ON EXTENDED EIGENVALUES AND EXTENDED EIGENVECTORS OF TRUNCATED SHIFT

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Abstract. We give a complete description of the set of extended eigenvectors of truncated shifts defined on the model spaces $K^2_a := H^2 \oplus uH^2$, in the case of $u$ is a Blaschke product.

1. Introduction and preliminaries

Let $H$ be a complex Hilbert space, and denote by $\mathcal{L}(H)$ the algebra of all bounded linear operators on $H$. If $T$ is an operator in $\mathcal{L}(H)$, then a complex number $\lambda$ is an extended eigenvalue of $T$ if there is a nonzero operator $X$ such that $TX = \lambda XT$. We denote by the symbol $\sigma_{\text{ext}}(T)$ the set of extended eigenvalues of $T$. The set of all extended eigenvectors corresponding to $\lambda$ will be denoted as $E_{\text{ext}}(\lambda)$. Obviously $1 \in \sigma_{\text{ext}}(T)$ for any operator $T$. Indeed, one can take $X$ being the identity operator.

Let $T$ in $\mathcal{L}(H)$, and let $\sigma(T)$ and $\sigma_p(T)$ denote the spectrum and the point spectrum of $T$ respectively. By a theorem of Rosenblum [4], it was established in [2] that

\[(1.1) \quad \sigma_{\text{ext}}(T) \subset \{ \lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset \}.\]

Moreover, when $H$ is finite dimensional, in [2] the set of extended eigenvalues has been characterized by the following theorem

Theorem 1.1. Let $T$ be an operator on a finite dimensional Hilbert space $H$. Then $\sigma_{\text{ext}}(T) = \{ \lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset \}$.

Proof. First we consider the case when $T$ is not invertible. In this situation both $T$ and $T^*$ have nontrivial kernels. Let $X'$ be a nonzero operator from kernel of $T^*$ to kernel of $T$. Define $X = X'P$ where $P$ denotes the orthogonal projection on kernel of $T^*$. Clearly, $X \neq 0$, and $TX = 0 = \lambda XT$ for any $\lambda \in \mathbb{C}$. Consequently, $\sigma_{\text{ext}}(T) = \mathbb{C}$. On the other hand, since $T$ is not invertible, for any complex number $\lambda$, $0 \in \sigma(T) \cap \sigma(\lambda T)$. Thus

$\sigma_{\text{ext}}(T) = \mathbb{C} = \{ \lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \neq \emptyset \}$.

Now assume that $T$ is invertible so that $0 \notin \sigma(T)$. In view of [14] it suffices to show that $\{ \lambda \in \mathbb{C} : \sigma(T) \cap \sigma(\lambda T) \} \subset \sigma_{\text{ext}}(T)$. So suppose that $\alpha$ is a (necessarily nonzero) complex number such that $\alpha \in \sigma(T)$ and $\alpha \in \sigma(\lambda T)$. Since $\alpha \in \sigma(T)$ there exists a vector $a$ such that $Ta = \alpha a$. On the other hand, $\alpha \in \sigma(\lambda T)$ implies that $\lambda \neq 0$ so $\alpha/\lambda \in \sigma(T)$. Therefore, $\frac{\alpha}{\lambda} \in \sigma(T^*)$ and there is a vector $b$ such that $T^*b = (\frac{\alpha}{\lambda})b$. Let $X = a \otimes b$. Then $TX = \lambda XT$ and consequently $\lambda \in \sigma(T)$. \hfill \Box

From this theorem it follows the following consequences

Corollary 1.2. Let $T$ be an operator on a finite dimensional Hilbert space $H$. Then

1. If $T$ is invertible then $\sigma_{\text{ext}}(T) = \{ \alpha/\beta : \alpha, \beta \in \sigma(T) \}$, and if $Ta = \alpha a$, $T^*b = \beta b$ then $a \otimes b \in E_{\text{ext}}(\alpha/\beta)$.
2. $\sigma_{\text{ext}}(T) = \{1\}$ if and only if $\sigma(T) = \{\alpha\}, \alpha \neq 0$. 

\[ \]
(3) \( \sigma_{ext}(T) = \mathbb{C} \) if and only if \( 0 \in \sigma(T) \). Moreover, this assertion remains available in infinite dimensional Hilbert spaces if \( 0 \in \sigma_p(T) \cap \sigma_p(T^*) \).

The next section contains the needed background on the spaces \( K^2_u \).

2. BACKGROUND ON \( K^2_u \)

Nothing in the section is new, and the bulk of it can be found in standard sources, for example [3], [1], [6] and [5].

