THE GLOBAL WEAK SOLUTIONS TO LANDAU-LIFSHTIZ TYPE EQUATIONS

ZONGLIN JIA, YOUDE WANG

Abstract. In this paper, we consider a generalized Landau-Lifshitz equation on an n-dimensional closed Riemannian manifold $\mathcal{T}$ and we call it GLLE for short. The unknown map $u$ is from $\mathcal{T}$ to a compact Lie algebra $\mathfrak{g}$. The equation is transformed from classical Landau-Lifshitz equation. We prove the existence of weak solution to GLLE.

1. Introduction

In physics, the Landau-Lifshitz equation, which is deduced by Lev Landau and Evgeny Lifshitz, is used to describe the precessional motion of magnetization in a solid. Then we will explain it roughly.

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ which is the Euclidean space of 3 dimension. $u$, denoting magnetization vector, is a mapping from $\Omega$ to $\mathbb{R}^3$. The energy of $u$ is defined by

$$ E(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{R}^3} u|^2 \, dx $$

where the $\nabla_{\mathbb{R}^3}$ and $\Delta_{\mathbb{R}^3}$ denotes the gradient operator and laplace operator on $\mathbb{R}^3$ and $dx$ is the volume element. The well known Landau-Lifshitz equation is

$$ u_t = u \times \Delta_{\mathbb{R}^3} u $$

Here $\times$ is vector cross product in $\mathbb{R}^3$. If we add damping term to the equation, it can be written as

$$ u_t + \alpha u \times u_t = u \times \Delta_{\mathbb{R}^3} u $$

Here $\alpha$ is the damping parameter which is characteristic of the material.

If the energy is

$$ E(u) = \frac{1}{2} \int_{\Omega} |\nabla_{\mathbb{R}^3} u|^2 f \, dx $$

the related equation should be

$$ u_t + \alpha u \times u_t = u \times (f \Delta_{\mathbb{R}^3} u + \nabla_{\mathbb{R}^3} f \cdot \nabla_{\mathbb{R}^3} u) $$

where $f$ is a positive smooth function.

Now let us review something about compact Lie algebra. If $G$ is a compact Lie group and $\mathfrak{g}$ is its Lie algebra, there is an $Ad(G)$-invariant inner product, denoted by $\langle \cdot, \cdot \rangle$, on $\mathfrak{g}$ such that for any $X, Y, Z \in \mathfrak{g}$

$$ \langle Y, [X, Z] \rangle + \langle [X, Y], Z \rangle = 0 $$

where $[\cdot, \cdot]$ is the Lie bracket. So

$$ \langle X, [X, Z] \rangle = 0 $$
The details can be found in chapter 4 of [XWY].

It’s easy to see that \((\mathbb{R}^3, \times)\) is a compact Lie algebra. So we consider the following equation:

\[
\begin{aligned}
\begin{cases}
u_t + \alpha [u, u_t] = [u, f \Delta u \cdot \nabla f \cdot \nabla u] + F(x, t, u) \\ u(\cdot, 0) = u_0 : \mathbb{T} \rightarrow S_g(1) & \quad u_0 \in H^1(\mathbb{T}, \mathfrak{g})
\end{cases}
\end{aligned}
\]

We call it generalized Landau-Lifshitz equation or GLLE for short, where \((\mathbb{T}, h)\) is a n-dimensional closed Riemannian manifold with metric \(h = (h_{ij})\), \(\mathfrak{g}\) is a m-dimensional compact Lie algebra associated with a compact Lie group \(G\), \([\cdot, \cdot]\) is the Lie bracket, \(S_g(1)\) denotes the unit sphere in \(\mathfrak{g}\) centered at the origin. \(\mathbb{T} \rightarrow \mathbb{R}^+\) is a \(C^1\)-function. \((\cdot, \cdot)\) is the Ad(G)-invariant inner product and we always omit it. \(F\) is a smooth mapping from \(\mathbb{T} \times \mathbb{R}^+ \times \mathfrak{g}\) to \(\mathfrak{g}\) and satisfies the relation:

\[
(F(x, t, z), z) = 0 \quad \text{for any } z \in \mathfrak{g}
\]

\(u : \mathbb{T} \times \mathbb{R}^+ \rightarrow \mathfrak{g}\) is an unknown mapping. \(\Delta\) and \(\nabla\) are Laplace-Beltrami operator and gradient operator on \(\mathbb{T}\). In local coordinates \((x^1, \ldots, x^n)\) on \(\mathbb{T}\) and \((u_1, \ldots, u_m)\) on \(\mathfrak{g}\),

\[
\Delta u = (\Delta u_1, \ldots, \Delta u_m)
\]

where

\[
\Delta u_k = \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^i} \left( h_{ij} \sqrt{h} \frac{\partial u_k}{\partial x^j} \right) \quad \text{for } k = 1, 2, \ldots, m
\]

and \((h_{ij})\) is the inverse of \((h_{ij})\).

\[
\nabla f \cdot \nabla u = (\nabla f \cdot \nabla u_1, \ldots, \nabla f \cdot \nabla u_m)
\]

where

\[
\nabla f \cdot \nabla u_k = \frac{\partial f}{\partial x^i} \frac{\partial u_k}{\partial x^p} h_{ip} \quad \text{for } k = 1, 2, \ldots, m
\]

