Spin glass transition in a magnetic field: a renormalization group study

I.R. Pimentel
Department of Physics and CFMC, University of Lisbon,
Av. Prof. Gama Pinto, 2, 1649 Lisboa, Portugal

T. Temesvári
HAS Research Group for Theoretical Physics, Eötvös University,
H-1117 Pázmány Péter sétány 1/A, Budapest, Hungary

C. De Dominicis
Service the Physique Théorique, CEA Saclay,
F-91191 Gif-sur-Yvette, France

(Dated: February 12, 2002)

Abstract
We study the transition of short range Ising spin glasses in a magnetic field, within a general replica symmetric field theory, which contains three masses and eight cubic couplings, that is defined in terms of the fields representing the replicon, anomalous and longitudinal modes. We discuss the symmetry of the theory in the limit of replica number \( n \to 0 \), and consider the regular case where the longitudinal and anomalous masses remain degenerate. The spin glass transitions in zero and non-zero field are analyzed in a common framework. The mean field treatment shows the usual results, that is a transition in zero field, where all the modes become critical, and a transition in nonzero field, at the de Almeida-Thouless (AT) line, with only the replicon mode critical. Renormalization group methods are used to study the critical behavior, to order \( \varepsilon = 6 - d \). In the general theory we find a stable fixed-point associated to the spin glass transition in zero field. This fixed-point becomes unstable in the presence of a small magnetic field, and we calculate crossover exponents, which we relate to zero-field critical exponents. In a finite magnetic field, we find no physical stable fixed-point to describe the AT transition, in agreement with previous results of other authors.

PACS numbers: 75.50.Lk, 75.40.Cx
I INTRODUCTION

Spin glasses are specially interesting disordered magnetic systems, with competing interactions which generate frustration. Concepts and techniques developed in the study of these complex systems have had impact in a variety of other subjects, like combinatorial optimization, neural networks, prebiotic evolution and protein folding. Here we concentrate on the simplest spin glass model, that is the Ising spin glass with a Gaussian distribution of interactions in a uniform magnetic field. Despite the enormous amount of work dedicated over the past twenty years, to the study of spin glasses, no consensus has yet been reached on the most fundamental properties of these systems, namely the nature and complexity of the glassy phase and the existence of a transition in a nonzero magnetic field.

In their seminal paper Edwards and Anderson introduced a lattice model for the spin glass, with short-range interactions, and used a replica method to perform the average over quenched disorder. Two different pictures have since then been proposed for the spin glass. One is the mean-field theory for the spin glasses, provided by the Parisi solution for the infinite-range, Sherrington-Kirkpatrick (SK) model, which predicts a glassy phase described by an infinite number of pure states organized in an ultrametric structure, and a phase transition occurring in a magnetic field. The alternative is the "droplet" model, which claims that the real, short-range spin systems behave quite differently, the glassy phase being described by only two pure states, related by a global inversion of the spins, and no phase transition occurring in a magnetic field. The first picture results from replica symmetry breaking, while in the second picture there is no replica symmetry breaking. The spin glass transition in a field, found by de Almeida and Thouless in the SK model, occurs along a line in the field-temperature plane, the AT-line, which marks the instability of the replica-symmetric solution against replica symmetry breaking. In the mean-field theory, it represents a line of second-order transitions. Via the replica method the order parameter is represented by a field $Q_{\alpha\beta}$, (where $\alpha = 1, \ldots, n$ is a replica label), which has $n(n-1)/2$ independent components (since $Q_{\alpha\alpha} \equiv 0$, $Q_{\alpha\beta} \equiv Q_{\beta\alpha}$). Linear combinations of these components define three different sets of modes: the replicon, the anomalous and the longitudinal. In zero-field all the modes become critical at the transition temperature, while in a finite-field only the replicon modes become critical at the AT-line.

The existence, or not, of a spin glass transition in a field, is a crucial issue in the characterization of the spin glass. It has been disputed both from the theoretical as well as the experimental side. Extensive numerical simulations have also been carried out to clarify the situation, but have neither been conclusive. A fundamental step towards the clarification of the controversy, and therefore the understanding of the spin glass, lies in the investigation of how the fluctuations, associated to the finite-range interac-
ations, will modify the mean-field picture, and in particular the AT-transition. Green, Moore and Bray\cite{17} made a one-loop perturbation calculation of the AT-line in a magnetic field, however considering the effect of the field only in the masses, the couplings remaining the same as in zero-field. It becomes then most important to study the problem of the spin glass in a field by a powerful analytical method, such as the renormalization group (RG), the aim being to find a fixed-point that controls the spin glass transition in a non-zero field. The spin glass transition in zero-field was already studied within the RG by Harris et al.\cite{18} Bray and Roberts\cite{19} considered the case of non-zero field and carried out a RG study, in which they retained only the replicon modes to calculate the critical behavior at the AT-line. They found no physical fixed-point in a field.

In this paper we study the spin glass transition in a field, presenting the complete set of RG equations for an Ising spin glass in a uniform magnetic field, which contains the replicon, anomalous and longitudinal modes, and explicit dependence on the number of replicas $n$. This allow us to discuss, in a common framework, the transitions in zero and non-zero field, and the crossover region around the zero-field critical point, thus investigating the role that a small magnetic field plays in the transition. We keep the replica number $n$ finite to discuss the limit $n \to 0$ in the calculation, which is a delicate and important matter when the anomalous and longitudinal modes are considered. Here we analyze the case where the limit $n \to 0$ corresponds to a theory with a symmetry that involves equality between the anomalous and longitudinal masses, as it occurs in the spin glass mean field theory. By contrast, the case where there may be singularities in the limit $n \to 0$, which may lead to the breaking up of the equality between the anomalous and longitudinal masses, will be considered in a separate publication.\cite{20} The main purpose of our work is then to search for the possible existence of a fixed-point associated to the spin glass transition in a field, within the complete set of RG equations. To avoid the complexity of the glassy phase, we approach the transition from the high-temperature, replica-symmetric phase, like Bray and Roberts.\cite{19}

In a previous publication,\cite{21} we presented the derivation of the replica field theory, relevant to the study of the spin glass transition of short-range models in a field. We argued that for studying this transition around the upper critical dimension $d = 6$, it is necessary to use the generic theory, with all the three bare masses and the eight cubic couplings involving the replica fields, which correspond to all the possible invariants of the replica symmetric Lagrangian. It is however more convenient to work directly with a field theory defined in terms of the eigenfields, which corresponds to a block-diagonalization of the mass operator into the replicon, anomalous and longitudinal modes, and we introduced then the appropriate representation of the cubic interaction. The mass operator and the cubic couplings for the eigenfields, are in fact simply obtained by performing a formal decom-


position of the replica fields into the replicon, anomalous and longitudinal fields, and use the generic properties of the fields. However, in order to do a perturbation calculation one still has to deal with the problem of having non-diagonal free propagators. Such a difficulty was overcome in Ref. 21, by introducing a particular non-orthogonal basis and its biorthogonal counter-part, in which a set of vertices were computed, the perturbation computation being alleviated through selection rules. In this work we present a different approach, in which we introduce projection operators into the non-diagonal propagators, the projectors being simply derived from the generic properties of the fields. This method presents a generalization, for the complete set of modes, of the method used by Bray and Roberts.\(^{19}\) We have then a field theory for the spin glass that works directly with the longitudinal, anomalous and replicon fields, and allows to perform a standard perturbation expansion. An alternative formulation of the spin glass field theory can be obtained using Replica Fourier Transform methods, which turn out particularly efficient in the study of the replica symmetry broken phase.\(^{22}\)

The outline of the paper is as follows. In Sec. II we present the cubic field theory written in terms of the replicon, anomalous and longitudinal fields. In Sec. III we derive the mean-field results. In Sec. IV we present the renormalization group equations, calculated to lowest order in \(\varepsilon = 6 - d\) (where \(d\) is the spatial dimension), and discuss the spin glass transitions in zero and non-zero field. Sec.V contains the conclusions of our work.

## II CUBIC FIELD THEORY

We consider a short-range Ising spin glass in a uniform magnetic field \(H\), described by the Edwards-Anderson model,

\[
\mathcal{H} = -\sum_{ij} J_{ij} S_i S_j - H \sum_i S_i
\]

for \(N\) spins, \(S_i = \pm 1\), located on a regular \(d\)-dimensional lattice, where the bonds \(J_{ij}\), which couple nearest-neighbor spins only, are independent random variables with a Gaussian distribution, characterized by zero mean and variance \(\Delta^2\). The summations are over pairs \((ij)\) of distinct sites on the lattice and over the lattice sites \(i\). The replica method allows to calculate the average of the free energy over the quenched disorder, in terms of the average of \(n\) replicas of the partition function \(Z^n\), with \(n\) positive integer, which provides the spin glass behavior in the analytically continued limit \(n \rightarrow 0\). This procedure transforms the originally disordered system into a uniform one, described by an effective Hamiltonian for the replica spins \(S_i^\alpha, \alpha = 1, \ldots, n\). A field theoretical continuum representation of the spin glass lattice model can be built using a standard Hubbard-Stratonovich transformation, which leads to the averaged replicated partition function, expressed as an
integral over replica fields \(Q_{i}^{\alpha\beta}\). The fields \(Q_{i}^{\alpha\beta}\) are defined on a \((n(n-1)/2)\)-dimensional replica space, of the pairs \((\alpha, \beta)\) of distinct replicas.

In order to construct a perturbation expansion around the mean-field solution, which corresponds to the infinite range or infinite dimensional (i.e. spin coordination number \(z \to \infty\)) model, one separates the field \(Q_{i}^{\alpha\beta}\) into

\[
Q_{i}^{\alpha\beta} = Q^{\alpha\beta} + \phi_{i}^{\alpha\beta}
\]

where \(Q^{\alpha\beta}\) represents the uniform, mean-field value of the order parameter, and \(\phi_{i}^{\alpha\beta}\) are the fluctuations around it.

In addition, since we approach the transition from the high-temperature phase, one can assume a replica symmetric mean-field solution

\[
Q^{\alpha\beta} = Q
\]

The partition function then takes the form

\[
Z^{n} \sim \int D\phi \exp \left(-L^{(1)} - L^{(2)} - L^{(3)} - \ldots \right)
\]

where, after Fourier transform into momenta space, one has, for the contributions up to cubic order,

\[
L^{(1)} = \sqrt{N} \sum_{(\alpha\beta)} \left( Q \Theta^{-1} - \ll S^{\alpha} S^{\beta} \gg \right) \phi_{\mathbf{p}=0}^{\alpha\beta}
\]

\[
L^{(2)} = \frac{1}{2} \sum_{(\alpha\beta)(\gamma\delta)} \sum_{\mathbf{p}} \phi_{\mathbf{p}}^{\alpha\beta} M_{\alpha\beta,\gamma\delta}(\mathbf{p}) \phi_{-\mathbf{p}}^{\gamma\delta}
\]

\[
L^{(3)} = -\frac{1}{3!} \frac{1}{\sqrt{N}} \sum_{(\alpha\beta)(\gamma\delta)(\mu\nu)} \sum_{\mathbf{p}_{1}, \mathbf{p}_{2}, \mathbf{p}_{3}} \phi_{\mathbf{p}_{1}}^{\alpha\beta} \phi_{\mathbf{p}_{2}}^{\gamma\delta} \phi_{\mathbf{p}_{3}}^{\mu\nu} W_{\alpha\beta,\gamma\delta,\mu\nu}
\]

the prime in the sum in Eq. (7) indicating constraint to momentum conservation, \(\mathbf{p}_{1} + \mathbf{p}_{2} + \mathbf{p}_{3} = 0\). The quantity \(\Theta^{-1} = (k_{B}T/\Delta)^{2}/z\), where the coordination number is \(z = 2d\), essentially represents the temperature squared. The average \(\ll \ldots \gg\) is defined with the Boltzmann weight, \(\exp \left(\sum_{(\alpha\beta)} Q S^{\alpha} S^{\beta} + h \sum_{\alpha} S^{\alpha}\right)\), where \(h = H/k_{B}T\). The masses \(M_{\alpha\beta,\gamma\delta}\) and the couplings \(W_{\alpha\beta,\gamma\delta,\mu\nu}\) are expressed in terms of spin correlations. The sum in the first Brillouin zone are confined to the range \(0 < |\mathbf{p}| < \Lambda\), with cutoff \(\Lambda \simeq 1\), and \(M_{\alpha\beta,\gamma\delta}(\mathbf{p})\) is expanded for \(\mathbf{p} \ll 1\), keeping as usual only the terms up to second order. Then, an appropriate rescaling of the fields \((\phi(z\Theta)^{-1/2} \to \phi)\), with a corresponding rescaling of the masses and the couplings, allows to write the mass operator in a standard form, i.e. with the coefficient of the momentum equal to unity,

\[
M_{\alpha\beta,\gamma\delta}(\mathbf{p}) = \mathbf{p}^{2} \delta_{\alpha\beta,\gamma\delta} + z \left[ \delta_{\alpha\beta,\gamma\delta} - \right.
\]
the Kronecker-delta being defined in the space of replica pairs. For the interaction operator one has,

\[ W_{\alpha\beta,\gamma\delta,\mu\nu} = (z\Theta)^{3/2} \left[ \langle S^\alpha S^\beta S^\gamma S^\delta \rangle - \langle S^\alpha S^\beta \rangle \langle S^\gamma S^\delta \rangle \right. \]
\[ \left. - \langle S^\gamma S^\delta \rangle \langle S^\alpha S^\beta \rangle \langle S^\mu S^\nu \rangle + 2 \langle S^\alpha S^\beta \rangle \langle S^\gamma S^\delta \rangle \langle S^\mu S^\nu \rangle \right]. \]  

(9)

The expansion in the fluctuations corresponds to an expansion in \(1/z\).

