ADJOINT PAIRS AND UNBOUNDED NORMAL OPERATORS

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Abstract. An adjoint pair is a pair of densely defined linear operators \(\{A, B\}\) on a Hilbert space such that \(\langle Ax, y \rangle = \langle x, By \rangle\) for \(x \in D(A), y \in D(B)\). We consider adjoint pairs for which 0 is a regular point for both operators and associate a boundary triplet to such an adjoint pair. Proper extensions of the operator \(B\) are in one-to-one correspondence \(T_C \leftrightarrow C\) to closed subspaces \(C\) of \(N(A^*) \oplus N(B^*)\). In the case when \(B\) is formally normal and \(D(A) = D(B)\), the normal operators \(T_C\) are characterized. Next we assume that \(B\) has an extension to a normal operator with bounded inverse. Then the normal operators \(T_C\) are described and the case when \(N(A^*)\) has dimension one is treated.

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1. Introduction

This paper deals with various notions and constructions in unbounded operator theory on Hilbert space. The basic objects studied in this paper are adjoint pairs \(\{A, B\}\) (see Definition 2) of densely defined operators \(A, B\) on a Hilbert space \(\mathcal{H}\) for which the number 0 is a regular point of \(A\) and \(B\).

In Section 3 we use a result of M.I. Vishik \cite{Vi} (stated as Theorem 5 below) to associate to such a pair a boundary triplet (see Definition 1) for the operator matrix

\[
\mathfrak{A} = \begin{pmatrix}
0 & A \\
B & 0
\end{pmatrix}
\]

acting as symmetric operator with domain \(D(B) \oplus D(A)\) on \(\mathcal{H} \oplus \mathcal{H}\) (Theorem 6). Then the theory of boundary triplets allows one to describe the proper extensions of the symmetric operator \(\mathfrak{A}\) in terms of closed relations.

In the remaining Sections 4-6 we assume in addition that the operator \(B\) is formally normal and \(D(A) = D(B)\). Our aim is to study normal extensions of the operator \(B\).

The proper extensions of \(B\) (that is, closed operators \(T\) satisfying \(B \subseteq T \subseteq A^*\)) can be described in terms of closed subspaces \(C\) of the Hilbert space \(N(A^*) \oplus N(B^*)\). Let \(T_C\) denote the corresponding operator.

In Section 4 the normality of the operator \(T_C\) is characterized in terms of the subspace \(C\) (Theorems 10 and 12). In Section 5 and 6 we assume that the formally normal operator \(B\) admits an extension to a normal operator \(R^*\) with bounded inverse. Then there exists a unitary operator \(W\) satisfying \(R^{-1}W = (R^*)^{-1}\) which leads to simplifications of the normalcy criteria for the operator \(T_C\). The case when \(N(A^*)\) has dimension one is treated in Section 6 and the result is Theorem 17.

Throughout the whole paper, \(\{A, B\}\) denotes an adjoint pair such that 0 is a regular point for \(A\) and \(B\).

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Let us add a few bibliographical comments and hints. Adjoint pairs are treated in [Vi] and [EE, Section III.3]. Boundary triplets have been invented by Kochubei [Ko] and Bruk [Bk]; a fundamental paper on boundary triplets is [DM]. Boundary triplets associated with adjoint pairs were constructed and studied in [MM]. Pioneering work on formally normal operators and their extensions to normal operators was done by Biruik and Coddington [BC], [Cd2]. The existence of formally normal operators which have no normal extension was discovered by Coddington [Cd1]; a very simple example can be found in [Sch86]. Unbounded normal operators have been extensively studied by Stochel and Szafraniec, see e.g. [SS1], [SS2], [SS3]. Concerning the theory of unbounded operators on Hilbert space we refer to the author’s graduate text [Sch12], see also [EE].

2. Some operator-theoretic notions

In this short section we collect a few concepts and notations from operator theory which are crucial in what follows.

Let \( T \) be a linear operator on a Hilbert space. We denote by \( D(T) \) its domain, by \( R(T) \) its range und by \( N(T) \) its kernel.

The symbol \( \dot{+} \) refers to the direct sum of vector spaces.

The bounded operator on \( H \) are denoted by \( B(H) \).

A number \( \lambda \in \mathbb{C} \) is called regular for \( T \) if there exists a constant \( \gamma > 0 \) (depending on \( \gamma \) in general) such that
\[
\| (T - \lambda I)\varphi \| \geq \gamma \| \varphi \| \quad \text{for} \quad \varphi \in D(T).
\]

A densely defined operator \( A \) is called formally normal if \( D(A) \subseteq D(A^*) \) and
\[
\| Ax \| = \| A^* x \| \quad \text{for} \quad x \in D(A).
\]

By polarization, condition \( \mathbf{3} \) implies that
\[
\langle Ax, Ax' \rangle = \langle A^* x, A^* x' \rangle \quad \text{for} \quad x, x' \in D(A) = D(A^*).
\]

A formally normal operator \( A \) is called normal if \( D(A) = D(A^*) \).

Next we recall the notion of a boundary triplet.

Definition 1. Suppose that \( T \) is a densely defined symmetric operator on \( H \). A boundary triplet for \( T^* \) is a triplet \( (K, \Gamma_0, \Gamma_1) \) of a Hilbert space \( (K, \langle \cdot, \cdot \rangle) \) and linear mappings \( \Gamma_0 : D(T^*) \to K \) and \( \Gamma_1 : D(T^*) \to K \) such that:

(i) \( \langle T^* x, y \rangle - \langle x, T^* y \rangle = \langle \Gamma_1 x, \Gamma_0 y \rangle - \langle \Gamma_0 x, \Gamma_1 y \rangle \) for \( x, y \in D(T^*) \),

(ii) the mapping \( D(T^*) \ni x \mapsto (\Gamma_0 x, \Gamma_1 x) \in K \oplus K \) is surjective.

3. Adjoint pairs and boundary triplets

The main concept occurring in this paper is the following.

Definition 2. An adjoint pair is a pair \( \{A, B\} \) of densely defined linear operators \( A \) and \( B \) on Hilbert space \( H \) such that
\[
\langle Ax, y \rangle = \langle x, By \rangle \quad \text{for} \quad x \in D(A), y \in D(B).
\]

Clearly, \( \mathbf{5} \) is equivalent to the relations
\[
A \subseteq B^* \quad \text{and} \quad B \subseteq A^*.
\]

Thus, any pair of densely defined operators \( A, B \) satisfying \( \mathbf{5} \) is an adjoint pair.

In the literature, “adjoint pairs” often appear as “dual pairs” (for instance, in [MM]). Since the latter notion is used in different context in other parts of mathematics, we prefer to speak about “adjoint pairs” (as in [EE]).

We mention two examples.
Example 3. Suppose $A$ is a densely defined closed operator. Then $\{A, A^*\}$ and
$\{A^*, A\}$ are adjoint pairs. Note that in both cases we have equalities in $[12]$.

Example 4. Suppose $T$ is a densely defined closed symmetric operator and $\alpha \in \mathbb{C}$. Then the operators
$A_0 := T + \alpha I$ and $B_0 := T^* + \overline{\alpha} I$ form an adjoint pair.
Likewise, $A := T + \alpha I$ and $B := (T^* + \overline{\alpha} I)[D(T)] = T + \overline{\alpha} I$ are an adjoint pair.
Set $a = \text{Re}\alpha$ and $b = \text{Im}\alpha$. For $x \in D(T)$ we compute
\[
\|Ax\|^2 = \|(T + aI)x\|^2 + b^2\|x\|^2 = \|Bx\|^2.
\]
Hence $A$ and $B$ are formally normal operators. Further, if $b \neq 0$, it follows from (7) that $0$ is a regular point for $A$ and $B$.

