The Statistical Mechanics of Microscopic Long-Range Bulk-Boundary Dependence in Black-Hole Physics and Holography

Manfred Requardt

Institut für Theoretische Physik
Universität Göttingen
Friedrich-Hund-Platz 1
37077 Göttingen Germany
(E-mail: requardt@theorie.physik.uni-goettingen.de)

Abstract

We argue in the following that the entropy-area law of black-hole physics and the various holographic bounds are the consequences of the microscopic dynamics of elementary degrees of freedom living on or near the Planck scale. We locate them both in the interior and on the boundary of, for example, the black hole with the strange area-behavior of various quantities being the result of a long-range bulk-boundary dependence among these degrees of freedom. In contrast to other approaches we regard the vacuum fluctuations on microscopic scales as the relevant elementary building blocks. In so far certain relations to old ideas of Sakharov, Zeldovich et al are acknowledged (induced gravity). Most importantly, we prove that the existence of a large energy gap between a few low-lying excitation patterns and the majority of the other (in principle) possible excitation patterns in a subvolume with given boundary excitation is crucial for this area-dependence. We also remark that this is an indication that some particular entangled space-time geometry of a somewhat non-local character prevails in the microscopic (Planck) regime. Our findings are corroborated by the explanation of a number of open questions in the field (see the table of contents at the end of the introduction).
1 Introduction

In [1] Bekenstein remarked that the deeper meaning of black-hole entropy (henceforth abbreviated by BH) remains mysterious. He asks, is it similar to that of ordinary entropy, i.e. the log of a counting of internal BH-states, associated with a single BH-exterior? ([2], [3] or [4]). Or, similarly, is it the log of the number of ways, in which the BH might be formed. Or is it the log of the number of horizon quantum states? ([5], [6]). Does it stand for information, lost in the transcendence of the hallowed principle of unitary evolution? ([7], [8]). He then claims that the usefulness of any proposed interpretation of BH-entropy depends on how well it relates to the original “statistical” aspect of entropy as a measure of disorder, missing information, multiplicity of microstates compatible with a given macrostate, etc.

We think, the latter statement is a very important remark which, in our view, is sometimes lost sight of in the discussion. It is in particular the dynamical aspect of disorder which is important in statistical mechanics and which goes beyond the frequently invoked but physically somewhat empty pure information-entropy point of view (subjective ignorance). See in this context the remarks on p. 4545 of [9].

A large group of researchers in the field view the question of the localisation of the microscopic degrees of freedom (henceforth DoF), generating BH-entropy in a seemingly rather geometric way and locate them on or in the vicinity of the BH-horizon (to mention a few, [10], [11], [12]). In e.g. [11] thesis 1 reads: “. . . S resides on the horizon”, while in thesis 4 it is stated: “The idea is wrong that the DoF are inside the BH”. An argument to this effect is given for example in [12] and goes as follows: “. . . The coupling from outside to inside is not weak but very strong, while the reverse coupling is not so much weak as nonexistent! Indeed this last observation points up the fact that conditions in the interior should be irrelevant, almost by definition, to what goes on outside. . . . that it should have anything to do with counting interior states”.

As to the latter point, we must admit that we are a little bit sceptical as, in our view, it seems to be a classical or quasiclassical argument. We will show that at or near the Planck-scale, which is in our approach (see below) the appropriate environment, there exists a marked correlation connecting the interior and the exterior of the BH through the horizon. We already provided strong arguments in favor of the existence of such a non-local and long-range collective behavior in the second part of [13] and enter into a quite detailed quantitative analysis of the microscopic correlation structure in section 4.1 of the present paper. We argue that making a few reasonable
assumptions about the fine structure of the vacuum fluctuation spectrum which can be observationally confirmed and amalgamating this with the holographic principle in scenarios where the latter can be confirmed, we can rigorously prove that the vacuum fluctuations are long-range (anti)correlated in (quantum)space-time on microscopic scales (see the following for more details).

The idea that the relevant DoF can be thought of as being essentially confined to a thin halo about the horizon, may have been inspired by the apparent behavior of the physical concept of entanglement-entropy. This is a concept with wide applications also in statistical mechanics and/or quantum information theory (see e.g. [14] for more references); the two papers which are relevant in our present context are [15] and [16]; see also the more recent [17]. In these papers certain arrays of coupled harmonic oscillators and their continuum limit (a Klein-Gordon field theory) have been analysed. After some delicate and tedious calculations it was shown that the entanglement-entropy of the groundstate (i.e. the vacuum in the continuum limit), when traced over a subvolume, $V_1$, with $V = V_1 \cup V_2$, is proportional to the area of the dividing surface while the correct calculation of the prefactor is more delicate. These calculations were however performed in flat Minkowski space but it was argued that the results have a certain bearing also for the BH-situation. As a consequence of these findings there now seems to be a certain tendency to associate BH-entropy (at least to some degree) with entanglement-entropy and having its origin in the DoF near the horizon.

To put these various results in perspective we would like to make the following points clear. In [14] we showed that under the condition that interactions and correlations are short-ranged, groundstates lead to an entanglement-entropy which is proportional to the area of the dividing surface for a wide range of Hamiltonians. On the other hand, for systems being in a state which displays long-range correlations (e.g. a (quantum) critical state) this does not hold even for the groundstate. This shows that the assumption of short-range correlations is important. Furthermore, even in the short-range case, for eigenstates which are highly or lowly excited, we proved that the entropy depends linearly on the volume or the log of the volume of the subsystems times the area of the dividing surface. This implies that in these latter situations the entropy is no longer localized near the dividing boundary.

This is supported by yet another interesting result. With the state of a subsystem, defined over a subvolume, $V_1$, being a canonical Gibbs state, we can extend this temperature state to a pure vector state, $\Psi$, on the
larger volume, \( V_1 \cup V_2 \), in such a way that its restriction, i.e. its partial trace, to \( V_1 \) is the Gibbs state, we started from. This implies that the entanglement-entropy of the total vector state, \( \Psi \), with respect to \( V_1 \) is the original thermodynamic entropy of the Gibbs state. The latter happens to be proportional to the volume \( V_1 \) in the generic case, which by the same token holds for the entanglement-entropy of \( \Psi \). We thus see that quite a lot of vector states have entanglement-entropies being proportional to the respective subvolumes. In this context see also the observations in [18] section II.D and [19]

What does this mean for the understanding of BH-entropy? Due to the work of Bekenstein and many others we have learned that the maximal entropy or information which can be stored in a BH is proportional to the area of the horizon (which is a stronger result as the one referring only to some particular state). Making an educated speculation, we model the interior of the BH as some kind of (quantum)statistical subsystem on a sufficiently microscopic scale with the local vacuum fluctuations on this presumed fundamental scale as its degrees of freedom. This idea is actually not so far-fetched; cf. the old ideas of e.g. Sakharov and Wheeler (see below). It should then be possible in principle to put this system in a higher excited state by making e.g. the fluctuation spectrum less correlated, i.e. increasing its disorder. In case the fluctuation spectrum happens to be short-range correlated, our above cited rigorous results ([14]) show that the maximal entropy should be proportional to the volume and not the area. We hence arrive at the preliminary result

**Conclusion 1.1** The assumption of short-range correlations is necessarily violated in the BH-context. On the other hand, the conclusion that entanglement-entropy is proportional to the area was (among other things) based on this (tacit) assumption. But even in the case where this situation would prevail, the result does not hold for the maximal entropy. Therefore we conjecture that an explanation of BH-entropy is more delicate and seems to need some more prerequisites.

In the following we want to provide some of these missing prerequisites and undertake to formulate a theory of bulk-boundary-statistical mechanics which extends to some extent ordinary statistical mechanics.

There is a thoughtful discussion of such problems in [20] p.31ff which points in a similar direction concerning the problem of the localisation of the responsible DoF. It is argued that, in the end, all the different suggestions like e.g. thermal atmosphere, horizon, interior, may come out to be complementary aspects of the same physical DoF, a working hypothesis we
try to substantiate in the following. Two other well-written papers of partly
review character and dealing with these questions and the respective con-
text are [21], [22]. In [22], for example, the idea is contemplated that pure
information may underlie ultimately all of physics. This point of view is
corroborated by the observation that the holographic principle betrays little
about the character of the microscopic DoF ([22]). We are following a similar
working philosophy for already quite some time (see e.g. [13], [42], [60], [61],
[66] and further earlier references given there).

This whole approach relates in an interesting way to other fundamental
theories being in vogue presently as e.g. string theory or loop quantum
gravity, which are not really of this mentioned character. There may be
two modes of relation between these, at first glance, not entirely compatible
approaches. Either, the theory we are developing in the following, and which
builds on the excitation patterns of vacuum fluctuations on primordial scales
as fundamental building blocks (extending old ideas of Sakharov et al), lives
on a finer scale than these two other theories, which have then to be regarded
as derived theories. Or, on the contrary, this framework is itself kind of
an effective theory which uses some gross features of a more fundamental
theory. Both cases are in principle possible while we personally favor the
first alternative. Anyhow, the phenomenon that the entropy-area law can
apparently be derived both in string theory and loop quantum gravity for at
least certain extremal cases, and, on the other hand, can also be understood
in our more model independent framework, may be an indication that it is
perhaps the consequence of somewhat more general features, a point of view
we explicitly forward in our approach.

As the paper is relatively long and deals with quite a variety of subjects
which are grouped around some central sections, we conclude this intro-
duction, for convenience of the reader, with a table of contents. A central
role is played by the quantitative analysis in section 4 while in the sections
after section 4 the preceding results are applied to a variety of important
questions in the field.

