The Effects of Light-Wave Nonstaticity on Accompanying Geometric-Phase Evolutions

Jeong Ryeol Choi

Department of Nanoengineering, Kyonggi University,
Yeongtong-gu, Suwon, Gyeonggi-do 16227, Republic of Korea

Abstract
Quantum mechanics allows the emergence of nonstatic quantum light waves in the Fock state even in a transparent medium of which electromagnetic parameters do not vary over time. Such wave packets become broad and narrow in turn periodically in time in the quadrature space. We investigate the effects of wave nonstaticity arisen in a static environment on the behavior of accompanying geometric phases in the Fock states. In this case, the geometric phases appear only when the measure of nonstaticity is not zero and their time behavior is deeply related to the measure of nonstaticity. While the dynamical phases undergo linear decrease over time, the geometric phases exhibit somewhat oscillatory behavior where the center of oscillation linearly increases. In particular, if the measure of nonstaticity is sufficiently high, the geometric phases abruptly change whenever the waves become narrow in the quadrature space. The understanding for the phase evolution of nonstatic light waves is necessary in their technological applications regarding wave modulations.

Keywords: wave nonstaticity; geometric phase; light wave; Hannay angle; wave function
1. INTRODUCTION

The Berry’s seminal discovery [1] to the appearance of an additional phase evolution in eigenstates of the Hamiltonian during a slow variation of a quantum system in the parameter space had triggered extensive research for such extra phases in both theoretical and experimental spheres. The Berry phase is a geometrical character of quantum waves, which corresponds to a holonomy transformation in state space. It is impossible to gauge out the geometrical character in the phase because the Berry phase or, in general, the geometric phase is a gauge invariant. For this reason, the geometric phase is non-negligible and, especially, inevitable for analyzing the transmission of light waves in media with time-varying parameters since it reflects the geometry of a quantum wave evolution. It has been proved that the original concept of the Berry’s geometric phase can be extended to more general cases which are nonadiabatic, non-cyclic and/or non-unitary evolutions of light waves [2, 3].

The geometric phase is a promising research subject that has been widely investigated with the purpose of manipulating quantum phases of light waves and controlling their behaviors. The scientific fields that the geometric phase can be applicable are plentiful: they include a holonomic quantum computation with geometric gates [4], a stellar interferometry [5], the testing of CPT (charge conjugation, parity, and time reversal) invariance in particle physics [6], entanglement of atoms [7] analysis of Aharonov-Bohm effect [8], etc. Among them, a geometric quantum computation enables us to carry out quantum logic operations by means of multi-qubit gates, which is a main technique for realizing quantum computers [4].

It is assumed that the geometric phase appears for nonstatic states of a quantum system, whereas it always vanishes for stationary states [9]. As is well known, the ordinary waves in the Fock state are static in time, resulting in no emergence of the geometric phase. However, if we prepare a quantum wave with time-varying eigenfunctions, there will appear geometric phases even for a simple situation where the parameters of the medium do not vary over time. For instance, the eigenfunctions in coherent and squeezed states are expressed in terms of time regardless that the parameters of the media depend on time or not [10]. This leads to the
appearance of the geometric phase in such states \[10, 11\].

Meanwhile, it was reported from our recent work \[12\] that nonstatic quantum light waves can also appear in the Fock state in a static environment associated with a transparent medium. During the time evolution of such wave packets, the waves exhibit a peculiar behavior as a manifestation of their nonstaticity, which is that they become narrow and broad in turn periodically in quadrature space. Subsequently, we also analyzed the mechanism underlain in such a phenomenon from a fundamental point of view \[13\].

