Painlevé property, local and nonlocal symmetries and symmetry reductions for a (2+1)-dimensional integrable KdV equation

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Abstract

The Painlevé property for a (2+1)-dimensional Korteweg-de Vries (KdV) extension, the combined KP3 (Kadomtsev-Petviashvili) and KP4 (cKP3-4) is proved by using Kruskal’s simplification. The truncated Painlevé expansion is used to find the Schwartz form, the Bäcklund/Levi transformations and the residual nonlocal symmetry. The residual symmetry is localized to find its finite Bäcklund transformation. The local point symmetries of the model constitute a centerless Kac-Moody-Virasoro algebra. The local point symmetries are used to find the related group invariant reductions including a new Lax integrable model with a fourth order spectral problem. The finite transformation theorem or the Lie point symmetry group is obtained by using a direct method.

Keywords: Painlevé property, residual symmetry, Schwartz form, Bäcklund transforms, D’Alembert waves, symmetry reductions, Kac-Moody-Virasoro algebra, (2+1)-dimensional KdV equation

1 Introduction

Recently, a novel (2+1)-dimensional Korteweg-de Vries (KdV) extension, the combined KP3 (Kadomtsev-Petviashvili) and KP4 (cKP3-4) equation

\begin{align}
  u_{xt} &= a [(6u u_x + u_{xxx})_x - 3u_{yy}] + b (2v u_x + v_{xxx} + 4u u_y)_x - v_{yy}, \\
  u_y &= v_x, \\
\end{align}

is proposed by one of the present authors (Lou) [1]. KdV equation [2] and its (2+1)-dimensional extensions such as the KP equation [3], the Nizhnik-Novikov-Veselov (NNV) equation [4, 5, 6], the asymmetric NNV equation (ANNV) [7, 8, 9] and the Ito equation are fundamental nonlinear integrable models in mathematical physics [10].

The Lax integrability of the cKP3-4 equation is guaranteed by the existence of the Lax pair [1] ($w_x = v_y$)

\begin{align}
  \psi_y &= i(\psi_{xx} + u \psi), \quad i \equiv \sqrt{-1}, \\
  \psi_t &= 2ib\psi_{xxxx} + 4a\psi_{xxx} + 4ibu\psi_{xx} + 2(3au + 2ibux + bv)\psi_x \\
  &\quad - i(3av + bw - 2bu^2 + 3aiux - 2bu_{xx} + ibvx)\psi, \\
\end{align}

and the dual Lax pair

\begin{align}
  \phi_y &= -i(\phi_{xx} + u \phi), \\
  \phi_t &= -2ib\phi_{xxxx} + 4a\phi_{xxx} - 4ibu\phi_{xx} + 2(3au - 2ibux + bv)\phi_x \\
  &\quad + i(3av + bw - 2bu^2 - 3aiux - 2bu_{xx} - ibvx)\phi.
\end{align}

*Corresponding author: loujyue@nbu.edu.cn. Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.
In Ref. [1], the multiple solitons of the model (1) are obtained by using Hirota’s bilinear approach. Applying the velocity resonant mechanism [12] [13] to the multiple soliton solutions, the soliton molecules with arbitrary number of solitons are also found in [1]. It is further discovered that the model permits the existence of the arbitrary D’Alembert type waves which implies that there are one special type of solitons and soliton molecules with arbitrary shapes but fixed model dependent velocity.

In this paper, we investigate other significant properties such as the Painlevé property (PP), Schwartz form, Bäcklund transformations, infinitely local and nonlocal symmetries, Kac-Moody-Virasoro symmetry algebras, group invariant solutions and symmetry reductions for the cKP3-4 equation [11]. To study the PP of a nonlinear partial differential equation system, there are some equivalent ways such as the Weiss-Tabor-Carnevale (WTC) approach [14], Kruskal’s simplification, Conte’s invariant form [15] and Lou’s extended method [16]. In the section 2 of this paper, the PP of (1) is tested by using the Kruskal’s simplification, Conte’s invariant form and Lou’s extended method [16]. In the section 2, the nonlocal symmetry (the residual symmetry) is localized by introducing a prolonged system. Whence a nonlocal symmetry is localized, it is straightforward to find the truncated Painlevé expansion, one can find many interesting results for integrable systems including the Bäcklund/Levi transformation, Schwarz form, bilinearization and Lax pair. In Ref. [17], it is found that the nonlocal symmetries, the residual symmetries can also be directly obtained from the truncated Painlevé expansion. The residual symmetries can be used to find Darboux transformations [18] [19] and the interaction solutions between a soliton and another nonlinear wave such as a cnoidal wave and/or a Painlevé wave [20] [21]. In the section 3, the nonlocal symmetry (the residual symmetry) is localized by introducing a prolonged system. Whence a nonlocal symmetry is localized, it is straightforward to find its finite transformation which is equivalent to the Bäcklund/Levi transformation. In section 4, it is found that similar to the usual KP equation, the general Lie point symmetries of the cKP3-4 equation possess also three arbitrary functions of the time $t$ and constitute a centerless Kac-Moody-Virasoro symmetry algebra. Using the general Lie point symmetries, two special types of symmetry reductions are found. The first type of (1+1)-dimensional reduction equation is Lax integrable with fourth order spectral problem. The second type of symmetry reduction equation is just the usual KdV equation. In section 5, we study the finite transformation theorem of the general Lie point symmetries via a simple direct method instead of the traditional complicated method by solving an initial value problem. The last section includes a short summary and some discussions.

