TRACTABILITY OF NON-HOMOGENEOUS TENSOR PRODUCT PROBLEMS IN THE WORST CASE SETTING

RONG GUO AND HEPING WANG

Abstract. We study multivariate linear tensor product problems with some special properties in the worst case setting. We consider algorithms that use finitely many continuous linear functionals. We use a unified method to investigate tractability of the above multivariate problems, and obtain necessary and sufficient conditions for strong polynomial tractability, polynomial tractability, quasi-polynomial tractability, uniformly weak tractability, \((s,t)\)-weak tractability, and weak tractability. Our results can apply to multivariate approximation problems with kernels corresponding to Euler kernels, Wiener kernels, Korobov kernels, Gaussian kernels, and analytic Korobov kernels.

1. Introduction

Recently, there has been an increasing interest in \(d\)-variate computational problems \(S = \{S_d\}_{d \in \mathbb{N}}\) with large or even huge \(d\). In order to solve these problems we usually consider algorithms using finitely many information operations. For a given error threshold \(\varepsilon \in (0, 1)\), the information complexity \(n(\varepsilon, S_d)\) is defined to be the minimal number of information operations for which the approximation error of some algorithm is at most \(\varepsilon\). It is interesting to study how the information complexity \(n(\varepsilon, S_d)\) depends on \(\varepsilon^{-1}\) and \(d\). This is the subject of tractability. Research on tractability of multivariate problems started in 1994 (see [24]) and they quickly gained the popularity. Nowadays, tractability form an area of an intensive research with wide scope of unexpected results and open problems. The fundamental references about tractability are books [14, 15, 16]. There are new papers on the subject constantly coming out (see e.g. [1, 2, 3, 8, 12, 19, 22, 23]).

Various notions of tractability have been studied recently for many multivariate problems. We recall some of the basic tractability notions (see [14, 15, 16, 19, 22]). Let \(S = \{S_d\}_{d \in \mathbb{N}}\). We say the problem \(S\) is

- **strongly polynomially tractable (SPT)** iff there exist non-negative numbers \(C\) and \(p\) such that

\[
n(\varepsilon, S_d) \leq C(\varepsilon^{-1})^p \quad \text{for all} \quad d \in \mathbb{N}, \quad \varepsilon \in (0, 1).
\]

The exponent \(p^*\) of SPT is defined as the infimum of \(p\) for which (1.1) holds;

- **polynomially tractable (PT)** iff there exist non-negative numbers \(C\), \(p\) and \(q\) such that

\[
n(\varepsilon, S_d) \leq Cd^q(\varepsilon^{-1})^p \quad \text{for all} \quad d \in \mathbb{N}, \quad \varepsilon \in (0, 1);
\]

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\begin{itemize}
\item quasi-polynomially tractable (QPT) iff there exist positive numbers $C$ and $t$ such that
\begin{equation}
\label{eq:QPT}
 n(\varepsilon, S_d) \leq C \exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1})) \quad \text{for all } \, d \in \mathbb{N}, \, \varepsilon \in (0, 1).
\end{equation}
The exponent $t^*$ of QPT is defined as the infimum of $t$ for which (1.2) holds;
\item uniformly weakly tractable (UWT) iff for all $\alpha, \beta > 0$
\[
\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{(\varepsilon^{-1})^\alpha + d^\beta} = 0;
\]
\item weakly tractable (WT) iff
\[
\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0;
\]
\item $(s,t)$-weakly tractable ($(s,t)$-WT) for fixed positive $s$ and $t$ iff
\[
\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n(\varepsilon, S_d)}{(\varepsilon^{-1})^s + d^t} = 0.
\]
\end{itemize}

Clearly, $(1,1)$-WT is the same as WT. If the problem $S$ is not WT, then $S$ is called intractable. We say that the problem $S$ suffers from the curse of dimensionality if there exist positive numbers $C, \varepsilon_0, \alpha$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and infinitely many $d \in \mathbb{N}$,
\[
n(\varepsilon, S_d) \geq C(1 + \alpha)^d.
\]

This paper is devoted to studying tractability of non-homogeneous tensor product problems with some special properties in the worst case setting. Such approximation problems were investigated in [17, 20] for Korobov kernels, in [4] for Gaussian kernels, and in [8, 9, 12] for analytic Korobov kernels. In the average case setting, there are many papers devoted to discussing tractability of non-homogeneous tensor product problems with covariance kernels being a product of univariate kernels and satisfying some special properties (see [1, 2, 5, 10, 11, 13, 20, 21, 25, 26]).

In this paper, we use a unified method to study tractability of the above multivariate problems in the worst case setting, and obtain necessary and sufficient conditions for strong polynomial tractability, polynomial tractability, quasi-polynomial tractability, uniformly weak tractability, $(s,t)$-weak tractability, and weak tractability. Our results can apply to multivariate approximation problems with kernels corresponding to Euler kernels, Wiener kernels, Korobov kernels, Gaussian kernels, and analytic Korobov kernels. For the first three kernels the obtained results are new.

The paper is organized as follows. In Section 2 we give preliminaries about non-homogeneous tensor product problems in the worst case setting and present the main results, i.e., Theorem 2.1. Section 3 is devoted to proving Theorem 2.1. In Section 4, we give the applications of Theorem 2.1 to the problems with kernels corresponding to Euler kernels, Wiener kernels, Korobov kernels, Gaussian kernels, and analytic Korobov kernels.

\section{Preliminaries and main results}

In this section, we recall the definition of non-homogeneous linear tensor product problem in the worst case setting (see [17, 20]).
A linear tensor product problem in the worst case setting is a sequence of linear operators

\[ S = \{S_d\}_{d \in \mathbb{N}} \]

such that for every \( j \in \mathbb{N} \) there exists a separable Hilbert space \( \mathcal{H}_j \), a Hilbert space \( \mathcal{G}_j \) and a continuous linear operator \( S_j : \mathcal{H}_j \to \mathcal{G}_j \) such that

\[ S_d = \bigotimes_{j=1}^d S_j : \mathcal{H}_d \to \mathcal{G}_d, \]

where \( \mathcal{H}_d = \bigotimes_{j=1}^d \mathcal{H}_j \) and \( \mathcal{G}_d = \bigotimes_{j=1}^d \mathcal{G}_j \) for every \( d \in \mathbb{N} \).

If \( \mathcal{H}_j = \mathcal{H}_1 \), \( \mathcal{G}_j = \mathcal{G}_1 \) and \( S_j = S_1 \) for all \( j \in \mathbb{N} \), then the linear tensor product problem \( S \) is called homogeneous.

Without loss of generality we assume that all operators \( S_k \), \( k \in \mathbb{N} \) are compact. Then the operators

\[ W_k = S_k^* S_k : \mathcal{H}_k \to \mathcal{H}_k \quad k \in \mathbb{N} \]

are compact, self-adjoint, and non-negative definite, where \( S_k^* \) are the adjoint operators of \( S_k \). Let \( \{ (\lambda(k,j), \eta(k,j)) \}_{j \in \mathbb{N}} \) denote the eigenpairs of \( W_k \) satisfying

\[ W_k(\eta(k,j)) = \lambda(k,j) \eta(k,j) \quad \text{with} \quad \lambda(k,1) \geq \lambda(k,2) \geq \cdots \geq 0, \quad \eta(k,j) \in \mathcal{H}_k. \]

Then the eigenpairs of the operators

\[ W_d = S_d^* S_d = \bigotimes_{j=1}^d W_j, \quad d \in \mathbb{N} \]

are given by

\[ \{ (\lambda_{d,j}, \eta_{d,j}) \}_{j=(j_1,j_2,\ldots,j_d) \in \mathbb{N}^d}, \]

where

\[ \lambda_{d,j} = \prod_{k=1}^d \lambda(k,j_k), \quad \eta_{d,j} = \prod_{k=1}^d \eta(k,j_k). \]

Let the sequence \( \{ \lambda_{d,j} \}_{j \in \mathbb{N}} \) be the nonincreasing rearrangement of \( \{ \lambda_{d,j} \}_{j \in \mathbb{N}^d} \).

Then we have

\[ \sum_{j=1}^\infty \lambda_{d,j}^\tau = \prod_{k=1}^d \sum_{j_k=1}^\infty \lambda(k,j)\tau, \quad \text{for any} \quad \tau > 0. \] (2.1)

Note that the above sums are not always finite for any \( \tau > 0 \).

