Color Superconducting State of Quarks

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Abstract
An introductory review of physics of color superconducting state of matter is presented. Comparison with superconductivity in electron systems reveals difficulties involved in formulating color superconductivity theory at moderately ultra-nuclear density.

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1 Introduction

During the last 2-3 years color superconductivity became a compelling area of QCD. The burst of interest to this topic was triggered by papers [1, 2], though the subject has about two decades history [3, 4]. At present the number of publications on color superconductivity is a two digit one. Our list of references includes only a minor part of them. The range of questions related to the field became so wide that it can not be elucidated and even touched in the present brief review. The reader willing to get deeper and broader knowledge of the subject may address review papers [5, 6].

The phenomenon of color superconductivity develops in the high density regime of QCD when the interaction of quarks starts to feel the presence of the Fermi surface. Attractive quark interaction makes the Fermi surface unstable to the formation of the condensate of quark pairs (diquarks) with nontrivial color structure. The term ”color superconductivity” reflects the similarity to the behavior of electrons in ordinary superconductors. The analogy is however not so complete as it will be discussed in what follows.

In Section 2 we sketch the QCD phase diagram in order to locate the color superconductivity region in the density-temperature coordinates.

In Section 3 the symmetries of color superconducting phase are discussed. Section 4 is devoted to the dynamics of the condensate and to the question to what extent the methods used in Bardeen, Cooper and Schrieffer (BCS) theory are applicable. Finally in Section 5 we present two unrelated topics chosen to illustrate how rich the physics of color superconductivity is.

2 The phases of QCD

It is well known that QCD in ”normal” conditions (zero temperature and density) still contains compelling open questions such as the origin and dynamics of confinement. Despite important lacunas in QCD at normal conditions substantial efforts have been undertaken investigating QCD at nonzero temperature. These studies were primarily motivated by heavy ion collision experiments. Much less attention has been devoted to QCD at nonzero density, i.e. to the question ”what happens to the matter as you squeeze it harder and harder?” [6].

The reason why until recently high temperature region has obtained much more attention than the high density one is twofold. As already mentioned
the development of QCD at $T > 0$ was motivated by heavy ion collision experiments while the nonzero density regime is more difficult to reach in the laboratory since the temperature in collisions is much hotter than the critical one for superconductivity. In nature this state is realized in neutron star interior which is beyond direct experimental probes. On the theoretical side our main knowledge of $T > 0$ QCD comes from lattice Monte Carlo simulations. This powerful theoretical tool does not work at nonzero density since in this case the determinant of the Dirac operator is complex resulting in nonpositive measure of the corresponding path integral and the failure of Monte Carlo simulation procedure. Recent breakthrough in understanding of the high density QCD phase has been achieved using models (like Nambu Jona-Lasinio or instanton gas) and perturbation theory which is applicable in the limit of ultra high density.

Before we start to draw the phase diagram the following remark is in order. As we know from statistical mechanics the actual variable representing finite density in all equations in the chemical potential $\mu$. For free massless quarks at $T = 0$ the connection between the density $n$ and the chemical potential $\mu$ reads

\[
n = N_c N_f \frac{\mu^3}{3\pi^2},
\]

where $N_c$ and $N_f$ are the number of quark colors and flavors respectively.

We begin the discussion of the phase diagram by reminding what occurs along the vertical axis in the $(T, \mu)$ plane, i.e. at $\mu = 0$ [5, 7]. The central event happening along this line in the transition from hadronic to the quark-gluon phase. The hadronic phase is characterized by the chiral symmetry breaking and confinement. At the critical temperature which is about 170 MeV [8] a phase transition occurs, gluons and quarks become deconfined and the chiral symmetry is restored. Such a description of the temperature phase transition is a bird’s-eye view. One should keep in mind at least two remarks. First, the fact that chiral symmetry restoration and deconfinement occur at the same critical temperature is confirmed by Monte Carlo simulations but not yet rigorously proved. Second, the value of the critical temperature and the order of the phase transition depend upon the number of flavors and the quark constituent masses.

Now let us turn to our main subject, namely what happens when one moves to the right along the horizontal $\mu$ axis of the $(T, \mu)$ plane. The vacuum state of QCD $(T = 0, \mu = 0)$ is characterized by the existence of the chiral quark condensate $\langle \bar{\psi}_L \psi_R \rangle$. With $\mu$ increasing we enter the
nuclear (hadron) matter phase. Normal nuclear density \( n_0 \approx 10^{-3} \) GeV\(^3\) corresponds to \( \mu_0 \approx 0.3 \) GeV. The behavior of the chiral condensate at such densities depends on whether the formation of two quark droplets simulating hadrons is taken into account explicitly. If not, the chiral condensate remains practically at its vacuum value \( \langle \bar{u}u \rangle \). If yes, the phase is broken into droplets (hadrons) within which the chiral symmetry is restored surrounded by empty space with condensate at its vacuum value \( \langle \bar{p} \rangle \). Another condensate which is present both in the vacuum and hadron phases is the gluon one. Its value decreases with increasing \( \mu \) \[10\].

