THREE-MANIFOLDS CLASS FIELD THEORY
(HOMOLOGY OF COVERINGS FOR A NON-VIRTUALLY HAKEN MANIFOLD)

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I am moving so slowly,  
A yard per five minutes  
How can one reach one’s destination  
If the shoes are of small size? [O]

This is a first in a series of papers in which we explore a deep conceptual relation between three-manifolds and number fields, a subject which may be given a name of “arithmetic topology”. An immediate and transparent reason for such a relation is the fact that the cohomological dimension of $Gal(\bar{K}/K)$ equals three for any number field $K$.

This paper resulted from the author’s attempt to settle the Thurston’s covering conjecture - that is, that any irreducible three-manifold $M$ with infinite $\pi_1$ has a finite covering with positive Betti number. In the process of study it became clear, however, that conjecturally non-existing non-virtually Haken manifolds show “a strong wish to survive”, that is, the information which one can derive about them organizes itself into a harmonious picture of a non-contradictory nature. So instead of trying to show that such manifolds do not exist, we adopt a more positive approach to study the homology of their finite coverings. This is parallel to studying ideal class groups of finite extensions of a given number field. In fact, the techniques we develop will be very useful for this number-theoretic problem.

The fundamental questions in three-manifold class field theory that we ask are:

- **Class towers.** Suppose $M$ is a non-virtually Haken three-manifold. Fix a prime $p$. How fast the $p$-component of the first homology group $H_1(N,\mathbb{Z})$ grows for finite coverings $N$ of $M$?

- **Ideal class modules.** What is the structure of $H_1(N)_{(p)}$ as a Galois module over $G = \pi_1(M)/\pi_1(N)$ for $N$ normal?

- **Maximal unramified $p$-extensions.** What is a structure of pro-$p$ completion of $\pi_1(M)$ as a pro-$p$ group?

We also ask about the multiplication in cohomology for a virtually non-Haken three-manifold. (We give an answer in case of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ a 2-torsion in $H_1$).

A number of algebraic techniques, some well-known, some less known and some new, is needed to answer our fundamental questions. These include:

- Standard Lyndon-Serre-Hochschild spectral sequence in group cohomology

- Cohomological theory for $\mathbb{Z}_p$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$ -actions on rational homology spheres

- Linking form $H_1 \times H_1 \to \mathbb{Q}/\mathbb{Z}$ and the classification of coverings to isotropic and non-isotropic
Theorem 9.1. If $M$ satisfies the condition $(R)$ then either $M$ is virtually Haken or there exists a Zariski dense representation of $\pi_1(M)$ in $SL_2(\mathbb{C})$.

Theorem 9.2. If $M$ satisfies $(R)$ then either $M$ is virtually Haken or for any prime $p$ there exists a finite normal covering of $M$ with rank $(H_1(N,\mathbb{F}_p)) \geq 4$.

Yet the Theorem 9.2 follows from the Theorem 9.1 and simple argument in strong approximation theory for algebraic groups as described by Lubotzky in [Lu], we describe also an alternative way using the spectral sequences in group cohomology. By the reason which will become clear later, the Theorem 9.2 represents the non-abelian step in blowing up $H_1$. The next is the abelian step.

Theorem 10.1 (Class towers). Let $p$ be a prime such that rank $(H_1(M,\mathbb{F}_p)) \geq 4$. Suppose $M$ is not virtually Haken. Let $M_{i+1}, i \geq 1$ be the maximal abelian $p$-covering of $M_{i-1}$ (the Hilbert class covering). Let $r_i = \text{rank}(H_1(M_i,\mathbb{F}_p))$. Then

(i) $r_{i+1} \geq \frac{r_i^2 - r_i}{2}$.

(ii) Let $\tilde{M}_{i+1}$ be maximal elementary abelian $p$-covering of $\tilde{M}_i$ ($\tilde{M}_1 = M_1$). The group $H_1(\tilde{M}_{i+1},\mathbb{Z}(p))$ has exponent $\geq p^{r_{i-1}}$. In particular, $r_i$ has superexponential growth and the class tower is infinite.

The next result answers the problem about the pro-$p$ completion of $\pi_1(M)$. Philosophically, it says that a non-virtually Haken 3-manifold gives rise to a “$p$-adic 3-manifold”. On the other hand, it shows that if the Thurston conjecture is false, then there are many Poincaré duality pro-$p$ groups in dimension 3. (It is very hard to construct a Poincaré duality pro-$p$ group which is not $p$-adic analytic.)

Theorem 10.2. (The structure of the pro-$p$ completion). In conditions of theorem 10.1, let $\mathcal{G}$ be the pro-$p$ completion of $\pi_1(M)$. Then $\mathcal{G}$ is a Poincaré duality pro-$p$ group of dimension 3.

The second type of results deal with manifolds with small $H_1(M)(p)$. For $H_1(M)(p)$ cyclic it is easy to prove that $M \approx N/(\mathbb{Z}/p^n\mathbb{Z})$ where $N$ is a $p$-homology sphere. For $H_1(M)(2) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ we prove $M \approx N/Q_{2^n}$ where $N$ is a 2-homology sphere and $Q_{2^n}$ is the (generalized) quaternionic group. Moreover, $n = 3$ if and only if the linking form is hyperbolic. For $H_1(M) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ we determine the structure of multiplication in cohomology as follows:
Theorem 15.4. Let $M$ be a non-virtually Haken manifold with $H_1(M)_2 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then there exists a basis $x, y, z$ in $H^1(M, \mathbb{F}_2)$ such that either $xy = xz = yz = 0$ or $x^2 = yz, y^2 = xz, z^2 = xy$.

We then apply this analysis to determine the 2-torsion in $\mathbb{Z}_2$- and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$- coverings of $M$ if $H_1(M)_2 \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. This is the most difficult part of the paper. Its understanding requires all techniques used in the theory.

In course of our study in chapter 13, we introduce a new spectral sequence, with $E_1$-term $H(G, \mathbb{F}_p)$, converging to $H(K, \mathbb{F}_p)$ where $1 \to K \to G \to C_p \to 1$ is an exact sequence of pro-$p$ groups. In case $p = 2$ it reduces to a long exact sequence

$$H^i(G, \mathbb{F}_2) \xrightarrow{res} H^i(K, \mathbb{F}_2) \xrightarrow{f} H^i(G, \mathbb{F}_2) \xrightarrow{x} H^i(G, \mathbb{F}_2) \to \ldots$$

which is equally important for three-manifold theory and number theory. We give some immediate applications to inequalities in group cohomology (see Adem [Ad]) and strong ring structure results about $H^*(G, \mathbb{F}_2)$ in spirit of Serre’s theorem [Se1], [Se2].

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1. Preliminary results is group cohomology

In this section, we collect some miscellaneous facts from group cohomology, which will be at use in further study.

