Abstract
In this paper, we solve the Dirichlet problem for Sobolev maps between singular metric spaces that extends the corresponding result of Guo and Wenger (Commun Anal Geom 28(1):89–112, 2020). The main new ingredient in our proofs is a suitable extension of the theory of trace for metric valued Sobolev maps developed by Korevaar and Schoen (Commun Anal Geom 1(3–4):561–659, 1993). We also develop a theory of trace in the borderline case, which investigates a sharp condition to characterize the existence of traces.

Keywords Metric valued Sobolev spaces · Dirichlet problem · Upper gradients · Hajlasz–Sobolev space · Trace operator

Mathematics Subject Classification 58E20 · 46E35 · 49Q10

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1 Introduction

The nonlinear Dirichlet problem associated to the $p$-harmonic mapping system in an Euclidean domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, asks for a continuous map $u : \Omega \to \mathbb{R}^m$ so that

\[
\begin{cases}
\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]

The case $p = 2$ corresponds to the classical Dirichlet boundary value problem associated to the harmonic mapping system. An equivalent way to formulate the general Dirichlet problem is to consider energy minimizing mappings via the Euler–Lagrange equations. To be more precise, one considers minimizers of the $p$-energy

\[
E_p(u) := \int_\Omega |\nabla u|^p dx.
\]

The above two formulations are not necessarily equivalent in general when we move from Euclidean spaces to Riemannian manifolds. Given two Riemannian manifolds $(M, g)$ and $(N, h)$, there is a natural $p$-energy functional acting on smooth maps

\[
E_p(u) := \int_M |\nabla u|^p d\mu,
\]

where $|\nabla u|$ is the Riemannian length of the gradient of $u$ and $\mu$ is Riemannian volume induced by $g$ on $M$. Minimizers of the $p$-energy functional are called minimizing $p$-harmonic mappings, while critical points are called weakly $p$-harmonic mappings. In general, we only have the one-side inclusion:

\[
\{\text{minimizing } p \text{ harmonic mappings}\} \subset \{\text{weakly } p \text{ harmonic mappings}\}.
\]

The case $p = 2$ corresponds to the classical harmonic mappings. We refer the interested readers to [34] for the theory of harmonic mappings, and to [4, 9, 14, 30] for the theory of $p$-harmonic mappings, between Riemannian manifolds.

One of the classical methods to solve the Dirichlet problem in the smooth setting is the direct method from the calculus of variations. To apply it, one essentially needs the following four ingredients:

- A suitable $L^p$ theory for traces of manifold valued Sobolev maps;
- A global $p$-Poincaré inequality for Sobolev maps with zero trace;
- The Rellich–Kondrachov compactness theorem for manifold valued Sobolev maps;
- Lower semicontinuity of the energy functional $E_p$ with respect to $L^p$-convergence.

With all these ingredients at hand, the proof goes roughly as follows: Let $\{u_k\} \subset W^{1,p}(\Omega, N)$ be an energy minimizing sequence subordinate to the Dirichlet boundary condition $Tu_k = T\phi$ on $\partial \Omega$, where $\phi \in W^{1,p}(\Omega, N)$ is a fixed map and $T : W^{1,p}(\Omega, N) \to L^p(\partial \Omega, N)$ is the trace operator. The an easy application of the global Poincaré inequality, together with the characterization of traces of Sobolev maps, would give the boundedness of $\{u_k\}$...
in $W^{1,p}(\Omega, N)$ (with respect to the Sobolev norm) and thus by the Rellich–Kondrachov compactness theorem for Sobolev spaces, we know that there exists a limiting map $u \in W^{1,p}(\Omega, N)$ such that a further subsequence $\{u_k\}$ converges in $L^p(\Omega, N)$ to $u$. The lower semicontinuity of the energy functional implies the $p$-energy of $u$ would attain the minimum, and at the same time, the convergence result for traces of Sobolev maps shall imply $Tu = T\phi$ on $\partial\Omega$. Therefore, $u$ is a proper solution to the Dirichlet problem.

Now, consider a mapping $u: \Omega \to Y$, where $X = (X, d_X, \mu)$ is a metric measure space and $Y = (Y, d_Y)$ a metric space. Unlike the smooth Riemannian case, there is no natural $p$-energy functional associated to a sufficiently regular map. Indeed, there are several well-known (and generally different) $p$-energy functionals existing in the literature: the Korevaar–Schoen energy functional [22], the Jost energy functional [20], the Hajlasz energy functional [11], the upper gradient energy functional [15, 35], the Cheeger energy functional [3] and the Kuwae–Shioya energy functional [27]; see [12, 16] for more energy functionals and the associated Sobolev spaces of metric valued maps. We would like to remark that the general interest in considering harmonic mappings in the singular metric setting dates back to the remarkable work of Gromov–Schoen [6], where the authors found important applications to rigidity problems for certain discrete groups; see [7] for a detailed survey on the theory of harmonic mappings between singular metric spaces.

In this article, we shall focus on the upper gradient energy functional and solve the associated Dirichlet problem. Throughout this paper, $X = (X, d_X, \mu)$ is assumed to be a complete metric measure space, $Y = (Y, d_Y)$ a complete metric space, $\Omega \subset X$ a bounded domain and $\mathcal{H}$ a $\sigma$-finite Borel regular measure on $\partial\Omega$. For notational simplicity, we sometimes drop the subscripts $X, Y$ from the distances $d_X, d_Y$ and simply write $d$.

Before the statement of our main results, we recall a couple of definitions. One of the key concepts we shall need is the following definition of trace for metric valued functions.

**Definition 1.1** Let $u: \Omega \to Y$ be a $\mu$-measurable function. Fix a point $x \in \partial\Omega$. If for some point $Tu(x) \in Y$, it holds

$$\lim_{r \to 0^+} \frac{1}{\mu(B(x,r) \cap \Omega)} \int_{B(x,r) \cap \Omega} d_Y(u, Tu(x)) \, d\mu = 0,$$

(1.1)

then we say that the trace $Tu(x)$ of $u$ at $x \in \partial\Omega$ exists. Also, we say that $u$ has a trace $Tu$ on $\partial\Omega$ if $Tu(x)$ exists for $\mathcal{H}$-almost every $x \in \partial\Omega$.

As in the smooth setting, we need to separate a class of admissible domains so that the Dirichlet problem is solvable.

**Definition 1.2** We say that the triple $(\Omega, \mu, \mathcal{H})$ is weakly $(q, \theta)$-admissible, $1 < q < \infty$ and $\theta > 0$, if

- $\mu$ is a doubling measure on $\Omega$;
- $\mathcal{H}$ is upper codimension-$\theta$ regular on $\partial\Omega$;
- $\Omega$ supports a local $q$-Poincaré inequality.

We say that $(\Omega, \mu, \mathcal{H})$ is $(q, \theta)$-admissible if in addition $\Omega$ supports a global $p$-Poincaré inequality for all $p \geq q$, that is, for $u \in H^{1,p}(\Omega)$ with $Tu = 0$ $\mathcal{H}$-almost everywhere on $\partial\Omega$, it holds

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|g_u\|_{L^p(\Omega)}.$$

(1.2)

It is clear that if $(\Omega, \mu, \mathcal{H})$ is (weakly) $(q, \theta)$-admissible, then it is (weakly) $(p, \theta)$-admissible for any $p > q$. 

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It is clear from the definition that admissible domains form a natural extension of the class of bounded Lipschitz domains (with the induced Hausdorff measure on the boundary) in a smooth Riemannian manifold. There are many other examples, for instance by [33], the class of bounded $C^{1,1}$ hypersurfaces in the Heisenberg group $\mathbb{H}^1$ (or more generally in any step 2 homogeneous groups) with the induced boundary measure (which can be chosen as either the 3-dimensional Hausdorff measure, or the 3-dimensional Minkowski content, or the perimeter measure) is $(q, 1)$-admissible for any $q > 1$; see also Example 5.5 below for a sophisticated example of admissible domain.

For the next concept, we refer to Sect. 2.2 below for the notion of a non-principal ultrafilter $\omega$ on $\mathbb{N}$ and the definition of ultra-limit $\lim_\omega a_m$ of a bounded sequence $\{a_m\}$ of real numbers. Let $(Y, d)$ be a metric space and $\omega$ a non-principal ultrafilter on $\mathbb{N}$. Denote by $Y_\omega$ the set of equivalent classes $[(y_m)]$ with the sequence $\{y_m\}$ in $Y$ satisfying $\sup_m d(y_1, y_m) < \infty$, where sequences $\{y_m\}$ and $\{y'_m\}$ are identified if $\lim_\omega d(y_m, y'_m) = 0$. The metric space obtained by equipping $Y_\omega$ with the distance $d_\omega([(y_m)], [(y'_m)]) = \lim_\omega d(y_m, y'_m)$ is called the ultra-completion or ultra-product of $Y$ with respect to $\omega$. It is clear that $Y$ isometrically embeds into $Y_\omega$ via the map $\iota: Y \to Y_\omega$, which assigns to $x$ the equivalent class $[(x)]$ of the constant sequence $\{x\}$. The following definition was introduced in [8].

**Definition 1.3** A metric space $Y$ is said to be 1-complemented in some ultra-completion of $Y$ if there exists a non-principal ultrafilter $\omega$ on $\mathbb{N}$ for which there is a 1-Lipschitz retraction from $Y_\omega$ to $Y$.

The class of metric spaces that are 1-complemented in some ultra-completion includes all proper metric spaces, all dual Banach spaces, some non-dual Banach spaces such as $L^1$, all Hadamard spaces and injective metric spaces; see [8, Proposition 2.1].

Let $N^{1,p}(\Omega, Y)$ be the Sobolev space based on upper gradients and $E^p(u)$ the $p$-upper gradient energy functional of $u$ (see Sect. 2 below for precise definition). Our first main result can be formulated as follows.

