Evolution of interfaces for the non-linear parabolic p-Laplacian type reaction–diffusion equations

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We present a full classification of the short-time behaviour of the interfaces and local solutions to the nonlinear parabolic $p$-Laplacian type reaction–diffusion equation of non-Newtonian elastic filtration:

$$u_t - \left(|u_x|^{p-2} u_x\right)_x + bu^\beta = 0, \quad p > 2, \beta > 0.$$  

The interface may expand, shrink, or remain stationary as a result of the competition of the diffusion and reaction terms near the interface, expressed in terms of the parameters $p, \beta, \text{sign} \, b,$ and asymptotics of the initial function near its support. In all cases, we prove the explicit formula for the interface and the local solution with accuracy up to constant coefficients. The methods of the proof are based on non-linear scaling laws and a barrier technique using special comparison theorems in irregular domains with characteristic boundary curves.

Key words: nonlinear degenerate parabolic PDE; parabolic $p$-Laplacian; reaction-diffusion equation; interface; nonlinear scaling laws; super- and subsolutions

1 Introduction

We consider the Cauchy problem (CP) for the non-linear degenerate parabolic equation:

$$Lu \equiv u_t - \left(|u_x|^{p-2} u_x\right)_x + bu^\beta = 0, \quad x \in \mathbb{R}, 0 < t < T,$$

with

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

where $p > 2, \ b \in \mathbb{R}, \ \beta > 0, \ 0 < T \leq +\infty,$ and $u_0$ is non-negative and continuous. We assume that $b > 0$ if $\beta < 1,$ and $b$ is arbitrary if $\beta \geq 1$ (see Remark 1.1). Equation (1.1) arises in many applications, such as the filtration of non-Newtonian fluids in porous media [9] or non-linear heat conduction [10] in the presence of the reaction term expressing additional release ($b > 0$) or absorption ($b < 0$) of energy.

The goal of this paper is to analyse the behaviour of interfaces separating the regions where $u = 0$ and where $u > 0.$ We present full classification of the short-time evolution
of interfaces and local structure of solutions near the interface. Due to invariance of (1.1) with respect to translation, without loss of generality, we will investigate the case when \( \eta(0) = 0 \), where

\[
\eta(t) = \sup \{ x : u(x,t) > 0 \},
\]

and precisely, we are interested in the short-time behavior of the interface function \( \eta(t) \) and local solution near the interface. We shall assume that

\[
u_0 \sim C(-x)^{\alpha} \quad \text{as} \quad x \to 0^- \quad \text{for some} \quad C > 0, \ \alpha > 0.
\]

The direction of the movement of the interface and its asymptotic is an outcome of the competition between the diffusion and reaction terms and depends on the parameters \( p, b, \beta, C, \) and \( \alpha \). Since the main results are local in nature, without loss of generality we may suppose that \( u_0 \) either is bounded or satisfies some restriction on its growth rate as \( x \to -\infty \), which is suitable for existence, uniqueness, and comparison results (see Section 3). The special global case

\[
u_0(x) = C(-x)^{\alpha}, \quad x \in \mathbb{R},
\]

will be considered when the solution to the problem (1.1), (1.4) is of self-similar form. Our estimations are global in time in these special cases.

Initial development of interfaces and structure of local solution near the interfaces is very well understood in the case of the reaction–diffusion equations with porous medium type diffusion term:

\[
u_t - (u^m)_{xx} + bu^\beta = 0 \quad x \in \mathbb{R}, 0 < t < T.
\]

Full classification of the evolution of interfaces and the local behaviour of solutions near the interfaces in CP (1.5), (1.2), (1.3) was presented in [1] for the slow diffusion (\( m > 1 \)) case, and in [2] for the fast diffusion (\( 0 < m < 1 \)) case. The major obstacle in solving the interface development problem for non-linear degenerate parabolic equations is a problem of non-uniform asymptotics in the sense of singular perturbations theory, namely that the dominant balances as \( t \to 0^+ \) between the terms in (1.1), (1.5) on curves that approach the boundary of the support on the initial line depending on how they do so. The general theory, including existence, boundary regularity, uniqueness, and comparison theorems, for the reaction–diffusion equations of type (1.5) in general non-cylindrical and non-smooth domains is developed in [3] in the one-dimensional case, and in [5–7] in the multi-dimensional case. Comparison theorems proved in [3] were essential tools in developing the rigorous proof method in [1,2] for solving interface problem for the reaction–diffusion equation (1.5). The rigorous proof method developed in [1,2] is based on a barrier technique using special comparison theorems in irregular domains with characteristic boundary curves. In this paper, we apply the method developed in [1] to solve the interface problem for the PDE (1.1).

The structure of the paper is as follows: In Section 2, we outline the main results. In Section 3, we apply rescaling and prove for some preliminary estimations that are necessary for using our barrier technique. Finally, in Section 4, we prove the results of Section 2. To avoid technical difficulties, we give explicit values of some of lengthy constants in the appendix.
Remark 1.1 We are not interested in the special case $p = 2$ of semi-linear heat equation. This case was completed in [16,17] (see also [1]). However, we will mention when our results extend to the limit case $p = 2$. In general, the case $p = 2$ is in some sense a singular limit. For example, if $b > 0, 0 < \beta < 1, \alpha < \frac{p}{p-\beta}$, then we prove that the interface initially expands and

$$\eta(t) \sim C_1 t^{1/(p-\alpha(p-2))} \quad \text{as } t \to 0^+.$$ 

By passing to the limit as $p \downarrow 2$ formally, this yields a false result. In fact, from [17] it follows that if $p = 2$, then

$$\eta(t) \sim C_2 \left( t \log \frac{1}{t} \right)^{\frac{1}{2}} \quad \text{as } t \to 0^+.$$ 

2 Description of main results

In Figure 1, we present classification diagram in the $(\alpha, \beta)$ plane for the initial interface development in CP (1.1)–(1.3) if $b > 0$.

- **Region (1):** $\alpha < p/(p - 1 - \min\{1, \beta\})$; diffusion dominates and interface expands.
- **Region (2):** $\alpha = p/(p - 1 - \beta), 0 < \beta < 1$; diffusion and absorption are in balance in this borderline case. There is a critical constant $C_*$ such that interface expands for $C > C_*$, and shrinks for $C < C_*$.
- **Region (3):** $\alpha > p/(p - 1 - \beta), 0 < \beta < 1$; absorption term dominates and interface shrinks.
- **Region (4):** $\alpha \geq p/(p - 2), \beta \geq 1$; interface has initial ‘waiting time’.

To describe the asymptotic properties of the interface and local solution near the interface, we divide the results into two different subcases:

(I) $b \neq 0$ (either $b > 0, \beta > 0$ or $b < 0, \beta \geq 1$) and $p > 2$; and (II) $b = 0$.  

(1) In this case there are four different subcases, as shown in Figure 1 and itemized above. (In view of our assumptions, the case \( b < 0 \) relates to the part of the \((\alpha, \beta)\) plane with \( \beta \geq 1 \).)

**Region (1)**

**Theorem 1** Let \( u_0 \) satisfies (1.3) with \( \alpha < \frac{p}{p-1-\min\{1,\beta\}} \). Then, interface initially expands and

\[
\eta(t) \sim \xi_* t^{1/(p-\alpha(p-2))} \quad \text{as} \quad t \to 0^+,
\]

where

\[
\xi_* = C^{\frac{p-2}{p-\alpha(p-2)}} \xi_*'
\]

and \( \xi_*' > 0 \) depends on \( p \) and \( \alpha \) only (see Lemma 5). For arbitrary \( \rho < \xi_* \) there exists \( f(\rho) > 0 \) depending on \( C, p, \) and \( \alpha \) such that

\[
u(x, t) \sim f(\rho)t^{(\alpha/p-\alpha(p-2))} \quad \text{as} \quad t \to 0^+,\]

along the curve \( x = \xi_\rho(t) = \rho t^{1/(p-\alpha(p-2))} \).

A function \( f \) is a shape function of the self-similar solution of (1.1), (1.4) with \( b = 0 \) (see Lemma 5):

\[
u_\ast(x, t) = t^{\alpha/p-\alpha(p-2)} f(\xi), \quad \xi = xt^{-\frac{1}{p-\alpha(p-2)}}.
\]

In fact, \( f \) is a unique solution of the following non-linear ODE problem:

\[
\begin{cases}
\left( |f'(\xi)|^{p-2} f'(\xi) \right)' + \frac{1}{p-\alpha(p-2)} \xi f'(\xi) - \frac{\alpha}{p-\alpha(p-2)} f(\xi) = 0, & -\infty < \xi < \xi_* \\
f(-\infty) \sim C(-\xi^2), f(\xi_\ast) = 0, f(\xi) \equiv 0, & \xi \geq \xi_*.
\end{cases}
\]

Its dependence on \( C \) is given through the following relation:

\[
f(\rho) = C^{p/(p-\alpha(p-2))} f_0 \left( C^{(p-2)/(\alpha(p-2)-p)} \rho \right), \quad (2.6a)
\]

\[
f_0(\rho) = w(\rho, 1), \quad \xi_*' = \sup\{ \rho : f_0(\rho) > 0 \} > 0, \quad (2.6b)
\]

where \( w \) is a solution of (1.1), (1.4) with \( b = 0, C = 1 \). Lower and upper estimations for \( f \) are given in (2.28). Moreover,

\[
\xi_*' = A_0^{\frac{p-2}{p-1}} \left[ \frac{(p-1)^{p-1}(p-\alpha(p-2))}{(p-2)^{p-1}} \right]^{\frac{1}{2}} \xi_*'',
\]

where \( A_0 = w(0, 1) \) and \( \xi_*'' \) is some number in \([\xi_1, \xi_2]\), where

\[
\xi_1 = (p-1)^{\frac{1}{p-2}} \left( \alpha(p-2) \right)^{-\frac{1}{p}}, \quad \xi_2 = 1 \quad \text{if} \quad (p-1)(p-2)^{-1} \leq \alpha < p(p-2)^{-1},
\]

\[
\xi_1 = 1, \quad \xi_2 = (p-1)^{\frac{1}{p-2}} \left( \alpha(p-2) \right)^{-\frac{1}{p}}, \quad \text{if} \quad 0 < \alpha \leq (p-1)(p-2)^{-1}. \quad (2.8)
\]
In particular, if \( z = (p - 1)(p - 2)^{-1} \) and \( p > 1 + (\min\{1, \beta\})^{-1} \), then the explicit solution of the problem (1.1), (1.4) with \( b = 0 \) is given by (2.24), and we have

\[
\zeta_1 = \zeta_2, \quad \zeta_1^* = (p - 1)^{p-1}(p - 2)^{1-p}, \quad f_0(x) = (\zeta_1^* - x)^{\frac{p-1}{(p-2)}}. 
\tag{2.9}
\]

The explicit formulae (2.1) and (2.3) mean that the local behavior of the interface and solution along \( x = \zeta_\rho(t) \) coincides with that of the problem (1.1), (1.4) with \( b = 0 \).