Even if the environment is static in that case, the periodical time variation of the waves is accompanied by the evolution of the geometric phases. This is due to the fact that the eigenfunctions also vary in time along the nonstaticity of the wave. We will investigate the characteristics of the geometric phases in the Fock state arisen in such a situation in this work. It will be focused on analyzing how the geometric phases evolve in time in relation with the wave nonstaticity. We will compare the behavior of the geometric phases with that of the dynamical phases. The Hannay angle \[14\] of the system, which is the classical analogues of the geometric phases, will also be investigated by utilizing the relation between them and its physical meanings will be addressed. Based on the Hannay angle, we can obtain an insight on the classical geometric structure of light and its connection with the quantum geometric-phase structure \[15, 16\].

2. DESCRIPTION OF NONSTATIC WAVES

To establish the geometric phases for a nonstatic wave, we first show how to describe nonstatic quantum light waves in a static environment. The Hamiltonian for a light is given by

\[ \hat{H} = \frac{\hat{p}^2}{2\epsilon} + \epsilon \omega^2 q^2 / 2, \]  

(1)

where \( \hat{q} \) is the quadrature operator, \( \hat{p} = -i\hbar \partial/\partial q \), and \( \epsilon \) is the electric permittivity of the medium. The geometric phases are dependent on the preparation of the wave functions \[17\]. If we consider elementary static wave functions in the Fock state, of which eigenfunctions are given by \( \langle q | \phi_n \rangle = (\alpha/\pi)^{1/4} (\sqrt{2^n n!})^{-1} H_n (\sqrt{\alpha} q) \exp \left[ -\alpha q^2 / 2 \right] \) where \( \alpha = \epsilon \omega / \hbar \) and \( H_n \) are Hermite polynomials, the geometric phases do not take place \[10, 18\]. However, for the case of
the wave functions whose eigenfunctions are time-dependent, the geometric phases are nonzero because the geometric phases are given in terms of the time derivative of the eigenfunctions. Notice that, for a time-independent Hamiltonian, including the case regarded here, there are Schrödinger solutions associated with wave nonstaticity as well as the ones that correspond to static waves.

Instead of $\langle q|\phi_n \rangle$, nonstatic waves in this context can be described with generalized eigenfunctions of the form [12]

$$
\langle q|\Phi_n \rangle = \left( \frac{\beta(t)}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n \left( \sqrt{\beta(t)}q \right) \exp \left[ -\frac{\beta(t)}{2} \left( 1 - i \frac{\dot{f}(t)}{2\omega} \right) q^2 \right],
$$

where $\beta(t) = \epsilon \omega / [\hbar f(t)]$ and $f(t)$ is a time function which is given by

$$
f(t) = A \sin^2 \tilde{\varphi}(t) + B \cos^2 \tilde{\varphi}(t) + C \sin[2\tilde{\varphi}(t)],
$$

under an auxiliary condition, $AB - C^2 = 1$ with $AB \geq 1$, while $\tilde{\varphi}(t) = \omega(t - t_0) + \varphi$ whereas $\varphi$ is a real constant. It is now possible to establish time-dependent wave functions associated with the nonstatic wave in terms of $\langle q|\Phi_n \rangle$, such that [12]

$$
\langle q|\Psi_n(t) \rangle = \langle q|\Phi_n(t) \rangle \exp \left[ -i \omega(n + 1/2) \int_{t_0}^{t} f^{-1}(t')dt' + i \gamma_n(t_0) \right],
$$

where $\gamma_n(t_0)$ are phases at $t_0$. We note that the wave functions given above satisfy the Schrödinger equation associated with the Hamiltonian represented in Eq. (1), and $f(t)$ used here follows the nonlinear differential equation of the form $\ddot{f} - (\dot{f})^2/(2f) + 2\omega^2 (f - 1/f) = 0$.

We will restrict the classical angle within the range $-\pi/2 \leq \varphi < \pi/2$, because the research in this range is enough owing to the fact that Eq. (3) is a periodic function with the angle period $\pi$.