**Theorem 1.4** Suppose $(\Omega, \mu, H)$ is a $(q, \theta)$-admissible domain with $q > \theta$ and $Y$ is a metric space that is 1-complemented in some ultra-completion of $Y$. Then for each $\phi \in N^{1,p}(\Omega, Y)$, $1 < q < p$, there exists a mapping $u \in N^{1,p}(\Omega, Y)$ with $Tu = T\phi$ at $H$-a.e. in $\partial \Omega$ such that

$$E^p(u) = \inf \left\{ E^p(v) : v \in N^{1,p}(\Omega, Y) \text{ and } T v = T\phi \text{ at } H\text{-a.e. in } \partial \Omega \right\}.$$ 

To the best of our knowledge, Theorem 1.4 seems to be the most general setting for the solvability of the Dirichlet problem. In particular, it can be viewed as a natural extension of [36, Theorem 5.6], [8, Theorem 1.4] and [7, Theorem 1.1]. In the formulation of Theorem 1.4, we need the fact that the trace operator $T$ is well-defined on $N^{1,p}(\Omega, Y)$. When $\Omega \subseteq X$ is a bounded Lipschitz domain in a smooth Riemannian manifold and $Y$ is a complete metric space, this fact was established by Korevaar–Schoen in [22, Section 12].

Our second main result extends it to the more general singular setting.

**Theorem 1.5** Suppose $(\Omega, \mu, H)$ is a weakly $(p, \theta)$-admissible domain with $p > \theta$ and $Y$ is a complete metric space embedded isometrically into some Banach space. Then the trace operator

$$T : N^{1,p}(\Omega, Y, d\mu) \to L^p(\partial \Omega, Y, dH)$$

is bounded.
Note that in general Theorem 1.5 fails for the borderline case \( p = \theta \). In Sect. 5, we shall deduce sharper result in the borderline case by adding a weight \( \omega \) to the underlying measure \( \mu \); see Theorems 5.1 and 5.3 below.

Another crucial fact that we shall need in the proof of Theorem 1.4 is the following convergence result for traces of Sobolev maps with uniformly bounded energy. When \( \Omega \subset \tilde{X} \) is a bounded Lipschitz domain in a smooth Riemannian manifold and \( Y \) is a complete metric space, this fact was established by Korevaar–Schoen in [22, Theorem 1.12.2].

**Theorem 1.6** Suppose \((\Omega, \mu, \mathcal{H})\) is weakly \((p, \theta)\)-admissible with \( p > \theta \) and \( Y \) is complete. Let \( \{u_i\} \subset \mathcal{N}^{1,p}(\Omega, Y) \) be a sequence with uniformly bounded energy, that is,

\[
\sup_{i \in \mathbb{N}} \int_{\Omega} g_{u_i}^p \, d\mu < \infty.
\]

If \( u_i \) converges to some \( u \in \mathcal{N}^{1,p}(\Omega, Y) \) in \( L^p(\Omega, Y) \), then \( Tu_i \to Tu \) in \( L^p(\partial \Omega, Y, d\mathcal{H}) \).

We next briefly comment on the ideas used in the proofs of our main theorems. As pointed out before, the proof of Theorem 1.4 relies essentially on the direct method from the calculus of variations. In the setting of Theorem 1.4, a version of the Rellich–Kondrachov compactness theorem for metric valued Sobolev maps was obtained in [8] and lower semicontinuity of the upper gradient energy is well-known, and thus the essential missing ingredient is a suitable \( L^p \) theory for traces of metric valued Sobolev maps.

The definition of trace in [22, Section 1.12] relies on the Lipschitz differentiable structure of \( \Omega \), which looks apparently different than what we have introduced here. When the underlying spaces are metric measure spaces with much less geometric properties, Definition 1.1 becomes a more natural way to define the trace. When the target space \( Y \) is \( \mathbb{R} \), the trace results using Definition 1.1 are under developing; see [26, 28, 29, 31, 32]. An useful observation in the proof of Theorem 1.5 is that by Lemma 3.2, the isometric embedding (of \( Y \) into some Banach space) and the trace operator commute, thus we only need to focus on the case when the target space \( Y \) is a Banach space.

In Theorem 1.5, it requires that \( p > \theta \). It is natural to consider the borderline case when \( \theta = p \). Theorem 5.1 and Theorem 5.3 deal with this borderline case. Especially from Theorem 5.3 and Example 5.5, we obtain a sharp condition to full characterize the existence of traces if additionally \( \Omega \) is a John domain with compact closure.

In this paper, we mainly consider the existence result for the Dirichlet problem. A natural question would be the interior regularity of the solutions. In the case when the target metric space \( Y = \mathbb{R} \), there are local Lipschitz regularity results for solutions of the Dirichlet problem associated to the Cheeger energy and the upper gradient energy functional; see [1, 19, 24]. For general metric valued target space, there is a recent remarkable work due to Zhang and Zhu [39], where the authors derived local Lipschitz regularity of solutions of the Dirichlet problem associated to the Korevaar–Schoen energy functional; see also [38]. Recently, Guo and Xiang [10] established local Hölder continuity of solutions of the Dirichlet problem associated to a variant of the Korevaar–Schoen \( p \)-energy functional. However, the method there relies crucially on the structure of the energy functional and does not extend to the upper gradient energy functional. We thus formulate it as an open question below.

**Open question** Under the assumptions of Theorem 1.4, can we further establish local Hölder regularity result of the solution \( u \)? If so, under some kind of curvature assumption for \( \Omega \) as in [39] or [24], can we establish local Lipschitz regularity of \( u \) for the harmonic case \( p = 2 \)?

The paper is organized as follows. In Sect. 2, we recall the necessary definitions concerning metric valued Sobolev maps via upper gradients and ultra-completion of metric spaces. In
 Sect. 3, we give an extension of the trace theory of Korevaar–Schoen and prove our trace theorem. Section 4 is devoted to the proof of Theorem 1.4. In the final Sect. 5, we present a refined theory of trace in the borderline case.

2 Preliminaries

Let \((X, d_X, \mu)\) be a complete metric measure space and \((Y, d_Y)\) a complete metric space. Let \(\Omega \subset X\) be a bounded domain. We say that the measure \(\mu\) is a doubling measure on \(\Omega\) if there exists a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r) \cap \Omega) \leq C_d \mu(B(x, r) \cap \Omega) < \infty
\]

for all \(x \in \tilde{\Omega}\) and \(r > 0\), where \(B(x, r) := \{y \in X : d(y, x) < r\}\) denotes an open ball centered at \(x\) with radius \(r\).

Given a set \(F \subset \tilde{\Omega}\) endowed with a \(\sigma\)-finite Borel regular measure \(\mathcal{H}\), we say that \(\mathcal{H}\) is upper codimension-\(\theta\) regular on \(F\) for some \(\theta > 0\) if \(\mathcal{H}\) is a doubling measure on \(F\) and there exists a constant \(C_F > 0\) such that

\[
\mathcal{H}(B(x, r) \cap F) \leq C_F \frac{\mu(B(x, r) \cap \Omega)}{r^\theta}
\]

(2.1)

for all \(x \in F\) and \(0 < r \leq \text{diam}(\Omega)\). If in addition, there exists a constant \(C'_F > 0\) such that

\[
\mathcal{H}(B(x, r) \cap F) \geq C'_F \frac{\mu(B(x, r) \cap \Omega)}{r^\theta}
\]

(2.2)

for all \(x \in F\) and \(0 < r \leq \text{diam}(\Omega)\), then \(\mathcal{H}\) is called Ahlfors condimension-\(\theta\) regular on \(F\). Note that under the condition (2.1) and (2.2), \(\mathcal{H}\) is doubling on \(F\) provided that \(\mu\) is doubling on \(\Omega\).

2.1 Metric-valued Sobolev spaces via upper gradients

Let \(X = (X, d, \mu)\) be a metric measure space and \(Z = (Z, d_Z)\) be a complete metric space. Let \(\Omega \subset X\) be a domain with \(\mu(\Omega) < \infty\).

For \(p \geq 1\), we denote by \(L^p(\Omega, Z)\) the space of all \(\mu\)-measurable and essentially separably valued map \(u : \Omega \to Z\) such that for some \(z_0 \in Z\), the function \(x \mapsto d(u(x), z_0) \in L^p(\Omega)\). A sequence \(\{u_k\} \subset L^p(\Omega, Z)\) is said to converge to \(u \in L^p(\Omega, Z)\) if

\[
\int_{\Omega} d_Z^p(u(x), u_k(x)) d\mu(x) \to 0 \quad \text{as } k \to \infty.
\]

When \((Z, d_Z) = (V, |\cdot|)\) is a Banach space, we may endow \(L^p(\Omega, V)\) with a natural norm

\[
\|f\|_{L^p(\Omega, V)} := \left(\int_{\Omega} |f|^p \, d\mu\right)^{1/p}.
\]

Similarly, we can define \(L^p(\partial \Omega, V) := L^p(\partial \Omega, V, d\mathcal{H})\). If \(V\) is \(\mathbb{R}\), we set \(L^p(\Omega, \mathbb{R}) := L^p(\Omega)\) and \(L^p(\partial \Omega, \mathbb{R}, d\mathcal{H}) := L^p(\partial \Omega)\) for brevity.

We next introduce metric valued Sobolev spaces based on upper gradients. This concept was first introduced in [15] and then functions with \(p\)-integrable upper gradients were studied in [23]. Later, the theory of real-valued Sobolev spaces based on upper gradients was explored in-depth in [35]. Here we only give a very brief introduction and refer the interested readers to the recent monograph [16] for more information.
Let $\Gamma$ be a family of nonconstant rectifiable curves in $\Omega$ and $F(\Gamma)$ the family of all Borel measurable functions $\varrho : \Omega \to [0, \infty]$ such that for every $\gamma \in \Gamma$,

$$\int_{\gamma} \varrho ds \geq 1.$$ 

For $1 \leq p < \infty$, we define the $p$-modulus of the family $\Gamma$ as

$$\text{Mod}_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{\Omega} \varrho^p d\mu.$$ 

We say that a property holds for $p$-almost every curve in $\Omega$ if the family $\Gamma$ of all nonconstant rectifiable curves in $\Omega$ for which the property fails has zero $p$-modulus.

**Definition 2.1** (Upper gradients) A Borel function $g : \Omega \to [0, \infty]$ is called an upper gradient for a map $u : \Omega \to Z$ if for every rectifiable curve $\gamma : [a, b] \to \Omega$, we have the inequality

$$d_Z(u(\gamma(b)), u(\gamma(a))) \leq \int_{\gamma} g\, ds. \quad (2.3)$$

If inequality $(2.3)$ holds for $p$-almost every curve in $\Omega$, then $g$ is called a $p$-weak upper gradient for $u$.

