Dynamical analysis of quantum linear systems driven by multi-channel multi-photon states

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Abstract

In this paper, we investigate the dynamics of quantum linear systems where the input signals are multi-channel multi-photon states, namely states determined by a definite number of photons superposed in multiple input channels. In contrast to most existing studies on separable input states in the literature, we allow the existence of quantum correlation (for example quantum entanglement) in these multi-channel multi-photon input states. Due to the prevalence of quantum correlations in the quantum regime, the results presented in this paper are very general. Moreover, the multi-channel multi-photon states studied here are reasonably mathematically tractable. Three types of multi-photon states are considered: 1) \( m \) photons are superposed among \( m \) channels, 2) \( N \) photons are superposed among \( m \) channels where \( N \geq m \), and 3) \( N \) photons are superposed among \( m \) channels where \( N \) is an arbitrary positive integer. Formulae for intensities and states of output fields are presented. Examples are used to demonstrate the effectiveness of the results.

Key words: quantum linear systems, multi-photon states, intensity.

1 Introduction

Dynamical response analysis is an essential ingredient of control engineering, and is also the basis of controller design. For example, impulse response, step response, and frequency response are standard materials in modern control textbooks, see, e.g., [26], [2], [59], [12], [40]. Fluctuation analysis of a dynamic system driven by white noise underlies the celebrated Kalman filter and linear quadratic Gaussian (LQG) control. Likewise, in the quantum regime, the response of quantum linear systems to quantum Gaussian white noise is the basis of quantum filtering and measurement-based feedback control, see, e.g., [6], [7], [23], [8], [9], [45], [47], [15], [42], [1], [58] and references therein.

In addition to quantum Gaussian noise commonly treated in quantum optical laboratories, in recent years, highly nonclassical quantum signals such as single-photon states, multi-photon states, and Schrodinger’s cat states have been attracting increasing interest due to their promising applications in quantum information technology, [32]. Loosely speaking, an \( \ell \)-photon state of a light beam means that the light field contains exactly \( \ell \) photons. In this paper, we are concerned with continuous-mode \( \ell \)-photon states, that is, these photons are determined by their frequency (or equivalently temporal) profiles centered at the carrier frequency of the light field. Continuous-mode single- and multi-photon states have found important applications in quantum computing, quantum communication, and quantum metrology, [18], [32], [29], [36], [30], [14], [21], [22], [4], [5], [52], [57], [53], [49], [35].

In the quantum control community, the responses of quantum systems to single-photon states and multi-photon states have been studied in the past few years. The phenomenon of cross phase shift on a coherent signal induced by a single photon pulse was investigated in [31]. Gough et al. derived quantum filters for Markov quantum systems

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driven by single-photon states or Schrödinger’s cat states, [21], [22]. This theory has been applied to the study of phase modulation in [13]. Quantum master equations for an arbitrary quantum system driven by multi-photon states were derived in [4]. Quantum filters (stochastic master equations) for multi-photon states have been derived in [44], for both homodyne detection and photodetection. Numerical simulations carried out in [44] for a two-level system driven by a 2-photon state revealed interesting and complicated nonlinear behavior in this photon-atom interaction. When a two-level atom, initialized in the ground state, is driven by a single photon, the exact form of the output field state was derived in [37]. More discussions can be found in, e.g., [16], [28], [35] and references therein.

In [57], the analytic expression of the output field state of a quantum linear system driven by a single-photon state was derived. Specifically, a class of $m$-channel $m$-photon states was given in [57, Eq. (44)]. For such states, each input channel has exactly one photon whose pulse shape is determined by a single-variable function $\nu_k$, $(k = 1, \ldots, m)$. Moreover, there exists no statistical correlation among photons in different channels; that is, these $m$-photon states are separable states. A more general class of $m$-channel $m$-photon states was given in [57, Eq. (95)]. For this class of states, quantum correlations are allowed to exist among different channels. Unfortunately, because photon pulses are functions of two variables, $\xi_{jk}$, the extent of quantum correlation is severely limited. The research initialized in [57] has been continued in [53], where more general forms of multi-photon input states were considered. In the study of [53], different channels may have different numbers of photons. To be specific, as shown in [53, Eq. (22)], the $j$th channel may have $\ell_j$ photons. The $m$-channel multi-photon states defined in [53, Eq. (22)] belong to the class of $m$-channel multi-photon states defined in [53, Eq. (34)]. This class of states also contain those defined in [57, Eq. (95)] as special cases (when $\ell_j = 1$ for all $j = 1, \ldots, m$). However, because photon pulses are functions of three variables, $\xi_{jk}$, the extent of quantum correlation among the input channels is still limited. An $m$-channel multi-photon state is then proposed in [53, Eq. (41)], where the $j$th channel is an $\ell_j$-photon state whose pulse shape is given by a function of $\ell_j$ variables, $\Psi_j(t_1, \ldots, t_{\ell_j})$. Therefore, the states defined in [53, Eq. (41)] somehow is more general than those in [53, Eq. (34)]. It is worth noting that the states defined in [53, Eq. (41)] are separable states, that is, there exists no correlation among different channels. The $m$-channel multi-photon states defined in [53, Eq. (41)] is subsequently extended to a broader class of states as given in [53, Eq. (43)], which allow quantum correlations among different input channels. Unfortunately, The states defined in [53, Eq. (34)] and [53, Eq. (43)] appear rather abstract and mathematically intractable. In fact, all the examples studied in [53] focused on separable input states. In other words, none of these examples is for the general multi-channel multi-photon states defined in [53, Eq. (34)] or [53, Eq. (43)]. Actually, in the existing literature it appears hard to find multi-channel multi-photon states that are described by [53, Eq. (34)] or [53, Eq. (43)]. It is fair to say that the multi-photon states studied in [53] are either too simple (separable states) or too complicated (such as those given in [53, Eq. (43)]).

The purpose of this paper is to provide a direct study of the dynamical response of quantum linear systems to initially entangled $m$-channel multi-photon states. Unlike those separable states defined in [57, Eq. (44)] and [53, Eq. (41)], and those states defined in [57, Eq. (95)] and [53, Eq. (34)] whose pulse shapes are functions of two or three variables, the pulse shapes of the states defined in this paper are characterized by functions of $m$ (or $N \geq m$) variables, more detail is given in Eqs. (41) and (119). Examples presented in this paper demonstrate that these types of $m$-channel multi-photon states can be easily processed by quantum linear systems. The states in Eqs. (41) and (119) are subsequently extended to more general classes of states Eqs. (99) and (143), respectively. These classes of states are very general as they contain many forms of multi-channel multi-photon states as special cases, see, e.g., [29, Chapter 6], [41], [11]. Moreover, these states are mathematically more tractable than those in [53, Eq. (43)]. Therefore, the study carried out in this paper is more relevant to quantum linear feedback networks and control.

Three types of multi-channel multi-photon states are studied in this paper. Case 1): $m$ photons are superposed among $m$ channels. Specifically, the $m$-channel $m$-photon states are defined in Subsection 3.1. When the underlying quantum linear system is passive, the analytic expression of the output intensity is presented in Subsection 3.2, see Theorem 1. Moreover, the steady-state output field state is investigated in Subsections 3.3 and 3.4, see Theorems 2 and 3. When the underlying quantum linear system is non-passive, the steady-state output state is no longer an $m$-channel $m$-photon state, see Theorems 5 and 6. Case 2): $N$ photons are superposed among $m$ channels where $N \geq m$. For this case, we assume the underlying quantum linear system is passive. Then the analytical expressions of the output state are derived, see Theorems 5 and 6. Case 3): $N$ photons are superposed among $m$ channels where $N$ is an arbitrary positive integer. The $m$-channel $N$-photon states are presented in Subsection 5.1. And in Subsection 5.2, the steady-state output state of a quantum linear passive system driven by an $m$-channel $N$-photon input state is derived, see Theorem 7.

Notation. The complex unit $\sqrt{-1}$ is denoted by $i$. Given a column vector of complex numbers or operators $x = [x_1 \cdots x_k]^T$, denote $x^\# = [x_1^\dagger \cdots x_k^\dagger]^T$, where the superscript “*” stands for complex conjugation or Hilbert space
adjoint. Denote \( x^\dagger = (x^\#)^T \). Define the doubled-up column vector \( \tilde{x} \triangleq [x^T, x^\dagger]^T. \) Let \( I_k \) be an identity matrix and \( 0_k \) a zero square matrix, both of dimension \( k \). Denote \( J_k = \text{diag}(I_k, -I_k) \). (The subscript “\( k \)” may be omitted when it causes no confusion.) Given a matrix \( X \in \mathbb{C}^{2k \times 2k} \), define \( X^\dagger \triangleq J_k X^\dagger J_k. \) Given a matrix \( A \), let \( A^{jk} \) denote the entry on the \( j \)th row and \( k \)th column. Let \( m \) be the number of input channels. Let \( n \) be the number of degrees of freedom of a given quantum linear system, namely the number of quantum harmonic oscillators. The ket \( |\phi\rangle \) denotes the initial state of the system, and \( |0\rangle \) stands for the vacuum state of free fields. The convolution of two functions \( f \) and \( g \) is \( f \ast g \).

Given two matrices \( U, V \in \mathbb{C}^{r \times k} \), define a doubled-up matrix \( \Delta(U, V) \triangleq [U V; V^\# U^\#] \). Given two operators \( A \) and \( B \), their commutator is defined to be \( [A, B] \triangleq AB - BA. \) The \( m \)-fold integral \( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m \) is sometimes denoted by \( \int d\tau \). Given a function \( f(t) \) in the time domain, define its two-sided Laplace transform \([43, \text{Eq. (13)}]\) to be \( F[s] = \mathcal{L}_b\{f(t)\}(s) \triangleq \int_{-\infty}^{\infty} e^{-st} f(t) dt \). The \( m \) dimensional Fourier transform is, \([10] \),

\[
\int_{-\infty}^{\infty} \frac{1}{(2\pi)^m/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m \ e^{-i(\omega_1 t_1 + \cdots + \omega_m t_m)} f(t_1, \ldots, t_m).
\]

We set \( \hbar = 1 \) throughout this paper.

2 Preliminaries

In this section, quantum linear systems are briefly introduced; more discussions can be found in, e.g., \([17], [50], [51], [46], [19], [47], [24], [38], [55], [56], [48], [33], \) and \([54] \). Besides, some tensors and their associated operations are also discussed.

2.1 Quantum linear systems

A quantum linear system \( G \) is shown schematically in Fig. 1. In this model, the quantum linear system \( G \) consists of a collection of \( n \) interacting quantum harmonic oscillators \( a = [a_1, \ldots, a_n]^T \). Here, \( a_j \) (\( j = 1, \ldots, n \)) defined on a Hilbert space \( \mathcal{H}_G \), is the annihilation operator of the \( j \)th quantum harmonic oscillator. The adjoint operator of \( a_j \), denoted by \( a_j^\dagger \), is called a creation operator. These operators satisfy the canonical commutation relations \( [a_j, a_k^\dagger] = \delta_{jk} \). The input light fields are represented by a vector of annihilation operators \( b_{in}(t) = [b_{in,1}(t), \ldots, b_{in,m}(t)]^T \); the entry \( b_{in,j}(t) \) (\( j = 1, \ldots, m \)), defined on a Fock space \( \mathcal{F} \), is the annihilation operator for input channel \( j \). The adjoint operator of \( b_{in,j}(t) \), denoted by \( b_{in,j}^\dagger(t) \), is also called a creation operator. However, unlike \( a_j \) and \( a_j^\dagger \), these annihilation and creation operators satisfy the following singular commutation relations, \([17], [20, \text{Eq. (20)}]\),

\[
\begin{align*}
[b_{in,j}(t), b_{in,k}^\dagger(r)] &= \delta_{jk} \delta(t - r), \\
[b_{in,j}(t), b_{in,k}(r)] &= [b_{in,j}^\dagger(t), b_{in,k}^\dagger(r)] = 0, \quad j, k = 1, \ldots, m, \forall t, r \in \mathbb{R}.
\end{align*}
\]  

(2)

Notice the presence of the Dirac delta function \( \delta(t - r) \) in Eq. (2). Mathematically, it is often more convenient to work with integrated annihilation and creation operators, which are defined respectively to be \( B_{in}(t) \triangleq \int_{t_0}^{t} b_{in}(r) dr \) and \( B_{in}^\#(t) \triangleq \int_{t_0}^{t} b_{in}^\#(r) dr \), where the lower limit \( t_0 \) of the integral is the initial time, namely the time when the system and the fields start interaction. The gauge process (also called number process) is defined by the following \( m \)-by-\( m \) matrix function, \([17, \text{Chapter 11}], [20, \text{Section III.A}], [57, \text{Eq. (11)}]\),

\[
\Lambda(t) \triangleq \int_{t_0}^{t} b_{in}^\#(\tau)b_{in}^T(\tau) d\tau.
\]

(3)
In this paper, we deal with *canonical* quantum input fields, that is, the only non-zero Ito products are, \[17, \text{Chapter } 11, \, [19], \, [20], \, [57, \text{Eq. (12)}], \]
\[
d B_{in,j}(t)dB_{in,k}^*(t) = \delta_{jk}dt, \quad dB_{in,j}(t)dB_{in,l}^*(t) = \delta_{kl}dt, \quad dB_{in,j}(t) = \delta_{jk}dt, \quad dB_{in,l}(t) = \delta_{kl}dt. \quad (4)
\]