We approximate \( S_d f \) by algorithms \( A_{n,d} f \) of the form

\[ A_{n,d} f = \phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)), \]

where \( L_1, L_2, \ldots, L_n \) are continuous linear functionals on \( H_d \), and \( \phi_{n,d} : \mathbb{R}^n \to G_d \) is an arbitrary measurable mapping. The worst case approximation error for the algorithm \( A_{n,d} \) is defined as

\[ e(A_{n,d}) = \sup_{\|f\|_{H_d} \leq 1} \| S_d f - A_{n,d} f \|_{G_d}. \]

The \( n \)th minimal worst case error, for \( n \geq 1 \), is defined by

\[ e(n, S_d) = \inf_{A_{n,d}} e(A_{n,d}), \]
where the infimum is taken over all algorithms of the form \(2.2\). According to [14], the \(n\)th minimal worst case error is given by

\[
e(n, S_d) = \frac{\lambda_{d,n+1}}{2},
\]

and is achieved by the \(n\)th optimal algorithm

\[
A_{n,d}^* f = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{H_d} S_d \eta_{d,j}.
\]

For \(n = 0\), we use \(A_{0,d} = 0\). We remark that the so-called initial error \(e(0, S_d)\) is defined by

\[
e(0, S_d) = \sup_{\|f\|_{H_d} \leq 1} \|S_d f\|_{G_d} = \|S_d\| = \frac{\lambda_{d,1}}{2}.
\]

The information complexity for \(S_d\) can be studied using either the absolute error criterion (ABS), or the normalized error criterion (NOR). We define the information complexity \(n^X(\varepsilon, S_d)\) for \(X \in \{\text{ABS}, \text{NOR}\}\) as

\[
n^X(\varepsilon, S_d) = \min\{n : e(n, S_d) \leq \varepsilon CRI_d\},
\]

where

\[
CRI_d = \begin{cases} 1, & \text{for } X = \text{ABS}, \\ e(0, S_d), & \text{for } X = \text{NOR}. \end{cases}
\]

In this paper we consider a special class of non-homogeneous linear tensor product problems \(S = \{S_d\}_{d \in \mathbb{N}}\) in the worst case setting. In the average case setting the similar non-homogeneous linear tensor product problems were investigated in [2]. Assume that the eigenvalues

\[
\prod_{k=1}^{d} \lambda(k, j_k) \text{ for } (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d
\]

of the operator \(W_d = S_d^* S_d\) of the problem \(S = \{S_d\}_{d \in \mathbb{N}}\) satisfy the following three conditions:

1. \(\lambda(k, 1) = 1, k \in \mathbb{N}\);
2. the sequence \(\{h_k\}_{k \in \mathbb{N}}\) is nonincreasing, where \(h_k = \frac{\lambda(k, 2)}{\lambda(k, 1)} \in (0, 1]\);
3. there exist a positive constant \(\tau > 0\) such that

\[
(2.3) \quad \sup_{k \in \mathbb{N}} H(k, \tau) = H(1, \tau) < \infty,
\]

where

\[
H(k, \tau) = \sum_{j=2}^{\infty} \left( \frac{\lambda(k, j)}{\lambda(k, 2)} \right)^{\tau}.
\]

Let

\[
\tau_0 = \inf \{\tau : \tau \text{ satisfies (2.3)}\}.
\]

Then we say that the problem \(S = \{S_d\}_{d \in \mathbb{N}}\) has Property (P).

We are ready to present the main result of this paper.

**Theorem 2.1.** Let \(S = \{S_d\}_{d \in \mathbb{N}}\) be a non-homogeneous tensor product problem with Property (P) in the worst case setting. Then for the absolute error criterion or the normalized error criterion, we have
(i) $S$ is strongly polynomially tractable iff
\[
A_* = \lim \inf_{d \to \infty} \frac{\ln h^{-1}_d}{\ln d} > 0,
\]
and the exponent of SPT is
\[
p^* = \max \{ \frac{2}{A_*}, 2\tau_0 \}.
\]
(ii) $S$ is strongly polynomially tractable iff it is polynomially tractable.
(iii) $S$ is quasi-polynomially tractable iff
\[
B := \lim_{d \to \infty} \ln h^{-1}_d > 0.
\]
This is equivalent to that $h_k \not\equiv 1$. Furthermore, the exponent of QPT is
\[
t^* = \max \{ \frac{2}{B}, 2\tau_0 \}.
\]
(iv) $S$ is quasi-polynomially tractable iff $S$ is uniformly weakly tractable, iff $S$ is $(s,t)$-weakly tractable with $s > 0$ and $t \in (0,1]$, and iff $S$ is weakly tractable.
(v) $S$ is $(s,t)$-weakly tractable with $s > 0$ and $t > 1$.
(vi) $S$ suffers from the curse of dimensionality iff $h_k \equiv 1$.

Remark 2.2. Let $S = \{ S_d \}_{d \in \mathbb{N}}$ be a non-homogeneous tensor product problem. If the eigenvalues of the operator $W_d = S_d^* S_d$ satisfy Conditions (2) and (3) of Property (P), then for NOR, Theorem 2.1 holds.

Indeed, let $\tilde{S} = \{ \tilde{S}_d \}_{d \in \mathbb{N}}$ be the non-homogeneous tensor product problem which the eigenvalues $\{ \prod_{k=1}^d \tilde{\lambda}(k,j_k) \}_{(j_1,j_2,\ldots,j_d) \in \mathbb{N}_d}$ of the corresponding operator $\tilde{W}_d = \tilde{S}_d^* \tilde{S}_d$ of satisfy
\[
\tilde{\lambda}(k,j) = \frac{\lambda(k,j)}{\lambda(k,1)}, \quad j \in \mathbb{N}, \quad k = 1, \ldots, d.
\]
Then $\tilde{S}$ has Property (P) with the same $h_k$ and Theorem 2.1 is applicable. Also for NOR, the problems $S$ and $\tilde{S}$ have the same tractability. Hence, for NOR, Theorem 2.1 holds.

In order to prove Theorem 2.1 we need the following lemmas.

Lemma 2.3. (See [13, Theorem 5.2].) Consider the non-zero problem $S = \{ S_d \}_{d \in \mathbb{N}}$ in the worst case setting, where $S_d : H_d \to G_d$ is a compact linear operator between two Hilbert spaces $H_d$ and $G_d$. Assume that the eigenvalues $\{ \lambda_{d,j} \}_{j \in \mathbb{N}}$ of the operator $W_d = S_d^* S_d$ satisfying
\[
\lambda_{d,1} \geq \lambda_{d,2} \geq \cdots \geq 0.
\]
Then for the normalized error criterion, we have
(a) $S$ is polynomially tractable iff there exist two constants $q \geq 0$ and $\tau > 0$ such that
\[
C_{\tau,q} := \sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_{d,j}}{\lambda_{d,1}} \right)^{\frac{1}{\tau}} d^{-q} \right)^{\frac{\tau}{\tau+1}} < \infty.
\]
(b) $S$ is strongly polynomially tractable iff (2.8) holds with $q = 0$. If so then the exponent of SPT is

$$p^* = \inf \{2\tau \mid \tau \text{ satisfies (2.8) with } q = 0\}.$$  

**Lemma 2.4.** (See [16, Theorem 23.2].) Consider the problem $S = \{S_d\}_{d \in \mathbb{N}}$ in the worst case setting as in Lemma 2.3. Then for the normalized error criterion, $S$ is quasi-polynomially tractable iff there exists a $\tau > 0$ such that

$$B_\tau := \sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_{d,j}}{\lambda_{d,1}} \right)^{\tau(1+\ln d)} \right) ^\frac{1}{\tau} d^{-2} < \infty.$$  

If so then the exponent of QPT is

$$t^* = \inf \{2\tau \mid \tau \text{ satisfies (2.9)}\}.$$  

**Lemma 2.5.** Consider the problem $S = \{S_d\}_{d \in \mathbb{N}}$ in the worst case setting as in Lemma 2.3. For any fixed $t > 0$ if there exists a positive $\tau$ such that

$$\lim_{d \to \infty} d^{t-\tau} \ln \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_{d,j}}{\lambda_{d,1}} \right)^{\tau} \right) = 0,$$  

then $S$ is $(s, t)$-weakly tractable for any $s > 0$ and this $t$ for the normalized error criterion.

Lemma 2.5 is likely not new, however we cannot find its proof. For readers’ convenience we give its proof.

**Proof.** Without loss of generality we assume that $\lambda_{d,1} = 1$. Since

$$n \lambda_{d,n}^* \leq \sum_{k=1}^{n} \lambda_{d,k}^* \leq \sum_{k=1}^{\infty} \lambda_{d,k}^*,$$

we get

$$\lambda_{d,n} \leq n^{-\frac{1}{2}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^* \right)^{\frac{1}{2}}.$$  

We note that $\lambda_{d,n} \leq \varepsilon^2$ holds for

$$n = \left\lceil \sum_{k=1}^{\infty} \lambda_{d,k}^* \varepsilon^{-2\tau} \right\rceil.$$  

It follows that

$$n(\varepsilon, S_d) = \min \{n \mid \lambda_{d,n+1} \leq \varepsilon^2\} \leq \left\lceil \sum_{k=1}^{\infty} \lambda_{d,k}^* \varepsilon^{-2\tau} \right\rceil.$$  

Since $[x] \leq x + 1 \leq 2\max\{1, x\}$ and $\ln \max\{x, 1\} \leq \max\{0, \ln x\} = (\ln x)_+$, we have

$$\ln n(\varepsilon, S_d) \leq \frac{\ln 2 + \left( 2\tau \ln \varepsilon^{-1} + \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k}^* \right) \right)_+}{\varepsilon^{-s} + d^t},$$  

which goes to 0 as $\varepsilon^{-1} + d$ tends to infinity. Hence $S$ is $(s, t)$-weakly tractable with $s > 0$ and this $t$. 

□
Proof of Theorem 2.1

Since $\lambda_{d,1} = \prod_{k=1}^{d} \lambda(k,1) = 1$ we know that tractability for $S = \{S_d\}_{d \in \mathbb{N}}$ is the same for both the absolute and normalized error criteria.

