SESHADRI STRATIFICATION FOR SCHUBERT VARIETIES AND
STANDARD MONOMIAL THEORY

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To the memory of C. S. Seshadri, a friend and a wonderful teacher

Abstract. The theory of Seshadri stratifications has been developed by the authors with the intention to build up a new geometric approach towards a standard monomial theory for embedded projective varieties with certain nice properties. In this article, we investigate the Seshadri stratification on a Schubert variety arising from its Schubert subvarieties. We show that the standard monomial theory developed in [32] is compatible with this new strategy.

1. Introduction

According to Seshadri, standard monomial theory (for short SMT) deals with the construction of nice bases of finite-dimensional representations of a semi-simple algebraic group $G$, and more generally of what are called Demazure modules in a symmetrizable Kac-Moody setup, see [44]. The term standard monomial theory might be a bit misleading because there is no definition of what a standard monomial theory really is. Despite this ambiguity, the idea of a standard monomial theory stimulated a wealth of research projects connecting the geometry of flag varieties with many aspects in representation theory, K-theory and algebraic combinatorics.

Already in [29], see also [44], Seshadri, Musili and Lakshmibai pointed out the close connection between a nice filtration of the vanishing ideal $I(\partial X(\tau))$ in $O_{X(\tau)}$, standard monomial theory, and their indexing system by admissible pairs. Here $X(\tau)$ is a Schubert variety in a (partial) flag variety $G/Q$ for a maximal parabolic subgroup of classical type $Q$, and $\partial X(\tau)$ is the reduced union of all Schubert varieties properly contained in $X(\tau)$. This indication of a connection between standard monomial theory and valuation theory was the starting point for [14]. In ibidem, the second and third author tested out in the simplest (but nontrivial) case, the Grassmann variety, how to join ideas from standard monomial theory and associated semi-toric degenerations [7, 12, 44] together with the theory of Newton-Okounkov bodies [21] and its associated toric degenerations [4]. The next aim was to get a Newton-Okounkov type interpretation of standard monomial theory and its associated combinatorics, the Lakshmibai-Seshadri path model [33].

The task to find a setup like in [14] for a larger class of embedded projective varieties $X \hookrightarrow \mathbb{P}(V)$ was accomplished in [8] in a surprisingly general framework: Seshadri stratifications.

A Seshadri stratification of an irreducible embedded projective variety $X \hookrightarrow \mathbb{P}(V)$ comes with a collection of projective subvarieties $X_p$ in $X$, indexed by a finite set $A$. 
The set $A$ inherits naturally a partial order $\leq$ defined by: for $p, q \in A$, $p \leq q$ if and only if $X_p \subseteq X_q$. In addition, one needs for each $p \in A$ a homogeneous function $f_p \in \text{Sym}(V^*)$ of degree larger or equal to one. These subvarieties and functions have to satisfy certain compatibility conditions, see Definition 2.1. The example studied in this article is the case of a (generalized) Schubert variety embedded in the projective space over a Demazure module. The subvarieties are the Schubert subvarieties which are contained in the given one, and the functions are the extremal weight vectors in the dual of the Demazure module (see Section 3).

Fix a total order $\geq_t$ on $A$ (compatible with the given partial order, see Section 4.3). The existence of a Seshadri stratification on $X$ implies the existence of a quasi-valuation (which depends on the choice of $\geq_t$) with values in $\mathbb{Q}^A$. The quasi-valuation in turn defines a filtration of the homogeneous coordinate ring $\mathbb{K}[X]$, which can be used to construct a semi-toric degeneration of $X$ (see Section 4). The degenerate variety is a reduced union of equidimensional projective toric varieties, one irreducible component for each maximal chain in $A$. The geometry of the degenerate variety is completely governed by the image $\Gamma \subseteq \mathbb{Q}^A$ of the quasi-valuation, which is a finite union of finitely generated monoids, one for each maximal chain in $A$. This is the reason why $\Gamma$ is called the associated fan of monoids.

Now having this tool, there are several natural tasks to accomplish. In this sense the present paper is one in a series of articles. Particularly nice Seshadri stratifications are those which are balanced and normal [8], it is described in ibidem how to construct a standard monomial theory in this case. Other aspects of normal stratifications are discussed in [10]: for example, a Gröbner basis of the defining ideal of the semi-toric degeneration of the variety can be lifted to define the embedded projective variety.

Another task is to investigate the compatibility with already established structures like Hodge algebras [12], LS-algebras [7], or the SMT developed in [32]. The case of the Grassmann variety and its classical standard monomial theory was studied in [14], which was the starting point for us to develop the notion of a Seshadri stratification. The close connection of Seshadri stratifications with LS-algebras is studied in [9].

The connection between the SMT in [32] and Seshadri stratifications is one topic of this article. The approach we take here is a mixture of the methods developed in [8] and [32]. We start by defining a basis $\mathcal{B}(V(\lambda)_r)$ of $V(\lambda)^*_r$, the dual of the Demazure module in which we have embedded $X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_r)$ the Schubert variety. This basis has the following nice property: if $\mathcal{V}$ is a quasi-valuation associated to the Seshadri stratification (which depends on the choice of a total order $\geq_t$), then the value of $\mathcal{V}$ on the elements in $\mathcal{B}(V(\lambda)_r)$ is independent of the choice of the total order!

This is the starting point for the SMT in Section 7. But note that the basis $\mathcal{B}(V(\lambda)_r)$ is far from being unique. Roughly speaking: we define a filtration indexed by a partially ordered set such that the associated graded components, which are called leaves, are one-dimensional. Choosing a nice basis $\mathcal{B}(V(\lambda)_r)$ corresponds to picking representatives of these leaves. It would be interesting to get a representation theoretic interpretation of this combinatorially defined collection of subspaces.

Now the connection with [32] is the fact that the basis of $V(\lambda)^*_r$ constructed in ibidem is an example of a such a nice basis $\mathcal{B}(V(\lambda)_r)$. This is proved in Appendix III,
Section 13. To see that the SMT for generalized flag varieties developed in [32] can be viewed as a special example for the theory developed in [8], it remains to see that the notion of a standard monomial is the same. The proof of this purely combinatorial part can be found in Appendix I, Section 11. Here we show that the fan of monoids coming from our Newton-Okounkov type construction is a reincarnation of the path model given by the LS-paths [33]. So one gets in addition the desired geometric interpretation of the LS-path model as sequences of renormalized vanishing multiplicities of certain functions.

Another task is to use the new algebro-geometric tool to construct in a purely geometric way a standard monomial theory for generalized flag varieties, avoiding the representation theoretic tools used in [32]. This aspect is discussed in [11], where a standard monomial theory for Schubert varieties is constructed exploiting: the geometry of the Seshadri stratifications of Schubert varieties by their Schubert subvarieties and the combinatorial LS-path character formula for Demazure modules. In the same article, the general theory of Seshadri stratifications is somehow improved by: (1) allowing arbitrary linearizations in the definition of the quasi-valuation (see Section 2.6 in [11]) and (2) weakening the definition of balanced Seshadri stratification (see Section 2.9 in [11]).

We give now a short summary of the results of this article. The wording of the following theorems is rather informal, the precise formulations can be found later in this article:

**Theorem A.** The Seshadri stratification of an embedded Schubert variety \( X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)) \) arising from its Schubert subvarieties is balanced and normal (see Section 7).

The main part of this article is dedicated to the proof of the following theorem. As in the case of Newton-Okounkov theory, one of the main problems is to get an explicit description of the semigroups associated to the (quasi-)valuation.

**Theorem B.**
1) The associated quasi-valuation \( V \) on the homogeneous coordinate ring \( \mathbb{K}[\tilde{X}(\tau)] \) has as fan of monoids a lattice type realization of the LS-path model (Section 7).
2) The standard monomial theory for \( X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)) \) constructed in [32] is an example of a standard monomial theory for a balanced and normal Seshadri stratification in [8] (Section 8).

Part 2) of the above theorem is proved also in [9] in the algebraic context of LS-algebras.

By the general theory of Seshadri stratifications one recovers known results on semitoric degenerations of Schubert varieties and their properties [7], and polytopal realizations of the LS-path models [13] turn up as Newton-Okounkov simplicial complexes, see Sections 8, 9 and 10:

**Theorem C.**
1) There exists a flat degeneration of \( X(\tau) \) to \( X_0 \), a union of projective toric varieties. Moreover, the special fibre \( X_0 \) is equidimensional, it is isomorphic to \( \text{Proj}(\text{gr}_V^\mathbb{K}[\tilde{X}(\tau)]) \), and its irreducible components are normal toric varieties.
2) Schubert varieties are projectively normal, so one has a standard monomial theory also for the ring of sections \( \bigoplus_{s \geq 0} H^0(X(\tau), \mathcal{L}_{s\lambda}) \).

3) The Newton-Okounkov simplicial complex associated to the quasi-valuation coincides with the polytope with an integral structure defined in [13].

The article is structured as follows: we start in Section 2 with a quick review on the concept of a Seshadri stratification. In Section 3 we present the setup: Schubert varieties embedded into the projective space over a Demazure module for a symmetrizable Kac-Moody group over an algebraically closed field \( \mathbb{K} \). In particular, we show that the stratification by Schubert subvarieties is a Seshadri stratification.

In Section 4 we recall the construction of the associated quasi-valuation in the case of Schubert varieties. To determine in such a general setting an explicit description of the image of the quasi-valuation (the fan of monoids), one needs to make a good guess. The favorite candidate is presented in Section 5. The connection with the path model and a conjecture by Lakshmibai is presented in Appendix I (Section 11).

We present in Section 6 a new approach towards a construction of a basis of the dual \( V(\lambda)_\tau^* \) of the Demazure module \( V(\lambda)_\tau \), these elements are called path vectors. Note that the notion of a path vector in this article is more general than that in [32]. So for the convenience of the reader we give in Appendix II and III (Sections 12 and 13) a detailed and adapted review of the constructions in [32]. We have rewritten the formulation and the proofs in a way more adjusted to the point of view of this article.

As a consequence we describe in Section 7 a standard monomial theory. In addition, we show that the Seshadri stratification is balanced and normal. In Section 8 we discuss some applications: Koszul property, Khovanskii basis, compatibility of the standard monomial theory with the strata, compatibility with the SMT in [32], straightening relations.

In Section 9 we give (yet another) proof of the projective normality of embedded Schubert varieties. The associated Newton-Okounkov simplicial complex is described in Section 10. Here we recover the polytopes with integral structure constructed by Dehy in [13] in connection with the LS-path model.

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We learned so much of what we know about flag and Schubert varieties from Seshadri. The stimulating, humorous and encouraging discussions with him have always been a special event.

2. Seshadri stratifications

Throughout the article we fix \( \mathbb{K} \) to be an algebraically closed field.

For a partially ordered set \((A, \leq)\) (in the following called a poset) and \( p \in A \), we denote by \( A_p \) the subset \( \{ q \in A \mid q \leq p \} \); \((A_p, \leq)\) is again a poset.

Let \( V \) be a finite dimensional vector space over \( \mathbb{K} \). The hypersurface defined as the vanishing set of a homogeneous polynomial function \( f \in \text{Sym}(V^*) \) will be denoted by \( H_f := \{ [v] \in \mathbb{P}(V) \mid f(v) = 0 \} \).
A variety \(X\) is always assumed to be irreducible. The affine cone in \(V\) over a projective variety \(X \subseteq \mathbb{P}(V)\) will be denoted by \(\hat{X}\).

2.1. Seshadri stratifications. We start with a brief recollection on Seshadri stratifications, as introduced in [8].

Let \(X \subseteq \mathbb{P}(V)\) be an embedded projective variety with graded homogeneous coordinate ring \(R := \mathbb{K}[\hat{X}]\). Let \(X_p, p \in A\), be a collection of projective subvarieties \(X_p\) in \(X\), indexed by a finite set \(A\). The set \(A\) inherits naturally a partial order \(\leq\) defined by: for \(p, q \in A\), \(p \leq q\) if and only if \(X_p \subseteq X_q\). We assume that there exists a unique maximal element \(p_{\text{max}}\) in \(A\) with \(X_{p_{\text{max}}} = X\).

For each \(p \in A\), we fix a homogeneous function \(f_p \in \text{Sym}(V^*)\) of degree larger or equal to one.

**Definition 2.1 ([8])**. The collection of subvarieties \(X_p\) and homogeneous functions \(f_p\) for \(p \in A\) is termed a Seshadri stratification, if the following conditions are satisfied:

- (S1) the projective varieties \(X_p, p \in A\), are smooth in codimension one; if \(q < p\) is a covering relation in \(A\), then \(X_q \subseteq X_p\) is a codimension one subvariety;
- (S2) for any \(p \in A\) and any \(q \not\leq p\), the function \(f_q\) vanishes on \(X_p\);
- (S3) for \(p \in A\), the set-theoretical intersection satisfies

\[ \mathcal{H}_{f_p} \cap X_p = \bigcup_{q \text{ covered by } p} X_q. \]

In a Seshadri stratification, the functions \(f_p\) are called extremal functions.

**Remark 2.2.** In the axiom (S3), if \(p \in A\) is a minimal element, there is no element covered by \(p\). Hence the right hand side is the empty set.

Throughout the paper, the following lemma will be used often without mention.

**Lemma 2.3 ([8])**. If the collection of subvarieties \(X_p\) of \(X\) and homogeneous functions \(f_p\) for \(p \in A\) defines a Seshadri stratification, then

i) the function \(f_p\) does not identically vanish on \(X_p\),

ii) all maximal chains in \(A\) have the same length, which coincides with \(\dim X\). In particular, the poset \(A\) is graded.

**Definition 2.4.** Let \(p \in A\). The length \(\ell(p)\) of \(p\) is the length of a (hence any) maximal chain joining \(p\) with a minimal element in \(A\).

According to the above lemma, the length is well-defined and satisfies \(\ell(p) = \dim X_p\).

**Example 2.5 ([8])**. For a fixed \(p \in A\), the collection of varieties \(X_q\) and the extremal functions \(f_q\) for \(q \in A_p\) satisfies the conditions (S1)-(S3), and hence defines a Seshadri stratification for \(X_p \leftrightarrow \mathbb{P}(V)\).

**Remark 2.6.** Later in the article we will consider the affine cones of the subvarieties in a Seshadri stratification. It is useful to extend the notation one step further. For a minimal element \(p \in A\), the affine cone \(\hat{X}_p \cong \mathbb{A}^1\). We set \(\hat{A} := A \cup \{p_{-1}\}\) with \(\hat{X}_{p_{-1}} := \{0\} \in V\). Since the variety \(\hat{X}_{p_{-1}}\) is contained in the affine cone \(\hat{X}_p\) for any minimal element \(p \in A\), the set \(\hat{A}\) inherits a poset structure by requiring \(p_{-1}\) to be the unique minimal element.
2.2. A Hasse diagram with bonds. We associate an edge-coloured directed graph to a Seshadri stratification of a projective variety $X$ consisting of subvarieties $X_p$ and extremal functions $f_p$ for $p \in A$.

The Hasse diagram $G_A$ of the poset $A$ is a directed graph on $A$ whose edges are covering relations, pointing to the larger element.

For a covering relation $p > q$ in $A$, $\hat{X}_q$ is a prime divisor in $\hat{X}_p$. According to (S1), the local ring $O_{\hat{X}_p, \hat{X}_q}$ is a discrete valuation ring. Let $\nu_{p,q} : O_{\hat{X}_p, \hat{X}_q} \setminus \{0\} \to \mathbb{Z}$ be the associated valuation (see also Section 4.1). Let $R_p := \mathbb{K}[\hat{X}_p]$ denote the homogeneous coordinate ring of $X_p$. For $f \in R_p \setminus \{0\}$, the value $\nu_{p,q}(f)$ is the vanishing multiplicity of $f$ in the divisor $\hat{X}_q$. The integer $b_{p,q} := \nu_{p,q}(f_p)$ will be called the bond between $p$ and $q$. By (S3), we have $b_{p,q} \geq 1$.

The Hasse diagram with bonds is the diagram with edges coloured with the corresponding bonds: $q \xrightarrow{b_{p,q}} p$.

Remark 2.7. We extend the construction to the poset $\hat{A}$ (Remark 2.6) and the associated extended Hasse diagram $G_{\hat{A}}$. For a minimal element $p \in A$, the bond $b_{p,p^{-1}}$ is defined to be the vanishing multiplicity of $f_p$ at $\hat{X}_{p^{-1}} = \{0\}$, which coincides with the degree of $f_p$.

2.3. The generic hyperplane stratification. The definition of a Seshadri stratification looks restrictive, but it has been shown in [8]:

Proposition 2.8. Every embedded projective variety $X \subseteq \mathbb{P}(V)$, smooth in codimension one, admits a Seshadri stratification.

The proof uses Bertini’s Theorem [20]. If $X$ is of dimension $r$, then one can find linear functions $f_r, \ldots, f_1 \in V^*$, so that the inductively defined intersections: $X_r = X$, and for $j = 1, \ldots, r - 1$: $X_j = X_{j+1} \cap H_{f_{j+1}}$, are reduced varieties which are moreover smooth in codimension one. Now $X_1$ is a smooth curve, and one can find $f_1 \in V^*$ such that the intersection $X_0 = X_1 \cap H_{f_1}$ is a union of $s = \deg X$ many points. By choosing appropriate functions $f_{0,1}, \ldots, f_{0,s}$ for the (0-dimensional) irreducible components $X_{0,1}, \ldots, X_{0,s}$, one gets in this way a Seshadri stratification for $X$.

3. Seshadri stratification on Schubert varieties

The Seshadri stratification on an embedded flag variety $G/B \hookrightarrow \mathbb{P}(V(\lambda))$ consisting of Schubert subvarieties and the extremal weight functions is briefly discussed in [8]. In this section we introduce these stratifications on Schubert varieties in full generality.

3.1. Let $A$ be a symmetrizable generalized Cartan matrix and let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of the root system. We denote by $s_i \in W$ the simple reflection in the Weyl group associated to the simple root $\alpha_i$. Denote by $G$ the associated maximal Kac-Moody group [36, 37]. This is in general not an algebraic group but an ind-scheme. The group $G$ comes equipped with a (positive) Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. The unipotent radicals of $B$ and its opposite will be denoted by $U$ and $U^-$. The action of $B$ on $G$ from the right is free, which makes it possible to talk about
the flag variety $G/B$ as in the finite type case [37]. We assume the root datum to be simply connected.

Denote by $\Lambda = \Lambda(T)$ the weight lattice and, according to the choice of $B$, let $\Lambda^+$ be the monoid of integral dominant weights. For $\lambda \in \Lambda^+$ let $V_{\mathbb{C}}(\lambda)$ be the irreducible highest weight representation for the complex version $\mathfrak{g}_\mathbb{C}$ of the associated Kac-Moody algebra. Fixing a highest weight vector $v_\lambda \in V_{\mathbb{C}}(\lambda)$, we get an admissible lattice $V_{\mathbb{Z}}(\lambda) \subseteq V_{\mathbb{C}}(\lambda)$ by applying the Kostant integral $\mathbb{Z}$-form $U(\mathfrak{g}_\mathbb{C})_{\mathbb{Z}}$ of the enveloping algebra of $\mathfrak{g}_\mathbb{C}$ to the fixed highest weight vector $v_\lambda \in V_{\mathbb{C}}(\lambda)$. The tensor product $V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{K}$ with the field $\mathbb{K}$ gives the desired $G$-module $V(\lambda)$ of highest weight $\lambda$. The module $V(\lambda)$ is called the Weyl module associated to the dominant weight $\lambda$. See [1, 18, 37] for more information about Weyl modules, their universal property and a realization as the dual representation of the module of global sections $H^0(G/B, \mathcal{L}_\lambda^\vee)$, where the dominant weight $\lambda^*$ is appropriately chosen.

Let $Q \subseteq G$ be the parabolic subgroup generated by $B$ and the root subgroups $U_{-\alpha} \subseteq G$ associated to simple roots $\alpha$ such that $\langle \lambda, \alpha^\vee \rangle = 0$. Denote by $W_Q$ the Weyl group of $Q$. We associate to $\tau \in W/W_Q$ an extremal weight vector $v_\tau \in V_{\mathbb{Z}}(\lambda)$ of weight $\tau(\lambda)$, see (20) for an explicit construction.

According to the choice of $B \subseteq G$ let $\mathfrak{b}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ be a Borel subalgebra, and let $U(\mathfrak{b})_{\mathbb{Z}} \subseteq U(\mathfrak{g})_{\mathbb{Z}}$ be a corresponding Kostant integral $\mathbb{Z}$-form of the enveloping algebra. Let $U(\mathfrak{b})_{\mathbb{K}}$ be the algebra obtained by tensoring with $\mathbb{K}$.

We denote by $V(\lambda)_\tau$ the corresponding Demazure module: it is the linear span $V(\lambda)_\tau = \langle B \cdot v_\tau \rangle$ of the $B$-orbit through $v_\tau$. The Demazure module is finite dimensional, and it is equipped with an action of $B$. More precisely, there exists a normal subgroup $B' \subseteq B$, such that $B'$ acts trivially on $V(\lambda)_\tau$ and $B/B'$ is a finite dimensional affine algebraic group acting on $V(\lambda)_\tau$. The orbit closure $X(\tau) = \overline{B \cdot [v_\tau(\lambda)]} \subseteq \mathbb{P}(V(\lambda)_\tau)$ is the Schubert variety associated to $\tau$. This construction of the Schubert variety is independent of the choice of $\lambda$ [37, 38], i.e. for any dominant weight $\lambda'$ having the same associated parabolic subgroup $Q$, the above construction leads to an isomorphic variety. In particular, the Schubert variety contains only finitely many $B$-orbits.

3.2. **The homogeneous coordinate ring of Schubert varieties.** The following connection between Demazure modules and the homogeneous coordinate ring of Schubert varieties is well known:

**Lemma 3.1.** We have a graded epimorphism of $B$-modules:

$$\bigoplus_{s \geq 0} V(s\lambda)^* \to \mathbb{K}[\hat{X}(\tau)] = \bigoplus_{s \geq 0} \mathbb{K}[\hat{X}(\tau)]_s.$$

**Proof.** For $s \geq 1$, we consider the diagonal embedding of $X(\tau)$ into its $s$-fold product with itself, followed by the Segre embedding:

$$\begin{align*}
X(\tau) &\hookrightarrow X(\tau) \times \cdots \times X(\tau) \\
\mathbb{P}(V(\lambda)_\tau) &\hookrightarrow \mathbb{P}(V(\lambda)_\tau) \times \cdots \times \mathbb{P}(V(\lambda)_\tau) \xrightarrow{\text{Segre}} \mathbb{P}(V(\lambda)^{\otimes s}).
\end{align*}$$

(1)
Every homogeneous function of degree \( s \) on \( V(\lambda)_\tau \) can be written as a composition of the map \( v \mapsto v^\otimes s \) and a linear function on \( V(\lambda)^{\otimes s} \). So we get by restriction a surjective morphism \( (V(\lambda)^{\otimes s})_\tau \to \mathbb{K}[V(\lambda)_\tau]_s \).

Let \( v_\lambda \in V(\lambda) \) and \( v_{s\lambda} \in V(s\lambda) \) be highest weight vectors. One has a natural \( G \)-equivariant morphism \( \phi : V(s\lambda) \to V(\lambda)^{\otimes s} \), sending \( v_{s\lambda} \) to \( v^\otimes s \) and \( v_{\tau(s\lambda)} \) to \( v^\otimes s \) (up to a nonzero scalar multiple). The morphism \( \phi \) induces hence a natural \( B \)-equivariant morphism \( \phi_\tau : V(s\lambda)_\tau \to V(\lambda)^{\otimes s}_\tau \).

The inclusion \( X(\tau) \to \mathbb{P}(V(\lambda)^{\otimes s}) \) described in (1) is \( B \)-equivariant, and it sends \( [v_\tau] \in \mathbb{P}(V(\lambda)_\tau) \) to the element \( [v^\otimes s] \) in \( \mathbb{P}(V(\lambda)^{\otimes s}) \). It follows by the \( B \)-equivariance that the image of the Schubert variety in (1) lies in \( \mathbb{P}(\phi_\tau(V(s\lambda)_\tau)) \subseteq \mathbb{P}(V(\lambda)^{\otimes s}_\tau) \). Since \( V(s\lambda)_\tau \) is by definition the linear span of the \( B \)-orbit \( B.v_{\tau(s\lambda)} \), the \( B \)-equivariance implies that \( \phi_\tau(V(s\lambda)_\tau) \) is indeed the linear span of the cone over the image of the Schubert variety.

The surjective morphism \( (V(\lambda)^{\otimes s})^*_\tau \to \mathbb{K}[V(\lambda)_\tau]_s \) induces hence a surjective \( B \)-equivariant morphism \( V(s\lambda)^*_\tau \to \phi_\tau(V(s\lambda)^*_\tau) \simeq \mathbb{K}[\check{X}(\tau)]_s \).

If \( \mathbb{K} \) is of characteristic zero, then it is well known that the morphism \( \phi_\tau \) is in fact injective. In positive characteristic one can either use Frobenius splitting (see [2, 37, 39, 41, 42]) or the quantum Frobenius splitting and its applications (see [24, 25, 32]) to show that the map \( \phi_\tau \) is always injective. We will use later, usually without mention, the following representation theoretic interpretation of the homogeneous coordinate ring \( \mathbb{K}[\check{X}(\tau)] \) of \( X(\tau) \subseteq \mathbb{P}(V(\lambda)_\tau) \):

**Corollary 3.2.** We have a graded isomorphism of \( B \)-modules:

\[
\mathbb{K}[\check{X}(\tau)] = \bigoplus_{s \geq 0} \mathbb{K}[\check{X}(\tau)]_s \simeq \bigoplus_{s \geq 0} V(s\lambda)^*_\tau.
\]

3.3. Partial orders. Let \( \ell \) be the length function on the Weyl group \( W \). The value \( \ell(w) \) is the minimal length of a reduced decomposition of \( w \) as a product of simple reflections.

We often identify \( W/W_Q \) with the subset \( W^Q \subseteq W \) of representatives of elements of \( W/W_Q \) in \( W \) of minimal length. The Weyl group is naturally endowed with a partial order by viewing the pair consisting of \( W \) and the simple reflections as a Coxeter system. This partial order is called the Bruhat order on \( W \). We get an induced Bruhat order on \( W/W_Q \) via the identification with the subset \( W^Q \subseteq W \). We view the length function \( \ell(\cdot) \) as a function on \( W/W_Q \) as follows: we define \( \ell(\tau) \) for \( \tau \in W/W_Q \) to be \( \ell(\hat{\tau}) \), where \( \hat{\tau} \in W^Q \) is the unique minimal representative of \( \tau \). This is the same as the length in the graded poset \( W/W_Q \).

The partial order and the length function have the following geometric interpretation: for \( \tau \in W/W_Q \), the length \( \ell(\tau) \) is the dimension \( \dim X(\tau) \) of the corresponding Schubert variety, and if \( \kappa \in W/W_Q \) is a second element, then \( X(\kappa) \subseteq X(\tau) \) if and only if \( \kappa \leq \tau \) in the Bruhat order.

3.4. Roots and subgroups. Let \( \Phi \) be the set of real roots of \( G \). Having fixed the Borel subgroup \( B \), we can divide \( \Phi \) into the set of positive real roots \( \Phi^+ \) and negative real roots \( \Phi^- \). Let \( \Delta \) denote the set of simple roots.
For a real root $\beta$ denote by $U_\beta \subseteq G$ the one-dimensional root subgroup corresponding to $\beta$. The parabolic subgroup $Q$ is determined by a subset $Y$ of the simple roots $\Delta$. Let $\Phi_T \subseteq \Phi$ be the subset of real roots spanned by $Y$. The Weyl group $W_Q$ is the subgroup of $W$ generated by the simple reflections $s_\alpha$, $\alpha \in Y$.

Since $\Phi_T$ is stable under $W_Q$, the following conventions make sense: for $\tau \in W/W_Q$ and $\gamma \in \Phi^+$ we write $\tau^{-1}(\gamma) \in \Phi_Q$ if $w^{-1}(\gamma) \in \Phi_Q$ for one, and hence every representative $w \in W$ of $\tau$. Similarly, write $\tau^{-1}(\gamma) \not\in \Phi_Q$ and $\tau^{-1}(\gamma) < 0$ if $w^{-1}(\gamma) \not\in \Phi_Q$ and $w^{-1}(\gamma) < 0$ for one, and hence every representative $w$ of $\tau$ in $W$. For an element $\sigma \in W/W_Q$ set

$$\Phi^+_{\sigma} := \{ \gamma \in \Phi^+ | \sigma^{-1}(\gamma) \not\in \Phi_Q, \ \sigma^{-1}(\gamma) < 0 \}.$$ 

The set $\Phi^+_{\sigma}$ is finite and closed under addition, so the product of the root subgroups is a subgroup $U_{\sigma}$ of $U$. The decomposition $U_{\sigma} = \prod_{\gamma \in \Phi^+_{\sigma}} U_{\gamma}$ as product of root subgroups holds for any chosen ordering of the elements in $\Phi^+_{\sigma}$. The orbit map $U_{\sigma} \to \mathbb{P}(V(\lambda)_{\sigma})$, $u \mapsto u \cdot [v_\sigma]$, is an isomorphism onto the Schubert cell $C(\sigma)$, viewed as a subset of $\mathbb{P}(V(\lambda)_{\sigma})$.

3.5 Smooth in codimension one. Our goal is to endow the embedded Schubert variety $X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_{\tau})$ with a Seshadri stratification and use the associated valuations and quasi-valuations to study the corresponding homogeneous coordinate rings. In this section we start from the requirement of smoothness in codimension one. Such a property is well known, we provide a short proof for completeness.

