The distributional hyper-Jacobian determinants in fractional Sobolev spaces

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Abstract: In this paper we give a positive answer to a question raised by Baer-Jerison in connection with hyper-Jacobian determinants and associated minors in fractional Sobolev spaces. Inspired by recent works of Brezis-Nguyen and Baer-Jerison on the Jacobian and Hessian determinants, we show that the distributional $m$th-Jacobian minors of degree $r$ are weak continuous in fractional Sobolev spaces $W^{m-\frac{m}{r},m}$, and the result is optimal, satisfying the necessary conditions, in the framework of fractional Sobolev spaces. In particular, the conditions can be removed in case $m = 1, 2$, i.e., the $m$th-Jacobian minors of degree $r$ are well defined in $W^{s,p}$ if and only if $W^{s,p} \subseteq W^{m-\frac{m}{r},m}$ in case $m = 1, 2$.

Key words: Hyper-Jacobian, Higher dimensional determinants, Fractional Sobolev spaces, Distributions.

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1 Introduction and main results

Fix integer $m \geq 1$ and consider the class of non-smooth functions $u$ from $\Omega$, a smooth bounded open subset of $\mathbb{R}^N$, into $\mathbb{R}^n$ $(N \geq 2)$. The aim of this article is to identify when the hyper($m$th)-Jacobian determinants and associated minors of $u$, which were introduced by Olver in [16], make sense as a distribution.

In the case $N = n$ and $m = 1$, starting with seminal work of Morrey[14], Reshetnyak[15] and Ball[1] on variational problems of non-linear elasticity, it is well known that the distributional (1th-)Jacobian determinant $\text{Det}(Du)$ of a map $u \in W^{1,\frac{N^2}{N+2}}(\Omega, \mathbb{R}^N)$ (or $u \in L^q \cap W^{1,p}(\Omega, \mathbb{R}^N)$ with $\frac{N-1}{p} + \frac{1}{q} = 1$ and $N-1 < p \leq \infty$) is defined by

$$\text{Det}(Du) := \sum_j \partial_j (u^i (\text{adj} Du)_j^i),$$

where $\text{adj} Du$ means the adjoint matrix of $Du$. Furthermore, Brezis-Nguyen [5] extended the range of the map $u \mapsto \text{Det}(Du)$ in the framework of fractional Sobolev spaces. They showed that the distributional Jacobian determinant $\text{Det}(Du)$ for any $u \in W^{1,\frac{N}{N+2}}(\Omega, \mathbb{R}^N)$ can be defined as

$$\langle \text{Det}(Du), \psi \rangle := \lim_{k \to \infty} \int_{\Omega} \text{det}(Du_k) \psi dx \quad \forall \psi \in C_c^1(\Omega, \mathbb{R}),$$

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where \( u_k \in C^1(\Omega, \mathbb{R}^N) \) such that \( u_k \to u \) in \( W^{1-\frac{1}{N}} \). They pointed out that the result recovers all the definitions of distributional Jacobian determinants mentioned above, except \( N = 2 \), and the distributional Jacobian determinants are well-defined in \( W^{s,p} \) if and only if \( W^{s,p} \subseteq W^{1-\frac{1}{N}} \) for \( 1 < p < \infty \) and \( 0 < s < 1 \).

In the case \( n = 1 \) and \( m = 2 \), similar to the results in [3], the distributional Hessian(2th-Jacobian) determinants are well-defined and continuous on \( W^{2-\frac{2}{N}}(\mathbb{R}^N) \) (see [13, 14]). Baer-Jersion [1] pointed out that the continuous results of Hessian determinant in \( W^{2-\frac{2}{N}}(\mathbb{R}^N) \) with \( N \geq 3 \) implies the known continuity results in space \( W^{1,p}(\mathbb{R}^N) \cap W^{2,q}(\mathbb{R}^N) \) with \( 1 < p, r < \infty, \frac{2}{p} + \frac{2}{q} = 1, N \geq 3 \) (see [16, 17]). Furthermore they showed that the distributional Hessian determinants are well-defined in \( W^{s,p} \) if and only if \( W^{s,p} \subseteq W^{2-\frac{2}{N}} \) for \( 1 < p < \infty \) and \( 1 < s < 2 \).

For \( m > 2 \), mth-Jacobian, as a generalization of ordinary Jacobian, was first introduced by Escherich [8] and Gegenbauer [11]. In fact, the general formula for hyper-Jacobian can be expressed by using Cayley’s theory of higher dimensional determinants. All these earlier investigations were limited to polynomial functions until Olver [16] turn his attention to some non-smooth functions. Especially he showed that the mth-Jacobain determinants (minors) of degree \( r \) can be defined as a distribution provided

\[
\frac{r - t}{\gamma} + \frac{t}{\delta} \leq 1, t := m \mod r
\]

or

\[
u \in W^{m-\frac{\mu}{\gamma}}(\Omega, \mathbb{R}^n) \text{ with } \gamma \geq \max\left\{ \frac{N r}{N + r} \right\}.
\]

Bare-Jersion [1] raised an interesting question: whether do there exist fractional versions of this result? I.e., is the mth-Jacobian determinant of degree \( r \) continuous from space \( W^{m-\frac{\mu}{\gamma}} \) into the space of distributions? Our first results give a positive answer to the question. We refer to Sec. 2 below for the following notation.

**Theorem 1.1.** Let \( q, n, N \) be integers with \( 2 \leq q \leq n := \min\{n, N\} \), for any integer \( 1 \leq r \leq q \), multi-indices \( \beta \in I(r, n) \) and \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \) \( (j = 1, \cdots, m) \), the mth-Jacobian \((\beta, \alpha)\)-minor operator \( u \mapsto M_{\alpha}^\beta(D^m u) \) (see (2.6)) : \( C^m(\Omega, \mathbb{R}^n) \to \mathcal{D}'(\Omega) \) can be extended uniquely as a continuous mapping \( u \mapsto Div_{D^{\alpha}}^\beta(D^m u) : W^{m-\frac{\mu}{\gamma}}(\Omega, \mathbb{R}^n) \to \mathcal{D}'(\Omega) \). Moreover for all \( u, v \in W^{m-\frac{\mu}{\gamma}}(\Omega, \mathbb{R}^n), \psi \in C^\infty_c(\Omega, \mathbb{R}), \) we have

\[
|\langle Div_{\alpha}^\beta(D^m u) - Div_{\alpha}^\beta(D^m v), \psi \rangle| \leq C_{r, q, n, N} \| u - v \|_{W^{m-\frac{\mu}{\gamma}}} \left( \| u \|_{W^{m-\frac{\mu}{\gamma}}}^{r-1} + \| v \|_{W^{m-\frac{\mu}{\gamma}}}^{r-1} \right) \| D^m \psi \|_{L^\infty}.
\]  

(1.1)

We recall that for \( 0 < s < \infty \) and \( 1 \leq p < \infty \), the fractional Sobolev space \( W^{s,p}(\Omega) \) is defined as follows: when \( s < 1 \)

\[
W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \mid \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}} < \infty \right\},
\]

and the norm

\[
\| u \|_{W^{s,p}} := \| u \|_{L^p} + \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

When \( s > 1 \) with non-integer,

\[
W^{s,p}(\Omega) := \{ u \in W^{[s],p}(\Omega) \mid D^s u \in W^{s-[s],p}(\Omega) \}.
\]
the norm
\[ \|u\|_{W^{s,p}} := \|u\|_{W^{s,p}[\Omega]} + \left(\int_{\Omega} \int_{\Omega} \frac{|D^{|s|}u(x) - D^{|s|}u(y)|^p}{|x-y|^{N+|s|p}} dxdy\right)^{\frac{1}{p}}. \]

**Remark 1.2.** It is worth pointing out that we may use the same method to get a similar result, see Corollary 3.5 for \( u \in W^{m-\frac{m}{p},q}(\Omega) \) with \( m \geq 2 \). Theorem 1.1 and Corollary 3.5 recover not only the definitions of Jacobian and Hessian determinants mentioned above, but also the definitions of \( m \)-th Jacobian in [16] since the following facts

(i) \( W^{m-\frac{m}{r}}(\Omega, \mathbb{R}^n) \cap W^{m-\frac{m}{r}-1,\delta}(\Omega, \mathbb{R}^n) \subset W^{m-\frac{m}{r},r}(\Omega, \mathbb{R}^n) \) with continuous embedding if \( \frac{r+1}{\gamma} + \frac{\delta}{\gamma} \leq 1 \) (1 < \( \delta < \infty \), 1 < \( r \leq N \)), where \( t := m \text{ mod } r \).

(ii) \( W^{m-\frac{m}{r}}(\Omega, \mathbb{R}^n) \subset W^{m-\frac{m}{r},r}(\Omega, \mathbb{R}^n) \) (1 < \( r \leq N \)) with continuous embedding if \( \gamma \geq \max\{ \frac{N_0}{N+t} \} \).

