Quantization as a functor

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Abstract

Notwithstanding known obstructions to this idea, we formulate an attempt to turn quantization into a functorial procedure. We define a category \textbf{Poisson} of Poisson manifolds, whose objects are integrable Poisson manifolds and whose arrows are isomorphism classes of regular Weinstein dual pairs; it follows that identity arrows are symplectic groupoids, and that two objects are isomorphic in \textbf{Poisson} iff they are Morita equivalent in the sense of P. Xu. It has a subcategory \textbf{LPoisson} that has duals of integrable Lie algebroids as objects and cotangent bundles as arrows. We argue that naive $\mathbb{C}^*$-algebraic quantization should be functorial from \textbf{LPoisson} to the well-known category \textbf{KK}, whose objects are separable $\mathbb{C}^*$-algebras, and whose arrows are Kasparov’s KK-groups. This limited functoriality of quantization would already imply the Atiyah–Singer index theorem, as well as its far-reaching generalizations developed by Connes and others. In the category \textbf{KK}, isomorphism of objects implies isomorphism of K-theory groups, so that the functoriality of quantization on all of \textbf{Poisson} would imply that Morita equivalent Poisson algebras are quantized by $\mathbb{C}^*$-algebras with isomorphic K-theories. Finally, we argue that the correct codomain for the possible functoriality of quantization is the category \textbf{RKK}(I), which takes the deformation aspect of quantization into account.
“First quantization is a mystery, but second quantization is a functor” (E. Nelson)

Comme l’on sait la “quantification géométrique” consiste à rechercher un certain foncteur de la catégorie des variétés symplectiques et symplectomorphismes dans celle des espaces de Hilbert complexes et des transformations unitaires (…) Il est bien connu qu’un tel foncteur n’existe pas. (from A. Crumeyrolle’s review MR81g:58016 of [19])

1 Introduction

The functoriality of second quantization (a construction involving exponential Hilbert spaces or Fock spaces) mentioned in our opening quote of Nelson is an almost trivial matter. The deep problem suggested by this quote is the possible functoriality of “first” quantization, which simply means the quantization of Poisson manifolds $\mathcal{P}$.

The simplest example is probably $\mathcal{P} = T^*(\mathbb{R}^n)$ with the usual Poisson structure. Defining quantization either by the Schrödinger representation $U_\hbar^S$ of the Heisenberg group $H_n$ in dimension $2n$, or by the Weyl–Moyal prescription $Q_\hbar^W$ (which points of view are essentially equivalent), it follows either way that the quantization of $T^*(\mathbb{R}^n)$ is functorial with respect to affine linear symplectomorphisms and unitary intertwiners; see, e.g., [18] or [32]. Taking Weyl–Moyal quantization to be concrete, this statement specifically means that one has

$$Q_\hbar^W(f \circ L^{-1}) = U_\hbar^M(L)Q_\hbar^W(f)U_\hbar^M(L)^*,$$

where $f \in C_0^\infty(T^*(\mathbb{R}^n))$ (for simplicity), $L$ is an affine linear symplectomorphism, and $U_\hbar^M$ is the representation of the affine symplectic group composed of the metaplectic representation of the linear symplectic group and the (projective) Schrödinger representation $U_\hbar^S$ of the translation group in dimension $2n$. As is well known, the Groenewold–Van Hove theorem (cf. [20, 21] for an up-to-date treatment) precludes functoriality under a larger class of classical transformations [19]. This seems about all that is known about the functoriality of (first) quantization.

The above example has a number of instructive features. Firstly, $T^*(\mathbb{R}^n)$ has a large amount of symmetry, which is fully exploited by the Weyl–Moyal quantization prescription. The rather meager functoriality properties are a direct consequence of this symmetry. Indeed, the Berezin–Toeplitz quantization prescription on $T^*(\mathbb{R}^n)$ (relying on its Kähler structure), which is physically as acceptable as the Weyl–Moyal prescription, and is much better behaved analytically [32], enjoys even less functoriality. Since both prescriptions hinge on rather special properties of $T^*(\mathbb{R}^n)$, for the sake of generalization it would seem wise not to let the notion of functoriality of quantization rely on the precise details of a quantization prescription, but rather on a certain equivalence class to which it belongs.
Secondly, the Groenewold–Van Hove no-go theorem suggests that taking unitary transformations on the quantum side does not leave enough room to manoeuvre in the codomain category of a potential quantization functor. Hence one needs a different class of arrows at least in the quantum category. It is convenient to work with C*-algebras rather than concrete Hilbert spaces; instead of unitary operators one should then speak of *-automorphisms. For example, eq. (1) defines conjugation by $U^M_k(L)$ as a *-automorphism of the C*-algebra of compact operators on $L^2(\mathbb{R}^n)$.

Furthermore, it is extremely unnatural to only work with simple C*-algebras (like the compact operators), and once one has decided to work with general C*-algebras, it goes without saying that one should consider general Poisson manifolds, instead of merely symplectic ones. The conclusion so far, then, is that the naive idea that quantization ought to be functorial with respect to isomorphisms of Poisson manifolds and *-isomorphisms of C*-algebras, let alone the stronger requirement of functoriality with respect to Poisson maps and *-homomorphisms, respectively, has to be given up.

More suitable categories of C*-algebras necessarily have a weaker notion of isomorphism than *-isomorphism. To obtain powerful results, and also to restore a certain parallel between the classical and the quantum categories, we will accordingly use a classical category in which isomorphism of objects is weaker than isomorphism of Poisson manifolds in the usual sense.

Our original idea was that one should use Morita equivalence on both sides; for Poisson manifolds this notion was developed by Xu [59], whereas the older C*-algebraic theory is due to Rieffel [51, 52]. If one has categories in which isomorphism of objects comes down to Morita equivalence, then the possible functoriality of quantization would imply that quantization preserves Morita equivalence. Such categories are easily defined [35, 36]. On the classical side one has the category Poisson, whose objects are integrable Poisson manifolds, and whose arrows are isomorphism classes of regular Weinstein dual pairs. On the quantum side one has a category C* whose objects are C*-algebras and whose arrows are unitary equivalence classes of Hilbert bimodules (for the latter see also [56, 15]).

