A Positive Test
for Fermi-Dirac Distributions
of Quark-Partons

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Abstract

By describing a large class of deep inelastic processes with standard parameterization for the different parton species, we check the characteristic relationship dictated by Pauli principle: broader shapes for higher first moments. Indeed, the ratios between the second and the first moment and the one between the third and the second moment for the valence partons is an increasing function of the first moment and agrees quantitatively with the values found with Fermi-Dirac distributions.
Four experimental facts conspire to indicate that the Pauli exclusion principle plays a role in the quark-parton distributions in the nucleons. They are the defect in the Gottfried sum rule, the asymmetry in the Drell-Yan processes on proton and deuteron targets, and the high $x$ behaviour of the ratios $F_2^p(x)/F_2^n(x)$ and $g_1^p(x)/F_2^p(x)$. The first two facts imply the inequality $\bar{d} > \bar{u}$ in the sea of the proton, which has been advocated long time ago. From the other two one can deduce the dominance of the $u^\uparrow$ parton in the high $x$ region; this is an indication of the relationship between the shape of the distribution as a function of $x$ and the first moment of a parton required by Pauli principle: broader shapes for higher first moments.

More recently, by assuming the approximate relationship $u^\downarrow(x) = \frac{1}{2} d(x)$, it has been possible to successfully relate the structure functions $F_2^p(x) - F_2^n(x)$ and $x g_1^p(x)$ for $x \geq 0.2$.

Pauli principle suggests to assume Fermi-Dirac functions for the quark distributions,
\begin{equation}
q^{\uparrow(\downarrow)}(x) = \frac{f(x)}{\exp\left\{\frac{x - \bar{x}(q^{\uparrow(\downarrow)})}{\bar{x}}\right\} + 1},
\end{equation}
and Bose-Einstein functions for the gluons,
\begin{equation}
G^{\uparrow(\downarrow)}(x) = \frac{8}{3} \frac{f(x)}{\exp\left\{\frac{x - \bar{x}(G^{\uparrow(\downarrow)})}{\bar{x}}\right\} - 1},
\end{equation}
where $\bar{x}$ plays the role of the temperature, the $\bar{x}$ are the thermodynamical potentials of each parton species and $f(x)$ is a weight function with the usual form $A x^\alpha (1 - x)^\beta$.

To account for the increase of $F_2^p(x)$ at small $x$ we have to add to Eq. an additional contribution $q_L(x) = \bar{q}_L(x)$, which should be unpolarized and isospin-invariant to get finite quark-parton model sum rules (QPMSR). This leads to a satisfactory description of a large class of deep inelastic data in Ref. and its updated version, including the E154 data by SLAC.

Here we want to test a specific property of the quantum statistical distributions given by Eq. namely the shape-first moment relationship previously mentioned for fermionic partons. The QPMSR imply
\begin{equation}
u^\uparrow \gg d^\downarrow \geq u^\downarrow > d^\uparrow.
\end{equation}
The defect in the Gottfried sum rule yields
\begin{equation}\bar{d} > \bar{u}.
\end{equation}
We can argue the same pattern than Eq. and for the corresponding broadnesses. To be more precise we expect the ratio between the second and the first moment, that is the mean value of $x$, as well as the one between the third and the second moment of each quark parton to be an increasing function of the first moment. For the gluons the deviations from the Boltzmann distribution (implying the shape to be independent from the first moment) are in the opposite direction; thus, one expects $\Delta G(x)$ softer than $G(x)$, and $G(x)$ softer than $\bar{q}(x)$, if the same $f(x)$ is taken for quarks and gluons.
To establish whether experiments confirm this shape-first moment relationship, we try to describe the same data considered in [11], by taking, for each light quark-parton species of a given flavour and spin, at \( Q^2 = 3 \text{GeV}^2 \), the form \( (p = u^\uparrow, u^\downarrow, d^\uparrow, d^\downarrow, \bar{u}^\downarrow, \bar{d}^\uparrow) \):

\[
p(x, Q^2_0) = A_p x^\alpha (1-x)^{\beta_p} + A_L x^{\alpha_L} (1-x)^{\beta_L}.
\] (5)

Going on with the \textit{thermodynamical} language, we name \textit{gas} and \textit{liquid} the two terms in Eq. (5).