For $A = B = 1$ and $C = 0$, Eq. (3) reduces to $f(t) = 1$ which corresponds to the case that gives static wave functions; All other choices for the set of $A$ and $B$ give nonstatic wave functions. We have compared static and nonstatic wave packets in Fig. 1. The nonstatic wave packets shown in Fig. 1(B) vary periodically over time. The degree of such time variation caused by nonstaticity is determined by the quantitative measure of nonstaticity. For detailed expression of the nonstaticity measure, see Appendix A. The nonstatic-wave packets described up until now will be used in order to investigate the geometric phases in the subsequent subsection.
FIG. 1: Comparison between probability densities for static (A) and nonstatic (B) wave packets. We have chosen the time function \( f(t) \) for A as 1 and for B as Eq. (3) with \( A = 2.5 \) and \( B = 0.5 \). We take only positive values for \( C \) throughout all figures in this work for convenience. Then, the value of \( C \) is automatically determined from \( A \) and \( B \) through the auxiliary condition given below Eq. (3). Other values that we have chosen are \( n = 5 \), \( \omega = 1 \), \( \epsilon = 1 \), \( h = 1 \), \( t_0 = 0 \), and \( \varphi = 0 \). All variables are chosen to be dimensionless for convenience; this rule will also be applied to subsequent figures. For the nonstatic case, the probability density undergoes cyclic evolution with time period \( T = \pi/\omega \), i.e., the width of the wave packet becomes broad and narrow in turn over time.

3. GEOMETRIC PHASE

The geometric phases are examples of holonomy which gives additional phase evolutions of the
quantum wave over time. The development of the geometric phases for one-dimensional simple
wave description offers essential ideas which enable us to demonstrate topological features in
quantum mechanics. The geometric phases for the nonstatic waves can be evaluated from
\[ \gamma_{G,n}(t) = \int_{t_0}^{t} \langle \Phi_n(t') | i \frac{\partial}{\partial t'} | \Phi_n(t') \rangle dt' + \gamma_{G,n}(t_0). \] (5)
These are parts of the phases of quantum wave functions at time \( t \), which have geometric origin.
On one hand, there is a concept of the geometric phase whose definition is a little different: it
is the geometrical part of the phase acquired during only one cycle evolution of the eigenstate
through a closed path in the circuit [11]. In what follow, we will use the former concept of the
geometric phase associated with Eq. (5) throughout this paper. We assume that the initial
phases are zero for convenience from now on: \( \gamma_{G,n}(t_0) = 0 \). It is possible to evaluate Eq. (5) by
using Eq. (2) with the consideration of \( f(t) \) given in Eq. (3). Hence, we have (see METHODS
section which is the last section)
\[ \gamma_{G,n}(t) = \frac{1}{2} \left( n + \frac{1}{2} \right) \left\{ (A + B)\omega(t - t_0) - 2[\tan^{-1} Z(t) - \tan^{-1} Z(t_0) + G(t)] \right\}, \] (6)
for \( t \geq t_0 \), where \( Z(\tau) = C + A \tan[\omega(\tau - t_0) + \varphi] \), and \( G(t) \) is a time function that is expressed
in terms of the unit step function (Heaviside step function) \( u[t] \) as \( G(t) = \pi \sum_{m=0}^{\infty} u[t - t_0 - (2m + 1)\pi/(2\omega) + \varphi/\omega] \). If we regard the periodical discontinuities of tangent functions in Eq.
(6) with a period of \( \pi \), \( G(t) \) is necessary in order to compensate them in a way that \( \gamma_{G,n}(t) \)
become continuous functions. We note that Eq. (6) holds within the considered range for \( \varphi \),
\(-\pi/2 \leq \varphi < \pi/2 \) (see the previous subsection for this range). Thus we have obtained the
geometric phases for the nonstatic light waves. The formula of the geometric phases given
above may provide a deeper insight for the understanding of the nature of the nonstatic waves.

