Cosmic Degeneracy with Dark Energy Equation of State

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Abstract

We discuss a degeneracy between the geometry of the universe and the dark energy equation of state $w_X$ which exists in the power spectrum of the cosmic microwave background. In particular, for the case with $w_X < -1$, this degeneracy has interesting implications to a lower bound on $w_X$ from observations. It is also discussed how this degeneracy can be removed using current observations of type Ia supernovae.
1 Introduction

Almost all observations strongly suggest that the present universe is dominated by an enigmatic component called dark energy. The simplest candidate for dark energy is the cosmological constant; other models such as quintessence [1], k-essence [2], that of stringy origin [3, 4], the generalized Chaplygin gas [5] and Cardassian models [6] have also been discussed. It is very important to consider whether we can differentiate these models of dark energy from current observations. Many authors have studied this subject for specific models or in a phenomenological way. Phenomenologically, the dark energy is characterized by its equation of state $w_X = p_X/\rho_X$ where $\rho_X$ and $p_X$ are its energy density and pressure respectively. For the cosmological constant, $w_X$ is equal to $-1$; however, in other models, it can possibly deviate from $-1$. Several authors have studied constraints on $w_X$ from observations such as cosmic microwave background (CMB), large scale structure (LSS) and type Ia Supernovae (SNeIa) [7, 8, 9]. Since observations of CMB strongly suggest that the universe is flat [10, 11], most analyses were done with the prior of a flat universe to study constraints on $w_X$. However, it is important to notice that almost all works which obtained constraints on the geometry of the universe from CMB observations assumed a cosmological constant as the dark energy, i.e., equation of state $w_X = -1$. It is well-known that the peak location of the power spectrum of CMB is sensitive to the geometry of the universe. However, it is also known that the equation of state of dark energy affects the location of acoustic peaks. Thus it is interesting to study the power spectrum of CMB with varying $\Omega_k$ and $w_X$ where $\Omega_k$ is the energy density of curvature normalized by the critical energy density.

In this paper, we discuss observational consequences of varying the geometry of the universe $\Omega_k$ and the equation of state of dark energy $w_X$. Although many models have time-dependent equation of state and some papers investigate observational consequences of such cases [12], to study it phenomenologically, we assume that $w_X$ is independent of time throughout this paper. By studying the angular power spectrum of the CMB varying $\Omega_k$ and $w_X$, one can find that there is a degeneracy between $\Omega_k$ and $w_X$ in CMB. This means that we cannot know precisely the geometry of the universe without knowledge of the equation of state of the dark energy from observations of CMB. This also means that the constraints on $w_X$ obtained so far which assume a flat universe is changed if we also consider the case with a non-flat universe. In most analyses so far, it is assumed the equation of state as $w_X \geq -1$ since the case with $w_X < -1$ violates the weak energy condition. However, some models which predict $w_X < -1$ have been proposed [2] and some authors discuss constraints on $w_X$ for such cases [8, 9]. Thus, in this paper, we discuss the case with $w_X \leq -1$ in which the degeneracy occurs in a closed universe. It is interesting to notice that the degeneracy in the region where $w_X < -1$ affects a observational lower bound on $w_X$. Compared to the one obtained assuming a flat universe, a lower bound on $w_X$ becomes less stringent.

The organization of this paper is as follows. In the next section, we examine the
Degeneracy between $\Omega_k$ and $w_X$ in CMB. Then, in section 3, we study implications to this degeneracy from the observation of type Ia supernovae. The last section is devoted to summary.

2 Degeneracy between $\Omega_k$ and $w_X$ in CMB

Observations of the CMB can measure cosmological parameters very precisely as seen from the recent WMAP result [11]. However, it is known that there exist some degeneracies in certain sets of cosmological parameters which cannot be removed from the observations of the CMB alone [14, 15, 16]. One such example is a degeneracy between $\Omega_m$ and $h$ where $h$ is the Hubble parameter (in units of 100 km/s/Mpc). Although we cannot break this degeneracy with observations of CMB, the Hubble parameter can be independently determined by other observations such as from the Hubble Space Telescope (HST) [17]. Once $h$ is determined independently of CMB observations, the degeneracy is lifted.

