ON THE FLOQUET ANALYSIS OF COMMUTATIVE PERIODIC LINDBLADIANS IN FINITE DIMENSION

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Abstract. We consider the Markovian Master Equation over matrix algebra $M_d$, governed by periodic Lindbladian in standard (Kossakowski-Lindblad-Gorini-Sudarshan) form. It is shown that under simplifying assumption of commutativity, the Floquet normal form of resulting completely positive dynamical map is not guaranteed to be given by simultaneously globally Markovian maps. In fact, the periodic part of the solution is even shown to be necessarily non-Markovian. Two examples in algebra $M_2$ are explicitly calculated: a periodically modulated random qubit dynamics, being a generalization of pure decoherence scheme, and a classically perturbed two-level system, coupled to reservoir via standard ladder operators.

1. Introduction

Periodically controlled open quantum systems recently began gaining increasing attention, mainly for their applicability in quantum information processing, error correction, quantum thermodynamics and general description of dephasing processes in presence of external quasi-classical perturbations. The general microscopical construction of Markovian Master Equation (MME) describing an open quantum system with periodic Hamiltonian, weakly interacting with reservoir of infinite degrees of freedom, was established in [1] and later extended in [2]. The MME was obtained with application of celebrated Floquet theory in the usual regime of weak coupling limit. This approach proved itself to be of particular importance for laser spectroscopy and quantum thermodynamics [3, 4, 5], ultimately leading to major advancements in description of quantum heat engines, solar cells and related ideas [6, 7, 8, 9, 10].

In this paper, we elaborate on general properties of the solution of MME on algebra $M_d$ of complex matrices of size $d$,

$$\frac{d\rho_t}{dt} = L_t(\rho_t),$$

where $\rho_t$ (for $t \in \mathbb{R}_+ = [0, \infty)$) is a time-dependent density matrix, i.e. a Hermitian, positive semi-definite matrix of trace one, and $L_t$ is the time-periodic Lindbladian in celebrated standard (Kossakowski-Lindblad-Gorini-Sudarshan) form [11, 12, 13, 14, 15],

$$L_t(\rho) = -i[H_t, \rho] + \sum_j \left( V_{j,t} \rho V_{j,t}^* - \frac{1}{2} \{ V_{j,t}^*, V_{j,t} \} \right),$$

with all matrices $H_t, V_{j,t} \in M_d$ periodic with some period $T > 0$ and $H_t$ being Hermitian (we put $\hbar = 1$ for convenience); $\{a,b\} = ab + ba$ stands for the anticommutator. Lindbladian (1.2) generates evolution $\rho_t = \Lambda_t(\rho_0)$, where $\{\Lambda_t : t \in \mathbb{R}_+\}$ is a one-parameter family of quantum dynamical maps, each of them completely positive, trace preserving and a trace norm contraction. Structure of $L_t$ guarantees that $\Lambda_t$ satisfies a much stronger condition of being CP-divisible, or Markovian [15, 16]:

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Definition 1. Completely positive trace preserving linear map \( \Lambda_t, t \in \mathbb{R}_+ \) on \( \mathbb{M}_d \) will be called CP-divisible or Markovian over interval \( \mathcal{I} \subseteq \mathbb{R}_+ \), if and only if the associated two-parameter family of propagators

\[
V_{t,s} = \Lambda_t \Lambda_s^{-1}
\]

is also completely positive and trace preserving for all \( t, s \in \mathcal{I}, s \leq t \).

In this work, we will be focusing mainly on CP-divisibility (Markovianity) of certain evolution maps understood as in definition 1. The notation will be fairly standard. From here onwards, we endow \( \mathbb{M}_d \) with Frobenius (Hilbert-Schmidt) inner product

\[
\langle a, b \rangle_F = \text{tr} a^* b.
\]

\( \mathbb{M}_d \) is spanned by orthonormal Frobenius basis \( \{ F_i : i = 1, \ldots, d^2 \} \), where we conveniently choose \( F_{d^2} = \frac{1}{\sqrt{d}} I \), \( I \) standing for identity matrix, with all remaining \( F_i \) traceless \cite{17, 18}. By \( \| a \|_1 \) and \( \| a \| \), \( a \in \mathbb{M}_d \), we will respectively denote the trace norm and induced operator norm of matrix \( a \) and its Hermitian conjugate will be \( a^* \). For linear space \( \mathcal{X} \), \( \mathcal{B}(\mathcal{X}) \) will be the algebra of bounded linear maps over \( \mathcal{X} \) (complete with respect to supremum norm). For \( A \in \mathcal{B}(\mathcal{X}) \), symbols \( E_A(\lambda) \) will denote the eigenspace of \( A \) corresponding to eigenvalue \( \lambda \) and geometric multiplicity of \( \lambda \) will be \( k_\lambda = \dim E_A(\lambda) \). To shorten the notation, we will simply write \( A \in \text{CP}_{1,p}(\mathcal{X}) \) (resp. \( A \in \text{CP}_{u,u} \)) if \( A \) is completely positive and trace preserving over ordered space \( \mathcal{X} \) (resp. completely positive unital over unital algebra \( \mathcal{A} \)). We will write \( a \geq 0 \), or \( a \in \mathcal{A}^+ \), to indicate that \( a \) lays in positive cone in \( \mathcal{A} \) (i.e. is positive semi-definite).

2. Floquet approach to CP-divisible dynamics

Since \( \mathbb{M}_d \) and \( \mathbb{C}^{d^2} \) are naturally isomorphic as linear spaces, Master Equation (1.1) may be always vectorized \cite{19, 20, 21}, i.e. represented as ordinary differential equation for vector-valued function \( t \mapsto p_t \in \mathbb{C}^{d^2} \); in such case, \( L_t \) is a periodic matrix in \( \mathbb{M}_{d^2} \) and hence, the MME is expressible as ordinary differential equation (ODE) with periodic matrix coefficient. Dynamical map \( \Lambda_t \) generated by (1.1) also admits a bijective representation in \( \mathbb{M}_{d^2} \) (often called the superoperator in this context) and satisfies a matrix counterpart of MME,

\[
\frac{d}{dt} \Lambda_t = L_t \Lambda_t, \quad \Lambda_0 = \text{id},
\]

being therefore the \textit{principal fundamental solution} of (1.1). By general characterization of ODEs with periodic coefficients provided by celebrated \textit{Floquet’s theorem} \cite{22}, \( \Lambda_t \) admits a product structure

\[
\Lambda_t = P_t e^{tX},
\]

such that function \( t \mapsto P_t \in \mathcal{B}(\mathbb{M}_d) \) is periodic, \( P_0 = \text{id} \), and \( X \in \mathcal{B}(\mathbb{M}_d) \) is constant. Both maps of the pair \( (P_t, e^{iX}) \) are invertible and differentiable and the (non-unique) pair itself is called the \textit{Floquet normal form} of solution \( \Lambda_t \).

In this section we present some results concerning conditions for CP-divisibility of Floquet normal form, partially in general case (in section 2.1) and especially, in case of commutative Lindbladian families (in section 3). Commutativity is a severe simplification, however still of practical applicability for various quantum models and of conceptual and mathematical importance, as it provides an exactly solvable case. In particular, we show that one \textit{may not} expect simultaneous CP-divisibility of Floquet pair, even despite their composition is perfectly Markovian as a legitimate quantum dynamics. Some general remarks regarding asymptotic properties of solutions are also addressed (in section 2.2). Two exemplary applications of such commutative periodic Lindbladian families are then presented in section 4.
2.1. General considerations. Finding the explicit form of a pair \((P_t, e^{tX})\) may be a very challenging task, as it clearly requires one to find an actual solution of the ODE first. In fact, no universal methods of obtaining the solution exist apart from some perturbative approaches, including Dyson, Magnus or Fer expansions \([23, 24, 25]\); these however are rarely exactly summable. Some properties may be sometimes deduced from the stroboscopic form of the fundamental matrix solution: given a solution \(\Lambda_t = P_t e^{tX}\), the stroboscopic dynamics is \(\Lambda_{nT} = e^{nTX}\), which is easily implied by periodicity of \(P_t\); clearly, \(\Lambda_{nT} \in \text{CP}_{t.p}(\mathbb{M}_d)\). Putting \(n = 1\), we obtain the so-called monodromy matrix \(\Lambda_T = e^{TX}\) which allows to find

\[
X = \frac{1}{T} \log \Lambda_T,
\]

where existence of the logarithm is assured by invertibility of \(\Lambda_T\). The problem arises with complete positivity of a semigroup \(\{e^{tX} : t \in \mathbb{R}_+\}\) as a priori there is no guarantee that \(\Lambda_T\) lays in the range of any Markovian semigroup, i.e. any branch of \(\log \Lambda_T\) in Lindblad form exists. Problem of accessibility of set \(\text{CP}_{t.p}(\mathbb{M}_d)\) (i.e. quantum channels) by Lindblad semigroups is surprisingly non-trivial even in low-dimensional matrix algebras and is subject to active research \([26, 27, 28, 29, 30]\); we will not however address it here directly.

We call a map \(\Phi\) on \(C^*\)-algebra \(\mathcal{A}\) a \(*\)-map, if and only if it is Hermiticity preserving, i.e. satisfies \(\Phi(x^*) = \Phi(x)^*\) for all \(x \in \mathcal{A}\). The following simple claim holds:

**Proposition 1.** Let the Floquet normal form \((P_t, e^{tX})\) satisfy MME (2.1) for periodic Lindbladian (1.2). If one of the maps of the pair is trace preserving and a \(*\)-map over \(\mathbb{M}_d\), so is the second one.

The proof is basic and relies on invertibility of both maps in Floquet pair. Let us now introduce few additional notions. By expanding matrices \(V_{j,t}\) in expression (1.2) in Frobenius basis, one obtains Lindbladian in so-called first standard form \([11, 13, 14, 15]\),

\[
L_t(\rho) = -i[H_t, \rho] + \sum_{j,k=1}^{d^2-1} a_{jk}(t) \left( F_j \rho F^*_k - \frac{1}{2} \{ F^*_k F_j, \rho \} \right),
\]

where the Kossakowski matrix \(a_t = [a_{jk}(t)] \in \mathbb{M}_{d^2-1}\) is positive semi-definite. Then, \(L_t\) generates a CP-divisible, trace preserving dynamical map iff it is of a form (1.2), which is true iff it is of a form (2.4) for Hermitian \(H_t\) and \(a_t \succeq 0\). For later use, we also introduce

\[
D_{jk}(x) = F_j x F^*_k - \frac{1}{2} \{ F^*_k F_j, x \},
\]

to shorten the notation a little bit.

