New class of hybrid metric-Palatini scalar-tensor theories of gravity

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Abstract. A class of scalar-tensor theories (STT) including a non-metricity that unifies metric, Palatini and hybrid metric-Palatini gravitational actions with non-minimal interaction is proposed and investigated from the point of view of their consistency with generalized conformal transformations. It is shown that every such theory can be represented on-shell by a purely metric STT possessing the same solutions for a metric and a scalar field. A set of generalized invariants is also proposed. This extends the formalism previously introduced in [1]. We then apply the formalism to Starobinsky model, write down the Friedmann equations for three possible cases: metric, Palatini and hybrid metric-Palatini, and compare some inflationary observables.

Keywords: modified gravity, inflation

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1 Introduction

$F(R)$ theories of gravity have been conceived as the simplest modification of Einstein’s General Relativity (GR) [2]–[9]. The modification is achieved by a straightforward replacement of the Einstein-Hilbert action with a function of the curvature scalar, and the main aim of such an alternation is to create a theory which would encompass phenomena that cannot be satisfactorily explained by GR only, such as the accelerated expansion of the universe [10]–[21]. The present-day cosmic speed-up is explained by the presence of a cosmological constant $\Lambda$, accounting for the ‘dark’ energy content, amounting to as much as 68.3% of the total matter-energy density [15]. The exact nature of the dark energy remains still unknown. The cosmic speed-up might be also explained by a modification of General Relativity different from adding the cosmological constant $\Lambda$, during which the universe underwent an accelerated expansion, which led to the observed homogeneity and resolved the flatness problem [17]–[29]. An $F(R)$ model remaining with a very good agreement with the Planck satellite observations is the Starobinsky model, in which the Einstein-Hilbert action was supplemented with a quadratic correction [17]–[29].

So far, $F(R)$ theories have been mostly analyzed in the metric [30]–[40] and Palatini approaches [41]–[51]. In the metric approach, one treats the metric field as the only dynamical variable entering the action, whereas in the Palatini approach, the connection is now independent of the metric tensor, and the field equations are obtained by performing variation with respect to both the metric and the connection. Based on the equations, one determines the relation between these two objects. In the case of $F(R)$ theories, the connection turns
out to be an auxiliary field, and the theory becomes effectively metric. An interesting feature of $F(R)$ gravity is its equivalence to some classes of scalar-tensor theory. By performing a Legendre transformation, one can introduce a scalar field non-minimally coupled to the curvature, and analyze the theory using mathematical machinery developed for scalar-tensor gravity. It turns out that in case of Palatini $F(R)$ theory, unlike in the metric version, the scalar field has no dynamics, which means that the formalism introduces no additional degree of freedom.

An interesting generalization of the Palatini and metric $F(R)$ theories of gravity are so-called hybrid metric-Palatini theories, which were devised to avoid certain shortcomings manifested by both theories [52]–[60]. For example, metric $F(R)$ theories introduce an additional degree of freedom behaving as a scalar field. To have an impact on large scales, it should have a low mass. Presence of such field, however, affects dynamics at a shorter scale as well, and it should be possible to observe its influence on our Solar System. Because no such effect was observed, one must introduce a screening mechanism [61, 62]. On the other hand, the field is just an algebraic function of the trace of the energy-momentum tensor, so that it introduces no additional degrees of freedom [47]. This leads to very serious drawbacks, for example to infinite tidal forces at surfaces of compact objects. The hybrid metric-Palatini theory, however, introduces long-range forces without being in conflict with local measurements and the need to invoke screening mechanisms. It also predicts viable formation of large-scale structures in accelerating cosmologies [52, 53, 56].

All issues described above are usually discussed in a more general framework of STT (see also [63]–[84]) to which any $F(R)$ theory can be transform. Metric, Palatini and hybrid metric-Palatini theories have all a scalar-tensor representation, which, for the hybrid case, will be presented in the first section of the paper. This is not, however, equivalent to saying that any scalar-tensor theory arises from some $F(R)$ gravity. For such an equivalence (in a mathematical sense) to be present, certain conditions must be satisfied.

Our idea is to present a new approach to scalar-tensor theories of gravity that unifies three previously investigated in the literature: metric, Palatini, and hybrid. Such an approach will encompass within one family of theories not only metric, but also Palatini scalar-tensor theories of gravity, and will be a natural extension of the hybrid metric-Palatini gravity. The proposed formalism will also allow one to determine if a given STT is equivalent to some metric, Palatini or hybrid $F(R)$ gravity.

The paper is organized as follows: in the first section, we discuss shortly how one can obtain scalar-tensor representation of $F(R)$ gravity. Next, we review hybrid metric-Palatini theories and then present their generalization, postulating an action functional, writing the equations of motion and solving for connection. In the last section, we switch our attention to cosmological applications of the theory and write Friedmann equations for metric, Palatini and hybrid $F(R)$ theories. As an example, we analyze the Starobinsky model and compare inflationary parameters. Some more technical aspects are presented in two appendices. The first one is focused on formal properties of the Legendre transformation and some (partially new) examples of inflationary potentials. The second one extends the formalism of frame transformations and their invariants [1] to a new hybrid metric-Palatini STT case.

Throughout the paper we work with general spacetime of dimension $n$ with a metric of the Lorentzian signature $(-,+,\ldots,+)$. We stick to the following convention when writing the curvature scalar: $R$ is a general curvature, $\mathcal{R}$ denotes curvature built from the metric only, and $\hat{R}$ is Palatini curvature, i.e. constructed both from the metric and connection. Our notational conventions are borrowed from [1].
2 From $F(R)$ to scalar-tensor gravity

In this subsection we review known facts concerning metric, Palatini as well as hybrid $F(R)$-gravity (see e.g. [52]–[60]).

Consider the action of minimally coupled $F(R)$-gravity:

$$S_F[g_{\mu\nu}.,.] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} F(R) + S_{\text{matter}}(g_{\mu\nu}, \chi), \quad (2.1)$$

where $F(R)$ is a function of either a Ricci $R = \mathcal{R}(g)$ or a Palatini-Ricci $R = \tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu}(\Gamma)$ scalar and $\Gamma$ denotes torsionless connection. The matter part of the action $S_{\text{matter}}$ is assumed to be metric-dependent (independent of the connection).

In both cases the action (2.1) is dynamically equivalent to the constrained system with linear gravitational Lagrangian

$$S[g_{\mu\nu}.,\Xi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} (F'(\Xi)(R - \Xi) + F(\Xi)) + S_{\text{matter}}(g_{\mu\nu}, \chi). \quad (2.2)$$

Introducing further a scalar field $\Phi = F'(\Xi)$ and taking into account the constraint equation $\Xi = R$, one arrives at the dynamically equivalent STT action with a non-dynamical scalar field

$$S[g_{\mu\nu},.,\Phi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} (\Phi R - U_F(\Phi)) + S_{\text{matter}}(g_{\mu\nu}, \chi) \quad (2.3)$$
either in metric or Palatini case. The potential:

$$U_F(\Phi) = R(\Phi)\Phi - F'(R(\Phi)) \quad (2.4)$$
is the result of Legendre transformation (see appendix A for details), where $\Phi = \frac{dF(R)}{dR}$ and $R = \frac{dU_F(\Phi)}{d\Phi}$.

It is known that both cases can be realized as Brans-Dicke (BD) theories with different values of the parameter $\omega_{\text{BD}}$. Original BD is a metric scalar-tensor theory determined by the gravitational action:

$$S_{\text{BD}}[g_{\mu\nu},\Phi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} \left( \Phi R - \frac{\omega_{\text{BD}}}{\Phi} \partial_\mu \Phi \partial^\mu \Phi - U(\Phi) \right), \quad (2.5)$$

where BD parameter $\omega_{\text{BD}} \in \mathbb{R}$ and $U(\Phi)$ denotes the self-interaction potential.

The action (2.5) is cast in so-called Jordan frame. The Jordan frame is characterized by a non-minimal coupling between the curvature and the scalar field, with the matter part of the action depending on the metric and matter fields only. One may also make use of a conformal transformation of the metric tensor, defined by (B.1a), in order to switch to a frame in which the theory will be easier to analyze. Most commonly, one will choose the transforming function $\gamma_1$ in such a way, that the curvature and scalar field will no longer be coupled. One calls such frame the 'Einstein frame'. This, however, comes at a price of an anomalous coupling between the scalar and matter fields, leading to a violation of the Weak Equivalence Principle. The issue of whether Jordan or Einstein frame is the physical one remains open and has been widely discussed in the literature [85]–[101].

\footnote{One should stress that Palatini $F(R)$-gravity is not dynamically equivalent to a metric one with the same function $F(R)$.}
3 Hybrid metric-Palatini theory

In the hybrid metric-Palatini theory, one adds to the metric Einstein-Hilbert action a function of the Palatini curvature scalar $\hat{R}(g, \Gamma)$, which can be treated as a correction term [52]. The theory was devised to serve as a bridge between metric and Palatini theories, allowing one to avoid certain drawbacks of the latter.

The action functional is given by [60]:

$$S[g_{\mu\nu}, \Gamma_{\alpha}^{\mu\nu}] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g}[\Omega_A R(g) + F(\hat{R}(g, \Gamma)) + S_{\text{matter}}[g_{\mu\nu}, \chi]]$$

(3.1)

where $\Omega_A$ is a coupling constant.

We will be interested in the scalar-tensor representation of the theory. In order to switch to desired form of the action, we follow the standard procedure and perform a Legendre transformation of the $F(\hat{R})$ function, introducing a scalar field (see eq. (2.2)) and defining a potential $U_F(\Phi)$ as in (2.4) (cf. (A.3)). We will arrive at the following form of the action functional:

$$S[g_{\mu\nu}, \Gamma_{\alpha}^{\mu\nu}, \Phi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g}[\Omega_A R(g) + \Phi \hat{R}(g, \Gamma) - U_F(\Phi)] + S_{\text{matter}}[g_{\mu\nu}, \chi]$$

(3.2)

It is clear now that variation w.r.t. the connection will produce exactly the same result as in case of purely Palatini $F(\hat{R})$ gravity, i.e. the curvature scalar $\hat{R}$ will turn out to be a function of the conformally related metric $\bar{g}_{\mu\nu} = \Phi^2 g_{\mu\nu}$. Therefore, we can use the result of [52] and express the action (3.2) as a function of the metric and scalar field only:

$$S[g_{\mu\nu}, \Phi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} \left[ (\Omega_A + \Phi) R(g) + \frac{n-1}{(n-2)} \partial_\mu \Phi \partial^\mu \Phi - U_F(\Phi) \right] + S_{\text{matter}}[g_{\mu\nu}, \chi].$$

(3.3)

Let us now perform a shift (re-definition, cf. (B.1c)) in the scalar field and introduce $\psi = \Omega_A + \Phi$; this will yield:

$$S[g_{\mu\nu}, \psi] = \frac{1}{2\kappa^2} \int_{\Omega} d^n x \sqrt{-g} \left( \psi R(g) - \frac{(n-1)}{(n-2)} \psi \partial_\mu \psi \partial^\mu \psi - U_F(\psi) \right) + S_{\text{matter}}[g_{\mu\nu}, \chi].$$

(3.4)

where the Brans-Dicke parameter $\omega_{BD}$ is now a function of the scalar field:

$$\omega_{BD}(\psi) = -\frac{(n-1)}{(n-2)(\psi - \Omega_A)}$$

One can observe that when $\Omega_A \to 0$ then the theory becomes Palatini $F(\hat{R})$ gravity. In the limit $\Omega_A \to \infty$, however, it reproduces GR. Any value of the parameter $\Omega_A$ lying in between these two values gives a mixture of the two approaches [52].

Let us now compute the integral invariant $I_M^\psi$ as defined in [85] (cf. its generalization presented in appendix B):

$$I_M^\psi(\psi) = \frac{1}{\sqrt{n}} \int_{\psi_0}^\psi \sqrt{\pm\frac{(n-2)A(\psi')B(\psi') + (n-1)(A'(\psi'))^2}{A^2(\psi')}} d\psi'$$

(3.5)

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2The generalization of this transformation will be introduced in the next section.

