ANALOGUES OF LUSZTIG'S HIGHER ORDER RELATIONS
FOR THE $q$–ONSAGER ALGEBRA

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Abstract. Let $A, A^*$ be the generators of the $q$–Onsager algebra. Analogues of Lusztig’s $r$–th higher order relations are proposed. In a first part, based on the properties of tridiagonal pairs of $q$–Racah type which satisfy the defining relations of the $q$–Onsager algebra, higher order relations are derived for $r$ generic. The coefficients entering in the relations are determined from a two-variable polynomial generating function. In a second part, it is conjectured that $A, A^*$ satisfy the higher order relations previously obtained. The conjecture is proven for $r = 2, 3$. For $r$ generic, using an inductive argument recursive formulae for the coefficients are derived. The conjecture is checked for several values of $r \geq 4$. Consequences for coideal subalgebras and integrable systems with boundaries at $q$ a root of unity are pointed out.

MSC: 81R50; 81R10; 81U15; 81T40.

Keywords: $q$–Onsager algebra; Quantum group; Higher $q$–Serre relations; Tridiagonal algebra

1. Introduction

Consider the quantum universal enveloping algebras for arbitrary Kac-Moody algebras $\hat{g}$ introduced by Drinfeld [D] and Jimbo [J]. Let $\{a_{ij}\}$ be the extended Cartan matrix. Fix coprime integers $d_i$ such that $d_i a_{ij}$ is symmetric. Define $q_i = q^{d_i}$. The quantum universal enveloping algebra $U_q(\hat{g})$ is generated by the elements $\{h_i, e_j, f_j\}$, $j = 0, 1, \ldots, \text{rank}(g)$, which satisfy the defining relations:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \quad [e_i, f_j] = q_i^{-j+h_i} - q_i^{-j-h_i}, \quad i, j = 0, 1, \ldots, \text{rank}(g),$$

together with the so-called quantum Serre relations $(i \neq j)$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} f_i^{1-a_{ij}+k} f_j f_i^k = 0.$$

In the mathematical literature [L], generalizations of the relations (1.1) - the so-called higher order quantum $(q-)$Serre relations - have been proposed. For $\hat{g} = \hat{s}l_2$, they read [L]:

$$\sum_{k=0}^{2r+1} (-1)^k \left[ \begin{array}{c} 2r+1 \\ k \end{array} \right]_q e_i^{2r+1-k} e_j^k e_i^k = 0,$$

$$\sum_{k=0}^{2r+1} (-1)^k \left[ \begin{array}{c} 2r+1 \\ k \end{array} \right]_q f_i^{2r+1-k} f_j^k f_i^k = 0 \quad \text{for} \quad i \neq j, \quad i, j = 0, 1.$$

Note that the higher order $q$–Serre relations (1.2) appear in different contexts. On one hand, they determine the commutation relations among the elements of the $U_q(\hat{g})$-subalgebra generated by the divided powers of $e_i, f_i$ which can be also obtained using the braid group action of $U_q(\hat{g})$. They also arise in the discussion of the quantum Frobenius homomorphism [L]. On the other hand, as will be mentioned in the last Section, in the integrable systems literature the higher order $q$–Serre relations (1.2) play a central role in the identification of the symmetry of quantum integrable models at $q$ a root of unity [DFM, KM, AYP, ND]. For instance, using (1.2) one shows that the XXZ spin chain with periodic boundary conditions enjoys a $\hat{s}l_2$ loop symmetry at $q$ a root of unity [DFM]. Also,

1As usual, we denote: $\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{[n]_q!}{[m]_q! [n-m]_q!}, \quad [n]_q^! = \prod_{n=1}^{\infty} [n]_q, \quad [n]_q = q^{n-\frac{n(n-1)}{2}}, \quad [0]_q = 1$.

2For $\hat{g} = \hat{s}l_2$, recall that $a_{ii} = 2, a_{ij} = -2$ with $i, j = 0, 1$.  

arXiv:1312.3433v2 [math-ph] 18 Dec 2013.
they are essential to derive the Serre relations for the basic generators of the superintegrable chiral Potts model [ND] AYP.

In recent years, a new algebraic structure called the $q$–Onsager algebra that is closely related with $U_q(sl_2)$ has appeared in the mathematical [Ter] and integrable systems [Bas] literature. In particular, a relation between the $q$–context of $U_q(sl_2)$ has been exhibited [TT2, Bas, BK2, BS1, BB1, K]. Recall that the tridiagonal algebra introduced in the context of $P$– and $Q$–polynomial association schemes [Ter] is an associative algebra with unit which consists of two generators $A$ and $A^*$ called the standard generators. In general, the defining relations depend on five scalars $\rho, \rho^*, \gamma, \gamma^*$ and $\beta$. The $q$–Onsager algebra is a special case of the tridiagonal algebra: it corresponds to the reduced parameter sequence $\gamma = 0, \gamma^* = 0, \beta = q^2 + q^{-2}$ and $\rho = \rho_0, \rho^* = \rho_1$ which exhibits all interesting properties that can be extended to more general parameter sequences. The defining relations of the $q$–Onsager algebra read

\begin{equation}
[A, [A, [A, A^*]]_{q^{-1}}] = \rho_0[A, A^*], \quad [A^*, [A^*, [A^*, A]]_{q^{-1}}] = \rho_1[A^*, A],
\end{equation}

which can be seen as $\rho_1$–deformed analogues of the $q$–Serre relations (1.1) associated with $g \equiv s_l 2$. For $q = 1$, $\rho_0 = \rho_1 = 16$, note that they coincide with the Dolan-Grady relations [DG].

In the study of tridiagonal algebras and the representation theory associated with the special case $\rho_0 = \rho_1 = 0$, higher order $q$–Serre relations (1.2) play an important role in the construction of a basis of the corresponding vector space. As suggested in [TT1 Problem 3.4], for $\rho_0 \neq 0, \rho_1 \neq 0$ finding analogues of the higher order $q$–Serre relations for the $q$–Onsager algebra is an interesting problem. Another interest for the case of periodic boundary conditions [DFM], such relations should play a central role in the identification of the symmetry of the Hamiltonian of the XXZ open spin chain at $q$ a root of unity and special boundary parameters.

Motivated by these open problems, in the present paper we propose the $r - th$ higher order tridiagonal relations associated with the $q$–Onsager algebra (1.3), which we refer as the $r - th$ higher order $q$–Dolan-Grady relations. As will be argued, they can be written in the form:

\begin{equation}
\sum_{p=0}^{r} \sum_{j=0}^{2r+1-2p} (-1)^{j+p} \rho_0^p c_{j}^{[r,p]} A^{2r+1-2p-j} A^* A^j = 0 ,
\end{equation}

\begin{equation}
\sum_{p=0}^{r} \sum_{j=0}^{2r+1-2p} (-1)^{j+p} \rho_1^p c_{j}^{[r,p]} A^*^{2r+1-2p-j} A^* A^* A^j = 0
\end{equation}

where $c_{j}^{[r,p]} = c_{2(r-p)+1-j}^{[r,p]}$ and

\begin{equation}
c_{j}^{[r,p]} = \sum_{k=0}^{j} \binom{r-p}{\frac{j-k}{2}} \binom{p}{\frac{j-k}{2}} \sum_{s_1, s_2, \ldots, s_p} s_1^2 \ldots s_p^2 \left[ s_{p+1} \ldots s_{p+k} \right]^2 s_1 \ldots s_{p+1} \ldots s_{p+k}
\end{equation}

with

\begin{align}
\left\{ \begin{array}{l}
\quad j = 0, r - p , \quad s_i \in \{ 1, 2, \ldots, r \} , \\
\quad p : \quad s_1 < \cdots < s_p ; \quad s_{p+1} < \cdots < s_{p+k} , \quad \\
\quad \{ s_1, \ldots, s_p \} \cap \{ s_{p+1}, \ldots, s_{p+k} \} = \emptyset
\end{array} \right.
\end{align}

For $\rho_0 = \rho_1 = 0$, the coefficients reduce to the ones in [NT].

The paper is organized as follows. In the next Section, we briefly recall the notion of tridiagonal (TD) pairs and tridiagonal systems (see [NT] and references therein). Based on the properties of TD pairs of $q$–Racah type [NT], higher order tridiagonal relations satisfied by TD pairs are constructed (cf. Theorem 2). For the reduced parameter sequence (see above comments), it is known that the basic tridiagonal relations reduce to (1.3). In this case, it is shown that the higher order tridiagonal relations associated with (1.3) take the form (1.3) for $r$ generic. See Example 2. A two-variable polynomial generating function for the coefficients is proposed, which gives (1.5). Motivated by these results, in Section 3 it is more generally conjectured that given $A, A^*$ satisfying (1.3) the $r - th$

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3 The $q$–commutator $[X, Y]_q = qXY - q^{-1}YX$ is introduced, where $q$ is called the deformation parameter.

4 Here $\{x\}$ denotes the integer part of $x$. Let $j, m, n$ be integers, we write $j = \underline{m, n}$ for $j = m, m + 1, \ldots, n - 1, n$. 
higher order $q$–Dolan-Grady relations \[1.3\] with \[1.3\] are satisfied (see conjecture\[1\]). First, we prove the conjecture in the simplest examples for $r = 2, 3$. Then, by induction the existence and structure of relations of the form \[1.4\] for generic values of $r$ are studied, leading to explicit recursive formulæ for the coefficients $c^{r,p}_k$. For several values of $r \geq 4$, the coefficients obtained by both approaches - from the properties of TD pairs or by induction - are found to coincide, giving another support for the proposal. In the last Section, potential applications of the higher $q$–Dolan-Grady relations \[1.4\] to the theory of coideal subalgebras and integrable systems with boundaries at $q$ a root of unity are pointed out. In Appendices A,B useful recursion relations are reported.

