Geometry of generalized higher order fields and applications to classical linear electrodynamics

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Abstract
Motivated by obtaining a consistent mathematical description for the radiation reaction of point charged particles in linear classical electrodynamics, a theory of generalized higher order tensors and differential forms is introduced. The generalization of some fundamental notions of the differential geometry and the theory of differential forms is presented. In particular, the cohomology and integration theories for generalized higher order forms are developed, including the Cartan calculus, a generalization of de Rham cohomology and a version of Thom’s isomorphism theorem. We consider in detail a special type of generalized higher order tensors associated with bounded maximal $n$-acceleration and use it as a model of spacetime. A generalization of electrodynamics theory with higher order fields is introduced. Although the theory is non-local in the usual sense, it is free of some of the pathologies appearing in the standard linear classical electrodynamics. Indeed, we show that combining the generalized higher order fields with maximal acceleration geometry the evolution of a point charged particle interacting with the generalized higher order fields can be described by solutions of an implicit second order ordinary differential equation. In flat space such equation is Lorentz invariant, does not have pre-accelerated solutions of Dirac’s type or run-away solutions, it is compatible with Newton’s first law of dynamics and with the covariant Larmor’s power radiation law. A generalization of the Maxwell-Lorentz theory is also introduced. The theory is linear in the field sector and it reduces to the standard Maxwell-Lorentz electrodynamics when the maximal acceleration is infinite. Finally, we discuss the assumptions of our framework in addition to some predictions of the theory. Apart from the many open questions already leave in this work, we indicate further research directions, including the full development of the cohomology theory of generalized forms, its relation with calibrated geometry and a theory of curvature of generalized metrics. From the physical side, we emphasize the extension to non-linear Yang-Mills theory and to gravity as well as the problem of quantizing theories with generalized higher order fields.

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1 Introduction

1.1 Motivation

The theory of classical electrodynamics of point charged particles suffers from severe theoretical problems when one considers the coupled dynamics of charged particles with the total external and its own radiation electromagnetic field. In such circumstances, the standard theory of classical electrodynamics, based on Maxwell’s equations and the Lorentz-Dirac equation, makes un-physical predictions [40]. A main theoretical difficulty is that although the derivation of the standard theory is based on general principles, the equation has run-away (solutions with acceleration un-bounded, even if there is no external force) and pre-accelerated solutions (depending of the future sector of the world-line of the particle). Therefore, the problematic situation on which the classical electrodynamics theory lies permanently and the practical implications that a consistent theory of radiation reaction may be found in accelerator science and in modeling very dense, non-neutral, relativistic plasmas, demonstrates the relevance of having a consistent classical theory of charged particles and radiation reaction.

One can argue that such problems are of less relevance for fundamental physics, since from the advent of quantum field theory, classical field theories describing fundamental interactions should be considered as the classical limit of an appropriate quantum field theory. Following this line of thought, one could expect that the unwanted problems of the classical electromagnetic theory (divergent Coulomb fields, run away solutions and pre-accelerated solutions of the Lorentz-Dirac force equation [25]) are cured in the framework of a convenient quantum theory. This point of view is supported by the fact that some of the unwanted effects of the classical theory are believed to be significant at scales where quantum effects become relevant.

Despite the fact that such problems make the classical theory unsatisfactory from a theoretical viewpoint, the development of a quantum field theory able to overcome them is yet to be realized. The situation is further complicated by the absence of a well-defined regime of validity for the classical theory; without the security of a valid standard classical electromagnetic theory up to the energy scale where quantum effects become relevant, a top-down approach to constructing a quantum theory is inherently problematic. These comments specially apply to the work of Moniz and Sharp [50], where several hypothesis are not clearly justified (in particular, the asymptotic condition of the states and some requirements on analysity of the solutions). Thus, we consider that the conclusions drawn in [50] are only of partial validity and under too restrictive conditions.

One proposed solution to the foundational problems of classical electromagnetism was detailed by Landau and Lifshitz [44] and cast in the framework of singular perturbation theory (and developed thereafter) by H. Spohn and co-workers [59, 60]. Despite the success of this second order differential equation in describing the dynamics of a point charged particle (instead of the third order Dirac equation obtained from Dirac’s theory [25]), the theory is still laden with theoretical difficulties. In par-
ticular, the starting point in Landau-Lifshitz theory is the Lorentz-Dirac equation, which is substituted by a more convenient yet equivalent equation. This procedure is seemingly ad-hoc (one starts from a differential equation which is physically unacceptable) and does not appeal to fundamental principles. Furthermore, it is unclear whether pre-accelerated solutions can be obtained from the Landau-Lifshitz equation. Although these deficiencies may be overcome by future theoretical developments, one cannot be sure a priori whether this will be achieved within the framework of standard classical electromagnetic theory.

Another approach with the of flavor the standard classical theory was introduced more than an century by Larmor and reviewed later by W. Bonnor [6] (see also [39]). The main idea was that the observable rest mass $m$ of a point charged particle can vary with time, providing the origin of the energy-momentum radiated and that in effective terms eliminates the Schott’s term in the Lorentz-Dirac equation. The main difficulty with Bonnor’s theory relies on the physical motivation and the interpretation of a variable mass for a point charged particle, that usually is interpreted as a fundamental particle. Although Bonnor’s proposal is consistent with experiment, the idea of varying mass for elementary particles is disfavored by the complications in the interpretation of fundamental particle, for example as labeled by a irreducible representation of the Poincaré’s group.

T. S. Mo and C. H. Papas introduced the idea that the external electromagnetic force on a point charged particle could be dissipative [49]. Thus, a modification of the Lorentz force in such a way that a term proportional to the acceleration was introduced. However, apart from the required non-local modification of the energy-momentum tensor, the theory is lacking of a clear foundation from fundamental principles. In Mo-Papas’ theory it is unclear the origin of the radiation term as well as the particular form of the generalized Lorentz force. Nevertheless, the proposed differential equation does not have pre-acceleration, run away solutions and is compatible with radiation damping.

A very different approach to the foundations of classical electrodynamics was developed by R. P. Feynman and J. A. Wheeler [28], based on previous work of K. Schwarzschild, H. Tetrode and V. Fokker. For instance, in the Wheeler-Feynman theory, the fields are still living on the manifold $M$ and the particle-field interaction is formulated in a time-symmetric way. As a result, the radiation damping term is the same than in the standard Abraham-Lorentz theory. Such damping term it is the problematic one. Thus, Wheeler-Feynman’s appears more as a justification for the theory of Abraham-Lorentz-Dirac than a solution to the problem.

Without being exhaustive in the review of the different ideas and theories aiming to solve the problems of classical electrodynamics, it is clear that the current frameworks are not entirely satisfactory. The experimental difficulties in testing fundamental postulates of theories of classical electrodynamics, particularly in relation to the problems described before, partially explain the lack of a solution to the foundational problems of the theory. It is therefore reasonable to investigate new perspectives and investigate the possible new experimental consequences.
In this work we present a theory of generalized higher order jet bundle tensors and differential forms and we apply it to classical electrodynamics, in order to find a model of electrodynamics of point charged particles free from the problems of run away solutions and pre-accelerated solutions of Dirac’s type. Motivated by experimental testability and economy of postulates, the formalism for our theory is based on a phenomenological description of the electromagnetic field $\mathbf{F}$, as determined by its effect on the motion of point charged particles. Point charged particles are characterized by certain smooth curves on the spacetime $M$ that are interpreted as the world-line curves of point charged particles interacting with the electromagnetic field that we want to measure. It is the freedom that one has in the description of the electromagnetic field that in combination with the bound on $n$-acceleration sufficient to find a consistent field-particle dynamics. Each of the both new elements alone does not provide a consistent theory.

The theory proposed does not aim to solve the problem of the singularities of the electromagnetic fields, yet we will use the more phenomenological technique of renormalization of mass. In this way, although the problems of the infinities in classical electrodynamics is a difficult one, we show that it is possible to understand other problems. Indeed, we believe that the problem of infinities in electrodynamics is of different nature than the run-away and pre-accelerated solutions and probably requires a better understanding of the ultra-violet limits of classical field theories. Possible avenues to solve the divergent problems could be higher order modifications of the non-linear electrodynamics of Born and Infeld, Bopp-Podolsky theory or action at a distance theories on the way of Wheeler-Feynman theory. However, we do not consider that topic here and we show that indeed one can have an effective, consistent description of classical electrodynamics, if one consider maximal acceleration geometry structure, generalized higher order fields and mass renormalization.

1.2 Criticism of the notions of external electromagnetic field and point test particle

In the description of point charged test particle dynamics influenced by the action of an external field, an strategy that one can adopt is the following. One starts with the hypothesis that the test particle only ‘see’ the external field that we wish to measure (called the external field). The standard description of the trajectory of the test particle is then obtained as a solution to the Lorentz force equation, determined by the external field. Such a naïve approach, based on experimental determination of the evolution of the test particle, would however yield an inaccurate description of the external field. This is because an accelerating charged particle radiates electromagnetic waves, and this radiation of energy-momentum would have a twofold effect on the motion of the test particle. Firstly, the total electromagnetic field would now be a linear combination of the external and radiation fields, as opposed to the sole

\footnote{The point particle will be considered structureless, therefore disregarding spin, etc...}
external field. Secondly, the radiation of energy by the test particle would change its energy-momentum four-vector, thereby producing a dynamical effect on the motion of the test particle. Since the motion of the test particle is determined by the total field (rather than the sole external field), this suggests that the above description of the motion of a test particle is not accurate enough to describe the behavior of point charged in external electromagnetic fields.

Let us focus our attention initially on the related issue of how an electromagnetic field is defined in terms of physical measurements on the motion of point charged particles (note we have replaced the denomination of ‘test particle’ with ‘point charged particle’). The discussion of the previous paragraph indicates that the motion of a point charged particle is affected not only by the external field, but also by the radiation of the particle emitted as it accelerates. Therefore, the notion of defining an external electromagnetic field based on the measurement of the motion of test particles is not as clear-cut as we would like, since it cannot be characterized by measurements on the motion of charged particles without a complete description of the self-field. Both the point charged particle and the total field form a coupled dynamical system. In such systems it is difficult to abstract a notion of an external electromagnetic field that is consistent with a phenomenological characterization. Further, we observe that it is natural to interpret the standard theory of classical electrodynamics as an effective theory. With this interpretation in mind, the notion of an external field must appear along with an associated notion of a test particle. In addition, a covariant notion of weak-strong fields should be possible.

1.3 A theory of generalized higher order electromagnetic fields

In order to describe the classical electrodynamics of point charged particles interacting with electromagnetic fields, a modification of the notion of electromagnetic field can be useful. We propose that the description of an electromagnetic field should depend not only on the macroscopic source, but also on the state of motion of the charged particle which is used in each particular measurement setting. The motion of the particle is not prescribed a priori, but we assumed that such evolution exists and that it is regular in the domain of definition, assumptions that in the classical domain are reasonable. We then follow fundamental physical and geometric principles to obtain a mathematical formalism for the electromagnetic field, the equation of motion of the point charged particle and the equation of evolution for the fields. The guiding principles when pursuing this strategy are as follows:

1. A minimal higher order extension of the notion of the field. We think that a formalism capable of accommodating the dynamical system of point charged particles interacting with electromagnetic fields can be constructed if the elec-

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The pair (particle, external field) is a convenient representation if the external field has a much stronger effect than the radiative-reaction in the motion of the test particle. A notion for strong or week field can be introduced in a covariant way using the norm-operator associated with the Riemannian metric determined by an observer.
tromagnetic and other fields are described as sections of certain sub-bundles of higher order jet bundles over the spacetime manifold $M$. In particular, the generalization of the notion of a field introduces degrees of freedom in such a way that one can obtain consistent dynamics of particles and fields. This procedure is performed in a minimal way, introducing the minimal number of new degrees of freedom, attempting to be as conservative as possible in the process of the generalization. In this case, the generalized electromagnetic field lives in the third jet $J^3_0(M)$.

2. **Geometry of maximal acceleration.** In the Theory of General Relativity, the geometry of the spacetime is dynamical. On the other hand, if the modified electromagnetic field enters in Einstein’s equations, then in an analogous way the substitute for the metric field needs to be formulated in the same framework of higher order jet bundles. Therefore, the substitute of the spacetime metric tensor will be a tensor defined on higher order bundles over the spacetime manifold $M$. One way to achieve this is by using geometries of maximal acceleration. This provides a minimum extension such that the principles third to five described below are fulfilled. Furthermore, a description of geometries with maximal covariant acceleration requires the use of generalized metrics, with coefficients living on the jet bundle $J^2_0(M)$. This can be seen as a justification of the formalism of generalized higher order fields.

3. We assume that the theory is an effective theory, in the sense that it depends on a small parameter in such a way that in the limit when such parameter goes to zero, one obtains the standard classical Maxwell-Lorentz theory. This parameter is related to the inverse square of the maximal covariant acceleration described in the point before. However, we do not discuss a particular mechanism producing the maximal acceleration and only general considerations are presented.

4. Conservation of the energy-momentum of the system. This implies that the loss of energy-momentum of the particle must be compensated by its emitted radiation, following a covariant Larmor formula [40, 56]. That the covariant Larmor’s law is still valid is currently an assumption. However, it is consistent with the fact that Maxwell’s equation is a good effective description of the electromagnetic fields in the regime that accelerations are small compared with the maximal acceleration.

5. The equation of motion of the point charged particles must be a second order ordinary differential equation. This is a requirement that we follow, in order to avoid the problems that plague the standard formulation of classical mechanics. This principle is very restrictive, as we will see.

We construct a theory where all the physical fields (electromagnetic excitation, electromagnetic field, density currents and generalized metrics in the case of electrodynamics) have coefficients living on higher order jet bundles over the spacetime
manifold $M$. A maximal covariant $n$-acceleration is introduced, providing a book-keeping parameter for the theory. We show that in the limit of large maximal $n$-acceleration, there is a convenient definition of external field and test particle which recovers the standard notions.

As the result of our analysis of the extension of electromagnetic fields to higher order jet bundles on the spacetime manifold $M$ and with point particles moving with bounded speed and acceleration, we derive an effective theory living on $M$ for point charged particles and electromagnetic fields such that it does not contain some of the pathologies of the standard classical theory. In particular, the electromagnetic fields are described by a set of equations analogous to Maxwell’s equations, whereas the point charged particle is described by the covariant equation (6.7) (or in its normal coordinate version form, equation (6.6)), which is of second order and does not suffer most of the problems found with the Lorentz-Dirac equation. The effective full dynamics of fields and point charged particles is described in a general covariant way by equations (6.7) and the generalized Maxwell equations (7.8), (7.10) together with the charge conservation law (7.11).

Our considerations will involve localized point charged particles with fixed charge $e$ and mass $m$. The electromagnetic media will be the vacuum, and therefore the permittivity and permeability are constant fields living on the spacetime manifold $M$. The study of other more convenient media and charge distributions will be postponed to subsequent investigations.

1.4 Structure of the work

In this paper we explore a classical electrodynamic theory with electromagnetic fields to be sections of the bundle $\Lambda^2(M, F(J^k_0(M)))$. These are differential forms over $M$ that, when applied to sections of $\Gamma TM$ gives a section of $F(J^k_0(M))$. In section 2 such differential forms are introduced. Particular attention is payed in the case of generalized metric, since it will be of relevance to the construction of geometries of maximal acceleration. The fundamental notion of non-linear connection that we need to define natural objects in the relevant bundles is discussed. We describe the fundamental geometric and cohomological notions of generalized forms required in later developments. Then we consider the Cartan’s calculus of differential forms, the rudiments (including a version of Stokes’ theorem) and the corresponding de Rham cohomology (including a version of Poincaré’s lemma and Thom’s isomorphism theorem). Although the section is primarily of mathematical character, we have restricted ourself to developed the notions and results that are strictly necessary for the application to generalized higher order electrodynamics.

In section 3 the notion and fundamental properties of maximal acceleration geometry are briefly presented. Maximal covariant acceleration introduces a natural perturbation parameter (the inverse of the square of the norm of the maximal co-

\*An alternative description for generalized higher order fields is as sections of the bundle of forms $\Lambda^p(J^k_0(M))$ that are horizontal, in the sense that will be indicated later. Fortunately both descriptions are equivalent.
variant acceleration), which is fundamental for our treatment and will eventually forbid run-away solutions in a natural way. We did not pursue a more fundamental explanation for maximal acceleration, restricting our considerations to a kinematic formulation of the new geometry.

The generalized higher order fields are introduced in section 4 from the cohomological perspective discussed in section 2. Also, we discuss the behavior of the singularities of the fields. The combination of the cohomology theory of section 2 with the analytical structure that we assume for the electromagnetic fields provides an unambiguous way of generalizing the notion of electromagnetic field in section 4.

In section 5 the Lorentz-Dirac equation is discussed in the standard framework of classical electrodynamics, in the new setting of generalized higher order fields and in the contest of geometries of maximal acceleration. In particular, we use a simple way to obtain the Lorentz-Dirac equation, in a similar way as it was discussed by Rohrlich [56], but for each of the possible different frameworks (standard electrodynamics in Minkowski spacetime, standard electromagnetic fields but with a maximal acceleration geometry and generalized electromagnetic fields in Minkowski spacetime).

In section 6 the second order differential equation (6.7) describing the motion of a charged particle is obtained and its basic properties discussed. The argument is based strongly on the notions presented in sections 3 and section 4 and the method explained in section 5, applied in this case to generalized electromagnetic fields in a maximal acceleration geometry. We show that for generalized higher order fields, maximal acceleration and conservation of energy momentum are compatible. The resulting differential equation for a point charged particle is general covariant, free of run-away and pre-accelerated solutions of Dirac’s type.

In section 7 a set of differential equations for an effective electromagnetic theory on $M$ are obtained and the skeleton of a generalized gauge theory described. In particular, the explicit form of the generalized electromagnetic field and current is obtained, and prove that follow a generalized Maxwell’s equations. Gauge invariance, the notion of vacuum and boundary conditions are also briefly discussed.

In section 8, the results of the paper are briefly discussed and some relations with other approaches to electrodynamics highlighted. Finally, some directions of further investigation are indicated like the development of the cohomology theory of generalized forms, its relation with calibrated geometry, the need of a theory of curvature of generalized metrics, the extension to non-linear Yang-Mills theory and to gravity as well as the quantization of the theories with generalized higher order fields.

1.5 Conventions

$M$ is an $n$-dimensional, Hausdorff, second countable, smooth manifold (in applications to physics $n = 4$, although this restriction is not necessary for most of the mathematical results that we present). That $M$ hold such technical properties $M$ assures the existence and uniqueness of natural geometric operators, in particular the existence of exterior algebra of differential forms and partitions of unity. An
arbitrary point in the manifold \( M \) is denoted by Latin letters, usually by \( x \) or \( p \). However, in the case of \( x \), the same symbol will be used to designate its coordinates in a local coordinate chart on \( M \) or for curves on \( M \). An arbitrary local coordinate chart on \( M \) is denoted by \((U, x)\), where \( U \) is an open neighborhood. A curve on \( M \) will be a smooth map \( x : I \to M \), being \( I \) an interval of \( R \). The distinction between the three different meanings of \( x \) will be clear from the context, although parameterized curves will be often denoted by \( x(\sigma) \) or by \( x(\tau) \), indicating the dependence in the given parameter. Greek letters are used for spacetime indices and run from 1 to \( n \). General smooth differential forms will also be denoted by Greek letters. However, for the electromagnetic fields, currents and electromagnetic potential 1-form we follow the traditional notation and denoted by \( F, G, A, J \) the Faraday, excitation, potential and current density forms. When considering fiber bundles over \( M \), indices of the fiber over \( x \) in the jet bundle are denoted by capital Latin letters. For instance for fibers over \( x \) of the jet bundle \( J_k^0(M) \) they run in \( \{1, \ldots, nk\} \). Einstein’s summation convention is used, if anything else is not stated. In some occasions we will use multi-index notation, in order to simplify algebraic expressions. The manifold \( M \) is equipped with a pseudo-Riemannian metric \( \eta \) with signature \((-1, 1, \ldots, 1)\). In more concrete applications, the spacetime manifold will be four dimensional and will have signature \((-1, 1, 1, 1)\). In this case, when a 3+1 decomposition is possible, three dimensional objects are indicated by arrowed vectors like \( \vec{V}, \vec{A}, \vec{B} \), etc...

2 Differential geometry of jet bundles and cohomology of generalized differential forms

In this section we introduce the notions generalized tensors and generalized differential forms that we will use later. Firstly, a short introduction to \( k \)-jets bundles \( J_k^0(M) \) is necessary, since the geometric objects that we will consider are constructed on such bundles. Then a relevant class of connections defined on each \( k \)-jet bundle are explained. Once we have chosen a connection, the notion of generalized tensors and forms is given: they will be tensor and forms along maps on \( M \) with values on the algebra of functions \( \mathcal{F}(J_k^0(M)) \). The justification to introduce such sophisticated objects is based on physical considerations and will be fully explained in later sections. Notice that it allows a notion of electromagnetic field which is capable to accommodate the changes produce by radiating point charged particles.

We will show that the fundamental notions of differential geometry can be transplanted to the framework of generalized tensors and forms. For instance, a notion of generalized metric is introduced as generalized tensor, as well as associated notions as distance, isometries, Hodge operator and isometry. The fundamental notions of the causal theory for generalized metrics with Lorentzian signature are also discussed. The foundations of the Cartan’s calculus for generalized forms is developed. With a notion of generalized metric and the Cartan’s calculus for generalized forms on hand, the fundamental results of the cohomology of generalized differential forms are proved. In particular, integration along the fiber implies a \( k \)-jet bundle version
of Thom’s isomorphism theorem. We finish this section introducing the fundamental
notions of a theory of integration of generalized forms. We provide the fundamental
properties of the integral, including a version of Stokes’ theorem.

2.1 Jet bundles and natural lifts

Jet theory is a natural framework for the study of the geometry of ordinary and
partial differential equations (see for instance for a introduction to jet theory the
references [13, 24, 42, 46, 58]). Given a smooth curve \( x: I \to M \), the set of
derivatives \( (x(0), \frac{dx}{d\sigma}|_0, \ldots, \frac{d^k x}{d\sigma^k}|_0) \) determines a point in the space of jets of curves
\( J^k_0(p) \) in a neighborhood of the point \( p = x(0) \in M \),

\[
J^k_0(p) := \left\{ (x(0), \frac{dx}{d\sigma}|_0, \ldots, \frac{d^k x}{d\sigma^k}|_0), \forall C^k x: I \to M, x(0) = p \in M, 0 \in I \right\}.
\]

The jet bundle \( J^k_0(M) \) over \( M \) is then the disjoint union

\[
J^k_0(M) := \bigsqcup_{x \in M} J^k_0(x).
\]

The projection map is

\[
k\pi: J^k_0(M) \to M, \quad (x(0), \frac{dx^\mu}{d\sigma}|_0, \frac{d^2 x^\mu}{d\sigma^2}|_0, \ldots, \frac{d^k x^\mu}{d\sigma^k}|_0) \mapsto x(0).
\]

Since it is smooth, the projection \( k\pi: J^k_0(M) \to M \) determines the fiber bundle
\( (J^k_0(M), M, k\pi) \). The fiber over \( p \in M \), \( J^k_0(p) := k\pi^{-1}(p) \) is a manifold of dimension \( nk \).

Local coordinates on the fiber \( J^k_0(p) \) are induced from the local coordinate system
\( (U, x) \) over \( M \) such that a natural system of local coordinates associated with
the \( k \)-jet at \( x(0) \) of the map \( x: I \to M \) is \((x^\mu(\sigma), \frac{dx^\mu}{d\sigma}, \frac{d^2 x^\mu}{d\sigma^2}, \ldots, \frac{d^k x^\mu}{d\sigma^k})|_{\sigma=0} \). A natural
system of local coordinates of a point in \( J^k_0(M) \) is denoted by \((x^1, \ldots, x^n, y^1, \ldots, y^{nk})\).

The linear map \( k\pi_\ast: J^k_0(M) \to M \) is differentiable and the differential of the projection
\( k\pi_\ast \) at \( u \in k\pi^{-1}(x) \) is the linear map

\[
(k\pi_\ast)_u: T_u J^k_0(M) \to T_x M.
\]

We will denote by \( k\pi_\ast \) the projection \( (k\pi_\ast): T J^k_0(M) \to TM \) such that at each \( u \)
it is the linear map \( (k\pi_\ast)_u \).

Given a curve \( x: I \to M \), a \( k \)-lift is the curve \( k_x: I \to J^k_0(M) \) such that the following diagram commutes,

\[
\begin{array}{ccc}
J^k_0(M) \\
\downarrow k\pi \\
M.
\end{array}
\]
In order to fix a $k$-lift one needs to specified each point $(x^1(\sigma),...,x^n(\sigma),y^1,...,y^{nk}(\sigma))$ of the $k$-lift: each point on the lift $kx : I \to J^k_0(M)$ has local coordinates given by $(x^\mu(\sigma), \frac{dx^\mu(\sigma)}{d\sigma},...\frac{dx^{nk}(\sigma)}{d\sigma})$, where $\sigma$ is the parameter of the curves $x : I \to M$ and $kx : I \to J^k_0(M)$.

There are also the notions of lift of tangent vectors and smooth functions:

- Let $X \in T_xM$ be a tangent vector at $x \in M$. A lift $k\pi_*(X)$ at $u \in k\pi^{-1}(x)$ is a tangent vector at $u$ such that $k\pi_*(k\pi_*(X)) = X$.

- Let us denote by $\mathcal{F}(J^k_0(M))$ the algebra of real smooth functions over $J^k_0(M)$. Then there is defined the lift of a function $f \in \mathcal{F}(M)$ to $\mathcal{F}(J^k_0(M))$ by $k\pi^*(f)(u) = f(x), \forall u \in k\pi^{-1}(x)$.

The kernel of $(k\pi_*)_x$ at $x \in M$ is the vector space $k\mathcal{V}_x := (k\pi_*)^{-1}(0_x)$, where $0_x$ is the zero vector in $T_xM$. Then vertical bundle over $M$ is the manifold $k\mathcal{V} := \bigsqcup_{x \in M} k\mathcal{V}_x$ with the induced projection $k\tilde{\pi}_V : k\mathcal{V} \to M$. The vertical bundle over $J^k_0(M)$ is determined by the surjection $k\pi_V : k\mathcal{V} \to J^k_0(M), (k\pi_*)^{-1}(0_x) \ni \xi^u \mapsto u \in (\pi^k)^{-1}(x)$.

$k\mathcal{V}$ is a real vector bundle over $J^k_0(M)$, since it is the kernel of $k\pi_*$. The composition of $k\pi_V$ with $k\pi$ determines also a real vector bundle over $M$,

$k\pi \circ k\pi_V : k\mathcal{V} \to M$.

One can introduce the notation $k\tilde{\pi}_V$ and the vertical bundle over $J^k_0(M)$ is determined by the surjection

$k\pi_V : k\mathcal{V} \to J^k_0(M), (k\pi_*)^{-1}(0_x) \ni \xi^u \mapsto u \in (\pi^k)^{-1}(x)$.

$k\mathcal{V}$ is a real vector bundle over $J^k_0(M)$, since it is the kernel of $k\pi_*$. The composition of $k\pi_V$ with $k\pi$ determines also a real vector bundle over $M$,

$k\pi \circ k\pi_V : k\mathcal{V} \to M$. 

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2.2 Connections on $TJ^k_0(M)$ and a covariant derivative on $TM$

There are several general notions of connections in differential geometry. Let us recall the notion of general connection that will be particularly useful for us (see for instance [42]). On an arbitrary fiber bundle $\pi_E : E \to M$, the vertical bundle is $\mathcal{V}E := \ker((\pi_E)_*)$.

**Definition 2.1** A connection on the bundle $\pi_E : E \to M$ is a vector valued 1-form $\Phi_E : T_E \to \mathcal{V}E$ such that

1. $\Phi^2_E = \Phi_E$,
2. $\operatorname{Im} \Phi = \mathcal{V}E$.

Therefore, a connection is a projection operator on $T_E$. For the map $h_E = (Id - \Phi_E) : T_E \to T_E$ one has that $h_E^2 = h_E$.

$\ker(\Phi)$ is a sub-bundle $\mathcal{H}E$ of $E$ called the horizontal bundle. Also, it is direct that the following decomposition holds,

$$T_E = \Phi_E(T_E) \oplus h_E(T_E) := \mathcal{V}E \oplus \mathcal{H}E.$$  

Thus for each $u \in \mathcal{E}$ there is a unique decomposition $T_u\mathcal{E} = \mathcal{H}u\mathcal{E} \oplus \mathcal{V}u\mathcal{E}$. Given the connection $\mathcal{H}E$, the horizontal lift of a vector field $X \in \Gamma TM$ is a horizontal field $^hX \in \Gamma \mathcal{H}E$ such that $^k\pi_u(^hX) = X$.

**Remark 2.2** It is equivalent to give a connection $\Phi_E$ or give a distribution $\mathcal{H}E$ satisfying 2.1. The distribution $\mathcal{H}E$ is not necessarily integrable and the integrability obstruction is given by the curvature of the connection.

**Definition 2.3** A connection on $^k\pi : J^k_0(M) \to M$ is a global splitting

$$TJ^k_0(M) = \mathcal{V}J^k_0(M) \oplus \mathcal{H}J^k_0(M),$$

and with $\mathcal{V}_uJ^k_0(M) = ^k\mathcal{V}_u = \ker(^k\pi_u)$ for each $u \in J^k_0(M)$.

We will denote by $^k\mathcal{H} = \mathcal{H}J^k_0(M)$ to the horizontal distributions.

Let us consider the Levi-Civita connection $D$ of the semi-Riemannian metric $\eta$ on $M$. Our objective in this sub-section is to introduce a natural connection $\mathcal{H}D \hookrightarrow TT\mathcal{M}$ on $TM$. Related with this connection, it will be shown later the existence of connections on the jet bundles $J^k_0(M)$.  

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In order to introduce the connection $\mathcal{H}_D$ on $TM$, let us consider the commutative diagram

$$
\begin{array}{ccc}
TTM & \xrightarrow{\pi_2} & TM \\
\pi_1 & \downarrow & \\
TM & \xrightarrow{\pi} & M
\end{array}
$$

and let $\mathcal{V} = \ker(\pi_1) \subset TTM$ be the vertical sub-bundle. The vertical lift of a vector field is canonically defined [21] by the relations

$$
^vl : TM \to \mathcal{V}, \quad \hat{X} \mapsto (^v\hat{X})_{u=(x,y)} := \frac{d(x, y + t\hat{X})}{dt}|_{t=0}.
$$

The geodesic spray $S$ is the section $S \in \Gamma TTM$ such that $\pi_* S = \pi_1 S$ and the geodesic curves of $S$ is the geodesic flow of $\eta$. Then the connection $\mathcal{H}_D$ is determined by the distribution of $TTM$ determined by the conditions [21][22].

1. It is torsion free,

$$
[^hX, ^vY] - [^hY, ^vX] - ^v[X,Y] = 0,
$$

where $^hX$ is the horizontal lift of the connection $D$.

2. The geodesic spray

$$
S = y^i \frac{\partial}{\partial x^i} + \gamma^i_{jk}(x) y^j y^k
$$

of the Riemannian connection $D$ determines the horizontal lift by the formula

$$
^hX_u := \frac{1}{2}([^vX, S]_u + (^cX)_u), \quad X \in T_x M, u \in \pi^{-1}(x).
$$

One has the following

**Proposition 2.4** The set $\{(^hX)_u, X \in T_x M, u \in \pi^{-1}(x), x \in M\}$ determines a vector space of sections that defines a connection $\mathcal{H}_D$ on $TM$.

Since $\dim(\mathcal{V}) = n$, one has that $\dim(\mathcal{H}_D) = n$. The connection $\mathcal{H}_D$ has associated the horizontal projection operator $h_D : TTM \to \mathcal{H}_D$ and $\mathcal{H}_D = Im(h_D)$. The horizontal lift of an arbitrary vector field $X \in \Gamma TM$ is denoted by $^hX \in \Gamma TTM$. Given $\hat{X} \in \Gamma TTM$, its horizontal component is denoted by $^h\hat{X}$.

**Example 2.5** Let $(M, \eta)$ be a Lorentzian manifold. Given an holonomic local frame $\{\frac{\partial}{\partial x_\mu}, \mu = 1, ..., n\}$ the connection coefficients of the Levi-Civita connection $D$ are given by the expression

$$
(D_{\partial_\nu} \partial_\rho)^\mu := \gamma^\mu_{\nu\rho} = \frac{1}{2} \eta^{h\mu s} (\frac{\partial \eta_{s\nu}}{\partial x^\rho} - \frac{\partial \eta_{s\rho}}{\partial x^\nu} + \frac{\partial \eta_{s\rho}}{\partial x^\nu}), \quad \mu, \nu, \rho, s = 1, ..., n.
$$
A connection on $TTM$ associated with the connection $D$ on $TM$

Given the Levi-Civita connection $D$ of the metric $\eta$, one can introduce the homomorphism

$$N : TTM \to TTM, \quad \frac{\partial}{\partial x^\mu} \mapsto N^\nu_{\mu} \frac{\partial}{\partial y^\nu}, \quad \frac{\partial}{\partial y^\mu} \mapsto \frac{\partial}{\partial y^\mu}; \quad \mu, \nu = 1, \ldots, n.$$ 

such that the projector $h_D$ is given by

$$h_D : TTM \to TTM, \quad \hat{X} \mapsto (Id - N)(\hat{X}). \quad (2.5)$$

Then $h_D$ is determined by the components of $N$,

$$N^\mu_{\nu}(u) = \gamma^\mu_{\nu\rho}(x) y^\rho, \quad \mu, \nu, \rho = 1, \ldots, n. \quad (2.6)$$

**Proposition 2.6** The homomorphism $(2.5)$ determines a connection in the sense of the Definition [2.1] on the bundle $TM$.

An adapted basis for $\mathcal{H}_D$ is given by the globally defined distribution

$$\left\{ \left. \frac{\delta}{\delta x^1}|_u, \ldots, \frac{\delta}{\delta x^n}|_u, \frac{\partial}{\partial y^1}|_u, \ldots, \frac{\partial}{\partial y^n}|_u \right| \right\}, \quad \frac{\delta}{\delta x^\nu}|_u = \frac{\partial}{\partial x^\nu}|_u - N^\mu_{\nu}(u) \frac{\partial}{\partial y^\mu}|_u \quad (2.7)$$

for $\mu, \nu = 1, \ldots, n$. Then the set of local sections

$$\left\{ \left. \frac{\delta}{\delta x^1}|_u, \ldots, \frac{\delta}{\delta x^n}|_u \right| u \in TU \right\} \quad (2.8)$$

generates the local horizontal distribution $\mathcal{H}_D|_U$, while the set of local sections

$$\left\{ \left. \frac{\partial}{\partial y^1}|_u, \ldots, \frac{\partial}{\partial y^n}|_u \right| u \in TU \right\} \quad (2.9)$$

generates the local vertical distribution $\mathcal{V}|_U$.

**The covariant derivative of $D$**

Given a connection on $TM$, there is associated a covariant derivative on $M$, defined in local coordinates by the expression

$$(D_{\dot{\sigma}} Z)^\nu := \dot{Z}^\nu + N^\nu_{\rho}(u)Z^\rho, \quad u = (x, \dot{x}) \in TM, \quad Z \in \Gamma TM. \quad (2.10)$$

The derivatives are performed respect to $\sigma$ (in this case, an affine parameter). The geodesic equation for the linear connection $D$ is given as the curve on $M$ solution of the differential equation

$$D_{\dot{\sigma}} \dot{x} = 0, \quad (2.11)$$

Equation $(2.11)$ corresponds to the geodesic equation of the pseudo-Riemannian metric $\eta$. 

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2.3 Induced connections on $TJ_0^k(M)$ from connections on $TM$

Given a connection $\mathcal{H}$ on $TM$, we can show the existence of a connection $^k\hat{\mathcal{H}}$ on $J_0^k(M)$ related with $\mathcal{H}$. The construction is similar to the induced connection of the references [41, 42]. The connections $^k\hat{\mathcal{H}}$ will be used in the construction of geometric objects and some relevant natural operators on $J_0^k(M)$.

Let us consider the commutative diagram,

$$
\begin{array}{ccc}
J_0^k(M) & \overset{^k\pi}{\longrightarrow} & TM \\
\downarrow{^k\pi} & & \downarrow{\pi} \\
M & \overset{id}{\longrightarrow} & M \\
\end{array}
$$

(2.12)

The commutative diagram (2.12) induces another commutative diagram for each $^kx \in J_0^k(M)$; if $(x,y) \in TM$ and $x \in M$, with $pr(^kx) = (x,y)$ and $\pi(x,y) = x$, then one has that

$$
\begin{array}{ccc}
T^k_0(M) & \overset{^kpr_*}{\longrightarrow} & T(x,y)TM \\
\downarrow{^k\pi_*} & & \downarrow{\pi_*} \\
T_xM & \overset{id}{\longrightarrow} & T_xM \\
\end{array}
$$

(2.13)

is also commutative.