We are not very sensitive to the strange quark distribution. Indeed, the only observables depending on it that we are considering are \( F_2^n(x)/F_2^p(x) \) and \( F_2^n(x) \). For the first one isospin invariance implies the same contribution for both members of the ratio, the latter being near to one in the small \( x \) region, where strange partons are concentrated. In that region the second observable, \( F_2^p(x) \), receives a large contribution from the second term of Eq. (5), responsible for its increase at small \( x \). Therefore, in order to avoid the introduction of new parameters, we take, according to Ref. [13],

\[
s(x) = \bar{s}(x) = \frac{\bar{u}(x) + \bar{d}(x)}{4.2}.
\] (6)

As in Ref. [11] we consider \( F_3(x) \), measured at \( Q^2 = 3 \text{GeV}^2 \) [14], \( F_2^p(x) \) and \( F_2^n(x) \) at \( Q^2 = 4 \text{GeV}^2 \) [1, 5, 15] and the polarized structure functions measured at SLAC \( (g_1^p(x) \text{ and } g_1^n(x) \text{ at } Q^2 = 3 \text{GeV}^2 \text{ [16]} \text{ and } Q^2 = 2 \text{GeV}^2 \text{ [17]} \text{ and } Q^2 = 5 \text{GeV}^2 \text{ [18]}) \). To report all the data at the same \( Q^2_0 = 3 \text{GeV}^2 \) we assume the \( Q^2 \) dependence found by NMC, while the good agreement, in the common \( x \) range, between the data regarding \( g_1^n(x) \) at \( Q^2 = 2 \text{GeV}^2 \) and at \( Q^2 = 5 \text{GeV}^2 \) makes us confident that they both give a good approximation of that quantity at \( Q^2 = 3 \text{GeV}^2 \).

The unpolarized structure functions are given by:

\[
\begin{align*}
F_2^{p,n}(x) &= x \left\{ \pm \frac{1}{6} [u(x) + \bar{u}(x) - d(x) - \bar{d}(x)] + \frac{5}{18} [u(x) + \bar{u}(x) + d(x) + \bar{d}(x)] \\
&+ \frac{1}{9} [s(x) + \bar{s}(x)] \right\}, \\
x F_3(x) &= x [u(x) + d(x) + s(x) - \bar{u}(x) - \bar{d}(x) - \bar{s}(x)].
\end{align*}
\] (7)

As long as for the polarized distribution we recall that the presence of a gluon isoscalar contribution, related to the anomaly, has been advocated [18] to explain the defect in the Ellis-Jaffe sum rule [19] for the proton found by the EMC experiment [5]. This contribution, up to first order in \( \alpha_s \), is given by the convolution [20]

\[
S(x) = \frac{\alpha_s}{6\pi} \int_x^1 \frac{dz}{z} (1 - 2z) \left( \ln \frac{1 - z}{z} - 1 \right) \Delta G \left( \frac{x}{z} \right).
\] (8)

In the framework of a complete NLO analysis, we should consider all possible \( \alpha_s \) contributions, but this is beyond the purposes of the present work. However, in a factorization scheme where chiral symmetry is respected (a choice closer to the picture of the nucleons in term of constituent quarks [21]), the high
value of $\Delta G$ required to explain the experimental results suggests that the gluon polarization gives the dominant contribution proportional to $\alpha_s$. Therefore, we shall give the following expressions for the nucleon polarized structure functions:

$$g_{1,1}^{n,n}(x) = \pm \frac{1}{12} [\Delta u(x) - \Delta d(x)] + \frac{5}{36} [\Delta u(x) + \Delta d(x)] + S(x),$$

with

$$\Delta G(x, Q_0^2) = A_{\Delta G} x^{\alpha_{\Delta G}} (1 - x)^{\beta_{\Delta G}}. \tag{10}$$

