PATH-DEPENDENT HAMILTON-JACOBI EQUATIONS
WITH SUPER-QUADRATIC GROWTH IN THE GRADIENT
AND THE VANISHING VISCOSITY METHOD

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Abstract. The non-exponential Schilder-type theorem in Backhoff-Veragucas, Lacker and Tangpi [Ann. Appl. Probab., 30 (2020), pp. 1321–1367] is expressed as a convergence result for path-dependent partial differential equations with appropriate notions of generalized solutions. This entails a non-Markovian counterpart to the vanishing viscosity method.

We show uniqueness of maximal subsolutions for path-dependent viscous Hamilton-Jacobi equations related to convex super-quadratic backward stochastic differential equations.

We establish well-posedness for the Hamilton-Jacobi-Bellman equation associated to a Bolza problem of the calculus of variations with path-dependent terminal cost. In particular, uniqueness among lower semi-continuous solutions holds and state constraints are admitted.

1. Introduction

Backhoff-Veraguas, Lacker and Tangpi [2] derived a non-exponential Schilder-type theorem, which they used to obtain new limit theorems for backward stochastic differential equations (BSDEs) and, in the Markovian case, for the corresponding partial differential equations (PDEs). They posed the question whether it is possible to have a corresponding PDE result in the non-Markovian case as well. Purpose of our work is to provide an answer to this question.

We establish well-posedness for (second order) path-dependent viscous Hamilton-Jacobi equations and for (first-order) path-dependent Hamilton-Jacobi-Bellman (HJB) equations with possibly super-quadratic growth in the gradient. Together with a modification of the Schilder type theorem in [2], we obtain a non-Markovian vanishing viscosity result for path-dependent PDEs and thereby address the mentioned open problem in [2].

The notions of solutions for our path-dependent PDEs are in the spirit of contingent solutions for PDEs (see, e.g., [24]), also known as Dini solutions (see, e.g., [3] and [50]) or minimax solutions (see, e.g., [46]).

In the context of contingent or Dini solutions for first-order standard PDEs related to Bolza problems, [13] and also [39] are very close to our approach. More recent works in this direction are [37], [4] and [6]. Regarding the possible use of viscosity solution techniques, we refer the reader to the remarks on p. 1202 in [7].

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In particular, fast growth in the gradient, discontinuity of the Lagrangian, and extended real-valued lower semi-continuous terminal data (to allow right-end point constraints in optimal control problems) cause non-trivial issues. For example, solutions of HJB equations can be expected then to be only lower semi-continuous. In the second-order case, the only works we are aware of that use contingent-type solutions are [48] and [47], where Isaacs equations corresponding to Markovian stochastic differential games with drift control and bounded control spaces are studied. However, similar constructions (stochastic or Gaussian derivatives) are also used in [26] and [27]. In the context of first-order path-dependent PDEs, [29] is most relevant. It deals with a calculus of variations problem involving a path-dependent terminal cost and the related path-dependent HJB equation. The setting is very close to our problem (DOC) below. The main difference is that in [29] the terminal cost is required to be Lipschitz continuous, which leads to Lipschitz continuity of the corresponding value function and also makes it possible in [29] to develop a viscosity solution theory. In our work, we require only continuity resp. lower semi-continuity of the terminal cost, which is one of the reasons why we establish a Dini resp. minimax solution theory. For the current state of the art for first-order path-dependent PDEs and for further relevant works, see [25] and the references therein.

In the literature [11, 19–23, 31, 38, 40–45] on viscosity solutions of second-order path-dependent PDEs, the Hamiltonian is required to grow at most linearly in the gradient (the same condition is also needed in [12], where a notion of strong-viscosity solutions is used). Overcoming this restriction for any notion of generalized solutions has been a longstanding open problem. By proving wellposedness of maximal (Dini) subsolutions for a class of second-order path-dependent PDEs with quadratic and even super-quadratic growth in the gradient, we establish first results related to this problem.

Non-Markovian large deviation problems and their connections to path-dependent PDEs are also studied in [36]. In contrast to our work, in [36] only the (limiting) rate function is characterized as a solution of a (first-order) path-dependent PDE. Moreover, the terminal condition is required to be Lipschitz continuous whereas we need only continuity.

2. Setup

2.1. Notation. Let \( \Omega = C([0, T], \mathbb{R}^d) \). The canonical process on \( \Omega \) is denoted by \( X \), i.e., \( X(t, \omega) = \omega(t) \) for each \( (t, \omega) \in [0, T] \times \Omega \). Let \( \mathbb{F} = \{ \mathcal{F}_t \}_{t \in [0, T]} \) be the (raw) filtration generated by \( X \). Given a probability measure \( P \) on \( (\Omega, \mathcal{F}_T) \), denote by \( \mathbb{F}^P = \{ \mathcal{F}^P_t \}_{t \in [0, T]} \) the \( P \)-completion of the right-limit of \( \mathbb{F} \).

We equip \( \Omega \) with the supremum norm \( \| \cdot \|_\infty \) and \([0, T] \times \Omega \) with the pseudo-metric \( d_\infty \) defined by

\[
d_\infty((t_1, \omega_1), (t_2, \omega_2)) := |t_1 - t_2| + \sup_{s \in [0, T]} |\omega_1(s \wedge t_1) - \omega_2(s \wedge t_2)|.
\]

Continuity and semi-continuity of functions defined on \( \Omega \) (resp. \([0, T] \times \Omega \)) are to be understood with respect to \( \| \cdot \|_\infty \) (resp. \( d_\infty \)). Note that semi-continuous functions on \([0, T] \times \Omega \) are \( \mathbb{F} \)-progressive. From now on, we write l.s.c. (resp. u.s.c.) instead of lower semi-continuous (resp. upper semi-continuous).

With slight abuse of notation, we also use the notation \( \| \cdot \|_\infty \) to express the sup-norm for functions belonging to other function spaces.
We often identify vectors with constant functions, e.g., given a map \( h : \Omega \to \mathbb{R} \), a vector \( z \in \mathbb{R}^d \), and a path \( \omega \in \Omega \), we write \( h(\omega + z) \) instead of \( h(\omega + z I_{[0,T]}) \).

Given \((t_0, x_0, n) \in [0, T] \times \Omega \times \mathbb{N}\), denote by \( P_{t_0, x_0, n} \) be the unique probability measure on \((\Omega, \mathcal{F}_T)\) such that \( X_{[t_0,T]}^{(n)} \) is a d-dimensional standard \((P_{t_0, x_0, n}, \mathbb{F})\)-Wiener process starting at \( z(t_0) \) and that \( P_{t_0, x_0, n}(X_{[0,t_0]} = x_0|_{[0,t_0]}) = 1 \). We write \( E_{t_0, x_0, n} \) for the corresponding expected value. Moreover, \( P_{t_0, x_0} := P_{t_0, x_0, 1} \) and \( F_{t_0, x_0, n} := \mathbb{R}^{P_{t_0, x_0, n}} \).

As space of controls, the set \( L_b \) of all bounded \( \mathbb{F} \)-progressive processes from \([0, T] \times \Omega\) to \( \mathbb{R}^d \) is used (whereas in [2] the controls are \( \mathbb{R}^{P_0} \)-progressive).

We denote by \( \text{dom} \) the effective domain of an extended real-valued function.

2.2. Data. Let \( h : \Omega \to \mathbb{R} \cup \{ \infty \} \) and \( \ell : [0, T] \times \mathbb{R}^d \to \mathbb{R} \cup \{ \infty \} \) be measurable functions. We use the following hypotheses for \( \ell \).

(H1) The function \( \ell = \ell(t, a) \) satisfies the Tonelli-Nagumo condition, i.e., there is a function \( \phi : [0, \infty) \to \mathbb{R} \) bounded from below with \( \phi(r)/r \to \infty \) as \( r \to \infty \) such that \( \ell(t, a) \geq \phi(|a|) \) on \([0, T] \times \mathbb{R}^d \). Moreover, \( \ell(t, \cdot) \) is l.s.c., proper, and convex for every \( t \in [0, T] \).

(H2) \( \int_0^T \ell(t, 0) dt < \infty \).

These hypotheses are nearly identical with the corresponding condition (TI) for the Lagrangian in [2] (where it is denoted by \( g \)). In some of our main results, we will, in addition to (H1) and (H2), also assume that \( \ell \) is continuous and finite-valued. In those cases, (TI) is satisfied as pointed out in [2].

2.3. The optimal control problems and HJB equations. Let \( n \in \mathbb{N} \). The value for our stochastic optimal control problem (SOC\(_n\)) with initial data \((t_0, x_0) \in [0, T] \times \Omega\) is given by

\[
v_n(t_0, x_0) := \inf_{a \in \mathcal{L}_b^{t_0}} \mathbb{E}_{t_0, x_0, n} \left[ \int_{t_0}^T \ell(t, a(t)) dt + h(X + A^a) \right],
\]

where \( \mathcal{L}_b^{t_0} = \{ a \in \mathcal{L}_b : a|_{[0,t_0]} = 0 \} \) and \( A^a \) is a continuous process on \([0, T] \times \Omega\) defined by \( A^a(t) := \int_0^t \ell(s, a(s)) ds \). The terminal value problem involving the corresponding HJB equation is

\[
(TVP \ n) \quad \left( \partial_t + \frac{1}{2n} \partial_{xx} \right) u(t, x) + \inf_{a \in \mathbb{R}^d} [a \cdot \partial_x u(t, x) + \ell(t, a)] = 0 \quad \text{in } [0, T] \times \Omega,
\]

\[
u(T, x) = h(x) \quad \text{on } \Omega.
\]

Remark 2.1. If (H1) and (H2) hold and \( h \) is bounded, then \( \sup_{n \geq 1} \| v_n \| < \infty \).

The value for our deterministic optimal control problem (DOC) with initial data \((t_0, x_0) \in [0, T] \times \Omega\) is given by

\[
v_0(t_0, x_0) := \inf_{x \in \mathcal{X}^{1,1}(t_0, x_0)} \left[ \int_{t_0}^T \ell(t, x(t)) dt + h(x) \right],
\]

where \( \mathcal{X}^{1,1}(t_0, x_0) := \{ x \in \Omega : x|_{[0, t_0]} = x_0|_{[0, t_0]} \text{ and } x|_{[t_0, T]} \in W^{1,1}(t_0, T; \mathbb{R}^d) \} \).

