THE CAP SET PROBLEM: UP TO DIMENSION 7

HENRY (MAYA) ROBERT THACKERAY

Abstract. An $s$-cap $n$-flat is given by a set of $s$ points, no three of which are on a common line, in an $n$-dimensional affine space over the field of three elements. The cap set problem in dimension $n$ is: what is the maximum $s$ such that there is an $s$-cap $n$-flat?

The first two papers in this series of articles considered the cap set problem in dimensions up to and including 5. In this paper, which is the third in the series, we consider dimensions 6 and 7: we prove that every 110-cap 6-flat is a 112-cap 6-flat minus two cap points, and that there are no 289-cap 7-flats.

Keywords: cap, cap set, combinatorics, affine space, projective space
MSC2020: 51E20, 05B40, 05D99, 05B25, 51E15

1. Introduction

This paper is the third in a series that investigates the cap set problem. The cap set problem in dimension $n$ is: what is the maximum $s$ such that there is an $s$-cap $n$-flat (also called an $n$-dimensional cap of size $s$), that is, a pair $(S, F)$ such that $F$ is an $n$-dimensional affine space over $\mathbb{Z}/3\mathbb{Z}$ and the size-$s$ subset $S$ of $F$ contains no three points on a common line? For dimensions up to and including 6, the cap set problem is solved; see Davis and Maclagan [1], Edel et al. [2], Ellenberg and Gijswijt [3], and Potechin [4] for background reading and previously known results for the problem.

The first paper of this series (Thackeray [5]) classified certain caps in dimensions up to and including 4, and the second paper (Thackeray [6]) classified caps of size at least 41 in dimension 5; we take those two papers as known. The current paper investigates dimensions 6 and 7: we prove that every 110-cap 6-flat is a 112-cap 6-flat minus two cap points, and that there are no 289-cap 7-flats.

2. Lemmas

We start by recalling known results and proving useful lemmas.

Theorem 2.1 (Large 5-dimensional caps). Let $C$ be a 5-dimensional cap. If we have $|C| \geq 43$, then $C$ is a 45-cap 5-flat minus at most two cap points; if $|C| = 42$, then $C$ is a $\triangle 686$ or $C$ is a 45-cap 5-flat minus three cap points. In a $\triangle 686$, the hyperplane point counts are as in Table 1.

Proof. See Thackeray [6]; the last sentence was checked directly using a representative $\triangle 686$. \hfill $\Box$

Lemma 2.2 (Hyperplane directions of 45-cap 5-flat). Let $C$ be a 45-cap 5-flat. Of all the 3-flat directions $D$ of $C$, exactly 45 are such that the nine 3-dimensional caps in the 3-flats of $D$ are eight square pyramids and one tetrahedron plus centre.
There is a unique line direction $L$ of $C$ such that in each of those 45 directions $D$, each of the nine 3-flats is the union of some lines in $L$. The cap $C$ has

(i) exactly 10 \{18, 9, 18\} hyperplane directions in which the 18-cap 4-flats are both $882A_1$ caps,
(ii) exactly 45 \{18, 9, 18\} hyperplane directions in which the 18-cap 4-flats are both $882A_2$ caps,
(iii) exactly 30 \{15, 15, 15\} hyperplane directions in which each 4-flat is a union of lines in $L$, and
(iv) exactly 36 \{15, 15, 15\} hyperplane directions in which each 4-flat is not a union of lines in $L$,
and for each two hyperplane directions in the same one of those four categories, some symmetry of $C$ sends one of the two hyperplane directions to the other.

Proof. This was checked directly using a representative 45-cap 5-flat.

Lemma 2.3 (Replacing cap points in an 18-cap 4-flat). Let $(S, F)$ be an $882A_2$. Let $n \in \{1, 2, 3\}$. If $A$ and $B$ are subsets of $S$ and $F - S$ respectively such that $|A| = |B| = n$ and $((S - A) \cup B, F)$ is a cap, then that cap contains a 9-cap 3-flat (and therefore is neither an $882A_1$ nor an $882A_2$).

Proof. Without loss of generality, let $(S, F)$ be the $882A_2$ in Thackeray [5, Figure 42, “Another $882A_2$”]. For each $n \in \{1, 2, 3\}$, and for each subset $S_-$ of $S$ such that $|S_-| = n$, a computer search found each possible subset $S_+$ of $F - (S - S_-)$ such that $|S_+| = n$ and $((S - S_-) \cup S_+, F)$ is a cap with no 9-cap 3-flat (which corresponds to $A = S_- - S_+$ and $B = S_+ - S_-$. In the cases $n = 1$, $n = 2$, and $n = 3$, there are respectively $18 \choose 1$, $18 \choose 2$, and $18 \choose 3$ solutions such that $S_+ = S_-$ (so $A = B = \emptyset$); the computer search confirmed that there are no more solutions for each of those values of $n$. (A further computer search confirmed that for $n = 4$, additional solutions do exist.)

Each $882A_2$ has the following properties.

(a) The $882A_2$ has exactly one 2-flat direction in which each 2-flat has exactly two cap points. This is the nine-2s 2-flat direction.
(b) The $882A_2$ has exactly one \{8, 5, 5\} hyperplane direction in which the 8-cap 3-flat is a cube. This direction is the 855 cube direction.
(c) The $882A_2$ has exactly one \{8, 8, 2\} hyperplane direction in which both 8-cap 3-flats are cubes. This direction is the 882 cube direction.
(d) There is some independent pair $(x_1, x_2)$ of co-ordinates (determined up to negating $x_1$ and/or $x_2$) such that the $x_1$-hyperplane (respectively, $x_2$-hyperplane) direction is the 855 cube direction (respectively, the 882 cube direction).

\begin{table}
\centering
\begin{tabular}{|l l| l l|}
\hline
Point count & Hyperplane directions & Point count & Hyperplane directions \\
\hline
{20,16,6} & 3 & {16,15,11} & 24 \\
{18,18,6} & 4 & {16,14,12} & 36 \\
{18,17,7} & 18 & {15,15,12} & 3 \\
{18,12,12} & 6 & {14,14,14} & 27 \\
\hline
\end{tabular}

Table 1. Hyperplane point counts of a $\Delta 686$.
\end{table}
direction) and the point count of the $882A_2$ for $(x_1, x_2)$ is

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix}.$$  

For such a pair $(x_1, x_2)$, the 4-cap 2-flats $(x_1, x_2) = (0, \pm 1)$ are translations of each other; they are the standard squares of the $882A_2$. The midpoints of the cap line segments in the four 2-flats $(x_1, x_2) = (\pm 1, \pm 1)$ form a square – which is the square of midpoints of the $882A_2$ – and the 2-flat containing that square also contains the centres of the standard squares and the cap points in the 2-flats $(x_1, x_2) = (\pm 1, 0)$.

**Lemma 2.4** (Hyperplanes that are 18-cap 4-flats). In each 45-cap 5-flat, for each $\{18, 18, 9\}$ hyperplane direction $D$ in which the 18-cap 4-flats are $882A_2$ caps, we have the following.

(a) The nine 2s 2-flat directions of the 18-cap 4-flats in $D$ are parallel.

(b) The $882$ cube direction of each 18-cap 4-flat in $D$ is parallel to the $855$ cube direction of the other 18-cap 4-flat in $D$.

(c) The side directions of each standard square of each 18-cap 4-flat in $D$ are parallel to the diagonal directions of each standard square of the other 18-cap 4-flat in $D$.