The considerations of this paper are essentially based on an important theorem of Vishik [Vi, Theorems 1 and 2]. It is an extension of a result of Calkin [Ck] for symmetric operators. We state this result as Theorem 5 and add an number of useful formulas.

A crucial part is the existence of the operator $R$ with bounded inverse; a nice proof of this assertion can be also found in [16, Theorem 3.3].

Theorem 5. Suppose $\{A, B\}$ is a adjoint pair and $0$ is a regular number for the
operators $A$ and $B$, that is, exists a constant $\gamma > 0$ such that
\[
\|Ax\| \geq \gamma \|x\| \quad \text{and} \quad \|By\| \geq \gamma \|y\| \quad \text{for} \quad x \in D(A), y \in D(B).
\]
Then there exists a closed operator $R$ on $\mathcal{H}$ such that $R$ and $R^*$ have inverses $R^{-1} \in \mathcal{B}(\mathcal{H})$, $(R^*)^{-1} \in \mathcal{B}(\mathcal{H})$,
\[
A \subseteq R \subseteq B^*, \quad B \subseteq R^* \subseteq A^*
\]
and
\[
\mathcal{D}(A^*) = \mathcal{D}(B)\oplus (R^*)^{-1}\mathcal{N}(B^*)\oplus \mathcal{N}(A^*),
\]
\[
\mathcal{D}(B^*) = \mathcal{D}(A)\oplus R^{-1}\mathcal{N}(A^*)\oplus \mathcal{N}(B^*).
\]
For $x_0 \in D(A)$, $y_0 \in D(B)$, $u \in \mathcal{N}(A^*)$, $v \in \mathcal{N}(B^*)$, we have
\[
A^*(x_0 + (R^*)^{-1}v + u) = Bx_0 + v, \quad B^*(y_0 + R^{-1}u + v) = Ay_0 + u.
\]

Let us adopt the following notational convention: Elements of $\mathcal{N}(A^*)$ are denoted by $u, u', v_1, v_2, u_1, u_2, u_1', u_2'$, while symbols $v, v', v_1, v_2, v_1', v_2'$ always refer to vectors of $\mathcal{N}(B^*)$.

As throughout, $\{A, B\}$ denotes an adjoint pair such that $0$ is a regular point for the
operators $A$ and $B$.

We define an operator $\mathfrak{A}$ with domain $\mathcal{D}(\mathfrak{A}) = \mathcal{D}(B)\oplus \mathcal{D}(A)$ on the direct sum
Hilbert space $\mathcal{H} \oplus \mathcal{H}$ by the operator matrix
\[
\mathfrak{A} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.
\]
From (5) it follows at once that the operator $\mathfrak{A}$ is symmetric. It is easily verified that the adjoint operator $\mathfrak{A}^*$ has the domain $\mathcal{D}(\mathfrak{A}^*) = \mathcal{D}(A^*) \oplus \mathcal{D}(B^*)$ and is given by the matrix
\[
\mathfrak{A}^* = \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix}.
\]
Let $(x, y), (x', y') \in \mathcal{D}(\mathfrak{A}^*)$. Then $x, x' \in \mathcal{D}(A^*)$ and $y, y' \in \mathcal{D}(B^*)$. Therefore, by (10) and (11), $x, y, x', y'$ are of the form
\[
x = x_0 + (R^*)^{-1}v_1 + u_1, \quad x' = x_0' + (R^*)^{-1}v'_1 + u'_1,
\]
\[
y = y_0 + R^{-1}u_2 + v_2, \quad y' = y_0' + R^{-1}u'_2 + v'_2.
\]
with \( x_0, x'_0 \in \mathcal{D}(B) \), \( y_0, y'_0 \in \mathcal{D}(A) \), \( v_1, v_2, v'_1, v'_2 \in \mathcal{N}(B^*) \), \( u_1, u_2, u'_1, u'_2 \in \mathcal{N}(A^*) \). Using equation (20) we derive

\[
(17) \quad \langle Ay_0, (R^*)^{-1}v'_1 \rangle = (Ry_0, (R^*)^{-1}v'_1) = \langle y_0, R^*(R^*)^{-1}v'_1 \rangle = \langle y_0, v'_1 \rangle
\]

and similarly

\[
(18) \quad ((R^*)^{-1}v_1, Ay'_0) = \langle v_1, y'_0 \rangle.
\]

Replacing (17) by (18) the same reasoning yields

\[
(19) \quad \langle Bx_0, R^{-1}u'_2 \rangle = \langle x_0, u'_2 \rangle, \quad \langle R^{-1}u_2, Bx'_0 \rangle = \langle u_2, x'_0 \rangle.
\]

Further, since \( u'_1, u_1 \in \mathcal{N}(A^*) \) and \( v'_2, v'_2 \in \mathcal{N}(B^*) \), we have

\[
(20) \quad \langle Ay_0, u'_1 \rangle = \langle u_1, Ay'_0 \rangle = \langle Bx_0, v'_2 \rangle = \langle v_2, Bx'_0 \rangle = 0.
\]

Now we apply the preceding formulas (15), (17), (18), (19), (17) and compute

\[
(21) \quad \langle \mathfrak{A}^*(x, y), (x', y') \rangle = \langle (x, y), \mathfrak{A}^*(x, y) \rangle
\]

Next we introduce an auxiliary Hilbert space

\[
\mathcal{K} = \mathcal{N}(A^*) \oplus \mathcal{N}(B^*),
\]

with scalar product \( \langle \cdot, \cdot \rangle \) defined by

\[
(22) \quad \langle (u, v), (u', v') \rangle = \langle u, u' \rangle + \langle v, v' \rangle, \quad u, u' \in \mathcal{N}(A^*), \ v, v' \in \mathcal{N}(B^*),
\]

and linear mappings \( \Gamma_0 : \mathcal{D}(\mathfrak{A}^*) \to \mathcal{K} \) and \( \Gamma_1 : \mathcal{D}(\mathfrak{A}^*) \to \mathcal{K} \) by

\[
(23) \quad \Gamma_0(x, y) = (u_1, v_1) \quad \text{and} \quad \Gamma_1(x, y) = (u_2, -v_2),
\]

where \( x, y \) are of the form (15) and (17). Then

\[
(24) \quad \langle (u_2, -v_2), (u'_1, v'_1) \rangle = \langle (u_1, v_1), (u'_2, -v'_2) \rangle
\]

for \( (x, y), (x', y') \in \mathcal{D}(\mathfrak{A}^*) \). Comparing (24) with (21) we get

\[
(\mathfrak{A}^*(x, y), (x', y')) = \langle (x, y), (\mathfrak{A}^*(x, y) \rangle = \langle \Gamma_1(x, y), (x', y') \rangle - \langle \Gamma_0(x, y), \Gamma_1(x', y') \rangle,
\]

which is condition (i) of Definition 1 for \( T = \mathfrak{A}^* \). Condition (ii) of Definition 1 is obvious from the description of domains \( \mathcal{D}(A^*) \) and \( \mathcal{D}(B^*) \) given in Proposition 5. Summarizing the preceding we have proved the following

**Theorem 6.** Suppose that \( \{A, B\} \) is an adjoint pair such that 0 is a regular point for \( A \) and \( B \). Then the triplet \( (\mathcal{K}, \Gamma_0, \Gamma_1) \) of the Hilbert space \( \mathcal{K} = \mathcal{N}(A^*) \oplus \mathcal{N}(B^*) \) and the mappings \( \Gamma_0 \) and \( \Gamma_1 \), defined in equation (20), is a boundary triplet for the operator \( \mathfrak{A}^* \).
Remark 7. Suppose $A$ and $B$ are operators on a Hilbert space and $\alpha \in \mathbb{C}$. Set
\[ A' := A - \alpha I \quad \text{and} \quad B' := B - \overline{\alpha} I. \]
From Definition 2 it follows at once that $\{A, B\}$ is an adjoint pair if and only if $\{A', B'\}$ is an adjoint pair. Obviously, 0 is a regular number for $A$ and $B$ if and only if $\alpha$ is a regular number for $A'$ and $\overline{\alpha}$ is a regular number for $B'$. Further, $A$ and $B$ are formally normal (resp. normal) if and only if $A'$ and $B'$ are formally normal (resp. normal). Using these facts we can treat adjoint pairs $\{A', B'\}$ for which $\alpha$ is a regular number for $A'$ and $\overline{\alpha}$ is a regular number for $B'$ by reducing them to pairs $\{A, B\}$ studied in this paper. Note that in the corresponding formulas we have to replace $N(A)$ by $N(A' + \alpha I)$ and $N(B)$ by $N(B' + \overline{\alpha} I)$.

