Generalization of the Taylor formula

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Abstract

The following expression

\[ y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \cdots + \frac{(x - x_0)^{n-1}}{(n-1)!} y^{(n-1)}(x_0) + \int_{x_0}^{x} K(x, s) F(y(s)) ds \]

is obtained as a generalization of the classical Taylor formula. The functions \( y_1(x), y_2(x), \ldots, y_n(x) \) are the solutions of the differential equation \( F(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0 \) fulfilling the initial conditions \( y_i^{(k)}(x_0) = \delta_{ik} \), where \( \delta_{ik} \) is the Kronecker symbol. \( K(x, s) \) is the solution of the equation \( F(y) = 0 \) satisfying \( K^{(i)}(x, s)_{x=s} = \delta_{i,n-1} \).

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I. The theory of the adjoint differential equation to a given \( n \)th order linear differential equation [1] allows to obtain a formula which generalizes the classical formula of the Taylor expansion:

\[ y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \cdots + \frac{(x - x_0)^{n-1}}{(n-1)!} y^{(n-1)}(x_0) + \int_{x_0}^{x} (x - s)^{n-1} (n-1)! y^{(n)}(s) ds . \] (1)

Let

\[ F(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0 \] (2)

be a homogeneous linear differential equation of the \( n \)th order whose coefficients \( a_i(x), i = 1, 2, \ldots, n, \) are continuous functions on the interval \([a, b]\).

Let also

\[ G(z) = (-1)^n |z|^{(n)} - (a_1 z)^{(n-1)} - \cdots - (-1)^n a_n z = 0 \] (3)

be the adjoint equation of eq. (2).

The following identity is known to be true

\[ zF(y) - yG(z) = \frac{d}{dx} T(y, z) , \] (4)

where

\[ T(y, z) = zy^{(n-1)} + [(a_1 z) - z']y^{(n-2)} + [(a_2 z) - (a_1 z)']y^{(n-3)} + \cdots + [(a_n z) - (a_{n-1} z)]y^{(n-2)} + \cdots + (-1)^{n-1} z^{(n-1)} y \]. (5)

Let us integrate both sides of (4) from \( x_0 \) to \( x \), where both of them are points belonging to the interval \([a, b]\). We get

\[ \int_{x_0}^{x} \{z(s)F(y(s)) - y(s)G(z(s))\} ds = \int_{x_0}^{x} \frac{d}{ds} T(y(s), z(s)) ds . \] (6)

Let

\[ z(s) = \varphi(x, s) \] (7)
be a solution of the adjoint equation $G(z) = 0$ which depends on the parameter $x$ and fulfills the initial conditions
\[
\begin{align*}
z(s)|_x &= \varphi(x, x) = 0, \quad z'(s)|_x = \varphi'_s(x, x) = 0, \ldots, \quad z^{(n-2)}(s)|_x = \varphi^{(n-2)}_s(x, x) = 0, \\
z^{(n-1)}(s)|_x &= \varphi^{(n-1)}_s(x, x) = 1.
\end{align*}
\]
Equation (8) becomes
\[
\int_{x_0}^x \varphi(x, s) F[y(s)]ds = \Upsilon[y(s), z(s)|_x].
\]
It is worth noticing that $\Upsilon(y, z)$ is a linear and homogeneous function in $z, z', \ldots, z^{(n-1)}$, the coefficient of $z^{(n-1)}$ being $(-1)^{n-1}y$. For $x = s$, $\Upsilon(y, z)$ reduces to $(-1)^{n-1}y(s)$ because of (8). On the other hand, $\Upsilon(y, z)$ is a linear and homogeneous function in $y, y', \ldots, y^{(n-1)}$ whose coefficients are displayed in (8). They depend on $x$ and $s$ because $z = \varphi(x, s)$. For $s = x_0$, the coefficients of $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ are functions of $x$ that we denote by
\[
(-1)^{n-1}\lambda_1(x), (-1)^{n-1}\lambda_2(x), \ldots, (-1)^{n-1}\lambda_n(x).
\]
Equation (11) becomes
\[
\int_{x_0}^x \varphi(x, s) F[y(s)]ds = (-1)^{n-1}y(x) + (-1)^{n-1}[\lambda_1(x)y(x_0) + \lambda_2(x)y'(x_0) + \cdots + \lambda_n(x)y^{(n-1)}(x_0)]
\]
Solving for $y(x)$, one gets
\[
y(x) = \lambda_1(x)y(x_0) + \lambda_2(x)y'(x_0) + \cdots + \lambda_n(x)y^{(n-1)}(x_0) + (-1)^{n-1}\int_{x_0}^x \varphi(x, s) F[y(s)]ds.
\]
Let us give the meaning of the functions $\lambda_i(x)$. For this, in eq. (11) we substitute $y(x)$ by the solution $y_i(x)$ of the differential equation $F(y) = 0$ which satisfies the initial conditions
\[
y_0(x_0) = 0, \ldots, y_i^{(i-2)}(x_0) = 0, \quad y_i^{(i-1)}(x_0) = 1, \quad y_i^i(x_0) = 0, \ldots, \quad y_i^{(n-1)}(x_0) = 0,
\]
where $i = 1, 2, \ldots, n$. Thus, we obtain
\[
y_i(x) = \lambda_i(x).
\]
We conclude that $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$ in eq. (11) are the solutions of the differential equation $F(y) = 0$ with the initial conditions shown in the following Table:

| $y_1$ | 1, 0, 0, 0 |
| $y_2$ | 0, 1, 0, 0 |
| \ldots | \ldots |
| $y_n$ | 0, 0, 0, 1 |

It is known that in this case $y_1(x), y_2(x), \ldots, y_n(x)$ is a fundamental set of solutions of the differential equation $F(y) = 0$. Thus, equation (11) turns into
\[
y(x) = y_1(x)y(x_0) + y_2(x)y'(x_0) + \cdots + y_n(x)y^{(n-1)}(x_0) + (-1)^{n-1}\int_{x_0}^x \varphi(x, s) F[y(s)]ds.
\]
In the particular case when
\[
F(y) = y^{(n)},
\]
one obtains
\[
G(z) = (-1)^nz^{(n)}.
\]
The solution of the differential equation $G(z) = 0$ satisfying the initial conditions is
\[
\varphi(x, s) = (-1)^{n-1}(x-s)^{n-1}/(n-1)!
\]
and the solutions of the differential equation $y^{(n)} = 0$ which satisfy the initial conditions given for each of them in the Table are

\[
\begin{align*}
y_1(x) &= 1 \\
y_2(x) &= \frac{x - x_0}{1!} \\
&\quad \ldots \ldots \\
y_n(x) &= \frac{(x - x_0)^{n-1}}{(n-1)!}.
\end{align*}
\]

Formula (12) turns in this case into Taylor’s expansion in (1). This explains why the formula (12) on which we want to draw the attention of the reader of this work is a generalization of the Taylor formula.

II. The method just described allows to express the fundamental system of solutions $y_1(x), y_2(x), \ldots, y_n(x)$ of the differential equation $F(y)$ which satisfy the initial conditions written for each of them in the Table above by means of the functions $\phi(x,s)$.

Indeed, taking into account the expression (11) of $\Upsilon(y,z)$, we find out that

\[
y_1(x) = (-1)^n \left[ a_{n-1}(s)\phi(x,s) - \frac{\partial(a_{n-2}(s)\phi(x,s))}{\partial s} + \cdots + (-1)^{n-1}\frac{\partial^{n-1}\phi(x,s)}{\partial s^{n-1}} \right]_{s=x_0}
\]

\[
y_{n-2}(x) = (-1)^n \left[ a_2(s)\phi(x,s) - \frac{\partial(a_1(s)\phi(x,s))}{\partial s} + \frac{\partial^2\phi(x,s)}{\partial s^2} \right]_{s=x_0}
\]

\[
y_{n-1}(x) = (-1)^n \left[ a_1(s)\phi(x,s) - \frac{\partial\phi(x,s)}{\partial s} \right]_{s=x_0}
\]

\[
y_n(x) = (-1)^{n-1}\phi(x,s)|_{s=x_0}
\]

Focusing our attention on the last formula of this set, we infer that if $s$ is any point in $[a, b]$, the integral $K(x,s)$ of $F(y) = 0$ which depends on the parameter $s$, and for which the initial conditions

\[
K(x,s)|_{s=x} = 0, \quad K'(x,s)|_{s=x} = 0, \quad \ldots, K^{(n-2)}(x,s)|_{s=x} = 0, \quad K^{(n-1)}(x,s)|_{s=x} = 1,
\]

hold, is $(-1)^{n-1}\phi(x,s)$. Thus, we have the identity

\[
K(x,s) = (-1)^{n-1}\phi(x,s).
\]

One consequence of this identity is that the fundamental formula (12) can also be written as follows

\[
y(x) = y_1(x)y(x_0) + y_2(x)y'(x_0) + \cdots + y_n(x)y^{(n-1)}(x_0) + \int_{x_0}^{x} K(x,s)F[y(s)]ds.
\]

III. Applications

1. Let us consider the nonhomogeneous differential equation

\[
F(y) = f(x).
\]

Its solution for which the following initial conditions

\[
y(x_0) = 0, \quad y'(x_0) = 0, \ldots, y^{(n-1)}(0) = 0
\]

hold is given by eq. (16) where in the right-hand side $F[y(s)]$ is substituted by $f(s)$. One gets in this way

\[
y(x) = \int_{x_0}^{x} K(x,s)f(s)ds.
\]
which is the well-known Cauchy formula.