Replica symmetry allows three distinct components for the mass, Eq. (8), and eight distinct components for the cubic interaction, Eq. (9). Hence,

\[ \mathcal{L}^{(2)} = \frac{1}{2} \sum_p \left\{ (p^2 + M_1) \sum_{(\alpha\beta)} \phi_{p}^{\alpha\beta} \phi_{-p}^{\alpha\beta} + M_2 \sum_{(\alpha\beta\gamma)} (\phi_{p}^{\alpha\beta\gamma} \phi_{-p}^{\alpha\beta\gamma} + \phi_{p}^{\alpha\beta} \phi_{-p}^{\alpha\beta}) \right. \]
\[ \left. + M_3 \sum_{(\alpha\beta\gamma\delta)} \phi_{p}^{\alpha\beta\gamma\delta} \phi_{-p}^{\alpha\beta\gamma\delta} \right\} \]  

(10)

and

\[ \mathcal{L}^{(3)} = -\frac{1}{\sqrt{N}} \sum_{p_1, p_2, p_3} \left\{ \frac{1}{6} W_1 \sum_{(\alpha,\beta,\gamma)} \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\beta\gamma} \phi_{p_3}^{\gamma\alpha} + \frac{1}{12} W_2 \sum_{(\alpha\beta)} \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\alpha\beta} \phi_{p_3}^{\alpha\beta} \right. \]
\[ \left. + \frac{1}{2} W_3 \sum_{(\alpha,\beta,\gamma,\delta)} \phi_{p_1}^{\alpha\beta\gamma\delta} \phi_{p_2}^{\alpha\beta\gamma\delta} \phi_{p_3}^{\alpha\beta\gamma\delta} \right. \]
\[ \left. + \frac{1}{8} W_4 \sum_{(\alpha,\beta,\gamma,\delta)} \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\alpha\beta} \phi_{p_3}^{\alpha\beta} \right. \]
\[ \left. + \frac{1}{6} W_5 \sum_{(\alpha,\beta,\gamma,\delta)} \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\alpha\beta} \phi_{p_3}^{\alpha\beta} \right. \]
\[ \left. + \frac{1}{12} W_6 \sum_{(\alpha,\beta,\gamma,\delta)} \phi_{p_1}^{\alpha\beta} \phi_{p_2}^{\alpha\beta} \phi_{p_3}^{\alpha\beta} \right. \]
\[ \left. + \frac{1}{6} W_7 \sum_{(\alpha,\beta,\gamma,\delta,\mu)} \phi_{p_1}^{\alpha\beta \gamma \delta} \phi_{p_2}^{\alpha\beta \gamma \delta} \phi_{p_3}^{\alpha\beta \gamma \delta} \right. \]
\[ \left. + \frac{1}{12} W_8 \sum_{(\alpha,\beta,\gamma,\delta,\mu)} \phi_{p_1}^{\alpha\beta \gamma \delta} \phi_{p_2}^{\alpha\beta \gamma \delta} \phi_{p_3}^{\alpha\beta \gamma \delta} \right\} , \]  

(11)

the sums in Eqs. (10)-(11) being restricted to distinct replicas. One can rewrite \(\mathcal{L}^{(2)}\) and \(\mathcal{L}^{(3)}\) in terms of sums over unrestricted replicas, for which we obtain forms similar to those, with new masses and new couplings, defined as linear combinations of \(M_i\) and \(W_i\). From a symmetry point of view, Eqs. (10)-(11) contain all the possible quadratic and cubic invariants of the symmetry group of the system, which corresponds to invariance of the Lagrangian under any permutation of the \(n\) replicas.

We now wish to write the quadratic part \(\mathcal{L}^{(2)}\) in terms of the mass eigenvalues, which corresponds to a block diagonalization into the longitudinal
(L), anomalous (A) and replicon (R) subspaces. Any replica field \( \phi^{\alpha\beta} \) can be decomposed in its projections onto the L, A, R subspaces,

\[
\phi^{\alpha\beta} = \phi^{\alpha\beta}_L + \phi^{\alpha\beta}_A + \phi^{\alpha\beta}_R. \tag{12}
\]

The field projections \( \phi^{i}_{\alpha\beta} \) can be obtained via the projection operators \( P^{i}_{\alpha\beta,\gamma\delta} \), onto the different subspaces,

\[
\phi^{i}_{\alpha\beta} = \sum_{(\gamma\delta)} P^{i}_{\alpha\beta,\gamma\delta} \phi^{\gamma\delta}, \quad i = L, A, R. \tag{13}
\]

The matrix elements of the projection operators are simply derived from the general properties of the replicon, anomalous and longitudinal fields, as shown in the Appendix. The three subspaces are characterized by different symmetries, as follows.

The longitudinal subspace is 1-dimensional. The longitudinal eigenvector \( e^L \) is symmetric under interchange of all replica indices. This implies a replica independent longitudinal field

\[
\phi^{L}_{\alpha\beta} = \phi^{L} \tag{14}
\]

given by

\[
\phi^{L}(p) = a^{L}(p)e^L \tag{15}
\]

and a replica independent longitudinal projector

\[
P^{L}_{\alpha\beta,\gamma\delta} = P^{L} \tag{16}
\]

defined by

\[
P^{L} = e^Le^L, \tag{17}
\]

with \( e^L \) in Eqs. (15) and (17), representing a component of the replica independent eigenvector, \( e^L_{\alpha\beta} = e^L \). One finds for the longitudinal projector the explicit form, Eq. (A3),

\[
P^{L} = \frac{2}{n(n-1)}. \tag{18}
\]

The longitudinal subspace is hence described by a single scalar field \( \phi_L \).

The anomalous subspace is \((n - 1)\)–dimensional. The anomalous eigenvectors are symmetric under interchange of all but one of the replica indices. This implies that a generic anomalous field can be represented by a one-replica field \( \phi^{A}_{\alpha} \), i.e. it can be expressed as,

\[
\phi^{A}_{\alpha\beta} = \frac{1}{2}(\phi^{A}_{\alpha} + \phi^{A}_{\beta}), \tag{19}
\]
with the condition

$$\sum_{\alpha} \phi_{\alpha}^{A} = 0.$$  \hspace{1cm} (20)

One can write the one-replica field $\phi_{\alpha}^{A}$ in terms of the set of anomalous eigenvectors $e_{A,\mu}$, as

$$\phi_{\alpha}^{A}(p) = \sum_{\mu=1}^{n-1} a_{A,\mu}(p) e_{\alpha}^{A,\mu},$$  \hspace{1cm} (21)

with the normalization $\sum_{\alpha} e_{\alpha}^{A,\mu} e_{\alpha}^{A,\mu'} = \frac{4}{(n-2)} \delta_{\mu\mu'}$. One then defines an anomalous projector operator

$$P_{A}^{\alpha,\beta} = \sum_{\mu=1}^{n-1} e_{\alpha}^{A,\mu} e_{\beta}^{A,\mu},$$  \hspace{1cm} (22)

which has the property

$$\sum_{\alpha} P_{A}^{\alpha,\beta} = 0.$$  \hspace{1cm} (23)

The projectors in Eqs. (13) and (22) are simply related by

$$P_{A}^{\alpha,\beta,\gamma,\delta} = \frac{1}{4} \left[ P_{A}^{\alpha,\gamma} + P_{A}^{\alpha,\delta} + P_{A}^{\beta,\gamma} + P_{A}^{\beta,\delta} \right].$$  \hspace{1cm} (24)

The explicit form of the anomalous projector $P_{A}^{\alpha,\beta}$ is given by, Eq. (A8),

$$P_{A}^{\alpha,\beta} = \frac{4}{(n-2)} \left( \delta_{\alpha\beta} - \frac{1}{n} \right).$$  \hspace{1cm} (25)

The anomalous subspace is then described by the set of fields $\phi_{\alpha}^{A}$, $\alpha = 1, \ldots, n$, with the constraint Eq. (20), which leaves $(n-1)$ independent fields.

The replicon subspace is $n(n-3)/2$-dimensional. The replicon eigenvectors are symmetric under interchange of all but two replica indices. This implies that the replicon fields depend on two replica indices and verify the conditions$^{23}$

$$\sum_{\alpha(\neq \beta)} \phi_{\alpha\beta}^{R} = 0, \quad \alpha = 1, \ldots, n.$$  \hspace{1cm} (26)

A replicon field can be written in terms of the set of replicon orthonormal eigenvectors $e_{R,\nu}$ as

$$\phi_{\alpha\beta}^{R}(p) = \sum_{\nu=1}^{n(n-3)} a_{R,\nu}(p) e_{\alpha\beta}^{R,\nu}.$$  \hspace{1cm} (27)

The replicon projector is defined by
\[ P^R_{\alpha\beta,\gamma\delta} = \sum_{\nu=1}^{\frac{n(n-3)}{2}} e^R_{\alpha\beta} e^R_{\gamma\delta} \]  \hspace{1cm} (28) 

and has the property

\[ \sum_{\alpha(\neq \beta)} P^R_{\alpha\beta,\gamma\delta} = 0. \]  \hspace{1cm} (29) 

The explicit form of the replicon projector is given by, Eq. (A14),

\[ P^R_{\alpha\beta,\gamma\delta} = (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - (\delta_{\alpha\gamma} + \delta_{\alpha\delta} + \delta_{\beta\gamma} + \delta_{\beta\delta}) \frac{1}{n-2} \]
\[ + \frac{2}{(n-1)(n-2)} \]  \hspace{1cm} (30) 

naturally, with \( \alpha \neq \beta \) and \( \gamma \neq \delta \). Eq. (30) is equivalent to the replicon projector as introduced by Bray and Roberts. The replicon subspace is then described by the set of \( \phi^R_{\alpha\beta} \) fields, with the constraints in Eq. (26), which leaves \( n(n-1)/2 - n = n(n-3)/2 \) independent fields. The characteristic property of projection operators \( P^2 = P \) is readily observed for the longitudinal, anomalous and replicon projectors in Eqs. (18), (25) and (30).

