Cluster Structures of the Ground and Excited States of $^{12}$Be Studied with Antisymmetrized Molecular Dynamics

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The structures of the ground and excited states of $^{12}$Be were studied with antisymmetrized molecular dynamics. The ground state was found to be a state with a developed $2\alpha$ core with two neutrons occupying the intruder orbits. The energy levels of the newly measured spin-assigned states were described well, except for the $1^{-}$ state. The calculations indicated that many exotic cluster structures appear in the low-energy region. The widths concerning $\alpha$ and $^8\text{He}$ decays were discussed by using reduced width amplitudes.

I. INTRODUCTION

Owing to the progress of experimental techniques, information concerning the excited states of light unstable nuclei has rapidly increased. Recently, exotic clustering in the light unstable nuclei has become one of the attractive subjects in 3experimental and theoretical research. Since, in light stable nuclei, it has already been known that clustering is one of the essential features of nuclear dynamics, not only in excited states, but also in ground states, it is natural to expect cluster features in light unstable nuclei. Pioneering theoretical studies have suggested the development of cluster structures with a $2\alpha$ core in Be and B isotopes 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{B}^+_4,\text{B}^+_5,\text{B}^+_6,\text{B}^+_7,\text{B}^+_8,\text{B}^+_9,\text{B}^+_{10},\text{B}^+_{11},\text{B}^+_{12},\text{B}^+_{13},\text{B}^+_{14},\text{B}^+_{15},\text{B}^+_{16}
\]

especially, highly excited states with developed cluster structures in $^{10}$Be and $^{11}$Be have been studied by microscopic calculations 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

In the case of $^{12}$Be, the existence of cluster states was suggested in experimental measurements of the excited states 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

In recent experiments of $^6\text{He}^+\text{Be}$ and $^8\text{He}^+\text{Be}$ breakup reactions 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

many new excited states were discovered above the threshold energies. Some of the states are candidates of exotic cluster states, since the measured spin-parities of those excited states indicate a rotational band with a large moment of inertia. It is an interesting subject to investigate clustering aspects in $^{12}$Be. Although the molecular states in $^{12}$Be were theoretically suggested with a potential model with He clusters 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

they have not yet been studied by microscopic calculations. In the recent investigations with the Generator Coordinate Method of a coupled-channel two-body cluster model, using microscopic $^6\text{He}^+\text{He}$ and $^8\text{He}^+\text{He}$ wave functions, a strong mixing of both configurations was predicted 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

The ground and the low-lying states of $^{12}$Be present us with other attractive subjects concerning the vanishing of a neutron magic number, 8. The vanishing of the neutron magic number is already known in a neighboring nucleus, $^{11}$Be. The vanishing in $^{12}$Be was predicted in the early theoretical works 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

The abnormal configurations((sd)$^2$) in the low-lying states were experimentally studied by $^{10}\text{Be}(t, p)^{12}\text{Be}$ reaction 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

Moreover, the vanishing in $^{12}$Be was supported by a recent measurement of the spin-parity for a low-lying $1^{-}$ state 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

A new low-lying $0^+_2$ state 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

is also important to solve the inversion mechanism. In the recent theoretical studies, the structure of low-lying states were microscopically described by a molecular-orbital model by N. Itagaki et al. 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

and by a method of antisymmetrized molecular dynamics by one of the authors (Y.K.) 

\[
\text{Be}^+_2,\text{Be}^+_3,\text{Be}^+_4,\text{Be}^+_5,\text{Be}^+_6,\text{Be}^+_7,\text{Be}^+_8,\text{Be}^+_9,\text{Be}^+_10,\text{Be}^+_11,\text{Be}^+_12,\text{Be}^+_13,\text{Be}^+_14,\text{Be}^+_15,\text{Be}^+_16
\]

Our aim is to conduct systematic research of the structures of the ground and excited states of $^{12}$Be based on microscopic theoretical calculations while focusing on clustering aspects. First of all, we have studied the systematics of the level structures, including the experimentally observed excited states. We have searched for cluster and non-cluster states to solve the following problems concerning clustering aspects in $^{12}$Be. Do their cluster structures appear in $^{12}$Be? If they do appear, what are the characteristics of their structures in unstable nuclei compared with those of stable nuclei? The roles of the valence nucleons in the cluster states are interesting problems. We have also investigated the mechanism of the development and breaking of clustering.

The important point is that the theoretical approach should be free from model assumptions, such as the stability of the mean-field and the existence of inert cores or clusters, because we have to describe various structures covering developed cluster structures as well as shell-model-like structures in the ground and excited states. It is difficult to study developed clustering in excited states with such mean-field approaches as the traditional shell model and the Hartree-Fock model. The cluster structures of the excited states of $^9\text{Be}$ and $^{10}\text{Be}$ were successfully explained by
particular, the formulation of the present calculations is same as that described in Ref. [7]. In this section, I briefly explain the formulation of AMD for a study of the nuclear structure of excited states. The adopted effective interactions are calculation after spin-parity projection based on the AMD method. In the next section (Sec. II), we explain the theoretical approach for the nuclear structure [2,5,7,11,25,26]. Within the AMD framework, we do not need such model assumptions as inert cores, clusters, or axial symmetries, because the wave function of the nuclear system is written by Slater determinants, where the spatial part of each single-particle wave function for a nucleon is expressed by a localized Gaussian wave packet. Due to the flexibility of the AMD wave function, we successfully described the structure changes between shell-model-like states and cluster states as functions of the neutron number in light unstable nuclei. Owing to progress in computational power, it has become possible to study excited states by extended AMD calculations. This method is based on variational calculation after spin-parity projection (VAP calculation) within the framework of AMD, which has already been confirmed to be powerful for studying the excited states of light nuclei, as shown in studies of the stable nucleus $^{12}$C [26], and also unstable nuclei ($^{10}$Be and $^{11}$Be) [7,11]. The authors and their collaborators succeeded to describe various structures of the excited states and to reproduce many kinds of experimental data for nuclear structures of these nuclei with the AMD method.

In the present work, the structures of the ground and excited states of $^{12}$Be were analyzed by performing a variational calculation after spin-parity projection based on the AMD method. In the next section (Sec. III), we explain the formulation of AMD for a study of the nuclear structure of excited states. The adopted effective interactions are explained in Sec. III. In Sec. IV, we present the calculated results concerning such observables as the energy levels, radii and $\beta$ decays as well as the E1 and E2 transitions compared with the experimental data. In the discussion (Sec. V), the intrinsic structures and the rotational band structures are described. The single-particle behavior of the valence neutrons is analyzed. We discuss the systematics of the development of cluster states in Be isotopes, which can be classified according to the neutron orbits surrounding the 2\,$\alpha$ core. We investigate the inter-cluster motions between He clusters extracted from the obtained $^{12}$Be wave functions, and calculate the decay widths for the $^4$He and $^6$He channels by using the method of reduced width amplitudes.

II. FORMULATION

The formulation of AMD for a nuclear structure study of ground and excited states is explained in [2,26]. In particular, the formulation of the present calculations is same as that described in Ref. [2]. In this section, I briefly review the formulation.

The wave function of a system is written by a superposition of the AMD wave functions($\Phi_{AMD}$). The AMD wave function of a nucleus with a mass number $A$ is a Slater determinant of Gaussian wave packets. The $i$th single-particle wave function is a product of the spatial wave function, the intrinsic spin function and the iso-spin function. The spatial part is presented by variational complex parameters, $X_{X_1}$, $X_{X_2}$, $X_{X_3}$, which indicate the center of the Gaussian wave packets. The orientation of the intrinsic spin part is expressed by a variational complex parameter $\xi$, and the iso-spin function is fixed to be up(proton) or down(neutron) in the present calculations. Thus, an AMD wave function is expressed by a set of variational parameters, $Z \equiv \{X_n, \xi_i\}$ ($n = 1, 2, 3$ and $i = 1, \cdots, A$), which indicate the centers of Gaussians of the spatial part and the spin orientations of the intrinsic spin part of the single-particle wave functions.

When we consider a parity-eigen state projected from an AMD wave function, the total wave function is written by two Slater determinants. In the case of a total-angular-momentum eigen state, the wave function of the state is represented by the integral of the rotated AMD wave functions. The expectation values of a given tensor operator for the structure changes between shell-model-like states and cluster states as functions of the neutron number in light unstable nuclei. Owing to progress in computational power, it has become possible to study excited states by extended AMD calculations. This method is based on variational calculation after spin-parity projection (VAP calculation) within the framework of AMD, which has already been confirmed to be powerful for studying the excited states of light nuclei, as shown in studies of the stable nucleus $^{12}$C [26], and also unstable nuclei ($^{10}$Be and $^{11}$Be) [7,11]. The authors and their collaborators succeeded to describe various structures of the excited states and to reproduce many kinds of experimental data for nuclear structures of these nuclei with the AMD method.

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In order to obtain the wave function for the lowest $J^\pm$ state, we varied the parameters $X_i$ and $\xi_i$ ($i = 1 \sim A$) to minimize the energy expectation value of a spin-parity projected AMD wave function, $\Phi = P_{MK}^{J^\pm} \Phi_{AMD}(Z)$, using the fractional cooling method. That is to say, we performed energy variation after spin-parity projection (VAP) for an AMD wave function.

With the VAP calculation for the $J^\pm$ eigen state, $\Phi_1^{J^\pm}(Z) = P_{MK}^{J^\pm} \Phi_{AMD}(Z)$, with an appropriate $K'$, we obtained a set of parameters, $Z = Z_{1}^{J^\pm}$, which presents the wave function of the first $J^\pm$ state. In order to search for the parameters $Z = Z_{n}^{J^\pm}$ of the $n$th $J^\pm$ state, the wave functions were superposed so as to be orthogonal to the lower states. The parameters $Z_{n}^{J^\pm}$ of the $n$th $J^\pm$ state were provided by varying $Z$ so as to minimize the energy of the wave function orthogonalized to the lower states.

After the VAP calculation of the $J^+_n$ states for various $J$, $n$ and $\pi = \pm$, we obtained the optimum intrinsic states, $\Phi_{AMD}(Z_{n,\pi}^{J^+})$, which approximately describe the corresponding $J^+_n$ states. In order to obtain more precise wave functions, we superposed the spin-parity eigen wave functions projected from all of the obtained intrinsic states. Namely, we determined the final wave functions for the $J^+_n$ states by simultaneously diagonalizing the Hamiltonian matrix, $\langle P_{MK}^{J^+} \Phi_{AMD}(Z_{n,\pi}^{J^+}) | H | P_{MK}^{J^+} \Phi_{AMD}(Z_{n,\pi}^{J^+}) \rangle$, and the norm matrix, $\langle P_{MK}^{J^+} \Phi_{AMD}(Z_{n,\pi}^{J^+}) | P_{MK}^{J^+} \Phi_{AMD}(Z_{n,\pi}^{J^+}) \rangle$, with regard to $(i,j)$ for all of the obtained intrinsic states, and to $(K', K''')$. Compared with the experimental data, such as the energy levels and $E2$ transitions, the expectation values were calculated with the final states after diagonalization.

### III. INTERACTIONS

The adopted interaction is the sum of the central force, the spin-orbit force and the Coulomb force. The central force is chosen to be the MV1 force of case 3 [27], which contains a zero-range three-body force, and $\nu$ is chosen to be the MV1 force of case 3 [27], which contains a zero-range three-body force, and $\nu$ is chosen to be the MV1 force of case 3 [27], which contains a zero-range three-body force, and $\nu$ is chosen to be the MV1 force of case 3 [27], which contains a zero-range three-body force.