Increasing \( \mu \) further one reaches the point \( \mu_1 \) where diquark condensate is formed and color superconducting phase arises. The difference \( (\mu_1 - \mu_0) \) is of the order of the QCD scale \( \Lambda_{QCD} \approx 0.2 \) GeV \[4\]. The transition at \( \mu_1 \) is of the first order. The main signature of this transition is the breaking of the color gauge group SU(3)\(_c\). At this point we remind that the statement that local gauge invariance is spontaneously broken is a convenient fiction reflecting the fact that derivations are performed within a certain fixed gauge \[11\]. The pattern of breaking differs for 2 and 3 flavors and this will be discussed in Section 3. The situation with chiral symmetry as also quite different for \( N_f = 2 \) and \( N_f = 3 \). In the first case it is restored in superconducting phase while in the second it remains broken due to a nontrivial mixing of color and flavor variables (color-flavor locking). The discussion of this point is also postponed till Section 3. To stress the difference between superconducting phase for \( N_f = 2 \) and \( N_f = 3 \) the first one got the name 2SC (two flavors superconducting) while the second one is called CFL (color-flavor locked).

Despite essential differences of superconducting phases for \( N_f = 2 \) and \( N_f = 3 \) the transition from one sector to another is performed by the variation of a single parameter. This is the mass of a strange quark: at \( m_s = \infty \) we have a world with two flavors, while at \( m_s = 0 \) – the one with three. The physical value of the strange quark mass \( m_s \approx 0.15 \) GeV is in between these two extremes. If one can dial \( m_s \) increasing it at fixed \( \mu \), one finds a first order transition from CFL phase into a 2SC one. This happens because \( \langle \bar{u}u \rangle \) and \( \langle ds \rangle \) condensates gradually become smaller than \( \langle \bar{u}d \rangle \). On the contrary at asymptotically high densities \( (\mu \to \infty) \) the system for any \( m_s \neq \infty \) is certainly in the CLF phase. At \( \mu = 0 \) the chiral symmetry restoration at \( T_c \) occurs via second order transition for \( m_s > m_s^c \) (\( N_f = 2 \) regime) and via the first order for \( m_s < m_s^c (N_f = 3 \) regime). The value of \( m_s^c \) is estimated from lattice calculations as half of the physical mass of the strange quark.

The final observation concerns the existence of the tricritical point \( E \).
Consider the two-flavor case with zero mass $u$ and $d$ quarks. Then the phase transition at $\mu = 0$, $T = T_c$ is, as we already know, of the second order, while the transition to the 2CS phase at $\mu = \mu_1$, $T = 0$ is of the first order. This means that the phase diagram features a tricritical point $E$ to which a first order line approaches from the large $\mu$ side and the second order line emanates towards lower values of $\mu$.

3 Symmetries of color superconducting state

The appearance of the diquark condensate at nonzero density leads to drastic changes in symmetries characterizing the system.

The pairing of the two quarks embodies a new feature somewhat unfamiliar for particle physics but inherent for BCS superconductivity. This is the existence of anomalous averages of two creation or annihilation operators which must be zero in any state with a fixed number of particles. The microscopic Hamiltonian proportional to $\bar{\psi}\psi$ which according to the Wick’s theorem is factorized as $\langle \bar{\psi}\psi \rangle \bar{\psi}\psi$ conserves the number of particles and doesn’t lead to superconductivity. Superconducting state corresponds to an alternative factorization of the form $\langle \bar{\psi}\psi \rangle \bar{\psi}\psi$ where $\langle \bar{\psi}\psi \rangle$ is the anomalous average which is zero in the normal state. Such factorization corresponds to off-diagonal long range order, a concept introduced by Yang. There is a connection between the ”nonconservation” of the number of particles and the breaking of the local gauge invariance mentioned before. Both are artifacts of a certain way of description – the mean field Hamiltonian in the first case fixed gauge in the other. The physically significant quantity is not the product of field operators but modules of the gap. Its phase is significant only if one deals with actually open systems like in Josephson effect. At this point one may ask a question ”How far the analogy between delocalized Cooper pairs and diquarks extends?” In terms of condensed matter physics the question is whether diquarks resemble Cooper pairs or delute gas of more compact Schafroth pairs. In current literature the first option is taken almost for granted. In our view the situation is not so obvious and we shall return to this question in Section 4.

