CONSTRUCTING NON-UNIQUELY ERGODIC ARATIONAL TREES

BRIAN MANN AND PATRICK REYNOLDS

Abstract. In this technical note, we adapt an idea of Gabai to construct non-uniquely ergodic, non-geometric, arational trees.

1. Introduction

Let $F = F_N$ denote the rank-$N$ free group, with $N \geq 4$. A factor is a conjugacy class of non-trivial, proper free factors of $F_N$; a conjugacy class of elements of $F_N$ is primitive if any of its representatives generates a representative of a factor. A curve is a one edge $\mathbb{Z}$-splitting of $F_N$ with primitive edge group; every curve is a very small tree. Two curves $T, T'$ are called disjoint if there is a two edge simplicial tree $Y$ such that both $T$ and $T'$ can be obtained by collapsing the components of an orbit of edges in $Y$ to points; this is the same as having that $l_T + l_{T'}$ is a length function for a very small tree, namely $Y$.

A measure on a tree $T$ is a collection of finite Borel measures $\{\mu_I\}$, where $I$ runs over finite arcs of $T$, that is invariant under the $F_N$-action and is compatible with restriction to subintervals; if $T$ has dense orbits, then the set of measures on $T$ is a finite dimensional convex cone \cite{9}. The measure of a finite subtree $K \subseteq T$ is the sum of measures of the arcs in any partition of $K$ into finitely many arcs. The co-volume of $T$ is the infimum of measures of a finite forests $K$ such that $T \subseteq \cup_{g \in F_N} gK$. If $H \leq F_N$ is finitely generated and does not fix a point of $T$, then there is a unique minimal $H$ invariant subtree $T_H \subseteq T$; define the co-volume of $H$, denoted $\text{covol}(H)$, to be the co-volume of $T_H$, for $T$ fixed. A tree $T$ is called arational if $\text{covol}(F') > 0$ for every factor $F'$; this is the same as having every factor act freely and simplicially on some invariant subtree of $T$, see \cite{15,3}. An arational tree $T$ is called non-uniquely ergodic if there are non-homothetic measures $\mu$ and $\mu'$ for $T$; see \cite{9,14,5} for more about measures on trees. Our main result is:

Theorem: Let $T, T'$ be disjoint curves with neighborhoods $U, U'$. There is a 1-simplex of non-uniquely ergodic, arational, non-geometric trees with one endpoint in each of $U, U'$.

Notions around geometric trees are reviewed in Section 2. Examples of non-uniquely ergodic arational trees dual to measured foliations on surfaces

Date: November 8, 2013.
are well-known, but all such trees are geometric; R. Martin contructed one example of a non-uniquely ergodic tree that is geometric and of Levitt type \cite{13}. Our procedure gives the first examples of non-uniquely ergodic, arational trees that are non-geometric.

1.1. Analogy with Gabai’s Construction and Outline of Proof. Our proof of the Theorem is an adaptation of an idea of Gabai, and there are two main technical steps; before explaining those, we recall the proof of the following:

**Theorem 1.1.** \cite{7} Theorem 9.1] Let $\Sigma$ be a $k$-simplex of disjoint curves in $\mathcal{PML}$, and $U_1, \ldots, U_{k+1}$ be neighborhoods of the extreme points. There is a $k$-simplex $\Sigma'$ of non-uniquely ergodic minimal and filling laminations with extreme points in the $U_j$’s.

Theorem 1.1 follows at once by induction after showing:

**Proposition 1.2.** \cite{7} Let $\alpha_1, \ldots, \alpha_r \in \mathcal{PML}(S_{g,p})$ be a collection of disjoint curves; let $U_j$ be a neighborhood of $\alpha_j$, and let $c$ be a curve. There are disjoint curves $\alpha'_1, \ldots, \alpha'_s$ so that $\alpha'_j \in U_j$ with neighborhoods $U'_j \subseteq U_j$, such that $i(\beta, c) > d_c > 0$ for any $\beta \in \cup_j U'_j$.