A $p$-weak upper gradient $g$ of $u$ is minimal if for every $p$-weak upper gradient $\tilde{g}$ of $u$, $\tilde{g} \geq g$ $\mu$-almost everywhere. If $u$ has an upper gradient in $L^p_{\text{loc}}(\Omega)$, then $u$ has a unique (up to sets of $\mu$-measure zero) minimal $p$-weak upper gradient. We denote the minimal upper gradient by $g_u$. The Sobolev space $N^{1,p}(\Omega, Z)$ consists of all $u \in L^p(\Omega, Z)$ with an $L^p$-integrable minimal $p$-weak upper gradient $g_u \in L^p(\Omega)$. For each $u \in N^{1,p}(\Omega, Z)$, we shall use $E^p(u)$ to denote the upper gradient energy functional of $u$, that is,

$$E^p(u) = \int_{\Omega} g_u^p d\mu.$$ 

An alternative way to introduce $N^{1,p}(\Omega, Z)$ is to use isometric embedding $Z \subset V$ and then define $N^{1,p}(\Omega, Z)$ as the Banach space-valued Sobolev spaces $N^{1,p}(\Omega, V)$. As this will be convenient for us later in establishing the theory of trace, we briefly record Banach space valued Sobolev spaces $N^{1,p}(\Omega, V)$ here.

The Dirichlet space $D^{1,p}(\Omega, V)$ consists of all measurable functions $u : \Omega \to V$ that have an upper gradient belonging to $L^p(\Omega)$. We can equip the Dirichlet space $D^{1,p}(\Omega, V)$ with the seminorm

$$\|u\|_{D^{1,p}(\Omega, V)} := \inf_{g} \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all $p$-weak upper gradient $g$ of $u$.

Let

$$\tilde{N}^{1,p}(\Omega, V) = D^{1,p}(\Omega, V) \cap L^p(\Omega, V)$$

be equipped with the seminorm

$$\|u\|_{\tilde{N}^{1,p}(\Omega, V)} = \|u\|_{L^p(\Omega, V)} + \|u\|_{D^{1,p}(\Omega, V)}.$$ 

We obtain a normed space $N^{1,p}(X, V)$, which is called the Sobolev space of $V$-valued functions on $\Omega$, by passing to equivalence classes of functions in $\tilde{N}^{1,p}(\Omega, V)$, where $u_1 \sim u_2$ if and only if $\|u_1 - u_2\|_{\tilde{N}^{1,p}(\Omega, V)} = 0$. Thus,

$$N^{1,p}(\Omega, V) := \tilde{N}^{1,p}(\Omega, V)/\{u \in \tilde{N}^{1,p}(\Omega, V) : \|u\|_{\tilde{N}^{1,p}(\Omega, V)} = 0\}. $$
Since we may embed the metric space $Z$ isometrically into some Banach space $L^\infty(Y)$, we can alternatively define $N^{1,p}(\Omega, Z)$ via $N^{1,p}(\Omega, Z) := N^{1,p}(\Omega, L^\infty(Z))$; see [16, Section 7]. If $V$ or $Z$ is $\mathbb{R}$, we set $N^{1,p}(\Omega, \mathbb{R}) =: N^{1,p}(\Omega)$. We shall denote by $N^{1,p}_0(\Omega, \mathbb{R})$ the space of functions $u \in N^{1,p}(\Omega)$ such that $Tu(x) = 0$ on $\partial\Omega$ for $\mathcal{H}$-almost every $x \in \Omega$.

We say $\Omega$ supports a local $q$-Poincaré inequality, $1 \leq q < \infty$, if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\int_{B(x,r) \cap \Omega} |u - u_{B(x,r) \cap \Omega}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r) \cap \Omega} g^q \, d\mu \right)^{1/q} \quad (2.4)$$

holds for all $x \in \Omega$ and $r > 0$, for every function $u : \Omega \to \mathbb{R}$ belonging to $L^1_{loc}(\Omega)$, and for every upper gradient $g$ of $u$.

**Remark 2.2**

(i) If $\mu$ is a doubling measure on $\Omega$, then the inequality (2.4) holds not only for $x \in \Omega$, but also for $x \in \partial\Omega$; see [31, Remark 2.13] for a discussion.

(ii) It follows from [15, Theorem 8.1.42] that for doubling metric measure spaces, the validity of a Poincaré inequality is independent of the target Banach spaces. Hence if $\mu$ is a doubling measure on $\Omega$ and $V$ is a Banach space, then that $\Omega$ supports a local $q$-Poincaré inequality implies that (2.4) holds for for all $x \in \Omega$ and $r > 0$, for every function $u : \Omega \to V$ belonging to $N^{1,p}(\Omega, V)$ with every upper gradient $g$ of $u$.

(iii) If we instead consider metric space valued functions, by embedding the target metric space into a Banach space, that $\Omega$ supports a local $q$-Poincaré inequality implies that there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\int_{B(x,r) \cap \Omega} \int_{B(x,r) \cap \Omega} dZ(u(z), u(y)) \, d\mu(z) \, d\mu(y) \leq Cr \left( \int_{B(x,\lambda r) \cap \Omega} g^q \, d\mu \right)^{1/q}$$

holds for all $x \in \Omega$ and $r > 0$, for every map $u : \Omega \to Z$ belonging to $L^1_{loc}(\Omega, Z)$, and for every Borel function $g$ being upper gradients of $u$.

In Definition 1.2, we imposed a global $p$-Poincaré inequality on $\Omega$ for Sobolev functions with zero trace. When $\Omega$ is a bounded Lipschitz domain in a Riemannian manifold, this condition is easily verified. When $\Omega$ is a bounded domain in a general metric measure space $X$, it seems to be not yet clear what should be a reasonable geometric assumption imposed on $\Omega$. One essentially needs that the zero extension of a Sobolev function on $\Omega$ with zero trace shall be a global Sobolev function on $X$. For this, we need to analyze the size of the exceptional set of points on $\partial\Omega$ such that (1.1) fails. This involves a careful study of the pointwise behavior of a Sobolev function in singular metric spaces and it goes beyond the scope of this paper. We refer the interested readers to [21] for results and discussions along this direction.

### 2.2 Ultra-completions of metric spaces

We briefly recall the relevant definitions concerning ultra-completions and ultra-limits of metric spaces. Details can be found for instance in [2].

A non-principal ultrafilter on $\mathbb{N}$ is a finitely additive probability measure $\omega$ on $\mathbb{N}$ such that every subset of $\mathbb{N}$ is measurable and such that $\omega(A)$ equals 0 or 1 for all $A \subset \mathbb{N}$ and $\omega(A) = 0$ whenever $A$ is finite. Given a compact Hausdorff topological space $(Z, \tau)$ and a sequence $\{z_m\} \subset Z$ there exists a unique point $z_\infty \in Z$ such that $\omega(\{m \in \mathbb{N} : z_m \in U\}) = 1$ for every $U \ni \tau$ containing $z_\infty$. We denote the point $z_\infty$ by $\lim_\omega z_m$. 

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Let $Y = (Y, d)$ be a metric space and $\omega$ a non-principal ultrafilter on $\mathbb{N}$. A sequence $\{y_m\} \subset Y$ is bounded if $\sup_m d(y_1, y_m) < \infty$. Define an equivalence relation $\sim$ on bounded sequences in $Y$ by considering $\{y_m\}$ and $\{y'_m\}$ equivalent if $\lim_{\omega} d(y_m, y'_m) = 0$. Denote by $[(y_m)]$ the equivalence class of $\{y_m\}$. The ultra-completion $Y_{\omega}$ of $Y$ with respect to $\omega$ is the metric space given by the set

$$Y_{\omega} := \{[(y_m)]: \{y_m\} \text{ bounded sequence in } Y\},$$

equipped with the metric

$$d_\omega([(y_m)], [(y'_m)]) := \lim_{\omega} d(y_m, y'_m).$$

The ultra-completion $Y_{\omega}$ of $Y$ is a complete metric space, even if $Y$ itself is not complete.

### 3 Extension of the trace theory of Korevaar–Schoen

Let $(X, d_X, \mu)$ be a complete metric measure space, $(V, |\cdot|)$ be a Banach space and $\Omega \subset X$ be a bounded domain. Assume $\partial \Omega$ is endowed with an upper codimension-$\theta$ regular measure $\mathcal{H}$ with $\theta > 0$.

We give an alternative definition of the trace for Banach valued maps.

**Definition 3.1** Let $u : \Omega \to V$ be a $\mu$-measurable function. Then $Tu(x) \in V$ is the trace of $u$ at $x \in \partial \Omega$ if the following equation holds:

$$\lim_{r \to 0^n} \int_{B(x, r) \cap \Omega} |u - Tu(x)|\, d\mu = 0. \quad (3.1)$$

We say that $u$ has a trace $Tu$ on $\partial \Omega$ if $Tu(x)$ exists for $\mathcal{H}$-almost every $x \in \partial \Omega$.

We next show that Definition 3.1 is consistent with Definition 1.1.

**Lemma 3.2** Let $(Y, d_Y)$ be a complete metric space and $h : Y \to L^\infty(Y)$ be an isometric embedding. For any $\mu$-measurable function $u : \Omega \to Y$, if the function $h \circ u : \Omega \to L^\infty(Y)$ has a trace in the sense of Definition 3.1, then the function $u$ has a trace in the sense of Definition 1.1.

**Proof** Let $T(h \circ u)$ be the trace of $h \circ u$ in the sense of Definition 3.1. Then for $\mathcal{H}$-almost every $x \in \partial \Omega$, the Eq. (3.1) holds for $T(h \circ u)$. Note that (3.1) implies that for each $k \in \mathbb{N}$, we can find a ball $B(x_k, r_k)$ centered at $x \in \partial \Omega$ with radius $r_k > 0$ such that there exists a point $x_k \in B(x, r_k) \cap \Omega$ with

$$|T(h \circ u)(x) - h \circ u(x_k)| < 2^{-k}.$$

Hence $\{h \circ u(x_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty(Y)$ that converges to $T(h \circ u)(x)$.

Since $h$ is an isometric embedding, $\{u(x_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $Y$ and hence has a limit in the complete metric space $Y$, for which we denote by $Tu(x)$. Moreover, it follows from the isometric property of $h$ that $h \circ Tu(x) = T(h \circ u)(x)$ and (1.1) is satisfied with $T(h \circ u)$ for $\mathcal{H}$-almost every $x \in \partial \Omega$. Thus, $Tu$ is the trace of the function $u$ in the sense of Definition 1.1. \qed

The proof of Lemma 3.2 actually tells that the isometric embedding $h$ and the trace operator $T$ commute. Thus to develop a theory of trace, we shall not distinguish the traces operators

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in Definitions 1.1 and 3.1. From now on, we shall focus on the case when \((Y, d_Y) = (V, | \cdot |)\) is a Banach space.