3Such invariants are determined up to an integration constant and can be normalized in various ways.
for this theory, characterized by the following set of functions of the scalar field: \( A(\psi) = \psi, B(\psi) = -\frac{(n-1)}{(n-2)(-\Omega A + \psi)}, V(\psi) = V(\psi), \alpha(\psi) = 0 \). The invariant is given by:

\[
\mathcal{I}_M^n(\psi) = \sqrt{\frac{4(n-1)}{n}} \left[ \arctan \left( \sqrt{\frac{\Omega A - \psi}{\Omega A}} \right) - \arctan \left( \sqrt{\frac{\Omega A - \psi_0}{\Omega A}} \right) \right]
\]

For the metric \( F(R) \) theory, the invariant takes the following form:

\[
\mathcal{I}_M^n(\psi) = \sqrt{\frac{n-1}{n}} \ln \left( \frac{\psi}{\psi_0} \right), \quad (3.6)
\]

and for the Palatini \( F(\hat{R}) \):

\[
\mathcal{I}_M^n(\psi) = 0. \quad (3.7)
\]

Three different values of this invariant enable us to distinguish between three different cases of \( F(R) \)-gravity (hybrid, Palatini and metric) in a frame independent way. This is due to the fact that transformations (B.1a), (B.1c) do not change invariant quantities characterizing metric STT’s.

### 4 Hybrid metric-Palatini generalization

In order to encompass both metric and Palatini scalar-tensor theories of gravity within one hybrid approach, we postulate the following action functional:

\[
S[g_{\mu\nu}, \Gamma^\alpha_{\mu\nu}, \Phi] = \frac{1}{2\kappa^2} \int d^n x \sqrt{-g} \left[ A_1(\Phi) R(g) + A_2(\Phi) \hat{R}(g, \Gamma) - B(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \
- Q^\mu(\gamma, \Gamma) C_1(\Phi) \partial_\mu \Phi - \hat{Q}^\mu(\gamma, \Gamma) C_2(\Phi) \partial_\mu \Phi - \mathcal{V}(\Phi) \right] + S_{\text{matter}}[e^{2\alpha(\Phi)} g_{\mu\nu}, \chi]. \quad (4.1)
\]

depending on the choice of seven functions of one-variable \( A_1, A_2, B, V, C_1, C_2, \alpha \), which determine the so-called frame. We stick to the convention that all quantities with 'hat' are calculated using the independent connection \( \Gamma \). In particular, the quantities \( Q_\mu = g^{\alpha\beta} \nabla_\mu g_{\alpha\beta} \) and \( \hat{Q}_\mu = -g^{\alpha\beta} \nabla_\alpha g_{\beta\mu} \) depend on the non-metricity of the connection \( \Gamma \) and vanish only when \( \Gamma^\alpha_{\mu\nu} = \left\{ \frac{\alpha}{\mu\nu} \right\}_g \), i.e. in the Levi-Civita case.

The action is covariant with respect to generalized conformal transformations as described in appendix B. These transformations (B.2a)–(B.2f) divide all frames into mathematically equivalent classes in such a way that \( (g, \Gamma, \Phi) \) — corresponding solutions of field equations (see below), transforms each other by (B.1a)–(B.1c).

In particular, setting \( A_1(\Phi) = \Omega A, A_2(\Phi) = \Phi, B(\Phi) = C_1(\Phi) = C_2(\Phi) = 0 \) one gets the theory stemming from the typical hybrid metric-Palatini action. For \( A_1(\Phi) = 0, A_2(\Phi) > 0 \) we can recover Palatini STT class. On the other hand, setting \( A_1(\Phi) > 0, A_2(\Phi) = C_1(\Phi) = C_2(\Phi) = 0 \) we are in a purely metric subclass. Moreover, the action (4.1) is preserved under the generalized conformal transformations (B.1a)–(B.1c) (see appendix).
The equations of motion are obtained by varying w.r.t. the independent variables: metric, connection and scalar field. The metric equations of motion have the following form:

\[
\mathcal{A}'_1(\Phi)\mathcal{G}_{\mu\nu}(g) + \mathcal{A}_2(\Phi)\dot{\mathcal{G}}_{\mu\nu}(g, \Gamma) + \left(\mathcal{A}'_1(\Phi) + \frac{1}{2} \mathcal{B}(\Phi) - \mathcal{C}_1(\Phi)\right) (\partial \Phi)^2 g_{\mu\nu} + \frac{1}{2} \mathcal{V}(\Phi) g_{\mu\nu} \\
- \left(\mathcal{A}'_1(\Phi) + \mathcal{B}(\Phi) - \mathcal{C}_2(\Phi)\right) \partial_\mu \Phi \partial_\nu \Phi + (\mathcal{C}_2(\Phi) \nabla_\mu \partial_\nu - \mathcal{C}_1(\Phi) g_{\mu\nu} \Box) \Phi - \mathcal{A}'_1(\Phi) (\nabla_\mu \partial_\nu - g_{\mu\nu} \Box^g) \Phi \\
+ Q_{\beta \lambda \zeta} \left[\frac{1}{2} C_2(\Phi) \delta_{(\nu \mu)}^\alpha \delta^{\lambda \zeta} - C_1(\Phi) \left(\frac{1}{2} g_{\mu\nu} g^{\sigma \beta} g^{\lambda \zeta} - g_{\mu\nu} g^{\lambda \zeta} + \delta^\sigma_{(\nu \mu)} g^{\lambda \zeta}\right)\right] \partial_\sigma \Phi = \kappa^2 T_{\mu\nu},
\]

(4.2)

where \( T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \) denotes simply the matter stress-energy contribution and \( \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \).

The scalar field equation of motion reads as:

\[
\mathcal{A}'_1(\Phi) \mathcal{R}(g) + \mathcal{A}_2(\Phi) \dot{\mathcal{R}}(g, \Gamma) + B'(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2B(\Phi) \Box g^{\mu\nu} \partial_\mu \Phi Q_{\rho \alpha \beta} \\
\times \left(2 \frac{g^{\mu\nu} g^{\alpha \beta} - g^{\mu \alpha} g^{\beta \nu}}{\mathcal{A}_2(\Phi)}\right) + C_1(\Phi) \nabla_\mu Q^\mu + C_2(\Phi) \nabla_\mu \dot{Q}^\mu - \mathcal{V}'(\Phi) = -2\kappa^2 a'(\Phi) T,
\]

(4.3)

where the right hand side is due to non-minimal coupling in the action and \( T = g^{\mu\nu} T_{\mu\nu} \).

The last set of equations comes from varying with respect to the connection:

\[
\nabla_\alpha \left[\frac{\sqrt{-g}}{g} \left( g^{\alpha (\zeta \lambda)} - g^{\lambda \zeta} \delta^\alpha_\beta \right)\right] = \\
\sqrt{-g} \partial_\alpha \Phi \left[ g^{\alpha (\zeta \lambda)} \left(\frac{C_2(\Phi) - C_1(\Phi) - \mathcal{A}'(\Phi)}{\mathcal{A}_2(\Phi)}\right) - g^{\lambda \zeta} \delta^\alpha_\beta \left(\frac{-C_2(\Phi) - \mathcal{A}'(\Phi)}{\mathcal{A}_2(\Phi)}\right)\right].
\]

(4.4)

It admits the following generic solutions for the connection:

\[
\Gamma^\alpha_{\mu\nu} = \left\{ \begin{array}{c} \Gamma^\alpha_{\mu\nu} \\
\mathcal{F}_1(\Phi) \delta^\alpha_{(\mu} \partial_{\nu)} \Phi - \mathcal{F}_2(\Phi) g^{\alpha \beta} \partial_\beta \Phi, \end{array} \right.
\]

(4.5)

and the non-metricity:

\[
Q^\alpha_{\mu\nu} = \nabla_\alpha g^{\mu\nu} = 2 \left(\mathcal{F}_1(\Phi) - \mathcal{F}_2(\Phi)\right) \delta^\alpha_{(\mu} g^{\nu) \rho} \partial_\rho \Phi + 2 \mathcal{F}_1(\Phi) g^{\mu\rho} \partial_\rho \Phi,
\]

where

\[
\mathcal{F}_1(\Phi) = \frac{2C_1(\Phi) + (n - 3)C_2(\Phi) + (n - 1)\mathcal{A}'(\Phi)}{\mathcal{A}_2(\Phi)(n - 1)(n - 2)},
\]

and

\[
\mathcal{F}_2(\Phi) = \frac{2C_1(\Phi) - C_2(\Phi) + \mathcal{A}'(\Phi)}{\mathcal{A}_2(\Phi)(n - 2)}.
\]

The fact that the connection can be expressed in terms of metric tensor and scalar field only means that it introduces no additional degrees of freedom in the theory. Particularly, the connection is dynamically (on-shell) metric (Levi-Civita)\(^4\) if and only if the following condition is satisfied \((\mathcal{F}_1(\Phi) = \mathcal{F}_2(\Phi) = 0)\)

\[
C_1(\Phi) = C_2(\Phi) = -\mathcal{A}'(\Phi)
\]

\(^4\)More generally, the connection is Levi-Civita with respect to a conformal metric \(\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\) iff \(C_1 = C_2\) which follows from the condition \(\mathcal{F}_1 = \mathcal{F}_2 = (\ln \Omega)'\). In this case, the non-metricity \(Q^\alpha_{\mu\nu}\) is of the Weyl type, i.e. \(\nabla_\alpha g^{\mu\nu} = W_{\alpha} g^{\mu\nu}\).
Otherwise, one can always choose the parameters
\[ \gamma_2(\Phi) = -\mathcal{F}_1(\Phi), \quad \gamma_3(\Phi) = -\mathcal{F}_2(\Phi) \]
of the transformation (B.1b) in such a way that the connection \( \Gamma \) becomes metric for the original metric \( g \). This changes the frame parameters \((\mathcal{B}, \mathcal{C}_1, \mathcal{C}_2)\) to the new one (cf. (B.2a)–(B.2f)) in such a way that \( \mathcal{C}_1(\Phi) = \mathcal{C}_2(\Phi) = -\mathcal{A}_2(\Phi) \) and
\[
\mathcal{B}(\Phi) = \frac{(n - 2)A_2(\Phi)\mathcal{B}(\Phi) - (n - 1)(A_2(\Phi))^2 + 2A_2(\Phi)[C_2(\Phi) - nC_1(\Phi)]}{(n - 2)A_2(\Phi)}
\]
+ \( \frac{(n^2 - 5)C_2(\Phi)^2 - 4C_1(\Phi)^2 + 2(4 + n - n^2)C_1(\Phi)C_2(\Phi)}{(n - 2)(n - 1)A_2(\Phi)} \) \quad (4.6)

while the remaining ones \((A_1, A_2, V, \alpha)\) are unchanged. Particularly, in the case \( C_1(\Phi) = C_2(\Phi) = -\mathcal{A}_2(\Phi) \) one gets \( \mathcal{B}(\Phi) = \mathcal{B}(\Phi), \mathcal{C}_1(\Phi) = \mathcal{C}_2(\Phi) = -\mathcal{A}_2(\Phi) \).

Therefore, one can get rid of the auxiliary connection from the action (4.1) and replace it (on-shell) with the Christoffel symbols and functions of scalar field and its first derivatives. This will effectively lead to a metric scalar-tensor theory of gravity described by the action:
\[
S[g_{\mu\nu}, \Phi] = \frac{1}{2\kappa^2} \int d^n x \sqrt{-g} \left[ \mathcal{A}(\Phi)\mathcal{R}(g) - \mathcal{B}(\Phi)g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \mathcal{V}(\Phi) \right] + S_{\text{matter}}[e^{2\alpha(\Phi)}g_{\mu\nu}, \chi]. \quad (4.7)
\]
where (now on-shell \( Q_{\alpha\mu\nu}(g, \Gamma) = 0 \) and \( \mathcal{R}(g) = \mathcal{R}(g, \Gamma) \)):
\[
\mathcal{A}(\Phi) = A_1(\Phi) + A_2(\Phi). \quad (4.8)
\]
The corresponding field equations in this frame can be recast into the form
\[
\mathcal{A}(\Phi)\mathcal{G}_{\mu\nu}(g) - (\nabla^\mu \mathcal{A}(\Phi)\nabla^\nu g_{\mu\nu} - g_{\mu\nu}\Box g_{\mu\nu})\mathcal{A}(\Phi) = T^\Phi_{\mu\nu} + \kappa^2 T_{\mu\nu} \quad (4.9)
\]
\[
\mathcal{A}'(\Phi)\mathcal{R}(g) + \mathcal{B}(\Phi)(\partial^\Phi)^2 + 2\mathcal{B}(\Phi)\Box g_{\mu\nu} - \mathcal{V}'(\Phi) = -2\kappa^2\alpha'(\Phi)T \quad (4.10)
\]
where \((\partial^\Phi)^2 = g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\) and
\[
T^\Phi_{\mu\nu} = \mathcal{B}(\Phi)\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}(\mathcal{B}(\Phi)(\partial^\Phi)^2 + \mathcal{V}(\Phi))g_{\mu\nu} \quad (4.11)
\]
\[= -(\partial^\Phi)^2 \mathcal{B}(\Phi) u_\mu u_\nu - \frac{1}{2}(\mathcal{B}(\Phi)(\partial^\Phi)^2 + \mathcal{V}(\Phi))g_{\mu\nu}, \]
mimics a perfect fluid with velocity \( u_\mu = \partial_\mu\Phi / \sqrt{-(\partial^\Phi)^2} \) determined by the normalized gradient co-vector \( (u_\mu u^\mu = -1) \). In fact, equations (4.9), (4.10) can be obtained from (4.2), (4.3) if one takes into account \( C_1 = C_2 = -\mathcal{A}' \) and \( Q_{\alpha\mu\nu} = 0 \). Moreover, the solutions for the metric and the scalar field in both frames (4.1) and (4.7) are exactly the same while the solution for the connection is changed.