Before dealing with the problem, let us review some previous results. In case \(\alpha = 0\), \(f = 1\), and \([\cdot, \cdot]\) is cross product in \(\mathbb{R}^3\), Wang has established the existence of global solution in \([W]\). In \([T]\), Tilioua consider the weak solution to

\[
\begin{aligned}
\begin{cases}
u_t - \alpha u \times u_t = -\gamma u \times [\Delta_{\mathbb{R}^3} u - \mathbb{P}(u) + \beta (J \cdot \nabla u) u] \quad \text{in } (0, T) \times \Omega \\ u(0, \cdot) = u_0 : \Omega \rightarrow S^2
\end{cases}
\end{aligned}
\]

with parameters \(\alpha, \gamma, \beta\), where \(\mathbb{P}\) denotes the orthogonal projector onto the closure of the space of gradients of smooth functions in \(L^2\)-topology and \(\Omega\) is a bounded domain in \(\mathbb{R}^3\). He used penalized function. We use Wang’s method: first, we consider the auxiliary equation

\[
\begin{aligned}
\begin{cases}
u_t^\varepsilon + \alpha \left[ \frac{\nu^\varepsilon}{\max \{|\nu^\varepsilon|, 1\}} \right] \cdot v_t^\varepsilon \
= \varepsilon (f \Delta v^\varepsilon + \nabla f \cdot \nabla v^\varepsilon) + \left[ \frac{v^\varepsilon}{\max \{|v^\varepsilon|, 1\}} \right] \cdot f \Delta v^\varepsilon + \nabla f \cdot \nabla v^\varepsilon + F(x, t, \frac{v^\varepsilon}{\max \{|v^\varepsilon|, 1\}})
\end{cases}
\end{aligned}
\]

\(v^\varepsilon(0, \cdot) = u_0\)
By Galerkin method, it is not difficult to prove that above equation has a solution in $L^\infty([0, T], H^1(\mathbb{T}, g)) \cap W^{1,2}_2([0, T] \times \mathbb{T}, g)$; second, we take a special test function $v^\epsilon - v^\epsilon \min\{1, |v^\epsilon|\}$ to multiply the two sides of above equation and integrate on $\mathbb{T}$ to show that $|v^\epsilon| \leq 1$; at last, taking any test function $\varphi$ and integrating by parts, we prove the theorem. Our main result is following.

**Definition 1.1.** $u$, the weak solution of GLLE with initial value $u_0$, is a function in $L^\infty([0, T], H^1(\mathbb{T}, g)) \cap W^{1,2}_2([0, T] \times \mathbb{T}, g)$ which satisfies the relation:

$$\int_\mathbb{T} u(T) \varphi(T) d\mathbb{T} - \int_\mathbb{T} u_0 \varphi(0) d\mathbb{T} + \alpha \int_0^T \int_\mathbb{T} [u, u_t] \varphi d\mathbb{T} dt$$

$$= \int_0^T \int_\mathbb{T} u \varphi_t d\mathbb{T} dt - \int_0^T \int_\mathbb{T} [u, f \nabla u] \nabla \varphi d\mathbb{T} dt + \int_0^T \int_\mathbb{T} F(x, t, u) \varphi d\mathbb{T} dt$$

**Theorem 1.2.** If $u_0 \in H^1(\mathbb{T}, g)$ and $|u_0| = 1$ a.e and $f : \mathbb{T} \to \mathbb{R}^+$ is a $C^1$-function and $F$ is a smooth mapping from $\mathbb{T} \times \mathbb{R}^+ \times g$ to $g$ which satisfies the relation:

$$\langle F(x, t, z), z \rangle = 0 \text{ for any } z \in g$$

then the GLLE has weak solution with initial value $u_0$.

2. **The Proof of Theorem**

**Proof:** Let $\lambda_i (i = 1, 2, \ldots)$ be the eigenvalues of the operator $-f \Delta - \nabla f \cdot \nabla$ on the domain $H^2(\mathbb{T})$. $\omega^i$ is the normalized eigenfunction corresponding to $\lambda_i$. That is to say,

$$-f \Delta \omega^i - \nabla f \cdot \nabla \omega^i = \lambda_i \omega^i$$

The details of the eigenvalues of $-f \Delta - \nabla f \cdot \nabla$ can be found in chapter 2.4 of [WMX]. According to Galerkin approximation, let

$$u^N(x, t) = \sum_{i=1}^N \beta_i^N(t) \omega^i(x)$$
Here \( \{\beta_i^N(t)\} \) are unknown functions which take values in \( \mathfrak{g} \) and assumed to satisfies the following ordinary differential equation:

\[
\begin{align*}
\frac{d\beta_i^N}{dt} + \alpha \sum_{k=1}^{N} \int_T \left[ \frac{\sum_{j=1}^{N} \beta_j^N \omega^j}{\max\{|\sum_{j=1}^{N} \beta_j^N \omega^j|, 1\}} \right] \omega^i \omega^k \, dT \\
= \int_T \left[ \frac{\sum_{j=1}^{N} \beta_j^N \omega^j}{\max\{|\sum_{j=1}^{N} \beta_j^N \omega^j|, 1\}} \right] \sum_{k=1}^{N} \beta_k^N (-\lambda_k \omega^k) \omega^i \, dT \\
+ \int_T F(t, x, \max\{|\sum_{j=1}^{N} \beta_j^N \omega^j|, 1\}) \omega^i \, dT + \varepsilon \int_T \sum_{k=1}^{N} \beta_k^N (-\lambda_k \omega^k) \omega^i \, dT \\
\beta_j^N(0) = \int_T u_0 \cdot \omega^j \, dT
\end{align*}
\]