Here, \( W^{1,p}(t_0, T; \mathbb{R}^d), p \in [1, \infty], \) is the Sobolev space of all \( x \in L^p(t_0, T; \mathbb{R}^d) \) that have a weak derivative \( x' \in L^p(t_0, T; \mathbb{R}^d) \). The terminal value problem involving
the corresponding HJB equation is
\[
(TVP) \quad \partial_t u(t, x) + \inf_{a \in \mathbb{R}^d} \left[ a \cdot \partial_x u(t, x) + \ell(t, a) \right] = 0 \quad \text{in } [0, T) \times \Omega,
\]
\[
u(T, x) = h(x) \quad \text{on } \Omega.
\]

3. Notions of solutions of path-dependent HJB equations

We call a function \( u : [0, T] \times \Omega \to \mathbb{R} \cup \{\infty\} \) non-anticipating if \( u(t, x) = u(t, x(\cdot \wedge t)) \) for every \( (t, x) \in [0, T] \times \Omega \). Note that whenever a function on \([0, T] \times \Omega\) is l.s.c. or u.s.c. (with respect to \( d_{\infty} \)), then it is automatically non-anticipating.

3.1. Dini solutions. Given a non-anticipating function \( u : [0, T] \times \Omega \to \mathbb{R} \cup \{\infty\} \), we define the lower and upper Dini derivative
\[
d_{-}u(t_0, x_0)(1, a) := \lim_{\delta \downarrow 0} \frac{u(t_0 + \delta, x_0(\cdot \wedge t_0) + A^a(\cdot \vee t_0) - A^a(t_0)) - u(t_0, x_0)}{\delta},
\]
\[
d_{+}u(t_0, x_0)(1, a) := \lim_{\delta \downarrow 0} \frac{u(t_0 + \delta, x_0(\cdot \wedge t_0) + A^a(\cdot \vee t_0) - A^a(t_0)) - u(t_0, x_0)}{\delta}
\]
at points \((t_0, x_0) \in [0, T] \times \Omega\) in direction \((1, a) \in \mathbb{R} \times \mathbb{R}^d\). Here, in unison with the process \( A^a \) in Subsection 2.3, \( A^a(t) = at \).

The following (path-dependent) notion of Dini semi-solutions is motivated by the notion of (contingent) solutions used in Theorem 4.1 of [13] for HJB equations related to Bolza problems. Our notion is also related to the infinitesimal version of minimax solutions for path-dependent Isaacs equations used in [34].

**Definition 3.1.** Let \( u : [0, T] \times \Omega \to \mathbb{R} \cup \{\infty\} \).

(i) We call \( u \) a Dini supersolution of \((TVP)\) if \( u \) is l.s.c., \( u(T, \cdot) \geq h \), and, for every \((t_0, x_0) \in \text{dom}(u)\) with \( t_0 < T \),
\[
\inf_{a \in \mathbb{R}^d} \left[ d_{-}u(t_0, x_0)(1, a) + \ell(t_0, a) \right] \leq 0.
\]

(ii) We call \( u \) a Dini subsolution of \((TVP)\) if \( u \) is u.s.c., \( u(T, \cdot) \leq h \), and, for every \((t_0, x_0) \in \text{dom}(u)\) with \( t_0 < T \),
\[
\inf_{a \in \mathbb{R}^d} \left[ d_{+}u(t_0, x_0)(1, a) + \ell(t_0, a) \right] \geq 0.
\]

We call \( u \) a maximal Dini subsolution of \((TVP)\) if \( u \) is a Dini subsolution of \((TVP)\) and, for every Dini subsolution \( v \) of \((TVP)\), we have \( u \geq v \).

**Example 3.2.** Let \( d = 1, \ h = \infty.1_K, \) where \( K := \{ t \mapsto t^{1/2} \} \subset \Omega, \) and \( \ell \) be defined by \( \ell(t, a) = |a|^{3/2} \). Then \( \nu_0 \) satisfies
\[
\nu_0(t_0, x_0) = \begin{cases} 
\int_{t_0}^{T} 2^{-3/2}t^{-3/4} \ dt = 2^{1/2} T^{1/4} - t_0^{1/4} & \text{if } x_0|_{[0,t_0]}(t) = t^{1/2}, \\
\infty & \text{otherwise}.
\end{cases}
\]

Note that, for each \((t_0, x_0) \in \text{dom}(\nu_0)\) with \( t_0 < T \) and each \( a \in \mathbb{R} \), we have \( d_{-} \nu_0(t_0, x_0)(1, a) = \infty \) for which the constant perturbation involving \( a \) is responsible. (To obtain a finite value for our Dini derivative, we would need to permit the non-constant perturbation \( t \mapsto t^{1/2} \).) Thus \( \inf_{a \in \mathbb{R}^d} [d_{-} \nu_0(t_0, x_0)(1, a) + |a|^{3/2}] \leq 0 \) is never satisfied. Hence, in our example there does not exist a Dini supersolution of \((TVP)\). This justifies the need for an appropriate weaker notion of solution (see Subsection 3.2).
Given a non-anticipating function $u : [0, T] \times \Omega \to \mathbb{R}$, we define the upper stochastic Dini derivative
\[
\lim_{\delta \downarrow 0} \frac{E_{t_0, x_0, n}[u(t_0 + \delta, X + A^a(\cdot, t_0) - A^a(t_0)) - u(t_0, x_0)]}{\delta}.
\]
at points $(t_0, x_0) \in [0, T] \times \Omega$ in direction $(1, a, n^{-1} I_d) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$ (cf. [26, 27, 47, 43]).

The following notion of subsolutions for second order path-dependent PDEs is motivated by the minimax solutions used in Theorem 5.1 of [47, 48] in a Markovian framework.

**Definition 3.3.** Let $u : [0, T] \times \Omega \to \mathbb{R}$. We call $u$ a Dini subsolution of $(TVP_n)$ if $u$ is u.s.c., $u(T, \cdot) \leq h$, and, for every $(t_0, x_0) \in [0, T] \times \Omega$,
\[
\inf_{a \in \mathbb{R}^d} \left[ d^{1,2}_+ u(t_0, x_0)(1, a, n^{-1} I_d) + \ell(t_0, a) \right] \geq 0.
\]
We call $u$ a maximal Dini subsolution of $(TVP_n)$ if $u$ is a Dini subsolution of $(TVP_n)$ and, for every Dini subsolution $v$ of $(TVP_n)$, we have $u \geq v$.

**Remark 3.4.** In our specific setting, the use of Dini type semiderivatives such as those introduced in this section suffices. This motivates us to call our generalized solutions Dini solutions. For more general data, path-dependent counterparts of contingent derivatives such as the Clio derivatives in [4] need to be utilized. Corresponding generalized solutions would be called contingent solutions.

**3.2. Minimax solutions.** Here, we introduce a weaker notion of solution, which is an adjustment of the infinitesimal notion of minimax solutions in [1]. It is also motivated by the notion of (l.s.c. contingent) solutions used in Theorem 5.1 of [13]. The problem in Example 3.2, which partially motivated this weaker notion, is overcome by allowing non-constant perturbations (see also the notion of contingent solutions in [3] that are defined via contingent derivatives with function-valued directions).

**Definition 3.5.** Let $u : [0, T] \times \Omega \to \mathbb{R} \cup \{\infty\}$ be non-anticipating.

(i) We call $u$ a minimax supersolution of $(TVP)$ if $u$ is l.s.c., $u(T, \cdot) \geq h$, and, for every $(t_0, x_0) \in \text{dom}(u)$ with $t_0 < T$,
\[
\inf_{x \in \mathcal{X}^{1,2}(t_0, x_0)} \lim_{\delta \downarrow 0} \left[ u(t_0 + \delta, x) - u(t_0, x_0) + \int_{t_0}^{t_0 + \delta} \ell(s, x'(s)) \, ds \right] \delta^{-1} \leq 0.
\]

(ii) We call $u$ an l.s.c. minimax subsolution of $(TVP)$ if $u$ is l.s.c., $u(T, \cdot) \leq h$, and, for every $(t_0, x_0) \in [0, T] \times \Omega$ and every $(t, x) \in \left( (t_0, T] \times \mathcal{X}^{1,2}(t_0, x_0) \right) \cap \text{dom}(u)$ with $\int_{t_0}^{t} \ell(s, x'(s)) \, ds < \infty$,
\[
\lim_{\delta \downarrow 0} \left[ u(t - \delta, x) - u(t, x) - \int_{t - \delta}^{t} \ell(s, x'(s)) \, ds \right] \delta^{-1} \leq 0.
\]

(iii) We call $u$ an l.s.c. minimax solution of $(TVP)$ if $u$ is a minimax super- and an l.s.c. minimax subsolution of $(TVP)$.
3.3. Consistency with classical solutions. First, we provide the definitions for path derivatives. The first-order ones are due to Kim [32] and the second-order ones are due to Dupire [13]. Our presentation follows [21] and [35].

**Definition 3.6.** Let \( u : [0, T] \times \Omega \to \mathbb{R} \).

(i) We write \( u \in C^{1,1}([0, T] \times \Omega) \) if \( u \in C([0, T] \times \Omega, \mathbb{R}) \) and if there are functions \( \partial_t u \in C([0, T] \times \Omega, \mathbb{R}) \) and \( \partial_x u \in C([0, T] \times \Omega, \mathbb{R}^d) \) called \textit{first-order path derivatives} such that, for every \((t_0, x_0) \in [0, T] \times \Omega\), \( x \in X^{1,1}(t_0, x_0) \) and every \( t \in (t_0, T] \), we have
\[
u(t, x) - u(t_0, x_0) = \int_{t_0}^t [\partial_t u(s, x) + x'(s) \cdot \partial_x u(s, x)] \, ds.
u(t, x) - u(t_0, x_0) = \int_{t_0}^t [\partial_t u(s, x) + x'(s) \cdot \partial_x u(s, x)] \, ds.
\]

(ii) We write \( u \in C^{1,2}([0, T] \times \Omega) \) if \( u \in C^{1,1}([0, T] \times \Omega) \) with corresponding first-order path derivatives \( \partial_t u \) and \( \partial_x u \) and if there is a function \( \partial_{xx} u \in C([0, T] \times \Omega, \mathbb{R}^{d \times d}) \) called \textit{second-order path derivative} such that, for every \((t_0, x_0) \in [0, T] \times \Omega\), every probability measure \( P \) on \( \mathcal{F}_T \) such that \( X \) is a \((P, \mathbb{F})\)-Itô-semimartingale after time \( t_0 \) with bounded characteristics and with \( P(X|_{[0,t_0]} = x_0|_{[0,t_0]}) = 1 \), and every \( t \in (t_0, T] \), we have
\[
u(t, X) - u(t_0, x_0) = \int_{t_0}^t \partial_t u(s, X) \, ds + \int_{t_0}^t \partial_x u(s, X) \cdot dX(s)
u(t, X) - u(t_0, x_0) = \int_{t_0}^t \partial_t u(s, X) \, ds + \int_{t_0}^t \partial_x u(s, X) \cdot dX(s)
\]
\[+ \int_{t_0}^t \frac{1}{2} \partial_{xx} u(s, X) : d\langle X(s) \rangle, \quad P\text{-a.s.}\]
\[
+ \int_{t_0}^t \frac{1}{2} \partial_{xx} u(s, X) : d\langle X(s) \rangle, \quad P\text{-a.s.}\]

Here, \( \langle X(\cdot) \rangle \) is the quadratic variation of \( X|_{[t_0,T]} \times \Omega \) and, given matrices \( M, N \in \mathbb{R}^{d \times d} \), \( M : N \) is the trace of \( MN^\top \).