2.1. Basic notation, model spaces and kernel functions. Let \( H^2 \) be the standard Hardy space, the Hilbert space of holomorphic functions in the open unit disk \( D \subset \mathbb{C} \) having square-summable Taylor coefficients at the origin. We let \( S \) denote the unilateral shift operator on \( H^2 \). Its adjoint, the backward shift, is given by

\[
S^* f(z) = \frac{f(z) - f(0)}{z}.
\]

For the remainder of the paper, \( u \) will denote a non-constant inner function. the subspace \( K^2_u = H^2 \ominus uH^2 \) is a proper nontrivial invariant subspace of \( S^* \), the most general one by the well-known theorem of A. Beurling. The compression of \( S \) to \( K^2_u \) will be denoted by \( S_u \). Its adjoint, \( S_u^* \), is the restriction of \( S^* \) to \( K^2_u \). For \( \lambda \) in \( D \), the kernel function in \( H^2 \) for the functional of evaluation at \( \lambda \) will be denoted by \( k_\lambda \); it is given explicitly by

\[
k_\lambda(z) = \frac{1}{1 - \lambda z}.
\]

2.2. Riesz bases of \( K^2_u \). It is known that the model space \( K^2_u \) is finite dimensional if and only if \( u \) is finite Blaschke product

\[
B(z) = \prod_{i=1}^{n} b_{\alpha_i}^{p_i}, \quad \text{with} \quad b_\lambda = \frac{\lambda - z}{1 - \lambda z} \quad \text{for} \quad \lambda \in D, \quad p_i, n \in \mathbb{N}^*, \quad \text{and} \quad \alpha_i \neq \alpha_j \quad \text{for} \quad i \neq j.
\]

In the general case, if \( B \) is an infinite Blaschke product defined by

\[
B(z) = \prod_{i=1}^{\infty} \frac{\alpha_i}{\alpha_i} b_{\alpha_i}^{p_i}, \quad p_i \in \mathbb{N}^*,
\]

then the following Cauchy kernels

\[
e_{i,l}(z) = \frac{b_{\alpha_i}^{p_i} z^l}{(1 - \alpha_i z)^{l+1}}, \quad \forall i \geq 1, \quad l = 0, \ldots, p_i - 1,
\]

span the space \( K^2_B \). In particular, if \( p_i = 1 \) for \( i \) in \( \mathbb{N}^* \), then \( e_{i,0} \) will be denoted by \( e_i \), i.e.,

\[
e_i(z) = k_{\alpha_i}^{B_i}(z).
\]

If we denote by \( \{ e_{i,l} : i \geq 1, \ l = 0, \ldots, p_i - 1 \} \) (see [5]) the dual set of \( \{ e_{i,l} : i \geq 1, \ l = 0, \ldots, p_i - 1 \} \), (i.e., the set of kernels verifying

\[
\langle e_{i,k}, e_{j,l} \rangle = \delta_{kl} \delta_{ij}, \quad \forall i, j \geq 1, \ k = 0, \ldots, p_i - 1, \ l = 0, \ldots, p_j - 1,
\]
where \((\cdot, \cdot)\) denotes the inner product in \(L^2\), and \(\delta_{ij}\) denotes the well-known Kronecker \(\delta\)-symbol, then we have the following lemma

**Lemma 2.1.** If \(B\) is a Blaschke product defined by \((2.3)\), then

\[
S_B^*e_{i,l} = \begin{cases} \overline{\alpha_i}e_{i,0} & \text{if } l = 0 \\ le_{i,l-1} + \overline{\alpha_i}e_{i,l} & \text{otherwise}, \end{cases}
\]

and

\[
S_Be_{i,l} = \begin{cases} \alpha_i e_{i,p_{i-1} - 1} & \text{if } l = p_i - 1 \\ \alpha_i e_{i,l} + (l + 1)e_{i,l+1} & \text{otherwise}. \end{cases}
\]

**Proof.** For the first equality, if \(l = 0\), then

\[
S_B^*e_{i,0}(z) = \frac{k_B^i(z) - k_B^i(0)}{z} = \frac{\overline{\alpha_i}}{1 - \overline{\alpha_i}z} = \overline{\alpha_i}e_{i,0}(z).
\]