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2 Some Fundamental Issues

One reason why the question of the microscopic origin of BH-entropy is still not yet settled after so many years of intense discussions lies in the fact that this topic is (so to speak) situated just at the horizon between a region in “theory space” which is understood at least in principle, i.e. quantum field theory (Q.F.Th.) in curved space-time, and a region which is still sort of a terra incognita, i.e. quantum gravity (Q.Gr.) (irrespective of the claims of string theory). Therefore most attempts tried to attack the open problems from the better known exterior region (the low-energy end) by exploiting the methods and tools of Q.F.Th. and, consequently, located the microscopic origin of BH-entropy mainly in the region exterior to the horizon or the horizon itself (either thermal atmosphere or horizon states or both).

One feature which appears in traditional Q.F.Th. is the existence of zero-point oscillations (or excitations) of the Fourier modes in the expansion of, for example, the Hamiltonian of free quantum field theories. These excitations were frequently neglected to some extent (“renormalized away”) and viewed as “unphysical” but have come to a certain prominence again in the context of BH-entropy and related fields like e.g. the Casimir or the Unruh effect where they prove their objective existence. We think that the
(re)emergence of these notions is pointing in the right direction but are sceptical if they can really provide a quantitatively correct explanation if dealt with in the usual, somewhat narrow traditional way. First, these virtual vacuum excitations typically show up in a perturbational treatment of Q.F.Th. in a specific and model dependent way. This implies that a Klein-Gordon theory vacuum looks, so to speak, different from a QED-vacuum etc., each having its particular vacuum excitation modes. In principle, the real vacuum outside a BH should then support all these different excitation modes and in addition all the vacuum excitations showing up in all the effective (and at the moment mostly unknown) theories living on higher energy scales up to the Planck scale and, last but not least, all the geometric and possibly topological excitations of space-time itself. After all, this seems to represent a great higgledy-piggledy of superficially different excitation patterns, while we think, given the well-known limited value of the particle picture in fully developed Q.Gr., that they will look all alike near the Planck scale and supposedly do no longer have a (virtual) particle character of whatever type.

The history of the idea of vacuum fluctuations and/or zero-point energies is both involved and fascinating. Nice recent historical reviews are [25] and [26]. It is particularly noteworthy that already Nernst discussed in quite some detail the concept of zero-point energy which he considered as being situated in the aether ([27]). He even had the idea that energy conservation may only hold in a statistical sense and that particles do perform what was later called \textit{Zitterbewegung}.

Pauli came back to the problem and related it (presumably for the first time) to general relativity and the \textit{cosmological constant} (but apparently did not realize the necessity of a negative pressure!, which has a repulsive effect, as he stated that his calculations show that the universe would not even stretch to the moon). For these reasons he seemed to ignore the possibility of zero-point energies (cf. the remarks in [28]). By and large, zero-point energies were not taken very seriously at that time anyhow, as can be inferred from a letter by Bohr to Pauli (quoted in [26]). In that letter Bohr rightly remarks that in quantum theory such effects can only be observed by making measurements, implying the interference of quantum objects with \textit{macroscopic} objects. From a strict logical point of view one can therefore never decide if e.g. field fluctuations were already present in the pure vacuum or, on the other hand, have been created by the interference with a measuring apparatus. We should emphasize that this ambiguity besets the whole field of quantum theory and is the reason why such fundamental questions are difficult to settle once for all. It is here not the place to enter into
a necessarily involved epistemological debate. We should however point to related epistemological problems in general relativity. There the question is, is space-time “really” curved in an objective sense or are, on the other hand, the measuring instruments deformed by gravity. It is actually the (aesthetic) question which theory is more coherent and satisfying as Einstein used to point out.

More recent reviews of the cosmological constant problem and its relation to vacuum fluctuations are e.g. [29], [30] and [31]. Concerning our own line of thought, a very clearly written contribution is [32]. Zeldovich in particular points out that one of the typical arguments that zero-point energies are artefacts is wrong. That means, the (standard) reasoning that a quantum-field-system vacuum four-vector with, say, energy \((E,0)\) must go over into some \((E',p')\) under a Lorentz transformation with \(p' \neq 0\) unless \(E = 0\) (i.e. it is not Lorentz-invariant). This conclusion is not correct for various reasons (actually, \(E\) is infinite in our case). But we should in particular emphasize that in a complete theory including gravitation energy-momentum is rather part of the energy-momentum two-tensor and an object like \(\text{const} \cdot \text{Diag}(1,-1,-1,-1)\) or rather \(\text{const} \cdot g_{\mu\nu}\) is covariant and yields the negative contribution to the pressure. We can however conclude the following:

**Conclusion 2.1** We can learn from the preceding remarks that the inclusion of gravity is crucial if one wants to deal consistently with zero-point energies.

To conclude this brief interlude about zero-point energies one should make it clear that all the zero-point energies occurring in models do arise from pure fluctuations of some observables, thus underpinning (at least in our view) their real existence. In the most simple example, the harmonic oscillator, the Hamiltonian is essentially the sum of \(P^2\) and \(Q^2\), i.e.

\[
H = P^2/2m + m\omega/2 \cdot Q^2
\]

and with

\[
\langle P \rangle_0 = \langle Q \rangle_0 = 0
\]

in the groundstate, \(\psi_0\), we have

\[
h \cdot \omega/2 = \langle H \rangle_0 = 1/2m \cdot \langle (P - \langle P \rangle_0)^2 \rangle_0 + m\omega/2 \cdot \langle (Q - \langle Q \rangle_0)^2 \rangle_0
\]

with

\[
\langle (P - \langle P \rangle_0)^2 \rangle_0 \cdot \langle (Q - \langle Q \rangle_0)^2 \rangle_0 \geq h^2/4
\]

which follows from \([P,Q] = -i\hbar\).
In the same way we have in (matter-free) QED:

\[ H = \text{const} \cdot (E^2 + B^2) \]  

with \[ \langle E \rangle_0 = \langle B \rangle_0 = 0 \]  

so that again \( \langle H \rangle_0 \) is a sum over pure vacuum fluctuations of the non-commuting quantities \( E \) and \( B \).

Another point is the problem of continuous space in general in connection with, for example, entanglement-entropy. It is easy to make relatively rigorous calculations within the framework of quantum lattice theories (see e.g. [14]). It is in particular unproblematical to divide a quantum system into two subsystems in this framework. The total Hilbert-space becomes the natural tensor product of the two subspaces while the total algebra of observables can be uniquely represented as the tensor product of the corresponding subalgebras., i.e.

\[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \]  

This is certainly the reason why both [15] and [16] start from a discretized version of a continuous theory. Taking then the continuum limit is difficult and not free of ambiguities since infinities do arise which are not so easy to get rid of in a non-adhoc manner.

On the other hand, when starting directly from a continuous Q.F.Th. model, it is not obvious in the general situation how to make a natural division of this model system into two subsystems (without losing certain contributions!), e.g. divide a total Fock space into a tensor product of the Fock spaces of two subsystems. For special situations this can be done in a reasonable way like the Rindler wedge (see the seminal paper [33] or [34] for a concise discussion), but in general appropriate explicit coordinate transformations are not at our disposal which allow to represent a subsystem with the help of two different mode expansions, the one belonging to the total system, the other being restricted to the subspace. In this context the existence of time-like Killing-vector fields play an important role.

Remark: We mention in passing the important results of axiomatic Q.F.Th. which show how difficult it is to isolate subsystems, defined by local algebras of observables and the physical consequences of Tomita-Takesaki-theory ([35]).

Another point which is not really clear to us is the range of validity of this Q.F.Th. approach when it comes to the really microscopic DoF of the
quantum vacuum, i.e. when we approach the Planck-scale, as we surmise that the bulk of DoF contributing to e.g. the BH-entropy is of this very microscopic nature. Furthermore, in the continuum approach one has to introduce an adhoc UV-cutoff in order to keep the calculated value of the entropy finite. Note in this respect the choice of a “brick wall” in [10] in order to make an effective division of interior and exterior of the BH. It would be advantageous in our view to have a framework which generates such a cutoff in a natural way.

In the following we undertake to develop such a “bottom-up” approach. We start from a microscopic regime in which higher complex structures like e.g. quantum fields and particle excitations emerge as derived and extended excitation patterns living on this primordial array of elementary DoF (see also [13] and further references given there). In this respect we want to mention the old ideas of Sakharov ([36],[38],[33] or [41], p.426ff).

To put it briefly, Sakharov argued that for example the gravitational field is a derived effect of deformations in the spectrum of vacuum fluctuations, thus yielding sort of a “metric elasticity of space”; in [39] this is called induced gravity, see also [40]. This represents in our view a deep shift in emphasis concerning the fundamental constituents of our understanding of the structure of our hierarchy of physical theories. I.e., the traditional order of analysis is reversed. Instead of starting from particles or fields and their interaction and frequently only indirectly and at a later stage, arriving at the analysis of e.g. the fine structure of the physical vacuum, we do not regard the vacuum as sort of a stage but rather as the real source of all the objects and phenomena emerging in it.

We developed and described such an approach in the papers, mentioned in the introduction. As a further motivation we would like to cite a passage from [41], p.2202f, just to show that such a point of view is by no means entirely far-fetched: “... A particle means as little to the physics of the vacuum as a cloud means to the physics of the sky. In other words, elementary particles do not form a really basic starting point for the description of nature. Instead they represent a first-order correction to vacuum physics. The vacuum, that zero-order state of affairs, with its enormous densities of virtual photons and virtual positive-negative pairs and virtual wormholes, has to be described properly before one has a fundamental starting point for a proper perturbation-theoretic analysis”.