**Remark 3.7.** If \( u \in C^{1,1}([0, T] \times \Omega) \), then its first-order path derivatives are uniquely determined. If, in addition, \( u \in C^{1,2}([0, T] \times \Omega) \), then its second-order path-derivative is uniquely determined as well. We refer to Section 2.3 of [21] for more details.

**Definition 3.8.** Let \( u : [0, T] \times \Omega \to \mathbb{R} \).

(i) We call \( u \) a \textit{classical subsolution} (resp. classical supersolution, classical solution) of \((\text{TVP})\) if \( u \in C^{1,1}([0, T] \times \Omega) \), \( u(T, \cdot) \leq h \) (resp. \( u(T, \cdot) \geq h \), \( u(T, \cdot) = h \)), and, for every \((t, x) \in [0, T] \times \Omega \),
\[
\partial_t u(t, x) + \inf_{a \in \mathbb{R}^d} [a \cdot \partial_x u(t, x) + \ell(t, a)] \geq (\text{resp. } \leq, =) 0.\]
\[
\partial_t u(t, x) + \inf_{a \in \mathbb{R}^d} [a \cdot \partial_x u(t, x) + \ell(t, a)] \geq (\text{resp. } \leq, =) 0.\]

(ii) We call \( u \) a \textit{classical subsolution} (resp. classical supersolution, classical solution) of \((\text{TVP})\) if \( u \in C^{1,2}([0, T] \times \Omega) \) (resp. \( u(T, \cdot) \geq h \), \( u(T, \cdot) = h \)) and, for every \((t, x) \in [0, T] \times \Omega \),
\[
\partial_t u(t, x) + \frac{1}{2n} \partial_{xx} u(t, x) + \inf_{a \in \mathbb{R}^d} [a \cdot \partial_x u(t, x) + \ell(t, a)] \geq (\text{resp. } \leq, =) 0.\]
\[
\partial_t u(t, x) + \frac{1}{2n} \partial_{xx} u(t, x) + \inf_{a \in \mathbb{R}^d} [a \cdot \partial_x u(t, x) + \ell(t, a)] \geq (\text{resp. } \leq, =) 0.\]

The following result follows immediately from Definitions 3.1, 3.3 and 3.8.

**Proposition 3.9** (Consistency of Dini solutions with classical solutions). \textit{Assume that \( \ell \) is continuous.}

(i) Let \( u \in C^{1,1}([0, T] \times \Omega) \). Then \( u \) is a classical subsolution (resp. classical supersolution, classical solution) of \((\text{TVP})\) if and only if \( u \) is a Dini subsolution (resp. Dini supersolution, Dini solution of \((\text{TVP})\)).
(ii) Let \( u \in C^{1,2}([0,T] \times \Omega) \). Then \( u \) is a classical subsolution of \((\text{TVP}_m)\) if and only if \( u \) is a Dini subsolution of \((\text{TVP}_m)\).

**Proposition 3.10** (Partial consistency of l.s.c. minimax solutions with classical solutions). Assume that \( \ell \) is continuous and real-valued. Let \( u \in C^{1,1}([0,T] \times \Omega) \).

(a) If \( u \) is an l.s.c. minimax subsolution of \((\text{TVP})\), then \( u \) is a classical subsolution of \((\text{TVP})\).

(b) If \( u \) is a classical supersolution of \((\text{TVP})\), then \( u \) is a minimax supersolution of \((\text{TVP})\).

**Remark 3.11.** The converse of Proposition 3.10 (b) cannot expected to be valid in general because the infimum over \( \mathcal{X}^{1,1}(t_0, x_0) \) in (3.3) can be strictly less than the corresponding infimum over \( \mathbb{R}^d \). For similar reasons, we cannot expect the converse of Proposition 3.10 (a) to be valid in general.

**Proof of Proposition 3.10.** Part (b) follows immediately from Definition 3.5 (i) and Definition 3.8 (i). It remains to prove part (a). To this end, fix \((t_0, x_0) \in [0, T) \times \Omega \) and assume that \( u \) is an l.s.c. minimax subsolution of \((\text{TVP})\). Fix \( t \in (t_0, T) \). Fix \( a \in \mathbb{R}^d \). Then, for any \( x(\cdot) \in \mathcal{X}^{1,1}(t_0, x_0) \) with a continuous derivative \( x' \) that satisfies \( x'(t) = a \), we have

\[
0 \geq \lim_{\delta \to 0} \left( u(t-\delta, x) - u(t, x) - \int_{t-\delta}^{t} \ell(s, x'(s)) \, ds \right) \delta^{-1} \\
= \lim_{\delta \to 0} \left( -\left( \int_{t-\delta}^{t} \partial_s u(s, x) + \partial_x u(s, x) \cdot x'(s) \, ds \right) - \int_{t-\delta}^{t} \ell(s, x'(s)) \, ds \right) \delta^{-1} \\
= -\partial_t u(t, x) - \partial_x u(t, x) \cdot a - \ell(t, a).
\]

Since \( u \in C^{1,1}([0,T] \times \Omega) \), \( \ell \) is continuous, and \( a \) was arbitrary in \( \mathbb{R}^d \), we have

\[
\inf_{a \in \mathbb{R}^d} \left[ \partial_t u(t_0, x_0) + \partial_x u(t_0, x_0) \cdot a + \ell(t_0, a) \right] \geq 0,
\]

i.e., \( u \) is classical subsolution of \((\text{TVP})\). \( \square \)

4. **Main results**

**Theorem 4.1.** Assume \((H1)\) with \( \phi(r) = |r|^p \) for some \( p > 1 \) and \((H2)\). Let \( h \) be continuous and bounded. Then \( (v_n) \) converges to \( v_0 \) uniformly on compacta and \( v_0 \) is continuous. Moreover, \( v_0 \) is the unique l.s.c. minimax solution of \((\text{TVP})\) that is bounded from below. If, in addition, \( \ell \) is continuous and finite-valued, then we have the following:

(i) For each \( n \in \mathbb{N} \), the function \( v_n \) is the unique bounded maximal Dini subsolution of \((\text{TVP}_m)\).

(ii) The function \( v_0 \) is the unique bounded maximal Dini subsolution of \((\text{TVP})\).

**Proof.** The uniform convergence of \((v_n)\) to \( v_0 \) on compacta and the continuity of \( v_0 \) are proven in Section 3. Theorem 4.3 characterizes \( v_0 \) as the unique minimax solution resp. as the unique bounded maximal Dini subsolution of \((\text{TVP})\). Theorem 7.2 addresses the remaining part, i.e., wellposedness of \((\text{TVP}_m)\). \( \square \)

**Remark 4.2.** Well-posedness of \((\text{TVP}_m)\) requires \( h \) to be only u.s.c. and bounded (see Theorem 7.2). The corresponding result in the Markovian case treated in [2] is of similar strength (well-posedness holds for maximal viscosity supersolutions of the corresponding viscous Hamilton-Jacobi equations, which is due to [17]).
Theorem 4.3. Assume (H1).

(a) Let $h$ be l.s.c., proper, and bounded from below. Then the value function $v_0$ is the unique l.s.c. minimax solution of (TVP) that is bounded from below.

(b) Assume (H2). Let $\ell$ be continuous and finite-valued. Let $h$ be u.s.c. and bounded. Then $v_0$ is the unique maximal bounded Dini subsolution of (TVP).

Proof. See Section 8.

5. PROOF OF THE CONVERGENCE RESULT

Consider the semicontinuous envelopes $v_*$ and $v^*$ defined by

$$v_*(t_0, x_0) := \sup_{\delta > 0, \ n \in \mathbb{N}} \inf_{m \geq n} v_m(t, x),$$

$$v^*(t_0, x_0) := \inf_{\delta > 0, \ n \in \mathbb{N}} \sup_{m \geq n} v_m(t, x)$$

for every $(t_0, x_0) \in [0, T] \times \Omega$. Here, $O_\delta(t_0, x_0)$ is the open $\delta$-neighborhood of $(t_0, x_0)$ in $([0, T] \times \Omega, d_\infty)$.

First, we establish an auxiliary result.

Lemma 5.1. Assume (H1). Let $t_n \to t_0$ in $[0, T]$ as $n \to \infty$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(a_n)$ be a sequence in $L^1 = L^1([0, T] \times \Omega, dt \otimes d\mathbb{P}; \mathbb{R}^d)$ that converges weakly to some $a \in L^1$. Then $\mathbb{E}_\mathbb{P} \int_{t_0}^T \ell(t, a(t)) \, dt \leq \lim\mathbb{E}_\mathbb{P} \int_{t_n}^T \ell(t, a_n(t)) \, dt$.