(i). If $S$ is strongly polynomially tractable with the exponent $p^*$ of SPT, then by Lemma 2.3 (b) we know that for any fixed $x > p^*/2$, we have

$$C_x := \sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right)^{\frac{1}{x}} < \infty.$$  

It follows from (2.1) and (2.3) that for any $d \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} \lambda_{d,j}^x = \prod_{k=1}^{d} \sum_{j=1}^{\infty} \lambda(k,j)^x = \prod_{k=1}^{d} (1 + \sum_{j=2}^{\infty} \lambda(k,j)^x)$$

$$= \prod_{k=1}^{d} (1 + H(k,x)h_k^x) \leq C_x^x < \infty. \quad (3.1)$$

This means that for $1 \leq k \leq d$, $H(k,x) < \infty$. This means that $x \geq \tau_0$.

Clearly for any $x > p^*/2$, we have

$$H(k,x) = 1 + \sum_{j=3}^{\infty} \left( \frac{\lambda(k,j)}{\lambda(k,2)} \right)^x \geq 1.$$  

It follows from (3.1) that

$$\prod_{k=1}^{d} (1 + h_k^x) \leq \prod_{k=1}^{d} (1 + H(k,x)h_k^x) \leq C_x^x. \quad (3.2)$$

Using the inequality $\ln(1+x) \geq x \ln 2$, $x \in [0,1]$, Condition (2), and (3.2), we get that

$$d \ln 2h_d^x \leq \ln 2 \sum_{k=1}^{d} h_k^x \leq \sum_{k=1}^{d} \ln(1 + h_k^x) = \ln \left( \prod_{k=1}^{d} (1 + h_k^x) \right) \leq \ln C_x^x, \quad (3.3)$$

which means that

$$\ln d + \ln(\ln 2) - x \ln h_d^{-1} \leq \ln(\ln C_x^x).$$

It follows that

$$\frac{\ln h_d^{-1}}{\ln d} \geq \frac{\ln d + \ln(\ln 2) - \ln(\ln C_x^x)}{x \ln d}. \quad (3.4)$$

Letting $d \to \infty$ in (3.4), we get that

$$A_* = \liminf_{d \to \infty} \frac{\ln h_d^{-1}}{\ln d} \geq \frac{1}{x} > 0, \quad (3.5)$$

i.e., (5.1) holds. Since (3.5) and the inequality $x \geq \tau_0$ hold for any $x > p^*/2$ we obtain that

$$p^* \geq \max \left\{ \frac{2}{A_*}, 2\tau_0 \right\}. \quad (3.6)$$
On the other hand, assume that (2.4) holds. From (2.3) we know that for $x > \tau_0$,

(3.7)  
$$1 \leq \sup_{k \in \mathbb{N}} H(k, x) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda(k, j)}{\lambda(k, 2)} \right)^x \leq M_x < \infty.$$  

It follows from (3.1), (3.7), and the inequality $\ln(1 + x) \leq x$ $(x \geq 0)$ that for any $x > \max \left\{ \frac{1}{A_*}, \tau_0 \right\}$ and $d \in \mathbb{N}$,

(3.8)  
$$\sum_{j=1}^{\infty} \lambda_{d,j}^x \leq \prod_{k=1}^{d} (1 + M_x h_k^x) = \exp \left( \sum_{k=1}^{d} \ln(1 + M_x h_k^x) \right) \leq \exp \left( M_x \sum_{k=1}^{\infty} h_k^x \right).$$  

For any $x > \max \left\{ \frac{1}{A_*}, \tau_0 \right\}$, by (2.4) we have

$$h_k^x = k \frac{x \ln h_k^{-1}}{\ln k} \quad \text{and} \quad \liminf_{k \to \infty} \frac{x \ln h_k^{-1}}{\ln k} = x A_* > 1,$$

which implies that

$$D_x := \sum_{k=1}^{\infty} h_k^x < \infty.$$  

From (3.8) we get that

$$\sup_{d \in \mathbb{N}} \sum_{j=1}^{\infty} \lambda_{d,j}^x \leq \exp(D_x M_x) < \infty.$$  

By Lemma 2.3 (b) we obtain that $S$ is strongly polynomially tractable, and the exponent of $SPT$ satisfies

(3.9)  
$$p^* \leq \max \left\{ \frac{2}{A_*}, 2\tau_0 \right\}.$$  

From (3.6) and (3.9) we obtain (2.5). The proof of (i) is complete.

(ii). We know that $SPT$ can deduce $PT$. So it suffices to show that $PT$ implies $SPT$. Assume that $S$ is $PT$. By Lemma 2.3 (a) there exist numbers $x > 0$, $q \geq 0$ such that

$$C_{x, q} := \sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right)^{\frac{1}{d}} d^{-q} < \infty.$$  

It follows from (2.1) that

(3.10)  
$$\sum_{k=1}^{d} \ln(1 + h_k^x) \leq \sum_{k=1}^{d} \ln(1 + H(k, x)h_k^x) = \sum_{j=1}^{\infty} \lambda_{d,j}^x \leq \ln(C_{x, q}^{x} d^{2x}).$$  

Similar to the proof of (3.3), by (3.10) we get

$$d \ln 2 h_d^x \leq \ln \sum_{k=1}^{d} h_k^x \leq \sum_{k=1}^{d} \ln(1 + h_k^x) \leq \ln(C_{x, q}^{x}) + qx \ln d,$$

which implies that

$$\ln d + \ln(\ln 2) - x \ln h_d^{-1} \leq \ln(C_{x, q}^{x} + qx \ln d).$$

It follows that

(3.11)  
$$\frac{\ln h_d^{-1}}{\ln d} \geq \frac{\ln d + \ln(\ln 2) - \ln(C_{x, q}^{x} + qx \ln d)}{x \ln d}.$$
Letting $d \to \infty$ in (3.11), we get that
\[
\liminf_{d \to \infty} \frac{\ln h_d^{-1}}{\ln d} \geq \frac{1}{x} > 0.
\]
This implies that (2.14) holds and hence by the proved (i) we obtain that $S$ is strongly polynomially tractable. This completes the proof of (ii).

(iii). Assume that $S$ is quasi-polynomially tractable with the exponent $t^*$ of QPT. Then by Lemma 2.4 we know that for any fixed $x > t^*/2$, we have
\[
B_x := \sup_{d \in \mathbb{N}} \left( \sum_{j=1}^{\infty} \lambda_{d,j}^{x(1+\ln d)} \right)^{1/2} d^{-2} < \infty.
\]
It follows from (2.11) that
\[
\sum_{k=1}^{d} \ln \left( 1 + h_k^{x(1+\ln d)} \right) \leq \sum_{k=1}^{d} \ln \left( 1 + H(k, x(1 + \ln d)) h_k^{x(1+\ln d)} \right)
\]
\[
= \ln \left( \sum_{j=1}^{\infty} \lambda_{d,j}^{x(1+\ln d)} \right) \leq \ln(B_x^x) + 2x \ln d.
\]
(3.12)

If $d = 1$, then the above inequality gives that
\[
\sum_{j=1}^{\infty} \lambda_{1,j}^{x(1+\ln 1)} = \sum_{j=1}^{\infty} \lambda(1,j)^x = 1 + H(1, x)h_1^x < \infty.
\]
This means that $x \geq \tau_0$. Using (3.12), the inequality $\ln(1 + x) \geq x \ln 2$ ($x \in [0, 1]$), and the monotonicity of the sequence of $\{h_k\}$, we get
\[
\ln 2 d h_d^{x(1+\ln d)} \leq \ln 2 \sum_{k=1}^{d} h_k^{x(1+\ln d)} \leq \sum_{k=1}^{d} \ln \left( 1 + h_k^{x(1+\ln d)} \right) \leq \ln(B_x^x) + 2x \ln d,
\]
which implies that
\[
\ln 2 + \ln d - x(1 + \ln d) \ln h_d^{-1} \leq \ln \left( \ln(B_x^x) + 2x \ln d \right).
\]
This yields that
\[
\ln h_d^{-1} \geq \frac{\ln d + \ln \ln 2 - \ln \left( \ln(B_x^x) + 2x \ln d \right)}{x(1 + \ln d)}.
\]
(3.13)

Letting $d \to \infty$ in (3.13) and noting the monotonicity of $\{h_k\}$, we get
\[
B := \lim_{d \to \infty} \ln h_d^{-1} \geq \frac{1}{x} > 0.
\]
Since the inequalities $B \geq 1/x$ and $x \geq \tau_0$ hold for any $x > t^*/2$ we obtain that
\[
t^* \geq \max \left\{ \frac{2}{B}, 2\tau_0 \right\}.
\]
(3.14)