**Notations:** Throughout this paper $K$ denotes a field. $q$ is assumed not to be a root of unity.

### 2. Higher order relations from the theory of tridiagonal pairs

The main result of this Section is Theorem\[2\] which follows from previous works on tridiagonal pairs. Note that the basic material (Definitions 2.1-2.2, Lemma 2.2 and Theorem 1) introduced in the beginning of this Section is essentially taken from \[ITT\], \[NT\], \[TI\]. As an application of Theorem 2, the higher order relations from the theory of tridiagonal pairs are derived in Example 2 for $r$ generic.

#### 2.1. Tridiagonal pairs of $q$–Racah type

Let $V$ denote a vector space over $K$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call $W$ an *eigenspace* of $A$ whenever $W \neq 0$ and there exists $\theta \in K$ such that $W = \{v \in V | Av = \theta v\}$, $\theta$ is the *eigenvalue* of $A$ associated with $W$. $A$ is *diagonalizable* whenever $V$ is spanned by the eigenspaces of $A$.

**Definition 2.1.** \[ITT\], Definition 1.1] Let $V$ denote a vector space over $K$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on $V$ we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

1. Each of $A, A^*$ is diagonalizable,
2. There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that
   \[
   A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d,
   \]
   where $V_{-1} = 0$ and $V_{d+1} = 0$.
3. There exists an ordering $\{V^*_i\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that
   \[
   A V^*_i \subseteq V^*_i - V^*_i + V^*_{i+1}, \quad 0 \leq i \leq \delta,
   \]
   where $V^*_{-1} = 0$ and $V^*_{\delta+1} = 0$.
4. There does not exist a subspace $W$ of $V$ such that $AW \subseteq W$, $A^* W \subseteq W$, $W \neq 0$, $W \neq V$. We say the pair $A, A^*$ is over $K$. We call $V$ the *underlying vector space*.

**Note 1.** According to a common notational convention, for a linear transformation $A$ the *conjugate-transpose* of $A$ is denoted $A^*$. We emphasize we are not using this convention. In a TD pair $(A, A^*)$, the linear transformations $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

Let $A, A^*$ denote a TD pair on $V$, as in Definition\[2.1\] By \[ITT\] Lemma 4.5] the integers $d$ and $\delta$ from (ii), (iii) are equal and called the *diameter* of the pair. An ordering of the eigenspaces of $A$ (resp. $A^*$) is said to be *standard* whenever it satisfies \[2.1\] (resp. \[2.2\]). Let $\{V_i\}_{i=0}^d$ (resp. $\{V^*_i\}_{i=0}^\delta$) denote a standard ordering of the eigenspaces of $A$ (resp. $A^*$). For $0 \leq i \leq d$, let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) associated with $V_i$ (resp. $V^*_i$).

In what follows, we assume:

\[
\theta_i = \alpha + bq^{2i-d} + cq^{d-2i}, \quad 0 \leq i \leq d,
\]

\[
\theta_i^* = \alpha^* + b^* q^{2i-d} + c^* q^{d-2i}, \quad 0 \leq i \leq d,
\]

where $\alpha, \alpha^*$ are scalars, $q, b, b^*, c, c^*$ are nonzero scalars such that $q^4 \neq 1, q^2 \neq 1, q^2 \neq -1$. In this case, we say that $A, A^*$ is a tridiagonal pair of $q$–Racah type \[NT\], Theorem 5.3].

**Lemma 2.1.** For each positive integer $s$, there exists scalars $\beta_s, \gamma_s, \gamma^*_s, \delta_s, \delta^*_s$ in $K$ such that

\[
\theta_i^2 - \beta_s \theta_i \theta_j + \theta_j^2 - \gamma_s (\theta_i + \theta_j) - \delta_s = 0, \quad (0 \leq i, j \leq d).
\]

\[
\theta_i^* - \beta_s \theta_i \theta_j + \theta_j^* - \gamma^*_s (\theta_i + \theta_j) - \delta^*_s = 0, \quad |i - j| = s.
\]
2.2. Tridiagonal systems. For the analysis presented below, it will be convenient to recall the notion of TD system [ITT]. Let \( \text{End}(V) \) denote the \( \mathbb{K} \)-algebra of all linear transformations from \( V \) to \( V \). Let \( A \) denote a diagonalizable element of \( \text{End}(V) \). Let \( \{V_i\}_{i=0}^d \) denote an ordering of the eigenspaces of \( A \) and let \( \{\theta_i\}_{i=0}^d \) denote the corresponding ordering of the eigenvalues of \( A \). For \( 0 \leq i \leq d \), define \( E_i \in \text{End}(V) \) such that \( (E_i - I)V_i = 0 \) and \( E_i V_j = 0 \) for \( j \neq i \) \( (0 \leq j \leq d) \). Here \( I \) denotes the identity of \( \text{End}(V) \). We call \( E_i \) the primitive idempotent of \( A \) corresponding to \( V_i \) (or \( \theta_i \)). Observe that (i) \( I = \sum_{i=0}^d E_i \); (ii) \( E_i E_j = \delta_{ij} E_i \) \((0 \leq i, j \leq d)\); (iii) \( V_i = E_i V \) \((0 \leq i \leq d)\); (iv) \( A = \sum_{i=0}^d \theta_i E_i \). Here \( \delta_{ij} \) denotes the Kronecker delta. Now let \( A, A^* \) denote a TD pair on \( V \). An ordering of the primitive idempotents of \( A \) (resp. \( A^* \)) is said to be standard whenever the corresponding ordering of the eigenspaces of \( A \) (resp. \( A^* \)) is standard.

**Definition 2.2.** [ITT] Definition 2.1 Let \( V \) denote a vector space over \( \mathbb{K} \) with finite positive dimension. By a tridiagonal system (or TD system) on \( V \) we mean a sequence

\[
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
\]

that satisfies (i)–(iii) below.

(i) \( A, A^* \) is a TD pair on \( V \).

(ii) \( \{E_i\}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A \).

(iii) \( \{E_i^*\}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A^* \).

We say that \( \Phi \) is over \( \mathbb{K} \).

**Lemma 2.2.** [ITT] Lemma 2.2 Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a TD system. Then the following hold for \( 0 \leq i, j, r \leq d \).

(i) \( E_i^* A^* E_j^* = 0 \) if \( |i - j| > r \),

(ii) \( E_i A^* E_j = 0 \) if \( |i - j| > r \).

2.3. Higher order tridiagonal relations. By [ITT] Theorem 10.1] the pair \( A, A^* \) satisfy two polynomial equations called the tridiagonal relations. These generalize the \( q \)-Serre relations [ITT] and the Dolan-Grady relations [DG].

**Theorem 1.** (see [ITT] Theorem 10.1)] Let \( \mathbb{K} \) denote a field, and let \( A, A^* \) denote a TD pair of \( q \)-Racah type over \( \mathbb{K} \). Then, there exists a sequence of scalars \( \beta, \gamma, \gamma^*, \delta, \delta^* \) taken from \( \mathbb{K} \) such that both

\[
A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* = 0,
\]

\[
A^*, A^2A - \beta A^* AA^* + AA^*^2 - \gamma^*(A^*A + AA^*) - \delta^* A = 0.
\]

**Proof.** Let \( \Delta_1 \) denote the expression of the left-hand side of (2.6). We show \( \Delta_1 = 0 \). For \( 0 \leq i, j \leq d \), one finds \( E_i \Delta_1 E_j = p_1(\theta_i, \theta_j) \) \( E_i A^* E_j \) with

\[
p_1(x, y) = (x - y)((x^2 - \beta xy + y^2 - \gamma(x + y) - \delta) ) .
\]

Take \( \beta = \beta_1, \gamma = \gamma_1 \) and \( \delta = \delta_1 \). For each \( i, j \), according to Lemma 2.1 \( p_1(\theta_i, \theta_j) = 0 \) if \( |i - j| \leq 1 \) and according to Lemma 2.2 \( E_i A^* E_j = 0 \) if \( |i - j| > 1 \). It implies (2.6). Similar arguments are used to show (2.7). \( \square \)

Higher order relations can be constructed along the same line, using the properties of tridiagonal pairs described in the previous subsections.

**Definition 2.3.** Let \( x, y \) denote commuting indeterminates. For each positive integer \( r \), we define the polynomials \( p_r(x, y), p_r^*(x, y) \) as follows:

\[
p_r(x, y) = (x - y) \prod_{s=1}^r(x^2 - \beta_s xy + y^2 - \gamma_s(x + y) - \delta_s),
\]

\[
p_r^*(x, y) = (x - y) \prod_{s=1}^r(x^2 - \beta_s xy + y^2 - \gamma_s^*(x + y) - \delta_s^*).
\]

We observe \( p_r(x, y) \) and \( p_r^*(x, y) \) have a total degree \( 2r + 1 \) in \( x, y \).
Lemma 2.3. For each positive integer \( r \),
\[
(2.11) \quad p_r(\theta_i, \theta_j) = 0 \quad \text{and} \quad p_r^*(\theta_i^*, \theta_j^*) = 0 \quad \text{if} \quad |i - j| \leq r \quad (0 \leq i, j \leq d).
\]

Proof. Immediate.