Next we restate some facts from the theory of boundary triplets adapted to the present situation (see e.g. [Sch12, Section 14.2]). Recall that a closed operator $\mathfrak{A}$ on $\mathcal{H}_2 := \mathcal{H} \oplus \mathcal{H}$ is called a proper extension of the symmetric operator $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{T} \subseteq \mathfrak{A}^\ast$.

A closed relation on $\mathcal{K}_2$ is a closed linear subspace of $\mathcal{K}_2 := \mathcal{K} \oplus \mathcal{K}$.

Lemma 8. Suppose $\mathfrak{C}$ is a closed relation on $\mathcal{K}_2 = \mathcal{K} \oplus \mathcal{K}$. Then there exists a unique proper extension $\mathfrak{A}_\mathfrak{C}$ of $\mathfrak{A}$ defined by $\mathfrak{A}_\mathfrak{C} := \mathfrak{A}^\ast [\mathcal{D}(\mathfrak{A}_\mathfrak{C})]$, where
\begin{equation}
\mathcal{D}(\mathfrak{A}_\mathfrak{C}) = \{ (\mathfrak{A}^\ast) : (\Gamma_0(x, y), \Gamma_1(x, y)) \in \mathfrak{C} \}.
\end{equation}
Each proper extension of $\mathfrak{A}$ is of this form. Further, the extension $\mathfrak{A}_\mathfrak{C}$ of $\mathfrak{A}$ is self-adjoint if and only the relation $\mathfrak{C}$ is self-adjoint.

Proof. [Sch12, Proposition 14.17].

Clearly, $\mathfrak{T}$ is a proper extension of $\mathfrak{A}$ if and only if there are closed operators $S, T$ on $\mathcal{H}$ such that
\[ A \subseteq S \subseteq B^\ast, B \subseteq T \subseteq A^\ast, \]
and
\begin{equation}
\mathfrak{T} = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}.
\end{equation}
It is clear that $\mathfrak{T}$ is self-adjoint operator on $\mathcal{H}_2$ if and only if $T = S^\ast$.

For an adjoint pair $\{A, B\}$, a closed operator $T$ on $\mathcal{H}$ satisfying $B \subseteq T \subseteq A^\ast$ is called a proper extension of $B$. Likewise, by a proper extension of $A$ we mean a closed operator $S$ such that $A \subseteq S \subseteq B^\ast$.

Then, as discussed in the paragraph before last, self-adjoint extensions of $\mathfrak{A}$ on $\mathcal{H} \oplus \mathcal{H}$ are in one-to-one correspondence to proper extensions $T$ of the operator $B$ on $\mathcal{H}$, and equivalently, to proper extensions $S$ of $A$ on $\mathcal{H}$. These operators $T$ and $S$ will be studied in the next section.

The passage to $2 \times 2$ operator matrices is an old and powerful trick in operator theory which was used in many papers and different contexts, see e.g. [AI] or [GS].

4. Formally normal operators and normal operators

In this section we continue the considerations of the previous section and assume in addition that $B$ is a closed formally normal operator and $\mathcal{D}(A) = \mathcal{D}(B)$.

Recall that $B$ is formally normal means that $\mathcal{D}(B) \subseteq \mathcal{D}(B^\ast)$ and $\|Bx\| = \|B^\ast x\|$ for $x \in \mathcal{D}(B)$. Since (15) holds by assumption, we have $A = B^\ast [\mathcal{D}(B)]$ and therefore $\|Bx\| = \|Ax\|$ for $x \in \mathcal{D}(A) = \mathcal{D}(B)$.

Hence, in particular, the operator $A$ is also formally normal and closed. In fact, the above assumption is symmetric in the operators $B$ and $A$. 

"ADJOINT PAIRS AND UNBOUNDED NORMAL OPERATORS \(^5\)"
Let $x \in D(A^*)$ and $y \in D(B^*)$. As noted above (see \((\text{10})\) and \((\text{11})\)), $x$ and $y$ are of the form
\begin{equation}
 x = x_0 + (R*)^{-1}v_1 + u_1, \quad y = y_0 + R^{-1}u_2 + v_2,
\end{equation}
where $x_0 \in D(B)$, $y_0 \in D(A)$, $v_1$, $v_2 \in \mathcal{N}(B^*)$, $u_1$, $u_2 \in \mathcal{N}(A^*)$. Then, setting $x' = 0$, $y = 0$ and renaming $y'$ by $y$ in formula \((\text{21})\) we obtain
\begin{equation}
 \langle A^*x, y \rangle - \langle x, B^*y \rangle = \langle v_1, v_2 \rangle - \langle u_1, u_2 \rangle.
\end{equation}

Now we suppose that $C$ is a closed subspace of $K = \mathcal{N}(A^*) \oplus \mathcal{N}(B^*)$. We define linear operators $T_C$ and $S_C$ on the Hilbert space $H$ by
\begin{equation}
 T_C = A^*[D(T_C^*)], \quad S_C = B^*[D(S_C)],
\end{equation}
where
\begin{equation}
 D(T_C) = \{x_0 + (R*)^{-1}v_1 + u_1 : x_0 \in D(B), \quad (u_1, v_1) \in C \},
\end{equation}
\begin{equation}
 D(S_C) = \{y_0 + R^{-1}u_2 + v_2 : y_0 \in D(A), \quad (u_2, v_2) \in C \}.
\end{equation}

Further, let $C'$ denote the closed linear subspace of $K$ given by
\begin{equation}
 C' = \{(u_2, v_2) \in K : \langle v_1, v_2 \rangle = \langle u_1, u_2 \rangle \text{ for all } (u_1, v_1) \in C \}.
\end{equation}

Recall that for any relation $C$ on $K = \mathcal{N}(A^*) \oplus \mathcal{N}(B^*)$ the adjoint relation $C^*$ is the relation on $K' := \mathcal{N}(B^*) \oplus \mathcal{N}(A^*)$ defined by
\begin{equation}
 C^* = \{(v_2, u_2) \in K' : \langle v_2, v_1 \rangle = \langle u_2, u_1 \rangle \text{ for } (u_1, v_1) \in C \}.
\end{equation}

Comparing \((\text{32})\) and \((\text{33})\) we conclude that
\begin{equation}
 C' = \{(u_2, v_2) : (v_2, v_1) \in C^* \}.
\end{equation}

Hence, since $C^*$ is a closed linear relation, $C'$ is a closed linear subspace of $K$.

**Lemma 9.** $T_C$ is a proper extension of $B$, that is, $T_C$ is a closed linear operator such that $B \subseteq T_C \subseteq A^*$. Each proper extension of $B$ is of this form. Further,
\begin{equation}
 (T_C^*)^* = S_C^*, \quad \text{and} \quad (S_C^*)^* = T_C^*.
\end{equation}

**Proof.** By a general result on boundary triplets \cite[Lemma 14.13]{Sch12}, the mappings $\Gamma_0, \Gamma_1$ of $D(\mathcal{A}^*)$, endowed with the graph norm, into $K$ are continuous. Since the operators $A, B$ are closed and the subspaces $C, C'$ of $K$ are closed, it follows easily from this result that $T_C$ and $S_C$ are closed operators. The inclusions $B \subseteq T_C \subseteq A^*$ are obvious from the definition of $T_C$. Thus, $T_C$ is a proper extension of $B$.