Introducing the field decomposition in Eq. (12), with the definitions in Eqs. (14) and (19), into Eq. (10), and using repeatedly the conditions in Eqs. (20) and (26), one obtains the quadratic part in the diagonalized form

\[ \mathcal{L}^{(2)} = \frac{1}{2} \sum_p \left\{ \left( p^2 + m_L \right) \frac{n(n-1)}{2} \phi^L(p)\phi^L(-p) \right. \]
\[ + \left. \left( p^2 + m_A \right) \frac{(n-2)}{4} \sum_\alpha \phi^A_\alpha(p)\phi^A_\alpha(-p) \right. \]
\[ + \left. \left( p^2 + m_R \right) \sum_{(\alpha\beta)} \phi^R_{\alpha\beta}(p)\phi^R_{\alpha\beta}(-p) \right\} \]  \hspace{1cm} (31) 

where \( m_L, m_A, m_R \) are the longitudinal, anomalous and replicon mass eigenvalues, which are given by

\[ m_L = M_1 + 2(n-2)M_2 + \frac{1}{2}(n-2)(n-3)M_3. \]
\[ m_A = M_1 + (n-4)M_2 - (n-3)M_3 \]  \hspace{1cm} (32) 
\[ m_R = M_1 - 2M_2 + M_3. \]

We notice that the anomalous and longitudinal masses are equal in the limit \( n \to 0 \). For later discussion we define the variable
\[ \bar{m}_{AL} = \frac{m_A - m_L}{n}, \]  

which is given by

\[ \bar{m}_{AL} = -M_2 - \frac{1}{2}(n - 3)M_3. \]

Now, because of Eqs. (20) and (26), the fields \( \phi^A_\alpha \) and \( \phi^{R\beta}_\alpha \) are still not independent. One can however construct an expansion in terms of the orthonormal eigenvectors \( e^L_\alpha, e^{A,\mu}_\alpha, e^{R,\nu}_\beta \). Introducing Eqs. (15), (21) and (27) into Eq. (31), one gets

\[
\mathcal{L}^{(2)} = \frac{1}{2} \sum_{\mathbf{p}} \left\{ (\mathbf{p}^2 + m_L) a^L(\mathbf{p}) a^L(-\mathbf{p}) + (\mathbf{p}^2 + m_A) \sum_{\mu=1}^{n-1} a^{A,\mu}(\mathbf{p}) a^{A,\mu}(-\mathbf{p}) + (\mathbf{p}^2 + m_R) \sum_{\nu=1}^{n(n-3)} a^{R,\nu}(\mathbf{p}) a^{R,\nu}(-\mathbf{p}) \right\}
\]

and then

\[
< a^L(\mathbf{p}) a^L(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_L \right)^{-1}
\]
\[
< a^{A,\mu}(\mathbf{p}) a^{A,\mu'}(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_A \right)^{-1} \delta_{\mu,\mu'}
\]
\[
< a^{R,\nu}(\mathbf{p}) a^{A,\nu'}(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_R \right)^{-1} \delta_{\nu,\nu'}.
\]

Using the expressions for the fields in Eqs. (15), (21), (27), together with Eq. (36), and the definitions in Eqs. (17), (22), (28), one finds the bare propagators

\[
< \phi^L(\mathbf{p}) \phi^L(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_L \right)^{-1} P^L
\]
\[
< \phi^A_\alpha(\mathbf{p}) \phi^A_\alpha(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_A \right)^{-1} P^{A,\beta}_{\alpha,\beta}
\]
\[
< \phi^{R\beta}_\alpha(\mathbf{p}) \phi^{R\gamma}_\delta(-\mathbf{p}) > = \left( \mathbf{p}^2 + m_R \right)^{-1} P^{R,\beta}_{\alpha,\beta,\gamma,\delta}.
\]

The cubic interaction can be written in terms of the longitudinal, anomalous and replicon fields, by following a procedure similar to the one that lead to Eq. (31), that is, introducing the decomposition in Eq. (12), with Eqs. (14) and (19), into Eq. (11) and repeatedly applying the conditions in Eqs. (20) and (26), one obtains
\[
\mathcal{L}^{(3)} = -\frac{1}{\sqrt{N}} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3} \left\{ \frac{1}{6} g_1 \sum_{\alpha,\beta,\gamma} \phi^R_{\alpha\beta}(\mathbf{p}_1) \phi^R_{\beta\gamma}(\mathbf{p}_2) \phi^R_{\gamma\alpha}(\mathbf{p}_3) \\
+ \frac{1}{12} g_2 \sum_{\alpha,\beta} \phi^R_{\alpha\beta}(\mathbf{p}_1) \phi^R_{\alpha\beta}(\mathbf{p}_2) \phi^R_{\alpha\beta}(\mathbf{p}_3) \\
+ \frac{1}{4} g_3 \sum_{\alpha,\beta} \phi^R_{\alpha\beta}(\mathbf{p}_1) \phi^R_{\alpha\beta}(\mathbf{p}_2) \left( \phi^A_{\alpha}(\mathbf{p}_3) + \phi^A_{\beta}(\mathbf{p}_3) \right) \\
+ \frac{1}{2} g_4 \sum_{\alpha,\beta} \phi^R_{\alpha\beta}(\mathbf{p}_1) \phi^R_{\alpha\beta}(\mathbf{p}_2) \phi^L(\mathbf{p}_3) + \frac{1}{2} g_5 \sum_{\alpha,\beta} \phi^R_{\alpha\beta}(\mathbf{p}_1) \phi^A_{\alpha}(\mathbf{p}_2) \phi^A_{\beta}(\mathbf{p}_3) \\
+ \frac{1}{6} g_6 \sum_{\alpha} \phi^A_{\alpha}(\mathbf{p}_1) \phi^A_{\alpha}(\mathbf{p}_2) \phi^A_{\alpha}(\mathbf{p}_3) + \frac{1}{2} g_7 \sum_{\alpha} \phi^A_{\alpha}(\mathbf{p}_1) \phi^A_{\alpha}(\mathbf{p}_2) \phi^L(\mathbf{p}_3) \\
+ \frac{1}{6} g_8 \phi^L(\mathbf{p}_1) \phi^L(\mathbf{p}_2) \phi^L(\mathbf{p}_3) \right\} 
\]

with the couplings,

\[
\begin{align*}
    g_1 &= W_1 - 3W_5 + 3W_7 - W_8 \\
    g_2 &= W_2 - 6W_3 + 3W_4 + 6W_5 + 4W_6 - 12W_7 + 4W_8 \\
    g_3 &= -W_1 + \frac{1}{2} W_2 + \frac{1}{2} \left( n - 8 \right) W_3 - \frac{1}{2} \left( n - 5 \right) W_4 - \left( n - 8 \right) W_5 \\
    &\quad - \frac{1}{2} \left( n - 6 \right) W_6 + \frac{1}{2} \left( 5n - 28 \right) W_7 - \left( n - 5 \right) W_8 \\
    g_4 &= -W_1 + \frac{1}{2} W_2 + \left( n - 4 \right) W_3 + \frac{1}{4} \left( n^2 - 5n + 10 \right) W_4 - \left( n - 4 \right) W_5 \\
    &\quad - \left( n - 3 \right) W_6 - \frac{1}{2} \left( n - 4 \right) \left( n - 7 \right) W_7 + \frac{1}{4} \left( n - 4 \right) \left( n - 5 \right) W_8 \\
    g_5 &= \frac{1}{4} \left( n - 4 \right) W_1 + \frac{1}{4} W_2 + \frac{1}{2} \left( n - 5 \right) W_3 - \frac{1}{4} \left( 2n - 7 \right) W_4 \\
    &\quad + \frac{1}{4} \left( n - 5 \right) \left( n - 6 \right) W_5 - \frac{1}{2} \left( n - 4 \right) \left( n - 6 \right) W_6 - \frac{1}{4} \left( n - 4 \right) \left( 2n - 13 \right) W_7 \\
    &\quad + \frac{1}{4} \left( n - 4 \right) \left( n - 5 \right) W_8 \\
    g_6 &= -\frac{1}{4} \left( 3n - 8 \right) W_1 + \frac{1}{8} \left( n - 4 \right) W_2 + \frac{3}{8} \left( n - 4 \right)^2 W_3 \\
    &\quad - \frac{3}{8} \left( n - 3 \right) \left( n - 4 \right) W_4 - \frac{3}{2} \left( n - 3 \right) \left( n - 4 \right) W_5 \\
    &\quad + \frac{1}{8} \left( n - 3 \right) \left( n^2 - 6n + 16 \right) W_6 - \frac{3}{8} \left( n - 3 \right) \left( n - 4 \right) \left( n - 8 \right) W_7 \\
    &\quad + \frac{1}{4} \left( n - 3 \right) \left( n - 4 \right) \left( n - 5 \right) W_8 \\
    g_7 &= \left( n - 2 \right) \left[ \frac{1}{4} \left( n - 4 \right) W_1 + \frac{1}{4} \left( n - 3 \right) W_2 + \frac{1}{8} \left( n - 3 \right) \left( n - 6 \right) W_4 \\
    &\quad - \frac{3}{2} \left( n - 3 \right) \left( n - 4 \right) W_5 \\
    &\quad + \frac{1}{8} \left( n - 3 \right) \left( n^2 - 6n + 16 \right) W_6 - \frac{3}{8} \left( n - 3 \right) \left( n - 4 \right) \left( n - 8 \right) W_7 \\
    &\quad + \frac{1}{4} \left( n - 3 \right) \left( n - 4 \right) \left( n - 5 \right) W_8 \right] 
\end{align*}
\]
\[ + \frac{1}{2} (n - 3) (n - 6) W_5 + \frac{1}{4} (n - 3) (n - 4) W_6 \\
+ \frac{1}{8} (n - 3) (n - 4) (n - 12) W_7 - \frac{1}{8} (n - 3) (n - 4) (n - 5) W_8 \]

\[ g_8 = n(n - 1) \left[ (n - 2) W_1 + \frac{1}{2} W_2 + 3 (n - 2) W_3 \\
+ \frac{3}{4} (n - 2) (n - 3) W_4 + 3 (n - 2) (n - 3) W_5 \\
+ (n - 2) (n - 3) W_6 + \frac{3}{2} (n - 2) (n - 3) (n - 4) W_7 \\
+ \frac{1}{8} (n - 2) (n - 3) (n - 4) (n - 5) W_8 \right] . \]

Again, we notice that the couplings \( g_3 \) and \( g_4 \), \( g_6 \) and \( g_7 \), \( g_7 \) and \( g_8 \)/n, are equal in the limit \( n \to 0 \). For later discussion we define the variables

\[ \bar{g}_4 = \frac{4}{n} (g_4 - g_3) \]
\[ \bar{g}_7 = \frac{2}{n} (g_7 - g_6) \]
\[ \bar{g}_8 = \frac{1}{n^3} (g_8 - 3ng_7 + 2ng_6) \]

for which we have

\[ \bar{g}_4 = 2W_3 + (n - 3)W_4 - 4W_5 - 2W_6 - 2(n - 6)W_7 + (n - 5)W_8 \]
\[ \bar{g}_7 = \frac{1}{2} (n - 3)W_1 + \frac{1}{4} W_2 + \frac{1}{4} (5n - 16)W_3 + \frac{1}{4} (n - 3)(n - 5)W_4 \\
+ (n - 3)(n - 5)W_5 + \frac{1}{4} (n - 3)(n - 6)W_6 \\
+ \frac{1}{4} (n - 3)(n - 4)(n - 11)W_7 - \frac{1}{4} (n - 3)(n - 4)(n - 5)W_8 \]
\[ \bar{g}_8 = \frac{1}{4} W_1 + \frac{3}{4} W_3 + \frac{3}{8} (n - 3)W_4 + \frac{3}{2} (n - 3)W_5 + \frac{1}{2} (n - 3)W_6 \\
+ \frac{9}{8} (n - 3)(n - 4)W_7 + \frac{1}{8} (n - 3)(n - 4)(n - 5)W_8 . \]

Hence, we have derived a Lagangean for the spin glass that is directly defined in terms of longitudinal, anomalous and replicon fields, and allows to perform a standard perturbation expansion. In summary, this field theory is defined by the quadratic mass term \( \mathcal{L}^{(2)} \) in Eq. (31), the cubic interaction \( \mathcal{L}^{(3)} \) in Eq. (38), and the bare propagators in Eq. (37), which involve the longitudinal, anomalous and replicon projectors given in Eqs. (18), (25) and (30), that verify the constraints in Eqs. (23) and (29). This in fact represents a general field theory that contains the spin glass symmetry and also other symmetries as will be discussed.
III EQUATION OF STATE, BARE MASSES AND BARE COUPLINGS

The replica symmetric mean field value of the order parameter $Q$ is determined by the stationary condition $\mathcal{L}^{(1)} = 0$, which from Eq. (5), provides the implicit form

$$Q\Theta^{-1} = \langle S^\alpha S^\beta \rangle$$

$$= \frac{Tr_{\{S^\alpha\}} \left[ S^\alpha S^\beta \exp \left( \sum_{(\alpha\beta)} Q S^\alpha S^\beta + h \sum_{\alpha} S^\alpha \right) \right]}{Tr_{\{S^\alpha\}} \exp \left( \sum_{(\alpha\beta)} Q S^\alpha S^\beta + h \sum_{\alpha} S^\alpha \right)}$$

(42)

for the equation of state, $Q = Q(\Theta, h, n)$.