(2.1) can be written as

\[
(Id + A(\beta)) \frac{d\beta}{dt} = B(\beta)
\]

Here \( \beta = (\beta_1^N, \beta_2^N, ..., \beta_N^N)^T \), \( Id \) is the unit matrix, and \( A(\beta) \) is an antisymmetric matrix. So \( Id + A(\beta) \) is an invertible matrix. We obtain:

\[
\begin{align*}
\frac{d\beta}{dt} = (Id + A(\beta))^{-1} B(\beta) \\
\beta(0) = (\int_T u_0 \cdot \omega^1 \, dT, \int_T u_0 \cdot \omega^2 \, dT, ..., \int_T u_0 \cdot \omega^N \, dT)^T
\end{align*}
\]

(2.2)

Since \( F \) is smooth, the right side of (2.2) is locally Lipschitz continuous and there is a \( \tau > 0 \) so that the solution of (2.2) exists in \([0, \tau]\). Then we get:

\[
\begin{align*}
\int_T u_t^N \cdot \omega^i \, dT + \alpha \int_T \left[ \frac{u^N}{\max\{|u^N|, 1\}} \right] \omega^i \, dT \\
= \varepsilon \int_T (f \Delta u^N + \nabla f \cdot \nabla u^N) \omega^i \, dT \\
+ \int_T \left[ \frac{u^N}{\max\{|u^N|, 1\}} \right] f \Delta u^N + \nabla f \cdot \nabla u^N \omega^i \, dT \\
+ \int_T F(x, t, \max\{|u^N|, 1\}) \omega^i \, dT \\
u^N(0, \cdot) = u_0^N := \sum_{i=1}^{N} (\int_T u_0 \cdot \omega^i \, dT) \omega^i
\end{align*}
\]

(2.3)
Multiplying the two sides of (2.3) by $\beta_i^N(t)$ and summing $i$ from 1 to $N$ and integrating by parts, we get:

$$\frac{1}{2} \frac{d}{dt} \int_T |u|^2 d\mathbb{T} + \varepsilon \int_T f |\nabla u|^2 d\mathbb{T} = 0$$

so

$$\int_T |u|^2 d\mathbb{T} + 2\varepsilon \int_0^t \int_T f |\nabla u|^2 d\mathbb{T} dt$$

$$= \int_T |u_0|^2 d\mathbb{T} = \sum_{i=1}^N (\int_T u_0 \cdot \omega^i d\mathbb{T})^2$$

$$\leq \sum_{i=1}^\infty (\int_T u_0 \cdot \omega^i d\mathbb{T})^2 = \int_T |u_0|^2 d\mathbb{T} = vol(\mathbb{T})$$

where $vol(\mathbb{T})$ is the volume of $\mathbb{T}$.

Since

$$\int_T |u|^2 d\mathbb{T}$$

$$= \int_T \left( \sum_{i=1}^N \beta_i^N(t) \omega^i \right) \left( \sum_{j=1}^N \beta_j^N(t) \omega^j \right) d\mathbb{T}$$

$$= \sum_{i=1}^N \sum_{j=1}^N \beta_i^N(t) \beta_j^N(t) \int_T \omega^i \omega^j d\mathbb{T}$$

$$= \sum_{i=1}^N \sum_{j=1}^N \beta_i^N(t) \beta_j^N(t) \cdot \delta_{ij}$$

$$= \sum_{i=1}^N |\beta_i^N(t)|^2$$

then for any $T > 0$ and any $i$, $\beta_i^N(t)$ can be extended to $[0, T]$. That is to say, $u^N$ can be extended to $[0, T]$.

Multiplying the two sides of (2.3) by $-\lambda_i \beta_i^N$ and summing $i$ from 1 to $N$, we get:

$$\int_T u^N (f \Delta u^N + \nabla f \cdot \nabla u^N) d\mathbb{T} + \alpha \int_T \left[ \frac{u^N}{\max\{ |u^N|, 1 \}} , u^N_i \right] (f \Delta u^N + \nabla f \cdot \nabla u^N) d\mathbb{T}$$

$$= \varepsilon \int_T |f \Delta u^N + \nabla f \cdot \nabla u^N|^2 d\mathbb{T}$$

$$+ \int_T F \left( x, t, \frac{u^N}{\max\{ |u^N|, 1 \}} \right) (f \Delta u^N + \nabla f \cdot \nabla u^N) d\mathbb{T}$$
so

\[
\frac{1}{2} \frac{d}{dt} \int_T f |\nabla u^N|^2 \, d\mathcal{T} + \varepsilon \int_T |f \nabla u^N + \nabla f \cdot \nabla u^N|^2 \, d\mathcal{T}
\]

\[
= \alpha \int_T \left[ \frac{u^N}{\max\{|u^N|, 1\}}, u^N \right] (f \nabla u^N + \nabla f \cdot \nabla u^N) \, d\mathcal{T}
\]

\[
- \int_T F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) (f \nabla u^N + \nabla f \cdot \nabla u^N) \, d\mathcal{T}
\]