On the other hand, assume that (2.6) holds. From (2.3) we know that for $x > \tau_0$,
\[
1 \leq \sup_{k \in \mathbb{N}} H(k, x(1 + \ln d)) \leq \sup_{k \in \mathbb{N}} H(k, x) \leq M_x < \infty.
\]
(3.15)
It follows from (3.12), (3.15), and the inequality ln(1 + x) ≤ x (x ≥ 0) that for any
x > max \{B, \tau_0\} and d ∈ \mathbb{N},
\[\sum_{j=1}^{\infty} \lambda_{d,j} x^{(1+\ln d)} \leq \prod_{k=1}^{d} (1 + M_x h_k^{x(1+\ln d)}) = \exp \left( \sum_{k=1}^{d} \ln(1 + M_x h_k^{x(1+\ln d)}) \right) \]
\[\leq \exp \left( M_x \sum_{k=1}^{d} h_k^{x(1+\ln d)} \right) \leq \exp \left( M_x \sum_{k=1}^{d} h_k^{x(1+\ln k)} \right) \]
(3.16)
\[\leq \exp \left( M_x \sum_{k=1}^{\infty} h_k^{x(1+\ln k)} \right).\]

For any x > max \{\frac{1}{B}, \tau_0\}, by (2.6) we have
\[h_k^{x(1+\ln k)} = k^{-\frac{x(1+\ln k)\ln h_k^{-1}}{\ln k}} \quad \text{and} \quad \liminf_{k \to \infty} \frac{x(1 + \ln k) \ln h_k^{-1}}{\ln k} = xB > 1,\]
which implies that
\[E_x := \sum_{k=1}^{\infty} h_k^{x(1+\ln k)} < \infty.\]

From (3.16) we get that
\[\sup_{d \in \mathbb{N}} \sum_{j=1}^{\infty} \lambda_{d,j} x^{(1+\ln d)} \leq \exp(E_x M_x) < \infty.\]

By Lemma 2.4 we obtain that S is quasi-polynomially tractable, and the exponent of QPT satisfies
(3.17)
\[t^* \leq \max \{\frac{\tau_0}{2}, 2\tau_0\}.\]

From (3.14) and (3.17) we obtain (2.7). This completes the proof of (iii).

(iv). Since QPT ⇒ UWT ⇒ (s, t)-WT (or WT), it suffices to show that (s, t)-WT ⇒ QPT with t ∈ (0, 1] and s > 0. Assume that S is (s, t)-weakly tractable with t ∈ (0, 1] and s > 0. First we show that \(h_k \not\equiv 1\). If \(h_k \equiv 1\), then \(e(n, S_d) = 1\) for 1 ≤ n ≤ 2d. This means that
\[\epsilon(n, S_d) \geq 2^d - 1, \quad \epsilon \in (0, 1),\]
and
\[\lim_{d \to \infty} \frac{\ln n(1/2, S_d)}{2^d + d^t} \geq \lim_{d \to \infty} d^{1-t} \ln 2 > 0, \quad t \in (0, 1].\]

Hence S suffers from the curse of dimensionality and is not (s, t)-weakly tractable with t ∈ (0, 1] and s > 0. This leads to a contradiction. Hence \(h_k \not\equiv 1\). By the monotonicity of \(\{h_k\}\) we get
\[B = \lim_{k \to \infty} \ln h_k^{-1} > 0.\]

By the proved (iii) we obtain that S is quasi-polynomially tractable. (iv) is proved.

(v). It follows from (3.8) that for x > \(\tau_0\),
\[\ln \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right) \leq M_x \sum_{k=1}^{d} h_k^x \leq M_x d.\]
We have for $t > 1$,

$$\lim_{d \to \infty} d^{-t} \ln \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right) \leq \lim_{d \to \infty} M_x d^{1-t} = 0.$$ 

By lemma 2.5 we get that $S$ is $(s,t)$-weakly tractable with $s > 0$ and $t > 1$. This complete the proof of (v).

(vi). We have proved in (iv) that if $h_k \equiv 1$ then $S$ suffers from the curse of dimensionality, and if $h_k \not\equiv 1$ then $S$ is quasi-polynomially tractable and does not suffer from the curse of dimensionality. This completes the proof of (vi).

The proof of Theorem 2.1 is finished.

**Remark 3.1.** Let $S = \{S_d\}_{d \in \mathbb{N}}$ be a non-homogeneous tensor product problem. If the eigenvalues of the operator $W_d = S_d^* S_d$ satisfy Condition (3) of Property (P) and the following Condition (2)′: there exist a nonincreasing positive sequence $\{f_k\}_{k \in \mathbb{N}}$ and two positive constants $A_1, A_2$ such that such that for all $k \in \mathbb{N}$, we have

$$A_1 f_k \leq h_k \leq A_2 f_k.$$ 

Then for NOR, Theorem 2.1 (i), (ii), (v) hold.

Indeed, in the proof of Theorem 2.1 (v), we only used Condition (3) of Property (P). Hence for NOR, Theorem 2.1 (v) holds. In the proofs of Theorem 2.1 (i) and (ii), we used Condition (3) of Property (P) and the monotonicity of $\{h_k\}$. If Condition (2) is replaced by Condition (2)′, then the inequalities (3.3) and (3.8) can be replaced by the following inequalities

$$\ln 2 A_1^x d f_d^x \leq \ln 2 A_1^x \sum_{k=1}^{d} f_k^x \leq \ln 2 \sum_{k=1}^{d} h_k^x \leq \ln \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right),$$

and

$$\ln \left( \sum_{j=1}^{\infty} \lambda_{d,j}^x \right) \leq M_x \sum_{k=1}^{\infty} h_k^x \leq M_x A_2^x \sum_{k=1}^{\infty} f_k^x.$$ 

Using the above inequalities and the methods in the proofs of Theorem 2.1 (i) and (ii), noting that

$$\liminf_{d \to \infty} \frac{\ln h_d^{-1}}{\ln d} = \liminf_{d \to \infty} \frac{\ln f_d^{-1}}{\ln d},$$

we can prove that for NOR, Theorem 2.1 (i) and (ii) hold.

### 4. Applications of Theorem 2.1

Consider the approximation problem APP = \{APP$_d$\}$_{d \in \mathbb{N}}$, 

$$\text{APP}_d : H_{K_d} \to L_2([0,1]^d) \quad \text{with} \quad \text{APP}_d f = f,$$

where $H_{K_d}$ is a Hilbert space related to the kernels $K_d$ which are of tensor product and correspond to Euler kernels, Wiener kernels, Korobov kernels, Gaussian kernels, and analytic Korobov kernels. This section is devoted to giving the applications of Theorem 2.1 to these cases.
4.1. Function approximation with Euler kernels.

In this subsection we consider multivariate approximation problems with Euler kernels. Assume that \( r = \{r_k\}_{k \in \mathbb{N}} \) is a sequence of nondecreasing nonnegative integers satisfying

\[
0 \leq r_1 \leq r_2 \leq r_3 \leq \ldots.
\]

Let \( H(K^{E}_{d,r}) \) be the reproducing kernel Hilbert space with reproducing kernel being the Euler kernel

\[
K^{E}_{d,r}(x, y) = \prod_{k=1}^{d} K^{E}_{1,r_k}(x_k, y_k), \quad x, y \in [0, 1]^d,
\]

where

\[
K^{E}_{1,r}(x, y) = \int_{[0,1]^r} \min(x, s_1) \min(s_1, s_2) \ldots \min(s_r, y) ds_1 ds_2 \ldots ds_r, \quad r \in \mathbb{N}_0
\]

is the Euler kernel (see [16, pp. 222-226]). If \( r = 0 \), we get the standard Wiener kernel

\[
K^{E}_{1,0}(x, y) = \min(x, y).
\]

If \( X^{E}_r(t) \) is a Gaussian random process with zero mean and covariance kernel \( K^{E}_{1,r} \), then it is called the univariate integrated Euler process.

The reproducing kernel Hilbert space \( H(K^{E}_{1,r}) \) is equal to the space of functions \( f: [0, 1] \rightarrow \mathbb{R} \) such that the \( r \)-th derivative of \( f \) is absolutely continuous and the \( (r+1) \)-th derivative of \( f \) belongs to \( L^2[0, 1] \) and \( f \) satisfies the following boundary conditions

\[
f(0) = f'(1) = f''(0) = \cdots = f^{(r)}(s_r) = 0,
\]

where \( s_r = 0 \) if \( r \) is even and \( s_r = 1 \) if \( r \) is odd. The inner product of \( H(K^{E}_{1,r}) \) is given by

\[
\langle f, g \rangle_r = \int_0^1 f^{(r+1)}(x)g^{(r+1)}(x)dx \quad \text{for all} \quad f, g \in H(K^{E}_{1,r}).
\]

Obviously, the space \( H(K^{E}_{d,r}) \) is a tensor product of the univariate integrated Euler process.