**Proof.** This follows from Thackeray [6, Lemma 3.1], which indicates that a 45-cap 5-flat with a chosen $\{18, 18, 9\}$ hyperplane direction $D$ in which the 18-cap 4-flats are $882A_2$ caps is unique up to isomorphisms under which the image of each hyperplane in $D$ is a hyperplane in $D$. \qed

**Lemma 2.5** (Two 18-cap 4-flats minus cap points). Consider a 5-dimensional cap $C$ with co-ordinate $x_1$. Let the cap $\tilde{C}$, with the same underlying 5-flat as $C$, be the union of two $882A_2$ caps in the 4-flats $x_1 = \pm 1$ respectively. Suppose that the union of the two 4-dimensional subcaps $x_1 = \pm 1$ of $C$ is obtained from that of $\tilde{C}$ by removing $n$ cap points, for some nonnegative integer $n$.

(a) Suppose that the $882A_2$ caps $x_1 = \pm 1$ in $\tilde{C}$ are translations of each other. If $n$ is 0, 1, 2, or 3, then the number of cap points in the 4-flat $x_1 = 0$ of $C$ is 0, at most 1, at most 2, or at most 3 respectively.

(b) Suppose that the $882A_2$ caps $x_1 = \pm 1$ in $\tilde{C}$ are point reflections of each other. If $n$ is 0, 1, 2, or 3, then the number of cap points in the 4-flat $x_1 = 0$ of $C$ is at most 4, at most 4, at most 5, or at most 5 respectively.

Each of those upper bounds cannot be lowered.

**Proof.** Consider $\tilde{C}$. For each point $Q$ in the 4-flat $x_1 = 0$, find the number $n(Q)$ of cap line segments $L$ with one cap point in each of the 4-flats $x_1 = \pm 1$ such that $Q$ is the midpoint of $L$, and for each $a \in \{\pm 1\}$, let $S_a(Q)$ be the set of cap points $P$ in the $882A_2$ cap $x_1 = a$ such that the midpoint of $PQ$ is a cap point in the $882A_2$ cap $x_1 = -a$.

For part (a), it was checked that if the $882A_2$ cap $x_1 = 1$ is the image of the $882A_2$ cap $x_1 = -1$ under the translation map $U$, then the following statements hold:
(i) there are exactly 18 points \( Q \) with \( n(Q) = 1 \), namely, the images under \( U \) of the cap points in \( x_1 = 1 \); for these points \( Q \), the cap line segment with midpoint \( Q \) is \( U(Q)U^{-1}(Q) \);

(ii) there are exactly four points \( Q \) with \( n(Q) = 2 \), which form the image under \( U \) of the square of midpoints of the \( 82A_2 \) cap \( x_1 = 1 \); these four points \( Q \) are the midpoints of eight disjoint cap line segments \( L \), and for each of these four points \( Q \) and each \( a \in \{\pm 1\} \), the cap line segment \( S_a(Q) \) is not contained in any 2-flat in the nine-2s 2-flat direction of the \( 82A_2 \) cap \( x_1 = a \), and \( Q \) is the midpoint of the image of \( S_a(Q) \) under \( U^n \); and

(iii) all other points \( Q \) satisfy \( n(Q) \geq 4 \).

The result in part (a) follows. (After at most three cap points are removed from \( \tilde{C} \) to obtain the 4-flats \( x_1 = \pm 1 \) of \( C \), consider which points \( Q \), and how many points \( Q \), can be cap points in the 4-flat \( x_1 = 0 \) of \( C \).)

For part (b), it was checked that if the square of midpoints of the \( 82A_2 \) cap \( x_1 = 1 \) is the image of the square of midpoints of the \( 82A_2 \) cap \( x_1 = -1 \) under the translation map \( U \), then the following statements hold:

(i) there are exactly four points \( Q \) with \( n(Q) = 0 \), namely, the images under \( U \) of the cap points in the square of midpoints of \( x_1 = 1 \);

(ii) there are exactly four points \( Q \) with \( n(Q) = 2 \), they are the midpoints of eight disjoint cap line segments \( L \), those points \( Q \) form a square \( S \) that is a translation of each of the standard squares of the \( 82A_2 \) caps \( x_1 = \pm 1 \), and the centre of \( S \) is the image under \( U \) of the centre of the square of midpoints of \( x_1 = 1 \);

(iii) there are exactly 24 points \( Q \) with \( n(Q) = 3 \); for each \( Q_2 \) and \( Q_3 \) with \( n(Q_2) = 2 \) and \( n(Q_3) = 3 \), the intersection of \( S_1(Q_2) \cup S_{-1}(Q_2) \) and \( S_1(Q_3) \cup S_{-1}(Q_3) \) has at most one point or is a cap line segment with midpoint \( Q_3 \); for each two different points \( Q_3 \) and \( Q_4 \) with \( n(Q_3) = n(Q_4) = 3 \), if the intersection of \( S_1(Q_3) \cup S_{-1}(Q_3) \) and \( S_1(Q_4) \cup S_{-1}(Q_4) \) has more than two points, then

1. that intersection has exactly three points,
2. that intersection determines \( \{Q_3, Q_4\} \), and
3. the midpoint \( Q_0 \) of the line segment \( Q_3Q_4 \) satisfies \( n(Q_0) = 0 \); and

(iv) all other points \( Q \) satisfy \( n(Q) \geq 4 \).

The result in part (b) follows as in part (a). \( \square \)

3. Dimension 6

Throughout this section, \( C \) is a 6-dimensional cap with some co-ordinate \( \tilde{x}_1 \).

Potechin \([4]\) proved that there is a unique 112-cap 6-flat up to isomorphism. A representative 112-cap 6-flat is shown in Figure \([3]\).

Lemma 3.1 (The 112-cap 6-flat). Suppose that \( C \) is a 112-cap 6-flat.

(a) The numbers of \( \{45,45,22\} \) hyperplane directions and \( \{40,36,36\} \) hyperplane directions of \( C \) are 56 and 308 respectively.

(b) For every \( \{40,36,36\} \) hyperplane direction \( D \) of \( C \), there are co-ordinates \( y_1 \) and \( y_2 \) of \( C \) with \( (y_1, y_2) \) independent such that each of the \( y_1 \) - and \( y_2 \) - hyperplane directions of \( C \) has point count \( \{45,45,22\} \) and the \( (y_1 - y_2) \) - hyperplane direction of \( C \) is \( D \).
### Figure 1

A 112-cap 6-flat. Each of the four 18-cap 4-flats 
$\tilde{x}_1, x_1 = (\pm 1, \pm 1)$ is an $882A_2$.

(c) In each 4-flat direction of $C$, there are (i) four 18-cap 4-flats, four 9-cap 4-flats, and one 4-cap 4-flat; (ii) six 15-cap 4-flats, one 10-cap 4-flat, and two 6-cap 4-flats; or (iii) one 16-cap 4-flat and eight 12-cap 4-flats. The numbers of 4-flat directions in cases (i), (ii), and (iii) are respectively 
\[ \binom{56}{2} = 1540, \binom{36}{2} = 3360, \text{ and } \binom{57}{2} = 3255. \]

(d) In part (c), the 4-flat directions in case (i) are precisely the 4-flat directions $D$ that are obtained from the unordered pairs $\{D_1, D_2\}$ of different \{45,*,45\} hyperplane directions of $C$ by letting each 4-flat in $D$ be the intersection of a hyperplane in $D_1$ with a hyperplane in $D_2$. 