For any \(f \in L^p_{\text{loc}}(\Omega, V)\), we define the centered fractional maximal operator as

\[
M_{\theta, p} f(z) = \sup_{0 < r < 2\text{diam}(\partial \Omega)} \left( r^\theta \int_{B(z, r) \cap \Omega} |f|^p \, d\mu \right)^{1/p}, \quad \text{for each } z \in \partial \Omega. \quad (3.2)
\]

Then it is easy to see that this fractional maximal operator maps \(L^p_{\text{loc}}(\Omega, V)\) into the space of real-valued lower semicontinuous functions on \(\partial \Omega\).

The following boundedness result follows from [31, Lemma 4.2].

**Lemma 3.3** Let \(1 \leq p < \infty\). Then the fractional maximal operator \(M_{\theta, p}\) is bounded from \(L^p(\Omega)\) to weak-\(L^p(\partial \Omega, d\mathcal{H})\) provided if \(p > \theta\).

We are ready to prove the boundedness of the trace operator for Banach space valued Sobolev maps. When the Banach space is \(\mathbb{R}\), the result was obtained in [31]. The essential idea of the proof is similar with the one used in [31].

**Theorem 3.4** Suppose \((\Omega, \mu, \mathcal{H})\) is weakly \((p, \theta)\)-admissible for some \(p > 1\) and \(p > \theta\). Then the trace operator \(T : N^{1, p}(\Omega, V) \to L^p(\partial \Omega, V)\) is bounded and linear.

**Proof** Let \(u \in N^{1, p}(\Omega, V)\) and \(R = 2\text{diam}(\Omega)\) be fixed. For any \(z \in \partial \Omega\) and \(k \in \mathbb{N}\), we define

\[
T_k u(z) = \int_{B(z, 2^{-k} R) \cap \Omega} u \, d\mu.
\]

We first show that the limits

\[
\widetilde{T} u = \lim_{k \to \infty} T_k u
\]

exist \(\mathcal{H}\)-almost everywhere on \(\partial \Omega\). It suffices to show that the function

\[
u^* = \sum_{k \geq 0} |T_{k+1} u - T_k u| + |T_0 u|
\]

belongs to \(L^p(\partial \Omega)\), since \(u^* \in L^p(\partial \Omega)\) implies that \(u^*(z) < \infty\) for \(\mathcal{H}\)-almost everywhere \(z \in \partial \Omega\). Then it suffices to show that

\[
\|u^*\|_{L^p(\partial \Omega)} \leq \|T_0 u\|_{L^p(\partial \Omega, V)} + \sum_{k \geq 0} \|T_{k+1} u - T_k u\|_{L^p(\partial \Omega, V)} < \infty.
\]

Notice that \(T_0 u(z) = \int_{B(z, R)} u \, d\mu = \int_{\Omega} u \, d\mu\) for any \(z \in \partial \Omega\), since \(\Omega \subset B(z, R)\) for any \(z \in \partial \Omega\). It follows from the upper codimension relation (2.1) that

\[
\|T_0 u\|_{L^p(\partial \Omega, V)}^p \leq \int_{\partial \Omega} \int_{B(z, R) \cap \Omega} |u|^p \, d\mu \, d\mathcal{H}(z) \leq \int_{\partial \Omega} \frac{R^\theta}{\mathcal{H}(\partial \Omega)} \int_{\Omega} |u|^p \, d\mu \, d\mathcal{H} = R^\theta \|u\|_{L^p(\Omega, V)}^p.
\]

For any \(k \geq 0\), it follows from the doubling property of \(\mu\), the upper codimension relation (2.1) and the local \(p\)-Poincaré inequality that

\[
\|T_{k+1} u - T_k u\|_{L^p(\partial \Omega, V)}^p \leq \int_{\partial \Omega} \left( \int_{B(z, 2^{-k} R) \cap \Omega} |u - u_{B(z, 2^{-k} R) \cap \Omega}| \, d\mu \right)^p \, d\mathcal{H}
\]

\[
\leq \int_{\partial \Omega} \left( \int_{B(z, 2^{-k} R) \cap \Omega} |u - u_{B(z, 2^{-k} R) \cap \Omega}| \, d\mu \right)^p \, d\mathcal{H}
\]

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\[
\begin{align*}
\int_{\partial \Omega} \frac{(2^{-k} R)^p}{\mu(B(z, 2^{-k} \lambda R) \cap \Omega)} &\int_{B(z, 2^{-k} \lambda R) \cap \Omega} g_u(x)^p \, d\mu(x) \, dH(z) \\
\leq &\int_{\partial \Omega} \frac{(2^{-k} \lambda R)^{p-\theta}}{H(B(z, 2^{-k} \lambda R) \cap \partial \Omega)} \int_{B(z, 2^{-k} \lambda R) \cap \Omega} g_u(x)^p \, d\mu(x) \, dH(z) \\
&- \int_{\Omega(2^{-k} \lambda R)} g_u(x)^p \int_{B(z, 2^{-k} \lambda R) \cap \Omega} \frac{2^{-k} \lambda R)^{p-\theta}}{H(B(z, 2^{-k} \lambda R) \cap \partial \Omega)} \, dH(z) \, d\mu(x) \\
\leq &\int_{\Omega(2^{-k} \lambda R)} g_u(x)^p \int_{B(z, 2^{-k} \lambda R) \cap \Omega} \frac{(2^{-k} \lambda R)^{p-\theta}}{H(B(z, 2^{-k} \lambda R) \cap \partial \Omega)} \, dH(z) \, d\mu(x) \\
\end{align*}
\]

(3.3)

where \( \Omega(r) := \{x \in \Omega: d(x, \partial \Omega) < r\} \) and the second last inequality used the fact that \( H \) is doubling while \( B(x, 2^{-k} \lambda R) \cap \partial \Omega \subseteq B(z, 2^{-k+1} \lambda R) \cap \partial \Omega \) whenever \( z \in B(x, 2^{-k} \lambda R) \cap \partial \Omega \).

Since \( p > \theta \), combining the estimates of \( \|T_0 u\|_{L^p(\partial \Omega)} \) and \( \|T_{k+1} u - T_k u\|_{L^p(\partial \Omega, \nu)} \), we obtain that

\[\|u^*\|_{L^p(\partial \Omega)} \lesssim \|u\|_{L^p(\partial \Omega)} + \sum_{k \geq 0} (2^{-k} R)^{1-\theta/p} \|g_u\|_{L^p(\partial \Omega)} \lesssim \|u\|_{N^{1,p}(X, \nu)} < \infty.\]

Thus, \( \tilde{T}u \) exists \( H \)-almost everywhere on \( \partial \Omega \). Moreover, since \( |\tilde{T}u| \leq u^* \), we have

\[\|\tilde{T}u\|_{L^p(\partial \Omega, \nu)} \leq \|u^*\|_{L^p(\partial \Omega)} \lesssim \|u\|_{N^{1,p}(X, \nu)}.\]

The proof will be complete once we show \( \tilde{T}u = Tu \) on \( \partial \Omega \). For this, it suffices to show that the equation (3.1) holds with \( \tilde{T}u(z) \) for \( H \)-almost every \( z \in \partial \Omega \). Set

\[E = \{z \in \partial \Omega: M_{\theta, p} g_u(z) < \infty \text{ and } T_k u(z) \to \tilde{T}u(z) \text{ as } k \to \infty\}.\]

Then the existence of \( \tilde{T}u \) and Lemma 3.3 imply that \( H(\partial \Omega \setminus E) = 0 \).

For any \( 0 < r \leq R \), let \( k_r \in \mathbb{N} \) such that \( 2^{-k_r-1} R < r \leq 2^{-k_r} R \). Then it follows from the doubling property of \( \mu \) that for any \( z \in E \) and \( 0 < r \leq R \),

\[\int_{B(z, r) \cap \Omega} |u - \tilde{T}u(z)| \, d\mu \leq \int_{B(z, r) \cap \Omega} |u - T_{k_r}(z)| \, d\mu + |T_{k_r}(z) - \tilde{T}u(z)| \]

\[\lesssim \int_{B(z, 2^{-k_r} R) \cap \Omega} |u - u|_{B(z, 2^{-k_r} R) \cap \Omega} \, d\mu + |T_{k_r}(z) - \tilde{T}u(z)| \]

\[\lesssim 2^{-k_r} R \left( \int_{B(z, 2^{-k_r} R) \cap \Omega} g_u(x)^p \, d\mu(x) \right)^{1/p} + |T_{k_r}(z) - \tilde{T}u(z)| \]

\[\lesssim (2^{-k_r} R)^{1-\theta/p} M_{\theta, p} g_u(z) + |T_{k_r}(z) - \tilde{T}u(z)|.\]

Since \( z \in E \) and \( k_r \to \infty \) as \( r \to 0 \), we have

\[\int_{B(z, r) \cap \Omega} |u - \tilde{T}u(z)| \, d\mu \to 0, \text{ as } r \to 0.\]

Hence (3.1) holds with \( \tilde{T}u(z) \) for \( H \)-almost every \( z \in \partial \Omega \). The proof is complete.

**Proof of Theorem 1.5** This is a direct consequence of Theorem 3.4 and Lemma 3.2.

As a consequence of the proof of Theorem 3.4, we obtain the following convergence result for traces of metric valued Sobolev spaces, which in particular gives Theorem 1.6.
Theorem 3.5 Suppose \((\Omega, \mu, \mathcal{H})\) is \((p, \theta)\)-admissible with \(p > \theta\). Let \(\{u_i\} \subset N^{1,p}(\Omega, Y)\) be a sequence with uniformly bounded energy, that is,

\[
\sup_{i \in \mathbb{N}} \int_{\Omega} g_{u_i}^p d\mu < \infty.
\]

If \(u_i\) converges to some \(u \in N^{1,p}(\Omega, Y)\) in \(L^p(\Omega, Y)\), then \(T u_i \to T u\) in \(L^p(\partial \Omega, Y)\). Furthermore, two maps \(u, v \in N^{1,p}(\Omega, Y)\) have the same trace if and only if \(d(u, v) \in N^{1,p}(\Omega, \mathbb{R})\) and has zero trace.