After these general remarks we turn to concrete symmetries of the quark system with nonzero condensate. Pairs of quarks cannot be color singlets, they may be either in color triplet or sextet state. The basic single gluon exchange, which is the QCD ”Coulomb force”, is attractive in 3 color channel
with color "wave function" proportional to $\lambda^\alpha \lambda^\beta \varepsilon_{\alpha\beta\gamma}$, where $\lambda$ are the Gell-Mann matrices. Instanton interaction also leads to attraction in $\mathfrak{F}$ channel. Interaction in the sextet channel is believed to be weaker or even repulsive. Thus the condensate in $\mathfrak{3}$ channel picks the color direction which means that gauge symmetry is broken. The situation reminds spontaneous magnetization below Curie point. The breaking pattern turns out to be quite different for $N_f = 2$ and $N_f = 3$. Therefore the two cases are considered separately in the next two subsections.

3.1 Two flavors

At $\mu = \mu_1$ the first order phase transition from hadronic to 2 SC phase occurs. At $\mu < \mu_1$ the system features the condensate $\langle \bar{\psi}_R \psi_L \rangle$ which breaks chiral symmetry. At $\mu > \mu_1$ the color superconducting condensate energetes which has the following structure

$$\Delta \propto \langle \psi^{\alpha i}_L \psi^{\beta j}_L \varepsilon_{ij} \varepsilon_{\alpha\beta3} \rangle \propto -\langle \psi^{\alpha i}_R \psi^{\beta j}_R \varepsilon_{ij} \varepsilon_{\alpha\beta3} \rangle. \quad (3.1)$$

Here $\varepsilon$ are antisymmetric tensors, $\alpha$ and $\beta$ – color indices, $i$ and $j$ -flavor ones, indices $L, R$ are Lorentz indices. Pairing in this condensate is among quarks with the same helicities, the pairs are $(ud - du)$ flavor singlets, and therefore the condensate does not break chiral symmetry. Transition is of the first order and therefore the two condensates may compete within a certain interval of $\mu$. However as it was recently shown [15] this is not the case and as soon as diquark condensate is formed the chiral one is extinguished in the sense that its influence on the thermodynamic properties of the system (the critical temperature and the gap) becomes negligible. Resorting to the analogy from solid state physics one may compare chiral condensate with nonmagnetic impurities in ordinary superconductor. According to Anderson theorem [16] such impurities do not alter the properties of superconductor in the first order in their concentration.

The color wave function of the condensate (3.1) is proportional to $\varepsilon_{\alpha\beta3}$. This means that the first two colors (say red and green) are paired while the third one (blue) "remains in cold". The condensate (3.1) is invariant under the $SU(2)$ subgroup of color rotations which do not affect the third (blue) quark. Thus the color gauge symmetry is broken down to $SU(3)_c \rightarrow SU(2)_c$. This symmetry pattern implies that five of the eight gluons acquire mass via Anderson-Higgs mechanism. One of the three massless gluons is mixed with
a photon. This phenomenon called "rotated" electromagnetism [17] will be discussed in Section 5.

To summarize, in the 2SC phase color symmetry is reduced, chiral symmetry is restored, photon is mixed with one of the eight original gluons.

3.2 Three Flavors

Consider now QCD with three flavors of massless quarks. The condensate is approximately of the form

$$\Delta \propto \langle \psi_L^{\alpha i} \psi_L^{\beta j} \varepsilon_{ijA} \varepsilon_{\alpha\beta A} \rangle = -\langle \psi_R^{\alpha i} \psi_R^{\beta j} \varepsilon_{ijA} \varepsilon_{\alpha\beta A} \rangle. \quad (3.2)$$

Summation over the index $A$ links color and flavor. This is the famous color-flavor locking suggested in [18]. The condensate (3.2) is invariant under neither color nor left-handed flavor or right-handed flavor separately. It remains invariant only under global $SU(3)$ rotation, so that the symmetry breaking pattern is $SU(3)_c \times SU(2)_L \times SU(2)_R \rightarrow SU(3)_{c+L+R}$. Color-flavor locking has a direct analogy in condensed matter physics. This is the so-called $^3He - B$ phase of superfluid helium 3. In this phase the two atoms forming the pair are in spin-triplet $S = 1$ state with orbital momentum $l = 1$. The corresponding condensate has the form

$$\Delta \propto (-p_x + ip_y)\chi(S_z = +1) + p_z\chi(S_z = 0) + (p_x + ip_y)\chi(S_z = -1), \quad (3.3)$$

where $\chi$ is the spin wave function, and $\Delta$ is invariant under combined orbital and spin rotation corresponding to the total momentum $j = 0$.