We fix the Schubert variety $X(\tau)$ in $\mathbb{P}(V(\lambda)_{\tau})$ and let $X(\kappa) \subseteq X(\tau)$ be a Schubert variety of codimension one. There exists a positive real root $\beta$ such that $s_\beta \kappa = \tau$.

**Lemma 3.3.** Let $\tau > \kappa$ be two elements in $W/W_Q$ such that $\ell(\tau) = \ell(\kappa) + 1$. Let $\beta$ be the positive real root such that $s_\beta \kappa = \tau$. The product map

$$U_\kappa \times U_{-\beta} \to \mathbb{P}(V(\lambda)_{\tau}), \quad (u, v) \mapsto uv \cdot [v_\kappa],$$

induces an isomorphism of $U_\kappa \times U_{-\beta}$ with a $T$-stable affine dense subset of $X(\tau)$, containing $\kappa$ as the only $T$-fixed point.

**Proof.** Denote by $U^-_Q$ the subgroup of $G$ generated by the one-parameter subgroups $U_\gamma$ corresponding to negative real roots $\gamma$ which are not in $\Phi_Q$, it is an affine ind-group. The orbit map $\kappa U^-_Q \kappa^{-1} \to G/Q$, $\kappa u \kappa^{-1} \mapsto \kappa u \kappa^{-1} \cdot \kappa$, defines an isomorphism of the subgroup $\kappa U^-_Q \kappa^{-1} \subseteq G$ onto an open $T$-stable affine ind-subvariety in $G/Q$, containing $\kappa$ as the only $T$-fixed point. By definition, $U_\kappa$ is contained in $\kappa U^-_Q \kappa^{-1}$. Since $U_\beta \not\subseteq U_\kappa$, it follows $U_{-\beta} \subseteq \kappa U^-_Q \kappa^{-1}$. So one can regard $U_\kappa \times U_{-\beta}$ as a $T$-stable ($T$-acting by conjugation) affine subspace in $\kappa U^-_Q \kappa^{-1}$, and the restriction of the map $\kappa U^-_Q \kappa^{-1} \to G/Q$ to $U_\kappa \times U_{-\beta}$ induces a $T$-equivariant isomorphism of the latter to an affine subspace in $G/Q$.

Let $\text{SL}_2(\beta)$ be the $\text{SL}_2$-copy in $G$ corresponding to the positive real root $\beta$. Since

$$U_{-\beta} \cdot \kappa = \text{SL}_2(\beta) \cdot \kappa = \text{SL}_2(\beta) \cdot \tau = U_{\beta} \cdot \tau,$$

it follows $U_{-\beta} \cdot \kappa \subseteq X(\tau)$. The Schubert variety $X(\tau)$ is $U$-stable and hence $U_\kappa U_{-\beta} \cdot \kappa$ is an affine subspace in $X(\tau)$ of the same dimension as $X(\tau)$. 

\[\blacksquare\]
Denote the image of $U_\kappa \times U_{-\beta}$ by $C(\tau, \kappa)$, it is a cell. Let $C(\kappa) := U_\kappa \cdot \kappa \subset X(\kappa)$ and $C(\tau) := U_\tau \cdot \tau \subset X(\tau)$ be the usual Schubert cells. By construction one has

$$C(\kappa) \subset C(\tau, \kappa), \text{ and } C(\tau) \cap C(\tau, \kappa) \text{ is dense in } X(\tau).$$

**Remark 3.4.** The product $U_\kappa \times U_{-\beta}$ is isomorphic to $\mathbb{A}^{t(\tau)}$. By choosing a parameterization of the root subgroups of $U_\kappa$ by parameters $t_\gamma, \gamma \in \Phi_\kappa^+$, and $U_{-\beta}$ by $t_\beta$, we get a parameterization of an open and dense subset of $X(\tau)$, which contains the open cell $C(\sigma)$ in $X(\sigma)$. The lemma above implies that a generic element in the image of $U_\kappa \times U_{-\beta}$ in $\mathbb{P}(V(\lambda)_\tau)$ is of the form

$$[d^{[\tau(\lambda), \beta^\vee]} v_{\tau(\lambda)} + \text{sum of weight vectors of weight } \tau(\lambda)].$$

Here $c$ is a nonzero constant and $v_{\tau(\lambda)}$ is a weight vector of weight $\tau(\lambda)$. Denote by $f_\tau \in (V(\lambda)_\tau)^*$ the corresponding dual vector, i.e. $f_\tau$ is a $T$-eigenvector of weight $-\tau(\lambda)$, and $f_\tau(v_{\tau(\lambda)}) = 1$. It follows that $f_\tau$ vanishes on $X(\sigma)$ with multiplicity $|\langle \tau(\lambda), \beta^\vee \rangle|$.

**Corollary 3.5.** Schubert varieties are smooth in codimension one.

**Proof.** The Borel subgroup $B$ acts on $X(\tau)$ with a finite number of orbits. The set of singular points is stable under this action, so we have only to check that the codimension one orbits contain no singular points of $X(\tau)$. But such an orbit contains a $T$-fixed point $\kappa$ so that the associated Schubert variety $X(\kappa)$ is of codimension one in $X(\tau)$. By the above construction, there exists an open and dense cell $C(\tau, \kappa)$ in $X(\tau)$ containing the orbit $B \cdot \kappa$ and meeting the open orbit $B \cdot \tau$ in a dense subset. It follows that the $B$-orbit through $\kappa$ contains only smooth points in $X(\tau)$. \hfill \bullet

Let $\kappa, \tau$ und $\beta$ be as in Lemma 3.3.

**Corollary 3.6.** The following hold:

i) $\beta$ is not in the non-negative real span of $\Phi_\kappa^+$.

ii) The $T$-weight spaces in $V(\lambda)_\tau$ of weight $\kappa(\lambda) + \ell\beta$, $\ell \geq 0$, are at most one-dimensional.

**Proof.** An element $\delta$ in the non-negative real span of $\Phi_\kappa^+$ has the property that $\kappa^{-1}(\delta)$ is a non-negative real span of negative roots, but $\kappa^{-1}(\beta)$ is a positive root. Since $V(\lambda)_\tau = \langle U_\kappa U_{-\beta} \cdot v_\kappa \rangle$ by Lemma 3.3, it follows that all $T$-weight spaces in $V(\lambda)_\tau$ of weight $\kappa(\lambda) + \ell\beta$, $\ell \geq 0$, are at most one-dimensional. \hfill \bullet

### 3.6. A Seshadri stratification for $X(\tau)$

As before, let $Q \supseteq B$ be the standard parabolic subgroup of $G$ associated to $\lambda$, i.e. $Q$ is generated by $B$ and the root subgroups $U_{-\alpha}$ for all simple roots $\alpha$ such that $\langle \lambda, \alpha^\vee \rangle = 0$. Let $\tau \in W/W_Q$, in the following we consider the Schubert variety $X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_\tau)$ embedded in the projective space $\mathbb{P}(V(\lambda)_\tau)$ associated to the Demazure module $V(\lambda)_\tau$.

The set $A_\tau := \{ \sigma \in W/W_Q \mid \sigma \leq \tau \}$, endowed with the Bruhat order, is a poset. To each $\sigma \in A_\tau$ we associate the Schubert variety $X(\sigma)$, which is a closed subvariety of $X(\tau)$. So we have a collection of subvarieties $X(\sigma)$ of $X(\tau)$, indexed by the partially ordered $A_\tau$ such that $\kappa \leq \sigma$ if an only if $X(\kappa) \subseteq X(\sigma)$. In addition, all the subvarieties are smooth in codimension one by Corollary 3.5, and it is well-known that the covering...
relations correspond to codimension one subvarieties. The length function defined in Definition 2.4 corresponds to the usual length function on $W$ as a Coxeter group (see Section 3.3).

To get a Seshadi stratification, we need in addition a collection of homogeneous functions $f_\sigma \in \mathbb{K}[V(\lambda)_\tau]$. We have fixed for all $\sigma \in A_\tau$ a $T$-eigenvector $v_\sigma \in V(\lambda)_\tau$ of weight $\sigma(\lambda)$. The corresponding weight space is one dimensional, so the vector is unique up to scalar multiple. Denote by $f_\sigma \in (V(\lambda)_\tau)^*$ the corresponding dual vector, i.e. $f_\sigma$ is a $T$-eigenvector of weight $-\sigma(\lambda)$, and $f_\sigma(v_\sigma) = 1$.

**Proposition 3.7.** The collection of subvarieties $X(\sigma)$ and linear functions $f_\sigma$, $\sigma \in A_\tau$, defines a Seshadi stratification for $X(\tau)$.

**Proof.** By Corollary 3.5, it remains to prove that the functions satisfy the conditions (S2) and (S3). To show (S2), we prove in fact a bit more: $f_\kappa |_{X(\tau)}$ is identically 0 if and only if $\kappa \not\leq \tau$.

If $f_\kappa |_{X(\tau)} \not= 0$, then there exists an element $z \in C(\tau)$ such that $f_\kappa(z) \not= 0$. Let $\hat{z} \in \hat{X}(\tau) \setminus \{0\}$ be an element in the affine cone over $X(\tau)$ such that $[\hat{z}] = z$, and let $\hat{z} = \sum_{\mu \in \Lambda} w_\mu$ be a decomposition into weight vectors. Since $f_\kappa(\hat{z}) \not= 0$, we know $w_{\kappa(\lambda)} \not= 0$. Now $\kappa(\lambda)$ is an extremal weight (it is a vertex of the polytope obtained as the convex hull of all weights occuring in $V(\lambda)_\tau$), so there exists a one-parameter subgroup $\eta : \mathbb{K}^* \to T$ such that $\lim_{t \to 0} \eta(t)z = [v_\kappa]$. We find $[v_\kappa] \in \overline{C(\tau)} = X(\tau)$, which implies $X(\kappa) \subseteq X(\tau)$ and hence $\kappa \leq \tau$. Vice versa, suppose $\kappa \leq \tau$. But then we know $f_\kappa |_{C(\kappa)} \not= 0$ by construction. Since $X(\kappa) \subseteq X(\tau)$, it follows $f_\kappa |_{X(\tau)} \not= 0$.

Note that $f_\tau \in (V(\lambda)_\tau)^*$ spans a $B$-stable line, so the intersection $H_{f_\tau} \cap X(\tau)$ is closed and $B$-stable. Indeed, $f_\tau(z) \not= 0$ for $z \in X(\tau)$ if and only if $[z] \in C(\tau)$. The intersection is hence the union of all Schubert varieties contained in, but not equal to $X(\tau)$, i.e. it is the union of all Schubert varieties of codimension one in $X(\tau)$, which proves (S3). 

The corresponding Hasse diagram with bonds is just the usual Bruhat graph for the set $A_\tau$: one has an edge between $\sigma, \kappa \in A_\tau$, $\sigma > \kappa$, if and only if there exists a positive real root $\beta$ such that $\sigma = s_\beta \kappa$ and $\ell(\sigma) = \ell(\kappa) + 1$. Lemma 3.3 implies that the bond is equal to $a_{\sigma, \kappa} = (\kappa(\lambda), \beta^{\vee})$, where $\beta^{\vee}$ is the coroot associated to $\beta$. The Hasse diagram with bonds for the adjoint representation of SL$_3$ is depicted below, with the Weyl group $W$ identified with the symmetric group $S_3$:

![Hasse Diagram](attachment:image.png)

**Remark 3.8.** As in Remark 2.6 and 2.7, we let $\hat{X}(\tau_{-1})$ denote the origin of $\hat{X}(\tau)$: it is contained in all affine cones $\hat{X}(\tau)$. The bond $b_{id, \tau_{-1}}$ is the vanishing multiplicity of
For 

Let \( X(\tau) \subseteq G/Q \) be a Schubert variety of dimension \( \ell(\tau) = r \). Fix a maximal chain \( C \) in \( A_r \), i.e., we have a sequence of Weyl group elements

\[
\mathcal{C} : \quad \tau = \tau_r \leftarrow \tau_{r-1} \leftarrow \tau_{r-2} \leftarrow \cdots \leftarrow \tau_2 \leftarrow \tau_1 \leftarrow \tau_0 = \text{id},
\]

such that \( \tau_j > \tau_{j-1} \), \( \ell(\tau_j) = \ell(\tau_{j-1}) + 1 \) and the \( \beta_j \) are positive roots such that \( s_{\beta_j} \tau_j = \tau_{j-1} \) for all \( j = 1, \ldots, r \). Or, in geometric terms, we have a sequence of Schubert varieties

\[
(3) \quad X(\tau) = X(\tau_r) \supseteq X(\tau_{r-1}) \supseteq X(\tau_{r-2}) \supseteq \cdots \supseteq X(\tau_2) \supseteq X(\tau_1) \supseteq X(\tau_0) = \text{id},
\]

successively contained in each other of codimension one, and \( s_{\beta_j} \tau_j = \tau_{j-1} \) for \( j = 1, \ldots, r \).

To such a chain of subvarieties we associate a higher rank valuation (see [8]).

4. Basics on valuations and quasi-valuations.

**Definition 4.1.** Let \( R \) be a \( \mathbb{K} \)-algebra. A quasi-valuation on \( R \) with values in a totally ordered abelian group \( G \) is a map \( \nu : R \setminus \{ 0 \} \to G \) satisfying the following conditions:

(a) \( \nu(x + y) \geq \min\{\nu(x), \nu(y)\} \) for all \( x, y \in R \setminus \{ 0 \} \) with \( x + y \neq 0 \);

(b) \( \nu(\lambda x) = \nu(x) \) for all \( x \in R \setminus \{ 0 \} \) and \( \lambda \in \mathbb{K}^* \);

(c) \( \nu(xy) \geq \nu(x) + \nu(y) \) for all \( x, y \in R \setminus \{ 0 \} \) with \( xy \neq 0 \).

The map \( \nu \) is called a valuation if the inequality in (c) can be replaced by an equality:

(c') \( \nu(xy) = \nu(x) + \nu(y) \) for all \( x, y \in R \setminus \{ 0 \} \) with \( xy \neq 0 \).

Quasi-valuations on \( R \) can be thought of as algebra filtrations on \( R \) (see [22, Section 2.4]). The proof of the following lemma is straightforward, see [8].

**Lemma 4.2.** Let \( \nu, \nu_1, \ldots, \nu_k : R \setminus \{ 0 \} \to G \) be quasi-valuations and let \( x, y \in R \setminus \{ 0 \} \).

i) If \( \nu(x) \neq \nu(y) \), then \( \nu(x + y) = \min\{\nu(x), \nu(y)\} \).

ii) If \( x + y \neq 0 \) and \( \nu(x + y) > \nu(x) \), then \( \nu(x) = \nu(y) \).

iii) The map \( R \setminus \{ 0 \} \to G, x \mapsto \min\{\nu_j(x) \mid j = 1, \ldots, k\} \) defines a quasi-valuation on \( R \).

Natural examples of valuations arise from vanishing orders of functions. By Corollary 3.5, Schubert varieties are smooth in codimension one. The local ring \( \mathcal{O}_{X(\tau_i), \hat{X}(\tau_{i-1})} \) is hence a discrete valuation ring for all \( i = 0, \ldots, r \) (see also Section 2.2). Since \( \mathbb{K}(\hat{X}(\tau_i)) \) is the quotient field of \( \mathcal{O}_{\hat{X}(\tau_i), \hat{X}(\tau_{i-1})} \), we obtain a \( \mathbb{Z} \)-valued valuation

\[
\nu_i : \mathbb{K}(\hat{X}(\tau_i)) \setminus \{ 0 \} \to \mathbb{Z}.
\]

For \( g \in \mathbb{K}(\hat{X}(\tau_i)) \setminus \{ 0 \} \), the value \( \nu_i(g) \) is called the vanishing order of \( g \) along the divisor \( \hat{X}(\tau_{i-1}) \).
4.2. Higher rank valuations. We come back to the case of a chain of Schubert varieties as in (3). In the following we consider the affine cones $\hat{X}(\tau_j)$, $j = 0, \ldots, r$, over the embedded Schubert varieties.

Since $\mathcal{C}$ is a chain, the elements in $\mathcal{C}$ are totally ordered. The vector space $\mathbb{Q}^\mathcal{C}$ has as standard basis the elements $e_{\tau_j}$, $j = 0, \ldots, r$. The lexicographic order on $\mathbb{Q}^\mathcal{C}$ is defined as follows: for $g = \sum_{j=0}^r c_j e_{\tau_j}$ and $g' = \sum_{j=0}^r c'_j e_{\tau_j} \in \mathbb{Q}^\mathcal{C}$, $g' > g$ if there exists $0 \leq j \leq r$ such that $c'_j = c_r$, $\ldots$, $c'_{j+1} = c_{j+1}$ and $c'_j > c_j$. From now on we take on $\mathbb{Q}^\mathcal{C}$ this lexicographic order.

Let $N$ be the l.c.m. of all bonds appearing in the Hasse diagram with bonds $G_{A_r}$.

Let $g_r := g \in K(\hat{X}(\tau_r))$ be a non-zero rational function with vanishing order $o_r \in \mathbb{Z}$ along the prime divisor $\hat{X}(\tau_{r-1})$ in $\hat{X}(\tau_r)$. We define a rational function

$$h := \frac{g_r^N}{f_r N_{c}} \in K(\hat{X}(\tau_r)),$$

where $b_r$ is the vanishing order of $f_r$ along $\hat{X}(\tau_{r-1})$. In other words, $b_r$ is the bond in the Hasse diagram $G_{A_r}$ associated to the edge joining $\tau_r$ and $\tau_{r-1}$.

The restriction of $h$ to $\hat{X}(\tau_{r-1})$, denoted by $g_{r-1}$, gives rise to a well-defined non-zero rational function in $K(\hat{X}(\tau_{r-1}))$ ([8, Lemma 4.1]). This procedure can henceforth iterated, yielding a sequence of rational functions $g_i := (g_r, g_{r-1}, \ldots, g_0$) with $g_k \in K(\hat{X}(\tau_k)) \setminus \{0\}$. The vanishing order of $f_{\tau_k}$ on $\hat{X}(\tau_{k-1})$ will be denoted by $b_k$.

In view of the $N$-th powers appearing in the sequence of rational functions, we define a map $\nu_\mathcal{C} : K[\hat{X}(\tau)] \setminus \{0\} \rightarrow \mathbb{Q}^\mathcal{C}$ in the following way:

$$g \mapsto \frac{\nu_r(g_r)}{b_r} e_{\tau_r} + \frac{1}{N} \frac{\nu_{r-1}(g_{r-1})}{b_{r-1}} e_{\tau_{r-1}} + \ldots + \frac{1}{N^r} \frac{\nu_0(g_0)}{b_0} e_{\tau_0}.$$

It is proved in [8, Section 6]:

**Proposition 4.3.** $\nu_\mathcal{C}$ is a $\mathbb{Q}^\mathcal{C}$-valued valuation on $K[\hat{X}(\tau)]$.

Denote by $\nu_\mathcal{C}(X(\tau)) = \{\nu_\mathcal{C}(f) \mid f \in K[\hat{X}(\tau)] \setminus \{0\}\} \subseteq \mathbb{Q}^\mathcal{C}$ the image of the valuation map $\nu_\mathcal{C}$, it is a monoid called the *valuation monoid*. The valuation $\nu_\mathcal{C}$ defines a filtration of $K[\hat{X}(\tau)]$ by subspaces, for $a \in \mathbb{Q}^\mathcal{C}$ we set:

$$K[\hat{X}(\tau)]^\mathcal{C}_{\geq a} := \{ g \in K[\hat{X}(\tau)] \setminus \{0\} \mid \nu_\mathcal{C}(g) \geq a \} \cup \{0\},$$

respectively

$$K[\hat{X}(\tau)]^\mathcal{C}_{\geq a} := \{ g \in K[\hat{X}(\tau)] \setminus \{0\} \mid \nu_\mathcal{C}(g) > a \} \cup \{0\}.$$ 

The subquotient $K[\hat{X}(\tau)]^\mathcal{C}_{\geq a}/K[\hat{X}(\tau)]^\mathcal{C}_{\geq a}$ is called a *leaf* of the valuation. It has been shown in [8, Section 6]:

**Proposition 4.4.** The valuation $\nu_\mathcal{C}$ has at most one dimensional leaves.
4.3. A higher rank quasi-valuation. There is no obvious reason why one maximal chain should better reflect the geometry of the Schubert variety than another. Moreover, it is not at all clear why in general the associated valuation monoid $\mathcal{V}_\epsilon(X(\tau))$ should be finitely generated.

We fix on $A_r$ a total order $\leq_\ell$ refining the partial order which respects the length, i.e. if $\ell(\sigma) > \ell(\kappa)$, then $\sigma >_\ell \kappa$. The set $Q^{A_r}$ is thus endowed with a total order by taking the lexicographic order as in Section 4.2. Such a total order is compatible with the addition in $Q^{A_r}$.

In the following we consider for a maximal chain $C$ the vector space $Q^C$ as a subspace of $Q^{A_r}$ spanned by the coordinate functions $e_\sigma$, $\sigma \in C$. The total order on $Q^{A_r}$ induces a total order on $Q^C$. Note that this order coincides with the total order on $Q^C$ fixed in Section 4.2.

Since we regard $Q^C$ as a subspace of $Q^{A_r}$, it makes sense to write $\mathcal{V}_\epsilon(g) \in Q^{A_r}$ for a regular function $g \in \mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$.

Denote by $\mathcal{C}$ the set of all maximal chains in $A_r$. By Lemma 4.2, the minimum over a finite list of valuations is a quasi-valuation.

**Definition 4.5.** i) We define the quasi-valuation $\mathcal{V} : \mathbb{K}[\hat{X}(\tau)] \setminus \{0\} \to Q^{A_r}$ by

$$\mathcal{V}(g) := \min\{\mathcal{V}_\epsilon(g) \mid C \in \mathcal{C}\}.$$

ii) For $\mathbf{a} = (a_\sigma)_{\sigma \in A_r} \in Q^{A_r}$, the support of $\mathbf{a}$ is defined by

$$\text{supp} \mathbf{a} := \{\sigma \in A_r \mid a_\sigma \neq 0\}.$$

**Remark 4.6.** Let $g \in \mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$. Unless $\text{supp} \mathcal{V}(g)$ is a maximal chain, there might be several maximal chains $\mathcal{C}$ such that $\mathcal{V}(g) = \mathcal{V}_\epsilon(g)$.

As an example let us consider an extremal function $f_\kappa$ for $\kappa \in A_r$.

**Lemma 4.7.** For any $\kappa \in A_r$, $\mathcal{V}(f_\kappa) = e_\kappa$.

**Proof.** Let $\mathcal{C} : \tau =: \tau_r > \tau_{r-1} > \ldots > \tau_0 = \text{id}$ be a maximal chain in $A_r$, and denote by $b_i$ the bond between $\tau_i$ and $\tau_{i-1}$, $i = 0, \ldots, r$ (recall that, as in Remark 3.8, $\tau_{r-1}$ is the additional element in the extended set $\hat{A}_r$.) Suppose first that $\kappa = \tau_j$ for some $0 \leq j \leq r$. By Lemma 2.3, $f_\kappa$ does not vanish identically on $\hat{X}(\tau_i)$ for $i \geq j$. The inductive procedure to determine $(f_\kappa)\epsilon = (g_r = f_\kappa, g_{r-1}, \ldots, g_0)$ gives:

$$g_r = f_\kappa, \quad g_{r-1} = f_\kappa^N, \quad \ldots, \quad g_j = f_\kappa^{N^{r-j}}.$$

The function $g_j = f_\kappa^{N^{r-j}}$ vanishes on the divisor $\hat{X}(\tau_{j-1})$ with the multiplicity $a_j = N^{r-j}b_j$, where $b_j$ is the bond between $\kappa = \tau_j$ and $\tau_{j-1}$. So we have

$$g_{j-1} = \frac{g_j^N}{(f_{\tau_j})^{N^{\tau_{j-1}}}} = \frac{f_{\tau_j}^{N^{\tau_{j-1}}+1}}{f_{\tau_j}^{N^{\tau_{j-1}}+1}} = 1.$$

This function $g_{j-1}$ does not vanish on any of the Schubert varieties, so the procedure implies $g_i = 1$ for all $i < j$. We have proved:

$$(f_\kappa)\epsilon = (f_\kappa, f_\kappa^N, f_\kappa^{N^2}, \ldots, f_\kappa^{N^{r-j}}, 1, \ldots, 1).$$
For the valuation $\mathcal{V}_e$ this implies:

$$\mathcal{V}_e(f_\kappa) = \sum_{i=0}^{r} \frac{1}{N_i} \nu_i(g_i)b_i = \frac{1}{N^{r-j}} \nu_j(f^{N^r-j})b_j c_\kappa = c_\kappa = e_\kappa.$$ 

If $\kappa \notin \mathcal{C}$, then let $\tau_k \in \mathcal{C}$ be the unique element such that $\ell(\tau_k) = \ell(\kappa)$. Since $\tau_k$ and $\kappa$ are not comparable with respect to the Bruhat order on $A_r$, $f_\kappa$ vanishes on $\hat{X}(\tau_k)$. But $f_\kappa$ is not the zero function, so there exist elements $\tau_j > \tau_{j-1} \geq \tau_k$ in $\mathcal{C}$, such that $\tau_j$ covers $\tau_{j-1}$, $f_\kappa|_{\hat{X}(\tau_j)} \neq 0$, but $f_\kappa$ vanishes on $\hat{X}(\tau_{j-1})$. It follows with the same arguments as above:

$$(f_\kappa)e = (f_\kappa, f_\kappa^N, \ldots, f_\kappa^{N^r-j}, \ldots) \implies \mathcal{V}_e(f_\kappa) = \frac{\nu_j(f^{N^r-j})}{N^r j b_j} c_{p_j} + \sum_{i<j} c_i c_{p_i}$$

for some rational numbers $c_i \in \mathbb{Q}$, $0 \leq i \leq j-1$. By assumption we have $\nu_j(f_\kappa) > 0$ and $\ell(\tau_j) > \ell(\kappa)$, so $\mathcal{V}_e(f_\kappa) = e_\kappa$ with respect to the total order on $\mathbb{Q}^A_r$. Since $\mathcal{V}(f_\kappa)$ is the minimum of the $\mathcal{V}_e(f_\kappa)$, $\mathcal{C}$ running over all maximal chains in $A_r$, we get $\mathcal{V}(f_\kappa) = e_\kappa$. 

**Remark 4.8.** The proof shows in fact more precisely: the property $\mathcal{V}(f_\kappa) = e_\kappa$ holds for every choice of a total order $\geq_t$ that respects the length.

Let $\Gamma := \{\mathcal{V}(g) \mid g \in \mathbb{K}[X(\tau)] \setminus \{0\}\} \subseteq \mathbb{Q}^A_r$ be the image of the quasi-valuation. Since $\mathcal{V}$ is only a quasi-valuation, $\Gamma$ is in general not anymore a monoid as in the case of valuations.

But we have the following compensation: for a maximal chain $\mathcal{C} \in \mathcal{C}$, we set

$$(6) \quad \Gamma_e := \{a \in \Gamma \mid \text{supp } a \subseteq \mathcal{C}\}.$$ 

By Proposition 8.7 in [8] we have $\mathcal{V}(g) = \mathcal{V}_e(g)$ if and only if $\text{supp } \mathcal{V}(g) \subseteq \mathcal{C}$. So if $g, h \in \mathbb{K}[X(\tau)] \setminus \{0\}$ are such that $\text{supp } \mathcal{V}(g), \text{supp } \mathcal{V}(h) \subseteq \mathcal{C}$, then

$$\mathcal{V}_e(g) + \mathcal{V}_e(h) = \mathcal{V}_e(gh) \geq \mathcal{V}(gh) \geq \mathcal{V}(g) + \mathcal{V}(h) = \mathcal{V}_e(g) + \mathcal{V}_e(h),$$

which implies equality everywhere and hence:

**Lemma 4.9.** $\Gamma_e$ is a monoid.

Being a union of monoids, $\Gamma = \bigcup_{e \in \mathcal{C}} \Gamma_e$ gets a name:

**Definition 4.10.** $\Gamma$ is called the fan of monoids associated to the quasi-valuation $\mathcal{V}$.

The following is a summary of some of the results in [8], reformulated in the setting of Schubert varieties. Recall that a quasi-valuation defines a filtration of $\mathbb{K}[X(\tau)]$ by subspaces in the same way as in (4) and (5). For $a \in \mathbb{Q}^A_r$ we set:

$$(7) \quad \mathbb{K}[X(\tau)]_{\geq_t a} := \{g \in \mathbb{K}[X(\tau)] \setminus \{0\} \mid \mathcal{V}(g) \geq_t a\} \cup \{0\},$$

respectively

$$(8) \quad \mathbb{K}[X(\tau)]_{> t a} := \{g \in \mathbb{K}[X(\tau)] \setminus \{0\} \mid \mathcal{V}(g) > t a\} \cup \{0\}.$$ 

The subquotients of these subspaces are called leaves.
Theorem 4.11 ([8]). Let $\mathcal{V}$ be the quasi-valuation on $\mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$ associated to the Seshadri stratification defined in Section 3.6 and the total order "$\leq_t$" fixed Section 4.3. The following statements hold:

i) $\mathcal{V}$ has at most one-dimensional leaves.

ii) The fan of monoids $\Gamma$ is contained in $\mathbb{Q}^{A_r}_{\geq 0}$.

iii) The fan of monoids $\Gamma$ is the union of the finitely generated monoids $\Gamma_\mathfrak{C}$, where the union is running over all maximal chains $\mathfrak{C}$.

iv) The quasi-valuation is additive if the supports of both functions are contained in a maximal chain: for $g, h \in \mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$, $\mathcal{V}(gh) = \mathcal{V}(g) + \mathcal{V}(h)$ if and only if there exists a maximal chain $\mathfrak{C}$ such that $\text{supp} \mathcal{V}(g), \text{supp} \mathcal{V}(h) \subseteq \mathfrak{C}$.