2 Painlevé property, Bäcklund transformation and Schwartz form of the cKP3-4 equation

According to the standard WTC approach, if the model (1) is Painlevé integrable, all the possible solutions of the model can be written as

$$u = \sum_{j=0}^{\infty} u_j \phi^{i-\alpha}, \quad v = \sum_{j=0}^{\infty} v_j \phi^{i-\beta},$$

(6)

with four arbitrary functions among $u_j$ and $v_j$ in addition to the fifth arbitrary function, the arbitrary singular manifold $\phi$, where $\alpha$ and $\beta$ should be positive integers. In other words, all the solutions of the model are single valued about the arbitrary movable singular manifold $\phi$.

To fix the constants $\alpha$ and $\beta$, one may use the standard leading order analysis. Substituting $u \sim u_0 \phi^{-\alpha}$ and $v \sim v_0 \phi^{-\beta}$ into (1), and comparing the leading terms for $\phi \sim 0$, we get the only possible branch with

$$u_0 = -2\phi_x^2, \quad v_2 = -2\phi_x \phi_y, \quad \alpha = \beta = 2.$$

(7)

Substituting (6) with (7) into (1) yields the recursion relation on the functions $\{u_j, v_j\}$

$$\begin{bmatrix} J_{11} & J_{12} \\ (j-2)f_y & -(j-2)f_x \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} u_j \\ v_j \end{bmatrix} = \begin{bmatrix} F_1(u_0, u_1, \ldots, u_{j-1}, v_0, v_1, \ldots, v_{j-1}) \\ F_2(u_0, u_1, \ldots, u_{j-1}, v_0, v_1, \ldots, v_{j-1}) \end{bmatrix},$$

(8)

where $J_{11} = (j-5)(j-6)$ and $J_{12} = -bf_y^2(\phi^2 + 4\phi_x^2 - a(j+1)(j-4)f_x^2), \quad F_1 = -bf_y(j-5)f_y, \quad F_2$ are dependent only on $u_0, u_1, \ldots, u_{j-1}, v_0, v_1, \ldots, v_{j-1}$ and the derivatives of $\phi$ with respect to $x, y$ and $t$. The determinant of the matrix $J$ reads

$$\det J = f_y^2(j+1)(j-2)(j-4)(j-5)(j-6)(af_x + bf_y).$$

(9)
From (8) and (9), we have

\[
\begin{pmatrix} u_j \\ v_j \end{pmatrix} = J^{-1} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}
\quad (10)
\]

for \( j \neq -1, 2, 4, 5 \) and 6. At the resonance points \( j = -1, 2, 4, 5 \) and 6, five arbitrary functions, say, \( \{\phi, u_2, u_4, u_5, u_6\} \), can be included in the formal solution (6) if all the resonant conditions from (8) are satisfied. To verify these resonant conditions, one can use the Kruskal’s simplification for the singular manifold \( \phi \sim 0 \) replaced by \( \phi \sim x + \psi(y, t) \sim 0 \). Under the Kruskal’s simplification, it is straightforward to find

\[
\begin{align*}
u_0 = -2, & \quad \nu_1 = -2\psi_y, \quad u_1 = 0, \quad u_3 = -\frac{1}{2}\psi_{yy}, \\
u_2 = \frac{3a}{2b}(\psi_y - 2u_2) + \frac{1}{2}\psi_{yy} - 2u_2\psi_y + \frac{1}{2b}\psi_t, & \quad \nu_3 = -\frac{1}{2}\psi_{yy}\psi_y + u_2y, \\
u_4 = u_4\psi_y - \frac{1}{4}\psi_{yyy}, & \quad \nu_5 = u_5\psi_y + \frac{1}{3}\psi_{y}y, \quad \nu_6 = u_6\psi_y + \frac{1}{4}u_5y
\end{align*}
\quad (11)
\]

while \( \psi, u_2, u_4, u_5 \) and \( u_6 \) are arbitrary functions of \( y \) and \( t \). Thus the model (11) is not only Lax integrable but also Painlevé integrable.