Apart from the (flawed) idea in the previous paragraph, the use of the categories Poisson and C* was in addition motivated by the fact that the possible functoriality of quantization as a map from Poisson to C* would imply the “quantization commutes with reduction” principle (see [12] and references therein). Hence this perhaps somewhat mysterious principle would appear in a canonical mathematical light. For let $Q \leftarrow S_1 \rightarrow P$ and $P \leftarrow S_2 \rightarrow R$ be regular Weinstein dual pairs, quantized by an $A$-$B$ Hilbert bimodule $Q(S_1)$ and a $B$-$C$ Hilbert bimodule $Q(S_2)$, respectively. Here $B_0 \cong C_0(P)$. Functoriality of quantization implies

$$Q(S_1 \oplus_P S_2) = Q(S_1) \hat{\otimes}_B Q(S_2).$$

Now composition of arrows $\oplus$ in Poisson is given by symplectic reduction, whereas the interior tensor product of Rieffel that defines arrow composition
\( \odot \) in \( C^* \) is a quantized version of the classical reduction procedure \([31, 32, 35]\). Hence the left-hand side is “quantization after reduction,” whereas the right-hand side stands for “reduction after quantization.”

As will be recalled below, \( \text{Poisson} \) has a subcategory \( \text{LPoisson} \) whose objects are duals of integrable Lie algebroids, and whose arrows are cotangent bundles. The point is that there indeed exists a functor from \( \text{LPoisson} \) to \( C^* \) resembling quantization on the object side, so that Morita equivalent Poisson manifolds in \( \text{LPoisson} \) are quantized by Morita equivalent \( C^* \)-algebras. Although this is a nontrivial result, it was pointed out to the author by Alan Weinstein that it is rather untypical, and that it would be mistake to conjecture an extension of this result to all (integrable) Poisson manifolds (as we did in a previous draft of this paper, of which Weinstein was the referee). He actually pointed out a class of counterexamples, as follows. Take any two tori of the same (even) dimension, but carrying different symplectic structures. These will always be Morita equivalent as Poisson manifolds \([59]\). However, one can easily choose the symplectic structures in such a way that their respective quantizations (as defined in \([53, 54]\)) fail to be Morita equivalent as \( C^* \)-algebras (for any value of \( \hbar \)); cf. \([55]\).

These counterexamples show that Morita equivalence is still not coarse enough on the quantum side to be preserved by quantization, and suggests that it might be more appropriate to use K-equivalence of \( C^* \)-algebras (i.e., isomorphism of K-groups). The natural codomain of a possible quantization functor is then clearly the category \( \text{KK} \), whose objects are separable \( C^* \)-algebras, and whose arrows are Kasparov’s KK-groups \([8, 26]\). Although isomorphism of objects in \( \text{KK} \) is not the same as isomorphism of their K-groups, the latter is implied by the former, and the category \( \text{KK} \) has the enormous computational advantage (for example, over \( C^* \)) that the Hom-spaces \( KK(A, B) \) are abelian groups. The results in \([36, 24]\) then strongly suggest that quantization should be functorial from \( \text{LPoisson} \) to \( \text{KK} \).

It should be mentioned that even this limited functoriality of quantization would already imply the Atiyah–Singer index theorem as well as its generalization to foliations due to Connes \([8, 10]\). Moreover, it further motivates the generalization of the latter to a general index theorem for Lie groupoids called for in \([57]\). One can only marvel at the possible implications that the complete functoriality of quantization would have.

The use of \( \text{KK} \) is still unsatisfactory, in that its objects are single \( C^* \)-algebras; one effectively works at some fixed value of Planck’s constant \( h \) (like in geometric quantization). In the spirit of deformation quantization, it is much better to use continuous fields of \( C^* \)-algebras as the target of the quantization operation, as first proposed by Rieffel \([53]\). This suggests the use of the category \( \text{RKK}(I) \) as the codomain of a possible quantization functor. For technical reasons this category has upper semicontinuous fields of separable \( C^* \)-algebras over the interval \( I = [0, 1] \) (seen as the parameter space of \( \hbar \)) as objects, and the so-called representable KK-groups \( \mathcal{R}KK(I, -,-) \) \([27]\) as arrows.

The plan of this paper is as follows. In Section \([\text{2}]\) we review the construction of the “classical” category \( \text{Poisson} \) \([32, 30]\), which is the domain of the alleged
quantization functor in any of the approaches we discuss. Section 3 describes the simplest candidate $C^*$ for the codomain category of this functor \cite{15, 35, 50}. This category also lies at the basis of the construction of the more sophisticated codomains used further on. In Section 4 we prove that quantization is functorial from the subcategory $L\text{Poisson}$ of $\text{Poisson}$ to $C^*$. Section 5 recalls Kasparov’s category $\text{KK}$ \cite{3, 26}, and refines the previous result so as to apply to $\text{KK}$ rather than $C^*$. In Section 6 we turn to deformation quantization. In particular, we indicate how the original ideas of formal deformation quantization \cite{2} can be realized in the context of $C^*$-algebras, so as to motivate both Rieffel’s axioms for $C^*$-algebraic quantization \cite{53} and the author’s modification of these \cite{32}. As outlined in Section 7, these considerations immediately lead to a refinement $C^*(I)$ of the category $C^*$, in such a way that the step from $C^*$ to $\text{KK}$ is analogous to the passage from $C^*(I)$ to a category $R\text{KK}(I)$. The latter, then, is our proposal for the codomain of a potential functorial quantization procedure. Thus quantization should be a functor from $\text{Poisson}$ to $R\text{KK}(I)$.

Finally, let us note that, in view of the audience towards which these lectures were directed, some definitions are given in greater detail than others. Most participants were familiar with the likes of derived categories and symplectic groupoids, whereas elementary knowledge of operator algebras seemed lacking.