The adopted interaction parameters in the present work are those used in Refs. [7,11], which reproduced the G3RS force [28], as a density-dependent term in addition to the two-body interaction, $V^{(2)}$, of the modified Volkov No.1 force. The spin-orbit force of the G3RS force [28] is adopted.

### IV. RESULTS

The structures of the excited states of $^{12}$Be were studied based on the VAP calculation within the framework of AMD. In this section, we present theoretical results, such as the energy levels, radii, and transitions ($\beta$, $E2$, $E1$ and $E0$), while comparing them with the experimental data. A detailed analysis of the structures of the states is given in the next section.

The adopted interaction parameters in the present work are those used in Refs. [7,11], which reproduced the abnormal spin-parity $1/2^+$ of the ground state of $^{11}$Be. Namely, the Majorana, Bartlett and Heisenberg parameters in the central force are $m = 0.65$, $b = h = 0$, and the strength of the spin-orbit force is chosen to be $u_l = -u_{11} = 3700$ MeV. The width parameter ($\nu$) was chosen to be 0.17 fm$^{-2}$, which gave a minimum energy of $^{12}$Be in a simple AMD calculation without the spin projection.

The wave functions for the lowest $J^\pm$ states were obtained by the VAP calculation of $P_{MK}^{J^\pm} \Phi_{AMD}$ by choosing $(J^\pm, K') = (0^+, 0), (2^+, 0), (4^+, 0), (6^+, 0), (8^+, 0), (1^-, 1), (2^-, 1), (3^-, 1), (4^-, 1), (5^-, 1), (6^-, 1), (3^+, 2), (5^+, 2), (7^+, 2), (0^-, 0), (1^+, 1), (1^+, 0)$. After obtaining the lowest states($J_n^\pm$), we calculated the second and third $J^\pm$ states ($2^-, 0$, $2^+, 1$, $4^-, 1$, $6^+, 2$, $1^-$) with the VAP calculation of the superposed wave functions orthogonal to the obtained lower $J^\pm$ states. The obtained AMD wave functions are considered to approximately describe the intrinsic states of the corresponding $J^+_n$ states. The final wave functions of the $J^+_n$ states were determined by superposing the spin-parity eigen states projected from these obtained AMD wave functions so as to simultaneously diagonalize the Hamiltonian matrix and the norm matrix. In principle, the number of the superposed AMD wave functions is 24, which is the number of calculated levels. In the present calculations, we omitted several of them which are not important, just to save computational time. Namely, we diagonalize the positive-parity(negative-parity) states projected from 22(18) AMD wave functions.

#### A. Energies

The theoretical binding energy of $^{12}$Be was found to be 61.9 MeV (60.4 MeV) after (before) diagonalization, which underestimated the experimental value, 68.65 MeV. We found that the binding could be improved by changing the Majorana parameter ($m$) of the interaction to a smaller value. For example, a theoretical value of 66.1 MeV was obtained by using $m = 0.62$ with a simple VAP calculation before diagonalization. Nevertheless, we adopted the parameter $m = 0.65$ in the present work, because this value reproduces the parity inversion of $^{11}$Be [11], which should...
be important for describing the ground-state properties of the neighboring nucleus, $^{12}\text{Be}$. We checked that the change in the $m$ parameter has no significant effect on the excitation energies, $E(0^{+}_{1})$ and $E(8^{+}_{2})$, of $^{12}\text{Be}$, at least. In fact, the calculated excitation energies of $E(0^{+}_{2})$ before diagonalization are 3.0 MeV and 2.8 MeV, in the cases of $m = 0.62$ and $m = 0.65$, respectively.

The theoretical level scheme is shown in Fig. 1. The calculations suggest that many excited states appear in the low-energy region. This is the first theoretical work which systematically reproduces the experimental energy levels of all the spin-assigned states, except for the $1^{-}$ state. By analyzing the intrinsic AMD wave functions of the states, we can consider that there exist rotational bands ($K\pi = 0^{+}_{1}$, $0^{+}_{2}$, $0^{+}_{3}$ and $1^{-}_{1}$), which consist of the following states: 

$\{0^{+}_{1}, 2^{+}_{1}, 4^{+}_{1}, 6^{+}_{1} \text{ and } 8^{+}_{1}\}$, 

$\{0^{+}_{2}, 2^{+}_{2}, 4^{+}_{2}, 6^{+}_{2} \text{ and } 8^{+}_{2}\}$, and 

$\{1^{-}_{1}, 2^{-}_{1}, 3^{-}_{1}, 4^{-}_{1}, 5^{-}_{1}\}$, respectively.

The energy-spin systematics for positive-parity states are shown in Fig. 3. It is surprising that the newly observed levels, $4^{+}$ at 13.2 MeV and $6^{+}$ at 16.1 MeV, correspond well to the $4^{+}_{2}$ and $6^{+}_{2}$ states obtained in the present results. An interesting point is that these excited states belong not to the yrast band, $K\pi = 0^{+}_{1}$, but to the new excited $K\pi = 0^{+}_{2}$ band with developed cluster structure. The theoretical results predict the existence of many positive-parity states belonging to the $K\pi = 0^{+}_{1}$ and $K\pi = 0^{+}_{2}$ bands in the lower energy region. Even though $^{12}\text{Be}$ has a neutron magic number of 8, the calculated ground $K\pi = 0^{+}_{1}$ band has a large moment of inertia, because the band head $0^{+}_{1}$ state is not the ordinary state with a closed neutron $p$-shell, but an intruder state with a developed cluster structure. It has a prolate deformed structure, which is dominated by $2p-2h$ configurations, and reaches the band terminal at the $8^{+}_{1}$ state, because the $J^{\pm} = 8^{+}$ state is the highest-spin state in the $2h\omega$ configurations. The third $0^{+}$ band ($K\pi = 0^{+}_{3}$) was found to be represented by other $2p-2h$ configurations than those in the $K\pi = 0^{+}_{2}$ band. The $K\pi = 0^{+}_{3}$ band has an extremely large moment of inertia because of a remarkably developed $6\text{He}+6\text{He}$ clustering, and reaches the band terminal $6^{+}_{3}$ state accompanying the spin-alignment of nucleons in the high-spin region. On the other hand, the main components of the $0^{+}_{2}$ and $2^{+}_{2}$ states are of the $0h\omega$ configurations, with the closed neutron $p$-shell, and constitute the $K\pi = 0^{+}_{2}$ band. The band head $0^{+}_{2}$ state of this band, which was theoretically predicted to appear just above the intruder ground band by Itagaki et al. and Kanada-En’yo et al., was recently discovered in an observation of coincidence gamma rays by Shimoura et al.

The theoretical excitation energy of the $1^{-}_{3}$ state is larger than the measured $1^{-}$ state at 2.68 MeV. In spite of the overestimation of the excitation energy, we regard this $1^{-}_{3}$ state in the $K\pi = 1^{-}$ as the $1^{-}$ state at 2.68 MeV because of the large $E1$ transition strength, as shown later. In order to improve the excitation energy of this $1^{-}$ state, it is important to take the mixing of the other configuration into account. For example, by mixing a $1^{-}$ state with $K\pi = 0^{-}$, which is obtained at slightly higher energy than the $1^{-}$ state with $K\pi = 1^{-}$ in VAP calculations, the $1^{-}_{3}$ state gains about 1 MeV. Another reason for the overestimation may be connected to the fact that the present interactions give a too small value for the effective energy difference between the $s_{1/2}$ and $d_{5/2}$ orbits, which was found in the calculations of $^{11}\text{Be}$ in Ref. [11].

**B. Radii**

The theoretical values of the root-mean-square radii of the density distributions of point-like nucleons, protons, and neutrons are listed in Table 1 together with the experimental matter radius deduced from the reaction cross sections. Because of deformations, the proton and neutron radii in the ground state are larger compared with those in the $0^{+}_{2}$ state. Since the theoretical values concerning the difference $\Delta r = r_{n} - r_{p}$ between the proton and neutron radii are about 0.3 fm, $^{12}\text{Be}$ is a candidate of the neutron skin nucleus. Compared with the experimental data, the present result for $0^{+}_{1}$ is larger because the present parameter $m = 0.65$ is considered to be too large to quantitatively reproduce the radii of $p$-shell nuclei.

The proton and neutron densities as functions of the radius are shown in Fig. 3. In both the $0^{+}_{1}$ and $0^{+}_{2}$ states, the excess neutrons enhance the neutron density in the surface region. In the $0^{+}_{1}$ state, due to cluster development, the proton density increases in the surface region while it decreases at the center of the nucleus. It should be noted that, even if a nucleus has a neutron halo (the long tail of the neutron density in the outer region), the details of the halo structure are not expressed in the present model space because of a limitation of the Gaussian forms in AMD wave functions.

**C. $\beta$ decay strength**

The strength of the $\beta$ decay from $^{12}\text{Be}(0^{+}_{1})$ to $^{12}\text{B}(1^{+})$ provides helpful experimental evidence for breaking of the neutron $p$-shell closure in $^{12}\text{Be}(0^{+}_{1})$. T. Suzuki et al. suggested that weak $\beta$ decay is experimental evidence for an admixture of non-$0h\omega$ configurations in $^{12}\text{Be}(0^{+}_{1})$, because the $\beta$ decays from a normal $p$-closed state of $^{12}\text{Be}(0^{+}_{1})$ must
FIG. 1. Excitation energies of the levels in $^{12}$Be. The experimental data are from the Table of Isotopes and Refs. [14,21,13,15,22].

FIG. 2. Energy-spin systematics for positive-parity states in $^{12}$Be. The excitation energies of the natural spin states are plotted as functions of $J(J+1)$. The circle symbols are the theoretical results, while the cross symbols are the experimental data of the spin-parity assigned states, which are taken from the Table of Isotopes and Refs. [14,22].
TABLE I. The root-mean-square radii of the density distribution of point-like nucleons, protons, and neutrons. The experimental matter radius deduced from the reaction cross sections is taken from Ref. [29].

| $^{12}$Be ($0^+_1$) | matter | proton | neutron |
|---------------------|--------|--------|---------|
|                     | 2.85 fm | 2.67 fm | 2.94 fm |
| $^{12}$Be ($0^+_2$) | 2.75 fm | 2.56 fm | 2.84 fm |
| exp.                | 2.59 ± 0.06 | - | - |

FIG. 3. Densities as functions of the radial coordinate ($r$). The densities of the protons and neutrons in the $0^+_1$ state (solid) and in the $0^+_2$ state (dashed) are shown.
be stronger. We calculated the Gamow-Teller transition strength, $B(GT) \equiv \langle |\sigma \tau| \rangle^2$, where the wave function for the daughter state, $^{12}\text{Be}(1^+_1)$, was obtained by a VAP calculation with $(J^\pi, K') = (1^+, 1)$. The $B(GT)$ values are given shown in Table II. The present result, $B(GT) = 0.9$, is almost as small as the experimental data, $B(GT) = 0.59$, because the component of the $2h\omega$ configurations in the parent $^{12}\text{Be}(0^+_1\pi)$ makes the transition matrix element of the Gamow-Teller operator to be small. In fact, the original $0^+_1\pi$ state of $^{12}\text{Be}$ obtained by a simple VAP calculation before diagonalization has very weak $\beta$ transitions as $B(GT) = 0.02$. On the other hand, the decay from $0^+_2\pi$ is strong as $B(GT) = 3.0$ before diagonalization, because the parent state has the ordinary $0\hbar\omega$ configuration. After state mixing with diagonalization, the GT strength from $^{12}\text{Be}(0^+_1\pi)$ is distributed to that from $^{12}\text{Be}(0^+_2\pi)$. As a result, after the diagonalization the $B(GT) = 3.0$ before diagonalization, because the parent state has the ordinary $0\hbar\omega$ configuration. After state mixing with diagonalization, the GT strength from $^{12}\text{Be}(0^+_1\pi)$ decreases to 2.1. According to the present results, the GT transitions from $^{12}\text{Be}(0^+_1\pi)$ are weak, while the $B(GT) = 0.004$ is largest among the $2^+$ states because the $2^+_2$ state is the other $0\hbar\omega$ state belonging to the $K^\pi = 0^+_2$ band.