We can recover the degree of a homogeneous function from its quasi-valuation. Notice that $\deg(f_\kappa) = 1$ for all $\kappa \in A_r$.

Proposition 4.12 ([8]). Let $g \in \mathbb{K}[\hat{X}(\tau)] \setminus \{0\}$ and suppose $\mathcal{V}(g) = \sum_{\kappa \in A_r} a_\kappa e_\kappa$. Let $m$ be such that $ma_\kappa \in \mathbb{N}$ for all $\kappa \in A_r$. Then there exist $\lambda \in \mathbb{K}^*$ and $g' \in \mathbb{K}[\hat{X}(\tau)]$ such that

$$g^m = \lambda \prod_{\kappa \in A_r} f_{\kappa}^{ma_\kappa} + g'$$

with $\mathcal{V}(g') > \mathcal{V}(g^m)$ when $g' \neq 0$. If $g$ is homogeneous and $g' \neq 0$, then $\prod_{\kappa \in A_r} f_{\kappa}^{ma_\kappa}$ and $g'$ are homogeneous of the same degree as $g$. In particular, if $g$ is homogeneous, then:

$$\deg g = \sum_{\kappa \in A_r} a_\kappa.$$

Let $C$ be a (not necessarily maximal) chain in $A_r$. We define a cone $K_C \subseteq \mathbb{R}^{A_r}$ by:

$$K_C := \sum_{\kappa \in C} \mathbb{R}^{\geq 0} e_\kappa.$$

The positivity property of the quasi-valuation implies that the subspaces defined by the filtration are ideals, so we can consider the associated graded algebra $\text{gr}_\mathcal{V} \mathbb{K}[\hat{X}(\tau)]$. To describe this degenerate algebra, we introduce the notion of a fan algebra:

Definition 4.13 ([8]). The fan algebra $\mathbb{K}[\Gamma]$ associated to the fan of monoids $\Gamma$ is defined as

$$\mathbb{K}[\Gamma] := \mathbb{K}[x_a \mid a \in \Gamma]/I(\Gamma)$$

where $I(\Gamma)$ is the ideal generated by the following elements:

$$\begin{cases} x_a \cdot x_b - x_{a+b}, & \text{if there exists a chain } C \subseteq A_r \text{ such that } a, b \in K_C; \\ x_a \cdot x_b, & \text{if there exists no such a chain}. \end{cases}$$

To simplify the notation, we shall write $x_a$ also for its class in $\mathbb{K}[\Gamma]$ when there is no ambiguity. For a maximal chain $\mathfrak{C}$ denote by $\mathbb{K}[\Gamma_{\mathfrak{C}}]$ the following subalgebra:

$$\mathbb{K}[\Gamma_{\mathfrak{C}}] := \bigoplus_{\underline{a} \in \Gamma_{\mathfrak{C}}} \mathbb{K}x_{\underline{a}} \subseteq \mathbb{K}[\Gamma],$$
then \( K[\Gamma] \) is naturally isomorphic to the usual semigroup algebra associated to the monoid \( \Gamma \). We endow the algebra \( K[\Gamma] \) with a grading inspired by Proposition 4.12: for \( a \in \Gamma \subseteq \mathbb{Q}_{\geq 0}^d \), the degree of \( x_a \) is defined by

\[
\deg x_a = \sum_{p \in A} a_p.
\]

One of the main results in [8], when applied to this Seshadri stratification on \( X(\tau) \), gives:

**Theorem 4.14 ([8]).** There exists a flat degeneration \( \Psi : \mathcal{A} \to \mathbb{A}^1 \) such that the generic fibre is isomorphic to \( X(\tau) \) and the special fibre \( X_0 \) is isomorphic to \( \mathrm{Proj}(\mathrm{gr}_V K[X(\tau)]) \). The degenerate variety \( X_0 \) is a reduced union of equidimensional projective toric varieties, one irreducible component for each maximal chain in \( A_\tau \). The irreducible component associated to a maximal chain \( \mathcal{C} \) is isomorphic to \( \mathrm{Proj}(K[\Gamma_{\mathcal{C}}]) \).

5. **The fan of monoids \( \Gamma \) and the Lakshmibai-Seshadri lattice**

The fan of monoids \( \Gamma \) is a central object in the theory of Seshadri stratifications. For example, by Theorem 4.14 the fan algebra completely describes the semi-toric degeneration of the variety \( X \). So it is important to get a concrete description of the monoids \( \Gamma_{\mathcal{C}} \) for all maximal chains \( \mathcal{C} \).

We consider the Seshadri stratification of a Schubert variety \( X(\tau) \) defined in section 3.6. We will see later (Theorem 7.7), that the fan of monoids \( \Gamma \) associated to the quasi-valuation \( V \) coincides with the fan of monoids \( L_{\lambda}^+ \) we present now.

The fan of monoids \( L_{\lambda}^+ \) can be described as a reformulation of a conjecture made by Lakshmibai. In [31], see also Appendix C of [43], she made a conjecture on a possible index system for a basis of the space of global sections \( H^0(X(\tau), L_{\lambda}) \). The conjecture of Lakshmibai was reformulated later in terms of the Lakshmibai-Seshadri path model for representations and was proved in [33].

More details about the connection between Lakshmibai’s conjecture, the path model, as well as the lattices and monoids defined below are explained in Appendix I, Section 11.

5.1. **The LS-lattice.**

We fix a maximal chain \( \mathcal{C} : \tau = \tau_r > \ldots > \tau_0 = \text{id} \) in \( A_\tau \). For such a fixed maximal chain, we simplify the notation by writing \( b_i \) instead of \( b_{\tau_i, \tau_{i-1}} \) for the bonds. Note that all extremal functions \( f_{\tau_k} \) have degree 1 and hence \( b_0 = 1 \).

**Definition 5.1.** The lattice \( L_{\mathcal{C}, \lambda} \subseteq \mathbb{Q}_\mathcal{C} \) defined underneath is called the *Lakshmibai-Seshadri lattice* (for short *LS-lattice*) associated to \( \lambda \) and \( \mathcal{C} \):

\[
L_{\mathcal{C}, \lambda} := \left\{ a = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}_\mathcal{C} \mid \begin{array}{c}
 b_r a_r \in \mathbb{Z} \\
 b_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\
 \vdots \\
 b_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\
 a_0 + a_1 + \ldots + a_r \in \mathbb{Z}
\end{array} \right\}.
\]

Motivated by Theorem 4.11 and Proposition 4.12 we set:
Definition 5.2. The monoid obtained as the intersection $\text{LS}_{\varepsilon, \lambda} \cap \mathbb{Q}_{\geq 0}$ of the LS-lattice with the positive quadrant is denoted by $\text{LS}_{\varepsilon, \lambda}^+$.

For $\underline{a} \in \text{LS}_{\varepsilon, \lambda}$ we call the sum $a_0 + a_1 + \ldots + a_r$ the degree of $\underline{a}$. For $m \geq 0$ let $\text{LS}_{\varepsilon, \lambda}^+(m)$ be the subset of elements in $\text{LS}_{\varepsilon, \lambda}^+$ of degree $m$.

5.2. The LS-fan of monoids. As in the case of the fan of monoids associated to a quasi-valuation (Definition 4.10) we set:

Definition 5.3. Denote by $\text{LS}^{\lambda} \subseteq \mathbb{Q}^A$ the set theoretic union of the lattices $\text{LS}_{\varepsilon, \lambda}$ over all maximal chains $\mathcal{C} \in \mathcal{C}$. The intersection $\text{LS}^{\lambda} \cap \mathbb{Q}_{\geq 0}$ is denoted by $\text{LS}^{\lambda,+}$, which is called the Lakshmibai-Seshadri fan of monoids (for short: the LS-fan of monoids) associated to $\lambda$ and $\tau$.

The notion of the degree of an element and its support is extended to $\text{LS}^{\lambda}$ in the obvious way. Let $\text{LS}^{\lambda,+}(m)$ for $m \geq 0$ be the subset of elements of degree $m$. For $\underline{a} = (a_\kappa)_{\kappa \in A_\tau} \in \text{LS}^{\lambda,+}(m)$ we define

$$\text{weight}(\underline{a}) := \sum_{\kappa \in A_\tau} a_\kappa \lambda(\kappa).$$

To avoid too many indices, we do not add an index $\tau$ to the fan of monoids $\text{LS}^{\lambda,+}$ as long as it is clear from the context that the fan is associated to the partially ordered set $A_\tau$ with the corresponding bonds.

For example, right now we add the index $\text{LS}^{\lambda,\tau}$, just to point out: if $\tau' > \tau$, then the inclusion $A_\tau \hookrightarrow A_{\tau'}$ induces an inclusion $\mathbb{Q}^{A_\tau} \hookrightarrow \mathbb{Q}^{A_{\tau'}}$ sending $e_\kappa \in \mathbb{Q}^{A_\tau}$ for $\kappa \in A_\tau$ to $e_\kappa \in \mathbb{Q}^{A_{\tau'}}$. It follows directly from the definition of the fan of monoids that this inclusion induces an inclusion of fans of monoids:

$$\mathbb{Q}^{A_\tau} \hookrightarrow \mathbb{Q}^{A_{\tau'}}$$

(10)

$$\text{LS}^{\lambda,\tau} \hookrightarrow \text{LS}^{\lambda,\tau'}.$$

We have more precisely: $\text{LS}^{\lambda,\tau} = \text{LS}^{\lambda,\tau'} \cap \mathbb{Q}^{A_\tau}$. This “extension” $\text{LS}^{\lambda,\tau} \hookrightarrow \text{LS}^{\lambda,\tau'}$ of the fan of monoids makes it possible to use induction arguments.

5.3. Panorama on the next sections. Consider the Seshadri stratification defined in Section 3.6 for the embedded Schubert variety $X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_\tau)$ and the associated quasi-valuation defined in Section 4.3. We have associated to a maximal chain in (6) a monoid $\Gamma_\varepsilon \subseteq \mathbb{Q}^\varepsilon$, let $\mathcal{L}^\varepsilon \subseteq \mathbb{Q}^\varepsilon$ be the lattice generated by the monoid $\Gamma_\varepsilon$.

The aim of the next sections is to show:

i) the lattice $\mathcal{L}^\varepsilon$ coincides with the lattice $\text{LS}_{\varepsilon, \lambda}$;
ii) the monoid $\Gamma_\varepsilon$ coincides with the monoid $\text{LS}^{\varepsilon, \lambda}$;
iii) the path vectors, which will be described in the next section, define representatives for all the leaves of the quasi-valuation $\mathcal{V}$.
6. Some special vectors and functions

We come back to the situation as before: a Schubert variety \( X(\tau) \subset \mathbb{P}(V(\lambda)_\tau) \) embedded in the projective space over a Demazure submodule \( V(\lambda)_\tau \subset V(\lambda) \). To describe a filtration on the Demazure modules \( V(\lambda)_\tau \), we need the following partial order on \( \mathbb{Q}^{A_r} \).

For \( \underline{a} \in \mathbb{Q}^{A_r} \) denote by \( \{\text{supp } \underline{a}\}_j \) the subset of elements in \( \text{supp } \underline{a} \) of length \( j \). We declare \( \underline{a} \) is thin if \( \# \{\text{supp } \underline{a}\}_j \leq 1 \) for all \( j \geq 0 \). For example, all the elements in \( \text{LS}^+_{\lambda} \) are thin.

**Definition 6.1.** Let \( \underline{a}, \underline{b} \in \mathbb{Q}^{A_r}_{\geq 0} \) be both thin and of the same degree \( m \). Let \( \text{supp } \underline{a} = \{\tau_1, \tau_2, \ldots\} \) and \( \text{supp } \underline{b} = \{\kappa_1, \kappa_2, \ldots\} \) be enumerated in such a way that \( \ell(\tau_1) > \ell(\tau_2) > \ldots \) and \( \ell(\kappa_1) > \ell(\kappa_2) > \ldots \).

We say \( \underline{a} \succ \underline{b} \) if \( \tau_1 > \kappa_1 \), or \( \tau_1 = \kappa_1 \) and \( a_{\tau_1} > b_{\kappa_1} \), or \( \tau_1 = \kappa_1 \) and \( a_{\tau_1} = b_{\kappa_1} \) and \( \tau_2 > \kappa_2 \), or \( \tau_1 = \kappa_1 \) and \( a_{\tau_1} = b_{\kappa_1} \) and \( \tau_2 = \kappa_2 \) and \( a_{\tau_2} > b_{\kappa_2} \) or \( \ldots \).

An immediate consequence for the fixed total order \( \geq_t \) on \( \mathbb{Q}^{A_r} \) is the following:

**Lemma 6.2.** If \( \underline{a}, \underline{b} \in \text{LS}^+_\lambda(s) \), then \( \underline{a} \succ \underline{b} \) implies \( \underline{a} \geq_t \underline{b} \) for all possible choices of \( \geq_t \).

Given \( \underline{a} \in \text{LS}^+_\lambda(1) \), let \( \underline{\sigma} \) be a reduced decomposition of the maximal element \( \sigma \) in \( \text{supp } \underline{a} \). (Note that for \( \underline{a} \in \text{LS}^+_\lambda \) the support is linearly ordered with respect to “\( \geq \)”, so the maximal element is independent of the choice of \( \geq_t \).

In Section 12.3 we describe a purely combinatorial algorithm how to associate to such a pair \((\underline{a}, \underline{\sigma})\) a sequence of integers \((n_1, \ldots, n_t)\) and a sequence of simple roots \((\alpha_{i_1}, \ldots, \alpha_{i_t})\). The vector \( v_{\underline{a}, \underline{\sigma}} \in V(\lambda)_\tau \) associated to \( \underline{a} \in \text{LS}^+_\lambda(1) \) and the reduced decomposition \( \underline{\sigma} \) is defined by:

\[
v_{\underline{a}, \underline{\sigma}} = X_{-i_1}^{(n_1)} \cdots X_{-i_t}^{(n_t)} v_{\lambda}.
\]

For every \( \underline{a} \in \text{LS}^+_\lambda(1) \) fix a reduced decomposition \( \underline{\sigma}^2 \) of its maximal element in \( \sigma \) in \( \text{supp } \underline{a} \). It was shown in [32] that the collection of vectors \( \{v_{\underline{a}, \underline{\sigma}^2} \mid \underline{a} \in \text{LS}^+_\lambda(1)\} \) is a basis of the Demazure module \( V(\lambda)_\tau \). The basis was actually constructed over \( \mathbb{Z} \), so that it specializes to a basis for any algebraically closed field. For more details see *ibidem*, respectively Appendix II, Section 12.

The basis is far from being canonical. What is canonical about this construction is the following collection of subspaces defined by these vectors. For more details see the Appendix II, Section 12. For \( \underline{a} \in \text{LS}^+_\lambda(1) \) set

\[
(11) \quad V(\lambda)_{\tau, \preceq \underline{a}} = \left\langle v_{\underline{a}', \underline{\sigma}^2} \mid \underline{a} \preceq \underline{a}' ; \underline{a}' \in \text{LS}^+_\lambda(1), \text{\underline{\sigma}^2 reduced decomposition of maximal element in supp } \underline{a}' \right\rangle_K,
\]

and let

\[
(12) \quad V(\lambda)_{\tau, \preceq \underline{a}} = \left\langle v_{\underline{a}', \underline{\sigma}^2} \mid \underline{a} \succ \underline{a}' ; \underline{a}' \in \text{LS}^+_\lambda(1), \text{\underline{\sigma}^2 reduced decomposition of maximal element in supp } \underline{a}' \right\rangle_K.
\]

The following is known about these subspaces, see Appendix II, Section 12 for details:

**Proposition 6.3.**

i) For \( \underline{a} \in \text{LS}^+_\lambda(1) \), the subspaces \( V(\lambda)_{\tau, \preceq \underline{a}} \) of \( V(\lambda)_\tau \) are \( U(\mathfrak{g})^+_K \)-stable.
\[ \text{ii) For } \underline{a} \in \text{LS}_\lambda^+(1), \text{ the leaves } V(\lambda)_{\tau, \underline{a}} / V(\lambda)_{\tau, \underline{a}} \text{ are one dimensional.} \]

\[ \text{iii) For any choice of a reduced decomposition } \underline{a} \text{ of the maximal element } \sigma \text{ in the support } \text{supp } \underline{a}, \text{ if } \underline{a} \in \text{LS}_\lambda^+(1), \text{ } v_{\underline{a}} \text{ is a representative of the leaf } V(\lambda)_{\tau, \underline{a}} / V(\lambda)_{\tau, \underline{a}}, \text{ and its class } \overline{v}_{\underline{a}} \text{ is a generator of the leaf.} \]

\[ \text{iv) If } \underline{a'} \text{ and } \underline{a} \text{ are different reduced decompositions of the maximal element } \sigma \text{ in the support } \text{supp } \underline{a}, \text{ then } v_{\underline{a}} = v_{\underline{a'}} + \sum b_{\underline{a}} v_{\underline{a''}}, \text{ where the sum is running over elements } \underline{a''} \in \text{LS}_\lambda^+(1) \text{ such that } \underline{a'} \geq \underline{a} \text{.} \]

So it makes sense to write just \( v_{\underline{a}} \) instead of \( v_{\underline{a}, \sigma} \) if no confusion is possible. For later purpose it is important to note that neither the definition of the fan of monoids \( \text{LS}_\lambda^+ \) nor the definition of the vectors \( v_{\underline{a}, \sigma} \) respectively of the subspaces \( V(\lambda)_{\tau, \underline{a}} \) and \( V(\lambda)_{\tau, \underline{a}} \) involves or depends on the choice of the total order \( \geq \). We will now define a kind of dual basis which again induces a filtration, but this time on the dual space. Note that the following definition of a path vector is more general than the one in [32] and does not use the quantum Frobenius splitting.

**Definition 6.4.** Let \( \underline{a} \in \text{LS}_\lambda^+(1) \). A path vector associated to \( \underline{a} \) is a linear function \( p_\underline{a} \in V(\lambda)^*_\tau \) which is a \( T \)-eigenvector of weight \( -\text{weight}(\underline{a}) \), and such that

\[ \begin{align*}
\text{i) there exists a reduced decomposition } \underline{a} \text{ of the maximal element } \sigma \text{ in } \text{supp } \underline{a} \text{ such that } p_\underline{a}(v_{\underline{a}, \sigma}) = 1; \\
\text{ii) for } \underline{a'} \in \text{LS}_\lambda^+(1) \text{ and for some reduced decomposition } \underline{a'} \text{ of the maximal element } \sigma' \text{ in } \text{supp } \underline{a}, \text{ } p_\underline{a}(v_{\underline{a'}, \sigma'}) = 0 \text{ implies } \underline{a'} \geq \underline{a}. 
\end{align*} \]

**Lemma 6.5.** The definition of a path vector is independent of the choice of the reduced decompositions of \( \sigma \) and \( \sigma' \).

**Proof.** We have to show that if the properties in the definition hold for one choice of a reduced decomposition, then it holds for any choice of a reduced decomposition.

We start with property ii) and prove it by induction over the partial order \( \geq \). The set \( \text{LS}_\lambda^+(1) \) has a unique minimal element with respect to \( \geq \), it is the element \( \underline{a}'' = (0, \ldots, 0, 1) \). It can also be characterized as the unique element of weight \( \lambda \), and hence \( p_\underline{a}(v_{\underline{a''}, \underline{a}'}) = 0 \) unless \( \underline{a} = \underline{a}'' \).

Suppose now \( \underline{a}'' \in \text{LS}_\lambda^+(1) \) is given and \( \underline{a}'' \not\geq \underline{a} \). Note if \( \underline{a'''}, \underline{a''} \in \text{LS}_\lambda^+(1) \) is such that \( \underline{a''} \not\geq \underline{a'''} \), then \( \underline{a'''} \not\geq \underline{a} \) and we may assume by induction: \( p_\underline{a}(v_{\underline{a'''}, \underline{a'''}}) = 0 \) for any reduced decomposition \( \underline{a'''} \) of the maximal element \( \sigma''' \) in the support of \( \underline{a'''} \). By the definition of a path vector, we know there exists at least one reduced decomposition \( \underline{a''} \) of the maximal element in the support of \( \underline{a''} \) such that \( p_\underline{a}(v_{\underline{a''}, \underline{a'''}}) = 0 \). Now Proposition 6.3, part iv), and induction implies: \( p_\underline{a}(v_{\underline{a''}, \underline{a'''}}) = 0 \) for any reduced decomposition \( \underline{a''} \) of \( \underline{a'''} \).

To prove part i), let \( p_\underline{a} \) be a path vector associated to \( \underline{a} \in \text{LS}_\lambda^+(1) \). By Proposition 6.3, part iv), and part ii) of Lemma 6.5, \( p_\underline{a}(v_{\underline{a}, \sigma}) = 1 \) for any reduced decomposition \( \underline{a} \) of the maximal element \( \sigma \) in \( \text{supp } \underline{a} \).

The following lemma is helpful when using induction arguments. Notations in (10) will be adopted without mention. Since now and in the following lemma we have two Schubert varieties: \( X(\tau) \) and \( X(\sigma) \), it makes sense to add the index \( \tau \) respectively \( \sigma \) to the monoids: \( \text{LS}_\lambda^+ \) respectively \( \text{LS}_\lambda^+ \), to avoid confusion. Note that \( \text{supp } \underline{a} \) for
$a \in \text{LS}_{\lambda,\tau}^+$ is contained in a maximal chain $C \subseteq A_r$, hence it is linearly ordered and the maximal element is independent of the choice of $\geq_t$.

Recall that if $a$ is such that the maximal element in $\text{supp } a$ is smaller or equal to $\sigma$, then by (10), $a$ can be naturally viewed as an element in $\text{LS}_{\lambda,\tau}^+(1)$.

**Lemma 6.6.** Suppose $\sigma < \tau$, and for $a \in \text{LS}_{\lambda,\tau}^+(1)$ and let $p_a \in V(\lambda)^*_{\sigma}$ be a path vector associated to $a$. If $a$ is such that the maximal element in $\text{supp } a$ is smaller or equal to $\sigma$, then the restriction $p_a|_{V(\lambda)_{\sigma}}$ is a path vector in $V(\lambda)^*_{\sigma}$ associated to $a \in \text{LS}_{\lambda,\tau}^+(1)$.

**Proof.** For all $a' \in \text{LS}_{\lambda,\tau}^+(1)$ fix a reduced decomposition $a'$ of the maximal element $\sigma'$ in $\text{supp } a'$. By Theorem 12.16, the set $\{v_{a'} \mid a' \in \text{LS}_{\lambda,\tau}^+(1)\}$ forms a basis of the Demazure module $V(\lambda)^*_{\sigma}$. It follows by the definition of a path vector that the restriction $p_{a'}|_{V(\lambda)_{\sigma}}$ is still a weight vector of weight $(-\text{weight}(a))$. This restricted function satisfies $p_{a'}|_{V(\lambda)_{\sigma}}(v_{a'}) = 1$, and for all $a' \in \text{LS}_{\lambda,\tau}^+(1)$ we have: $p_{a'}|_{V(\lambda)_{\sigma}}(v_{a'}) \neq 0$ only if $a' \triangleright a$. Hence $p_a$, restricted to $V(\lambda)^*_{\sigma}$, is a path vector in $V(\lambda)^*_{\sigma}$ associated to $a \in \text{LS}_{\lambda,\tau}^+(1)$.

The path vectors have another special property which helps to apply induction arguments. Again suppose $\sigma < \tau$. We get an induced Seshadri stratification for $X(\sigma)$, which has as associated partially ordered set the subset $A_\sigma \subseteq A_r$ (Example 2.5). To define the quasi-valuation $V$, we have fixed a total order $\geq_t$ on $A_r$ (satisfying the conditions in Section 4.3). We denote by the same symbol $\geq_t$ the induced total order on $A_\sigma$. Let $V_\sigma$ be the corresponding quasi-valuation on the homogeneous coordinate ring $\mathbb{K}[\hat{X}(\sigma)]$ given by the embedding $X(\sigma) \hookrightarrow X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_{\tau})$. Note that the image of $X(\sigma)$ is contained in $\mathbb{P}(V(\lambda)_{\sigma})$.

**Lemma 6.7.** Let $a \in \text{LS}_{\lambda,\tau}^+(1)$ be such that $a$ is the largest elements in $\text{supp } a$, so $a$ can be naturally viewed as an element in $\text{LS}_{\lambda,\tau}^+(1)$. For a path vector $p_a$ associated to $a$ the quasi-valuation $V(p_a) \in \mathbb{Q}^A_{r}$ is equal to the extension by zeros of $V_\sigma(p_a|_{\hat{X}(\sigma)}) \in \mathbb{Q}^A_{r}$.

**Proof.** The Demazure module $V(\lambda)^*_{\sigma}$ is the linear span of $\hat{X}(\sigma)$. Let $a$ be a reduced decomposition of $\sigma$. By assumption, $v_{a\sigma} \in V(\lambda)^*_{\sigma}$, so $p_{a\sigma}|_{V(\lambda)_{\sigma}}$ does not vanish identically, which implies $p_{a\sigma}|_{\hat{X}(\sigma)}$ does not vanish identically.

A maximal chain $C$ in $A_\sigma$ can always be extended to a maximal chain $C$ in $A_r$. From the definition of the valuation in Section 4.2, $V_\sigma(p_a|_{\hat{X}(\sigma)}) = V_\sigma(p_a)$ for such a maximal chain.

Suppose now $C$ in $A_r$ is a maximal chain that does not contain $\sigma$. Let $\sigma' \in C$ be the unique element in $C$ having the same length as $\sigma$. The linear span of $\hat{X}(\sigma')$ is the Demazure module $V(\lambda)^*_{\sigma'}$, which is spanned by vectors $v_{a\sigma'}$, where $a\sigma' \in \text{LS}_{\lambda,\sigma'}^+(1) \subseteq \text{LS}_{\lambda,\tau}^+(1)$. Again, the inclusion is given by extension by zeros.

By the definition of a path vector we know $p_{a\sigma'}(v_{a\sigma'}) \neq 0$ only if $a\sigma' \triangleright a$. But this would imply that the maximal element in the support of $a\sigma'$ (which is smaller or equal to $\sigma'$) would be larger or equal to $\sigma$, which is impossible. It follows: $p_{a\sigma'}(v_{a\sigma'}) \equiv 0$. Since $p_a$ is not the zero function on $\hat{X}(\tau)$, there exists elements $\eta > \zeta \geq \sigma'$ in the maximal chain $C$ such that $p_a$ does not vanish identically on $\hat{X}(\eta)$, but vanishes on the divisor $\hat{X}(\zeta)$. It follows that the coefficient of $e_\eta$ in $V_\sigma(p_a)$ is strictly positive. Hence for any
choice of the total order $\geq_t$ (satisfying the rules in Section 4.3), $V_{\mathcal{C}}(p_\mathcal{C}) \geq_t V_{\mathcal{C}'}(p_\mathcal{C})$, where $\mathcal{C}'$ is any maximal chain in $A_\tau$ containing $\sigma$. In the following we consider the inclusion $Q^{A_\tau} \hookrightarrow Q^{A_\varrho}$ as given by extension by zeros. It follows:

$$\begin{align*}
V_\sigma(p_\mathcal{A}|X(\sigma)) &= \min \{ V_{\mathcal{C}'}(p_\mathcal{A}|X(\sigma)) \mid \mathcal{C}' \text{ maximal chain in } A_\sigma \} \\
&= \min \{ V_{\mathcal{C}}(p_\mathcal{A}) \mid \mathcal{C} \text{ maximal chain in } A_\tau, \sigma \in \mathcal{C} \} \\
&= \min \{ V_{\mathcal{C}}(p_\mathcal{A}) \mid \mathcal{C} \text{ maximal chain in } A_\tau \} = V(p_\mathcal{A}).
\end{align*}$$

\[ \blacksquare \]

We drop now the index $\tau$ again and use the same notation as before: $LS^+_\lambda$ instead of $LS^+_{\lambda,\tau}$. In connection with the Seshadri stratification, the following “dual filtration” is important. For $a \in LS^+_\lambda(1)$ denote by $V(\lambda)^*_{r,r\cdot a}$ the subspace defined as follows

$$V(\lambda)^*_{r,r\cdot a} = \left\langle p_{a'} \mid p_{a'} \text{ a path vector, } a' \succeq a, a' \in LS^+_\lambda(1) \right\rangle_{\mathbb{K}},$$

and set

$$V(\lambda)^*_{r,\cdot a} = \left\langle p_{a'} \mid p_{a'} \text{ a path vector, } a' \succeq a, a' \in LS^+_\lambda(1) \right\rangle_{\mathbb{K}}.$$

The following is known about these subspaces, see Appendix III, Section 13:

**Proposition 6.8.**

i) For every $a \in LS^+_\lambda(1)$ fix a path vector $p_a$ associated to $a$. The set $\mathbb{B}(V(\lambda)^*_{r}) = \{ p_a \mid a \in LS^+_\lambda(1) \}$ is a basis for $V(\lambda)^*_{r}$.

ii) For $a \in LS^+_\lambda(1)$, the leaves $V(\lambda)^*_{r,\cdot a}/V(\lambda)^*_{r,r\cdot a}$ are one dimensional, and any path vector $p_a$ is a representative of such a leaf.

The following is the most important result needed to start a standard monomial theory.

**Theorem 6.9.** If $p_\mathcal{A} \in V(\lambda)^*_{r} \cong \mathbb{K}[\hat{X}(\tau)]_1$ is a path vector associated to $a \in LS^+_\lambda(1)$, then $V(p_\mathcal{A}) = a$, independent of the choice of the total order $\geq_t$.

As an immediate consequence we get the following relationship between the collection of subspaces defined in (13) and (14) and the filtration of $\mathbb{K}[\hat{X}(\tau)]$ induced by the quasi-valuation, see (7) and (8). Recall that the latter has at most one-dimensional leaves (Theorem 4.11).

