Notes on direct images of pluricanonical bundles

Sho Ejiri

Abstract
We generalize slightly Popa–Schnell’s theorem regarding the direct images of pluricanonical bundles to the case when the ample line bundle is not globally generated. We also treat the case of positive characteristic.

Keywords Pluricanonical bundles · Fujita’s freeness conjecture · Positive characteristic · Direct images

Mathematics Subject Classification 14D06 · 14E30

1 Introduction
Popa and Schnell [15] proposed a conjecture that relativizes Fujita’s freeness conjecture:

Conjecture 1.1 ([15, Conjecture 1.3]) Let the base field be an algebraically closed field of characteristic zero. Let $f : X \to Y$ be a morphism from a smooth projective variety $X$ to a smooth projective variety $Y$ of dimension $n$. Let $\mathcal{L}$ be an ample line bundle on $Y$. Then for every $m \geq 1$, the sheaf

$$f_* \omega_X^m \otimes \mathcal{L}^l$$

is generated by its global sections for $l \geq m(n+1)$.

They proved the conjecture affirmatively when $|\mathcal{L}|$ is free:

Theorem 1.2 ([15, Theorem 1.4]) Let the base field be an algebraically closed field of characteristic zero. Let $f : X \to Y$ be a morphism from a smooth projective variety $X$ to a smooth projective variety $Y$ of dimension $n$. Let $\mathcal{L}$ be an ample line bundle on $Y$. Then for every $m \geq 1$, the sheaf

$$f_* \omega_X^m \otimes \mathcal{L}^l$$

is generated by its global sections for $l \geq m(n+1)$.
X to a projective variety Y of dimensions n. Let \( L \) be an ample line bundle on Y with \( |L| \) free. Then for every \( m \geq 1 \), the sheaf

\[
f_* \omega_X^m \otimes L^l
\]

is generated by its global sections for \( l \geq m(n+1) \).

Several results regarding to the above theorem are known [3–5, 10–12]. In this note, we generalize slightly the above theorem to the case when \( |L| \) is not necessarily free.

**Theorem 3.1** Let the base field be an algebraically closed field of characteristic zero. Let \( f : X \to Y \) be a morphism from a smooth projective variety X to a projective variety Y of dimension n. Let \( L \) be an ample line bundle on Y. Let \( j \) be the smallest positive integer such that \( |L^j| \) is free. Then for every \( m \geq 1 \), the sheaf

\[
f_* \omega_X^m \otimes L^l
\]

is generated by its global sections for \( l \geq m(jn+1) \).

When \( j = 1 \), i.e. \( |L| \) is free, then the above theorem is equivalent to Theorem 1.2. When \( j \geq 2 \), Theorem 3.1 cannot be concluded simply from Theorem 1.2. Indeed, applying Theorem 1.2 in the situation of Theorem 3.1, we can only deduce that the sheaf

\[
f_* \omega_X^m \otimes L^{jl}
\]

is generated by its global sections for \( l \geq m(n+1) \), but \( jm(n+1) = jmn + jm > jmn + m = m(jn+1) \).

We furthermore treat the case of positive characteristic. Note that there exist counterexamples to Conjecture 1.1 and Theorem 1.2 in positive characteristic (cf. [9, 16]). In the following three theorems, the relative ampleness of the canonical divisor (on the generic fiber) is assumed, which is not a too strong assumption, since it is satisfied in the situation of Fujita’s freeness conjecture (the case of \( f = \text{id} \)).

The following is an analog of Theorem 1.2 in positive characteristic:

**Theorem 1.3** ([6, Theorem 1.5]) Let the base field be an algebraically closed field of positive characteristic. Let \( f : X \to Y \) be a surjective morphism from a smooth projective variety X to a projective variety Y of dimension n. Let \( L \) be an ample line bundle on Y with \( |L| \) free. Suppose that \( \omega_X \) is \( f \)-ample. Then there exists an integer \( m_0 \geq 1 \) such that the sheaf

\[
f_* \omega_X^m \otimes L^l
\]

is generated by its global sections for each \( m \geq m_0 \) and \( l \geq m(n+1) \).

