Abstract. We study stabilization of finite-dimensional representations of the periplectic Lie superalgebras $p(n)$ as $n \to \infty$.

The paper gives a construction of the tensor category $\text{Rep}(P)$, possessing nice universal properties among tensor categories over the category $\mathfrak{svect}$ of finite-dimensional complex vector superspaces.

First, it is the "abelian envelope" of the Deligne category corresponding to the periplectic Lie superalgebra, in the sense of [EHS15].

Secondly, given a tensor category $\mathcal{C}$ over $\mathfrak{svect}$, exact tensor functors $\text{Rep}(P) \to \mathcal{C}$ classify pairs $(X, \omega)$ in $\mathcal{C}$ where $\omega : X \otimes X \to \Pi \mathbb{I}$ is a non-degenerate symmetric form and $X$ not annihilated by any Schur functor.

The category $\text{Rep}(P)$ is constructed in two ways. The first construction is through an explicit limit of the tensor categories $\text{Rep}(p(n))$ ($n \geq 1$) under Duflo-Serganova functors. The second construction (inspired by P. Etingof) describes $\text{Rep}(P)$ as the category of representations of a periplectic Lie supergroup in the Deligne category $\mathfrak{svect} \boxtimes \text{Rep}(GL_t)$.

An upcoming paper will give results on the abelian and tensor structure of $\text{Rep}(P)$.

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### 1. Introduction

1.1. The (complex) periplectic Lie superalgebra $p(V)$ is the Lie superalgebra of endomorphisms of a complex vector superspace $V$ possessing a non-degenerate symmetric form $\omega : V \otimes V \to \Pi \mathbb{I} \cong \mathbb{C}^{0|1}$, where $\Pi$ denotes the parity shift on vector superspaces (this is also referred to as an "odd form"). An example of such superalgebra is $p(n) = p(\mathbb{C}^{n|n})$, where $\omega_n : \mathbb{C}^{n|n} \otimes \mathbb{C}^{n|n} \to \mathbb{C}^{0|1}$ pairing the even and odd parts of the vector superspace $\mathbb{C}^{n|n}$.

The periplectic Lie superalgebras possess an interesting non-semisimple representation theory; some results on the category $\text{Rep}(p(n))$ of finite-dimensional integrable representations of $p(n)$ can be found in [BDE+16, Che15, DLZ15, Gor01, Moo03, Ser02].

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In this paper, we propose studying the stabilization properties of finite-dimensional integrable representations of \( p(n) \) as \( n \) grows large by considering a certain “limit category” \( \text{Rep}(P) \). This category is also important in its own right, possessing nice universal properties.

1.2. Deligne categories. The inspiration for the construction of \( \text{Rep}(P) \) comes from the theory of Deligne categories. In [DMS2], Deligne and Milne constructed a family of categories \( Tilt(GL_t) \), parameterized by \( t \in \mathbb{C} \).

These are Karoubian additive rigid SM categories. Among such categories, \( Tilt(GL_t) \) is the universal category generated by a dualizable object \( V \) of categorical dimension \( t \), and can be considered an interpolation of the family of categories \( \text{Rep}(GL_n(\mathbb{C})), n \geq 0 \). The categories \( Tilt(GL_t) \) are called Deligne categories for the general linear group.

For \( t \) non-integer, these are semisimple tensor categories; for \( t \in \mathbb{Z} \), the category \( Tilt(GL_t) \) is Karoubian but not abelian. An abelian envelope \( \text{Rep}(GL_t) \) of \( Tilt(GL_t) \), possessing nice universal properties, was constructed in [EHS15].

An important feature of the category \( \text{Rep}(GL_t) \) (and of \( Tilt(GL_t) \), for \( t \notin \mathbb{Z} \)) is that it is not equivalent to the category of representations of any affine algebraic group, nor supergroup. More examples of non-Tannakian categories of this form were constructed in [Del07, Et14].

1.3. Periplectic Deligne category. In the periplectic case, a Karoubian additive Deligne category \( \mathfrak{P} \) was constructed in [CE17b, KT14]. This is a Karoubian additive rigid SM category, and is a module category over \( s\text{Vect} \) (hence endowed with an endofunctor \( \Pi \) such that \( \Pi^2 = \text{Id} \)).

Among such categories, it is the universal category generated by a dualizable object \( \tilde{V} \) and a non-degenerate symmetric form \( \omega_{\tilde{V}} : \tilde{V} \otimes \tilde{V} \to \Pi^2 \).

The Karoubian structure of the category \( \mathfrak{P} \) was studied in [CE17b], and its tensor ideals were classified in [Con17].

Remark 1.3.1. The endomorphism algebras
\[
\text{End}_{\mathfrak{P}}(\tilde{V}^{\otimes k}) := \text{End}_{\mathfrak{P}}(\tilde{V}^{\otimes k}) \oplus \text{Hom}_{\mathfrak{P}}(\tilde{V}^{\otimes k}, \Pi \tilde{V}^{\otimes k})
\]
are known as the periplectic (or marked) Brauer algebras, and have been studied in [BDE+18, CP16, Con16, CE17a, KT14, Tha17].

Remark 1.3.2. Unlike the general linear case, here we do not have a parameter \( t \): the categorical dimension of \( \tilde{V} \) is automatically zero, since \( \tilde{V} \cong \Pi \tilde{V}^* \) and hence \( \text{dim} \tilde{V} = -\text{dim} \tilde{V} \).

1.4. Construction of the category \( \text{Rep}(P) \). Let \( s\text{Vect} \) denote the symmetric monoidal category of finite-dimensional vector superspaces with parity-preserving maps.

In what follows, we work in the 2-category of module categories over \( s\text{Vect} \). The bottom line of this is that all the categories we consider are endowed with an autoequivalence called the parity shift and denoted by \( \Pi \), and the functors we consider respect intertwine with the parity shifts. In particular, any symmetric monoidal structure on the categories considered below respects the \( s\text{Vect} \)-module structure.

We construct a tensor category \( \text{Rep}(P) \) as a certain limit of categories \( \text{Rep}(p(n)) \) with respect to the Duflo-Serganova functors \( DS_x : \text{Rep}(p(n)) \to \text{Rep}(p(n-2)) \). Such functors are defined as follows (see [DS05]).

Given a Lie superalgebra \( \mathfrak{g} \), let \( x \in \mathfrak{g} \) be an odd element satisfying \([x,x] = 0\). Let \( \mathfrak{g}_x := \text{Ker} \text{ad}_x / \text{Im} \text{ad}_x \). This is again a Lie superalgebra, and we can define a functor
\[
DS_x : \text{Rep}(\mathfrak{g}) \to \text{Rep}(\mathfrak{g}_x), \ M \mapsto M_x := \text{Ker} x / \text{Im} x.
\]
It is easy to see that this functor is symmetric monoidal. In our case, taking \( \mathfrak{g} = p(n) \) and \( x \) of rank 2, we obtain \( \mathfrak{g}_x \cong p(n-2) \).

The functors \( DS_x \) are in general not exact, but turn out to be exact on certain subcategories of \( \text{Rep}(p(n)) \), allowing us to consider a limit of \( \text{Rep}(p(n)) \) over \( n \).

\(^1\text{Denoted also by } \text{Rep}(GL_t), D_t.\)
As in \cite{EHS15}, the obtained category Rep($P$) is not Tannakian: that is, it is not equivalent to the category of representations of any supergroup. Furthermore, it is an interesting example of a tensor $sVect$-module category which is not split, in the sense that we cannot present Rep($P$) as $sVect \otimes V$ for any tensor category $V$.

The category Rep($P$) is a lower highest weight category (see Definition 7.0.1): namely, we have a filtration $\operatorname{Rep}(P) = \bigcup_{k \geq 0} \operatorname{Rep}^k(P)$ by full highest weight subcategories, whose standard, costandard and tilting objects play the same role in each $\operatorname{Rep}^k(P)$ for $k' \geq k$. The full subcategory of tilting objects in $\operatorname{Rep}(P)$ is then precisely $\Psi$. The highest weight structure of $\operatorname{Rep}(P)$ is described in Section 9 and will be further investigated in an upcoming paper.

1.5. **Universal property.** Let us now describe the universal properties of Rep($P$) proved in this paper.

Let $T$ be a tensor category (more generally, an abelian symmetric monoidal category with biexact bilinear bifunctor $\otimes$ and a simple unit object $1$).

Our first theorem establishes that Rep($P$) is universal among tensor categories generated by an object with a non-degenerate symmetric odd form (see Theorems 7.0.2 and 8.2.1):

**Theorem 1.** Let $X \in T$ be a dualizable object with a non-degenerate symmetric form $\omega_X : X \otimes X \to \Pi_1$.

Assume $X$ is not annihilated by any Schur functor. There exists an essentially unique exact SM functor $\operatorname{Rep}(P) \to T$ carrying $V$ to $X$ and $\omega_V$ to $\omega_X$.

This result is proved using the explicit construction of Rep($P$) given in Section 6 and the techniques developed in \cite{EHS15}.

Furthermore, we show that the collection of categories $\langle \operatorname{Rep}(P), \operatorname{Rep}(p(n))\rangle_{n \geq 1}$ forms the “abelian envelope” of the Karoubian Deligne category $\Psi$, in the sense of \cite{Del07} \cite{EHS15}:

**Theorem 2.** Let $X \in T$ be a dualizable object with a non-degenerate symmetric form $\omega_X : X \otimes X \to \Pi_1$.

Consider the canonical SM functor $F_X : \Psi \to T$ sending the generator $\bar{V}$ of $\Psi$ to $X$ and the form $\omega_{\bar{V}}$ on $\bar{V}$ to $\omega_X$.

1. If $X$ is not annihilated by any Schur functor then $F_X$ factors through the embedding $I : \Psi \to \operatorname{Rep}(P)$ and gives rise to a faithful exact symmetric monoidal functor

$$F_X : \operatorname{Rep}(P) \to T,$$

$$(V \mapsto X, \omega_V \mapsto \omega_X).$$

2. If $X$ is annihilated by some Schur functor then there exists a unique $n \in \mathbb{Z}_+$ such that $F_X$ factors through the symmetric monoidal functor $\Psi \to \operatorname{Rep}(p(n))$ and gives rise to a faithful exact symmetric monoidal functor

$$F_X : \operatorname{Rep}(p(n)) \to T,$$

$$(C^n \mapsto X)$$

with the canonical form on $C^n$ sent to $\omega_X$.

This result is stated in stronger form in Theorem 8.2.1 and is proved in Section 8 using previous results on extension of functors from $\Psi$ to $\operatorname{Rep}(P)$ (see Theorem 7.0.2) and Tannakian formalism in tensor categories, discussed in Section 8 and Appendix 11.

As a corollary, we obtain an alternative construction of Rep($P$), inspired by \cite{Et14}. It is much shorter and more compact than the first construction of Rep($P$), although not as explicit.

For any $t \in \mathbb{C}$, consider the tensor Deligne category $\operatorname{Rep}(GL_t)$ constructed in \cite{EHS15}.

We denote its $t$-dimensional generator by $(X_t)_0$.

Let $\operatorname{Rep}(GL_t)$ be the corresponding module category over $sVect$, and consider the object

$$X_t := (X_t)_0 \oplus \Pi(X_t)_0^*$$

\footnote{In \cite{EHS15}, this category is denoted by $\mathcal{V}_t$; here we consider a “doubled” version $\operatorname{Rep}(GL_t) = sVect \otimes \mathcal{V}_t$ which is a module category over $sVect$.}
in Rep(\(GL_n\)).

The object \(X_t\) is not annihilated by any Schur functor, and comes equipped with a canonical non-degenerate symmetric form \(\omega_{X_t} : X_t \otimes X_t \to \Pi 1\).

By Theorem 2, we obtain an exact symmetric monoidal functor

\[
F : \text{Rep}(P) \to \text{Rep}(GL_n), \quad V \mapsto X_t, \quad \omega_V \to \omega_{X_t}.
\]

Let \(P(X_t)\) be the group scheme in \(\text{Rep}(GL_n)\) preserving the form \(\omega_{X_t}\): namely, we consider the universal commutative Hopf algebra object \(O(P(X_t)) \subset \text{Sym}(X_t \oplus \Pi X_t)\) in \(\text{Ind} - \text{Rep}(GL_n)\) whose action on \(X_t\) preserves \(\omega_{X_t}\). We then have:

**Corollary 3.** The functor \(F\) induces an equivalence of tensor categories

\[
\text{Rep}(P) \to \text{Rep}(P(X_t)).
\]

This result is proved in Section 8.3.

**Remark 1.5.1.** Let \(\text{End}(X_t) \cong X_t \otimes X_t^*\) be internal endomorphism algebra of \(X_t\), seen as a Lie algebra object in \(\text{aRep}(GL_n)\). Let \(p(X_t) \subset \text{End}(X_t)\) be Lie subalgebra preserving \(\omega_{X_t}\). Then \(p(X_t) = \text{Lie}(P(X_t))\), and \(\mathfrak{gl}((X_t)_0) \subset p(X_t)\). Unlike in the classical representation theory of Lie superalgebras, we do not expect that any representation of \(p(X_t)\) whose restriction to \(\mathfrak{gl}((X_t)_0)\) is integrable (that is, given by objects in \(\text{Rep}(GL_n)\)) integrates to a representation of the group scheme \(P(X_t)\). See Remark 8.3.4.

1.6. **Structure of the paper.** In Section 3, we present some results about the representation theory of the Lie superalgebra \(p(\infty)\) used in the construction of \(\text{Rep}(P)\).

In Section 4 we define full highest weight subcategories \(\text{Rep}^k(p(n)) \subset \text{Rep}(p(n))\), which will be the building blocks for \(\text{Rep}(P)\). In Section 5 we define the \(DS\) functor, and prove some stabilization results for the subcategories \(\text{Rep}^k(p(n))\) under \(DS\). In Section 6 we construct the category \(\text{Rep}(P)\) itself, and present some of its basic properties.

Sections 7 and 8 are devoted to the proofs of Theorems 1 and 2 above.

Section 8 uses heavily the terminology of affine algebraic groups in pre-Tannakian tensor categories; a short summary of the necessary definitions and results in this subject is given in Appendix 11.

Section 9 describes the lower highest weight structure on \(\text{Rep}(P)\).

Finally, in Appendix 10 we give a combinatorial proof of a result of [DLZ15] describing the a spanning set of the superspace \(\text{Hom}_{p(n)}(V_n^{\otimes k}, 1)\).

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2. **Preliminaries and notation**

2.1. **General.** Throughout this paper, we will work over the base field \(\mathbb{C}\), and all the categories considered will be \(\mathbb{C}\)-linear.

A **vector superspace** will be defined as a \(\mathbb{Z}/2\mathbb{Z}\)-graded vector space \(V = V_0 \oplus V_1\). The **parity** of a homogeneous vector \(v \in V\) will be denoted by \(p(v) \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}\) (whenever the notation \(p(v)\) appears in formulas, we always assume that \(v\) is homogeneous).
2.2. **Tensor categories.** In the context of symmetric monoidal (SM) categories, we will denote by \( \mathbb{1} \) the unit object, and by \( \sigma \) the symmetry morphisms.

A functor between symmetric monoidal categories will be called a **SM functor** if it respects the SM structure in the sense of [Del90, 2.7] (there it is called "foncteur ACU"); similarly, an \( \otimes \)-natural transformation between SM functors is a transformation respecting the monoidal structure in the sense of [Del90, 2.7].

**Remark 2.2.1.** Any natural transformation between SM functors on rigid SM categories is an isomorphism (see [DM82]).

In this paper a **tensor category** is a rigid SM abelian \( \mathbb{C} \)-linear category, where the bifunctor \( \otimes \) is bilinear on morphisms, and \( \text{End}(\mathbb{1}) \cong \mathbb{C} \). An explicit definition can be found in [Del90, EGNO15]. Note that in such a category the bifunctor \( \otimes \) is biexact.

A **tensor functor** between two tensor categories is an exact SM functor.

**Remark 2.2.2.** Any tensor functor is automatically faithful (see [DM82]).

As in [Del90, 2.12], a **pre-Tannakian category** is a tensor category satisfying finiteness conditions: namely, every object has finite length and every Hom-space is finite-dimensional over \( \mathbb{C} \).

Given an object \( V \) in a SM category, we will denote by \( \text{coev} : \mathbb{1} \to V \otimes V^* \), \( \text{ev} : V^* \otimes V \to \mathbb{1} \) the coevaluation and evaluation morphisms for \( V \). We will also denote by \( \mathfrak{gl}(V) := V \otimes V^* \) the internal endomorphism space with the obvious Lie algebra structure on it. The object \( V \) is then a module over the Lie algebra \( \mathfrak{gl}(V) \); we denote the action by \( \text{act} : \mathfrak{gl}(V) \otimes V \to V \). The notation \( \text{Rep}(\mathfrak{gl}(V)) \) will stand for the category of all integrable finite-dimensional super-representations of the Lie superalgebra \( \mathfrak{gl}(V) \), even if this algebra is purely even.

2.3. **Super structure on SM categories.** Let \( s\text{Vect} \) be the SM category of supervectorspaces and even morphisms. The categories appearing in this paper will be module categories over the SM category \( s\text{Vect} \): this essentially means that they will be equipped with a \( \mathbb{C} \)-linear endofunctor \( \Pi \) (exact if the category is abelian) and an isomorphism \( \Pi^2 \to \text{Id} \).

Our categories will be enriched over \( \text{Vect} \) but not over \( s\text{Vect} \) (for instance, we will only consider “even morphisms” in the categories of representation of superalgebras). On the few occasions when we would like to consider \( \mathbb{Z}/2\mathbb{Z} \)-graded Hom-spaces, we will denote these by

\[
 s\text{Hom}(X,Y) := \text{Hom}(X,Y) \oplus \text{Hom}(X,\Pi Y).
\]

Moreover, any monoidal structure considered on such categories will be compatible with the module structure over \( s\text{Vect} \), making them analogues of algebras over \( \mathbb{Z}/2\mathbb{Z} \). Such categories will be called \( s\text{Vect}\)-categories for short. All functors between \( s\text{Vect}\)-categories will respect the \( s\text{Vect}\)-module structure, and will be called \( s\text{Vect}\)-functors; these are functors commuting with the parity shift functors \( \Pi \). When all the categories involved are tensor \( s\text{Vect}\)-categories, we will drop the “\( s\text{Vect}\)-” and just use the term “tensor functor” for short.

Given any additive linear category \( \mathcal{T} \), we will denote by \( s\mathcal{T} \) the corresponding tensor \( s\text{Vect}\)-category

\[
 s\mathcal{T} := s\text{Vect} \boxtimes_{\text{Vect}} \mathcal{T}.
\]

This can be viewed as a special case of the Deligne tensor product of abelian categories, described in [Del90].

2.4. **The periplectic Lie superalgebra.**

\[ ^3\text{Caveat: this notation differs from [EHS15].} \]
2.4.1. **Definition of periplectic Lie superalgebra.** Let \( n \in \mathbb{Z}_{>0} \), and let \( V_n \) be an \((n|n)\)-dimensional vector superspace equipped with a non-degenerate odd symmetric form

\[
\beta : V_n \otimes V_n \rightarrow \mathbb{C}, \quad \beta(v, w) = \beta(w, v), \quad \text{and} \quad \beta(v, w) = 0 \text{ if } p(v) = p(w).
\]

Then \( \text{End}_\mathbb{C}(V_n) \) inherits the structure of a vector superspace from \( V_n \). We denote by \( \mathfrak{p}(n) \) the Lie superalgebra of all \( X \in \text{End}_\mathbb{C}(V_n) \) preserving \( \beta \), i.e. satisfying

\[
\beta(Xv, w) + (-1)^{p(X)p(v)} \beta(v, Xw) = 0.
\]

Let \( V_{0,n} \) be the even part of \( V_n \), and let \( V_{1,n} \) be the odd part of \( V_n \):

\[ V_n = V_{0,n} \oplus V_{1,n}. \]

The form \( \beta \) induces a non-degenerate pairing of the underlying vector spaces \( V_{0,n} \) and \( V_{1,n} \).

**Remark 2.4.1.** Choosing dual bases \( v_1, v_2, \ldots, v_n \) in \( V_{0,n} \) and \( v'_1, v'_2, \ldots, v'_n \) in \( V_{1,n} \), we can write the matrix of \( X \in \mathfrak{p}(n) \) as \((A \bigtriangledown B)\) where \( A, B \) are \( n \times n \) matrices such that \( B^t = B, C^t = -C \).

We will occasionally denote \( \mathfrak{g} = \mathfrak{p}(n) \), with triangular decomposition \( \mathfrak{g} \cong \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) where \( \mathfrak{g}_0 \cong \mathfrak{gl}(n), \mathfrak{g}_- \cong \Pi S^2 V_{1,n}, \mathfrak{g}_1 \cong \Pi \Lambda^2 V_{1,n} \).

Then the action of \( \mathfrak{g}_{\pm 1} \) on any \( \mathfrak{g} \)-module is \( \mathfrak{g}_0 \)-equivariant.

2.4.2. **Weights for the periplectic superalgebra.** The integral weight lattice for \( \mathfrak{p}(n) \) will be \( \mathbb{Z}^{2n} \).

* We fix a set of simple roots \(-\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_n - \varepsilon_{n-1}\), the first root is odd and all others are even.

Hence the dominant integral weights will be given by \( \lambda = \sum_i \lambda_i \varepsilon_i, \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \).

* We fix an order on the weights of \( \mathfrak{p}(n) \): for weights \( \mu, \lambda \), we say that \( \mu \geq \lambda \) if \( \mu_i \leq \lambda_i \) for each \( i \).

**Remark 2.4.2.** It was shown in \cite{BDE+16} Section 3.3 that if \( \leq \) corresponds to a highest weight structure on the category of finite-dimensional representations of \( \mathfrak{p}(n) \).

* The simple finite-dimensional representation of \( \mathfrak{p}(n) \) with dominant integral highest weight \( \lambda \) whose highest weight vector is even will be denoted by \( L_n(\lambda) \).

**Example 2.4.3.** The natural representation \( V_n \) of \( \mathfrak{p}(n) \) has highest weight \(-\varepsilon_1 \), with odd highest weight vector; hence \( V_n \cong \Pi \Lambda^2(n)(-\varepsilon_1) \). The representation \( \Lambda^2 V_n \) has highest weight \(-2\varepsilon_1 \), and the representation \( S^2 V_n \) has highest weight \(-\varepsilon_1 - \varepsilon_2 \) for \( n \geq 2 \); both have even highest weight vectors, and

\[ \Lambda^2 V_n \rightarrow L_n(-2\varepsilon_1), \quad L_n(-\varepsilon_1 - \varepsilon_2) \hookrightarrow S^2 V_n. \]

* Set \( \rho(n) = \sum_{i=1}^n (i - 1)\varepsilon_i \), and for any weight \( \lambda \), denote

\[ \bar{\lambda} = \lambda + \rho(n). \]

* We will associate to \( \lambda \) a weight diagram \( \Delta(\lambda) \), defined as a labeling of the integer line by symbols \( \bullet \) ("black ball") and \( \circ \) ("empty") such that such that \( j \) has label \( \bullet \) if \( j \in \{\bar{\lambda}_i | i = 1, 2, \ldots\} \), and label \( \circ \) otherwise.

* We denote \( |\lambda| = -\sum_i \lambda_i \).

* When \( \lambda \geq 0 \) and \( |\lambda| \leq k \), we say that \( \lambda \) is \( k \)-admissible. For such weights, we have:

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} = \lambda_{k+2} = \cdots = 0. \]

* Let \( k \geq 0 \). We denote by \( \lambda \) the Young diagram corresponding to \( \lambda \) (so the columns of \( \lambda \) have lengths \(-\lambda_1, -\lambda_2, \ldots\)); then the number of boxes in \( \lambda \) is \( |\lambda| \). By abuse of notation, \( \lambda \) will also stand for the corresponding irreducible representation of the symmetric group \( S_k \). We will also denote by \( \lambda^\top \) the transpose of the Young diagram \( \lambda \), and by \( \lambda^\Diamond \) the corresponding weight (hence

\[ \lambda^\top_1 = -2\{i : \lambda_i \leq -1\}, \quad \lambda^\Diamond_2 = -2\{i : \lambda_i \leq -2\} \]
and so forth). In terms of diagrams, \( d_{\lambda^\vee} \) is obtained from \( d_\lambda \) by reflecting the diagram with respect to the point \(-1/2\), and then reversing the colors in the diagram.

### 2.4.3. Representations of \( p(n) \)

We denote by \( \text{Rep}(p(n)) \) the category of finite-dimensional representations of \( p(V) \) whose restriction to \( p(V)_0 \cong gl(n) \) integrates to an action of \( GL(n) \).

By definition, the morphisms in \( \text{Rep}(p(n)) \) will be \textit{grading-preserving} \( p(V) \)-morphisms, i.e., \( \text{Hom}_{\text{Rep}(p(n))}(X, Y) \) is a vector space and not a vector superspace. This is important in order to ensure that the category \( \text{Rep}(p(n)) \) be abelian.

The category \( \text{Rep}(p(n)) \) is not semisimple. In fact, this category is a highest weight category, having simple, standard, costandard, and projective modules (these are also injective and tilting, per [BKN]). Given a simple module \( L_n(\lambda) \) in \( \text{Rep}(p(n)) \), we denote the corresponding standard, costandard, and projective modules by \( \Delta_n(\lambda), \nabla_n(\lambda), P_n(\lambda) \) respectively.

This paper heavily relies on the results in [BDE16]. In particular, we use the following two results throughout the paper:

**Theorem 2.4.4** (See [BDE16].) There exists a natural \( \mathbb{Z} \)-grading on the functor \( - \otimes V \cong \bigoplus_{j \in \mathbb{Z}} \Theta_j \), given by generalized eigenspaces of the tensor Casimir. The relations on the translation functors \( \Theta_j, j \in \mathbb{Z} \) induce a representation of the infinite Temperley-Lieb algebra \( TL_\infty(q = i) \) on the Grothendieck ring on \( \text{Rep}(p(n)) \).

**Theorem 2.4.5** (See [BDE16].) Indecomposable projectives in \( \text{Rep}(p(n)) \) satisfy the following properties:

1. Indecomposable projectives \( P_n(\lambda) \) are multiplicity-free for any \( \lambda \).
2. For any \( \lambda \) and any \( i \), \( \Theta_i P_n(\lambda) \) is indecomposable projective or zero.