Otherwise,

\[
S_B^*e_{i,l}(z) = \frac{ll^{l-1}}{(1 - \overline{\alpha_i}z)^{l+1}} = \frac{l}{(1 - \overline{\alpha_i}z)} \frac{l^l}{(1 - \overline{\alpha_i}z)^l} + \frac{\overline{\alpha_i}}{1 - \overline{\alpha_i}z} \frac{l^l}{(1 - \overline{\alpha_i}z)^l} = le_{i,l-1}(z) + \overline{\alpha_i}e_{i,l}(z).
\]

For the second equality, it is sufficient to use the first one together with the fact that

\[
\langle S_B^*e_{i,k}, e_{j,l} \rangle = \langle e_{i,k}^*, S_Be_{j,l} \rangle, \quad \forall \ i, j \geq 1, \ k = 0, ..., p_i - 1, \ l = 0, ..., p_j - 1.
\]

\(\square\)

If we denote by \(E_i = \text{span}\{e_{i,0}, ..., e_{i,p_i-1}\}\) and by \(E_i^* = \text{span}\{e_{i,0}^*, ..., e_{i,p_i-1}^*\}\), for \(i\) in \(\mathbb{N}^*\). Then Lemma 2.1 derives the following consequences

**Corollary 2.2.** For each \(i\) in \(\mathbb{N}^*\), we have

1. The subspaces \(E_i\) and \(E_i^*\) are invariant of \(S_B^*\) and \(S_B\) respectively.
2. Let \(l \in \{0, 1, ..., p_i - 1\}\). For each \(k = 0, 1, ..., l\), we have

\[
(S_B - \alpha_i I)^k e_{i,p_i-1,l}^* \neq 0, \text{ and } (S_B - \alpha_i I)^{l+1} e_{i,p_i-1,l}^* = 0.
\]

In particular, \(\ker(S_B - \alpha_i I)^{l+1} = \text{span}\{e_{i,p_i-1,l}^*, ..., e_{i,p_i-1}^*\}\), and for all

\[
k \geq p_i, \text{ we have } \ker(S_B - \alpha_i I)^k = \ker(S_B - \alpha_i I)^{p_i} = E_i^*.
\]

**Proof.** The first point is trivial. For the second one, we will argue by induction. This result is trivial for \(l = 0\). We assume that it is true for all \(k = 0, 1, ..., l-1\), i.e.,

\[
x := (S_B - \alpha_i I)^{l-1} e_{i,p_i-1}^* \neq 0, \text{ and } (S_B - \alpha_i I)x = 0.
\]

It is enough to show that

\[
(S_B - \alpha_i I)^l e_{i,p_i-1,l}^* \neq 0, \text{ and } (S_B - \alpha_i I)^{l+1} e_{i,p_i-1,l}^* = 0.
\]

By using Lemma 2.1 and the induction hypothesis, we have that

\[
(S_B - \alpha_i I)^l e_{i,p_i-1,l}^* = (p_i - l)x \neq 0,
\]

and

\[
(S_B - \alpha_i I)^{l+1} e_{i,p_i-1,l}^* = (p_i - 1)(S_B - \alpha_i I)x = 0.
\]

Consequently, \(\text{span}\{e_{i,p_i-1,l}^*, ..., e_{i,p_i-1}^*\} \subset \ker(S_B - \alpha_i I)^{l+1}\) and \((S_B - \alpha_i I)^{l+1}\) is injective on \(\text{span}\{e_{i,p_i-1,l}^*, ..., e_{i,p_i-1}^*\}\). To complete the proof, we shall show that \((S_B - \alpha_i I)^{l+1}\) is injective on

\[
\text{span}\{E_j^* : j \geq 1 \text{ and } j \neq i\}.
\]

But the subspaces \(E_j^*\) are invariant of \((S_B - \alpha_i I)^{l+1}\). Thus, it is sufficient to show that \((S_B - \alpha_i I)^{l+1}\) is injective on \(E_j^*\) for any \(j \neq i\). To do so, suppose to the contrary
Thus, which contradicts the fact that \((S_B - \alpha_i I)^t x = 0 \) for \( x \in E^*_j \) and \( j \neq i \), then \((S_B - \alpha_i I)^t x \in \text{span}\{e^*_i, p_{i-1}\} \), which contradicts the fact that \(E^*_j\) is invariant of \((S_B - \alpha_i I)^t\).

\[\square\]

Bivas and Petrovic determine in [2] the extended spectrum of truncated shift. Our main result, that is Theorem 3.3, gives a complete description of the set of extended eigenvectors of truncated shift \(S_B\). Moreover, it affirm the result of Bivas and Petrovic for the set \(\sigma_{\text{ext}}(S_B)\) without using the Sz.-Nagy-Foias commutant lifting theorem. Consequently, it strengthens [2, Theorem 3.10].