The present results concerning the weak $GT$ decay from the ground state is consistent with the discussion given in Refs. [8,20].

### D. E2,E1,E0 transition strength

Although the data concerning the $E2$ transition strength provide good information about proton deformations, we must take care of the following points in the analysis of light neutron-rich nuclei. Firstly, it is dangerous to assume the same deformation of the proton density as that of the neutron density in light unstable nuclei. Secondly, the classical relations between the intrinsic deformation ($Q_0$) and the observables, which are often used in a simple analysis of heavy nuclei, do not necessarily work in light nuclei. Therefore, it is necessary to analyze the transition strength based on a microscopic calculation. As shown in previous AMD studies [2,26,7], the experimental $Q$-moments and $B(E2)$ values of light nuclei were reproduced well by using bare charges, because of the advantage of the AMD method, which can directly express the proton and neutron deformations.

Table II gives the theoretical $B(E2)$, $B(E1)$ and $B(E0)$ values calculated by the VAP calculation after diagonalization. The intra-band $E2$ transitions in the $K^\pi = 0^+_1$ band are strong, as seen in $B(E2; 2^+_1 \rightarrow 0^+_1) = 14 \, e^2\text{fm}^4$, due to the deformed intrinsic state. On the other hand, $B(E2)$ is smaller in the transition between the $2^+_2$ and $0^+_1\pi$ states in the $K^\pi = 0^+_2$ band, which have rather spherical shapes compared to those in the $K^\pi = 0^+_1$ band. Although the deformations of the $1^+_1$ and $2^+_2$ states are suggested to exist just above the $0^+_1\pi$ and $2^+_1\pi$ states, we should more carefully investigate the energy difference and the state mixing between the $K^\pi = 0^+_1$ and $K^\pi = 0^+_2$ bands before making conclusion.

Table II. The strength of $\beta$ decays. $B(GT)$ is defined as $\langle |\sigma \tau| \rangle^2$. The experimental data are taken from [30].

| initial $(J^\pi, E_x)$ (MeV) | final $(J^\pi, E_x)$ (MeV) | B(GT) before diagonalization | B(GT) after diagonalization |
|-----------------------------|-----------------------------|-------------------------------|----------------------------|
| $^{12}\text{Be}(0^+_1\pi)$  | $^{12}\text{B}(1^+_1\pi)$  | 0.2                          | 0.9                        |
| $^{12}\text{Be}(0^+_2\pi)$  | $^{12}\text{B}(1^+_1\pi)$  | 3.0                          | 2.1                        |
| $^{12}\text{Be}(1^+_1\pi)$  | $^{12}\text{B}(1^+_1\pi)$  | 0.05                         | 0.07                       |
| $^{12}\text{Be}(2^+_1\pi)$  | $^{12}\text{B}(1^+_1\pi)$  | 0.004                        | 0.1                        |
| $^{12}\text{Be}(2^+_2\pi)$  | $^{12}\text{B}(1^+_1\pi)$  | 0.5                          | 0.4                        |

Compared with the experimental data, $B(E2; 2^+_1 \rightarrow 0^+_1) = 10.5 \pm 1.1 \, (e^2\text{fm}^4)$ of $^{10}\text{Be}$, the present results predict a larger $B(E2)$ of $^{12}\text{Be}$ as $B(E2; 2^+_1 \rightarrow 0^+_1) = 14 \, e^2\text{fm}^4$. We should point out that the $E2$ transition strength is sensitive to the mixing ratio with the $0^+_1$ and $2^+_2$ states as well as deformations. Since the $0^+_1$ and $2^+_2$ states are suggested to exist just above the $0^+_1\pi$ and $2^+_1\pi$ states, we should more carefully investigate the energy difference and the state mixing between the $K^\pi = 0^+_1\pi$ and $K^\pi = 0^+_2\pi$ bands before making conclusion.

The $E1$ transition strength was recently measured as $B(E1; 0^+_1\pi \rightarrow 1^+_1\pi) = 0.05 \, e^2\text{fm}^2$ [21], which is rather large compared with other light nuclei. The calculated result, $B(E1; 0^+_1\pi \rightarrow 1^+_1\pi) = 0.02 \, e^2\text{fm}^2$, reasonably agrees with this.
TABLE III. Theoretical results of $E_2$, $E_1$ and $E_0$ transition strength, $B(E\lambda)$. $B(E0)$ is defined as $| < f | 1 3 E r^2 | i > |$. 

| transitions          | band($K^+$) | Mult. | present                  |
|----------------------|-------------|-------|--------------------------|
| $^{12}$Be$^{+}_{1/2} \rightarrow 0^{-}_{1}$ | in $0^{-}_{1}$ | $E2$ | $14 \ (e^2 \text{ fm}^4)$ |
| $^{12}$Be$^{+}_{3/2} \rightarrow 0^{-}_{1}$ | in $0^{-}_{1}$ | $E2$ | $8 \ (e^2 \text{ fm}^4)$  |
| $^{12}$Be$^{+}_{1/2} \rightarrow 0^{-}_{2}$ | in $0^{-}_{2}$ | $E2$ | $8 \ (e^2 \text{ fm}^4)$  |
| $^{12}$Be$^{+}_{3/2} \rightarrow 0^{-}_{2}$ | in $0^{-}_{2}$ | $E2$ | $14 \ (e^2 \text{ fm}^4)$ |
| $^{12}$Be$^{+}_{1/2} \rightarrow 2^{+}_{1}$ | in $0^{+}_{1}$ | $E2$ | $10 \ (e^2 \text{ fm}^4)$  |
| $^{12}$Be$^{+}_{3/2} \rightarrow 2^{+}_{1}$ | in $0^{+}_{1}$ | $E2$ | $5 \ (e^2 \text{ fm}^4)$   |
| $^{12}$Be$^{+}_{1/2} \rightarrow 2^{+}_{2}$ | in $0^{+}_{2}$ | $E2$ | $5 \ (e^2 \text{ fm}^4)$   |
| $^{12}$Be$^{+}_{3/2} \rightarrow 2^{+}_{2}$ | in $0^{+}_{2}$ | $E2$ | $6 \ (e^2 \text{ fm}^4)$   |
| $^{12}$Be$^{+}_{1/2} \rightarrow 1^{-}_{1}$ | $E1$        | $2 \times 10^{-2} \ (e^2 \text{ fm}^4)$|
| $^{12}$Be$^{+}_{3/2} \rightarrow 1^{-}_{1}$ | $E1$        | $2 \times 10^{-2} \ (e^2 \text{ fm}^4)$|
| $^{12}$Be$^{+}_{1/2} \rightarrow 0^{-}_{1}$ | $E0$        |       | $1.7 \ (e^2 \text{ fm}^2)$|

Experimental data. Before diagonalization, the deformed $0^{-}_{1}$ state possesses the strength of the $E1$ transition from the $1^{-}_{1}$ state, while the transition to the spherical $0^{-}_{2}$ state is weak. After diagonalization, the strength in the $0^{-}_{1}$ state distributes in the $0^{-}_{2}$ state due to state mixing between the $0^{-}_{1}$ and the $0^{-}_{2}$ states. In other words, the rather large $B(E1; 0^{-}_{1} \rightarrow 1^{-}_{1})$ is caused by a deformation of the ground state. If the mixing of the $K^\pi = 0^-$ component in the $1^{-}_{1}$ state is taken into account as described in Table III, theoretical value of the $E1$ strength become large as $B(E1; 0^{-}_{1} \rightarrow 1^{-}_{1})$ = 0.14 $e^2 \text{fm}^2$, because the $K^\pi = 0^-$ component has a similar intrinsic structure as that of the $0^{-}_{1}$ state.

V. DISCUSSIONS

In the present calculations, many deformed rotational bands appeared in $^{12}$Be. It was found that $2\pi$ core is formed in most of the states as well as in other neutron-rich Be isotopes: $^{10}$Be and $^{11}$Be $^{[11]}$. In this section, we consider the intrinsic structures of $^{12}$Be while focusing on the clustering aspects and the behavior of the valence neutrons. The features of the intrinsic structures, the roles of valence neutrons and the inter-cluster motions in the cluster states are discussed.

A. Systematics of rotational bands

By analyzing the structures of the intrinsic states, we can classify the ground and excited states into rotational bands. We superpose the wave functions, $P_{MK}^{J_\pi,\Phi_{AMD}}$, projected from all of the intrinsic wave functions, $\Phi_{AMD}(Z_{i}^{j_\pi,\pi_i})$, obtained by VAP so as to diagonalize the Hamiltonian matrix. Although the final wave function $\Phi_{J_\pi}^{(12)Be}$ for the $J^+_n$ state after diagonalization is the linear combination of spin-parity projected AMD wave functions $P_{MK}^{J_\pi,\Phi_{AMD}}(Z_{i}^{j_\pi,\pi_i})$, we consider the AMD wave function before diagonalization as being the intrinsic state of the corresponding $J^\pm$ state, because $P_{MK}^{J_\pm,J^\pm,\Phi_{AMD}}(Z_{i}^{j_\pm})$ is found to be the major component of the final result $\Phi_{J^\pm}^{(12)Be}$ of the $J^+_n$ state, except for the $2^+_1$ and $2^+_4$ states. In Table III, the squared amplitudes, $\langle \Phi_{J^\pm}^{(12)Be}| P_{MK}^{J_\pm,\Phi_{AMD}}(Z_{i}^{j_\pm}) \rangle^2$ are listed. The amplitudes are larger than 0.6, except for the $1^-_2$, $2^+_3$ and $2^+_4$ states. In the negative-parity state of $1^-_2$, the component of the original state becomes less than 0.6 after diagonalization. Considering that this state might be unstable, we neglect the $1^-_2$ level in the present results. In the case of the $2^+_2$ state, the main component of the $2^+_2$ state is not the projected AMD state, $P_{MK}^{2^+_2,\Phi_{AMD}}(Z_{i}^{2^+_2})$, obtained by a VAP calculation for $J^+_n = 2^+_3$, but is the $P_{MK}^{2^+_2,\Phi_{AMD}}(Z_{i}^{2^+_2})$, which is projected from the intrinsic state of the $3^+_1$ state. This means that a new $2^+_3$ state appears in addition to three $2^+_1$ states obtained by the VAP calculation, $Z_{1}^{2^+_1}$, $Z_{2}^{2^+_1}$ and $Z_{3}^{2^+_1}$. The energy of this new $2^+_3$ state, $P_{MK}^{2^+_3,\Phi_{AMD}}(Z_{i}^{2^+_3})$, is slightly lower than that of $P_{MK}^{2^+_3,\Phi_{AMD}}(Z_{i}^{2^+_3})$. As a result, the wave functions, $\Phi_{AMD}(Z_{1}^{3^+_1})$ and $\Phi_{AMD}(Z_{3}^{3^+_1})$, are regarded to be the intrinsic states of the $2^+_3$ and $2^+_4$ states, respectively, although the amplitude of the original VAP state, $P_{MK}^{3^+_1,\Phi_{AMD}}(Z_{i}^{3^+_1})$, in the $2^+_4$ state is as small as 0.4 after diagonalization because of mixing the $2^+_3$ state with the $2^+_4$ state. The reason why the intrinsic state of the $2^+_3$ state is not obtained in the VAP calculation with $(J^\pm, K^\pm) = (2^+, 0^-)$ is because the $2^+_3$ state belongs to the $K^\pi = 2^+$ band.