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2 The classical category

We recall the definition of the category of integrable Poisson manifolds introduced in \cite{35, 36}. This category relies on the theory of symplectic groupoids (cf. \cite{7, 11} and refs. therein, as well as the forthcoming 2nd edition of \cite{40}). The objects in $\text{Poisson}$ satisfy the following condition.

Definition 1 A Poisson manifold $P$ is called integrable when there exists a symplectic groupoid over $P$.

This definition is due to \cite{11}. The integrability assumption is necessary in order to have identities in $\text{Poisson}$; see below. In this paper we assume for simplicity
that the symplectic groupoid in question is Hausdorff. This assumption can be dropped at the expense of considerable technical complications, which can be overcome if the definition of continuous or smooth functions on non-Hausdorff manifolds in [13] is used.

The arrows in Poisson are isomorphism classes of certain Weinstein dual pairs. Recall that, given two Poisson manifolds \( P \) and \( Q \), a Weinstein dual pair \( Q \leftarrow S \rightarrow P \), simply called a dual pair in what follows, consists of a symplectic manifold \( S \) and Poisson maps \( q : S \rightarrow Q \) and \( p : S \rightarrow P \), such that \( \{q^*f, p^*g\} = 0 \) for all \( f \in C^\infty(Q) \) and \( g \in C^\infty(P) \) [23, 58]. Two Q-P dual pairs \( Q \leftarrow S_1 \rightarrow P, i = 1, 2, \) are isomorphic when there is a symplectomorphism \( \varphi : S_1 \rightarrow S_2 \) for which \( q_2 \varphi = q_1 \) and \( p_2 \varphi = p_1 \).

The notion of regularity for dual pairs is explained in [35, 36]; its goal is to guarantee the existence of the following symplectic quotients. Part of the regularity condition is the stipulation that the maps \( p \) and \( q \) be complete, and that \( q \) is a surjective submersion. Let \( R \) be a third integrable Poisson manifold, and let \( Q \leftarrow S_1 \rightarrow P \) and \( P \leftarrow S_2 \rightarrow R \) be regular dual pairs. The embedding \( S_1 \times_P S_2 \subset S_1 \times S_2 \) is coisotropic [32]; we denote the corresponding symplectic quotient by \( S_1 \circ P S_2 \). This is the middle space of a regular dual pair \( P \leftarrow S_1 \circ_P S_2 \rightarrow R \). The operation \( \circ \) is associative up to isomorphism.

For suitable choices of dual pairs, the product \( \circ \) is the same as Marsden–Weinstein reduction [32]; this should not be surprising in view of its general definition in terms of symplectic reduction.

Using results in [42] and [11], it can be shown that if \( P \) is integrable, then there exists an \( s \)-connected and \( s \)-simply connected symplectic groupoid \( \Gamma(P) \) whose base space is isomorphic to \( P \) as a Poisson manifold. Moreover, \( \Gamma(P) \) is unique up to isomorphism of symplectic groupoids. Cf. Lemma 5.6 in [35]. The upshot of this is that the isomorphism class \([P \leftarrow \Gamma(P) \rightarrow P]\) is a two-sided identity for \( \circ_P \). (We denote the source and target maps in a groupoid by \( s \) and \( t \), respectively.)

**Definition 2** The category Poisson has integrable Poisson manifolds as objects, and isomorphism classes of regular dual pairs as arrows.

The original reason for the introduction of this category was the fact that two Poisson manifolds are Morita equivalent in the sense of Xu [59] iff they are isomorphic objects in Poisson; see Prop. 5.13 in [35]. In particular, a Poisson manifold is integrable iff it is Morita equivalent to itself (as already observed by Xu). The category Poisson is a classical analogue of the category of \( C^* \)-algebras with unitary equivalence classes of Hilbert bimodules as arrows [33, 30].

We now introduce a subcategory LPoisson of Poisson on which we will be able to define a quantization functor. This subcategory is not full, though in an informal sense it is large and interesting. Recall that a Lie groupoid \( G \) over \( G_0 \) has an associated Lie algebroid \( A(G) \), which is a vector bundle over \( G_0 \) [41]. The dual vector bundle \( A^*(G) \) is equipped with a canonical Poisson structure \([11, 12]\) (also cf. [11, 32] for a review). This Poisson structure is linear, in that the Poisson bracket of two (fiberwise) linear functions is again linear. Conversely,
any linear Poisson structure is the dual of some Lie algebroid [12] (but this Lie
algebroid need not be integrable). Poisson manifolds of the form $A^\ast(G)$ include
all cotangent bundles, all duals of Lie algebras, all manifolds with zero Poisson
bracket, all semidirect product Poisson structures, and all Poisson manifolds
defined by a foliation.

The objects of $LPoisson$ are the Poisson manifolds $A^\ast(G)$ associated to ar-
bitrary Lie groupoids $G$. The arrows in $LPoisson$ are isomorphism classes of
regular dual pairs that are of the following form. Let $G$ and $H$ be Lie groupoids,
and suppose that a manifold $M$ is a $G$-$H$ bibundle; we write $G \leftarrow M \rightarrow H$.
This means that $G$ and $H$ act smoothly on $M$ on the left and on the right,
respectively, in such a way that the actions commute; cf. [35, 42, 44, 45]. A
construction in [34], generalizing the momentum map of symplectic geo-
mery from Lie groups to Lie groupoids, associates a dual pair
\[
A^\ast(G) \overset{J_L}{\leftarrow} T^\ast(M) \overset{J_R}{\rightarrow} A^\ast(H)
\] (2)
to such a bibundle. For a dual pair of this form to be regular, it suffices that the
bibundle be principal [12, 14, 15] (also see [35] for a review); this means that the
base map $\pi : M \rightarrow G_0$ of the $G$-action on $M$ is a surjective submersion, and that
$H$ acts freely and transitively on the fibers of $\pi$. It follows that $M/H \cong G_0$.
In foliation theory principal bibundles are seen as generalized maps between
leaf spaces (see, e.g., [24, 44]), and, more generally, principal bibundles are
sometimes called generalized maps between groupoids.