Proof. We follow the lines of the proof of the corresponding deterministic closure theorem 10.8.ii in [10]. First, note that, by lower semi-continuity and convexity of $\ell$, for every $\delta > 0$, we have $\mathbb{E}_\mathbb{P} \int_{t_0 + \delta}^T \ell(t, a(t)) \, dt \leq \lim\mathbb{E}_\mathbb{P} \int_{t_0}^T \ell(t, a_n(t)) \, dt$ (more details can be found the proof of Lemma A.1 of [2]). Next, fix $\varepsilon > 0$. Since $\ell$ is bounded from below, there is a $\delta_0 > 0$ independent from $n$ such that, for all $\delta \in (0, \delta_0)$, we have $\mathbb{E}_\mathbb{P} \int_{t_0 + \delta}^T \ell(t, a_n(t)) \, dt > -\varepsilon$ and thus $\lim\mathbb{E}_\mathbb{P} \int_{t_0}^T \ell(t, a_n(t)) \, dt + \varepsilon \geq \mathbb{E}_\mathbb{P} \int_{t_0}^T \ell(t, a(t)) \, dt$. Again, as $\ell$ is bounded from below, either the right-hand side of the previous inequality converges to $\mathbb{E}_\mathbb{P} \int_{t_0}^T \ell(t, a(t)) \, dt$ as $\delta \to 0$ or otherwise the left-hand side equals $\infty$. This concludes the proof as $\varepsilon$ was chosen arbitrarily.

The following two statements adapt Theorem 2.2 in [2] to our slightly more general setting.

Lemma 5.2. Assume (H1) with $\phi(r) = |r|^p$ for some $p > 1$ and (H2). Let $h$ be l.s.c. and bounded from below. Then $v_0 \leq v_*$.

Proof. Let $(t_0, x_0) \in [0, T] \times \Omega$. It suffices to consider the case $v_*(t_0, x_0) < \infty$. Let $(t_n, x_n)$ be a sequence in $[0, T] \times \Omega$ that converges to $(t_0, x_0)$ in $d_\infty$ and that satisfies $v_*(t_0, x_0) = \lim_n v_*(t_n, x_n)$. Let $(a_n)$ be a sequence in $\mathcal{L}$ such that each $a_n$ belongs to $\mathcal{L}_b$ and is an $n^{-1}$-minimizer of $(SOC_n)$ with initial data $(t_n, x_n)$. Then there exists a subsequence $(\theta_k)_k := (t_{nk}, x_{nk}, n_k)_k$ of $(t_n, x_n, n)_n$ with

$$v_*(t_0, x_0) = \lim_k \mathbb{E}_{\theta_k} \left( \int_{t_{nk}}^T \ell(t, a_{nk}(t)) \, dt + h(X + A_{nk}) \right)$$
and, for all \( k \in \mathbb{N} \),
\[
v_*(t_0, x_0) - 1 \leq \mathbb{E}_{\theta_k} \int_{t_n_k}^{T} \ell(t, a_{n_k}(t)) \, dt + \mathbb{E}_{\theta_k} h(X + A^{a_{n_k}}) \leq v_*(t_0, x_0) + 1.
\]

Since \( \ell \) and \( h \) are bounded from below, we can assume that
\[
v^1 = \lim_k \mathbb{E}_{\theta_k} \int_{t_n_k}^{T} \ell(t, a_{n_k}(t)) \, dt \quad \text{and} \quad v^2 = \lim_k \mathbb{E}_{\theta_k} h(X + A^{a_{n_k}})
\]
for some \( v^1, v^2 \in \mathbb{R} \) with \( v_*(t_0, x_0) = v^1 + v^2 \) (cf. Theorem 11.1.i and its proof in [11]). As \( \sup_k \mathbb{E}_{\theta_k} \int_{t_n_k}^{T} \ell(t, a_{n_k}(t)) \, dt < \infty \) and \( a_{n_k}|_{[0,t_n_k]} = 0 \) for all \( k \in \mathbb{N} \), one can proceed nearly exactly as in the proof of Lemma A.1 in [2] to show that the probability measures \( (P_{\theta_k} \circ (A^{a_{n_k}})^{-1})_k \) are tight. Let us point out the differences to [2]. Thanks to our additional requirement that the function \( \phi \) from Hypothesis (H1) satisfies \( \phi(r) = |r|^p \) for some \( p > 1 \), we can estimate \( \int_0^T |a_{n_k}(t)|^p \, dt \) instead of \( \int_0^T |a_{n_k}(t)| \, dt \) (cf. with the first displayed equation in the proof of Lemma A.1 in [2]).

As \( \sup_{k} \mathbb{E}_{\theta_k} \int_{t_n_k}^{T} \ell(t, a_{n_k}(t)) \, dt < \infty \) and \( a_{n_k}|_{[0,t_n_k]} = 0 \) for all \( k \in \mathbb{N} \), one can proceed nearly exactly as in the proof of Lemma A.1 in [2] to show that the probability measures \( (P_{\theta_k} \circ (A^{a_{n_k}})^{-1})_k \) are tight. Let us point out the differences to [2]. Thanks to our additional requirement that the function \( \phi \) from Hypothesis (H1) satisfies \( \phi(r) = |r|^p \) for some \( p > 1 \), we can estimate \( \int_0^T |a_{n_k}(t)|^p \, dt \) instead of \( \int_0^T |a_{n_k}(t)| \, dt \) (cf. with the first displayed equation in the proof of Lemma A.1 in [2]).

Thus, by Skorohod’s representation theorem, there exists a probability space \((\bar{\Omega}, \bar{F}, \bar{P})\) with a sequence of \( \bar{\Omega} \times \Omega \)-valued random variables \((A_{n_k}, \bar{X}_{n_k})_k\) that satisfies \( \bar{P} \circ (A_{n_k}, \bar{X}_{n_k})^{-1} = P_{\theta_k} \circ (A^{a_{n_k}}, X)^{-1} \) for each \( k \in \mathbb{N} \) and that converges (after passing to a subsequence), \( \bar{P} \)-a.s., to some random variable \( \bar{A}_0, \bar{X}_0 \). Next, define a sequence \( (\bar{a}_k) \) of \( \mathbb{R}^d \)-valued processes on \([0, T] \times \bar{\Omega} \) by \( \bar{a}_k(t) := a_{n_k}(t, \bar{X}_{n_k}) \). Again as in the proof of Lemma A.1 in [2], one can deduce that \( (\bar{a}_k) \) is equi-absolutely integrable and thus has a subsequential weak limit in \( L^1([0, T] \times \bar{\Omega}, dt \otimes d\bar{P}; \mathbb{R}^d) \) that we denote by \( \bar{a}_0 \) and that satisfies, by Lemma 5.1
\[
\mathbb{E}^{\bar{P}} \int_{t_0}^{T} \ell(t, \bar{a}_0(t)) \, dt \leq \lim_{\ell \to \infty} \mathbb{E}^{\bar{P}} \int_{t_n_k}^{T} \ell(t, \bar{a}_k(t)) \, dt = \lim_{\ell \to \infty} \mathbb{E}_{\theta_k} \int_{t_n_k}^{T} \ell(t, a_{n_k}(t)) \, dt
\]
as well as \( \bar{A}_0(t) = \int_0^t \bar{a}_0(s) \, ds \), \( \bar{P} \)-a.s., for every \( t \in [0, T] \). Moreover, \( \bar{A}_0|_{[0,t_0]} = 0 \), \( \bar{P} \)-a.s. Hence, together with \( h \) being l.s.c. and \( \bar{X}_0 = x_0(\cdot \wedge t_0) \), \( \bar{P} \)-a.s., we have
\[
(5.1) \quad v_*(t_0, x_0) = v^1 + v^2 \geq \mathbb{E}^{\bar{P}} \left[ \int_{t_0}^{T} \ell(t, \bar{a}_0(t)) \, dt + h(x_0(\cdot \wedge t_0) + \bar{A}_0) \right].
\]

To conclude the proof, it suffices to note that there exists some \( x \in X^{1,1}(t_0, x_0) \) such that the right-hand side of (5.1) is greater than or equal to \( \int_{t_0}^{T} \ell(t, x'(t)) \, dt + h(x) \)
(cf. also Remark 2.6 of [28]).

\begin{lemma}
Assume (H1) and (H2). Let \( h \) be u.s.c. and bounded. Then \( v^* \leq v_0 \).
\end{lemma}
Proof. It suffices to follow the arguments of the first paragraph of the proof of Theorem 2.2 in \[2\] and make the obvious adjustments. For the convenience of the reader, we quickly go over it. Fix \((t_0, x_0) \in [0, T] \times \Omega\) and \(a \in L^1(0, T; \mathbb{R}^d)\) with \(\int_0^T \ell(t, a(t)) dt < \infty\). Given \(N \in \mathbb{N}\), define \(a^N(t) \coloneqq a(t)\) for \(|a(t)| \leq N\) and \(a^N(t) \coloneqq (N/|a(t)|) a(t)\) for \(|a(t)| > N\). Next, consider a sequence \((t_n, x_n)\) that converges to \((t_0, x_0)\) in \([0, T] \times \Omega\) and satisfies \(v^*(t_0, x_0) = \lim_n v_n(t_n, x_n)\). Then
\[
v^*(t_0, x_0) \leq \lim_{n \to \infty} E_{t_n, x_n, n} \left[ \int_{t_n}^T \ell(t, a^N(t)) dt + h(X + A^N \cdot (t_n - A^N(t_n)) \right]
\]
\[
\leq \lim_{n \to \infty} \int_{t_n}^T \ell(t, a^N(t)) dt
\]
\[
+ \lim_{n \to \infty} E_{0, 0} h \left( x_n (\cdot \wedge t_n) + \frac{1}{\sqrt{n}} (X - X(t_n)1_{[t_n, T]}) + A^N (\cdot \wedge t_n) - A^N(t_n) \right)
\]
\[
\leq \int_{t_0}^T \ell(t, a^N(t)) dt + h(x_0 (\cdot \wedge t_0) + A^N (\cdot \wedge t_0) - A^N(t_0))
\]
as \(h\) is u.s.c. as well as bounded and \(\int_0^T \ell(t, a^N(t)) dt < \infty\), which follows from \(\int_0^T \ell(t, a(t)) dt < \infty\), \((H2)\), and the convexity of \(\ell(t, \cdot)\) (cf. the first displayed equation after (38)) in \(\mathbb{R}^d\). Finally, letting \(N \to \infty\) concludes the proof as in \(\mathbb{R}^d\). \(\square\)

Remark 5.4 (No Lavrentiev phenomenon). Assume \((H1)\) and \((H2)\). Let \(h\) be continuous and bounded. Then
\[
v_0(t_0, x_0) = \inf_{x \in X^{1, \infty}(t_0, x_0)} \left[ \int_{t_0}^T \ell(t, x(t)) dt + h(x(x)) \right],
\]
where
\[
X^{1, \infty}(t_0, x_0) := \{ x \in \Omega : x|_{[0, t_0]} = x_0|_{[0, t_0]} \text{ and } x|_{[t_0, T]} \in W^{1, \infty}(t_0, T; \mathbb{R}^d) \}.
\]
This result follows from the proofs of Lemmata 5.2 and 5.3 but with each \(P_{t_0, x_0, n}\), \(n \in \mathbb{N}\), replaced by the unique probability measure under which \(X = x(\cdot \wedge t_0)\) a.s. For a more direct proof, it suffices to slightly modify the proof of Proposition 4.1 in \(\mathbb{R}^d\) (this result is due to \(\mathbb{R}^d\)), where the case \(h \equiv 0\) is treated.