Because the cKP3-4 equation (11) is Painlevé integrable, we can use the truncated Painlevé expansion,

\[
u = \frac{\nu_0}{\phi^2} + \frac{\nu_1}{\phi} + u_2, \quad \psi = \nu_0 \frac{\nu_1}{\phi^2} + \frac{\nu_1}{\phi} + \nu_2,
\quad (12)
\]

to find other interesting properties of the cKP3-4 equation (11).

It is known that using the relations (12) with \( u_2 = \nu_2 = 0 \), the cKP3-4 equation can be bilinearized (11) to

\[
[D_x D_z + a(3D_y - D_z^2)] f \cdot f = 0,
\quad (13)
\]

and

\[
[a(2bD_z D_y - 3D_x D_t + 3D_z D_x) + bD_y D_z] f \cdot f = 0,
\quad (14)
\]

with help of the auxiliary variable \( \tau \), where the Hirota’s bilinear operators \( D_z, z = x, y, t, \tau \) are defined by

\[
D_z f \cdot g = (\partial_x - \partial_z) \frac{\partial}{\partial z} f(z) g(z')|_{z = x}.
\quad (15)
\]

After introducing Möbius transformation \( (\phi \rightarrow \frac{c_0 + c_1 \phi}{b_0 + b_1 \phi}) \) with \( c_0 b_1 \neq c_1 b_0 \) invariants,

\[
g = \frac{\phi_t}{\phi_x}, \quad h = \frac{\phi_y}{\phi_x}, \quad S = \frac{\phi_{xxx}}{\phi_x} - 3\frac{\phi_x^2}{2\phi_x^2},
\quad (16)
\]

and substituting (12) with \( u_2 \nu_2 = 0 \) into (11), one can directly obtain the auto and/or non-auto Bäcklund transformation (BT) theorem and the residual symmetry theorem:

**Theorem 1.** Bäcklund transformation theorem. If \( \phi \) is a solution of the Schwartz cKP3-4 equation

\[
h^{2}(h_{xx}^{-1}w_{xx})_{x} = b[S_{xx}(3h_{xx}h_{xxx} - 5h_{xx}^{2}) + S_{x}(h_{xx}h_{xxx} - 4h_{xx}h_{xx}) - h_{xx}S_{xxx} - h_{xx}h_{xxxx} + h_{xx}h_{xxxxx} - (3h_{xx}h_{y} + h_{xx}h_{yy})h_{xxx} + 5h_{xx}h_{xx}^{2} + h_{xx}(4h_{xx}h_{xy} + h_{xx}h_{yy})],
\quad (17)
\]

then both

\[
\begin{align*}
u_{a} = \frac{h^{2}}{4} + \frac{aw_{x}}{4b h_{xx}} - S - \frac{\phi_{x}^2}{\phi_{x}^2} - \frac{3h_{xx}S_{xx} + h_{xx}S_{xxx} + h_{xx}h_{xxxx} + h_{xx}h_{xxxxx} - h_{xx}h_{xxxxx} + h_{xx}h_{xxxxx} - (3h_{xx}h_{y} + h_{xx}h_{yy})h_{xxx} + 5h_{xx}h_{xx}^{2} + h_{xx}(4h_{xx}h_{xy} + h_{xx}h_{yy})}{4h_{xx}},
\quad (18)
\end{align*}
\]

and

\[
\begin{align*}
u_{b} = u_{a} = -\frac{2\phi_{x}^2}{\phi_{x}^2} + \frac{2\phi_{x}^2}{\phi_{x}^2},
\quad v_{a} = v_{b} = -\frac{2\phi_{x}^2}{\phi_{x}^2} + \frac{2\phi_{x}^2}{\phi_{x}^2}.
\quad (19)
\end{align*}
\]
are solutions of the cKP3-4 equation \( \text{(1)} \).