**Proof** For both assertions, embedding \(Y\) isometrically into some Banach space \(V\) if necessary, we may assume \(Y = V\) is a Banach space.

For the first claim, recall that in the proof of Theorem 3.4, we proved that \(T f = \tilde{T} f\) for any \(f \in N^{1,p}(X, V)\), where

\[
\tilde{T} f = \lim_{k \to \infty} T_k f.
\]

It follows from the estimate (3.4) that

\[
\|T f - T_k f\|_{L^p(\partial \Omega, V)} \leq \sum_{j \geq k} \|T_{j+1} f - T_j f\|_{L^p(\partial \Omega, V)} \lesssim \sum_{j \geq k} (2^{-j} R)^{1-\theta/p} \|g_f\|_{L^p(\Omega)} \lesssim (2^{-k} R)^{1-\theta/p} \|g_f\|_{L^p(\Omega)},
\]

where \(g_f\) is the minimal upper gradient of \(f\).

Hence for any two functions \(f, h \in N^{1,p}(X, V)\) and any \(k \in \mathbb{N}\), we have

\[
\|T f - T h\|_{L^p(\partial \Omega, V)} \leq \|T f - T_k f\|_{L^p(\partial \Omega, V)} + \|T h - T_k h\|_{L^p(\partial \Omega, V)} + \|T_k f - T_k h\|_{L^p(\partial \Omega, V)} \lesssim (2^{-k} R)^{1-\theta/p} (\|g_f\|_{L^p(\Omega)} + \|g_h\|_{L^p(\Omega)}) + \|T_k f - T_k h\|_{L^p(\partial \Omega, V)},
\]

where \(g_f\) and \(g_h\) are minimal upper gradients of \(f\) and \(h\), respectively. Notice that for any \(z \in \partial \Omega\), we have

\[
T_k f(z) = \int_{B(z, 2^{-k} R) \cap \Omega} f \, d\mu \quad \text{and} \quad T_k h(z) = \int_{B(z, 2^{-k} R) \cap \Omega} h \, d\mu.
\]

Thus

\[
\|T_k f - T_k h\|_{L^p(\partial \Omega, V)} = \int_{\partial \Omega} |T_k f(z) - T_k h(z)|^p \, d\mathcal{H}(z)
\]

\[
\leq \int_{\partial \Omega} \left( \int_{B(z, 2^{-k} R) \cap \Omega} |f(x) - h_{B(z, 2^{-k} R) \cap \Omega}|^p \, d\mu(x) \right)^{1/p} \, d\mathcal{H}(z)
\]

\[
\lesssim \int_{\partial \Omega} \int_{B(z, 2^{-k} R) \cap \Omega} |f(x) - h(x)|^p \, d\mu(x) \, d\mathcal{H}(z) + \int_{\partial \Omega} \left( \int_{B(z, 2^{-k} R) \cap \Omega} |h(x) - h_{B(z, 2^{-k} R) \cap \Omega}|^p \, d\mu(x) \right)^{1/p} \, d\mathcal{H}(z) =: I_1 + I_2.
\]

Using similar arguments as that in (3.4), we obtain

\[
I_2 \lesssim (2^{-k} R)^{\theta} \|g_h\|_{L^p(\Omega)}^p.
\]

For the estimate of \(I_1\), it follows from the upper codimension relation (2.1) that

\[
I_1 = \int_{\partial \Omega} \frac{1}{\mu(B(z, 2^{-k} R) \cap \Omega)} \int_{B(z, 2^{-k} R) \cap \Omega} |f(x) - h(x)|^p \, d\mu(x) \, d\mathcal{H}(z).
\]
from the definition of trace that for $H$ γ \mid sequence has uniformly bounded energy, then $Tu_i$ i.e., $Tu$

The above inequality shows that if the sequence $u_i$ converges to $u$ in $L^p(\Omega, V)$ and if the sequence has uniformly bounded energy, then $Tu_i$ converges to $Tu$ in $L^p(\partial \Omega, V)$. Indeed, if we choose $f = u$ and $h = u_i$ in the above inequality, we know form the lower semicontinuity of energy (see [16, Theorem 7.3.9]) that the energy of $f$ is fixed, the second term can be made small by choosing $k$ large.

We now turn to the second claim and assume that $u, v \in N^{1,p}(\Omega, V)$ have the same trace, i.e., $Tu(x) = Tv(x)$ for $H$-almost every $x \in \partial \Omega$. We first show that $d(u, v) = |u - v| \in N^{1,p}(\Omega)$. Since $|u - v| \leq |u| + |v|$, $|u - v| \in L^p(\Omega)$. The minimal upper gradient $g_{u-v}$ of $|u - v|$ is controlled by $g_u + g_v$, where $g_u$ and $g_v$ are minimal upper gradients of $u$ and $v$. Indeed, for any rectifiable curve $\gamma$ connecting $x, y \in \Omega$, by triangle inequality, we have that

$$
|u(x) - v(x)| - |u(y) - v(y)| \leq |u(x) - u(y)| + |v(x) - v(y)| \leq \int_\gamma g_u + g_v \, ds.
$$

Thus, $|u - v| \in N^{1,p}(\Omega)$. Since $Tu(x) = Tv(x)$ for $H$-almost every $x \in \partial \Omega$, it follows from the definition of trace that for $H$-almost every $x \in \partial \Omega$, we have

$$
\lim_{r \to 0^+} \int_{B(x, r) \cap \Omega} |u - v| \, d\mu \leq \lim_{r \to 0^+} \int_{B(x, r) \cap \Omega} |u - Tu(x)| \, d\mu + |Tu(x) - Tv(x)| + \lim_{r \to 0^+} \int_{B(x, r) \cap \Omega} |Tv(x) - v(x)| \, d\mu = 0.
$$

Hence $|u - v|$ has trace zero.

For the converse, assume that $|u - v| \in N^{1,p}(\Omega)$ has trace zero. Notice that for any $x \in \partial \Omega$ and any $y \in B(x, r) \cap \Omega$, we have that

$$
|Te(x) - Tv(x)| \leq |Te(x) - u(x)| + |u(x) - v(y)| + |v(y) - Tv(x)|.
$$

It follows from the definition of trace that for $H$-almost every $x \in \partial \Omega$, we have

$$
|Te(x) - Tv(x)| \leq \int_{B(x, r) \cap \Omega} |Te(x) - u| \, d\mu + \int_{B(x, r) \cap \Omega} |u - v| \, d\mu + \int_{B(x, r) \cap \Omega} |v - Tv(x)| \, d\mu \to 0, \text{ as } r \to 0.
$$

Thus, $u$ and $v$ have the same trace. □
4 Solution to the Dirichlet problem

4.1 Hajlasz–Sobolev spaces and consequences

Let $\Omega \subset X$ be a domain and $Y$ a complete metric space.

**Definition 4.1 (Hajlasz–Sobolev spaces)** A measurable map $u : \Omega \rightarrow Y$ belongs to the Hajlasz–Sobolev space $M^{1,p}(\Omega, Y)$ if $u \in L^p(X, Y)$ and there exists a nonnegative function $g \in L^p(\Omega)$ such that the Hajlasz gradient inequality

$$d_Y(u(x), u(z)) \leq d_X(x, z)(g(x) + g(z))$$

holds for all $x, y \in \Omega \setminus N$ for some $N \subset \Omega$ with $\mu(N) = 0$. For each $u \in M^{1,p}(\Omega, Y)$, the associated Hajlasz energy $E_H^p(u)$ is defined as

$$E_H^p(u) := \inf_g \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all Hajlasz gradient $g$ of $u$, that is, $g$ such that (4.1) holds.

The following equivalence of metric valued Sobolev spaces is well-known.

**Proposition 4.1** [16, Corollary 10.2.9] Suppose $\mu$ is doubling and $\Omega$ supports a $q$-Poincaré inequality for some $1 \leq q < p$. Then $M^{1,p}(\Omega, Y) = N^{1,p}(\Omega, Y)$. Furthermore, there exists a constant $C \geq 1$, depending only on the data associated to $\Omega$, such that for each $u \in M^{1,p}(\Omega, Y)$,

$$C^{-1}E^p(u) \leq E_H^p(u) \leq CE^p(u).$$

The proof of Theorem 4.3 requires the following Rellich compactness result, which was proved in [8, Theorem 3.1] when $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain.

**Theorem 4.2** (Generalized Rellich compactness) Suppose $\mu$ is doubling and $\Omega$ supports a $q$-Poincaré inequality for some $1 \leq q < p$. For every $m \in \mathbb{N}$, let $(Y_m, d_m)$ be a complete metric space, $K_m \subset Y_m$ compact and $\{u_m\} \subset N^{1,p}(\Omega, Y_m)$. Suppose that $(K_m, d_m)$ is uniformly compact and

$$\sup_{m \in \mathbb{N}} \int_{\Omega} d_m^p(u_m(x), y_m)d\mu(x) + E^p(u_m) < \infty$$

for some and thus every $y_m \in K_m$. Then after possibly passing to a subsequence, there exist a complete metric space $Z$, a compact subset $K \subset Z$, isometric embeddings $\varphi_k : Y_m \rightarrow Z$, and $v \in N^{1,p}(\Omega, Z)$ such that $\varphi_m(K_m) \subset K$ for all $m \in \mathbb{N}$ and $\varphi_m \circ u_m$ converges to $v$ in $L^p(\Omega, Z)$.

Recall that a sequence of compact metric space $(B_m, d_m)$ is called uniformly compact if $\sup_m \text{diam} B_m < \infty$ and if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that every $B_m$ can be covered by at most $N$ balls of radius $\varepsilon$.

**Proof of Theorem 4.2** The proof is essentially contained in [8, Theorem 3.1] and thus we only point out the necessary changes. The key ingredient of the proof is that under our assumptions on $\Omega$, we have by Proposition 4.1 that $N^{1,p}(\Omega, Y) = M^{1,p}(\Omega, Y)$ and each $u_m \in N^{1,p}(\Omega, Y)$ satisfies the pointwise inequality (4.1)

$$d_m(u_m(x), u_m(x')) \leq d(x, x')(h_m(x) + h_m(x'))$$

almost everywhere for some $h_m \in L^p(\Omega)$ with $\|h_m\|_{L^p(\Omega)}^p \leq C(p, \Omega)E^p(u_m)$.

\[\square\]
4.2 Proof of Theorem 1.4

In this section, we provide the proof of Theorem 1.4, which is very similar to [8, Proof of Theorem 1.4]. In the first step, we prove the following result on ultra-limits of subsequences of Sobolev maps, which extends [8, Theorem 1.6].

