Berry phase for oscillating neutrinos

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Abstract

We show the presence of a topological (Berry) phase in the time evolution of a mixed state. For the case of mixed neutrinos, the Berry phase is a function of the mixing angle only.

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I. INTRODUCTION

Particle mixing and oscillations play a relevant role in high energy physics. In particular, in recent years a growing interest in neutrino mixing and oscillations \cite{1} has been developed which manifests itself both in a strong experimental effort and in a renewed theoretical research activity. Contributions towards the correct theoretical understanding of the field mixing have been recently presented \cite{2,3}.

In the present paper we show how the notion of Berry phase \cite{4} enters the physics of mixing by considering the example of neutrino oscillations.

Since its discovery \cite{4}, the Berry phase has attracted much interest \cite{5} at theoretical as well as at experimental level. This interest arises because the Berry phase reveals geometrical features of the systems in which it appears, which go beyond the specific dynamical aspects and as such contribute to a deeper characterization of the physics involved. The successful experimental findings in many different quantum systems \cite{5} stimulate further search in this field.

Aimed by these motivations, we show that the geometric phase naturally appears in the standard Pontecorvo formulation of neutrino oscillations.

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Our result shows that the Berry phase associated to neutrino oscillations is a function of the mixing angle only. We suggest that such a result has phenomenological relevance: since geometrical phases are observable, the mixing angle can be (at least in principle) measured directly, i.e. independently from dynamical parameters as the neutrino masses and energies.

Although in the following we treat the neutrino case, we stress that our result holds in general, also in the case of mixed bosons (Kaons, $\eta's$, etc.).

II. BERRY PHASE FOR OSCILLATING NEUTRINOS

Let us first consider the two flavor case [1]:

\[
|\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \\
|\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle.
\]

The electron neutrino state at time $t$ is [1]

\[
|\nu_e(t)\rangle \equiv e^{-iHt}|\nu_e(0)\rangle = e^{-i\omega_1 t} \left( \cos \theta |\nu_1\rangle + e^{-i(\omega_2 - \omega_1)t} \sin \theta |\nu_2\rangle \right),
\]

where $H|\nu_i\rangle = \omega_i|\nu_i\rangle$, $i = 1, 2$. Our conclusions will also hold for the muon neutrino state, with due changes which will be explicitly shown when necessary.

The state $|\nu_e(t)\rangle$, apart from a phase factor, reproduces the initial state $|\nu_e(0)\rangle$ after a period $T = \frac{2\pi}{\omega_2 - \omega_1}$:

\[
|\nu_e(T)\rangle = e^{i\phi}|\nu_e(0)\rangle, \quad \phi = -\frac{2\pi \omega_1}{\omega_2 - \omega_1}.
\]

We now show how such a time evolution does contain a purely geometric part, i.e. the Berry phase. It is a straightforward calculation to separate the geometric and dynamical phases following the standard procedure [6]:

\[
\beta_e = \phi + \int_0^T \langle \nu_e(t) | i\partial_t | \nu_e(t) \rangle \, dt \\
= -\frac{2\pi \omega_1}{\omega_2 - \omega_1} + \frac{2\pi}{\omega_2 - \omega_1} (\omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta) = 2\pi \sin^2 \theta.
\]

We thus see that there is indeed a non-zero geometrical phase $\beta$, related to the mixing angle $\theta$, and that it is independent from the neutrino energies $\omega_1, \omega_2$ and masses $m_1, m_2$. In a similar fashion, we obtain the Berry phase for the muon neutrino state:

\[
\beta_\mu = \phi + \int_0^T \langle \nu_\mu(t) | i\partial_t | \nu_\mu(t) \rangle \, dt = 2\pi \cos^2 \theta.
\]