**Theorem 2.** Residual symmetry theorem. If \( \phi \) is a solution of the Schwartz cKP3-4 equation \( \text{(1)} \), and the fields \( \{u, v\} = \{u_a, v_a\} \) are related to the singular manifold \( \phi \) by \( \text{(18)} \), then

\[
\{\sigma^u, \sigma^v\} = \{2\phi_{xx}, 2\phi_{xy}\}
\]

(20) is a nonlocal symmetry (residual symmetry) of the cKP3-4 equation \( \text{(1)} \). In other words, \( \text{(20)} \) solve the symmetry equations, the linearized equations of \( \text{(1)} \)

\[
\sigma^u_x = a[(6u^wux + 6u\sigma_x^u + \sigma_{xxx})_x - 3\sigma^u_y] + b(2\nu \sigma^u_x + 2\sigma^w_x + \sigma_{xxx}^w + 4\sigma^u uy + 4u\sigma^w_y)_x - \sigma^v_y.
\]

(21)

\[\sigma^v_y = \sigma^v_x.\]

From \( \text{(17)} \), one can find that when \( b = 0 \), the Schwartz cKP3-4 is reduced back to the usual Schwartz KP equation

\[w = 0.\]

The BT \( \text{(18)} \) is a non-auto BT because it changes a solution of the Schwartz cKP3-4 equation \( \text{(1)} \) to that of the usual cKP3-4 equation \( \text{(1)} \). The BT \( \text{(19)} \) may be considered as a non-auto BT if \( u_a, v_a \) are replaced by \( u_b, v_b \) for the same equation \( \text{(1)} \).

From the auto-BT \( \text{(19)} \) and the trivial seed solution \( \{u_a = 0, v_a = 0\} \), one can obtain some interesting exact solutions. Substituting \( \{u_a = 0, v_a = 0\} \) into \( \text{(15)} \), we have

\[
\frac{\lambda^2}{4} + \frac{aw_x}{4h_{xx}} - S - \frac{\phi_x^2}{\phi_y^2} = 0,
\]

(22)

\[
g + bh^3 + 3ah^2 - \frac{b}{2}(a + bh) S - h_{xx} - \frac{3(a + bh)}{2\phi_x^2} h_x \phi_{xx} - h_x \phi_{xx} = 0.
\]

(23)

After solving the over determined system \( \text{(17)}, (22) \) and \( (23) \), one can find various exact solutions from the BT \( \text{(19)} \) with \( \{u_a = 0, v_a = 0\} \). Here, we discuss only for the travelling wave solutions of the system \( \text{(17)}, \)

\[
(\phi_{xxxx} + (5\Phi_{xxxx} + 4\Phi_{xxxx})\Phi^{2}_x + 17\Phi_x \Phi^{2}_x \Phi_{xxxx} - 9\Phi^{4}_x = 0,
\]

(24)

\[
(3akp^2 + bp^3 + k^2 \omega) \Phi^{2}_x - k^4(ak + bp)(4\Phi_x \Phi_{xxxx} - 3\Phi^{2}_x) = 0.
\]

(25)

Here we list three special solution examples of the cKP3-4 equation \( \text{(1)} \) related to \( \text{(24)} \) and \( \text{(25)} \).

**Example 1.** D’Alembert type arbitrary travelling waves moving in one direction with a fixed model dependent velocity.

\[
\Phi = \Phi(x), \quad x = b^2x - 2a^3t - aby, \quad p = -ak/b, \quad \omega = -2a^3/b^2,
\]

(26)

\[
u = -bv/a = 2b^4[\ln(\Phi)]_{xx},
\]

where \( \Phi \) is an arbitrary function of \( \xi = b^2x - 2a^3t - aby. \)

Because of the arbitrariness of \( \Phi \), the localized excitations with special fixed model dependent velocity \( \{-2a^3/b^2, -2a^3/b \} \) possess rich structures including kink shapes, plateau shapes, molecule forms, few cycle periods, periodic solitons, etc. in addition to the usual sech \(^2 \) form \( \text{(1)} \).

**Example 2.** Rational wave.

\[
\Phi = kx + py - p^2k^{-2}(3ak + bp)t + \zeta_0,
\]

(27)

\[
u = kv/p = -\frac{2k^6}{(3akp^2 + bp^3t - k^3x - k^2py - \zeta_0k^2)^2}
\]

with arbitrary constants \( k, p \) and \( \zeta_0 \).