By analyzing the intrinsic states, we find the rotational bands $K^\pi = 0^+_1, 0^+_2, 0^+_3, 2^+_4, 1^+_1$, which consist of the states $(0^+_1, 2^+_4, 4^+_1, 6^+_4, 8^+_1), (0^+_2, 2^+_4, (0^+_3, 2^+_4, 4^+_1, 6^+_4) (2^+_3), 3^+_4, 5^+_4, 7^+_1), (1^-_1, 2^-_1), 3^-_4, 4^-_1, 5^-_1$, respectively. We analyze the
TABLE IV. The amplitudes of the corresponding VAP wave functions $P_{MK}^{J_{K}^{\pm}} \Phi_{AMD}(Z_{i_{n_{i}}}^{I_{K}^{\pm}})$ in the final wave functions $(\Phi_{J_{K}^{\pm}(^{12}\text{Be})})$ obtained after the diagonalization. Total intrinsic spins $(S_{p}^{2})$ and $(S_{n}^{2})$ for protons and neutrons in the spin-parity projected states $P_{MK}^{J_{K}^{\pm}} \Phi_{AMD}(Z_{i_{n_{i}}}^{I_{K}^{\pm}})$ before the diagonalization are also listed.

| $J_{n_{i}}^{\pm}$ after diagonalization | $(J^{e_{i}}, |K|, n_{i})$ in VAP | $(\Phi_{J_{K}^{\pm}(^{12}\text{Be})}|P_{MK}^{J_{K}^{\pm}} \Phi_{AMD}(Z_{i_{n_{i}}}^{I_{K}^{\pm}}))^{2}$ | $(S_{p}^{2})$ | $(S_{n}^{2})$ |
|----------------------------------------|---------------------------------|---------------------------------|--------|--------|
| 0$^{+}_{i}$                            | $(0^{+}, 0, 1)$                 | 0.8                             | 0.1   | 1.0   |
| 2$^{+}_{i}$                            | $(2^{+}, 0, 1)$                 | 0.8                             | 0.1   | 1.1   |
| 4$^{+}_{i}$                            | $(4^{+}, 0, 1)$                 | 0.9                             | 0.1   | 1.2   |
| 6$^{+}_{i}$                            | $(6^{+}, 0, 1)$                 | 0.9                             | 0.0   | 1.5   |
| 8$^{+}_{i}$                            | $(8^{+}, 0, 1)$                 | 0.9                             | 0.1   | 1.6   |
| 0$^{-}_{i}$                            | $(0^{-}, 0, 2)$                 | 0.6                             | 0.2   | 0.1   |
| 2$^{-}_{i}$                            | $(2^{-}, 0, 2)$                 | 0.7                             | 0.9   | 0.1   |
| 0$^{+}_{i}$                            | $(0^{+}, 0, 3)$                 | 0.7                             | 0.0   | 0.5   |
| 2$^{+}_{i}$                            | $(2^{+}, 0, 3)$                 | 0.4                             | 0.0   | 1.0   |
| 4$^{+}_{i}$                            | $(4^{+}, 0, 2)$                 | 0.6                             | 0.0   | 1.0   |
| 6$^{+}_{i}$                            | $(6^{+}, 0, 2)$                 | 0.6                             | 0.0   | 1.2   |
| 1$^{-}_{i}$                            | $(1^{-}, 1, 1)$                 | 0.9                             | 0.1   | 0.8   |
| 2$^{-}_{i}$                            | $(2^{-}, 1, 1)$                 | 0.9                             | 0.1   | 0.8   |
| 3$^{-}_{i}$                            | $(3^{-}, 1, 1)$                 | 0.9                             | 0.1   | 1.0   |
| 4$^{-}_{i}$                            | $(4^{-}, 1, 1)$                 | 0.7                             | 0.1   | 1.1   |
| 5$^{-}_{i}$                            | $(5^{-}, 1, 1)$                 | 0.9                             | 0.5   | 1.2   |
| 2$^{+}_{i}$                            | $(3^{+}, 2, 1)$                 | 0.8                             | 0.0   | 2.1   |
| 3$^{+}_{i}$                            | $(3^{+}, 2, 1)$                 | 0.8                             | 0.0   | 2.1   |
| 5$^{+}_{i}$                            | $(5^{+}, 2, 1)$                 | 0.8                             | 0.0   | 2.0   |
| 7$^{+}_{i}$                            | $(7^{+}, 2, 1)$                 | 0.9                             | 0.0   | 1.2   |
| 0$^{+}_{i}$                            | $(0^{+}, 0, 1)$                 | 1.0                             | 0.1   | 1.7   |
| 1$^{+}_{i}$                            | $(1^{+}, 1, 1)$                 | 0.9                             | 0.0   | 1.3   |
| 1$^{+}_{i}$                            | $(1^{+}, 0, 1)$                 | 1.0                             | 1.9   | 0.3   |
| 6$^{+}_{i}$                            | $(6^{-}, 1, 1)$                 | 0.9                             | 0.0   | 2.0   |
parity-eigen components in the single-particle wave functions of the intrinsic states, and find that each rotational band is dominated by either the 0hω, 1hω or 2hω excited configuration. The states in the three positive-parity bands ($K^\pi = 0^+_1$, 0$^+_2$, 2$^+_1$) have the dominant 2hω excited configurations, while the states in the $K^\pi = 0^+_2$ band are dominated by the 2hω configurations. The first point is that the states in the ground band are not the neutron shell-closed states, but are the prolate deformed intruder states with developed clustering, even though this $^{12}$Be nucleus has a neutron magic number of N = 8. This is consistent with discussions of the vanishing of the neutron magic number in $^{12}$Be given by Refs. [18,19,20]. The $K^\pi = 0^+_1$ band starting from the 0$^+_1$ state terminates with the 8$^+_1$ state, because the $J^\pi = 8^+$ is the highest spin in the 2hω configurations. On the other hand, the present results predict that the 0$^+_2$ and 2$^+_2$ states with the neutron p-shell closed structures constitute the excited band, $K^\pi = 0^+_2$, just above the ground band. The second interesting point is that the newly observed levels, 4$^+_1$, are considered to belong to the higher band of the two bands($K^\pi = 0^+_1$ and $K^\pi = 0^+_2$), both of which are the 2hω excited states with developed cluster structures. The $K^\pi = 2^+_1$ band consists of the 2$^+_3$, 3$^+_1$, 5$^+_1$, 7$^+_1$ states, which have the other 2hω configurations. Although spin-parity eigen states, 4$^+_1$ and 6$^+_1$, are able to be projected from the intrinsic states of this $K^\pi = 2^+_1$ band in principle, the 4$^+_1$ and 6$^+_1$ states in the $K^\pi = 2^+_1$ band cannot be identified, because these states strongly mix with the 4$^+_1$, 4$^+_2$, 6$^+_1$ and 6$^+_2$ states in the $K^\pi = 0^+_1$ and $K^\pi = 0^+_2$ bands. We find a negative-parity band, $K^\pi = 1^-_1$, in which the states(1$^-,2^-,3^-,4^-,5^-$) are dominated by 1hω configurations.

### B. Intrinsic structures

We discuss the features of the intrinsic structures, such as the deformations and clustering aspects. The density distributions of the intrinsic AMD wave functions are shown in Fig. [3].

In most of the states, except for the 2$^+_2$, 5$^+_1$ and 1$^+_2$ states, the densities of protons have dumbbell-like shapes, which indicate the formation of a 2α core. On the other hand, 2α core breaking occurs in the 2$^+_2$, 5$^+_1$ and 1$^+_2$ states. For the quantitative discussions, we can estimate the degree of core breaking based on the non-zero values of the squared total-intrinsic spin of protons, $\langle S_p^2 \rangle$, given in Table IV. In most of the states, the values $\langle S_p^2 \rangle$ almost equal to 0, which indicates only a slight breaking of the 2α core. However, as seen in the large $\langle S_p^2 \rangle$ values in the 2$^+_2$ and 5$^+_1$ states, the 2α core structures somehow dissociate in these band terminal states of the $K^\pi = 0^+_1$ and 1$^-_1$ bands because of the spin-alignment effect. On the other hand, the 1$^-_1$ state has a quite different structure from that of the 2α-core state. It is a spin-aligned state with proton spin, $S_p \sim 1$, where only one of the 2α clusters is completely broken. In fact, no dumbbell shape is seen in the proton density of this state (Fig. [3]). The alignment of the intrinsic spins is the origin of the unnatural spin parity of this state. This 1$^-_1$ state is analogous to the 3/2$^-$ state at 8.4 MeV in $^{15}$Be, and also to the 1$^-_1$ state in $^{12}$C, in which cluster breaking occurs due to the the aligned intrinsic spins.

The deformation parameter ($\beta$) of the proton density shown in Fig. [3] is a useful quantity in order to quantitatively discuss the developments of clustering in the 2α core states. Here, we explain the cluster developments in relation to the neutron structure, such as the neutron deformation and neutron intrinsic spin.

In the $K^\pi = 0^+_1$ band, the states have a prolate deformation with the developed 2α core, as shown in Fig. [3]. In the 0$^+_1$, 2$^+_1$ and 4$^+_1$ states, the 2α core well develops following prolate deformation of the neutron density, which is caused by the 2 neutrons in a longitudinal sd-orbit. The ground 0$^+_1$ state has an approximately axial symmetric shape (Fig. [3]), which is considered to be caused by the symmetric neutron configuration described by a $p_{3/2}$ sub-shell closure and the axial symmetric sd-orbit with ($\lambda\mu$) = (20) symmetry in the SU3 limit. The non-zero total intrinsic spin of neutrons of the ground 0$^+_1$ state shown in Table [V] is considered to come from a sub-shell closure effect of the neutron $p_{3/2}$-shell. As the total spin increases, the alignment of the intrinsic spins of the sd-shell neutrons further increases the total intrinsic spin of neutrons (Table [V]). The aligned sd-shell neutrons make the neutron-density deformation of the nucleus to be smaller in the high-spin region, $J \geq 6$. As a result, the clustering become weak in the 6$^+_1$ and 8$^+_1$ states, as found in the reduction of the proton deformation parameter (Fig. [3]).

In the $K^\pi = 0^+_2$ band with the dominant 0hω configurations, the neutron deformation is smaller than that in the $K^\pi = 0^+_1$ band. In the 0$^+_2$ state, 4 neutron pairs make a small tetrahedron-shape structure, which is approximately equivalent to the p-shell closed structure. Because of the shell effect of the neutron p-shell closure, the clustering is smaller than that in the 0$^+_1$ state. Although the intrinsic state of the band terminal 2$^+_1$ state in this band has the proton density as seen in Fig. [3], the spin-parity $J^\pi = 2^+$ eigen state projected from this intrinsic state is dominated by the neutron p-shell closure component. The total spin $J = 2$ in this 2$^+_1$ state is composed of the aligned total-angular momentum of the p-shell protons. The overlap $|\langle P_{MK=0}^{2^+_1,0} | P_{MK=0}^{2^+_1,0} \rangle |^2$ is still large as about 0.6. The j-j coupling feature in this band terminal state is found in the non-zero value of the squared total-intrinsic spin of protons (Table [V]) and also in the disappearance of the dumbbell-like shape of proton density (Fig. [3]).

