On hypercomplexifying real forms of arbitrary rank

Dennis Hou
dhou@uci.edu

Abstract

For certain problems involving vector fields, it is possible to find an associated imaginary field that, in conjunction with the first, forms a complex field for which the equation can be solved. This result is generalized to arbitrary Clifford algebras, followed by quaternionic vectors as a special case. All results are shown to reduce to the established method of complexifying vector fields. For simplicity, differential forms are used rather than vector notation.

Key words: hypercomplex forms, topological duals

1 Introduction

The major feature of the algorithm in [1] for solving the Trkalian equation is writing a vector field as the real part of a complex field. Specifically, given the Monge potentials $f$, $g$, and $h$, a vector field $v$ can be written as follows:

$$v = f\nabla g + \nabla h.$$ 

Here, $v$ is the real part of a vector field $c = e^{ig}\nabla F$, where $F$ is defined by

$$F = (h + if)e^{-ig}.$$ 

It follows that the imaginary part, $w$, the “topological dual” (henceforth “Baldwin dual”) of $v$, is given by

$$w = -h\nabla g + \nabla f.$$ 

In terms of differential forms [2],

$$A = f\, dg + dh, \quad A^\sim = -h\, dg + df.$$ 

It is possible to generalize this to higher dimensions. For instance, a vector field of five to eight dimensions can be thought of as the real part of a quaternionic field.

2 Generalized Baldwin Dual

Consider the one-form $A$ of rank $2n - 1$, where $n = 2^k - 1$ for some integer $k$. The extended non-unique Clebsch decomposition is

$$A = f_1 df_2 + f_3 df_4 + \ldots + df_{2n-1}.$$ 

Then define $F$ such that
\[ F = (f_{2n-1} + e_1 f_1 + e_2 f_3 + \ldots + e_{n-1} f_{2n-3}) \exp(-e_1 f_2 - e_2 f_4 - \ldots - e_{n-1} f_{2n-2}). \]

\[ \bar{F} = (f_{2n-1} - e_1 f_1 - e_2 f_3 - \ldots - e_{n-1} f_{2n-3}) \exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}), \]

\[ dF = ((df_{2n-1} + e_1 df_1 + e_2 df_3 + \ldots + e_{n-1} df_{2n-3}) - (f_{2n-1} + e_1 f_1 + e_2 f_3 + \ldots + e_{n-1} f_{2n-3}))(e_1 df_2 + e_2 df_4 + \ldots + e_{n-1} df_{2n-2}) \exp(-e_1 f_2 - e_2 f_4 - \ldots - e_{n-1} f_{2n-2}), \]

\[ \bar{dF} = ((df_{2n-1} - e_1 df_1 - e_2 df_3 - \ldots - e_{n-1} df_{2n-3}) + (f_{2n-1} - e_1 f_1 - e_2 f_3 - \ldots - e_{n-1} f_{2n-3}))(e_1 df_2 + e_2 df_4 + \ldots + e_{n-1} df_{2n-2}) \exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}). \]

This immediately leads to a generalization of the main result, namely that \( A \) can be represented as the real part of a one-form acting upon an \( n \)-dimensional Clifford algebra. Specifically,

\[ A = (\exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}) \ dF + \exp(-e_1 f_2 - e_2 f_4 - \ldots - e_{n-1} f_{2n-2}) \ d\bar{F})/2, \]

which is equivalent to \( \text{Re}(\exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}) \ dF) \). Defining the dual as the entire non-real portion fails because of the presence of multiple bases, so let \( A^{\sim} \) denote the Baldwin dual of \( A \) that corresponds to the \( k \)th base (excluding unity) on the algebra associated with the rank of the form. Extracting from \( \exp(e_1 f_2 + e_2 f_4 + \ldots + e_{2n-1} f_{2n-2}) \ dF \),

\[ A^{\sim 1} = df_1 - f_{2n-1} df_2 - f_3 df_4 + f_5 df_6 - f_7 df_{10} + f_9 df_8 - \ldots - f_{2n-5} df_{2n-2} + f_{2n-3} df_{2n-4}, \]

\[ A^{\sim 2} = df_3 - f_{2n-1} df_4 + f_1 df_6 - f_5 df_2 + f_7 df_{12} - f_9 df_{14} + \ldots + f_{2n-5} df_{2n-8} + f_{2n-3} df_{2n-6}, \]

and so on. For \( n = 2 \), the imaginary dual reduces to the known form. There is also a natural extension of the set of conditions that corresponds to the known criteria for Baldwin duality over a complex field with respect to \( g \),

\[ dA + A^{\sim} \ dg = 0, \]

\[ A^{\sim} \wedge dA = 0, \]

\[ A \wedge dA^{\sim} = 0, \]

\[ A \wedge dA = A^{\sim} \wedge dA^{\sim}. \]

Specifically,

\[ dA + f_2 A^{\sim 1} + f_4 A^{\sim 2} + f_6 A^{\sim 3} + \ldots + f_{2n-2} A^{\sim n-1} = 0. \]

However, it is difficult to write down a generalization for

\[ dA + A^{\sim} \ dg = 0. \]

Given the value of \( n \), though, generating equations for the duals becomes simple. Using
\[ d(\exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}) \; dF) = dA + e_1 dA \sim 1 + e_2 dA \sim 2 + \ldots + e_{n-1} dA \sim n-1 = (e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}) (\exp(e_1 f_2 + e_2 f_4 + \ldots + e_{n-1} f_{2n-2}) \; dF) \]

and equating parts, the analogues quickly arise. Restraints on the duals can be determined from this:

\[ f_2 dA \sim 1 + f_4 dA \sim 2 + f_6 A \sim 3 + \ldots + f_{2n} dA \sim n-1 = 0, \]

\[ dA \wedge A \wedge A \wedge A \wedge A = 0, \]

\[ A \wedge dA \wedge A \wedge A = 0. \]