Similarly to the optimal results for the ordinary distributional Jacobian and Hessian determinants in [3] [11], an natural question is that whether the results in Theorem 1.1 is optimal in the framework of the space \( W^{s,p} \)? I.e., is the distributional \( m \)-th Jacobian minors of degree \( r \) well-defined in \( W^{s,p}(\Omega, \mathbb{R}^n) \) if and only if \( W^{s,p}(\Omega, \mathbb{R}^n) \subset W^{m-\frac{m}{r},r}(\Omega, \mathbb{R}^n) \)? Such a question is connected with the construction of counter-examples in some special fractional Sobolev spaces. Indeed, the above conjecture is obviously correct in case \( r = 1 \). Our next results give a partial positive answer in case \( r > 1 \).

**Theorem 1.3.** Let \( m, r \) be integers with \( 1 < r \leq n \), \( 1 < p < \infty \) and \( 0 < s < \infty \) be such that \( W^{s,p}(\Omega, \mathbb{R}^n) \not\subset W^{m-\frac{m}{r},r}(\Omega, \mathbb{R}^n) \). If the condition

\[ 1 < r < p, s = m - m/r \text{ non-integer} \tag{1.2} \]

fails, then there exist a sequence \( \{u_k\}_{k=1}^{\infty} \subset C^m(\overline{\Omega}, \mathbb{R}^n) \), multi-indices \( \beta \in I(r, n) \), \( \alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \) with \( \alpha^j \in I(r, N) \) and a function \( \psi \in C^\infty_c(\Omega) \) such that

\[ \lim_{k \to \infty} \|u_k\|_{s,p} = 0, \quad \lim_{k \to \infty} \int_{\Omega} M^\beta_\alpha(D^m u) \psi dx = \infty, \tag{1.3} \]

one still unanswered question is whether the above optimal results hold in case (1.2). We give some discuss in Sec. 4 and give positive answers in case \( m = 1 \) and 2. Indeed

**Theorem 1.4.** Let \( m = 1 \) or 2 and \( r, s, p \) be as in Theorem 1.3. Then there exist a sequence \( \{u_k\}_{k=1}^{\infty} \subset C^m(\overline{\Omega}, \mathbb{R}^n) \), multi-indices \( \beta \in I(r, n) \), \( \alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m) \) with \( \alpha^j \in I(r, N) \) and a function \( \psi \in C^\infty_c(\Omega) \) such that (1.3) holds.

Furthermore, we give reinforced versions of optimal results, see Theorem 4.9 for \( u \in W^{2-\frac{2}{r},r}(\Omega) \) with \( 1 < r \leq N \), we expect that there are reinforced versions of optimal results for \( W^{m-\frac{m}{r},r}(\Omega)(m > 2) \), for instance there exist a sequence \( \{u_k\}_{k=1}^{\infty} \subset C^m(\overline{\Omega}) \) and a function \( \psi \in C^\infty_c(\Omega) \) such that

\[ \lim_{k \to \infty} \|u_k\|_{s,p} = 0, \quad \lim_{k \to \infty} \int_{\Omega} M_\alpha(D^m u) \psi dx = \infty \tag{1.4} \]

for any \( s, p \) with \( W^{s,p}(\Omega) \not\subset W^{m-\frac{m}{r},r}(\Omega) \).

This paper is organized as follows. Some facts and notion about higher dimensional determinant and hyper-Jacobian are given in Section 2. In Section 3 we establish the weak continuity results and definitions for distributional hyper-Jacobian minors in fractional Sobolev space. Then we turn to the question about optimality and get some positive results in Section 4.
2 Higher dimensional determinants

In this section we collect some notation and preliminary results for hyper-Jacobian determinants and minors. First we recall some notation and facts about about ordinary determinants and minors, whereas further details can be found in [12].

Fix $0 \leq k \leq n$, we shall use the standard notation for ordered multi-indices

$$I(k,n) := \{ \alpha = (\alpha_1, \cdots, \alpha_k) \mid \alpha_i \text{ integers}, 1 \leq \alpha_1 < \cdots < \alpha_k \leq n \},$$

where $n \geq 2$. Set $I(0,n) = \{0\}$ and $|\alpha| = k$ if $\alpha \in I(k,n)$. For $\alpha \in I(k,n)$,

(i) $\overline{\alpha}$ is the element in $I(n-k,n)$ which complements $\alpha$ in $\{1,2,\cdots,n\}$ in the natural increasing order.

(ii) $\alpha - i$ means the multi-index of length $k - 1$ obtained by removing $i$ from $\alpha$ for any $i \in \alpha$.

(iii) $\alpha + j$ means the multi-index of length $k + 1$ obtained by adding $j$ to $\alpha$ for any $j \notin \alpha$.

(iv) $\sigma(\alpha, \beta)$ is the sign of the permutation which reorders $(\alpha, \beta)$ in the natural increasing order for any multi-index $\beta$ with $\alpha \cap \beta = \emptyset$. In particular set $\sigma(\overline{0}, 0) := 1$.

Let $n, N \geq 2$ and $A = (a_{ij})_{n \times N}$ be an $n \times N$ matrix. Given two ordered multi-indices $\alpha \in I(k, N)$ and $\beta \in I(k, n)$, then $A^\beta_\alpha$ denotes the $k \times k$-submatrix of $A$ obtained by selecting the rows and columns by $\beta$ and $\alpha$, respectively. Its determinant will be denoted by

$$M^\beta_\alpha(A) := \det A^\beta_\alpha,$$

and we set $M^0_\emptyset(A) := 1$. The adjoint of $A^\beta_\alpha$ is defined by the formula

$$(\text{adj } A^\beta_\alpha)^i_j := \sigma(i, \beta - i) \sigma(j, \alpha - j) \det A^\beta_{\alpha-j}, \quad i \in \beta, j \in \alpha.$$ 

So Laplace formulas can be written as

$$M^\beta_\alpha(A) = \sum_{j \in \alpha} a_{ij} (\text{adj } A^\beta_\alpha)^i_j, \quad i \in \beta. $$

Next we pay attention to the higher dimensional matrix and determinant.

An $m$-dimensional matrix $A$ of order $N^m$ is a hypercubical array of $N^m$ as

$$A = (a_{l_1l_2\cdots l_m})_{N \times \cdots \times N},  \quad (2.3)$$

where the index $l_i \in \{1, \cdots N\}$ for any $1 \leq i \leq m$.

**Definition 2.1.** Let $A$ be an $m$-dimensional matrix, then the (full signed) determinant of $A$ is the number

$$\det A = \sum_{\tau_2, \cdots, \tau_m \in S_N} \Pi_{s=2}^m \sigma(\tau_s) a_{1\tau_2(1)\cdots \tau_m(1)} a_{2\tau_2(2)\cdots \tau_m(2)} \cdots a_{N\tau_2(N)\cdots \tau_m(N)},$$

where $S_N$ is the permutation group of $\{1, 2, \cdots, N\}$ and $\sigma(\cdot)$ is the sign of $\cdot$.

For any $1 \leq i \leq m$ and $1 \leq j \leq N$, the $j$-th $i$-layer of $A$, the $(m-1)$-dimensional matrix denoted by $A_{l_i=j}$, which generalizing the notion of row and column for ordinary matrices, is defined by

$$A_{l_i=j} := (a_{l_1l_2\cdots l_{i-1}jl_{i+1}\cdots l_m})_{N \times \cdots \times N}.$$ 

According to Definition 2.1 we can easily obtain that
Lemma 2.2. Let $A$ be an $m$-dimensional matrix and $1 \leq i \leq m$. $A'$ is a matrix such that a pair of $i$-layers in $A$ is interchanged, then

$$
\text{det } A' = \begin{cases} (-1)^{m-1} \text{det } A & i = 1, \\ - \text{det } A & i \geq 2. \end{cases}
$$

For any $A$ and $1 \leq i < j \leq m$, the $(i, j)$-transposition of $A$, denoting by $A^{T(i,j)}$, is a $m$-dimensional matrix defined by

$$
a'_{l_1, \ldots, l_i, \ldots, l_j, \ldots, l_m} = a_{l_1, \ldots, l_j, \ldots, l_i, \ldots, l_m}
$$

for any $l_1, \ldots, l_m = 1, \ldots, N$.