1.1 Proposition (Cartan). Let $p$ be a prime and let $G = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p$. Consider the graded ring $A = H^*(G, \mathbb{Z})$. Then

1) $A_n$ is a vector space over $\mathbb{F}_p$ for all $n \geq 1$

2) the natural graded ring homomorphism $A \to H^*(G, \mathbb{F}_p)$ is injective for all $n \geq 1$

Proof. Recall that $H^*(G, \mathbb{F}_p) \approx S(x_1, \ldots, x_n) \oplus \wedge(y_1, \ldots, y_n)$, deg $x_i = 2$, deg $y_j = 1$ if $p$ is odd and $H^*(G, \mathbb{F}_2) \approx S(y_1 \ldots y_n)$ if $p = 2$. So the Poincaré series of $H^*(G, \mathbb{F}_p)$ is $(\frac{1}{1-t})^n(1+t)^n = \frac{1}{(1-t)^n}$. Let $K = \mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p < G$. Write the Lyndon-Serre-Hochschild spectral sequence for $H^*(G, \mathbb{Z})$:...
\[
\begin{array}{c}
\downarrow : \\
0 \\
E_2 \quad H^*(K, \mathbb{Z}_p) \implies H^*(G, \mathbb{Z}) \\
\downarrow 0 \\
H^*(K, \mathbb{Z})
\end{array}
\]

We get \( P_G(t) \preceq P_K(t) + \frac{t^2}{1-t} \frac{1}{(1-t)^n-1} \), where \( P_G(t) \) is the Poincaré series \( 1 + \sum_{k \geq 1} \log(p(|H^k(G, \mathbb{Z})|)) \). Observe that equality will imply that the spectral sequence above is degenerate at \( E_2 \). Now, the short exact sequence \( 0 \rightarrow \mathbb{Z} \times_p \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow F_p \rightarrow 0 \) implies the long exact sequence

\[
H^i(G, \mathbb{Z}) \times_p H^i(G, \mathbb{Z}) \rightarrow H^i(G, F_p) \rightarrow H^{i+1}(G, \mathbb{Z}) \rightarrow \ldots ,
\]

for \( i \geq 1 \), so \( |H^i(G, F_p)| \leq |H^i(G, \mathbb{Z})||H^{i+1}(G, \mathbb{Z})| \) and the equality implies that \( H^i(G, \mathbb{Z}) \) is a vector space over \( F_p \). This gives

\[
\sum_{\ell \geq 1} \dim_{F_p}(H^i(G, F_p)) \leq \frac{1+t}{t}(P_G(t)-1) \leq \frac{1+t}{t}(P_K(t)-1) + \frac{t}{(1-t)^n}
\]

Now, suppose by induction that the statement of the theorem is valid for \( (n-1) \), then \( \frac{1+t}{t}(P_K(t)-1) = \left( \sum_{i \geq 1} \dim_{F_p}(H^i(K, F_p)) \right) = \frac{1}{(1-t)^n} - 1 \), so \( \sum_{i \geq 1} \dim_{F_p}(H^i(G, F_p)) \leq \frac{1}{(1-t)^n} - 1 \).

Since this is actually an equality, we have equalities elsewhere above, which proves the induction step, hence the theorem.

1.2 Corollary. Let \( n + 2 \). Then:

1) if \( p = 2 \) then \( \tilde{A} \subset S(y_1, y_2) \) is a subring \( F_2[y_1^2, y_2^2, y_1y_2(y_1 + y_2)] \)

2) if \( p > 2 \) then \( \tilde{A} \approx (F_p[x_1, x_2])[\epsilon] \) where \( \epsilon^2 = 0 \) and \( \deg \epsilon = 3 \). Here \( \tilde{A} = F_p \oplus_{i \geq 1} A_i \)

3) \( \dim_{F_p} A^{2m} = m + 1, \dim_{F_p} A^{2m+1} = m \)

Proof. The s.s. above looks like

\[
\begin{array}{cccccccc}
\vdots \\
\downarrow \\
Z_p & Z_p & Z_p & Z_p & Z_p & Z_p & Z_p & 0 \\
E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots \\
Z_p & Z_p & Z_p & Z_p & Z_p & Z_p & Z_p & Z_p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots \\
Z & 0 & Z & 0 & Z & 0 & Z & 0
\end{array}
\]
so $A^3 \equiv \mathbb{F}_p$. In 1) the only element in $\mathbb{F}_2^{(3)}(y_1, y_2)$, invariant under the natural action of $GL_2(\mathbb{F}_2)$ is $y_1y_2(y_1 + y_2)$. 2), 3) an obvious from the spectral sequence.

1.3 Proposition. Let $W$ be a $\mathbb{Z}_p$-module, finitely generated as an abelian group. Then:

1) $\oplus_{i \geq 1, \text{odd}} H^i(\mathbb{Z}_p, W)$ is a free $\mathbb{F}_p[t]$ module of rank $\dim_{\mathbb{F}_p} H^1(\mathbb{Z}_p, W)$

2) $\oplus_{i \geq 2, \text{even}} H^i(\mathbb{Z}_p, W)$ is a free $\mathbb{F}_p[t]$ module of rank $\dim_{\mathbb{F}_p} H^2(\mathbb{Z}_p, W)$

where the module structure comes from the natural coupling $H^*(\mathbb{Z}_p, W) \otimes H^*(\mathbb{Z}_p, \mathbb{Z}) \to H^*(\mathbb{Z}_p, W)$.

Proof. See [CE].

1.4 Proposition. Let $G$ be a group acting freely and discontinuously in the connected sphere $X$. Let $Y = X/G$. Consider the 1.1. of the covering $X \to Y$:

\[ H^*(G, H^2(Y, \mathbb{Z})) \]

\[ E_2 \to H^*(G, H^1(Y, \mathbb{Z})) \]

\[ \to H^*(G, \mathbb{Z}) \]

Then at any $E_r$ all rows are graded $H^*(G, \mathbb{Z})$ modules and all differentials are module homomorphism.

Proof. This follows from the multiplicative structure of the equivalent s.s. of the Borel’s fibration

\[ X \longrightarrow X_G \]

\[ \downarrow \]

\[ BG \]

1.5. Let $W$ be a module over $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Consider the $E_2$-term of the Lyndon-Serre-Hochschild spectral sequence $H^i(\mathbb{Z}_p, H^i(\mathbb{Z}_p, W)) \Rightarrow H^{i+1}(\mathbb{Z}_p \oplus \mathbb{Z}_p, W)$. There is a natural action of the ring without unit $(X, Y) \subset \mathbb{Z}_p[X, Y]$ in all $E_r$ with the properties:

1) multiplication by $X : E_2^{p,q} \to E_2^{p+2,q}$ is an isomorphism for $p > 0$

2) multiplication by $Y : E_2^{p,q} \to E_2^{p,q+2}$ is an isomorphism for $q > 0$

3) $d_r$ is an endomorphism of $E_r$ as a graded $(X, Y)$-module,

4) the action of $(X, Y)$ in $E_\infty$ agrees with the module structure of $H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W)$ our $H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, \mathbb{Z})$.

Proof. The proof is immediate from the double complex, which is a resolution of $\mathbb{Z}$ over the sum of two cyclic groups.
2. Free $\mathbb{Z}_p$-actions in three-manifolds

This and the next section are devoted to the cohomological study of elementary abelian group actions on rational homology three-spheres.