The dynamical phases can also be derived from their definition using the same wave functions
and are given by \( \gamma_{D,n}(t) = -(1/2)(n + 1/2)(A + B)\omega(t - t_0) \) (see METHODS section). If we
regard that the measure of nonstaticity shown in Appendix A is nearly proportional to \( A + B \)
provided that \( A + B \gg 4 \), \( \gamma_{D,n}(t) \) at a certain time is linearly proportional to the measure of
nonstaticity for highly nonstatic waves.
FIG. 2: Time evolution of the geometric phase in addition to the dynamical phase for several different values of the parameters. The values of \((A, B, n, \omega)\) are \((1, 1, 0, 0.5)\) for \(A\), \((0.5, 2.5, 5, 0.5)\) for \(B\), and \((0.1, 10.0, 10, 1)\) for \(C\). We have chosen other parameters as \(t_0 = 0\), and \(\varphi = 0\). The pink-white graphics in the upper part of the panels are the time evolution of the corresponding probability density. The measure of nonstaticity is 0.00 for \(A\), 0.79 for \(B\), and 3.50 for \(C\).
Whereas the dynamical phases linearly decrease over time from their initial values, the time behavior of the geometric phases is not so simple. Let us divide the geometric phase in Eq. (6) into two parts for the convenience of analyses. We call the term that involves \( (A + B) \) as the first part and the remaining terms the second part. We readily confirm that the first part, which increases in a monotonic manner over time, exactly cancels the dynamical phase. Hence, the second part of the geometric phase is the same as the total phase of the wave. However, it may be not so easy to completely estimate the evolution of the geometric phases since the second part is somewhat intricate.

To understand the overall time behavior of the geometric phases, we plotted the evolution of the geometric phase in Fig. 2 together with the dynamical phase for several different values of parameters. Let us first examine the effects of \( A \) and \( B \) on the geometric phase. For a trivial case where \( A = B = 1 \) and \( C = 0 \), which corresponds to Fig. 2(A), \( f(t) \) becomes unity and the eigenfunctions, Eq. (2), reduce to time-independent ones as have seen from the previous subsection. As a consequence, the geometric phases result in \( \gamma_{G,n}(t) = 0 \) whereas the dynamical phases become \( \gamma_{D,n}(t) = -(n + 1/2)\omega(t - t_0) \). Hence, the geometric phases do not appear in this case as it should be, while the dynamical parts are the well known formula. We confirm from Fig. 2(A), which is the case of the static wave, that the second part of the geometric phases linearly decreases over time, leading exact cancelling with the first part. In addition, the second part of the geometric phase is the same as the dynamical phase only when \( A = B = 1 \).

If at least one of \( A \) and \( B \) is not unity, it becomes the case of a nonstatic wave as shown in Figs. 2(B) and 2(C). Then, the width of the corresponding probability density periodically varies over time with the period of \( T = \pi/\omega \). Hence, the frequency in its periodic change is large when \( \omega \) is high. We can confirm from Figs. 2(B) and 2(C) that the second part of the geometric phase changes depending on the width of the wave packet. In a moment when the width is large, the second part nearly monotonically decreases over time. However, when the width is small, the second part somewhat abruptly decrease. Thus, the geometric phase, which is the addition of the first and the second parts in the figure, varies according to the evolution of the wave packet. We can more concretely demonstrate these situations from Fig. 3 which
FIG. 3: Behavior for the time derivative of the geometric phase $d\gamma_{G,n}(t)/dt$, the dynamical phase $d\gamma_{D,n}(t)/dt$, and the total phase $d\gamma_n(t)/dt$. All chosen parameters for A and B are the same as those for B and C in Fig. 2, respectively.

shows that the time derivative of the geometric phase abruptly drops when the width of the wave packet is narrow. By the way, the dynamical phases vary in a monotonic manner in all situations. In this way, the system will pick up a memory of its time evolution in the form of the geometric phase which will contribute to an observable shift of the phase in the wave function.
FIG. 4: The evolution of the second part of the geometric phase for several values of $A$ (A) and $B$ (B), where $B = 1$ for A, $A = 1$ for B, $n = 5$, $\varphi = 0$, and $t_0 = 0$. The conventions for colors designated for solid lines in the legends are also applied to the dashed and the short dashed lines within the figure panels. The measure of nonstaticities in turn from red to violet curve are 0.00, 0.79, 2.00, 3.82, 7.39, 14.48, and 35.70 for both A and B.