Then we have

Lemma 2.3. Let $A$ be an $m$-dimensional matrix and $1 \leq i < j \leq m$, if $m$ is odd and $1 < i < j \leq m$ or $m$ is even, then

$$
\text{det } A^{T(i,j)} = \text{det } A.
$$

Proof. According to the definition of the $m$-dimensional determinant, we only to show the claim in case $m$ is even, $i = 1$ and $j = 2$.

$$
\text{det } A = \sum_{\tau_2, \ldots, \tau_m \in S_N} \Pi_{s=2}^m \sigma(\tau_s)a_{\tau_2(1)}a_{\tau_2(2)}a_{\tau_m(1)}a_{\tau_m(2)} \cdots a_{N_{\tau_2}(N)}a_{N_{\tau_m}(N)}
$$

More generally, suppose $A$ be an $m$-dimensional matrix of order $N_1 \times \cdots \times N_m$, $1 \leq r \leq \min\{N_1, \ldots, N_m\}$, and an type of multi-index $\alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m)$ where $\alpha^j := (\alpha^j_1, \cdots, \alpha^j_{N_j})$, $\alpha^j_1, \cdots, \alpha^j_{N_j} \in \{1, 2, \cdots, N_j\}$ and $\alpha^j_1 \neq \alpha^j_2$ for $i_1 \neq i_2$. Define the $\alpha$-minor of $A$, denoted by $A_{\alpha}$, to be the $m$-dimensional matrix of order $r^m$ as

$$
A_{\alpha} = (b_{l_1l_2 \cdots l_m})_{r^m},
$$

where $b_{l_1l_2 \cdots l_m} := a_{\alpha^1_1 \alpha^1_2 \cdots \alpha^m_m}$. Its determinant will be denoted by

$$
M_{\alpha}(A) := \text{det } A_{\alpha}.
$$

If $\alpha^j$ is not increasing, let $\tilde{\alpha}^j$ be the increasing multi-indices generated by $\alpha^j$ and $\tilde{\alpha} := (\tilde{\alpha}^1, \cdots, \tilde{\alpha}^m)$, then Lemma 2.2 implies that $M_{\alpha}(A)$ and $M_{\tilde{\alpha}}(A)$ differ only by a sign. Without loss of generality, we can assume $\alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m)$ with $\alpha^j \in I(r, N_j)$. Moreover we set $M_{\alpha}(A) := 1$. 

\[ \square \]
Next we pay attention to hyper-Jacobian determinants and minors for a map \( u \in C^m(\Omega, \mathbb{R}^n) \). We will denote by \( D^m u \) the hyper-Jacobian matrix of \( u \), more precisely, \( D^m u \) is a \((m + 1)\)-dimensional matrix with order \( n \times N \times \cdots \times N \) given by

\[
D^m u := (a_{i_1i_2\cdots i_{m+1}})_{n\times N\times \cdots \times N}
\]

where

\[
a_{i_1i_2\cdots i_{m+1}} = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m+1}} u^j.
\]

Then for any \( \beta \in I(r, n) \), \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \) and \( 1 \leq r \leq \min\{n, N\} \), the \( m \)-th Jacobian \((\beta, \alpha)\)-minor of \( u \), denoted by \( M^\beta_\alpha(D^m u) \), is the determinant of the \((\beta, \alpha)\)- minor of \( D^m u \), i.e.,

\[
M^\beta_\alpha(D^m u) := M_{(\beta, \alpha)}(D^m u).
\] (2.6)

In particular if \( N = n \) and \( \beta = \alpha^1 = \cdots = \alpha^m = \{1, 2, \cdots, N\} \), \( \det(D^m u) \) is called the \( m \)-th Jacobian determinant of \( u \). Similarly, the hyper-Jacobian matrix \( D^m u \) of \( u \in C^m(\Omega) \) is a \( m \)-dimensional matrix with order \( N \times \cdots \times N \) and the \( m \)-th Jacobian \( \alpha \)-minor of \( u \) is defined by \( M_\alpha(D^m u) \).

In order to prove the main results, some lemmas, which can be easily manipulated by the definition of hyper-Jacobians, are introduced as follows.

**Lemma 2.4.** Let \( u = (v, \cdots, v) \in C^m(\Omega, \mathbb{R}^n) \) with \( v \in C^m(\Omega) \). For any \( \beta \in I(r, n) \) and \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \), \( 1 \leq r \leq n \)

\[
M^\beta_\alpha(D^m u) = \begin{cases} r!M_\alpha(D^m v) & m \text{ is even}, \\ 0 & m \text{ is odd}. \end{cases}
\]

**Lemma 2.5.** Let \( u \in C^m(\Omega, \mathbb{R}^n) \), \( \beta \in I(r, n) \) and \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \), \( 1 \leq r \leq n \). Then for any \( 1 \leq i \leq m \)

\[
M^\beta_\alpha(D^m u) = \sum_{\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \cdots, \tau_m \in S_r} \Pi_{s \in I} \sigma(\tau_s) M^\beta_{\alpha^i}(Dv(i)),
\] (2.7)

where \( M^\beta_{\alpha^i}(\cdot) \) is the ordinary minors and \( v(i) \in C^1(\Omega, \mathbb{R}^r) \) can be written as

\[
v^j(i) = \partial_{\alpha^1_{\tau_1(j)}} \cdots \partial_{\alpha^i_{\tau_{i-1}(j)}} \partial_{\alpha^i_{\tau_{i+1}(j)}} \cdots \partial_{\alpha^m_{\tau_m(j)}} u^\beta_j, \quad j = 1, \cdots, r.
\]

### 3 Hyper-jacobians in fractional Sobolev spaces

In this section we establish the weak continuity results for the Hyper-jacobian minors in the fractional Sobolev spaces \( W^{m-\frac{m}{r}\beta}(\Omega, \mathbb{R}^n) \).

Let \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \), we set

\[
\tilde{\alpha} = (\alpha^1 + (N + 1), \cdots, \alpha^m + (N + m)), R(\tilde{\alpha}) := \{(i_1, \cdots, i_m) \mid i_j \in \alpha^j + (N + j)\}.
\]

For any \( I = (i_1, \cdots, i_m) \in R(\tilde{\alpha}) \),

\[
\tilde{\alpha} - I := (\alpha^1 + (N + 1) - i_1, \cdots, \alpha^m + (N + m) - i_m); \\
\sigma(\tilde{\alpha} - I, I) := \Pi_{s=1}^m \sigma(\alpha^s + (N + s) - i_s, i_s); \\
\partial_I := \partial_{x_{i_1}} \cdots \partial_{x_{i_m}}; \quad \tilde{x} := (x_1, \cdots, x_N, x_{N+1}, \cdots, x_{N+m}).
\]

We begin with the following simple lemma:
Lemma 3.1. Let $u \in C^m(\Omega, \mathbb{R}^n)$, $\psi \in C^m_{c}(\Omega)$, $0 \leq r \leq n := \min\{n, N\}$, $\beta \in I(r, n)$ and $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m)$ with $\alpha^j \in I(r, N)$ $(1 \leq j \leq m)$, then

$$
\int_{\Omega} M^\beta_{\alpha}(D^m u) \psi dx = \sum_{I \in R(\alpha)} (-1)^m \sigma(\vec{\alpha} - I, I) \int_{\Omega \times [0,1]^m} M^\beta_{\alpha-I}(D^m U) \partial_I \Psi d\bar{x},
$$

(3.1)

for any extensions $U \in C^m(\Omega \times [0,1]^m, \mathbb{R}^n) \cap C^{m+1}(\Omega \times (0,1)^m, \mathbb{R}^n)$ and $\Psi \in C^m_{c}(\Omega \times [0,1]^m, \mathbb{R})$ of $u$ and $\psi$, respectively.

Proof. It is easy to show the results in case $r = 0, 1$ or $n = 1$. So we give the proof only for the case $2 \leq r \leq n$. Denote

$$
U_i := \begin{cases} U |_{x_{N+i+1} = \cdots = x_{N+m} = 0}, & 1 \leq i \leq m-1, \\ U, & i = m. \end{cases}, \quad \Psi_i := \begin{cases} \Psi |_{x_{N+i+1} = \cdots = x_{N+m} = 0}, & 1 \leq i \leq m-1, \\ \Psi, & i = m. \end{cases}
$$

$$
\Omega_i := \Omega \times [0,1)_{x_{N+i+1}} \times \cdots \times [0,1)_{x_{N+i}}, \quad \bar{x}_i := (x, x_{N+i+1}, \ldots, x_{N+i}).
$$

Applying the fundamental theorem of calculus and the definition of $M^\beta_{\alpha}(D^m u)$, we have

$$
\int_{\Omega} M^\beta_{\alpha}(D^m u) \psi dx = -\int_{\Omega_1} \partial_{N+1} (M^\beta_{\alpha}(D^m U_1) \Psi_1) d\bar{x}_1 = -\int_{\Omega_1} \partial_{N+1} M^\beta_{\alpha}(D^m U_1) \Psi_1 d\bar{x}_1 - \int_{\Omega_1} M^\beta_{\alpha}(D^m U_1) \partial_{N+1} \Psi_1 d\bar{x}_1.
$$

(3.2)