Similarly to the case of characteristic zero, we generalize the above theorem:
**Theorem 1.4** (see Theorem 3.3) Let the base field be an algebraically closed field of positive characteristic. Let $f : X \to Y$ be a surjective morphism from a smooth projective variety $X$ to a projective variety $Y$ of dimension $n$. Let $\mathcal{L}$ be an ample line bundle on $Y$. Let $j$ be the smallest positive integer such that $|\mathcal{L}^j|$ is free. Suppose that $\omega_X$ is $f$-ample. Then there exists an integer $m_1 \geq 1$ such that the sheaf

$$f_* \omega_X^m \otimes \mathcal{L}^l$$

is generated by its global sections for each $m \geq m_1$ and $l \geq m(jn + 1)$. Even when the canonical bundle on $X$ is not $f$-ample, if it is ample on the generic fiber, then we can prove a similar result to the above theorem:

**Theorem 1.5** (see Theorem 3.3) Let the base field be an algebraically closed field of positive characteristic. Let $f : X \to Y$ be a surjective morphism from a smooth projective variety $X$ to a projective variety $Y$ of dimension $n$. Let $\mathcal{L}$ be an ample line bundle on $Y$. Let $j$ be the smallest positive integer such that $|\mathcal{L}^j|$ is free. Suppose that $\omega_X$ is ample, where $X_\eta$ is the generic fiber of $f$. Then there exists an integer $m_2 \geq 1$ such that the sheaf

$$f_* \omega_X^m \otimes \mathcal{L}^l$$

is generically generated by its global sections for each $m \geq m_2$ and $l \geq m(jn + 1)$. Theorems 1.4 and 1.5 follow from Theorem 3.3 whose proof is based on the idea of the referee of [6].

**2 Preliminaries**

In this section, we collect several terminologies and definitions that are used in this note.

Let $k$ be a field. We mean by a variety an integral separated scheme of finite type over $k$.

For a normal variety $X$ and a $\mathbb{Q}$-Weil divisor $\Delta = \sum_i \delta_i \Delta_i$, where each $\Delta_i$ is a prime divisor on $X$, we denote by $[\Delta]$ (resp. $\{\Delta\}$) the Weil divisor $\sum_i \lfloor \delta_i \rfloor \Delta_i$ (resp. the effective $\mathbb{Q}$-Weil divisor $\Delta - [\Delta]$).

For a variety $X$ of positive characteristic, $F^e_X : X \to X$ denotes the $e$-times iterated Frobenius morphism of $X$.

For a coherent sheaf $\mathcal{G}$ on a variety and a positive integer $m$, we denote by $S^m(\mathcal{G})$ the $m$-th symmetric product of $\mathcal{G}$.

**Definition 2.1** Let $Y$ be a projective variety over a field $k$. Let $\mathcal{G}$ be a coherent sheaf on $Y$.

- Let $V$ be an open subset of $Y$. We say that $\mathcal{G}$ is *generated by its global sections on $V$* (or *globally generated on $V$*) if the natural morphism

$$H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \to \mathcal{G}$$

is surjective.
is surjective on $V$.

- We say that $\mathcal{G}$ is \textit{generically generated by its global sections} (or \textit{generically globally generated}) if $\mathcal{G}$ is generated by its global sections on a dense open subset of $Y$.

**Definition 2.2** Let $X$ be a normal variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$. We say that the pair $(X, \Delta)$ is $F$-pure if the composite of

$$
\mathcal{O}_X \xrightarrow{F^c_X \#} F^c_X \mathcal{O}_X \xleftrightarrow{\cdot (\lfloor p^c - 1 \Delta \rfloor)} \mathcal{O}_X
$$

splits locally as an $\mathcal{O}_X$-module homomorphism for each $e \geq 1$.

### 3 Proofs of theorems

In this section, we prove the main theorems. First, we consider the case of characteristic zero.

**Theorem 3.1** Let the base field be an algebraically closed field of characteristic zero. Let $X$ be a normal projective variety and let $\Delta$ be an effective $\mathbb{Q}$-Weil divisor on $X$ such that $i(K_X + \Delta)$ is Cartier for an integer $i \geq 1$ and that $(X, \Delta)$ is log canonical. Let $Y$ be a projective variety of dimension $n$ and let $f : X \to Y$ be a morphism. Let $L$ be an ample line bundle on $Y$. Let $j$ be the smallest positive integer such that $|L|^j$ is free. Then for each $m \geq 1$ with $i | m$, the sheaf

$$
f_* \mathcal{O}_X(m(K_X + \Delta)) \otimes L^j
$$

is generated by its global sections for each $l \geq m(jn + 1)$.

**Proof** The proof is almost the same as that of [15, Theorem 1.7], and our argument follows [8, Proof of Theorem 3.2.1].

