Crasovskii approach to construct Lyapunov function and its derivative function for analyzing stability of non-linear systems

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Abstract. This paper aims at constructing Lyapunov function and its derivative function to analyze and judge the stability of non-linear time-invariant systems by Crasovskii approach. A candidate Lyapunov function is selected first according to the system function. Then a replacement matrix function, which is used to judge the negative definiteness of derivative function of candidate Lyapunov function, is derived by solving Jacobi matrix of the system function. When this replacement matrix function is determined to be negative definite, in light of Lyapunov stability theory, thereby the non-linear time-invariant system is asymptotically stable. Examples indicate that this approach is valid for judging the stability of some non-linear time-invariant systems, and it still can be used to determine the unknown parameters for some non-linear time-invariant systems. This approach provides another way to analyze the stability of non-linear time-invariant systems.

1 Introduction
Lyapunov stability theory is widely used to analyze the stability or design the stabilization controller for control systems [1, 2]. For the linear time-invariant system, its stability can be judged with the first Lyapunov method or with the second Lyapunov method in which Lyapunov function can be determined by solving Lyapunov equation. With regard to the non-linear systems, however, the first Lyapunov method cannot be used because there has no any Eigen equation. And the second Lyapunov method can neither be used because it is difficult to find or construct an appropriate Lyapunov function and its derivative function for non-linear systems. Thus, there has to construct Lyapunov function by other ways so that the second Lyapunov method can be used [3, 4].

In this paper, in order to judge the stability of the non-linear control system, Lyapunov function is constructed on the basis of Crasovskii principle [5, 6]. By observing and citing the system state equation, there selects a candidate Lyapunov function, which is a positive definite function about state variables. Then deriving and proving the derivative function of this candidate Lyapunov function is a negative definite function. In accordance with the second Lyapunov method, thus, the non-linear system is proved to be asymptotically stable [7, 8]. There are a few application examples to verify this stability judging approach is effective and it can still be used to determine the undecided parameters of some non-linear time-invariant systems. In the last of the paper, there are some discussions on the application range of Crasovskii approach.
2 Crasovskii approach based on Lyapunov stability theory

Considering the continuous non-linear time-invariant control system:

\[ \dot{x} = f(x), \quad t \geq 0 \]  

where \( x = [x_1, x_2, \ldots, x_n]^T \), it is \( n \)-dimensional state vector. And \( f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T \) is system function, it is the non-linear vector function about state vector \( x \).

For the non-linear control system, there may be solitary one or many equilibrium states. Yet these equilibrium states all can be translated to the original point of the state space by the appropriate coordinate transformation. Therefore, the single equilibrium state \( x = 0 \) is considered hereafter:

\[
\begin{cases}
    f(x) = 0, \quad \forall x = 0 \\
    f(x) \neq 0, \quad \forall x \neq 0
\end{cases}
\]  

2.1 Derivation of Crasovskii approach

For the non-linear system Eq. 1 represents, selecting a candidate Lyapunov function to be:

\[ V(x) = f^T(x) f(x) \]  

Because \( V(x) = f^T(x) f(x) = f^T(x) I f(x) \) and \( I \) is a unit matrix, so candidate Lyapunov function \( V(x) \) is necessarily a positive definite function:

\[
\begin{cases}
    V(x) = f^T(x) f(x) = 0, \quad \forall x = 0 \\
    V(x) > 0, \quad \forall x \neq 0
\end{cases}
\]  

Solving the derivative function of the candidate Lyapunov function \( V(x) \) and denoted by \( \dot{V}(x) \):

\[
\dot{V}(x) = \frac{df^T(x) f(x)}{dt} = \frac{df(x)}{dt} f(x) + f^T(x) \frac{df(x)}{dt} = f^T(x) \frac{df(x)}{dt} f(x) + f^T(x) \frac{df(x)}{dx} \frac{df(x)}{dt} f(x)
\]

\[
\dot{V}(x) = f^T(x) \left[ \frac{\partial f(x)}{\partial x} f(x) + \frac{\partial f(x)}{\partial x} f(x) \right] + f^T(x) \left[ \frac{\partial f(x)}{\partial x} f(x) \right] f(x)
\]

\[
\dot{V}(x) = f^T(x) \left[ \frac{\partial f(x)}{\partial x} f(x) \right] + f^T(x) \left[ \frac{\partial f(x)}{\partial x} f(x) \right] f(x)
\]