Multiplying the two sides of (2.3) by \( \frac{d\beta^N}{dt} \) and summing i from 1 to N and integrating by parts, we get:

\[
\int_T |u^N_t|^2 \, d\mathcal{T} + \varepsilon \frac{d}{dt} \int_T f |\nabla u^N|^2 \, d\mathcal{T}
\]

\[
= - \int_T \left[ \frac{u^N}{\max\{|u^N|, 1\}}, u^N \right] (f \nabla u^N + \nabla f \cdot \nabla u^N) \, d\mathcal{T}
\]

\[
+ \int_T F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) u^N_t \, d\mathcal{T}
\]

Multiplying the two sides of (2.6) by \( \alpha \) and then adding the two sides of (2.5) to its two sides and integrating by parts, we obtain:

\[
\frac{1 + \alpha \varepsilon}{2} \frac{d}{dt} \int_T f |\nabla u^N|^2 \, d\mathcal{T} + \varepsilon \int_T |f \nabla u^N + \nabla f \cdot \nabla u^N|^2 \, d\mathcal{T} + \alpha \int_T |u^N_t|^2 \, d\mathcal{T}
\]

\[
= \alpha \int_T F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) u^N_t \, d\mathcal{T} + \int_T \nabla \left( F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right) \cdot \nabla u^N \cdot f \, d\mathcal{T}
\]

Note that

\[
\left| \nabla \left( F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right) \right|
\]

\[
\leq \left| \nabla_x F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right| + \left| \frac{\partial F}{\partial z} \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right| \cdot \nabla \left( \frac{u^N}{\max\{|u^N|, 1\}} \right)
\]

and

\[
\nabla \left( \frac{u^N}{\max\{|u^N|, 1\}} \right) = \left( \nabla \left( \frac{u_1^N}{\max\{|u^N|, 1\}} \right), \nabla \left( \frac{u_2^N}{\max\{|u^N|, 1\}} \right), \ldots, \nabla \left( \frac{u_m^N}{\max\{|u^N|, 1\}} \right) \right)
\]
and for $k=1,2,...,m$

$$\nabla \left( \frac{u_k^N}{\max\{|u^N|, 1\}} \right)$$

$$= \frac{\nabla u_k^N}{\max\{|u^N|, 1\}} - \chi_{\{|u^N|>1\}} \frac{u_k^N (\sum_{i=1}^{m} \nabla u_i^N \cdot u_i^N)}{|u|^3}$$

where

$$\chi_{\{|u^N|>1\}}(x) = \begin{cases} 0 & |u^N(x)| \leq 1 \\ 1 & |u^N(x)| > 1 \end{cases}$$

so there exists a $C_1$ such that

$$\left| \nabla \left( F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right) \right| \leq C_1 |\nabla u^N| + C_1$$

Since

$$\left| F \left( x, t, \frac{u^N}{\max\{|u^N|, 1\}} \right) \right| \leq C_2$$

so

$$\frac{1 + \alpha \varepsilon}{2} \frac{d}{dt} \int_T |\nabla u^N|^2 d\mathbb{T} + \varepsilon \int_T |f \Delta u^N + \nabla f \cdot \nabla u^N|^2 d\mathbb{T} + \alpha \int_T |u_t^N|^2 d\mathbb{T}$$

$$\leq \alpha C_2 \int_T |u_t^N|^2 d\mathbb{T} + \int_T (C_1 + C_1 |\nabla u^N|) \cdot |\nabla u^N| \cdot f d\mathbb{T}$$

$$\leq \frac{\alpha}{2} \int_T |u_t^N|^2 d\mathbb{T} + \frac{\alpha C_2}{2} \frac{1}{vol(\mathbb{T})} + C_1 \int_T (1 + |\nabla u^N|^2 + |\nabla u^N|^2) f d\mathbb{T}$$

$$\leq \frac{\alpha}{2} \int_T |u_t^N|^2 d\mathbb{T} + \frac{\alpha C_2}{2} \frac{1}{vol(\mathbb{T})} + \frac{3}{2} C_1 \int_T (1 + |\nabla u^N|^2) f d\mathbb{T}$$

$$\leq \frac{\alpha}{2} \int_T |u_t^N|^2 d\mathbb{T} + \frac{\alpha C_2}{2} \frac{1}{vol(\mathbb{T})} + \frac{3}{2} C_1 \int_T |\nabla u^N|^2 d\mathbb{T}$$
moreover
\[
\frac{1 + \alpha \varepsilon}{2} \int_{\mathbb{T}} f |\nabla u^N|^2 \, d\mathbb{T} + \varepsilon \int_{0}^{t} \int_{\mathbb{T}} |f \triangle u^N + \nabla f \cdot \nabla u^N|^2 \, d\mathbb{T} \, dt \\
+ \frac{\alpha}{2} \int_{0}^{t} \int_{\mathbb{T}} |u_t^N|^2 \, d\mathbb{T} \, dt \\
\leq \left( \frac{\alpha C_2^2}{2} + \frac{3C_1}{2} ||f||_{\infty} \right) \text{vol}(\mathbb{T}) t + \frac{3}{2} C_1 \int_{0}^{t} \int_{\mathbb{T}} |\nabla u^N|^2 \, d\mathbb{T} \, dt \\
+ \frac{1 + \alpha \varepsilon}{2} \int_{\mathbb{T}} f |\nabla u_0^N|^2 \, d\mathbb{T}
\]
\] 
(2.7)