The evolution of the second part of the geometric phase is illustrated in Fig. 4. Although the first part of the geometric phase becomes large as the value of $A + B$ (or the measure of nonstaticity) increases, the envelope of the second part is not so significantly affected by the
values of $A$ and $B$. Actually, the gradient of the envelope of the second part is irrelevant to the measure of nonstaticity; such a gradient is determined by $\omega$ instead. The envelope of the second part decreases more rapidly as $\omega$ grows, whereas the first part increases more rapidly at the same condition.

Figure 4 shows that, when both $A$ and $B$ are unity, the second parts linearly decrease as time goes by. On the other hand, if $A$ and/or $B$ deviate from the unity, the gradient in the phase evolution is not constant over time. We can confirm from Fig 4(A) that the second parts abruptly drop when $\omega t$ is $\pi$, $2\pi$, $3\pi$, etc. provided that $A$ is very large while $B$ is unity; however, except for these moments, the second parts almost do not vary over time. As a consequence, the bottom envelope of the second parts is identical to the standard value which is $-(n + 1/2)\omega(t - t_0)$. Similar behaviors in the phase evolution can also be seen from specific curves in Fig. 4(B), which correspond to the case where $B$ is very large while $A$ is unity; in this case, the second parts drop when $\omega t$ is $\pi/2$, $3\pi/2$, $5\pi/2$, etc.

The dependence of the geometric-phase evolution on $\varphi$ within the considered region $-0.5\pi \leq \varphi < 0.5\pi$ is shown in Fig. 5. All geometric phases in the figure start from zero, but the interval of phase between the adjacent geometric phases is $\Delta \varphi = 0.15\pi$. Because the geometric phase periodically varies over $\varphi$ where the period of such a variation is $\pi$, it is not difficult to know the pattern of the geometric phase outside the considered region for $\varphi$. More precisely speaking, the value of the Berry phase for $\varphi = \pi$ is exactly the same as that for $\varphi = 0$; the value of the Berry phase for $\varphi = 1.15\pi$ is exactly the same as that for $\varphi = 0.15\pi$, etc.

The overall phases of the waves are given as $\gamma_{n}(t) = \gamma_{G,n}(t) + \gamma_{D,n}(t)$. Hence, by adding our two results for geometric and dynamical phases, we have the total phases such that $\gamma_{n}(t) = -\omega(n + 1/2) \int_{t_0}^{t} f^{-1}(t') dt'$. These formulae are exactly the same as the phases having appeared in the wave functions, Eq. [4], under the condition that the initial phases are zero, $\gamma_{n}(t_0) = 0$.

Hannay confirmed that the phase similar to the geometric phase also appears in the classical domain as an analogy with the geometric phase [14]. This is a geometrical angle for a classical wave which allows to estimate its holonomy effect from the corresponding geometrical interpretation. This is an important theoretical concept and, hence, it may be instructive to
FIG. 5: Time evolution of the geometric phase for various values of $\varphi$. We have chosen parameters as $A = 2.5$, $B = 0.5$, $n = 0$, $\omega = 0.5$, and $t_0 = 0$.