Theorem 2. For each positive integer \( r \),
\[
(2.12) \quad \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij} A_i^r A_j^r = 0 , \quad \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij}^* A_i^r A_j^r = 0
\]
where the scalars \( a_{ij}, a_{ij}^* \) are defined by:
\[
(2.13) \quad p_r(x,y) = \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij} x_i y_j \quad \text{and} \quad p_r^*(x,y) = \sum_{i,j=0}^{i+j \leq 2r+1} a_{ij}^* x_i y_j.
\]

Proof. Let \( \Delta_r \) denote the expression of the left-hand side of the first equation of (2.12). We show \( \Delta_r = 0 \). For \( 0 \leq i, j \leq d \), one finds \( E_i \Delta_r E_j = p_r(\theta_i, \theta_j) E_i A_i^r E_j \) with (2.13). According to Lemma 2.3 and Lemma 2.2 it follows \( \Delta_r = 0 \). Similar arguments are used to show the second equation of (2.12).

2.4 Higher order \( q \)-Dolan-Grady relations. As a straightforward application, we are now interested by the \( r \)-th higher order tridiagonal relations which special cases for \( r = 1 \) produce the defining relations of the \( q \)-Onsager algebra (1.3).

Definition 2.4. Consider a TD pair \( A, A^* \) of \( q \)-Racah type with eigenvalues such that \( \alpha = \alpha^* = 0 \). Assume \( r = 1 \). The corresponding tridiagonal relations (2.6), (2.7) are called the \( q \)-Dolan-Grady relations.

Example 1. For a TD pair \( A, A^* \) of \( q \)-Racah type with eigenvalues such that \( \alpha = \alpha^* = 0 \), the parameter sequence is given by \( \beta_1 = q^2 + q^{-2}, \quad \gamma_1 = \gamma_1^* = 0, \quad \delta_1 = -bc(q^2 - q^{-2})^2, \quad \delta_1^* = -b^*c^*(q^2 - q^{-2})^2 \). Define \( \delta_1 = \rho_0, \delta_1^* = \rho_1 \). The \( q \)-Dolan-Grady relations are given by:
\[
(2.14) \quad \sum_{j=0}^{3} (-1)^j \binom{3}{j}_q A^{3-j} A^* A^j - \rho_0 (AA^* - A^* A) = 0,
\]
\[
(2.15) \quad \sum_{j=0}^{3} (-1)^j \binom{3}{j}_q A^{3-j} A A^* A^j - \rho_1 (A^* A - AA^*) = 0.
\]

Remark 1. The relations (2.14), (2.15) are defining relations of the \( q \)-Onsager algebra (1.3).

Definition 2.5. Consider a TD pair \( A, A^* \) of \( q \)-Racah type with eigenvalues such that \( \alpha = \alpha^* = 0 \). For any positive integer \( r \), the corresponding higher order tridiagonal relations (2.12) are called the higher order \( q \)-Dolan-Grady relations.

Example 2. For a TD pair \( A, A^* \) of \( q \)-Racah type with eigenvalues such that \( \alpha = \alpha^* = 0 \), the higher order \( q \)-Dolan-Grady relations are given by (1.4) with the identification \( \rho_0 = \delta_1 \) and \( \rho_1 = \delta_1^* \).

Proof. For \( \alpha = \alpha^* = 0 \) in (2.3), (2.4), from Lemma 2.1 one finds \( \beta_s = q^{2s} + q^{-2s}, \quad \gamma_s = \gamma_s^* = 0, \quad \delta_s = -bc(q^{2s} - q^{-2s})^2, \quad \delta_s^* = -b^*c^*(q^{2s} - q^{-2s})^2 \). Then, the first polynomial generating function (2.4) reads:
\[
(2.16) \quad p_r(x,y) = (x-y) \prod_{s=1}^{r} \left( x^2 - \frac{[2s]_q^2}{[s]_q^2} xy + y^2 - [s]_q^2 \rho_0 \right),
\]
where the notation \( \beta_s = [2s]_q^2/[s]_q^2 \) and \( \delta_s/\rho_0 = [s]_q^2 \) has been introduced. Expanding the polynomial in the variables \( x, y \) as (2.13), one shows that the coefficients \( a_{ij} \) in (2.13) take the form:
\[
(2.17) \quad a_{2r+1-2p-j} = (-1)^{j+p} \rho_0 [r,p] c_j^{[r,p]}
\]
where \( c_j^{[r,p]} \) solely depend on \( q \), and are vanishing otherwise. By induction, one finds that they are given by (1.5). Replacing \( \rho_0 \to \rho_1 \), the second relation in (1.4) follows.

For \( r = 2,3 \), the higher order \( q \)-Dolan-Grady relations (1.3) can be constructed in a straightforward manner:
Example 3. The first example of higher order \( q \)-Dolan-Grady relations is given by (1.4) for \( r = 2 \) with:

\[
\begin{align*}
\ell_{0}^{[2,0]} &= 1, \quad \ell_{1}^{[2,0]} = 1 + [2]_{q^2} + [4]_{q^2} \equiv \left[ \begin{array}{c} 5 \\ 1 \end{array} \right]_{q}, \quad \ell_{2}^{[2,0]} = 2 + [2]_{q^2} + [4]_{q^2} \equiv \left[ \begin{array}{c} 5 \\ 2 \end{array} \right]_{q}, \\
\ell_{0}^{[2,1]} &= 1 + [2]_{q^2} \equiv q^4 + q^{-4} + 3, \quad \ell_{1}^{[2,1]} = 1 + [2]_{q^2} + [4]_{q^2} + [2]_{q^2} \equiv [5]_{q} [3]_{q}, \\
\ell_{0}^{[2,2]} &= [2]_{q^2} \equiv (q^2 + q^{-2})^2.
\end{align*}
\]

Example 4. The second example of higher order \( q \)-Dolan-Grady relations is given by (1.4) for \( r = 3 \) with:

\[
\begin{align*}
\ell_{j}^{[3,0]} &= \left[ \begin{array}{c} 7 \\ j \end{array} \right]_{q}, \quad j = 0, \ldots, 7, \\
\ell_{0}^{[3,1]} &= 1 + [2]_{q^2}^2 + [3]_{q^2}^2, \\
\ell_{1}^{[3,1]} &= 1 + [2]_{q^2}^2 + [3]_{q^2}^2 + (1 + [3]_{q^2}^2) [4]_{q^2} + ([2]_{q^2} + [3]_{q^2}) [2]_{q^2} + (1 + [2]_{q^2}^2) [6]_{q^2} \equiv [3]_{q^2}, \\
\ell_{2}^{[3,1]} &= 2(1 + [2]_{q^2}^2 + [3]_{q^2}^2) + (1 + [3]_{q^2}^2) [4]_{q^2} + ([2]_{q^2} + [3]_{q^2}) [2]_{q^2} + (1 + [2]_{q^2}^2) [6]_{q^2} \\
&\quad + \frac{[4]_{q^2} [6]_{q^2}}{[2]_{q^2}^3 [3]_{q^2}^2} + \frac{[2]_{q^2} [3]_{q^2}}{[3]_{q^2}^2} + \frac{[3]_{q^2}^2}{3} [4]_{q^2}, \\
\ell_{0}^{[3,2]} &= [2]_{q^2}^2 + [3]_{q^2}^2 + [2]_{q^2}^2 + [3]_{q^2}^2, \\
\ell_{1}^{[3,2]} &= [2]_{q^2}^2 + [3]_{q^2}^2 + [2]_{q^2}^2 + [3]_{q^2}^2 + [2]_{q^2}^2 [6]_{q^2} + [3]_{q^2}^2 [4]_{q^2} + [2]_{q^2}^2 + [3]_{q^2}^2, \\
\ell_{0}^{[3,3]} &= [2]_{q^2}^2 + [3]_{q^2}^2.
\end{align*}
\]

To end up this Section, let us consider the family of relations satisfied by a TD pair of \( q \)-Racah type such that\(^5\) \( \rho_0 = \rho_1 = 0 \). In this special case, the polynomial generating function for the coefficients (2.10) factorizes:

\[
(2.18) \quad p_r(x, y) = x^{2r+1} \prod_{s=1}^{r} \left( 1 - q^{2s} \frac{y}{x} \right).
\]

According to the \( q \)-binomial theorem, one has:

\[
(2.19) \quad (1 - u)(1 - q^2u) \cdots (1 - q^{2r}u) = \sum_{j=0}^{2r+1} \left[ \begin{array}{c} 2r+1 \\ j \end{array} \right]_{q} (1)^{j} q^{2jr} u^j.
\]

If we denote the l.h.s. of (2.19) by \( g_{q}(u) \), observe \( p_r(x, y) = x^{2r+1} g_{q}(q^{2r} \frac{y}{x}) \). As a consequence, for \( \rho_0 = \rho_1 = 0 \) the higher order \( q \)-Dolan-Grady relations satisfied by the corresponding TD pair simplify to the well-known Lusztig’s higher order \( q \)-Serre relations \([1]\):

\[
(2.20) \quad \sum_{j=0}^{2r+1} (-1)^{j} \left[ \begin{array}{c} 2r+1 \\ j \end{array} \right]_{q} A^{2r+1-j} A^{*r} A^{j} = 0, \\
\sum_{j=0}^{2r+1} (-1)^{j} \left[ \begin{array}{c} 2r+1 \\ j \end{array} \right]_{q} A^{*2r+1-j} A^{r} A^{*j} = 0.
\]

\(^5\)For instance, choose \( b, b^{*} = 0 \) and/or \( c, c^{*} = 0 \) in (2.3), (2.4).
3. Higher order relations for the $q$–Onsager algebra; recursion for generating the coefficients

In the previous Section, it was shown that every TD pair of $q$–Racah type such that $\alpha = \alpha^* = 0$ satisfies the $r$–th higher order $q$–Dolan-Grady relations (1.3) with (1.5). For the special case $r = 1$, these relations coincide with the defining relations of the $q$–Onsager algebra (1.3). According to these results, we propose the following conjecture:

**Conjecture 1.** Let $A, A^*$ be the fundamental generators of the $q$–Onsager algebra (1.3). The relations (1.4) hold.

The purpose of this Section is to prove the conjecture for $r = 2, 3$. Then, using an inductive argument we will study the general structure and derive recursion formulae – independently of the results of the previous Section – for the coefficients $c^{[r,p]}$. As we will discuss at the end of this Section, a detailed comparison with the coefficients obtained in the previous Section for several values of $r \geq 4$ supports the conjecture.