Fix $m \geq 1$ such that $m(K_X + \Delta)$ is Cartier. Put $\mathcal{B} := \mathcal{O}_X(m(K_X + \Delta))$. We may assume that $f_* \mathcal{B} \neq 0$. Then the image of the natural morphism $f^* f_* \mathcal{B} \to \mathcal{B}$ is isomorphic to $b \otimes B$ for an ideal sheaf $b$. Taking the log resolution of $b$ and $\Delta$, we may assume that $b \cong \mathcal{O}_X(-E)$ for an effective divisor $E$ such that $\text{Supp}(\Delta + E)$ is simple normal crossing. Set $\mathcal{M} := \mathcal{B}(-E)$. Then we have

$$
f^* f_* \mathcal{B} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{B}
$$

and

$$
f_* \mathcal{B} \xrightarrow{id} f_* f^* f_* \mathcal{B} \xrightarrow{f_* \alpha} f_* \mathcal{M} \xrightarrow{f_* \beta} f_* \mathcal{B},
$$
so \( f^*_s \mathcal{M} \cong f^*_s \mathcal{B} \). We show that there is a divisor \( 0 \leq E' \leq E \) such that each coefficient in \( \Delta' := \Delta + \frac{m-1}{m} E - E' \) is in \((0, 1]\). We can write

\[
\Delta = \sum_{i=1}^{\mu} d_i D_i, \quad d_i \in \mathbb{R} \cap (0, 1], \quad \text{and} \quad E = E_1 + \sum_{i=1}^{\mu} e_i D_i, \quad e_i \in \mathbb{Z}_{\geq 0},
\]

where \( \text{Supp}(E_1) \) and \( \text{Supp}(\Delta) \) have no common components. Put

\[
E' := \left\lfloor \frac{m-1}{m} E_1 \right\rfloor + \sum_{i=1}^{\mu} \left[ d_i + \frac{m-1}{m} e_i - 1 \right] D_i.
\]

Then one can check that \( E' \) is what we wanted. Let \( s \) be the smallest integer such that \( f^*_s \mathcal{B} \otimes \mathcal{L}^s \) is globally generated. Then, we see from the morphism

\[
f^*_s \mathcal{B} \otimes f^* \mathcal{L}^s \rightarrow \mathcal{M} \otimes f^* \mathcal{L}^s
\]

that \( \mathcal{M} \otimes f^* \mathcal{L}^s \) is globally generated. Then by the Bertini theorem, there is a smooth member \( D \in |\mathcal{M} \otimes f^* \mathcal{L}^s| \) such that \( \text{Supp}(D) \) and \( \text{Supp}(\Delta') \) have no common components and \( \text{Supp}(\Delta' + D) \) is simple normal crossing. Let \( L \) be a Cartier divisor on \( Y \) such that \( \mathcal{O}_Y(L) \cong \mathcal{L} \). We now have

\[
m(K_X + \Delta) + sf^* L \sim D + E,
\]

from which we obtain

\[
(m - 1)(K_X + \Delta) \sim Q \frac{m-1}{m} D + \frac{m-1}{m} E - \frac{m-1}{m} sf^* L,
\]

and so we get

\[
m(K_X + \Delta) - E' + lf^* L
\]

\[
\sim Q K_X + \Delta - E' + \frac{m-1}{m} D + \frac{m-1}{m} E + \left( l - \frac{m-1}{m} s \right) f^* L
\]

\[
= K_X + \Delta' + \frac{m-1}{m} D + \left( l - \frac{m-1}{m} s \right) f^* L
\]

for an \( l \geq 1 \). Then it follows from the projection formula and the vanishing theorem ([1, Theorem 3.2] or [7, Theorem 6.3]) that

\[
H^i(Y, f^*_s \mathcal{B} \otimes \mathcal{L}^l) = 0
\]

for each \( i > 0 \) and \( l > (m - 1)/m \). Note that since \( 0 \leq E' \leq E \) we have

\[
f^*_s \mathcal{M} \cong f^*_s \mathcal{O}_X(m(K_X + \Delta) - E') \cong f^*_s \mathcal{B}.
\]
Hence, if \( l > (m - 1)/m + jn \), then \( f_* B \otimes \mathcal{L}^l \) is 0-regular with respect to \( \mathcal{L}^j \) in the sense of Castelnuovo–Mumford, and so it is globally generated \([13, \text{Theorem 1.8.3}]\). By the definition of \( s \), we obtain

\[
s \leq \frac{m - 1}{m} s + jn + 1,
\]

and thus \( s \leq m(jn + 1) \). Therefore, if \( l \geq m(jn + 1) \), then

\[
\frac{m - 1}{m} s + jn + 1 \leq \frac{m - 1}{m} m(jn + 1) + jn + 1 = m(jn + 1) \leq l,
\]

so \( f_* B \otimes \mathcal{L}^l \) is globally generated, which is our claim. 

