SHEAVES WITH CONNECTION ON ABELIAN VARIETIES

MITCHELL ROTHSTEIN

1. Introduction

Let $X$ and $Y$ be abelian varieties over an algebraically closed field $k$, dual to one another, and let $\text{Mod}(\mathcal{O}_X)$ and $\text{Mod}(\mathcal{O}_Y)$ be their respective categories of quasicoherent $\mathcal{O}$-modules. Mukai proved in [Muk] that the derived categories $D\text{Mod}(\mathcal{O}_X)$ and $D\text{Mod}(\mathcal{O}_Y)$ are equivalent via a transform now known as the Fourier-Mukai transform,

$$S_1(\mathcal{F}) = \alpha_2^* (P \otimes \alpha_1^*(-1)^*(\mathcal{F})),$$

(1.1)

where $P$ is the Poincaré sheaf and $\alpha_1$ and $\alpha_2$ are the projections from $X \times Y$ to $X$ and $Y$ respectively. A few years earlier, Krichever [K] rediscovered a construction due originally to Burchnall and Chaundy [BC], by which the affine coordinate ring of a projective curve minus a point may be imbedded in the ring of formal differential operators in one variable. The construction involves the choice of a line bundle on the curve, and Krichever took the crucial step of asking, in the case of a smooth curve, how the imbedding varies when the line bundle moves linearly on the Jacobian. The answer is now well-known, that the imbeddings satisfy the system of differential equations known as the KP-hierarchy. In fact, the Krichever construction is an instance of the Fourier-Mukai transform, with the crucial addition that the transformed sheaf is not only an $\mathcal{O}_Y$-module but a $\mathcal{D}_Y$-module, where $\mathcal{D}_Y$ is the sheaf of linear differential operators on $Y$, [N1] [N2] [R].

This example serves as the inspiration for the present work, which is concerned with the role of the Fourier-Mukai transform in the theory of sheaves on $Y$ equipped with a connection. The main point is that in the derived category, all sheaves on $Y$ with connection are constructed by the Fourier-Mukai transform in a manner directly generalizing the

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Krichever construction. The connection need not be integrable, though the paper focuses mostly on that case.

The basic idea is the following. Set

$$g = H^1(X, \mathcal{O}) .$$

(1.2)

Then there is a tautological extension

$$0 \longrightarrow g^* \otimes \mathcal{O} \longrightarrow \mathcal{E}^\mu \longrightarrow \mathcal{O} \longrightarrow 0$$

(1.3)

given by the extension class $1 \in \text{End}(g^*) = \text{Ext}^1(\mathcal{O}, g^* \otimes \mathcal{O})$. Now let $\mathcal{F}$ be any quasicoherent sheaf of $\mathcal{O}_X$-modules, and tensor the sequence (1.3) with $\mathcal{F}$:

$$0 \longrightarrow g^* \otimes \mathcal{F} \longrightarrow \mathcal{E} \otimes \mathcal{F}^\mu \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow 0 .$$

(1.4)

We refer to any splitting of sequence (1.4) as a splitting on $\mathcal{F}$. Let $\text{Mod}(\mathcal{O}_X)_{sp}$ denote the category of pairs $(\mathcal{F}, \psi)$, where $\mathcal{F}$ is a sheaf on $X$ and $\psi : \mathcal{F} \longrightarrow \mathcal{E} \otimes \mathcal{F}$ is a splitting, with the obvious morphisms.

Let $\text{Mod}(\mathcal{O}_Y)_{cxn}$ denote the category of quasicoherent sheaves on $Y$ equipped with a connection. In section 2 we use the Fourier-Mukai transform to establish an equivalence of bounded derived categories:

$$D^b\text{Mod}(\mathcal{O}_X)_{sp} \leftrightarrow D^b\text{Mod}(\mathcal{O}_Y)_{cxn} .$$

(1.5)

Note that the extension class of (1.4) belongs $g^* \otimes H^1(X, \mathcal{E}\text{nd}(\mathcal{F}))$ and is therefore a linear map

$$H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{E}\text{nd}(\mathcal{F})) .$$

(1.6)

It is easily seen to be the map on cohomology induced by the map $\mathcal{O} \longrightarrow \mathcal{E}\text{nd}(\mathcal{F})$. Thus

**Proposition 1.1.** Given an $\mathcal{O}$-module $\mathcal{F}$, there is a splitting on $\mathcal{F}$ if and only if the natural map

$$\mathcal{O} \longrightarrow \mathcal{E}\text{nd}(\mathcal{F})$$

induces the 0-map

$$H^1(X, \mathcal{O}) \longrightarrow H^1(X, \mathcal{E}\text{nd}(\mathcal{F})) .$$

(1.7)

Moreover, if (1.8) holds, then the set of such splittings is an affine space over $g^* \otimes H^0(X, \mathcal{E}\text{nd}(\mathcal{F}))$.

The intuitive idea behind the equivalence (1.5) is the following. Let $g = \text{dim}(Y)$. Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ be an affine open cover of $X$ and for $i = 1, \ldots, g$, let $\{c(i)_{m,n}\} \in Z^1(\{\mathcal{U}\}, \mathcal{O})$ be a 1-cocycle, such that the classes $[c(1)], \ldots, [c(g)]$ form a basis for $H^1(X, \mathcal{O})$. Let $\xi_1, \ldots, \xi_g$ denote this basis, and let $\omega^1, \ldots, \omega^g$ denote the dual basis for $H^1(X, \mathcal{O})^*$. In
light of proposition 1.1, a splitting on $F$ amounts to a collection of endomorphisms $\psi(i)_n \in \Gamma(U_n, \mathcal{E}nd((−1_X)^*(F)))$ such that

\[
\psi(i)_n - \psi(i)_m = \text{multiplication by } -c(i)_{n,m}.
\]

Let $G = S_1(F)$. Then the collection of endomorphisms $\psi = \{\psi(i)_n\}$ endows $G$ with a connection in the following way. For each $n$, there is a connection $\nabla_n$ relative to $U_n$ on $\mathcal{P}|_{U_n \times Y}$, such that on the overlaps,

\[
\nabla_n - \nabla_m = c_{nm}.
\]

Therefore, one gets a connection relative to $X$ on $\alpha_1^*(-1_X)^*(F) \otimes \mathcal{P}$ by defining

\[
\nabla(\phi \otimes \sigma) = \phi \otimes \nabla_n(\sigma) + \sum i \omega^i \psi(i)_n(\phi) \otimes \sigma
\]

for $\phi \in (-1_X)^*(F)$ and $\sigma \in \mathcal{P}$. Now one applies $\alpha_2^*$ to produce a connection on $G$.

In the Krichever construction, $X$ is the Jacobian of a smooth curve $C$ with a base point $P$, and $F$ is $\mathcal{O}_C(*P)$, regarded as a sheaf on $X$ by the abel map. The case where $X$ is an arbitrary abelian variety and $F$ is of the form $G \otimes \mathcal{O}_X(*D)$ for a coherent sheaf $G$ and an ample hypersurface $D \subset X$ has been studied in [N1] and [N2].

Now consider the curvature tensor. To each object $(F, \psi) \in \text{Ob Mod}(\mathcal{O}_X)_{sp}$ we associate a section

\[
[S_1(\psi, \psi)] \in \wedge^2 g^* \otimes \text{End}(F),
\]

simply by taking the commutator $[\psi(i)_n, \psi(j)_n]$, which, by (1.9), is independent of the chart. Applying the Fourier-Mukai transform to morphisms, one has

\[
S_1([\psi, \psi]) \in \wedge^2 g^* \otimes \text{End}(S_1(F)).
\]

Letting $[\nabla, \nabla]$ denote curvature, one has (Proposition 4.1),

\[
S_1([\psi, \psi]) = [\nabla_\psi, \nabla_\psi].
\]

In particular, $S_1$ restricts to a functorial correspondence

\[
(F, \psi) \text{ with } [\psi, \psi] = 0 \leadsto \mathcal{D}\text{-module structure on } S_1(F).
\]

We prove that this also induces an equivalence of bounded derived categories (theorem 4.3).

The main point regarding the integrable case is the following. Let

\[
X^2 \xrightarrow{\pi} X
\]

denote the $g^*$-principal bundle associated to the extension $\mathcal{E}$. It is known that $X^2$ is the moduli space of line bundles on $Y$ equipped with
an integrable connection. For a discussion of \(X\) in greater generality, see [Mc], [Ros], and [S]. Let \(A = \pi_*(\mathcal{O}_X)\). Since \(\pi\) is an affine morphism, the category of \(\mathcal{O}_{X^\natural}\)-modules is equivalent to category of \(A\)-modules. Then the subcategory of \(\text{Mod}(\mathcal{O}_X)_{sp}\) whose objects satisfy \([\psi, \psi] = 0\) is precisely \(\text{Mod}(A) = \text{Mod}(\mathcal{O}_{X^\natural})\) (Proposition 4.1). Thus we have an equivalence of categories

\[
D^b\text{Mod}(\mathcal{O}_{X^\natural}) \leftrightarrow D^b\text{Mod}(\mathcal{D}_Y)
\] (1.17)

The outline of the paper is as follows. In section 2 we prove the basic equivalence theorem. By way of illustration, section 3 offers a new proof of a theorem of Matsushima on vector bundles with a connection. In section 4 the equality (1.14) is established and the equivalence (1.17) is proved. Section 5 gives some examples. In particular, we establish the formula

\[
\hat{A} = \mathcal{D}_{[0] \rightarrow Y}.
\] (1.18)

Sections 6 and 7 contain general results about coherence, holonomicity and the characteristic variety. Section 8 illustrates the theory in the case of the Krichever construction. In sections 9 and 10 we refine and extend several results of Nakayashiki on characteristic varieties of \(BA\)-modules and commuting rings of matrix partial differential operators. This last topic is a natural setting for the further study of integrable systems; some brief remarks on this relationship are included at the end. Further applications to nonlinear partial differential equations will appear in a future work.

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2. First equivalence theorem

We adopt the following sign conventions for the Fourier-Mukai transform:

\[
\text{Mod}(\mathcal{O}_X) \xrightarrow{\mathcal{S}_1} \text{Mod}(\mathcal{O}_Y) \\
\mathcal{S}_1(\mathcal{F}) = \alpha_{2*}(\mathcal{P} \otimes \alpha_1^*(-1)^*(\mathcal{F}))
\]

(2.1)

\[
\text{Mod}(\mathcal{O}_Y) \xrightarrow{\mathcal{S}_2} \text{Mod}(\mathcal{O}_X) \\
\mathcal{S}_2(\mathcal{G}) = \alpha_{1*}(\mathcal{P} \otimes \alpha_2^*(\mathcal{G}))
\]

(2.2)

The fundamental result in [Muk] is

\footnote{The referee informs us that this equivalence also appears in an unpublished preprint by Laumon [L]}
Mukai’s Theorem. The derived functors

\[ D^b \text{Mod}(\mathcal{O}_X) \xrightarrow{RS} D^b \text{Mod}(\mathcal{O}_Y) \]

\[ D^b \text{Mod}(\mathcal{O}_Y) \xrightarrow{RS} D^b \text{Mod}(\mathcal{O}_X) \]  

(2.3)

are defined, and

\[ Rs_1 Rs_2 = T^{-g} \]  

(2.4)

\[ Rs_2 Rs_1 = T^{-g} \]  

(2.5)

where \( T \) is the shift automorphism on the derived category,

\[ (T^F)^n = F^{n+1}. \]

Proof. \[ \text{[Muk, p. 156]} \]

The other key result of [Muk] for our purposes is that the Fourier-Mukai transform exchanges tensor product and Pontrjagin product. Letting \( \beta_1 \) and \( \beta_2 \) denote the projections from \( Y \times Y \) to \( Y \) and denoting the group law on \( Y \) by \( m \), the Pontrjagin product is defined by

\[ \mathcal{F} \ast \mathcal{G} = m_*(\beta_2^*(\mathcal{F}) \otimes \beta_1^*(\mathcal{G})). \]  

(2.6)

Theorem Mukai 2.

\[ Rs_2(g_1 \ast g_2) = Rs_2(g_1) \otimes^{L} Rs_2(g_2) \]  

(2.7)

\[ Rs_1(f_1 \otimes f_2) = T^g(Rs_1(f_1) \ast^{R} Rs_1(f_2)). \]  