The dynamics of the open quantum linear system \(G\) can be described conveniently in the \((S_-, L, H)\) formalism \[19, \, [56]\]. Here, \(S_-\) is a constant unitary matrix of dimension \(m\), which can be used to model static devices such as phase shifters and beamsplitters. The operator \(L\) describes how the system is coupled to the fields, and is of the form \(L = C_- a + C_+ a^\dagger\) with \(C_-, C_+ \in \mathbb{C}^{m \times n}\). For example, when an optical cavity is driven by a light field, \(L\) can be of the form \(L = \sqrt{n} a\), where \(a\) is the annihilation operator of the quantum harmonic operator for the cavity (the cavity mode) and \(\kappa > 0\) is the coupling strength. The operator \(H\) stands for the initial system Hamiltonian, which can be written as \(H = \frac{1}{2} \hat{a}^\dagger \Delta \left( \Omega_- , \Omega_+ \right) \hat{a}\) with constant matrices \(\Omega_- , \Omega_+ \in \mathbb{C}^{m \times n}\) satisfying \(\Omega_- = \Omega_-^\dagger\) and \(\Omega_+ = \Omega_+^T\). For example, for the cavity just mentioned, upon a constant shift \(\frac{1}{2} \omega_d\), \(H = \omega_d a^\dagger a\), where \(\omega_d\) is the detuning frequency between the cavity mode and the carrier frequency of the input light field. Then, in Ito form the Schrodinger’s equation for the temporal evolution of the open quantum linear system in Fig. 1 is, \[23, \, [19, \text{Eq. (30)}], \, [20, \, \text{Eq. (22)}], \, [57, \text{Eq. (13)}], \]
\[
d U(t, t_0) = \left\{ \begin{array}{ll} \text{Tr} \left[ (S_- - I_m) d\Lambda(t)^T \right] + dB_{in}(t)L - L^T S_- dB_{in}(t) - \frac{1}{2} L^T L + iH \right\} U(t, t_0), \quad t \geq t_0 \tag{5} \end{array} \]

with \(U(t, t_0) = I\) (identity operator) for all \(t \leq t_0\).

In the Heisenberg picture, system operators evolve according to \(\hat{a}(t) = U(t, t_0)^* \hat{a} U(t, t_0)\) (component-wise on the components of \(\hat{a}\)). The output field \(\hat{b}_{out}(t)\) carries away information of the system after interaction, and is defined by \(\hat{b}_{out}(t) \triangleq U(t, t_0)^* \hat{b}_{in}(t) U(t, t_0)\) (component-wise on the components of \(\hat{b}_{in}(t)\)). Consequently, by Eq. (5) and Ito calculus, Heisenberg’s equation for the system in Fig. 1 is, \[20, \, \text{Eq. (26)}], \, [57, \, \text{Eqs. (14)-(15)}], \]
\[
\hat{a}(t) = A \hat{a}(t) + B S \hat{b}_{in}(t), \quad \hat{b}_{out}(t) = C \hat{a}(t) + S \hat{b}_{in}(t), \quad \hat{a}(t_0) = \hat{a}, \quad (6)
\]
\[
(7)
\]

where the constant system matrices are
\[
C = \Delta(C_-, C_+), \quad B = -C^\phi, \quad A = -\frac{1}{2} C^\phi C - i J_{in} \Delta(\Omega_-, \Omega_+), \quad S = \Delta(S_-, 0). \tag{8}
\]

The gauge process \(\Lambda_{out}(t)\) of the output fields,
\[
\Lambda_{out}(t) \triangleq \int_{t_0}^{t} \hat{b}_{out}^\dagger(\tau) b_{out}^T(\tau) d\tau = U(t, t_0)^* \Lambda(t) U(t, t_0), \quad (9)
\]
satisfies the following quantum stochastic differential equation (QSDE), \[19, \, [56, \, \text{Eq. (16)}], \]
\[
d \Lambda_{out}(t) = S^\# d\Lambda(t) S^T + S^\# dB_{in}^\dagger(t)L^T(t) + L^\#(t)dB_{in}^T(t)S^T + L^\#(t)L^T(t) dt. \tag{10}
\]

The diagonal elements of \(\Lambda_{out}(t)\) are operators for the total number of photons in each of the \(m\) output channels, counted from time \(t_0\) to \(t\). The intensity of the output field, namely the rate of change of the number process \(\Lambda_{out}(t)\), is defined to be, \[57, \, \text{Eq. (45)}],
\[
\bar{n}_{out}(t) \triangleq \langle \phi \Psi | \hat{b}_{out}^\dagger(t) b_{out}^T(t) | \phi \Psi \rangle. \tag{11}
\]

In Eq. (11), \(|\phi\rangle\) is the initial system state and \(|\Psi\rangle\) is the initial input field state, respectively. Therefore, the ket vector \(|\phi \Psi\rangle\) is the joint system-field state. The bra vector \(|\phi \Psi\rangle\) is the Hilbert space conjugate of the ket vector \(|\phi \Psi\rangle\). In this paper, \(|\phi\rangle\) is assumed to be the vacuum state, whereas the specific form of \(|\Psi\rangle\) will be given in due course.
The quantum linear system $G$ is said to be *asymptotically stable* if the matrix $A$ is Hurwitz stable [55, Sec. III-A]. In analogy to classical (namely non-quantum) control theory, the impulse response function of the system $G$ is, [57, Eq. (18)],

$$g_G(t) \triangleq \begin{cases} \delta(t)S - Ce^{\Lambda t}C^\dagger S, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

which enjoys the following block form

$$g_G(t) = \Delta(g_G^-(t), g_G^+(t)),$$

with matrices

$$g_G^-(t) \triangleq \begin{cases} \delta(t)S - [C_- \ C_+]e^{\Lambda t} \begin{bmatrix} C_+^\dagger \\ -C_-^\dagger \end{bmatrix} S_-, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$g_G^+(t) \triangleq \begin{cases} -[C_- \ C_+]e^{\Lambda t} \begin{bmatrix} -C_+^T \\ C_-^T \end{bmatrix} S^#, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Solving Eqs. (6)-(7) we have

$$\bar{b}_{\text{out}}(t) = Ce^{\Lambda(t-t_0)}\bar{a} + \int_{t_0}^{t} g_G(t-r)\bar{b}_{\text{in}}(r)dr.$$

Sending $t_0 \to -\infty$ in Eq. (15) yields

$$\bar{b}_{\text{out}}(t) = \int_{-\infty}^{t} g_G(t-r)\bar{b}_{\text{in}}(r)dr = g_G \circ \bar{b}_{\text{in}}(t).$$

**Remark 1** If the interaction starts in the remote past, namely $t_0 \to -\infty$, and if the system is asymptotically stable, Eq. (16) indicates that the initial system information has no influence on the output field. This is also true in classical control theory, see, e.g., [26].

Define a matrix function

$$g_{G^{-1}}(t) \triangleq \Delta(g_{G^-}(-t)^\dagger, -g_{G^+}(-t)^T).$$

It can be verified that

$$g_G \circ g_{G^{-1}} \circ f(t) = g_{G^{-1}} \circ g_G \circ f(t) = f(t)$$

holds for any function $f(t)$ of suitable dimension. That is, $g_{G^{-1}}(t)$ is the inverse function of the impulse response function $g_G(t).$ According to Eqs. (16) and (18), in the limit $t_0 \to -\infty$ we have

$$\bar{b}_{\text{in}}(t) = g_{G^{-1}} \circ \bar{b}_{\text{out}}(t).$$

A class of quantum linear passive systems is obtained when $C^+ = 0$ and $\Omega^+ = 0$ in Eq. (8). In this context, it is sufficient to work in the annihilation-operator representation. To be specific, it suffices to study

$$\dot{a}(t) = Aa(t) + BS_-\bar{b}_{\text{in}}(t),$$

$$\bar{b}_{\text{out}}(t) = Ca(t) + S_-\bar{b}_{\text{in}}(t),$$

where

$$A = -i\Omega_- - \frac{1}{2}C_-^\dagger C_-, \quad B = -C_-^\dagger, \quad C = C_-.$$

In this case, Eq. (14) becomes

$$g_G^-(t) = \begin{cases} \delta(t)S_- - C_-e^{\Lambda t}C_+^\dagger S_-, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$g_G^+(t) = 0.$$
Finally, we cite the following result for the Gaussian transfer of quantum linear systems.

**Lemma 1** [57, Theorem 2] Let the quantum linear system $G$ be initialized in the vacuum state $|\phi\rangle$ and let the input field be in the vacuum state $|0\rangle$. Then, the steady-state output field state is a Gaussian state with the spectral density

$$ R_{\text{out}}[i\omega] = G[i\omega] \left[ \begin{array}{cc} I_m & 0 \\ 0 & 0_m \end{array} \right] G[i\omega]^\dagger, $$

(22)

where $G[i\omega]$ is the two-sided Laplace transform of $g_G(t)$, as introduced in the Notation part. In particular, if the system is passive, then the output is also in a vacuum state, that is,

$$ R_{\text{out}}[i\omega] = \left[ \begin{array}{cc} I_m & 0 \\ 0 & 0_m \end{array} \right]. $$

(23)

### 2.2 Tensors

The concept of tensors and their associated operations are essential mathematical machinery for the study carried out in this paper [39], [25], [53]. In this subsection, we briefly discuss several tensors.

Given an $m \times m$ matrix function $A(t)$ and an $m$-way $m$-dimensional tensor function $\varphi(t_1, \ldots, t_m) = (\varphi_{j_1, \ldots, j_m}(t_1, \ldots, t_m))$, $(j_1, \ldots, j_m = 1, \ldots, m)$, define another $m$-way $m$-dimensional tensor function $\psi(r_1, \ldots, r_m) = (\psi_{i_1, \ldots, i_m}(r_1, \ldots, r_m))$ in such a way that, for all $i_1, \ldots, i_m = 1, \ldots, m$,

$$ \psi_{i_1, \ldots, i_m}(r_1, \ldots, r_m) = \sum_{j_1, \ldots, j_m=1}^m \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m A^{i_1 j_1}(r_1 - t_1) \cdots A^{i_m j_m}(r_m - t_m) \varphi_{j_1, \ldots, j_m}(t_1, \ldots, t_m). $$

(24)

Eq. (24) may be re-written in a more compact form

$$ \psi = \varphi \otimes_t^m A, $$

(25)

where the subscript “$t$” indicates the time domain, while the superscript “$m$” implies the $m$-fold convolution. Applying the $m$-dimensional Fourier transform (1) to Eq. (24), we get

$$ \psi_{i_1, \ldots, i_m}(i\omega_1, \ldots, i\omega_m) = \sum_{j_1, \ldots, j_m=1}^m A^{i_1 j_1}[i\omega_1] \cdots A^{i_m j_m}[i\omega_m] \varphi_{j_1, \ldots, j_m}(i\omega_1, \ldots, i\omega_m), $$

(26)

where

$$ A^{i_k j_k}[i\omega_k] = \int_{-\infty}^{\infty} e^{-i\omega_k t} A^{i_k j_k}(t) dt $$

is the two-sided Laplace transform of $A^{i_k j_k}(t)$. In analogy to Eq. (25), we may also write Eq. (26) in the following compact form

$$ \psi = \varphi \otimes_\omega^m A, $$

(27)

where the subscript “$\omega$” indicates the frequency domain.

Given an $m$-way $m$-dimensional tensor function $\varphi(i\omega_1, \ldots, i\omega_m)$, its norm is defined to be

$$ \|\varphi(i\omega_1, \ldots, i\omega_m)\| \triangleq \sqrt{\sum_{j_1, \ldots, j_m=1}^m |\varphi_{j_1, \ldots, j_m}(i\omega_1, \ldots, i\omega_m)|^2}. $$

(28)

We end this subsection by citing the following result.
Lemma 2  [39, Theorem 3.3] Let two tensors $\psi$ and $\varphi$ be related by Eq. (26), or equivalently Eq. (24). If $A[i\omega]$ is unitary for all $\omega \in \mathbb{R}$, then
\[
\|\psi(i\omega_1, \ldots, i\omega_m)\| = \|\varphi(i\omega_1, \ldots, i\omega_m)\|, \ \forall \omega_1, \ldots, \omega_m \in \mathbb{R}. \tag{29}
\]

More tensors and related operations will be discussed in due course.

3  $m$ photons superposed among $m$ channels

In this section, we investigate how a quantum linear system responds to $m$-photon input states. We first define $n$-photon states in Subsection 3.1, then derive the output intensity in Subsection 3.2, after that, we present the analytic form of the output field state when the underlying quantum linear system is passive in Subsections 3.3 and 3.4, finally we turn to the non-passive case in Subsection 3.5.