**Example 3.** Soliton solution.

\[
\Phi = 1 + \exp(\xi), \quad \xi = kx + py - \frac{1}{k^2}(-ak^5 + bk^4p + 3akp^2 + bp^3)t + \zeta_0
\]
with arbitrary constants $k$, $p$ and $\xi_0$.

Different from the D’Alembert wave \cite{20}, the soliton solution \cite{28} possesses arbitrary velocity \{-p/k, \(-ak^2-bkp+3ak^{-2}p^2+bp^4k^{-3}\)\} but fixed sech$^2$ shape.

### 3 Localization of nonlocal symmetry \cite{20}

Similar to the usual KP equation \cite{21} and the supersymmetric KdV equation \cite{22}, the nonlocal symmetry (residual symmetry) \cite{20} can be localized by introducing auxiliary variables

$$
\psi_1 = \phi_x, \quad \psi_2 = \phi_y, \quad \psi_3 = \phi_{1x}, \quad \psi_4 = \phi_{2x}.
$$

It is straightforward to verify that the nonlocal symmetry of the cKP3-4 equation \cite{1} becomes a local one for the prolonged system \cite{1}, \cite{17}, \cite{18} with \{\(u = u, \quad v = v\)\} and \cite{20}. The vector form of the localized symmetry of the prolonged system can be written as

$$
V = 2\psi_3\partial_u + 2\psi_4\partial_v - \phi^2\partial_\phi - 2\phi\psi_1\partial_\psi_3 - 2\phi\psi_2\partial_\psi_4 - 2(\phi^2 + \phi\psi_3)\partial_\phi_3 - 2(\phi\psi_2 + \phi\psi_4)\partial_\phi_4.
$$

According to the closed prolongation structure \cite{20}, one can readily obtain the finite transformation (auto Bäcklund transformation) theorem by solving the initial value problem

$$
\begin{align*}
\frac{du(e)}{de} &= 2\phi_3(e), \quad \frac{dv(e)}{de} = 2\phi_4(e), \quad \frac{d\phi_1(e)}{de} = -\phi(e)^2, \quad \frac{d\phi_2(e)}{de} = -2\phi(e)\phi_1(e), \\
\frac{d\phi_3(e)}{de} &= -2[\phi_1(e)^2 + \phi(e)\psi_3(e)], \quad \frac{d\phi_4(e)}{de} = -2[\phi_1(e)\phi_2(e) + \phi(e)\psi_4(e)], \\
\{u(e), v(e), \phi(e), \phi_1(e), \phi_2(e), \phi_3(e), \phi_4(e)\}_{e=0} &= \{u, v, \phi, \phi_1, \phi_2, \phi_3, \phi_4\}.
\end{align*}
$$

**Theorem 3.** \textit{Auto Bäcklund transformation theorem.} If \{\(u, \quad v, \quad \phi, \quad \phi_1, \quad \phi_2, \quad \phi_3, \quad \phi_4\)\} is a solution of the prolonged system \cite{1}, \cite{17}, \cite{18} with \{\(u = u, \quad v = v\)\} and \cite{20}, so is \(\{u(e), v(e), \phi(e), \phi_1(e), \phi_2(e), \phi_3(e), \phi_4(e)\}\) with

$$
\begin{align*}
\phi(e) &= \frac{\phi}{1 + \epsilon \phi}, \quad \phi_1(e) = \frac{\phi_1}{(1 + \epsilon \phi)^2}, \quad \phi_2(e) = \frac{\phi_2}{(1 + \epsilon \phi)^2}, \\
\phi_3(e) &= \frac{\phi_3}{(1 + \epsilon \phi)^2}, \quad \phi_4(e) = \frac{2\epsilon \phi_1^2}{(1 + \epsilon \phi)^2}, \quad \phi_1(e) = \frac{\phi_1(1 + \epsilon \phi)}{(1 + \epsilon \phi)^2}, \\
u(e) &= u + \frac{2\epsilon \phi_3}{1 + \epsilon \phi}, \quad v(e) = v + \frac{2\epsilon \phi_4}{1 + \epsilon \phi}.
\end{align*}
$$

Comparing the theorem 2 and the theorem 3, one can find that for the cKP3-4 equation \cite{1}, the transformation \cite{33} is equivalent to \cite{17} by using the transformation \(1 + \epsilon \phi \to \phi\).