(2.8)

Proof. \[ \text{[Muk, p. 160]} \]

We want to apply this to the following situation. Let \( \mathcal{I} \subseteq \mathcal{O}_Y \) be the ideal sheaf of \( 0 \in Y \), and let \( k(0) \) be the skyscraper sheaf at the origin with fiber \( k \). Then \( R^0s_2(k(0)) = \mathcal{O}_X \) and \( R^i s_2(k(0)) = 0 \) for \( i > 0 \). Thus the Fourier-Mukai transform takes the short exact sequence

\[ 0 \rightarrow g^* \otimes_k k(0) \rightarrow \mathcal{O}/\mathcal{I}^2 \rightarrow k(0) \rightarrow 0 \]  

(2.9)

to a short exact sequence of vector bundles on \( X \), and it is easy to see that it is precisely the sequence \([1.3]\). Thus by theorem Mukai 2, if \( \mathcal{F} \rightarrow_{\psi} \mathcal{E} \otimes \mathcal{F} \) is a splitting on \( \mathcal{F} \), \( \psi \) induces a morphism

\[ Rs_1(f) \xrightarrow{S_1(\psi)} T^{-g}(\mathcal{O}/\mathcal{I}^2) \ast RS_1(f). \]  

(2.10)

(We identify objects in an abelian category with complexes concentrated in degree 0.) If we now take \( g^{th} \) cohomology, we get

\[ S_1(f) \xrightarrow{H_gS_1(\psi)} \mathcal{O}/\mathcal{I}^2 \ast S_1(f). \]  

(2.11)
The point is this. If $\mathcal{G}$ is any $\mathcal{O}_Y$-module, there is a prolongation sequence
\[
0 \longrightarrow \Omega^1 \otimes_{\mathcal{O}} \mathcal{G} \longrightarrow j(\mathcal{G}) \xrightarrow{\nu_\mathcal{G}} \mathcal{G} \longrightarrow 0,
\] (2.12)
such that a splitting of $\nu_\mathcal{G}$ is precisely a connection on $\mathcal{G}$. As a sheaf of abelian groups,
\[
j(\mathcal{G}) = \mathcal{G} \oplus (\Omega^1 \otimes_{\mathcal{O}} \mathcal{G}),
\] (2.13)
with $\mathcal{O}$-module structure
\[
f(\phi, \omega \otimes \psi) = (f\phi, f\omega \otimes \psi + df \otimes \phi).
\] (2.14)
Thus a connection on $\mathcal{G}$ is a splitting of (2.12). Since $Y$ is an abelian variety, there is a characterization of $j(\mathcal{G})$ in terms of the Pontrjagin product.

**Lemma 2.1.** For any $\mathcal{O}_Y$-module $\mathcal{G}$,
\[
j(\mathcal{G}) = (\mathcal{O}/\mathcal{I}^2)^* \mathcal{G}.
\] (2.15)

*Proof.* Let $Y_1 \subset Y \times Y$ denote the first order neighborhood of the diagonal, and let $\pi_i : Y_1 \longrightarrow Y$, $i = 1, 2$, denote the two projections. Then
\[
j(\mathcal{F}) = \pi_{2*} \pi_1^*(\mathcal{F}).
\] (2.16)
(This holds for any variety.) Let $\tilde{Y} = \text{Spec}(\mathcal{O}/\mathcal{I}^2)$, the first order neighborhood of 0 in $Y$. Then $Y_1$ may be identified with $Y \times \tilde{Y}$ in such a way that $\pi_1$ corresponds to projection onto the first factor and $\pi_2$ corresponds to the group law, $\tilde{m}$. Let $\iota : \tilde{Y} \to Y$ denote the inclusion map. Then
\[
\mathcal{O}/\mathcal{I}^2 * \mathcal{G} = m_* (\beta_2^* \iota_* (\mathcal{O}_{\tilde{Y}}) \otimes \beta_1^* (\mathcal{G}))
= m_* (1 \times \iota)_* (\beta_2^* (\mathcal{O}_{\tilde{Y}}) \otimes \beta_1^* (\mathcal{G}))
= \tilde{m}_* \tilde{\beta}_1^* (\mathcal{G}) = j(\mathcal{G}).
\] (2.17)

Combining this lemma with the map (2.11), we see that a splitting on $\mathcal{F}$ induces a splitting of the prolongation sequence of $\mathcal{S}_1(\mathcal{F})$, i.e., a connection on $\mathcal{S}_1(\mathcal{F})$. So we have a functor
\[
\text{Mod}(\mathcal{O}_X)_{sp} \xrightarrow{S_1} \text{Mod}(\mathcal{O}_Y)_{cxn}.
\] (2.18)
We will check later that this description is equivalent to the one given in the introduction.
Conversely, if we apply $S_2$ to a splitting of the prolongation sequence, $\mathcal{G} \rightarrow j(\mathcal{G}) = \mathcal{O}/I^2 \ast \mathcal{G}$, theorem Mukai 2 gives a splitting
\[ S_2(\mathcal{G}) \xrightarrow{\psi} \mathcal{E} \otimes S_2(\mathcal{G}) . \] (2.19)

So we have
\[ \text{Mod}(\mathcal{O}_Y)_{\text{cxn}} \xrightarrow{S_2} \text{Mod}(\mathcal{O}_X)_{\text{sp}} . \] (2.20)

The categories $\text{Mod}(\mathcal{O}_X)_{\text{sp}}$ and $\text{Mod}(\mathcal{O}_Y)_{\text{cxn}}$ are abelian. Moreover, objects in either $\text{Mod}(\mathcal{O}_X)_{\text{sp}}$ or $\text{Mod}(\mathcal{O}_Y)_{\text{cxn}}$ may be resolved by a Čech resolution with respect to an affine open cover of $X$ or $Y$. Thus the derived functors
\[ D^b\text{Mod}(\mathcal{O}_X)_{\text{sp}} \xrightarrow{RS_1} D^b\text{Mod}(\mathcal{O}_Y)_{\text{cxn}} \] (2.21)
\[ D^b\text{Mod}(\mathcal{O}_Y)_{\text{cxn}} \xrightarrow{RS_2} D^b\text{Mod}(\mathcal{O}_X)_{\text{sp}} \] (2.22)
exist.

The main result of this section is

**Theorem 2.2.**

\[ RS_1RS_2 = T^{-g} , \] (2.23)
\[ RS_2RS_1 = T^{-g} . \] (2.24)

**Proof.** Let $\zeta$ denote the functor $T^gRS_1RS_2$. Let $\text{for}$ denote the forgetful functor from $D^b\text{Mod}(\mathcal{O}_Y)_{\text{cxn}}$ to $D^b\text{Mod}(\mathcal{O}_Y)$. Then
\[ \text{for} \; \zeta = \text{for} \] (2.25)

by Mukai’s theorem. In particular, for any object $(\mathcal{F}, \nabla) \in \text{Ob} \; \text{Mod}(\mathcal{O}_Y)_{\text{cxn}}$, $H^i\zeta(\mathcal{F}, \nabla) = 0$ for $i > 0$. Thus $\zeta(\mathcal{F}, \nabla) = (\mathcal{F}, \nabla')$ for some new connection $\nabla'$.

Let $\mathcal{F} \xrightarrow{\tau} j(\mathcal{F})$ denote the splitting associated to $\nabla$, and let $\mathcal{F} \xrightarrow{\tau'} j(\mathcal{F})$ denote the splitting associated to $\nabla'$. Let $\psi$ denote the corresponding splitting on $S_2(\mathcal{F})$. Then $\psi$ is the $0^{th}$ cohomology of
\[ RS_2(\mathcal{F}) \xrightarrow{RS_2(\tau)} \mathcal{E} \otimes RS_2(\mathcal{F}) , \] (2.26)
from which it follows that $\tau'$ is the $0^{th}$ cohomology of
\[ \mathcal{F} \xrightarrow{T^gRS_1RS_2(\tau)} j(\mathcal{F}) . \] (2.27)

Thus $\tau = \tau'$, again by Mukai’s Theorem.

Similarly, if $(\mathcal{G}, \psi)$ is an object in $\text{Mod}(\mathcal{O}_X)_{\text{sp}}$,
\[ T^gRS_2RS_1(\mathcal{G}, \psi) = (\mathcal{G}, \psi) . \] (2.28)

The next lemma then completes the proof of the theorem. \qed
Lemma 2.3. Let $C_1$ and $C_2$ be abelian categories, and let

$$D^b C_1 \xrightarrow{F_1} F_2 \xrightarrow{F_2} D^b C_2$$

be $\delta$-functors. If $F_1$ and $F_2$ are isomorphic when restricted to the subcategory $C_1 \subset D^b C_1$, then they are isomorphic.

Proof. This follows by induction on the cohomological length of an object in the bounded derived category, using \cite[lemme 12.6, p.104]{Bo} and the triangle axiom TR3, \cite[p.28]{Bo}, once it is noted that the constructions used there are functorial. \hfill \square

Remark Let $\gamma_i, \gamma_{i,j}$ denote the projections on $X \times Y \times Y$. The key to Mukai’s theorem is the elementary formula

$$\gamma^*_{1,2}(P) \otimes \gamma^*_{1,3}(P) = (1 \times m)^*(P).$$

This formula also plays a crucial but hidden role in theorem 2.2, which we would like to make explicit.

Let $\tilde{Y} = \text{Spec}(\mathcal{O}/\mathcal{I}^2)$. Then $P|_{X \times \tilde{Y}}$ is a line bundle on $X \times \tilde{Y}$, trivial on $X \times \{0\}$. Set $\tilde{P} = P|_{X \times \tilde{Y}}$, and let $\tilde{\alpha}_1 : X \times \tilde{Y} \to X$ be the projection. Then $\mathcal{E} = \tilde{\alpha}_1*(\tilde{P})$ and $\mu : \mathcal{E} \to \mathcal{O}$ is the morphism which restricts a section of $\tilde{P}$ to $X \times \{0\}$. Let $\tilde{\gamma}_i, \tilde{\gamma}_{i,j}$ denote the projections on $X \times Y \times \tilde{Y}$. Then we get an infinitesimal form of (2.30),

$$\tilde{\gamma}^*_{1,2}(P) \otimes \tilde{\gamma}^*_{1,3}(\tilde{P}) = (1 \times \tilde{m})^*(P).$$

If $\mathcal{G}$ is a sheaf on $X \times Y$, then a connection on $\mathcal{G}$ relative to $X$ is an isomorphism

$$\tilde{\gamma}^*_{1,2}(\mathcal{G}) \approx (1 \times \tilde{m})^*(\mathcal{G})$$

restricting to the identity on $X \times Y$. Thus (2.31) says that $\tilde{\gamma}^*_{1,3}(\tilde{P})$ is the obstruction to endowing $\mathcal{P}$ with a connection relative to $X$. Given a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, a splitting on $\mathcal{F}$ is precisely what is needed to cancel this obstruction. Indeed, a splitting may be regarded as an isomorphism

$$\tilde{\alpha}_1^*(\mathcal{F}) \xrightarrow{\psi} \tilde{P} \otimes \tilde{\alpha}_1^*(\mathcal{F})$$

restricting to the identity on $X$. Applying $(-1_X, -1_{\tilde{Y}})$, we get

$$\tilde{\alpha}_1^*(-1)^*(\mathcal{F}) \longrightarrow \tilde{P} \otimes \tilde{\alpha}_1^*(-1)^*(\mathcal{F})$$.
If we then apply \( \tilde{\gamma}^{*}_{1,2} \) to \( P \otimes \alpha^{*}_{1}(-1)^{*}(F) \) we find
\[
\tilde{\gamma}^{*}_{1,2}(P \otimes \alpha^{*}_{1}(-1)^{*}(F)) = \tilde{\gamma}^{*}_{1,2}(P) \otimes \tilde{\gamma}^{*}_{1,3} \tilde{\alpha}^{*}_{1}(-1)^{*}(F) \\
\approx \tilde{\gamma}^{*}_{1,2}(P) \otimes \tilde{\gamma}^{*}_{1,3}(P) \otimes \tilde{\gamma}^{*}_{1,3} \tilde{\alpha}^{*}_{1}(-1)^{*}(F) \\
= (1 \times \tilde{m}^{*})(P) \otimes \tilde{\gamma}^{*}_{1,3}(P) \tilde{\alpha}^{*}_{1}(-1)^{*}(F) \\
= (1 \times \tilde{m}^{*} \otimes \alpha^{*}_{1}(-1)^{*}(F)),
\]
which is a relative connection on \( P \otimes \alpha^{*}_{1}(-1)^{*}(F) \). Then apply \( \alpha_{2}^{*} \) to get a sheaf with connection on \( Y \), and this is our functor \( S_{1} \).