**Theorem 4.3** Suppose $(\Omega, \mu, \mathcal{H})$ is a $(p, \theta)$-admissible domain with $p > \theta$ and $Y_\omega$ is an ultra-completion of the complete metric space $Y$. If $\{u_k\} \subset N^{1,p}(\Omega, Y)$ is a bounded sequence for some $p > 1$, then, after possibly passing to a subsequence, the map $\phi(z) := [(u_m(z))]$ belongs to $W^{1,p}(\Omega, Y_\omega)$ and satisfies

$$E^p(\phi) \leq \liminf_{k \to \infty} E^p(u_k).$$

Moreover, if $Tu_k$ converges to some map $\rho \in L^p(\partial\Omega, Y)$ $\mathcal{H}$-almost everywhere on $\partial\Omega$, then $T\phi = \iota \circ \rho$ $\mathcal{H}$-almost everywhere on $\partial\Omega$.

**Proof of Theorem 4.3** The proof is essentially contained in [8, Proof of Theorem 1.6] and we present it again for the convenience of the readers. After possibly passing to a subsequence, we may assume that

$$E^p(u_k) \to \liminf_{m \to \infty} E^p(u_m)$$

as $k \to \infty$.

Fix $y_0 \in Y$ and apply the Rellich compactness Theorem 4.2. After possibly passing to a subsequence, there exist a complete metric space $Z = (Z, d_Z)$, a compact subset $K \subset Z$, and isometric embeddings $\varphi_k : Y \to Z$ and $v \in N^{1,p}(\Omega, Z) = M^{1,p}(\Omega, Z)$ such that $\varphi_k(y_0) \subset K$ for all $k$ and $v_k := \varphi_k \circ u_k$ converges in $L^p(\Omega, Z)$ to $v$ as $k \to \infty$. After passing to a further subsequence, we may assume that $v_k$ converges almost everywhere to $v$ on $\Omega$. Let $N \subset \Omega$ be a set of $\mu$-measure zero such that $v_k(z) \to v(z)$ for all $z \in \Omega \setminus N$.

Define a subset of $Z$ by $B := \{v(z) : z \in \Omega \setminus N\}$.

The map $\psi : B \to Y_\omega$, given by $\psi(v(z)) = [(u_k(z))]$ when $z \in \Omega \setminus N$ is well-defined and isometric by [8, Lemma 2.2]. Since $Y_\omega$ is complete, there exists a unique extension of $\psi$ to $\overline{B}$, which we denote again by $\psi$. After possibly redefining the map $v$ on $N$, we may assume that $v$ has image in $\overline{B}$ and hence $v$ is an element of $M^{1,p}(\Omega, \overline{B})$. Now, we define a mapping by

$$\phi(z) := \psi(v(z)) = [(u_k(z))]$$

and then $\phi$ belongs to $M^{1,p}(\Omega, Y_\omega)$ and by the lower semicontinuity of upper gradient energy [16, Theorem 7.3.9] it satisfies

$$E^p(\phi) \leq E^p(v) \leq \liminf_{k \to \infty} E^p(v_k) = \lim_{k \to \infty} E^p(u_k). \quad (4.3)$$

It remains to prove the trace equality. Suppose $Tu_k$ converges to some map $\rho \in L^p(\partial\Omega, Y)$ almost everywhere on $\partial\Omega$. Arguing as in [8, Page 104], we can find compact subsets $C_1 \subset C_2 \subset \cdots \subset Y$, isometric embeddings $\varphi_k : Y \to Z$ and $v \in N^{1,p}(\Omega, Z)$ such that $v_k := \varphi_k \circ u_k$ converges in $L^p(\Omega, Z)$ to $v$ as $k \to \infty$. Furthermore, if we set $C = \bigcup_{i=1}^\infty C_i$, then after passing to a further subsequence if necessary we may assume that $v_k$ converges to $v$ almost everywhere on $\Omega$ and $\varphi_k|_{C_i}$ converges pointwise to an isometric embedding $\varphi : C \to Z$, with the convergence being uniform on each $C_k$. Let $N \subset \Omega$ be a set of $\mu$-measure zero such that $v_k(z) \to v(z)$ for all $z \in \Omega \setminus N$. 

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Define a subset of $Z$ by
\[ B := \{v(z) : z \in \Omega\} \cup \varphi(C). \]

The map $\psi : B \to Y_\omega$ given by
\[
\begin{cases}
\psi(v(z)) = [(u_k(z))] & \text{if } z \in \Omega\setminus N, \\
\psi(\varphi(x)) = \iota(x) = [(x)] & \text{if } x \in C,
\end{cases}
\]
is well-defined and an isometric embedding by [8, Lemma 2.2]. Since $Y_\omega$ is complete, there exists a unique isometric extension of $\psi$ to $\overline{B}$, which we denote again by $\psi$. After possibly redefining the map $v$ on $N$, we may assume $v \in M^{1,p}(\Omega, \overline{B})$. The map $\phi(z) := \psi(v(z)) = [(u_k(z))]$ then belongs to $M^{1,p}(\Omega, Y_\omega)$ and satisfies (4.3). Moreover, by Proposition 3.5, we have that $Tu_k = \varphi_k \circ Tu_k$ converges to $\varphi \circ \rho$ almost everywhere on $\partial \Omega$ and a subsequence of $Tu_k$ converges to $T v$ almost everywhere. It thus follows that $T v = \varphi \circ \rho$ and hence
\[ T \phi = \psi \circ T v = \psi \circ \varphi \circ \rho = \iota \circ \rho. \]

The proof is complete. \hfill \Box

With Theorem 4.3 at hand, the proof of Theorem 1.4 is immediate.

**Proof of Theorem 1.4** Let $\phi \in N^{1,p}(\Omega, Y)$ and let $\{u_k\} \subset N^{1,p}(\Omega, Y)$ be an energy minimizing sequence with $Tu_k = T \phi$ for each $k$. Then by the characterization of trace from Proposition 3.5, $h_k(x) = d(u_k(x), \phi(x)) \in N^{1,1}_{0}(\Omega, \partial \Omega, \mathcal{H})$. Since $\sup_k E^p(h_k) < \infty$, it follows from the global $p$-Poincaré inequality (1.2) that $\sup_k \|h_k\|_{L^p(\Omega)} < \infty$. Hence
\[
\sup_k \int_\Omega d^p(u_k(x), y_0) d\mu(x) + E^p(u_k) < \infty.
\]

Thus $\{u_k\}$ is a bounded sequence in $N^{1,p}(\Omega, Y)$. Let $Y_\omega$ be an ultra-completion of $Y$ such that $Y$ admits a 1-Lipschitz retraction $P : Y_\omega \to Y$. After possibly passing to a subsequence, we may assume by Theorem 4.3 that the map $v(z) := [(u_k(z))]$ belongs to $N^{1,p}(\Omega, Y_\omega)$ and satisfies $T v = \iota \circ T \phi$ and
\[
E^p(v) \leq \lim_{k \to \infty} E^p(u_k).
\]

Since $P : Y_\omega \to Y$ is a 1-Lipschitz retraction, the map $u := P \circ v$ belongs to $N^{1,p}(\Omega, Y)$ and satisfies $Tu = T \phi$ and $E^p(u) \leq \lim_{k \to \infty} E^p(u_m)$. The proof is complete. \hfill \Box

## 5 The theory of trace in the borderline case

In Theorem 3.4, $(\Omega, \mu, \mathcal{H})$ is assumed to be weakly $(p, \theta)$-admissible with the requirement that $p > \theta$. It is natural to ask for what happens if $p = \theta$. We shall address this problem in this section. To simplify our exposition, we say that $(\Omega, \mu, \mathcal{H})$ is weakly $\theta$-admissible if it is weakly $(p, \theta)$-admissible with $p = \theta$.

For the borderline case when $\Omega$ is weakly $\theta$-admissible for some $\theta > 1$, the traces of $N^{1,p}(X, V, d\mu)$ may not exist. To characterize the existence of the traces, we give an addition weight on the measure $\mu$ and investigate the relationship between the existence of the traces and the properties of the weight function.
For a locally integrable weight function $\rho : \Omega \to (0, \infty)$, we define $L^p(\Omega, V, \rho d\mu)$ and $N^{1,p}(\Omega, V, \rho d\mu)$ by replacing $d\mu$ with the weighted measure $\rho \mu$ in the integrals of the norms.

The following result follows essentially from the proof of Theorem 3.4, whose core idea is similar with the one used in [31]

**Theorem 5.1** Suppose that $(\Omega, \mu, \mathcal{H})$ is $\theta$-admissible for some $\theta > 1$. Let $\omega : (0, \infty) \to [1, \infty)$ be a non-increasing function and $\rho(x) := w(\text{dist}(x, \partial \Omega))$. Then the trace operator

$$T : N^{1,\theta}(\Omega, V, \rho d\mu) \to L^\theta(\partial \Omega, V)$$

is bounded and linear, provided that $\int_0^1 t^{-1} \omega(t)^{-1/\theta} \, dt < \infty$.