Let us assume that periodic part of Floquet normal form \(P_t\) is a trace preserving \(*\)-map. Applying lemma 1 (available in A.1), it may be then cast in the form

\[
P_t(x) = \sum_{j,k=1}^{d^2} p_{jk}(t) F_j x F^*_k, \quad x \in \mathbb{M}_d,
\]

for Hermitian, periodic matrix \([p_{jk}(t)] \in \mathbb{M}_{d^2}\). This allows us to formulate a following result, which can be considered as a partial answer to the question of simultaneous complete positivity of Floquet pair in general case:

**Proposition 2.** Let \((P_t, e^{tX})\) be the Floquet normal form for \(L_t\) (2.4) s.t. \(P_t\) is a trace preserving \(*\)-map over \(\mathbb{M}_d\), admitting a form (2.6), and let \(\tilde{P}_t = [p_{jk}(t)]_{j,k=1}^{d^2-1}\).
Then \( \{ e^{t X} : t \in \mathbb{R}_+ \} \), is a Markovian contraction semigroup if and only if

\[
a_0 - \frac{d\tilde{P}_t}{dt} \bigg|_0 \in M_{d^2-1}^+. \tag{2.7}
\]

**Proof.** Showing the claim involves simple algebra, therefore we only sketch the proof. As \( \Lambda_t \) satisfies the operator MME (2.1), after differentiating \( \Lambda_t = P_t e^{t X} \) one easily obtains \( \frac{dP_t}{dt} = L_t P_t - P_t X \) which, after putting \( t = 0 \) and reordering, yields

\[
X = L_0 - \frac{dP_t}{dt} \bigg|_0. \tag{2.8}
\]

Therefore, \( \{ e^{t X} : t \in \mathbb{R}_+ \} \) is CP-divisible if and only if (2.8) is of standard form. According to lemma 1, \( P_t \) can be given as

\[
P_t(x) = x + i [G_t, x] - \{ K_t, x \} + \sum_{j,k=1}^{d^2-1} p_{jk}(t) F_j x F_k^* , \tag{2.9}
\]

where \( G_t \) and \( K_t \) are Hermitian matrices,

\[
G_t = \frac{1}{2\sqrt{d}} \sum_{j=1}^{d^2-1} \left( p_{ja^2(t)} F_j - p_{a^2(j)} F_j^* \right), \quad K_t = \frac{1}{2} \sum_{j,k=1}^{d^2-1} p_{jk}(t) F_k^* F_j. \tag{2.10}
\]

Differentiating (2.9) and substituting to (2.8) leads, after some algebra, to

\[
X(x) = -i [H_0 - \frac{dG(t)}{dt} \bigg|_0, x] + \sum_{j,k=1}^{d^2-1} \left( a_{jk}(0) - \frac{dp_{jk}(t)}{dt} \bigg|_0 \right) D_{jk}(x), \tag{2.11}
\]

for \( D_{jk} \) given via (2.5). Matrix \( \tilde{P}_t \) is clearly Hermitian, so \( H_0 - \frac{dG(t)}{dt} \bigg|_0 \) is also; hence, (2.11) defines a generator of completely positive contraction semigroup, i.e. is Markovian, if and only if a matrix \( \begin{bmatrix} a_{jk}(0) - \frac{dp_{jk}(t)}{dt} \bigg|_0 \end{bmatrix}_{j,k} \) is positive semi-definite. \( \Box \)

2.2. **Stability and asymptotic behavior of solutions.** Stability of solutions remains a significant matter of classical theory of ODEs. Naturally, it is equally important in context of Floquet analysis as we are very often interested in qualitative asymptotic behavior of solutions to certain initial value problems, i.e. after very long evolution time. In particular, asymptotic behavior of Floquet solutions is fully deducible from analysis of so-called characteristic multipliers of the system (see below) and it is known that solutions exhibit dramatically different characteristics depending on the multipliers, ranging from almost-exponential decaying to 0, through formation of periodic limit cycles to even unbounded growth, or “blowing up”, as \( t \to \infty \). Fortunately, in our case of Markovian dynamics, the infinite growth scenario is forbidden (loosely speaking, by contractivity of dynamical maps), however other possibilities remain.

Let us now assume that \( X \), given in the Floquet normal form is diagonalizable, i.e. satisfies an eigenequation \( X(\varphi_j) = \mu_j \varphi_j \) for \( \mu_j \in \mathbb{C}, \ j \in \{1, 2, ..., d^2\} \), such that set \( \{ \varphi_j \} \) is linearly independent and hence a basis in \( \mathbb{M}_d \). Monodromy matrix \( \Lambda_T = e^{TX} \) satisfies the eigenequation for the same set of matrices,

\[
\Lambda_T(\varphi_j) = \lambda_j \varphi_j, \quad \lambda_j = e^{\mu_j T}, \ j \in \{1, 2, ..., d^2\}, \tag{2.12}
\]

and by spectral mapping theorem, \( e^{tX}(\varphi_j) = e^{\mu_j t} \varphi_j, \ t \in \mathbb{R}_+ \). We then call spec(\( \Lambda_T \)) = \{\( \lambda_j \)\} the set of characteristic multipliers and spec(\( X \)) = \{\( \mu_j \)\} the
set of characteristic exponents of the system. Note that \( \{ \mu_j \} \) is not uniquely defined by monodromy matrix, as shifting transformation \( \{ \mu_j \} + \frac{2\pi i}{k}, k \in \mathbb{Z}^d \), leaves \( \text{spec}(\Lambda_T) \) unchanged (simply, log \( \Lambda_T \) is non-unique). Now, define a set of functions
\[
\rho_j(t) = \Lambda_t(\varphi_j) = e^{\mu_j t} \phi_j(t), \quad \phi_j(t) = P_j(\varphi_j),
\]
which are naturally solutions to the MME in question, i.e. states. By diagonalizability of \( X \), set \( \{ \rho_j \} \) is a fundamental set of solutions. The general solution for (1.1) is then expressible as a linear combination
\[
\rho_t = \sum_{j=1}^{d^2} c_j \rho_j(t) = \sum_{j=1}^{d^2} c_j e^{\mu_j t} \phi_j(t),
\]
where coefficients \( c_j \) are prescribed by initial condition \( \rho_0 = \sum_{j=1}^{d^2} c_j \varphi_j \). Evidently, by periodicity of Floquet states \( \phi_j(t) \), we have also \( \rho_j(t + nT) = \lambda_j^n \rho_j(t) \). As a result, the stroboscopic dynamics \( \Lambda_{nT} \) simply multiplies the initial state by factor \( \lambda_j^n = e^{n \mu_j T} \) and a long-time behavior of solution is directly influenced by properties of characteristic multipliers. One quickly checks that the following result holds:

**Proposition 3.** Let \( |\lambda_j| \leq 1 \). Then \( \rho_j(t) \) vanishes as \( t \to \infty \) if \( |\lambda_j| < 1 \), oscillates periodically if \( \lambda_j = \pm 1 \), or quasi-periodically if \( \lambda_j \notin \{ -1, 1 \} \), \( |\lambda_j| = 1 \). On the contrary, if \( |\lambda_j| > 1 \), then \( \rho_j(t) \) grows infinitely in norm.

Naturally, \( |\lambda_j| \leq 1 \) (which is satisfied if \( \text{Re} \mu_j < 0 \)) guarantees that the solution is stable and if \( |\lambda_j| > 1 \), unstable (“blows up” at large times); hence, a general solution (2.14) will be called asymptotically stable, if and only if all \( |\lambda_j| \leq 1 \). Fortunately, in case of quantum dynamics, unstability of solutions is disallowed by spectral properties of monodromy matrix:

**Proposition 4.** Spectrum of \( \Lambda_T \) lays inside unit disc \( \mathbb{D}^1 \). In the result, all solutions of MME (1.1) for periodic \( L_t \) in standard form (2.4) are asymptotically stable.

**Proof.** Since \( \Lambda_T \in \text{CP}_{u.p.}(M_d) \), then its dual \( \Lambda_T' \in \text{CP}_u(M_d) \) and is of the same spectrum. It is known [31, 28, 32] that spectrum of any linear map \( \Phi \in \text{CP}_u(\mathcal{A}^t) \), \( \mathcal{A}^t \) being C*-algebra, admits a decomposition into two disjoint parts, \( \text{spec}(\Phi) = \{ 1 \} \cup \mathcal{P} \), where \( \mathcal{P} \) is closed under complex conjugation and contained in \( \mathbb{D}^1 \).

Let us show this explicitly in case of \( \Lambda_T' \). Since it is unital, \( \Lambda_T'(I) = I \) and identity matrix is an eigenvector for eigenvalue 1. Moreover, since \( \Lambda_T' \) is completely positive, it attains its norm at \( I [33, \text{Proposition 3.6}] \), \( \| \Lambda_T' \| = \| \Lambda_T'(I) \| = \| I \| = 1 \), which is therefore equal to spectral radius of \( \Lambda_T' \). Hence, \( \text{spec}(\Lambda_T'), \text{spec}(\Lambda_T) \subset \mathbb{D}^1 \).

As \( \Lambda_T' \) is also a *-map, taking the Hermitian adjoint of eigenequation \( \Lambda_T'(x) = \lambda x \) for any \( \lambda \in \text{spec}(\Lambda_T) \setminus \{ 1 \} \) and some \( x \in M_d \), yields \( \overline{\lambda} \) also an eigenvalue for eigenvector \( x^* \). This shows that \( \text{spec}(\Lambda_T') \setminus \{ 1 \} \) is either real or consists of pairs \( \{ \lambda, \overline{\lambda} \} \), \( |\lambda| \leq 1 \), i.e. \( \text{spec}(\Lambda_T) \) is invariant w.r.t. complex conjugation. This finally shows that solutions of a form \( \rho_j(t) = e^{\mu_j t} \phi_j(t) \), as well as any general solution (2.14), are all stable by proposition 3.