For example, the canonical $G$-$G$ bibundle $G \leftarrow G \rightarrow G$ gives rise to the dual pair
\[
A^\ast(G) \overset{\epsilon}{\leftarrow} T^\ast(G) \overset{\pi}{\rightarrow} A^\ast(G),
\] (3)
where $T^\ast(G)$ is the cotangent bundle defined in [11]. This is precisely the
symplectic groupoid associated to the Poisson manifold $A^\ast(G)$.

Let $LG$ be the category of Lie groupoids, whose arrows are isomorphism
classes of principal bibundles (see [33, 34, 35]). Composition of arrows is defined
as follows. Suppose one has principal bibundles $G \rightarrow M \leftarrow H$ and $H \rightarrow N \leftarrow K$. The fiber product $M \times_{H_0} N$ carries a right $H$ action, given by
$h : (m, n) \mapsto (mh, h^{-1} n)$ (defined as appropriate). The orbit space
\[
M \circledast H N = (M \times_H N)/H
\] (4)
is a $G$-$K$ bibundle in the obvious way. This defines a product on matched
bibundles, which becomes associative on isomorphism classes. We define $LG$ as
the full subcategory of $LG$ whose objects are $s$-connected and $s$-simply connected
Lie groupoids.

According to Thm. 3 and eq. (4.30) in [36], the above procedure defines
a functor $A^\ast$ from $LG$ to $Poisson$. That is, on objects one has $G \rightarrow A^\ast(G)$,
whereas on arrows the functor in question maps the isomorphism class of a $G$-
$H$ bibundle $M$ to the isomorphism class of the dual pair [2]. The operation $A^\ast$
may also be defined on $LG$, but it fails to be functorial because identities are not
always preserved. Note that $A^*$ indeed maps $\tilde{L}G$ into Poisson: $A^*(G)$ is actually integrable, with associated symplectic groupoid $T^*(G)$.

**Definition 3** The category $\text{LPoisson}$ is the image of the functor

$$A^*: \tilde{L}G \to \text{Poisson}.$$ 

Thus $\text{LPoisson}$ has Poisson manifolds of the form $A^*(G)$, where $G$ is a Lie groupoid, as objects, and isomorphism classes of cotangent bundles of the form \((\mathcal{M})\), where $M$ is a principal bibundle, as arrows. Note that $\text{LPoisson}$ contains all identities as appropriate, since the symplectic groupoid $T^*(G)$ is $s$-connected and $s$-simply connected whenever $G$ is.

### 3 The simplest quantum category

Within the Hilbert space formalism for quantum mechanics, it is natural to assume that the observables of a quantum system form a $C^*$-algebra [22, 32]. Recall that a $C^*$-algebra $A$ is a complex associative algebra with involution, equipped with the structure of a Banach space, such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in A$. A $C^*$-algebra can always be faithfully represented as a norm-closed involutive algebra of bounded operators on a Hilbert space, on which the involution is just the adjoint, and the norm is the usual operator norm. See [9] for an overview of the use of $C^*$-algebras in modern mathematics.

At some fixed value of Planck’s constant $\hbar$, quantization then associates a $C^*$-algebra to a Poisson algebra. This view of quantization is not satisfactory, and ought to be replaced by the idea of deformation quantization, but for the moment we stick to it for pedagogical reasons. Thus, to a first approximation, the objects of the codomain category of a possible quantization functor should be $C^*$-algebras. In order to assemble these into a category, the most obvious choice would be to take the arrows to be *-homomorphisms, but the pertinent isomorphisms would then be *-isomorphisms. As mentioned in the Introduction, this choice is inappropriate for quantization theory. A more suitable class of arrows between $C^*$-algebras is formed by (isomorphism classes of) so-called Hilbert bimodules.

A Hilbert bimodule is the $C^*$-algebraic analogue of a bimodule for algebras over a given ring [1, 2]. A new feature compared to the purely algebraic situation is that an $A$-$B$ Hilbert bimodule is endowed with a $B$-valued inner product. The complete definition is as follows.

**Definition 4** Let $A$ and $B$ be $C^*$-algebras. An $A$-$B$ Hilbert bimodule is an $A$-$B$ bimodule $E$ (where $A$ and $B$ are seen as complex algebras, so that $E$ is a complex linear space) with a compatible $B$-valued inner product. Thus there is a sesquilinear map $\langle \cdot, \cdot \rangle : E \times E \to B$, linear in the second and antilinear in the first entry, satisfying $\langle x, y \rangle^* = \langle y, x \rangle$, $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ iff $x = 0$. The compatibility of the inner product with the remaining structures means that firstly $E$ has to be complete in the norm $\|x\|^2 = \|\langle x, x \rangle\|$, secondly that $\langle x, yb \rangle = \langle x, y \rangle b$, and thirdly that $\langle a^*x, y \rangle = \langle x, ay \rangle$ for all $x, y \in E$, $b \in B$. 


and \( a \in A \). Finally, the left action of \( A \) on \( E \) is required to be nondegenerate in the sense that \( AE \) is dense in \( E \).

Note that \( A \)-\( C \) Hilbert bimodules are just Hilbert spaces carrying a nondegenerate representation of \( A \). On the other hand, a \( C \)-\( B \) Hilbert bimodule under the obvious action of \( C \) (by multiples of the unit operator) is called a Hilbert \( B \) module.

The basic example of an \( A \)-\( A \) Hilbert bimodule is \( E = A \) with the obvious actions and the inner product \( \langle a, b \rangle = a^* b \). See \[30\] for the basic theory, cf. \[35\] for a comparison between Hilbert bimodules and analogous structures in mathematics, and have a look at \[32\] for the use of Hilbert bimodule in quantization theory. One feature of algebraic bimodules that survives in the Hilbert case is the existence of a bimodule tensor product \[51, 52\]: from an \( A \)-\( B \) Hilbert bimodule \( E \) and a \( B \)-\( C \) Hilbert bimodule \( \tilde{E} \) one can form an \( A \)-\( C \) Hilbert bimodule \( E \hat{\otimes}_B \tilde{E} \), called the interior tensor product of \( E \) and \( \tilde{E} \). For the following definition \[15, 35, 56\] we also need the notion of unitary equivalence, which the reader may guess (see \[30\], p. 24).