**Proof** The proof is a minor modification of that used in Theorem 3.4 and thus we use the same notations here. Repeat the proof of Theorem 3.4 until the estimate (3.3). Since $\omega$ is non-increasing, by considering the weighted measure $\rho(x) d\mu$ instead of the measure $d\mu$ in the the estimate (3.3), we obtain that

$$\|T_{k+1} u - T_k u\|_{L^\theta(\partial \Omega, V)}^\theta \leq \int_{\Omega(2^{-k} \lambda R)} g(x) \rho(x) \int_{B(x, 2^{-k} \lambda R) \cap \Omega} \frac{d\mathcal{H}(z)}{\omega(2^{-k} \lambda R) \mathcal{H}(B(z, 2^{-k} \lambda R) \cap \partial \Omega)} d\mu(x)$$

$$\leq \frac{1}{\omega(2^{-k} \lambda R)} \int_{\Omega(2^{-k} \lambda R)} g(x) \rho(x) d\mu = \frac{1}{\omega(2^{-k} \lambda R)} \|g\|_{L^\theta(\Omega, \rho d\mu)}^\theta,$$

Since $\omega(t) \geq 1$, we have

$$\|T_0 u\|_{L^\theta(\Omega)}^\theta \leq R^\theta \|u\|_{L^\theta(\Omega, V)}^\theta \leq R^\theta \|u\|_{L^\theta(\Omega, V, \rho d\mu)}^\theta.$$

Thus, it follows from $\int_0^1 t^{-1} \omega(t)^{-1/\theta} \, dt < \infty$ that

$$\|u^*\|_{L^\theta(\Omega)} \lesssim R^\theta \|u\|_{L^\theta(\Omega, V, \rho d\mu)} + \sum_{k \geq 0} \frac{1}{(\omega(2^{-k} \lambda R))^{1/p}} \|g\|_{L^p(\Omega, \rho d\mu)}^p \int_{2^{-k+1} \lambda R}^{2^k \lambda R} \frac{dt}{t}$$

$$\lesssim \|u\|_{L^\theta(\Omega, V, \rho d\mu)} + \sum_{k \geq 0} \|g\|_{L^p(\Omega, \rho d\mu)} \int_{2^{-k+1} \lambda R}^{2^k \lambda R} \frac{dt}{t} \omega(t)^{1/\theta} \lesssim \|g\|_{L^p(\Omega, \rho d\mu)} \int_0^1 \frac{dt}{tw(t)^{1/\theta}},$$

Hence $\tilde{T} u$ exists $\mathcal{H}$-almost everywhere on $\partial \Omega$ and we have the estimate

$$\|\tilde{T} u\|_{L^\theta(\partial \Omega)} \leq \|u^*\|_{L^\theta(\partial \Omega)} \lesssim \|u\|_{N^{1,\theta}(\Omega, V, \rho d\mu)}.$$

The proof will be complete once we show $\tilde{T} u = Tu$. We shall use the similar idea as in the proof of Theorem 3.4, and set

$$E = \{ z \in \partial \Omega : M_{\theta, \theta}(g \rho^{1/\theta})(z) < \infty \text{ and } T_k u(z) \to \tilde{T} u(z) \text{ as } k \to \infty \}. $$
Note that Lemma 3.2 also works for \( p = \theta \). Then \( \mathcal{H}(\partial \Omega \setminus E) = 0 \) and for every \( z \in E \) and every \( 2^{−kr−1}R < r \leq 2^{−kr}R \), we have

\[
\int_{B(z, r) \cap \Omega} |u - \tilde{T}u(z)| \, d\mu \lesssim 2^{−kr} R \int_{B(z, 2^{−kr}R) \cap \Omega} g(x)^\rho \, d\mu(x) + |Tk_zu(z) - \tilde{T}u(z)| \\
\leq 2^{−kr} R \left( \int_{B(z, 2^{−kr}R) \cap \Omega} g(x)^\rho \, d\mu(x) \right) + |Tk_zu(z) - \tilde{T}u(z)| \\
\leq \frac{1}{w(\lambda/2)} M_{\theta, \omega}(g \rho^1(\theta)) \, (z) + |Tk_zu(z) - \tilde{T}u(z)|.
\]

since \( \int_0^1 t^{-1} w(t)^{-1/\theta} \, dt < \infty \), the non-increasing function \( w(t) \to \infty \) as \( t \to 0 \), and so it follows

\[
\int_{B(z, r) \cap \Omega} |u - \tilde{T}u(z)| \, d\mu \to 0, \quad \text{as } r \to 0.
\]

This means that (3.1) holds with \( \tilde{T}u(z) \) for \( \mathcal{H} \)-almost every \( z \in \partial \Omega \) and the proof is thus complete. \( \square \)

The integrability condition on \( \omega \) can be relaxed if \( \Omega \) has certain nice geometry. For this, we shall introduce the so-called John domains, which plays an important role in geometric analysis in metric spaces \([13]\).

**Definition 5.2** A bounded domain \( \Omega \subseteq X \) is called a John domain with John constant \( c_J \in (0, 1] \) and John center \( a \in \Omega \) if every \( x \in \Omega \) can be joined to \( a \) by a rectifiable curve \( \gamma : [0, \ell_\gamma] \to \Omega \) parametrized by arc-length such that \( \gamma(0) = x, \gamma(\ell_\gamma) = a \) and

\[
dist(\gamma(t), X \setminus \Omega) \geq c_J t \quad \text{for all } t \in [0, \ell_\gamma]. \tag{5.1}
\]

If \( \Omega \) is a John domain with compact closure in \( X \), the it follows from the Arzela–Ascoli theorem that every \( z \in \partial \Omega \) can be joined to the John center by a rectifiable curve such that (5.1) holds.

**Theorem 5.3** Let \( \Omega, \theta, w \) and \( \rho \) be as in Theorem 5.1. Assume additionally that \( \Omega \) is a John domain with compact closure. Then the trace operator

\[
T : N^{1,\theta}(\Omega, V, \rho \, d\mu) \to L^\theta(\partial \Omega, V)
\]

is bounded and linear provided

\[
\int_0^1 t^{-1} \omega(t)^{-1/(\theta-1)} \, dt < \infty. \tag{5.2}
\]

**Proof** We first show that the trace operator \( T \) is bounded and linear if (5.2) holds. Let \( u \in N^{1,\theta}(\Omega, V) \) be fixed. Let \( \delta = dist(a, \partial \Omega) > 0 \), where \( a \in \Omega \) is the John center of \( \Omega \). For any point \( z \in \partial \Omega \), let \( \gamma_z \) be the arc-length parametrized curve that connects the points \( a \) and \( z \), and satisfies (5.1).

Let \( t_k = \delta(1 - \frac{cJ}{2M})^k \) and \( r_k = \frac{cJ}{2M} t_k \) for \( k = 0, 1, \ldots \). Next, we define a chain of balls \( \{B_z^k\}_{z \in \mathbb{N}} \) by setting

\[
B_z^k = B(\gamma_z(t_k), r_k), \quad \text{for all } k = 0, 1, \ldots.
\]

The chain \( \{B_z^k\}_{z \in \mathbb{N}} \) consists of balls of bounded overlap, where the upper bound on the number of overlapping balls depends only on the John constant \( c_J \). Moreover, abbreviating

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\( \alpha := (2 - \frac{C_J}{2}) \), we have \( B_{z}^{k+1} \subset \alpha B_{z}^{k} \). By triangle’s inequality and direct computations, for any \( x \in \alpha \lambda B_{z}^{k} \), we have
\[
\beta_{1}r_{k} =: \frac{c_{J}r_{k}}{2} \leq d(x, z) \leq \left( \alpha \lambda + \frac{2\lambda}{c_{J}} \right) r_{k} := \beta_{2}r_{k}
\tag{5.3}
\]
and
\[
\beta_{1}r_{k} = \frac{c_{J}r_{k}}{2} \leq d(x, \partial \Omega) \leq \left( \alpha \lambda + \frac{2\lambda}{c_{J}} \right) r_{k} = \beta_{2}r_{k}.
\tag{5.4}
\]

Then we define \( T_{k}u(z) := \int_{B_{z}^{k}} u \, d\mu \).

In the next step, we will show that \( \tilde{T}u(z) := \lim_{k \to +\infty} T_{k}u(z) \) exists for \( \mathcal{H} \)-almost every \( z \in \partial \Omega \) and that \( Tu = \tilde{T}u \in L^{p}(\partial \Omega, V) \). It suffices to show that the function
\[
u^{*} = \sum_{k \geq 0} |T_{k+1}u - T_{k}u| + |T_{0}u| =: |T^{*}u| + |T_{0}u|
\]
belongs to \( L^{p}(\partial \Omega) \). We claim that
\[
\|\nu^{*}\|_{L^{p}(\partial \Omega)} \leq \|T_{0}u\|_{L^{p}(\partial \Omega, V)} + \|T^{*}u\|_{L^{p}(\partial \Omega)} < \infty.
\]

By the doubling property of \( \mu \) and the relation (5.3), we know that
\[
\|T_{0}u\|_{L^{p}(\partial \Omega)}^{\theta} \leq \frac{\int_{\partial \Omega} \int_{B_{z}^{k}} |u(x)|^{\theta} \, d\mu(x) \, d\mathcal{H}(z)}{\int_{\partial \Omega} \int_{\mu(B_{z}^{k})} |u(x)|^{\theta} \, d\mu(x) \, d\mathcal{H}(z)}
\]
\[
\approx \int_{\partial \Omega} \int_{\mu(B_{z}^{k})} \frac{|u(x)|^{\theta} \chi_{B_{z}^{k}}}{\mu(B_{z}^{k})} \, d\mu(x) \, d\mathcal{H}(z)
\]
\[
\leq \int_{\partial \Omega} |u(x)|^{\theta} \int_{\mu(B_{z}^{k})} \chi_{B_{z}^{k}} \, d\mu(x) \, d\mathcal{H}(z) \, d\mu(x)
\]
\[
\approx \int_{\partial \Omega} |u(x)|^{\theta} \int_{\mu(B_{z}^{k})} \frac{\delta^{-\theta} \, d\mathcal{H}(z)}{\mu(B_{z}^{k})} \, d\mu(x) \, d\mathcal{H}(z)
\]
\[
\lesssim \delta^{-\theta} \int_{\partial \Omega} |u(x)|^{\theta} \, d\mu(x) \leq \delta^{-\theta} \int_{\partial \Omega} |u(x)|^{\theta} \rho(x) \, d\mu(x) \approx \|u\|^{\theta}_{L^{p}(\Omega), V, \rho d\mu}.
\]

Here we used the fact that \( \mathcal{H} \) is doubling and \( B(x, \beta_{2}\delta) \cap \Omega \subset B(z, 2\beta_{2}\delta) \cap \Omega \) whenever \( z \in B(x, \beta_{2}\delta) \).