**Proof.** Parts (a) to (c) were checked directly using a representative 112-cap 6-flat. From parts (a) and (c), we deduce that each 4-flat direction $D$ has co-ordinates $y_1$ and $y_2$ such that the 4-flat point count of $C$ for $(y_1, y_2)$ is
\[
\begin{pmatrix}
18 & 9 & 18 \\
9 & 4 & 9 \\
18 & 9 & 18
\end{pmatrix}
, \quad
\begin{pmatrix}
15 & 6 & 15 \\
15 & 10 & 15 \\
15 & 6 & 15
\end{pmatrix}
, \quad
\begin{pmatrix}
12 & 12 & 12 \\
12 & 16 & 12 \\
12 & 12 & 12
\end{pmatrix}.
\]

Part (d) follows. \(\square\)

**Lemma 3.2** (Refining \{45,*,45\}). Suppose that the two 5-flats $\tilde{x}_1 = \pm 1$ have exactly 45 cap points each. It follows that for some co-ordinate $x_1$ of $C$, the point count of $C$ for $(\tilde{x}_1, x_1)$ is
\[
\begin{pmatrix}
18 & * & 18 \\
9 & * & 9 \\
18 & * & 18
\end{pmatrix}
\]
and the four 18-cap 4-flats $(\tilde{x}_1, x_1) = (\pm 1, \pm 1)$ are all $882A_2$ caps.

**Proof.** Each 45-cap 5-flat $\tilde{x}_1 = \pm 1$ has exactly 90 dual vectors that correspond to \{18,9,18\} hyperplane directions in which the 18-cap 4-flats are two $882A_2$ caps (two dual vectors for each of 45 such hyperplane directions). Translate (the arguments of) such dual vectors of $\tilde{x}_1 = -1$ (respectively, $\tilde{x}_1 = 1$) to obtain a set $S_{-1}$
(respectively, $S_1$) of 90 dual vectors of the 5-flat $\bar{x}_1 = 0$. Write $F$ for the space of all $3^5$ dual vectors of the 5-flat $\bar{x}_1 = 0$.

For each $a \in \{\pm 1\}$, among the 121 hyperplane directions of $(S_a, F)$, the numbers of $\{45, 45, 0\}$, $\{36, 36, 18\}$, $\{30, 30, 30\}$, and $\{27, 27, 36\}$ hyperplane directions are respectively 1, 10, 90, and 20. The $\{45, 45, 0\}$ hyperplane direction $D$ of $(S_a, F)$ corresponds via duality to the axis line direction $L$ of the 45-cap 5-flat $\bar{x}_1 = a$: two different dual vectors $f_1$ and $f_2$ in $F$ are in the same hyperplane in $D$ if and only if, in the hyperplane direction corresponding to the dual vector $f_1 - f_2$, each hyperplane is a union of lines in $L$. It was verified using a representative 45-cap 5-flat that if $y$ and $z$ are co-ordinates of $F$ corresponding to the $\{45, 45, 0\}$ hyperplane direction and any $\{27, 27, 36\}$ hyperplane direction respectively of $(S_a, F)$ such that $y$ and $z$ take the value 0 at the origin of $F$, then the following statements hold: the point count of $(S_a, F)$ for $(y, z)$ is

$$
\begin{pmatrix}
9 & 0 & 18 \\
18 & 0 & 18 \\
18 & 0 & 9
\end{pmatrix}
$$
or

$$
\begin{pmatrix}
18 & 0 & 9 \\
18 & 0 & 18 \\
9 & 0 & 18
\end{pmatrix}
$$
each 18-point 3-flat of the form $(y, z) = (b, c)$ is the complement of a union of three disjoint lines, and each 9-point 3-flat of the form $(y, z) = (b, c)$ is a 9-cap 3-flat.

Suppose that $S_{-1}$ and $S_1$ are disjoint; we derive a contradiction.

If the 45-cap 5-flats $\bar{x}_1 = \pm 1$ have parallel axis line directions, then $(S_{-1}, F)$ and $(S_1, F)$ have the same hyperplane direction $D$ as a $\{45, 45, 0\}$ hyperplane direction, so in each 4-flat of $D$ that does not contain 0 in $F$, there are 45 points in each of $S_{-1}$ and $S_1$, which is impossible since the number of points in each 4-flat is $3^4 = 81 < 90 = 2(45)$. So the axis line directions of the 45-cap 5-flats $\bar{x}_1 = \pm 1$ are not parallel.

Let the co-ordinates $y_{-1}$ and $y_1$ of $F$ correspond via duality to the axis line directions of $\bar{x}_1 = -1$ and $\bar{x}_1 = 1$ respectively, and suppose that $y_{-1}$ and $y_1$ take the value 0 at the origin of $F$. For each $a \in \{\pm 1\}$, in each hyperplane $y_a = \pm 1$: there are 45 points in $S_a$ and at least 27 points in $S_{-a}$, so there are at most 9 points in neither $S_a$ nor $S_{-a}$ (because $S_{-1}$ and $S_1$ are disjoint). Therefore, among the $3^5 - 2(90) = 63$ points in $F - S_{-1} - S_1$, at least $63 - 4(9) = 27$ points are in the 3-flat $(y_{-1}, y_1) = (0, 0)$, which has exactly $3^3 = 27$ points (as every 3-flat does); it follows that all the inequalities in this paragraph are equalities.

Therefore, the $y_{-1}$-hyperplane direction of $(S_1, F)$ and the $y_1$-hyperplane direction of $(S_{-1}, F)$ have point count $\{27, 27, 36\}$, so the point counts of $(S_{-1}, F)$ and $(S_1, F)$ for $(y_{-1}, y_1)$ are

$$
\begin{pmatrix}
a & 0 & 27 - a \\
18 & 0 & 18 \\
27 - a & 0 & a
\end{pmatrix}
$$
and

$$
\begin{pmatrix}
b & 18 & 27 - b \\
0 & 0 & 0 \\
27 - b & 18 & b
\end{pmatrix}
$$
respectively, for some $a \in \{9, 18\}$ and some $b \in \{9, 18\}$. Since $S_{-1}$ and $S_1$ are disjoint and each 3-flat has 27 points in total, we have $a + b \leq 27$ and $(27 - a) + (27 - b) \leq 27$, so $27 \leq a + b \leq 27$, so $a + b = 27$, so $\{a, b\} = \{9, 18\}$.

Now the 3-flat $(y_{-1}, y_1) = (-1, 1)$ in $(S_{-1}, F)$ and the 3-flat $(y_{-1}, y_1) = (-1, 1)$ in $(S_1, F)$ are, in some order, a 9-cap 3-flat and the complement of a union of three disjoint lines, so their 9- and 18-point sets are not disjoint, so $S_{-1}$ and $S_1$ are not disjoint. We obtain a contradiction. \qed
Lemma 3.3. (a) Suppose that the 4-flat point count of $C$ for some independent pair $(x_1, x_2)$ of co-ordinates is

$$
\begin{pmatrix}
18 & 7 & 18 \\
7 & a & * \\
18 & * & *
\end{pmatrix},
$$

and that each of the 18-cap 4-flats $(x_1, x_2) = (-1, -1)$, $(x_1, x_2) = (-1, 1)$, and $(x_1, x_2) = (1, 1)$ is an 882$A_2$. It follows that $a \leq 4$. Moreover, if $a \geq 1$, then the relative positions of those three 18-cap 4-flats are determined up to isomorphism.

