The Approximate Solution of Fractional Damped Burger's Equation and its Statistical Properties

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Abstract. The aim of this paper is to extend the variational iteration method (VIM) to find the approximate solution of fractional damped Burger's equation and finding its statistical concepts.

Keywords Fractional calculus, Fractional damped Burger's equation, Caputo derivative, variational iteration method (VIM), statistical concepts.

1. Introduction
Nonlinear fractional partial differential equations are widely used to describe many important phenomena and dynamic processes such as engineering, acoustic, viscoelasticity, electromagnetics [4, 13]. In general, there is no method that yields an exact solution for nonlinear fractional differential equations.

Burger's equation is a fundamental partial differential equation in fluid mechanics. It is also a very important model encountered in several areas of applied mathematics such as heat conduction, acoustic waves, gas dynamics and traffic flow[9]. The one dimensional nonlinear Burger's equation was first introduced in 1915 by Bateman H., [1], who found its steady solutions descriptive of certain viscous flows. It was later proposed by Burger J. M.(1948) [3] as one of a class of equation describing mathematical models of turbulence. Later the Burger's equation was studied by Cole J. D., (1951) [5] who gave a theoretical solution, based on Fourier series analysis, using the appropriate initial and boundary conditions. Gorguis A., (2005) [6] gives comparison between Cole - Hopf transformation and Decomposition method for solving Burger's equation. Momani S., (2006) [10] has presented nonperturbative analytical solutions of the space-and time-fractional Burger's equations by Adomian decomposition method. Inc (2008) [7] used variational iteration method for solving space-time fractional Burger's equations. Wang Qi. (2008) [14] extend the application of the homotopy perturbation and Adomian decomposition methods to construct approximate solutions for the nonlinear fractional KdV-Burger's equation. Biazar J. and Aminikhah H., (2009) [2] solve Burger's equation by using variational iteration method (VIM) by which Approximate solution can be found and which is better than ADM. In (2011) Pandey
K. and Verma L., [11] gave a note on Crank Nicolson scheme for Burger’s equation without Hopf-Cole transformation solutions are obtained by ignoring nonlinear term.

2. Preliminaries
In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper.

2.1. Definition [8]
A real valued function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p$, $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space $C_\mu^n$ if $f^{(n)}(x) \in C_\mu, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.2. Definition [8]
The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(x) \in C_\mu, \mu \geq -1$ is defined as:
\[
J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \ x > 0
\]
\[
J^0 f(x) = f(x)
\]
Properties of the operator $J^\alpha$ can be found for $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $y > -1$ as follows:
1. $J^\alpha J^\beta f(x) = J^{\alpha + \beta} f(x) = J^\beta J^\alpha f(x)$;
2. $J^\alpha \mathcal{C} = \frac{\mathcal{C}}{\Gamma(1+\alpha)}x^\alpha$, $\mathcal{C}$ is constant;
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

2.3. Definition [7,8]
The fractional derivative of $f(x)$ in the Caputo sense is defined as:
\[
D^\alpha f(x) = J^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt, \ a > 0
\]
For $n - 1 < \alpha \leq n, n \in \mathbb{N}, x > 0$ and $\Gamma(.)$ is the gamma function.

2.4. Definition [7,8]
For $n$ be the smallest integer that exceeds $\alpha$, the Caputo time-fractional derivative of a function $u(x, t)$ of order $\alpha > 0$ is defined as:
\[
D^\alpha_t u(x, t) = \frac{\partial^{\alpha u(x,t)}}{\partial t^\alpha} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^{\alpha n u(x,\tau)}}{\partial \tau^n} d\tau, n - 1 < \alpha < n
\end{array} \right.
\]
and the space-fractional derivative operator of order $\beta > 0$ is defined as:
\[
D^\beta_x u(x, t) = \frac{\partial^{\beta u(x,t)}}{\partial x^\beta} = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n-\beta)} \int_0^x (x-\theta)^{n-\beta-1} \frac{\partial^{\beta n u(x,\theta)}}{\partial \theta^n} d\theta, n - 1 < \beta < n
\end{array} \right.
\]
Satisfies the following properties:
1. $D^0_C = 0, \mathcal{C}$ constant;
2. $D^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma - \alpha + 1)} x^{\gamma - \alpha}, \ x > 0, \gamma > -1$;
3. \( D^a \left( \sum_{i=0}^{m} c_i f_i(x, t) \right) = \sum_{i=0}^{m} c_i D^a f_i(x, t) \), where \( c_0, c_1, \ldots, c_m \) are constant.