3.1  $m$-photon states

In this subsection we introduce $m$-photon states. We begin with the single-channel single-photon state case. In this case, $m = 1$. A single-channel single-photon state can be defined by
\[
|1_\xi\rangle \triangleq \int_{-\infty}^{\infty} dt \xi(t) b^*_m(t) |0\rangle. \tag{30}
\]
Here, $\xi$ is a square-integral function, i.e., $\xi \in L_2(t, \mathbb{C})$. The Euclidian norm of $\xi$, $\|\xi\| \triangleq \sqrt{\int_{-\infty}^{\infty} |\xi(t)|^2 dt}$, is equal to 1. Consequently, $\langle 1_\xi | 1_\xi \rangle = 1$. That is, $|1_\xi\rangle$ is a normalized state. Moreover, it can be easily shown that
\[
\lim_{t_0 \to -\infty, t \to +\infty} \langle 1_\xi | \Lambda(t) | 1_\xi \rangle = 1, \tag{31}
\]
where $\Lambda(t)$ is the gauge operator defined in Eq. (3). Eq. (31) shows that there is only one photon in the field. On the other hand, it can be readily shown that
\[
\langle 1_\xi | b_m(t) | 1_\xi \rangle = \langle 1_\xi | b^*_m(t) | 1_\xi \rangle = 0. \tag{32}
\]
That is, the average field amplitude is zero. It is worth noting that $|1_\xi\rangle$ is not a single-photon coherent state that can be defined by
\[
|\alpha_\xi\rangle \triangleq \exp \left( \int_{-\infty}^{\infty} dt \alpha \xi(t) b^*_m(t) - \int_{-\infty}^{\infty} dt (\alpha \xi(t))^* b_m(t) \right) |0\rangle, \tag{33}
\]
where $\alpha = e^{i\theta}$ is a complex number. Actually, for $|\alpha_\xi\rangle$ Eq. (31) still holds, but Eq. (32) does not.

Next, we discuss single-channel two-photon states, which can be defined to be
\[
|2_\xi\rangle \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi(t_1, t_2) b^*_m(t_1) b^*_m(t_2) dt_1 dt_2 |0\rangle. \tag{34}
\]
Here, the ordinary function $\xi(t_1, t_2)$ is required to normalize the state, namely $\langle 2_\xi | 2_\xi \rangle = 1$. Moreover, if we swap $t_1$ and $t_2$ in Eq. (34), we get
\[
|2_\xi\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi(t_2, t_1) b^*_m(t_1) b^*_m(t_2) dt_1 dt_2 |0\rangle. \tag{35}
\]
Comparing Eqs. (34) and (35) we learn that $\xi(t_1, t_2) = \xi(t_2, t_1)$, that is, $\xi$ is symmetric.

Now, let us look at two-channel two-photon states, which can be defined to be
\[
|\Psi_{in,2}\rangle \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi(t_1, t_2) b^*_m(t_1) b^*_m(t_2) dt_1 dt_2 |0_1\rangle \otimes |0_2\rangle. \tag{36}
\]
Again, the ordinary function $\xi(t_1, t_2)$ is required to normalize the state, namely $\langle \Psi_{\text{in,2}} | \Psi_{\text{in,2}} \rangle = 1$. This is guaranteed by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \ |\xi(t_1, t_2)|^2 = 1.$$  \hspace{1cm} (37)

(Notice that in this case, the symmetry condition $\xi(t_1, t_2) = \xi(t_2, t_1)$ is not necessary.) It can be readily shown that

$$\lim_{t_0 \to -\infty, t_1 \to \infty} \langle \Psi_{\text{in,2}} | \Lambda(t) | \Psi_{\text{in,2}} \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$  \hspace{1cm} (38)

where $\Lambda(t)$ is the gauge process defined in Eq. (3). Eq. (38) means that each channel contains one photon. Moreover, if we use the single-photon state $\int_{-\infty}^{\infty} \gamma(t) b_{\text{in,2}}^* dt |0_2\rangle$ to measure the second channel, we will get a single-photon state for the first channel, which is given by

$$\int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \gamma^*(t) \xi(t, r) dr \right] b_{\text{in,1}}^* dt |0_1\rangle.$$  \hspace{1cm} (39)

In general, Eq. (36) defines a state for which the two photons are entangled. However, for the special case that $\xi(t_1, t_2) = \xi_1(t_1) \xi_2(t_2)$, we end up with a product state

$$|\Psi_{\text{in,2}}\rangle = \int_{-\infty}^{\infty} \xi_1(t_1) b_{\text{in,1}}^* (t_1) dt_1 |0_1\rangle \otimes \int_{-\infty}^{\infty} \xi_2(t_2) b_{\text{in,2}}^* (t_2) dt_2 |0_2\rangle.$$

(40)

That is, there exists no statistical correlation between these two photons.

We are ready to introduce general $m$-channel $m$-photon states. Such states can be of the form

$$|\Psi_{\text{in}}\rangle \triangleq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m \ \psi_{\text{in}}(t_1, \ldots, t_m) b_{\text{in,1}}^* (t_1) \cdots b_{\text{in,m}}^* (t_m) |0_1\rangle \otimes \cdots \otimes |0_m\rangle.$$  \hspace{1cm} (41)

That is, $m$ photons are superposed among $m$ input channels. By analogy with Eq. (37), it can be readily shown that the normalization condition for $|\Psi_{\text{in}}\rangle$ is

$$\|\psi_{\text{in}}\|^2 = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m \ |\psi_{\text{in}}(t_1, \ldots, t_m)|^2 = 1.$$  \hspace{1cm} (42)

The bra vector $\langle \Psi_{\text{in}} |$, namely the conjugate of the ket vector $|\Psi_{\text{in}}\rangle$, is

$$\langle \Psi_{\text{in}} | = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dt_1 \cdots dt_m \ \psi_{\text{in}}^*(t_1, \ldots, t_m) b_{\text{in,1}} (t_1) \cdots b_{\text{in,m}} (t_m) |0_1\rangle \otimes \cdots \otimes |0_m\rangle.$$  \hspace{1cm} (43)

For the $m$-photon state $|\Psi_{\text{in}}\rangle$, it is clear that

$$\langle \Psi_{\text{in}} | \bar{b}_{\text{in}}(t) |\Psi_{\text{in}}\rangle = 0.$$  \hspace{1cm} (44)

That is, the average field amplitude of the input light field in the $m$-photon state $|\Psi_{\text{in}}\rangle$ is 0. Next, we look at two-time correlations. For each $k = 1, \ldots, m$, define a function of two variables $\zeta_k(t, r)$ to be

$$\zeta_k(t, r) \triangleq \psi_{\text{in}}(t_1, \ldots, t_{k-1}, r, t_{k+1}, \ldots, t_m).$$  \hspace{1cm} (45)

Also, define a diagonal matrix function

$$\Lambda(t, r) \triangleq \begin{bmatrix} \int d\tau_2 \cdots d\tau_m \ \zeta_1(\tau, t)^* \zeta_1(\tau, r) \\
\cdots \\
\int d\tau_1 \cdots d\tau_{m-1} \ \zeta_m(\tau, t)^* \zeta_m(\tau, r) \end{bmatrix}, \ \forall t, r \in \mathbb{R}.$$  \hspace{1cm} (46)
The output intensity is given by

$$\langle \Psi_{\text{in}} | \tilde{b}_{\text{in}}(t) \tilde{b}_{\text{in}}(r) | \Psi_{\text{in}} \rangle = \delta(t-r) \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix} + \begin{bmatrix} \Lambda(r,t) & 0 \\ 0 & \Lambda(t,r)^\# \end{bmatrix}. \quad (47)$$

**Remark 2** If all the input fields are in the vacuum state, it is well-known that

$$\langle \Psi_{\text{in}} | \tilde{b}_{\text{in}}(t) \tilde{b}_{\text{in}}(r) | \Psi_{\text{in}} \rangle = \delta(t-r) \begin{bmatrix} I_m & 0 \\ 0 & 0_m \end{bmatrix}. \quad (48)$$

In this case, the field is Markovian. The second term on the right-hand side of Eq. (47) reveals the non-Markovian nature of the m-photon input fields. Moreover, due to the presence of the pulse shape \( \psi_{\text{in}} \) in all the diagonal entries of \( \Lambda(t,r) \), the inputs can be regarded as correlated non-Markovian noise inputs.

For convenience, in the sequel we use the shorthand notation \( |0^\otimes m\rangle \) for \( |0_1\rangle \otimes \cdots \otimes |0_m\rangle \).

### 3.2 The passive case: output intensity

In this subsection, for the quantum linear passive system (20) driven by the m-photon state \( |\Psi_{\text{in}}\rangle \) defined in Eq. (41), we derive a formula for the output intensity \( \bar{n}_{\text{out}}(t) \) defined in Eq. (11).

Recall that in the passive case the matrix \( C_+ = 0 \). Substitution of \( L(t) = C_- \delta(t) \) into Eq. (10) yields

$$d\Lambda_{\text{out}}(t) = S^#d\Lambda(t)S_T^T + S^#d\bar{B}_{\text{in}}^#(t)a^T(t)C_T^- + C^#a^#(t)d\bar{B}_{\text{in}}^T(t)S_T^- + C^#a^#(t)a^T(t)C_T^- dt. \quad (49)$$

Define a matrix function of dimension \( n \) to be

$$\Sigma(t) \triangleq \langle \phi_{\text{in}} | a(t)a^T(t) | \phi_{\text{in}} \rangle, \quad t \geq t_0. \quad (50)$$

Define an \( n \)-by-\( m \) matrix function

$$f(t) \triangleq \langle \phi_{\text{in}} | a(t)db_{\text{in}}^t(t) | \phi_{\text{in}} \rangle. \quad (51)$$

The following theorem is the main result of this subsection, which gives an explicit procedure for computing the output intensity \( \bar{n}_{\text{out}}(t) \).

**Theorem 1** Assume the underlying quantum linear passive system \( G \) is asymptotically stable. The matrix function \( f(t) \) defined in Eq. (51) is the solution to a system of ordinary differential equations (ODEs)

$$f(t) = \int_{t_0}^t e^{A(t-r)}C_+^T \begin{bmatrix} \langle \zeta_1(t) | \zeta_1(r) \rangle \\ \vdots \\ \langle \zeta_m(t) | \zeta_m(r) \rangle \end{bmatrix} dr, \quad (52)$$

with the initial condition \( f(t_0) = 0 \), where

$$|\zeta_j(t) \rangle \triangleq \int dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_m \psi_{\text{in}}(t_1, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_m) \prod_{k \neq j} b_{\text{in}}^T(t_k) |0^\otimes m\rangle, \quad j = 1, \ldots, m. \quad (53)$$

The output intensity is given by

$$\bar{n}_{\text{out}}(t) = S^# \begin{bmatrix} \langle \zeta_1(t) | \zeta_1(t) \rangle \\ \vdots \\ \langle \zeta_m(t) | \zeta_m(t) \rangle \end{bmatrix} S_T^T \quad (54)$$

$$+ S^#f(t)^T C_T^- + C^#f(t)^# S_T^- - C^#C_T^- + C^# \Sigma(t)^T C_T^-,$$
in which the covariance function \( \Sigma(t) \) solves the following matrix equation

\[
\Sigma(t) = A\Sigma(t) + \Sigma(t)A^\dagger + C^\dagger C_\perp - C^\dagger S_- \dot{f}(t) + \dot{f}(t)S_-^\dagger C_\perp,
\]

with the initial condition \( \Sigma(t_0) = I_n \).

**Proof.** We prove this theorem in three steps.

**Step 1.** We establish Eq. (52). Firstly, it can be readily shown that

\[
b_{in}(t)|\Psi_{in} = \begin{bmatrix}
|\zeta_1(t)\rangle \\
|\zeta_2(t)\rangle \\
\vdots \\
|\zeta_m(t)\rangle 
\end{bmatrix},
\]

where \( |\zeta_j(t)\rangle \) are those defined in Eq. (53), \( j = 1, \ldots, m \). As a result,

\[
\langle \phi|\Psi_{in}|b_{in}(t)|\Psi_{in}\rangle = \langle \Psi_{in}|b_{in}(t)|\Psi_{in}\rangle = \begin{bmatrix}
\langle \zeta_1(t)|\zeta_1(t)\rangle \\
\vdots \\
\langle \zeta_m(t)|\zeta_m(t)\rangle
\end{bmatrix},
\]

where, for all \( j = 1, \ldots, m \), the inner products are

\[
\langle \zeta_j(t)|\zeta_j(r)\rangle = \int dt_1 \cdots dt_{j-1} dt_{j+1} \cdots dt_m \psi_{in}(t_1, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_m)^* \psi_{in}(t_1, \ldots, t_{j-1}, r, t_{j+1}, \ldots, t_m).
\]

Moreover, by Eqs. (51) and (56) we have

\[
f(t) = \langle \phi|\Psi_{in}|a(t)|dB_{in}(t)|\phi|\Psi_{in}\rangle
\]

\[
= \left[ \langle \phi|\Psi_{in}|dB_{in,1}(t)|a(t)\rangle|\phi|\Psi_{in}\rangle \cdots \langle \phi|\Psi_{in}|dB_{in,m}(t)|a(t)\rangle|\phi|\Psi_{in}\rangle \right] dt,
\]

where, in the second step, the fact \( [dB_{in,j}(t), a(t)] = 0 \) has been used \( j = 1, \ldots, m \). Eq. (59) can be re-written as

\[
\dot{f}(t) = \left[ \langle \phi|\zeta_1(t)|a(t)\rangle|\phi|\Psi_{in}\rangle \cdots \langle \phi|\zeta_m(t)|a(t)\rangle|\phi|\Psi_{in}\rangle \right].
\]

Secondly, solving Eq. (20) we get

\[
a(t) = e^{A(t-t_0)}a(t_0) - \int_{t_0}^t e^{A(t-r)}C^\dagger b_{in}(r)dr, \quad t \geq t_0.
\]

Partition the \( n \) by \( m \) matrix function \( e^{At}C^\dagger \) into \( m \) columns,

\[
e^{At}C^\dagger = \begin{bmatrix}
c_1(t) \\
c_2(t) \\
\vdots \\
c_m(t)
\end{bmatrix}.
\]
By Eqs. (62), (56), and (61), we have

\[
\begin{align*}
&\left[ \langle \phi \zeta_1(t) | a(t) | \phi \Psi_{in} \rangle \cdots \langle \phi \zeta_m(t) | a(t) | \phi \Psi_{in} \rangle \right] \\
&= - \left[ \langle \phi \zeta_1(t) | \int_t^t dr \ e^{A(t-r)} C_1^\dagger b_n(r) | \phi \Psi_{in} \rangle \cdots \langle \phi \zeta_m(t) | \int_t^t dr \ e^{A(t-r)} C_1^\dagger b_n(r) | \phi \Psi_{in} \rangle \right] \\
&= - \int_t^t \sum_{j=1}^m c_j (t-r) \left[ \langle \phi \zeta_1(t) | b_{in,j}(r) | \phi \Psi_{in} \rangle \cdots \langle \phi \zeta_m(t) | b_{in,j}(r) | \phi \Psi_{in} \rangle \right] dr \\
&= - \int_t^t \left[ \langle \phi \zeta_1(t) | c_1(t-r) b_{in,1}(r) | \phi \Psi_{in} \rangle \cdots \langle \phi \zeta_m(t) | c_m(t-r) b_{in,m}(r) | \phi \Psi_{in} \rangle \right] dr \\
&= - \int_t^t \left[ \langle \zeta_1(t) | c_1(t-r) \phi \zeta_1(r) \cdots \langle \zeta_m(t) | c_m(t-r) \phi \zeta_m(r) \rangle \right] dr \\
&= - \int_t^t e^{A(t-r)} C_1^\dagger \left[ \langle \zeta_1(t) | \zeta_1(r) \rangle \cdots \langle \zeta_m(t) | \zeta_m(r) \rangle \right] dr. 
\end{align*}
\]

(63)

Substituting Eq. (63) into Eq. (60) gives Eq. (52).