3 Special Case: Quaternions

The above generalization finds the Baldwin dual for any form of rank \(2n-1\) to a \(n\)-dimensional algebra. Forms of rank \(2n\) can be easily constructed simply by making the modulus of the hypercomplex form something other than unity. For other ranks, there at first seems to be no standard method of construction. Consider the special case \(n = 3\):

\[ A = f_1 df_2 + f_3 df_4 + f_5 df_6 + df_7, \]

\[ F = (f_7 + if_1 + jf_3 + kf_5) \exp(-if_2 - jf_4 - kf_6), \]

\[ \bar{F} = (f_7 - if_1 - jf_3 - kf_5) \exp(if_2 + jf_4 + kf_6), \]

\[ dF = (df_7 + i df_1 + j df_3 + k df_5) \exp(-if_2 - jf_4 - kf_6) - (f_7 + if_1 + jf_3 + kf_5)(i df_2 + j df_4 + k df_6) \exp(-if_2 - jf_4 - kf_6), \]

\[ d\bar{F} = (df_7 - i df_1 - j df_3 - k df_5) \exp(if_2 + jf_4 + kf_6) - (f_7 - if_1 - jf_3 - kf_5)(i df_2 + j df_4 + k df_6) \exp(if_2 + jf_4 + kf_6), \]

\[ A = \text{Re}(\exp(if_2 + jf_4 + kf_6) \; dF). \]

The Baldwin duals are as follows:

\[ A \sim 1 = df_1 - f_7 df_2 - f_3 df_4 + f_5 df_6, \]

\[ A \sim 2 = df_3 - f_7 df_4 + f_1 df_6 - f_5 df_2, \]

\[ A \sim 3 = df_5 - f_7 df_6 - f_1 df_4 + f_3 df_2. \]

Now let \(B = A + iA \sim 1 + jA \sim 2 + kA \sim 3\). Then

\[ dB = (if_2 + jf_4 + kf_6) \; B, \]

which leads to,

\[ dA + f_2 A \sim 1 + f_4 A \sim 2 + f_6 A \sim 3 = 0, \]

\[ dA \sim 1 - f_2 A + f_6 A \sim 2 - f_4 A \sim 3 = 0, \]

\[ dA \sim 2 - f_4 A + f_6 A \sim 1 + f_2 A \sim 3 = 0, \]

\[ dA \sim 3 - f_6 A + f_4 A \sim 1 - f_2 A \sim 2 = 0. \]
Using the exterior derivative and product on this set of equations leads to more conditions satisfied by $A$ and its Baldwin duals.

\[
\begin{align*}
&f_2 dA^1 + f_4 dA^2 + f_6 dA^3 = 0, \\
&f_2 dA - f_6 dA^2 + f_4 dA^3 = 0, \\
&f_4 dA + f_6 dA^1 - f_2 dA^3 = 0, \\
&f_6 dA - f_4 dA^1 + f_2 dA^2 = 0, \\
&dA \wedge A^1 \wedge A^2 \wedge A^3 = 0, \\
&dA^1 \wedge A \wedge A^2 \wedge A^3 = 0, \\
&dA^2 \wedge A \wedge A^1 \wedge A^3 = 0, \\
&dA^3 \wedge A \wedge A^1 \wedge A^2 = 0, \\
&dA \wedge A^3 = A^1 \wedge (dA^1)^3 = A^2 \wedge (dA^2)^3 = A^3 \wedge (dA^3)^3.
\end{align*}
\]

Special cases that reduce the rank of $A$ to six are not interesting, because they can only arise by eliminating one of the zero-form potentials, (or else the rank would drop to five,) but none of the constraints rely on these. However, for $f_5 = f_6 = 0$, the equations are reduced as follows:

\[
\begin{align*}
&dA + f_2 A^1 + f_4 A^2 = 0, \\
&dA^1 - f_2 A - f_4 A^3 = 0, \\
&dA^2 - f_4 A + f_2 A^3 = 0, \\
&dA^3 + f_4 A^1 - f_2 A^2 = 0, \\
&f_2 dA^1 + f_4 dA^2 = 0, \\
&f_2 dA + f_4 dA^3 = 0, \\
&f_4 dA - f_2 dA^3 = 0, \\
&f_4 dA^1 - f_2 dA^2 = 0, \\
&dA \wedge A^1 \wedge A^2 = 0, \\
&dA^1 \wedge A \wedge A^3 = 0, \\
&dA^2 \wedge A \wedge A^3 = 0, \\
&dA^3 \wedge A \wedge A^2 = 0, \\
&A \wedge dA^2 = A^1 \wedge (dA^1)^2 = A^2 \wedge (dA^2)^2 = A^3 \wedge (dA^3)^2.
\end{align*}
\]

For rank-three forms,

\[
\begin{align*}
&f_3 = f_4 = f_5 = 0, \\
&A^2 = A^3 = 0.
\end{align*}
\]

Furthermore, most of the constraints vanish, leaving

\[
\begin{align*}
&dA + f_2 A^1 = 0, \\
&dA^1 - f_2 A = 0, \\
&dA \wedge A^1 = 0, \\
&dA^1 \wedge A = 0, \\
&A \wedge dA = A^1 \wedge dA^1,
\end{align*}
\]

which are precisely the equations specified in [1].
4 Acknowledgments

This paper is dedicated to Charlotte Hwa, of whom it is written\footnote{Proverbs xxvii, 17.}, “Iron sharpeneth iron; so a man sharpeneth the countenance of his friend.”

5 References

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