Ray Chaos in a Photonic Crystal

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The ray dynamics in a photonic crystal was investigated. Chaos occurs for perfectly periodic crystals, the rays dynamics being very sensitive to the initial conditions. Depending on the filling factor, the ray dynamics can exhibit stable paths near (fully) chaotic motion. The degree of chaoticity is quantified through the computation of Lyapunov exponents. It results that the more diluted is the geometry, the more chaotic is the dynamic. Therefore, despite the perfect periodicity of the geometry, light transport is a diffusive process which can be tuned from normal diffusion (brownian motion) to anomalous diffusion because of the existence of Lévy flights.

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Photonic crystals, i.e. artificial periodic structures exhibiting a photonic band gaps [1], have been studied at very different angles since the pioneering works of S. John and E. Yablonovitch [2, 3]. From the sole effect of acting like mirrors, the engineering of band structures and the recognition that they behave as metamaterials [4], they have shown a wide range of physical phenomena. However, they have been hardly considered in the high frequency domain: it is indeed at most the first three bands which are considered for band structure engineering or effective properties.

In this work, we aim at initiating a new direction of research by considering the propagation of light in periodic dielectric structures when the wavelength is very small with respect to the scatterers. In this limit, band diagrams become highly complicated because of a huge number of eigenmodes. This is the domain of wave chaos where puzzling phenomena such as random lasing [5] or localization [6, 7] may be expected. Wave chaos is not restricted to optics and electromagnetism but has been extensively studied in quantum mechanics [8] where it is known as “quantum chaos”. Many quantum systems exhibit chaotic footprints: hydrogen atoms in magnetic field [9], quantum chaotic cavities (quantum dots billiards) [10, 11] or chaotic quantum scattering [12]. In electromagnetism or optics, most of the studies dealt with chaotic cavities [2, 3, 7] but only few of them with open chaotic systems such as optical fibers [18] or photonic crystals [19, 20]. In the case of a two dimensional photonic crystal, chaos is revealed by the help of the statistical properties of the energy level spacing [19, 20]. They are computed from the eigenfrequencies of the Helmholtz equation solved with proper boundary conditions. These approaches might indicate that light dynamics in a 2D photonic-crystal is “quasi-integrable” [20] meaning that the light dynamics exhibits regular and chaotic trajectories characterized through different level-spacing distributions. However, the dynamical properties of light are not a direct output of such an approach. Here we show that the dynamics of light and the transition from regular to chaotic trajectories can be understood with the help of light rays. We show that this transition is imprinted in the dynamics of the transmitted rays, the dynamics of the reflecting rays being always chaotic. From the knowledge of the dynamics, we are able to compute the diffusive properties of light in the photonic crystal. We show a transition from a brownian diffusion to an “anomalous” regime.

Figure 1: A small portion of the photonic crystal and notations used in the text. "A_p" referred to the pth incoming point on the cylinder surface at the pth step, "B_p" is the pth point at the cylinder surface after refraction. α denotes the angle between the ray and the horizontal axis. θ_{A_p} (resp. θ_{B_p}) the angle of the point A_p (resp. B_p) and the horizontal axis. θ_{A_1} (resp. θ_{B_1}) the angle of the point A_p (resp. B_p) the angle corresponding to the direction under which the ray intersect the cylinder. The dashed red lines are the axis of symmetry of the crystal.