Eqs. (8) and (10) give a three parameter description of $S(x)$ ($\alpha_s A_{\Delta G}, \alpha_{\Delta G}, \beta_{\Delta G}$), which is general enough for the gluon isoscalar term that one has to add to the valence term to obtain $g_{1}^{p}(x)$ and $g_{1}^{n}(x)$. However, the fact that the presence of this contribution, which is important in the small $x$ region, influence only a little the determination of the shapes of the valence quarks, which depend on the $\beta_q$’s, makes the debate on this issue purely academic with respect to the main purpose of this paper.

As usual, we assume

$$g_{1}^{d}(x) = \frac{1}{2} \left(1 - \frac{3}{2} \omega_{D}\right) (g_{1}^{p}(x) + g_{1}^{n}(x)), \tag{11}$$

where $\omega_{D}$ is the amount of D-wave in the deuteron ground state (0.058) \cite{22}.

We have then four parton distributions ($u, d, \bar{d}$ and $\bar{u}$) to describe the three unpolarized structure functions ($F_2^p(x), F_2^n(x)$, and $F_3(x)$), and three parton distributions ($\Delta u, \Delta d$ and $\Delta G$) for the two polarized ones ($\Delta u(x)$ and $\Delta d(x)$). To overcome the ambiguity arising from this situation, we require the unpolarized distributions to comply with the Adler sum rule \cite{23} and with the NA51 \cite{3} measurement of the asymmetry in the Drell-Yan production of muon pairs,

$$\alpha_{DY}(x = .18) = -0.09 \pm 0.04 \pm 0.02. \tag{12}$$

We also require the $\beta_q$ and $\beta_{\Delta G}$ to be larger than all the $\beta_q$’s and

$$-0.8 < \alpha, \alpha_{\Delta G} \leq 0. \tag{13}$$

We consider the cases with $\Delta G = 0$ and with $\Delta G$ free. At small $x$ it is difficult to separate the two terms in Eq. (3). Since the second term is expected to give a small contribution ($\sim 10\%$) \cite{11} to the second moment of each parton, but an infinite one to the first moment, we expect the second moment of the first term to be less affected than the second one by the ambiguity in disentangling the two contributions. However, the second term gives an equal contribution to each quark-parton: therefore the existence of a correlation between the contribution to the first moment and the shape of the first term in Eq. (3) is not affected by this uncertainty, which has the same influence on the first and second moment of the gas component of each parton. Moreover, since the low $x$ behaviour of the
two contributions is very different (in the case of Ref. [10], their power behaviours differ by almost one unity), we expect that also for the first moment the ambiguity is not so large.

As we have previously said, the Fermi-Dirac function for the quark distributions gives an automatic correlation between the first moment and the shape of each quark-parton distribution, namely broader shapes for higher first moments, with both quantities depending on \( \bar{x} \), once fixed \( f(x) \) and \( \bar{x} \). Instead, for the first term of Eq. (5), once imposed a common value for \( \alpha \), there are two parameters for each parton, \( A_p \) and \( \beta_p \), to reproduce the first two moments of the first term in Eq. (5), which we shall indicate with \( p \) and \( p^{(2)} \) respectively. The second term in Eq. (5), responsible of the increase of \( F_2^p(x) \) at low \( x \) and apparently power-like, gives an infinite contribution to any \( p \) (equal for each light parton, should we start from a finite positive value of \( x \)) and an equal contribution to \( p^{(2)} \).