**Lemma 2.7** Let us consider a distribution $^k\hat{\mathcal{H}} \subset TJ_0^k(M)$ such that $pr_*(^k\hat{\mathcal{H}}) \subset \mathcal{H}$ and that the diagram (2.13) commutes. Then it holds that $^k\hat{\mathcal{H}} \cap ^k\mathcal{V} = 0$.

**Proof.** If the vector field $\tilde{X}_0 \in ^k\hat{\mathcal{H}}$ and is vertical, then $(^k\pi_*)(\tilde{X}_0) = 0$. The commutative diagram (2.13) implies that $^kpr_*(\tilde{X}_0) \in \mathcal{H}_D$ and $^k\pi_*(^kpr_*(\tilde{X}_0)) = 0$. Thus the relation $^kpr_*(\tilde{X}_0) = 0$ holds, since the only horizontal vector $U$ such that $^kpr_*(U)_i = 0$ is the null vector. $\square$

**Lemma 2.8** Let us consider $^k\hat{\mathcal{H}}$ such that $^kpr_*(^k\hat{\mathcal{H}}) \subset \mathcal{H}$ and that the diagrams (2.13) commute. Then $\dim(^k\hat{\mathcal{H}}) \leq \dim(M) = n$.

**Proof.** Assume that it has dimension $n + 1$. Then there is a basis $\{\tilde{X}_1, ..., \tilde{X}_n, \tilde{Z}\}$ of $^k\hat{\mathcal{H}}$ such that $\tilde{X}_i = X_i$ for each $i = 1, ..., n$. Therefore,

$$
^kpr_*(\tilde{Z}) = a^i X_i = a^i ^kpr_*(\tilde{X}_i) = ^kpr_*(a^i \tilde{X}_i),
$$

from what follows that $^kpr_*(\tilde{Z} - a^i \tilde{X}_i) = 0$. Since diagrams (2.13) are commutative, it follows that $\tilde{Z} - a^i \tilde{X}_i$ is vertical. Therefore, it must be zero (by Lemma 2.7). $\square$

**Proposition 2.9** Let us consider $^k\hat{\mathcal{H}}_{m}$ of maximal dimension such that $^kpr_*(^k\hat{\mathcal{H}}) \subset \mathcal{H}$. Then it has dimension $\dim(^k\hat{\mathcal{H}}_{m})n$.  

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Proof. We have that \( \dim(TJ^k_0(M)) = n(k + 1) \), \( \dim(\ker(k\pr_*)) = nk \). On the other hand, \( k\pr_* \) is surjective. Therefore, \( \dim(\text{Im}(pr_*)) = n \). Assume that there is \( X_0 \in \mathcal{H}_m \) such that it is not the image of some \( \hat{X}_0 \in k\hat{\mathcal{H}} \). This is in contradiction with the following facts (just counting dimensions):

1. \( k\hat{\mathcal{H}}_m \) is of maximal dimension,
2. \( pr_*(k\hat{\mathcal{H}}_m) \subset \mathcal{H} \),
3. The projection \( k\pr_* \) is surjective.

Therefore, it follows that \( \dim(k\hat{\mathcal{H}}_m) = n \).

\[ \square \]

**Theorem 2.10** A distribution \( k\hat{\mathcal{H}}_m \) of maximal dimension such that \( pr_*(k\hat{\mathcal{H}}) \subset \mathcal{H} \) and that the diagrams (2.13) commutes is a connection on \( J^k_0(M) \).

**Proof.** Let \( k\hat{\mathcal{H}}_m \) a maximal distribution of dimension \( n \). As we showed in Lemma 2.7 \( k\hat{\mathcal{H}}_m \cap k\mathcal{V} = 0 \). This shows that \( TJ^k_0(M) = k\hat{\mathcal{H}}_m \oplus k\mathcal{V} \).

\[ \square \]

**Definition 2.11** Given \( \mathcal{H} \) a connection on \( TM \), a connection \( k\hat{\mathcal{H}} \) on \( J^k_0(M) \) as in Theorem 2.10 is an induced connection on \( J^k_0(M) \).

We considered an induced connection \( k\hat{\mathcal{H}} \) on \( J^k_0(M) \) induced by the connection \( \mathcal{H}_D \), determined from the Levi-Civita connection \( D \). Let us note that Theorem 2.10 does not fix the connection \( k\hat{\mathcal{H}}_m \), but only shows the existence of such connections. Indeed, because of the difference between the dimensionality between \( T_{kx}J^k_0(M) \) and \( T(x,y)TM \), it is not possible to fix the connection in such a way and a bunch of connections exists that full-fill Theorem 2.10. However, one can characterize such connections as follows,

**Proposition 2.12** Let us consider two connections \( k\hat{\mathcal{H}}_1, k\hat{\mathcal{H}}_2 \) constructed as in Theorem 2.10. If the corresponding horizontal lifts \( h_1X \in k\hat{\mathcal{H}}_1|_{xX} \) and \( h_2X \in k\hat{\mathcal{H}}_2|_{xX} \) of \( X \in TM \) are such that \( pr_*(h_1X - h_2X) = 0 \), then the vector field \( h_1X - h_2X \in TJ^k_0(M) \) is vertical.

**Proof.** It is a direct consequence of the fact that the diagrams (2.13) commute.

The following notion captures when a property is independent of the connection:

**Definition 2.13** Equivariant relations.

- Two connections \( k\hat{\mathcal{H}}_1 \) and \( k\hat{\mathcal{H}}_2 \) are said to be equivalent iff \( h_1X - h_2X \in k\mathcal{V}, \forall X \in TM \).

- A property is connection independent if only depends on the induced connection as in in Theorem 2.10 and if it is true for any set of connection independent connections.
If we denote by $^kH$ the set of connections as in Theorem 2.10, the following result holds:

**Proposition 2.14** The relation of being connection independent for induced connections 2.13 is an equivalence relation in $^kH$.

Each induced connection $^k\hat{H}$ has associated a projection map $\hat{h}_k$,

$$\hat{h}_k : TJ^k_0(M) \rightarrow ^k\hat{H}$$

with the property that $(\hat{h}_k)^2 = \hat{h}_k$.

**Remark 2.15** We have the following remarks:

- One way to define a connection on $J^k_0(M)$ is by using a semi-spray of order $k$ plus additional conditions on the variation equations [13]. However, for the problem that we are interested in this paper (the dynamics of point charged particles with the electromagnetic field), we do not have any natural semi-spray of order $k$ which is free of pathological solutions. We have two natural semi-sprays defined in the problem. The first one corresponds to the Lorentz-Dirac equation. This is a spray of third order. However, the Lorentz-Dirac spray is not physically acceptable: it is well known that it contains un-physical solutions. The natural spray corresponds to the geodesic equation of the pseudo-Riemannian metric $\eta$. This is a spray of second order $k = 2$. It determines geodesics describing the dynamics of free point charged particles without interacting with an electromagnetic field, thus not suitable to our problem (description of the motion of a point charged particle).

- Kaluza-Klein dimensional reduction theories does not provides a natural spray to define a convenient connection. The reduction reproduces the Lorentz force equation, that brings to the game several connections (see for instance [47, 48, 32]). However, it does not take into account radiation reaction effects.

- The equation governing the dynamics of a point charged particle is not necessarily a (semi)-spray. It will turn out that is an implicit differential equation of second order and does not have the form of a spray.

Given a connection $^k\hat{H}$ on $J^k_0(M)$, the local connection coefficients of the endomorphism $N$ in the natural basis

$$\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{nk}}\right\} \quad (2.14)$$

are $\{N^A_\mu, \mu = 1, \ldots, A = 1, \ldots, nk\}$. The vertical distribution is generated locally by the local frame,

$$\left\{\frac{\partial}{\partial y^1}|_x, \ldots, \frac{\partial}{\partial y^{nk}}|_x, x \in U \subset M\right\}.$$
Let us adopt a connection as in Theorem 2.10. Then there is a splitting of any local frame of \( T^k_0(M) \) such that the horizontal distribution is locally generated by

\[
\left\{ \frac{\delta}{\delta x^1} |_x, \ldots, \frac{\delta}{\delta x^n} |_x, \frac{\partial}{\partial y^1} |_x, \ldots, \frac{\partial}{\partial y^{nk}} |_x \right\}, \quad \frac{\delta}{\delta x^\nu} |_x = \frac{\partial}{\partial x^\nu} |_x - N^A \nu \frac{\partial}{\partial y^A} |_x \tag{2.15}
\]

with \( \nu = 1, \ldots, n, A = 1, \ldots, nk \). Therefore, the functions \( kn^2 \) functions \( N^A \nu(x, y) \) determines the connection in the given coordinate system. From Theorem 2.10, \( n^2 \) of those functions are fixed. Therefore, there are still \((k-1)n^2\) to be fixed by imposing additional conditions.

A way to impose conditions on the connection is the following,

**Definition 2.16** Let us consider a connection \( ^k\mathcal{H} \) of \( J^k_0(M) \). An horizontal \( p \)-form is such that \( \omega(..., V, ...) = 0 \) for any vertical vector field \( V \in \hat{V} \).

A local frame of vertical forms is determined by exterior product of vertical 1-forms

\[
\delta y^A = dy^A + N^A \nu \, dx^\nu, \quad \nu = 1, \ldots, n, \quad A = 1, \ldots, nk, \tag{2.16}
\]

where the dual co-frame of 1-forms \( \{dx^1, \ldots, dx^n, dy^1, \ldots, dy^{nk}\} \) defined by the relations

\[
dx^i \left( \frac{\partial}{\partial x^j} \right) = \delta^i_j, \quad dx^i \left( \frac{\partial}{\partial x^A} \right) = 0, \\
dx^A \left( \frac{\partial}{\partial x^i} \right) = 0, \quad dy^A \left( \frac{\partial}{\partial y^B} \right) = \delta^A_B, \quad i, j = 1, \ldots, n, \quad A, B = 1, \ldots, nk.
\]

**Proposition 2.17** Every horizontal \( p \)-form is connection independent.

**Proof.** Given two equivalent connections \( ^k\mathcal{H}_1 \) and \( ^k\mathcal{H}_2 \), if a vector \( Z \) is horizontal respect to the first, it must be re-written in the form \( ^hZ_1 = ^hZ_2 + V(\, ^hZ_1) \), with \( ^hZ \in ^k\mathcal{H}_2 \) and \( V(\, ^hZ_1) \) vertical. Thus

\[
\omega(..., \, ^hZ_1, ...) = \omega(..., \, ^hZ_2 + V(Z), ...) \\
= \omega(..., \, ^hZ_2, ...) + \omega(..., \, V(Z), ...) \\
= \omega(..., \, ^hZ_2, ...).
\]

The exterior product of horizontal forms is also horizontal and connection independent, forming the algebra of horizontal differential forms.

**Example 2.18** A tensor field of type \((p, q)\) \( T \) is a section of the bundle \( T^{(p,q)}J^k_0(M) \). Given a connection \( ^k\mathcal{H} \) on \( J^k_0(M) \), one can define the notions of horizontal and vertical tensors of any order.
A tensor field $T$ of type $(2, 0)$ is a section of the bundle $T^{(2,0)}J^k_0(M)$. Locally, the sections of $T^{(2,0)}J^k_0(M)$ are spanned (with coefficients on $\mathcal{F}(J^k_0(M))$) by the tensor product of elements of the frame $\{e_1(\text{k}x), \ldots, e_{n(k+1)}(\text{k}x)\}$

$$T^{(2,0)}J^k_0(M) = \text{span}\left\{ e_i(\text{k}x) \otimes e_j(\text{k}x), \ i, j = 1, \ldots, n(k+1) \right\}.$$ 

If there is defined a connection $k\hat{\mathcal{H}}$ as in Theorem 2.10, the tensor bundle of $(0, 2)$ horizontal tensors is locally spanned as

$$T^{(2,0)}hJ^k_0(M) = \text{span}\left\{ \frac{\delta}{\delta x^\mu}|_{kx} \otimes \frac{\delta}{\delta y^\nu}|_{kx}, \mu, \nu = 1, \ldots, n \right\}.$$

Similarly, the tensor bundle of $(2, 0)$ vertical tensors is locally spanned as

$$T^{(2,0)}vJ^k_0(M) = \text{span}\left\{ \frac{\partial}{\partial y^A}|_{kx} \otimes \frac{\partial}{\partial y^B}|_{kx}, \ A, B = 1, \ldots, nk \right\}.$$

The bundle over $J^k_0(M)$ of hv tensors of type $(1, 1)$ is locally spanned

$$T^{(1,1)}J^k_0(M) = \text{span}\left\{ \frac{\delta}{\delta x^\mu}|_{kx} \otimes \frac{\partial}{\partial y^A}|_{kx}, \mu = 1, \ldots, n, A = 1, \ldots, nk \right\}.$$

One can consider an horizontal $(1, 1)$ tensor, that generically will have the following expression in local coordinates

$$T(\text{k}x) = T^i_j(\text{k}x)\frac{\delta}{\delta x^i}|_{kx} \otimes \frac{\partial}{\partial y^j}|_{kx}, T^i_j(\text{k}x) \in \mathcal{F}(J^k_0(M)).$$

Horizontal $p$-forms can be spanned locally in a similar way. For instance, a 2-form horizontal can be expressed in local coordinates as

$$\omega(\text{k}x) = \omega_{ij}(\text{k}x)dx^i|_{kx} \wedge dx^j|_{kx}.$$ 

Note that horizontal forms does not depend on the specific connection $k\hat{\mathcal{H}}$ that we can choose.

The tensor product of horizontal tensors is an horizontal tensor. In a similar way, the exterior product of horizontal forms is an horizontal form. We will show below that there is a notion of horizontal exterior derivative, for which definition we need a connection on $J^k_0(M)$. We will also show that the notion of horizontal exterior derivative is connection independent.

### 2.4 Generalized forms and tensors fields

Let $J^k_0(M)$ be the $k$-jet bundle over $M$ and $\eta$ a pseudo-Riemannian metric on $M$. Let us consider a connection $k\hat{\mathcal{H}}$ on $J^k_0(M)$ as in Theorem 2.10.
Definition 2.19 Let $pr : T^{(p,q)}J^k_0(M) \to J^k_0(M)$ be a tensor bundle over $J^k_0(M)$. A $k$-tensor along the curve $k^x : I \to M$ (in short, a tensor along the curve $k^x : I \to M$) is a map $\hat{S} : I \to T^{(p,q)}J^k_0(M)$ such that for the lift $k^x : I \to M$ to $J^k_0(M)$, the following diagram commutes:

\[
\begin{array}{ccc}
T^{(p,q)}J^k_0(M) & \xrightarrow{pr} & J^k_0(M) \\
\hat{S} & & \\
I & \xrightarrow{k^x} & J^k_0(M).
\end{array}
\]

A tensor $\hat{S}$ along the curve $k^x : I \to M$ is horizontal if when acting on any arbitrary vertical vector or vertical 1-form the result is zero. A tensor $\hat{S}$ is vertical if when acting on any horizontal vector the result is zero.

One defines in a similar way differential forms along the curve $k^x : I \to M$ as a map $\hat{\omega} : M \to \Lambda^p J^k_0(M)$ such that the following diagram commutes,

\[
\begin{array}{ccc}
\Lambda^p J^k_0(M) & \xrightarrow{pr} & J^k_0(M) \\
\hat{\omega} & & \\
I & \xrightarrow{k^x} & J^k_0(M).
\end{array}
\]

Let us define the following projections:

\[
\hat{h}_k : T^{(p,q)}(J^k_0(M)) \to T^{(p,q)}h(J^k_0(M)), \quad T \mapsto \hat{h}_k(T)
\]

such that on horizontal vectors and forms $T = \hat{h}_k(T)$, but if $\hat{h}_k(T)$ acting over any vertical vector or differential 1-form is zero. Then the horizontal tensor bundle is

\[
\rho^{(p,q)} : T^{(p,q)}h(J^k_0(M)) \to J^k_0(M).
\]

In a similar way, one can define the horizontal forms bundle $\Lambda^p h(J^k_0(M))$ and the corresponding projection $pr : \Lambda^p h(J^k_0(M)) \to J^k_0(M)$. We have the following direct generalization of Proposition 2.17.

Proposition 2.20 Vertical tensors $T \in T^{(p,0)}J^k_0(M)$ are connection independent.

Proof. For vertical tensors $T \in T^{(0,q)}J^k_0(M)$ acting on horizontal 1-forms $\theta = \theta_\mu dx^\mu$ we have that

\[
T(..., \theta, ...) = T(..., \theta_\mu dx^\mu, ...) = 0
\]

and is clear that the result does not depend on the connection. \qed

The tensorial product of connection independent tensors is connection independent, thus forming the sub-algebra of connection independent tensors of the tensor algebra $(\sum_p T^{(p,0)}J^k_0(M), \otimes)$. However, similar calculations show that the notion of vertical form and horizontal contravariant tensor is not connection independent.

22
Weak definition of generalized tensors and forms

**Definition 2.21** Let \((M, \eta)\) be a pseudo-Riemannian manifold. A \(k\)-generalized \((p,q)\)-tensor \(\hat{S}\) is a map such that:

1. To each causal curve \(x : I \to M\) associates a horizontal tensor \(\hat{S} : I \to T^{(p,q)}_h J^k_0(I)\) along \(kx : I \to M\) as in definition 2.19.

2. If two lifted curves \(kx_1 : I \to J^k_0(M)\) and \(kx_2 : I \to J^k_0(M)\) intersect at the point \(kx(s_0) \in kx_1 \subset kx_2\), then the value at the intersection of the generalized tensor coincides,

\[
\hat{S}(kx_1(s_0)) = \hat{S}(kx_2(s_0)).
\]

(2.17)

A \(k\)-generalized \(p\)-form is a map such that to each timelike curve \(x : I \to M\) associates a horizontal differential form \(\hat{\omega} : I \to \Lambda^p_h J^k_0(I)\) along \(x : I \to M\).

Therefore, given a curve \(x : I \to M\) and the lift of the curve (with a prescribed initial point over the initial point \(x(s_0)\)) \(kx : I \to J^k_0(M)\) a \(k\)-generalized \((p,q)\)-tensor associates a unique section of \(T^{(p,q)}_h J^k_0(M)\) along the map \(kx : I \to J^k_0(M)\).

**Remark 2.22** We give some remarks on definition 2.21:

- The second condition in definition 2.21 is related with locality character of physical fields and generalization of the idea of local field to higher order jet bundles. The physical interpretation is that the fields are determined by the trajectories of the probe particles.

- We restrict to consider horizontal fields only. These fields have un-ambiguous interpretations in terms macroscopic flux across spatial surfaces, which is how energy-momentum variations are computed.

We will restrict our attention to causal curves \(x : I \to M\) in definition 2.21 because its relevance for physical applications.

Strong definition of generalized tensors and forms

Definition 2.21 provides a notion of tensors along lifted curves to \(J^k_0(M)\). However, this notion is not enough to define some natural operations on generalized tensors and generalized differential forms (in particular to consider an exterior derivative operator and a generalized Lie derivative). In order to be able to define such operations, one needs to define them on open neighborhoods of \(J^k_0(M)\) and on variations.

One can achieve a more convenient definition of generalized field by considering fields over lifts to \(J^k_0(M)\) of tubes in \(M\). A way to construct convenient tubes is the following. Let \(\Sigma^{n-1} \hookrightarrow M\) be a sub-manifold of \(M\) and

\[
\Delta : I \times \Sigma^{n-1} \to M,
\]

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be a congruence of curves such that the lift
\[ k\Delta := k\pi^{-1}(\Delta(I \times \Sigma^{n-1})) \]  
(2.18)
is a tube in \( J^k_0(M) \) with not intersections between the lifted curves in the sense that
\[ \Delta(\cdot, \delta_1) \cap \Delta(\cdot, \delta_2) = \emptyset. \]
Note that two curves can intersect at different times.

**Definition 2.23** Let \((M, \eta)\) be a pseudo-Riemannian manifold. A \(k\)-generalized 
\((p,q)\)-tensor \(\hat{S}\) is a map such that:

1. To each tube \(\Delta : I \times \Sigma^{n-1} \to M\) associates a horizontal tensor along the lifted 
tube \(k\Delta\), which is a map \(\hat{S} : I \to T^0_{h} J^k_0(\Delta)\).

2. If two tubes intersect, the value of the tensor \(\hat{S}\) coincides in the intersection,
\[ \hat{S}(k\Delta_1) = \hat{S}(k\Delta_2), \text{ for any } kx \in k\Delta_1 \cap k\Delta_2. \]  
(2.19)
Similarly, a \(k\)-generalized \(p\)-form is a map such that to each tube \(\Delta : I \times \Sigma^{n-1} \to M\) 
associates a horizontal differential form along the lifted \(k\Delta\), which is a map \(\hat{S} : I \to \Lambda^p J^k_0(\Delta)\).

It is clear that the set of generalized tensors form a real vector space, that we denote 
by \(T^0_{h} J^k_0(M)\) (respectively, \(\Lambda^p J^k_0(M)\)) for differential forms.

**Example 2.24** If \(k = 0\), \(\hat{S}\) is a standard tensor or form defined in a tube \(\hat{S} \in \Gamma T^0(M)\).

In the case that the manifold \((M, \eta)\) is Lorentzian, one can specify to work with 
tubes composed by causal curves. Also It is also interesting the possibility to work 
with congruences of curves, solutions of a given differential equation on \(M\). Then one needs to check that given an arbitrary point \(kx \in J^k_0(M)\), it can be surrounded 
by a tube composed by lifting to \(J^k_0(M)\) local solutions of the differential equation.

**Remark 2.25** The notion of tube in \(J^k_0(M)\) that we use assumes that there are 
non-intersections in the curves composing the tube, at fixed time. This is natural 
from the point of view of locality, that becomes even a more restrictive notion than 
in usual geometry. In this sense, one can speak about *local properties up to k-order.*

### 2.5 Tensors and forms with values on \(\mathcal{F}(J^k_0(M))\)

**Definition 2.26** A generalized tensor \(T\) of type \((p,q)\) with values on \(\mathcal{F}(J^k_0(M))\) is 
a smooth section of the bundle of \(\mathcal{F}(M)\)-linear homomorphisms
\[ T^{(p,q)}(M, \mathcal{F}(J^k_0(M))) := \text{Hom}(T^p M \times ... \times T^p M \times TM \times ... \times TM, \mathcal{F}(J^k_0(M))). \]  
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A \( p \)-form \( \omega \) with values on \( \mathcal{F}(J^k_0(M)) \) is a smooth section of the bundle of \( \mathcal{F}(M) \)-linear completely alternate homomorphisms

\[ \Lambda^p(M, \mathcal{F}(J^k_0(M))) := \text{Alt}(TM \times \cdots \times TM, \mathcal{F}(J^k_0(M))). \]

The space of 0-forms is \( \Gamma \Lambda^0(M, \mathcal{F}(J^k_0(M))) := \mathcal{F}(J^k_0(M)) \).

Given a connection \( \hat{k}^k \mathcal{H} \) on \( J^k_0(M) \), there is an alternative way to describe generalized tensors (for both weak and strong definitions of generalized tensors and forms). The connection \( \hat{k}^k \mathcal{H} \) determines the horizontal lifts

\[ T_xM \ni X \mapsto hX \in T_{\hat{x}}J^k_0(M) \]

and

\[ T^*_xM \ni \alpha \mapsto h\alpha \in T^*_{\hat{x}}J^k_0(M). \]

Then the tensor \( \hat{S} \in \Gamma T^{(p,q)}(M, \mathcal{F}(J^k_0(M))) \) defines a \( \mathcal{F}(J^k_0(M)) \)-multilineal form

\[ \hat{S} : T^*M \times \cdots \times T^*M \times TM \times \cdots \times TM \to \mathcal{F}(J^k_0(M)) \]

by the rule

\[ (\alpha_1, \ldots, \alpha_p, X_1, \ldots, X_q) \mapsto \hat{S}(h\alpha_1, \ldots, h\alpha_p, hX_1, \ldots, hX_k, \ldots, hX_q). \quad (2.20) \]

This rule determines the homomorphism

\[ \phi^{-1} : T^{(p,q)}_h(J^k_0(M)) \to T^{(p,q)}(M, \mathcal{F}(J^k_0(M))), \quad \hat{S} \mapsto \hat{S}. \]

(2.21)

There is a similar construction for generalized forms,

\[ \phi^{-1} : \Lambda^p_h(J^k_0(M)) \to \Lambda^p(M, \mathcal{F}(J^k_0(M))), \quad \hat{\omega} \mapsto \hat{\omega} \]

such that it is an homomorphism of graded algebras,

\[ \phi^{-1}(\alpha_1 \wedge \alpha_2) = \phi^{-1}(\alpha_1) \wedge \phi^{-1}(\alpha_2), \quad \alpha_i \in \Lambda^p_h(J^k_0(M)). \]

(2.22)

The relation between definition 2.26 and this new alternative definition are given by the following two results,

**Proposition 2.27** The definition of horizontal forms \( \mathcal{F}(J^k_0(M)) \)-valued is connection independent.

**Proof.** Given two equivalent connections as the specified in Theorem 2.10, for any vector field \( X \in \Gamma TM, \ h_1X - h_2X \) is vertical (similarly for the lifting of differential forms). Therefore, since the tensor \( T \) is horizontal,

\[ \hat{\omega}(h_1X_1, \ldots, h_1X_q) = \hat{\omega}(h_2X_1, \ldots, h_2X_q). \]

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Given a connection \( kH \), it is possible to define the horizontal component of forms by using the projection
\[
\hat{h}_k : \Lambda^p(J^h_0(M)) \to \Lambda^p(J^h_0(M))
\]
that associates to each form \( \hat{\omega} \in \Gamma \Lambda^p(J^h_0(M)) \) its horizontal component. Thus we have the following Proposition,

**Proposition 2.28** Fixed a connection \( kH \), the homomorphisms (2.27) and (2.22) are \( \mathcal{F}(J^h_0(M)) \)-isomorphisms.

**Proof.** Let us fix our attention in the first homomorphism (2.21). Since \( \hat{S} \in T^h_{(p,q)}(J^h_0(M)) \), it is horizontal and the fact that the horizontal lift is an injective homomorphism, we have that (2.21) is injective. To prove that it is also surjective, one can consider an arbitrary \( \hat{S} \in \Gamma T^h_{(p,q)}(M, \mathcal{F}(J^h_0(M))) \) defined at each point \( kx \) in the following way. For the horizontal \( (h\alpha_1, \ldots, h\alpha_p) \) 1-forms and horizontal \((X_1, \ldots, X_p)\) vector fields on \( M \) one has that,

\[
\hat{S}(h\alpha_1, \ldots, h\alpha_p, hX_1, \ldots, hX_q)(kx) := \hat{S}(k\pi_s(h\alpha_1), \ldots, k\pi_s(h\alpha_p), k\pi_s(hX_1), \ldots, k\pi_s(hX_q))(x);
\]

for any other vectors and forms on \( J^h_0(M) \) one has \( \hat{S}(\beta_1, \ldots, \beta_p, Y_1, \ldots, Y_q) = 0 \), if any vector \( Y \) or 1-form \( \beta \) is vertical. Then \( \hat{S} \) is horizontal and such that \( \phi(\hat{S}) = \hat{S} \). Therefore, \( \phi \) is also surjective. Note that the computations are done pointwise. The localization property follows in a similar way as in Proposition 3.1 in [41].

Because of the equivalence (2.28) we will use the two definitions of generalized field, as sometimes one could be more convenient than the other. Any section of \( \text{Hom}(T^*M \times \ldots \times T^*M \times TM \times \ldots \times TM, \mathcal{F}(J^h_0(M))) \) is defined point-wise as

\[
(\alpha_1, \ldots, \alpha_p, X_1, \ldots, X_q) \mapsto \hat{T}(h\alpha_1, \ldots, h\alpha_p, hX_1, \ldots, hX_q) \in \mathcal{F}(J^h_0(M)),
\]

that is linear in any of its arguments, for any \( x \in M \) and \( \alpha_i \in T^*_xM \), \( X_i \in T_xM \) and with \( \hat{T} \in T_{(p,q)}(J^h_0)(M) \). Similarly, any section \( \hat{\omega} \) of \( \text{Hom}(TM \times \ldots \times TM, \mathcal{F}(J^h_0(M))) \) is defined point-wise as

\[
(X_1, \ldots, X_p) \mapsto \hat{\omega}(hX_1, \ldots, hX_p) \in \mathcal{F}(J^h_0(M)),
\]

for a unique such that it is alternate for any \( x \in M \) and \( X_i \in T_xM \) and with \( \hat{\omega} \in \Lambda^p_h(J^h_0(M)) \).

The isomorphism \( \phi \) defined by (2.22) induces the injection (on the image of \( \phi \))
\[
k\zeta := \phi^{-1} : \Gamma \Lambda^p(M, \mathcal{F}(J^h_0(M))) \to \Gamma \Lambda^p_h(J^h_0(M))
\]
and is defined by the formula (2.20). In local coordinates equation (2.25) is such that

\[
k\zeta : \Gamma \Lambda^p(M, \mathcal{F}(J^h_0(M))) \to \Gamma \Lambda^p_h(J^h_0(M))
\]

\[
\theta_I(kx) \, dx^I \mapsto \theta_I(kx) \, \phi^{-1}(dx^I),
\]

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where \( I \) is a multi-index. \( k\zeta \) is well defined, independent of coordinates and is an algebra homomorphism,
\[
k\zeta(\alpha_1 \wedge \alpha_2) = k\zeta(\alpha_1) \wedge k\zeta(\alpha_2).
\]
Note that \( k\zeta \) is an isomorphism on the image. Therefore, one has that
\[
k^{-1}(\hat{\omega}_1 \wedge \hat{\omega}_2) = k^{-1}(\hat{\omega}_1) \wedge k^{-1}(\hat{\omega}_2), \quad \hat{\omega}_1, \hat{\omega}_2 \in \zeta(\Lambda^p(M, F(J^k_0(M)))) \subset \Lambda^p(J^k_0(M)).
\]

**Proposition 2.29** Let \( k\mathcal{H}_1 \) and \( k\mathcal{H}_2 \) be two connections on \( J^k_0(M) \). Then
\[
k\zeta_1 = k\zeta_2.
\]

**Proof.** Writing the homomorphisms \( k\zeta_1 \) and \( k\zeta_2 \) in local coordinate, it is clear that it does not depend on the connection: in local coordinates and using multi-index notation for a form \( \omega = \theta_I d\bar{x}^I \), \( k\zeta_i, i = 1, 2 \) are given by expressions of the form
\[
k\zeta_i(\theta_I(kx)d\bar{x}^I) = \theta_I(kx)d\bar{x}^I = k\zeta_2(\theta_I(kx)d\bar{x}^I).
\]
\( \square \)

**Corollary 2.30** The projection \( \hat{h}_k : \Lambda^p(J^k_0(M)) \to \Lambda^p_0(J^k_0(M)) \) does not depend on the connection \( k\mathcal{H}_\cdot \).

Thus, for two different connections \( k\mathcal{H}_1 \) and \( k\mathcal{H}_2 \) we have the same projection operator, \( \hat{h}^1_k(\hat{\omega}) = \hat{h}^2_k(\hat{\omega}) \).

**Remark 2.31** We have the following remarks:

- The generalized higher order fields considered are smooth sections of the generalized forms
  \[
  \bar{\alpha} \in \Gamma \Lambda^p(M, F(J^k_0(M)))
  \]
or sections of a given generalized tensor bundle
  \[
  \bar{T} \in \Gamma T^{(p,q)}(M, F(J^k_0(M))),
  \]
for some specific natural number \( k \), indicating the \( k \)-jet order where these fields take values.

- When applied to classical electrodynamics, there is the same number of horizontal degrees of freedom that in the standard electromagnetic theory.
Generalized fields can be understood as weak generalized tensor or strong generalized tensors as was described before. When one needs to speak about exterior derivatives or other differential operators acting on generalized higher order fields, they will be understood in a strong way.

The manifold $M$ is of relevance, since it is a requirement to define tensor along curves or along tubes that contain a specific $x \in M$.

The generalized tensor algebra is
\[
T^*(M, \mathcal{F}(J^{k}_0(M))) := \sum_{p,q} \oplus T^{(p,q)}(M, \mathcal{F}(J^{k}_0(M))),
\]
where the product is induced from the tensor product on
\[
T^*(J^{k}_0(M)) = \sum_{p,q} \oplus T^{(p,q)}(J^{k}_0(M)).
\]

In a similar way, one defines the generalized exterior algebra,
\[
\Lambda(M, \mathcal{F}(J^{k}_0(M))) := \sum_{p=0}^{n} \oplus \Lambda^p(M, \mathcal{F}(J^{k}_0(M))),
\]
where the exterior product is induced from exterior product of the standard exterior algebra over $M$,
\[
\Lambda(J^{k}_0(M)) = \sum_{p=0}^{n} \oplus \Lambda^p(J^{k}_0(M)).
\]

Therefore, the product in the exterior algebra is defined as
\[
\alpha^{(k)x} \wedge \beta^{(k)x} = (\alpha I^{(k)x}e^I(x)) \wedge (\beta J^{(k)x}e^J(x)) := \alpha I^{(k)x} \beta J^{(k)x} e^I(x) \wedge e^J(x).
\]

Given a local frame $\{e_I(x), I = 1, ..., \dim(\Lambda^p(M))\}$ for $\Lambda^p(M)$, the homomorphism implies that a local frame for $\Lambda^p(M, \mathcal{F}(J^{k}_0(M)))$ is obtained as the linear closure of $\{e_I(x), I = 1, ..., \dim(\Lambda^p(M))\}$ with coefficients in $\mathcal{F}(J^{k}_0(M))$.

Given a $p$-form $\alpha \in \Lambda^p M$, there is a form $\varphi(\alpha) \in \Lambda^p(M, \mathcal{F}(J^{k}_0(M)))$ such that for $(X_1, ..., X_p) \subset \Gamma TM$ is defined by
\[
\varphi(\alpha)(X_1, ..., X_p) = \alpha(X_1, ..., X_p).
\]

Therefore, one can establish the homomorphism of vector spaces
\[
\varphi : \Lambda^p M \to \Lambda^p(M, \mathcal{F}(J^{k}_0(M))), \quad \varphi(\alpha)_u(X_1, ..., X_p) = \alpha_x(X_1, ..., X_p).
\]

The value $\varphi(\alpha)_u(X_1, ..., X_p)$ is constant along the fiber $\forall u \in \pi^{-1}(x)$. 

28
2.6 Generalized metric structures

A relevant type of generalized tensor that we consider in some detail are generalized metrics. For the purposes of this work a generalized metric \( \hat{g} \) will be a weak generalized higher order tensor.

**Definition 2.32** A generalized metric \( \hat{g} \) is a section \( \hat{g} \in \Gamma T_{h}^{(0,2)}(J_{0}^{k}(M)) \), a week generalized tensor such that:

1. It is smooth: given a curve \( x: I \to M \) and a lift \( kx: I \to J_{0}^{k}(M) \), for any two smooth vector fields \( \hat{X}_{1}, \hat{X}_{2} \) along \( kx: I \to J_{0}^{k}(M) \), the function \( \hat{g}(kx)(\hat{X}_{1}, \hat{X}_{2}): I \to \mathbb{R} \) is smooth, except if \( \hat{g}(kx)(\hat{X}_{1}, \hat{X}_{2}) = 0 \).

2. It is homogeneous of degree zero in the following sense: if the lift \( kx: I \to M \) has local coordinates \((x^{\mu}(s), \dot{x}^{\mu}(s), \ddot{x}^{\mu}(s)...)\), then

\[
\hat{g}(x^{\mu}(s), \lambda_{1}\dot{x}^{\mu}(s), \lambda_{2}^{2}\ddot{x}^{\mu}(s) + \lambda_{3}\dddot{x}^{\mu}(s) + 3\lambda_{2}\lambda_{3}\ddot{x}^{\mu}(s) + \lambda_{3}^{3}\dddot{x}^{\mu}(s)\) = \hat{g}(x^{\mu}(s), \dot{x}^{\mu}(s), \ddot{x}^{\mu}(s), ...x^{(k)}(s))(X, X)
\]

for all \( \hat{X} \) along \( x: I \to M \) and \( \lambda_{i} > 0, i = 1, ..., k \).

3. It is a symmetric form in the sense that \( \hat{g}(kx)(\hat{X}_{1}, \hat{X}_{2}) = \hat{g}(kx)(\hat{X}_{2}, \hat{X}_{1}) \) for any smooth pair of vector fields \( \hat{X}_{1}, \hat{X}_{2} \) along \( x: I \to M \).

4. It is bilinear in the sense that

\[
\hat{g}(kx)(\hat{X}_{1} + f(kx)\hat{X}_{2}, \hat{X}_{3}) = \hat{g}(kx)(\hat{X}_{1}, \hat{X}_{3}) + f(kx)\hat{g}(kx)(\hat{X}_{1}, \hat{X}_{3}), \quad (2.30)
\]

with \( \hat{X}_{1}, \hat{X}_{2}, \hat{X}_{3} \) vectors fields along the curve \( kx: I \to J_{0}^{k}(M) \) and \( f \in \mathcal{F}(J_{0}^{k}(M)) \).

5. It is weak non-degenerate in the following sense: if the condition

\[
\hat{g}(kx)(\hat{X}, \hat{Z}) = 0, \quad (2.31)
\]

holds for any horizontal smooth vector field \( \hat{Z} \) along any curve \( kx: I \to J_{0}^{k}(M) \), then the horizontal field \( \hat{X} = 0 \).