Here is how Gabai proves Proposition 1.2. The surface $S$ is cut into pieces $P_1, \ldots, P_k$ by the $\alpha_j$’s, and gluing back along a fixed $\alpha_j$ gives a surface $\sigma$ that is at least as complex as a 4-punctured sphere or a punctured torus; $\alpha_j$ is essential and non-peripheral on $\sigma$. A generic choice $\gamma$ of a curve in $\sigma$ intersects $\alpha_j$ and all arcs of $c$ cutting $\sigma$ (to be more concrete, one can apply a high power of a pseudo-Anosov on $\sigma$ to $\alpha_j$ to get $\gamma$); now apply a high power of a Dehn twist in $\alpha_j$ to $\gamma$ to get a curve $\alpha'_j$ in $U_j$, and replace $\alpha_j$ with $\alpha'_j$.

To ensure positive intersection of every $\alpha'_j$ with $c$, one begins by modifying $\alpha_j$, chosen so that either $c$ meets $\sigma$ in an essential arc or else $c = \alpha_j$; after finding $\alpha'_j$, move to a boundary component $\alpha'_j$ of $\sigma$ and continue. Since $i(c, \alpha'_j) > 0$, continuity of $i(\cdot, \cdot)$ ensures that we can find a neighborhoods $U'_j \subseteq U_j$ such that $i(c, \beta) > 0$ for any $\beta \in \cup_j U'_j$.

We proceed essentially in the same way, with our “curves” serving as surrogates of curves on a surface. The analogue, for a tree $T$, of having positive intersection with every simple closed curve on a surface is that every factor acts with positive co-volume in $T$ (this is the same as every factor acting freely and simplicially on its minimal subtree of $T$ \cite{15}), and our analogue of continuity of $i(\cdot, \cdot)$ is the continuity of the Kapovich-Lustig intersection function combined with continuity of the restriction map from the space of very small $F_N$-trees to the space of $H$-trees for $H$ a finitely generated subgroup of $F_N$. Since our aim is to construct limiting trees that are both arational and non-geometric, we have two versions of Proposition 1.2; these are the two main technical steps in our argument and appear as Propositions 3.1 and 4.2 below.
This paper is organized as follows: In Section 2 we give relevant background about trees, currents, and laminations. In Section 3 we give our analogue of Proposition 1.2 that ensures arational limiting trees, while in Section 4 we give our analogue of Proposition 1.2 that ensures non-geometric limiting trees. Section 5 contains the proof the main result.

Acknowledgements: We wish to thank the participants, Mladen Bestvina in particular, of the Fall 2011 “working group on Out(\(F_N\))” at the University of Utah for pleasant discussions about Lemma 0.

2. Background

Fix a basis \(B\) for \(F_N\). Use \(cv_N\) to denote the set of very small \(F_N\)-trees; \(cv_N\) denotes the subset of free and simplicial very small trees; and \(\partial cv_N = cv_N \setminus cv_N\). If \(F'\) is another finite rank free group, we use \(cv(F')\) and \(\partial cv(F')\) to denote the corresponding spaces of \(F'\)-trees. If \(T\) is a very small tree, then \(l_T\) denotes its length function; spaces of trees get the length functions topology, which coincides with the equivariant Gromov-Hausdorff topology.

2.1. Currents and Laminations. Use \(\partial F_N\) to denote the boundary of some \(T_0 \in cv_N\), and put \(\partial^2 F_N := (\partial F_N \times \partial F_N \setminus \text{diag.})/\mathbb{Z}_2\); this can be thought of as the space of unoriented geodesic lines in \(T_0\). The obvious action of \(F_N\) on \(\partial F_N\) gives an action of \(F_N\) on \(\partial^2 F_N\). A lamination is a non-empty, invariant, closed subset \(L \subseteq \partial^2 F_N\). A current is a non-zero invariant Radon measure \(\nu\) on \(\partial^2 F_N\); the support of a current \(\nu\), denoted \(\text{Supp}(\nu)\), is a lamination; the set of currents gets the weak-* topology and is denoted by \(\text{Curr}_N\). There is a continuous function, called intersection,

\[
\langle \cdot, \cdot \rangle : cv_N \times \text{Curr}_N \to \mathbb{R}_{\geq 0}
\]

that is homogeneous in the first coordinate and linear in the second coordinate [10].