\( \Box \)

**Remark 3.2** The author learned from Professor Masataka Iwai that if \( X \) and \( Y \) are smooth and if \( f : X \to Y \) is smooth, then for every ample line bundle \( \mathcal{L} \) on \( Y \),

\[
f_* \omega^m_X \otimes \mathcal{L}^l
\]

is generated by its global sections for each \( l \geq m(n + 1) + n(j - 1) \), where \( j \) is the smallest positive integer such that \( |\mathcal{L}^j| \) is free. The proof is based on an analytic method that is similar to that of \([10]\), which uses a singular Hermitian metric constructed in \([2, \text{Section 11}]\).

Next, we treat the case of positive characteristic.

**Theorem 3.3** Let the base field be an algebraically closed field of characteristic \( p > 0 \). Let \( X \) be a normal projective variety and let \( \Delta \) be an effective \( \mathbb{Q} \)-Weil divisor on \( X \) such that \( i(K_X + \Delta) \) is Cartier for an integer \( i > 0 \) not divisible by \( p \). Let \( Y \) be a projective variety of dimension \( n \) and let \( f : X \to Y \) be a morphism. Suppose that there exists a dense open subset \( V \subseteq Y \) such that

- \( (U, \Delta|_U) \) is \( F \)-pure, where \( U := f^{-1}(V) \), and
- \( K_U + \Delta|_U \) is ample over \( V \).

Let \( \mathcal{L} \) be an ample line bundle on \( Y \) and let \( j \) be the smallest positive integer such that \( |\mathcal{L}^j| \) is free. Then there exists an integer \( m_0 \geq 1 \) such that the sheaf

\[
f_* \mathcal{O}_X(m(K_X + \Delta)) \otimes \mathcal{L}^l
\]

is generated by its global sections for each \( m \geq m_0 \) with \( i|m \) and each \( l \geq m(jn + 1) \).

To prove the theorem, we need the following lemma:

**Lemma 3.4** Let the base field be a field of characteristic \( p > 0 \). Let \( Y \) be a projective variety of dimension \( n \) and let \( \mathcal{E} \) be a coherent sheaf on \( Y \). Let \( \mathcal{L} \) be an ample Cartier divisor on \( Y \) and let \( j \) be a positive integer such that \( |j\mathcal{L}| \) is free. Let \( \{a_e\}_{e \geq 1} \) be a
sequence of positive integers such that $a_e/p^e$ converges to $\varepsilon + jn$ for an $\varepsilon \in \mathbb{R}_{>0}$. Then there exists an $e_0 \geq 1$ such that

$$F^e_Y(\mathcal{E}(a_e L))$$

is generated by its global sections for each $e \geq e_0$.

**Proof** We prove that $F^e_Y(\mathcal{E}(a_e L))$ is 0-regular with respect to $jL$ in the sense of Castelnuovo–Mumford. If this holds, then the assertion follows from [13, Theorem 1.8.3]. For $0 < i \leq n$, we have

$$H^i(Y, (F^e_Y(\mathcal{E}(a_e L)))(-ij L)) \cong H^i(Y, F^e_Y(\mathcal{E}((a_e - p^e ij)L))))$$

$$\cong H^i(Y, \mathcal{E}((a_e - p^e ij)L))) =: V_e,i$$

by the projection formula. Since $L$ is ample, there is an $m_0 \geq 1$ such that

$$H^i(Y, \mathcal{E}(mL)) = 0$$

for each $m \geq m_0$ and $i > 0$. By the assumption, there is an $e_0 \geq 1$ such that

$$a_e - p^e ij = p^e\left(\frac{a_e}{p^e} - ij\right) \geq p^e\left(\frac{\varepsilon}{2} + jn - ij\right) \geq \frac{p^e \varepsilon}{2} \geq m_0$$