### 4 Symmetry reductions of the cKP3-4 equation

Using the standard Lie point symmetry method or the formal series symmetry approach \cite{23, 24} to the cKP3-4 equation, it is straightforward to find the general Lie point symmetry solutions of \cite{21} are generated by the following three generators,

$$
\begin{align*}
\left( \sigma^u \over \sigma^v \right) &= K_0(\alpha) = \left( \begin{array}{c}
\alpha u_x \\
\alpha v_x + \frac{1}{2\alpha} \alpha_t
\end{array} \right), \\
\left( \sigma^u \over \sigma^v \right) &= K_1(\beta) = \left( \begin{array}{c}
\beta u_y + \frac{1}{2\beta} \beta_t \\
\beta v_y - \frac{1}{2\beta} \beta_t
\end{array} \right).
\end{align*}
$$
and

$$K_2(\theta) = \left( \begin{array}{c} \theta u_t + \frac{a}{2b}(ay + bx)u_x + \frac{b}{2}\theta_t(yu)_y + \frac{3a^2}{4b^2}\theta_t + \frac{a}{2b}\theta_{tt} \\ \theta v_x + \frac{a}{2b}(ay + bx)v_x - \frac{1}{2}\theta_t(yu)_y + \frac{2a^2 - 2b}{4b^2}\theta_t + \frac{bx - 2ay}{4b^2}\theta_{tt} - \frac{a}{8b}\theta_t \end{array} \right),$$

(36)

where \(\alpha, \beta\) and \(\theta\) are arbitrary functions of \(t\).

The symmetries \(K_0(\alpha), K_1(\beta)\) and \(K_0(\theta)\) constitute a special Kac-Moody-Virasoro algebra with the nonzero commutators

$$[K_2(\theta), K_0(\alpha)] = K_0(\theta\alpha), \quad [K_2(\theta), K_1(\beta)] = K_1(\theta\beta), \quad [K_2(\theta_1), K_2(\theta_2)] = K_2(\theta_1\theta_2 - \theta_1\theta_2),$$

(37)

where the commutator \([F, G]\) with \(F = (F_1(u), v), F_2(u), v))^T\) and \(G = (G_1(u), v), G_2(u), v))^T\), where the superscript \(T\) means the transposition of matrix, is defined by

$$[F, G] \equiv \left( \begin{array}{c} F'_{1u} \\ F'_{2u} \\ F'_{2v} \\ G'_{1u} \\ G'_{2u} \\ G'_{2v} \end{array} \right) \left( \begin{array}{c} G'_{1u} \\ G'_{2u} \end{array} \right) F,$$

and \(F'_{1u}, F'_{1v}, F'_{2u}, F'_{2v}, G'_{1u}, G'_{1v}, G'_{2u}, G'_{2v}\) are partial linearized operators, say,

$$F'_{1u}G_1 \equiv \left. \frac{d}{\epsilon} F_1(u + \epsilon G_1, v) \right|_{\epsilon=0}.$$

From (37), we know that \(K_0\) and \(K_1\) constitute the usual Kac-Moody algebra and \(K_2\) constitutes the Virasoro algebra if we fix the arbitrary functions \(\alpha, \beta\) and \(\theta\) as special exponential functions \(\exp(mt)\) or polynomial functions \(t^m\) for \(m = 0, \pm 1, \pm 2, \ldots\).

Applying the Lie point symmetries \(K_0(\alpha), K_1(\beta)\) and \(K_0(\theta)\) to the cKP3-4 equation (1), we can get two nontrivial symmetry reductions.

**Reduction 1:** \(\theta \neq 0\). For \(\theta \neq 0\), we rewrite the arbitrary functions in the form

$$\theta \equiv \rho^\alpha \neq 0, \quad \alpha \equiv \rho^\alpha \omega_1, \quad \beta \equiv \rho^\beta \omega_1, \quad \theta_1 \equiv \rho \beta_1.$$

(38)

Under the new definitions (38), the group invariant condition

$$K_0(\alpha) + K_1(\beta) + K_2(\theta) = 0$$

yields the first type of group invariant solutions in the form

$$u = U(\xi, \eta) \left( \rho^\omega \right)^2 - \frac{\rho \rho_y}{2b} \frac{3a^2}{4b} - \frac{2b}{2b} - \frac{\beta_1 \rho}{4b},$$

(39)

$$v = V(\xi, \eta) \left( \rho^\omega \right)^3 - \frac{\rho \rho_x}{2b} \frac{(2ay - bx)\rho_x}{2b^2} - \frac{7a^3}{4b} - \rho \omega_1 \frac{3a \rho \beta_1}{2b} + \frac{3a \rho \beta_1}{4b^2},$$