If \(T \in \partial cv_N\), then either \(T\) is not free or else \(T\) is not simplicial, hence for any \(\epsilon > 0\), there is \(g \in F_N\) with \(l_T(g) < \epsilon\). Define

\[
L(T) := \cap_{\epsilon > 0}\{(g^{-\infty}, g^{\infty}) | l_T(g) < \epsilon\}
\]

The set \(L(T)\) is a lamination [6]. Kapovich and Lustig gave a complete characterization of when a tree and a current have intersection equal to zero: \(\langle T, \nu \rangle = 0\) if and only if \(\text{Supp}(\nu) \subseteq L(T)\) [11].

2.2. Geometric Trees. The topology on \(\partial cv_N\) is metrizable, and we fix a compatible metric. We record a lemma that follows immediately from the definition of the Gromov-Hausdorff topology.

Lemma 2.1. For any finitely generated \(H \leq F_N\), the function

\[
\text{cv}_N \to \text{cv}(H) : T \mapsto T_H
\]

is continuous.
Considering the Gromov-Hausdorff topology, one gets a function $cv_N : T \mapsto x \in T$ that is “continuous” in the following sense: given a finite subset $S \subseteq F_N$ and $\epsilon > 0$, there is a $\delta > 0$ so that if $T'$ is $\delta$-close to $T$, then the partial action of $S$ on the convex hull of $S \times T$ is $\epsilon$-approximated by the partial action of $S$ on the convex hull of $S \times T'$; this point is explained in [16], in particular, see Skora’s discussion of Proposition 5.2. We call this function a continuous choice of basepoint on $cv_N$.

We very quickly recall some notions around geometric trees; see [2] for details. Every $T \in \partial cv_N$ admits resolutions as follows: fix $x \in T$ and let $B_n$ be the $n$-ball in the Cayley tree for $F_N$ with respect to $\mathcal{B}$. The partial isometries induced by elements of $\mathcal{B}$ on the convex hull $K(T, x_T, n)$ of $B_n x$ form a pseudo-group, which can be suspended to get a band complex $Y = Y(T, x, n)$, which is dual to a very small tree $T_n$, and $T_n \to T$ as $n \to \infty$. The geometric trees $T_n$ come with morphisms $f_n : T_n \to T$, and $T$ is geometric if and only if $f_n$ is an isometry for $n >> 0$. The band complex $Y$ decomposes via Imanishi’s theorem into a union of finitely many maximal families of parallel compact leaves, called families, and finitely many minimal components, which are glued together along singular leaves; each family $\mathcal{C}$ has a well-defined width, denoted $w(\mathcal{C})$, see also [8]. The family $\mathcal{C}$ is called a pseudo-annulus if every leaf contains an embedded copy of $S^1$; the family $\mathcal{C}$ is called non-annular if it is not a pseudo-annulus.

2.3. Dehn Twists. Let $T$ be a curve; for simplicity, assume $T/F_N$ is a circle. Choosing an edge $e$ in $T$ gives $F_N$ the structure of an HNN-extension $F_N = \langle a_1, \ldots, a_{N-1}, t, w'|w' = w^t \rangle$, where $w \in \langle a_1, \ldots, a_{N-1} \rangle$. The subgroup $V = \langle a_1, \ldots, a_{N-1}, w^t \rangle$ is the stabilizer of one of the endpoints of $e$, and the subgroup $\langle w \rangle$ is the stabilizer of $e$. The element $t$ is called the stable letter for this HNN-structure. Given this choice of $e$ one gets a Dehn twist automorphism $\tau$ of $F_N$, defined by $\tau(t) = tw$ and $\tau(a_i) = a_i$; the element $w$ is called the twistor. The class of $\tau$ in $Out(F_N)$ does not depend on the choice of $e$ and also is called a Dehn twist automorphism. Cohen and Lustig prove the following; see [4, Theorem 13.2].

**Proposition 2.2.** [4] Let $\tau$ be a Dehn twist with twistor $w$ corresponding to a curve $T$. If $T' \in \partial CV_N$ satisfies $l_{T'}(w) > 0$, then $\lim_{k \to \pm \infty} T'^{\tau_k} = T$.

If $T'$ satisfies the hypotheses of Proposition 2.2, then we simply say that $T'$ intersects $T$. We call $\tau$ as above the Dehn twist associated to the curve $T$; in light of Proposition 2.2, the ambiguity of replacing $\tau$ with $\tau^{-1}$ is not important. Notice that if $T'$ is a curve that is disjoint from $T$, then $T' \tau = T'$. If $\tau$ is the Dehn twist associated to a curve $T$, then we write $\tau = \tau(T)$; dually, we define $T_\tau$ to be the unique curve satisfying $\tau(T_\tau) = \tau$.