**Region (2)**

**Theorem 2** Let \( b > 0, 0 < \beta < 1, p \geq 2, z = p/(p - 1 - \beta) \) and

\[
C_* = \left[ \frac{|b|(p - 1 - \beta)^p}{(1 + \beta)p^{p-1}(p - 1)} \right]^{\frac{1}{p-1}}.
\]

If \( u_0 \) satisfies (1.3), then interface expands or shrinks according as \( C > C_* \) or \( C < C_* \) and

\[
\eta(t) \sim \zeta_*^\frac{p-1}{p} t^{\frac{\beta-1}{\beta}} \quad \text{as} \quad t \to 0+, \tag{2.10}
\]

where \( \zeta_* \leq 0 \) if \( C \leq C_* \), and for arbitrary \( \rho < \zeta_* \) there exists \( f_1(\rho) > 0 \) such that

\[
u(x, t) \sim f_1(\rho)t^{1/(1-\beta)} \quad \text{for} \quad x = \rho t^{\frac{p-1}{1-\beta}}, \quad t \to 0+. \tag{2.11}
\]

Assume that \( u_0 \) is defined by (1.4). If \( \beta(p - 1) = 1 \), then the explicit solution to (1.1), (1.4) is

\[
u(x, t) = C(\zeta_* t - x)^{\frac{p-1}{p}} = b(1 - \beta)C^{\beta-1}((C/C_*)^{p-1-\beta} - 1). \tag{2.12}
\]

It has an expanding interface if \( C > C_* \), a shrinking interface if \( 0 < C < C_* \), and is a stationary solution if \( C = C_* \).

Let \( \beta(p - 1) \neq 1 \). If \( C = C_* \), then \( u_0 \) is a stationary solution to (1.1), (1.4). If \( C \neq C_* \), then the solution to (1.1), (1.4) is of the self-similar form:

\[
u(x, t) = t^{1/(1-\beta)} f_1(\zeta), \quad \zeta = x t^{-\frac{p-1}{p-\beta}}, \tag{2.13}
\]

\[
\eta(t) = \zeta_* t^{\frac{p-1}{p}}, \quad 0 \leq t < +\infty. \tag{2.14}
\]

If \( C > C_* \), then the interface expands, \( f_1(0) = A_1 > 0 \) (see Lemma 7), and

\[
C_1 t^{\frac{1}{1-\beta}} (\zeta_1 - \zeta)^{\mu} \leq u \leq C_2 t^{\frac{1}{1-\beta}} (\zeta_2 - \zeta)^{\frac{p}{p-1-\beta}}, \quad 0 \leq x < +\infty, \quad 0 < t < +\infty, \tag{2.15}
\]

where

\[
\mu = (p - 1)(p - 2)^{-1} \quad \text{if} \quad \beta(p - 1) > 1; \quad \mu = p(p - 1 - \beta)^{-1} \quad \text{if} \quad \beta(p - 1) < 1,
\]

which implies

\[
\zeta_1 \leq \zeta_* \leq \zeta_2. \tag{2.16}
\]

The right-hand side of (2.15) (2.16), respectively) may be replaced by \( \bar{C}_2 t^{\frac{1}{1-\beta}} (\zeta_2 - \zeta)^{\frac{p}{p-1}} (\zeta_2 \) respectively); see the appendix for the description of all the relevant constants. Let \( \beta(p -
1) \( \neq 1 \) and \( 0 < C < C_* \). Then, the interface shrinks and if \( \beta(p - 1) > 1 \), then

\[
[C^{1-\beta}(-x)]_+^{\frac{p(1-\beta)}{1-p}} - b(1 - \beta)t [\frac{1}{p}]_+^{1-p} \leq u
\]

\begin{align*}
\leq [C^{1-\beta}(-x)]_+^{\frac{p(1-\beta)}{1-p}} - b(1 - \beta)(1 - \left( \frac{C}{C_*} \right)^{p-1-\beta}) t [\frac{1}{p}]_+^{1-p}, & \quad x \in \mathbb{R}, \quad 0 \leq t < +\infty, \\
\text{(2.17)}
\end{align*}

which again implies (2.16), where \( \zeta_1 \) (\( \zeta_2 \), respectively) is replaced with

\[
-C^{-\frac{p(1-\beta)}{1-p}} (b(1 - \beta))^{\frac{p-1}{1-p}}
\]

\[
\left( -C^{-\frac{p(1-\beta)}{1-p}} (b(1 - \beta))(1 - (C/C_*)^{p-1-\beta}) \right)^{\frac{p-1}{1-p}}, \quad 0 \leq t < +\infty,
\]

(2.18)

However, if \( \beta(p - 1) < 1 \), then

\[
C_* \left( -x - \zeta_3 t^{\frac{1-\beta}{p-1}} \right) [\frac{1}{p}]_+^{1-p} \leq u \leq C_3 (-\zeta_4 t^{\frac{1-\beta}{p-1}} - x) [\frac{1}{p}]_+^{1-p}, \quad 0 \leq t < +\infty,
\]

(2.18)

where the left-hand side is valid for \( x \geq -\ell_0 t^{\frac{1}{p-1}} \), whereas the right-hand side is valid for \( x \geq -\ell_1 t^{\frac{1}{p-1}} \). From (2.18), (2.16) follows if we replace \( \zeta_1 \) and \( \zeta_2 \) with \( -\zeta_3 \) and \( -\zeta_4 \), respectively.

If \( \beta(p - 1) \neq 1 \), in general, the precise value \( \zeta_* \) can be found only by solving the ODE \( L^0 f_1 = 0 \) (see (4.4b)) below and calculating \( \zeta_* = \sup \{ \zeta : f_1(\zeta) > 0 \} \).

The right-hand side of (2.11) ((2.10), respectively) relates to the self-similar solution (2.13) (to its interface, as in (2.14), respectively). If \( \beta(p - 1) = 1 \), we then have explicit values of \( \zeta_* \) and \( f_1(\rho) \) via (2.12), whereas in general we have lower and upper bounds via (2.15)–(2.18). If \( u_0 \) satisfies (1.3) with \( x = p/(p - 1 - \beta), C = C_* \), then the small-time behaviour of the interface and local solution depends on the terms smaller than \( C_*(x)^{p/(p-1-\beta)} \) in the expansion of \( u_0 \) as \( x \to 0^- \).

**Region (3)**

**Theorem 3** Let \( b > 0 \), \( 0 < \beta < 1, p \geq 2, \alpha > p/(p - 1 - \beta) \). If \( u_0 \) satisfies (1.3), then interface shrinks and

\[
\eta(t) \sim -\ell_* t^{1/(\alpha(1-\beta))} \quad \text{as} \quad t \to 0+,
\]

(2.19)

where \( \ell_* = C^{-1/\alpha} (b(1 - \beta))^{1/\alpha(1-\beta)} \). For arbitrary \( \ell > \ell_* \), we have

\[
u(x, t) \sim \left[ C^{1-\beta}(-x) \right]_+^{\alpha(1-\beta)} - b(1 - \beta)t [\frac{1}{p}]_+^{1/(1-\beta)} \quad \text{as} \quad t \to 0+
\]

(2.20)

along the curve \( x = \eta(t) = -\ell t^{1/\alpha(1-\beta)} \).

Hence, the interface initially coincides with that of the solution

\[
\bar{u}(x, t) = \left[ C^{1-\beta}(-x) \right]_+^{\alpha(1-\beta)} - b(1 - \beta)t [\frac{1}{p}]_+^{1/(1-\beta)}
\]
to the problem
\[
\tilde{u}_t + b \tilde{u}^\gamma = 0, \quad \tilde{u}(x, 0) = C(-x)_+^z.
\]
Respective lower and upper estimations are given in Section 4 (see (4.16) and (4.19)).

**Region (4)**

In this case, the interface initially has a waiting time. We divide the results into four different subcases (see Figure 1).

(4a) Let \( \beta = 1, \alpha = p/(p - 2) \). This case reduces to the case \( b = 0 \) by a simple transformation (see Section 3). If \( u_0 \) is defined by (1.4), then the unique solution to (1.1), (1.4) is

\[
u_C(x, t) = C(-x)^{p/(p - 2)} \exp(-bt) \left[ 1 - (C/C^+)p^2b^{-1}(1 - \exp(-b(p - 2)t)) \right]^{1/(p - 2)}, \tag{2.21}
\]

for \( x \in \mathbb{R}, \ t \in [0, T) \), where

\[
T = +\infty \text{ if } b \geq (C/C^+)p^2/2,
\]

\[
T = (b(2 - p))^{-1} \ln[1 - b(C/C^+)p^2], \text{ if } -\infty < b < (C/C^+)p^2/2,
\]

\[
C = [(p - 2)p/(2(p - 1)p^2 - 1)]^{1/(p - 2)}.
\]

If \( u_0 \) satisfies (1.3), then lower and upper estimations are given by \( u_{C \pm \epsilon} \).

(4b) Let \( \beta = 1, \alpha > p/(p - 2) \). Then, for arbitrary \( \epsilon > 0 \) there exists \( x_\epsilon < 0 \) and \( \delta_\epsilon > 0 \) such that

\[
(C - \epsilon)(-x)^z \exp(-bt) \leq u(x, t) \leq (C + \epsilon)(-x)^z \exp(-bt)
\]

\[
\times \left[ 1 - e b^{-1}(p - 2)^{-1} \left( 1 - \exp(-b(p - 2)t) \right) \right]^{1/(p - 2)}, \quad x > x_\epsilon, \ 0 \leq t \leq \delta_\epsilon.
\]

(4c) Let \( 1 < \beta < p - 1, \ \alpha \geq p/(p - 1 - \beta) \). Then, for any \( \epsilon > 0 \) \( x_\epsilon < 0 \) and \( \delta_\epsilon > 0 \) such that

\[
g_{-\epsilon}(x, t) \leq u(x, t) \leq g_{\epsilon}(x, t), \quad x \geq x_\epsilon, \ 0 \leq t \leq \delta_\epsilon,
\]

where

\[
g_{\epsilon}(x, t) = \begin{cases} \left[ (C + \epsilon)^{1 - \beta}x^{1 - \beta} + b(\beta - 1)(1 - d_\epsilon)t \right]^{1/(1 - \beta)}, \quad x \leq x_\epsilon < 0, \\ 0, \quad x \geq 0, \end{cases}
\]

\[
d_\epsilon = \begin{cases} \epsilon \text{ sign } b \quad \text{ if } \alpha > p/(p - 1 - \beta), \\ \left( \left( (C + \epsilon)/C^+ \right)^{p - 1 - \beta} + \epsilon \right) \text{ sign } b \quad \text{ if } \alpha = p/(p - 1 - \beta), \end{cases}
\]

and the constant \( C^+ \) is defined in Region (2) of (1).

(4d) Let either \( 1 < \beta < p - 1, \ p/(p - 2) \leq \alpha < p/(p - 1 - \beta) \), or \( \beta \geq p - 1, \ \alpha \geq p/(p - 2) \). If \( \alpha = p/(p - 2) \), then for arbitrary \( \epsilon > 0 \) there exists \( x_\epsilon < 0 \) and \( \delta_\epsilon > 0 \) such that

\[
(C - \epsilon)(-x)^{p/(p - 2)}(1 - \gamma_\epsilon t)^{1/(2 - p)} \leq u \leq (C + \epsilon)(-x)^{p/(p - 2)}(1 - \gamma_\epsilon t)^{1/(2 - p)}, \tag{2.24}
\]

where

\[
\gamma_\epsilon = \left[ 2(p - 1)p^2(C + \epsilon)^{p - 2}/(p - 2)^{1 - p} \right] + \epsilon.
\]
However, if \( \alpha > p/(p - 2) \), then for arbitrary \( \epsilon > 0 \) there exists \( x_\epsilon < 0 \) and \( \delta_\epsilon > 0 \) such that

\[
(C - \epsilon)(-x)_+^p \leq u \leq (C + \epsilon)(-x)_+^p (1 - \epsilon t)^{1/(2 - p)}, \quad x \geq x_\epsilon, \quad 0 \leq t \leq \delta_\epsilon.
\] (2.25)

(II) \( b = 0 \). We divide this case into three subcases.