We consider the multivariate approximation problem \( \text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}} \) which is defined via the embedding operator

\[
\text{APP}_d: H(K^{E}_{d,r}) \rightarrow L_2([0, 1]^d) \quad \text{with} \quad \text{APP}_d f = f.
\]

For the above approximation problem \( \text{APP} = \{\text{APP}_d\} \), the eigenvalues of the operator \( W_d = \text{APP}_d^* \text{APP}_d \) are given by (see [7] or [16, pp. 222-226])

\[
\{\lambda^{E}_{d,j}\}_{j \in \mathbb{N}^d} = \{\lambda^{E}(1, j_1)\lambda^{E}(2, j_2) \ldots \lambda^{E}(d, j_d)\}_{(j_1, \ldots, j_d) \in \mathbb{N}^d},
\]

where

\[
\lambda^{E}(k, j) = \left[ \frac{1}{\pi(j - \frac{1}{2})} \right]^{2r_k+2}, \quad j \in \mathbb{N}.
\]
Now we verify that the above approximation problem APP satisfies Conditions (2) and (3) of Property (P). First we note that the sequence \( \{ \lambda_k \} \), \( \lambda_k = \frac{\lambda(2 + k)}{\lambda(1)} \) is nonincreasing due to (4.1). Next we have for \( x > \frac{1}{3r_1+2} \),

\[
\sup_{k \in \mathbb{N}} H^E(k, x) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda(2 + k)}{\lambda(1)} \right)^{x(j+1)} \leq \sum_{j=2}^{\infty} \left( \frac{3}{2j-1} \right)^{2x(j+1)} < \infty.
\]

It follows that APP satisfies Condition (3) of Property (P) with \( h_k = \frac{1}{3r_1+2} \). By Remark 2.2, we have the following corollary.

**Corollary 4.1.** Consider the approximation problem \( APP = \{ APP_d \}_{d \in \mathbb{N}} \) in the space \( H(K^E_d) \) with sequence \( r = \{ r_k \}_{k \in \mathbb{N}} \) satisfying (4.1). Then for the normalized error criterion, we have

1. \( APP \) is strongly polynomially tractable iff
   \[
   A_* = \liminf_{k \to \infty} \frac{(2r_k + 2) \ln 3}{\ln k} = 2 \ln 3 \liminf_{k \to \infty} \frac{r_k}{\ln k} > 0,
   \]
   and the exponent of SPT is
   \[
   p^* = \max \left\{ \frac{2}{A_*}, \frac{1}{r_1+1} \right\}.
   \]

2. \( APP \) is strongly polynomially tractable iff it is polynomially tractable.

3. \( APP \) is quasi-polynomially tractable for all sequences \( r = \{ r_k \}_{k \in \mathbb{N}} \) satisfying (4.1). Furthermore, the exponent of QPT is
   \[
   t^* = \max \left\{ \frac{2}{\lim_{k \to \infty} (2r_k + 2) \ln 3 \cdot \frac{1}{r_1+1}} \right\} = \frac{1}{r_1+1}.
   \]

4. Obviously, QPT implies UWT, WT, and \((s, t)\)-WT for any positive \( s \) and \( t \), for all sequences \( r = \{ r_k \}_{k \in \mathbb{N}} \) satisfying (4.1).

**Remark 4.2.** It is easy to see that for \( x > \frac{1}{3r_1+2} \),

\[
\sup_{d \in \mathbb{N}} \sum_{j=1}^{\infty} \left( \lambda^E_{d,j} \right)^x = \sup_{d \in \mathbb{N}} \prod_{k=1}^{d} \sum_{j=1}^{\infty} \left[ \frac{1}{\pi(j - \frac{1}{2})} \right]^{2x} \leq \sup_{d \in \mathbb{N}} \prod_{k=1}^{d} \sum_{j=1}^{\infty} \left[ \frac{1}{\pi(j - \frac{1}{2})} \right]^2 < 1.
\]

Here, we use the equality that

\[
\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8}.
\]

It follows from [14] Theorem 5.1] that for ABS, APP is strongly polynomially tractable for all sequences \( r = \{ r_k \}_{k \in \mathbb{N}} \) satisfying (4.1).

Next we consider the exponent of SPT. Set

\[
G(x) = \sum_{j=1}^{\infty} \left[ \frac{1}{\pi(j - \frac{1}{2})} \right]^{x}, \quad \tau = \lim_{k \to \infty} r_k.
\]
Then $G(x)$ is decreasing on $(1, +\infty)$, and $G(2) = 1/2$. So there exists a unique $\xi_0 \in (1, 2)$ such that $G(\xi_0) = 1$. Then
\[
\sup_{d \in \mathbb{N}} \sum_{j=1}^{\infty} (\lambda_{d,j}^G)^x = \sup_{d \in \mathbb{N}} \prod_{k=1}^{d} G(2r_k + 2)x \begin{cases} < \infty, & \text{if } x > \max \left\{ \frac{\xi_0}{2r_1+2}, \frac{1}{2r_1+2} \right\}, \\
= \infty, & \text{if } x < \min \left\{ \frac{\xi_0}{2r_1+2}, \frac{1}{2r_1+2} \right\}. \end{cases}
\]
This means that the exponent of SPT is
\[
\max \left\{ \frac{\xi_0}{r+1}, \frac{1}{r_1+1} \right\}.
\]

### 4.2. Function approximation with Wiener kernels.

In this subsection we consider multivariate approximation problems with Wiener kernels. Let $H(K_d^W)$ be the reproducing kernel Hilbert space with reproducing kernel being the Wiener kernel
\[
K_{d,r}(x, y) = \prod_{k=1}^{d} K_{1,r_k}(x_k, y_k), \quad x, y \in [0, 1]^d,
\]
where
\[
K_{1,r}(x, y) = \int_{0}^{\min(x,y)} \frac{(x-u)^r}{r!} \frac{(y-u)^r}{r!} du = \int_{0}^{1} \frac{(x-u)^r}{r!} \frac{(y-u)^r}{r!} du
\]
is the Wiener kernel, $x, y \in [0, 1]$, $t_+ = \max\{t, 0\}$, and $r = \{r_k\}_{k \in \mathbb{N}}$ is a sequence of nondecreasing nonnegative integers satisfying (4.1). If $r = 0$, we also get the standard Wiener kernel
\[
K_{1,0}(x, y) = \min(x, y).
\]
If $X_r^W(t)$ is a Gaussian random process with zero mean and covariance kernel $K_{1,r}$, then it is called the univariate integrated Wiener process. Integrated Euler and Wiener processes have many applications in probability theory, statistics, physics.

The reproducing kernel Hilbert space $H(K_d^W)$ is equal to the space of functions $f : [0, 1] \to \mathbb{R}$ such that $f$ is absolutely continuous and satisfies the following boundary conditions
\[
f(0) = f'(0) = f''(0) = \cdots = f^{(r)}(0) = 0.
\]
The inner product of $H(K_d^W)$ is given by
\[
(f, g)_r = \int_{0}^{1} f^{(r+1)}(x)g^{(r+1)}(x)dx \quad \text{for all } f, g \in H(K_d^W).
\]

Obviously, the space $H(K_d^W)$ is a tensor product of the univariate $H(K_1^W)$, i.e.,
\[
H(K_d^W) = H(K_{1,r_1}) \otimes H(K_{1,r_2}) \otimes \cdots \otimes H(K_{1,r_d}).
\]

We consider the multivariate approximation problem $APP = \{APP_d\}_{d \in \mathbb{N}}$ which is defined via the embedding operator
\[
APP_d : H(K_d^W) \to L_2([0, 1]^d) \quad \text{with } APP_d f = f.
\]

For the above approximation problem $APP = \{APP_d\}$, the eigenvalues of the operator $W_d = APP_d^{\dagger} APP_d$ are given by (see [10][11])
\[
\{\lambda_{d,j}^W\}_{j \in \mathbb{N}^d} = \{\lambda^W(1, j_1)\lambda^W(2, j_2)\cdots\lambda^W(d, j_d)\}_{(j_1, \ldots, j_d) \in \mathbb{N}^d},
\]
where
\[ \lambda^W(k, j) = \left[ \frac{1}{\pi(j - \frac{1}{2})} \right]^{2r_k+2} + \mathcal{O}(j^{-(2r_k+3)}), \ j \to \infty, \]
and for two positive sequences \( f, g : \mathbb{N} \to [0, \infty) \),
\[ f(k) = \mathcal{O}(g(k)), \ k \to \infty \]
means that there exists two constants \( C > 0 \) and \( k_0 \in \mathbb{N} \) for which \( f(k) \leq Cg(k) \) holds for any \( k \geq k_0 \),
\[ f(k) = \Theta(g(k)), \ k \to \infty \]
mean that
\[ f(k) = \mathcal{O}(g(k)) \] and \( g(k) = \mathcal{O}(f(k)), \ k \to \infty \)
Now we verify that the above approximation problem APP satisfies Conditions (2) and (3) of Property (P). It was proved in [11] that the sequence \( \{h^W_k\} \) satisfies
\[ h^W_k = \lambda^W(k, 2) / \lambda^W(k, 1) = \Theta(r_k^{-2}) = \Theta((1 + r_k)^{-2}), \ k \to \infty. \]
We conclude that the approximation problem APP satisfies Condition (2)' defined in Remark 3.1 with \( f^W_k = (1 + r_k)^{-2}, \ k \in \mathbb{N}. \)
It follows from [11, Thm. 4.1] that for \( x \in (3/5, 1] \),
\[ A_x := \sup_{k \in \mathbb{N}} \sum_{j=3}^{\infty} \left( \frac{\lambda^W(k, j)}{\lambda^W(k, 2)} \right)^x < \infty. \]
We conclude that for \( x > \frac{2}{3} \)
\[ \sup_{k \in \mathbb{N}} H^W(k, x) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda^W(k, j)}{\lambda^W(k, 2)} \right)^x = 1 + \sup_{k \in \mathbb{N}} \sum_{j=3}^{\infty} \left( \frac{\lambda^W(k, j)}{\lambda^W(k, 2)} \right)^x = 1 + A_x < \infty. \]
It follows that APP satisfies Condition (3) of Property (P) with \( \tau_0 \in [0, \frac{3}{5}] \). By Remark 3.1, we have the following corollary.