Now let $T$ be an arbitrary proper extension of $B$. Then the matrix $\Xi$, defined by \((\text{20})\) with $S := T^*$, is a self-adjoint operator on $H_2$. Hence, by Lemma \(\Xi = \mathcal{A}_e^\ast\) for some closed relation $\mathcal{C}$ on $K_2$. Let $C$ denote the set of vectors $\Gamma_0(x, y)$ for $(\Gamma_0(x, y), \Gamma_1(x, y)) \in \mathcal{C}$. Then $C$ is a closed subspace of $K$ and from the definition of $\Xi = \mathcal{A}_e^\ast$ it follows that $T = T_C$.

Finally, we prove \((\text{35})\). From \((\text{28})\), \((\text{20})\) and \((\text{32})\) we conclude that
\begin{equation}
 \langle T_C x, y \rangle = \langle x, S_C^* y \rangle \quad \text{for} \quad x \in D(T_C), \quad y \in D(S_C^*).
\end{equation}

This equation implies that $T_C \subseteq (S_C^*)^*$ and $S_C \subseteq (T_C)^*$. Combining both relations yields $T_C \subseteq (S_C^*)^* \subseteq ((T_C)^*)^* = T_C^*$, because the operator $T_C$ is closed. Thus $T_C = (S_C^*)^*$. Applying the adjoint yields $(T_C^*)^* = ((S_C^*)^*)^* = S_C^*$.

The following theorem characterizes the case when the operator $T_C$ is normal. Condition (i) ensures that the domains $D(T_C)$ and $D((T_C)^*)$ coincide, while condition (ii) implies the equality of norms $\|T_C z\|$ and $\|(T_C)^* z\|$.

**Theorem 10.** Suppose $B$ is a closed formally normal operator and $D(A) = D(B)$. Let $C$ be a closed linear subspace of $K$ satisfying the following two conditions:
\begin{itemize}
  \item[(i)] $\{(R^*)^{-1}v_1 + u_1 : (u_1, v_1) \in C \} = \{R^{-1}u_2 + v_2 : (u_2, v_2) \in C' \}$,
  \item[(ii)] $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$.
\end{itemize}
By assumption, the operator we conclude that on
Then, by (40),
Similarly, by (38)
with (40)
Since (39) we derive
Proof. First we prove that the conditions (i) and (ii) imply that \( T_C \) is a normal operator. Since \( (T_C)^* = S_C \), by Lemma 3, condition (i) ensures that \( D(T_C) = D((T_C)^*) \).
Let \( z \in D(T_C) = D((T_C)^*) \). Then, by (30) and (31), the vector \( z \) is of the form
\[
z = x_0 + (R^*)^{-1}v_1 + u_1 = y_0 + R^{-1}u_2 + v_2,
\]
with \( x_0 \in D(B), y_0 \in D(A) \) and \( (u_1, v_1) \in \mathcal{C}, (u_2, v_2) \in \mathcal{C} \). By condition (i), \( R^{-1}u_2 + v_2 \) is of the form \( R^{-1}u_2 + v_2 = (R^*)^{-1}v_1 + u_1' \) for some vector \( (u_1', v_1') \in \mathcal{C} \).
Then, by (40),
\[
x_0 - y_0 + (R^*)^{-1}(v_1 - v_1') + (u_1 - u_1') = 0.
\]
Since the decomposition \( D(A^*) = D(B) + (R^*)^{-1}N(B^*) + N(A^*) \) is a direct sum, we conclude that \( x_0 = y_0 \).
We have \( T_C z = A^*z = Bx_0 + v_1 \) and \( (T_C)^*z = S_C z = A^*z = Ay_0 + u_2 = Ax_0 + u_2 \). Using that \( B^*v_1 = 0 \) we compute
\[
\|T_C z\|^2 = \langle Bx_0 + v_1, Bx_0 + v_1 \rangle = \|Bx_0\|^2 + \|v_1\|^2
\]
Similarly, by \( A^*u_2 = 0 \),
\[
\|(T_C)^*z\|^2 = \langle Ax_0 + u_2, Ax_0 + u_2 \rangle = \|Ax_0\|^2 + \|u_2\|^2.
\]
By assumption, \( B \) is formally normal and \( A = B^* [D(B) \cap \mathcal{D}(A) = D((T_C)^*) - N(A^*) \) holds. Therefore, by (38) and (39) yields \( \|T_C z\|^2 = \|(T_C)^*z\|^2 \). This proves that \( T_C \) is normal. By definition, \( D(B) \subseteq D(T_C) \) and \( B = T_C [D(B) \cap \mathcal{D}(B) \) is a direct sum, \( B \) on \( \mathcal{H} \). Clearly, \( B \subseteq N \) implies \( N^* \subseteq B^* \), so that \( N^* [D(A) = B^* [D(A) = A \) and hence \( A \subseteq N^* \subseteq B^* \). Taking the adjoint operation we get \( B \subseteq N \subseteq A^* \).
Instead, \( N \) is a closed operator such that \( B \subseteq N \subseteq A^* \), it follows from Lemma 3 that \( N = T_C \) for some closed subspace \( \mathcal{C} \) of \( \mathcal{K} \). Then \( D(T_C) = D((T_C)^*) = D(S_C) \) by the normality of \( N = T_C \) and \( S_C \). From (37) it follows the elements of \( D(T_C) = D(S_C) \) are precisely the vectors \( z \) of the form
\[
z = x_0 + (R^*)^{-1}v_1 + u_1 = y_0 + R^{-1}u_2 + v_2,
\]
with \( x_0, y_0 \in D(B) = D(A) \) and \( (u_1, v_1) \in \mathcal{C}, (u_2, v_2) \in \mathcal{C} \).
The crucial step is to show that \( x_0 = y_0 \). Recall that \( T_C^*z = Bx_0 + v_1 \) and \( (T_C)^*z = S_C z = B^*z = Ay_0 + u_2 \). Therefore, since \( T_C \) is normal, using equation (41) we derive
\[
\langle Bx_0 + v_1, B(x_0 - y_0) \rangle = \langle T_C z, T_C (x_0 - y_0) \rangle
\]
\[
= \langle (T_C)^*z, (T_C)^* (x_0 - y_0) \rangle = \langle Ay_0 + u_2, A(x_0 - y_0) \rangle.
\]
By \( B^*v_1 = 0 \) and \( A^*z_2 = 0 \) this implies
\[
\langle Bx_0, B(x_0 - y_0) \rangle = \langle Ay_0, A(x_0 - y_0) \rangle.
\]
By assumption, the operator \( B \) is formally normal and \( A \subseteq B^* \). Hence, combining (37) and (38), we obtain
\[
\langle Bx_0, B(x_0 - y_0) \rangle = \langle By_0, B(x_0 - y_0) \rangle
\]
so \( B(x_0 - y_0), B(x_0 - y_0) \rangle = 0 \) and hence \( B(x_0 - y_0) = 0 \). Since \( B \subseteq R^* \) and \( R^* \) is invertible, we conclude that \( x_0 = y_0 \).
Inserting the equality $x_0 = y_0$ into (10), we get $(R^*)^{-1}v_1 + u_1 = R^{-1}u_2 + v_2$. Note that all vectors $(R^*)^{-1}v_1 + u_1$ with $(u_1, v_1) \in \mathcal{C}$ and all vectors $R^{-1}u_2 + v_2$ with $(u_2, v_2) \in \mathcal{C}'$ are in $\mathcal{D}(T_u) = \mathcal{D}((T_u)^*)$. Thus, it follows that condition (i) is fulfilled.

