Abstract. For an $N$-partite quantum system we show that separability implies inequalities on Bell correlations which are stronger than the local reality inequalities by a factor $2^{(N-1)/2}$.
1. **Introduction:** Striking features of quantum entanglement were brought into sharp focus by the landmark papers of Einstein, Podolsky and Rosen and Schrödinger. The Bohm-Aharonov version of the EPR paradox with two spin half particles in a singlet state led to Bell’s theorem that quantum theory violates ‘local realism’. Subsequently other entangled states, e.g. Greenberger-Horne-Zeilinger (GHZ) state of four spin ½ particles and its $N$-particle generalizations led to optimal local reality inequalities derived by Mermin (for odd $N$) and Roy and Singh (all $N$) and others. The Mermin-Roy-Singh (MRS) inequalities are violated by suitable entangled or non-separable states by a factor $2^{(N-1)/2}$. This is now recognised to be the maximal violation possible, due to Cirel’son’s theorem for $N = 2$, and recent work of Werner and Wolf for general $N$.

The physical origins of the ideas of local realism and quantum separability are very different. The exponential violation of local realism by suitable entangled states is therefore an extremely interesting but indirect consequence of entanglement. The current explosive interest in applications of quantum entanglement to quantum information theory prompts us to seek direct qualitative and quantitative signatures of quantum entanglement. Here we present separability inequalities on Bell correlations which are exponentially stronger than local reality inequalities for large $N$. We show that for $N$-partite systems there are entangled states violating separability by a factor $2^{(N-1)/2}$. It is natural to conjecture that this exponential violation of separability, being a quantitative measure of quantum parallelism, is intimately connected to the exponential speed-ups achievable in quantum computation.

2. **Local Realism Versus Separability:** Consider a composite system which breaks up into $N$ components. The $k$th component is measured with apparatus specified by a set of parameters $a_k$ to determine the value of a variable $A^{(k)}(a_k)$ which by its very definition must lie between $\nu_k$ and $\mu_k$,

$$\nu_k \leq A^{(k)}(a_k) \leq \mu_k.$$  

Repeated simultaneous measurements of $A^{(k)}(a_k)$ yield their correlation function as the expectation value $\langle A(a) \rangle$, where

$$A(a) = \prod_{k=1}^{N} A^{(k)}(a_k),$$  

(1)
and \( a \equiv (a_1, a_2, \cdots, a_N) \). A Bell correlation function \( \langle B \rangle \) is a linear combination of such correlation functions, with the Bell variable \( B \) defined by

\[
B = \sum_a c(a) A(a),
\]

where \( c(a) \) are real numbers.

According to Bell’s formulation of Einstein locality or local realism, if all pairs made out of the \( N \) sub-systems are mutually spacelike separated, then in a ‘Local Hidden Variable’ theory,

\[
\langle A(a) \rangle_{LHV} = \int d\lambda \rho(\lambda) \prod_{k=1}^N A^{(k)}(\lambda, a_k),
\]

where ‘\( \lambda \)’ are hidden variables which determine the outcomes \( A^{(k)}(\lambda, a_k) \) in individual runs, and \( \rho(\lambda) \) their probability distribution. Local reality means that for each \( \lambda \) the outcome \( A^{(k)}(\lambda, a_k) \) is independent of all other orientations \( a_\ell \) and outcomes \( A^{(\ell)}(\lambda, a_\ell) \) for \( \ell \neq k \) observed at spacelike separation.

On the other hand in quantum theory each \( A^{(k)}(a_k) \) becomes a self-adjoint operator in a Hilbert space \( H^{(k)} \) with eigenvalues in the interval \([\nu_k, \mu_k]\), and

\[
\langle A(a) \rangle = \text{Tr} \rho \prod_{k=1}^N A^{(k)}(a_k),
\]

where \( \rho \) is the density operator for the quantum state defined on \( \bigotimes_{k=1}^N H^{(k)} \).

In defining quantum separability, there is no reference to spatial separation of the subsystems. For pure states of bipartite systems, \( \rho = |\psi\rangle \langle \psi| \), where any \( \psi \) can be written as a Schmidt biorthogonal sum

\[
\psi = \sum_{i=1}^M \sqrt{p_i} \psi_i^{(1)} \psi_i^{(2)},
\]

with \( p_i > 0, \sum_i p_i = 1 \). The state is called separable if the Schmidt rank \( M = 1 \) and called entangled or EPR correlated if \( M > 1 \). For \( N \)-partite systems with \( N > 2 \), and for mixed states of bipartite systems we must use the following more general definition. A density operator \( \rho \) on the tensor product of \( N \) Hilbert spaces \( \bigotimes_{k=1}^N H^{(k)} \) is called separable or disentangled or
classically correlated if it can be written as a convex combination of tensor product states
\[ \rho = \sum_i r_i \bigotimes_{k=1}^N \rho_i^{(k)} \] (6)
(where the sum converges in trace class norm), with \( r_i > 0, \sum_i r_i = 1 \).
Otherwise it is called entangled or EPR correlated.