**Corollary 4.3.** Consider the approximation problem APP = \{APP_d\}_{d \in \mathbb{N}} in the space \( H(K^W_d, r) \) with sequence \( r = \{r_k\}_{k \in \mathbb{N}} \) satisfying (4.1). Then for the normalized error criterion, we have

1. APP is strongly polynomially tractable iff
   \[ A_* = \liminf_{k \to \infty} \frac{2 \ln(1 + r_k)}{\ln k} = 2 \liminf_{k \to \infty} \frac{\ln r_k}{\ln k} > 0, \]
   and the exponent of SPT is
   \[ p^* = \max \{ \frac{2}{A_*}, 2\tau_0 \}. \]
2. APP is strongly polynomially tractable iff it is polynomially tractable.
3. APP is \( (s, t) \)-weakly tractable with \( s > 0 \) and \( t > 1 \) for all sequences \( r = \{r_k\}_{k \in \mathbb{N}} \) satisfying (4.1).

**Remark 4.4.** We do not know the exact value of \( \tau_0 \). It is open.
4.3. Function approximation with Korobov kernels.

In this subsection we consider multivariate approximation problems with Korobov kernels. First we recall definition of Korobov spaces (see [14, Appendix A]). Let \( r = \{ r_k \}_{k \in \mathbb{N}} \) and \( g = \{ g_k \}_{k \in \mathbb{N}} \) be two sequences satisfying
\[
1 \geq g_1 \geq g_2 \geq \cdots \geq g_k \geq \cdots > 0,
\]
and
\[
0 < r_1 \leq r_2 \leq \cdots \leq r_k \leq \cdots.
\]
For \( d = 1, 2, \cdots \), we define the spaces
\[
H_{d,r,g} = H_{1,r_1,g_1} \otimes H_{1,r_2,g_2} \otimes \cdots \otimes H_{1,r_d,g_d}.
\]
Here \( H_{1,\alpha,\beta} \) is the Korobov space of univariate complex valued functions \( f \) defined on \([0, 1]\) such that
\[
\| f \|_{H_{1,\alpha,\beta}}^2 := | \hat{f}(0) |^2 + \beta^{-1} \sum_{h \in \mathbb{Z}, h \neq 0} |h|^{2\alpha} |\hat{f}(h)|^2 < \infty,
\]
where \( \beta \in (0, 1] \) is a scaling parameter, and \( \alpha > 0 \) is a smoothness parameter,
\[
\hat{f}(h) = \int_0^1 \exp(-2\pi i h x) f(x) dx \text{ for } h \in \mathbb{Z}
\]
are the Fourier coefficients of \( f \), \( i = \sqrt{-1} \). If \( \alpha > \frac{1}{2} \), then \( H_{1,\alpha,\beta} \) consists of 1-periodic functions and is a reproducing kernel Hilbert space with reproducing kernel
\[
R_{\alpha,\beta}(x, y) := 1 + 2\beta \sum_{j=1}^{\infty} j^{-2\alpha} \cos(2\pi j (x - y)), \ x, y \in [0, 1].
\]
If \( \alpha \) is an integer, then \( H_{1,\alpha,\beta} \) consists of 1-periodic functions \( f \) such that \( f^{(\alpha - 1)} \) is absolutely continuous, \( f^{(\alpha)} \) belongs to \( L_2([0, 1]) \), and
\[
\| f \|_{H_{1,\alpha,\beta}}^2 = | \int_{[0, 1]} f(x) dx |^2 + (2\pi)^{2\alpha} \beta^{-1} \int_{[0, 1]} |f^{(\alpha)}(x)|^2 dx.
\]
For \( d \geq 2 \) and two sequences \( r = \{ r_k \} \) and \( g = \{ g_k \} \) satisfying (4.3) and (4.4), the space \( H_{d,\alpha,\beta} \) is a Hilbert space with the inner product
\[
\langle f, g \rangle_{H_{d,r,g}} = \sum_{h \in \mathbb{Z}^d} \rho_{d,r,g}(h) \hat{f}(h) \overline{\hat{g}(h)},
\]
where
\[
\rho_{d,r,g}(h) = \prod_{j=1}^d (\delta_{0,h_j} + g_j^{-1}(1 - \delta_{0,h_j})) |h_j|^{2r_j},
\]
\( \delta_{0,i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \) and
\[
\hat{f}(h) = \int_{[0, 1]^d} \exp(-2\pi i h \cdot x) f(x) dx \text{ for } h \in \mathbb{Z}^d
\]
are the Fourier coefficients of \( f, x \cdot y = x_1 y_1 + \cdots + x_d y_d \). If \( r_1 > \frac{1}{2} \), then \( H_{d,r,g} \) consists of 1-periodic functions on \([0, 1]^d\) and is a reproducing kernel Hilbert space.
with reproducing kernel
\[
K_{d,r,g}(x,y) = \prod_{k=1}^{d} R_{r_k,g_k}(x_k, y_k)
= \prod_{k=1}^{d} \left( 1 + 2g_k \sum_{j=1}^{\infty} j^{-2r_k} \cos(2\pi j(x_k - y_k)) \right), \quad x, y \in [0,1]^d.
\]

For integers \( r_j \), the inner product of \( H_{d,r,g} \) can be expressed in terms of derivatives, see [14, Appendix A]. If \( \{r_k\} \) is nondecreasing and \( g_k = (2\pi)^{-2r_k} \), then we obtain the specific Korobov spaces \( H_{d,r} \) which were given in [17, 20].

In this subsection, we consider the multivariate approximation problem \( \text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}} \) which is defined via the embedding operator
\[
\text{APP}_d : H_{d,r,g} \rightarrow L^2([0,1]^d) \quad \text{with} \quad \text{APP}_d f = f.
\]

For the above approximation problem \( \text{APP} = \{\text{APP}_d\} \), the eigenvalues of the operator \( W_d = \text{APP}_d^* \text{APP}_d \) are given by (see [14, p. 184])
\[
\{\lambda_{d,j}\}_{j \in \mathbb{N}^d} = \{\lambda(1,j_1)\lambda(2,j_2)\ldots\lambda(d,j_d)\}_{(j_1,...,j_d) \in \mathbb{N}^d},
\]
where \( \lambda(k,1) = 1 \), and
\[
\lambda(k,2j) = \lambda(k,2j+1) = g_k j^{-2r_k} \in (0,1], \quad j \in \mathbb{N}.
\]

Now we verify that the above approximation problem \( \text{APP} \) satisfies Property (P). First we note that \( \lambda(k,1) = 1 \), and the sequence \( \{h_k\}, \quad h_k = \frac{\lambda(k,2)}{\lambda(k,1)} = g_k \) is nonincreasing due to (4.3). Next we have for \( x > \frac{1}{2r_1} \),
\[
\sup_{k \in \mathbb{N}} H(k,x) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda(k,j)}{\lambda(k,2)} \right)^x J \Lambda_{j-1} = \sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} j^{-2r_1 x} = H(1,x) = 2 \sum_{j=1}^{\infty} j^{-2r_1 x} = 2 \zeta(2r_1 x) < \infty,
\]
where \( \zeta(s) = \sum_{j=1}^{\infty} j^{-s} \) is the Riemann zeta function which is well-defined only for \( s > 1 \). This means that the above approximation problem \( \text{APP} \) has Property (P) with \( h_k = g_k \) and \( \tau_0 = \frac{1}{2r_1} \). By Theorem 2.1, we have the following corollary.

**Corollary 4.5.** Consider the above approximation problem \( \text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}} \) in the worst case setting with the sequences \( r \) and \( g \) satisfying (4.3) and (4.4). Then for the absolute error criterion or the normalized error criterion, we have

1. \( \text{APP} \) is strongly polynomially tractable iff
   \[
   A_* = \liminf_{k \to \infty} \frac{\ln g_k^{-1}}{\ln k} > 0,
   \]
   and the exponent of SPT is
   \[
   p^* = \max \left\{ \frac{2}{A_*}, \frac{1}{\tau_0} \right\}.
   \]
2. \( \text{APP} \) is strongly polynomially tractable iff it is polynomially tractable.
(3) APP is quasi-polynomially tractable iff $B := \lim_{k \to \infty} \ln g_k^{-1} > 0$ iff $g_k \neq 1$. Furthermore, the exponent of QPT is
\[ t^* = \max \left\{ \frac{2}{B}, \frac{1}{r_1} \right\}. \]

(3) APP is quasi-polynomially tractable iff APP is uniformly weakly tractable, iff APP is $(s,t)$-weakly tractable with $s > 0$ and $t \in (0,1]$, and iff APP is weakly tractable.

(4) APP is $(s,t)$-weakly tractable with $s > 0$ and $t > 1$ for all sequences $r$ and $g$ satisfying (4.3) and (4.4).

(5) If $g_k \equiv 1$, then APP suffers from the curse of dimensionality.