**Definition 5** The category \( C^* \) has \( C^* \)-algebras as objects, and unitary equivalence classes of Hilbert bimodules as arrows. The arrows are composed by the interior tensor product.

It follows that the identity arrow at \( A \) is the canonical Hilbert bimodule \( A \) defined above. In addition, it turns out that two \( C^* \)-algebras are Morita equivalent \[52\] iff they are isomorphic as objects in \( C^* \) \[17, 35, 56\]. This property suggests that the category \( C^* \) should be regarded as a quantum version of Poisson.

## 4 Functoriality of simple quantization

We are now in a position to state the first result on the functoriality of quantization.

**Theorem 1** There exists a functor \( Q : \text{LPoisson} \rightarrow C^* \) that on objects maps \( A^*(G) \) to \( C^*(I) \).

The reason why \( C^*(G) \) may indeed be seen as the quantization of \( A^*(G) \) at some fixed value of \( \hbar \) is actually to be found in deformation quantization \[32, 33, 37, 49\].

**Proof.** The functor \( Q \) is the composition of the following functors:

\[
\text{LPoisson} \overset{(A^*)^{-1}}{\longrightarrow} \text{LG} \overset{C^*}{\longrightarrow} C^*(I). \tag{5}
\]

We discuss the functors \((A^*)^{-1}\) and \(C^*\) in turn.

Firstly, it follows from Props. 3.3 and 3.5 in \[43\] that for \( s \)-connected and \( s \)-simply connected Lie groupoids \( G \) the association \( G \mapsto A^*(G) \) is invertible. Hence \( A^* : \text{LG} \rightarrow \text{LPoisson} \) is an isomorphism of categories, with inverse \((A^*)^{-1}\).
(It would have been sufficient for \( A^* : LG \to LPoisson \) to define an equivalence of categories, for even then it would possess an inverse up to natural isomorphism, which would be enough for our purposes.)

Secondly, on objects the map \( G \mapsto C^*(G) \) from Lie groupoids to \( C^* \)-algebras is the well-known association of a convolution \( C^* \)-algebra to a Lie groupoid \( \mathfrak{g} \) (or, more generally, to a locally compact groupoid with Haar system \( [50] \)). Following a special case in \( [46] \) (in which the arrows were taken to be Morita equivalences of groupoids), the map \( G \mapsto C^*(G) \) was extended to a functor from \( LG \) to the category \( C^* \) in \( [36] \).

It follows that \( Q = C^* \circ (A^*)^{-1} : LPoisson \to C^* \) is a functor. ■

We can illustrate this result by noting that an identity \((3)\) in \( LPoisson \) is mapped to the canonical \( C^*(G)-C^*(G) \) Hilbert bimodule \( C^*(G) \). Hence the symplectic manifold \( T^*(G) \) and the Poisson manifold \( A^*(G) \) are both mapped into the \( C^* \)-algebra \( C^*(G) \), but \( T^*(G) \) is seen as an arrow, and \( A^*(G) \) is regarded as an object in \( LPoisson \), so that the former is mapped to \( C^*(G) \) as (the middle space of) a Hilbert bimodule, whereas the latter is sent to \( C^*(G) \) seen as a \( C^* \)-algebra.

5 The category \( KK \)

As mentioned in the Introduction, the preceding theorem cannot be extended to all of \( Poisson \), because noncommutative tori provide counterexamples. We therefore propose to replace the category \( C^* \) by an analogous category \( KK \), in which at least these counterexamples are circumvented, and whose use is very attractive in many ways. Perhaps the main motivation for looking at quantization as a functor taking values in \( KK \) is that the well-known relationship between quantum mechanics and index theory \([16, 17, 57]\) would be clarified by such functoriality.

The category \( KK \) emerged from Kasparov’s work on K-theory \([26]\), and is discussed in detail in \([3]\). Here we only sketch the main points that allow one to understand that \( KK \) is a subtle and deep modification of \( C^* \). Given two separable \( C^* \)-algebras \( A \) and \( B \), one defines \( E(A, B) \) as the collection of \( A \)-\( B \) Hilbert bimodules \( E \) that are countably generated in \( B \), and are equipped with the following additional structure.

Firstly, \( E \) should be of the form \( E = E_1 \oplus E_2 \), where each \( E_i \) is an \( A \)-\( B \) Hilbert bimodule. Secondly, and this is the main feature, there should be an operator \( F : E \to E \) that is adjointable (i.e., there is \( F^* : E \to E \) such that \( \langle F^*x, y \rangle = \langle x, Fy \rangle \)) and odd (in that \( F(E_1) \subseteq E_2 \) and \( F(E_2) \subseteq E_1 \)). This \( F \) should be an almost unitary intertwiner of \( A \mid E_1 \) and \( A \mid E_2 \), in that for each \( a \in A \) the operators \( [F, a] \), \( (F^2 - 1)a \), and \( (F - F^*)a \) be compact. (Here an operator on an \( A \)-\( B \) Hilbert bimodule \( E \) is said to be compact when it can be approximated in norm by linear combinations of rank one operators of the form \( z \mapsto x(y, z) \) for \( x, y \in E \). In noncommutative geometry, compact operators are treated as infinitesimals \([3]\).)

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This setting was mainly motivated by index theory, in which $A = C(X)$ for some compact manifold $X$, $B = \mathbb{C}$, $E$ is the Hilbert space of $L^2$-sections of some spinor bundle over $X$, and $F$ is a pseudodifferential operator; see [23]. In general, the category $\mathbb{C}^*$ is enormously enriched by requiring the presence of $F$. However, when the $A$ action on some $A$-$B$ Hilbert bimodule $E_1$ happens to be by compact operators, one may choose $E = E_1 \oplus 0$ and $F = 0$ so as to obtain an element of $E(A, B)$. Such elements may even survive the step to be explained next.