Proof of the first conclusion of Theorem 4.1. By Lemmata 5.2 and 5.3 \(v_0 \leq v^* \leq v_0\). Thus \((v_n)\) converges to \(v_0\) uniformly on compacta and \(v_0\) is continuous (cf. Lemmata V.1.5 and V.1.9 of \(\mathbb{R}^d\) and keep Remark \(\mathbb{R}^d\) in mind). \(\square\)

6. Connections to BSDEs

We present parts of the theory of (convex) superquadratic BSDEs from \(\mathbb{R}^d\) that are relevant for our work. Fix \((t_0, x_0) \in [0, T] \times \Omega\), \(t_1 \in [t_0, T]\), \(n \in \mathbb{N}\), and a \(\mathcal{F}_{t_1}\)-measurable random variable \(\xi : \Omega \to [0, T] \cup \{\infty\}\). Consider the BSDE
\[
dY(t) = - \sup_{a \in \mathbb{R}^d} \left[ a \cdot Z(t) - \ell(t, a) \right] dt + Z(t) dX(t), \text{ on } [t_0, t_1], \text{ and } P_{t_0, x_0, n}\text{-a.s.},
\]
(6.1)
\[
Y(t_1) = \xi.
\]

Definition 6.1. A pair \((Y, Z)\) is a supersolution of (6.1) if

- \(Y : [t_0, t_1] \times \Omega \to \mathbb{R}\) is a càdlàg and \(\mathcal{F}_{t_0, x_0, n}\)-adapted process,
• \( Z : [t_0, t_1] \times \Omega \to \mathbb{R}^d \) is an \( \mathbb{F}^{t_0, x_0, n} \)-predictable process with
\[
\mathbb{E}^{t_0, x_0, n}_{t_0, x_0, n} \int_{t_0}^{t_1} |Z(t)|^2 \, dt < \infty,
\]
• \((t, \omega) \mapsto \left[ \int_{t_0}^{t} Z(r) \, dX(r) \right](\omega), [t_0, t_1] \times \Omega \to \mathbb{R} \) is an \( \mathbb{F}^{t_0, x_0, n}, \mathbb{P}^{t_0, x_0, n} \)-supermartingale, and,
• for every \( t, s \in [t_0, t_1] \) with \( t \leq s \), we have, \( P^{t_0, x_0, n-}\)-a.s.,
\[
Y(s) \geq Y(t) + \int_{t}^{s} \left\{ -\sup_{a \in \mathbb{R}^d} [a \cdot Z(r) - \ell(r, a)] \right\} \, dr + \int_{t}^{s} Z(r) \, dX(r),
\]
\( Y(t_1) \geq \xi \).

A pair \( (Y, Z) \) is a minimal supersolution of \( (6.1) \) if it is a supersolution of \( (6.1) \) and, for every supersolution \( \bar{Y}, \bar{Z} \) of \( (6.1) \), we have \( Y \leq \bar{Y}, dt \otimes dP^{t_0, x_0, n} \)-a.e.

**Proposition 6.2.** Assume \((H1)\) and \((H2)\). Let \( \xi \) be bounded. Then \( (6.1) \) has a unique minimal supersolution \((Y, Z)\) and we write
\[
\ell^{t_0, x_0, n}(\xi) := Y_t, \quad t \in [t_0, t_1].
\]

**Proof.** By the proof of Theorem 3.4 in [10] and by Theorem A.1 in [10], the set of supersolutions of \((6.1)\) is non-empty. Thus we can apply Theorem 4.17 in [15], which concludes the proof. \( \square \)

From now on, let \( T = 1 \) in this section for the sake of simplicity. Fix \( (t_0, x_0) \in [0, T] \times \Omega \). Recall that \( \mathcal{L}^{t_0}_b = \{ a \in \mathcal{L}_b : a|_{[0,t_0]} = 0 \} \). Given \( a \in \mathcal{L}_b \), define \( A^a : [0, T] \times \Omega \to \mathbb{R}^d \) by \( A^a(t, \omega) := \int_0^t a(s, \omega) \, ds \). We also identify a control \( a \in \mathcal{L}_b \) with the function \( \omega \mapsto a(\omega) \), where \( |a(\omega)|(t) = a(t, \omega) \). Recall that \( X(t, \omega) = \omega(t) \), \( (t, \omega) \in [0, T] \times \Omega \), and that \( \mathbb{E}^{t_0, x_0} = \mathbb{E}^{t_0, x_0, n} \).

For the next statement, we borrow the following constructions from [2]. Given \( t \in [0, 1) \), \( \omega_1, \omega_2 \in \Omega \) with \( \omega_2(0) = 0 \), a path \( \omega_1 \odot \omega_2 \in \Omega \) is defined by
\[
(\omega_1 \odot \omega_2)(s) := \omega_1(s \wedge t) + \sqrt{1 - t} \omega_2 \left( \frac{s - t}{1 - t} \right) 1_{[t, 1]}(s)
\]
(note that in \( 2 \odot \) is used instead of \( \odot \) and a path \( \omega_1^{(t)} \in \Omega \) is defined by
\[
\omega_1^{(t)}(s) := \frac{1}{\sqrt{1 - t}} [\omega_1(t + s(1 - t)) - \omega_1(t)].
\]
Moreover, we use the function \( \ell^{(t_0)} : [0, 1] \times \mathbb{R}^d \to \mathbb{R} \cup \{ \infty \} \) defined by
\[
\ell^{(t_0)}(t, a) := (1 - t_0) \ell \left( t_0 + t(1 - t_0), \frac{a}{\sqrt{1 - t_0}} \right).
\]
Also set \( (\omega_1 \odot \omega_2) := \omega_1, \omega_1^{(1)} \equiv 0, \) and \( \ell^{(1)} \equiv 0 \).

**Lemma 6.3.** Assume \((H1)\) and \((H2)\). Let \( h \) be bounded. Then
\[
-\nu_1(t_0, x_0) = \sup_{a \in \mathcal{L}_b} \mathbb{E}_{0,0} \left[ (-h)(x_0 \odot t_0 \left[ X + A^{\hat{a}(X)} \right]) - \int_0^1 \ell^{(t_0)}(t, \hat{a}(t, X)) \, dt \right].
\]

The calculations in our proof are essentially the same as those in the proof of Lemma 5.1 in [2]. Although of similar nature, the corresponding statements are different. This permits a more elementary proof in our case.
Proof of Lemma 6.3. One should keep in mind, that $X = x_0 \circ_{t_0} X(t_0)$, $P_{t_0,x_0}$-a.s. We only consider the case $t_0 < 1$ as otherwise (6.5) is clearly satisfied.

Step 1. Fix $a \in C_{b,l}^0$. Note that

$$\mathbb{E}_{t_0,x_0} \left[ \int_{t_0}^1 \ell(t, a(t, x_0 \circ_{t_0} X(t_0))) dt \right] = \mathbb{E}_0,0 \left[ \int_{t_0}^1 \ell(t, a(t, x_0 \circ_{t_0} X(t_0))) dt \right]$$

$$= \mathbb{E}_0,0 \left[ (1 - t_0) \int_{t_0}^1 \ell(t_0 + t(1 - t_0), a(t_0 + t(1 - t_0), x_0 \circ_{t_0} X(t_0))) dt \right]$$

$$= \mathbb{E}_0,0 \left[ \int_{0}^1 \ell(t_0) (t, a(t, X(t))) dt \right],$$

where $\tilde{a} \in \mathcal{L}_b$ is defined by

$$\tilde{a}(t, \omega) := \sqrt{1 - t_0} a(t_0 + t(1 - t_0), x_0 \circ_{t_0} \omega).$$

Also note that

$$\mathbb{E}_{t_0,x_0} \left[ (-h)(x_0 \circ_{t_0} [X + A^n(x_0 \circ_{t_0} X(t_0))]) \right]$$

$$= \mathbb{E}_0,0 \left[ (-h)(x_0 \circ_{t_0} [X + (A^n(x_0 \circ_{t_0} X)]) \right]$$

$$= \mathbb{E}_0,0 \left[ (-h)(x_0 \circ_{t_0} [X + \tilde{a}]) \right]$$

because, for every $t \in [0,1]$,

$$(A^n(x_0 \circ_{t_0} X(t_0))(t) = \frac{1}{\sqrt{1 - t_0} \int_{t_0}^{t_0 + t(1 - t_0)}} a(s, x_0 \circ_{t_0} X) ds = \int_{0}^{t} \tilde{a}(s, X) ds.$$

Consequently, the left-hand side of (6.5) is less than or equal to the right-hand side of (6.5).

Step 2. Fix $\tilde{a} \in \mathcal{L}_b$. Define $a \in C_{b,l}^0$ by

$$a(t, \omega) := \frac{1}{\sqrt{1 - t_0} \int_{t_0}^{t_0 + t(1 - t_0)}} \tilde{a} \left( \frac{t - t_0}{1 - t_0}, \omega \right) 1_{[t_0,1] \times \{x_0|_{[0,t_0]}\}}(t, \omega|_{[0,t_0]}).$$

Going over the calculations in Step 1 backward, one can deduce that the right-hand side of (6.5) is less than or equal to the left-hand side of (6.5).

The following statement together with Theorem 4.1 provide a non-Markovian non-linear Feynman-Kac formula that connects maximal Dini subsolutions of pathdependent PDEs with minimal supersolutions of convex superquadratic BSDEs, for which we use the notation (6.2).

**Theorem 6.4.** Assume (H1) and (H2). Let $\ell$ be continuous and finite-valued. Let $h$ be bounded. Fix $(t_0, x_0) \in [0, T] \times \Omega$ and $n \in \mathbb{N}$. Then $v_n(t_0, x_0) = -\mathbb{E}_{t_0,x_0,n}^{\mathcal{E}_{t_0,x_0,n}(-h)}$. Moreover, for $P_{t_0,x_0,n}$-a.e. $\omega \in \Omega$ and every $t \in [t_0, T]$, we have $v_n(t, \omega) = -\mathcal{E}_{t,T}^{t_0,x_0,n}(-h)(\omega)$.