3. Extended eigenvalues and extended eigenvectors of \(S_B\)

If \(B\) is a Blaschke product defined by (2.5), it was shown in [3] that \(\sigma(B) = \{\alpha_i\}_{i=1}^n\) and \(\sigma_p(B) = \{\alpha_i\}_{i\geq 1}\). For the remainder of this paper, the zeros \(\{\alpha_i\}_{\geq 1}\) are all nonzero. Before showing our main result, we give theorem 3.1 as a direct application of Theorem 1.1 and Lemma 2.1. If \(B\) is a finite Blaschke product defined by (2.4) with \(p_i = 1\) for all \(i\), then by Corollary 1.2, \(\sigma_{\text{ext}}(S_B) = \{\alpha_i/\alpha_j : i, j = 1, \ldots, n\}\) and \(e^*_i \otimes e_j \in E_{\text{ext}}(\alpha_i/\alpha_j)\). It is natural to ask weather this eigenvector is unique or not. The following theorem answers this question affirmatively.

**Theorem 3.1.** If \(B\) is a finite Blaschke product defined in (2.4) with \(p_i = 1\) for all \(i\), then \(\sigma_{\text{ext}}(S_B) = \{\alpha_i/\alpha_j : i, j = 1, \ldots, n\}\) and \(E_{\text{ext}}(\alpha_i/\alpha_j) = \text{span}\{e^*_i \otimes e_j : \alpha_k/\alpha_l = \alpha_i/\alpha_j\}\).

**Proof.** Since \(\{e_i\}_{i=1}^n\) and \(\{e^*_i\}_{i=1}^n\) are bases Riesz for \(K_B^2\), the set \(\{E_{ij} := e^*_i \otimes e_j\}_{i,j=1}^n\) is a basis Riesz for \(L(K_B^2)\). Now assume that \(X \in L(K_B^2)\) is a solution to the equation

\[S_B X = \frac{\alpha_i}{\alpha_j} X S_B,\]

then there are a family of complex numbers \(\{a_{ij}\}_{i,j=1}^n\) such that

\[S_B \left( \sum_{k,l=1}^n a_{kl} E_{kl} \right) = \frac{\alpha_i}{\alpha_j} \left( \sum_{k,l=1}^n a_{kl} E_{kl} \right) S_B,\]

hence

\[\left( \sum_{k,l=1}^n \frac{\alpha_k}{\alpha_l} a_{kl} E_{kl} \right) S_B = \left( \sum_{k,l=1}^n \frac{\alpha_i}{\alpha_j} a_{kl} E_{kl} \right) S_B,\]

Since \(S_B\) is invertible and \(\{E_{ij} := e^*_i \otimes e_j\}_{i,j=1}^n\) is a Riesz basis for \(L(K_B^2)\),

\[\frac{\alpha_k}{\alpha_l} a_{kl} = \frac{\alpha_i}{\alpha_j} a_{kl}, \ \forall k, l = 1, \ldots, n,\]

thus

\[E_{\text{ext}}(\frac{\alpha_i}{\alpha_j}) = \text{span}\{e^*_i \otimes e_j : \alpha_k/\alpha_l = \alpha_i/\alpha_j\}.\]

\[\square\]

**Remark 3.2.** If \(\alpha_k/\alpha_l \neq \alpha_i/\alpha_j\) for all \((k,l) \neq (i,j)\), then

\[E_{\text{ext}}(\frac{\alpha_i}{\alpha_j}) = \{e^*_i \otimes e_j\},\]

that is why we have said that this solution is unique.

Now, let \(B\) be an infinite Blaschke product as in (2.5), and let \(\{\gamma_i\}_{i \in \mathbb{T}}\) be the set of limit points of \(\{\alpha_i\}_{i \geq 1}\) on the circle \(\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}\). By (1.1), we have

\[\sigma_{\text{ext}}(S_B) \subset \left\{ \frac{\alpha_i}{\alpha_j} : i, j \geq 1 \right\} \cup \left\{ \frac{\alpha_i}{\gamma_j} : i \geq 1, \ j \in I \right\} \cup \left\{ \frac{\gamma_i}{\alpha_j} : i \in I, \ j \geq 1 \right\}.\]