As to our concrete enterprise, we adopt the following strategy. We assume certain qualitative and, as we think, well-founded properties concerning the underlying microscopic substratum and which are expected to hold in any case, irrespective of the concrete shape of a possible future theory
of Q.Gr. In so far the approach is the same as in [13] and in some of the papers cited there. In a sense one may describe this working philosophy as being similar to a foundation of phenomenological thermodynamics which is based on certain general properties of a supposed underlying theory of statistical mechanics but not on any details or models of the latter (as it is e.g. done in [43]). One may it also call a principle theory in the sense of Einstein (cf. [44]), that is, one postulates certain general principles, to start from, without making too detailed or uncontrolled model assumptions.

To be more specific, by making a few and observationally well-founded assumptions about the behavior of the spectrum and correlations of vacuum fluctuations, we derive a couple of almost model-independent and general results which lead to surprisingly strong constraints for the physics on this microscopic scale. In this way we hope to infer general and structural results about a regime to which we, at the moment, do not have direct experimental access. By the way, we think it is perhaps a funny side remark that in both fields, i.e. phenomenological thermodynamics and our BH-context the almost universal concept of entropy is the really crucial analytic tool to infer some deep results about an underlying microscopic theory, the details of which are not yet known or remain unresolved.

3 The Fundamental Postulates

We presume the two main reasons why the possibility that the entropy-area law may be generated by DoF sitting inside the BH, i.e. within the horizon, has never been seriously taken into account in most of the representations, are the following:

- Superficially, the interior of a BH seems to be an essentially empty and inactive area.

- The area-law has been mainly viewed as an indication of its geometric origin; entropy proportional to horizon area meaning: DoF located on or near the horizon.

On the other hand, people with a stronger statistical-mechanical background may come to a different conclusion. Given that in a first approximation a large part of the BH interior (with the possible exception of its central singularity and its immediate neighborhood) is not entirely different from an arbitrary piece of empty space or space-time being exposed to some gravitational field, we assume that, apart from possible finite distortions (see
e.g. the Unruh-effect), the fluctuation spectrum of the vacuum excitations is at least qualitatively similar in both cases.

In [13] we performed the following thought experiment. The assumption that on very small scales the extent of vacuum fluctuations is very large and extremely large if we approach the Planck scale, even if perhaps not shared by every worker in the field, seems to be corroborated by a wide spectrum of more or less independent inferences. The following model assumptions seems in our view therefore reasonable.

**Postulate 3.1** *It is allowed to replace a piece of space by a coarse-grained statistical model which is composed of microscopic grains of, to supply a typical scale, Planck-size which support elementary DoF which, individually, can strongly fluctuate. In energy units the elementary fluctuations are assumed to be of Planck-energy size.*

In sect.4 of [13] we made a calculation which shows that, given the huge number of roughly Planck-size grains in such a piece of space and provided that the individual grains are allowed to fluctuate almost independently, the total fluctuations in a macroscopic or mesoscopic piece of space of typical physical quantities are still large enough (i.e. macroscopic) as to be observable. More precisely, with \( q_i \) some physical quantity belonging to a microscopic grain (e.g. energy, momentum, some charge etc.) and \( Q_V := \sum_i q_i \) the observable belonging to the volume \( V \), the fluctuation of the latter behaves as

\[
\langle Q_V Q_V \rangle^{1/2} \sim V^{1/2}
\]

with \( N \sim V \) the number of grains in \( V \). This is a consequence of the central limit theorem. As such large integrated fluctuations in a macroscopic region of the physical vacuum are not observed (they are in fact rather microscopic on macroscopic scales), we conclude:

**Conclusion 3.2** *The individual grains or supposed elementary DoF do not fluctuate approximately independently.*

Remark: We note that this fact is also corroborated by other, independent observations.

We can refine the result further (cf. [13]) by assuming that the fluctuations in the individual grains are in fact correlated over a certain distance or are *short-range correlated*. In mathematical form this is expressed as *integrable correlations*. This allows that “positive” and “negative” deviations from the mean value can compensate each other. Letting e.g. \( q(x) \) be the
density of a certain physical observable and \( Q_V := \int_V q(x) d^n x \) the integral over \( V \). In order that
\[
\langle Q_V Q_V \rangle^{1/2} \ll V^{1/2}
\]
we proved in [13] that it is necessary that
\[
\int_V d^n y \langle q(x)q(y) \rangle \approx 0
\]
Remark: For convenience we normalize \( q(x) \) so that \( \langle q(x) \rangle = 0 \) with \( \langle \cdot \rangle \) denoting the expectation in the vacuum state (or, in another context, a thermodynamic equilibrium state).

We made a more detailed analysis in [13] under what conditions property (10) can be achieved. In any case we can again conclude:

**Conclusion 3.3** Nearly vanishing fluctuations in a macroscopic volume, \( V \), together with short-range correlations imply that the fluctuations in the individual grains are anticorrelated in a fine-tuned way, i.e. positive and negative fluctuations strongly compensate each other which technically is expressed by property (10).

We now come to implications derived from the so-called holographic principle. For the time being, we only deal with situations where the spacelike holographic principle holds. That means for example, (quasi)static backgrounds. The reason is that, as our approach develops a relatively new point of view concerning this context, we would like to keep the scenario in a first step free from additional technical complications. We will however briefly comment upon dynamical aspects and time-dependent backgrounds in subsection 7.3 where we discuss a variety of examples. The range of its validity is discussed in e.g. [22]; see also [23] and [24]. We note however that our following analysis will shed a new light on this principle and its true range of validity and will, furthermore, unearth presumably interesting relations to the ideas of Zeldovich and Sakharov mentioned above.

The BH-scenario in asymptotic flat space-time belongs to this class. This leads to the next postulate.

**Postulate 3.4** There exists a class of scenarios in which the maximal amount of information or entropy which can be stored in a spherical volume is proportional to the area of the bounding surface. This is the spacelike holographic principle.
Remark: In some treatments this property is translated into the statement that the number of available DoF is of order (area) or, even stronger, the relevant DoF are situated on the bounding surface. In our view this conclusion is wrong or at least premature as we try to show in the following.

Putting now together the content of postulate 3.1 and postulate 3.4, we already reasoned in [13] that what all this is really implying is that the DoF in the volume, \( V \), are long-range (anti)correlated in a very peculiar way (cf. also certain remarks in section 7 of [15]).

**Conclusion 3.5** From postulate 3.4 we infer that each fluctuation pattern in \( V \) is fixed by the corresponding pattern situated on the bounding surface or in a thin shell about this surface.

This now leads to our final conclusion:

**Conclusion 3.6** The fluctuation pattern in \( V \) is long-range anticorrelated in a fine-tuned way on a microscopic scale and is essentially fixed by the state of the fluctuations on the bounding surface.

Remark: The peculiar relation between long-range-anticorrelations on a microscopic scale and the correlations among extended excitation patterns on a larger (e.g. mesoscopic) scale like e.g. ordinary quantum mechanics is discussed in section 6.

What remains to be done in a next step is to clarify the subtle details of this fluctuation structure. This is a non-trivial task as it turns out that, while the phenomenon of long-range anticorrelations as such is certainly an important property, it is not! the crucial and characterizing property in this specific context. What is really peculiar is the fixation of the bulk DoF by the surface DoF as this latter property is not already implied by the former correlation result. We will show that to achieve this we need yet another prerequisite.

But before we will do this we want to show that the preceding reasoning already allows us to draw some simple but important geometric conclusions. For convenience we assume that the sets of DoF, occurring in the following are countable. We now consider two concentric spheres, \( S_1, S_2 \) with radii \( R_1 < R_2 \). The DoF on \( S_1 \) by assumption determine the DoF inside \( S_1 \). The DoF on \( S_2 \) determine the DoF inside \( S_2 \) and in particular on \( S_1 \). Put differently, in for example three dimensions, we have

\[
\#(\text{modes in } B_i) = \text{const} \cdot R_i^2
\]  

(11)
where $B_i$ is the interior of $S_i$. Furthermore, there are $(R_2/R_1)^2$ different modes on $S_2$ per mode on $S_1$. We denote the individual modes on $S_i$ or in $B_i$ by $\omega_{S_i,j}$ and $\omega_{B_i,j}$ respectively, with mode $\omega_{B_i,j}$ uniquely fixed by mode $\omega_{S_i,j}$. It follows that to each $\omega_{S_1,j}$ corresponds a class $[\omega_{S_1,j}]_{S_2}$ of modes on $S_2$ with cardinality $(R_2/R_1)^2$. By the same token, these latter modes fix $(R_2/R_1)^2$ different modes in $B_2$. As the modes in $B_2$ are standing in a 1:1 relation to modes on $S_1$, it follows from this typical example what additional property is needed to yield this strong and quite unusual result when looked upon with a statistical mechanical eye (subsection 4.3). But before we come to that issue, we want in a first step to clarify in more quantitative detail the type of long-range anticorrelation is already sufficient to understand the entropy-area-law. An instructive example (standing however for many others), taken from the statistical mechanics of phase transitions, shows that this is not the case (see subsection 4.2). But what is perhaps more important, we can learn from this typical example what additional property is needed to yield this result when looked upon with a statistical mechanical eye (subsection 4.3). But before we come to that issue, we want in a first step to clarify in more quantitative detail the type of long-range correlation or influence and in particular its weak spatial decay which seems to be prevalent in this context. We emphasize that we take some pains to approach this question without making ad hoc assumptions like e.g. assuming some particular underlying dynamical model. We only will employ the facts we have described above. The results will have, among other things, some bearing on the range of validity of the various holographic bounds discussed in section [7].