By equations (38) and (39) and the normality of the operator $T_u$, we have
\[ \|Bx_0\|^2 + \|v_1\|^2 = \|(T_u)z\|^2 = \|(T_u)^*z\|^2 = \|Ax_0\|^2 + \|u_2\|^2. \]
Recall that $\|Bx_0\| = \|Ax_0\|$, because $B$ is formally normal. Hence $\|v_1\| = \|u_2\|$, which proves that condition (ii) holds.

Now we consider the special case where $\mathcal{C}$ is the the graph of a *densely defined closed linear operator* $C$ of the Hilbert space $\mathcal{N}(B^*)$ into the Hilbert space $\mathcal{N}(A^*)$:
\[ (42) \quad \mathcal{C} = \{(Cv_1, v_1) : v_1 \in \mathcal{D}(C)\}. \]

**Lemma 11.** $\mathcal{C}' = \{(u_2, C^*u_2) : u_2 \in \mathcal{D}(C^*)\}$.

**Proof.** Let $(u_2, v_2) \in \mathcal{K}$. By the definitions (12) and (32) of $\mathcal{C}$ and $\mathcal{C}'$, we have $(u_2, v_2) \in \mathcal{C}'$ if and only if $(v_1, v_2) = (Cv_1, u_2)$ for all $v_1 \in \mathcal{D}(C)$. The latter holds if and only if $u_2 \in \mathcal{D}(C^*)$ and $v_2 = C^*u_2$, which proves the assertion.

The following is a reformulation of Theorem 10 for subspaces (42).

**Theorem 12.** Suppose $B$ is a closed formally normal operator and $\mathcal{D}(A) = \mathcal{D}(B)$. Let $\mathcal{C}$ be a closed subspace of $\mathcal{K}$ of the form (42).

Then the operator $T_u$ is normal if and only if there exists an isometric linear operator $U$ of $\mathcal{D}(C^*)$ onto $\mathcal{D}(C)$ such that
\[ (43) \quad R^{-1}u_2 + C^*u_2 = (R^*)^{-1}Uu_2 + CUu_2 \quad \text{for} \quad u_2 \in \mathcal{D}(C^*). \]

**Proof.** First we suppose that $T_u$ is normal. Let $u_2 \in \mathcal{D}(C^*)$. Then $(u_2, C^*u_2) \in \mathcal{C}'$, so by condition (i) there exists a vector $v_1 \in \mathcal{D}(C)$ such that $R^{-1}u_2 + v_2 = (R^*)^{-1}v_1 + u_1$. Since (11) is a direct sum, $v_1$ is uniquely determined by $u_2$. Clearly, the map $u_2 \mapsto v_1$ is linear. Since $\|u_2\| = \|v_1\|$ by condition (ii) and equation (42), there is an isometric linear map $U : \mathcal{D}(C^*) \to \mathcal{D}(C)$ given by $Uu_2 = v_1$. Inserting $Uu_2 = v_1, v_2 = C^*u_2, u_1 = Cv_1 = CUu_2$ into condition (i), we obtain (43).

Now we prove the converse implication. Let $(u_2, v_2) \in \mathcal{C}'$. Then $v_2 = C^*u_2$ by Lemma 11 and $v_1 := Uu_2 \in \mathcal{D}(C)$, so $(Cv_1, v_1) = (CUu_2, Uv_1) \in \mathcal{C}$, and $Uu_2 = v_1$. Inserting $Uu_2 = v_1, v_2 = C^*u_2, u_1 = Cv_1 = CUu_2$ into condition (i), we obtain (43).

Now let $(u_1, v_1) \in \mathcal{C}$. Since $U$ is surjective, there is $u_2 \in \mathcal{D}(C^*)$ such that $Uu_2 = v_1$. Then $(u_2, C^*u_2) \in \mathcal{C}'$, $u_1 = Cv_1$ by (42) and (43) yields $R^{-1}u_2 + v_2 = (R^*)^{-1}v_1 + u_1$. This proves that condition (i) is satisfied. Since $U$ is isometric, condition (ii) holds as well.

5. **Normal operators**

In this section, we assume in addition that the operator $R$ is normal. Recall that we assumed throughout that $R$ has a bounded inverse $R^{-1} \in \mathcal{B}(\mathcal{H})$.

Note that $R$ is normal if and only if $R^*$ is normal, or equivalently, $R^{-1}$ (resp. $(R^*)^{-1}$) is normal. In particular, the assumption is symmetric in the operators $A$ and $B$.

Next we consider the *polar decomposition* of the operator $R$:
\[ R = U|R|. \]
Here $|R| := [R^*R]^{1/2}$ is the *modulus* of $R$ and the *phase operator* $U$ of $R$ is a partial isometry with initial space $\mathcal{N}(T)^\perp$ and final space $\mathcal{N}(T^*)$. Since $R$ is normal with
bounded inverse, $U$ is a unitary operator which commutes with $R$. We have

$$\|R^*\| = |R|, \quad R^* = U^* R U, \quad R = U R U = R U R^{-1} = U R^{-1} = U R^{-1}. \tag{45}$$

Since $A \subseteq R$, we have $\mathcal{D}(A) \subseteq \mathcal{D}(R) = \mathcal{D}(|R|)$. Let

$$T := |R| |\mathcal{D}(A)|$$

denote the restriction of $|R|$ to the domain $\mathcal{D}(A)$. Then $T$ is a densely defined positive symmetric operator such that $T \geq (\|R^{-1}\|)^{-1} I$.

**Lemma 13.** $\mathcal{N}(A^*) = U \mathcal{N}(T^*)$ and $\mathcal{N}(B^*) = U^* \mathcal{N}(T^*)$.

**Proof.** Let $x \in \mathcal{N}(T^*)$. Using (46) we derive

$$\langle Ux, Ay \rangle = \langle x, U^* Ay \rangle = \langle x, U^* R y \rangle = \langle x, U^* U R |R| y \rangle = \langle x, |R| y \rangle = \langle x, T y \rangle = \langle T^* x, y \rangle = 0$$

for $y \in \mathcal{D}(A) = \mathcal{D}(B)$. From this equation it follows that $U x \in \mathcal{D}(A^*)$ and $A^* U x = 0$, that is, $U x \in \mathcal{N}(A^*)$. This shows that $U \mathcal{N}(T^*) \subseteq \mathcal{N}(A^*)$.

Conversely, suppose $z \in \mathcal{N}(A^*)$. For $y \in \mathcal{D}(A) = \mathcal{D}(B)$, we obtain

$$\langle U^* z, Ty \rangle = \langle U^* z, |R| y \rangle = \langle z, U R |R| y \rangle = \langle z, R y \rangle = \langle z, A y \rangle = \langle A^* z, y \rangle = 0.$$

Therefore, $U^* z \in \mathcal{D}(T^*)$ and $T^* U^* z = 0$. That is, we have $U^* z \in \mathcal{N}(T^*)$ and hence $z \in U \mathcal{N}(T^*)$, so that $\mathcal{N}(A^*) \subseteq U \mathcal{N}(T^*)$.

Putting the preceding together, we have proved that $U \mathcal{N}(T^*) = \mathcal{N}(A^*)$.

The proof of the second equality $\mathcal{N}(B^*) = U^* \mathcal{N}(T^*)$ is similar. \qed

Next we introduce another unitary operator $W$. Since $R$ is normal, the equation

$$WR^* x = Rx, \quad x \in \mathcal{D}(R) = \mathcal{D}(R^*), \tag{46}$$

defines an isometric linear operator on $\mathcal{H}$ with dense domain and dense range. Hence it extends to unitary operator, denoted again by $W$, on $\mathcal{H}$. Then

$$WR^* = R, \quad W^* R = R^*, \quad \text{and } (R^*)^{-1} = R^{-1} W, \quad W = R(R^*)^{-1}. \tag{47}$$

Applying the adjoint to $(R^*)^{-1} = R^{-1} W$, we get $R^{-1} = W^* (R^*)^{-1} = W^* R^{-1} W$, so $W R^{-1} = R^{-1} W$ and $W^* R^{-1} = R^{-1} W$. This implies that the unitary $W$ commutes with $R^{-1}$ and $(R^*)^{-1}$.