Since it is difficult to disentangle the gas and liquid component in Eq. (5), we parameterize each light parton distribution with the same \( \alpha \); in this way the shape of the corresponding distribution is dictated by \( \beta_p \), which is related to the high \( x \) behaviour, where the contribution of the liquid part is expected to become negligible. To make the comparison between Eqs. (1) and (5) more appropriate, we fix the liquid part to the value previously found with the parton statistical distributions, namely

\[
A_L \ x^{\alpha_L} (1 - x)^{\beta_L} = \begin{cases} 
0.12 \ x^{-1.19} (1 - x)^{0.8} & \Delta G = 0, \\
0.12 \ x^{-1.19} (1 - x)^{0.6} & \Delta G \text{ free.}
\end{cases}
\]

In tables [1, 2 and 3] the values of the parameters found in the fit and the first three moments of the gas component, with their ratio for each parton, with and without gluons, are reported and compared with the updated fit (whose results are almost identical to the one in Ref. [11]) with quantum statistical distributions.

An attentive glance to the numbers in the tables brings us to the following conclusions:

i) for the quantum statistical as well as for the present parameterization the parameters and the moments of the quark-partons have a very weak dependence on the gluon contribution and the agreement between the moments, obtained with the different parameterization, is good with the exception of the \( \bar{u} \) parton and \( \Delta G \).

ii) using parameterization (5) we get, for the ratios \( p^{(2)}/p \) and \( p^{(3)}/p^{(2)} \) as a function of \( p \) for the quarks, a pattern very similar to the one obtained with Fermi-Dirac functions, which have the property

\[
\begin{align*}
\frac{u^{(2)}}{u} & \gg \frac{d^{(2)}}{d} \geq \frac{u^{(2)}}{u^2} > \frac{d^{(2)}}{d^2}, \\
\frac{u^{(3)}}{u^{(2)}} & \gg \frac{d^{(3)}}{d^{(2)}} \geq \frac{u^{(3)}}{u^{(2)}} > \frac{d^{(3)}}{d^{(2)}},
\end{align*}
\]

this shows that the ratios \( p^{(2)}/p \) and \( p^{(3)}/p^{(2)} \) are increasing functions of \( p \) according to Eq. (3). Also, \( \bar{d}^{(2)}/\bar{d} > \bar{u}^{(2)}/\bar{u} \), as we expect from Eq. (3).
The differences found for $\bar{u}$ and $\Delta G$ are not unexpected. As for the first one, which is the lighter parton with the lowest first moment, the difference is large in percentage but small in amount. The different shape found, which is a consequence of the high value of $\beta_{\bar{u}}$, is explained by the small precision in the determination of a parameter which is related to the shape of a distribution with a very low first moment. Concerning the gluons, we are not surprised to find a $\Delta G$ about more than twice larger than with the Bose-Einstein form, which has a less divergent extrapolation to small $x$, and a different shape: the fact that the fit is equally good with and without gluons shows that we are not able to get, within our approach, relevant information on them. However, the value found is consistent with the NLO result of Ref. [25], $\Delta G = 1.1 \pm 0.4$, where the gluons are also constrained by the $Q^2$ dependence of the data. The value found for $\alpha_{\Delta G}$ is at the lower limit of the one in Ref. [25].

The fact that the ratios $d^{(2)}_{\uparrow}/d^{(2)}_{\downarrow}$ and $d^{(3)}_{\uparrow}/d^{(2)}_{\downarrow}$, for the valence quark with the smallest first moment, are smaller than the ones for the other valence quarks confirms the correlation among broader shape and higher first moment, which shows up in the dominance of $u^\uparrow$ at high $x$. Needless to say, $u^{(2)}_{\uparrow}/u^{\uparrow}$ and $u^{(3)}_{\uparrow}/u^{(2)}_{\uparrow}$ come out larger than the corresponding ratio of the other valence partons, as we expect from the high $x$ behaviour of $F_{2g}(x)/F_{2p}(x)$ and $A_{1p}(x)$. Indeed, already in Ref. [26], by taking the single parton distributions from a set of deep inelastic data, the author has been able to find a similar trend with a distribution for $u^\uparrow$ and for $d^\uparrow$ broader and narrower respectively than for the other valence partons.