From Eq. (8) we have the bare masses, (in units of $z$),

$$M_1 = -\Theta \left[ 1 - \Theta^{-1} - \langle S^\alpha S^\beta \rangle^2 \right]$$

$$M_2 = -\Theta \left[ \langle S^\alpha S^\beta \rangle - \langle S^\alpha S^\beta \rangle^2 \right]$$

$$M_3 = -\Theta \left[ \langle S^\alpha S^\beta S^\gamma S^\delta \rangle - \langle S^\alpha S^\beta \rangle^2 \right]$$

(43)

and from Eq. (9) we have the bare cubic couplings, (in units of $z^{3/2}$),

$$W_1 = \Theta^{3/2} \left[ 1 - 3 \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 \right]$$

$$W_2 = \Theta^{3/2} \left[ -2 \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 \right]$$

$$W_3 = \Theta^{3/2} \left[ -2 \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 \right]$$

$$W_4 = \Theta^{3/2} \left[ 2 \langle S^\alpha S^\beta \rangle^3 - 2 \langle S^\alpha S^\beta \rangle \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right]$$

$$W_5 = \Theta^{3/2} \left[ \langle S^\alpha S^\beta \rangle - 2 \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 \right.$$

$$- \langle S^\alpha S^\beta \rangle \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right]$$

$$W_6 = \Theta^{3/2} \left[ -3 \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 + \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right]$$

$$W_7 = \Theta^{3/2} \left[ - \langle S^\alpha S^\beta \rangle^2 + 2 \langle S^\alpha S^\beta \rangle^3 + \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right.$$

$$- 2 \langle S^\alpha S^\beta \rangle \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right]$$

$$W_8 = \Theta^{3/2} \left[ 2 \langle S^\alpha S^\beta \rangle^3 - 3 \langle S^\alpha S^\beta \rangle \langle S^\alpha S^\beta S^\gamma S^\delta \rangle \right.$$

$$+ \langle S^\alpha S^\beta S^\gamma S^\delta S^\mu S^\nu \rangle \right].$$

The masses and the couplings, respectively in Eqs. (43) and (44), depend on the field $h$, and also the temperature $\Theta$, through $Q$. 

13
One finds that the spin correlations are given by

\[ \langle S^{\alpha} S^{\beta} \rangle = \tanh^2(\sqrt{Qy} + h) \]
\[ \langle S^{\alpha} S^{\beta} S^{\gamma} S^{\delta} \rangle = \tanh^4(\sqrt{Qy} + h) \]
\[ \langle S^{\alpha} S^{\beta} S^{\gamma} S^{\delta} S^{\mu} S^{\nu} \rangle = \tanh^6(\sqrt{Qy} + h) \] (45)

with the notation

\[ \tanh^k(...) = \frac{\int dy e^{-\frac{1}{2}y^2} \tanh^k(...) \cosh^n(\ldots)}{\int dy e^{-\frac{1}{2}y^2} \cosh^n(\ldots)} \]

For small magnetic fields \( h \), hence small \( Q \), one gets, keeping terms up to order \( Q^3, h^2 \) and \( n \),

\[ \langle S^{\alpha} S^{\beta} \rangle = Q + (n - 2)Q^2 + \left( \frac{17}{3} - 5n \right) Q^3 + h^2 + \ldots \]
\[ \langle S^{\alpha} S^{\beta} S^{\gamma} S^{\delta} \rangle = 3Q^2 - 2(10 - 3n)Q^3 + \ldots \] (46)
\[ \langle S^{\alpha} S^{\beta} S^{\gamma} S^{\delta} S^{\mu} S^{\nu} \rangle = 15Q^3 + \ldots \]

Introducing the first of Eq. (46) into the stationary condition Eq. (42), gives

\[ 0 = -tQ + (n - 2)Q^2 + \left( \frac{17}{3} - 5n \right) Q^3 + h^2 + \ldots \] (47)

where we defined the temperature variable \( t = (\Theta^{-1} - 1) \).

Eqs. (32), (34), (43) and (46) give for the replicon, anomalous and longitudinal bare masses,

\[ m_R = (1 + t)^{-1} \left[ t + 2(2 - n)(1 + t)Q - \frac{1}{2} (3(6 - 5n)) Q^2 \right. \]
\[ + n(1 - n)(1 + t)^2 \left. \right] Q^2 + 4 (15 - 17n) Q^3 + \ldots \] (48)

\[ m_A = (1 + t)^{-1} \left[ t + 2(1 + t)Q - 3Q^2 + 2(10 - 3n)Q^3 + \ldots \right] \]
\[ + 2 (30 - 19n) Q^3 + \ldots \]
\[ m_L = (1 + t)^{-1} \left[ t + 2(1 + t)Q - 3Q^2 + 2(10 - 3n)Q^3 + \ldots \right] \]

and

\[ \bar{m}_{AL} = (1 + t)^{-1} \left[ (1 + t)Q + \frac{1}{2} \left( 3(n - 3) + (1 - n)(1 + t)^2 \right) Q^2 \right. \]
\[ + (30 - 19n) Q^3 + \ldots \] (49)
Eqs. (39), (41), (44) and (46) give for the bare couplings, up to order $Q$,

\begin{align}
g_1 &= (1 + t)^{-3/2} [1 - 3Q + \ldots] \\
g_2 &= (1 + t)^{-3/2} [4Q + \ldots] \\
g_3 &= (1 + t)^{-3/2} [-1 + (7 - n)Q + \ldots] \\
g_4 &= (1 + t)^{-3/2} [-1 + (7 - 2n)Q + \ldots] \\
g_5 &= (1 + t)^{-3/2} \left[ -\left(1 - \frac{1}{4}n\right) + \left(7 - \frac{11}{4}n\right)Q + \ldots \right] \\
g_6 &= (1 + t)^{-3/2} \left[ \left(2 - \frac{3}{4}n\right) - \left(17 - \frac{41}{4}n\right)Q + \ldots \right] \\
g_7 &= (1 + t)^{-3/2} (n - 2) \left[ -\left(1 - \frac{1}{4}n\right) - \frac{1}{2}(17 - 9n)Q + \ldots \right] \\
g_8 &= (1 + t)^{-3/2} n(n - 1) \left[ -(2 - n) + (17 - 15n)Q + \ldots \right]
\end{align}

and

\begin{align}
\bar{g}_4 &= (1 + t)^{-3/2} [-4Q + \ldots] \\
\bar{g}_7 &= (1 + t)^{-3/2} \left[ -\frac{1}{2}(3 - n) + \frac{1}{2}(29 - 8n)Q + \ldots \right] \\
\bar{g}_8 &= (1 + t)^{-3/2} \left[ \frac{1}{4} - \frac{1}{2}(9 - 3n)Q + \ldots \right].
\end{align}

In zero magnetic field, $h = 0$, Eq. (47) has the physical solutions, $Q \geq 0$:

\begin{align}
Q &= 0 \quad \text{for} \quad t > 0, \\
Q &= -\frac{1}{(2 - n)}t + \ldots \quad \text{for} \quad t < 0.
\end{align}

One thus finds a spin glass transition in mean field at $t_c = 0$. The three masses $m_R, m_A, m_L$ vanish at the transition point, the replicon, anomalous and longitudinal modes, becoming then simultaneously critical. Above the transition, $t > 0$,

$$m_R = m_A = m_L \simeq t$$

and, below the transition, $t < 0$,

$$m_R \simeq -\frac{n}{(2 - n)}t$$
$$m_A \simeq -\frac{2}{(2 - n)}t$$
$$m_L \simeq -t.$$
We have then that, above the transition the three masses are equal, vanishing at $t_c = 0$, whereas below the transition an anisotropy develops, the replicon mass being zero while the anomalous and longitudinal masses are finite and degenerate in the limit $n \to 0$. For the couplings above the transition one has

\[
g_1 = -g_3 = -g_4 = -\frac{4}{(4-n)}g_5 = \frac{4}{(8-3n)}g_6
\]

\[
g_2 = 0
\]

\[
g_7 = \frac{1}{(n-2)(n-4)}g_7 = \frac{1}{n(n-1)(n-2)}g_8 = 1
\]  

(55)

and

\[
\bar{g}_4 = 0
\]

\[
g_1 = -\frac{2}{(3-n)}\bar{g}_7 = 4\bar{g}_8 = 1
\]  

(56)

In zero field and above the transition, where $Q = 0$, there is indeed only one mass and one coupling, since

\[
M_1 = t(1 + t)^{-1}, \quad M_2 = M_3 = 0
\]

\[
W_1 = (1 + t)^{-3/2}, \quad W_i = 0, \quad i = 2, \ldots, 8.
\]  

(57)

In a finite magnetic field, $h \neq 0$, $Q$ is finite, which generates a splitting in the masses and the couplings. A spin glass transition occurs at the AT-line, $h = h_c(T)$, where the replicon modes become critical while the anomalous and longitudinal modes remain non-critical. The AT-line is then characterized by the vanishing of the replicon mass

\[
m_R = 0.
\]  

(58)

The stationarity condition Eq. (47) together with Eqs. (48) and (58), determine the AT-line,

\[
h^2 = -\frac{1}{6}t^3 - \frac{1}{4}nt^2.
\]  

(59)

On the AT-line, $Q \simeq h^{2/3}$, and

\[
m_A \simeq \left(1 - \frac{n}{2}\right)h^{2/3}
\]

\[
m_L \simeq (1 - n)h^{2/3}
\]  

(60)

Likewise one can obtain the couplings from Eqs. (50)-(51).

The expressions derived above for the bare masses and bare couplings provide the initial conditions for the renormalization group study.
IV RENORMALIZATION GROUP

We obtain the renormalization group equations by standard methods of integration of degrees of freedom over an infinitesimal momentum shell, $e^{-d\Lambda} |p| < \Lambda$, at the cutoff $\Lambda$.

The masses are determined by the zero momentum limit of the two-point vertex functions, $\Gamma^{(2)}_i(p)$, $i = R, A, L$. The leading, one-loop, order approximation for the two-point vertex functions corresponds to the "bubble" diagrams. The vertices are determined by the three-point vertex functions, $\Gamma^{(3)}_j(p_1, p_2, p_3)$, $j = 1, \ldots, 8$, calculated at zero momenta, with the definition: $\Gamma^{(3)}_1 = \Gamma_{RRR}$, $\Gamma^{(3)}_2 = \Gamma_{RRR}$, $\Gamma^{(3)}_3 = \Gamma_{RRA}$, $\Gamma^{(3)}_4 = \Gamma_{RL}$, $\Gamma^{(3)}_5 = \Gamma_{RRL}$, $\Gamma^{(3)}_6 = \Gamma_{RRL}$, $\Gamma^{(3)}_7 = \Gamma_{LAA}$, $\Gamma^{(3)}_8 = \Gamma_{LAL}$. The leading, one-loop, order approximation for the three-point vertex functions corresponds to the "triangle" diagrams.