**Corollary 10.** For all $a \in LS^+_\lambda(1)$, the subspaces $V(\lambda)^*_{r,\cdot a}$ respectively $V(\lambda)^*_{r,r\cdot a}$ are compatible with the filtration induced by the quasi-valuation. That is to say, independently of the choice of the total order $\geq_t$, we have

$$V(\lambda)^*_{r,\cdot a} \subseteq \mathbb{K}[\hat{X}(\tau)]_{\geq a}, \quad V(\lambda)^*_{r,r\cdot a} \subseteq \mathbb{K}[\hat{X}(\tau)]_{> a},$$

and $V(\lambda)^*_{r,r\cdot a}/V(\lambda)^*_{r,\cdot a} \cong \mathbb{K}[\hat{X}(\tau)]_{\geq a}/\mathbb{K}[\hat{X}(\tau)]_{> a}$.

**Proof.** The proof is by induction of $\ell(\tau)$, the case $\ell(\tau) = 0$ being obvious. So suppose now $\ell(\tau) \geq 1$, and the theorem holds for all smaller Schubert varieties.

To be more precise, if $\sigma < \tau$, then we have an induced Seshadri stratification for the Schubert variety $X(\sigma)$, with associated partially ordered set $A_\sigma \subseteq A_\tau$, the fixed total
order $\geq_t$ on $A_\tau$ induces a total order $\geq_t$ on $A_\sigma$, we get an induced quasi-valuation $\mathcal{V}_\sigma$ on $\mathbb{K}[\hat{X}(\sigma)]$, and an associated LS-fan of monoids $\text{LS}^+_\lambda\sigma$.

We view this LS-fan of monoids $\text{LS}^+_\lambda\sigma$ as being embedded into $\text{LS}^+_1$ via extension by zeros, see (10). For $\underline{a} \in \text{LS}^+_1(1)$ let $p_{\underline{a}}$ be a corresponding path vector. If the largest element in $\text{supp} \underline{a}$ is $\sigma$ and $\sigma < \tau$, then $\underline{a} \in \text{LS}^+_{\lambda\sigma}(1)$. We know by Lemma 6.6 that the restriction $p_{\underline{a}\mid \hat{X}(\sigma)}$ is a path vector for the Seshadri stratification on $X(\sigma)$.

And, by Lemma 6.7, we know $\mathcal{V}(p_{\underline{a}}) = \mathcal{V}_\sigma(p_{\underline{a}\mid \hat{X}(\sigma)})$, where the equality has to be read as: $\mathcal{V}_\sigma(p_{\underline{a}\mid \hat{X}(\sigma)})$ is treated as an element in $\mathbb{Q}A^c$ via extension by zeros. So our induction assumption gives: $\mathcal{V}_\sigma(p_{\underline{a}\mid \hat{X}(\sigma)}) = \underline{a}$, independent of the choice of the total order on $A_\sigma$.

Now in the proof of Lemma 6.7 we have seen if $\mathcal{C}$ is a maximal chain in $A_\tau$ such that $\sigma \in \mathcal{C}$, then $\mathcal{V}_\mathcal{C}(p_{\underline{a}}) = \mathcal{V}_\mathcal{C}(p_{\underline{a}\mid \hat{X}(\sigma)})$, where $\mathcal{C}'$ is obtained from $\mathcal{C}$ by omitting all elements larger than $\sigma$. And if $\mathcal{C}$ is a maximal chain in $A_\tau$ not containing $\sigma$, then we have seen in the proof of Lemma 6.7 that, independent of the choice of the total order $\geq_t$, (satisfying the rules in Section 4.3), $\mathcal{V}_\mathcal{C}(p_{\underline{a}}) = \mathcal{V}_\mathcal{C}(p_{\underline{b}})$, where $\mathcal{C}'$ is any maximal chain in $A_\tau$ containing $\sigma$. It follows: $\mathcal{V}(p_{\underline{a}}) = \underline{a}$, independent of the choice of the total order $\geq_t$.

It remains to consider the case: the largest element in $\text{supp} \underline{a}$ is $\sigma$. In this case we use Corollary 13.13. Let $m \geq 1$ be such that $ma_\tau \in \mathbb{N}$. Then we know that $p_{\underline{a}^m}$ is, up to multiplication by a root of unity, equal to $p_{ma_\tau}\underline{p}_{\underline{a}}$. Here $\underline{b} = m\underline{a} - ma_\tau e_\tau \in \text{LS}^+_\lambda(m - ma_\tau)$ and $\underline{p}_{\underline{b}} \in V((m - ma_\tau)\lambda)_{1}^*$ is a path vector associated to the leaf $\underline{b}$.

By Theorem 4.11, we know the quasi-valuation is additive if the support of the functions is contained in a common maximal chain. Now by induction we know: $\mathcal{V}(p_{\underline{a}}) = \underline{b}$, and this holds independent of the choice of the total order. By Lemma 4.7 we know: $\mathcal{V}(p_{ma_\tau}) = ma_\tau e_\tau$, and this holds independent of the choice of the total order on $A_\tau$. Since $\text{supp} \underline{a} = \text{supp} \underline{b} \cup \{e_\tau\}$ is contained in a maximal chain, we see:

$$\mathcal{V}(p_{\underline{a}}) = \frac{1}{m}\mathcal{V}(p_{\underline{a}^m}) = \frac{1}{m}\mathcal{V}(p_{ma_\tau}\underline{p}_{\underline{a}}) = \frac{1}{m} \left(\mathcal{V}(p_{ma_\tau}) + \mathcal{V}(p_{\underline{a}})\right) = a_\tau e_\tau + (\underline{a} - a_\tau e_\tau) = \underline{a},$$

and this holds independent of the choice of the total order. 

7. Standard monomial theory

7.1. Standard monomial basis. Recall that the elements in the support of $\underline{a} \in \text{LS}^+_1(\tau)$ are linearly ordered with respect to the partial order $\geq_t$ on $A_\tau$, so there exists always a unique maximal element in the support: $\text{max supp} \underline{a}$, and a unique minimal element in the support: $\text{min supp} \underline{a}$.

**Definition 7.1.** A monomial $p_{\underline{a}^1} \cdots p_{\underline{a}^m} \in \mathbb{K}[\hat{X}(\tau)]$ of path vectors with $\underline{a}^1, \ldots, \underline{a}^m \in \text{LS}^+_1(1)$ is called **standard** if for each $1 \leq j \leq m - 1$ we have $\text{min supp} \underline{a}^j \geq \text{max supp} \underline{a}^{j+1}$.

An important property of the elements of the LS-fan of monoids is stated in the Appendix I in Lemma 11.8 (see also [7]): Every element $\underline{a} \in \text{LS}^+_1(m)$ has a unique decomposition $\underline{a} = \underline{a}^1 + \ldots + \underline{a}^m$ into $m$ elements $\underline{a}^i \in \text{LS}^+_1(1), i = 1, \ldots, m$, such that $\text{supp} \underline{a}^1 \geq \text{supp} \underline{a}^2 \geq \ldots \geq \text{supp} \underline{a}^m$.

For every $\underline{a} \in \text{LS}^+_1(1)$ fix a path vector $p_{\underline{a}^1}$, so this collection $\mathbb{B}(V(\lambda)_\tau)$ of path vectors forms a basis for $V(\lambda)_\tau^*$ (see Proposition 6.8).
So we start with the fixed basis \( \mathbb{B}(V(\lambda)_\tau) \) consisting of path vectors. Given \( a \in \text{LS}_\lambda^+(m) \), \( m \geq 1 \), we have a unique decomposition \( a = a^1 + \ldots + a^m \) with \( a^i \in \text{LS}_\lambda^+(1) \), \( i = 1, \ldots, m \). We associate to \( a \) the standard monomial

\[
p_a = p_{a^1} \cdots p_{a^m}.
\]

By Theorem 4.11, we know the quasi-valuation is additive if the support of the functions is contained in a common maximal chain. By Theorem 6.9 we know \( \mathcal{V}(p_{a^i}) = a^i \), independent of the choice of the total order \( \geq_t \). As a consequence we see:

\[
\mathcal{V}(p_a) = \mathcal{V}(p_{a^1}) + \ldots + \mathcal{V}(p_{a^m}) = a^1 + \ldots + a^m = a,
\]

independent of the choice of the total order \( \geq_t \), and hence:

**Corollary 7.2.** For the standard monomial \( p_a \), we have \( \mathcal{V}(p_a) = a \), independent of the choice of the total order \( \geq_t \).

This leads us directly to a vector space basis of \( \mathbb{K}[\check{X}(\tau)] \) by standard monomials:

**Theorem 7.3.** Let \( \mathbb{B}(V(\lambda)_\tau) \) be a fixed basis of \( V(\lambda)_\tau^* = \mathbb{K}[\check{X}(\tau)]_1 \) consisting of path vectors. The set of standard monomials in the elements of \( \mathbb{B}(V(\lambda)_\tau) \) forms a vector space basis for \( \mathbb{K}[\check{X}(\tau)] \).

**Proof.** For \( a \in \text{LS}_\lambda^+(m) \) we have \( \mathcal{V}(p_a) = a \) by Corollary 7.2, which implies that the set of standard monomials \( \{p_a \mid a \in \text{LS}_\lambda^+(m)\} \) is a set of linearly independent vectors. On the other hand, by Lemma 3.1, the dimension of \( \mathbb{K}[\check{X}(\tau)]_m \) is bounded above by the dimension of \( V(m\lambda)_\tau \), and by Corollary 3.2 we have \( \dim V(m\lambda)_\tau = \mathbb{K}[\check{X}(\tau)]_m \). And one can either use Demazure’s character formula and its combinatorial version in [33] (see also Theorem 11.3 in Appendix I), or one can use [32] (see also Appendix II, summary before Theorem 12.16) to see that \( \dim V(m\lambda)_\tau \) is equal to the cardinality of \( \text{LS}_\lambda^+(m) \). This implies that \( \{p_a \mid a \in \text{LS}_\lambda^+(m)\} \) is a basis for \( \mathbb{K}[\check{X}(\tau)]_m \).

In [8] we have introduced two special properties of a Seshadri stratification. Recall that in the process of associating a quasi-valuation to a Seshadri stratification, the only choice we made is a total order \( \leq_t \) on \( A \) refining the given partial order and preserving the length function.

**Definition 7.4.** A Seshadri stratification of the embedded projective variety \( X \hookrightarrow \mathbb{P}(V) \), with homogeneous coordinate ring \( R \), is called balanced if the following two properties hold:

(i) the fan of monoids \( \Gamma \) associated to the quasi-valuation \( \mathcal{V} \) is independent of the choice of the total order \( \geq_t \);

(ii) for each \( a \in \Gamma \) there exists a regular function \( x_a \in R \) such that \( \mathcal{V}(x_a) = a \) for all possible choices of a total order \( \geq_t \).

The second special property introduced in [8] is related to the question whether the monoids \( \Gamma_\mathcal{C} \) are saturated, that is to say: if \( K_\mathcal{C} \) is the real cone generated by \( \Gamma_\mathcal{C} \) and \( \mathcal{L}_\mathcal{C} \) is the lattice generated by \( \Gamma_\mathcal{C} \), then \( \mathcal{L}_\mathcal{C} \cap K_\mathcal{C} = \Gamma_\mathcal{C} \) for all maximal chains \( \mathcal{C} \).

Algebraically this is equivalent to say that the algebras \( \mathbb{K}[\Gamma_\mathcal{C}] \) (Section 4.3) are normal for all maximal chains \( \mathcal{C} \). Geometrically this condition is equivalent to the normality...
of all the irreducible components of the degenerate variety \( \text{Spec}(\text{gr}_V \mathbb{K}[\hat{X}(\tau)]) \) (see Theorem 4.14). We give a name to such a situation:

**Definition 7.5.** A Seshadri stratification is called normal if for all maximal chains \( \mathcal{C} \) the monoid \( \Gamma_{\mathcal{C}} \) is saturated.

**Remark 7.6.** In the above definition, we have chosen implicitly a total order \( \geq^t \) refining the given partial order on \( A \). Different such choices would produce different monoids \( \Gamma_{\mathcal{C}} \). When the Seshadri stratification is balanced, the monoids \( \Gamma_{\mathcal{C}} \) are independent of the choice of the refinement of the partial order. In this case, being normal is indeed a property associated to the stratification.

**Theorem 7.7.** The Seshadri stratification of the Schubert variety \( X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)) \) defined in Section 3.6 is normal and balanced. The fan of monoids associated to the quasi-valuation \( V \) coincides with the fan of monoids \( \text{LS}^+_{\lambda} \).

**Proof.** Fix a basis \( \mathbb{B}(V(\lambda)_\tau) \) of \( V(\lambda)_\tau^+ = \mathbb{K}[\hat{X}(\tau)]_1 \) consisting of path vectors. By Theorem 7.3 we know that \( \mathbb{K}[\hat{X}(\tau)]_1 \) has a vector space basis consisting of the standard monomials in the elements of \( \mathbb{B}(V(\lambda)_\tau) \). This vector space basis of \( \mathbb{K}[\hat{X}(\tau)]_1 \) is, by construction (and Lemma 11.8), indexed by the elements of \( \text{LS}^+_{\lambda} \), and Theorem 7.3 states: for \( a \in \text{LS}^+_{\lambda} \) we have \( V(p_a) = a \), independent of the choice of the total order \( \geq^t \).

This property has several consequences. Since the quasi-valuation differs for pairwise different basis elements, Lemma 4.2 implies \( V(p) \in \text{LS}^+_{\lambda} \) for any nonzero element in \( \mathbb{K}[\hat{X}(\tau)]_1 \). In particular, the fan of monoids associated to the quasi-valuation \( V \) coincides with the fan of monoids \( \text{LS}^+_{\lambda} \), and this holds independent of the choice of the total order \( \geq^t \). Moreover, the basis given by the standard monomials has the properties desired in Definition 7.4, so the Seshadri stratification is balanced.

For each maximal chain, the monoid \( \text{LS}^+_{\mathcal{C},\lambda} \) is defined (see Definition 5.2) as the intersection of the lattice \( \text{LS}_{\mathcal{C},\lambda} \) with the positive quadrant \( \mathbb{Q}_{\geq 0}^\mathcal{C} \), in particular, the monoid is saturated. The stratification is hence normal.

**8. Some applications**

The fact that the fan of monoids coincides with \( \text{LS}^+_{\lambda} \) implies that the stratification is of LS-type in the terminology of [10]. Hence Theorem 3.2 and Theorem 3.4 in [10] imply:

**Corollary 8.1.** The homogeneous coordinate ring of \( X(\tau) \) admits a quadratic Gröbner basis and is a Koszul algebra.

The positivity property of the quasi-valuation implies that the subspaces defined by the filtration associated to \( V \) (see Section 4) are ideals, so we can consider the associated graded algebra \( \text{gr}_V \mathbb{K}[\hat{X}(\tau)]_1 \). Theorems 11.1 and 12.2 in [8] imply:

**Theorem 8.2.**

i) The degenerate algebra \( \text{gr}_V \mathbb{K}[\hat{X}(\tau)]_1 \) is isomorphic to the fan algebra \( \mathbb{K}[\text{LS}^+_{\lambda}] \) (see Definition 4.13).

ii) There exists a flat degeneration of \( X(\tau) \) into \( X_0 \), a union of projective toric varieties. Moreover, the special fibre \( X_0 \) is equidimensional, it is isomorphic to
Proj(gr_V[K[X](\tau)]), and its irreducible components are normal varieties and in bijection with maximal chains in A_\tau.

8.1. Straightening laws. Let B(V(\lambda)_\tau) be a fixed basis of V(\lambda)* = K[X(\tau)]_1 consisting of path vectors. For the special class of path vectors constructed in [32] one can find relations in [32] and [26]. But it turns out that these relations actually fit perfectly into the framework of normal and balanced Seshadri stratifications. The following is just a reformulation of part of Proposition 2.20 in [11] in terms of the Seshadri stratification of Schubert varieties and shows that the above mentioned relations can be viewed as a special case:

Proposition 8.3. (i) If a monomial p_{a_1} \cdots p_{a_n} of path vectors is not standard, then there exists a straightening relation expressing it as a linear combination of standard monomials

p_{a_1} \cdots p_{a_n} = \sum_h u_h p_{a_{h,1}} \cdots p_{a_{h,n}},

where u_h \neq 0 only if a_1 + \cdots + a_n \aleq a_{h,1} + \cdots + a_{h,n}.

(ii) If in (i) there exists a chain C such that supp a_i \subseteq C for all i = 1, \ldots, n, and a'_1 + \cdots + a'_n is the decomposition of a_1 + \cdots + a_n \in LS^+, then the standard monomial p_{a'_1} \cdots p_{a'_n} appears in the right side of the equation in (i) with a non-zero coefficient.

To rewrite a monomial in K[X(\tau)] that is not standard as a linear combination of standard monomials, one can use the subduction algorithm (see for example [22]). The appropriate notation in this context is in the language of Khovanskii basis. For details concerning the subduction algorithm in the context of Seshadri stratifications we refer to [8]. Recall that the quasi-valuation V depends on the choice of a total order \geq_t on A_\tau refining the Bruhat order and preserving the length function. To emphasize this, we write V_{\geq_t}.

Definition 8.4. (i) A subset G \subseteq K[X(\tau)] is called a Khovanskii basis for the quasi-valuation V_{\geq_t}, if the image of G in gr_V_{\geq_t} K[X(\tau)] generates the algebra gr_V_{\geq_t} K[X(\tau)].

(ii) A subset G \subseteq K[X(\tau)] is called a Khovanskii basis for a Seshadri stratification, if it is a Khovanskii basis for all possible V_{\geq_t}, where \geq_t is a linear extensions of \geq satisfying: if \ell(p) > \ell(q) then p \succ_t q.

Let B(V(\lambda)_\tau) be a fixed basis of V(\lambda)* = K[X(\tau)]_1 consisting of path vectors. Theorem 6.9, Theorem 7.7 and Theorem 7.3 together imply:

Corollary 8.5. The set B(V(\lambda)_\tau) is a Khovanskii basis for the Seshadri stratification defined in Section 3.6.

8.2. Compatibility with the strata. We refer to [8] for more details, we just quote the results which hold in general for Seshadri stratifications. Since we consider two Schubert varieties at the same time, we add a \tau or \kappa as index, for example we write
Set $\mathbb{L}^+_{\lambda, \tau}$ instead of $\mathbb{L}^+_{\lambda, \tau}$. We fix basis $\mathbb{B}(V(\lambda)_\tau)$ of $V(\lambda)_\tau^* = \mathbb{K}[\hat{X}(\tau)]_1$ consisting of path vectors.

**Definition 8.6.** Let $\kappa \in A_\tau$. A standard monomial $p_{\underline{a}} \cdots p_{\underline{a}}^m \in \mathbb{K}[\hat{X}(\tau)]$ on $\hat{X}(\tau)$ is called standard on $\hat{X}(\kappa)$ if $\text{max supp } \underline{a}^1 < \kappa$.

Let $\kappa \in A_\tau$ be such that $\kappa < \tau$. By Example 2.5, we know that the collection of subvarieties $X(\delta)$, $\delta \leq \kappa$, and the extremal functions $p_\delta$ for $\delta \in A_\kappa$ satisfy the conditions (S1)-(S3), and hence defines a Seshadri stratification for $X(\kappa) \hookrightarrow \mathbb{P}(V(\lambda)_\tau)$. Let $\geq_t$ be the total order on $A_\tau$ chosen in the construction of $\mathcal{V}$. We keep the notation $\geq_t$ for the induced total order on $A_\kappa$. Let $\mathcal{V}_\kappa$ be the associated quasi-valuation on $\mathbb{K}[\hat{X}(\kappa)]$. In this setup, one gets various natural objects associated to the subset $A_\kappa \subseteq A_\tau$:

- $\mathbb{L}^+_{\lambda, \kappa}(1) := \{\underline{a} \in \mathbb{L}^+_{\lambda, \tau}(1) \mid \text{supp } \underline{a} \in A_\kappa\}$,
- $\mathbb{B}(V(\lambda)_\kappa) = \{p_{\underline{a}}|_{\hat{X}(\kappa)} \mid \underline{a} \in \mathbb{L}^+_{\lambda, \kappa}(1)\}$, and
- $\mathbb{L}^+_{\lambda, \kappa} := \{\underline{a} \in \mathbb{L}^+_{\lambda, \tau} \mid \text{supp } \underline{a} \subseteq A_\kappa\}$.

A natural question arises: What is the connection between these objects and the fan of monoids associated to $\mathcal{V}_\kappa$, its generating set, etc?

We consider the vector space $\mathbb{Q}^{A_\kappa}$ as a subspace of $\mathbb{Q}^{A_\tau}$. Since every maximal chain in $A_\kappa$ is a chain in $A_\tau$, by abuse of notation we write $\mathcal{V}_\kappa(f) \in \mathbb{Q}^{A_\tau}$ for a non-zero function $f \in \mathbb{K}[\hat{X}(\kappa)]$. For a proof see [8].

**Theorem 8.7.** Since the Seshadri stratification of $X(\tau)$ is balanced and normal, the following holds:

i) for all $\kappa \in A_\tau$, the induced Seshadri stratification on $X(\kappa)$ is balanced and normal;

ii) the fan of monoids associated to $\mathcal{V}_\kappa$ is equal to $\mathbb{L}^+_{\lambda, \kappa}$; $\mathbb{L}^+_{\lambda, \kappa}(1)$ is its generating set of indecomposables and $\mathbb{B}(V(\lambda)_\kappa)$ is a Khovanskii basis for the Seshadri stratification of $X(\kappa)$;

iii) if $p_{\underline{a}} \cdots p_{\underline{a}}^m$ is a standard monomial, standard on $X(\kappa)$, then $\mathcal{V}_\kappa(p_{\underline{a}} \cdots p_{\underline{a}}^m|_{\hat{X}(\kappa)}) = V(p_{\underline{a}} \cdots p_{\underline{a}}^m) = \sum_{i=1}^m \underline{a}^i$;

iv) the restrictions of the standard monomials $p_{\underline{a}} \cdots p_{\underline{a}}^m|_{\hat{X}(\kappa)}$, standard on $X(\kappa)$, form a basis of $\mathbb{K}[\hat{X}(\kappa)]$;

v) a standard monomial $p_{\underline{a}} \cdots p_{\underline{a}}^m$ on $\hat{X}(\tau)$ vanishes on the subvariety $\hat{X}(\kappa)$ if and only if $p_{\underline{a}} \cdots p_{\underline{a}}^m$ is not standard on $\hat{X}(\kappa)$;

vi) the vanishing ideal $I(\hat{X}(\kappa))$ of $\hat{X}(\kappa)$ in $\mathbb{K}[\hat{X}(\tau)]$ is generated by the elements in $\mathbb{B}(V(\lambda)_\tau) \setminus \mathbb{B}(V(\lambda)_\kappa)$, and the ideal has as vector space basis the set of all standard monomials on $\hat{X}(\tau)$ which are not standard on $\hat{X}(\kappa)$;

vii) for all pairs of elements $\kappa, \delta \in A_\tau$, the scheme theoretic intersection $X(\kappa) \cap X(\delta)$ is reduced. It is the union of those subvarieties $X(\xi)$ such that $\xi \leq \kappa$ and $\xi \leq \delta$, endowed with the induced reduced structure.

### 8.3. Compatibility with the standard monomial theory in [32].

Let $\mathbb{B}(V(\lambda)_\tau)$ be a fixed basis of $V(\lambda)_\tau^* = \mathbb{K}[\hat{X}(\tau)]_1$ consisting of path vectors. All the results stated in Sections 7 and 8 so far assume that one fixes at the beginning such a basis consisting of path vectors, no other property is assumed.
In [32] too, using the quantum Frobenius morphism, a basis of $V(\lambda)^*_\tau = \mathbb{K}[\hat{X}(\tau)]_1$ was constructed, which was the foundation for the standard monomial theory in ibidem. By Lemma 13.3, these vectors $p_{\mu,\ell}$ (for the notation see Appendix III, Section 13) are path vectors in the sense of Definition 6.4. It follows that the basis constructed in ibidem is a special choice of a basis $\mathbb{B}(V(\lambda)_\tau)$ of $V(\lambda)_\tau^* = \mathbb{K}[\hat{X}(\tau)]_1$ consisting of path vectors.

The notion of a standard monomial used in [32] is based on the definition of a standard concatenation of LS-paths, see Definition 11.4, and the fact that an LS-path of shape $m\lambda$ can always be written as a standard concatenation of LS-paths of shape $\lambda$, see Proposition 11.5. Now the bijection $\Theta$ between the LS-paths and the fan of monoids described in Appendix I, Section 13, preserves the notion of standardness. That is to say, it sends a standard concatenation of LS-paths to a standard sum of elements in the fan of monoids $LS^+_\lambda$, see Definition 11.7 and the text thereafter. It follows that the notion of a standard monomial in [32] for special products of the vectors $p_{\mu,\ell}$ coincides with the notion of a standard monomial in Definition 7.1. Summarizing we have:

**Proposition 8.8.** The standard monomial theory described in [32] is the same as in Theorem 7.3, only the possible choice of path vectors is restricted in [32] to those obtained via the quantum Frobenius splitting (see Appendix III, Section 13).

See [9] for another approach to this proposition in the framework of LS-algebras.

9. Projective normality

There are meanwhile many different proofs of the normality of Schubert varieties respectively the projective normality of embedded Schubert varieties. A geometric proof can be found, for example, in [46] (finite type case), the proofs in [42] (finite type case) and [38] (Kac-Moody groups) use (the algebraic geometric) Frobenius splitting, the proof in [32] uses the standard monomial theory developed in the same paper.

We give here another proof, which is a direct application of the theory of Seshadri stratifications. Let $SR(A_\tau)$ be the Stanley-Reisner algebra of the poset $A_\tau$. By definition

$$SR(A_\tau) := \mathbb{K}[t_\sigma | \sigma \in A_\tau]/(t_\sigma t_\kappa | \sigma, \kappa \text{ not comparable}).$$

By [5, 6], the Stanley-Reisner algebra $SR(A_\tau)$ is Cohen-Macaulay over any field $\mathbb{K}$. By Theorem 7.7 we know that the Seshadri stratification is normal. By Theorem 14.1 in [8], the normality of the stratification and the Cohen-Macaulayness of the Stanley-Reisner algebra $SR(A_\tau)$ imply:

**Theorem 9.1.** The embedded Schubert variety $X(\tau) \hookrightarrow \mathbb{P}(V(\lambda)_\tau)$ is projectively normal.

Fix a dominant weight $\lambda$ and let $Q \supset B$ be the parabolic subgroup of $G$ associated to $\lambda$. We identify the dual space $V(\lambda)^*$ with the space of global sections $H^0(G/Q, L_{\lambda})$ of the line bundle $L_{\lambda} := G \times Q \mathbb{K}_{-\lambda}$. Let $\phi : G/Q \hookrightarrow \mathbb{P}(V(\lambda))$ be the corresponding embedding.

The projective normality implies: the natural morphisms $\mathbb{K}[X(\tau)]_m \to H^0(X(\tau), L_{m\lambda})$ are isomorphisms for all $m \geq 0$. So one can reformulate the standard monomial theory in terms of sections. We use the same notation as in the section before.
Theorem 9.2.  

i) The standard monomials \( p_{a_1} \cdots p_{a_m} \) of degree \( m \) with \( a_1, \ldots, a_m \in \text{LS}_X^+(1) \) form a basis for \( H^0(X(\tau), L_{m\lambda}) \).

ii) A standard monomial \( p_{a_1} \cdots p_{a_m} \) on \( X(\tau) \) vanishes on the subvariety \( X(\kappa) \) for \( \kappa \leq \tau \) if and only if the monomial is not standard on \( \hat{X}(\kappa) \).

iii) For \( \kappa \leq \tau \), the restrictions of the standard monomials \( p_{a_1} \cdots p_{a_m} \mid_{\hat{X}(\kappa)} \) of degree \( m \), standard on \( X(\kappa) \), form a basis of \( H^0(X(\tau), L_{m\lambda}) \).

iv) For \( \kappa \leq \tau \) and all \( m \geq 1 \), the restriction map \( H^0(X(\tau), L_{m\lambda}) \to H^0(X(\kappa), L_{m\lambda}) \) is surjective.

v) For all \( m \geq 1 \), the multiplication map \( S^mH^0(X(\tau), L_{\lambda}) \to H^0(X(\tau), L_{m\lambda}) \) is surjective.

Proof.  Part i)–iii) of the theorem are just reformulations of the corresponding results in Theorem 8.7. Part v) is an immediate consequence of i), part iv) is an immediate consequence of i) and ii).

Remark 9.3.  Another consequence of standard monomial theory is the vanishing of the higher cohomology: for all \( m \geq 1 \) and \( i \geq 1 \), \( H^i(X(\tau), L_{m\lambda}) = 0 \). For a proof see, for example, [32], Theorem 7, or [26], Theorem 6.4, where the proof is given even for unions of Schubert varieties.

10. Newton-Okounkov simplicial complex

Newton-Okounkov bodies generalize Newton polytopes from toric geometry to a more general algebro-geometric as well as representation-theoretic setting. The Newton-Okounkov body is constructed from an embedded projective variety \( X \hookrightarrow \mathbb{P}(V) \), its homogeneous coordinate ring \( R \), and a valuation \( \nu \) on \( R \). Roughly speaking, in the nicest of all cases, the Newton-Okounkov body associated to a projective variety is a polytope. It is known that in such a case there exists a flat toric degeneration from \( X \) to the toric variety corresponding to this polytope. The Newton-Okounkov body stores a lot of information about the original variety, for example the degree of the embedded variety.

Since instead of a valuation, we have only a quasi-valuation, it is not straightforward to define a Newton-Okounkov body. But one can get an appropriate replacement. We formulate the construction here only for the case of Schubert varieties. See [8] for more details in the general case.

