A New Way to Derive the Taub-NUT Metric with Positive Cosmological Constant

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Abstract

We investigate a biaxial Bianchi IX model with positive cosmological constant, which is sometimes called the Λ-Taub-NUT spacetime, whose exact solution is well known. The minisuperspace of biaxial Bianchi IX models admits two non-trivial Killing tensors that play an important role for deriving the Taub-NUT metric. We also give a brief discussion about the asymptotic behaviour of Bianchi IX models.

1 Introduction

The metric of a Bianchi IX spacetime is given as

$$ds^2 = -dt^2 + a^2(t)\omega_1^2 + b^2(t)\omega_2^2 + c^2(t)\omega_3^2,$$

(1.1)

where \((t, a, b, c)\) have dimensions of time or length, since we are setting the speed of light \(c\) (not the \(c(t)\) above) equal to unity, as well as Newton’s gravitational constant \(G\). \(\{\omega_k\}\) is a set of \(S^3\)-invariant one-forms,

$$\begin{align*}
\omega_1 &= \cos \psi \, d\theta + \sin \psi \sin \theta \, d\phi, \\
\omega_2 &= \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi, \\
\omega_3 &= \, d\psi + \cos \theta \, d\phi,
\end{align*}$$

(1.2)
obeying
\[ d\omega_1 = \omega_2 \land \omega_3 \quad \text{et cyc.,} \quad (1.3) \]
where \(0 \leq \theta \leq \pi\), \(0 \leq \phi < 2\pi\), \(0 \leq \psi < 4\pi\). The three principal circumferences of the distorted \(S^3\) are then \((4\pi a, 4\pi b, 4\pi c)\). On the other hand, the \(\Lambda\)-Taub-NUT spacetimes are originally introduced by \[1\] as spacetimes whose spatial topology is a biaxial \(S^3\) and which satisfy the Einstein equations with positive cosmological constant,
\[ R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (1.4) \]
The form of the solution depends on the choice of time coordinate; for example it is given by \[2, 3\] with an arbitrary constant \(C_0\) and arbitrary positive constant \(D_0\) as
\[ ds^2 = \frac{3D_0}{\Lambda} \left( -\frac{d\tau^2}{f(\tau)} + \frac{f(\tau)}{4} \omega_3^2 + \frac{\tau^2 + 1}{4} (\omega_1^2 + \omega_2^2) \right), \quad (1.5) \]
\[ f(\tau) = \frac{D_0 \tau^4 + 2(3D_0 - 2)\tau^2 + C_0 \tau + 4 - 3D_0}{1 + \tau^2}. \quad (1.6) \]
Thus the \(\Lambda\)-Taub-NUT spacetime is none other than a biaxial Bianchi IX spacetime with positive cosmological constant. In this paper we present a new derivation of (1.5) by considering the minisuperspace defined below.

## 2 Minisuperspace

The orthonormal components of the Ricci tensor of a Bianchi IX spacetime are given by \[5\] as
\[ R^0_0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}, \quad (2.1) \]
\[ R^1_1 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{a^4 - (b^2 - c^2)^2}{2a^2b^2c^2}. \quad (2.2) \]
Here an overdot is a derivative with respect to the proper time \(t\) in the metric (1.1). \(R^2_2\) and \(R^3_3\) are just permutations of \(R^1_1\), and off-diagonal elements of the Ricci tensor are zero. Let us write the dimensional variables \((t, a, b, c)\) in terms of dimensionless Misner variables \((\zeta, \alpha, \beta_a, \beta_b, \beta_c, \beta, \gamma)\) \[4\] as
\[ t = \sqrt{\frac{3}{\Lambda}} \zeta, \quad a = \sqrt{\frac{3}{\Lambda}} e^{\alpha + \beta_a}, \quad b = \sqrt{\frac{3}{\Lambda}} e^{\alpha + \beta_b}, \quad c = \sqrt{\frac{3}{\Lambda}} e^{\alpha + \beta_c}, \quad (2.3) \]
\[ \beta_a = \beta + \sqrt{3}\gamma, \quad \beta_b = \beta - \sqrt{3}\gamma, \quad \beta_c = -2\beta, \quad (2.4) \]
where \(\alpha\) tells how spatially large the model is, since the \(S^3\)-volume is \(16\pi^2 (3/\Lambda)^{3/2} e^{3\alpha}\), while \(\beta\) and \(\gamma\) describe how distorted \(S^3\) is.

