Abstract—In this paper, a stochastic dynamic control strategy is presented to prevent the spread of an infection over a homogeneous network. The infectious process is persistent, i.e., it continues to contaminate the network once it is established. It is assumed that there is a finite set of network management options available such as degrees of nodes and promotional plans to minimize the number of infected nodes while taking the implementation cost into account. The network is modeled by an exchangeable controlled Markov chain, whose transition probability matrices depend on three parameters: the selected network management option, the state of the infectious process, and the empirical distribution of infected nodes (with not necessarily a linear dependence). Borrowing some techniques from mean-field team theory the optimal strategy is obtained for any finite number of nodes using dynamic programming decomposition and the convolution of some binomial probability mass functions. For infinite-population networks, the optimal solution is described by a Bellman equation. It is shown that the infinite-population strategy is a meaningful sub-optimal solution for finite-population networks if a certain condition holds. The theoretical results are verified by an example of rumor control in social networks.

I. INTRODUCTION

Networks are ubiquitous in today’s world, connecting people and organizations in various ways to improve the quality of day-to-day life in terms of, for example, health services [1], consumers demand [2], energy management [3], and social activities [4], to name only a few. There has been a growing interest in the literature recently on network analysis, and in particular, on enhancing network reliability and security [5], [6]. This problem has been on the spotlight ever since the dramatic influence of social media on public opinion was observed in a number of major events.

Controlling the spread of undesirable phenomena such as disease and misinformation over a network is an important problem for which different approaches are proposed in the literature [7], [8]. The dynamics of an infection propagating in a network of \( n \) nodes, where each node has a binary state (susceptible and infected), can be modeled by a Markov chain with a \( 2^n \times 2^n \) transition probability matrix. Since the computational complexity of such a model is exponential with respect to \( n \), mean-field theory has proved to be effective in approximating a large-scale dynamic network by an infinite population one. For this purpose, the dynamics of the probability distribution of infected nodes can be described by a differential equation (called diffusion equation) [9], [10].

In the analysis of infection spread, the main objective is to study the dynamics of the states of nodes, specially after a sufficiently long time, in order to determine the rate of convergence to the steady state [11], [12], [13]. It is shown in [11] that if the rate of spread of infection over the rate of cure is less than the inverse of the largest eigenvalue of the adjacency matrix, the infinite-population network reaches to an absorbing state in which all states are healthy, i.e., the infection is eventually cleared.

In the control of infection spread, on the other hand, the objective is to derive the transition probabilities such that a prescribed performance index, which is a function of implementation cost and the number of infected nodes, is minimized [14], [15], [16], [17], [18]. This problem is computationally difficult to solve, in general. However, in the special case when the diffusion equation and cost function have certain structures, the optimal strategy can be obtained analytically. For example, in [14], [15], [16] it is assumed that the network dynamics and cost are linear in control action (that is immunization or curing rates), which leads to bang-bang control strategy. The interested reader is referred to [17], [18] for more details on optimal resource allocation methods.

This paper studies the optimal control of a network consisting of an arbitrary number of nodes that are influenced (coupled) by the empirical distribution of infected nodes (such couplings are not necessarily linear). The infectious process is assumed to be persistent, in the sense that the infection does not disappear after the initial time. In contrast to the papers cited in the previous paragraph which consider a continuous action set, in this paper it is assumed that there is a limited number of resources available, which means that the action set is finite. In addition, we raise a practical question that when the solution of an infinite-population network constructs a meaningful approximation for the finite-population one. Inspired by existing techniques for mean-field teams [19], [20], [21], [22], [23], [24], we first compute the optimal solution of a finite-population network for the case where the empirical distribution of infected nodes is observable. Next, we derive an infinite-population Bellman equation that requires no observation of infected nodes, and identify a stability condition under which the solution of the infinite-population network constitutes a near-optimal solution for the finite-population one.

The paper is structured as follows. In Section II the
problem is formulated and the objectives are subsequently described. The optimal control strategies, as the main results of the paper, are derived on micro and macro scales in Section III. An illustrative example of a social network is presented in Section V. The results are finally summarized in Section VI.