According to the Lemma 2.5, $M^\beta_{\alpha}(D^m U_1)$ can be written as

$$
M^\beta_{\alpha}(D^m U_1) = \sum_{r_2, \ldots, r_m \in S_r} \Pi^{m-2}_s \sigma(\tau_s) M^0_{\alpha^{(1)}}(DV),
$$

where $\bar{0} := \{1, 2, \ldots, r\}$ and

$$
V_1(\bar{x}_1) := (V_1^1(\bar{x}_1), \ldots, V_1^r(\bar{x}_1)), \quad V_j^j = \partial_{\alpha^{(2)}(j)} \cdots \partial_{\alpha^{m(r)}(J)} u^j.
$$

Then

$$
\int_{\Omega} M^\beta_{\alpha}(D^m u) \psi dx = \sum_{r_2, \ldots, r_m \in S_r} \Pi^{m-2}_s \sigma(\tau_s) \left\{ -\int_{\Omega_1} \partial_{N+1} M^\bar{0}_{\alpha^{(1)}}(DV_1) \Psi_1 d\bar{x}_1 - \int_{\Omega_1} M^\bar{0}_{\alpha^{(1)}}(DV_1) \partial_{N+1} \Psi_1 d\bar{x}_1 \right\}.
$$

(3.3)

We denote the first part integral on the right-hand side by $I$, Laplace formulas of the 2-dimensional minors imply that

$$
I = -\sum_{i \in \alpha^1} \sum_{j=1}^r \int_{\Omega_1} \sigma(i, \alpha^1 - i) \sigma(j, \bar{0} - j) \partial_{N+1} \partial_{\alpha^{(1)}} V_j^i M^\bar{0}^i_{\alpha^{(1)}}(DV_1) \Psi_1 d\bar{x}_1
$$

$$
= \sum_{i \in \alpha^1} \sum_{j=1}^r \int_{\Omega_1} \sigma(i, \alpha^1 - i) \sigma(j, \bar{0} - j) \partial_{N+1} V_j^i \left( \partial_{\alpha^{(1)}} M^\bar{0}^i_{\alpha^{(1)}}(DV_1) \Psi_1 + M^\bar{0}^i_{\alpha^{(1)}}(DV_1) \partial_{\alpha^{(1)}} \Psi_1 \right) d\bar{x}_1.
$$

(3.4)

Since

$$
\sum_{i \in \alpha^1} \sigma(i, \alpha^1 - i) \sigma(j, \bar{0} - j) \partial_{\alpha^{(1)}} M^\bar{0}^i_{\alpha^{(1)}}(DV_1) = 0
$$

we have

$$
\int_{\Omega} M^\beta_{\alpha}(D^m u) \psi dx = \sum_{i \in \alpha^1} \sum_{j=1}^r \int_{\Omega_1} \sigma(i, \alpha^1 - i) \sigma(j, \bar{0} - j) \partial_{\alpha^{(1)}} M^\bar{0}^i_{\alpha^{(1)}}(DV_1) \Psi_1 d\bar{x}_1 = 0.
$$
for any \( j \), it follows that
\[
I = \sum_{i_1 \in \alpha^1} \sum_{j=1}^r \int_{\Omega_1} \sigma(i, \alpha^1 - i) \sigma(j, \alpha^1 - j) \, \partial_{N+1} V_1^j M_{\alpha^1-i}^{-j} (DV_1) \partial_i \Psi_1 d\vec{x}_1
\]
\[
= \sum_{i_1 \in \alpha^1} \int_{\Omega_1} \sigma(i, \alpha^1 - i) \sigma(N + 1, \alpha^1 - i) M_{\alpha^1+(N+1)-i}^{-i} (DV_1) \partial_i \Psi_1 d\vec{x}_1
\]
\[
= - \sum_{i_1 \in \alpha^1} \int_{\Omega_1} \sigma(\alpha^1 + (N + 1) - i, i) M_{\alpha^1+(N+1)-i}^{\alpha^1} (DV_1) \partial_i \Psi_1 d\vec{x}_1 .
\]

Combining with (3.3), we obtain that
\[
\int_{\Omega} M_\alpha^\beta (D^m u) \psi d\vec{x} = - \sum_{i_1 \in \alpha^1 + (N + 1)} \sigma(\alpha^1 + (N + 1) - i_1, i_1) \sum_{\tau_2, \ldots, \tau_m \in S_r} \Pi_{s=2}^m \sigma(\tau_s) \int_{\Omega_1} M_{\alpha^1+(N+1)-i_1}^\tau (DV_1) \partial_1 \Psi_1 d\vec{x}_1 .
\]

For any \( i_1 \in \alpha^1 + (N + 1) \), we denote \( \gamma := \alpha^1 + (N + 1) - i_1 \), then
\[
\sum_{\tau_2, \ldots, \tau_m \in S_r} \Pi_{s=2}^m \sigma(\tau_s) M_{\alpha^1+(N+1)-i_1}^\tau (DV_1) = \sum_{\tau_1, \tau_2, \ldots, \tau_m \in S_r} \Pi_{s=1}^m \sigma(\tau_s) \partial_{\gamma \tau_1 (1)} V_1^1 \cdots \partial_{\gamma \tau_1 (r)} V_1^r
\]
\[
= \sum_{\tau_1, \tau_2, \ldots, \tau_m \in S_r} \Pi_{s=1}^m \sigma(\tau_s) \left( \partial_{\gamma \tau_1 (1)} \partial_{\tau_2 (1)} \cdots \partial_{\alpha_m} U_1^{\tau_1} \right) \cdots \left( \partial_{\gamma \tau_1 (r)} \partial_{\alpha_2} U_1^{\tau_2} \cdots \partial_{\alpha_m} U_1^{\tau_r} \right)
\]
\[
= M_\alpha(i_1) (D^m U_1) ,
\]

where \( \alpha(i_1) := (\alpha^1 + (N + 1) - i_1, \alpha^2, \ldots, \alpha^m) \). Hence
\[
\int_{\Omega} M_\alpha^\beta (D^m u) \psi d\vec{x} = - \sum_{i_1 \in \alpha^1 + (N + 1)} \sigma(\alpha^1 + (N + 1) - i_1, i_1) \int_{\Omega_1} M_{\alpha(i_1)}^\beta (D^m U_1) \partial_i \Psi_1 d\vec{x}_1
\]
\[
= \sum_{i_1 \in \alpha^1 + (N + 1)} \sigma(\alpha^1 + (N + 1) - i_1, i_1) \int_{\Omega_2} \partial_{N+2} \left( M_{\alpha(i_1)}^\beta (D^m U_2) \partial_i \Psi_2 \right) d\vec{x}_2
\]
\[
(3.7)
\]

An easy induction and the argument similar to the one used in (3.2)-(3.6) shows that
\[
\int_{\Omega} M_\alpha^\beta (D^m u) \psi d\vec{x} = \sum_{s=1}^j \sum_{i_s \in \alpha^s + (N + s)} \Pi_{s=1}^j \sigma(\alpha^s + (N + s) - i_s, i_s) \int_{\Omega_j} M_{\alpha(i_1\cdots\iota_j)}^\beta (D^m U_j) \partial_{i_1\cdots\iota_j} \Psi_j d\vec{x}_j
\]
\[
(3.8)
\]

for any \( 1 \leq j \leq m \), where
\[
\alpha(i_1\iota_2\cdots\iota_j) := (\alpha^1 + (N + 1) - i_1, \cdots, \alpha^j + (N + j) - i_j, \alpha^{j+1}, \cdots, \alpha^m).
\]

\[\square\]

**Lemma 3.2.** Let \( u, v \in C^m(\Omega, \mathbb{R}^n) \) and \( \psi \in C^m_c(\Omega) \) and \( 2 \leq q \leq n \). Then for any \( 1 \leq r \leq q \), \( \beta \in I(r, n) \) and \( \alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m) \) with \( \alpha^j \in I(r, N) \),
\[
\left| \int_{\Omega} M_\alpha^\beta (D^m u) \psi d\vec{x} - \int_{\Omega} M_\alpha^\beta (D^m v) \psi d\vec{x} \right| \leq C \| u - v \|_{W^m - \frac{m}{q}, \psi} \left( \| u \|_{W^m - \frac{m}{q}, \psi} + \| v \|_{W^m - \frac{m}{q}, \psi} \right) \| D^m \psi \|_{L^\infty},
\]
\[
(3.9)
\]
the constant \( C \) depending only on \( q, r, m, n \) and \( \Omega \).
Proof. Let \( \tilde{u} \) and \( \tilde{v} \) be extensions of \( u \) and \( v \) to \( \mathbb{R}^N \) such that
\[
\|\tilde{u}\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)} \leq C\|u\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}, \quad \|\tilde{v}\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)} \leq C\|v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}
\]
and
\[
\|\tilde{u} - \tilde{v}\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)} \leq C\|u - v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)},
\]
where \( C \) depending only on \( q, m, n, N \) and \( \Omega \).