The scalar curvature of the distorted \(S^3\) at one time is
\[ (3)R = \frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{2a^2b^2c^2}. \quad (2.5) \]
It is interesting to note from Heron’s formula that when the numerator is positive, it is the square of the area of a triangle of edge lengths \((2a, 2b, 2c)\), which are the principal circumferences divided by \(2\pi\). Then \((3) R\) is 8 times the square of the area of a triangle of edge lengths the principal circumferences \((4\pi a, 4\pi b, 4\pi c)\), divided by the square of the \(S^3\)-volume that is \(16\pi^2 abc\). When the numerator is negative, so \((3) R < 0\), the edge lengths do not obey the triangle inequality. Furthermore, when the triangle exists and is acute, all three of the \(S^3\) Ricci tensor eigenvalues, \((3) R^1, (3) R^2,\) and \((3) R^3\), are positive, but when the triangle is obtuse or does not exist, two of the Ricci tensor eigenvalues are negative.

Multiplying the \(S^3\) scalar curvature by a quantity proportional to the two-thirds power of the \(S^3\) volume gives the dimensionless quantity

\[
V = \frac{1}{6} (abc)^{\frac{4}{3}} R = \frac{1}{2\Lambda} e^{2\alpha} (3) R = \frac{1}{12} \left( 4e^{-2\beta} \cosh 2\sqrt{3}\gamma - 4e^{4\beta} \sinh 2\sqrt{3}\gamma - e^{-8\beta} \right). \tag{2.6}
\]

Now letting an overdot denotes a derivative with respect to the dimensionless time coordinate \(\zeta\), the Einstein equations (1.4) give three dimensionless 2nd-order equations

\[
\ddot{\alpha} = 3 - 3\dot{\alpha}^2 - 2V e^{-2\alpha}, \tag{2.7}
\]
\[
\ddot{\beta} = -3\dot{\alpha} \dot{\beta} + \frac{1}{2} \frac{\partial V}{\partial \beta} e^{-2\alpha}, \tag{2.8}
\]
\[
\ddot{\gamma} = -3\dot{\alpha} \dot{\gamma} + \frac{1}{2} \frac{\partial V}{\partial \gamma} e^{-2\alpha}, \tag{2.9}
\]

and one dimensionless 1st-order constraint equation,

\[
\dot{\alpha}^2 - \dot{\beta}^2 - \dot{\gamma}^2 = 1 - V e^{-2\alpha}. \tag{2.10}
\]

Note that by combining (2.10), its time derivative, and any two of (2.7)-(2.9), one can derive the remaining 2nd-order equation, so only the 1st-order constraint (2.10) and any two of the three 2nd-order equations are independent.

Note also that if we choose \(\gamma = \dot{\gamma} = 0\) as an initial condition, then \(\partial V / \partial \gamma = 0\) and \(\ddot{\gamma} = 0\), so \(\gamma\) remains zero for all time, which is just a biaxial model.

One notices that (2.7)-(2.10) are reproduced by the following action:

\[
S = \frac{1}{2} \int d\tau \left( N^{-1} e^{3\alpha} (-\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2) - N \left( e^{3\alpha} - e^\alpha V \right) \right), \tag{2.11}
\]

where now the dot denotes the derivative with respect to \(\tau\), and \(N\) is a Lagrange multiplier. The relation between \(\zeta\) and \(\tau\) is

\[
\frac{d}{d\zeta} = \frac{1}{N} \frac{d}{d\tau}. \tag{2.12}
\]