(b) Suppose that the 4-flat point count of $C$ for some independent pair $(x_1, x_2)$ of co-ordinates is

$$
\begin{pmatrix}
18 & 7 & 18 \\
7 & a & * \\
17 & * & *
\end{pmatrix},
$$

and that each of the 18-cap 4-flats $(x_1, x_2) = (\pm 1, 1)$ is an 882$A_2$. It follows that $a \leq 4$.

Proof. (a) The 43-cap 5-flat $x_1 = -1$ is a 45-cap 5-flat minus two cap points that are in the 4-flat $(x_1, x_2) = (-1, 0)$, and the 43-cap 5-flat $x_2 = 1$ is a 45-cap 5-flat minus two cap points that are in the 4-flat $(x_1, x_2) = (0, 1)$. From Lemma 2.4 it follows that the 18-cap 4-flats $(x_1, x_2) = \pm (1, 1)$ are translations or point reflections of each other. The former case yields $a = 0$, and the latter case yields $a \leq 4$ (in each case, avoid a line of three cap points).

(b) There is a unique noncap point $P$ in the 4-flat $(x_1, x_2) = (-1, -1)$ such that $P$ and the cap points in the 5-flat $x_1 = -1$ together form a cap in the 5-flat $x_1 = -1$. In that new cap, the 18-cap 4-flat $(x_1, x_2) = (-1, -1)$ is an 882$A_2$ by Thackray [6]. Lemma 3.1]. Now argue as in part (a). \qed

Proposition 3.4 (Refining $\{43, \ast, 43\}$). Choose an independent pair $(x_1, x_2)$ of co-ordinates for $C$. Suppose that the point count of $C$ for $(x_1, x_2)$ is among the following, where $a$, $b$, and $c$ are nonnegative integers, and each of the four 4-flats $(x_1, x_2) = (\pm 1, \pm 1)$ can be obtained from an 882$A_2$ by removing at most two cap points.

\begin{align*}
(a) \quad & \begin{pmatrix} 18 & a & 18 \\ 7 & b & 7 \\ 18 & c & 18 \end{pmatrix} \\
(b) \quad & \begin{pmatrix} 18 & a & 18 \\ 7 & b & 8 \\ 18 & c & 17 \end{pmatrix} \\
(c) \quad & \begin{pmatrix} 18 & a & 18 \\ 8 & b & 8 \\ 17 & c & 17 \end{pmatrix} \\
(d) \quad & \begin{pmatrix} 18 & a & 18 \\ 8 & b & 9 \\ 17 & c & 16 \end{pmatrix} \\
(e) \quad & \begin{pmatrix} 18 & a & 18 \\ 7 & b & 9 \\ 18 & c & 17 \end{pmatrix} \\
(f) \quad & \begin{pmatrix} 18 & a & 17 \\ 7 & b & 9 \\ 18 & c & 17 \end{pmatrix} \\
(g) \quad & \begin{pmatrix} 18 & a & 18 \\ 8 & b & 9 \\ 17 & c & 16 \end{pmatrix} \\
(h) \quad & \begin{pmatrix} 18 & a & 17 \\ 8 & b & 9 \\ 17 & c & 17 \end{pmatrix} \\
(i) \quad & \begin{pmatrix} 18 & a & 16 \\ 8 & b & 9 \\ 17 & c & 18 \end{pmatrix}
\end{align*}

In each case, $a + b + c \leq 22$ holds. The following restrictions hold for specific matrices.

(a) If $a + b + c \geq 19$, then $b \leq 4$.

(b) If $a + b + c \geq 20$, then $b \leq 4$.

(c) If $a + b + c = 22$, then $(a, b, c) = (6, 7, 9)$ or $(a, b, c) = (9, 4, 9)$.

(d) If $a + b + c \geq 21$, then $b \leq 4$. 

(e) If \(a + b + c = 22\), then \((a, b, c) = (4, 9, 9)\) or \((a, b, c) = (9, 4, 9)\).

(f) If \(a + b + c = 22\), then \((a, b, c) = (9, 4, 9)\).

(g) If \(a + b + c = 22\), then \((a, b, c) is one of \((4, 9, 9), (6, 7, 9), and (9, 4, 9)\).

(h) If \(a + b + c = 22\), then \((a, b, c) is one of \((9, 4, 4), (9, 6, 7), and (9, 4, 9)\).

Proof. For all cases, each of \(a, b, \) and \(c \) is at most 9 (avoid a 5-dimensional cap of size at least 43 with an impossible hyperplane point count).

We consider matrix (c); argue similarly for the other matrices. Suppose \(b \geq 8\) and \(c \geq 9\). By Lemma 3.3, the cap points in the 5-flats \(x_1 = -1, x_2 = -1\), and \(x_1 = -x_2\), together with some point in \((x_1, x_2) = (-1, -1)\) and some point in \((x_1, x_2) = (1, -1)\), form a cap with point count
\[
\begin{pmatrix}
18 & 0 & 0 \\
8 & b & 0 \\
18 & 9 & 18
\end{pmatrix},
\]
which contradicts Lemma 3.3.

Therefore, we have the following. If \(b \geq 8\), then \(c \leq 8\) and (by a similar argument) \(a \leq 4\), so \(a + b + c \leq 4 + 9 + 8 = 21\). If \(5 \leq b \leq 7\), then \(a \leq 6\) by Lemma 3.3, so \(a + b + c \leq 6 + 7 + 9 = 22\). If \(b \leq 4\), then \(a + b + c \leq 9 + 4 + 9 = 22\). \(\square\)

Proposition 3.5 (Refining 5-flat directions). We have the following.

(a) In \(C\), suppose that the two 5-flats \(\vec{x}_1 = \pm 1\) have exactly 45 cap points each. It follows that if the two 45-cap 5-flats \(\vec{x}_1 = \pm 1\) are not point reflections of each other, then the number of cap points in the 5-flat \(\vec{x}_1 = 0\) is at most 6. That figure of 6 cannot be lowered.

(b) In \(C\), suppose that the two 5-flats \(\vec{x}_1 = \pm 1\) combined have at least 88 cap points in total. It follows that if the two 45-cap 5-flats obtained by completing the 5-flats \(\vec{x}_1 = \pm 1\) are not point reflections of each other, then the number of cap points in the 5-flat \(\vec{x}_3 = 0\) is at most 14.

Proof. (a) In \(C\), a co-ordinate \(x_1\) can be chosen as in Lemma 3.2. Therefore, without loss of generality, there are co-ordinates \(x_2\) to \(x_5\) of \(C\), with \((\vec{x}_1, x_1, \ldots, x_5)\) independent, such that (i) the 5-flat \(\vec{x}_1 = -1\) is Figure 3 of Thackeray [6] in \((x_1, \ldots, x_5)\) co-ordinates and (ii) the 5-flat \(\vec{x}_1 = 1\) is the image under an invertible linear map \(T\) of the 5-flat \(\vec{x}_1 = -1\) in \((x_1, \ldots, x_5)\) co-ordinates such that \(T\) sends each 4-flat \((\vec{x}_1, x_1) = (-1, a)\) to the 4-flat \((\vec{x}_1, x_1) = (1, a)\).