2.5. Lemma [7]
If \( n-1 < \alpha \leq n, f \in C^n, n \in \mathbb{N} \) and \( \geq -1 \), then;
\[
D^a f(x) = f(x), \quad \text{for } x > 0
\]
\[
j^a D^a f(x) = f(x) - \sum_{k=0}^{n-1} f^k (0^+) \frac{\alpha^k}{k!}, \quad \text{where } x > 0
\]

3. Finding Approximate Solution and Using Statistical Tests for Reliability of the Solution
The fractional damped Burger’s equation is:
\[
u_t^{(\alpha)} + uu_x - u_{xx} + \lambda u = 0 \quad , 0 < \alpha \leq 1, \ x \in \mathbb{R}, \ t \geq 0, \lambda > 0
\]
(3)

We can reduce equation (3), by take the transformation;
\[
x(\xi, t) = kx + \frac{ct^a}{\Gamma(\alpha + 1)} + \xi_0
\]
(4)

Where \( k, c, \xi_0 \) are constants, by using this transformation to get:
\[
u_t^{(\alpha)}(\xi) = u'(\xi)D^\alpha \xi = u'(\xi)\frac{c}{\Gamma(\alpha + 1)} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \alpha + 1)} = cu'(\xi)
\]

And
\[
u u_x = u(\xi)(u(\xi))_x = ku(\xi)u'(\xi)
\]

Also
\[
u u_{xx} = (u(\xi))_{xx} = ku'\xi)(\xi) = k(ku''(\xi)) = k^2u''(\xi)
\]

So by this transformation for \( \xi \), equation (3) can be turned into the following nonlinear ordinary differential equation of second order with respect to the variable \( \xi \);
\[
u cu'(\xi) + ku(\xi)u'(\xi) - k^2u''(\xi) + u(\xi) = 0, \ u'(\xi) = \frac{du}{d\xi}
\]
(5)

Integrating (5) from 0 to \( \xi \) to get:
\[
u cu(\xi) - cu(0) + k \frac{1}{2} u^2(\xi) - k \frac{1}{2} u^2(0) - k^2u'(0) + k^2u'(0) + \lambda \int_0^\xi u(t)dt = 0
\]

Let \( L = k^2u'(0) - cu(0) - k \frac{1}{2} u^2(0) \)

So to obtain;
\[
u cu(\xi) + k \frac{1}{2} u^2(\xi) - k^2u'(\xi) + \lambda \int_0^\xi u(t)dt + L = 0
\]
(6)

Now, integrating (6) from 0 to \( \xi \) to get:
\[
\int_0^\xi cu(t)dt + k \frac{1}{2} \int_0^\xi u^2(t)dt - k^2u(\xi) + k^2u(0) + \lambda \int_0^\xi (\xi - t)u(t)dt + L\xi = 0
\]
So

\[ u(\xi) = u(0) + \frac{L}{k^2} + \frac{1}{k^2} \int_0^\xi [cu(t) + \frac{k}{2} u^2(t) + \lambda(\xi - t)u(t)] dt \] (7)