**Step 2.** We establish Eq. (55). By Ito calculus and Eq. (51), algebraic manipulations give

\[
\begin{align*}
\text{d} \Sigma(t) &= d\langle \phi \Psi_{in} | a(t) a^\dagger(t) | \phi \Psi_{in} \rangle \\
&= \langle \phi \Psi_{in} | (da(t)) a^\dagger(t) | \phi \Psi_{in} \rangle + \langle \phi \Psi_{in} | a(t)(da^\dagger(t)) | \phi \Psi_{in} \rangle + \langle \phi \Psi_{in} | (da(t))(da^\dagger(t)) | \phi \Psi_{in} \rangle \\
&= A \Sigma(t) dt + \Sigma(t) A^\dagger dt + C_1^\dagger C_- dt \\
&\quad - C_1^\dagger S f(t)^\dagger \quad - f(t) S_1^\dagger C_. 
\end{align*}
\]

(64)

Differentiating both sides of Eq. (64) with respect to \( t \) yields Eq. (55).

**Step 3.** We establish Eq. (54). By the canonical commutation relation \( [a_j, a_k^\dagger] = \delta_{jk} \ (j, k = 1, \ldots, n) \), we have

\[
\begin{align*}
\Sigma(t) &= \langle \phi \Psi_{in} | a(t) a^\dagger(t) | \phi \Psi_{in} \rangle \\
&= \langle \phi \Psi_{in} | \left[ I + (a^\#(t) a^\dagger(t))^T \right] | \phi \Psi_{in} \rangle \\
&= I + \langle \phi \Psi_{in} | (a^\#(t) a^\dagger(t))^T | \phi \Psi_{in} \rangle \\
&= I + \langle \phi \Psi_{in} | (a^\#(t)) a^\dagger(t) | \phi \Psi_{in} \rangle. 
\end{align*}
\]

(65)

This, together with Eq. (49), yields

\[
\begin{align*}
\langle \phi \Psi_{in} | d\Lambda_{out}(t) | \phi \Psi_{in} \rangle &= S_1^\# \langle \phi \Psi_{in} | d\Lambda(t) | \phi \Psi_{in} \rangle S^T_1 + S_2^\# \langle \phi \Psi_{in} | dB_{in}(t) a^\dagger(t) | \phi \Psi_{in} \rangle C^T \\
&\quad + C_1^\# \langle \phi \Psi_{in} | a^\#(t) dB_{in}(t) | \phi \Psi_{in} \rangle S^T_1 + C_2^\# \langle \phi \Psi_{in} | a^\#(t) a^\dagger(t) | \phi \Psi_{in} \rangle C^T dt \\
&= S_1^\# \langle \phi \Psi_{in} | d\Lambda(t) | \phi \Psi_{in} \rangle S^T_1 + S_2^\# f(t)^T C^T + C_1^\# f(t)^\# S^T_1 - C_2^\# C_- dt + C_2^\# \Sigma(t)^T C^T dt. 
\end{align*}
\]

(66)

By Eq. (57),

\[
\langle \phi \Psi_{in} | d\Lambda(t) | \phi \Psi_{in} \rangle = \begin{bmatrix}
\langle \zeta_1(t) | \zeta_1(t) \rangle \\
\vdots \\
\langle \zeta_m(t) | \zeta_m(t) \rangle
\end{bmatrix} dt, 
\]

(67)
in which the diagonal terms \((\zeta_j(t)|\zeta_j(t))\) are given in Eq. (58). Substituting Eq. (67) into Eq. (66) and differentiating both sides of the resulting equation with respect to \(t\) yields Eq. (54).\(\square\)

### 3.3 The passive case: state transfer

In this subsection, we derive the analytical form of the output state of a quantum linear passive system driven by the \(m\)-photon state \(|\Psi_{in}\rangle\) defined in Eq. (41).

The following is the main result of this subsection.

**Theorem 2** Let \(G\) be an asymptotically stable quantum linear passive system which is initialized in the vacuum state and is driven by the \(m\)-photon input \(|\Psi_{in}\rangle\) defined in Eq. (41). The steady state \((t_0 \to -\infty)\) of the output field is an \(m\)-photon state of the form

\[
|\Psi_{out}\rangle = \sum_{j_1,\ldots,j_m=1}^{m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dr_1 \cdots dr_m b_{out,j_1}^*(r_1) \cdots b_{out,j_m}^*(r_m) \psi_{out,j_1,\ldots,j_m}(r_1,\ldots,r_m)|0^{\otimes m}\rangle,
\]

where the output pulse is given by the \(m\)-fold convolution

\[
\psi_{out} = \psi_{in} \circledast G -
\]

with the transfer function \(G_{G^-}(t)\) given in Eq. (21).

**Proof.** In the passive case, sending \(t_0 \to -\infty\), by Eqs. (17) and (19) we get

\[
\begin{bmatrix}
  b_{in}(t) \\
  \dot{b}_{in}(t)
\end{bmatrix} = \int_{-\infty}^{\infty} \begin{bmatrix}
  G_{G^-}(r-t) b_{out}(r) \\
  G_{G^-}(r-t) \dot{b}_{out}(r)
\end{bmatrix} dr.
\]

(70)

Consequently,

\[
|\Psi_{in}\rangle \\
= \int dT \psi_{in}(t_1,\ldots,t_m) b_{in,1}^*(t_1) \cdots b_{in,m}^*(t_m)|0^{\otimes m}\rangle \\
= \int dT \psi_{in}(t_1,\ldots,t_m) \sum_{j_1=1}^{m} \int_{-\infty}^{\infty} g_{G^-}^{j_1}(r_1-t_1) b_{out,j_1}^*(r_1) dr_1 \cdots \sum_{j_m=1}^{m} \int_{-\infty}^{\infty} g_{G^-}^{j_m}(r_m-t_m) b_{out,j_m}^*(r_m) dr_m |0^{\otimes m}\rangle \\
= \sum_{j_1=1}^{m} \cdots \sum_{j_m=1}^{m} \int dT b_{out,j_1}^*(r_1) \cdots b_{out,j_m}^*(r_m) \int dT \psi_{in}(t_1,\ldots,t_m) |0^{\otimes m}\rangle,
\]

(71)

where \(\psi_{out,j_1,\ldots,j_m}(r_1,\ldots,r_m) = \int dT g_{G^-}^{j_1}(r_1-t_1) \cdots g_{G^-}^{j_m}(r_m-t_m) \psi_{in}(t_1,\ldots,t_m), \forall j_1,\ldots,j_m = 1,\ldots,m.\) (72)

By means of notation defined in Eq. (25), in compact form Eq. (72) becomes Eq. (69). As a result, the steady-state output state \(|\Psi_{out}\rangle\) is that given in Eq. (68).\(\square\)

**Remark 3** In particular, when the input pulse is of a product form

\[
\psi_{in}(t_1,\ldots,t_m) = \xi_1(t_1) \cdots \xi_m(t_m),
\]

(73)
the input state $|\Psi_{\text{in}}\rangle$ in Eq. (41) becomes a separable state

$$|\Psi_{\text{in}}\rangle \equiv \prod_{k=1}^{m} B^*_{\text{in},k}(\xi_k)|0_k\rangle,$$

where the notation

$$B^*_{\text{in},k}(\xi) \equiv \int_{-\infty}^{\infty} \xi(t)b^*_{\text{in},k}(t)dt$$

has been used. In this case, Eq. (72) reduces to

$$\psi_{\text{out},j_1,\ldots,j_m}(r_1,\ldots,r_m) = \prod_{k=1}^{m} \int_{-\infty}^{\infty} dt_k \ g_{G-}^{j_k}(r_k - t_k)\xi_k(t_k).$$

Define

$$\xi_{\text{out},jk}(r) \equiv \int_{-\infty}^{\infty} g_{G-}^{j_k}(r - t)\xi_k(t)dt.$$

Then, by Theorem 2,

$$|\Psi_{\text{out}}\rangle = \sum_{j_1,\ldots,j_m=1}^{m} \prod_{k=1}^{m} B^*_{\text{out},jk}(\xi_{\text{out},jk})|0^\otimes m\rangle = \prod_{k=1}^{m} \sum_{j=1}^{m} B^*_j(\xi_{\text{out},jk})|0^\otimes m\rangle,$$

where

$$B^*_{\text{out},k}(\xi) \equiv \int_{-\infty}^{\infty} \xi(t)b^*_{\text{out},k}(t)dt, \quad k = 1,\ldots,m.$$

Interestingly, $|\Psi_{\text{out}}\rangle$ in Eq. (78) can also be derived by means of [57, Theorem 5]. Therefore, Theorem 2 generalizes the main result in [57]. Finally, it is worth pointing out that, in general, even if the input state $|\Psi_{\text{in}}\rangle$ is a separable state (74), the output state $|\Psi_{\text{out}}\rangle$ in Eq. (78) is not a separable state any more, as illustrated by the following two examples.

**Example 1 (beamsplitter)** A beamsplitter is a static passive device widely used in optical laboratories, [27], [3], [34], see Fig. 2. In terms of the $(S_-, L, H)$ formalism, a beamsplitter can be modeled by $L = 0$, $H = 0$, and

$$S_- = \begin{bmatrix} R & T \\ T & R \end{bmatrix}, \quad R, T \in \mathbb{C}, \quad |R|^2 + |T|^2 = 1.$$  

Let the 2-photon input state be

$$|\Psi_{\text{in}}\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 \psi_{\text{in}}(t_1,t_2)b^*_{\text{in},1}(t_1)b^*_{\text{in},2}(t_2)|0_10_2\rangle.$$  

---

**Fig. 2. Schematic representation of a beamsplitter**
If the input pulse shape \( \psi_{in}(t_1,t_2) \) is not factorizable, then the two input channels are initially entangled. By (68), the output state is

\[
|\Psi_{out}\rangle = RT \int dr_1 dr_2 b_{in,1}^*(r_1)b_{in,1}^*(r_2)\psi_{in}(r_1,r_2)|0_1\rangle \otimes |0_2\rangle + R^2 \int dr_1 dr_2 b_{in,1}^*(r_1)b_{in,2}^*(r_2)\psi_{in}(r_1,r_2)|0_10_2\rangle \\
+ T^2 \int dr_1 dr_2 b_{in,1}^*(r_2)b_{in,2}^*(r_1)\psi_{in}(r_1,r_2)|0_10_2\rangle + RT|0_1\rangle \otimes \int dr_1 dr_2 b_{in,2}^*(r_1)b_{in,2}^*(r_2)\psi_{in}(r_1,r_2)|0_2\rangle,
\]

which is exactly [29, Eq. (6.8.7)].