**Proof.** We prove the theorem only for the case $n = 1$ and $T = 1$.

We will employ expressions $\rho^\ell$, $f$ being a measurable function from $[0,1] \times \mathbb{R}^d$ to $\mathbb{R} \cup \{\infty\}$ that satisfies (H1) and (H2) in place of $\ell$, from Section 2 of [2], which are defined by

$$\rho^\ell(\xi) := \sup_{Q \in \mathcal{Q}} \mathbb{E}^Q \left[ \xi - \int_{0}^{1} f(s, a^Q(s)) ds \right], \xi : \Omega \to \mathbb{R} \text{ bounded and } \mathcal{F}_T \text{-measurable},$$

$$f(x, a) = \frac{1}{\sqrt{1 - t_0} \int_{t_0}^{t_0 + t(1 - t_0)}} a \left( \frac{t - t_0}{1 - t_0}, \omega \right) 1_{[t_0,1] \times \{x_0|_{[0,t_0]}\}}(t, \omega|_{[0,t_0]}).$$
where \( Q \) is the set of all probability measures on \((\Omega, \mathcal{F}_T)\) that are absolutely continuous with respect to \( P_{0,0} \) and \( a^Q: [0,1] \times \Omega \to \mathbb{R}^d \) is the unique \( \mathbb{F} \)-progressive process with \( \int_0^1 |a^Q(s)|^2 \, ds < \infty \), \( P_{0,0} \)-a.s., that satisfies

\[
dQ = \exp \left( \int_0^1 a^Q(s) \, dX(s) - \frac{1}{2} \int_0^1 |a^Q(s)|^2 \, ds \right) \, dP_{0,0}
\]

(Section 2 of [2]).

Moreover, by a straightforward adjustment of the proof of Lemma 5.1 in [2], \( \rho^{(t_0)}[\cdot; \omega \circ_t X] \) equals \( \rho^{(t_0)}[\cdot; (x_0 \circ_{t_0} X)] \) according to (BBD) in [2]. Moreover, by a straightforward adjustment of the proof of Lemma 6.3 in [2], \( \rho^{(t)}[\cdot; \omega \circ_t X] = Y(t, \omega) \) for every \( t \in [t_0, T] \) and \( P_{t_0,x_0} \)-a.e. \( \omega \in \Omega \). In this context, note that Lemma 5.1 and Lemma A.2, both in [2], formally require continuity of \( h \), which, however, is not necessary (cf. the corresponding material in [14]). Using Lemma 6.3 we can deduce that \( v_1(t, \cdot) = -Y(t), P_{t_0,x_0} \)-a.s., for every \( t \in [t_0, T] \).

\[ \square \]

7. The Second Order HJB Equations

Lemma 7.1. Let \( n \in \mathbb{N} \). Assume (H1). Let \( \ell = \ell(t,a) \) be continuous in \( t \). A bounded u.s.c. function \( u: [0,T] \times \Omega \to \mathbb{R} \) is a Dini subsolution of \((\text{TVP}_n)\) if and only if \( u(T,\cdot) \leq h \) and, for every \((t_0,x_0) \in [0,T] \times \Omega, t \in (t_0,T], a \in \mathbb{R}^d\),

\[
u(t_0,x_0) \leq E_{t_0,x_0,n} \left[ \int_{t_0}^t \ell(s,a) \, ds + u(t,X + A^n(\cdot \vee t_0) - A^0(t)) \right].
\]

Proof: We adapt the proof of Theorem V.3 in [47], which is situated in a Markovian context with bounded control spaces, to our non-Markovian setting with the unbounded control space \( \mathbb{R}^d \).

(i) Let \( u \) be a Dini subsolution of \((\text{TVP}_n)\). Fix \((t_0,x_0) \in [0,T] \times \Omega \). Put \( E = E_{t_0,x_0,n} \) and \( P = P_{t_0,x_0,n} \). Assume that there are \( t_1 \in (t_0,T] \) and \( a \in \mathbb{R}^d \), and \( \varepsilon > 0 \) such that

\[
u(t_1,X + A^n(\cdot \vee t_0) - A^0(t)) - u(t_0,x_0) + \int_{t_0}^{t_1} \ell(s,a) \, ds < -\varepsilon.
\]

Consider the set \( S \) of all \( \mathbb{F} \)-stopping times \( \tau \) that satisfy \( t_0 \leq \tau \leq t_1 \), \( P \)-a.s., and

\[
u(\tau,X + A^n(\cdot \vee t_0) - A^0(t)) - u(t_0,x_0) + \int_{t_0}^\tau \ell(s,a) \, ds \geq \frac{E[\tau] - t_0}{t_1 - t_0} \cdot (-\varepsilon).
\]

We equip \( S \) with an order \( \preceq \) defined by \( \tau_1 \preceq \tau_2 \) if and only if \( \tau_1 \leq \tau_2, P \)-a.s. Now, let \( \bar{S} \) be a totally ordered non-empty subset of \( S \). Note that there exists a sequence \((\tau_k) \) in \( \bar{S} \) such that \( E[\tau_k] \uparrow \sup_{\tau \in \bar{S}} E[\tau] \) as \( k \to \infty \). Since \( \bar{S} \) is totally ordered and \( E \) is linear, \((\tau_k) \) is increasing and converges to \( \bar{\tau} := \sup_k \tau_k \). We have \( \bar{\tau} \in \bar{S} \) because

\[
u(\bar{\tau},X + A^n(\cdot \vee t_0) - A^0(t)) + \int_{t_0}^{\bar{\tau}} \ell(s,a) \, ds \geq \frac{E[\bar{\tau}] - t_0}{t_1 - t_0} \cdot (-\varepsilon).
\]
the sets $M(t, \omega, y)$ are non-empty because they contain $0$ and they are compact because, for any sequence $(\delta_k)_{k \in [0, t_1 - t]}$ that converges to some $\delta$, we have

$$\lim_k E_{t, \omega, n} \left[ u(t + \delta, X + A^\delta(\cdot \lor t) - A^a(t)) + \int_t^{t + \delta} \ell(s, a) \, ds \right] \geq \lim_k E_{t, \omega, n} \left[ u(t + \delta_k, X + A^{\delta_k}(\cdot \lor t) - A^a(t)) + \int_t^{t + \delta_k} \ell(s, a) \, ds \right]$$

as $u$ is u.s.c. and bounded. Moreover, for every sequence $(s_k, \omega_k, y_k)_k$ that belongs to histo $u|_{[t_0, t_1] \times \Omega}$ and that converges to some $(t, \omega, y)$ with respect to the metric $(|s_1, \omega_1, y_1, s_2, \omega_2, y_2|) \equiv |s_1 - s_2| + ||\omega_1 - \omega_2||_\infty + |y_1 - y_2|$ and hence also with respect to $(|s_1, \omega_1, s_2, \omega_2, y_2|) \equiv d_\infty((s_1, \omega_1, s_2, \omega_2), |y_1 - y_2|$, every sequence $(\delta_k)_{k \in [0, t_1 - t_0]}$ with $\delta_k \in M(s_k, \omega_k, y_k)$ has a subsequential limit that belongs to $M(t, \omega, y)$ because for every subsequential limit $\delta$ of $(\delta_k)$ (there is at least one), we have, by possibly passing to a subsequence,

$$\lim_k E_{t, \omega, n} \left[ u(t + \delta, X + A^\delta(\cdot \lor t) - A^a(t)) + \int_t^{t + \delta} \ell(s, a) \, ds \right] = \lim_k E_{t, \omega, n} \left[ u(t + \delta, s_k, \omega_k(\cdot \lor s_k) + \frac{1}{\sqrt{n}}(X - X(s_k))1_{[s_k, T]} + A^\delta(\cdot \lor t) - A^a(t)) + \int_t^{t + \delta} \ell(s, a) \, ds \right]$$

Hence, $M$ considered as a map from histo $u|_{[t_0, t_1] \times \Omega}$ equipped here with the metric $(|s_1, \omega_1, y_1, s_2, \omega_2, y_2|) \equiv |s_1 - s_2| + ||\omega_1 - \omega_2||_\infty + |y_1 - y_2|$ into the set of all compact non-empty subsets of $[0, t_1 - t_0]$ is u.s.c. (Theorem 2.2 on p. 31 in [33]) and thus measurable (Theorem 2.1 on p. 29 in [33]), i.e., all sets of the form $\{u(t, \omega, y) \in hypo u|_{[t_0, t_1] \times \Omega} : M(t, \omega, y) \cap E \neq \emptyset\}$, $E$ being a closed subset of $[0, t_1 - t_0]$, belong to $\mathcal{B}(hypo u|_{[t_0, t_1] \times \Omega})$ (see p. 41 in [33]). Consequently, by Theorem 3.13 on p. 49 in...
there exists a \(B(\text{hypo } u_{[t_0, t_1] \times \Omega})\)-measurable selector \(\delta(\cdot, \cdot, \cdot)\) of \(M\) that satisfies

\[
|t_1 - t_0 - \delta(t, \omega, y)| = \text{dist}(t_1 - t_0, M(t, \omega, y)) \text{ for every } (t, \omega, y) \in \text{hypo } u_{[t_0, t_1] \times \Omega}.
\]

Therefore, the map

\[
\omega \mapsto \delta[\omega] := \delta(\tau(0)(\omega), \omega(\cdot \land \tau(0)(\omega))) + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0),
\]

\[
u(\tau(0)(\omega), \omega(\cdot \land \tau(0)(\omega))) + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0))
\]
is \(\mathcal{F}_{\tau_0}\)-measurable. Also note that, since, by [33] and the continuity of \(\ell(\cdot, a)\),
every set \(M(t, \omega, u(t, \omega))\) contains a strictly positive element, we have the inequality

\[
|t_1 - t_0 - \delta(t, \omega, u(t, \omega))| < |t_1 - t_0| \text{ and hence } \delta(t, \omega, u(t, \omega)) > 0.
\]