2.1. Suppose $\mathbb{Z}_p$ acts freely in a closed oriented three-manifold $N$, which is a rational homology sphere, that is, $H_1(N, \mathbb{Z})$ is torsion. Then $W = H_1(N, \mathbb{Z})$ is a $\mathbb{Z}_p$-module. We wish to understand the cohomology $H^i(\mathbb{Z}_p, W)$. Put $M = N/\mathbb{Z}_p$

**Proposition 2.1.**

1) either $\dim_{\mathbb{F}_p} H^1(\mathbb{Z}_p, W)$ and $\dim_{\mathbb{F}_p} H^2(\mathbb{Z}_p, W)$ are both 1, or they are both zero.

2) there is a natural exact sequence $0 \to \mathbb{F}_p \to H^2(M, \mathbb{Z}) \to H^0(\mathbb{Z}_p, W) \to 0$.

**Proof.** Write the cohomological s.s. for $N \to M$, observing that $H^2(N, \mathbb{Z}) \approx W$ by Poincaré duality

$$
\begin{array}{cccccccc}
\mathbb{Z} & 0 & \mathbb{Z}_p & 0 & \mathbb{Z}_p & 0 & \mathbb{Z}_p \\
E_2 & H^0(\mathbb{Z}_p, W) & H^1 & H^2 & H^3 & H^4 & H^5 & H^6 \Rightarrow H^*(M, \mathbb{Z}) \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& \mathbb{Z} & 0 & \mathbb{Z}_p & 0 & \mathbb{Z}_p & 0 & \mathbb{Z}_p \\
\end{array}
$$

Since $H^k(M, \mathbb{Z}) = 0$ for $k \geq 4$ all rows should be killed. That means that either the $d_2$ kills all odd $H^i(\mathbb{Z}_p, W)$ and $d_3$ kills all even $H^i(\mathbb{Z}_p, W)$ or $d_4$ kills the first and the fourth row by Proposition 1.3 and Proposition 1.4. That proves 1), and 2) is obvious.

3. Free $\mathbb{Z}_p \oplus \mathbb{Z}_p$-actions in three-manifolds

3.1. Suppose now that a rational homology sphere $N^3$ is acted upon freely by $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Then $W = H^2(N, \mathbb{Z})$ is a $\mathbb{Z}_p \oplus \mathbb{Z}_p$ module, and we are interested to understand $H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W)$. Write the group extension s.s. [CE] using the information of Proposition 2.1:

$$
\begin{array}{ccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
E_2 & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \Rightarrow H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \\
& H^*(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \\
\end{array}
$$

or

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
E_2 & 0 & 0 & 0 & 0 & \Rightarrow H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \\
\end{array}
$$
\[ H^*(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \]

We will call this alternative case A and case B. In any case, we wish to understand the first row, namely, \( H^*(\mathbb{Z}_p, (H^0(\mathbb{Z}_p, W)) \). By Proposition 2.1 2), we have a short sequence of \( \mathbb{Z}_p \)-modules \( 0 \rightarrow \mathbb{F}_p \rightarrow H^2(M, \mathbb{Z}) \rightarrow H^0(\mathbb{Z}_p, W) \rightarrow 0 \). Now, the first factor of \( \mathbb{Z}_p \oplus \mathbb{Z}_p \) acts freely on \( M = N/ \) (action of the second factor), and \( M \) is a rational homology sphere, because \( N \) is (recall that \( H^*(M, \mathbb{Q}) \approx H_{inv}(N, \mathbb{Q}) \). So by proposition 2.1 1), either \( H^i(\mathbb{Z}_p, H^2(M, \mathbb{Z})) = \mathbb{Z}_p \) or 0 for all \( i \geq 1 \). Now, the long exact sequence

\[ \ldots \rightarrow H^i(\mathbb{Z}_p, \mathbb{F}_p) \rightarrow H^i(\mathbb{Z}_p, H^2(M, \mathbb{Z})) \rightarrow H^i(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \rightarrow H^{i+1}(\mathbb{Z}_p, \mathbb{F}_p) \rightarrow \ldots \]

reduces to either \( H^i(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \approx H^{i+1}(\mathbb{Z}_p, \mathbb{F}_p) \approx \mathbb{F}_p \), or to \( \ldots \mathbb{F}_p \rightarrow \mathbb{F}_p \rightarrow H^i(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \rightarrow \mathbb{F}_p \rightarrow \mathbb{F}_p \rightarrow H^{i+1}(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \rightarrow \ldots \)

Now, in the latter case the map \( H^*(\mathbb{Z}_p, \mathbb{F}_p) \rightarrow H^*(\mathbb{Z}_p, H^2(M, \mathbb{Z})) \) is a \( \mathbb{F}_p[t] \)-module homomorphism (deg \( t = 2 \)), so it is either zero or an isomorphism for all \( i \) of the same parity. This implies immediality that all \( H^i(\mathbb{Z}_p, H^0(\mathbb{Z}_p, W)) \) are of the same dimension 0,1, or 2 for \( i \geq 1 \).

So the \( E_2 \) term for \( H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \) looks like:

### Case A

\[
\begin{array}{cccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
E_2 & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & 0 \leq m \leq 2., & \Rightarrow & H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \\
* & \mathbb{Z}_p^m & \mathbb{Z}_p^m & \mathbb{Z}_p^m & \ldots ,
\end{array}
\]

### Case B

\[
\begin{array}{cccc}
0 & 0 \ldots & 0 \leq m \leq 2.
\end{array}
\]

In particular, \( \dim_{\mathbb{F}_p} H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \leq m + \ell \) (case A) and \( \leq m \) (case B).

3.2. Let \( Q = N/\mathbb{Z}_p \oplus \mathbb{Z}_p \) and write the s.s. for the covering \( N \rightarrow Q \):

\[
H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, Z) \approx \mathcal{A}
\]

\[
E_2 \searrow H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \rightarrow H^*(Q, Z)
\]

\[
0
\]

\[
H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, Z) \approx \mathcal{A}
\]

Since \( H^i(M, \mathbb{Z}) = 0 \) for \( i \geq 4 \), all rows should be killed by differentials in high dimensions. There are exactly three nontrivial differentials which are \( \mathcal{A} \)-homomorphisms: \( d_2 : \mathcal{A} \rightarrow H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W), d_3 : H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W)/Im d_2 \rightarrow \mathcal{A} \) and \( d_4 : \text{Ker } d_2 \rightarrow \mathcal{A}/Im d_3 \).

We first observe:
Proposition 3.1. $d_3$ is nontrivial in high dimension.

Proof. Suppose the opposite. Then $d_4 : \ker d_2 \to A_{n+4}$ should be an isomorphism, which is impossible by Corollary 1.2.3).

Proposition 3.2. $d_4 = 0$ in positive dimensions.

Proof. Suppose the opposite. Assume first $p = 2$. Then $0 \neq \ker d_2 \approx A/\text{Im} d_3$. Ker $d_2$ is an ideal in $A$, so it contains a principal ideal and since $A$ is a domain by Proposition 1.1, 1.2.1), the dimension of $(\ker d_2)_n$ grows as $n$. On the other hand, $\text{Im} d_3$ is an ideal in $A$. We may look at $A$ as a graded ring of an irreducible projective curve over $\mathbb{F}_2$. Then $A/\text{Im} d_3$ is a graded ring of a null-dimensional scheme, so dim$(A/\text{Im} d_3)$ stays bounded, a contradiction.