For more details on the structure of \( \text{Rep}(p(n)) \) we refer the reader to [BDE16].

### 2.5. The Deligne category \( \text{Rep}(GL_t) \)

The Deligne categories \( \text{Rep}(GL_t) \) \( (t \in \mathbb{C}) \) are a family of tensor categories possessing a number of remarkable properties.

Such categories were constructed in [DMS2] for \( t \in \mathbb{Z} \), and in [EHS15] for \( t \in \mathbb{Z} \). We will denote the category constructed in [DMS2] [EHS15] by \( \text{Rep}_{\text{halved}}(GL_t) \), and by \( \text{Rep}(GL_t) \) the “doubled” Deligne category

\[
\text{Rep}(GL_t) := s\text{Vect} \otimes_{\text{vect}} \text{Rep}_{\text{halved}}(GL_t).
\]

The original motivation of [DMS2] was to construct an example of a tensor category which is not Tannakian, i.e. not the category of representations of any algebraic group (nor supergroup).

Below we give a short overview of the construction of \( \text{Rep}(GL_t) \).

#### 2.5.1. Tensor category \( \text{Rep}(GL_t) \)

Let \( t \in \mathbb{C} \). We begin with a category\(^4\) \( \mathcal{D}^0_t \) which is freely generated, as a \( \mathbb{C} \)-linear category, by one dualizable object of dimension \( t \) (we will denote it by \( V \) in this section).

The category \( \mathcal{D}^0_t \) can be embedded naturally into a Karoubian additive rigid symmetric monoidal category. The Karoubian additive envelope of \( \mathcal{D}^0_t \) will be denoted \( \text{Tilt}(GL_t) \), for reasons which will be explained later on.

Yet a priori it is not clear how to embed it into a tensor (abelian) category. For \( t \notin \mathbb{Z} \), the category \( \text{Tilt}(GL_t) \) is indeed abelian and even semisimple. We will set \( \text{Rep}(GL_t) := \text{Tilt}(GL_t) \) when \( t \notin \mathbb{Z} \).

For \( t \in \mathbb{Z} \), this is not the case. Yet it turns out the one can construct a tensor category \( \text{Rep}(GL_t) \) into which \( \text{Tilt}(GL_t) \) embeds as a full additive rigid SM subcategory, as was done in [EHS15].

The main idea behind the construction is using the Duflo-Serganova functors defined in [DS05]. This is a collection of SM functors, defined for each \( m, n \geq 1 \):

\[
DS_{x} : \text{Rep}(gl(m|n)) \to \text{Rep}(gl(m - 1|n - 1)).
\]

\(^4\)Deligne denotes this category by \( \text{Rep}_t(gl_t) \), see [Del02] Section 10; also known as the oriented Brauer category with parameter \( t \), see [BCNRT1].
Although the functors $DS_x$ are not exact, but are exact on certain subcategories of $\text{Rep}(\mathfrak{gl}(m|n))$. This allows us to construct a new tensor category $\text{Rep}(GL_t)$, $t := m - n$, together with a collection of SM functors $F_{m,n} : \text{Rep}(GL_t) \to \text{Rep}(\mathfrak{gl}(m|n))$ which are compatible with the functors $DS_x$. Note that the SM functors $F_{m,n}$ are not exact.

The category $\text{Rep}(GL_t)$ should be seen as an inverse limit of the system $(\text{Rep}(\mathfrak{gl}(m|n)), DS_x)$. 

**Remark 2.5.1.** The functors $F_{m,n}$ are “local” equivalences: the categories $\text{Rep}(GL_t)$, $\text{Rep}(\mathfrak{gl}(m|n))$ have natural $\mathbb{Z}_+$-filtrations on objects which are preserved by the functors $F_{m,n}$, and the latter induce an equivalence between each $\text{Rep}(GL_t)^k$ and $\text{Rep}^k(\mathfrak{gl}(m|n))$ for $m, n \gg k$.

The obtained category $\text{Rep}(GL_t)$ is a lower highest weight category: namely, for each $k \geq 0$, the subcategory $\text{Rep}(GL_t)^k$ is a highest weight category, whose standard, costandard and tilting objects play the same role in each $\text{Rep}(GL_t)^k$ for $k' \geq k$. The full subcategory of tilting objects in $\text{Rep}(GL_t)$ is then precisely $\text{Tilt}(GL_t)$.

Finally, we mention the “interpolation property” of $\text{Rep}(GL_t)$.

**Theorem 2.5.3 ([EHSL14]).** Let $C$ be a tensor category, and let $X$ be an object in $C$ of integral dimension $t$. Consider the canonical SM functor

$$F_X : \text{Tilt}(GL_t) \to C$$

 carrying the $t$-dimensional generator $V$ of $\text{Tilt}(GL_t)$ to $X$.

(a) If $X$ is not annihilated by any Schur functor then $F_X$ uniquely factors through the embedding $I : \text{Tilt}(GL_t) \to \text{Rep}(GL_t)$ and gives rise to an exact SM functor

$$\text{Rep}(GL_t) \to C.$$

(b) If $X$ is annihilated by some Schur functor then there exists a unique pair $m, n \in \mathbb{Z}_+$, $m - n = t$, such that $F_X$ factors through the SM functor $\text{Tilt}(GL_t) \to \text{Rep}(\mathfrak{gl}(m|n))$ and gives rise to an exact SM functor

$$\text{Rep}(\mathfrak{gl}(m|n)) \to C$$

 sending the standard representation $\mathbb{C}^{m|n}$ to $X$.

2.6. **The Deligne category $\mathcal{P}$.** Let $\mathcal{G}$ be a $\mathbb{C}$-linear category on two objects: $\mathcal{C}, \Pi \mathcal{C}$, both having one-dimensional endomorphism spaces and no morphisms between them. The category $\mathcal{G}$ has a rigid SM structure: considered as the full subcategory of $\text{sVect}$ whose objects are $\mathcal{C}, \Pi \mathcal{C}$, it inherits the SM structure on vector superspaces.

The Deligne category $\mathcal{P}_0$ is the universal SM $\mathcal{G}$-module category which is generated by an object $\hat{V}$ with a fixed symmetric non-degenerate form $\omega_{\hat{V}} : \hat{V} \otimes \hat{V} \to \Pi \mathcal{C}$. The objects in $\mathcal{P}_0$ are enumerated by pairs $[(n, \epsilon)]$ where $n \in \mathbb{Z}_{\geq 0}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z} = \{0,1\}$. In particular, $[(1,0)] = \hat{V}$. The endomorphism spaces in $\mathcal{P}_0$ described by the signed Brauer algebras (see e.g. [BDE+18, CE17b, KT14]), and the monoidal structure is given by

$$[(n, \epsilon)] \otimes [(n', \epsilon')] := [(n + n', \epsilon + \epsilon')].$$

**Example 2.6.1.** In this notation, $\mathcal{1} = [(0,0)]$ and $[(1,1)]$ is a generator (for any $\epsilon$).

The action of $\mathcal{G}$ given by $\Pi[(n, \epsilon)] = [(n, \mathcal{1} - \epsilon)]$.

A formal definition of this monoidal category via generators and relations can be found in [BDE+18, CE17b, KT14].

We will denote by $\mathcal{P}$ the Karoubian additive envelope of $\mathcal{P}_0$. For details on $\mathcal{P}$, see [CE17b].

---

5In [KT14], this category is called the marked Brauer category for $\delta = 0, \epsilon = -1$. 
Let $\mathfrak{p}(\infty)$ be the category of algebraic representations of the Lie superalgebra $\mathfrak{p}(\infty)$; for more details, see [Ser14]. We also prove some additional results which will be used later.

3.1. **Definition.** Let $V_\infty = \mathbb{C}^{\infty|\infty}$ be a countable-dimensional vector superspace, with even part $V_{0,\infty}$ and odd part $V_{1,\infty}$ (that is, $V_\infty = V_{0,\infty} \oplus V_{1,\infty}$). We fix a non-degenerate symmetric pairing form $\omega : V_\infty \otimes V_\infty \to \Pi \mathbb{C}$. Consider the Lie superalgebra $\mathfrak{p}(\infty) \subset \mathfrak{gl}(\infty|\infty)$ of finite-rank operators on $V_\infty$ preserving $\omega$.

For each $n \geq 1$, consider an orthogonal decomposition $V_\infty = V_n \oplus V_n^\perp$, where $V_n = \mathbb{C}^n|n$, such that $\omega$ restricts to non-degenerate forms on both $V_n$ and $V_n^\perp$. This decomposition induces an inclusion $\mathfrak{p}(n) \subset \mathfrak{p}(\infty)$ (in fact, $\mathfrak{p}(\infty) = \varinjlim_n \mathfrak{p}(n)$), and defines a Lie super subalgebra

$$\mathfrak{p}(n)^\perp = \mathfrak{p}(\infty) \cap \text{End}(V_n^\perp).$$

This subalgebra commutes with $\mathfrak{p}(n)$.

Let $\text{Rep}(\mathfrak{p}(\infty))$ be the abelian SM category of algebraic $\mathfrak{p}(\infty)$-modules: namely, modules occurring as subquotients of finite direct sums of tensor powers of $V_\infty$. This category is studied extensively in [Ser14]; see also [NSS].

**Remark 3.1.1.** Let $\text{Rep}(\mathfrak{sp}(\infty))$ be the category of algebraic super-representations of the Lie algebra $\mathfrak{sp}(\infty) = \bigcup_{n \geq 1} \mathfrak{sp}(2n)$. This category is equivalent to the category of algebraic super-representations of $\mathfrak{sp}(\infty|\infty)$ (see [Ser14]).

The category $\text{Rep}(\mathfrak{p}(\infty))$ is Koszul dual to $\text{Rep}(\mathfrak{sp}(\infty))$ and not equivalent to it, as stated in [Ser14] (cf. [NSS]). The Koszul duality can be seen as follows. Consider the category $\text{Rep}(\mathfrak{gl}(\infty|\infty))$ of algebraic super-representations of $\mathfrak{gl}(\infty)$ (see [PS11, Ser12]). This category is equivalent to the category of algebraic super-representations of $\mathfrak{gl}(\infty|\infty)$.

Let $\mathbb{C}^\infty$ be the defining representation of $\mathfrak{gl}(\infty)$, and $\mathbb{C}_\infty$ its restricted dual, with the obvious pairing $\mathbb{C}^\infty \otimes \mathbb{C}_\infty \to \mathbb{C}$.

Consider the object $V_\infty = \mathbb{C}^\infty \oplus \Pi \mathbb{C}_\infty^\ast$ in $\text{Rep}(\mathfrak{gl}(\infty))$.

Then the category $\text{Rep}(\mathfrak{p}(\infty))$ can be described as the category of finite-length $\mathfrak{gl}(\infty)$-equivariant (super-)modules over the Lie algebra object $\Pi S^2(V_\infty)$ in $\text{Rep}(\mathfrak{gl}(\infty))$.

In the same spirit, let $E = (\mathbb{C}^\infty \otimes \mathbb{C}_\infty^\ast) \oplus \Pi \mathbb{C}_\infty^\ast$, and consider a symplectic form on $E$ given by a symplectic form on the even part $\mathbb{C}^\infty \otimes \mathbb{C}_\infty$, and a symmetric form on the odd part $\mathbb{C}_\infty^\ast$.

---

**Remark 2.6.2.** Caveat: the definitions in [BE17, CE17b, KT14] produce a monoidal supercategory, namely a category enriched over $\text{sVect}$. In particular, the objects $[(n, \pm \varepsilon)]$ are identified for any $n$.

To pass from the supercategory $\tilde{\mathfrak{P}}$ to our version of $\mathfrak{P}$, we define $\text{Hom}(\mathfrak{P}([(n_1, \varepsilon_1)], [(n_2, \varepsilon_2)])$ to be the morphisms in $\tilde{\mathfrak{P}}$ between the corresponding objects $[n_1], [n_2]$ whose parity is $(-1)^{\varepsilon_1 + \varepsilon_2}$.

Finally, we define the functor $I_n : \mathfrak{P} \to \text{Rep}(\mathfrak{p}(n))$ to be the $C$-linear SM $\text{sVect}$-functor which sends the generator $\tilde{V}$ of $\mathfrak{P}$ to $V_n$, and the form $\omega_{\tilde{V}}$ to the form $\omega_n : V_n \otimes V_n \to \Pi \mathbb{C}$. Such a functor is unique up to a $\otimes$-isomorphism of $\text{sVect}$-functors.

**Lemma 2.6.3.** The functor $I_n$ is full.

This statement is a direct consequence of the following result:

**Proposition 2.6.4.** The $\mu_2$-graded space $\text{Hom}_{\mathfrak{p}(n)}(V_n^\otimes k, 1)$ is non-zero only when $k$ is even, and is spanned by morphisms given by partitioning the $k$ factors into (disjoint) pairs, and considering a tensor product of $k/2$ contraction maps $V_n^\otimes 2 \to \Pi \mathbb{C}$ on these pairs.

This result is proved in [DLZ15] using geometric methods. We present an alternative proof of their result in Appendix 10 (see Corollary 10.1.1), using translation functors and Temperley-Lieb relations between them.

3. **The infinite perplectic Lie superalgebra**

In this section we recall some facts about the category of algebraic representations of Lie superalgebra $\mathfrak{p}(\infty)$; for more details, see [Ser14]. We also prove some additional results which will be used later.

---

6The group of the automorphisms of $I_n$ is $\mathbb{Z}/2\mathbb{Z}$. 
The superalgebra preserving such a form is \( \mathfrak{spo}(\infty|\infty) \), whose algebraic representations can be described as follows: they form the category of finite-length \( \mathfrak{gl}(\infty) \)-equivariant (super-)modules over the Lie algebra object \( S^2(E) \) in \( \text{Rep}(\mathfrak{gl}(\infty)) \). Since \( E \cong V_\infty \), we can consider these as (super-)modules over the Lie algebra object \( S^2(V_\infty) \) in \( \text{Rep}(\mathfrak{gl}(\infty)) \).

The corresponding enveloping algebras of the Lie algebras \( \Pi S^2(V_\infty) \), \( S^2(V_\infty) \) will then be \( \wedge(S^2(V_\infty)) \) and \( \text{Sym}(S^2(V_\infty)) \). These are clearly Koszul dual.

We have a left-exact functor
\[
\Phi_n : \text{Rep}(\mathfrak{p}(\infty)) \longrightarrow \text{Rep}(\mathfrak{p}(n)), \ M \mapsto M^{p(n)}
\]
as well as an exact SM functor
\[
\text{Res} : \text{Rep}(\mathfrak{p}(\infty)) \longrightarrow \text{Rep}(\mathfrak{gl}(\infty))
\]
restricting to the even part \( \mathfrak{gl}(\infty) \) of \( \mathfrak{p}(\infty) \).

**Lemma 3.1.2.** The functor \( \Phi_n \) is a SM functor.

**Proof.** We need to check that
\[
M^{p(n)} \otimes K^{p(n)} \cong (M \otimes K)^{p(n)}
\]
for any \( M, K \). We clearly have an inclusion \( \subset \), so we only need to check that we have an isomorphism of vector spaces. Let us now forget about the action of \( \mathfrak{p}(n) \) and consider \((-)^{p(n)}\) as composition of functors
\[
\text{Res}_{p(n)}^{\mathfrak{p}(\infty)} : \text{Rep}(\mathfrak{p}(\infty)) \longrightarrow \text{Rep}(\mathfrak{p}(\infty))
\]
(notice that we use here that \( \mathfrak{p}(n)^{1} \cong \mathfrak{p}(\infty) \)) and invariants \( \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, -) \). Hence it is enough to check that the functor of taking invariants \( \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, -) \) in \( \text{Rep}(\mathfrak{p}(\infty)) \) is a SM \( \text{sVect} \)-functor.

We now use the fact that \( \mathbb{C} \) is a simple injective object in \( \text{Rep}(\mathfrak{p}(\infty)) \), as is any tensor power of \( V_\infty \) (these generate the subcategory of injective objects under taking \( \oplus \) and direct summands; see \cite{Scr14}). This allows us to write for any \( M, K \in \text{Rep}(\mathfrak{p}(\infty)) \)
\[
M = \mathbb{C}^{\oplus m} \oplus M', \ K = \mathbb{C}^{\oplus k} \oplus K'
\]
where
\[
\dim \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, M') = \dim \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, K') = 0
\]
Hence it remains to check that \( \dim \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, M' \otimes K') = 0 \).

Indeed, let \( I, J \) be direct sums of tensor powers of \( V_\infty \) such that \( M' \subset I, K' \subset J \). By (2), we may choose \( I, J \) which do not have \( \mathbb{C} \) as direct summands. Hence \( I \otimes J \) is also a direct sum of positive tensor powers of \( V_\infty \), which implies that
\[
\dim \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, M' \otimes K') = \dim \text{Hom}_{\mathfrak{p}(\infty)}(\mathbb{C}, I \otimes J) = 0.
\]
This completes the proof of the lemma.

\[\square\]

3.2. **Restriction functors.** Let \( p, q \geq 0 \) such that \( p + q = n \). Let \( W_{p,q} \subset V_n \) be an isotropic subspace of dimension \((p|q)\) and \( W'_{q,p} \) be another isotropic subspace of dimension \((q|p)\) such that \( V_n = W_{p,q} \oplus W'_{q,p} \). Note that the non-degenerate symmetric form \( \beta \) establishes an isomorphism \( W'_{p,q} \cong \text{H}W^*_{p,q} \).

Let
\[
\mathfrak{g}_{p,q} = \{ X \in \mathfrak{p}(n) \mid X(W_{p,q}) \subset W_{q,p}, X(W'_q) \subset W'_q \}.
\]

**Remark 3.2.1.** Choose a basis \( e_1, \ldots, e_n, e'_1, \ldots, e'_{n'} \) for \( V_n \) such that
\[
W_{p,q} = \text{span}\{e_1, \ldots, e_p, e_{(p+1)'}, \ldots, e_{n'}\}.
\]
Then \( X \in \mathfrak{g}_{p,q} \) would be of the form

\[
\begin{pmatrix}
A & B \\
-D & C \\
-A & D
\end{pmatrix}
\]

One can easily see that \( \mathfrak{g}_{p,q} \) is isomorphic to \( \mathfrak{gl}(p|q) = \mathfrak{gl}(W_{p,q}) \), where the isomorphism is defined by

\[
\mathfrak{gl}(p|q) \to \mathfrak{g}_{p,q}, \quad x \mapsto (x \oplus \Pi x^*) : W_{p,q} \oplus W'_{q,p} \to W_{p,q} \oplus W'_{q,p} = V_n.
\]

Consider the inclusion of Lie superalgebras \( \mathfrak{g}_{p,q} \simeq \mathfrak{gl}(p|q) \subset \mathfrak{p}(n) \). This induces a tensor \( \text{sVect}\)-functor

\[
\text{Res} : \text{Rep}(\mathfrak{p}(n)) \longrightarrow \text{Rep}(\mathfrak{gl}(p|q))
\]

Applying this functor to the standard representation \( V_n = \mathbb{C}^{n|n} \) of \( \mathfrak{p}(n) \), we have: \( \text{Res}(V) = \mathbb{C}^{p|q} \oplus \Pi(\mathbb{C}^{p|q})^* \).

**Lemma 3.2.2.** We have a natural isomorphism

\[
\begin{array}{ccc}
\text{Rep}(\mathfrak{p}(\infty)) & \xrightarrow{\text{Res}} & \text{Rep}(\mathfrak{gl}(\infty|\infty)) \\
\Phi_n \downarrow & & \Gamma_{p|q} \downarrow \\
\text{Rep}(\mathfrak{p}(n)) & \xrightarrow{\text{Res}} & \text{Rep}(\mathfrak{gl}(p|q))
\end{array}
\]

for any \( p+q = n \); here \( \Gamma_{p|q} : \text{Rep}(\mathfrak{gl}(\infty)) \to \text{Rep}(\mathfrak{gl}(p|q)) \) is the functor of invariants with respect to the corresponding Lie super subalgebra

\[
\mathfrak{gl}(p|q)^+ = \text{End} \left( \left( \mathbb{C}^{p|q} \right)^\perp \right) \subset \mathfrak{gl}(\infty|\infty).
\]

Until the end of this section, we will use the following shorthand:

**Notation 3.2.3.** We denote \( \tilde{\mathfrak{g}} = \mathfrak{p}(\infty) \), with triangular decomposition \( \tilde{\mathfrak{g}} \cong \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \) where

\[
\tilde{\mathfrak{g}}_0 \cong \mathfrak{gl}(\infty), \quad \tilde{\mathfrak{g}}_{-1} \cong \wedge^2 V_{1,\infty}, \quad \tilde{\mathfrak{g}}_1 \cong \Lambda^2 V_{0,\infty}.
\]

Then the action of \( \tilde{\mathfrak{g}}_{\pm 1} \) on any \( \tilde{\mathfrak{g}} \)-module is \( \tilde{\mathfrak{g}}_0 \)-equivariant.

We will also use the notation \( \mathfrak{g} := \mathfrak{p}(n) \), with triangular decomposition \( \mathfrak{g} \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \).

**Proof.** In the spirit of the proof of Lemma 3.1.2, it is enough to check this for \( p = q = 0 \): namely, it is enough to check that for any \( \tilde{\mathfrak{g}} \)-module \( M \) and any \( \tilde{\mathfrak{g}}_0 \)-map \( f : \tilde{\mathfrak{g}}_1 \to M \), this map is in fact a map of \( \tilde{\mathfrak{g}} \)-modules.

In particular, the \( \tilde{\mathfrak{g}}_1 \)-module structure on \( M \) will give a \( \tilde{\mathfrak{g}}_0 \)-equivariant map \( \tilde{f} : \tilde{\mathfrak{g}}_1 \otimes \mathbb{C} \to M \); the \( \tilde{\mathfrak{g}}_0 \)-module \( \tilde{\mathfrak{g}}_1 \) is irreducible, hence this map is either 0 (which means that the original map \( f \) was \( \tilde{\mathfrak{g}}_1 \)-equivariant, as required) or injective. In the latter case, we see that \( \tilde{\mathfrak{g}}_1 \) acts faithfully on the image of \( f \). Yet this contradicts the “large annihilator condition” given in [Ser14, Proposition 4.2] on the objects of \( \text{Rep}(\mathfrak{p}(\infty)) \): every vector in \( M \) must be annihilated by some finite-corank subalgebra of \( \tilde{\mathfrak{g}} \), and such a subalgebra must have a non-trivial intersection with \( \tilde{\mathfrak{g}}_1 \). Hence \( f \) was \( \tilde{\mathfrak{g}}_1 \)-equivariant, and similarly one shows that \( f \) is \( \tilde{\mathfrak{g}}_{-1} \)-equivariant as well. This proves the result of the lemma.

**Corollary 3.2.4.** We have a natural isomorphism

\[
\begin{array}{ccc}
\text{Rep}(\tilde{\mathfrak{g}}) & \xrightarrow{\text{Res}} & \text{Rep}(\tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_{\pm 1}) \\
\Phi_n \downarrow & & \Gamma_n \downarrow \\
\text{Rep}(\tilde{\mathfrak{g}}) & \xrightarrow{\text{Res}} & \text{Rep}(\tilde{\mathfrak{g}}_0)
\end{array}
\]

induced by the functors in Lemma 3.2.2.
The isomorphism classes of simple modules in $\text{Rep}(\mathfrak{p}(\infty))$ are parametrized (up to parity shift) by non-decreasing integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_n = 0$ for $n > 0$. Let $L_\infty(\lambda)$ denote the simple $\mathfrak{p}(\infty)$-representation corresponding to $\lambda$, with even highest weight vector.

Alternatively, these can be parametrized by partitions of arbitrary size, as done in [Ser14], with sequence $\lambda$ as above corresponding to Young diagram $\lambda$ (notation as in Section 2.4).

Let $\lambda^-$ denote the set of non-decreasing integer sequences $\mu = (\mu_1, \mu_2, \ldots)$ which differ from $\lambda$ by exactly one entry, and $\lambda^+$ denote the set of non-decreasing integer sequences $\mu = (\mu_1, \mu_2, \ldots)$ which differ from $\lambda$ by exactly one entry, and $\sum_i \lambda_i - \mu_i = -1$.

This parametrization of simples is non-orthodox, see examples below.

**Example 3.2.5.** The natural representation $V_\infty$ of $\mathfrak{p}(\infty)$ has highest weight $-\varepsilon_1$, with odd highest weight vector; hence

$$V_\infty \cong \Pi L_\infty(-\varepsilon_1) = \Pi L_\infty(\emptyset).$$

Similarly, we have

$$\wedge^2 V_\infty \cong L_\infty(2\varepsilon_1) = L_\infty(\square), \quad S^2 V_\infty \cong L_\infty(-\varepsilon_1 - \varepsilon_2) = L_\infty(\bullet).$$

The following lemma is proved in [Ser14]:

**Lemma 3.2.6.** For any $\lambda$ as above, we have a short exact sequence

$$0 \to \bigoplus_{\mu \in \Delta^-} \Pi L_\infty(\mu) \to L_\infty(\lambda) \otimes V_\infty \to \bigoplus_{\mu \in \Delta^+} L_\infty(\mu) \to 0.$$

**Lemma 3.2.7.** Every simple object in $\text{Rep}(\mathfrak{p}(\infty))$ occurs in the cosocle of some injective object.

**Proof.** Recall that the isomorphism classes of simple objects (up to parity shift) in $\text{Rep}(\mathfrak{p}(\infty))$ may be enumerated by Young diagrams of arbitrary size. For any Young diagram $\beta$, let $L_\infty(\beta)$ denote the corresponding simple object with even highest weight vector, and $Y(\beta)$ its injective hull.

We use Koszul duality between $\text{Rep}(\mathfrak{p}(\infty))$ and $\text{Rep}(\mathfrak{sp}(\infty))$ (see Remark 3.1.1) to compute the multiplicities of composition factors in the socle filtration of the indecomposable injective objects of $\text{Rep}(\mathfrak{p}(\infty))$. The injective resolutions of simple objects in $\text{Rep}(\mathfrak{p}(\infty))$ are given in [SS] 4.3.5, 4.3.9; this immediately implies that for any Young diagrams $\beta, \gamma$, and any $k \geq 0$, we have:

$$[\text{soc}^k Y(\gamma) : L_\infty(\beta)] = \sum_{\gamma \vdash k, \gamma \in \text{QSym}} N_{\gamma, \beta}^\gamma,$$

where $\text{QSym}$ is the set of all quasi-symmetric partitions (such that $\gamma_i^\gamma = \gamma_i - 1$) and $N_{\gamma, \beta}^\gamma$ is the Littlewood-Richardson coefficient.