Whereas local realism is a concept independent of quantum theory, separability is formulated entirely in terms of quantum theory. What is the precise connection between them? Consider first \( N = 2 \). The LHV representation implies the Bell-CHSH inequalities, but it is known\(^{14}\) that these inequalities are not sufficient to derive the LHV representation. Fortunately, it is easy to show\(^{12}\), without going via Bell inequalities, that for separable quantum states, the Bell correlations \( A(a) \) obey a LHV representation. For separable states
\[ \text{Tr} \rho A^{(1)}(a_1) A^{(2)}(a_2) \cdots A^{(N)}(a_n) = \sum_i r_i A^{(1)}(a_1, i) A^{(2)}(a_2, i) \cdots A^{(N)}(a_n, i), \] (7)
where \( A^{(k)}(a_k) \) are observables on \( \mathcal{H}^{(k)} \) depending on parameter sets \( a_k \), and
\[ A^{(k)}(a_k, i) = \text{Tr} \rho_i^{(k)} A^{(k)}(a_k). \] (8)

The decomposition (7) is exactly of the Bell Local Hidden Variables (LHV) form and readily shows that all Bell correlations in separable states must obey the Bell local-realism inequalities. There exist partial results in the reverse direction. Gisin and Peres\(^{15}\) showed using the Schmidt decomposition that for every pure entangled state of a bipartite system one can find observables whose Bell correlations violate local realism inequalities. However, Werner\(^ {12}\) constructed a class of mixed entangled states for \( N = 2 \) which nevertheless admit a LHV representation for Bell correlations of all observables. Thus there is no one to one correspondence between separable quantum states and those admitting a LHV representation for Bell correlations. The focus of the present work will be to show that maximal violations of separability inequalities for \( N \)-partite systems are exponentially higher than the maximal violations of local reality.

3. **Summary of MRS Local Reality Inequalities For \( N \)-qubit systems**: Consider the operators \( A^{(k)}(a_k) \) to be \( \sigma_x^{(k)} \) or \( \sigma_y^{(k)} \), the Pauli spin operators for
the \( k \)th qubit. Define \( \sigma_{\pm}^{(k)} = \sigma_x^{(k)} \pm i\sigma_y^{(k)} \), and the Bell operators

\[
B_+ = \frac{1}{2} \left( \bigotimes_{k=1}^{N} \sigma_+^{(k)} + \bigotimes_{k=1}^{N} \sigma_-^{(k)} \right) \tag{9}
\]

\[
B_- = \frac{1}{2i} \left( \bigotimes_{k=1}^{N} \sigma_+^{(k)} - \bigotimes_{k=1}^{N} \sigma_-^{(k)} \right) \tag{10}
\]

which are of the general form given by (1) and (2) when reexpressed in terms of \( \sigma_x^{(k)}, \sigma_y^{(k)} \). Can the quantum Bell correlations which are expectation values of the operators \( B_{\pm} \) be reproduced by the corresponding LHV expressions,

\[
\langle B_+ \rangle_{\text{LHV}} = \Re \int d\lambda \rho(\lambda) \prod_{k=1}^{N} \left( \sigma_x^{(k)}(\lambda) + i\sigma_y^{(k)}(\lambda) \right) \tag{11}
\]

\[
\langle B_- \rangle_{\text{LHV}} = \Im \int d\lambda \rho(\lambda) \prod_{k=1}^{N} \left( \sigma_x^{(k)}(\lambda) + i\sigma_y^{(k)}(\lambda) \right), \tag{12}
\]

where \(-1 \leq \sigma_{x,y}^{(k)}(\lambda) \leq 1\)? The \( \langle B_{\pm} \rangle_{\text{LHV}} \) are linear in each \( \sigma_x^{(k)}(\lambda) \) and their extreme values must be reached when \( \sigma_{x,y}^{(k)}(\lambda) = \pm 1 \). The MRS procedure quickly yields the bounds,

\[
|\langle B_{\pm} \rangle_{\text{LHV}}| \leq 2^{(N-1)/2}, \ N \text{ odd} \tag{13}
\]

\[
|\langle B_+ \rangle_{\text{LHV}}| + |\langle B_- \rangle_{\text{LHV}}| \leq 2^{N/2}, \ N \text{ even,} \tag{14}
\]

which are known to be violated by quantum correlations by a factor \( 2^{(N-1)/2} \).