The renormalization equations are given, to lowest order in $\epsilon = 6 - d$, by

\[
\frac{\partial m_R}{\partial l} = (2 - \eta_R)m_R - g_1^2 \frac{(n^4 - 8n^3 + 19n^2 - 4n - 16)}{n(n - 1)(n - 2)^2} + 2(3n^2 - 15n + 16) \frac{(n^3 - 9n^2 + 26n - 22)}{2(n(n - 1)(n - 2)^2)} I_{RR}
\]

\[
- g_3 \left( \frac{8(n - 1)(n - 4)}{n(n - 2)^2} \right) I_{RA} - g_4 \left( \frac{8}{n(n - 1)} \right) I_{RL} - g_5 \left( \frac{16}{n(n - 2)^2} \right) I_{AA}
\]

\[
\frac{\partial m_A}{\partial l} = (2 - \eta_A)m_A - g_3 \left( \frac{2(n - 3)(n - 4)}{n - 2)^2} \right) I_{RR} - g_5 \left( \frac{16n(n - 3)}{(n - 1)(n - 2)^2} \right) I_{RA} - g_6 \left( \frac{32}{n(n - 2)^2} \right) I_{AA} - g_7 \left( \frac{32}{n(n - 1)(n - 2)^2} \right) I_{LA}
\]

\[
\frac{\partial m_L}{\partial l} = (2 - \eta_L)m_L - g_4 \left( \frac{2(n - 3)}{(n - 1)} \right) I_{RR} - g_7 \left( \frac{16}{n(n - 2)^2} \right) I_{AA} - g_8 \left( \frac{4}{n^3(n - 1)^3} \right) I_{LL}
\]

\[
\frac{\partial g_1}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta_R)g_1 + g_1^3 \frac{(n^5 - 10n^4 + 33n^3 - 8n^2 - 104n + 112)}{(n - 1)(n - 2)^3}
\]
\[ \frac{\partial g_2}{\partial l} = \frac{1}{2} (\varepsilon - 3 \eta_R) g_2 - \left[ g_1^2 g_2 \right] \frac{24 n}{(n - 2)^2} \] \[ \frac{\partial g_3}{\partial l} = \frac{1}{2} (\varepsilon - 2 \eta_R - \eta_A) g_3 + \left[ g_2^2 g_3 \right] \frac{n^5 - 12 n^4 + 47 n^3 - 44 n^2 - 48 n + 64}{(n - 1)(n - 2)^3} \] \[ \frac{\partial g_4}{\partial l} = \frac{1}{2} (\varepsilon - 2 \eta_R - \eta_L) g_4 + \left[ g_1^2 g_4 \right] \frac{2 n^4 - 16 n^3 + 38 n^2 - 8 n - 32}{(n - 1)(n - 2)^2} \]
\[ \frac{\partial g_5}{\partial t} = \frac{1}{2}(\varepsilon - \eta_A - 2\eta_A)g_5 - \left[ g_1g_3 \frac{(n^3 - 13n^2 + 48n - 48)}{(n - 2)^3} \right] I_{RRR} - g_2g_3 \frac{n^3 - 10n^2 + 31n - 28}{(n - 2)^3} I_{RRR} + \left[ g_1g_5 \frac{4(n^4 - 8n^3 + 19n^2 - 4n - 16)}{(n - 1)(n - 2)^3} + g_2g_5 \frac{2(6n^2 - 30n + 32)}{(n - 1)(n - 2)^3} \right] I_{RRA} + g_3g_5 \frac{8(n^2 - 5n + 4)}{(n - 2)^3} I_{RLA} + g_4g_5g_7 \frac{32(n^2 - 3n + 4)}{n(n - 1)(n - 2)^2} I_{RAL} - g_5g_6 \frac{64}{n(n - 2)^3} I_{AAA} + g_5g_7^2 \frac{32}{n(n - 1)(n - 2)^2} I_{AAL} \]

\[ \frac{\partial g_6}{\partial t} = \frac{1}{2}(\varepsilon - 3\eta_A)g_6 + g_3 \left[ \frac{(n^4 - 14n^3 + 69n^2 - 140n + 96)}{(n - 2)^3} \right] I_{RRR} \]

\[ + g_3g_5 \left[ \frac{12n(n - 3)(n - 4)}{(n - 2)^3} \right] I_{RRA} - g_5g_6 \left[ \frac{48n(n - 3)}{(n - 1)(n - 2)^3} \right] I_{RAA} + g_6 \left[ \frac{64(n - 3)}{n(n - 2)^3} \right] I_{AAA} + g_6g_7^2 \left[ \frac{96}{n(n - 1)(n - 2)^2} \right] I_{AAL} \]

\[ \frac{\partial g_7}{\partial t} = \frac{1}{2}(\varepsilon - 2\eta_A - \eta_L)g_7 + \frac{1}{2} \left[ \frac{2(n^2 - 7n + 12)}{(n - 2)} \right] I_{RRR} + g_4g_5 \frac{8n(n - 3)}{(n - 1)(n - 2)} I_{RRA} + g_4g_7 \frac{16n(n - 3)}{(n - 1)(n - 2)^2} I_{RAA} + g_7^2 \frac{64}{n(n - 2)^2} I_{AAA} + g_7^3 \frac{32}{n(n - 1)(n - 2)^2} I_{AAL} + g_7^2g_8 \frac{16}{n^2(n - 1)^2(n - 2)} I_{ALL} \]
\[\frac{\partial g_8}{\partial l} = \frac{1}{2} (\varepsilon - 3\eta_L) g_8 + g_3^4 [4n(n-3)] I_{RRR} + g_7^3 \left[ \frac{64(n-1)}{(n-2)^3} \right] I_{AAA} + g_8^3 \left[ \frac{8}{n^3(n-1)^3} \right] I_{LLL} \]  

with

\[\eta_R = \frac{1}{3} \left\{ g_1^2 \frac{(n^4 - 8n^3 + 19n^2 - 4n - 16)}{(n-1)(n-2)^2} + g_2^2 \frac{2(3n^2 - 15n + 16)}{n-1)(n-2)^2} + g_3^2 \frac{(n^3 - 9n^2 + 26n - 22)}{2(n-1)(n-2)^2} \right\} T_{RR} + \frac{1}{3} \left\{ g_4^2 \frac{4(n-1)(n-4)}{n(n-2)^2} (T_{RA} + T_{AR}) + g_5^2 \left[ \frac{4}{n(n-1)} \right] (T_{RL} + T_{LR}) + g_6^2 \left[ \frac{16}{(n-2)^2} \right] T_{AA} \right\} \]  

\[\eta_A = \frac{1}{3} \left\{ g_2^3 \frac{[2(n-3)(n-4)]}{(n-2)^2} T_{RR} + g_2^2 \left[ \frac{8n(n-3)}{(n-1)(n-2)^2} \right] (T_{RA} + T_{AR}) + g_5^2 \left[ \frac{32}{n(n-2)^2} \right] T_{AA} + g_7^2 \left[ \frac{16}{n(n-1)(n-2)^2} \right] (T_{AL} + T_{LA}) \right\} \]  

\[\eta_L = \frac{1}{3} \left\{ g_4^2 \left[ \frac{2(n-3)}{(n-1)} \right] T_{RR} + g_5^2 \left[ \frac{16}{n(n-2)^2} \right] T_{AA} + g_8^2 \left[ \frac{4}{n^3(n-1)^3} \right] T_{LL} \right\} \]  

defining

\[I_{ij} = \frac{1}{(1 + m_i)(1 + m_j)}, \]  

\[I_{ijk} = \frac{1}{(1 + m_i)(1 + m_j)(1 + m_k)}, \]  

\[T_{ij} = \frac{1 + 3m_j}{(1 + m_i)(1 + m_j)^3}. \]
In Eqs. (61)-(74) the usual geometrical factor $K_d = 2/(4\pi d/2)$ is absorbed in the couplings, i.e., $K_d^{1/2} g_i \to g_i$. The one-loop perturbation equations for the two-point and three-point vertex functions contain $1/n$ -factors, which consequently appear in the renormalization equations. Those factors, which come always associated with terms involving the anomalous and longitudinal modes, could in principle, lead to divergences in the limit $n \to 0$. The spin glass symmetry does however prevent this problem, as will be seen.

As mentioned before, the field theory considered, and hence the renormalization equations just derived, are quite general, containing different symmetries, which depend on $n$. The spin glass is obtained in the limit $n \to 0$. Here we assume that perturbation theory works in a regular way as presented above. The possibility that subdominant dangerous terms may take over (with as a consequence a breakdown of the symmetry involved here) will be considered separately. With that assumption there are now two important features. First, at the mean-field level, i.e. in 0-loop order, the spin glass is characterized by a degeneracy in the masses and the couplings, i.e. $m_A$, $g_3$ and $g_4$, $g_6$ and $g_7$, $g_8/n$, are equal, when $n \to 0$; we then defined the variables, $\bar{m}_{AL}$, $\bar{g}_4$, $\bar{g}_7$, and $\bar{g}_8$, in Eqs. (33) and (40), which although incorporating $1/n$ -factors, are in 0-loop order, well defined quantities in terms of the spin correlations functions, when $n \to 0$, see Eqs (34) and (41). Secondly, starting with the set of variables $m_R$, $m_A$, $\bar{m}_{AL}$, $g_1$, $g_2$, $g_3$, $g_4$, $g_5$, $g_6$, $g_7$, $g_8$, one can built a set of equations for the 1-loop two-point and three-point vertex functions corresponding to those variables (defining $\bar{\Gamma}_{AL} = (\Gamma_A - \Gamma_L)/n$, $\bar{\Gamma}_4 = 4(\Gamma_4 - \Gamma_3)/n$, $\bar{\Gamma}_7 = 2(\Gamma_7 - \Gamma_6)/n$, $\bar{\Gamma}_8 = (\Gamma_8 - 3n\Gamma_7 + 2n\Gamma_6)/n^3$), which is free of $1/n$ -factors, and hence is well behaved in the limit $n \to 0$. This means that the degeneracy that exists, in the masses and the couplings in 0-loop order, is also present in 1-loop order, and consequently, the renormalization group transformation preserves this symmetry. One can then write the renormalization equations for the new set of variables, and $n = 0$:

\[
\frac{\partial m_R}{\partial l} = (2 - \eta_R)m_R - \left[ 4g_1^2 - 8g_1g_2 + \frac{11}{4}g_2^2 \right] I_{RR} + \left[ 10g_3^2 + 4g_3\bar{g}_4 \right] I_{RA} - 4g_3^2 I_{AA} + 8g_3^2 \bar{m}_{AL} I_{RAA}
\] (75)

\[
\frac{\partial m_A}{\partial l} = (2 - \eta_A)m_A - 6g_3^2 I_{RR} + 8 \left[ g_3^2 + g_6g_7 \right] I_{AA} + 8g_3^2 \bar{m}_{AL} I_{AAA}
\] (76)

\[
\frac{\partial \bar{m}_{AL}}{\partial l} = (2 - \eta_A)\bar{m}_{AL} - \bar{\eta}_{AL}m_A + \frac{3}{2} \left[ g_3^2 + 2g_3\bar{g}_4 \right] I_{RR}
\] (77)
\[-12g_2^2I_{RA} - \left[5g_6^2 + 16g_6\bar{g}_7 + 8g_6\bar{g}_8 + 6\bar{g}_7^2\right]I_{AA} \]
\[-8\left[g_6^2 + 2g_6\bar{g}_7\right]\bar{m}_{AL}I_{AAA}\]

\[
\frac{\partial g_1}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta_R)g_1 + \left[14g_1^2 - 18g_1^2g_2 + \frac{9}{2}g_1g_2^2 + \frac{1}{8}g_2^3\right]I_{RRR} \tag{78}
- \left[18g_1g_3^2 + \frac{3}{2}g_2g_3^2 + 12g_1g_3\bar{g}_4\right]I_{RRA} + 6g_3g_5I_{RAA} - 8g_3^3I_{AAA}\]

\[
\frac{\partial g_2}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta_R)g_2 + \left[24g_1^2g_2 - 30g_1g_2^2 + \frac{17}{2}g_2^3\right]I_{RRR} \tag{79}
+ \left[24g_1g_3^2 - 36g_2g_3^2 - 12g_2g_3\bar{g}_4\right]I_{RRA} + 24g_3^2g_5I_{RAA}\]