Equation (7) can be solved by variational iteration method (VIM) and because of the uniform convergence, few terms are enough for good accuracy [12], to obtain;

\[ u_0(\xi) = u(0) + \frac{L}{k^2} \]

\[ u_1(\xi) = \left( \frac{cu(0)}{k^2} + \frac{u^2(0)}{2k} \right) \xi + \left( \frac{ct \alpha}{(\xi(\alpha + 1))} \right) \xi^2 + \left( \frac{L^2}{2k^2} + \frac{\lambda L}{2k^2} \right) \xi^3 \]

\[ u_2(\xi) = \left( \frac{cu(0)}{2k^4} + \frac{u^2(0)}{4k^3} \right) \xi^2 + \left( \frac{ct \alpha}{(\xi(\alpha + 1))} \right) \xi + \left( \frac{L^2}{8k^2} + \frac{\lambda L}{8k^2} \right) \xi^3 \]

\[ u_3(\xi) = \left( \frac{cu(0)}{6k^6} + \frac{u^2(0)}{12k^5} \right) \xi^3 + \left( \frac{ct \alpha}{(\xi(\alpha + 1))} \right) \xi^2 + \left( \frac{L^2}{40k^5} + \frac{\lambda L}{40k^5} \right) \xi^4 \]

... 

And so on.

Substituting these quantities into (4) we have:

\[ u_0(x, t) = u(0) + \frac{L}{k^2}(kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ u_1(x, t) = \left( \frac{cu(0)}{k^2} + \frac{u^2(0)}{2k} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ + \left( \frac{L^2}{2k^2} + \frac{\lambda L}{2k^2} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0)^3 \]

\[ u_2(x, t) = \left( \frac{cu(0)}{2k^4} + \frac{u^2(0)}{4k^3} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ + \left( \frac{L^2}{8k^2} + \frac{\lambda L}{8k^2} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ u_3(x, t) = \left( \frac{cu(0)}{6k^6} + \frac{u^2(0)}{12k^5} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ + \left( \frac{L^2}{40k^5} + \frac{\lambda L}{40k^5} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

... 

And so on.

The solution of fractional damped Burger’s equation is:

\[ u(x, t) = \sum_{n=0}^\infty u_n(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots \]

\[ = u(0) + \left( \frac{L}{k^2} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0) \]

\[ + \left( \frac{ct \alpha}{(\xi(\alpha + 1))} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0)^2 \]

\[ + \left( \frac{L^2}{2k^2} + \frac{\lambda L}{2k^2} \right) (kx + \frac{ct \alpha}{(\xi(\alpha + 1))} + \xi_0)^3 \] (8)
\[ + \left( \frac{L^2}{8k^2} + \frac{\lambda u(0)}{8k^6} \right) \left( kx + \frac{c t^{\alpha}}{r'(a+1)} + \xi_0 \right)^4 + \left( \frac{cu(0)}{6k^3} + \frac{u^2(0)}{12k^5} \right) \left( kx + \frac{c t^{\alpha}}{r'(a+1)} + \xi_0 \right)^3 + \frac{c t}{i_2k^8} + \frac{\lambda u(0)}{24k^8} \left( kx + \frac{c t^{\alpha}}{r'(a+1)} + \xi_0 \right)^4 + \left( \frac{L^2}{40k^9} + \frac{\lambda L}{40k^9} \right) \left( kx + \frac{c t^{\alpha}}{r'(a+1)} + \xi_0 \right)^5 + \ldots \]

As a special case let \( \lambda = u(0) = 1 \) and \( \xi_0 = 0 \). Then

\[ u(x, t) = 1 + \frac{c}{k^2} \left( kx + \frac{c t^{\alpha}}{r'(a+1)} \right)^2 + \frac{1}{2k^4} \left( kx + \frac{c t^{\alpha}}{r'(a+1)} \right)^3 + \frac{1}{8k^6} \left( kx + \frac{c t^{\alpha}}{r'(a+1)} \right)^4 + \frac{c}{2k^8} \left( kx + \frac{c t^{\alpha}}{r'(a+1)} \right)^5 + \ldots \]

(9)