3. Matsushima’s Theorem

As an application of theorem 2.2, we will give a new proof of Matsushima’s Theorem on the homogeneity of vector bundles admitting a connection \([\text{Mat}]\). The key is the following lemma, which is of interest in its own right. Here we assume \( \text{char}(k) = 0 \).

**Lemma 3.1.** Let \( F \) be a coherent \( \mathcal{O}_{X} \)-module with a splitting. Then \( F \) is finitely supported.

**Proof.** The proof is similar to that of lemma 3.3 in \([\text{Muk}]\). Assuming \( \dim(\text{supp}(F)) > 0 \), let \( C \) be a curve contained in \( \text{supp}(F) \), and let \( \tilde{C} \xrightarrow{\pi} C \) be its normalization. Let \( F' = \pi^{*}(F) \). Then we get a non-zero vector bundle \( F'' \) on \( \tilde{C} \) upon taking the quotient of \( F' \) by its torsion part. Let \( E' \) denote the pullback of \( E|_{C} \) to \( \tilde{C} \). Then any splitting on \( F \) induces a splitting
\[
F' \xrightarrow{\psi} E' \otimes F'.
\]
Since \( \text{Tor}(E' \otimes F') = E' \otimes \text{Tor}(F') \), we get a splitting of the sequence
\[
0 \rightarrow g^{*} \otimes F'' \rightarrow E' \otimes F'' \rightarrow F'' \rightarrow 0.
\]
But the extension class of the sequence \([3.2]\) is the bottom arrow in the commutative diagram
\[
\begin{array}{ccc}
H^{1}(\tilde{C}, \mathcal{O}) & \xrightarrow{a} & H^{1}(X, \mathcal{O}) \\
& \searrow b & \downarrow e \xrightarrow{\beta} H^{1}(\tilde{C}, \mathcal{E}_{\text{End}}(F''))
\end{array}
\]
where \( b \) is induced by the natural inclusion \( \mathcal{O} \xrightarrow{\beta} \mathcal{E}_{\text{End}}(F'') \) and \( a \) is the derivative of the natural map \( \text{Pic}(X) \rightarrow \text{Pic}(\tilde{C}) \). In particular, \( a \) is not the 0-map. Moreover, in characteristic 0, \( \beta \) splits by the trace, so \( b \) is injective. This is a contradiction, for now \( e \neq 0 \) so that \([3.2]\) does not split.

We then have
Theorem 3.2 (Matsushima). Any vector bundle on $Y$ admitting a connection is homogeneous.

Proof. Let $\mathcal{G}$ be such a vector bundle. Then the cohomology sheaves $R^iS^2_2(\mathcal{G})$ are $\mathcal{O}_X$-coherent and admit splittings. By lemma 3.1, they are all finitely supported. As in [Muk, example 3.2, p. 158], we then have $R^iS^2_2(\mathcal{G}) = 0$ for $i \neq g$, and $\mathcal{G}$ is then the Fourier-Mukai transform of a finitely supported sheaf. By [Muk, 3.1], $\mathcal{G}$ is homogeneous. \qed

Remark The converse of the statement above is also part of Matsushima’s theorem. That the converse can be proved by the Fourier-Mukai transform is already noted in [N2, prop 5.9].

4. Curvature tensor and the integrable case

If $(\mathcal{G}, \nabla)$ is a sheaf with connection on $Y$, then its curvature tensor is a linear map

$$[
\nabla, \nabla]: \wedge^2(\mathfrak{g}) \longrightarrow \text{End}(\mathcal{G}) .$$

Before explaining how the curvature can be read off from the transform of $\mathcal{G}$, we want to show that the functor $S_1$ has the Čech description given in the introduction. Let $\psi_1$ and $\psi_2$ be two splittings on a given sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, and let $\eta = \psi_1 - \psi_2$. By proposition 1.1, $\eta$ is a map

$$\eta: \mathfrak{g} \longrightarrow \text{End}(\mathcal{F}) .$$

Applying the Fourier-Mukai transform to $\text{End}(\mathcal{F})$, we get

$$S_1(\eta): \mathfrak{g} \longrightarrow \text{End}(S_1(\mathcal{F})) .$$

Denoting the two connections on $S_1(\mathcal{F})$ by $\nabla_1$ and $\nabla_2$, it is easy to check that

$$\nabla_2 = \nabla_1 - S_1(\eta) .$$

Now let $U \hookrightarrow X$ be an affine open subset, equipped with a section $\rho \in \Gamma(\mathcal{U}, \mathcal{E})$ such that $\mu(\rho) = 1$. Then $(\iota_* (\mathcal{O}_U), \rho)$ is an object in $\text{Mod}(\mathcal{O}_X)_{sp}$. The corresponding object in $\text{Mod}(\mathcal{O}_Y)_{cxn}$ is a connection on $\alpha_2_*(\mathcal{P}|_{U \times Y})$. Since functions on $\mathcal{U}$ act as endomorphisms of $(\iota_* (\mathcal{O}_U), \rho)$, this connection is linear over $\alpha_2_*(\mathcal{O}_U)$. So we in fact have a connection relative to $U$ on $\mathcal{F}|_{U \times Y}$. Call this connection $\nabla^\rho$. If $\mathcal{F}$ is any $\mathcal{O}_X$-module, $\rho$ induces a splitting on $\mathcal{F}|_U$. Then if $\psi$ is any other splitting on $\mathcal{F}$, it must take the form

$$\psi(\cdot) = \cdot \otimes \rho + \sum_{i=1}^q \omega^i \psi(i)^\rho(\cdot) ,$$

(4.5)
for some endomorphisms $\psi(i)^\rho$. Again the corresponding sheaf with connection on $Y$ is the direct image of a sheaf with relative connection on $X \times Y$, and by (4.4) it has the form of (1.11).

It is now easy to read off the curvature tensor of $S_1(F, \psi)$. There is an exact sequence

$$0 \longrightarrow \wedge^2 g^* \otimes \mathcal{O} \longrightarrow \wedge^2 \mathcal{E} \xrightarrow{\mu_2} g^* \otimes \mathcal{O} \longrightarrow 0,$$

(4.6)

If we iterate $\psi$ and then skew-symmetrize, we get a map

$$F \left[\psi, \psi\right] \longrightarrow \wedge^2 (E) \otimes F.$$  

(4.7)

One sees easily that $(\mu_2 \otimes 1) \circ [\psi, \psi] = 0$, so

$$[\psi, \psi] \in \wedge^2 g^* \otimes \text{End}(F).$$  

(4.8)

As noted in the introduction, in terms of the family of endomorphisms $\psi(i)_n$, one simply has

$$[\psi, \psi] = \sum \omega^i \wedge \omega^j [\psi(i)_n, \psi(j)_n].$$  

(4.9)

Applying the functor $S_1$ to morphisms, we get $S_1([\psi, \psi]) \in \wedge^2 g^* \otimes \text{End}(S_1(F)).$ Set $S_1(F, \psi) = (S_1(F), \nabla_\psi)$. Let $[\nabla, \nabla]$ denote curvature.

**Proposition 4.1.** $[\nabla_\psi, \nabla_\psi] = S_\infty([\psi, \psi]).$

**Proof.** First we claim that the relative connections $\nabla_n$ on $\mathcal{P}|_{\mathcal{U}_n \times Y}$ are integrable. Indeed, from (1.11), the curvature tensor of $\nabla_n$ is independent of the index $n$. To show that it is 0, we have only to consider the case where $0 \in \mathcal{U}_n$ and note that all connections on the trivial bundle $\mathcal{O}_Y$ are integrable.

Denoting the relative connection on $\mathcal{P} \otimes \alpha^*_i(-1)^*(\mathcal{F})$ by $\nabla = \sum \omega^i \nabla(i)$,

$$[\nabla(i), \nabla(j)] = [\nabla(i)_n \otimes 1 - 1 \otimes \psi(i)_n, \nabla(j)_n \otimes 1 - 1 \otimes \psi(j)_n]$$

$$= 1 \otimes [\psi(i)_n, \psi(j)_n].$$  

(4.10)

Similarly, given $(\mathcal{G}, \nabla)$ in $\text{Ob Mod}(\mathcal{O}_Y)_{\text{c.x.n}}$, we have $S_\varepsilon([\nabla, \nabla]) \in \wedge^6 (g^*) \otimes \mathcal{E}\backslash(\mathcal{S}_\varepsilon(\mathcal{G}))$. Setting $S_2(\mathcal{G}, \nabla) = (S_2(\mathcal{G}), \psi_\nabla)$, one proves

**Proposition 4.2.** $[\psi_\nabla, \psi_\nabla] = S_2([\nabla, \nabla]).$

Turning now to the integrable case, consider the group extension

$$0 \longrightarrow g^* \longrightarrow X^2 \pi \longrightarrow X \longrightarrow 0$$

(4.11)
mentioned in the introduction. The morphism \( \pi \) being affine, \( X^\natural \) is characterized by \( \pi_* \) of its structure sheaf. Define a sheaf \( \mathcal{A} \) of \( \mathcal{O}_X \)-modules as follows. For each affine open \( U \subset X \) and each \( \rho \in E(U) \) such that \( \mu(\rho) = 1 \), introduce independent variables \( x_1^\rho, \ldots, x_g^\rho \). Then

\[
\mathcal{A}|_U = \mathcal{O}|_U[x_1^\rho, \ldots, x_g^\rho].
\]

Glue these sheaves together by the rule that if \( \tilde{\rho} = \rho + \sum_{i=1}^g \omega_i f_i \), then

\[
x_i^\rho = x_i^{\tilde{\rho}} + f_i
\]

as sections of \( \mathcal{A} \).

**Definition 4.3.** \( X^\natural = \text{Spec}(\mathcal{A}) \).

**Proposition 4.4.** The full subcategory of \( \text{Mod}(\mathcal{O}_X)_{sp} \) whose objects \( (\mathcal{F}, \psi) \) satisfy \([\psi, \psi] = 0\) is canonically isomorphic to the category of \( \mathcal{O} \)-modules on \( X^\natural \), which is to say the category of \( \mathcal{A} \)-modules on \( X \).

**Proof.** Let \( \rho \) be as above. Since \( \mathcal{A} \) is locally \( \mathcal{O}[x_1^\rho, \ldots, x_g^\rho] \), to give an \( \mathcal{A} \)-module structure to an \( \mathcal{O}_X \)-module \( \mathcal{F} \) is to choose, for every \( \rho \), a set of commuting endomorphisms \( \psi_1^\rho, \ldots, \psi_g^\rho \in \Gamma(U, \mathcal{E}nd(\mathcal{F})) \), such that if

\[
\tilde{\rho} = \rho + \sum_{i=1}^g \omega_i f_i,
\]

then \( \psi_i^\rho - \psi_i^{\tilde{\rho}} = \text{multiplication by } f_i \). This is clearly the same as lifting \( \mathcal{F} \) to an object \( (\mathcal{F}, \psi) \in \text{Ob } \text{Mod}(\mathcal{O}_X)_{sp} \) such that \([\psi, \psi] = 0\). \( \square \)

Now we may restrict our functors \( S_1 \) and \( S_2 \) to the subcategories \( \text{Mod}(\mathcal{O}_X^\natural) \subset \text{Mod}(\mathcal{O}_X)_{sp} \) and \( \text{Mod}(\mathcal{D}_Y) \subset \text{Mod}(\mathcal{O}_Y)_{cxn} \) respectively, to get functors

\[
\begin{align*}
\text{Mod}(\mathcal{O}_X^\natural) & \xrightarrow{S_1} \text{Mod}(\mathcal{D}_Y) \\
\text{Mod}(\mathcal{D}_Y) & \xrightarrow{S_2} \text{Mod}(\mathcal{O}_X^\natural)
\end{align*}
\]

(4.15)

We then get our second equivalence theorem, whose proof is the same as that of the first.