4.1 The Long-Range Character of Influence and Correlations on the Primordial Scale

Superficially considered one may be inclined to think that the property of long-range anticorrelation is already sufficient to understand the entropy-area-law. An instructive example (standing however for many others), taken from the statistical mechanics of phase transitions, shows that this is not the case (see subsection 4.2). But what is perhaps more important, we can learn from this typical example what additional property is needed to yield this strong and quite unusual result when looked upon with a statistical mechanical eye (subsection 4.3). But before we come to that issue, we want in a first step to clarify in more quantitative detail the type of long-range correlation or influence and in particular its weak spatial decay which seems to be prevalent in this context. We emphasize that we take some pains to approach this question without making ad hoc assumptions like e.g. assuming some particular underlying dynamical model. We only will employ the facts we have described above. The results will have, among other things, some bearing on the range of validity of the various holographic bounds discussed in section [7].

4. The Statistical Mechanics of Bulk-Boundary Dependence

Observation 3.7 The modes in $B_2$, induced by the class $[\omega_{S_1,j}]_{S_2}$, are different from each other in $B_2 \setminus B_1$ but coincide within $B_1$. We denote the individual modes on $S_1$ or in $B_1$ by $\omega_{S_1}$ and $\omega_{B_1}$ respectively.
bound does not hold. The possible physical reasons for this exceptional behavior are then isolated in the other subsections.

In principle there are two possible modes of discussion, first: in a more classical statistical way, second: relying more on the arsenal of tools provided by quantum (field) theory. In the litterature one frequently finds a mixture of arguments taken from both fields, even in the same paper. This is however not entirely unreasonable. For one, we are mainly talking about model theories which are designed to elucidate some typical behavior, not about a final fundamental theory. For another, it is by no means clear that quantum theory still holds sway unaltered on those fundamental scales we are interested in. After all, it may only be yet another effective theory which is only correct within certain boundaries as to resolution of space-time.

This being said, we will analyse the above question mainly with the help of ordinary probability theory. We think, the quantum case behaves in a similar way but it is easier (for the time being) to treat the DoF under discussion as ordinary random variables since they can be taken to commute among each other. Furthermore, it is sometimes overlooked that a large part of ordinary quantum theory can be cast into a form of ordinary probabilistic correlation analysis (without for example employing Hilbert space methods), as has so successfully been demonstrated by Bell (16).

That is, in the following we treat the DoF in a subvolume, $V$, plus the ones being situated on the boundary, $\partial V$, as a statistical system, the statistical behavior being mainly induced by the openness of the systems relative to the outside regime (similar to a heat-bath). We restrict our analysis to a coarse-grained view and treat the set of DoF as countably discrete with their respective values also assumed discrete (but this is only a matter of convenience).

**Definition 4.1** We regard the DoF as random variables living on a probability (or sample) space $\Omega$, the sample points, $\omega$, being individual fluctuation patterns at a given time in $V$ and/or $\partial V$ while the DoF are the elementary states of fluctuations located in certain grains, $C_i$ of this extended fluctuation pattern. The DoF in the interior, $V$, associated with the $C_i$ are denoted by $X_i$, the ones on the boundary by $Y_j$. Events, usually denoted by $A, \ldots$ are certain aggregates of sample points (excitation patterns), i.e. admissible (as to technical details of probability theory see below) subsets in $\Omega$.

In the following, particular types of such sets are important. In general probability theory or in the path integral formalism such sets are called cylinder sets. Let $M_\nu$ be an arbitrary subset of elementary random variables $X_i$, i.e. DoF, and let their respective momentary values be $x_i,$
Definition 4.2 We denote by $A(\{M_\nu \ni X_i = x_i\})$ the subset of sample points $\omega$, i.e. fluctuation patterns with the local fluctuations (elementary DoF), $X_i$, in the grains $C_i$ belonging to the selected set $M_\nu$ having the (discrete) values $\{x_i\}$. We assume that the probability of such sets is non-vanishing provided the sets are not empty (see below), that is

$$\text{pr}(A(\{M_\nu \ni X_i = x_i\})) > 0 \quad \text{if} \quad A \neq \emptyset$$

(12)

Corollary 4.3 For a single elementary random variable, $X$ (DoF), one can then define the associated probability distribution by

$$d(x) := \text{pr}(X = x)$$

(13)

with $x$ varying. By the same token we get its expectation value and mean square deviation

$$\bar{x} := \sum_x x \cdot d(x) \quad , \quad \Delta x^2 := \sum_x (x - \bar{x})^2 \cdot d(x)$$

(14)

That is, the respective sets consist of excitation patterns in $V$ with the local fluctuation, $X$, in some $C$ having the fixed value $x$. In the same vein one can define more general types of events respectively subsets. For these and other elementary probabilistic concepts see e.g. [47] or [48].

The following observation is important. The DoF on the boundary, $\partial V$, already generate the full algebra of events in the interior.

Observation 4.4 The holographic principle in the way we are using it in this paper tells us that a full specification of the momentary values of the DoF on the boundary, $Y_j = y_j$ for all $Y_j$ on $\partial V$ fix the values of all the $X_i$ in $V$. For our stochastic analysis this implies the following.

i) Each random variable, $X_i$, is a function of the set of $Y_j$, i.e.

$$X_i = F_i(\{Y_j\}_{\partial V}) \quad , \quad x_i = f_i(\{y_j\}_{\partial V})$$

(15)

and the $f_i$ define a map from the sample space $\Omega_{\partial V}$ to $\Omega_V$

$$f : \{y_j\}_{\partial V} \rightarrow \{x_i = f_i(\{y_j\})\}$$

(16)

ii) By the same token, every event $A(\{X_i = x_i\}_V)$, $X_i$ a single DoF, is either the one-point set $A(\{Y_j = y_j\}_{\partial V})$ for some configuration $\{y_j\}$ on the boundary if $\{x_i\}$ is lying in the image set of $f$ or it is empty. In this way each general event in $V$ can be formed from corresponding sets in $\partial V$ by using the usual set-theoretic constructions employed in probability theory.
We now want to estimate the strength of the statistical influence the boundary DoF’s are having on the DoF’s in the interior. This turns out to be a really subtle point. As we want to avoid adhoc model assumptions, concepts like forces or direct interactions among the DoF’s are presumably too primitive, while the information encoded in correlation functions is too limited for our purposes. Therefore, we will introduce a new concept, which amalgamates the virtues of the preceding concepts while avoiding the respective drawbacks. We will call it influence function.

Remember that in previous sections we found that the DoF are long-range correlated. Traditionally, in statistical physics one expresses such a property by means of correlation functions. We found however, that in our context, where we have only a very particular (and limited) type of information at our disposal, correlation functions are a too crude and inflexible tool as a starting point (this will become clearer in the following analysis). A, in our view, better starting point are conditional probabilities. An elementary example is the following.

**Definition 4.5** We denote by \( \text{pr}(X_i = x_i | Y_j = y_j) \) the probability of the event \( X_i = x_i \) given \( Y_j = y_j \). In more detail it reads

\[
\text{pr}(X_i = x_i | Y_j = y_j) := \frac{\text{pr}(X_i = x_i, Y_j = y_j)}{\text{pr}(Y_j = y_j)} \tag{17}
\]

provided that \( \text{pr}(Y_j = y_j) \neq 0 \). In a similar vein one can define more general conditional probabilities. For example instead of a definite value one can allow the respective values to lie in certain sets, or one can define

\[
\text{pr}(X_i = x_i | \{M_{\nu} \ni Y_j = y_j\}) \tag{18}
\]

with \( M_{\nu} \) a subset of boundary random variables.

We know the following.

**Observation 4.6** Denoting by \( M_V = \{X_i\}_V \) the full set of elementary random variables in \( V \), that is, the full set of local fluctuations in \( V \), and by \( M_{\partial V} = \{Y_j\}_{\partial V} \) the full set of elementary random variables on the boundary, we have

\[
\text{pr}(X_i = x_i | \{M_{\nu} \ni Y_j = y_j\}) \to \delta(x - x_0) \tag{19}
\]

for a sequence of increasing sets \( \{M_{\nu} \ni Y_j = y_j\} \) with \( M_{\nu} \to M_{\partial V} \) and

\[
x_0 = f_i((\{y_j\}_{\partial V}) \tag{20}
\]
Definition 4.7 We abbreviate the above conditional probability distribution by
\[ d_\nu(x) := \Pr(X_i = x | \{M_\nu \ni Y_j = y_j\}) \]  
(21)

(omitting some of the indices for notational convenience)

Now the following will happen. For each member of the increasing sets \((M_\nu, \{y_j\})\) we can calculate the expected value of \(X_i\) and its variance, i.e.
\[ x_\nu := \sum_x x \cdot d_\nu(x), \quad \Delta_\nu x^2 := \sum_x (x - x_\nu)^2 \cdot d_\nu(x) \]  
(22)

The values \(x_\nu\) will approach the above limit value \(x_0\) while the mean square deviation will become smaller and smaller with increasing \(\nu\) and vanishes in the limit. In other words, the whole distribution becomes more and more concentrated around \(x_0\).

Remark: Note that due to the huge number of constituents in each macroscopic volume (compared to the Planck scale) we expect to observe an almost continuous behavior.