The unitary $W$ is the square of the phase operator $U$. Indeed, since $W U^* |R| x = U |R| x$ by (46) and the range of $|R|$ is dense, we get $W U^* = U$, so that

$$W = U^2.$$

Combined with Lemma 13 we conclude that

$$W \mathcal{N}(B^*) = \mathcal{N}(A^*) \quad \text{and} \quad W^* \mathcal{N}(A^*) = \mathcal{N}(B^*). \tag{48}$$

From (47) and (48) we obtain $(R^*)^{-1} \mathcal{N}(B^*) = R^{-1} W \mathcal{N}(B^*) = R^{-1} \mathcal{N}(A^*)$. Inserting this into (49) we get

$$\mathcal{D}(A^*) = \mathcal{D}(B) \upharpoonright R^{-1} \mathcal{N}(A^*) \mathcal{N}(A^*), \tag{49}$$

$$\mathcal{D}(B^*) = \mathcal{D}(A) \upharpoonright R^{-1} \mathcal{N}(A^*) \mathcal{N}(B^*). \tag{50}$$

Further, by (48),

$$(R^*)^{-1} \mathcal{N}(B^*) = U |R|^{-1} U^* \mathcal{N}(T^*) = \mathcal{N}(T^*), \quad R^{-1} \mathcal{N}(A^*) = U^* |R|^{-1} U \mathcal{N}(T^*) = \mathcal{N}(T^*)$$

and therefore by Lemma 13,

$$\mathcal{D}(A^*) = \mathcal{D}(B) \upharpoonright |R|^{-1} \mathcal{N}(T^*) \mathcal{N}(T^*) \upharpoonright U \mathcal{N}(T^*), \tag{51}$$

$$\mathcal{D}(B^*) = \mathcal{D}(A) \upharpoonright |R|^{-1} \mathcal{N}(T^*) \mathcal{N}(T^*) \upharpoonright U^* \mathcal{N}(T^*). \tag{52}$$
The formulas (49), (50), (51), (52) are useful descriptions of the domains \( \mathcal{D}(A^*) \) and \( \mathcal{D}(B^*) \).

Using the unitaries \( W \) and \( U \) we can reformulate and slightly simplify Theorems 10 and 12 under the assumption that \( R \) is normal. We do not carry out these restatements and mention only the corresponding changes in the case of \( W \). Then condition (i) in Theorem 10 should be replaced by

\[
\{ R^{-1}Wv_1 + u_1 : (u_1, v_1) \in C \} = \{ R^{-1}u_2 + v_2 : (u_2, v_2) \in C' \},
\]

and in Theorem 12 equation (53) becomes

\[
R^{-1}u_2 + C^*u_2 = R^{-1}WUu_2 + CUu_2 \quad \text{for} \quad u_2 \in \mathcal{D}(C^*).
\]

**Example 14.** In this example we consider the special case

\[
\mathcal{C} = \{(0, v_1) : v_1 \in \mathcal{N}(B^*)\}.
\]

Then \( C' = \{(u_2, 0) : u_2 \in \mathcal{N}(A^*)\} \). Since \( R^{-1}W = (R^*)^{-1} \) and \( WN(B^*) = \mathcal{N}(A^*) \), condition (i) of Theorem 10 is fulfilled. Condition (ii) holds trivially, so the operator \( T_C \) is normal. From (50) and (51) we conclude easily that \( T_C = R^* \) and \( S_C = R \).

To construct examples we now reverse our considerations and begin with a normal operator \( Z \) on \( \mathcal{H} \) with trivial kernel.

Then \( \mathcal{N}(Z^*) = \mathcal{N}(Z) = \{0\} \) and \( R := Z^{-1} \) is a normal operator with adjoint \( R^* = (Z^*)^{-1} \). Further, assume that \( R \) is unbounded. Then \( \mathcal{D}(R) \subseteq \mathcal{R}(Z) \neq \mathcal{H} \).

From now on suppose that \( U \neq \{0\} \) is a close linear subspace of \( \mathcal{H} \) such that

\[
U \cap \mathcal{D}(R) = \{0\}.
\]

Since \( \mathcal{D}(R) \neq \mathcal{H} \), such spaces exist; one can even show that there are infinite-dimensional closed subspaces \( U \) satisfying (53).

We denote by \( P \) the orthogonal projection of \( \mathcal{H} \) on \( U \) and by \( W \) the unitary operator defined by (49). Then equation (47) holds. In particular, \( Z = Z^*W^* \).

Further, we define

\[
A := R[\mathcal{D}(A)] \quad \text{and} \quad B := R^*[\mathcal{D}(B)],
\]

\[
\mathcal{D}(A) = \mathcal{D}(B) := Z(I - P)\mathcal{H} = Z^*W^*(I - P)\mathcal{H}.
\]

**Proposition 15.** \( A \) and \( B \) are densely defined closed formally normal operators and 0 is a regular point for both operators. They form an adjoint pair and we have

\[
\mathcal{D}(A^*) = P\mathcal{H} = U \quad \text{and} \quad \mathcal{D}(B^*) = W^*P\mathcal{H} = W^*U.
\]

**Proof.** First we show that \( \mathcal{D}(A) \) is dense. Let \( x \in \mathcal{H} \). Assume that \( x \perp \mathcal{D}(A) \). Then we have

\[
0 = \langle x, Z(I - P)y \rangle = \langle Z^*x, (I - P)y \rangle
\]

for all \( y \in \mathcal{H} \), so \( Z^*x \in P\mathcal{H} = U \). Since \( Z^*x \in \mathcal{D}(R^*) = \mathcal{D}(R) \), we obtain \( Z^*x = 0 \) by (53). Hence \( x = 0 \), which proves that \( \mathcal{D}(A) \) is dense.

The operators \( A \) and \( B \) are formally normal, because they are restrictions of the normal operators \( R \) and \( R^* \), respectively. Since \( R \) and \( R^* \) have bounded inverses, 0 is a regular point for \( A \) and \( B \). Clearly, \( A \) and \( B \) form an adjoint pair.

We prove that \( \mathcal{N}(A^*) = P\mathcal{H} \) and \( \mathcal{N}(B^*) = W^*P\mathcal{H} \). Let \( y \in \mathcal{H} \). Since

\[
\langle AZ(I - P)x, y \rangle = \langle (I - P)x, y \rangle
\]

for all \( x \in \mathcal{H} \), it follows that \( y \in \mathcal{N}(A^*) \) if and only if \( y \in P\mathcal{H} \). Similarly,

\[
\langle BZ^*W^*(I - P)x, y \rangle = \langle (I - P)x, Wy \rangle, \quad x \in \mathcal{H},
\]

implies that \( y \in \mathcal{N}(B^*) \) if and only if \( Wy \in P\mathcal{H} \), that is, \( y \in W^*P\mathcal{H} \). \( \square \)
6. The one-dimensional case

In this section we remain the setup and the assumptions of the preceding section. We shall treat the simplest case when \( \mathcal{U} = \mathcal{P}\mathcal{H} = \mathcal{N}(A^*) \) has dimension one. Throughout this section, we suppose that \( \mathcal{U} = C \cdot \xi \), where \( \xi \) is a fixed vector of \( \mathcal{H} \) such that \( \xi \notin D(R) \).