Another important fact related to the action (4.7) comes from the matter energy-momentum conservation. It follows from (20) that \( \nabla^\mu T^\Phi_{\mu\nu} = -\frac{1}{2}(\mathcal{A}'(\Phi)\mathcal{R}(g) + 2\kappa^2\alpha'(\Phi)T)^2 \nabla^\nu\Phi. \)
Now, using the identity given in [84]: \((\Box^\nu \nabla^\mu - \nabla^\nu \Box g_{\mu\nu})\mathcal{A}(\Phi) = \mathcal{R}_{\mu\nu}(g)\nabla^\mu\mathcal{A}(\Phi) \) we conclude that the matter-energy is conserved
\[
\nabla^\mu T_{\mu\nu} = \alpha'(\Phi)T \nabla^\nu\Phi \quad (4.12)
\]
provided that there is a minimal coupling with the matter. It means that the stress-energy tensor is conserved (on-shell) in all frames such that $\alpha'(\Phi) = 0$ independently of the other frame parameters $(A_1, A_2, B, C_1, C_2, V)$. Otherwise, one can always change the metric $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu} = e^{-2\alpha(\Phi)}g_{\mu\nu}$ in order to obtained the matter stress-energy conservation $\nabla^{\mu}\bar{T}_{\mu\nu} = 0$.

Finally, combining (20) with the trace of (19), one can replace (20) (cf. [85]) by
\[
2([n - 1]2\bar{\Phi})^2 + (n - 2)\bar{\Phi}\bar{B}(\Phi)](2\bar{\Phi} + \frac{d\bar{\Phi}}{d\Phi}) \]
\[
- (n - 2)\bar{\Phi}V(\Phi) = 2\kappa^2 T(\bar{\Phi}) - (n - 2)\alpha'(\Phi)\bar{\Phi}(\Phi). \tag{4.13}
\]
It shows that the scalar field has no dynamics in a metric frame if and only if
\[
(n - 1)(\bar{\Phi})^2 + (n - 2)\bar{\Phi}\bar{B}(\Phi) = 0 \tag{4.14}
\]
i.e. $(2 - n)\bar{B}(\Phi) = (n - 1)\bar{\Phi}'(\Phi)[\ln \bar{\Phi}(\Phi)]'$. It turns out that this property is conformally-invariant and can be expressed by vanishing of $\frac{dM}{d\Phi} = 0$, where $\mathcal{I}_M(\Phi)$ denotes the integral invariant (3.5). It will be shown later on that (4.14) uniquely characterizes BD models arising from the $F(R)$-Palatini action: $\mathcal{I}_M = \mathcal{I} = \text{const}$ (cf. (B.4)).

We remark that the action (4.7) does not remember the initial action (4.1) from which it has been obtained. In order to perform the inverse (off-shell) transformation one has to assume some function $A_2(\Phi)$ (or $A_1(\Phi)$, or the invariant $\mathcal{I}_A(\Phi)$). Then applying all possible transformations (B.1b) one can recover all generalized frames which project (on-shell) onto a given metric frame. In other words, the totality of all generalized frames (4.1) indicates the transformations (B.1b) one can recover all generalized frames which project (on-shell) onto a given metric frame.

In this way different decompositions (4.8) provide a family of different off-shell (solution equivalent) actions in the form (4.1).

Having done the projection to the metric theory we are left with the possibility of using the conformal (B.1a) as well as diffeomorphism (B.1c) transformations in order to reach simpler (e.g. Einstein canonical) forms [85]. We recall that only the Palatini case satisfies the conformally-invariant condition (4.14).

Let us now calculate the $(\bar{A}, \bar{B})$ functions for $R + F(R)$ theories of gravity. We will consider all three possible approaches: metric, Palatini and hybrid metric-Palatini. For these three theories, in the scalar-tensor representation, the frame functions of the scalar field are shown in the table 1. The potential $U_F(\Phi)$ is defined by means of the Legendre transformation as $U_F(\Phi) = \Phi R(\Phi) - F(R(\Phi))$, with $\Phi = dF(R)/dR$ (for more details see appendix A). These theories have different (not related by the transformations (B.1a), (B.1c)) solutions since their invariants are different (cf. appendix B). In order to investigate the solutions for a metric and a scalar field it is convenient to switch to the corresponding metric ST representation: making use of the definitions (4.6) and (4.8) of the frame functions $\bar{A}(\Phi)$ and $\bar{B}(\Phi)$, one gets the corresponding metric ST representation as shown in the table 2.

---

This data determines uniquely the splitting (4.8) as well as the functions $\tilde{C}_1 = \tilde{C}_2 = -A_2'$ when $\bar{A}$ is given.
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
  & $A_1$ & $A_2$ & $B$ & $C_1$ & $C_2$ & $\mathcal{V}$ & $\alpha$ \\
\hline
metric & $\Phi$ & 0 & 0 & 0 & 0 & $U_F(\Phi - 1)$ & 0 \\
Palatini & 0 & $\Phi$ & 0 & 0 & 0 & $U_F(\Phi - 1)$ & 0 \\
hybrid & $\Omega_A$ & $\Phi$ & 0 & 0 & 0 & $U_F(\Phi)$ & 0 \\
\hline
\end{tabular}
\end{center}

Table 1. Different frame parametrizations of $R + F(R)$ gravity.

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
  & $\dot{A}$ & $\dot{B}$ & $\mathcal{V}$ & $\alpha$ \\
\hline
metric & $\Phi$ & 0 & $U_F(\Phi - 1)$ & 0 \\
Palatini & $\Phi$ & $-\frac{n-1}{n-2} \Phi$ & $U_F(\Phi - 1)$ & 0 \\
hybrid & $\Omega_A + \Phi$ & $-\frac{n-1}{n-2} \Phi$ & $U_F(\Phi)$ & 0 \\
\hline
\end{tabular}
\end{center}

Table 2. The corresponding metric SST frames for three cases of $R + F(R)$ gravity.

5 Cosmological applications

We may now attempt to write down Friedmann equations for the action (4.1). This task is pretty straightforward, as the theory turns out to be fully metric. The equations of motion will be the same as in case of metric scalar-tensor theories; the only difference is the definition of the parameters ($\bar{A}$, $\bar{B}$), since now they differ from the ones we started with.

For four-diemnsional Friedmann-Robertson-Walker metric:

$$g_{\mu\nu} = \text{diag} \left( -1, \frac{a^2(t)}{kr^2}, a^2(t)r^2, a^2(t)r^2 \sin^2 \theta \right),$$

where $k$ is the spatial curvature, we get the following Friedmann equations (assuming that $\Phi = \Phi(t)$ and a barotropic $p = w\rho$ perfect fluid as a source):

$$3H^2 = \frac{\kappa^2 \rho}{\mathcal{A}(\Phi)} - \frac{3k}{a^2} \frac{\mathcal{B}(\Phi)}{\mathcal{A}(\Phi)} \dot{\Phi}^2 - 3 \frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)} H \dot{\Phi} + \frac{1}{2} \frac{\mathcal{V}(\Phi)}{\mathcal{A}(\Phi)},$$

$$2\dot{H} + 3H^2 = -\frac{\kappa^2 \rho}{\mathcal{A}(\Phi)} \left[ \frac{k}{a^2} \frac{\mathcal{B}(\Phi) + 2 \mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)} \dot{\Phi}^2 - \frac{\mathcal{A}'(\Phi)}{\mathcal{A}(\Phi)} (2H \dot{\Phi} + \ddot{\Phi}) + \frac{\mathcal{V}(\Phi)}{2 \mathcal{A}(\Phi)} \right].$$

$$(3(\mathcal{A}'(\Phi))^2 + 2\mathcal{A}(\Phi)\mathcal{B}(\Phi)) \ddot{\Phi} = -3 \left( 3(\mathcal{A}'(\Phi))^2 + 2\mathcal{A}(\Phi)\mathcal{B}(\Phi) \right) H \dot{\Phi}$$

$$- \left( (\mathcal{A}(\Phi)\mathcal{B}(\Phi))' + 3\mathcal{A}'(\Phi)\mathcal{A}'(\Phi) \right) \dot{\Phi}^2 + (2\mathcal{V}(\Phi)\mathcal{A}(\Phi) - \mathcal{V}'(\Phi)\mathcal{A}(\Phi))$$

$$+ \kappa^2 \rho (1 - 3w) \left[ \ddot{\mathcal{A}}(\Phi) - 2a'(\Phi)\dot{\mathcal{A}}(\Phi) \right].$$

Here, prime denotes differentiaton w.r.t. the scalar field, dot — w.r.t. the cosmic time, and $H = \dot{a}/a$, as usual.

Combining first two expressions one can infer deceleration/acceleration formula for the scale factor

$$\frac{\ddot{a}}{a} \equiv \ddot{H} + H^2 = -\frac{\kappa^2 \rho (1 + 3w)}{6 \mathcal{A}(\Phi)} + \frac{2\mathcal{B}(\Phi) + 3\mathcal{A}'(\Phi)}{6 \mathcal{A}(\Phi)} \dot{\Phi}^2 - \frac{\mathcal{A}'(\Phi)}{2 \mathcal{A}(\Phi)} (2H \dot{\Phi} + \ddot{\Phi}) + \frac{\mathcal{V}(\Phi)}{6 \mathcal{A}(\Phi)}.$$

It shows that only the last term (if positive) supports acceleration explicitly. Otherwise some more complicated scenarios are needed in order to get the right hand side positive.
If we act with the covariant derivative on the energy-momentum tensor, we get:

$$\nabla_{\mu} T^{\mu\nu} = \alpha'(\Phi) T \partial^{\nu} \Phi \tag{5.4}$$

If the anomalous coupling between the scalar field and the matter part of the action is not present, then the energy-momentum tensor is conserved. In this case, the energy density can be obtained by solving the following equation:

$$\dot{\rho} + 3H(1 + w)\rho = 0, \tag{5.5}$$

which gives:

$$\rho(a) = \rho_0 a^{-3(1+w)} \tag{5.6}$$

where $\rho_0$ is the energy density at the present time. In fact, one can take into account more than one energy density source and write:

$$\rho_i(a) = \rho_{i,0} a^{-3(1+w_i)}, \tag{5.7}$$

with $i$ indexing different components of the total energy density, such as dust ($\rho_m, w = 0$), radiation ($\rho_r, w = \frac{1}{3}$) or dark energy ($\rho_\Lambda, w = -1$).

In the next section, we will demonstrate, on concrete examples of the Starobinsky model $F(R) = R + \beta R^2$, the differences between three approaches: metric, Palatini, and hybrid metric-Palatini.

