RATIONAL AND NON-RATIONAL ALGEBRAIC VARIETIES:
LECTURES OF JÁNOS KOLLÁR

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INTRODUCTION

Rational varieties are among the simplest possible algebraic varieties. Their study is as old as algebraic geometry itself, yet it remains a remarkably difficult area of research today. These notes offer an introduction to the study of rational varieties. We begin with the beautiful classical geometric approach to finding examples of rational varieties, and end with some subtle algebraic arguments that have recently established non-rationality of varieties that otherwise share many of their traits.

In lecture one, rationality and unirationality are defined, and illustrated with a series of examples. We also introduce some easily computable invariants, the plurigenera, that vanish for all rational varieties. Using the plurigenera, one develops a sense how rare rational varieties are. For example, we will immediately see that no smooth projective hypersurface whose degree exceeds its embedding dimension can be rational: no plane cubic curve is rational, no space quartic surface is rational, and so on.

The second and third lectures focus on the rationality question for smooth cubic surfaces over arbitrary fields, an issue thoroughly explored by B. Segre in the forties. This is a pretty story, depending subtly on the field of definition. It was already understood one hundred years ago that every cubic surface is rational over the complex numbers; however, the situation is quite complicated over the rational numbers. In the second lecture, we construct examples of non-rational smooth cubics over \( \mathbb{Q} \) by considering the orbits of the Galois group of \( \overline{\mathbb{Q}}/\mathbb{Q} \) on the twenty seven lines on the cubic surface. The construction makes use of a beautiful theorem of Segre: no smooth cubic surface of Picard number one can be rational. In the third lecture, we prove this theorem of Segre. Using the same techniques, we...
also prove the following stronger theorem of Manin: two smooth cubic surfaces of Picard number one are birationally equivalent if and only if they are projectively equivalent.

The cubic surface case is especially interesting in light of the fact that there exist no smooth rational cubic threefolds. This longstanding— and sometimes controversial— question was settled in the early seventies by Clemens and Griffiths; see Section 3.4. Still to this day, no one knows whether or not there exists a smooth rational quartic fourfold.

When the obvious numerical obstructions to rationality vanish, it can be difficult to determine whether or not a variety is rational. The purpose of the final two lectures is to show that indeed, there are abundant examples of non-rational, and even non-ruled, varieties of every dimension that would appear to be rational from the point of view of naïve numerical invariants. The examples of non-ruled smooth Fano varieties (definition 4.1) are defined over fields of characteristic zero, but the only known proof that they are non-ruled uses the technique of reduction to prime characteristic. The characteristic $p$ argument, presented in Lecture 4, relies heavily on the special nature of differentiation in prime characteristic. In Lecture 5, the characteristic zero theorems are deduced from the characteristic $p$ results.

These techniques are applied in an appendix written by Joel Rosenberg to construct new explicit examples of varieties defined over $\mathbb{Q}$ that are Fano but not ruled.

The notes have been written with the goal of making them accessible to students with a basic training in algebraic geometry at the level of [H]. The first lecture is to be relatively easy, with subsequent lectures requiring more of the reader. The first three lectures are in the realm of classical algebraic geometry, while a scheme theoretic approach is indispensible for the last lecture.

Finally, included are detailed solutions to nearly all the exercises scattered throughout the notes. The exercises formed an important part of the summer course, as they form an important part of the written version. Many of the solution ideas were worked out together with the summer school participants, especially Sándor Kovács, Gábor Megyesi, and Endre Szabó. Finally, special thanks are due to Károly Böröczky Jr. for organizing the summer school, and to the European Mathematical Society for funding it.

**Notation.**

Varieties defined over non-algebraically closed fields occupy a central position in our study. The notation for the ground field is suppressed when the field is clear from the context or irrelevant, but occasionally we write $X/k$ to emphasize that the variety $X$ is defined over the ground field $k$. The algebraic closure of $k$ is denoted $\bar{k}$.

Varieties are assumed reduced and irreducible, except where explicitly stated otherwise. Because we are concerned with birational properties, there is no loss of generality in assuming all varieties to be quasi-projective. In any case, our main interest is in smooth projective varieties.

Morphisms and rational maps between varieties are always assumed to be defined
over the ground field, except where explicitly stated otherwise. Likewise, linear
systems on a variety $X/k$ are assumed defined over $k$.

Morphisms are denoted by solid arrows $\rightarrow$ and rational maps by dotted arrows $\dashrightarrow$. The “image” of a rational map is the closure of the image of the morphism obtained by restricting the rational map to some non-empty open set where it is defined; in the same way, we define the image of a subvariety under a rational map, provided that the map is defined at its generic point. In the case of birational maps, the image of a subvariety is also called the birational transform. Likewise, the graph of a rational map $X \phi \rightarrow Y$ is the closure in $X \times Y$ of the graph of the morphism obtained by restricting $\phi$ to a non-empty open set where it is defined.

If $L$ is any field containing $k$, the symbol $X(L)$ denotes set of the $L$-rational points, or $L$-points for short, of $X$.

The final lecture deals with schemes quasi-projective over an arbitrary affine base scheme $S$. In this case, all morphisms and all rational maps are assumed defined over $S$. Likewise, all products are defined over $S$.

1. **First Lecture**

We begin with the precise definitions of rationality and unirationality.

**1.1. Definition.** A variety $X$ is rational if there exists a birational map $\mathbb{P}^n \phi \rightarrow X$, that is, if $X$ is birationally equivalent to projective space.

According to the conventions agreed upon in the introduction, implicit in the above definition is an unnamed ground field $k$ and the birational map $\phi$ is assumed to be defined over $k$. Occasionally, we will emphasize the ground field by saying that “$X$ is rational over $k$.” Of course, if $X$ is defined over $k$ and $L$ is any field extension of $L$, then $X$ may also be considered to be defined over $L$. If $X$ is rational over $k$, then $X$ is also rational over $L$.

**1.2. Definition.** A variety $X$ is unirational if there exists a generically finite dominant rational map $\mathbb{P}^n \phi \rightarrow X$.

Roughly speaking, a variety is unirational if a dense open subset is parameterized by projective space, and rational if such a parametrization is one-to-one.

The disconcerting use of the prefix “uni” in referring to a map which is finite-to-one makes more sense when viewed in historical context. Rational varieties were once called “birational,” in reference to the rational maps between them and projective space in each direction. “Unirationality” thus refers to the map from $\mathbb{P}^n$ to the variety, defined in one direction only.

These lectures treat the following general question: Which varieties are rational or unirational?

It is important to realize that the rationality or unirationality of a variety may depend on what we take to be the field of definition. For example, a variety $X$ defined over $\mathbb{Q}$ may be considered as a variety defined over $\mathbb{R}$. It is possible that
there is a birational map given by polynomials with real coefficients from projective space to $X$, but there is no such birational map from given by polynomials with rational coefficients. Our first example nicely illustrates this point.

1.3. Quadrics. A smooth quadric hypersurface in projective space is rational over $k$ if and only if it has at least one $k$-point. In particular, every smooth quadric over an algebraically closed field is rational.

Proof. If a variety $X$ is unirational over $k$, then it has a $k$-point. This obvious when $k$ is infinite: the map $\mathbb{P}^n \dasharrow X$ is defined on some Zariski open subset of $\mathbb{P}^n$ and because $\mathbb{P}^n$ has plenty of $k$-points in every open set, the image of any one of them under $\phi$ will be a $k$-point of $X$. This is not so obvious when $k$ is finite, because $\mathbb{P}^n$ has open sets with no $k$-points. The finite field case follows from Nishimura’s Lemma (Exercise 1 below).

Conversely, let $X$ be a smooth quadric in $\mathbb{P}^{n+1}$, defined over $k$, and with a $k$-point $P \in X(k)$. Let $\phi$ be the projection from $P$ onto any $n$-plane not containing $P$. Choosing coordinates so that $P = (0 : 0 : \cdots : 0 : 1)$, we have

$$\mathbb{P}^{n+1} \dasharrow \mathbb{P}^n$$

$$(x_0 : \cdots : x_{n+1}) \mapsto (x_0 : \cdots : x_n).$$

Restricting $\pi$ to $X$, we expect a generically one-to-one map of $X - \{P\}$ to $\mathbb{P}^n$; the exceptions occur when an entire line on $X$ is collapsed to a point by the projection. But most points can not lie on lines that are collapsed by this map, as this would force $P$ to be a singular point of $X$. □

Exercise 1: Prove Nishimura’s Lemma: If $Y$ is smooth, $Y'$ is projective, and there is a rational map $Y \dasharrow Y'$, then if $Y$ has a $k$-point, so does $Y'$. Also, find a counterexample when $Y$ is not smooth.

Every smooth cubic surface in $\mathbb{P}^3$ defined over an algebraically closed field is rational, since it is isomorphic to the blowup of $\mathbb{P}^2$ at six points [H, p 395]. The rationality question for cubic surfaces over a non-algebraically closed field is more subtle. This will be our main focus in lectures two and three. For now, we discuss a few simple examples to indicate their richness.

1.4. A non-rational cubic surface. Let $X$ be a cubic surface in real projective three space, defined by an equation (in affine coordinates) $x^2 + y^2 = f_3(z)$, where $f_3$ has three distinct real roots. Then $X$ is not rational over $\mathbb{R}$.

Proof. Consider the graph of $f_3$. The set of points in the source $\mathbb{R}$ where $f_3$ takes positive values has two disjoint components. The equation $x^2 + y^2 = f_3(z)$ has real solutions (in fact, a circle’s worth) if and only if $f_3 \geq 0$, so we see that as a real manifold $X(\mathbb{R})$ has two distinct components. But if $X$ is birationally equivalent to $\mathbb{P}^2$ over $\mathbb{R}$, then because $\mathbb{P}^2(\mathbb{R})$ is connected as a real manifold, so would be $X(\mathbb{R})$. □
However, it turns out that the cubic surface $X$ defined above is unirational over $\mathbb{R}$. We leave the reader the pleasure of finding a map $\mathbb{P}^2_{\mathbb{R}} \dashrightarrow X$ that is two-to-one onto one of the manifold components and misses the other component entirely. Indeed, the preimage of a point in the missed manifold component can be interpreted as a pair of complex conjugate points in the complex manifold $\mathbb{P}^2(\mathbb{C})$.

1.5. Singular cubics. An irreducible cubic hypersurface in projective space (that is not a cone over a cubic hypersurface of lower dimension) is rational over $k$ if it has a singular $k$-point.

Proof. Let $X$ be the cubic hypersurface in $\mathbb{P}^{n+1}$, defined over the ground field $k$. Project from the singular point $P \in X(k)$ onto a general hyperplane defined over $k$. Since $P$ has multiplicity two on $X$, any line through $P$ has a unique third point of intersection with $X$. Its projection onto the hyperplane gives the one-to-one map from $X$ to $\mathbb{P}^n$. Of course, this makes sense only when the line through $P$ does not lie on $X$. This is where we use the assumption that $X$ is not a cone: because $X$ is not a cone, the generic line through $P$ does not lie on $X$. □

1.6. Rationality of cubic hypersurfaces. If a smooth cubic hypersurface of even dimension contains two disjoint linear spaces, each of half the dimension, then the cubic hypersurface is rational. In particular, a smooth cubic surface is rational over $k$ if it contains two skew lines defined over $k$ (of the twenty seven lines on the surface defined over $\bar{k}$).

Proof. Let $X \subset \mathbb{P}^{2n+1}$ be the cubic hypersurface, and let $L_1$ and $L_2$ be the two linear spaces on $X$. Consider the map

$$L_1 \times L_2 \xrightarrow{\phi} X$$

$$(P, Q) \mapsto \text{third intersection point } X \cap \overline{PQ}.$$ 

This defines a birational map from $L_1 \times L_2$ to $X$. The map is well defined because each line intersects $X$ in exactly three points (counting multiplicities). This map is birational: if the pre-image of $x \in X$ includes two distinct pairs $(P_1, Q_1)$ and $(P_2, Q_2)$ on $L_1 \times L_2$, then the projections of the linear spaces $L_1$ and $L_2$ from $x$ onto a general hyperplane would intersect each other in more than one point, which is impossible (see 1.7 for a more general discussion). Because

$$\mathbb{P}^{2n} \dashrightarrow \mathbb{P}^n \times \mathbb{P}^n \dashrightarrow L_1 \times L_2 \dashrightarrow X$$

are birational equivalences, we conclude that $X$ is rational. Note that all maps above are defined over the ground field $k$. □

Exercise 2:

1. Find examples of smooth cubic hypersurfaces in $\mathbb{P}^{2n+1}$ containing two disjoint $n$-planes.
2. What is the dimension of the variety of all such cubics?
3. Why have we not considered linear spaces of non-equal dimension?
1.7. Discussion. More generally, given any two subvarieties, $U$ and $V$, of a degree three hypersurface $X$, one is tempted to form a similar map:

$$
\phi : U \times V \rightarrow X
$$

$$(u, v) \mapsto \text{third intersection point } X \cap \overline{uv}.
$$

If $U$ and $V$ are disjoint, this map is a morphism except at pairs of points $(u, v)$ spanning a line on $X$; in general, it is not defined on $U \cap V$.

The map $\phi$ can not be dominant unless $\dim U + \dim V \geq \dim X$ and it can not be generically finite unless $\dim U + \dim V = \dim X$. When $\phi$ is finite, how does one compute its degree?

To determine the pre-image of a general point $x \in X$, consider the projection $\pi_x$ from $x \in X$ to a general hyperplane. The set $\pi_x(U) \cap \pi_x(V)$ consists of all points $\pi_x(u) = \pi_x(v)$, with $u, v,$ and $x$ collinear. In this case, assuming that $u \neq v$, the points $(u, v) \in U \times V$ are the pre-images of $x$ under $\phi$. So, if $U \cap V = \emptyset$, we expect that the degree of $\phi$ is the cardinality of $\pi_x(U) \cap \pi_x(V)$. More generally, we must subtract something for the intersection points of $U$ and $V$.

In Example 1.6, we applied this idea with $U$ and $V$ linear subspaces and deduced that cubic hypersurfaces are rational if they contain two disjoint linear subvarieties of half the dimension. More generally, the idea is useful for detecting unirationality of some cubics, as the next example shows.

1.8. Unirationality of Cubic Surfaces. It is easy to see that a smooth cubic surface in $\mathbb{P}^3$ containing two non-coplanar rational curves is unirational. (We remind the reader that implicit in this statement is that both curves are defined over the ground field $k$, and that the surface is unirational over $k$.)

Indeed, let $C_1$ and $C_2$ be rational curves on the surface $X$, and define the map $C_1 \times C_2 \rightarrow X$ as above. Because $C_1$ and $C_2$ do not lie in the same plane, their join (meaning the locus of points lying on lines joining points on $C_1$ to points on $C_2$) must be all of $\mathbb{P}^3$. This ensures that the map $\phi$ is dominant, and hence finite. Because $C_1$ and $C_2$ are rational (over $k$), we conclude that $X$ is unirational (over $k$).

1.8.1. Using 1.8, we can easily deduce that a sufficiently general cubic surface containing two $k$-points will be unirational. Indeed, two rational curves $C_1/k$ and $C_2/k$ can be found by intersecting $X$ with the tangent plane at each of the two $k$-points. Assuming both $C_1$ and $C_2$ are irreducible over $\bar{k}$ (as usually happens), each is a plane cubic curve with a singular $k$-point, and hence rational by Example 1.5. Furthermore, $C_1$ and $C_2$ are not coplanar; otherwise, their union is a plane section of the cubic surface, and hence a plane cubic, so it could have at most one singular point. Using the map $C_1 \times C_2 \rightarrow X$ defined in 1.7, we conclude that $X$ is unirational. Finally, note that $\phi$ is degree six—the plane projections of $C_1$ and $C_2$ are both cubic plane curves, so they intersect in nine points, but three of these nine points come from the intersection points of $C_1$ and $C_2$.

Even in the degenerate case where $C_1$ or $C_2$ is reducible over $\bar{k}$, the argument often goes through unchanged. If either $C_1$ or $C_2$ is a union of a line and an
irreducible quadric, then each of these components is defined and rational over $k$, and the above argument goes through using these rational curves—assuming they are not coplanar. The only way in which they can be coplanar is when $C_1 = C_2$ is a plane section of $X$ consisting of three lines intersecting in three distinct points. In this case, the line through the original two $k$-rational points is a $k$-rational line on $X$, and so at least when $k$ is infinite it contains many $k$-rational points on $X$. Applying the argument to a generic pair of these points, we again see that $X$ is unirational.

There is one honest exceptional case where this argument breaks down: the tangent section curve $C_1$ or $C_2$ could be a union of three lines with none of these lines defined over $k$. The only way this can happen is when the three lines meet in a single $k$-rational point, which actually happens in some interesting examples. A point $P$ on a cubic surface such that $T_PX \cap X$ is a union of three lines is called an Eckardt point. A cubic surface can have at most finitely many Eckardt points, since they occur exactly when three of the twenty seven lines on the surface intersect in a single point. Indeed, cubic surfaces with Eckardt points are rather special among all cubics; see [Ec].

**1.8.2.** The above argument shows that a smooth cubic surface containing a single $k$-rational point (that is not an Eckardt point) is unirational, at least over an infinite field. The point is that that tangent plane to this point intersects the surface in a singular cubic, giving rise to a rational curve on $X$. This curve contains plenty of $k$-points, and so we can apply 1.8.1. This argument is due to B. Segre [S43]. In fact, Segre later showed that if a smooth cubic surface (over an infinite field $k$) contains a $k$-point, then it contains infinitely many $k$-points [S51]. It follows from the argument described above that any smooth cubic surface containing a $k$-point is unirational.

**1.8.3.** An interesting variation on the map discussed in 1.7 is when we allow $U = V$. For example, suppose that $X$ is a smooth cubic four-fold in $\mathbb{P}^5$ containing a smooth surface $S$.

Consider the map

$$\frac{S \times S}{\sim} \xrightarrow{\phi} X$$

$$(P, Q) \mapsto \text{third point of intersection } X \cap PQ.$$ 

Here, $\frac{S \times S}{\sim}$ is the symmetric product of $S$, the quotient variety of $S \times S$ by the action of the two-element group interchanging the factors. If $S$ is unirational over $k$, then so is $S \times S$, and hence so is the image $\frac{S \times S}{\sim}$ under the generically two-to-one quotient map.

Consider a general point $x \in X$, say not on $S$. When is $x$ in the image of $\phi$? Consider the family of lines $\{sx\}_{s \in S}$. The point $x$ is in the image of $\phi$ precisely when at least one of these lines intersects $S$ in a point other than $s$. In particular, the projection from $x$, $S \xrightarrow{\pi_x} S' \subset \mathbb{P}^4$ can not be one-to-one. Indeed, $x$ has a unique pre-image under $\phi$ precisely when the projection $\pi_x$ collapses exactly two points of
S to a single point. On the other hand, if $S \xrightarrow{\pi_x} S'$ is not of degree one, then $x$ will have infinitely many preimages under $\phi$.

Thus $\phi$ is finite and dominant if and only if the generic projection of $S$ from a point $x \in X$ is one-to-one except on a finite set. The next exercise provides one case where this condition can be verified.

**Exercise 3.** Find examples of smooth surfaces in $\mathbb{P}^5$ such that the generic projection to $\mathbb{P}^4$ has exactly one singular point. Use this to give some more examples of unirational cubic four-folds. (Hint: Consider the linear system of plane cubics through four points.)

1.9. **Numerical Constraints.** Rationality and unirationality force strong numerical constraints on a variety. Let $\Omega_X$ be the sheaf of differential forms (Kähler differentials) of $X$ over $k$.

1.10. **Theorem.** If a smooth projective variety $X$ is rational, then $H^0(X, (\Omega_X)^\otimes m)$ is zero for all $m \geq 1$. The same holds for unirational $X$, provided the ground field has characteristic zero.

**Proof.** Suppose we have a generically finite, dominant map $\mathbb{P}^n \xrightarrow{\phi} X$. Let $U \subset \mathbb{P}^n$ be an open set over which $\phi$ is defined; its complement may be assumed to have codimension at least two.

Non-zero differential forms on $X$ pull back to non-zero differential forms on $U$, that is, we have an inclusion $(\phi^* \Omega^1_X)^\otimes m \hookrightarrow (\Omega^1_U)^\otimes m$. This is obvious when $\phi$ is birational, and easy to check when $\phi$ is finite (assuming $k$ has characteristic zero). Because the complement of $U$ has codimension at least two, the differential forms on $U$ extend uniquely to forms on $\mathbb{P}^n$, so that

$$H^0(U, (\Omega^1_U)^\otimes m) = H^0(\mathbb{P}^n, (\Omega^1_{\mathbb{P}^n})^\otimes m).$$

Therefore $H^0(X, (\Omega_X^1)^\otimes m) \subset H^0(\mathbb{P}^n, (\Omega^1_{\mathbb{P}^n})^\otimes m)$, and the problem is reduced to proving the vanishing for $\mathbb{P}^n$, left as Exercise 4. □

**Exercise 4.** Complete the proof by showing that $H^0(\mathbb{P}^n, (\Omega^1_{\mathbb{P}^n})^\otimes m)$ is zero.

1.10.1. **Remark.** In prime characteristic, unirationality of $X$ does not necessarily force the vanishing of the invariants $H^0(\Omega_X^\otimes m)$. Indeed, the pull back map for differential forms can be the zero map, so the argument above fails. For example, consider the Frobenius map $F$ on $\mathbb{A}^n$ sending $(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1^p, \ldots, \lambda_n^p)$, where $p > 0$ is the characteristic of the ground field. The induced map of differential forms $F^* \Omega \rightarrow \Omega$ sends every differential $dx$ to $d(x^p) = px^{p-1}dx = 0$.

In characteristic $p$, we are led to the more sensible notion of separable unirationality. A variety $X$ is separably unirational if there is a dominant generically finite map $\mathbb{P}^n \rightarrow X$ such that the induced inclusion of function fields is separable. In other words, separably unirational is equivalent to “unirational by a generically étale map.” By the definition of étale, the pull back map of differential forms is injective (in fact, an isomorphism) on a dense open set, so the proof of 1.10 shows that $H^0(X, \Omega_X^\otimes m) = 0$ for a smooth projective separably unirational variety $X$ of arbitrary characteristic.
1.11. Corollary. If $X$ is a smooth projective variety that is separably unirational (for example, rational), then $H^0(X, \mathcal{O}(mK_X)) = 0$ for all $m \geq 1$.

Here $K_X$ denotes the canonical class of $X$, that is, the divisor class representing the canonical sheaf $\omega_X = \wedge^n \Omega_X$ of highest differential forms. The dimension of the $k$-vector space $H^0(X, \mathcal{O}(mK_X))$ is called the $m^{th}$ plurigenus of $X$, and is denoted $h^0(mK_X)$. The plurigenera are easily computable obstructions to rationality.

It is conjectured that for a smooth projective variety, the plurigenera are the only obstructions to a related property of varieties called uniruledness; see Definition 4.2. A variety for which all plurigenera vanish is said to have negative Kodaira dimension (or Kodaira dimension $-\infty$). An understanding of these varieties is an essential feature of Mori’s Minimal Model Program for the birational classification of algebraic varieties; see [KaMM]. Miyaoka proved that uniruledness is equivalent to negative Kodaira dimension for smooth three-folds of characteristic zero [Mi]; see also [S-B]. The conjecture remains open in higher dimension.

