Approximately Lie ternary \((\sigma, \tau, \xi)\)-derivations on Banach ternary algebras

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Abstract

Let \(A\) be a Banach ternary algebra over a scalar field \(\mathbb{R}\) or \(\mathbb{C}\) and \(X\) be a ternary Banach \(A\)-module. Let \(\sigma, \tau\) and \(\xi\) be linear mappings on \(A\), a linear mapping \(D: (A, [\[,\]_A]) \to (X, [\[,\]_X])\) is called a Lie ternary \((\sigma, \tau, \xi)\)-derivation, if

\[
D([abc]) = [D(a)bc]_X(\sigma, \tau, \xi) + [D(b)ac]_X(\sigma, \tau, \xi) + [D(c)ba]_X(\sigma, \tau, \xi),
\]

for all \(a, b, c \in A\), where \([abc]_{(\sigma, \tau, \xi)} = a\tau(b)\xi(c) - \sigma(c)\tau(b)a\).

In this paper, we investigate the generalized Hyers–Ulam–Rassias stability of Lie ternary \((\sigma, \tau, \xi)\)-derivations on Banach ternary algebras.

1. Introduction

In the 19th century, many mathematicians considered ternary algebraic operations and their generalizations. A. Cayley ([7]) introduced the notion of cubic matrix. It was later generalized by Kapranov, Gelfand and Zelevinskii in 1990 ([12]). Below, a composition rule includes a simple example of such non-trivial ternary operation:

\[
\{a, b, c\}_{ijk} = \sum_{i,j,k=1}^N a_{ij}b_{jm}c_{kn}, \quad i, j, k = 1, 2, ..., N.
\]

There are a lot of hopes that ternary structures and their generalization will have certain possible applications in physics. Some of these applications are (see [2,3],[5],[10],[13,14,15]). A ternary (associative) algebra \((A, [\[,\]])\) is a linear space \(A\) over a scalar field \(F = (\mathbb{R} or \mathbb{C})\) equipped with a linear mapping, the so-called ternary product, \([\[,\]]\): \(A \times A \times A \to A\) such that \([[abc]de] = [abd]ce\) for all \(a, b, c, d, e \in A\). This notion is a natural generalization of the binary case. Indeed if \((A, \odot)\) is a usual (binary) algebra then \([abc] := (a \odot b) \odot c\) induces a ternary product making \(A\) into a ternary algebra which will be called trivial. It is known that unital ternary algebras are trivial and finitely generated ternary algebras are ternary subalgebras of trivial ternary algebras [6]. There are other types of ternary algebras in which one may consider other versions of associativity. Some examples of ternary algebras are (i)
"cubic matrices" introduced by Cayley [7] which were in turn generalized by Kapranov, Gelfand and Zelevinskii [12]; (ii) the ternary algebra of polynomials of odd degrees in one variable equipped with the ternary operation \([p_1 p_2 p_3] = p_1 \odot p_2 \odot p_3\), where \(\odot\) denotes the usual multiplication of polynomials.

By a Banach ternary algebra we mean a ternary algebra equipped with a complete norm \(\|\cdot\|\) such that \(\|abc\| \leq \|a\|\|b\|\|c\|\). If a ternary algebra \((A,\{\} )\) has an identity, i.e. an element \(e\) such that \(a = [ace] = [eca] = [cea]\) for all \(a \in A\), then \(a \odot b := [aeb]\) is a binary product for which we have

\[(a \odot b) \odot c = [aeb]ec = [aceb]c = a \odot (b \odot c)\]

and

\[a \odot e = [ace] = a = [eca] = e \odot a,\]

for all \(a, b, c \in A\) and so \((A,\{\} )\) may be considered as a (binary) algebra. Conversely, if \((A,\{\} )\) is any (binary) algebra, then \([abc] := a \odot b \odot c\) makes \(A\) into a ternary algebra with the unit \(e\) such that \(a \odot b = [aeb]\).