(1) Let \( p > 2 \), \( 0 < \alpha < p/(p - 2) \). In this case, the interface expands. First, assume that \( u_0 \) is defined by (1.4). Then, if \( \alpha = (p - 1)/(p - 2) \), the explicit solution to the problem (1.1), (1.4) is

\[
u(x, t) = C(\xi_\ast t - x)^{(p-1)/(p-2)}, \quad \xi_\ast = C^{p-\alpha} \left( \frac{p - 1}{p - 2} \right)^{p-1}.
\] (2.26)

If \( 0 < \alpha < p/(p - 2) \), then the solution to (1.1), (1.4) has the self-similar form (2.4)

\[
\eta(t) = \xi_\ast t^{\frac{p - \alpha}{p - 2}}, \quad 0 \leq t < +\infty,
\] (2.27)

where \( \xi_\ast \) and \( f \) satisfy (2.2), (2.5)–(2.8). Moreover, we have

\[
C_4 t^{\frac{1}{p - 3}} (\xi_3 - \xi_2)^{p-1} \leq u \leq C_5 t^{\frac{1}{p - 2}} (\xi_4 - \xi_3)^{p-1},
\] (2.28)

where \( \xi_3 \) (\( \xi_4 \), respectively) is defined by the right-hand side of (2.7), where we replace \( \xi_\ast'' \) with \( C \frac{\xi_2^{p-2}}{p-2} \xi_1 \) (with \( C \frac{\xi_2^{p-2}}{p-2} \xi_2 \), respectively) and

\[
C_4 = C^{p/(p - 3)} A_0 \xi_3^{(p-1)/(2-p)} , \quad C_5 = C^{p/(p - 3)} A_0 \xi_4^{(p-1)/(2-p)}.
\]

In the particular case \( \alpha = (p - 1)(p - 2)^{-1} \), when an explicit solution is given by (2.26), we have \( \xi_3 = \xi_4 = \xi_\ast \) and both lower and upper estimations in (2.28) lead to the explicit solution (2.26). In general, when \( \alpha \neq (p - 1)(p - 2)^{-1} \) the precise value \( \xi_\ast \) relates to the similarity ODE for \( f(\xi) \) from (2.5), namely, \( \xi_\ast = \sup \{ \xi : f(\xi) > 0 \} \). If \( u_0 \) satisfies (1.3) with \( 0 < \alpha < p/(p - 2) \), then (2.1) and (2.3) are valid. Lower and upper bounds for \( f(p) \) follow from (2.28).

(2) Let \( p > 2 \), \( \alpha = p/(p - 2) \). In this case, the interface initially has a waiting time. If \( u_0 \) is defined by (1.4), then the explicit solution to (1.1), (1.4) is

\[
u_C(x, t) = C(-x)_+^p \left[ 1 - (C/\tilde{C})^{p-2} (p - 2) t \right]^{1/(2-p)} \quad x \in \mathbb{R}, \quad 0 \leq t < T,
\] (2.29)

where

\[
T = (\tilde{C}/C)^{p-2} (p - 2)^{-1}
\]

and the constant \( \tilde{C} \) is defined in Region (4) of (I).

If \( u_0 \) satisfies (1.3) with \( \alpha = p/(p - 2) \), then lower and upper estimations are given by \( u_{C \pm \epsilon} \).
(3) Let $p > 2, \alpha > p/(p - 2)$. In this case also the interface initially remains stationary and for arbitrary $\varepsilon > 0$ there exists $x_\varepsilon < 0$ and $\delta_\varepsilon > 0$ such that

$$(C - \varepsilon)(-x)_+^\alpha \leq u \leq (C + \varepsilon)(-x)_+^\alpha (1 - \varepsilon t)^{1/2 - p}, \quad x_\varepsilon \leq x, \quad 0 \leq t \leq \delta_\varepsilon.$$

\[ (2.30) \]

3 Preliminary results

The mathematical theory of non-linear $p$-Laplacian type degenerate parabolic equations is well developed. Throughout this paper we shall follow the definition of weak solutions and of supersolutions (or subsolutions) of equation (1.1) in the following sense.

**Definition 3.1** A measurable function $u \geq 0$ is a local weak solution (respectively sub- or supersolution) of (1.1) in $\mathbb{R} \times (0, T]$ if

- $u \in C^1_{loc}(0, T; L^2_{loc}(\mathbb{R}) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\mathbb{R}) \cap L^{1+\beta}_{loc}(\mathbb{R})))$;

- for all subinterval $[t_0, t_1] \subset (0, T]$ and for all $\mu_i \in C^1([t_0, t_1]), i = 1, 2$ such that $\mu_1(t) < \mu_2(t)$ for $t \in [t_0, t_1]$

\[ \int_{\mu_i(t)}^{\mu_j(t)} u \phi dx \bigg| _{t_0}^{t_1} + \int_{t_0}^{t_1} \int_{\mu_i(t)}^{\mu_j(t)} ( - u \phi_t + |u_x|^{p-2}u_x \phi_x + bu^\beta \phi ) dx dt = 0 \ (\text{resp. } \leq \text{ or } \geq 0), \quad (3.1) \]

where $\phi \in C^{2,1}_{x,t}(\overline{D})$ is an arbitrary function that equals zero when $x = \mu_i(t), t_0 \leq t \leq t_1, i = 1, 2$, and

$$D = \{ (x, t) : \mu_1(t) < x < \mu_2(t), t_0 < t < t_1 \}.$$  

The questions of existence and uniqueness of initial boundary value problems for (1.1), comparison theorems, and regularity of weak solutions are known due to [12–15,21,22,26], etc. Qualitative properties of free boundaries for the quasi-linear degenerate parabolic equations were studied via energy methods in [8]. The proof of the following typical comparison result is standard.

**Lemma 4** Let $g$ be a non-negative and continuous function in $\overline{Q}$, where

$$Q = \{ (x, t) : \eta_0(t) < x < +\infty, 0 < t < T \leq +\infty \},$$

$f$ is in $C^{2,1}_{x,t}$ in $Q$ outside a finite number of curves $x = \eta_j(t)$, which divide $Q$ into a finite number of subdomains $Q_j$, where $\eta_j \in C[0, T]$: for arbitrary $\delta > 0$ and finite $\Delta \in (\delta, T]$ the function $\eta_j$ is absolutely continuous in $[\delta, \Delta]$. Let $g$ satisfy the inequality

$$Lg \equiv g_t - \left( |g_x|^{p-2}g_x \right)_x + bg^\beta \geq 0, \quad (\leq 0)$$

at the points of $Q$, where $g \in C^{2,1}_{x,t}$. Assume also that the function $|g_x|^{p-2}g_x$ is continuous in $Q$ and $g \in L^\infty(Q \cap (t \leq T_1))$ for any finite $T_1 \in (0, T]$. Then, $g$ is a supersolution
(subsolution) of \((1.1)\). If, in addition we have
\[ g \bigg|_{x=\eta_0(t)} \geq (\leq) \, u \bigg|_{x=\eta_0(t)}, \quad g \bigg|_{t=0} \geq (\leq) \, u \bigg|_{t=0}, \]
then
\[ g \geq (\leq) \, u, \quad \text{in} \quad \overline{Q}. \]

Suppose that \( b \geq 0 \) and that \( u_0 \) may have unbounded growth as \(|x| \to +\infty\). It is well known that in this case some restriction must be imposed on the growth rate for existence, uniqueness and comparison results in the CP \((1.1), (1.2)\). Optimal growth condition for the equation \((1.1)\) with \( b = 0, p > 2 \) was derived in \([13, 14]\). If initial data may be majorised by power law function \((1.4)\), then there exists a unique solution (with \( T = +\infty \)) and a comparison principle is valid if \( 0 < \alpha < p/(p - 2) \). If \( \alpha = p/(p - 2) \), then existence, uniqueness, and comparison results are valid only locally in time. In particular, from \([13, 14]\) it follows that the unique explicit solution to \((1.1), (1.4)\) with \( b = 0, \alpha = p/(p - 2), T = (\bar{C}/C)^{p-2}(p-2)^{-1} \) is \( u_C(x,t) \) from \((2.29)\).

If the function \( u(x,t) \) is a solution to CP \((1.1), (1.4)\) with \( b = 0 \), then the function
\[ \tilde{u}(x,t) = \exp(-bt)u(x,(b(2-p))^{-1}(\exp(b(2-p)t) - 1)) \]
is a solution to \((1.1)\) with \( b \neq 0, \beta = 1 \). Hence, from the above mentioned result it follows that the unique solution to CP \((1.1), (1.4)\) with \( p > 2, b \neq 0, \beta = 1, \alpha = p/(p - 2) \) is the function \( \tilde{u}_C(x,t) \) from \((2.21)\).

We are not interested in necessary and sufficient conditions on the growth rate at infinity for existence, uniqueness, and comparison results for the CP \((1.1), (1.2)\) with \( b > 0, p > 2, \beta > 0 \); for our purposes, it is enough to mention that if \( u_0 \) may be majorised by the function \((1.4)\) with \( \alpha \) satisfying \( 0 < \alpha < p/(p - 2) \), then the CP \((1.1), (1.2)\) with \( b > 0, p > 2, \beta > 0, T = +\infty \) has a unique solution and for this class of initial data a comparison principle is valid. This easily follows from the fact that the solution of the CP \((1.1), (1.2)\) with \( b = 0 \) is a supersolution of the CP with \( b > 0 \), and hence it becomes a global locally bounded uniform upper bound for the increasing sequence of approximating bounded solutions of the CP with \( b > 0 \).

In the next four lemmas, we apply rescaling to establish some preliminary estimations of the solution to CP.

**Lemma 5** If \( b = 0 \) and \( p > 2, 0 < \alpha < p/(p - 2) \), then the solution \( u \) of the CP \((1.1), (1.4)\) has a self-similar form \((2.4)\), where the self-similarity function \( f \) satisfies \((2.6)\). If \( u_0 \) satisfies \((1.3)\), then the solution to CP \((1.1), (1.2)\) satisfies \((2.1)-(2.3)\).

**Lemma 6** Let \( u \) be a solution to the CP \((1.1), (1.2)\) and \( u_0 \) satisfy \((1.3)\). Let one of the following conditions be valid:

(a) \( b > 0, \quad 0 < \beta < 1 < p, \quad 0 < \alpha < p/(p - 1 - \beta) \);
(b) \( b \neq 0, \quad \beta \geq 1, \quad p > 2, \quad 0 < \alpha < p/(p - 2) \).

Then, \( u \) satisfies \((2.3)\).
Lemma 7 Let $u$ be a solution to the CP (1.1), (1.4) with $b > 0$, $0 < \beta < 1$, $p > 2$, $\alpha = p/(p - 1 - \beta)$. Then, the solution $u$ has the self-similar form (2.13). If $C > C_\ast$, then $f_1(0) = A_1$, where $A_1$ is a positive number depending on $p$, $\beta$, $C$, and $b$. If $u_0$ satisfies (1.3) with $\alpha = p/(p - 1 - \beta), C > C_\ast$, then $u$ satisfies

$$u(0, t) \sim A_1 t^{1/(1 - \beta)} \text{ as } t \to 0 + .$$

Lemma 8 Let $u$ be a solution to the CP (1.1)–(1.3) with $b > 0, 0 < \beta < 1, \alpha > p/(p - 1 - \beta)$. Then, for arbitrary $\ell > \ell_\ast$ (see (2.19)) the asymptotic formula (2.20) is valid with $x = \eta_\ell(t) = -\ell t^{1/(\alpha - 1)}$.