Now, let $p$ be odd. Recall $A = \mathbb{F}_p[x,y][\epsilon]$, $\epsilon^2 = 0$. Any homogeneous ideal $I$ contains $(f) \cdot \epsilon$, where $0 \neq f \in \mathbb{F}_p[x,y]$, and so dim$I_{2n-1}$ grows as $2n$. So dim$(\ker d_2)_{2n-1} \approx \dim(A/\text{Im} d_3)_{2n+3} \approx 2n$. On the other hand again, $\text{Im} d_3$ is nontrivial in high dimension, so dim$(A/\text{Im} d_3)_{2n+3}$ stays bounded by the same argument as above. So $d_4 = 0$.

Observe that $E_\infty$ term of the spectral sequence above should be

$$
\begin{array}{cccc}
& p^2\mathbb{Z} & 0 & 0 & 0 \\
E_\infty & 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 \\
& \mathbb{Z} & 0 & \mathbb{Z}_p \oplus \mathbb{Z}_p \\
\end{array}
$$

since $H^3(M, \mathbb{Z}) \approx \mathbb{Z}$ does not have torsion.

That implies the following.

3.3 Corollary. For $n \geq 3$ we have an exact sequence $0 \to A_{n-2} \xrightarrow{d_2} H^n(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \xrightarrow{d_3} A_{n+3} \to 0$.

In particular,

$$
a_{2n} = \dim_{\mathbb{Z}_p} H^{2n}(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) = n + (n + 1) = 2n + 1 \\
a_{2n-1} = \dim_{\mathbb{Z}_p} H^{2n-1}(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) = (n - 2) + (n + 2) = 2n, \ n \geq 1
$$

$$
\dim_{\mathbb{Z}_p} H^{1}(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \leq 3, \ \dim H^{2}(\mathbb{Z}_p \oplus, \mathbb{Z}_p, W) \leq 4.
$$

4.5. Now recall the s.s. for $H^*(\mathbb{Z}_p \otimes \mathbb{Z}_p, W)$ from section 2. We see immediately that case B is impossible because of the inequality dim$(H^2(\mathbb{Z}_p \oplus \mathbb{Z}_p, W)) \leq m$. So we get the s.s.

$$
\begin{array}{cccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
E_2 & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \Rightarrow H^*(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \\
& \mathbb{Z}_p^m & \mathbb{Z}_p^m & \mathbb{Z}_p^m & \ldots
\end{array}
$$
and \( a_\ell = \dim_{\mathbb{F}_2} H^\ell(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \leq \ell + m \).

In particular, since for big \( n, a_{2n-1} = 2n \), we have \( m \geq 1 \), so \( m = 1 \) or \( m = 2 \).
Assume \( m = 2 \). If \( a_2 = 3 \) there should be a nontrivial differential, killing something from the second diagonal \( (\mathbb{Z}_p, \mathbb{Z}_p, \mathbb{Z}_p^2) \). We have the following choices

\[
\begin{array}{ccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p
\end{array}
\]

(case I)
\[d_2 \neq 0\]
\*[\[\begin{array}{cc}
\mathbb{Z}_p^2 & \mathbb{Z}_p^2 \\
\mathbb{Z}_p^2 & \mathbb{Z}_p^2
\end{array}\]

\[
\begin{array}{ccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p
\end{array}
\]

(case II)
\[d_2 \neq 0\]
\*[\[\begin{array}{cc}
\mathbb{Z}_p^2 & \mathbb{Z}_p^2 \\
\mathbb{Z}_p^2 & \mathbb{Z}_p^2
\end{array}\]

\[
\begin{array}{ccc}
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\
\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p
\end{array}
\]

(case III)
\*[\[\begin{array}{cc}
\mathbb{Z}_p^2 & \mathbb{Z}_p^2 \\
\mathbb{Z}_p^2 & \mathbb{Z}_p^2
\end{array}\]

either \( d_2 \) or \( d_3 \neq 0 \)

If the case I is not realized, then \( a_1 = \dim H^1(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) = 3 \), but that means that in the spectral sequence of page 6, \( d_3 : H^1(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) \to A_4 \) is an isomorphism, so \( d_4 : \text{Ker} \ d_3 \to A_4 \) is zero. Because of the form of \( E^\infty \) that means that \( |A_0 : \text{Ker} \ d_2| = p^2 \), so \( \dim \ker d_2 = 2 \), hence \( \dim H^2(\mathbb{Z}_p \oplus \mathbb{Z}_p, W) = 2 + \dim A_5 = 4 \), a contradiction. So we have case I. Moreover, since \( a_1 = 2 \) and \( a_2 = 3 \), \( d_4 : p\mathbb{Z} \to A_4 \) is nontrivial and \( \dim(\ker d_2) = 1 \). If \( a_2 = 4 \), then there may be no differential, touching the second diagonal, in particular \( a_1 = 3 \) and \( |\ker d_2| = p^2 \) as above.

4. Algebraic study of linking forms

Let \( W \) be a finite abelian group. A linking form is a symmetric bilinear form

\[(\cdot, \cdot) : W \otimes_{\mathbb{Z}} W \to \mathbb{Q}/\mathbb{Z}\]

for which an induced map

\[W \to \text{Hom}(W, \mathbb{Q}/\mathbb{Z}) = \hat{W}\]

is an isomorphism.
Lemma (4.1). If \( W = \oplus_p W(p) \) be the canonical decomposition as a direct sum of \( p \)-groups, then \( W(p) \) is orthogonal to \( W(q) \) for \( p \neq q \).

The proof is obvious. From now on we assume \( W \) to be a \( p \)-group.

Lemma (4.2). If \( x \in W, \text{ord}(x) = p^k \), then \( \max_y \text{ord}(x,y) = p^k \).

The proof is again obvious.

Let \( V = \mathbb{Z}/p^\ell \mathbb{Z} \) and define for \( u,v \in V \)
\[
(u,v)_V = p^\ell (x,y) = (u,y) = (x,v)
\] (*)

where \( p^\ell x = u \) and \( p^\ell y = u \).

Lemma (4.3). The formula (*) defines a linking form in \( V \).

Proof. If \( \bar{x} \neq x \) with \( p^\ell \bar{x} = u \), then \( p^\ell (\bar{x},y) - p^\ell (x,y) = (u,y) - (u,y) = 0 \), so \( (u,v)_V \) is well-defined. On the other hand, if \( (u,v)_V = 0 \) for all \( v \), then \( (u,v) = 0 \) for all \( y \), so \( u = 0 \). Q.E.D.

Let \( W = \mathbb{Z}/p^{k_1} \mathbb{Z} \oplus \mathbb{Z}/p^{k_2} \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{k_s} \mathbb{Z} \) with \( k = k_1 \geq k_2 > \ldots > k_s \). Let \( U = W/\text{Ker} \) (multiplication by \( p^{k-1} \)). Then we have an isomorphism of \( \mathbb{F}_p \)-vector spaces
\[
U \xrightarrow{p^{k-1}} p^{k-1}W,
\]
and \( \langle \cdot, \cdot \rangle_{p^{k-1}W} \) induces a nondegenerate \( \mathbb{F}_p \)-valued scalar product in \( U \). If \( p \) is odd there exists an element \( \bar{x} \in U \) with \( \langle \bar{x}, x \rangle_U \neq 0 \). Let \( x \) be an element of \( W \) of maximal order \( p^k \), which projects to \( \bar{x} \), then \( p^{k-1}(x,x) \neq 0 \), so \( (x,x) \) has order \( p^k \).