A computer search examined each \(T\) in turn. For each \(T\), the program found

(i) the number \(n_0\) of points \(P\) in the 5-flat \(\vec{x}_1 = 0\) such that \(P\) is not a midpoint of a line segment with one endpoint in the 5-flat \(\vec{x}_1 = -1\) and the other endpoint in the 5-flat \(\vec{x}_1 = 1\), and

(ii) the number \(n_2\) of points \(Q\) in the 5-flat \(\vec{x}_1 = 0\) such that \(Q\) is a midpoint of at most two such line segments.

For each \(T\), the program checked whether \(n_0 \geq 6\) holds and \(n_2 \geq 14\) holds; if at least one of those was found to hold, then the program displayed the data of \(T\) as well as \(n_0\) and \(n_2\). It was thus determined that for each \(T\), exactly one of the following is true: (i) \((n_0, n_2) = (0, 45)\) (there are 8 maps \(T\) in this case), (ii) \((n_0, n_2) = (22, 22)\) (there are 8 maps \(T\) in this case), (iii) \((n_0, n_2) = (6, 6)\) (there are 32 maps \(T\) in this case), (iv) \((n_0, n_2) = (2, 14)\) (there are 176 maps \(T\) in this case), or (v) both \(n_0 \leq 5\) and \(n_2 \leq 13\) hold.
For each of the 8 maps $T$ in case (i), the 45-cap 5-flat $\tilde{x}_1 = 1$ is the image of the 45-cap 5-flat $\tilde{x}_1 = -1$ under some translation map $U$ from the 6-flat of $C$ to itself. Since each of the 45-cap 5-flats $\tilde{x}_1 = \pm 1$ is complete (that is, it cannot be enlarged to form a new cap of the same dimension and greater size by adding cap points), it follows that the 45 points $Q$ form the image of the 45-cap 5-flat $\tilde{x}_1 = 1$ under $U$, each of the 45 points $Q$ is the midpoint of exactly one cap line segment with one endpoint in each of the two 5-flats $\tilde{x}_1 = \pm 1$ (namely, the line segment $U(Q)U^2(Q)$), and those 45 cap line segments are pairwise disjoint.

For each of the 8 maps $T$ in case (ii), the two 45-cap 5-flats $\tilde{x}_1 = \pm 1$ are point reflections of each other, the 22 points $P$ form a cap, and if we combine that cap with the 45-cap 5-flats $\tilde{x}_1 = \pm 1$, then we obtain a 112-cap 6-flat.

An example in case (iii) is in Figure 2. For each of the 32 maps $T$ in case (iii), the six points $P = Q$ form a cap and we obtain a complete 96-cap 6-flat. Using the computer search results, it was verified that those 96-cap 6-flats are all pairwise isomorphic via isomorphisms that preserve $\tilde{x}_1$ and preserve or negate $x_1$; therefore, given any $\{45, 45, 6\}$ hyperplane direction $D$ of any complete 96-cap 6-flat, there is an isomorphism from that cap to Figure 2 that sends the hyperplanes in $D$ to the 5-flats $\tilde{x}_1 = -1, \tilde{x}_1 = 0$, and $\tilde{x}_1 = 1$ respectively.

![Figure 2. A 96-cap 6-flat.](image_url)

Part (a) of the Proposition follows, using the information about points $P$ above.

(b) This follows from Theorem 2.1 and the information about points $Q$ in the proof of part (a).

**Proposition 3.6** (Refining 5-flat directions). We have the following.

(a) In $C$, suppose that the 5-flats $\tilde{x}_1 = -1$ and $\tilde{x}_1 = 1$ have exactly 45 and 40 cap points respectively, the 40-cap 5-flat $\tilde{x}_1 = 1$ is the 40-cap 5-flat in a $\{40, 36, 36\}$ hyperplane direction of some 112-cap 6-flat, and that 40-cap 5-flat has an $\{18, 18, 4\}$ hyperplane direction in which the 18-cap 4-flats are 882 $A_2$ caps. It follows that the number of cap points in the 5-flat $\tilde{x}_1 = 0$ is at most 3. That figure of 3 cannot be lowered.

□
(b) In $C$, suppose that the union of the two 5-flats $\vec{x}_1 = \pm 1$ can be obtained from the union of those two 5-flats in part (a) by removing at most two cap points. It follows that the number of cap points in the 5-flat $\vec{x}_1 = 0$ is at most 13.

**Proof.** It was verified that if $C$ is the 40-cap 5-flat $\vec{x}_1 = 1$ as specified in part (a), then the number of $\{18, 18, 4\}$ hyperplane directions of $C$ is 10 and the symmetries of $C$ act transitively on the set of $\{18, 18, 4\}$ hyperplane directions of $C$.

Computer searches were performed as in the proof of Proposition 3.5, where the point count of $C$ for $(\vec{x}_1, x_1)$ is

$$\begin{pmatrix} 18 & * & 18 \\ 9 & * & 4 \\ 18 & * & 18 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 15 & * & 18 \\ 15 & * & 4 \\ 15 & * & 18 \end{pmatrix}.$$

A representative $\{18, 4, 18\}$ direction of the 40-cap 5-flat $\vec{x}_1 = 1$ was used as the $x_1$-hyperplane direction of that 5-dimensional subcap, and each of Lemma 2.2's four options (i) to (iv) for the $x_1$-hyperplane direction of the 45-cap 5-flat $\vec{x}_1 = -1$ was considered in turn by the computer search.

For each $T$, the number of points $P$ was at most 3, and the number of points $Q$ was at most 13. The search found examples such that there are exactly three points $P$ and those points are not collinear. $\square$

**Proposition 3.7** (Refining $\{45,*,42\}$, part 1). In $C$, suppose that the numbers of cap points in the two 5-flats $\vec{x}_1 = -1$ and $\vec{x}_1 = 1$ are 45 and 42 respectively, and that the 42-cap 5-flat $\vec{x}_1 = 1$ is a $\Delta686$. It follows that the number of cap points in the 5-flat $\vec{x}_1 = 0$ is at most 18.

**Proof.** The 42-cap 5-flat $\vec{x}_1 = 1$ has a $\{20, 16, 6\}$ hyperplane direction, so there is a co-ordinate $x_1$ of $C$ with $(\vec{x}_1, x_1)$ independent such that the point count of $C$ for $(\vec{x}_1, x_1)$ is

$$\begin{pmatrix} 18 & a & 20 \\ 9 & b & 6 \\ 18 & c & 16 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 15 & a & 20 \\ 15 & b & 6 \\ 15 & c & 16 \end{pmatrix}.$$

The computer search of Thackeray [6, Table 1], the classification of 44-cap 5-flats in Thackeray [6], and the classification of 42-cap 5-flats above imply that

(i) The first matrix yields $a \leq 2$, $b \leq 2$, and $c \leq 9$, so $a + b + c \leq 13$; and
(ii) The second matrix yields $a \leq 6$, $b \leq 6$, and $c \leq 6$, so $a + b + c \leq 18$. $\square$

**Proposition 3.8** (Refining $\{45,*,42\}$, part 2). In $C$, suppose that the numbers of cap points in the 5-flats $\vec{x}_1 = -1$, $\vec{x}_1 = 1$, and $\vec{x}_1 = 0$ are respectively 45, 42, and at least 20. Suppose that the 42-cap 5-flat $\vec{x}_1 = 1$ is a 45-cap 5-flat minus three cap points. It follows that (i) the 5-flat $\vec{x}_1 = 0$ has at most 22 cap points, (ii) the 42-cap 5-flat $\vec{x}_1 = 1$ is a point reflection of the 45-cap 5-flat $\vec{x}_1 = -1$ minus three cap points, and (iii) $C$ is a subcap of a 112-cap 6-flat.