Example 2 (optical cavity) An optical cavity is a system composed of totally reflecting and/or partially transmitting mirrors [3, Chapter 5.3], [46, Chapter 7], [34]. A widely used type of optical cavities is the so-called Fabry-Perot cavity. A single-mode Fabry-Perot cavity with two input channels, as shown in Fig. 3, can be modeled by parameters

\[
(I, L = \begin{bmatrix} \sqrt{\kappa_1} a \\ \sqrt{\kappa_2} a \end{bmatrix}, H = \omega_d a^* a).
\]

Here, \( \kappa_1 \) and \( \kappa_2 \) are coupling strengths between the cavity and the external fields, and \( \omega_d \) is the detuning between the resonant frequency of the cavity and the external fields. By Eq. (20) we have the following QSDEs

\[
\dot{a}(t) = (-\kappa_1 + \kappa_1 + i\omega_d)a(t) - \sqrt{\kappa_1} b_{in,1}(t) - \sqrt{\kappa_2} b_{in,2}(t), \\
b_{out,1}(t) = \sqrt{\kappa_1} a(t) + b_{in,1}(t), \\
b_{out,2}(t) = \sqrt{\kappa_2} a(t) + b_{in,2}(t).
\]

Let the input state be that given in Eq. (81). In what follows we calculate the steady state of the output field. Define functions

\[
\Phi_1(r,t_2) \triangleq \int_{t_2}^{\tau} dt_1 e^{-(i\omega_d + \frac{\kappa_1 + \kappa_2}{2})(r-t_1)}\psi_{in}(t_1,t_2),
\]

\[
\Phi_2(t_1,r) \triangleq \int_{-\infty}^{t_1} dt_2 e^{-(i\omega_d + \frac{\kappa_1 + \kappa_2}{2})(r-t_2)}\psi_{in}(t_1,t_2),
\]

and

\[
\Phi(r,\tau) \triangleq \int_{-\infty}^{\tau} dt_1 \int_{-\infty}^{\tau} dt_2 e^{-(i\omega_d + \frac{\kappa_1 + \kappa_2}{2})(r+\tau-t_1-t_2)}\psi_{in}(t_1,t_2).
\]

By Theorem 2, the output state is

\[
|\Psi_{out}\rangle = \sqrt{\kappa_1 \kappa_2} \int_{-\infty}^{\infty} dr_1 dr_2 b_{out,1}^*(r_1)b_{out,2}^*(r_2) [\kappa_1 \Phi(r_1,r_2) - \Phi_2(r_1,r_2)] |0_1\rangle \otimes |0_2\rangle \\
+ \int_{-\infty}^{\infty} dr_1 dr_2 b_{out,1}^*(r_1)b_{out,2}^*(r_2) [\psi_{in}(r_1,r_2) - \kappa_1 \Phi_1(r_1,t_2) - \kappa_2 \Phi_2(r_1,r_2) \\
+ \kappa_1 \kappa_2 (\Phi(r_1,r_2) + \Phi(r_2,r_1))] |0_1\rangle \otimes |0_2\rangle \\
+ \sqrt{\kappa_1 \kappa_2} \int_{-\infty}^{\infty} dr_1 dr_2 b_{out,2}^*(r_1)b_{out,2}^*(r_2) [\kappa_2 \Phi(r_1,r_2) - \Phi_1(r_1,t_2)] |0_1\rangle \otimes |0_2\rangle.
\]
In particular, if
\[ \psi_{in}(t_1, t_2) = \xi_1(t_1)\xi_2(t_2), \] (89)
that is, the input is a tensor product state of two single-photon states, one for each channel, then Eqs. (85)-(87) reduce to
\[ \Phi_1(r_1, r_2) = \xi_2(r_2)\eta_1(r_1), \] (90)
\[ \Phi_2(r_1, r_2) = \xi_1(r_1)\eta_2(r_2), \] (91)
and
\[ \Phi(r_1, r_2) = \eta_1(r_1)\eta_2(r_2), \] (92)
where
\[ \eta_i(t) \triangleq \int_{-\infty}^{t} e^{-i(\omega_d + \frac{\Gamma_1 + \Gamma_2}{2})(t-r)}\xi_i(r)dr, \quad i = 1, 2. \] (93)
As a result, Eq. (88) becomes
\[ |\Psi_{out}\rangle = (B^*_{out,1}(\xi_1) - \kappa_1 B^*_{out,2}(\eta_1) - \sqrt{\kappa_1\kappa_2}B^*_{out,2}(\eta_2)) (B^*_{out,2}(\xi_2) - \sqrt{\kappa_1\kappa_2}B^*_{out,1}(\eta_1) - \kappa_2 B^*_{out,1}(\eta_2)) |0_1\rangle \otimes |0_2\rangle, \] (94)
where the convention in Eq. (79) has been used. Sending \( \kappa_1 \to 0 \) in Eq. (94) yields
\[ |\Psi_{out}\rangle = B^*_{out,1}(\xi_1) |0_1\rangle \otimes B^*_{out,2}(\xi_2 - \kappa_2 \eta_2) |0_2\rangle. \] (95)
This means that the influence of the system on the first channel is negligible and the output fields are almost in a product state. This is quite reasonable: when the coupling strength \( \kappa_1 \to 0 \), the first channel has no interaction with the system, so the state of channel one does not change. On the other hand, in the limit \( \kappa_1 \to 0 \), the non-separable state in Eq. (88) becomes
\[ |\Psi_{out}\rangle = \int dr_1 dr_2 b^*_{out,1}(r_1) b^*_{out,2}(r_2) \{ \psi_{in}(r_1, r_2) - \kappa_2 \Phi_2(r_1, r_2) \} |0_1\rangle \otimes |0_2\rangle. \] (96)
Notice that
\[ \psi(r_1, r_2) - \kappa_2 \Phi_2(r_1, r_2) \\
= \psi(r_1, r_2) - \kappa_2 \int_{-\infty}^{r_1} dt_2 e^{-i(\omega_d + \frac{\Delta}{2})(r_2-t_2)} \psi_{in}(r_1, t_2) \\
= g * \psi_{in}(r_1, r_2), \] (97)
where
\[ g(t) \triangleq \begin{cases} \\
\delta(t) - \kappa_2 e^{-i(\omega_d + \frac{\Delta}{2})t}, & t \geq 0, \\
0, & t < 0. \end{cases} \] (98)
That is, the output state is still an entangled state. And the system does have influence on the first channel. This cannot happen when the two input channels are separable, as shown in Eq. (95) above.

3.4 The passive case: the invariant set

Define a class of \( m \)-channel \( m \)-photon states of the form
\[ \mathcal{F}_1 \triangleq \{ |\Psi\rangle = \sum_{j_1, \ldots, j_m = 1}^{m} \int \bigotimes_{i=1}^{m} (\psi_{j_1,\ldots,j_m}(t_1, \ldots, t_m) b^*_{j_1}(t_1) \cdots b^*_{j_m}(t_m) |0^\otimes m\rangle \}, \] (99)
the \( m \) way \( m \) dimensional tensor function \( \psi \) normalizes \( |\Psi\rangle \).

Here, \( b_{j_k}^* (t) \) (\( k = 1, \ldots, m \)) is a creation operator. In the definition of the class \( \mathcal{F}_1 \), we don’t specify whether \( b_{j_k}^* (t) \) is for input or for output. In fact, in this subsection we show that \( \mathcal{F}_1 \) is invariant under the linear action of a quantum linear passive system. That is, both input and output states are elements of \( \mathcal{F}_1 \).
Recall the \(m\)-photon input state defined in Eq. (41). Define an \(m\)-way \(m\)-dimensional tensor function \(\psi_{in}^+(r_1, \ldots, r_m)\) with entries

\[
\psi_{in,j_1,\ldots,j_m}^+ \triangleq \begin{cases} 
\psi_{in}(r_1, \ldots, r_m), & j_1 = 1, j_2 = 2, \ldots, j_m = m, \\
0, & \text{elsewhere}.
\end{cases}
\]

(100)

Clearly, \(|\Psi_{in}\rangle = |\psi_{in}^+\rangle \in \mathcal{F}_1\). On the other hand, by Theorem 2, the output state \(|\psi_{out}\rangle \in \mathcal{F}_1\) too. This motivates us to study more general pulse shape transfer than that in Theorem 2. Actually, we have the following result.

**Theorem 3** Let the input state for an asymptotically stable quantum linear passive system \(G\) be an element \(|\Psi_{in}\rangle \in \mathcal{F}_1\) with pulse shape parametrized by an \(m\)-way \(m\)-dimensional tensor function \(\psi_{in}\). Then the steady-state output state \(|\Psi_{out}\rangle\) is also an element in \(\mathcal{F}_1\) with pulse shape given by

\[
|\Psi_{out}\rangle = |\Psi_{in}\rangle \otimes \omega^m G.
\]

(101)

Alternatively, in the frequency domain,

\[
|\psi_{out}(i\omega_1, \ldots, i\omega_m)\rangle = \omega^m G. 
\]

(102)

Moreover, we have

\[
\|\psi_{out}(i\omega_1, \ldots, i\omega_m)\| = \|\psi_{in}(i\omega_1, \ldots, i\omega_m)\|.
\]

(103)

**Proof.** By analogy with the proof of Theorem 2, we know that for all \(k = 1, \ldots, m\) and \(j_k = 1, \ldots, m\),

\[
b_{in,j_k}(t_k) = \sum_{j_1=1}^{m} \int_{-\infty}^\infty g_{G}^{i_k,j_k}(r_k - t_k) b_{out,i_k}(r_k) dr_k.
\]

(104)

Consequently,

\[
|\Psi_{in}\rangle = \sum_{j_1,\ldots,j_m=1}^m \int d^m t \ b_{in,j_1}(t_1) \cdots b_{in,j_m}(t_m) \psi_{j_1,\ldots,j_m}(t_1, \ldots, t_m) |0^\otimes m\rangle \\
= \sum_{j_1,\ldots,j_m=1}^m \int d^m t \ \psi_{j_1,\ldots,j_m}(t_1, \ldots, t_m) \sum_{i_1=1}^m \int_{-\infty}^\infty g_{G}^{i_1,j_1}(r_1 - t_1) b_{out,i_1}(r_1) dr_1 \\
\times \cdots \sum_{i_m=1}^m \int_{-\infty}^\infty g_{G}^{i_m,j_m}(r_m - t_m) b_{out,i_m}(r_m) dr_m |0^\otimes m\rangle \\
= \sum_{i_1=1}^m \cdots \sum_{i_m=1}^m \int d^m t \ b_{out,i_1}(r_1) \cdots b_{out,i_m}(r_m) \\
\times \int d^m t \ g_{G}^{i_1,j_1}(r_1 - t_1) \cdots g_{G}^{i_m,j_m}(r_m - t_m) \psi_{j_1,\ldots,j_m}(t_1, \ldots, t_m) |0^\otimes m\rangle \\
= \sum_{i_1=1}^m \cdots \sum_{i_m=1}^m \int d^m t \ \psi_{out,i_1,\ldots,i_m}(r_1, \ldots, r_m) b_{out,i_1}(r_1) \cdots b_{out,i_m}(r_m) |0^\otimes m\rangle,
\]

(105)

where

\[
\psi_{out,i_1,\ldots,i_m}(r_1, \ldots, r_m) \triangleq \sum_{j_1,\ldots,j_m=1}^m \int d^m t \ g_{G}^{i_1,j_1}(r_1 - t_1) \cdots g_{G}^{i_m,j_m}(r_m - t_m) \psi_{in,j_1,\ldots,j_m}(t_1, \ldots, t_m).
\]

(106)

In the compact form, Eq. (106) is exactly Eq. (101). Therefore, Eq. (101) is established. Applying the \(m\)-dimensional Fourier transform (1) to Eq. (101) gives Eq. (102). Because the system is passive, \(G[i\omega]\) is a unitary matrix for all \(\omega \in \mathbb{R}\). Consequently, Eq. (103) follows Lemma 2 immediately. ■
3.5 The non-passive case

A quantum linear system is said to be non-passive if $C_+ \neq 0$ and (or) $\Omega_+ \neq 0$ in Eq. (8). Non-passive elements, like optical parametric oscillators (OPOs), are key ingredients of quantum optical systems. [27], [3], [34]. In this subsection, we study the output state of a non-passive linear system driven by an $m$-photon input state.

Firstly, we introduce some notation. Define operators

$$b_j^d(t) \triangleq \begin{cases} b_{m,j}^d(t), & d = -1, \\ b_{m,j}^d(t), & d = 1, \end{cases}, \quad j = 1, \ldots, m. \tag{107}$$

Then, define an $m \times \cdots \times m \times 2 \times \cdots \times 2$ tensor operator $b = b_{j_1, \ldots, j_m}^{d_1, \cdots, d_m} (t_1, \ldots, t_m)$, whose entries are

$$b_{j_1, \ldots, j_m}^{d_1, \cdots, d_m} (t_1, \ldots, t_m) \triangleq b_{j_1}^{d_1}(t_1) \cdots b_{j_m}^{d_m}(t_m), \quad j_1, \ldots, j_m = 1, \ldots, m, \ d_1, \ldots, d_m = \pm 1. \tag{108}$$

Denote

$$g_{G^d}^{kj}(t) \triangleq \begin{cases} g_{G^d}^{kj}(t), & d = -1, \\ -g_{G^d}^{kj}(t)^*, & d = 1, \end{cases}, \quad j, k = 1, \ldots, m. \tag{109}$$

Then, define an $m \times \cdots \times m \times 2 \times \cdots \times 2$ tensor function $\psi$ with entries

$$\psi_{j_1, \ldots, j_m}^{d_1, \cdots, d_m} (r_1, \ldots, r_m) \triangleq \int_{-\infty}^\infty dt_1 \cdots dt_m \ g_{G^d}^{j_1}(r_1 - t_1) \cdots g_{G^d}^{j_m}(r_m - t_m) \psi_{in}(t_1, \ldots, t_m). \tag{110}$$

Finally, define the following operation between tensors $b$ and $\psi$

$$(b, \psi) \triangleq \sum_{j_1, \ldots, j_m = 1}^m \sum_{d_1, \ldots, d_m = \pm 1} \int dr_1 \cdots dr_m \ \psi_{j_1, \ldots, j_m}^{d_1, \cdots, d_m} (r_1, \ldots, r_m) b_{j_1}^{d_1}(r_1) \cdots b_{j_m}^{d_m}(r_m). \tag{111}$$

The following result shows how a quantum non-passive linear system processes multi-photon states.