A similar conjecture, attributed to Mumford, predicts that the vanishing of $H^0(X, \Omega_X^{\otimes m})$ for all $m \geq 0$ is the only obstruction to a smooth projective variety $X$ being rationally connected, at least in characteristic zero. A variety is said to be rationally connected if every pair of points can be joined by a rational curve. See [K96, p202] for a discussion of this conjecture, which has recently been proved in dimension three by Kollár, Miyaoka and Mori [KoMM].

Exercise 5.

(1) Prove Corollary 1.11.

(2) Show that the plurigenera of a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ do not vanish when $d > n$. Conclude that no smooth hypersurface whose degree exceeds its embedding dimension is separably unirational.

Of course, in characteristic zero, separable unirationality is the same as unirationality. In prime characteristic, unirational but not separably unirational varieties exist, and in fact, there are unirational hypersurfaces of arbitrary degree:

Exercise 6.

(1) Show that a purely inseparable cover of a unirational variety over a perfect field is unirational.

(2) Show that there exist unirational hypersurfaces of arbitrarily large degree relative to their embedding dimension.

The next four exercises provide more examples of rational and non-rational varieties.

Exercise 7.

(1) Let $k = \mathbb{C}(t)$, and let $X$ be a degree $d$ hypersurface in $\mathbb{P}^n$ defined over $k$. Prove that if $d \leq n$, then $X$ has at least one $k$ point. Find an example
with exactly one \(k\)-point. For \(d > n\), find a hypersurface with no \(k\)-points. Explain why such a hypersurface is non-rational.

(2) Same as (1), but with \(k = \mathbb{C}(t, s)\) and \(d^2 \leq n\).

**Exercise 8.**

(1) Let \(X_{a,2} \subset \mathbb{P}^1 \times \mathbb{P}^n\) be a smooth hypersurface of bi-degree \((a, 2)\). For \(n \geq 2\), show that \(X_{a,2}\) is rational over \(\mathbb{C}\).

(2) Let \(X_{a,2} \subset \mathbb{P}^2 \times \mathbb{P}^n\) be a smooth hypersurface of bi-degree \((a, 2)\). For \(n \geq 4\), show that \(X_{a,2}\) is rational over \(\mathbb{C}\).

**Exercise 9.**

(1) Prove that the variety of \(m \times n\) matrices of rank at most \(t\) is rational over any field. Find its non-smooth locus.

(2) Consider an \(n \times n\) array of general linear forms on \(\mathbb{P}^n\). Prove that the hypersurface defined by the determinant of this array is a rational variety. When is this variety smooth?

(3) Prove that every smooth cubic surface over an algebraically closed field is determinantal.

**Exercise 10.** Let \(X\) be a smooth hypersurface in \(\mathbb{P}^n\) of degree \(d \leq n\). Assuming the field is algebraically closed, find a rational curve passing through every point of \(X\).

It is an open question whether or not such a smooth hypersurface has a rational surface through every point.

## 2. Second Lecture

In lecture one, we saw that smooth hypersurfaces of large degree relative to their dimension are never rational. On the other extreme, a linear hypersurface is obviously rational, and we observed that a quadric hypersurface is rational over \(k\), provided it has a \(k\)-point. For cubics, however, we saw both rational and non-rational examples, indicating that the rationality question for cubics is more subtle.

The purpose of this lecture is to consider the rationality question for smooth cubic surfaces in detail. Because every smooth cubic surface over an algebraically closed field is isomorphic to a blowup of the plane at six points,\(^1\) every such surface is rational. The interesting issue is the rationality or non-rationality of cubic surfaces defined over non-algebraically closed fields.

Rationality for cubic surfaces has interesting applications to Diophantine equations. For example, consider a cubic \(f(x, y, z) = 0\) defined over \(\mathbb{Z}\) or some other number ring. Let \(S/\mathbb{Q}\) be the cubic surface it defines. If \(S\) is unirational, then the

\(^1\) For this, and other basic properties of cubic surfaces, the reader is referred to [H, V 4], or the more elementary account in [Ger]. The discussion in [R] is also quite fun and informative, although lacking in proofs.
rational map $\mathbb{A}^2 \dashrightarrow S$ can be used to find rational, and hence integer, solutions of the equation: for each $(s, t) \in \mathbb{A}^2(\mathbb{Q})$, the image

$$\phi(s, t) = (\phi_1(s, t), \phi_2(s, t), \phi_3(s, t))$$

is a $\mathbb{Q}$-solution. If $S/\mathbb{Q}$ is actually rational, then $\phi$ is invertible on an open subset and we have an essentially complete parameterization of the solutions: except on a locus of finitely many curves, every solution of $f_3 = 0$ is uniquely described by the parameterization $(\phi_1(s, t), \phi_2(s, t), \phi_3(s, t))$. Segre’s 1943 paper [S43] contains applications of this idea to problems posed by Mordell regarding the representation of rational numbers by ternary cubic forms. A nice survey of recent developments in this direction is offered by Colliot-Thélène in [C86].

**Exercise 11.** (Swinnerton-Dyer [S-D]) Consider the cubic surface defined by

$$t(x^2 + y^2) = (4z - 7t)(z^2 - 2t^2).$$

Prove that

1. The real points of this surface, considered as a real two-manifold, consist of two connected components.
2. On one manifold component, $\mathbb{Q}$-points are dense.
3. On the other manifold component, there are no $\mathbb{Q}$-points.

Hint: Let $a$ and $b$ be integers, and let $\prod p_i^{m_i}$ be a prime factorization of $a^2 + b^2$. Then if $p_i = 3$ modulo 4, then $m_i$ is even.

Rationality for cubic surfaces is quite subtle. Our goal is to clarify the situation by proving the following theorem of Beniamino Segre.

**2.1. Segre’s Theorem.** *If the Picard number of a smooth cubic surface is one, then the surface is not rational.*

By definition, the Picard number $\rho_k$ of a normal projective variety over $k$ is the rank of its Néron-Severi group, the group of divisors up to numerical equivalence. Of course, this depends on the ground field. The Picard number of a smooth cubic surface $S$ over an algebraically closed field is seven: thinking of $S$ as the blowup of $\mathbb{P}^2$ at six points, the Picard group (and hence the Néron-Severi group) is freely generated by the six exceptional lines and the pull back of the hyperplane class. On the other hand, the cubic surface $S$ may be defined over some non-algebraically closed field $k$, even if the individual points we blow up are not defined over $k$. In this case, the Picard number of $S/k$ may be less than seven.

Both the hypothesis that $X$ is smooth and the hypothesis that its Picard number is one are essential in Segre’s theorem. See Remark 3.2.

---

²Do not be misled to believe that if a map $\phi$ defined over $\mathbb{Q}$ is dominant, essentially all $\mathbb{Q}$-solutions are parameterized by $\phi$. Indeed, it is possible to miss most of them. For example, the squaring map $\mathbb{A}_\mathbb{Q}^1 \dashrightarrow \mathbb{A}_\mathbb{Q}^1$ sending $x$ to $x^2$ is dominant, but its image, while Zariski dense, is fairly sparse, consisting only of the perfect squares.
Before proving Theorem 2.1, we establish a criterion for detecting when a cubic surface has Picard number one. Recall that every cubic surface over an algebraically closed field contains exactly twenty-seven distinct lines.

**2.2. Theorem.** Let \( S/k \) be a smooth cubic surface in \( \mathbb{P}^3 \) and consider the action of the Galois group of \( \bar{k}/k \) on the twenty seven lines of \( S/k \). The following are equivalent.

1. The Picard number \( \rho_k(S) \) is one.
2. The sum of the lines in each orbit is linearly equivalent to a multiple of the hyperplane class on \( S \).
3. No orbit consists of disjoint lines on \( S \).

**2.2.1.** The proof of Theorem 2.2 makes use of the following general principle. If \( k \subset L \) is a Galois extension of fields, and \( X \) is quasi-projective variety defined over \( k \), then the Galois group \( G \) of \( L/k \) acts on the \( L \)-points of \( X \). An \( L \)-point of \( X \) is a \( k \)-point if and only if it is fixed by this action of \( G \). Likewise, the group \( G \) permutes around the subschemes of \( X \) defined over \( L \), and such a subscheme is defined over \( k \) if and only if it is fixed by this action. Indeed, the subschemes of \( X \) defined over \( k \) can be interpreted as orbits of the action of \( G \) on the \( L \)-subschemes of \( X \). All these facts follow easily from the simple observation that \( G \) acts on a polynomial ring \( L[X_1, \ldots, X_n] \) in such a way that the fixed subring is the polynomial ring \( k[X_1, \ldots, X_n] \), and the induced map of schemes

\[
\text{Spec } L[X_1, \ldots, X_n] \to \text{Spec } k[X_1, \ldots, X_n]
\]

is a quotient map: lying over each prime ideal in \( k[X_1, \ldots, X_n] \) are prime ideals of \( L[X_1, \ldots, X_N] \) making up an orbit under the action of \( G \). See [Na, 10.3], or [Se; p 108].

**Proof of Theorem 2.2.** The twenty seven lines \( \{L_i\}_{i=1, \ldots, 27} \) on \( S/\bar{k} \) span the Néron-Severi group of \( S/\bar{k} \). In fact, these lines generate the “cone of curves” for \( S/\bar{k} \), meaning that every effective curve on \( S/\bar{k} \) is numerically equivalent to a non-negative integer combination the \( L_i \). See, for example, [H p405].

Now, without loss of generality, the ground field \( k \) may be assumed perfect. Indeed, if \( k^\infty \) denotes the perfect closure \( \cup_{e} k^{1/p^e} \) of \( k \), then the automorphism groups \( \text{Aut} \bar{k}/k \) and \( \text{Aut} \bar{k}/k^\infty \) are identical, because \( k^\infty \) is the precisely the fixed field of \( G = \text{Aut} \bar{k}/k \). The cubic surface \( S \) is smooth whether regarded over \( k \) or \( k^\infty \), and the orbits of the action of \( G \) on the twenty-seven lines are independent of this choice. Furthermore, any divisor on \( S \) defined over \( k \) is a priori defined over \( k^\infty \), while any divisor defined over \( k^\infty \) is defined over some purely inseparable extension \( k^{1/p^e} \) of \( k \), so that some \( p^{e\text{th}} \) multiple is defined over \( k \). This implies that the Néron Severi group has the same rank over \( k \) or \( k^\infty \). In particular, conditions (1), (2), and (3) are equivalent over \( k \) if and only if they are equivalent over \( k^\infty \).

Assuming \( k \) is perfect, we can understand \( S \) over the non-algebraically closed field \( k \) by considering the action of the Galois group \( G \) of \( \bar{k}/k \) on the \( \bar{k} \) points of \( S \). The extension \( k \hookrightarrow \bar{k} \) is Galois, so divisors of \( S \) defined over \( k \) can be identified with
the $G$-invariant divisors of $S/\bar{k}$, which in turn can be identified with the $G$-orbits of divisors defined over $\bar{k}$. In particular, the Picard group of $S/k$ is the subgroup of the Picard group of $S/\bar{k}$ invariant under the natural action of $G$. If $\{L_{i_1}, \ldots, L_{i_t}\}$ is an orbit of $G$ on the twenty-seven lines, then $L_{i_1} + \cdots + L_{i_t}$ is an effective curve on $S/k$. We leave it as an exercise to check that these orbit sums span the Picard group (and hence Néron-Severi group) of $S/k$, and in fact, they generate the cone of curves for $S/k$.

Thus, the Picard number is one if and only if these orbit sums are all multiples of each other. Equivalently, $\rho_k = 1$ if and only if every orbit sum is a multiple of the hyperplane class. This establishes the equivalence of (1) and (2).

To see that (1) implies (3), suppose that $\{L_1, \ldots, L_t\}$ is an orbit consisting of non-intersecting lines. If the Picard number is one, then any other orbit sum $L_{t+1} + \cdots + L_r$ is a positive $\mathbb{Q}$-multiple of $L_1 + \cdots + L_t$. Then

$$(L_1 + \cdots + L_t)^2 = \frac{m}{n} (L_1 + \cdots + L_t) \cdot (L_{t+1} + \cdots + L_r) \quad \text{with} \quad \frac{m}{n} \geq 0.$$ 

This produces a contradiction: the left-hand-side is $\sum_{i=1}^t L_i^2 = -t$, while the right-hand-side is non-negative because only distinct effective divisors are intersected. This contradiction proves that (1) implies (3).

Finally, to prove (3) implies (1), we invoke the following easy fact whose proof is left as an exercise: An effective curve with positive self-intersection lies in the interior of the cone of curves. Assuming that the Picard number of $S/k$ is at least two, the cone of curves has dimension at least two, and therefore has a non-zero non-interior point. This forces one of the orbit sum generators for the cone of curves, say $L_1 + \cdots + L_t$, to have non-positive self-intersection.

Consider an orbit $\{L_1, \ldots, L_t\}$, and assume that $L_1$ intersects $L_2$. Note that $L_1$ can intersect no other line $L_i$ in this orbit, because

$$(L_1 + \cdots + L_t)^2 = \sum_{i=1}^t L_i^2 + 2 \sum_{i<j} L_i \cdot L_j$$

$$= (-t) + 2 (\text{number of pairs of intersecting lines in the orbit})$$

is non-positive, and $G$ acts transitively on $\{L_1, \ldots, L_t\}$. Thus the orbit $\{L_1, \ldots, L_t\}$ is partitioned into pairs of intersecting lines, with all pairs disjoint from each other, and $G$ acts transitively on the set of these pairs.

Now consider the degenerate conic $L_1 + L_2$ on $S/\bar{k}$. The plane $H$ spanned by $L_1$ and $L_2$ intersects $S$ in a third line $L$. Consider the linear system on $S/\bar{k}$ of plane sections containing $L$. Throwing away the fixed component $L$, this gives a base-point-free pencil of conics on $S/\bar{k}$. Obviously $L_1 + L_2$ belongs to this linear system, but we claim that in fact, every pair of intersecting lines in the orbit $\{L_1, \ldots, L_t\}$ determines a conic in this linear system. To see this, note that the linear system determines a morphism $S/k \to \mathbb{P}^1$ whose fibers are the conics of the pencil. For $i \neq 1, 2$, each $L_i$ is disjoint from $L_1 + L_2$, so it must be contained in a fiber of this morphism. From this we conclude that for every pair $\{L_i, L_j\}$ of intersecting lines
in the orbit \( \{ L_1, \ldots, L_t \} \), we have a linear equivalence \( L + L_1 + L_2 = L + L_i + L_j \). Because \( G \) permutes around the intersecting pairs \( \{ L_i, L_j \} \), the line \( L \) must be fixed by \( G \). This is true even if there are no intersecting pairs in the orbit besides \( L_1 \) and \( L_2 \), for then \( G \) fixes \( L_1 + L_2 \), so that \( G \) fixes \( L \). In either case, \( L \) makes up its own orbit, contrary to assumption (3). □

**Exercise 12.** Complete the proof of Theorem 2.2 by proving the following two lemmas.

1. The sums of the lines in each orbit of the action of Galois group of \( \overline{k}/k \) on the twenty seven lines on a cubic surface \( S \) over \( k \) generate the cone of curves.
2. On a non-singular projective surface, an effective curve with positive self intersection lies in the interior of the cone of curves.

Specific non-rational cubic surfaces over \( \mathbb{Q} \) are constructed in the next exercise.

**Exercise 13.**

1. Find all lines on a smooth cubic surface given by an equation of the form \( u^3 = f_3(x, y) \) in affine coordinates. In particular, find the lines on the Fermat hypersurface \( a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = a_0 \). Do the same for \( u^2 = f_3(x, y) \).
2. Show that if \( a \) is a rational number that is not a perfect cube, then the rational surface defined by \( x_1^3 + x_2^3 + x_3^3 = a \) has Picard number one over \( \mathbb{Q} \). Conclude that such surfaces are not rational.

In fact, Segre showed that a surface over \( \mathbb{Q} \) defined by the equation \( a_0x_0^3 + a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0 \) has Picard number one if and only if, for all permutations \( \sigma \) of four letters, the rational number

\[
\frac{a_{\sigma(0)}a_{\sigma(1)}}{a_{\sigma(2)}a_{\sigma(3)}}
\]

is not a cube [S43]. The proof of Exercise 13 (on page 50) easily generalizes to yield this stronger result.

**2.3. Maps to Projective Space.** To prove Segre’s theorem, we need a good understanding of what is involved in mapping a smooth surface to the plane, so we digress to consider this general problem.

A rational map \( S \xrightarrow{\phi \gamma} \mathbb{P}^2 \) is given by a two-dimensional fixed-component-free linear system of curves on a surface \( S \), say \( \gamma \), contained in some complete linear system \( |C| \). If the map were everywhere defined, we could compute the self intersection multiplicity of \( \gamma \), by which we mean the intersection multiplicity of two general members of \( \gamma \), from the self intersection of its image. In particular, when \( \phi, \gamma \) is a morphism, \( \gamma^2 \) is its degree, assuming the map is finite.

However, the map may not be defined everywhere. The points where it is not defined are precisely the base points of \( \gamma \), call them \( P_1, \ldots, P_r \) and let their multiplicities be \( m_1, \ldots, m_r \). In general, the expected contribution of the base point \( P_i \)
to the self intersection multiplicity of $\gamma$ is $m_2$. However, this is valid only when two
general curves in $\gamma$ have distinct tangents at $P_i$. The number will be even higher if
the curves share tangents at $P_i$: this is the case where $\gamma$ has “infinitely near” base
points.

To make this precise, let $S' \xrightarrow{\pi} S$ be the blowup of $S$ at a base point $P$ of $\gamma$. The
birational transform of $\gamma$ is the linear system $\gamma'$ on $S'$ whose generic member is the
birational transform of the generic member of $\gamma$. The base points of $\gamma'$ that lie in
the exceptional fiber are called base points of $\gamma$ infinitely near $P$. They represent
the tangent directions at $P$ that are shared by all members of $\gamma$.

Let $C$ be a general member of $\gamma$ and let $C'$ be its birational transform on the
blowup $S'$. It is easy to verify that $\pi^* C = mE + C'$ where $m$ is the multiplicity of
$\gamma$ at the base point $P$ and $E$ is the exceptional fiber of the blowup $\pi$. The linear
system $\gamma'$ on $S'$ is obtained by pulling back the linear system $\gamma$ and throwing away
the fixed component $mE$. It particular, to the extent to which $S$ and $S'$ are “the
same,” $\gamma$ and $\gamma'$ determine “the same” map to $\mathbb{P}^2$.

Exercise 14. With notation as above, verify that

$$C'^2 = C^2 - m^2 \quad \text{and} \quad C' \cdot K_{S'} = C \cdot K_S + m,$$

where $C$ is a general member of $\gamma$ and $C'$ is a general member of $\gamma'$. We write this
also as $\gamma'^2 = \gamma^2 - m^2$ and $\gamma' \cdot K_{S'} = \gamma \cdot K_S + m$.

The process of blowing up base points can be iterated until the multiplicities of
all base points drop to zero. In this way, we arrive at a smooth surface $\bar{S}$, and a
base point free linear system $\bar{\gamma}$ defining the “same map” (ie, composed with the
blowing up map) to projective space.

This process is called resolving the 0 of the rational map $\phi_{\gamma}$.

Beginning with a birational map $S \xrightarrow{\phi_{\gamma}} \mathbb{P}^2$, the linear system $\bar{\gamma}$ on $\bar{S}$ defines a
birational morphism to $\mathbb{P}^2$. In this case, $C^2 = 1$ and $C \cdot K_{\bar{S}} = -3$, where $C$ is a
general member of \( \hat{\gamma} \). Repeated applications of Exercise 14 express this in terms of the divisor \( C \) on \( S \):

\[
(2.3.1) \quad C^2 - \sum m_i^2 = 1 \quad \text{and} \quad K_S \cdot C + \sum m_i = -3
\]

where the sums are taken over all base points, including the infinitely near ones, and the \( m_i \) are their multiplicities.

This leads to the following theorem.

2.4. Theorem. Let \( S \) be a smooth projective surface over \( k \). Then \( S \) is rational over \( k \) if and only if \( S \) admits a fixed-component-free two-dimensional linear system \( \gamma \) defined over \( k \) satisfying

\[
\gamma^2 - \sum m_i^2 = 1
\]

and

\[
K_S \cdot \gamma + \sum m_i = -3,
\]

where the \( m_i \) are the multiplicities of all base points of \( \gamma \), including the infinitely near ones.

Proof. Assume that \( S \) is rational over \( k \), and let \( S \xrightarrow{\phi} \mathbb{P}^2 \) be a birational map. Let \( \gamma \) be the fixed-component-free linear system obtained by pulling back the linear system of hyperplanes on \( \mathbb{P}^2 \). The dimension of \( \gamma \) is two and the desired numerical conditions have been computed already in 2.3.1.

Conversely, given a linear system \( \gamma \) satisfying the given numerical conditions, it determines a rational map \( S \xrightarrow{\phi} \mathbb{P}^2 \) defined over \( k \). We need only verify that this map is actually birational. Because the map is a priori defined over \( k \), to check that it is birational we are free to assume that \( k \) is algebraically closed since whether or not the map is dominant and degree one is unaffected by replacing \( k \) by its algebraic closure.

Blow up the base points of \( \gamma \), including the infinitely near ones, to obtain a morphism \( \bar{S} \xrightarrow{\bar{\phi}} \mathbb{P}^2 \) resolving the indeterminacies of \( \phi \). The dimension of \( \bar{\gamma} \) is 2, and the numerical conditions \( \bar{\gamma}^2 = 1 \) and \( \bar{\gamma} \cdot \bar{K}_{\bar{S}} = -3 \) hold. Because \( S \) and \( \bar{S} \) are birationally equivalent, it is sufficient to show that the morphism \( \bar{S} \xrightarrow{\bar{\phi}} \mathbb{P}^2 \) is a birational equivalence.

To check that the map \( \bar{S} \xrightarrow{\bar{\phi}} \mathbb{P}^2 \) is surjective, assume, on the contrary, that its image is a plane curve, \( B \). The fibers of \( \bar{S} \to B \) would then be the elements of the linear system \( \bar{\gamma} \). But thinking of the elements \( \bar{C} \in \bar{\gamma} \) as the fibers of this map, we would have \( \bar{\gamma}^2 = 0 \). This contradicts the fact that \( \bar{\gamma}^2 = 1 \), so that \( \phi_{\bar{\gamma}} \) is surjective.

Finally, the morphism determined by \( \bar{\gamma} \) is generically one-to-one because its degree is determined by the formula \( \bar{\gamma}^2 = (\deg \phi_{\bar{\gamma}})H^2 \). Because \( \bar{\gamma}^2 = 1 \), we conclude that the map must be one-to-one.  □
2.5. Cautionary Example. Let \( F(X) \in \mathbb{Q}[X] \). As \( \lambda \) and \( \mu \) vary through \( \mathbb{C} \), the linear system \( |\lambda Y + \mu F(X)| \) is a one dimensional linear system on \( \mathbb{A}^2 \) defined over \( \mathbb{Q} \). The zeros of \( F(X) \) determine the base points, since \((x, y)\) is a base point if and only if \((x, y) = (0, \alpha)\) where \( \alpha \) is a root of \( F \). These base points may not be defined over \( \mathbb{Q} \), although the linear system is defined over \( \mathbb{Q} \). The map to projective space determined by this linear system is defined over \( \mathbb{Q} \) even when its base points are not.

3. Third Lecture

In this lecture we will prove Segre’s Theorem. Essentially the same argument, with minor modifications to be made afterwards, will prove the following stronger theorem of Manin [M66].