for each $e \geq e_0$, so $V_{e,i} = 0$ for each $e \geq e_0$. \qed

**Proof of Theorem 3.3** The proof is based on the idea suggested by the referee of [6]. Let $e_0 \geq 1$ be an integer such that $i \mid (p^{e_0} - 1)$. For $m \geq 1$ with $i \mid m$ and $e \geq 1$ with $e_0 \mid e$, applying $\mathcal{H}om_{\mathcal{O}_X}(\cdot, m(K_X + \Delta))$ to the composite of

$$\mathcal{O}_X \xrightarrow{F^e_X} F^e_X \mathcal{O}_X \xleftarrow{\mathcal{O}_X} F^e_X \mathcal{O}_X ((p^e - 1)\Delta),$$

we obtain the morphism

$$F^e_X(\mathcal{O}_X(((m - 1)p^e + 1)(K_X + \Delta))) \to \mathcal{O}_X(m(K_X + \Delta)) \quad (1)$$

by the Grothendieck duality. This is surjective on $U$ by the $F$-purity of $(U, \Delta|_U)$. Set $\mathcal{G}(m) := f_*\mathcal{O}_X(m(K_X + \Delta))$ for each $m \geq 1$ with $i \mid m$. Pushing morphism (1) forward by $f$, we get

$$F^e_Y(\mathcal{G}((m - 1)p^e + 1)) \to \mathcal{G}(m). \quad (2)$$

There is an $m_0 \geq 1$ such that the above morphism is surjective on $V$ for each $m \geq m_0$ with $i \mid m$, since $(K_X + \Delta)|_U$ is ample over $V$ (this is proved by the same argument
as that of [14, Corollary 2.23]). Also, since $(K + \Delta)|_U$ is ample over $V$, there are a $d \geq m_0$ with $i \mid d$ and $n_0 \geq 1$ such that the natural morphism

$$\mathcal{G}(d) \otimes \mathcal{G}(n) \to \mathcal{G}(d + n)$$

is surjective on $V$ for each $n \geq n_0$ with $i \mid n$. For $e, m \geq 1$, let $q_{e,m}$ and $r_{e,m}$ be integers such that $(m - 1)p^e + 1 = q_{e,m}d + r_{e,m}$ and $n_0 \leq r_{e,m} < n_0 + d$. Note that if $e_0 \mid e$ and $i \mid m$ then $i \mid r_{e,m}$. Then, for each $m \geq 1$ with $i \mid m$ and $e \geq 1$ with $e_0 \mid e$, the natural morphism

$$S^{q_{e,m}}(\mathcal{G}(d)) \otimes \mathcal{G}(r_{e,m}) \to \mathcal{G}((m - 1)p^e + 1)$$

(3)

is surjective on $V$, where $S^m(\cdot)$ denotes $m$-th symmetric product. Combining morphisms (2) and (3), we obtain the morphism

$$F_{Y*}^e(S^{q_{e,m}}(\mathcal{G}(d)) \otimes \mathcal{G}(r_{e,m})) \to \mathcal{G}(m)$$

(4)

for each $m \geq m_0$ with $i \mid m$ and $e \geq 1$ with $e_0 \mid e$. This is surjective on $V$ by the construction. Put $E := \bigoplus_{n_0 \leq r < n_0 + d, i \mid r} \mathcal{G}(r)$. By the definition of $E$ and morphism (4), we have the morphism

$$F_{Y*}^e(S^{q_{e,m}}(\mathcal{G}(d)) \otimes E) \to F_{Y*}^e(S^{q_{e,m}}(\mathcal{G}(d)) \otimes \mathcal{G}(r_{e,m})) \to \mathcal{G}(m)$$

(5)

for each $m \geq m_0$ with $i \mid m$ and $e \geq 1$ with $e_0 \mid e$. For each $m \geq 1$ with $i \mid m$, let $s_m$ be the smallest integer such that $\mathcal{G}(m) \otimes \mathcal{L}^{s_m}$ is globally generated. Then $S^{q_{e,m}}(\mathcal{G}(d))(q_{e,m}s_dL)$ is globally generated on $V$. Taking the tensor product of morphism (5) and $\mathcal{O}_X(lL)$, where $l \geq 1$ and $L$ is a Cartier divisor on $Y$ with $\mathcal{O}_Y(L) \cong \mathcal{L}$, we get the sequence of morphisms

$$\mathcal{G}(m)(lL) \leftarrow F_{Y*}^e(S^{q_{e,m}}(\mathcal{G}(d)) \otimes E(lp^eL))$$