The photonic crystal under study is depicted in Fig. 1: it is a biperiodic set of dielectric cylinders, in which the trajectories of a ray are studied, by applying Snell-Descartes relations at the boundary of the cylinders. If the cylinders are perfectly reflecting instead of trans-
parent then the system is nothing else than a periodic Lorentz gas. It has an extensive bibliography both in the physical and mathematical literatures (see e.g. [21] and references hereby). It consists in an ensemble of noninteracting point particles moving freely with elastic collisions on fixed scatterers (the cylinders). The photonic billiard can be seen as the refractive extension of the periodic Lorentz gas. Upon this analogy a light ray represents a particle trajectory and, instead of having infinite walls, the repulsive potential is finite with a value given by the optical index of the cylinders. Whereas there exists a huge literature about the periodic Lorentz gas with hard-wall scatterers, we have not been able to find references on the ”refractive periodic Lorentz gas” [22]. The Lorentz gas is an unfolding of the Sinai Billiard [23] for which the dynamics is always chaotic, whatever the photonic crystal parameters, making impossible long-range precision concerning a single trajectory. So, despite its deterministic character, this system can only be studied from a statistical point of view. Transport is diffusive and the square character, this system can only be studied from a statistical point of view. Transport is diffusive and the square character, this system can only be studied from a statistical point of view. Transport is diffusive and the square character, this system can only be studied from a statistical point of view. The Lorentz gas exhibits superdiffusion due to the existence of Levy flights. The dynamical properties of rays in the crystal are now described. Numerical computations were performed with an optical index of the cylinders equal to \( n = 1.5 \), which leads to a coefficient of reflection of the order of 10 %. In Fig:1 of the Supplementary Material (SM I), we show the propagation of a laser beam computed by solving rigorously the Maxwell equations into the photonic crystal. Because of the low optical-index contrast between the cylinder and the surrounding medium, light propagation is dominated by the transmitted rays. So we focus only on rays which are refracted and thus transmitted through the cylinders. The ray dynamic has a Hamiltonian formulation in terms of the eikonal equation: \( \frac{d}{ds} \left( n \frac{d\theta}{ds} \right) = \nabla n \) [31], therefore the phase space is four dimensional. Because \( n \) is constant inside the cylinder, the solutions to this equation, i.e. the ray trajectories, are piecewise linear. In order to characterize them, we compute the Poincaré surface of section. Poincaré surface of section are a convenient way to represent as a 2D-graph a higher dimensionality phase-space by representing continuous evolution in a discrete way. They preserve informations about the dynamics [32]. As an example, periodic orbits in the phase space appears as periodic points in the Poincaré surface of section [33]. For the refractive Lorentz gas, the Poincaré surface of section is the set of points \((\theta_A, \theta_A')\) defined by the intersection of the ray with the boundary of a cylinder: \(\theta_A\) is the angle between the horizontal axis and the light ray when it crosses the boundary of the cylinder and \(\theta_A'\) is the angle between the incident ray and the normal to the cylinder boundary (see Fig:1). The Poincaré map is constructed in two steps (see Supplementary Materials II):

1. An incident ray intersects a cylinder, defining a point \( A_p \) given by the system of equations (2a and 2b),
2. The ray is refracted twice at the boundary, giving the point $B_0$. The set of equations (1a and 1b) describes these refraction steps.

The ray propagates in air until it hits a new cylinder, this gives $A_{p+1}$ and the process goes on. The equations are solved numerically in order to compute the ray trajectory in the whole crystal.

Examples of Poincaré surfaces of section are given in Fig.2 for different values of the normalized period $t^*$ ($t^*$ varying from 2.2 to 6). It will be shown in the following that this parameter governs the stability properties of the ray dynamics.

The most prominent point is the existence of islands of stability characterized by closed curves in the Poincaré surface of section, surrounded by a chaotic sea. Islands of stability characterize ballistic trajectories in the crystal i.e. rays that oscillate around the axis of symmetry of the crystal. Cylinders act as cylindrical lenses: they focus rays in the vicinity of the axis of symmetry of the crystal. As a result, these trajectories are periodic or quasi-periodic in real space as explained in the SM (III).

The area covered by these islands of stability decreases when $t^*$ increases and they reduce to single points above the threshold $t_{\text{thres}}^* > \frac{\pi}{2}$, as shown in the SM (IV). In the example of Fig.2, the threshold is $t_{\text{thres}}^* = 6$ for $n = 1.5$. At small angles, the islands of stability are ellipses. This corresponds to Gauss conditions: $\sin(\theta) \sim \theta$ and $\cos(\theta) \sim 1$. When the angles increase, the islands of stability are deformed because of non-linear effects, although rays still propagate along the crystal axis. At the edges of the transition between the islands of stability and the chaotic sea, Cantori [26] can be observed (see Fig.3) because of an increase of non-linear effects: this is a direct consequence of KAM theory [34, 35]. Again rays propagate along the crystal axis. When non-linear effects still increase rays leave crystal axis and motion is completely chaotic with sometimes segments of ballistic motion. In that case rays diffuse in the entire crystal. We conclude that the "quasi-integrable" character of the light dynamics in a photonic crystal [24] is a consequence of the dynamical properties of the transmitted rays.