**Proposition 5.** The following claims hold:

(1) If \( \text{tr} \varphi \neq 0 \) or if \( \varphi \geq 0 \), then \( \varphi \in E_{\Lambda_T}(1) \). Equivalently, if \( \lambda \neq 1 \), then the corresponding eigenvector \( \varphi \in E_{\Lambda_T}(\lambda) \) is necessarily traceless and non-positive semi-definite;

(2) There exists \( \varphi \in E_{\Lambda_T}(1) \), such that \( \varphi \geq 0 \);

(3) If \( \varphi \in E_{\Lambda_T}(\lambda) \) for \( \lambda \in \text{spec}(\Lambda_T) \setminus \mathbb{R} \), then \( \varphi^* \in E_{\Lambda_T}(\overline{\lambda}) \). Eigenvalue \( \lambda \in \mathbb{R} \) is simple (i.e. \( k_\lambda = 1 \)) if and only if \( \varphi \) is Hermitian.
Proof. For claim 1, note that as \( \Lambda_T \) satisfies eigen equation \( \Lambda_T(\varphi) = \lambda \varphi \), trace preservation condition demands

\[
(1 - \lambda) \operatorname{tr} \varphi = 0. 
\]  
(2.15)

If \( \varphi \geq 0 \), then necessarily \( \operatorname{tr} \varphi > 0 \) (otherwise we would have \( \|\varphi\|_1 = \operatorname{tr} \varphi \leq 0 \)) so in both cases (tr \( \varphi \neq 0 \) or \( \varphi \geq 0 \)) under consideration, \( \varphi \) is of non-zero trace. Condition (2.15) is then satisfied if and only if \( \lambda = 1 \). Second part of the claim is then a simple contraposition. For claim 2, we will utilize another known result which states that if \( \Phi \) is positive on finite dimensional \( \mathcal{C}^* \)-algebra \( \mathcal{A} \) and \( r \) is its spectral radius, then there exists eigenvector \( x \in \mathcal{A} \) such that \( \Phi(x) = rx \) and \( x \geq 0 \) [31, Theorem 2.5]. As spectral radius of \( \Lambda_T \) was shown to be 1 in the proof of theorem 4, demanding \( \lambda = 1 \) yields that eigen equation \( \Lambda_T(\varphi) = \varphi \) is satisfied for (at least one) matrix \( \varphi \geq 0 \). Finally, claim 3 is a direct consequence of Hermiticity preservation: given \( \varphi \in \mathcal{E}_\lambda(\lambda) \), taking the adjoint of eigenequation \( \Lambda(\varphi) = \lambda \varphi \) gives \( \varphi^* \) is an eigenvector for eigenvalue \( \lambda \) as earlier. If \( \lambda \in \mathbb{R} \), then both \( \varphi, \varphi^* \in \mathcal{E}_\lambda(\lambda) \) and \( k_\lambda \geq 2 \), unless \( \varphi = \varphi^* \).

Finally, we summarize by noticing that in certain scenario, completely positive dynamics will always admit a periodic steady state:

**Theorem 1.** Every \( \Lambda_t \in CP_{t.p.}(\mathbb{M}_d) \) given by Floquet pair \( (P_t, e^{iX}) \) admits an asymptotic quasi-periodic limit cycle in \( \mathbb{M}_d. \) If in addition \( \operatorname{spec}(\Lambda_T) \cap (S^1 \setminus \{-1, 1\}) = \emptyset \), then \( \Lambda_t \) admits an asymptotic periodic limit cycle, i.e. a periodic steady state.

**Proof.** Properties of spectrum of monodromy matrix allow to decompose \( \operatorname{spec}(X) \) into three disjoint subsets, \( \operatorname{spec}(X) = \bigcup_{i=1}^3 \mathcal{E}_i \) such that

\[
\mathcal{E}_1 = \{ \mu_j \in \pi i/T \}, \quad \mathcal{E}_2 = \{ \mu_j \in i \left( \mathbb{R} \setminus \pi Z \right) \}, \quad \mathcal{E}_3 = \{ \operatorname{Re} \mu_j < 0 \}. 
\]  
(2.16)

Then, it suffices to rewrite (2.14) by re-grouping terms

\[
\rho_t = \sum_{\mu_j \in \mathcal{E}_1} c_je^{ik\pi t/T} \phi_j(t) + \sum_{\mu_j \in \mathcal{E}_2} c_je^{i\mu_j t} \phi_j(t) + \sum_{\mu_j \in \mathcal{E}_3} c_je^{i\mu_j t} e^{i\mu_j t} \phi_j(t). 
\]  
(2.17)

Of course the limit cycle will be simply a function \( \rho_t^\infty \) constructed by deprecating the very last sum,

\[
\rho_t^\infty = \sum_{\mu_j \in \mathcal{E}_1} c_je^{ik\pi j/T} \phi_j(t) + \sum_{\mu_j \in \mathcal{E}_2} c_je^{i\mu_j t} \phi_j(t). 
\]  
(2.18)

The first sum in (2.18) is easily seen to be periodic by identity \( e^{k\pi i} = (-1)^k \), \( k \in \mathbb{Z} \) and second one is quasi-periodic, since time-shifting the sum to \( t + T \) shifts coefficients \( c_j \) by phase factors, \( c_j \mapsto c_j e^{i\mu_j T} \). Quasi-periodicity becomes a simple periodicity if \( \operatorname{Im} \mu_j = 0 \) for \( \mu_j \in \mathcal{E}_2 \), i.e. if \( \mathcal{E}_2 \) is empty. Evidently, since all terms in sum over \( \mathcal{E}_3 \) in (2.17) vanish exponentially (\( \operatorname{Re} \mu_j < 0 \)), one always finds such \( t_0 \in \mathbb{R}_+ \) large enough, that sup \( t > t_0 \) \( \|\rho_t - \rho_t^\infty\|_1 \) is as small as desired, i.e. functions \( [t_0, \infty) \mapsto \rho_t, \rho_t^\infty \) are arbitrarily close to each other in supremum norm in space of continuous matrix-valued functions \( \mathcal{C}_0([t_0, \infty), \mathbb{M}_d) \); by equivalence of norms, the result remains true regardless of matrix norm used. This shows that \( \rho_t^\infty \) is indeed a limit cycle for \( \rho_t \), periodic if no eigenvalues \( \lambda \in S^1 \setminus \{-1, 1\} \) exist.

### 3. Commutative Lindbladian families

Here we inspect a simplified class of commutative Lindbladian, which provides an exactly solvable case. We assume that the family \( \{ L_t : t \in \mathbb{R}_+ \} \) of periodic Lindbladians in standard form (2.4) satisfies commutativity condition

\[
L_t L_s(x) = L_s L_t(x), \quad t, s \in \mathbb{R}_+, \ x \in \mathbb{M}_d. 
\]  
(3.1)
3.1. CP-divisibility of Floquet normal form. The core result of this section, presented in form of theorems 2 and 3 below, shows that for special case of commutative Lindbladians (3.1), both maps of Floquet pair \((P_t, e^{iX})\) can be simultaneously Markovian over some intervals in \(\mathbb{R}_+\) and the semigroup part \(e^{iT}\) in fact is Markovian in whole \(\mathbb{R}_+\). However, it is not true for the periodic part \(P_t\) as an interesting property is revealed: it is \textit{impossible} for \(P_t\) to be uniformly Markovian over a whole time of evolution. The question of simultaneous CP-divisibility of Floquet pair, stated in the Introduction, is hence answered negatively.

**Theorem 2.** Let \(L_t\) be of standard form (2.4), periodic and obeying the commutativity condition (3.1). Then, it generates a CP-divisible quantum dynamical map \(\Lambda_t\) admitting Floquet normal form \((P_t, e^{iX})\) such that:

1. \(\{e^{iX} : t \in \mathbb{R}_+\} \subset CP_{t.p.}(M_d)\) and is CP-divisible contraction semigroup (i.e. a quantum dynamical semigroup);
2. \(P_t, t \in \mathbb{R}_+,\) is a trace preserving \(*\)-map;
3. \(P_t\) is CP-divisible in interval \(T \subset \mathbb{R}_+\) if and only if
   \[a_t = \frac{1}{T} \int_0^T a_{t'} dt' \in M_{d^2-1}^+\] for every \(t \in T;\) (3.2)
4. \(P_t\) is completely positive for some \(t \in \mathbb{R}_+;\)
   \[\int_0^t a_{t'} dt' - \frac{t}{T} \int_0^T a_{t'} dt' \in M_{d^2-1}^+.\] (3.3)

**Theorem 3.** Map \(P_t\) governed by Lindbladian (2.4) satisfies the following:

1. \(P_t\) is CP-divisible everywhere in \(\mathbb{R}_+\) iff Kossakowski matrix \(a_t\) is constant;
2. If \(a_t\) is constant, then \(P_t \in CP_{t.p.}(M_d)\) for all \(t \in \mathbb{R}_+;\)
3. If \(a_t\) is continuous and non-constant, then there exists a non-empty union of intervals \(N \subset \mathbb{R}_+\) such that \(P_t\) is not CP-divisible (non-Markovian) in \(N.\)

**Proof of theorem 2.** Commutativity condition (3.1) allows to avoid cumbersome time-ordering procedure (like in Dyson expansion) and solution to MME (1.1) is exactly obtainable. For brevity, let us introduce three antiderivatives

\[H_t = \int_0^t H_{t'} dt', \quad A_{jk}(t) = \int_0^t a_{jk}(t') dt', \quad A_t = [A_{jk}(t)]_{jk} = \int_0^t a_{t'} dt'.\] (3.4)

Define map \(\Phi_t\) on \(M_d\) via

\[\Phi_t = \exp \int_0^t L_{t'} dt' = \exp \left(-i[H_t, \cdot] + \sum_{j,k=1}^{d^2-1} A_{jk}(t) D_{jk} \right).\] (3.5)