**Remark 2.33** The homogeneity condition implies that \( \hat{g}(kx)(\hat{X}, \hat{Z}) \) is invariant under the action of the chain rule \( \left( \frac{d}{ds} \right)^{k} = \left( \frac{d}{dp} \frac{d}{dp} \right)^{k} \). Therefore, it is invariant under positive parameterization of the curve \( x: I \to M \).

Given a generalized metric, there is defined a generalized bilinear form \( g_{u} \) at the point \( u \in \pi^{-1}(x) \),

\[
\hat{g}_{u}: T_{x}M \times T_{x}M \to \mathbb{R}, \quad (X_{1}, X_{2}) \mapsto g_{u}(h_{1}X_{1}, h_{2}X_{2})
\]
such that it is positive homogeneous in the sense that \( g_u(ax, ay) = a^2 g(y, y) \) for any \( a \in \mathbb{R}^+ \) and symmetric. Note that \( \hat{g} \) does not depend on the particular non-linear connection. Then following Proposition 2.28, it is possible to interpret a generalized metric \( \mathcal{F}(J_0^k(M)) \) as the bilinear form
\[
\tilde{g} : TM \times TM \to R, \quad (X_1, X_2) \mapsto \hat{g}_u(h X_1, h X_2).
\] (2.32)

**Proposition 2.34** The generalized metric \( \hat{g} \in \Gamma T^{(0,2)} h J_0^k(M) \) has associated a unique section \( \bar{g} \in \Gamma T^{(0,2)}(M, \mathcal{F}(J_0^k(M))) \) such that conditions 1 to 5 in definition ?? hold.

1. \( \bar{g} \) is smooth in the sense that for all \( X_1, X_2 \) smooth vector fields along the curve \( x : I \to M \), the function \( \bar{g}(X_1, X_2) \) is smooth except when it takes the zero value.

2. It is homogeneous of degree zero: if \( k x : I \to M \) has local coordinates \( (x^\mu(s), \dot{x}^\mu(s), \ddot{x}^\mu(s), ...) \), then
\[
\bar{g}(x^\mu(s), \lambda_1 \dot{x}^\mu(s), \lambda_2 \ddot{x}^\mu(s) + 3 \lambda_2 \lambda_1 \ddot{x} + \lambda_3 \dddot{x} + ...)(X, X) = \hat{g}(x^\mu(s), \dot{x}^\mu(s), \ddot{x}(s), ... x^{(k)}(s))(X, X)
\]
for all \( X \in T_x M \) and \( \lambda_i > 0, i = 1, ..., k \).

3. \( \bar{g} \) is symmetric, in the sense that
\[
\bar{g}(X_1, X_2) = \bar{g}(X_2, X_1)
\]
for all \( X_1, X_2 \) smooth vector fields along the curve \( x : I \to M \).

4. It is bilinear,
\[
\bar{g}(X_1 + f(k x) X_2, X_3) = \bar{g}(X_1, X_3) + f(k x)\bar{g}(X_2, X_3),
\]
for all \( X_1, X_2, X_3 \) arbitrary smooth vector fields along \( x : I \to M \) and \( f \in \mathcal{F}(M) \).

5. It is non-degenerate, in the sense that if \( \bar{g}(X, Z) = 0 \) for all \( Z \) smooth along \( x : I \to M \), then \( X = 0 \).

**Proof.** By Proposition 2.28 it is enough to show that the above properties follow one by one from the corresponding properties in definition 2.32. Let us check the non-degeneracy condition. The other properties follow using similar calculations. If \( \hat{g} \) is non-degenerate, then \( \hat{g}(X, Z) = \hat{g}(h X, h Z) = 0 \) for all \( Z \in \Gamma TM \) implies that, since \( h : T_x M \to \mathcal{H} J_0^k(x) \) is an isomorphism of real vector spaces, \( X(x) = 0 \) at any \( x \in M \). \( \square \)
From the bilinear property it follows that locally the generalized metric can be written as
\[ \bar{g} = g_{\mu\nu} d^4x^\mu \otimes d^4x^\nu. \]

The forms \( \{ d^4x^\mu, \mu = 1, ..., n \} \) act linearly on vectors and define an adequate basis.

**Example 2.35** We can consider the following examples of generalized metrics:

- A relevant example that will appear in later sections is a generalized metric of the form
  \[ \lambda \bar{g} = \lambda(kx) \eta(x), \quad (2.33) \]
  where \( \lambda \) is a positive, homogeneous of degree zero element of \( \mathcal{F}(J^0_k(M)) \) for some fixed positive integer \( k \), \( \eta \) is the Minkowski metric and \( kx : I \to J^0_k(M) \) the lift of the curve \( x : I \to M \).

- Generalized metrics are provided by Finsler structures \([1, 2, 3]\). In this case the vertical hessian \( g_{ij}(u) \) of the Finsler function \( F \) is smooth on \( TM \setminus \{0\} \). This example is of positive signature.

Of particular interest for the considerations in this paper are the generalized metrics of the form \((2.33)\),
\[ g(u) = \lambda(u) \eta(x), \quad x = k\pi(u) \subset J^0_k(M). \]
where \( \eta \) is a Lorentzian metric on \( M \). The exact form of the conformal factor \( \lambda \) will be elucidated later in the context of geometries of maximal covariant acceleration. Therefore, the bilinear form \( g \) is not a (pseudo)-Riemannian metric, since \( \lambda \) is not a function on \( M \) but a function on \( J^2_0(M) \). In particular, we assume that \( \lambda \in \mathcal{F}(J^2_0(M)) \) is invariant under local transformations of coordinates on \( M \).

**Definition 2.36** Fixed a curve \( x : I \to M \), the signature of the generalized metric \( \bar{g} \) at \( u = kx(s) \) is the signature of the symmetric bilinear metric \( \bar{g}_u : T_xM \times T_xM \to \mathbb{R} \).

Thus the signature of a generalized metric will depend on the curve \( kx : I \to M \) where it is evaluated. This implies a natural classification of curves \( \mathcal{F}(I, J^0_k(M)) \) with the corresponding curves have the same signature: fixed a curve, \( x : I \to M \), we denote by the signature sector associated with \( x : I \to M \) to the set of all the generalized metrics with the same along corresponding curves \( z : I' \to M \) with the same signature \( \bar{g} \) on \( x : I \to M \).

\(^5\)In order to simplify the treatment and the calculations, we are considering flat spacetime in the sense that \( \eta \) is flat. Therefore, \( \eta \) is Minkowski and \( M \) is a flat domain of \( \mathbb{R}^n \) of dimension \( n \). In physical applications the action of the Lorentz group as transformation group is the standard one. For \( n = 4 \), the factor \( \lambda \) will be proved that is Lorentz invariant.
Example 2.37 If one considers the generalized metric $\lambda \hat{g}$ (2.33) in the first case in the example 2.35: if $\lambda$ is positive, the metric (2.33) has the same signature than the metric $\eta$.

Given a curve $x : I \to M$ and a vector field $W$ along the curve, if the signature of $g$ is $(-1, 1, \ldots, 1)$ one can define a bilinear, positive definite, symmetric form

$$g_+ \in \Gamma T^{(0,2)}(M, \mathcal{F}(J^k_0(M)))$$

such that along the curve

$$g_+(X, X)(u) = g(X, X)(u) - 2 \frac{g^2(X, W)}{g(W, W)}(u), \quad g(W, W) < 0,$$  

(2.34)

with $u = k_x(s) \in J^k_0(M)$, $X \in \Gamma TM$. For a generalized metric of the form $g = \lambda \eta$, the bilinear, positive form (2.34) is

$$g_+(X, X)(u) = \lambda(u) \eta_+(X, X)(x), \quad g(W, W) < 0, \quad u \in J^k_0(M), \quad X \in \Gamma TM,$$

with

$$\eta_+(X, X)(x) = \eta(X, X)(x) - 2 \frac{g^2(X, W)}{\eta(W, W)}(x).$$

Both generalized Riemannian metrics $\eta_+$ and $g_+$ can be used to provide a definition of length along the curve $x : I \to M$.

2.7 Group of isometries of a generalized metric

Let $\psi : J^k_0(M) \to J^k_0(M)$ be a diffeomorphism of $M$ and denote the tangent map by $\psi_* : TJ^k_0(M) \to TJ^k_0(M)$. Then one can define the following homomorphism of the tangent bundle $TJ^k_0(M)$,

$$\hat{\psi}_x : T_x M \to T_{k\pi(\psi(u))}(M), \quad X \mapsto k_{\pi}(\psi_*)|_u \circ k_X,$$

with $k_{\pi}(u) = x$ and $k_{\pi}, \psi_*$ the differential maps.

Definition 2.38 An isometry of a generalized metric $\bar{g}$ is a diffeomorphism $\psi : J^k_0(M) \to J^k_0(M)$ such that on any arbitrary curve $x : I \to M$ it holds that

$$\hat{g}_{k_x}(X_1, X_2) = \hat{g}_{\psi(k_x)}(\hat{\psi}_x(X_1), \hat{\psi}_x(X_2)),$$  

(2.35)

for any two vector fields along $x : I \to M$.

The following Proposition follows easily from the definition 2.38.

Proposition 2.39 The group of isometries of a generalized metric does not depend on the non-linear connection $k\mathcal{H}$ used in the definition of the corresponding diffeomorphisms.
Proof. Since a generalized metric can be locally written as \( \bar{g} = \bar{g}_{\mu\nu}(kx) dx^\mu \otimes dx^\nu \), it is clear that the isometry group does not depend on the connection. \( \square \)

Remark 2.40 The set of isometries \( Iso(\bar{g}) \) of the generalized metric \( \bar{g} \) is a subgroup of the group of diffeomorphism \( Diff(J^k_0(M)) \). However, it is not so clear if it is a Lie group.

Example 2.41 We can consider the isometry groups of the following generalized metrics:

- Isometries of the metric of type \( \lambda \bar{g}(kx) = \lambda (kx)\eta(x) \). Let \( Iso(\lambda \bar{g}) \) be the group of isometries of \( g \), and let \( iso(\lambda) \) the group of diffeomorphisms
  
  \[
  iso(\lambda) := \left\{ \phi : J^k_0(M) \to J^k_0(M), \text{ s.t. } \lambda \circ \phi = \lambda \right\}.
  \]

The isometry group of \( \eta \) is \( Iso(\eta) \). Then one can write

\[
Iso(\lambda \bar{g}) = (Iso(\eta) \cap iso(\lambda)) \times D
\]

where \( D \) is the multiplicative group of generalized dilatations,

\[
D := \left\{ f : J^k_0(M) \to J^k_0(M), \text{ s.t. } \lambda \to f^{-1} \lambda, \eta \to f \eta \right\}.
\]

It is clear that \( iso(\lambda) \subset Iso(\eta) \) and that it is a subgroup of \( Iso(\eta) \).

In the particular case when \( \lambda \) is invariant under \( Iso(\eta) \), one has that

\[
Iso(\eta) \cap iso(\lambda) = Iso(\eta).
\]

Thus in this case the isometry group is

\[
Iso(\lambda \bar{g}) = Iso(\eta) \times D.
\]

In general, the group of generalized dilatations \( D \) is not necessarily a Lie group of transformations of \( M \). However, if \( D \) is a Lie group, the full group \( Iso(\eta) \times D \) is a Lie group too.

- Isometries of a Finsler metric. In this case, the group of isometries is a Lie group \([23]\). In particular, the group of transformations that preserve the Finsler function \( F \) (see for instance \([2, 3]\) for standard notation in Finsler geometry) coincides with the group of transformations that leave the Finslerian distance invariant. They also prove that the group of isometries is a differentiable manifold and a Lie group.

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Let us consider the pull-back bundle
\[
\begin{array}{ccl}
k\pi^*TM & \overset{\rho_2}{\longrightarrow} & TM \\
\rho_1 \downarrow & & \downarrow \pi \\
J^k_0(M) & \overset{k\pi}{\longrightarrow} & M
\end{array}
\]

Then a generalized metric \( g(u) \) (without "hat" from now on) can be understood as a metric on the fiber \( \rho_1^{-1}(u) \subset k\pi^*TM \). Each fiber is the pull-back is a vector space generated by the pull-back of the local frame on \( M \)
\[
\{k\pi^*_u(\tilde{e}_1(x)), \ldots, k\pi^*_u(\tilde{e}_n(x)), \tilde{e}_i(x) \in \Gamma TM \}.
\]

In such local frame for the pull-back fibers, the fiber metric \( h \) has components
\[
\begin{align*}
g(u)_{\mu\nu} &:= g_{\mid u}(k\pi^*_u(\tilde{e}_\mu(x)), k\pi^*_u(\tilde{e}_\nu(x))).
\end{align*}
\]

One can diagonalize the metric component matrix \( g_{\mu\nu}(u = kx(x)) \) along \( x : I \to M \) using the Gram-Schmidt’s procedure at each point. The difference with the Gram-Schmidt for pseudo-Riemannian manifolds is that the matrix of the transformation lives on \( J^k_0(M) \). In such orthonormal frame
\[
\begin{align*}
\{\rho_2(k\pi^*_u(\tilde{e}_1(x))), \ldots, \rho_2(k\pi^*_u(\tilde{e}_n(x)))\}
\end{align*}
\]
along the curve \( x : I \to M \) such that the generalized metric \( g(x(s)) \) is diagonal,
\[
g_u(e_\mu, e_\nu) := (g(u))_{\mu\nu} = \delta_{\mu\nu}.
\]

The inverse of the matrix \( g(u)_{\mu\nu} \) has components \( g(u)^{\mu\nu} \) and determines an element of \( \Gamma T^{(2,0)}(M, \mathcal{F}(J^k_0(M))) \) which is non-degenerate and symmetric. The raising and lowering indices operations are defined using \( g^{-1} \) and \( g \) (if anything else is specified). \( g^{-1} \) denotes the inverse matrix components of \( g \). Thus, if we fix a frame on \( M \), one obtains \( g^{-1} = (g^{-1})^{\mu\nu} e_\mu \otimes e_\nu \) such that \((g^{-1})^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu \).

The generalized metric \( g \) determines the isomorphism,
\[
g : \Gamma T^{(1,0)}(M, \mathcal{F}(M, J^k_0(M))) \to \Gamma T^{(0,1)}(M, \mathcal{F}(M, J^k_0(M))), \quad X \mapsto \tilde{X} := g(X, \cdot).
\]

In similar way, the generalized tensor \( g^{-1} \) determines the following canonical isomorphism,
\[
g^{-1} : \Gamma \Lambda^1(M, \mathcal{F}(M, J^k_0(M))) \to \Gamma T^{(1,0)}(M, \mathcal{F}(M, J^k_0(M))), \quad \omega \mapsto g^{-1}(\omega, \cdot).
\]
and a lowering index operation
\[ \kappa : \Gamma \Lambda^2(M, \mathcal{F}(J_0^k(M))) \to \Gamma T^{(1,1)}(M, \mathcal{F}(J_0^k(M))), \quad \kappa(\alpha)(\omega, X) := \alpha(g^{-1}(\omega, \cdot), X), \]
with \( X \in \Gamma TM, \alpha \in \Gamma \Lambda^2(M, \mathcal{F}(J_0^k(M))), \omega \in \Gamma \Lambda^1(M, \mathcal{F}(J_0^k(M))). \)

### 2.9 A generalized star operator

In order to introduce a star operator \( \star_g \) associated with the generalized metric \( g \), we will use local coordinate expressions (see for instance [35]). The generalized metric \( g \) determines a star operator \( \star_g \) acting on the algebra \( \Lambda^p(M, \mathcal{F}(J_0^k(M))) := \bigoplus_{p=0}^{n} \Lambda^p(M, \mathcal{F}(J_0^k(M))) \) in the following way. Let \( \{ e^\mu(u), \mu = 1, \ldots, n \} \) be a local, orthonormal frame respect of \( g \) defined as before. The Levi-Civita symbol is denoted by \( \epsilon_{\mu_1 \cdots \mu_n} \). Then the \( \star_g \) operator of the algebra \( \Lambda^p(M, \mathcal{F}(J_0^k(M))) \) is the \( \mathcal{F}(J_0^k(M)) \)-multilineal map determined by the image on the elements \( e_{\mu_1} \wedge \cdots \wedge e_{\mu_p} \),
\[ \star_g : \Gamma \Lambda^p(M, \mathcal{F}(J_0^k(M))) \to \Gamma \Lambda^{n-p}(M, \mathcal{F}(J_0^k(M))) \]
\[ (e_{\mu_1}^{(k)} x) \wedge \cdots \wedge e_{\mu_p}^{(k)} x) \mapsto \epsilon_{\nu_1 \cdots \nu_n} g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} e_{\nu_{p+1}}^{(k)} x \wedge \cdots \wedge e_{\nu_n}^{(k)} x \]
and then it is extended to an arbitrary generalized form
\[ \alpha = \alpha_{\mu_1 \cdots \mu_p}^{(k)} x e_{\mu_1}^{(k)} x \wedge \cdots \wedge e_{\mu_p}^{(k)} x \in \Gamma \Lambda^p(M, \mathcal{F}(J_0^k(M))) \]
by the multilineal property [35],
\[ \star \alpha^{(k)} x = \alpha e_{\nu_1 \cdots \nu_n}^{(k)} x g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p} x e_{\nu_{p+1}}^{(k)} x \wedge \cdots \wedge e_{\nu_n}^{(k)} x. \]

By direct computation in an orthogonal frame, one can prove the following generalization of the Hodge star operator as in standard Riemannian geometry,

**Proposition 2.42**
\[ \star \star \alpha = (-1)^{p(n-1)+s(g)} \alpha, \quad \alpha \in \Gamma \Lambda^p(M, \mathcal{F}(J_0^k(M))), \]
where \( s(g) \) is the signature of the generalized metric \( g \).

If \( M \) is a four dimensional manifold, the \( \star_g \) operator on 2-forms is invariant under conformal transformations of \( g \). Therefore, when acting on an ordinary differential 2-form on \( M \), the operators \( \star_g \) and \( \star_\eta \) determined by \( g = \lambda \eta \) and \( \eta \) respectively coincide: if \( \{ \tilde{e}^\nu, \nu = 1, \ldots, 4 \} \) is a local orthonormal dual frame for \( \eta \) and \( \{ e^\nu, \nu = 1, \ldots, 4 \} \) for \( \eta \).
1,...,4} is a local orthonormal dual frame for $g$, the relation between both is the conformal relation

$$\tilde{e}^\nu = \lambda^{-1} e^\nu, \nu = 1,...,4.$$ 

Thus, one has that

$$\star_g \alpha(kx) = \alpha_{\mu_1 \mu_2} (kx) \epsilon_{\nu_1...\nu_4} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} e^{\nu_3} (kx) \wedge e^{\nu_4} (kx)$$

$$= \alpha_{\mu_1 \mu_2} (kx) \epsilon_{\nu_1...\nu_4} \lambda^{-1} \eta^{\mu_1 \nu_1} \lambda^{-1} \eta^{\mu_2 \nu_2} \lambda \tilde{e}^{\nu_3} (kx) \wedge \lambda \tilde{e}^{\nu_4} (kx)$$

$$= \alpha_{\mu_1 \mu_2} (kx) \epsilon_{\nu_1...\nu_4} \eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} \tilde{e}^{\nu_3} (kx) \wedge \tilde{e}^{\nu_4} (kx) = \star_\eta \alpha(kx).$$

### 2.10 Length and proper time associated with generalized metrics

Generalized metrics do not have well defined locally signature: given two tangent vectors $X_1, X_2 \in T_xM$, the value of $g(X_1, X_2)$ will depend on the curve along it is evaluated and therefore, the notion of signature depends at each point $x$ of the given curve. Therefore, it is interesting to restrict to sectors of curves, where the metric $g$ has a well defined signature along all the curves. The main two relevant types of sectors are the following:

- If the generalized tensor $g$ is positive definite, a distance function on $M$ is defined in the following way. Let $x : [\sigma_1, \sigma_2] \to M$ be a smooth curve. Then the length $L[x]$ is

$$L[x] = \int_{\sigma_1}^{\sigma_2} \sqrt{g(x, \frac{dx}{d\sigma}, \frac{dx}{d\sigma},...,)} \left( \frac{dx(\sigma)}{d\sigma}, \frac{dx(\sigma)}{d\sigma} \right) d\sigma. \quad (2.43)$$

**Definition 2.43** Let $M$ be a connected manifold. The distance function between two points $p, q \in M$ is defined to be

$$d(p, q) = \inf \{ L[x], x : I \to M, \quad \text{s.t.} \quad x(\sigma_1) = p, \ x(\sigma_2) = q \}. \quad (2.44)$$

Given a point $p$ and a submanifold $N \hookrightarrow M$, the distance between $p$ and $N$ is

$$d(p, N) := \inf \{ d(p, q), q \in N \}. \quad (2.45)$$

The notions of boundness, metric balls, completeness etc... can be extended from Riemannian geometry to the category of generalized metrics in a direct way.

- If $g$ is of Lorentzian signature, some of the standard notions of causal curves can be translated in complete analogy from Lorentzian geometry [b]. For instance, there is a well defined notion of causal curve:
A vector field \( X \in \Gamma TM \) is timelike if \( g(X, X) < 0 \). A curve \( x : I \to M \) is timelike if the tangent vector field is timelike.

A vector field is lightlike if \( g(X, X) = 0 \); a curve on \( M \) is lightlike if the tangent vector field is lightlike.

A vector field \( X \in \Gamma TM \) is spacelike if \( g(X, X) > 0 \). A curve on \( M \) is spacelike if the tangent vector field is spacelike.

The proper-time of a non spacelike curve is

\[
L[x] := \int_{\sigma_1}^{\sigma_2} \sqrt{-g_{x(\sigma)}(\frac{dx}{d\sigma}, \frac{dx}{d\sigma})} d\sigma.
\]  

(2.46)

The Lorentzian distance between two causal connected points (that be connected by a non spacelike curve) is

\[
d(p, q) = \sup \{ L[x], x(\sigma_1) = p, x(\sigma_2) = q \}.
\]  

(2.47)

The proper-time parameter along a non spacelike curve \( x : [\sigma_1, \sigma_2] \to M \) is the function

\[
\tau[\xi] := \int_{\sigma_1}^{\xi} \sqrt{-g_{x(\sigma)}(\frac{dx}{d\sigma}, \frac{dx}{d\sigma})} d\sigma.
\]  

(2.48)

The notion of length of a curve in the positive case and of proper time in the Lorentzian case have geometric meaning.

**Proposition 2.44** Given a generalized metric structure \((M, g)\), the Weierstrass functional acting on an arbitrary curve \( x : [\sigma_1, \sigma_2] \to M \),

\[
W[x] := \int_{\sigma_1}^{\sigma} \sqrt{|g_{x(\sigma)}(\frac{dx}{d\sigma}, \frac{dx}{d\sigma})|} d\sigma,
\]  

(2.49)

is independent of the parametrization.

**Proof.** It is a consequence of the homogeneity condition of \( g \). \( \square \)

Given a non spacelike curve \( x : I \to M \), the proper time function \( \tau[x](\sigma) \) (or arc-length function \( L[x](\sigma) \) in the positive case) depends only on the point \( x(\sigma) \) and the initial point \( x(\sigma_1) \). Therefore, one can use \( \tau(\sigma) \) as a parameter of the curve.

**Example 2.45** We can consider the following examples:

- Let us consider the generalized metric \( g(\kappa x) = \lambda(\kappa x)\eta(x) \), with \( \eta \) a Lorentzian metric. Then if \( \lambda \) is re-parametrization invariant, \( g \) is re-parametrization invariant too and the proper-time (or arc-length) is also re-parametrization invariant.
The fundamental tensor of a Finsler spacetime structure \([4]\) is homogeneous of degree zero and is defined the expression
\[
g_{\mu\nu} = \frac{\partial^2 L(x,y)}{\partial y^\mu \partial y^\nu}.
\] (2.50)

When one evaluates the tensor \(g\) on each curve \(x : I \rightarrow M\) it determines a generalized metric \(g(x, \dot{x})\). Also, it is clear that the proper time parameter is reparameterization invariant.

**Example 2.46** Let \(g\) be a generalized metric of signature \((-1,1,1,1)\). For generalized metrics of the type \(g = \lambda \eta\), with \((M, \eta)\) a geodesic complete Lorentzian spacetime and \(0 < \lambda\) bounded on \(M\), the causal theory of \(\eta\) and \(g\) are related. For instance, it is not difficult to see that the notions of global hyperbolicity for \((M, g)\) and \((M, \eta)\) coincide. A notion of time orientable is natural in this case.

The investigation of the critical points of the functional (2.49) can be performed by standard techniques of analysis. The critical points of (2.49) provide important information on the local and global properties of geometric spaces. In addition, for specific generalized metrics (for instance, for metrics of maximal acceleration studied later), the signature sector is the same for all the geodesics.

### 2.11 Cartan calculus for generalized forms

In this subsection we generalize the fundamental notions of the Cartan calculus to the algebra of generalized differential forms viewed as sections of \(\Lambda^p(M, \mathcal{F}(J^k_0(M)))\).

#### The interior product homomorphism for generalized forms

The interior product of the exterior algebra \(\Lambda M\) is denoted by
\[
\iota_X : \Gamma \Lambda^p M \rightarrow \Gamma \Lambda^{p-1} M, \quad \Lambda^p M \ni \alpha \mapsto \iota_X \alpha \in \Lambda^{p-1} M,
\]
with \(X \in \Gamma TM\) and defined by the relation
\[
(\iota_X \alpha)(X_1, \ldots, X_{p-1}) := \alpha(X, X_1, \ldots, X_{p-1}), \quad X, X_1, \ldots, X_{p-1} \in \Gamma TM.
\]

The interior product is an \(F(M)\)-endomorphism
\[
\iota_X : \Gamma \Lambda^p(M, \mathcal{F}(J^k_0(M))) \rightarrow \Gamma \Lambda^{p-1}(M, \mathcal{F}(J^k_0(M))), \quad \tilde{\alpha} \mapsto {^k \zeta^{-1}} \iota_{\bar{X}}^k \zeta \tilde{\alpha}. \tag{2.51}
\]

**Proposition 2.48** The interior product \(\iota_X\) has the following properties,
1. It is linear,
\[ \iota_X(\bar{\alpha}_1 + f(x)\bar{\alpha}_2) = \iota_X \bar{\alpha}_1 + f(x)\iota_X f(x)\bar{\alpha}_2, \quad \bar{\alpha}_i \in \Gamma \Lambda^p J^k_0(M), \quad f \in \mathcal{F}(M), \]

2. For every \( X \in \Gamma TM \), \( \iota_X \circ \iota_X \bar{\alpha} = 0 \), \( \forall \bar{\alpha} \in \Gamma \Lambda^p(M, \mathcal{F}(J^k_0(M))) \).

3. It is an skew-derivation; for any \( \bar{\alpha}_i \in \Gamma \Lambda^p_i(M, \mathcal{F}(J^k_0(M))) \), \( i = 1, 2 \),
\[ \iota_X (\bar{\alpha}_1 \wedge \bar{\alpha}_2) = \iota_X \bar{\alpha}_1 \wedge \bar{\alpha}_2 + (-1)^{p_1} \bar{\alpha}_1 \wedge \iota_X \bar{\alpha}_2. \]

4. It is zero when acting on sections of \( \Gamma \Lambda^0(M, \mathcal{F}(J^k_0(M))) \).

**Proof.** For the linearity property, the proof is a direct computation:
\[
\iota_X(\bar{\alpha}_1 + f(x)\bar{\alpha}_2) = k_\zeta^{-1} \iota_{h_X} k_\zeta(\bar{\alpha}_1 + f(x)\bar{\alpha}_2)
\]
\[
= k_\zeta^{-1} \iota_{h_X} (k_\zeta \bar{\alpha}_1 + k_\zeta f(x)k_\zeta \bar{\alpha}_2)
\]
\[
= k_\zeta^{-1} (\iota_{h_X} k_\zeta \bar{\alpha}_1 + k_\zeta f(x)k_\zeta \bar{\alpha}_2)
\]
\[
= k_\zeta^{-1} \iota_{h_X} k_\zeta \bar{\alpha}_1 + k_\zeta^{-1} k_\zeta f(x)k_\zeta^{-1} \iota_{h_X} k_\zeta \bar{\alpha}_2
\]
\[
= \iota_X \bar{\alpha}_1 + f(x)\iota_X f(x)\bar{\alpha}_2.
\]

For the second property follows from the standard interior product,
\[
\iota_X \iota_X \bar{\alpha} = k_\zeta^{-1} \iota_{h_X} k_\zeta k_\zeta^{-1} \iota_{h_X} k_\zeta \bar{\alpha}
\]
\[
= k_\zeta^{-1} \iota_{h_X} k_\zeta \bar{\alpha}
\]
\[
= 0.
\]

For the third property, if \( \bar{\alpha} \in \Lambda^{p_1}(M, \mathcal{F}(J^k_0(M))) \),
\[
\iota_X(\bar{\alpha}_1 \wedge \bar{\alpha}_2) = k_\zeta^{-1} \iota_{h_X} k_\zeta \bar{\alpha}_1 \wedge \bar{\alpha}_2
\]
\[
= k_\zeta^{-1} \iota_{h_X} \left(k_\zeta \bar{\alpha}_1 \wedge k_\zeta \bar{\alpha}_2 + (-1)^{p_1} k_\zeta \bar{\alpha}_1 \wedge \iota_{h_X} k_\zeta \bar{\alpha}_2\right)
\]
\[
= \iota_X \bar{\alpha}_1 \wedge \bar{\alpha}_2 + (-1)^{p_1} \bar{\alpha}_1 \wedge \iota_X \bar{\alpha}_2.
\]

Similarly, the proof of the last property is a direct calculation: if \( \bar{\alpha} \in \Lambda^{0}(M, \mathcal{F}(J^k_0(M))) \),
\[
\iota_X \bar{\alpha} = k_\zeta^{-1}(\iota_{h_X} k_\zeta \bar{\alpha}) = k_\zeta^{-1}(0) = 0.
\]

\[\square\]

**Proposition 2.49** The definition of the interior products (2.47) is connection independent.
Proof. Given two horizontal distributions $k\hat{\mathcal{H}}_1$ and $k\hat{\mathcal{H}}_2$ as in Theorem 2.10, there are two definitions of interior products of generalized forms,

$$ 1_{tX} \bar{\alpha} = k^2 \zeta^2 \iota_1 k \bar{\zeta} \alpha, \quad 2_{tX} \bar{\alpha} = k^2 \zeta^2 \iota_1 k \bar{\zeta} \alpha. $$

Comparing the two interior products,

$$ 1_{tX} \bar{\alpha} - 2_{tX} \bar{\alpha} = k^2 \zeta^2 \iota_1 k \bar{\zeta} \alpha - k^2 \zeta^2 \iota_1 k \bar{\zeta} \alpha $$

Using local coordinates, one can see that $k^2 \zeta^2 - 1 = Id$. Therefore,

$$ 1_{tX} \bar{\alpha} - 2_{tX} \bar{\alpha} = k^2 \zeta^2 \iota_1 k \bar{\zeta} \alpha - \iota_1 k \bar{\zeta} \alpha $$

On the other hand, $h_1 X - h_2 X = V$ is vertical (the easiest way to check this is using local coordinates). Then one gets

$$ \hat{\alpha}(h_2 X, \cdot) - \hat{\alpha}(h_1 X, \cdot) = 0. $$

\[\square\]

The exterior covariant derivative for generalized forms

The generalization of the exterior differential operator $d$ on smooth forms over $M$

$$ d : \mathcal{A}^p M \to \mathcal{A}^{p+1} M $$

to sections of $\mathcal{A}^p (M, \mathcal{F}(J_0^k (M)))$ will be a $\mathcal{F}(M)$-anti-derivation of the algebra $\mathcal{A}(M, \mathcal{F}(J_0^k (M)))$ of degree 1,

$$ d_4 : \mathcal{A}^p (M, \mathcal{F}(J_0^k (M))) \to \mathcal{A}^{p+1} (M, \mathcal{F}(J_0^k (M))). $$

In order to introduce a natural operator with the properties of the usual exterior differentiation, we will consider the exterior derivative of forms defined on the jet bundle $J_0^k (M)$,

$$ d_J : \mathcal{A}^p (J_0^k (M)) \to \mathcal{A}^{p+1} (J_0^k (M)). \quad (2.52) $$

and the horizontal component $\tilde{h}_k (d_J k \zeta \alpha)$. 