Let the traveling wave solution (9) be a probability density function if:

\[ \int_0^1 u(x, t) \, dx \, dt = 1 \]

\[ 1 + \left( \frac{1+c}{k^2} + \frac{1}{2k^4} \right) \left( k + \frac{c t^{\alpha}}{r'(a+1)} \right) + \left( \frac{1}{2k^2} + \frac{1}{4k^3} \right) \left( k^2 + \frac{c k^2}{r'(a+1)^2} \right) + \left( \frac{1}{3k^4} + \frac{c}{12k^6} \right) \left( k^4 + \frac{c k^2}{r'(a+1)^3} \right) + \left( \frac{1}{4k^8} \right) \left( k^4 + \frac{c k^2}{r'(a+1)^5} \right) + \left( \frac{5k^2 c^2}{2(a+1)(r'(a+1)^2)} + \frac{10k^2 c^3}{3(a+1)(r'(a+1)^3)} \right) + \ldots = 1 \]

(10)

Now, let \( k = -1 \), \( \alpha = \frac{9}{10} \), then by using numerical analysis (Newton-Raphson method) we find the value of \( c \) from equation (10) which has one solution

\[ f(c) = -0.4292 - 0.4486c + 0.0967c^2 + 0.5142c^3 + 0.0934c^4 + 0.0073c^5 \]

\[ c = 1.0873 \]

Substituting these values into (9) then the probability density function is:

\[ u(x, t) = 1 - x + 1.1302t^{\frac{9}{10}} - 0.5873x + 0.6638t^{\frac{9}{10}} + 1.5873x^2 - 3.5881xt^{\frac{9}{10}} + 2.0278t^{\frac{9}{10}} + 0.2937x^2 - 0.6639xt^{\frac{9}{10}} + 0.2299t^{\frac{9}{10}} - 0.5291x^3 + 0.7639t^{\frac{27}{10}} \]
\[ +1.7940x^2 t^{\frac{9}{10}} - 2.0278xt^{\frac{9}{10}} - 0.0979x^3 + 0.1413t^{\frac{27}{10}} + 0.3319x^2 t^{\frac{9}{10}} \]
\[ -0.3752xt^{\frac{9}{10}} + 0.1323x^4 + 0.2159t^{\frac{18}{5}} - 0.5981x^3 t^{\frac{9}{10}} + 1.0141x^2 t^{\frac{5}{2}} \]
\[ -0.7641xt^{\frac{27}{10}} \]

**Remark**

We will choose the probability density functions \( u_1^*(x) \) and \( u_1^{**}(t) \) which satisfy the conditions \( E(x) > 0, E(x^2) > 0 \) and \( E(x) > E(x^2) \). Also \( E(t) > 0, E(t^2) > 0 \) and \( E(t) > E(t^2) \). \( var(x) > 0, var(t) > 0 \). The correlation coefficient lies in \([-1, 1]\).

\[ u_1(x, t) = 1 - x + 1.1302t^{\frac{9}{10}} - 0.5873x + 0.6638t^{\frac{9}{10}} + 1.5873x^2 - 3.5881xt^{\frac{9}{10}} \]
\[ +0.2937x^2 - 0.6639xt^{\frac{9}{10}} + 0.2299t^{\frac{9}{5}} - 0.5291x^3 + 0.7639t^{\frac{27}{10}} \]
\[ +1.7940x^2 t^{\frac{9}{10}} - 2.0278xt^{\frac{9}{10}} - 0.0979x^3 + 0.1413t^{\frac{27}{10}} + 0.3319x^2 t^{\frac{9}{10}} \]
\[ -0.3752xt^{\frac{9}{10}} + 0.1323x^4 + 0.2159t^{\frac{18}{5}} - 0.5981x^3 t^{\frac{9}{10}} + 1.0141x^2 t^{\frac{5}{2}} \]
\[ -0.7641xt^{\frac{27}{10}} \]