### 5.1 Example: Starobinsky model

Starobinsky model is a simple modification of the General Relativity. In this model, the Einstein-Hilbert action is supplemented with a quadratic correction. There are three possible approaches one can take in order to analyze the theory: treating the curvature as a function of the metric, of the metric and the connection, or assuming that only the correction term, $\beta R^2$, is constructed from both the metric and the connection:

- **Case 1**: $F(\mathcal{R}) = \mathcal{R}(g) + \beta \mathcal{R}(g)^2$ — metric;
- **Case 2**: $F(\hat{R}) = \hat{R}(g, \Gamma) + \beta \hat{R}(g, \Gamma)^2$ — Palatini;
- **Case 3**: $F(\mathcal{R}, \hat{R}) = \Omega_A \mathcal{R}(g) + \beta \hat{R}(g, \Gamma)^2$ — hybrid metric-Palatini.

Let us notice that it does not make much sense to analyze the case when the Einstein-Hilbert action (i.e. the curvature itself) is constructed à la Palatini, and the correction is metric, as the Palatini Einstein-Hilbert action always turns out to be fully metric.

The next step will be to transform the theory to the scalar-tensor representation. For the first two cases, the potential $U_F(\Phi) = \Phi R(\Phi) - F(R(\Phi))$ will be exactly the same, since the procedure does not differ for the metric and Palatini approaches. However, one will end up with different potential when considering the third case. All three cases generate the same quadratic potential which due the presence of linear term is shifted $U_F(\Phi) \rightarrow U_F(\Phi - \Omega_A)$ for the metric and the Palatini cases (cf. appendix A) Also, the coupling between the field and the curvature will not be the same as in the first two cases, because now the scalar field is defined as $\Phi = \frac{dF(\mathcal{R}, \hat{R})}{d\mathcal{R}}$. The differences and similarities between these three cases are shown in table 1. As one can see, metric and Palatini cases are almost identical, the difference being the value of the couplings ($A_1, A_2$).
For the $\bar{A}(\Phi)$ and $\bar{B}(\Phi)$ functions shown in the table 1, the Friedmann equations will read as follows:

**Metric.**

\[
3H^2 = \frac{\kappa^2}{\Phi} \sum_i \rho_i - 3 \frac{k}{a^2} - 3H \frac{\dot{\Phi}}{\Phi} + \frac{1}{8\beta} \left( \frac{\Phi - 1}{\Phi} \right)^2, \tag{5.8a}
\]

\[
2\dot{H} + 3H^2 = -\frac{\kappa^2}{\Phi} \sum_i w_i \rho_i - \frac{k}{a^2} - 2H \frac{\dot{\Phi}}{\Phi} - \frac{\ddot{\Phi}}{\Phi} + \frac{1}{8\beta} \left( \frac{\Phi - 1}{\Phi} \right)^2, \tag{5.8b}
\]

\[
\ddot{\Phi} = \frac{\kappa^2}{3} \sum_i \left(1 - 3w_i\right) \rho_i - 3H \dot{\Phi} - \frac{\Phi - 1}{6\beta}. \tag{5.8c}
\]

**Palatini.** In case of the Palatini approach, scalar field has no dynamics, so it introduces no additional degrees of freedom. This is caused by the fact that the denominator in the eq. (5.2c) vanishes. The equations are given by:

\[
3H^2 = \frac{\kappa^2}{\Omega + \Phi} \sum_i \rho_i - 3 \frac{k}{a^2} - 3H \frac{\dot{\Phi}}{\Phi} + \frac{1}{8\beta} \left( \frac{\Phi - 1}{\Phi} \right)^2, \tag{5.9a}
\]

\[
2\dot{H} + 3H^2 = -\frac{\kappa^2}{\Omega + \Phi} \sum_i w_i \rho_i - \frac{k}{a^2} + 3 \frac{\dot{\Phi}^2}{4\Phi^2} - 2H \frac{\dot{\Phi}}{\Phi} - \frac{\ddot{\Phi}}{\Phi} + \frac{1}{8\beta} \left( \frac{\Phi - 1}{\Phi} \right)^2, \tag{5.9b}
\]

\[
0 = \kappa^2 \sum_i \left(1 - 3w_i\right) \rho_i - \frac{\Phi - 1}{2\beta}. \tag{5.9c}
\]

The third equation means that, in principle, one can express $\Phi$ in terms of $\rho_i$, as the relation between these objects is algebraic. Therefore, the evolution equation for the scale factor turns out to be of second order and can be described as a two-dimensional dynamical system of Newtonian type with an effective potential function (cf. [44, 46, 55]). This becomes even more transparent in another conformally-equivalent Einstein ($A = 1$, $B = 0$, $\alpha = \frac{1}{2}\ln \Phi$) frame, cf. eqs. (27)–(28) in [46].

**Hybrid metric-Palatini.**

\[
3H^2 = \frac{\kappa^2}{\Omega + \Phi} \sum_i \rho_i - 3 \frac{k}{a^2} - 3 \frac{\dot{\Phi}^2}{4\Phi(\Omega + \Phi)} - 3H \frac{\dot{\Phi}}{\Omega + \Phi} + \frac{1}{8\beta} \frac{\Phi^2}{\Omega + \Phi}, \tag{5.10a}
\]

\[
2\dot{H} + 3H^2 = -\frac{\kappa^2}{\Omega + \Phi} \sum_i w_i \rho_i - \frac{k}{a^2} + 3 \frac{\dot{\Phi}^2}{4\Phi(\Omega + \Phi)} - 2H \frac{\dot{\Phi}}{\Omega + \Phi} - \frac{\ddot{\Phi}}{\Omega + \Phi} + \frac{1}{8\beta} \frac{\Phi^2}{\Omega + \Phi}, \tag{5.10b}
\]

\[
\ddot{\Phi} = -\frac{\kappa^2}{3\Omega} \sum_i \left(1 - 3w_i\right) \rho_i - 3H \dot{\Phi} + \frac{\dot{\Phi}^2}{2\Phi} + \frac{\Phi^2}{6\beta}. \tag{5.10c}
\]

This shows that the dynamics of the scale factor, as well as the scalar field, is different in all three cases. Only in the Palatini case, the scalar field has no dynamics. More precisely, three cases are mathematically different and cannot be related by a conformal transformation of the metric and a scalar field redefinition. We are going to illustrate now how some physical prediction can differ for the models presented above. To this aim let us consider inflationary parameters.
Within the context of scalar-tensor theories of gravity, where non-minimal coupling between curvature and scalar field might be present, one must keep in mind that certain physical predictions, such as the number of e-folds, may strongly depend on the choice of conformal frame. However, certain observables can be expressed in a way that is frame-independent. For example, as it was shown in [88], it is possible to express slow-roll parameters characterizing inflation in a manifestly frame-independent way, making use of an invariant generalization of the scalar field potential (cf. appendix B). There is, however, a caveat in this way of thinking. As Karam et al. are showing in the paper [88], even though the spectral indices become functions of the invariant potential, which has the same form in every conformal frame, their numerical values might be different due to the fact that in different frames, the inflation lasts for different number of e-folds. Since we are not interested here in comparing conformal frames, but rather in comparing values of spectral indices for different theories, we decide to carry out all calculations in what is called ‘Einstein frame’.

In the Palatini case, there is no additional degree of freedom related to the scalar field (as it can be expressed as a function of matter, which is negligible during the inflation), so it cannot give rise to any dynamical fluctuations. Only metric and hybrid theories will be of any interest to us.\(^6\) We start by computing the spectral indices making use of notation introduced in [94]. First, one must compute the invariant potential and express it in terms of an invariant generalization of the scalar field (B.4). Having obtained the potential, one can compute the invariant slow-roll parameters and the number of e-folds, and then substitute the result in the formula for spectral indices.

One can write the following invariant slow-roll parameters characterizing cosmic inflation [94]:

\[
\hat{\kappa}_0^{(V)} = \frac{1}{4I_2^2} \left( \frac{dI_2}{dI} \right)^2, \\
\hat{\kappa}_1^{(V)} = 4\hat{\kappa}_0^{(V)} - \frac{1}{I_2} \frac{d^2I_2}{dI^2}. 
\]

where in four dimensions (cf. (3.5))

\[
I \equiv I_M^4, \\
I_2 = \frac{V(\Phi)}{A(\Phi)^2}.
\]

Slow-roll conditions are given by \( |\hat{\kappa}_i^{(V)}| \ll 1 \), and the inflation ends when \( \hat{\kappa}_0^{(V)} = 1 \). In the beginning, we analyse the metric Starobinsky model. Invariant \( I \) is given by:

\[
I = \frac{\sqrt{3}}{2} \ln \left( \frac{\Phi}{\Phi_0} \right) 
\]

so that

\[
\Phi(I) = \Phi_0 e^{\frac{2}{\sqrt{3}}I}
\]

The invariant potential can be written as:

\[
I_2 = \left( e^{\frac{2}{\sqrt{3}}I} - 1 \right)^2 
\]

---

\(^6\)The method proposed below does not apply in the Palatini case where \( I = 0 \). As it was shown in our earlier papers [45, 46] the inflationary effects manifest themselves differently in both Jordan as well as in Einstein frame. They are the results of some singularities in an effective Newtonian-type potential governing the universe evolution.
Upon substitution in (5.11), one gets:

\[
\hat{\kappa}_1^{(V)} = \frac{8e^{2\sqrt{3} I}}{3 \left(e^{2\sqrt{3} I} - 1 \right)^2}, \quad \hat{\kappa}_0^{(V)} = \frac{4}{3 \left(e^{2\sqrt{3} I} - 1 \right)^2}.
\] (5.15)

The number of e-folds in the Einstein frame can be computed from the following formula:

\[
\hat{N} = \int_{I_0}^{I_{\text{end}}} \frac{1}{\sqrt{2\hat{\kappa}_0^{(V)}}} dI \approx \frac{3}{4} e^{2\sqrt{3} I_0},
\] (5.16)

where \(I_0\) is the value of the scalar field at the beginning of inflation, and \(I_{\text{end}}\) — at the end. We additionally assumed that \(I_{\text{end}} \ll I_0\). Finally, we can compute the spectral index for the scalar field (up to the first order in slow-roll parameters):

\[
\hat{n}_s = 1 - 2\hat{\kappa}_0^{(V)} - \hat{\kappa}_1^{(V)} = \frac{-5 - 14e^{2\sqrt{3} I_0} + 3e^{4\sqrt{3} I_0}}{3 \left(e^{2\sqrt{3} I_0} - 1 \right)^2} \approx \frac{-5 - 56\hat{N}^3 + 16\hat{N}^2}{3(4\hat{N}^3 - 1)^2}.
\] (5.17)

For \(\hat{N} = 50\), one gets \(\hat{n}_s \approx 0.958\), and for \(\hat{N} = 60\), \(\hat{n}_s \approx 0.965\), in agreement with the Planck satellite result \(n_s = 0.968 \pm 0.006\) [102].

In the hybrid case, the invariant generalization of the scalar field is given by:

\[
I = \sqrt{3} \left(\arctan \sqrt{\frac{\Phi}{\Omega_A}} - \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right),
\] (5.18)

so that:

\[
\Phi(I) = \Omega_A \tan^2 \left(\frac{I}{\sqrt{3}} + \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right).
\] (5.19)

The invariant potential can be expressed as:

\[
I_2 = \frac{\tan^4 \left(\frac{I}{\sqrt{3}} + \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right)}{4\beta \left(1 + \tan^2 \left(\frac{I}{\sqrt{3}} + \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right)\right)^2}.
\] (5.20)

The slow-roll parameters can be now computed easily:

\[
\hat{\kappa}_0^{(V)} = \frac{4}{3} \cot^2 \left(\frac{I}{\sqrt{3}} + \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right), \quad \hat{\kappa}_1^{(V)} = \frac{4}{3} \csc^2 \left(\frac{I}{\sqrt{3}} + \arctan \sqrt{\frac{\Phi_0}{\Omega_A}}\right).
\] (5.21)

As we can see, already at this point we encounter a problem for the hybrid Starobinsky model. The second slow-roll parameter, \(\hat{\kappa}_1^{(V)}\), is given by the squared cosecans function, which does not take values smaller than 1. Therefore, it is impossible to satisfy the condition \(|\hat{\kappa}_1^{(V)}| < 1\), and further calculation reveals that, for the theory, the scalar spectral index is equal to \(-\frac{1}{3}\), which is in a very strong disagreement with observations. Therefore, the hybrid Starobinsky model is disfavoured by experimental data. More detailed qualitative analysis and the confrontation with observational data will be presented elsewhere.
6 Conclusions and perspectives

In this paper, we presented a possible generalization of hybrid metric-Palatini theories. Our idea was to add a function of a scalar field non-minimally coupled to the curvature built entirely from the metric tensor to an action functional for general Palatini scalar-tensor theories of gravity introduced in [1]. In such a way, one will create a self-consistent theory being a minimal extension of the metric, Palatini and hybrid metric-Palatini gravity with the freedom of transforming the scalar field, metric tensor and affine connection independently, using the formulae (B.1a)–(B.1b). Under this transformation, the action functional must remain form-invariant; to achieve this, one needs to transform the functions of the scalar field \((A_1, A_2, B, C_1, C_2, \nu, \alpha)\) defining, together with the dynamical variables \((g, \Gamma, \Phi)\), the conformal frame. Knowing how the functions transform, one is able to come up with certain combinations of them remaining invariant under the conformal change. Invariants can be used to check if two arbitrary scalar-tensor theories can be linked with the transformations (B.1a)–(B.1b). If so, such theories should be considered mathematically equivalent by means of the generalized conformal transformation combined with a diffeomorphism of the scalar field.