Let $\beta$ be any Young diagram. We wish to find $\gamma$ such that $L_\infty(\beta) \subset \text{cosoc} Y(\gamma)$.

Take $\delta$ to be a rectangular partition of length $|\beta|$ and width $|\beta| + 1$, and let $\gamma = \delta + \delta$ be the Young diagram with $\gamma_i = \delta_i + \delta_i$ for any $i$.

Then $N_{\delta, \beta}^\gamma = 1$, so $[\text{soc}^{k} Y(\gamma) : L_\infty(\beta)] = 1$ for $k = |\delta|$.

Let us show that $\text{soc}^{k} Y(\gamma)$ is contained in the cosocle; that is, we want to show that for any $\alpha$ such that $|\alpha| < |\beta|$, we have $[Y(\gamma) : L_\infty(\alpha)] = 0$.

Indeed, assume that $N_{\delta, \beta}^\gamma \neq 0$ for some $\gamma \in \text{QSym}$. Then $\ell(\gamma) \leq \ell(\gamma) = \ell(\delta)$, where $\ell()$ denotes the length of the partition, that is $\ell(\gamma) = \gamma_i^\gamma$. Since $\gamma \in \text{QSym}$, we have

$$\gamma_1 = \ell(\gamma) + 1 = \ell(\delta) = 1 = \delta_1.$$

Hence $\gamma$ can be embedded into $\delta$. On the other hand, $|\gamma| = |\gamma| - |\alpha| > |\gamma| - |\beta| = |\delta|$. This is a contradiction, thus the required statement is proved. \[\square\]

---

7For any $\mu \in \Delta^\pm$, $\mu$ is obtained from $\lambda$ by adding or removing one box.
4. The subcategories \( \text{Rep}^k(\mathfrak{p}(n)) \)

4.1. Definition. Throughout this subsection, we will work with fixed \( n \geq 1 \), and will omit it from the notation.

**Definition 4.1.1.** Let \( \text{Rep}^k(\mathfrak{p}(n)) \subset \text{Rep}(\mathfrak{p}(n)) \) be the full subcategory of \( \mathfrak{p}(n) \)-modules occurring as subquotients in finite direct sums of \( V^k \) (or their parity shifts) for \( j = 0, \ldots, k \).

Clearly, \( \text{Rep}^k(\mathfrak{p}(n)) \) is an abelian subcategory of \( \text{Rep}(\mathfrak{p}(n)) \). Moreover, although it is not closed under \( \otimes \), it is closed under (tensor) duality: for any \( M \in \text{Rep}^k(\mathfrak{p}(n)) \), its dual \( M^* \) also belongs to \( \text{Rep}(\mathfrak{p}(n)) \).

**Remark 4.1.2.** By Corollary 8.1.12, the category \( \text{Rep}(\mathfrak{p}(n)) \) is a direct limit of the subcategories \( \text{Rep}^k(\mathfrak{p}(n)) \) when \( k \to \infty \).

**Lemma 4.1.3.** Let \( k < n + 2 \). The simple modules \( L_n(\lambda) \in \text{Rep}^k(\mathfrak{p}(n)) \) are precisely those for which \( \lambda \) is \( k \)-admissible.

**Proof.** We prove the statement by induction on \( k \in \mathbb{Z}_{\geq 0} \), with the trivial base case \( k = 0 \). Assume the claim is true for \( k \); we will prove it for \( k + 1 \).

First of all, we show that for any \((k + 1)\)-admissible \( \lambda \), the module \( L_n(\lambda) \) lies inside \( \text{Rep}^{k+1}(\mathfrak{p}(n)) \). Indeed, we only need to check this for weights \( \lambda \) such that \( |\lambda| = k + 1, \lambda \geq 0 \), in which case this follows directly from the construction above.

Now, we need to check that for any \( k \)-admissible \( \lambda \) and any subquotient \( L_n(\mu) \) of \( \mathfrak{p}(n) \), its highest weight \( \mu \) is \((k + 1)\)-admissible.

**Remark 4.1.4.** The category \( \text{Rep}^k(\mathfrak{p}(n)) \) has enough projective and injective objects.

**Proof.** Consider the inclusion functor \( i : \text{Rep}^k(\mathfrak{p}(n)) \to \text{Rep}(\mathfrak{p}(n)) \). This functor is exact and has a left adjoint \( i^* \) and a right adjoint \( i^! \). Then \( i \) takes projective objects to projective objects, hence \( i^*(P_n(\lambda)) \) is projective in \( \text{Rep}^k(\mathfrak{p}(n)) \) for any \( \lambda \). Furthermore, \( i^*(P_n(\lambda)) \to L(\lambda) \) for any \( L(\lambda) \in \text{Rep}^k(\mathfrak{p}(n)) \), which means that there are enough projective objects in \( \text{Rep}^k(\mathfrak{p}(n)) \).

4.2. Standard objects in \( \text{Rep}^k(\mathfrak{p}(n)) \). We now describe the standard (highest weight) objects in the category \( \text{Rep}^k(\mathfrak{p}(n)) \).

Let \( k \leq n \).
Consider \( V_{1,n}^\otimes k \subset V_n^\otimes k \). Then \( g_{-1} \) acts on \( V_{1,n}^\otimes k \) by zero. Consider the \( GL(V_{0,n}) \times S_k \) decomposition

\[
V_{1,n}^\otimes k \cong \bigoplus_{|\lambda|=k} S^\lambda V_{1,n} \otimes \lambda \cong \Pi^k \bigoplus_{|\lambda|=k} S^\lambda V_{0,n}^* \otimes \lambda.
\]

By definition of \( \Delta(\lambda) \) (see BDE\textsuperscript{16} Section 3.1), we have a map of \( p(n) \otimes \mathbb{C}[S_k] \)-modules

\[
\bigoplus_{|\lambda|=k} \Delta_n(\lambda) \otimes \lambda^V \to \Pi^k V_n^\otimes k
\]

which is non-zero on each of the summands.

**Definition 4.2.1.** Let \( \lambda \geq 0, |\lambda| = k \). We denote the image of the map

\[
\Delta_n(\lambda) \longrightarrow \Pi^k S^\lambda V_n
\]

by \( \overline{\Delta}_n^k(\lambda) \).

More generally, for any \( \lambda \geq 0, |\lambda| \leq k \), we denote

\[
\overline{\Delta}_n^k(\lambda) := \overline{\Delta}_n^{|\lambda|}(\lambda).
\]

Similarly, we define the objects \( \nabla_n^k(\lambda) \):

Consider the \( g_0 \oplus g_1 \)-map \( V_n^\otimes k \to V_{1,n}^\otimes k \) with \( g_1 \) acting trivially on the latter. As before, we have an isomorphism of \( GL(V_{0,n}) \times S_k \)-modules

\[
V_{1,n}^\otimes k \cong \Pi^k \bigoplus_{|\lambda|=k} S^\lambda V_{0,n}^* \otimes \lambda.
\]

Hence we have a map of \( p(n) \otimes \mathbb{C}[S_k] \)-modules

\[
\bigoplus_{|\lambda|=k} V_n^\otimes k \to \Pi^k \nabla_n(\lambda) \otimes \lambda^V
\]

which is non-zero on each of the summands.

**Definition 4.2.2.** Let \( \lambda \geq 0, |\lambda| = k \). We denote the image of the map

\[
\Pi^k S^\lambda V_n \to \nabla_n(\lambda)
\]

by \( \nabla_n^k(\lambda) \).

More generally, for any \( \lambda \geq 0, |\lambda| \leq k \), we denote

\[
\nabla_n^k(\lambda) := \nabla_n^{|\lambda|}(\lambda).
\]

Clearly the highest weight module \( \overline{\nabla}_n(\lambda) \) lies in \( \text{Rep}^k(p(n)) \), and so does its simple head \( L_n(\lambda) \). The latter is also the socle of \( \nabla_n^k(\lambda) \).

4.3. **Connection to representations of** \( p(\infty) \). Let \( \text{Rep}^k(p(\infty)) \) be defined as the full subcategory of \( \text{Rep}(p(\infty)) \) whose objects occur as subquotients in finite direct sums of \( \mathbb{1}, V_\infty, V_\infty^2, \ldots, V_\infty^\otimes k \).

Clearly, for any \( n \geq 1 \), \( \Phi_n(\text{Rep}^k(p(\infty))) \subset \text{Rep}^k(p(n)) \).

**Lemma 4.3.1.** For any \( n > 2k, \geq 0 \), the restriction of \( \Phi_n \) to \( \text{Rep}^k(p(\infty)) \) is exact and faithful.

**Proof.** The functor

\[
\text{Res} : \text{Rep}(p(\infty)) \to \text{Rep}(gl(\infty))
\]

is an exact monoidal functor taking \( V_\infty \) to \( \mathbb{C}^\infty \otimes \Pi \mathbb{C}^\infty \), where \( \mathbb{C}^\infty \) is the tensor generator of \( \text{Rep}(gl(\infty)) \). Hence the image of \( \text{Rep}^k(p(\infty)) \) under \( \text{Res} \) lies in \( \text{Rep}^k(gl(\infty)) \), which is the full subcategory whose objects are (up to change of parity) subquotients of finite direct sums of \( (\mathbb{C}^\infty)^\otimes r \otimes (\mathbb{C}^\infty)^\otimes s \) for \( r + s \leq k \).
Let \( p = \left\lfloor \frac{n}{2} \right\rfloor \) and let \( q = n - p \). Consider the monoidal functor \( \Gamma_{p/q} : s\text{Rep}(\mathfrak{gl}(\infty)) \to \text{Rep}(\mathfrak{gl}(p/q)) \). This functor is exact on the subcategory \( \text{Rep}^k(\mathfrak{gl}(\infty)) \) when \( k < \min(p, q) \), i.e. when \( 2k < n \) \( \text{see } \) [EHS15, Theorem 6.1.3], and hence we have a natural isomorphism

\[
\begin{array}{ccc}
\text{Rep}^k(\mathfrak{gl}(\infty)) & \xrightarrow{\text{Res}} & \text{Rep}(\mathfrak{gl}(p/q)) \\
\Phi_n & \xrightarrow{\text{Res}} & \Gamma_n \\
\text{Rep}(\mathfrak{p}(n)) & \xrightarrow{\text{Res}} & \text{Rep}(\mathfrak{gl}(p/q))
\end{array}
\]

where the functors are all exact and faithful except perhaps \( \Phi_n : \text{Rep}^k(\mathfrak{p}(\infty)) \to \text{Rep}(\mathfrak{p}(n)) \), which is known to be left exact. Hence the latter is exact and faithful as well. \( \square \)

Throughout the paper, we will use the following notation:

**Notation 4.3.2.** A \( \mathfrak{p}(n) \)-weight \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) such that \( \lambda \geq 0 \) will be interpreted as a weight for \( \mathfrak{p}(\infty) \) by considering the infinite sequence \( (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots) \) (adding an infinite tail of zeroes). The latter sequence will also be denoted \( \lambda \).

For \( \lambda \not\geq 0 \), we set \( L_{\infty}(\lambda) := 0 \).

**Proposition 4.3.3.** Let \( n > 2k \geq 0 \), and let \( \lambda \) be a \( k \)-admissible weight for \( \mathfrak{p}(n) \). Consider the corresponding simple module \( L_{\infty}(\lambda) \in \text{Rep}^k(\mathfrak{p}(\infty)) \). Then

\[
\Phi_n(L_{\infty}(\lambda)) \cong \overline{\Delta}^k_n(\lambda).
\]

**Proof.** It is enough to check this statement for \( \lambda \) such that \( k = |\lambda| \).

Recall Notation 3.2.3. By [Ser14, Lemma 17] we have:

\[
L_{\infty}(\lambda) = \bigcap_{\psi \in \text{Hom}_{\mathfrak{p}(\infty)}(V^\otimes_k, V^\otimes_{k-2})} \text{Ker}\left( \psi|_{S^{\lambda}\lambda V_{\infty}} \right)
\]

Since \( \Phi_n \) is exact on \( \text{Rep}^k(\mathfrak{p}(\infty)) \), we have:

\[
\Phi_n(L_{\infty}(\lambda)) = \bigcap_{\psi \in \text{Hom}_{\mathfrak{p}(\infty)}(V^\otimes_k, V^\otimes_{k-2})} \text{Ker}\left( \Phi_n(\psi)|_{S^{\lambda}\lambda V_n} \right)
\]

Next, consider the map \( f : \Delta_n(\lambda) \to V^\otimes_k \) described in Definition 4.2.1. For any \( s < k \), we have:

\[
\text{Hom}_{\mathfrak{p}(n)}(\Delta_n(\lambda), V^\otimes_{s}) = \text{Hom}_{\mathfrak{g}(\mathfrak{g})/\mathfrak{g}-1}(S^{\lambda}\lambda V_{1,n}, R_{\mathfrak{g}(\mathfrak{g})/\mathfrak{g}-1}^s V^\otimes_{s}) = 0
\]

Hence \( \text{Im}(f) \subset \text{Ker}(\psi) \) for any \( \psi \in \text{Hom}_{\mathfrak{p}(n)}(V^\otimes_k, V^\otimes_{s}) \). Thus \( \overline{\Delta}^k_n(\lambda) \subset \Phi_n(L_{\infty}(\lambda)) \).

We now consider the surjective map of \( \mathfrak{g} \)-modules

\[
\psi : U(\tilde{\mathfrak{g}}_1) \otimes_{U(\tilde{\mathfrak{g}}_1)} S^{\lambda}\lambda V_{1,\infty} \to L_{\infty}(\lambda)
\]

as in [Ser14, Section 4.5]. As a \( \tilde{\mathfrak{g}}_0 \)-map, this can be written as

\[
\psi : \Lambda(\tilde{\mathfrak{g}}_1) \otimes S^{\lambda}\lambda V_{1,\infty} \to L_{\infty}(\lambda), \quad \forall x \in \tilde{\mathfrak{g}}_1, v \in S^{\lambda}\lambda V_1, \ x \otimes v \mapsto x.v
\]

Furthermore, we have:

\[
L_{\infty}(\lambda) \subset V^\otimes_k
\]

and so

\[
\Lambda^\leq_k(\tilde{\mathfrak{g}}_1) \otimes S^{\lambda}\lambda V_{1,\infty} \subset \text{Ker}(\psi)
\]

This implies that the map \( \psi \) restricts to the surjective map of modules in \( \text{Rep}(\tilde{\mathfrak{g}}_0) \):

\[
\psi : \Lambda^\leq_k(\tilde{\mathfrak{g}}_1) \otimes S^{\lambda}\lambda V_{1,\infty} \to L_{\infty}(\lambda), \quad \forall x \in \tilde{\mathfrak{g}}_1, v \in S^{\lambda}\lambda V_{1,\infty}, \ x \otimes v \mapsto x.v
\]

When we take \( \mathfrak{p}(n) \)-invariants in the above picture, we obtain a surjective map

\[
\Lambda^\leq_k(\mathfrak{g}_1) \otimes S^{\lambda}\lambda V_{n,1} \to L_{\infty}(\lambda)|_{\mathfrak{p}(n)-} = \Phi_n(L_{\infty}(\lambda)), \quad \forall x \in \mathfrak{g}_1, v \in S^{\lambda}\lambda V_{n,1}, \ x \otimes v \mapsto x.v
\]

Hence the map \( f : \Delta_n(\lambda) \to \Phi_n(L_{\infty}(\lambda)) \) is surjective, and thus \( \overline{\Delta}^k_n(\lambda) \cong \Phi_n(L_{\infty}(\lambda)) \), as required. \( \square \)
Similarly, we have:

**Proposition 4.3.4.** Let \( n > 2k \geq 0 \), and let \( \lambda \) be a \( k \)-admissible weight for \( p(n) \). Then

\[
\Phi_n(L_\infty(\lambda))^* \cong \Pi^\lambda \nabla_n^k(\lambda')
\]

with the weight \( \lambda' \) interpreted as in Section 2.4.2.

**Proof.** Clearly, it is enough to prove the statement for \( k := |\lambda| \).

Recall that the module \( \Phi_n(L_\infty(\lambda))^* \) is a quotient of \( \Pi^k S_{\lambda'} \nabla_n^k \cong S_{\lambda'} \nabla_n^k \).

Let \( f : S_{\lambda'} \nabla_n^k \rightarrow \Pi^k \nabla_n^k(\lambda') \) be the map described in Section 4.2.

For any \( s < k \), we have:

\[
\text{Hom}_{p(n)}(V_n^{\otimes s}, \Pi^k \nabla_n^k(\lambda')) = \text{Hom}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}(Res^{p(n)}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1} V_n^{\otimes s}, S_{\lambda'} \nabla_n^k) = 0
\]

Hence \( \psi \circ f = 0 \) for any \( \psi \in \text{Hom}_{p(n)}(V_n^{\otimes k}, V_n^{\otimes s}) \), and thus we have a map

\[
\Phi_n(L_\infty(\lambda))^* \rightarrow \Pi^k \nabla_n^k(\lambda) \hookrightarrow \Pi^k \nabla_n^k(\lambda').
\]

Now, to prove the required statement, we only need to show that \( \Phi_n(L_\infty(\lambda))^* \) is generated by a lowest weight vector with respect to the usual Borel in \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \). In other words, we would like to show that \( \Phi_n(L_\infty(\lambda)) \) is a highest weight module with respect to the same Borel. This is a direct consequence of [Ser14, Section 4.5], inferred as in the proof of Proposition 4.3.3. \( \Box \)

### 4.4. Further properties

We now give some immediate corollaries of Propositions 4.3.3 and 4.3.4.

**Corollary 4.4.1.** Let \( n > 2k \geq 0 \), and let \( \lambda \) be a \( k \)-admissible weight for \( p(n) \). Then

\[
\Delta_n^k(\lambda)^* \cong \Pi^\lambda \nabla_n^k(\lambda').
\]

**Corollary 4.4.2.** Let \( n > 2k \), and \( \lambda \) be a \( k \)-admissible weight such \( |\lambda| \leq k - 1 \). Then we have short exact sequences

\[
0 \rightarrow \bigoplus_{\mu \in \{\lambda-\epsilon_i | i \geq 1\}, \mu \text{ is } k\text{-admissible}} \Pi \Delta_n^k(\mu) \rightarrow \Delta_n^k(\lambda) \otimes V_n \rightarrow \bigoplus_{\mu \in \{\lambda+\epsilon_i | i \geq 1\}, \mu \text{ is } k\text{-admissible}} \Delta_n^k(\mu) \rightarrow 0.
\]

**Proof.** This is a direct consequence of Lemma 3.2.6 together with the fact that \( \Phi_n \) is a SM functor and is exact on \( \text{Rep}^k(p(\infty)) \). \( \Box \)

Similarly, we have:

**Corollary 4.4.3.** Let \( n > 2k + 2 \), \( k \geq 0 \), and \( \lambda \) be a \( k \)-admissible weight such \( |\lambda| \leq k - 1 \). Then we have short exact sequences

\[
0 \rightarrow \bigoplus_{\mu \in \{\lambda+\epsilon_i | i \geq 1\}, \mu \text{ is } k\text{-admissible}} \nabla_n^k(\mu) \rightarrow \nabla_n^k(\lambda) \otimes V_n \rightarrow \bigoplus_{\mu \in \{\lambda-\epsilon_i | i \geq 1\}, \mu \text{ is } k\text{-admissible}} \nabla_n^k(\mu) \rightarrow 0.
\]

The following statement is the analogue of [EHS15, Lemma 6.2.4]:

**Proposition 4.4.4.** Let \( n > 2k + 2 \), \( k \geq 0 \), and \( \lambda \) be a \( k \)-admissible weight, and \( \mu \geq 0 \). Then

\[
[\Delta_n(\lambda) : L_n(\mu)] = [\Delta_n^k(\lambda) : L_n(\mu)].
\]

**Proof.** For any module \( M \in \text{Rep}(p(n)) \), let \([M]\) denote the corresponding element of the reduced\(^8\) Grothendieck group of \( \text{Rep}(p(n)) \). We wish to prove that

\[
[\Delta_n(\lambda)] - [\Delta_n^k(\lambda)] \in \text{span}_{\mathbb{Z}_+} \{[L_n(\nu)] | \nu \geq 0\}.
\]

\(^8\)The reduced Grothendieck group (resp. ring) of an \( \mathbf{Vect} \) category is the quotient of the usual Grothendieck group (resp. ring) by the subspace (resp. ideal) generated by \([X] - [\Pi X]\) for every object \([X]\).
We prove this statement by induction on $|\lambda|$. The base of the induction is $\lambda = 0$, in which case $\Delta_k^k(\lambda) = \mathbb{C}$. Furthermore, by [BDE+16] Theorem 6.3.1 all composition factors $L_n(\nu)$ of $\Delta_n(0)$ satisfy $\nu \not\geq 0$ except $\nu = 0$; this implies the required statement for $\lambda = 0$.

We now proceed to the step of induction. Assume the statement is true for all $\lambda$ such that $|\lambda| = s$. Then for each such $\lambda$, we have:

$$[\Delta_n(\lambda) \otimes V_n] = [\Delta_k^k(\lambda) \otimes V_n] \in \text{span}_{\mathbb{Z}_+} \{[L_n(\nu) \otimes V_n]|\nu \not\geq 0\}.$$  

**Sublemma 4.4.5.** For any $\tau \not\geq 0$, we have

$$[L_n(\tau) \otimes V_n] \in \text{span}_{\mathbb{Z}_+} \{[L_n(\nu)]|\nu \not\geq 0\}.$$  

**Proof.** Assume we have $[L_n(\tau) \otimes V_n : L_n(\nu)] > 0$ for some $\nu \geq 0$. Then

$$\dim \text{Hom}(P_n(\nu) \otimes V_n, L_n(\tau)) = \dim \text{Hom}(P_n(\nu), L_n(\tau) \otimes V_n) > 0$$

and hence $P_n(\nu) \otimes V_n$ has a direct summand $P_n(\tau)$. But the translation rules for indecomposable projectives [BDE+16] Section 7] show that for any direct summand $P_n(\tau')$ of $P_n(\nu) \otimes V_n$, we would have: $\tau' \geq \nu$ and hence $\tau' > 0$. Contradiction. \hfill $\square$

Hence

$$[\Delta_n(\lambda) \otimes V_n] - [\Delta_k^k(\lambda) \otimes V_n] \in \text{span}_{\mathbb{Z}_+} \{[L_n(\nu)]|\nu \not\geq 0\}.$$  

Next, recall from [BDE+16] Section 4.3] that we have short exact sequences

$$0 \to \bigoplus_{\mu \in \{\lambda - \varepsilon_i| i \geq 1\}} \Pi \Delta_n(\mu) \longrightarrow \Delta_n(\lambda) \otimes V_n \longrightarrow \bigoplus_{\mu \in \{\lambda + \varepsilon_i| i \geq 1\}} \Delta_n(\mu) \to 0.$$

Using these, as well as Corollary 4.4.2 we conclude that

$$[\Delta_n(\lambda) \otimes V_n] - [\Delta_k^k(\lambda) \otimes V_n] = \sum_{\mu \in \{\lambda \pm \varepsilon_i| i \geq 1\}} [\Delta_n(\mu)] - [\Delta_k^k(\mu)]$$

where we use the convention $\Delta_k^k(\mu) = 0$ for non-$k$-admissible $\mu$.

The statement now follows for any $\mu \in \{\lambda \pm \varepsilon_i| i \geq 1\}$; since we started with an arbitrary $k$-admissible $\lambda$ such that $|\lambda| = s$, the above argument shows that the statement holds for any $\mu$ such that $|\mu| = s + 1$. This completes the proof of the lemma. \hfill $\square$

**Corollary 4.4.6.** Let $n > 2k + 2$, $k \geq 0$, and $\lambda$ be a $k$-admissible weight. Then $\Delta_k^k(\lambda)$ is the maximal quotient of $\Delta_n(\lambda)$ belonging to $\text{Rep}^k(p(n))$.

An analogous statement holds for $\nabla_k^k(\lambda)$ as well.

Finally, we compute the extensions between $\Delta_k^k(\lambda)$ and $\nabla_k^k(\mu)$. This will be used later in Section 5.2.

**Lemma 4.4.7.** Let $n > 2k + 2$, $k \geq 0$, and $\lambda$, $\mu$ be $k$-admissible weights. Then

$$\dim \text{Hom}_{p(n)}(\Delta_k^k(\lambda), \nabla_k^k(\mu)) = \delta_{\lambda,\mu}$$

and

$$\text{Ext}_{p(n)}^1(\Delta_k^k(\lambda), \nabla_k^k(\mu)) = 0.$$  

**Proof.** Let

$$K := \text{Ker} \left( \Delta_n(\lambda) \to \Delta_k^k(\lambda) \right) \quad C := \text{Coker} \left( \nabla_k^k(\mu) \to \nabla_n(\mu) \right).$$

We have long exact sequences

$$0 \to \text{Hom}_{p(n)}(\Delta_k^k(\lambda), \nabla_k^k(\mu)) \to \text{Hom}_{p(n)}(\Delta_n(\lambda), \nabla_k^k(\mu)) \to \text{Hom}_{p(n)}(K, \nabla_k^k(\mu)) \to$$

$$\text{Ext}_{p(n)}^1(\Delta_k^k(\lambda), \nabla_k^k(\mu)) \to \text{Ext}_{p(n)}^1(\Delta_n(\lambda), \nabla_k^k(\mu)) \to \text{Ext}_{p(n)}^1(K, \nabla_k^k(\mu)) \to \ldots$$
and

\[ 0 \to \text{Hom}_{p(n)} \left( \Delta_n(\lambda), \nabla_n^k(\mu) \right) \to \text{Hom}_{p(n)} \left( \Delta_n(\lambda), \nabla_n(\mu) \right) \to \text{Hom}_{p(n)} \left( \Delta_n(\lambda), C \right) \]

\[ \to \text{Ext}^1_{p(n)} \left( \Delta_n(\lambda), \nabla_n^k(\mu) \right) \to \text{Ext}^1_{p(n)} \left( \Delta_n(\lambda), \nabla_n(\mu) \right) \to \text{Ext}^1_{p(n)} \left( \Delta_n(\lambda), C \right) \to \ldots \]

First, we use the fact that \( \text{Hom}_{p(n)} \left( \Delta_n(\lambda), \nabla_n(\mu) \right) = \delta_{\lambda,\mu} \) with the image of \( \Delta_n(\lambda) \to \nabla_n(\lambda) \) being \( L_n(\lambda) \).