Note that $\beta_e + \beta_\mu = 2\pi$. We can thus rewrite (3) as

\[
|\nu_e(T)\rangle = e^{i2\pi \sin^2 \theta} e^{-i\omega_eeT}|\nu_e(0)\rangle,
\]

where we have used the notation
\[\langle \nu_e(t) | i \partial_t | \nu_e(t) \rangle = \langle \nu_e(t) | H | \nu_e(t) \rangle = \omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta \equiv \omega_{ee}. \tag{7}\]

We will also use
\[\langle \nu_\mu(t) | i \partial_t | \nu_\mu(t) \rangle = \langle \nu_\mu(t) | H | \nu_\mu(t) \rangle = \omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta \equiv \omega_{\mu\mu}, \tag{8}\]
\[\langle \nu_\mu(t) | i \partial_t | \nu_e(t) \rangle = \langle \nu_\mu(t) | H | \nu_e(t) \rangle = \frac{1}{2} (\omega_2 - \omega_1) \sin 2\theta \equiv \omega_{\mu e}, \tag{9}\]

with \(\omega_{\mu e} = \omega_{ee}\).

In order to better understand the meaning of (4)-(6), we observe that, as well known, \(|\nu_e\rangle\) is not eigenstate of the Hamiltonian, and
\[\langle \nu_e(0) | \nu_e(t) \rangle = e^{-i\omega_1 t} \cos^2 \theta + e^{-i\omega_2 t} \sin^2 \theta. \tag{10}\]

Thus, as an effect of time evolution, the state \(|\nu_e\rangle\) “rotates” as shown by eq.(10). However, at \(t = T\),
\[\langle \nu_e(0) | \nu_e(T) \rangle = e^{i\phi} = e^{i\beta_e} e^{-i\omega_{ee} T}, \tag{11}\]
i.e. \(|\nu_e(T)\rangle\) differs from \(|\nu_e(0)\rangle\) by a phase \(\phi\), part of which is a geometric “tilt” (the Berry phase) and the other part is of dynamical origin. In general, for \(t = T + \tau\), we have
\[\langle \nu_e(0) | \nu_e(t) \rangle = e^{i\phi} \langle \nu_e(0) | \nu_e(\tau) \rangle = e^{i2\pi \sin^2 \theta} e^{-i\omega_{ee} \tau} (e^{-i\omega_1 \tau} \cos^2 \theta + e^{-i\omega_2 \tau} \sin^2 \theta). \tag{12}\]

Also notice that \(\langle \nu_\mu(t) | \nu_e(t) \rangle = 0\) for any \(t\). However,
\[\langle \nu_\mu(0) | \nu_e(t) \rangle = \frac{1}{2} e^{i\phi} e^{-i\omega_1 \tau} \sin 2\theta (e^{-i(\omega_2 - \omega_1) \tau} - 1), \quad \text{for } t = T + \tau, \tag{13}\]

which is zero only at \(t = T\). Eq.(13) expresses the fact that \(|\nu_e(t)\rangle\) “oscillates”, getting a component of muon flavor, besides getting the Berry phase. At \(t = T\), neutrino states of different flavor are again each other orthogonal states.

Generalization to \(n\)-cycles is also interesting. Eq.(4) (and (3)) can be rewritten for the \(n\)-cycle case as
\[\beta_e^{(n)} = \int_0^{nT} \langle \nu_e(t) | i \partial_t - \omega_1 | \nu_e(t) \rangle dt = 2\pi n \sin^2 \theta, \tag{14}\]

and eq.(12) becomes
\[\langle \nu_e(0) | \nu_e(t) \rangle = e^{in\phi} \langle \nu_e(0) | \nu_e(\tau) \rangle, \quad \text{for } t = nT + \tau. \tag{15}\]

Similarly eq.(13) gets the phase \(e^{in\phi}\) instead of \(e^{i\phi}\). Eq.(14) shows that the Berry phase acts as a “counter” of neutrino oscillations, adding up \(2\pi \sin^2 \theta\) to the phase of the (electron) neutrino state after each complete oscillation.