If we define

\[
\eta = N \left( e^{3\alpha} - V e^\alpha \right), \tag{2.13}
\]

then (2.11) becomes

\[
S = -\frac{1}{2} \int d\tau \left( \eta^{-1} \left( e^{6\alpha} - V e^{4\alpha} \right) (-\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2) - \eta \right). \tag{2.14}
\]
This is a relativistic point-particle action in three dimensions \((\alpha, \beta, \gamma)\) with mass \(m = 1\) and the minisuperspace metric
\[
    ds^2 = (e^{6\alpha} - V e^{4\alpha})(-d\alpha^2 + d\beta^3 + d\gamma^2).
\]  
This three-dimensional (or two-dimensional for biaxial case) curved space obtained from the four-dimensional Bianchi IX space is an example of a metric on minisuperspace whose geodesics give solutions of Einstein equations \([6]\). Therefore, time evolution of a Bianchi IX space with \(\Lambda\) is equivalent to particle motion along a geodesic curve in this minisuperspace. A more rigorous way to obtain (2.14) is shown for example in \([7]\).

3 Killing Tensors

We now investigate geometrical properties of the minisuperspace associated with the biaxial Bianchi IX model with \(\gamma = 0\), so \(a = b = (3/\Lambda)^{1/2}e^{\alpha+\beta}\) and \(c = (3/\Lambda)^{1/2}e^{2\alpha-2\beta}\). Let us first define null coordinates \((u, v)\) as
\[
    u = \alpha - \beta + \frac{1}{2} \ln 3 - \frac{2}{3} \ln 2, \quad v = \alpha + \beta + \frac{1}{2} \ln 3.
\]  
Then the minisuperspace metric, the nonzero terms of the Levi-Civita connection, and the Ricci scalar are respectively
\[
    ds^2 = -\frac{4}{27} U(u,v) du dv \text{ with } U(u,v) = e^{3u+3v} - e^{3u+v} + e^{6u-2v},
\]
\[
    \Gamma^u_{uu} = U^{-1} \partial_u U, \quad \Gamma^u_{vv} = U^{-1} \partial_v U,
\]
\[
    R = 81 U^{-3} e^{9u}(3 e^{-v} - 5 e^v).
\]

3.1 Killing Vectors

We first show the non-existence of a Killing vector. If \(K\) is a Killing vector, its components satisfy the Killing equations
\[
    \nabla_u K_u = \partial_u K_u - \Gamma^u_{uu} K_u = U \partial_u (U^{-1} K_u) = 0,
\]
\[
    \nabla_v K_v = \partial_v K_v - \Gamma^v_{vv} K_v = U \partial_v (U^{-1} K_v) = 0,
\]
\[
    \nabla_v K_u + \nabla_u K_v = \partial_v K_u + \partial_u K_v = 0.
\]  
The first two give, for arbitrary functions \(f(v)\) and \(g(u)\),
\[
    K_u = f(v) U, \quad K_v = g(u) U,
\]  
or equivalently
\[
    K^u = -\frac{27}{2} g(u), \quad K^v = -\frac{27}{2} f(v).
\]
In two dimensions, the fact that a Killing vector obeys \( \nabla_\mu K^\mu = 0 \) implies that the dual one-form

\[
L = \varepsilon_{\mu\nu} K^\mu dx^\nu = -f(v)Udu + g(u)Udv \tag{3.10}
\]

is closed, \( dL = 0 \), so that locally it is exact, \( L = dM(u, v) \). Integrating this along a line of constant \( v \) gives

\[
M = -f(v) \int Udu = -f(v) \left( \frac{e^{3u+v}}{3} - \frac{e^{3u} + e^{6u-2v}}{6} \right) + C(v), \tag{3.11}
\]

whereas integrating this along a line of constant \( u \) gives

\[
M = g(u) \int Udv = g(u) \left( \frac{e^{3u+v}}{3} - \frac{e^{3u+v} - e^{6u-2v}}{2} \right) + D(u). \tag{3.12}
\]

Only for \( f(v) = g(u) = 0 \) do these agree, so there is no nonzero Killing vector.

