The order of the product of two elements in the periodic groups

Mohsen Amiri\(^a\) and Igor Lima\(^b\)

\(^a\)Departamento de Matemática – ICE-UFAM, Universidade Federal do Amazonas, Manaus, AM, Brazil; \(^b\)Departamento de Matemática, Universidade de Brasília, Campus Universitário Darcy Ribeiro, Brasília, DF, Brazil

**ABSTRACT**

Let \(G\) be a periodic group, and let \(\text{LCM}(G)\) be the set of all \(x \in G\) such that \(o(x^n)\) divides the least common multiple of \(o(x^n)\) and \(o(z)\) for all \(z \in G\) and all integers \(n\). In this paper, we prove that the subgroup generated by \(\text{LCM}(G)\) is a locally nilpotent characteristic subgroup of \(G\) whenever \(G\) is a locally finite group.

**1. Introduction**

One of the oldest problems in group theory is, given two elements \(x, y\) in a group \(G\), of orders \(m\) and \(n\), respectively, to find information on the order of the product \(xy\). Understanding even the easier problem of when \(ab\) has finite order would have great implications in group theory, for instance in the study of finitely-generated groups in which the generators have finite order. Burnside’s problem is a good example of a difficult problem which asks whether a finitely generated periodic group is necessarily finite. A negative answer to this problem has been provided in 1964 by Golod and Shafarevich [8, 9], although many variants of this question still remain unsolved to this day. For more information on this subject, see Kostrikin [13], Novikov and Adian [1], Ivanov and Ol’shanskiĭ [11, 12, 18], Zelmanov [24, 25], and Lysënok [14]. Let \(G\) be a periodic group, and let \(H\) and \(R\) be two non-empty subsets of \(G\). Let \(\text{LCM}(H, R)\) be the set of all \(x \in H\) such that \(o(x^n)\) divides the least common multiple \(\text{lcm}(o(x^n), o(z))\) for all \(z \in R\) and all integers \(n\). In particular, we denote the set \(\text{LCM}(G, G)\) by \(\text{LCM}(G)\) and the subgroup \(\langle \text{LCM}(G) \rangle\) by \(\text{LC}(G)\). In this paper, we prove that the subgroup generated by \(\text{LCM}(G)\) is a nilpotent characteristic subgroup of \(G\) whenever \(G\) is a finite group. As consequence of this result, we prove the following theorem:

**Theorem 1.1.** Let \(G\) be a locally finite periodic group. Then \(\text{LC}(G)\) is a locally nilpotent subgroup of \(G\).

A subgroup series of a group \(G\) is a finite chain of subgroups of \(G\) contained in each other. Subgroup series can simplify the study of a group to the study of simpler subgroups and their
relations, and several subgroup series can be invariantly defined and are important invariants of groups. For more information on this subject, see [3, 5, 16, 17, 23]. Let \( G \) be a group. Let \( L C_i(G) = L C(G) \) and for \( i = 2, 3, \ldots \), define \( L C_i(G) = \frac{L C_{i-1}(G)}{L C_{i-1}(G)} \). We say the group \( G \) is a \( L C \)-nilpotent group, whenever there exists a finite \( L C \)-series
\[
L C_0(G) := 1 \leq L C_1(G) \leq L C_2(G) \leq \cdots \leq L C_k(G) = G
\]
such that \( \frac{L C_i(G)}{L C_{i-1}(G)} \) is a nilpotent group for all \( i = 1, 2, \ldots, k \). In this case, the \( L C \)-series \((1.1)\) is called a nilpotent \( L C \)-series for \( G \). The smallest integer \( k \) such that \( L C_{i+k}(G) = L C_k(G) \) for all integers \( i = 1, 2, \ldots \) is called length of series and \( G \) is called \( L C \)-nilpotent of class \( k \). As application of this definition, we prove the following proposition:

**Proposition 1.2.** Let \( G \) be a finite \( L C \)-nilpotent group of class \( t \), and let \( H = P_1 \times \cdots \times P_k \) where \( P_i \in Syl_p(G) \) and \( |G| = p_1^{s_1}p_2^{s_2}\ldots p_k^{s_k} \) where \( p_1 < \cdots < p_k \) are prime numbers. If \( L C_i(G) = L C_{i+1}(G) \) for all \( i = 2, \ldots, t \), then there exists a bijection \( f \) from \( G \) to \( H \) such that \( o(xy) = \text{max}(o(x), o(y)) \) for all \( x \neq y \in G \). Let \( CP2 \) be the class of finite groups \( G \) such that \( o(xy) = \text{max}(o(x), o(y)) \) for all \( x \neq y \in G \). For a \( p \)-group \( G \) and an integer \( n \), we denote the sets \( \{ x \in G \mid x^n = 1 \} \) and \( \{ x^n \in G \mid x \in G \} \) by \( \Omega_n(G) \) and \( \cup_n(G) \), respectively. Note that the class \( CP2 \) of \( p \)-groups is larger than the class of abelian \( p \)-groups, regular \( p \)-groups (see Theorem 3.14 of [21], II, p. 47) and \( p \)-groups whose subgroup lattices are modular (see Lemma 2.3.5 of [20]). Moreover, from the main theorem in [26], we infer that powerful \( p \)-groups for \( p \) odd also belong to \( CP2 \).

In what follows, we adopt the notation established in the Isaacs’ book on finite groups [15].

### 2. \( L C(G) \)

We shall need the following theorem about the groups belonging to \( CP2 \).

**Theorem 2.1** (Theorem D in [22]). A finite group \( G \) is contained in \( CP2 \) if and only if one of the following statements holds:

1. \( G \) is a \( p \)-group and \( \Omega_n(G) = \{ x \in G \mid x^n = 1 \} \) for all integers \( n \).
2. \( G \) is a Frobenius group of order \( p^aq^b \), \( p < q \), with kernel \( F(G) \) of order \( p^a \) and cyclic complement.

**Definition 2.2.** Let \( G \) be a periodic group, and let \( H \) and \( R \) be two non-empty subsets of \( G \). Let \( L C(M, R) \) be the set of all \( x \in H \) such that \( o(x^y) \mid \text{lcm}(o(x^a), o(y)) \) for all \( y \in R \) and all integers \( n \). In addition, if \( L C(G) = G \), then we say \( G \) is an \( L C \)-group. The subgroup generated by \( L C(G) \) is denoted by \( L C(G) \).

**Example 2.3.** Let \( G = F \rtimes H \) be a Frobenius group. Let \( P \in Syl_p(F) \) such that \( P \in CP2 \). Let \( x \in P \), and let \( n \) be an integer. For all \( h \in G \setminus F \), we have \( o(x^ah) = o(h) \mid \text{lcm}(o(x^a), o(h)) \). Since \( F \) is a nilpotent group and \( P \in CP2 \), for all \( z \in F \), we have \( o(x^az) \mid \text{lcm}(o(x^a), o(z)) \). It follows that \( P \subseteq L C(G) \). In particular, if \( F \) is an abelian group, then \( L C(G) = F \).

**Lemma 2.4.** Let \( G \) be a periodic group. Let \( H \) and \( R \) be two \( G \)-invariant subsets of \( G \). Then \( L C(H, R) \) is a \( G \)-invariant subset of \( G \). Also, if \( H^{-1} = H \) and \( R^{-1} = R \), then for all \( x \in L C(H, R) \), we have \( x^{-1} \in L C(H, R) \).

**Proof.** Let \( \sigma \in \text{Aut}(G) \), and let \( v \in L C(H, R) \), and let \( x \in \langle v \rangle \). For all \( y \in R \), we have \( o(xy) = o(\sigma(xy)) = o(\sigma(x)\sigma(y)) \). Since \( \sigma(R) = R \), there exists \( g \in R \) such that \( \sigma(g) = z \). Clearly, \( o(z) = o(\sigma(g)) \). For all \( z \in R \), we have
\[ o((\sigma(x)z) = o(\sigma(xg)) = o(xg) \mid \text{lcm}(o(x), o(g)) = \text{lcm}(o(\sigma(x)), o(z)). \]

So \( \sigma(x) \in LCM(H, R) \). Also, for all \( z \in R \), we have
\[ o(x^{-1}z) = o((z^{-1}x)^{-1}) = o(z^{-1}x). \]

It follows that
\[ o(x(z^{-1}x)^{-1}) = o(xz^{-1}) \mid \text{lcm}(o(x), o(z^{-1})) = \text{lcm}(o(x^{-1}), o(z)). \]

Hence \( x^{-1} \in LCM(H, R) \).