**Proof.** By Lemma 3.2, there is a co-ordinate $x_1$ of $C$ with $(\vec{x}_1, x_1)$ independent such that the union of the two 5-dimensional caps $\vec{x}_1 = \pm 1$ in $C$ is obtained by removing at most three cap points from a 90-cap 6-flat $C$, with the same underlying 6-flat as
C, such that the point count of $\tilde{C}$ for $(\tilde{x}_1, x_1)$ is
\[
\begin{pmatrix}
18 & 0 & 18 \\
9 & 0 & 9 \\
18 & 0 & 18
\end{pmatrix}
\]
and the four 18-cap 4-flats $(\tilde{x}_1, x_1) = (\pm 1, \pm 1)$ in $\tilde{C}$ are all 882$A_2$ caps. From the (18,9,18) columns of that point count matrix, it follows that for each $a \in \{\pm 1\}$, the 882 cube direction of each of the two 882$A_2$ caps $(\tilde{x}_1, x_1) = (a, \pm 1)$ in $\tilde{C}$ is parallel to the 855 cube direction of the other 882$A_2$.

It follows that without loss of generality (possibly negating $x_1$), the point count of $C$ for $(\tilde{x}_1, x_1)$ is among
\[
\begin{pmatrix}
18 & a & 18 \\
9 & b & 6 \\
18 & c & 18
\end{pmatrix}, \quad
\begin{pmatrix}
18 & a & 18 \\
9 & b & 7 \\
18 & c & 17
\end{pmatrix}, \quad
\begin{pmatrix}
18 & a & 18 \\
9 & b & 8 \\
18 & c & 16
\end{pmatrix},
\]
In each case, we have $a \leq 9$ and $c \leq 9$ (no 5-dimensional cap of size at least 43 has a hyperplane direction with point count \{18, *, 10\}).

In $C$, if each of the 5-flats $x_1 = 1$, $\tilde{x}_1 - x_1 = 0$, $\tilde{x}_1 + x_1 = 0$, and $x_1 = -1$ has at most 41 cap points, then each of the six point count matrices yields $a + b + c \leq 19$ and a contradiction. (For example, the last of the six matrices yields $a \leq 41 - (18 + 17) = 6$, $b \leq 41 - (18 + 17) = 6$, and $c \leq 41 - (18 + 16) = 7$, so $a + b + c \leq 6 + 6 + 7 = 19$.) Therefore, at least one of the four caps in the 5-flats $x_1 = 1$, $\tilde{x}_1 - x_1 = 0$, $\tilde{x}_1 + x_1 = 0$, and $x_1 = -1$ of $C$ has at least 42 cap points.

Let $C_5$ be such a 5-dimensional cap. From the point count matrices above, $C_5$ has an \{18, 18, *, \} \{18, 17, *, \} \{18, 16, *, \} or \{18, 15, *, \} hyperplane direction in which the 18-, 17-, 16-, and 15-cap 4-flats are 882$A_2$ caps minus at most three cap points each. It follows that $C_5$ is a 45-cap 5-flat minus at most three cap points: in each \{18, 18, 6\} or \{18, 17, 7\} hyperplane direction of a $\triangle 686$, each 18-cap 4-flat is a $963B$ or a $981C$ respectively, not an 882$A_2$, and a $\triangle 686$ has neither \{18, 16, 8\} nor \{18, 15, 9\} hyperplane directions.

By Lemma 2.3, there is a unique way to add cap points to the 4-flats $(\tilde{x}_1, x_1) = (1, \pm 1)$ of $C$ to obtain 882$A_2$ caps, so those 882$A_2$ caps must be the caps in the same positions in $\tilde{C}$. Therefore, by Lemma 2.4, the nine-2s 2-flat directions of the 882$A_2$ caps $(\tilde{x}_1, x_1) = (\pm 1, \pm 1)$ in $\tilde{C}$ are parallel to one another, and the 882 and 855 cube directions of each of those four 882$A_2$ caps are parallel to those directions, in some order, of each other such 882$A_2$ cap.

Suppose that the underlying 5-flat of $C_5$ is $x_1 = 1$ or $x_1 = -1$. We have $b \leq 9$, and Lemma 2.4 implies the following: the two 882$A_2$ caps $(\tilde{x}_1, x_1) = (\pm 1, -1)$ of $\tilde{C}$ are translations or point reflections of each other, and the two 882$A_2$ caps $(\tilde{x}_1, x_1) = (\pm 1, 1)$ of $\tilde{C}$ are translations or point reflections of each other. Lemma 2.5 yields $a \leq 4$ and $c \leq 5$, so $a + b + c \leq 4 + 9 + 5 = 18$, so we have a contradiction.

Therefore, the underlying 5-flat of $C_5$ is $x_1 = 1$ or $x_1 = -1$. Lemma 2.4 implies that the 882$A_2$ caps $(\tilde{x}_1, x_1) = (\pm 1, -1)$ in $\tilde{C}$ are translations or point reflections of each other, and the 882$A_2$ caps $(\tilde{x}_1, x_1) = (\pm 1, 1)$ in $\tilde{C}$ are translations or point
reflections of each other. We have \( b \leq 4 \) by Lemma 2.5 so \( a + b + c \leq 9 + 4 + 9 = 22 \), which yields (i).

We have \( a + c \geq 20 - 4 = 16 \), so \( (a, c) \) is among \((7, 9), (8, 8), (9, 7), (8, 9), (9, 8), \) and \((9, 9)\). If \( a + c = 16 \), then \( b \geq 20 - 16 = 4 \); if \( (a, c) = (8, 9) \), then \( b \geq 20 - 17 = 3 \); and if \( a = c = 9 \), then \( b \geq 20 - 18 = 2 \).

By Lemma 2.5 the 882A2 caps \( (\bar{x}_1, x_1) \) are point reflections of each other (in each of the six point count matrices, the bottom-left-to-top-right diagonal is \((18, b, 18)\) or \((18, b, 17)\), and \( b \geq 2 \)). Therefore, the square of midpoints of \((\bar{x}_1, x_1) = (1, 1)\) in \( \bar{C} \) is the image of the square of midpoints of \((\bar{x}_1, x_1) = (-1, -1)\) in \( C \) under some translation map \( U \), and all cap points in the 4-flat \((\bar{x}_1, x_1) = (0, 0)\) in \( C \) are in the image \( M \) under \( U \) of the square of midpoints of the 882A2 cap \((\bar{x}_1, x_1) = (1, 1)\) in \( \bar{C} \). The 2-flat \( F \) that contains \( M \) is a translation of a 2-flat in the nine-2s 2-flat directions of the 882A2 caps \((\bar{x}_1, x_1) = (\pm 1, \pm 1)\) in \( \bar{C} \).