Here we discuss another degeneracy. As we will show in the following, there is a degeneracy between $\Omega_k$ and $w_X$. To illustrate this degeneracy, we plot the CMB angular power spectrum $C_l$ for the cases with $\Omega_k = 0, w_X = -1$ and with $\Omega_k = -0.05, w_X = -5$ in Fig. 1. To calculate the angular power spectrum $C_l$, we used a modified version of CMBFAST [18], and in the present analysis, we consider constant $w_X$ and included fluctuation of dark energy according to [19]. Here we took the sound speed in the rest frame of the dark energy as $c_{\text{eff}} = 1$. In this figure, other cosmological parameters are fixed as $\Omega_m = \Omega_b + \Omega_c = 0.27, \Omega_b h^2 = 0.024$ and $n = 0.99$ and $\tau = 0.166$ which correspond to the best-fit values for the power law $\Lambda$CDM model from WMAP results [11], where $\Omega_m, \Omega_b$ and $\Omega_c$ are the energy density of matter, baryon and cold dark matter (CDM) respectively, $n$ is the initial spectral index and $\tau$ is the optical depth of reionization. From this figure, we can see the degeneracy between $\Omega_k$ and $w_X$. Since the current constraint on $w_X$ assuming the flat universe is $-1.38 < w_X < -0.82$ (at 95 % C.L.) [9], $w_X = -5$ is completely disfavored in the flat universe. However if we consider such a value in the non-flat universe, the resulting angular power spectrum can become almost the same as the best-fit model. When we take the equation of state as $w_X < -1$, effects of dark energy becomes significant only at late times, thus the structure of the acoustic peaks is mostly determined by the energy density of matter (i.e., $\Omega_c$ and $\Omega_b$). Therefore the shape of the acoustic peaks is almost the same even if we change the values of $\Omega_k$ and $w_X$. However, the location of the acoustic peaks is affected by $\Omega_k$ and $w_X$. In a closed universe (i.e. $\Omega_k < 0$), the peak location is shifted to lower multipole. On the other hand, by decreasing $w_X$, it is shifted to higher multipole. If we take the values of $\Omega_k$ and $w_X$ to cancel these effects, we can have almost the same angular power spectrum for different sets of these parameters. Notice that, since the angular power spectrum at low multipole region is affected by the integrated Sachs-Wolfe (ISW) effect, the different sets of $\Omega_k$ and $w_X$ may give different structure at low multipoles even if they have indistinguishable structure of acoustic peaks. However, also notice that even in such a region, it may be difficult to distinguish them
due to the cosmic variance.

Changing $\Omega_k$ and $w_X$, the location of the peaks can be shifted because the peak position is mostly determined by the angular diameter distance to the last scattering surface which can be affected by $\Omega_k$ and $w_X$. The location of the acoustic peak is inversely proportional to the angular diameter distance \#2 20, 21

\[ l_{\text{peak}} \propto \frac{1}{d_A(z_{\text{rec}})} \quad (1) \]

where $d_A$ is given by

\[ d_A(z) = \frac{1}{(1 + z)|\Omega_k|^{1/2}H_0} \times \sin \left( |\Omega_k|^{1/2} \int_0^z dz' \left[ \Omega_r (1 + z')^4 + \Omega_m (1 + z')^3 + \Omega_k (1 + z')^2 + \Omega_X (1 + z')^{3(1+w)} \right]^{-1/2} \right). \quad (2) \]

Here "$\sin$" is defined as "sin" for $\Omega_k < 0$, "1" for $\Omega_k = 0$ and "sinh" for $\Omega_k > 0$. In Fig. 2 we plot the angular diameter distance in the $\Omega_k - w_X$ plane. The points shown in the figure indicate the same parameters which are used in Fig. 1. We can clearly see that the parameter sets which give the same angular diameter distance also give almost the same peak position.