As it turned out, any hybrid scalar-tensor theory can be projected using \((\gamma_2, \gamma_3)\) functions to a theory which is fully metric, i.e. its coefficients satisfy the relation \(C_1(\Phi) = C_2(\Phi) = -A_2'(\Phi)\). The vectors \(Q^\mu\) and \(\bar{Q}^\mu\) built from non-metricity vanish on-shell and one gets a metric theory with the functions \((\bar{A}, \bar{B})\) given by (4.6) and (4.8). Conversely, if one is given a metric scalar-tensor theory, such as the Brans-Dicke theory in the metric approach, then it is impossible to reconstruct the hybrid theory equivalent to it without further specification of the value of the invariant \(I_A\). In other words, hybrid theories that are not mathematically equivalent, i.e. have different invariants, can be dynamically equivalent to the same metric scalar-tensor theory.

The class of STT considered in the present paper is not a particular case of more general theory with two scalar fields and arbitrary functional dependence \(F(R, \hat{R})\) presented in [103, 104] since it is singular in their terminology. It will be a task for our future investigation to analyze \(F(R, \hat{R})\) from the point of view of solution-equivalent classes and describe them in the form of invariants. Another future task would be to study more general nonminimal coupling which takes into account nonmetricity (cf. [105]). In this context, the idea of non-metricity driven inflation [106] should be reconsidered.

There are exactly-solvable cosmological models in metric STT for some special choices of potential functions (see e.g. [33, 107] and references therein). They can be used to generate, by applying conformal transformations (B.1a)–(B.1a), new exact solutions in Palatini and hybrid STT cases.

As it has been already mentioned, the solutions for the metric and for the scalar field in both frames (4.1) and (4.7) are exactly the same while ones for the connection change according to (4.5). This property can be used to capture some dark energy effects related with galactic curves [108, 109] or applied to stellar structure descriptions [110]–[113].

This shows that there is a renewing interest and ongoing activity in applications of Palatini STT in astrophysics, which is enforced due to the recent developments in solving dark matter, dark energy and cosmic inflation problems (cf. [69]–[81] and [114]–[120]).

Our finding has also practical meaning. As shown, when applicable, it allows the calculation and comparison of some inflationary observables based on the metric STT. Secondly, in order to solve equations of motion in an arbitrary frame it might be more convenient to find solutions for the metric and scalar field in the simpler projected metric frame or one
of the conformally equivalent frames and then transform them to the initial frame getting a solution for the connection directly from (4.4). So each metric ST model can be enriched by adding arbitrary nonmetricity, extending the Levi-Civita connection in a dynamical way. For these reasons, the formalism introduced here allows for getting better insight and deeper understanding of mutual relationships among different STT both on the operational as well the conceptual level.

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A Scalar field potential from $F(R)$ modified gravity

A.1 Few remarks on the Legendre transformation

As it was already mentioned, in the purely gravitational $F(R)$ action (2.1) the scalar field potential $U(\Phi)$ encoding information about the function $F(R)$ is given by

$$U_F(\Phi) \equiv U(\Phi) = R(\Phi)\Phi - F(R(\Phi))$$

(A.1)

where $\Phi = \frac{dF(R)}{dR}$ and $R(\Phi)$ denotes the inverse relation. More exactly, for a given $F$ the potential $U_F$ is a (singular) solution of the Clairaut’s differential equation [121]:

$$U(\Phi) = \Phi \frac{dU}{d\Phi} - F\left(\frac{dU}{d\Phi}\right).$$

(A.2)

In fact, due to the inverse function theorem, such differentiable solution exists around each point $F''(R) \neq 0$. A remarkable property of such solution (A.1) is that it can be always plotted on the $(\Phi, U)$-plane in the parametric form $R_1 < R < R_2$:

$$\Phi = F'(R),$$

(A.3)

$$U = RF'(R) - F(R)$$

(A.4)

even if an explicit functional dependence $U(\Phi)$ remains unknown. Conversely, having done the potential $U(\Phi)$ one can plot as well on the $(R, F)$-plane the corresponding $F(R)$ function $\Phi_1 < \Phi < \Phi_2$:

$$R = U'(\Phi),$$

(A.5)

$$F = \Phi U'(\Phi) - U(\Phi).$$

(A.6)

\footnote{\text{A family of real lines } U(\phi) = c\phi - F(c) \text{ evolving around the singular solution and parameterized by the integration constant } c \text{ consists of regular solutions.}}
Moreover, the functional transformation \( F(R) \mapsto U_F(\phi) \), a.k.a. the Legendre transform, possesses the following useful properties (which can be checked by straightforward calculations):

- It is involutive, i.e. it is own inverse. It means that the function \( F(R) \) is a Legendre transform of \( U_F(\Phi) \).
- Trivial, i.e. constant, potential \( U(\Phi) \) corresponds to the linear Lagrangian \( F(R) = bR + c \).
- More generally, for a given \( F(R) \) and the corresponding Legendre transform \( U_F(\Phi) \) one considers \( \tilde{F}(R) = aF(AR) + bR + c \), where \( A, a, b, c \) are numerical constants\(^8\) and \( \tilde{\Phi} = \frac{dF(R)}{dR} \). Then

\[
\tilde{U}_F(\Phi) = aU_F \left( \frac{\Phi - b}{Aa} \right) - c. \tag{A.7}
\]

- Similarly, modifying linearly the potential (A.7)

\[
\tilde{U}_F(\Phi) \rightarrow \tilde{U}_F(\Phi) + B\tilde{\Phi} = aU_F \left( \frac{\Phi - b}{Aa} \right) + B\tilde{\Phi} - c, \tag{A.8}
\]

one finds that it results from

\[
\tilde{F}(R) = aF(A(R - B)) + b(R - B) + c. \tag{A.9}
\]

- In the case of inverse function \( F^{-1} \) one finds

\[
U_{F^{-1}}(\Phi) = -\Phi U_F(\Phi^{-1}) \tag{A.10}
\]

or equivalently \( F_{U^{-1}}(R) = -RF_U(R^{-1}) \).

- Assuming \( F(R) = \int f^{-1}(R)dR \) we obtain

\[
U(\Phi) = \int f(\Phi)d\Phi \tag{A.11}
\]

and vice versa.

For the purpose of this section it is convenient to introduce the following terminology: *We say that two functions are weakly equivalent \( F \sim \tilde{F} \) if they differ by the linear transformation (both dependent and independent variables) in the form (A.8).*

*We say that two functions are weakly related by the Legendre transformation \( F \rightsquigarrow \tilde{U} \) if \( F \mapsto U \) and \( U \sim \tilde{U} \).*

\(^8\) \( c = -2\Lambda \) plays a role of cosmological constant.
A.2 Some viable example

Our purpose now is to illustrate how the above works on some concrete examples.\footnote{Shortcut symbols, e.g. LFI, DWI, refers to table I on pages 14-16 in \cite{22}, listing viable inflationary potentials discussed later in the paper.}

**Example 1.** (Power law Lagrangian) For $F(R) = R^p$, $p \neq 1, 0$ we get $U(\Phi) = \frac{1}{q-1} \left( \frac{q-1}{q} \Phi \right)^q$, where $q = \frac{p}{p-1}$, \footnote{This is equivalent to $p = \frac{1}{q-1}$ or in more symmetric form $\frac{1}{q} + \frac{1}{q} = 1$. This expression is involutive and possesses the following asymptotic: $p \to 0^\pm$ if $q \to 0^\mp$; $p \to 1^\pm$ if $q \to \pm\infty$.} \(\text{(see also DWI)}\). The more general form is $F(R) = a(A(R-B))^p + b(R-B) + c$ with the potential $U(\Phi) = \frac{a}{q-1} \left( \frac{q-1}{q} \Phi - \frac{b}{a} \right)^q - 1 + b \Phi - c$. These cover the cases: LFI, SFI, CSI, IMI, BI, UHI, DSI from \cite{22}. In particular, taking Starobinsky type Lagrangian $F(R) = R - 2\Lambda + \gamma R^2$ one finds $U(\Phi) = \frac{1}{2\gamma} (\Phi - 1)^2 + 2\Lambda$ (see also DWI).

**Example 2.** The exponential function $F(R) = e^R$ provides the logarithmic potential $U(\Phi) = \Phi(\ln \Phi - 1)$. Thus $F(R) = ae^{A(R-B)} + b(R-B) + c$ leads to $U(\Phi) = \frac{2}{A} (\ln(\frac{\Phi - b}{A}) - 1) + b \Phi - c$. It should be mentioned that the exponential gravity model has been already proposed in several papers (see e.g. \cite{21}).

**Example 3.** Conversely, exchanging $F$ and $U$ in the above example one gets: $F(R) = aA(R-B)(\ln(A(R-B)) - 1) + c$ leads to $U(\Phi) = ae^{\frac{(\Phi-b)}{A}} + b \Phi - c$ (cf. RCHI, ESI, PLI).

**Example 4.** Replacing exp $R$ by its inverse one gets $U_{\ln R}(\Phi) = 1 + \ln \Phi$ (cf. WRI).

**Example 5.** If $F(R) = (R-1)e^R$ then $U(\Phi) = \Phi(W(\Phi) - 1) + \frac{\Phi}{W(\Phi)}$, where $W$ denotes the Lambert $W$-function. This generalizes to $F(R) = (R-1)e^R$ and $U(\Phi) = \Phi(W(\Phi) - 1) + \frac{\Phi}{W(\Phi)} = \Phi(W(\Phi) - 1) + \exp W(\Phi)$.

**Example 6.** It is possible to generalize the above case: $F(R) = e^{pW(R)} \left(W(R) + \frac{1}{q}\right)$ gives $U(\Phi) = \left(\frac{\Phi}{p}\right)^q \left(\ln\left(\frac{\Phi}{p}\right) - \frac{1}{q}\right)$, where $\frac{1}{p} + \frac{1}{q} = 1$; $p, q \neq 1, 0$ (see e.g. RCHI, OSTI). After the field redefinition $\phi = q \ln \left(\frac{\Phi}{p}\right)$ we arrive to $U(\phi) = \frac{1}{q} e^{\phi}(\phi - 1)$.

**Example 7.** Conversely, taking $U(\Phi) = e^{pW(\Phi)} \left(W(\Phi) + \frac{1}{q}\right)$ one gets $F(R) = \left(\frac{\Phi}{p}\right)^q \left(\ln\left(\frac{R}{p} - 1\right) - \frac{1}{q}\right)$.

**Example 8.** $F(R) = R \arcsin R + \sqrt{1 - R^2}$ gives $U(\Phi) = -\cos \Phi$ (cf. NI). Thus, the inverse $\bar{U}(\Phi) = \arccos(-\Phi)$ is obtained from $F(R) = -\sqrt{R^2 - 1} - \arcsin \frac{1}{R}$.

**Example 9.** $F(R) = R \ln(R + \sqrt{1 + R^2}) - \sqrt{1 + R^2}$ gives $U(\Phi) = \cosh \Phi$. Thus, the inverse $\bar{U}(\Phi) = \text{arcCosh} \Phi$ is obtained from $F(R) = \sqrt{1 + R^2} - \text{arcSinh} \frac{1}{R}$.