Lemma (4.5). Suppose

1) \( W \) has a linking form

2) \( W \) admits an orthogonal action of \( C_p \)

Then for \( \zeta \) a generator of \( C_p \)-action, \( \dim \mathbb{F}_p \text{Im}(1 - \zeta) \) is even.

Proof. Consider a pairing on \( \text{Im}(1 - \zeta) \), defined by \( \langle (1 - \zeta)x, (1 - \zeta)y \rangle = (x, \zeta^{p-1} (1 - \zeta)y) \). It is immediately seen to be nondegenerate. Moreover, since \( \zeta \) acts orthogonally, we have
\[
\langle (1 - \zeta)y, (1 - \zeta)x \rangle = (y, \zeta^{p-1} (1 - \zeta)x) = (\zeta^{p-1} (1 - \zeta^{-1})y, x) = -\langle (1 - \zeta)x, (1 - \zeta)y \rangle,
\]
so \( \langle \cdot, \cdot \rangle \) is a symplectic structure. Arguing like in 1.4, we see that \( W \approx \sum_i W_i \oplus \hat{W}_i \) so rank \( W \) is even and \( \dim \mathbb{F}_p W \) is even, too.
5. Linking forms and transfer non-vanishing theorem

5.1 Fundamentals. Let \( M^{2m-1} \) be a closed oriented manifold. The linking form (see, for example, [vD])

\[
(\ , \ ) : H^{t_{\text{tors}}}_{m-1}(M) \otimes \mathbb{Z} H^{t_{\text{tors}}}_{m-1}(M) \to \mathbb{Q}/\mathbb{Z}
\]

is defined as follows: by universal coefficients formula one has \( H^{t_{\text{tors}}}_m(M) \approx \text{Ext}^1(H^{t_{\text{tors}}}_{m-1}(M), \mathbb{Z}) \approx \text{Hom}(H^{t_{\text{tors}}}_{m-1}(M), \mathbb{Q}/\mathbb{Z}) \). On the other hand, \( H^{t_{\text{tors}}}_m(M) \approx H^{t_{\text{tors}}}_{m-1}(M) \) by Poincaré duality, so one gets an isomorphism

\[
H^{t_{\text{tors}}}_{m-1}(M) \approx \text{Hom}(H^{t_{\text{tors}}}_{m-1}(M), \mathbb{Q}/\mathbb{Z}),
\]

which means that there is a nondegenerate form (1), which is easily seen to be symmetric.

Here is a more geometric description. Let \( x, y \in H^{t_{\text{tors}}}_{m-1}(M) \), so that \( N \cdot x = 0 \).

Realize \( x \) by a smooth chain \( \tilde{x} \) and find a smooth chain \( \tilde{z} \) such that \( \partial \tilde{z} = N \tilde{x} \).

Realize \( y \) by a smooth chain \( \tilde{y} \), disjoint from \( \tilde{x} \) and intersecting \( z \) transversally. Let \( \sharp(\tilde{z} \cap \tilde{y}) \) be a number of intersection points, counted with sign. Then

\[
(x, y) = \frac{1}{N} \sharp(\tilde{z} \cap \tilde{y}) \pmod{\mathbb{Z}}
\]

5.2 Reciprocity formula. Let \( N \xrightarrow{\pi} M \) be a finite covering of \( 2m-1 \)-dimensional manifolds. Let \( t : H_*(M) \to H_*(N) \) be the transfer map. The reciprocity formula reads: for any \( x \in H^{t_{\text{tors}}}_{m-1}(N) \), \( y \in H^{t_{\text{tors}}}_{m-1}(M) \),

\[
(\pi_* x, y)_M = (x, ty)_N,
\]

or, in other words, \( t = - (\pi_*)^* \), as linear maps between abelian groups with nondegenerate \( \mathbb{Q}/\mathbb{Z} \)-valued scalar product.

\textbf{Proof.} Let \( \partial \tilde{z} = N \cdot \tilde{y} \), then \( \partial(t \tilde{z}) = N \cdot t \tilde{y} \). Realize \( x \) by a smooth chain disjoint from \( t \tilde{y} \) and transversal to \( t \tilde{z} \). Then \( \pi_* \tilde{z} \) is a smooth chain, disjoint from \( \tilde{y} \), and transversal to \( \tilde{z} \).

Next, \( \sharp(\pi_* \tilde{z}, \tilde{y}) = \sharp(\tilde{z}, t \tilde{y}) \), which proves the formula.

One notices that the reciprocity formula follows from the formula \( t = PD \circ \pi_* \circ PD \) for the Gysin homomorphism, c.f. [Ka].

\textbf{Transfer non-vanishing Theorem (5.3).} Suppose \( H_*(N) = 0 \) for \( 0 < i < m-1 \), \( H_{m-1}(N) \) is torsion, \( \pi : N \to M \) is a normal covering with the Galois group \( G \) and suppose \( H_{m-1}(G) = 0 \). Then \( t : H_{m-1}(M) \to H_{m-1}(N) \) is injective.

\textbf{Proof.} Write the homological spectral sequence of the covering \( N \to M \):

\[
\begin{array}{cccc}
(H_{m-1}(N))_G & 0 & \mathbb{Z} & H_1(G, \mathbb{Z}) \ldots H_{m-1}(G, \mathbb{Z}) & H_m(G, \mathbb{Z}) \ldots \\
\| & & & 0 & \\
\end{array}
\]
We see that $H_{m-1}(M) \approx (H_{m-1}(N))_G$, in particular, $H_{m-1}(N) \xrightarrow{\pi_*} H_{m-1}(M)$ is onto, hence $t = (\pi_*)^*$ is injective, since $\langle \cdot \rangle_M$ and $\langle \cdot \rangle_N$ are nondegenerate.

**Corollary (5.4).** Let $N \to M$ be a normal covering of three-manifolds with the Galois group $G$. If $b_1(M) = b_1(N) = 0$ and $G/[G,G] = 1$, then $t : H_1(M) \to H_1(N)$ is injective. In particular, this is true for $G = SL(2, \mathbb{F}_p)$ for $p \geq 5$.

6. The structure of anisotropic extension, I

6.1. Let $M$ be a closed oriented three-manifold with $H_1(M)(p) = \mathbb{Z}/p^{k_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$, where the decomposition is orthogonal with respect to the linking form. Fix a generator $z$ of the last component. Consider the homomorphism

$$(z, \cdot) : H_1(M) \to \mathbb{Z}_p$$

and denote by $N$ the cyclic covering of $M$ with respect to this homomorphism. We assume $b_1(M) = b_1(N) = 0$. In this chapter we study the structure of $(H_1(N))(p)$.

