\beta_1+1}{\beta})^{-1} \) we obtain that the principal terms are summable which implies summability of \( \bar{\nu}(G_n(\epsilon)) \).

Now define
\[ E_n := \{(x,0) : \text{for all } j < n : \sum_{r=0}^{n^*} \varphi(f^{R_{n^*}}(x)+j+x,0) \leq (\alpha - \frac{\epsilon}{2})n^*\}, \]
where \( R_l = \sum_{i=0}^{l-1} R \circ f^i \) is the \( l \)-th ergodic sum of \( R \). As \( E_n(\epsilon) \subseteq G_n(\epsilon), \bar{\nu}(G_n(\epsilon)) \) summable implies that \( \sum_{n=1}^{\infty} \bar{\nu}(E_n(\epsilon)) < \infty \).

By Birkhoff’s ergodic theorem
\[ \lim_{n \to \infty} \frac{R_n(x,j)}{n} = \bar{R} = \frac{1}{\nu_\Delta(\Lambda)} \]
for \( \nu_\Delta \) a.e. \((x,j) \in \Delta\), and so the theorem follows. \( \Box \)
Example 7.3. The condition $\tau < \frac{1}{1+\gamma \beta}$ is close to optimal in that, taking $\gamma = \beta$, we require $\tau < \frac{1}{2 \beta}$. We may construct a Young Tower and observable $\varphi$, $\int_{\Delta} \varphi \, d\nu_\Delta = 0$ and $\alpha > 0$ such that $\nu_\Delta(S_n, \varphi(x, j) \geq n^\gamma \alpha) \leq C n^{-\tau}$, yet for all $\tau > \frac{1}{\beta+1}$,

$$\lim_{n \to \infty} \max_{0 \leq m \leq n-n^\tau} n^{-\tau} S_{nm} \circ T^m = 0$$

We sketch the main idea of the tower and observable and make a couple of technical adjustments to ensure the tower is mixing and that the observable is not a coboundary. The construction is based on that of [1]. The base partition consists of disjoint intervals $\Lambda_i$ of length $i^{-\beta-2}$ and height $2i$. Above the base element $\Lambda_i$, the levels of the tower consist of $\{(x, j) : 0 \leq j \leq 2i - 1\}$. We define $\varphi$ on the Tower by, if $x \in \Lambda_i$,

$$\varphi(x, j) = \begin{cases} -1 & \text{if } 0 \leq j < i; \\ 1 & \text{if } i \leq j < 2i. \end{cases}$$

Clearly $\nu_\Delta(\varphi) = 0$.

Let $0 < \alpha < 1$. Note that $S_n, \varphi(x, j) \geq n^\gamma \alpha$ only if $(x, j) \in (R > n^\gamma)$ and in fact $\nu_\Delta(S_n, \varphi(x, j) \geq n^\gamma \alpha) \geq C \nu_\Delta(R > 2n^\gamma) = \sum_{j=2n^\gamma}^{\infty} (2j)^{1-2\gamma} \leq C n^{-\gamma\beta}$.

However if $\tau > \frac{1}{2 \beta+1}$ then $\sum_{j>n^\gamma} \nu(\Lambda_j) \leq \sum_{n=1}^{\infty} n^{-\tau(\beta+1)} < \infty$. Hence by the Borel-Cantelli lemma $f^n(x, 0) \in \bigcup_{j>n^\gamma} \Lambda_j$ only finitely many times for $\nu$ a.e. $(x, 0)$. This implies that for $\nu$ a.e. $(x, 0)$ there exists an $N(x)$ such that for all $n \geq N(x)$

$$\text{for all } j < n : \sum_{r=0}^{n^\gamma} \varphi(f^{j+r}x, 0) < \alpha n^\gamma.$$ 

Hence for $\mu$ a.e. $x \in M$

$$\lim_{n \to \infty} \max_{0 \leq m \leq n-n^\tau} n^{-\tau} S_{nm} \circ T^m < \alpha$$

for every $\alpha > 0$.

The same argument shows for $\nu_\Delta$ a.e. $(x, j)$

$$\lim_{n \to \infty} \max_{0 \leq m \leq n-n^\tau} n^{-\tau} S_{nm} \circ T^m = 0$$

and

$$\lim_{n \to \infty} \min_{0 \leq m \leq n-n^\tau} n^{-\tau} S_{nm} \circ T^m = 0$$

The heights of the levels in the tower above are all multiples of 2. Furthermore the observable $\varphi$ is a coboundary. If we define

$$\psi(x, j) = \begin{cases} j & \text{if } x \in \Lambda_k, 0 \leq j \leq k \\ 2k - j & \text{if } x \in \Lambda_k, k < j \leq 2k - 1. \end{cases}$$