Minimal non-nilpotent groups are characterized by Schmidt as follows:

**Theorem 2.5** (See (9.1.9) of [17]). Assume that every maximal subgroup of a finite group \( G \) is nilpotent but \( G \) itself is not nilpotent. Then:

(i) \( G \) is solvable.
(ii) \( |G| = p^m q^n \) where \( p \) and \( q \) are unequal prime numbers.
(iii) There is a unique Sylow \( p \)-subgroup \( P \) and a Sylow \( q \)-subgroup \( Q \) is cyclic. Hence \( G = QP \) and \( P \leq G \).

In the following theorem, we give a necessary and sufficient condition for a finite group to be an \( LCM \)-group.

**Theorem 2.6.** Let \( G \) be a finite group. Then \( G \) is an \( LCM \)-group if and only if \( G \) is a nilpotent group and each Sylow subgroup of \( G \) belongs to \( CP2 \).

**Proof.** If \( G \) is a nilpotent group and all Sylow subgroups of \( G \) belong to \( CP2 \), then clearly, \( LCM(G) = G \). For the other side suppose that \( LCM(G) = G \). Suppose for a contradiction that there exists a minimal non-nilpotent subgroup \( A \) of \( G \). From Theorem 2.5, \( A = S \times \langle a \rangle \) where \( \text{gcd}(|S|, |\langle a \rangle|) = 1 \). Let \( s \in S \) such that \( s \not\in N_G(\langle a \rangle) \). We have \( o(a(a^{-1}s)) = o(s) \). Then \( o(s) \mid \text{lcm}(o(a), o(a^{-1}s)) = o(a) \), which is a contradiction. Hence \( G \) is a nilpotent group. Let \( P \in Syl_p(G) \). Since for all \( x, y \in P \), we have \( o(xy) \mid \text{lcm}(o(x), o(y)) = \max\{o(x), o(y)\} \), we have \( P \in CP2 \) by definition of \( CP2 \).

Let \( G \) be a periodic group. Let \( p \) be a positive integer, and let \( R_p(G) = \{x \in G : \gcd(o(x), p) \neq 1\} \) and \( H_p(G) = \{x \in G : \gcd(o(x), p) = 1\} \).

The following lemma is useful to prove Theorems 2.10 and 2.8.

**Lemma 2.7.** Let \( G \) be a periodic group and \( p \) a positive integer. Let \( A \) be a subset of \( G \) such that \( H_p(G) \subseteq A \).

(i) If \( x \in LCM(H_p(G), A) \), then \( \exp (\langle x^G \rangle) = o(x) \).
(ii) If \( x \in LCM(R_p(G), G) \), then \( \exp (\langle x^G \rangle) = o(x) \).

In particular, \( \exp (LCM(G)) = \exp (LCM(G)) \).

**Proof.** Let \( S \) be a subset of \( x^G \) such that \( F := \langle x^G \rangle = \langle S \rangle \) but \( F \neq \langle T \rangle \) for all proper subsets \( T \) of \( S \). Any \( w \in F \) is just a finite sequence \( w = s_1 \ldots s_r \), whose entries \( s_1, \ldots, s_r \) are elements of \( S \cup S^{-1} \). The integer \( r \) is called the length of the element \( w \) and its norm \( |w| \) with respect to the generating set \( S \) is defined to be the shortest length of \( w \) over \( S \).

(i) Let \( y \in F \). We proceed by induction on \( |y| \). The case \( |y| = 0 \) is trivial. So suppose that the result is true for all \( a \in F \) with \( |a| < |y| \). There are \( s_1, \ldots, s_k \in S, \epsilon_i \in \{1, -1\} \) and positive integers \( n_1, \ldots, n_k \) such that \( y = (s_1)^{\epsilon_1 n_1} \ldots (s_k)^{\epsilon_k n_k} \). We may assume that \( n_1 > 0 \). By induction hypothesis, we have \( o((s_1)^{\epsilon_1 n_1 - 1} \ldots (s_k)^{\epsilon_k n_k}) \mid o(x) \). It follows that \( \gcd(o((s_1)^{\epsilon_1 n_1 - 1} \ldots (s_k)^{\epsilon_k n_k}), p) = 1 \). Since
Lemma 2.7. Suppose that \(\text{exp}(G) = 1\). Then \(\text{exp}(G) = 1\). From Lemma 2.7, \(\text{exp}(G) = 1\). Let \(G\) be a locally finite periodic group. Then \(\text{Lem}(G)\) is a nilpotent characteristic subgroup of \(G\).