Then, by direct calculation one can check, by expanding matrix exponentials into power series and applying commutativity condition (3.1), that \(L_t\) commutes with \(\Phi_t\) and \(\Phi_t\) satisfies differential equation

\[\frac{d}{dt} \Phi_t = \Phi_t L_t = L_t \Phi_t, \quad \Phi_0 = id,\] (3.6)

which is simply the MME in question; hence we have \(\Phi_t = \Lambda_t\) as \(\Phi_t\) must be a unique solution and the monodromy matrix is \(\Lambda_T = \exp \int_0^T L_{t'} dt'.\) Finding \(X\) requires one to solve an equation \(\Lambda_T = e^{iT}\) by computing a logarithm of monodromy matrix (which is achieved by seeking for Jordan normal form of \(\Lambda_T;\) see e.g. [34] for details), which cannot be uniquely determined. In effect, one obtains an infinite family of
valid logarithms; for our purpose however, it totally suffices to choose
\[
X = \frac{1}{T} \int_0^T L \cdot dt' = -i \frac{L}{T} [\mathcal{H}_T, \cdot] + \frac{1}{T} \sum_{j,k=1}^{d^2-1} A_{jk}(T) D_{jk}.
\] (3.7)
Clearly, \( \mathcal{H}_T \) is Hermitian. Moreover, for any \( x = (x_i) \in \mathbb{C}^{d^2-1} \) and \( t \in \mathbb{R}_+ \),
\[
(x, A_t x) = \sum_{j,k=1}^{d^2-1} A_{jk}(t) x_j x_k = \int_0^T \left( \sum_{j,k=1}^{d^2-1} a_{jk}(t') x_j x_k \right) dt' \geq 0,
\] (3.8)
since \( [a_{jk}(t)]_{jk} \geq 0 \); therefore, also \( A_t \geq 0 \) for all \( t \in \mathbb{R}_+ \) and \( X \) chosen in (3.7) is of standard form. In other words, if commutativity condition holds then there always exists map \( X \) solving equation \( \Lambda_T = e^{TX} \), which generates a Markovian semigroup; this proves claim 1. For claim 2, note that since \( \{ e^{tx} : t \in \mathbb{R}_+ \} \) is a legitimate Markovian dynamics, then \( P_t \) is also trace preserving \(*\)-map via proposition 1.

By formula (3.7), \( X \) commutes with any integral of a form \( \int_{t_0}^t L \cdot dt' \) and therefore \( \Lambda_t X = X \Lambda_t \). As in any associative algebra we have \( e^{A}e^B = e^{A+B} \) provided \( AB = BA \),
\[
P_t = \Lambda_t e^{-tX} = \exp \left( \int_0^t L_v dt' - t \frac{I}{T} \int_0^T L_v dt' \right) = \exp \int_0^t (L_v - X) dt'
\] (3.9)
which further yields an explicit formula for \( P_t \),
\[
P_t = \exp \left[ -i [\mathcal{H}_t - \frac{t}{T} \mathcal{H}_T, \cdot] + \sum_{j,k=1}^{d^2-1} \left( A_{jk}(t) - \frac{A_{jk}(T)}{T} \right) D_{jk} \right].
\] (3.10)

By inspection, \( P_t \) is clearly periodic. To show claim 3, note that (3.9) implies
\[
\frac{dP_t}{dt} = (L_t - X) P_t, \quad P_0 = \text{id},
\] (3.11)
since \( X \) and \( P_t \) commute. By general considerations [15, 16], if some map \( \Phi_t \)
satisfies an ODE of a form \( \frac{d}{dt} \Phi_t = \mathcal{G}_t \Phi_t \), then \( \Phi_t \) is CP-divisible in interval \( \mathcal{I} \subset \mathbb{R}_+ \)
if and only if \( \mathcal{G}_t \) is of standard form for every \( t \in \mathcal{I} \). This shows that sufficient and necessary condition for CP-divisibility of \( P_t \) is
\[
L_t - X = -i [\mathcal{H}_t - \frac{1}{T} \mathcal{H}_T, \cdot] + \sum_{j,k=1}^{d^2-1} \left( A_{jk}(t) - \frac{A_{jk}(T)}{T} \right) D_{jk}
\] (3.12)
being of standard form which, by obvious hermiticity of \( \mathcal{H}_t - \frac{1}{T} \mathcal{H}_T \), leads to condition (3.2). Finally, claim 4 can be shown by appropriately putting \( P_t \) in Choi-Kraus form (we will follow a general idea of a proof of [15, Theorem 4.2.1]). Let us define positive semi-definite matrix \( B_{jk}(t) = A_{jk}(t) - \frac{A_{jk}(T)}{T} t \). Set also maps \( \mathcal{K}(t) \) and \( \mathcal{M}_{jk}(t) \) defined over \( x \in \mathbb{M}_d \) by
\[
\mathcal{K}(t)(x) = -i [\mathcal{K}_t, x] - \frac{1}{2} \sum_{j,k=1}^{d^2-1} B_{jk}(t) F_k^* F_j x - \frac{1}{2} \sum_{j,k=1}^{d^2-1} B_{jk}(t) F_k^* F_j,
\] (3.13a)
\[
\mathcal{M}_{jk}(t)(x) = B_{jk}(t) F_j x F_k^*.
\] (3.13b)
\( P_t \) may be then rewritten as
\[
P_t = \exp \left( \mathcal{K}(t) + \sum_{j,k=1}^{d^2-1} \mathcal{M}_{jk}(t) \right) = \lim_{N \to \infty} \left( \exp \frac{\mathcal{K}(t)}{N} \cdot \exp \sum_{j,k=1}^{d^2-1} \frac{\mathcal{M}_{jk}(t)}{N} \right)^N,
\] (3.14)
where the last expression comes after applying Lie-Trotter product formula \cite{35}. Pick an arbitrary \( t \in \mathbb{R}_+ \). Define a matrix

\[
U_{N,\tau} = \exp \frac{\tau}{N} \left( -iK_t - \frac{1}{2} \sum_{j,k=1}^{d^2-1} B_{jk}(t) F_k^* F_j \right). \tag{3.15}
\]

and function \( \tau \mapsto \phi_{N,\tau} \in B(\mathbb{M}_d) \) by \( \phi_{N,\tau}(x) = U_{N,\tau} x U_{N,\tau}^* \). By inspection, \( U_{N,\tau} \) is unitary (since \([B_{jk}(t)]_{jk} \) is Hermitian) and \( \phi_{N,\tau} \) is completely positive. Moreover, it satisfies a differential equation

\[
\frac{d}{d\tau} \phi_{N,\tau} = \frac{K(t)}{N} \phi_{N,\tau}, \tag{3.16}
\]

therefore \( \phi_{N,\tau} = \exp \frac{K(t)}{N} \tau \) and it is a semigroup of completely positive maps. Hence, the first exponential \( \exp \frac{K(t)}{N} \tau \) appearing in (3.14) is simply \( \phi_{N,\tau} \) evaluated at \( \tau = 1 \) and is completely positive.

As \([B_{jk}(t)]_{jk} \geq 0\), one has \([B_{jk}(t)]_{jk} = C_t^* C_t \) for some \( C_t = [C_{jk}(t)]_{jk} \in \mathbb{M}_d^{d^2-1} \); then, we have

\[
\exp \sum_{j,k=1}^{d^2-1} \frac{M_{jk}(t)}{N} = \exp \left( \frac{1}{N} \sum_{j,k=1}^{d^2-1} \sum_{l=1}^{d^2-1} C_{jl}(t) C_{kl}(t) F_j^* F_l \right) = e^{F_N(t)}, \tag{3.17}
\]

where \( \Gamma_N(t)(x) = \sum_{l,j=1}^{d^2-1} \frac{\hat{C}_l(t) x \hat{C}_j(t)^*}{\sqrt{N}} \) and \( \hat{C}_l(t) = \sum_{j=1}^{d^2-1} C_{jl}(t) F_j \). By induction, the \( n \)-th power of \( \Gamma_N(t) \) is defined via

\[
\Gamma_N(t)^n(x) = \sum_{l_1,...,l_n} \Pi_{l_1,...,l_n}(t) x \Pi_{l_1,...,l_n}(t)^*, \tag{3.18}
\]

where \( \Pi_{l_1,...,l_n}(t) = \prod_{j=1}^{n} \frac{\hat{C}_l(t)}{\sqrt{N}} \); this allows to express the action of (3.17) on \( \mathbb{M}_d \),

\[
e^{F_N(t)}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_n(t)^n(x) = \sum_{n=0}^{\infty} \sum_{l_1,...,l_n} \frac{\Pi_{l_1,...,l_n}(t)}{\sqrt{n!}} \frac{\Pi_{l_1,...,l_n}(t)^*}{\sqrt{n!}} x, \tag{3.19}
\]

which is of Choi-Kraus form and therefore completely positive for all \( t \in \mathbb{R}_+ \). Thus, \( P_t \) is also completely positive (as a composition).

**Proof of theorem 3.** Notice, that if \( a = a \geq 0 \) is constant, conditions (3.2) and (3.3) given in theorem 2 are automatically satisfied so \( P_t \in \text{CP}_{t.p.}(\mathbb{M}_d) \) and is CP-divisible everywhere; this proves claim 2 as well as necessity stated in claim 1. For sufficiency, let us assume \( P_t \) is CP-divisible everywhere in \([0, T)\) (and in \( \mathbb{R}_+ \) in consequence). Then, for any \( x \in \mathbb{C}^{d^2-1} \), define non-negative continuous function \( f_x(t) \) by

\[
f_x(t) = \langle x, a_t x \rangle = \sum_{j,k=1}^{d^2-1} a_{jk}(t) x_j \overline{x_k} \tag{3.20}
\]

and denote its restriction to \([0, T)\) by the same symbol. Everywhere CP-divisibility of \( P_t \) yields, by theorem 2, that condition (3.2) is met for every \( t \in [0, T) \), i.e.

\[
f_x(t) - \frac{1}{T} \int_0^T f_x(t') dt' \geq 0 \quad \text{for all } x \in \mathbb{C}^{d^2-1}. \tag{3.21}
\]

Take any \( x \neq 0 \); as \( f_x \) is continuous on \([0, T)\), the mean value theorem for definite integrals implies existence of some \( \tau_x \in (0, T) \) such that \( \int_0^T f_x(t) dt = T f_x(\tau_x) \).
and hence, by (3.21), also $f_\mathbf{x}(t) \geq f_\mathbf{x}(\tau_\mathbf{x})$ for every $t \in [0, T)$, i.e. $f_\mathbf{x}(\tau_\mathbf{x}) = \min_{t \in [0, T)} f_\mathbf{x}(t)$. Then however,

$$\int_0^T f_\mathbf{x}(t) dt = T \min_{t \in [0, T)} f_\mathbf{x}(t)$$

(3.22)

which is possible if and only if $f_\mathbf{x}$ is constant for any $\mathbf{x}$. But this in turn implies $a_t$ must also be constant (see proposition 6 in A.1 for explanation) and claim 1 is shown. Finally, for claim 3, assume $a_t$ is not constant. Then, $P_t$ is not everywhere CP-divisible via claim 1, or equivalently, inequality (3.21) is not satisfied for all $\mathbf{x} \in \mathbb{C}^{d^2-1}$. Denote now

$$\mathcal{P}_x = \{ t \in [0, T) : f_\mathbf{x}(t) \geq \frac{1}{T} \int_0^T f_\mathbf{x}(t') dt' \}, \quad \mathcal{P} = \bigcap_{x \in \mathbb{C}^{d^2-1}} \mathcal{P}_x.$$  

(3.23)

Under such notion, CP-divisibility of $P_t$ is allowed only over subset $\mathcal{P} \subsetneq [0, T)$ and hence, its complement $\mathcal{N} = [0, T) \setminus \mathcal{P}$ is non-empty. By continuity of $f_\mathbf{x}$, both $\mathcal{P}$ and $\mathcal{N}$ must be (possibly uncountable) unions of intervals in $[0, T)$.