Let \(A\) be a Banach ternary algebra and \(X\) be a Banach space. Then \(X\) is called a ternary Banach \(A\)-module, if module operations \(A \times A \times X \to X\), \(A \times X \times A \to X\), and \(X \times A \times A \to X\) are \(\mathbb{C}\)-linear in every variable. Moreover satisfy:

\[[abc]_A \cdot dx\big|_X = [a[bed]_A x]\big|_X = [ab[cdx]_A]_X\]
\[[abc]_A \cdot xdf\big|_X = [a[bcx]_A d]\big|_X = [ab[cdx]_A]_X,\]
\[[xab]_A \cdot xcf\big|_X = [x[abc]_A d]\big|_X = [xa[bcx]_A]_X,\]
\[[axb]_A \cdot xcf\big|_X = [ax[bcx]_A d]\big|_X = [ab[cdx]_A]_X,\]
\[[abx]_A \cdot xcf\big|_X = [ab[bcx]_A d]\big|_X = [ab[cdx]_A]_X,\]

for all \(x \in X\) and all \(a, b, c, d \in A\), and

\[\max\{\|[xab]_A\|, [[axb]_A]_X, [[abx]_A]_X\} \leq \|a\|\|b\|\|x\|\]

for all \(x \in X\) and all \(a, b \in A\).

Let \(A\) be a normed algebra, \(\sigma\) and \(\tau\) two mappings on \(A\) and \(X\) be an \(A\)-bimodule. A linear mapping \(L : A \to X\) is called a Lie \((\sigma, \tau)\)-derivation, if

\[L([a, b]) = [L(a), b]_{\sigma, \tau} - [L(b), a]_{\sigma, \tau}\]

for all \(a, b \in A\), where \([a, b]_{\sigma, \tau}\) is \(\sigma\tau(b) - \sigma(b)\sigma\) and \([a, b]\) is the commutator \(ab - ba\) of elements \(a, b\).

Now, let \((A, [\cdot]_A )\) be a Banach ternary algebra over a scalar field \(\mathbb{R}\) or \(\mathbb{C}\) and \((X, [\cdot]_X )\) be a ternary Banach \(A\)-module. Let \(\sigma, \tau\) and \(\xi\) be linear mappings on \(A\). A linear mapping \(D : (A, [\cdot]_A ) \to (X, [\cdot]_X )\) is called a Lie ternary \((\sigma, \tau, \xi)\)-derivation, if

\[D([abc]_A) = [[D(a)bc]_X]_{\sigma, \tau, \xi} + [[D(b)ac]_X]_{\sigma, \tau, \xi} + [[D(c)ba]_X]_{\sigma, \tau, \xi}\]

(1.1)

for all \(a, b, c \in A\), where \([abc]_{\sigma, \tau, \xi} = \sigma\tau(b)\xi(c) - \sigma(c)\tau(b)a\).

If a Banach ternary algebra \(A\) has an identity \(e\) such that \(\|e\| = 1\), as we said above, \(A\) may be considered as a (binary) algebra. Now let \(X\) be a ternary Banach \(A\)-module, then \(X\) may be considered as a Banach \(A\)-module by following module product:

\[a x = [aex]_X\]

\[x a = [xea]_X\]

for all \(a \in A, x \in X\).

Let \(A\) be a unital Banach ternary algebra and \(X\) be a ternary Banach \(A\)-module. If \(D : A \to X\) is a Lie ternary \((\sigma, \tau, \xi)\)-derivation such that \(\sigma, \tau\) and \(\xi\) are linear mappings on \(A\), additionally, \(\tau(e) = e\), then it is easy to prove that \(D\) is a Lie \((\sigma, \xi)\)-derivation.
The stability of functional equations was started in 1940 with a problem raised by S. M. Ulam [19]. In 1941 Hyers affirmatively solved the problem of S. M. Ulam in the context of Banach spaces. In 1950 T. Aoki [4] extended the Hyers' theorem. In 1978, Th. M. Rassias [16] formulated and proved the following Theorem:

Assume that $E_1$ and $E_2$ are real normed spaces with $E_2$ complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$ the mapping $t \to f(tx)$ is continuous on $\mathbb{R}$, and let there exist $\epsilon \geq 0$ and $p \in [0, 1]$ such that $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \leq \epsilon\|x\|^p(1 - 2^p)$ for all $x \in E_1$.

The equality $\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$ has provided extensive influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations [8,11,15,17,18]. In 1994, a generalization of Rassias' theorem was obtained by Gavruta [9], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function.

2. Lie ternary $(\sigma, \tau, \xi)$–derivations on Banach ternary algebras

In this section our aim is to establish the Hyers–Ulam–Rassias stability of Lie ternary $(\sigma, \tau, \xi)$–derivations.