### 3.2 Rank-2 Killing Tensors

For a rank-2 Killing tensor, the Killing equations are

\[
\nabla_u K_{uu} = 0, \tag{3.13}
\]
\[
\nabla_v K_{vv} = 0, \tag{3.14}
\]
\[
\nabla_u K_{uv} + 2 \nabla_v K_{uu} = 0, \tag{3.15}
\]
\[
\nabla_v K_{uv} + 2 \nabla_u K_{vv} = 0. \tag{3.16}
\]

Similar to the calculations for a Killing vector, the first two equations give

\[
K_{uu} = f(v)U^2, \quad K_{vv} = g(u)U^2, \tag{3.17}
\]

while the last two indicate

\[
K_{uv} = A(u, v)U, \tag{3.18}
\]

\[
\partial_v A = -\frac{1}{2} \frac{dg}{du} U - g \partial_u U, \tag{3.19}
\]
\[
\partial_u A = -\frac{1}{2} \frac{df}{dv} U - f \partial_v U. \tag{3.20}
\]

If one integrates \((3.19)\) with respect to \( v \), \( A \) becomes

\[
A = a(u) - \frac{1}{2} \frac{dg}{du} \left( \frac{e^{3u+3v}}{3} - \frac{e^{3u} + e^{6u-2v}}{2} \right) - g(u) \left( e^{3u+3v} - 3e^{3u+v} - 3e^{6u-2v} \right), \tag{3.21}
\]

while from \((3.20)\), we have

\[
A = b(v) - \frac{1}{2} \frac{df}{dv} \left( \frac{e^{3u+3v}}{3} - \frac{e^{3u+v}}{3} + \frac{e^{6u-2v}}{6} \right) - f(v) \left( e^{3u+3v} - \frac{e^{3u+v}}{3} - \frac{e^{6u-2v}}{3} \right). \tag{3.22}
\]
One notices that the choice \( f(v) = 0 \) and \( g(u) = 2F_1 e^{-6u} \) is consistent with both (3.21) and (3.22) as follows:

\[
a(u) = F_0, \quad b(v) = A = F_0 + 3F_1 e^{-2v},
\]

where \( F_0 \) and \( F_1 \) are constants. Setting \( F_1 = 0 \) gives a Killing tensor proportional to the metric, whereas \( F_1 \neq 0 \) gives a nontrivial Killing tensor. \( F_0 = 0 \) and \( F_1 = 1 \) give the non-trivial rank-2 Killing tensor as

\[
K_{uu} = 0, \quad K_{vv} = 2 e^{-6u} U^2, \quad K_{uv} = 3 e^{-2v} U.
\]

### 3.2.1 Killing-Yano Tensors

Since the minisuperspace has two dimensions, the maximum rank of Killing-Yano tensors is also two, and there exists at most only one rank-2 Killing-Yano tensor. The Killing-Yano equations \( \nabla_{(\mu} f_{\nu)\rho} = 0 \) for \( f_{uv} = -f_{vu} \) give

\[
\nabla_u f_{uv} = \nabla_v f_{vu} = 0.
\]

These equations give

\[
\begin{align*}
\partial_u f_{uv} - \Gamma^u_{uu} f_{uv} &= 0 \Rightarrow f_{uv} = f(v) U, \\
\partial_v f_{vu} - \Gamma^v_{vv} f_{vu} &= 0 \Rightarrow f_{uv} = g(u) U.
\end{align*}
\]

Therefore \( f(v) = g(u) \) is a constant, and the associated Killing tensor \( K_{\mu\nu} = f_{\mu\rho} f_{\nu}^\rho \) is just proportional to the metric:

\[
K_{11} = K_{22} = 0, \quad K_{12} = K_{21} \propto U.
\]

### 3.3 Rank-4 Killing Tensors

Similar to the case of rank-2, tedious calculations show that the following rank-4 symmetric tensor with constants \( G_k \) is a Killing tensor