We next estimate the \( L^{p} \)-norm of \( |T^{*}u| \). For any \( k \geq 0 \), by the local \( \theta \)-Poincaré inequality and the fact that \( B_{z}^{k+1} \subset B_{z}^{k} \), we know that
\[
|T_{k}u(z) - T_{k+1}u(z)| = |u_{B_{z}^{k}} - u_{B_{z}^{k+1}}| \leq |u_{B_{z}^{k}} - u_{\alpha B_{z}^{k}}| + |u_{\alpha B_{z}^{k}} - u_{B_{z}^{k+1}}|
\]
\[
\lesssim r_{k} \left( \int_{\alpha B_{z}^{k}} g(x)^{\theta} \, d\mu(x) \right)^{1/\theta} \lesssim r_{k} \omega(\beta_{2}r_{k})^{-1/\theta} \left( \int_{\alpha B_{z}^{k}} g(x)^{\theta} \rho(x) \, d\mu(x) \right)^{1/\theta}.
\]

Then, by Hölder’s inequality, we have
\[
\left( \sum_{k=0}^{\infty} |T_{k}u(z) - T_{k+1}u(z)| \right)^{\theta} \lesssim \left( \sum_{k=0}^{\infty} \omega(\beta_{2}r_{k})^{-1/(\theta-1)} \right)^{\theta-1} \left( \sum_{k=0}^{\infty} r_{k}^{\theta} \int_{\alpha B_{z}^{k}} g(x)^{\theta} \rho(x) \, d\mu(x) \right).
\]
Since \( \omega \) satisfies
\[
\sum_{k=0}^{\infty} \omega(\beta_{2r_k})^{-1/(\theta-1)} \approx C + \sum_{k=1}^{\infty} \omega(\beta_{2r_k})^{-1/(\theta-1)} \int_{\beta_{2r_k}}^{\beta_{2r_{k+1}}} \frac{dt}{t} \leq C + \int_0^1 \frac{dt}{t \omega(t)^{1/(\theta-1)}} < +\infty,
\]

it follows from the relation (5.4) that
\[
\|T^* u\|_{L^p(\partial \Omega)}^{\theta} = \int_{\partial \Omega} \left( \sum_{k=0}^{\infty} |T_k u(z) - T_{k+1} u(z)| \right)^\theta d\mathcal{H}(z)
\]
\[
\leq \int_{\partial \Omega} \sum_{k=1}^{\infty} r_k^\theta \int_B g(x)^\theta \rho(x) \, d\mu(x) \, d\mathcal{H}(z)
\]
\[
\leq \sum_{k=0}^{\infty} \int_{\Omega(\beta_{2r_k}) \setminus \Omega(\beta_{1r_k})} g(x)^\theta \rho(x) \int_{\partial \Omega} \frac{r_k^\theta \chi_{B(x, \beta_{2r_k}) \cap \partial \Omega}}{\mu(B(z, \beta_{2r_k}) \cap \partial \Omega)} \, d\mathcal{H}(z) \, d\mu(x)
\]
\[
\leq \sum_{k=0}^{\infty} \int_{\Omega(\beta_{2r_k}) \setminus \Omega(\beta_{1r_k})} g(x)^\theta \rho(x) \, d\mu(x)
\]
\[
\leq \int_{\Omega} g(x)^\theta \rho(x) \, d\mu(x) = \|g\|_{L^p(\Omega, \rho d\mu)}^p.
\]

Hence \( \tilde{T} u \) exist \( \mathcal{H} \)-almost everywhere on \( \partial \Omega \) and
\[
\|\tilde{T} u\|_{L^p(\partial \Omega, \nu)} \leq \|u^*\|_{L^p(\partial \Omega)} \lesssim \|u\|_{N^{1,\theta}(X, V, \rho d\mu)}.
\]

It is left to show \( \tilde{T} u = T u \). This follows by a similar argument as in the proof of Theorem 5.1, upon noticing that the convergence assumption \( \int_0^1 t^{-1} w(t)^{-1/(\theta-1)} dt < \infty \) implies the non-increasing function \( w(t) \to \infty \) as \( t \to 0 \). The proof is thus complete. \( \square \)

We recall the following well-known lemma from [37].

**Lemma 5.4** [37] Let \( (K, d_K, \mu_K) \) be a \( \sigma \)-finite metric measure space. Then the following conditions on \( (K, d_K, \mu_K) \) are equivalent:

(i) \( L^p(K) \subset L^q(K) \) for all \( p, q \in (0, \infty) \) with \( p > q \);

(ii) \( \mu_K(K) < +\infty \).

The integrability assumption (5.2) is sharp as the following example demonstrates.

**Example 5.5** Let \( \Omega = \mathbb{D} \subset \mathbb{R}^2 \) with the measure \( \mu \) in \( \Omega \) given by \( d\mu = \text{dist}(x, \partial \mathbb{D})^{\theta-1} dx \), \( \theta > 1 \). On the boundary \( \partial \Omega \), let \( \mathcal{H} \) be the 1-dimensional Hausdorff measure. By [18, Theorem 3.4], \( \Omega \) supports a local 1-Poincaré inequality and hence supports a local \( \theta \)-Poincaré inequality. The doubling property of \( \mu \) and upper codimension-\( \theta \) regularity of \( \mathcal{H} \) follow by direct computations. Thus \( (\Omega, \mu, \mathcal{H}) \) is weakly \( \theta \)-admissible. Let \( w \) and \( \rho \) be as in Theorem 5.1 with

\[
\int_0^1 t^{-1} \omega(t)^{-1/(\theta-1)} dt = \infty. \quad (5.5)
\]

Then there exists a function \( u \in N^{1,\theta}(\Omega, \rho d\mu) \) such that \( T u(z) \) does not exist for all \( z \in \partial \Omega \).
Proof To find such a function $u$, it suffices to construct a function $f : (0, 1) \to [0, \infty)$ such that

\[
\begin{align*}
    f(t) dt &= +\infty, \\
    f(t)^\theta t^{\theta-1} \omega(t) dt &< +\infty.
\end{align*}
\]  

(5.6)

Indeed, if such a function $f$ exists, then we may define $u$ by setting $u(0) = 0$ and

\[
u(x) = \int_0^{|x|} f(t) dt.\]

(5.7)

Then the Borel function $g : \Omega \to [0, \infty)$ given by $g(x) = f(|x|)$ is an upper gradient of $u$. The relation (5.6) implies that

\[
\|g\|_{L^\theta(\Omega, \rho d\mu)}^\theta = \int_\Omega g(x)^\theta \operatorname{dist}(x, \partial \Omega)^{\theta-1} \omega(\operatorname{dist}(x, \partial \Omega)) dx
\]

\[
\lesssim \int_0^1 f(t)^\theta t^{\theta-1} w(t) dt < \infty.
\]

thus it follows from the Hölder inequality that for any $t_0 \in (0, 1),

\[
\int_{t_0}^1 f(t) dt \leq \left( \int_{t_0}^1 f(t)^\theta t^{\theta-1} \omega(t) dt \right)^{1/\theta} \left( \int_{t_0}^1 w(t)^{1/(\theta-1)} dt \right)^{(\theta-1)/\theta} < \infty,
\]

since $w(t) \geq 1$ implies that $\int_{t_0}^1 t^{-1} w(t)^{1/(\theta-1)} dt \leq \int_{t_0}^1 t^{-1} dt < \infty$. Hence for any $t_0 \in (0, 1)$, we have

\[
\int_0^{t_0} f(t) dt = \infty.
\]

(5.8)

Then for the function $u$ defined by (5.7), the trace $Tu(z)$ does not exist for all $z \in \partial \Omega$, since as $x$ goes to the boundary $\partial \Omega$, the function $u(x)$ goes to infinity uniformly.

As one might notice, there is a gap here, that is we do not know if the function $u$ belongs to $L^\theta(\Omega, \rho d\mu)$ or not. But this gap could be fixed by modifying the function to be an oscillatory function with values in $[0, 1)$, instead of an increasing function with respect to $|x|$, such that there is a sequence $\{t_i\} \in \mathbb{N}$ with

\[
\int_{B(z, \Delta x_k) \cap \Omega} |u| \rho d\mu \geq \frac{2}{3}, \quad \int_{B(z, \Delta x_k+1) \cap \Omega} |u| \rho d\mu \leq \frac{1}{3}
\]

for all $z \in \partial \Omega$. We omit the details here but refer to [25, Remark 3.6] and [26, Lemma 3.6] for details of a similar modification.

Let us go back to find the function $f$ satisfying relation (5.6). Let

\[
h(t) = f(t) \cdot t^{1/(\theta-1)}.
\]

Then to find a function $f$ satisfying relation (5.6) is equivalent to find a function $h$ satisfying

\[
\begin{align*}
    \int_0^1 h(t) \cdot t^{-1} \omega(t)^{-1/(\theta-1)} dt &= +\infty, \\
    \int_0^1 h(t)^\theta \cdot t^{-1} \omega(t)^{-1/(\theta-1)} dt &< +\infty.
\end{align*}
\]

(5.9)

Consider the metric measure space $((0, 1), d_E, \mu_1)$ with $d_E$ being the Euclidean metric and $d\mu_1 = t^{-1} \omega(t)^{-1/(\theta-1)} dt$. Since for any $t_0 \in (0, 1)$, we have

\[
\mu_1([t_0, 1)) = \int_{t_0}^1 t^{-1} w(t)^{-1/(p-1)} dt < \infty.
\]
μ1 is a σ-finite measure. Moreover, since the relation (5.5) implies that μ1((0, 1)) = ∞, it follows from Lemma 5.4 that Lθ((0, 1), μ1) ⊈ L1((0, 1), μ1), i.e., there exists a function h: (0, 1) → [0, ∞) such that h ∈ Lθ((0, 1), μ1) but h \∉ L1((0, 1), μ1). Such an h satisfies (5.9).

In conclusion, we have constructed a function u ∈ N1,θ(Ω, ρdμ) such that Tu(z) does not exist for any z ∈ ∂Ω.

Remark 5.6 • The combination of Theorem 5.3 and Example 5.5 actually shows that the condition (5.2) characterizes the existence of traces of Sobolev spaces N1,θ(Ω, V, ρdμ). This characterization is new even when the Banach space V = R, extending and improving the corresponding result of [31].

• Theorem 5.3 and Example 5.5 are inspired by the recent works from [17] and [25]. A full characterization of the existence of traces on regular trees was given in [25]. In [17], the upper half space Rd++ with measure \( d\mu = |x_0|^{θ−1}dx \) and weight function \( ϕ(t) = \log^λ(4/τ) \) was considered. It was shown that the traces of the weighted Sobolev space exist if and only if λ > θ − 1 for θ > 1, which coincides with the relation (5.2).

We refer the interested readers to [17, Theorem 1.2 and Example 1.1] as a special case to understand the more general Theorem 5.3 and Example 5.5.

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