II. PROBLEM FORMULATION

A. Notational convention

Throughout this article, \( \mathbb{R} \) is the set of real numbers and \( \mathbb{N} \) is the set of natural numbers. For any \( n \in \mathbb{N} \), let \( N_n \) and \( \mathcal{M}_n \) represent finite sets \( \{1, \ldots, n\} \) and \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \), respectively, and \( x_{1:n} \) denote the vector \( (x_1, \ldots, x_n) \). In addition, \( \mathbb{E}[\cdot] \), \( \mathbb{P}(\cdot) \) and \( \mathbb{I}(\cdot) \) refer to the expectation, probability and indicator operators, respectively. For any \( n \in \mathbb{N} \) and \( p \in [0, 1] \), \( \text{Bino}(\cdot, n, p) \) is the binomial probability distribution function of \( n \) binary trials with success probability \( p \).

B. Model

Consider a population of \( n \in \mathbb{N} \) homogeneous users that are exposed to an infectious process (e.g., disease or fake news). Let \( x^t \in \{S, I\} \) be the state of user \( i \in N_n \) at time \( t \in \mathbb{N} \), where \( S \) and \( I \) stand for “susceptible” and “infected”, respectively. Denote by \( m_t \in \mathcal{M}_n \) the empirical distribution of the infected users at time \( t \in \mathbb{N} \), i.e.,

\[
m_t = \frac{1}{n} \sum^n_{i=1} \mathbb{I}(x^t_i = I).
\]

1) Resources: Let \( \mathcal{U} \) denote the set of finite options available to the network manager (e.g., a company or a government). The objective of the network manager is to minimize the effect of the infectious process on the users by employing the available options effectively. For instance, one possible option is the degree of nodes and by varying the degree (i.e., topology), the spread of an infection can be impeded. Alternatively, the option may be an action plan such as vaccination or health promotion, influencing the rates of infection and cure. Denote by \( u_t \in \mathcal{U} \) the option taken by the network manager at time \( t \in \mathbb{N} \).

2) Infectious process: Let \( z_t \in Z \) be the state of an infectious process at time \( t \in \mathbb{N} \), where \( Z \) is a finite set consisting of all possible states. Denote by \( \mathbb{P}(z_{t+1} \mid z_t, u_t) \) the transition probability according to which state \( z_t \in Z \) transits to state \( z_{t+1} \in Z \) under option \( u_t \in \mathcal{U} \), \( \forall t \in \mathbb{N} \). Note that the level of persistence of the infectious process is incorporated in the above transition probability matrix.

3) Dynamics of users: Suppose that the state of user \( i \) is susceptible, the state of the infectious process is \( z_t \), option \( u_t \) is chosen and the number of infected users is \( n \), \( m_t, t \in \mathbb{N} \). Then, user \( i \) becomes infected with the following probability:

\[
\mathbb{P}(x^t_{i+1} = I \mid x^t_i = S, u_t, m_t, z_t) := f^0(u_t, m_t, z_t), \quad \text{(1)}
\]

where \( f^0: \mathcal{U} \times \mathcal{M}_n \times Z \rightarrow [0, 1] \). In addition, when the state of user \( i \) is infected, it changes to susceptible according to the following probability:

\[
\mathbb{P}(x^t_{i+1} = S \mid x^t_i = I, u_t, m_t, z_t) := f^1(u_t, m_t, z_t), \quad \text{(2)}
\]

where \( f^1: \mathcal{U} \times \mathcal{M}_n \times Z \rightarrow [0, 1] \). It is to be noted that the network topology is implicitly described in transition probabilities (1) and (2).

4) Per-step cost: Let \( c(u, m, z) \in \mathbb{R}_{> 0} \) be the cost associated with implementing option \( u \in \mathcal{U} \) when the empirical distribution of the infected users is \( m \in \mathcal{M}_n \) and the state of the infectious process is \( z \in Z \). For practical purposes, the per-step cost function is considered to be an increasing function of the empirical distribution of the infected users, i.e., the more infection, the higher cost.