Proof. First we prove that \(\text{Lem}(H_p(G), G) \subseteq \text{Fit}(G)\). Let \(x \in \text{Lem}(H_p(G), G)\), and let \(H = \langle x^G \rangle\). From Lemma 2.7, \(\text{exp}(H) = 1\). If \(o(x)\) is a power of prime number \(p\), then clearly, \(H\) is a \(p\)-group. So suppose that \(o(x) = p_1^{a_1}p_2^{a_2}...p_k^{a_k}\) where \(p_1 < p_2 < ... < p_k\) are prime numbers. Then \(x = v_1v_2...v_k\) where \(o(v_i) = p_i^{b_i}\) for all \(i = 1, 2, ..., k\) and \(v_iv_j = v_jv_i\) for all \(1 \leq i \leq j \leq k\). Let \(H_i = \langle v_i^G \rangle\) for all \(1 \leq i \leq j \leq k\). For the reason that each \(H_i\) is nilpotent normal \(p_i\)-subgroups of \(G\), we have \(H = H_1H_2...H_k\) is a nilpotent subgroup. Let \(H_{x_i} = \langle x_i^G \rangle\) where \(x_i \in \text{Lem}(H_p(G), G) = \{x_1, ..., x_k\}\). Then \(C = H_{x_1}H_{x_2}...H_{x_k}\) is a nilpotent subgroup of \(G\). Since \(\text{Lem}(H_p(G), G) \leq C\), we have \(\text{Lem}(H_p(G), G) \subseteq \text{Fit}(G)\).

Similarly, we can prove that \(\text{Lem}(H_p(G), G) \subseteq \text{Fit}(G)\).

Corollary 2.9. Let \(G\) be a finite group. Then \(\text{Lem}(G)\) is a nilpotent characteristic subgroup of \(G\).

Proof. Let \(p\) be a prime number which is co-prime to \(|G|\). From Theorem 2.8, \(\langle \text{Lem}(H_p(G), G) \rangle = \langle \text{Lem}(G, G) \rangle = \text{Lem}(G)\) is a nilpotent characteristic subgroup of \(G\).

Theorem 2.10. Let \(G\) be a locally finite periodic group. Then \(\text{Lem}(G)\) is a locally nilpotent subgroup of \(G\).

Proof. Suppose for a contradiction that \(\text{Lem}(G)\) is not a locally nilpotent subgroup of \(G\). Then there exist \(z_1, ..., z_t \in \text{Lem}(G)\) such that \(\langle z_1, ..., z_t \rangle\) is not a nilpotent group. Let \(H = \langle z_1, ..., z_t \rangle\). Since \(G\) is locally finite, \(H\) is a finite group. From Corollary 2.9, \(\langle z_1, ..., z_t \rangle \leq \text{Lem}(G)\) is a nilpotent, which is a contradiction.

The following example shows that the condition \(x^n \in \text{Lem}(G)\) for all integers \(n\) in Corollary 2.9 is necessary.

Example 2.11. Let \(G = C_2 \times C_2 \times A_4\). Then \(\text{exp}(G) = 6\). There exists an element \(x\) of order 6 in \(G\), such that \(x^2 = (1, 2, 3)\). But

\[o((1, 2, 3)(2, 3, 4)) = o((1, 3)(2, 4)) = 2 \not| \text{lcm}(o(1, 2, 3), o(2, 3, 4)).\]

In the following example, we introduce an infinite group \(G\) such that its \(\text{Lem}(G)\) subgroup is not a nilpotent subgroup.
Example 2.12. The idea of the following example works with the theory of hyperbolic groups (in the Gromov sense [10]) and Kazhdan’s property (T), which the reader can see the definitions in [4, 19]. Fix a prime \( p \) \( > 2 \) and denote by \( G \) the free product of two copies of \( \mathbb{Z}/p\mathbb{Z} \). This group is hyperbolic (see Lemma 1.18 in [19]). Via iterated small cancelation in hyperbolic groups (see Section 4 in [19]), we may construct periodic quotients of that group (see proof of Corollary 2 in [19]), and this leads to the following. For \( n = p^k \) with \( k \) large enough, the quotient \( F(p, k) \) of \( G \) by the smallest normal subgroup of \( G \) that contains all elements \( g^n, g \in G \), is an infinite group. It is also a non-amenable group. To see this, we may argue as follows. The Ol’shanskiĭ common quotient theorem says that two hyperbolic groups \( H \) and \( H' \) always have a common quotient (see Theorem 5.7 and Corollary 5.8 in [6]), and one can moreover assume that the torsion elements in the quotient come from the torsion elements of \( H \), or of \( H' \). In particular, there is a quotient \( H' \) of \( G \) with Kazhdan’s property (T) whose torsion elements have order \( p \). Then, if \( k \) is large enough, the quotient of \( H' \) by the smallest normal subgroup containing all elements \( h^n \) with \( h \in H \) is an infinite Kazhdan group; and this implies that \( F(p, k) \) is not amenable (indeed since by compactness the only discrete groups that satisfy both property (T) and amenability are finite). By definition, \( < \text{LCM}(F(p, k)) > \) is \( F(p, k) \) itself; in particular it is not nilpotent.