Let \( C \) be the set of all maximal chains in \( A_\tau \).

Definition 10.1.  The Newton-Okounkov simplicial complex \( \Delta_V \) associated to the quasi-valuation \( V \) is defined as

\[
\Delta_V := \bigcup_{\varepsilon \in C} \bigcup_{m \geq 1} \left\{ \frac{1}{m} a \mid a \in \Gamma_\varepsilon, \deg a = m \right\} \subseteq \mathbb{R}^{A_\tau}.
\]

Another object associated to the Seshadri stratification is the order complex \( \Delta(A_\tau) \) attached to the poset \( (A_\tau, \leq) \). It is the simplicial complex having the set \( A_\tau \) as vertices and all chains \( C \subseteq A_\tau \) as faces, i.e. \( \Delta(A_\tau) = \{ C \subseteq A_\tau \mid C \text{ is a chain} \} \). A geometric
realization of $\Delta(A_\tau)$ can be constructed by intersecting the cones $K_C$ (see Section 4.3) with appropriate hyperplanes: for a chain $C \subseteq A_\tau$ denote by $\Delta_C \subseteq \mathbb{R}^{A_\tau}$ the simplex:

$$\Delta_C := \text{convex hull of } \{e_p \mid p \in C\}.$$  

The union of the simplexes

$$|\Delta(A_\tau)| := \bigcup_{C \subseteq A_\tau \text{ chain}} \Delta_C \subseteq \mathbb{R}^{A_\tau}$$

is the desired geometric realization of $\Delta(A_\tau)$. The maximal simplexes are those $\Delta_\xi$ arising from maximal chains $\xi$ in $A_\tau$. It has been shown in [8], Proposition 13.3:

**Proposition 10.2.** The Newton-Okounkov simplicial complex $\Delta_V$ coincides with the geometric realization of $\Delta(A_\tau)$: $\Delta_V = |\Delta(A_\tau)|$. In particular, $\Delta_V$ is a homogeneous simplicial complex of dimension $r$.

In the construction of a toric variety associated to a polytope $P \subseteq \mathbb{R}^l$, the sequence of integral points one gets as the intersections $mP \cap \mathbb{Z}^l$, $m \geq 1$, plays an important role. Instead of multiplying the polytope one could also “shrink” the lattice and consider the sets $P \cap \frac{1}{m}\mathbb{Z}^l$, $m \geq 1$. This leads to the definition of an integral structure on our simplicial complex $\Delta_V$.

**Definition 10.3.** An integral structure on $\Delta_V$ is a collection of subsets $\Delta_V(n) \subset \Delta_V$ for all $n \in \mathbb{N}$ and an affine embedding $i_\Delta : \Delta \to \mathbb{R}^{\dim \Delta}$ for each simplex $\Delta$ in $\Delta_V$ such that:

- the vertices of $i(\Delta)$ have integral coordinates,
- if we denote $\Delta(n) = \Delta \cap \Delta_V(n)$, then
  $$i_\Delta(\Delta(n)) = \{v \in i_\Delta(\Delta(n)) \mid nv \in \mathbb{Z}^{\dim \Delta}\}.$$

The goal is of course to have an integral structure on $\Delta_V$ such that for all $m \in \mathbb{N}$:

$$\Delta_V(m) = \left\{ \frac{1}{m}a \mid a \in \Gamma_\xi, \deg a = m \right\},$$

and to generalize in this way the tools provided by the Newton-Okounkov theory to the setting of Newton-Okounkov simplicial complexes.

In [8] we have constructed such an integral structure in the general case. We will see that in the case of Schubert varieties, the construction recovers the one given by Dehy in [13].

Let $\xi = \{\tau = \tau_r > \ldots > \tau_0 = \text{id}\}$ be a maximal chain in $A_\tau$, and for $k = 1, \ldots, r$ let $\beta_k$ be the positive real root such that $\tau_k = s_{\beta_k} \tau_{k-1}$ and let $b_k = \langle \tau_{k-1}(\lambda), \beta_k^\vee \rangle$ be the bond.

Denote by $\Delta_\xi$ the maximal simplex associated to the given maximal chain. This means in term of the order complex: $\Delta_\xi \subseteq \mathbb{R}^\xi \subseteq \mathbb{R}^{A_\tau}$ is the convex hull of the $e_{\tau_i}$, $i = 0, \ldots, r$. So a typical element in $\Delta_\xi$ is of the form $\sum_{i=0}^r a_i e_{\tau_i}$, where the $a_i \geq 0$ are such that $\sum_{i=0}^r a_i = 1$.

Dehy defined the map $i_{\Delta_\xi} : \Delta_\xi \to \mathbb{R}^r$ as follows:

$$i_{\Delta_\xi}(e_{\tau_j}) = \begin{cases} 0, & \text{if } j = 0, \\ \sum_{k=1}^j b_k e_k, & \text{otherwise,} \end{cases}$$
and the map is extended to the simplex by:

\[ i_{\Delta e} \left( \sum_{k=0}^{r} a_j e_{r_j} \right) = \sum_{k=0}^{r} a_j i_{\Delta e} (e_{r_j}). \]

If \( \Delta' \) is not a maximal simplex, then it is contained in some maximal simplex \( \Delta e \) and we define \( i_{\Delta'} = i_{\Delta e} |_{\Delta'} \). Dehy showed in [13]: these maps glue together and provide a well defined integral structure on \( \Delta e \) in the sense of Definition 10.1.

**Lemma 10.4.** If \( \Delta e \) is the maximal simplex in \( \Delta V \) associated to the maximal chain \( \mathcal{C} \), then for the integral structure constructed above holds:

\[ \Delta_e(m) = \left\{ \frac{1}{m} a \mid a \in \Gamma_, \deg a = m \right\}. \]

**Proof.** Let \( \mathcal{C} = \{ \tau = \tau_r > \ldots > \tau_0 = \text{id} \} \) be the maximal chain, and for \( k = 1, \ldots, r \) let \( \beta_k \) be the corresponding positive real root and \( b_k \) the bond. We have

\[ \Delta_e(m) = \left\{ a = \sum_{i=0}^{r} a_i e_{r_i} \in \Delta_e \mid m(i_{\Delta e}(a)) \in \mathbb{Z} \right\}. \]

Now for \( a = \sum_{i=0}^{r} a_i e_{r_i} \in \Delta e \) we have

\[ m(i_{\Delta}(\sum_{i=0}^{r} a_i e_{r_i})) = m(\sum_{j=1}^{r} a_j(\sum_{k=1}^{j} b_k e_k)) = m(\sum_{k=1}^{r} b_k(a_k + \ldots + a_r)e_k) \in \mathbb{Z} \]

if and only if \( ma \in \text{LS}^+_{\lambda}(m) \). Indeed, recall that \( \sum_{i=0}^{r} a_i = 1 \) and \( a_i \geq 0 \) for all \( i = 0, \ldots, r \). So the degree condition \( m(a_0 + \ldots + a_r) = m \) is automatically satisfied.

**Corollary 10.5.** The Newton-Okounkov simplicial complex \( \Delta_V \) admits an integral structure such that \( \Delta_V(m) = \left\{ \frac{1}{m} a \mid a \in \Gamma(m) \right\} \).

The degree formula in [8], Theorem 13.6, has in this setting the following formulation:

**Proposition 10.6.** The degree of the embedded Schubert variety \( X(\tau) \subseteq \mathbb{P}(V(\lambda)_{\tau}) \) is equal to the sum \( \sum_{\mathcal{C}} \prod_{j=1}^{r} b_j \) running over all maximal chains in \( A_{\tau} \), and \( \prod_{j=1}^{r} b_j \) is the product of all bonds along the maximal chain \( \mathcal{C} \).

The degree formula can also be found in [7], and in [23] in a symplectic context.

11. **Appendix I: LS-paths and the LS-lattice**

We fix a dominant weight \( \lambda \in \Lambda^+ \). Let \( Q \subseteq G \) be the standard parabolic subgroup associated to \( \lambda \), i.e., \( Q \) is generated by the Borel subgroup \( B \) and the root subgroup \( U_{-\alpha} \) for all simple roots \( \alpha \) such that \( \langle \lambda, \alpha^\vee \rangle = 0 \). The Weyl group of \( G \) is denoted by \( W \), the Weyl group of \( Q \) by \( W_Q \).

In this section we recall the notion of a Lakshmibai-Seshadri path of shape \( \lambda \), or, for short an \( \text{LS-path} \) of shape \( \lambda \), and we explain the connection with the fan of monoids \( \text{LS}^+_\lambda \) introduced in Section 5.
11. Chains of elements in $W/W_Q$. We start with the notion of a $(d, \lambda)$-chain as it was introduced by Lakshmibai [31] (see also [43, 44]).

A maximal chain joining two elements $\kappa$ and $\sigma \in W/W_Q$, $\kappa > \sigma$, is a pair of sequences $(\kappa_t, \ldots, \kappa_0; \beta_t, \ldots, \beta_1)$, where

- $\kappa = \kappa_t > \cdots > \kappa_0 = \sigma$ is a chain of elements in $W/W_Q$ and
- $\beta_t, \ldots, \beta_1$ are positive real roots such that
- $s_{\beta_t} \kappa_{i-1} = \kappa_i$ and $\ell(\kappa_i) = \ell(\kappa_{i-1}) + 1$, $i = 1, \ldots, t$.

**Definition 11.1.** Given a positive rational number $d$, the chain $(\kappa_t, \ldots, \kappa_0, \beta_t, \ldots, \beta_1)$ is called a $(d, \lambda)$-chain joining $\kappa$ and $\sigma$ if in addition $d(\kappa_i(\lambda), \beta_i, \kappa_{i-1}) \in \mathbb{Z}$ for all $i = 1, \ldots, t$.

It has been shown in [13] that if one maximal chain between $\kappa$ and $\sigma$ has the property of being a $(d, \lambda)$-chain, then all maximal chains joining $\kappa$ and $\sigma$ are $(d, \lambda)$-chains.

In many jointed papers, Seshadri, Lakshmibai, Musili and others (see [27, 28, 29, 30, 31, 44]) developed in the framework of the standard monomial theory case by case a combinatorial character formula, leading Lakshmibai to the notion of what is now called an LS-path. The name $LS$-path came up in connection with the theory of path models of representations, which was developed in [33, 34].

**Definition 11.2.** An $LS$-path $\pi = (\sigma_p, \sigma_{p-1}, \ldots, \sigma_1; 0, d_p, \ldots, d_1 = 1)$ of shape $\lambda$ is a pair of sequences of elements in $W/W_Q$ and rational numbers such that

- $\sigma : \sigma_p > \sigma_{p-1} > \cdots > \sigma_1$ is a linearly ordered sequence of elements in $W/W_Q$,
- $d : 0 < d_p < \ldots < d_1 = 1$ is a sequence of rational numbers,
- for all $i = 2, \ldots, p$, there exists a $(d_i, \lambda)$-chain joining $\sigma_i$ and $\sigma_{i-1}$.

The support $\text{supp} \pi$ of an $LS$-path $\pi$ is the set $\{\sigma_p, \sigma_{p-1}, \ldots, \sigma_1\}$.

11. Interpretation in terms of a path model. In [33], the definition of LS-paths was transferred from the context of Weyl group combinatorics into the setting of combinatorics of piecewise linear paths in $\Lambda_\mathbb{R} = \Lambda \otimes \mathbb{R}$. Let $\Pi$ be the set of all piecewise linear paths $\phi : [0, 1] \to \Lambda_\mathbb{R}$ starting in $0$ and ending in $\phi(1) \in \Lambda$. (Two paths $\pi_1, \pi_2$ are identified if there exists a nondecreasing, surjective, continuous map $\psi : [0, 1] \to [0, 1]$ (for short: a reparameterization) such that $\pi_1 = \pi_2 \circ \psi$). The endpoint $\phi(1)$ of a path is called the weight of the path. In [33, 34], for all simple roots $\alpha$ folding operators $e_\alpha, f_\alpha$ are defined on the set $\Pi$. For a sufficiently nice path $\pi_0$ ending in a dominant integral weight $\lambda = \pi_0(1)$, let $\mathbb{B}(\pi_0)$ be the smallest set of piecewise linear paths which contains the chosen starting path $\pi_0$ and is stable under all folding operators. This set is called a path model for the representation $V(\lambda)$.

An example for a “nice” path is the straight line path $\pi_\lambda$ which joins the origin with the dominant weight $\lambda$. It turns out that in this special case the set $\mathbb{B}(\lambda)$ (or rather $\mathbb{B}(\pi_\lambda)$ to be precise) is equal to the set of LS-paths of shape $\lambda$ as in Definition 11.2. To make this more precise, we have to give an interpretation of a LS-path as a piecewise linear path in $\Lambda_\mathbb{R}$.

Let $\pi = (\sigma_p, \sigma_{p-1}, \ldots, \sigma_1; d_p+1 = 0, d_p, \ldots, d_1 = 1)$ be an LS-path of shape $\lambda$. We consider $\pi : [0, 1] \to \Lambda_\mathbb{R}$ as the piecewise linear path [33] defined by:

$$t \mapsto (d_p - d_{p+1})\sigma_p(\lambda) + \ldots + (d_j - d_{j+1})\sigma_j(\lambda) + (t - d_j)\sigma_{j-1}(\lambda) \quad \text{for} \quad d_j \leq t \leq d_{j-1}.$$
The endpoint of the path is

\[ \pi(1) = \sum_{i=1}^{p} (d_i - d_{i+1}) \sigma_i(\lambda). \]

The proof of the bijection:

LS-paths of shape \( \lambda \) as in Definition 11.2 \( \longleftrightarrow \) path model \( \mathbb{B}(\lambda) \)
can be found in [33]. The folding operators play a crucial role in the proof.

The following combinatorial character formula was conjectured by Lakshmibai, proved
in many special cases by Seshadri, Lakshmibai, Musili and others in [27, 28, 29, 30, 31],
and proved in full generality in [33, 34] in the framework of the path model theory. Let \( \rho \in \Lambda^+ \) be an element such that \( \langle \rho, \alpha \rangle = 1 \) for all simple roots \( \alpha \), and let \( D_{\alpha} \) be the
Demazure operator on the group ring \( \mathbb{Z}[\Lambda] = \mathbb{Z}[e^\mu \mid \mu \in \Lambda] \) associated to a simple root \( \alpha \):

\[ D_{\alpha}(e^\mu) := e^{\mu+\rho} - e^{\sigma_{\alpha}(\mu+\rho)} \frac{1}{1 - e^{-\alpha} e^{-\rho}}. \]

Let

\[ \mathbb{B}(\lambda)_\tau := \{ \pi = (\sigma_p, \sigma_{p-1}, \ldots, \sigma_1; 0, d_p, \ldots, d_1 = 1) \mid \pi \text{ LS-path of shape } \lambda, \sigma_p \leq \tau \} \]

be the set of all LS-paths of shape \( \lambda \) starting with an element \( \sigma_p \) which is smaller or
equal to \( \tau \). The following character formula was proved in [33]:

**Theorem 11.3.** If \( \tau = s_{i_1} \cdots s_{i_r} \) be a reduced decomposition, then

\[ D_{\alpha_{i_1}} \circ \cdots \circ D_{\alpha_{i_r}} e^\lambda = \sum_{\pi \in \mathbb{B}(\lambda)_\tau} e^{\pi(1)}. \]

It is well known that the projective normality of Schubert varieties is strongly related
to Demazure’s character formula, see, for example [45] and [19]. Or, in other words:

\[ \text{Char } (H^0(X(\tau), L_\lambda)^*) = D_{\alpha_{i_1}} \circ \cdots \circ D_{\alpha_{i_r}} e^\lambda, \]

holds for all very ample line bundles \( L_\lambda \) on \( G/Q \) if and only if \( X(\tau) \subseteq \mathbb{P}(V(\lambda)_\tau) \) is
projectively normal for all these \( \lambda \). One has to be a bit cautious with the formulation
if \( G \) is not of finite type. Over an arbitrary algebraically closed field, the proof of the
projective normality is an essential part in the construction of Schubert varieties, see
[37, 38].

Assuming the normality of Schubert varieties, this formula shows in connection with
Demazure’s character formula: the set of LS-paths in \( \mathbb{B}(\lambda)_\tau \) provides a combinatorial
character formula for Demazure modules, and it shows that this set provides an indexing
system for a basis of the Demazure module \( V_\tau(\lambda) \). Or, in its dual version, \( \mathbb{B}(\lambda)_\tau \) provides
an indexing system for a basis of \( H^0(X(\tau), L_\lambda) \), which is of course what Seshadri and
Lakshmibai had in mind.

On the set \( \Pi \) one has a product, the concatenation of paths. Given \( \pi_1, \pi_2 \in \Pi \), by
the product \( \pi := \pi_1 \ast \pi_2 \) we mean the piecewise linear path obtained by concatenation:

\[ \pi(t) := \begin{cases} 
\pi_1(2t), & \text{if } 0 \leq t \leq \frac{1}{2}; \\
\pi_1(1) + \pi_2(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases} \]
Given an LS-path $\pi \in \mathbb{B}(\lambda)$, then let $i(\pi)$ be the maximal element in $\text{supp} \pi$ and let $e(\pi)$ be the minimal element in $\text{supp} \pi$. We write $\text{supp} \pi_1 \geq \text{supp} \pi_2$ if $e(\pi_1) \geq i(\pi_2)$.

**Definition 11.4.** Given two LS-paths $\pi_1, \pi_2 \in \mathbb{B}(\lambda)$, the concatenation $\pi_1 \ast \pi_2$ is called *standard* if $\text{supp} \pi_1 \geq \text{supp} \pi_2$.

Let $\pi_\lambda$ be again the straight line path joining the origin with the dominant weight $\lambda$. The path $\pi_{2\lambda}$ coincides with the concatenation $\pi_\lambda \ast \pi_\lambda$. It has been shown in [33] that the path model $\mathbb{B}(2\lambda)$ coincides with the path model $\mathbb{B}(\pi_\lambda \ast \pi_\lambda)$ obtained by applying the folding operators to the concatenation $\pi_\lambda \ast \pi_\lambda$. And the elements in $\mathbb{B}(\pi_\lambda \ast \pi_\lambda)$ are exactly the concatenations $\pi := \pi_1 \ast \pi_2, \pi_1, \pi_2 \in \mathbb{B}(\lambda)$, which are standard. Proceeding by induction one can show:

**Proposition 11.5.** Every LS-paths of shape $m\lambda$ can be decomposed as a standard product of $m$ LS-paths of shape $\lambda$. In other words: for any $\pi \in B(m\lambda)$ one can find LS-paths $\pi_1, \ldots, \pi_m \in B(\lambda)$ such that $\text{supp} \pi_1 \geq \text{supp} \pi_2 \geq \ldots \geq \text{supp} \pi_m$, and $\pi$ is (up to reparameterization) equal to $\pi_1 \ast \pi_2 \ast \ldots \ast \pi_m$.

### 11.3. Interpretation in terms of a fan of monoids

Recall the situation in Section 4: we have a Schubert variety $X(\tau) \subseteq G/Q$, embedded in the projective space over a Demazure module $\mathbb{P}(V(\lambda)_\tau)$, endowed with a Seshadri stratification. The partially ordered set $A_\tau$ is the set $\{\sigma \in W/W_Q \mid \sigma \leq \tau\}$.

Let $\mathcal{V}$ be the associated quasi-valuation, which depends on a choice of a total order $!$ of $A_\tau$, see Section 4.3. We introduced in Section 5 the fan of monoids $\text{LS}_\lambda^+ = \text{LS}_\lambda \cap Q^A_{\geq 0}$ as a candidate for the fan of monoids $\Gamma$ associated to $\mathcal{V}$. Note that $\text{LS}_\lambda^+$ does not depend of the choice of the total order!

For $a \in \text{LS}_\lambda$, we denote by $\text{supp} a$ the support $\{\kappa \in A_\tau \mid a_\kappa \neq 0\}$ of $a$. Denote by $\text{LS}_\lambda^+(m)$ the subset of elements of degree $m$, i.e.:

$$\text{LS}_\lambda^+(m) = \left\{ a = \sum_{\kappa \in A_\tau} a_\kappa e_\kappa \in \text{LS}_\lambda^+ \mid \sum_{\kappa \in A_\tau} a_\kappa = m \right\}.$$  

We define now a map $\Theta$ between

$$\mathbb{B}(m\lambda)_\tau = \left\{ \text{LS-paths } \pi \text{ of } \text{shape } m\lambda, \ i(\pi) \leq \tau \right\} \xrightarrow{\Theta} \text{LS}_\lambda^+(m) = \left\{ \text{elements of degree } m \text{ in the fan of monoids } \text{LS}_\lambda^+ \right\}.$$  

The map resembles the weight map in (16). The difference is that the map remembers the Weyl group elements. For $m \geq 1$, we define the map by

$$\pi = (\sigma_p, \ldots, \sigma_1; 0, d_p, \ldots, d_1 = 1) \mapsto \Theta(\pi) := \sum_{j=1}^{p} (d_j - d_{j+1}) me_{\sigma_j},$$

where we set $d_{p+1} = 0$.

See [7], Proposition 1 and the discussion before Proposition 32 for a different proof of the following result.

**Proposition 11.6.** The map $\Theta$ induces a bijection between the set of LS-paths $\mathbb{B}(m\lambda)_\tau$ of shape $m\lambda$ and the set $\text{LS}_\lambda^+(m)$ of elements of degree $m$ in the fan of monoids $\text{LS}_\lambda^+$. The map respects the support, i.e. $\text{supp} \pi = \text{supp} \Theta(\pi)$.
Proof. The map $\Theta$ is obviously an injective map from $\mathbb{B}(m\lambda)_r$ to $\mathbb{Q}^{d_1}$. For a given path $\pi = (\sigma_p, \sigma_{p-1}, \ldots, \sigma_1; 0, d_p, \ldots, d_1 = 1) \in \mathbb{B}(m\lambda)_r$, the condition $0 < d_p < \ldots < d_1 = 1$ implies $\Theta(\pi) \in \mathbb{Q}_{\geq 0}^{d_1}$, the sum of the coefficients of $\Theta(\pi)$ is equal to $m$, and $\text{supp} \pi = \text{supp} \Theta(\pi)$. It remains to show that the image is in $\text{LS}_{\lambda}$, and that the map is surjective.

Let $\pi$ be as above. Fix a maximal chain $\mathcal{C} = (\kappa_r > \ldots > \kappa_0) \in A_r$ such that $\text{supp} \pi \subseteq \mathcal{C}$. Since $\pi$ is an LS-path of shape $m\lambda$, for all $i = 1, \ldots, p$ there exists a $(d_i, m\lambda)$-chain between $\sigma_i$ and $\sigma_{i-1}$. By [13] we know that if one maximal chain between $\sigma_i$ and $\sigma_{i-1}$ is a $(d_i, m\lambda)$-chain, then all maximal chains joining $\sigma_i$ and $\sigma_{i-1}$ are $(d_i, m\lambda)$-chains. So we can fix the $(d_i, m\lambda)$-chains to be subchains of $\mathcal{C}$.

We define an “extended” path $\hat{\pi} = (\kappa_r, \ldots, \kappa_0; d_{r+1} = 0, d_r, \ldots, d_0 = 1)$ as follows: the sequence of Weyl group elements is the entire maximal chain $\mathcal{C}$, and the sequence of rational numbers is now only weakly increasing. The rational numbers $d_j$ are given by the following rules: for $j = 1, \ldots, r$ set:

If $\kappa_j > \sigma_p$, then set $\hat{d}_j = 0$. If $\sigma_i \geq \kappa_j > \sigma_{i-1}$ for some $i = 2, \ldots, p$, then set $\hat{d}_j = d_i$. If $\sigma_1 \geq \kappa_j$, then set $\hat{d}_j = 1$.

From the fact that the “squeezed in” subchains are $(d_i, m\lambda)$-chains, we conclude that the condition on $\pi$ to be an LS-path implies for $\hat{\pi}$:

$$\forall j = 0, \ldots, r: \quad \hat{d}_j (\kappa_j(m\lambda), \beta_j^\gamma) \in \mathbb{Z},$$

where $\beta_j$ is the positive real root such that $s_{\beta_j \kappa_{j-1}} = \kappa_j$.

We adapt the definition of $\Theta$ and set: $\hat{\Theta}(\hat{\pi}) := \sum_{j=1}^r (\hat{d}_j - \hat{d}_{j+1})m e_{\kappa_j}$. By the choice of the $\hat{d}_j$, $j = 1, \ldots, r$, we have $\hat{\Theta}(\hat{\pi}) = \Theta(\pi)$. Now by comparing the coefficients in the equation:

$$\Theta(\pi) = \hat{\Theta}(\hat{\pi}) = \sum_{j=1}^r (\hat{d}_j - \hat{d}_{j+1})m e_{\kappa_j} = \sum_{j=0}^r a_j e_{\kappa_j}$$

we get:

$$m \hat{d}_r = a_r,$$

$$m \hat{d}_{r-1} = m \hat{d}_r + a_{r-1} = a_r + a_{r-1},$$

$$m \hat{d}_{r-2} = m \hat{d}_{r-1} + a_{r-2} = a_r + a_{r-1} + a_{r-2}.$$ 

$$\vdots$$

$$m \hat{d}_0 = m \hat{d}_1 + a_0 = \sum_{i=0}^r a_i$$

Since $b_j = (\kappa_{j-1}(\lambda), \beta_j^\gamma) = |(\kappa_j(\lambda), \beta_j^\gamma)|$ is the bond between $\kappa_{j-1}$ and $\kappa_j$, we see that the LS-path conditions are equivalent to the lattice conditions for $\text{LS}_{\kappa,\lambda}$. We have for all $i = 1, \ldots, r$:

$$\hat{d}_j (\kappa_j(m\lambda), \beta_j^\gamma) = m \hat{d}_j (\kappa_j(\lambda), \beta_j^\gamma) \in \mathbb{Z} \iff (a_r + \ldots + a_j) b_j \in \mathbb{Z}. \quad (19)$$

It follows that $\Theta$ is indeed a well defined injective map from $\mathbb{B}(m\lambda)_r$ to $\text{LS}_{\lambda}^+(m) \subseteq \mathbb{Q}^{d_1}$.

The arguments used above work also vice versa: starting with an element in $\underline{a} \in \text{LS}_{\lambda}^+(m)$, fix a maximal chain such that $\underline{a} \in \text{LS}_{\kappa,\lambda}^+(m)$. One attaches to $\underline{a}$ an “extended” LS-path $\hat{\pi}$ having as sequence of Weyl group elements the elements of the maximal chain, and the rational numbers $\hat{d}_i$ are given by $\hat{d}_i = (a_r + \ldots + a_i)/m$, $i = 0, \ldots, r$. In
addition, set \( d_{r+1} = 0 \). Let \( \pi \) be the pair of sequences obtained by omitting those \( \kappa_{j+1} \) and \( d_{j+1} \) such that \( d_{j+1} = d_j \), \( j = 0, \ldots, r \). The equivalence in (19) shows: this is a LS-path \( \pi \) of shape \( m\lambda \) contained in \( \mathbb{B}(m\lambda)_\tau \).

It follows: the map \( \Theta \) induces a bijection between the LS-paths \( \pi \) in \( \mathbb{B}(m\lambda)_\tau \) and the elements in \( \text{LS}_\lambda^+(m) \).

The set \( \text{LS}_\lambda^+ \) is in general a fan of monoids, but a single monoid. The sum of two elements \( \underline{a}_1, \underline{a}_2 \in \text{LS}_\lambda^+ \) makes sense (meaning is again an element in \( \text{LS}_\lambda^+ \)) only if there exists a maximal chain \( \mathcal{C} \) in \( A_\tau \) such that \( \text{supp} \underline{a}_1, \text{supp} \underline{a}_2 \subseteq \mathcal{C} \). This leads to the following definition:

**Definition 11.7.** A sum \( \underline{a} = \underline{a}_1 + \ldots + \underline{a}_m \) of elements \( \underline{a}_i, \ldots, \underline{a}_m \in \text{LS}_\lambda^+ \) is called standard if \( \text{supp} \underline{a}_1 \geq \text{supp} \underline{a}_2 \geq \ldots \geq \text{supp} \underline{a}_m \). In particular, a standard sum is again an element in \( \text{LS}_\lambda^+ \).

The map \( \Theta \) respects the support, so it turns a concatenation \( \pi_1 \cdots \pi_m \), which is standard, into a sum which is standard: \( \Theta(\pi_1 \cdots \pi_m) = \sum_{i=1}^m \Theta(\pi_i) \). An important consequence: by Proposition 11.5, an LS-path \( \pi \) of shape \( m\lambda \) can always be written in a unique way as a concatenation of LS-paths of shape \( \lambda \): \( \pi = \pi_1 \cdots \pi_m \) such that \( \text{supp} \pi_1 \geq \text{supp} \pi_2 \geq \ldots \geq \text{supp} \pi_m \). For an element in \( \text{LS}_\lambda^+ \) this implies:

**Lemma 11.8.** Every element \( \underline{a} \in \text{LS}_\lambda^+(m) \) has a unique decomposition \( \underline{a} = \underline{a}_1 + \ldots + \underline{a}_m \) into \( m \) elements \( \underline{a}_i \in \text{LS}_\lambda^+(1), \ i = 1, \ldots, m \), such that \( \text{supp} \underline{a}_1 \geq \text{supp} \underline{a}_2 \geq \ldots \geq \text{supp} \underline{a}_m \).

See also [7] for a different proof of Lemma 11.8.