![Figure 3: Details of Fig.2 for $t^* = 3$ showing Cantori surrounded by the chaotic sea.](image)

The strength of chaos, i.e., the sensitivity to initial conditions, can be quantified through the computation of Lyapunov exponents. In Hamiltonian systems with two degrees of freedom, the Lyapunov exponents come in pairs with a null sum because of area preservation in the phase-space [34] (this is Liouville theorem). The positive exponent denoted $\lambda$, and called in the following Lyapunov exponent, quantifies the exponential sensitivity to initial conditions. The Lyapunov exponents $\lambda$ were computed per step, that is, each time a ray hits a cylinder. Because of islands of stability the dynamics is clearly not ergodic and the Lyapunov exponent may depend on initial conditions. The Lyapunov exponent takes low value for ballistic paths, as expected (blue color in the lower inset of Fig. 4) and the value sharply increases outside the islands of stability. As shown in Fig.2, the size of the islands of stability decreases as $t^*$ increases.

Above this threshold, the Lyapunov exponent is almost constant and large, whatever the initial conditions are. In Fig.2 the Lyapunov exponents averaged over the initial conditions are represented (plain line). The averaged exponents increase monotonically when increasing the normalized period $t^*$.

In the example of Fig.2, the Lyapunov exponents averaged over the initial conditions are represented (plain line). The averaged Lyapunov exponents come in pairs with a null sum because of area preservation in the phase-space [34] (this is Liouville theorem). The positive exponent denoted $\lambda$, and called in the following Lyapunov exponent, quantifies the exponential sensitivity to initial conditions. The Lyapunov exponents $\lambda$ were computed per step, that is, each time a ray hits a cylinder. Because of islands of stability the dynamics is clearly not ergodic and the Lyapunov exponent may depend on initial conditions. The Lyapunov exponent takes low value for ballistic paths, as expected (blue color in the lower inset of Fig. 4) and the value sharply increases outside the islands of stability. As shown in Fig.2, the size of the islands of stability decreases as $t^*$ increases.
Figure 4: Lyapunov exponent versus the normalized period $t^\star$ of the crystal. The plain line is the Lyapunov exponent for the refractive Lorentz gas whereas the dashed blue curve corresponds to the reflective Lorentz gas. Lyapunov exponents are given per step unit. The inset shows the Lyapunov exponent for the refractive Lorentz gas as a function of $t^\star$ and of the initial angle between the ray and the horizontal axis (launching angle). It takes low values (blue color) for launching angles around the crystal axis of symmetry. For $t^\star \geq t^\star_{\text{thres}} = 6$ the Lyapunov exponent is independent of the launching angle.

Figure 5: Rays density after 300 simulation steps for a) $t^\star = 3$, b) $t^\star = 5$ and c) $t^\star = 6$. The simulation was running with $3 \times 10^4$ rays starting from $(0,0)$ with random initial conditions under a uniform distribution. In that case one can clearly notice a non-isotropic density of rays. Most of the rays concentrate in 8 spots localized on the principal axis of the photonic crystal. These rays propagate in the photonic crystal through regular trajectories. When $t^\star$ is above the threshold $t^\star \geq t^\star_{\text{thres}}$, the dynamics only admits chaotic motion (see Fig 4c) with $t^\star = 6$) and the density of rays becomes isotropic with a maximum at the center of the photonic crystal. The density of rays follows a Gaussian distribution law, which indicates that rays undergo a brownian motion. For intermediate values of the normalized period $t^\star$, the density of rays is characterized by spots on the crystal axis plus an isotropic distribution which is maximum at the photonic crystal center. For this intermediate situation, some rays follow regular paths which lead to the spots in the rays density and some others follow chaotic motion giving the isotropic distribution.

Hamiltonian systems are known to lead to superdiffusion [24]. It has been shown that in a two-dimensional periodic potential [24, 25] the mean square displacement varies as $<r^2_t> \sim t^\alpha$ as a results of ballistic flights, random motion and the "sticky" barriers formed by Cantori [27]. The mean square displacement was computed for an ensemble of $10^4$ rays with initial conditions chosen randomly (with an uniform distribution) and the exponent $\alpha$ of the diffusion law was determined. It is plotted on Fig 6 as a function of the normalized period $t^\star$. Its values are larger than 1 when the dynamics contains chaotic regions and islands of stability which confirms the superdiffusive behavior. It reaches values close to 2 when the gap between the cylinders is small ($t^\star \sim 2$) i.e. when the dynamic is dominated by islands of stability and it decreases to 1 as the area of stability islands decreases. For $t^\star \geq 6$, that is to say when the dynamic is completely chaotic, diffusion is close to normal. Nevertheless, the exponent remains slightly larger than 1 because of ballistic paths exactly along the crystal principal axis.