For intuitive understanding, we consider the effects of the equation of state and the geometry of the universe with an example. Now we compare the case with $w_X = -1$ and $w_X = -2$. If $w_X = -2$, the energy density of the dark energy $\rho_X$ grows proportional to $a^3$. When the present energy density of the dark energy is fixed as $\Omega_X = 0.7$, this means that the total energy density of the universe in the past is smaller than that for the case with $w_X = -1$ since the energy density of dark energy grows faster than for the $w_X = -1$ case. In other words, the expansion of the universe is slower than that for the case with $w_X = -1$. This means that the distance to the last scattering surface becomes larger. Thus, if we take $w_X$ more negative, the locations of peaks are shifted to higher multipoles. On the other hand, in a closed universe (i.e., $\Omega_k < 0$), the locations of the acoustic peaks are shifted to lower multipoles compared to the case with a flat universe because of the geodesic effects on the trajectories of photons. So, if we consider the dark energy with super-negative $w_X$ (i.e., $w_X < -1$) in a closed universe, these effects can cancel each other, and the resulting location of the peaks becomes the same as that of the flat universe with a cosmological constant. Conversely, when we consider dark energy with $w_X > -1$ in an open universe, this kind of cancellation can also happen. But in this case, the dark energy affects the history of the universe from earlier times than for the case with $w_X < -1$. Thus the degeneracy is milder compared to the case with a super-negative equation of state, $w_X < -1$.

\#2It also depends on the sound horizon at the surface of last scattering. However, since we assume that the equation of state is constant here, the dark energy affects the CMB only at late time, i.e., the sound horizon at the surface of last scattering is not affected by changing $w_X$. Thus we do not consider the effect of the sound horizon on the location of the peaks here.
Notice that, if we take the equation of state as \( w_X \ll -1 \), the dark energy is less relevant to the angular diameter distance. As we can see from Eq. (3), for the case with \( w_X \ll -1 \), the term related to the dark energy becomes less significant in the integral. This means that the shift of the location of the acoustic peaks saturates at some point. Thus even if we take very negative values of \( w_X \), the value of \( \Omega_k \) which is needed to realize the same peak location as the \( \Lambda \)CDM model does not change much. This means that we cannot obtain a significant lower bound on \( w_X \) from CMB observations alone.

To study this degeneracy quantitatively, we show contours of \( \Delta \chi^2 \) in the \( \Omega_k - w_X \) plane in Fig. 3. In this figure, for an illustration, we fixed the other cosmological parameters except \( \Omega_k \) and \( w_X \) as \( \Omega_m h^2 = 0.14, \Omega_b h^2 = 0.024, h = 0.71, n = 0.99 \) and \( \tau = 0.166 \) and the normalization of spectra is marginalized. The value of \( \chi^2 \) is calculated using the code provided by WMAP team [23, 24]. Here we used the TT mode data only. Since locations of the acoustic peaks are strongly dependent on the angular diameter distance, contours of constant \( \Delta \chi^2 \) look similar to \( d_A \). As we can see from the figure, the allowed region extends to more negative \( w_X \) region, which indicates that we cannot obtain a significant lower bound on \( w_X \) from CMB data alone without specifying the value of \( \Omega_k \) as we mentioned above. This is because effects of decreasing \( w_X \) on the peak location saturate at some points. This also means that we cannot determine the geometry of the universe without the knowledge of the equation of state of the dark energy. However if we use other cosmological data such as SNeIa, we can lift the degeneracy to some extent. We will discuss this issue in the next section.

### 3 Implications from SNe observations

In this section, we see whether we can break the degeneracy which exists in the power spectrum of CMB. It is well-known that confidence level contours in the \( \Omega_m - \Omega_\Lambda \) plane from CMB observations lie almost along the flat universe line, while contours from SNe observations are roughly orthogonal to those from CMB observations. So, in the \( \Omega_m - \Omega_\Lambda \) plane, observations of CMB and SNeIa are complementary. Thus we can expect that this kind of complementarity exists in the \( \Omega_k - w_X \) plane. In the following, we study the constraint with current SNe data in this plane.