Secondly, Corollary [1.4.6] implies that \((C : L_n(\lambda)) = (K : L_n(\mu)) = 0\) so \( \text{Hom}_{p(n)} \left( \Delta_n(\lambda), C \right) = 0 \), \( \text{Hom}_{p(n)} \left( K, \nabla_n^k(\mu) \right) = 0 \).

Substituting these into the long exact sequences above, we conclude that

\[ \dim \text{Hom}_{p(n)} \left( \Delta_n^k(\lambda), \nabla_n^k(\mu) \right) = \delta_{\lambda,\mu}. \]

Next, we recall that

\[ \text{Ext}^1_{p(n)} \left( \Delta_n(\lambda), \nabla_n(\mu) \right) = 0. \]

The second long exact sequence then implies that \( \text{Ext}^1_{p(n)} \left( \Delta_n(\lambda), \nabla_n^k(\mu) \right) = 0 \) and hence

\[ \text{Ext}^1_{p(n)} \left( \Delta_n^k(\lambda), \nabla_n^k(\mu) \right) = 0. \]

\[ \square \]

5. The DS functor

5.1. Definition and basic properties. Let \( p, q > 0 \). We continue with the notation of Section 3.2. Let \( y \in \mathfrak{gl}(p|q) \) be an odd element of rank 1 such that \( y(W_{p,q}) \subset (W_{p,q})_0 \) and \( x = (y, \Pi y^*) \in \mathfrak{g}_{p,q} \subset \mathfrak{p}(n) \) be the corresponding odd element.

Let \( M \in \text{Rep}(\mathfrak{p}(n)) \). We define

\[ DS_x(M) = \frac{\text{Ker}(x|M)}{\text{Im}(x|M)}, \quad \text{and} \quad DS_y(M) = \frac{\text{Ker}(y|M)}{\text{Im}(y|M)}. \]

Lemma 5.1.1. We have an isomorphism of Lie superalgebras \( DS_x(\mathfrak{p}(n)) \cong \mathfrak{p}(n - 2) \) and \( DS_y(\mathfrak{gl}(p|q)) \cong \mathfrak{gl}(p - 1|q - 1) \).

Proof. The first statement follows from the fact that \( DS_x(M \otimes N) \cong DS_x(M) \otimes DS_x(N) \) as supervector spaces (see [Ser11a]). This means that \( DS_x \) defines a SM \text{sVect}-functor \( DS_x : \text{Rep}(\mathfrak{p}(n)) \to \text{sVect} \); one immediately sees that \( DS_x(V_n) \) is an \((n - 2|n - 2)\)-dimensional superspace with an (induced) odd symmetric form, and hence \( DS_x(\mathfrak{p}(n)) \) is a superalgebra preserving that form.

To prove that it is isomorphic to \( \mathfrak{p}(n - 2) \), we use the fact that \( \mathfrak{p}(n) \cong \bigwedge^2 V_n \), and hence

\[ DS_x(\mathfrak{p}(n)) \cong \bigwedge^{2} \mathbb{C}^{n-2|n-2} \cong \mathfrak{p}(n - 2) \]

as supervector spaces.

The second statement can be proved analogously (see also [DS05]).

\[ \square \]

Hence we obtain functors

\[ DS_x : \text{Rep}(\mathfrak{p}(n)) \to \text{Rep}(\mathfrak{p}(n - 2)) \]

and

\[ DS_y : \text{Rep}(\mathfrak{gl}(p|q)) \to \text{Rep}(\mathfrak{gl}(p - 1|q - 1)). \]

These functors are again SM \text{sVect}-functors.
Lemma 5.1.2. The functors $DS_x$ commute with the functors $Res$; that is, we have a natural isomorphism:

$$\begin{align*}
\text{Rep}(\mathfrak{p}(n)) & \xrightarrow{\text{Res}} \text{Rep}(\mathfrak{gl}(p|q)) \\
\text{Rep}(\mathfrak{p}(n-2)) & \xrightarrow{\text{Res}} \text{Rep}(\mathfrak{gl}(p-1|q-1))
\end{align*}$$

\text{Proof.} This follows directly from the fact that $y$ goes to $x$ under the inclusion $\mathfrak{gl}(p|q) \subset \mathfrak{p}(n)$. □

Finally, we state the following result. Although it is straightforward, will be very useful later on.

Lemma 5.1.3. Let $\mathfrak{g}$ be $\mathfrak{p}(n)$ or $\mathfrak{gl}(p|q)$, $z$ be $x$, $y$ as above, respectively, and $DS_z$ be $DS_x$ or $DS_y$ respectively. Consider the adjoint representation of $\mathfrak{g}$ (which we denote by $\mathfrak{g}$ as well). This is a Lie algebra object in $\text{Rep}(\mathfrak{g})$, and any $M \in \text{Rep}(\mathfrak{g})$ has a natural module structure over it (in $\text{Rep}(\mathfrak{g})$), given by a natural transformation $\text{act}^\mathfrak{g} : \mathfrak{g} \otimes - \to \text{Id}$.

Let $\mathfrak{g}_z := DS_z(\mathfrak{g}_z)$.

Then $DS_z(\text{act}^\mathfrak{g}) \cong \text{act}^\mathfrak{g}_z$, where the latter is the natural action of the adjoint representation of $\mathfrak{g}_z$ on any $\mathfrak{g}_z$-module.

Remark 5.1.4. Clearly, the restriction of the functor $\text{Res}$ to the full subcategory $\text{Rep}^k(\mathfrak{p}(n)) \subset \text{Rep}(\mathfrak{p}(n))$ has image in the full subcategory $\text{Rep}^k(\mathfrak{gl}(p|q))$.

5.2. Main result on $DS$ functor. We now prove a central result on $DS$ functors for periplectic Lie superalgebras. Now, fix $k \geq 0$.

Proposition 5.2.1. For $n \geq 8k + 2$, the functor $DS_x : \text{Rep}^k(\mathfrak{p}(n)) \to \text{Rep}^k(\mathfrak{p}(n-2))$ is an equivalence.

\text{Proof.} Let $p, q \geq 0$ be such that $n = p + q$ and $\min(p, q) > 4k$; by Theorem 7.1.1 the functor $DS_y : \text{Rep}^k(\mathfrak{gl}(p|q)) \to \text{Rep}^k(\mathfrak{gl}(p-1|q-1))$ becomes an equivalence of categories (in particular, exact and faithful).

By Lemma 5.1.2, the functor $DS_x : \text{Rep}^k(\mathfrak{p}(n)) \to \text{Rep}^k(\mathfrak{p}(n-2))$ is exact and faithful as well, so we only need to show that it is full and essentially surjective. Below we describe the main steps of the proof, with details given in Lemmas 5.2.3, 5.2.4.

\textbf{Proof that $DS_x$ is full:}

In Lemma 5.2.3, we show that for any simple $L$, $DS_x(L)$ is simple. We use this fact to prove Lemma 5.2.4 for any $M \in \text{Rep}^k(\mathfrak{p}(n))$ we have

$$[\text{soc}(M) : \mathbb{C}] = [\text{soc}(DS_x(M)) : \mathbb{C}]$$

Clearly, this implies

$$\dim \text{Hom}_{\mathfrak{p}(n)}(\mathbb{C}, M) = \dim \text{Hom}_{\mathfrak{p}(n-2)}(\mathbb{C}, DS_x M)$$

Next, let $M, N \in \text{Rep}^k(\mathfrak{p}(n))$. We have isomorphisms

$$\text{Hom}_{\mathfrak{p}(n)}(N, M) \cong \text{Hom}_{\mathfrak{p}(n)}(\mathbb{C}, N^* \otimes M), \quad \text{and} \quad \text{Hom}_{\mathfrak{p}(n-2)}(DS_x N, DS_x M) \cong \text{Hom}_{\mathfrak{p}(n-2)}(\mathbb{C}, (DS_x N)^* \otimes DS_x M) \cong \text{Hom}_{\mathfrak{p}(n-2)}(\mathbb{C}, DS_x (N^* \otimes M))$$

Hence (5) implies:

$$\dim \text{Hom}_{\mathfrak{p}(n)}(N, M) = \dim \text{Hom}_{\mathfrak{p}(n-2)}(DS_x N, DS_x M)$$

which means that $DS_x$ is full.

\textbf{Proof that $DS_x$ is essentially surjective:}

We wish to show that for any $M' \in \text{Rep}^k(\mathfrak{p}(n-2))$ there exists $M \in \text{Rep}^k(\mathfrak{p}(n))$ such that $DS_x M \cong M'$.

We prove this by induction on the length of $M'$.
For \( \ell(M') = 0 \), this is trivial. Assume now that the statement holds for any module in \( \text{Rep}^k(p(n - 2)) \) of length strictly less that \( \ell(M') \).

Let \( L' \) be a simple submodule of \( M' \):

\[
0 \to L' \to M' \to N' \to 0
\]

By Lemma 5.2.3 there exists a simple module \( L \in \text{Rep}^k(p(n - 2)) \) such that \( DS_x(L) = L' \). By induction assumption, there exists \( N \in \text{Rep}^k(p(n - 2)) \) such that \( DS_x(N) = N' \).

Now we recall that any module in \( \text{Rep}^k(p(n - 2)) \) is a subquotient of a certain finite direct sum \( T' \) of tensor powers \( V_{n-2}^{s} \), \( 0 \leq s \leq k \). Let \( T' \) be the appropriate direct sum for \( L' \).

It is easy to see that \( T' \) lies in the essential image of \( DS_x \). Indeed, let us write

\[
T' = \bigoplus_{0 \leq s \leq k} \left(V_{n}^{s}\right)^{m_s}
\]

for some \( m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0} \), and set

\[
T := \bigoplus_{0 \leq s \leq k} \left(V_{n}^{s}\right)^{m_s}
\]

Using the fact that \( DS_x \) respects direct sums and tensor products, as well as \( DS_x(V_n) \equiv V_{n-2} \), we obtain: \( DS_x(T') \equiv T' \).

Denote \( Q' := T'/L' \):

\[
\begin{array}{ccc}
0 & \rightarrow & L' \\
\downarrow & & \downarrow \\
0 & \rightarrow & T'
\end{array}
\]

Notice that \( N' \hookrightarrow M' \twoheadrightarrow T' \) is the pullback of the arrows \( N' \hookrightarrow Q' \twoheadrightarrow T' \).

Since \( DS_x \) is fully faithful and exact, there is an injective map \( L \hookrightarrow T \). Denote the cokernel of this map by \( Q \). Then \( DS_x(Q) = Q' \) and there is an injective map \( N \hookrightarrow Q \), such that the diagram below is taken by \( DS_x \) to the corresponding maps in the diagram above:

\[
\begin{array}{ccc}
L & \downarrow & N \\
0 & \rightarrow & T \\
0 & \rightarrow & Q
\end{array}
\]

Consider the pullback \( N \leftarrow M \leftarrow T \) of the arrows \( N \leftarrow Q \leftarrow T \). Since \( DS_x \) is exact, it preserves pullbacks, and hence \( DS_x(M) = M' \), as required.

This completes the proof of Proposition 5.2.1. \(
\\square
\)

**Remark 5.2.2.** The fact that \( DS_x \) is full can also be concluded from the result of [DLZ15] (see Proposition 2.6.4).

**Lemma 5.2.3.** Let \( |\lambda| \leq k \) and \( n > 8k + 2, k \geq 0 \). Consider the simple module \( L_n(\lambda) \), and the highest weight module \( \Delta_n^k(\lambda) \) in \( \text{Rep}^k(p(n)) \). We have:

1. \( DS_x \Delta_n^k(\lambda) = \Delta_{n-2}^k(\lambda) \).
2. \( DS_x L_n(\lambda) = L_{n-2}(\lambda) \).

**Proof.**

(1) We prove the statement by induction on \( |\lambda| \).

For \( |\lambda| = 0 \), we have \( \lambda = 0 \) and \( \Delta_0^k(0) = \Phi_{n}(1), \Delta_{n-2}^k(0) = \Phi_{n-2}(1) \). Hence the statement holds in this case.

The induction step follows directly from Lemma 4.1.2.

(2) Consider the surjective map \( f : \Delta_n^k(\lambda) \rightarrow L_n(\lambda) \). Since \( DS_x \) is exact, we have: \( DS_x L_n(\lambda) \) is the quotient of \( DS_x \Delta_n^k(\lambda) = \Delta_{n-2}^k(\lambda) \), and hence \( DS_x L_n(\lambda) \) has \( L_{n-2}(\lambda) \) as quotient.
Moreover, \([DS_x L_n(\lambda) : L_{n-2}(\lambda)] = 1\), since
\[
 [DS_x L_n(\lambda) : L_{n-2}(\lambda)] \leq [\Delta_{n-2}(\lambda) : L_{n-2}(\lambda)] \leq [\Delta_{n-2}(\lambda) : L_{n-2}(\lambda)] = 1
\]
On the other hand, we have the transposed map
\[
f^t : L_n(\lambda)^* \rightarrow (\Delta_k^n(\lambda)^*)^*
\]
Denote by \(\lambda^\vee\) the highest weight of the simple module \(L_n(\lambda)^*\). Clearly, \(L_n(\lambda^\vee) = L_n(\lambda)^*\) sits in \(\text{Rep}^k(p(n))\) as well.

Then
\[
L_n(\lambda) \subset (\Delta_k^n(\lambda^\vee))^*
\]
and hence
\[
DS_x L_n(\lambda) \subset DS_x ((\Delta_k^n(\lambda^\vee))^*) = (DS_x \Delta_k^n(\lambda^\vee))^* = (\Delta_{n-2}(\lambda^\vee))^*
\]
In particular, the socle of \(DS_x L_n(\lambda)\) coincides with the socle of \((\Delta_{n-2}(\lambda^\vee))^*\), which is \(L_{n-2}(\lambda)\). Hence \(DS_x L_n(\lambda) = L_{n-2}(\lambda)\), as required.

\[\square\]

**Lemma 5.2.4.** Let \(n > 8k + 2\), \(k \geq 0\) and \(M \in \text{Rep}^k(p(n))\). Then \([\text{soc}(M) : \mathbb{C}] = [\text{soc}(DS_x(M)) : \mathbb{C}].\)

**Proof.** Recall that \(DS_x(\mathbb{C}) = \mathbb{C}\). Moreover, by Lemma 5.2.3 \(DS_x\) takes simple modules to simple modules, and induces a bijection on the set of isomorphism classes of simple modules in \(\text{Rep}^k(p(n)), \text{Rep}^k(p(n-2))\). Hence in order to prove the statement of the lemma, it is enough to show that given a non-split short exact sequence

\[
0 \rightarrow \mathbb{C} \rightarrow M \rightarrow N \rightarrow 0
\]
its image
\[
0 \rightarrow \mathbb{C} \rightarrow DS_x M \rightarrow DS_x N \rightarrow 0
\]
will not split either.

Assume the contrary, and consider a module \(M\) of minimal length for which a non-split short exact sequence

\[
0 \rightarrow \mathbb{C} \rightarrow M \rightarrow N \rightarrow 0
\]
is sent by \(DS_x\) to a split sequence
\[
0 \rightarrow \mathbb{C} \rightarrow DS_x M \rightarrow DS_x N \rightarrow 0
\]
Let \(L\) be a simple quotient of \(N\), and denote:

\[
M' := \text{Ker}(M \twoheadrightarrow L), \quad N' := \text{Ker}(N \twoheadrightarrow L)
\]
Then we have a short exact sequence
\[
0 \rightarrow \mathbb{C} \rightarrow M' \rightarrow N' \rightarrow 0
\]
is sent by \(DS_x\) to a split sequence
\[
0 \rightarrow \mathbb{C} \rightarrow DS_x M' \rightarrow DS_x N' \rightarrow 0
\]
By assumption, the sequence (6) is split, since \(\ell(M') < \ell(M)\). Hence we have a section \(M' \twoheadrightarrow \mathbb{C}\).

Consider the pushout of the maps \(M' \twoheadrightarrow \mathbb{C}\), \(M' \twoheadrightarrow M\):

\[
\begin{array}{ccc}
M' & \rightarrow & K \\
\downarrow & & \downarrow \\
\mathbb{C} & \rightarrow & L
\end{array}
\]
Then
\[
0 \rightarrow \mathbb{C} \twoheadrightarrow K \twoheadrightarrow L \rightarrow 0
\]
is a short exact sequence, whose image under $DS_x$ is clearly split. We now consider two possibilities:

**Case $N \neq L$.** In this case $N' \neq 0$ and hence $\ell(K) < \ell(M)$. This implies that we have a section $K \to \mathbb{C}$, inducing a section $M \to \mathbb{C}$, which in turn contradicts the assumption that the original short exact sequence was not split.

**Case $N = L$.** In this case, we have a non-split extension

$$0 \to \mathbb{C} \to M \to L \to 0$$

and we claim that $M$ is a highest weight module whose highest weight coincides with that of $L$.

Indeed, denote by $\lambda$ the highest weight of $L$. Consider the weights of the module $M$. These are the weights of $L$ together with the trivial weight. Since $\lambda$ is $k$-admissible, we have: $\lambda_i \leq 0$ for any $i$. Hence weight 0 is not higher than $\lambda$, and thus $M$ has no weights higher than $\lambda$. This means that we have a map $\Delta(\lambda) \to M$. Since $M$ is not split, this map is surjective, and we conclude that $[\Delta(\lambda) : \mathbb{C}] > 0$. Now [BDE+16 Theorem 6.3.3] implies that there exists $i$ such that $\lambda_i \geq 0$, which contradicts our conditions on $\lambda$ such that $L_n(\lambda) \in \text{Rep}^k(p(n))$.

This completes the proof of Lemma 5.2.4.

This completes the proof of Lemma 5.2.4.

5.3. **Compatatability with the specialization functors.** Let $n \geq 3$. Consider the functors $\Phi_n : \text{Rep}(p(\infty)) \to \text{Rep}(p(n))$ and $\Phi_{n-2} : \text{Rep}(p(\infty)) \to \text{Rep}(p(n-2))$.

Let $x \in p(n)_1$ be such that $x$ lies in the centralizer of $p(n-2)$. Then for any $M \in \text{Rep}(p(\infty))$ we have:

$$M^{p(n-2)^+} \subset \text{Ker} \, x |_{M^p(n)}$$

and hence we can consider the composition

$$M^{p(n-2)^+} \hookrightarrow \text{Ker} \, x |_{M^p(n)} \to DS_x \left( M^{p(n)} \right)$$

which we denote by $\eta_M$. This gives a natural $\otimes$-transformation $\eta : \Phi_{n-2} \to DS_x \circ \Phi_n$.

**Remark 5.3.1.** Applying the functor $\text{Res} : \text{Rep}(p(n-2)) \to \text{Rep}(\mathfrak{gl}(p-1|q-1))$ for $p + q = n$ to $\eta$, we obtain a natural $\otimes$-transformation $\Gamma_{p-1|q-1} \to DS_y \Gamma_{p|q}$ which is described in [EHS15 Section 7.2].

**Lemma 5.3.2.** Let $n > 4k + 2$. The restriction of $\eta$ to $\text{Rep}^k(p(\infty))$ is an isomorphism.

**Proof.** As it is mentioned in the Remark above, $\Gamma_{p-1|q-1} \to DS_y \Gamma_{p|q}$ is the natural transformation described in [EHS15 Section 7.2]. In particular, by [EHS15 Lemma 7.2.1], the restriction of $\text{Res}(\eta)$ to $\text{Rep}^k(\mathfrak{gl}(p-1|q-1))$ for $p, q > 2k$ is an isomorphism. Since $\text{Res}$ is a faithful exact functor, we conclude that $\eta$ is both surjective and injective, and hence an isomorphism as well.

6. **Construction of the category $\text{Rep}(P)$**

6.1. **Construction.** For each $n \geq 4, n \in 2\mathbb{Z}$, fix $x_n \in p(n)$ an odd element of rank 2 (as in Section 4). Recall that by Proposition 5.2.1, the functors

$$DS_x : \text{Rep}^k(p(n)) \to \text{Rep}^k(p(n-2))$$

are equivalences of SM categories for $n \gg k$. Furthermore, for each $k, n \geq 1$ we have a fully faithful exact SM embedding $\text{Rep}^k(p(n)) \to \text{Rep}^{k+1}(p(n))$. This allows us to define

$$\text{Rep}^k(P) := \lim_{n \in 2\mathbb{Z}, n \to \infty} \text{Rep}^k(p(n)), \quad \text{Rep}(P) := \lim_{k \to \infty} \text{Rep}^k(P)$$

with respect to the functors $DS_x$ and the inclusions above.

The category $\text{Rep}(P)$ will be called the Deligne category for the periplectic Lie superalgebra. The category $\text{Rep}(P)$ is clearly a $\text{sVect}$-category. Furthermore, we have:
Lemma 6.1.1. The category $\text{Rep}(\mathcal{P})$ is a tensor $\text{sVect}$-category.

Proof. The categories $\text{Rep}^k(p(n))$ are abelian for any $k$, so $\text{Rep}^k(\mathcal{P})$ is abelian as well, and so is $\text{Rep}(\mathcal{P})$.

For any $n, k, m$, the tensor structure on $\text{Rep}(p(n))$ induces a biexact bilinear bifunctor

$$\text{Rep}^k(p(n)) \times \text{Rep}^m(p(n)) \longrightarrow \text{Rep}^{k+m}(p(n)).$$

This induces a bifunctor

$$\text{Rep}^k(\mathcal{P}) \times \text{Rep}^m(\mathcal{P}) \longrightarrow \text{Rep}^{k+m}(\mathcal{P})$$

for any $k, m \geq 0$, and hence a rigid SM structure on $\text{Rep}(\mathcal{P})$, which satisfies all the requirements of a tensor category structure. □

By definition, $\text{Rep}^k(\mathcal{P})$ comes equipped with SM functors

$$F_n : \text{Rep}^k(\mathcal{P}) \longrightarrow \text{Rep}^k(p(n))$$

for each $n \in 2\mathbb{Z}$ and $k$, which are equivalences when $n > 8k + 2$. These induce SM functors

$$F_n : \text{Rep}(\mathcal{P}) \longrightarrow \text{Rep}(p(n))$$

for any $n \in 2\mathbb{Z}$.

We denote by $V$ the object of $\text{Rep}(\mathcal{P})$ corresponding to the natural representation:

$$V = (V_n)_{n \in 2\mathbb{Z}} \in \text{Rep}^1(\mathcal{P}) \subset \text{Rep}(\mathcal{P}).$$

Clearly, for any $n \in 2\mathbb{Z}$, we have $F_n(V) = V_n$. The object $V$ is equipped with a pairing $\omega_V : V \otimes V \rightarrow \Pi_1$.

Remark 6.1.2. We can also define $\text{Rep}^k(\mathcal{P}') := \lim_{\rightarrow n \in 2\mathbb{Z} + 1, n \rightarrow \infty} \text{Rep}^k(p(n))$, $\text{Rep}(\mathcal{P}') := \lim_{\rightarrow k \rightarrow \infty} \text{Rep}^k(\mathcal{P}')$ and set $V'$ to be the corresponding “natural representation” with a pairing $\omega_{V'}$.

We will show in Corollary 7.0.3 that these two constructions are equivalent, i.e. we have a (unique) equivalence of tensor categories $\text{Rep}(\mathcal{P}) \cong \text{Rep}(\mathcal{P}')$ with $V$ sent to $V'$ and $\omega_V$ sent to $\omega_{V'}$.

This allows us to consider $F_n : \text{Rep}(\mathcal{P}) \longrightarrow \text{Rep}(p(n))$ for any $n \in \mathbb{Z}$. For this reason, from now on, we will impose any conditions on the parity of $n$, although such assumption would not influence the results below.

Consider the morphism

$$F_n : \text{Hom}_{\text{Rep}(\mathcal{P})}(V^\otimes l, 1) \longrightarrow \text{Hom}_{p(n)}(F_n(V^\otimes l), F_n(1)) = \text{Hom}_{p(n)}(V_n^\otimes l, \mathbb{C}).$$

Lemma 6.1.3. The morphism $F_n$ above is surjective.

Proof. By the results of [DLZ15] (also proved in Corollary 10.1.1), the spaces $\text{Hom}_{p(n)}(V_n^\otimes l, \mathbb{C})$, $\text{Hom}_{p(n-2)}(V_{n-2}^\otimes l, \mathbb{C})$ are spanned by contraction maps for any $n \geq 3$ and any $l \geq 0$. Hence the map

$$DS_{x_n} : \text{Hom}_{p(n)}(V_n^\otimes l, \mathbb{C}) \rightarrow \text{Hom}_{p(n-2)}(V_{n-2}^\otimes l, \mathbb{C})$$

is surjective. The lemma now follows. □

Corollary 6.1.4. The functor $F_n$ is full on the (full) Karoubian additive SM subcategory generated by $V$. 

6.2. Translation functors and the Casimir. Recall the notation \( \text{act} : \mathfrak{gl}(V) \otimes V \rightarrow V \) which stands for the action of the Lie algebra object \( \mathfrak{gl}(V) \) in \( \text{Rep}(P) \) on \( V \) (see Section 2.2).

**Definition 6.2.1.** Let
\[
\triangle : \mathfrak{gl}(V) \otimes V \otimes V \rightarrow V \otimes V, \quad \triangle := \text{act} \otimes \text{Id} + (\text{Id} \otimes \text{act}) \circ (\sigma_{\mathfrak{gl}(V), V} \otimes \text{Id})
\]
and let \( p(V) \in \text{Rep}(P) \) be the maximal subobject of \( \mathfrak{gl}(V) \) such that the map
\[
\omega \circ \triangle : p(V) \otimes V \otimes V \rightarrow V \otimes V \rightarrow \Pi^1
\]
is zero.

That is, \( p(V) \) is the subobject of \( \mathfrak{gl}(V) \) preserving the form \( \omega \).