**Theorem 4** Let $G$ be an asymptotically stable quantum linear system which is initialized in the vacuum state and is driven by the $m$-photon input $|\Psi_in\rangle$ defined in Eq. (41). The steady state ($t_0 \to -\infty$) of the output field is

$$\rho_{out} = (b, \psi) \langle \phi | \rho_\infty | \phi \rangle (b, \psi)^*, \tag{112}$$

where $(b, \psi)$ is that given in Eq. (111), and $\rho_\infty$ is a zero-mean Gaussian state for the joint system whose covariance function is given by Eq. (22) in Lemma 1.

**Proof.** In steady state, the joint system-field state is

$$\rho_\infty = \lim_{t_0 \to -\infty} \lim_{t \to \infty} U(t, t_0) |\phi \Psi_{in}\rangle \langle \phi \Psi_{in}| U(t, t_0)^*. \tag{113}$$

The steady-state output field state is obtained by taking the partial trace of $\rho_\infty$ with respect to the system, that is,

$$\rho_{out} = \lim_{t_0 \to -\infty} \lim_{t \to \infty} \langle \phi | U(t, t_0) |\phi \Psi_{in}\rangle \langle \phi \Psi_{in}| U(t, t_0)^* |\phi \rangle. \tag{114}$$

According to Eq. (19),

$$\tilde{b}_{in}(t) = g_{G^{-1}} \ast \tilde{b}_{out}(t) = g_{G^{-1}} \ast U(t, t_0)^* \tilde{b}_{in}(t) U(t, t_0). \tag{115}$$

So,

$$U(t, t_0) \tilde{b}_{in}(t) U(t, t_0)^* = g_{G^{-1}} \ast \tilde{b}_{in}(t). \tag{116}$$
This, together with Eq. (17), yields
\[
\lim_{t_0 \to -\infty} \lim_{t \to \infty} U(t, t_0) |\phi \Psi_{in}\rangle
= \lim_{t_0 \to -\infty} \int dt_1 \cdots dt_m \psi_{in}(t_1, \ldots, t_m) U(t, t_0) b_{in,1}^*(t_1) \cdots b_{in,m}^*(t_m) |\phi \otimes^m\rangle
= \lim_{t_0 \to -\infty} \int dt_1 \cdots dt_m \psi_{in}(t_1, \ldots, t_m) U(t, t_0) b_{in,1}^*(t_1) U(t, t_0)^* \cdots U(t, t_0) b_{in,m}^*(t_m) U(t, t_0)^* U(t, t_0) |\phi \otimes^m\rangle
= \int dt_1 \cdots dt_m \psi_{in}(t_1, \ldots, t_m) \int_{-\infty}^{\infty} \left[ -g_{G+}^1 (r_1 - t_1) + g_{G-}^1 (r_1 - t)^T \right] b_{in}(r_1) dr_1
\times \cdots
\times \int_{-\infty}^{\infty} \left[ -g_{G+}^m (r_m - t_m) + g_{G-}^m (r_m - t)^T \right] b_{in}(r_m) dr_m U(t, t_0) |\phi \otimes^m\rangle
= \int dt_1 \cdots dt_m \psi_{in}(t_1, \ldots, t_m) \left\{ \sum_{j_1=1}^m \int_{-\infty}^{\infty} -g_{G+}^{j_1} (r_1 - t_1) b_{in,j_1}(r_1) dr_1 + \sum_{j_1=1}^m \int_{-\infty}^{\infty} g_{G-}^{j_1} (r_1 - t_1) b_{in,j_1}(r_1) dr_1 \right\}
\times \cdots
\times \left\{ \sum_{j_1=1}^m \int_{-\infty}^{\infty} -g_{G+}^{j_1} (r_m - t_m) b_{in,j_m}(r_m) dr_m + \sum_{j_m=1}^m \int_{-\infty}^{\infty} g_{G-}^{j_1} (r_m - t_m) b_{in,j_m}(r_m) dr_m \right\} U(t, t_0) |\phi \otimes^m\rangle
= \langle b, \psi | U(t, t_0) |\phi \otimes^m\rangle.
\] (117)

Substituting it into Eq. (114) we obtain
\[
\rho_{out} = \lim_{t_0 \to -\infty} \lim_{t \to \infty} \langle \phi | U(t, t_0) |\phi \Psi_{in}\rangle \langle \phi \Psi_{in} | U(t, t_0)^* |\phi \rangle
= \langle b, \psi \rangle \lim_{t_0 \to -\infty} \lim_{t \to \infty} \langle \phi | U(t, t_0) |\phi \otimes^m\rangle \langle \phi \otimes^m | U(t, t_0)^* |\phi \rangle \langle b, \psi \rangle^*
= \langle b, \psi \rangle \langle \phi | \rho_{\infty} |\phi \rangle \langle b, \psi \rangle^*,
\] (118)

which is exactly Eq. (112). ■

4 \quad \textit{N (N ≥ m) photons superposed among m channels}

In this section, we study how quantum linear passive systems respond to \(N\) photons that are superposed over \(m\) channels, thus generalizing the results in Section 3.

4.1 \quad \textit{The case when \(N \geq m\)}

Let the input field be in a state where \(N\) photons are superposed among \(m\) input channels share. Specifically, the input state is defined as
\[
|\Psi_{in}\rangle \triangleq \frac{1}{\sqrt{\mathcal{N}^m}} \int dt_1^m \psi_{in}(t_1^1, \ldots, t_1^m, \ldots, t_m^1, \ldots, t_m^m) b_{in,1}^*(t_1^1) \cdots b_{in,1}^*(t_1^m) \cdots b_{in,m}^*(t_m^1) \cdots b_{in,m}^*(t_m^m) |0^\otimes^m\rangle.
\] (119)

In Eq. (119), the positive integers \(k_i\) satisfy \(\sum_{i=1}^m k_i = N\), and \(\mathcal{N} > 0\) is the normalization coefficient. Clearly, the \(i\)th input channel has \(k_i\) photons. In what follows, we derive the output field state.
Theorem 5  The steady-state output field state of an asymptotically stable quantum linear passive system $G$ driven by the $N$-photon input state $|\Psi_{in}\rangle$ is

$$
|\Psi_{out}\rangle = \frac{1}{\sqrt{N_N}} \sum_{t^1_1=1}^m \sum_{t^1_{k_1}=1}^m \sum_{t^m_l=1}^m \sum_{t^m_{k_m}=1}^m f_{t^1_1 \ldots t^m_{k_m}}(r^1_1, \ldots, r^1_{k_1}, \ldots, r^m_l, \ldots, r^m_{k_m})b^*_{t^1_1}(r^1_1) \cdots b^*_{t^1_{k_1}}(r^1_{k_1}) \cdots b^*_{t^m_l}(r^m_l) \cdots b^*_{t^m_{k_m}}(r^m_{k_m}) |\phi^\otimes m\rangle,
$$

where, for all $l^1_1, \ldots, l^1_{k_1}, \ldots, l^m_l, \ldots, l^m_{k_m} = 1, \ldots, m$, the pulse shapes are

$$
\psi_{out}^{l^1_1 \ldots l^m_{k_m}}(r^1_1, \ldots, r^1_{k_1}, \ldots, r^m_l, \ldots, r^m_{k_m})
\triangleq \int d t^1 \cdots g_G^{-1}(r^1_1 - t^1_1) \cdots g_G^{-1}(r^1_{k_1} - t^1_{k_1}) \cdots g_G^{-m}(r^m_l - t^m_l) \cdots g_G^{-m}(r^m_{k_m} - t^m_{k_m}) \psi_{in}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_l, \ldots, t^m_{k_m}).
$$

Proof.  We prove this result by means of an approach different from that for Theorem 2. In the Schrödinger picture, the output field state can be obtained by tracing out the system. That is,

$$
|\Psi_{out}\rangle = \langle \phi | \lim_{t_0 \to -\infty} U(t, t_0) |\phi \Psi_{in}\rangle
= \frac{1}{\sqrt{N_N}} \int d t^1 \cdots \psi_{in}(t^1_1, \ldots, t^1_{k_1}, \ldots, t^m_l, \ldots, t^m_{k_m})
\times \langle \phi | b^*(t^1_1, -\infty) \cdots b^*(t_{k_1}^1, -\infty) \cdots b^*_m(t^m_l, -\infty) \cdots b^*_m(t^m_{k_m}, -\infty) |\phi \rangle^\otimes m\rangle,
$$

where

$$
\tilde{b}^{-}(t, -\infty) \triangleq U(t, -\infty)\tilde{b}_{in}(t)U(t, -\infty)^*.
$$

By Eq. (19),

$$
\tilde{b}_{in}(t) = g_{G^{-1}} \oplus U(t, -\infty)^* \tilde{b}_{in}(t)U(t, -\infty).
$$

Alternatively,

$$
U(t, -\infty)\tilde{b}_{in}(t)U(t, -\infty)^* = g_{G^{-1}} \oplus \tilde{b}_{in}(t).
$$

Substituting Eq. (125) into Eq. (123) yields

$$
\tilde{b}^{-}(t, -\infty) = g_{G^{-1}} \oplus \tilde{b}_{in}(t).
$$

Consequently,

$$
\tilde{b}^{-}_j(t, -\infty) = \sum_{k=1}^m \int_{-\infty}^{k_j} g_{G^{-1}}(r_j - t)\tilde{b}_{in,k}^*(r_j)dr_j, \quad j = 1, \ldots, m.
$$
Substituting Eq. (127) into Eq. (122) gives

\[
\int d^3 \vec{r} \psi_{in}(t_1^1, \ldots, t_1^{k_1}, \ldots, t_m^1, \ldots, t_m^{k_m}) b_1^{-*}(t_1^1, - \infty) \cdot \cdot \cdot b_m^{-*}(t_m^1, - \infty) \cdot \cdot \cdot b_m^{-*}(t_m^{k_m}, - \infty)
\]

\[
= \int d^3 \vec{r} \psi_{in}(t_1^1, \ldots, t_1^{k_1}, \ldots, t_m^1, \ldots, t_m^{k_m}) \sum_{l_1^1=1}^m \int_{-\infty}^\infty g_{G^{-1}}(r_1^1 - t_1^1) b_{in,l_1^1}^*(r_1^1) dr_1^1 \cdot \cdot \cdot \sum_{l_m^{k_m}=1}^m \int_{-\infty}^\infty g_{G^{-1}}(r_m^{k_m} - t_m^{k_m}) b_{in,l_m^{k_m}}^*(r_m^{k_m}) dr_m^{k_m}
\]

\[
= \sum_{l_1^1=1}^m \cdot \cdot \cdot \sum_{l_m^{k_m}=1}^m \int d^3 \vec{r} b_{in,l_1^1}^*(r_1^1) \cdot \cdot \cdot b_{in,l_m^{k_m}}^*(r_m^{k_m})
\]

\[
\times \int d^3 \vec{r} g_{G^{-1}}(r_1^1 - t_1^1) \cdot \cdot \cdot g_{G^{-1}}(r_m^{k_m} - t_m^{k_m}) \psi_{in}(t_1^1, \ldots, t_1^{k_1}, \ldots, t_m^1, \ldots, t_m^{k_m})
\]

\[
= \sum_{l_1^1=1}^m \cdot \cdot \cdot \sum_{l_m^{k_m}=1}^m \int d^3 \vec{r} b_{in,l_1^1}^*(r_1^1) \cdot \cdot \cdot b_{in,l_m^{k_m}}^*(r_m^{k_m}) \psi_{out}(r_1^1, \ldots, r_1^{k_1}, \ldots, r_m^1, \ldots, r_m^{k_m}),
\]

(128)

where, in the last step, the definition of \( \psi_{out}^{l_1^1,\ldots,l_m^{k_m}}(r_1^1,\ldots,r_1^{k_1},\ldots,r_m^1,\ldots,r_m^{k_m}) \) in Eq. (121) has been used. Substitution of Eq. (128) into Eq. (122) yields Eq. (120). The proof is completed. \( \square \)

**Example 3** Given a beamsplitter

\[
S_- = \begin{bmatrix} R & -T \\ T & R \end{bmatrix}, \quad R, T \in \mathbb{C}, \quad |R|^2 + |T|^2 = 1,
\]

(129)

and a 3-photon input state of the form

\[
|\Psi_{in}\rangle = \frac{1}{\sqrt{N_3}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \psi_{in}(t_1, t_2, t_3) b_{in,1}^*(t_1) b_{in,2}^*(t_2) b_{in,3}^*(t_3) |0_1\rangle \otimes |0_2\rangle,
\]

(130)

Clearly, in this case, there are two input channels \((m = 2)\), the total number of photons is \(N = 3\), while there is one photon in the first channel \((k_1 = 1)\) and two photons in the second channel \((k_2 = 2)\). Moreover, the element of the impulse response \(g_{G^{-1}}(t) \equiv S_-\). Simple calculation yields

\[
\langle \Psi_{in}|\Psi_{in}\rangle = \frac{1}{N_3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \left( \psi_{in}^*(t_1, t_3, t_2) \psi_{in}(t_1, t_2, t_3) + |\psi_{in}(t_1, t_2, t_3)|^2 \right).
\]

(131)

So the normalization constant \(N_3\) is

\[
N_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 \left( \psi_{in}^*(t_1, t_3, t_2) \psi_{in}(t_1, t_2, t_3) + |\psi_{in}(t_1, t_2, t_3)|^2 \right).
\]