2. Assume that

\[ F(y) = y'' + y \]

We will have

\[ G(z) = z'' + z \]

and the solution of the equation

\[ z'' + z = 0 \]

which fulfills the conditions

\[ z(s)|_{s=x} = 0, \quad z'(s)|_{s=x} = 1, \]

is

\[ z = \phi(x, s) = -\sin(x - s). \]

Equations (13) provide

\[ y_1(x) = \cos(x - x_0), \quad y_2(x) = \sin(x - x_0) \]

and eq. (12) becomes

\[ y(x) = y(x_0) \cos(x - x_0) + y'(x_0) \sin(x - x_0) + \int_{x_0}^x \sin(x - s) [y''(s) + y(s)] ds. \]

The following formula:

\[ y(x) = y(x_0) \cosh(x - x_0) + y'(x_0) \sinh(x - x_0) + \int_{x_0}^x \sinh(x - s) [y''(s) - y(s)] ds \]

can be similarly proved.

3. Let us take

\[ F(y) = y'''' + 5y'' + 4y, \]

which implies:

\[ G(z) = z'''' + 5z'' + 4z. \]

The solution of \( G(z) = 0 \) which fulfills the initial conditions

\[ z(s)|_{s=x} = 0, \quad z'(s)|_{s=x} = 0, \quad z''(s)|_{s=x} = 0, \quad z'''(s)|_{s=x} = 1 \]

is

\[ z = \phi(x, s) = \frac{\sin 2(x - s) - 2 \sin(x - s)}{6}. \]

Applying equations (13), one finds

\[ y_1(x) = \frac{1}{3} \left[ 4 \cos(x - x_0) - \cos 2(x - x_0) \right], \]
\[ y_2(x) = \frac{1}{6} \left[ 8 \sin(x - x_0) - \sin 2(x - x_0) \right], \]
\[ y_3(x) = \frac{1}{3} \left[ \cos(x - x_0) - \cos 2(x - x_0) \right], \]
\[ y_4(x) = \frac{1}{6} \left[ 2 \sin(x - x_0) - \sin 2(x - x_0) \right]. \]
Formula (12) becomes
\[
y(x) = \frac{1}{3} \{4 \cos(x - x_0) - \cos(2(x - x_0))\} y(x_0) + \frac{1}{6} \{8 \sin(x - x_0) - \sin(2(x - x_0))\} y'(x_0)
+ \frac{1}{3} \{\cos(x - x_0) - \cos(2(x - x_0))\} y''(x_0) + \frac{1}{6} \{2 \sin(x - x_0) - \sin(2(x - x_0))\} y'''(x_0)
- \int_{x_0}^{x} \frac{1}{6} \{\sin(2(x - s) - 2 \sin(x - s))\} \[y'''(s) + 5y''(s) + 4y(s)] ds .
\]

4. Let us consider the integro-differential equation
\[
y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = \int_{x_0}^{x} N(x, s)y(s)ds + f(x) ,
\]
where \(a_1(x), \ldots, a_n(x)\) are continuous functions in the interval \([a, b], x_0 \in [a, b]\), and \(N(x, s)\) is a continuous function in the domain \(a \leq x \leq b, a \leq s \leq b\). We want to show that the solution of (18) fulfilling the initial conditions
\[
y(x_0) = y_0, \quad y'(x_0) = y_0', \quad \ldots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)}
\]
is the solution of an integral Volterra equation of the second kind which we will obtain in the following. Let us apply eq. (16), where \(F\) is the solution of an integral Volterra equation of the second kind which we will obtain in the following. Let \(Y(x)\) be the solution of the differential equation \(F(y) = f(x)\) which satisfies the initial conditions (19).

Let \(Y(x)\) be the solution of the differential equation \(F(y) = f(x)\) which satisfies the initial conditions (19). Its expression is given by
\[
Y(x) = y_1(x)y_0 + y_2(x)y_0' + \cdots + y_n(x)y_0^{(n-1)} + \int_{x_0}^{x} K(x, s)ds \int_{x_0}^{s} N(x, t)y(t)dt + \int_{x_0}^{x} K(x, s)f(s)ds .
\]

On the other hand, we notice that
\[
N_1(x, t) = \int_{x_0}^{x} K(x, s)N(s, t)dt .
\]

Therefore, eq. (21) becomes
\[
y(x) = \int_{x_0}^{x} N_1(x, t)y(t)dt + Y(x) .
\]

Thus, we have shown that the solution of the integro-differential equation (18) which fulfills the initial conditions (19) is the solution of the integral Volterra equation of the second kind with the kernel \(N_1(x, t)\) given by (22), while \(Y(x)\) is the solution of the differential equation \(F(y) = f(x)\) with the same initial conditions (19).

References

[1] V.V. Stepanov, Lectures on Differential Equations, Moscow, 1953, pp. 205-214.