**Lemma 6.2.2.** The following is true:

1. The object \( p(V) \) is a Lie algebra subobject of \( \mathfrak{gl}(V) \) in \( \text{Rep}(P) \).
2. For any \( n \geq 1 \), \( F_n(p(V)) \cong p(n) \) (adjoint representation) as Lie algebra objects in \( \text{Rep}(p(n)) \).
3. We have an orthogonal decomposition
\[
\mathfrak{gl}(V) \cong p(V) \oplus p(V)^*
\]
of \( \text{Rep}(P) \) objects with respect to the form
\[
tr := \text{ev} \circ \sigma_{V, V^*} : \mathfrak{gl}(V) \cong V \otimes V^* \rightarrow 1.
\]
4. We have an isomorphism: \( p(V) \cong \Pi S^2V \).
5. There is a natural transformation of functors \( \text{act} : p(V) \otimes (-) \rightarrow \text{Id} \) making any \( M \in \text{Rep}(P) \) a module over the Lie algebra object \( p(V) \) in \( \text{Rep}(P) \).

**Proof.** The statement (1) is straightforward. For (2), recall that for \( n >> 0 \), \( F_n \) is both a SM functor and an equivalence on \( \text{Rep}^4(P) \) to which all the objects appearing in Definition 6.2.1 belong. Hence \( F_n(p(V)) \) would be the maximal Lie subalgebra of \( F_n(\mathfrak{gl}(V)) \) satisfying conditions as in Definition 6.2.1, i.e. preserving the form \( F_n(\omega) : V_n \otimes V_n \rightarrow \Pi^1 \).

This implies that \( F_n(p(V)) \cong p(n) \) (as Lie algebras in \( \text{Rep}(p(n)) \)). This proves the statement for large \( n \). Now \( DS_n(p(n)) \cong p(n - 2) \) (as Lie algebras) for any \( n \geq 3 \), which proves (2) for any \( n \).

The decomposition in (3) now follows from an analogous decomposition for \( \mathfrak{gl}(n|n) \) and \( p(n) \). To prove (4), let \( \eta : \Pi V \rightarrow V^* \) be the isomorphism induced by the form \( \omega \).

Consider the isomorphism \( f : \Pi V \otimes V \rightarrow \mathfrak{gl}(V) = V \otimes V^* \) given by
\[
f = \sigma_{V, V^*} \circ (\eta \otimes \text{Id}) + (\text{Id} \otimes \eta) \circ \sigma_{\Pi^2V}
\]
Now \( f \) induces an isomorphism \( p(V) \cong \Pi S^2V \), which can be verified by applying \( F_n \) to both sides, for large enough \( n \), and obtaining \( p(n) \cong \Pi S^2V_n \), which is known to be true.

Finally, (5) follows from Lemma 5.1.3.

We now define a natural endomorphism \( \Omega \) of the endofunctor \( V \otimes (-) \) on \( \text{Rep}(P) \).

**Definition 6.2.3.** For any \( M \in \text{Rep}(P) \), let \( \Omega_M \) be the composition
\[
V \otimes M \xrightarrow{\text{Id} \otimes \text{coer} \otimes \text{Id}} V \otimes p(V)^* \otimes p(V) \otimes M \xrightarrow{i_* \otimes \text{Id}} V \otimes \mathfrak{gl}(V) \otimes p(V) \otimes M \xrightarrow{(\text{act} \otimes \text{act})} V \otimes M
\]
where \( i_* : p(V)^* \rightarrow \mathfrak{gl}(V) \) is the isomorphism defined in Lemma 6.2.2.

**Definition 6.2.4.** For \( k \in \mathbb{C} \), we define a functor \( \Theta_k : \text{Rep}(P) \rightarrow \text{Rep}(P) \) as the functor \( V \otimes (-) \) followed by the projection onto the generalized \( k \)-eigenspace for \( \Omega \). That is, we set
\[
(7) \quad \Theta_k(M) := \bigcup_{n > 0} \text{Ker}(\Omega - k \text{Id})_{|_{M \otimes V}}^n
\]
The fact that \( \Theta_k(M) \) is a direct summand of \( V \otimes M \), is proved in the same way as for operators on finite-dimensional vector spaces.

Finally, we set \( \Theta := \Pi^k \Theta_k \) in case \( k \in \mathbb{Z} \).
Lemma 6.2.5. Let $n \geq 1$. We have: $F_n(\Omega) = \Omega^n F_n$, where $\Omega^n$ is the tensor Casimir for $p(n)$ (see [BDE+16 4.1.1]). That is, for any $M \in \text{Rep}(P)$, $F_n(\Omega M) = \Omega^n F_n(M)$ as endomorphisms of $V_n \otimes F_n(M)$.

Proof. This follows directly from the fact that $F_n$ is a SM functor, together with [15].

Corollary 6.2.6.

(1) For any $k \in \mathbb{Z}$, we have $\Theta_k' \cong 0$.

(2) We have a natural isomorphism of functors

$$F_n \Theta_k \cong \Theta_{k(n)} F_n$$

where $\Theta_{k(n)} : \text{Rep}(p(n)) \to \text{Rep}(p(n))$ is the translation functor defined in [BDE+16 4.1.7].

Proof. Follows directly from Lemma 6.2.5 together with the definition of $\Theta_{k(n)}$ as the generalized $k$-eigenspace of $\Omega^n$ on $- \otimes V_n$ (up to a twist by $\Pi^k$).

Remark 6.2.7. A similar idea is used to prove the Kac-Wakimoto conjecture for $p(n)$, as will be shown in an upcoming work by the authors.

Lemma 6.2.8. If $Y$ has simple socle, then $\Theta_i(\text{soc}(Y))$ is either zero or has simple socle. Similarly, if $Y$ has simple cosocle, then $\Theta_i(\text{cosoc}(Y))$ is either zero or has simple cosocle.

Proof. We prove the second statement (the first one is proved analogously). Let $k \geq 0$ be such that $Y \in \text{Rep}^k(P)$. Let $n >> k$ be such that $\text{Rep}^{k+1}(P) \cong \text{Rep}^{k+1}(p(n))$. Then $F_n(Y)$ has simple socle and hence is a quotient of an indecomposable projective. Applying translation functors to an indecomposable projective in $\text{Rep}(p(n))$, one obtains either zero or once again an indecomposable projective. Thus $\Theta_i F_n(Y)$ is either zero, or has simple socle as well. Now, $\Theta_i F_n(Y) \in \text{Rep}^{k+1}(p(n)) \cong \text{Rep}^{k+1}(P)$ so $\Theta_i(Y)$ is either zero, or has simple cosocle, as required.

6.3. Properties of the category $\text{Rep}(P)$. We list below some “local” and “global” properties of the category $\text{Rep}(P)$.

We begin with the “global” properties:

(1) The isomorphism classes of simple objects in $\text{Rep}(P)$ (up to parity shift) are parametrized by infinite non-decreasing integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_i = 0$ for $i >> 0$. Such sequences will be called weights for $\text{Rep}(P)$, and the set of weights will be denoted by $\Lambda$.

Every simple object is isomorphic to $L(\lambda) \in \text{Rep}^{\vert \lambda \vert}(P)$ which we define as the inverse limit of simple $p(n)$-modules $L_n(\lambda)$.

(2) Any object in $\text{Rep}(P)$ has finite length.

(3) For any infinite non-decreasing integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\vert \lambda \vert \leq k$ we define $\Delta(\lambda)$ (standard objects) as the inverse limit of $\text{Rep}^k(\lambda)$, and similarly, we define $\nabla(\lambda)$ (costandard objects) as the inverse limit of $\text{Rep}^k(\lambda)$. Then the socle of $\Delta(\lambda)$ and the socle of $\nabla(\lambda)$ are isomorphic to $L(\lambda)$.

(4) We have

$$\dim \text{Hom}_{\text{Rep}(P)}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda, \mu}, \quad \text{Ext}^1_{\text{Rep}(P)}(\Delta(\lambda), \nabla(\mu)) = 0.$$  

(see Lemma 4.4.7).

(5) For any weight $\lambda$ in $\text{Rep}(P)$, we have (see Corollary 4.4.1):

$$\Delta(\lambda)^* \cong \Pi^{\vert \lambda \vert} \nabla(\lambda^\vee) \quad \text{and} \quad L(\lambda)^* \cong \Pi^{\vert \lambda \vert} L(\lambda^\vee)$$

and have maps

$$\Delta(\lambda) \to S^{\lambda^\vee} V, \quad S^{\lambda^\vee} V \to \nabla(\lambda).$$

(6) For each $\lambda$, the functor $V \otimes (-)$ takes $\Delta(\lambda)$ to a $\Delta$-filtered object, and $\nabla(\lambda)$ to a $\nabla$-filtered object, according to Corollaries 4.3.2 4.4.3. In particular, the tensor powers of $V$ have both standard and costandard filtration.
Below are some “local” properties of $\text{Rep}(P)$, namely, properties of the subcategories $\text{Rep}^k(P)$:

1. Simple objects $L(\lambda)$ lying in $\text{Rep}^k(P)$ are those for which $\lambda$ satisfies: $|\lambda| \leq k$. By abuse of notation, we will call such weights $k$-admissible.

2. The category $\text{Rep}^k(P)$ has enough projectives and injectives (see Lemma 6.4.1); the projective cover of $L(\lambda)$ in $\text{Rep}^k(P)$ will be denoted by $P_k(\lambda)$.

3. The tensor structure on $\text{Rep}(P)$ is given by functors $\text{Rep}^k(P) \otimes \text{Rep}^l(P) \to \text{Rep}^{k+l}(P)$ and $\text{Rep}^k(P)$ is closed under the tensor duality contravariant functor $(\cdot)^*$.

6.4. Connection to the category $\mathcal{P}$.

Lemma 6.4.1. There exists a fully faithful SM $\mathcal{P}$-linear $s\text{Vect}$-functor $I : \mathcal{P} \to \text{Rep}(P)$ which sends the generator $\tilde{V}$ of $\mathcal{P}$ to $V$, and the form $\omega_{\tilde{V}}$ on $\tilde{V}$ to $\omega_V$. Furthermore, there is a natural $\otimes$-isomorphism making the following diagram of functors commutative:

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{I} & \text{Rep}(P) \\
\downarrow{I_n} & & \downarrow{F_n} \\
\text{Rep}(p(n)) & & 
\end{array}
$$

Proof. Consider the additive SM $s\text{Vect}$-functor $I : \mathcal{P} \to \text{Rep}(P)$ defined by sending $\tilde{V}$ to $V$, and the form on $\tilde{V}$ to $\omega_V$. Such a functor is uniquely defined (up to a unique natural $\otimes$-isomorphism).

The composition $F_n \circ I$ is then an additive SM functor $\mathcal{P} \to \text{Rep}(p(n))$ satisfying similar conditions, and hence it is isomorphic to $I_n$ (by the uniqueness of $I_n$) under some natural $\otimes$-isomorphism, making the above diagram commutative.

It remains to check that $I$ is fully faithful. Indeed, it is enough to check that for any $r, s \geq 0$,

$$I : \text{Hom}_\mathcal{P}(\tilde{V}^\otimes r, \tilde{V}^\otimes s) \to \text{Hom}_{\text{Rep}(P)}(V^\otimes r, V^\otimes s)$$

is an isomorphism. Taking some $n >> r, s$ we see that

$$I_n \cong F_n \circ I : \text{Hom}_\mathcal{P}(\tilde{V}^\otimes r, \tilde{V}^\otimes s) \to \text{Hom}_{\text{Rep}(p(n))}(V_n^\otimes r, V_n^\otimes s)$$

is surjective, implying that the previous map was surjective as well. To check that it is also injective, it remains to check that

$$\dim \text{Hom}_\mathcal{P}(\tilde{V}^\otimes r, \tilde{V}^\otimes s) = \dim \text{Hom}_{\text{Rep}(P)}(V^\otimes r, V^\otimes s)$$

which follows directly from Proposition 2.6.4.

6.5. Connection to representations of $p(\infty)$.

Proposition 6.5.1. There exists an exact faithful SM $s\text{Vect}$-functor $\Phi : \text{Rep}(p(\infty)) \to \text{Rep}(P)$ taking $V_\infty$ to $V$, and the form $V_\infty \otimes V_\infty \to \Pi \mathcal{P} \to \omega_V$. Furthermore, there is a natural $\otimes$-isomorphism making the following diagram of functors commutative:

$$
\begin{array}{ccc}
\text{Rep}(p(\infty)) & \xrightarrow{\Phi} & \text{Rep}(P) \\
\downarrow{\Phi_n} & & \downarrow{F_n} \\
\text{Rep}(p(n)) & & 
\end{array}
$$

Proof. Recall that for $n >> k$, the SM functor $\Phi_n : \text{Rep}(p(\infty)) \to \text{Rep}(p(n))$ restricts to an exact functor $\text{Rep}^k(p(\infty)) \to \text{Rep}^k(p(n))$. This functor commutes with the $DS_{x_n}$ functor, as shown in Lemma 6.3.2. Hence we have a well-defined exact functor $\Phi : \text{Rep}^k(p(\infty)) \to \text{Rep}^k(P)$ extending to a functor $\Phi : \text{Rep}(p(\infty)) \to \text{Rep}(P)$. This functor clearly satisfies the statement of the proposition.

By Lemma 6.3.3, we also have:
Corollary 6.5.2. Let $L_\infty(\lambda)$ be a simple object in $\text{Rep}(\mathfrak{p}(\infty))$. Then $\Phi(L_\infty(\lambda)) \cong \Delta(\lambda)$.

Corollary 6.5.3. Let $E$ be an injective object in $\text{Rep}(\mathfrak{p}(\infty))$. Then $\Phi(E) \in I(\Psi)$.

Proof. The subcategory of injective objects in $\text{Rep}(\mathfrak{p}(\infty))$ is generated, as an additive Karoubian category, by the tensor powers of $V_\infty$. Since $\Phi$ is $\mathbb{C}$-linear and monoidal, for any injective object $E \in \text{Rep}(\mathfrak{p}(\infty))$, $\Phi(E)$ belongs to the full additive Karoubian subcategory generated by tensor powers of $V$. This subcategory is precisely $I(\Psi)$. □

7. The first universal property

To prove the universality of category $\text{Rep}(P)$, we use the following theorem, proved in [EHS15].

Consider a SM functor $I: D \to T$ from an additive $\mathbb{C}$-linear rigid SM category $D$ to a tensor $\mathbb{C}$-linear category $T$.

We assume the following conditions hold:
1. The functor $I: D \to T$ is fully faithful and $\mathbb{C}$-linear.
2. Any object $X \in T$ can be presented as an image of a map $I(f)$ for some $f: P \to Q$ in $D$.
3. For any epimorphism $X \to Y$ in $T$ there exists a nonzero $T \in D$ such that the epimorphism $X \otimes I(T) \to Y \otimes I(T)$ splits.

Theorem 7.0.1 ([EHS15]). Under these assumptions the functor $I$ induces for any tensor $s\text{Vect}$-category $A$ an equivalence of the following categories
- $\text{Fun}^{\text{ex}}(V, A)$, the category of faithful exact SM functors $V \to A$,
- $\text{Fun}^{\text{faith}}(P, A)$, the category of faithful SM functors $P \to A$.

We will apply this theorem to $D = \Psi$, $V = \text{Rep}(P)$ to obtain the following results:

Theorem 7.0.2. For any tensor $s\text{Vect}$-category $A$ we have an equivalence of the following categories
- $\text{Fun}^{\text{ex}}(\text{Rep}(P), A)$, the category of exact SM $s\text{Vect}$-functors $T \to A$,
- $\text{Fun}^{\text{faith}}(P, A)$, the category of faithful SM $s\text{Vect}$-functors $P \to A$.

Proof. We only need to check that there is a fully faithful SM embedding $\Psi \hookrightarrow \text{Rep}(P)$ satisfying Conditions [7]. This will be done in the remainder of this Section. □

In particular, we have:

Corollary 7.0.3. The categories $\text{Rep}(P)$ and $\text{Rep}(P')$, defined in Section 6.1 are equivalent as tensor $s\text{Vect}$-categories. This equivalence commutes with the embeddings of $\Psi$ into both categories.

7.1. Overview of the proof. The main ingredient in proving Theorem 7.0.2 is showing that there exists a SM functor $I: \Psi \to \text{Rep}(P)$ satisfying the Conditions [7].

Condition (1) has been proved in Lemma 6.4.1. Condition (2) will be proved in Section 7.2 below, and Condition (3) will be proved in Section 7.3.

7.2. Presentation of objects. In this section we prove the following Proposition, required for Theorem 7.0.2.

Proposition 7.2.1. For any $M \in \text{Rep}(P)$, there exist objects $T_1, T_2 \in \Psi$ (finite direct summands of direct sums of tensor powers of $V$) and maps $T_1 \to M \hookrightarrow T_2$.

The outline of the proof follows that of [EHS15 Proposition 8.4.1]:

Proof. Due to the existence of duality $(\cdot)^*$ in $\text{Rep}(P)$ which preserves $I(\Psi)$, it is enough to prove the existence of $T_1$ as above.

We will prove the statement in several steps.

Step 1: We prove the statement for $M = P_k(0)$, the projective cover of $1$ in $\text{Rep}^k(P)$ (see Lemma 7.2.1).
Step 2: We prove the statement for any standard object $\Delta(\lambda)$ in $\text{Rep}(\mathcal{P})$ (see Lemma 8.1.12).

Step 3: We prove the statement for any projective object $P$ in $\text{Rep}^k(\mathcal{P})$, for any $k \geq 0$.

Step 4: We prove the statement for any object $M$ in $\text{Rep}(\mathcal{P})$.

**Proof of Step 3:** It is enough to prove the statement for indecomposable objects $P_k(\lambda)$ where $|\lambda| \leq k$.

Consider the inclusion functor $i^k : \text{Rep}^k(\mathcal{P}) \to \text{Rep}(\mathcal{P})$ and its left adjoint $i^{k,*} : \text{Rep}(\mathcal{P}) \to \text{Rep}^k(\mathcal{P})$. Let $P_{2k}(0)$ be the projective cover of $1$ in $\text{Rep}^{2k}(\mathcal{P})$ and let $\lambda(\lambda) \in \text{Rep}(\mathcal{P})$. Consider

$$Y := i^{k,*}(P_{2k}(0) \otimes L(\lambda)),$$

the maximal quotient of $P_{2k}(0) \otimes L(\lambda)$ lying in $\text{Rep}^k(\mathcal{P})$. Then for any $M \in \text{Rep}^k(\mathcal{P})$, we have:

$$\text{Hom}_{\text{Rep}^k(\mathcal{P})}(Y, M) \cong \text{Hom}_{\text{Rep}(\mathcal{P})}(P_{2k}(0) \otimes L(\lambda), M) \cong \text{Hom}_{\text{Rep}^{2k}(\mathcal{P})}(P_{2k}(0), L(\lambda) \otimes M)$$

which means that $Y$ is a projective object in $\text{Rep}^k(\mathcal{P})$ (see also [EHS15, Lemma 8.2.1]).

The covering epimorphism $P_{2k}(0) \to 1$ induces an epimorphism

$$P_{2k}(0) \otimes L(\lambda) \to L(\lambda)$$

which factors through an epimorphism $Y \to L(\lambda)$.

By definition of projective cover, the latter induces a split epimorphism

$$Y \twoheadrightarrow P_k(\lambda).$$

We now consider the composition

$$P_{2k}(0) \otimes \Delta(\lambda) \rightarrow P_{2k}(0) \otimes L(\lambda) \rightarrow Y \rightarrow P_k(\lambda)$$

where the first map is induced by the epimorphism $\Delta(\lambda) \rightarrow L(\lambda)$.

Applying Steps 1 and 2, we conclude that there exists an epimorphism $T \to P_k(\lambda)$ where $T$ is in the image of $\mathcal{Q}$.

**Proof of Step 4:** Let $k$ be such that $M$ belongs to $\text{Rep}^k(\mathcal{P})$. The category $\text{Rep}^k(\mathcal{P})$ has enough projectives, so there exists an epimorphism $P \to M$ where $P$ is a projective object in $\text{Rep}^k(\mathcal{P})$. Applying Step 3, we obtain an epimorphism $T \to P$, with $T$ in the image of $\mathcal{Q}$; composed with the former, it gives an epimorphism

$$T \to M$$

as wanted. \hfill \Box

**Lemma 7.2.2.** Let $n \geq 0$. Then there exists $X_n \in I(\mathcal{Q})$ such that $F_n(X_n) = P_n(0)$ (projective cover of trivial module in $\text{Rep}(\mathfrak{p}(n))$), and the cosocle of $X_n$ is 1.

**Proof.** We will use the following auxiliary statement.

**Sublemma 7.2.3.** There exist translation functors $\Theta_{i_1}, \ldots, \Theta_{i_t}$ on $\text{Rep}(\mathfrak{p}(n))$ such that $P_n(0) = \Theta_{i_t} \circ \cdots \circ \Theta_{i_1} \mathbb{C}$.

**Proof.** Recall that by Corollary 6.1.2, any object in $\text{Rep}(\mathfrak{p}(n))$ is a subquotient of a direct sum of tensor powers of $V_n$. Hence any indecomposable projective is an indecomposable direct summand of some tensor power of $V_n$. It remains to check that summands of the form $\Theta_{i_t} \circ \cdots \circ \Theta_{i_1} \mathbb{C}$ are indeed indecomposable. Indeed, translation functors take indecomposable projectives either to zero or to indecomposable projectives (see Theorem 6.2.4), and hence take objects with simple cosocle to either zero or objects with simple cosocle. Thus objects of the form $\Theta_{i_t} \circ \cdots \circ \Theta_{i_1} \mathbb{C}$ have simple cosocle and are indecomposable. \hfill \Box

We now define $X_n := \Theta_{i_t} \circ \cdots \circ \Theta_{i_1} 1 \in \text{Rep}(\mathcal{P})$ where $i_1, \ldots, i_t$ are as in the Sublemma above. Then $F_n(X_n) \cong P_n(1)$ by Corollary 6.2.6.

The object $X_n$ has simple cosocle by Lemma 6.2.8 so we only need to check that there exists a non-zero map $X_n \to 1$, which follows from Lemma 6.1.3. \hfill \Box
Lemma 7.2.4. Let \( k \geq 0 \) and let \( P_k(0) \) be the projective cover of \( 1 \) in \( \text{Rep}^k(P) \). Then there exists a pair \((T, f)\) where \( T \in I(\mathfrak{B}) \) and \( f : T \to P_k(0) \) is a surjective map.

The proof is completely analogous to [EHST19] Lemma 8.4.3. We present it here for completeness.

Proof. Let \( n \gg k \) be such that \( \text{Rep}^k(P) \cong \text{Rep}^k(p(n)) \). Let \( P_n(0) \) be the projective cover of the trivial module in \( p(n) \) and let \( X_n \in I(\mathfrak{B}) \) be as in Lemma 7.2.2. Let \( P_k(0) \) be the projective cover of \( 1 \) in \( \text{Rep}^k(P) \), and let \( \pi : P_n(0) \to F_n(P_k(0)) \) be the quotient map.

Denote by \( Z := i^k,\kappa(X_n) \) the maximal quotient of \( X_n \) lying in \( \text{Rep}^k(P) \), with projection \( \phi : X_n \to Z \). Then \( \text{cosoc}(Z) \cong 1 \), and hence we have a surjective map \( p : P_k(0) \to Z \) making the diagram

\[
\begin{array}{ccc}
P_k(0) & \xrightarrow{p} & Z \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
\]

commutative. Clearly it is enough to prove that \( p \) is an isomorphism.

We will prove that \( p \) is a monomorphism (and hence an isomorphism). Indeed, recall that \( P_k(0) \) is a projective object in \( \text{Rep}^k(P) \); the objects in this subcategory are subquotients of objects in \( I(\mathfrak{B}) \cap \text{Rep}^k(P) \). Hence there exists \( D \in I(\mathfrak{B}) \cap \text{Rep}^k(P) \) such that \( P_k(0) \subset D \). We denote this inclusion by \( f \). Composing \( F_n(f) \) with \( \pi : F_n(X_n) \cong P_n(0) \to F_n(P_k(0)) \), we obtain a map \( g = F_n(f) \circ \pi : F_n(X_n) \to F_n(D) \). By Corollary 6.1.4, the functor \( F_n \) is full on \( I(\mathfrak{B}) \), hence there exists a map \( g : X_n \to D \) in \( \text{Rep}(P) \) such that \( F_n(g) = g \). Since \( D \in \text{Rep}^k(P) \), this map factors through \( X_n \to Z \), inducing a map \( \alpha : Z \to D \) such that \( \bar{\alpha} = \alpha \circ \phi \).

Hence we have: \( g = F_n(f) \circ \pi = F_n(\alpha) \circ F_n(p) \circ \pi \). The map \( \pi \) is surjective, so by cancellation law, \( F_n(f) = F_n(\alpha) \circ F_n(p) \). Since \( F_n(f) \) is injective, so is \( F_n(p) \), and so is \( p \) (here we use again that \( p \in \text{Rep}^k(P) \cong \text{Rep}^k(p(n)) \)).

To sum up our constructions: we have commutative diagrams in \( \text{Rep}(P) \) (left) and \( \text{Rep}(p(n)) \) (right):

\[
\begin{array}{ccc}
X_n & \xrightarrow{\bar{\phi}} & D \\
\downarrow & & \downarrow \\
P_k(0) & \xrightarrow{p} & Z \\
\downarrow & & \downarrow \\
1 & & 1
\end{array}
\]

\[
\begin{array}{ccc}
P_n(0) & = F_n(X_n) & \xrightarrow{g} & F_n(D) & \xrightarrow{F_n(\alpha)} & F_n(Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_n(P_k(0)) & \xrightarrow{F_n(f)} & F_n(D) & \xrightarrow{F_n(p)} & F_n(Z) & \xrightarrow{F_n(\phi)} & F_n(C)
\end{array}
\]

Lemma 7.2.5. Let \( \lambda \) be a partition. There exists \( T \in \mathfrak{B} \) such that \( T \to \Delta(\lambda) \) in \( \text{Rep}(P) \).

Proof. Recall that by Lemma 3.2.7 every simple in \( \text{Rep}(p(\infty)) \) occurs in the cosocle of some injective object, i.e., as a cosocle of some direct summand of a tensor power of \( V_{\infty} \). Together with Corollary 6.5.2, this implies that \( \Delta(\lambda) \) is a quotient of some direct summand of a tensor power of \( V \) in \( \text{Rep}(P) \), as required.