Proof of Lemma 5 If we consider a function

$$u_k(x, t) = ku(k^{-1/\alpha}x, k^{(x(p - 2) - p)/\alpha}), \quad k > 0,$$

it may easily be checked that this satisfies (1.1), (1.4). From [13,14], it follows that under the condition of the lemma there exists a unique global solution to (1.1), (1.4). Therefore, we have

$$u(x, t) = ku(k^{-1/\alpha}x, k^{(x(p - 2) - p)/\alpha}), \quad k > 0.$$  

If we choose $k = t^{\alpha/(p - \alpha(p - 2))}$, then (3.4) implies (2.4) for $u$ with $f(\xi) = u(\xi, 1)$. In fact, $f$ is a unique non-negative and differentiable weak solution of the boundary value problem:

$$\begin{cases}
|f'(\xi)|^{p - 2}f'(\xi) + \frac{1}{p - 2(\alpha - \beta)}\xi f'(\xi) - \frac{\alpha}{p - 2(\alpha - \beta)}f(\xi) = 0, & -\infty < \xi < +\infty \\
f(-\infty) \sim C(-\xi)^{\alpha}, \quad f(+\infty) = 0,
\end{cases}$$

and there exists an $\xi_\ast > 0$ such that $f$ satisfies (2.5): it is positive and smooth for $\xi < \xi_\ast$ and $f = 0$ for $\xi \geq \xi_\ast$ (9). Thus, (2.27) is valid. To find the dependence of $f$ on $C$, we can again use scaling. Namely, let $w$ be a solution of the CP (1.1), (1.4) with $C = 1$. Then, it may be easily checked that for arbitrary $k > 0$

$$u(x, t) = kw(C^{1/\alpha}k^{-1/\alpha}x, C^{p/\alpha}k^{(x(p - 2) - p)/\alpha}t).$$

By choosing $k = (C^{p/\alpha}t)^{\alpha/(p - \alpha(p - 2))}$, we then have

$$u(x, t) = C^{-\frac{\alpha}{p - 2(\alpha - \beta)}}w(C^{-\frac{\alpha}{p - 2(\alpha - \beta)}}\xi, 1)t^{\alpha/(p - \alpha(p - 2))}.$$  

Formulae (2.6) and (2.2) follow from (3.6) and (2.4).

Now assume that $u_0$ satisfies (1.3). Then, for arbitrary sufficiently small $\epsilon > 0$, there exists $x_\epsilon < 0$ such that

$$(C - \varepsilon/2)(-x)^2_+ \leq u_0(x) \leq (C + \varepsilon/2)(-x)^2_+, \quad x \geq x_\epsilon.$$  

Let $u_\epsilon(x, t)$ be a solution to the CP (1.1), (1.2) with initial data $(C + \varepsilon)(-x)^2$ and $((C - \varepsilon)(-x)^2$, respectively. Since the solution to the CP (1.1), (1.2) is continuous,
there exists a number $\delta = \delta(\epsilon) > 0$ such that

$$u_\epsilon(x, t) \geq u(x, t), \quad u_{-\epsilon}(x, t) \leq u(x, t) \text{ for } 0 \leq t \leq \delta. \quad (3.8)$$

From (3.7), (3.8), and a comparison principle, it follows that

$$u_{-\epsilon} \leq u \leq u_\epsilon \text{ for } x \geq x_\epsilon, \quad 0 \leq t \leq \delta. \quad (3.9)$$

Obviously,

$$u_{\pm\epsilon}(\xi_\rho(t), t) = f(\rho; C \pm \epsilon)t^{2/(p-2)}(p-2), \quad t \geq 0. \quad (3.10)$$

(Furthermore, we denote the right-hand side of (2.6a) by $f(\rho, C)$.) Now taking $x = \xi_\rho(t)$ in (3.9), after multiplying to $t^{2/(p-2)}(p-2)$ and passing to the limit, first as $t \to 0$ and then as $\epsilon \to 0$, we can easily derive (2.3). Similarly, from (3.9), (2.27), and (2.2), (2.1) easily follows.

**Proof of Lemma 6** As in the previous proof, (3.7)–(3.9) follow from (1.3). Let the conditions of one of the cases, (a) or (b), with $b > 0$ be valid. Then, from the results mentioned earlier it follows that the existence, uniqueness, and comparison results of the CP (1.1), (1.2) with $u_0 = (C \pm \epsilon)(-x)^2_+$, $T = +\infty$ hold. Now, if we rescale

$$u_{k_1}^{\pm\epsilon}(x, t) = ku_{\pm\epsilon}(k^{-1/2} x, k^{(2p-2)/p} t), \quad k > 0, \quad (3.11)$$

then $u_{k_1}^{\pm}(x, t)$ satisfies the following problem:

$$u_t - (|u_x|^{p-2}u_x)_x + b k^{(2p-2)/p}u^p = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3.12a)$$

$$u(x, 0) = (C \pm \epsilon)(-x)^2_+, \quad x \in \mathbb{R}. \quad (3.12b)$$

There exists a unique solution to CP (3.12), which also obeys a comparison principle. Since $\alpha(p-1-\beta) - p < 0$, by using a comparison principle in Lemma 5 it follows that

$$u_{k_1}^{\pm\epsilon}(x, t) \leq u_{k_2}^{\pm\epsilon}(x, t) \leq \cdots \leq v_{\pm}(x, t), \quad x \in \mathbb{R}, \quad t \geq 0; \quad \text{if } k_1 < k_2, \quad (3.13)$$

where $v_{\pm\epsilon}$ is a solution to CP (1.1), (1.2) with $b = 0$, $u_0 = (C \pm \epsilon)(-x)^2_+$, $T = +\infty$. From the results of [13,26], it follows that the sequence of non-negative and locally bounded solutions $\{u_{k}^{\pm\epsilon}\}$ is locally uniformly Hölder continuous, and weakly pre-compact in $W_{loc}^{1, p}(\mathbb{R} \times (0, T))$. Since $\alpha(p-1-\beta) - p < 0$, passing to limit as $k \to +\infty$, from (3.1) it follows that the limit function is a solution of the CP (1.1), (1.2) with $b = 0$, $u_0 = (C \pm \epsilon)(-x)^2_+$, $T = +\infty$. Due to uniqueness we have

$$\lim_{k \to +\infty} u_{k}^{\pm\epsilon}(x, t) = v_{\pm}(x, t), \quad x \in \mathbb{R}, \quad t \geq 0. \quad (3.14)$$

Hence, $v_{\pm\epsilon}$ satisfies (3.10). If we now take $x = \xi_\rho(t)$, where $\rho$ is an arbitrary fixed number satisfying $\rho < \xi_{\epsilon}$, then from (3.14) it follows that

$$\lim_{k \to +\infty} ku_{\pm\epsilon}(k^{-1/2} \xi_\rho(t), k^{(2p-2)/p} t) = f(\rho; C \pm \epsilon)t^{2/(p-2)}(p-2), \quad t > 0. \quad (3.15)$$
If we take \( \tau = k^{(z(p-2) - p)/2}t \), then (3.15) implies
\[
u_{\pm e}(x_\tau(t), \tau) \sim f(\nu; C \pm e)\tau^{(z(p-2) - p)/2}, \quad \text{as } \tau \to +0. \tag{3.16}
\]
As before, (2.3) follows from (3.9), (3.16).

Now consider the case (b) with \( b < 0 \). Suppose that \( u_{\pm e} \) is a solution of the Dirichlet problem
\[
u_t - (|u|^{p-2}u)_x + bu^\beta = 0, \quad |x| < |x_\epsilon|, \quad 0 < t < \delta, \tag{3.17a}
\]
\[
u(x,0) = (C \pm e)(-x)^2, \quad |x| \leq |x_\epsilon|, \tag{3.17b}
\]
\[
u(x_\epsilon, t) = (C \pm e)(-x)^2, \quad u(-x_\epsilon, t) = 0, \quad 0 \leq t \leq \delta. \tag{3.17c}
\]
The function \( u_k^{+\pm e} \), defined as in (3.11), satisfies the Dirichlet problem:
\[
u_t - (|u|^{p-2}u)_x + bk^{(z(p-1) - p)/2}u^\beta = 0 \quad \text{in } D^k_\epsilon, \tag{3.18a}
\]
\[u(k^{1/2}x_\epsilon, t) = k(C \pm e)(-x)^2, \quad u(-k^{1/2}x_\epsilon, t) = 0, \quad 0 \leq t \leq k^{(p-2)/2} \delta \tag{3.18b}
\]
\[u(x,0) = (C \pm e)(-x)^2, \quad |x| \leq k^{1/2}|x_\epsilon|, \tag{3.18c}
\]
where
\[D^k_\epsilon = \{(x, t) : |x| < k^{1/2}|x_\epsilon|, \quad 0 < t \leq k^{(p-2)/2} \delta \}.
\]
There exists a number \( \delta > 0 \) (which does not depend on \( k \)) such that both (3.17a)–(3.17c) and (3.18a)–(3.18c) have a unique solution (see discussion preceding Lemma 5). In view of finite speed of propagation, \( \delta = \delta(e) > 0 \) may be chosen such that
\[u(-x_\epsilon, t) = 0, \quad 0 \leq t \leq \delta. \tag{3.19}
\]
Applying the comparison theorem, from (3.7), (3.8), and (3.19), (3.9) follows for \( |x| \leq |x_\epsilon|, \quad 0 \leq t \leq \delta \).

