Probing the quark-gluon interaction with hadrons

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We present a unified picture of mesons and baryons in the Dyson-Schwinger/Bethe-Salpeter approach, wherein the quark-gluon and quark-(anti)quark interaction follow from a systematic truncation of the QCD effective action.

The masses of both the ground state mesons and baryons are found to be in reasonable agreement with contemporary ‘quark-core rainbow-ladder approximations’, suggesting a partial insensitivity to the details of the quark-gluon interaction. However, discrepancies remain in the meson sector, and for excited baryons, that may require the further investigation of higher order corrections, following the methods outlined herein.

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Hadrons provide a rich experimental environment for the study of the strong interaction, from details of the resonance spectrum to form factors and transition decays via electromagnetic probes. These reflect the underlying substructure of bound states by resolving, in a non-trivial way, the quarks and gluons of which they are composed. A theoretical understanding of hadrons in terms of these underlying degrees of freedom, interacting as dictated by quantum chromodynamics (QCD), is an on-going effort. Many approaches tackle it in different ways, simplifying certain aspects of the theory. Probing sensibly our theoretical constructs with experimental input thus provides understanding of the theory itself.

In continuum approaches to QCD, it is not possible in general to include all possible correlation functions in a calculation, as there are infinitely many of them. Although this can be viewed a priori as a limitation of continuum approaches, only a finite number of these correlation functions have a significant role in the observable properties of hadrons. Therefore, continuum methods provide an ideal framework to unravel the underlying mechanisms that generate observable effects from the elementary and non-observable degrees of freedom of QCD. This is in contrast to lattice QCD calculations, which can be viewed as theoretical experiments in the sense that, although they contain in principle all the dynamics of QCD, it is challenging to single out individual contributions to a particular measurement. This makes these two approaches complementary.

Amongst the different continuum approaches, the combination of Dyson–Schwinger (DSE) and Bethe–Salpeter equations (BSE) has proven to be extremely useful in the calculation of hadronic properties from QCD [1–3]. Typically, solutions of DSEs constitute the building blocks (propagators and vertices) of bound-state calculations using BSEs, which provide the bridge between QCD and observables. As described in more detail below, the particular interaction terms kept in the corresponding DSEs determine the interaction kernels among constituents in the BSEs, thereby defining a particular truncation of the DSE/BSE system. One works towards a model independent truncation by including a larger set of interaction terms; although this programme is obviously not achievable in its totality, significant progress can be made.

The construction of a reliable truncation is a challenging task. However, one is guided by various symmetries – notably that of chiral symmetry – that provide for stringent constraints. Due to the technical difficulties in finding an explicit implementation of these symmetries at the level of the quark and gluon interaction, simplifications are necessary and these typically fall into three categories: (i) the quark-gluon vertex is truncated to its tree-level component times a momentum-dependent effective coupling with the quark DSE and hadron BSEs solved self-consistently [4–24]; (ii) a more sophisticated model for the quark-gluon vertex is used, with the contribution from its different tensor structures modelled, hence abandoning self-consistency but gaining instead flexible insight into the relative importance of each of these structures [25–30]; (iii) some non-perturbative effects of the quark and gluon interaction are taken into account by solving the quark-gluon vertex DSE self-consistently, but potentially introducing some truncation artifacts. We follow the latter approach here with our motivation being two-fold: on one hand we demonstrate that it is technically possible to study both mesons and baryons using realistic QCD Green’s functions, thus removing the technical barriers that forced in the past a simplification of both the gauge and quark sector of QCD in order to enable the calculation of hadron properties using BSEs; on the other hand we wish to illustrate that the calculation of hadron properties from BSEs allows the size and relevance of truncation artifacts to be pinned down. Of course, such an approach is meaningful only because there exists a systematic procedure to introduce, order by order, interaction terms in the DSEs [31–34], akin to the loop expansion in perturbation theory. It is worth mentioning here the somewhat hybrid possibility of supplementing some of QCDs degrees of freedom in favor of effective ones, such as pions, has also been explored [35–39].

In the present work, we incorporate the results of a recent study of the quark-gluon vertex from its DSE [40] in the calculation of meson and baryon masses. Although
on the technical side this is no novelty for meson calculations [41, 42], it is the first time that a covariant three-body baryon calculation is performed beyond the rainbow-ladder (RL) approximation. While there exist other recent investigations of the quark-gluon vertex [43, 44], these have not yet been confronted with the challenge of reproducing hadron phenomena for reasons we discuss below.