(132)
According to Eq. (120), the steady-state output state is

$$
|\Psi_{\text{out}}\rangle = \frac{\sqrt{T}}{\sqrt{3}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle \\
- T \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \left[ R^2 \psi_{\text{in}}(t_1, t_2, t_3) + R^2 \psi_{\text{in}}(t_1, t_3, t_2) - T^2 \psi_{\text{in}}(t_1, t_2, t_3) \right] |0_1\rangle \otimes |0_2\rangle \\
- R \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \left[ T^2 \psi_{\text{in}}(t_1, t_2, t_3) + T^2 \psi_{\text{in}}(t_1, t_2, t_3) - R^2 \psi_{\text{in}}(t_1, t_2, t_3) \right] |0_1\rangle \otimes |0_2\rangle \\
+ \frac{R^2 T}{\sqrt{3}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle.
$$

(133)

In particular, if $R = T = \frac{1}{\sqrt{2}}$, Eq. (133) reduces to

$$
|\Psi_{\text{out}}\rangle = \frac{1}{2\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle \\
- \frac{1}{2\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \left[ \psi_{\text{in}}(t_1, t_2, t_3) + \psi_{\text{in}}(t_1, t_3, t_2) - \psi_{\text{in}}(t_1, t_2, t_3) \right] |0_1\rangle \otimes |0_2\rangle \\
- \frac{1}{2\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \left[ \psi_{\text{in}}(t_1, t_2, t_3) + \psi_{\text{in}}(t_1, t_3, t_2) - \psi_{\text{in}}(t_1, t_2, t_3) \right] |0_1\rangle \otimes |0_2\rangle \\
+ \frac{1}{2\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle.
$$

(134)

Furthermore, if $\psi_{\text{in}}(t_1, t_2, t_3)$ is real-valued and fully symmetric, and $\|\psi_{\text{in}}(t_1, t_2, t_3)\| = 1$, then by Eq. (132), $N_3 = 2$. And Eq. (134) becomes

$$
|\Psi_{\text{out}}\rangle = \frac{1}{4} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle \\
- \frac{1}{4} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle \\
- \frac{1}{4} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle \\
+ \frac{1}{4} \int d\tilde{t}_1 b_{\text{out}, 2}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle.
$$

(135)

Define 3-photon states

$$
|\Pi_{30}\rangle \triangleq \frac{1}{\sqrt{6}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle,

|\Pi_{31}\rangle \triangleq \frac{1}{\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle,

|\Pi_{32}\rangle \triangleq \frac{1}{\sqrt{2}} \int d\tilde{t}_1 b_{\text{out}, 1}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_1\rangle \otimes |0_2\rangle,

|\Pi_{33}\rangle \triangleq \frac{1}{\sqrt{6}} \int d\tilde{t}_1 b_{\text{out}, 2}(t_1) b_{\text{out}, 2}(t_2) b_{\text{out}, 3}(t_3) \psi_{\text{in}}(t_1, t_2, t_3) |0_2\rangle.
$$

(136)

It is easy to show that all the states in Eq. (136) are normalized. Moreover, $|\Pi_{30}\rangle$ is a 3-photon state for the first channel, $|\Pi_{33}\rangle$ is a 3-photon state for the second channel, and $|\Pi_{31}\rangle$ and $|\Pi_{32}\rangle$ are states where two channels share three photons. With these new notations, Eq. (135) can be re-written as

$$
|\Psi_{\text{out}}\rangle = \frac{\sqrt{6}}{4} |\Pi_{30}\rangle \otimes |0_2\rangle - \frac{\sqrt{2}}{4} |\Pi_{31}\rangle - \frac{\sqrt{2}}{4} |\Pi_{32}\rangle + |0_1\rangle \otimes \frac{\sqrt{6}}{4} |\Pi_{33}\rangle.
$$

(137)
Finally, if
\[ \psi_{in}(t_1, t_2, t_3) = \xi_1(t_1)\xi_2(t_2)\xi_3(t_3), \]
that is, the input is a product state where each channel contains exactly one photon. In this case, Eq. (136) reduce to
\[ |\Pi_{30}\rangle = \frac{1}{\sqrt{6}} B_{out,1}^*(\xi_1) B_{out,1}^*(\xi_2) B_{out,1}^*(\xi_3)|0\rangle \otimes |0\rangle \]
\[ |\Pi_{21}\rangle = \frac{1}{\sqrt{2}} B_{out,1}^*(\xi_1) B_{out,1}^*(\xi_2)|0\rangle \otimes B_{out,2}^*(\xi_3)|0\rangle, \]
\[ |\Pi_{12}\rangle = \frac{1}{\sqrt{2}} B_{out,1}^*(\xi_1)|0\rangle \otimes B_{out,2}^*(\xi_2) B_{out,2}^*(\xi_3)|0\rangle, \]
\[ |\Pi_{03}\rangle = \frac{1}{\sqrt{6}} |0\rangle \otimes B_{out,2}^*(\xi_1) B_{out,2}^*(\xi_2) B_{out,2}^*(\xi_3)|0\rangle. \]
That is, all the states become product states. If we ignore pulse shapes, we may identify \(|\Pi_{30}\rangle\) with \(|3_1\rangle\), \(|\Pi_{03}\rangle\) with \(|3_2\rangle\), \(|\Pi_{21}\rangle\) with \(|2_1\rangle\otimes |1_2\rangle\), and \(|\Pi_{12}\rangle\) with \(|1_1\rangle\otimes |2_2\rangle\). Accordingly, the state reduces to
\[ |\Psi_{out}\rangle = \frac{\sqrt{6}}{4} |3_1\rangle \otimes |0_2\rangle - \frac{\sqrt{3}}{4} |2_1\rangle \otimes |1_2\rangle - \frac{\sqrt{3}}{4} |1_1\rangle \otimes |2_2\rangle + \frac{\sqrt{6}}{4} |0_1\rangle \otimes |3_2\rangle. \]

### 4.2 The invariant set

In this subsection, we define a set of \(N\)-photon states and show that this set is invariant under the steady-state linear action of a quantum linear passive system \(G\). The discussions in the subsection generalizes those in subsection 3.4.

Update the function \(\psi_{in}(t_1, \ldots, t_{k_1}, \ldots, t^m_{k_m})\) in Eq. (119) to a tensor \(\psi_{in}^\dagger(t_1, \ldots, t_{k_1}, \ldots, t^m_{k_m})\), whose elements are defined as
\[
\psi_{in}^\dagger_{i_1, \ldots, i_{k_1}, \ldots, i^m_{k_m}}(t_1^{i_1}, \ldots, t_{k_1}^{i_{k_1}}, \ldots, t^m_{k_m}) \triangleq \begin{cases} \psi_{in}(t_1^{i_1}, \ldots, t_{k_1}^{i_{k_1}}, \ldots, t^m_{k_m}), & i_1 = 1, \ldots, i_{k_1} = k_1, \ldots, i^m_{k_m} = m; \\ 0, & \text{elsewhere.} \end{cases}
\]
Then the output pulse shape defined in Eq. (121) can be expressed in the following tensor form
\[ \psi_{out} = \psi_{in}^\dagger \otimes_i^N g_G. \]

Motivated by this, define a class of \(N\)-photon state
\[
\mathcal{F}_2 \triangleq \left\{ |\Psi\rangle = \sum_{i_1=1}^{m} \cdots \sum_{i_{k_1}=1}^{m} \cdots \sum_{i^m_{k_m}=1}^{m} \int d^3 \mathbf{t} \left( \psi_{i_1}^\dagger(t_1^{i_1}) \cdots \psi_{i_{k_1}}^\dagger(t_{k_1}^{i_{k_1}}) \cdots \psi_{i^m_{k_m}}^\dagger(t^m_{k_m}) |0\rangle \otimes |0^m\rangle \right) \times b_{i_1}^*(t_1^{i_1}) \cdots b_{i_{k_1}}^*(t_{k_1}^{i_{k_1}}) \cdots b_{i^m_{k_m}}^*(t^m_{k_m}) \right\}.
\]
The following result shows that the set \(\mathcal{F}_2\) is invariant under the steady-state action of a quantum linear passive system \(G\).
Theorem 6  The steady-state output state of the quantum linear passive system $G$ driven by an $N$-photon input $|\psi_{\text{in}}\rangle \in \mathcal{F}_2$ with pulse information encoded by an $m$-way $m$-dimensional tensor matrix $\psi_{\text{in}}$ is another element $|\psi_{\text{out}}\rangle \in \mathcal{F}_2$, whose pulse information is encoded by an $m$-way $m$-dimensional tensor matrix $\psi_{\text{out}}$ with elements,

$$
\psi_{\text{out}}^{i_1\cdots i_l} |r_1, r_2, \ldots, r_l\rangle = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_l=1}^{m} \sum_{i_m=1}^{m} \int dt_1 \cdots dt_l g_{1G}^{-1} (r_1 - t_1) \cdots g_{lG}^{-1} (r_l - t_l) \times \\
\times c_{i_1\cdots i_l}^{1\cdots l} g_{mG}^{-1} (r_m - t_m) \cdots g_{lG}^{-1} (r_m - t_m) \psi_{\text{in}}^{i_1\cdots i_l} |r_1, r_2, \ldots, r_l\rangle.
$$

In compact form, Eq. (144) can be written as

$$
|\psi_{\text{out}}\rangle = |\psi_{\text{in}}\rangle \otimes_{l=1}^{m} g_{-l}.
$$

This result can be established in a similar way as Theorem 3. So the proof is omitted.

5 An arbitrary number of photons superposed in $m$ input channels

In all the previous discussions, we have implicitly assumed that the total number of photons is no less than the number of input channels. In this section, we remove this constraint.

5.1 $m$-channel $N$-photon input states

In this subsection, we first present a class of $m$-channel $N$-photon input states where $N$ is an arbitrary integer. Some illustrative examples are also given.

Let an $N$-photon input state be

$$
|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dt_j \sum_{k=1}^{m} \xi_{jk}(t_j) b_{\text{in},k}^*(t_j) |0^\otimes m\rangle,
$$

(146)

where $N$ is an arbitrary positive integer and $\mathcal{N}_N$ is the corresponding normalization coefficient. The input state $|\psi_{\text{in}}\rangle$ is parametrized by the pulse shapes $\xi_{jk}(t) (k = 1, \ldots, m$ and $j = 1, \ldots, N)$. Clearly, different combinations of $\xi_{jk}(t)$ give rise to different multi-photon multi-channel states. By the notation in Eq. (75), the $N$-photon input state in Eq. (146) is

$$
|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \sum_{k=1}^{m} B_{\text{in},k}^*(\xi_{jk}) |0^\otimes m\rangle.
$$

(147)

**Remark 4** A class of photon-Gaussian states has been defined in [57, Eq. (95)]. If $\rho_{R}$ used there is of the form $\rho_{R} = |\phi 0^\otimes m\rangle$, then the resulting states are $m$-channel $m$-photon states. They are in fact the special case of the $m$-channel $N$-photon above-defined (the case when $N = m$).

We first study two examples of this type of multi-photon states.

**Example 4** When $N = 1$ and $m = 2$, by Eq. (147), the input state is

$$
|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{N}} B_{\text{in},1}^*(\xi_{11}) |0_1\rangle \otimes |0_2\rangle + |0_1\rangle \otimes \frac{1}{\sqrt{N}} B_{\text{in},2}^*(\xi_{12}) |0_2\rangle.
$$

(148)

That is, a single photon is superposed over two input channels. If $\xi_{11} = \xi_{12} \equiv \xi$ and $||\xi|| = 1$, then the normalization condition requires that $\mathcal{N}_2 = 1$. In this case, Eq. (148) becomes

$$
|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}} B_{\text{in},1}^*(\xi) |0_1\rangle \otimes |0_2\rangle + \frac{1}{\sqrt{2}} |0_1\rangle \otimes B_{\text{in},2}^*(\xi) |0_2\rangle.
$$

(149)
For the state in Eq. (149) it can be readily shown that
\[
\lim_{{t_0 \to -\infty, t \to \infty}}\langle \Psi_{\text{in}}|\Lambda(t)|\Psi_{\text{in}}\rangle = \int_{-\infty}^{\infty} d\tau \left[\langle \Psi_{\text{in}}|b_1^{*}(\tau)b_1(\tau)|\Psi_{\text{in}}\rangle \langle \Psi_{\text{in}}|b_2^{*}(\tau)b_2(\tau)|\Psi_{\text{in}}\rangle\right] = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\] (150)

That is, the photon is not localized in one of the two channels. Instead, it is indeed shared by two channels. This reveals the particle property of photons.

On the other hand, if \(\xi_{11} \equiv 0\), then
\[
|\Psi_{\text{in}}\rangle = |0_1\rangle \otimes \frac{1}{\sqrt{N_1}}B_2^*(\xi_{12})|0_2\rangle.
\] (151)

In this case, the first channel is in the vacuum state and the second channel is in a single-photon state.