\[
\frac{\partial g_3}{\partial l} = \frac{1}{2}(\varepsilon - 2\eta_R - \eta_A)g_3 + \left[8g_1^2g_3 + \frac{11}{2}g_2g_3^2 - 16g_1g_2g_3\right]I_{RRR} \tag{80}
- \left[10g_3^2 + 4g_3^2\bar{g}_4\right]I_{RRA} + \left[14g_3^2g_6 + 8g_3\bar{g}_7 + 4g_3\bar{g}_4g_6\right]I_{RAA}
- 8g_5^2g_6I_{AAA}\]

\[
\frac{\partial \bar{g}_4}{\partial l} = \frac{1}{2}(\varepsilon - 2\eta_R - \eta_A)\bar{g}_4 + 2\bar{m}_{AL}g_3 + \left[16g_1^2g_3 + 8g_1^2g_4 + \frac{7}{2}g_2g_3^2\right] \tag{81}
- 12g_1g_3g_5 + \frac{11}{2}g_2g_4 - 16g_1g_2g_4\right]I_{RRR} + \left[48g_1g_3g_5 - 28g_2g_3g_5\right.
- 2g_3^2 - 10g_3^2\bar{g}_4 - 4g_3\bar{g}_4\right]I_{RRA} + \left[16g_3^2g_5 + 8g_3g_6^2 + 4g_3^2\bar{g}_7\right.
+ 8g_3\bar{g}_4g_6 + 32g_3^2\bar{g}_8 + 16g_3\bar{g}_4\bar{g}_7 + 2g_4^2g_6\right]I_{RAA} - 16g_5g_7I_{AAA}\]

\[
\frac{\partial g_5}{\partial l} = \frac{1}{2}(\varepsilon - \eta_R - 2\eta_A)g_5 - \left[6g_1g_3^2 - \frac{7}{2}g_2g_3^2\right]I_{RRR} + \left[-8g_1g_5^2\right] \tag{82}
+ 8g_2g_5^2 - 4g_3^2g_5\right]I_{RRA} + \left[8g_3^3 + 20g_3g_5g_6 + 8g_3g_5\bar{g}_7\right.
+ 4\bar{g}_4g_5g_6\right]I_{RAA} - \left[4g_5g_6^2 + 8g_5g_6\bar{g}_7\right]I_{AAA}\]

\[
\frac{\partial g_6}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta_A)g_6 - 12g_3^3I_{RRR} - \left[20g_6^3 + 24g_6^2\bar{g}_7\right]I_{AAA} \tag{83}\]

\[
\frac{\partial \bar{g}_7}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta_A)\bar{g}_7 + \bar{m}_{AL}g_6 + \left[3g_3^3 - 6g_3^2\bar{g}_4\right]I_{RRR} + 12g_3g_5^2I_{RRA} \tag{84}
- 12g_5^2g_6I_{RAA} - \left[4g_3^3 + 36g_5^2\bar{g}_7 + 28g_6^2g_7^2 + 16g_6\bar{g}_8\right]I_{AAA}\]
\[ \frac{\partial \bar{g}_8}{\partial \ell} = \frac{1}{2} (\epsilon - 3 \eta_A) \bar{g}_8 + \frac{3}{2} \bar{\eta}_{AL} \bar{g}_7 + \frac{3}{4} g^3 + 3g^2 \bar{g}_4 - 3g_3 \bar{g}_4^2 \right] I_{RRR} \]

\[ + 9 \left[ g_3 g_5^2 + \bar{g}_4 g_5^2 \right] I_{RRA} + 18 \left[ g_6^2 g_6 - g_5^2 \bar{g}_7 \right] I_{RAA} - \left[ 12g_6 \bar{g}_7 \right]
\]

\[ + g_6^3 + 42g_6 \bar{g}_7 + 14 \bar{g}_7 + 12g_6 \bar{g}_7 \bar{g}_8 + 48 \bar{g}_6 \bar{g}_7 \bar{g}_8 \right] I_{AAA} \]

with

\[ \eta_R = \frac{1}{3} \left\{ 4g_1^2 - 8g_1 g_2 + \frac{11}{4} g_2 \right\} T_{RR} \]

\[ - \left[ 5g_3^2 + 2g_3 \bar{g}_4 \right] \left( T_{RA} + T_{AR} \right) + 4g_2^2 T_{AA} \}

\[ \eta_A = \frac{1}{3} \left\{ 6g_3^2 T_{RR} - 8 \left[ g_6^2 + g_6 \bar{g}_7 \right] T_{AA} \right\} \]

\[ \bar{\eta}_{AL} = -\frac{1}{2} \left( g_3^2 + 2g_3 \bar{g}_4 \right) T_{RR} + 2g_5^2 \left( T_{RA} + T_{AR} \right) \]

\[ + \frac{1}{3} \left[ 5g_6^2 + 16g_6 \bar{g}_7 + 8g_6 \bar{g}_8 + 6 \bar{g}_7 \right] T_{AA} \].

We remark that the theory with \( n \) finite, allows the three masses, as well as the eight couplings \( g_j \), to be different, while the theory with \( n = 0 \), imposes \( m_A = m_L \) and \( g_3 = g_4, \ g_6 = g_7 = g_8/\eta \). The two sets of renormalization equations, Eqs (61)-(74) and Eqs. (75)-(88), imply two different procedures, i.e. the first corresponds to keeping \( n \) finite along the renormalization iteration, and only at the end set it zero, while the second corresponds to imposing \( n = 0 \) all the way along the renormalization iteration. The two procedures may lead to different results, as \( n \) influences the flow equations. When using the first procedure to study the spin glass one has to carefully take into account the symmetry of the problem, in order to obtain the relevant results. The latter procedure naturally enforces the spin glass symmetry. We will discuss the cases of zero and small magnetic field using the two procedures, for illustration. Let us then look for the fixed-points (f.p.) of the renormalization group equations.

In zero magnetic field, \( h = 0 \), and \( t > 0 \), the three masses, \( m_i, i = R, A, L \), are equal, see Eq. (53), and the eight couplings, \( g_j, j = 1, \ldots, 8 \) are simply related, as in Eqs. (55)-(56). So, if one imposes this symmetry

\[ m_R = m_A = m_L = m \]

\[ g_1 = -g_3 = -g_4 = \frac{4}{(n-4)} g_5 = \frac{4}{(8-3n)} g_6 \]

\[ = \frac{4}{(n-2)(n-4)} g_7 = \frac{1}{n(n-1)(n-2)} g_8 = g \]

\[ g_2 = 0 \].
on the renormalization equations, Eqs. (61)-(74), they reduce to

\[ \frac{\partial m}{\partial l} = (2 - \eta)m + (2 - n)g^2 \frac{1}{(1 + m)^2} \]  

(91)

\[ \frac{\partial g}{\partial l} = \frac{1}{2}(\varepsilon - 3\eta)g - (2 - n)g^3 \frac{1}{(1 + m)^3} \]  

(92)

with

\[ \eta = -\frac{1}{3}(2 - n)g^2 \frac{1 + 3m}{(1 + m)^4}. \]  

(93)

Equivalently, if one imposes the zero-field symmetry, with \( n = 0 \),

\[ m_R = m_A = m, \quad \bar{m}_{AL} = 0 \]  

(94)

\[ g_1 = -g_3 = -g_5 = \frac{1}{2}g_6 = -\frac{2}{3}\bar{g}_7 = 4\bar{g}_8 = g \]  

(95)

on the renormalization equations, Eqs.(75)-(88), they reduce to Eqs. (91)-(93), naturally with \( n = 0 \). For the Eqs. (91)-(93), one finds:

(i) the trivial Gaussian f.p.

\[ m^* = 0, \quad g^* = 0, \]  

which is unstable for \( \varepsilon > 0 \), with eigenvalues \( \lambda_1^G = 2, \lambda_2^G = \frac{1}{2}\varepsilon \);

(ii) the nontrivial zero-field f.p.

\[ m^* = -\frac{\varepsilon}{2}, \quad g^{*2} = \frac{\varepsilon}{2 - n}, \]  

(97)

which is stable for \( \varepsilon > 0 \), with eigenvalues \( \lambda_1^{ZF} = 2(1 - \varepsilon), \lambda_2^{ZF} = -\varepsilon \). The zero-field f.p. has then the associated critical exponents

\[ \nu = \frac{1}{2} + \frac{5}{12}\varepsilon, \quad \eta = -\frac{1}{3}\varepsilon \]  

(98)

in agreement with the results of Harris et al.\(^{18}\)

In the generalized parameter space, with the renormalization equations (61)-(74), or (75)-(88), one can get the fixed-points associated to zero-field by searching for fixed-points with the same symmetry as in Eqs. (89)-(90), or (94)-(95), and indeed one then finds two fixed-points, which correspond to Eqs. (96)-(97), and \( \bar{m}_{AL}^* = 0 \). For \( \varepsilon > 0 \), the Gaussian f.p. is unstable in all the directions, while the non-trivial zero-field f.p. is stable in two but unstable in six directions in the coupling-space, see Table I, concerning Eqs. (75)-(88). The appearance of unstable directions for the couplings in this situation, arises from the fact that the generalized parameter space has a
lower symmetry than the characteristic of zero-field. However, of crucial importance is the fact that the initial values of the couplings, Eqs. (55)-(56), lie along one of the stable directions, i.e. Eqs. (61)-(74) and Eqs. (75)-(88), have an eigenvector (associated to the eigenvalue \(-1\)), which has precisely the symmetry in, respectively, Eq. (90) and Eq. (95), see Table I for the latter. So, the system will start and will remain in a stable direction, the unstable directions playing no role. That stable direction in fact represents an invariant manifold.

In a nonzero magnetic field, \(h \neq 0\), an anisotropy develops in the masses, and in the couplings. To discuss the effect of a small magnetic field, \(h \ll 1\), we linearize the renormalization equations about the zero-field f.p. For the mass equations, Eqs. (61)-(63), we obtain the eigenvalues

\[
\begin{align*}
\lambda_1 &= 2 - \frac{5}{3} \epsilon \\
\lambda_2 &= 2 - \frac{2}{3} \epsilon \\
\lambda_3 &= 2 + \frac{(n - 14)}{3(n - 2)} \epsilon
\end{align*}
\]

and the scaling fields

\[
\begin{align*}
u_1 &= \delta m_R + \frac{2}{(n - 1)} \delta(m_A - m_R) - \frac{2}{(n - 1)} \delta \left( \frac{m_A - m_L}{n} \right) \\
u_2 &= -\frac{2(n - 3)}{(n - 1)(n - 2)} \delta(m_A - m_R) + \frac{4}{(n - 1)} \delta \left( \frac{m_A - m_L}{n} \right) \\
u_3 &= -\frac{2}{(n - 1)(n - 2)} \delta(m_A - m_R) - \frac{2}{(n - 1)} \delta \left( \frac{m_A - m_L}{n} \right)
\end{align*}
\]

where \(\delta m_i = m_i - m^*\), \(i = R, A, L\). Similar results, with \(n = 0\), are obtained from the linearization of the mass equations, Eqs. (75)-(77), around the zero-field f.p. The field \(u_1\) represents the average mass, i.e. \(u_1 = [n(n - 3)/2\delta m_R + (n - 1)\delta m_A + \delta m_L]/[n(n - 1)/2]\), while the fields \(u_2\) and \(u_3\) are determined by the anisotropy variables, \((m_A - m_R)\) and \((m_A - m_L)/n\).

One has that

\[
\begin{align*}
u_1 &\simeq (t - t_c) \\
u_2 &\simeq 2Q \\
u_3 &\simeq -3Q^2
\end{align*}
\]

So, we identify the critical exponent for the correlation length,

\[
\nu = \lambda_1^{-1} = \frac{1}{2} + \frac{5}{12} \epsilon
\]
and have the crossover exponents, for \( n = 0 \),

\[
\begin{align*}
\phi_2 &= \lambda_2 \nu = 1 + \frac{1}{2} \varepsilon \\
\phi_3 &= \lambda_3 \nu = 1 + 2 \varepsilon,
\end{align*}
\]

which turn out to be, to order \( \varepsilon \), the critical exponent for the order parameter \( \beta = \phi_2 \), and the critical exponent for the specific heat \( \alpha = -\phi_3 \). Here, \( \nu, \beta \) and \( \alpha \) are the zero-field critical exponents.