4. **Separability Inequalities For \( N \) Qubit Systems**: Consider first the factorized state

\[
|\psi\rangle = \bigotimes_{k=1}^{N} |\psi^{(k)}\rangle, \ |\psi^{(k)}\rangle = \left( \begin{array}{c} \alpha^{(k)} \\ \beta^{(k)} \end{array} \right)
\]

with \(|\alpha^{(k)}|^2 + |\beta^{(k)}|^2 = 1\). This yields

\[
\left|\langle \psi| \bigotimes_{k=1}^{N} \sigma_-^{(k)} |\psi\rangle\right| = \left| \prod_{k=1}^{N} 2\beta^{(k)*} \alpha^{(k)} \right| \leq 1 \tag{15}
\]

Hence, for the Bell correlations in factorized states,

\[
|\langle \psi| B_{\pm} |\psi\rangle| \leq 1,
\]
\[ |\langle \psi | B_+ | \psi \rangle| + |\langle \psi | B_- | \psi \rangle| \leq \sqrt{2}, \]

where \( B_{\pm} \) are the operators defined by Eqs. (9) and (10). We now show that the expectation values of \( B_{\pm} \) in a general separable state (6) must obey the same inequalities. Each density operator \( \rho_i^{(k)} \) in (6) is a convex combination of pure states,

\[ \rho_i^{(k)} = \sum_s c_{is}^{(k)} |\psi_i^{(k)} \rangle \langle \psi_i^{(k)}|, \]

with \( c_{is}^{(k)} > 0, \sum_s c_{is}^{(k)} = 1 \). We readily deduce by using the convexity properties and a relabelling of indices that a general separable density operator (6) can also be written as a convex combination of tensor products of pure states

\[ \rho = \sum_I r_I \bigotimes_{k=1}^N |\psi_I^{(k)} \rangle \langle \psi_I^{(k)}|, \]

with \( r_I > 0, \sum_I r_I = 1 \). Hence,

\[
\left| \text{Tr} \rho \bigotimes_{k=1}^N \sigma_-^{(k)} \right| = \left| \sum_I r_I \prod_{k=1}^N \langle \psi_I^{(k)} | \sigma_-^{(k)} | \psi_I^{(k)} \rangle \right| \\
\leq \sum_I r_I = 1 \tag{16}
\]

where we have used the positivity of \( r_I \) and the result (15) for factorized states. This immediately implies that the Bell correlations in arbitrary separable states must obey

\[ |\text{Tr} \rho B_{\pm}| \leq 1, \tag{17} \]

and

\[ |\text{Tr} \rho B_+| + |\text{Tr} \rho B_-| \leq \sqrt{2}, \tag{18} \]

for every \( N \)-partite separable density operator \( \rho \). These are the announced “Separability inequalities”. Comparison with Eqs. (13), (14) shows that the separability inequalities are stronger than the local reality inequalities by a factor \( 2^{(N-1)/2} \). We expect the experimental violations of (17), (18) by entangled states to be useful signatures of non-separability. We also hope that a theoretical study of these exponential violations will illuminate the mechanism of speed-up achieved in quantum computation.
5. **Quantum Violations of Separability and Local Reality:** For the entangled pure state

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|^\uparrow\cdots\uparrow + e^{i\theta} |\downarrow\cdots\downarrow), \]

where \( \uparrow \) and \( \downarrow \) denote eigenstates of \( \sigma_z \) with eigenvalues \( \pm 1 \), we have,

\[
\langle \psi | B_+ | \psi \rangle = \cos \theta \ \frac{2^{N-1}}{2^{N-1}} \tag{19}
\]

\[
\langle \psi | B_- | \psi \rangle = \sin \theta \ \frac{2^{N-1}}{2^{N-1}} \tag{20}
\]

Taking successively \( \theta = 0, \pi/2 \) and \( \pi/4 \) we see that the values of \( \langle B_+ \rangle \), \( \langle B_- \rangle \) and \( |\langle B_+ \rangle| + |\langle B_- \rangle| \) violate the separability inequalities by a factor \( 2^{N-1} \), and the local reality inequalities by a factor \( 2^{(N-1)/2} \). We can prove that these violations are maximal by a generalization of Cirel’son’s theorem to \( N \)-partite systems\(^{16} \) or by the variational methods of Werner and Wolf\(^{8} \).

The present work can be generalized in several directions. The inequalities can be generalized to the case of partially separable density matrices of \( N \)-partite systems and compared with partial local hidden variable theory inequalities obtained by Svetlichny and others\(^{17} \). We have also obtained separability inequalities for \( N \)-Qudit systems\(^{18} \).

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