12. **Appendix II: A filtration on \( V_\mathbb{Z}(s\lambda) \)**

12.1. **Some comments on the Appendix II and III.** The reason to add this and the following section as appendices is the slightly different approach compared to the one in [32]. The main goal in *ibidem* was to show that a standard monomial theory in the sense of Lakshmibai, Musili and Seshadri exists for all Schubert varieties in a generalized flag variety, and this in the setting of a symmetrizable Kac-Moody group. So it was sufficient to construct one standard monomial theory. For this the path vectors \( p_\pi \) in \( V(\lambda)^* \) have been constructed, and to prove their linear independence, a basis \( \{ v_\pi \mid \pi \text{ LS path} \} \) of \( V_\mathbb{Z}(\lambda)^* \) was constructed. Both bases depend on certain choices, which, at that time, was not relevant to achieve the main goal.

To show that the construction of a standard monomial theory via a Seshadri stratification is compatible with the construction in [32], one has to take care of these possible choices. For this reason we introduce the filtration of \( V_\mathbb{Z}(\lambda)_\tau \) in (25) respectively (26), and its dual filtration in (36) respectively (37). Note that the definition of a path vector in Definition 13.1 (respectively Definition 6.4) is more general than the corresponding definition in [32]. Indeed, Corollary 13.6 shows how these two definitions are related. Theorem 6.9 shows that the path vectors are representatives of the filtration associated to the Seshadri stratification, independently of the choice of a compatible total order. The main ingredient in the proof is Corollary 13.13, which enables us to proceed with the proof by induction.
Many of the tools and some of the rather technical statements in this and the following section can be, at least implicitly, found in [32]. But the more general notion of a path vector and the idea to use filtrations instead of a fixed ordered basis made it necessary to adapt some of the proofs and definitions. For example, the preorder defined in Definition 12.8 is slightly different from the one used in [32].

In order to present a text written in a uniform style (and not always refer to [32] with some explanation why the results still hold in this slightly different setting) we have decided to rewrite part of [32]. We have declared this part as appendices because apart from minor changes and the adaption to the new setting, none of the results is new except for the following statements: part iii) of Theorem 12.16, as well as Proposition 13.10 and the statements after. They are new because they are now formulated for path vectors in the sense of Definition 13.1.

To conclude, one last observation. The filtration defined in (11) and (12) (see also (25) and (26)) is by construction of combinatorial nature, and so is its dual version defined in (13) and (14). These filtrations are strongly related to the definition of the path vectors. The role of the quantum Frobenius splitting is now reduced to that of a tool proving the linear independence of the vectors $v_{\alpha^\vee}$. So it would be interesting to have a purely representation theoretic interpretation of the filtration defined in (11) and (12).

12.2. Notation. We start with the complex version of the symmetrizable Kac-Moody algebra $\mathfrak{g}_C$. Let $U(\mathfrak{g})_\mathbb{Z}$ be the Kostant-$\mathbb{Z}$-form of the enveloping algebra $U(\mathfrak{g})$, and for a ring $\mathbb{R}$ set $U(\mathfrak{g})_{\mathbb{R}} = U(\mathfrak{g})_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$.

For a simple root $\alpha_i$, $1 \leq i \leq n$, denote by $X_i$ the Chevalley generator of weight $\alpha_i$ and let $X_{-i}$ be the Chevalley generator of weight $-\alpha_i$. We denote by $U(\mathfrak{g})_\mathbb{Z}^+$ respectively $U(\mathfrak{g})_\mathbb{Z}^-$ the part of the Kostant-$\mathbb{Z}$-form generated by the divided powers of the $X_i$ respectively the $X_{-i}$. For a ring $\mathbb{R}$ we set $U(\mathfrak{g})_{\mathbb{R}}^+ = U(\mathfrak{g})_\mathbb{Z}^+ \otimes_\mathbb{Z} \mathbb{R}$, $U(\mathfrak{g})_{\mathbb{R}}^-$ is defined similarly.

We fix a dominant integral weight $\lambda$ and choose a highest weight vector $v_\lambda$ in the complex irreducible $\mathfrak{g}_C$-module $V_C(\lambda)$. Let $V_\mathbb{Z}(\lambda) = U(\mathfrak{g})_\mathbb{Z} v_\lambda$ be the corresponding Kostant-$\mathbb{Z}$-form of the module. We associate to $\tau \in W/W_Q$ in a canonical way an extremal weight vector $v_\tau$ of weight $\tau(\lambda)$ as follows: let $\tau = s_{\alpha_1} \cdots s_{\alpha_r}$ be a reduced decomposition and set $n_r = \langle \lambda, \alpha_i^\vee \rangle$, $n_{r-1} = \langle s_{\alpha_i}(\lambda), \alpha_i^\vee \rangle$, ..., and $n_1 = \langle (s_{\alpha_2} \cdots s_{\alpha_r})(\lambda), \alpha_i^\vee \rangle$. The numbers are all positive, and we set

$$v_\tau = X_{-\alpha_1}^{(n_1)} \cdots X_{-\alpha_{r-1}}^{(n_{r-1})} X_{-\alpha_r}^{(n_r)} v_\lambda.$$  

(20)

Here $X^{(n)}$ is the usual abbreviation for the divided power $X^n_{m!}$. The fact that $v_\tau$ is independent of the chosen reduced decomposition follows from the Verma identities, see [47] respectively [35], Section 39.3: if $\tau = s_{\alpha_1} \cdots s_{\alpha_i} = s_{\alpha_{j_1}} \cdots s_{\alpha_{j_r}}$ are two reduced decompositions and $(n_1, \ldots, n_r)$ respectively $(m_1, \ldots, m_r)$ are the associated tupels of integers, then one has

$$X_{-\alpha_1}^{(n_1)} \cdots X_{-\alpha_{r-1}}^{(n_{r-1})} X_{-\alpha_r}^{(n_r)} = X_{-\alpha_1}^{(m_1)} \cdots X_{-\alpha_{r-1}}^{(m_{r-1})} X_{-\alpha_r}^{(m_r)}$$  

(21)

in the enveloping algebra of $\mathfrak{g}$. The weight vector $v_\tau$ generates the weight space $V_\mathbb{Z}(\lambda)_{\tau(\lambda)}$, which is free of rank one.
We denote by \( V_Z(\lambda)_\tau \) the corresponding Demazure module \( V_Z(\lambda)_\tau = U(g)_Z v_\tau \). For a ring \( \hat{R} \) let \( V_{\hat{R}}(\lambda) = V_Z(\lambda) \otimes_Z \hat{R} \) be the corresponding module for \( U(g)_{\hat{R}} \) over the ring \( \hat{R} \). Similarly, for a ring \( \hat{R} \) we write \( V_{\hat{R}}(\lambda)_\tau = V_Z(\lambda)_\tau \otimes_Z \hat{R} \) for the Demazure module over the ring \( \hat{R} \).

### 12.3. Some combinatorics.

A first step in the construction of the filtration of \( V_Z(s\lambda)_\tau \) is a combinatorial procedure which associates to a given element \( a \in LS^+_\lambda(s) \) a sequence of pairs of simple roots and nonnegative integers.

Let \( \supp a = \{ \tau_q, \tau_{q-1}, \ldots, \tau_1 \} \) be the support of \( a \) with \( \tau_q > \tau_{q-1} > \ldots > \tau_1 \). We fix a reduced decomposition \( \tau_q(s) = s_{i_1} \cdots s_{i_t} \). We have \( s_{i_1} \tau_q < \tau_q \). Let now \( 0 \leq j < q \) be minimal such that \( s_{i_1} \tau_i \leq \tau_i \) for all \( i > j \). Or, in other words, \( j \) is maximal such that \( s_{i_1} \tau_j > \tau_j \), and one knows hence: \( s_{i_1} \tau_j + 1 \geq \tau_j \).

Let \( a' \in \mathbb{Q}^{s_\lambda} \) be the element obtained from \( a \) as follows:

\[
\begin{align*}
\mathbb{Q}' = \left\{ \begin{array}{ll}
\sum_{h=j+1}^q a_{e_{s_i}} e_{\tau_h} + \sum_{h=1}^j a_{e_{\tau_h}}, & \text{if } s_{i_1} \tau_{j+1} \neq \tau_j; \\
\sum_{h=j+2}^q a_{e_{s_i}} e_{\tau_h} + (a_{e_{\tau_j+1}} + a_{\tau_j}) e_{\tau_j} + \sum_{h=1}^{j-1} a_{e_{\tau_h}}, & \text{if } s_{i_1} \tau_{j+1} = \tau_j.
\end{array} \right.
\end{align*}
\]

**Lemma 12.1.** There exists a maximal chain \( \mathcal{C}' \) such that \( a' \in LS^+_{\lambda, \mathcal{C}'}(s) \).

**Proof.** Let \( \alpha \) be a simple root and let \( \sigma \in W/W_Q \) be such that \( s_{\sigma} \alpha < \sigma \). If \( k < \sigma \) is of length one less than \( \sigma \), then either \( s_{\sigma} \alpha = \kappa \), or \( s_{\sigma} \kappa < \kappa \). And, in the latter case, one has \( s_{\sigma} \kappa < s_{\alpha} \kappa \), and the bond between \( s_{\sigma} \kappa \) and \( s_{\alpha} \kappa \). Starting with a maximal chain \( \mathcal{C} \) such that \( a \in LS^+_{\lambda, \mathcal{C}}(s) \), one uses the procedure just described to construct stepwise out of \( \mathcal{C} \) a maximal chain \( \mathcal{C}' \) such that \( a' \in LS^+_{\lambda, \mathcal{C}'}(s) \).

**Remark 12.2.** A proof of Lemma 12.1 in the language of paths can be found in [32].

By the bijection in Proposition 11.6 between LS-paths of shape a multiple of \( \lambda \) and elements of the fan of monoids \( LS^+_\lambda \), we can associate to \( a \in LS^+_\lambda(s) \) an element in the weight lattice. We set:

\[
\text{weight}(a) = \sum_{i=1}^q a_i \tau_i(\lambda) \in \Lambda.
\]

It is the weight of the LS-path \( \Theta^{-1}(a) \) (see (16)) of shape \( s \lambda \) associated to \( a \) via the bijection in (18). In the same way we can associate an element in the weight lattice to \( a' \) (see (22)), and, by construction, we have

\[
\text{weight}(a') - \text{weight}(a) = n_1 \alpha_{i_1} = (a_q |(\tau_q(\lambda), \alpha_{i_1}^\vee)| + \ldots + a_{j+1}|(\tau_{j+1}(\lambda), \alpha_{i_1}^\vee)|)\alpha_{i_1}.
\]

So by the procedure above we associate to \( a \in LS^+_\lambda(s) \) the pair \( ((n_1, \alpha_{i_1}), a') \), where \( n_1 \in \mathbb{N}, \alpha_{i_1} \) is a simple root and \( a' \in LS^+_\lambda(s) \).

Note that \( s_{i_1} \tau_q = s_{i_2} \cdots s_{i_t} \) is a reduced decomposition of \( s_{i_1} \tau_q \), the maximal element in \( \supp a' \). So by repeating the procedure with \( a' \), we associate to \( a \) the tuple \( ((n_1, \alpha_{i_1}), (n_2, \alpha_{i_2}), a'') \), where \( n_1, n_2 \in \mathbb{N}, \alpha_{i_1}, \alpha_{i_2} \) are simple roots and \( a'' \in LS^+_\lambda(s) \).

By repeating the procedure again and again, we arrive at a sequence of integers and simple roots: \( ((n_1, \alpha_{i_1}), (n_2, \alpha_{i_2}), \ldots, (n_t, \alpha_{i_t}), a''') \), where \( a''' \in LS^+_\lambda(s) \) is equal to \( s e id \). The simple roots in the sequence are determined by the reduced decomposition...
\( \tau_q = s_{i_1} \cdots s_{i_t} \) and the integers are determined by \( \underline{a} \) through the procedure explained above.

The procedure associates hence to \( \underline{a} \) and the chosen reduced decomposition \( \underline{r} \) of \( \tau_q \) a sequence of pairs of simple roots and nonnegative integers:

\[
S(\underline{a}, \underline{r}) := ((n_1, \alpha_{i_1}), (n_2, \alpha_{i_2}), \ldots, (n_t, \alpha_{i_t})�)
\]

12.4. Some vectors and a filtration. Given \( \underline{a} \in \text{LS}^+(s) \), let \( \sigma \in \text{supp} \underline{a} \) be the maximal element. We fix a reduced decomposition \( \underline{r} \), i.e. \( \sigma = s_{i_1} \cdots s_{i_t} \), and let \( S(\underline{a}, \underline{r}) = ((n_1, \alpha_{i_1}), (n_2, \alpha_{i_2}), \ldots, (n_t, \alpha_{i_t}) \) be the associated sequence of simple roots and integers as in (24).

**Definition 12.3.** The vector \( v_{\underline{a}, \underline{r}} \in V_\mathbb{Z}(s \lambda)_\tau \) associated to \( \underline{a} \in \text{LS}^+(s) \) and the reduced decomposition \( \underline{r} \) is defined by:

\[
v_{\underline{a}, \underline{r}} = X_{-i_1}^{(n_1)} \cdots X_{-i_t}^{(n_t)} v_{s \lambda}.
\]

**Remark 12.4.** One may view this procedure as a generalization of the construction of extremal weight vectors in (20). Indeed, it is easy to check that if \( \underline{a} \in \text{LS}^+(1) \) is of weight \( \sigma(\lambda) \) for some \( \sigma \in W/W \), then \( v_{\underline{a}, \underline{r}} = v_\sigma \), independent of the chosen reduced decomposition. But in general, note that \( v_{\underline{a}, \underline{r}} \) depends on the choice of the reduced decomposition \( \underline{r} \) of \( \sigma \), see Remark 12.6. The procedure to get the sequence \( S(\underline{a}, \underline{r}) \) was inspired by [40]. K. N. Raghavan and P. Sankaran define in this article a procedure to attach such a sequence to the Young tableaux defined by Lakshmibai, Musili and Seshadri in [29],[30]. These tableaux exist, for example, for all representations of the simple groups of classical type \( A, B, C, D \).

**Remark 12.5.** Let \( \tau = s_{j_1} \cdots s_{j_r} \) be a reduced decomposition. It is well known that \( V_\mathbb{Z}(s \lambda)_\tau \) is spanned by the vectors of the form \( X_{-j_1}^{(p_1)} \cdots X_{-j_r}^{(p_r)} v_{s \lambda} \). By choosing for \( \sigma \preceq \tau \) a compatible reduced decomposition, one sees easily that \( V_\mathbb{Z}(s \lambda)_\sigma \subseteq V_\mathbb{Z}(s \lambda)_\tau \). It follows hence: \( v_{\underline{a}, \underline{r}} \in V_\mathbb{Z}(s \lambda)_\tau \).

We denote by \( V_\mathbb{Z}(s \lambda)_\tau, \underline{a} \) the \( \mathbb{Z} \)-submodule:

\[
V_\mathbb{Z}(s \lambda)_\tau, \underline{a} = \left\{ v_{\underline{a'}, \underline{r'}} \bigg| \underline{a} \triangleright \underline{a'}, \underline{a'} \in \text{LS}^+(s), \underline{r'} \text{ reduced decomposition of the maximal element in supp} \underline{a'} \right\},
\]

and by \( V_\mathbb{Z}(s \lambda)_\tau, \underline{a} \) the \( \mathbb{Z} \)-submodule:

\[
V_\mathbb{Z}(s \lambda)_\tau, \underline{a} = \left\{ v_{\underline{a'}, \underline{r'}} \bigg| \underline{a} \triangleright \underline{a'}, \underline{a'} \in \text{LS}^+(s), \underline{r'} \text{ reduced decomposition of the maximal element in supp} \underline{a'} \right\}.
\]

We refer to Definition 6.1 for the definition of \( \triangleright \).

**Remark 12.6.** In general, \( v_{\underline{a}, \underline{r}} \) depends on the chosen reduced decomposition \( \underline{r} \) of \( \sigma \). So in \( V_\mathbb{Z}(s \lambda)_\tau, \underline{a} \) one may find several vectors \( v_{\underline{a}, \underline{r}} \) and \( v_{\underline{a}, \underline{r'}} \) indexed by the same \( \underline{a} \in \text{LS}^+(s) \), but with different reduced decompositions \( \underline{r} \) and \( \underline{r'} \) of the maximal element \( \sigma \in \text{supp} \underline{a} \). The next aim is to analyze the difference \( v_{\underline{a}, \underline{r}} - v_{\underline{a}, \underline{r'}} \).
12.5. **The Frobenius splitting trick.** To have a closer look at the \( \mathbb{Z} \)-submodules \( V_{\mathbb{Z}}(s\lambda)_{r,\underline{a}} \) defined above, we use representation theory of quantum groups at a root of unity. To keep the notation simple, we assume \( \mathfrak{g} \) to be of type \( \mathbb{A}, \mathbb{D} \) or \( \mathbb{E} \), so the associated Cartan matrix is symmetric. The construction works in the same way for an arbitrary symmetrizable Kac-Moody algebra, only the details are more technical. In particular, the Frobenius splitting involves the Langlands dual Kac-Moody algebra. The details can be found in [32].

Let \( \ell \) be an even positive integer. Let \( A := \mathbb{Z}[q,q^{-1}] \) be the ring of Laurent polynomials and denote by \( R \) the ring \( A/I \), where \( I \) is the ideal generated by the \( 2\ell \)-th cyclotomic polynomial. Let \( \bar{R} \) be the ring obtained by adjoining all roots of unity to \( \mathbb{Z} \). We fix an embedding \( R \hookrightarrow \bar{R} \). If \( \mathbb{K} \) is an algebraically closed field and \( \text{Char} \mathbb{K} = 0 \), then we consider \( \mathbb{K} \) as an \( R \)- respectively \( \bar{R} \)-module by the inclusions \( R \subseteq \bar{R} \subseteq \mathbb{K} \). If \( \text{Char} \mathbb{K} = p > 0 \), then we consider \( \mathbb{K} \) as an \( R \)- respectively \( \bar{R} \)-module by extending the canonical map \( \mathbb{Z} \to \mathbb{K} \) to a map \( R \to \bar{R} \to \mathbb{K} \) (where the first map is given by the projection \( \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) and the inclusion \( \mathbb{Z}/p\mathbb{Z} \subset \mathbb{K} \)).

Similar to the classical case (see [24, 25, 32, 35] for more details), let \( U_q(\mathfrak{g}) \) be the quantum group associated to the symmetrizable Kac-Moody algebra \( \mathfrak{g} \) over the field \( \mathbb{C}(q) \). Denote by \( U_{q,A}(\mathfrak{g}) \) the corresponding Kostant-Lusztig form of the quantum group [35] and let \( U_{\xi}(\mathfrak{g})_{\bar{R}} := U_{q,A}(\mathfrak{g}) \otimes \bar{R} \) be the corresponding quantum group over \( \bar{R} \) at a \( 2\ell \)-th root of unity \( \xi \).

For a simple root \( \alpha_i \) denote by \( E^{\pm}_i \) the generators of the quantum group \( U_{\xi}(\mathfrak{g})_{\bar{R}} \) corresponding to the root \( \alpha_i \) respectively \( -\alpha_i \). It has been shown by Lusztig (see [35]) that the assignment

\[
X_i^{(k)} \mapsto E^{(k\ell)}_i,
\]

extends to an algebra homomorphism \( \text{Fr} : U(\mathfrak{g})^+_{\bar{R}} \to U_{\xi}(\mathfrak{g})^+_{\bar{R}} \), called the splitting of the Frobenius map. Here \( U_{\xi}(\mathfrak{g})^\pm_{\bar{R}} \) denotes the part of the Lusztig form generated by the divided powers of the \( E^{\pm}_i \). There is a corresponding Frobenius splitting map \( \text{Fr}' : U(\mathfrak{g})^-_{\bar{R}} \to U_{\xi}(\mathfrak{g})^-_{\bar{R}} \) sending \( X_i^{(k)} \mapsto E^{-k\ell}_i \).

**Remark 12.7.** The reason to restrict to quantum groups at a \( 2\ell \)-th root of unity for an even \( \ell \) is just for convenience. A splitting also exists for other roots of unity, but one has to be careful because there are restrictions for certain types. For our considerations the case \( 2\ell \)-th root of unity for an even \( \ell \) is sufficient, and the assumptions made here makes it possible to present a uniform approach.

For a dominant weight \( \lambda \in \Lambda^+ \) let \( M(\lambda) \) be the simple \( U_q(\mathfrak{g}) \)-module of highest weight \( \lambda \) over the field \( \mathbb{C}(q) \). We fix an \( A \)-lattice in \( M(\lambda) \) by choosing a highest weight vector \( m_{\lambda} \), and we set: \( M_A(\lambda) = U_{q,A}(\mathfrak{g}) m_{\lambda} \). Set \( M_R(\lambda) := M_A(\lambda) \otimes_A \bar{R} \), then \( M_R(\lambda) \) is a \( U_{\xi}(\mathfrak{g})_{\bar{R}} \)-module such that its character is given by the Weyl character formula. It is called the Weyl module of highest weight \( \lambda \) for \( U_{\xi}(\mathfrak{g})_{\bar{R}} \), see [3, 24, 25] for details and further properties.

The extremal weight spaces are free of rank one. Fix a highest weight vector \( m_{\lambda} \) and, using the quantum Verma identities [35], one can associate to \( \tau \in W/W_Q \) a corresponding extremal weight vector \( m_\tau \in M_R(\lambda) \) of weight \( \tau(\lambda) \), it generates the corresponding rank one \( \bar{R} \)-module. Denote by \( M_R(\lambda)_{\tau} := U_{\xi}(\mathfrak{g})^+_{\bar{R}} \cdot m_\tau \) the Demazure
submodule associated to \( \tau \). Via the Frobenius splitting \( \text{Fr}' \), the \( U_\xi(\mathfrak{g})_R^+ \)-module \( M_{\tilde{R}}(\lambda) \) and its tensor powers become \( U(\mathfrak{g})_{\tilde{R}}^+ \)- respectively \( U(\mathfrak{g})_{\tilde{R}}^- \)-modules. In general, one can not glue these actions together to get a global \( U(\mathfrak{g})_{\tilde{R}} \)-representation.

The module \( M_{\tilde{R}}(\lambda)^{\otimes \ell} \) has a weight space decomposition \( \bigoplus_{\mu \in \Lambda} (M_{\tilde{R}}(\lambda)^{\otimes \ell})_\mu \). Denote by \( (M_{\tilde{R}}(\lambda)^{\otimes \ell})_{1/\ell} \) the \( \tilde{R} \)-submodule \( \bigoplus_{\mu \in \ell \Lambda} (M_{\tilde{R}}(\lambda)^{\otimes \ell})_\mu \). The \( \tilde{R} \)-submodule \( (M_{\tilde{R}}(\lambda)^{\otimes \ell})_{1/\ell} \) is not a \( U_\xi(\mathfrak{g})_{\tilde{R}} \)-submodule, but it is, via the Frobenius splitting map, a \( U(\mathfrak{g})_{\tilde{R}}^+ \)- respectively \( U(\mathfrak{g})_{\tilde{R}}^- \)-submodule. It has been shown in [32]: on this submodule one can glue these two module structures to get a \( U(\mathfrak{g})_{\tilde{R}} \)-module structure.

In particular, we have a natural inclusion of \( U(\mathfrak{g})_{\tilde{R}}^+ \)- respectively of \( U(\mathfrak{g})_{\tilde{R}}^- \)-modules:

\[
V_{\tilde{R}}(\lambda) \hookrightarrow (M_{\tilde{R}}(\lambda)^{\otimes \ell})_{1/\ell} \subseteq M_{\tilde{R}}(\lambda)^{\otimes \ell},
\]

which sends the highest weight vector \( v_\lambda \) to the highest weight vector \( m_{\lambda}^{\otimes \ell} \), and, more generally, an extremal weight vector \( v_\kappa \in V_{\tilde{R}}(\lambda)_\tau \) to \( m_{\kappa}^{\otimes \ell} \). More generally, we consider natural inclusions of \( U(\mathfrak{g})_{\tilde{R}}^+ \)- respectively of \( U(\mathfrak{g})_{\tilde{R}}^- \)-modules:

\[
V_{\tilde{R}}(s\lambda) \hookrightarrow ((M_{\tilde{R}}(s\lambda)^{\otimes \ell})_{1/\ell} \subseteq M_{\tilde{R}}(s\lambda)^{\otimes \ell} \hookrightarrow M_{\tilde{R}}(\lambda)^{\otimes s\ell}.
\]

In the comultiplication formula \( \Delta(E_{\pm j}^{(m)}) \) for the quantum group [35], the generators \( K_j^\pm \) associated to the simple roots \( \alpha_j \) play an important role:

\[
\begin{align*}
\Delta(E_j^{(p)}) &= \sum_{p' + p'' = p} q^{p'p''} E_j^{(p')} K_j^{p''} \otimes E_j^{(p'')} \\
\Delta(E_{-j}^{(p)}) &= \sum_{p' + p'' = p} q^{-p'p''} E_{-j}^{(p')} \otimes K_{-j}^{p''} E_{-j}^{(p'')}. 
\end{align*}
\]

Recall that we consider quantum groups at a root of unity, so \( q = \xi \) is a \( 2\ell \)-th root of unity. For the representations we consider, the generators \( K_{\pm j} \) operate on weight spaces by multiplication with some root of unity. As long as the precise value is not important, we use the notation \( \nu^* \) for “some root of unity”.

12.6. The vectors \( v_{a,\sigma} \) and their image in \( M_{\tilde{R}}(\lambda)^{\otimes s\ell} \). Given \( a \in \text{LS}_\lambda^+(s) \), let \( \sigma \in \text{supp} \ a \) be the maximal element. We fix a reduced decomposition \( \sigma \), i.e. \( \sigma = s_{i_1} \cdots s_{i_t} \).

As a next step we consider the image of \( v_{a,\sigma} \) in \( M_{\tilde{R}}(\lambda)^{\otimes s\ell} \) with respect to the embedding (for an appropriate \( \ell \)):

\[
V_{\tilde{R}}(s\lambda) \hookrightarrow (M_{\tilde{R}}(s\lambda)^{\otimes \ell})_{1/\ell} \hookrightarrow M_{\tilde{R}}(s\lambda)^{\otimes \ell} \hookrightarrow M_{\tilde{R}}(\lambda)^{\otimes s\ell}.
\]

These are embeddings of \( U(\mathfrak{g})^\pm \)-modules, acting on the quantum group modules via \( \text{Fr}' \). We will use the notation \( \circ \) to denote these module structures. For example, for \( m \in M_{\tilde{R}}(\lambda)^{\otimes s\ell} \) and \( X \in U(\mathfrak{g})^\pm \), \( X \circ m := \text{Fr}'(X)(m) \).

We will see that the image of \( v_{a,\sigma} \) (see Definition 12.3) has a natural “maximal term” which is independent of the choice of the reduced decomposition \( \sigma \). To make this statement more precise, we define a preorder on tuples of weight vectors in \( M_{\tilde{R}}(\lambda) \).

Let \( \triangleright \) be the usual partial order on the set of weights: \( \nu \succ \chi \) if the difference \( \nu - \chi \) is a non-negative sum of positive roots. In the following we define a preorder on the set of ordered tuples of non-zero weight vectors. Since rescaling does not affect the preorder, we write the tuples as tensor products.
Definition 12.8. Let $m_{\nu_1}, \ldots, m_{\nu_t}$ and $m_{\chi_1}, \ldots, m_{\chi_t}$ be weight vectors in $M_{\tilde{R}}(\lambda)$ of weight $\nu_1, \ldots, \nu_t$ respectively $\chi_1, \ldots, \chi_t$. We say $m_{\nu_1} \otimes \cdots \otimes m_{\nu_t}$ is smaller than $m_{\chi_1} \otimes \cdots \otimes m_{\chi_t}$, in symbols:

$$m_{\nu_1} \otimes \cdots \otimes m_{\nu_t} \preceq m_{\chi_1} \otimes \cdots \otimes m_{\chi_t},$$

- if $\nu_1 > \chi_1$ and $m_{\nu_1} \in U_{\xi}(\mathfrak{g})_{\tilde{R}}^+ \cdot m_{\chi_1}$,
- or $\nu_1 = \chi_1$, $m_{\nu_1}, m_{\chi_1}$ are linearly dependent, $\nu_2 > \chi_2$ and $m_{\nu_2} \in U_{\xi}(\mathfrak{g})_{\tilde{R}}^+ \cdot m_{\chi_2}$,
- or $\nu_1 = \chi_1$, $\nu_2 = \chi_2$, $m_{\nu_1}, m_{\chi_1}$ are linearly dependent, $m_{\nu_2}, m_{\chi_2}$ are linearly dependent, $\nu_3 > \chi_3$ and $m_{\nu_3} \in U_{\xi}(\mathfrak{g})_{\tilde{R}}^+ \cdot m_{\chi_3}$,
- and so on...

Note that $\succ$ turns into $\preceq$. It is easy to see that for $\alpha, \alpha' \in LS^+(s)$: $m_{\alpha} \succ m_{\alpha'}$ if and only if $\alpha' \succ \alpha$. We use sometimes the following variation of the preorder.

Definition 12.9. Given weight vectors $m_{\nu_1}, \ldots, m_{\nu_t}, m_{\chi_1}, \ldots, m_{\chi_t}$ in $M_{\tilde{R}}(\lambda)$ as above, and a permutation $s \in S_t$, we write

$$m_{\nu_1} \otimes \cdots \otimes m_{\nu_t} \prec_s m_{\chi_1} \otimes \cdots \otimes m_{\chi_t}.$$

- if $\nu_{s(1)} > \chi_{s(1)}$ and $m_{\nu_{s(1)}} \in U_{\xi}(\mathfrak{g})_{\tilde{R}}^+ \cdot m_{\chi_{s(1)}}$,
- or $\nu_{s(1)} = \chi_{s(1)}$, $m_{\nu_{s(1)}}$ and $m_{\chi_{s(1)}}$ are linearly dependent, and $\nu_{s(2)} > \chi_{s(2)}$ and the weight vector $m_{\nu_{s(2)}} \in U_{\xi}(\mathfrak{g})_{\tilde{R}}^+ \cdot m_{\chi_{s(2)}}$,
- and so on...