It is easy to check that

$$\varphi = \psi \circ F - \psi$$

We will modify the tower and the observable so that the greatest common denominator of the return time function $R$ is 1 (to ensure the tower is mixing) and that the new observable is not a coboundary. We change $\Lambda_3$ to have height 3. This entails that the tower is mixing. On the levels above $\Lambda_3$ we modify $\varphi$ to $\varphi_1$ so that $\varphi_1(x, j) = \kappa > 0$, $j = 0, 1, 2$, $x \in \Lambda_3$ where $\kappa > 0$ is small but $\varphi_1 = \varphi$ elsewhere. This entails $r_1 := \nu_\Delta(\varphi_1) = \kappa \nu_\Delta(\Lambda_3) > 0$. We subtract $r_1/(\nu_\Delta(\Lambda_2))$ from the value of $\varphi_1$ on $\Lambda_2$ to form a new observable $\varphi_2$ such that $\nu_\Delta(\varphi_2) = 0$. Since $F$ has a fixed point $p$ on $\Lambda_3$ and since $\sum_{j=0}^{2} \varphi_2(x, j) \neq 0$ we conclude $\varphi_2$ is not a coboundary (by the Lišvic theorem [17]). The new tower with observable $\varphi_2$ we defined has the properties of the former pertinent to our example.
References

[1] Bryc, W and Smolenski, W. On the Convergence of Averages of Mixing Sequences. Journal of Theoretical Probability, Vol 6, (1993), No. 3.

[2] Chen, H and Yu, M. A generalization of the Erdős-Rényi limit theorem and the corresponding multifractal analysis. J. Number Theory 192 (2018), 307-327.

[3] Chen, H, Ding, D and Long, X. The Hausdorff dimension of level sets described by Erdős-Rényi average. J. Math. Anal. Appl. 458 (2018), no. 1, 464-480.

[4] Chernov, N and Zhang, H. K. A family of chaotic billiards with variable mixing rates. Stoch. Dyn. 5 (2005) 535-553.

[5] Chazottes, J.-R.; Collet, P.: Almost-sure central limit theorems and the Erdős-Rényi law for expanding maps of the interval. Ergodic Theory Dynam. Systems 25, (2005), 419–441.

[6] Deheuvels, P.; Devroye, L.; Lynch, J.: Exact convergence rate in the limit theorems of Erdős-Rényi and Shepp. Ann. Probab. 14 (1986), 209–223.

[7] Denker, M.; Kabluchko, Z.: An Erdős-Rényi law for mixing processes. Probab. Math. Statist. 27 (2007), 139–149.

[8] Denker, M and Nicol, M. Erdős-Rényi limit laws for hyperbolic dynamical systems. Journal LMS 87(2) (2013), 497–508.

[9] Erdős, P.; Rényi, A.: On a new law of large numbers. J. Analyse Math. 23 (1970), 103–111.

[10] Grigull, J.: Große Abweichungen und Fluktuationen für Gleichgewichtsmaße rationaler Abbildungen. Dissertation Georg-August-Universität zu Göttingen 1993.

[11] Kifer, Y. Functional Erdős-Rényi law of large numbers for nonconventional sums under weak dependence., Elect. J. Probab. 22 (2017), no.23, 1-17.

[12] Kifer,Y. Erdős-Rényi law of large numbers in the averaging setup., Stoch. Dyn. 18 (2018), no. 3, 1850018, 26 pp.

[13] A. Korepanov, Z. Kosloff and I. Melbourne. Explicit coupling argument for nonuniformly hyperbolic transformations. Proc. Roy. Soc. Edinburgh Sect. A 149 (2019), no. 1, 101-130.

[14] Melbourne, I.: Large and moderate deviations for slowly mixing dynamical systems. Proc. Amer. Math. Soc. 137 (2009), 1735–1741.

[15] Melbourne, I and Varandas, P. A note on statistical properties for non-uniformly hyperbolic systems with slow contraction and expansion. Stoch. Dyn. 16 (2016), no. 3, 1660012, 13 pp.

[16] Melbourne, I.; Nicol, M.: Large deviations for nonuniformly hyperbolic systems. Trans. Amer. Math. Soc. 360 (2008), 6661–6676.

[17] Nicol, M; Scott,A. Livšic theorems and stable ergodicity for group extensions of hyperbolic systems with discontinuities. Ergodic Theory Dynam. Systems 23 (2003), no. 6, 1867-1889.

[18] Pollicott, M; Sharp, R.: Large deviations for intermittent maps. Nonlinearity 22(9) (2009), 2079–2090.
[19] Rey-Bellet, L.; Young, L.-S.: Large deviations in nonuniformly hyperbolic dynamical systems. Ergodic Theory Dynam. Systems 28 (2008) 587–612.

[20] Stein, E: *Singular integrals and differentiability properties of functions*, Princeton University Press, 1986.

[21] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math.* 147 (1998), 585–650.

[22] L.-S. Young. Recurrence times and rates of mixing. *Israel J. Math.* 110 (1999), 153–188.