3.1. Theorem. Two smooth cubic surfaces defined over a perfect field, each of Picard number one, are birationally equivalent if and only if they are projectively equivalent.

Caution: Manin’s theorem does not assert that every birational equivalence is a projective equivalence. It guarantees only that if two surfaces are birationally equivalent, then there exists a projective equivalence between them.

3.2. Remark. Manin’s theorem holds under weaker hypotheses: the ground field need not be perfect and smoothness can be weakened to non-singularity. Because the proof in this case requires some distracting technicalities, we do not bother with it here; see [K97, 5.3.3].

However, the hypothesis of smoothness can not be weakened to include singular varieties. For instance, consider a plane conic defined over \( k \), together with six points on it conjugate, but not individually defined, over \( k \). By blowing up the six points and then contracting the conic, we achieve a singular cubic surface with Picard number one. All such surfaces are birationally equivalent to each other, but two such are projectively equivalent if and only if the corresponding six-tuples of points are projectively equivalent in \( \mathbb{P}^2 \).

Similarly, the hypothesis that the Picard number is one is essential. For example, there exist smooth cubic surfaces of Picard number two containing exactly one line defined over \( \mathbb{Q} \). Because the line contains plenty of \( \mathbb{Q} \)-points, the surface is unirational by 1.8.1, so it contains a Zariski dense sets of \( \mathbb{Q} \)-points. Contract the \( \mathbb{Q} \)-rational line and then blow up a general \( \mathbb{Q} \)-point. The resulting surface is a smooth cubic surface of Picard number two, birationally equivalent over \( \mathbb{Q} \) to the original surface. However, two such surfaces are not isomorphic in general. Any isomorphism between them would amount to an automorphism of the original cubic surface interchanging the two distinguished points, but because the automorphism group of the cubic surface is finite, this is not possible in general.

The proof of Segre’s (and Manin’s) theorem begins with the general observation that the Picard group of a smooth cubic surface of Picard number one is generated by the class of a hyperplane section. Indeed, the Picard group of \( S \) is torsion-free,
because \( \text{Pic}(S) \subset \text{Pic}(S_k) \cong \mathbb{Z}^3 \). Also, the hyperplane class \( H \) is not divisible, for otherwise \( H = mD \) for some \( D \), and this would force \( H^2 = 3 = m^2D^2 \), which forces \( m = 1 \).

Segre's theorem asserts that no cubic surface with Picard number one can be rational. If this were false, there would be a birational map \( S \stackrel{\phi}{\longrightarrow} \mathbb{P}^2_k \) defined over \( k \). The pull back of the hyperplane linear system on \( \mathbb{P}^2 \) is base point free on the dense open set of \( S \) where \( \phi \) is defined, and the closure of this linear system in \( S \) is the linear system \( \gamma \) defining the rational map \( \phi \). Because the Picard group is generated by the hyperplane class \( H \), the linear system \( \gamma \) must be contained in the complete linear system \( |dH| \) for some \( d \). Therefore, the proof of Segre's theorem will be complete upon proving the following theorem.

3.3. Theorem. If \( S \subset \mathbb{P}^3_k \) is a smooth cubic surface, then there is no fixed-component-free linear system on \( S \) contained in \( |dH| \) that defines a birational map to the projective plane.

Although the statement of this theorem is less appealing than Segre's Theorem, we have, in effect, reduced the proof of Segre's theorem to the case where the ground field is algebraically closed: if such a linear system is defined over \( k \), then it is defined over the algebraic closure of \( k \). Note that a naïve reduction to the algebraically closed field case is not possible, as the Picard number is never one over an algebraically closed field.

Proof of 3.3. Suppose that such a linear system, \( \gamma \), exists and defines the birational map \( S \mathrel{\wedgearrow{\gamma}} \mathbb{P}^2_k \). Without loss of generality, we assume \( k \) is algebraically closed, as explained above.

Let \( P_1, \ldots, P_r \) be the base points of \( \gamma \), including the infinitely near ones, and let \( m_1, \ldots, m_r \) be their multiplicities. From the computation 2.3.1, we have

\[
\sum m_i^2 = \gamma^2 - 1 = 3d^2 - 1
\]

\[
\sum m_i = -K_S \cdot \gamma - 3 = 3d - 3.
\]

(3.3.1)

If all \( m_i \) are less than or equal to \( d \), then

\[
3d^2 - 1 = \sum m_i^2 \leq d \sum m_i = 3d^2 - 3d < 3d^2 - 1.
\]

This contradiction ensures that at least one \( m_i \) is greater than \( d \).

Let \( P \) be a base point of \( \gamma \) of multiplicity greater than \( d \). There is no loss of generality in assuming that \( P \in S \), that is, that \( P \) is an actual base point, not an infinitely near one. This is because the multiplicity of a base point is greater than or equal to the multiplicity of any base point infinitely near it. Indeed, the multiplicity of \( P \) is at least the sum of the multiplicities of all the base points infinitely near to \( P \) to first order. (We leave as an exercise the following fact: the multiplicity of \( P \in S \) as a base point of the linear system \( \gamma \) is greater than or equal to the sum of the multiplicities of all base points of \( \gamma' \) which lie over \( P \), where \( \gamma' \) is the birational transform of the linear system \( \gamma \) under the blowing up map of \( S \) at \( P \).)
Furthermore, the high multiplicity base point $P$ cannot lie on any line on $S$. Indeed, since $\gamma \subset |dH|$, we must have that $L \cdot C \leq d$ for all lines $L$ on $S$ and all $C \in \gamma$. Computing $C \cdot L$ as the sum over all points (with multiplicities) in $C \cap L$, we see that $C$ cannot have a multiple point of order more than $d$ on $L$.

The proof of Theorem 3.3 proceeds by induction on $d$. The inductive step is accomplished by finding a birational self map of $S$ that takes $\gamma$ to a linear system contained in the linear system $|d'H|$ with $d' < d$.

How can we find a birational self-map of the cubic surface? First, recall the following involution of a plane cubic curve $E$: fixing a point $P$ on $E$, define the map $\tau$ which sends $Q \in E$ to the third point of intersection of $E$ with $\overline{PQ}$. The map $\tau$ extends to an involution defined everywhere on $E$ by sending the point $P$ to the intersection of $E$ with the tangent line through $P$.

We attempt to construct a similar involution of the cubic surface $S \subset \mathbb{P}^3$. Define a self-map $\tau$ of $S$ as follows: fix a point $P$ on $S$ and for each $Q \in S$, let $\tau(Q)$ be the third point of intersection of $S$ with the line $\overline{PQ}$. This defines a rational map $S \dashrightarrow S$ such that $\tau^2 = id$. If we assume that $S$ contains no lines through $P$, then $\tau$ is defined everywhere on $S$, except at $P$. However, unlike the situation of the plane cubic, there is a whole plane of tangent lines to $S$ at $P$, so there is no way to extend $\tau$ to a morphism at $P$. Indeed, $\tau$ contracts the entire curve $D = T_P S \cap S$ to the point $P$ on $S$.

As usual, the best way to sort out different tangent directions at a point is to blow up. Let $S' \xrightarrow{\pi} S$ be the blowup of $S$ at $P$, let $E$ be the exceptional fiber, and let $D'$ be the birational transform of the curve $D = T_P S \cap S$.

By definition, the blowup of $\mathbb{P}^3$ at $P$ consists of those points $\{(x, L)\}$ in $\mathbb{P}^3 \times \mathbb{P}^2$, where $\mathbb{P}^2 = \mathbb{P}(T_P(\mathbb{P}^3))$, such that $x \in L$. The blowup of $S$ at $P$ is identified with the birational transform of $S$. The blowing up map $\pi$ is the projection of $S'$ onto the first factor $S \subset \mathbb{P}^3$. Let $q$ denote the projection of $S'$ onto the second factor $\mathbb{P}^2$:

\[
\begin{array}{ccc}
S' & \xrightarrow{\pi} & S \times \mathbb{P}^2 \\
\downarrow & & \downarrow q \\
S & \xrightarrow{\tau} & \mathbb{P}^2
\end{array}
\]

**Exercise 15.** Show that $q$ is two-to-one and ramified along a smooth curve of degree 4. Find the equation of this branch locus.

The rational map $\tau$ of $S$ extends to a morphism $\tilde{\tau}$ of $S'$. The following facts about $\tilde{\tau}$ are easily verified:

1. $\tilde{\tau}$ is the unique non-trivial Galois automorphism of the degree two cover $S'$ of $\mathbb{P}^2$. 

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(2) \( \tilde{\tau}(E) = D' \) and \( \tilde{\tau}(D') = E \).

(3) \(|\pi^*H - E| = |q^*L| \), where \( L \) is a line in \( \mathbb{P}^2 \); that is \( \pi^*\mathcal{O}_S(1)(-E) \cong q^*\mathcal{O}_{\mathbb{P}^2}(1) \).

To complete the proof of Theorem 3.3, let \( P \) be a base point of the linear system \( \gamma \), and suppose that the multiplicity of \( P \) is \( m > d \). Let \( \gamma' = \pi^*\gamma - mE \) be the birational transform of \( \gamma \) via the blowup \( S' \xrightarrow{\pi} S \) at \( P \), as discussed in 2.3. Because \( \gamma \subset |dH| \), we have

\[
\gamma' + (m - d)E = \pi^*\gamma - dE \subset |\pi^*(dH) - dE| = |d(\pi^*H - E)| = |q^*(dL)|.
\]

Applying the automorphism \( \tilde{\tau} \) to \( S' \), the elements of \( \gamma' + (m - d)E \) are taken to another linear system inside \( |q^*(dL)| \) because \( \tilde{\tau} \in \text{Gal}(S'/\mathbb{P}^2) \) preserves any linear system pulled back from \( \mathbb{P}^2 \). Therefore

\[
\tilde{\tau}(\gamma' + (m - d)E) = \tilde{\tau}(\gamma') + (m - d)D' \subset |q^*(dL)| = |d(\pi^*H - E)| \subset |\pi^*(dH)|.
\]

Pushing back down to \( S \), we have

\[
\tau(\gamma) + (m - d)D \subset |dH|.
\]

Because \( D \) is a hyperplane section of \( S \), we conclude that

\[
\tau(\gamma) \subset |(d - (m - d))H|.
\]

Because \( m > d \), the linear system \( \tau(\gamma) \) is contained in \( |d'H| \), with \( d' < d \). By induction, we conclude eventually that \( \gamma \subset |H| \), that is, that \( d = 1 \). But now considering the two useful formulas in 3.3.1, we see that \( \sum m_i = 3d - 3 = 0 \) and \( \sum m_i^2 = 3d^2 - 1 = 2 \), where the \( m_i \) are the multiplicities of the base points of \( \gamma \).

The first equation forces all the \( m_i \) to be zero, while the second forces two of the \( m_i \) to be exactly one. This contradiction completes the proof of Segre’s theorem.

□

The proof Segre’s theorem is easily altered to produce the following proof of Manin’s theorem.

**Proof of Theorem 3.1.** Assume that \( S \xrightarrow{\phi} S' \) is a birational equivalence. Let \( \gamma \) be the linear system on \( S \) obtained by pulling back the hyperplane system on \( S' \) via \( \phi \). Thus \( \phi = \phi_\gamma \), and \( \gamma \) is a linear system of dimension 3.

Because the Picard number of \( S \) is one, we can assume, as before, that \( \gamma \subset |dH| \) for some \( d \), where \( H \) is a hyperplane section of \( S \). Let \( P_1, \ldots, P_r \) be the base points of \( \gamma \), including the infinitely near ones, and suppose their multiplicities are \( m_1, \ldots, m_r \). Because \( H^2 = 3 \) and \( K_{S'} \cdot H = -3 \), we again compute using Exercise 14 that

\[
\sum m_i = 3d - 3 \quad \text{and} \quad \sum m_i^2 = 3d^2 - 3.
\]

If all \( m_i \leq d \), then \( 3d^2 - 3 = \sum m_i^2 \leq d(\sum m_i) = 3d^2 - 3d \). This is possible only if \( d = 1 \), in which case \( \phi \) is an automorphism of \( \mathbb{P}^3 \).
Therefore, we may assume without loss of generality that some \( m_i > d \), and the corresponding \( P_i \) may be assumed to be on \( S \). The same trick that was used to accomplish the inductive step in the proof of Segre’s theorem works here too. The only problem is that the involution \( \tau \) will not be defined over \( k \) unless the base point \( P \) is defined over \( k \). A priori, the argument shows only that if \( S \) and \( S' \) are birationally equivalent over \( k \), then they are projectively equivalent over \( \bar{k} \).

To see that \( S \) and \( S' \) are actually projectively equivalent over \( k \), we need to construct an involution \( \tau \) defined over \( k \). Because the Galois group of \( \bar{k} \) over \( k \) acts on the \( P_i \) preserving multiplicities \( m_i \), it follows from the fact that \( \sum m_i = 3d - 3 \) that at most two of the base points \( P_i \) can have multiplicity greater than \( d \). If exactly one, say \( P_1 \), has multiplicity greater than \( d \), then the Galois group fixes this base point. Because \( k \) is perfect, this implies that \( P_1 \) is defined over \( k \), so the involution \( \tau \) is defined over \( k \) and the proof of Manin’s Theorem is complete. If exactly two base points, say \( P_1 \) and \( P_2 \), have multiplicity exceeding \( d \), then the Galois group must fix their union, and so \( P_1 \cup P_2 \) is defined over \( k \). As before, \( P_1 \) may be assumed to be on \( S \), but it is possible that \( P_2 \) is infinitely near \( P_1 \). The existence of an involution defined over \( k \) needed to complete the proof is left as an exercise. \( \square \)

**Exercise 16.** Let \( S \) be a cubic surface defined over \( k \) and let \( P_1 \) and \( P_2 \) be distinct \( \bar{k} \)-points of \( S \) such that \( P_1 \cup P_2 \) is defined over \( k \). Assuming that the line through \( P_1 \) and \( P_2 \) does not lie on \( S \), construct an involution \( \tau \) of \( S \) defined over \( k \). Similarly, interpret and prove a version of this statement in the case where \( P_2 \) is an point on the blowup of \( S \) at \( P_1 \).

Use this involution to complete the proof of Manin’s theorem.

**3.4. Some Historical Remarks and Subsequent Developments.** The geometry of cubic surfaces and the configuration of the twenty seven lines on them occupied a tremendous amount of attention of the nineteenth century algebraic geometers. However, many of the beautiful geometric arguments presented here in our discussion of rationality of cubic surfaces go back to Segre in the middle of this century, in the series of papers listed in the bibliography. Segre was motivated, at least at first, by arithmetic questions of Mordell on representing integers by ternary forms. The arithmetic applications are still important today; see [C86], [C92], [K96a].

Manin’s theorem— that any two birationally equivalent smooth cubic surfaces of Picard number one are actually projectively equivalent— has a stronger analog for three-folds, at least over an algebraically closed field of characteristic zero. Indeed, in 1971, Iskovskikh and Manin proved that any birational equivalence between smooth hypersurfaces of degree four in \( \mathbb{P}^4 \) must be a projective equivalence [IM]. This implies that no smooth quartic threefold is rational: the birational automorphism group of the quartic threefold is the same as its group of projective automorphisms; since the latter is finite, the threefold can not be birationally equivalent to \( \mathbb{P}^2 \), which has an infinite automorphism group. Iskovskikh and Manin acknowledge their indebtedness to Fano, who, despite some serious errors, had laid
the foundations of this investigation, using ideas of Max Noether.

This theorem of Iskovskikh and Manin settled the longstanding Lüroth problem:

Is every unirational variety actually rational?

For curves, Lüroth’s theorem states that every unirational curve is rational; Castelnuovo and Enriques later showed that every unirational surface is rational (with some help from Zariski in prime characteristic, where “unirational” should be read “separably unirational”). Fano and others had long expected a negative answer to the higher dimensional Lüroth problem, but the question remained open for many years, despite some erroneous counterexamples proposed by prominent mathematicians. The theorem of Iskovskikh and Manin on the non-existence of rational quartic three-folds settled—negatively—the Lüroth problem, because undisputed examples of unirational quartic three-folds had already been constructed by Segre [S60].

Around the same time, Clemens and Griffiths also resolved the Lüroth problem, by showing that there exist no smooth rational cubic three-folds [CG]. (It is not hard to see that every smooth cubic threefold is unirational, an elementary fact Clemens and Griffiths attribute to Max Noether; see [CG, p 352].) Clemens and Griffiths approach was entirely different, based on a study of the Intermediate Jacobian of the cubic threefold.

Also in the early seventies, Artin and Mumford gave a simple proof of the existence of non-rational unirational varieties in all dimensions three or higher, using the observation that the torsion subgroup of the third integral singular cohomology group of a non-singular projective variety over $\mathbb{C}$ is a birational invariant [AM].

We now have a reasonably complete understanding of rationality for smooth three-folds; see the papers of Sarkisov, Iskovskikh, Bardelli and Beauville listed in the bibliography. Meanwhile, Colliot-Thélène and Ojanguren developed further the examples of Artin and Mumford in all dimensions [CO].

For four-folds, considerably less is known. Pukhlikov generalized the methods of [IM] to show that a birational equivalence between two smooth quintics in $\mathbb{P}^5$ is actually a projective equivalence. Again the corollary follows: there exist no smooth rational quintic four-folds [P]. The short paper of Tregub [Tr] gives some nice examples of rational cubic hypersurfaces in $\mathbb{P}^5$. But there is not a single known example of a smooth rational quartic hypersurface of dimension four or higher; nor is there any proof that one can not exist.

Although we now know that not every unirational variety is rational, it is natural to wonder about other classes of varieties that are “close to rational.” Recent work of Kollár has established the existence of abundant examples of non-rational varieties with various other nice properties, such as rational connectivity [K95], [K97]. A smooth projective variety is rationally connected if every two points are joined by a rational curve.

We have seen that a smooth hypersurface whose degree is no greater than its embedding dimension is covered by rational curves. In fact, such a hypersurface has the even stronger property of rational connectivity. This is a special case of
a theorem of Kollár, Miyaoka and Mori stating that every smooth Fano variety of characteristic zero is rationally connected [KoMM]. Kollár’s book [K96] presents an in-depth study of rational curves on algebraic varieties.

4. Fourth Lecture

It is easy to find smooth projective varieties that are not rational: a high degree hypersurface in $\mathbb{P}^n$ is not even separably unirational, as we observed in the first lecture. It is much harder to find examples of non-rational varieties when the obvious obstructions to rationality, such as the plurigenera, vanish.

4.1. Definition. A Fano variety is a smooth projective variety whose anti-canonical bundle is ample.

By anti-canonical bundle we mean the dual of the sheaf of differential $n$-forms, where $n$ is the dimension. More generally, it is not really necessary to assume the variety is smooth in the definition of Fano, provided one can make sense of the sheaf of differential $n$-forms as a line bundle.

The plurigenera vanish for any Fano variety, so it is natural to wonder whether all Fano varieties might be rational. In the final two lectures, we construct specific examples of non-rational Fano varieties. In this lecture, we focus mostly on the prime characteristic case. We will construct hypersurfaces with ample anti-canonical divisor that are not even ruled in characteristic $p$. The argument makes heavy use of the special nature of derivations in characteristic $p$, and is not valid in characteristic zero. Unfortunately, our hypersurfaces are not smooth. In Lecture 5, we will show how to deduce the existence of smooth projective Fano varieties that are not ruled. In particular, we will produce explicit and abundant examples of non-rational smooth Fano hypersurfaces. This complements the rather special examples previously known; see 3.4.

It will be necessary to consider geometric conditions weaker than rationality, so we recall the following definitions.

4.2. Definition. A variety $X$ is ruled if there exists a variety $Y$ and a birational map $Y \times \mathbb{P}^1 \dashrightarrow X$.

A variety $X$ is uniruled if there exists a variety $Y$ and a generically finite dominant map $Y \times \mathbb{P}^1 \dashrightarrow X$. The variety $X$ is separably uniruled if this birational map is generically étale, that is, if the function field extension is separable.

Loosely speaking, a variety is uniruled if it is covered by rational curves. Of course, every rational variety $X$ is ruled, since $\mathbb{P}^n$ is birationally equivalent to $\mathbb{P}^{n-1} \times \mathbb{P}^1$. Furthermore, every ruled variety is separably uniruled, and every unirational variety $X$ is uniruled. In characteristic zero, separably uniruled is equivalent to uniruled.

In this lecture, we prove the following theorem.

4.3. Theorem. For any prime $p > 0$, there exists a projective variety of characteristic $p$ that is not separably uniruled, but whose anti-canonical sheaf is an
ample invertible sheaf. In fact, there are examples given by hypersurfaces with local
equation of the form \( y^p = f(x_1, \ldots, x_n) \).

We will prove even more: for every sufficiently general \( f \) whose degree is in a certain
range (described in 4.5.18), the affine hypersurface defined by \( y^p - f(x_1, \ldots, x_n) \)
has a natural compactification that is not separably uniruled but whose anti-canonical sheaf is ample invertible.

It is relatively easy to find singular non-rational varieties with ample anti-canonical sheaf over any field: the cone over an elliptic curve is an immediate example. It is harder to produce non-ruled examples. The point here is not so much the mere existence of such varieties, but rather the explicit construction of them in 4.5. Essentially the same construction yields smooth Fano varieties that are not ruled in characteristic zero, but we will need the characteristic \( p \) results to deduce this.

Non-ruledness will be proved by appealing to the next proposition. Recall that
a line bundle \( \mathcal{M} \) on a variety \( X \) is big if some multiple defines a birational map
\( X \not\rightarrow X' \subset \mathbb{P}^n \) to its image in projective space. Intuitively, big line bundles are
"birationally ample."

4.4. Proposition. If \( X \) is a smooth projective variety admitting a big line bundle
\( \mathcal{M} \subset \bigwedge^i \Omega_X \), then \( X \) is not separably uniruled.

Remark. If \( X \) is a smooth projective variety of characteristic zero, the Bogomolov-
Sommese vanishing theorem guarantees that the only sheaf of differential forms that
can contain any big line bundle is the line bundle \( \bigwedge^\dim X \Omega_X \) itself; see [EV, p 58].
In particular, in characteristic zero, Proposition 4.4 degenerates to the following
statement: no smooth projective variety of general type is uniruled.

Proof of 4.4. Suppose that \( X \) is separably uniruled, and let \( Y \times \mathbb{P}^1 \not\rightarrow X \) be a
generically étale map. Without loss of generality, \( Y \) may be replaced by a smooth
affine open subset on which \( \phi \) is a morphism. The pull back map on differential forms
\( \phi^* \Omega_X \not\rightarrow \Omega_{Y \times \mathbb{P}^1} \)
is an isomorphism on a dense open set because \( \phi \) is generically étale. In fact, all
we need is that \( \phi^* \) is injective, so that for the subsheaf \( \mathcal{M} \) of \( \bigwedge^i \Omega_X \), we have the
inclusion \( \phi^* \mathcal{M} \subset \bigwedge^i \Omega_{Y \times \mathbb{P}^1} \).