3. Forcing Arational Limits

Here is the first part of our adaptation of Gabai’s procedure; we use this result to construct arational trees. Throughout this section, we blur the
distinction between a factor and its representatives, arguing with subgroups and their conjugacy classes as needed.

**Proposition 3.1.** Let $F$ be a factor, and let $T$ and $T'$ be disjoint curves with neighborhoods $T \in U$, $T' \in U'$. There are disjoint curves $T_1, T'_1$ with neighborhoods $T_1 \in U_1 \subseteq U$, $T'_1 \in U'_1 \subseteq U'$, such that for any $S \in U_1 \cup U'_1$, $S_F$ is free and simplicial.

*Proof.* We will do the proof in the case where both $T$ and $T'$ are splittings with one loop-edge, and the common refinement is a graph with one vertex and two loop edges. The other cases are similar, are easier, and are left as exercises to the reader.

Let $V$ and $V'$ be the vertex groups of $T$ and $T'$, respectively, and let $A$ be the vertex group of the refinement. Let $w$ and $w'$ be the generators of the edge groups of $T$ and $T'$ respectively, and let $t$ and $t'$ be the respective stable letters. Let $A = \langle a_1, \ldots, a_{N-3}, w, w', w'' \rangle$ with $w \in \langle a_1, \ldots, a_{N-3} \rangle$, and where $\{a_1, \ldots, a_{N-3}, t, t', w'\}$ is a basis for $F_N$. Note that $V = \langle A, t' \rangle$ and $V' = \langle A, t \rangle$.

It is not the case that $F$ can contain $V$ or $V'$; indeed, both $V$ and $V'$ strictly contain co-rank 1 factors.

We need to modify $T, T'$ to get new curves in $U, U'$, respectively, so that $F$ does not intersect the vertex groups of the new curves. We accomplish this in several steps, each of which removes certain kinds of intersections; we will appeal to Proposition 2.2 to move curves into $U, U'$.

First suppose that $F$ contains (after choosing a conjugacy representative) $\langle a_1, \ldots, a_{N-3}, t \rangle$. Note that $F$ cannot also contain both $w'$ and $t'$, or else $F = F_N$. Let $\varepsilon$ be whichever of these letters is not contained in $F$. Let $f$ be the automorphism that sends $a_1 \mapsto a_1 \varepsilon$ and that is the identity on the other basis elements. Replace $T$ with the tree $Tf^{-1}$, and replace $a_1$ by $f(a_1)$; note that $Tf^{-1}$ satisfies the hypotheses of the proposition. Furthermore, $F$ does not contain $\langle a_1, \ldots, a_{N-3}, t \rangle$.

**Step 1:** By Howson’s Theorem (see [17]), there are only finitely many conjugacy classes of intersection of $F$ with $\langle a_1, \ldots, a_{N-3}, t \rangle$. Hence, by applying a sufficiently high power of a fully irreducible automorphism on the factor $\langle a_1, \ldots, a_{N-3}, t \rangle$, say $\varphi$, and extending $\varphi$ to $F_N$ by sending $w' \mapsto w'$ and $t' \mapsto t'$, we can guarantee that in the tree $T\varphi^{-1} := T_{1/2}$, conjugates of $F$ intersect the vertex group $\varphi(V) = V_{1/2} = \langle \varphi(a_1), \ldots, \varphi(a_{N-3}), w', t', \varphi(w') \rangle$ only in elements which contain instances of $w', t'$, and $\varphi(w')$ (i.e. no element in the intersection can be contained in any subfactor of $\varphi(\langle a_1, \ldots, a_{N-3}, t \rangle)$).

The edge group of $T_{1/2}$ is generated by $\varphi(w)$, whose reduced form must be a word containing some instances of $t$, and hence it is hyperbolic in $T$. Also, by construction of $\varphi$, $T_{1/2}$ remains commonly refined with $T'$. The vertex group of the refinement is $\varphi(A) = A_{1/2}$.