In the $K^\pi = 0^+_3$ band, the 0$^+_3$ and 2$^+_3$ states have extremely developed cluster structures, like $^6$He+$^6$He. One of
FIG. 4. Density distribution of the intrinsic state $\Phi_{AMD}(Z_{\pm}^0)$ before projection and diagonalization. The intrinsic system is projected onto a plane which contains the approximately longitudinal axis of the intrinsic state. The density is integrated along a transverse axis perpendicular to the plane. The densities for matter, protons and neutrons are presented at the left, middle and right, respectively. The deformation parameters $\beta$ are written at the bottom of the figures. The expectation values of $xx$, $yy$, $zz$ for the matter, proton and neutron densities are written below the figures. The size of the frame box is 10 fm×10 fm.
the reasons for the development of clustering is considered to be the orthogonal condition to the $0^+_1$ and $2^+_1$ states in the lower band. The inter-cluster distance shrinks with increasing total spin, due to the spin alignment. In the $0^+_3$, $2^+_3$ and $4^+_3$, low matter density regions between the clusters are seen (Fig. 4), while they disappear in the $6^+_3$ state. The intrinsic structure of the $K^\pi = 0^+_3$ band changes rather rapidly with an increase of the total spin, as can be seen in the decrease of the deformation parameter $\beta$ (Fig. 3) and also in the spin alignment of the neutron intrinsic spins (Table V). Even though the $^6\text{He}+^6\text{He}$ cluster structure changes due to the spin alignment in this $K^\pi = 0^+_3$ band, it is helpful to consider the spatial symmetry in the $SU_3$ representation. Since the intrinsic state of the band head $0^+_3$ has $(\lambda\mu) = (24)$ symmetry in the $SU_3$ limit, which provides a $K = 0$ band from $J^\pi = 0^+ \rightarrow 6^+$, it is reasonable that $K^\pi = 0^+_3$ band terminates at the $6^+_3$ state.

In spite of the mixing of the states and the spin alignments, all of the states in the $K^\pi = 0^+_1$, $0^+_3$ and $2^+_1$ bands are dominated by $2\hbar\omega$ configurations with 2 neutrons in the $sd$-shell. It is another interesting problem to search for the weak-coupling cluster structure, which is described by the relative motion between the $2^+\text{He}$ clusters. In pioneering work by M. Ito et al. [16], weak coupling states with $2^-\text{He}$ clusters in $^{12}\text{Be}$ were studied. Although the calculation was not fully microscopic, because they did not make antisymmetrization of the neutrons between clusters, it is interesting that He-cluster states were suggested to appear near and above the threshold energies. The presently predicted $0^+_3$ state is a candidate of the weak coupling cluster state, because of the large relative distance between clusters. However, in the $4^+_3$ and $6^+_3$ states in the $K^\pi = 0^+_3$ bands, the components of the weak coupling cluster state may not be large, since the He-cluster structure becomes weak due to the spin alignments. Instead, it is expected that the molecular resonances with a weak-coupling cluster structure may exist above those spin-aligned states. We performed a simple VAP calculation of the higher states, and found that candidates of the weak-coupling states appear above those spin-aligned states with $2\hbar\omega$ configurations obtained in the present calculations. We find a $4^+_3$ state with a developed $^9\text{He}+^6\text{He}$-cluster structure at a $0.6$ MeV higher energy than the $4^+_1$ state. Although the $6^+_3$ state was unstable for cluster escaping in the present VAP calculations using $m = 0.65$, we obtained a $6^+_3$ state with a developed $^9\text{He}+^6\text{He}$ structure at a few MeV higher energy than the $6^+_2$ state by using $m = 0.62$. Such weakly bound states should be carefully investigated by taking care of the threshold energies and the stability for particle decays.

The negative-parity states $1^-_1$, $2^-_1$, $3^-_1$, $4^-_1$ and $5^-_1$ in the $K^\pi = 1^-_1$ band are deformed states with cluster structures. The deformed structures of the neutron density are made from the dominant $1\hbar\omega$ excited configurations with one neutron in the $sd$-shell. The degree of prolate deformation of the neutron density in this band is smaller than the states in the $K^\pi = 0^+_1$ band. The smaller neutron deformation causes the weaker cluster development in the $K^\pi = 1^-_1$ band compared with those in the $K^\pi = 0^+_1$ band. The $1^-_1$, $2^-_1$, $3^-_1$, $4^-_1$ states have axial-asymmetric neutron structures, while the band terminal $5^-_1$ state has approximately axial-symmetric neutron density. The reason for $K^\pi = 1^-_1$ of the $1^-_1$ state can be naturally understood by a weak coupling picture of $^{11}\text{Be}$ core and a neutron. In the simple weak coupling picture, since the $1^-_1$ state of $^{12}\text{Be}$ can be described by the coupling of a $^{11}\text{Be}(1/2^-)$ core with $(\lambda\mu) = (21)$ $SU_3$ symmetry and a $s_{1/2}$-orbit neutron, the $K^\pi = 1^-_1$ in the $^{12}\text{Be}(1^-_1)$ state come from the $K^\pi = 1^-_1$ in the $^{11}\text{Be}$ core.

In the recent work with a three-body($n+n+^{10}\text{Be}$) model [23], in which the low-lying states of $^{12}\text{Be}$ were studied from the weak coupling picture, the excitation energy of the $1^-_1$ state was predicted to be about $3$ MeV. Another possible negative parity band is $K^\pi = 0^-$, which is a parity doublet band of a $^6\text{He}+^6\text{He}$ cluster structure, as is suggested in a He+He cluster model by P. Descouvemont et al. [7]. We found a $1^-_1$ state with the $^8\text{He}+\alpha$ cluster structure by the VAP calculation for $K^\pi = 0^-$ at slightly higher energy than that for $K^\pi = 1^-_1$. Even though the mixing of this $K^\pi = 0^-$ component is important to lower the excitation energy of the $1^-_1$ state as mentioned before in [15], the major component of the $1^-_1$ state is the $K^\pi = 1^-$ state. The improved results including the $K^\pi = 0^-$ configuration suggest the higher state($1^-_2$) with the significant $K^\pi = 0^-$ component may appear at $9$ MeV excitation energy, which is consistent with the prediction in Ref. [7].

Next, we discuss the structures of the unnatural spin-parity states. As mentioned above, the unnatural spin-parity of the $1^-_1$ state is caused by the aligned intrinsic spins of protons. On the other hand, the unnatural spin-parity of the other states ($3^-_1$, $5^-_1$, $7^-_1$, $1^-_2$ and $0^-_2$) originates from the aligned intrinsic spins of the $sd$-shell neutrons. Alignments of the neutron intrinsic spins were found in the $2^+_3$, $3^+_3$ and $5^+_1$ states, as seen in the expectation values of the squared total neutron spins, $(S^2_n) \sim 2$, presented in Table IV. The spin alignment is mainly caused by 2 $sd$-shell neutrons. The deformation and the density distribution of the intrinsic state of the $7^-_1$ state are very similar to those of the $3^+_1$ and $5^+_1$, though $(S^2_n)$ in the $7^-_1$ state is slightly smaller. The $1^-_1$ and $0^-_1$ states with unnatural spin-parity have aligned intrinsic spins of neutrons, like the states in the $K^\pi = 2^+_1$ band. Although a $^6\text{He}+^6\text{He}$-like structure is found in the $1^-_1$ state, one of the $^4\text{He}$ clusters is an excited $^6\text{He}$ with $S_n = 1$ components.

Here, we stress that it is essential to consider the orthogonality to the lower states as well as antisymmetrization of all the nucleons in a study of the excited states of $^{12}\text{Be}$, because the present results suggest the existence of many $2\alpha$ core states in the low-energy region. Although these states are not necessarily written in terms of $^6\text{He}+^6\text{He}$ clusters, such lower states must have effects on the higher weak-coupling cluster states. The present results indicate the importance of treating degrees of freedom of the intrinsic spins for all the nucleons in a study of $^{12}\text{Be}$. The reasons
can be summarized as follows. First of all, a closed $p_{3/2}$-shell of neutrons plays an important role for the energy gain of the intruder ground state. Secondly, the alignments of intrinsic spins as $S = 1$ are necessary to describe the unnatural spin-parity states. In the third point, the spin alignment of the neutron intrinsic spins occurs in high spin states.

C. Picture of molecular orbit

As mentioned in the previous subsection, the $2\alpha$ core structure appears in many of the states. The development of clustering is sensitive to the prolate deformation of the neutron density. We inspected the neutron structure based on the single-particle behavior of the valence neutrons surrounding the $2\alpha$ core. We also considered the relation between the neutron orbits and the cluster development. We analyzed the single-particle wave functions of the valence neutrons in the intrinsic states. The detailed formulation of how to extract the single-particle energies and wave functions from an AMD wave function is described in Refs. [5,7].

According to the analyses by the single-particle orbits and the single-particle energies, the $2\alpha$ core is composed of nucleons occupying the lowest 4 proton orbits and 4 neutron orbits. The higher 4 neutron orbits correspond to those of the valence neutrons surrounding the $2\alpha$ core. By extracting the positive and negative components in the orbits, many of the single-particle orbits are found to be approximately parity-eigen states, as the amplitude of the dominant parity-eigen state is more than 70% in each orbit. Since the negative and positive-parity orbits of the valence neutrons are associated with the $p$-orbits and $sd$-orbits, respectively, we can classify the ground and excited states of $^{12}\text{Be}$ in terms of the $n\hbar\omega$ excitation, where $n$ is the number of the valence neutrons with the dominant positive-parity components. The $K^\pi = 0^+_2$ band is approximately described by $0\hbar\omega$ configurations, while the $K^\pi = 0^+_1, 0^+_3, 2^+_1$ bands are dominated by $2\hbar\omega$ configurations with 2 neutrons in $sd$-like orbits. On the other hand, the main components of the $K^\pi = 1^+_1$ band are $1\hbar\omega$ configurations.

The idea of molecular orbits surrounding a $2\alpha$ core is helpful to understand the roles of the valence neutrons in neutron-rich Be isotopes. The molecular orbits in Be isotopes were suggested in a study of $^9\text{Be}$ with a $2\alpha + n$ cluster model [24]. They assumed $\sigma$-orbits and $\pi$-orbits which are made from linear combinations of the $p$-orbits around the $\alpha$ cores (see Fig.5). This idea was applied to neutron-rich Be isotopes by Seya et al. a long time ago [1]. In the 1990’s von Oertzen et al. [3,31] revived this kind of research to understand the rotational bands of neutron-rich Be isotopes, and Itagaki et al. [8,9] described the structures of the low-lying states of $^{10}\text{Be}$ and $^{12}\text{Be}$ by assuming $2\alpha$ core and valence neutrons in the molecular orbits. The formation of the $2\alpha$ and valence neutron structures in neutron-rich Be isotopes was first guaranteed theoretically by the AMD calculation [2,5,6,7,11], where the existence of any clusters or molecular orbits was not assumed. In these AMD studies, the viewpoint of the molecular orbit was found to be useful to understand the cluster development in $^{10}\text{Be}$ and $^{12}\text{Be}$. Therefore, it is an interesting problem whether the states of $^{12}\text{Be}$ can be described by the molecular orbits.