Replacing \( Y \) by an even smaller affine open subset if necessary, we may assume
its sheaf of differential forms is free. In particular,
\[
\Omega_{Y \times \mathbb{P}^1} \cong \Omega_Y \oplus \Omega_{\mathbb{P}^1} \cong \mathcal{O}_Y^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2),
\]
where we have abused notation slightly by omitting the symbols for pull back of
\( \Omega_Y \) and \( \Omega_{\mathbb{P}^1} \) to the product \( Y \times \mathbb{P}^1 \). We have
\[
\phi^* \mathcal{M} \subset \bigwedge^i (\Omega_{Y \times \mathbb{P}^1}) \cong \bigwedge^i (\mathcal{O}_Y^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)).
\]
If $\mathcal{M}$ is big, so is its pull back under any generically finite map, so in particular, the powers of $\phi^*\mathcal{M}$ should have enough global sections to separate points on an open set of $Y \times \mathbb{P}^1$. But this is impossible, since the symmetric powers of

$$\bigwedge^i (\mathcal{O}_{Y^m}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$$

obviously have very few global sections. □

4.4.1. Remark. The inclusion $\phi^*\mathcal{M} \subset \Omega_{Y \times \mathbb{P}^1}$ always holds in characteristic zero. But in characteristic $p$ it can fail when the map is inseparable. In fact, the varieties to which we apply Proposition 4.4 to verify non-separably uniruledness are uniruled. See also Exercise 6.

4.5. The Construction. Consider an affine hypersurface in $\mathbb{A}^n$ defined by the equation $y^p = f_{mp}(x_1, \ldots, x_n)$, where $f_{mp}$ is a (non-homogeneous) polynomial of degree $mp$.

**Exercise 17:** Prove that the non-smooth points of the affine hypersurface defined by $y^p - f$ are described in terms of the critical points of $f$ as follows. In the case where the ground field has characteristic $p$, the non-smooth points are in one-to-one correspondence with the critical points of $f$, with $(y, x_1, \ldots, x_n) = (\lambda, \lambda_1, \ldots, \lambda_n)$ a non-smooth point if and only if $(\lambda_1, \ldots, \lambda_n)$ is a critical point of $f$. In the case where the ground field has characteristic not equal to $p$, $(\lambda, \lambda_1, \ldots, \lambda_n)$ is a non-smooth point if and only if $(\lambda_1, \ldots, \lambda_n)$ is a critical point of $f$ of critical value zero. Show that for sufficiently general choice of $f$, the hypersurface has only isolated non-smooth points in the characteristic $p$ case, and is everywhere smooth in characteristic zero case.

The non-rational Fano variety will be a hypersurface defined by an equation of the form $y^p - f_{mp}$. We need a compactification\(^3\) of this affine hypersurface. The obvious one, namely its closure in $\mathbb{P}^{n+1}$, is insufficient for our purposes; the next exercise indicates why.

**Exercise 18:** Prove that the projective closure in $\mathbb{P}^{n+1}$ of the affine hypersurface defined by $y^p - f_{mp}$ is never smooth in any characteristic (whenever $m \geq 2$).

4.5.1. Instead, we compactify the hypersurface by taking its closure in a weighted projective space. Let $F_{mp}(X_0, \ldots, X_n)$ be a homogeneous polynomial of degree $mp$ in the variables $X_0, \ldots, X_n$. Let $P$ be the weighted projective space with coordinates

$$Y, X_0, X_1, \ldots, X_n,$$

where $Y$ is degree $m$ and the $X_i$ are all degree one; that is $P = \text{Proj } k[Y, X_0, \ldots, X_n]$. A common notation for this weighted projective space is $\mathbb{P}(m, 1, 1, \ldots, 1)$, but we will shorten this to just $P$.

\(^3\)More accurately, we need a completion of our variety, not a compactification. The Zariski topology is always compact in any case. However, over the complex numbers, a variety is complete precisely when it is compact as a complex manifold. For this reason, it is customary to refer to “compactification” of a variety.
Consider the closed subscheme $Z$ in $\mathbb{P}$ defined by $Y^p - F_{mp}$. In the affine chart of $\mathbb{P}$ where $X_i$ does not vanish, it is an affine hypersurface of the type considered above. Because these charts cover $Z$, the projective scheme $Z$ is smooth for generic choice of $F_{mp}$ (at least when the characteristic is not $p$). Our goal is to show that we can choose the integers $m, n$ and $p$ such that $Z$ is Fano but not separably uniruled.

4.5.2. The variety $Z$ has an alternate description as a cyclic cover of projective space. Let $\mathbb{P} = \mathbb{P}^n$ be the projective space with homogeneous coordinates $X_0 : \cdots : X_n$, and let $V_i$ be the open affine set where $X_i \neq 0$. The affine coordinates on $V_i$ are $X_j / X_i$.

Consider the line bundle $\mathcal{O}_{\mathbb{P}}(m)$ on $\mathbb{P}$. Fixing local generators $s_i$ on $V_i$, we have

$$s_i = \left( \frac{X_i}{X_j} \right)^m s_j.$$  

Let $U$ be the variety formed by the union of the open sets $U_i = V_i \times \mathbb{A}^1$, patched together by the relations $y_i = \left( \frac{X_i}{X_j} \right)^m y_j$, where $y_i$ is the local coordinate for the copy of $\mathbb{A}^1$ in $U_i$. The natural projection $U \to \mathbb{P}^n$ defines an $\mathbb{A}^1$-bundle over $\mathbb{P}$. This is the total space of the line bundle whose sheaf of sections is $\mathcal{O}_{\mathbb{P}}(m)$; it is neither affine nor projective.

Now let $F_{mp}(X_0, \ldots, X_N)$ be a homogeneous polynomial of degree $mp$, and let $Z$ be the subvariety of $U$ defined locally by the equations $y_i^p - F_{mp} X_i^m$ in the open subset $U_i$. In each patch, the variety $Z$ has exactly the form of the affine hypersurface $y_i^p = f_{mp}$. Thus, for a generic choice of the homogeneous polynomial $F_{mp}$, the variety $Z$ is smooth, at least assuming $k$ is not of characteristic $p$. This construction obviously produces a variety isomorphic to the $Z$ constructed in 4.5.1 as a subscheme of the weighted projective space $\mathbb{P}$.

4.5.3. There is a natural isomorphism $\mathcal{O}_U(-Z) = \pi^* \mathcal{O}_{\mathbb{P}}(-mp)$. Indeed, the patching data for $\mathcal{O}_U(-Z)$, the defining ideal for $Z$ as a closed subvariety of $U$, has the same transition functions as $\pi^* \mathcal{O}_{\mathbb{P}}(-mp)$: a local generator for either sheaf on the affine neighborhood $U_j$ is transformed into a local generator on $U_i$ by multiplication by $\left( \frac{X_i}{X_j} \right)^{-mp}$.

The map $Z \to \mathbb{P}$, obtained by restricting the natural projection $U \to \mathbb{P}$, is a finite surjective map, of degree $p$. This is easily checked locally: each point of $\mathbb{P}$ has precisely $p$ preimages. Of course, when $k$ has characteristic $p$, the preimage of a point in $\mathbb{P}$ is a single point of multiplicity $p$, which is to say, $Z$ is purely inseparable over $\mathbb{P}$.

4.5.4. The Fano range. For appropriate choices of the integers $m, n$ and $p$, the variety $Z$ will be Fano. Specifically, the anti-canonical sheaf of $Z$ is an ample invertible sheaf whenever $m, n$ and $p$ satisfy

$$mp - m < n + 1,$$

26
whether or not $Z$ is smooth. There are several different ways to see this. We explain
the least succinct way first, because the computation will be of the most use later.

**Method 1.** Compute $K_Z$ using the adjunction formula for $Z \subset U$. First compute $K_U$ using the exact sequence

\[
0 \to \pi^*\Omega_P \to \Omega_U \to \pi^*\mathcal{O}_P(-m) \to 0.
\]

The exactness of 4.5.5 is easily verified by the local computation

\[
dy_i = d\left[\left(\frac{X_i}{X_j}\right)^{-m}y_j\right] = \left(\frac{X_i}{X_j}\right)^{-m}dy_j + d\left[\left(\frac{X_i}{X_j}\right)^{-m}\right]y_j,
\]

observing that $d\left[\left(\frac{X_i}{X_j}\right)^{-m}\right]$ is pulled back from $\Omega_P$ and that $\left(\frac{X_i}{X_j}\right)^{-m}dy_j$ maps to a local generator $\pi^*\mathcal{O}_P(-m)$. Therefore,

\[
\omega_U = \bigwedge^{n+1} \Omega_U = \left(\bigwedge^n \pi^*\Omega_P\right) \otimes \pi^*\mathcal{O}_P(-m) = \pi^*\mathcal{O}_P(-n-1-m).
\]

By adjunction, therefore,

\[
\omega_Z = (\omega_U \otimes \mathcal{O}_U(Z))|_Z = \pi_Z^*\mathcal{O}_P(-n-1-m) \otimes \pi_Z^*\mathcal{O}_P(mp) = \pi_Z^*\mathcal{O}_P(mp-n-1-m),
\]

where $\pi_Z$ is the restriction of $\pi$ to $Z$. Because $Z \xrightarrow{\pi_Z} \mathbb{P}$ is a finite map, the pull back of an ample line bundle on $\mathbb{P}$ is ample. This proves that $\omega_Z^{-1}$ is ample whenever the numerical condition $mp - m < n + 1$ is satisfied.

**Method 2.** Compute $K_Z$ by the adjunction formula for $Z$ in the weighted projective space $\mathbb{P}$. This yields $K_Z = (K_{\mathbb{P}} + Z)|_Z$, so that $\omega_Z = \mathcal{O}_Z(-m-n-1+mp)$. This sheaf is invertible, because $Z$ is locally a hypersurface in a smooth variety, and hence Gorenstein. It follows that $\omega_Z^{-1}$ is ample if and only if $mp-m < n+1$ is satisfied.

**Method 3.** Compute $K_Z$ using the Hurwitz formula for the finite map of $Z$ to $\mathbb{P}^n$. This formula predicts that the canonical class for $Z$ is the pullback of the canonical class of $\mathbb{P}^n$ plus the ramification divisor. This can not be applied when the characteristic is $p$, however, because $Z \to \mathbb{P}$ is inseparable, ie, the map is everywhere ramified.

4.5.6. **A special subbundle of differential forms.** Our goal is to apply Proposition 4.2 to conclude that the variety $Z$ is not ruled. In order to do so, we need to find a big line sub-bundle of a sheaf of differential forms on $Z$. This will be an exterior power of a special subsheaf $Q$ of $\Omega_Z$, which we now construct.

Consider the familiar exact sequence

\[
(4.5.7) \quad \mathcal{O}_U(-Z)|_Z \xrightarrow{d} \Omega_U|_Z \to \Omega_Z \to 0.
\]
(This is the “conormal” or the “second exact sequence;” see \[H, II 8.12\].) Let us scrutinize the map \(d\). In the chart \(U_0 = \{X_0 \neq 0\}\), set \(x_i = \frac{X_i}{X_0}\), \(y = \frac{Y}{X_0}\), and \(\frac{F_{mp}}{X_0} = f(x_1, \ldots, x_n)\). The map

\[
\mathcal{O}_U(-Z)|_Z \xrightarrow{d} \Omega_U|_Z
\]

sends the local generator \(y^p - f(x_1, \ldots, x_n)\) to

\[
d(y^p - f(x_1, \ldots, x_n)) = -\frac{\partial f}{\partial x_1} dx_1 - \cdots - \frac{\partial f}{\partial x_n} dx_n + py^{p-1} dy.
\]

Something very interesting happens in characteristic \(p\): the image of \(d\) is contained in the subsheaf of \(\Omega_U|_Z\) generated by the differentials \(dx_1, \ldots, dx_n\), that is,

\[
d(\mathcal{O}_U(-Z)|_Z) \subset \pi^* \Omega_P.
\]

Making use of the \(\mathcal{O}_Z\)-module isomorphism \(\mathcal{O}_U(-Z)|_Z \cong \pi_Z \mathcal{O}_P(-mp)\) derived in 4.5.3, we can define a \(\mathcal{O}_Z\)-module map in characteristic \(p\) only

\[
(4.5.8) \quad \pi_Z \mathcal{O}_P(-mp) \xrightarrow{d} \pi_Z \Omega_P
\]

sending a local generator \(f\) to \(df = \sum \frac{\partial f}{\partial x_i} dx_i\) and extending \(\mathcal{O}_Z\)-linearly. The use of the symbol \(d\) to denote this map is somewhat misleading, since the map is not a derivation, but is \(\mathcal{O}_Z\)-linear. Miraculously, this is a well defined \(\mathcal{O}_Z\)-module map in characteristic \(p\), because the transition functions for \(\mathcal{O}_P(-mp)\) are \(p^{th}\) powers, and are therefore killed by \(d\).

Let \(Q\) be the cokernel of the \(\mathcal{O}_Z\)-module map 4.5.8. There is an exact sequence of \(\mathcal{O}_Z\)-modules

\[
(4.5.9) \quad \pi_Z \Omega_P \to \Omega_U|_Z \to \pi_Z \mathcal{O}_P(-m) \to 0.
\]

obtained by restricting sequence 4.5.5 to \(Z\). Combining this with the exact sequence 4.5.7, we get an exact sequence of \(\mathcal{O}_Z\)-modules:

\[
(4.5.10) \quad 0 \to Q \to \Omega_Z \to \pi_Z \mathcal{O}(-m) \to 0.
\]

An exterior power of \(Q\) will give us the desired big subbundle of a sheaf of differential forms (at least after desingularizing). It is important to realize that the assumption that \(k\) has characteristic \(p\) is essential: nothing like this is possible in characteristic zero.

The next exercise is not essential for our computation, but it should help clarify what is going on in the construction of \(Q\).

**Exercise 19:** Let \(X\) be an arbitrary variety over a field \(k\). A connection on an invertible sheaf \(L\) of \(\mathcal{O}_X\)-modules is a \(k\)-linear map

\[
L \xrightarrow{\nabla} L \otimes \Omega_X
\]
satisfying $\nabla(fs) = f\nabla(s) + s \otimes df$ for local sections $s \in \mathcal{L}$ and $f \in \mathcal{O}_X$.

(1) Explain how a connection can be interpreted as a rule for differentiating sections of line bundles.

(2) Show that if $k$ has prime characteristic $p$, then any line bundle that is a $p^{th}$ power admits a connection.

(3) Observe that the map $d$ above in 4.5.8 can be constructed from the composition

$$
\mathcal{O} \xrightarrow{\text{mult by } s} \mathcal{L}^p \xrightarrow{\nabla} \mathcal{L}^p \otimes \Omega_X
$$

where $s$ is a global section of $\mathcal{L}^p$, using the identifications $H^0(\mathcal{L}^p \otimes \Omega_X) = \text{Hom}(\mathcal{O}_X, \mathcal{L}^p \otimes \Omega_X) = \text{Hom}(\mathcal{L}^{-p}, \Omega_X)$.

4.5.11. Bigness of the special subbundle of differential forms. With an eye towards applying Proposition 4.4, we hope to find integers $m, n$ and $p$ so that $\bigwedge^{n-1} Q$ will be a big invertible sheaf. Assume, for a moment, that $Z$ is smooth. The sequence 4.5.10 would imply that $Q$ is locally free of rank $n - 1$, and that $\bigwedge^{n-1} Q \hookrightarrow \bigwedge^{n-1} \Omega_Z$. We could easily determine the range of values for $m, n$ and $p$, for which $\bigwedge^{n-1} Q$ is big. Indeed,

$$
\omega_Z = \bigwedge^n \Omega_Z = \bigwedge^{n-1} Q \otimes \pi^*_Z \mathcal{O}_P(-m),
$$

so that

$$
\bigwedge^{n-1} Q = \omega_Z \otimes \pi^*_Z \mathcal{O}_P(m) = \pi^* \mathcal{O}_P(mp - m - 1),
$$

using the isomorphism $\omega_Z = \pi^* \mathcal{O}_P(mp - m - n - 1)$ verified in 4.5.4. From this we could conclude that $\bigwedge^{n-1} Q$ is ample (and hence big) whenever $n + 1 < mp$.

Because there are plenty of choices of integers $m, p$ and $n$ for which the constraints

$$
mp - m < n + 1 < mp
$$

hold, we would expect to be able to use Proposition 4.4 to find plenty of non-rational Fano varieties. Unfortunately, however, this argument fails because the variety $Z$ is virtually never smooth: this is the price to be paid for the characteristic $p$ trickery that allowed us to construct $Q$. Indeed, $\bigwedge^{n-1} Q$ is not even an invertible sheaf in general.

It is easy to alter $\bigwedge^{n-1} Q$ so as to get big invertible sheaf. On the smooth locus of $Z$, the sheaf $\bigwedge^{n-1} Q$ is naturally isomorphic to $\pi^* \mathcal{O}_P(mp - n - 1)$ Now, because $Z$ is a hypersurface in a smooth variety, it is normal, so that any invertible sheaf defined on the complement of a codimension two closed subscheme extends uniquely to a reflexive sheaf of $\mathcal{O}_Z$-modules. Since $\bigwedge^{n-1} Q$ agrees with $\pi^*_Z \mathcal{O}_P(mp - n - 1)$ on the smooth locus, its "reflexive hull"

$$
(\bigwedge^{n-1} Q)^{**} = \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{H}om_{\mathcal{O}_Z}(\bigwedge^{n-1} Q, \mathcal{O}_Z), \mathcal{O}_Z)
$$
is an invertible sheaf of $\mathcal{O}_Z$-modules isomorphic to $\pi^*_Z\mathcal{O}_\mathcal{P}(mp-n-1)$. This sheaf is ample when $mp > n+1$, and is a subsheaf of $(\wedge^{n-1}\Omega_Z)^\ast\ast$. On the smooth locus of $Z$, this sheaf restricts to an invertible subsheaf of differential $n-1$ forms on $Z$.

4.5.12. Desingularizing $Z$. Proposition 4.4 is only valid for smooth varieties, so we must resolve the singularities of $Z$. Bigness is preserved under birational pull back, so pulling back $(\wedge^{n-1}Q)^\ast\ast$ to a desingularization, it is still a big invertible sheaf. But we must then check that this pull back is a subsheaf of some sheaf of differential forms. We accomplish this by choosing the polynomial $F_{mp}$ so as to make an explicit resolution straightforward.

Recall that the non-smooth points of $Z$ are are given precisely by the critical points of the polynomials $f_{mp}(x_1, \ldots, x_n)$ (Exercise 17). It is easy to desingularize $Z$ under the the following non-degeneracy assumption:

**Assumption 4.5.13:** Each dehomogenization of $F_{mp}$ is a polynomial with only non-degenerate critical points.

This means that for each $i = 0, \ldots, n$, the polynomial

$$f = F_{mp}(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)$$

in $n$ variables has only non-degenerate critical points. As usual, a critical point of a polynomial $f$ is a point where all the partial derivatives of $f$ vanish, and the critical point is non-degenerate if the determinant of the Hessian matrix of second derivatives does not vanish there. Here, “point” means point defined over the algebraic closure of the ground field. Such $F_{mp}$ exist over any infinite field (with some exceptions in characteristic two); see 4.5.16.

Assuming now that $F_{mp}$ has the non-degeneracy condition described above, we now complete the proof of Theorem 4.3 by desingularizing $Z$ and verifying that $(\wedge^{n-1}Q)^\ast\ast$ pulls back to a subsheaf of regular differential forms.

The advantage of non-degenerate critical points is that, after possibly enlarging the ground field, the affine equation of the hypersurface $Z$ can be assumed of the form

$$y^p = c + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + f_3$$

if $n$ is even, or

$$y^p = c + x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} + x_n^2 + f_3$$

if $n$ is odd and $p > 2$.

where the $y, x_i$'s are local coordinates at a non-smooth point of $Z$, $c$ is a constant, and $f_3 = f_3(x_1, \ldots, x_n)$ is a polynomial of order three or more in the $x_i$ (see Exercise 21). Fortunately, desingularizing such a hypersurface is easy.

**Exercise 20:** Show that if $f$ has only isolated non-degenerate critical points, then the affine hypersurface defined by $y^p - f$ becomes smooth upon blowing up each non-smooth point (over the algebraic closure of the ground field), regardless of the characteristic of the ground field.
4.5.14. Verification that \( \mathcal{M} \) is a subsheaf of differential forms. Having shown that \( Z \) can be smoothed by blowing up points, let \( Z' \xrightarrow{q} Z \) be this desingularization of \( Z \). Assuming that the ground field is characteristic \( p \), consider the sheaf

\[
\mathcal{M} = q^*(\bigwedge^{n-1} Q)^{**} = q^*\pi^*\mathcal{O}_\mathbb{P}(mp - n - 1)
\]

as in 4.5.6. We know that \( \mathcal{M} \) is big, and we wish to show that it is contained in \( \bigwedge^{n-1} \Omega_{Z'} \). This is just a matter of computing local generators for \( \mathcal{M} \) and comparing them to local generators for \( \bigwedge^{n-1} \Omega_{Z'} \).

We return to the somewhat mysterious definition of \( Q \). Recall that \( Q \) is the cokernel of the very special \( \mathcal{O}_Z \)-module map

\[
\pi_\mathbb{P}^*\mathcal{O}_Z(-mp) \xrightarrow{d} \pi_\mathbb{P}^*\Omega_\mathbb{P},
\]

defined in 4.5.6. Think of \( d \) as the pull back of a map of \( \mathcal{O}_\mathbb{P} \)-modules \( \mathcal{O}_\mathbb{P}(-mp) \xrightarrow{d'} \Omega_\mathbb{P} \), sending the local generator \( f_{mp} \) to

\[
df_{mp} = \sum_{i=1}^{n} \frac{\partial f_{mp}}{\partial x_i} dx_i.
\]

(We reiterate that this map is deceptively subtle: its existence is a very special consequence of the fact that the ground field is characteristic \( p > 0 \); Cf 4.5.6.)

Taking the \((n - 1)^{st}\) exterior power of the cokernel \( Q \) of \( d = \pi_\mathbb{P}^*d' \), we have convenient local generators

\[
\eta_i = (-1)^i \frac{dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n}{\partial df_{mp}/\partial x_i}
\]

for \( \bigwedge^{n-1} Q \) on the open set where \( \partial f_{mp}/\partial x_i \) is non-zero. Note that \( \eta_i = \eta_j \) whenever both are defined. The locus where no \( \eta_i \) is defined is precisely the non-smooth locus of \( f_{mp} \). Since this set has codimension at least two, this sheaf extends uniquely to a sheaf \( (\bigwedge^{n-1} Q)^{**} \) on all of \( Z \). The extension can be defined as a subsheaf of the constant sheaf of rational differential forms on \( Z \) generated by the \( \eta_i \). By definition of \( \mathcal{M} \), these pull back to local generators of \( \mathcal{M} \) on the desingularization \( Z' \).

To check that \( \mathcal{M} \subset \bigwedge^{n-1} \Omega_{Z'} \), we only need check what happens along the exceptional fibers of \( Z' \rightarrow Z \), since we already know the inclusion holds on the smooth locus of \( Z \). This is a straightforward computation; we work it out in one case below.

Let \( y, x_1, \ldots, x_n \) be local coordinates for \( Z \) near a non-smooth point, and let \( y, x'_1, \ldots, x'_n \) denote local coordinates on the blowup \( Z' \) of the ideal \((y, x_1, \ldots, x_n)\), with \( x_i = yx'_i \). Computing the pull back of, say, \( \eta_n \) when \( n \) is even and \( p = 2 \), we have

\[
q^*\eta_n = \frac{d(yx'_1) \wedge \cdots \wedge d(yx'_{n-1})}{\partial(y^2 + x_1x_2 + \cdots + x_{n-1}x_n + g)/\partial x_n}
\]
where \( g \) has order 3 or more in \((y, x_1, \ldots, x_n)\). Performing the differentiation, we see that the denominator is \( x_{n-1}^n + h \), where \( h \) is order 2 or more in \((y, x_1, \ldots, x_n)\), which we write as \( y(x_{n-1}^n + yh') \) in local coordinates on the blowup, with \( h' \) in \((y, x_1', \ldots, x_n')\). Thus

\[
q^* \eta_n = \frac{y^{n-1}(dx_1' \wedge \cdots \wedge dx_{n-1}') + \sum_{j=1}^{n-1} y^{n-2}(dx_1' \wedge \cdots \wedge dy \wedge \cdots \wedge dx_{n-1}')}{y(x_{n-1}^n + yh')}
= y^{n-3} \left( \frac{y(dx_1' \wedge \cdots \wedge dx_{n-1}')}{{x_{n-1}^n} + yh'} \right),
\]

where the \( j \)th term in the sum has \( dy \) in the \( j \)th position.