The starting point for all truncations relevant to hadronic observables is the quark propagator DSE, see Fig. 1

\[ S^{-1}(p; \mu) = Z_2 S_0^{-1}(p) + \Sigma(p; \mu) , \]

with quark self-energy

\[ \Sigma(p; \mu) = g^2 Z_1 C_F \int_k \gamma^\mu S(q) \Gamma^\nu(q, p) D_{\mu\nu}(k) . \] (2)

Here \( q = k + p \), the integral measure is \( \int_k = \int d^4k/(2\pi)^4 \) and \( Z_1, Z_2 \) are renormalization constants. It is clearly dependent upon both the gluon propagator \( D_{\mu\nu}(k) \) and the quark-gluon vertex \( \Gamma^\nu(q, p) \). (The Landau gauge) propagators are

\[ S^{-1}(p) = Z_f^{-1}(p^2) \left( i\gamma^\mu M(p^2) \right) , \] (3)

\[ D_{\mu\nu}(k) = T_{(k)}^{\mu\nu} Z(k^2)/k^2 , \] (4)

with quark wavefunction \( Z_f^{-1}(p^2) \), dynamical mass \( M(p^2) \) and gluon dressing \( Z(k^2) \). The transverse projector is \( T_{(k)}^{\mu\nu} = \delta^{\mu\nu} - k^\mu k^\nu/k^2 \). The above-mentioned systematic construction of truncation schemes starts by using a 2PI effective action up to a certain loop order and deriving from it a truncated DSE for the full quark-gluon vertex \( \Gamma^\nu(q, p) \), which in turn determines the quark self-energy (2). A quark-antiquark interaction kernel is defined via a functional derivative

\[ [K^{\bar{q}q}]_{ik;lj} = -\frac{\delta}{\delta[S(x, y)]_{kl}} [\Sigma(x', y')]_{ij} , \] (5)

followed by a Fourier transform to momentum space [31–34]. This is the kernel that appears in the BSE description of a meson as a quark-antiquark bound state

\[ [\Gamma_M(p, P)]_{ij} = \int_k [K^{\bar{q}q}]_{ik;lj} [\chi_M(k, P)]_{kl} , \] (6)

where \( \Gamma_M(p, P) \) is the Bethe–Salpeter amplitude and \( \chi_M(k, P) = S(k_+) \Gamma_M(k, P) S(-k_-) \) its wavefunction.

The quantum numbers of the state are defined by its covariant decomposition [11, 23, 45]. This gives access to both the mass of the bound-state as well as details of its internal structure. The crucial motivation for such a definition of the Bethe–Salpeter kernel is that chiral symmetry, as expressed via the axial-vector Ward–Takahashi identity (axWTI)

\[ [\Sigma(p_+\gamma_5 + \gamma_5 \Sigma(p_-))]_{ij} = \int_k [K^{\bar{q}q}]_{ik;lj} [\Sigma(k_+)\gamma_5 + \gamma_5 \Sigma(k_-)]_{kl} , \] (7)

is correctly implemented in the calculation of meson properties [46]. This guarantees, in particular, the identification of the pion as a Goldstone boson in the chiral limit.

For example, consider the rainbow-ladder truncation where the dressed quark-gluon vertex is replaced by \( \gamma^\mu \) and the gluon propagator is modelled by a phenomenological function. Then, suppressing constants and color factors for simplicity, we have

\[ [\Sigma(p)]_{ij} \simeq \int_k [\gamma^\mu S(q)\gamma^\nu]_{ij} D_{\mu\nu}(k) , \] (8)

from which a functional derivative provides the kernel

\[ [K^{\bar{q}q}]_{ik;lj} \simeq [\gamma^\mu]_{ik} [\gamma^\nu]_{lj} D_{\mu\nu}(q) . \] (9)

Since we wish to also consider bound-states of three-quarks, it proves useful as an intermediate step to formulate the diquark bound-state

\[ [\Gamma_D(p, P)]_{ij} = \int_k [K^{\bar{q}q}]_{ik;lj} [\chi_D(k, P)]_{kl} , \] (10)

with diquark amplitude \( \Gamma_D(p, P) \) and wavefunction \( \chi_D(k, P) = S(k_+) \Gamma_D(k, P) S^T(-k_-) \). The superscript \( T \) denotes transposition; the covariant decomposition of a \( J^{++} \) diquark \( \Gamma_D \) is that of a \( J^{++} \) meson \( \Gamma_M C \) together with a charge conjugation matrix. The corresponding diquark kernel is then

\[ [K^{\bar{q}q}]_{ik;lj} = [K^{\bar{q}q}]_{ik;lj} , \] (11)

that is, it amounts to a transposition of the lower spin-line. For general ladder-like kernels, such as that in (9) the color factors are \((N_c^2 - 1)/2N_c\) and \(-(N_c + 1)/2N_c\) for a meson and diquark, respectively.