The masses are given in terms of the scaling fields by

\[
\begin{align*}
\delta m_R &= u_1 + u_2 + u_3 \\
\delta m_A &= u_1 + \frac{1}{2} (4 - n) u_2 + (3 - n) u_3 \\
\delta m_L &= u_1 + (2 - n) u_2 + \frac{1}{2} (2 - n)(3 - n) u_3
\end{align*}
\]

and

\[
\delta \left( \frac{m_A - m_L}{n} \right) = \frac{1}{2} [u_2 + (3 - n) u_3].
\]

When \( n \) is finite, \( m_A \) and \( m_L \), as well as \( (m_A - m_L)/n = \bar{m}_{AL} \), will grow under the renormalization group iteration, and we have then \( m_A \neq m_L \), with different regimes occurring, according to the relative amplitude of the two masses. However, for \( n = 0 \), one has that \( m_A = m_L \), even though \( \bar{m}_{AL} \) grows, and that single mass controls the behavior of the system.

We now concentrate on the set of renormalization group equations with \( n = 0 \), Eqs. (75)-(88), to investigate the fixed-points in a finite magnetic field \( h \neq 0 \). One has essentially two masses, \( m_R \) and \( m_A = m_L \). On the AT-line, the bare masses take the values, \( m_R = 0 \) and \( m_A = m_L \approx h^{2/3} \). Fixing the replicon modes critical, as characteristic of the spin glass transition, we have that the anomalous and longitudinal modes will scale out of the problem, i.e. their mass diverges under the renormalization group transformation. After a few iterations, we effectively obtain the reduced set of renormalization equations

\[
\begin{align*}
\frac{\partial m_R}{\partial l} &= (2 - \eta_R)m_R - \left[ 4 g_1^2 - 8 g_1 g_2 + \frac{11}{4} g_2^2 \right] I_{RR} \\
\frac{\partial g_1}{\partial l} &= \frac{1}{2} (\varepsilon - 3 \eta_R) g_1 + \left[ 14 g_1^3 - 18 g_1^2 g_2 + \frac{9}{2} g_1 g_2^2 + \frac{1}{8} g_2^3 \right] I_{RRR} \\
\frac{\partial g_2}{\partial l} &= \frac{1}{2} (\varepsilon - 3 \eta_R) g_2 + \left[ 24 g_1^2 g_2 - 30 g_1 g_2^2 + \frac{17}{2} g_2^3 \right] I_{RRR}
\end{align*}
\]
with

\[ \eta_R = \frac{1}{3} \left[ 4g_1^2 - 8g_1g_2 + \frac{11}{4}g_2^2 \right] T_{RR} \]  

(109)

which are equivalent to the ones studied by Bray and Roberts.\(^\text{19}^\) We find the same type of fixed-points as in their work:

(i) \( g_1^* = 0, g_2^* = 0 \). This is the usual Gaussian f.p. which is unstable for \( \varepsilon > 0 \);

(ii) \( g_1^2 = -\varepsilon/24, g_2^* = 0 \). This f.p. is stable but unphysical, because it is complex and hence inaccessible from the domain of physical initial conditions via the renormalization group equations;

(iii) \( g_1^* = (0.415 \pm 0.090)\sqrt{\varepsilon}, g_2^* = (0.708 \mp 0.018)\sqrt{\varepsilon}; g_1^* = i0.010\sqrt{\varepsilon}, g_2^* = i0.283\sqrt{\varepsilon}. \) These fixed-points are unphysical, and also turn out to be unstable.

Hence no physical stable fixed-point is found to describe the AT transition.

We present now a set of fixed-points which we found for Eqs.(75)-(88), when searching for fixed-points with critical masses \( m_R \) and \( m_A = m_L \), i.e., by dropping \( m_R \) and \( m_A \) in the denominators of those equations. In addition to the zero-field fixed-point, discussed above, which has the symmetry \( m_R^* = m_L^*, m_{AL}^* = 0 \), we find the other following ones:

(I) \( g_1^* = \frac{1}{2}\sqrt{\varepsilon}, g_2^* = \sqrt{\varepsilon}, g_3^* = -\frac{1}{4}\sqrt{\varepsilon}, g_6^* = 0, g_4^* = -2u\sqrt{\varepsilon}, g_8^* = \frac{9}{4}u\sqrt{\varepsilon}; m_R^* = 0, m_{AL}^* = \frac{3}{8}\varepsilon. \) This in fact represents a line of equivalent fixed-points, \( u \) being an arbitrary real number, that parametrizes the line. The eigenvalues, both in the coupling and mass spaces, remain unchanged along the line. Those fixed-points are stable in two directions, unstable in two directions, and marginal in one direction in the coupling-space, see Table II. In the mass-space the eigenvalues are, \( \lambda_1 = 2 - \frac{\varepsilon}{2}, \lambda_2 = \lambda_3 = 2. \) These fixed-points are in fact related to a fixed-point that is obtained for a different symmetry (\( m_R = m_A = 0, m_L \neq 0 \)), which will be discussed in a separate publication.\(^\text{20}^\)

(II) \( g_1^* = g_2^* = g_5^* = 0, g_3^* = -0.350\sqrt{\varepsilon}, g_4^* = 0.656\sqrt{\varepsilon}, g_6^* = 0.519\sqrt{\varepsilon}, g_7^* = -0.308\sqrt{\varepsilon}, g_8^* = 0.067\sqrt{\varepsilon}; m_R^* = -0.153\varepsilon, m_A^* = -0.073\varepsilon, m_{AL}^* = 0.072\varepsilon. \) This f.p. is stable in five directions and unstable in three directions in the coupling-space, see Table III. The eigenvalues in the mass-space are, \( \lambda_1 = 2 - 0.795\varepsilon, \lambda_2 = 2 - 0.524\varepsilon, \lambda_3 = 2 - 0.151\varepsilon. \)

(III) \( g_1^* = -0.031\sqrt{\varepsilon}, g_2^* = 0.540\sqrt{\varepsilon}, g_3^* = 0.311\sqrt{\varepsilon}, g_4^* = 0.237\sqrt{\varepsilon}, g_6^* = 0.214\sqrt{\varepsilon}, g_7^* = -0.368\sqrt{\varepsilon}, g_8^* = 0.134\sqrt{\varepsilon}; m_R^* = -0.068\varepsilon, m_A^* = -0.557\varepsilon, m_{AL}^* = 0.002\varepsilon. \) This f.p. is stable in five directions and unstable in three directions in the coupling-space, see Table IV. The eigenvalues in the mass-space are, \( \lambda_1 = 2 - 0.519\varepsilon, \lambda_2 = 2 - 0.325\varepsilon, \lambda_3 = 2 - 0.096\varepsilon. \)

So, although one can find different real fixed-points involving \( m_R \) and \( m_A = m_L \), none is fully stable in coupling-space, excluding the zero-field f.p. discussed above. Each of the three fixed-points above attracts its own
critical manifold which is spanned by the eigenvectors associated to the ir-
relevant eigenvalues ($\lambda < 0$), i.e. the stable directions. The initial conditions
in a field, correspond in the coupling-space to a vector where all the compo-
nants are finite, see Eqs. (55)-(56). Comparing the structure of the critical
manifolds associated to each of the fixed points (I), (II) and (III), as given in
respectively Table II, III and IV, with the structure of the initial conditions,
one may say the following. F.p. (I) is quite disconnected, f.p. (II) has some
connection, f.p. (III) is the most connected one to the initial conditions.
Fixed-point (II) suggests a separation of the coupling-space into two sectors
containing, respectively, $g_1, g_2, g_5$ and $g_3, \bar{g}_4, g_6, \bar{g}_7, \bar{g}_8$, neither of those
sectors being however completely stable. Fixed-point (III) is the most likely
to have some influence on the flows in the coupling-space in a field. Notice
that the dimension of its critical manifold is equal to the number of distinct
unbarred couplings ($g_j, j = 1, 2, 3, 5, 6$). It is yet difficult to give a physical
meaning to those three fixed points, and in particular connect them to a
spin glass transition in a field. They may reflect different symmetries that
are included in the general field theory considered.

V CONCLUSIONS

We present the general field theory, appropriate to study the transition of
a short-range spin glass in a magnetic field, which contains three masses
and eight couplings, explicitly written in terms of the fields for the replicon,
anomalous and longitudinal modes. This theory allows a standard pertur-
bation expansion, using propagators which involve projector operators. The
structure of the masses and the couplings depends on the mean field value of
the order parameter, and the number of replicas $n$. We discuss the symmetry
of the theory in the limit $n \rightarrow 0$, and consider the regular case where there
is a degeneracy between the anomalous and longitudinal masses, and a de-
genearcy in the couplings involving the anomalous and longitudinal modes.
We calculate the equation of state, the bare masses and the bare couplings
in mean-field theory. The mean-field calculation shows a transition in zero
field, where all the modes become critical, and a transition in nonzero field,
along the AT-line, with only the replicon mode critical. We then study
the problem using the renormalization group, to order $\varepsilon = 6 - d$. The
renormalization group transformation preserves the degeneracy between the
anomalous and longitudinal masses, and the degeneracy in the couplings
involving the anomalous and longitudinal modes, when $n$ is fixed to zero.
Within the general field theory we find a fixed-point associated to the tran-
sition in zero-field, which provides the zero-field critical exponents. This
fixed-point which is stable in zero-field becomes unstable in the presence of
a small magnetic field, and we calculate crossover exponents, which we relate
to zero-field critical exponents. It is, however, not clear where the crossover
leads to. For a finite magnetic field, we find no physical stable fixed-point, that would describe the AT transition, in agreement with the results of Bray and Roberts.\textsuperscript{19} The absence of stable fixed-points, accessible from the domain of physical initial Hamiltonians, can have different interpretations in the renormalization group, as discussed in Ref. 19: the fluctuations may drive the transition first order, the transition may be first order even within mean-field theory, or the fluctuations may destroy the transition. To conclude, the present study shows that a theory with degeneracy between the anomalous and longitudinal masses, as imposed by keeping $n$ equal zero, leaves no place for a second order spin glass transition outside zero-field.

**ACKNOWLEDGMENTS**

We are grateful to E. Brézin for helpful discussions. This work has been supported by the Hungarian Science Fund (OTKA), grant No. T032424.

**APPENDIX**

We present here the derivation of the expressions for the longitudinal, anomalous and replicon projectors given in Eqs. (18), (25) and (30). We obtain those expressions following a procedure similar to the one used by Bray and Roberts to get the components of the replicon projector in their work.\textsuperscript{19}

The longitudinal projector is defined by, Eq. (17),

$$P^L = e^L e^L$$

(A1)

where $e^L$ represents a component of the replica independent unit vector, $e^L \equiv \sqrt{2/n(n-1)} \left[1, \ldots, 1\right]$, 

$$e^L_{\alpha \beta} = e^L$$

(A2)

The longitudinal projector has then only one component, which is obviously given by

$$P^L = \frac{2}{n(n-1)}.$$ 

(A3)

The anomalous projector is defined by, Eq. (26),

$$P^A_{\alpha, \beta} = \sum_{\mu=1}^{n-1} e^A_{\alpha, \mu} e^A_{\beta, \mu}$$

(A4)

where the vectors $e^A_{\alpha, \mu}$ have the property.
\[ \sum_\alpha e^{A,\mu}_\alpha = 0 \quad (A5) \]

and normalization \[ \sum_\alpha e^{A,\mu}_\alpha e^{A,\mu}_\alpha = \frac{4}{(n-2)} \] (which follows from the fact that \( e^{A,\mu} \) is a unit vector in the space of replica pairs and \( e^{A,\mu}_{\alpha\beta} = \frac{1}{2}(e^{A,\mu}_{\alpha} + e^{A,\mu}_{\beta}) \)).