Remark 4.6. In [17, 20], the authors investigated tractability of the approximation problem
\[ \text{APP}_d : H_{d,r} \to L_2([0,1]^d) \text{ with } \text{APP}_d f = f, \]
where $H_{d,r} = H_{d,r,g}$, $r = \{r_k\}$ and $g = \{g_k\}$ satisfy $0 < r_1 \leq r_2 \leq \ldots$, $g_k = (2\pi)^{-2r_1}k, k = 1,2,\ldots$. They obtained the following results.

For the absolute error criterion or the normalized error criterion, we have
- SPT holds iff PT holds iff
  \[ R = \limsup_{k \to \infty} \frac{\ln k}{r_k} < +\infty, \]
with the exponent of SPT
  \[ p^* = \max \left\{ \frac{1}{r_1}, \frac{R}{\ln 2\pi} \right\}. \]
- QPT holds iff UWT holds iff WT holds iff
  \[ r_1 > 0. \]
Furthermore, the exponent of QPT is
  \[ t^* = \frac{1}{r_1}. \]

We remark that the above results follows directly from Corollary 4.5. Hence Corollary 4.5 is a generalization of the above conclusion.

4.4. Function approximation with Gaussian kernels.
In this subsection we consider multivariate approximation problems with Gaussian kernels. Let $H(K_{d,\gamma})$ be the reproducing kernel Hilbert space with the Gaussian kernel
\[ K_{d,\gamma}(x,y) = \prod_{j=1}^{d} K_{1,\gamma_j}(x_j, y_j), \quad x,y \in \mathbb{R}^d, \]
where
\[ K_{1,\gamma}(x,y) = \exp(-\gamma^2(x-y)^2), \quad \gamma > 0, \quad x,y \in \mathbb{R}, \]
and $\gamma = \{\gamma_j\}_{j \in \mathbb{N}}$ is a given sequence of shape parameters not depending on $d$ and satisfying
\[ \gamma_1^2 \geq \gamma_2^2 \geq \ldots > 0. \]
Obviously, the space $H(K_{d,\gamma})$ is a tensor product of the univariate $H(K_{1,\gamma})$, i.e.,
\[ H(K_{d,\gamma}) = H(K_{1,\gamma_1}) \otimes H(K_{1,\gamma_2}) \otimes \cdots \otimes H(K_{1,\gamma_d}). \]
The reproducing kernel Hilbert space $H(K_{d,\gamma})$ has been widely used in many fields such as numerical computation, statistical learning, and engineering (see e.g., [4, 6, 18]).

We consider the multivariate approximation problem $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ which is defined via the embedding operator

$$\text{APP}_d : H(K_{d,\gamma}) \to L_{2,d} \text{ with } \text{APP}_d f = f,$$

where

$$L_{2,d} = \left\{ f \mid \|f\|_{L_{2,d}} = \left( \int_{\mathbb{R}^d} |f(x)|^2 \prod_{j=1}^d \frac{\exp(-x_j^2)}{\sqrt{\pi}} \, dx \right)^{1/2} < \infty \right\}$$

is a separable Hilbert space of real-valued functions on $\mathbb{R}^d$ with inner product

$$\langle f, g \rangle_{L_{2,d}} = \int_{\mathbb{R}^d} f(x)g(x) \prod_{j=1}^d \frac{\exp(-x_j^2)}{\sqrt{\pi}} \, dx.$$

For the above approximation problem $\text{APP} = \{\text{APP}_d\}$, the eigenvalues of the operator $W_d = \text{APP}_d^* \text{APP}_d$ are given by (see [4, 18, 23])

$$\{\lambda_{d,j}\}_{j \in \mathbb{N}^d} = \{\lambda(1,j_1)\lambda(2,j_2)\ldots\lambda(d,j_d)\}_{(j_1,\ldots,j_d) \in \mathbb{N}^d},$$

where

$$\lambda(k,j_k) = (1 - \omega_{\gamma_k})\omega_{\gamma_k}^{-1}, \quad \text{and} \quad \omega_{\gamma} = \frac{2\gamma^2}{1 + 2\gamma^2 + \sqrt{1 + 4\gamma^2}} \text{ for } \gamma > 0.$$  

Clearly, $0 < \omega_{\gamma} < 1$ for $\gamma > 0$ and $\omega_{\gamma}$ is an increasing function of $\gamma$ and $\omega_{\gamma}$ tends to $0$ iff $\gamma$ tends to $0$. We also have

$$\lim_{\gamma \to 0} \frac{\omega_{\gamma}}{\gamma^2} = 1, \quad \text{and} \quad \lim_{\gamma \to 0} \frac{\ln \omega_{\gamma}^{-1}}{\ln \gamma^{-2}} = 1. \tag{4.7}$$

Now we verify that the above approximation problem $\text{APP}$ satisfies Conditions (2) and (3) of Property (P). First we note that the sequence $\{h_k\}$, $h_k = \frac{\lambda(k,2)}{\lambda(k,1)} = \omega_{\gamma_k}$ is nonincreasing due to (11.3) and (11.6). Next we have for any $x > 0$,

$$H(k,x) = \sum_{j=2}^{\infty} \frac{\lambda(k,j)}{\lambda(k,2)} x = \sum_{j=2}^{\infty} \left( \frac{\omega_{\gamma_k}^{-1}}{\omega_{\gamma_k}} \right) x = \sum_{j=2}^{\infty} \omega_{\gamma_k}^{-1} x^{j-2} = \frac{1}{1 - \omega_{\gamma_k} x},$$

so that for $x > 0$,

$$\sup_{k \in \mathbb{N}} H(k,x) = H(1,x) = \frac{1}{1 - \omega_{\gamma_1} x} < \infty.$$ 

It follows that $\text{APP}$ satisfies Condition (3) of Property (P) with $h_k = \omega_{\gamma_k}$ and $\tau_0 = 0$. By Remark 2.2 and (4.7), we have the following corollary.

**Corollary 4.7.** Consider the approximation problem $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ in the space $H(K_{d,\gamma})$ with shape parameters $\gamma = \{\gamma_j\}$ satisfying (11.5). Then for the normalized error criterion, we have

(1) $\text{APP}$ is strongly polynomially tractable iff

$$r(\gamma) = \lim_{k \to \infty} \frac{\ln \omega_{\gamma_k}^{-1}}{\ln k} = \lim_{k \to \infty} \frac{\ln \gamma_k^{-2}}{\ln k} > 0,$$

and the exponent of SPT is $\frac{2}{r(\gamma)}.$
(2) APP is strongly polynomially tractable iff it is polynomially tractable.

(3) APP is quasi-polynomially tractable for all shape parameters. Furthermore, the exponent of QPT is

\[ t^* = \lim_{k \to \infty} \frac{2}{\ln \frac{1+2\gamma_j^2 + \sqrt{1+4\gamma_j^2}}{2\gamma_j^2}}. \]

(4) Obviously, APP is quasi-polynomially tractable implies \((s, t)\)-weakly tractable for any positive \(s\) and \(t\), as well as uniformly weakly tractable and weakly tractable for all shape parameters.

Remark 4.8. In Corollary 4.7 we obtain the exact value of the exponent of QPT. This result is new. The other results in Corollary 4.7 were obtained in \([4]\). Indeed, the authors in \([4]\) obtained complete results about the tractability of APP as follows.

For the absolute error criterion, we have

- SPT holds for all shape parameters with the exponent \(\min \left\{ 2, \frac{2}{r(\gamma)} \right\}\), where \(r(\gamma)\) is defined by

\[
\begin{align*}
    r(\gamma) & = \sup \left\{ \delta > 0 \mid \sum_{j=1}^{\infty} \gamma_j^{2/\delta} < \infty \right\} \\
    & = \sup \left\{ \beta > 0 \mid \lim_{j \to \infty} \frac{\gamma_j^2}{\beta^2} = 0 \right\} = \liminf_{j \to \infty} \frac{\ln \gamma_j^{-2}}{\ln j},
\end{align*}
\]

where the last equality follows from \([1]\).

- Obviously, SPT implies all PT, QPT, WT, \((s, t)\)-WT for any positive \(s\) and \(t\), as well as UWT and WT, for all shape parameters.

For the normalized error criterion, we have

- SPT holds iff PT holds iff \(r(\gamma) > 0\), with the exponent \(\frac{2}{r(\gamma)}\).

- QPT holds for all shape parameters with the exponent \(t^* \leq \frac{2}{\ln \frac{1+2\gamma_j^2 + \sqrt{1+4\gamma_j^2}}{2\gamma_j^2}}. \)

- Obviously, QPT implies \((s, t)\)-WT for any positive \(s\) and \(t\), as well as UWT and WT for all shape parameters.