Corollary 2.13. Let \( G \) be a finite group.

(i) If \( \text{Fit}(G) = 1 \), then \( \text{LC}(G) = 1 \).

(ii) If \( N \triangleleft G \), then \( \text{LC}(N) \leq \text{Fit}(G) \).

**Proof.** From Theorem 2.8, \( \text{LC}(G) \) is a normal nilpotent subgroup of \( G \).

(i) Since \( \text{Fit}(G) = 1 \), we have \( \text{LC}(G) = 1 \).

(ii) From Theorem 2.8, \( \text{LC}(N) \) is a nilpotent characteristic subgroup of \( N \). Hence \( \text{LC}(N) \leq \text{Fit}(G) \).

Corollary 2.14. Let \( G \) be a finite solvable group. If \( \text{Fit}(G) \neq G \) is a Hall subgroup such that for each Sylow subgroup \( P \) of \( \text{Fit}(G) \), we have \( P \in \text{CP}2 \), then \( \text{Fit}(G) = \text{LC}(G) \).

**Proof.** Let \( y \in G \setminus \text{Fit}(G) \) and let \( x \in \text{Fit}(G) \). We have \( (yx)^m = x^{r_1}x^{r_2}...x^{r_m}y^m \). If \( m = o(y) \), then \( (yx)^{o(y)} = x^{r_1}x^{r_2}...x^{r_m} \in \text{Fit}(G) \). Now, since every Sylow subgroup of \( \text{Fit}(G) \) belongs to \( \text{CP}2 \), we have \( (x^{r_1}x^{r_2}...x^{r_m})^p(x) = 1 \). It follows that \( o(xy) = o(xy) | o(x)o(y) = \text{LC}(o(x), o(y)) \).

Corollary 2.15. Let \( G \) be a non-nilpotent finite group such that \( 2 \nmid |\text{Fit}(G)| \). Then the following statements are equivalent:

(i) All proper subgroups of \( G \) are LCM-groups.

(ii) \( G \) is a minimal non-nilpotent group.

**Proof.** (i) \( \Rightarrow \) (ii): From Theorem 2.8, all subgroups of \( G \) are nilpotent. So \( G \) is a minimal non-nilpotent group.

(ii) \( \Rightarrow \) (i): If \( G \) is a minimal non-nilpotent group, then from Theorem 2.5, there is a unique Sylow \( p \)-subgroup \( P \) and a cyclic Sylow \( q \)-subgroup \( Q \) such that \( G = P \rtimes Q \). From Lemma 2.3 of [2], \( \frac{G}{\text{Fit}(G)} \) is Frobenius group such that the Frobenius kernel is elementary abelian and the Frobenius complement is of prime order. Hence the nilpotency classes of \( P \) is at most two. Since \( 2 \nmid |\text{Fit}(G)| \), \( p \geq 2 \), so \( P \) is a regular group, and then \( P \in \text{CP}2 \). Now, from Corollary 2.14, all subgroups of \( G \) are LCM-groups.
Note that Corollary 2.15, in general, is not true. For example, a GAP [7] check yields, the SmallGroup(160, 199) is a minimal non-nilpotent group such that its Sylow 2-subgroup is not a LCM-group.

**Theorem 2.16.** Let $G$ be a finite group and $H$ be a locally solvable periodic group.

(i) If $G \setminus \text{LCM}(G) \subseteq \{x \in G : o(x) = \exp(G)\}$, then $G$ is a nilpotent group.