Thus, by Lemma V.3 in [47], \(\tilde{\tau}_0 := \tau_0 + \delta[\cdot]\) is an \(\mathcal{F}\)-stopping time with \(\tilde{\tau}_0 > \tau_0\) on \(\{\tau_0 < t_1\}\). Finally, since

\[
\mathbb{E} \left[ u(\tilde{\tau}_0, X + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0)) + \int_{t_0}^{\tilde{\tau}_0} \ell(s, a) \, ds \right]
\]

\[
= \int_\Omega \left( \mathbb{E} \left[ u(\tilde{\tau}_0, X + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0)) + \int_{t_0}^{\tilde{\tau}_0} \ell(s, a) \, ds \right] \mathbb{F}_{\tau_0} \right)(\omega)
\]

\[
+ \int_{t_0}^{\tau_0(\omega)} \ell(s, a) \, ds \right) P(d\omega)
\]

\[
= \int_\Omega \left( \mathbb{E}_{\tau_0(\omega): \omega} \left[ u(\tilde{\tau}_0(\omega), X + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0)) + \int_{t_0}^{\tilde{\tau}_0(\omega)} \ell(s, a) \, ds \right] \right)
\]

\[
+ \int_{t_0}^{\tau_0(\omega)} \ell(s, a) \, ds \right) P(d\omega)
\]

and since, for each \(\omega \in \Omega\), thanks to \(\delta[\omega] \in M(\tau_0(\omega), \omega(\cdot \land \tau_0(\omega))) + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0)\),
\(u(\tau_0(\omega), \omega(\cdot \land \tau_0(\omega)) + A^\alpha(\cdot \lor t_0) - A^\alpha(t_0))\) and thanks
to the definition of \(M\),

\[
\mathbb{E}_{\tau_0(\omega): \omega} \left[ u(\tilde{\tau}_0(\omega), X + A^\alpha(\cdot \lor \tau_0(\omega)) - A^\alpha(\tau_0(\omega))) + \int_{t_0}^{\tilde{\tau}_0(\omega)} \ell(s, a) \, ds \right]
\]

\[
= \mathbb{E}_{\tau_0(\omega): \omega} \left[ u(\tau_0(\omega) + \delta[\omega], X + A^\alpha(\cdot \lor \tau_0(\omega)) - A^\alpha(\tau_0(\omega))) + \int_{t_0}^{\tau_0(\omega)} \ell(s, a) \, ds \right]
\]
we will use \( (\cdot, \cdot) \) for every \( \tau \geq 0 \). To establish upper semi-
continuity, we can assume, without loss of generality, that \( \tau = \tau_0 \) is bounded and u.s.c. \( P \)-a.s., i.e., we have a contradiction to \( \tau_0 \) being the maximal element in \( S \).

Hence, \( \tau_0 = \tau_1 \), \( P \)-a.s. However, this in turn contradicts (7.2), which concludes the proof of this direction.

(ii) Showing the remaining direction is straight-forward.

\[ \square \]

**Theorem 7.2.** Assume (H1) and (H2). Fix \( n \in \mathbb{N} \). Let \( h \) be u.s.c. and bounded. Let \( \ell \) be continuous and finite-valued. Then \( v_n \) is the unique bounded maximal Dini subolution of \((TVP_n)\). Moreover, each bounded Dini subolution of \((TVP_n)\) is dominated from above by \( v_n \) even if we dispense with the assumption that \( h \) is u.s.c.

**Proof:** (i) Existence and regularity: First, we show that \( v_n \) is bounded and u.s.c. Boundedness of \( h, (H1) \), and (H2) yield boundedness of \( v_n \). To establish upper semi-
continuity, we will use \( \cdot \) defined by (6.3) and the notation (6.4). For this reason, we assume that \( T = 1 \) and \( n = 1 \) but this assumption is not restrictive. We continue by fixing a pair \((t_0, x_0) \in [0, T] \times \Omega\) and considering a sequence \((t_k, x_k)_{k \geq 1} \in [0, T] \times \Omega\) that converges to \((t_0, x_0)\) and satisfies \( \lim_k v_n(t_k, x_k) = \lim_{(t,x) \to (t_0,x_0)} v_n(t, x) \). We distinguish between two cases.

**Case 1:** \( t_0 < 1 \). Fix an \( \varepsilon < 0 \) such that \( t_0 + \varepsilon < 1 \). We can assume, without loss of generality, that \( t_k \leq t_0 + \varepsilon \) for all \( k \in \mathbb{N} \). Let \( a \in \mathcal{L}_b \). Then the map (see (6.4))

\[
(s, t, \omega) \mapsto \ell^\ast(t, a(t, \omega)) = (1 - s) \ell \left( s + t(1 - s), \frac{a(t, \omega)}{\sqrt{1 - s}} \right)
\]

from \( [0, t_0 + \varepsilon] \times [0, T] \times \Omega \) to \( \mathbb{R} \) has a real upper bound due to \( \ell \) being continuous and \( a \) being bounded. Note that \( (t, x) \mapsto (x \circ t, \omega), [0, T] \times \Omega \to \Omega \), is continuous for every \( \omega \in \Omega \) with \( \omega(0) = 0 \). Thus, by Lemma 6.3

\[
\lim_k v_n(t_k, x_k) \leq \lim_k \mathbb{E}_{0,0,n} \left[ h(x_k \circ t_k \circ t [X + A^n]) + \int_0^T \ell(t_k)(t, a(t)) \, dt \right]
\]

\[
\leq \mathbb{E}_{0,0,n} \left[ h(x_0 \circ t_0 \circ t [X + A^n]) + \int_0^T \ell(t_0)(t, a(t)) \, dt \right].
\]
Since \( a \in \mathcal{L}_b \) was arbitrary, we can deduce after again invoking Lemma 6.3 that
\[
\lim_k v_n(t_k, x_k) \leq v_n(t_0, x_0).
\]

Case 2: \( t_0 = 1 \). Then, proceeding similarly as in Case 1 but using the constant control \((t, \omega) \mapsto a(t, \omega) = 0\), we have
\[
\lim_k v_n(t_k, x_k) = \lim_k E_{0, 0, n} \left[ h(x_k \circ t_k [X + A^n] \right] + \int_0^T \ell(t_k)(t, 0) \, dt \right] \leq h(x_0) = v_n(t_0, x_0).
\]

We can conclude that \( v_n \) is u.s.c.

Next, we establish the subsolution property via the BSDE connection in Theorem 6.4. To this end, we use the notation \( (6.2) \) and fix \((t_0, x_0) \in [0, T) \times \Omega, t \in [t_0, T], a \in \mathbb{R}^d \). Note first that, by Proposition 3.6 (1) in [15],
\[
\left( 7.4 \right) \quad \mathcal{E}_{t_0, T}^{t_0, x_0, n}(-h) = \mathcal{E}_{t_0}^{t_0, x_0, n} \left( \mathcal{E}_{t_0, T}^{t_0, x_0, n}(-h) \right).
\]

Since \( v_n \) is bounded and, by Theorem 6.4, \( v_n(t, \cdot) = -\mathcal{E}_{t, T}^{t_0, x_0, n}(-h), P_{t, x_0, n}-a.s., \) we can apply Theorem 3.4 in [10] (see also Lemma A.2 in [2]) together with (3) in [2] to deduce that, with \( Q_{t_0, x_0, n} := \mathbb{P}_{t_0, x_0, n} \circ (X + A^n(\cdot \vee t_0) - A^n(t_0))^{-1} \),
\[
\mathbb{E}_{t_0, x_0, n} \left[ \mathcal{E}_{t_0}^{t_0, x_0, n} \left( \mathcal{E}_{t_0, T}^{t_0, x_0, n}(-h) \right) \right] \geq \mathbb{E}_{t_0, x_0, n} \left[ \mathcal{E}_{t_0, x_0, n}^{t, x}(X + A^n(\cdot \vee t_0) - A^n(t_0)) - \int_{t_0}^t \ell(s, a) \, ds \right].
\]

Thus, by (7.4) and Theorem 6.4, \( v_n \) satisfies (7.1) in place of \( u \). Hence, by Lemma 7.1, \( v_n \) is a Dini subsolution of (IVP n).

(ii) Uniqueness: Let \( u \) be a bounded Dini subsolution of (IVP n). Without loss of generality, let \( n = 1 \). Fix \((t_0, x_0) \in [0, T) \times \Omega \) and \( \varepsilon > 0 \). It suffices to show that \( u(t_0, x_0) \leq v_n(t_0, x_0) + \varepsilon \). By (a slight modification) of (36) and (BBD), both in [2], there exists an \( \alpha \in \mathcal{L}_b \) with \( a(t, \omega) = \sum_{i=1}^n a_i(\omega), 1_{[s_{i-1}, s_i]}(t) \), where \( t_0 = s_0 < s_1 < \cdots < s_m = T \) and each \( a_i : \Omega \to \mathbb{R}^d \) is \( F_{s_i-1} \)-measurable, such that
\[
\tilde{E}_{t_0, x_0} \left[ \int_{t_0}^T \ell(t, a(t)) \, dt + h(X) \right] \leq v_n(t_0, x_0) + \varepsilon. \tag{7.1}
\]
Here, for any \((s, x) \in [t_0, T) \times \Omega, P_{s, x} := P_{s, x} \circ (\tilde{X}^{s, x})^{-1}, \tilde{E}_{s, x} := \mathbb{E}^{\tilde{P}_{s, x}}, \text{ and } \tilde{X}^{s, x} : [0, T) \times \Omega \to \mathbb{R}^d \text{ is the unique solution of } \tilde{X}^{s, x}(t) = \tilde{X}^{s, x}(s) + \int_s^t a(r, \tilde{X}^{s, x}) \, dr + X(t) - X(s), t \in [s, T], \text{ with initial condition } \tilde{X}^{s, x}|_{[0, s]} = x|_{[0, s]} \rangle. \]
Next, note that, \( a_1 \circ \tilde{X}^{t_0, x_0} = a_1 \circ x_0 =: \tilde{a}_1 \in \mathbb{R}^d \) and \( X + A^{a_1}(\cdot \vee t_0) - A^{a_1}(t_0) = \tilde{X}^{t_0, x_0} \) on \([0, s_1] \), \( P_{t_0, x_0}-a.s. \). Thus, by Lemma 7.1,
\[
u(t_0, x_0) \leq \tilde{E}_{t_0, x_0} \left[ \int_{t_0}^{s_1} \ell(t, \tilde{a}_1) \, dt + u(s_1, \tilde{X}^{t_0, x_0}) \right] = \tilde{E}_{t_0, x_0} \left[ \int_{t_0}^{s_1} \ell(t, a(t)) \, dt + u(s_1, X) \right].
\]