To prove the convergence of the sequences \( \{u_k^{+\pm e}\} \) as \( k \to +\infty \), we need to prove uniform boundedness. Consider a function
\[g(x, t) = (C + 1)(1 + x^2)^{2z}(1 - vt)^{(1-z)/2}, \quad x \in \mathbb{R}, \quad 0 \leq t \leq t_0 = \frac{v-1}{2},
\]
where
\[v = h_* + 1, \quad h_* = h_*(x; p) = \max_{x \in \mathbb{R}} h(x),
\]
\[h(x) = (p-2)x^{p-1}(C+1)p^{-2}(1+x^2)^{\frac{(p-2)(z-1)}{2z}}x^2 |x|^{p-2} \left( \frac{1+x^2}{x^2} + (p-2) \frac{1+x^2}{|x|^2} + (z-2)(p-1) \right).
\]
Then, we have
\[L_k g \equiv g_t - (|g|^{p-2}g)_x + bk^{(z(p-1)-p)/2}g^\beta = (C + 1)(p-2)^{-1}(1 + x^2)^{z}(1 - vt)^{p-1} S \quad \text{in } D^k_\epsilon,
\]
\[S = v - h(x) + b(p-2)(C + 1)^{p-1}k^{(z(p-1)-p)/2} \left( 1 + x^2 \right)^{p-1} \left( 1 - vt \right)^{\frac{p-1}{z-p}},
\]
and hence
\[S \geq 1 + R \quad \text{in } D^k_\epsilon = D^k_\epsilon \cap \{0 < t \leq t_0\}, \tag{3.20}
\]
where
\[ R = O\left(k^{p-2-p/2}\right) \quad \text{uniformly for } (x,t) \in D_{0^e}^k \quad \text{as } k \to +\infty. \]
Moreover, we have for \(0 < \epsilon \ll 1\)
\[
g(x,0) \geq u_{k}^{\pm\epsilon}(x,0) \quad \text{for } |x| \leq k^{1/2}|x_\epsilon|, \quad (3.21a)
\]
\[
g(\pm k^{1/2}x_\epsilon,t) \geq u_{k}^{\pm\epsilon}(\pm k^{1/2}x_\epsilon,t) \quad \text{for } 0 \leq t \leq t_0. \quad (3.21b)
\]
Hence, \(\exists k_0 = k_0(\alpha; p)\) such that for all \(k \geq k_0\) the comparison theorem implies
\[
0 \leq u_{k}^{\pm\epsilon}(x,t) \leq g(x,t) \quad \text{in } \bar{D}_{0^e}^k. \quad (3.22)
\]
Let \(G\) be an arbitrary fixed compact subset of
\[
P = \{(x,t) : x \in \mathbb{R}, \quad 0 < t \leq t_0\}.
\]
We take \(k_0\) so large that \(G \subset D_{0^e}^k\) for \(k \geq k_0\). From (3.22), it follows that the sequences \(\{u_{k}^{\pm\epsilon}\}, k \geq k_0\), are uniformly bounded in \(G\). As before, from the results of [13,26] it follows that the sequence of non-negative and locally bounded solutions \(\{u_{k}^{\pm\epsilon}\}\) is locally uniformly Hölder continuous, and weakly pre-compact in \(W_{loc}^{1,p}(\mathbb{R} \times (0, T))\). It follows that for some subsequence \(k'\)
\[
\lim_{k' \to +\infty} u_{k'}^{\pm\epsilon}(x,t) = v_{\pm\epsilon}(x,t), \quad (x,t) \in P. \quad (3.23)
\]
Since \(\alpha(p - 1 - \beta) - p < 0\), passing to limit as \(k' \to +\infty\), from (3.1) for \(u_{k'}^{\pm\epsilon}\) it follows that \(v_{\pm\epsilon}\) is a solution to the CP (1.1), (1.2) with \(b = 0, T = t_0, u_0 = (C \pm \epsilon)(-x_\epsilon)^{\alpha}_+\). As before, from (3.10), (3.15), (3.16), and (3.9), the required estimation (2.3) follows.

The first assertion of Lemma 7 has been proved in [24] for the case \(p > 2\). If \(u_0\) satisfies (1.3), the estimation (3.2) may be proved exactly as estimation (2.3) was proved in Lemma 5.

**Proof of Lemma 8** Asymptotic behaviour (1.3) implies (3.7) and (3.8). Assume that \(v_{\pm\epsilon}\) solves the problem:
\[
v_t - (|v|^p - 2v)v_x + b v^\beta = 0, \quad |x| \leq |x_\epsilon|, \quad 0 < t \leq \delta,
\]
\[
v(x,0) = (C \pm \epsilon)(-x)^{\alpha}_+, \quad |x| \leq |x_\epsilon|,
\]
\[
v(x_\epsilon,t) = (C \pm \epsilon)(-x_\epsilon)^{\alpha}_+, \quad v(-x_\epsilon,t) = u(-x_\epsilon,t), \quad 0 \leq t \leq \delta.
\]
According to comparison result from (3.7) and (3.8), (3.9) follows for \(|x| \leq |x_\epsilon|, \quad 0 \leq t \leq \delta\). If we rescale
\[
u_{k}^{\pm\epsilon}(x,t) = ku_{\pm\epsilon}(k^{-\frac{1}{2}}x,k^{\beta-1}t), \quad k > 0,
\]
then \( u_k^{\pm \epsilon} \) satisfies the Dirichlet problem

\[
v_t - k \frac{p-\alpha(p-1-\beta)}{p-1} \left( |v_x|^{p-2}v_x \right)_x + bv^\beta = 0 \text{ in } E^k_{\epsilon},
\]

\[
v(x,0) = (C \pm \epsilon)(-x)^2, \quad |x| \leq k^{1/\epsilon}|x_\epsilon|,
\]

\[
v(k^{1/2}x_\epsilon, t) = k(C \pm \epsilon)(-x_\epsilon)^2, \quad v(-k^{1/2}x_\epsilon, t) = ku(-x_\epsilon, k^{\beta-1}t), \quad 0 \leq t \leq k^{1-\beta} \delta,
\]

where

\[
E^k_{\epsilon} = \{ |x| < k^{1/2}|x_\epsilon|, \quad 0 < t \leq k^{1-\beta} \delta \}.
\]

The goal is to prove the convergence of the sequence \( \{u_k^{\pm \epsilon}\} \) as \( k \to +\infty \). To establish uniform bound consider \( g(x, t) = (C + 1)(1 + x^2)^{\beta/2} \exp t \). We have

\[
\tilde{L}_k g \equiv g_t - k \frac{p-\alpha(p-1-\beta)}{p-1} \left( |g_x|^{p-2}g_x \right)_x + bg^\beta \geq g \left[ 1 - k \frac{p-\alpha(p-1-\beta)}{p-1} x^{p-1}(C + 1)^{\beta-2} \exp((p-2)}
\]

\[
\times (1 + x^2) \frac{p-\alpha(p-1-\beta)}{p-1} \right) \frac{1 + x^2}{|x|^2} \frac{1 + x^2}{(p-2)(p-1)} \right) \right] \] in \( E^k_{\epsilon} \). (3.24)

Let \( t_0 > 0 \) be fixed and let \( E^k_{0\epsilon} = E^k_{\epsilon} \cap \{(x, t) : \quad 0 < t \leq t_0\} \). From (3), it follows that

\[
\tilde{L}_k g \geq (1 + R) \quad \text{in } E^k_{0\epsilon},
\]

where

\[
R = O(k^\beta) \quad \text{uniformly for } (x, t) \in E^k_{0\epsilon} \quad \text{as } k \to +\infty
\]

\[
\theta = (p - \alpha(p - 1 - \beta)/\alpha), \quad \text{if } \alpha < p/(p - 2),
\]

\[
\theta = \beta - 1, \quad \text{if } \alpha \geq p/(p - 2).
\]

We have for \( 0 < \epsilon \ll 1 \) that

\[
g(x,0) = u_k^{\pm \epsilon}(x,0), \quad \text{for } |x| \leq k^{1/\epsilon}|x_\epsilon|,
\]

and

\[
u_k^{\pm \epsilon}(-k^{1/2}x_\epsilon, t) = o(k), \quad 0 \leq t \leq t_0 \text{ as } k \to \infty,
\]

\[
g(\pm k^{1/2}x_\epsilon, t) \geq u_k^{\pm \epsilon}(\pm k^{1/2}x_\epsilon, t), \quad \text{for } 0 \leq t \leq t_0,
\]

if \( k \) is chosen large enough. Therefore, the comparison principle implies (3.22) in \( E^k_{0\epsilon} \), where the respective functions \( u_k^{\pm \epsilon} \) and \( g \) apply in the context of this proof. As before, from the interior regularity results [13, 26], it follows that the sequence of non-negative and locally bounded solutions \( \{u_k^{\pm \epsilon}\} \) is locally uniformly Hölder continuous, and weakly pre-compact in \( W^{1,p}_{loc}(\mathbb{R} \times (0, T)) \). It follows that for some subsequence \( k' \), (3.23) is valid. Since \( \alpha > p/(p - 1 - \beta) \), it follows that the limit functions \( v^{\pm \epsilon} \) are solutions to the problem:

\[
v_t + bv^\beta = 0, \quad x \in \mathbb{R}, \quad 0 < t \leq t_0; \quad v(x,0) = (C \pm \epsilon)(-x)^2, \quad x \in \mathbb{R},
\]

i.e.,

\[
v^{\pm \epsilon}(x, t) = \left[ (C \pm \epsilon)\frac{1}{p}(-x)^{\alpha(1-\beta)} - b(1-\beta)t \right]^{1/\beta}.
\]
Let \( l > l_* \) be an arbitrary number and \( \epsilon > 0 \) be chosen such that
\[
(C - \epsilon)^{1 - \beta} \rho^{2(1 - \beta)} > b(1 - \beta).
\]
If we now take \( x = \eta(t) \) and \( \tau = k^{\beta - 1}t \), it follows from (3.23) that
\[
u_{\pm, \epsilon}(\eta(t), \tau) \sim \left[ (C \pm \epsilon)^{1 - \beta} \rho^{2(1 - \beta)} - b(1 - \beta) \right]^{1/(1 - \beta)} \tau \rightarrow \infty \text{ as } \tau \rightarrow 0^+.
\]
(3.25)
Since \( \epsilon > 0 \) is arbitrary, from (3.9) and (3.25), (2.20) follows.

4 Proofs of the main results

In this section, we prove the main results described in Section 2.

(I) \( b \neq 0 \) and \( p > 2 \).

Region (1)

Proof of Theorem 1 Assume \( \alpha < p/(p - 1 - \min\{1, \beta\}) \). The formula (2.3) follows from Lemma 5. Since \( \rho \) is arbitrary, it implies
\[
limit_{t \rightarrow 0^+} \inf \eta(t)t^{1/(\alpha(p - 2) - p)} \geq \xi_*.
\]
(4.1)
Take an arbitrary sufficiently small number \( \epsilon > 0 \). Let \( u_\epsilon \) be a solution of the CP (1.1), (1.4) with \( b = 0 \) and with \( C \) replaced by \( C + \epsilon \). As before, the second inequality of (3.7) and the first inequality of (3.8) follow from (1.3). Suppose that \( b > 0 \). In this case, \( u_\epsilon \) is a supersolution of (1.1). From (3.7), (3.8), and a comparison principle, the second inequality of (3.9) follows. By Lemma 3.1, we then have
\[
\eta(t) \leq (C + \epsilon)^{\frac{2 - p}{p - 2 - \alpha}} \xi_* t^{1/(p - 2(p - 2))}, \quad 0 \leq t \leq \delta,
\]
and hence
\[
limit_{t \rightarrow 0^+} \sup \eta(t)t^{\frac{1}{p - 2 - \alpha}} \leq \xi_*. \quad (4.2)
\]
Assume now that \( b < 0 \) and \( \beta \geq 1 \). The function
\[
\bar{u}_e(x, t) = \exp(-bt)u_e \left( x, \frac{1}{b(2 - p)} \left[ \exp(b(2 - p)t) - 1 \right] \right)
\]
is a solution to the CP (1.1), (1.4) with \( \beta = 1 \) and with \( C \) replaced by \( C + \epsilon \). As before, from (1.3) the first inequality of (3.8) follows, where we replace \( u_\epsilon \) with \( \bar{u}_e \). Choose \( |x_\epsilon| \) and \( \delta \) so small that
\[
\bar{u}_e < 1 \text{ in } B = \{(x, t) : x \geq x_\epsilon, \ 0 < t \leq \delta\}.
\]
Obviously, \( \bar{u}_e \) is a supersolution of (1.1) in \( B \). From (3.7), (3.8), and a comparison principle, the second inequality of (3.9), with \( u_\epsilon \) replaced by \( \bar{u}_e \), follows. Thus, we have
\[
\eta(t) \leq (C + \epsilon)^{\frac{2 - p}{p - 2 - \alpha}} \xi_* \left\{ \left( b(2 - p) \right)^{1/(p - 2(p - 2))} \left[ \exp(b(2 - p)t) - 1 \right] \right\}^{1/(p - 2(p - 2))}, \quad 0 \leq t \leq \delta.
\]
which again implies (4.2). From (4.1) and (4.2), (2.1) follows. Finally, (2.7)–(2.9) follow from (2.28), which will be proved later in this section.

