Minimal Cycles, Black Holes and QFT's

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We review some aspects of minimal cycles in string compactifications and their role in constructing new critical theories in six and lower dimensions as well as in accounting for black hole entropy. (Based on a talk presented at the Salam Memorial Meeting, the Abdus Salam International Center for Theoretical Physics, Fall 1997)
1. Some Reflections on Abdus Salam

Before I start with the scientific aspect of this note, I would like to mention a few words about my impressions of Abdus Salam and his contributions to physics and to the advancement of science in developing countries.

Science and scientific tradition has a long history. At different times different countries have led the cause of advancing human knowledge. Many of the countries that today are called “developing countries” or “third world countries” have at an earlier time been at the forefront of knowledge and have made important contributions to science. Indeed the present scientific tradition is based on the foundations set by these earlier advances. The very fact that science has developed in this way also clearly demonstrates that science does not belong to a restricted group of people. It belongs to all humanity. Thus all humanity and in particular scientists share a responsibility of making pursuit of science as much of a global, borderless endeavor, as they can.

In the present century if one is to name a single person who has contributed most to the development of science in the developing countries the natural choice would be Abdus Salam. With the creation of the International Center for Theoretical Physics he has contributed tremendously to making modern science accessible to a much larger audience of humanity. Through this center and through numerous visits to developing countries he encouraged and challenged the citizens of these countries to take up the cause of advancing science. In fact the ICTP has gone far beyond a center for developing countries, and has become a center which attracts first rate scientists from all over the world and has become a regular sight for important conferences in physics and in mathematics.

The creation of the ICTP was a rather difficult task. He always had a tough task of convincing authorities for the need of having such a center and the need for improvement of science in the developing countries. The irony of this is that he had an easier time convincing authorities of developed countries in contributing in this direction than those of the developing countries. To the policy makers of the developing countries he emphasized an important fact: It may appear that advancing modern abstract science in developing countries is a futile task, when one can import any advanced technological product. However the progress of a country will ultimately come from having a strong scientific base, even though it may appear in the short run to be far from any concrete application.

Even though this aspect of his life has had such a crucial impact in the lives of so many scientists, his contributions to theoretical physics are no less important. In particular his
contributions to particle physics resulted in his discovery of what we now call the Salam-Weinberg standard model. Since this aspect of his life and his scientific contributions in general are already very well known I will not write any further on this point.

2. Basic Idea

Perhaps the most important aspect of the recent revolution in string theory has been the appreciation of the role that extended objects play in physics. These objects can come in various dimensionalities, and are typically viewed as solitonic degrees of freedom. If they have $p$ spatial dimensions, they are called $p$-branes (extending the terminology from membrane where $p = 2$). The existence of such states, together with the properties of the internal geometry $M$ where the string is compactified upon leads to many interesting and novel physical phenomena. In particular the $p$-brane can wrap around a $q$-dimensional cycle $C \subset M$ of the internal manifold leading to a $p - q$ dimensional brane in the uncompactified space. The basic idea is to investigate what kinds of $C$’s there are and what physical consequences they lead to. There are basically two classes of cycles that we will consider; the cycles $C$ that can shrink to zero size, and are thus called the vanishing cycles, and cycles that are typically big and cannot shrink beyond a certain minimum size. The type that can be shrunk are typically “rigid”, which means that they do not have any moduli associated with them. The second type, typically come in a family, i.e. there is a moduli space associated with them (what this means is that if we look at minimal cycles in a given homology class there is more than one, and they can be parameterized by some space called the moduli space). We shall see that the first type will lead to interesting quantum field theory questions whereas the latter type have bearing on questions involving black holes and their entropy.

3. Vanishing Cycles and QFT Interpretations

Consider a string compactification with a $p$-dimensional vanishing cycle. If we wrap a $p$-brane around such a cycle, the resulting state will be a 0-brane, i.e. a particle, from the viewpoint of the uncompactified spacetime. Moreover the mass of this state is given by