As we already remarked above, our aim is to quantify the influence a DoF on, say, the boundary, \(\partial V\), exerts on a given fixed DoF in the interior, \(V\), and, in particular, its dependence on the spatial distance between the DoF. In this enterprise we only want to use the holographic information we described above (observation 4.6) plus very few, as we think, well-motivated simplifying, assumptions. The natural candidates are the conditional probability distributions, \(d_\nu(x)\), which we can (at least in principle; on our macroscopic scale such observations or measurements can at the moment presumably not yet be carried out) compare with \(d(x)\), i.e. the unconditioned probability distribution (that is, without making any assumptions about possible outcomes of observations of boundary-DoF). The idea is to study the change of \(d_\nu(x)\) when the sequence of increasing sets, \((M_\nu, \{y_j\})\), approaches the limit set, \((M_{\partial V}, \{y_j\}_{\partial V})\).

It is reasonable to add (in a thought experiment!) one DoF, \((Y_j, y_j)\) after another, i.e.
\[ (Y_1; y_1) \rightarrow (Y_1, Y_2; y_1, y_2) \rightarrow \ldots \rightarrow (Y_1, \ldots, Y_{N_{\partial V}}; y_1, \ldots, y_{N_{\partial V}}) \]  
(23)

Furthermore, we will choose a simple geometric set-up, i.e. we assume that \(V\) is a ball, \(B_R\), of radius \(R\), centered at the origin and hence \(\partial V\) a sphere, \(S_R\) of radius \(R\).
The functions \( d_\nu(x) \), considered as a whole, have however the disadvantage that they will move around in the respective variable space (at least for small \( \nu \)), so that it may be difficult to use their changing shape as a quantitative measure of influence. Note that there exists a, at first glance, natural distance-measure in function space, namely the \( L^2 \)-norm:

\[
\| d_\nu(x) - d_\mu(x) \|^2 := \sum_x |d_\nu(x) - d_\mu(x)|^2
\]

However, while this norm is widely used in general, it does not really reflect the particular property that the variance vanishes in the limit, which we think is actually the characteristic property and provides us with a good measure of influence. So we employ the following, as we think, characteristic measure of influence.

**Observation 4.8** We assume that the variance, \( \Delta_\nu x^2 \), is a good candidate for measuring the influence of the boundary on a bulk-DoF.

From what we have said above, we start from \( \Delta_\nu=0 x^2 = \Delta x^2 \), i.e. the variance with unconstrained boundary-DoF’s and end up with \( \Delta_N\partial V x^2 = 0 \), that is, with all boundary-DoF’s fixed.

Remark: Note that in our analysis the random variables are assumed to be discrete, i.e. \( N_{\partial V} \) is a very large but finite number.

Now the following seems to be natural while it cannot of course be rigorously proved. With \( N_{\partial V} \) large and

\[
\Delta_N\partial V x^2 - \Delta x^2 = -\Delta x^2
\]

a finite number, each DoF, \( (Y_j, y_j) \), will add a small amount to the vanishing of the variance in the limit when we perform the process described in equation (23). We have however to take the following into account. Taking in the general case three events \( A, B, C \) the following holds

\[
pr(A \cap B \cap C) = pr(A|B \cap C) \cdot pr(B|C) \cdot pr(C)
\]

Only in the case where \( B \) and \( C \) are statistically independent do we have

\[
pr(B|C) = pr(B)
\]

Applied to our case we have for example

\[
pr(X = x, Y_1 = y_1, Y_2 = y_2) = \]

\[
pr(X = x|Y_1 = y_1, Y_2 = y_2) \cdot pr(Y_2 = y_2|Y_1 = y_1) \cdot pr(Y_1 = y_1)
\]
and in general

\[ pr(Y_2 = y_2|Y_1 = y_1) \neq pr(Y_2 = y_2) \] (29)

In other words, if we fix \( Y_1 \) this will also have an effect on the next boundary-DoF \( Y_2 \) in our arbitrary but fixed selection, the value distribution of which will no longer be completely independent of the previous elements. This means, the summed effect of a large number of such boundary-DoF on an interior DoF may be somewhat more involved as in the case of complete independence. On the other hand, we are only interested in the average influence and, in particular, on the spatial dependence of this influence. Therefore we proceed in the following way.

We make the following, as we think reasonable, assumption.

**Assumption 4.9** We assume that, starting from the original unconstrained variance, \( \Delta x^2 \), of some arbitrary but fixed DoF in the interior, \( V \), we can represent its ultimate vanishing after all boundary-DoF have been fixed by a sum over the individual influences or effects of these DoF, modified by a correction term which (see above) depends on the position in the selection process of the respective boundary-DoF. I.e. we write

\[ \Delta x^2 = \sum_{(Y_j, y_j)} I(|r_j - r_i|; \xi_j) \] (30)

The meaning of the various terms is the following: We call \( I \) an influence function. It depends on the distance between the respective DoF on the boundary and the DoF in the interior and on a set of parameters, abbreviated by \( \xi \), which encode its position in the selection process as indicated above. \( \xi \) contains in particular the distances of \( Y_j \) to the preceding boundary-DoF in the selection. We make then the further simplifying assumption that we are allowed to extract the spatial dependence of \( I(|r_j - r_i|; \xi_j) \) on \( |r_j - r_i| \) and write

\[ I(|r_j - r_i|; \xi_j) = I_1(|r_j - r_i|) \cdot g_j(\xi_j) \] (31)

Remark: It is the function \( I_1(|r_j - r_i|) \) we are mainly interested in. It represents the individual influence of \( Y_j \) on \( X_i \) when no other boundary-DoF have been fixed. So it comes nearest to what one views as correlation.

Note now that the whole microscopic and presumably messy details of the behavior of the above sum are contained in the functions \( g_j(\xi_j) \) while the spatial part, \( I_1(|r_j - r_i|) \), should be quite robust. In the \( g_j(\xi_j) \) is, among other things, also encoded the details of the behavior on the chosen values.
\{y_i\} of the DoF preceeding \(Y_j\). All these details should only marginally affect the gross spatial behavior. So we feel encouraged to write

\[
\sum_{\mathbf{r}_j \in S_R} I_1(|\mathbf{r}_j - \mathbf{r}_i|) = O(\Delta x^2)
\]  \(32\)

with the number of \(\mathbf{r}_j\) being proportional to the area of the sphere \(S_R\). Due to the huge number of involved DoF we can go over to an integral and get

**Conclusion 4.10** The relevant numerical relation is

\[
\int_{S_R} I_1(|\mathbf{r} - \mathbf{r}_0|) d\mathbf{r} = O(\Delta x^2)
\]  \(33\)

with \(\mathbf{r}_0\) some point in the interior.

We are interested in the decay behavior of \(I_1(|s|)\) for large \(|s|\). As usual one makes an ansatz like

\[
I_1(|s|) = |s|^{-\alpha} \cdot i(s)
\]  \(34\)

for non-vanishing \(s\) with \(i(s)\) being some numerical, bounded and non-decaying function. One could equally well make e.g. the choice

\[
|s|^{-\alpha} \to (1 + |s|^\alpha)^{-1}
\]  \(35\)

to avoid a singularity for vanishing \(s\), but such details are not really important. Our central observation is now the following.

**Observation 4.11** In

\[
\int_{S_R} I_1(|\mathbf{r} - \mathbf{r}_0|) d\mathbf{r} = O(\Delta x^2)
\]  \(36\)

the rhs is essentially a finite number not depending on \(R\) as it represents the variance of some unconstrained bulk DoF, while the lhs may in principle depend on \(R\). For large \(R\) we hence can infer

\[
\int_{S_R} (|\mathbf{r} - \mathbf{r}_0|)^{-\alpha} d\mathbf{r} \approx \text{const}
\]  \(37\)

independent of the radius \(R\).
Due to the inherent symmetry we can make the special choice \( r_0 = (0, 0, z_0) \) with \( z_0 = R \cdot k \) and \( 0 \leq k \leq k_0 < 1 \). We have to evaluate the integral (in three space dimensions):

\[
R^2 \cdot \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos \theta (x^2 + y^2 + (z - z_0)^2)^{-\alpha/2}
\]

\((x = R \sin \theta \sin \phi, \ldots)\) With the above choice for \( z_0 \) this yields

\[
2\pi \cdot R^{(2-\alpha)} \cdot \int_{-1}^{+1} du ((1 + k^2) - 2ku)^{-\alpha/2}
\]

Remark: The integrand is always positive. This follows already from the structure of the integrand, we started from. On the other hand, we have

\[
(1 + k^2) - 2ku = (1 - k)^2 + (2k - 2ku) > 0
\]

as \( k^2 < 1 \) and \( 2k - 2ku \geq 0 \) since \( |u| \leq 1 \).

**Conclusion 4.12** We see that we can only avoid a contradiction if we have in leading order

\[
\alpha = 2
\]

This implies

\[
I_1(|r - r_0|) \approx |r - r_0|^{-2}
\]

in three space dimensions and analogous results in other dimensions. In other words, the influence decays like a Coulomb force-law for sufficiently large distances.

For later use we can calculate the integrated influence of a sphere on a DoF in the exterior, i.e. for \( k \geq k_0 > 1 \). For \( \alpha = 2 \) the integral

\[
\int_{-1}^{+1} du((1 + k^2) - 2ku)^{-1}
\]

is dominated for large \( k \) by the first term in the denominator. We thus have

**Corollary 4.13** For points outside the sphere \( S_R \) with (for convenience \( r_0 = (0, 0, k \cdot R) \)) the above integral decays as \( \sim k^{-2} \) for large distances \( k \) from the sphere.

This has important consequences which will be discussed in section 7.