**Theorem 2.1.** Suppose $f : A \to X$ is a mapping with $f(0) = 0$ for which there exist mappings $g, h, k : A \to A$ with $g(0) = h(0) = k(0) = 0$ and a function $\varphi : A \times A \times A \times A \to [0, \infty]$ such that

$$\bar{\varphi}(x, y, u, v, w) = \frac{1}{2} \sum_{n=0}^{\infty} \varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w) < \infty$$

$$\|f(\lambda x + \lambda y + [uvw]_A) - \lambda f(x) - \lambda f(y) - [[f(u)v]x]_{(g, h, k)} + [[f(v)w]x]_{(g, h, k)} + [[f(w)u]x]_{(g, h, k)}\| \leq \varphi(x, y, u, v, w)$$

for all $\lambda \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and for all $x, y, u, v, w \in A$. Then there exist unique linear mappings $\sigma, \tau$ and $\xi$ from $A$ to $A$ satisfying

$$\|g(x) - \sigma(x)\| \leq \bar{\varphi}(x, x, 0, 0, 0)$$

$$\|h(x) - \tau(x)\| \leq \bar{\varphi}(x, x, 0, 0, 0)$$

and

$$\|k(x) - \xi(x)\| \leq \bar{\varphi}(x, x, 0, 0, 0)$$

for all $x \in A$.

**Proof.** One can show that the limits

$$\sigma(x) := \lim_n \frac{1}{2^n} g(2^n x)$$

$$\tau(x) := \lim_n \frac{1}{2^n} h(2^n x)$$

$$\xi(x) := \lim_n \frac{1}{2^n} k(2^n x)$$
exist for all \( x \in A \), also \( \sigma, \tau \) and \( \xi \) are unique linear mappings which satisfy (2.4), (2.5) and (2.6) respectively (see [17]).

Put \( \lambda = 1 \) and \( u = v = w = 0 \) in (2.3) to obtain

\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y, 0, 0, 0) \quad (x, y \in A).
\]  

Fix \( x \in A \). Replace \( y \) by \( x \) in (2.8) to get

\[
\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0, 0, 0).
\]

One can use the induction to show that

\[
\left\| \frac{f(2^k x)}{2^k} - \frac{f(2^q x)}{2^q} \right\| \leq \frac{1}{2^n} \sum_{k=q}^{p} \varphi(2^k x, 2^k x, 0, 0, 0) \quad (2.9)
\]

for all \( x \in A \), and all \( p > q \geq 0 \). It follows from the convergence of series (2.2) that the sequence \( \left\{ \frac{f(2^n x)}{2^n} \right\} \) is Cauchy. By the completeness of \( X \), this sequence is convergent. Set

\[
D(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]

for all \( x \in A \). Putting \( u = v = w = 0 \) and replacing \( x, y \) by \( 2^n x \) and \( 2^n y \) in (2.3) respectively, and divide the both sides of the inequality by \( 2^n \) we get

\[
\|2^{-n} f(2^n (\lambda x + \lambda y)) - 2^{-n} \lambda f(2^n x) - 2^{-n} \lambda f(2^n y)\| \leq \frac{1}{2^n} \varphi(2^n x, 2^n x, 0, 0, 0).
\]

Passing to the limit as \( n \to \infty \) we obtain \( D(\lambda x + \lambda y) = \lambda D(x) + \lambda D(y) \).

Put \( q = 0 \) in (2.9) to get

\[
\left\| \frac{f(2^k x)}{2^k} - f(x) \right\| \leq \frac{1}{2^n} \sum_{k=0}^{p-1} \varphi(2^k x, 2^k x, 0, 0, 0)
\]

for all \( x \in A \).

Taking the limit as \( p \to \infty \) we infer that

\[
\|f(x) - D(x)\| \leq \varphi(x, x, 0, 0, 0)
\]

for all \( x \in A \). Next, let \( \gamma \in \mathbb{C}(\gamma \neq 0) \) and let \( N \) be a positive integer number greater than \( |\gamma| \). It is shown that there exist two numbers \( \lambda_1, \lambda_2 \in \mathbb{T} \) such that \( 2^N = \lambda_1 + \lambda_2 \), since \( D \) is additive, we have \( D(\frac{1}{2} x) = \frac{1}{2} D(x) \) for all \( x \in A \). Hence

\[
\begin{align*}
D(\gamma x) &= D(\frac{N}{2} \frac{\gamma}{N} x) = ND(\frac{1}{2} \frac{\gamma}{N} x) = \frac{N}{2} D(\frac{\gamma}{N} x) \\
&= \frac{N}{2} D(\lambda_1 x + \lambda_2 x) = \frac{N}{2} (D(\lambda_1 x) + D(\lambda_2 x)) \\
&= \frac{N}{2} (\lambda_1 + \lambda_2) D(x) = \frac{N}{2} \frac{\gamma}{N} D(x) = \gamma D(x)
\end{align*}
\]

for all \( x \in A \). Thus \( D \) is linear.