$$\cong F_{Y*}^e(S^{q_{e,m}}(\mathcal{G}(d))(q_{e,m}s_dL) \otimes E((lp^e - q_{e,m}s_d)L))$$

$$\leftarrow F_{Y*}^e\left((\bigoplus \mathcal{O}_Y) \otimes E((lp^e - q_{e,m}s_d)L)\right)$$

$$\cong \bigoplus F_{Y*}^e(E((lp^e - q_{e,m}s_d)L)),$$

(6)

which are surjective on $V$. Put $a_e := lp^e - q_{e,m}s_dL$. Since

$$\frac{a_e}{p^e} = \frac{lp^e - q_{e,m}s_d}{p^e} \quad e \to +\infty \quad l - \frac{m - 1}{d} s_d,$$

if $l > (m - 1)s_d/d +jn$ and $e \gg 0$, then the sheaf (6) is globally generated by Lemma 3.4, and hence $\mathcal{G}(m)(lL)$ is globally generated on $V$. By the definition of $s_m$, we have

$$s_m \leq \frac{m - 1}{d} s_d + jn + 1.$$
When \( m = d \), we get \( s_d \leq d(jn + 1) \). Therefore, when \( m \) is not necessarily equal to \( d \), if \( l \geq m(jn + 1) \), then

\[
\frac{m - 1}{d} s_d + jn + 1 \leq \frac{m - 1}{d} d(jn + 1) + jn + 1 = m(jn + 1) \leq l,
\]

so \( \mathcal{O}(m)(L) \) is globally generated on \( V \), which is our claim. \( \square \)

**Acknowledgements**  The author wishes to express his thanks to Professors Osamu Fujino, Mihnea Popa and Christian Schnell for valuable comments. He is grateful to Professor Masataka Iwai for informing him of the result mentioned in Remark 3.2. He also would like to thank the referee of [6] for the permission to use the referee’s idea in this note.

**References**

1. Ambro, F.: Quasi-log varieties. Proc. Steklov Inst. Math. 240, 214–233 (2003)
2. Angehrn, U., Siu, Y.T.: Effective freeness and point separation for adjoint bundles. Invent. Math. 122(2), 291–308 (1995)
3. Deng, Y.: Applications of the Ohsawa–Takegoshi extension theorem to direct image problems. Int. Math. Res. Not. IMRN 2021(23), 17611–17633 (2021)
4. Dutta, Y.: On the effective freeness of the direct images of pluricanonical bundles. Ann. Inst. Fourier (Grenoble) 70(4), 1545–1561 (2020)
5. Dutta, Y., Murayama, T.: Effective generation and twisted weak positivity of direct images. Algebra Number Theory 13(2), 425–454 (2019)
6. Ejiri, S.: Direct images of pluricanonical bundles and Frobenius stable canonical rings of generic fibers (2019). arXiv:1909.07000 (to appear in Algabr. Geom.)
7. Fujino, O.: Fundamental theorems for the log minimal model program. Publ. Res. Inst. Math. Sci. 47(3), 727–789 (2011)
8. Fujino, O.: Iitaka conjecture: An Introduction. SpringerBriefs in Mathematics. Springer, Singapore (2020)
9. Gu, Y., Zhang, L., Zhang, Y.: Counterexamples to Fujita’s conjecture on surfaces in positive characteristic. Adv. Math. 400, Art. No. 108271 (2022)
10. Iwai, M.: On the global generation of direct images of pluri-adjoint line bundles. Math. Z. 294(1–2), 201–208 (2020)
11. Kawamata, Y.: On a relative version of Fujita’s freeness conjecture. In: Bauer, I., et al. (eds.) Complex Geometry, pp. 135–146. Springer, Berlin (2002)
12. Kollár, J.: Higher direct images of dualizing sheaves I. Ann. Math. 123(1), 11–42 (1986)
13. Lazarsfeld, R.K.: Positivity in Algebraic Geometry. I. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48. Springer, Berlin (2004)
14. Patakfalvi, Zs.: Semi-positivity in positive characteristics. Ann. Sci. Éc. Norm. Supér. 47(5), 991–1025 (2014)
15. Popa, M., Schnell, C.: On direct images of pluricanonical bundles. Algebra Number Theory 8(9), 2273–2295 (2014)
16. Shentu, J., Zhang, Y.: On the simultaneous generation of jets of the adjoint bundles. J. Algebra 555, 52–68 (2020)