By Proposition 2.2 we can apply a high power of the Dehn twist $\tau = \tau(T)$ to $T_{1/2}$ to move $T_{1/2}$ into $U$; we use $T_{1/2}$ to denote this new curve as well;
note that by applying \( \tau \) we could not introduce new intersections of \( F \) with \( V_{1/2} \) that meet \( V \) non-trivially, as \( V \) is fixed by \( \tau \).

**Step 2:** Now we perturb the tree \( T' \). Consider an automorphism \( \varphi \) of the factor \( \langle w', t', t \rangle \) which sends \( w' \mapsto w' \varphi(t) \), \( t' \mapsto t' \varphi(t) \) and \( \varphi(t) \mapsto \varphi(t) \). Extend \( \varphi \) to \( F_N \). Since the intersection of any conjugate of \( F \) with \( V_{1/2} \) cannot contain \( \varphi(t) \), it follows that conjugates of \( F \) intersect the \( A_{1/2} \) only in elements whose reduced form must contain instances of \( \varphi(w') \) (that is, any elements in the intersection of \( F \) with \( \langle \varphi(a_1), \ldots, \varphi(a_{N-3}), \varphi(w'), \varphi(w^t), \varphi(w^t') \rangle \) cannot be contained in \( \langle \varphi(a_1), \ldots, \varphi(a_{N-3}), \varphi(w'), \varphi(w^{t}) \rangle \)).

Denote the tree \( T' \varphi^{-1} \) by \( T'_{1/2} \), so \( T'_{1/2} \) is disjoint from \( T_{1/2} \). Denote by \( A'_{1/2} := \varphi(A_{1/2}) \) the vertex group of the common refinement.

**Step 3:** Now we go back to \( T_{1/2} \). By the remark at the end of Step 2, if we apply an automorphism \( g \) of \( V_{1/2} = \langle A'_{1/2}, \varphi(t') \rangle \) by \( t \mapsto t \varphi(t') \) and extending to \( F_N \), in the resulting tree \( T_{1/2}g^{-1} = T_1 \), no conjugate of \( F \) non-trivially intersects the vertex group \( V_1 = g(V_{1/2}) \).

Again, using Proposition 2.2 with \( \tau = \tau(T_{1/2}) \), we can \( T_1 \) from the previous paragraph into \( U \); call the new curve \( T_1 \) as well. By construction, \( F \) does not meet the vertex group of \( T_1 \) non-trivially.

Repeat the same process for \( T'_{1/2} \) to obtain a tree \( T'_1 \) in \( U' \), with \( T'_1 \) disjoint from \( T_1 \) and such that \( F \) does not non-trivially intersect the vertex group of \( T'_1 \).

To finish we apply Lemma 2.1 along with the fact that \( \text{cn}(F) \) is open in \( \text{cn}(F) \) to find neighborhoods \( U_1, U'_1 \) with \( T_1 \in U_1 \subseteq U \) and \( T'_1 \in U'_1 \subseteq U' \), as desired.

\[ \square \]

4. **Forcing Non-Geometric Limits**

In this section we bring the second part of our adaptation of Gabai’s procedure; we use the main result of this section to construct non-geometric trees as limits of curves. The reader is assumed to be familiar with Rips theory [2]. First, we record the following:

**Lemma 4.1.** Suppose that \( Y(T, x_T, n) \) contains a non-annular family of width \( w \). For any \( \epsilon > 0 \), there is \( \delta > 0 \) such that if \( T' \) is \( \delta \)-close to \( T \), then \( Y(T', x_{T'}, n) \) contains a non-annular family of width \( w - \epsilon \).

**Proof.** Let \( C \) be a non-annular family of width \( w \). Suppose \( C \) intersects \( K(T, x_T, n) \) in the intervals \( b_0, b_1, b_2, \ldots, b_k \). Use \( \varphi_{i,j} \) to denote the partial isometry, which corresponds to an element of \( \mathbb{B}_\pm \), that maps \( b_i \) to \( b_j \).

As \( C \) is a family, the interior of each \( b_i \) is disjoint from the set of extremal points of \( \text{dom}(\varphi_b) \), for every partial isometry \( \varphi_b \) corresponding to \( b \in \mathbb{B}_\pm \); further, the orbit of each point of \( b_i \) is finite and is contained in \( C \), and the length of each \( b_i \) is \( w \). Note that by definition of the Gromov-Hausdorff topology, for any \( \eta > 0 \), there is \( \delta' > 0 \), such that for \( T' \) \( \delta' \)-close to \( T \),
there is a $(1 + \eta)$ bi-Lipschitz, $B_n$-equivariant map $f = f(T, T', n)$ from an
$\eta$-dense subtree of $K(T, x_T, n)$ onto an $\eta$-dense subtree of $K(T', x_{T'}, n)$; this
uses that the $K(\cdot, \cdot)$’s are trees.