(40)

where \(U(\xi, \eta) \equiv U\) and \(V(\xi, \eta) \equiv V\) are group invariant functions of the group invariant variables \(\xi\) and \(\eta\) with

$$\xi = \frac{x}{\rho} - \frac{ay}{b \rho} - \alpha_1 + \frac{a \theta_1}{b}, \quad \eta = \frac{y}{\rho^\omega} - \beta_1.$$

(41)

Substituting (39) and (40) into (1), we can find the group invariant reduction equations for the group invariant functions \(U\) and \(V\),

$$U_\eta = V_\xi,$$

$$V_{\eta \eta} = (V_{\xi \xi} + 4UV_\xi + 2UV_\xi)\xi.$$

(42)

It is interesting that the reduction system (42) is Lax integrable with the fourth order spectral problem

$$\lambda \Psi = 2\Psi_{\xi \xi \xi} + 4U\Psi_\xi + 2(2U_\xi - iV)\Psi_\xi - \left( V_\xi d\xi - 2U_\xi - 2U_\xi + iV_\xi \right) \Psi,$$

(43)

$$\Psi_\eta = i(\Psi_{\xi \xi} + U\Psi).$$

(44)
### Reduction 2: $\beta \neq 0$. For $\beta \neq 0$ case, the group invariant condition

$$K_0(\alpha) + K_1(\beta) = 0$$

and $v_x = u_y$ yield the usual KdV reduction

$$(U_T + U_{XXX} + 6U_x)_X = 0, \quad X = \frac{1}{\sqrt{\beta}} \left( \frac{\alpha}{\beta} y - x \right), \quad T = \int \beta^{-5/2} (a\beta - ba) dt$$

with

$$u = \frac{U(X,T)}{\beta} + \frac{\beta_1 y}{4b\beta} + \frac{\alpha^2 (3a\beta - ba)}{6b^2 (a\beta - ba)}$$

$$v = -\frac{\alpha U(X,T)}{\beta^2} - \frac{\beta_1 x}{4b\beta} + \left[ \frac{(3a\beta + ba)\beta_2}{4b^2 \beta^2} - \frac{\alpha_2}{2b\beta} \right] y - \frac{\alpha^3 (3a\beta - ba)}{6b^3 (a\beta - ba)}.$$ 

### 5 Finite transformation theorem of $K_0(\alpha) + K_1(\beta) + K_2(\theta)$ via direct method

The finite transformation of $K_0(\alpha) + K_1(\beta) + K_2(\theta)$ may be obtained by solving the initial value problem $\{x', y', t', u', v'\} = \{x'(t), y'(t), t'(t), u'(t), v'(t)\}$, $\{\alpha', \beta', \theta'\} = \{\alpha(t'), \beta(t'), \theta(t')\}$, when

$$\frac{dt'}{\hat{\epsilon}} = \theta', \quad \frac{dy'}{\hat{\epsilon}} = \left( \beta' + \frac{1}{2} \theta', y' \right), \quad \frac{dx'}{\hat{\epsilon}} = \left[ \alpha' + \frac{1}{4b} \theta', ay' + bx' \right].$$

$$\frac{du'}{\hat{\epsilon}} = -\frac{1}{4b} \beta' + \frac{1}{2} \theta'_t - \frac{a}{8b} \theta'_t y' - \frac{3a^2}{8b^2} \theta'_{tt},$$

$$\frac{dv'}{\hat{\epsilon}} = -\frac{1}{2b} \alpha' + \frac{3a}{4b^2} \beta' - \frac{a}{4b} \theta'_t u' - \frac{3}{4} \theta'_t v' + \frac{9a^3}{8b^3} \theta'_{tt} + \frac{2ay' - bx'}{8b^2} \theta'_{ttt},$$

where $t'(0) = t$, $y'(0) = y$, $x'(0) = x$, $u'(0) = u$, $v'(0) = v$.

However, the exact solution of the initial value problem is very complicated and quite awkward even for the pure KP ($a = 0$) case. An alternative simple method is to find symmetry group via a direct method by using a priori ansatz

$$u = A + BU(x', y', t'), \quad v = A_1 + B_1 U(x', y', t') + B_2 V(x', y', t'),$$

(51)

to determine the functions $\{A, B, A_1, B_1, B_2, x', y', t'\} = \{A, B, A_1, B_1, B_2, x', y', t'(x, y, t)\}$ such that both $\{u, v\}$ and $\{U, V\}$ are solutions of the cKP3-4 equation. Substituting the ansatz into $\{U, V\}$ and requiring $U$ and $V$ satisfying the same cKP3-4 equation but with different variables $\{x', y', t'\}$, one can readily determine all the undetermined functions $\{A, B, A_1, B_1, B_2, x', y', t'\}$. The final result can be summarized to the following finite transformation theorem.