(ii) Suppose that $[x, y] \in \text{LCM}(G)$ for all $x, y \in G$. Then $o(uv) \mid \text{lcm}(o(u), o(v)) \cdot o([u, v])$ for all $u, v \in G$.

(iii) Suppose that $[x, y] \in \text{LCM}(H)$ for all $x, y \in H$. Then $o(uv) \mid \text{lcm}(o(u), o(v)) \cdot o([u, v])$ for all $u, v \in H$.

**Proof.**

(i) Clearly, we may assume that $G$ is not a $p$-group. Then $\exp(G)$ has at least two distinct prime divisors. Let $P \in \text{Syl}_p(G)$. By our assumption $P \subseteq \text{LCM}(G)$. From Theorem 2.8, $\text{LC}(G)$ is normal and nilpotent subgroup of $G$, so $P \triangleleft G$. It follows that every Sylow subgroup of $G$ is normal in $G$, so $G$ is a nilpotent group.

(ii) Let $u, v \in G$, and let $m = \text{lcm}(o(u), o(v))$. From Hall-Petresco formula, we have $(uv)^m = u^m v^m c_2^m \cdots c_{m-1}^m c_m^m$ where $c_i \in \gamma_i([u, v])$. Since $\gamma_1([u, v]) = \langle [u, v]^g : g \in [u, v] \rangle$ and $[u, v] \in \text{LCM}(G)$, from Lemma 2.7, we have $\exp(\gamma_1([u, v])) = o([u, v])$. Then $(uv)^m \cdot o([u, v]) = (c_2^m \cdots c_{m-1}^m c_m^m)^{o([u, v])} = 1$, as claimed.

(iii) It is similar to the case (ii). \hfill \Box

In the following example, we show that in general, the equality $\text{Fit}(G) = \text{LC}(G)$ is not true.

**Example 2.17.** Let $G$ be a $p$-group of maximal class and order $p^n > p^{p+1}$ such that $\exp(G) > p^3$. Let $M$ be the regular maximal subgroup of $G$. Let $x \in M$ of order $\exp(G)$ and $y \in G \setminus M$. Then $o(xy^{-1}) \leq p^2$ and $o(y) \leq p^2$. Since $o(xy^{-1}y) = \exp(G) \neq p^2 = \text{lcm}(o(xy^{-1}, o(y))$, we have $y \not\in \text{LCM}(G)$. Consequently, $\text{LCM}(G) \subseteq M$, and so $\text{LC}(G) \leq M \neq G$.

### 3. LC-Series

Let $G$ be a group. Define $\text{LC}_1(G) = \text{LC}(G)$ and for $i = 2, 3, \ldots$, define $\text{LC}(\frac{G}{\text{LC}_{i+1}(G)}) = \frac{\text{LC}(G)}{\text{LC}_{i+1}(G)}$. The series

$$\text{LC}_1(G) \leq \text{LC}_2(G) \leq \cdots \leq \text{LC}_i(G) \leq \cdots \tag{3.1}$$

is called LC-series of $G$. We say the LC-series (3.1), is finite whenever there exists a positive integer $k$ such that $\text{LC}_{i+k}(G) = \text{LC}_k(G)$ for all integers $i = 1, 2, \ldots$. The smallest integer $k$ such that $\text{LC}_{i+k}(G) = \text{LC}_k(G)$ for all integers $i = 1, 2, \ldots$ is called length of series.

**Definition 3.1.** We say the group $G$ is a LC-nilpotent group, whenever there exists a finite LC-series

$$\text{LC}_0 := 1 \leq \text{LC}_1(G) \leq \text{LC}_2(G) \leq \cdots \leq \text{LC}_k(G) = G \tag{3.2}$$

such that $\frac{\text{LC}(G)}{\text{LC}_{i+1}(G)}$ is a nilpotent group for all $i = 1, 2, \ldots, k$. In this case the LC-series (3.2), is called a LC-nilpotent series for $G$.

**Example 3.2.** Let $G$ be a non-abelian group of order $pq$ where $p < q$ are primes. Then $G$ has a nilpotent LC-series of length two. Also, any nilpotent groups is a LC-nilpotent group. Furthermore, $G$ is a finite LC-nilpotent group of class one if and only if $G$ is a nilpotent group and all Sylow subgroups of $G$ are in CP2.
We denote by $X_{lc}$ the set of all finite LC-nilpotent groups $G$ of class $t_G$ such that $LC(G) = LCM(G)$ for all $i = 2, ..., t_G$.