□

4. Exemplary Applications

In this section, we examine two examples of Master Equations governed by commutative periodic Lindbladian families. For clarity of presentation, we will limit our analysis to the simplest case of algebra $\mathbb{M}_2$, however generalizations to higher dimensional systems are naturally obtainable. In all the following, we denote by $\{\sigma_j\}_{j=1}^4$, $\sigma_4 = I$, the set of Pauli matrices, spanning $\mathbb{M}_2$. The orthonormal Frobenius basis in $\mathbb{M}_2$ is then constructed by defining

$$F_j = \frac{\sigma_j}{\sqrt{2}}, \quad j \in \{1, \ldots, 4\}.$$  

(4.1)

The solutions of differential equations over $\mathbb{M}_2$ appearing in this section will always be obtained by the so-called vectorization procedure, i.e. by applying some arbitrarily chosen isomorphism $\mathbb{M}_2 \to \mathbb{C}^4$. For simplicity, we choose it as

$$x \mapsto \mathbf{x} = (x_1, x_2, x_3, x_4)^T, \quad x_j = (F_j, x)_F = \frac{1}{\sqrt{2}} \text{tr} \sigma_j x,$$  

(4.2)

i.e. we map each matrix to a vector of its components in Frobenius basis. Note, that $x_4 = \frac{1}{\sqrt{d}} \text{tr} x$. Then, every map $W \in B(\mathbb{M}_2)$ is then expressed as a matrix

$$W = [W_{jk}]_{jk} \in \mathbb{M}_4, \quad W_{jk} = (F_j, W(F_k))_F = \frac{1}{2} \text{tr} [\sigma_j W(\sigma_k)].$$  

(4.3)

In particular, $W$ is trace preserving iff $W_{\delta} = \delta_{\delta,j}$. If a Hermitian basis $\{F_i\}$ is used (which is the case here), then $W$ is a *-map iff $[W_{jk}]_{jk}$ is real. Likewise, we make bijective replacements $\rho_t \mapsto r(t)$, $\Lambda_t \mapsto \Lambda(t) = [\Lambda_{jk}(t)]_{jk}$ and $L_t \mapsto L(t) = [L_{jk}(t)]_{jk}$, such that the MME transforms into linear ODE of a form

$$\frac{dr(t)}{dt} = L(t)r(t).$$  

(4.4)

4.1. Periodically modulated random dynamics. As a first simple, yet popular example, we will briefly analyze a random dynamics with additional assumption of time-periodicity of decoherence rates, i.e. a generalization of pure decoherence model of a qubit, involving all Pauli channels. We take the Master Equation in a following form [36]

$$\frac{d\rho_t}{dt} = L_t(\rho_t) = \frac{1}{2} \sum_{j=1}^3 \gamma_j(t)(\sigma_j \rho_t \sigma_j - \rho_t).$$  

(4.5)
We assume all functions $\gamma_j(t)$ are non-negative, periodic and continuous. Exploiting a useful property of Pauli matrices $\sigma_j^2 = \sigma_j^T \sigma_j = I$, (4.5) is quickly seen to be of form (2.4) for $H_t = 0$ and Kossakowski matrix $a_t = [\delta_{jk}\gamma_j(t)]_{jk}$. In such case, the derived $\Lambda_t$ is a convex combination of Pauli channels.

Invoking the vectorization procedure mentioned earlier, matrix $L(t)$ is found to be diagonal in Frobenius basis,

$$L(t) = -\text{diag}\{\gamma_2(t) + \gamma_3(t), \gamma_1(t) + \gamma_3(t), \gamma_1(t) + \gamma_2(t), 0\}. \quad (4.6)$$

Note, that $L_{44}(t) = 0$ which is required for trace preservation. Solution to (4.5) is then again given by diagonal matrix $\Lambda(t)$,

$$\Lambda(t) = \text{diag}\{e^{-\Gamma_{2+3}(t)}, e^{-\Gamma_{1+3}(t)}, e^{-\Gamma_{1+2}(t)}, 1\}, \quad (4.7)$$

where functions $\Gamma_{j+k}(t) = \Gamma_j(t) + \Gamma_k(t)$ are the antiderivatives,

$$\Gamma_j(t) = \int_0^t \gamma_j(t')dt' \quad (4.8)$$

and are all non-negative. Here, again $\Lambda_{44}(t) = 1$ is simply the trace preservation condition. Validity of (4.7) can be easily verified by checking that indeed $\frac{d}{dt} \Lambda(t) = L(t) \Lambda(t)$.

The Floquet pair $(P_t, e^{tX})$ determined by $\Lambda_t$ can then be calculated by finding its matrix counterpart $(P(t), e^{tX})$ and transforming back to $B(M_2)$. By (2.2) and (2.3),

$$X = -\frac{1}{T} \text{diag}\{\Gamma_{2+3}(T), \Gamma_{1+3}(T), \Gamma_{1+2}(T), 0\}, \quad (4.9a)$$

$$P(t) = \Lambda(t)e^{-tX} = \text{diag}\{e^{-\vartheta_{2+3}(t)}, e^{-\vartheta_{1+3}(t)}, e^{\vartheta_{1+2}(t)}, 1\}, \quad (4.9b)$$

for functions $\vartheta_{j+k}(t)$ being the shorthand for

$$\vartheta_{j+k}(t) = \Gamma_{j+k}(t) - \frac{\Gamma_{j+k}(T)}{T} t \quad (4.10)$$

By (4.8), functions $\Gamma_j(t)$ satisfy additivity property $\Gamma_j(t + T) = \Gamma_j(t) + \Gamma_j(T)$ and so, $P(t)$ is easily shown to be periodic as required by Floquet theorem. Inverting the vectorization and performing some mild algebra, one recovers original maps over $M_2$,

$$X(x) = \begin{pmatrix} -\beta_1(x_{11} - x_{22}) & \beta_2 x_{21} - \beta_3 x_{12} \\ \beta_2 x_{12} - \beta_3 x_{21} & \beta_1(x_{11} - x_{22}) \end{pmatrix}, \quad (4.11a)$$

$$P_t(x) = \begin{pmatrix} \xi_1(t) x_{11} + \xi_2(t) x_{22} & \chi_1(t) x_{12} - \chi_2(t) x_{21} \\ \chi_1(t) x_{21} - \chi_2(t) x_{12} & \xi_2(t) x_{11} + \xi_1(t) x_{22} \end{pmatrix}, \quad (4.11b)$$

where the following notation was introduced for brevity,

$$\beta_{1,2} = \frac{1}{2T} (\Gamma_1(T) \pm \Gamma_2(T)), \quad \beta_3 = \frac{1}{2T} (\beta_1 + 2\Gamma_3(T)), \quad (4.12)$$

$$\xi_{1,2}(t) = \frac{1}{2} \left(1 + e^{-\vartheta_1(t) - \vartheta_2(t)}\right), \quad \chi_{1,2}(t) = \frac{1}{2} \left(e^{-\vartheta_1(t) - \vartheta_3(t)} \pm e^{-\vartheta_2(t) - \vartheta_3(t)}\right). \quad (4.13)$$

Clearly, by diagonal structure of (4.9a), we have $\text{spec}(X)$ consisting of all diagonal elements of $X$; its eigenbasis is simply a canonical orthonormal basis $\{e_j\}_{j=1}^4$ of $\mathbb{C}^4$, $(e_j)_k = \delta_{jk}$, which consists (after inverting the vectorization) to eigenmatrices of maps $X, \Lambda_T$ in form $\varphi_j = \sum_{k=1}^4 \delta_{jk} F_k = F_j$, i.e. each Pauli matrix $\sigma_j$ spans one-dimensional eigenspace. This gives rise to set of characteristic multipliers

$$\text{spec}(\Lambda_T) = \exp\{T \cdot \text{spec}(X)\} = \{1, e^{-\Gamma_{2+3}(T)}, e^{-\Gamma_{1+3}(T)}, e^{-\Gamma_{1+2}(T)}\}, \quad (4.14)$$
asymptotic stability of solutions. claims of proposition 5. where both eigenvalues are or multiplicity 2. all of the above facts correspond to which, since Γ

CP-divisibility of semigroup part

then,

\[
\rho_t = \frac{1}{\sqrt{2}} \left( P_t(I) + \sum_{\pi \text{ even}} c_{\pi(1)} e^{-\frac{1}{2} \Gamma_{\pi(2)+\pi(3)}(T)} P_t(\sigma_{\pi(1)}) \right)
\]

for even permutations \( \pi \) in symmetric group \( S_3 \). Let us now verify properties of a solution, as they appeared in proposition 5 and theorems 1–3.