**Theorem 4.5.** The derived functors

\[
\begin{align*}
D^b \text{Mod}(\mathcal{O}_X^\natural) & \xrightarrow{RS_1} D^b \text{Mod}(\mathcal{D}_Y) \\
D^b \text{Mod}(\mathcal{D}_Y) & \xrightarrow{RS_2} D^b \text{Mod}(\mathcal{O}_X^\natural)
\end{align*}
\]

(4.16) \hspace{1cm} (4.17)

exist, and satisfy

\[
\begin{align*}
RS_1 RS_2 &= T^{-g}, \\
RS_2 RS_1 &= T^{-g}.
\end{align*}
\]
5. Some examples

5.1. Using theorem 4.5, one easily recovers the observation that $X^\natural$ is the moduli space of degree-0 line bundles on $Y$ equipped with a connection [Me]. Indeed, let $\sigma : Z \to X^\natural$ be any morphism and let $\sigma' = \pi \circ \sigma$. Let $P_{\sigma'}$ be the line bundle on $Z \times Y$ induced by $\sigma'$. Then the $\cD_Y$-module $S_1(\sigma_*(\cO_Z))$ is in fact the direct image of a flat connection relative to $Z$ on $\cP_{\sigma'}$. In particular, we have the 1:1 correspondence

$$\text{points of } X^\natural \leftrightarrow \text{line bundles with connection on } Y .$$

5.2. Let $J \subset \text{Sym}(\mathfrak{g})$ be an ideal. Then $\text{Spec}(\text{Sym}(\mathfrak{g})/J)$ is a subvariety of the fiber $\pi^{-1}(0)$, and we may apply $S_1$ to its structure sheaf. This gives a $\cD$-module structure on $\cO_Y \otimes_k \text{Sym}(\mathfrak{g})/J$. Writing sections of $\cD$ in the standard format with functions on the left and invariant differential operators on the right, make the identification

$$\cD \simeq \cO_Y \otimes_k \text{Sym}(\mathfrak{g}) . \quad (5.1)$$

Then $\cO_Y \otimes_k J$ is a left ideal, and $S_1(k(0) \otimes_k \text{Sym}(\mathfrak{g})/J)$ is the quotient $\cD$-module. In particular,

$\cD_Y$ is the Fourier-Mukai transform of the structure sheaf of the subgroup $\mathfrak{g}^* = \pi^{-1}(0) \subset X^\natural$. 

5.3. Going the other way, we may ask which $\cD_Y$-module transforms to $\cA$. By theorem 4.5 this is the same as asking for the Fourier-Mukai transform of $\cA$. Observe first that there is an important filtration $\{\cA(m)\}$ on $\cA$ coming from the local identifications of $\cA$ with a polynomial algebra over $\cO$. One has exact sequences

$$0 \to \cA(m) \to \cA(m + 1) \to \text{Sym}^{m+1}(\mathfrak{g}) \otimes_k \cO_X \to 0 . \quad (5.2)$$

We then have

**Lemma 5.1.** $R^i S_1(\cA) = 0$ for $i \neq g$, and $R^g S_1(\cA)$ is supported at the origin.

**Proof.** By (5.2) the question reduces to the corresponding assertion about $\cO$. The latter is proved in [Mum, p.126]. $\square$

It remains to identify the $\cD_Y$-module $R^g S_1(\cA)$. If $\cG$ is an $\cO_Y$-module, we may form $\cD \otimes_\cO \cG$, using right multiplication by functions as the $\cO$-module structure on $\cD$. Then $\cD \otimes_\cO \cG$ has a left $\cD$-module structure. In particular, define

$$\cD_{(0)} \overset{\text{def}}{=} \cD \otimes k(0) . \quad (5.3)$$
Proposition 5.2. $R^g S_1(\mathcal{A}) = \mathcal{D}_{(0)\to Y}$.

First proof. By theorem [4.3], it suffices to prove that $R^g S_2(\mathcal{D}_{(0)\to Y}) = \mathcal{A}$. Explicitly, $\mathcal{D}_{(0)\to Y}$ is the sheaf of $k$-vector spaces $\text{Sym}(g)(0)$, with the following $\mathcal{D}$-module structure. Given $L \in \text{Sym}(g)$ and $\xi \in g$, $\xi$ acts on $L$ by $\xi \cdot L = \xi L$. To give the action of $\mathcal{O}_Y$, we need only consider $f \in \mathcal{O}_{Y,0}$. Expand the differential operator $fL$ in the form $\sum \xi^I g_I$, $g_I \in \mathcal{O}_{Y,0}$. Define $fL := \sum \xi^I g_I(0) \in \text{Sym}(g)$.

Then $fL =: fL$. Let $\mathcal{F} = R^g S_2(\mathcal{D}_{(0)\to Y})$. We want to prove that $\mathcal{F}$ is a free rank-1 $\mathcal{A}$-module. We have the global section $1 \in \text{Sym}(g) = H^0(Y, \mathcal{D}_{(0)\to Y})$.

Let $U \subset X$ be an open subset and let $\sigma$ be a meromorphic section of $\mathcal{P}$ on $U \times Y$ such that $\sigma(0) = 1$, where $\sigma(0) = \sigma|_{U \times \{0\}}$. Since $\sigma$ is regular on the support of $\alpha^2_2(\mathcal{D}_{(0)\to Y})$, there is a well-defined section $\sigma \otimes 1 \in \Gamma(U \times Y, \mathcal{P} \otimes \alpha^2_2(\mathcal{D}_{(0)\to Y})) = \Gamma(U, \mathcal{F})$.

If $f$ is a meromorphic function on $U \times Y$, regular on $U \times \{0\}$, then $f \sigma \otimes 1 = \sigma \otimes : f 1 : = f(0) \sigma \otimes 1$.

Thus $\sigma \otimes 1$ is independent of $\sigma$, and defines a global section of $\mathcal{F}$. Now let $\rho \in \Gamma(U, \mathcal{E})$ satisfy $\mu(\rho) = 1$. Then $\rho$ determines both a relative connection $\nabla$ on $\mathcal{P}|_{U \times Y}$ and a set of sections $x(1), \ldots, x(g) \in \Gamma(U, \mathcal{A})$, such that $\mathcal{A}|_{U} = \mathcal{O}[x(1), \ldots, x(g)]$. By definition of the $\mathcal{A}$-module structure on $\mathcal{F}$,

$$x(i_1) \ldots x(i_k)(\sigma \otimes L) = x(i_1) \ldots x(i_{k-1})(\nabla_{i_k}(\sigma) \otimes L - \sigma \otimes \xi_{i_k} L).$$

(5.4)

It now follows by induction on $k$ that $\sigma \otimes 1$ generates $\mathcal{F}$ freely as a $\mathcal{A}$-module. □

A second proof will be given in section [3]

6. Coherence, Duality and Holonomicity

6.1. If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, we may form $\mathcal{D} \otimes \mathcal{O} \mathcal{G}$ using right multiplication by functions as the $\mathcal{O}_Y$-module structure on $\mathcal{D}$. This leaves us with left multiplication by elements of $\mathcal{D}$ to give a left $\mathcal{D}$-module structure. Similarly, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, we may form the $\mathcal{A}$-module $\mathcal{A} \otimes \mathcal{F}$. Note that

$$\text{Hom}_\mathcal{D}(\mathcal{D} \otimes \mathcal{O} \mathcal{G}, \cdot) = \text{Hom}_\mathcal{O}(\mathcal{G}, \text{for}(\cdot));$$

(6.1)

$$\text{Hom}_\mathcal{A}(\mathcal{A} \otimes \mathcal{F}, \cdot) = \text{Hom}_\mathcal{O}(\mathcal{F}, \text{for}(\cdot)).$$

(6.2)
where \( \text{for} \) is the forgetful functor.

**Theorem 6.1.**

\[
RS_2(D \otimes (\cdot)) = A \otimes RS_2(\cdot) . \tag{6.3}
\]

\[
RS_1(A \otimes (\cdot)) = D \otimes RS_1(\cdot) . \tag{6.4}
\]

**Proof.** By theorem 4.5, it suffices to prove the first equality. Let \( F_1 \) be an object in \( D_b \text{Mod}(A) \) and let \( F_2 \) be an object in \( D_b \text{Mod}(O_X) \). Using (6.1), (6.2), Mukai’s theorem and theorem 2.2,

\[
\text{Hom}(A \otimes F_2, F_1) = \text{Hom}(F_2, \text{for}(F_1))
\]

\[
= \text{Hom}(RS_\infty(F_\infty), \{\nabla(RS_\infty(F_\infty))\})
\]

\[
= \text{Hom}(D \otimes RS_1(F_2), RS_1(F_1))
\]

\[
= \text{Hom}(RS_2(D \otimes RS_\infty(F_\infty)), T^{-1}(F_\infty))
\]

\[
= \text{Hom}(RS_2(D \otimes T^g RS_1(F_2)), F_1) . \tag{6.5}
\]

Writing \( F_2 \) as \( RS_2(G) \) for some \( G \), we get

\[
\text{Hom}(A \otimes RS_2(G), F_1) = \text{Hom}(RS_2(D \otimes G), F_1) . \tag{6.6}
\]

The theorem is proved. \( \Box \)

**Remark** This result appears to conflict with the interchange of tensor and pontrjagin product stated in theorem Mukai2. Note, however, that in the definition of \( D \otimes G \), one is using the right \( O \)-module structure on \( D \) to define the tensor product and the left \( O \)-module structure on the resulting sheaf.

We digress to give a

**Short proof of Proposition 5.2.** By theorem 5.1,

\[
RS_2(D_{\{0\} \to Y}) = RS_2(D \otimes k(0))
\]

\[
= A \otimes RS_2(k(0))
\]

\[
= A \otimes O_X = A . \tag{6.7}
\]

\( \Box \)
A more important corollary is the following. Let \( D_{\text{coh}}^b \text{Mod}(A) \) (resp. \( D_{\text{coh}}^b \text{Mod}(D_Y) \)) denote the subcategory of complexes with \( A \)-coherent (resp. \( D \)-coherent) cohomology.

**Theorem 6.2.** The functors \( RS_1 \) and \( RS_2 \) restrict to equivalences between the categories \( D_{\text{coh}}^b \text{Mod}(A) \) and \( D_{\text{coh}}^b \text{Mod}(D_Y) \).

**Proof.** We give the proof in one direction, the other direction being the same. The category \( D_{\text{coh}}^b \text{Mod}(A) \) of complexes with coherent cohomology is generated by sheaves of the form \( A \otimes F \), where \( F \) is a coherent \( \mathcal{O}_X \)-module. It is well-known that \( RS_\infty(F) \) belongs to \( D_{\text{coh}}^b \text{Mod}(\mathcal{O}_Y) \). By theorem 6.1,

\[
RS_1(A \otimes F) = D \otimes RS_1(F) . \tag{6.8}
\]

This completes the proof, for \( D \otimes (-) \) maps \( D_{\text{coh}}^b \text{Mod}(\mathcal{O}_Y) \) to \( D_{\text{coh}}^b \text{Mod}(D_Y) \).