The fact that chiral symmetry in the CFL phase is broken makes it difficult to distinguish it from hypernuclear matter, i.e. hadron phase made of quarks with 3 flavors. This observation is called quark-hadron continuity [19]. There is also pairing in hypernuclear matter, this time in dibaryon channels $\Lambda\Lambda$, $\Sigma\Sigma$ and $N\Xi$ resulting in superfluidity phenomenon.

Complete breaking of color gauge group in CFL phase implies that all eight gluons become massive. Again as in the 2SC case photon combines with one of the gluons.

In nature the two quarks are light and the third, the strange one, is of middle weight. As already mentioned at $\mu \to \infty$ the CFL phase is realized. With $\mu$ decreasing, or with $m_s$ increasing at fixed $\mu$, there is a critical point at which strange quark decouples and two-flavor chiral symmetry is restored. This unlocking phase transition (from CFL to 2SC) is of the first order [20].
4 Dynamics of color superconducting phase.

From BCS theory of superconductivity we know that this phenomena arises under the following three conditions

(i) Attraction between particles

(ii) Existence of the Fermi surface

(iii) Interaction of the particles must be concentrated within a thin layer of momentum space around the Fermi surface, i.e. the Debye frequency has to be much smaller than the chemical potential, $\omega_D \ll \mu$. This requirement may be called Thin Shell Condition (TSC).

All the three points listed above are in principle met by quark system at finite density. However one should keep in mind certain reservations and substantial distinctions from electrons in metal. We address these issues in the next section.

4.1 Comparing Superconductivity in Electron and Quark Systems

QCD tells us that quarks in $\bar{3}$ color channel attract each other while QED gives Coulomb repulsion between electrons. Effective electron attraction is, as we know, a result of a complicated mechanism involving lattice phonons. The relevant feature of this interaction is that it satisfies TSC, i.e. involves only electrons with energies $\omega$ close to Fermi energy $\mu$, $|\omega - \mu| < \omega_D \ll \mu$, where $\omega_D$ is the Debye energy. Under this condition Cooper pairs completely overlap, $n\xi^3 \sim (10^8 - 10^{10}) \gg 1$, where $n$ is the density of electrons, $\xi$ is the Pippard coherence length which measures the spatial extension of the pair wavefunction.

According to Section 2, in QCD with two massless flavors transition to superconducting 2SC phase occurs at $\mu \simeq 0.4$ GeV \cite{21} which recalling Eq. (2.1) corresponds to $n \simeq 1.3 \cdot 10^{-2}$ GeV$^3$. Pippard coherence length may be estimated as $\xi \simeq 1/\pi\Delta$ \cite{22}, where $\Delta$ - is the value of the gap, $\Delta \simeq 0.1$ GeV \cite{22}. This yields $\xi \simeq 0.6$ fm which agrees with the estimate $\xi \simeq 0.8$ fm given in \cite{23}. Thus in the ”newly born” 2SC phase $n\xi^3 \lesssim 1$. The corresponding Debye frequency (parameter $\Lambda$ of Ref. \cite{21}) is only $\omega_D \simeq 2\mu$ instead of $\omega_D \gg \mu$ in BCS. In solid state physics the limit $n\xi^3 \ll 1, \omega_D \simeq \mu$ is known.
as Schafroth regime [14] of Bose condensation in the dilute gas limit. With \( \xi \simeq 0.6 \text{ fm} \) the BCS regime settles from \( \mu \simeq (200-300) \text{ GeV} \).

In a more formal way the role of TSC will be displayed in the next subsection. We shall see that TSC provides weak coupling solution of the gap equation. This is due to the fact that under TSC particles interact in two dimensions instead of three and the sum \( \sum_k k^{-2} \) logarithmically diverges at small momenta. Weak coupling solution is the cornerstone of superconductivity theory.

Now we turn to another distinction of color superconductivity from the BCS picture. The distinguishing feature of ordinary superconductor is that it is a perfect diamagnet, i.e. magnetic field \( B = 0 \) inside it. This is the famous Meissner effect. On the other hand from QCD we know that color-electric and color-magnetic field are "frozen" into the vacuum in the form of the gluon condensate. With the quark density increasing the gluon condensate is expected to "burn out" with qualitative description of this process still lacking. Some insight may be gained from the dilaton model [24] which suggests that at \( \mu \simeq 0.4 \text{ GeV} \) when the 2SC phase arises the gluon condensate decreases by (10-20)% from its vacuum value [25].