4.5. Function approximation with analytic Korobov kernels.

In this subsection we consider multivariate approximation problems with analytic Korobov kernels. Let \(H(K_{d,a,b})\) be the reproducing kernel Hilbert space with the analytic Korobov kernel

\[ K_{d,a,b}(x, y) = \prod_{k=1}^{d} K_{1,a_k,b_k}(x_k, y_k), \quad x, y \in [0, 1]^d, \]

where \(a = \{a_k\}_{k \in \mathbb{N}}\) and \(b = \{b_k\}_{k \in \mathbb{N}}\) are two sequences of positive weights satisfying

\[ 0 < a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots, \quad \text{and} \quad b_* := \inf_{k \in \mathbb{N}} b_k > 0. \]
$K_{1,a,b}$ are univariate analytic Korobov kernels,
\[ K_{1,a,b}(x, y) = \sum_{k \in \mathbb{Z}} \omega^{|h|} \exp(2\pi ih(x - y)), \ x, y \in [0, 1]. \]

Here $\omega \in (0, 1)$ is a fixed number, $i = \sqrt{-1}, a, b > 0$. Hence, we have
\[ K_{d,a,b}(x, y) = \sum_{h \in \mathbb{Z}} \omega_h \exp(2\pi ih(x - y)), \ x, y \in [0, 1]^d, \]
with
\[ \omega_h = \omega_{\sum_{k=1}^d a_k |h_k|^{\tau_k}}, \ \forall \ h = (h_1, h_2, \cdots, h_d) \in \mathbb{Z}^d, \]
for fixed $\omega \in (0, 1)$.

Obviously, the space $H(K_{d,a,b})$ is a tensor product of the univariate $H(K_{1,a,b})$, i.e.,
\[ H(K_{d,a,b}) = H(K_{1,a_1,b_1}) \otimes H(K_{1,a_2,b_2}) \otimes \cdots \otimes H(K_{1,a_d,b_d}). \]
The reproducing kernel Hilbert space $H(K_{d,a,b})$ has been widely used in the study of tractability and exponential convergence-tractability (see [3, 8, 9, 12]).

We consider the multivariate approximation problem $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ which is defined via the embedding operator
\[ \text{APP}_d : H(K_{d,a,b}) \to L_2([0, 1]^d) \text{ with } \text{APP}_d f = f. \]

For the above approximation problem $\text{APP} = \{\text{APP}_d\}$, the eigenvalues of the operator $W_d = \text{APP}_d$ $\text{APP}_d$ are given by
\[ \{\lambda_{d,j}\}_{j \in \mathbb{N}^d} = \{\lambda(1, j_1)\lambda(2, j_2) \cdots \lambda(d, j_d)\}_{(j_1, \ldots, j_d) \in \mathbb{N}^d}, \]
where $\lambda(k, 1) = 1$, and
\[ \lambda(k, 2j) = \lambda(k, 2j + 1) = \omega^{a_k j}, \ j \in \mathbb{N}. \]

Now we verify that the above approximation problem $\text{APP}$ satisfies Property (P). First we note that $\lambda(k, 1) = 1$, and the sequence $\{h_k\}$, $h_k = \frac{\lambda(k, 2)}{\lambda(k, 1)} = \omega^a$ is nonincreasing due to [14,8]. Next we have for any $x > 0$,
\[ \sup_{k \in \mathbb{N}} H(k, x) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda(k, j)}{\lambda(k, 2)} \right)^x = 2 \sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} \omega^{a_k(j^h - 1)} \leq 2 \sum_{j=1}^{\infty} \omega^{a_k(j^h - 1)}. \]

Since
\[ \omega^{a_k(j^h - 1)} = \frac{1}{\omega^a j} \ln \omega^{-1} \text{ and } \lim_{j \to \infty} \frac{\omega^a j}{\ln j} = \infty, \]
we conclude that for $x > 0$,
\[ \sup_{k \in \mathbb{N}} H(k, x) \leq M_x := 2 \sum_{j=1}^{\infty} \omega^{a_k(j^h - 1)} < \infty. \]

This means that the above approximation problem $\text{APP}$ has Property (P) with $h_k = \omega^a$ and $\tau_0 = 0$. By Theorem 2.1, we have the following corollary.

**Corollary 4.9.** Consider the above approximation problem $\text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}}$ in the space $H(K_{d,a,b})$ with the sequences $a$ and $b$ satisfying [14,8]. Then for the absolute error criterion or the normalized error criterion, we have
(1) APP is strongly polynomially tractable iff
\[ A_* = \liminf_{k \to \infty} \frac{a_k \ln \omega^{-1}}{\ln k} = \ln \omega^{-1} \liminf_{k \to \infty} \frac{a_k}{\ln k} > 0, \]
and the exponent of SPT is
\[ p^* = \frac{2}{A_*}. \]

(2) APP is strongly polynomially tractable iff it is polynomially tractable.

(3) APP is quasi-polynomially tractable for all sequences \( a \) and \( b \) satisfying (4.8).
Furthermore, the exponent of QPT is
\[ t^* = \frac{2}{B}, \quad B = \ln \omega^{-1} \lim_{k \to \infty} a_k. \]

(4) Obviously, quasi-polynomially tractable implies \((s, t)\)-weakly tractable for any positive \( s \) and \( t \), as well as uniformly weakly tractable and weakly tractable for all sequences \( a \) and \( b \) satisfying (1.8).

Remark 4.10. Corollary 4.9 is not new. All results in Corollary 4.9 were obtained in [12, Theorem 3.1]. See [8, 9, 12] for background and more information.

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REFERENCES
[1] J. Chen, H. Wang, Average case tractability of multivariate approximation with Gaussian kernels, J. Approx. Theory 239 (2019) 51-71.
[2] J. Chen, H. Wang, J. Zhang, Average case \((s, t)\)-weak tractability of non-homogeneous tensor product problems, J. Complexity 49 (2018) 27-45.
[3] J. Dick, P. Kritzer, F. Pillichshammer, H. Woźniakowski, Approximation of analytic functions in Korobov spaces, J. Complexity 30 (2014) 2-28.
[4] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, On dimension-independent rates of convergence for function approximation with Gaussian kernels, SIAM J. Numer. Anal. 50 (2012) 247-271.
[5] G. E. Fasshauer, F. J. Hickernell, H. Woźniakowski, Average case approximation: convergence and tractability of Gaussian kernels, Monte Carlo and Quasi-Monte Carlo 2010, eds. L. Plaskota and H. Woźniakowski, Springer Verlag, 2012, 329-345.
[6] A. I. J. Forrester, A. Sobester, and A. J. Keane, Engineering Design via Surrogate Modelling: A Practical Guide, Wiley, Chichester, 2008.
[7] F. Gao, J. Hanning, F. Torcaso, Integrated Brownian motions and exact \( L_2 \)-small balls, Ann. Probab. 31 (2003) 1329-1337.
[8] C. Irrgeher, P. Kritzer, F. Pillichshammer, H. Woźniakowski, Tractability of multivariate approximation defined over Hilbert spaces with exponential weights, J. Approx. Theory 207 (2016) 301-338.
[9] P. Kritzer, F. Pillichshammer, H. Woźniakowski, Tractability of multivariate analytic problems, in: P. Kritzer, H. Niederreiter, F. Pillichshammer, A. Winterhof (Eds.), Uniform Distribution and Quasi-Monte Carlo Methods. Discrepancy, Integration and Applications, De Gruyter, Berlin, 2014, pp. 147-170.
[10] M. A. Lifshits, A. Papageorgiou, H. Woźniakowski, Average case tractability of non-homogeneous tensor product problems, J. Complexity 28 (2012) 539-561.
[11] M. A. Lifshits, A. Papageorgiou, H. Woźniakowski, Tractability of multi-parametric Euler and Wiener integrated processes, Probab. Math. Statist. 32 (2012) 131-165.
[12] Y. Liu, G. Xu, A note on tractability of multivariate analytic problems, J. Complexity 34 (2016) 42-49.
[13] Y. Liu, G. Xu, Average case tractability of a multivariate approximation problem, J. Complexity, 43 (2017) 76-102.
[14] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, EMS, Zürich, 2008.
[15] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume II: Standard Information for Functionals, EMS, Zürich, 2010.
[16] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, Volume III: Standard Information for Operators, EMS, Zürich, 2012.
[17] A. Papageorgiou, H. Woźniakowski, Tractability through increasing smoothness, J. Complexity 26 (2010) 409-421.
[18] C. E. Rasmussen, C. Williams, Gaussian Processes for Machine Learning, MIT Press, 2006 (online version at http://www.gaussianprocess.org/gpml/).
[19] P. Siedlecki, Uniform weak tractability, J. Complexity 29 (6) (2013) 438-453.
[20] P. Siedlecki, Uniform weak tractability of multivariate problems with increasing smoothness, J. Complexity 30 (2014) 716-734.
[21] P. Siedlecki, $(s, t)$-weak tractability of Euler and Wiener integrated processes, J. Complexity 45 (2018) 55-66.
[22] P. Siedlecki, M. Weimar, Notes on $(s, t)$-weak tractability: a refined classification of problems with (sub)exponential information complexity, J. Approx. Theory 200 (2015) 227-258.
[23] I. H. Sloan, H. Woźniakowski, Multivariate approximation for analytic functions with Gaussian kernels, J. Complexity 45 (2018) 1-21.
[24] H. Woźniakowski, Tractability and strong tractability of multivariate tensor product Problems, J. Comput. Inform. 4 (1994) 1-19.
[25] G. Xu, Quasi-polynomial tractability of linear problems in the average case setting, J. Complexity 30 (2014) 54-68.
[26] G. Xu, Tractability of linear problems defined over Hilbert spaces, J. Complexity 30 (2014) 735-749.

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.

E-mail address: wanghp@cnu.edu.cn.