Now, choose $\eta$ small enough so that $2k\eta(1 + \eta) << \epsilon$, and let $T'$ be $\delta'$-close
to $T$ with $\delta'$ as in the previous paragraph. Use $b'_i$ for the $f$-image of $b_i$, $\varphi'_{i,j}$
for the corresponding partial isometries of $K(T', x_{T'}, n)$. Note that $\varphi'_{i,i+1}$ is
defined on a central segment $I'_i$ of length at least $w/(1 + \eta) - 2\eta(1 + \eta)$
and that $\varphi'_{i,i+1}(I_i)$ overlaps with $b'_i$ in a central segment of width at least
$w/(1 + \eta) - 4\eta(1 + \eta)$. Hence, there is a central segment $J_0 \subseteq b'_0$ of length at
least $w/(1 + \eta) - 2k\eta(1 + \eta)$ with $\varphi'_{0,j}(J_0)$ contained in the central segment
of length $w/(1 + \eta) - 2\eta(1 + \eta)$ of $b'_j$. Hence, no $\varphi'_{0,j}$-image of $J_0$ can meet
an extremal point of $\text{dom}(\varphi'_{i,i+1})$.

Let $C'$ be the union of leaves in $Y(T', x_{T'}, n)$ containing the points of $J_0$.
Note that by choosing $T'$ $\delta'$-close to $T$, we have ensured that $\varphi'_{0,1}$ is
the only partial isometry from $B^\pm$ that is defined on $J_0$, since this is true in
$Y(T, x_T, n)$; similarly, $(\varphi'_{k-1,k})^{-1}$ is the only element of $B^\pm$ that is defined
on $\varphi'_{0,k}(J_0)$. Hence, $C'$ is contained in a family, which has width at least
$w - \epsilon$, as desired. \hfill \Box

\begin{proposition}
Let $T$ and $T'$ be disjoint curves with neighborhoods $T \in U$, $T' \in U'$, and let $n \in \mathbb{N}$ be given. There are disjoint curves $T_1, T'_1$ with
neighborhoods $T_1 \in U_1 \subseteq U$, $T'_1 \in U'_1 \subseteq U'$, such that for any $S \in U_1 \cup U'_1$, $Y(S, x_S, n)$ contains a non-annular family of width bounded away from zero.
\end{proposition}

\begin{proof}
We begin with an observation: if $A \in \partial c\gamma_N$ and if $y \in A$ is a point
that is fixed by a subgroup $H \leq F_N$, then $H$ fixes a point in $A_n$ if and only
if $H^g = \langle h_1, \ldots, h_r \rangle$ for $\{h_1, \ldots, h_r\} \subseteq B_n$, where $g \in F_N$ is chosen so that
$H^g$ fixes $gy \in K(A, x_0, 1)$ ($H^g$ is cyclically reduced with respect to $B$). It
follows that if $A$ is the curve with vertex group $V = \langle a_1, \ldots, a_{N-2}, w, w' \rangle$, edge stabilizer $\langle w \rangle$, and stable letter $t$, then $A_n$ contains an edge with non-
trivial stabilizer if and only if $w \in H$ and $w' \in H_t$, with $H, H_t$ fixing points
in $A$ and having generating sets contained in $B_n$.

In light of the discussion in the above paragraph, we proceed along the
lines of the proof of Proposition 3.1. For simplicity, we assume that $B = \{a_1, \ldots, a_{N-3}, w', t, t' \}$ is a basis for $F_N$ so that the vertex groups $V, V'$ of
$T, T'$ are $V = \langle a_1, \ldots, a_{N-3}, w, w' \rangle$, $V' = \langle a_1, \ldots, a_{N-3}, t, w', w t' \rangle$; any
other basis gives a quasi-isometric word metric, and the below argument is
more cumbersome with arbitrary $B$.