The values of the first and second moment of the gas part in Eqs. (1) and (5) are in good agreement. (The exception of $\bar{u}$ does not contradict this conclusion, since the small value of its first moment makes less relevant its shape.) This agreement is, at our advice, an additional argument in favour of the Fermi-Dirac distributions, for which, once fixed $\bar{x}$ and $f(x)$ for each parton, $p$, $p^{(2)}$, and $p^{(3)}$ depend on one parameter, $\bar{x}(p)$; to say it in a more clear way, the fit with the present parameterization might give equal values of $\beta_q$’s and distributions which differ only by a constant factor, which would be, within our statistical language, the Boltzmann limit. The agreement found for the ratio $p^{(3)}/p^{(2)}$ for the valence quarks with the two different parameterizations gives a positive test, which is very little affected by the difficulty of disentangling the gas from the liquid contribution, since the last one has little relevance in the evaluation of the second moments of the valence quarks and even a minor one for the third moments.

We wish to compare our evaluation of the first two moments of the non-singlet polarized distributions with a previous analysis of the same data [27] and with the result found in the NLO analysis of Ref. [25]. To this purpose, in table 4 these moments and their ratio are reported for $g^{p(NS)}_1$. One can see a very good agreement of the value found for the second moment of $g^{p(NS)}_1$ between the quantum statistical distributions, the present parameterization and the one in [25], while the difference found for the first moment depends on the small $x$ behaviour.
The gluons are expected to have a softer distribution than the $\bar{q}$’s if the same $f(x)$ is taken. The great uncertainty on their distribution does not allow to draw meaningful conclusions from the fact that table 3 agrees with this expectation. A typical property of the Bose-Einstein function is that $\Delta G(x)/G(x)$ is a decreasing function of $x$ (soft polarization for the gluons), therefore $\Delta G^{(2)}/\Delta G < G^{(2)}/G$. In order to show this property one needs a NLO analysis suitable to determine the shape of $\Delta G$ and $G$. We plan to perform a consistent NLO analysis to include all the contributions proportional to $\alpha_s(Q^2)$ and to evolve the distributions found in this way to the higher $Q^2 (\sim 10 GeV^2)$ of the CERN experiments on polarized proton and deuteron targets. We do not expect a too challenging test for the resulting predictions, since the precision of the measurements, especially in the low $x$ region not reached at SLAC, is worse for CERN data.

In fact, we tried to extend our analysis to the CERN data at $<Q^2> = 10 GeV^2$ for $g_1^p(x)$ and $g_1^d(x)$ [28], together with the unpolarized distributions evaluated at $Q^2 = 10 GeV^2$. Unfortunately the absence of data on $g_1^n (g_1^{He^3})$ and the minor precision of the data for $g_1^d$, together with the fact that the unpolarized structure functions at $Q^2 = 10 GeV^2$ are known with a smaller precision, do not allow to extract from the data the trend of $p^{(2)}/p$ and $p^{(3)}/p^{(2)}$, which come out different from equally good fits.

To get an idea of quality and differences of the two fits we compare in Fig. 1 and 2 $x F_3(x)$ measured in Ref. [14] and $x g_1^n(x)$ measured in Ref. [12] with the quantum statistical and present parameterization in the case $\Delta G$ free. As we see, the data are very well reproduced and there is very small difference between the two fits.

In conclusion, as a consequence of our analysis, we can add a new fact to the ones mentioned at the beginning of this paper, supporting a role of the Pauli principle: $d^\uparrow$, the valence quark with the lowest first moment, is also the one with the narrower shape, and the dependence of the shape on the first moment for each parton is in good agreement with the one found using Fermi-Dirac functions.

Our preliminary work on NLO analysis of the SLAC data confirms the trend with a slightly narrower distribution for $d^\uparrow$ and, of course, a largely broader one for $u^\uparrow$ with respect to $u^\downarrow$ and $d^\downarrow$.