**Example 10.** In order to find out $F(R)$-lagrangian for the Higgs potential $U(\Phi) = (\Phi^2 - v^2)^2$ (see DWI) one has to solve a quibic algebraic equation in the form

$$\Phi^3 - v^2 \Phi - \frac{R}{4} = 0.$$
Seeking real solutions of this equation one has to distinguish two cases. Either \( R > \frac{8v^3}{3\sqrt{3}} \) and
\[
\Phi = \frac{1}{2} \left[ R + \sqrt{R^2 - \frac{64}{27}v^6} \right] \frac{1}{3} + \frac{1}{2} \left[ R - \sqrt{R^2 - \frac{64}{27}v^6} \right] \frac{1}{3} = \frac{2v}{\sqrt{3}} \cosh \left( \frac{1}{3} \ln \frac{3\sqrt{3}(R + \sqrt{R^2 - \frac{64}{27}v^6})}{8v^3} \right)
\]
or \( R < \frac{8v^3}{3\sqrt{3}} \) and \( \Phi = \frac{2v}{\sqrt{3}} \cos \left( \frac{1}{3} \left[ \arccos \frac{3\sqrt{3}R}{8v^3} + 2k\pi \right] \right) \), \( k = 0, 1, 2 \). Thus \( f(R) = \int \Phi(R) dR \), where \( \Phi \) is given by one of the formulas above.

Alternatively, we can start with the quadratic (Starobinsky) action and then perform the field redefinition (B.1c): \( \Phi \rightarrow \Phi^2 \). In such case one has to change the other frame parameters as well (cf. (B.2)).

**Example 11.** (Nojiri-Odintsov [18])\(^{11}\) \( F(R) = \frac{1}{2}R^2 + \frac{\alpha}{R} \). Then \( R^3 - \Phi R^2 - \alpha = 0 \) and
\[
R = \frac{\Phi}{3} + \left[ \frac{\Phi^3}{27} + \frac{\alpha}{2} + \sqrt{\frac{\alpha \Phi^3}{27} + \frac{\alpha^2}{4}} \right] \frac{1}{3} + \left[ \frac{\Phi^3}{27} + \frac{\alpha}{2} - \sqrt{\frac{\alpha \Phi^3}{27} + \frac{\alpha^2}{4}} \right] \frac{1}{3},
\]
which in the limit \( \alpha \rightarrow 0 \) gives \( R = \Phi \). The explicit form of the potential \( U(\Phi) = \int R(\Phi) d\Phi \) is rather complicated for \( \alpha \neq 0 \).

**Example 12.** (Hu-Sawicki [19])\(^{12}\) This general class of Lagrangians is given by \( F(R) = R + \frac{\alpha R^n}{\beta + R^n} \sim -(\beta + R^n)^{-1} \sim U(\Phi) = \frac{\beta + (n-1)R^n}{\beta + R^n} \), where \( \Phi = nR^{n-1}(\beta + R^n)^{-2} \). Particularly, for \( n = 1 \) we get \( U(\Phi) = 2\sqrt{\Phi} - \beta \Phi \) while for \( n = 2 \) one has to solve the quartic equation \( R^4 + 2\beta R^2 - \frac{2}{\Phi} R + \beta^2 = 0 \) (see also RGI for inverse relation).

**Example 13.** (Tsujikawa [20]) Consider \( F(R) = \tanh R = \int \frac{dR}{\cosh^2 R} \) then \( U(\Phi) = \int \arccosh \frac{1}{\sqrt{1 - \Phi}} \).

### B Transformation formulae in hybrid metric-Palatini scalar-tensor theories of gravity

Unlike in the metric approach, where the connection is Levi-Civita w.r.t. metric tensor, and the conformal change of the latter results in a transformation of the former, in the Palatini formalism one needs to transform these two objects separately. We postulate the following transformation formulae, defined by three functions \( \gamma_i(\Phi) \) and an additional diffeomorphism of the scalar field, for the variables entering the action functional:

\[
\begin{align*}
\bar{g}_{\mu\nu} &= e^{2\gamma_1(\Phi)} g_{\mu\nu}, \\
\bar{\Gamma}^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + 2\delta^\alpha_{(\mu} \partial_{\nu)} \gamma_2(\Phi) - g_{\mu\nu} g^{\alpha\beta} \partial_\beta \gamma_3(\Phi), \\
\Phi &= f(\Phi).
\end{align*}
\]

\(^{11}\)In the most general form the Lagrangian is determined by \( F(R) = R + \alpha R^n + \beta R^{-n} \).
\(^{12}\)These two classes of lagrangians [19] and [18] are shown to pass the Solar system tests.
following way:

\[
\begin{align*}
\mathcal{A}_1(\bar{\Phi}) &= e^{(n-2)\gamma_1(\Phi)} A_1(\bar{f}(\bar{\Phi})), \quad \mathcal{A}_2(\bar{\Phi}) = e^{(n-2)\gamma_1(\Phi)} A_2(\bar{f}(\bar{\Phi})) \\
\mathcal{B}(\bar{\Phi}) &= e^{(n-2)\gamma_1(\Phi)} \left[ \mathcal{B}(\bar{f}(\bar{\Phi})) + (n-1) \left( n A_2(\bar{f}(\bar{\Phi})) \gamma_2^{(\Phi)} + A_2(\bar{f}(\bar{\Phi})) (\gamma_2^{(\Phi)})^2 ight) 
- A_2(\bar{f}(\bar{\Phi})) (\gamma_2^{(\Phi)})^2 - 2 \frac{d A_1(\bar{f}(\bar{\Phi}))}{d\Phi} (\gamma_2^{(\Phi)}) - 2 \frac{d A_1(\bar{f}(\bar{\Phi}))}{d\Phi} (\gamma_3^{(\Phi)}) 
- (n-2) A_2(\bar{f}(\bar{\Phi})) (\gamma_2^{(\Phi)}) (\gamma_3^{(\Phi)}) + (n-2) A_1(\bar{f}(\bar{\Phi})) (\gamma_1^{(\Phi)}) (\gamma_2^{(\Phi)})^2 
+ \bar{f}'(\bar{\Phi}) \left( C_1(\bar{f}(\bar{\Phi})) (2 n \gamma_1^{(\Phi)} - 2(n+1)\gamma_2^{(\Phi)} + 2 \gamma_3^{(\Phi)}) 
- C_2(\bar{f}(\bar{\Phi})) (2 \gamma_1^{(\Phi)} - (n+3)\gamma_2^{(\Phi)} + (n+1)\gamma_3^{(\Phi)}) \right) \right], \\
\mathcal{C}_1(\bar{\Phi}) &= e^{(n-2)\gamma_1(\Phi)} \left[ f'(\bar{\Phi}) C_1(\bar{f}(\bar{\Phi})) - A_2(\bar{f}(\bar{\Phi})) \left( \frac{n-1}{2} \gamma_2^{(\Phi)} + \frac{n-3}{2} \gamma_3^{(\Phi)} \right) \right], \\
\mathcal{C}_2(\bar{\Phi}) &= e^{(n-2)\gamma_1(\Phi)} \left[ f'(\bar{\Phi}) C_2(\bar{f}(\bar{\Phi})) - A_2(\bar{f}(\bar{\Phi})) ( (n-1) \gamma_2^{(\Phi)} - \gamma_3^{(\Phi)} ) \right], \\
\mathcal{V}(\bar{\Phi}) &= e^{n\gamma_1(\Phi)} V(\bar{f}(\bar{\Phi})), \\
\bar{\alpha}(\bar{\Phi}) &= \alpha(\bar{f}(\bar{\Phi})),
\end{align*}
\]

preserving the form of the action, where \( \bar{f}(\bar{\Phi}) = \Phi \) and \( \bar{\gamma}(\bar{\Phi}) = -\gamma(\bar{f}(\bar{\Phi})) \) and \( \bar{\gamma}'(\bar{\Phi}) = \frac{d\bar{\gamma}(\bar{\Phi})}{d\Phi} \), etc. (cf. [1] for more detailed explanation from a group-theoretical point of view).

In this way, the formulae above proclaim the invariance of the action (4.1) under the transformations (B.1a)–(B.1c) establishing the mathematical (or solution) equivalence between transformed frames. It means that changing the frame \( (A_1, A_2, B, C_1, C_2, V, \alpha) \) to \( (\bar{A}_1, \bar{A}_2, \bar{B}, \bar{C}_1, \bar{C}_2, \bar{V}, \bar{\alpha}) \) according to (B.2a)–(B.2f), one should transform solutions of the corresponding field equations by the formulas (B.1a)–(B.1c).

One should distinguish three cases.

Setting \( A_2 = C_1 = C_2 = 0 \) into the action as well as in the formulae above one reconstructs well-known metric scalar-tensor theories which contain metric \( F(R) \)-subclass. (the formalism introduced in [85], slightly generalized to arbitrary dimension \( n > 2 \) [88]). In this case the transformation (B.1b) is not active.

Similarly, setting \( A_1 = 0 \) one finds Palatini scalar-tensor theories introduced in [1] which contain \( F(R) \)-subclass.

The most general case with \( A_1, A_2 \neq 0 \) which contains hybrid \( f(R) \)-subclass has not been studied before. Moreover, it has been shown that any generalized frame \( (A_1, A_2, B, C_1, C_2, V, \alpha) \) is on-shell solution equivalent to the purely metric frame \( \{A, B, V, \alpha \} \) with \( A = A_1 + A_2 \) and \( B \) given by the formulae (4.6).

Analogously to both metric and Palatini cases, it is convenient to introduce invariant quantities, i.e. quantities such that their functional form is independent of the conformal frame we are using:

\[
\begin{align*}
\mathcal{I}_A(\Phi) &= \frac{A_1(\Phi)}{A_2(\Phi)}, \\
\mathcal{I}_V^{(1)}(\Phi) &= \frac{V(\Phi)}{(A_1(\Phi))^\frac{n}{2}}, \quad \mathcal{I}_V^{(2)}(\Phi) = \frac{V(\Phi)}{(A_2(\Phi))^\frac{n}{2}}, \\
\mathcal{I}_\alpha^{(1)}(\Phi) &= \frac{A_1(\Phi)}{e^{(n-2)\alpha(\Phi)}}, \quad \mathcal{I}_\alpha^{(2)}(\Phi) = \frac{A_2(\Phi)}{e^{(n-2)\alpha(\Phi)}},
\end{align*}
\]
and also an integral invariant generalizing (3.5) (we assume $A_1(\Phi) + A_2(\Phi) > 0$):\(^\text{13}\)

\[
\mathcal{I}(\Phi) = \int_{\Phi_0}^{\Phi} \frac{d\Phi'}{(A_1(\Phi') + A_2(\Phi'))} \left[ (n-1)(n-2)B(\Phi')(A_1(\Phi') + A_2(\Phi')) \\
+ (1+\mathcal{I}_{A}^{-1}(\Phi')) \left[ (n-1)A_1(\Phi')^2 + (\mathcal{I}_B(\Phi') + 1) \left[ -4C_1(\Phi') + (n^2 - 5)C_2(\Phi') \right] \\
- 2(n^2 - n - 4)C_1(\Phi')C_2(\Phi') + 2(n-1)A_2(\Phi')(C_2(\Phi') - nC_1(\Phi')) \right] \right]^{\frac{1}{2}}.
\]

One can notice that not all invariants are independent and some of them might be singular. However in the limiting cases ($A_2 = 0$ or $A_1 = 0$) they reproduce correspondingly the metric or Palatini ones. In fact, the integral invariant can be extend further to two parameter family ($a_1, a_2 \in \mathbb{R}$):

\[
\mathcal{I}^{(a_1,a_2)}(\Phi) = \int_{\Phi_0}^{\Phi} \frac{d\Phi'}{(a_1A_1(\Phi') + a_2A_2(\Phi'))} \left[ (n-1)(n-2)B(\Phi')(a_1A_1(\Phi') + a_2A_2(\Phi')) \\
+ (a_1 + a_2\mathcal{I}_{A}^{-1}(\Phi')) \left[ (n-1)A_1(\Phi')^2 + (a_1\mathcal{I}_B(\Phi') + a_2) \left[ -4C_1(\Phi') + (n^2 - 5)C_2(\Phi') \right] \\
- 2(n^2 - n - 4)C_1(\Phi')C_2(\Phi') + 2(n-1)A_2(\Phi')(C_2(\Phi') - nC_1(\Phi')) \right] \right]^{\frac{1}{2}}.
\]

References

[1] A. Kozak and A. Borowiec, Palatini frames in scalar-tensor theories of gravity, *Eur. Phys. J. C* 79 (2019) 335 [arXiv:1808.05598] [csSPIRE].