First, we explain some technical details which are needed to consider constraints from SNeIa observations. Observations of SNeIa give a measured distance modulus \( \mu_0 \) which is given by

\[
\mu_0 = 5 \log \left( \frac{d_L(z)}{\text{Mpc}} \right) + 25, \tag{3}
\]

where \( d_L \) is the luminosity distance in units of Mpc. The luminosity distance is calculated as

\[
d_L(z) = \frac{1 + z}{|\Omega_k|^{1/2} H_0} \sin \left\{ |\Omega_k|^{1/2} \int_0^z dz' \left[ \Omega_m (1 + z')^3 + \Omega_k (1 + z')^2 + \Omega_X (1 + z')^{3(1+w)} \right]^{-1/2} \right\}. \tag{4}
\]
in which definition of “sinn” is given in the previous section.

To find constraints from SNeIa observations, we used data from High-z Supernova Search Team [25] and Supernova Cosmology Project (SCP) [26]. We also included the data from SNeIa discovered recently [27, 28, 29]. The published data of High-z Supernova Search Team gives the distance modulus of each SNeIa defined in Eq. (3), while the SCP team gives the estimated effective B-band magnitude of each SN Ia $m_{\text{eff}}^B$. The effective B-band magnitude $m_{\text{eff}}^B$ and the distance modulus $\mu_0$ are related by

$$m_{\text{eff}}^B = M_B + \mu_0,$$

where $M_B$ is the peak absolute magnitude of a “standard” SN Ia in the B-band. Wang [30] found the relation between the data of SCP team and High-z SN team

$$M_B^{\text{MLCS}} \equiv m_{\mu_0}^{\text{MLCS}} - \mu_0^{\text{MLCS}} = -19.33 \pm 0.25,$$

where all quantities in this equation are estimated with the multi light curve shape (MLCS) method.

We use this relation to combine data of High-z SN Search team and SCP team. The data set of High-z SN team and SCP team consist 50 SNe and 42 SNe. In their data set, 18 SNe are the same, thus, including recently discovered 3 SNe, 95 SNe are used in total in our analysis.

The likelihood for the parameters are determined from a $\chi^2$ statistics,

$$\chi^2 = \sum_i \frac{[\mu_0^{\text{th}}(z_i; \Omega_m, \Omega_X, h, w) - \mu_0^{\text{exp}}]^2}{\sigma_{\mu_0,i}^2 + \sigma_v^2},$$

where $\mu_0^{\text{th}}(z_i; \Omega_m, \Omega_X, h, w)$ is a predicted distance modulus for a given cosmological parameters, $\mu_0^{\text{exp}}$ is the measured distance modulus for each SN, $\sigma_{\mu_0,i}$ is the measurement error of the distance modulus and $\sigma_v$ is the velocity dispersion in galaxy redshift in units of the distance modulus. For the data of High-z SN team, following [25], we adopted $\sigma_v = 200$ km/s and added 2500 km/s in quadrature for high-redshift SN Ia whose redshift were determined from the broad features in the SN spectrum. We calculated the likelihood function as $L = L_0 \exp[-\chi^2/2]$, where $L_0$ is an arbitrary normalization constant.

First, we show the likelihood contours in the $\Omega_m - \Omega_X$ plane in Fig. 4. In this figure, the equation of state of dark energy is fixed as $w_X = -1, -2$ and $-3$, respectively. As $w_X$ decreases, the allowed region is shifted to lower values of $\Omega_X$. This is because a lower value of $w_X$ can result in a more accelerating universe for the same value of $\Omega_X$. Also notice that, similarly to the case with the angular diameter distance, the dark energy is less relevant to the luminosity distance if we take the equation of state $w_X$ at a very negative value. Thus the likelihood contours for the cases with $w_X = -2$ and $-3$ are less different compared to than those for the $w_X = -1$ and $w_X = -2$ cases.