To check that the local generator \( q^* \eta_n \) has no pole along the exceptional fibers, we can compute compute in any open set which intersects the exceptional divisor \( E \). In the neighborhood considered above, the exceptional divisor \( E \) is defined by \( y \), so the generator \( q^* \eta_n \) vanishes along \( E \) to at least order \( n - 3 \). So \( M \) has no poles along \( E \) whenever \( n \geq 3 \). Because the computation along each exceptional divisor is essentially the same, we conclude that \( M \) is a subsheaf of \( \bigwedge^{n-1} \Omega_{Z'} \), whenever \( n \geq 3 \).

4.5.15. \( Z \) is not ruled. Now we are in position to apply Proposition 4.4. The blowup variety \( Z' \) is smooth and carries the big invertible sheaf \( M \) that is a subsheaf of a sheaf of differential forms on \( Z' \). Proposition 4.4 implies that \( Z' \) cannot be separably uniruled. Because \( Z' \) is birationally equivalent to \( Z \), it follows also that \( Z \) is not separably uniruled. In particular, \( Z \) is not ruled.

This essentially completes the proof of Theorem 4.3. Any subvariety \( Z \) of \( P \) defined by an equation of the form \( Y^p - F_{mp} \), where \( F_{mp} \) satisfies Assumption 4.5.13 (ensuring the singular points of \( Z \) are non-degenerate) and where the numerical constraints \( pm - m < n + 1 < pm \) hold, is an example of a non-ruled projective variety whose anti-canonical sheaf is ample invertible. We now only need to observe that such \( F_{mp} \) satisfying this assumption do exist.

Exercise 21: A critical point \( P \) of a polynomial \( f \) is non-degenerate if the determinant of the Hessian matrix \( \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \) does not vanish at \( P \), or equivalently, if \( \left\{ \frac{\partial f}{\partial x_i} \right\}_{i=1}^n \) generate the maximal ideal of \( P \). Prove the following Morse Lemma for polynomials over an infinite field \( k \).

1. If the characteristic of \( k \) is greater than two, then a sufficiently general polynomial function of degree \( d \) in \( n \) variables over \( k \) has only non-degenerate critical points.
2. If \( k \) has characteristic two, then every critical point of a polynomial in an odd number of variables is degenerate, where as the general polynomial function of an even number of variables has only non-degenerate critical points.

4.5.16. When the ground field is infinite, the Morse Lemma ensures that every sufficiently general choice of \( F_{mp} \) satisfies Assumption 4.5.13. Thus, over an infinite
field, there are many examples of non-ruled \( Z \). It is not obvious that such \( F_{mp} \) exist over finite fields. Fortunately, explicit examples, due to Joel Rosenberg, show that polynomials \( F_{mp} \) satisfying Assumption 4.5.13 and also satisfying the numerical constraints \( mp - m < n + 1 < pm \) exist over any finite field. A specific example, over any field of characteristic \( p \), is the hypersurface in the weighted projective space \( \mathbf{P} \) defined by

\[
Y^p - \sum_{i=0}^{n} X_i^{mp-1} X_{i+1}
\]

(4.5.17)

where the subscripts are taken modulo \( n + 1 \). Here, \( n \) is any integer greater than two satisfying \( pm - m < n + 1 < pm \). See Appendix I.

4.5.18. Summary. We have established the following. Fixing positive integers \( p, m \) and \( n \) satisfying

\[ pm - m < n + 1 < pm \]

with \( p \) prime and \( n \) at least three, let \( F_{mp} \) be a homogeneous polynomial of degree \( mp \) in \( n+1 \) variables. If \( Z \) denotes the hypersurface defined by \( Y^p - F_{mp} \) in weighted projective space \( \mathbf{P} \), the following hold for sufficiently general choices of \( F_{mp} \):

1. The anti-canonical sheaf of \( Z \) is ample invertible;
2. When the characteristic is not \( p \), the variety \( Z \) is smooth;
3. When the characteristic is \( p \), the variety \( Z \) is not ruled (even after arbitrary base extension).

The precise meaning of “sufficiently general” here is that \( F_{mp} \) should define a projective hypersurface with only isolated non-smooth points, all of which are non-degenerate; that is, \( F_{mp} \) must satisfy Assumption 4.5.13.

Unfortunately, however, the varieties \( Z \) are non-smooth in characteristic \( p \), precisely the case where we have proven them non-ruled. In Lecture 5, we will see that the same polynomial defines a non-ruled smooth Fano variety in characteristic zero. In particular, these are examples of non-rational smooth Fano varieties.

5. Final Lecture

The goal of this lecture is to establish the existence of a multitude of smooth non-rational Fano varieties in every dimension. In fact, we construct specific families of non-ruled smooth Fano varieties. The main point is the method of the proof. Here we content ourselves with simple applications, producing only some examples of non-rational Fano varieties. With minor modifications the method produces many more such examples, see [K95] or [K97, V 5] for details.

For a homogeneous polynomial \( F_{mp} \) of degree \( mp \) let \( Z = Z(F) \) denote the hypersurface \( Y^p - F_{mp} \) in the weighted projective space \( \mathbf{P} = \mathbb{P}(m, 1, 1, \ldots, 1) \). This was constructed in 4.5, and its key properties are summarized in 4.5.18.

5.1. Theorem. Fix an arbitrary ground field \( k \) of characteristic zero. Fix integers \( m \geq 1, n \geq 3 \) and a prime \( p \) satisfying

\[ (p - 1)m < n + 1 < pm. \]
Then there exists a homogeneous polynomial $F_{mp} \in k[X_0, \ldots, X_n]$ such that the corresponding variety $Z(F)$ is a smooth projective Fano variety that is not ruled over $k$.

5.2. Remarks.

5.2.1. We will prove here only a weaker form Theorem 5.1 in full detail: we will show that $Z(F)$ is not rational.

5.2.2. It is quite likely that if $(p - 1)m < n + 1 < pm$ then every smooth variety $Z(F)$ as above is nonrational. Unfortunately this is not known.

5.2.3. For most values of $m, n$, and $p$, one can use Theorem 5.1 to write down explicit examples of smooth Fano varieties over $\mathbb{Q}$ that are not ruled. For instance, the examples of Rosenberg (see Appendix I) show that if $n \not\equiv -1 \mod p$ and $mp \geq 3$

\[
\{Y^p - \sum_{i=0}^{n} X_i^{mp-1} X_{i+1} = 0\} \subset P
\]

is a smooth Fano variety that is not ruled. See also [K96, V 5.16.3] for other examples deduced by a related theorem.

5.2.4. Let $V_{mp}$ be the vector space of homogeneous polynomials $F_{mp} \in k[X_0, \ldots, X_n]$. By [K97 IV.1.8.3], there are countably many subvarieties $W_i \subset V_{mp}$ such that if $F \notin \cup_i W_i$ then $Z(F)$ is not ruled. Thus if $k$ is uncountable, then finding just one non-ruled example guarantees that most of the varieties $Z(F)$ are not ruled, hence also not rational.

In any case, if $k$ is not algebraic over $\mathbb{Q}$, then the Morse Lemma guarantees that a generic choice of $F_{mp}$ produces a non-ruled example; see the explanation 5.3.2.

Lecture 5 is devoted to the proof of Theorem 5.1.

The idea of the proof is simple. We already know that $Z$ is a smooth Fano variety in characteristic zero 4.5.4. The proof that $Z$ is non-ruled uses reduction to characteristic $p$. To get a rough idea how this is done, suppose first that we wish to construct an example defined over $\mathbb{Q}$. Consider $F_{mp} \in \mathbb{Z}[X_0, \ldots, X_n]$, with $m, n$ and $p$ as in Theorem 5.1. Consider the scheme

\[
\text{Proj} \frac{\mathbb{Z}[Y, X_0, \ldots, X_n]}{Y^p - F_{mp}} = Z_\mathbb{Z} \subset P_\mathbb{Z},
\]

a closed subscheme of the weighted projective space $P_\mathbb{Z} = \text{Proj} \mathbb{Z}[Y, X_0, \ldots, X_n]$. There is a morphism

\[
Z_\mathbb{Z} \rightarrow \text{Spec} \mathbb{Z},
\]

whose special fiber over $(p)$ is the singular Fano variety as constructed in Lecture 4. Furthermore, we can arrange this so the special fiber is non-ruled: we need only choose $F_{mp}$ so that its reduction modulo $p$ has only non-degenerate critical points in each affine patch. Now the idea is to apply the following theorem of Matsusaka [Mats] about the behavior of ruledness in families.
5.3. Theorem. Let $V$ be a discrete valuation ring that is a localization of a finitely generated algebra over a field or over the integers.\(^4\) Let $K$ (respectively, $k$) denote its quotient field (respectively, residue field). Let $Z_S$ be a normal irreducible projective scheme over $S = \text{Spec } V$. If the generic fiber of the natural projection $Z_S \to S$ is ruled over $K$, then each irreducible component of the special fiber is ruled over $k$.

**Cautionary Remark.** Theorem 5.3 underscores the reason we are led to consider non-ruled varieties in our quest for non-rational ones: ruledness is better behaved in families than rationality. We can not conclude a special member of a family is rational when we know the generic member is rational. For example, a family of degree three hypersurfaces in $\mathbb{P}^3$ has a smooth cubic surface as its generic member, but it can have singular members that are cones over elliptic curves. The generic member is rational, whereas the special member of this family is only ruled.

5.3.1. Remark. To keep things elementary, we prove here only the following weak form of Matsusaka’s theorem: If the general fiber above is rational, then the components of the special fiber are ruled. This will be sufficient to conclude the existence of non-rational $Z(F)$. For the full proof of Theorem 5.3, the reader is referred to [K96, p 184].

Before proving Theorem 5.3, we show how to use it to deduce the existence of non-rational Fano varieties in characteristic zero.

**Deduction of Theorem 5.1 from Theorem 5.3.** Let $Z$ be the hypersurface in $\mathbb{P}$ defined over $k$. Choose a finitely generated $\mathbb{Z}$-algebra $A$ contained in $k$ over which $Z$ is defined, that is, containing all the coefficients of the defining equation for $Z$. Having chosen $A$, let $S = \text{Spec } A$, and observe that there is an $S$-scheme $Z_S$ such that the scheme $\text{Spec } k \times_S Z_S$ is naturally isomorphic to $Z$.

Assuming $Z$ were ruled, fix a birational map

$$W \times_k \mathbb{P}^1_k \overset{\phi}{\to} Z.$$ 

Now choose $A$ large enough so that it contains all the elements of $k$ necessary to describe the $k$-scheme $W$ and the map $\phi$. This gives rise to an $S$-scheme $W_S$ and an $S$-scheme map

$$W_S \times_S \mathbb{P}^1_S \overset{\phi}{\to} Z_S$$

which is a birational equivalence.

The base scheme $S$ may be replaced by the spectrum of a discrete valuation ring as follows. Normalizing if necessary, there is no loss of generality in assuming the base ring $A$ is normal. Now localizing $A$ at a minimal prime of $(pA)$, we achieve a discrete valuation ring, say $V$. By base change, we replace $S = \text{Spec } A$ with the two-point scheme $S = \text{Spec } V$, and assume the map $Z_S \to S$ is a family over spectrum of the discrete valuation ring $V$.

\(^4\)In fact, the base $V$ can be any excellent discrete valuation ring, although we do not need to apply the theorem in such generality. For the definition of excellent, see [M, p260].
The generic fiber of $Z_S \to S$ would clearly be ruled. However, the special fiber is a variety of prime characteristic $p$, defined in $\mathbb{P}$ by an equation of the form $Y^p - F_{mp}$. We proved in Theorem 4.3 that such a variety is not ruled in general. Indeed, if $F_{mp}$ is any homogeneous polynomial having only non-degenerate critical points locally (i.e., satisfying Assumption 4.5.13) over the residue field $V/uV$, then the special fiber $Z_p$ is not ruled. So lifting $F_{mp}$ to a polynomial defined over $V$, we have a scheme $Z_S$ in $\mathbb{P}_S$ whose fiber over the generic point of $S$ is a smooth Fano variety $Z \subset \mathbb{P}$, defined over the fraction field of $V$, which can not be ruled by Theorem 5.3. This contradiction completes the proof that $Z_k$ is not ruled. □

5.3.2. Remark. If $k$ is non-algebraic over $\mathbb{Q}$, we expect some of the coefficients of $F_{mp}$ to be non-algebraic over $\mathbb{Q}$. This forces $V/uV$ to be an infinite field in general. Now the Morse Lemma implies that sufficiently general choices of $F_{mp}$ modulo $uV$ satisfy Assumption 4.5.13 (see Exercise 21). This means that all sufficiently general choices of $F_{mp}$ defined over $k$ give rise to varieties $Z$ that are smooth Fano non-ruled varieties.

We now complete the proof of the existence of non-rational Fano varieties by proving the weak form of Theorem 5.3. With some extra work, Theorem 5.3 can be proved in full, but we refer to [K96, p. 184] for the details.

Proof Theorem 5.3 in the weak form 5.3.1. Let $\xi$ be the generic point of $S$ and assume the fiber $Z_\xi$ of $Z_S \to S$ over $\xi$ is rational. Since $S$ is birationally equivalent to $\xi$, a birational map

$$\mathbb{P}^n_S \times_S \xi = \mathbb{P}^n_\xi \dashrightarrow Z_\xi = Z_S \times_S \xi$$

defines a birational map

$$\mathbb{P}^n_S \phi \to Z_S$$

over $S$. (If the generic fiber of $Z_S \to S$ is only ruled, we can find a $S$-scheme $\mathbb{P}^n_S \times_S W$ mapping birationally onto $Z_S$ over $S$. But to make the following proof work, this scheme must be both regular and proper over $S$.)

Let $\Gamma_S$ be the normalization of the (closure of the) graph (in $\mathbb{P}^n_S \times_S Z_S$) of the birational map $\phi$. The normalization is a finite, birational morphism. So composing with the natural projections, we have proper birational morphisms $\Gamma_S \pi_1 \to Z_S$ and $\Gamma_S \pi_2 \to \mathbb{P}^n_S$.

Consider the special fiber $Z_p$ of $Z_S \to S$. Being defined by a single equation, namely the pullback of the uniformizing parameter $u$, all components of $Z_p$ have codimension one. Likewise, the components of the special fiber $\Gamma_p$ of $\Gamma_S \to S$ are all codimension one. Now, because $\Gamma_S \pi_1 \to Z_S$ is a proper birational map of normal schemes, it is an isomorphism in codimension one (on the base). This means that for each irreducible component of $Z_p$, there corresponds a unique irreducible

\[^{5}\text{In general, the normalization map of an arbitrary scheme can fail to be finite, although it is finite for schemes of essentially of finite type over } \mathbb{Z}, \text{ or more generally for any excellent scheme.}\]
component $\Gamma_0$ of $\Gamma_p$ mapping birationally to it. Therefore, it suffices to show that the reduced irreducible divisor $\Gamma_0$ is ruled.

Consider the restriction of $\pi_2$ to $\Gamma_0$. This gives a morphism $\Gamma_0 \rightarrow \mathbb{P}^n$, where 

$$\mathbb{P}^n_p = \mathbb{P}^n_S \times_S \text{Spec } (V/uV)$$

is the special fiber of $\mathbb{P}^n_S \rightarrow S = \text{Spec } V$. If $\Gamma_0 \rightarrow \mathbb{P}^n_p$ is birational, the proof of Theorem 5.3 is complete: then $\Gamma_0$, and hence the birationally equivalent variety $Z_p$, would be birationally equivalent to $\mathbb{P}^n_p$.

On the other hand, the map $\Gamma_S \xrightarrow{\pi_2} \mathbb{P}^n_S$ is a proper birational morphism, so if its restriction to the divisor $\Gamma_0$ is not birational, then $\Gamma_0$ must be an exceptional divisor for this map. But exceptional divisors of proper birational maps to regular schemes are always ruled, as the following theorem of Abhyankar shows [Ab, p336].

5.4. Theorem. Let $Y \xrightarrow{\pi} X$ be a proper birational morphism of irreducible schemes, with $Y$ normal and $X$ regular. Then every exceptional divisor of $\pi$ is ruled over its image. That is, if $E$ is an integral subscheme of $Y$ of codimension one, whose image $E'$ has codimension greater than one in $X$, then $E$ is birationally equivalent over $E'$ to a scheme $W \times_{E'} \mathbb{P}^1_{E'}$.

Abhyankar’s proof uses valuation theory. Rather than reproduce his proof here in full generality, we provide a nice geometric proof in the case where $Y$ is of finite type over $X$, which is certainly sufficient for our purposes.

We first point out that when $Y$ and $X$ are algebraic varieties defined over a field $k$ of characteristic zero, Theorem 5.4 follows easily from Hironaka’s theorem on resolution birational maps.\footnote{Again, some very mild reasonability condition on $X$ is required: it is sufficient if $X$ is of finite type over a localization of a finitely generated algebra over a field or $\mathbb{Z}$. Indeed, $X$ can be any excellent scheme.\footnote{Although Abhyankar’s 1956 proof came first.}} In this case, the morphism $\pi$ factors through a sequence of blow-ups along smooth centers:

\[\begin{array}{c}
Y \\
\pi \\
X_0 \\
\sigma_1 \\
X_1 \\
f \\
\vdots \\
\sigma_r \\
X_r
\end{array}\]
Here, each $X_{i+1} \xrightarrow{\sigma_{i+1}^{-1}} X_i$ is a blowing up along a nonsingular center, and $X_r \xrightarrow{f} Y$ is a birational morphism from the non-singular variety $X_r$. To see that $E$ is ruled, there is no harm in replacing $Y$ by $X_r$, and $E$ by its birational transform on $X_r$. Because $E$ is exceptional for the composition of the blowups $X_r \to X$, its image on some $X_i$ must be an exceptional divisor for some blowup $\sigma_i$. But the exceptional divisor of a blow-up of a non-singular variety along a non-singular center is a projective space bundle over the center; in particular, such exceptional divisors, including $E$, must be ruled.

Because we are interested only in birational properties, it is not actually necessary to use Hironaka’s deep theorem. The idea can be adapted as follows. We construct the tower of blowups $X_i \xrightarrow{\sigma_i} X_{i-1}$ by blowing up the image $E_{i-1}$ of $E$ on $X_{i-1}$, but always restricting to the regular (non-singular) loci of the $E_{i-1}$, so that each $X_i$ is regular. Again, the exceptional fiber of the blow-up of a regular scheme along a regular subscheme is a projective space bundle over the center of the blowup. So again, we can show that $E$ is ruled by showing that the image of $E$ on some $X_i$ must be the exceptional divisor for some blowup $\sigma_i$.

This is shown by keeping track of a numerical invariant that drops with each non-trivial blowup. This numerical invariant can be taken to be the order of vanishing along $E$ of the pull back of a local generator of $\omega_{X_i}$ to $\omega_Y$, provided that one can make sense of the sheaves $\omega_{X_i}$ and $\omega_Y$ and that one can define a pullback map $\pi^*$ to $\omega_Y$. For example, when $Y$ and $X$ are of finite type over an algebraically closed field, there is no problem making sense of $\omega_X$ and $\omega_Y$ and the argument is easily adapted to this case.

In carrying out this argument, complications arise when the scheme $X$ (and $Y$) are not defined over some base field. When $X$ and $Y$ are both smooth over some base scheme $S$, one can try to work with the relative canonical modules $\omega_{X/S}$. This sometimes works (for instance, if $E$ is flat over $S$), but unfortunately, it breaks down precisely in the case we need it. The trouble arises because we must work with regular schemes that may not be smooth over the base scheme. Indeed, the following situation is typical: the scheme $X$ may be $\mathbb{A}^n_S$ with $S$ the two-point scheme $\text{Spec } \mathbb{Z}_p$. We will blow up a regular subscheme $E_0$ which maps to the closed point of $S$, say the closed point defined by $(p, x_1, \ldots, x_n)$. The resulting blow-up scheme $X_1$ is regular, but it is not smooth over $S$. In this case, it is hard to define a relative canonical module $\omega_{X_1/S}$ that has the properties need to carry out the argument along the lines suggested above.

However, we will be able to adapt the proof of Abhyankar’s theorem in our case by working with the relative canonical modules for the birational maps $X_i \to X_0$. In fact, the duals of these canonical modules, the so-called Jacobian ideals, are more convenient to work with.

5.4.1. A Digression on Relative Canonical Modules and Jacobian Ideals. Let $Y$ be a scheme of finite type over $X$, and suppose that $Y \to X$ has relative dimension $d$. We say that $Y$ is smooth over $X$ if the sheaf of relative Kähler differentials $\Omega_{Y/X}$
is a locally free $\mathcal{O}_Y$ module of rank $d$. In this case, the relative canonical module $\omega_{Y/X}$ is defined to be the invertible sheaf $\bigwedge^d \Omega_{Y/X}$.

When $Y$ is normal and smooth in codimension one over $X$, the relative canonical module $\omega_{Y/X}$ can be defined as the unique reflexive $\mathcal{O}_Y$ module that agrees with the above construction on the smooth locus of $Y \to X$. Equivalently, $\omega_{Y/X}$ is the double dual of the $\mathcal{O}_Y$ module $\bigwedge^d \Omega_{Y/X}$. Although this canonical module is not necessarily invertible, it still can be interpreted as the “determinant” of $\Omega_{Y/X}$ via the natural map

$$\bigwedge^d \Omega_{Y/X} \to \omega_{Y/X} = (\bigwedge^d \Omega_{Y/X})^{**}$$

which is neither injective nor surjective in general.

This method of defining the canonical module fails when $Y$ is not smooth in codimension one over $X$ (for example, when $Y \to X$ is a blowup). Let us compute the “determinant” of $\Omega_{Y/X}$ in a different way so that it will generalize to this case. Fix an embedding $Y \hookrightarrow W$ of $Y$ in a smooth $X$ scheme $W$, for instance, $W$ may be taken to be an open subset of affine space over $X$. Consider the conormal complex:

$$\mathcal{I}_Y / \mathcal{I}_Y^2 \xrightarrow{d} i^* \Omega_{W/X} \to \Omega_{Y/X} \to 0$$

where $\mathcal{I}_Y \subset \mathcal{O}_W$ is the ideal sheaf of $Y$ in $W$. If $\mathcal{I}_Y$ is locally generated by a regular sequence, as it must be, for instance, when both $X$ and $Y$ are regular, then the conormal sequence is exact also on the left:

$$0 \to \mathcal{I}_Y / \mathcal{I}_Y^2 \xrightarrow{d} i^* \Omega_{W/X} \to \Omega_{Y/X} \to 0.$$  

This suggests a method for computing the “determinant” of $\Omega_{Y/X}$ when $X$ and $Y$ are regular. In this case $\mathcal{I}_Y / \mathcal{I}_Y^2$ and $i^* \Omega_{W/X}$ are both locally free, hence we can propose the definition

$$\omega_{Y/X} := \bigwedge^n i^* \Omega_{W/X} \otimes (\bigwedge^{n-d} \mathcal{I}_Y / \mathcal{I}_Y^2)^{-1},$$

where $n$ is the relative dimension of $W$ over $X$ and $d$ is the relative dimension of $Y$ over $X$. This agrees with our previous definition, but makes sense even when $Y \to X$ is not smooth in codimension one. The module $\omega_{Y/X}$ is invertible (provided $Y$ is generically smooth over $X$ so that ranks are as expected). We do not need to worry about the dualizing properties of the sheaf.