But
\[
\int_{\mathbb{T}} |\nabla u_0|^2 \, d\mathbb{T} = \int_{\mathbb{T}} -\triangle u_0 \cdot u_0 \, d\mathbb{T} \\
= \int_{\mathbb{T}} -\sum_{i=1}^{\infty} (\int_{\mathbb{T}} \triangle u_0 \cdot \omega^i \, d\mathbb{T}) \omega^i \cdot u_0 \, d\mathbb{T} \\
= \int_{\mathbb{T}} \sum_{i=1}^{\infty} (\int_{\mathbb{T}} u_0 \cdot (-\triangle \omega^i) \, d\mathbb{T}) \omega^i \cdot u_0 \, d\mathbb{T} \\
= \sum_{i=1}^{\infty} \lambda_i \left| \int_{\mathbb{T}} u_0 \cdot \omega^i \, d\mathbb{T} \right|^2
\]

and
\[
\int_{\mathbb{T}} |\nabla u_0^N|^2 \, d\mathbb{T} \\
= \int_{\mathbb{T}} \sum_{i=1}^{N} \sum_{j=1}^{N} (\int_{\mathbb{T}} u_0 \cdot \omega^i \, d\mathbb{T}) (\int_{\mathbb{T}} u_0 \cdot \omega^j \, d\mathbb{T}) \nabla \omega^i \cdot \nabla \omega^j \, d\mathbb{T} \\
= \sum_{i=1}^{N} \sum_{j=1}^{N} (\int_{\mathbb{T}} u_0 \cdot \omega^i \, d\mathbb{T}) (\int_{\mathbb{T}} u_0 \cdot \omega^j \, d\mathbb{T}) \int_{\mathbb{T}} -\triangle \omega^i \cdot \omega^j \, d\mathbb{T} \\
= \sum_{i=1}^{N} \lambda_i \left| \int_{\mathbb{T}} u_0 \cdot \omega^i \, d\mathbb{T} \right|^2
\]

Let \( E(t) = \int_{0}^{t} \int_{\mathbb{T}} f |\nabla u_N|^2 \, d\mathbb{T} \, dt \), we infer from (2.7) that
\[
\frac{1 + \alpha \varepsilon}{2} \frac{d}{dt} E \leq \left( \frac{\alpha C_2^2}{2} + \frac{3C_1}{2} ||f||_{\infty} \right) \text{vol}(\mathbb{T}) t + \frac{3}{2} C_1 E + \frac{1 + \alpha \varepsilon}{2} ||f||_{\infty} \int_{\mathbb{T}} |\nabla u_0|^2 \, d\mathbb{T}
\]

By Gronwall inequality, we obtain:
\[
E(t) \leq C_8(T)
\]
Here $C_8(T)$ is independent of $\varepsilon$. So

$$\frac{1 + \alpha \varepsilon}{2} \int_T |f| |\nabla u^N|^2 dT + \varepsilon \int_0^t \int_T |f \triangle u^N + \nabla f \cdot \nabla u^N|^2 dT dt \leq \frac{\alpha C_1^2}{2} + \frac{3C_1}{2} ||f||_\infty \text{vol}(\mathbb{T}) t + \frac{3}{2} C_1 \cdot C_8(T) + \frac{1 + \alpha \varepsilon}{2} ||f||_\infty \int_T |\nabla u_0|^2 dT$$

since $0 < m \leq f(x) \leq ||f||_\infty < \infty$, 

$$\frac{1 + \alpha \varepsilon}{2} m \int_T |\nabla u^N|^2 dT \leq \frac{1 + \alpha \varepsilon}{2} \int_T |f| |\nabla u^N|^2 dT$$

$$\leq \frac{\alpha C_1^2}{2} + \frac{3C_1}{2} ||f||_\infty \text{vol}(\mathbb{T}) t + \frac{3}{2} C_1 \cdot C_8(T) + \frac{1 + \alpha \varepsilon}{2} ||f||_\infty \int_T |\nabla u_0|^2 dT$$

and there exists a $C_{10}(T)$ which is independent of $\varepsilon$ such that

$$\int_T |\nabla u^N|^2 dT \leq C_{10}(T)$$

and

$$\int_0^t \int_T |\triangle u^N|^2 dT dt \leq 2 \int_0^t \int_T |f \triangle u^N + \nabla f \cdot \nabla u^N|^2 dT dt + 2 \int_0^t \int_T \left| -\frac{\nabla f}{f} \nabla u^N \right|^2 dT dt$$

$$\leq \frac{2}{m^2} \int_0^t \int_T |f \triangle u^N + \nabla f \cdot \nabla u^N|^2 dT dt + 2 \frac{1}{m^2} \int_0^t \int_T |\nabla u^N|^2 dT dt$$

$$\leq \frac{\alpha C_1^2}{\varepsilon m^2} + \frac{3C_1}{\varepsilon m^2} ||f||_\infty \text{vol}(\mathbb{T}) t + \frac{3}{\varepsilon m^2} C_1 \cdot C_8(T) + \frac{1 + \alpha \varepsilon}{\varepsilon m^2} ||f||_\infty \int_T |\nabla u_0|^2 dT$$

$$+ 2 \left| -\frac{\nabla f}{f} \right|^2 \frac{1}{m} \int_0^t \int_T |\nabla u^N|^2 dT dt$$

$$\leq \frac{\alpha C_1^2}{\varepsilon m^2} + \frac{3C_1}{\varepsilon m^2} ||f||_\infty \text{vol}(\mathbb{T}) t + \frac{3}{\varepsilon m^2} C_1 \cdot C_8(T) + \frac{1 + \alpha \varepsilon}{\varepsilon m^2} ||f||_\infty \int_T |\nabla u_0|^2 dT$$