see the Hannay angle of the system. For an integrable classical system, action-angle variables theory at the semiclassical level from torus quantization gives the fact that the Hannay angle $\Theta_H(t)$ is related to the geometric phase as $\Theta_H(t) = -\partial \gamma_{G,n}(t)/\partial n$ [21, 22]. From this, we can easily confirm that the Hannay angle in this case is of the form $\Theta_H(t) = -2\gamma_{G,0}(t)$. This natural relation reveals that the evolution of the classical additional angle can be represented in terms of the geometric phase associated with the zero-point quantum wave packet. This simple clear picture provides a significant geometric meaning for unification of the gauge structure of the wave propagation for quantum and classical mechanics, leading in effect to Bohr’s correspondence principle [23]. If we develop a method to generate the generalized wave packet given by Eq. (4) in the future, the demonstration of this consequence may be possible by measuring Hannay angle using an averaging technique introduced in Refs. [24–26]. The reason why the Hannay angle takes place in the classical system is that the structure of the quantum Hilbert space is quite the same as the classical phase space [27, 28].
4. CONCLUSION

Ordinary quantum light waves, which propagate through a transparent medium in which electromagnetic parameters do not vary, are static in time, resulting in no appearance of the geometric phase in Fock states. However, if one or both of the parameters $A$ and $B$ deviates from unity, the waves in the Fock states in that medium become nonstatic and, as a consequence, the geometric phases emerge. Not only the appearance of such nonstatic waves but also the characteristics of the resultant geometric phases may be noteworthy [12]. We have analyzed the influence of the wave nonstaticity on the evolution of the geometric phases in such a case.

The geometric phases of the light waves exhibit periodical oscillatory behavior with the period of $\omega T = \pi$, where the center of such oscillation linearly increases over time. On the other hand, the dynamical phases always show linear decrease. Because the scale of the geometric phases is smaller than that of the dynamical phases, the total phases evolve toward opposite direction over time. If the measure of nonstaticity is high, both the geometric and the dynamical phases rapidly evolve in time. Although the geometric phases increase on the whole, they periodically drop with the angle period of $\pi$ provided that the measure of nonstaticity is sufficiently high. Such a variation of the geometric phases is quite significant when the measure of nonstaticity is extremely large.

We have shown that Hannay angle, which is the classical analogue of the geometric phase, is represented in terms of the geometric phase associated to the zero-point wave function. This elegant outcome shows a unified picture of the interpretation of the geometric character of light waves in the quantum and the classical regime. This not only bridges the quantum and classical world, but can also be extended to more generalized quantum light waves, such as the light in a squeezed state and the Gaussian wave packet propagating in time-varying media [29, 30]. The understanding of the evolution of the geometric phases for nonstatic quantum light waves in a static environment is necessary in quantum optics, especially in relation with wave modulations. Practical utilization of the geometric phases in diverse scientific areas may be available under the fundamental knowledge associated with the behavior of quantal phases.
On one hand, the geometric phase of quantum systems which evolve in a non-unitary way and its relation with wave nonstaticity as well as accompanying non-unitary effects such as decoherence and dissipation may be worthwhile to be explored in a next research. Kinematic approach to the geometric phase of non-unitarily evolving systems is important in the pursuing of robustness of geometric quantum computation.

5. METHODS

Let us see how to evaluate the geometric and the dynamical phases. We first derive the geometric phases. From a minor computation in the configuration space after inserting Eq. (2) into Eq. (5), we have under the condition $\gamma_{G,n}(t_0) = 0$:

$$\gamma_{G,n}(t) = \frac{1}{2} \left(n + \frac{1}{2}\right) \Gamma_G,$$

where

$$\Gamma_G = \omega [g_1(t) - g_2(t)] + \frac{g_3(t)}{4\omega},$$

with

$$g_1(t) = \int_{t_0}^{t} f(t') dt',$$

$$g_2(t) = \int_{t_0}^{t} \frac{1}{f(t')} dt',$$

$$g_3(t) = \int_{t_0}^{t} \frac{\dot{f}(t')^2}{f(t')} dt'.$$

Straightforward evaluations of $g_i(t)$ ($i = 1, 2, 3$) using Eq. (3) yield

$$g_i(t) = G_i(t) - G_i(t_0),$$

for $t_0 \leq t < t_0 + \pi/(2\omega) - \varphi/\omega$, where

$$G_1(\tau) = \frac{1}{4\omega} \{2(A + B)\omega \tau - (A - B) \sin\{2[\omega(\tau - t_0) + \varphi]\} - 2C \cos\{2[\omega(\tau - t_0) + \varphi]\}\},$$