Region (2)

**Proof of Theorem 2** First, consider the global case of (1.4). The problem (1.1), (1.4) has a unique global solution and for this class of initial data a comparison principle is valid [13,14].

If $\beta(p-1) = 1$, it may be easily checked that the explicit solution to (1.1), (1.4) is given by (2.12).

Let $\beta(p-1) \neq 1$. The self-similar form (2.13) follows from Lemma 7. Let $C > C_\ast$. Consider a function

$$g(x,t) = t^{1/(1-\beta)}f_1(\zeta), \quad \zeta = xt^{-p/(p-\beta)}.$$  \hfill (4.3)

We then have

$$Lg = t^{\beta/(p-\beta)}L^0f_1,$$

\hfill (4.4a)

$$L^0f_1 = \frac{1}{1-\beta}f_1 - \left( |f'|^p - 2p \frac{1-\beta}{p(1-\beta)} f' \right) + b f^p_1.$$ \hfill (4.4b)

Choose as a function $f_1$

$$f_1(\zeta) = C_0(\zeta_0 - \zeta)_{+}^{\gamma_0}, \quad 0 < \zeta < +\infty,$$

where $C_0, \zeta_0, \gamma_0$ are some positive constants. Taking $\gamma_0 = p/(p-1-\beta)$, from (4.4b) we have

$$L^0f_1 = bC_0^\beta(\zeta_0 - \zeta)_{+}^{\frac{\beta p}{p-1-\beta}} \left\{ 1 - \left( \frac{C_0}{C_\ast} \right)^{p-1-\beta} + \frac{C_0^{1-\beta}}{b(1-\beta)}(\zeta_0 - \zeta)_{+}^{\frac{\beta(1-p)+1}{1-\beta}} \right\}.$$ \hfill (4.5)

To prove an upper estimation, we take $C_0 = C_2, \zeta_0 = \zeta_2$ (see the appendix). If $\beta(p-1) > 1$, then we have

$$L^0f_1 \geq bC_2^{\beta}(\zeta_2 - \zeta)_{+}^{\frac{\beta p}{p-1-\beta}} \left\{ 1 - \left( \frac{C_2}{C_\ast} \right)^{p-1-\beta} + \frac{C_2^{1-\beta}}{b(1-\beta)}(\zeta_2 - \zeta)_{+}^{\frac{\beta(1-p)+1}{1-\beta}} \right\} = 0, \quad \text{for } 0 \leq \zeta \leq \zeta_2,$$

whereas if $\beta(p-1) < 1$, we have

$$L^0f_1 \geq bC_2^{\beta}(\zeta_2 - \zeta)_{+}^{\frac{\beta p}{p-1-\beta}} \left\{ 1 - \left( \frac{C_2}{C_\ast} \right)^{p-1-\beta} \right\} = 0, \quad \text{for } 0 \leq \zeta \leq \zeta_2.$$

From (4.4a), it follows that

$$Lg \geq 0 \quad \text{for } 0 < x < \zeta_2 t^{\frac{p-1-\beta}{1-\beta}}, \quad 0 < t < +\infty,$$ \hfill (4.6a)

$$Lg = 0 \quad \text{for } x > \zeta_2 t^{\frac{p-1-\beta}{1-\beta}}, \quad 0 < t < +\infty.$$ \hfill (4.6b)

Lemma 4 implies that $g$ is a supersolution of (1.1) in $\{(x,t) : x > 0, t > 0\}$. Since

$$g(x,0) = u(x,0) = 0 \quad \text{for } 0 \leq x < +\infty,$$ \hfill (4.7a)

$$g(0,t) = u(0,t) \quad \text{for } 0 \leq x < +\infty,$$ \hfill (4.7b)
the right-hand side of (2.15) follows. If \( \beta(p - 1) < 1 \), then to prove the lower estimation we take \( C_0 = C_1, \zeta_0 = \zeta_1, \gamma_0 = p/(p - 1 - \beta) \). Then, from (4.5) we derive

\[
\mathcal{L}^0 f_1 \leq b C_1^{\beta}(\zeta_1 - \zeta) \frac{p - \beta}{p - 1} \left\{ 1 - \left( \frac{C_1}{C_c} \right)^{p - 1 - \beta} + \frac{C_1^{1-\beta}}{b(1-\beta)} \frac{\beta^{p-\beta}}{\zeta_1^{p-1}} \right\} = 0 \text{ for } 0 \leq \zeta \leq \zeta_1,
\]

and from (4.4a) it follows that

\[
\begin{align*}
\mathcal{L}g & \leq 0 \quad \text{for } 0 < x < \zeta_1 t \frac{p - 1 - \beta}{p - \beta}, \quad 0 < t < +\infty, \quad (4.8a) \\
\mathcal{L}g & = 0 \quad \text{for } x > \zeta_1 t \frac{p - 1 - \beta}{p - \beta}, \quad 0 < t < +\infty. \quad (4.8b)
\end{align*}
\]

As before, from Lemma 4 and (4.7a), (4.7b), the left-hand side of (2.15) follows.

If \( \beta(p - 1) > 1 \), then to prove the lower estimation we take \( C_0 = C_1, \zeta_0 = \zeta_1, \gamma_0 = (p - 1)/(p - 2) \). Then, from (4.4b) we have

\[
\begin{align*}
\mathcal{L}^0 f_1 & = C_1(1-\beta)^{-1}(\zeta_1 - \zeta) \frac{1}{(p-\beta)(p-2)} \left\{ \zeta_1 - \left( \frac{\beta(p-1) - 1}{p(p-2)} \right) (1-\beta) C_1^{p-2} \left( \frac{p-1}{p-2} \right)^p \right. \\
& \quad + b(1-\beta) C_1^{\beta-1}(\zeta_1 - \zeta) (p - 1) \right\} \\
& \leq C_1(1-\beta)^{-1}(\zeta_1 - \zeta) \frac{1}{(p-\beta)(p-2)} \left\{ \zeta_1 - C_1^{p-2} (1-\beta) (p-1) \frac{(p-1)^p}{p-2} + b(1-\beta) C_1^{\beta-1} \frac{\beta^{p-\beta}}{\zeta_1^{p-1}} \right\} \\
& = 0 \quad \text{for } 0 < \zeta < \zeta_1,
\end{align*}
\]

which again implies (4.8a), (4.8b). From Lemma 4, the left-hand side of (2.15) follows.

By applying the same analysis, it may easily be checked that the alternative upper estimation is valid if \( C_0 = C_2, \zeta_0 = \zeta_2, \gamma_0 = (p - 1)/(p - 2) \).

Let \( \beta(p - 1) > 1 \) and \( 0 < C < C_* \). Consider a function

\[
\begin{align*}
g(x, t) & = \left[ C_1^{1-\beta} (-x) \frac{\beta^{p-\beta}}{p(p-1)} - b(1-\beta)(1-\gamma) t \right]^{-1}_{+}, \quad x \in \mathbb{R}, \quad t > 0,
\end{align*}
\]

where \( \gamma \in [0, 1] \). Let us estimate \( \mathcal{L}g \) in

\[
M = \{(x, t) : -\infty < x < \mu(t), \; t > 0\}, \quad \mu(t) = -[b(1-\beta)(1-\gamma) C_1^{\beta-1} t]^{-1}_{-}\frac{p - 1 - \beta}{p - \beta}.
\]

We have

\[
\begin{align*}
\mathcal{L}g & = b g S, \\
S & = \gamma - p^{-1} \beta (1-p) + (p-1) b^{-1} (p-1-\beta)^{-p} C^{p-1-\beta} \left[ 1 - \frac{b(1-\beta)(1-\gamma) t}{C^{\beta-1} (-x) \frac{\beta^{p-\beta}}{p(p-1)}} \right]^{-\frac{p-\beta}{1-\beta}} \\
& \quad - p^\beta (p-1) b^{-1} (p-1-\beta)^{-p} C^{p-1-\beta} \left[ 1 - \frac{b(1-\beta)(1-\gamma) t}{C^{\beta-1} (-x) \frac{\beta^{p-\beta}}{p(p-1)}} \right]^{-\frac{p-\beta}{1-\beta}}. \quad (4.9a)
\end{align*}
\]

Hence,

\[
S|_{t=0} = \gamma - \left( \frac{C}{C_c} \right)^{p-1-\beta}, \quad S|_{x=\mu(t)} = \gamma. \quad (4.9b)
\]
Moreover,

\[
S_t = \frac{p^{p-1}(p-1)(1-\gamma)C^{p-2}}{(p-1-\beta)p}\left(\frac{p-1}{p}\right)^{\frac{p-1}{p-\beta}} \left[1 - C^{\beta-1}(-x)^{\frac{p-1}{p-\beta}} b(1-\beta)(1-\gamma)t\right]^{\frac{p-2}{p-\beta}}
\]

\[
\times \left[\left(\beta(p-1) - 1\right)\beta(p-2)C^{\beta-1}b(1-\beta)(-x)^{\frac{p-1}{p-\beta}} (1-\gamma)t + \left(\beta(p-1) - 1\right)(2\beta)\right] \geq 0 \quad \text{in } M.
\]

Thus,

\[
\gamma - \left(\frac{C}{C_\ast}\right)^{p-1-\beta} \leq S \leq \gamma \quad \text{in } M.
\]

If we take \( \gamma = \left(\frac{C}{C_\ast}\right)^{p-1-\beta} \) (\( \gamma = 0 \), respectively), then we have

\[
Lg \geq 0 \quad (Lg \leq 0, \text{respectively}) \quad \text{in } M, \quad (4.10a)
\]

and the estimation (2.17) follows from Lemma 4.

Let \( \beta(p-1) < 1 \) and \( 0 < C < C_\ast \). First, we can establish the following rough estimation:

\[
\left[C^{1-\beta}(-x)^{\frac{p-1}{p-\beta}} - b(1-\beta)\left(1 - \left(\frac{C}{C_\ast}\right)^{p-1-\beta}\right)t\right]^{\frac{1}{p-\beta}} \leq u(x,t) \leq C(-x)^{\frac{p-1}{p-\beta}}
\]

\[
x \in \mathbb{R}, 0 \leq t < +\infty. \quad (4.11)
\]

To prove the left-hand side, we consider the function \( g \) as in the case when \( \beta(p-1) > 1 \) with \( \gamma = \left(\frac{C}{C_\ast}\right)^{p-1-\beta} \). As before, we then derive (4.9a) and, since

\[
S_t \leq 0 \quad \text{in } M,
\]

we have \( S \leq 0 \) in \( M \). Hence, (4.10a), (4.10b) are valid with reversed inequality. As before, from Lemma 4, the left-hand side of (4.11) follows. Since

\[
Lu_0 = bu_0^\beta \left(1 - \left(\frac{C}{C_\ast}\right)^{p-1-\beta}\right) \geq 0 \quad \text{for } x \in \mathbb{R}, t \geq 0,
\]

the second inequality in (4.11) follows. Using (4.11), we can now establish a more accurate estimation (2.18). Consider a function

\[
g(x,t) = \mathcal{C}_0(\xi - \xi t^\frac{p-1-\beta}{p-\beta} - x)^{\frac{1}{p-\beta}} \quad \text{in } G_{t},
\]

\[
G_{t} = \{(x,t) : \xi(t) = -\xi t^\frac{p-1-\beta}{p-\beta} < x < +\infty, \, 0 < t < +\infty\},
\]

where \( C_0 > 0, \, \xi > 0, \, \ell > \xi_0 \) are some constants. Calculating \( Lg \) in

\[
G_{t}^+ = \{(x,t) : \xi(t) < x < -\xi_0 t^\frac{p-1-\beta}{p-\beta}, \, 0 < t < +\infty\},
\]
we have

\[
Lg = bg^\beta S, \quad S = 1 - \left( \frac{C_0}{C_s} \right)^{p-1-\beta} - (b(1 - \beta))^{-1} C_0^{1-\beta} \zeta_0 t^{\frac{p-1-\beta}{p-1-\beta}} \times (-\frac{1}{\zeta_0 t^{p-1-\beta}} - x)^{\frac{p-1-\beta}{p-1-\beta}}.
\]