\[
\begin{align*}
K_{uuuu} &= 0, \quad (3.29) \\
K_{uuuv} &= 81 G_3 e^{-2v} U^3 \quad (3.30) \\
K_{uuvv} &= \left(G_0 + 3G_1 e^{-4v} + 2G_2 e^{-2v} \\
&\quad + G_3 \left(2 e^{6v} - 12 e^{4v} + 18 e^{2v} + 54 e^{6u-4v} - 72 e^{3u+v}\right)\right) U^2, \quad (3.31) \\
K_{uvuu} &= \left(3G_1 e^{-6u-2v} + G_2 e^{-6u} \\
&\quad + G_3 \left(18 e^{-3u+v} + 27 e^{-2v} - 6 e^{-3u+3v}\right)\right) U^3, \quad (3.32) \\
K_{vvuu} &= (2G_1 e^{-12u} + 12G_3 e^{-6u}) U^4. \quad (3.33)
\end{align*}
\]
Note that if \( G_0 \) alone is nonzero, the Killing tensor is proportional to the symmetric product of two metrics; if \( G_1 \) alone is nonzero, the Killing tensor is proportional to the symmetric product of \( K_{\mu\nu} \) from (3.24) with itself; if \( G_2 \) alone is nonzero, the Killing tensor is proportional to the symmetric product of \( g_{\mu\nu} \) and \( K_{\mu\nu} \); but if \( G_3 \neq 0 \), one gets a new nontrivial rank-4 Killing tensor. Here we shall set \( G_0 = G_1 = G_2 = 0, G_3 = 16 \).

4 Exact Solution

As shown above, the two nontrivial invariants of motion are

\[
E_1 = K_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 6 e^{-2v} U \left( \frac{dv}{d\tau} \right)^2 + 2 e^{-6u} U^2 \left( \frac{dv}{d\tau} \right)^2, \tag{4.1}
\]

\[
E_2 = K_{\mu\nu\rho\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 2^6 \cdot 3^4 e^{-2v} U^3 \left( \frac{du}{d\tau} \right)^3 \left( \frac{dv}{d\tau} \right)^3 + 2^6 \cdot 3^4 e^{-6u} U^4 \left( \frac{dv}{d\tau} \right)^4
\]

\[
+ 2^6 \cdot 3^4 \left( e^{6v} - 6 e^{4v} + 9 e^{2v} + 27 e^{6u-4v} - 36 e^{3u+v} \right) U^2 \left( \frac{du}{d\tau} \right)^2 \left( \frac{dv}{d\tau} \right)^2
\]

\[
+ 2^6 \cdot 3^4 \left( 6 e^{-3u+v} + 9 e^{-2v} - 2 e^{-3u+3v} \right) U^3 \frac{du}{d\tau} \left( \frac{dv}{d\tau} \right)^3, \tag{4.2}
\]

where we are setting the Lagrange multiplier \( \eta = 1 \) so that \( \tau \) becomes the proper time along timelike geodesics in the minisuperspace metric, giving

\[
\frac{4}{27} \frac{du}{d\tau} \frac{dv}{d\tau} U = 1. \tag{4.3}
\]

This \( \tau \) is not to be confused with the \( \Lambda \)-Taub-NUT time coordinate in the metric (1.5). By using (4.3), we can simplify the expressions of \( E_1 \) and \( E_2 \) to

\[
E_1 = \frac{81}{2} e^{-2v} + 2 e^{-6u} U^2 \left( \frac{dv}{d\tau} \right)^2, \tag{4.4}
\]

\[
E_2 = 3^{13} e^{-2v} \left( \frac{dv}{d\tau} \right)^2
\]

\[
+ 4 \cdot 3^7 \left( e^{6v} - 6 e^{4v} + 9 e^{2v} + 27 e^{6u-4v} - 36 e^{3u+v} \right) U^2 \left( \frac{dv}{d\tau} \right)^2
\]

\[
+ 2^4 \cdot 3^4 \left( 6 e^{-3u+v} + 9 e^{-2v} - 2 e^{-3u+3v} \right) U^3 \frac{du}{d\tau} \left( \frac{dv}{d\tau} \right)^3
\]

\[
+ 2^6 \cdot 3 e^{-6u} U^4 \left( \frac{dv}{d\tau} \right)^4. \tag{4.5}
\]
The right-hand side of (4.4) shows that \( E_1 > 0 \). The original constraint (2.10) in null coordinates is

\[
\frac{du}{d\zeta} \frac{dv}{d\zeta} = e^{-(3u+3v)} U.
\]  

(4.6)

Thus by comparing this with (4.3), the relation between \( d\zeta \) and \( d\tau \) is

\[
\frac{d}{d\zeta} = \frac{2U e^{-\frac{2}{3}(u+v)}}{3\sqrt{3}} \frac{d}{d\tau}.
\]  

(4.7)