From Section 3 we know that in color superconducting phase at least part of the gluon degrees of freedom becomes massive. It means effective screening of low-frequency modes and therefore it is reasonable to assume that the formation of diquark condensate should lead to the decrease of the gluon condensate. This may be considered as nonabelian Meissner effect [10]. Alternative way of reasoning is the following. Suppose one performs calculation of color superconducting state neglecting gluon condensate and obtains the value of the chemical potential \( \mu_1 \) from which the condensation starts. With gluon condensate included the system will remain in color superconducting state only if this is thermodynamically favorable, i.e. provided the energy gain due to transition into superconducting state exceeds the gain due to the formation of the gluon condensate. As it was shown on general grounds in [10] and later confirmed by model calculations in [26] the superconducting state which has appeared at the values of \( \mu \) just above \( \mu_1 \) is destroyed by gluon condensate. This means that the transition into superconducting state is shifted to higher values of the chemical potential than those predicted by calculations presented up to now. Solution of the complete problem with the inclusion of finite \( T, \mu \) and nonperturbative gluon fields remains a challenge for future work.

To summarize, we can say that comparison of color and electron super-
conductivities leads to rather surprising conclusions. Namely, the onset of the superconducting phase at low values of $\mu (\mu \simeq 0.4 \text{ GeV for 2SC})$ becomes questionable. The reason is twofold. First, unlike the situation in BCS theory the parameter $n_\xi^3$ is not large. Diquarks resemble Schafroth pairs rather than Cooper ones. The existence of the weak coupling solution of the gap equation is not obvious since interacting quarks are not necessarily confined to the two-dimensional layer around the Fermi surface. Another reservation concerns the role played by nonperturbative gluon fields. Due to gluon condensate color superconductor is not a color diamagnet. At moderately low chemical potential transition to the superconducting phase may be blocked by gluon condensate.

4.2 Gap Equation

In previous sections our presentation was very qualitative. From now on we shall be more formal but still avoid complicated equations to be found in the original papers. Again our starting point will be the BCS theory. Its formulation in the form close to particle physics language may be found e.g. in Refs. [27]-[29]. One starts with partition function written as

$$Z = \exp \{V_4 \Omega(\Delta, \mu, T)\}, \quad (4.1)$$

where $V_4$ is the 4-volume of the system, $\Delta \sim \langle \psi \psi \rangle$ is the condensate, or gap, $\Omega$ is what is called effective potential in the language of field theory or thermodynamic potential in statistical physics. In BCS theory at $T = 0$ the frequency-momentum representation for $\Omega$ reads [29]

$$\Omega_e = \frac{\Delta^2}{4g^2} + \frac{i}{(2\pi)^4} \int d\omega d^3p \ln\{1 - \frac{\Delta^2}{\omega^2 - \xi^2 + i\varepsilon}\}, \quad (4.2)$$

where $g^2$ is the interaction-constant, $\xi = p^2/2m - \mu$. The stationary point of $\Omega_e$, i.e. the condition $\partial \Omega_e / \partial \Delta = 0$ gives the gap equation

$$1 = \frac{g^2\mu^2}{2\pi^2} \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}}, \quad (4.3)$$

In passing from (4.2) to (4.3) one integrates the second term in (4.2) over $\omega$ and then resorts to TSC by presenting the momentum integral as

$$\int \frac{d^3p}{(2\pi)^3} = \int d\xi N_e(\xi) \simeq N_e(0) \int d\xi, \quad (4.4)$$
where $N_e(0)$ is the density of states at the Fermi surface:

$$N_e(0) = \frac{1}{2\pi^2} \int dp \frac{dp^2}{d\xi} = \frac{mp_F^2}{2\pi^2}. \quad (4.5)$$

The famous weak-coupling solution of Eq. (4.3) reads

$$\Delta = 2\omega_D \exp \left\{-\frac{2\pi^2}{g^2 mp_F}\right\}. \quad (4.6)$$

Now we turn to color superconductivity and consider the 2SC case. The corresponding effective potential has been derived in [21] (see also [15] for details) and reads

$$\Omega_q = \frac{\Delta^2}{4g^2} - \frac{1}{2} tr \ln \left\{1 + \frac{\Delta^2}{\tilde{p}^+ \tilde{p}^-}\right\}, \quad (4.7)$$

where $tr$ implies summation over color, flavor and Lorentz indices and for $T \neq 0$ integration over $dp_4$ is replaced by summation over fermionic Matsubara modes $p_4 = (2n + 1)\pi T$. The operator $\varphi$ in (4.7) has the form

$$\varphi = \varepsilon_{\alpha\beta3} \varepsilon_{ij} C\gamma_5 = \varepsilon_{\alpha\beta3} \varepsilon_{ij} \gamma_2 \gamma_4 \gamma_5, \quad (4.8)$$

and $\tilde{p} = p_\mu \gamma_\mu$, $p_\pm = (p_k, p_4 \pm i\mu)$.