Suppose that there exists another ternary \( (\sigma, \tau, \xi) \)-derivation \( D' : A \to X \) satisfying (2.7). Since \( D' (x) = \frac{1}{\lambda_0} D(2^n x) \), we see that

\[
\begin{align*}
\|D(x) - D'(x)\| &= \frac{1}{2^n} \|D(2^n x) - D'(2^n x)\| \\
&\leq \frac{1}{2^n} (\|f(2^n x) - D(2^n x)\| + \|f(2^n x) - D'(2^n x)\|) \\
&\leq 40 \frac{2^p}{2} \frac{2^n}{2^n} 2^{n(p-1)} |x|^p,
\end{align*}
\]

which tends to zero as \( n \to \infty \) for all \( x \in A \). Therefore \( D' = D \) as claimed. Similarly one can use (2.4), (2.5) and (2.6) to show that there exist unique linear mappings \( \sigma, \tau \) and \( \xi \) defined


by \( \lim_{n \to \infty} \frac{g(2^n x)}{2^n}, \lim_{n \to \infty} \frac{h(2^n x)}{2^n} \) and \( \lim_{n \to \infty} \frac{k(2^n x)}{2^n} \), respectively.

Putting \( x = y = 0 \) and replacing \( u, v, w \) by \( 2^n u, 2^n v \) and \( 2^n w \) in (2.3) respectively, we obtain

\[
\begin{align*}
\|f([2^n uw]_A) - &\, [f(2^n u)2^{2n} uv]_X]_{(g,h,k)} + [f(2^n v)2^{2n} uw]_X]_{(g,h,k)} + [f(2^n w)2^{2n} vu]_X]_{(g,h,k)} \| \\
\leq &\, \varphi(0, 0, 2^n u, 2^n v, 2^n w),
\end{align*}
\]

then

\[
\begin{align*}
\frac{1}{2^{2n}}\|f([2^n uw]_A) - &\, [f(2^n u)2^{2n} uv]_X]_{(g,h,k)} + [f(2^n v)2^{2n} uw]_X]_{(g,h,k)} + [f(2^n w)2^{2n} vu]_X]_{(g,h,k)} \| \\
\leq &\, \frac{1}{2^{2n}}\varphi(0, 0, 2^n u, 2^n v, 2^n w)
\end{align*}
\]

for all \( u, v, w \in A \), hence,

\[
\begin{align*}
&\lim_{n \to \infty} \frac{1}{2^{2n}}\|f([2^n uw]_A) - [f(2^n u)2^{2n} uv]_X]_{(g,h,k)} + [f(2^n v)2^{2n} uw]_X]_{(g,h,k)} \\
&\quad + [f(2^n w)2^{2n} vu]_X]_{(g,h,k)} \| \leq \lim_{n \to \infty} \frac{1}{2^{2n}}\varphi(0, 0, 2^n u, 2^n v, 2^n w) \\
&\quad = 0
\end{align*}
\]

therefore

\[
\begin{align*}
D([uw]_A) = &\, \lim_{n \to \infty} \frac{f([2^n uw]_A)}{2^{2n}} = \lim_{n \to \infty} \frac{f([2^n u2^n v2^n w]_A)}{2^{2n}} \\
= &\, \lim_{n \to \infty} \left( \frac{[f(2^n u)2^{2n} uv]_X]_{(g,h,k)} - [f(2^n v)2^{2n} uw]_X]_{(g,h,k)} - [f(2^n w)2^{2n} vu]_X]_{(g,h,k)} \right) \\
= &\, \lim_{n \to \infty} \left( \frac{f(2^n u)h(2^n v)k(2^n w) - g(2^n w)h(2^n v)f(2^n u) - f(2^n v)h(2^n u)k(2^n w)}{2^{2n}} \\
&\quad + \frac{g(2^n u)h(2^n v)f(2^n w) - f(2^n v)h(2^n u)k(2^n w) + g(2^n w)h(2^n u)f(2^n v)}{2^{2n}} \right) \\
= &\, (D(u)\tau(v)\xi(w) - \sigma(w)\tau(v)D(u)) - (D(v)\tau(u)\xi(w) - \sigma(w)\tau(u)D(v)) \\
&\quad - (D(w)\tau(u)\xi(v) - \sigma(u)\tau(v)D(w)) \\
= &\, ([D(u)\xi]_x]_{(\sigma, \tau, \xi)} - [D(v)\nu]_x]_{(\sigma, \tau, \xi)} - [D(w)\nu]_x]_{(\sigma, \tau, \xi)}
\end{align*}
\]