It is perhaps worth mentioning that a clustering of correlations \( \sim |x|^{-2} \) in four space-time dimensions occurs also in the context of vacuum-Bell-inequalities in [49], theorem 4.1 or in massless quantum field theories (cf. [50]).
4.2 Long-Range Correlations and Goldstone Phenomenon in Ordinary Physics

In this subsection we want to show by means of counterexamples that long-range (anti)correlations alone are not sufficient to entail an area-law behavior of entropy. It turns out that frequently the underlying reason seems to be the existence of Goldstone modes or other collective excitations, representing small fluctuations around the ordered state.

In [13] we discussed in some detail the example of the (3-dimensional) harmonic crystal in a pure phase. This means in this context that its global position is assumed to be fixed in space and does not fluctuate. In this and related systems we observe the phenomenon of spontaneous symmetry breaking of a continuous group. In this particular case the translation group is broken with the phonons as Goldstone-modes. In such a situation we have long-range (anti)correlations between certain observables, in our case these are the atomic positions and the respective deviations from their equilibrium positions.

The crucial estimate in [13] having a certain bearing on our present discussion is equation (35) in section 4, the meaning of which is the following. With the global position of the crystal being fixed, the fluctuations of the individual atomic positions can be bounded by

\[ \delta x_i^2 := \langle (x_i - \langle x_i \rangle)^2 \rangle \lesssim a^2 \] (44)

with \( a \) the lattice spacing. For simplicity we take a row of \( j \) atoms on, say, the x-axis, starting from \( x_0 \) and ending in \( x_j \). Denoting \( x_k - x_{k-1} \) by \( u_k \) with \( \langle u_k \rangle = a \), we showed in [13] that due to (44) we can infer that

\[ \langle \sum_{k \neq k' = 1}^j (u_k - a)(u_{k'} - a) \rangle \approx -\langle \sum_{k=1}^j (u_k - a)^2 \rangle \lesssim -j \cdot (2a)^2 \] (45)

In other words, the left hand side of the equation which contains a double sum over the correlations of the relative atomic positions, \( u_k \), is strongly negative, meaning that a positive elongation at site \( k \) has to be compensated by negative elongations at other sites so that the fluctuation of \( x_j \) which is essentially a sum over all the \( u_k \) remains small. This is an example of long-range anticorrelation.

Nevertheless, irrespective of the presence of this long-range order the entropy of subvolumes is proportional to the volume and not! the area of the boundary. That is, while it is clear that an ordered phase has an entropy which is lower than in the unordered phase, it is still an extensive quantity.
Remark: Note that the standard thermodynamic explanation for the (almost) additivity of entropy relies on short-range correlations and the possibility to divide a large system into a (large) number of weakly interacting small but still macroscopic subsystems. In the presence of long-range correlations this is no longer possible.

The (perhaps surprising) deeper reason for this volume behavior of e.g. the entropy in the face of long-range order can be understood in our view in the following way. Phase transitions as a consequence of some spontaneous symmetry breaking fall more or less in two broad classes. First, the class with the broken symmetry belonging to a continuous (Lie-) group. Second, the class where the symmetry happens to be discrete, a typical example being the Ising model. In the first case, the Goldstone theorem (in fact the consequence of the spontaneous breaking of a continuous symmetry) tells us that, at least in the case of short-range interactions, there exist gapless Goldstone modes, i.e. modes with the energy-momentum dispersion law, $\varepsilon(p)$, passing through $\varepsilon = 0$ for $p = 0$.

Remark: Strictly speaking, in the non-relativistic regime these modes usually have, except in the interaction-free case, a finite lifetime, i.e. a certain $p$-dependent width. We neglect these details in the following.

Employing the Landau picture of elementary excitations in many-body theory (see e.g. [51] or [52]), meaning that approximately the Hamiltonian or the excitation spectrum can be understood as a sum over weakly interacting collective or elementary excitations (the “normal modes” of the system), we can write

$$ H \approx \sum_i H_i \quad \text{with} \quad H_i := \int \varepsilon_i(p) \cdot a_i^\dagger(p)a_i(p) \, dp $$

We can now infer that the total entropy, $S$, is roughly

$$ S \approx \sum_i S_i \quad \text{or at least} \quad S \geq \sum_i S_i $$

with $S_i$, belonging to the free system built over $\varepsilon_i(p)$, being extensive, i.e. being proportional to the volume. In our case there does exist at least one such excitation branch (the Goldstone excitations). This means that while there is of course an increase of order in the system below a phase transition point or line, resulting in a decrease of entropy, the latter is still homogeneous in the volume as it contains the contributions of basically free systems.

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In our above example the Goldstone modes are the phonons, the approximative system is a free gas of phonons. In classical terms, they represent the normal coordinates belonging to collective oscillation modes of the lattice atoms about their equilibrium positions. To make a clearer connection to our BH-topic, we may assume that for example the atoms on the boundary of a macroscopic subvolume are held fixed. But nevertheless, due to the fact that there can exist phonons with arbitrarily low energy (small oscillations), for non-vanishing temperature $T$, the modes belonging to the enclosed bulk system fill a full region of phase space. That is, the phase space volume of the interior bulk system with the boundary values held fixed still fulfills

$$\Omega_{\text{bd}}^\text{bd} \sim e^V \quad \text{i.e.} \quad S \sim V$$  \hspace{1cm} (48)

This latter procedure is for example one of the methods to generate a pure phase in the bulk interior. In the case of a spin system like e.g.,

$$H(S) = \sum_{ik} J_{ik} \mathbf{S}_i \cdot \mathbf{S}_j$$  \hspace{1cm} (49)

one proceeds by fixing the spins on the boundary of some large subvolume and study the interior system in, say, the limit $V \to \infty$. Below a critical point the system will display some preferred direction of magnetisation (or some other kind of order parameter), that is, develop long-range order with the thermodynamic entropy being proportional to the volume. The gapless excitations are the magnons in this case.

That our explanation is reasonable, can be tested by analysing the situation for the second class of systems displaying spontaneous symmetry breaking. A typical representative is the Ising model with the spins having for example the two orientations $\pm 1$. The broken symmetry is now discrete; consequently there are in general no gapless Goldstone excitations. On the other hand, one can show ([53],[54],[55]) that below the critical point (the realm of spontaneous magnetisation) the truncated correlation functions decay exponentially. That is, we have the situation, discussed earlier, of short-range correlated systems with the usual volume behavior of entropy.

There are a few examples where systems develop a gap due to long-range interactions (e.g. Coulomb). Another case in point is the famous BCS-model of superconductivity. In this example we have long-range correlations and Goldstone modes not passing through $E = 0$ for $p = 0$. However, in all of these cases the energy gap above the groundstate is of atomic size and in general much smaller than the energy scale given by the temperature $T$. As in the canonical or grandcanonical ensemble, the energy is allowed
to stretch over, in principle, arbitrary scales, entropy is still linear in the
volume. Matters may become different however, if these two scales become
comparable. This last observation leads over with almost necessity to the
topic addressed in the next subsection.

4.3 Systems with a large Energy Gap

In subsection 4.1 we analysed the fine structure of the long-range anticorre-
lations which show up in connection with the holographic principle in our
approach. Now we provide a deeper physical reason for this strange and
counterintuitive behavior in form of a dynamical or spectral property of
certain Hamiltonians which may play a role in this field.

We have finally (cf. subsection 4.2) located the essential ingredient which
was still missing.

**Conjecture 4.14** The crucial precondition for systems of statistical me-
chanics having an entropy which is proportional to the area of the bound-
ning surface is a sufficiently large gap in the energy spectrum of the bulk-
Hamiltonian, defined on \( V \) with fixed boundary conditions on \( \partial V \), above the
ground state energy or between a few low-lying excited states and the rest
of the energy-spectrum. More precisely, the gap has to be so large that it
exceeds the typical excitation energies being considered to be present in the
respective situation.

That is, given a system-Hamiltonian, \( H_V \), over the sub-volume \( V \) with, for
reasons of simplicity, discrete spectrum, \( \{ \varepsilon_i \} \), we assume that

\[
\varepsilon_0 \leq \varepsilon_1 \leq \ldots \leq \varepsilon_N \ll \varepsilon_{(N+1)} \leq \varepsilon_{(N+2)} \ldots
\]  

I.e., we assume a large energy gap, \( \Delta := \varepsilon_{(N+1)} - \varepsilon_N \) between the energy
values \( \varepsilon_N \) and \( \varepsilon_{(N+1)} \) with \( N = O(1) \), which is so large that the typical
energies, \( E \), existing in the system fulfill

\[
E < \varepsilon_{(N+1)}
\]  

Physically this means that in the usual situation only finitely many energy
levels are occupied while in principle the total number of eigenvalues of \( H_V \)
may nevertheless be infinite.

This observation needs however some more specifications. A Hamilton-
ian defined over a finite volume, \( V \), needs for its complete specification
boundary conditions on its bounding surface, \( \partial V \). That is, we state the
following:
Observation 4.15 For our purpose Hamiltonians, $H_{V}^{bd}$, are relevant which for each selected boundary condition, bd, on $\partial V$ out of a class of admissible conditions, Bd, have spectral properties as assumed above, i.e.

$$
\varepsilon_{0}^{bd} \leq \varepsilon_{1}^{bd} \leq \ldots \leq \varepsilon_{N_{bd}}^{bd} \ll \varepsilon_{(N+1)bd}^{bd} \leq \varepsilon_{(N+2)bd}^{bd} \leq \ldots
$$

with $N_{bd} = O(1)$ and $\Delta H_{V}^{bd} = \varepsilon_{(N+1)bd}^{bd} - \varepsilon_{N_{bd}}^{bd} > E$ uniformly in $bd \in Bd$, with $E$ some typical energy scale available in the respective scenario.