[2] S. Capozziello and M. De Laurentis, Extended Theories of Gravity, *Phys. Rept.* 509 (2011) 167 [arXiv:1108.6266] [csSPIRE].

[3] T.P. Sotiriou and V. Faraoni, $f(R)$ Theories Of Gravity, *Rev. Mod. Phys.* 82 (2010) 451 [arXiv:0805.1726] [csSPIRE].

[4] A. De Felice and S. Tsujikawa, $f(R)$ theories, *Living Rev. Rel.* 13 (2010) 3 [arXiv:1002.4928] [csSPIRE].

[5] S. Capozziello and M. Francaviglia, Extended Theories of Gravity and their Cosmological and Astrophysical Applications, *Gen. Rel. Grav.* 40 (2008) 357 [arXiv:0706.1146] [csSPIRE].

[6] S. Carloni, P.K.S. Dunsby, S. Capozziello and A. Troisi, Cosmological dynamics of $R^m$ gravity, *Class. Quant. Grav.* 22 (2005) 4839 [gr-qc/0410046] [csSPIRE].

[7] T. Clifton, P.G. Ferreira, A. Padilla and C. Skordis, Modified Gravity and Cosmology, *Phys. Rept.* 513 (2012) 1 [arXiv:1106.2476] [csSPIRE].

[8] S. Nojiri and S.D. Odintsov, Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models, *Phys. Rept.* 505 (2011) 59 [arXiv:1011.0544] [csSPIRE].

[9] S. Nojiri, S.D. Odintsov and V.K. Oikonomou, Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution, *Phys. Rept.* 692 (2017) 1 [arXiv:1705.11098] [csSPIRE].

[10] S.M. Carroll, V. Duvvuri, M. Trodden and M.S. Turner, Is cosmic speed-up due to new gravitational physics?, *Phys. Rev. D* 70 (2004) 043528 [astro-ph/0306438] [csSPIRE].

\(^\text{13}\) We have a freedom in choosing integration constant and normalization condition.
[11] K. Bamba, S. Capozziello, S. Nojiri and S.D. Odintsov, Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests, *Astrophys. Space Sci.* **342** (2012) 155 [arXiv:1205.3421] [inSPIRE].

[12] G. Allemandi, A. Borowiec and M. Francaviglia, Accelerated cosmological models in first order nonlinear gravity, *Phys. Rev. D* **70** (2004) 043524 [hep-th/0403264] [inSPIRE].

[13] G. Allemandi, A. Borowiec, M. Francaviglia and S.D. Odintsov, Dark energy dominance and cosmic acceleration in first order formalism, *Phys. Rev. D* **72** (2005) 063505 [gr-qc/0504057] [inSPIRE].

[14] S. Nojiri and S.D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, [hep-th/0601213] [inSPIRE].

[15] T. Clifton and P.K.S. Dunsby, On the Emergence of Accelerating Cosmic Expansion in \( f(R) \) Theories of Gravity, *Phys. Rev. D* **91** (2015) 103528 [arXiv:1501.04004] [inSPIRE].

[16] D.N. Vollick, \( 1/R \) Curvature corrections as the source of the cosmological acceleration, *Phys. Rev. D* **68** (2003) 063510 [astro-ph/0306630] [inSPIRE].

[17] A.A. Starobinsky, A New Type of Isotropic Cosmological Models Without Singularity, *Adv. Ser. Astrophys. Cosmol.* **3** (1987) 130 [inSPIRE].

[18] S. Nojiri and S.D. Odintsov, Modified gravity with negative and positive powers of the curvature: Unification of the inflation and of the cosmic acceleration, *Phys. Rev. D* **68** (2003) 123512 [hep-th/0307288] [inSPIRE].

[19] W. Hu and I. Sawicki, Models of \( f(R) \) Cosmic Acceleration that Evade Solar-System Tests, *Phys. Rev. D* **76** (2007) 064004 [arXiv:0705.1158] [inSPIRE].

[20] S. Tsujikawa, Observational signatures of \( f(R) \) dark energy models that satisfy cosmological and local gravity constraints, *Phys. Rev. D* **77** (2008) 023507 [arXiv:0709.1391] [inSPIRE].

[21] S.D. Odintsov, D. Sáez-Chillón Gómez and G.S. Sharov, Is exponential gravity a viable description for the whole cosmological history?, *Eur. Phys. J. C* **77** (2017) 862 [arXiv:1709.06800] [inSPIRE].

[22] J. Martin, C. Ringeval and V. Vennin, *Encyclopædia Inflationaris*, *Phys. Dark Univ.* **5-6** (2014) 75 [arXiv:1303.3787] [inSPIRE].

[23] M. Rinaldi, G. Cognola, L. Vanzo and S. Zerbini, Reconstructing the inflationary \( f(R) \) from observations, *JCAP* **08** (2014) 015 [arXiv:1406.1096] [inSPIRE].

[24] M. Artyomowski and Z. Lalak, Inflation and dark energy from \( f(R) \) gravity, *JCAP* **09** (2014) 036 [arXiv:1405.7818] [inSPIRE].

[25] H. Sami, J. Ntahompagaze and A. Abebe, Inflationary \( f(R) \) Cosmologies, *Universe* **3** (2017) 73 [arXiv:1709.04860] [inSPIRE].

[26] T. Tenkanen, Resurrecting Quadratic Inflation with a non-minimal coupling to gravity, *JCAP* **12** (2017) 001 [arXiv:1710.02758] [inSPIRE].

[27] P. Carrilho, D. Mulryne, J. Ronayne and T. Tenkanen, Attractor Behaviour in Multifield Inflation, *JCAP* **06** (2018) 032 [arXiv:1804.10489] [inSPIRE].

[28] J.P.B. Almeida, N. Bernal, J. Rubio and T. Tenkanen, Hidden Inflaton Dark Matter, *JCAP* **03** (2019) 012 [arXiv:1811.09640] [inSPIRE].

[29] T. Takahashi and T. Tenkanen, Towards distinguishing variants of non-minimal inflation, *JCAP* **04** (2019) 035 [arXiv:1812.08492] [inSPIRE].

[30] T.P. Sotiriou, \( f(R) \) gravity and scalar-tensor theory, *Class. Quant. Grav.* **23** (2006) 5117 [gr-qc/0604028] [inSPIRE].
[31] P. Teyssandier and P. Tourrenc, *The Cauchy problem for the $R + R^2$ theories of gravity without torsion*, *J. Math. Phys.* 24 (1983) 2793 [SPIRE].

[32] S. Capozziello, M. De Laurentis and V. Faraoni, *A Bird’s eye view of $f(R)$-gravity*, *Open Astron. J.* 3 (2010) 49 [arXiv:0909.4672] [SPIRE].

[33] V. Faraoni, *Cosmology in scalar-tensor gravity*, Kluwer Academic Publisher (2004).

[34] V. Faraoni, *$f(R)$ gravity: Successes and challenges*, in 18th SIGRAV Conference, 10, 2008, arXiv:0810.2602 [SPIRE].

[35] H. Motohashi and A.A. Starobinsky, *Constant-roll inflation in scalar-tensor gravity*, *JCAP* 11 (2019) 025 [arXiv:1909.10883] [SPIRE].

[36] H. Motohashi and A.A. Starobinsky, *$f(R)$ constant-roll inflation*, *Eur. Phys. J. C* 77 (2017) 538 [arXiv:1704.08188] [SPIRE].

[37] G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani and S. Zerbini, *A Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion*, *Phys. Rev. D* 77 (2008) 046009 [arXiv:0712.4017] [SPIRE].

[38] S. Nojiri and S.D. Odintsov, *Modified $f(R)$ gravity consistent with realistic cosmology: From matter dominated epoch to dark energy universe*, *Phys. Rev. D* 74 (2006) 086005 [hep-th/0608008] [SPIRE].

[39] S. Nojiri and S.D. Odintsov, *Modified $f(R)$ gravity unifying $R^n$ inflation with Lambda CDM epoch*, *Phys. Rev. D* 77 (2008) 026007 [arXiv:0710.1738] [SPIRE].

[40] S. Capozziello, C. Corda and M.F. De Laurentis, *Massive gravitational waves from $f(R)$ theories of gravity: Potential detection with LISA*, *Phys. Lett. B* 669 (2008) 255 [arXiv:0812.2272] [SPIRE].

[41] A. Borowiec, W. Godlowski and M. Szydlowski, *Accelerated cosmological models in modified gravity tested by distant supernovae snia data*, *Phys. Rev. D* 74 (2006) 043502 [astro-ph/0602526] [SPIRE].

[42] B. Li, K.-C. Chan and M.-C. Chu, *Constraints on $f(R)$ Cosmology in the Palatini Formalism*, *Phys. Rev. D* 76 (2007) 024002 [astro-ph/0610794] [SPIRE].

[43] A. Borowiec, W. Godlowski and M. Szydlowski, *Dark matter and dark energy as a effects of Modified Gravity*, *eConf C* 0602061 (2006) 09 [astro-ph/0607639] [SPIRE].

[44] A. Borowiec, M. Kamionka, A. Kurek and M. Szydlowski, *Cosmic acceleration from modified gravity with Palatini formalism*, *JCAP* 02 (2012) 027 [arXiv:1109.3420] [SPIRE].

[45] A. Borowiec, A. Stachowski, M. Szydlowski and A. Wojnar, *Inflationary cosmology with Chaplygin gas in Palatini formalism*, *JCAP* 01 (2016) 040 [arXiv:1512.01199] [SPIRE].

[46] A. Stachowski, M. Szydlowski and A. Borowiec, *Starobinsky cosmological model in Palatini formalism*, *Eur. Phys. J. C* 77 (2017) 406 [arXiv:1608.03196] [SPIRE].

[47] G.J. Olmo, *Palatini Approach to Modified Gravity: $f(R)$ Theories and Beyond*, *Int. J. Mod. Phys. D* 20 (2011) 413 [arXiv:1101.3864] [SPIRE].

[48] S. Tsujikawa, K. Uddin and R. Tavakol, *Density perturbations in $f(R)$ gravity theories in metric and Palatini formalisms*, *Phys. Rev. D* 77 (2008) 043007 [arXiv:0712.0082] [SPIRE].

[49] K. Uddin, J.E. Lidsey and R. Tavakol, *Cosmological perturbations in Palatini modified gravity*, *Class. Quant. Grav.* 24 (2007) 3951 [arXiv:0705.0232] [SPIRE].

[50] B. Li and M.-C. Chu, *CMB and Matter Power Spectra of Early $f(R)$ Cosmology in Palatini Formalism*, *Phys. Rev. D* 74 (2006) 104010 [astro-ph/0610486] [SPIRE].
51] G. Allemandi, M. Capone, S. Capozziello and M. Francaviglia, Conformal aspects of Palatini approach in extended theories of gravity, Gen. Rel. Grav. 38 (2006) 33 [hep-th/0409198] [inSPIRE].
52] S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo and G.J. Olmo, Hybrid metric-Palatini gravity, Universe 1 (2015) 199.
53] S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo and G.J. Olmo, Cosmology of hybrid metric-Palatini f(X)-gravity, JCAP 04 (2013) 011 [arXiv:1209.2895] [inSPIRE].
54] N.A. Lima and V. Smer-Barreto, Constraints on hybrid metric-Palatini models from background evolution, arXiv:1501.05786.
55] A. Borowiec et al., Invariant solutions and Noether symmetries in Hybrid Gravity, Phys. Rev. D 91 (2015) 023517 [arXiv:1407.4313] [inSPIRE].
56] N.A. Lima, Dynamics of Linear Perturbations in the hybrid metric-Palatini gravity, Phys. Rev. D 89 (2014) 083527 [arXiv:1402.4458] [inSPIRE].
57] C.G. Böhmer, F.S.N. Lobo and N. Tamanini, Einstein static Universe in hybrid metric-Palatini gravity, Phys. Rev. D 88 (2013) 104019 [arXiv:1305.0025] [inSPIRE].
58] S. Capozziello and S. Tsujikawa, Solar system and equivalence principle constraints on f(R) gravity by chameleon approach, Phys. Rev. D 77 (2008) 107501 [arXiv:0712.2268] [inSPIRE].
59] J. Khoury and A. Weltman, Chameleon cosmology, Phys. Rev. D 77 (2008) 107501 [arXiv:0712.2268] [inSPIRE].
60] U. Lindström, The Palatini Variational Principle and a Class of Scalar-Tensor Theories, Nuovo Cim. B 35 (1976) 130 [inSPIRE].
61] U. Lindström and M. Roček, A Gravitational First Order Action for the Bosonic String, Class. Quant. Grav. 4 (1987) L79 [inSPIRE].
62] Y. Fujii and K.M. Maeda, The Scalar-Tensor Theory of Gravitation, Cambridge University Press (2003).
63] A. Iglesias, N. Kaloper, A. Padilla and M. Park, How (Not) to Palatini of scalar-tensor gravity, Phys. Rev. D 76 (2007) 104001 [arXiv:0708.1163] [inSPIRE].
64] F. Bauer, Filtering out the cosmological constant in the Palatini formalism of modified gravity, Gen. Rel. Grav. 43 (2011) 1733 [arXiv:1007.2546] [inSPIRE].
65] P. Wang, P. Wu and H. Yu, A new extended quintessence, Eur. Phys. J. C 72 (2012) 2245 [arXiv:1301.5832] [inSPIRE].
66] A. Racioppi, Coleman-Weinberg linear inflation: metric vs. Palatini formulation, JCAP 12 (2017) 041 [arXiv:1710.04853] [inSPIRE].
67] A. Racioppi, New universal attractor in nonminimally coupled gravity: Linear inflation, Phys. Rev. D 97 (2018) 123514 [arXiv:1801.08810] [inSPIRE].
68] L. Järv, A. Racioppi and T. Tenkanen, Palatini side of inflationary attractors, Phys. Rev. D 97 (2018) 083513 [arXiv:1712.08471] [inSPIRE].