(4.12)

Hence, if we take \( C_0 = C_s \), then

\[
Lg \leq 0 \quad \text{in } G^+_\ell; \quad Lg = 0 \quad \text{in } G^\ell \setminus \bar{G}^+_\ell.
\]

(4.13)

To obtain a lower estimation, we now choose \( \zeta_0 = \zeta_3, \ell = \ell_0 \) (see the appendix). Using (4.11), we have

\[
g(\zeta(t), t) = C_s(\ell - \zeta_3)^{\frac{p}{p-1-\beta} t^{\frac{1}{p-1-\beta}}} = \left( b(1 - \beta) \theta_s t \right)^{\frac{1}{p-1-\beta}}
\]

\[
= \left[ C^{1-\beta} \ell_0^{\frac{p}{p-1-\beta}} - b(1 - \beta) \left( 1 - \left( \frac{C}{C_s} \right)^{p-1-\beta} \right) \right]^{\frac{1}{p-1-\beta}} t^{\frac{1}{p-1-\beta}} \leq u(\zeta(t), t), \quad t \geq 0,
\]

(4.14a)

\[
g(x, 0) = u(x, 0) = 0, \quad 0 \leq x \leq x_0,
\]

(4.14b)

\[
g(x_0, t) = u(x_0, t) = 0, \quad t \geq 0,
\]

(4.14c)

where \( x_0 > 0 \) is an arbitrary fixed number. By using (4.13), (4.14a)–(4.14c), we can apply Lemma 4 in

\[
G'_{\ell_0} = G_{\ell_0} \cap \{ x < x_0 \}.
\]

Since \( x_0 > 0 \) is arbitrary number, the desired lower estimation from (2.18) follows.

Let us now prove the right-hand side of (2.18). Since

\[
S_x \geq 0, \quad \text{for } \zeta(t) < x < -\zeta_0 t^{\frac{p-1-\beta}{p-1-\beta}}, \quad t > 0,
\]

from (4.12) it follows that

\[
S \geq S_{|x=\zeta(t)|} = 1 - \left( \frac{C_0}{C_s} \right)^{p-1-\beta} - (b(1 - \beta))^{-1} C_0^{1-\beta} \zeta_0 (\ell - \zeta_0)^{\frac{p-1-\beta+1}{p-1-\beta}}.
\]

Taking now \( C_0 = C_3, \zeta_0 = \zeta_4, \ell = \ell_1 \) (see the appendix), we have

\[
S_{|x=\zeta(t)|} = 0;
\]

hence (by using (4.11))

\[
Lg \geq 0 \quad \text{in } G'_{\ell_1}, \quad Lg = 0 \quad \text{in } G_{\ell_1} \setminus \bar{G}'_{\ell_1},
\]

\[
u(\zeta(t), t) \leq C^{\frac{p}{p-1-\beta}} \ell^{\frac{1}{p-1-\beta}} = C_{3}(\ell - \zeta_4)^{\frac{p}{p-1-\beta} t^{\frac{1}{p-1-\beta}}} = g(\zeta(t), t), \quad t \geq 0,
\]

and, for arbitrary \( x_0 > 0 \), (4.14b) and (4.14c) are valid. As before, applying Lemma 4 in \( G'_{\ell_1} \), we then derive the right-hand side of (2.18), since \( x_0 > 0 \) is arbitrary. From (2.15), (2.17), and (2.18), it follows that

\[
\zeta_1 t^{\frac{p-1-\beta}{p-1-\beta}} \leq \eta(t) \leq \zeta_2 t^{\frac{p-1-\beta}{p-1-\beta}}, \quad 0 \leq t < +\infty,
\]
where the constants $\zeta_1$ and $\zeta_2$ are chosen according to relevant estimations for $u$. If $u_0$ satisfies (1.3) with $\alpha = p/(p - 1 - \beta)$ and with $C \neq C_*$, then the asymptotic formulae (2.10) and (2.11) may be proved as the similar estimations (2.1) and (2.3) were in Lemma 5.

**Region (3)**

**Proof of Theorem 3** Take an arbitrary sufficiently small number $\epsilon > 0$. From (1.3), (3.7) follows. Then, consider a function

$$g_\epsilon(x, t) = [(C + \epsilon)^{1-\beta}(-x)^{p(1-\beta)} - b(1 - \beta)(1 - \epsilon)t]^{-1/(1-\beta)}. \quad (4.15)$$

We estimate $Lg$ in

$$M_1 = \{(x, t) : x_\epsilon < x < \eta(t), \quad 0 < t < \delta_1\},$$

$$\eta(t) = -\ell t^{1/(\alpha(1-\beta))}, \quad \ell(\epsilon) = (C + \epsilon)^{-1/\alpha} [b(1 - \beta)(1 - \epsilon)]^{1/(\alpha(1-\beta))},$$

where $\delta_1 > 0$ is chosen such that $\eta(\epsilon)(\delta_1) = x_\epsilon$. We have

$$Lg_\epsilon = bg_\epsilon^\beta \{\epsilon + S\},$$

$$S = -b^{-1}(p - 1)x^{p-1}(z(1 - \beta) - 1)(C + \epsilon)^{p-1-\beta}(-x)^{p(p-1-\beta)-p} \left\{ g_\epsilon x^{-\alpha} / (C + \epsilon) \right\}^{(p-2)}/\alpha,$$

$$= -b^{-1}x^{p-1}(C + \epsilon)^{p-1-\beta}(-x)^{p(p-1-\beta)-p} \left\{ g_\epsilon x^{-\alpha} / (C + \epsilon) \right\}^{(p-1)-1} S_1,$$

$$S_1 = \{(x(1 - \beta) - 1)(p - 1) \left\{ g_\epsilon x^{-\alpha} / (C + \epsilon) \right\}^{1-\beta} + x\beta(p - 1)\}.$$

If $\beta(p - 1) \geq 1$, then we can choose $x_\epsilon < 0$ such that (with sufficiently small $|x_\epsilon|$)

$$|S| < \frac{\epsilon}{2} \quad \text{in} \quad M_1.$$

Thus, we have

$$Lg_\epsilon > b(\epsilon/2)g_\epsilon^\beta \quad (Lg_{-\epsilon} < -b(\epsilon/2)g_{-\epsilon}^\beta, \text{respectively}) \quad \text{in} \quad M_1,$$

$$Lg_{\pm\epsilon} = 0 \quad \text{for} \quad x > \eta(\pm\epsilon)(t), \quad 0 < t \leq \delta_1,$$

$$g_\epsilon(x, 0) \geq u_0(x) \quad (g_{-\epsilon}(x, 0) \leq u_0(x), \text{respectively}), \quad x \geq x_\epsilon.$$

Since $u$ and $g$ are continuous functions, $\delta = \delta(\epsilon) \in (0, \delta_1]$ may be chosen such that

$$g_\epsilon(x_\epsilon, t) \geq u(x_\epsilon, t) \quad (g_{-\epsilon}(x_\epsilon, t) \leq u(x_\epsilon, t), \text{respectively}), \quad 0 \leq t \leq \delta.$$
From comparison with Lemma 5, it follows that
\[ g_{-\varepsilon} \leq u \leq g_\varepsilon \quad x \geq x_\varepsilon, \quad 0 \leq t \leq \delta, \tag{4.16a} \]
\[ \eta_{\ell,-\varepsilon}(t) \leq \eta(t) \leq \eta_{\ell,\varepsilon}, \quad 0 \leq t \leq \delta, \tag{4.16b} \]
which imply (2.19) and (2.20).

Let $\beta(p - 1) < 1$. In this case the left-hand side of (4.16a), (4.16b) may be proved similarly. Moreover, we can replace $1 + \varepsilon$ with $1$ in $g_{-\varepsilon}$ and $\eta_{\ell,-\varepsilon}$.

To prove a relevant upper estimation, consider a function
\[ g(x, t) = C_6 (x - \frac{\beta t}{1 - \beta} - x)^\frac{\varepsilon}{\alpha} \text{ in } G_{\ell, \delta}, \]
where $\ell \in (\ell_*, \infty)$ and
\[
\zeta_5 = (\ell_*/\ell)^{\alpha(1 - \beta)}(1 - \varepsilon)\ell, \\
C_6 = \left[ 1 - (\ell_*/\ell)^{\alpha(1 - \beta)}(1 - \varepsilon) \right]^{-\alpha} \left[ C^{1 - \beta} - \ell^{-\alpha(1 - \beta)}b(1 - \beta)(1 - \varepsilon) \right]^{1/(1 - \beta)}.
\]

From (2.20), it follows that for arbitrary $\ell > \ell_*$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, \ell) > 0$ such that
\[ u(\eta_{\ell, t}, t) \leq [C^{1 - \beta} \ell^{\alpha(1 - \beta)} - b(1 - \beta)(1 - \varepsilon)]^{1/\alpha} t^{1/\alpha}, \quad 0 \leq t \leq \delta. \tag{4.17} \]
Calculating $Lg$ in
\[ G_{\ell, \delta}^+ = \{(x, t) : \eta_{\ell, t} < x < -\zeta_5't^{-\frac{\varepsilon}{\alpha(1 - \beta)}}, \quad 0 < t < \delta\}, \]
we have
\[ Lg = bg^\beta S, \]
\[ S = 1 - ((1 - \beta))^{-1} \zeta_5 C_6^{1/\alpha} \left( \frac{\beta t}{1 - \beta} \right)^{1 - \beta - 1/\alpha} - b^{-1}(\alpha - 1)(p - 1)x^{p - 1} C^{p/\alpha} g^{p - 1 - \beta - (p/\alpha)}. \]

Since
\[ S \geq 0 \text{ in } G_{\ell, \delta}^+, \]

\[ S \geq S_{x = \eta_{\ell, t}} = 1 - ((1 - \beta))^{-1} \zeta_5 C_6^{1 - \beta} (\ell - \zeta_5)^{\alpha(1 - \beta)} - b^{-1}(\alpha - 1)(p - 1)x^{p - 1} C_6^{p - 1 - \beta} (\ell - \zeta_5)^{1/\alpha} (1 - \beta)^{2(1 - \beta)} - p. \]

Then, we have
\[ S \geq \varepsilon - b^{-1} C_6^{p - 1 - \beta}(\alpha - 1)(p - 1)x^{p - 1} (\ell - \zeta_5)^{1/\alpha} (1 - \beta)^{2(1 - \beta)} - p \text{ in } G_{\ell, \delta}^+. \]

Hence, we can choose $\delta = \delta(\varepsilon) > 0$ so small that
\[ Lg \geq b(\varepsilon/2)g^\beta \text{ in } G_{\ell, \delta}^+. \tag{4.18a} \]
Using (4.17), we can apply Lemma 4 in $G'_{\ell,\delta} = G_{\ell,\delta} \cap \{x < x_0\}$, for $\forall x_0 > 0$. We have

$$Lg = 0 \text{ in } G'_{\ell,\delta} \setminus \overline{G}^+_{\ell,\delta}, \quad (4.18b)$$

$$u(\eta(t), t) \leq \left[C^{1-\beta}e^{x(1-\beta)} - b(1-\beta)(1-\epsilon)\right]^{\frac{1}{1-p}} - \xi = C_6(\ell - \xi t)^{\frac{1}{1-p}} = g(\eta(t), t), \quad 0 \leq t \leq \delta. \quad (4.18c)$$

$$u(x_0, t) = g(x_0, t) = 0, \quad 0 \leq t \leq \delta, \quad u(x_0) = g(x_0) = 0, \quad 0 \leq x \leq x_0. \quad (4.18d)$$

Since $x_0 > 0$ is arbitrary, from (4.18a)–(4.18d) and comparison principle it follows that for all $\ell > \ell_*$ and $\epsilon > 0$ there exists $\delta = \delta(\epsilon, \ell) > 0$ such that

$$u(x, t) \leq C_6(-\xi t)^{\frac{1}{1-p}} \text{ in } \overline{G}^+_{\ell,\delta}. \quad (4.19)$$

Since (2.20) is valid along $x = \eta(t)$, $\delta$ may be chosen so small that

$$-\ell t^{1/(1-\beta)} \leq \eta(t) \leq -\xi t^{1/(1-\beta)}, \quad 0 \leq t \leq \delta. \quad (4.20)$$

Since $\ell > \ell_*$ and $\epsilon > 0$ are arbitrary numbers, (2.19) follows from (4.20).