Expressions (4.2) and (4.7) differ in two points. First, quarks have additional degrees of freedom - color and flavor. Traces over color and flavor indices result in additional factors

$$tr_{c\varepsilon\varepsilon} = tr(\delta_{\alpha\beta} - \delta_{3\alpha}\delta_{3\beta}) = 2, \quad tr_{f\varepsilon\varepsilon} = tr\delta_{ij} = 2. \quad (4.9)$$

Second, electrons in (4.2) are nonrelativistic particles with mass $m$ while (4.7) is written for massless relativistic quarks. The density of states for massless quarks with color and flavor (see (4.7)) is

$$N_q(0) = \frac{2\mu^2}{\pi^2}, \quad (4.10)$$

instead of (4.3) for electrons. The propagator of the relativistic quark has two poles corresponding to positive and negative frequencies. These two poles show up in the trace over Lorentz indices of the term $(\tilde{p}^+ \tilde{p}^-)^{-1}$ in (4.7). One has

$$tr_L \frac{1}{\tilde{p}^+ \tilde{p}^-} = \frac{1}{p_4^2 + (p - \mu)^2} + \frac{1}{p_4^2 + (p + \mu)^2}. \quad (4.11)$$
The second term (antiparticles contribution) is usually neglected \[5\] since its denominator is everywhere greater than \(\mu\).

The gap equation is again obtained from stationary point of \(\Omega_q\) by making use of TSC and (4.9), (4.10) and (ref4.12). It has the form

\[
1 = \frac{8g^2\mu^2}{\pi^2} \int_0^{\omega_D} \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}}. \tag{4.12}
\]

the weak coupling solution reads

\[
\Delta = 2\omega_D \exp\left\{-\frac{\pi^2}{8g^2\mu^2}\right\}. \tag{4.13}
\]

In the models like NJL the Debye frequency \(\omega_D\) and the coupling constant \(g^2\) are fitted simultaneously but no unique ”standard” solution exists so far \[30\]. As an educated guess one may take \(g^2 = 2\) GeV\(^{-2}\), \(\omega_D = 0.8\) GeV and \(\mu = 0.5\) GeV. This yields

\[
\Delta \simeq 0.15\text{GeV} \tag{4.14}
\]

in perfect agreement with numerical calculations performed in various models, see \[5\] for references to the original works.

We wish to remind what are the main approximations and assumptions on which the naive estimate (4.14) is based. In writing the expression (4.7) for \(\Omega_q\) the contribution from the chiral condensate \(\langle \bar{\psi}\psi\rangle\) was omitted. This step is justified by model-independent Anderson theorem \[15\]. The neglect of antiparticle contribution is also a reliable approximation. The applicability of TSC in getting weak-coupling solution (4.13) is questionable at low \(\mu\) values as discussed at length in the previous section. The role of nonperturbative gluons in (4.7) was completely ignored and this might be a serious flaw of all approaches to color superconductivity developed up to now. As it was shown in\[10\] the contribution of the gluon condensate to \(\Omega_q\) is at low values of \(\mu\) very close to the contribution from diquark condensate and the crucial point is which force will win. With these considerations in mind the estimates (4.13) - (4.14) do not seem exceedingly oversimplified since complicated calculations within different models are still plagued by the tacit neglect of the factors listed above.

### 4.3 Ginzburg-Landau Free Energy

Expansion of the thermodynamic potential \(\Omega_q\) containing the second and fourth order terms is called Ginzburg-Landau Functional. The first order
term vanishes because of stationarity condition while the sixth order term is important near the tricritical point. Ginzburg-Landau functional describes the behavior of the system in the vicinity of the critical temperature.