Elements $(E, F)$ of $E(A, B)$ are called Kasparov cycles. Elements of $KK(A, B)$ are equivalence classes of such cycles under the following relations: unitary equivalence, translation of $F$ along norm-continuous path, and the addition of degenerate Kasparov cycles. The latter are those for which the operators $[F, a]$, $(F^2 - 1)a$, and $(F - F^*)a$ are 0 for all $a$. The ensuing equivalence relation may be reexpressed in a number of alternative forms [3].

It is not difficult to see that $KK(A, B)$ is an abelian group; the group operation is the direct sum of both bimodules and operators $F$, and the inverse of the class of a Kasparov cycle $(E, F)$ is the class of $(E^{op}, -F)$ (where $E^{op}$ is $E$ with the opposite grading). Moreover, the association $(A, B) \mapsto KK(A, B)$ is contravariantly functorial in the first entry, and covariantly functorial in the second. One recovers $K$-theory as a special case of $KK$-theory by $K_0(A) \cong KK(\mathbb{C}, A)$, with topological $K$-theory as the special case $K^0(X) \cong K_0(C(X))$ whereas $K$-homology [23] emerges as $K^0(A) \cong KK(A, \mathbb{C})$.

The deepest aspect of Kasparov’s theory is the existence of the so-called intersection product

$$KK(A, B) \times KK(B, C) \to KK(A, C),$$

which is functorial in all conceivable ways. This leads to the category $KK$, whose objects are separable $\mathbb{C}^*$-algebras, and whose arrows are the $KK$-groups. More precisely, the Hom-space of arrows from $B$ to $A$ is $KK(A, B)$. In addition, $KK(A, B)$ defines a space of homomorphisms from $K_0(A)$ to $K_0(B)$ through the intersection product

$$KK(\mathbb{C}, A) \times KK(A, B) \to KK(\mathbb{C}, B) \cong K_0(B).$$

In particular, if two $\mathbb{C}^*$-algebras are isomorphic in $KK$, then their $K$-groups are isomorphic.

Refining Theorem [1], we now conjecture that there exists a functor from Poisson to $KK$ that on objects maps $A^*(G)$ to $C^*(G)$. The evidence for this idea comes from the noncommutative geometry approach to index theory [8] [3] [10], as follows. The crucial step in the proof of the $K$-theoretic version of the Atiyah–Singer index theorem [1] is the association of a Gysin or wrong-way map $f! : K^0(X) \to K^0(Y)$ to a continuous (and usually smooth) map $f : X \to Y$ between locally compact spaces (usually manifolds) $X$ and $Y$. For example, an embedding $M \hookrightarrow \mathbb{R}^n$ with pullback $T^*(M) \hookrightarrow \mathbb{R}^{2n}$ induces a map $K^0(T^*(M)) \to K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}$, which is the topological index of Atiyah and Singer [1]. In $KK$-theory, $f!$ is seen as an element of $KK(C_0(X), C_0(Y))$,
inducing the map between $K^0(X)$ and $K^0(Y)$ through the intersection product as explained above. The proof of the index theorem hinges on the property

$$(g \circ f)! = f! g!,$$  \hspace{2cm} (6)$$

where the right-hand side is given by the intersection product.

To generalize this, it is useful to regard a space $X$ as a groupoid (in which $X$ is both the base and the total space of the groupoid, and the source, target, and object inclusion maps are all equal to the identity map), so that $C_0(X)$ is the $C^*$-algebra $C^*(X)$ of the groupoid $X$. One may then attempt to generalize the setting of the preceding paragraph to construct a functor from $LG$ to $KK$. In other words, an object $G$ is mapped into $C^*(G)$, and a principal $G$-$H$ bibundle, which we now call $F$ with some abuse of notation, is mapped into $f! \in KK(C^*(G), C^*(H))$. Then (6) is to hold, along with the preservation of identities.

For the longitudinal index theorem for foliations \cite{8, 9, 10} it is sufficient to do this for the case that $G$ is a space $X$ and $H$ is the holonomy groupoid of a foliation. The more symmetric case that $G$ and $H$ are both holonomy groupoids was treated in \cite{24}; in both situations one has to impose an additional technical condition (of $K$-orientability) on $F$. The case that $G$ and $H$ are both arbitrary Lie groupoids has not been treated yet in the literature, but this should be possible. The ensuing functor from $LG$ to $KK$ could then be composed with the functor $(A^*)^{-1}$ from the proof of Theorem \cite{1} to obtain the desired functor from $L\text{Poisson}$ to $KK$. Our conjecture then asks for an extension of this functor from $L\text{Poisson}$ to Poisson.

6 From formal to $C^*$-algebraic deformation quantization

As mentioned in the Introduction, the use of the category $KK$ is still unsatisfactory. Its objects are single $C^*$-algebras, describing quantum-mechanical algebras of observables at some fixed value of $\hbar$. However, in the context of quantization theory it is important to study quantum theory for a range of values of Planck’s “constant” $\hbar$, and to control the classical limit. This can be done in a purely algebraic way \cite{2}, or in an analytic $C^*$-algebraic way, as first proposed by Rieffel \cite{53} (also cf. \cite{32}).

We start with some remarks on the purely algebraic approach, called formal deformation quantization or star-product quantization, which serve the purpose of stressing the analogy between formal and $C^*$-algebraic deformation quantization.