According to a well known trace theorem of Stein in [17, 18], where \( W^{m, \frac{m}{q}}(\mathbb{R}^N) \) is identified as the space of traces of \( W^{m,q}(\mathbb{R}^N \times (0, +\infty)^m) \), there is a bounded linear extension operator
\[
E : W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n) \to W^{m,q}(\mathbb{R}^N \times (0, +\infty)^m, \mathbb{R}^n).
\]

Let \( U \) and \( V \) be extensions of \( \tilde{u} \) and \( \tilde{v} \) to \( \mathbb{R}^N \times (0, +\infty)^m \), respectively, i.e.,
\[
U = E\tilde{u}, \quad V = E\tilde{v}.
\]

We then have
\[
\|D^m U\|_{L^q(\Omega \times (0,1)^m)} \leq C\|u\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}, \quad \|D^m V\|_{L^q(\Omega \times (0,1)^m)} \leq C\|v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}
\]
and
\[
\|D^m U - D^m V\|_{L^q(\Omega \times (0,1)^m)} \leq C\|u - v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}.
\]
Let \( \Psi \in C_c^m(\Omega \times [0,1)^m) \) be an extension of \( \psi \) such that
\[
\|D^m \Psi\|_{L^\infty(\Omega \times [0,1)^m)} \leq C\|D^m \psi\|_{L^\infty(\Omega)}.
\]
According to Lemma 3.1, we have
\[
\left| \int_{\Omega} M^\alpha(D^m u)\psi dx - \int_{\Omega} M^\alpha(D^m v)\psi dx \right| \leq \sum_{I \in R(\tilde{\alpha})} \int_{\Omega \times (0,1)^m} \left| M^\alpha_{\tilde{\alpha}-I}(D^m U) - M^\alpha_{\tilde{\alpha}-I}(D^m V) \right| \|\partial_I \Psi\| dx^r.
\]
Let \( \Psi \in C_c^m(\Omega \times [0,1)^m) \) be an extension of \( \psi \) such that
\[
\|D^m \Psi\|_{L^\infty(\Omega \times [0,1)^m)} \leq C\|D^m \psi\|_{L^\infty(\Omega)}.
\]
According to Lemma 3.1, we have
\[
\left| \int_{\Omega} M^\alpha(D^m u)\psi dx - \int_{\Omega} M^\alpha(D^m v)\psi dx \right| \leq \sum_{I \in R(\tilde{\alpha})} \int_{\Omega \times (0,1)^m} \left| M^\alpha_{\tilde{\alpha}-I}(D^m U) - M^\alpha_{\tilde{\alpha}-I}(D^m V) \right| \|\partial_I \Psi\| dx^r.
\]
Note that for any \( I \in R(\tilde{\alpha}) \)
\[
\left| M^\alpha_{\tilde{\alpha}-I}(D^m U) - M^\alpha_{\tilde{\alpha}-I}(D^m V) \right| \leq \sum_{\tau_1, \ldots, \tau_m \in S_r} \left| \partial_{\tau_1(1)} \cdots \tau_{m(1)} U^{\beta_1} \cdots \partial_{\tau_1(r)} \cdots \tau_{m(r)} V^{\beta_1} \cdots \partial_{\tau_1(r)} \cdots \tau_{m(r)} V^{\beta_1} \right|
\]
\[
\leq \sum_{\tau_1, \ldots, \tau_m \in S_r} \sum_{s=1}^r |D^m U|^{s-1}|D^m U - D^m V| |D^m V|^{r-s}
\]
\[
\leq C|D^m U - D^m V|(|D^m U|^{r-1} + |D^m V|^{r-1}).
\]
Combining with (3.10), we can easily obtain
\[
\left| \int_{\Omega} M^\alpha(D^m u)\psi dx - \int_{\Omega} M^\alpha(D^m v)\psi dx \right| \leq C \int_{\Omega \times (0,1)^m} |D^m U - D^m V|(|D^m U|^{r-1} + |D^m V|^{r-1}) dx^r \|D^m \Psi\|_{L^\infty(\Omega \times (0,1)^m)}
\]
\[
\leq C\|u - v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}(\|u\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}^{r-1} + \|v\|_{W^{m, \frac{m}{q}}(\mathbb{R}^N, \mathbb{R}^n)}^{r-1}) \|D^m \psi\|_{L^\infty}.}
\]
According to the above lemma, we can give the definitions of distributional $m$th-Jacobian minors of $u$ with degree less that $q$ when $u \in W^{m-\frac{m}{q},q}(\Omega,\mathbb{R}^n)$ ($2 \leq q \leq n$).

**Definition 3.3.** Let $u \in W^{m-\frac{m}{q},q}(\Omega,\mathbb{R}^n)$ with $2 \leq q \leq n$. For any $0 \leq r \leq q$, $\beta \in I(r,n)$ and $\alpha = (\alpha^1,\alpha^2,\cdots,\alpha^n)$ with $\alpha^j \in I(r,N)$, the distributional $m$th-Jacobian $(\beta,\alpha)$-minors of $u$, denoted by $\text{Div}_\alpha(D^m u)$, is defined by

$$
\langle \text{Div}_\alpha(D^m u), \psi \rangle := \begin{cases} 
\int_{\Omega} \psi(x)dx, & r = 0; \\
\lim_{k \to \infty} \int_{\Omega} M_{\alpha}^\beta(D^m u_k)\psi dx, & 1 \leq r \leq q 
\end{cases}
$$

(3.11)

for any $\psi \in C_0^\infty(\Omega)$ and any sequence $\{u_k\}_{k=1}^\infty \subset C^m(\overline{\Omega},\mathbb{R}^n)$ such that $u_k \to u$ in $W^{m-\frac{m}{q},q}(\Omega,\mathbb{R}^n)$.

Obviously this quantity is well-defined since Lemma 3.2 and the fact that $C^m(\overline{\Omega},\mathbb{R}^n)$ is dense in $W^{m-\frac{m}{q},q}(\Omega,\mathbb{R}^n)$.

**Proof of Theorem 1.1.** It is clear that Theorem 1.1 is a consequence of Lemma 3.2 and Definition 3.3. \hfill \square

According to the trace theory and the approximate theorem, we obtain a fundamental representation of the distributional $m$th Jacobian minors in $W^{m-\frac{m}{q},q}$.

**Proposition 3.4.** Let $u \in W^{m-\frac{m}{q},q}(\Omega,\mathbb{R}^n)$ with $2 \leq q \leq n$. For any $0 \leq r \leq q$, $\beta \in I(r,n)$ and $\alpha = (\alpha^1,\alpha^2,\cdots,\alpha^n)$ with $\alpha^j \in I(r,N)$,

$$
\int_{\Omega} \text{Div}_\alpha(D^m u)\psi dx = \sum_{I \in I(\alpha)} (-1)^m \sigma(\tilde{\alpha} - I,I) \int_{\Omega \times [0,1)^m} M_{\tilde{\alpha} - I}(D^m U)\partial_1 \Psi d\bar{x}
$$

for any extensions $U \in W^{m,q}(\Omega \times [0,1)^m,\mathbb{R}^n)$ and $\Psi \in C_0^\infty(\Omega \times [0,1)^m)$ of $u$ and $\psi$, respectively.

Note that the $m$-dimensional matrix $D^m u$ is symmetric if $u \in C^m(\Omega)$, i.e., $(D^m u)^{T(i,j)} = D^m u$ for any $1 \leq i < j \leq m$. An argument similar to the one used in Lemma 3.1 and 3.2 show that

**Corollary 3.5.** Let $u \in W^{m-\frac{m}{q},q}(\Omega)$ with $2 \leq q \leq N$ and $m \geq 2$. For any $0 \leq r \leq q$ and $\alpha = (\alpha^1,\alpha^2,\cdots,\alpha^n)$ with $\alpha^j \in I(r,N)$, Then the $m$th Jacobian $\alpha$-minor operator $u \mapsto M_{\alpha}(D^m u) : C^m(\Omega) \to \mathcal{D}'(\Omega)$ can be extended uniquely as a continuous mapping $u \mapsto \text{Div}_\alpha(D^m u) : W^{m-\frac{m}{q},q}(\Omega) \to \mathcal{D}'(\Omega)$. Moreover for all $u,v \in W^{m-\frac{m}{q},q}(\Omega)$, $\psi \in C_0^\infty(\Omega,\mathbb{R})$ and $1 \leq r \leq q$, we have

$$
|\langle \text{Div}_\alpha(D^m u) - \text{Div}_\alpha(D^m v), \psi \rangle| \leq C_{r,q,N,\Omega} \|u - v\|_{W^{m-\frac{m}{q},q}} \left( \|u\|_{W^{m-\frac{m}{q},q}}^{r-1} + \|v\|_{W^{m-\frac{m}{q},q}}^{r-1} \right) \|D^m \psi\|_{L^\infty},
$$