At any time \( t \in \mathbb{N} \), the network manager chooses its option according to the control law \( g_t : (\mathcal{M}_n \times Z)^t \rightarrow \mathcal{U} \) as follows:

\[
u_t = g_t(m_{1:t}, z_{1:t}), \quad t \in \mathbb{N}.
\]

Note that \( g := \{g_1, g_2, \ldots\} \) is the strategy of the network manager.

C. Problem statement

Assumption 1 The transition probabilities and cost function are time-homogeneous. In addition, the underlying primitive random variables of users as well as the infectious process are mutually independent in both space and time. Furthermore, the primitive random variables of users are identically distributed.

Given a discount factor \( \beta \in (0, 1) \), define the total expected discounted cost:

\[
J_n(g) = \mathbb{E}^g \left\{ \sum_{t=1}^{\infty} \beta^{t-1} c(u_t, m_t, z_t) \right\},
\]

where the above cost function depends on the choice of strategy \( g \) and the number of users \( n \).

Problem 1 Find an optimal strategy \( g^* \) such that for any strategy \( g \),

\[
J^*_n := J_n(g^*) \leq J_n(g).
\]

Problem 2 Find a sub-optimal strategy \( \tilde{g} := \{\tilde{g}_1, \tilde{g}_2, \ldots\} \), \( \tilde{g}_t : Z^t \rightarrow U \), \( t \in \mathbb{N} \), such that its performance converges to the optimal performance of the infinite-population as the number of users increases, i.e.,

\[
|J_n(\tilde{g}) - J^*_n| \leq \epsilon(n),
\]

where \( \lim_{n \to \infty} \epsilon(n) = 0 \).

III. THEORETICAL RESULTS

Prior to solving Problems 1 and 2, it is necessary to understand the dynamics of the empirical distribution of the infected users, and more importantly, the way it evolves over time according to each option of the network manager and state of the infectious process. To this end, the following theorem is needed.

Theorem 1 Let Assumption 1 hold. Given any \( m_t \in \mathcal{M}, u_t \in \mathcal{U} \) and \( z_t \in Z, t \in \mathbb{N} \), the transition probability matrix of the empirical distribution of the infected users is
characterized as:
\[ P(m_{t+1}|m_t = 0, u_t, z_t) = \text{Bino}(nm_{t+1}, f^0(u_t, m_t, z_t), n), \]
\[ P(m_{t+1}|m_t = 1, u_t, z_t) = \text{Bino}(nm_{t+1}, 1 - f^1(u_t, m_t, z_t), n), \]
\[ P(m_{t+1}|m_t \notin \{0,1\}, u_t, z_t) = \left( \text{Bino}(\cdot, f^0(u_t, m_t, z_t), n-nm_t) \right) (nm_{t+1}). \]

**Proof** The proof proceeds in three steps. In the first step, suppose \( m_t = 0 \), i.e., \( x_t^i = S, \forall i \in N_n \). In such a case, \( nm_{t+1} = \sum_{i=1}^{n} I(x_{t+1}^i = 1) \) is a random variable consisting of \( n \) i.i.d. Bernoulli random variables with the success probability \( f^0(u_t, m_t, z_t) \). In the second step, suppose \( m_t = 1 \), i.e., \( x_t^i = I, \forall i \in N_n \). Therefore, \( nm_{t+1} = \sum_{i=1}^{n} I(x_{t+1}^i = 1) \) is a random variable consisting of \( n \) i.i.d. Bernoulli random variables with the success probability \( 1 - f^1(u_t, m_t, z_t) \). In the last step, suppose that \( m_t \notin \{0,1\} \). Then, \( nm_{t+1} = \sum_{i=1}^{n} I(x_{t+1}^i = 1) \) is the sum of two independent random variables, where the first one is comprised of \( n-nm_t \) i.i.d. Bernoulli random variables with the success probability \( f^0(u_t, m_t, z_t) \) while the second one is comprised of \( nm_t \) i.i.d. Bernoulli random variables with the success probability \( 1 - f^0(u_t, m_t, z_t) \). The proof is now complete, on noting that the probability mass function of two independent random variables is the convolution of their probability mass functions.