Eq.(14) is interesting especially because it can be rewritten as
\[\beta_e^{(n)} = \int_0^{nT} \langle \nu_e(t) | U^{-1}(t) i \partial_t (U(t) | \nu_e(t) \rangle dt = \int_0^{nT} \langle \tilde{\nu}_e(t) | i \partial_t | \tilde{\nu}_e(t) \rangle dt = 2\pi n \sin^2 \theta, \tag{16}\]
with \( U(t) = e^{-if(t)} \), where \( f(t) = f(0) - \omega_1 t \), and

\[
|\tilde{\nu}_e(t)\rangle \equiv U(t)|\nu_e(t)\rangle = e^{-if(0)} \left( \cos \theta |\nu_1\rangle + e^{-i(\omega_2-\omega_1)t} \sin \theta |\nu_2\rangle \right) .
\]

Eq. (17) actually provides an alternative way for defining the Berry phase \([3]\), which makes use of the state \(|\tilde{\nu}_e(t)\rangle\) given in eq. (17). In contrast with the state \(|\nu_e(t)\rangle\), \(|\tilde{\nu}_e(t)\rangle\) is not “tilted” in its time evolution:

\[
\langle \tilde{\nu}_e(0)|\tilde{\nu}_e(t)\rangle = \langle \tilde{\nu}_e(0)|\tilde{\nu}_e(\tau) \rangle , \quad \text{for } t = nT + \tau ,
\]

which is to be compared with eq. (15). From eq. (17) we also see that time evolution only affects the \(|\nu_2\rangle\) component of the state \(|\tilde{\nu}_e(t)\rangle\), so that we have

\[
i\partial_t|\tilde{\nu}_e(t)\rangle = (\omega_2 - \omega_1)e^{-if(0)}e^{-i(\omega_2-\omega_1)t} \sin \theta |\nu_2\rangle \\
= (H - \omega_1)e^{-if(0)} \left( \cos \theta |\nu_1\rangle + e^{-i(\omega_2-\omega_1)t} \sin \theta |\nu_2\rangle \right) \\
= (H - \omega_1)|\tilde{\nu}_e(t)\rangle .
\]

We thus understand that eq. (16) directly gives us the geometric phase because the quantity \( i\langle \tilde{\nu}_e(t)|\tilde{\nu}_e(t)\rangle dt \) is the overlap of \(|\tilde{\nu}_e(t)\rangle\) with its “parallel transported” at \( t + dt \).

Another geometric invariant which can be considered is

\[
s = \int_0^{nT} \omega_{\mu e} \, dt = \pi n \sin 2\theta .
\]

Since \( \omega_{\mu e} \) is the energy shift from the level \( \omega_{ee} \) caused by the flavor interaction term in the Hamiltonian \([4]\), it is easily seen that

\[
\omega_{\mu e}^2 = \Delta E^2 \equiv \langle \nu_e(t)|H^2|\nu_e(t)\rangle - \langle \nu_e(t)|H|\nu_e(t)\rangle^2 ,
\]

and then we recognize that eq. (21) gives the geometric invariant discussed in ref. \([4]\), where it is defined quite generally as \( s = \int \Delta E(t)dt \). It has the advantage to be well defined also for systems with non-cyclic evolution.

We now consider the case of three flavor mixing. Consider again the electron neutrino state at time \( t \) \([2]\):

\[
|\nu_e(t)\rangle = e^{-i\omega_1t} \left( \cos \theta_{12} \cos \theta_{13} |\nu_1\rangle + e^{-i(\omega_2-\omega_1)t} \sin \theta_{12} \cos \theta_{13} |\nu_2\rangle + e^{-i(\omega_3-\omega_1)t} e^{i\delta} \sin \theta_{13} |\nu_3\rangle \right) ,
\]

where \( \delta \) is the analogous of the CP violating phase of the CKM matrix. Let us consider the particular case in which the two frequency differences are proportional: \( \omega_3 - \omega_1 = q(\omega_2 - \omega_1) \), with \( q \) a rational number. In this case the state \( (22) \) is periodic over a period \( T = \frac{2\pi}{\omega_2 - \omega_1} \), and we can use the previous definition of Berry phase:

\[
\beta = \phi + \int_0^T \langle \nu_e(t)|H|\nu_e(t)\rangle dt = 2\pi \left( \sin^2 \theta_{12} \cos^2 \theta_{13} + q \sin^2 \theta_{13} \right) ,
\]

which of course reduces to the result \([4]\) for \( \theta_{13} = 0 \). Eq. (23), however, shows that \( \beta \) is not completely free from dynamical parameters since the appearance in it of the parameter \( q \).

Although because of this, \( \beta \) is not purely geometric, nevertheless it is interesting that it does not depend on the specific frequencies \( \omega_i, \ i = 1, 2, 3 \), but on the ratio of their differences only. This means that we have now (geometric) classes labelled by \( q \).

It is in our plan to calculate the geometric invariant \( s \) for the three flavor neutrino state: this requires consideration of the projective Hilbert space in the line of ref. \([4,8]\).
III. FINAL REMARKS AND CONCLUSIONS

The geometric phase is generally associated with a parametric dependence of the time evolution generator. In such cases, the theory exhibits a gauge-like structure which may become manifest and characterizing for the physical system, e.g. in the Bohm-Aharonov effect [4].

It is then natural to ask the question about a possible gauge structure in the case considered in this paper. Let us see how, indeed, a covariant derivative may be here introduced.

Let us consider the evolution of the mass eigenstates

$$i\partial_t|\nu_i(t)\rangle = H|\nu_i(t)\rangle,$$

(24)

where $i = 1, 2$. These equations are invariant under the following (local in time) gauge transformation

$$|\nu_i(t)\rangle \rightarrow |\tilde{\nu}_i(t)\rangle \equiv U(t)|\nu_i(t)\rangle = e^{-if(t)}|\nu_i(t)\rangle,$$

(25)

provided $[H, U(t)] = i\partial_t U(t)$, i.e.

$$U^{-1}(t)HU(t) = H + U^{-1}(t)i\partial_t U(t) = H + \partial_t f(t).$$

(26)

This suggests that, by rewriting (24) as

$$(i\partial_t - H)|\nu_i(t)\rangle = 0,$$

(27)

we can consider $D_t \equiv \partial_t + iH$ as the “covariant derivative”:

$$D_t \rightarrow D'_t = U(t)D_t U^{-1}(t).$$

(28)

We have indeed

$$iD'_t|\tilde{\nu}_i(t)\rangle = iD'_t U(t)|\nu_i(t)\rangle = iU(t)D_t|\nu_i(t)\rangle = 0,$$

(29)

which in fact expresses the invariance of eq.(24) under (25).

We thus see that the time dependent canonical transformation of the Hamiltonian eq.(26) and eq.(25) play the role of a local (in time) gauge transformation. Note that the state $|\tilde{\nu}_e(t)\rangle$ of eq.(17) is a superposition of the states $|\tilde{\nu}_i(t)\rangle$.

The role of the “diabatic” force arising from the term $U^{-1}(t)i\partial_t U(t)$ has been considered in detail elsewhere [10].

Summarizing, we have shown that there is a Berry phase built in in the neutrino oscillations, we have explicitly computed it in the cyclic two-flavour case and in a particular case of three flavor mixing. The result also applies to other (similar) cases of particle oscillations.

We have noticed that a measurement of this Berry phase would give a direct measurement of the mixing angle independently from the values of the masses.

The above analysis in terms of “tilting” of the state in its time evolution, parallel transport and covariant derivative also suggests that field mixing may be seen as the result of a curvature in the state space. The Berry phase appears to be a manifestation of such a curvature.

Finally, we remark that the recognition of the geometric phase associated to mixed states also suggests to us that a similar geometric phase also occurs in entangled quantum states which can reveal to be relevant in completely different contexts than particle oscillations, namely in quantum computation [11].
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