1. Properties of Floquet states. Eigenbasis of \( \Lambda_T \) is simply the Frobenius (Pauli) basis, \( \varphi_j = \frac{1}{\sqrt{2}} F_j \). If all functions \( \gamma_j(t) \neq 0 \), then necessarily \( \Gamma_j(T) > 0 \) for all \( j, k \in \{1, 2, 3\} \) and spec(\( X \)) is non-degenerate. If this is the case, only one of eigenvectors, \( \varphi_0 = \frac{1}{\sqrt{2}} I \), spans eigenspace \( E_{\Lambda_T}(1) \) and is of non-zero trace, being also positive semi-definite. If, on the other hand, there exists \( \Gamma_j(T) = 0 \), then necessarily \( \Gamma_j(T) = \Gamma_k(T) = 0 \). Assume, without loss of generality, \( j = 1, k = 2 \). Then,

\[
\text{spec}(X) = -\frac{1}{T} \{0, \Gamma_3(T)\}, \quad \text{spec}(\Lambda_T) = \{1, e^{-\Gamma_3(T)}\},
\]

where both eigenvalues are or multiplicity 2. All of the above facts correspond to claims of proposition 5.

2. Asymptotic stability of solutions. As clearly \( \text{spec}(\Lambda_T) \subset \mathbb{D}^1 \), all solutions \( \rho_j(t), \rho_t \) are stable. immediately, \( \text{(4.15)} \) yields a unique periodic limit cycle

\[
\rho_t^\infty = \frac{1}{\sqrt{2}} P_t(I) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

being, in this simple case, a trivial limit point in \( M_2 \), the maximally mixed state.

3. CP-divisibility of semigroup part \( \{e^{tX} : t \in \mathbb{R}_{+}\} \). This can be shown by checking that the expression \( \text{(4.11a)} \) for map \( X \) can be cast into

\[
X(x) = \sum_{j=1}^{3} \frac{\Gamma_j(T)}{T} \left( \sigma_j x \sigma_j - \frac{1}{2} \sigma_j \sigma_j, x \right)
\]

which, since \( \Gamma_j(t) \geq 0 \), is of standard form; therefore \( \{e^{tX}\} \subset \text{CP}_{t,p}(M_2) \) and is a CP-divisible contraction semigroup.

4. Trace and Hermiticity preservation of \( P_t \) \( \text{(4.11b)} \). One checks that if Frobenius basis \( \{F_i\} \) spanning \( M_d \) is Hermitian, then linear map \( T : M_d \to M_d \) is a *-map iff its corresponding matrix \( T_{ij} = (F_i, T(F_j))_F \) is real; this is exactly the case with matrix \( X(t) \) \( \text{(4.9b)} \). Moreover, \( T \) will preserve the trace iff \( T_{\delta_{ij}} = \delta_{ij} \), which also holds.

5. CP-divisibility of \( P_t \). To explicitly check whenever \( P_t \) is CP-divisible, we construct its corresponding propagator \( \mathcal{V}_{t,s} = P_t P_s^{-1} \) for any \( t \geq s \), and find sufficient and necessary conditions which guarantee \( \mathcal{V}_{t,s} \in \text{CP}_{t,p}(M_2) \). This is achieved by examining Choi matrix of \( \mathcal{V}_{t,s} \) and under additional requirement of infinite divisibility, imposed on Markovian dynamical maps; the result, explicitly presented in A.2, shows that \( \mathcal{V}_{t,s} \in \text{CP}_{t,p}(M_2) \) for all \( t \geq s \) in some interval \( I \subset \mathbb{R}_{+} \), if and only if

\[
\gamma_j(t) - \frac{\Gamma_j(T)}{T} \geq 0
\]

for all \( t \in I \) and \( j \in \{1, 2, 3\} \). This is equivalent to positive semi-definiteness of matrix

\[
a_t - \frac{1}{T} \int_0^T a_e dt' = \left[ \delta_{jk} \left( \gamma_j(t) - \frac{\Gamma_j(T)}{T} \right) \right]_{jk},
\]

exactly as claim 3 of theorem 2 states.
(6) **Complete positivity of $P_t$.** This is similarly verified by examining Choi matrix of $P_t$ (4.11b) in A.2, where not only sufficient, but also an explicit necessary condition for Markovianity of $P_t$ over $\mathbb{M}_2$ is found. In particular, it is shown that

$$\Gamma_j(t) - \frac{\Gamma_j(T)}{T} t \geq 0, \quad j \in \{1, 2, 3\}$$

(4.21)

is a sufficient condition for complete positivity at given $t \in \mathbb{R}_+$. This is equivalent to condition (3.3) in remaining claim 4 of theorem 2.

### 4.2. Periodically driven two-level system.

The second example concerns a **two-level system with periodically modulated Hamiltonian**, coupled to external reservoir via standard ladder operators constructed from Pauli matrices. We utilize the MME in usual standard form [3], however with time-dependent Hamiltonian part,

$$\frac{dp_t}{dt} = L_t(\rho_t) = -\frac{i\omega(t)}{2}[\sigma_3, \rho_t] + \gamma_\uparrow D_{\sigma_+}(\rho_t) + \gamma_\downarrow D_{\sigma_-}(\rho_t),$$

(4.22)

where $D_A$ is defined as $D_A(\rho) = A\rho A^* - \frac{1}{2}\{A^*A, \rho\}$, matrices $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \in M_2(\mathbb{R})$, $\sigma_\pm = \sigma_\mp^T$, are the usual ladder operators and $\gamma\downarrow, \gamma\uparrow > 0$ stand for pumping and dumping transition rates, respectively. System’s self Hamiltonian is $H_t = \frac{i}{2}\omega(t)\sigma_3$ and is diagonal in eigenvectors $e_0 = (0, 1)$, $e_1 = (1, 0)$. These eigenvectors denote the ground and exited state, respectively. Real function $\omega(t)$ is the energy difference between states $e_1$ and $e_0$, **periodically modulated** by some external quasi-classical source, $\omega(t) = \omega(t + T)$.

The corresponding Kossakowski matrix of Lindbladian $L_t$ in (4.22) is

$$a = \frac{1}{2} \left( \begin{array}{ccc} \gamma_\downarrow + \gamma_\uparrow & i(\gamma_\downarrow - \gamma_\uparrow) & 0 \\ -i(\gamma_\downarrow - \gamma_\uparrow) & \gamma_\downarrow + \gamma_\uparrow & 0 \\ 0 & 0 & 0 \end{array} \right) \geq 0,$$

(4.23)

which is constant. We next obtain solution in a form of Floquet pair by utilizing the same vectorization procedure as in previous example (we omit calculations for brevity, as the whole procedure is similar),

$$P_t(x) = \left( \begin{array}{c} e^{i\varpi(t)}x_{11} \\ e^{-i\varpi(t)}e^{i\varpi(T)}x_{12} \\ e^{-i\varpi(T)}x_{21} \\ e^{i\varpi(T)}x_{22} \end{array} \right),$$

(4.24a)

$$X(x) = \left( \begin{array}{c} x_{21} - \gamma_\downarrow x_{11} + \gamma_\uparrow x_{22} \\ \frac{-\gamma_\downarrow + \gamma_\uparrow}{2} + i\frac{\varpi(T)}{T} x_{21} - i\frac{\varpi(T)}{T} x_{12} \\ \frac{-\gamma_\downarrow + \gamma_\uparrow}{2} + i\frac{\varpi(T)}{T} x_{21} - i\frac{\varpi(T)}{T} x_{12} \\ x_{22} - \gamma_\downarrow x_{11} + \gamma_\uparrow x_{22} \end{array} \right),$$

(4.24b)

for antiderivative $\varpi(t) = \int_0^T \omega(t)dt'$. With some effort, $X$ can be then put in standard form

$$X = -i\frac{\varpi(T)}{2T}[\sigma_3, \cdot] + \gamma_\downarrow D_{\sigma_-} + \gamma_\uparrow D_{\sigma_+},$$

(4.25)

i.e. $\{e^{iX} : t \in \mathbb{R}_+\}$ is CP-divisible and claim 1 of theorem 2 holds; since $P_t$ does not alter diagonal elements of density matrix and $P_t(x)$ is Hermitian for Hermitian $x$, the same is true for claim 2.

One finds the spectrum of Choi matrix of $P_t$ to be $\text{spec}(C[P_t]) = \{0, 2\}$ ($k_0 = 3$), so $P_t \in \text{CP}_{t,p}(\mathbb{M}_2)$ for all $t \in \mathbb{R}_+$. Curiously, Choi matrix of its propagator, $V_{t,s} = P_t P_s^{-1}, t \geq s$, yields the same spectrum regardless of $t, s$ so map $P_t$ is CP-divisible **globally**, i.e. in whole $\mathbb{R}_+$. This is then confirmed by theorem 2, since, as $a$ is constant, inequalities (3.2) and (3.3) are always satisfied. We remark that this observation remains consistent with theorem 3 as global Markovianity of $P_t$ was allowed only if Kossakowski matrix was constant.
Eigendecomposition of matrix counterpart of map $X$ allows also to find $\text{spec}(X)$ and $\text{spec}(\Lambda_T)$, i.e. sets of characteristic exponents and multipliers,

$$\text{spec}(X) = \{ \mu_1 = 0, \mu_2 = -\gamma_l - \gamma_l, \mu_{3,4} = -\frac{1}{2}(\gamma_l + \gamma_l) \pm \frac{\imath \pi(\mathcal{D})}{\mathcal{D}} \},$$  \hspace{1cm} (4.26a)

$$\text{spec}(\Lambda_T) = \{ \lambda_1 = 1, \lambda_2 = e^{-T(\gamma_l + \gamma_l)}, \lambda_{3,4} = e^{-\frac{T}{2}(\gamma_l + \gamma_l)}e^{\pm \imath \pi(\mathcal{D})} \},$$  \hspace{1cm} (4.26b)

along with eigenvectors (put in corresponding order)

$$\varphi_1 = \frac{\sqrt{2}}{\gamma_l + \gamma_l}\text{diag}\{\gamma_l, \gamma_l\}, \quad \varphi_2 = \frac{1}{\sqrt{2}}\sigma_3, \quad \varphi_{3,4} = \pm \imath \sqrt{2}\sigma_\tau. \hspace{1cm} (4.27)$$

Hence, subset $\mathcal{E}_2$ of $\text{spec}(X)$ is again empty. We emphasize here, that the eigenbasis $\{\varphi_j\}$ is not orthogonal (w.r.t. Frobenius inner product) since $X$ is not normal. Again, $\text{spec}(\Lambda_T) \subset \mathbb{D}^1$ and is closed under complex conjugation. All eigenvectors apart from $\varphi_1$, i.e. those spanning eigenspaces $E_{\lambda_T}(\lambda)$ for $\lambda \neq 1$, are then traceless and non-positive semi-definite, as proposition 5 states. Two real multipliers $\lambda_{1,2}$ are simple eigenvalues and so $\varphi_{1,2}$ are Hermitian; naturally, $\varphi_1 \geq 0$ and $\varphi_3 = \varphi_4^*$, as $\lambda_3 = \bar{\lambda}_4$.