Let $\mathfrak{c} = (\tau_r, \ldots, \tau_0)$ be a maximal chain in $A_r$ and let $\underline{a} \in LS^+_{\xi, \lambda}(s)$. Let $\ell$ be an even number such that $\ell a_j \in \mathbb{N}$ for all $j = 0, \ldots, r$. Let $\sigma = \tau_q \in \text{supp} \underline{a}$ be the unique maximal element. We associate to $\underline{a}$ the vector

$$m_{\underline{a}} := m_{\tau_q} \otimes \cdots \otimes m_{\tau_0} \in \left( M_{\tilde{R}}(\lambda) \right)^{\otimes \sigma \ell}.$$

Sometimes it is convenient to permute the tensor factors. If $s \in S_{s \ell}$ is a permutation, then let

$$m_{\underline{a}}^s := m_{\kappa_1} \otimes \cdots \otimes m_{\kappa_\ell} \in \left( M_{\tilde{R}}(\lambda) \right)^{\otimes s \ell}$$

be the tensor product of the same factors as in $m_{\underline{a}}$, with the first factor $m_{\kappa_1}$ be the same as the $s(1)$-th factor in $m_{\underline{a}}$, and the second factor $m_{\kappa_2}$ be the same as the $s(2)$-th factor in $m_{\underline{a}}$ and so on.

Remark 12.10. The following proposition provides an important combinatorial tool, we will refer to it later quite often. For this reason it is important to keep the following in mind: the weight space decomposition of $M_{\tilde{R}}(\lambda)$ makes the module $M_{\tilde{R}}(\lambda)^{\otimes s \ell}$ being multigraded, so an element can be written as a linear combination of its multihomogeneous components.

The summands presented in (29) are multihomogeneous, but they do not correspond necessarily to a multihomogeneous decomposition. Indeed, several different terms might be of the same multihomogeneous degree. The summands listed in (29) are coming from the successive application of the root vectors $X_{-i}$ respectively the corresponding elements $E_{-i}$ in the quantum group.
There is one case we will encounter later, where this distinction is not necessary. Let \( \kappa_1, \ldots, \kappa_{sl} \) be elements in \( W/W_Q \). The multihomogeneous weight space of weight \( (\kappa_1(\lambda), \ldots, \kappa_{sl}(\lambda)) \) has rank one and is spanned by \( m_{\kappa_1} \otimes \cdots \otimes m_{\kappa_{sl}} \). So if in some multihomogeneous component of \( v_{\underline{a}, \sigma} = X^{(n_1)}_{-i_1} \cdots X^{(n_r)}_{-i_r} \circ m^{\otimes st}_\lambda \), viewed as an element in \( M_R(\lambda)^{\otimes st} \), a summand of the form \( m_{\kappa_1} \otimes \cdots \otimes m_{\kappa_{sl}} \) shows up, then also one of the terms in (29) is, up to a scalar multiple, equal to \( m_{\kappa_1} \otimes \cdots \otimes m_{\kappa_{sl}} \). Recall that elements \( m^a \) for \( \underline{a} \in LS^+_\lambda(s) \) are of this form.

**Proposition 12.11.** Let \( \underline{a} \in LS^+_\lambda, s, \) let \( \sigma = s_{i_1} \cdots s_{i_r} \) be a reduced decomposition of the maximal element in \( \text{supp} \underline{a} \), and let \( \ell \) be an even number such that \( la_\sigma \in \mathbb{N} \) for all \( \sigma \in \text{supp} \underline{a} \).

i) In the expression for \( v_{\underline{a}, \sigma} = X^{(n_1)}_{-i_1} \cdots X^{(n_r)}_{-i_r} \circ m^{\otimes st}_\lambda \) as an element in \( M_R(\lambda)^{\otimes st} \) we get: \( v_{\underline{a}, \sigma} = m^a + \text{summands of the form} \)

\[
(E^{(h_1)}_{-i_1} \cdots E^{(h_r)}_{-i_r} m_\lambda) \otimes \cdots \otimes (E^{(p_1)}_{-i_1} \cdots E^{(p_r)}_{-i_r} m_\lambda). 
\]

which either vanish or are strictly smaller than \( m^a \):

\[
m^a \blacktriangleright (E^{(h_1)}_{-i_1} \cdots E^{(h_r)}_{-i_r} m_\lambda) \otimes \cdots \otimes (E^{(p_1)}_{-i_1} \cdots E^{(p_r)}_{-i_r} m_\lambda).
\]

In particular, one of these summands is a non-zero multiple of \( m^a'' \) for \( \underline{a}'' \in LS^+_\lambda(s) \) only if \( \underline{a} \succ \underline{a}'' \).

ii) Let \( s \in S_{st} \) be a permutation. In the expression for \( v_{\underline{a}, \sigma} \) as an element in \( M_R(\lambda)^{\otimes st} \) we get: \( v_{\underline{a}, \sigma} = \nu^s v_{\underline{a}, \sigma^s} \) + summands strictly smaller than \( m^a \) with respect to \( \blacktriangleright_s \). In particular, one of these summands is a non-zero multiple of \( m^{a'', \sigma} \) for \( \underline{a}'' \in LS^+_\lambda(s) \) only if \( \underline{a} \blacktriangleright \underline{a}'' \).

Before we come to the proof, let us point out one important consequence. The summands occurring in (29) may become linearly dependent, cancel each other or add up to something. So recall that \( \blacktriangleright \) is a preorder on the associated tuples in (29) and not on the elements of \( M_R(\lambda)^{\otimes st} \).

Let \( \underline{a}, \underline{a}'' \in LS^+_\lambda(s) \), \( \underline{a} \neq \underline{a}'' \), and suppose we have written \( v_{\underline{a}, \sigma} \) as a linear combination of multihomogeneous tensors. As pointed out in Remark 12.10, a non-zero multiple of \( m^{a''} \) occurs in such a sum with a non-zero coefficient only if at least one of the terms in (29) is a non-zero multiple of \( m^{a''} \). Summarizing we have:

**Corollary 12.12.** In an expression of \( v_{\underline{a}, \sigma} \) as a linear combination of multihomogeneous elements in \( M_R(\lambda)^{\otimes st} \), the coefficient in front of \( m^{a''} \) is different from zero only if \( \underline{a} \succ \underline{a}'' \).

**Proof.** (of Proposition 12.11) The claim in i) holds obviously if the maximal element in \( \text{supp} \underline{a} \) is of length zero. We proceed by induction on the length of the maximal element in \( \text{supp} \underline{a} \).

Let \( \underline{a}' \) be obtained from \( \underline{a} \) as in (22). By induction, we know that \( v_{\underline{a}', s_{i_1} \sigma} \) has, viewed as an element in \( M_R(\lambda)^{\otimes st} \), a unique maximal summand which is equal to \( m^a \). This element has the form

\[
m^a' := m_{s_{i_1} \tau_q} \otimes \cdots \otimes m_{s_{i_1} \tau_q} \otimes \cdots \otimes \left( \text{remaining part same as in } m^a \right).
\]
(i) If we apply \( X^{(n_i)}_{-i_1} \) to the term \( m^\alpha \), then we get a sum that runs over all possible \((s \cdot \ell)\)-tuples:

\[
X^{(n_i)}_{-i_1} \circ m^\alpha = \sum \mathbf{v}^*(E^{(k_1)}_{-i_1} m_{s_1 \tau_0}) \otimes \ldots \otimes (E^{(k_n)}_{-i_1} m_{\tau_0}),
\]

such that \( n_1 \ell = k_1 + \ldots + k_n \ell \). Since all tensor factors are weight vectors and the \( K_{-i} \) acts by multiplication with some root of unity, we omit the \( K_{-i} \) in the tensor product formula above and summarize the action of the powers of the \( K_{-i} \) by the symbol \( \mathbf{v}^* \) for “some root of unity”.

For an extremal weight vector \( m_\kappa \) in \( M_\mathcal{R}(\lambda) \) we know: \( E^{(k)}_{-i} m_\kappa = 0 \) if \( k > |\langle \kappa(\lambda), \alpha_\ell^\vee \rangle| \). So if one of the \( k \)'s is too big, then such a term is zero. With respect to the chosen ordering, for a maximal term in the sum (31) the \( k_1 \) will be necessarily maximal, and hence \( k_1 = \langle s_1 \tau_q(\lambda), \alpha_{i_1}^\vee \rangle \). A summand in (31) will be a maximal term with respect to “\( \bigcirc \)” only if the first tensor factor is (up to rescaling) equal to \( m_{\tau_0} \).

If a summand in (31) is such that \( k_1 \) is strictly smaller than \( \langle s_1 \tau_q(\lambda), \alpha_{i_1}^\vee \rangle \), then we get a weight vector in \( M_\mathcal{R}(\lambda) = U_\xi(g)^+ \cdot m_{\tau_0} \), say of weight \( \mu \), such that \( \mu \succ \tau_q(\lambda) \). It follows that, independent of the form of the remaining part of the tensor product, this summand is either zero or strictly smaller than \( m^\alpha \) with respect to the preorder “\( \bigcirc \)”.

The same arguments apply to the exponents \( k_2, k_3 \) etc. up to \( k_{\max} \), where \( k_{\max} = \ell(a_q + \ldots + a_{j+1}) \). Recall that \( n_1 = a_q |\langle \tau_q(\lambda), \alpha_{i_1}^\vee \rangle| + \ldots + a_{j+1} |\langle \tau_{j+1}(\lambda), \alpha_{i_1}^\vee \rangle| \) by (23). So applying \( X^{(n_i)}_{-i_1} \) to the term \( m^\alpha \) gives (up to a root of unity) the desired maximal term \( m^\alpha \) (and this one shows up exactly once), plus terms which are strictly smaller than \( m^\alpha \) with respect to the preorder “\( \bigcirc \)”.

It remains to show that the coefficient in front of \( m^\alpha \) is equal to 1 and not just some root of unity. We write \( m^\alpha \) as \( m'_1 \otimes m'_2 \), where the first part has \( \ell(a_q + \ldots + a_{j+1}) \)-tensor factors and is of the form \( m_{s_{i_1} \tau_q} \otimes \ldots \otimes m_{s_{i_1} \tau_{j+1}} \). For the tensor product we get (in the comultiplication rule (28) we have \( p'' = 0 \) for the first term):

\[
X^{(n_i)}_{-i_1} \circ m^\alpha = E^{(\ell_{n_1})}_{-i_1} \cdot (m'_1 \otimes m'_2) = (E^{(\ell_{n_1})}_{-i_1} \cdot m'_1) \otimes (K^{\ell_{n_1}}_{-i_1} \cdot m'_2) + \text{smaller terms}.
\]

Since the tensor factors in \( m'_1 \) are all extremal weight vectors and the divided power of \( E_{-i_1} \) is maximal, rewriting \( (E^{(\ell_{n_1})}_{-i_1} \cdot m'_1) \) as a sum as in (31), only one summand is non-zero, and this is \( \mathbf{v}^* m_{\tau_q} \otimes \ldots \otimes m_{\tau_{j+1}} \). But now a simple induction procedure shows that in this case the root of unities in front of the term in (28) get cancelled by the powers of the roots of unities coming from the action of the powers of \( K_{-i} \) on the tensor factors. So we get \( \mathbf{v}^* = 1 \) for this part of the tensor product.

It remains to consider the second part: \( (K^{\ell_{n_1}}_{-i_1} \cdot m'_2) \). We know the weight \( \mu = \text{weight}(\underline{\alpha}) \) is an element in \( \Lambda \). In particular, \( \langle \mu, \alpha_{i_1}^\vee \rangle \in \mathbb{Z} \). Let \( \mu_1 \) be the weight \( a_q \tau_q(\lambda) + \ldots + a_{j+1} \tau_{j+1}(\lambda) \) and set \( \mu_2 = \mu - \mu_1 \). These are in general just rational weights, but we know: \( \langle \tau_q(\lambda), \alpha_{i_1}^\vee \rangle, \ldots, \langle \tau_{j+1}(\lambda), \alpha_{i_1}^\vee \rangle \leq 0 \) and \( \langle \tau_j(\lambda), \alpha_{i_1}^\vee \rangle > 0 \). By the local integrality property (see [33]), this implies...
\langle \mu_1, \alpha^{\vee}_{t_1} \rangle \in \mathbb{Z} \text{ and hence } \langle \mu_2, \alpha^{\vee}_{t_1} \rangle \in \mathbb{Z}. \text{ It follows: } (K^{e_{t_1} - 1} \cdot m_2') = \xi^{e_{t_1} (\mu_2, \alpha^{\vee}_{t_1})} \cdot m_2' = m_2' \text{ because: } \langle \mu_2, \alpha^{\vee}_{t_1} \rangle \in \mathbb{Z}, \ell \text{ is even and hence } \ell^2 \text{ is divisible by } 2\ell, \text{ and } \xi \text{ is a } 2\ell\text{-th root of unity.}

(ii) Next consider a summand $m$ in the expression of $v_{a', s_{t_1}, \sigma}$ such that $m \neq m^2$, but the first $\ell(a_q + \ldots + a_{j+1})$ tensor factors of $m$ and $m^2$ coincide. In other words, up to some non-zero constant,

$$m = m_{s_{t_1} \tau} \otimes \cdots \otimes m_{s_{t_1} \tau_{j+1}} \otimes m'.$$

The same procedure as in 1) shows: applying $X^{(n_1)}_{-i_1}$ to the term $m$ gives a sum of tensor products of weight vectors which has a unique maximal element with respect to “$\triangleright$”, and this element is

$$\hat{m} = m_{s_{t_1} \tau_{q}} \otimes \cdots \otimes m_{s_{t_1} \tau_{j+1}} \otimes m',
\ell(a_q + \ldots + a_{j+1})\text{-tensor factors}
$$

so the first $\ell(a_q + \ldots + a_{j+1})$ tensor factors of $\hat{m}$ and $m^2$ coincide. Now the assumption $m^2 \triangleright m$ implies $m^2 \triangleright \hat{m}$, and hence $m^2$ is larger with respect to “$\triangleright$” than all summands in this sum.

(iii) It remains to consider the following case: $m$ is a summand in the expression of $v_{a', s_{t_1}, \sigma}$ and there exists a $1 \leq d \leq \ell(a_q + \ldots + a_{j+1})$, such that (up to non-zero constants) the first $(d - 1)$ tensor factors of $m$ and $m^2$ coincide, but the $d$-th tensor factor does not agree. The $d$-th tensor factor of $m^2$ is an extremal weight vector, say $m_{s_{t_1} \tau_i}$ for some $q \geq i \geq j + 1$. So $m^2 \triangleright m$ implies for the $d$-th tensor factor $m_{\nu_d}$ of $m$: for the weight $\nu_d$ we have $s_{t_1} \tau_i(\lambda) < \nu_d$, and for the weight vector $m_{\nu_d}$ we have $m_{\nu_d} \in M_{\hat{R}}(\lambda)_{s_{t_1} \tau_i} = U_{\xi}(0)^{+}_{\hat{R}} \cdot m_{s_{t_1} \tau_i}$.

We apply now $X^{(n_1)}_{-i_1}$ to the term $m$. We consider first what happens to the first $(d - 1)$ tensor factors. The same procedure as in 1) shows: if one does not apply the maximal possible divided power of $E_{-i_1}$ to the first, second, $\ldots$, $(d - 1)$-th tensor factor (maximal means applying gives still a non-zero result), then the resulting tensor product is a weight vector which is automatically strictly smaller than $m^2$ with respect to “$\triangleright$”.

So let us assume we have a summand $\hat{m}$ in $X^{(n_1)}_{-i_1} \circ m$ where one has applied the maximal possible divided power of $E_{-i_1}$ to the first $(d - 1)$ tensor factors of $m$. But whatever divided power of $E_{-i_1}$ one applies to $m_{\nu_d}$ (= the $d$-th tensor factor): $m_{\nu_d} \in M_{\hat{R}}(\lambda)_{s_{t_1} \tau_i}$ implies $E^{(k)}_{-i_1} \cdot m_{\nu_d} \in M_{\hat{R}}(\lambda)_{\tau_i}$ for all $k \geq 0$. And, if the result for $k > 0$ is not equal to zero, then the $\alpha_{t_1}$-string through $\nu_d$ will never meet $\tau_i(\lambda)$, which implies $m^2 \triangleright \hat{m}$.

Now for $a'' \in \text{LS}^+_s(\sigma)$ the inequality $m^2 \triangleright m^{a''}$ implies automatically $a \triangleright a''$, which finishes the proof of part one of the proposition.

It remains to prove the second part of the proposition. As we have seen in (28), up to multiplication by a root of unity (coming from the action of some $K_j$), the summands
in (29) showing up in the expression of \( v_{\underline{a}, \underline{a}} \) are symmetric with respect to permutation of the factors.

Since we argued always only using the summands as in (29) (remember, we look at them rather as tuples and not as elements in a \( \tilde{R} \)-module), we can apply the arguments as before also to prove part two.

The only difference is the change of the order in which we look at the tensor factors. Instead of using the lexicographic indexing of the tensor factors: first the left most factor, then the second most left factor etc., we take the enumeration given by the permutation \( s \): the “first” factor is the \( s(1) \)-th, the “second” factor is the \( s(2) \)-th etc. Now we use the preorder \( \triangleright_s \), and the same arguments as before to show that in the expression for \( v_{\underline{a}, \underline{a}} \), with respect to the preorder \( \triangleright_s \), there is a unique maximal element \( m^\underline{a}, s \) showing up as a summand with a root of unity as a coefficient. And all the other terms are strictly smaller than \( m^\underline{a}, s \) with respect to \( \triangleright_s \).

If one of the terms in (29) is, up to rescaling, equal to \( m^\underline{a}, s'' \), then, by the symmetry, up to rescaling, \( m^\underline{a}, s'' \) appears as one of the terms in (29). Now \( m^\underline{a}, s \triangleright_s m^\underline{a}, s'' \) implies \( m^\underline{a} \triangleright m^\underline{a}, s'' \) and hence \( \underline{a} \triangleright \underline{a}, s'' \).

12.7. A \( U_{\tilde{Z}(\mathfrak{g})^+} \)-stable filtration with one dimensional leaves. Let \( \mu \in \Lambda \) be an integral weight. Denote by \( V_{\tilde{Z}}(s\lambda)_{\mu} \) the corresponding weight space in \( V_{\tilde{Z}}(s\lambda) \). Given \( \tau \in W/W_Q \), let \( \text{LS}^+_1(s) \) be the set of elements of degree \( s \) in the fan of monoids \( \text{LS}^+_1 \) (associated to the partially ordered set with bonds \( A_r \) in Section 5).

The vectors \( v_{\underline{a}, \underline{a}} \) for \( \underline{a} \in \text{LS}^+_1(s) \) are weight vectors of weight \( \text{weight}(\underline{a}) \) in \( V_{\tilde{Z}}(s\lambda)_{\tau} \). Recall that \( \underline{a} \) is a reduced decomposition of the maximal element in \( \text{supp} \underline{a} \).

If \( V_{\tilde{Z}}(s\lambda)_{\mu} \neq \{0\} \), then there exists a large enough element \( \tau \in W/W_Q \), such that the weight space \( V_{\tilde{Z}}(s\lambda)_{\mu} \) is completely contained in the Demazure module \( V_{\tilde{Z}}(s\lambda)_{\tau} \).

In the following we fix \( \mu \) such that \( V_{\tilde{Z}}(s\lambda)_{\mu} \neq \{0\} \), and \( \tau \in W/W_Q \) such that \( V_{\tilde{Z}}(s\lambda)_{\mu} \) is completely contained in the Demazure module \( V_{\tilde{Z}}(s\lambda)_{\tau} \).

For every \( \underline{a} \in \text{LS}^+_1(s) \) of weight \( \text{weight}(\underline{a}) = \mu \) fix a reduced decomposition \( \underline{a} \) of the maximal element \( \sigma \) in \( \text{supp} \underline{a} \). We denote

\[
\mathbb{B}(s\lambda)_{\mu} := \{ v_{\underline{a}, \underline{a}} | \underline{a} \in \text{LS}^+_1(s), \text{weight}(\underline{a}) = \mu \}.
\]

Lemma 12.13. The set \( \mathbb{B}(s\lambda)_{\mu} \) form a basis for \( V_{\tilde{Z}}(s\lambda)_{\mu} \).

Proof. The set \( \text{LS}^+_1(s) \) is finite, so we can fix an even number \( \ell \) such that \( \ell a_\kappa \in \mathbb{N} \) for all \( \underline{a} \in \text{LS}^+_1(s) \). We consider the embedding \( V_{\tilde{R}}(s\lambda)_{\mu} \hookrightarrow V_{\tilde{R}}(s\lambda)_{\tau} \hookrightarrow M_{\tilde{R}}(s\lambda)^{\otimes \text{st}} \). By Proposition 12.11, an element \( v_{\underline{a}, \underline{a}} \in \mathbb{B}(s\lambda)_{\mu} \) has as maximal term \( m^\underline{a} \) in its expression in \( M_{\tilde{R}}(s\lambda)^{\otimes \text{st}} \), independent of the choice of the reduced decomposition \( \underline{a} \). It follows immediately that the elements in \( \mathbb{B}(s\lambda)_{\mu} \) are linearly independent over \( \tilde{R} \). The coefficient of the leading term is 1, so they remain linearly independent after base change with any algebraically closed field \( \mathbb{K} \). Now the Kac-Weyl character formula for \( V_{\tilde{Z}}(s\lambda) \) and the character formula for the path model ([34], Theorem 9.1) implies that the vectors \( v_{\underline{a}, \underline{a}} \) of weight \( \mu \) are not only linearly independent but form a basis for the weight space \( V_{\tilde{Z}}(s\lambda)_{\mu} \). By construction, the dimensions of the weight spaces are independent of the choice of the algebraically closed field \( \mathbb{K} \), so the (images of the) vectors \( v_{\underline{a}, \underline{a}} \) form a basis for \( V_{\tilde{Z}}(s\lambda)_{\mu} \) for any algebraically closed field \( \mathbb{K} \).
As a consequence we see that the $\mathbb{Z}$-lattice $L$ spanned by the $v_{\underline{a}, \sigma}$ of weight $\mu$ in $V^\ast_Z(s \lambda)_{\mu}$ has the same rank as $V^\ast_Z(s \lambda)_{\mu}$. It follows that the quotient $V^\ast_Z(s \lambda)_{\mu}/L$ is a torsion module. Let $\bar{v} \in V^\ast_Z(s \lambda)_{\mu}/L \setminus \{0\}$ and fix a representative $v \in V^\ast_Z(s \lambda)_{\mu}$. Let $d > 1$ be the minimal positive integer such that $dv \in L$, say $dv = \sum_d a_{d, \underline{a}, \sigma}$, where the sum is running over all $\underline{a}$ of weight $\mu$.

Since $d$ is supposed to be minimal, there exists a prime $p$ dividing $d$ and an index $\underline{a}'$ such that $d_{\underline{a}'} \neq 0$ and $p$ does not divide $d_{\underline{a}'}$. If $\mathbb{K}$ is an algebraically closed field of characteristic $p$, then the image of $dv$ in $V^\ast_Z(s \lambda)_{\mu}$ is zero, which implies that the linear combination $\sum_{\underline{a}} d_{\underline{a}} v_{\underline{a}, \sigma}$ in $V^\ast_Z(s \lambda)_{\mu}$ is equal to zero. Since the images of the $v_{\underline{a}, \sigma}$ in $\mathbb{K}$ are linearly independent, this implies that the coefficients $d_{\underline{a}'}$ are all equal to zero, in contradiction to the assumption that $d_{\underline{a}'}$ is not divisible by $p$. It follows: $V^\ast_Z(s \lambda)_{\mu}/L = \{0\}$, and hence $\mathbb{B}(s \lambda)_{\mu}$ spans $V^\ast_Z(s \lambda)_{\mu}$. 

Given $\underline{a} \in \text{LS}_\lambda^+(s)$ and the vectors $v_{\underline{a}, \sigma} \in V^\ast_Z(s \lambda)_{\sigma}$, it is natural to inspect the dependence on the choice of the reduced decompositions $\sigma$. Let $\sigma'$ be a different reduced decomposition of the maximal element in $	ext{supp}\, \underline{a}$, and let $v_{\underline{a}, \sigma'} \in V^\ast_Z(s \lambda)_{\sigma'}$ be the corresponding vector.

**Lemma 12.14.** We have:

$$v_{\underline{a}, \sigma'} = v_{\underline{a}, \sigma} + \sum_{\underline{a}'' \in \text{LS}_\lambda^+(s)_{\mu}, \text{weight}(\underline{a}'') = \mu, \underline{a} \not> \underline{a}''} b_{\underline{a}''} v_{\underline{a}'' \sigma''}.$$ 

**Proof.** Let $\mu = \text{weight}(\underline{a})$. We assume first $\tau' \geq \tau$ is large enough so that $V^\ast_Z(s \lambda)_{\mu}$ is completely contained in $V^\ast_Z(s \lambda)_{\tau'}$. We fix a basis $\mathbb{B}(s \lambda)_{\mu}$ of $V^\ast_Z(s \lambda)_{\mu}$ as in Lemma 12.13. Without loss of generality we can fix the basis so that $v_{\underline{a}, \sigma} \in \mathbb{B}(s \lambda)_{\mu}$. Lemma 12.13 implies then: we can write $v_{\underline{a}, \sigma'}$ as a linear combination of the elements in $\mathbb{B}(s \lambda)_{\mu}$.

To be more precise, we use the Frobenius splitting trick. The set $\text{LS}_\lambda^+(s)$ is finite, so we can fix an even number $\ell$ such that $\ell a_{\lambda} \in \mathbb{N}$ for all $\underline{a} \in \text{LS}_\lambda^+(s)$. We consider the embedding $V^\ast_{\tilde{R}}(s \lambda)_{\mu} \hookrightarrow V^\ast_{\tilde{R}}(s \lambda)_{\tau'} \hookrightarrow M^\ast_{\tilde{R}}(\lambda)_{r \ell}$. As an immediate consequence of Proposition 12.11 and Corollary 12.12 we get

$$v_{\underline{a}, \sigma'} = v_{\underline{a}, \sigma} + \sum_{\underline{a}'' \sigma'' \in \mathbb{B}(s \lambda)_{\mu}, \underline{a} \not> \underline{a}''} b_{\underline{a}''} v_{\underline{a}'' \sigma''}.$$ 

It remains to replace $\tau'$ by $\tau$ again. Now $\underline{a} \not> \underline{a}''$ implies in particular that $\sigma$, the largest element in $\text{supp}\, \underline{a}$, is greater or equal in the Bruhat order than $\sigma''$, the largest element in $\text{supp}\, \underline{a}''$. It follows that $v_{\underline{a}'' \sigma''} \in \mathbb{B}(s \lambda)_{\mu}$ and $\underline{a} \not> \underline{a}''$ implies automatically $\underline{a} \in \text{LS}_\lambda^+(s)$, weight$(\underline{a}) = \mu$ and $\underline{a} \not> \underline{a}''$, which finishes the proof. 

Because of Lemma 12.14 it makes sense to write just $v_{\underline{a}}$ instead of $v_{\underline{a}, \sigma}$ if no confusion is possible. It means we fix for $\underline{a} \in \text{LS}_\lambda^+(s)$ a reduced decomposition $\sigma$ for the maximal element in $\text{supp}\, \underline{a}$, and the difference $v_{\underline{a}, \sigma} - v_{\underline{a}, \sigma'}$ we get by choosing a different decomposition $\sigma'$ can be neglected in the corresponding context.

**Lemma 12.15.** Let $\underline{a} \in \text{LS}_\lambda^+(s)$ be of weight $\mu$ and $1 \leq i \leq n$. If $k \geq 1$, then

$$X_i^{(k)} v_{\underline{a}} = \sum_{\underline{a} \not> \underline{a}'} d_{\underline{a}, \underline{a}'} v_{\underline{a}'}$$ 

for some $\underline{a}' \in \text{LS}_\lambda^+(s)$. 

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Proof. If necessary, we choose first an element $\tau' \geq \tau$ which is large enough so that for $\mu' = \mu + k\alpha_i$ with $k \geq 0$ we have: $V_\mathbb{Z}(s\lambda)_{\mu'} \subseteq V_\mathbb{Z}(s\lambda)_{\tau'}$.

The set $\text{LS}_+^{\tau',r}(s)$ associated to $A_{\tau'}$ is finite, so we can fix an even integer $\ell$ such that $\ell a_\alpha \in \mathbb{N}$ for all $a_\alpha \in \text{LS}_+^{\tau',r}(s)$ (we add a $\tau'$ as index to emphasize that we consider the set $A_{\tau'}$).

We consider the embedding $V_\mathbb{Z}(s\lambda)_{\mu} \hookrightarrow V_{\mathbf{R}}(s\lambda)_{\tau'} \hookrightarrow M_{\mathbf{R}}(\lambda)_{\tau'}^\otimes k\ell$. By Proposition 12.11 we can write $v_{a_\alpha}$ in $M_{\mathbf{R}}(\lambda)_{\tau'}^\otimes k\ell$ as $m_{a_\alpha}$ plus multihomogeneous terms which are strictly smaller than $m_{a_\alpha}$ with respect to $\triangleright$. Applying $X_i^{(k)}$ to one of the latter terms gives again a sum of tensor products of weight vectors, all strictly smaller than $m_{a_\alpha}$.

It remains to consider $X_i^{(k)} \circ m_{a_\alpha}$. By the definition of $\triangleright$, the result is a sum of tensor products of weight vectors, all strictly smaller than $m_{a_\alpha}$.