In the high frequency limit, light propagation in a photonic crystal exhibits complex propagation patterns with regular paths and chaotic motion. For this limiting case, light transport is best described as a diffusive process that can be tuned from a superdiffusive regime to a brownian motion. An interesting feature of the system is its simplicity, allowing for experimental investigations of wave chaos footprint in optics. It paves the way to studies about the quantum aspects of light such as quantum correlations and entanglement in the case of a chaotic propagation.
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I. GAUSSIAN BEAM PROPAGATION

We have compute the propagation of a laser beam through a sample of \(7 \times 7\) cylinders of the photonic crystal. The Maxwell equations are exactly solved and the result is presented on Fig. 1. The parameters used in the simulation are:

- optical index \(n = 1.5\)
- wavelength \(\lambda = 500\) nm
- cylinders radius \(R = 20\lambda\)
- photonic crystal period \(T = 5R\)

This simulation indicates that the laser beam can be modeled by rays, a ray being the path of the light energy. The initial gaussian beam is split into many light rays during propagation. It also indicates that the field distribution is dominated by the transmitted rays as a result of the low optical index of the cylinders.

II. RAY DYNAMICS

The ray dynamics is splitted in two parts. At the \(n\)-step, an incoming ray impact on a cylinder. It is first refracted at the dioptric air/glass then propagate into the cylinder and finally is refracted at the dioptric glass/air. This path leads as seen on Fig:1 to the point \(B_p\). \((\theta_{B_p}, \theta_{1_{B_p}})\) can be linked to \((\theta_{A_p}, \theta_{1_{A_p}})\) by the set of equations:

\[
\begin{align*}
\theta_{1_{B_p}} &= -\theta_{1_{A_p}} \quad (1a) \\
\theta_{B_p} &= \theta_{A_p} + 2 \arcsin \left( \frac{\sin (\theta_{1_{A_p}})}{n} \right) \quad (1b)
\end{align*}
\]

where \(n\) is the optical index of the cylinders.

From the point \(B_p\) the ray propagate in air until it intersects a new cylinder defining then the point \(A_{p+1}\). The couple \((\theta_{A_{p+1}}, \theta_{1_{A_{p+1}}} )\) is found with the help of equations:

\[
\begin{align*}
f_B(x) &= \tan(\alpha) \left[ x - R \cos(\theta_{B_p}) \right] + R \sin(\theta_{B_p}) \quad (2a) \\
(x_{A_{p+1}} - x_{p+1}^c)^2 + (f_B(x_{A_{p+1}}) - y_{p+1}^c)^2 &= R^2 \quad (2b)
\end{align*}
\]

with \(\alpha\) the angle between the horizontal axis and the ray. It verifies: \(\alpha = \theta_{B_p} + \theta_{1_{B_p}} = \theta_{A_{p+1}} + \theta_{1_{A_{p+1}}},\)

\((x_{p+1}^c, y_{p+1}^c)\) are the cylinder coordinates at the \((p+1)\)th collision.

The Jacobian \(J_p\) of the total transformation \((\theta_{A_p}, \theta_{1_{A_p}}) \rightarrow (\theta_{A_{p+1}}, \theta_{1_{A_{p+1}}})\) is given by \(J_p = J_{p+1} J_{p+1}^B J_p^{BA}\) where \(J_{p+1}^B\) is the Jacobian of the transformation \((\theta_{B_p}, \theta_{1_{B_p}}) \rightarrow (\theta_{A_{p+1}}, \theta_{1_{A_{p+1}}})\)

\[
J_{p+1}^B = \begin{pmatrix} m_n - 1 & m_n \\ 2 - m_p & 1 - m_p \end{pmatrix} \quad (3)
\]

with

\[
m_p = \frac{1}{\cos(\theta_{B_p} + \theta_{1_{B_p}} - \theta_{A_{p+1}})} \frac{l_{jump}}{R} \quad (4)
\]

where \(R\) is the cylinder radius and \(l_{jump}\) the distance between the points \(B_p\) and \(A_{p+1}\).