Let $A, A^*$ be the fundamental generators of the $q$–Onsager algebra (1.3). Observe that the defining relations can be written:

\[
(3.1) \quad \sum_{i=0}^{3} (-1)^i \binom{3}{i}_q A^{3-i} A^i A^i - \rho_0 (AA^* - A^* A) = 0 ,
\]

\[
(3.2) \quad \sum_{i=0}^{3} (-1)^i \binom{3}{i}_q A^{3-i} AA^{*i} - \rho_1 (A^* A - AA^*) = 0 .
\]

By analogy with Lusztig’s higher order $q$–Serre relations, we are interested in more complicated linear combinations of monomials of the type $A^n A^m A^m$, $n + m = 2r + 1, 2r - 1, ... , 1$, that are vanishing. The defining relations (3.1), (3.2) correspond to the case $r = 1$ of (1.3). Below, successively we derive the relations (1.4) for $r = 2, 3$ and study the generic case by induction.

### 3.1. Proof of the relations for $r = 2$.

Consider the simplest example beyond (3.1): we are looking for a linear relation between monomials of the type $A^n A^2 A^m$, $n + m = 5, 3, 1$. According to the defining relations (3.1), note that the monomial $A^3 A^*$ can be written as:

\[
(3.3) \quad A^3 A^* = \alpha A^2 A^* A - \alpha AA^* A^2 + A^* A^3 + \rho_0 (AA^* - A^* A) \quad \text{with} \quad \alpha = [3]_q .
\]

Multiplying from the left by $A$ or $A^2$, the corresponding expressions can be ordered as follows: each time a monomial of the form $A^n A^* A^m$ with $n \geq 3$ arise, it is reduced using (3.3). It follows:

\[
A^4 A^* = (\alpha^2 - \alpha) A^2 A^* A^2 + (1 - \alpha^2) AA^* A^2 + A^2 A^4 + \rho_0 (A^2 A^* - \alpha A^* A^2 + (\alpha - 1) AA^* A) ,
\]

\[
A^5 A^* = (\alpha^3 - 2\alpha^2 + 1) A^2 A^* A^3 + \alpha (-\alpha^2 + \alpha + 1) AA^* A^4 + \alpha (\alpha - 1) A^* A^5
\]

\[
+ \rho_0 \left( 2\alpha^{-1} A^2 A^* A + \alpha (\alpha - 3) AA^* A^2 - (\alpha^2 - \alpha - 1) A^* A^3 \right)
\]

\[
+ \rho_0^2 (AA^* - A^* A) .
\]

For our purpose, four different types of monomials may be now considered: $A^5 A^* A^2$, $A^4 A^2 A^2$, $A^3 A^2 A^2$ and $A^3 A^2 A^3$. Following the ordering prescription, each of these monomials can be reduced as a combination of monomials of the type $(n, m, p, s, t \geq 0)$:

\[
(3.4) \quad A^n A^p A^s A^t A^m \quad \text{with} \quad n \leq 2 , \ n + m = 5, 3, 1 ,
\]

\[
A^n A^* A^p A^s A^t \quad \text{with} \quad p \leq 2 , \ s \geq 1 , \ p + s + t = 5, 3, 1 .
\]
For instance, the monomial $A^5A^{*2}$ is reduced to:

$$A^5A^{*2} = (\alpha^3 - 2\alpha^2 + 1) \left( \alpha A^2 A^* A^2 A - \alpha A^2 A^* A A^* A^2 + A^2 A^{*2} A^3 \right)$$

$$- (\alpha^3 - \alpha^2 - \alpha) \left( (\alpha^2 - \alpha) AA^* A^2 A^* A^2 + (1 - \alpha^2) AA^* AA^* A^3 + \alpha AA^* A^4 \right)$$

$$+ (\alpha^2 - \alpha) \left( (\alpha^3 - 2\alpha^2 + 1) A^* A^2 A^* A^3 - (\alpha^3 - \alpha^2 - 1) A^* AA^* A^4 + \alpha (\alpha - 1) A^{*2} A^5 \right)$$

$$+ \rho_0(\alpha^3 - 2\alpha^2 + 2\alpha) (A^2 A^* AA^* - AA^* A^2 A^* + A^* A^2 A^*)$$

$$+ \rho_0(-\alpha^3 + 2\alpha^2 - 1) \left( A^2 A^{*2} A + \alpha AA^* AA^* A \right)$$

$$+ \rho_0(\alpha^4 - \alpha^3 - \alpha^2) \left( AA^* A^2 \right)$$

$$+ \rho_0(\alpha^4 - 3\alpha^3 + 2\alpha^2 - \alpha) \left( A^* AA^* A^2 \right)$$

$$+ \rho_0(-\alpha^4 + 2\alpha^3 - \alpha^2 + 1) \left( A^{*2} A^3 \right)$$

$$+ \rho_0^3 \left[ A, A^{*2} \right] .$$

The two other monomials $A^4A^{*2} A$, $A^3A^{*2} A^2$ are also ordered using (3.3). One obtains:

$$A^4A^{*2} A = (\alpha^2 - \alpha) \left( A^2 A^* A^2 A + \alpha AA^* A^2 A^3 \right) + \alpha^2 A^{*2} A^5$$

$$+ (\alpha^2 - 1) \left( \alpha AA^* AA^* A^3 - \alpha AA^* A^2 A^2 - \alpha AA^* A^4 - AA^* A^4 \right)$$

$$+ \rho_0 \left( A^2 A^{*2} A - (1 - \alpha^2) AA^* A^2 - \alpha^2 A^{*2} A^3 \right)$$

$$+ \rho_0(\alpha^2 - \alpha) \left( A^* AA^* A^2 - AA^* AA^* A \right) ,$$

$$A^2 A^* A^2 A^2 = \alpha \left( A^2 AA^* A^2 - AA^* A^2 A^2 + A^2 A^2 A^3 - AA^* A^4 \right) + A^{*2} A^5 + \rho_0(\alpha A^2 A^2 - A^{*2} A^3) .$$

The ordered expression for the fourth monomial $A^3A^{*2}$ directly follows from (3.3). Having the explicit ordered expressions of $A^5A^{*2}$, $A^4A^{*2} A$, $A^3A^{*2} A^2$ and $A^3A^{*2}$ in terms of monomials of the type (3.1), let us consider the combination:

$$f_2(A, A^*) = c_0^{[2,0]} A^{*2} A^2 - c_1^{[2,0]} A^4 A^{*2} A + c_2^{[2,0]} A^4 A^{*2} A^2 - \rho_0 c_0^{[2,1]} A^3 A^{*2}$$

with unknown coefficients $c_j^{[2,0]}$, $j = 1, 2, c_0^{[2,1]}$, and normalization $c_0^{[2,0]} = 1$. After simplifications, the combination takes the ordered form:

$$f_2(A, A^*) = c_3^{[2,0]} A^2 A^{*2} A^3 - c_4^{[2,0]} AA^* A^2 A^4 + c_5^{[2,0]} A^{*2} A^5 + g_2(A, A^*)$$

where

$$c_3^{[2,0]} = \alpha^3 - 2\alpha^2 + 1 , \quad c_4^{[2,0]} = \alpha^2(\alpha^2 - \alpha - 1) + c_1^{[2,0]}(1 - \alpha^2) , \quad c_5^{[2,0]} = (\alpha^2 - \alpha)^2 - \alpha^2 c_1^{[2,0]} + c_2^{[2,0]} .$$

Inspired by the structure of Lusztig’s higher order $q$–Serre relations, consider the conditions under which the combination $g_2(A, A^*)$ never contains monomials of the form $A^p A^* A^s A^t (p \leq 2, s \geq 1)$. At the lowest order in $\rho_0$, given a particular monomial the condition under which its coefficient in $g_2(A, A^*)$ is vanishing is given by:

$$A^2 A^* A^2 A^* A^2 : \alpha^3 - 2\alpha^2 + 1 - c_1^{[2,0]}(\alpha - 1) = 0 ,$$

$$A^2 A^* AA^* A^2 : \alpha^3 - 2\alpha^2 + 1 + c_2^{[2,0]} = 0 ,$$

$$AA^* A^2 A^* A^3 : (\alpha - 1)(-\alpha^3 + \alpha^2 + \alpha) - c_1^{[2,0]}(1 - \alpha^2) - c_2^{[2,0]} = 0 ,$$

$$AA^* AA^* A^3 : (1 - \alpha^2)(-\alpha^3 + \alpha^2 + \alpha) + c_1^{[2,0]} \alpha(1 - \alpha^2) = 0 ,$$

$$A^* A^2 A^* A^3 : (\alpha - 1)(\alpha^3 - 2\alpha^2 + 1) - c_1^{[2,0]}(\alpha^3 - \alpha) + c_2^{[2,0]} = 0 ,$$

$$A^* AA^* A^4 : (\alpha - 1)(-\alpha^3 + \alpha^2 + \alpha) - c_1^{[2,0]}(1 - \alpha^2) - c_2^{[2,0]} = 0 ,$$
Recall that $\alpha = [3]_q$. The solution $\{c_j^{[2,0]}, j = 1, 2\}$ to this system of equations exists, and it is unique. In terms of $q$–binomials, it reads:

$$(3.7) \quad c_j^{[2,0]} = \left[ \begin{array}{c} 5 \\ \n \end{array} \right]_q, \quad c_j^{[2,0]} = \left[ \begin{array}{c} 5 \\ \n \end{array} \right]_q.$$ 