However one can see from the form of (4.4) that it becomes simplified if a new time coordinate \( T \) is chosen as

\[
\frac{d}{dT} = \frac{2}{9} e^{-3u+2v} U \frac{d}{d\tau} = \frac{1}{\sqrt{3}} e^{-\frac{2}{3}u+\frac{7}{3}v} \frac{d}{d\zeta} = 2\sqrt{\Lambda} a(t)^3 c(t)^{-1} \frac{d}{dt}.
\]  

(4.8)

Then (4.4) can be rewritten as

\[
E_1 = \frac{81}{2} e^{-2v} + \frac{81}{2} e^{-4v} \left( \frac{dv}{dT} \right)^2
\]

\[
= \frac{81}{2} e^{-2v} + \frac{81}{8} \left( \frac{d}{dT} e^{-2v} \right)^2,
\]  

(4.9)

or

\[
\frac{1}{4} \left( \frac{d}{dT} e^{-2v} \right)^2 = \frac{2}{81} E_1 - e^{-2v}.
\]  

(4.10)

One can obtain the solution as

\[
e^{-2v} = C^2 - (T - T_0)^2 \quad \left( C = \sqrt{\frac{2E_1}{9}} \right),
\]  

(4.11)

where the range of \( T \) is

\[
T_0 - C \leq T \leq T_0 + C.
\]  

(4.12)

Note that \( T_0 \) is just a shift of time, so we choose \( T_0 = 0 \), and the inequalities above become equalities at past and future infinity for the proper time \( t \) of the biaxial Bianchi IX spacetime metric.

The constraint equation (4.3) then gives the solution for \( u \). In terms of the \( T \) coordinate, it becomes

\[
3 e^{3u} e^{-7v} \frac{du}{dT} \frac{dv}{dT} \equiv -\frac{1}{2} e^{-5v} \left( \frac{d}{dT} e^{3u} \right) \left( \frac{d}{dT} e^{-2v} \right)
\]

\[
= 1 - e^{-2v} + e^{3u-5v}.
\]  

(4.13)

By using (4.11), one gets

\[
\left( \frac{d}{dT} e^{3u} \right) = \frac{e^{3u} + e^{5v} - e^{3v}}{T},
\]  

(4.14)

which has the solution

\[
e^{3u} = B T + \left( \frac{6C^2 - 8)T^4 + (12C^2 - 9C^4)T^2 + 3C^6 - 3C^4}{3C^6 (C^2 - T^2)^{\frac{3}{2}}} \right),
\]  

(4.15)
where $B$ is another constant which is related to $E_2$ by
\[
E_2 = 4 \cdot 3^7 \cdot \frac{9B^2C^{12} + 36C^4 - 96C^2 + 64}{C^6}.
\]

Since $u$ and $v$ are given as explicit functions of $T$ by (4.11) and (4.15), the biaxial Bianchi IX metric (1.1) can be written explicitly in terms of $T$ and the two parameters as
\[
ds^2 = \frac{3}{Λ} \left[ -\frac{1}{3} e^{7v - 3u} dT^2 + \frac{1}{3} e^{2v} (dθ^2 + \sin^2 θ \ dφ^2) + \frac{4}{3} e^{3u - v} (dψ + \cos θ \ dφ)^2 \right]
\]
\[
= \frac{3}{Λ} \left[ -\frac{1}{3} (C^2 - T^2)^{-\frac{7}{2}} \left( BT + \frac{6T^4C^2 - 9T^2C^4 + 3C^6 - 8T^4 + 12T^2C^2 - 3C^4}{3C^6 (C^2 - T^2)^{\frac{1}{2}}} \right)^{-1} dT^2
\]
\[
+ \frac{1}{3(C^2 - T^2)} (dθ^2 + \sin^2 θ \ dφ^2)
\]
\[
+ \frac{4}{3} \sqrt{C^2 - T^2} \left( BT + \frac{6T^4C^2 - 9T^2C^4 + 3C^6 - 8T^4 + 12T^2C^2 - 3C^4}{3C^6 (C^2 - T^2)^{\frac{1}{2}}} \right) (dψ + \cos θ \ dφ)^2 \right].
\]