**Theorem 2** Let Assumption 1 hold. Problem 1 admits an optimal stationary strategy characterized by the following Bellman equation: for any \( m \in M_n \) and \( z \in Z \),

\[
V(m, z) = \min_{u \in U} \left( c(u, m, z) + \beta \sum_{z' \in Z} \sum_{m' \in M} P(z' | z, u) P(m' | m, u, z) V(m', z') \right). 
\]

Let \( g^* \) be a minimizer of the above Bellman equation; then the optimal action at time \( t \in \mathbb{N} \) is given by \( u_t^* = g^*(m_t, z_t) \).

**Proof** From the proof of Theorem 1 and the fact that the infectious process evolves in a Markovian manner with a transition probability independent of the states of users, it follows that:

\[
P(m_{t+1}, z_{t+1} | m_{1:t}, z_{1:t}, u_{1:t}) = P(z_{t+1} | z_t, u_t) \times P(m_{t+1} | m_t, u_t, z_t),
\]

where the left-hand side of equation (4) does not depend on the control laws \( g_{1:t} \). Hence, one can find the optimal solution of Problem 1 via the dynamic programming principle [25], and this leads to the Bellman equation (3).

According to Theorem 2, the optimal strategy does not depend on the history of infected users and infectious process, i.e., it is sufficient to know the current values in order to optimally control the network.

**Remark 1** The cardinality of the space of the Bellman equation (3), i.e., \( M_n \), is linear in the number of users \( n \).

In the special case of \( n = \infty \), the probability mass function of \( m_{t+1} \) becomes a Dirac measure. In such a case, there is no loss of optimality in restricting attention to the dynamics of the controlled differential equations. More precisely, define

\[
p_t := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} I(x_t^i = I).
\]

According to [20, Lemma 4], the following equality holds with probability one for any trajectory \( z_{1:\infty} \),

\[
p_{t+1} = (1 - p_t) f^0(u_t, p_t, z_t) + p_t (1 - f^1(u_t, p_t, z_t)),
\]

with \( p_t = \mathbb{E}[m_1] \). By incorporating the macro-scale (infinite-population) dynamics (5) into the Bellman equation (3), one arrives at the following Bellman equation:

\[
\tilde{V}(p, z) = \min_{u \in U} \left( c(u, p, z) + \beta \sum_{z' \in Z} P(z' | z, u) \times \tilde{V} \left( (1 - p) f^0(u, p, z) + p (1 - f^1(u, p, z)), \tilde{z} \right) \right),
\]

for any \( p \in [0,1] \) and \( z \in Z \). Let \( g : [0,1] \times Z \to U \) be a minimizer of the right-hand side of equation (6), and define the following action at time \( t \):

\[
\tilde{u}_t := g(p_t, z_t).
\]

Notice that \( p_{1:t+1} \) is a stochastic process adapted to the filtration \( z_{1:t} \) for any \( t \in \mathbb{N} \), i.e.,

\[
p_{t+1} = (1 - p_t) f^0(g(p_t, z_t), p_t, z_t) + p_t (1 - f^1(g(p_t, z_t), p_t, z_t)).
\]

To establish the convergence result, the following assumptions are imposed on the model.

**Assumption 2** There exist positive constants \( k^0, k^1, k^c \in \mathbb{R}_{\geq 0} \) such that given any \( u \in U, z \in Z \) and \( m^1, m^2 \in [0,1], \)

\[
|f^0(u, m^1, z) - f^0(u, m^2, z)| \leq k^0|m^1 - m^2|,
\]

\[
|f^1(u, m^1, z) - f^1(u, m^2, z)| \leq k^1|m^1 - m^2|,
\]

\[
|c(u, m^1, z) - c(u, m^2, z)| \leq k^c|m^1 - m^2|.
\]

**Assumption 3** The parameters introduced in Assumption 2 satisfy the inequality \( \max(k^0, k^1) < \frac{1}{\beta} \).