Also note that, for \( P_{t_0, x_0}-a.e. \omega \in \Omega, \) we have, with \( x_1 = \tilde{X}^{t_0, x_0}(\omega), a_2 \circ \tilde{X}^{s_1, x_1} = a_2 \circ x_1 \in \mathbb{R}^d \) and \( X + A^{a_2}(\cdot \vee s_1) - A^{a_2}(s_1) = \tilde{X}^{s_1, x_1} \) on \([0, s_2] \), \( P_{s_1, x_1}-a.s., \) and thus, by Lemma 7.1,
\[
u(s_1, x_1) \leq \tilde{E}_{s_1, x_1} \left[ \int_{s_1}^{s_2} \ell(t, \tilde{a}_2) \, dt + u(s_2, \tilde{X}^{s_1, x_1}) \right].
\]
Assume (H1). An l.s.c. function \( \tilde{u} \) from below is a minimax supersolution of (TVP) and its extension to unbounded domains, which is treated in Section 11.2 of [10]. Next, consider the (non-void) set \( S \). The first order HJB equation

\[
\frac{1}{2} \int_0^T \ell(t, x', (s, x')) ds + u(t, x) = \sup_{(s, x) \in S} \{ \int_0^1 \ell(s, x'(s)) ds + u(t, x) \}
\]

for all \( (t, x, (s, x')) \in \mathbb{R} \times S \). The supremum is attained. To see this, consider a sequence \((s_n, x_n)_n\) in \( S \) with \( s_n \to \tilde{s} \). Since \( \int_0^{s_n} \ell(s, x_n'(s)) ds < \infty \) and \( u \) is bounded from below, one can show as in Sections 11.1 and 11.2 of [10] that there is a subsequence \((s_{n_k}, x_{n_k})_k\) such that \( x_{n_k} \) converges to some \( \tilde{x} \) in \( X^{1,1}(t_0, x_0) \), i.e.,

\[
\|x_{n_k} - \tilde{x}\|_\infty + \|x_{n_k}' - \tilde{x}'\|_{L^1(t_0, T; \mathbb{R}^d)} \to 0.
\]

Lower semi-continuity of \( u \) together with Theorem 10.8.ii in [10]

\[
\liminf_{k \to \infty} u(s_{n_k}, x_{n_k}) \
\leq \frac{\tilde{s} - t_0}{t_1 - t_0} \cdot \varepsilon + u(t_0, x_0).
\]
Thus $\hat{s}, \hat{x} \in \text{dom}(u)$ and, by (8.2), $\hat{s} < t_1$. Hence, by (8.3), there is a $\delta > 0$ and an $x \in X^{1,1}(\hat{s}, \hat{x})$ such that $u(\hat{s} + \delta, x) - u(\hat{s}, \hat{x}) + \int_{\hat{s}}^{\hat{s} + \delta} \ell(s, x'(s)) ds \leq \frac{\delta}{t_1 - t_0} \cdot \varepsilon$ and $\hat{s} + \delta < t_1$. Thus $(\hat{s} + \delta, x) \in S$, which is a contradiction to the maximality of $\hat{s}$.

(ii) Fix $(t_0, x_0) \in \text{dom}(u)$ with $t_0 < T$. Suppose that, for every $(t_0, \tilde{x}_0) \in [t_0, T) \times X^{1,1}(t_0, x_0)$ for every $t \in (t_0, T)$, there exists an $x \in X^{1,1}(t_0, \tilde{x}_0)$ such that (8.1) holds with $(t_0, x_0)$ replaced by $(t_0, \tilde{x}_0)$. Then one can proceed similarly as in the proof of Lemma 3.6 in [4] to show that there exists a sequence $(x_n)$ in $X^{1,1}(t_0, x_0)$ and an increasing sequence $(A_n)$ of finite subsets of $[t_0, T]$ whose union $A$ is dense in $[t_0, T]$ such that, for each $n \in \mathbb{N}$ and every $t \in A_n$, (8.1) holds with $x$ replaced by $x_n$. Thus $\sup_{t \in A_n} \int_{t_0}^{t} \ell(s, x_n(s)) ds \leq u(t_0, x_0) + c < \infty$, where $-c$ is a lower bound of $u$. Note that we are in a similar situation as in part (i) of this proof and thus it can be shown in the same way that there is a subsequence $(x_{n_k})$ of $(x_n)$ that converges to some $\tilde{x}$ in $X^{1,1}(t_0, x_0)$. Now, fix $t \in (t_0, T)$ and a sequence $(s_k)$ in $A$ with $s_k \in A_{n_k}$ for each $k$ and with $s_k \to t$ as $k \to \infty$. By Theorem 10.8.ii in [10] and lower semi-continuity of $u$,

$$u(t_0, x_0) \geq \lim_{k \to \infty} \left[ \int_{t_0}^{s_k} \ell(s, x_{n_k}(s)) ds + u(s_k, x_{n_k}) \right] \geq \int_{t_0}^{t} \ell(s, \tilde{x}'(s)) ds + u(t, \tilde{x}),$$

i.e., there is an $x \in X^{1,1}(t_0, x_0)$ such that, for every $t \in [t_0, T)$, (8.1) holds. From this point, (8.3) follows easily.

**Lemma 8.2.** Assume (H1). An l.s.c. function $u : [0, T] \times \Omega \to \mathbb{R} \cup \{\infty\}$ is an l.s.c. minimax subsolution of (LVP) if and only if $u(T, \cdot) \leq h$ and, for every $(t_0, x_0) \in [0, T) \times \Omega$, $t \in (t_0, T)$, and $x \in X^{1,1}(t_0, x_0)$, we have

$$u(t_0, x_0) \leq \int_{t_0}^{t} \ell(s, x'(s)) ds + u(t, x).$$

**Proof.** (i) Let $u$ be an l.s.c. minimax subsolution of (LVP). For the sake of a contradiction, assume that there exist a $(t_0, x_0) \in [0, T) \times \Omega$, a $t_1 \in (t_0, T]$, an $x \in X^{1,1}(t_0, x_0)$, and an $\varepsilon > 0$ such that

$$u(t_0, x_0) - u(t_1, x) - \int_{t_0}^{t_1} \ell(s, x'(s)) ds > \varepsilon$$

as well as $(t_1, x) \in \text{dom}(u)$ and $\int_{t}^{t_1} \ell(s, x'(s)) ds < \infty$ for all $t \in [t_0, t_1]$. Put $\hat{s} := \inf \left\{ t \in (t_0, t_1] : u(t, x) - u(t_1, x) - \int_{t_0}^{t_1} \ell(s, x'(s)) ds \leq \frac{t_1 - t}{t_1 - t_0} \cdot \varepsilon \right\}$. We show that this infimum is attained. To this end, consider a sequence $(s_n)$ in $(t_0, t_1]$ with $s_n \to \hat{s}$ and $u(s_n, x) - u(t_1, x) - \int_{s_n}^{t_1} \ell(s, x'(s)) ds \leq \frac{t_1 - s_n}{t_1 - t_0} \cdot \varepsilon$. Then

$$u(\hat{s}, x) - \int_{s_n}^{t_1} \ell(s, x'(s)) ds \leq \lim_{n \to \infty} u(s_n, x) - \lim_{n \to \infty} \int_{s_n}^{t_1} \ell(s, x'(s)) ds \leq \frac{t_1 - \hat{s}}{t_1 - t_0} \cdot \varepsilon + u(t_1, x),$$

i.e., $\hat{s}$ is a minimum and $(\hat{s}, x) \in \text{dom}(u)$. Moreover, by (8.3), $t_0 < \hat{s} \leq t_1$. Finally, by (8.4), there is a $\delta \in [0, \hat{s} - t_0)$ such that $u(\hat{s} - \delta, x) - u(\hat{s}, x) - \int_{\hat{s} - \delta}^{\hat{s}} \ell(s, x'(s)) ds \leq \frac{\delta}{t_1 - t_0} \cdot \varepsilon$. Hence, $u(\hat{s} - \delta, x) - u(t_1, x) - \int_{\hat{s} - \delta}^{t_1} \ell(s, x'(s)) ds \leq \frac{\delta}{t_1 - t_0} \cdot \varepsilon$, which is a contradiction to the minimality of $\hat{s}$.\]

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(ii) Showing the remaining direction is straight-forward. □

Proof of Theorem 4.3. (a) Establishing the lower semi-continuity of $v_0$ is quite standard. It is very similar to the proof of Lemma 5.2 and actually slightly easier as no probability is involved (cf. also Proposition 3.1 in [13] for the non-path-dependent case). To deduce that $v_0$ is an l.s.c. minimax solution, it suffices to apply the dynamic programming principle with the existence of a minimizer for (DOC) (cf. Theorem 11.1 in [10]) together with Lemmata 8.1 and 8.2. Finally, we can apply Lemmata 8.1 and 8.2 again to obtain a comparison principle between l.s.c. minimax subsolutions and minimax supersolutions, from which uniqueness follows.

(b) Taking Remark 5.4 into account, one can see that the proof follows essentially from the content of Section 7. The measures $\mathbb{P}_{t_0,x_0,n}$ need to be replaced by the Dirac measures under which $X = x(\cdot \land t_0)$ a.s. for each $(t_0, x_0) \in [0, T] \times \Omega$ and one should note that the domains of the controls are different ([0, T] here vs. [0, T] × Ω in Section 7). Moreover, the BSDE argument in part (i) of the proof of Theorem 7.2 needs to be replaced by the deterministic dynamic programming principle. □

9. Conclusion

The main contributions of this paper are a non-Markovian vanishing viscosity result for path-dependent PDEs (PPDEs) that corresponds to the non-exponential Schilder theorem in [2], well-posedness for new notions of generalized solutions of PPDEs that can have quadratic or even super-quadratic growth in the gradient, and a non-Markovian Feynman-Kac formula for convex superquadratic BSDEs.

We want to emphasize that, here, control-theoretic methods, or equivalently (in our case), results from the theory of large deviations have been applied to obtain stability results for PPDEs (corresponding PDEs results have been obtained in [2] in the Markovian case). Of great interest would be research that investigates the opposite direction, i.e., to establish stronger results (in particular, suitable stability results) for PPDEs with (only) quadratic growth in the gradient in order to derive non-Markovian large deviation results similarly as it has been successfully done in the Markovian case via PDEs.

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