$$+ 2 \left| -\frac{\nabla f}{f} \right|^2 \frac{1}{m} C_8(T)$$

so we get

- $\{u^N\}$ is a bounded sequence in $L^\infty([0, T], H^1(\mathbb{T}, g))$;
- $\{u_N\}$ is a bounded sequence in $L^2([0, T], L^2(\mathbb{T}, g))$;
- $\{\triangle u^N\}$ is a bounded sequence in $L^2([0, T], L^2(\mathbb{T}, g))$;
- $\{\nabla u^N\}$ is a bounded sequence in $L^\infty([0, T], L^2(\mathbb{T}, TT \otimes g))$, where $TT$ is the tangent bundle of $\mathbb{T}$.

By the property of weak limits and Aubin-Lions Lemma, there exists a $u^\varepsilon \in W^{1,2}_2(\mathbb{T} \times [0, T], g)$ and a subsequence of $\{u^N\}$ which is also denoted by $\{u^N\}$ such that
\[ u^N \to v^\varepsilon \text{ weakly* in } L^\infty([0, T], H^1(\mathbb{T}, g)); \]
\[ u^N \to v^\varepsilon \text{ strongly in } L^\infty([0, T], L^2(\mathbb{T}, g)); \]
\[ u^N \to v^\varepsilon \text{ a.e. } \mathbb{T} \times [0, T]; \]
\[ u^N_t \to v^\varepsilon_t \text{ weakly in } L^2([0, T], L^2(\mathbb{T}, g)); \]
\[ \Delta u^N \to \Delta v^\varepsilon \text{ weakly in } L^2([0, T], L^2(\mathbb{T}, g)); \]
\[ \nabla u^N \to \nabla v^\varepsilon \text{ weakly* in } L^\infty([0, T], L^2(\mathbb{T}, TT \otimes g)). \]

Since
\[ ||u^N||_{L^\infty([0, T], H^1(\mathbb{T}, g))} \leq C_{12} \]
and
\[ u^N \to v^\varepsilon \text{ weakly* in } L^\infty([0, T], H^1(\mathbb{T}, g)), \]
\[ \|v^\varepsilon\|_{L^\infty([0, T], H^1(\mathbb{T}, g))} \leq C_{12} \]

By the same method, we can get:
\[ \int_0^T \int_T |v^\varepsilon_t|^2 d\mathbb{T} \leq C_{13} \]
so
\[ \left[ \frac{u^N}{\max\{|u^N|, 1\}}, u^N_t \right] \rightharpoonup \left[ \frac{v^\varepsilon}{\max\{|v^\varepsilon|, 1\}}, v^\varepsilon_t \right] \]
weakly in \( L^2([0, T], L^2(\mathbb{T}, g)) \).
\[ \left[ \frac{u^N}{\max\{|u^N|, 1\}}, \Delta u^N \right] \rightharpoonup \left[ \frac{v^\varepsilon}{\max\{|v^\varepsilon|, 1\}}, \Delta v^\varepsilon \right] \]
weakly in \( L^2([0, T], L^2(\mathbb{T}, g)) \).
\[ \left[ \frac{u^N}{\max\{|u^N|, 1\}}, \nabla f \cdot \nabla u^N \right] \rightharpoonup \left[ \frac{v^\varepsilon}{\max\{|v^\varepsilon|, 1\}}, \nabla f \cdot \nabla v^\varepsilon \right] \]
weakly* in \( L^\infty([0, T], L^2(\mathbb{T}, g)) \).

Fix \( r \in \mathbb{Z}^+ \) and take any \( N \geq r \). Then we multiply two sides of (2.3) by \( \eta^i(t) \) which belongs to \( C^\infty([0, T], g) \) and sum i from 1 to r. Letting
\[ \Phi^r(x, t) = \sum_{i=1}^r \omega^i(x)\eta^i(t) \]
and integrating on \([0, T]\), we get:
\[ \int_0^T \int_\mathbb{T} u^N_t \cdot \Phi^r d\mathbb{T} dt + \alpha \int_0^T \int_\mathbb{T} \left[ \frac{u^N}{\max\{|u^N|, 1\}}, u^N_t \right] \Phi^r d\mathbb{T} dt \]
\[ = \varepsilon \int_0^T \int_\mathbb{T} (f \Delta u^N + \nabla f \cdot \nabla u^N) \Phi^r d\mathbb{T} dt \]
\[ + \int_0^T \int_\mathbb{T} \left[ \frac{u^N}{\max\{|u^N|, 1\}}, f \Delta u^N + \nabla f \cdot \nabla u^N \right] \Phi^r d\mathbb{T} dt \]
\[ + \int_0^T \int_\mathbb{T} F\left(x, t, \frac{u^N}{\max\{|u^N|, 1\}}\right) \Phi^r d\mathbb{T} dt \]
Letting $N$ tends to $\infty$, we get:

\[
\int_0^T \int_\Omega \nu^\varepsilon \cdot \Phi^r \, d\Omega \, dt + \alpha \int_0^T \int_\Omega \left[ \max\{\nu^\varepsilon, 1\}, \nu^\varepsilon \right] \Phi^r \, d\Omega \, dt
\]

\[
= \int_0^T \int_\Omega \left[ \frac{\nu^\varepsilon}{\max\{\nu^\varepsilon, 1\}}, f \Delta \nu^\varepsilon + \nabla f \cdot \nabla \nu^\varepsilon \right] \Phi^r \, d\Omega \, dt
\]

\[
+ \int_0^T \int_\Omega F(x, t, \frac{\nu^\varepsilon}{\max\{\nu^\varepsilon, 1\}}) \Phi^r \, d\Omega \, dt
\]

\[
+ \varepsilon \int_0^T \int_\Omega (f \Delta \nu^\varepsilon + \nabla f \cdot \nabla \nu^\varepsilon) \Phi^r \, d\Omega \, dt
\]