**Theorem 4.** Finite transformation theorem. If $\{U, \quad V\} = \{U(x, y, t), V(x, y, t)\}$ is a solution of $\{U, \hat{\epsilon}\}$, so is $\{u, v\}$ with

$$u = \sqrt{\tau t} U(x', y', t') + \frac{\tau a y}{8b \tau t} + \frac{y_0}{4b \sqrt{\tau t}} + \frac{3a^2}{4b^2} (\sqrt{\tau t} - 1),$$

$$v = \frac{\alpha}{\sqrt{b}} \left( \sqrt{\tau t} - 1 \right) U(x', y', t') + \sqrt{\tau t} V(x', y', t') + \frac{\tau a (bx - 2ay)}{8b^2 \tau t}$$

$$+ \frac{a^3}{4b^3} \left( 7 - 3\sqrt{\tau t} - 4 \sqrt{\tau t} \right) + \frac{b x_0 - ay_0}{2b^2 \sqrt{\tau t}} - \frac{ay_0}{4b^2 \sqrt{\tau t}},$$

$$x' = \sqrt{\tau t} x - \frac{a}{b} \left( \sqrt{\tau t} - \sqrt{\tau t} \right) y + x_0, \quad y' = \sqrt{\tau t} y + y_0, \quad t' = \tau,$$

(54)

where $x_0 = x_0(t)$, $y_0 = y_0(t)$ and $\tau = \tau(t)$ are three arbitrary functions of $t$. 

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To verify the correctness one can directly substitute (52)–(54) into (1). In fact, one can take the arbitrary functions \(x_0, y_0\) and \(\tau\) in the forms

\[
\tau = t + \epsilon \theta, \quad x_0 = \epsilon \alpha, \quad y_0 = \epsilon \beta,
\]

where \(\theta, \alpha\) and \(\beta\) are arbitrary functions of \(t\). Substituting (55) into (52) and (53) yields

\[
\left( \begin{array}{c}
u \\
\end{array} \right) = \left( \begin{array}{c}
U \\
V
\end{array} \right) + \epsilon [K_2(\theta) + K_0(\alpha) + K_1(\beta)] + O(\epsilon^2),
\]

which means theorem 4 is just the finite transformation theorem of the symmetry \(K_2(\theta) + K_0(\alpha) + K_1(\beta)\).

Applying theorem 4 to the D’Alembert wave (26), we get a new solution

\[
u = 2b^4 \sqrt{\tau} [\ln(\Phi)]_{\zeta \zeta} + \frac{7\tau y}{8b^4} + \frac{y_0}{4b^2 \sqrt{\tau}} + \frac{3a^2}{4b^2} (\sqrt{\tau} - 1),
\]

\[
\zeta = b \sqrt{\tau} (bx - ay) + \zeta_0
\]

with \(\tau, y_0\) and \(\zeta_0\) being arbitrary functions of \(t\) and \(\Phi\) being an arbitrary function of \(\zeta\).

6 Conclusions and discussions

In summary, the cKP3-4 equation (1) is a significant (2+1)-dimensional KdV extension with various interesting integrable properties. In this paper, the Painlevé property, auto- and nonauto- Bäcklund transformations, local and nonlocal symmetries, Kac-Moody-Virasoro symmetry algebra, finite transformations related to the local and nonlocal symmetries, and the Kac-Moody-Virasoro group invariant reductions are investigated.

Usually, starting from the trivial vacuum solution \((u = 0)\), the Bäcklund transformation will lead to one soliton solution. However, for the cKP3-4 equation (1), the trivial vacuum solution and Bäcklund transformations will lead to abundant solutions including rational solutions, arbitrary D’Alembert type waves, solitons with a fixed form (sech\(^2\) form) and arbitrary velocity, and solitons and soliton molecules with fixed velocity but arbitrary shapes (special D’Alembert waves).

There are two important (1+1)-dimensional symmetry reductions of the cKP3-4 equation (1). The first type of reduction equation is Lax integrable with fourth order spectral problem. The second reduction is just the KdV equation. The more about the cKP3-4 equation (1) and its special reduction (42) will be reported in our future studies.

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