For superconducting quark matter this functional was first written down in Ref. [4]. Since then several authors addressed the subject, see e.g. [31, 32, 15]. In the simplest case of homogeneous condensate when the gradient terms are absent one has

$$\Omega_q = \Delta^2 N_q(0) \left( \frac{T - T_c}{T_c} \right) + \Delta^4 N_q(0) \frac{7\zeta(3)}{16\pi^2}.
$$

(4.15)

The critical temperature $T_c$ is about half the gap, i.e. $T_c \sim 50 MeV$ [5]. Comparing this value with the critical chemical potential $\mu_1 \sim 400 MeV$ one concludes that quark matter is a high temperature superconductor [5]. On the other hand $T_c$ is too low to be easily accessible in heavy ion collisions.

The simplicity of the expression (4.15) for Ginzburg-Landau functional should not cover some underlying problems. From BCS theory we know that in the Ginzburg-Landau phenomenological approach it is assumed that magnetic field varies slowly over the coherence length which is $\xi \leq 1 fm$ for color superconductor (see Section 4.1). On the other hand we know that nonperturbative QCD fields have the characteristic correlation length $T_g \sim 0.2 fm$ [33]. This is the distance at which the QCD string is formed and the width of the string between quarks. Again, as in Sections 4.1-4.2 we encounter an open question concerning the role of nonperturbative gluons.

5 Miscellaneous results

The number of problems discussed in relation to color superconductivity phenomena is already very wide. Two of them will be touched upon in this concluding Section of our short introductory review. These are ultra-high density limit [34] and modified electromagnetism [17].

5.1 The Ultra-High Density Limit

In our previous discussion we paid attention to somewhat shaky grounds on which the low $\mu$ color superconductivity theory is based. The region of low and moderate chemical potential is of great interest since there are hopes that this is the regime in which color superconductivity might be realized.
in heavy ion collisions and in the interior of the neutron starts. From the theoretical standpoint the ultra-high energy limit is certainly more refined.

With the chemical potential increasing the Fermi momentum becomes large and asymptotic freedom implies that the interaction between quarks is weak. In this case the gap can be calculated in perturbation theory. The resulting solution drastically differs from the classical BCS result (4.6) and the analogous expression (4.13) assumed in low density regime of color superconductivity. We remind that (4.13) is written in terms of the coupling constant $g^2$ with the dimension $m^{-2}$. This constant may be considered as induced by one-gluon-exchange with gluon line becoming very hard resulting in four-fermion interaction. Although this gluon-line roughening mechanism is far from being clear [35, 36] one is tempted to extrapolate (4.13) for $\mu \to \infty$ in terms of dimensionless QCD coupling constant $g(QCD)$ as $\Delta \sim \{-c/g^2(QCD)\}$. The correct answer however is ($N_c = 3, N_f = 2$)

$$\Delta \simeq \frac{\mu}{g^5(QCD)} b \exp \left( -\frac{3\pi^2}{\sqrt{2g}} \right), \quad (5.1)$$

where $b = 512\pi^42^{-1/3}(2/3)^{5/2}$. The different $g$ dependence in the exponents (4.13) and (5.1) is due to relativistic and retardation effects, While (4.13) is derived for contact interaction, (5.1) fully accounts for the structure of the gluon propagator. The result (5.1) is based on the Debye screening of electric gluons and Landau damping of magnetic ones. What has been neglected in (5.1) is the effect of vertex corrections and hence (5.1) is reliable only for $\mu \gg 10^8$ MeV. With the account of the $g(QCD)$ behavior dictated by asymptotic freedom the result (5.1) implies logarithmic growth of $\Delta$ as $\mu \to \infty$. This in turn means that the strange quark mass becomes irrelevant and the CFL phase becomes more favorable than the 2SC one. The derivation of (5.1) from the first principles of the QCD is probably the most significant achievement of the color superconductivity theory.

5.2 Rotated Electromagnetism

In Section 3 we have already mentioned how the photon couples to color superconducting matter [17], namely that the original photon combines with one of the gluons. Consider the CFL phase. Evidently the condensate (3.2) breaks electromagnetic gauge invariance since pairing occurs between differently charged quarks. It is straightforward to find a combination of charge
and color generators for which all quark pairs are neutral. This ”new charge” is
\[ Q = Q + \eta \lambda_8 \] 
(5.2)
with \( \eta = 1/\sqrt{3} \). From (5.2) it is easy to see that all quarks have integer \( \tilde{Q} \) charge. The massless photon is
\[ \tilde{A}_\mu = \frac{g a_\mu + \eta e G^8_\mu}{\sqrt{\eta^2 e^2 + g^2}} = \cos \alpha_0 A_\mu + \sin \alpha_0 G^*_\mu. \]
(5.3)

The mixing angle \( \alpha_0 \approx \eta e/g \) is small. The magnetic field corresponding to \( \tilde{A} \) experience the ordinary Meissner effect.