One can check that (1.3) and (4.17) coincide with each other by identifying their time coordinates and parameters as follows:
\[
D_0 = \frac{4}{3} (τ^2 + 1) = \frac{1}{3} e^{2v} = \frac{1}{3(C^2 - T^2)},
\]
\[
D_0 = \frac{4}{3C^2}, \quad C_0 = 4BC^4.
\]

5 Asymptotic Behaviour of the Bianchi IX model

5.1 Definition of Asymptotic States

Let us consider the time evolution of a Bianchi IX spacetime without singularity, that is, a solution that is regular for all $-∞ < ζ < ∞$. The motion in phase space is, in principle, determined by six initial conditions, namely $α, β, γ$ and their conjugate momenta $π_α, π_β, π_γ$. Since they are constrained by (2.10), and since it is always possible to shift the time coordinate by a constant, one can get rid of two constants from the initial conditions. Hence the actual number of parameters is only four.

Consider those four parameters in the region where $α → ∞$. We assume that two of them are $β$ and $γ$, in other words, we assume that $β$ and $γ$ are asymptotically constants, and $\dot{β}, \dot{γ} → 0$. Let us denote the other two asymptotic constants by $c_β, c_γ$. As shall be shown shortly, $π_β$ and $π_γ$ diverge as $α → ∞$. In order to define $c_β$ and $c_γ$, we first investigate the asymptotic behaviour of $α$ and $\dot{α}$, and then we calculate how $\dot{β}$ and $\dot{γ}$ damp as $α$ grows.

Since $\dot{β}, \dot{γ} → 0$, (2.7) and (2.10) give
\[
\ddot{α} = 3 - 3α^2 - 2Ve^{-2α}
\]
\[
\rightarrow 1 - \dot{α}^2,
\]
\[
(5.1)
\]
where an overdot denotes the derivative with respect to the dimensionless time coordinate $\zeta$. The asymptotic solution is

$$\alpha = \ln(De^{\zeta} + Ee^{-\zeta}),$$  \hspace{1cm} (5.2)

where $D$ and $E$ are constants that each depends on the choice of where $\zeta = 0$, though their product $DE$ is invariant under constant shifts of $\zeta$ and is approximately the asymptotic value of $V/4$. In fact, a slightly better asymptotic form, with errors of the order of $e^{-5\alpha}$, can be shown to be

$$\alpha = \frac{1}{2} \ln(Fe^{2\zeta} + G + He^{-2\zeta}),$$  \hspace{1cm} (5.3)

where $G$ is the asymptotic value of $V/2$ and where $FH$ is the asymptotic value of $(1/16)V^2 - (1/8)[(\partial V/\partial \beta)^2 + (\partial V/\partial \gamma)^2]$. Note that $D$ and $E$, or $F$, $G$, and $H$, at $\zeta \to \infty$ are in general different from $D$ and $E$, or $F$, $G$, and $H$, at $\zeta \to -\infty$. One can integrate (2.8) to get

$$e^{3\alpha} \dot{\beta} = \int d\zeta \left( \frac{1}{2} \frac{\partial V}{\partial \beta} e^{\alpha} \right).$$  \hspace{1cm} (5.4)

By assumption $\beta$ and $\gamma$ are asymptotically constants, so that one can approximate (5.4) by the following form:

$$e^{3\alpha} \dot{\beta} \sim \frac{1}{2} \frac{\partial V}{\partial \beta} \dot{\alpha} e^{-2\alpha}. $$  \hspace{1cm} (5.5)

When (5.2) is used, this is simplified to

$$\dot{\beta} \sim \frac{1}{2} \frac{\partial V}{\partial \beta} \dot{\alpha} e^{-2\alpha}. $$  \hspace{1cm} (5.6)