Now, in order to prove Abhyankar’s theorem, we must consider the case where $Y \to X$ is a birational morphism of regular schemes. The relative dimension is zero. The map of rank $n$ locally free $\mathcal{O}_Y$-modules

$$\mathcal{I}_Y / \mathcal{I}_Y^2 \xrightarrow{d} i^* \Omega_{W/X}$$

gives rise to a map of invertible $\mathcal{O}_Y$-modules

$$\bigwedge^n \mathcal{I}_Y / \mathcal{I}_Y^2 \xrightarrow{d} \bigwedge^n i^* \Omega_{W/X}.$$
Tensoring with \((\bigwedge^n i^*\Omega_{W/X})^{-1}\) we get an exact sequence

\[0 \to \bigwedge^n I_Y/I_Y^2 \otimes (\bigwedge^n i^*\Omega_{W/X})^{-1} \to \mathcal{O}_Y \to \mathcal{Q} \to 0.\]

Here, \(\omega_{Y/X} = \bigwedge^n i^*\Omega_{W/X} \otimes (\bigwedge^n I_Y/I_Y^2)^{-1}\) so that its dual, \(\omega_{Y/X}^{-1} = \bigwedge^n I_Y/I_Y^2 \otimes (\bigwedge^n i^*\Omega_{W/X})^{-1}\), is a sheaf of ideals in \(\mathcal{O}_Y\). It is often called the Jacobian ideal and denoted by \(J_{Y/X}\). Note also that \(\mathcal{Q}\) is some torsion \(\mathcal{O}_Y\)-module supported on the non-smooth locus of \(Y \to X\), so that the Jacobian ideal defines the non-smooth locus of \(Y \to X\).

To explain the name, choose local coordinates \(x_1, \ldots, x_n\) for \(W\) over \(X\), such that the \(dx_i\) are a free basis for \(\Omega_{W/X}\). Suppose that \(I_Y\) is defined locally by the regular sequence \(f_1, \ldots, f_n\). Then the map of free \(\mathcal{O}_Y\) modules

\[I_Y/I_Y^2 \xrightarrow{d} i^*\Omega_{W/X}\]

sends the class of a generator \(\bar{f}_i\) to \(\sum_{i=1}^n \frac{\partial f_i}{\partial x_j} dx_j\). In other words, the map is defined by the Jacobian matrix \(\left(\frac{\partial f_i}{\partial x_j}\right)\), so that the map

\[\bigwedge^n I_Y/I_Y^2 \hookrightarrow \bigwedge^n i^*\Omega_{W/X}\]

is defined by its determinant. In particular, the Jacobian ideal \(J_{Y/X}\) is locally generated by this Jacobian determinant.

The proof of Theorem 5.4 rests on the simple observation that Jacobian ideals are multiplicative.

**Exercise 22:** If \(Z \xrightarrow{\pi} Y \to X\) are finite type birational maps of regular schemes, then

\[J_{Z/X} = (J_{Z/Y})(J_{Y/X} \mathcal{O}_Z)\]

as ideals of \(\mathcal{O}_Z\). Here \(J_{Y/X} \mathcal{O}_Z\) denotes the ideal of \(\mathcal{O}_Z\) generated by the pullbacks of generators of \(J_{Y/X}\); because \(J_{Y/X}\) is invertible, this is the same as \(\pi^*J_{Y/X}\).

We can now give an easy proof of Abhyankar’s theorem; the idea is from a paper of B. Johnston [Jo].

**Proof of Theorem 5.4 assuming \(Y\) is of finite type over \(X\).** Because \(Y\) is normal, it is regular in codimension one. So we are free to replace \(Y\) by an open set containing the generic point of \(E\) so as to assume that \(Y\) is regular.\(^8\)

The divisor \(E\) is exceptional for the given birational map \(Y \xrightarrow{\pi} X\), so its image under \(\pi\) is a subscheme of codimension at least two. Let us denote this image

\[^8\text{This is where the excellence hypothesis is used: we must assume that the regular locus is open, which holds, of course, when \(Y\) is of finite type over a localization of a finitely generated algebra over a field or \(\mathbb{Z}\).}\]
subscheme by $E_0$. Note that $E_0$ is a reduced and irreducible closed subscheme of the regular scheme $X$.

The subscheme $E_0$ need not be regular. However, because it is reduced, the locus of its non-regular points is a proper closed subscheme. So we can replace $X$ by an open set $X_0$ in which $E_0$ is regular. Let $X_1 \xrightarrow{\sigma_1} X_0$ be the blow-up of the regular scheme $X_0$ along the regular subscheme $E_0$. The resulting scheme $X_1$ will be regular, and the exceptional fiber of the blowup will be a projective space bundle over the center $E_0$ of the blowup. In particular, the exceptional divisor is ruled.

Let $E_1$ be the image of $E \subset Y$ in $X_1$ under the rational map $Y \xrightarrow{\pi_1^{-1} \circ \pi} X_1$. Of course, $E_1$ must be contained in the exceptional set for $\sigma_1$, but it may be strictly smaller. If $E_1$ is codimension one in $X_1$, then $E_1$ must be this exceptional divisor. In this case, $E$ is ruled and the proof is complete.

Otherwise, $E_1$ has codimension larger than one in $X_1$ and we repeat the process of replacing $X_1$ and $E_1$ by an open subset on which $E_1$ is regular and blowing up along $E_1$. In this way, we construct a sequence of blowups $X_i \xrightarrow{\sigma_i} X_{i-1}$. Each $X_i$ is regular (but not necessarily smooth over any base scheme) and each exceptional fiber is ruled and contains the image $E_i$ of $E$. The process terminates (meaning $\sigma_{i+1}$ is an isomorphism) if and only if $E_i$ is codimension one in $X_i$. If the process terminates, the proof is complete, because then $E$ is birational to the exceptional divisor of some $\sigma_i$, and so $E$ must be ruled.

5.4.2. Termination of the process. If the process does not terminate, we have a sequence of blowings up of regular schemes

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} X_3 \ldots$$

where no $\sigma_i$ is an isomorphism (we say “$\sigma_i$ is a non-trivial blow-up”). Such a non-trivial blow-up is never smooth (nor even flat!). Since the Jacobian ideal $\mathcal{J}_{X_i/X_{i-1}} \subset \mathcal{O}_{X_i}$ defines the non-smooth locus of the blowup $X_i \to X_{i-1}$, none of these Jacobian ideals can be the unit ideal. Indeed, because the blowup $X_i \to X_{i-1}$ is not smooth along the exceptional divisor, the Jacobian ideal remains a proper ideal after localizing along any component of an exceptional divisor.

The rational map $Y \xrightarrow{\pi_1} X_i$ is a morphism on some open set $Y_i$ containing the generic point of $E$. In particular, by Exercise 22, the morphisms

$$Y_i \to X_i \to X_0$$

induce a multiplicative relation of Jacobian ideals in $\mathcal{O}_{Y_i}$:

$$\mathcal{J}_{Y_i/X_0} = (\mathcal{J}_{Y_i/X_i})(\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y_i}).$$

Localizing along $E$, we have a multiplicative relation of proper ideals

$$\mathcal{J}_{Y_i/X_0}\mathcal{O}_{Y,E} = (\mathcal{J}_{Y_i/X_i}\mathcal{O}_{Y,E})(\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y,E})$$
in the discrete valuation ring $\mathcal{O}_{Y,E}$. In particular, the pullback of each Jacobian ideal $\mathcal{J}_{X_i/X_0}$ to $\mathcal{O}_{Y,E}$ strictly contains the fixed ideal $\mathcal{J}_{Y/X_0}\mathcal{O}_{Y,E}$. Since this latter ideal of the local ring $\mathcal{O}_{Y,E}$ depends only on a small neighborhood of $E$ in $Y$, we denote it by $\mathcal{J}_{Y/X_0}$.

Likewise, using the multiplicative property for Jacobian ideals for the blowups $X_{i+1} \to X_i \to X_0$, we see that after pulling back to $Y$ and localizing along $E$, the ideal $\mathcal{J}_{X_i/X_0}\mathcal{O}_{Y,E}$ strictly contains the ideal $\mathcal{J}_{X_{i+1}/X_0}\mathcal{O}_{Y,E}$. We are led to a sequence of proper inclusions

$$\mathcal{J}_Y \subsetneq \cdots \subsetneq \mathcal{J}_{X_i} \subsetneq \mathcal{J}_{X_{i-1}} \subsetneq \cdots \subsetneq \mathcal{J}_{X_1} \subsetneq \mathcal{J}_{X_0}$$

in the discrete valuation ring $\mathcal{O}_{Y,E}$ (the notation for “relative to $X_0$” and “localize along $E$” has been suppressed).

This leads immediately to a contradiction: fixing a uniformizing parameter $t$ for $\mathcal{O}_{Y,E}$, and setting $\mathcal{J}_Y = (t^m)$, it is obvious that at most $m$ ideals can be properly contained between $\mathcal{J}_Y$ and $\mathcal{O}_{Y,E}$. The process must terminate after at most $m$ blowups, and the proof is complete. □

5.4.3. Remark. One can associate the numerical invariant given by the length of

$$\frac{\mathcal{J}_{X_i}}{\mathcal{J}_Y}$$

to each blowup. The proof showed that this number is strictly decreasing for a non-trivial blowup. This number can be interpreted as the “discrepancy along $E$” between differentials on $Y$ and on $X_i$. In this sense, the proof we have given above is very close in spirit to the proof we suggested in the classical case. The difference is that the differentials here are relative to the scheme $X_0$, whereas in the classical case, the differentials are relative to the ground field.

We now know that there are a host of smooth non-ruled Fano varieties of every dimension greater than two. In particular, there are a host of non-rational Fano varieties. We have a procedure for constructing examples. In Appendix I, some explicit examples over $\mathbb{Q}$ are described.

There are many variations on this basic method for constructing non-ruled varieties. For example, in [K95], Kollár shows that a very general hypersurface of degree $d$ in $\mathbb{P}^n_{\mathbb{C}}$ is not ruled whenever $d$ satisfies

$$d \geq 2 \left\lceil \frac{n+3}{3} \right\rceil .$$

Here, $[x]$ denotes the least integer greater than or equal to $x$, and “very general” means that there is a countable union of subvarieties in the space all hypersurfaces in $\mathbb{P}^n$ that must be avoided. This result, however, does not produce any specific example of a non-ruled hypersurface in $\mathbb{P}^n$. Further applications of this method appear in [K97].
Appendix I: Polynomials with non-degenerate critical points over finite fields
by Joel Rosenberg

In this appendix, specific examples of polynomials over finite fields with only non-degenerate critical points are recorded. This establishes the existence of non-ruled Fano varieties over every field of characteristic zero.

Proposition. Given a prime \( p \), and integers \( n \) and \( m \) with \( n > 0, n \not\equiv -1 \mod p \), and \( mp \geq 3 \), let \( F \in \mathbb{F}_p[x_0, \ldots, x_n] \) be the homogeneous polynomial of degree \( mp \)

\[
F(x_0, \ldots, x_n) = \sum_{i=0}^{n} x_i^{mp-1} x_{i+1},
\]

where we understand subscripts to be taken mod \( n+1 \). Then any dehomogenization \( f \) of \( F \)

\[
f(x_0, \ldots, \hat{x}_i, \ldots, x_n) = F(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)
\]

will have only isolated critical points in \( \bar{\mathbb{F}}_p \), and all of them will be non-degenerate.

Proof. From the cyclic symmetry of \( F \), it is clear we need only consider the dehomogenization

\[
f(x_1, \ldots, x_n) = F(1, x_1, \ldots, x_n).
\]

Then we have

\[
\frac{\partial F}{\partial x_i} = x_{i-1}^{mp-1} - x_i^{mp-2} x_{i+1},
\]

\[
\frac{\partial^2 F}{\partial x_i \partial x_{i+1}} = -x_i^{mp-2},
\]

\[
\frac{\partial^2 F}{\partial x_i^2} = 2x_i^{mp-3} x_{i+1},
\]

and all other second partials of \( F \) are zero. We find that any critical points of \( f \) will have

\[
1 - x_1^{mp-2} x_2 = 0,
\]

\[
x_1^{mp-1} - x_2^{mp-2} x_3 = 0,
\]

\[
x_2^{mp-1} - x_3^{mp-2} x_4 = 0,
\]

\[
\vdots
\]

\[
x_{n-2}^{mp-1} - x_{n-1}^{mp-2} x_n = 0
\]

\[
x_{n-1}^{mp-1} - x_n^{mp-2} = 0
\]

We conclude that the critical points of \( f \) are exactly those points with

\[
x_i = \zeta^{(\sum_{j=0}^{i-1}(1-mp)^j)}
\]
for some $\zeta$ with
\[
\zeta \left( \sum_{j=0}^{n} (1 - mp)^j \right) = 1.
\]
In particular, all critical points will be isolated and will have all their coordinates nonzero.

To see that these points are non-degenerate, we write down the Hessian of $f$,
\[
H = \begin{pmatrix}
2x_1^{mp-3}x_2 & -x_1^{mp-2} \\
-x_1^{mp-2} & 2x_2^{mp-3}x_3 & -x_2^{mp-2} \\
& -x_2^{mp-2} & 2x_3^{mp-3}x_4 \\
& & & \ddots \\
& & & & & \ddots \\
& & & & & & & 2x_{n-1}^{mp-3}x_n & -x_{n-1}^{mp-2} \\
& & & & & & & -x_{n-1}^{mp-2} & 2x_n^{mp-3}
\end{pmatrix}
\]
and compute its determinant at each of the critical points. If we let $H_j$ be the upper left $j \times j$ submatrix of $H$, and $h_j = \det(H_j)$, we see that we have a recursion for $\det(H) = h_n$:

\[
h_0 = 1, \\
h_1 = 2x_1^{mp-3}x_2, \\
h_j = 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{2mp-4}h_{j-2}, \text{ for } 1 < j \leq n,
\]
where we understand $x_{n+1}$ to equal 1. We will show that modulo the ideal of first partials ($\partial f/\partial x_i$), this reduces to

\[
h_j \equiv (j + 1) \prod_{i=1}^{j} x_i^{mp-3}x_{i+1}.
\]
Clearly this is true for $j = 0$ and $j = 1$. Now, inductively, for $1 < j \leq n$,

\[
h_j = 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{2mp-4}h_{j-2} \\
\equiv 2x_j^{mp-3}x_{j+1}h_{j-1} - x_{j-1}^{mp-3}x_j^{mp-2}x_{j+1}h_{j-2} \\
\equiv 2jx_j^{mp-3}x_{j+1} \prod_{i=1}^{j-1} x_i^{mp-3}x_{i+1} - (j - 1)x_{j-1}^{mp-3}x_j^{mp-3}x_{j+1} \prod_{i=1}^{j-2} x_i^{mp-3}x_{i+1} \\
= (j + 1) \prod_{i=1}^{j} x_i^{mp-3}x_{i+1},
\]
as desired. In particular, $\det(H) = h_n$ is equal to $n + 1$ times a monomial, which is nonzero for any critical point of $f$. So $f$, and similarly any dehomogenization of $F$, has no degenerate critical points. $\square$
These examples show that the methods discussed in the text can be used to prove the existence of non-ruled smooth projective Fano varieties over arbitrary fields of characteristic zero. Without these explicit examples, our methods would not have established the existence of non-rational Fano varieties over, say, \( \mathbb{Q} \). The reason is that the Morse Lemma fails over finite fields, so after reducing modulo \( p \), we could not guarantee any polynomial with only non-degenerate critical points (i.e., satisfying Assumption 4.5.13) exists with degrees satisfying the required constraints.

The Proposition of this appendix shows that the polynomial

\[
\sum_{i=0}^{n} X_i^{2m-1} X_{i+1}
\]

satisfies Assumption 4.5.13 of Lecture 4. Furthermore, considered over \( \mathbb{Q} \), it is easy to check that none of the critical points has critical value zero, so that the hypersurface defined by

\[
Y^p = \sum_{i=0}^{n} X_i^{2m-1} X_{i+1}
\]

in the weighted projective \( \mathbb{P} \) is smooth over \( \mathbb{Q} \).

This produces a host of specific examples of non-rational smooth projective Fano varieties of every dimension. For example, consider the equation

\[
Y^2 - \sum_{i=0}^{n} X_i^{2m-1} X_{i+1} + 2G,
\]

for any even \( n \) in the range \( m < n + 1 < 2m \), and \( G \) is (weighted) homogeneous of degree \( 2m \). In the weighted projective space \( \mathbb{P} \) where \( Y \) has weight \( m \) and the \( X_i \)'s have weight one, this polynomial defines a non-rational Fano variety over \( \mathbb{Q} \), which is smooth for generic choices of \( G \). In fact, it is smooth when \( G = 0 \), giving a truly explicit example.
Solutions for Exercises

**Exercise 1.** This was originally proved by Nishimura in 1955 [Ni]. The following proof is due to Endre Szabó.

Use induction on the dimension of $Y$. If $Y$ has dimension one, rational maps are morphisms defined everywhere, and the result is obvious. If $Y$ is a smooth variety with a $k$-point $P$, blow up $P$ to get a variety $\tilde{Y}$. The blowup map $\tilde{Y} \to Y$ is defined over $k$, and the exceptional fiber, being isomorphic to a projective space $\mathbb{P}$, has lots of $k$-points. Any rational map $Y \dasharrow Y'$ defined over $k$ determines a rational map $\tilde{Y} \dasharrow \tilde{Y}'$. Because $\tilde{Y}$ is smooth and $Y'$ is projective, the locus of indeterminacy has codimension at least two. This means that $\tilde{\phi}$ restricts to a rational map of the exceptional fiber $\mathbb{P}$. Because this variety has smaller dimension, we are done by induction.

If $Y$ is not smooth, Nishimura’s Lemma can fail. Indeed, let $Y$ be the projective closure of the affine cone over a smooth projective variety $X$ with no $k$-points. Then $Y$ has exactly one $k$ point, the vertex of the cone. Blowing up the vertex, we achieve a smooth projective variety $\tilde{Y}$ with no $k$-points, since the exceptional fiber is $k$-isomorphic to $X$. The rational map $Y \dasharrow \tilde{Y}$ gives the counterexample to Nishimura’s Lemma in the case where the source is not smooth.

**Exercise 2.** (1) Choose coordinates so that the disjoint $n$ planes $L_1$ and $L_2$ are given by \(\{X_0 = X_1 = \cdots = X_n = 0\}\) and \(\{X_{n+1} = X_{n+2} = \cdots = X_{2n+1} = 0\}\) respectively.

A cubic given by an equation of the form

\[
\sum_{i \leq n, j > n} a_{ijk} X_i X_j X_k
\]

obviously contains both planes. The generic member in this linear system of cubics in $\mathbb{P}^{2n+1}$ is smooth, because it admits the following smooth special member:

\[
\sum_{i=0}^{n} (X_i^2 X_{i+n+1} + X_i X_{i+n+1}^2).
\]

This is easily checked by the Jacobian criterion (assuming the characteristic is not 3).

(2) To count the dimension of the linear system of cubics containing a fixed pair of disjoint planes $L_1$ and $L_2$, we count the number of monomials of degree 3 in $2n + 2$ variables minus the number of those monomials involving only variables generating the ideal of $L_1$ or of $L_2$. The total is

\[
\binom{2n+1+3}{3} - \binom{n+3}{3} - \binom{n+3}{3} = (n+1)^2(n+2),
\]

so the dimension of the linear system is $(n+1)^2(n+2) - 1$. 

\[46\]
Finally, to find the dimension of the space of all cubics containing any pair of disjoint planes, we need to add the dimension of the space of such pairs of planes. As a generic pair of planes are disjoint, this is twice the dimension of the Grassmannian of $n$ planes in $\mathbb{P}^{2n+1}$, or $2(n+1)^2$. So the dimension of the space of all smooth $2n$ dimensional cubic hypersurfaces containing a pair of disjoint $n$-planes is $(n+1)^2(n+4) - 1$.

(3) A cubic surface containing a linear subspace of dimension greater than $n$ is never smooth. Indeed, choosing coordinates so that $X$ contains the space defined by $\{X_0 = X_1 = \cdots = X_{n-1} = 0\}$,

$$X = \left\{ \sum_{i=0}^{n-1} X_i f_i = 0 \right\}$$

where $f$ are degree two. Now $X$ can not be smooth along the locus of points where $\{X_0 = X_1 = \cdots = X_{n-1} = 0\}$ and $\{f_0 = f_1 = \cdots = f_{n-1} = 0\}$, because the Zariski tangent space at these points is $2n+1$ dimensional. But because this locus is defined by only $2n$ equations, it must have non-empty intersection with $X$.

**Exercise 3.** Recall that a linear system of plane cubics with up to seven as signed base points (including possibly one infinitely near another), no four collinear and no seven on a conic, has no unassigned base points [H, p399]. Also recall that a linear system on a smooth surface is very ample if and only if imposing two more base points (including one infinitely near another) causes the dimension to drop by exactly two [H p396].

Consider four general points $P_1, \ldots, P_4$ in $\mathbb{P}^2$ and let $\beta = |3H - P_1 - P_2 - P_3 - P_4|$ be the linear system of cubics in $\mathbb{P}^2$ passing through these points. Using the criterion above, we see that (the pull back of) this linear system, $\beta = |3H - E_1 - E_2 - E_3 - E_4|$ to the blowup of $\mathbb{P}^2$ at the four points is very ample. Using this linear system embed the blowup as a surface $S$ in $\mathbb{P}^5$.

**Claim:** A generic projection of $S$ to $\mathbb{P}^4$ is a surface $S'$ with exactly one singular point.

First, for two general points $P$ and $Q$ on $\mathbb{P}^2$, consider the following two linear subsystems of $\beta$ on $S$. Fixing defining equations $s_0, \ldots, s_5$ for generators of $\beta$, consider the linear subsystems whose defining equations satisfy:

$$\gamma := \left\{ s \in \beta \mid \frac{s(P)}{s_0(P)} = \frac{s(Q)}{s_0(Q)} \right\}$$

$$\alpha := \left\{ s \in \beta \mid s(P) = s(Q) = 0 \right\}.$$

Note that $\alpha \subset \gamma \subset \beta$, and the dimensions drop by exactly one with each successive condition imposed. The linear system $\gamma$ determines a projection $\pi$ of $\mathbb{P}^5$ to $\mathbb{P}^4$, sending $S$ to, say, $S'$. By definition of $\gamma$, we have $[s_0(P) : s_1(P) : \cdots : s_5(P)] = [s_0(Q) : s_1(Q) : \cdots : s_5(Q)]$, so that $\pi$ sends $P$ and $Q$ to the same point of $S'$.