Let $e_1, \ldots, e_{s-1}$ be the generators of all components except the last. Then $(e_i, e_j) = 0$ and $(e_i, e_i)$ has the order $p^{k_i}$ in $\mathbb{Q}/\mathbb{Z}$. Since all $e_i$ lie in the kernel of (*) the transfer $te_i$ splits as $v_i + \zeta v_i + \ldots + \zeta^{p-1}v_i$, where $\pi_*(v_i) = e_i$ and $\zeta$ is the deck transformation corresponding to (*). Indeed, we may find a loop in $\pi_1(M)$, representing $e_i$, and lying in the kernel of the composition $\pi_1(M) \to \pi_1(M) \to \mathbb{Z}_p$, so that its preimage in $N$ splits to $p$ connected components. We denote $W = H_1(N)(p)$, $V = H_1(M)(p)$ and $W_i$ the subgroup of $W$ generated by $v_1, \zeta v_1, \ldots, \zeta^{p-1}v_1$. We start with the following (obvious) remark.

**Lemma (6.1).** $W_i$ is a cyclic $\mathbb{Z}[C_p]$ -module, i.e. the quotient of $\mathbb{Z}[C_p]$.

**Lemma (6.2).** The element $p_i = v_i + \zeta v_i + \ldots + \zeta^{p-1}v_i = te_i$ is of order $p^{k_i}$ in $W_i$.

**Proof.** On one hand, the order of $p_i = te_i$ is no more that the order of $e_i$, that is, $p^{k_i}$. On the other, by the reciprocity formula of 5.2,

$$(p_i, v_i)_N = (e_i, e_i)_M,$$

and the RHS is of order $p^{k_i}$, so $\text{ord}(p_i) \geq p^{k_i}$, as claimed.

**Lemma (6.3).** The restriction of $t : H_1(M) \to H_1(N)$ on $\bigoplus_{i=1}^{s-1} \mathbb{Z}/p^{k_i}\mathbb{Z}$ is injective. Moreover $t(\bigoplus_{i=1}^{s-1} \mathbb{Z}/p^{k_i}\mathbb{Z}) = (H_1(N))(\mathbb{Z}_p$).

**Proof.** The first statement follows from $(p_i, v_j)_N = (e_i, e_j)_M$ the orthogonality of the decomposition of $H_1(M)$ and nondegeneracy of $\langle \cdot, \cdot \rangle_M$. To prove the second, write the homology spectral sequence of the covering $N \to M$,

$$
\begin{array}{cccc}
H_i(N)_{\mathbb{Z}_p} & d_2 & \vdots \\
\mathbb{Z} & \mathbb{Z}_p & 0
\end{array}
$$

from which we deduce the exact sequence
0 \to (H_1(N))_{\mathbb{Z}_p} \to H_1(M) \to \mathbb{Z}_p \to 0

so \dim_{\mathbb{Z}_p} H_1(N)_{\mathbb{Z}_p} = k_1 + \ldots + k_{s-1}, hence \dim_{\mathbb{Z}_p}(H_1(N))_{\mathbb{Z}_p} = k_1 + \ldots + k_{s-1}, and so the injectivity of \( t \) implies the surjectivity.

**Corollary (6.3).** Let \( N \to M \) be a cyclic covering, corresponding to the homomorphism \((\cdot, z) : H_1(M) \to \mathbb{Z}_p \) with \((z, z) \neq 0\). Then \( H^i(\mathbb{Z}_p, H_1(N)) = 0 \) for \( i \geq 1 \).

**Proof.** Since \( \bigoplus_{i=1}^{s-1} \mathbb{Z}/p^i\mathbb{Z} = \pi_*(H_1(N)) \) we have \( (H_1(N))_{\mathbb{Z}_p} = t(\bigoplus_{i=1}^{s-1} \mathbb{Z}/p^i\mathbb{Z}) = t \circ \pi_*(H_1(N)) = \text{Im}(1 + \zeta + \ldots + \zeta^{p-1}) \) so that \( H^i(\mathbb{Z}_p, H_1(N)) = 0 \). But then \( H^1(\mathbb{Z}_p, H_1(N)) = 0 \) for all \( i \geq 1 \) by proposition 2.1.

**Lemma (6.4).** The restriction of \((\cdot, \cdot)_N\) on \( W_1 + \ldots + W_{s-1} \) is nondegenerate.

**Proof.** Let \( y \in W_1 + \ldots + W_{s-1} \). There exists a minimal \( N \) such that \((1 - \zeta)^N y = 0\), so \( 0 \neq x = (1 - \zeta)^N y \) and \( x \in (H_1(N))_{\mathbb{Z}_p} \). By the previous lemma, \( x = \alpha_1 p_1 + \ldots + \alpha_s p_s \) such that for one \( i \), \( \alpha_i \notin p^k\mathbb{Z} \) by lemma 3.2. Now

\[(x, v_i)_N = (\alpha_1 p_1 + \ldots + \alpha_s p_s, v_i)_N = \alpha_i (p_i, v_i)_N = \alpha_i (e_i, e_i)_M \neq 0.\]

But

\[(x, v_i)_N = ((1 - \zeta)^N y, v_i)_N = (y, (1 - \zeta^{-1})^N v_i)_N,\]

and \((1 - \zeta^{-1})^N v_i \in W_i\), so \((\cdot, \cdot)_N\) is nondegenerate on \( W_1 + \ldots + W_{s-1} \).

**Corollary (6.5).** \( H_1(N) = W_1 + \ldots + W_{s-1} \)

**Proof.** By the previous lemma, \( H_1(N) \) splits as \( (W_1 + \ldots + W_S) \oplus (W_1 + \ldots + W_S)^\perp \). Suppose the latter space is nontrivial. Since any action of \( C_p \) in an abelian \( p \)-group has fixed points, we would have \( H_1(N)_{\mathbb{Z}_p} \) strictly contains \( t(H_1(M)) \), which contradicts lemma 3.3.

**6.6 Theorem (Structure theorem for anisotropic extension).** The group \( H_1(N) \) is a sum of cyclic modules: \( H_1(N) = W_1 \oplus \ldots \oplus W_{s-1} \) with \( W_i \approx \Lambda/\alpha_i \) where \( \Lambda = \mathbb{Z}[\zeta]/(\zeta^p - 1) \approx \mathbb{Z}[C_p] \) and \( \alpha_i \) ideals in \( \Lambda \). Moreover \( (\Lambda/\alpha_i) \otimes \mathbb{Z}_p \approx \mathbb{Z}[p^k]\mathbb{Z}_\Lambda \) and \( \text{Ext}^1_\Lambda(\mathbb{Z}, W) = 0 \).

**Proof.** We need only to check the last statement. It follows from the lemma 6.3 that \( H^{\text{even}} > 0(\mathbb{Z}_p, H_1(N)) = 0 \), so by 2.1, also \( H^{\text{odd}}(\mathbb{Z}_p, H_1(N)) = 0 \), hence \( H^{\geq 0}(\mathbb{Z}_p, W_i) = 0 \).

## 7. Shrinking

Shrinking is a process leading to a \( \mathbb{Z}/p\mathbb{Z} \) split component in \( H_1 \) of a specially chosen covering, as described below.