where \( \frac{\partial f(x)}{\partial x} \) is Jacobi matrix of \( f(x) \):
Let $F(x) = \frac{\partial f(x)}{\partial x}$, there can yield:

$$V(x) = f^T(x)\left[ F^T(x) + F(x) \right] f(x) \quad (7)$$

Observing from Eq. 7, if $F^T(x) + F(x)$ is negative definite:

$$\left[ F^T(x) + F(x) \right] = 0, \forall x = 0$$

then $V(x)$ is negative definite, too. Consequently, in accordance with Lyapunov stability theory, $V(x)$ is positive definite and $V(x)$ is negative definite, and so non-linear system Eq. 1 represents is asymptotically stable.

Above mentioned approach to judge the stability of non-linear system is proposed based on Lyapunov stability theory by Crasovskii in 1960’s. It is often concluded as Crasovskii principle.

2.2 Generalization of Crasovskii approach

For the continuous non-linear time-invariant system Eq. 1 represents, original point $x = 0$ is the solitary equilibrium state, and $F(x)$ is Jacobi matrix of $f(x)$. If $F^T(x) + F(x)$ is negative definite, then the solitary equilibrium state $x = 0$ is asymptotically stable and $V(x) = f^T(x) f(x)$ is a Lyapunov function. Furthermore, if when the Euclid norm, $||x||$, of state vector $x$ approaches infinity, i.e. $||x|| \to \infty$, Lyapunov function $V(x)$ approaches infinity too:

$$V(x) = f^T(x) f(x) \to \infty$$

then the solitary equilibrium state $x = 0$ is even globally asymptotically stable.

When Crasovskii approach is used to judge the stability of the non-linear time-invariant system, the first step is to solve the Jacobi matrix $F(x)$ of $f(x)$ and calculate $F^T(x) + F(x)$. And the second step is to determine wether $F^T(x) + F(x)$ is negative definite. If $F^T(x) + F(x)$ is negative definite, the considered non-linear system can be determined to be asymptotically stable and $V(x) = f^T(x)/f(x)$ is right a Lyapunov function. Nevertheless, if $F^T(x) + F(x)$ cannot be determined to be negative definite, there can conclude neither stability nor unstability about the non-linear time-invariant system.

3 Application examples

By Crasovskii approach, the asymptotical stability of some non-linear time-invariant systems can be analyzed and judged. In addition, the undetermined parameters can be determined so as to make the non-linear time-invariant systems are asymptotically stable in the sense of Lyapunov.

3.1 Judging the stability of non-linear systems

Considering the second order continuous non-linear time-invariant system:

$$\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= x_1 - x_2 - x_2^3
\end{align*} \quad (10)$$
it can be solved that \( x_1 = x_2 = 0 \) is the solitary equilibrium state in real number domain of state space. Jacobi matrix \( F(x) \) and \( F^T(x) + F(x) \) can also be solved:

\[
F(x) = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} -1 & 0 \\ 1 & -1 - 3x_2^2 \end{bmatrix}
\]

\[
F^T(x) + F(x) = \begin{bmatrix} -2 & 1 \\ 1 & -2 - 6x_2^2 \end{bmatrix}
\]

For \( F^T(x) + F(x) \), its first and the second order principal determinant \( \Delta_1, \Delta_2 \) respectively are:

\[
\begin{cases} 
\Delta_1 = -2 < 0 \\
\Delta_2 = 12x_2^2 + 3 > 0 
\end{cases}
\]

According to Sylvester criterion [9, 10], \( F^T(x) + F(x) \) is negative definite. Furthermore, when the Euclid norm, \(|x|\), of state vector \( x \), approaches infinity: \(|x| \to \infty\), Lyapunov function \( V(x) \) approaches to infinity, too:

\[
V(x) = f^T(x) f(x) = x_1^2 + (x_1 - x_2 - x_2^3)^2 \to \infty
\]

Consequently, the non-linear system Eq. 10 represents is globally asymptotically stable at the solitary equilibrium state \( x_1 = x_2 = 0 \).

### 3.2 System parameter determining

Taking into account of the second order continuous non-linear time-invariant system with undetermined parameters:

\[
\begin{cases} 
\dot{x}_1 = ax_1 + x_2 \\
\dot{x}_2 = x_1 - x_2 + bx_2^3 
\end{cases}
\]

now determining the value ranges of parameters \( a \) and \( b \), so as to make the system is globally asymptotically stable at the solitary equilibrium state \( x_1 = x_2 = 0 \).