**Region (4)**

(4a) This case is immediate.

(4b) Let $\beta = 1, \; \alpha > p/(p-2)$. As before, from (1.3), (3.7) follows. Then, consider a function

$$g(x, t) = (C - \epsilon)(-x)^{\frac{p}{p-1}} \exp(-bt),$$

which satisfies

$$Lg \leq 0 \text{ for } x_0 < x < 0, \; t > 0; \quad Lg = 0 \text{ for } x > 0, \; t > 0.$$

We can choose $\delta = \delta(\epsilon) > 0$ such that

$$g(x_0, t) \leq u(x_0, t), \quad 0 \leq t \leq \delta,$$

and from a comparison principle, the left-hand side of (2.22) follows. To prove the right-hand side, consider

$$g(x, t) = (C + \epsilon)(-x)^{\frac{p}{p-1}} \exp(-bt) \left[1 - \epsilon \left(b(p-2)\right)^{-1} \left(1 - \exp(-b(p-2)t)\right)\right]^{1/2-p}.$$

We have

$$Lg = (p-2)^{-1}(C + \epsilon)(-x)^{\frac{p}{p-1}} \exp(-b(p-1)t)g^{p-1}$$

$$\times \left\{\epsilon - (p-2)x^{p-1}(p-1)(C + \epsilon)^{p-2}(-x)^{\frac{p-2}{p-1}}\right\}, \quad x < 0, \; t > 0,$$

and hence, if $|x_0|$ is small enough,

$$Lg \geq 0 \text{ for } x_0 < x < 0, \; t > 0; \quad Lg = 0 \text{ for } x > 0, \; t > 0.$$
As before, a comparison principle implies the right-hand side of (2.22). The estimations (2.23)–(2.25) in the cases (4c) and (4d) may be proved similarly.

(II) $b = 0$.

(1) Let $p > 2$, $0 < \alpha < p/(p - 2)$.

First assume that $u_0$ is defined by (1.4). The self-similar form (2.4) and the formula (2.27) are well-known results (see Lemma 5). To prove (2.28), consider a function

$$g(x, t) = t^{\alpha/(p - \alpha(p - 2))}f(\xi).$$

We have

$$Lg = t^{\alpha(p - 1) - \alpha(p - 2)}Lf,$$

$$Lf = \frac{\alpha}{p - \alpha(p - 2)}f - \frac{1}{p - \alpha(p - 2)}\xi f' - (|f'|^{p - 2}f')'.$$

Choose

$$f(\xi) = C_0(\xi_0 - \xi)^{(p - 1)/(p - 2)}, \quad 0 < \xi < +\infty,$$

where $C_0$ and $\xi_0$ are some positive constants. Then, we have

$$Lf = (p - \alpha(p - 2))^{-1}(p - 1)(p - 2)^{-1}C_0(\xi_0 - \xi)^{1/p - 2}R(\xi) \quad \text{for } 0 \leq \xi \leq \xi_0, \quad t > 0$$

$$R(\xi) \equiv \alpha(p - 2)(p - 1)^{-1}\xi_0 + (1 - \alpha(p - 2)(p - 1)^{-1})\xi - (p - 1)^{p - 2}/(p - \alpha(p - 2))C_0^{p/2} C_0^{p - 2}$$

To prove an upper estimation, we take $C_0 = C_4$. Then, we have

$$R(\xi) \geq \nu_\alpha \xi_4 - (p - 1)^{p - 1}/(p - 2)^{-1}(p - \alpha(p - 2))C_0^{p - 2} = 0 \quad \text{for } 0 \leq \xi \leq \xi_4,$$

where

$$\nu_\alpha = \{1 \quad \text{if } \alpha \geq (p - 1)(p - 2)^{-1} \quad \text{if } \alpha < (p - 1)(p - 2)^{-1} \}.$$

Hence,

$$Lg \geq 0 \quad \text{for } 0 < x < \xi_4t^{1/p - \alpha(p - 2)}, \quad t > 0,$$

$$Lg = 0 \quad \text{for } 0 > \xi_4t^{1/p - \alpha(p - 2)}, \quad t > 0,$$

$$u(0, t) = g(0, t), \quad t \geq 0; \quad u(x, 0) = g(x, 0), \quad x \geq 0,$$

and a comparison principle imply the right-hand side of (2.28). The left-hand side of (2.28) may be established similarly if we take $C_0 = C_4$. $\xi_0 = \xi_3$. Equations (2.2) and (2.6) follow from Lemma 5. Finally, (2.7)–(2.9) easily follow from (2.27) and (2.28). If $u_0$ satisfies (1.3) with $0 < \alpha < p/(p - 2)$, then (2.1)–(2.3) follow from Lemma 5.

The cases (2) and (3) are immediate.

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Appendix

We give here explicit values of the constants used in Section 2 in the outline of the results for Region (2) of (1) and later in Section 4 during the proof of these results:

\[ \zeta_1 = A \zeta_1^{ \frac{1}{\beta} } (1 - \beta)^{ \frac{1}{\beta} } (p - 1)(1 + b(1 - \beta)A_1^{\beta - 1})^{-\frac{1}{\beta}} (p - 2)^{-1}, \]

\[ C_1 = A_1 \zeta_1^{-\mu}, \quad \text{if} \quad \beta(p - 1) > 1, \]

\[ \zeta_1 = A_1 \zeta_1^{ \frac{1}{\beta} } (1 - \beta) (1 + \beta) p^{\beta - 1} (p - 1) \left( 1 + b(1 - \beta)A_1^{\beta - 1} \right)^{-\frac{1}{\beta}} (p - 1 - \beta)^{-1}, \]

\[ C_1 = A_1 \zeta_1^{- \frac{1}{\beta - p}}, \quad \text{if} \quad \beta(p - 1) < 1, \]

\[ \zeta_2 = A_1 \zeta_2^{ \frac{1}{\beta} } (1 - \beta) (1 + \beta) p^{\beta - 1} (p - 1) \left( 1 + b(1 - \beta)A_1^{\beta - 1} \right)^{-\frac{1}{\beta}} (p - 1 - \beta)^{-1}, \]

\[ C_2 = A_1 \zeta_2^{- \frac{1}{\beta - p}}, \quad \text{if} \quad \beta(p - 1) > 1, \]

\[ \zeta_2 = \left( A_1 / C_* \right) \zeta_2^{ \frac{1}{\beta} } p^{\beta - 1}, \quad C_2 = C_*, \quad \text{if} \quad \beta(p - 1) < 1, \]

\[ \bar{\zeta}_2 = A_1^{\frac{1}{\beta} } \left( \frac{(p - 1)(p - 2)}{\beta(p - 1 - \beta)} \right)^{\frac{1}{\beta}}, \quad \bar{C}_2 = A_1 \bar{\zeta}_2^{ \frac{1}{\beta - p} }, \quad \text{if} \quad \beta(p - 1) > 1, \]

\[ \bar{\zeta}_2 = A_1^{\frac{1}{\beta} } (1 - \beta)^{ \frac{1}{\beta} } (p - 1)(1 + b(1 - \beta)A_1^{\beta - 1})^{-\frac{1}{\beta}} (p - 2)^{-1}, \]

\[ \bar{C}_2 = A_1 \bar{\zeta}_2^{ \frac{1}{\beta - p} }, \quad \text{if} \quad \beta(p - 1) < 1, \]

\[ \ell_0 = C_* \left( \frac{(p - 1)(p - 2)}{\beta(p - 1 - \beta)} \right)^{\frac{1}{\beta - p}} (1 - \beta) \theta_*, \]

\[ \zeta_3 = C_* \left( \frac{(p - 1)(p - 2)}{\beta(p - 1 - \beta)} \right)^{\frac{1}{\beta - p}} \left( C_* \right)^{\frac{1}{\beta - p}} - 1 \left[ (1 - \beta) \theta_* \right]^{\frac{1}{\beta - p}}, \]

\[ \theta_* = \left[ 1 - \left( C/C_* \right)^{p - 1 - \beta} \right] 
\left[ \left( C/C_* \right)^{\frac{(1 - \beta)(p - 1)}{\beta - p}} - 1 \right]^{-1}, \]

\[ \ell_1 = C_* \left( \frac{(p - 1)(p - 2)}{\beta(p - 1 - \beta)} \right)^{\frac{1}{\beta - p}} \left( b(1 - \beta)(\delta_* \Gamma)^{-1} \left( (1 - \delta_* \Gamma) - (1 - \delta_* \Gamma)^{1 - p} \left( C/C_* \right)^{p - 1 - \beta} \right) \right)^{\frac{1}{\beta - p}}, \]

\[ \zeta_4 = \delta_* \Gamma \ell_1, \quad \Gamma = 1 - \left( C/C_* \right)^{\frac{1}{\beta - p}} \quad \text{and} \quad C_3 = C \left( 1 - \delta_* \Gamma \right)^{\frac{1}{\beta - p}}, \]
where $\delta_* \in (0, 1)$ satisfies

$$g(\delta_*) = \max_{[0, 1]} g(\delta), \quad g(\delta) = \delta^{\frac{1+\beta(1-p)}{\alpha(1-\beta)}} \left[ (1 - \delta \Gamma) - \left( \frac{C}{C_*} \right)^{\beta-1-\beta} (1 - \delta \Gamma)^{1-\beta} \right],$$

$$\ell_* = C^{-\frac{1}{\alpha}} \left( b(1 - \beta) \right)^{1/(\alpha(1-\beta))},$$

$$\zeta_5 = \left( \frac{\ell_*}{\ell} \right)^{\alpha(1-\beta)} (1 - e) \ell, \quad \text{if } \beta(p - 1) < 1,$$

$$C_6 = (1 - \left( \frac{\ell_*}{\ell} \right)^{\alpha(1-\beta)} (1 - e))^{-\frac{1}{\alpha}} \left[ C^{1-\beta} - \ell^{-\alpha(1-\beta)} b(1 - \beta)(1 - e) \right]^{\frac{1}{\alpha}}.$$