---

6.2. Given a complex \( F \in \text{Ob} \ D_{\text{coh}}^b \text{Mod}(D_Y) \), its dual complex is

\[
\Delta^{D_Y}(F) = R\text{Hom}_{D_Y}(F, T^g(D_Y)) . \tag{6.9}
\]

Note that \( \Delta^{D_Y}(F) \) is naturally a complex of right \( D \)-modules, but we regard it as a complex of left \( D \)-modules, using the antiinvolution

\[
D \rightarrow D
\]

\[
L = \sum f_I \xi^I \mapsto \sum (-1)^{|I|} \xi^I f_I . \tag{6.10}
\]

Given \( F \in \text{Ob} \ D^b \text{Mod}(A) \), define \( \Delta^A F = R\text{Hom}(F, T^g(A)) \). In particular,

\[
\Delta^A(A \otimes F) = A \otimes \Delta^{\mathcal{O}_X}(F) ; \tag{6.11}
\]

\[
\Delta^D(D \otimes F) = D \otimes \Delta^{\mathcal{O}_Y}(F) . \tag{6.12}
\]

**Proposition 6.3.**

\[
RS_2 \Delta^{D_Y} = T^{-g} \Delta^A RS_2 , \tag{6.13}
\]

\[
\Delta^{D_Y} RS_1 = RS_1 \Delta^A T^{-g} . \tag{6.14}
\]

**Proof.** By theorem 4.5 it suffices to prove the first equality. It also suffices to consider the case where \( F \) is an object in \( D^b \text{Mod}(D_Y) \) of the form \( F = D \otimes_{\mathcal{O}} G \), for some object \( G \in \text{Ob} \ D^b \text{Mod}(\mathcal{O}_Y) \). Then \( \Delta^D(F) = D \otimes \Delta^{\mathcal{O}_Y}(G) \). By [Muk, p. 161]

\[
RS_2 \Delta^{\mathcal{O}_Y} = T^{-g} \Delta^{\mathcal{O}_X} RS_2 . \tag{6.15}
\]
Then
\[ RS_2 \Delta^D(y(F)) = RS_2(D \otimes \Delta^O(y(G))) = A \otimes RS_2 \Delta^O(y(G)) \] (by theorem 6.1)
\[ = A \otimes T^{-g} \Delta \Omega RS_\varepsilon(G) \]
\[ = A \otimes T^{-g} R\text{Hom}_O(RS_2(G), T^g(O_X)) \]
\[ = T^{-g} R\text{Hom}_A(A \otimes RS_2(G), T^g(A)) \]
\[ = T^{-g} R\text{Hom}_A(RS_2(F), T^g(A)) = T^{-g} \Delta A T^{-g} RS_2(F). \]
(6.17)

6.3. For the rest of this section we assume \( \text{char}(k) = 0 \). Recall that a \( D \)-module \( F \) is said to be holonomic if its characteristic variety has the least possible dimension, namely \( g \).

**Proposition 6.4.** Let \( F \) be a coherent \( D \)-module. Then \( F \) is holonomic if and only if \( H^i(\Delta^D(F)) = 0 \) for \( i \neq 0 \).

**Proof.** [Bo, p. 230] \( \square \)

Since holonomicity of a \( D \)-module is a local condition, one expects it to be encoded globally when one takes the Fourier transform.

**Theorem 6.5.** Let \( F \in \text{Ob} \ D^b \text{Mod}(A) \) be a complex such that the cohomology of \( RS_1(F) \) is concentrated in a single degree \( i \) and \( R^i S_1(F) \) is \( D \)-coherent. Then \( R^i S_1(F) \) is holonomic if and only if the cohomology of \( RS_1 \Delta^A(F) \) is concentrated in degree \( g - i \).

**Proof.** Let \( \hat{F} = R^i S_1(F) \). Regarding \( \hat{F} \) as a complex in degree 0,
\[ H^j(\Delta^D(\hat{F})) = H^j(\Delta^D T^i RS_1(F)). \] Then by proposition 6.4, \( \hat{F} \) is holonomic if and only if \( H^j(\Delta^D T^i RS_1(F)) = 0 \) for \( j \neq 0 \). By proposition 6.3,
\[ H^j(\Delta^D T^i RS_1(F)) = H^j(\Delta^D RS_1 T^i(F)) \]
\[ = H^j(RS_1 \Delta^A T^{i-g}(F)) = H^j(RS_1 T^{g-i} \Delta^A(F)). \]
(6.18)

This vanishes for \( j \neq 0 \) if and only if \( R^l S_1 \Delta^A(F) \) vanishes for \( l \neq g - i \). \( \square \)

We leave the detailed study of this condition to a future work.
7. Characteristic Variety

In many important examples, it is possible to be quite explicit about the characteristic variety of the transform of a coherent \( A \)-module.

Let \( \{ A(m) \} \) be the filtration in example 5.3. If \( F \) is a coherent \( A \)-module, then one has the usual notion of good filtration with respect to \( \{ A(m) \} \). As the Fourier-Mukai transform exchanges local data for global data, it is worth noting that good filtrations exist globally.

**Proposition 7.1.** Let \( Z \) be a projective scheme of finite type over \( k \), and let \( A \) be a sheaf of \( \mathcal{O}_Z \)-algebras. Regarding \( A \) as an \( \mathcal{O} \)-module by letting \( \mathcal{O} \) act on the right, assume \( A \) is quasicoherent. If \( L \) is an ample line bundle on \( Z \) and \( M \) is a sheaf of coherent left \( A \)-modules, then there is presentation of \( M \) of the form

\[
(A \otimes L^{n_2})^r_2 \rightarrow (A \otimes L^{n_1})^r_1 \rightarrow M \rightarrow 0 .
\]

(7.1)

**Proof.** The proof is the same as for the case \( A = \mathcal{O} \). (cf. [H, p. 122])

In particular,

**Corollary 7.2.** If \( F \) is a sheaf of coherent \( A \) (resp. \( D_Y \)) modules, then \( M \) has a global good filtration by coherent \( \mathcal{O}_X \) (resp. \( \mathcal{O}_Y \)) modules.

Let \( F \) be a coherent \( A \)-module, and let \( \{ F_m \} \) be a good filtration. It follows from (5.2) that

\[
Gr A = Sym(g) \otimes \mathcal{O}_X .
\]

(7.2)

Thus, for each \( \xi \in g \) we have a homogeneous map of degree 1

\[
Gr F \xrightarrow{\xi} Gr F .
\]

(7.3)

Identifying \( A \) with \( \mathcal{O}[x(1), ..., x(g)] \) on a sufficiently small open set, we get a commutative diagram

\[
\begin{array}{ccc}
F_m & \xrightarrow{x(i)} & F_{m+1} \\
\downarrow & & \downarrow \\
Gr_m F & \xrightarrow{\xi_i} & Gr_{m+1} F
\end{array}
\]

(7.4)

Set \( R^j S_1(F, \psi) = (R^j S_1(F), \nabla^j) \), and set \( \nabla^j = \sum \omega^j \nabla^j(i) \). It follows from the explicit formula for \( S_1 \) that for all \( j \), the diagram

\[
\begin{array}{ccc}
R^j S_1(F_m) & \xrightarrow{\nabla^j(i)} & R^j S_1(F_{m+1}) \\
\downarrow & & \downarrow \\
R^j S_1(Gr_m F) & \xrightarrow{R^j S_1(\xi_i)} & R^j S_\infty(\mathcal{G} \nabla^{\infty}_{\infty+} F)
\end{array}
\]

(7.5)
commutes.

Let \( \mathcal{K}_m \) denote the kernel of the natural map
\[
\mathfrak{g} \otimes Gr_m\mathcal{F} \rightarrow Gr_{m+1}\mathcal{F}.
\] (7.6)

Let us say that the filtered \( \mathcal{A} \)-module \( \mathcal{F} \) satisfies filtered W.I.T with index \( i \) (cf. [Muk, p.156]) if \( R^j\mathcal{S}_1(\mathcal{F}) = 0 \) for \( j \neq i \) and the same is also true of \( \mathcal{F}_m \) and \( \mathcal{K}_m \) for \( m \) sufficiently large. Following Mukai, we denote the surviving cohomology sheaf of \( R\mathcal{S}_1(\mathcal{F}) \) by \( \hat{\mathcal{F}} \). Then we have
\[
0 \rightarrow \hat{\mathcal{K}}_m \rightarrow \mathfrak{g} \otimes \hat{Gr}_m\mathcal{F} \rightarrow \hat{Gr}_{m+1}\mathcal{F} \rightarrow 0 \quad (7.7)
\]
for \( m \) sufficiently large. By the preceding remarks, we therefore have

**Proposition 7.3.** Let \( \mathcal{F} \) be a filtered \( \mathcal{A} \)-module satisfying filtered W.I.T.. Then

1. \( \{\hat{\mathcal{F}}_m\} \) is a good \( \mathcal{D} \)-filtration on \( \hat{\mathcal{F}} \).
2. \( \hat{Gr}_m\mathcal{F} = Gr_m\hat{\mathcal{F}} \) for \( m \) sufficiently large.

Assume \( \mathcal{F} \) satisfies filtered W.I.T. with index \( i \), and let \( \mathcal{I}(\hat{\mathcal{F}}) \subset Sym(\mathfrak{g}) \otimes \mathcal{O}_Y \) denote the characteristic ideal sheaf of \( \hat{\mathcal{F}} \). Fix an affine open subset \( \mathcal{U} \subset Y \), let \( A = \Gamma(\mathcal{U}, \mathcal{O}_Y) \) and let \( I = \Gamma(\mathcal{U}, \mathcal{I}(\hat{\mathcal{F}})) \). This ideal may be described as follows. We have a map
\[
Sym(\mathfrak{g}) \xrightarrow{gr\psi} H^0(X, End(Gr\mathcal{F}))
\] (7.8)
coming from the \( GrA \)-module structure on \( Gr\mathcal{F} \). Redefining the filtration if necessary, we may assume that \( \mathcal{K}_m \) and \( \mathcal{F}_m \) satisfy W.I.T. for all \( m \). Since \( \mathcal{U} \) is affine, proposition \( \square \) gives
\[
\Gamma(\mathcal{U}, Gr\hat{\mathcal{F}}) = H^i(X \times \mathcal{U}, \mathcal{P} \otimes \alpha_\infty^*(G\nabla\mathcal{F}))
\] (7.9)
= \( H^i(X, \alpha_{1*}(\mathcal{P}|_{X \times U}) \otimes (Gr\mathcal{F})) \).

It is clear from the construction that the \( Sym(\mathfrak{g}) \)-module structure on \( \Gamma(\mathcal{U}, Gr\hat{\mathcal{F}}) \) is given by the composition
\[
Sym(\mathfrak{g}) \xrightarrow{gr\psi} H^0(X, End(Gr\mathcal{F})) \rightarrow H^0(X, \mathcal{E} \otimes (G\nabla\mathcal{F} \otimes \alpha_\infty^*(\mathcal{P}|_{X \times \mathcal{U}})))
\] (7.10)
\rightarrow End(H^i(X, Gr\mathcal{F} \otimes \alpha_{1*}(\mathcal{P}|_{X \times \mathcal{U}}))) .

Putting this together with the \( \mathcal{A} \)-module structure, we have a map
\[
A \otimes Sym(\mathfrak{g}) \xrightarrow{\lambda_\mathcal{U}} End(\Gamma(\mathcal{U}, Gr\hat{\mathcal{F}})) \quad (7.11)
\]
Thus we have

**Theorem 7.4.**
\[
\Gamma(\mathcal{U}, \mathcal{I}(\hat{\mathcal{F}})) = \sqrt{ker(\lambda_\mathcal{U})} \quad (7.12)
\]
We will make use of this result in section \( \square \).
8. The Krichever Construction

Let us briefly explain how the Krichever construction fits into the present framework. Assume now that $X = Y = \text{Jac}(C)$, where $C$ is a smooth curve of positive genus. Pick a base point $P \in C$ and let $C \xrightarrow{a} X$ be the associated abel map. Taking $\mathcal{F} = a_* (\mathcal{O}_C(*P))$, it is easy to see that $\mathcal{F}$ admits a $\mathcal{A}$-module structure. It suffices to take representative cocycles $\{c_{nm}(i)\}$ for a basis of $H^1(C, \mathcal{O})$ with respect to an open cover $\{U_n\}$, and solve the equations

$$c_{nm}(i) = f_n(i) - f_m(i) \quad (8.1)$$

with $f_n(i) \in \Gamma(U_n, \mathcal{O}(nP))$. (Note that we are identifying $\mathcal{O}(nP)$ with its endomorphism sheaf.) An important ingredient here is the flag on $H^1(C, \mathcal{O})$ coming from the natural maps

$$H^0(C, \mathcal{O}(nP)/\mathcal{O}) \rightarrow H^1(C, \mathcal{O}). \quad (8.2)$$

Denoting the images of these maps by $V_n$, we have a sequence of subsheaves of $\mathcal{D}$,

$$\mathcal{O}_V = \mathcal{D}_0 \subset \mathcal{D}_1 \subset ... , \quad (8.3)$$

where $\mathcal{D}_n$ is the $\mathcal{O}$-algebra generated by the vector fields belonging to $V_n$. In particular,

$$\mathcal{D}_1 = \mathcal{O}[\xi], \quad (8.4)$$

where $\xi$ is a basis of the one-dimensional space $V_1$. Now we may also filter $\mathcal{A}$ by subalgebras

$$\mathcal{O}_X = \mathcal{A}_0 \subset \mathcal{A}_1 \subset ... \quad (8.5)$$

in the same way, so that if we set

$$\mathcal{A}_i(m) = \mathcal{A}_i \cap \mathcal{A}(m) , \quad (8.6)$$

then

$$\text{Gr}_i \mathcal{A} = \text{Sym}(V_i) \otimes \mathcal{O}. \quad (8.7)$$

Theorem 4.5 extends to this more general situation:

The Fourier-Mukai transform gives an equivalence of categories

$$D^b\text{Mod}(\mathcal{A}_i) \leftrightarrow D^b\text{Mod}(\mathcal{D}_i). \quad (8.8)$$

Now there is an essentially canonical $\mathcal{A}_1$-module structure on $\mathcal{O}(nP)$. Indeed, let $z$ be a local parameter at $P$. Then $a^*(\mathcal{A}_1)$ is the subsheaf
of $\mathcal{O}(nP)[x]$ characterized by the following growth condition: If $U$ is a neighborhood of $P$, then
\begin{equation}
\Gamma(U, a^*(\mathcal{A}_1)) = \{ \sum f_i x^i \in \Gamma(U, \mathcal{O}(nP)[x]) \mid \text{for all } j, \sum_{i \geq j} \binom{i}{j} \frac{f_i}{z^{i-j}} \in \Gamma(U, \mathcal{O}) \} .
\end{equation}
(8.8)

Then $\mathcal{O}(nP)$ is in fact a sheaf of $\mathcal{A}_1|_C$-algebras under
\begin{equation}
\mathcal{A}_1|_C \rightarrow \mathcal{O}(nP)
\end{equation}
\begin{equation}
f(x) \mapsto f(x = 0) .
\end{equation}
(8.9)

The key to the Krichever construction is

**Proposition 8.1.** Let $\mathcal{G} = \overline{\mathcal{O}(nP)}$ regarded as $\mathcal{D}_1$-module. Then $\mathcal{G}|_{X-\Theta}$ is canonically isomorphic to $\mathcal{D}_1$.

This proposition is simply a translation into the language of this paper of well-known results. The canonical generator is the Baker-Akhiezer function. The discussion given here is similar to that of [R]. The important point is that
\begin{equation}
H^0(C, \mathcal{O}(nP)) = \text{End}_{\mathcal{A}_1}(\mathcal{O}(nP)) = \text{End}_{\mathcal{D}_1}(\mathcal{G}) .
\end{equation}
(8.10)

Thus, if we let $\chi$ denote the canonical generator of $\mathcal{G}|_{X-\Theta}$, then for all $f \in H^0(C, \mathcal{O}(nP))$, there exists a unique $L_f \in H^0(X-\Theta, \mathcal{D}_1)$ such that
\begin{equation}
f\chi = L_f \chi .
\end{equation}
(8.11)

This is the Burchnall-Chaundy representation of $H^0(C, \mathcal{O}(nP))$, done for all line bundles at once. The famous result is that the $L_f$'s satisfy the KP-hierarchy.

9. **Further Examples**

Nakayashiki has studied generalizations of the Krichever construction in which the curve and point are replaced by an arbitrary variety together with an ample hypersurface. In this section we will illustrate the results of section 2 using a somewhat more general version of his examples. In particular we will obtain some refinements of his results about characteristic varieties.

Let $Z$ be a smooth projective variety, and take $X$ to be its albanese variety. Assume for simplicity that the albanese map
\begin{equation}
Z \xrightarrow{a} X
\end{equation}
(9.1)
is an imbedding. Let $\mathcal{A}_Z = a^*(\mathcal{A})$. Then an $\mathcal{A}_Z$-module is the same as an $\mathcal{A}$-module supported on $Z$, so we have a functor
\begin{equation}
\mathcal{A}_Z\text{-modules} \rightarrow \mathcal{D}_Y\text{-modules} .
\end{equation}
Let $D \subset Z$ be an ample hypersurface. As in the Krichever construction, $\mathcal{O}(\ast D)$ may be given the structure of an algebra over $\mathcal{A}_Z$, making $\tilde{\mathcal{O}}(\ast D)$ a $\mathcal{D}$-module. (Note that $\mathcal{O}(\ast D)$ is W.I.T. of index 0.) Nakayashiki refers to such $\mathcal{D}$-modules as BA-modules.

In general, when one has a splitting on a sheaf $\mathcal{R}$, there is an induced map

$$g \rightarrow H^0(X, \mathcal{E}nd(\mathcal{R})/\mathcal{O}) \quad (9.2)$$

by virtue of (1.9). In the present example, this is a map

$$g \psi' \rightarrow H^0(Z, \mathcal{O}(\ast D)/\mathcal{O}) \quad (9.3)$$

splitting the natural map

$$H^0(Z, \mathcal{O}(\ast D)/\mathcal{O}) \rightarrow H^1(Z, \mathcal{O}) = g \quad (9.4)$$

Conversely, let $g \rightarrow H^0(Z, \mathcal{O}(rD)/\mathcal{O})$ be any left inverse of the natural map $H^0(Z, \mathcal{O}(rD)/\mathcal{O}) \rightarrow H^1(Z, \mathcal{O}) = g$. Then there is a splitting $\psi$ on $\mathcal{O}(\ast D)$ such that $\psi' = T$. We say that the splitting is associated to $T$. Equivalently, let $D_r$ denote the scheme $(D, \mathcal{O}/\mathcal{O}(rD))$ and let $\mathcal{M} = \mathcal{O}(\nabla D)/\mathcal{O}$, which we regard as a line bundle on $D_r$. Then we think of $\psi$ as being associated to the rational map

$$D_r \rightarrow \mathbb{P}(\mathfrak{g}^*) \quad (9.5)$$

induced by the linear system $T(\mathfrak{g})$.

Now let $\mathcal{O}(\ast D)$ be endowed with a $\mathcal{A}_Z$-module structure associated to a fixed map $T$. From $T$ we get a map

$$\mathfrak{g} \otimes \mathcal{O}_{D_r} \rightarrow \mathcal{M} \quad , \quad (9.6)$$

from which we may construct a Koszul complex (where $\mathcal{O} = \mathcal{O}_{D_r}$)

$$0 \rightarrow \wedge^g \mathfrak{g} \otimes \mathcal{M}^{-g+1} \rightarrow \cdots \rightarrow \wedge^2 \mathfrak{g} \otimes \mathcal{M}^{-1} \rightarrow \wedge^1 \mathfrak{g} \otimes \mathcal{O} \rightarrow \mathcal{M} \rightarrow 0 \quad . \quad (9.7)$$

Now let $\mathcal{H}$ be a sheaf of coherent $\mathcal{O}_Z$-modules, and set

$$\mathcal{F} = \mathcal{H} \otimes \mathcal{O}(\ast D) \quad , \quad (9.8)$$

regarded as a $\mathcal{A}_Z$-module. We make the simplifying hypothesis that the injections

$$\mathcal{O}(krD) \rightarrow \mathcal{O}((k + 1)rD) \quad (9.9)$$

induce injections

$$\mathcal{F}_k \rightarrow \mathcal{F}_{k+1} \quad , \quad (9.10)$$

where $\mathcal{F}_k = \mathcal{H} \otimes \mathcal{O}(krD)$. This is equivalent to
Assumption

\[ \text{Tor}^1(\mathcal{H}, \mathcal{O}_D) = 0. \]  (9.11)

This gives us a filtration on \( \mathcal{F} \), with associated graded sheaf

\[ \text{Gr}_k \mathcal{F} = \mathcal{H}|_{D_r} \otimes \mathcal{M}^k, \]  (9.12)

upon which \( \mathfrak{g} \) acts through the map \( T \). We want this to be a good filtration.

**Lemma 9.1.** The following are equivalent.

1. \( \mathfrak{g} \otimes \mathcal{H} \to \mathcal{M} \otimes \mathcal{H} \to 0 \) is exact.
2. The base locus of \( T(\mathfrak{g}) \) does not meet the support of \( \mathcal{H} \).
3. The complex \( (\wedge^i \mathfrak{g} \otimes \mathcal{M}^{1-i}) \otimes \mathcal{H} \) is exact
4. If we set \( \mathcal{R}_j = \ker(\wedge^j t) \), then
   \[ 0 \to \mathcal{R}_j \otimes \mathcal{H} \to \wedge^j \mathfrak{g} \otimes \mathcal{M}^{1-j} \otimes \mathcal{H} \to \mathcal{R}_{j-1} \otimes \mathcal{H} \to 0 \]
   is exact for all \( j \).
5. The filtration \( \{ \mathcal{F}_k \} \) is a good \( \mathcal{A}_Z \)-filtration.

**Proof.** The equivalence of 1 and 2 follows from Nakayama’s lemma. If 2 holds, then the Koszul complex is exact at points in the support of \( \mathcal{H} \), from which 3 follows. Since 1 is part of 3, the first three statements are equivalent. Then 4 follows because \( \wedge^j \mathfrak{g} \otimes \mathcal{M}^{1-j} \to \mathcal{R}_{j-1} \to 0 \) is exact at all points in \( \text{supp}(\mathcal{H}) \), and the \( \mathcal{R}_j \)’s are all projective. But 1 is part of 4, so 1 through 4 are equivalent. Now the filtration \( \mathcal{F}_k \) is good if and only if

\[ \mathfrak{g} \otimes \text{Gr}_k \mathcal{F} \to \text{Gr}_{k+1} \mathcal{F} \to 0 \]  (9.13)

is exact for large \( k \). But \( \text{Gr}_k \mathcal{F} = \mathcal{H}|_{D_r} \otimes \mathcal{M}^k \), so 1 and 5 are equivalent.

Assume now that \( \{ \mathcal{F}_k \} \) is a good \( \mathcal{A}_Z \)-filtration.

**Lemma 9.2.** \( \mathcal{F} \) satisfies filtered W.I.T. with index 0.

**Proof.** It is clear that \( \mathcal{F}_k \) satisfies W.I.T. with index 0 if \( k \) is sufficiently large. Let \( \mathcal{K}_k \) denote the kernel of (9.13). Since \( \mathcal{M} \) is invertible,

\[ \mathcal{K}_k = \mathcal{K}_1 \otimes \mathcal{M}^k. \]  (9.14)

Since \( \mathcal{M} \) is ample on \( D_r \), there exists \( k \) such that

\[ H^i(D_r, \mathcal{K}_j \otimes \mathcal{L}) = 0 \]  (9.15)

for any line bundle \( \mathcal{L} \) which is the pullback of a degree-0 line bundle on \( X \), any \( j \geq k \) and all \( i > 0 \). Thus for large \( k \), \( \mathcal{K}_k \) also satisfies W.I.T. with index 0. 

\[ \square \]
This puts us in the position to apply theorem 7.4. Let $U \subset Y$ be affine open and let $A = \Gamma(U, \mathcal{O}_Y)$. Let $\epsilon_1$ and $\epsilon_2$ denote the projections on $D_r \times Y$. We must study the graded $A \otimes \text{Sym}(g)$-module

$$H \overset{\text{def}}{=} \bigoplus_{k=0}^{\infty} H^0(D_r, \mathcal{H}|_{D_r} \otimes \epsilon_1^*(\mathcal{P}|_{D_r \times U}) \otimes \mathcal{M}^k).$$ (9.16)

Let

$$S = H^0(D_r, \bigoplus_{k=0}^{\infty} \mathcal{M}^k).$$ (9.17)

Then $H$ is a graded $S$-module. Moreover, we have the map

$$g \overset{T}{\rightarrow} H^0(Z, \mathcal{O}(rD)/\mathcal{O}) = H^0(D_r, \mathcal{M}),$$ (9.18)

which induces

$$\text{Sym}(g) \rightarrow \text{Sym}(H^0(D_r, \mathcal{M})).$$ (9.19)

Then $\text{Sym}(g)$ acts on $H$ through the composition of (9.19) with the natural homomorphism

$$\text{Sym}(H^0(D_r, \mathcal{M})) \rightarrow S.$$ (9.20)

If we apply Proj to the composite map $\text{Sym}(g) \rightarrow S$, we recover the rational map

$$D_r \overset{\Psi}{\rightarrow} \mathbb{P}(g^*).$$ (9.21)

associated to the linear map $T$, (9.18). Moreover, applying Proj to the graded $A \otimes S$-module $H$, we get the sheaf

$$\overline{H} = (\alpha_1^*(\mathcal{H}) \otimes \mathcal{P})|_{D_r \times U}$$ (9.22)

on $D_r \times U$. Thus, the consideration of $H$ as an $A \otimes \text{Sym}(g)$-module may be viewed on the sheaf level as taking the direct image of $(\alpha_1^*(\mathcal{H}) \otimes \mathcal{P})|_{D_r \times U}$ under the map $\Psi \times 1$. (Recall that by lemma 9.1, $\Psi$ is defined on the support of $\mathcal{H}|_{D_r}$.)