At the next order $\rho_0$, the conditions such that monomials of the type $A^pA^qA_sA^t$ with $p + s + t \leq 3$ and $s \geq 1$ yield to:

$$c_j^{[2,1]} = q^4 + q^{-4} + 3.$$ 

All other coefficients of the monomials $A^pA^qA_sA^t$ for $n + m = 5, 3, 1$ are explicitly determined in terms of $c_j^{[2,0]}$ ($j = 0, 1, 2$), $c_0^{[2,1]}$, $\rho_0$ and $\rho_0^3$. Based on these results, we conclude that the $q$–Dolan-Grady relation $(3.1)$ implies the existence of a unique linear relation between monomials of the type $A^pA^qA_sA^t$ with $n + m = 5, 3, 1$. This relation can be seen as a $\rho_0$–deformed analogue of the simplest higher order $q$–Serre relation. Explicitly, one finds:

$$(3.8) \quad \sum_{j=0}^{5} (-1)^j \left[ \begin{array}{c} 5 \\ \n \end{array} \right] A^{5-j}A^sA^t = \rho_0 \left( (q^4 + q^{-4} + 3)(A^3A^2 - A^2A^3) - [5]_q \left( A^2A^2A - AA^2A^2 \right) \right) - f_0(q^2 + q^{-2})^2 (AA^2 - A^2A).$$ 

Using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$ which exchanges $(3.1)$ and $(3.8)$, the second relation generalizing $(3.2)$ is obtained. The coefficients coincide with the ones given in Example 3 which proves conjecture 1 for $r = 2$. For the special undeformed case $\rho_0 = \rho_1 = 0$, note that both relations reduce to the simplest examples of higher order $q$–Serre relations.

### 3.2. Proof of the relations for $r = 3$

Following a similar analysis, the next example of higher order $q$–Dolan-Grady relations can be also derived. To this end, one is looking for a linear relation between monomials of the type $A^pA^qA_sA^t, n + m = 7, 5, 3, 1$. Assume the $q$–Dolan-Grady relation $(3.1)$ and its simplest consequence $(3.8)$. Write the four monomials:

$$A^7A^3 = (A^7A^2)A^3, \quad A^6A^3 = (A^6A^2)A^3, \quad A^5A^3A^2 = (A^5A^2)A^3, \quad A^5A^3 = (A^5A^2)A^3.$$ 

Using $(3.8)$ and then $(3.1)$, they can be expressed solely in terms of monomials of the type:

$$(3.9) \quad A^nA^3A^m \quad \text{with} \quad n \leq 4, \quad n + m = 7, 5, 3, 1,$$

$$A^pA^2A^sA^t \quad \text{with} \quad p \leq 4, \quad s \geq 1, \quad p + s + t = 7, 5, 3, 1.$$ 

Then, introduce the combination

$$(3.10) \quad f_3(A, A^*) = c_0^{[3,0]} A^7A^3 - c_1^{[3,0]} A^6A^3A + c_2^{[3,0]} A^5A^3A^2 - \rho_0 c_0^{[3,1]} A^5A^3$$

with unknown coefficients $c_j^{[3,0]}$, ($j = 1, 2$), $c_0^{[3,1]}$ and normalization $c_0^{[3,0]} = 1$. By straightforward calculations using the ordered expressions of $A^7A^3, A^6A^3A, A^5A^3A^2$ and $A^5A^3$, $f_3(A, A^*)$ is reduced to a combination of monomials of the type $(3.9)$. Note that the coefficients of the monomials $A^nA^3A^m$ for $n + m = 5, 3, 1$ are of order $\rho_0, \rho_0^2, \rho_0^3$, respectively. Identifying the conditions under which the coefficient of any monomial of the form

$$A^pA^2A^sA^t \quad \text{with} \quad p \leq 4, \quad s \geq 1, \quad p + s + t = 7, 5, 3, 1,$$

is vanishing, one obtains a system of equations for the coefficients, which solution is unique. Simplifying $(3.10)$ according to the explicit solutions $c_j^{[3,0]}$, $j = 1, 2$ and $c_0^{[3,1]}$, one ends up with the next example of higher order $q$–Dolan-Grady relations. Using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$, the second relation follows. One finds:

$$(3.11) \quad \sum_{p=0}^{3} \rho_0^{7-2p} \sum_{j=0}^{3} (-1)^{j+p} c_j^{[3,p]} A^{7-2p-j}A^sA^t = 0,$$

$$\sum_{p=0}^{3} \rho_1^{7-2p} \sum_{j=0}^{3} (-1)^{j+p} c_j^{[3,p]} A^{7-2p-j}A^sA^t = 0$$
where \( c_j^{[3,p]} = c_j^{[3,p]} \), \( j = \begin{bmatrix} 7 \\ j \end{bmatrix}_q \) and
\[
\begin{align*}
c_0^{[3,1]} &= \left( q^8 + 3q^4 + 6 + 3q^{-4} + q^{-8} \right), \\
c_1^{[3,1]} &= \left[ 7 \right]_q \left( q^6 + q^4 + q^2 + 4 + q^{-2} + q^{-4} + q^{-6} \right), \\
c_2^{[3,1]} &= \left[ 7 \right]_q \left( q^2 - 1 + q^{-2} \right) \left( q^6 + 2q^4 + 4 + 2q^{-2} + q^{-4} \right), \\
c_0^{[3,2]} &= \left( q^6 + 2q^4 + 3q^2 + q^{-6} \right) \left( 3q^6 + 3q^4 + 2q^2 + q^{-6} \right), \\
c_1^{[3,2]} &= \left[ 7 \right]_q \left( q^8 + q^6 + 4q^4 + q^2 + 7 + q^{-2} + 4q^{-4} + q^{-6} + q^{-8} \right), \\
c_0^{[3,3]} &= \left( 2q^2 + 3q^2 \right).
\end{align*}
\]

It is straightforward to compare the coefficients above with the ones obtained from the expansion of the polynomial generating function \( p_3(x, y) \). Although the coefficients above look different, they coincide exactly with the ones reported in Example 3. This proves the conjecture for \( r = 3 \). Again, for the special case \( \rho_0 = \rho_1 = 0 \) Lusztig’s higher order \( q \)-Serre relations are recovered.

### 3.3. Relations for \( r \) generic

Above examples \( 3.3, 3.11 \) suggest that conjecture \( 1 \) holds for \( r \) generic. Looking for a linear relation between monomials of the type \( A^r A^{*r} A^m, n + m = 2r + 1, 2r - 1, ..., 1 \), for \( r \geq 1 \), relations of the form
\[
\sum_{p=0}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{2(r-p)+1-j} A^{*r} A^j = 0, \\
\sum_{p=0}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{2(r-p)+1-j} A^r A^{*j} = 0,
\]
are expected, provided the elements \( A, A^* \) satisfy the \( q \)-Dolan-Grady relations \( 3.1, 3.2 \). Our aim is now to study these relations in details and obtain recursive formulae for the coefficients \( c_j^{[r,p]} \).

In order to study the higher order \( q \)-Dolan-Grady relations \( 3.12 \) for generic values of \( r \), we proceed by induction. First, assume the basic relation \( 3.1 \) holds and implies all relations \( 3.12 \) up to \( r \) which explicit coefficients \( c_j^{[r,p]} \) in terms of \( q \) are assumed to be known. It is the case for \( r = 2, 3 \) as shown above. Our aim is to construct the higher order relation associated with \( r + 1 \) and express the coefficients \( c_j^{[r+1,p]} \), \( (j = 0, 1, ..., 2r + 3 - 2p, p = 0, ..., r + 1) \) in terms of \( c_j^{[r,p']} \) \( (j' = 0, 1, ..., 2r + 1 - 2p', p' = 0, ..., r) \). Following the steps described for \( r = 2, 3 \), from the relation \( 3.12 \) we first deduce:
\[
\begin{align*}
A^{2r+1} A^{*r} &= - \sum_{j=1}^{2r+1} (-1)^j c_j^{[r,0]} A^{2r+1-j} A^{*r} A^j - \sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+1} (-1)^{j+p} c_j^{[r,p]} A^{2(r-p)+1-j} A^{*r} A^j, \\
A^{2r+2} A^{*r} &= - \sum_{j=2}^{2r+2} (-1)^j M_j^{(r,0)} A^{2r+2-j} A^{*r} A^j - \sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+2} (-1)^{j+p} M_j^{(r,p)} A^{2(r-p)+2-j} A^{*r} A^j, \\
A^{2r+3} A^{*r} &= - \sum_{j=3}^{2r+3} (-1)^j N_j^{(r,0)} A^{2r+3-j} A^{*r} A^j - \sum_{p=1}^{r} \rho_0^p \sum_{j=0}^{2(r-p)+3} (-1)^{j+p} N_j^{(r,p)} A^{2(r-p)+3-j} A^{*r} A^j,
\end{align*}
\]
where the coefficients \( M_j^{(r,p)}, N_j^{(r,p)} \) are determined recursively in terms of \( c_j^{[r,p]} \) (see Appendix A). Now, write the four monomials:
\[
A^{2r+3} A^{*r+1} = (A^{2r+3} A^{*r}) A^*, \\
A^{2r+2} A^{*r+1} A = (A^{2r+2} A^{*r}) A^* A, \\
A^{2r+1} A^{*r+1} A^2 = (A^{2r+1} A^{*r}) A^* A^2, \\
A^{2r+1} A^{*r+1} = (A^{2r+1} A^{*r}) A^*.
\]
Using \( 3.14-3.16 \), they can be expressed solely in terms of:
\[
\begin{align*}
A^n A^{*r+1} A^m & \quad \text{with} \quad n \leq 2r, n + m = 2r + 3, 2r + 1, ..., 1, \\
A^p A^{*r} A^s A^t & \quad \text{with} \quad p \leq 2r, s \geq 1, p + s + t = 2r + 3, 2r + 1, ..., 1.
\end{align*}
\]
It is however clear from (3.14)-(3.16) that each monomial $A^{2r+3}A^{r+1}$, $A^{2r+2}A^{r+1}A$, $A^{2r+1}A^{r+1}A^2$ and $A^{2r+1}A^{r+1}$ can be further reduced using (3.11). For instance,