Replace $t$ with $tu$, where $u$ is a long random word in $\langle w', t' \rangle$. Next, as in
Step 1 of the proof of Proposition 3.1, find an irreducible automorphism $\varphi$
of $\langle a_1, \ldots, a_{N-3}, t \rangle$ with large dilatation and extend $\varphi$ to $F_N$ in the obvious
way. Use $\psi$ to denote the composition of these two automorphisms. Apply
a high power of $\tau(T)$ to $T \psi$ to get $T_{1/2} \in U$. Note that the stable letter for
$T_1$ has very long length with respect to $B$. \hfill \Box
Now, repeat the procedure from the above paragraph to $T'$ to get $T'_{1/2}$ in $U'$ with stable letter having very long length with respect to $\mathcal{B}$. Now repeat both these operations on $T_{1/2}$ and $T'_{1/2}$ to get $T_1$ and $T'_1$.

By the discussion in the first paragraph of this proof, there is $w > 0$ so that both $Y(T_1, x_{T_1}, n)$ and $Y(T'_1, x_{T'_1}, n)$ contain a non-annular family of width at least $w$. Choose $\varepsilon << w$, and let $\delta > 0$ be as given by Lemma 4.1. The intersections of the $\delta$-balls around $T_1, T'_1$ with $U, U'$ give neighborhoods $U_1, U'_1$ of $T, T'$ satisfying the conclusions of the statement. \hfill \qed

5. The Proof of the Main Result

We will use Lemma 4.1 along with the following characterization of non-geometric trees with dense orbits. Note that arational trees have dense orbits [15].

**Lemma 5.1.** Let $T \in \partial \mathcal{CV}_N$ have dense orbits. The tree $T$ is non-geometric if and only if $Y(T, x_T, n)$ contains a non-annular family for every $n$.

**Proof.** By Imanishi’s theorem, if $T$ has dense orbits and is geometric, then for $n >> 0$, $Y(T, x_T, n)$ is a union of minimal components. Further, the space $Y(T, x_T, n)$ can contain an annular family only if $T$ contains a non-degenerate arc with non-trivial stabilizer, which is impossible if $T$ has dense orbits; see, for instance, [12]. \hfill \qed

Having $T_n \to T$ not exact can be thought of as $T$ being non-geometric on the scale $B_n$. Our interest in Lemmas 4.1 and 5.1 can be paraphrased as follows for trees $T$ with dense orbits: if $T$ is non-geometric on scale $B_n$, then for $T'$ close enough to $T$, $T'$ also is non-geometric on scale $B_n$; and if $T$ is non-geometric on the scale $B_n$ for every $n$, then $T$ is non-geometric. Now, we are in position to prove our main result.

**Theorem 5.2.** Let $T, T'$ be disjoint curves with neighborhoods $U, U'$. There is a 1-simplex of non-uniquely ergodic, arational, non-geometric trees with one extreme point in each of $U, U'$.

**Proof.** Enumerate all factors of $F_N$ as $F^1, F^2, \ldots, F^k, \ldots$. Set $T_0 = T$, $T'_0 = T'$ and $U_0 = U, U'_0 = U'$. We proceed inductively, defining for $k > 0$ $T_k, T'_k$ with neighborhoods $U_k \subseteq U_l, U'_k \subseteq U'_l$ for $l \leq k$, with $\{F^j\}_{j \leq l}$ acting freely and simplicially on any $S \in U_k \cup U'_k$ and with $Y(S, x_S, l)$ containing a non-annular family of width greater than $w(k) > 0$ for any $S \in U_k \cup U'_k$. Assume that $T_{k-1}, T'_{k-1}$ are defined. To define $T_k, T'_k, U_k, U'_k$, first apply Proposition 3.1 to $T_{k-1}, T'_{k-1}, U_{k-1}, U'_{k-1}$, and then apply Proposition 4.2 to the result. By shrinking the neighborhoods from the conclusion of Proposition 3.1 slightly we can assume that they are contained in compact neighborhoods satisfying the same conclusions.

Note that each $U_k, U'_k$ is contained in a ball by construction, and the radii of these balls must go to zero (for example, by density of non-arational trees in $\partial \mathcal{CV}_N$). Hence, $T_k$ converge as to some $\lambda \in \partial \mathcal{CV}_N,$ and $T'_k$ converge to
some $\lambda'$. By construction, any factor of $F_N$ acts freely and simplicially on $\lambda$ and $\lambda'$. Further, by Lemma 5.1, $\lambda$ and $\lambda'$ are non-geometric.