Suppose \( b \geq 3 \). The 882A2 caps \((\bar{x}_1, x_1) = \pm(1, -1)\) in \( \bar{C} \) must be point reflections of each other (if they were translations of each other, then at most two of the cap points of \( C \) in \((\bar{x}_1, x_1) = (0, 0)\) could be in \( F \)). Without loss of generality, let the 5-flat \( \bar{x}_1 = -1 \) of \( C \) be as in Figure 11. The 18-cap 4-flat \((x_1, x_2) = (1, 1)\) of \( \bar{C} \) is as in that figure without loss of generality, and the 18-cap 4-flat \((x_1, x_2) = (1, \pm 1)\) of \( \bar{C} \) is a translation of the 18-cap 4-flat \((x_1, x_2) = (1, 1)\) in the figure by some vector \( v \). The square \( M \) consists of the cap points in the 4-flat \((x_1, x_2) = (0, 0)\) in the figure. All cap points in the 4-flat \((x_1, x_2) = (0, 0)\) of \( C \) are among those of \( M \), and among those in the image of \( M \) under translation by \( -v \). If two squares that are translations of each other intersect in at least three points, then they coincide, so \( v = 0 \). That implies parts (ii) and (iii).

We may therefore assume \( b = 2 \); it follows that \( a = c = 9 \).

If the 882A2 caps \((\bar{x}_1, x_1) = \pm(1, -1)\) in \( \bar{C} \) are translations of each other, then the 9-cap 4-flats of \( \bar{C} \) in \((\bar{x}_1, x_1) = (-1, 0)\) and \((\bar{x}_1, x_1) = (0, -1)\) determine the positions of the standard squares of the 882A2 cap of \( C \) in \((\bar{x}_1, x_1) = (1, 1)\), and in all cases we obtain at least one line of three cap points and a contradiction.

Therefore, the 882A2 caps \((\bar{x}_1, x_1) = \pm(1, -1)\) in \( \bar{C} \) are point reflections of each other, and arguing as before using the 4-flats \((\bar{x}_1, x_1) = (-1, 0)\) and \((\bar{x}_1, x_1) = (0, -1)\), we obtain parts (ii) and (iii).

\[\text{Theorem 3.9.} \quad \text{Every 110-cap 6-flat is a 112-cap 6-flat minus two cap points.}\]

\[\text{Proof.} \quad \text{Let } C \text{ be a 110-cap 6-flat. From the standard diagram in Figure 11 every 110-cap 6-flat has at least one } \{45, 45, 20\}, \{45, 44, 21\}, \{45, 43, 22\}, \{44, 44, 22\}, \{45, 42, 23\}, \{44, 43, 23\}, \text{ or } \{43, 43, 24\} \text{ hyperplane direction. Choose } \bar{x}_1 \text{ to be a co-ordinate of such a hyperplane direction } D \text{ so that the numbers of cap points in the 5-flats } \bar{x}_1 = -1, \bar{x}_1 = 1, \text{ and } \bar{x}_1 = 0 \text{ are in nonincreasing order.} \]

If \( D \) has point count \( \{45, 45, 20\}, \{45, 44, 21\}, \{45, 43, 22\}, \text{ or } \{44, 44, 22\}, \text{ then by Proposition 5.5 the 5-flats } \bar{x}_1 = \pm 1 \text{ are subcaps of 45-cap 5-flats that are point reflections of each other, and we can argue as before to show that (i) for some co-ordinate } x_1 \text{ of } C, \text{ the point count of } C \text{ for } (\bar{x}_1, x_1) \text{ can be obtained from} \]

\[
\begin{pmatrix}
18 & 9 & 18 \\
9 & 4 & 9 \\
18 & 9 & 18
\end{pmatrix}
\]
by subtracting 2 in total from the nine entries combined, and (ii) $C$ is a 112-cap 6-flat minus two cap points.

If $D$ has point count \{45, 42, 23\}, then Propositions 3.7 and 3.8 yield a contradiction.

If $D$ has point count \{44, 43, 23\}, then by Lemma 3.2, there is a co-ordinate $x_1$ of $C$ such that the point count of $C$ for $(\bar{x}_1, x_1)$ (possibly after negating $\bar{x}_1$) can be obtained from some point count matrix in Proposition 3.4 by adding 1 to some entry in the left or right column, so by that theorem, the 5-flat $\bar{x}_1 = 0$ of $C$ has at most 22 cap points, and we obtain a contradiction.

Suppose that $D$ has point count \{43, 43, 24\}. Arguing as before, for some co-ordinate $x_1$ of $C$, the point count of $C$ for $(\bar{x}_1, x_1)$ (possibly after negating $\bar{x}_1$) is, without loss of generality, either among the point count matrices in Proposition 3.4 – in which case, by that theorem, the 5-flat $\bar{x}_1 = 0$ of $C$ has at most 22 cap points, and we have a contradiction – or among the following four point count matrices where each of the 4-dimensional caps $(\bar{x}_1, x_1) = (\pm 1, \pm 1)$ can be completed to an 882A2:

\[
\begin{pmatrix}
17 & a & 17 \\
9 & b & 9 \\
17 & c & 17
\end{pmatrix},
\begin{pmatrix}
17 & a & 18 \\
9 & b & 9 \\
17 & c & 16
\end{pmatrix},
\begin{pmatrix}
16 & a & 18 \\
9 & b & 9 \\
18 & c & 16
\end{pmatrix},
\begin{pmatrix}
18 & a & 18 \\
9 & b & 9 \\
16 & c & 16
\end{pmatrix}.
\]

In each case, we have $b \leq 9$, and if $b \geq 6$ then the completed 882A2 caps in $(\bar{x}_1, x_1) = (\pm 1, 1)$ are translations or point reflections of each other, and similarly for $(\bar{x}_1, x_1) = (\pm 1, -1)$, so $a \leq 5$ and $c \leq 5$, so $a + b + c \leq 5 + 9 + 5 = 19$, which gives a contradiction. Therefore, in each case, we have $b \leq 5$, $a \leq 9$, and $c \leq 9$ (for the last of the four point count matrices, note that no 42-cap 5-flat has a \{16, 16, 10\} hyperplane direction), so $a + b + c \leq 23$, which gives a contradiction. \(\square\)

4. Dimension 7

**Proposition 4.1** (Refining 6-flat directions). Let the 7-dimensional cap $C$ have co-ordinate $x_1$.

(a) Suppose that the two 6-flats $x_1 = \pm 1$ have exactly 112 cap points each. It follows that the 6-flat $x_1 = 0$ has at most 34 cap points.

(b) Suppose that the two 6-flats $x_1 = \pm 1$ combined have at least 222 points in total. It follows that the 6-flat $x_1 = 0$ has at most 55 cap points.

**Proof.** (a) For some co-ordinate $x_2$ of $C$ with $(x_1, x_2)$ independent, the point count of $C$ for $(x_1, x_2)$ is one of the following two options.
Option 1: \[
\begin{pmatrix}
45 & a & 45 \\
22 & b & 22 \\
45 & c & 45
\end{pmatrix}
\]

The 45-cap 5-flats \((x_1, x_2) = (-1, \pm 1)\) are point reflections of each other, and the 45-cap 5-flats \((x_1, x_2) = (1, \pm 1)\) are point reflections of each other.

If neither of \((x_1, x_2) = (1, \pm 1)\) is a point reflection of \((x_1, x_2) = (-1, -1)\), then each of \(a, b,\) and \(c\) is at most 6, so \(a + b + c \leq 3(6) = 18\).