The following theorem shows that this inclusion is proper, more precisely
Theorem 3.3. If $B$ is an infinite Blaschke product defined by $(2.5)$, then

$$\sigma_{ext}(SB) = \left\{ \frac{\alpha_i}{\alpha_j} : i, j \geq 1 \right\},$$

and for any $i, j \geq 1$, we have

$$E_{ext} \left( \frac{\alpha_i}{\alpha_j} \right) = \text{span} \left[ \sum_{k=0}^{l} \sum_{r=0}^{m} c_{k-r} \frac{(m-r-k)!(p_m-r-1)!}{(l-k)!(p_m-1)!} \alpha^*_{m-p_m-r-1} \otimes e_{n,l-k} \right],$$

where $c_{k-r} = c_{k-r}(1)$ for all $m, n \geq 1$ where $\frac{\alpha_m}{\alpha_n} = \frac{\alpha_i}{\alpha_j}$, $l = 0, \ldots, \min(p_m-1, p_n-1)$, $c_{k-r} \in \mathbb{C}$ and $c_0 \neq 0$.

Proof. Let $\lambda \in \mathbb{C}$ and $X \in \mathcal{L}(K_B^2)$ be such that

$$SBX = \lambda XSB,$$

then by Lemma 2.3 for all $j \geq 1$ we have

$$SBXe^*_{j,l} = \left\{ \begin{array}{ll}
\frac{\lambda \alpha_j X e^*_{j,l}}{\alpha_j} & \text{if } l = p_j - 1 \\
\frac{\lambda \alpha_j X e^*_{j,l+1}}{\alpha_j} & \text{if } l = 0, \ldots, p_j - 2.
\end{array} \right.$$

If $X \neq 0$, then necessarily there are $i, j \geq 1$, $l$ in $\{0, 1, \ldots, p_j - 1\}$ and $(c_0 \neq 0)$ in $\mathbb{C}$ such that

$$\lambda = \frac{\alpha_i}{\alpha_j} \text{ and } X e^*_{j,l} = c_0 e^*_{i,p_i-1}.$$

Then

$$SBXe^*_{j,l-1} = \frac{\alpha_i}{\alpha_j} X (\alpha_j e^*_{j,l-1} + le^*_{j,l}),$$

$$\left(SB - \alpha_i I\right)X e^*_{j,l-1} = \frac{\alpha_i}{\alpha_j} l c_0 e^*_{i,p_i-1},$$

consequently there exist complex numbers $(c_0^{(1)} \neq 0)$ and $c_1$ such that

$$X e^*_{j,l-1} = c_0^{(1)} e^*_{i,p_i-2} + c_1 e^*_{i,p_i-1},$$

moreover, by (3.1)

$$e_0^{(1)} (\alpha_i e^*_{i,p_i-2} + (p_i - 1) e^*_{i,p_i-1}) + c_1 \alpha_j e^*_{i,p_i-1} = \alpha_i (c_0^{(1)} e^*_{i,p_i-2} + c_1 e^*_{i,p_i-1}) + \frac{\alpha_i}{\alpha_j} l c_0 e^*_{i,p_i-1},$$

hence

$$c_0^{(1)} = \frac{\alpha_i}{\alpha_j} l \frac{1}{p_i - 1} c_0.$$

By repeating the same calculation a number of times equal to $\min(p_i - 2, l - 1)$, we obtain that

$$X e^*_{j,l-k} = \sum_{r=0}^{k} c_0^{(r)} e^*_{i,p_i-1-r}, \text{ where }$$

$$c_0^{(r)} = \frac{\alpha_i}{\alpha_j} \frac{(l + r - k)!(p_i - r - 1)!}{(l - k)!(p_i - 1)!} c_{k-r}, \text{ } k = 2, \ldots, \min(p_i - 1, l),$$

thus, if $l \geq p_i$, we have

$$\left(SB - \alpha_i I\right)X e^*_{j,l-p_i} = \frac{\alpha_i}{\alpha_j} l (l - p_i + 1) \sum_{r=0}^{p_i-1} c_0^{(r)} e^*_{i,p_i-1-r}, \text{ where } c_0^{(p_i-1)} \neq 0,$$

therefore

$$\left(SB - \alpha_i I\right)^{p_i}X e^*_{j,l-p_i} \neq 0 \text{ and } \left(SB - \alpha_i I\right)^{p_i+1}X e^*_{j,l-p_i} = 0,$$
and that contradicts Corollary 2.2. Thus, if \( \lambda = \frac{\alpha_i}{\alpha_j} \) and \( X \neq 0 \), then \( l \) must be in the range \( \{0, 1, \ldots, \min(p_i - 1, p_j - 1)\} \), and the operator

\[
X_{i,j} := \sum_{k=0}^{l} \sum_{r=0}^{k} c_{k-r} \left( \frac{\alpha_i}{\alpha_j} \right)^r \frac{(l + r - k)!(p_i - r - 1)!}{(l - k)!(p_j - 1)!} e_{i,p_i-r-k} \otimes e_{j,l-k}
\]

where \( c_{k-r} \in \mathbb{C} \), \( c_0 \neq 0 \) and \( l = 0, \ldots, \min(p_i - 1, p_j - 1) \), is a nonzero solution of

\[
S_B X = \frac{\alpha_i}{\alpha_j} X S_B.
\]