Consider a linear subspace \( C \) of \( \mathcal{K} = C \cdot \xi \oplus C \cdot W^* \xi \). It is obvious that the operator \( T_C \) is not normal if \( \dim C = 0 \) or \( \dim C = 2 \). If \( C = C \cdot W^* \xi \), we know from Example 15 that \( T_C = (R^*)^{-1} \). Thus it remains to study that case

\[
C = C \cdot (\xi, \alpha W^* \xi) \quad \text{for some} \quad \alpha \neq 0.
\]

Then, by (32), a vector \((\beta_1 \xi, \beta_2 W^* \xi) \in \mathcal{K} \), with \( \beta_1, \beta_2 \in C \), belongs to \( \mathcal{C}' \) if and only if \((\alpha W^* \xi, \beta_2 W^* \xi) = (\xi, \beta_1) \), or equivalently, \( \beta \beta_2 = \beta_1 \). Therefore,

\[
\mathcal{C}' = C \cdot (\pi \xi, W^* \xi).
\]

Clearly, condition (ii) is equivalent to \( \|\alpha W^* \xi\| = \|\pi \pi \xi\| \), that is, \( |\gamma| = 1 \). Recall that \( Z^* W^* = Z \). Therefore, by the preceding, Theorem 10 yields the following:

\[
T_C \text{ is a normal operator if and only if there is a number} \quad \gamma \in C \quad \text{such that} \quad (\alpha - \pi \gamma)Z \xi = (\gamma W^* - I) \xi,
\]

\[
|\gamma| = 1.
\]

Before we continue we illustrate this statement in a very special case.

Example 16. Suppose \( S \) is a densely defined symmetric operator with equal non-zero deficiency indices. Then \( S \) has a self-adjoint extension \( X \) on \( \mathcal{H} \). Clearly, \( A := S + iI \) and \( B := S - iI \) are formally normal operators with domain \( D(S) \) and \( R := X + iI \) is a normal extension of \( A \) with bounded inverse \( Z \). We choose a vector \( \xi \in \mathcal{H} \) such that \( \xi \notin D(X) \) and define \( C \) and \( \mathcal{C}' \) by (59) and (77), respectively. Then we are in the setup described above.

Let us consider equation (55). Since \( W^* - I = (X - iI)(X + iI)^{-1} \), we have

\[
(\gamma W^* - I) \xi = -2i\gamma (X + iI)^{-1} \xi + (\gamma - 1) \xi = -2i\gamma Z \xi + (\gamma - 1) \xi.
\]

Therefore, since \( Z \xi \in D(X) \) and \( \xi \notin D(X) \), (77) is fulfilled if and only if \( \gamma = 1 \) and \( \alpha - \pi \gamma = -2i \), or equivalently, \( \gamma = 1 \) and \( \alpha = a - i \) with a real number \( a \). Therefore, by the preceding, the operator \( T_C \) is normal if and only if

\[
C = C \cdot (\xi, (a - i)W^* \xi) \quad \text{for some} \quad a \in \mathbb{R}.
\]

That is, the normal extensions of \( B \) are parametrized by the real number \( a \). It can be shown that for \( C \) as in (77) the corresponding operator \( T_C + iI \) is self-adjoint and hence a self-adjoint extension of the symmetric operator \( S = B + iI \).

Now we return to the general case. The normal operator \( R = Z^{-1} \) can be written as \( R = X + iY \), where \( X \) and \( Y \) are strongly commuting self-adjoint operators. Since \( R \) is unbounded, at least one of the operators \( X \) and \( Y \) has to be unbounded. Our aim is to reformulate conditions (59) and (60) in terms of \( X \) and \( Y \). For this we need some notation.

Suppose \( |\gamma| = 1 \) and \( \gamma \neq 1 \). Then we define numbers \( t_\gamma \) and \( s_\gamma \) by

\[
t_\gamma := i(\gamma + 1)(\gamma - 1)^{-1},
\]

\[
s_{\gamma, \alpha} := (\alpha - \pi \gamma)(\gamma - 1)^{-1}.
\]
Both numbers $t_\gamma$ and $s_{\gamma,\alpha}$ are real, because $|\gamma| = 1$. In the case $\gamma = 1$ we set
\begin{equation}
\tag{64}
t_1 := \infty, \quad s_{1,\alpha} := \text{Im} \alpha.
\end{equation}
Formulas \((62)\) and \((63)\) express the real numbers $t_\gamma$ and $s_{\gamma,\alpha}$ in terms of $\gamma$ and $\alpha$.

Now we want to reverse these transformations. Let $\alpha = a + ib$ with $a, b$ real. Then we obtain
\begin{align}
\gamma &= (t_\gamma + i)(t_\gamma - i)^{-1}, \\
\alpha &= a + ib = bl_\gamma - s_{\gamma,\alpha} + ib.
\end{align}
Thus, given $t, s \in \mathbb{R}$, $\gamma$ is uniquely determined by \((65)\) and there is a one-parameter family of numbers $\alpha$ in \((66)\), with $b \in \mathbb{R}$ as real parameter, satisfying the equations \((62)\) and \((63)\). Likewise, if $t = \infty, s \in \mathbb{R}$, we have $\gamma = 1$ and $\alpha = a + is$ is a one-parameter family with real parameter $a$ such that \((64)\) holds.

Further, we define operators
\begin{equation}
\tag{67}
R_t := X - tY, \quad t \in \mathbb{R}, \quad R_\infty := -Y.
\end{equation}
Note that $R_\infty$ and the closure of $R_t$, $t \in \mathbb{R}$, are self-adjoint operators.

**Theorem 17.** Suppose $C$ is a subspace of $K$ which is of the form \((59)\). Then the operator $T_C$ is a normal operator if and only if there exists $t \in \mathbb{R} \cup \{\infty\}$ such that $\xi$ is eigenvector of the operator $R_t$. In this case, if $s$ denotes the eigenvalue of eigenvector $\xi$ of $R_t$, the pair $(t, s)$ is uniquely determined by $C$. More precisely, if conditions \((59)\) and \((60)\) are satisfied, then $t = t_\gamma$ and $s = s_{\gamma,\alpha}$.

**Proof.** First we rewrite the right-hand side of equation \((59)\). From the definition of $W$ we get $W^* = R^* Z$, so $W^* = (X - iY)(X + iY)^{-1}$ and therefore
\begin{equation}
\tag{68}
\gamma W^* - I = \gamma(X - iY)(X + iY)^{-1} - (X + iY)(X + iY)^{-1} = [(\gamma - 1)X - i(\gamma + 1)Y](X + iY)^{-1}.
\end{equation}

Suppose $\gamma \neq 1$. Then, combining \((68)\) and \((62)\) we obtain
\begin{equation}
\gamma W^* - I = (\gamma - 1)[X - t_\gamma Y](X + iY)^{-1} = (\gamma - 1)R_{t_\gamma}(X + iY)^{-1},
\end{equation}
Hence, since $(\alpha - \bar{\alpha} \gamma)Z = s_{\gamma,\alpha}(\gamma - 1)(X + iY)^{-1}$, condition \((59)\) is equivalent to
\begin{equation}
\tag{69}
(s_{\gamma,\alpha} - R_{t_\gamma})(X + iY)^{-1}\xi = 0.
\end{equation}