1) **The moments**

To evaluate the expected values \( E(x) \), \( E(t) \), \( E(x^2) \) and the second moments \( E(x^2), E(t^2) \) we need \( u_1^*(x) \) and \( u_1^{**}(t) \). We find them as follows:

\[ u_1^*(x) = \int_0^1 u_1(x, t) dt \]
\[ = 1 - x + 0.5948 - 0.5873x + 0.3494 + 1.5873x^2 - 1.8885x + 0.2937x^2 \]
\[ -0.3494x + 0.0821 - 0.5291x^3 + 0.2065 + 0.9442x^2 - 0.7242x \]
\[ -0.0979x^3 + 0.0382 + 0.1747x^2 - 0.1340x + 0.1323x^4 + 0.0469 \]
\[ -0.3148x^3 + 0.3622x^2 - 0.2065x \]

\[ u_1^{**}(t) = \int_0^1 u_1(x, t) dx \]
\[ = 1 - 0.5000 + 1.1302t^{\frac{9}{10}} - 0.2937 + 0.6638t^{\frac{9}{10}} + 0.5291 - 1.7941t^{\frac{9}{10}} \]
\[ +0.0979 - 0.3319t^{\frac{9}{10}} + 0.2299t^{\frac{5}{2}} - 0.1323 + 0.7639t^{\frac{27}{10}} - 0.5980t^{\frac{9}{10}} \]
\[ -1.0139t^{\frac{9}{5}} - 0.0245 + 0.1413t^{\frac{27}{10}} + 0.1106t^{\frac{9}{10}} - 0.1876t^{\frac{9}{5}} + 0.0265 \]
\[ +0.2159t^{\frac{18}{5}} - 0.1495t^{\frac{9}{10}} + 0.3380t^{\frac{9}{5}} - 0.3821t^{\frac{27}{10}} \]
(1) Expected Value of $x$

\[ E(x) = \int_0^1 x u_1^*(x) dx \]
\[ = 0.2035 \]

(2) Expected Value of $t$

\[ E(t) = \int_0^1 t u_1^*(t) dt \]
\[ = 0.4127 \]

It means that the first expected length of the wave is concentrated at value 0.2035, which is the middle of the wave which agrees with nature.

Also the first expected time of wave is concentrated at 0.4127, which means that the wave takes a long time which agrees with nature.

(3) The Second Moment of $x$

\[ E(x^2) = \int_0^1 x^2 u_1^*(x) dx \]
\[ = 0.0843 \]

(4) The Second Moment of $t$

\[ E(t^2) = \int_0^1 t^2 u_1^*(t) dt \]
\[ = 0.2850 \]

E($x$) > E($x^2$) shows that the first wave is stronger than the second.

The second expected length of the wave is concentrated at 0.0843, which means that the length begins to disperse (scatter) which agrees with nature.

The second expected time of the wave is concentrated at 0.2850, which means that the wave stay for a short time which agrees with nature.

(5) The Expected Value of $xt$

\[ E(xt) = \int_0^1 \int_0^1 xt u_1^*(x,t) dx dt \]
\[ = 0.0791 \]

This joint expected value for length and time of the wave is 0.0791, which means short wave and which agrees with nature.

(II) The Variance

(1) Variance of $x$

\[ \sigma_x^2 = E(x^2) - [E(x)]^2 \]
\[ = 0.0429 \]

The variation for length of wave is 0.0429, so that the separation is very small. This means that the power of the wave is focused in the middle of the wave and separation begins from the first wave which agrees with nature.

The variation for time of wave is 0.1147, so that the separation is small. This means that the time of a separated wave begins from the first wave and so on which agrees with nature.

(III) The Covariance

\[ Cov(x, t) = E(xt) - E(x)E(t) \]
\[ = -0.0048 \]

The range of deviation of the length and time of the wave from its expected values is very small which agrees with nature.

(IV) The Correlation Coefficient

\[ \rho = \frac{cov(x, t)}{\sqrt{var(x)\sqrt{var(t)}}} \]
\[ = -0.0685 \]
This means that the relation between the amplitude of the wave and time is strong (high amplitude corresponding to the beginning of the wave in terms of length and time) and vice versa which agrees with nature.

Figure 1 represents the traveling wave solution when \(-4 < x < 4\), \(1 < t < 3\) and \(-10 < u < 10\).

Figure 2 represents the traveling wave solution when \(-0.5 < x < 4\), \(0.5 < t < 4\) and \(-7.5 < u < 10\).
Figure 3 represents the traveling wave solution when \(0 < x < 2\), \(0 < t < 2\) and \(-7.5 < u < 10\).

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