**Lemma 3.3.** Let $G \in X_{lc}$ of order $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ where $p_1 < \cdots < p_k$ are prime numbers. Let $H = P_1 \times \cdots \times P_k$ where $P_i \in Syl_{p_i}(G)$ for all $i = 1, ..., k$. Then there exists a bijection $f$ from $G$ to $H$ such that $o(x) | o(f(x))$ for all $x \in G$.

**Proof.** We proceed by induction on $|G|$. Let $N$ be a normal minimal subgroup of $G$ such that $N \leq LG_1(G)$. We may assume that $N \leq P_1$. Let $Q_i = \frac{P_i}{P_i \cap N}$ for $i = 1, 2, ..., k$. By induction hypothesis, there exists a bijection $\theta$ from $G$ onto $Q_1 \times \cdots \times Q_k$ such that $o(xN) | o(\theta(xN))$ for all $xN \in \frac{G}{N}$. Let $\theta(N) = M$. Let $x \in G$. Since $G$ is a solvable group, we may assume that $G = P_1P_2...P_k$. Then $x = x_1x_2...x_k$ where $x_i \in P_i$ for all $i = 1, 2, ..., k$. Let $H = N \rtimes \langle x \rangle$. Set $o(x) = m$, and $gcd(m, |P_1|) = c$. We have $H = (N(x^{m/c})) \rtimes \langle x^c \rangle$. Let $h \in N$. If $x_1h = 1$, then $o(xh) = o(x_1h, x_2, ..., x_k)$. So suppose that $x_1h \neq 1$. Clearly, $m/c | o(xh)$. Then $(xh)^{m/c} = h^{x^{-1}h^{x^{-2}}...h^{x^{-m/c}}} \in H(x^{m/c})$. If $c > p_i$, then $N \leq LG_1(N(x^{m/c}))$, and so $o(xh) = o(x) = o(x_1h, x_2, ..., x_k)$. So suppose that $c = p_i$. If $h^{x^{-1}h^{x^{-2}}...h^{x^{-m/c}}} = x^{m/c}$, then $h \in \langle x \rangle$, and so $o(xh) = o(x) = o(x_1h, x_2, ..., x_k)$. If $h^{x^{-1}h^{x^{-2}}...h^{x^{-m/c}}} \neq x^{m/c}$, then $o(xh) = o(x) | o(x_1h, x_2, ..., x_k)$. Hence we may define a bijection $f_x$ from $xN$ onto $\theta(x)M$ such that $o(xh) | o(f_x(xh))$ for all $h \in N$. Let $X$ be a left transversal for $N$ in $G$. Then $f = \bigcup_{x \in X} f_x$ has the desired property.

Given a finite group $G$, let $\psi(G) = \sum_{x \in G} o(x)$. Many studies have been done on the function $\psi$, to find an exact upper or lower bound for sums of element orders in non-cyclic finite groups. Hence the following corollaries could be useful to find some bound for the value of function $\psi$.

**Corollary 3.4.** Let $G \in X_{lc}$ be a finite group of order $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$ where $p_1 < \cdots < p_k$ are prime numbers. Let $H = P_1 \times \cdots \times P_k$ where $P_i \in Syl_{p_i}(G)$ for all $i = 1, ..., k$. Then $\psi(G) \leq \psi(H)$.

**Corollary 3.5.** Let $G \in X_{lc}$ such that $G$ is not a nilpotent group. Then there exists $H \in X_{lc}$ such that $H$ is nilpotent and $\psi(G) \leq \psi(H)$.

**Remark 3.6.** Let $G$ be a finite group. Let $x, y \in LCM(G)$ such that $o(xy) = lcm(o(x), o(y))$. Let $h \in G$. We have

$$o(xyh) = lcm(o(x), o(yh)),$$

$$= lcm(o(x), o(y), o(h)).$$

It follows that $xy \in LCM(G)$.

With the computational group theory system GAP, the $G:=\text{SmallGroup}(16,13)$ the $LC(G)$ is different from $LCM(G)$. In view of this example, answering to the following questions would be interesting.

**Question 3.7.**

1. What is the set of all finite groups $G$, with $LC(G) = LCM(G)$?
2. What is the set of all LC-nilpotent groups of class two?

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ORCID

Mohsen Amiri http://orcid.org/0000-0003-3314-0301
Igor Lima http://orcid.org/0000-0002-0346-2716

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