Stellar Oscillons

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Received _______________; accepted ________________
ABSTRACT

We study the weakly nonlinear evolution of acoustic instability of a plane-parallel polytrope with thermal dissipation in the form of Newton’s law of cooling. The most unstable horizontal wavenumbers form a band around zero and this permits the development of a nonlinear pattern theory leading to a complex Ginzburg-Landau equation (CGLE). Numerical solutions for a subcritical, quintic CGLE produce vertically oscillating, localized structures that resemble the oscillons observed in recent experiments of vibrated granular material.
1. Introduction

The excitation of sound waves in the envelopes of stars has been extensively studied for its diagnostic importance as well as for its intrinsic physical interest. The most familiar mechanism of stellar acoustic instability is the Eddington valve mechanism or kappa mechanism, so-called because it relies on the dependence of opacity, $\kappa$, on physical conditions. This mechanism, which is basically thermal, resembles the phenomenon of negative differential resistivity\(^1\) familiar in condensed matter physics. However, sound waves can become unstable even when there is no kappa mechanism operating, and we here discuss a simple version of such instability.

The case of optically thin perturbations to a fluid layer stratified under gravity is one where the kappa mechanism cannot operate and yet it does show instabilities of sound waves under suitable conditions\(^2\). The space available for this work is not sufficient for a discussion of the conditions under which such instabilities can occur (but see Umurhan\(^3\)). Moreover this paper is a contribution to a symposium on nonlinear astrophysics, so our aim is to describe some nonlinear aspects of these acoustic instabilities, with pauses on the way for only a few indispensable remarks about the linear theory in section 3, following an introduction to the basic equations in section 2. In section 4 we outline the nonlinear procedures and conclude in section 5.

2. Equations and Equilibria

We consider the dynamics of a plane-parallel fluid subject to some form of radiative heat exchange but not to viscosity. The equations of motion for this system are,

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)$$

$$\rho (\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \rho g \mathbf{\hat{z}} \quad (2)$$

$$C_v \rho (\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla) T + p \nabla \cdot \mathbf{u} = Q(T, \rho) \quad (3)$$

$$p = \mathcal{R} \rho T \quad (4)$$
where the variables $\rho$, $\mathbf{u}$, $T$ and $p$ are respectively density, vector velocity, temperature and pressure and $Q(T,\rho)$ represents the thermal sources and sinks of the medium due to radiation and possibly mechanical effects. The vertical coordinate $z$ is measured positively downward and $\hat{z}$ is a downward-pointing unit vector.

In equilibrium $\mathbf{u} = 0$ and the state variables depend only on $z$. The governing equations are the hydrostatic equation, the equation of state, and the thermal equilibrium condition, $Q(T_0,\rho_0) = 0$, where the subscript naught denotes equilibrium values. Rather than go into the details of the transfer problem in the equilibrium state, we simply postulate that there is an equilibrium in which $T_0(z) = \beta z$, where $\beta$ is a constant; this is in fact the state one obtains in the diffusion limit with no heating and with a fairly general form for the opacity. With the hydrostatic condition and the equation of state, we then find that $\rho_0(z) = \rho_*(z/z_*)^m$ and $p_0(z) = p_*(z/z_*)^{m+1}$ where $\rho_*$ and $p_*$ are constants and $m = g/(\mathcal{R}\beta) - 1$ is an atmospheric analogue of the polytropic index. We let $T_0 = \beta z_*$, and choose $p_0 = p_0(z_*)$, $\rho_0 = \rho_0(z_*)$, so we have $p_* = \mathcal{R}\rho_* T_*$. We introduce natural units so that there remain only nondimensional equations in evidence. We let $z_*$ be the unit of length, the speed of sound ($c_a = \sqrt{\gamma \mathcal{R} T_*}$) be the unit of speed, $\rho_*$ be the unit of density and so on. The equilibrium temperature is then $T_0 = z$ and similarly for the other quantities.

We assume that the atmosphere is truncated so that the equilibrium thermodynamic quantities are nowhere zero. The fluid is confined between $z = 1$ below and $z = z_0 > 0$ above, where $(1 - z_0)$ is the nondimensional layer thickness. We require that the vertical velocity vanishes on $z = z_0$ and $z = 1$.