Definition 2.50 The exterior derivative operator of generalized forms is the homomorphism

$$ d_4 : \mathcal{A}^p (M, \mathcal{F}(J_0^k (M))) \to \mathcal{A}^{p+1} (M, \mathcal{F}(J_0^k (M))), \quad \tilde{\alpha} \mapsto k \zeta^{-1} \tilde{h}_k (d_J k \zeta \tilde{\alpha}). \quad (2.53) $$

Remark 2.51 The definition of the exterior derivative $d_4$ only makes sense for strongly defined generalized forms, as in Definition 2.23 and for $\Sigma \hookrightarrow M$ of codimension zero.
Lemma 2.52 The isomorphism (2.25) implies the relations

1. For any $p + 1$ vector fields $\{X_1, \ldots, X_{p+1}\}$ on $M$, it holds that

$$k \zeta^{-1} \hat{h}_k (d_j k \zeta \hat{\alpha})(X_1, \ldots, X_{p+1}) = \hat{h} (d_j \hat{\alpha})(\hat{h} X_1, \ldots, \hat{h} X_{p+1}).$$  \hspace{1cm} (2.54)

2. For any form $\hat{\alpha} \in \Lambda^p(J^k_0(M))$ and $p + 1$ vector fields $\{X_1, \ldots, X_{p+1}\}$ on $M$, it holds that

$$d_j \hat{h}_k \hat{\alpha}(\hat{h} X_1, \ldots, \hat{h} X_{p+1}) = \hat{h} d_j \hat{\alpha}(\hat{h} X_1, \ldots, \hat{h} X_{p+1}).$$  \hspace{1cm} (2.55)

Proof. The first property follows directly from the homomorphism (2.25) and from the formula (2.20),

$$k \zeta^{-1} \hat{h}_k (d_j k \zeta \hat{\alpha})(X_1, \ldots, X_{p+1}) = \hat{h} (d_j \hat{\alpha})(\hat{h} X_1, \ldots, \hat{h} X_{p+1}).$$

The second property requires the use of Cartan’s formula for the exterior differential $d_j \hat{\alpha}$,

$$d_j \hat{\alpha}(\hat{h} X_1, \ldots, \hat{h} X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} h X_i(\hat{\alpha}(\hat{h} X_1, \ldots, \hat{h} X_i, \ldots, \hat{h} X_{p+1}))$$

$$+ \sum_{i,j=1}^{p+1} (-1)^{i+j+1} \hat{\alpha}([h X_i, h X_j], h X_1, \ldots, \hat{h} X_i, \ldots, \hat{h} X_j, \ldots, h X_{p+1}).$$  \hspace{1cm} (2.56)

Note that $\hat{h}_k$ is linear. Therefore,

$$d_j(\hat{h}_k(\hat{\alpha}))(\hat{h} X_1, \ldots, \hat{h} X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} h X_i(\hat{h}_k \hat{\alpha}(\hat{h} X_1, \ldots, \hat{h} X_i, \ldots, \hat{h} X_{p+1}))$$

$$+ \sum_{i,j=1}^{p+1} (-1)^{i+j+1} \hat{h}_k \hat{\alpha}([h X_i, h X_j], h X_1, \ldots, \hat{h} X_i, \ldots, \hat{h} X_j, \ldots, h X_{p+1}).$$

Also, the curvature vector field is

$$R(X_i, X_j) := [h X_i, h X_j] - h[X_i, X_j]$$  \hspace{1cm} (2.57)

is a vertical vector field. Therefore,

$$\hat{h}_k \hat{\alpha}([h X_i, h X_j] - h[X_i, X_j], h X_1, \ldots, \hat{h} X_i, \ldots, \hat{h} X_j, \ldots, h X_{p+1}) = 0.$$
Cartan’s formula can be re-written as

\[ d_J(\hat{h}_k(\hat{\alpha}))(hX_1, \ldots, hX_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} hX_i((\hat{h}_k\hat{\alpha})(hX_1, \ldots, h\hat{X}_i, \ldots, hX_{p+1})) \]

\[ + \sum_{i,j=1}^{p+1} (-1)^{i+j+1}\hat{h}_k\hat{\alpha}(h[X_i, hX_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}) \]

\[ = \sum_{i=1}^{p+1} (-1)^{i+1} hX_i(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_p)) \]

\[ + \sum_{i,j=1}^{p+1} (-1)^{i+j+1}\hat{h}_k\hat{\alpha}(h[X_i, X_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}). \]

Note that since

\[ d_J(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_p))(hX_i) = (hX_i) \cdot (\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_{p+1})), \]

Cartan’s formula can be re-written as

\[ d_J\hat{\alpha}(hX_1, \ldots, hX_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} d_J(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_{p+1}))(hX_i) \]

\[ + \sum_{i,j=1}^{p+1} (-1)^{i+j+1}\hat{\alpha}(h[X_i, X_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}). \]

Now we compute the right hand side of (2.55). Since \( \hat{h}_k \) is linear, one has that

\[ \hat{h}_k d_J\hat{\alpha}(hX_1, \ldots, hX_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \hat{h}_k(d_J(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_{p+1}))(hX_i) \]

\[ + \sum_{i,j=1}^{p+1} (-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i, hX_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}). \]

Each of the terms in the first line can be computed as

\[ \hat{h}_k d_J(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_{p+1}))(hX_i) = d_J(\hat{\alpha}(hX_1, \ldots, h\hat{X}_i, \ldots, hX_p))(hX_i). \]

As before, we use that \([hX_i, hX_j] - h[X_i, X_j] \) is vertical. Therefore,

\[ \hat{h}_k(\sum_{i,j=1}^{p+1} (-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i, hX_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1})) \]

\[ = \sum_{i,j=1}^{p+1} (-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i, X_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}) \]

\[ = \sum_{i,j=1}^{p+1} (-1)^{i+j+1}(\hat{\alpha})(h[X_i, X_j], hX_1, \ldots, h\hat{X}_i, \ldots, h\hat{X}_j, \ldots, hX_{p+1}). \]

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Gluing all together,

\[
\hat{h}_kd_f(\hat{\alpha}(hX_1,...,hX_{p+1})) = \sum_{i=1}^{p+1}(-1)^{i+1}\hat{h}_kd_f(\hat{\alpha}(hX_1,...,hX_{i},...,hX_{p+1}))(hX_i)
+ \sum_{i,j}^{p+1}(-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i,X_j],hX_1,...,h\hat{X}_i,...,h\hat{X}_j,...,hX_{p+1})
= \sum_{i=1}^{p+1}(-1)^{i+1}(hX_i)(d_f(\hat{\alpha}(hX_1,...,h\hat{X}_i,...,hX_{p+1})))
+ \sum_{i,j}^{p+1}(-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i,X_j],hX_1,...,h\hat{X}_i,...,h\hat{X}_j,...,hX_{p+1}).
\]

This calculation proves the second property. □

From the prove of Lemma 2.52, one obtains an explicit formula for the exterior differential \(d_4\hat{\alpha} \in \Lambda^{p+1}(M,J^k_0(M))\) acting on \(p+1\) vector fields \(X_1,...,X_{p+1}\) on \(M\),

\[
d_4(\hat{\alpha}(X_1,...,X_{p+1})) = \sum_{i=1}^{p+1}(-1)^{i+1}hX_i(\hat{\alpha}(hX_1,...,h\hat{X}_i,...,hX_{p+1}))
+ \sum_{i,j}^{p+1}(-1)^{i+j+1}(\hat{h}_k\hat{\alpha})(h[X_i,X_j],hX_1,...,h\hat{X}_i,...,h\hat{X}_j,...,hX_{p+1}).
\]

The operator \(d_4\) is related with the exterior covariant derivative \(D_{\nu\eta}^{12}\),

\[
(D_{\nu\eta}\omega)(X_1,...,X_{p+1}) := (d_f\omega)(hX_1,...,hX_{p+1})
= \sum_{i=1}^{p+1}(-1)^{i+1}hX_i(\omega \cdot (hX_1,...,h\hat{X}_i,...,hX_{p+1}))
+ \sum_{i,j}^{p+1}(-1)^{i+j+1}\omega(hX_i,hX_j),(hX_1,...,h\hat{X}_i,...,h\hat{X}_j,...,hX_{p+1}),
\]

\(\omega \in \Gamma \Lambda^p(J^k_0(M)).\)

For horizontal forms \(\hat{\alpha} \in \Lambda^p_h(J^k_0(M))\), this formula reduces to

\[
(D_{\nu\eta}\hat{\alpha})(X_1,...,X_{p+1}) := \sum_{i=1}^{p+1}(-1)^{i+1}hX_i(\hat{\alpha} \cdot (hX_1,...,h\hat{X}_i,...,hX_{p+1}))
+ \sum_{i,j}^{p+1}(-1)^{i+j+1}\hat{\alpha}(h[X_i,X_j],hX_1,...,h\hat{X}_i,...,h\hat{X}_j,...,hX_{p+1}),
\]

from which follows the
Proposition 2.53 Given \( \tilde{\alpha} = k\xi^{-1}(\dot{\alpha}) \in \Lambda^p(M, F(M, J^k_0(M))) \), then it holds that
\[
d_4\tilde{\alpha}(X_1, ..., X_{p+1}) = k\xi^{-1}(D\kappa\dot{\alpha})(X_1, ..., X_{p+1}). \tag{2.58}
\]

**Proof.** The first property is proven after a short calculation, completely analogous to the proof of the linearity of the interior product, except that \( \lambda \) is a scalar instead of an arbitrary smooth function. The second property follows from the following calculation. If \( \tilde{\alpha}_1, \tilde{\alpha}_2 \in \Gamma\Lambda^p(M, F(J^k_0(M))) \), then
\[
d_4(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2) = k\xi^{-1}d_J(k\xi(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2)) = k\xi^{-1}(\dot{h}_k(d_Jk\xi(\tilde{\alpha}_1)) \wedge k\xi(\tilde{\alpha}_2) + (-1)^p \tilde{\alpha}_1 \wedge \dot{h}_k(d_Jk\xi(\tilde{\alpha}_2))) = d_4(\tilde{\alpha}_1 \wedge \tilde{\alpha}_2 + (-1)^p \tilde{\alpha}_1 \wedge d_4(\tilde{\alpha}_2)).
\]
The third property is proved using local coordinates. If \((U, x)\) is a local coordinate system on \( M \), we have that
\[
d_4f(kx) = k\xi^{-1}\dot{h}_k(d_Jk\xi(f(kx))).
\]
This relation is equivalent to
\[
d_4f(kx) = k\xi^{-1}\dot{h}_k\left(\frac{\partial f}{\partial x^\mu}k\xi(dx^\mu) + \frac{\partial f}{\partial y^A}\delta y^A\right) = k\xi^{-1}\frac{\partial f}{\partial x^\mu}k\xi(dx^\mu) = \frac{\partial f}{\partial x^\mu}k\xi^{-1}\dot{h}_k(k\xi(dx^\mu)) = \frac{\partial f}{\partial x^\mu}d_4x^\mu.
\]
After a short calculation, using Lemma 2.52 one obtains
\[
d_4 d_4 \bar{\alpha}(X_1, ..., X_{p+2}) = k \zeta^{-1} \hat{h}_k(d_j k \hat{h}_k(d_j k \zeta(\bar{\alpha}))) (X_1, ..., X_{p+2}) \\
= \hat{h}_k \hat{h}_k d_j (d_j \bar{\alpha}) (h X_1, ..., h X_{p+2}) \\
= \hat{h}_k d_j (\bar{\alpha})(h X_1, ..., h X_{p+2}) = 0,
\]
since \( d_j d_j (\bar{\alpha}) = 0 \).

\[\square\]

**Corollary 2.55** For any \( k \zeta(\bar{\alpha}) = \bar{\alpha} \in \Lambda^n(J^*_0(M)) \) the relation
\[
D_{k^1} \circ D_{k^2} k^1 \zeta\bar{\alpha}(X_1, ..., X_{p+2}) = 0
\]
(2.60)
holds.

**Proof.** The nilpotent condition \( d_4 d_4 = 0 \) can be re-written as
\[
0 = d_4 d_4 \bar{\alpha} = k \zeta^{-1} D_{k^1} k \zeta k \zeta^{-1} D_{k^2} k \zeta(\bar{\alpha}) \\
= k \zeta^{-1} D_{k^1} D_{k^2} k \zeta(\bar{\alpha}).
\]
This implies \( 0 = D_{k^1} D_{k^2} k \zeta(\bar{\alpha}). \)

\[\square\]

**Remark 2.56** It is well known that in general the exterior covariant derivative is not nilpotent [42]. However, the nilpotent property (2.60) holds for only horizontal forms and not for arbitrary forms.

**Proposition 2.57** The exterior differential operator (2.53) is connection independent.

**Proof.** Given to connections \( \hat{h}_{k_i} \), \( i = 1, 2 \), one can define two anti-derivations,
\[
\bar{\alpha} \mapsto k \zeta_1^{-1} h_1 (d_j k \zeta_1(\bar{\alpha})), \quad \bar{\alpha} \mapsto k \zeta_2^{-1} h_2 (d_j k \zeta_2(\bar{\alpha})).
\]
We compare both anti-derivations as follows,
\[
k \zeta_1^{-1} h_1 (d_j k \zeta_1(\bar{\alpha}))(X_1, ..., X_{p+1}) - k \zeta_2^{-1} h_2 (d_j k \zeta_2(\bar{\alpha}))(X_1, ..., X_{p+1}) = \\
\hat{h}_k (d_j k \zeta_1(\bar{\alpha}))(h X_1, ..., h X_{p+1}) - \hat{h}_k (d_j k \zeta_2(\bar{\alpha}))(h X_1 + V_1, ..., h X_{p+1} + V_{p+2}),
\]
for some vertical vector fields \( V_i \) such that \( h_j X_i = h^1 X_i + V_i, i = 1, ..., p + 1 \). Then by Proposition 2.26 the second pieces above is
\[
\hat{h}_k (d_j k \zeta_2(\bar{\alpha}))(h X_1 + V_1, ..., h X_{p+2} + V_{p+1}) = \hat{h}_k (d_j \bar{\alpha})(h X_1, ..., h X_{p+1}).
\]
Note the relation (by Corollary 2.30)
\[
\hat{\alpha}_2 := \hat{h}_k (d_j \hat{\alpha}_2) = \hat{h}_k (d_j \hat{\alpha}_2) = \hat{\alpha}_1.
\]
Thus one has that
\[ k \zeta_2^{-1} h_k^2 (d_J k \zeta_2) (h_1 X_1 + V_1, \ldots, h_1 X_{p+2} + V_{p+1}) = \hat{h}_k^1 (d_J \hat{\alpha}_2) (h_1 X_1, \ldots, h_1 X_{p+1}). \]

Again, note that \( \hat{\alpha}_1 - \hat{\alpha}_2 \) is zero,
\[ \hat{h}_k^1 (d_J \hat{\alpha}_2) (h_1 X_1, \ldots, h_1 X_{p+1}) = \hat{h}_k^1 (d_J \hat{\alpha}) (h_1 X_1, \ldots, h_1 X_{p+1}) \]
from where the result follows. \( \square \)

Thus, the exterior derivative \( d_4 \) does not depend on the covariant derivative that we use in its definition.

**Example 2.58** A 1-form \( \phi \in \Gamma \Lambda^1(M, \mathcal{F}(\mathcal{J}_0^k M)) \) can be written in local coordinates as
\[ \phi = \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \, d_4 x^i. \]

Then its exterior derivative is
\[
d_4 \phi = d_4 \left( \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \, d_4 x^i \right)
= d_4 \left( \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \right) \wedge d_4 x^i + \left( \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \right) \wedge (d_4)^2 x^i
= \partial_i \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \, d_4 x^i \wedge d_4 x^i.
\]

It is clear that \( d_4 \phi \) does not depend on the local coordinate system on \( M \). If we calculate \( d_2^2 \) one obtains the expression
\[ d_4 (d_4 \phi) = \partial_j \partial_i \phi_i (x(s), \dot{x}, \ddot{x}, \ldots, x^{(k)}) \, d_4 x^j \wedge d_4 x^i \wedge d_4 x^i = 0. \]

It is natural to ask whether there is a generalization of Cartan’s formula. Indeed, there is such natural generalization:

**Proposition 2.59** The following formula holds:
\[ (\iota_X \circ d_4 + d_4 \circ \iota_X)(\bar{\alpha}) = k \zeta k \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right). \]  \hspace{1cm} (2.61)
for all \( \bar{\alpha} \in \Gamma \Lambda^p(M, \mathcal{F}(\mathcal{J}_0^k M)), \ X \in \Gamma TM. \)

**Proof.** By direct calculation we have that
\[
(\iota_X \circ d_4 + d_4 \circ \iota_X)(\bar{\alpha}) = k \zeta^{-1} \iota_{\mathcal{J}_X} k \zeta k \zeta^{-1} \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right) + k \zeta^{-1} \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right)
= k \zeta^{-1} \iota_{\mathcal{J}_X} \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right) + k \zeta^{-1} \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right)
= k \zeta^{-1} \hat{h}_k \left( \mathcal{L}_{\mathcal{J}_X} k \zeta (\bar{\alpha}) \right).
\]

An additional property that will be of relevance in the generalization of exterior differential equations to higher order forms is the following,
Corollary 2.60 Let \( \bar{f} \in \mathcal{F}(J^k_0(M)) \) be a function such that is constant on \( M \). Then in any coordinate system it holds the relation
\[
d_4 \bar{f}(y^{(1)}\mu, y^{(2)}\nu, \ldots, y^{(k)}\rho) = 0. \tag{2.62}
\]

Example 2.61 It is natural to ask whether the above formula (2.61) reduces to the known case when we take \( k = 0 \) and therefore \( J^0_0(M) \simeq M \). In this case, \( \hat{X} = X \) and \( \hat{H}_0 = Id \). Therefore, it is clear that equation (2.61) coincides with the usual Cartan’s formula.

Other standard relations of the Cartan calculus can be extended to the algebra of generalized forms by similar considerations. Therefore, we have a complete Cartan’s calculus for generalized forms.

2.12 Vertical volume forms

Let us consider a connection \( k\hat{H} \) on \( J^k_0(M) \). We can construct a non-zero vertical \( nk \)-form
\[
dvol^V(kx) : \in \Gamma \Lambda^{nk} J^k_0(M)
\]
at each point of a local trivialization of \( k\pi^{-1}(U) \), with \( U \subset M \) being an open subset on \( M \). Given the local natural coordinate system \( (x^\mu, y^A, \mu = 1, \ldots, k, A = 1, \ldots, kn) \) on a local trivialization \( U \times R^{nk} \) of \( J^k_0(M) \), the vertical volume form can be constructed using the non-linear connection \( k\hat{H} \) and the corresponding covariant vertical frame \( \{\delta y^A, A = 1, \ldots, nk\} \) (2.16) by the expression
\[
dvol^V(kx) := w^{(kx)} \delta y^1 \wedge \cdots \wedge \delta y^{nk}. \tag{2.63}
\]
on the trivialization \( U \times R^{nk} \) of \( J^k_0(M) \).

Proposition 2.62 The \( nk \)-form \( dvol^V \) has the following properties:

1. \( dvol^V \) is a vertical form, \( dvol^V(\ldots, X, \ldots) = 0 \) for any \( X \) horizontal.

2. It is natural and globally defined on \( J^k_0(M) \).

Proof. The first statement follows from the fact that \( dvol^V \) is constructed with vertical forms and the notion of being vertical is connection independent. The second property is immediate using the connection \( k\hat{H} \) on the fiber bundle \( k\pi : J^k_0(M) \to M \).

More generally, one can introduce a general notion of vertical volume form,

Definition 2.63 Given a connection \( k\hat{H} \), a vertical volume form is a section of \( \Lambda^{nk} J^k_0(M) \) such that it is zero when evaluated on any horizontal vector \( \hat{X} \) and is non-zero everywhere.
**Remark 2.64** The vertical volume form (2.63) is not connection independent. However, given two connections $^{k}\hat{H}_1$ and $^{k}\hat{H}_2$, if $dvol_V(1)$ is the vertical volume form constructed with $^{k}\hat{H}_2$ and $X(2) \in \Gamma^{k}\hat{H}_2$, one has

$$dvol_V(1)(..., \dot{X}(1),...) = 0, \quad dvol_V(1)(..., \dot{X}(2),...) = 0.$$ 

**Example 2.65** The vertical vector sub-bundle $^{k}\tilde{\pi} : ^{k}\mathcal{V} \to \mathcal{M}$ embedded in a fiber orientable tangent bundle $TJ_0^k(\mathcal{M})$ admits a vertical volume form. Let $\sigma_V : ^{k}\mathcal{V} \to TJ_0^k(\mathcal{M})$ be a fiber preserving embedding. If $J_0^k(\mathcal{M})$ is orientable, let $dvol_f$ be a volume form on $J_0^k(\mathcal{M})$ and $^{k}\hat{H}$ a connection such that $\{^{h}X_1, ..., ^{h}X_n\}$ generates locally a horizontal distribution. Then the form

$$\iota_{^{h}X_1} \cdots \iota_{^{h}X_n} dvol_f$$

is a vertical volume form and the form

$$\sigma_V^* (\iota_{^{h}X_1} \cdots \iota_{^{h}X_n} dvol_f)$$

(2.64)

is a $nk$-volume form on $^{k}\mathcal{V}$.

**Example 2.66** We can consider the following constructions for vertical forms,

1. Let $E = \mathcal{M} \times V$, with $V$ be a vertical vector space of dimension $d$ and basis $\{e_1, ..., e_d\}$, whose dual basis is $\{e^1, ..., e^d\}$. A vertical form along the fiber is given by the exterior product of 1-forms

$$dvol_V = e^1 \wedge \cdots \wedge e^d.$$ 

Then any top-form $dvol_V$ of degree equal to $\dim(V)$ and non-zero everywhere determines a vertical top form on $E$ by $(\pi_E)^*(dvol_V)$, with $\pi_E : E \to \mathcal{M}$.

2. Let $\pi_E : E \to \mathcal{M}$ be an orientable vector bundle of dimension $n + d$ with $dvol_E$ a $n + d$-volume form on $E$. Let us also assume that $(\mathcal{M}, \eta)$ is a flat pseudo-Riemannian manifold. Then there is a globally defined orthonormal frame $\{e_1, ..., e_n\}$ on $\mathcal{M}$ and then

$$\iota_{e_1} \cdots \iota_{e_n} dvol_E$$

is a globally defined vertical volume form.

3. An orientable fiber bundle $\mathcal{E}$ with an horizontal distribution $\mathcal{H}_\mathcal{E}$ admits a vertical volume form, constructed in a similar way as the volume form (2.63).

---

$^6$The fact that the vertical and horizontal distributions are in general globally defined implies that the embedding $\sigma_V : ^{k}\mathcal{V} \to TJ_0^k(\mathcal{M})$ is globally defined as well.
The normalization function \( w(kx) \) in equation (2.63) is defined by the normalization condition,

\[
\int_{k\pi^{-1}(x)} d\text{vol}_V = 1.
\] (2.65)

**Proposition 2.67** Given the vertical volume form \( d\text{vol}_V \) as in (2.63), there is a local frame \( \{V_1, \ldots, V_{nk}\} \) such that \( d\text{vol}_V(V_1, \ldots, V_{nk}) = 1 \).

**Proof.** Let \( \{\tilde{V}_1, \ldots, \tilde{V}_n\} \) be an arbitrary local frame for the vertical bundle. Since the volume form acting on a basis is different than zero and finite, the result is obtained dividing by the factor \( d\text{vol}_V(V_1, \ldots, V_{nk}) \neq 0 \) the first element \( \tilde{V}_1 \) of the local frame. \( \square \)

**Example 2.68** If \((M, \eta)\) is flat, the existence of a vertical volume form on the bundle \( J^k_0(M) \) is assured. Then a global holonomic frame along the fiber \( k\pi^{-1}(x) \) is \( \{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{nk}}\} \) and the vertical volume form can be written locally as

\[
d\text{vol}_V(kx) := w(kx) \delta y_1 \wedge \cdots \wedge \delta y_{nk}.
\]

### 2.13 Closed vertical volume forms

We will consider vertical volume forms that are closed in the sense that

\[
d_J(d\text{vol}_V) = 0.
\] (2.66)

The existence of a closed vertical volume form implies some constraints on the connections. If we write down the condition (2.66), one gets an expression of the form

\[
d_J(d\text{vol}_V) = \frac{1}{w} \frac{\partial w}{\partial x^\rho} dx^\rho \wedge d\text{vol}_V + w d_J(\delta y_1 \wedge \cdots \wedge \delta y_{nk}).
\]

Each of the exterior derivatives of the vertical 1-forms \( \delta y^A \) can be written as

\[
d_J(\delta y^A) = d_J(d_J y^A + N^A_\rho dx^\rho) = d_J d_J y^A + d_J(N^A_\rho) \wedge dx^\rho = d_J(N^A_\rho) \wedge dx^\rho.
\]

Each of the differentials

\[
d_J(N^A_\rho) = \frac{\partial N^A_\rho}{\partial x^\sigma} dx^\sigma + N^A_\rho B_\rho \delta y^B.
\]

After some elementary algebra one ends up with the following to constraints,

- A constraint involving only the connection,

\[
\sum_{A=1}^{kn} (-1)^A \frac{\partial N^A_\rho}{\partial x^\sigma} dx^\sigma \wedge dx^\rho \wedge d\text{vol}_V(\frac{\partial}{\partial y^A}, \cdot) = 0.
\] (2.67)
This constraint is satisfied if one imposes the strong condition of $k\mathcal{H}$ being symmetric,

$$\frac{\partial N^A}{\partial x^\sigma} = \frac{\partial N^A}{\partial x^\rho}. \quad (2.68)$$

This is a generalization of the torsion-free condition for the Levi-Civita connection.

• The second constraint is on the volume form: it must satisfy the condition

$$\frac{1}{w} \frac{\partial w}{\partial x^\rho} + \sum_{A=1}^{kn} N^A A_\rho = 0, \quad \rho = 1, \ldots, n. \quad (2.69)$$

Note that because of the definition of $N^A_{B\rho}$, the components $\frac{\partial w}{\partial x^\rho}$ and $N^A A_\rho$ transform in the same way under local coordinate transformations. Thus the equation (2.69) is covariant.

**Example 2.69** Let us consider the case that

1. The 1-form $\varpi = N^A A_\rho \, dx^\rho$ lives on $M$,

2. The 1-form is closed in the sense that $d\varpi = 0$.

Then the partial differential equation (2.69) has a (local) solution: locally there is also a 0-form $\nu$ such that $\varpi = d\nu$.

Closed vertical volume forms are $nk$-calibration forms,

**Proposition 2.70** A closed vertical volume form $d\text{vol}_V$ on $J_0^k(M)$ is a $nk$-calibration in the sense of Harvey and Lawson [37].

**Proof.** Let $T_{nk}(x)$ be any $nk$-dimensional vector sub-space of $T_u J_0^k(M)$. If there is a $\xi \in T_{nk}(u) \cap k\mathcal{H}_u \neq \emptyset$, then $d\text{vol}_V|_{T_{nk}} = 0$. If $T_{nk} = k\mathcal{V}$, then there is a $nk$-vector $\xi_{nk}$ with $d\text{vol}_V(\xi_{nk}) = 1$. The result follows by linearity of the action of $d\text{vol}_V$ on $nk$-vectors. $\square$

### 2.14 The de Rham cohomology of differential forms $\Lambda^p(M, \mathcal{F}(J_0^k(M)))$

The fact that the operator $d_4$ is nilpotent implies the existence of some interesting cohomology theories,

**Definition 2.71** A $p$-form $\alpha \in \Lambda^p(M, \mathcal{F}(J_0^k(M)))$ is **closed** if $d_4 \alpha = 0$; it is **exact** if there is a $(p-1)$-form $\beta \in \Lambda^{p-1}(M, \mathcal{F}(J_0^k(M)))$ such that $d_4 \beta = \alpha$.

The vector space of $d_4$-closed forms is

$$Z^p(M, \mathcal{F}(J_0^k(M))) := \left\{ \alpha \in \Gamma \Lambda^p(M, \mathcal{F}(J_0^k(M))) \, \text{s.t.} \, d_4 \alpha = 0 \right\}. $$
The vector space of $d_4$-exact forms is
\[ B^p(F(M, J^k_0(M))) := \{ \alpha \in \Gamma \Lambda^p(M, F(J^k_0(M))) \mid \exists \beta \in \Gamma \Lambda^{p-1}(M, F(J^k_0(M))) \text{ s.t. } d_4 \beta = \alpha \}. \]

The $p$-cohomology group is
\[ H^p(M, F(J^k_0(M))) := Z^p(M, F(J^k_0(M)))/B^p(M, F(J^k_0(M))). \]

Therefore, the cohomology $H^*(M, F(J^k_0(M)))$ is defined from the differential structure of the manifold $M$.

One can also define the vertical compact cohomology $p$-cohomology group of the $k$-jet bundle $J^k_0(M)$,
\[ H^p_{cv}(M, F(J^k_0(M))) := \{ \alpha \in \Gamma \Lambda^p(M, F(J^k_0(M))) \mid \alpha_x \text{ has compact support on each fiber } k^{\pi-1}(x) \}. \]

Then the compact vertical cohomology $H^*_{cv}(M, F(J^k_0(M)))$ is defined in a similar way.

**Remark 2.72** We will prove later (see below subsection 2.16) that $H^*_{cv}(M, F(J^k_0(M)))$ is invariant under homotopy: if $M$ and $\tilde{M}$ are homotopy equivalent, then $H^*_{cv}(M, F(J^k_0(M))) \simeq H^*_{cv}(\tilde{M}, F(J^k_0(\tilde{M})))$.

### 2.15 Bounded vertical compact cohomology

A further restriction in the cohomology forms is the following,

**Definition 2.73** Let $(M, \eta)$ be a Lorentzian manifold. Then the covariantly bounded $k$-jet bundle is the fiber bundle $J^k_{0b}(M) \to M$ such that

1. The curves $x : I \to M$, $x(0) = p$ are smooth,
2. Each of the covariant derivatives is bounded,
\[ |\eta(D^i_x \dot{x}, D^j_x \dot{x})| \leq c_{i+1} \in R^+, \text{ } i = 1, ..., k - 1, \]

where $D$ is the Levi-Civita connection of $\eta$.

Note that the values of $c_i$ could be different for each $i = 1, ..., k$ and some of the values could be un-bounded, but by definition at least one of the constants $c_i$ should be finite. Thus the specification of the bounded jet bundle $J^k_{0b}(M)$ will be given by a finite collection of constants $\{(c_1, ..., c_k)\}$.

**Example 2.74** The physical interpretation of **Definition 2.73** is better understood after the following two different examples,
1. For \( k = 1 \), the second condition is equivalent to the requirement of a \textit{maximal speed}. Note that it only indicates a bound on the value of the allowed covariant speed. In this case \( c_1 = 1 \) in natural coordinates and is \( J^1_{0b} \).

2. For \( k = 2 \) and for time like curves of \( \eta \), the bound \( c_2 = A^2_{\text{max}} < \infty \) is equivalent to the requirement of maximal acceleration. Note that one can have \( c_1 = \infty \), dropping the requirement of maximal covariant speed. It corresponds to \( J^2_{0b}(M) \) with \( (c_1 = \infty, c_2 < A^2_{\text{max}}) \).

3. Other example of relevance is \( k = 3 \) with \( (c_1 < \infty, c_2 < \infty, c_3 < \infty) \). The bounded jet bundle with such specification will be denoted by \( J^3_{0b}(M) \). This bundle is of relevance for the electromagnetic theory that we will develop in this work.

We can consider the following cohomology theory,

\textbf{Definition 2.75} Let \((M, \eta)\) be a Lorentzian manifold and consider the fiber bundle \( J^k_{0b}(M) \). The covariantly bounded compact vertical cohomology is

\[ H_p^{cv}(M, \mathcal{F}(J^k_{0b}(M))) := \{ \alpha \in \Gamma \Lambda^p(M, \mathcal{F}(J^k_{0b}(M))) \mid \alpha_x \text{ has compact support on each fiber } k^{\pi-1}(x) x \in M \}. \]

In a similar way, one can consider the de Rham cohomologies \( H^*_{dR}(J^k_{0b}(M)) \) and \( H^*_{cv}(J^k(M)) \).

\textbf{Remark 2.76} We have the following remarks:

- Note the restriction to \( J^k_{0b}(M) \) and the associated cohomology \( H^*(M, \mathcal{F}(J^k_{0b}(M))) \), even for non-compact vertical forms.

- The cohomologies \( H^*(M, \mathcal{F}(J^k_{0b}(M))) \), \( H^*_{cv}(M, \mathcal{F}(J^k_{0b}(M))) \), \( H^*_{cv}(M, \mathcal{F}(J^k_{0b}(M))) \) and \( H^*(J^k_{dR}(M)) \) are in principle different from each other.

- From the different types of cohomologies that one can construct with generalized differential forms, the relevant four our applications in generalized field theory is \( H^*_{cv}(M, \mathcal{F}(\tilde{J}^2_{0b}(M))) \). We will prove in subsection 2.16 that \( H^*_{cv}(M, \mathcal{F}(\tilde{J}^3_{0b}(M))) \) is homotopy invariant of \( M \), for the set of constants \( \{c_1 < \infty, c_2 < \infty, c_3 > \infty\} \).

The bounded cohomology is not only defined on jet bundles. For instance, it is possible to define it for homotopy equivalent bundles

\[ E^k \hookrightarrow J^k_{0b}(M), \quad (2.70) \]

for a given \( k \), such that there is an \( J^k_{0b}(M) \) with the homotopy equivalence

\[ E^k \simeq J^k_{0b}(M). \quad (2.71) \]
Later, we will see how to relate the cohomologies $H_\ast$ where $\mathcal{D}$. Thom’s isomorphism theorem and homotopy invariance of $J$ in order to prove the homotopy invariance of $H_\ast$ we note that there is also an homotopy between $\mathcal{J}$. Proposition 2.78 admits straightforward generalizations, as the following example shows. Let us consider $\tilde{\mathcal{J}}^{3}_{0b}(M)$ with the set of constants ($c_1 = \infty, c_2 < \infty, c_3 < \infty$). Then the following homotopies hold,

$$\tilde{\mathcal{J}}^3_{0b}(M) \simeq \mathcal{J}^{3}_{0b}(M) \simeq \mathcal{J}^1_{0b}(M) \simeq TM.$$  

As a direct consequence of Proposition 2.78 we have the following
Proposition 2.80 Let \((M, \eta)\) be a Lorentzian manifold. Then

1. It holds the following isomorphism:

\[
H^*(J^k_0(M)) \simeq H^*(J^{k-1}_0(M)).
\]  (2.74)

2. It holds the following isomorphism:

\[
H^*_{cv}(J^k_0(M)) \simeq H^*_{cv}(J^{k-1}_0(M)).
\]  (2.75)

Proof. We construct a homotopy between \(J^k_0 M\) and \(J^{k-1}_0(M)\). Since both cohomologies \(H^*_{cv}\) and \(H^*\) are invariant under homotopy, the result follows. \(\blacksquare\)

Corollary 2.81 The following relation holds,

\[
H^p_{cv}(J^k_0) = 0, \quad p = kn + 1, \ldots, (k + 1)n.
\]  (2.76)

Proof. It is enough to consider the isomorphism (2.75) for the cohomology groups

\[
H^p_{cv}(J^{k-1}_0(M)), \quad \text{for} \quad p > \dim(J^{k-1}_0(M)),
\]

which are identically zero. \(\blacksquare\)

As a consequence of (2.75) and Example (2.73) we have the following

Corollary 2.82 Let us consider \(J^3_0(M)\) with \((c_1 = \infty, c_2 < \infty, c_3 < \infty)\). Then the following isomorphisms hold,

\[
H^*_{cv}(J^3_0(M)) \simeq H^*_{cv}(J^2_0(M)) \simeq H^*_{cv}(TM).
\]  (2.77)

Remark 2.83 Corollary 2.82 can be extended easily to \(k > 1\). Based on this Corollary, from now on we will contemplate vertical bounds of the form \(J^3_0(M)\) with \(\{c_1 = \infty, c_2 < \infty, \ldots, c_k < \infty\}\). This is indeed the jet bundle that we will find in the generalized electrodynamic theory of point charged particles.

Definition 2.84 Let \(\pi_E : E \to B\) be a vector bundle with orientable fiber over with \(\pi_B^{-1}(x)\) of dimension \(l\). Then the averaging along the fiber of a \(l + k\) form is the homomorphism

\[
\langle \cdot \rangle : \Gamma \Lambda^{l+k}E \to \Gamma \Lambda^kB, \quad \alpha \mapsto \int_{\pi_B^{-1}(x)} \alpha.
\]  (2.78)

Given a vector bundle \(\pi_E : E \to B\), let us consider the cohomology of the differential form \(\Lambda^*(E)\) with compact vertical support \(H^*_{cv}(E)\). Then one has the following isomorphism \([8]\).
Theorem 2.85 Let $\pi_E : E \to M$ be a vector bundle of finite type over a $n$-dimensional manifold $M$. Then if the dimension of the fibers $\pi^{-1}_B(x)$ is $l$, one has the isomorphism
\[ H^*_c(E) \simeq H^*_{dR}(M). \] (2.79)

Let us construct the following vector bundle structure on jet bundles. We consider a symmetric connection $N$ on $J^k_0(M)$. Then to each $k$-jet $(x, \dot{x}, \ddot{x}, \ldots, x^{(k)})$ one associates the corresponding $(x, D_x \dot{x}, D_x^2 \dot{x}, \ldots, D_x^{(k)} \dot{x})$, where $D$ is covariant the covariant derivative associated with $N$. Let $j^k_0(M) = \pi^{-1}_B(x)$ be the fiber over $x$. Then the map
\[ \psi : j^k_0(x) \to \bigoplus^k T_x M, \quad (x, \dot{x}, \ddot{x}, \ldots, x^{(k)}) \mapsto (x, D_x \dot{x}, D_x^2 \dot{x}, \ldots, D_x^{(k)} \dot{x}) \] (2.80)
is a bijection: if one considers normal coordinates containing $n$, $\psi$ can be written as
\[ (x, \dot{x}, \ddot{x}, \ldots, x^{(k)}) \mapsto (x, \dot{x}, \ddot{x}, \ldots, x^{(k)}), \]
which shows clearly that it is injective and surjective. Therefore, one can define a vector space structure $(j^k_0(x), +, \cdot)$, defined by
\begin{itemize}
  \item $+ : j^k_0(x) \times j^k_0(x) \to j^k_0(x), \quad j_1 + j_2 := \psi^{-1}(\psi(j_1) + \psi(j_2))$,
  \item $\cdot : j^k_0(x) \times R \to j^k_0(x), \lambda \cdot j_1 := \psi^{-1}(\lambda \psi(j_1))$.
\end{itemize}

Let us consider the Whitney sum $\bigoplus^k W^k TM$ of $k$ tangent bundles $T^k M$.

Proposition 2.86 Given a symmetric connection $N$ on $M$, the fiber bundle $J^k_0(M)$ is furnished with a vector bundle structure, which is fibrewise induced from (2.80) and such that the following isomorphism holds:
\[ (J^k_0(M), +, \cdot) \simeq \bigoplus^k W^k TM. \] (2.81)

In particular, one can use the Levi-Civita connection $D$ associated with the metric $\eta$ and its extension of the Levi-Civita connection to higher order jet bundles. However, such a choice is not necessary and there are other possibilities for a natural choice of $N$, for instance, with connection associated with a given spray.

As an application of Theorem 2.85 and Proposition 2.86, we found the following isomorphism:

Proposition 2.87 Let $H^*_c(J^k_0(M))$ be as before and $H^*_{dR}(M)$ be the real de Rham cohomology of $M$. Then the following isomorphism holds:
\[ H^*_c(J^k_0(M)) \simeq H^*_{dR}^{n-kn}(M). \] (2.82)
Proof. One can apply Thom’s isomorphism theorem to the vector bundle \( k\pi : J^k_0(M) \to M \), obtaining by fiber averaging the following isomorphism:

\[
H^*_c(J^k_0(M)) \xrightarrow{\langle \cdot \rangle} H^{*-nk}_dR(M),
\]
from which follows the result. \( \square \)

Corollary 2.88. The cohomology \( H^*_c(J^k_0(M)) \) is an homotopy invariant of \( M \).

Proposition 2.89. Let \((M, \eta)\) be a Lorentzian manifold and \( H^*_c(J^k_0(M), F(J^k_0(M))) \) as before. Then any element of \([\alpha] \in H^*_c(M, F(J^k_0(M)))\) is bounded.

Proof. Using the corresponding operator norm, one obtains the result. \( \square \)

Let us recall the injective homomorphism \( \varphi \) defined by (2.29),

\[ \varphi : \Lambda^p M \to \Lambda^p(M, F(J^k_0(M))), \quad \varphi(\alpha)_u(X_1, ..., X_p) = \alpha_x(X_1, ..., X_p). \]

Then the following result is a consequence from the isomorphism (2.87),

Lemma 2.90. Let us assume the hypothesis as in Thom’s theorem. Then each element of a class \([\hat{\alpha}] \in H^{p+kn}_c(J^k_0(M))\) admits a decomposition,

\[
3\zeta(\hat{\alpha}) = k\zeta(\varphi(\langle \hat{\alpha} \rangle)) + dJ A, \quad A \in \Lambda^{p-1}_c(J^k_0(M)), \quad \hat{\alpha} \in [\hat{\alpha}]. \tag{2.83}
\]

This decomposition is unique.