A star-product on a Poisson manifold $P$ endows the free module

$$C^\infty(P)[[\hbar]] = C^\infty(P, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$$

over the commutative ring $\mathbb{C}[[\hbar]]$ of complex formal power series in one variable with the structure of an associative unital algebra over $\mathbb{C}[[\hbar]]$ (whose product is
conventionally written as $*$), in such a way that
\[ C^\infty(P)[[\hbar]]/\hbar C^\infty(P)[[\hbar]] \cong C^\infty(P) \]
as algebras over $\mathbb{C}$ (so that $f \ast g - g \ast f = 0$ in $C^\infty(P)[[\hbar]]/\hbar C^\infty(P)[[\hbar]]$), and that Dirac’s condition
\[ f \ast g - g \ast f + i\hbar\{f, g\} = 0 \]
holds in $C^\infty(P)[[\hbar]]/\hbar^2 C^\infty(P)[[\hbar]]$. To state these axioms, it is crucial that there is a canonical map $f \mapsto f = f + 0 \cdot \hbar + 0 \cdot \hbar^2 + \cdots$ from $C^\infty(P)$ to $C^\infty(P)[[\hbar]]$. Now a unital algebra $A$ over $\mathbb{C}[[\hbar]]$ is nothing but a $\mathbb{C}[[\hbar]]$-algebra $A$ in the sense that there is an injective ring homomorphism from $\mathbb{C}[[\hbar]]$ into the center of $A$; cf. [32, p. 121]. Hence, one could define a generalized star-product on a Poisson manifold $P$ as an associative unital $\mathbb{C}[[\hbar]]$-algebra $A$ such that
\begin{enumerate}
\item $A/\hbar A \cong C^\infty(P)$ as algebras over $\mathbb{C}$;
\item there is a cross-section $Q : C^\infty(P) \to A$ of the canonical projection $\pi : A \to A/\hbar A$ for which Dirac’s condition holds in the sense that
\[ Q(f) \ast Q(g) - Q(g) \ast Q(f) + i\hbar Q(\{f, g\}) = 0 \]
in $A/\hbar^2 A$.
\end{enumerate}

Rieffel’s analytic approach [53], based on the use of continuous fields of $C^*$-algebras, was a direct analogue of the original definition of a star-product, in that his fiber algebras $A_\hbar$ were obtained by putting an $\hbar$-dependent product $\ast_\hbar$ as well as an $\hbar$-dependent norm $\| \cdot \|_\hbar$ on $C^\infty(P)$ (assuming, for simplicity, that $P$ is compact), and completing. Hence also here one has a canonical map $f \mapsto f$, this time from $C^\infty(P)$ to $A_\hbar$ (for each value of $\hbar$), in terms of which Rieffel formulated Dirac’s condition as
\[ \lim_{\hbar \to 0} \frac{i}{\hbar} (f \ast_\hbar g - g \ast_\hbar f - \{f, g\})|_\hbar = 0. \]
It was subsequently realized that more general continuous fields of $C^*$-algebras were needed in order to incorporate examples related to Berezin–Toeplitz quantization; cf. [38, p. 121] and references therein. In the present context, such fields are best described using the formalism of $C(X)$ $C^*$-algebras, which we now recall.

The following definition is due to Kasparov [27] (in the more general case of locally compact $X$). We will only need the case $X = I$.

**Definition 6** Let $X$ be a compact Hausdorff space. A $C(X)$ $C^*$-algebra is a $C^*$-algebra $A$ with a unital embedding of $C(X)$ in the center of its multiplier algebra. In other words, $A$ comes equipped with a unital injective $^*$-homomorphism $C(X) \to \mathcal{Z}(\mathcal{M}(A))$.

The structure of $C(X)$ $C^*$-algebras was fully clarified by Nilsen [17], as follows. A field of $C^*$-algebras is a triple $(X, \{A_x\}_{x \in X}, A)$, where $\{A_x\}_{x \in X}$ is some family of $C^*$-algebras indexed by $X$, and $A$ is a family of sections (that is, maps $f : X \to \coprod_{x \in X} A_x$ for which $f(x) \in A_x$) that is
1. a $C^*$-algebra under pointwise operations and the natural norm
\[ \|f\| = \sup_{x \in X} \|f(x)\|_{A_x}; \]

2. closed under multiplication by $C(X)$;

3. full, in that for each $x \in X$ one has \{ $f(x) \mid f \in \Gamma$ \} = $A_x$.

The field is said to be continuous when for each $f \in A$ the function $x \mapsto \|f(x)\|$ is in $C(X)$ (this is equivalent to the corresponding definition of Dixmier [14]; cf. [28]). The field is upper semicontinuous when for each $f \in A$ and each $\varepsilon > 0$ the set \{ $x \in X \mid \|f(x)\| \geq \varepsilon$ \} is compact.

Thm. 2.3 in [47] now states that a $C(X) C^*$-algebra $A$ defines a unique upper semicontinuous field of $C^*$-algebras
\[(X, \{ A_x = A/C(X, x)A \}_{x \in X}, A). \]

Here
\[ C(X, x) = \{ f \in C(X) \mid f(x) = 0 \}, \]
and, with abuse of notation, $f \in A$ is identified with the section
\[ x \mapsto \pi_x(f), \]
where $\pi_x : A \to A_x$ is the canonical projection.

Moreover, Blanchard [5] proved that a $C(X) C^*$-algebra $A$ defines a continuous field of $C^*$-algebras whenever the map $x \mapsto \|\pi_x(f)\|$ is continuous for each $f \in A$. Thus a continuous field of $C^*$-algebras over $X$ may be described as a $C(X) C^*$-algebra with this additional continuity condition.

It should be noted that, unlike vector bundles, continuous fields of $C^*$-algebras may well fail to be locally trivial. Here the restriction of a continuous field $(X, \{ A_x \}_{x \in X}, A)$ to some open subset $Y \subset X$ is said to be trivial when $A_x = B$ for all $x \in Y$, and $A$ contains $C_0(Y, B)$. In $C^*$-algebraic deformation quantization, where $X = I$, both the situation that the field is trivial at $(0, 1]$ and the case that all fiber algebras $A_\hbar$ are pairwise non-isomorphic occur! The former happens, for example, in Weyl–Moyal quantization and its generalizations, whereas the latter occurs for certain noncommutative tori.

In any case, we see that a continuous field of $C^*$-algebras over the interval $I$ is nothing but a $C(I) C^*$-algebra $A$ with an additional continuity property. Hence, in analogy with the notion of a generalized star-product introduced above, we may reformulate Def. II.1.2.5 in [32] as follows.