The question is of course, do there exist physical mechanisms which generate such a peculiar spectral behavior?

It turns out that a closer inspection of this problem leads to surprising consequences and ramifications which go far beyond the, at first glance, seemingly isolated technical question we are posing. To put it briefly, we found that behind this question is lurking the question of the microscopic (causal) organisation of our space-time. Or stated differently, what is called for are new types of space forms which go substantially beyond our ordinary classical (continuum) geometries.

We should note that the existence or necessity of energy gaps shows up also elsewhere in modern high-energy physics. In Kaluza-Klein theories, for example, we have a tower of widely separated energy scales, labelled by the excitation modes of the small internal space. In supersymmetry it is also argued that we do not see (most of) the supersymmetric partners because they are so heavy.

But here the situation is different and much more involved. In our case the distribution of energy (eigen)values in the bulk follows a surface-law because the effect of the surface is strongly felt in the interior. Put differently, the underlying reason is not some internal small space but rather the entangled or in some sense non-local microscopic fine structure of the real geometric space-time.

The reason is the following. It is obvious that the holographic principle introduces a specific kind of quantum non-locality into the framework which seems to extend the many forms of “non-locality”, which are almost ubiquitous in ordinary quantum theory (whereas they are frequently “discussed away” as it is felt that they are in conflict with the locality dogma). The most prominent is the pure quantum phenomenon of entanglement.

Conjecture 4.16 We conjecture that the kind of quantum non-locality, observed in the area-law of BH-physics and the holographic principle, and the various aspects of non-locality and (long-range) entanglement being present in quantum theory as such, are of exactly the same nature. They are both
the result of a particular non-local (with respect to the classical realm of space-time!) microscopic organisation of quantum space-time.

We described the deep structure of space-time in recent work in more detail (see e.g. [42], [60], [61] and further references given there). One should also note that closely related phenomena do occur in the small-world scenario (see e.g. [62] for a rigorous discussion and more references).

Recently we came upon a promising geometric generalisation of Riemannian geometry in pure mathematics, called subriemannian geometry or Carnot-Caratheodory spaces, which seems to be able to encode some aspects of this geometric “double structure” we have in mind and which we called wormhole spaces. That is, a classical continuum surface structure with an ordinary distance metric embedded in an ambient hyperspace which allows for short-cuts between classically widely separated regions (cf. e.g. [63] or [64]).

On the other hand, we have to provide arguments that the long-range entanglement structure, encoded in the microscopic structure of space-time (at or near the Planck level) has as one of its consequences this mentioned gap-structure in the excitation spectrum. As this requires a quite extensive (and technical) investigation of its own we will give the necessary details elsewhere.

5 The Interior of the BH on the Microscopic Scale

From the results of the preceding section we can now develop a picture of the microscopic state of a BH (for reasons of simplicity we only discuss the Schwartschild-BH). We remarked several times in this paper that we regard our framework among other things as an extension and generalisation of old ideas of Sakharov, Zeldovich et al (induced gravity). The BH-interior is in our view a particularly instructive example.

In contrast to ordinary regions of, for example, the Minkowski vacuum, the characteristic property of the BH-interior on a microscopic scale is in our approach that the vacuum fluctuation structure is deformed by the central singularity. While macroscopically this singularity is essentially structure-less (the only characteristic being the central mass, \( M \)), we think that the microscopic deformation structure of the vacuum fluctuation pattern in the BH-interior expresses the history of the formation of the BH. This was already mentioned as a possible source of BH-entropy in the literature. Our microscopic analysis supports this point of view.
We will come to the interesting possibility of relating the macroscopic (classical) solutions of the Einstein equations to the corresponding microscopic vacuum fluctuation patterns in subsection 7.3. As to the BH-solution we can formulate the following conclusion

**Conclusion 5.1** From our analysis the following picture does naturally emerge. The different possible ways of creating the same macroscopic BH are expressed in the different microscopic fluctuation patterns existing in the BH-interior. An observer being located in the exterior of the static BH has in general no knowledge of this microscopic fine structure. It follows that for him all these microstates have the same probability. This implies that for him the BH is in a state of maximal entropy.

### 6 Mesoscopic Excitation Patterns and their Correlations

In the introductory sections we argued that the fluctuation spectrum has to be long-range (anti)correlated. Otherwise it would be possible to observe the integrated fluctuations in macroscopic subvolumes, which does not conform with our general experience. One should note that this argument applies, in the first place, to the state we call vacuum on a macroscopic or mesoscopic scale; put differently, to the ensemble of microscopic excitation patterns which look macroscopically like the vacuum. Here we employ the same philosophy as in statistical mechanics, where whole subclasses of microstates are subsumed under the corresponding macrostates.

The situation is slightly different for states containing particle excitations. These particle excitation patterns are extended regions of small but coherent deformations of the vacuum fluctuation structure and last for macroscopic times. Here we share the working philosophy of e.g. Sakharov mentioned above. But according to the general holographic philosophy and due to the long-range correlations we described in the preceding sections, every particle excitation, localized for example deep in the interior of $V$, has its counterpart in form of a characteristic excitation pattern on the boundary of $V$.

**Observation 6.1** From the above argument of a correspondence of an internal particle excitation and a unique boundary excitation (microscopic one-one correspondence or with respect to equivalence classes) it follows that only a certain amount of particles can be stored in a finite volume, with this number depending on the area of the bounding surface. More specifically, an
$N$-particle state, mesoscopically localized in the interior, has to be expressed by a boundary excitation pattern which corresponds uniquely to this interior state (or in the sense of classes of microstates). But the number of different possible boundary states is proportional to the area of the boundary.

In connection with such mesoscopic (or macroscopic) excitation patterns there exists another interesting question which has to be clarified in order to show the consistency of the framework. We learned that on the primordial level the respective fluctuations are long-range correlated. We argued that the spatial influence decays like $\sim |r|^{-2}$ in three space dimensions. On the other hand, in mesoscopic or macroscopic model theories, with constituents such particle-like excitations, we observe on these scales of lesser resolution of space-time all possible kinds of decay of correlations, from short-range to long-range. That is, we have for instance to explain how on a coarser scale short-range correlations between the respective constituents of a model theory do emerge from long-range correlation among the elementary DoF on a microscopic scale.

**Observation 6.2** The task consists of explaining a partial decoupling between these different scales of resolution. Note that compared to e.g. the Planck scale all ordinary scales are of macroscopic size. As may be expected, the decoupling will be a result of a certain coarse-graining and averaging-out of finer details.

To discuss this problem in a more concise form we introduce some notation.

**Definition 6.3** The subset of microstates, being compatible with the macroscopic state we call vacuum we denote by $\Omega_{\text{vac}}$

\[ \Omega \supset \Omega_{\text{vac}} = \{ \omega_{\text{vac}}^j \} \tag{53} \]

with $\Omega$ the total set of possible microstates. By the same token the set of microstates belonging to a certain particle state, $\psi$ (e.g. a wave function), is denoted by

\[ \Omega_{\psi} = \{ \omega_{\psi}^j \} \tag{54} \]

etc. Alternatively we denote these subsets or classes by $[\text{vac}]$, $[\psi]$ etc.

It is easier to discuss the notion of correlations in the realm of statistical mechanics and ensembles. That means, we have a macrostate consisting of a number, $N$, of particles confined to some volume in space (e.g. a temperature state). To discuss correlations one typically uses the ensemble
picture. In other words we assume to have an ensemble, \( \Lambda = \{ \psi^N_i \} \), of \( N \)-particle (quantum)-states belonging to the macroscopic state. Now, on the two levels of resolution, the mesoscopic or macroscopic level, denoted by \( II \), and the microscopic level, denoted by \( I \), the situation is described in the following way.

Let us for example assume that we want to describe the correlations between the individual localisations (positions) of the \( N \) particles. This is a natural concept in statistical mechanics. Each \( \psi^N_i \) on level \( II \) comprises a class of microscopic configurations on level \( I \), i.e.

\[
[\psi^N_i] = \{ \omega^i_j \}
\]

(55)

On level \( I \) the mesoscopic ensemble, \( \Lambda \), is represented by

\[
\Lambda = \{ \psi^N \} = \{ [\psi^N_i] \}
\]

(56)

Remark: We can as well discuss correlations in pure quantum many-particles states. Then we start directly from states like \( \psi^N \) instead of ensembles like \( \Lambda \).

The important point is that in the model theories on level \( II \) the microscopic fine structure, being present in the primordial states, \( \omega_j \), is not encoded or at most in an averaged sense. This holds in particular for the long-range nature of the correlations on level \( I \) and the correspondence of bulk and boundary excitations. One can state this in a possibly more illuminating way. The observables on level \( II \) and level \( I \) are entirely different and only loosely coupled to each other. While on level \( II \) we are for example interested in positions of particles, i.e. weak large-scale deformations of fluctuation patterns on level \( I \), and their correlations, the long-range correlations on level \( I \) exist mainly between the individual elementary fluctuations or DoF. These latter correlations are averaged or washed out in the transition from level \( I \) to level \( II \).

Conclusion 6.4 From the above it follows that the correlation structures of level \( I \) and level \( II \) are decoupled to a large extent or are related only in a relatively subtle way. As to this latter point we think e.g. of the phenomenon of entanglement and similar quantum phenomena.

7 Applications

In this section we want to apply our framework to a variety of issues and problems which have been raised in connection with the holographic principle and related entropy bounds, with the aim of supplying answers from
our point of view or illuminate the problem under discussion from another angle. The topics to be discussed will be: i) the existence of a natural cutoff, ii) the species problem, iii) cosmological backgrounds like closed-static or time-dependent which seem to imply that the simple spatial holographic principle has to be generalized, iv) the problem of unitarity.