7.3. Splitting of epimorphism.

Proposition 7.3.1. For any epimorphism \( X \to Y \) in \( \text{Rep}(P) \) there exists a nonzero \( T \in \mathfrak{B} \) such that the epimorphism \( X \otimes I(T) \to Y \otimes I(T) \) splits.

Proof. We will start with the case that \( Y = 1 \) or \( \Pi \). It suffices to prove the statement for \( X = V^\otimes 2d \) since every \( X \) is a quotient of a finite direct sum of \( V^\otimes k \) and \( \text{SHom}(V^\otimes k, 1) \neq 0 \) implies that \( k \) is even.
Proposition 2.6.4 implies that
\[ \text{SHom}(V^\otimes k, 1) = \text{SHom}_{\text{Rep}(p(\infty))}(V^\otimes k, \mathbb{C}). \]
Consider the $S_{2d}$-decomposition in $\text{Rep}(p(\infty))$
\[ V^\otimes_{2d} = \oplus |\lambda| = 2d S^\lambda V_\infty \otimes Y_\lambda. \]
The cosocle of $S^\lambda V_\infty$ contains $\mathbb{C}$ if and only if $\lambda$ is quasisymmetric and in this case the multiplicity of $\mathbb{C}$ in $cosoc(S^\lambda V_\infty)$ is 1. Therefore it suffices to show that for any quasisymmetric $\lambda$ an epimorphism $\tau : S^\lambda V \to 1$ splits.
As a right $\mathbb{C}[S_{2d}]$-module the space $\text{SHom}(V^\otimes k, 1)$ is generated by the map
\[ \gamma : V^\otimes_{2d} \xrightarrow{\varphi} (\Pi V^*)^\otimes d \otimes V^\otimes d \xrightarrow{\text{ev}} 1, \]
where $\varphi = \omega^\otimes d \otimes id^\otimes d$ and $\omega : V \to \Pi V^*$ is the isomorphism induced by the form. Moreover, $\text{SHom}(V^\otimes k, \mathbb{C})$ as a right $\mathbb{C}[S_{2d}]$ module is induced from one-dimensional $H$-module $\mathbb{C}\gamma$, where $H$ is the stabilizer of $\mathbb{C}\gamma$ in $S_{2d}$ and is isomorphic to the Coxeter group of type $B_d$. In particular, this module has a basis which consists of $\gamma \sigma$ for some permutations $\sigma \in S_{2d}$. Let $\pi_{\lambda}$ be a Young projector associated with some tableau of shape $\lambda$ for a quasisymmetric partition $\lambda$. Then $\pi_{\lambda}\sigma\gamma \neq 0$ for some $\sigma$. Therefore, using the conjugation by $\sigma$ we may assume without loss of generality that $\pi_{\lambda}\gamma \neq 0$. Then $\tau$ is a composition
\[ S^\lambda(V) \xrightarrow{i_\lambda} V^\otimes_{2d} \xrightarrow{\varphi} (\Pi V^*)^\otimes d \otimes V^\otimes d \xrightarrow{\text{ev}} 1 \]
for an embedding $i_\lambda$ such that $\pi_{\lambda} \circ i_\lambda = i_\lambda$. Then the epimorphism
\[ V^\otimes d \otimes S^\lambda(V) \xrightarrow{id \otimes \tau} V^\otimes d \]
splits with splitting morphism given by $(id \otimes \pi_{\lambda}) \circ (id \otimes \varphi^{-1}) \circ (\text{coev} \otimes id)$. \hfill \qed

8. The second universal property

In this section we prove another universality property, describing the collection of categories $(\text{Rep}(P), \text{Rep}(p(n)))_{n \geq 1}$ as universal tensor $\mathbf{sVect}$-categories generated by a single object with a pairing into $\Pi \mathbb{1}$. This result is stated in Theorem 8.2.1. The statement and proof of the theorem requires the language of affine group schemes in pre-Tannakian tensor categories. The necessary definitions and results are given in Appendix II.

From now on, we will consider pre-Tannakian tensor $\mathbf{sVect}$-categories (see Section 2 for definitions), and will omit the pre-Tannakian assumption, as well as write tensor functor instead of “tensor $\mathbf{sVect}$-functor”.

8.1. Affine group scheme $P(X)$.

8.1.1. Definition. Let $T$ be a $\mathbf{sVect}$-tensor category and let $X \in T$ be an object together with a non-degenerate symmetric form $\omega X : X \otimes X \to \Pi \mathbb{1}$. We denote by $\psi X : X \to \Pi X^*$ the induced isomorphism.

Consider the functor $\text{Alg}_T \to \text{Grps}$ defined by the formula
\[ A \mapsto \text{Aut}_{A-\text{Mod}}(i_A(X), i_A(\omega X)). \]
That is, $A$ is sent to the group of $A$-module automorphisms $\theta : A \otimes X \to A \otimes X$ such that the diagram below is commutative:
\[ \begin{array}{ccc}
A \otimes X & \xrightarrow{i_A(\psi X)} & A \otimes \Pi \mathbb{1} \otimes_A A \otimes X^* \\
\theta \downarrow & & \downarrow \text{Id} \otimes_A \theta^\vee \\
A \otimes X & \xrightarrow{i_A(\psi X)} & A \otimes \Pi \mathbb{1} \otimes_A A \otimes X^* 
\end{array} \]

where \( \theta^r : A \otimes X^* \to A \otimes X^* \) is given by

\[
A \otimes X^* \xrightarrow{\Id \otimes \coev_X} A \otimes X^* \otimes X \otimes X^* \xrightarrow{\sigma_X, X^*} A \otimes X \otimes X^{* \otimes 2} \xrightarrow{\theta \otimes \Id} A \otimes X \otimes X^{* \otimes 2} \to A \otimes X^*
\]

and \( \sigma_A : A \otimes A^* \to A^* \otimes A \) is the symmetry isomorphism.

As before, this functor is corepresentable. The Hopf \( T \)-algebra representing this functor is the largest Hopf \( T \)-algebra quotient of \( \mathcal{O}(GL(X)) \) which preserves the form \( \omega \). It can be written explicitly as a quotient of \( \text{Sym}(X \otimes X^*) \) by a certain ideal, similar to \([Et14, \text{Definition 2.3}]\).

**Definition 8.1.1.** Let \( P(X) \in \text{Grps}_T \) be the \( T \)-group representing the above functor.

We have an obvious \( T \)-group inclusion \( P(X) \to GL(X) \), so \( X \) is a faithful representation of \( P(X) \).

By Lemma \([11.6.1]\) this immediately implies:

**Lemma 8.1.2.** Any representation of \( P(X) \) in \( T \) is a subquotient of \( \bigoplus_{i \in I} X^{\otimes a_i} \otimes U \) for some finite set \( I \), \( a_i \in \mathbb{Z}_{\geq 0} \) and some \( U \in T \) considered with a trivial \( P(X) \)-action.

We now want to consider such representations when \( U \) belongs to the essential image of \( s\text{Vect} \) in \( T \):

**Notation 8.1.3.** We denote by \( \text{Rep}(P(X)) \) be the tensor \( s\text{Vect} \)-category of \( P(X) \)-representations generated by tensor powers of \( X \) and their parity shifts. Namely, the objects in this category are subquotients of the \( P(X) \)-representation \( \bigoplus_{i \in I} X^{\otimes a_i} \otimes (\Pi_1)^{\otimes b_i} \) for some finite set \( I \), and some \( a_i \in \mathbb{Z}_{\geq 0} \), \( b_i \in \{0, 1\} \). The morphisms are \( P(X) \)-equivariant \( T \)-morphisms.

**Remark 8.1.4.** By Lemma \([8.1.2]\) for \( T = s\text{Vect} \) and \( X \) a finite dimensional vector superspace, the category \( \text{Rep}(P(X)) \) contains all the finite dimensional (super-)representations of \( P(X) \).

Let \( \tilde{P}_X \) denote the fundamental group of \( \text{Rep}(P(X)) \).

**Lemma 8.1.5.** We have an inclusion \( i : P(X) \to \tilde{P}_X \) (\( P(X) \) considered with conjugation action) and a quotient \( q : \tilde{P}_X \to \mu_2 \), with \( q \circ i = 1 \).

**Remark 8.1.6.** The canonical \( T \)-group homomorphism \( \pi_X : \pi(T) \to GL(X) \) does not factor through the inclusion \( P(X) \to GL(X) \); this is manifested in the fact that \( \tilde{P}_X \) does not split into a direct product of \( P(X) \) and \( \mu_2 \), and the category \( \text{Rep}(P(X)) \) cannot be “halved”, unlike for example \( \text{Rep}(GL(mn)) \) (see \([EHSS15]\)).

**Proof.** Let \( \tilde{P}_X \) denote the fundamental group of \( \text{Rep}(P(X)) \). The action of \( P(X) \) on its own representations induces a \( \text{Rep}(P(X)) \)-group inclusion \( i : P(X) \to \tilde{P}_X \).

We have a tensor functor \( J : s\text{Vect} \to \text{Rep}(P(X)) \) taking a vector superspace \( V \) to \( V \otimes \mathbb{1} \) with trivial \( P(X) \)-action. This functor induces a \( \text{Rep}(P(X)) \)-group homomorphism \( q : \tilde{P}_X \to J(\pi(s\text{Vect})) \cong \mu_2 \). This homomorphism is surjective by Lemma \([11.4.2]\).

Since \( P(X) \) acts trivially on the objects in the image of \( J \), the composition of the homomorphisms \( P(X) \to \tilde{P}_X \to \mu_2 \) is trivial, as required.

**Remark 8.1.7.** In fact, one can show that \( \tilde{P}_X \) is an extension of \( P(X) \) and \( \mu_2 \) whenever all trivial representations of \( P(X) \) in \( \text{Rep}(P(X)) \) belong to the essential image of \( J \).

8.1.2. Functoriality.

**Lemma 8.1.8.** Let \( F : T' \to T \) be a tensor functor between two tensor \( s\text{Vect} \)-categories. Let \( X \in T' \), and \( \omega_X : X \otimes X \to \Pi_1 \) a symmetric form. Then we have an isomorphism of \( T \)-group schemes \( F(P(X)) \cong P(F(X)) \).
Proof. Let $G = P(X)$. The $\mathcal{T}$-group scheme $F(G)$ is defined by the functor of points $\text{Alg}_\mathcal{T} \to \text{Grps}$ given by the formula

$$F(G)(B) = G(F^!(B)).$$

where $F^! : \text{Ind} - \mathcal{T} \to \text{Ind} - \mathcal{T}'$ is the right adjoint to $F$, which is automatically lax symmetric monoidal. We can define a $\mathcal{T}$-group homomorphism

$$F(G) \to P(F(X))$$
on the functors of points in the following way: for a fixed $B \in \text{Alg}_\mathcal{T}$, we would like to define a group homomorphism

$$\text{Aut}_{\text{Mod}_{\mathcal{T}'(B)}} \left( i_{F^!(B)}(X), i_{F^!(B)}(\omega_X) \right) \to \text{Aut}_{\text{Mod}_{\mathcal{T}B}} (i_B(F(X)), i_B(F(\omega_X)))$$

For

$$\alpha \in \text{Aut}_{\text{Mod}_{\mathcal{T}'(B)}} \left( i_{F^!(B)}(X), i_{F^!(B)}(\omega_X) \right) = \text{Aut}_{\text{Mod}_{\mathcal{T}'(B)}} \left( F^!(B) \otimes X, F^!(B) \otimes \omega_X \right)$$

we have a map $F(\alpha) \in \text{Aut}_{\text{Mod}_{\mathcal{T}'(B)}} \left( FF^!(B) \otimes X, FF^!(B) \otimes \omega_X \right)$. Under base change $FF^!(B) \to B$ we obtain an element of $\text{Aut}_{\text{Mod}_{\mathcal{T}B}} (i_B(F(X)), i_B(F(\omega_X)))$. Hence we built a $\mathcal{T}$-group homomorphism $F(G) \to P(F(X))$. One can then check that it is an isomorphism by writing explicitly $O(G) = O(P(X))$ as a quotient of $\text{Sym}(X \otimes X^*)$ (see [ET14]).

Let us now consider the following situation.

Let $\mathcal{T}'$ be a tensor $\mathbf{sVect}$-category, and let $X \in T'$ with a symmetric form $\omega_X : X \otimes X \to \Pi 1$. Assume $\mathcal{T}'$ is generated by $X, \omega_X$ as a tensor $\mathbf{sVect}$-category; that is, assume we have a short exact sequence of $\mathcal{T}'$-groups

$$1 \to P(X) \to \pi(\mathcal{T}') \to \mu_2 \to 1$$

and any object $M \in T'$ is a subquotient of $\bigoplus_{i \in I} X^{\otimes a_i} \otimes (\Pi 1)^{\otimes b_i}$ for some finite set $I$, and some $a_i \in \mathbb{Z}_{\geq 0}, b_i \in \{0,1\}$.

Lemma 8.1.9. Let $F : \mathcal{T}' \to \mathcal{T}$ be a tensor functor into a tensor $\mathbf{sVect}$-category $\mathcal{T}$. Let $Y := F(X)$. Then $F$ induces an equivalence of categories $\mathcal{T}' \to \text{Rep}(P(Y))$.

Proof. Let $G = \pi(\mathcal{T}), G' = \pi(\mathcal{T}')$ and $\varepsilon : G \to F(G')$ be the induced morphism (as in Section 11.2). By Theorem 11.3.1 we have an equivalence of tensor $\mathbf{sVect}$-categories

$$\mathcal{T}' \cong \text{Rep}(F(G'), \varepsilon)$$

(note as in Theorem 11.3.1).

Now, by Lemma 8.1.8 $F(P(X)) \cong P(Y)$, hence we have a $\mathcal{T}$-group inclusion $P(Y) \to F(G')$. The corresponding restriction functor is a tensor functor

$$R : \text{Rep}_\mathcal{T}(F(G'), \varepsilon) \to \text{Rep}(P(Y))$$

and we wish to show that this is an equivalence. For this, let $\tilde{P}_Y$ denote the fundamental group of $\text{Rep}(P(Y))$, and $\varepsilon' : G \to \tilde{P}_Y$ the corresponding $\mathcal{T}$-group homomorphism.

By Theorem 11.3.1 we have a tensor equivalence $\text{Rep}(P(Y)) \cong \text{Rep}(\tilde{P}_Y, \varepsilon')$, and hence commuting $\mathcal{T}$-group homomorphisms

$$\tilde{P}_Y \xrightarrow{r} F(G') \xrightarrow{\varepsilon'} G$$

Clearly, it is enough to check that $r : \tilde{P}_Y \to F(G')$ is an isomorphism.

We begin by showing that $r$ is an inclusion. Indeed, $Y$ is a faithful representation of both $F(G')$ and $\tilde{P}_Y$ (in the sense of Lemma 11.6.1), and hence both $\mathcal{T}$-groups embed into $\text{GL}(Y)$, their embeddings compatible with maps $r, \varepsilon, \varepsilon'$. Hence $r$ is injective.
By assumption, the $\mathcal{T}'$-group $G'$ is an extension of $P(X)$, $\mu_2$; hence $F(G')$ is an extension of $P(Y) \cong F(P(X))$, $\mu_2 \cong F(\mu_2)$. By Lemma 8.1.5, the $\mathcal{T}$-group $\tilde{P}_Y$ comes with an inclusion $i : P(Y) \to \tilde{P}_Y$ and a quotient $q : \tilde{P}_Y \to \mu_2$ such that $q \circ i = 1$. Hence the $\mathcal{T}$-group inclusion $r : \tilde{P}_Y \to F(G')$ is an isomorphism, and the lemma is proved.

In particular, we obtain a stronger version of the result in Lemma 8.1.5.

**Corollary 8.1.10.** Let $\mathcal{T}$ be a $s\text{Vect}$-tensor category, and let $X \in \mathcal{T}$ be an object together with a symmetric form $\omega_X : X \otimes X \to \Pi$. Let $P_X$ be the fundamental group of $\text{Rep}(P(X))$. Then $P_X$ is an extension of $P(X)$ (with conjugation action) and $\mu_2$ (with trivial $P(X)$ action).

**Proof.** This follows directly from Proposition 11.5.1 (alternatively, from Lemma 8.1.5 and the equivalence $\text{Rep}(P_n) \cong \text{Rep}(P(n))$).

**Corollary 8.1.11.** The fundamental group of $\mathcal{T} = \text{Rep}(P(n))$ is a semidirect product of $P_n$ and $\mu_2$, where $V_n$ is the defining representation of $p(n)$ with isomorphism $\omega_n : V_n \otimes V_n \to \Pi$.

**Proof.** Throughout the proof, we will consider $\text{Rep}(P)$-groups.

Let $A \in \text{Alg}_{\text{Rep}(P)}$. Recall that the functor $A \mapsto P(A)$ is given by $P(A) = \text{Aut}_\mathcal{T}(i_A)$ where $i_A : \text{Rep}(P) \to A \otimes \text{Mod}$, $M \mapsto A \otimes M$ is a tensor functor between tensor categories, Theorem 7.0.2 implies that $\text{a } \otimes \text{-automorphism } \theta : i_A \to i_A$ is uniquely defined by its restriction to the full subcategory $I(\mathfrak{P}) \subset \text{Rep}(P)$. In turn, the restriction is completely determined by the maps

$$\theta_{\Pi} : A \otimes \Pi \to A \otimes \Pi$$

and

$$\theta_V : A \otimes V \to A \otimes V$$

such that the diagrams below commute:

\[
\begin{array}{ccc}
A \otimes \Pi \otimes A \otimes \Pi & \to & A \\
\theta_{\Pi} \downarrow & & \downarrow \text{Id}
\end{array}
\quad
\begin{array}{ccc}
A \otimes V & \overset{i_A(\psi)}{\to} & A \otimes \Pi \otimes_A A \otimes V^* \\
\theta \downarrow & & \downarrow \text{Id} \\
A \otimes V & \overset{i_A(\psi)}{\to} & A \otimes \Pi \otimes_A A \otimes V^*
\end{array}
\]

(here $\psi_V : V \to \Pi V^*$ is the isomorphism induced by $\omega_V$, as before). We now construct the maps in the extension (10).

First, consider the group homomorphism $\phi : P(V) \to P$ defined as follows. Let $\eta \in P(V)(A)$. This is a map $\eta : A \otimes V \to A \otimes V$ preserving $\omega_V$ (and hence $\psi_V$).

We then define the homomorphism between (usual) groups $\phi_A : P(V)(A) \to P(A)$ as

$$\eta \in P(V)(A) \mapsto \tilde{\eta} \in P(A)$$

with $\tilde{\eta}_{\Pi} = \text{Id}$ and $\tilde{\eta}_V = \eta$. The map $\phi_A$ is clearly injective.
Next, consider the tensor functor \( J : \text{sVect} \to \text{Rep}(\mathcal{P}) \).

We have a homomorphism \( q : \mathcal{P} \to \mu_2 \cong J(\pi(\text{sVect})) \) given by
\[
\theta \in \mathcal{P}(A) \mapsto \theta_{11} \in \mu_2(A)
\]
(see also [De90], Section 8 and the proof of Lemma 8.1.5). This homomorphism is surjective by Lemma 11.4.2.

Thus we have and inclusion \( \phi : \mathcal{P}(V) \to \mathcal{P} \) and a quotient \( q : \mathcal{P} \to \mu_2 \). By the constructions above, the composition of these maps is zero. Furthermore, the fact that \( \theta \in \mathcal{P}(A) \) is determined by \( \theta_{11} \) and \( \theta_V \) implies that \( q \) is indeed the cokernel map of \( \phi \).

The statement of the Proposition follows.

\[\square\]

8.2. Universal property.

**Theorem 8.2.1.** Let \( A \) be a pre-Tannakian SM \( \text{sVect} \)-category, and let \( X \in A \) be a dualizable object with a non-degenerate symmetric form
\[
\omega_X : X \otimes X \to \Pi_1 A.
\]
Consider the canonical SM functor \( F_X : \mathfrak{F} \to A \) sending the generator \( \mathcal{V} \) of \( \mathfrak{F} \) to \( X \) and the form \( \omega_V \) on \( \mathcal{V} \to \omega_X \).

1. If \( X \) is not annihilated by any Schur functor then \( F_X \) factors through the embedding \( I : \mathfrak{F} \to \text{Rep}(\mathcal{P}) \) and gives rise to a faithful tensor functor
\[
F_X : \text{Rep}(\mathcal{P}) \to T, \quad V \mapsto X, \omega_V \to \omega_X.
\]
Furthermore, the functor \( F_X \) factors through an equivalence of categories \( \text{Rep}(\mathcal{P}) \to \text{Rep}(P(X)) \).

2. If \( X \) is annihilated by some Schur functor then there exists a unique \( n \in \mathbb{Z}_+ \) such that \( F_X \) factors through the SM functor \( \mathfrak{F} \to \text{Rep}(\mathfrak{p}(n)) \) and gives rise to a faithful tensor functor
\[
F_X : \text{Rep}(\mathfrak{p}(n)) \to T, \quad V_n \mapsto X
\]
with the form on \( V_n \) sent to \( \omega_X \). Furthermore, the functor \( F_X \) factors through an equivalence of categories \( \text{Rep}(\mathfrak{p}(n)) \to \text{Rep}(P(X)) \).

**Lemma 8.2.2.** If \( X \) is not annihilated by any Schur functor then the functor \( F_X \) is faithful.

**Proof.** In this proof, we will write
\[
\mathfrak{s}\text{Hom}_{\text{Rep}(\mathcal{P})}(U, W) := \text{Hom}_{\text{Rep}(\mathcal{P})}(U, W) \oplus \text{Hom}_{\text{Rep}(\mathcal{P})}(U, \Pi W)
\]
to avoid unnecessary repetitions.

Recall that for any \( m, k \geq 0 \), the space \( \mathfrak{s}\text{Hom}_{\text{Rep}(\mathcal{P})}(V^\otimes m, V^\otimes k) \) is zero iff \( m + k \notin 2\mathbb{Z} \). If \( m + k \in 2\mathbb{Z} \), let \( r = (m + k)/2 \). We have an isomorphism
\[
\mathfrak{s}\text{Hom}_{\text{Rep}(\mathcal{P})}(V^\otimes m, V^\otimes k) \cong \mathfrak{s}\text{Hom}_{\text{Rep}(\mathcal{P})}(V^\otimes r, V^\otimes r)
\]
defined via the isomorphism \( V^* \cong \Pi V \). This isomorphism is preserved by \( F_X \), since the latter is a SM functor. Hence it is enough to check that \( F_X \) is injective on the algebra \( \mathfrak{s}\text{End}_{\text{Rep}(\mathcal{P})}(V^\otimes r) \).

Since \( F_X \) is SM, it commutes with symmetry isomorphisms, meaning that
\[
F_X : \mathfrak{s}\text{End}_{\text{Rep}(\mathcal{P})}(V^\otimes r) \to \text{End}_T(X^\otimes r)
\]
is an \( S_r \times S_r \)-equivariant map.

The algebra \( \mathfrak{s}\text{End}(V^\otimes r) \) is called the signed (or odd) Brauer algebra, and has a diagrammatic basis \( \mathcal{B} \), each element describing a string diagram on \( 2r \) endpoints, located in 2 rows (\( r \) dots in each row). We refer the reader to [BDE+18] for a detailed description of the basis, and to [CE17b, KTI14] for more details on the algebra.

We will use the fact that such a basis can be described using compositions and tensor products of (both even and odd!) maps \( \text{Id}_V : V \to V \), \( \sigma : V \otimes V \to V \otimes V \) (the symmetry morphism) \( \omega_V : V \otimes V \to \Pi V \) and \( \omega_V : \Pi V \to V \otimes V \). In terms of diagrams, these correspond respectively to vertical strings, crossings, caps and cups.
Now let $\sum_{D \in B} \alpha_D D$ be a linear combination of diagrams such that $F_X (\sum_{D \in B} \alpha_D D) = 0$.

Let $B' \subset B$ be the set of $D$ such that $\alpha_D \neq 0$. We may assume that $B'$ has the minimal possible size, and will show that we must have $B' = \emptyset$.

First, recall that $F_X$ does not annihilate any Schur functor, hence it is injective on the subalgebra $\mathbb{C} S_\cdot \subset \mathfrak{s} \text{End}(\mathcal{V} \otimes r)$ given by permutations of the factors. This means that at least one of $D \in B'$ has a cup or a cap.

**Sublemma 8.2.3.** All $D \in B'$ have caps and caps in the same positions.

**Proof.** Indeed, let $D_0 \in B'$, and assume it has a cup connecting positions $i, j$ (i.e. it has a tensor factor $\omega_X : \Pi_1 \to \mathcal{V} \otimes \mathcal{V}$ in positions $i, j$). Let us compose $\sum_{D \in B} \alpha_D D$ with $f = \omega_{i,j}^1 \circ \omega_{i,j} \in \mathfrak{s} \text{End}(\mathcal{V} \otimes r)$, where $\omega_{i,j}$ indicates the pairing $\omega_X$ is performed on the $i$-th and $j$-th factors (and similarly for $\omega_{i,j}^1$).

Since $F_X$ is monoidal, $F_X$ commutes with $\omega_X, \omega_X^1$, and hence $\sum_{D \in B} \alpha_D F(f \circ D) = 0$.

Now, for each $D \in B$, the morphism $f \circ D$ is either zero an element of the basis $\mathcal{B}$, up to a sign (see [BDE+18]). Moreover, $f \circ D = 0$ precisely when $D$ has a cup connecting positions $i, j$ (so $f \circ D_0 = 0$). Hence $\sum_{D \in B} \alpha_D f \circ D$ is a linear combination having less summands than $\sum_{D \in B} \alpha_D D$, and hence must have all coefficients equal to zero. This implies that all the diagrams in $B'$ have a cup connecting positions $i, j$.