\(J_{p+1}^B\) is the Jacobian of transformation \((\theta_{A_p}, \theta_{1_{A_p}}) \rightarrow (\theta_{B_p}, \theta_{1_{B_p}})\)

\[
J_p = \begin{pmatrix} \frac{1}{n} \cos(\theta_{A_p}) \cos(\theta_{1_{A_p}}) \\ 0 \end{pmatrix} \quad (5)
\]

In the Gauss conditions all Jacobians become independent of \(n\) and the Jacobian of the total transformation \(J\) reduced to:

\[
J_{Gauss} = \begin{pmatrix} 2 \tan(\alpha) + 1 - \tan^2(\alpha) \\ 2 \tan(\alpha) + 1 - \tan^2(\alpha) \end{pmatrix} \quad (6)
\]

where \(\alpha = T/R\) is the normalized period of the crystal \((T\) is the crystal period, \(R\) is the cylinders radius).

The eigenvalues of the Jacobian in the Gauss conditions are given by:

\[
\Lambda_{\pm} = 1 + \tan^2(\alpha) \pm \frac{1}{n} \sqrt{\tan^2(\alpha) - 1} \quad (7)
\]
III. PERIOD OF THE BALLISTIC TRAJECTORIES IN THE GAUSS CONDITIONS AND QUASI-PERIODIC TRAJECTORIES;

From Eq. 7 it is possible to find the period $P(t^*)$ of the ballistic trajectories in the Gauss conditions. It can be defined as the number of cylinders after which the couple $(\theta_{A_p + P(t^*)}, \theta_{A_1 + P(t^*)})$ is equal to $(\theta_{A_p}, \theta_{A_1})$. When they are complexes the eigenvalues read:

$$\Lambda_{\pm} = 1 + \frac{t^*}{n} - t^* \pm \frac{1}{n} \sqrt{t^*(n-1)[2n-(n-1)t^*]}$$

Their modulus is 1 (the map is area preserving). In the eigenbase the map is given by:

$$\begin{pmatrix} \theta_{A_p + P(t^*)} \\ \theta_{A_1 + P(t^*)} \end{pmatrix}_{EB} = \begin{pmatrix} e^{i\arg[\Lambda_+]} & 0 \\ 0 & e^{i\arg[\Lambda_-]} \end{pmatrix}^{P(t^*)} \begin{pmatrix} \theta_{A_p} \\ \theta_{A_1} \end{pmatrix}_{EB}$$

where $EB$ refers to the eigenbase.

Finally we find that the period of the ballistic trajectories in the Gauss conditions is:

$$P(t) = \frac{2\pi}{\arctan[\sqrt{t^*(n-1)[2n-(n-1)t^*]}/n-t^*(n-1)]}$$

(8)

It is plotted in Fig 2 where we compare numerical results in the Gauss conditions and the analytical formula. We can notice an excellent agreement between the numerical results (green dots in Fig 2) and the analytical result (plain line in Fig 2).

The period of the oscillations $P(t^*)$ is a real number whereas the number of cylinders is of course an integer number. This leads to quasi-periodicity. As a matter of fact, in the eigenbase, for example $\theta_{A_n}$ writes $\theta_{A_n} = \theta_{A_0} e^{2\pi i t^*/t^*}$ where $\theta_{A_0}$ is the initial value. If we now
write $P(t^*) = I(t^*) - F(t^*)$ where $I(t^*)$ is the nearest integer from $P(t^*)$ and $F(t^*)$ the fractional part. An easy way to understand the origin of quasi-periodic is to assume that $F(t^*) \ll I(t^*)$. Then we can write $\theta_{A_n} = \theta_{A_0} e^{2i\pi \frac{m}{m-1}}(1 + 2i\pi \frac{n}{m})$. The first term $e^{2i\pi \frac{m}{m-1}}$ is a periodic function. The second term $(1 + 2i\pi \frac{n}{m})$ leads to a slipping in the phase space and the trajectory appears as quasi-periodic. It is illustrated by the Fig. for $t^* = 3.2$ and $t^* = 4.2$. The period is $P(3.2) = 3.84$ then $I(3.2) = 4$ and $F(3.2) = 0.16$ and $P(4.2) = 3.17$ with $I(4.2) = 3$ and $F(4.2) = -0.17$. After respectively 4 or 3 steps the point $A_{n+P(t^*)}$ is close to the point $A_n$ but shift from a quantity due to $F(t^*)$ which leads to quasi-periodicity.