**7.1.** Assume that in the decomposition \( H_1(M) = \mathbb{Z}/p^k_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^k_s\mathbb{Z}, k_s \geq 2 \) and consider a map \( H_1(M) \to \mathbb{Z}/p\mathbb{Z} \) sending \( e_s \) to a generator and \( e_i \) to zero, \( i < s \). Let \( N \) be the corresponding covering. Let \( v_i, i < s \) be defined as above and let \( v_s = t(e_s) \). Again denote by \( W_i \) the subgroup of \( H_1(N) \) generated by \( v_i \) as \( \Lambda \)-module. Since \( v_s \) is invariant, in fact \( W_s \) is cyclic of order \( \leq p^{k_s} \). We claim that in fact that \( \text{ord} v_s = p^{k_s - 1} \). Indeed, \( (v_s, v_s)_N = (e_s, \pi_* v_s)_M = p(e_s, e_s) \),
Proof. The proof of completely parallel to that of lemma 6.2. Let us prove for $N$ and $v$ as above, let $(v, s) = (e_s, \pi_s(z))$ and since $\pi_s(z) \in \text{Ker} \epsilon$ we see that $\text{ord}(e_s, \pi_s(z)) \leq p^{k_s-1}$, so $\text{ord} v \leq p^{k_s-1}$. Hence $W_1 \approx \mathbb{Z}/p^{k_s-1}$. Now we claim that one has an exact sequence

$$0 \rightarrow \mathbb{Z}_p \xrightarrow{\psi} H_1(M) \xrightarrow{t} H_1(N)^{\mathbb{Z}_p} \rightarrow 0,$$

where $\text{Im} \psi = p^{k_s-1}e_1$.

Indeed, the exactness in the middle term follows from the reciprocity in the same manner as in lemma 6.3. The exactness in the last term follows from $|H_1(N)^{\mathbb{Z}_p}| = |H_1(N)_{\mathbb{Z}_p}| = |\text{Ker} \epsilon|$. The rest of the argument of section 3 goes unchanged and we arrive to the following result.

7.2 **Theorem (Splitting theorem for shrinking).** The group $H_1(N)$ splits as a direct orthogonal sum

$$H_1(N) = (W_1 + \ldots + W_{s-1}) \oplus W_s$$

where $W_s \approx \mathbb{Z}/p^{k_s-1}\mathbb{Z}$ with the trivial action and $W_i \approx \Lambda/\alpha_i$ for some ideals $\alpha_i$.

**Remark 7.2.** The statement of the theorem holds true if $p = 2$ and $H_1(M) = V \oplus \mathbb{Z}/2^k\mathbb{Z}$, $k \geq 2$, with the same proof.

8. **Isotropic extension**

8.1. Assume that $H_1(M) = \mathbb{Z}/p^{k_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{k_s}\mathbb{Z} \oplus (\mathbb{Z}_p \oplus \mathbb{Z}_p)$ is the orthogonal decomposition and the form in the last summand is hyperbolic, given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We study the covering, corresponding to the map $\epsilon : H_1(M) \rightarrow \mathbb{Z}_p$ defined by $(\epsilon_i, e_{s+1})$ and, sending $e_i$ to zero, $i \leq s$ or $i = s + 2$ and $e_{s+1}$ to a generator. Let $v_i$, $i \leq s$ be as above, let $T = t(v_{s+1})$ and let $v$ be such that $\pi_s v = e_{s+2}$ and $N(v) = t e_{s+2}$, such $v$ exists because $e_{s+2} \in \text{Ker} \epsilon$. For $i \leq s$ we keep on denoting $p_i = t e_i$.

**Lemma (8.2).**

1) $\text{ord} p_i = p^{k_i}$ for $i \leq s$

2) $\text{ord} T = p$

3) $p_i$ and $T$ generate $H_1(N)^{\mathbb{Z}_p}$ and $\sum a_i p_i + aT = 0$ implies $p^{k_i} | a_i$ and $p | a$.

**Proof.** The proof of completely parallel to that of lemma 6.2. Let us prove for example, 3). Suppose $\sum a_i p_i + aT = 0$. Then for $j \leq s$, $0 = (v_j, \sum a_i p_i + aT)_N = \sum a_i (e_i, e_j) + a(e_j, e_{s+1}) = a_j (e_j, e_j)$, so $p^{k_i} | a_j$, hence $\sum a_i p_i = 0$ and $aT = 0$, and we deduce that also $p | a$. Q.E.D.

Let $W_i$, $i \leq s$ be the cyclic submodule, generated by $v_i$. Then $(\sum W_i)^{\mathbb{Z}_p}$ contains a subgroup generated by $p_i$, and since $|(\sum W_i)^{\mathbb{Z}_p}| = |(\sum W_i)_{\mathbb{Z}_p}| = \oplus_{i \leq s} \mathbb{Z}/p^{k_i} \mathbb{Z}$, $T \not\in \sum W_i$. It follows, exactly as in lemma 6.4, that $(\cdot, \cdot) | \sum W_i$ is nondegenerate.
Let $H_1(N) = (\sum W_i) \oplus (\sum W_i)\perp$ be the orthogonal decomposition. Observe that $T \in (\sum W_i)\perp$ by reciprocity. Now, it follows easily, that one can modify $v$ to $\tilde{v}$ such that $\tilde{v} \in (\sum W_i)\perp$. Let $W$ be the cyclic submodule, generated by $\tilde{v}$. There should be an invariant element in $W$, and since by lemma 8.2. 3), $p_i$ and $T$ generate $H_1(N)e_r$, we conclude that $T \in W$.

We claim that $N\tilde{v} = (1+\zeta+\ldots+\zeta^{p-1})\tilde{V} = 0$. Indeed, for any $w \in H_1(N)$, $(N\tilde{V}, w)_N = (e_{s+2}, \pi_s W)_N$ but $\pi_s W \in \text{Ker} e$ and $e_{s+2}$ is in the kernel of $(\cdot, \cdot)_M|_{\text{Ker} e}$. We sum up all the information in the following theorem

**Theorem (8.3) (Splitting theorem for isotropic extensions).** The group $H_1(N)$ splits orthogonally as

$$H_1(N) = \sum_{i \leq s} W_i \oplus W,$$

where $W$ is a $\Lambda/\Lambda \cdot (1 + \zeta + \ldots + \zeta^{p-1})$ -module.

Moreover, $H^i(\mathbb{Z}_p, H_1(N)) \approx \mathbb{Z}_p$ for $i \geq 1$

**Proof.** Only the last statement needs proof. It is enough to show that $T \notin \text{Im} N$ (recall that $N = 1 + \zeta + \ldots + \zeta^{p-1}$). But if $T = Nw$, then $(T, v) = (Nw, v) = (w, Nv) = 0$. However $(T, v) = (e_{s+1}, e_{s+2})_M = 1$, a contradiction.

9. **Blowing up $H_1$: the non-abelian step**

In this chapter we prove that any “generic” 3-manifold $M$ has a finite covering $N$ with large $H_1(N)_p$. We first consider the case of homology sphere $M$ with the following “genericity” assumption:

(R). The Casson invariant $|\lambda(M)| > \# (\text{representation of } \pi_1(M) \text{ onto } SL_2(\mathbb{F}_5))$.

Our first result is:

**Theorem (9.1).** Let $M$ be a homology sphere satisfying (R). Then either $M$ is virtually Haken or $\pi_1(M)$ admits a Zariski dense representation in $SU(2)$.