Proof. By the invariance of integration operation along the fiber \( \langle \cdot \rangle \), one has that

\[
H^{p+kn}_c(J^k_0(M)) \simeq H^{p}_dR(M).
\]

Note that for each cohomology class \([\hat{\alpha}] \in H^{p+kn}_c(J^k_0(M))\), one has that

\[
\langle \hat{\alpha} \rangle = \langle \varphi(\langle \hat{\alpha} \rangle) \rangle.
\]

Therefore, by Thom’s isomorphism theorem it follows that

\[
[\hat{\alpha}] = [\varphi(\langle \hat{\alpha} \rangle)]
\]

and one obtains the relation (2.83). \( \square \)

We prove now an isomorphism which is of relevance for physical interpretation of the theory of generalized electromagnetic fields,

Proposition 2.91. Let \( J^k_{0b}(M) \) be the bounded \( k \)-jet bundle over \( M \) with constants \((c_1 = \infty, c_i < \infty, 1 < i \leq k)\). Then there is the following isomorphism,

\[
\varphi^*_b : H^*_dR(M) \to H^*_c(M, F(J^k_{0b}(M))). \tag{2.84}
\]
Proof. Let us consider the homomorphism \( \varphi_b : \Gamma \Lambda^p(M) \rightarrow \Gamma \Lambda^p_{cv}(M, F(J^k_{0b}(M))) \), as a restriction of \( \varphi \). The homomorphism \( \varphi_b \) commutes with \( d_4 \),

\[ \varphi_b (d\alpha) = d_4 (\varphi_b (\alpha)) \]  

(2.85)

Therefore, it induces an homomorphism of cohomologies:

\[ \varphi^*_b : H^*_dR(M) \rightarrow H^*_c (M, F(J^k_{0b}(M))). \]

\( \varphi^*_b \) is injective, which follows from the injectivity of \( \varphi_b \). The isomorphism (2.84) is also surjective. Let \([\bar{\alpha}] \in H^p_{cv}(M, F(J^k_{0b}(M)))\) be a class of cohomology such that for each \( \bar{\alpha} \in [\alpha], \) \( d_4(\alpha) = 0 \) holds. Let us define the homomorphism

\[ \tilde{\varphi}_b : H^*_c (M, F(J^k_{0b}(M))) \rightarrow H^{*+kn}_{dvvol}(J^k_{0}(M)), \quad [\bar{\alpha}] \mapsto [k\xi(\bar{\alpha}) \wedge dvvol]. \]  

(2.86)

Then \( \langle k\xi(\bar{\alpha}) \wedge dvvol \rangle \) is in some cohomology class in \( H^*_dR(M) \). Thus the image \( \varphi^*_b (\langle k\xi(\bar{\alpha}) \wedge dvvol \rangle) \) is in the same cohomology class of \( \bar{\alpha} \) and the result is proved.

\[ \blacksquare \]

Theorem 2.92 The cohomology \( H^*_c (M, F(J^k_{0b}(M))) \) is a homotopy invariant of \( M \).

Proof. Since it is related with the usual de Rham cohomology \( H^*_dR(M) \), it is an homotopy invariant of \( M \). \[ \square \]

As a consequence of Proposition (2.91) and the homotopy invariance of the cohomology, we have that

Corollary 2.93 Let \( \pi_B : B \rightarrow M \) be a sub-bundle of \( \pi_V : V \rightarrow M \) with fibers \( \pi_B^{-1}(x) \) homotopic to \( \pi_V^{-1}(x) \). Then

\[ H^*_c (B) \simeq H^*_c (M, F(J^k_{0b}(M))). \]

As a relevant example we have the following,

Proposition 2.94 One has the following isomorphism

\[ H^*_c (M, F(J^3_{0b}(M))) \simeq H^*_dR(M). \]  

(2.87)

Proof. Note that the fiber \( \tilde{j}^3_{0b}(x) \) has one extra dimension more than \( j^3_{0b}(M) \), corresponding to the direction of the globally defined coordinate

\[ \beta^{-1} : \tilde{j}^3_{0b}(x) \rightarrow R, \quad \beta^3 \mapsto \beta_2(kx) := |\eta(D^2_{x, \bar{x}})|. \]  

(2.88)

Then there is the following isomorphism

\[ \psi : \Lambda^* j^3_{0b}(M) \rightarrow \Lambda^*+1 j^3_{0b}(M), \quad \hat{\alpha} \mapsto \hat{\alpha} \wedge \beta d(\beta^{-1}). \]  

(2.89)
This induces the following isomorphism,
\[ \psi^* : H^{*+2n}J^2_{0b}(M) \to \Lambda^{*+2n+1}J^3_{0b}(M), \quad [\hat{\alpha} \wedge dvol_V] \mapsto [\hat{\alpha} \wedge dvol_V \wedge \beta \delta(\beta^{-1})]. \]
(2.90)

From which follows the following isomorphism (by homotopy and fiber integration),
\[ \vartheta : H^{*+2n+1}cvJ^3_{0b}(M) \to H^*cv(M, F(J^3_{0b}(M))), \quad [\hat{\alpha} \wedge dvol_V \wedge \beta \delta(\beta^{-1})] \mapsto [\beta \bar{\alpha}], \]
(2.91)

Thus we have the following isomorphisms
\[ H^{*+2n+1}cvJ^3_{0b}(M) \simeq H^{*+2n}cv(J^2_{0b}(M)) \simeq H^*cv(J^1_{0b}(M)) \simeq H^*_dR(M). \]

\[ \square \]

**Remark 2.95** We have the following remarks:

- The closed vertical volume form in \( J^3_{0b}(M) \) is \( dvol_V \wedge \beta \delta(\beta^{-1}) \).
- There is a natural map (not depending on coordinates)
  \[ \Lambda^p(M, F(J^3_{0b}(M))) \to \Lambda^p(M, F(J^3_{0b}(M))), \quad \hat{\alpha} \wedge \beta \delta(\beta^{-1}) \mapsto \beta \bar{\alpha} \]  
  (2.92)
  which is injective.

**Corollary 2.96** There is the following isomorphism,
\[ H^*_cv(M, F(J^1_{0b}(M))) \simeq H^*_dR(M). \]  
(2.93)

We remark that the two ingredients, vertical compact support and bounded jet bundle manifold are of relevance to our application to electrodynamics: the compact vertical domain is relevant to have consistence with bounded jet bundle base manifold and to avoid infinite kinetic world-lines; bounded cohomology is useful to avoid run-away solutions.

### 2.17 Integration theory of generalized forms

**Definition 2.97** The integral of a generalized form \( \bar{\alpha} \in \Gamma \Lambda^p(M, F(J^1_{0b}(M))) \) on the \( p \)-dimensional submanifold \( M_p \) of \( M \) is
\[ \int_{M_p} \bar{\alpha} := \int_{M_p} \langle \bar{\alpha} \rangle. \]
(2.94)
We note the commutativity of the integration operations,

\[ \int_{M_p} \langle \hat{\alpha} \rangle = \langle \int_{M_p} \hat{\alpha} \rangle. \] (2.95)

There is a direct formula for the integral (2.94) in terms of integral of forms \( \hat{\alpha} \),

\[ \int_{M_p} \langle \hat{\alpha} \rangle = \int_{\nu^{-1}(M_p)} \hat{\alpha} \wedge dvol_v, \] (2.96)

where \( dvol_v \) is a vertical volume form. At this point, it is not required that \( dvol_v \) is closed, but we will require later such property to prove the corresponding Stokes’ theorem for generalized forms.

From the definition (2.97), the following is direct,

**Proposition 2.98** For the integral operation (2.94) the following properties hold:

- It reduces to the standard definition for the case of standard differential forms.
- It is direct that the integral of generalized forms is linear,

\[ \int_{M_p} (\hat{\alpha} + \lambda \hat{\beta}) = \int_{M_p} \hat{\alpha} + \lambda \int_{M_p} \hat{\alpha}, \] (2.97)

for any \( \hat{\alpha}, \hat{\beta} \in \Gamma \Lambda^*(M, \mathcal{F}(J^k(J^0_k(M)))) \) and \( \lambda \in \mathbb{R} \).

**Lemma 2.99** If \( \zeta \hat{\alpha} = \hat{\alpha} \), then

\[ \int_{M_p} d_4 \hat{\alpha} = \int_{M_p} \langle d_J \hat{\alpha} \rangle. \] (2.98)

**Proof.** We make use of Lemma 2.52,

\[ \int_{M_p} d_4 \hat{\alpha} = \int_{M_p} \langle \hat{h}_k d_J \hat{\alpha} \rangle = \int_{M_p} \langle d_J \hat{\alpha} \rangle. \] (2.99)

Generalization of Stokes’ Theorem

There is a version of Stokes’ theorem for generalized forms and an invariance under diffeomorphisms of differential forms,

**Proposition 2.100** Let \( \bar{\alpha} \in \Gamma \Lambda^p(M, J^0_k(M)) \) such that \( d_4(\bar{\alpha}) = 0 \) and consider a \( p \)-dimensional submanifold \( M_p \) as before. Let also \( d_J(dvol_v = 0) \). Then the following formula holds,

\[ \int_{M_p} d_4 \bar{\alpha} = \int_{\partial M_p} \bar{\alpha}. \] (2.100)
Proof. From the definition,
\[ \int_{M_p} d\hat{\alpha} = \langle \left( \int_{M_p} dJ \hat{\alpha} \right) \rangle = \int_{k\pi^{-1}(M_p)} (dJ \hat{\alpha}) \wedge dvol = \ast. \]
Then note that the exterior derivative of \( dvol_J \) is zero by hypothesis. Thus we have that \( dJ(\hat{\alpha} \wedge dvol) = dJ \hat{\alpha} \wedge dvol \). Therefore,
\[ \ast = \int_{k\pi^{-1}(M_p)} (dJ \hat{\alpha} \wedge dvol). \]
Then one can use Stokes’ theorem, showing that
\[ \int_{k\pi^{-1}(M_p)} dJ(\hat{\alpha} \wedge dvol) = \int_{k\pi^{-1}(\partial(M_p))} \hat{\alpha} \wedge dvol. \]
Taking into account the relation
\[ \partial(k\pi^{-1}(M_p)) = k\pi^{-1}(\partial(M_p)) \]
and again the definition of the integral \[ \ref{2.97} \] we have that
\[ \int_{k\pi^{-1}(\partial(M_p))} \hat{\alpha} \wedge dvol = \int_{\partial M_p} \langle \hat{\alpha} \rangle = \int_{\partial M_p} \hat{\alpha}. \]
\[ \blacksquare \]

Invariance under diffeomorphisms of the integral
Let \( f : \tilde{M} \rightarrow M \) be a differential function. One can define the pull-back of a generalized form,
\[ f^*\bar{\alpha}(X_1, \ldots, X_p) := \bar{\alpha}(f_1(X_1), \ldots, f_p(X_p)). \tag{2.101} \]
One has the following relations,
\[ f^*\langle \hat{\alpha} \rangle(X_1, \ldots, X_p) = \langle f^*\hat{\alpha} \rangle(X_1, \ldots, X_p) = \langle f^*\hat{\alpha} \rangle(X_1, \ldots, X_p) \]
\[ = \langle f^*(\hat{\alpha})(X_1, \ldots, X_p) \rangle = \langle f^*(\hat{\alpha})(X_1, \ldots, X_p) \rangle = \langle f^*(\hat{\alpha})(X_1, \ldots, X_p) \rangle = \langle f^*(\hat{\alpha})(X_1, \ldots, X_p) \rangle. \]
Then one obtains the following invariance under diffeomorphism,

Proposition 2.101 Let \( f : \tilde{M}_p \rightarrow M_p \) be a diffeomorphism between \( p \)-dimensional manifolds. Then
\[ \int_{\tilde{M}_p} f^*\bar{\alpha} = \int_{M_p} \bar{\alpha}. \tag{2.102} \]

Proof. Using the above computation, one has
\[ \int_{\tilde{M}_p} f^*\bar{\alpha} = \int_{\tilde{M}_p} f^*\langle \hat{\alpha} \rangle = \int_{M_p} \langle \hat{\alpha} \rangle = \int_{M_p} \hat{\alpha}. \]
\[ \blacksquare \]
3 Elements of maximal acceleration spacetimes

The second principal assumption adopted in this work is that the $n$-acceleration of a point charged particle is bounded (in a covariant way, that is, independent of local coordinate system). Let us briefly summarize the idea of maximal acceleration, prior to see where our point of view departs from previous ones. The original idea of maximal acceleration starts with E. Caianiello’s work (see for instance the review [15] and references therein and also the works [11] and [61] for original developments of the idea of maximal acceleration) in the contest of a geometrization of quantum mechanical systems. Thus, in Caianiello’s theory, uncertainty in the observables is related with the curvature of a Sasaki-type metric in the relevant phase space. It was as a consequence of this and that speed of light in vacuum is an upper limit for causal interactions that one arrives to maximal proper acceleration.

Caianiello’s theory was not general covariance of the theory: it was formulated on flat spacetime, using coordinates and for the proper acceleration. A covariant formalism for geometries of maximal acceleration was developed in [31] and explores instead the possibility of bounded covariant $n$-acceleration. Although it was motivated by the non-covariance problem of Caianiello’s quantum geometry [15], the formalism developed in [31] was independent of the details of the mechanism generating the bound in the covariant acceleration and also from the particular value that the maximal acceleration $A_{\text{max}}$ can take. Thus, the framework developed in [31] can be used in a more general context than Caianiello’s quantum geometry. That the maximal proper acceleration is bounded is a direct consequence from the fact that $n$-acceleration is bounded in our formalism.

However, [31] presupposed the existence of the Lorentzian metric $\eta$, from where one constructs the metric of maximal acceleration. In the theory presented in this section we revers the situation. We first introduce a generalized metric $\bar{g} \in \Gamma T^{(0,2)}(M)F(J^2(M))$ as the natural object associated with measurements of the proper time of physical clocks at rest with the generic point particle $x : I \rightarrow M$. Then the Lorentzian proper time along a timelike curve is associated with the proper time of the average of the generalized metric $\bar{g}$. Thus, the Lorentzian spacetime metric emerges as averaged description of the geometry of maximal acceleration.

A main distinction between our approach and other theories and models of maximal acceleration is that in our theory the kinematical formalism contains a maximal $n$-acceleration respect to a given Lorentzian metric from the beginning. We will not attempt to provide the mechanism for the bound of the $n$-acceleration of point charged particles. Such mechanism should be based on a deepest description of spacetime as a discrete, quantum spacetime and will be developed elsewhere. Instead, we provide an heuristic argument for the existence of a maximal acceleration, based on maximal speed and minimal characteristic length. Moreover, we speak of maximal bound $n$-acceleration, in contrast with Caianiello and others approach, that consider bound of the proper acceleration. Our approach to maximal acceleration, has the benefit that it is covariant respect to the Lorentz group and by the introduction of a connection, can be made general covariant, as reference [31].
showed.

The notion of maximal acceleration appears in other theories in Physics. In string theory, it appears as a consequence of the formation of Jean's instability when the strings reach a critical temperature, that makes the strings disconnected [51, 10]. Very recently, maximal proper acceleration was found as a natural consequence in covariant loop gravity [57]. A dramatic consequence of maximal acceleration for the gravitational theory and the equivalence principle is that the corresponding theories in such geometries should be free of singularities, a point first noted by Caianiello.

Maximal acceleration in classical electrodynamics

In classical electrodynamics, there are several scenarios where maximal acceleration appears:

- The Abraham and Lorentz's electron models are theoretically valid under the assumption that acceleration have a value less than a threshold value (see reference [60] for a discussion of those models), in order to preserve causality.
- In the extended model of the electron proposed by P. Caldirola [16] it appears a maximal acceleration, when a maximal speed of interaction and the hypothesis of the minimal unit of time chronon are used in the definition of acceleration [17].
- The existence of a maximal value for the electric field in Born-Infeld electrodynamics [9] suggests that the four dimensional force on a point electron is also bounded in such theory. Thus, it is natural to conjecture the existence of a bounded proper acceleration in such a case.

These three examples bring to light that the hypothesis of maximal acceleration could be of relevance for the solution of some of the important problems in the foundations of classical electrodynamics.

3.1 An heuristic argument for maximal acceleration

There is an heuristic argument for the existence of a maximal acceleration based on the assumption that there is a minimal length $L_{\text{min}}$ and a maximal speed. This argument, first used in the context of classical electrodynamics by Caldirola [17], is here expressed in complete generality. The minimal length is assumed to be scale of the domain in the spacetime that affects the individual particle in changing the dynamical state. This idea is not necessarily related with a quantification of spacetime, but requires a notion of extended local domain where cause-effect relations are originated. Therefore, the maximal acceleration could be relational, depending on the physical system. This is in contrast with universal maximal acceleration. However, we will require that the maximal acceleration is very large compare with the acceleration of the probe particle. In this way, our perturbative scheme will be perfectly applicable.
By adopting the above hypotheses, the effect on a particle done by its surrounding is bounded by a maximal work

\[ L_{\text{min}} m a \sim \delta m v_{\text{max}}^2, \]

where \( a \) is the value of the acceleration in the direction of the total exterior effort is done. Then one associates this value to the work over any fundamental degree of freedom evolving in \( M \), caused by rest of the system. Since the speed must be bounded, \( v_{\text{max}} \leq c \). Also, the maximal work produced by the system on a point particle is \( \delta m = -m \). Thus, there is a bound for the value of the acceleration,

\[ a_{\text{max}} \simeq \frac{c^2}{L_{\text{min}}}. \]  

Thus, the existence of maximal propagation speed and minimal length are of fundamental relevance in the argument for maximal acceleration. This will be of relevance when we investigate the possibility of superluminal motion in this section.

### 3.2 Principle of maximal \( n \)-acceleration and the clock hypothesis in classical mechanics

The notion of physical clock should be related with the general form of the principles of classical dynamics. The first dynamical principle corresponds to the principle of inertia and the notion of inertial coordinate system,

**Definition 3.1** An inertial coordinate system \( (U, \mathbb{R}^n) \) is such that the world-line of any free particle is described by a parameterized straight line of \( \mathbb{R}^n \).

Thus, if the world-line describing a point particle is not a straight line in some given coordinate system, it must be because the point particle is not moving free or the coordinate system is not inertial or both. Note that the geodesic motion is not free motion, even if there is a coordinate system (normal coordinate system) where the geodesic is described by straight line in \( \mathbb{R}^n \).

Even if strictly speaking, there are not perfect inertial coordinate system in real experiments, Definition 3.1 is not empty of content, as it serves as a natural foundation for the second law of dynamics in terms of differential equations. The simplest possibility of a equation of motion for point particles, in concordance with experience and the principle of inertia, is that such differential equations are of second order respect to the time parameter in the inertial coordinate system. Moreover, one can consider approximate inertial systems, that relate with experimental settings. This shows that the notion of inertial coordinate system is useful.

There is some arbitrariness in the choice of the time parameter in the differential equation for a given point particle. However, there are some parameters that appear more natural that others, in the context of the discussion before.
Definition 3.2Given a world-line $\tilde{x} : I \rightarrow M$ corresponding to a physical point particle, a physical time parameter $s$ is such that the curve $\tilde{x} : I \rightarrow M$ satisfies a second order differential equation respect to $s$.

If the world-line $x : I \rightarrow M$ satisfies a second order differential equation respect to a physical time parameter, a natural definition of co-moving physical clock is the following:

Definition 3.3A co-moving physical clock associated with the world-line $\tilde{x} : I \rightarrow M$ of a point particle is a map $\tau : \tilde{x}(\tilde{I}) \rightarrow R, \tau \mapsto x(\tau)$ such that the following diagram

![Diagram]

commutes for any physical time parameterization $\tilde{x} : \tilde{I} \rightarrow M$ of the un-parameterized curve $x(I)$.

We need to prove the existence of co-moving physical clock parameters. In Special and General Relativity, the existence of a co-moving physical time parameter is guaranteed by the clock hypotheses, the Principle of Relativity and the principle of constancy of the speed of light in vacuum. As a result, the parameter $\tau$ can be chosen to be the proper-time of a Lorentzian metric $\eta$. Such proper-time of $\eta$ corresponds to the time of a co-moving physical clock as in Definition 3.2 that is independent of the acceleration respect to an inertial coordinates system.

The clock hypothesis

For world-lines whose acceleration vector is different than zero, it was found useful to make an additional assumption about the rate of clocks associated with the world-line. The point was, to consider the clocks as un-affected by the acceleration of the curve, in such a way that they work exactly as a relativistic clock will do (in the corresponding instantaneous inertial frame). The clock hypothesis can be formulated as follows (see [26], p. 64 and [55], p. 65):

*There exist ideal clocks, that is, clock that are completely unaffected by acceleration; that is, as one whose instantaneous rate depends only on its instantaneous speed in accordance with the time dilatation formula of Special Relativity. Thus, one can adopt such clocks as the co-moving proper clocks.*

This hypothesis is of fundamental relevance for the foundation of gravity theory in terms of a metric structure and in Special Relativity to define instantaneously inertial systems and its relation with inertial coordinate systems. However, the clock hypothesis it does not necessarily hold for arbitrary co-moving physical clocks. Indeed, for clocks that measure time according to a generalized metric, the clock hypothesis does not necessarily holds, since the rate of the proper clocks could
depend on the acceleration. Indeed, we will see later that there are other options that the proper time associated with a Lorentzian metric that are compatible with Definition 3.3.

**Definition 3.4** Given a physical world-line $\tilde{x} : \tilde{I} \to M$, parameterized by a physical parameter $t$, an instantaneous inertial coordinate system associated to the world-line $\tilde{x}$ is a local coordinate system such that, respect to a given inertial system $(U, R^n)$, it moves with constant speed $\frac{d\tilde{x}^\mu}{dt}$.

**Principle of maximal $n$-acceleration**

We can formulate the principle in the following way,

For each physical world-line, there is associated a physical co-moving clock such that for the time parameter $\tau$ of such clock, the $n$-acceleration measured is bounded by a constant value $A_{\text{max}}$ that does not depend on the world-line in the sense that

$$\eta(\ddot{x}, \dot{x}) < A_{\text{max}}.$$  \hspace{1cm} (3.2)

for the Lorentzian metric $\eta$. Therefore, we reverse the previous theories of maximal acceleration and introduce the principle of maximal $n$-acceleration from the beginning. The principle, stated in such a way, will convey the modifications of the field theories as well as the classical equation of motion of point particles.

In the next section we will prove that it is possible to construct a generalized metrics $g$ from a Lorentzian metric $\eta$, such that principle of maximal $n$-acceleration holds. On the other hand, the clock hypothesis does not hold for geometries of maximal acceleration. This is a form of converse result that the one in [29], where the starting point is the non-validity of the clock hypothesis, they concluded the existence of a maximal (universal) acceleration [29].

**3.3 General covariant formulation of the metric of maximal $n$-acceleration**

Since we are assuming the existence of the Lorentzian metric $\eta$, we can construct its Levi-Civita connection $D$ and its derivative operator along $x : I \to M$. Then the four covariant acceleration vector $D_x \dot{x} \in T_{\tau}x M$ is bounded by the Lorentzian metric $\eta$ (note that the covariant acceleration is spatial like vector). This is defined on the bundle $TM \setminus NC$, where $\pi_{NC} : NC \to M$ is the null-cone bundle determined by $\eta$ and

$$NC := \bigcup_{x \in M} NC_x, \quad NC_x := \{ y \in T_x M \text{ s.t. } g(y, y) = 0 \}.$$  \hspace{1cm} (3.3)

Let $(M, \eta)$ be a Lorentzian $n$-dimensional spacetime. Then there is defined a Sasaki-type metric on the bundle $TM \setminus NC$,

$$g_S = \eta_{\mu\nu} dx^\mu \otimes dx^\nu + \frac{1}{A^2_{\text{max}}} \eta_{\mu\nu} \left( \delta y^\mu \otimes \delta y^\nu \right).$$  \hspace{1cm} (3.3)
Theorem 3.5 Let \( x : I \to M \) be a curve such that

1. If the tangent vector at the point \( 1^x \in J^1_0(M) \) is \( T = (\dot{x}, \ddot{x}) \) and
2. It holds that \( \eta(\dot{x}, \ddot{x}) \neq 0 \).

Then there is a non-degenerate, symmetric form \( g \) obtained from the embedding isometric embedding \( e : x(I) \to TM \) from \((TM, g_S)\) such that acting on the tangent vector \( \dot{x} \) the value is

\[
g_{\mu\nu}(x(\tau)) = \left( 1 + \frac{\eta(D_{\tau} \dot{x}(\tau) D_{\tau} \dot{x}(\tau))}{A_{\max}^2 \eta(\dot{x}, \ddot{x})} \right) \eta_{\mu\nu}. \tag{3.4}
\]

Proof. The tangent vector at the point \( (x(\tau), \dot{x}(\tau)) = 1^x \in TM \) is \( (\dot{x}, \ddot{x}) \in TTM \). The metric \( g_S \) acting on the vector field \( T = (\dot{x}, \ddot{x}) \in T(x(\tau), \dot{x}(\tau))N \) has the value

\[
g_S(T, T) = \left( \eta_{\mu\nu} dx^\mu \otimes dx^\nu + \frac{1}{A_{\max}^2} \eta_{\mu\nu}(\delta y^\mu \otimes \delta y^\nu) \right)(T, T)
= \left( \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{A_{\max}^2} \eta_{\mu\nu}(\dot{x}^\mu - N^\mu_{\rho}(x, \dot{x}) \dot{x}^\rho) (\dot{x}^\nu - N^\nu_{\lambda}(x, \dot{x}) \dot{x}^\lambda) \right) = \star.
\]

By the second hypothesis, one obtains that the following expression holds:

\[
\star = \left( 1 + \frac{1}{A_{\max}^2 \eta(\dot{x}, \ddot{x})} \eta_{\mu\nu} D_{\tau} \dot{x}^\mu D_{\tau} \dot{x}^\nu \right) \eta(\dot{x}, \ddot{x}),
\]

that coincides with the value \( (g_{\mu\nu} dx^\mu \otimes dx^\nu)(\dot{x}, \ddot{x}) \), with the components \( g_{\mu\nu} \) given by the formula (3.4).

The bilinear form

\[
g(2x) = g_{\mu\nu}(2x) dx^\mu \otimes dx^\nu \tag{3.5}
\]

with components given by the covariant formula (3.4) is the metric of maximal acceleration in general coordinates. It determines the proper time along the curve \( x : I \to M \) and also provides a generalization of the notion of angle.

Corollary 3.6 Let \( x : I \to M \) such that:

- It holds that \( g(\dot{x}, \ddot{x}) < 0, \eta(\dot{x}, \ddot{x}) < 0 \),
- The covariant condition

\[
\eta(D_{\tau} \dot{x}, D_{\tau} \dot{x}) \geq 0. \tag{3.6}
\]

holds. Then one has the natural bound

\[
0 \leq \eta(D_{\tau} \dot{x}, D_{\tau} \dot{x}) < A_{\max}^2. \tag{3.7}
\]
Because this property, the bilinear form \([3.4]\) is a metric of maximal acceleration, since the covariant \(n\)-acceleration it turns out to be bounded by the maximal value \(A_{\text{max}}\).

**Definition 3.7** A curve of maximal acceleration is a curve \(x : I \rightarrow M\) such that
\[
\eta(D_x \dot{x}, D_x \ddot{x}) = A_{\text{max}}^2. \tag{3.8}
\]

**Corollary 3.8** For a curve of maximal acceleration \(x : I \rightarrow M\), one has the relation
\[
g(\dot{x}, \ddot{x}) = 1 + \eta(\dot{x}, \dot{x}). \tag{3.9}
\]

This result indicates that for maximal acceleration curves, the proper parameters associated to \(\eta\) and \(g\) differ considerably.

If the covariant derivative \(D\) is the induced connection on \(TM \setminus NC\) induced by the Levi-Civita connection of the Minkowski metric \(\eta\) as it was defined in section 2, there is defined globally a coordinates system where the connection coefficients \(\gamma^\mu_{\nu\rho}\) are zero. In such coordinate system \(N^\mu_{\rho} = 0\) holds and therefore \(D_\xi \dot{x} = \ddot{x}\). In such coordinate system it also holds \(\gamma^\mu_{\nu\rho} = 0\). Therefore, the metric coefficients\([3.4]\) can be written as
\[
g_{\mu\nu}(x(\tau)) := \left(1 + \frac{\eta_{\sigma\lambda}\ddot{x}^\sigma(\tau)\ddot{x}^\lambda(\tau)}{A_{\text{max}}^2 \eta(\dot{x}, \dot{x})}\right) \eta_{\mu\nu} \, dx^\mu \otimes dx^\nu, \tag{3.10}
\]
that defines an element \(g \in \Gamma T^{(0,2)}(M, \mathcal{F}J^2_0(M))\) in the following way: given a curve \(2x : I \rightarrow J^2_0(M)\), the value of \(g\) along the curve \(s \mapsto x(s)\) is \(g(2^2x(\tau)) = g(\tau)\).

**Remark 3.9** The appearance of the metric \(\eta\) avoids a fully general invariant theory of geometry of maximal acceleration. It should be much more natural to obtain \(\eta\) from the basic fundamental object that is a generalized metric \(\bar{g}\). However, we hope that a future form of the theory could provide a natural origin to the metric \(\eta\) from fundamental principles.

### 3.4 Perturbation scheme

Let us denote by \(\tau\) the proper-time parameter along a given curve \(x : I \rightarrow M\) respect to \(g\). The acceleration square function is defined by the expression
\[
a^2(\tau) := \eta_{\mu\nu} \ddot{x}^\mu \ddot{x}^\nu. \tag{3.11}
\]

In our considerations the curves are far from the maximal acceleration condition, \(a^2(\tau) \ll A_{\text{max}}^2\). To make precise meaning to such statement we need of a perturbation theory. Let us consider the difference
\[
\delta(\tau) := \eta(\dot{x}, \dot{x}) - g(\dot{x}, \dot{x}).
\]
Then the following approximations hold,

\[
\left( 1 + \frac{\eta \sigma \lambda \ddot{x}^\lambda (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2 \eta(\dot{x}, \dot{x})} \right) \eta(\dot{x}, \dot{x}) = \left( 1 + \frac{\eta \sigma \lambda \ddot{x}^\sigma (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2 (g(\dot{x}, \dot{x}) + \delta(\dot{x}, \dot{x}))} \right) \eta(\dot{x}, \dot{x}) \\
\simeq \left( 1 - \frac{\eta \sigma \lambda \ddot{x}^\sigma (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2 (1 - \delta(\dot{x}, \dot{x}))} \right) \eta(\dot{x}, \dot{x}) \\
\simeq \left( 1 - \frac{\eta \sigma \lambda \ddot{x}^\sigma (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2} \right) \eta(\dot{x}, \dot{x}) + \frac{\eta(\ddot{x}(\tau), \ddot{x}(\tau))}{A_{\text{max}}^2} \delta \eta(\dot{x}, \dot{x}).
\]

Since the form \( \delta \) is \textit{small} on curves far from the maximal acceleration curves, we neglect the second term

\[
\left| \left( 1 - \frac{\eta \sigma \lambda \ddot{x}^\sigma (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2} \right) \eta(\dot{x}, \dot{x}) \right| >> \left| \frac{\eta(\ddot{x}(\tau), \ddot{x}(\tau))}{A_{\text{max}}^2} \delta \eta(\dot{x}, \dot{x}) \right|. \tag{3.12}
\]

We will keep the above approximation \( \delta \approx 0 \) when \( a^2 << A_{\text{max}}^2 \) in the calculations performed in this work. This is because the function \( \delta(\dot{x}, \dot{x}) \) adds a higher order term in the perturbative expressions that we will consider. Thus for instance, the expression of the generalized metric without such approximation will be \( \delta \approx 0 \); with the approximation, the metric is instead

\[
g_{\mu\nu} = (1 - \frac{\eta(\dot{D}_2 \dot{x}, \dot{D}_2 \dot{x})}{A_{\text{max}}^2}) \eta(\dot{x}, \dot{x}) + \text{higher order terms}.
\]

In a normal coordinate system for \( \eta \), the function \( \epsilon \) is defined by the relation

\[
\epsilon(\tau) := \left( \frac{\eta \sigma \lambda \ddot{x}^\sigma (\tau) \dot{\lambda}^\lambda (\tau)}{A_{\text{max}}^2} \right).
\tag{3.13}
\]

The covariant definition of the function \( \epsilon \) requires of a non-linear connection,

\[
\epsilon(\tau) := \left( \frac{\eta(\dot{D}_2 \dot{x}, \dot{D}_2 \dot{x})}{A_{\text{max}}^2} \right).
\tag{3.14}
\]

For timelike trajectories and from the relation \( (3.10) \), the relation between \( g \) and \( \eta \)

\[
g(\tau) = (1 - \epsilon(\tau)) \eta. \tag{3.15}
\]

It follows that the relation between the proper parameter of \( \eta \) and \( g \) is

\[
ds = (1 - \dot{\epsilon})^{-1} d\tau. \tag{3.16}
\]

Therefore, given a timelike curve respect to \( \tau \) the relation between \( s \) and \( \tau \) is not strictly speaking a re-parameterization \( r : R \rightarrow R \). This is related with the fact that \( g \) measures the physical proper times, while \( \eta \) appears as a derived, although convenient object.
The function $\epsilon(\tau)$ determines a bookkeeping parameter $\epsilon_0$ by where

$$\epsilon(\tau) = \epsilon_0 h(\tau), \quad \epsilon_0 = \max\{\epsilon(\tau), \tau \in I\}. \quad (3.17)$$

For compact curves, the bookkeeping parameter always exists. However, we will need to bound the value of higher order derivatives in order to keep such parameter bound for non compact curves. A generator set for asymptotic expansions is \{\epsilon^l, l = -\infty, ..., -1, 0, 1, ...\}. Now we can make sense of the statement that we will consider curves that are far from the curves of maximal acceleration. What that means is that the effects of order $\epsilon^2$ or higher in any analytical function on $\epsilon_0$ are negligible compared with first order. Then the monomials in powers of the derivatives of $\epsilon$ define a basis for asymptotic expansions.

Also, for curves of maximal acceleration the generalized metric (3.15) is degenerated,

$$g(Z, Z)|_{a=A_{\text{max}}} = (1 - \epsilon)|_{a=A_{\text{max}}}, \eta(Z, Z) = 0,$$

for all vector field along $\dot{x} : I \rightarrow M$. Therefore, more of our considerations will not be applicable to such curves. Indeed, we will assume that all the derivatives ($\epsilon, \dot{\epsilon}, \ddot{\epsilon}, ...$) are small.

The generalized metric $g$ defines different kinematical relations than $\eta$. The parametrization of the world-line curves are such that $g(\dot{x}, \dot{x}) = -1$, implying that $\eta(\dot{x}, \dot{x}) \neq -1$ in general. Indeed, we have the following

**Proposition 3.10** In a maximal acceleration geometry space $(M, g)$ the following kinematical conditions hold:

1. $g(\dot{x}, \dot{x}) = -1$, \quad (3.18)
2. $g(\dot{x}, \ddot{x}) = \frac{\dot{\epsilon}}{2} \eta(\dot{x}, \dot{x}) + \text{higher order terms}$ \quad (3.19)
3. $g(\dddot{x}, \ddot{x}) + g(\ddot{x}, \dddot{x}) = \frac{d}{d\tau} \left( \frac{\dot{\epsilon} \eta(\dot{x}, \ddot{x})}{2} \right) + \dot{\epsilon} \eta(\ddot{x}, \ddot{x}) + \text{higher order terms}$ \quad (3.20)

and analogous conditions hold for higher derivatives obtained by derivation of the previous ones.

**Proof.** The first condition holds by definition. The second relation is obtained by taking the derivative of (3.18):

$$2 g_{\mu\rho} \dddot{x}^\mu \dddot{x}^\rho + \frac{d}{d\tau} \left( g_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu = 0.$$

From the definition of $\dot{\epsilon}$ and $g_{\mu\nu}$, one has that

$$\frac{d}{d\tau} (g_{\mu\nu}) = \dot{\epsilon} \eta_{\mu\nu} + \text{higher order terms}.$$
Therefore,
\[ 2g_{\mu\rho} \dddot{x}^{\mu} \dot{x}^{\rho} - \dot{\epsilon} \eta_{\mu\nu} \dddot{x}^{\mu} \ddot{x}^{\nu} + \text{higher order terms} = 0, \]
from which follows (3.19). The third relation is obtained by deriving (3.19) and taking into account (3.18):
\[ \frac{d}{d\tau} (g_{\mu\nu}) \dddot{x}^{\mu} \dot{x}^{\nu} + g_{\mu\nu} \dddot{x}^{\mu} \ddot{x}^{\nu} + g_{\mu\nu} \dddot{x}^{\mu} \dddot{x}^{\nu} = \frac{d}{d\tau} \left( \frac{\epsilon}{2} \eta(\dot{x}, \dot{x}) \right) + \text{higher order terms}, \]
from which follows the third relation (3.20),
\[ g(\dddot{x}^{\mu} \dot{x}^{\rho}) + g(\dddot{x}^{\mu} \dot{x}^{\rho}) = g_{\mu\nu} \dddot{x}^{\mu} \ddot{x}^{\nu} + g_{\mu\nu} \dddot{x}^{\mu} \dddot{x}^{\nu} = \frac{d}{d\tau} \left( \frac{\epsilon}{2} \eta(\dot{x}, \dot{x}) \right) + \dot{\epsilon} \eta(\dddot{x}, \dot{x}) + \text{higher order terms}. \]
The conditions for higher derivatives are obtained from previous ones by derivation and algebraic manipulations.