(3.12)

where the constant depending only on $r,q,N$ and $\Omega$. In particular, the distributional minor $\text{Div}_\alpha(D^m u)$ can be expressed as

$$
\int_{\Omega} \text{Div}_\alpha(D^m u)\psi dx = \sum_{I \in I(\alpha)} (-1)^m \sigma(\tilde{\alpha} - I,I) \int_{\Omega \times [0,1)^m} M_{\tilde{\alpha} - I}(D^m U)\partial_1 \Psi d\bar{x}
$$

for any extensions $U \in W^{m,q}(\Omega \times [0,1)^m)$ and $\Psi \in C_0^\infty(\Omega \times [0,1)^m)$ of $u$ and $\psi$, respectively.
4 The optimality results in fractional Sobolev spaces

In this section we establish the optimality results of Theorem 1 in the framework of spaces $W^{s,p}$. Before proving the main results, we state some interesting consequences (see [4, Theorem 1 and Proposition 5.3]):

Lemma 4.1. For $0 \leq s_1 < s_2 < \infty$, $1 \leq p_1, p_2, p \leq \infty$, $s = \theta s_1 + (1 - \theta) s_2$, $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2}$ and $0 < \theta < 1$, the inequality
\[ \|f\|_{W^{s,p}(\Omega)} \leq C\|f\|^\theta_{W^{s_1,p_1}(\Omega)}\|f\|^{1 - \theta}_{W^{s_2,p_2}(\Omega)}. \]
holds if and only if the following condition fails
\[ s_2 \geq 1 \text{ is an integer, } p_2 = 1 \text{ and } s_2 - s_1 \leq 1 - \frac{1}{p_1}. \]

Proposition 4.2. The following equalities of spaces holds:

(i) $W^{s,p}(\Omega) = F^{s,p,p}(\Omega)$ if $s > 0$ is a non-integer and $1 \leq p \leq \infty$.

(ii) $W^{s,p}(\Omega) = F^{s,p,2}(\Omega)$ if $s \geq 0$ is an integer and $1 < p < \infty$.

Remark 4.3. The definition of Triebel-Lizorkin spaces $F^{s,p,q}$ can be seen in [4, 19].

Remark 4.4. If $1 < r \leq N$, according to the embedding properties of the Triebel-Lizorkin spaces $F^{s,p,q}$, see e.g. [19, page 196], and Proposition 4.2, we consider all possible cases:

(i) $s - m + \frac{m}{r} > \max\{0, N - \frac{N}{r}\}$, then the embedding $W^{s,p}(\Omega) \subset W^{m-\frac{m}{r},r}(\Omega)$ holds;

(ii) $s - m + \frac{m}{r} < \max\{0, N - \frac{N}{r}\}$, the embedding fails;

(iii) $s - m + \frac{m}{r} = \max\{0, N - \frac{N}{r}\}$, there are three sub-cases:

(a) if $p \leq r$, then the embedding $W^{s,p}(\Omega) \subset W^{m-\frac{m}{r},r}(\Omega)$ holds;

(b) if $p > r$ and $m - \frac{m}{r}$ integer, the embedding $W^{s,p}(\Omega) \subset W^{m-\frac{m}{r},r}(\Omega)$ holds;

(c) if $p > r$ and $m - \frac{m}{r}$ non-integer, the embedding fails.

In order to solve the optimality results, we just consider three cases:

(1) $1 < p \leq r, s + \frac{m}{r} < m + \frac{N}{p} - \frac{N}{r}$;

(2) $1 < r < p, 0 < s < m - \frac{m}{r}$;

(3) $1 < r < p, s = m - m/r$ non-integer.

Without loss of generality, one may assume that $n = N$, $(-8, 8)^N \subset \Omega$, and $\alpha' = (\alpha', \cdots, \alpha')$ with $\alpha' = (1, 2, \cdots, r)$. First we establish the optimality results in case $1 < r < p, 0 < s < m - \frac{m}{r}$.

Proposition 4.5. Let $m, r$ be integers with $1 < r \leq n$, $p > r$ and $0 < s < m - \frac{m}{r}$. Then there exist a sequence $\{u_k\}_{k=1}^{\infty} \subset C^m(\Omega, \mathbb{R}^N)$ and a function $\psi \in C^\infty_c(\Omega)$ such that
\[ \lim_{k \to \infty} \|u_k\|_{s,p} = 0, \quad \lim_{k \to \infty} \int_\Omega M_{\alpha'}(D^m u_k) \psi dx = \infty. \]
Proof. For any integer \( k \), we define \( u_k : \Omega \to \mathbb{R}^N \) as

\[
\begin{align*}
  u_k^i(x) &= k^{-\rho} \sin(kx_i), \quad 1 \leq i \leq r - 1; \\
  u_k^i(x) &= 0, \quad r < i \leq N
\end{align*}
\]

and

\[
  u_k^i(x) = k^{-\rho}(x_r)^m \prod_{j=1}^{r-1} \sin \left( \frac{m\pi}{2} + kx_j \right).
\]

Where \( \rho \) is a constant such that \( s < \rho < m - \frac{m}{r} \). Since \( \|D^{[s]+1}u_k\|_{L^\infty} \leq Ck^{[s]+1-\rho} \) and \( \|u_k\|_{L^\infty} \leq Ck^{-\rho} \), it follows that

\[
\|u_k\|_{s,p} \leq C \|u_k\|_{L^{[s]+1}}^{\theta} \leq C k^{s-\rho}.
\]

Let \( \psi \in C_c^\infty(\Omega) \) be such that \( \psi(x) = \prod_{i=1}^N \psi'(x_i) \), with \( \psi' \in C_c^1((0,\pi)) \), \( \psi' \geq 0 \) and \( \psi' = 1 \) in \( \left( \frac{1}{4}\pi, \frac{3}{4}\pi \right) \).

Then

\[
\int_\Omega M^\alpha_{\psi'}(D^m u_k)\psi dx \geq m! \int_{(\frac{1}{4}\pi, \frac{3}{4}\pi)^N} k^{mr-\rho m-\rho} \prod_{j=1}^{r-1} \sin \left( \frac{m\pi}{2} + kx_j \right) dx = C k^{mr-\rho m}.
\]

Hence the conclusion (4.2) holds.

Next we establishing the optimality results in case \( 1 < r < p, s = m - \frac{m}{r} \) non-integer by constructing a lacunary sum of atoms, which is inspired by the work of Brezis and Nguyen [5].

Proposition 4.6. Let \( m, r \) be integers with \( 1 < r \leq \frac{n}{p}, p > r \) and \( s = m - \frac{m}{r} \) non-integer. Then there exist a sequence \( \{u_k\}_{k=1}^\infty \subset C^m(\Omega, \mathbb{R}^N) \) and a function \( \psi \in C_c^\infty(\Omega) \) satisfying the conditions (4.2).

Proof. Fix \( k >> 1 \). Define \( v_k = (v_k^1, \ldots, v_k^N) : \Omega \to \mathbb{R}^N \) as follows

\[
\begin{align*}
  v_k^i(x) &= \sum_{l=1}^k \frac{1}{n_l(l+1)^{\rho}} \sin(n_l x_i), \quad 1 \leq i \leq r - 1; \\
  v_k^i(x) &= (x_r)^m \sum_{l=1}^k \frac{1}{n_l(l+1)^{\rho}} \prod_{j=1}^{r-1} \sin \left( \frac{m\pi}{2} + n_l x_j \right), \quad i = r; \\
  v_k^i(x) &= 0, \quad r + 1 \leq i \leq N.
\end{align*}
\]

Where \( n_l = k^{\frac{2}{\rho}} s^l \) for \( 1 \leq l \leq k \). Let \( \psi \in C_c^\infty(\Omega) \) be defined as (4.3). We claim that

\[
\|v_k\|_{s,p} \leq C, \quad \int_\Omega M^\alpha_{\psi'}(D^m v_k)\psi dx \geq C \ln k,
\]

where the constant \( C \) is independent of \( k \).