In the present results for $^{12}\text{Be}$, we find a new kind of molecular orbit besides the suggested $\pi$-orbit and $\sigma$-orbit. In the positive-parity orbits of the valence neutrons in $^{12}\text{Be}$, two kinds of molecular orbits appear, both of which are
FIG. 6. Density distributions of the single-particle wave functions of the valence neutrons in the intrinsic wave functions of the $0^+_1$, $0^+_3$, $6^+_1$ and $3^+_1$ states. The figures at the left (right) show the densities regarding the positive-parity components of the first (second) highest neutron orbits. The value $P_+$ in each orbit indicates the squared amplitude of the contained positive-parity component.
associated with $sd$-orbits. The first one is the $\sigma$-orbit (Fig. 3a), while the second one is a quite new molecular orbit, shown in Fig. 3c. This orbit is the other positive-parity orbit made from a linear combination of the $p$-orbits around the $\alpha$ cores. As shown in Fig. 3c, the combined $p$-orbits in this orbit are perpendicular to those in the $\sigma$-orbit. We call this new positive-parity orbit a $\delta'$-orbit in the present paper, although it has $(\lambda \mu) = (01)$ symmetry in the $SU_3$ limit, which is perpendicular to the so-called $\delta$-orbit in the field of the molecular physics. In the case of $^{12}$Be, the negative-parity orbit of the neutron surrounding $2\alpha$ does not necessarily correspond to the pure molecular $\pi$-orbit, because the $p_{3/2}$-shell closure cannot be described by simple $\pi$-orbits. Therefore, in the following discussions, we concentrate on the positive-parity orbits of the valence neutrons associated with the molecular $\sigma$-orbits and $\delta'$-orbits.

Fig. 3 shows the density distributions of the single-particle wave functions of the first and second highest neutron orbits. In the low-spin cluster states, the positive-parity orbits of the valence neutrons can be well associated with the two types of the molecular orbits ($\sigma$ and $\delta'$). In the $0^+_2$ state, two valence neutrons with up and down spins occupy the $\sigma$-like orbits, which have 2 nodes along the longitudinal axis. In the $0^+_3$ state, which is dominated by the other $2\hbar\omega$ configurations, the two neutrons occupy $\delta'$-like orbits. It is very surprising that the developed $^6$He+$^5$He cluster structure in the $0^+_3$ state is understood by the new molecular $\delta'$-orbits. It occurs when 2 deformed $^6$He clusters are attached in parallel. In the $0^+_2$ state, all of the 4 valence neutrons are in the negative-parity orbits. Comparing the energies of the $0^+_3$ state with those of the $0^+_1$ and $0^+_2$ states, the $\delta'$-orbit is the highest among the molecular orbits ($\sigma$, $\pi$ and $\delta'$).

The molecular $\sigma$-orbit is one of the reasons for the deformed ground state of $^{12}$Be with the $2\hbar\omega$ configurations, which is lower than the closed neutron-shell state. Since Be nuclei prefer prolate deformations because of the $2\alpha$-cluster core, the $\sigma$-orbit gains kinetic energy in the developed cluster system. In pioneering studies, $^{11}$Be states remind us of analogous states in $^9$Be, $^{10}$Be and $^{11}$Be. In terms of the molecular orbit around the $2\alpha$ core, the typical rotational bands in these Be isotopes are classified by the number of occupied $\sigma$-orbits. Here, we use the notation $\pi^m\sigma^n$ for those states having $m$ neutrons in the $\pi$-orbits and $n$ neutrons in the $\sigma$-orbits around a $2\alpha$ core. The valence neutron orbits in the lowest natural parity states($^9$Be($1/2^-$), $^{10}$Be($0^+_1$), $^{11}$Be($1/2^-$) and

D. Systematics of the $2\alpha$ clustering in Be isotopes

When we roughly regard the valence neutron orbits in the $0^+_1$, $1^-_1$, $0^+_2$ states of $^{12}$Be as the $\pi$-orbits and the $\sigma$-orbits, these $^{12}$Be states remind us of analogous states in $^9$Be, $^{10}$Be and $^{11}$Be. In terms of the molecular orbit surrounding the $2\alpha$ core, the typical rotational bands in these Be isotopes are classified by the number of occupied $\sigma$-orbits. Here, we use the notation $\pi^m\sigma^n$ for those states having $m$ neutrons in the $\pi$-orbits and $n$ neutrons in the $\sigma$-orbits around a $2\alpha$ core.
Comparing the $2\alpha$ systematics decreases as the number of the occupied $\sigma$-orbits increases. Especially, a drastic change in the $\sigma^0$ states from $^9\text{Be}$ to $^{10}\text{Be}$ is caused by the $p_{3/2}$ sub-shell closure effect. Also, in the $\sigma^1$ and $\sigma^2$ lines, there exist gaps between the states with a closed $p_{3/2}$-shell and those with an open $p_{3/2}$-shell. Thus, the closed $p_{3/2}$-shell plays an important role in weakening the $2\alpha$ development.

One of the interesting features of neutron-rich Be isotopes is the vanishing of the neutron magic number $N = 8$ in $^{11}\text{Be}$ and $^{12}\text{Be}$. In Fig. 7, the ground states are plotted as open circles. The intruder ground states in $^{11}\text{Be}$ and $^{12}\text{Be}$ correspond to points on the $\sigma^1$ and $\sigma^2$ lines, respectively. As shown in the Fig. 7, the $2\alpha$ distances in the ground state of $^{11}\text{Be}$ and $^{12}\text{Be}$ are about 3 fm. We may conjecture that the magnitude of 3 fm is the natural distance between $2\alpha$, which is energetically favored in these neutron-rich Be isotopes. Therefore, one of the reasons for the intruder ground states can be understood to be a restoration of the natural distance at about 3 fm in the very neutron-rich Be isotopes.

The $N = 6$ sub-shell closure effects are clearly seen in the energy gaps of 6 MeV as well as in the drastic structure changes between the $\sigma^0$ and $\sigma^1$ states in $^{10}\text{Be}$. The sub-shell effect is also reflected in the structure change between the $\sigma^1$ and $\sigma^2$ states in $^{11}\text{Be}$, as shown in the $2\alpha$ distance. This means that we can regard the vanishing of the magic number in neutron-rich Be isotopes as a shift of the neutron magic number from $N = 8$ to $N = 6$.

E. Inter-cluster motion and He decay width

In order to investigate the inter-cluster motion, we extracted the relative wave functions between clusters in the $^6\text{He} + ^4\text{He}$ and $^8\text{He} + ^4\text{He}$ channels. We assumed that the intrinsic wave functions of the He clusters are given by the $0^+$ states of $^4\text{He}$, $^6\text{He}$, $^8\text{He}$ in the SU$_3$ limit with the same width parameter as that of $^{12}\text{Be}$. The detailed structures of the $^8\text{He}$ nucleus, such as the neutron halo, are omitted in the present analysis for simplicity. We estimated the

FIG. 7. Relative distance between $2\alpha$ clusters in the intrinsic state of the major component described by a single AMD wave function. The distance $d$ is evaluated by the relative distance between the mean centers of 2 proton pairs, defined as $d \equiv \sqrt{\text{Re}(Z_1 + Z_2 - Z_3 - Z_4)/2\sqrt{7}}$. The dotted, thick solid and thin solid lines correspond to the distances in the band head states with zero($\sigma^0$), one($\sigma^1$) and two($\sigma^2$) neutrons in the $\sigma$ orbits. Namely, the lines ($\sigma^0$, $\sigma^1$, $\sigma^2$) correspond to $\{^9\text{Be}(1/2^-_1), ^{10}\text{Be}(0^+_1), ^{11}\text{Be}(1/2^-_1), ^{12}\text{Be}(0^+_1)\}$, $\{^9\text{Be}(1/2^+_1), ^{10}\text{Be}(1/2^-_1), ^{11}\text{Be}(1/2^+_1), ^{12}\text{Be}(1^-_1)\}$ and $\{^{10}\text{Be}(0^+_2), ^{11}\text{Be}(3/2^-_2), ^{12}\text{Be}(0^+_1)\}$, respectively. The open circles indicate the ground states.
inter-cluster motions between clusters in $^{12}$Be by the reduced width amplitudes ($\tilde{y}_L(r)$), which were calculated by projecting the $^{12}$Be wave functions to the cluster model space expressed by the superpositions of Brink functions. Moreover, the spectroscopic factors ($\tilde{S}$) and the total cluster probabilities ($\tilde{P}_c$) are helpful to quantitatively discuss the development of He$+\text{He}$ clustering. The detailed definitions and the practical calculation of $\tilde{y}_L(r)$, $\tilde{S}$ and $\tilde{P}_c$ are explained in Appendix A.

Figs. 3 and 4 show the R.W.A. ($\tilde{r}_L(r)$) in the $^{6}$He$+^{6}$He and $^{8}$He$+^{4}$He channels extracted from the obtained $J = L$ states of $^{12}$Be, respectively. The magnitudes $|\tilde{r}_L(r)|$ at the channel radius $a = 5$ fm, the spectroscopic factors $\tilde{S}$ and the total cluster probabilities ($\tilde{P}_c$) are presented in Table III.

In each band, even if a low spin state has a large amplitude $|\tilde{r}_L(r)|$ at the surface region and a large spectroscopic factor $\tilde{S}$, the amplitude becomes smaller in the high-spin states with $J \geq 6$. In particular, the amplitudes in the $J = 8$ state are very small. We consider that one of the reason for this reduction of the R.W.A. may be the effect of spin-alignments in the high spin states. In the $^{6}$He$+^{6}$He channel, the amplitude at the surface region is most remarkable in the $0^+_1$ state, which has the peak amplitude at the surface around $r = 5$ fm and a largest spectroscopic factor. It indicates the spatially developed $^{6}$He$+^{6}$He clustering in this state. Compared with that in the $0^+_1$ state, the amplitude in the $0^+_1$ state shifts inward and has a peak at about $r = 4$ fm. In the $0^+_2$ state, the surface amplitude is smallest among three $0^+$ states, which is consistent with this state has no developed clustering but the dominant neutron $p$-shell closure components. The amplitude $|\tilde{r}_L(r)|$ at about 4 fm in this $0^+_2$ state are caused by the mixing with the $0^+_1$ state. Concerning the $J^+_k = 2^+_1, 4^+_1$, and $6^+_2$ states, both the spectroscopic factors and the surface amplitudes $|ay_L(a)|/(a = 5$ fm) for the $^{6}$He$+^{6}$He channel are not small, but those for the $^{3}$He$+^{4}$He channel are small compared with the $^{6}$He$+^{6}$He channel. On the other hand, with respect to the R.W.A for the $^{8}$He$+^{4}$He channel, the surface amplitudes at $r = 4$ fm are found to be large in the states in the $K^\pi = 0^+_1$ band, while they are small in other bands such as $K^\pi = 0^+_2$. By analyzing the spectroscopic factors (Table II) and the R.W.A. (Fig. 3 and 4), we conclude that the $K^\pi = 0^+_1$ band has both components of $^{6}$He$+^{6}$He and $^{8}$He$+^{4}$He cluster structures at least in the low spin states. It is consistent with the result by the coupled-channel cluster model with $^{6}$He$+^{6}$He and $^{8}$He$+^{4}$He wave functions in Ref. [17]. On the other hand, in the $K^\pi = 0^+_2$ band, the $^{6}$He$+^{6}$He clustering appears predominantly. These results strongly suggested that the neutrons around the 2$\alpha$ core move about over the whole system in the $K^\pi = 0^+_1$ band states, while in the $K^\pi = 0^+_2$ band states they move about not over the whole system, but around either of the two $\alpha$ clusters.