7.1 The Problem of a Natural Cutoff

This problem becomes particularly virulent in practically all approaches which use quantum field theoretic methods. We do not intend to give a review of this problem as it is more or less ubiquitous (see for example [10], a catchword being brickwall). Usually such a high energy cutoff is introduced in a relatively adhoc manner whereas frequently the Planck scale is invoked in this context. It is necessary to render many continuum calculations finite, in particular the entropy itself.

We surmise however that these continuum-field calculations are only approximations, being correct only for low energies (i.e. exactly the opposite end of the usual scenario). The findings of our present paper are perhaps able to make the existence of such a cutoff more natural. We remind the reader that our starting point was the observation, almost rigorously proved (at least by physical standards), that by necessity the fluctuation spectrum in the (quantum) vacuum (of energy, momentum or what else) is both long-range and strongly anticorrelated on microscopic scales (which tacitly means, according to general folklore, the Planck scale).

That means that on already very small scales positive and negative deviations from some average, representing the macroscopic vacuum, have to compensate each other (anticorrelation) and that this compensation pattern is quite rigid over large distances (long-range correlated).

Conclusion 7.1 In the light of these two characteristics of the vacuum fluctuation pattern, it appears to be reasonable to associate the typical size of the grains of synchronous fluctuation with the Planck scale, thus leading to a relatively natural cutoff in length, energy etc.

7.2 The Species Problem

To discuss this point, we can use the notations introduced in section 6. The species problem is for example addressed in [11], section 2.3 or in [22], section II.C.4, to mention just a few sources. It consists roughly in the following. In the standard calculations of the various entropy bounds, made in the literature, the number of different species of particles play a role, which are
assumed to be confined to some ball, $V$ of radius $R$. One has for example for a gas of photons the sequence of relations

$$R \geq 2E, \ E \sim ZR^3T^4, \ S \sim ZR^3T^3$$

(57)

with $Z$ the number of particle species, $E$ the thermodynamic energy, $T$ the temperature and $S$ the entropy. From this one can infer

$$S \lesssim Z^{1/4}A^{3/4}$$

(58)

(see [22], loc.cit.) so that we can violate the spherical entropy bound if $Z$ is greater than $A$. In short, the calculation of material entropy depends on the number of different particle species, while the spherical entropy bound is purely geometric.

In our framework this problem is resolved in the following way. With the basis of everything being the microscopic (Planck-size) fluctuation pattern of the elementary DoF, we argued in e.g. section 6 that particle/field excitations happen to be large-scale deviations of this ground pattern. This implies that on a truely small scale the alleged different particle excitations in such an extremely densely packed ensemble of particles lose their individuality which they have on small energy scales and end up in the general background of excitations of the elementary DoF. That is, on the microscopic scale there do no longer exist different species. This then explains the purely geometric character of the area law.

### 7.3 The Range of Validity of the Spatial Holographic Bound

In [22] and elsewhere (e.g. [24] or [25]) counter-examples are given, which show that the simple spatial holographic bound in its original purely geometric form has only a limited range of application and has hence to be given a more general meaning (covariant holographic principle). We refer also to the recent [67] concerning cosmological entropy bounds. In the following we want to show that our approach sheds some new light on the unifying principles underlying these seemingly different bounds and, a fortiori, allows to draw important conclusions about the microscopic dynamics, going on in the deep-structure of the quantum vacuum and its relation to the cosmological solutions of the Einstein equations. Therefore, our microscopic analysis may be able to complement the various geometric extensions of the original, perhaps too narrow, idea.
7.3.1 The Closed Static Universe

One counter-example is a space-time, $\mathcal{M}$, containing a closed space-like hypersurface $\mathcal{O}$ (see [22], section IV.B.1). We divide $\mathcal{O}$ into

$$\mathcal{O} = V \cup Q$$

with $Q$ a small ball-like set in $\mathcal{O}$ (and thus having a small boundary). In $V$ we place a certain macroscopic amount of matter having a macroscopic entropy. By making $Q$ and thus the common boundary sufficiently small, the geometric area-law can easily be violated for the large volume $V$.

Our detailed analysis in section 4.1 shows that the area-law should not be regarded as a God-given geometric law, coming somehow from outside, but, quite to the contrary, is the result of a very subtle microscopic correlation and influence structure between, for example, bulk and boundary of a region. What is really crucial is the influence formula

$$I(|r - r'|) \approx |r - r'|^{-2}$$

in three space dimensions. For a ball like $Q$ with spherical boundary $S_R$, the area-law holds for the interior of $Q$. We showed at the end of section 4.1 that and why the situation is different for points lying outside of $S_R$.

For $S_r$ concentric with $S_R$ but $r > R$, the DoF on $S_R$ are not entirely fixed by the configuration on $S_R$ but only happen to be restricted statistically in their variance and this statistical influence becomes weaker and weaker with increasing $r$. More specifically

Conclusion 7.2 With $Q$ a small ball in the closed space $\mathcal{O}$ of radius $R$, the area-law in its simple form does not hold in the exterior of $Q$, i.e. in $V$. With $B_r$ a ball concentric with $Q$ and $r \gg R$ we have for the maximum entropies

$$S_{\text{max}}(B_r \setminus Q) \approx S_{\text{max}}(B_r) - S_{\text{max}}(Q) \approx S_{\text{max}}(B_r)$$

and hence

$$S_{\text{max}}(V) \geq S_{\text{max}}(B_r \setminus Q) \gg S_{\text{max}}(Q)$$

That is, in our framework this is physically a rather natural result.

7.3.2 An Expanding Non-Closed Universe

Considering our universe as expanding, practically flat and spatially un-bound, we can assume that it is more or less homogeneously filled with
matter of a, on average, low but non-vanishing density and hence, by the same token, non-vanishing entropy-density. That is, we can find sufficiently large spheres with an entropy content which exceeds the value, given by the spatial entropy bound.

In our approach this example and other time-dependent scenarios are particularly illuminating from a microscopic point of view, underlying the holographic principle. One should note that in the preceding section we have mostly dealt with the spatial holographic principle in approximately static space-times. In such scenarios we derived the above mentioned microscopic correlation result

$$I(|r - r'|) \approx |r - r'|^{-2}$$

(63)

On the other hand, we remarked in e.g. section 2 that we regard our paper (among other things) as an extension and generalisation of old ideas of Sakharov, Zeldovich and Wheeler (induced gravity). We remind the reader that in those contributions (macroscopic) gravity was viewed as a derived and secondary effect arising from (or, rather, being equivalent to) variations and deformations in the vacuum fluctuation spectrum.

The above observations, being made in the context of time-dependent cosmological scenarios, now clearly indicate the following.

**Observation 7.3** The fluctuation pattern, its correlation structure and dynamics in the deep structure of the physical vacuum or space-time (that is, on level I (cf. section 4)) is standing in correspondence to the respective solutions of the Einstein equations (on level II). Or phrased slightly differently, the classical solutions are the coarse-grained picture of their microscopic counterparts. If the former are dynamic and time-dependent, the same holds for the corresponding microscopic fluctuation and correlation patterns.

**Corollary 7.4** In such expanding solutions the correlations on level I decay, for example, faster than $\sim |r|^{-2}$, thus leading to an increased storage capacity of information per volume.

### 7.4 Unitarity

In connection with the formation and evaporation of a BH the problem of unitarity is strongly felt. This problem has been extensively discussed in the literature. In order not to blow up the representation too much, we only mention the discussions in e.g. [20], p.29ff, [22], sections III,F,G and [57], chapt. 9. The question is basically whether the information which
has gone into the BH (starting for example from some pure initial state) is still somehow encoded in the thermal spectrum which is emitted by the BH during its evaporation.

Conventional wisdom tells us that in thermal states information is lost. However, physical intuition, trained on relatively simple examples may be deceptive.

**Observation 7.5** In general there is no clear apriori distinction between pure vector states and mixtures, i.e. density matrices in quantum theory.

This means the following. A unique division between vector states and density matrices can be made in Hilbert space representations where the full set of bounded operators, $\mathcal{B}(\mathcal{H})$, corresponds to the algebra of observables. In the mathematical classification of operator algebras (originally given by Murray and v.Neumann) this is called the type-I case.

On the other hand, in cases where this is not so, existence of event horizons, only incomplete local information access etc., there exist general mathematical statements to the effect that it can happen that in the same Hilbert space representation each density matrix can be represented by a pure Hilbert space vector (see e.g. [35] or [56]) for a more physically motivated discussion or [58], Lemma 2.7.8, p.104 or [59], Corollary 1.12, p.295, for complete mathematical proofs. Helpful is perhaps also the discussion in [65].

In a similar directions points the observation we mentioned in [14], section 3. That is, a Gibbs state, given on a subsystem, $\mathcal{H}_1$, can be extended to a pure vector state on an extended system, $\mathcal{H}_1 \otimes \mathcal{H}_2$ for some suitable $\mathcal{H}_2$ with

$$\Psi := \sum \sqrt{p_i} \cdot \psi_i \otimes e_i$$

(64)

$\{e_i\}$ spanning a basis in $\mathcal{H}_2$, $\psi_i$ the eigenstates of the Hamiltonian in $\mathcal{H}_1$ and $p_i$ the Boltzmann weights.

Concerning our particular case of formation and evaporation of a BH, the question is of course the scale of resolution of space-time we want to employ and the amount of information a real or hypothetical observer is granted. For the truly microscopic scale, which interests us here, we refer to our remarks in section [5].

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