As a consequence one has the following approximate coordinate expressions,

**Corollary 3.11** For a geometry of maximal acceleration \((M, g)\), given the normalization \(g(\dot{x}, \dot{x}) = -1\), the following approximate expressions hold:

\(\dot{x}^{\rho} \dot{x}_{\rho} := g_{\mu\rho} \dot{x}^{\mu} \dot{x}^{\rho} = -1, \) \hspace{1cm} (3.21)

\(\dddot{x}^{\rho} \dddot{x}_{\rho} := g_{\mu\rho} \dddot{x}^{\mu} \dot{x}^{\rho} = \frac{\dot{\epsilon}}{2} + \mathcal{O}(\epsilon_0^2), \) \hspace{1cm} (3.22)

\(\dddot{x}^{\rho} \dddot{x}_{\rho} + \dddot{x}^{\rho} \dddot{x}_{\rho} = g_{\mu\rho} \dddot{x}^{\mu} \dot{x}^{\rho} + g_{\mu\rho} \dddot{x}^{\mu} \dddot{x}^{\rho} = \frac{d}{d\tau} \left( \frac{\dot{\epsilon} \eta(\dot{x}, \dot{x})}{2} \right) - \dot{\epsilon} + \mathcal{O}(\epsilon_0^2). \) \hspace{1cm} (3.23)

**Proof.** From Proposition 3.10 and the fact that \(g = (1 - \epsilon) \eta\), one gets the above expressions as the first order approximation in \(\epsilon_0\). \(\square\)

**Remark 3.12** Since we have disregarded from the beginning to consider higher orders in \(\epsilon\) and \(\delta\), our theory is a linear theory and not a complete perturbative theory and only with validity up to first order in \(\epsilon\).

### 3.5 Causal structure of the metric of maximal acceleration

For the analysis of the null sectors of the metric \(g\), the expression (3.21) cannot be used directly, since the factor in the denominator \(\eta(\dot{x}, \dot{x})\) is not allowed to be zero. It is more natural to use the relation (3.3), where \(\eta\) can be zero. Thus, the natural structure of a metric of maximal acceleration is the null set, that we define as follows,
Proposition 3.13 The null bundle $\pi_{NC} : NC \to M$ of the maximal acceleration metric $g$ is characterized by the following type of curves,

1. Null geodesics, characterized by $\eta(\dot{x}, \dot{x}) = 0$ and $\eta(D\dot{x}, D\dot{x}) = 0$.

2. Curves $x : I \to M$ characterized by the condition

$$
\left( 1 + \frac{\eta_{\sigma\lambda} \dddot{x}^\sigma(\tau) \dddot{x}^\lambda(\tau)}{A^2_{\max} \eta(\dot{x}, \dot{x})} \right) = 0, \quad \eta(\dot{x}, \dot{x}) \neq 0.
$$

Proof. From the formula of the Sasaki metric (3.3), it follows that $g(\dot{x}, \dot{x}) = \eta(\dot{x}, \dot{x}) + \frac{1}{A^2_{\max}} \eta(D\dot{x}, D\dot{x})$. Thus, if $g(\dot{x}, \dot{x}) = 0$ and $\eta(\dot{x}, \dot{x}) = 0$, it is necessary that:

1. $\eta(D\dot{x}, D\dot{x}) = 0$ or

2. $A^2_{\max}$ is not bounded.

Since we are assuming that the $n$-acceleration is bounded, only the first possibility is applicable, that corresponds to lightlike geodesics. On the other hand, if $\eta(\dot{x}, \dot{x}) \neq 0$, then the condition $g(\dot{x}, \dot{x}) = 0$ is equivalent to

$$
\left( 1 + \frac{\eta_{\sigma\lambda} \dddot{x}^\sigma(\tau) \dddot{x}^\lambda(\tau)}{A^2_{\max} \eta(\dot{x}, \dot{x})} \right) = 0,
$$

that corresponds to a curve of maximal acceleration. 

This implies that apart from the curves of maximal acceleration, the only null curves compatible with a maximal acceleration geometry, parameterized by the proper-time of $g$, are the null geodesics only.

The natural definition of time orientation in the space $(M, g)$ is the following,

Definition 3.14 A spacetime $(M, g)$ is time oriented if there is a vector field $W \in \Gamma TM$ such that at each point $x \in M$ and for each integral curve $x_W : I \to M$ of $W$ with initial condition $x_W(0) = x$, the vector field $W$ is timelike in the sense that $g(W, W) < 0$ along $x_W : I \to M$.

Thus, a future oriented timelike vector $Z$ is such that if the corresponding integral curve is $x_Z : I \to M$ and $W : I \to T x_Z$ is the restriction along the curve $x_Z$, then

$$
g(W, Z) := g_{\mu\nu}(k x_Z) Z^\mu W^\nu < 0.
$$

In a similar way, a curve $x : I \to M$ is future oriented if it is timelike and the tangent vector is future oriented respect to $g$.

The following result is direct,

Proposition 3.15 For curves $x : I \to M$ such that the condition (3.7) holds. Then:
Any time orientation \( T \in \Gamma TM \) of \( g \) is a time orientation of \( \eta \).

The causal character of any vector \( Z \in T_xM \) respect to \( g \) and \( \eta \) is the same.

**Proposition 3.16** The subset \( T_x^-M \subset TM \) of timelike, future oriented tangent vectors respect to \( g \) is an open set.

**Proof.** The future pointed sets respect to \( \eta \) is a strictly convex open cone on \( T_xM \). The condition \( \frac{a^2}{A_{\text{max}}} \) is open. Therefore, the intersections of both conditions, that is the constraint for \( g(\dot{x}, \dot{x})\eta(\dot{x}, \dot{x}) \) is positive and therefore \( T_x^-M \) is an open set. \( \square \).

The sector with \( \frac{a^2}{A_{\text{max}}} > 1 \) corresponds to a change in the causal structure of \( g \) respect to the causal structure of the averaged metric \( \eta \): any timelike vector with the metric \( \eta \) is spacelike with the metric \( g \) and vice versa, any timelike vector of \( g \) is a spacelike vector of \( \eta \). Thus curves of maximal acceleration define the boundary of different signature sectors.

### 3.6 Measurable Euclidean length and \( n \)-velocity in a geometry of maximal acceleration

By an observer we will mean a timelike, future oriented curve \( O : I \rightarrow M \). The notion of speed vector for a timelike trajectory can be defined unambiguously as follows. First, one considers the measurable Euclidean distance between an observer \( O \) and a point \( q \) as follows: when the observer \( O \) is at the spacetime point \( p \) sends a light signal. The signal reaches the point \( q \) and is reflected back to \( p' \). The reflected signal light is detected by the observer. The distance \( d(O, q) \) between the observer \( O \) and the point \( q \) is defined as one half times the speed of light in vacuum multiplied by the elapsed time \( T_{pp'} \) measured by the observer \( O \).

Now consider two points \( p, q \in M \).

**Definition 3.17** The measurable Euclidean distance between the points \( p, q \in M \) measured by the observer \( O : I \rightarrow M \) is defined as

\[
d_E(p, q) = |d(O, p) - d(O, q)|.
\]

(3.24)

This definition is applicable even if \( p \) and \( q \) are not simultaneous respect to \( O \). The formal consistency requirement for this for this operational definition of distance is that the speed of light is universally constant for all observer.

As a direct result we have that

**Proposition 3.18** For each observer \( O : I \rightarrow M \), the function

\[
d_E : M \times M \rightarrow \mathbb{R}, \quad (p, q) \mapsto |d(O, p) - d(O, q)|
\]

determines a metric function on \( M \).
Until now, the above definitions does not make use of any metric structure and only of the constancy of the speed of light. Therefore, we adopt the above as the definitions happening in a spacetime of maximal acceleration \((M, g)\).

In a maximal acceleration spacetime \((M, g)\), along a timelike curve \(x: I \to M\) the proper time elapsed from \(x(\tau)\) to \(x(\tau + \delta)\) is given by the formula
\[
\delta = \int_\tau^{\tau + \delta} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\tilde{\tau}.
\]

The fact that the geometry is of maximal acceleration changes the notion of proper time respect to the averaged Lorentzian geometry determined by the metric \(\eta = \langle g \rangle\).

Under the assumption that the observable geometry is the geometry of maximal acceleration, the observable averaged speed of a curve \(x: I \to M\) at the instant \(\tau\) measured by an observer \(O\) is defined to be
\[
v(\tau) = \lim_{\delta \to 0} \frac{1}{\delta} \frac{1}{\sqrt{1 - a^2 A_{\text{max}}^2}} T_{x(\tau)x(\tau + \delta)},
\]
where the distance \(T_{x(\tau)x(\tau + \delta)}\) is operationally defined as before for the observer \(O\). Thus for a geometry of maximal acceleration \((M, g)\), we adopt an analogous definition but replacing \(\eta\) by \(g\) in the new rule to calculate physical proper times.

In this way, we define

**Definition 3.19** Let \((M, g)\) be a maximal acceleration geometry and let \(x: I \to M\) be a timelike curve. Then the averaged speed between the points \(x(\tau)\) and \(x(\tau + \delta)\) along the trajectory \(x: I \to M\) is
\[
v(\tau) = \lim_{\delta \to 0} \frac{1}{\delta} \frac{1}{\sqrt{1 - a^2 A_{\text{max}}^2}} T_{x(\tau)x(\tau + \delta)}.
\] (3.26)

Using the formula for the maximal acceleration metric \((3.15)\), one can re-write the averaged speed as
\[
v(\tau) := \lim_{\delta \to 0} \frac{1}{\sqrt{1 - a^2 A_{\text{max}}^2}} \tilde{v}(s),
\]
where \(\tilde{v}(\tau)\) stands for the standard definition relativistic speed and with the condition that \(x(\tau) = \tilde{x}(s)\) using the proper-time of the Lorentzian metric \(\eta\),
\[
\tilde{v}(s) := \lim_{\delta \to 0} \frac{T_{\tilde{x}(s)\tilde{x}(s + \delta)}}{\int_{s}^{s+\delta} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\tilde{s}}.
\]

In order to simplify the treatment, let us consider \(a^2(\tau)\) to be constant. Then the above expression leads to
\[
v(\tau) := \frac{1}{\sqrt{1 - a^2 A_{\text{max}}^2}} \tilde{v}(s).
\] (3.27)

\(^{7}\)Note that the map \(t \mapsto \tau\) is not a diffeomorphism form \(R\) to \(R\). Therefore, although \(s\) and \(\tau\) are valid parameterizations of a curve, they are not reparameterizations of each other.

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The vector form of the notion of measurable averaged speed is the measurable $n$-velocity

$$v^\mu(\tau) = \frac{1}{\sqrt{1 - \frac{a^2}{A_{\text{max}}^2}}} \tilde{v}^\mu(s), \quad \mu = 1, \ldots, n.$$  (3.28)

The limit of Special Relativity is recovered by $A_{\text{max}} \to \infty$.

The decomposition of $v$ in temporal and spatial components respect to an inertial observer $O$ is

$$\frac{v}{c} = \frac{1}{\sqrt{1 - \frac{a^2}{A_{\text{max}}^2}}} \frac{1}{\sqrt{1 - \frac{\tilde{v}^2}{c^2}}} (1, \tilde{v}).$$

Taking into account the relation (3.27), we have that for the measurable $n$-velocity the formula

$$\frac{v}{c} = \frac{1}{\sqrt{1 - \frac{a^2}{A_{\text{max}}^2}}} \frac{1}{\sqrt{1 - \frac{\tilde{v}^2}{c^2}}} (1, \tilde{v}).$$  (3.29)

### 3.7 On the possibility of superluminal motion in spacetimes with metrics of maximal acceleration

From an abstract point of view, it seems possible to have superluminal motion in geometries of maximal acceleration. However, when the physical arguments in favour of maximal acceleration are considered, the impossibility of superluminal motion is concluded.

The mathematical conditions for superluminal motion

**Proposition 3.20** Let $(M, g)$ be a metric of maximal acceleration. Then

- Along a timelike trajectory with parameterizations $x : I \to M, \tau \mapsto x(\tau)$ and $\tilde{x} : I \to M, s \mapsto \tilde{x}(s) = x(\tau)$ as before, one has that
  $$v(\tau) \geq \tilde{v}(s)$$
  and equality only holds iff $D\dot{x} = 0$.
- The definition of definition of the measurable speed $v$ is the domain $a^2 < A_{\text{max}}^2$.

Condition (3.30) suggests the possibility of superluminal motion in spacetimes with maximal acceleration,

**Proposition 3.21** Let $(M, g)$ be a spacetime of maximal acceleration. Then a necessary condition for superluminal motion $\tilde{v}(\tau) > 1 = c$ consistent with the principle of maximal acceleration in the physical domain $\eta(\tilde{x}, \tilde{x}) < A_{\text{max}}^2, g(\tilde{x}, \tilde{x}) = -1$ is

$$a|\tilde{v}| > A_{\text{max}} c.$$  (3.31)
Proof. By the vector components in formula (3.29), one has that
\[ |\vec{v}| \frac{c}{c^2} = \frac{1}{\sqrt{1 - \left(1 - \frac{a^2}{A_{\text{max}}^2}\right) \frac{c^2}{c^2}}} > 1. \]

After a bit of algebra, this condition can be written as equation (3.31). □

This condition is compatible with \( \frac{a^2}{A_{\text{max}}^2} < 1 \) if \( |\vec{v}| > c \); it is compatible with \( \frac{a^2}{A_{\text{max}}^2} < 1 \) iff \( |\vec{v}| \gg c \). It is not possible for the case \( c = |\vec{v}| \) in the perturbative regime.

Some direct consequences of such kinematics are the following:

- It is not possible to have superluminal motion for geodesic motion. Thus a point particle in a pure gravitational field will not be superluminal.
- In the perturbative regime \( a^2 << A_{\text{max}}^2 \), superluminal motion is possible iff \( \beta^{-1} << 1 \).

Finally, let us mention that the quantity \( P_c = m A_{\text{max}} c \) has the dimensions of a power, that we can call critical power. Therefore, superluminal condition (3.31) requires the existence of powers on an electron bigger than the critical power.

The superluminal motion from the heuristic point of view

The heuristic argument in the beginning of the section, if one needs to consider it seriously, implies the existence of a maximal acceleration from two assumptions:

- Existence of a maximal speed for the propagation of physical interactions,
- Existence of a minimal length for the characteristic phenomena (a kind of generalized Debye length).

Thus, if maximal acceleration is a consistent requisite from a minimal length and maximal speed, it is not possible to have superluminal world-lines in a geometry of maximal acceleration. A further argument in the context of electromagnetic radiation will be given in section 6, when we obtain the value of the maximal acceleration for point charged particles.

3.8 Isometry group of the metric of maximal acceleration and null-structure

The assumption that the speed of light is constant and independent of the observer \( O \) has been of fundamental relevance to obtain a covariant notion of \( n \)-velocity vector which is independent of parameterizations. The proper parameterization of physical world-line curves is determined by the metric of maximal acceleration \( g \). Doing this, the principle of maximal acceleration is full-filled. The speed of light has been used as the standard to measure distances. Therefore, the Principle of constancy...
of the speed of light in vacuum also holds in our theory. Therefore, the generalized metric structure $g$ should have an isometry group that leaves invariant the null-cone structure. By Proposition 3.13, the null-cone of $g$ is the null-cone of $\eta$. Therefore,

**Proposition 3.22** Given a metric of maximal acceleration, the group leaving the null-cone structure of $g$ invariant is a Lie group (conformal group in dimension $n > 0$).

Also, the isometry group of the metric $g$ contains the isometry group of the Lorentzian metric $\eta$,

$$Iso(g) \supset Iso(\eta).$$

The factor $(1 - A_{\text{max}}^2)$ explicitly breaks the group of conformal transformations (that leaves invariant the light-cone). In the case that $A_{\text{max}} \to \infty$, the group leaving invariant the cone structure is the conformal group in $n$-dimensions. In the case that $\eta$ is the Minkowski metric, if $Iso(\eta) = O(1, n-1)$, then $Iso(g) \supset O(1, n-1)$ and then the proper-time of a metric of maximal acceleration is Lorentz invariant. Indeed, this is the largest group possible, since the acceleration factor

4 Higher order generalization of the electromagnetic field and current

In this section we introduce the notion of generalized electromagnetic fields as sections of a bundle $\Lambda^2(M, \mathcal{F}(J^k_0(M)))$ for an integer $k \geq 2$. The value of the integer $k$ will be fixed later. We will use the results on exterior algebra and cohomology from section 2. Due to the isomorphisms (2.86) and (2.87), the fields can also be considered as sections of $\Lambda^{2+nk}(J^k_0(M))$ and $\Lambda^{2+nk+1}(\tilde{J}^{k+1}_0(M))$. However, in order to keep a general formulation of generalized higher order field, we will develop in this section the formalism at the level of forms in $\Lambda^2(M, \mathcal{F}(J^k_0(M)))$.

It is well known that in standard Maxwell electrodynamics, the electromagnetic field of a point charged particle is divergent, with a singularity of Coulomb type. For the generalized higher order fields we will assume that also contain singularities of Coulomb type and that these are the only singularities that appear. Under such hypothesis, one can regularize the fields, with the result that the fields appearing in the equation of motion of a point charged probing particle are finite.

In Maxwell’s electrodynamics the fields are divergent along the world-line of the point charged particle, which is the submanifold the submanifold $e(I) = S \hookrightarrow M$, with $e : I \to M$ the world-line. $S$ is dynamically determined by the equation of motion of the point particle. In this case, the spacetime is $\tilde{M} := M \setminus S$ is by definition the region where the fields are finite.

After renormalization of mass procedure, all the fields that appear in the equations of motion are finite. Therefore, there is not need to subtract the submanifold $S$ from the domain of definition of the fields. This is the procedure that we will follow.
in this section. We first start with fields that are divergent on $S$ and after the introduction of a regulation procedure (that coincides with Dirac’s procedure), the fields will defined in the whole manifold $M$.

Finally, since we will use the exterior derivative of generalized forms, the notion of generalized higher order field that we will use is in the strong sense of definition 2.23. This will be implicitly understood in the rest of the work.

4.1 Generalization of the electromagnetic field as sections of $\Lambda^2(M, F(J^k_0(M)))$

The generalization of the Faraday form is given by the following

**Definition 4.1** Given a curve $x : I \rightarrow M$, the electromagnetic field $\bar{F}$ along the lift $k x : I \rightarrow J^k_0(M)$ is a closed 2-form $\bar{F} \in \Gamma\Lambda^2(M \setminus S, F(J^k_0(M \setminus S)))$.

Thus, in a local natural coordinate system, the generalized Faraday form $\bar{F}$ can be written as

$$\bar{F}(kx) = \bar{F}(x, \dot{x}, \ddot{x}, \ldots) = \left((F_{\mu\nu}(x)) + \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\right)dx^\mu \wedge dx^\nu, \quad (4.1)$$

with $F(x) \in \Gamma\Lambda^2 M$.

**Remark 4.2** We have the following remarks:

- The field $\bar{F} \in \Gamma\Lambda^2(M \setminus S, F(J^k_0(M \setminus S)))$. Other related fields appearing in the electromagnetic theory will be generalized to sections higher order jet bundles.

- $\tilde{\varphi}(\bar{F})$ is closed, $d_4 \tilde{\varphi}(\bar{F}) = 0$ and therefore it defines an element of the cohomology $H^*_c(J^k_0(M))$,

$$[\bar{F}] \mapsto [\tilde{\varphi}(\bar{F})] = [k\xi(\bar{F}) \wedge d\text{vol}_V].$$

These two notions for the electromagnetic field are equivalent.

- $x : I \rightarrow M$ must not be interpreted physically as the world-line of the particle generating the field. Indeed the curve $x : I \rightarrow M$ corresponds to the world-line of a point charged particle whose motion can experimentally be observed. Comparing such trajectories with the free particle world-lines, one should be able to identify the effect of the full electromagnetic field.

- Since the field $F(x)$ lives on $M$, it is clear that $d_4(\varphi(F)) = \varphi(dF)$.

Using the cohomology theory of forms on $\Lambda^p(M, F(J^k_0(M)))$ one can establish the following result,

**Theorem 4.3** Let $\bar{F} \in \Gamma\Lambda^2(M \setminus S, F(J^k_0(M \setminus S)))$, $d_4\bar{F}(kx) = 0$ be a generalized higher order electromagnetic field. Then the decomposition (4.1) with $d\bar{F} = 0$ is unique.
Proof. By the uniqueness in (2.90), the field \( F(x) := F_{\mu \nu}^{} dx^\mu \wedge dx^\nu \) in (4.1) can be identified with \( \varphi(\langle F \rangle) \) and since \( \varphi^* \) is an isomorphism, \( F(x) \) can also be identified with \( \langle F \rangle \) up to gauge in \( H_{2+\text{kn}}^2(M \setminus S, \mathcal{F}(J_{\text{c}}^k(M \setminus S))) \). This implies that

\[
F(x) = \langle F(x) \rangle + \langle d_J A \rangle, \quad A \in \Lambda^{1+\text{kn}}(M \setminus S, \mathcal{F}(J_{\text{c}}^k(M \setminus S))).
\]

Since \( \langle F(x) \rangle \) is unique, then \( \langle d_J A \rangle = 0 \),

\[
\langle F \rangle = \langle \varphi(\langle F \rangle) \rangle + \langle d_J A \rangle = \langle F \rangle + \langle d_J A \rangle = F(x).
\]

\( \square \)

Proposition 4.4 For a \( n \)-dimensional manifold, if \( [\bar{F} \wedge d\text{vol}_V] \in H_{2+\text{kn}}^2(J_{\text{c}}^k(M \setminus S)) \), then \( [F] \in H_2^dR(M \setminus S) \).

Proof. It is a consequence of Thom’s isomorphism theorem \( \square \)

From the above considerations, it follows the following formula for \( \bar{F} \),

\[
\bar{F} = \varphi(\bar{F}) + \Upsilon.
\]

Let us consider \( k\zeta(\bar{F}) \wedge d\text{vol}_V \). Then it follows that

\[
0 = d_J (k\zeta(\bar{F}) \wedge d\text{vol}_V) = d_J (k\zeta(\varphi(F)) + k\zeta(\Upsilon)) \wedge d\text{vol}_V = (k\zeta d_4(\varphi(F))) \wedge d\text{vol}_V + d_J (k\zeta(\Upsilon)) = k\zeta d_4(\varphi(F)) \wedge d\text{vol}_V + d_J (\Upsilon_{\mu \nu} dx^\mu \wedge dx^\nu \wedge d\text{vol}_V).
\]

Since \( d_4 \bar{F} = 0 \), it follows the relation

\[
d_4 \varphi(F) = - d_4 \Upsilon.
\]

4.2 The generalized excitation tensor \( \bar{G} \)

Similarly to the case of the generalized Faraday tensor \( \bar{F} \), the generalized excitation tensor is described by a generalized two form \( \bar{G} \),

Definition 4.5 The excitation tensor along the charged point particle \( kx \) is a 2-form \( \bar{G} \in \Gamma \Lambda^2(M \setminus S, \mathcal{F}(J_{\text{c}}^k(M \setminus S))) \),

\[
\bar{G}(kx) = \bar{G}(x, \dot{x}, \ddot{x}, \ldots) = (G_{\mu \nu}^{}(x) + \Xi_{\mu \nu}^{}(x, \dot{x}, \ddot{x}, \ldots)) dx^\mu \wedge dx^\nu.
\]

The determination of the form \( \bar{G} \) from the generalized Faraday tensor \( \bar{F} \) is provided by the constitutive relation,

\[
\bar{G} = \ast \bar{F}.
\]

The choice of this constitutive \( \square \) is for the vacuum. Other choices can be possible but will not be explored in this work. Note that the form \( \bar{G} \) is not necessarily closed with \( d_4 \).
Let us consider the star operator associated with \( g \). In this case, the field \( G(x) = G_{\mu\nu}dx^\mu \wedge dx^\nu \) is defined such that

\[
G(x) := \ast \varphi((G(u))).
\]

Therefore, the decomposition in the excitation tensor is unique. Note that an equivalent description of the excitation field is

\[
\tilde{\varphi}(\tilde{G}) = \tilde{G} \wedge d\text{vol}_V.
\]

There are some advantages in generalizing the electromagnetic field in this way. For instance, defining the tensors \( \Upsilon \) and \( \Xi \) as differential forms allow us to perform integrals that are diffeomorphism invariant and to use the machinery of exterior calculus in an analogous way as in standard classical electrodynamics. Also, it is straightforward to generalize Maxwell’s equations for the fields \( \tilde{F} \) and \( \tilde{G} \).

### 4.3 Higher order charge current density

The density current in electrodynamics is represented by a \( d_4 \)-closed 3-form

\[
J \in \Lambda^3(M \setminus S, \mathcal{F}(J^k_0(M \setminus S))).
\]

This can be generalized to

\[
\tilde{J}^k(x) = \tilde{J}(x, \dot{x}, \ddot{x}, ...) = J(x) + \Phi(x, \dot{x}, \ddot{x}, ...). \quad (4.6)
\]

For sources corresponding to point charged particles, the current density \( J \) is a distribution with support on the embedding \( S : I \to M \). Therefore, one requires that the support of \( \tilde{J} \) also lives on the lift \( kS : I \to J^k_0(M) \).

We will see that the current density \( \tilde{J} \) satisfies

\[
d_4 \tilde{J} = 0 \quad (4.7)
\]

is a consequence of the minimal extension of the fields (see section 7). This relation generalizes the charge conservation law in electrodynamics.

### 4.4 Geometric description of a point charged particle and other charge configurations

Let \( (M, \eta) \) be a time orientable spacetime and consider \( g \) the metric of maximal acceleration, defined as in Section 3. The space of world-line curves is

\[
\mathcal{C}(M) := \left\{ x : I \to M, \quad g(\frac{dx}{d\tau}, \frac{dx}{d\tau}) < 0, \quad \dot{x} \text{ future oriented} \right\}.
\]

One point particle is described by a curve \( x : I \to M \). In the case that a system has more than one point particle, each of them are described by the disjoint union of
curves, describing the evolution of each of such particles. The world-line $x : I \to M$ of a point charged particle has an associated lift

$$k x : I_x \to J^k_0(M).$$

Thus the set of lift to $J^k_0(M)$ corresponding to $C(M)$ is

$$kC(M) := \left\{ k x : I \to J^k_0(M), \quad g\left(\frac{dx}{d\tau}, \frac{dx}{d\tau}\right) < 0, \quad \dot{x} \text{ future oriented} \right\}. \quad (4.8)$$

### 4.5 Operator norms associated with generalized metrics

Let us first consider the operator norm associated with the generalized metric $g$ of the endomorphism $\bar{O} \in \Gamma T^{(1,1)}(M, \mathcal{F}(J^k_0(M)))$,

$$\|\bar{O}\|_g := \sup \left\{ \|\bar{O} u\|_g : u \neq 0 \right\}. \quad (4.9)$$

Defined in this way, $\|\bar{O}\|_g$ is intrinsic in the sense that is independent of the local frame or local coordinates that we use and only depends on $\bar{O}$, the symmetric form $g$ and the vector $V$.

**Proposition 4.6** For generalized metrics $g(kx) = \lambda(kx)\eta(x)$, it holds that

$$\|\bar{O}\|_g = \|\bar{O}\|_\eta.$$

**Definition 4.7** Given two operators $\bar{O}_1, \bar{O}_2 \in \Gamma T^{(1,1)}(M, \mathcal{F}(J^k_0(M)))$. Then we say that

$$\bar{O}_1 < \bar{O}_2 \iff \|\bar{O}_1\|_g < \|\bar{O}_2\|_g.$$

Let us consider the isomorphism induced by the generalized metric $g$ given by (2.39). In particular this expression determines the norm of the generalized 2-form $\bar{F} \in \Lambda^2(M, \mathcal{F}(J^k_0(M \setminus S)))$,

$$\|\bar{F}\|_g := \|\kappa(\bar{F})\|_g. \quad (4.10)$$

Also, note that the norm $\|, \|$ induces a pre-order relation in $\bar{F} \in \Lambda^2(M, \mathcal{F}(J^k_0(M \setminus S)))$. Such relation is useful to compare the strength of the 2-forms.
4.6 Definition of generalized electric and magnetic fields

Given a timelike vector field $W \in \Gamma TM$ associated to an observer, the generalized electric field is defined in a similar way as in the standard case,

$$E := \iota_W \tilde{F}. \quad (4.11)$$

The norm of $\|\kappa(\tilde{F})\|_g$ coincides with the norm $\| \cdot \|_g$ of the electric field,

$$\|E(u)\|_g = \|\iota_W \tilde{F}(u)\|_g = \|\kappa(\tilde{F})(u)\|_g, \quad u \in k\pi^{-1}(x).$$

The interpretation of $E = \iota_W \tilde{F}$ as the electric field depends on the local coordinate system (determined by the integral curves of $W$): if we change the observer to $\tilde{W}$, the electric field will be in principle different.

**Example 4.8** For a point charged particle, outside the world-line $S = x(\sigma)$, the electric field corresponding to $\tilde{F}$ measured in the coordinate system adapted to the motion of the charged point particle is

$$\|E_u\|_g := \|\tilde{F}\|_g(u) = \frac{1}{r^2},$$

where $r = d_g(x, S)$ is the distance from $x$ and $S$ measured with the positive definite generalized metric $g_+$ defined as in (2.34).

In a similar way, the definition of the generalized magnetic field is

$$\tilde{B} = \iota_W \star \tilde{F}. \quad (4.12)$$

**Proposition 4.9** An electromagnetic field is zero iff there is an observer $W$ such that the electric and magnetic field are both zero.

4.7 Short distance behavior of the electromagnetic field

Let $r = d(x, S)$ be the distance function from the point $x \in M$ to the world-line $S \hookrightarrow M$ using the Riemannian function $g_+$ as in equation (2.34). The expression of the Coulomb field for a point charged particle suggests that we should consider electromagnetic fields $\tilde{F}$ with the following development in powers of $r$,

$$\|\tilde{F}\|_g(x) = a_{-2} \frac{1}{r^2} + a_{-1} \frac{1}{r} + \sum_{k=0}^{+\infty} a_k r^k, \quad (4.13)$$

where each of the functions

$$a_i : J_0^k(M) \to M, i = -2, -1, 0, 1, ..., +\infty$$
are homogeneous of degree zero in $r$ and smooth functions on $J^0_k(M)$. For such generalized higher order fields, given a point $x(\sigma_0) \in S$ that is an isolated singularity of the field $\|\tilde{F}\|_{g}$ and a small sphere $S^2_{(x(\sigma_0), r)}$ surrounding $x(\sigma_0)$, one has the relation

$$4\pi Q = \int_{S^2_{(x(\sigma_0), r)}} \ast \tilde{F}$$

(4.14)

If one imposes that it must not depend on $r$ for $r$ finite and small enough, one has constraints on the averaged values of the coefficients,

$$\int_{S^2_{(x(\sigma_0), r)}} a_i(kx) d\Omega = 0, \quad i = -1, \ldots, \infty.$$  

(4.15)

If we impose the stronger condition that the integrals along any closed surface surrounding $x(\sigma_0)$ must be $4\pi Q$, one obtains the conditions

$$a_i(kx) = 0, \quad i = -1, \ldots, \infty.$$  

(4.16)

We adopt the convention that the quantity $Q$ in equation (4.14) corresponds to the total charge of the point charged particle whose world-line is the singularity $S$.

### 4.8 Analytic structure of the generalized electromagnetic fields

Given a charged point particle with world-line $x : I \to M$ with $k$-lift $k_x : I \to M$, the electromagnetic field outside from the singularity region $S$ can be decomposed as

$$\tilde{F}(x, \dot{x}, \ddot{x}, \ldots) = \varphi\left((F^C_{\mu\nu}(x) + F^D_{\mu\nu}(x))dx^\mu \wedge dx^\nu + F^{ext}_{\mu\nu}(x)dx^\mu \wedge dx^\nu\right) + \left(\Upsilon^{\text{div}}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots) + \Upsilon^{\text{reg}}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\right)dx^\mu \wedge dx^\nu.$$  

(4.17)

The piece $F^{ext}_{\mu\nu}(x)$ corresponds to the contribution to the field not generated by the singularity on $S$. The piece $\Upsilon^{\text{reg}}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)$ and $\Upsilon^{\text{div}}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)$ are the regular and divergent pieces of the field $\langle \tilde{F} \rangle$ on the singularity region $S$. $F^C_{\mu\nu}(x)$ is the divergent field on the world-line.

The behavior for the divergent field at short distances is

$$\|F^C_{\mu\nu}(x)\|_g = a_{-2} \frac{1}{r^2}.$$  

(4.18)

On the other hand, the field $F^D_{\mu\nu}$ is regular at short distances.

**Proposition 4.10** The following relations hold

1. The singularity of $\tilde{F}$ is such that

$$a_{-2} = \lim_{r \to 0} r^2 \tilde{F} = \lim_{r \to 0} r^2 F^C,$$
2. The singularities of $\Upsilon$ are of the form
\[ \lim_{r \to 0} r^2 \Upsilon(kx) = 0. \]

Then one has the relation
\[ F^C_{\mu\nu}(x) = a_{-2} \frac{1}{r^2} \theta_{\mu\nu}(x), \quad r \neq 0, \quad (4.19) \]
where the tensor $\theta_{\mu\nu}(x)$ is skew-symmetric and homogeneous of degree zero in $r$.

The higher order piece $\Upsilon^\text{div}_{\mu\nu}$ of $\bar{F}$ must be such that asymptotically
\[ (F^C_{\mu\nu}(x) + \Upsilon^\text{div}_{\mu\nu}(x, \dot{x}, \ddot{x}, ...)) \to F^C_{\mu\nu}(x), \quad (4.20) \]
when $r \to 0$. Therefore, we can assume that
\[ \Upsilon^\text{div}_{\mu\nu}(x, \dot{x}, \ddot{x}, ...) = 0. \quad (4.21) \]
This is in concordance with Dirac’s result on the regularity of the radiation field.

The generalized radiation field

The standard radiation field holds the relation
\[ F^{\text{rad}}(kx) = \bar{F}(kx) - F^C(x) - F^\text{ext}(x) - \Upsilon^\text{reg}(kx), \quad (4.22) \]
Following Dirac [25], for the standard Maxwell-Lorentz theory the radiation field is finite on the whole spacetime $M$. In such theory, the radiation reaction field,
\[ F^{\text{rad}}(x) = F^{\text{ret}} - F^{\text{adv}}, \quad (4.23) \]
with $F^{\text{ret}}$ and $F^{\text{adv}}$ are the retarded and advanced fields, obtained from the corresponding Liénard-Wiechert potentials. A balance equation analysis provides the following value for the radiation field on the world-line $x : I \to M$ [25],
\[ F^D_{\mu\nu} = \frac{4}{3} \left( \frac{d^3x_{\mu}}{ds^3} \frac{d^2x_{\nu}}{ds^2} - \frac{d^3x_{\nu}}{ds^3} \frac{d^2x_{\mu}}{ds^2} \right), \quad (4.24) \]
where $s$ is the proper-time along $S$ calculated using the metric $\eta$ and $\frac{dx_{\mu}}{ds} = \eta_{\mu\nu} \frac{dx_{\nu}}{ds}$.

This field is finite. However, the parameter $s$ must be substituted by the parameter $\tau$ in the expression of the radiating field. This is because we are considering that the standard clocks associated with a world-line are measured with $g$ and not with $\eta$. This provides the following expression for the radiation field,
\[ F^{\text{rad}}(3x) = \frac{4}{3} \left( \frac{d^3x_{\mu}}{d\tau^3} \frac{d^2x_{\nu}}{d\tau^2} - \frac{d^3x_{\nu}}{d\tau^3} \frac{d^2x_{\mu}}{d\tau^2} \right) + O(\epsilon_0^2). \quad (4.25) \]
We will adopt this value of the field for the generalized radiation field. Note that adopting this value of the field, we are assuring conservation of energy momentum.
and that the field is a solution of the corresponding Maxwell’s equations. Also, note that considering \( F^{\text{rad}} \in \Gamma \Lambda^2(M, \mathcal{F}(J^3_0(M))) \), we are reinterpreting the standard radiation field as a generalized field with values in a higher order jet bundle.

For the rest of the paper the regular part \( \Upsilon^{\text{reg}} \) will be just denoted by \( \Upsilon \) (and similarly for the regular part for \( \Xi, \Xi^{\text{reg}} \)). Let us consider the field generated by a point charged particle, which is also considered to be the absorber. If \( \mathcal{F}^{\text{ext}} = 0 \), out from the world-line \( x: I \to M \) all the fields in (4.1) are finite and one can write
\[
\bar{F}(x, \dot{x}, \ddot{x}, \ldots) = (F^C_{\mu\nu}(x) + F^{\text{rad}}_{\mu\nu}(3x) + \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)) dx^\mu \wedge dx^\nu. \tag{4.26}
\]
The pieces \( F^{\text{rad}}_{\mu\nu}(3x) \) and \( \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots) \) are analytic functions of the Euclidean distance to the world-line. Thus, at zero order approximation in the distance to the particle world line, one can write the relations for the regularized radiation field (the Coulomb field does not contribute to the radiation),
\[
\bar{F}^{\text{rad}}(x, \dot{x}, \ddot{x}, \ldots) := (F^{\text{rad}}_{\mu\nu}(3x) + \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)) dx^\mu \wedge dx^\nu. \tag{4.27}
\]
The constitutive relation \( G = \star F \) makes natural to consider
\[
G^{\text{rad}} := \star F^{\text{rad}}, \tag{4.28}
\]
where the star operator is associated with the metric of maximal acceleration \( g \).