The analogue of the Bethe–Salpeter equation for baryons can now be formulated. It contains the permuted sum of
This dominant self-energy correction has the form

\[ \Phi = \left[ K^{(qq)} \right] G_0^{(3)} \Phi + \sum_{a=1}^{3} \left[ K_a^{(qq)} \right] S^{-1}_a G_0^{(3)} \Phi, \quad (12) \]

see Fig. 2. We use here a compact notation, omitting indices, where implied discrete and continuous variables are summed or integrated over, respectively. Here \( G_0^{(3)} \) represents the product of three fully-dressed quark propagators \( S \). The subscript \( a \) labels the quark spectator to the two-body interaction.

At the truncation level of the 2PI effective action we will consider in this work, the irreducible three-body force is identically zero [34]. The two-body kernel follows from the quark and quark-gluon vertex DSE. The truncated vertex DSE can be given as a summation of self-energy contributions

\[ \Gamma^\mu(l, k) = Z_G \gamma^\mu + \Lambda_{\text{NA}}^\mu + \Lambda_{\text{AB}}^\mu + \ldots \quad (13) \]

with \( \Lambda_{\text{NA}}^\mu \) and \( \Lambda_{\text{AB}}^\mu \) the non-Abelian and Abelian contributions, respectively (see Fig. 3, and the ellipsis denotes higher loop contributions. It is important to stress that when the 2PI effective action is used to define the truncation at the level of vertex functions, the expansion must be chosen such that the functional derivative (5) can be formally performed to define the chiral-symmetry preserving BSE kernels. This implies that it must be possible to resolve diagrammatically the quark lines in the vertex self-energy (13). An alternative to this is using a 3PI or higher effective action to define the truncation, which means that vertices and propagators are independent objects [32, 34]. In Ref. [40] the Abelian correction \( \Lambda_{\text{AB}}^\mu \) and higher loop terms were not considered, retaining only the non-Abelian correction \( \Lambda_{\text{NA}}^\mu \) that connects the gauge to the matter sector. This dominant self-energy correction has the form

\[
\Lambda_{\text{NA}}^\mu(l, k) = \frac{N_c}{2} \int q \tilde{\Gamma}_\alpha(l_1, -q_1) S(q_3) \tilde{\Gamma}_\beta(l_2, -q_2) \\
\times \Gamma_{3\delta}^{\alpha'\beta'}(q_1, q_2, p_3) D^{\alpha\alpha'}(q_1) D^{\beta\beta'}(q_2); \quad (14)
\]

where the internal vertices \( \tilde{\Gamma}^\mu \) retain only the tree-level tensor structure in order to allow the cutting of quark lines in closed form. Further details on this equation and its solution can be found in [40].

With this truncation of the vertex DSE, the cutting process gives rise to a tractable quark-(anti)quark kernel, given in Fig. 4, whose topology is analogous to a gluon ladder

\[
\left[ K^{(q\bar{q})} \right]_{ij,mn} = D_{\mu\nu}(q) \left[ [\gamma^\mu]_{ij} [\gamma^\nu]_{mn} \\
+ [\gamma^\mu]_{ij} [\Lambda^\nu]_{mn} + [\Lambda^\mu]_{ij} [\gamma^\nu]_{mn} \right], \quad (15)
\]

This, taken with the truncated quark-gluon vertex above, satisfies chiral symmetry by construction [46]. Further corrections to the quark-gluon vertex, in particular those featuring more implicit quark propagators, yield different topologies and/or are higher loop order; they are beyond the scope of the present work but can in principle be included systematically.

A consistency check that chiral symmetry is correctly implemented is provided by calculating the pion mass in the chiral limit. Finding a massless pion, as we indeed do, serves as a verification that the numerics are under control. In [40] the internal vertices were adjusted such that the resulting quark running mass agrees with lattice calculations. Once the vertex truncation is defined, the only free parameter left is the current-quark mass. The light quark mass is then fixed to \( m_{u/d} = 3.7 \text{ MeV} \) so that the physical pion mass is obtained.

In Table I we show the ground-state meson masses below 1.4 GeV for two quark flavors for rainbow-ladder (using the Maris-Tandy interaction [5]) and beyond rainbow-ladder truncation presented here. The case of the scalar and vector states is remarkable in that their

| Meson          | RL | BRL | PDG [47] |
|---------------|----|-----|-----------|
| \( 0^{-+} (\pi) \) | 0.14 | 0.14 | 0.14       |
| \( 0^{++} (\sigma) \) | 0.64 | 0.52 | 0.48 (8)  |
| \( 1^{-+} (\rho) \)    | 0.74 | 0.77 | 0.78       |
| \( 1^{++} (a_1) \)    | 0.97 | 0.96 | 1.23 (4)  |
| \( 1^{++} (b_1) \)    | 0.85 | 1.1  | 1.23       |
| \( f_\pi \)            | 0.092 | 0.103 | 0.092     |
TABLE II. Baryon masses in GeV as calculated in rainbow-ladder [15, 18, 53, 54] and beyond rainbow-ladder.