In the same way, we can show that diagrams in $B'$ have caps in the same positions. $\square$

Assume $B'$ is not empty. Using the fact that $F_X$ is $S_\cdot \times S_{\cdot'}$-equivariant, we may rewrite $\sum_{D \in B} \alpha_D D$ as $\phi \otimes \psi$, where $\phi \in \mathbb{C} S_\cdot \subset \mathfrak{s} \text{End}(\mathcal{V} \otimes r')$ for some $r' > 0$ and $\psi \in \mathfrak{s} \text{End}(\mathcal{V} \otimes r)$ is of the form

$$(\omega_{1,2} \circ \omega_{1,2}) \otimes (\omega_{3,4} \circ \omega_{3,4}) \otimes \ldots$$

Now, by assumption on $F_X$, $F_X(\phi) \neq 0$. Moreover, $\omega_X$ is non-degenerate, so $F_X(\psi) \neq 0$. This implies that

$$F_X \left( \sum_{D \in B} \alpha_D D \right) \cong F_X(\phi \otimes \psi) \neq 0$$

leading to a contraction. $\square$

**Proof of Theorem 8.2.1.** First, assume $X$ is not annihilated by any Schur functor. We have seen that the functor $F_X$ is faithful. By Theorem 7.1.2, $F_X$ would then factor through $\mathcal{T}$, giving a tensor functor

$$F_X : \text{Rep}(\mathcal{P}) \to \mathcal{T}.$$ 

By Lemma 8.1.9 and Proposition 8.1.13, the functor $F_X$ factors through an equivalence of categories $\tilde{F}_X : \text{Rep}(\mathcal{P}) \cong \text{Rep}(\mathcal{P}(X))$.

Next, assume $X$ is annihilated by some Schur functor. Let $\mathcal{T}' := \text{Rep}(\mathcal{P}(X))$, and let $\tilde{P}_X = \pi(\mathcal{T}')$ be its fundamental group. By definition of $\text{Rep}(\mathcal{P}(X))$, it is generated by $X$ under taking tensor powers, direct sums, parity shifts and subquotients. Hence any object of $\mathcal{T}'$ is annihilated by some Schur functor. By Deligne’s theorem on super-Tannakian reconstruction (see [Del02]), $\mathcal{T}'$ is a (super-)Tannakian tensor category: namely, there exists a tensor functor

$$\Psi : \mathcal{T}' \to \text{sVect}$$

This functor induces an $\text{sVect}$-group homomorphism $\nu : \mu_2 \to \Psi(\tilde{P}_X)$.

Recall that the faithful tensor functor $\Psi$ preserves (categorical) dimensions, so $\Psi(X)$ is a non-zero vector superspace of (super-)dimension zero. Hence $\Psi(X) \cong V_n$ for some $n \in \mathbb{Z}_{\geq 1}$.

Now, the composition of $\Psi$ with $F_X : \Psi \to \mathcal{T}'$ is isomorphic to the composition of functors

$$\Psi \circ \text{Rep}(\mathcal{P}(n)) \cong \text{Forget} \to \text{sVect}$$

with the functor $\Psi \to \text{Rep}(\mathcal{P}(n))$ given by $V \mapsto V_n$.

Let $G := \Psi(\tilde{P}_X)$.

By Theorem 11.3.1, $\Psi$ induces an equivalence of categories $\mathcal{T}' \to \text{Rep}(G, \nu)$ (notation as in Section 11.3). By Corollary 8.1.10 and Lemma 8.1.8, $G$ is an extension of $\mathcal{P}(V_n)$ (as a subgroup)
and $\mu_2$ (as a quotient). Then homomorphism $\nu : \mu_2 \to G$ makes $G$ into a semidirect product of $P(V_n)$ and $\mu_2$. Hence we have an equivalence of tensor categories

$$F_X : \text{Rep}(P(V_n)) \to T' = \text{Rep}(P(X)).$$

Hence $\text{Rep}(P(X)) \cong \text{Rep}(p(n))$ for some (unique) $n$, and the functor $\Psi$ induces an equivalence of tensor categories $F_X : \text{Rep}(p(n)) \to \text{Rep}(P(X))$, whose composition with the functor $\Psi : \text{Rep}(p(n)) \to \text{Rep}(P(X))$ is isomorphic to $F_X$, as required.

8.3. An alternative construction of $\text{Rep}(P)$. As a corollary of Theorem 8.2.1 we present an alternative construction of the tensor category $\text{Rep}(P)$. Although this construction is less explicit than the one given in Section 6, it has the benefit of being much more compact.

Let $\text{Rep}(GL_t)$ be the abelian Deligne category described in Section 2.2 and denote by $X_0$ the $t$-dimensional generating object of $\text{Rep}(GL_t).

Let $T := \text{Rep}(GL_t)$. The fundamental group of $\text{Rep}(GL_t)$ is $GL_t := GL(X_0)$ (see [Del07]). This group acts naturally on any object $M$ in $T$ by a $T$-group homomorphism $\pi_M : GL_t \to GL(M)$.

The Lie algebra $gl_t$ of $GL_t$ is isomorphic to $X_0 \otimes X_0^*$ (see also Section 2.2), and acts naturally on any object $M$ of $T$ by

$$act_M : gl_t \otimes M \to M$$

coming from $\pi_M$.

The fundamental group of $T$ is

$$\pi(T) \cong GL_t \times \mu_2.$$ The natural action of this group on any object $M$ in $T$ again denoted by $\pi_M$.

Let $X_1 := \Pi X_0^*$, $X := X_0 \otimes X_1 \in \text{sRep}(GL_t)$ and consider the non-degenerate symmetric pairing $\omega : X \otimes X \to \Pi I$ given by

$$\omega_{i,j} : X_i \otimes X_j \to \Pi I \quad (i,j \in \{0,1\}), \quad \omega_{i,j} = \begin{cases} \Pi e_{i,j} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$ Let $P(X)$ be the corresponding $T$-group; we have a $T$-group inclusion

$$\eta : GL(X_0) \cong GL_t \hookrightarrow P(X).$$

Remark 8.3.1. The subgroup $GL_t$ can be seen as the “even part” of the $\text{Rep}(GL_t)$-supergroup $P(X)$.

Let $\tilde{\text{Rep}}(P(X))$ be the category of $P(X)$-representations $(M,\rho)$ in $T$ such that $\rho \circ \eta \cong \pi_M$. This is a tensor category.

Lemma 8.3.2. The category $\tilde{\text{Rep}}(P(X))$ is isomorphic to the category $\text{Rep}(P(X))$ of representations of $P(X)$, as defined in Notation 8.1.3.

Proof. Consider the $T$-group homomorphism $\varepsilon : GL_t \times \mu_2 \to P(X) \rtimes \mu_2$ given by $GL_t \to P(X)$. We can rewrite the definition of $\tilde{\text{Rep}}(P(X))$ as representations $\rho : P(X) \rtimes \mu_2 \to GL(M)$, $M \in T$ whose composition with $\varepsilon$ gives the natural action $\pi_M$.

Deligne’s theorem [11.3.1] then implies that $P(X) \rtimes \mu_2$ is the image of the fundamental group of $\tilde{\text{Rep}}(P(X))$ under the forgetful functor $F : \tilde{\text{Rep}}(P(X)) \to T$. Now Lemma 8.1.3 implies the required statement.

Proposition 8.3.3. There is an equivalence of tensor categories

$$\text{Rep}(P) \to \text{Rep}(P(X)), \ V \mapsto X, \ \omega_V \mapsto \omega_X.$$
Proof. We only need to check that $X$ is not annihilated by any Schur functor; if this holds, then by Theorem 5.2.1 we are done.

Assume $X$ is annihilated by some Schur functor. Then $X_0 \in \text{Rep}(GL_0)$ is annihilated by the same functor, which contradicts the construction of $\text{Rep}(GL_0)$ as a “free” rigid SM category generated by $X_0$ (cf. Section 2.3 and [EHS15]).

Remark 8.3.4. Let $\text{End}(X) \cong X \otimes X^*$ be the internal endomorphism algebra of $X$, seen as a Lie algebra object in $s\text{Rep}(GL_0)$. Let $p(X) \subseteq \text{End}(X)$ be Lie subalgebra preserving $\omega_X$. Then $p(X) = \text{Lie}P(X)$, and $\mathfrak{g}(X_0) \subseteq p(X)$ is a direct summand.

We have a “differentiation” functor

$$\text{Rep}(P(X)) \rightarrow \text{Rep}(p(X))$$

which can be shown to be SM and fully faithful, but is not essentially surjective (that is, surjective on objects). For example, given $a \in \mathbb{C} \setminus \mathbb{Z}$, we can define a homomorphism $\mathfrak{gl}(X_0) \rightarrow 1$ which does not integrate to an action of $GL_0$ on $1$; hence we have a homomorphism $p(X) \rightarrow 1$ which does not integrate to an action of $P(X)$ on $1$.

By analogy with the representation theory of (usual) supergroups, we may define $\text{Rep}(p(X))_{int}$ to be the full subcategory of “integrable” $p(X)$ representations: that is, representation of $p(X)$ whose restriction to $\mathfrak{gl}((X)_0)$ can be lifted to $\text{Rep}(GL_0)$. In other words, in the diagram

$$\begin{array}{ccc}
\text{Rep}(p(X))_{int} & \rightarrow & \text{Rep}(p(X)) \\
\downarrow & & \downarrow \text{Forget} \\
\text{Rep}(GL_0) & \xrightarrow{diff} & \text{Rep}(\mathfrak{gl}_t)
\end{array}$$

the objects $M$ of $\text{Rep}(p(X))$ which lie in $\text{Rep}(p(X))_{int}$ are precisely those for which $\text{Forget}(M)$ lies in the essential image of the functor $diff$ in the lower row.

Clearly, $\text{Rep}(p(X))_{int}$ contains the essential image of the functor $\text{Rep}(P(X)) \rightarrow \text{Rep}(p(X))$. One could ask whether this induces an equivalence $\text{Rep}(P(X)) \rightarrow \text{Rep}(p(X))_{int}$ as in the classical Lie superalgebra theory.

We expect the answer to be negative, due to the fact that the “odd” part of $p(X)$ is not nilpotent anymore, but do not have a counterexample at the moment.

More generally, given a tensor category $\mathcal{T}$, one can consider the group schemes and their Lie superalgebras in $s\mathcal{T} = s\text{Vect} \otimes \mathcal{T}$, and ask a similar question in this situation: consider a group scheme $G$ in $s\mathcal{T}$ with Lie algebra object $\mathfrak{g} = \mathfrak{g}_0 \oplus \Pi \mathfrak{g}_1$, $\mathfrak{g}_0, \mathfrak{g}_1 \in \mathcal{T}$, with $\mathfrak{g}_0 = \text{Lie}(G_0)$, $G_0 \in \text{AffSch}(\mathcal{T})$. Given a $\mathfrak{g}$-representation in $s\mathcal{T}$ with $\mathfrak{g}_0$ action which integrates to $G_0$, does the $\mathfrak{g}$-action integrate to a $G$-action?

Here is an example when the answer is “no”, as was provided to us by P. Etingof. Let $\mathcal{T} = s\text{Vect}$. Consider the abelian Lie superalgebra $\mathfrak{g}$ in $s\mathcal{T}$ with $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_1 = \mathbb{C}^{(0|1)}$ (the abelian Lie superalgebra in $\mathcal{T}$). Construct an affine group scheme $G$ by setting $O(G) := S(\mathfrak{g}^*)$ in $s\mathcal{T}$ with the comultiplication map $\Delta(x) := x \otimes 1 + 1 \otimes x$ where $x$ is a non-zero vector in $\mathfrak{g}^*$. Note that $O$ is isomorphic to the polynomial algebra $\mathbb{C}[x]$ and its Lie algebra is isomorphic to $\mathfrak{g}$. The group $G$ does not have any non-trivial one-dimensional representation while its Lie algebra $\mathfrak{g}$ has a representation $\rho_\lambda(x^*) = \lambda$ for any $\lambda \in \mathbb{C}$.

9. Lower highest weight structure

9.1. Weights in the category $\text{Rep}(P)$. The set of infinite non-decreasing integer sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_i = 0$ for $i >> 0$ will be called the set of weights for $\text{Rep}(P)$.

We associate to $\lambda$ a weight diagram $d_\lambda$, defined as a labeling of the integer line by symbols • (“black ball”) and ○ (“empty”) such that such that $i$ has label • if $i$ belongs to the sequence $\lambda + (0, 1, 2, \ldots)$, and label ○ otherwise.

Definition 9.1.1. Let $\lambda$ be a weight for $\text{Rep}(P)$. We will define its $n$-th truncation $\lambda^{(n)}$ as the weight of $p(n)$ given by the first $n$ entries of $\lambda$. 

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Then $d_{\lambda(n)}$ is obtained from $d_\lambda$ by taking only the first $n$ dots (starting from the leftmost one).

With the notation above, we have:
\[
F_n(L(\lambda)) \cong L_n(\lambda^{(n)}), \quad F_n(\Delta(\lambda)) \cong \sum_n^k(\lambda^{(n)}), \quad F_n(\nabla(\lambda)) \cong \nabla_n^k(\lambda^{(n)})
\]
for $n \gg k$.

9.2. **Lower highest weight structure.** We will now show that $\text{Rep}(\mathcal{P})$ is a lower highest weight category (i.e. “locally” highest weight).

**Definition 9.2.1.** A lower highest weight category is an artinian abelian $\mathbb{C}$-linear category $\mathcal{A}$ together with a poset $(\Lambda, \leq)$ (poset of weights) and a filtration $\Lambda = \bigcup_{k \in \mathbb{Z}_+} \Lambda^k$, such that the following conditions hold:

1. The set $\Lambda$ is in bijection with the set of isomorphism classes of simple objects in $\mathcal{A}$.
2. For each $\xi \in \Lambda$, the Serre subcategory $\mathcal{A}(\leq \xi)$ generated by simples $\{L(\lambda), \lambda \leq \xi\}$ contains a projective cover $\Delta(\xi)$ of $L(\xi)$, and an injective hull $\nabla(\xi)$ of $\xi$. The objects $\Delta(\xi), \nabla(\xi)$ are called standard and costandard objects in $\mathcal{A}$.
3. There exists precisely one isomorphism class of indecomposable objects $T(\xi)$ in $\mathcal{A}$ which has $\Delta(\xi)$ as a submodule, $T(\xi)/\Delta(\xi)$ has a filtration with standard subquotients, and $T(\xi)$ also has a filtration with costandard subquotients.

Such objects $T(\xi)$ are the indecomposable tilting objects in $\mathcal{A}$.

4. Let $k \geq 0$, and let $\mathcal{A}^k$ be the full subcategory of $\mathcal{A}$ whose objects are subquotients of finite direct sums of objects $T(\xi), \xi \in \Lambda^k$. Then each $\mathcal{A}^k$ is a highest weight category with poset $(\Lambda^k, \leq)$, simple objects $\{L(\xi), \xi \in \Lambda^k\}$, standard objects $\{\Delta(\xi), \xi \in \Lambda^k\}$, costandard objects $\{\nabla(\xi), \xi \in \Lambda^k\}$, and tilting objects $\{T(\xi), \xi \in \Lambda^k\}$. The category $\mathcal{A}^k$ also has enough projective and injective objects.

5. The subcategories $\mathcal{A}^k$ form a filtration on the category $\mathcal{A}$: $\mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}^k$.

**Remark 9.2.2.** The main difference between a highest weight category and a lower highest weight category is the possible lack of projectives and injectives in $\mathcal{A}$.

Consider the set of weights $\Lambda \times \pm \Lambda$ as in 2.4.2, where $\pm$ stands for the parity shift of the simple module. We consider the filtration and order on $\Lambda \times \pm$ induced by the filtration $\bigcup_{k \geq 0} \Lambda^k$ and the partial order $\leq$ on $\Lambda$, with
\[
\Lambda^k = \{\lambda \in \Lambda | |\lambda| \leq k\}.
\]
The partial order and the filtration on $\Lambda \times \pm$ disregard the possible parity shift.

**Proposition 9.2.3.** The category $\text{Rep}(\mathcal{P})$ has a lower highest weight structure given by the set of weights $\Lambda \times \pm$ with order $\leq$. The objects $\Delta(\lambda)$ play the role of standard objects, $\nabla(\lambda)$ play the role of costandard objects, and $I(\mathfrak{g})$ is the full subcategory of tilting objects. Furthermore, the tensor duality $(-)^*$ makes $\text{Rep}^k(\mathcal{P})$ into highest weight categories with duality, in the sense of [CPS89].

**Remark 9.2.4.** There is an important difference between $\text{Rep}(\mathcal{P})$ and $\text{Rep}(\mathfrak{p}(n))$ as lower highest weight categories: the category $\text{Rep}(\mathfrak{p}(n))$ has two obvious highest weight structures (one with standard objects of the form $\Delta_n(\lambda)$ and the other with standard objects of the form $\nabla_n(\lambda)$), but has no duality functor which maps standard objects to costandard objects. Meanwhile, the category $\text{Rep}(\mathcal{P})$ has only one obvious lower highest weight structure, as described above, but has a duality functor. The objects $\nabla(\lambda)$ cannot play the role of standard objects: for instance, both $\nabla(0)$ and $\nabla(-2\varepsilon_1)$ have cosocle $\mathbb{C}$. We thank K. Coulembier for pointing this out to us.

**Proof.** The only non-obvious statements we need to check are

1. For any $k \geq 0$ and any $\varepsilon$-admissible $\lambda$, $\mathcal{P}_k(\lambda)$ has a filtration with standard subquotients (“standard filtration”) $\Delta(\mu)$ such that $\mu \geq \lambda$ (in this case clearly the top quotient of the filtration would be $\Delta(\lambda)$).
(2) For any $\lambda$, $V_{\infty}^{\otimes s}$ has a filtration with standard subquotients, and a filtration with costandard subquotients ("costandard filtration"). Furthermore, each standard object appears as a subobject in such a standard filtration for some $s \geq 0$.

The proof of (1) will be given in Lemma 9.2.7 below.

To prove (2), recall from Corollary 6.5.2 that $\Phi : \text{Rep}(p(\infty)) \to \text{Rep}(\mathcal{P})$ takes the socle filtration of $V_{\infty}^{\otimes s}$ to a standard filtration of $V_{\infty}^{\otimes s}$. Furthermore, this implies for any weight $\lambda$ of $\text{Rep}(\mathcal{P})$ such that $|\lambda| = s$, $\Delta(\lambda)$ appears as a subobject in the induced standard filtration of $S^\lambda V$. The costandard filtration on $V_{\infty}^{\otimes s}$ is then obtained by applying the functor $(-)^*$ to the standard filtration. □

Definition 9.2.5. Let $T(\lambda)$ be an indecomposable tilting object such that $\Delta(\lambda) \subset T(\lambda)$ and the cokernel has a filtration with subquotients $\Delta(\mu)$, $\mu \leq \lambda$.

Remark 9.2.6. It is easy to see that $\Delta(\lambda) \subset T(\lambda) \subset S^\lambda V$, but the latter is not necessarily an isomorphism. For example, $V_{\infty}^{\otimes 3}$ has a direct summand isomorphic (up to change of parity) to $V$, and hence has more than three indecomposable direct summands.

Lemma 9.2.7. For any $k \geq 0$ and any $k$-admissible $\lambda$, $P_k(\lambda)$ has a filtration with standard subquotients $\Delta(\mu)$ such that $\mu \geq \lambda$.

Proof. We begin by showing that $P_k(\lambda)$ has some standard filtration. Let $X \in I(\mathfrak{P})$ be such that $P_k(\lambda)$ is the maximal quotient of $X$ lying in $\text{Rep}(\mathcal{P})$. The existence of such $X$ is shown in the proof of Lemma 7.2.4. We may assume that $X$ is indecomposable, and hence a direct summand of $V_{\infty}^{\otimes r}$ for some $r \geq 0$.

We may assume that $r > k$ (otherwise $P_k(\lambda)$ is a direct summand of $V_{\infty}^{\otimes r}$).

Consider objects

$$K := \bigcup_{g: V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq k} \ker(g)$$

in $\text{Rep}(\mathcal{P})$ and

$$K' := \bigcup_{f: V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq k} \ker(f)$$

in $\text{Rep}(p(\infty))$.

Since $\Phi$ is exact, we have:

$$\Phi(K') \cong \bigcup_{f: V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq k} \ker(\Phi(f))$$

Sublemma 9.2.8. We have: $\Phi(K') \cong K$.

Proof. Since $\Phi$ is monoidal, $\Phi(f) \in \text{Hom}_{\text{Rep}(\mathcal{P})}(V_{\infty}^{\otimes r}, V_{\infty}^{\otimes s})$ for any $f : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}$. Thus $K \subset \Phi(K')$.

On the other hand, we can choose a diagrammatic basis $\{g_i\}$ for $\text{Hom}_{\text{Rep}(\mathcal{P})}(V_{\infty}^{\otimes r}, V_{\infty}^{\otimes s})$ as in [BDE*18, KT14] consisting (up to change of parity) of string diagrams with $s$ dots in top row and $r$ dots in bottom row. The strings can be caps (connecting two dots in the bottom row), cups (connecting two dots in top row), or strings connecting 2 dots from different rows. Similarly, $\text{Hom}_{p(\infty)}(V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s})$ has a diagrammatic basis $\{f_i\}$ with the same diagrams, but now no caps are allowed.

On such diagrammatic morphisms, composition is defined by stacking diagrams (bottom to top) and then transforming the obtained diagram into an element of the basis by some rules described e.g. in [BDE*18, KT14].

Consider a diagrammatic morphism $g_i \in \text{Hom}_{\text{Rep}(\mathcal{P})}(V_{\infty}^{\otimes r}, V_{\infty}^{\otimes s})$. It has at most $s$ strings going from top to bottom, and so at least $\left\lceil \frac{s}{2} \right\rceil$ caps in bottom row. Thus can be written as

$$g_i = h \circ \Phi(f_j)$$
for some diagrammatic morphism $f_j \in \text{Hom}_{p(\infty)}(V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s'})$ where $s' \leq k$, and some diagrammatic morphism $h \in \text{Hom}_{\text{Rep}(P)}(V_{\infty}^{\otimes s'}, V_{\infty}^{\otimes r})$.

This implies that $\text{Ker}(g_i) \subset \text{Ker}(\Phi(f_j))$ and hence $\Phi(K') \subset K$. The sublemma is proved. \qed

Let $N := \text{Ker}(X \to P_k(\lambda))$.

**Sublemma 9.2.9.** $N = K \cap X$.

**Proof.** For any $g : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}$, $s \leq k$, $N \subset \text{Ker}(g|_{X})$, and hence $N \subset K \cap X$.

Vice versa, $P_k(\lambda)$ is a subobject of some $D \in I(\bar{\Omega})$, and hence $P_k(\lambda) \subset V_{\infty}^{\otimes s}$ for some $s \leq k$. This implies that $P_k(\lambda) = \text{Im}(g)$ for some $g : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}$ and thus $K \cap X \subset N$. This proves the statement of the sublemma. \qed

We now consider the socle filtration

$$\text{soc}^0 = 0 \subset \text{soc}^1 \subset \ldots \subset \text{soc}^{r'} = V_{\infty}^{\otimes r}$$

where $r' = \lfloor \frac{r}{4} \rfloor + 1$. By [Ser] Lemma 17, the $i$-th term is given by

$$\text{soc}^i = \bigcup_{f : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq r - 2i} \text{Ker}(f)$$

(here negative tensor powers are treated as zero) and hence $\text{soc}^{\lfloor \frac{r}{2} \rfloor} = K'$.

Taking the image of the socle filtration above under $\Phi$, we obtain a filtration

$$F^0(V_{\infty}^{\otimes r}) = 0 \subset F^1(V_{\infty}^{\otimes r}) \subset \ldots \subset F^i(V_{\infty}^{\otimes r}) = V_{\infty}^{\otimes r}$$

The subquotients in the latter filtration are direct sums of standard objects in $\text{Rep}(P)$ (by Corollary 9.5.2). As in Sublemma 9.2.8, we have:

$$F^i = \bigcup_{g : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq r - i} \text{Ker}(g)$$

and hence $F^{\lfloor \frac{r}{2} \rfloor}(V_{\infty}^{\otimes r}) = K$.

We claim that $F^i$ induces a unique filtration $F^0(X) = 0 \subset F^1(X) \subset \ldots \subset F^i(X) = X$ on $X$. Indeed, consider the projector $e \in \text{End}_{\text{Rep}(P)}(V_{\infty}^{\otimes r})$ onto $X$. For any $g : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}$, $g \circ e$ is also a map $V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}$ and hence for any $i$,

$$F^i \subset \bigcup_{g : V_{\infty}^{\otimes r} \to V_{\infty}^{\otimes s}, s \leq r - i} \text{Ker}(g \circ e).$$

This implies that $e(F^i) \subset F^i$, and hence $e(F^i) = X \cap F^i$, which establishes the uniqueness of the induced filtration on $X$, as required.

Hence the subquotients $F^i(X)/F^{i-1}(X)$ are be direct summands of $F^i(V_{\infty}^{\otimes r})/F^{i-1}(V_{\infty}^{\otimes r})$ and are thus direct sums of standard objects. In particular, this implies that $P_k(0) \cong F^i(X)/F^{i-k}(X)$ has a filtration with standard subquotients.

It remains to check that the standard subquotients $\Delta(\mu)$ of $P_k(\lambda)$ satisfy $\mu \geq \lambda$. Indeed, consider the obtained filtration $F^0 = 0 \subset F^1 \subset \ldots \subset F^i = P_k(\lambda)$ with standard subquotients $F^i/F^{i-1} = \Delta(\mu)$. Then for any $\mu$, we have an exact sequence

$$0 \to \text{Hom}(\Delta(\mu'), \nabla(\mu)) \to \text{Hom}(F^i, \nabla(\mu)) \to \text{Hom}(F^{i-1}, \nabla(\mu)) \to \text{Ext}^1(\Delta(\mu'), \nabla(\mu)) = 0$$

(the last equality follows from Lemma 4.4.7).

This implies that

$$\dim \text{Hom}(F^i, \nabla(\mu)) = \dim \text{Hom}(F^{i-1}, \nabla(\mu)) + \delta_{\mu', \mu}$$

for any $\mu$ and any $i > 0$. Hence the number of times a given $\mu$ appears among $\{\mu_i\}$ is precisely $\dim \text{Hom}(P_k(\lambda), \nabla(\mu)) = (\nabla(\mu) : L(\lambda))$ and the latter is non-zero only if $\mu \geq \lambda$. \qed

**Corollary 9.2.10** (BGG reciprocity). For any $\lambda, \mu \in \Lambda^k$, we have:

$$(P_k(\lambda) : \Delta(\mu)) = (\nabla(\mu) : L(\lambda)).$$
10. Appendix A: Direct summands of tensor powers

10.1. Temperley-Lieb action and statement of the result. The goal of this appendix is to give a combinatorial proof of the following result of [DLZ 15].