**Theorem 3** Let Assumptions 1, 2, and 3 hold. Then, \( \tilde{g} \) is a sub-optimal strategy for Problem 2 i.e.,

\[
|J_n(\tilde{g}) - J^*_\infty| \leq \frac{k^c}{(1 - \beta)(1 - \beta \max(k^0, k^1))} O \left( \frac{1}{\sqrt{n}} \right),
\]

where \( \lim_{n \to \infty} O \left( \frac{1}{\sqrt{n}} \right) = 0 \).

**Proof** Let \( \hat{m}_{t+1} \) denote the empirical distribution of the infected users under strategy \( g \) at time \( t \in \mathbb{N} \). For ease of display, let function \( \phi : U \times [0,1] \times Z \) denote the dynamics (5), i.e., \( p_{t+1} = \phi(u_t, p_t, z_t), t \in \mathbb{N} \). For any \( \tilde{m}_t, p_t \) and \( z_t \) at time \( t \in \mathbb{N} \), the following inequality holds as a result of the
triangle inequality, monotonicity of the expectation function, Assumptions [1] and [2] and equations (5), (7) and (8):

\[
E[\hat{m}_{t+1} - p_{t+1}] = E[\hat{m}_{t+1} - \phi(\hat{g}(p_t, z_t), p_t, z_t)] \\
\leq E[\hat{m}_{t+1} - \phi(\hat{g}(p_t, z_t), \hat{m}_t, z_t)] \\
+ E[\phi(\hat{g}(p_t, z_t), p_t, z_t)] - \phi(\hat{g}(p_t, z_t), \hat{m}_t, z_t)] \\
\leq O\left(\frac{1}{\sqrt{n}}\right) + \max(k^0, k^1)|\hat{m}_t - p_t|, \tag{9}
\]

where rate \(O\left(\frac{1}{\sqrt{n}}\right)\) is the rate of convergence to the infinite-population limit [20, Lemma 4]. On the other hand, from the triangle inequality, monotonicity of the expectation function, Assumptions [1] and [2] and equations (5), (7) and (8):

\[
J_n(\hat{g}) - J_n^\ast = E\left[\sum_{t=1}^{\infty} \beta^{t-1} c(\hat{g}(p_t, z_t), \hat{m}_t, z_t) - \sum_{t=1}^{\infty} \beta^{t-1} c(\hat{g}(p_t, z_t), p_t, z_t)\right] \\
\leq \sum_{t=1}^{\infty} \beta^{t-1} k^c [\hat{m}_t - p_t]. \tag{10}
\]

From [20, Lemma 2], we have that \(E[\hat{m}_1 - p_1] \leq O\left(\frac{1}{\sqrt{n}}\right)\). Then, the proof follows from Assumption [3] and successively using (9) in inequality (10).

**Remark 2** It is to be noted that no continuity assumption is imposed on the infinite-population strategy \([7]\) in order to derive Theorem [3]. In addition, an extra stability condition (i.e., Assumption [3]) is needed to ensure that the infinite-population strategy is stable when applied to the finite-population network.

Since the optimization in \([6]\) is over an infinite space \([0, 1]\), it is computationally difficult to find the exact solution. However, it is shown in [20, Corollary 1] that if the optimization problem is carried out over space \(\mathcal{M}_n\), the resultant solution will be a near-optimal solution for the finite-population case under Assumptions [1] and [3].

**Corollary 1** Let Assumptions [1] and [3] hold. Let also \(\hat{g}_n : \mathcal{M}_n \times \mathcal{Z}\) be a minimizer of the quantized version of the Bellman equation \([6]\), as proposed in [20, Corollary 1]. The performance of \(\hat{g}_n\) converges to \(J_n^\ast\) at the rate \(1/\sqrt{n}\).

**IV. SIMULATIONS: A SOCIAL NETWORK EXAMPLE**

Nowadays, many people get their daily news via social media, where a small piece of false information may propagate and lead to a widespread misinformation and potentially catastrophic consequences. As a result, it is crucial for network managers as well as governments to prevent large-scale misinformation on social media. Inspired by this objective, we present a simple rumor control problem, where the goal of a network manager is to minimize the number of misinformed users in the presence of a false rumor.