$$(A^{2r+3}A^{r+1})A^* = -\sum_{j=3}^{2r+3} (-1)^j N_j^{(r,0)} A^{2r+3-j} A^* A^r A^*$$

reducible

$$(3.20)$$

$$(A^{2r+3}A^{r+1})A^* = -\sum_{p=1}^{r} \sum_{j=0}^{2(r-p)+3} (-1)^j N_j^{(r,p)} A^{2(r-p)+3-j} A^r A^j A^*$$

reducible if $j \geq 3$

According to (3.11), observe that the monomials $A^i A^*$ (for $j$ even or odd) can be written as:

$$(3.19)\quad A^{2n+2} A^* = \sum_{k=0}^{n} \sum_{i=0}^{2} \rho_0^{n-k} \eta_{k,i}^{(2n+2)} A^{2-i} A^* A^{2k+i},$$

$$(3.20)\quad A^{2n+3} A^* = \sum_{k=1}^{n+1} \sum_{i=1}^{2} \rho_0^{n+1-k} \eta_{k,i}^{(2n+3)} A^{2-i} A^* A^{2k-1+i} + \rho_0^{n+1}(AA^* - A^* A),$$

where the coefficients $\eta_{k,i}^{(2n+2)}$, $\eta_{k,i}^{(2n+3)}$ are determined recursively in terms of $q$ (see Appendix A). It follows:

$$(3.21)\quad A^{2r+3} A^{r+1} = \sum_{i=1}^{r} N_i^{(r,0)} A^{2(r-i)+2} A^r (\sum_{k=1}^{i} \sum_{j=0}^{2} \rho_0^{i-k} \eta_{k,j}^{(2i+1)} A^{2-j} A^* A^{2k-1+j} + \rho_0( AA^* - A^* A))$$

$$(3.22)\quad f_{r+1}(A, A^*) = c_0^{[r+1,0]} A^{2r+3} A^{r+1} - c_1^{[r+1,0]} A^{2r+2} A^{r+1} A + c_2^{[r+1,0]} A^{2r+1} A^{r+1} A^2 - \rho_0 c_3^{[r+1,1]} A^{2r+1} A^{r+1}$$

with unknown coefficients $c_j^{[r+1,0]}$, $(j = 1, 2)$, $c_0^{[r+1,1]}$ and normalization $c_0^{[r+1,0]} = 1$. Combining all reduced expressions for $A^{2r+3} A^{r+1}$, $A^{2r+2} A^{r+1} A$, $A^{2r+1} A^{r+1} A^2$ and $A^{2r+1} A^{r+1}$ reported in Appendix B, one observes that $f_{r+1}(A, A^*)$ generates monomials either of the type (3.17) or (3.18). First, consider monomials of the type (3.18) which occur at the lowest order in $\rho_0$, namely $A^{2r} A^r A^2 A^* A$ and $A^{2r} A^r AA^* A^2$. The conditions under which their coefficients are vanishing read:

$$(3.23)\quad c_1^{[r+1,0]} = \left[ 2r + 3 \begin{array}{c} 2r + 3 \\ 1 \end{array} \right]_q, \quad c_2^{[r+1,0]} = \left[ 2r + 3 \begin{array}{c} 2r + 3 \\ 2 \end{array} \right]_q.$$
set of conditions:

\[ A^{2r-1} A^r A A^* A^3 : \quad N_4^{(r,0)} \eta_{1,1}^{(4)} + c_1^{[r+1,0]} M_3^{(r,0)} \eta_{1,1}^{(3)} = 0, \]

\[ A^{2r-1} A^r A A^* A^2 : \quad N_4^{(r,0)} \eta_{1,0}^{(4)} + c_1^{[r+1,0]} M_3^{(r,0)} \eta_{1,0}^{(3)} + c_2^{[r+1,0]} [r,0] = 0, \]

\[ A^{2(r-1)} A^r A A^* A^{2i+1} : \quad N_2^{(r,0)} (2i+3) + c_1^{[r+1,0]} M_2^{(r,0)} (2i+2) + c_2^{[r+1,0]} [r,0] = 0, \quad i = 1, r, \]

\[ A^{2(r-1)} A^r A A^* A^{2i+1} : \quad N_2^{(r,0)} (2i+3) + c_1^{[r+1,0]} M_2^{(r,0)} (2i+2) + c_2^{[r+1,0]} [r,0] = 0, \quad i = 1, r, \]

Using the recursion relations in Appendices A,B, we have checked that all above equations are satisfied, as expected.

More generally, one determines all other coefficients \( c_j^{[r+1,0]} \) for \( j \geq 3 \). One finds:

\[ c_3^{[r+1,0]} = N_3^{(r,0)} \mu_{1,2}^{(3)} = \begin{bmatrix} 2r + 3 \\ 3 \end{bmatrix}_q, \]

\[ c_4^{[r+1,0]} = N_4^{(r,0)} \mu_{1,2}^{(4)} + c_1^{[r+1,0]} M_3^{(r,0)} \mu_{1,2}^{(3)} = \begin{bmatrix} 2r + 3 \\ 4 \end{bmatrix}_q, \]

\[ c_{2k+1}^{[r+1,0]} = N_2^{(r,0)} (2k+1) + c_1^{[r+1,0]} M_2^{(r,0)} (2k) + c_2^{[r+1,0]} [r,0] = 0, \quad k = 2, r + 1, \]

\[ c_{2k+2}^{[r+1,0]} = N_2^{(r,0)} (2k+2) + c_1^{[r+1,0]} M_2^{(r,0)} (2k+1) + c_2^{[r+1,0]} [r,0] = 0, \quad k = 2, r. \]

For any \( j \geq 0 \), one finds that the coefficient \( c_j^{[r+1,0]} \) can be simply expressed as a \( q \)-binomial:

\[ c_j^{[r+1,0]} = \begin{bmatrix} 2r + 3 \\ j \end{bmatrix}_q. \]

All coefficients \( c_j^{[r+1,0]} \) being obtained, at the lowest order in \( \rho_0 \) one has to check that the coefficients of any unwanted term of the type \( (3.18) \) with \( p + s + t = 2r + 1, 2r - 1, \ldots, 1 \) are systematically vanishing. Using the recursion relations given in Appendices A.B, this has been checked in details. Then, following the analysis for \( r = 3 \) it remains to determine the coefficient \( c_0^{[r+1,1]} \) which contributes at the order \( \rho_0 \). The condition such that the coefficient of the monomial \( A^{2r} A^r A A^* \) is vanishing yields to:

\[ c_0^{[r+1,1]} = c_1^{[r,0]} - 2c_2^{[r,0]} + c_3^{[r,0]} - c_1^{[r,1]} + 2c_0^{[r,1]}. \]

Using the explicit expression for \( c_j^{[r+1,0]}, j = 0, 1, 2 \) and \( c_0^{[r+1,1]} \), we have checked in details that \( f_{r+1}(A, A^*) \) reduces to a combination of monomials of the type \( (3.17) \) only. The reduced expression \( f_{r+1}(A, A^*) \) determines uniquely all the remaining coefficients \( c_j^{[r+1,p]} \) for \( p \geq 1 \). For \( r \) generic, in addition to \( (3.24) \) and \( (3.25) \) one finally obtains:

\[ c_0^{[r+1,r+1]} = c_0^{[r+1,1]} c_0^{[r,r]} + N_0^{(r,r+1)}, \]

\[ c_0^{[r+1,p]} = N_0^{(r,p)} + c_0^{[r+1,1]} c_0^{[r,p-1]} \quad p = 2, r, \]
\[ c_{[r+1,1]}^{[r+1,1]} = N_0^{(r,0)} + c_1^{[r+1,0]} M_0^{(r,1)}, \]
\[ c_1^{[r+1,2]} = -N_5^{(r,0)} + N_3^{(r,1)} + c_0^{[r+1,1]} c_3^{[r,0]} + c_1^{[r+1,0]} M_0^{(r,2)}, \]
\[ c_1^{[r+1,r+1]} = \sum_{p=0}^{r-1} (-1)^{r+p+1} N_2^{(r,p)} + c_0^{[r+1,1]} \sum_{p=0}^{r-1} (-1)^{r+p+1} c_2^{[r+1,p]} c_{2(p-r-1)+1}^{[r+1,0]}, \]
\[ c_1^{[r+1,p]} = \sum_{j=0}^{p-1} (-1)^{j+p+1} N_2^{(r,j)} + c_0^{[r+1,1]} \sum_{j=0}^{p-2} (-1)^{j+p+1} c_{2(p-j)-1}^{[r+1,0]}, \quad p = 3, r, \]
\[ c_2^{[r+1,1]} = -N_4^{(r,0)} \eta_{1,0,2} + c_1^{[r+1,0]} M_3^{(r,0)} + c_2^{[r+1,0]} c_0^{[r,1]}, \]
\[ c_2^{[r+1,2]} = -N_6^{(r,0)} \eta_{1,0,2} - N_4^{(r,1)} \eta_{1,0,2} + c_1^{[r+1,1]} \eta_{0,1,2} + c_2^{[r+1,0]} \eta_{0,1,2} + c_2^{[r+1,0]} \eta_{0,1,2}, \]
\[ c_2^{[r+1,p]} = \sum_{j=0}^{p-1} (-1)^{j+p+1} N_2^{(r,j)} + c_1^{[r+1,0]} \sum_{j=0}^{p-2} (-1)^{j+p+1} c_{2(p-j)+2}^{[r+1]} \eta_{0,1,2}, \]
\[ c_3^{[r+1,1]} = \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+1}^{[r+1]} \eta_{1,2}, \]
\[ c_3^{[r+1,p]} = \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+2}^{[r+1]} \eta_{1,2} + c_1^{[r+1,1]} \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+1}^{[r+1]} \eta_{1,2} + c_2^{[r+1,0]} \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+2}^{[r+1]} \eta_{1,2}. \]