**Example 5** When \(N = 2\) and \(m = 3\), by Eq. (147), the input state is
\[
|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{N_2}} \sum_{k=1}^{3} B_k^*(\xi_{1k}) \sum_{k=1}^{3} B_k^*(\xi_{2k})|0^\otimes 3\rangle.
\] (152)

That is, three channels share two photons. If in particular \(\xi_{11} \equiv \xi_{22} \equiv 0\), then the input state becomes
\[
|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{N_2}}B_1^*(\xi_{21})|0_1\rangle \otimes B_2^*(\xi_{12})|0_2\rangle \otimes |0_3\rangle + \frac{1}{\sqrt{N_2}}B_1^*(\xi_{21})|0_1\rangle \otimes |0_2\rangle \otimes B_3^*(\xi_{13})|0_3\rangle
+ \frac{1}{\sqrt{N_2}}|0_1\rangle \otimes B_2^*(\xi_{12})|0_2\rangle \otimes B_3^*(\xi_{23})|0_3\rangle + \frac{1}{\sqrt{N_2}}|0_1\rangle \otimes |0_2\rangle \otimes B_3^*(\xi_{13})B_2^*(\xi_{23})|0_3\rangle.
\] (153)

if further \(\xi_{21} \equiv \xi_{13} \equiv 0\), then Eq. (153) reduces to
\[
|\Psi_{\text{in}}\rangle = \frac{1}{\sqrt{N_2}} |0_1\rangle \otimes B_2^*(\xi_{12})|0_2\rangle \otimes B_3^*(\xi_{23})|0_3\rangle.
\] (154)

which is a separable state. Finally, if \(\xi_{23}(t) \equiv 0\), then Eq. (154) reduces to
\[
|\Psi_{\text{in}}\rangle = |0_1\rangle \otimes \frac{1}{\sqrt{N_2}}B_2^*(\xi_{12})|0_2\rangle \otimes |0_3\rangle.
\] (155)

That is, the second channel has two photons while the first and third channels are in the vacuum states.

As demonstrated by the above two examples, Eq. (147) provides flexibility for specifying various types of multi-photon states. The following is the main result of this subsection, which shows the linear transfer of the pulse shapes from the input channels to the output channels.

### 5.2 Multi-photon output states

In this subsection, we derive the analytic form of the steady-state output state of a quantum linear passive system driven by an \(m\)-channel \(N\)-photon state defined in Eq. (146).

The following is the main result of this section.

**Theorem 7** Let \(G\) be an asymptotically stable quantum linear passive system which is initialized in the vacuum state and is driven by the \(N\)-photon input \(|\Psi_{\text{in}}\rangle\) defined in Eq. (147). The steady state \((t_0 \to -\infty)\) of the output field is another \(N\)-photon state of the form
\[
|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{N_N}} \prod_{j=1}^{N} \int_{-\infty}^{\infty} \sum_{i=1}^{m} \eta_{ij}(r_j)b_{\text{out},i}(r_j) dr_j |0^\otimes m\rangle,
\] (156)
where the output pulses are given by

\[ \eta_{lj}(t) \triangleq \sum_{k=1}^{m} \int_{-\infty}^{\infty} g_{G_{-}}^{lk}(t-r)\xi_{jk}(r)dr, \quad l = 1, \ldots, m, \quad j = 1, \ldots, N. \]  

(157)

Proof. The proof is similar to that for Theorem 2. In the limit \( t_0 \to -\infty \), by Eqs. (17) and (19) we get

\[ b_{in,k}^{*}(t) = \sum_{l=1}^{m} \int_{-\infty}^{\infty} g_{G_{-}}^{lk}(r-t)b_{out,l}(r)dr, \quad k = 1, \ldots, m. \]  

(158)

Substituting Eq. (158) into Eq. (147) we have

\[ |\Psi_{in}\rangle = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \left( \sum_{l=1}^{m} \int_{-\infty}^{\infty} g_{G_{-}}^{lk}(r_j-t_j)b_{out,l}(r_j)dr_j dt_j |0^{\otimes m}\rangle \right) \]

\[ = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \sum_{l=1}^{m} \left( \sum_{k=1}^{m} \int_{-\infty}^{\infty} g_{G_{-}}^{lk}(r_j-t_j)\xi_{jk}(t_j)dt_j \right) b_{out,l}(r_j)dr_j |0^{\otimes m}\rangle \]

\[ = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \int_{-\infty}^{\infty} \sum_{l=1}^{m} \eta_{lj}(r_j)b_{out,l}(r_j)dr_j |0^{\otimes m}\rangle \]

\[ = |\Psi_{out}\rangle, \]  

(159)

where the output pulse functions \( \eta_{lj}(t) \) are those given in Eq. (157). ■

6 Conclusion

In this paper, we have studied the dynamics of quantum linear systems in response to multi-channel multi-photon states. We have derived the intensity of the output field which can be used to investigate the influence of quantum linear systems on quantum correlations of light fields. We have also presented the explicit formula of the steady-state output field state for several classes of multi-channel multi-photon input states. The results presented here are very general and hold promising applications in photon-state-based quantum coherent feedback networks.

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References

[1] C. Altafini and F. Ticozzi. Modeling and control of quantum systems: an introduction. IEEE Trans. Automat. Contr., 57:1898–1917, 2012.
[2] B. D. O. Anderson and J. B. Moore. Optimal Filtering. Prentice-Hall, Englewood Cliffs, NJ, 1979.
[3] Hans-Albert Bachor and Timothy C Ralph. A guide to experiments in quantum optics. Wiley, 2004.
[4] B. Q. Baragiola, R. L. Cook, A. M. Branczyk, and J. Combes. N-photon wave packets interacting with an arbitrary quantum system. Phys. Rev. A., 86:013811, 2012.
[5] T. J. Bartley, G. Donati, J. B. Spring, X. M. Jin, M. Barbieri, A. Datta, B. J. Smith, and I. A. Walmsley. Multiphoton state engineering by heralded interference between single photons and coherent states. Phys. Rev. A., 86:043820, 2012.
[6] V. P. Belavkin. Quantum filtering of markov signals with white quantum noise. Radiotechnika i Electronika, 25:1445–1453, 1980.
[7] V. P. Belavkin. On the theory of control of observable quantum systems. Automat. Rem. Control, 44:178–188, 1983.
[8] V. P. Belavkin. Quantum stochastic calculus and quantum nonlinear filtering. J. Multivariate Anal., 42:171–201, 1992.
[9] V. P. Belavkin. Quantum diffusion, measurement and filtering. Theory Probab. Appl., 38:573–585, 1993.
[10] R. N. Bracewell. *The Fourier Transform and its Applications, 3rd edition.* McGraw Hill, 1999.

[11] B. Brecht, Dileep V. Reddy, C. Silberhorn, and M. G. Raymer. Photon temporal modes: A complete framework for quantum information science. *Phys. Rev. X*, 5:041017, Oct 2015.

[12] R. G. Brown and P. Y. C. Hwang. *Introduction to Random Signals and Applied Kalman Filtering, 3rd ed.* John Wiley & Sons, 1997.

[13] A. R. R. Carvalho, M. R. Hush, and M. R. James. Cavity driven by a single photon: conditional dynamics and nonlinear phase shift. *Phys. Rev. A.*, 86:023806, 2012.

[14] J. Cheung, A. Migdall, and M. L. Rastello. Special issue on single photon sources, detectors, applications, and measurement methods. *J. Modern Optics*, 56:139–140, 2009.

[15] D. Dong and I. R. Petersen. Quantum control theory and applications: a survey. *IET Control Theory & Applications*, 4:2651–2671, 2010.

[16] S. Fan, S. E. Kocabas, and J.-T. Shen. Input-output formalism for few-photon transport in one-dimensional nanophotonic waveguides coupled to a qubit. *Phys. Rev. A*, 82:066821, Dec 2010.

[17] C. W. Gardiner and P. Zoller. *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics.* Springer, 2004.

[18] K. M. Gheri, K. Ellinger, T. Pellizzari, and P. Zoller. Photon-wavepackets as flying quantum bits. *Fortschr. Phys.*, 46:401–415, 1998.

[19] J. E. Gough and M. R. James. The series product and its application to quantum feedforward and feedback networks. *IEEE Trans. Automat. Contr.*, 54:2530–2544, 2009.

[20] J. E. Gough, M. R. James, and H. I. Nurdin. Squeezing components in linear quantum feedback networks. *Phys. Rev. A*, 81:023804, 2010.

[21] J. E. Gough, M. R. James, and H. I. Nurdin. Quantum filtering for systems driven by fields in single photon states and superposition of coherent states using non-markovian embeddings. *Quantum Information Processing*, 12:1469–1499, 2013.

[22] J. E. Gough, M. R. James, H. I. Nurdin, and J. Combes. Quantum filtering for systems driven by fields in single-photon states or superposition of coherent states. *Phys. Rev. A*, 86:043819, Oct 2012.

[23] R. L. Hudson and K. R. Parthasarathy. Quantum itô’s formula and stochastic evolutions. *Communications in Mathematical Physics*, 93(3):301–323, 1984.

[24] M. R. James and J. E. Gough. Quantum dissipative systems and feedback control design by interconnection. *IEEE Trans. Automat. Control*, 55:1806–1821, 2010.

[25] T. G. Kolda and B. W. Bader. Tensor decompositions and applications. *SIAM Review*, 51:455–500, 2009.

[26] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems.* John Wiley and Sons, Inc, 1972.

[27] U. Leonhardt. Quantum physics of simple optical instruments. *Rep. Prog. Phys.*, 66:1207–1249, 2003.

[28] L.-Q. Liao and C. K. Law. Correlated two-photon scattering in cavity optomechanics. *Phys. Rev. A*, 87:043809, Apr 2013.

[29] R. Loudon. *The Quantum Theory of Light, 3rd ed.* Oxford University Press, Oxford, 2000.

[30] G. J. Milburn. Coherent control of single photon states. *Eur. Phys. J. Special Topics*, 159:113–117, 2008.

[31] W. J. Munro, K. Nemoto, and G. J. Milburn. Intracavity weak nonlinear phase shifts with single photon driving. *Optics Communications*, 283:741–746, 2010.

[32] A. Nysteen, P. T. Kristensen, D. P. S. McCutcheon, P. Kaer, and J. Mazrk. Scattering of two photons on a quantum emitter in a one-dimensional waveguide: exact dynamics and induced correlations. *New Journal of Physics*, 17(2):023030, 2015.

[33] Z. Y. Ou. Multi-photon interference and temporal distinguishability of photons. *Int. J. Modern Physics B*, 21:5033–5058, 2007.

[34] Y. Pan, G. Zhang, and M. R. James. Analysis and control of quantum finite-level systems driven by single-photon input states. *Automatica*, 69:18–23, 2016.

[35] Peter P Rohde, Wolfgang Mauerer, and Christine Silberhorn. Spectral structure and decompositions of optical states, and their applications. *New Journal of Physics*, 6(9):91, 2007.

[36] H. Sandberg, J. Delvenne, and J. C. Doyle. On lossless approximations, the fluctuation-dissipation theorem, and limitations of measurements. *IEEE Trans. Automat. Contr.*, 56:293–308, 2011.

[37] T. Sogo. On the equivalence between stable inversion for nonminimum phase systems and reciprocal transfer functions defined by the two-sided laplace transform. *Automatica*, 46:122–126, 2010.
[44] H. Song, G. Zhang, and Z. R. Xi. Continuous-mode multi-photon filtering. *SIAM Journal on Control and Optimization*, 54:1602–1632, 2016.

[45] R. van Handel, J. K. Stockton, and M. Mabuchi. Feedback control of quantum state reduction. *IEEE Trans. Automat. Contr.*, 50:768–780, 2005.

[46] D. F. Walls and G. J. Milburn. *Quantum Optics, 2nd ed.* Springer, 2008.

[47] H. W. Wiseman and G. J. Milburn. *Quantum Measurement and Control*. Cambridge University Press, Cambridge, UK, 2010.

[48] N. Yamamoto. Decoherence-free linear quantum subsystems. *IEEE Transactions on Automatic Control*, 59(7):1845–1857, 2014.

[49] N. Yamamoto and M. R. James. Zero-dynamics principle for perfect quantum memory in linear networks. *New Journal of Physics*, 16(7):073032, 2014.

[50] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part i: dynamics of quantum feedback systems. *IEEE Trans. Automat. Contr.*, 48:2107–2120, 2003.

[51] M. Yanagisawa and H. Kimura. Transfer function approach to quantum control-part ii: Control concepts and applications. *IEEE Transactions on Automatic Control*, 48(12):2121–2132, Dec 2003.

[52] M. Yukawa, K. Miyata, T. Mizuta, H. Yonezawa, P. Marek, R. Filip, and A. Furusawa. Generating superposition of up-to three photons for continuous variable quantum information processing. *Opt. Express*, 21(5):5529–5535, Mar 2013.

[53] G. Zhang. Analysis of quantum linear systems’ response to multi-photon states. *Automatica*, 50:442–451, 2014.

[54] G. Zhang, S. Grivopoulos, I. R. Petersen, and J. E. Gough. On the structure of quantum linear systems. *arXiv:1606.05719*, 2016.

[55] G. Zhang and M. R. James. Direct and indirect couplings in coherent feedback control of linear quantum systems. *IEEE Trans. Automat. Contr.*, 56:1535–1550, 2011.

[56] G. Zhang and M. R. James. Quantum feedback networks and control: a brief survey. *Chinese Science Bulletin*, 57:2200–2214, 2012.

[57] G. Zhang and M. R. James. On the response of quantum linear systems to single photon input fields. *IEEE Trans. Automat. Contr.*, 58:1221–1235, 2013.

[58] J. Zhang, Y-X Liu, R-B Wu, K. Jacobs, and F. Nori. Quantum feedback: theory, experiments, and applications. *arXiv:1407.8536v3 [quant-ph]*, 2015.

[59] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1996.