If $\pi$ collapses some other point $P'$ (possibly infinitely near $P$ or $Q$) to $\pi(P) = \pi(Q)$, then we have that whenever $s(P) = s(Q) = 0$ for some $s \in \beta$, the vanishing
\( s(P') \) is forced as well. This makes \( P' \) an unassigned base point of \( \alpha \), contradicting the genericty assumption. Likewise, if \( \pi \) collapses two other points \( P' \) and \( Q' \) (\( Q' \) may be infinitely near \( P' \)) to a single point of \( S' \), then the linear system \( |3H - P_1 - P_2 - P_3 - P_4 - P - Q - P'| \) has an unassigned base point \( Q' \), again a contradiction, since six general points and the one special point \( P' \) impose no extra conditions.

Thus any projection of \( S \subset \mathbb{P}^5 \) to \( \mathbb{P}^4 \) that collapses two general points of \( S \) to a single point of \( S' \) can collapse only these two points to a single point. The argument will be complete once we have shown that a general projection \( \mathbb{P}^5 \to \mathbb{P}^4 \) cannot be one-to-one on \( S \).

Consider the incidence correspondence

\[
\Gamma = \{(P, Q, x) \mid P, Q, x \text{ collinear}\} \subset S \times S \times \mathbb{P}^5.
\]

Through any two distinct points of \( S \), there is a unique line in \( \mathbb{P}^5 \), so the projection \( \Gamma \to S \times S \) is surjective, and its fibers are all one-dimensional. It follows that \( \Gamma \) is irreducible and of dimension 5.

Consider the other projection \( \Gamma \to \mathbb{P}^5 \). We know that if \( P \) and \( Q \) are collapsed to the same point of \( S' \) via \( \pi \), then these are the only two points collapsed under \( \pi \). This implies that the fiber over any point in the image of \( \Gamma \to \mathbb{P}^5 \) is simply the triple \((P, Q, x)\), so the fibers are zero-dimensional. From this we conclude that \( \Gamma \to \mathbb{P}^5 \) is surjective. This means a generic projection from any point in \( \mathbb{P}^5 \) cannot be one-to-one on \( S \).

This implies that a generic projection of \( S \subset \mathbb{P}^5 \) to a hyperplane in \( \mathbb{P}^5 \) collapses precisely two points of \( S \) to a single point \( S' \) in the image (which is therefore a singular point of \( S' \)). This completes the proof.

The surface we described is called a del Pezzo surface in \( \mathbb{P}^5 \). The reader familiar with rational quartic scrolls in \( \mathbb{P}^5 \) should be able to prove that these surfaces also have the property that a generic projection to \( \mathbb{P}^4 \) produces exactly one double point. In 1901, Severi claimed that these are the only two examples of such surfaces with a single “apparent double point,” as he called them [Sev, p 44]. Unfortunately, there was a gap in his argument. This gap is being considered in the developing thesis of Mariagrazia Violo (under the direction of Edoardo Sernesi) which also includes further generalizations of these ideas [V].

**Exercise 4.** Use the exact sequence

\[
0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0
\]

(see [H, p176] for the derivation of this sequence). Since \( \Omega_{\mathbb{P}^n} \subset \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \), it follows that \((\Omega_{\mathbb{P}^n})^{\otimes m} \subset (\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)})^{\otimes m} \cong \mathcal{O}_{\mathbb{P}^n}(-m)^{\oplus(n+1)m} \). Obviously now \( \Omega_{\mathbb{P}^n}^{\otimes m} \) has no non-zero global sections, since \( \mathcal{O}_{\mathbb{P}^n}(-m) \) has none.

**Exercise 5.** (1) To prove that the plurigenera of a separably unirational variety \( X \) vanish, reduce to the case of \( \mathbb{P}^n \) exactly as in the proof of Theorem 1.10. For \( \mathbb{P}^n \), we have \( \Lambda^n \Omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \) from the above sequence, whence the vanishing of \( H^0(\mathcal{O}_{\mathbb{P}^n}(-mn-m)) \) is immediate. This proves corollary 1.12.
(2) By the adjunction formula, the canonical class of a hypersurface \(X\) in \(\mathbb{P}^n\) is \(K_X = (K_{\mathbb{P}^n} + X)|_X\), so \(\mathcal{O}_X(K_X) = \mathcal{O}_X((-n - 1 + d)H)\) where \(X\) is degree \(d\). So for \(d > n\), the global sections of all powers are non-zero and the plurigenera do not vanish.

**Exercise 6.** (1) A variety \(X/k\) of dimension \(d\) is unirational if and only if its function field has an algebraic extension that is purely transcendental over \(k\). If \(X' \to X\) is purely inseparable, then by definition, the functional fields have the following relationship

\[
\{k(X')\}^{p^e} \subset k(X) \subset k(X')
\]

where \(p^e\) is some power of the characteristic \(p\). If \(X\) is unirational, then \(k(X)\) has an algebraic extension \(k(t_1, \ldots, t_d)\), and hence \(k(X') \subset k^{1/p^e}(t_1^{1/p^e}, \ldots, t_d^{1/p^e})\). Since \(k\) is perfect, \(k = k^{1/p^e}\), and \(k(X')\) is a subfield of a purely transcendental extension of \(k\). Thus \(X'\) is unirational over \(k\).

(2) To construct a unirational variety of arbitrary degree, fix any polynomial \(f\) of degree \(p^e\) in \(X_0, X_1, \ldots, X_n\), the homogeneous coordinates of \(\mathbb{P}^n\). Let \(Y\) be an indeterminate, and let \(g\) be the polynomial \(Y^{p^e} - f(X_0, X_1, \ldots, X_n)\). Then \(g\) defines a hypersurface in \(\mathbb{P}^{n+1}\), of degree \(p^e\), which is a purely inseparable cover of \(\mathbb{P}^n\). The previous computation shows that this hypersurface is unirational. Indeed, its function field is isomorphic to \(k(x_1, \ldots, x_n)(f^{1/p^e})\), a subfield of the purely transcendental extension \(k(x_1^{1/p^e}, \ldots, x_n^{1/p^e})\).

**Exercise 7.** This is sometimes called Tsen’s theorem [Ts]. The case \(d = n = 2\) is due to Max Noether in 1871 [No]. We seek solutions \(x_i = \sum_{j=0}^m a_{ij}t^j\), where the \(a_{ij}\) are unknown elements of \(\mathbb{C}\), to the degree \(d\) polynomial \(F(X_0, \ldots, X_n)\). Plugging in \(X_i = x_i\), and gathering up all terms \(t^r\), we see that the coefficient of \(t^r\) is a polynomial in the unknowns \(a_{ij}\). We have a solution if and only if we can choose the \(a_{ij}\) so that the coefficient of each \(t^r\) is zero. Note that a collection of complex numbers \(a_{ij}\) is a solution if and only if \(\lambda a_{ij}\) is solution, where \(\lambda\) is a non-zero scalar, so the solutions naturally live in a projective space over \(\mathbb{C}\).

Without loss of generality, we can assume that the coefficients of \(F\) are polynomials in \(t\), say of degree less than \(c\). In this case, the highest occurring power of \(t\) in the expansion of \(F(x_0, \ldots, x_n)\) is at most \(dm + c\). Thus to solve for the \(a_{ij}\) is to solve a system of \(dm + c\) equations in \(m(n + 1)\) unknowns. Since \(n \geq d\), there are more unknowns than equations when \(m \gg 0\), so there are solutions for the \(a_{ij}\) in projective space over \(\mathbb{C}\).

The argument for \(\mathbb{C}(t, s)\) is similar. Of course the same argument works with any algebraically closed field in place of \(\mathbb{C}\). If \(d > n\), the degree \(d\) hypersurface in \(\mathbb{P}^n\) may have no \(\mathbb{C}(t)\) points. For example, there are no \(\mathbb{C}(t)\) points of the variety defined by \(\sum_{i=0}^{d-1} t^iX_i^d = 0\) in \(\mathbb{P}^{d-1}\). By projectivizing the affine cone over this example, we get an example of a degree \(d\) hypersurface in \(\mathbb{P}^d\) that has exactly one \(\mathbb{C}(t)\)-point, \([0: 0: \ldots: 0: 1]\).

**Exercise 8.** Think of \(X_{a,2} \subset \mathbb{A}^1 \times \mathbb{P}^n\) as defined by an equation \(\sum_{i,j} a_{ij}(t)X_iX_j\) where the \(a_{ij} \in \mathbb{C}(t)\) have degree less than or equal to \(a\). This is a quadric in \(\mathbb{P}^n_{\mathbb{C}(t)}\)
By exercise 7, we know this quadric has a $\mathbb{C}(t)$-rational point provided $n \geq 2$. This makes the quadric rational over $\mathbb{C}(t)$, whence its function field is isomorphic to $\mathbb{C}(t)(x_1, \ldots, x_n) \cong \mathbb{C}(t, x_1, \ldots, x_n)$. This proves that $X_{a,2}$ is a rational $\mathbb{C}$-variety.

Geometrically, the point is that the projection $X_{a,2} \to \mathbb{P}^1$ has a section, making the family trivial on an open set.

The proof of (2) is similar.

**Exercise 9.** (1) The variety $Y$ of $m \times n$ matrices of rank at most $t$ is defined by the $t + 1$-minors of an $m \times n$ matrix of indeterminates. It is easy to check that its dimension is $mn - (m - t)(n - t)$. Define the rational map

$$Y \to \mathbb{A}^{mn-(m-t)(n-t)}$$

$$\lambda \mapsto \{(\ldots, \lambda_{ij}, \ldots) | i \text{ or } j \leq t\},$$

sending a matrix $\lambda$ to the indicated string of its entries. This map is a birational equivalence because, whenever the upper left hand $t$-minor $\Delta$ of the $m \times n$ matrix $\lambda$ is non-zero, we solve uniquely for each $\lambda_{ij}$ with both $i$ and $j$ greater than $t$. Indeed, since all $(t + 1)$-minors vanish, we can use the Laplace expansion to express any such $\lambda_{ij}$ with $i, j > t$ as a polynomial in the $\lambda$'s from the first $t$ columns and rows with denominators $\Delta$. This proves that $Y$ is a rational variety over any field.

The singular locus of $Y$ is the subvariety of matrices of rank strictly less than $t$. Indeed, if an $m \times n$ matrix has rank $t$, then some $(t - 1)$-minor is non-vanishing, so using the analogous map described above, we can map an open subset of $Y$ containing this matrix isomorphically to an open subset of affine space. Thus every full rank $t$ matrix in $Y$ is a smooth point. Conversely, if some matrix has rank less than $t$, all $t$-minors vanish, and considering the Laplace expansion of the $t + 1$-minors defining $Y$, we see easily that the Jacobian matrix is zero in this case. This says that the tangent space at such a point has dimension $mn$ and the point is a singular point of $Y$.

(2) Let $X$ be the subvariety of $\mathbb{P}^n$ defined by the vanishing of the determinant of the $n \times n$ matrix $L$ of general linear forms in $n + 1$ variables. For each $n + 1$ tuple $x = (x_0, x_1, \ldots, x_n)$, consider $L(x)$ as a linear map $k^n \to k^n$. This defines a rational map

$$X \subset \mathbb{P}^n \to \mathbb{P}^{n-1}$$

$$x = [x_0 : x_1 : \cdots : x_n] \mapsto \{\text{kernel of the matrix } L(x)\}.$$ 

The genericity assumption guarantees that the matrix $L$ has rank exactly $n - 1$ generically on $X$. Thus, for a generic $x \in X$, the kernel of the matrix $L(x)$ is a one dimensional subspace of $k^n$, and so determines a well defined point in $\mathbb{P}^{n-1}$.

It is easy to check that this map is birational: the genericity hypothesis on the linear forms guarantees that for distinct general elements $x, y$ in $X$, the matrices $L(x)$ and $L(y)$ have distinct null spaces.

As above, the singular locus of $X$ is defined by the vanishing of the $(n-1) \times (n-1)$ subdeterminants of the matrix of linear forms.
We now show that every smooth cubic surface is determinantal. The earliest proof of this fact appears to be in an 1866 paper of Clebsch, who credits Schröter [Cl]. We give here two different proofs.\footnote{Both are classical; I am grateful to I. Dolgachev for suggesting the first, which was worked out together with J. Keum, R. Lazarsfeld, and C. Werner, and to T. Geramita for suggesting the second, which can be found in [Ger].}

\begin{proof}
This proof uses the configuration of the twenty seven lines on the cubic surface $S$. We claim that there are nine lines on the surface that can be represented in two different ways as a union of three hyperplane sections. That is, there are six different linear functionals $l_1, l_2, l_3, m_1, m_2, m_3$ on $\mathbb{P}^3$ such that the hyperplane sections of $S$ determined by each is a union of three distinct lines, and the nine lines obtained as hyperplane sections with the $l_i$’s are the same nine lines obtained from the the $m_i$’s. Assuming this for a moment, the cubics $l_1l_2l_3$ and $m_1m_2m_3$ both define the same subscheme of $S$, which means that up to scalar, these cubics agree on $S$. In other words, the cubic

$$l_1l_2l_3 - \lambda m_1m_2m_3$$

is in the ideal generated by the cubic equation defining $S$, and hence it must generate it. On the other hand, the cubic $l_1l_2l_3 - \lambda m_1m_2m_3$ is obviously the determinant of the matrix

$$\begin{pmatrix}
    l_1 & m_1 & 0 \\
    0 & l_2 & m_2 \\
    -\lambda m_3 & 0 & l_3
\end{pmatrix}$$

The proof will complete upon establishing the existence of the special configuration of lines. First recall that $S$ is the blow-up of six points $P_1, P_2, \ldots, P_6$ in $\mathbb{P}^2$, no three on a line and no five on a conic. We embed $S$ in $\mathbb{P}^3$ using the linear system of plane cubics through these six points. The twenty seven lines on $S \subset \mathbb{P}^3$ are obtained as follows:

1. For each pair of two points $P_i$ and $P_j$, the birational transform of the line $\overline{P_iP_j}$ in $\mathbb{P}^2$ joining them;
2. For each point $P_i$, the birational transform of the conic $Q_i$ through the remaining five points;
3. For each point $P_i$, the fiber $E_i$ over $P_i$.

For any pair of indices $i, j$, the three lines $\overline{P_iP_j}$, $E_i$, and $Q_j$ form a (possibly degenerate) triangle on $S$; Indeed, thinking of the hyperplane sections of $S$ as cubics in the plane through the six points, this triangle is the hyperplane section given by the cubic obtained as the union of $\overline{P_iP_j}$ and $Q_j$. Now it is easy to find such a configuration. For instance, the nine lines

$$\{E_1, Q_2, \overline{P_1P_2}\} \cup \{E_2, Q_3, \overline{P_2P_3}\} \cup \{E_3, Q_1, \overline{P_1P_3}\}$$

are the same as the nine lines

$$\{Q_1, E_2, \overline{P_1P_2}\} \cup \{Q_2, E_3, \overline{P_2P_3}\} \cup \{Q_3, E_1, \overline{P_1P_3}\}.$$
with the groupings indicating the two different configurations of triangles.

Second Proof: This proof is more algebraic. Let $\gamma$ be the linear system of plane cubics through six the six points used to embed the blowup up of $\mathbb{P}^2$ at these six points as the cubic surface in $\mathbb{P}^3$. A set of defining equations $\{s_0, s_2, \ldots, s_3\}$ for a basis of $\gamma$ is necessarily the homogeneous ideal $I$ of functions vanishing at the six points. The blowup of the six points is by definition the (closure of the) graph of rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ given by $x \mapsto [s_0(x) : s_1(x) : s_2(x) : s_3(x)]$.

The homogeneous coordinate ring $R/I$ of the six points has a resolution

$$0 \to R^3 \xrightarrow{A} R^4 \to R \to R/I \to 0$$

where $R = k[X_0, X_1, X_2]$ is the homogeneous coordinate ring of $\mathbb{P}^2$, and $A$ is a $4 \times 3$ matrix whose 3-minors are precisely the generators $s_i$ for $I$. That defining ideals of codimension two subvarieties of projective space have resolutions of this form was proved by Hilbert in 1890 [Hil]; nowadays we recognize it as a special case of the well-known Hilbert-Burch theorem; see [E, p502].

Let $Y_0, Y_1, Y_2, Y_3$ be homogeneous coordinates for $\mathbb{P}^3$. The entries of the $3 \times 1$ matrix

$$A^t \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_1 \\ H_2 \end{bmatrix}$$

are bihomogeneous of degree $(1, 1)$ in each set of variables $X_i$ and $Y_i$, and the resulting closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^3$ is the graph of the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by $\gamma$, as one readily checks using Cramer’s rule for solving linear systems of equations.

The cubic surface $S$ is the projection of this graph to $\mathbb{P}^3$. Factoring above matrix equation differently:

$$\begin{bmatrix} H_1 \\ H_1 \\ H_2 \end{bmatrix} = B \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

where $B$ is a $3 \times 3$ matrix of linear polynomials in the $Y_i$. The determinant of $B$ is a degree three polynomial in the $Y_i$; this cubic obviously vanishes on the projection to $\mathbb{P}^3$ of the subvariety of $\mathbb{P}^2 \times \mathbb{P}^3$ defined by the $H_i$, and hence the determinant of $B$ must define the original cubic surface in $\mathbb{P}^3$.

**Exercise 10.** We first prove an even stronger result for lower degree hypersurfaces: if $d < n$, then there is a line through every point on $X$.

Without loss of generality, the point may be assumed to be the affine origin $P = [1 : 0 : \cdots : 0 : 0]$. Since $X$ passes through $P$, it has affine equation

$$f_1(x_1, \ldots, x_n) + f_2(x_1, \ldots, x_n) + \cdots + f_d(x_1, \ldots, x_n)$$

where each $f_i$ is homogeneous of degree $i$.  

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Any line through $P$ has the form 

$$(a_1 t, a_2 t, \ldots, a_n t); \ t \ \text{ranging through } k$$

where $(a_1, \ldots, a_n)$ is a non-zero point on the affine plane. To find such a line on $X$, we need

$$f_1(a_1 t, \ldots, a_n t) + f_2(a_1 t, \ldots, a_n t) + \cdots + f_d(a_1 t, \ldots, a_n t) = 0$$

for all $t$. Because each polynomial $f_i(a_1 t, \ldots, a_n t)$ is homogeneous of degree $i$ in $t$, we need $(a_1, \ldots, a_n)$ such that each $f_i(a_1, \ldots, a_n) = 0$. Because $d < n$, the equations $\{f_i = 0\}_{i=1}^d$ have a solution in $\mathbb{P}^{n-1}$, and we have found a line on $X$.

When $d = n$, there may not be a line on $X$. Instead, we prove the following: *Through every point on a degree $n$ hypersurface in $\mathbb{P}^n$, there passes a plane conic.*

Let $\mathcal{C}$ be the variety of plane conics in $\mathbb{P}^n$ passing through $P$. The linear system of plane conics forms a projective space of dimension 5, and those passing through $P$ is a hyperplane in this space. So the variety of conics through $P$ is a $\mathbb{P}^4$-bundle over the Grassmannian of planes in $\mathbb{P}^n$ through $P$. This Grassmannian has dimension $2(n-2)$, so the dimension of the variety $\mathcal{C}$ of conics is $2n$.

The hypersurfaces of degree $n$ in $\mathbb{P}^n$ passing through $P$ naturally form a hyperplane $\mathcal{X}$ in the $\binom{2n}{n} - 1$-dimensional projective space of all degree $n$ hypersurfaces in $\mathbb{P}^n$. Consider the incidence correspondence

$$\Gamma = \{(X, Q) | Q \subset X \} \subset \mathcal{X} \times \mathcal{C}$$

together with the two projections $\Gamma \xrightarrow{\pi} \mathcal{X}$ and $\Gamma \xrightarrow{\varphi} \mathcal{C}$.

The elements in the fiber of $\pi$ over a hypersurface $X \in \mathcal{X}$ can be identified with the conics on $X$ through $P$. In order to show that through every point on a degree $n$ (or less) hypersurface in $\mathbb{P}^n$ there passes a conic, we need to show that the projection $\Gamma \xrightarrow{\pi} \mathcal{X}$ is surjective.

We compute the dimension of $\Gamma$ using the other projection $\Gamma \xrightarrow{\varphi} \mathcal{C}$. Fix a conic $Q$ through $P$. We need to compute the dimension of $\pi^{-1}(Q)$, the space of degree $n$ hypersurfaces containing $Q$. Choose coordinates so that $Q$ is given by $x_3 = x_4 = \cdots = x_n = 0$ and a homogeneous degree 2 polynomial $g(x_0, x_1, x_2)$. The hypersurfaces of degree $m$ containing $Q$ can be written uniquely in the form:

$$x_n h_n(x_0, \ldots, x_n) + x_{n-1} h_{n-1}(x_0, \ldots, x_{n-1}) + x_{n-2} h_{n-2}(x_0, \ldots, x_{n-2}) + \cdots + x_3 h_3(x_0, x_1, x_2, x_3) + g \cdot h(x_0, x_1, x_2),$$

where the $h_i$ are homogeneous of degree $m - 1$ and $h$ is a homogeneous of degree $m - 2$. When $m = n$, the space of hypersurfaces of this form is of dimension $\binom{2n}{n} - 2$.

Now, because $\Gamma$ and $\mathcal{X}$ have the same dimension, the projection map $\Gamma \xrightarrow{\pi} \mathcal{X}$ is surjective if the fiber over some point in the image is of dimension zero. So the proof is complete upon exhibiting any particular hypersurface of degree $n$ containing only finitely many conics through a point $P$. We leave it to the reader to verify that the hypersurface defined by $X_0^n - X_1 X_2 \cdots X_n$ contains only finitely many conics through the point $[1 : 1 : \cdots : 1 : 1]$.
Exercise 11. This real cubic surface and its properties were considered in a 1962 paper of Swinnerton-Dyer [S-D].

To see that the surface has two real components, consider the affine chart where $t = 1$. Setting $v^2 = x^2 + y^2$, we see that the surface is a surface of revolution for the elliptic curve

$$v^2 = (4z - 7)(z^2 - 2).$$

Because the function $f(z) = (4z - 7)(z^2 - 2)$ has three distinct real roots, we know from Example 1.4 that the curve, and hence the surface of revolution, has two disjoint real components. The two real components correspond to $z \geq 7/4$ and $|z| \leq \sqrt{2}$ respectively.

On the real component where $z \geq 7/4$, the $\mathbb{Q}$-points are dense. Indeed, this component contains $[x : y : z : t] = [1 : 1 : 2 : 1]$. The tangent plane to the surface at this point intersects with the surface to produce an irreducible singular cubic on $S$. This curve is rational over $\mathbb{Q}$, and its $\mathbb{Q}$-points are dense among its $\mathbb{R}$-points. In particular, the $\mathbb{Q}$-values for $z$ are dense among all real values for $z \geq 7/4$. For each of these fixed $\mathbb{Q}$-values $z_0$, the plane $z = z_0$ intersects the surface $S$ in the circle $x^2 + y^2 = f(z_0)$. This conic is $\mathbb{Q}$-rational and its $\mathbb{Q}$-points are dense among its $\mathbb{R}$-points. Thus the $\mathbb{Q}$-points of $S$ are dense on the manifold component where $z/t \geq 7/4$.

There are no $\mathbb{Q}$-points on the component where $|z/t| \leq \sqrt{2}$. To see this, suppose that $[x : y : z : t]$ is such a $\mathbb{Q}$-point, where without loss of generality, $t$ and $z$ are assumed relatively prime integers, with $t > 0$. So

$$t(7t - 4z)(2t^2 - z^2) = (tx)^2 + (ty)^2$$

is an integer which is the sum of two rational squares. Thus any prime $p$ congruent to $3$ modulo $4$ that divides $t(7t - 4z)(2t^2 - z^2)$ must divide it an even number of times.