An actual solution is then obtained with formulas (2.14) and (2.17),

$$\rho_t = c_1\varphi_1 + c_2e^{-\left(\gamma_l + \gamma_l\right)t}\varphi_2 + e^{-\frac{T}{2}(\gamma_l + \gamma_l)}\left(c_3e^{\imath \pi(\mathcal{D})}t\phi_3(t) + c_4e^{-\imath \pi(\mathcal{D})}t\phi_4(t)\right), \hspace{1cm} (4.28)$$

where Floquet states $\phi_{3,4}(t) = P_t(\varphi_{3,4})$ are explicitly defined as

$$\phi_3(t) = -\imath \sqrt{2} e^{-\imath \pi(\mathcal{D})}e^{-\frac{T}{4}(\gamma_l + \gamma_l)}\sigma_+, \quad \phi_4(t) = \phi_3(t)^* = \imath \sqrt{2} e^{\imath \pi(\mathcal{D})}e^{-\frac{T}{4}(\gamma_l + \gamma_l)}\sigma_- \hspace{1cm} (4.29)$$

As $\rho_0 = \sum_{j=1}^{4} c_j\varphi_j$, one can express coefficients $c_j$ in terms of matrix elements $\rho_{jk}(0)$ of initial density matrix $\rho_0 = \sum_{j,k=1}^{4} \rho_{jk}(0)|e_j\rangle\langle e_k|$ calculated in Hamiltonian eigenbasis; these are found to be

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{\gamma_l \sqrt{2}}{\gamma_l + \gamma_l} - \sqrt{2}\rho_{11}(0), \quad c_3 = \frac{i}{\sqrt{2}}\rho_{12}(0), \quad c_4 = c_3^*, \hspace{1cm} (4.30)$$

where trace normalization and Hermiticity of $\rho_0$ were implicitly used. Clearly, solution (4.29) is stable and the asymptotic periodic orbit $\rho_0^\infty$ in this case is, similarly to previous example, also a single limit point, $\rho_t^\infty = \frac{1}{\sqrt{2}}\varphi_1$.

5. Conclusions

We presented an insight into general applicability of Floquet theory in description of Markovian Master Equations given by periodic, finite-dimensional Lindbladians in standard form. The performed analysis allowed for formulating some remarks on Floquet normal form of the induced quantum dynamical maps, partially in general case, and especially in simplified case of commutative Lindbladian families. In particular, it was shown that in generic case of periodic $L_t$, it is impossible for both maps of the Floquet pair to be globally simultaneously Markovian in commutative case. It was also shown that the traditional results of Floquet theory, like analysis of stability based on characteristic multipliers of the system, still possess an excellent application in case of completely positive dynamics. Two examples of possible non-trivial physical applicability of such Floquet-Lindblad theory were also briefly examined.

The question of exact properties of Floquet pair (i.e. simultaneous CP-divisibility, complete positivity and mutual connections between them etc.) in global case of non-commutative Lindbladian families – which clearly is the most interesting and important – remains an open problem. Unfortunately, even in simplest cases of low-dimensional matrix algebras the matter seems quite involved, mainly because
of various commutativity-related issues. However, it seems that global Markovianity of both maps of Floquet pair is rather unlikely to be possible, apart perhaps from some trivial cases.

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**Appendix A. Mathematical supplement**

### A.1. Secondary results and proofs.

**Lemma 1.** The following hold for every linear *-map T on M_d: a) T admits a unique Hermitian matrix [t_{jk}] ∈ M_d, such that T(x) = ∑_{j,k=1}^{d^2} t_{jk} F_j x F_k^* for every x ∈ M_d; b) T is completely positive iff [t_{jk}] ⩾ 0; c) if T is trace preserving, then there exist Hermitian matrices G, K ∈ M_d such that

\[ T(x) = x + i[G, x] - \{K, x\} + \sum_{j,k=1}^{d^2-1} t_{jk} F_j x F_k^*. \]  

(A.1)

**Proof.** Structure theorems by de Pillis [37], Jamiołkowski [38], Choi [39] and Hill [40, 41] allow to represent any *-map T in a form T(x) = ∑_{j} α_j X_j x X_j^*, where X_j ∈ M_d and α_j ∈ R (α_j ⩾ 0 iff T is completely positive). It suffices to expand X_j = ∑_{i,j} F_i x_i j F_j in Frobenius basis and collect expansion coefficients in form of new matrix, t_{jk} = ∑_{i,j} x_{i,j} x_{j,k}. Claims a) and b) then follow by examining properties of [t_{jk}]. For c), splitting sums in general decomposition of T allows one to write

\[ T(x) = Ex + x E^* + Ψ(x), \]  

(A.2)

for Ψ(x) = ∑_{j,k=1}^{d^2-1} t_{jk} F_j x F_k^* and matrix E = \frac{1}{d^2} I + \frac{1}{d^2} ∑_{j=1}^{d^2} t_{j} d^2 F_j, where we employed hermiticity of [t_{jk}]. E admits a unique Cartesian decomposition E =
M + iN, where $M = \frac{1}{2}(E + E^*)$ and $N = \frac{1}{2i}(E - E^*)$ are both Hermitian; therefore
\[
T(x) = i[N, x] + Mx + xM + \Psi(x).
\] (A.3)
Trace preservation condition imposed on (A.3) and cyclicity of trace imply Proposition 7.

**Proposition 6.** Let $n \geq 1$ and $a_\alpha = [a_{\mu\nu}(t)] \in M_n$ is Hermitian, $t \in \mathbb{R}$. If function $f_x(t) = (x, a, x)$ is constant for every $x \in \mathbb{C}^n$, then necessarily $a_\alpha$ is constant.

**Proof.** If $f_x$ is constant for every $x$, then in particular one can take $x = e_\alpha$, the canonical basis in $\mathbb{C}^n$. Each $e_\alpha$ contains 1 at $\alpha$-th place and zeros elsewhere; therefore
\[
f_{e_\alpha}(t) = \langle e_\alpha, a(t)e_\alpha \rangle = \sum_{\mu,\nu=1}^n a_{\mu\nu}(t)\delta_{\alpha\mu}\delta_{\alpha\nu} = a_{\alpha\alpha}(t),
\] (A.4)
and so each diagonal element $a_{\alpha\alpha}(t)$ is constant. On the other hand, let us take any $\alpha, \beta \in \{1, \ldots, n\}$, s.t. $\alpha \neq \beta$ and define a vector $y = e_\alpha + ie_\beta$. For such $y$, take also its componentwise-conjugate $\overline{y} = e_\alpha - ie_\beta$. One then checks, that
\[
f_y(t) = a_{\alpha\alpha}(t) - a_{\beta\beta}(t) + 2i \text{Re}a_{\alpha\beta}(t),
\] (A.5)
where Hermiticity of $a_\alpha$ was used; this however is constant iff $\text{Re}a_{\alpha\beta}(t)$ is constant (since we already shown $a_{\alpha\alpha}(t) = \text{const}$). Similarly, by calculating $f_{\overline{y}}(t)$ one concludes that also $\text{Im}a_{\alpha\beta}(t)$ must be constant, and hence all $a_{\mu\nu}(t) = \text{const}$. □

### A.2. Complete positivity and CP-divisibility of map $P_t$.
Here we provide justification for conditions (4.19) and (4.21), which are sufficient and necessary for complete positivity and Markovianity of map $P_t$ (4.11b). Proof will rely on determining geometrical conditions for positivity of certain Choi matrices, however with crucial help from infinite divisibility assumption of Markovian dynamics. For the following result, let us define a vector-valued function $\vartheta : \mathbb{R}_+ \to \mathbb{R}^3$,
\[
\vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \vartheta_3(t)), \quad \vartheta_j(t) = \gamma_j(t) - \frac{\Gamma_j(T)}{T},
\] (A.6)

**Proposition 7.** Map $P_t$ (4.11b) yielded by equation (4.5) over $M_2$ satisfies:

1. $P_t \in \text{CP}_{\Gamma,T}(M_2)$ iff $\vartheta(t) \in A = \bigcup_{j=1}^3 A_j$, where $A_j \subset \mathbb{R}^3$ are unbounded regions,
\[
A_1 = \{(x, y, z) : x, y \geq 0, z \geq \ln \frac{\cosh \frac{1}{2}(x - y)}{\cosh \frac{1}{2}(x + y)}\},
\] (A.7a)
\[
A_2 = \{(x, y, z) : x > x + y > 0, z \geq \ln \frac{\sinh \frac{1}{2}(x - y)}{\sinh \frac{1}{2}(x + y)}\},
\] (A.7b)
\[
A_3 = \{(x, y, z) : y > x + y > 0, z \geq \ln \frac{\sinh \frac{1}{2}(y - x)}{\sinh \frac{1}{2}(x + y)}\}
\] (A.7c)

2. $P_t$ is CP-divisible in some interval $\mathcal{I} \subset [0, T]$ iff $\vartheta(t) - \vartheta(s) \in \mathbb{R}^3$ for all $t, s \in \mathcal{I}, t \geq s$, which is the case iff $\gamma_j(t) - \frac{\Gamma_j(T)}{T} \geq 0$ for $j \in \{1, 2, 3\}$ and for all $t \in \mathcal{I}$.