Thus, for the excitation tensor one has the relation
\[
\bar{G}^{\text{rad}}(x, \dot{x}, \ddot{x}, \ldots) := (G^{\text{rad}}_{\mu\nu}(3x) + \Xi_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)) dx^\mu \wedge dx^\nu. \tag{4.29}
\]
All the fields in (4.27) and (4.29) are smooth on \( M \).

Note that although \( F^{\text{rad}}, G^{\text{rad}} \in \Gamma \Lambda^2(M, \mathcal{F}(J^3_0(M))) \), we did not fix the value of \( k \) where the full fields (4.27) and (4.29) are defined. However, the fact that \( F^{\text{rad}} \) and \( G^{\text{rad}} \) depends on the third order jet bundle strongly suggests that \( k = 3 \). Since there is not physical distinction (that is, an operational identification by experiment using prove particles) on the field \( \bar{F} \) between the contributions coming from \( F^{\text{rad}}(3x) \) and \( \Upsilon(x) \). This can be generalized to the case with external fields. On the other hand, the Coulomb singularity will be renormalized in the mass. The value \( k = 3 \) will be confirmed in section 3 after we obtain the consistent equation of motion for a point charged particle interacting with generalized higher order fields.

### 4.9 Physical interpretation of the generalized higher order fields

In section 1 we motivated the introduction of the generalized higher order fields based on the criticism of the concepts of external field-test particle system. The solution that we suggest is to substitute both notions by a new notion of field and probe particle such that the field depends on the state of motion of the probe particle. The notion of generalized higher order field accommodates naturally to the ideal of an operational field theory, in the sense that the mathematical structures must be linked with observables in a minimal economical way.
Thus, in the contest of Electrodynamics, generalized higher order fields are associated with the physical field that a probe particle will interact with. This is particularly clear for the generalized Faraday form $\bar{F}$, since it is that field which will entry on the equation of motion of the particle. On the other hand, the generalized excitation tensor $\bar{G}$ being also a generalized field is a postulate, which obviously natural from the point of view of the symmetries of the theory. However, that the excitation tensor $\bar{G}$ depends on the state of motion of the probe particle suggests that not only the electrodynamics but also matter fields should be considered from the perspective of generalized higher order fields.

Finally, let us clarify that our notion of generalized higher order field is not a generalization of the notion of Finsler field theories (see for instance [64, 65] among the recent contributions). While our theory is originated from an aim of maximal economical postulate in the formalism of field theory, the introduction of Finsler field theory is mainly based on arguments coming from quantum gravity phenomenology.

4.10 Physical interpretation of the relation $F = \langle k^{\zeta}(\bar{F}) \rangle$

Since $F(x)$ lives on the spacetime manifold $M$, it is natural to view $F$ as the standard electromagnetic field, outside the world-line $S$. However, this interpretation is not appropriated. First of all, such interpretation is in conflict with the original motivation to introduce the field $\bar{F}$. In the framework of generalized higher order fields, it is not longer valid the notion of external field and test particle. Only in special physical cases, when the field $F - \varphi(F)$ is small compared with $\bar{F}$ in the sense of the norm (4.9), one can approximate the pair $(\bar{F}, k^{x}(s))$ by the pair $(F, x(s))$ and ascribe to both independent physical reality. In such case, the language of external fields and test particles is useful to describe the physical phenomena.

An alternative interpretation of the field $F$ comes from its definition as a result of an integration along the fiber. If $F = \langle k^{\zeta}(\bar{F}) \rangle$, then it can be thought as the expected value of a physical measurement. In addition, the statistical distributions that one considers are compact distributions along the fiber. The integration is performed using a solution of a kinetic model but defined by probability distribution functions living in higher order jet bundles. Therefore, the physical interpretation of the relation $F = \langle k^{\zeta}(\bar{F}) \rangle$ is statistical, related with the measurement of fields using bunch of particles instead of individual point particles.

5 The Lorentz-Dirac equation in the framework of higher order electromagnetic fields and maximal acceleration geometry

In this section we derive the standard Lorentz-Dirac equation using a simple argument by F. Rorhlich. Then we reconsider the derivation in the framework of generalized higher order fields. In this two derivations the spacetime $M$ will be four dimensional and the metric is the Minkowski metric $\eta$ with signature $(-1,1,1,1)$. 85
Thus, all the contractions and lowering indices operations performed in this section are performed with $\eta$. The parameter $\tau$ is the proper time along a given curve with respect to the Minkowski metric $\eta$. The calculations are performed in a normal coordinate system of $\eta$, where it has the diagonal form $(-1, 1, 1, 1)$. Finally, we repeat the calculation but with a spacetime endowed with a metric of maximal acceleration $g$.

5.1 A simple derivation of the Lorentz-Dirac force equation

The Lorentz-Dirac equation describes the motion of a point charged particle interacting with its own electromagnetic and with an external field or force. In a normal coordinate system of $\eta$, if the external force is the Lorentz force on a particle, the Lorentz-Dirac is the third order differential equation

$$m \ddot{x}^\mu = e F_{\mu \nu} \dot{x}^\nu + \frac{2}{3} e^2 \left( \ddot{x}^\mu - (\dddot{x}^\rho \dddot{x}_\rho) \dot{x}^\mu \right), \quad \dddot{x}^\mu = \dddot{x}^\sigma \eta_{\mu \sigma}.$$  \hspace{1cm} (5.1)

This equation contains runaway and pre-accelerated solutions \cite{25}, both against what is observed in everyday experience and in contradiction with Newton’s first law of classical dynamics.

We present a simple derivation of the equation (5.1) based on the geometric method of adapted local frames. This derivation is what we have called Rohrlich’s argument \cite{56}. It illustrates a method that we will use in the next section in the context of generalized higher order fields. One starts with the Lorentz force equation for a point particle interacting with an electromagnetic field $F_{\mu \nu}$:

$$m_b \ddot{x}^\mu = e F_{\mu \nu} \dot{x}^\nu,$$  \hspace{1cm} (5.2)

where $m_b$ is the bare mass and $e$ the electric charge of the particle. Both sides of (5.2) are consistently orthogonal to $\dot{x}$. If one wants to generalize the equation (5.2) to have into account the radiation reaction, one can add to the right side a vector field along the curve $x : R \to M$. This is a map $Z : I \to J^k_0(M)$ such that the diagram

$$
\begin{array}{ccc}
J^k_0(M) & \rightarrow & I \\
\downarrow \pi & & \downarrow x \\
Z & \rightarrow & M
\end{array}
$$

commutes. The orthogonality condition

$$\eta(Z(\tau), \dot{z}(\tau)) = 0 \hspace{1cm} (5.3)$$

\footnote{Indeed one can use the same argument if instead of the Lorentz force there is an external force orthogonal to the 4-velocity.}

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implies the following general expression for $Z$,

$$
Z^\mu(\tau) = P^\mu_\nu(\tau)(a_1 \ddot{x}^\nu(\tau) + a_2 \dddot{x}^\nu(\tau) + a_3 \dddot{x}^\nu(\tau)), \quad P^\mu_\nu = \eta^\mu_\nu + \dot{x}_\mu(\tau) \dot{x}_\nu(\tau), \quad \dot{x}_\mu = \eta^\mu_\nu \ddot{x}^\nu
$$

(5.4)

with $a_1$, $a_2$ and $a_3$ a priori arbitrary. Using the orthogonality (5.3) we can write $a_1 = 0$, since this contribution will not appear in the right hand side if $P^\mu_\nu \ddot{x}^\nu = 0$. Then using the kinematical relations, one obtains

$$
\dddot{x}^\mu \eta^\rho_\sigma = -\dddot{x}^\rho \dddot{x}^\sigma \eta^\rho_\sigma, \quad \dddot{x}^\mu \ddot{x}_\mu = 0,
$$

from which follows the relations

$$
Z^\mu(\tau) = a_2 \dddot{x}^\mu(\tau) + a_3(\dddot{x}^\mu - (\dddot{x}^\rho \dddot{x}^\sigma \eta^\rho_\sigma) \ddot{x}^\mu)(\tau).
$$

The term $a_2 \dddot{x}^\mu$ combines with the left hand side to renormalize the mass

$$
(m_b - a_2)\dddot{x}^\mu = m \dddot{x}^\mu.
$$

(5.5)

The argument from Rohrlich is completed after realizing that in order to obtain the Lorentz-Dirac equation, one needs $a_3 = 2/3 e^2$. The same equation is obtained if instead of searching for a term containing the whole piece $a_3(\dddot{x}^\mu(\tau) - (\dddot{x}^\rho \dddot{x}^\sigma \eta^\rho_\sigma) \ddot{x}^\mu)(\tau)$, one requires that the right hand to be compatible with the relativistic Larmor’s law [40, 56],

$$
\dot{P}^\mu_\text{rad}(\tau) = \frac{2}{3} e^2 (\dddot{x}^\rho \dddot{x}^\sigma \eta^\rho_\sigma)(\tau) \ddot{x}^\mu(\tau).
$$

(5.6)

In order to fulfill this constraint, the minimal piece required in the equation of motion of a charged particle is $-2/3 e^2 (\dddot{x}^\rho \dddot{x}^\sigma \eta^\rho_\sigma) \ddot{x}^\mu$. The Schott term $2/3 e^2 \dddot{x}$ is a total derivative. It does not contribute to the averaged power emission of radiation. However, in Rohrlich’s argument, the radiation reaction term and the Schott term are necessary, due to the kinematical constraints of $\eta$ [25].

Rohrlich’s argument provides the Lorentz-Dirac equation in a short and elegant way, without introducing complicated integrations and balance relations. However, there are some points that make the Rohrlich argument not completely satisfactory. One difficulty is related with the structure of the vector $Z^\mu(\tau)$. In principle one can add pieces with higher derivatives and there is a lack of justification for the absence of such pieces. For instance, it is possible to introduce an additional term in the right hand side of the form

$$
Y^\mu(\tau) = (B^\mu \dot{x}^\nu - \dddot{x}^\mu B^\nu)(\tau) \dot{x}^\rho(\tau) \eta_{\nu \rho},
$$

(5.7)

The vector field $B^\mu(\tau)$ along $x(\tau)$ can be arbitrary. However, this new contribution can be written in the form (5.4). Therefore, Rohrlich’s argument is incomplete and several questions arise:

- It is not such that $\dddot{x} = 0$ in the whole interval $I$, 87
• Why one needs to start with the Lorentz force in defining the procedure?
• What is the origin of the additional terms like $Z^\mu(\tau)$ or $Y^\mu(\tau)$ to the Lorentz force in the equation of motion?
• Why are not there derivative terms higher than three?

One can consider an adapted (Frenet) frame to the curve. For curves embedded in $\mathbb{R}^n$ with the canonical flat connection, this method works if \{\dot{x}(s), \ddot{x}(s), \ldots, x^{(4)}(s)\} is a frame along $x : [0,1] \to M$; singularities must be treated individually. Thus, we do not expect higher order derivatives than four for describing arbitrary vector fields along $x : I \to M$. Finally, in order to keep the equation compatible with Larmor’s law, it is necessary that

$$a_4 = 0.$$  \hfill (5.8)

We have proved the following

**Proposition 5.1** In a four dimensional spacetime $(M, F)$, the only Lorentz covariant differential equation such that

- It is compatible with the covariant Larmor’s law,
- The kinematical constraint $\eta(\dot{x}, \ddot{x}) = -1$ holds,
- The constant observable mass condition $\dot{m} = 0$ holds,
- The external electromagnetic forces are not dissipative, $\eta(F_{L}^{ext}, \dot{x}) = 0$

is the Lorentz-Dirac equation.

### 5.2 Rohrlich’s derivation of the Lorentz-Dirac equation in the framework of generalized higher order fields

Let us consider Rohrlich’s argument in the framework of generalized electromagnetic fields introduced in section 4. In particular, the field defined by equation (4.27), when evaluated on the lift $k\dot{x}(s)$ of the world-line $x(\tau)$ of a charged particle is

$$\tilde{F}^{rad}(x, \dot{x}, \ddot{x}, \ldots, x^{(k)}) = \left(F^{\mu\nu}_{\mu\nu}(\dot{x}) + \Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\dot{x}^{\nu}\right) dx^\mu \land dx^\nu.$$

In the previous subsection we provided a solution to the required compatibility with the equation (4.27) that was $F^{rad}(S) = F^{D}(S), \Upsilon = 0$. In this subsection we consider $F^{rad}(S) = 0$ and we find a solution for $\lim_{k} S \Upsilon^{k}x$ which is consistent with (4.6). The piece $\Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\dot{x}^{\nu}$ must be formally like $Z^\mu(s)$. Let us write a formal series for $\Upsilon$ contracted with $\dot{x}$ (using $\eta$ to down indices),

$$\Upsilon_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots)\dot{x}^{\nu} = v_1 \ddot{x}_\mu + v_2 \dddot{x}_\mu + v_3 \dddot{x}_\mu + \ldots$$

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The minimal choice of the coefficients in the expansion that much such compatibility are

\[ v_1 = -\frac{2}{3}(\epsilon^2)\dddot{x}^\mu \dddot{x}^\sigma \eta_{\rho\sigma}, \quad v_2 = 0, \quad v_3 = \frac{2}{3}(\epsilon^2), \quad v_k = 0, \quad \forall k \geq 4. \]  

(5.9)

The value of \( v_1 \) is necessary to recover the standard radiation reaction term. The value of \( v_2 \) is zero by simplicity. Indeed, if one considers the electromagnetic mass originated by the Coulomb field, it will be compensate with a convenient \( v_2 \) term, producing a renormalization of the mass. \( v_3 \) has this value in order that the constraint \( Z^\mu \dddot{x}^\rho \eta_{\mu\rho} = 0 \) holds.

Apart from the arbitrary election \( v_k = 0, k \geq 4, \) there are more extra terms that we can add. For instance, a term like (5.7) it also possible to write down in the equation of motion. If we do this, the equation of motion has the following form

\[ m \dddot{x}^\mu = e F^{\mu \nu} \dot{x}^\nu + \frac{2}{3} e^2 \left( \dddot{x}^\mu + (\dddot{x}^\rho \dot{x}_\rho) \dddot{x}^\mu \right) + \left( B^\mu \dddot{x}^\nu - \dddot{x}^\mu B^\nu \right) \dot{x}_\nu, \]

\[ + \left( C^\mu \dddot{x}^\nu - \dddot{x}^\mu C^\nu \right) \dot{x}_\nu + \left( D^\mu \dddot{x}^\nu - \dddot{x}^\mu D^\nu \right) \dot{x}_\nu, \]  

(5.10)

with \( B(x, \dot{x}, \dddot{x}, \dddot{x}, \ldots), C(x, \dot{x}, \dddot{x}, \dddot{x}, \ldots) \) and \( D(x, \dot{x}, \dddot{x}, \dddot{x}, \ldots) \) elements of the jet bundle \( J^k_0(M) \). This equation can be written as

\[ m \dddot{x}^\mu = e F^{\mu \nu} \dot{x}^\nu + P^{\mu \nu} Z^\nu + \left( B^\mu \dot{x}_\nu - \dddot{x}_\nu B^\mu \right) \dot{x}_\nu + \left( C^\mu \dddot{x}^\nu - \dddot{x}^\mu C^\nu \right) \dot{x}_\nu \]

\[ + \left( D^\mu \dddot{x}^\nu - \dddot{x}^\mu D^\nu \right) \dot{x}_\nu. \]

However, the orthogonality condition \( \eta(\dddot{x}, \dot{x}) = 0 \) implies that

\[ B^\mu = 0, \quad C^\mu = 0, \quad D^\mu = 0. \]  

(5.11)

Therefore, in the same conditions than in [5.1], one has

**Proposition 5.2** With the same hypothesis than in [5.1], for generalized higher order fields, the differential equation for a point charged particle is the Lorentz-Dirac equation.

This result implies that only using generalized higher order fields is not enough to obtain a second order differential equation for point charged particles.

### 5.3 Rohrlich’s derivation of the Lorentz-Dirac equation with maximal acceleration

Using the kinematical constraints for metrics of maximal acceleration, we can repeat Rohrlich’s argument to obtain the Lorentz-Dirac equation. It will have a small modification due to the bound in the acceleration. Indeed, using the same notation as in section 4 and with \( Y = \sum_{i=1}^k \lambda_i \dot{x}^{(i)} \), one obtains

\[ Z^\mu(\tau) = P^{\mu \nu}(\tau)(\lambda_1 \dot{x}^\nu(\tau) + \lambda_2 \dddot{x}^\nu(\tau) + \lambda_3 \dddot{x}^\nu), \]
and by an analogous procedure as before, $\lambda_1$ is arbitrary and we can prescribe $\lambda_1 = 0$. The equation is consistent with Larmor’s law if

$$\frac{\dot{\xi}}{2} + \lambda_2 \frac{d}{d\tau}(g(\dot{x}, \ddot{x})) = 0, \quad \lambda_3 = \frac{2}{3} \varepsilon^2, \quad \lambda_k = 0, \quad \forall k \geq 4. \quad (5.12)$$

The corresponding modified ALD equation is

$$m \ddot{x} = eF_{\mu\nu} \dot{x}^\nu + \frac{2}{3} \varepsilon^2 \left( \ddot{x}^\mu - (\ddot{x}^p \dddot{x}^q \eta_{pq}) \dot{x}^p \right) + O(\varepsilon^2), \quad (5.13)$$

with $F_{\mu\nu} := \eta^{\mu\rho}F_{\rho\sigma}$. This equation is formally identical to the ALD equation,

**Proposition 5.3** With the same hypothesis than in [5.4] with a maximal acceleration geometry, the differential equation for a point charged particle is the Lorentz-Dirac equation.

This fact implies that only maximal acceleration hypothesis is not enough to solve the problem of the Schott term in the ALD equation. We have not considered the electrostatic contribution to the mass coming from the Coulomb field. If one wants to consider such contribution, an additional contribution $\lambda_2 \neq 0$ and a renormalization of the mass to eliminate the divergence that appears are needed.

### 6 A differential equation for point charged particles

In this section we obtain a relativistic dynamical model for point charged particles. The dynamics of the probe charged particle will be described by an implicit second order differential equation compatible with Larmor’s covariant formula (5.6). The fact that we impose the requirement of being a second order differential equation is to be in accordance with Newton’s first and second law. This requirement is a mayor difference with standard approaches to the electrodynamics of classical charged particles. Also, we do not require of the Lorentz-Dirac as first step in our theory. This is in sharp contrast the Landau-Lifshitz theory, where one finds an equivalent second order differential equation from the starting Lorentz-Dirac equation.

We will need to introduce maximal acceleration geometry. The justification to introduce such geometric framework is to be consistent with our philosophy of generalized higher order fields. Thus, if physical fields are generalized higher order fields and gravity should couple to them by a generalization of Einstein’s equations, then the spacetime metric structure should be in the same category of generalized higher order fields. Moreover, metrics of maximal acceleration provide a perturbative parameter. Also, since in such frameworks the acceleration is bounded, it is natural that there are no pre-accelerated solutions.
6.1 Derivation of the equation of motion

Let us assume that the spacetime metric is of the type of maximal acceleration $g$ and that the metric $\eta$, obtained by averaging, is the Minkowski metric. We will perform our calculations in a normal coordinate system $(x, U_n)$ of the Minkowski metric $\eta$. In such coordinate system the metric $\eta$ is globally constant and therefore

$$N_{\mu \nu} = 0, \quad \mu, \nu = 1, ..., 4, \quad x \in U_n.$$  

This is not a fundamental condition and will make easier some computations. Also, let us assume that the physical world-line of a point charged particle is a smooth curve of class $C^k$ such that $g(\dot{x}, \dot{x}) = -1$, $\dot{x}^1 > 0$ and such that the acceleration field is bounded from above. Using the generalized tensor fields introduced in section 2, one obtains the following general form for the differential equation,

$$\mathbf{T}_{\mu \nu}(x, \dot{x}, \ddot{x}, ...) = B_{\mu} \dot{x}_\nu - B_{\nu} \dot{x}_\mu + C_{\mu} \dot{x}_\nu - C_{\nu} \dot{x}_\mu + D_{\mu} \ddot{x}_\nu - D_{\nu} \ddot{x}_\mu + ... \quad \ddot{x}_\mu = g_{\mu \nu} \dddot{x}^\nu.$$  

This implies that we will have the expression

$$m_b \dddot{x}^\mu = e F_{\mu \nu} + (B^\mu \dot{x}_\nu - \dddot{x}^\mu B_\nu) \dddot{x}^\nu + (C^\mu \ddot{x}_\nu - \dddot{x}^\mu C_\nu) \dddot{x}^\nu + (D^\mu \dddot{x}_\nu - \dddot{x}^\mu D_\nu) \dddot{x}^\nu + ...,$$

and with $F_{\mu \nu} := g^{\mu \rho} F_{\rho \sigma} = \eta^{\mu \rho} F_{\rho \sigma}$. On the right hand side of the above expression all the contractions that appear in expressions like $(B^\mu \dot{x}_\nu - \dddot{x}^\mu B_\nu) \dddot{x}^\nu$, etc., are performed with the metric $g$ instead of the Minkowski metric $\eta$. The term $\delta m \dddot{x}^\mu$ corresponds to the electrostatic mass due to the Coulomb force $[10]$. The other terms come from the higher order terms of the expression (11) of the electromagnetic field.

The general form of a $k$-jet along a smooth curve $x : R \to M$ implies the relations

$$B^\mu(s) = \beta_1 \dot{x}^\mu(s) + \beta_2 \ddot{x}^\mu(s) + \beta_3 \dddot{x}^\mu(s) + \beta_4 \dddot{x}^\mu(s) + ...,$$

$$C^\mu(s) = \gamma_1 \dot{x}^\mu(s) + \gamma_2 \ddot{x}^\mu(s) + \gamma_3 \dddot{x}^\mu(s) + \gamma_4 \dddot{x}^\mu(s) + ...,$$

$$D^\mu(s) = \delta_1 \dot{x}^\mu(s) + \delta_2 \ddot{x}^\mu(s) + \delta_3 \dddot{x}^\mu(s) + \delta_4 \dddot{x}^\mu(s) + ....$$

In principle we can add higher derivative terms. However, the treatment of them will be the same than the fourth derivative term and eventually all of them will vanish. Also, for a 4-dimensional smooth manifold $\{ \dot{x}, \ddot{x}, \dddot{x}, \dddot{x} \}$ defines a local frame along the curve $x : I \to M$, except for the singularity points where some of the above derivatives are zero. We will assume that singularities only happens for isolated points. Otherwise, one needs to make a separate analysis (like we will see later for the uniform covariant motion).

Using the above expressions, one obtains

$$m_b \dddot{x}^\mu = \left( (\beta_1 \dot{x}^\mu(s) \dot{x}_\nu + \beta_2 \ddot{x}^\mu(s) \ddot{x}_\nu + \beta_3 \dddot{x}^\mu(s) \dddot{x}_\nu + \beta_4 \dddot{x}^\mu(s) \dddot{x}_\nu) \dot{x}^\nu - (\beta_1 \dot{x}_\nu(s) + \beta_2 \ddot{x}_\nu(s) + \beta_3 \dddot{x}_\nu(s) + \beta_4 \dddot{x}_\nu(s)) \dddot{x}^\nu \right) \dddot{x}^\nu$$

$$+ \left( (\gamma_1 \dot{x}^\mu(s) + \gamma_2 \ddot{x}^\mu(s) + \gamma_3 \dddot{x}^\mu(s) + \gamma_4 \dddot{x}^\mu(s)) \dddot{x}_\nu \right) \dddot{x}^\nu - (\gamma_1 \dddot{x}_\nu(s) + \gamma_2 \dddot{x}_\nu(s) + \gamma_3 \dddot{x}_\nu(s) + \gamma_4 \dddot{x}_\nu(s)) \dddot{x}^\nu$$

$$+ \left( (\delta_1 \dot{x}^\mu(s) + \delta_2 \ddot{x}^\mu(s) + \delta_3 \dddot{x}^\mu(s) + \delta_4 \dddot{x}^\mu(s)) \dddot{x}_\nu \right) \dddot{x}^\nu - (\delta_1 \dddot{x}_\nu(s) + \delta_2 \dddot{x}_\nu(s) + \delta_3 \dddot{x}_\nu(s) + \delta_4 \dddot{x}_\nu(s)) \dddot{x}^\nu.$$
Let us assume that there are not derivatives higher than 2 in the differential equation of a point charged particle. One way to achieve this is to impose that all the coefficients for higher derivations are equal to zero,

\[ \gamma_k = \delta_k = 0, \quad k \geq 0, \quad \beta_k = 0, \quad k > 3. \]  

(6.1)

With this choice and using the kinetic relations for \( g \), one obtains the expression

\[ m_b \dddot{x}^\mu = e F^\mu_\nu - \beta_2 \dddot{x}^\mu - \beta_3 \dddot{x}^\nu - \frac{1}{2} \beta_2 \dot{\epsilon} \dddot{x}^\mu - \beta_3 (-a^2(\tau) + \dot{\epsilon}^\prime) \dddot{x}^\mu. \]  

(6.2)

The differential equation governing the motion of a charged particle must be of second order and compatible with power radiation formula (5.6). Thus, if \( \dot{\epsilon}(\tau) \) is different than zero, one obtains the relations

\[ \beta_2 = \frac{4}{3} e^2 a^2(s) \frac{1}{\dot{\epsilon}}, \]  

(6.3)

\[ \beta_k = 0, \quad \forall k \geq 3. \]  

(6.4)

Therefore, at leading order in \( \epsilon_0 \) we obtain the differential equation for a charged particle in a higher order field in the case that \( a^2 \neq 0 \) and \( (A^2_{max})^{-1} \neq 0 \) to be

\[ m_b \dddot{x}^\mu = \frac{2}{3} e^2 a^2(s) \dddot{x}^\mu - \frac{2}{3} e^2 a^2(s) \frac{1}{\dot{\epsilon}} \dddot{x}^\mu. \]

There is a re-normalization of the bare mass. For \( \dot{\epsilon} \neq 0 \) and the renormalization of mass reads,

\[ m + \frac{2}{3} e^2 a^2(s) \frac{1}{\dot{\epsilon}} = m, \quad \dot{\epsilon} \neq 0. \]  

(6.5)

If we add an external field \( F^\mu_\nu \), interacting with the particle, we find the differential equation

\[ m \dddot{x}^\mu = e F^\mu_\nu \dddot{x}^\nu - \frac{2}{3} e^2 \eta_{\rho\sigma} \dddot{x}^\rho \dddot{x}^\sigma, \quad F^\mu_\nu = g^{\rho\nu} F_{\rho\nu}. \]  

(6.6)

In the case \( a^2 = 0 \), we postulate the same differential equation, which is equivalent to Newton’s first law.

One can express the equation of motion (6.6) in a covariant way as

\[ m D_{\dot{x}} \dot{x} = e \tilde{F}(x(s)) - \frac{2}{3} e^2 \eta(D_{\dot{x}} \dot{x}, D_{\dot{x}} \dot{x}), \]  

(6.7)

where \( D_{\dot{x}} \) is the non-linear covariant derivative along \( X = (\dot{x}, 0) \in T_{x(s), \dot{x}(s)} TM \) and \( \tilde{F} = g^{-1}(\iota_{\dot{x}} F, \cdot) \). We postulate that (6.7) is the differential equation that the particle follows.

**Remark 6.1** We have the following remarks:

- The factor \( \beta_3 \) is multiplying a factor \( \dddot{x}^\mu - (a^2) \dddot{x}^\mu \). However, we have seen that with the requirement of maximal acceleration, only \( k = 2 \) is needed to obtain a second order differential equation compatible with the power radiation formula. Note that although in the development in the derivatives \( k = 2 \), the corresponding generalized higher order fields are sections of \( \Lambda^2(M, \mathcal{F}(J^3_0(M))) \).
• Note that in the above derivation, $\dot{\epsilon}$ must be bounded and that the coefficient $\beta_2 \in \mathcal{F}(J^3_0(M))$.

• The derivation of equation (6.6) is not valid when $\dot{\epsilon}$ is zero. A separate discussion is necessary of that case. However, if the points where $\dot{\epsilon}$ is discrete, one can extend the validity of the differential equation by continuity.

• The derivation of equation (6.7) and (6.6) are independent of the Lorentz-Dirac equation. This is a point in favour of the consistency of the approach, that is not relegated to problematic equations.

If we assume that the maximal acceleration is infinite, then one has that $\dot{\epsilon} = 0$ identically. In this case the relation (6.2) reduces to

$$m \ddot{x}^\mu = e F^\mu_\nu - \beta_2 \ddot{x}^\mu - \beta_3 \dot{x}^\mu + \beta_3 a^2(\tau) \dot{x}^\mu \dot{x}^\mu.$$  \hspace{1cm} (6.8)

This is compatible with Larmor’s law if

$$\beta_3 = -\frac{2}{3} e^2.$$ 

In this case, the equation of motion should be the Lorentz-Dirac equation. This is the result from Proposition 5.1. This argument makes explicit that both ingredients, maximal acceleration and generalized higher order fields are necessary to obtain a second order differential equation for point charged particles.

### 6.2 General properties of the equation (6.6)

Let us consider a normal coordinate system for $\eta$. Let us multiply equation (6.6) by itself and contract with the metric $g$. Using the kinetic relations of Proposition 3.10 one obtains

$$m^2 a^2(1 - \epsilon) = e^2 F^\mu_\rho \dot{x}^\rho F^\nu_\lambda \dot{x}^\lambda (1 - \epsilon) \eta_{\mu\nu} + \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}$$

$$- 2e \frac{2}{3} e^2 F^\mu_\rho \dot{x}^\rho \dot{x}^\nu (1 - \epsilon) \eta_{\mu\nu}$$

$$= (1 - \epsilon) \left( F^2_L - \frac{1}{1 - \epsilon} \left( \frac{2}{3} e^2 \right)^2 (a^2)^2 \right).$$

with the magnitude of the Lorentz force $F_L$ given by

$$F^2_L = e^2 F^\nu_\mu F^\nu_\rho \dot{x}^\lambda \eta_{\lambda\rho}.$$ 

Proposition 6.2 For any curve solution of equation (6.6) one has the following consequences,
1. The Lorentz force is always spacelike or zero,
\[ F_L^2 \geq 0. \]

2. In the case the Lorentz force is zero, the magnitude of the acceleration is zero,
\[ F_L^2 = 0 \iff a^2 = 0. \]

3. If there is an external electromagnetic field, the acceleration is bounded by the strength of the corresponding Lorentz force.

Proof. In the limit \( \epsilon \ll 1 \), one can re-write the expression
\[ m^2 a^2 (1 - \epsilon) = (1 - \epsilon)(F_L^2 - \frac{1}{1 - \epsilon}(\frac{2}{3} e^2)^2 (a^2)^2) \]
as the following
\[ F_L^2 = \frac{1}{1 - \epsilon}(\frac{2}{3} e^2)^2 (a^2)^2 + m^2 a^2, \] (6.9)
from which follows the three consequences. \( \square \)

Solving the quadratic equation for \( u = a^2 \), one obtains
\[ u = 2m^{-2}C^{-1}(-1 + \sqrt{1 + F_L^2 C}), \quad C = \frac{(4e^2)^2}{m^4}. \] (6.10)

In the limit \( F_L^2 C \ll 1 \), one has
\[ u = 2m^{-2}C^{-1}(-1 + \sqrt{1 + F_L^2 C}) \approx 2m^{-2}C^{-1}(-1 + \frac{1}{2}F_L^2 C) = m^{-2}F_L^2, \]
in concordance with Newton second law of dynamics for the Lorentz force.

For the perturbative regime \( \epsilon \ll 1 \) and also in the complementary limit when \( F_L^2 C \gg 1 \), one has that
\[ u = 2m^{-2}C^{-1}(-1 + \sqrt{1 + F_L^2 C}) \approx 2m^{-2}C^{-1}F_L^2 C \approx 2m^{-2}F_L^2, \]
which is bigger by a factor \( \sqrt{2} \) of the expected magnitude for the acceleration following the Lorentz law. It is not difficult to show that this asymptotic value is maximal.

Theorem 6.3 Given the Lorentz force of magnitude \( F_L \), the maximal value attainable by particle following the equation of motion (6.6) is
\[ a = \sqrt{2} m^{-1} F_L. \] (6.11)
Proof. First, note that in the perturbative limit and for forces of electromagnetic origin, \(u \in [m^{-1} F_L, 2m^{-1} F_L]\). This is because the function

\[
S(F_L) = 2m^{-2}C^{-1}(-1 + \sqrt{1 + F_L^2C})
\]

is monotonically increasing with \(F_L\). Thus since we prove the existence of an asymptotic upper bound in the limit \(F_L^2C \to \infty\), in the perturbative regime there is the bound for the acceleration \([6.11]\). \(\Box\)

One way to read this result is that for point charged particles, the perturbative regime is characterized by the bound \(u \in [m^{-1} F_L, 2m^{-1} F_L]\). If the bound is violated, the perturbative regime is not valid and as a consequence, equation (6.6) is in principle, not valid.

Absence of run away solutions

One can prove the following version of Dirac’s asymptotic condition,

**Theorem 6.4** For solutions of the equation (6.6) it holds the following asymptotic condition,

\[
\lim_{\tau \to \infty} F_L(\tau) = 0 \Rightarrow \lim_{\tau \to \infty} a^2 = 0.
\]

**Proof.** From the equation (6.9) one has that the only non-negative solution for \(u\) is just \(u = 0\). Then from Proposition 6.2 follows the result. \(\Box\)

Run away solutions are solutions have the following peculiar behavior: even if the external forces have a compact domain in the spacetime, the charged particles follows accelerating without end. Theorem 6.4 implies that equation (6.6) is free of such pathological solutions,

**Corollary 6.5** Equation (6.6) does not have run-away solutions.

An alternative way to see this is the following. Let us assume that the external field is zero for some \(\tau > \tau_0\). Then the expression for the maximal acceleration is obtained again from equation (6.6),

\[
-m^2 \left(1 - \frac{a^2}{A_{\text{max}}^2}\right)a^2 = \left(\frac{2}{3} e^2\right)^2 (a^2)^2.
\]

This condition can be read as

\[
-m^2(1 - \epsilon) = A_{\text{max}}^2 \left(\frac{2}{3}\right)^2 \epsilon^2.
\]

This implies the following consequence,
Proposition 6.6 The equation \((6.6)\) does not have run away solutions iff
\[
A_{\text{max}} = \frac{3m}{2e^2} c^4. \tag{6.13}
\]

Thus, since we have proved already that \((6.6)\) does not have run away solutions, we have

Corollary 6.7 The maximal acceleration of a point charged particle whose world-line of the equation \((6.6)\) is determined by the formula \((6.13)\).

This acceleration is of the same order of magnitude than the maximal acceleration discovered by Caldirola [17].

Absence of pre-acceleration for the equation \((6.6)\)

In order to investigate the existence of pre-accelerated solutions of the equation \((6.6)\), let us consider the example of a pulsed electric field [25]. This example conveys the discovery of the pre-accelerated solutions in the Lorentz-Dirac equation. For a electric pulse
\[
\vec{E} = (\kappa \delta(\tau), 0, 0), \tag{6.14}
\]

the equation \((6.6)\) in the non-relativistic limit reduces to
\[
a \ddot{x}^0 = \kappa \delta(\tau), \quad a = \frac{3m}{2e^2}. \tag{6.15}
\]

The solution of this equation is the Heaviside function,
\[
a \dot{x} = \kappa, \quad \tau \geq 0, \tag{6.16}
0, \quad \tau < 0. \tag{6.17}
\]

This is not a pre-accelerated solution. Therefore,

Proposition 6.8 In the non-relativistic limit, equation \((6.6)\) does not have pre-accelerated solutions of Dirac’s type.

Since the theory is Lorentz covariant, equation \((6.6)\) does not have pre-accelerated solutions of Dirac’s type in any other coordinate system. Still, it is open the question if \((6.6)\) is free of any other type of pre-accelerated solutions. Also, note that the Dirac’s pulse should be considered as an approximation, since we are considering smooth electromagnetic fields.
Compatibility with the covariant power radiation law

The mechanical power performed by point charged particle whose world-line curve is a solution of the differential equation (6.6) is obtained by the contraction of $m\ddot{x}^\mu$ with the four dimensional velocity vector $\dot{x}$,

$$m\ddot{x}^\mu g_{\mu\nu} = F^\mu_\rho \dot{x}^\rho \dot{x}^\nu g_{\mu\nu} = \frac{2}{3}e^2a^2 g(\dot{x}, \dot{x}) = \frac{2}{3}e^2a^2.$$  \hspace{1cm} (6.18)

From this expression it follows the compatibility of the equation (6.6) with the covariant Larmor’s law. The point charged particle takes such energy from the higher order terms of the electromagnetic field, since $\beta_2 \dot{\epsilon} = -\frac{2}{3}e^2a^2$.