|        | RL | BRL | PDG [47] |
|--------|----|-----|----------|
| 1/2− (N) | 0.94 | 1.05 | 0.94 |
| 1/2− (N*) | 1.16 | 1.08 | 1.54(1) |
| 3/2+ (∆) | 1.22 | 1.24 | 1.23 |

masses agree well with the experimental value of the ρ- and σ- mesons; this is, however, an unexpected feature. On the one hand, it is well known that unquenching effects (absent in this calculation) such as pion-cloud corrections [36, 38, 48] and effects associated with decay processes [49, 50] provide extra attraction for vector and/or scalar channels. On the other hand, it is not yet settled whether the physical σ-meson originates from a qq state. Moreover, even if that was the case, the measured mass would be the result of the dynamical coupling of this bare state with several other channels [51, 52]. In this respect, it is interesting to note that several DSE/BSE calculations [23, 25, 29], including the present one, generate a relatively light scalar qq state and hence disfavour a purely dynamical origin of the σ-resonance.

The previous remarks can be summarized in that one would expect our quark-core calculation to lead, in general, to heavier meson masses. The fact that this is not the case is an indication that other non-Abelian mechanisms, absent in the present calculation, are of relevance even in the ground-state meson sector. Additional support for this is found in the a1/b1 − ρ and a1 − b1 splitting; whereas the former is improved with respect to the RL value, the latter is exceedingly large showing an imbalance in the different components of the present quark-gluon vertex.

Let us now turn our attention to baryons. In Table II we show the calculated nucleon and delta-baryon masses. We also calculated the mass of the nucleon parity partner 1/2−, as it is the first signature of the failure of RL in the baryon sector. Clearly, the masses for positive-parity states are slightly overestimated. This suggests that the simple quark-core picture supplemented with pion-cloud effects [35, 38, 39, 48, 55] could be realized for ground-state baryons. The possibility that ground-state baryons are, in contrast to mesons, less sensitive to the details of the quark-gluon interaction is an interesting one. It explains why the simple rainbow-ladder truncation has been thus far so successful in describing baryon properties [15, 18, 53, 54].

The situation for excited baryons does not appear to be so simple. In RL both the negative-parity nucleon and the Roper resonance come out extremely light [53] unless one inflates the interaction strength [56, 57], thus spoiling the agreement for other observables. The present calculation, using a genuine (albeit truncated) solution of a QCD-vertex DSE, does not improve on the situation of the nucleon’s negative-parity partner. This result by itself already makes a case for the systematic introduction of interaction mecha-

TABLE III. Sigma terms in MeV as calculated in rainbow-ladder and beyond rainbow-ladder.

|        | RL | BRL | Other [58–66] |
|--------|----|-----|---------------|
| 1/2− (N) | 30 | 60 | 25–60 |
| 3/2+ (∆) | 24 | 63 | 32–79 |

nisms, as it is not known a priori whether missing Abelian and/or non-Abelian mechanisms are leading in determining the spectrum of (bare) baryon resonances.

Another interesting aspect of using as interaction a solution of the quark-gluon vertex DSE is that it naturally features a quark-mass dependence of the interaction strength. One class of observables in which such a mass dependence is manifest are the baryon sigma-terms. Calculated using the Feynman-Hellmann theorem, they measure the dependence of the baryon mass on the quark mass. As shown in Table III, these are exceedingly small in RL, owing to the quark-mass independence of the effective interaction. This changes dramatically in the present calculation. For the nucleon we now obtain a value in reasonable agreement with the upper limit of the consensual range [58–62]. For the delta, although there is no well established value to serve as a reference, we find a similarly large result for the sigma term. However, this could be related to the absence of decay mechanisms that induce non-analytic behaviours in the baryon-mass curve near the physical point.

In summary, we presented the first calculation of both mesons and baryons in a mutually consistent truncation of the quark-gluon vertex beyond rainbow-ladder. In itself this is a huge technical achievement that is not tied to the particular vertex ansatz, and is in principle systematically extensible such that crossed ladder and higher order corrections can be incorporated [34]. The truncation naturally exhibits a dependence on the quark mass, which will form the basis of future investigations of the baryon octet-decuplet and meson nonet for the strange quark, as well as heavy-quark studies of charmonium and bottomonium. Additionally, the impact of corrections beyond rainbow-ladder on the internal structure of the hadrons must be tested through the calculation of electromagnetic form-factors.

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