**Proof.** For claim 1, calculate the Choi matrix of $P_t$,
\[
\mathcal{C}[P_t] = \begin{pmatrix}
\xi_1(t) & 0 & 0 & \chi_1(t) \\
0 & \xi_2(t) & -\chi_2(t) & 0 \\
0 & -\chi_2(t) & \xi_2(t) & 0 \\
\chi_1(t) & 0 & 0 & \xi_1(t)
\end{pmatrix},
\] (A.8)
Solution of this system may be then divided into three unbounded regions,

\begin{align}
\mathcal{B}_1 &= \{ \alpha_1, \alpha_2 \leq 1, \alpha_3 \leq \frac{1+\alpha_2}{\alpha_1+\alpha_2} \}, \\
\mathcal{B}_2 &= \{ \alpha_1 < 1, 1 < \alpha_2 < \alpha_1^{-1}, \alpha_3 \leq \frac{1+\alpha_2}{\alpha_1-\alpha_2} \}, \\
\mathcal{B}_3 &= \{ \alpha_1 > 1, \alpha_2 < \alpha_1^{-1}, \alpha_3 \leq \frac{1-\alpha_2}{\alpha_1-\alpha_2} \}.
\end{align}

Put also \( \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \). By reverting the \( \alpha_j \) substitution, regions \( \mathcal{A}_j \) show up as preimages of \( \mathcal{B}_j \) under a mapping \( (x, y, z) \mapsto (e^{-x}, e^{-y}, e^{-z}) \). A schematic plot of region \( \mathcal{A} \) is also presented in figure 1.

![Figure 1. Schematic plot of region \( \mathcal{A} \) for \( x, y, z \in [-5, 5] \). All bounding surfaces \( \mathcal{F}_{1,2,3} \) are tangent to appropriate planes \( x, y, z = 0 \) at the origin. The whole region is invariant w.r.t. rotations by angle \( 2\pi n / 3, \ n \in \mathbb{Z} \), around axis \( x = y = z \).](image)

Finally, claim 2 involves checking whether the propagator of \( P_t \), defined by simple expression \( \mathcal{V}_{t,s} = P_t P_s^{-1} \), is completely positive. This is achieved by computing its matrix counterpart \( \mathbf{P}(t) \mathbf{P}(s)^{-1} \) and transforming to \( B(\mathbb{M}_2) \). The result can be shown to be, due to diagonal structure of \( \mathbf{P}(t) \), similar to (4.11b),

\begin{align}
\mathcal{V}_{t,s}(x) &= \begin{pmatrix}
\xi_1(t, s) x_{11} + \xi_2(t, s) x_{22} & \chi_1(t, s) x_{12} - \chi_2(t, s) x_{21} \\
\chi_1(t, s) x_{21} - \chi_2(t, s) x_{12} & \xi_2(t, s) x_{11} + \xi_1(t, s) x_{22}
\end{pmatrix}, \\
&= \begin{pmatrix}
\frac{1}{2} \left( 1 + e^{-\delta_1(t, s)} - e^{-\delta_2(t, s)} \right) \\
\frac{1}{2} \left( 1 + e^{-\delta_2(t, s)} - e^{-\delta_1(t, s)} \right)
\end{pmatrix}
\end{align}

with a new set of two-variable functions \( \xi_{1,2}, \chi_{1,2} \) and \( \delta_j \),

\begin{align}
\xi_{1,2}(t, s) &= \frac{1}{2} \left( 1 + e^{-\delta_1(t, s)} - e^{-\delta_2(t, s)} \right), \quad (A.13a)
\end{align}
\[ \chi_{1,2}(t, s) = \frac{1}{2} \left( e^{-\delta_1(t,s) - \delta_3(t,s)} \pm e^{-\delta_2(t,s) - \delta_3(t,s)} \right), \]  
(A.13b) 
\[ \delta_j(t, s) = \vartheta_j(t) - \vartheta_j(s), \quad j \in \{1, 2, 3\}. \]  
(A.13c)

By its similarity to (4.11b), Choi matrix of \( V_{t,s} \) is of almost the same form as (A.8), however with \( \delta_j(t, s) \) in place of \( \vartheta_j(t) \). Requiring non-negativity of its spectrum leads, by introducing variables \( \alpha_j = e^{-\delta_j(t, s)} \), to exactly the same system of inequalities as (A.10). Therefore, \( C[V_{t,s}] \geq 0 \) and \( V_{t,s} \in \text{CP}_{t,p}(M_2) \) if and only if \( (\alpha_j) \in B \), or equivalently, if 
\[ \delta(t, s) = \vartheta(t) - \vartheta(s) = (\delta_1(t, s), \delta_2(t, s), \delta_3(t, s)) \in \mathcal{A}. \]  
(A.14)

However, the requirement of divisibility allows to greatly refine condition (A.14). First, notice that each \( P_t \in \text{CP}_{t,p}(M_2) \) is uniquely described by a vector \( \vartheta(t) \in \mathcal{A} \) and function \( t \mapsto P_t \) is represented by differentiable curve \( t \mapsto \vartheta(t) \). Similarly, every map of a form \( P_t P_{t'}^{-1} \) is bijectively determined by a vector \( \delta(t, t') \), with \( \delta(t, t) \) corresponding to identity map for any \( t \geq 0 \). Geometrically, function \( t \mapsto \delta(t, s) \) for some constant \( s \) is also a curve, created by translating curve \( \vartheta \) by constant vector \( -\vartheta(s) \), such that point \( \vartheta(s) \) is mapped into point \( 0 = (0, 0, 0) \), the origin. With any such curve, one associates its velocity,
\[ v(t) = \frac{d\vartheta(t)}{dt} = \frac{d\delta(t, s)}{dt}, \]  
(A.15)
which is tangent to it at point \( \vartheta(t) \). Suppose now \( P_t \) is CP-divisible in some interval \( I \subset \mathbb{R}_+ \). Then, for arbitrarily chosen \( t, t', s \in I \) such that \( t' \in [s, t] \), propagator \( V_{t,s} \) is a composition of two subsequent propagators, \( V_{t,s} = V_{t,t'} V_{t',s} \), both of them being again completely positive and divisible. As such, they are both uniquely described by some vectors \( \delta(t, t'), \delta(t', s) \in \mathcal{A} \). Divisibility condition is then equivalent to the addition rule
\[ \delta(t, s) = \delta(t, t') + \delta(t', s), \quad t' \in [s, t]. \]  
(A.16)
Suppose that the curve \( \vartheta \) corresponding to \( P_t \) is s.t. any component of its velocity, \( v_j(t) \), is negative anywhere in \( \mathbb{R}_+ \). Then, as \( t \mapsto \vartheta(t) \) is continuous, there exists an interval \([t_1, t_2]\) such that \( v_j(t_1) = v_j(t_2) = 0 \) and \( v_j(t) < 0 \) for all \( t \in (t_1, t_2) \), i.e. \( v(t) \) points in the direction outside of set \( \mathcal{A} \) within \( (t_1, t_2) \). Take any fixed \( s \in (t_1, t_2) \); necessarily, \( v_j(s) < 0 \). Then, a curve \( \delta(\cdot, s) \), starting at 0 is a geometrical representation of \( V_{t,s} \) for \( t \geq s \), as mentioned earlier. However, the velocity vector at the origin \( v(s) \notin \mathbb{R}^1_+ \) and so the curve \( \delta(\cdot, s) \) is initially directed outside of \( \mathbb{R}^1_+ \), i.e. there surely exists some \( t' > s \) small enough such that \( \delta(t', s) \notin \mathbb{R}^1_+ \). Moreover, it can be also shown that even \( \delta(t', s) \notin \mathcal{A} \): to achieve this, consider one of the boundary surfaces of region \( \mathcal{A} \) along one of the axes. Since \( \mathcal{A} \) is invariant with respect to rotations by angle \( 2n\pi/3 \), \( n \in \mathbb{Z} \) around axis \( x = y = z \), without loss of generality we can take the surface \( F_1 \), the lowest boundary of sub-region \( A_1 \) (A.7a).

Definition of \( \mathcal{A} \) yields that \( F_1 \) can be represented as a function \( F_1(x, y) \) given by formula
\[ F_1(x, y) = \ln \frac{\cosh \frac{\pi}{2}(x-y)}{\cosh \frac{\pi}{2}(x+y)}, \quad F_1 : \mathbb{R}^2_+ \rightarrow [0, -\infty). \]  
(A.17)
It is easy to notice \( \lim_{(x,y) \to (0,0)} F_1(x, y) = 0 \), with both \( x, y \) tending to zero from above. Let us consider any plane \( P_0 \) containing the \( z \) axis, spanned by vector \((0, 0, 1)\) and any vector \( n = (n_x, n_y, 0) \), \( n_x, n_y \geq 0 \), lying in plane \( z = 0 \) (see fig. 2).
Intersection of \( P_n \) and \( F_1 \) defines a convex curve \( \phi = (\phi_1, \phi_2, \phi_3) \) which may be given in parametric form as

\[
\phi_1(\xi) = n_x \xi, \quad \phi_2(\xi) = n_y \xi, \quad \phi_3(\xi) = F_1(x(\xi), y(\xi)) = \ln \frac{\cosh \frac{\xi}{2} (n_x - n_y)}{\cosh \frac{\xi}{2} (n_x + n_y)}.
\]

One can check, that the velocity vector \( \frac{d\phi(\xi)}{d\xi} \) of curve \( \phi \) simply evaluates to \( n \) for \( \xi = 0 \). In consequence, all vectors tangent to \( F_1 \) at \( 0 \) are also tangent to the plane \( z = 0 \). Likewise, all vectors tangent to surfaces \( F_2 \) and \( F_3 \) at \( 0 \) are also tangent to planes \( x = 0 \) and \( y = 0 \), respectively. Therefore, if velocity \( \mathbf{v}(s) \) of curve \( \delta(\cdot, s) \) at \( 0 \) has negative \( j \)-th component, then point \( t' > s \) can be chosen in such way that segment of curve \( \delta(t, s) \) for \( t \in [s, t'] \) is not enclosed by surface \( F_j \), and in the result, not in \( \mathcal{A} \). In consequence, \( \delta(t', s) \notin \mathcal{A} \) and so \( \mathcal{V}_{t', s} \notin \text{CP}_{t, p}(M_2) \). We have therefore found a division \( \mathcal{V}_{t, s} = \mathcal{V}_{t', t'} \mathcal{V}_{t', s} \) such that at least one of the propagators at the r.h.s. fails to be completely positive; therefore, \( P_t \) cannot be CP-divisible.

From this we imply that a curve \( \vartheta \) can represent a CP-divisible map iff \( \mathbf{v}(t) \in \mathbb{R}^3_+ \) for all \( t \in \mathcal{I} \), i.e. if condition

\[
\frac{d\vartheta_j(t)}{dt} = \gamma_j(t) - \frac{\Gamma_j(T)}{T} \geq 0,
\]

holds for all \( t \in \mathcal{I} \) and \( j \in \{1, 2, 3\} \). This concludes the proof. \( \square \)