3. Linear Theory

We now introduce small perturbations about the static basic state and linearize the resulting equations to study the stability of the equilibrium state. We shall consider a very simple version of the stability theory here since our aim is to bring out features of the
The most complicated physical issue is the treatment of the transfer problem in the general case. For weak, optically thin perturbations to the equilibrium, we have that

\[ Q(T, \rho) = Q(T_0, \rho_0) + Q_T(T_0, \rho_0)(T - T_0) + Q_\rho(T_0, \rho_0)(\rho - \rho_0) + \ldots \]  

where the subscripts represent differentiation. In Newton's law of cooling, \( Q_\rho = 0 \) and the higher order terms are neglected. We adopt that form here and write \( Q_T = -\rho_0 C_v q \), where \( q \) is a characteristic inverse time.

For a grey, optically thin medium, we may express \( q \) in terms of the absorption coefficient and the state variables. If we assume that this coefficient is proportional to a power of density times a power of temperature, the linear theory is straightforward and acoustic instabilities occur in several parameter regimes. In particular, the case of constant \( q \) is very simple and we shall adopt it here and write for the rest of this discussion that \( Q(T, \rho) = -q\rho_0 C_v (T - T_0) \) with \( q \) constant.

If we expand about the equilibrium solution and linearize, we obtain a set of equations that is tractable both analytically and numerically. These linear equations are separable and the general perturbation is written in the form

\[ f(x, y, z, t) = F(z) \exp[\sigma t + i(\omega t + k \cdot x)] \]  

where \( k \) is the horizontal wave vector and \( \sigma \) and \( \omega \) are real. This leads to a confluent hypergeometric equation if suitable dependent variables are introduced.

As usual, we find gravity (or convective) modes and acoustic modes, but we focus only on the fundamental acoustic mode for the purposes of this discussion. In Figure 1 we plot \( \omega \) and \( \sigma \) vs. \( k = |k| \) for the case of \( \gamma = 1.28 \), \( m = 1.5 \), \( z_0 = 0.1 \) and \( q = 3 \). We see that there is instability in a band of wavenumbers around zero. From this band, we then construct a nonlinear wave packet in the next section.
4. Acoustic Pattern Equations

The linear problem is characterized by a band of overstable wavenumbers around \( k = 0 \) and we have what is called a Hopf bifurcation in nonlinear stability theory. In this circumstance, the generic equation governing the nonlinear spatio-temporal evolution of a wave packet is a complex Ginzburg-Landau equation. To derive this equation, we use a multiple-scale analysis based on the availability of a small parameter, here the degree of instability of the layer.

The linear theory provides a condition on the parameters \( q \), \( m \), \( \gamma \), and \( z_0 \) for the onset of acoustic instability. We may fix three of these and treat the fourth, say \( \gamma \), as the control parameter governing the degree of instability. Thus, for fixed \( m \), \( z_0 \) and \( q \) we find that instability begins as \( \gamma \) passes below the critical value \( \gamma_c \) and we examine values

\[
\gamma = \gamma_c - \epsilon^2 \mu,
\]

where \( \mu \) is simply a fiducial quantity. It is \( \epsilon \) that measures the degree of instability and we take it to be small here.

The band of significantly unstable wavenumbers around zero has a width of order \( \epsilon \) and so we use this parameter to rescale the horizontal coordinate in the usual manner of nonlinear instability theory\(^8\). Similarly, we scale the time and we then look for solutions in terms of the scaled variables. In particular, the deviations from equilibrium temperature, density and vertical and horizontal velocity are of the forms

\[
\begin{align*}
\theta & \sim \epsilon \left[ A(\epsilon x, \epsilon y, \epsilon^2 t) \Theta(z) e^{i\omega t} + c.c. \right] + ... \\
\rho & \sim \epsilon \left[ A(\epsilon x, \epsilon y, \epsilon^2 t) P(z) e^{i\omega t} + c.c. \right] + ... \\
w & \sim \epsilon \left[ A(\epsilon x, \epsilon y, \epsilon^2 t) W(z) e^{i\omega t} + c.c. \right] + ... \\
u & \sim \epsilon^2 \left[ A(\epsilon x, \epsilon y, \epsilon^2 t) U(z) e^{i\omega t} + c.c. \right] + ...,
\end{align*}
\]

where \( A \) is the (generally complex) envelope function that describes the pattern of the instability. In linear theory, \( A \) is an arbitrary constant; in nonlinear theory it is a slowly varying function that is the focus of interest. The functions of \( z \), on substitution and suitable
asymptotic development, turn out to be the $z$-dependent parts of the linear eigenfunctions.

The asymptotic developments based on these scalings leads to an equation for $A$. On general grounds, an overstability of the kind we have here is known to lead to an equation of the form

$$
\partial_t A = \mu A + \alpha \Delta A + F(|A|^2)A
$$

(9)

where the Laplacian operates in the two dimensional space $(\epsilon x, \epsilon y)$. The linear part of this equation describes the linear stability theory while the quantity $F(|A|^2)$ represents the renormalization of the linear growth rate by nonlinear effects, with $F(0) = 0$.