Use $w_k$ to denote the edge stabilizer of $T_k$, and let $\eta_k$ be the counting current corresponding to $w_k$, so $\langle T_k, \eta_k \rangle = 0$. Since $T_k'$ is disjoint from $T_k$, we have that $\langle T_k', \eta_k \rangle = 0$ as well. Let $\eta$ be a representative of any accumulation point in projective current of the images of $\eta_k$. By continuity of $\langle \cdot, \cdot \rangle$, we have that $\langle \lambda, \eta \rangle = 0 = \langle \lambda', \eta \rangle$. Further, by the Kapovich-Lustig characterization of zero intersection, we have that $\emptyset \neq \text{Supp}(\eta) \subseteq L(\lambda) \cap L(\lambda')$.

By Theorem 4.4 of [3], we have that $L(\lambda) = L(\lambda')$, and by Theorem II of [5] any convex combination $\alpha \lambda + (1 - \alpha) \lambda'$ is the length function of a very small tree $T_\alpha$. On the other hand, from the definition of $L(\cdot)$, we certainly have that $L(\lambda) \subseteq L(T_\alpha)$; applying Theorem 4.4 of [3] again gives that $T_\alpha$ is arational with $L(T_\alpha) = L(\lambda)$. Hence, the segment $\{ T_\alpha | \alpha \in [0,1] \}$ satisfies the conclusion. \hfill $\square$

References

1. Mladen Bestvina and Mark Feighn, Outer limits (preprint), http://andromeda.rutgers.edu/feighn/papers/outer.pdf (1994).
2. __________, Stable actions of groups on real trees, Invent. Math. 121 (1995), no. 2, 287–321.
3. Mladen Bestvina and Patrick Reynolds, The boundary of the complex of free factors (preprint), arXiv:1211.3608 (2012).
4. Marshall Cohen and Martin Lustig, Very small group actions on R-trees and Dehn twist automorphisms, Topology 34 (1995), 575–617.
5. Thierry Coulbois, Arnaud Hilion, and Martin Lustig, Non-unique ergodicity, observers’ topology and the dual algebraic lamination for R-trees, Illinois J. Math. 51 (2007), no. 3, 897–911.
6. __________, R-trees and laminations for free groups. II. the dual lamination of an R-tree, J. Lond. Math. Soc. (2) 78 (2008), no. 3, 737–754.
7. David Gabai, Almost filling laminations and connectivity of ending lamination space, Geom. Top. 13 (2009), 1017–1041.
8. Damien Gaboriau, Gilbert Levitt, and Frédéric Paulin, Pseudogroups of isometries of R and Rips’ theorem on free actions on R-trees, Israel J. Math. 87 (1994), 403–428.
9. Vincent Guirardel, Dynamics of Out($F_n$) on the boundary of outer space, Ann. Sci. École Norm. Sup. 33 (2000), no. 4, 433–465.
10. Ilya Kapovich and Martin Lustig, Geometric intersection number and analogues of the curve complex for free groups, Geometry and Topology 13 (2009), 1805–1833.
11. __________, Intersection form, laminations and currents on free groups, Geom. Funct. Anal. 19 (2010), no. 5, 1426–1467.
12. Gilbert Levitt and Martin Lustig, Irreducible automorphisms of $F_n$ have north-south dynamics on compactified outer space, J. Inst. Math. Jussieu 2 1 (2003), 59–72.
13. Reiner Martin, Non-uniquely ergodic foliations of thin type, Ergodic Theory Dynam. Systems 17 (1997), no. 3, 667–674.
14. Patrick Reynolds, Dynamics of irreducible endomorphisms of $F_n$ (preprint), arXiv:1008.3659 (2010).
15. __________, Reducing systems for very small trees (preprint), arXiv:1211.3378 (2012).
16. Richard Skora, Deformations of length functions in groups, pre-print (1990).
17. John R. Stallings, Topology of finite graphs, Invent. Math. 71 (1983), no. 3, 551–565.
Department of Mathematics, University of Utah, 155 S 1400 E, Room 233, Salt Lake City, Utah 84112, USA

E-mail address: mann@math.utah.edu
E-mail address: reynolds@math.utah.edu