Ballistic trajectories in the crystal are characterized by closed curves in the phase space (stability islands). Stability islands on their own are characterized by complex eigenvalues of the Jacobian. Thus ballistic trajectories can propagate around the crystal axes ($\theta_A = 0, \pi/2$ mod $[\pi]$) if $t^* < \frac{2n}{n-1}$ and around the bisectors ($\theta_A = \pi/4$ mod $[\pi]$) if $t^*/\sqrt{2} < \frac{2n}{m+1}$. This gives the threshold above which islands of stability no more exist and above which the motion of light in the crystal is only chaotic.

V. LYAPUNOV EXPONENT BEHAVIOR AT LARGE $t^*$

Here we show that the Lyapunov exponent grows asymptotically as $\lambda \sim \ln(t^*)$.

For large normalized period of the crystal $t^* \gg 1$ collisions are uncorrelated. So the Jacobian become independent of the step number. Moreover the incident angle $\theta_A$ is close to $0$ mod $\pi$. Then $J_{AB}$ reduces to:

$$J_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $m_n$ reduces to $m = -\frac{l_{\text{jump}}}{R}$. The distance between two collisions is given by the mean free path $l_{\text{jump}} = \frac{1}{\nu} = \frac{(\pi \sigma)^2 R}{2}$. The density of disks $n^d \sim 1/T^2$ ($T$ is the crystal period) and $\sigma$ the cross section $\sigma = 2R$. The Jacobian $J$ of the total transformation $J = J_{AB} J_{BA'}$ is:

$$J = \begin{pmatrix} m-1 & m \\ m-2 & m-1 \end{pmatrix}$$

The eigenvalues are given by: $\Lambda_{\pm} = \frac{Tr(J) \pm \sqrt{Tr(J)^2 - 4}}{2}$. Within the assumption $t^* \gg 1$ the Lyapunov exponent is $\lambda = \ln(\Lambda_{\pm}) \sim \ln(t^*)$. So asymptotically it grows as $\ln(t^*)$. Of course the same dependence is found if expanding the positive eigenvalue of the Jacobian Eq.

VI. EXPERIMENTAL RESULTS

We have performed experiments to illustrate that light propagation in the photonic crystal is highly sensitive to the initials conditions. We shine a photonic crystal made of $20 \times 20$ glass cylinders with a laser beam (cylinder radius $\approx 2$ mm, photonic crystal period $T=1.1$ cm leading to $t^* = 2.2$, glass refractive index $n=1.78$, laser wavelength $\lambda=473$ nm). The beam diameter is on the order of $2$ mm. It corresponds to a collection of light rays. The dynamics of each of them is described by our model. We start with initial conditions $\theta = 24^\circ$ and $\theta_1 = 24^\circ$. In that case, initial conditions are such that all the laser beam follow a regular path (see Fig. a). It oscillates with a period close to 4 in agreement with the value given.
by formula $P(t^* = 2.2, n = 1.78) = 4.09$. We only increase the incident angle by $3^\circ$ ($\theta_1 = 37^\circ$). The laser beam starts to follow a regular path (see Fig: -b) but leave the crystal axis and finally diffuse in the entire crystal. By increasing again the initial conditions $\theta_1 = 34^\circ$, the laser beam splits after the first cylinder in several beams that diffuse in the photonic crystal.
Figure 5: Experimental results showing propagation of a laser beam in the refractive Lorentz gas. The cylinder radius is $R = 0.5 \text{ cm}$, the crystal period is $T = 1.1 \text{ cm}$ leading to $t = 2.2$. a) The initial conditions are $\theta = 24^\circ$ and $\theta_1 = 24^\circ$. The laser beam follows a regular path oscillating back and forth around the crystal axis. b) The initial conditions are $\theta = 24^\circ$ and $\theta_1 = 27^\circ$. The laser beam starts to follow a regular path but after hitting 4 cylinders it leaves the crystal axis and diffuses in the entire crystal. c) The initial conditions are $\theta_0 = 24^\circ$ and $\theta_0^\circ = 34^\circ$. The laser beam splits in several beams that diffuse in the photonic crystal.