Firstly, \( F(x) \) and \( F^T(x) + F(x) \) are solved:

\[
F(x) = \frac{\partial f(x)}{\partial x} = \begin{bmatrix} a & 1 \\ 1 & -1 + 5bx_2^4 \end{bmatrix}
\]

\[
F^T(x) + F(x) = \begin{bmatrix} 2a & 2 \\ 2 & -2 + 10bx_2^4 \end{bmatrix}
\]

Then solving the first and the second order principal determinant of replacement matrix function \( F^T(x) + F(x) \), \( \Delta_1 \) and \( \Delta_2 \), and let \( \Delta_1 < 0, \Delta_2 > 0 \):

\[
\begin{cases} 
\Delta_1 = 2a < 0 \\
\Delta_2 = 20abx_2^4 - 4a - 4 > 0 
\end{cases}
\]

Solving this inequality equation, there can get:

\[
a < 0, b < \frac{a + 1}{5a}
\]

For instance, let \( a = b = -1 \), there can be judged that Lyapunov function is positive definite:
\[ V(x) = f^T(x)f(x) \]
\[ = (-x_1 + x_2)^2 + (x_1 - x_2^2)^2 > 0 \] (18)

and \( F^T(x) + F(x) \) is negative definite:
\[ F^T(x) + F(x) = \begin{bmatrix} -2 & 2 \\ 2 & -2-10x_2^2 \end{bmatrix} < 0 \] (19)

Thus, the considered system is globally asymptotically stable at the solitary equilibrium state \( x_1=x_2=0 \).

4 Extended discussions on the application range of Crasovskii approach
Crasovskii approach is first faced toward analyzing and judging the stability of non-linear time-invariant systems. In fact, it is also applicable to analyze and judge linear time-invariant systems. For example, considering the linear time-invariant system:
\[ \begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = 2x_1 - 3x_2 \end{cases} \] (20)

there can solve the replacement matrix function of derivative function of Lyapunov function to be:
\[ F^T(x) + F(x) = \begin{bmatrix} -2 & 3 \\ 3 & -6 \end{bmatrix} \] (21)

It can be easily judged that \( F^T(x) + F(x) \) is negative definite. And Lyapunov function derived from Crasovskii approach is positive definite:
\[ V(x) = f^T(x)f(x) \]
\[ = (-x_1 + x_2)^2 + (2x_1 - 3x_2)^2 > 0 \] (22)

therefore, the linear system Eq. 20 represents is globally asymptotically stable at solitary equilibrium state \( x_1=x_2=0 \).

The reason why there usually doesn’t use Crasovskii approach to judge the stability of linear systems, is that the stability of linear systems can be directly judged by using the first or the second Lyapunov method.

On the other hand, Crasovskii approach cannot guarantee to solve the stability problems of all non-linear time-invariant systems. For instance, considering the non-linear time-invariant system as follows:
\[ \begin{cases} \dot{x}_1 = -2x_1 + 2x_2^4 \\ \dot{x}_2 = -x_2 \end{cases} \] (23)

there can solve \( F^T(x) + F(x) \) to be:
\[ F^T(x) + F(x) = \begin{bmatrix} -4 & 8x_2^3 \\ 8x_2^3 & -2 \end{bmatrix} \] (24)

its second order principal determinant \( \Delta_2 = 8-64x_2^3 \) cannot be judged as positive or negative. So \( F^T(x) + F(x) \) is indefinite and thereby the considered system Eq. 23 represents neither can be judged as stable nor can be judged as unstable. Whereas this non-linear system can be judged to be locally asymptotically stable at the solitary equilibrium state \( x_1=x_2=0 \) by linearization of non-linear system equation and using the first Lyapunov method.
5 Conclusions

Crasovskii approach to analyze and judge the stability of non-linear time-invariant systems is studied in this paper. First of all, a candidate Lyapunov function is selected based on the system function. Then a replacement matrix function, which can replace the derivative function of Lyapunov function, is constructed by solving the Jacobi matrix of the system function. And when the replacement matrix function is judged to be negative definite, the non-linear system under consideration is determined to be asymptotically stable and at the same times the candidate Lyapunov function is just a correct Lyapunov function. This approach can also be used to judge the stability of linear time-invariant systems. Unfortunately, for some other non-linear time-invariant systems, Crasovskii approach cannot judge their stability or instability. This investigation will be done in author’s next study.

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