The anomalous projector has two distinct components: \( P^A_{\alpha,\alpha} \) and \( P^A_{\alpha,\beta} \) with \( \alpha \neq \beta \). Setting \( \alpha = \beta \) in Eq. (A4) and summing both sides over \( \alpha \), gives

\[
n P^A_{\alpha,\alpha} = \frac{n-1}{n-2} \left( \sum_\mu e^{A,\mu}_\alpha e^{A,\mu}_\alpha \right) = \frac{4(n-1)}{(n-2)} ; \quad (A6)
\]

setting \( \alpha \neq \beta \) in Eq. (A4) and summing both sides over \( \alpha \neq \beta \), gives

\[
n(n-1) P^A_{\alpha,\beta} = \frac{n-1}{n-2} \left( \sum_\mu e^{A,\mu}_\alpha e^{A,\mu}_\beta \right) = \frac{4(n-1)}{(n-2)} , \quad (A7)
\]

where we used Eq. (A5). Eqs. (A6) and (A7) lead to the general form for the anomalous projector,

\[ P^A_{\alpha,\beta} = \frac{4}{(n-2)} \left( \delta_{\alpha,\beta} - \frac{1}{n} \right) \quad (A8) \]

The replicon projector is defined by, Eq. (31),

\[ P^R_{\alpha,\beta,\gamma,\delta} = \frac{1}{2}n(n-3) \sum_\nu e^{R,\nu}_{\alpha,\beta} e^{R,\nu}_{\gamma,\delta} . \quad (A9) \]

where \( e^{R,\nu} \) are unit vectors, which have the property

\[ \sum_{\alpha(\neq \beta)} e^{R}_{\alpha\beta} = 0. \quad (A10) \]

The replicon projector \( P^R_{\alpha,\beta,\gamma,\delta} \) has three different components: \( P^R_{\alpha,\beta,\alpha,\beta} \), \( P^R_{\alpha,\beta,\alpha,\delta} \) with \( \beta \neq \delta \), and \( P^R_{\alpha,\beta,\gamma,\delta} \) with \( \alpha \neq \gamma, \beta \neq \delta \). Setting \( \alpha = \gamma, \beta = \delta \) in Eq. (A9) and summing both sides over distinct \( \alpha, \beta \), gives
\[ n(n-1)P_{\alpha\beta,\alpha\beta}^R = - \sum_{\nu=1}^{n} \sum_{\alpha \neq \beta} e_{\alpha\beta}^R e_{\alpha\beta}^R \]
\[ = n(n-3); \quad (A11) \]
setting \( \alpha = \gamma, \beta \neq \delta \) in Eq. (A9) and summing both sides over distinct \( \alpha, \beta, \delta \), gives

\[ n(n-1)(n-2)P_{\alpha\beta,\alpha\delta}^R = \sum_{\nu=1}^{n-3} \sum_{\alpha, \beta, \delta} e_{\alpha\beta}^R e_{\alpha\delta}^R \]
\[ = -n(n-3) \quad (A12) \]
where we used Eq. (A10); setting \( \alpha \neq \gamma, \beta \neq \delta \) in Eq. (A9) and summing both sides over distinct \( \alpha, \beta, \gamma, \delta \), gives, using again Eq. (A10),

\[ n(n-1)(n-2)(n-3)P_{\alpha\beta,\gamma\delta}^R = \sum_{\nu=1}^{n-3} \sum_{\alpha, \beta, \gamma, \delta} e_{\alpha\beta}^R e_{\gamma\delta}^R \]
\[ = 2n(n-3) \quad (A13) \]
Eqs. (A11), (A12) and (A13) lead to the general form for the replicon projector,

\[ P_{\alpha\beta,\gamma\delta}^R = (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) - (\delta_{\alpha\gamma} + \delta_{\alpha\delta} + \delta_{\beta\gamma} + \delta_{\beta\delta}) \frac{1}{n-2} \]
\[ + \frac{2}{(n-1)(n-2)} \quad (A14) \]
with \( \alpha \neq \beta, \gamma \neq \delta \). The results in Eqs. (A.11), (A.12) and (A.13), are similar to the ones obtained by Bray and Roberts.\(^{19}\)
References

1. K. Binder and A.P. Young, Rev. Mod. Phys. 58, 801 (1986).
2. M. Mézard, G. Parisi, and M.A. Virasoro, Spin Glass Theory and Beyond (World Scientific, Singapore, 1987).
3. K.H. Fischer and J.H. Hertz, Spin Glasses (Cambridge University Press, Cambridge, 1991).
4. Spin Glasses and Random Fields, edited by A.P. Young (World Scientific, Singapore, 1998).
5. Spin Glasses and Biology, edited by D. Stein (World Scientific, Singapore 1992).
6. S.F. Edwards and P.W. Anderson, J. Phys. F 5, 965 (1975).
7. G. Parisi, Phys. Rev. Lett. 43, 1754 (1979); J. Phys. A 13, L115, 1101, 13, 1887 (1980); Phys. Rev. Lett. 50, 1946 (1983).
8. D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. 35, 1792 (1975).
9. D.S. Fisher and D.A. Huse, Phys. Rev. Lett. 56, 1601 (1986); Phys. Rev. B 38, 386 (1988).
10. W.L. McMillan, J. Phys. C 17, 3179 (1984).
11. A. J. Bray and M.A. Moore, in Proceedings of the Heidelberg Colloquium on Glassy Dynamics, edited by J.L. van Hemmen and I. Morgenstern (Springer Verlag, Heidelberg, 1986), p. 121; Phys. Rev. Lett. 58, 57 (1987).
12. J.R.L. de Almeida and D.J. Thouless, J. Phys. A 11, 983 (1977).
13. A.J. Bray and M.A. Moore, J. Phys. C 12, 79 (1979).
14. P. Nordblad and P. Svedlindh, in Spin Glasses and Random Fields, edited by A.P. Young (World Scientific, Singapore, 1998), p. 1.
15. P. Monod and H. Bouchiat, J. Physique Lett. 43, L45 (1982); V.S. Zotev and R. Orbach, (submitted), cond-mat/0201226.
16. E. Marinari, G. Parisi, and J.J. Ruiz-Lorenzo, in Spin Glasses and Random Fields, edited by A.P. Young (World Scientific, Singapore, 1998), p. 59; J. Houdayer and O.C. Martin, Phys. Rev. Lett. 82, 49, 34 (1999); F. Krzakala, J. Houdayer, E. Marinari, O. C. Martin, and G. Parisi, Phys. Rev. Lett. 87, 197204 (2001).
17. J.E. Green, M.A. Moore, and A.J. Bray, J. Phys. C 16, L815 (1983).
18. A.B. Harris, T.C. Lubensky, and J-H. Chen, Phys. Rev. Lett. 36, 415 (1976).
19. A.J. Bray and S.A. Roberts, J. Phys. C 13, 5405 (1980).
20. T. Temesvari, C. De Dominicis, and I.R. Pimentel, (submitted).
21. T. Temesvari, C. De Dominicis, and I.R. Pimentel, Eur. Phys. J. B 25, 361 (2002).
22. C. De Dominicis, D.M. Carlucci, and T. Temesvári, J. Phys. I France 7, 105 (1997).
23. E. Pytte and J. Rudnick, Phys.Rev. B 19, 3603 (1979).
24. K.G. Wilson and J.B. Kogut, Phys. Rep. 12C, 75 (1974).
TABLE I. Eigenvalues $\lambda$ and eigenvectors $E$, in coupling-space for the zero-field f.p..

| $\lambda$ | $-1$ | $-3.82$ | $0.74 \pm i2.54$ |
|-----------|------|---------|-----------------|
| $E$       | 1    | 0.51    | 0.09 \pm i0.06 |
|           | 0    | 0.22    | -0.50 \pm i0.36|
|           | -1   | -0.26   | -0.50 \pm i0.12|
|           | 0    | -0.84   | 0.34 \pm i0.05 |
|           | -1   | -0.48   | -0.41 \pm i0.01|
|           | 2    | 1       | 1               |
|           | -3/2 | -0.75   | -0.79 \pm i0.10 |
|           | 1/4  | 0.19    | 0.22 \pm i0.07 |
| $\lambda$ | 1    | 7       | $(10 \pm i\sqrt{23})/3$ |
| $E$       | 3/7  | 1/5     | $(13 \pm i\sqrt{23})/64$ |
|           | -16/7| -4/5    | $-(19 \mp i\sqrt{23})/16$ |
|           | -2   | -1      | $-(59 \mp i\sqrt{23})/48$ |
|           | 1    | 1       | 1               |
|           | -11/7| -1      | $-(205 \pm i\sqrt{23})/192$ |
|           | 26/7 | 3       | 3               |
|           | -41/14| -3     | $-(85 \pm i\sqrt{23})/32$ |
|           | 45/56| 3/2     | $(135 \pm i\sqrt{207})/128$ |
TABLE II. Eigenvalues $\lambda$ and eigenvectors $E$, in coupling-space for f.p. (I), \{degeneracy\}.

| $\lambda$ | $-1$ | $(2 \pm \sqrt{91})/6$ | 0 \{2\} | 1/2 \{3\} |
|-----------|------|----------------------|--------|----------|
|           | 1    | 1                    | 0      | 0        |
|           | 2    | $(14 \pm 2\sqrt{91})/3$ | 0      | 0        |
|           | 0    | 0                    | 0      | 0        |
| $E$       | $-1/2$ | $(11 \pm \sqrt{91})/6$ | 0      | 0        |
|           | 0    | 0                    | 0      | 0        |
|           | 0    | 0                    | $-9/8$ | 1        |
TABLE III. Eigenvalues $\lambda$ and eigenvectors $E$, in coupling-space for f.p. (II).

\[
\begin{array}{cccc}
\lambda & 0.606 & -1.692 & -1.001 & -0.319 \pm \text{i}1.818 \\
E & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.83 & 0.90 & 1.00 & 0 \\ 0.41 & 0.24 & 0.32 & -0.13 \pm \text{i}1.08 \\ 0 & 0 & 0 & 0 \\ 0.55 & 0.84 & 0.68 & -0.32 \pm \text{i}0.80 \\ -1.00 & 1.00 & 0.89 & -1.00 \pm \text{i}1.02 \\ -0.64 & 0.48 & 0.42 & -0.73 \pm \text{i}0.25 \end{bmatrix}
\end{array}
\]

\[
\begin{array}{cccc}
\lambda & -1.793 & 0.895 & 0.002 \\
E & \begin{bmatrix} 0.63 & 1.00 & -0.79 \\ -0.39 & -0.02 & 0.57 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.00 & 0.34 & 1.00 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}
\]
TABLE IV. Eigenvalues $\lambda$ and eigenvectors $E$, in coupling-space for f.p. (III).

| $\lambda$ | -1.45 | -1.00 | -0.65 | -0.37 $\pm$ 0.15 |
|------------|-------|-------|-------|------------------|
| $E$        |       |       |       |                  |
|            | 0.03  | -0.06 | 0.26  | 0.54 $\mp$ 0.49 |
|            | -1.00 | 1.00  | 0.08  | 1.00 $\mp$ 0.51 |
|            | -0.63 | 0.57  | 0.08  | -0.04 $\pm$ 0.21|
|            | -0.41 | 0.44  | -1.00 | 0.65 $\pm$ 1.93 |
|            | -0.48 | 0.40  | 0.14  | -0.17 $\pm$ 0.12|
|            | 0.93  | -0.68 | -0.58 | 0.48 $\pm$ 0.22 |
|            | -0.35 | 0.25  | 0.70  | -0.65 $\mp$ 0.56|
|            | -0.14 | 0.11  | -0.18 | 0.18 $\pm$ 0.25 |

| $\lambda$ | 0.31  | 0.16  | 0.04  |
|------------|-------|-------|-------|
| $E$        |       |       |       |
|            | -0.44 | -0.04 | 0.03  |
|            | -0.70 | 0.22  | -1.00 |
|            | 0.15  | 0.16  | -0.60 |
|            | 0.33  | -0.06 | 0.43  |
|            | 0.89  | 0.15  | -0.37 |
|            | -0.67 | -0.70 | 0.60  |
|            | 1.00  | 0.99  | -0.35 |
|            | -0.01 | -1.00 | -0.14 |