**Definition 7** A strict quantization of a Poisson manifold $P$ is a $C(I) C^*$-algebra $A$ such that

1. $A_0 = A/C(I, 0)A \cong C_0(P)$ as $C^*$-algebras;

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2. There exists a cross-section $Q$ of $\pi_0$, defined on a suitable Poisson subalgebra of $C_0(P)$, such that, in terms of $Q_h = \pi_h \circ Q$,

$$\lim_{h \to 0} \frac{i}{\hbar} (Q_h(f)Q_h(g) - Q_h(g)Q_h(f)) - Q_h(\{f, g\}) = 0;$$

3. For each $A \in A$, the function $h \mapsto \|\pi_h(A)\|$ from $I$ to $\mathbb{R}^+$ is continuous.

Here the norm and the product are taken in $A_h$.

The naive quantization of $A^*(G)$ by $C^*(G)$ can be amplified into a strict quantization \cite{22, 23, 25, 29}. The $C(I)$ $C^*$-algebra quantizing $A^*(G)$ turns out to be $C^*(G_T)$, where $G_T$ is the generalized tangent groupoid of $G$ \cite{24}.

7 The category RKK

We now define categories $C^*(I)$ and RKK, which are the appropriate substitutes of $C^*$ and KK, respectively, if one works with $C(I)$ $C^*$-algebras rather than merely with $C^*$-algebras. Our goal being a quantization functor with codomain RKK, for pedagogical reasons we first introduce $C^*(I)$.

We first generalize a definition of Blanchard \cite{1}, who considered the case $B = C(X)$ (also cf. \cite{39}).

**Definition 8** Let $A$ and $B$ be $C(X)$ $C^*$-algebras. An $A$-$B$ $C(X)$ Hilbert bimodule is an $A$-$B$ Hilbert bimodule for which the $A$ action is $C(X)$-linear.

The $C(X)$-linearity means the following: since the left action of $A$ on $E$ and the right action of $B$ on $E$ are both nondegenerate, they extend to the respective multiplier algebras, so that a priori one obtains two different actions of $C(X)$ on $E$, coming from $A$ and $B$ seen as $C(X)$ $C^*$-algebras. These actions must coincide. Consequently, one obtains a field $(E_x)_{x \in X}$ of $A_x$-$B_x$ Hilbert bimodules, where

$$E_x = E \hat{\otimes}_B B_x$$

is the interior tensor product of $E$ (as an $A$-$B$ Hilbert bimodule) and $B_x$ (as a $B$-$B_x$ Hilbert bimodule). The left action of $B$ on $B_x$ is defined through $\pi_x : B \to B_x$ and left multiplication, the right action of $B_x$ on itself is given by right multiplication, and the $B_x$-valued inner product on $B_x$ is $\langle A, B \rangle = A^*B$ as usual. The left action of $A_x$ on $E_x$ is well defined because of the $C(X)$-linearity of the given $A$ action on $E$. Thus one may think of an $A$-$B$ $C(X)$ Hilbert bimodule as a field of $A_x$-$B_x$ Hilbert bimodules with certain continuity properties following from the above definition; the special case $B = C(X)$ (so that $B_x = \mathbb{C}$) considered in \cite{1} corresponds to a field of nondegenerate representations of $A_x$ on a field of Hilbert spaces over $X$.

**Definition 9** Let $X$ be a compact Hausdorff space. The objects of the category $C^*(X)$ are $C(X)$ $C^*$-algebras. The arrows are unitary isomorphism classes of $C(X)$ Hilbert bimodules. Matched arrows are composed through Rieffel’s interior tensor product. The identity arrow $1_A$ at an object $A$ is the class of $A$, seen as the canonical $A$-$A$ Hilbert bimodule.
When $X$ is a point, one recovers the category $C^*$. When $X = I$, one has the category $C^*(I)$. The categories $C^*(X)$ are the appropriate $C^*$-algebraic analogues of the categories $k$-Alg in the purely algebraic setting, where $k$ is a commutative ring (cf. [35], Sect. 2.1): the objects of $k$-Alg are associative unital algebras over $k$, and the arrows are isomorphism classes of bimodules, composed using the obvious tensor product. In particular, $C^*(I)$ is the $C^*$-algebraic counterpart of the category $C[[\hbar]]$-Alg. It can be shown that two $C(X) C^*$-algebras are isomorphic as objects in $C^*(X)$ iff they are Morita equivalent as $C(X) C^*$-algebras (this means that they are Morita equivalent through an imprimitivity bimodule that is also a $C(X)$ Hilbert bimodule). The purely algebraic counterpart of this result is that two objects are isomorphic in $k$-Alg iff they are Morita equivalent (in the usual algebraic sense); see Prop. 2.4 in [35], and also cf. [35].

We now define the category $RKK(I)$, which refines $C^*(I)$ in the same way that $KK$ refines $C^*$. In fact, one may define a category $RKK(X)$ for any compact Hausdorff space $X$. As with $KK$, one starts with the notion of a Kasparov cycle. Given separable $C(X) C^*$-algebras $A$ and $B$, the elements of $R E(X; A, B)$ are those elements $(E, F) \in E(A, B)$ for which $E$ is an $A$-$B$ $C(X)$ Hilbert bimodule. There is no additional condition on $F$. The group $RKK(X; A, B)$ is then defined in precisely the same way as $KK(A, B)$, as the quotient of $R E(X; A, B)$ by the equivalence relation generated by unitary equivalence, translation of $F$ along norm-continuous path, and the addition of degenerate Kasparov cycles. There is an intersection product

$$RKK(X; A, B) \times RKK(X; B, C) \to RKK(X; A, C),$$

enabling one to define a category $RKK(X)$ in the obvious way; the objects are $C(X) C^*$-algebras, the arrows are the groups $RKK(X; -, -)$, composed through the intersection product. We are now in a position to state

**Conjecture 1** There is a functor from $\text{Poisson}$ to $RKK(I)$ that on objects defines a strict quantization.

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