Because $|z/t| \leq \sqrt{2}$, each of the integer factors

$$t, \ (7t - 4z), \ (2t^2 - z^2)$$

is positive. We claim that none is congruent to $3$ modulo $4$. Indeed, no prime $p$ congruent to $3$ modulo $4$ can divide any one of these factors to an odd power. For if some such $p$ does, then it must divide precisely two of the factors an odd number of times. But because $t$ and $z$ are relatively prime, it follows that $t$ and $2t^2 - z^2$ are relatively prime, and the only possible common prime factor of $t$ and $(7t - 4z)$ is $2$. Furthermore, if $p$ divides both $(7t - 4z)$ and $(2t^2 - z^2)$, then $p$ divides $(8t + 7z)(7t - 4z) - 28(2t^2 - z^2) = 17tz$. Since such $p$ divides neither $z$ nor $t$, the only possibility is $p = 17$, which is not congruent to $3$ modulo $4$.

Now if $t$ is even, then $z$ must be odd, but this would force $(2t^2 - z^2)$ to be congruent to $3$ modulo $4$. On the other hand, if $t$ is odd, then it must be congruent to $1$ modulo $4$, but this forces $(7t - 4z)$ to be congruent to $3$ modulo $4$. This contradiction implies that there is no $\mathbb{Q}$-rational point on the component of the surface where $|z/t| \leq \sqrt{2}$. 

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Exercise 12. (1). Let \( T_i = \sum_{L_j \in O(i)} L_j \) where \( O(1), \ldots, O(r) \) are the orbits of \( G \) on the twenty seven lines of \( S/k \). Suppose that \( C = \sum_{j=1}^{27} m_j L_j \) is a \( G \)-invariant effective curve, where without loss of generality \( G \) is assumed finite. Grouping the \( L_i \) into orbits, we write \( C = \sum_{i=1}^{r} C_i \) where \( C_i = \sum_{L_j \in O(i)} m_{ij} L_j \). Consider the curve class
\[
|G|C = \sum_{g \in G} gC = \sum_{i=1}^{r} \left( \sum_{g \in G} gC_i \right).
\]
Each term \( \sum_{g \in G} gC_i \) has each \( L_j \) appearing with exactly the same coefficient, namely \( c = \frac{|G|}{|O(i)|} \sum_{j \in O(i)} m_{ij} \). This implies that
\[
C = \frac{1}{|G|} \sum_{g \in G} gC = \sum_{i=1}^{r} cT_i.
\]
This proves that the orbit sums \( T_i \) generate the cone of curves for \( S/k \).

(2). Let \( C \) be an effective curve such that \( C^2 > 0 \). To show that \( C \) is in the interior of the cone of curves, it is sufficient to show that for any divisor \( D \), the divisor \((C + \varepsilon D)\) is also effective, for sufficiently small positive \( \varepsilon \).

Note that \((C + \varepsilon D)^2 = C^2 + 2\varepsilon C \cdot D + \varepsilon^2 D^2\), so this self intersection number is positive for \( \varepsilon \) sufficiently small. Also if \( H \) is any ample divisor, then \( H \cdot (C + \varepsilon D) = H \cdot C + \varepsilon (H \cdot D) \) is positive for small enough \( \varepsilon \), because \( H \cdot C > 0 \).

Because \((C + \varepsilon D)\) has positive self intersection and positive intersection with all ample divisors, it follows that \( C + \varepsilon D \) is effective, as is easily seen by applying Riemann-Roch to the divisors \( n(C + \varepsilon D) \) for \( n \gg 0 \) (see [H, p363]). This completes the proof.

Exercise 13. Consider a line \( L \) in the ambient three-space, together with the smooth cubic surface \( S \) defined by \( u^3 = f(x, y) \). If the line \( L \) lies on \( S \), then \( L \) projects to a line triply tangent to the smooth plane cubic curve defined by \( f(x, y) = 0 \) in the \( xy \)-plane. Let \( L' \) denote this projection, and suppose it has equation \( y = mx + b \). The line \( L' \) is triply tangent if and only if \( f(x, mx + b) \) is a cube of a linear form, say \((cx + d)^3\). Whenever we have this perfect cube, there are three lines on \( S \) projecting to \( L' \). These three lines have parametric equations
\[
(x, mx + b, \omega(cx + d))
\]
where \( \omega \) is one of the three cube roots of unity.

Because the plane cubic \( \{ f(x, y) = 0 \} \) has nine points of triple tangency, all twenty seven lines in \( S \) are constructed in this way.

We work out explicitly the lines on the Fermat surface given in homogeneous coordinates by
\[
a_0 X_0^3 + a_1 X_1^3 + a_2 X_2^3 + a_3 X_3^3.
\]
Factoring the first two two terms $a_0X_0^3 + a_1X_1^3$ completely into distinct homogeneous linear polynomials $l_1l_2l_3$, and likewise factoring $a_2X_2^3 + a_3X_3^3 = m_1m_2m_3$, the Fermat cubic has the form

$$l_1l_2l_3 + m_1m_2m_3 = 0.$$  

The linear factors are distinct because the surface is smooth. This produces nine lines on the surface, defined by the nine different pairs of planes containing it:

$$\{l_i = m_j = 0\}.$$  

By considering factorizations of the other two groupings of the terms $(a_0X_0^3 + a_2X_2^3) + (a_1X_1^3 + a_3X_3^3)$ and $(a_0X_0^3 + a_3X_3^3) + (a_1X_1^3 + a_2X_2^3)$ we can produce the remaining eighteen lines on the surface in similar fashion.

The case where $u^2 = f(x, y)$ is similar. However, there are three distinct lines on the surface in the plane at infinity (meeting in an Eckardt point). The remaining 24 lines on $S$ project to twelve lines in the plane $\{u = 0\}$ tangent to the smooth curve defined by $\{f = 0\}$. These twelve lines can be found by solving for $m$ and $b$ such that $f(x, mx + b)$ is a perfect square.

We show that the cubic surface defined by $x_1^3 + x_2^3 + x_3^3 = a$ (where $a$ is not a cube) is not rational over $\mathbb{Q}$. By Segre’s theorem, it suffices to show that Picard number is one, and by Theorem 2.2, it is enough to show that no Galois orbit consists of disjoint lines on the surface.

The computation above indicates that all lines are defined over the splitting field of $t^3 - a$ over $\mathbb{Q}$. This splitting field is degree six over $\mathbb{Q}$, and is generated by $\omega$, a primitive third root of unity and a real third root $\beta$ of $a$. The Galois group is the full group $S_3$ of all permutations of the roots $\beta, \beta\omega, \beta\omega^2$ of $t^3 - a$.

Factoring the equation for the cubic

$$(x_1^3 + x_2^3) + (x_3^3 + ax_0^3) = (x_1 + x_2)(x_1 + \omega x_2)(x_1 + \omega^2 x_2) + (x_3 + \beta x_0)(x_3 + \beta\omega x_0)(x_3 + \beta\omega^2 x_0)$$

we consider the lines as described above.

Any line defined by

$$\{x_1 + x_2 = 0; \ x_3 + \beta\omega^i x_0 = 0\}$$

for some $i = 0, 1, 2$ contains all other lines of this type in its orbit, since the Galois group acts transitively on the $\beta\omega^i$. This orbit consists of three line in the plane $\{x_1 + x_2 = 0\}$; in particular, the lines of this orbit are not disjoint.

Now consider the orbit of a line defined by

$$\{x_1 + \omega x_2 = 0; \ x_3 + \beta\omega^i x_0 = 0\}$$

for some $i = 0, 1, 2$. The cyclic permutation $\beta \mapsto \beta\omega \mapsto \beta\omega^2 \mapsto \beta$ fixes $\omega$. So the orbit of this line contains three lines in the plane $\{x_1 + \omega x_2 = 0\}$, and hence can
not consist of disjoint lines. Furthermore, the permutation interchanging $\beta \omega$ and $\beta \omega^2$ sends these lines to lines of the form

$$\{x_1 + \omega^2 x_2 = 0; \ x_3 + \beta \omega^i x_0 = 0\}.$$ 

The same cyclic permutation now takes this line to all others of this form, that is, to all others in the plane $\{x_1 + \omega^2 x_2 = 0\}$. So the six lines in these two planes constitute another orbit.

Considering the other two groupings of lines, we find that there are two more orbits consisting of three lines in the same plane, and two more orbits consisting of six lines in two planes. None of these six orbits consists of disjoint lines, so we conclude that the Picard number of the surface is one.

**Exercise 14.** Note that $E^2 = -1$, and that, since $P$ has multiplicity $m$ on $C$, $C' \cdot E = m$. Thus $C^2 = (C' + mE)(C' + mE) = (C')^2 + 2mC'.E + m^2E^2 = (C')^2 + m^2$.

For the other equality, first verify that $K_{S'} \cdot E = -1$ using the adjunction formula $\deg K_E = (K_{S'} + E) \cdot E$. Then compute that $C \cdot K_S = (C' + mE)(K_{S'} - E) = C' \cdot K_{S'} - C' \cdot E + mE \cdot K_{S'} - mE^2 = C' \cdot K_{S'} - m$.

**Exercise 15.** The map $S' \xrightarrow{q} \mathbb{P}^2 = \mathbb{P}(T_P \mathbb{P}^3)$ can be described as follows: for $Q \in S'$ but not in the exceptional fiber, thinking of $Q$ as a point in $S$, $q(Q)$ is the line $L$ through $P$ and $Q$; for $Q$ in the exceptional fiber, thinking of $Q$ as a direction at $P$, $q(Q)$ is the line through $P$ in the direction of $Q$.

Clearly the fiber of $q$ over a point $L \in \mathbb{P}^2$ consists of the two points $Q_1$ and $Q_2$ that, together with $P$, make up the intersection $L \cap S'$. Ramification occurs precisely when $Q_1 = Q_2$. To find the equation of this ramification locus, choose coordinates so that $P = [0 : 0 : 0 : 1]$ is the origin in an affine patch where the surface $S$ is given by an equation of the form

$$f_1(x, y, z) + f_2(x, y, z) + f_3(x, y, z)$$

with $f_i$ homogeneous of degree $i$.

A line $L$ through $P$ is given by parametric equations $(at, bt, ct)$ corresponding to a point $[a : b : c]$ in $\mathbb{P}^2$. The intersection points of this line with $S$ are given by the solutions of the equation

$$tf_1(a, b, c) + t^2f_2(a, b, c) + t^3f_3(a, b, c) = 0$$

The two solutions (other than $t = 0$) define the fiber over $L$ under the map $q$. Ramification occurs when the two solutions are identical, so is given by the discriminant. Thus the ramification locus in $\mathbb{P}^2$ is the quartic defined by the homogeneous equation $f_2^2 - 4f_1f_3$.

To see that this ramification locus is smooth, note that because $S'$ is a degree two cover of $\mathbb{P}^2$, locally $S'$ is defined by a quadratic polynomial of the form $u^2 - g(s, t)$, where $s, t$ are local coordinates on $\mathbb{P}^2$. The ramification locus is locally defined by $g = 0$. On the other hand, since the polynomial $u^2 - g(s, t)$ defines a smooth variety (namely $S'$), the Jacobian criterion implies that $g(s, t)$ also defines a smooth variety in $\mathbb{P}^2$. 

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Exercise 16. Consider the linear system $\gamma \subset |2H|$ of quadric sections on the cubic surface $S$ passing through both $P_1$ and $P_2$ with multiplicity at least two. If $P_2$ is infinitely near $P_1$, “passing through $P_2$” should be interpreted as “in the direction of $P_2$.” This linear system contains the symmetric square of the pencil of hyperplanes containing through $P_1$ and $P_2$, together with the divisor $T_{P_1}S \cap S + T_{P_2}S \cap S$ (interpreted as $2T_{P_i}S \cap S$ if $P_2$ is infinitely near) these divisors generate all of $\gamma$.

The only base points of $\gamma$ are $P_1$ and $P_2$. Indeed, since the line $\overline{P_1P_2}$ does not lie on $S$, the linear system of hyperplanes through $P_1$ and $P_2$ has a unique third base point $Q$ where $\overline{P_1P_2}$ intersects $S$. But since $Q$ can not lie in $T_{P_1}S \cap S + T_{P_2}S \cap S$, we see that $\gamma$ has exactly two base points, $P_1$ and $P_2$.

The dimension of $\gamma$ is three. Indeed, if $s_1$ and $s_2$ are defining equations for the linear system of hyperplanes through $\overline{P_1P_2}$, then defining equations for the generators of $\gamma$ are

$$s_1^2, s_1s_2, s_2^2, q$$

where $q$ is a defining equation for $T_{P_1}S + T_{P_2}S$.

It is now easy to check that the image of the rational map, which is defined over $k$,

$$S \dasharrow \mathbb{P}^3$$

is a (singular) quadric surface in $\mathbb{P}^3$, and that the map is two-to-one everywhere it is defined.

This two-to-one map allows us to define an involution of $S$: we interchange points of $S$ which map to the same point under the map given by $\gamma$. It is clearly defined over the ground field $k$.

Exercise 17. Regardless of the characteristic of the ground field, the non-smooth locus of an affine hypersurface defined by $G$ is the locus defined by $G$ and all its partial derivatives.

In particular, the non-smooth locus of the hypersurface defined by $y^p - f$ is the closed set defined by the ideal generated by $y^p - f$, $py^{p-1}$ and the partial derivatives of $f$. In characteristic zero, therefore, any non-smooth point will have $y$ coordinate zero. So a non-smooth point has the form $(y, x_1, \ldots, x_n) = (0, \lambda_1, \ldots, \lambda_n)$, where all the partial derivatives of $f(x_1, \ldots, x_n)$ vanish at $(\lambda_1, \ldots, \lambda_n)$. Since $y^p - f = 0$, it must be that $(\lambda_1, \ldots, \lambda_n)$ is a critical point of critical value zero. Equivalently, $(\lambda_1, \ldots, \lambda_n)$ is a non-smooth point of the hypersurface in $n$ space defined by $f$. But a sufficiently general polynomial $f$ defines a smooth hypersurface, so in characteristic zero we expect a general hypersurface of the form $y^p - f$ to be smooth.

In characteristic $p$, the derivative with respect to $y$ vanishes, so the non-smooth locus is defined by the ideal $(y^p - f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$. Every critical point $(\lambda_1, \ldots, \lambda_n)$ of $f$ determines exactly one non-smooth point, by setting $y_i = f(\lambda)^{1/p}$. (Of course, $f$ could fail to have critical points at all, as in the example $f = x_1 + x_2^{mp}$, but this means simply that the critical points are hiding at infinity.) For a general $f$, the expected dimension of the locus where all the $\frac{\partial f}{\partial x_i}$ vanish is zero. Thus a general hypersurface of the form $y^p - f$ in characteristic $p$ has only isolated non-smooth points.
Exercise 18. The hypersurface in $\mathbb{P}^{n+1}$ is given by a homogeneous polynomial

$$G = y^{p-1}t^{mp-p} - (F_0t^{mp} + F_1t^{mp-1} + \ldots + F_{mp-1}t + F_{mp}),$$

where each $F_i$ is a homogeneous polynomial in $x_1, \ldots, x_n$. The singular locus is defined by the homogeneous ideal

$$(G, \frac{\partial G}{\partial t}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n}).$$

Note that each derivative above is contained in the ideal $(t, x_1, \ldots, x_n)$. So the singular locus contains the point $[y : x_1 : \cdots : x_n : t] = [1 : 0 : \cdots : 0]$, regardless of the characteristic.

Exercise 19. Consider a connection $\mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega_X$ on $X$. On an open set $U$ where $\mathcal{L}$ is trivial, fix an isomorphism $\mathcal{O}_X(U) \cong \mathcal{L}(U)$, with $g \in \mathcal{L}$ corresponding to $1$ in $\mathcal{O}_X$. Set $\nabla(g) = g \otimes \eta$, for some one-form $\eta \in \Omega_X(U)$. On $U$, we have

$$\mathcal{L} \xrightarrow{\nabla} \mathcal{L} \otimes \Omega_X$$

$$fg \mapsto f\nabla(g) = g \otimes df = g \otimes (df + f\eta)$$

So we can think of $\nabla$ as a gadget that associates to the local section $f$, the one-form $df + f\eta$.

Now, we can “differentiate a section of $\mathcal{L}$ in any tangent direction.” Indeed, a tangent direction is interpreted as section of the sheaf of derivations of $\mathcal{O}_X$ to $\mathcal{O}_X$. Since $\text{Der}(\mathcal{O}_X, \mathcal{O}_X) = \text{Hom}(\mathcal{O}_X, \mathcal{O}_X)$, each derivation $\theta$ produces a homomorphism $\mathcal{O}_X \xrightarrow{\theta} \mathcal{O}_X$. Its value on the one form $df + f\eta$ can be considered the derivative of $f$ in the direction of $\theta$. So the local section $fg$ of $\mathcal{L}$ is sent to the section $\theta(df + f\eta)g$ of $\mathcal{L}$.

(2). For any line bundle $\mathcal{L}$, we can naively try to differentiate local sections as follows. Over an open set $U$ where $\mathcal{L}$ is trivial with fixed generator $g$, if we are given a section $\theta$ of the sheaf of derivations, we try sending each section $f \cdot g$ of $\mathcal{O}_X \cdot g$ to $(\theta f) \cdot g \in \mathcal{O}_X \cdot g \cong \mathcal{L}$. In general, of course, this does not lead to a globally well defined method for differentiating sections of $\mathcal{L}$, because patching fails. Indeed, let $g_1$ and $g_2$ be local generators for $\mathcal{L}$, related by the transition function $g_1 = \phi g_2$. If $s$ is a local section of $\mathcal{L}$, then writing $s = fg_1 = (\phi f)g_2$, we see that $\theta(s)$ is well defined if and only if $\theta(\phi f) = \phi \theta(f)$. Using the Leibnitz rule for derivations, we see that this is equivalent to $\theta(\phi) = 0$. Thus, this naive approach to differentiating sections gives a well defined global connection on $\mathcal{L}$ if and only if $\mathcal{L}$ admits transition functions that are killed by all derivations. Because derivations annihilate any function that is a $p^{th}$ power, it follows that any line bundle $\mathcal{L}$ that is a $p^{th}$ power of another line bundle $\mathcal{M}$ admits this natural connection, as the transition functions for $\mathcal{L}$ can be taken to be $p^{th}$ powers of transition functions for $\mathcal{M}$.

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(mp)$ and fix a global section $f$ of $\mathcal{L}$. A convenient choice of local trivialization for $\mathcal{L}$ is to let $U_i$ be the set where the homogeneous coordinate $x_i$ does not vanish. On $U_i$, we can consider $x_i^{mp}$ to be a local generator. Thinking of $f$ as a homogeneous polynomial of degree $mp$ in the homogeneous coordinates for $\mathbb{P}^n$, it has representation $(f/x_i^{mp})x_i^{mp}$ on $U_i$.  

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Exercise 20. The ground field may be assumed algebraically closed. We check the case where \( n \) is even; the case where \( n \) is odd is similar. At an isolated non-smooth point, after making suitable linear changes of coordinates, we can assume the equation has the form
\[
y^p - x_1x_2 - \cdots - x_{n-1}x_n = f_3(x_1, \ldots, x_n).
\]
Blowing up the ideal generated by \( y, x_1, \ldots, x_n \), the resulting scheme is covered by affine patches where \( x_i \neq 0 \). Consider one of these, say where \( x_1 \neq 0 \). There are local coordinates \( y', x_1, x'_2, x'_3, \ldots, x'_n \) where \( x_1y' = y \), and \( x_1x'_i = x_i \) for \( i > 1 \). The blown up hypersurface is defined by
\[
x_1^{p-2}y^p - x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n' - x_1f'_1(x_1, x_2', \ldots, x'_n)
\]
where \( f'_1 \) has order at least one in \( (x_1, x_2', \ldots, x'_n) \). This hypersurface is easily verified to be smooth, using the Jacobian criterion. Alternatively, it is sufficient to check that the exceptional divisor, namely the divisor defined by \( x_1 = 0 \) on this hypersurface, is smooth. This is obvious, since its equation is \( x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n' \) (or \( y'^p - x_2' - x_3'x_4' - \cdots - x_{n-1}'x_n' \) when \( p = 2 \)).

Exercise 21. Let \( x_1, \ldots, x_n \) be local coordinates around \( P \). In these coordinates, polynomials with a critical point at \( P \) all have the form
\[
f(x_1, \ldots, x_n) = a + \sum_{i \leq j} a_{ij}x_ix_j + \text{higher order terms}
\]
where \( a_{ij} \in k \) (by considering a Taylor series expansion, for instance). The Hessian of \( f \) is the symmetric matrix
\[
A = \begin{pmatrix}
2a_{11} & a_{12} & \cdots & a_{1n} \\
a_{12} & 2a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & 2a_{nn}
\end{pmatrix}
\]
and the invertibility of \( A \) is equivalent to the non-vanishing of the determinant of this symmetric matrix. (It is easy to verify that this is also equivalent to the condition that the \( \frac{\partial f}{\partial x_i} \)'s generate the maximal ideal \( (x_1, \ldots, x_n) \) of \( P \).)

In any characteristic other than two, the determinant of a symmetric matrix is a non-zero polynomial in the entries. Therefore, on a Zariski open subset of the finite dimensional vector space of quadratic polynomials in the \( x_i \), the coefficient matrix \( a_{ij} \) has non-zero determinant. Thus a “sufficiently general” polynomial over an infinite field has only non-degenerate critical points, assuming the characteristic is not two.

In characteristic two, however, a symmetric matrix is also an alternating matrix. In \( n \) is odd, it always has determinant zero, so all critical points of \( f \) are degenerate in this case. If \( n \) is even, however, the determinant of a symmetric \( n \times n \) matrix is a
non-zero polynomial, and again we conclude that a generic $f$ in an even number of variable has non-degenerate critical points. See [J. p 332-335] for these basic facts on alternating forms, convince yourself by looking at the cases $n \leq 4$.

**Exercise 22.** The question is local, so assume $Z \rightarrow Y \rightarrow X$ are maps of affine schemes corresponding to the ring maps $A \rightarrow B \rightarrow C$. By our finite type assumption, we know $B$ is a finitely generated $A$-algebra with generators, say $x_1, \ldots, x_n$ and relations, say $f_1, \ldots, f_n$. We can assume the same number of generators and relations because both $A$ and $B$ are regular of the same dimension. Likewise, $C$ is a finitely generated $B$ algebra with generators $y_1, \ldots, y_m$ and relations, say $g_1, \ldots, g_m$. Thus $x_1, \ldots, x_n, y_1, \ldots, y_m$ are $A$ algebra generators for $C$, with relations $f_1, \ldots, f_n, g_1, \ldots, g_m$.

In this case, the Jacobian ideal for $B$ over $A$ is the principal ideal given by the determinant of the $n \times n$ Jacobian matrix

$$\left( \frac{\partial f_i}{\partial x_j} \right)$$

and the Jacobian ideal for $C$ over $B$ is given by the determinant of the $m \times m$ matrix

$$\left( \frac{\partial g_i}{\partial y_j} \right).$$

Because the Jacobian ideal of $C$ over $A$ is the determinant of the $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix}
\frac{\partial f_i}{\partial x_j} & 0 \\
\frac{\partial g_i}{\partial x_j} & \frac{\partial g_i}{\partial y_j}
\end{pmatrix}$$

(in block form), the result is immediate.

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