This shows that for regular time evolution in which $\beta$ and $\gamma$ do not diverge as $\alpha$ goes to infinity, $\dot{\beta} \to \mathcal{O}(e^{-2\alpha})$ for large $\alpha$, and similarly $\dot{\gamma} \to \mathcal{O}(e^{-2\alpha})$. From the Lagrangian (2.11), the conjugate momenta of $\beta, \gamma$ are

$$\pi_\beta = -e^{3\alpha} \dot{\beta}, \quad \pi_\gamma = -e^{3\alpha} \dot{\gamma},$$

which diverge as $\alpha$ grows. However there is a canonical transformation that gives parameters that are asymptotically constants. Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\pi}_\alpha, \tilde{\pi}_\beta, \tilde{\pi}_\gamma)$ be a new canonical coordinates defined as

$$\tilde{\alpha} = e^{2\alpha} - V, \hspace{1cm} \tilde{\beta} = \beta, \hspace{1cm} \tilde{\gamma} = \gamma, \hspace{1cm} (5.8, 5.9, 5.10)$$

$$\tilde{\pi}_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = \frac{1}{2} e^{\alpha} \dot{\alpha}, \hspace{1cm} (5.11)$$

$$\tilde{\pi}_\beta = \frac{\partial L}{\partial \dot{\beta}} = -e^{3\alpha} \dot{\beta} + \frac{1}{2} \frac{\partial V}{\partial \beta} \dot{\alpha} e^{\alpha}, \hspace{1cm} (5.12)$$

$$\tilde{\pi}_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = -e^{3\alpha} \dot{\gamma} + \frac{1}{2} \frac{\partial V}{\partial \gamma} \dot{\alpha} e^{\alpha}, \hspace{1cm} (5.13)$$

where we keep $\dot{\alpha}$ since the equation of motion (2.7) is easier to evaluate than the analogous one for $\dot{\alpha}$. It can then be shown that $\tilde{\pi}_\beta$ and $\tilde{\pi}_\gamma$ damp as $\mathcal{O}(e^{-\alpha})$. Indeed the time derivative of $\tilde{\pi}_\beta$ is

$$\dot{\pi}_\beta = -\frac{\partial V}{\partial \beta} (\dot{\beta}^2 + \dot{\gamma}^2) e^{\alpha} + \frac{1}{2} \frac{\partial^2 V}{\partial \beta^2} \dot{\beta} \dot{\alpha} e^{\alpha} + \frac{1}{2} \frac{\partial^2 V}{\partial \beta \partial \gamma} \dot{\gamma} \dot{\alpha} e^{\alpha} \sim \mathcal{O}(e^{-\alpha}),$$  \hspace{1cm} (5.14)
and similar for $\tilde{\pi}_\gamma$. Therefore if we choose $c_\beta = \tilde{\pi}_\beta$ and $c_\gamma = \tilde{\pi}_\gamma$, all of the parameters $(\beta, \gamma, c_\beta, c_\gamma)$ are asymptotically constants. We call such a set of four asymptotic constants $(\beta, \gamma, c_\beta, c_\gamma)$ at $t \to -\infty$ an initial asymptotic state, while the set at $t \to +\infty$ is a final asymptotic state.

5.2 Map from Past to Future Infinity

Since asymptotic states are well-defined, one can consider a map $f$ from an initial asymptotic state to a final state,

$$f : (\beta, \gamma, c_\beta, c_\gamma)_- \to (\beta, \gamma, c_\beta, c_\gamma)_+.$$  \hspace{1cm} (5.15)

For the biaxial case, we know the exact solution, so that one can explicitly calculate $(\beta, c_\beta)_\pm$ as

$$\beta = \frac{v - u}{2},$$ \hspace{1cm} (5.16)

$$\beta_\pm = -\frac{1}{3} \ln \frac{4}{3} C^2 (T \to \pm C),$$ \hspace{1cm} (5.17)

$$c_\beta = -\frac{1}{3} \epsilon^{3u-2v} \left( \frac{dv}{dT} - \frac{du}{dT} \right) - \frac{1}{3} (\epsilon^{3u-4v} - 4 \epsilon^{6u-7v}) \left( \frac{dv}{dT} + \frac{du}{dT} \right),$$ \hspace{1cm} (5.18)

$$c_{\beta\pm} = -\frac{BC^2}{3} (T \to \pm C).$$ \hspace{1cm} (5.19)

Thus the map $f$ is the identity map in the biaxial case, which was suggested by the existence of two non-trivial Killing tensors in the minisuperspace. We perturbatively and numerically have found evidence that the map is no longer the identity in a triaxial model, but the exact form of the map has not yet been calculated.

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