Also interesting is to compute the non-relativistic limit. There is a coordinate system where the acceleration is $a = (0, \vec{a})$ and the velocity vector is $\dot{x} = (v^0, \vec{v})$. Two kinematical contractions $\ddot{x}^\mu \dot{x}^\nu g_{\mu\nu} = 1 - \epsilon \vec{a} \cdot \vec{v}$ and the contraction $g(\dot{x}, \dot{x}) = -(1-\epsilon)$.

Then one obtains the rule,

$$m\vec{a} \cdot \vec{v} = -\frac{1}{1-\epsilon} \frac{2}{3} \epsilon^2 (\vec{a} \cdot \vec{a}) \simeq -\frac{2}{3} \epsilon^2 (\vec{a} \cdot \vec{a}) + \epsilon \frac{2}{3} \epsilon^2 (\vec{a} \cdot \vec{a}).$$

Therefore,

$$m\vec{a} \cdot \vec{v} = \frac{2}{3} \epsilon^2 (\vec{a} \cdot \vec{a}) + O(\epsilon^3).$$ \hspace{1cm} (6.19)

Equation (6.19) is the mechanical power obtained from the forces acting on the particle by equation (6.6). In this sense, the non-relativistic limit of (6.6) is compatible the rate of energy loss by radiation. The same is true for the covariant equation of motion (6.7).

Compatibility with kinematic constraints

Using the kinetic relations from $g$, the relation (6.18) reduces to

$$mg(\ddot{x}, \dot{x}) = -\frac{2}{3} \epsilon^2 a^2 (\tau) + O(\epsilon^2).$$

Defining the characteristic time

$$\tau_0 := \frac{2}{3m} \epsilon^2,$$

the relation can be re-written as

$$g(\ddot{x}, \dot{x}) \simeq -\tau_0 a^2(\tau).$$ \hspace{1cm} (6.20)

For accelerations taking place in a time much more larger than $\tau_0$, the condition (6.20) is a natural substitute to the orthogonal condition $\eta(\ddot{x}, \dot{x}) = 0$. This interpretation also relates the maximal acceleration and the parameter $\tau_0$,

$$A_{\text{max}} \simeq \frac{c}{\tau_0} = \frac{3}{2} \frac{m c^4}{\epsilon^2},$$

where $c$ is the speed of light in vacuum. Thus, this value is consistent with the exact value $A_{\text{max}}$ obtained by consistency with the requirement of absence of run away solutions.
Estimate of the maximal acceleration for a point charged particle

The covariant Larmor’s formula implies an estimate for the maximal acceleration of a point charged particle. First, it is reasonable to assume that the maximal work that a point particle can realize is of the order of its rest mass $m c^2$. The characteristic time that this happens is of order of $\tau_0 = r_0/c$, with $r_0$ the classical radius of the point particle. Therefore, the maximal power emitted in form of radiation,

$$|P_{\text{rad}}^0(t)| = \frac{2}{3c^3} e^2 (\vec{x}^\rho \vec{x}^\sigma \eta_{\rho\sigma})(t) \dot{x}^0(t) < \frac{2}{3c^3} e^2 A_{\max}^2 = \frac{m c^2}{\tau_0}.$$ 

Taking the value $(\tau_0)^{-1} = \frac{2m^3 c^3 e^2}{3}$, one obtains

$$A_{\max} = \frac{3 m c^4}{2 e^2}.$$ 

This estimated value coincides with the exact value for $A_{\max}$ obtained by consistency in (6.13). From this value and (6.11) it follows the following bound for the value of the Lorentz force,

$$F_L \leq \frac{1}{3} e^2 m^2 c^4. \quad (6.21)$$

Finally, note that for such maximal acceleration, the maximal power is the critical power $P_c$ introduced in section 3,

$$P(A_{\max}) = \frac{2}{3c^3} e^2 A_{\max}^2 = \frac{2}{3c^3} e^2 \frac{3 m^4}{2 e^2} A_{\max}^2 = P_c. \quad (6.22)$$

Thus, since this value of the maximal acceleration is fixed by other arguments as do not have run-away solutions, we have that as consequence of Proposition 3.21

**Proposition 6.9** There cannot be superluminal solutions of the differential equation (6.6).

This is consistent with the argument in faubour of maximal acceleration based on the existence of a maximal speed of propagation for interactions and a minimal length. We conclude, that both speed and acceleration are bounded in our model of point charged particle.

7 An effective spacetime electrodynamic theory for generalized higher order fields and point charged particles

In this section we will introduce a generalization of Maxwell’s equations for the generalized higher order fields $\bar{F}, \bar{G}$ and the generalized current $\bar{J}$. The metric $\eta$ is assumed flat and $M$ is four dimensional. The theory uses an special type of
cohomology elements of $\Lambda^*(M, \mathcal{F}(J_{0b}^3(M)))$, namely $\Lambda^*(M, \mathcal{F}(\tilde{J}_{0b}^3(M)))$. This is by the restriction of the general class of elements $\Lambda^*(M, \mathcal{F}(J_{0b}^3(M)))$ to be compatible with the solutions of the equation of motion (6.6) (or its general covariant version (6.7)).

7.1 Minimal extension of the generalized electromagnetic fields

The minimal extension of the Faraday and excitation tensor fields are of the form (4.1) and (4.4). Except for very few terms, higher order terms do not contribute in the minimal extension theory. We assume that the constitutive relation $\bar{G} = \star \bar{F}$ holds. Since $\bar{G}$ and $\bar{F}$ are 2-forms, the $\star$ operators for $\eta$ and $g = \lambda \eta$ when acting on sections of $\Lambda^2(M, \mathcal{F}(J_{0b}^3(M)))$ coincide.

Let us consider the 1-forms

$$\{ \tilde{x}^{(i)} = x^{(i)}_{\mu} d_4 x^\mu, \quad i = 1, 2, 3 \}.$$ 

Then the electromagnetic and excitation fields $\bar{F}, \bar{G} \in \Gamma \Lambda^2(M, \mathcal{F}(J_{0b}^3(M)))$ can be expressed locally in the form

$$\bar{F}(x, \dot{x}, \ddot{x}) = (\varphi(F)(x) + \beta_2 \tilde{x} \wedge \tilde{x}),$$  

(7.1)

$$\bar{G}(x, \dot{x}, \ddot{x}) = \star (\varphi(F)(x) + \beta_2 \tilde{x} \wedge \tilde{x}),$$  

(7.2)

where the $\star$ operator is associated with $\eta$. The value of the coefficients in a normal coordinate system of $\eta$ is

$$\beta_2 = \frac{2}{3} e^2 A_{\text{max}}^2 \frac{(x^{(2)}_{\rho} x^{(2)}_{\lambda} \eta_{\rho \lambda})^{\frac{1}{2}}}{2 x^{(3)}_{\rho} x^{(2)}_{\lambda} \eta_{\rho \lambda}},$$

Since $d_4$ is a skew-derivation, the following relation holds:

$$d_4(\beta_i \tilde{x}^{(i)}) = \beta_i d_4(\epsilon) \wedge \tilde{x}^{(i)}_{\mu} d_4 x^\mu, \quad i = 1, 2, 3.$$ 

(7.3)

The expression for $\Upsilon$ in a normal coordinate system is

$$\Upsilon^{(k,x)} = \frac{2}{3} e^2 A_{\text{max}}^2 \frac{(x^{(2)}_{\rho} x^{(2)}_{\lambda} \eta_{\rho \lambda})^{\frac{1}{2}}}{2 x^{(3)}_{\rho} x^{(2)}_{\lambda} \eta_{\rho \lambda}} 	ilde{x}^{(2)} \wedge \tilde{x}^{(1)}.$$ 

(7.4)

Closeness of the form $\Upsilon$

**Proposition 7.1** The form $\Upsilon \in \Gamma \Lambda^2(M, \mathcal{F}(J_{0b}^3(M)))$ is closed,

$$d_4 \Upsilon = 0,$$ 

(7.5)

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Proof. It follows from the following calculation,
\[ d_4 \Upsilon(x) = d_4 \left( \frac{2}{3} \epsilon^2 A_{\text{max}}^2 \frac{(x^{(2)} \rho x^{(2)} \eta \lambda)}{2 x^{(3)} \rho x^{(2)} \lambda \eta \rho \lambda} x^{(2)} \wedge \tilde{x}^{(1)} \right) \\
= d_4 \left( \frac{2}{3} \epsilon^2 A_{\text{max}}^2 \frac{(x^{(2)} \rho x^{(2)} \eta \lambda)}{2 x^{(3)} \rho x^{(2)} \lambda \eta \rho \lambda} x^{(2)} \wedge \tilde{x}^{(1)} \right) \\
+ \left( \frac{2}{3} \epsilon^2 A_{\text{max}}^2 \frac{(x^{(2)} \rho x^{(2)} \eta \lambda)}{2 x^{(3)} \rho x^{(2)} \lambda \eta \rho \lambda} \right) d_4 \left( x^{(2)} \wedge \tilde{x}^{(1)} \right). \]

The first term is zero, since the operator \( d_4 \) acting on functions of higher order components is zero,
\[ d_4 \left( \frac{2}{3} \epsilon^2 A_{\text{max}}^2 \frac{(x^{(2)} \rho x^{(2)} \eta \lambda)}{2 x^{(3)} \rho x^{(2)} \lambda \eta \rho \lambda} x^{(2)} \wedge \tilde{x}^{(1)} \right) = 0. \]

The second contribution is zero, because \( d_4 \) is nil-potent,
\[ d_4 \left( x^{(2)} \wedge \tilde{x}^{(1)} \right) = d_4 x^{(2)} \wedge \tilde{x}^{(1)} - d_4 \left( x^{(2)} \wedge \tilde{x}^{(1)} \right) x^{(1)} \\
= d_4 (\epsilon) \wedge x^{(2)} \wedge \tilde{x}^{(1)} - d_4 (\epsilon) \wedge x^{(1)} = 0. \]

The following are direct consequences of the algebraic structure of the generalized higher order fields \( \bar{F} \) and \( \bar{G} \).

**Corollary 7.2** The generalized Faraday form \( \bar{F} \) and the excitation form \( \bar{G} \) are elements of \( \Lambda^2_{\text{cv}}(M, F(J_0^3(M))) \). In particular, \([\bar{F}] \in H^2_{\text{cv}}(J_0^3(M))\). Similarly, \([\Upsilon] \in H^3_{\text{cv}}(J_0^3(M))\).

**Proof.** It follows from the fact that for particles following the differential equation (6.6) have \( n \)-acceleration and the parameter \( \dot{\epsilon}^{-1} \) are bounded (if \( \dot{\epsilon} \neq 0 \)). For \( \epsilon = 0 \) it follows that \( \bar{F} = 0 \) and \( \bar{G} = 0 \). □

**Corollary 7.3** Under the same assumptions, the fibers of \( J_0^3 \) are such that
\[ j_0^3 := \{(x, \dot{x}, \ddot{x}, \dddot{x}) \in j_0^3(x), \text{s.t.}: \]

- The non-compact condition \( 0 < \eta(\dot{x}, \ddot{x}) < 1 \) holds,
- The non-compact condition \( 0 < \eta(\ddot{x}, \dddot{x}) < A_{\text{max}}^2 \) holds,
- The compact condition \( 0 < |\eta(\dddot{x}, \dddot{x})| < c_3 \) holds. \]
Proof. It is clear that the domain for the first constraint is not compact. The
domain for the second constraint , by the same reasons, the constraints 2 and 3 do
not have compact domain.

This result shows how maximal acceleration implies constraints in the higher con-
tributions to the generalized fields. Finally, it is natural to reconsider the isomor-
phism 2.82 since partially explains the existence of an effective theory where the
electromagnetic field is described by differential 2-forms living on $M$, obtained by
considering the averaged fields $\langle \vec{F} \rangle$ and $\langle \vec{G} \rangle$.

Electromagnetic vacuum

The electromagnetic vacuum is characterized by the absence of electromagnetic field,
$\vec{F} = 0$. This is a strong notion electromagnet of vacuum. In this context, we have
the following result,

**Proposition 7.4** If $\vec{F} = 0$, then $\Upsilon = 0$ and $\langle \vec{F} \rangle = 0$.

Proof. If $\vec{F} = 0$, by identifying the corresponding components in equation (7.1)
with equation (4.2), it follows that $\varphi \langle \vec{F} \rangle = 0$ and $\Upsilon = 0$.

The strong electromagnetic vacuum condition implies the weaker condition

$$\vec{F} = 0 \Rightarrow \langle \vec{F} \rangle = 0.$$ 

On the other hand, the weak vacuum condition does not imply the strong vacuum
condition. However, it implies

$$\langle \vec{F} \rangle \Rightarrow \langle \Upsilon \rangle = 0.$$ 

The difference between the week and strong vacuum condition is that in the weaker
condition one can have $\Upsilon \neq 0$,

$$\frac{2}{3} e^2 A_{\text{max}}^2 \frac{(x^{(2)\rho} x^{(2)\lambda} \eta_{\rho\lambda})}{2 x^{(3)\rho} x^{(2)\lambda} \eta_{\rho\lambda}} \frac{1}{x^{(2)}} \wedge \frac{1}{x^{(1)}} \neq 0 \Rightarrow x^{(2)\rho} x^{(2)\lambda} \eta_{\rho\lambda} \neq 0.$$

Therefore, under the assumption that at infinity the electromagnetic state is the
vacuum, the weak vacuum state is not compatible with the validity of the equation
of motion 6.6. Thus if equation 6.6 is valid for $r \to \infty$, the weak condition is not
enough and should be subjected to stronger constraints.

A similar argument shows that the strong vacuum condition implies

$$\vec{F} = 0 \Rightarrow \frac{2}{3} e^2 A_{\text{max}}^2 \frac{(x^{(2)\rho} x^{(2)\lambda} \eta_{\rho\lambda})}{2 x^{(3)\rho} x^{(2)\lambda} \eta_{\rho\lambda}} \frac{1}{x^{(2)}} \wedge \frac{1}{x^{(1)}} = 0 \Rightarrow x^{(2)\rho} x^{(2)\lambda} \eta_{\rho\lambda} = 0.$$

Thus the strong electromagnetic vacuum condition is compatible with physically
acceptable asymptotic properties of the equation 6.6.
7.2 Maxwell’s equations for higher order electromagnetic fields and currents

One of the motivations to develop the mathematical machinery of generalized forms in section 2 and section 4 is that it allows to generalize in an straightforward way Maxwell’s equations when written in covariant differential form. Thus, we propose the following Maxwell’s equations for \( \bar{F} \) and \( G = \star \bar{F} \). The homogeneous equations are

\[
d_4 \bar{F} = 0 \quad (7.6)
\]

Because equation (7.3) and (4.1), the homogeneous equation (7.6) is equivalent to the standard homogeneous Maxwell’s equations,

\[
dF = 0. \quad (7.7)
\]

This is in accordance with the isomorphism in Proposition (2.91),

\[
H_c^*(M, F(J_0^b(M))) \cong H^{* - 3n}_{dR}(M).
\]

This brings out the connection between the cohomology theory developed in section 2 and the theory of generalized higher order fields.

The non-homogeneous equations are

\[
d_4 \star \bar{F} = \bar{J} := \varphi(J) + d_4 \star \Upsilon. \quad (7.8)
\]

For the non-homogeneous equation we note that

\[
d_4 \star \bar{F} = d_4 \star \varphi(F) + d_4 \star \Upsilon = \varphi(J) + d_4 \star \Upsilon.
\]

It is natural to define

\[
\bar{J} = \varphi(J) + d_4 \star \Upsilon. \quad (7.9)
\]

Therefore, the current \( J \) is such that

\[
\varphi(J) = \bar{J} - d_4 \star \left( \frac{2}{3} e^2 A_{\text{max}}^2 \frac{x^{(2)}_{(\rho \lambda)} x^{(2)}_{(\lambda \rho)} \eta_{\rho \lambda}}{2 x^{(3)}_{(\rho \lambda \rho)}} \bar{x}^{(2)} \wedge \bar{x}^{(1)} \right).
\]

For such current, the non-homogeneous equations are equivalent to

\[
d_4 \star \varphi(F) = \varphi(J).
\]

Then one obtains an effective equation of the form

\[
d \star F = J, \quad (7.10)
\]

which are the standard non-homogeneous Maxwell equation.

If the current density \( J(x) \in \Lambda^3 M \) must be associated with physical systems, it must hold

\[
dJ = 0. \quad (7.11)
\]
Proposition 7.5 If (7.9) holds, then $d_4 \bar{J} = 0 \iff dJ = 0$.

Proof. It is a consequence of the commutation relation $d_4(\varphi(\alpha)) = \varphi(d\alpha)$, because of the definition 2.97 Integrating on domains $\partial U \subset M$ one obtains the same fluxes and total charge for $J$ and $\bar{J}$.

Boundary conditions for the generalized electromagnetic field

Let us pay attention on the boundary conditions for the generalized electromagnetic field $\bar{F}$. As in the discussion of the vacuum state before, the problem can be handle if one relates boundary conditions of the field $\bar{F}$ with the boundary conditions for the field $F$. For example, if the boundary of the domain $D \subset M$ the value $\bar{F}_0(\mathbf{x})$, then it is clear the following result,

Proposition 7.6 Given an admissible boundary condition $F_0(\mathbf{x})$ for $F(\mathbf{x})$, there is a corresponding admissible boundary condition for $\bar{F}_0$ given by

$$\bar{F}_0(k\mathbf{x}) = F_0(\mathbf{x}) + \Upsilon(k\mathbf{x}).$$

In particular, one can apply this result to the initial valued problem, where the Cauchy hypersurface $\Sigma \hookrightarrow M$ of the Maxwell’s problem is lifted in $J^0_M$. However, for a prescribed field $F_0(\mathbf{x})$, the boundary condition is not necessarily unique, as the example of the asymptotic vacuum conditions showed.

Theorem 7.7 Let (7.1) and (7.2) be the generalized Faraday and excitation tensor fields. If (7.4) holds, then the theory described by the system of equations (7.6), (7.8), (7.10) and (6.6) is equivalent to the theory described by the system of equations (7.7), (7.10), (7.11) and (6.6).

Proof. It is a direct consequence of the equivalence Propositions 7.5 and 7.5 and the equivalence between the possible boundary conditions for the corresponding set of equations.

Therefore, the dynamics of generalized electromagnetic fields can be reduced to the standard Maxwell dynamics. The differential equation for the dynamics of point charged particles is described by the equation (6.6) or the covariant version (6.7).

Potential 1-form for generalized higher order electromagnetic fields and gauge symmetry

Let us introduce local gauge potentials for (7.1). From equation (7.6), one can write

$$\bar{F} = d_4\varphi(A) + \Upsilon, \quad A \in \Lambda^1 M,$$

by the standard Poincaré’s lemma. Then equation. (7.8) can be expressed as

$$d_4 \ast (d_4 \varphi(A) + \Upsilon) = \varphi(J) + d_4 \ast \Upsilon$$

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with the potential $A$ satisfying the partial differential equation

$$d_4 \star d_4 \varphi(A) = \varphi(J). \quad (7.12)$$

The $\star$ operator coincides for both metrics $\eta$ or $g$, since it is applied acts on two forms and $\eta$ and $g$ are related by a scalar factor. Thus equation (7.12) is equivalent to the standard equation as in Maxwell’s theory for the electromagnetic potential $A$. Therefore, one expects the existence and uniqueness of solutions of equation (7.12).

As in standard electrodynamics, the potential is not fixed by the equations. Thus, it is necessary to consider an additional gauge fixing condition. For instance, the Lorentz gauge condition

$$\star \eta d_4 \star \eta \varphi(A) := \delta_\eta \varphi(A) = 0 \quad (7.13)$$

is a valid gauge fixing condition (indeed, it corresponds to the usual Lorentz’s condition for $\varphi(A) \in \lambda^1M$). Therefore,

**Corollary 7.8** If the Lorentz gauge condition holds, $\varphi(A)$ is a solution of the wave equation

$$\square \varphi(A) = \star_\eta J. \quad (7.14)$$

**Proof.** This is direct from the definition of the wave operator $\square = \delta_\eta d + d\delta_\eta$ and the Lorentz gauge fixing.

**Remark 7.9** Note that we have chosen the star operator $\star_\eta$ in order to simplify the corresponding wave equation and to have an admissible gauge fixing.

It is easy to prove that (7.6) and (7.8) are invariant under the gauge transformation

$$A \rightarrow A + d\phi,$$

with $\phi \in F((M))$. This gauge invariance corresponds with the standard gauge invariance of the equations (7.10) and (7.11).

**8 Discussion**

The theory presented in this work describes the motion in the vacuum of point charged particles interacting with the total electromagnetic field. It intensively uses generalized higher order fields and the differential geometry theory of them. These fields are functionals acting on ordinary vector fields over the spacetime manifold $M$ but where the co-domain are spaces of functions defined on $J^3_0(M)$. The generalized higher order fields are used to define an extension of the notion field from standard fields to fields with dependence on higher derivatives of the curve describing the particle that probes the field. Such extension provides additional degrees of freedom, allowing the fields to adapt the changes produced by the point charged particle used.
in the measurement of the field. With such mathematical tool on hand, we were able to find an effective electrodynamic theory (described by equations (7.6), (7.8), (7.11) and (6.6)) consistent with radiation and free from the pre-accelerated and runaway pathologies that plague the standard electrodynamic of classical point charged particles. Moreover, there is a general covariant version of the theory, where the differential equation of the point charged particle is the covariant version (6.7).

The effective theory for the usual Faraday field $F$ and excitation field $G$ living on $M$ is described by the equations (6.7), (7.7), (7.10) and (7.11). Such effective theory is based on the relations (7.5), (7.1) and (7.11).

8.1 Discussion of the assumptions of the theory

We have used of several assumptions in our considerations that because their novelty, we should discuss and motivate them further. Highlighted below are the most relevant hypothesis considered:

- **Generalized higher order fields as fields with values in higher order jet bundles.** This is the main new contribution of the theory. The notion is an hybrid between Mo-Papas’ theory, where dispersive forces were introduced and Wheeler-Feynman’s theory. As we said in the introduction, we do not abandon the concept of field as intermediate agent between charged particles. Also, we think that use the language of forms is useful to describe the field, in which case the electromagnetic force must be non-dissipative.

Even if we persists on the physical reality of fields, we should admit that they have a quite **ghostly character** in our theory: they depend of the state of motion of the probe particle. In some sense, there is some similarity between assigning a physical reality to such objects is like assigning physical reality to a quantum particle previous the measurement of a quantum observable. In the quantum case, we say that the quantum particle exists, but that its reality is not independent of the act of observation. Thus in a similar way, we say that the field exists, but its real value depend on how it is measured.

- **Generalized higher order fields are horizontal.** This is one of the main differences with other higher order field theories proposed in the literature (see for instance 11 14 24 52 63 65 66). However, there are several reasons for this choice:

  - If the generalized higher order fields are horizontal as defined in section 2, there is a natural geometric notion of **local macroscopic measurement device.** After an observer has been chosen, the measurement of a generalized higher order field can be magnified by using the effect on a congruence of probe particles instead of an individual probe particle. Such interaction

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9One still has to prove that there are not pre-accelerated solutions of any type, not only of Dirac’s type. In addition, the theory still uses of a mass renormalization procedure.
is mathematically described by the flux of the generalized higher order field through the elementary 2-dimensional element of $TM$ determined by each pair of tangent vectors. This is an useful construction, since the flux integral will increase the effect of the detection using a single probe particle. On the other hand, the isomorphism (2.84) implies that the flux is uniquely defined by the horizontal fields and viceversa.

- It allows a causal description of the radiation reaction phenomena, in the sense that changes in the fields in a region $U \subset M$ propagates with the speed of light (thanks to equation (7.14)). This property does not necessarily hold for arbitrary (not necessary horizontal) fields. This is again clearly possible using horizontal fields, by direct consequence of equation (7.14).

- For horizontal fields it is easier to generalize Maxwell’s equations, since one does not need to introduce additional dynamical equations for the non-horizontal pieces, which is at the present difficult to understand. This is again a consequence of the isomorphism (2.84).

- **Maximal $n$-acceleration and geometries of maximal acceleration.** Maximal $n$-acceleration was introduced through the metric of maximal acceleration $g$. It was assumed that the proper time is measured using $g$ and not the Minkowski metric $\eta$. This is an important departing point from Special Relativity and General Relativity, and implies that in our theory, the clock hypothesis does not hold. This is because the metric of maximal acceleration depends on the acceleration of the particle. Thus, this is implicit already in Caianiello’s work, although seemingly not explicitly stated.

The introduction of geometries of maximal acceleration was useful to eliminate runaway solutions and to define a perturbation scheme in terms of the parameter $\epsilon_0$. Since the asymptotic condition (6.12) holds, the book keeping parameter $\epsilon_0$ is related with the function

$$\epsilon(\tau) = \frac{a^2(\tau)}{A_{\text{max}}^2}. \quad (8.1)$$

In the limit when the maximal acceleration $A_{\text{max}}^2$ is infinite, the asymptotic expansions introduced in section 5 are trivially zero. In such case, the procedure used to obtain the second order differential equation (6.5) in section 6 fails. Thus, it is fundamental for our considerations to have a finite bound for the $n$-accelerations that a charged point particle can reach using electromagnetic fields.

We considered effective theories, where higher order terms greater than order one in $\epsilon_0$ were disregarded. Thus the effective electromagnetic theory developed in section 7 is independent of the nature of the maximal acceleration,
except for the fact that it must be constant for each charged particle world-line $^k x : I \to M$. However, we were able to provide a concrete value for the maximal acceleration, given by the formula (3.1). The values of such acceleration for the electron, are of other $10^{32} m/s^2$, far away from any acceleration measured in the laboratory or observed in astrophysical objects. Thus the effective theory should be a good candidate for the current regime of experience.

Maximal $n$-acceleration has two fundamental effects in electrodynamics: from one side, it provides a key ingredient to make the description of the motion of point charged particles with radiation reaction consistent, since avoid runaway solutions. On the other hand, it implies the kinematics of maximal acceleration geometry, where superluminal motion of accelerated massive particles is possible. Moreover, the hypothesis of maximal acceleration implies that is natural to consider fields as sections of $T^{(p,q)}(M, \mathcal{F}(J^k_0(M)))$ with $k \geq 2$, since generalized higher order field theory is linked with the existence of a maximal $n$-acceleration through a generalized metric.

There are effective models of the electron where the book-keeping parameter is the classical radius of the electron. It could be the case that both approximations, the existence of a universal bound in the acceleration and the existence of a finite radius of the electron are related, since extended charged models requires a limit for the acceleration, in order to preserve causality (see for instance [60]). However, the maximal proper acceleration $A_{\text{max}}$ must be very large to any possible acceleration at the scales where the theory is applied, as it is the case of our theory.

- **The equation of motion of a point charged particle must be of second order.**
  This is a natural assumption, since it allows to maintain the Principle of Inertia. Note that in the limit $A_{\text{max}} \to \infty$ the perturbative method used in section 7 does not work. Indeed, in the framework of standard fields over $M$ and for Maxwell’s electrodynamics, the equation of motion of a point charged particle must be Lorentz-Dirac equation, which is a third order differential equation. The Landau-Lifshitz equation, although second order, is not obtained from fundamental principles.

  The equation of motion must be consistent with the rate of energy-momentum lost by the emission of radiation. As a result, we find that (6.7) (and therefore (6.6)) is free of the pathologies of the Lorentz-Dirac equation.

- **Point charged particle as ideal model of probe charged particle.** We have assumed that the point charged particle is described by a one dimensional submanifold $x : I \to M$. However, one should consider other classes of submanifolds in the more realistic description of the motion of the charge distributions. In this context, the use of distributional sources can be relevant [62]. It can happen that extended models could provide a solution to the still infinite Coulomb singularity.
8.2 Phenomenological consequences of the theory

We have found several phenomenological consequences headlined below:

- Consequences to zero order in the book-keeping parameter $\epsilon_0$,
  - An implicit second order differential equation for point charged particles. The constraint (6.7) is form invariant under local coordinate transformations and it is free of run away and pre-accelerated solutions of Dirac’s type. The explicit solutions of that implicit equation are still to be fully explored, but in principle the equation of motion provides a falsifiable model for the dynamics of point charged particles.
  - Bound of the maximal covariant $n$-acceleration in the perturbative equation of motion. The $n$-acceleration $a^2 = \eta(\ddot{x}, \dot{x})$ was predicted to have the maximal value $A_{max}^2 = (\frac{3mc^4}{2e^2})^2$. This maximal acceleration, for the electron, takes the value,
    $$A_{max}(e^-) = \frac{3}{2} \frac{me^4}{e^2} = 3.126 \times 10^{32} m/s^2. \quad (8.2)$$
    This is an extraordinarily large value, which seems difficult to test in current facilities.\footnote{Several attempts to test in the laboratory the principle of maximal acceleration has been tried by Y. Freedman and co-workers. However, the values that the experiment try to check is of order $10^{19} m/s^2$. Thus, the acceleration (8.2) is not reachable in such attempts.}
  - A bound on the maximal value attainable by the Lorentz force. This is obtained as combination of the approximate bound of the Lorentz force $F_L \leq \frac{1}{3} e^2 m_e^2 c^4$.
    A direct consequence is that the electric field must be also bounded,
    $$E_{max} = \frac{1}{3} e (m c^2)^2. \quad (8.3)$$
    Notably, this bound is obtained in a linear theory, compared with the (undetermined) bound found in Born-Infeld electrodynamics.\footnote{\cite{9}}
  - Consequences for the speed of light. In our theory, the speed of light in vacuum is constant and it does not depend on the coordinate system. This is a fundamental assumption. As a consequence, the theory is Lorentz
invariant, in the case that the base manifold $M$ is a four dimensional flat spacetime. Thus, in the sense that any possible deviation from Lorentz invariance (in flat space) is not allowed.

The first three predictions do not depend on $A_{\text{max}}$ and are in principle falsifiable. One can explicitly solve (6.6) in several situations of interest (Penning trap for example) and in principle the results can be contrasted with experiment. For the third consequence-postulate, it is possible (in some weaker forms) to falsified or test it (see for instance the relevant test of the speed of light in [15]).

There are no predictions to higher order contributions in the parameter $\epsilon_0$, since by construction all higher orders in $\epsilon$ were disregarded.

### 8.3 Relation of the generalized higher order electrodynamics with other higher order field theories

The use of generalized higher order fields in electrodynamics and other field theories has been previously investigated in the literature. Without been exhaustive, some earlier work in the framework of Finsler or Finsler-Lagrange geometry can be found in [1, 14, 47, 48, 63, 65, 66]. However, distinctive from our approach is that we have followed a minimal generalization, from a particular motivation to introduce generalized higher order fields, that was to find a solution to the radiation reaction problem within a classical theory of electrodynamics. We have been able to prove the Theorem [74] and show that such dynamics is free of most of the pathologies of the original theory. Moreover, our theory does not have a direct relation with the Bopp and Podolsky’s theory, which is a higher order theory in the sense that contain partial differential equations with higher order derivatives than two, but with fields living on $M$.

### 8.4 Future developments

Some of the ideas presented in this paper have not been fully developed yet. We would like to mention some of them that are relevant for the framework presented in this work and that need further consideration. This includes a development of the cohomology theory of generalized forms and an extended investigation of the geometry of maximal acceleration. On the physical side, the experimental predictions of the theory were not developed in fully detail. Ranging in such spectrum, we headline below what could be further research of the framework of generalized higher order fields:

- From the physical point of view, we find interesting to explore the following points.
  - Experimental predictions of the theory presented in this paper. Currently, this is a challenging point, since radiation reaction effects are very small in current experiments. However, the Penning-trap is a standard
physical system where the equation of motion (6.6) can be predictive. In particular, equation (6.6) can be solved for the Penning trap and compare it with the non-relativistic QED prediction [12, 59].

– We have shown that the geometry of maximal acceleration allows for superluminal motion of massive point charged particles. However, that this can happen for generalized metrics is not entirely surprising. For instance, there are possible scenarios in Finsler spacetime models (see for instance [43, 53]) where the spacetime metric allows that to happens.

– Very strong constraints for superluminal motion in neutrinos systems have been found by analyzing the produced Cherenkov’s radiation mechanism [20]. However, it is also known that those constrains depend on the back-ground geometry (see for instance [19]). Thus, a natural question arises of which are the effects of Cherenkov’s radiation in geometries of maximal acceleration.

– Functional path integrals for the generalized electrodynam ic theory. Note that a main difficulty formulating an action functional for a given field theory is to define domain of the functional space where the fields are defined and to provide it with a reasonable measure. We can follow the example of functional integrals in the Finsler category, which are integrals on the whole $TM$ of a lagrangian $L : TM \to \mathbb{R}$ (see for instance [65, 52]). In a similar way, for generalized higher order fields, one expects to have an action formulation of the theory as

$$ S[\Phi(3x)] = \int_{j^3_0(M)} \text{dvol}_4(x) \wedge \text{dvol}_V(y) \, L(\Phi(3x)). \quad (8.4) $$

Each fiber $j^3_0$ is defined as in Corollary 7.3. Thus, apart from the constraint $\eta(\dot{x}, \dot{\dot{x}})$, there are additional constraints with non-compact domains. This is an additional problem for the definition of the integrals. There are several natural questions related with the properties of the functional (8.4),

* What are the Euler-Lagrange equations associated with (8.4)? What is the general action such that its Euler-lagrange equations are the (7.8) and (7.6) equations?

* It must be clarified the relation with the corresponding action functional of Maxwell’s theory,

$$ S[A(x)] = \int_M \left( - \frac{1}{2} F \wedge \ast F + J \wedge \ast \right). \quad (8.5) $$

Is it possible that $S[\Phi(3x)] - S[\varphi(A)]$ is a total derivative in $j^3_0(M)$? In the affirmative case, the Euler-Lagrange equations will be equivalent.
To show the consistency of the quantization by functional integration, with generating functional given by

\[ Z[A] = \int \mathcal{D}A \exp \left\{ i(S[\Phi(\mathcal{A}^3x)(A)] + jA) \right\}. \] (8.6)

We believe that at least at the formal level, this tentative formulation can bring some light on the quantization process.

- We have assumed that the electromagnetic medium is the standard vacuum. However, it seems natural to extend the theory to more general electromagnetic media compatible with generalized higher order fields and study the extension of the framework to models of electromagnetic media.

- We have assumed that the probe particles are point charged particles and described them by 1-dimensional embedded manifolds on \( M \) (world-line curves). This is however, one of the possible source configuration. One should be able to consider extended configurations, like sphere or charged planes. Although technically more complicated, it should be investigated in the view of potential applications in plasma physics.

- The extension of the framework to Yang-Mills field models is a natural problem. Indeed, from a unification perspective, this must be the case. In the Finslerian case such investigations has been performed for instance in [64].

- It is interesting to see if the generalized higher order fields correspond to a mathematical description in between the usual local representation of gauge variables by the potential \( A \) and the holonomy variables representation, which is the natural variable for quantum gauge theory [18].

- We did not discuss gravitational effects. We think that a proper discussion of them should not require additional methods. One direct issue is the following: we defined generalized electromagnetic fields as sections of \( \Lambda^2_{\nu^{\kappa}}(M, \mathcal{F}(J^3_0(M))) \), while the metric of maximal acceleration is a section of \( T^{(0,2)}(M, \mathcal{F}(J^2(M))) \). Thus, if the matter fields like the generalized electromagnetic field \( \bar{F} \) is related with the generalized metric \( g \), the embedding

\[ T^{(0,2)}(M, \mathcal{F}(J^3(M))) \hookrightarrow T^{(0,3)}(M, \mathcal{F}(J^2(M))) \] (8.7)

should play a relevant role.

- Probably related with the above point is to provide a mechanism for maximal acceleration. We suspect that the mechanism is of universal nature and that is of such magnitude that accelerations related with the classical radius of the electron, for instance, are small compared with such universal maximal acceleration. We suggest that the mechanism is related with the quantum nature of the spacetime. Thus, the resolution of this question will bring new insights on the problem of infinite self-energies.
From a mathematical point of view, we suggest the following research directions:

- A curvature theory for the non-linear connections on the spaces of jet bundles is another direction that could be of interest to develop. This will include the generalization of the fundamental results from Riemannian to generalized metrics.

- In particular, a curvature theory for geometries of maximal acceleration is missing. This can be introduced through the geodesic variation equation. More generally, it is of relevance for both the theory and practice, to develop the variational theory of metrics of maximal acceleration.

- We did not discuss in this work a natural formalism to describe generalized higher order fields as it is the theory of sheaves and sheaf cohomology [8, 67]. We think that the use of the sheaf cohomology language can be useful to understand the structure of the theory, in particular in the clarification of the relation of the several cohomologies introduced in section 2.

- The relation between the vertical geometry and the theory of calibrations [37] has been already mention. This has direct relevance for our research, since calibrations will be related with the way we perform the average of the higher order quantities.

- There are many other problems open to mathematical investigation in the geometry of generalized forms. We highlight here the generalization of Hodge harmonic theory and the generalization of de Rham current theory to generalized forms.

- A connection theory for metrics of maximal acceleration.

- The development of a representation theory for the isometry groups of metrics of maximal acceleration.

Conclusion

We have seen that in the framework of generalized higher order fields, it is possible to have a consistent theory of radiation-reaction of point charged particles at the classical level. It is also needed that the spacetime geometry is of maximal acceleration. Moreover, the theory can be extended to non-abelian theories and gravity. In a second step, one can consider the quantization of such fields theories.

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