Since functions which are like $\Phi^r(x, t)$ are dense in $L^2([0, T], L^2(\Omega, \mathfrak{g}))$, we get:

\[
\begin{cases}
\nu^\varepsilon_t + \alpha \left[ \frac{\nu^\varepsilon}{\max\{\nu^\varepsilon, 1\}}, \nu^\varepsilon \right]
\varepsilon (f \Delta \nu^\varepsilon + \nabla f \cdot \nabla \nu^\varepsilon) + \left[ \frac{\nu^\varepsilon}{\max\{\nu^\varepsilon, 1\}}, f \Delta \nu^\varepsilon + \nabla f \cdot \nabla \nu^\varepsilon \right] + F(x, t, \frac{\nu^\varepsilon}{\max\{\nu^\varepsilon, 1\}})
\nu^\varepsilon(0, \cdot) = u_0
\end{cases}
\]

choosing $\nu^\varepsilon - \varepsilon \frac{\min\{1, \nu^\varepsilon\}}{\nu^\varepsilon}$ as test function, we obtain:

\[
\int_\Omega \nu^\varepsilon \cdot (\nu^\varepsilon - \varepsilon \frac{\min\{1, \nu^\varepsilon\}}{\nu^\varepsilon}) \, d\Omega = \varepsilon \int_\Omega (f \Delta \nu^\varepsilon + \nabla f \cdot \nabla \nu^\varepsilon) \cdot (\nu^\varepsilon - \varepsilon \frac{\min\{1, \nu^\varepsilon\}}{\nu^\varepsilon}) \, d\Omega
\]

so

\[
\frac{1}{2} \frac{d}{dt} \int_{|\nu^\varepsilon| > 1} |\nu^\varepsilon|^2 (1 - \frac{1}{|\nu^\varepsilon|}) \, d\Omega
\]

\[
= \frac{1}{2} \int_{|\nu^\varepsilon| > 1} \frac{\nu^\varepsilon \cdot \nu^\varepsilon}{|\nu^\varepsilon|^2} \, d\Omega - \varepsilon \int_{|\nu^\varepsilon| > 1} f |\nabla \nu^\varepsilon|^2 (1 - \frac{1}{|\nu^\varepsilon|}) \, d\Omega - \varepsilon \int_{|\nu^\varepsilon| > 1} f \frac{|\nabla \nu^\varepsilon \cdot \nu^\varepsilon|^2}{|\nu^\varepsilon|^3} \, d\Omega
\]

Taking $\frac{\varepsilon \min\{\nu^\varepsilon, 1\}}{\max\{\nu^\varepsilon, 1\} - 1}$ as test function, we get:

\[
\int_{|\nu^\varepsilon| > 1} \frac{\nu^\varepsilon \cdot \nu^\varepsilon}{|\nu^\varepsilon|^2} \, d\Omega = \frac{1}{2} \int_{|\nu^\varepsilon| > 1} |\nu^\varepsilon|^2 (1 - \frac{1}{|\nu^\varepsilon|}) \, d\Omega
\]

\[
= - \varepsilon \int_T f \cdot \nabla \nu^\varepsilon \cdot \nabla \left[ \frac{\varepsilon \min\{\nu^\varepsilon, 1\}}{\max\{\nu^\varepsilon, 1\} - 1} \right] \, d\Omega
\]

\[
= - \varepsilon \int_{|\nu^\varepsilon| > 1} f \cdot |\nabla \nu^\varepsilon|^2 \frac{|\nu^\varepsilon| - 1}{|\nu^\varepsilon| (|\nu^\varepsilon| - 1 + \delta)} \, d\Omega - \varepsilon \int_{|\nu^\varepsilon| > 1} f \frac{|\nabla \nu^\varepsilon \cdot \nu^\varepsilon|^2}{|\nu^\varepsilon|^3} \cdot \frac{|\nu^\varepsilon|^2 + 2 |\nu^\varepsilon| - 1 + \delta}{|\nu^\varepsilon|^2 (|\nu^\varepsilon| + \delta - 1)^2} \, d\Omega
\]