If \((x_1, x_2) = (1, -1)\) or \((x_1, x_2) = (1, 1)\) is a point reflection of \((x_1, x_2) = (-1, -1)\), then up to isomorphism, the relative positions of the 6-flats \(x_1 = \pm 1\) are among \(2 \times 3^5\) options. (Fix the 6-flat \(x_1 = -1\); the 6-flat \(x_1 = 1\) is obtained by starting with a point reflection or translation of \(x_1 = -1\) and then, for some 5-dimensional vector \(v\), adding \(-v\), 0, and \(v\) respectively to the 5-flats \((x_1, x_2) = (1, -1), (x_1, x_2) = (1, 0),\) and \((x_1, x_2) = (1, 1)\) respectively.) A computer search verified that for each of those options,

(i) every point in the 6-flat \(x_1 = 0\) is the midpoint of a cap line segment with one endpoint in each of the 6-flats \(x_1 = \pm 1\), so there are no cap points in the 6-flat \(x_1 = 0\);
(ii) if the 112-cap 6-flats \(x_1 = \pm 1\) are not translations of each other, then there are at most two points \(Q\) in the 6-flat \(x_1 = 0\) such that \(Q\) is the midpoint of at most two cap line segments with one endpoint in each of the 6-flats \(x_1 = \pm 1\); and
(iii) if the 112-cap 6-flat \(x_1 = 1\) is the image of \(x_1 = -1\) under a translation map \(U\), then there are exactly 112 points \(Q\) in \(x_1 = 0\) such that \(Q\) is the midpoint of at most two cap line segments with one endpoint in each of \(x_1 = \pm 1\) (it follows that those 112 points \(Q\) are precisely the images of the cap points of \(x_1 = 1\) under \(U\), each of the 112 points \(Q\) is the midpoint of exactly one such cap line segment, namely \(U(Q)U^{-1}(Q)\), and those 112 cap line segments \(U(Q)U^{-1}(Q)\) are pairwise disjoint).

Option 2: \[
\begin{pmatrix}
45 & a & 36 \\
22 & b & 40 \\
45 & c & 36
\end{pmatrix}
\]

Without loss of generality, suppose that \(C\) is not included in Option 1. There are two cases.

- **Case 1**: It is impossible for the 40 in the point count of Option 2 to correspond to a 40-cap 5-flat with an \(\{18, 18, 4\}\) hyperplane direction in which the 18-cap 4-flats are \(882A_2\) caps: In this case, the relative positions of the 112-cap 6-flats \(x_1 = \pm 1\) are strongly restricted up to isomorphism (the 56 hyperplane directions of \(x_1 = -1\) with point count \(\{45, 22, 45\}\) are translations of the 56 hyperplane directions of \(x_1 = 1\) with point count \(\{40, 36, 36\}\) in which the 40-cap 5-flat has no \(\{18, 18, 4\}\) hyperplane direction in which the 18-cap 4-flats are \(882A_2\) caps, and that statement also holds with \(x_1 = -1\) and \(x_1 = 1\) swapped); a computer search verified that in this case, the 6-flat \(x_1 = 0\) has at most one cap point and at most one point \(Q\) such that \(Q\) is the midpoint of at most two cap line segments with one endpoint in each of \(x_1 = \pm 1\).
- **Case 2**: We may take the 40 in the point count of Option 2 to correspond to a 40-cap 5-flat with an \(\{18, 18, 4\}\) hyperplane direction in which the 18-cap 4-flats are \(882A_2\) caps: Taking that scenario, we have \(a \leq 3, c \leq 3,\)
and \( b \leq 109 - (45 + 36) = 28 \) (because no 110-cap 6-flat has a \{45, 36, 29\} hyperplane direction), so \( a + b + c \leq 3 + 28 + 3 = 34 \).

(b) The 6-flats \( x_1 = \pm 1 \) can be obtained from those in part (a) by removing at most two cap points in total.

Suppose that the 6-flats \( x_1 = \pm 1 \) together are obtained from those in Option 1 in part (a) by removing at most two cap points, and the numbers of cap points in the 5-flats \((x_1, x_2) = (0, 1), (x_1, x_2) = (0, 0), \) and \((x_1, x_2) = (0, -1)\) are respectively \( a, b, \) and \( c \).

If neither of the completions of \( (x_1, x_2) = (1, \pm 1) \) to 45-cap 5-flats is a point reflection of the completion of \( (x_1, x_2) = (-1, -1) \) to a 45-cap 5-flat, then each of \( a, b, \) and \( c \) is at most 14, so \( a + b + c \leq 3(14) = 42 \).

If the completion of \( (x_1, x_2) = (1, -1) \) or \( (x_1, x_2) = (1, 1) \) to a 45-cap 5-flat is a point reflection of the completion of \( (x_1, x_2) = (-1, -1) \) to a 45-cap 5-flat, then it follows from the information about points \( Q \) in part (a) that \( a + b + c \leq 2 \).

Suppose that the 6-flats \( x_1 = \pm 1 \) together are obtained from those in Option 2 in part (a) by removing at most two cap points, and define \( a, b, \) and \( c \) as before.

We have \( a \leq 13, c \leq 13, \) and \( b \leq 109 - (45 + 36 - 1) = 29 \) (in at least one of the two \( (45, b, 36) \) diagonals, the number of cap points removed is at most \( 2/2 = 1 \); no 110-cap 6-flat has a \{45, 36, 29\}, \{45, 35, 30\}, or \{44, 36, 30\} hyperplane direction), so \( a + b + c \leq 13 + 29 + 13 = 55 \).

\[ \square \]

**Theorem 4.2.** There are no 289-cap 7-flats.

*Proof.* By the standard diagram in Figure 4, each 289-cap 7-flat has at least one hyperplane direction with point count among \{112, 112, 65\}, \{112, 111, 66\}, \{112, 110, 67\}, and \{111, 111, 67\}. Each of those options is ruled out by Proposition 4.1 together with Theorem 3.9. \( \square \)

**Acknowledgements**

Many thanks to my postdoctoral supervisor Prof. James Raftery, to Prof. Roumen Anguelov, to Prof. Jan Harm van der Walt, to Prof. Mapundi Banda, to Prof. Anton Ströh, and to everyone else at the University of Pretoria, for all their generous support at this extraordinary time.
This work was supported by the UP Post-Doctoral Fellowship Programme administered by the University of Pretoria [grant number A0X 816].

Computer searches were carried out using Java programs on the Eclipse IDE software.

Many thanks to my mother Dr Anne Thackeray for letting me use her computer together with my own to perform some computer searches, and to all of my family for everything.

REFERENCES

[1] B. L. Davis and D. Maclagan. The card game Set. Math. Intell., 25(3):33–40, 2003. <https://doi.org/10.1007/BF02984846>.

[2] Y. Edel, S. Ferret, I. Landjev, and L. Storme. The classification of the largest caps in AG(5,3). J. Comb. Theory Ser. A, 99(1):95–110, 2002. <https://doi.org/10.1006/jcta.2002.3261>.

[3] J. S. Ellenberg and D. Gijswijt, On large subsets of \(F_q^n\) with no three-term arithmetic progression, Ann. Math. 2nd Ser., 185(1):339–343, 2017. <https://doi.org/10.4007/annals.2017.185.1.8>.

[4] A. Potechin, Maximal caps in AG(6,3), Des. Codes Cryptogr., 46(3):243–259, 2008. <https://doi.org/10.1007/s10623-007-9132-z>.

[5] H. (M.) R. Thackeray, The cap set problem and standard diagrams, Discrete Math., 344(11):2021. Article 112558, <https://doi.org/10.1016/j.disc.2021.112558>.

[6] H. (M.) R. Thackeray, The cap set problem: 41-cap 5-flats, 2022. Preprint.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF PRETORIA, PRETORIA, 0002 SOUTH AFRICA, maya.thackeray@up.ac.za, mayart314@outlook.com