The discussion above gives us the main result of this section.

**Theorem 9.3.** Let $\mathcal{G} = \mathcal{F}$, where $\mathcal{F} = \mathcal{H} \otimes \mathcal{O}_Z(*D)$, $\text{Tor}^1(\mathcal{H}, \mathcal{O}_D) = 0$, and the $\mathcal{A}_Z$-module structure on $\mathcal{O}(*D)$ is associated to $\Psi : D_r \rightarrow \mathbb{P}(g^*)$. Then the characteristic variety of $\mathcal{G}$, viewed as a subvariety of $\mathbb{P}(g^*) \times Y$, is the support of the sheaf

$$\text{Gr} \mathcal{G} = (\Psi \times 1)_*(\alpha_1^*(\mathcal{H}) \otimes \mathcal{P})|_{D_r \times Y}.$$ (9.23)
In [N2] the case $Z = X$ is considered, and it is proved that the codimension of the characteristic variety is $\dim(X) - \dim \text{Supp}(\mathcal{H})$. The more general theorem 9.3 yields quite detailed information in this case, and in particular has Nakayashiki’s result as a corollary. We are now dealing with a smooth, ample hypersurface $D \subset X$. Then there is a natural class of $\mathcal{A}$-module structures on $\mathcal{O}(\ast D)$. These are described in [N2] in terms of factors of automorphy, but may also be seen somewhat more geometrically. Let $\mathfrak{h}$ be the space of vector fields on $X$. We have the Gauss map

$$D \xrightarrow{\Psi} \mathbb{P}(\mathfrak{h}^*) .$$

(9.24)

The normal bundle to $D$ is $\Psi^*(\mathcal{O}(1))$, and thus we have a linear map

$$\mathfrak{h} \xrightarrow{\lambda} H^0(D, \mathcal{N}) .$$

(9.25)

The composition of $\lambda$ with the canonical map $H^0(D, \mathcal{N}) \rightarrow \mathfrak{g}$ is an isomorphism. Thus one may consider $\mathcal{A}$-module structures on $\mathcal{O}(\ast D)$ associated to the Gauss map. That is, we can choose the $\mathcal{A}$-module structure in such a way that the induced map

$$\mathfrak{g} \xrightarrow{\psi} H^0(X, \mathcal{O}(\ast D)/\mathcal{O})$$

(9.26)

is precisely the composition

$$\mathfrak{g} \simeq \mathfrak{h} \xrightarrow{\lambda} H^0(D, \mathcal{N}) \subset H^0(X, \mathcal{O}(\ast D)/\mathcal{O}) .$$

(9.27)

We will call such an $\mathcal{A}$-module structure canonical. In the language of this paper, Nakayashiki’s $\mathcal{D}$-modules are obtained as the Fourier-Mukai transform of $\mathcal{A}$-modules of the form $\mathcal{H} \otimes \mathcal{O}(\ast D)$, where $\mathcal{O}(\ast D)$ has a canonical $\mathcal{A}$-module structure. However, one may as well consider a more general class of $\mathcal{A}$-module structures, namely those associated to any $g$-dimensional linear system $V \subset H^0(D, \mathcal{N})$ which is basepoint-free and maps isomorphically onto $\mathfrak{g}$. The example of the Gauss map shows that this is the generic situation. As a corollary of theorem 9.3, we have

**Theorem 9.4.** Let $D \subset X$ be a smooth ample hypersurface. Let $V \subset H^0(D, \mathcal{N})$ be a $g$-dimensional basepoint-free linear system mapping isomorphically onto $\mathfrak{g}$, and let $\Psi : D \rightarrow \mathbb{P}(\mathfrak{g}^*)$ be the corresponding morphism. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module such that $\text{Tor}^1(\mathcal{H}, \mathcal{O}_D) = 0$, and set $\mathcal{G} = \mathcal{H} \otimes \mathcal{O}(\ast D)$. Give $\mathcal{G}$ the $\mathcal{D}$-module structure induced by a $\mathcal{A}$-module structure on $\mathcal{O}(\ast D)$ associated to $V$. Then the characteristic variety of $\mathcal{G}$ is

$$\text{ss}(\mathcal{G}) = \Psi(\text{Supp}(\mathcal{H}|_D)) \times Y .$$

(9.28)

(In particular, $\text{codim}(\text{ss}(\mathcal{G}) = \dim(X) - \dim \text{Supp}(\mathcal{H})$.)
Proof. Because $\mathcal{N}$ is ample, the morphism $\Psi$ is finite. Therefore, tensoring $\mathcal{H}|_D$ with a line bundle has no effect on the support of its direct image. The result then follows from theorem 9.3.

10. SOME REMARKS ON COMMUTING RINGS OF MATRIX PARTIAL DIFFERENTIAL OPERATORS

In some sense, the origins of the present subject date to work of Burchnall and Chaundy on commuting rings of ordinary differential operators [BC]. Such rings are always of dimension one. As Nakayashiki has observed in [N2], the Fourier-Mukai transform allows one to represent the ring $H^0(X, \mathcal{O}(*D))$, $X$ an abelian variety and $D$ a smooth ample hypersurface, by matrix valued partial differential operators in $g$ variables, the size of the matrix being the $g$-fold self-intersection number, $D^g$. We want to offer some further observations on this question.

Consider again the data $(Z, D, \mathcal{H})$ in section 8. Fix an integer $r$ and a subspace $V \subset H^0(Z, \mathcal{O}(rD)/\mathcal{O})$ mapping isomorphically onto its image under the natural map

$$H^0(Z, \mathcal{O}(rD)/\mathcal{O}) \longrightarrow \mathfrak{g}.$$  \hspace{1cm} (10.1)

As in section 8, we get a subsheaf $\mathcal{A}_1 \subset \mathcal{A}$ by imitating the construction of $\mathcal{A}$, replacing $\mathfrak{g}$ by $V$ throughout. Similarly, we have a subsheaf $\mathcal{D}_1 \subset \mathcal{D}_Y$ generated over $\mathcal{O}_Y$ by the vector fields belonging to the image of $V$. As in section 8, we have a rational map

$$D_r \xrightarrow{\Psi} \mathbb{P}(V^*) .$$  \hspace{1cm} (10.2)

Assuming now that $\text{supp}(\mathcal{H})$ does not meet the baselocus of $\Psi$, we can introduce a coherent $\mathcal{A}_1$-module structure on $\mathcal{F} = \mathcal{H} \otimes \mathcal{O}(rD)$, filtered by the submodules $\mathcal{H} \otimes \mathcal{O}(krD)$ exactly as before. We now have

$$\text{Gr}_1 \mathcal{A} = \text{Sym}(V) \otimes \mathcal{O}_Z$$  \hspace{1cm} (10.3)

$$D_r = \text{Spec}(\mathcal{O}/\mathcal{O}(-rD))$$  \hspace{1cm} (10.4)

$$\mathcal{M} = \mathcal{O}(rD)/\mathcal{O} \text{ (thought of as a line bundle on } D_r)$$  \hspace{1cm} (10.5)

$$\text{Gr} \mathcal{F} = \bigoplus_{l=0}^{\infty} \mathcal{H}|_{D_r} \otimes \mathcal{M}^l,$$  \hspace{1cm} (10.6)

where the $\text{Sym}(V)$-module structure on $\text{Gr} \mathcal{F}$ is defined by the inclusion

$$V \longrightarrow H^0(D_r, \mathcal{M}) .$$  \hspace{1cm} (10.7)

Set $\mathcal{G} = \hat{\mathcal{F}}$, regarded as a sheaf of $\mathcal{D}_1$-modules. We will examine conditions under which $\mathcal{G}$ is free in a neighborhood of a given line
bundle $\mathcal{L} \in Pic^0(Z)$. In order to have the equality

$$\widehat{Gr}\mathcal{F} = \text{Gr}\hat{\mathcal{F}}$$

(10.8)
in a neighborhood of $\mathcal{L}$, we make the assumptions

For all $k \geq 0$, $i > 0$, $H^i(Z, \mathcal{L} \otimes \mathcal{F}_k) = 0$. 

(10.9)

There exists $j \geq 2$ such that for all $i \neq j$, $H^i(Z, \mathcal{L} \otimes \mathcal{F}_{-1}) = 0$. 

(10.10)

For $\mathcal{G}$ to be free over $\mathcal{D}_1$ in a neighborhood of $\mathcal{L}$, it is sufficient that the fiber of $\text{Gr}\mathcal{G}$ at $\mathcal{L}$ be free over $\text{Sym}(V)$. We therefore have

**Proposition 10.1.** For $\mathcal{G}$ to be free over $\mathcal{D}_1$ in a neighborhood of $\mathcal{L}$, it is sufficient that

$$\bigoplus_{i=0}^{\infty} H^0(D_r, (\mathcal{L} \otimes \mathcal{H})|_{D_r} \otimes \mathcal{M}^i)$$

be freely generated as a module over $\text{Sym}(V)$.

Regarding the hypothesis of this proposition, we have the following purely algebraic lemma. Let $M$ be an arbitrary finitely generated graded $\text{Sym}(V)$-module, with $M_j = 0$ for $j < 0$. For all $j$ we have a complex

$$\wedge^2(V) \otimes M_{j-1} \xrightarrow{\beta_j} V \otimes M_j \xrightarrow{\alpha_j} M_{j+1} ,$$

(10.11)

where $\alpha_j$ is the action of $V$, and $\beta_j$ is defined by

$$v \wedge w \otimes m \mapsto v \otimes wm - w \otimes vm .$$

(10.12)

**Lemma 10.2.** $M$ is free over $\text{Sym}(V)$ if and only if the sequence (10.11) is exact for all $j$.

**Proof.** If $M = \bigoplus_{i=1}^{\ell} \text{Sym}(V)[n_i]$, where

$$\text{Sym}(V)[n_i] = \text{Sym}^{j+n_i}(V) ,$$

(10.13)

then the exactness of the complexes (10.11) follows from the well-known fact that the natural complex

$$\wedge^2(V) \otimes \text{Sym}^k(V) \to V \otimes \text{Sym}^{k+1}(V) \to \text{Sym}^{k+2}(V)$$

(10.14)
is always exact.
Conversely, suppose that (10.11) is always exact. For all \( j \), choose a subspace \( U_j \subset M_j \) complementary to the image of \( \alpha_{j-1} \). What we must show is that for all \( j \), the natural map
\[
\oplus_{i=1}^j \text{Sym}^i(V) \otimes U_{j-i} \xrightarrow{\gamma_j} M_j \tag{10.15}
\]
is injective, for then \( M \) is isomorphic to
\[
\oplus U_j \otimes_k \text{Sym}(V)[-j]. \tag{10.16}
\]
Given \( \ell \), assume \( \gamma_j \) is injective for \( j < \ell \). Then \( M_{\ell-1} \approx \bigoplus_{i=0}^{\ell-1} \text{Sym}^i(V) \otimes U_{\ell-1-i} \) and we have a commutative diagram
\[
\begin{array}{ccc}
V \otimes \left( \bigoplus_{i=0}^{\ell-1} \text{Sym}^i(V) \otimes U_{\ell-1-i} \right) & \xrightarrow{\delta} & \ell_{i=1} \text{Sym}^i(V) \otimes U_{\ell-1} \xrightarrow{\gamma_\ell} M_\ell \\
\downarrow & & \downarrow \alpha_{\ell-1} \\
\ell_{i=1} \text{Sym}^i(V) \otimes U_{\ell-i} & \xrightarrow{\gamma_\ell} & M_\ell
\end{array} \tag{10.17}
\]
Since \( M_{\ell-2} \approx \bigoplus_{i=0}^{\ell-2} \text{Sym}^i(V) \otimes U_{\ell-2-i} \), the exactness of (10.14) implies that \( \text{Ker}(\delta) = \text{Im}(\beta_{\ell-1}) \). Thus \( \ker(\delta) = \text{Ker}(\alpha_{\ell-1}) \), which says that \( \gamma_\ell \) is injective.