Letting $\delta \to 0$, by the controlled convergence theorem, we get:

\[
\int_{|\nu^\varepsilon| > 1} \frac{\nu^\varepsilon \cdot \nu^\varepsilon}{|\nu^\varepsilon|^2} \, d\Omega = - \varepsilon \int_{|\nu^\varepsilon| > 1} f \frac{|\nabla \nu^\varepsilon|^2}{|\nu^\varepsilon|^3} \, d\Omega + \varepsilon \int_{|\nu^\varepsilon| > 1} f \frac{|\nabla \nu^\varepsilon \cdot \nu^\varepsilon|^2}{|\nu^\varepsilon|^3} \, d\Omega
\]

so

\[
\frac{d}{dt} \int_{|\nu^\varepsilon| > 1} |\nu^\varepsilon|^2 \left(1 - \frac{1}{|\nu^\varepsilon|^2} \right) \, d\Omega \leq 0
\]
and

\[ q(t) := \int_{|v^\varepsilon| > 1} |v^\varepsilon|^2 \left( 1 - \frac{1}{|v^\varepsilon|} \right) d\mathbb{T} \]

which is non-negative and decreasing. Since \( |v^\varepsilon(\cdot, 0)| = |u_0| = 1 \), \( q(0) = 0 \). So \( |v^\varepsilon| \leq 1 \) and we obtain:

\[ v^\varepsilon_t + \alpha [v^\varepsilon, v^\varepsilon_t] = \varepsilon (f \Delta v^\varepsilon + \nabla f \cdot \nabla v^\varepsilon) + [v^\varepsilon, f \Delta v^\varepsilon + \nabla f \cdot \nabla v^\varepsilon] + F(x, t, v^\varepsilon) \]

Multiplying two sides by \( v^\varepsilon \) and integrating on \([0, T] \times \mathbb{T}, u \equiv \epsilon \leq 0 \) at \( T \) and by \( \varepsilon \), we get:

\[ \int_{\mathbb{T}} (|v^\varepsilon|^2 - 1) d\mathbb{T} + 2\varepsilon \int_{0}^{T} \int_{\mathbb{T}} |\nabla v^\varepsilon|^2 d\mathbb{T} dt = 0 \]  

(2.8)

Note that \( \{v^\varepsilon\} \) is a bounded sequence in \( L^\infty([0, T], H^1(\mathbb{T}, \mathfrak{g})) \) and \( \{v^\varepsilon_t\} \) is a bounded sequence in \( L^2([0, T], L^2(\mathbb{T}, \mathfrak{g})) \). So, by the property of weak limits and Aubin-Lions Lemma, there is a \( u \) and a subsequence of \( \{v^\varepsilon\} \) which is also denoted by \( \{v^\varepsilon\} \) such that:

- \( v^\varepsilon \rightharpoonup u \) weakly * in \( L^\infty([0, T], H^1(\mathbb{T}, \mathfrak{g})) \);
- \( v^\varepsilon \rightharpoonup u \) strongly in \( L^\infty([0, T], L^2(\mathbb{T}, \mathfrak{g})) \);
- \( v^\varepsilon_t \rightharpoonup u_t \) weakly in \( L^2([0, T], L^2(\mathbb{T}, \mathfrak{g})) \).

Letting \( \varepsilon \) in (2.8) tends to 0, we have:

\[ \int_{\mathbb{T}} (|u|^2 - 1) d\mathbb{T} = 0 \]

Since \( |v^\varepsilon| \leq 1, |u| \leq 1 \). So \( |u| = 1 \) a.e. \( \mathbb{T} \) and for all \( t \in [0, T] \). Because \( v^\varepsilon \rightharpoonup u \) a.e. \( [0, T] \times \mathbb{T}, F(x, t, v^\varepsilon) \rightharpoonup F(x, t, u) \) a.e. \( [0, T] \times \mathbb{T} \).

For any \( \varphi \in C^\infty([0, T] \times \mathbb{T}, \mathfrak{g}) \)

\[
\int_{\mathbb{T}} v^\varepsilon(T) \cdot \varphi(T) d\mathbb{T} - \int_{\mathbb{T}} u_0 \cdot \varphi(0) d\mathbb{T} + \alpha \int_{0}^{T} \int_{\mathbb{T}} [v^\varepsilon, v^\varepsilon_t] \cdot \varphi d\mathbb{T} = \\
\int_{0}^{T} \int_{\mathbb{T}} v^\varepsilon \varphi_t d\mathbb{T} - \int_{0}^{T} \int_{\mathbb{T}} [v^\varepsilon, f \nabla v^\varepsilon] \cdot \nabla \varphi d\mathbb{T} + \int_{0}^{T} \int_{\mathbb{T}} F(x, t, v^\varepsilon) \cdot \varphi d\mathbb{T} - \\
\varepsilon \int_{0}^{T} \int_{\mathbb{T}} f \nabla v^\varepsilon \cdot \nabla \varphi d\mathbb{T}
\]

Let \( \varepsilon \to 0 \) and by controlled convergence theorem,

\[ \int_{0}^{T} \int_{\mathbb{T}} F(x, t, v^\varepsilon) \cdot \varphi d\mathbb{T} dt \to \int_{0}^{T} \int_{\mathbb{T}} F(x, t, u) \cdot \varphi d\mathbb{T} dt \]

so we have:

\[
\int_{\mathbb{T}} u(T) \varphi(T) d\mathbb{T} - \int_{\mathbb{T}} u_0 \varphi(0) d\mathbb{T} + \alpha \int_{0}^{T} \int_{\mathbb{T}} [u, u_t] \varphi d\mathbb{T} dt = \\
\int_{0}^{T} \int_{\mathbb{T}} u \varphi_t d\mathbb{T} dt - \int_{0}^{T} \int_{\mathbb{T}} [u, f \nabla u] \nabla \varphi d\mathbb{T} dt + \\
\int_{0}^{T} \int_{\mathbb{T}} F(x, t, u) \varphi d\mathbb{T} dt
\]

This completes the proof.
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