Determination of $F$ is not possible in general, so it is represented by Taylor series and the asymptotic development allows the computation of the coefficients in this development at each order in $\epsilon$. Typically, only the leading order is needed for weak instability. Then equation (9) becomes the (cubic) complex Ginzburg-Landau equation (CGLE), which generally describes the patterns resulting from overstable systems with a continuous spectrum of horizontal wave numbers. The cubic term is able to saturate the linear growth when $Re(\beta) < 0$. However, if the leading nonlinearity, the cubic term, has a coefficient that does not allow for nonlinear saturation of the instability, higher order terms must be sought. This may require a modification of the scaling.

For situations without a strong symmetry in the vertical, nonlinear saturation by the cubic term often does not occur and we have what is called subcritical bifurcation. This resembles a phase transition of the first kind and it is what we see at the larger values of $q$. In that case, the acoustic pattern equation is of the form

$$
\partial_t A = \mu A + \alpha \Delta A + \beta |A|^2 A + \eta |A|^4 A
$$

(10)

The complex constants $\alpha$, $\beta$ and $\eta$ are functions of the physical parameters of our model atmosphere. Calculation of the coefficients requires a working out of the nonlinear perturbation theory and we indicate some results of such work for selected values in the table below.
Both subcritical and supercritical instabilities may occur and produce differing behavior. For the case of one-dimensional patterns, spatio-temporal disorder is the rule in the supercritical case\textsuperscript{2,9}. But in the subcritical state, stable isolated structures may be expected, as argued by Thual and Fauve\textsuperscript{10,11}. In the two-dimensional case of subcritical bifurcation, we find oscillating, stable, localized structures whose time dependence is shown in Figure 2 in a series of snapshots. This structure is robust and we have seen it with a large range of values of the CGLE coefficients, emerging from a wide variety of initial conditions.

Such oscillating pulses resemble the oscillons observed in recent experiments of vertically-shaken layers of granular material\textsuperscript{12,13,14}, and they also have emerged from other pattern equations\textsuperscript{15,16,17,18}. We have here used the term oscillon to characterize the similar object found with the subcritical CGLE. The ‘-on’ ending normally smacks of integrability and it may be that, in the astronomical context, the word spicule would be more apt. This is a matter for later discussion.

5. Discussion

Stratified layers with thermal dissipation frequently suffer acoustic instabilities for various reasons that have not all been clarified as yet. We have considered one of the simplest of these instabilities to show what they may lead to. In fact, it is well known that overstabilities in thin layers where the spectrum of horizontal wavenumbers is so dense as to be regarded as continuous have an amplitude function, or envelope, that satisfies an equation of the CGLE form when the amplitudes are not too large. As we shall discuss elsewhere, the inclusion of the effects of magnetic fields or rotation may not require much qualification of these remarks. So we are inclined to seek applications of these considerations.
to thin layers such as stellar chromospheres or slabs and disks. Global modes such as are familiar in helioseismology, being discrete, would require simpler treatments and call for ODEs for their nonlinear description. At any rate, we have been able to extract from this simple theory solitary structures of a kind that have been attributed to magnetic effects in discussions of solar atmospheric dynamics.

It is not even necessary that the acoustic waves be unstable for a description by the CGLE to be appropriate, as mechanical forcing could also be included in the theory. However, if the degree of instability or the amplitude of the forcing becomes too large, one may question the use of weakly nonlinear theory. The recently observed excitation by a solar flare of an expanding wave on the solar surface\textsuperscript{19} is a case in point and it remains to be seen whether simple pattern equations can be appropriate in such situations where the initial amplitudes are quite large. Still, it is also true that the wave amplitude decays quickly when the general background is stable and even then the weakly nonlinear theory may be brought to bear.

L.T. thanks NSF for a Postdoctoral Fellowship and Pierre Coullet for a helpful conversation.
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6. Figure Captions

Figure 1. Dispersion relation of the fundamental acoustic mode ($\gamma = 1.28$, $m = 1.5$, $z_0 = 0.1$ and $q = 3$). The curves are frequency and growth rate as a function of the horizontal wave number.

Figure 2. Evolution of an oscillon (CGLE parameters: $\alpha = 1.0$, $\beta = 3.0 + i$ and $\eta = -2.75 + i$). The domain is periodic in both spatial dimensions with a period of $10\pi$. The shaded surfaces are the real parts of the amplitude. The oscillation starts from the top left panel and continues to the right.