$$G_2(\tau) = \frac{1}{\omega} \tan^{-1}\{C + A \tan[\omega(\tau - t_0) + \varphi]\},$$

$$G_3(\tau) = \omega \{2(A + B)\omega \tau + (A - B) \sin\{2[\omega(\tau - t_0) + \varphi]\} + 2C \cos\{2[\omega(\tau - t_0) + \varphi]\} - 4 \tan^{-1}\{C + A \tan[\omega(\tau - t_0) + \varphi]\}\}. $$
By readjusting Eq. (8) with Eqs. (12)-(15), we have

\[ \Gamma_G = F_G(t) - F_G(t_0), \]  

(16)

where \( F_G(\tau) \) is given by

\[ F_G(\tau) = (A + B)\omega\tau - 2\tan^{-1}\{C + A\tan[\omega(\tau - t_0) + \varphi]\}. \]  

(17)

Equation (16) hold for the region \( t_0 \leq t < t_0 + \pi/(2\omega) - \varphi/\omega \), because we have considered \( t \geq t_0 \) and there is the first discontinuity in the tangent function at \( \pi/2 \). If we want to extend the expression in this equation to the whole region \( (t \geq t_0) \) that we have considered, it is necessary to compensate Eq. (16) by the unit step function \( u[t] \) to be

\[ \Gamma_G = F_G(t) - F_G(t_0) - 2\pi \sum_{m=0}^{\infty} u[t - t_0 - (2m + 1)\pi/(2\omega) + \varphi/\omega]. \]  

(18)

By rearranging Eq. (7) with Eqs. (18) and (17), we can easily have the formula of geometric phases which are given in Eq. (6) in the text.

Now we evaluate the dynamical phases. The definition of the dynamical phases are given by

\[ \gamma_{D,n}(t) = -\frac{1}{\hbar} \int_{t_0}^{t} \langle \Phi_n(t') | \hat{H}(\hat{q}, \hat{p}, t') | \Phi_n(t') \rangle dt' + \gamma_{D,n}(t_0). \]  

(19)

We also assume that \( \gamma_{D,n}(t_0) = 0 \) like the case of the geometrical part. Using the Hamiltonian of the simple harmonic oscillator and the Fock states given in Eq. (2), we can evaluate Eq. (19) and this results in

\[ \gamma_{D,n}(t) = \frac{1}{2} \left( n + \frac{1}{2} \right) \Gamma_D, \]  

(20)

where

\[ \Gamma_D = -\omega[g_1(t) + g_2(t)] - \frac{g_3(t)}{4\omega}. \]  

(21)

Using Eqs. (12)-(15), we have

\[ \Gamma_D = F_D(t) - F_D(t_0), \]  

(22)

where

\[ F_D(\tau) = -(A + B)\omega\tau. \]  

(23)
A minor readjustment of the above equations gives

\[ \gamma_{D,n}(t) = \frac{1}{2} \left( n + \frac{1}{2} \right) [F_D(t) - F_D(t_0)]. \]  

(24)

We can easily show that this is the same formula of the dynamical phases given in the text.

**Appendix A: Measure of nonstaticity**

In our previous paper \[12\], we have defined the quantitative measure of nonstaticity. In the Fock state, it is given by \[12\]

\[ D_F = \frac{\sqrt{(A + B)^2 - 4}}{2\sqrt{2}}. \]  

(A1)

Usually, the nonstatic properties of the light wave become significant as this measure increases.

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