Assuming the claim holds, we deduce \( u_k = (\ln k)^{-\frac{1}{\rho}} v_k \) and \( \psi \) satisfies the conditions (4.2). Hence it remains to prove (4.4).
On the one hand

\[ M_{\alpha'}^r(D^m v_k) = \left\{ \prod_{l=1}^{r-1} \left( \sum_{l_i=1}^{k} \frac{n_l}{(l_i+1)^2} \sin \left( \frac{m\pi}{2} + n_l x_i \right) \right) \right\} \times \left( m! \sum_{l_r=1}^{k} \frac{1}{n_r (l_r+1)^2} \prod_{j=1}^{r-1} \sin \left( \frac{m\pi}{2} + n_r x_j \right) \right) \]

\[ = m! \sum_{(l_1, \ldots, l_r) \in G} \frac{1}{n_r (l_r+1)^2} \prod_{i=1}^{r-1} \left( \frac{n_l}{(l_i+1)^2} \sin \left( \frac{m\pi}{2} + n_l x_i \right) \sin \left( \frac{m\pi}{2} + n_l x_i \right) \right) \]

\[ + m! \sum_{l=1}^{k} \frac{1}{l+1} \prod_{i=1}^{r-1} \sin^2 \left( \frac{m\pi}{2} + n_l x_i \right), \]

where

\[ G = \{(l_1, \ldots, l_r) \mid (l_1, \ldots, l_r) \neq (l, \ldots, l) \text{ for } l, l_1, \ldots, l_r = 1, \ldots, k\}. \]

Hence

\[ \int_{\Omega} M_{\alpha'}^r(D^m v_k) \psi \, dx \geq C \sum_{l=1}^{k} \frac{1}{l+1} \int_{(\pi, \frac{3\pi}{2})} \prod_{i=1}^{r-1} \sin^2 \left( \frac{m\pi}{2} + n_l x_i \right) \, dx - CI, \]

where

\[ I := \left| \int_{\Omega} \psi(x) \sum_{(l_1, \ldots, l_r) \in G} \frac{1}{n_r (l_r+1)^2} \prod_{i=1}^{r-1} \left( \frac{n_l}{(l_i+1)^2} \sin \left( \frac{m\pi}{2} + n_l x_i \right) \sin \left( \frac{m\pi}{2} + n_l x_i \right) \right) \, dx \right|. \]

Since \( n_l = k \pi^2 8^l \), it follows that

\[ \frac{n_l}{n_j} \leq |n_l - n_j| \text{ for any } l, l_j = 1, \ldots, k \text{ with } l \neq l_j, \]

\[ \min_{i \neq j} |n_l - n_j| \geq k^{m(r-1)} \]

and

\[ \{n_l \mid l = 1, \ldots, k\} \cap \{z \in \mathbb{R} \mid 2^{n-1} \leq |z| < 2^n\} \text{ has at most one element for any } n \in \mathbb{N}. \]

For any \( (l_1, \ldots, l_r) \in G \), there exists \( 1 \leq i_0 \leq r-1 \) such that \( l_{i_0} \neq l_r \), it follows from (4.3), (4.7) and (4.8) that

\[ \left| \frac{1}{n_r (l_r+1)^2} \int_{\Omega} \psi(x) \prod_{i=1}^{r-1} \left( \frac{n_l}{(l_i+1)^2} \sin \left( \frac{m\pi}{2} + n_l x_i \right) \sin \left( \frac{m\pi}{2} + n_l x_i \right) \right) \, dx \right| \]

\[ \leq \frac{C}{n_r (l_r+1)^2} \prod_{i=1}^{r-1} \frac{1}{n_l (l_i+1)^2} \left| \int_0^\pi \psi(x) \sin \left( \frac{m\pi}{2} + n_l x_i \right) \sin \left( \frac{m\pi}{2} + n_l x_i \right) \, dx \right| \]

\[ \leq C \prod_{i=1}^{r-1} \frac{1}{n_l (l_i+1)^2} \min \{|n_l - n_r|, 1\} \|D^m \psi\|_{L^\infty} \]

\[ \leq \frac{C}{|n_{i_0} - n_r|^m} \]

\[ \leq C k^{-r}. \]

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Combine with (1.6), we find
\[
\int_{\Omega} M_{\alpha}^v(D^m v_k) \psi dx \geq C \sum_{l=1}^k \frac{1}{l + 1} - C, \tag{4.11}
\]
which implies the second inequality of (1.4). On the other hand, in order to prove the first inequality of (4.4), it is enough to show that
\[
\|v_k^r\|_{s,p} \leq C, \tag{4.12}
\]
where \(v_k^r := (v_k^1, v_k^2, \cdots, v_k^{r-1}, \frac{v_k^r}{(x_r)^m})\). In fact, the Littlewood-Paley characterization of the Besov space \(B_{p,s}^r([0, 2\pi]^N)\) (e.g. [19]) implies that
\[
\|v_k^r\|_{s,p} \leq C \left( \|v_k^r\|_{L^p([0, 2\pi]^N)}^p + \sum_{j=1}^{\infty} 2^{sjp} \|T_j(v_k^r)\|_{L^p([0, 2\pi]^N)}^p \right)^{\frac{1}{p}}. \tag{4.13}
\]
Here the bounded operators \(T_j : L^p \to L^p\) are defined by
\[
T_j \left( \sum a_n e^{in \cdot x} \right) = \sum_{2^j |n| < 2^{j+1}} \left( \rho \left( \frac{|n|}{2j+1} \right) - \rho \left( \frac{|n|}{2j} \right) \right) a_n e^{in \cdot x},
\]
where \(\rho \in C_c^\infty(\mathbb{R})\) is a suitably chosen bump function. Then we have
\[
\|T_j(v_k^r)\|_{L^p([0, 2\pi]^N)}^p \leq C_p \sum_{l=1}^k \frac{1}{n_l^{sp}(l + 1)^{\frac{sp}{p}}} \|T_j(g_l, k)\|_{L^p([0, 2\pi]^N)}^p, \tag{4.14}
\]
where \(g_l, k = (\sin(n_1 x_1), \cdots, \sin(n_l x_{r-1}), \prod_{j=1}^{r-1} \sin(\frac{m r}{2} + n_l x_j))\). Indeed, since \(\sin(n_l x_i) = \frac{1}{2r}(e^{in_i x_1} - e^{-in_i x_1})\), \(g_l, k\) can be written as
\[
g_l, k(x) = \sum_{\varepsilon \in \{-1, 0, 1\}^{r-1}} a_\varepsilon e^{n_l \varepsilon \cdot \hat{x}},
\]
where \(\hat{x} = (x_1, \cdots, x_{r-1})\), \(|a_\varepsilon| \leq 1\) for any \(\varepsilon\). Set
\[
S(j, l) = \{ \varepsilon \in \{-1, 0, 1\}^{r-1} \mid 2^{j-1} \leq n_l |\varepsilon| < 2^{j+2} \}
\]
and
\[
\chi(j, l) = \begin{cases} 1 & S(j, l) \neq \emptyset \\ 0 & S(j, l) = \emptyset \end{cases}.
\]
Hence
\[
\|T_j(g_l, k)\|_{L^p([0, 2\pi]^N)}^p \leq C_{r,N} \chi(j, l). \tag{4.15}
\]
For any \(j\), if \(S(j, l) \neq \emptyset\), then \(\frac{2^{j-1}}{\sqrt{2}} \leq n_l < 2^{j+2}\), which implies that \(\sum_{l=1}^k \chi(j, l) < \left[ \frac{\log_2(r-1)}{6} \right] + 1\). Thus, applying (4.13), (4.14), and (4.15), we have
\[
\|v_k^r\|_{s,p} \leq C_{p,s,N,r} \left( \|v_k^r\|_{L^p([0, 2\pi]^N)}^p + \sum_{j=1}^{\infty} \sum_{l=1}^k \frac{2^{sjp}}{n_l^{sp}(l + 1)^{\frac{sp}{p}}} \chi(j, l) \right) \leq C_{p,s,N,r} \left( \|v_k^r\|_{L^p([0, 2\pi]^N)}^p + \sum_{l=1}^k \frac{1}{(l + 1)^{\frac{sp}{p}}} \left( \sum_{j=1}^{\infty} \chi(j, l) \right) \right), \tag{4.16}
\]
which implies (4.12) since \(\sum_{j=1}^{\infty} \chi(j, l) \leq \left[ \frac{\log_2(r-1)}{2} \right] + 4\) for any \(l\). \qed
Proof of Theorem 1.3. Clearly Theorem 1.3 is a consequence of Proposition 4.5 and 4.6 as explained in Remark 4.4.

Next we pay attention to the optimality results in case $1 < p \leq r$, $s + \frac{m}{r} < m + \frac{N}{p} - \frac{N}{r}$.

Proposition 4.7. Let $m, r$ be integers with $1 < p \leq r \leq n$ and $s + \frac{m}{r} < m + \frac{N}{p} - \frac{N}{r}$. If there exist a function $g \in C_c^\infty(B(0, 1), \mathbb{R}^n)$, $\beta \in I(r, n)$ and $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^m)$ with $\alpha^j \in I(r, N)$ such that

$$\int_{B(0, 1)} M^\alpha_k(D^m g(x)) |x|^m dx \neq 0. \quad (4.17)$$

Then there exist a sequence $\{u_k\}_{k=1}^\infty \subset C^m(\Omega, \mathbb{R}^N)$ and a function $\psi \in C_c^\infty(\Omega)$ satisfying the conclusions (1.3).

Proof. For any $0 < \varepsilon << 1$ we set

$$u_{\varepsilon} = \varepsilon^\theta g\left(\frac{x}{\varepsilon}\right), \quad (4.18)$$

where $\rho$ is a constant such that $s - \frac{N}{p} < \rho < m - \frac{N}{r} - \frac{m}{r}$.