– 23 –
[72] F. Bauer and D.A. Demir, *Inflation with Non-Minimal Coupling: Metric versus Palatini Formulations*, Phys. Lett. B **665** (2008) 222 [arXiv:0803.2664] [INSPIRE].

[73] S. Rasanen and P. Wahlman, *Higgs inflation with loop corrections in the Palatini formulation*, JCAP **11** (2017) 047 [arXiv:1709.07853] [INSPIRE].

[74] I. Antoniadis, A. Karam, A. Lykkas and K. Tamvakis, *Palatini inflation in models with an $R^2$ term*, JCAP **11** (2018) 028 [arXiv:1810.10418] [INSPIRE].

[75] S. Rasanen, *Higgs inflation in the Palatini formulation with kinetic terms for the metric*, Open J. Astrophys. **2** (2019) 1 [arXiv:1811.09514] [INSPIRE].

[76] V.-M. Enckell, K. Enqvist, S. Rasanen and L.-P. Wahlman, *Inflation with $R^2$ term in the Palatini formalism*, JCAP **02** (2019) 022 [arXiv:1810.05536] [INSPIRE].

[77] T. Markkanen, T. Tenkanen, V. Vaskonen and H. Veermäe, *Quantum corrections to quartic inflation with a non-minimal coupling: metric vs. Palatini*, JCAP **03** (2018) 029 [arXiv:1712.04874] [INSPIRE].

[78] N. Tamanini and C.R. Contaldi, *Inflationary Perturbations in Palatini Generalised Gravity*, Phys. Rev. D **83** (2011) 044018 [arXiv:1010.0689] [INSPIRE].

[79] D. Gal’tsov and S. Zhidkova, *Ghost-free Palatini derivative scalar-tensor theory: Desingularization and the speed test*, Phys. Lett. B **790** (2019) 453 [arXiv:1808.00492] [INSPIRE].

[80] A. Racioppi, *Non-Minimal (Self-)Running Inflation: Metric vs. Palatini Formulation*, arXiv:1912.10038 [INSPIRE].

[81] K. Shimada, K. Aoki and K.-i. Maeda, *Metric-affine Gravity and Inflation*, Phys. Rev. D **99** (2019) 104020 [arXiv:1810.05536] [INSPIRE].

[82] D.D. Canko, I.D. Gialamas and G.P. Kodaxis, *A simple $F(R, \phi)$ deformation of Starobinsky inflationary model*, Eur. Phys. J. C **80** (2020) 458 [arXiv:1901.06296] [INSPIRE].

[83] I. Antoniadis, A. Karam, A. Lykkas, T. Pappas and K. Tamvakis, *Rescuing Quartic and Natural Inflation in the Palatini Formalism*, JCAP **03** (2019) 005 [arXiv:1812.00847] [INSPIRE].

[84] T. Koivisto, *Covariant conservation of energy momentum in modified gravities*, Class. Quant. Grav. **23** (2006) 4289 [gr-qc/0505128] [INSPIRE].

[85] L. Järvi, P. Kuusk, M. Saal and O. Vilson, *Invariant quantities in the scalar-tensor theories of gravitation*, Phys. Rev. D **91** (2015) 024041 [arXiv:1411.1947] [INSPIRE].

[86] L. Järvi, P. Kuusk, M. Saal and O. Vilson, *Transformation properties and general relativity regime in scalar-tensor theories*, Class. Quant. Grav. **32** (2015) 235013 [arXiv:1504.02686] [INSPIRE].

[87] E.E. Flanagan, *The Conformal frame freedom in theories of gravitation*, Class. Quant. Grav. **21** (2004) 3817 [gr-qc/0403063] [INSPIRE].

[88] A. Karam, T. Pappas and K. Tamvakis, *Frame-dependence of higher-order inflationary observables in scalar-tensor theories*, Phys. Rev. D **96** (2017) 064036 [arXiv:1707.00984] [INSPIRE].

[89] S. Capozziello, S. Nojiri, S.D. Odintsov and A. Troisi, *Cosmological viability of $f(R)$-gravity as an ideal fluid and its compatibility with a matter dominated phase*, Phys. Lett. B **639** (2006) 135 [astro-ph/0604431] [INSPIRE].

[90] S. Bahamonde, S.D. Odintsov, V.K. Oikonomou and P.V. Tretyakov, *Deceleration versus acceleration universe in different frames of $F(R)$ gravity*, Phys. Lett. B **766** (2017) 225 [arXiv:1701.02381] [INSPIRE].
[91] S. Bahamonde, S.D. Odintsov, V.K. Oikonomou and M. Wright, Correspondence of $F(R)$ Gravity Singularities in Jordan and Einstein Frames, Annals Phys. 373 (2016) 96 [arXiv:1603.05113] [inSPIRE].

[92] A. Karam, A. Lykkas and K. Tamvakis, Frame-invariant approach to higher-dimensional scalar-tensor gravity, Phys. Rev. D 97 (2018) 124036 [arXiv:1803.04960] [inSPIRE].

[93] V. Faraoni, E. Gunzig and P. Nardone, Conformal transformations in classical gravitational theories and in cosmology, Fund. Cosmic Phys. 20 (1999) 121 [gr-qc/9811047] [inSPIRE].

[94] P. Kuusk, M. Rünkla, M. Saal and O. Vilson, Invariant slow-roll parameters in scalar-tensor theories, Class. Quant. Grav. 33 (2016) 195008 [arXiv:1605.07033] [inSPIRE].

[95] V. Faraoni and E. Gunzig, Einstein frame or Jordan frame?, Int. J. Theor. Phys. 38 (1999) 217 [astro-ph/9910176] [inSPIRE].

[96] S. Capozziello, P. Martin-Moruno and C. Rubano, Physical non-equivalence of the Jordan and Einstein frames, Phys. Lett. B 689 (2010) 117 [arXiv:1003.5394] [inSPIRE].

[97] N. Banerjee and B. Majumder, A question mark on the equivalence of Einstein and Jordan frames, Phys. Lett. B 754 (2016) 129 [arXiv:1601.06152] [inSPIRE].

[98] V. Faraoni and S. Nadeau, The (pseudo)issue of the conformal frame revisited, Phys. Rev. D 75 (2007) 023501 [gr-qc/0612075] [inSPIRE].

[99] X. Calmet and T.-C. Yang, Frame Transformations of Gravitational Theories, Int. J. Mod. Phys. A 28 (2013) 1350042 [arXiv:1211.4217] [inSPIRE].

[100] A.Y. Kamenshchik and C.F. Steinwachs, Question of quantum equivalence between Jordan frame and Einstein frame, Phys. Rev. D 91 (2015) 084033 [arXiv:1408.5769] [inSPIRE].

[101] D. Burns, S. Karamitsos and A. Pilaftsis, Frame-Covariant Formulation of Inflation in Scalar-Curvature Theories, Nucl. Phys. B 907 (2016) 785 [arXiv:1603.03730] [inSPIRE].

[102] Planck collaboration, Planck 2015 results. XX. Constraints on inflation, Astron. Astrophys. 594 (2016) A20 [arXiv:1502.02114] [inSPIRE].

[103] J.L. Rosa, S. Carloni, J.P. d. S.e. Lemos and F.S.N. Lobo, Cosmological solutions in generalized hybrid metric-Palatini gravity, Phys. Rev. D 95 (2017) 124035 [arXiv:1703.03336] [inSPIRE].

[104] J.L. Rosa, Cosmological and astrophysical applications of modified theories of gravity, other thesis, 11, 2019 [arXiv:1911.08257] [inSPIRE].

[105] A. Delhom, Minimal coupling in presence of non-metricity and torsion, arXiv:2002.02404.

[106] J. Stelmach, Nonmetricity driven inflation, Class. Quant. Grav. 8 (1991) 897 [inSPIRE].

[107] D. Fermi, M. Gengo and L. Pizzocchero, Integrable scalar cosmologies with matter and curvature, arXiv:2001.10448 [inSPIRE].

[108] C.A. Sporea, A. Borowiec and A. Wojnar, Galaxy Rotation Curves via Conformal Factors, Eur. Phys. J. C 78 (2018) 308 [arXiv:1705.04131] [inSPIRE].

[109] A. Wojnar, C.A. Sporea and A. Borowiec, A simple model for explaining Galaxy Rotation Curves, Galaxies 6 (2018) 70 [arXiv:1804.09620].

[110] A. Wojnar, Polytropic stars in Palatini gravity, Eur. Phys. J. C 79 (2019) 51 [arXiv:1808.04188] [inSPIRE].

[111] G.J. Olmo, D. Rubiera-Garcia and A. Wojnar, Minimum main sequence mass in quadratic Palatini $f(R)$ gravity, Phys. Rev. D 100 (2019) 044020 [arXiv:1906.04629] [inSPIRE].

[112] A. Sergyeyev and A. Wojnar, The Palatini star: exact solutions of the modified Lane-Emden equation, Eur. Phys. J. C 80 (2020) 313 [arXiv:1901.10448] [inSPIRE].
[113] G.J. Olmo, D. Rubiera-Garcia and A. Wojnar, Stellar structure models in modified theories of gravity: lessons and challenges, arXiv:1912.05202 [rsSPIRE].

[114] D. Coumbe, Asymptotically Weyl-Invariant Gravity, Int. J. Mod. Phys. A 34 (2019) 1950205 [arXiv:1910.05629] [rsSPIRE].

[115] A. Edery and Y. Nakayama, Palatini formulation of pure $R^2$ gravity yields Einstein gravity with no massless scalar, Phys. Rev. D 99 (2019) 124018 [arXiv:1902.07876] [rsSPIRE].

[116] I.D. Gialamas and A.B. Lahanas, Reheating in R2 Palatini inflationary models, arXiv:1911.11513.

[117] I. Antoniadis, A. Lykkas and K. Tamvakis, Constant-roll in the Palatini-$R^2$ models, JCAP 04 (2020) 033 [arXiv:2002.12681] [rsSPIRE].

[118] T. Tenkanen, Tracing the high energy theory of gravity: an introduction to Palatini inflation, Gen. Rel. Grav. 52 (2020) 33 [arXiv:2001.10135] [rsSPIRE].

[119] T. Tenkanen and E. Tomberg, Initial conditions for plateau inflation, arXiv:2002.02420.

[120] M. Shaposhnikov, A. Shkerin and S. Zell, Quantum Effects in Palatini Higgs Inflation, arXiv:2002.07105 [rsSPIRE].

[121] E.W.H. Kamke, Differentialgleichungen, Leipzig (1959), translated under the title Spravochnik po obyknovennym differentsial’nym uravneniyam, Moscow (1981).