We next mention the partial decay widths of the excited states above the threshold energies of the He decays. We calculated the theoretical values of the partial decay widths ($\Gamma_{\text{He}}$ and $\Gamma_{\text{He}}$) concerning the simple binary decays, $^{6}$He($0^+$)+$^{6}$He($0^+$) and $^{8}$He($0^+$)+$^{4}$He($0^+$), using the method of reduced width amplitudes:

$$\Gamma_{\text{He}} = 2P_L(a) \cdot \gamma^2_{\text{He}}(a), \quad P_L(a) = \frac{ka}{F_L(ka)}G_L^2(ka), \quad \gamma^2_{\text{He}}(a) = \frac{k^2}{2m}|ay_L(a)|^2,$$

where $k$ is the wave number of the resonance energy ($E^\ast$), $k = \sqrt{2\mu E^\ast/\hbar^2}$, $a$ is the channel radius, and $F_L$ and $G_L$ are the regular and irregular Coulomb functions. The resonance energies were evaluated by subtracting the experimental threshold energies of the channels from the theoretical excitation energies. In principle, since the present AMD wave functions are not sufficient to describe the long tail of the resonance states, we should smoothly connect the tail of the irregular Coulomb function ($G_L(kr)$) to the original relative wave function $\tilde{r}_L(r)$ for the resonance states at the surface region as usually done in the bound state approximation. We found that we can smoothly connect $\tilde{r}_L(r)$ with $G_L(kr)$ at the point around $r = 5$ fm in most of the resonance states in the present case. Therefore we chose the channel radius to be $a = 5$ fm to evaluate the partial decay widths. In Table III, the theoretical values of the reduced widths $\gamma^2_{\text{He}}(a)$, $\gamma^2_{\text{He}}(a)$, and the partial decay widths are listed. The width of the $0^+_1$ state for the $^{6}$He decay is broad because of the developed $^{6}$He$+^{6}$He cluster structure and the lack of a centrifugal barrier. Although the excited states seem to be stable for the partial decay width of the He channels, the stability of the resonance states should be carefully investigated by taking all of the other possible decay channels, such as the neutron decays and the excited He decays, into account.

VI. SUMMARY

We studied the structures of the ground and excited states of $^{12}$Be based on the framework of the AMD method. This was the first microscopic calculation which systematically reproduced the energy levels of all of the spin-
FIG. 8. The inter-cluster motion in the $^6$He+$^6$He channel in $^{12}$Be. The reduced width amplitudes (R.W.A), $r\tilde{y}_L(r)$ is defined in the text and appendix. The thick solid (dashed) lines are $r\tilde{y}_L(r)$ of the states in the $K^{\pi} = 0^+_3 (K^{\pi} = 0^+_1)$ band.
FIG. 9. The inter-cluster motion in the \(^8\)He-\(^4\)He channel in the ground and excited states of \(^{12}\)Be. The wave function \(\tilde{\gamma}_L(r)\) is defined in the text and appendix.
TABLE V. Approximated values for the $S$ factor, the total cluster probability of the He-cluster channels in $^{12}$Be. The magnitudes, $|a\tilde{y}_L(a)|$, of the reduced width amplitudes at the channel radius $a = 5$ fm are also shown.

| $J^+_n$ | $^6$He+$^6$He $\tilde{S}$ | $^8$He+$^4$He $\tilde{S}$ | $^6$He+$^6$He $\tilde{P}_c$ | $^8$He+$^4$He $\tilde{P}_c$ | $|a\tilde{y}_L(a)|$ fm$^{-1/2}$ |
|------|---------|---------|---------|---------|---------|
| 0$^-_1$ | 0.20 | 0.26 | 0.49 | 0.47 | 0.18 | 0.21 |
| 0$^-_2$ | 0.06 | 0.14 | 0.44 | 0.46 | 0.05 | 0.11 |
| 0$^-_3$ | 0.30 | 0.01 | 0.37 | 0.02 | 0.43 | 0.08 |
| 2$^-_1$ | 0.14 | 0.23 | 0.35 | 0.40 | 0.16 | 0.21 |
| 2$^-_2$ | 0.03 | 0.09 | 0.36 | 0.38 | 0.02 | 0.09 |
| 2$^-_3$ | 0.03 | 0.005 | 0.05 | 0.01 | 0.11 | 0.05 |
| 2$^-_4$ | 0.13 | 0.003 | 0.18 | 0.003 | 0.25 | 0.04 |
| 4$^-_1$ | 0.02 | 0.19 | 0.04 | 0.28 | 0.07 | 0.19 |
| 4$^-_2$ | 0.24 | 0.02 | 0.38 | 0.02 | 0.30 | 0.09 |
| 6$^-_1$ | 0.01 | 0.14 | 0.04 | 0.22 | 0.02 | 0.14 |
| 6$^-_2$ | 0.14 | 0.01 | 0.32 | 0.03 | 0.18 | 0.03 |
| 8$^-_1$ | $\sim$ 0 | 0.03 | $\sim$ 0 | 0.03 | $\sim$ 0 | 0.09 |

TABLE VI. The theoretical values of the partial decay widths and the reduced widths for the $^6$He+$^6$He and $^8$He+$^4$He decays from excited states of $^{12}$Be. The channel radius is chosen to be $a = 5$ fm.

| $J^+_n$ | $\gamma_{^6\text{He}}^2$ ($a$) | $\Gamma_{^6\text{He}}$ (keV) | $\gamma_{^8\text{He}}^2$ ($a$) | $\Gamma_{^8\text{He}}$ (keV) |
|------|---------|--------|---------|--------|
| 0$^-_1$ | $2.5 \times 10^{-3}$ | $7 \times 10^2$ | $1.3 \times 10^{-2}$ | $4 \times 10^1$ |
| 2$^-_1$ | $8.3 \times 10^{-2}$ | 1 | $3.5 \times 10^{-3}$ | 3 |
| 4$^-_1$ | $1.3 \times 10^{-3}$ | 7 | $1.7 \times 10^{-2}$ | 5 |
| 6$^-_2$ | $4.3 \times 10^{-2}$ | 16 | $2.2 \times 10^{-3}$ | 1 |
| 8$^-_1$ | $\sim$ | $1.6 \times 10^{-2}$ | $\sim$ | 1 |
states in $^{12}\text{Be}$, except for the $1^{-}$ state. One of the present discoveries is the $K^\pi = 0_3^+$ band with the $^6\text{He} + ^6\text{He}$ cluster structure, which corresponds well to the states recently observed in the He-He break-up reactions. The ground-state properties, such as the $\beta$-decay strength and the $E1$ transition strength, were reproduced by the present calculations, which suggest that the ground state is dominated by a deformed intruder state with the developed cluster structure. The theoretical results predicted many low-lying excited states. It was found that most of them have a $2\alpha$ core, while the $1_2^+$ state has a no $\alpha$-cluster structure.

We analyzed the structures and the single-particle wave functions of the intrinsic states, and found rotational bands with $K^\pi = 0_1^+, 0_2^+, 1_1^-, 0_3^+, 2_1^+$, which are dominated by the $2\hbar\omega$, $0\hbar\omega$, $1\hbar\omega$, $2\hbar\omega$ and $3\hbar\omega$ configurations, respectively. The $2\hbar\omega$ and $1\hbar\omega$ states have deformed structures with a $2\alpha$ core. In the positive-parity orbits occupied by the valence neutrons, we found the molecular orbits, $\sigma$-orbits and the new $\delta'$-orbits. The positive-parity molecular orbits play an important role in the development of cluster structure. Especially, the reason for the intruder ground state can be understood by the energy gain of the $\sigma$-orbit in the developed cluster state. Although the molecular orbits are helpful to understand the structures of the band head states, the model of the $2\alpha$ and $4\alpha$ neutrons in the molecular orbits is too simple to describe all of the excited states, because such phenomena as the intrinsic spin alignment, the dissociation or breaking of $2\alpha$ core and the mixing of the molecular orbits appear in $^{12}\text{Be}$.

We discussed the inter-cluster motions in the $^6\text{He} + ^6\text{He}$ and $^8\text{He} + ^4\text{He}$ channels, and estimated the partial width of these decay channels with the method of reduced width amplitudes. The analyzed results of the partial widths strongly suggest that the neutrons of the $K^\pi = 0_3^+$ band move around either of two $\alpha$ clusters, while the neutrons of the $K^\pi = 0_1^+$ band move over the whole system.

We also discussed the systematics of the cluster development in the neutron-rich Be isotopes according to the classification of the states with the number of neutrons in the $\sigma$-orbits. Concerning the vanishing of the neutron magic number $N = 8$ in the Be isotopes, we suggested that the reason for the intruder ground states may be explained by the restoration of the natural distance between the $2\alpha$ clusters.

**ACKNOWLEDGMENTS**

The authors would like to thank Dr. N. Itagaki for many discussions. They are also thankful to Prof. W. Von Oertzen for helpful comments. Valuable comments of Prof. S. Shimoura and Dr. A. Saito are also acknowledged. The computational calculations in this work were supported by the Supercomputer Project Nos. 58 and 70 of High Energy Accelerator Research Organization(KEK), and also supported by Research Center for Nuclear Physics in Osaka University and Yukawa Institute for Theoretical Physics in Kyoto University. This work was partly supported by Japan Society for the Promotion of Science and a Grant-in-Aid for Scientific Research of the Japan Ministry of Education, Science and Culture. This work was partially performed in the “Research Project for Study of Unstable Nuclei from Nuclear Cluster Aspects” sponsored by Institute of Physical and Chemical Research (RIKEN).

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APPENDIX A: INTER CLUSTER WAVE FUNCTIONS

The calculational methods of the reduced width amplitudes (R.W.A.), spectroscopic factors, and total cluster probability concerning the $^6$He($0^+$)+$^6$He($0^+$) and $^8$He($0^+$)+$^4$He($0^+$) channels are described in this appendix.

1. Reduced width amplitudes and cluster model space

The reduced width amplitudes, $y_L(a)$, are defined as follows:

$$y_L(a) \equiv q_1 \langle \frac{\delta(r-a)}{r} Y_{L0}(\hat{r}) \phi_0(C_1) \phi_0(C_2) | \Phi_L \rangle,$$

(A1)

where $\Phi_L$ is the internal wave function of a model wave function and $\phi_0(C_1)$ and $\phi_0(C_2)$ are the internal wave functions of the clusters $C_1$ and $C_2$. The mass numbers of the system and the clusters $C_1$ and $C_2$ are $A$, $A_1$, and $A_2$, respectively.

Here, we briefly review the R.W.A. in the system of two clusters written in the form of RGM (resonating group method) or GCM (generator coordinate method) wave functions. The detailed calculational methods of the RGM and GCM kernels are described, for example, in Ref. [33]. We assume the same width parameter ($\nu$) for clusters $C_1$ and $C_2$ described by harmonic oscillator shell model wave functions. When $\Phi_L$ is an RGM-type wave function expressed as,

$$\Phi_L = q_2 A \chi_L(r) Y_{L0}(\hat{r}) \phi_0(C_1) \phi_0(C_2),$$

(A3)

$$q_2 \equiv \frac{1}{\sqrt{(1+\delta_{C_1,C_2}) A A_1}},$$

(A4)

the R.W.A., $y_L(a)$, can be calculated based on the knowledge of the RGM norm kernel as follows. By expanding the relative motion $\chi_L(r)$ with the radial harmonic oscillator (H.O.) wave functions $R_{nL}(r, \nu')$ with the width parameter $\nu' = \frac{A_1 A_2}{A} \nu$ as

$$\chi_L(r) = \sum_n c_{nL} R_{nL}(r, \nu'),$$

(A5)
we obtain the R.W.A. as,

$$y_L(a) = \sum_n e_n L \mu_n L R_n L (a, \nu'),$$

(A6)

where $\mu_n L$ are the eigen values of the RGM norm kernel. If the RGM-type wave function $A_3$ is normalized to unity, the following equation is satisfied:

$$\sum_n e_n^2 \mu_n L = 1.$$  

(A7)

In the case of general $A$-nucleon wave functions, $\Phi_L$ can be separated into the cluster part and the non-cluster part,

$$\Phi_L = A\{\chi_L(r) Y_{L0}(\hat{r}) \phi_0 (C_1) \phi_0 (C_2)\} + \Phi_L^R,$$

(A8)

where $\Phi_L^R$ is the residual part after projection of the cluster model space, and satisfies $Y_{L0}(\hat{r})\phi_0 (C_1)\phi_0 (C_2) |\Phi_L^R\rangle = 0$. In this case, the R.W.A. of $\Phi_L$ can be obtained with equation $A_6$ by using an expansion of $\chi_L(r)$ with the H.O. functions shown in Eq.$A5$. The normal spectroscopic factors $S$ are calculated as

$$S = \int y_L(r)^2 r^2 dr = \sum_n \mu_n^2 L e_n^2 L.$$  

(A9)

It is useful to consider the total probability of the clustering component (total cluster probability) in $\Phi_L$ defined by the squared overlap with the cluster model space calculated as,

$$P_c = \sum_n \mu_n L e_n^2 L.$$  

(A10)

It is easily found that $P_c = 1$ for the normalized RGM-type wave functions from equation $A7$.