**Corollary 10.1.1.** The space $s\text{Hom}_{p(n)}(V_n^\otimes k, 1)$ is non-zero only when $k$ is even. It is spanned by morphisms given by partitioning the $k$ factors into (disjoint) pairs, and considering a tensor product of $k/2$ contraction maps $V_n^{\otimes 2} \rightarrow \Pi_1$ on these pairs.

The proof is based on the results on categorical action of the Temperley-Lieb algebra $TL_{\infty} (\sqrt{-1})$ on $\text{Rep}(p(n))$ obtained in [BDE+16], and standard Temperley-Lieb combinatorics; these combinatorics are fully described here for completeness of presentation.

10.2. Temperley-Lieb combinatorics. Consider the Temperley-Lieb algebra $TL_{\infty}(\sqrt{-1})$ over $\mathbb{Z}_{\geq 0}$, generated by elements $\theta_k$, $k \in \mathbb{Z}$, and the relations

(12) $\theta_k^2 = 0$
(13) $\theta_k \theta_{k+1} \theta_k = \theta_k$
(14) $\theta_k \theta_j = \theta_j \theta_k$ if $|k - j| > 1$.

By [BDE+16] Theorem 4.5.1, the translation functors $\Theta_k: \text{Rep}(p(n)) \rightarrow \text{Rep}(p(n))$, $k \in \mathbb{Z}$, satisfy categorical Temperley-Lieb relations, obtained from (12) by replacing $\theta_k$ by $\Theta_k$ and equalities by explicit isomorphisms.

**Notation 10.2.1.** Let $I = (i_1, \ldots, i_k)$ be a finite sequence of integers. Consider translation functors $\Theta_{i_j}$, $j = 1, \ldots, k$ in $\text{Rep}(p(n))$. Denote $\Theta_I := \Theta_{i_k} \ldots \Theta_{i_1}$ and by $\theta_I = \theta_{i_k} \ldots \theta_{i_1}$ the corresponding element in the Temperley-Lieb algebra. We will use the convention $\theta_{\emptyset} = 0$.

We say that two sequences $I, I'$ are equivalent if $\theta_I = \theta'_{I'}$, and that $I$ is reduced if $I$ has minimal length among the sequences in its equivalence class.

**Lemma 10.2.2.** Let $I$ be a reduced sequence. Let $m = \max \{ i \in I \}$. Then the value $m$ appears in the sequence $I$ exactly once. A similar statement holds for the minimal value in $I$.

**Proof.** We prove this statement by induction on $k$. The base case $k = 1$ is clear. Let $k > 1$ and assume the statement holds for any sequence of length less than $k$. Let $I = (i_1, \ldots, i_k)$ be a sequence as in the lemma, and consider $\theta_I$.

Assume $m = \max I$ appears more than once in $I$. Consider two consecutive appearances: $i_s = i_r = m$, $s < r$, and $i_j < m$ for $s < j < r$. Consider the subsequence $I' = (i_{s+1}, i_{s+2}, \ldots, i_{r-1})$. Then $\max I' \leq k - 1$, and we have, by the induction assumption, at most one appearance of the value $m - 1$ in the sequence $I'$.

If $m - 1$ does not appear in $I'$, then $\theta_I \cong 0$: one can “move” $\theta_{i_s}$ to be next to $\theta_{i_r}$ due to [10.2], and the result will be zero by .

If $m - 1$ appears in $I'$ exactly once, at position $i_j$, then one can “move” $\theta_{i_s}$, $\theta_{i_r}$ to be next to $\theta_{i_j}$ (on different sides of $\theta_{i_j}$) due to . Applying to $\theta_i \theta_j \theta_k = \theta_m \theta_{m-1} \theta_m$, we may replace this expression by $\theta_m$, giving a shorter expression for the element $\theta_I$. 

□
Proposition 10.2.3. Let \( I = (i_k, \ldots, i_1) \) be a sequence of integers. Assume there exists a surjection \( \Theta_I \mathcal{C} \to \mathcal{C} \) in \( \text{Rep}(\mathfrak{p}(n)) \). Then there exists an integer \( s \geq 0 \) and a natural isomorphism
\[
\Theta_I' \simarrow \Theta_I
\]
where \( I' \) is an integer sequence\(^9\) of length \( s(s+1) \) given by concatenation of \( s \) decreasing sequences of consecutive integers, where the first terms of the sequences grow from \( 1 \) to \( s-1 \):
\[
(1,0, \ldots, -s+1|2,1,0, \ldots, -s+2|3,2,1,0, \ldots, -s+3|\ldots|s, s-1, \ldots, 0)
\]
Proof. We may assume that \( I \) is a reduced sequence. The condition \( \Theta_I \mathcal{C} \to \mathcal{C} \) implies that there is a non-zero map \( \Theta_{I_{k-1}} \cdots \Theta_{I_{i_1}} \mathcal{C} \to \Theta_{I_{k-1}} \mathcal{C} \). Hence \( i_k = 1, i_1 = 0 \), since \( \Theta_0 \) is the only translation functor not annihilating the module \( \mathcal{C} \). Moreover, the same holds for any \( I' \sim I \).

We will use the following sublemma:

Sublemma 10.2.4. Let \( I = (i_k, \ldots, i_1) \) be a reduced sequence, and assume that for any \( I' = (\ldots, i'_2, i'_1) \sim I \), we have: \( i_1 = i'_1 \).

For any \( j \geq 1 \), consider the subsequence \( I_j = (i_j, i_{j-1}, \ldots, i_1) \) (in particular, \( m_1 = i_1 \)). Then there exists a reduced sequence \( I' \sim I \) obtained from \( I \) by a permutation of its elements, such that the rightmost \( m_k - m_1 + 1 \) terms of the sequence \( I' \) are \( m_k, m_k - 1, m_k - 2, \ldots, m_1 + 1, m_1 \).

Proof. Recall that \( m_j \) occurs exactly once in \( I_j \), due to Lemma 10.2.2. Let us denote its position by \( l \).

We need to show that for any \( j, m_j - 1 \) occurs exactly once in \( I_j \) to the right of \( m_j \), i.e. occurs exactly once in \( I_l \).

Indeed, \( I_j \) must contain \( m_j - 1 \) to the right of \( m_j \), otherwise one can move \( m_j \) to the rightmost part of \( I \), contradicting the maximality of \( i_1 \). By Lemma 10.2.2, \( m_j - 1 \geq m_{j-1} \) will occur in \( I_{l-1} \) at most once, which completes the proof of the sublemma.

We now return to the proof of the proposition. Denote \( s = \max I \).

For every \( 1 \leq r \leq s \), we will prove the following claim: \( I \) can be replaced by an equivalent reduced sequence \( I' = (i'_k, \ldots, i'_1) \) (obtained by a permutation of the elements of \( I \)), whose last \( (s+1)r \) elements are
\[
(s-r+1, s-r, \ldots, -r+1|s-r+2, s-r+1, \ldots, -r+2|\ldots|s-1, s-2, \ldots, -1|s, s-1, \ldots, 0),
\]
and \( \max \{i_l | (s+1)r < l \leq k \} = s-r \).

We prove this claim by induction on \( r \). For \( r = 1 \), this is a direct consequence of the above sublemma.

Now, assume the claim holds for some \( r < s \), and denote the corresponding reduced sequence by \( I' \). We prove the claim for \( r + 1 \).

Consider the subsequence \( J' = (i'_k, \ldots, i'_{(s+1)r+2}, i'_{(s+1)r+1}) \). Denote its rightmost element \( i'_{(s+1)r+1} \) by \( l \).

We begin by proving that \( l = -r \).

Indeed, assume \( l < -r \); then \( l = i'_{(s+1)r+1} \) can be moved to the right, obtaining an equivalent reduced sequence whose rightmost element is \(-r \), which contradicts \( i_1 = 0 \).

Next, assume \( l > -r \). Then it can be moved to the right to obtain an equivalent reduced sequence with a subsequence \( (l, l+1, l) \); by \((12)\), such a sequence is not reduced, leading to a contradiction.

Hence \( l = -r \); moreover, the above reasoning implies that for any reduced sequence equivalent to \( J' \), its rightmost element is \(-r \).

By induction assumption, the maximal element in \( J' \) is \( s-r \). Applying the sublemma to
\[
J' = (i'_k, \ldots, i'_{(s+1)r+2}, i'_{(s+1)r+1} = -r)
\]
\(\)We place delimiters | in the sequence only to stress the “building blocks” of the sequence; these do not carry any additional meaning.
we obtain an equivalent reduced sequence $J'$ whose rightmost $s + 1$ elements are $(s - r, s - r - 1, \ldots, -r)$, and hence a reduced sequence $I'$ equivalent to $I$ whose last $(s + 1)(r + 1)$ elements are as required.

Finally, we claim that for $r \leq s - 2$ there exists an element in $J'$ to the right of the maximal element $s - r$ which equals $s - r + 1$. Indeed, otherwise we would be able to move the maximal element $s - r \geq 2$ to the leftmost position in $J'$, contradicting $i_k = 1$.

The process stops exactly when $r = s$, and we obtain the desired form for $I'$.

reduced sequence $I' = (i'_k, \ldots, i'_1)$ whose rightmost $s + 1$ elements are $s, s - 1, \ldots, 0$.

The following lemma shows how to “take a square root of $\Theta_J$” whenever $\Theta_I : 1 \to 1$: namely, we show that we can find a sequence $J$ such that $\Theta_J \cong \ast(\Theta_J)\Theta_J$, and hence the morphism $\Theta_I : 1 \to 1$ is just the application of the counit of the adjunction $\ast(\Theta_J)\Theta_J$ to the object $1$.

**Lemma 10.2.5.** Let

$I = (1, 0, \ldots, -s + 1|2, 1, 0, \ldots, -s + 2|3, 2, 1, 0, \ldots, s, s - 1, \ldots, 0)$.

Consider

$J = (s - 1|s - 3, s - 2|s - 5, s - 4, s - 3| \ldots | -s + 5, \ldots, -1, 0, 1, 2| -s + 3, \ldots, -1, 0, 1| -s + 1, -s + 2, -s + 3, \ldots, -1, 0)$

which is equivalent (due to Sublemma above) to

$(-s + 1| -s + 3, -s + 2| \ldots | -s + 5, \ldots, 0, -1| -s - 3, \ldots, 0, -1| -s - 1, -s - 2, \ldots, 1, 0)$

Then the left adjoint functor to $\Theta_J$ is $\Theta J'$ where

$J' = (1, 0, \ldots, -s + 2|2, 1, 0, \ldots, -s + 4|3, 2, 1, 0, \ldots, -s + 6| \ldots | -s + 1, -s + 2|s)$,

and $\Theta J\Theta I \cong \Theta J$, $\Theta J'\Theta J \cong \Theta J$.

In particular, the last statement implies $\Theta J : 1 \neq 0$.

**Proof.** To prove the isomorphism $\Theta J\Theta I \cong \Theta J$, consider first the concatenation of $J, I$, with $J$ in its original form:

$(J|I) = (s - 1| \ldots | -s + 5, \ldots, -1, 0, 1, 2| -s + 3, \ldots, -1, 0, 1| -s + 1, -s + 2, -s + 3, \ldots, -1, 0| 1, 0, \ldots, -s + 1|2, 1, 0, \ldots, -s + 2|3, 2, 1, 0, \ldots, -s + 3| \ldots | -s + 1, -s + 2|s, s - 1, \ldots, 0)$.

The subsequence $(-s + 1, -s + 2, -s + 3, \ldots, -1, 0, 1, 0, \ldots, -s + 1)$ is equivalent to $(-s + 1)$, so we replace the former by the latter and obtain a sequence

$(J|I) \sim (s - 1| \ldots | -s + 5, \ldots, -1, 0, 1, 2| -s + 3, \ldots, -1, 0, 1| -s + 1, 2, 1, 0, \ldots, -s + 2|3, 2, 1, 0, \ldots, -s + 3| \ldots | -s + 1, -s + 2|s, s - 1, \ldots, 0)$.

Now $-s + 1$ can be moved to the leftmost position, and we can next consider the subsequence $(-s + 3, \ldots, -1, 0, 1|2, 1, 0, \ldots, -s + 3, -s + 2)$ which is congruent to $(-s + 3, -s + 2)$. Again, these can be moved to the right, and so on.

The sequence obtained at the end of this process, equivalent to the concatenation $(J|I)$, is just $J$ in its second incarnation,

$(-s + 1| -s + 3, -s + 2| \ldots | s - 5, \ldots, 0, -1| -s - 3, \ldots, 0, -1| -s - 1, -s - 2, \ldots, 1, 0)$.

Hence $\Theta J\Theta I \cong \Theta J$. The statement $\Theta J'\Theta J \cong \Theta J$ is verified directly. \(\square\)
11. Appendix B: Affine group schemes in tensor categories

We recall briefly the notion of an affine group scheme in a tensor category. This is discussed in detail in [Del90, Section 7] and [EHS15, Section 11.2].

11.1. Definition. Let $\mathcal{T}$ be a tensor category. Its ind-completion \( \text{Ind} - \mathcal{T} \) inherits a symmetric monoidal structure.

We denote by $\text{Alg}_\mathcal{T}$ the category of commutative algebra objects in $\text{Ind} - \mathcal{T}$. Its opposite category is called the category of $\mathcal{T}$-affine schemes.

Remark 11.1.1. The bifunctor $\otimes$ on $\mathcal{T}$ is biexact and faithful, so the all the schemes discussed below are faithfully flat.

Definition 11.1.2. We define $\mathcal{T}$-algebraic groups (called $\mathcal{T}$-groups for short) as group objects in the category of $\mathcal{T}$-affine schemes; that is, the category $\text{Grps}_\mathcal{T}$ is antiequivalent to the category of commutative Hopf algebra objects in $\text{Ind} - \mathcal{T}$.

We denote by $\mathcal{O}(G)$ the commutative Hopf algebra object corresponding to a $\mathcal{T}$-group $G$.

By Yoneda lemma, we may also identify $\mathcal{T}$-algebraic groups with the corresponding corepresentable functors $\text{Alg}_\mathcal{T} \to \text{Grps}_\mathcal{T}$ from commutative algebra objects in $\text{Ind} - \mathcal{T}$ to groups.

Given a SM functor $F : \mathcal{T} \to \mathcal{T}'$ and $G \in \text{Grps}_\mathcal{T}$, the image $F(G)$ is obtained by applying the functor $F$ to $\mathcal{O}(G) \in \text{Ind} - \mathcal{T}$.

When the SM functor $F : \mathcal{T} \to \mathcal{T}'$ is right exact, the $\mathcal{T}'$-algebraic group $F(G)$ can be also described in terms of the functor of points (see [EHS15, Section 11.2]).

11.2. Fundamental group. For $A \in \text{Alg}_\mathcal{T}$, the category of $A$-modules $A - \text{Mod}$ is defined in the standard way. The functor $i_A : \mathcal{T} \to A - \text{Mod}$ carries $X \in \mathcal{T}$ to $A \otimes X$, and is clearly monoidal.

The fundamental group of $\mathcal{T}$, denoted $\pi(\mathcal{T}) \in \text{Grps}_\mathcal{T}$, is defined by the functor of points

$$A \mapsto \text{Aut}^\otimes(i_A : \mathcal{T} \to A - \text{Mod}),$$

where $\text{Aut}^\otimes$ means that we are only considering monoidal natural transformations of $i_A$ (that is, $\eta \in \text{Aut}(i_A)$ such that $\eta_X \otimes Y \cong \eta_X \otimes \eta_Y$).

For the tensor categories considered in this paper, their fundamental groups turn out to be affine (that is, the functor of points is corepresentable). In general, this happens for instance when the base field is perfect and the category is pre-Tannakian (see Section 12.2), which holds for all the categories which will be considered in this paper.

Example 11.2.1. For $\mathcal{T} = \text{Vect}$, the category of $A$-modules $A - \text{Mod}$ is defined in the standard way. The functor $i_A : \mathcal{T} \to A - \text{Mod}$ carries $X \in \mathcal{T}$ to $A \otimes X$, and is clearly monoidal.

The fundamental group of $\mathcal{T}$, denoted $\pi(\mathcal{T}) \in \text{Grps}_\mathcal{T}$, is defined by the functor of points

$$A \mapsto \text{Aut}^\otimes(i_A : \mathcal{T} \to A - \text{Mod}).$$

where $\text{Aut}^\otimes$ means that we are only considering monoidal natural transformations of $i_A$ (that is, $\eta \in \text{Aut}(i_A)$ such that $\eta_X \otimes Y \cong \eta_X \otimes \eta_Y$).

For the tensor categories considered in this paper, their fundamental groups turn out to be affine (that is, the functor of points is corepresentable). In general, this happens for instance when the base field is perfect and the category is pre-Tannakian (see Section 12.2), which holds for all the categories which will be considered in this paper.

Example 11.2.1. For $\mathcal{T} = \text{Vect}$, the category of $A$-modules $A - \text{Mod}$ is defined in the standard way. The functor $i_A : \mathcal{T} \to A - \text{Mod}$ carries $X \in \mathcal{T}$ to $A \otimes X$, and is clearly monoidal.

The fundamental group of $\mathcal{T}$, denoted $\pi(\mathcal{T}) \in \text{Grps}_\mathcal{T}$, is defined by the functor of points

$$A \mapsto \text{Aut}^\otimes(i_A : \mathcal{T} \to A - \text{Mod}).$$

Again, if the base field is perfect and the category is pre-Tannakian, then this functor in corepresentable by a $\mathcal{T}$-algebraic group denoted $GL(X)$.

Given $G \in \text{Grps}_\mathcal{T}$, a representation $X$ of $G$ is an object $X \in \mathcal{T}$ endowed with a structure of left comodule of the appropriate Hopf algebra $\mathcal{O}(G)$. Alternatively, we can define a representation of $G$ as a $\mathcal{T}$-group homomorphism $G \to GL(X)$. We denote the category of representations of $G$ in $\mathcal{T}$ by $\text{Rep}_\mathcal{T}(G)$.

In particular, every object $X \in \mathcal{T}$ is endowed with a canonical action $\pi_X : \pi(\mathcal{T}) \to GL(X)$. This homomorphism is given, on the level of functors of points, by the assignment of $\theta(X) : i_A(X) \to i_A(X)$ to an automorphism $\theta$ of the functor $i_A : \mathcal{T} \to A - \text{Mod}$.

Let $F : \mathcal{T}' \to \mathcal{T}$ be a tensor functor between two tensor categories. Then $F(\pi(\mathcal{T}'))$ is a $\mathcal{T}$-group given by the functor of points

$$A \in \text{Alg}_\mathcal{T} \mapsto \text{Aut}^\otimes_A(A \otimes F(-)).$$
Such a tensor functor $F$ induces a morphism of $\mathcal{T}$-groups $\varepsilon : \pi(\mathcal{T}) \to F(\pi(\mathcal{T}'))$, which is given on the functors of points by

$$\text{Aut}_A^\otimes(A \otimes (-)) \xrightarrow{\varepsilon} \text{Aut}_A^\otimes(A \otimes F(-)).$$

The following theorem, due to Deligne, is a generalization of the Tannakian reconstruction theory. Let $\text{Rep}_\mathcal{T}(F(\pi(\mathcal{T}')), \varepsilon)$ the full subcategory of representations $F(\pi(\mathcal{T}')) \to GL(X)$ in $\mathcal{T}$ whose composition with $\varepsilon$ gives the natural action $\pi_X$ of $\pi(\mathcal{T})$ on $X$.

**Theorem 11.3.1 (Deligne, [Del90]).** The functor $F$ induces an equivalence of tensor categories $\mathcal{T}' \to \text{Rep}_\mathcal{T}(F(\pi(\mathcal{T}')), \varepsilon)$.

11.4. **Group homomorphisms and functors.** The following statement is proved in the same way as for classical group schemes (see for example [Mil17 Chapter X, Section 4]).

**Lemma 11.4.1.** Let $G, G'$ be $\mathcal{T}$-groups, and $f : G' \to G$ a $\mathcal{T}$-group homomorphism. Consider the corresponding tensor functor $F : \text{Rep}_\mathcal{T}(G) \to \text{Rep}_\mathcal{T}(G')$.

- The homomorphism $f$ is a quotient map iff the functor $F$ is full and the essential image of $F$ is closed under taking subobjects.
- The homomorphism $f$ is injective iff any $X \in \text{Rep}_\mathcal{T}(G)$ is a subquotient of an object in the essential image of $F$.

Now, let $F : \mathcal{T} \to \mathcal{T}'$ be a tensor functor between tensor categories. Denote $G' := \pi(\mathcal{T}')$, $G := F(\pi(\mathcal{T}))$. As in Theorem 11.3.1, we have a group homomorphism $\varepsilon : G' \to G$.

**Lemma 11.4.2.** The homomorphism $\varepsilon$ is a quotient iff the functor $F$ is full and the essential image of $F$ is closed under taking subobjects.

**Proof.** Recall that by Deligne’s Theorem 11.3.1, $\text{Forget} : \text{Rep}_\mathcal{T}'(G, \varepsilon) \to \mathcal{T}'$ satisfies the condition in the statement of the lemma iff $F$ does.

By Lemma 11.4.1, the homomorphism $\varepsilon$ is a quotient iff the functor

$$\tilde{F} : \text{Rep}_\mathcal{T}'(G) \to \text{Rep}_\mathcal{T}'(G')$$

induced by $\varepsilon$ is full, and its essential image closed under taking subobjects.

Now, the functors

$$\mathcal{T} \boxtimes \mathcal{T}' \xrightarrow{X \boxtimes Y \mapsto F(X) \otimes Y} \mathcal{T}'', \mathcal{T}' \boxtimes \mathcal{T}' \xrightarrow{Y \boxtimes Z \mapsto Y \otimes Z} \mathcal{T}'$$

induce equivalences

$$\mathcal{T} \boxtimes \mathcal{T}' \cong \text{Rep}_\mathcal{T}'(G), \mathcal{T}' \boxtimes \mathcal{T}' \cong \text{Rep}_\mathcal{T}'(G')$$

(cf. [Del90 Propositions 8.22, 8.23]), such that

$$\begin{array}{ccc}
\mathcal{T} \boxtimes \mathcal{T}' & \xrightarrow{F \boxtimes 1} & \mathcal{T}' \boxtimes \mathcal{T}' \\
\downarrow & & \downarrow \\
\text{Rep}_\mathcal{T}'(G) & \xrightarrow{\tilde{F}} & \text{Rep}_\mathcal{T}'(G')
\end{array}$$

Thus whenever $\tilde{F}$ satisfies the conditions in the lemma, so does $F$. The lemma is proved. \[\square\]

11.5. **The fundamental group of $\text{Rep}_\mathcal{T}(G)$.** Let $G$ be a $\mathcal{T}$-group, and consider the tensor category $\text{Rep}_\mathcal{T}(G)$ of $G$-representations in $\mathcal{T}$.

This category comes with equipped with two functors:

- A tensor functor $I : \mathcal{T} \to \text{Rep}_\mathcal{T}(G)$, where $M \mapsto M_{\text{triv}}$, considered with trivial $G$-action.
- A forgetful functor $F : \text{Rep}_\mathcal{T}(G) \to \mathcal{T}$, taking $M \in \text{Rep}_\mathcal{T}(G)$ to the underlying $\mathcal{T}$-object.
Clearly, we have a natural isomorphism $FI \cong \text{Id}$.

Let $\tilde{G}$ denote the fundamental group of $\text{Rep}_T(G)$.

By Deligne’s Theorem [11.3.1] the functor $I$ induces a homomorphism of $\text{Rep}_T(G)$-groups

$$q : \tilde{G} \rightarrow I(\pi(T))$$

(here $I(\pi(T))$ can be thought of as the group $\pi(T)$ with trivial $G$-action).

On the other hand, we have a homomorphism of $\text{Rep}_T(G)$-groups $i : G^{\text{adj}} \rightarrow \tilde{G}$ where $G^{\text{adj}}$ is the group $G$ with conjugation action.

**Proposition 11.5.1.** The group $\tilde{G}$ is an extension of $G^{\text{adj}}$ and $I(\pi(T))$. Moreover, the $T$-group $F(\tilde{G})$ is a semidirect product of $G$ and $\pi(T)$, the inclusion $\pi(T) \rightarrow F(\tilde{G})$ again given as in Theorem [11.3.1].

**Proof.** The fact that $i$ is injective and $q$ is surjective follows from Lemmata [11.4.1] and [11.4.2].

Next, we note that $G^{\text{adj}}$ acts trivially on an object $X$ in $\text{Rep}_T(G)$ iff this object belongs to the essential image of $I$. Hence for any object $X \in \text{Rep}_T(G)$, the action $\pi_X : \tilde{G} \rightarrow GL(X)$ factors through $q$ iff $\pi_X \circ i = 1$. Hence $\text{Ker}(q) = i$, as required. \(\square\)

**11.6. Faithful representations.** We recall the notion of faithful representation.

Let $G \rightarrow GL(X)$ be a representation of $G$.

**Lemma 11.6.1.** The following are equivalent:

1. The $T$-group homomorphism $G \rightarrow GL(X)$ is injective.
2. Any representation of $G$ in $T$ is a subquotient of $\bigoplus_{i \in I} X^{\otimes a_i} \otimes (X^*)^{\otimes b_i} \otimes U$ for some finite set $I$, $a_i, b_i \in \mathbb{Z}_{\geq 0}$ and some $U \in T$ considered with a trivial $G$-action.

**Proof.**

1. $\Rightarrow$ 2. Let $(M, \rho)$ be a representation of $G$. Then we have an injective map $M \xrightarrow{\rho} O(G) \otimes M_{\text{triv}}$ where $M_{\text{triv}}$ has trivial $G$ action. Recall that $O(G)$ is a quotient of $O(GL(X))$, which in turn is a quotient of $\text{Sym}(X \otimes X^*)$; the required statement now follows.

2. $\Rightarrow$ 1. Assume (2) holds. Consider the functor $F : \text{Rep}_T(GL(X)) \rightarrow \text{Rep}_T(G)$ induced by $\rho$. The essential image of this functor contains representations of the form $\bigoplus_{i \in I} X^{\otimes a_i} \otimes (X^*)^{\otimes b_i} \otimes U$ as in (2). Hence by Lemma [11.4.1] the homomorphism $\rho$ is injective.

The statement of the lemma is now proved. \(\square\)