The vector $X_i^{(k)} v_{a_\alpha}$ is of weight $\mu' = \text{weight}(a_\alpha) + k\alpha_i$. The weight space $V_\mathbb{Z}(s\lambda)_{\mu'}$ has the set $\mathbb{B}(s\lambda)_{\mu'}$ as basis (Lemma 12.13), it consists of vectors of the form $v_{a_\alpha}$, weight$(a_\alpha') = \mu'$.

Now Proposition 12.11 together with Corollary 12.12 implies: in an expression of $X_i^{(k)} v_{a_\alpha}$ as a linear combination of the elements in $\mathbb{B}(s\lambda)_{\mu'}$ only those $v_{a_\alpha'}$ can occur such that $a_\alpha \triangleright a_\alpha'$.

Now we come back to the assumption at the beginning: $\tau'$ is large enough so that $V_\mathbb{Z}(s\lambda)_{\mu'} \subseteq V_\mathbb{Z}(s\lambda)_{\tau'}$. But $a_\alpha \triangleright a_\alpha'$ implies in particular that the maximal element in $\text{supp} a_\alpha$ is greater or equal to the maximal element in $\text{supp} a_\alpha'$. But this implies automatically: if $\tau$ is greater or equal to the maximal elements in $\text{supp} a_\alpha$, then $v_{a_\alpha}$ and $v_{a_\alpha'}$ are already elements in $V_\mathbb{Z}(s\lambda)_{\tau}$. So the expression $X_i^{(k)} v_{a_\alpha} = \sum_{a_\alpha' \triangleright a_\alpha} d_{a_\alpha'} v_{a_\alpha'}$ holds already in $V_\mathbb{Z}(s\lambda)_{\tau}$.

We summarize the results of this section in the following (already known) statements and in a theorem:

i) The Demazure module $V_\mathbb{Z}(s\lambda)_{\tau}$ is a direct summand of the $\mathbb{Z}$-module $V_\mathbb{Z}(s\lambda)$, having as basis the set $\{v_{a_\alpha} \mid a_\alpha' \in \text{LS}_+^{\tau}(s)\}$.

ii) For $a_\alpha \in \text{LS}_+^{\tau}(s)$, the $\mathbb{Z}$-submodule $V_\mathbb{Z}(s\lambda)_{\tau,a_\alpha}$ of $V_\mathbb{Z}(s\lambda)_{\tau}$ is a $U(\mathfrak{g})_Z^+$-stable direct summand of $V_\mathbb{Z}(s\lambda)_{\tau}$, and it has as basis the set $\{v_{a_\alpha'} \mid a_\alpha' \in \text{LS}_+^{\tau}(s), a_\alpha \triangleright a_\alpha'\}$.

Theorem 12.16. The $\mathbb{Z}$-submodules $V_\mathbb{Z}(s\lambda)_{\tau,a_\alpha}$, $a_\alpha \in \text{LS}_+^{\tau}(s)$, define a $U(\mathfrak{g})_Z^+$-stable collection of subspaces of $V_\mathbb{Z}(s\lambda)_{\tau}$ with leaves $V_\mathbb{Z}(s\lambda)_{\tau,a_\alpha}/V_\mathbb{Z}(s\lambda)_{\tau,a_\alpha}$ free of rank 1.

Proof. Given a weight $\mu$, we can always choose an element $\tau'$ such that the weight space $V_\mathbb{Z}(s\lambda)_{\mu}$ is contained in the Demazure module $V_\mathbb{Z}(s\lambda)_{\tau'}$. By Lemma 12.13, we have a basis $\mathbb{B}(s\lambda)_{\mu}$ for each weight space $V_\mathbb{Z}(s\lambda)_{\mu}$, so the union of these bases provide a basis for the representation space $V_\mathbb{Z}(s\lambda)$.

We come now back to the fixed element $\tau$. The subset $\mathbb{B}_{\tau}(s) = \{v_{a_\alpha} \mid a_\alpha' \in \text{LS}_+^{\tau}(s)\} \subset \mathbb{B}$ consists of basis vectors contained in $V_\mathbb{Z}(s\lambda)_{\tau}$. By Lemma 12.15, the $\mathbb{Z}$-span of $\mathbb{B}_{\tau}(s)$ is stable under the action of $U(\mathfrak{g})_Z^+$. The extremal weight vector $v_{\tau}$ is an element in $\mathbb{B}_{\tau}(s)$, which implies that $\mathbb{B}_{\tau}(s)$ is a basis for $V_\mathbb{Z}(s\lambda)_{\tau}$, and hence the latter is a direct summand of $V_\mathbb{Z}(s\lambda)$.

By Lemma 12.14, the subset $\mathbb{B}_{\tau,a_\alpha}(s) = \{v_{a_\alpha} \in \mathbb{B}_{\tau}(s) \mid a_\alpha' \leq a_\alpha\}$ is a generating set for $V_\mathbb{Z}(s\lambda)_{\tau,a_\alpha}$ and, because of the linear independency, it is a basis. This implies also that
$V_Z(s\lambda)_{\tau,\underline{a}}$ is a direct summand of $V_Z(s\lambda)_\tau$. By Lemma 12.15, $V_Z(s\lambda)_{\tau,\underline{a}}$ is stable under the action of $U(\mathfrak{g})^+_\tau$. The subquotient $V_Z(s\lambda)_{\tau,\underline{a}}/V_Z(s\lambda)_{\tau,\underline{a'}}$ is spanned by the class $v_{\underline{a}}$ of $v_{\underline{a}}$, and hence is free of rank 1.

13. Appendix III: The path vectors and some relations

We keep the same notation as in Section 12.

In the Appendix II we defined a filtration of $V_Z(s\lambda)_{\tau}$ given by the subspaces $V_Z(s\lambda)_{\tau,\underline{a}}$, $\underline{a} \in LS^+(s)$. The vectors $v_{\underline{a}} \in V_Z(s\lambda)_{\tau,\underline{a}}$ defined in Definition 12.3 are representatives of the leaves associated to this filtration on $V_Z(s\lambda)_{\tau}$.

We will define a kind of dual basis which again induces a filtration, but this time on the dual space. We recall Definition 6.4:

**Definition 13.1.** Let $\underline{a} \in LS^+(s)$. A path vector associated to $\underline{a}$ is a linear function $p_{\underline{a}} \in V(s\lambda)^*_\tau$ which is a $T$-eigenvector of weight ($-\operatorname{weight}(\underline{a})$), and such that

i) there exists a reduced decomposition $\underline{a}$ of the maximal element $\sigma$ in $\supp \underline{a}$ such that $p_{\underline{a}}(v_{\underline{a},\sigma}) = 1$;

ii) for $\underline{a}' \in LS^+(s)$ and for some reduced decomposition $\underline{a}'$ of the maximal element $\sigma'$ in $\supp \underline{a}'$, $p_{\underline{a}}(v_{\underline{a}',\underline{a}'}) \neq 0$ implies $\underline{a}' \succ \underline{a}$.

**Remark 13.2.** This definition is independent of the choice of the reduced decomposition $\underline{a}$ (resp. $\underline{a}'$ of $\sigma'$), see Lemma 6.5.

More generally, for an appropriate ring $R$ (resp. an algebraically closed field $\mathbb{K}$), we say $p_{\underline{a}} \in V_R(s\lambda)^*_\tau$ (resp. $V_\mathbb{K}(s\lambda)^*_\tau$) is a path vector if it satisfies the same conditions with respect to $v_{\underline{a}',\underline{a}'}$ and $v_{\underline{a},\underline{a}} \in V_R(s\lambda)_{\tau}$ (resp. $V_\mathbb{K}(s\lambda)_{\tau}$).

We will use the Frobenius splitting trick to construct examples of such functions in $V_R(s\lambda)^*_\tau$. We fix a dominant weight $\lambda \in \Lambda^+$ and an element $\tau \in W/W_Q$. Given $\underline{a} \in LS^+(s)$, we fix an even number $\ell$ such that $\ell a_\kappa \in \mathbb{N}$ for all $\kappa \in \supp \underline{a}$. We have seen (see Section 12.5) that we have a natural inclusion of $U(\mathfrak{g})^+_\tau$-modules:

$$V_R(s\lambda)_{\tau} \hookrightarrow M_R(\lambda)^{\otimes s\ell},$$

(32)

The inclusion induces a restriction map for the corresponding dual modules:

$$\operatorname{res} : (M_R(\lambda)^*_\tau)^{\otimes s\ell} \twoheadrightarrow V_R(s\lambda)^*_\tau.$$

(33)

For an extremal weight vector $m_\kappa \in M_R(\lambda)_{\tau}$ denote by $x_\kappa \in M_R(\lambda)^*_\tau$ the dual vector. Let $\supp \underline{a} = \{\tau_q, \ldots, \tau_1\}$ with $\tau_q \ldots > \tau_1$. As a first step we associate to $\underline{a}$ the tensor product of the following extremal weight vectors:

$$x_\underline{a} := x_{\tau_q} \otimes \ldots \otimes x_{\tau_1} \otimes x_{\tau_{q-1}} \otimes \ldots \otimes x_{\tau_1} \otimes \ldots \otimes x_{\tau_1} \in (M_R(\lambda)^*_\tau)^{\otimes s\ell}.$$  

(34)

**Lemma 13.3.** The image $p_{\underline{a},\ell} := \operatorname{res}(x_\underline{a})$ in $V_R(s\lambda)^*_\tau$ is a path vector in $V_R(s\lambda)^*_\tau$.

**Proof.** This is a direct consequence of Proposition 12.11 and Corollary 12.12, which imply:

$$p_{\underline{a},\ell}(v_{\underline{a},\underline{a}}) = 1 \quad \text{and} \quad p_{\underline{a},\ell}(v_{\underline{a}',\underline{a}'}) \neq 0 \quad \text{only if} \quad \underline{a} \preceq \underline{a}'.$$

(35)
Definition 13.4. For $a \in LS^+_\lambda(s)$ and $\ell$ chosen as above, we call $p_{a,\ell} \in V_{\mathcal{R}}(s\lambda)_\tau^*$ the path vector associated to $a$ and $\ell$.

The set $LS^+_\lambda(s)$ is finite, so we can choose an even number $\ell$ such that $\ell a_\kappa \in \mathbb{N}$ for all $\kappa \in \text{supp } a$ and all $a \in LS^+_\lambda(s)$. For such a choice of $\ell$ we get:

Lemma 13.5. The collection of path vectors $\{p_{a,\ell} \mid a \in LS^+_\lambda(s)\}$ is a basis for $V_{\mathcal{R}}(s\lambda)_\tau^*$.

Proof. By fixing a linearization of the partial order $\leq$ on $LS^+_\lambda(s)$, (35) implies that the base change matrix between $\{p_{a,\ell} \mid a \in LS^+_\lambda(s)\}$ and the dual basis of $\{v_b \mid b \in LS^+_\lambda(s)\}$ in Theorem 12.16 is upper-triangular unipotent. $\blacksquare$

Corollary 13.6. Let $\ell$ be as above. For $a \in LS^+_\lambda(s)$ let $p_a \in V_{\mathcal{R}}(s\lambda)_\tau^*$ be a path vector. Then $p_a = p_{a,\ell} + \sum_{a' \triangleright a} m_{a,a'} a_{a'} \ell$.

Proof. Fix the same linearization as in the proof of Lemma 13.5. From the definition of path vectors, the base change matrix between $\{p_{a,\ell} \mid a \in LS^+_\lambda(s)\}$ and the dual basis of $\{v_b \mid b \in LS^+_\lambda(s)\}$ in Theorem 12.16 is also upper-triangular unipotent. $\blacksquare$

Therefore it makes sense to define the following $\mathcal{R}$-submodules of $V_{\mathcal{R}}(s\lambda)_\tau^*$ for $a \in LS^+_\lambda(s)$:

(36) $V_{\mathcal{R}}(s\lambda)_\tau^{|a,\ell} = \left\langle p_a \mid p_a \text{ a path vector, } a' \in LS^+_\lambda(s), a' \triangleright a \right\rangle_{\mathcal{R}}$

and set

(37) $V_{\mathcal{R}}(s\lambda)_\tau^{|a,a'} = \left\langle p_{a'} \mid p_{a'} \text{ a path vector, } a' \in LS^+_\lambda(s), a' \triangleright a \right\rangle_{\mathcal{R}}$.

Corollary 13.7. These subspaces define a filtration of $\mathcal{R}$-modules on $V_{\mathcal{R}}(s\lambda)_\tau^*$ with leaves $V_{\mathcal{R}}(s\lambda)_\tau^{|a,a'}/V_{\mathcal{R}}(s\lambda)_\tau^{|a,a''}$ free of rank 1. A path vector $p_a$ is a representative of such a leaf associated to $a$.

Remark 13.8. We fix an even number $\ell$ such that for all $a$ in $LS^+_\lambda(1)$ and all $\kappa \in \text{supp } a$ one has: $\ell a_\kappa \in \mathbb{N}$. If $a \in LS^+_\lambda(s)$ for some $s > 1$, then $a$ has a unique decomposition $a = a^1 + \ldots + a^s$, where $a^1, \ldots, a^s \in LS^+_\lambda(1)$ (Lemma 11.8). So the number $\ell$ chosen above has the property: for all $m \geq 1$ and all $a$ in $LS^+_\lambda(m)$ and all $\kappa \in \text{supp } a$ one has $\ell a_\kappa \in \mathbb{N}$.

Lemma 13.9. Let $\mathfrak{c}$ be a maximal chain, $a \in LS^+_{\mathfrak{c},\lambda}(s_1)$ and $b \in LS^+_{\mathfrak{c},\lambda}(s_2)$. Then $p_{a,\ell} p_{b,\ell} = c p_{a+b,\ell} + \sum_{a' \triangleright a+b} d_{a'} p_{a'}$, where $c$ is a root of unity, $p_{a+b,\ell}$ is the path vector associated to $\ell$ and $a + b \in LS^+_{\mathfrak{c},\lambda}(s_1 + s_2)$, and the $p_{a'}$ are path vectors for the leafs associated to $a' \in LS^+_{\mathfrak{c},\lambda}(s_1 + s_2)$.

Proof. This is an immediate consequence of part ii) of Proposition 12.11, subject to the following consideration: We view $p_{a,\ell} p_{b,\ell}$ as a function on $V_{\mathcal{R}}((s_1 + s_2)\lambda)_\tau$ via the restriction of $p_{a,\ell} \otimes p_{b,\ell} \in V_{\mathcal{R}}((s_1\lambda)_\tau \otimes V_{\mathcal{R}}(s_2\lambda)_\tau^*)$ to $V_{\mathcal{R}}((s_1 + s_2)\lambda)_\tau \otimes V_{\mathcal{R}}(s_2\lambda)_\tau$.

For the definition of $p_{a,\ell}$ (resp. $p_{b,\ell}$) we have viewed $V_{\mathcal{R}}((s_1\lambda)_\tau^*)$ (resp. $V_{\mathcal{R}}((s_2\lambda)_\tau^*)$) as being embedded in $M_{\mathcal{R}}(\lambda)_\tau^{|a,\ell}$ (resp. $M_{\mathcal{R}}(\lambda)_\tau^{|b,\ell}$). So we view $V_{\mathcal{R}}((s_1 + s_2)\lambda)_\tau^*$ and the tensor product $V_{\mathcal{R}}((s_1\lambda)_\tau \otimes V_{\mathcal{R}}(s_2\lambda)_\tau$ as being embedded in $M_{\mathcal{R}}(\lambda)_\tau^{|(s_1+s_2),\ell}$. 


Now one has to be careful with the coproduct. One can view $M(\lambda)^{\otimes (s_1+s_2)}_R$ as a $U(\mathfrak{g})^+_R$ module by viewing $M(\lambda)^{\otimes (s_1+s_2)}_R$ as a tensor product of $U(\mathfrak{g})^+_R$ modules, and one takes then Fr to turn it into a $U(\mathfrak{g})^+_R$-module. This is the way we view the module in the context of Proposition 12.11.

The way we look here at the module is different. We see $M(\lambda)^{\otimes (s_1+s_2)}_R$ and $M(\lambda)^{\otimes s_2}_R$ as tensor products of $U(\mathfrak{g})^+_R$-modules, and we make them via Fr into $U(\mathfrak{g})^+_R$-modules, and one takes then the tensor product of $U(\mathfrak{g})^+_R$-modules.

So we have on $M(\lambda)^{\otimes (s_1+s_2)}_R$ two different $U(\mathfrak{g})^+_R$-module structures, and this makes a difference for the image of a vector $v_{a,\sigma}, a \in L_S(s_1 + s_2)$, in $M(\lambda)^{\otimes (s_1+s_2)}_R$. But it is now easy to see that the only difference is: to get

$$v_{a,\sigma} = X^{(n_1)} \cdots X^{(n_r)} \circ (m^{s_1}_\lambda \otimes m^{s_2}_\lambda),$$

with respect to the second $U(\mathfrak{g})^+_R$-structure, one has first to apply the coproduct for the Lie algebra:

$$(38) \sum_{j_1=0}^{n_1} \cdots \sum_{j_r=0}^{n_r} (X^{(j_1)}_{-i_1} \cdots X^{(j_r)}_{-i_r} \circ m^{s_1}_\lambda) \otimes (X^{(n_1-j_1)}_{-i_1} \cdots X^{(n_r-j_r)}_{-i_r} \circ m^{s_2}_\lambda),$$

and then apply the Frobenius morphism. One checks now as before: the maximal term is still $m^a$, but the coefficient in front of it might be a root of unity. Since we use first the coproduct for the Lie algebra and then the Frobenius morphism, there are less summands than in the formula in (29).

But, up to multiplying by roots of unities, all terms $(E^{(h_1)}_{-i_1} \cdots E^{(h_r)}_{-i_r} m^r \lambda) \otimes \cdots \otimes (E^{(p_1)}_{-i_1} \cdots E^{(p_r)}_{-i_r} m^s \lambda)$ which show up in (38) after applying the Frobenius morphism, also occur also in (29). So we still can apply Proposition 12.11.

**Proposition 13.10.** Let $\mathfrak{c}$ be a maximal chain, $a \in L_S^+(s_1)$ and $b \in L_S^+(s_2)$. If $p_a$ and $p_b$ are path vectors associated to the leaves $a$ respectively $b$, then $p_\mathfrak{c}$ is a path vector associated to $a + b \in L_S^+(s_1 + s_2)$, up to multiplying by a root of unity.

**Proof.** Let $\ell$ be an even number such that $\ell a_\kappa \in \mathbb{N}$ for all $\kappa \in \text{supp } a$ and all $a \in L_S^+(1)$.

(1) The same property holds then for all $a \in L_S^+(s)$ and all $s \geq 1$, see Remark 13.8.

By Corollary 13.6, we know for $a \in L_S^+(s_1): p_a = p_{a,\ell} + \sum_{a' \geq a} d_{a',a} p_{a',\ell}$, and we have a similar expression for $p_b$.

Lemma 13.9 implies: $p_\mathfrak{c} \ell p_{b,\ell}$ is a path vector associated to $a + b \in L_S^+(s_1 + s_2)$. It remains to show: $a' \geq a$ and $b' \geq b$ (and strict inequality for at least one term) implies $p_\mathfrak{c} \ell p_{b',\ell}$ is a sum of path vectors $p_{a'}$ such that $a' \geq a + b$. This holds by Lemma 13.9 if there exists a maximal chain $\mathfrak{c}$ containing both $\text{supp } a'$ and $\text{supp } b'$.

If this is not the case, then we use the same arguments as in the proof of Lemma 13.9.

We view $p_\mathfrak{c} \ell p_{b',\ell}$ as a function on $V_R((s_1+s_2)\lambda)_\tau$ via the restriction of $p_\mathfrak{c} \ell p_{b',\ell} \in V_R(s_1\lambda)^{\otimes \tau} \otimes V_R(s_2\lambda)^{\otimes \tau}$ to $V_R((s_1+s_2)\lambda)_\tau$ for $V_R((s_1+s_2)\lambda)_\tau$. Now with the same argument as in the proof of Lemma 13.9, we can apply again Proposition 12.11. We conclude that $p_\mathfrak{c} \ell p_{b',\ell}(v_{a,\sigma}) \neq 0$ for some $\mathfrak{c}$ in $L_S^+(s_1 + s_2)$ only if $m^a + \ell':= \otimes_{\kappa \in A_{\tau}, m^{(a'_\kappa + \ell')}}$ occurs in the expression of $v_{a,\sigma}$ in Proposition 12.11.
The notation for $\mathcal{m}^{a'+b'}$ looks ambiguous at first glance. The sum $a'+b'$ makes sense in $Q^+_{\geq 0}$, but the support is now not anymore contained in a maximal chain. We fix an ordering of the tensor factors of $\mathcal{m}^{a'+b'}$. Recall (compare to the comments before Proposition 12.11) that $M_R(\lambda)^{\otimes (s_1+s_2)}e$ is multigraded, the homogeneous component of $M_R(\lambda)^{\otimes (s_1+s_2)}e$ having the same multidegree as $\mathcal{m}^{a'+b'}$ is just the span of $\mathcal{m}^{a'+b'}$. So if $\mathcal{m}^{a'+b'}$ shows up in a presentation $\nu_{\mathfrak{a}^a}$, then, up to multiplication by a unit, it must appear among one of the terms in (29).

But the tensor product is symmetric up to multiplying by roots of unit. Assume that $\mathcal{m}^{a'+b'}$ shows up in (29) with respect to the chosen ordering. Then for any other ordering to write down tensor factors in $\mathcal{m}^{a'+b'}$, this pure tensor appears as one of the terms in (29). This implies: if $\mathcal{m} \triangleright \mathcal{m}^{a'+b'}$ with respect to one ordering of the tensor factors of $\mathcal{m}^{a'+b'}$, then $\mathcal{m} \triangleright \mathcal{m}^{a'+b'}$ with respect any ordering of the tensor factors of $\mathcal{m}^{a'+b'}$. Note that equality (up to rescaling) is not possible by assumption.

As a next step we want to show that $\mathcal{a}' \triangleright \mathcal{a}$ and $\mathcal{b} \triangleright \mathcal{b}$ (and strict inequality for at least one term) imply: there exists an ordering of the tensor factors such that $\mathcal{m}^{a'+b'} \triangleright \mathcal{m}^{\mathcal{a}+\mathcal{b}}$.

Let $q$ be minimal such that there exist elements $\zeta_{q+1} < \ldots < \zeta_r = \tau$ in $\mathcal{A}_\tau$ with the properties:

$\{\xi \in \text{supp} \mathcal{a}' | \ell(\xi) > q\}, \{\xi \in \text{supp} \mathcal{a} | \ell(\xi) > q\}$,

$\{\xi \in \text{supp} \mathcal{b}' | \ell(\xi) > q\}, \{\xi \in \text{supp} \mathcal{b} | \ell(\xi) > q\}$

are all subsets of $\{\zeta_{q+1}, \ldots, \zeta_r = \tau\}$, and $a_{\zeta_j} = a'_{\zeta_j}$, $b_{\zeta_j} = b'_{\zeta_j}$ for $j \geq q + 1$.

Since $\mathcal{a}' \triangleright \mathcal{a}$ and $\mathcal{b} \triangleright \mathcal{b}$, the minimality of $q$ implies that we can find in $\text{supp} \mathcal{a}' \cup \text{supp} \mathcal{b}'$ an element $\zeta_q$ of length $q$. Without loss of generality we assume $\zeta_q \in \text{supp} \mathcal{a}'$ and $a'_{\zeta_q} > a_{\zeta_q} \geq 0$. If $b_{\zeta_q} = 0$, then set

$\mathcal{m}^{a'+b'} = m_{\zeta_0} \otimes (a_{\zeta_0} + b_{\zeta_0}) \otimes \cdots \otimes m_{\zeta_{q+1}} \otimes (a_{\zeta_{q+1}} + b_{\zeta_{q+1}}) \otimes m_{\zeta_q} \otimes \cdots$

and compare the tensor product with:

$\mathcal{m}^{\mathcal{a}+\mathcal{b}} = m_{\zeta_0} \otimes (a_{\zeta_0} + b_{\zeta_0}) \otimes \cdots \otimes m_{\zeta_{q+1}} \otimes (a_{\zeta_{q+1}} + b_{\zeta_{q+1}}) \otimes m_{\zeta_q} \otimes \cdots$.

Since $\zeta_q$ is greater than any of the elements in $\text{supp} \mathcal{a} \cup \text{supp} \mathcal{b}$ of length smaller than $q$, $a'_{\zeta_q} > a_{\zeta_q}$ implies for the tensor products: $\mathcal{m}^{a'+b'} \triangleright \mathcal{m}^{\mathcal{a}+\mathcal{b}}$.

If $b_{\zeta_q} > 0$, then necessarily $b'_{\zeta_q} \geq b_{\zeta_q}$, and the same arguments imply for

$\mathcal{m}^{a'+b'} = m_{\zeta_0} \otimes (a_{\zeta_0} + b_{\zeta_0}) \otimes \cdots \otimes m_{\zeta_{q+1}} \otimes (a_{\zeta_{q+1}} + b_{\zeta_{q+1}}) \otimes m_{\zeta_q} \otimes \cdots$

and

$\mathcal{m}^{\mathcal{a}+\mathcal{b}} = m_{\zeta_0} \otimes (a_{\zeta_0} + b_{\zeta_0}) \otimes \cdots \otimes m_{\zeta_{q+1}} \otimes (a_{\zeta_{q+1}} + b_{\zeta_{q+1}}) \otimes m_{\zeta_q} \otimes \cdots$

that $\mathcal{m}^{a'+b'} \triangleright \mathcal{m}^{\mathcal{a}+\mathcal{b}}$. By transitivity of $\triangleright$, it follows $\mathcal{m} \triangleleft \mathcal{m}^{\mathcal{a}+\mathcal{b}}$, which in turn implies $\mathcal{a} \triangleright \mathcal{a} + \mathcal{b}$.

By induction we get as an immediate consequence:

**Corollary 13.11.** If $p_\mathcal{a} \mathcal{a} \in \text{LS}^+_{\zeta,\lambda}(s)$, is a path vector, then $p_\mathcal{a}^m m$, $m \geq 1$, is a path vector associated to $m\mathcal{a} \in \text{LS}^+_{\zeta,\lambda}(ms)$, up to multiplying by a root of unity.

In the following we work over an algebraically closed field $\mathbb{K}$. 
Proposition 13.12. Let \( \underline{a} \in \text{LS}_{\xi, \lambda}^+(s) \) be such that \( a_\tau \geq m \) for some integer \( m > 0 \). If \( p_\underline{a} \) is a path vector associated to \( \underline{a} \), then \( p_\underline{a} = c p_m^r p_{\underline{b}} \), where \( c \) is some root of unity, \( \underline{b} = \underline{a} - me_\tau \in \text{LS}_{\xi, \lambda}^+(s - m) \), \( p_\underline{b} \) is a path vector associated to \( \underline{b} \) and \( p_r \) is the dual of the extremal weight vector \( v_\tau \in V(\lambda)_\tau \).

Proof. The proof is by decreasing induction along \( \rhd \). First assume \( \underline{a} = se_\tau \) is the unique maximal element in \( \text{LS}_{\xi, \lambda}^+(s) \). By weight considerations we know that, up to multiplication by a non-zero scalar, \( p_\underline{a} \) is equal to \( p_r^m \), and Proposition 13.10 implies that the scalar is a root of unity.

Suppose now \( \underline{a} \in \text{LS}_{\xi, \lambda}^+(s) \) is such that \( a_\tau \geq m > 0 \). After having chosen an appropriate \( \ell \) as in Corollary 13.6, we know \( p_\underline{a} = p_\underline{a}^\ell + \sum_{\underline{a}' \rhd \underline{a}} d_{\underline{a}' \ell} p_{\underline{a}' \ell} \). Now \( \underline{a}' \rhd \underline{a} \) implies \( a'_\tau \geq a_\tau \geq m \), so by induction, we can write (up to rescaling by a root of unity) \( p_\underline{a}^\ell \) as a product \( p_r^m p_{\underline{b}' \ell} \), where \( \underline{b}' = \underline{a}' - me_\tau \rhd \underline{b} \).

It remains to consider \( p_\underline{a}\ell \). We can write \( \underline{a} = me_\tau + \underline{b} \) where \( me_\tau \in \text{LS}_{\xi, \lambda}^+(m) \) and \( \underline{b} \in \text{LS}_{\xi, \lambda}^+(s - m) \). By Lemma 13.9, \( p_r^m p_{\underline{b}\ell} \) is, up to rescaling by a root of unity, equal to

\[
p_\underline{a}\ell + \sum_{\underline{a}'' \rhd \underline{a}} d_{\underline{a}'' \ell} p_{\underline{a}''}\ell
\]

Again by induction, we can write the \( p_{\underline{a}''} \ell \) as a product \( p_r^m p_{\underline{b}''} \ell \) where \( \underline{b}'' = \underline{a}'' - me_\tau \rhd \underline{b} \). It follows: we can write (up to multiplication by a root of unity)

\[
p_\underline{a} = p_r^m \left( p_{\underline{b}\ell} + \sum_{\underline{b}' \rhd \underline{b}} c_{\underline{b}'} p_{\underline{b}'} \right),
\]

which proves the claim.

By combining Corollary 13.11 and Proposition 13.12 we get:

Corollary 13.13. Let \( \underline{a} \in \text{LS}_{\xi, \lambda}^+(s) \) be such that \( a_\tau > 0 \) and let \( N \in \mathbb{N} \) be a positive integer such that \( Na_\tau \in \mathbb{N} \). If \( p_\underline{a} \) is a path vector associated to \( \underline{a} \), then \( p_\underline{a}^N \) is, up to multiplying by a root of unity, equal to \( p_r^{Na_\tau} p_{\underline{b}} \), where \( \underline{b} = Na_\tau - Na_\tau e_\tau \in \text{LS}_{\xi, \lambda}^+(Ns - Na_\tau) \) and \( p_{\underline{b}} \) is a path vector associated to \( \underline{b} \).

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