\[ c_4^{[r+1,1]} = \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+2}^{[r+1]} \eta_{1,2} + c_0^{[r+1,1]} \sum_{j=0}^{p-1} (-1)^{j+p+1} c_{2(p-j)+2}^{[r+1]} \eta_{1,2}. \]
According to above results and using the automorphism $A \leftrightarrow A^*$ and $\rho_0 \leftrightarrow \rho_1$, we conclude that if $A, A^*$ satisfy the defining relations (3.11), (3.2), then the higher order $q$–Dolan-Grady relations (3.12), (3.13) are such that the coefficients $c_{j}^{[r,p]}$ are determined recursively by (3.24), (3.25) and (3.26). For $p \geq 1$, they can be computed for practical purpose\footnote{As the reader may have noticed, for $r = 2, 3$ the coefficients $c_{j}^{[r,p]}$ are proportional to $[2r + 1]_q$ iff $j \neq 0$ or $2r + 1$. For a large number of values $r \geq 4$, this property holds too. As a consequence, the relations (1.4) drastically simplify for $q^{2r+1} = \pm 1$. This case is however not considered here.}. In particular, one observes that $c_{j}^{[r,p]} = c_{j}^{[2(r-p)+1]}$. For $r = 4, 5, \ldots \leq 10$, using a computer program we have checked in details that $r$–th higher order relations of the form (1.4) hold, and that the coefficients satisfy above recursive formula.

3.4. Comments. Let $A, A^*$ be the fundamental generators of the $q$–Onsager algebra with defining relations (1.3). Let $V$ denote an irreducible finite dimensional vector space and assume each of $A, A^*$ is diagonalizable on $V$. Then, it is known [Ter] Theorem 3.10 that $A, A^*$ act on $V$ as a TD pair. Assume conjecture\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}} holds. Then, the coefficients obtained from the two-variable polynomial (2.10) and given by (2.15) must coincide exactly with the coefficients satisfying above recursive formulae. For $r = 4, 5, \ldots \leq 10$, using a computer program we have checked the correspondence. Also, note that for the special case $\rho_0 = \rho_1 = 0$ Lusztig’s higher order $q$–Serre relations for generic values of $r$ are recovered: in this case the coefficients are given by (3.24) in agreement with [L]. Besides the proofs for $r = 2, 3$, these checks give another support for conjecture\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}}.

To conclude, let us mention that a proof of conjecture\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}} for generic values of $r$ - without using the properties of tridiagonal pairs - would be desirable. In this direction, by analogy with Lusztig’s work\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}} we expect that the construction of the braid group associated with the $q$–Onsager algebra\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}} will help.

4. Concluding remarks

An explicit relationship between the $q$–Onsager algebra (1.3) and a coideal subalgebra of $U_q(\widehat{sl}_2)$ is already known [Bas] (see also [IT2], [K]). Recall that there exists an algebra homomorphism from (1.3) to $U_q(\widehat{sl}_2)$ with scalars $c_i, \bar{c}_i, \epsilon_i \in K$ such that

\begin{align*}
A &= c_0 \epsilon_0 q^{h_0/2} + \bar{c}_0 \epsilon_0 q^{h_0/2} + \epsilon_0 q^{h_0}, \\
A^* &= c_1 \epsilon_1 q^{h_1/2} + \bar{c}_1 \epsilon_1 q^{h_1/2} + \epsilon_1 q^{h_1},
\end{align*}

where one identifies $\rho_i = c_i \bar{c}_i (q + q^{-1})^2$ for $i = 0, 1$. According to conjecture\footnote{\normalfont{P. Baseilhac and S. Kolb, in progress.}} the elements of the coideal subalgebra of $U_q(\widehat{sl}_2)$ generated by (4.1) satisfy the higher order $q$–Dolan-Grady relations (1.4). If $A, A^*$ act on a finite dimensional vector space $V$ [IN1] - see the explicit examples considered in [BK1] [BK2] - this is true according to Theorem 2. Also, for any of the special cases $c_i = 0$ or $\bar{c}_i = 0$, $\rho_i = 0$ so that the relations (1.4) reduce to the higher $q$–Serre relations (1.2).
A straightforward application of (1.4) concerns the theory of quantum integrable systems, in which case quantum groups provide efficient tools to solve a model. For instance, given a Hamiltonian which commutes with the elements of a quantum group, the structure of the Hamiltonian’s spectrum and eigenstates can be studied within the representation theory of the quantum group. For $q$ generic, this approach has been applied to several models. One of the most studied example is the XXZ spin-$s$ chain with periodic or special boundary conditions, and its thermodynamic limit’s analogue. For $q$ a root of unity, new features appear. For $s = 1/2$ the existence of additional properties which do not follow from the star-triangle equation has been early noticed by Baxter [Bax]. Numerically, additional degeneracies have been also observed in the spectrum for arbitrary spin (see references in [DFM, KM]) pointing out the existence of hidden symmetries. A breakthrough was made in [DFM] where it was shown that the model at $q$ a root of unity enjoys a $\hat{sl}_2$ loop algebra invariance. Remarkably, this property further extends to other integrable models [KM] (see also [ND, AYP]). For instance, the Fateev-Zamolodchikov XXZ spin chain ($s = 1$) [FZ] at $q = e^{i\pi/3}$ which is closely related with the 3–state superintegrable chiral Potts model. Importantly, in the works [DFM, KM, ND, AYP] the discovery of the hidden $\hat{sl}_2$ loop symmetry is essentially based on the higher order $q$–Serre relations (1.2) of $U_q(\hat{g})$. Having this in mind, a natural question is whether other types of loop symmetries can emerge for $q$ a root of unity and certain boundary conditions in open spin chain and related vertex models. Indeed, based on the relation between a certain coideal subalgebra of $U_q(\hat{sl}_2)$ and the reflection equation associated with the $U_q(\hat{sl}_2)$ $R$–matrix, the spectrum generating algebra associated with the XXZ open spin chain with generic boundary parameters and $q$ has been identified with the $q$–Onsager algebra (1.3) [BK1, BK2, BK3]. As a consequence, by analogy with the analysis in [DFM, KM, ND, AYP], the relations (1.4) here derived should play a central role in identifying the hidden symmetry of the open XXZ spin chain (as well as other models) at $q$ a root of unity and certain boundary conditions. We intend to study this problem elsewhere.

Finally, let us mention that higher rank generalizations of the $q$–Onsager algebra (1.3) have been proposed in [BB2] (see also [K]). By analogy with the case of $\hat{g} = \hat{sl}_2$ here presented, higher order relations can be constructed following a similar analysis [BV]. We expect these relations will find applications in the theory of tridiagonal algebras associated with higher rank affine Lie algebras $\hat{g}$.

Acknowledgments: We are indebted to P. Terwilliger for a careful reading of the first version of the manuscript, and sharing with us some of the results presented in Section 2. P.B thanks S. Baseilhac and S. Kolb for discussions.
APPENDIX A: Coefficients $\eta_{k,j}^{(m)}$, $M_j^{(r,p)}$, $N_j^{(r,p)}$

The coefficients that appear in eqs. (3.19), (3.20) are such that:
\[
\begin{align*}
\eta_{1,0}^{(3)} &= [3]_q, \\
\eta_{1,1}^{(3)} &= -[3]_q, \\
\eta_{1,2}^{(3)} &= 1, \\
\eta_{0,0}^{(4)} &= 1, \\
\eta_{0,1}^{(4)} &= q^2 + q^{-2}, \\
\eta_{0,2}^{(4)} &= -[3]_q, \\
\eta_{1,0}^{(4)} &= (q^2 + q^{-2})[3]_q, \\
\eta_{1,1}^{(4)} &= -(q^2 + q^{-2})[2]_q, \\
\eta_{1,2}^{(4)} &= [3]_q.
\end{align*}
\]