The R.W.A. of the wave function $\Phi_L^{(12\text{Be})}$ in the AMD framework is

$$y_L(a) = q_1 \frac{\delta (r-a)}{r^2} Y_{L0}(\hat{r}) \phi_0 (C_1) \phi_0 (C_2) |\Phi_L^{(12\text{Be})}\rangle,$$

(A11)

where $\Phi_L^{(12\text{Be})}$ is a normalized spin-parity projected AMD wave function, where the center of mass motion is extracted, and $\phi_0 (C_n)$ ($C_n = ^4\text{He}, \ ^6\text{He}, \ ^{8}\text{He}$) are the intrinsic wave functions of the $SU_3$-limit $0^+$ states of the He clusters. In order to project to the cluster model space from the $^{12}\text{Be}$ wave functions ($\Phi_L^{(12\text{Be})}$) and calculate the radial wave functions ($\chi_L(r)$) between the clusters, as mentioned in Eq.$A8$, we use the orthonormal sets of the GCM wave functions. Concerning the $k$-th cluster wave functions $\Psi^c (b_k)$ with the inter-cluster distances $\{b_k\}$, we adopt Brink-type functions, where two clusters($C_1 = ^4\text{He}$ and $C_2 = ^8\text{He}$) written in the $SU_3$-limit representation are located at the points $(0, 0, \frac{A_1}{A_1 + A_2} b_k)$ and $(0, 0, -\frac{A_2}{A_1 + A_2} b_k)$, respectively,

$$\Psi^c (b_k) = A\{\Psi(C_1, A_2 b_k) \Psi(C_2, -A_1 b_k)\}, \quad b_k = (0, 0, b_k).$$  

(A12)

The width parameters of the He clusters are chosen to be the same as that of the $^{12}\text{Be}$ wave functions for simplicity. Then we can rewrite $\Psi^c (b_k)$ as

$$\Psi^c (b_k) = \rho_0 (X_G) \cdot \Phi^c (b_k),$$

(A13)

$$\Phi^c (b_k) = q_{24} A\{\Gamma (r, b_k, \nu') \phi_0 (C_1) \phi_0 (C_2)\},$$

(A14)

$$\Gamma (r, b_k, \nu') = \left(\frac{2 \nu'}{\pi}\right)^{3/4} e^{-\nu' (r-b_k)^2}$$

(A15)

$$\rho_0 (X_G) = \left(\frac{2 \nu'}{\pi}\right)^{3/4} e^{-\nu' X_G^2}, \quad X_G = \frac{1}{A} \sum_{i=1, A} X_i,$$

(A16)

In each $L$, we make a set of the orthonormal basis $\tilde{\Phi}_{k,L}$ by linear combinations of the total-spin-projected GCM wave functions $\Phi_{k,L}^{(12\text{Be})}$:

$$\Phi_{k,L}^{(12\text{Be})} = q_{kL} n_{00} A_{kL}^L \Phi^c (b_k),$$

(A17)

$$\Phi_{k,L}^{(12\text{Be})} = \sum_{k'} A_{kL}^{(L)} \Phi_{k',L},$$

(A18)

$$\langle \tilde{\Phi}_{k,L} | \tilde{\Phi}_{k',L} \rangle = \delta_{k k'},$$

(A19)

$$\langle \tilde{\Phi}_{k,L} \rangle \langle \tilde{\Phi}_{k',L} \rangle = \delta_{k k'},$$

(A20)
where \( q_{kL} \) are the normalization factors \( q_{kL} = 1/(P_{nL}^e \Phi^e(b_k)|P_{nL}^f \Phi^f(b_k))^{1/2} \). By using the partial wave expansion of the functions \( \Gamma(r, b_k, \nu') \), the radial functions \( \chi^{(k)}_L(r) \) in the \( k \)-th wave functions \( \Phi_{kL} \) can be expressed as follows:

\[
\Phi_{kL} = \frac{1}{\sqrt{1+(\delta_{c1,c2})}} A\{ \chi^{(k)}_L(r) Y_{L0}(\hat{r}) \phi_0(C_1) \phi_0(C_2) \}, \tag{A21}
\]

\[
\chi^{(k)}_L(r) = q_{kL} \sqrt{\frac{2L+1}{4\pi}} \Gamma_L(r, b_k, \nu'), \tag{A22}
\]

\[
\Gamma_L(r, b_k, \nu') = \left( \frac{2\nu'}{\pi} \right)^{3/4} 4\pi i_L (2\nu'rb_k) e^{-\nu'(r^2+b_k^2)}, \tag{A23}
\]

where \( i_L \) is the modified spherical Bessel function. We assume that the projection operator \( P_L^c \) onto the cluster model space can be written by the orthonormal basis \( \tilde{\Phi}_{kL} \),

\[
P_L^c = \sum_k |\tilde{\Phi}_{kL}(\tilde{\Phi}_{kL})| = \sum_{k,k',k''} |\Phi_{k',L} A_{kk'}^{(L)} A_{kk''}^{(L)*} \langle \Phi_{k''L} | \Phi_{L}^{(12\text{Be})} \rangle|. \tag{A24}
\]

In this case, the radial function \( \chi_L(r) \) in Eq.A8 for \( \Phi_L^{(12\text{Be})} \) is written as

\[
\chi_L(r) = \sum_{k'} \chi^{(k')}_{L}(r) \{ \sum_{k,k'} A_{kk'}^{(L)} A_{kk''}^{(L)*} \langle \Phi_{k''L} | \Phi_{L}^{(12\text{Be})} \rangle \}. \tag{A25}
\]

### 2. Practical calculation of \( \mu_{nL} \)

In the present calculations, we chose \( b_k = 1, 2, \ldots, 9 \) fm for \( L = 0, 2 \) \((L = 0, 2, 4, 6, b_k = 2, 3, \ldots, 9 \) fm for \( L = 4, 6, \) and \( b_k = 3, \ldots, 9 \) fm for \( L = 8 \) \((L = 8) \) for the \(^{8}\text{He}^{+}^{4}\text{He}^{+}^{6}\text{He} \) channel. In the practical calculations, we approximate \( \mu_{nL} \) with \( \tilde{\mu}_{nL} \) calculated as follows. It is known that the eigen values \( \mu_{nL} \) equal to zero for the Pauli forbidden states with \( N = 2n + L < N_{\text{min}} \) \((N_{\text{min}} \) is the minimum allowed number) and \( \mu_{nL} \sim 1 \) with enough large numbers, \( N = 2n + L \). In order to obtain approximated values \( \tilde{\mu}_{nL} \), we truncate the quanta \( n \) with a finite number \( n = n_{\min}, \ldots, n_{\max} \) \((n_{\min} = 2n_{\min} + L, N_{\max} = 2n_{\max} + L) \) by assuming \( \tilde{\mu}_{nL} = 1 \) for \( N = 2n + L > N_{\max} \). With the use of the expansion of \( \chi_{L}^{(k)}(r) \) in the normalized cluster wave functions \( \Phi_{kL} \) by \( R_{nL}(r, \nu') \)

\[
\chi_{L}^{(k)}(r) = \sum_n c_{nL}^{(k)} R_{nL}(r, \nu'), \tag{A26}
\]

we obtain \( \tilde{\mu}_{nL} \) by following equations:

\[
\sum_n c_{nL}^{(k)*} c_{nL}^{(k)} = 1. \tag{A27}
\]

The approximated values \( \tilde{\mu}_{nL} \) for simple systems, \( \alpha + \alpha \) and \( \alpha + ^{16}\text{O} \) are compared with the exact eigen values \( \mu_{nL} \) in Table. \( \text{VI} \)

In the present calculation of \( ^{12}\text{Be} \), \( N_{\max} \) was chosen to be 14. By using \( \tilde{\mu}_{nL} \) determined by Eq.A27, we define the approximate R.W.A., \( \tilde{y}_L(a) \), the spectroscopic facors, \( \tilde{S} \), and the total cluster probability, \( \tilde{P}_c \) for \( \Phi_L^{(^{12}\text{Be})} \) by analogy with the relations in Eqs. \( \text{A6} \), \( \text{X} \) and \( \text{A10} \):

\[
\tilde{y}_L(a) = \sum_n c_{nL} \tilde{\mu}_{nL} R_{nL}(a, \nu'), \tag{A28}
\]

\[
\tilde{S} = \sum_n \tilde{\mu}_{nL}^2 c_{nL}^2, \tag{A29}
\]

\[
\tilde{P}_c = \sum_n \tilde{\mu}_{nL} c_{nL}^2, \tag{A30}
\]

where the coefficients \( c_{nL} \) were calculated by \( c_{nL} = \int r^2 \chi_L(r) R_{nL}(r) dr \) from \( \chi_L(r) \) given in Eq. \( \text{A25} \).
TABLE VII. The eigen values $\mu_{N=2n+L}$ and approximated values $\tilde{\mu}_{nL}(L = 0)$ of the RGM norm kernel for $\alpha + \alpha$ and $\alpha + ^{16}\text{O}$ systems. The values $\mu_N$ are taken from Ref. [33]. The width parameters for $\alpha$ and $^{16}\text{O}$ are assumed to be same.

| $N$ | $\alpha + \alpha$ | $\tilde{\mu}_{nL}$ | $\alpha + ^{16}\text{O}$ | $\mu_N$ | $\tilde{\mu}_{nL}$ |
|-----|-----------------|-----------------|-----------------|--------|-----------------|
| 0   | 0               | 0               | 0               | 0      | 0               |
| 2   | 0               | 0               | 2               | 0      | 0               |
| 4   | 0.7500          | 0.7500          | 4               | 0      | 0               |
| 6   | 0.9375          | 0.9375          | 6               | 0      | 0               |
| 8   | 0.9844          | 0.9844          | 8               | 0.2292 | 0.2292          |
| 10  | 0.9961          | 0.9962          | 10              | 0.5103 | 0.5063          |
| 12  | 0.9990          | 0.9987          | 12              | 0.7185 | 0.7538          |
|     |                 |                 | 14              | 0.8459 | 0.7399          |
|     |                 |                 | 16              | 0.9178 | 1               |
|     |                 |                 | 18              | 0.9568 | 1               |
|     |                 |                 | 20              | 0.9775 | 1               |
|     |                 |                 | 22              | 0.9884 | 1               |
|     |                 |                 | 24              | 0.9941 | 1               |
|     |                 |                 | 26              | 0.9970 | 1               |