for each \( u, v, w \in A \). Hence, the linear mapping \( D \) is a Lie ternary \((\sigma, \tau, \xi)\)-derivation. \( \square \)

**Corollary 2.2.** Suppose \( f : A \to X \) is a mapping with \( f(0) = 0 \) for which there exist mappings \( g, h, k : A \to A \) with \( g(0) = h(0) = k(0) = 0 \) and there exists \( \theta \geq 0 \) and \( p \in [0, 1) \) such that

\[
\begin{align*}
\|f(\lambda x + \lambda y + [uw]_A) - &\, \lambda f(x) - \lambda f(y) - [f(u)\nu]_X]_{(g,h,k)} + [f(v)\nu]_X]_{(g,h,k)} \\
&\quad + [f(w)\nu]_X]_{(g,h,k)} \| \leq \theta(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p),
\end{align*}
\]

\[
\begin{align*}
\|g(\lambda x + \lambda y) - &\, g(x) - g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \\
\|h(\lambda x + \lambda y) - &\, h(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \\
\|k(\lambda x + \lambda y) - &\, k(x) - k(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\end{align*}
\]

for all \( \lambda \in \mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and for all \( x, y \in A \). Then there exist unique linear mappings \( \sigma, \tau \) and \( \xi \) from \( A \) to \( A \) satisfying \( \|g(x) - \sigma(x)\| \leq \frac{\theta\|x\|^p}{1 - 2^p}, \|h(x) - \tau(x)\| \leq \frac{\theta\|x\|^p}{1 - 2^p} \)

and \( \|k(x) - \xi(x)\| \leq \frac{\theta\|x\|^p}{1 - 2^p} \), and there exists a unique Lie ternary \((\sigma, \tau, \xi)\)-derivation \( D : A \to X \) such that

\[
\|f(x) - D(x)\| \leq \frac{\theta\|x\|^p}{1 - 2^p}, \quad x, y, z \in A
\]

(2.11)
for all \( x \in A \).

**Proof.** Put \( \varphi(x, y, u, v, w) = \theta(||x||^p + ||y||^p + ||u||^p + ||v||^p + ||w||^p) \) in Theorem 2.1. \( \square \)

Note that a linear mapping \( D : (A, [\cdot, \cdot]_A) \to (X, [\cdot, \cdot]_X) \) is called a Jordan Lie ternary \((\sigma, \tau, \xi)\)-derivation, if

\[
D([aaa]_A) = [[D(a)aa]x]_{(\sigma, \tau, \xi)} + [[D(a)aa]x]_{(\sigma, \tau, \xi)} + [[D(a)aa]x]_{(\sigma, \tau, \xi)}
\]

for all \( a \in A \),

**Theorem 2.3.** Suppose \( f : A \to X \) is a mapping with \( f(0) = 0 \) for which there exist mappings \( g, h, k : A \to A \) with \( g(0) = h(0) = k(0) = 0 \) and a function \( \varphi : A \times A \times A \to [0, \infty] \) such that

\[
\varphi(x, y, u) = \frac{1}{2} \sum_{n=0}^{\infty} \varphi(2^n x, 2^n y, 2^n u) < \infty \quad (2.12)
\]

\[
||f(\lambda x + \lambda y + [uuu]_A) - \lambda f(x) - \lambda f(y) - [[f(u)uu]x]_{(g, h, k)} + [[f(u)uu]x]_{(g, h, k)} + [[f(u)uu]x]_{(g, h, k)}|| \leq \varphi(x, y, u) \quad (2.13)
\]

for all \( \lambda \in T^1 := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and for all \( x, y, u \in A \). Then there exist unique linear mappings \( \sigma, \tau \) and \( \xi \) from \( A \) to \( A \), and a unique Jordan Lie ternary \((\sigma, \tau, \xi)\)-derivation \( D : A \to X \) satisfying (2.4), (2.5), (2.6) and (2.7), respectively.