On the one hand, Lemma 4.1 implies that

$$||u_{\varepsilon}||_{s,p} \leq C ||u_{\varepsilon}||_{L^p}^{\theta} ||u_{\varepsilon}||_{[s]+1,p}^{1-\theta} \leq C \varepsilon^{\theta + \frac{s}{[s]+1}} ||\nabla||_{L^p}^{\theta} ||D^{[s]+1}g||_{L^p}^{1-\theta}, \quad (4.19)$$

where $\theta = \frac{[s]+1-s}{[s]+1}$. On the other hand, let $\psi \in C_c^\infty(\Omega)$ be such that $\psi(x) = |x|^m + O(|x|^{m+1})$ as $x \to 0$. Then

$$\int_{\Omega} M^\alpha_k(D^m u_{\varepsilon}) \psi dx = \varepsilon^{\rho r - rm + N} \int_{B(0, 1)} M^\alpha_k(D^m g(x)) \psi(\varepsilon x) dx$$

$$= \varepsilon^{\rho r - rm + N + m} \int_{B(0, 1)} M^\alpha_k(D^m g(x)) |x|^m dx + O(\varepsilon^{\rho r - rm + N + m + 1}). \quad (4.20)$$

Take $\varepsilon = \frac{1}{k}$ and hence the conclusion is proved. \hspace{1cm} \Box

In order to establishing the optimality results in case $1 < p \leq r$, $s + \frac{m}{r} < m + \frac{N}{p} - \frac{N}{r}$, a natural problem is raised whether there exists $g \in C_c^\infty(B(0, 1), \mathbb{R}^N)$ such that the conclusion (4.17) holds. We have positive answers to the problem in case $m = 1$ or 2, see Theorem 1.4 according the following Lemma:

Lemma 4.8. Let $g \in C_c^\infty(B(0, 1))$ be given as

$$g(x) = \int_{0}^{[x]} h(\rho) d\rho \quad (4.21)$$

for any $x \in \mathbb{R}^N$, where $h \in C_c^\infty((0, 1))$ and satisfies

$$\int_{0}^{1} h(\rho) d\rho = 0, \quad \int_{0}^{1} h^r(\rho) \rho^{-r+s-1} d\rho \neq 0.$$

Here $r \geq 2, s \geq 1$ are integers. Then for any $\alpha \in I(r, N)$, we have

$$\int_{B(0, 1)} M^\alpha_k(D^2 g(x)) |x|^s dx \neq 0. \quad (4.22)$$
Proof. It is easy to see that
\[ D^2 g = \frac{1}{|x|^d} (A + B), \]
where \( A = (a_{ij})_{N \times N} \) and \( B = (b_{ij})_{N \times N} \) are \( N \times N \) matrices such that
\[ a_{ij} = h(|x|)|x|^2 \delta_{ij}, \quad b_{ij} = (h'(|x|)|x| - h(|x|)) x_i x_j, \quad i, j = 1, \ldots, N. \]

Using Binet formula and the fact \( \text{rank}(B) = 1 \), one has
\[
M_\alpha^\theta(A + B) = M_\alpha^\theta(A) + \sum_{i \in \alpha} \sum_{j \in \alpha} \sigma(i, \alpha - i) \sigma(j, \alpha - j) b_{ij} M_{\alpha - j}^\theta(A) \\
= h^r(|x|)|x|^{2r} - h^r(|x|)|x|^{2r-2} \sum_{i \in \alpha} x_i^2 + h^{r-1}(|x|) h'(|x|)|x|^{2r-1} \sum_{i \in \alpha} x_i^2,
\]

Hence
\[
\int_{B(0,1)} M_\alpha^\theta(D^2 g)|x|^s \, dx = \int_{B(0,1)} |x|^{-3r+s} M_\alpha^\theta(A + B) \, dx = I - II + III,
\]
where
\[
I := \int_{B(0,1)} h^r(|x|)|x|^{-r+s} \, dx,
\]
\[
II := \int_{B(0,1)} h^r(|x|)|x|^{-r-2+s} \sum_{i \in \alpha} x_i^2 \, dx,
\]
and
\[
III := \int_{B(0,1)} h^{r-1}(|x|) h'(|x|)|x|^{-r-1+s} \sum_{i \in \alpha} x_i^2 \, dx.
\]

Then integration in polar coordinates gives
\[
III = \frac{r - N - s}{N} 2\pi \prod_{i=1}^{N-2} I(i) \int_0^1 h^r(\rho) \rho^{-r+N+s-1} \, d\rho,
\]
where \( I(i) = \int_0^\pi \sin^i \theta d\theta \). Similarly,
\[
II = \frac{r}{N} 2\pi \prod_{i=1}^{N-2} I(i) \int_0^1 h^r(\rho) \rho^{-r+N+s-1} \, d\rho,
\]
and
\[
I = 2\pi \prod_{i=1}^{N-2} I(i) \int_0^1 h^r(\rho) \rho^{-r+N+s-1} \, d\rho,
\]
which implies \([1.22]\), and then the proof is complete. \( \square \)

**Proof of Theorem 1.4.** Note that if \( m = 2 \) and \( g = (g', \cdots, g') \) with \( g' \in C^2(\Omega) \), then Lemma 2.4 implies
\[
M_\alpha^\theta(D^2 g) = r! M_{\alpha^2}^\theta(D^2 g')
\]
for any \( \alpha = (\alpha^1, \alpha^2), \alpha \in I(r, N) \). Hence Theorem 1.4 is the consequence of Proposition 4.5, 4.6, 4.7 and Lemma 4.8. \( \square \)
In particular, we can give a reinforced versions of optimal results in case $m = 2$.

**Theorem 4.9.** Let $1 < r \leq N$, $1 < p < \infty$ and $0 < s < \infty$ be such that $W^{s,p}(\Omega) \not\subseteq W^{2-\frac{2}{r},p}(\Omega)$. Then there exist a sequence $\{u_k\}_{k=1}^\infty \subset C^m(\Omega)$ and a function $\psi \in C_\infty(\Omega)$ such that

$$\lim_{k \to \infty} \|u_k\|_{s,p} = 0, \quad \lim_{k \to \infty} \int_\Omega M_\alpha^r(D^2 u_k) \psi \, dx = \infty, \quad (4.23)$$

**Proof.** We divide our proof in three case:

**Case 1:** $1 < p \leq r$ and $s + \frac{2}{r} < 2 + \frac{N}{p} - \frac{N}{r}$

Apply Lemma 4.8 and the argument similar to one used in Proposition 4.7.

**Case 2:** $r < p$ and $0 < s < 2 - \frac{2}{r}$

For $k >> 1$, we set

$$u_k := k^{-\rho}x_r \prod_{i=1}^{r-1} \sin^2(kx_i),$$

where $\rho$ is a constant with $s < \rho < 2 - \frac{2}{r}$. According to the facts that $\|u_k\|_{L^\infty} \leq Ck^{-\rho}$ and $\|D^2u_k\|_{L^\infty} \leq Ck^{2-\rho}$, it follows that

$$\|u_k\|_{s,p} \leq C\|u_k\|_{L^\frac{2}{p}} \|u_k\|_{L^\frac{2}{p}} \leq Ck^{s-\rho}.$$

On the other hand, Let $\psi \in C_\infty(\Omega)$ be defined as (4.3), the (4.1) in [1, Proposition 4.1] implies that

$$\left|\int_\Omega M_\alpha^r(D^2 u_k) \psi \, dx\right| \geq \int_\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)^N M_\alpha^r(D^2 u_k) \, dx$$

$$\geq k^{2r-2-\rho} 2^r \int_\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)^N x_r^{r-2} \left(\prod_{i=1}^{r-1} \sin(kx_i)\right)^{2r-2} \left(\sum_{j=1}^{r-1} \cos^2(kx_j)\right) \, dx$$

$$= Ck^{2r-2-\rho}. \quad (4.24)$$

**Case 3:** $2 < r < p$ and $s = 2 - \frac{2}{r}$

For any $k \in \mathbb{N}$ with $k \geq 2$, define $u_k$ with

$$u_k(x) = \frac{1}{(\ln k)^{2r}} x_r \sum_{i=1}^{k} \frac{1}{n_i^{2-\frac{2}{r}}} \prod_{i=1}^{r-1} \sin^2(n_i x_i) \quad x \in \mathbb{R}^N,$$

where $n_i = k^{\alpha_i}$. Let $\psi \in C_\infty(\Omega)$ be defined as (4.3). The argument similar to the one used in [1, Proposition 5.1] shows that

$$\|u_k\|_{W^{s,p}(\Omega)} \leq C\|u_k\|_{W^{s,p}(0,2\pi)^N} \leq C\frac{1}{(\ln k)^{2r}}$$

and

$$\left|\int_\Omega M_\alpha^r(D^2 u_k) \psi \, dx\right| = C \left|\int_{(0,2\pi)^r} M_\alpha^r(D^2 u_k) \prod_{i=1}^{r} \psi(x_i) \, dx_1 \cdots dx_r\right| \geq C(\ln k)^{\frac{1}{2}}. \quad \square$$

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