6 Acknowledgements

This introductory review resulted from a report made by the author at ITEP Wednesday seminar. The author is grateful to the participants for questions and remarks. Special thanks are to N.O.Agasian, V.I.Shevchenko and Yu.A.Simonov for numerous illuminating discussions and N.P.Igumnova for the help in preparing the manuscript. Support from grants RFFI-00-02-17836, RFFI-00-15-96786 and INAS CALL 2000 project # 110 is gratefully acknowledged.

References

[1] M.Alford, K.Rajagopal and F.Wilczek, Phys. Lett. B422, 247 (1998)
[2] R.Rapp, T.Schafer, E.V.Shuryak and M.Velkovsky, Phys. Rev. Lett. 81, 53 (1998)
[3] B.Barrois, Nucl. Phys. B129 ,390 (1977); S.Frautschi, Proceedings of workshop on hadronic matter at extreme density, Erice 1978.
[4] D.Bailin and A.Love, Phys. Rept. 107 , 325 (1984)
[5] K.Rajagopal and F.Wilczek, hep-ph/0011333
[6] M.Alford, hep-ph/0102047
[7] M.A.Halasz et al, Phys. Rev. D58, 096007 (1998)
[8] F.Karsch,
[9] G.W.Carter and D.Diakonov, hep-ph/9812445 v.2
[10] N.Agasian, B.Kerbikov and V.Shevchenko, Phys. Rep. 320, 131 (1999)
[11] S.Elitzur, Phys. Rev. D12, 3978 (1975)
[12] C.N.Yang, Rev. Mod. Phys. 34, 694 (1962)
[13] L.N.Cooper, Phys. Rev. 104, 1189 (1956)
[14] M.R.Schafroth, Phys. Rev. 96, 1442 (1954)
[15] B.Kerbikov, hep-ph/0106324
[16] P.Anderson, Journ. Phys. Chem. Sol. 11, 26 (1959); A.A.Abrinkosov and L.P.Gorkov, ZhETF 35, 1158 (1958); 36, 319 (1959)
[17] M.Alford, J.Berges and K.Rajagopal, Nucl. Phys. B571, 269 (2000)
[18] M.Alford, K.Rajagopal and F.Wilczek, Nucl. Phys. B537, 443 (1999)
[19] T.Schafer and F.Wilczek, Phys. Rev. Lett. 82, 3956 (1999)
[20] M.Alford, J.Berges and K.Rajagopal, Nucl. Phys. B558, 219 (1999)
[21] J.Berges and K.Rajagopal, Nucl. Phys. B538, 215 (1999)
[22] P.G.de Gennes, Superconductivity in Metals and Alloys, Perseus Books Publishing, Massachusetts, 1989
[23] T.Schafer, Nucl. Phys. A642, 45c (1998)
[24] A.A.Migdal and M.A.Shifman, Phys. Lett. B114, 445 (1982); J.Ellis and J.Lanik, Phys. Lett. B150, 289 (1985)
[25] N.O.Agasian, JETP Lett. 57, 208 (1993); N.O.Agasian, D.Ebert, E.-M.Ilgenfritz, Nucl. Phys. A637, 135 (1998)
[26] D.Ebert et al., hep-ph/0106110 v.2
[27] A.Svidzinsky, teor. Matem. Fiz. 9, 273 (1971)
[28] B.Sakita, Quantum Theory of Many-Variable Systems and Fields, World Scientific, Singapore, 1985

[29] S.Weinberg, Nucl. Phys. B413, 567 (1994)

[30] S.Klevansky, Rev. Mod. Phys. 64, 649 (1992); U.Vogl and W.Weise, Progr. Part. Nucl. Phys. 27 195 (1992); G.Ripka, Quarks Bound by Chiral Fields, Claredon Press, Oxford, 1997.

[31] D.Blaschke and A.Sedrakian, nucl-th/0006038

[32] Kei Iida and G.Baym, Phys. Rev. D63, 074018 (2001)

[33] H.G.Dosch and Yu.A.Simonov, Phys. Lett. B205, 339 (1988); L.Del Debbio, A.Di Giacomo, Yu.A.Simonov, Phys. Lett. B332, 11 (1994)

[34] D.T.Son, Phys. Rev. D59, 094019 (1999)

[35] K.Yamawaki and V.I.Zakharov, hep-ph/9406373

[36] Yu.A.Simonov, Phys. Lett. B412, 371 (1997)

[37] T.Schafer and F.Wilczek, Phys. Rev. D60, 114033 (1999)

[38] R.D.Pisarski and D.H.Rischke, Phys. Rev. D61, 051501 (2000)