The geometric interpretation of this lemma is the following. Let \( S \) be a scheme over \( k \), \( V \) a finite dimensional vector space over \( k \), \( \mathcal{M} \) a line bundle on \( S \), and \( V \xrightarrow{T} H^0(S, \mathcal{M}) \) a linear map. As in section 9, we associate to \( T \) a Koszul complex
\[
\cdots \rightarrow \text{Sym}^3 V \otimes \mathcal{M}^{-2} \xrightarrow{\text{Sym}^2} \text{Sym}^2 V \otimes \mathcal{M}^{-1} \xrightarrow{\text{Sym}^1} V \otimes \mathcal{O} \xrightarrow{T} \mathcal{M} \rightarrow 0 \tag{10.18}
\]
Let \( \mathcal{R}_i = \ker(\wedge i) \). Let \( \mathcal{H} \) be a sheaf of \( \mathcal{O}_S \)-modules. Then we have complexes
\[
0 \rightarrow \mathcal{R}_1 \otimes \mathcal{H} \rightarrow V \otimes \mathcal{H} \rightarrow \mathcal{M} \otimes \mathcal{H} \rightarrow 0 \tag{10.19}
\]
\[
0 \rightarrow \mathcal{R}_2 \otimes \mathcal{H} \rightarrow \text{Sym}^2 V \otimes \mathcal{M}^{-1} \otimes \mathcal{H} \rightarrow \mathcal{R}_1 \otimes \mathcal{H} \rightarrow 0. \tag{10.20}
\]
As in lemma 9.1, these sequences are exact if and only if
\[
V \otimes \mathcal{H} \rightarrow \mathcal{M} \otimes \mathcal{H} \rightarrow 0 \tag{10.21}
\]
is exact. We therefore have

**Theorem 10.3.** Assume (10.21) is exact. Consider the graded \( \text{Sym}(V) \)-module
\[
M = \bigoplus_{j \geq 0} H^0(S, \mathcal{H} \otimes \mathcal{M}^j). \tag{10.22}
\]
Then $M$ is free over $\text{Sym}(V)$ if and only if

$$0 \to H^1(S, \mathcal{R}_2 \otimes \mathcal{H} \otimes \mathcal{M}) \to \wedge^2 V \otimes H^1(S, \mathcal{H} \otimes \mathcal{M}^{j-1})$$

is exact for all $j \geq 1$, and

$$H^0(S, \mathcal{R}_1 \otimes \mathcal{H}) = r.$$  (10.24)

**Proof.** The maps $\alpha_j$ and $\beta_j$ in (10.11) are given in this case by tensoring (10.19) and (10.20) with $M_i$ and taking cohomology:

$$0 \to H^0(S, \mathcal{R}_1 \otimes \mathcal{H} \otimes \mathcal{M}^i) \to V \otimes M_j \xrightarrow{\alpha_j} M_{j+1} \xleftarrow{\beta_j} \wedge^2 V \otimes M_{j-1}$$  (10.25)

Therefore, (10.11) is exact for $j \geq 1$ precisely when

$$H^1(S, \mathcal{R}_2 \otimes \mathcal{H} \otimes \mathcal{M}^i) \to \wedge^2 V \otimes H^1(S, \mathcal{H} \otimes \mathcal{M}^{j-1})$$  (10.26)

is injective. The second condition appears because we have defined $M$ as a sum over nonnegative $j$, but have not assumed that $H^0(S, \mathcal{H} \otimes \mathcal{M}^{-1}) = 0$. The theorem then follows from lemma 10.2. \qed

Let us put all these ingredients together. We have a map

$$V \xrightarrow{T} H^0(D_r, \mathcal{M}) , \quad \mathcal{M} = \mathcal{O}(\nabla \mathcal{D})/\mathcal{O} ,$$  (10.27)

giving us subsheaves $\mathcal{A}_1 \subset \mathcal{A}$, $\mathcal{D}_1 \subset \mathcal{D}$, and a $\mathcal{A}_1$-module structure on $\mathcal{O}(\ast \mathcal{D})$. We have a sheaf $\mathcal{H}$ such that $\text{Tor}^1(\mathcal{H}, \mathcal{O}_D) = 0$ and $\text{supp}(\mathcal{H})$ does not meet the baselocus of $\text{Im}(T)$. We have $\mathcal{G} = \mathcal{H} \otimes \mathcal{O}(\ast \mathcal{D})$ and $\mathcal{F}_k = \mathcal{H} \otimes \mathcal{O}(kr \mathcal{D})$. The Koszul complex associated to $T$ is therefore exact on the support of $\mathcal{H}|_{D_r}$, and we can apply the previous theorem in combination with theorem 10.1.

**Theorem 10.4.** For $\mathcal{G}$ to be free over $\mathcal{D}_1$ in a neighborhood of $\mathcal{L} \in \text{Pic}^0(Z)$, the following conditions are sufficient:

1. For all $k \geq 0$, $i > 0$, $H^i(Z, \mathcal{L} \otimes \mathcal{F}_k) = 0$.
2. There exists $j \geq 2$ such that for all $i \neq j$, $H^i(Z, \mathcal{L} \otimes \mathcal{F}_{-1}) = 0$.
3. $0 \to H^1(D_r, \mathcal{R}_2 \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{M}^j) \to \wedge^2 V \otimes H^1(D_r, \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{M}^{j-\infty})$ is exact for all $j \geq 1$
4. $H^0(D_r, \mathcal{R}_1 \otimes \mathcal{L} \otimes \mathcal{H}) = r$.

Nakayashiki’s embedding may now be recovered. As in section 9, we take $Z$ to be $X$ itself, and we assume $D$ is smooth. We take an $\mathcal{A}$-module structure on $\mathcal{O}(\ast \mathcal{D})$ associated to a $g$-dimensional basepoint-free linear system $V \subset H^0(D, \mathcal{N})$ mapping isomorphically onto $\mathfrak{g}$. We
take $\mathcal{H} = \mathcal{O}(D)$. This affects only the filtration, not the $A$-module structure. Thus

$$\mathcal{F}_k = \mathcal{O}((k+1)D) .$$

(10.28)

**Theorem 10.5.** Set $\mathcal{G} = \hat{\mathcal{O}}(\ast D)$. Then $\mathcal{G}$ is a free $\mathcal{D}$-module in a neighborhood of any $\mathcal{L} \neq \mathcal{O}$.

**Proof.** By theorem 10.4, it suffices to verify the following:

1. For all $i > 0$, $k > 0$, $H^i(X, \mathcal{L}(kD)) = 0$.
2. There exists $j \geq 2$ such that for all $i \neq j$, $H^i(X, \mathcal{L}) = 0$.
3. For all $j \geq 1$, $0 \to H^1(D, \mathcal{R}_j \otimes \mathcal{L} \otimes \mathcal{N}^{j+1}) \to \wedge^2(V) \otimes H^1(D, \mathcal{L} \otimes \mathcal{N}^{j})$ is exact, where $\mathcal{R}_j$ are the kernel sheaves of the Koszul complex $\wedge^j(V) \otimes \mathcal{N}^{\infty-j}$.
4. $H^0(D, \mathcal{R}_1 \otimes \mathcal{L} \otimes \mathcal{N}) = 0$.

Items 1 and 2 are well-known. See, for example, [Mum, sec. 13, sec. 16]. It then follows from the exact sequence

$$0 \to \mathcal{L}((k-1)D) \to \mathcal{L}(kD) \to \mathcal{L} \otimes \mathcal{N}^k \to 0$$

(10.29)

that for all $k$ and all $0 < i < g-1$, $H^i(D, \mathcal{L} \otimes \mathcal{N}^k) = 0$. Since $\mathcal{R}_{g-1} = \mathcal{N}^{1-g}$, it follows by descending induction that

$$H^i(D, \mathcal{R}_j \otimes \mathcal{L} \otimes \mathcal{N}^{i}) = 0 \text{ for } 0 < i < j \text{ and all } k.$$  

(10.30)

In particular, 3 holds. Taking $k = 1$ we get a stronger statement, also by descending induction:

$$H^i(D, \mathcal{R}_j \otimes \mathcal{L} \otimes \mathcal{N}) = 0 \text{ for } i < j.$$  

(10.31)

Thus 4 holds. □

From the standpoint of integrable systems, the relevant feature of a $\mathcal{D}$-module is its endomorphism ring. As we saw in section 8, if $C$ is a curve embedded in its Jacobian, $\hat{\mathcal{O}(\ast P)}$ is a $\mathcal{D}_1$-module, where $\mathcal{D}_1 = \mathcal{O}[\xi]$. Extending this $\mathcal{D}_1$-module structure to a $\mathcal{D}$-module structure, we have

$$H^0(C, \mathcal{O}(\ast P)) = \text{End}_A(\mathcal{O}_{\hat{C}(\ast P)}) = \text{End}_D(\hat{\mathcal{O}_{\hat{C}(\ast P)})} .$$

(10.32)

It is the analysis of this endomorphism ring which leads to the KP hierarchy. Indeed, having trivialized $\hat{\mathcal{O}_{\hat{C}(\ast P)}}$ as a $\mathcal{D}_1$-module, its $\mathcal{D}$-endomorphisms are then differential operators in one variable with $g-1$ parameters, satisfying certain nonlinear differential equations.
More generally, one may hope to associate dynamics to the endomorphism ring of a $\mathcal{D}$-module coherent over a proper subalgebra, $\mathcal{O}[\xi_1, \ldots, \xi_n] \subset \mathcal{D}$, $n < g$. Those modules which are free over the smaller algebra provide a natural starting point for such an investigation. Note that the presence of any nontrivial endomorphisms in such a setting is already a strong condition on the $\mathcal{D}$-module, but one which can easily be satisfied by the methods presented here. Such examples will be the object of study in the sequel.

Finally, by way of advertisement, we mention

**Example: The Fano Surface.**

If $Z \subset \mathbb{CP}^d$ is a smooth cubic hypersurface, then its family of lines $S = \{ \ell \in \text{Gr}(2, 5) \mid \ell \subset Z \}$ is a smooth surface $[\mathcal{C}]$. For generic $s \in S$ corresponding to a line $\ell_s$, the set $\{ t \in S \mid \ell_t \cap \ell_s = \{ pt \} \}$ is a smooth ample hypersurface $D_s$, the incidence divisor. We have isomorphisms

$$\text{Alb}(S) \approx \text{Pic}^0(S) \approx J(Z),$$

(10.33)

where $J(Z)$ is the intermediate Jacobian of $Z$. The Albanese map is identified with the assignment $s \rightsquigarrow D_s$, which gives an embedding $S \subset \text{Pic}^0(S)$. The dimension of $\text{Pic}^0(S)$ is 5.

Setting $\mathcal{N} =$ normal bundle of $D_s$ in $S$ and $T_s(S) =$ tangent space to $S$ at $s$, we have an isomorphism $T_s(S) \approx H^0(D_s, \mathcal{N})$. The image of the natural map $H^0(D_s, \mathcal{N}) \to H^1(S, \mathcal{O})$, gives a subspace

$$\mathbb{C}^5 \approx \mathbb{V} \subset \mathbb{H}^{0\mathbb{C}}(S, \mathcal{O}) \approx \mathbb{C}^d.$$

(10.34)

Set $\mathcal{D}_1 = \mathcal{O}[\xi_1, \xi_2] \subset \mathcal{D}$, where $\xi_1$ and $\xi_2$ are a basis for $V$. Then the conditions of theorem 10.4 are fulfilled with $\mathcal{H} = \mathcal{O}(2D)$. Thus $\mathcal{H}(\ast D)$ is a $\mathcal{D}$-module, locally free as a $\mathcal{D}_\infty$-module at a generic point. The rank of $\mathcal{H}(\ast D)$ as a $\mathcal{D}_1$-module is the degree of the map $D_s \to \mathbb{P}^{\mathbb{C}}$ corresponding to the linear system $H^0(D_s, \mathcal{N})$. This degree is also 5. Thus we have a representation of $H^0(S, \mathcal{O}(\ast D))$ as $5 \times 5$ matrix partial differential operators in two variables, with $3(= \dim(H^1(S, \mathcal{O})) - \dim(V))$ parameters.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602

E-mail address: rothstei@math.uga.edu