**Example 1:** Consider \(n \in \mathbb{N}\) users and a matter of public interest with uncertain outcome such as an election. Let \(x_t^i = S\) mean that user \(i \in \mathbb{N}_n\) at time \(t \in \mathbb{N}\) is correctly informed about the topic and \(x_t^i = F\) mean otherwise. Denote

- The number of fake news may be viewed as the severity level of the misinformation induced by the rumor. In this example, we have implicitly assumed that when option “block” is taken by the manager, the source of rumor starts producing fake news cautiously from zero again. The initial states of users are identically and independently distributed with probability mass function \((0.85, 0.15)\), where 0.15 is the probability of being initially misinformation. At any time \(t \in \mathbb{N}\), given the empirical distribution of misinformation users \(m_t\), the number of fake news \(z_t\) and the option \(u_t\) taken by the network manager, an informed user is misled by the rumor with the following probability:

\[
f^0(u_t, m_t, z_t) = \begin{cases} 
0.2m_t, & u_t = 1, \\
0.2m_t, & u_t = 2, \\
0.1m_t^2, & u_t = 3.
\end{cases}
\]

where a larger number of misinformation users and fake news means a higher probability that a user becomes misinformation. On the other hand, a misinformation user becomes informed.
and convinced by the authenticated information provided by the network manager with a high probability. More precisely,

\[ f^1(u_t) = \begin{cases} 
0, & u_t = 1, 2, \\
0.8, & u_t = 3. 
\end{cases} \]

Denote by \( \ell : \mathcal{U} \times \mathcal{Z} \to \mathbb{R}_{\geq 0} \) the implementation cost of each option, and let:

\[ \ell(1, z) = 0, \quad \ell(2, z) = 0.2z + 1, \quad \ell(3, z) = 5, \]

for any \( z \in \{0, 1, 2, 3, 4\} \). It is desired to minimize the following cost function:

\[ \mathbb{E}\left[ \sum_{t=1}^{\infty} 0.9^{t-1} \left( 3.8m_t + \ell(u_t, z_t) \right) \right]. \]

To determine the optimal strategy, we first compute the transition probability matrix in Theorem 1 and then solve the Bellman equation (3) in Theorem 2 by using the value-iteration method. The optimal strategy for \( n = 200 \) is displayed in Figure 1 as a function of the empirical distribution of misinformed users and the number of fake news. Under this optimal strategy, one realization of Example 1 is depicted in Figure 2.

![Fig. 2. Trajectory of the solution for a network of size 200 in Example 1.](image)

To verify the results of Theorem 3 and Corollary 1, let the Bellman equation (6) be quantized with a step size \( 1/n \), and denote the resultant value function by \( V^Q(q_1, z_1) \), where \( q_1 \) is the closest number in \( \mathcal{M}_n \) to \( p_1 = 0.15 \). Subsequently, it is shown in Figure 3 that \( \mathbb{E}[V^Q(q_1, z_1)] \) converges to \( J_n^* = \mathbb{E}[V(m_1, z_1)] \), as \( n \) increases.

![Fig. 3. The quantized solution converges to the optimal solution as the number of users increases in Example 1.](image)

V. CONCLUSIONS

A stochastic dynamic control strategy was introduced over a homogeneous network to minimize the spread of a persistent infection. It was shown that the exact optimal solution can be efficiently computed by solving a Bellman equation whose state space increases linearly with the number of nodes. In addition, an approximate optimal solution was proposed based on the infinite-population network, where the approximation error was shown to be upper bounded by a term that decays to zero as the number of users tends to infinity. An example of a social network was then presented to verify the theoretical results.

As a future research direction, the obtained results can be extended to partially homogeneous networks wherein the nodes are categorized into several sub-populations of homogeneous nodes such as low-degree and high-degree nodes. In addition, various approximation methods may be used to further alleviate the computational complexity of the proposed solutions. The development of reinforcement learning algorithms based on the Bellman equations provided in this paper can be another interesting problem for future work.

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