Next we show that \((69)\) implies that
\begin{equation}
\tag{70}
R_{t_\gamma}\xi = s_{\gamma,\alpha}\xi.
\end{equation}
Let $E$ denote the spectral measure of the unbounded normal operator $R = X + iY$. Set $\mu_\xi$ denote the measure $\langle E(\cdot)\xi, \xi \rangle$ on $C$. From the functional calculus for self-adjoint operators we obtain
\begin{equation}
\tag{71}
\int_C |(s_{\gamma,\alpha} - (x - t_\gamma y))(x + iy)^{-1}|^2 \, d\mu_\xi(x + iy) = 0
\end{equation}
This implies that the function $(s_{\gamma,\alpha} - (x - t_\gamma y))(x + iy)^{-1}$ is zero $\mu_\xi$-everywhere. Hence $s_{\gamma,\alpha} - (x - t_\gamma y) = 0 \mu_\xi$-everywhere on $C$. Therefore,
\begin{equation}
\tag{72}
\int_C |s_{\gamma,\alpha} - (x - t_\gamma y)|^2 \, d\mu_\xi(x + iy) = 0
\end{equation}
Again by the spectral calculus, the integral on the left is
\[\|(s_{\gamma,\alpha} - (X - t_\gamma Y)\xi\|^2 = \|(s_{\gamma,\alpha} - R_{t_\gamma}\xi)\|^2.\]
Thus, by \((72)\), $(s_{\gamma,\alpha} - R_{t_\gamma})\xi = 0$, which proves \((70)\).
Now we consider the case $\gamma = 1$. Then, since $W^* - I = -2iY(X + iY)^{-1}$ and $(\alpha - \overline{\alpha})Z = 2i(\Im \alpha)(X + iY)^{-1}$, equation (53) reads as

\[(\Im \alpha + Y)(X + iY)^{-1}\xi = 0.\]  

(73)

Repeating the reasoning of the preceding paragraph, (72) yields $((\Im \alpha + Y)\xi = 0$. This means that $(s_{1,\alpha} - R_t)\xi = 0$, which proves (70) in the case $\gamma = 1$.

Summarizing, we have shown that conditions (59) and (60) imply $R_t, \xi = s_{\gamma,\alpha}\xi$ for all nonzero $\alpha$ and $\gamma$ of modulus one. Conversely, reversing the above reasoning it follows from the equation $R_t, \xi = s_{\gamma,\alpha}\xi$ that (59) and (60) hold.

It remains to show that the pair $(t, s)$, where $t \in \mathbb{R} \cup \{\infty\}$, $s \in \mathbb{R}$, is uniquely determined by the subspace $C$. Assume that $R_t\xi = s\xi$ and $R_{t'}\xi = s'\xi$.

First let $t = \infty$. Assume to the contrary that $t' \in \mathbb{R}$. Then it follows that $\xi \in \mathcal{D}(\gamma Y)$ and $\xi \in \mathcal{D}(X + iY)$, so $\xi \in \mathcal{D}(R)$, which contradicts the assumption. Thus $t' = \infty$. Then $-Y\xi = s\xi$ and $-Y\xi = s'\xi$ imply $s = s'$.

Now suppose $t \in \mathbb{R}$. Then, as shown in the preceding paragraph, $t' \in \mathbb{R}$. Again we assume to the contrary that $t \neq t'$. Then $(R_t - R_{t'})\xi = (t' - t)Y\xi = (s - s')\xi$, so $Y\xi = (t' - t)^{-1}(s - s')\xi$ and $(X + iY)\xi = (X - iY)\xi = (s - s')\xi$ with $c = s + (i + t)(t' - t)^{-1}(s - s')$. Arguing as in the preceding paragraph leads to a contradiction. Thus $t = t'$ and hence also $s = s'$. Thus we have shown in all cases that $(t, s) = (t', s')$.

We illustrate the preceding theorem by an example.

**Example 18.** Let $\mu$ be a Radon measure on $\mathbb{C}$ such that $\mu(\{z \in \mathbb{C} : |z| \leq \varepsilon\}) = 0$ for some $\varepsilon > 0$ and let $R$ denote the multiplication operator by the complex variable $z$ on $\mathcal{H} = L^2(\mathbb{C}; \mu)$. Clearly, $R$ and $R^*$ are normal operators with bounded inverses. We suppose that $R$ is unbounded and choose a function $\xi(z) \in L^2(\mathbb{C}; \mu)$ such that $z\xi(z) \notin L^2(\mathbb{C}; \mu)$. Then we are in the setup discussed above and $R^*$ is a normal extension of the operator $B$. By appropriate choices of $\mu$ and $\xi$, we can construct interesting cases.

First fix $s, t \in \mathbb{R}$ and suppose that $\mu$ is supported on the line $x - ty = s$, where $z = x + iy$, $x, y \in \mathbb{R}$. Then $(X - iY)\xi = s\xi$. Define $\gamma$ by (65), $\alpha$ by (67) with $b \in \mathbb{R}$, and the vector space $C$ by (57). Then, by Theorem 14, $T_C$ is normal and the operator $B$ has, in addition to $R^*$, precisely the one-parameter family of operators $T_C(b)$ (with real parameter $b$) as normal extensions on the Hilbert space $\mathcal{H}$.

A similar result is true for $t = \infty$, $s \in \mathbb{R}$.

Next we choose $\mu$ and $\xi$ such $\xi$ is not an eigenvector of some operator $R_t$ with $t \in \mathbb{R} \cup \{\infty\}$. (For instance, let $\mu$ be the Lebesgue measure outside of some ball around the origin and set $\xi := |z|^{-3}$. Then the operator $B$ has no other normal extension on $\mathcal{H}$ than the operator $R^*$.

**References**

[A] Ando, T.: Topics on Operator Inequalities, Hokkaido University, Sapporo, 1978.

[BC] Biriuik, G. and E.A. Coddington: Normal extensions of unbounded formally normal operators, J Math. Mech. 13(1964), 617–637.

[Bk] Bruck, V.M.: On a class of boundary value problems with a spectral parameter in the boundary condition, Mat. Sbornik 100(1976), 210–216.

[Ck] Calkin, J.W.: Abstract symmetric boundary conditions, Trans. Amer. Math. Soc. 45 (1939), 369–442.

[CD1] Coddington, E.A.: Formally normal operators having no normal extensions, Canad. J. Math. 17(1965), 1030–1040.

[CD2] Coddington, E.A.: Extension Theory of Formally Normal and Symmetric Subspaces, Mem. Amer. Math. Soc. 134(1973).

[DM] Derkach, V.A. and M.M. Malamud: Generalized resolvents and the boundary value problem for Hermitian operators with gaps, J. Funct. Anal. 95(1991), 1–95.
[EE] Edmunds, D.E. and W.D. Evans: *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

[GS] Gesztesy, F. and K. Schmüdgen: On a theorem of Z. Sebestyen and Zs. Tarsay, Acta Math. Sci. (Szeged) 85 (2019), 291–293.

[Ko] Kochubei, A.N.: Extensions of symmetric operators, Math. Notes 17 (1975), 25–28.

[MM] Malamud, M.M. and V.I. Mogilevskii: Krein type formula for canonical resolvents of dual pairs of linear relations. Methods Funct. Anal. Topol. 8(2002), 72–100.

[RS] Reed, M. and B. Simon: *Methods of Modern Mathematical Physics II. Fourier Analysis and Self-Adjointness*, Academic Press, New York, 1975.

[SS1] Stochel, J. and F.H. Szafraniec: On normal extensions of unbounded operators. I, J. Oper. Theory 14 (1985), 31–55.

[SS2] Stochel, J. and F.H. Szafraniec: On normal extensions of unbounded operators. II, Acta Sci. Math. (Szeged) 53 (1989), 153–177.

[SS3] Stochel, J. and Szafraniec, F.H.: On normal extensions of unbounded operators. III, Publ. RIMS Kyoto Univ. 25 (1989).

[Sch86] Schmüdgen, K.: A formally normal operator having no normal extension, Proc. Amer. Math. Soc. 96 (1985), 503–504.

[Sch12] Schmüdgen, K.: *Unbounded Self-adjoint Operators on Hilbert Space*, Springer-Verlag, Cham, 2012.

[Vi] Vishik, M.: On general boundary conditions for elliptic differential equations, Moskov. Mat. Obc. 1 (1952), 187–246; Amer. Math. Soc. Transl. 24 (1963), 107–172.

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