The study of the $Q^2$ evolution is in our plans, especially since the large $Q^2$ and $x$ excess found at Hera [29] might be explained by a modification of the evolution, which is reasonable if Pauli principle plays a role in parton distributions, consisting in a slower narrowing of the distribution of a fermionic parton with its levels almost completely occupied (in the case of the proton the $u^\uparrow$), as advocated in Ref. [30].
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Table 1: The values of the parameters found with the present parameterization (PP) are compared with the ones obtained using quantum statistical distributions (ST). The liquid part is given by Eq. (14). In the last two rows we report the results for the rhs of the Bjorken [24] and Gottfried [2] sum rules.
Table 2: The values of the first, second and third moment and of the ratios \( p^{(2)}/p \) and \( p^{(3)}/p^{(2)} \) for the gas component in Eq. (5) are reported in the case \( \Delta G = 0 \).
## Table 3

|       | $p$          | $p^{(2)}$     | $p^{(2)}/p$ | $p^{(3)}$     | $p^{(3)}/p^{(2)}$ |
|-------|--------------|---------------|-------------|--------------|------------------|
| $u^+$ | $1.254 \pm 0.010$ | $0.233 \pm 0.004$ | $0.186 \pm 0.004$ | $0.080 \pm 0.002$ | $0.345 \pm 0.010$ |
| $d^+$ | $0.661 \pm 0.011$ | $0.086 \pm 0.002$ | $0.130 \pm 0.004$ | $0.0220 \pm 0.0009$ | $0.256 \pm 0.013$ |
| $u^-$ | $0.592 \pm 0.009$ | $0.075 \pm 0.003$ | $0.127 \pm 0.005$ | $0.0189 \pm 0.0011$ | $0.25 \pm 0.02$ |
| $d^-$ | $0.350 \pm 0.011$ | $0.040 \pm 0.002$ | $0.115 \pm 0.007$ | $0.0093 \pm 0.0007$ | $0.23 \pm 0.02$ |
| $\bar{d}/2$ | $0.138 \pm 0.004$ | $0.0158 \pm 0.0006$ | $0.115 \pm 0.005$ | $0.0036 \pm 0.0002$ | $0.230 \pm 0.013$ |
| $\bar{u}/2$ | $0.055 \pm 0.005$ | $0.0023 \pm 0.0003$ | $0.041 \pm 0.007$ | $0.00021 \pm 0.00005$ | $0.09 \pm 0.02$ |
| $\Delta G$ | $1.9 \pm 1.3$ | $0.018 \pm 0.056$ | $0.009 \pm 0.030$ | $0.0010 \pm 0.0036$ | $0.05 \pm 0.26$ |

Table 3: Same as table 2 with $\Delta G$ free.
Table 4: The values of the first and second moment of the non-singlet part of $g_1^p$ with $\Delta G$ free, obtained with the present parameterization (PP), are compared with the same quantities from quantum statistical distributions (ST) and from Ref. [25, 27]. The values from Ref. [25] have been reported to $Q^2 = 3\text{GeV}^2$ with the NLO evolution equation for the moments.

|       | PP   | ST   | Ref. [25] | Ref. [27] |
|-------|------|------|-----------|-----------|
| $p$   | 0.0908 | 0.0927 | 0.0996     | 0.0989    |
| $p^{(2)}$ | 0.0201 | 0.0197 | 0.0201     | 0.0226    |
| $p^{(2)}/p$ | 0.221  | 0.213  | 0.202      | 0.228     |
Figure 1: The prediction for $xF_3(x)$ at $Q^2 = 3 \text{ GeV}^2$ in the case $\Delta G$ free is plotted and compared with the experimental data [14]. The solid and dashed line correspond to the fit with present parameterization and with quantum statistical distribution respectively.
Figure 2: The prediction for $xg_1^n(x)$ at $Q^2 = 3 \text{ GeV}^2$ in the case $\Delta G$ free is plotted and compared with the experimental data \cite{12}. The solid and dashed line corresponds to the fit with present parameterization and with quantum statistical distribution respectively.