**Proof.** By the same reasoning as the proof of Theorem 2.1, the limits

\[
D(x) := \lim_n \frac{1}{2^n} f(2^n x)
\]

\[
\sigma(x) := \lim_n \frac{1}{2^n} g(2^n x)
\]

\[
\tau(x) := \lim_n \frac{1}{2^n} h(2^n x)
\]

\[
\xi(x) := \lim_n \frac{1}{2^n} k(2^n x)
\]

exist for all \( x \in A \), also \( \sigma, \tau, \xi \) and \( D \) are unique linear mappings which satisfy (2.4), (2.5), (2.6) and (2.7) respectively. Putting \( x = y = 0 \) and replacing \( u \) by \( 2^n u \) in (2.13), we obtain
\[ \|D([uuu]_A) - [D(u)uu]_X\|_{(\sigma,\tau,\xi)} - [D(u)uu]_X\|_{(\sigma,\tau,\xi)} - [D(u)uu]_X\|_{(\sigma,\tau,\xi)} \| \\
= \lim_{n \to \infty} \| \frac{f(2^{3n}[uuu]_A)}{2^{3n}} - (D(u)\tau(u)\xi(u) - \sigma(u)\tau(u)D(u)) \\
- (D(u)\tau(u)\xi(u) - \sigma(u)\tau(u)D(u)) - (D(u)\tau(u)\xi(u) - \sigma(u)\tau(u)D(u)) \| \\
= \lim_{n \to \infty} \| \frac{f(2^{3n}[uuu]_A)}{2^{3n}} \\
- (\frac{f(2^{n}u)h(2^{n}u)k(2^{n}u) - g(2^{n}u)h(2^{n}u)f(2^{n}u) - f(2^{n}u)h(2^{n}u)k(2^{n}u)}{2^{3n}} \\
+ \frac{g(2^{n}u)h(2^{n}u)f(2^{n}u) - f(2^{n}u)h(2^{n}u)k(2^{n}u) + g(2^{n}u)h(2^{n}u)f(2^{n}u)}{2^{3n}}) \| \\
= \lim_{n \to \infty} \| \frac{1}{2^{3n}} \varphi(0,0,2^{n}u) \\
= 0 \]

for each \( u \in A \). Hence, the linear mapping \( D \) is a Jordan Lie ternary \((\sigma,\tau,\xi)\)-derivation. \( \square \)

**Corollary 2.4.** Suppose \( f : A \to X \) is a mapping with \( f(0) = 0 \) for which there exist mappings \( g, h, k : A \to A \) with \( g(0) = h(0) = k(0) = 0 \) and there exists \( \theta \geq 0 \) and \( p \in [0,1) \) such that

\[
\|f(\lambda x + \lambda y + [uuu]_A) - \lambda f(x) - \lambda f(y) - ([f(u)uu]_X)_{(g,h,k)} + ([f(u)uu]_X)_{(g,h,k)} \| \leq \theta(\|x\|^p + \|y\|^p + \|u\|^p),
\]

\[
\|g(\lambda x + \lambda y) - \lambda g(x) - \lambda g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \\
\|h(\lambda x + \lambda y) - \lambda h(x) - \lambda h(y)\| \leq \theta(\|x\|^p + \|y\|^p) \\
\|k(\lambda x + \lambda y) - \lambda k(x) - \lambda k(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]

for all \( \lambda \in \mathbb{T} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) and for all \( x, y \in A \). Then there exist unique linear mappings \( \sigma, \tau \) and \( \xi \) from \( A \) to \( A \) satisfying \( \|g(x) - \sigma(x)\| \leq \frac{\theta\|x\|^p}{1-\frac{1}{2^p}} \), \( \|h(x) - \tau(x)\| \leq \frac{\theta\|x\|^p}{1-\frac{1}{2^p}} \) and \( \|k(x) - \xi(x)\| \leq \frac{\theta\|x\|^p}{1-\frac{1}{2^p}} \), and there exists a unique Jordan Lie ternary \((\sigma,\tau,\xi)\)-derivation \( D : A \to X \) such that

\[
\|f(x) - D(x)\| \leq \frac{\theta\|x\|^p}{1-\frac{1}{2^p}}
\]

(2.11)

for all \( x \in A \).

**Proof.** Put \( \varphi(x,y,u) = \theta(\|x\|^p + \|y\|^p + \|u\|^p) \) in Theorem 2.3. \( \square \)

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