$$M = T \cdot V(C)$$

1 I will not include any references in this short note. The interested reader can consult the many good review articles already available.
where $T$ denotes the tension on the worldvolume of the $p$-brane and $V(C)$ denotes the volume of $C$. The assumption that the cycle is vanishing means that $V(C) \to 0$, which thus implies that we end up with massless particles. Depending on various cases and the different values of $p$ these massless particles will have different implications. For example if $p = 2$ and these vanishing 2 spheres arise in a type IIA string compactified on $K3$ down to six dimensions, this gives rise to massless vector multiplet charged under a $U(1)$, which thus naturally enhances the gauge symmetry from $U(1) \to SU(2)$. If there are more than one vanishing 2-cycle and they intersect one another according to the Dynkin diagrams of the $A - D - E$ groups, the resulting physics is an enhanced $A$-$D$-$E$ gauge symmetry. If we further compactify down to four dimensions and we have a 2-dimensional locus with $A_{k-1}$ singularity and another locus with $A_{p-1}$ singularity and they meet at a point, the mixed wrapped 2-cycles will now lead to $(k, p)$ bi-fundamental matter of $SU(k) \times SU(p)$ gauge group in four dimensions. In fact by arranging various intersecting singularities we can engineer and study the properties of a large class of quantum field theories in this way. Many non-perturbative questions of quantum field theories get translated to perturbative string questions.

Sometimes it may happen that a $q$-dimensional cycle shrinks, but the theory in question has no $q$-branes. In such a case the lightest states comes from a $q + r$ dimensional brane wrapped around it, with the smallest available $r$, which typically is $r = 1$. In this case we would end up with a tensionless string. An example of this is type IIB compactification on $K3$ with vanishing 2-cycle, in which case we have no 2-branes, but one can consider wrapping the available 3-brane of type IIB around the 2-cycle and this results in a tensionless string in 6 dimension. Such cases are a source of new phenomena in quantum field theories, and are believed to be resulting in new non-trivial conformal quantum field theories. The example just mentioned would be a non-trivial six dimensional critical quantum field theory. Similar examples can be constructed also upon compactification to lower dimensions.

4. Black Hole applications

Now we consider the case where the cycle $C$ cannot be made too small. These cases are also correlated with the fact that the cycle $C$ can be deformed inside the compactification geometry, while preserving its minimal area. Let us consider $K3$ compactification of type IIA with a fixed volume. Consider a genus $g$ Riemann surface $\Sigma$ in it with minimal area
(which implies that it is embedded holomorphically with some choice of complex structure on $K3$). In this case one can show that the volume of $C$ is always bigger than

$$V(C) \geq \text{const.} \sqrt{g}$$

Let us now consider wrapping a 2-cycle in type IIA theory compactified on $K3$ about a genus $g$ Riemann surface $\Sigma$. The resulting state will be a 0-brane (i.e. a particle) in 6 dimension with mass proportional to $V(C)$. Now let us consider the case where $g$ is large. In this case, using the above bound, we find that its mass becomes huge (of the order of $\sqrt{g}$). The resulting state can be viewed as a very massive charged particle in six dimensions. In fact it looks like a macroscopic black hole. One could then ask about comparisons between the area of the horizon and the number of such cycles $C$ and see if there is any relation, as would be expected based on Bekenstein-Hawking entropy formula. In this particular case the corresponding black hole solution has a singular horizon and so no exact reliable area can be extracted from it. Instead if we consider type IIB compactification on $K3 \times S^1$ and wrap a 3-brane around a Riemann surface $\Sigma$ of genus $g$ in $K3$ times the $S^1$ and in addition consider a state with some momentum $p$ along $S^1$ the corresponding black hole will have a non-singular horizon and one can compare the area of the horizon $A$ with the number of microscopic degrees of freedom of the 3-brane in that state.

The way to compute the microscopic degrees of freedom of this 3-brane is to first consider the effective 1+1 dimensional theory one gets after wrapping the 3-brane around the Riemann surface $\Sigma$. The number of degrees of freedom in the 1+1 dimensional theory will be related to number of ways the cycle $C$ (together with a choice of gauge field on it) can be deformed in $K3$. This gives a theory in 1+1 dimension with an effective $c = 6g$ degrees of freedom (4$g$ coming from bosonic modes and 2$g$ from fermionic modes). Then one looks at the number of states in this 1+1 dimensional theory on a circle with momentum $p$. This goes as

$$N \sim \exp[2\pi \sqrt{pc/6}] = \exp[2\pi \sqrt{pg}]$$

which agrees with the predicted entropy

$$S = \frac{1}{4} A = 2\pi \sqrt{pg}.$$
Similarly one can extend this analysis and construct other classes of black holes in 5 and 4 dimensions and compare the microscopic entropy with the predicted macroscopic entropy.

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