The recursion relations for $\eta_{k,j}^{(m)}$ are such that:
\[
\begin{align*}
\eta_{0,0}^{(2n+2)} &= 1, \\
\eta_{0,0}^{(2n+2)} &= [3]_q \eta_{k,0}^{(2n+1)} + \eta_{k,1}^{(2n+1)}, \\
\eta_{0,1}^{(2n+2)} &= \eta_{1,0}^{(2n+1)} - 1, \\
\eta_{k,1}^{(2n+2)} &= -[3]_q \eta_{k,0}^{(2n+1)} + \eta_{k+1,0}^{(2n+1)} + \eta_{k,2}^{(2n+1)}, \\
\eta_{n,1}^{(2n+2)} &= -[3]_q \eta_{n,0}^{(2n+1)} + \eta_{n,2}^{(2n+1)}, \\
\eta_{0,0}^{(2n+2)} &= -\eta_{1,0}^{(2n+1)}, \\
\eta_{2,0}^{(2n+2)} &= \eta_{k,0}^{(2n+1)} - \eta_{k+1,0}^{(2n+1)}, \\
\eta_{2,0}^{(2n+2)} &= \eta_{n,0}^{(2n+1)}, \\
\end{align*}
\]

and
\[
\begin{align*}
\eta_{k,0}^{(2n+3)} &= [3]_q \eta_{k-1,0}^{(2n+2)} + \eta_{k-1,1}^{(2n+2)}, \\
\eta_{k,1}^{(2n+3)} &= -[3]_q \eta_{k-1,0}^{(2n+2)} + \eta_{k,0}^{(2n+2)} + \eta_{k-2,2}^{(2n+2)}, \\
\eta_{n,1}^{(2n+3)} &= -[3]_q \eta_{n-1,0}^{(2n+2)} + \eta_{n,2}^{(2n+2)}, \\
\eta_{k,2}^{(2n+3)} &= \eta_{k-1,0}^{(2n+2)} - \eta_{k,1}^{(2n+2)}, \\
\eta_{n-1,2}^{(2n+3)} &= \eta_{n,0}^{(2n+2)}. \\
\end{align*}
\]

The coefficients that appear in eqs. (3.15), (3.16) are such that:
\[
\begin{align*}
M_j^{(r,0)} &= c_j^{[r,0]} - c_1^{[r,0]} c_{j-1}^{[r,0]}, \\
M_j^{(r,p)} &= -c_1^{[r,0]} c_{2r+1}^{[r,0]}, \\
M_j^{(r,p)} &= c_0^{[r,p]}, \\
M_j^{(r,p)} &= c_j^{[r,p]} - c_1^{[r,p]} c_{j-1}^{[r,p]}, \\
M_j^{(r,p)} &= -c_1^{[r,p]} c_{2(r-p)+1}^{[r,p]}, \\
\end{align*}
\]

and
\[
\begin{align*}
N_j^{(r,0)} &= c_j^{[r,0]} - c_1^{[r,0]} c_{j+1}^{[r,0]} + (c_1^{[r,0]} c_{j-2}^{[r,0]} - c_1^{[r,0]} c_{j-1}^{[r,0]} c_{j-2}^{[r,0]}), \\
N_j^{(r,0)} &= c_j^{[r,0]} c_{2r+1}^{[r,0]}, \\
N_j^{(r,0)} &= (c_1^{[r,0]} c_{2r+1}^{[r,0]} + c_1^{[r,0]} c_{2r+1}^{[r,0]}), \\
N_j^{(r,0)} &= 0, \\
N_j^{(r,0)} &= c_j^{[r,0]} c_{2r}^{[r,0]} c_{j-1}^{[r,0]} c_{j-2}^{[r,0]} + c_1^{[r,0]} c_{j-1}^{[r,0]} c_{j-2}^{[r,0]} + c_1^{[r,0]} c_{j-1}^{[r,0]} + c_0^{[r,0]} c_j^{[r,0]}, \\
N_j^{(r,0)} &= -c_1^{[r,0]} c_{0}^{[r,0]} c_{1}^{[r,0]} + c_1^{[r,1]} c_{0}^{[r,0]} c_{1}^{[r,0]}, \\
\end{align*}
\]
Higher order relations for the $q$-Onsager algebra

For $2 \leq p \leq r$,

\[ N^{(r,1)}_{2r} = \left( c_1^{[r,0]} - c_2^{[r,0]} \right) c_{2r-2} - c_1^{[r,1]} c_{2r-1} - c_0^{[r,1]} c_{2r}, \]
\[ N^{(r,1)}_{2r+1} = \left( c_1^{[r,0]} - c_2^{[r,0]} \right) c_{2r-1} - c_0^{[r,1]} c_{2r+1}, \]
\[ N^{(r,r+1)}_j = -c_0^{[r,1]} c_j^{[r,r]}, \quad j = 0, 1. \]

\[ N^{(r,p)}_j = \left( c_1^{[r,0]} - c_2^{[r,0]} \right) c_j^{[r,p]} - c_1^{[r,1]} c_{j-1} + c_0^{[r,1]} c_j^{[r,p-1]}, \quad j = 2, 2(r-p) + 1, \]
\[ N^{(r,p)}_0 = c_0^{[r,p]} - c_0^{[r,1]} c_0^{[r,p-1]}, \]
\[ N^{(r,p)}_1 = -c_1^{[r,0]} c_0^{[r,p]} + c_1^{[r,p]} - c_0^{[r,1]} c_0^{[r,p-1]}, \]
\[ N^{(r,p)}_{2(r-p)+2} = \left( c_1^{[r,0]} - c_2^{[r,0]} \right) c_{2(r-p)+1} - c_1^{[r,1]} c_{2(r-p)+2}, \]
\[ N^{(r,p)}_{2(r-p)+3} = \left( c_1^{[r,0]} - c_2^{[r,0]} \right) c_{2(r-p)+1} - c_0^{[r,1]} c_{2(r-p)+3}. \]
APPENDIX B: $A^{2r+2}A^{*r+1}A$, $A^{2r+1}A^{*r+1}A^2$, $A^{2r+1}A^{*r+1}$

In addition to \([3.21]\), the other monomials can be written as:

$$A^{2r+2}A^{*r+1}A = -\sum_{i=1}^{r} M_{2i+1} \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+2)} A^{-j} A^{*} A^{2k+j+1}$$

$$+ \sum_{i=1}^{r} M_{2i+1} A^{2(r-i)+1} A^{*r} \left( \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+1)} A^{-j} A^{*} A^{2k+j} + \rho_{0}^{i} (AA^{*} A - A^{*} A^{2}) \right)$$

$$- \sum_{i=1}^{r} (-\rho_{0})^{(r)} M_{0} A^{2(r-p)+2} A^{*r+1} A - M_{1} A^{2(r-p)+1} A^{*r} A^{*} A^{2} + M_{2} A^{2(r-p)} A^{*r} A^{2} A^{*} A$$

$$- M_{2} A^{2r} A^{*r+2} A^{*} A - \sum_{i=1}^{r-1} (-\rho_{0})^{p} M_{2i+1} \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+1)} A^{-j} A^{*} A^{2k+j+1}$$

$$+ \sum_{i=1}^{r-1} (-\rho_{0})^{p} \left( \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+2)} A^{-j} A^{*} A^{2k+j} + \rho_{0}^{i} (AA^{*} A - A^{*} A^{2}) \right),$$

$$A^{2r+1} A^{*r+1} A^{2} = c_{[r,0]}^{2} A^{2r} A^{*r} A A^{*} A^{2} - c_{[r,0]}^{2} A^{2r-1} A^{*r} A^{2} A^{*} A^{2}$$

$$- \sum_{i=1}^{r-1} (-\rho_{0})^{p} (c_{[r,0]}^{1} A^{2(r-p)+1} A^{*r+1} A^{2} - c_{[r,0]}^{1} A^{2(r-p)} A^{*r} A A^{*} A^{2} + c_{[r,0]}^{2} A^{2(r-p)-1} A^{*r} A^{2} A^{*} A^{2})$$

$$- (-\rho_{0})^{r} (c_{[r,0]}^{1} A^{*r+1} A A^{*} A^{2} - c_{[r,0]}^{1} A^{*r} A A^{*} A^{2})$$

$$+ \sum_{i=1}^{r-1} c_{2i+1} A^{2(r-i)} A^{*r} \left( \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+1)} A^{-j} A^{*} A^{2k+j+1} + \rho_{0}^{i} (AA^{*} A - A^{*} A^{3}) \right)$$

$$- \sum_{i=1}^{r-1} c_{2i+2} A^{2(r-i)-1} A^{*r} \left( \sum_{k=0}^{2} \sum_{j=0}^{i} \rho_{0}^{-k} \eta_{k,j}^{(2i+2)} A^{-j} A^{*} A^{2k+j+2} \right),$$

$$A^{2r+1} A^{*r+1} A^{2} = c_{[r,0]}^{1} A^{2r} A^{*r} A A^{*} - c_{[r,0]}^{1} A^{2r-1} A^{*r} A^{2} A^{*}$$

$$+ \sum_{k=1}^{r} c_{2k+1} A^{2(r-k)} A^{*r} \left( \sum_{i=0}^{k} \mu_{i,j}^{(2k+1)} A^{-j} A^{*} A^{2i-1+j} + \rho_{0}^{k} (AA^{*} - A^{*} A) \right)$$

$$- \rho_{0}^{r} (c_{[r,0]}^{1} A^{*r+1} - c_{[r,0]}^{1} A^{*r} A A^{*})$$

$$- \sum_{r=1}^{p} \rho_{0}^{p} (c_{[r,0]}^{1} A^{2(r-p)+1} A^{*r+1} A^{2} - c_{[r,0]}^{1} A^{2(r-p)} A^{*r} A A^{*} + c_{[r,0]}^{2} A^{2(r-p)-1} A^{*r} A^{2} A^{*})$$

$$+ \sum_{r=1}^{p} \rho_{0}^{p} \sum_{k=1}^{r} c_{2k+1} A^{2(r-p-k)} A^{*r} \left( \sum_{i=0}^{k} \mu_{i,j}^{(2k+1)} A^{-j} A^{*} A^{2i-1+j} + \rho_{0}^{k} (AA^{*} - A^{*} A) \right)$$

$$- \sum_{r=1}^{p} \rho_{0}^{p} \sum_{k=1}^{r} c_{2k+2} A^{2(r-p-k)-1} A^{*r} \left( \sum_{i=0}^{k} \mu_{i,j}^{(2k+2)} A^{-j} A^{*} A^{2i+j} \right).$$
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