Assume that \( n \) is a natural number different from \( j \) (i.e., \( \alpha_n \neq \alpha_j \)). Now, we find the image of \( e_{i,j}^* \) for \( l = 0, 1, \ldots, p_n - 1 \), under the operator \( X \) that verify (3.2), hence

\[
S_B X e_{n,l}^* = \begin{cases} \frac{\alpha_i}{\alpha_j} \alpha_n X e_{n,p_n-1}^* & \text{if } l = p_n - 1 \\ \frac{\alpha_i}{\alpha_j} \alpha_n X e_{n,l+1}^* + \frac{\alpha_i}{\alpha_j} (l + 1) X e_{n,l+1}^* & \text{if } l = 0, \ldots, p_n - 2. \end{cases}
\]

therefore, once again by Corollary 2.2 if there is \( l \) in \( \{0, 1, \ldots, p_n - 1\} \) such that \( X e_{n,l}^* \neq 0 \), then necessarily there is a natural number \( m \) (necessarily different from \( i \)) such that

\[
\frac{\alpha_i}{\alpha_j} = \frac{\alpha_m}{\alpha_n}, \text{ and } X e_{n,l}^* = c_0 e_{m,p_m-1}^*, \text{ (} c_0 \neq 0 \text{) } \in \mathbb{C}.
\]

So, in this case, \( X \) has the same behavior like the \( e_{j,l}^* \) case, i.e., \( X = X_{m,n} \) is a solution of (3.2).

Thus, we have exactly described the solution of (3.2) on a set which spans the space \( K_B^0 \). Consequently, \( E_{ext}(\alpha_i/\alpha_j) \) is given by

\[
E_{ext}\left( \frac{\alpha_i}{\alpha_j} \right) = \text{span}\{X_{m,n}, \forall m, n \geq 1 \text{ where } \frac{\alpha_m}{\alpha_n} = \frac{\alpha_i}{\alpha_j} \},
\]

as desired.

\[ \square \]

4. Concluding remarks

We finish this paper with some remarks which are summarized in the following. First, it is clear that Theorem 3.1 is a particular case of last theorem, nevertheless we have proved it as a direct result of Theorem 1.1.

In addition, if the set of zeros \( \{\alpha_i\}_{i \geq 1} \) satisfies the well-known Carleson condition (see [3]), then the set \( \{e_{i,l}^*\} \) forms a Riesz basis for \( K_B^0 \), and the solution of (3.2) is given in terms of this basis and the dual Riesz basis \( \{e_{l,i}\} \).

Also, if we suppose that \( \alpha_0 = 0 \) is a zero of \( B \), then by using the proof of Theorem 1.1 we have that \( \sigma_{ext}(S_B) = \mathbb{C} \). Indeed, the operator \( X = e_{0,p_0-1}^* \otimes e_{0,0} \) satisfies that \( S_B X = 0 = \lambda X S_B \), for all \( \lambda \in \mathbb{C} \).

And finally, as a direct result of (2) in Corollary 1.2 if

\[
B(z) = b^\alpha_n, \text{ where } n \in \mathbb{N}^* \text{ and } \alpha \in \mathbb{D},
\]

then \( \sigma_{ext}(S_B) = \{1\} \) and

\[
E_{ext}(1) = \text{span}\left\{ \sum_{k=0}^{l} \sum_{r=0}^{k} c_{k-r} \frac{(l + r - k)!(n - r - 1)!}{(l - k)!(n - 1)!} e_{\alpha,n-r-1}^* \otimes e_{\alpha,l-k}, l = 0, \ldots, n - 1 \right\}.
\]

Lastly, this paper gives a complete description of the set of extended eigenvectors of \( S_n \), in the case of \( n \) is a Blaschke product, and this leads naturally to the following question.
Problem 1. What is the set of extended eigenvectors of $S_u$ in the case of $u$ is a singular inner function?

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