In Fig. 5 the likelihood contours in the $\Omega_k - w_X$ plane are shown. In this figure, the density parameter for matter is fixed as $\Omega_m = 0.25, 0.3$ and 0.35. As we can see
from this figure, allowed regions extends to very negative values of $w_X$. The situation is similar to the case with CMB which we discussed in the previous section. However, the allowed parameter space which extends to more negative region in Fig. 5 is different from that in the case of CMB (see Fig. 3). In this sense, observations of SNe and CMB are complementary each other in the $\Omega_k - w_X$ plane too. Thus we can break the degeneracy in CMB to some extent with observations of SNeIa. But it is important to notice that, even if we can break the degeneracy between $\Omega_k$ and $w_X$, the constraint on $w_X$ without assuming the flat universe is less stringent than that obtained assuming the flat universe. Thus, to impose a more general lower bound on $w_X$ would require a new global fit to all cosmological parameters with the prior of the flat universe lifted. Detailed investigation of this issue is beyond the scope of this paper.

4 Summary

We discussed the degeneracy between the geometry of the universe and the equation of state of dark energy which exists in the CMB power spectrum. As is well-known, the geometry of the universe affects the location of the acoustic peaks. For the open universe, the peak position is shifted to higher multipole compared to that of the flat universe, while, for the closed universe, it is shifted to lower multipoles. It is also known that the equation of state of dark energy affects the location of the peaks. If we take a smaller value of $w_X$, the peak location is shifted to higher multipoles, and for a larger value of $w_X$, it becomes lower. If we take the values of $\Omega_k$ and $w_X$ to cancel these effects, we can have almost the same angular power spectrum for different sets of these parameters. For example, even if we take $w_X = -5$, we can have a power spectrum which has good agreement with CMB data for models which predict moderate ISW effect at low multipole by choosing the non-flat universe as $\Omega_k = -0.05$. Since effects of decreasing $w_X$ on the location of the acoustic peaks saturate at some points, we cannot obtain a significant lower bound on $w_X$ from CMB observations alone if we remove the assumption of the flat universe. However, observation of SNeIa can remove this degeneracy to some extent, as discussed in section 3. For the degeneracy that we discussed here, observations of CMB and SNeIa are complementary. But it is interesting to notice that, when we relax the assumption of the flat universe, the constraint on $w_X$ becomes less stringent compared to that assuming a flat universe. The full investigation of this issue will be the subject of a future work.

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#3 Even though we can see that allowed regions extends to very negative values of $w_X$ as in the case of CMB, this is not an intrinsic degeneracy. We can have a significant lower bound on $w_X$ using future SNe observations such as SNAP [31].
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Figure 1: The angular power spectra of CMB for $\Omega_k = 0, w = -1$ (solid line) and for $\Omega_k = -0.05, w_X = -5$ (dashed line). The other cosmological parameters are taken to be $\Omega_m h^2 = 0.14, \Omega_b h^2 = 0.024, h = 0.72, n = 0.99$ and $\tau = 0.166$. For most of the multipole $l$, the dashed and solid lines are almost indistinguishable.
Figure 2: Contours of constant $d_A$ are shown. The points in this figure indicate the parameters which are used in Fig. \[ \]

Figure 3: Contours of constant $\Delta \chi^2$ in the $\Omega_k - w_X$ plane. In this figure, we fixed the other cosmological parameters as in Fig. 1. The dotted and dashed lines indicate contour for $\Delta \chi^2 = 6.18$ and 11.83 respectively.
Figure 4: Contours of constant likelihood in the $\Omega_m - \Omega_X$ plane for $w_X = -1$ (top), $w_X = -2$ (middle) and $w_X = -3$ (bottom). The 95.4 % and 99.7 % confidence intervals are shown with dotted and dashed lines respectively.
Figure 5: Contours of constant likelihood in the $\Omega_k - w_X$ plane for $\Omega_m = 0.25$ (top), $\Omega_m = 0.3$ (middle) and $\Omega_m = 0.35$ (bottom). The 95.4 % and 99.7 % confidence intervals are shown with dotted and dashed lines respectively.