**Proof.** Since $\lambda(M) \neq 0$, there are some nontrivial representations of $\pi_1(M)$ in $SU(2)$ [A]. Suppose none of them is Zariski dense. Since $\pi_1(M)$ is perfect, we see that images of all representations should be finite. Moreover, among all finite subgroups of $SU(2)$ only $SL_2(\mathbb{F}_5)$ does not have abelian quotients [W]. Therefore, for any nontrivial representation $\rho : \pi_1(M) \to SU(2)$, $\rho(\pi_1(M)) \approx SL_2(\mathbb{F}_5)$. Let $W_1 \cup W_2$ be a Heegaard splitting of $M$. Let $R_1, R_2, R$ be representation varieties of $\pi_1(W_1), \pi_1(W_2)$ and $\pi_1(S)$ in $SU(2)$. Notice that set theoretically $R_1 \cap R_2$ is the representation variety of $\pi_1(M)$. We claim:

**Lemma (9.1).** For any $\rho \in R_1 \cap R_2$, $R_1$ and $R_2$ intersect transversally at $\rho$.

**Proof.** The Zariski tangent spaces of $R_i$ at $\rho$ are $H^1(\pi_1(W_i), su(2))$ with adjoint action and since $\pi_1(M) = \pi_1(W_1) \ast \pi_1(W_2)$, the twisted Myer Wietoris exact sequence of [JM] shows that $T_\rho R_1 \cap T_\rho R_2 \approx H^1(\pi_1(M), su(2))$. If the intersection is not transversal, then $H^1(\pi_1(M), su(2)) \neq 0$. Let $N$ be the covering of $M$ defined by the exact sequence

$$1 \to \pi_1(N) \to \pi_1(M) \xrightarrow{\rho} SL_2(\mathbb{F}_5) \to 1,$$
then $H^1(\pi_1(M), SU(2)) = (H^1(\pi_1(N), su(2))^{SL_2(\mathbb{F}_5)}$, so that $H^1(\pi_1(N), su(2)) \neq 0$. But the action of $\pi_1(N)$ in $su(2)$ is trivial, and so that $H^1(N, \mathbb{R}) \neq 0$, a contradiction.

Returning to the proof of the theorem, we see that $R_1$ and $R_2$ intersect transversally in a finite number of points, each being a representation of $\pi_1(M)$ to $SL_2(\mathbb{F}_5)$, so that $|\lambda(M)| \leq 2$ (representation of $\pi_1(M)$ to $SL_2(\mathbb{F}_5)$).

Q.E.D.

An application of strong approximation in algebraic groups, as described in [Lu], gives immediately the following result

**Theorem (9.2).** Let $M$ be a homology sphere, satisfying (R). Then for any prime $p$ there exists a finite normal covering $N \to M$ with rank $(H_1(N)_{(p)})$ arbitrarily large. In particular, there exists a covering with rank $(H_1(N)_{(p)}) \geq 4$.

We will now sketch an alternative approach, with slightly weaker statement, having in mind the application to manifolds without assumption (R). First, we may assume that the complex Zariski dense representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$ is defined over $\mathbb{Q}$, using the usual deformation argument [Re1]. So there is a subring $O_S$ in a number field $F$, such that $Im\rho \subset SL_2(O_S)$. Now, the Bass-Serre theory of groups, acting on trees [Se4] implies immediately, with the argument of Culler-Shalen [CS], that either $M$ is virtually Haken, or $\rho$ can be defined over $O \subset \mathbb{F}$. Let $p$ be a prime such that $H_1(M)_{(p)} = 0$ and for some prime $p$ over $p$ in $O$, $O/p = \mathbb{F}_p$; there are infinitely many such primes. We claim that the composite map $\rho_p: \pi_1(M) \to SL_2(\mathbb{O}) \to SL_2(\mathbb{F}_p)$ is on. Indeed, since $p \nmid |H_1(M)|$ and there are no proper perfect subgroups in $SL_2(\mathbb{F}_p)$, either $\rho_p$ is on, or $Im\rho_p = 1$, but the last option is impossible, since the kernel of reduction $SL_2(\mathbb{O}) \to SL_2(\mathbb{F}_p)$ is $p$-residually finite. Now, we look at the covering $N \to M$, defined by the short exact sequence $1 \to \pi_1(N) \to \pi_1(M) \xrightarrow{\rho_p} SL_2(\mathbb{F}_p)$. We assume $p > 5$. Our first claim is that $H_1(N)_{(p)} \neq 0$. Indeed, the s.s. of the covering $N \to M$ reads like

$$
\begin{align*}
H^*(SL_2(\mathbb{F}_p), \mathbb{Z}) & \\
H^*(SL_2(\mathbb{F}_p), W) & \\
0 & \Rightarrow H^{i+j}(M, \mathbb{Z}) \\
H^*(SL_2(\mathbb{F}_p), \mathbb{Z}) &
\end{align*}
$$

where $W = H^2(N) = \widehat{H}_1(N)$. Now, $H^*(SL_2(\mathbb{F}_5), \mathbb{Z})_{(p)}$ is $\mathbb{F}_p$ for $* = k(p - 1)$ and 0 otherwise, so $W = 0$ is impossible.

Now we will look at the same s.s. with $\mathbb{F}_p$-coefficients:

$$
\begin{align*}
H^*(SL_2(\mathbb{F}_p), \mathbb{F}_p) & \\
H^*(SL_2(\mathbb{F}_p), V^*) & \\
H^*(SL_2(\mathbb{F}_p), V) & \Rightarrow H^{i+j}(M, \mathbb{F}_p) \\
H^*(SL_2(\mathbb{F}_p), \mathbb{F}_p) &
\end{align*}
$$

where $V = H^1(N, \mathbb{F}_p)$. Since $p \nmid |H_1(M)|$, $V^{SL_2(\mathbb{F}_p)} = 0$. This eliminates the possibility of $\dim V = 1$. Now, if $\dim V \leq 3$, then $V$ is either the natural two-dimensional
module, or the adjoint module $\text{sl}_2(\mathbb{F}_p)$. In both cases, $V$ is cohomologically trivial for $p > 3$ as the restriction to the $p$-Silow subgroup shows, which is impossible by the same argument as above. Hence $\dim H_1(N, \mathbb{F}_p) \geq 4$ or $M$ is virtually Haken.

10. CLASS TOWERS OF THREE-MANIFOLDS AND THE STRUCTURE OF THE PRO-$p$ COMPLETION OF $\pi_1$

10.1. In this chapter we assume that $M$ is a three-manifold, not virtually Haken and such that $\text{rank } H_1(M)(p) \geq 4$ for some $p$. Our main result is as follows

Theorem (10.1). Let $M_1 = M = \tilde{M}_1$ and let $M_i$, (resp. $\tilde{M}_i$)$i \geq 2$ be the maximal abelian $p$-covering (resp. maximal elementary abelian $p$-covering) of $M_{i-1}$ (resp. $\tilde{M}_{i-1}$). Let $r_i = \text{rank } (H_1(M_i)(p))$ (resp. $\tilde{r}_i = \text{rank } (H_1(\tilde{M}_i)(p))$. Then

1) $r_{i+1} \geq \frac{r_i^2 - r_i}{2}$, and the same for $\tilde{r}_i$,

2) The group $H_1(\tilde{M}_{i+1})(p)$ has exponent $\geq p^{\tilde{r}_i - 1}$

In particular, the tower $\{M_i\}$ is infinite; that is, the pro-$p$ completion of $\pi_1(M)$ is infinite.

Proof. Let $H_1(M)(p) = \mathbb{Z}/p^{k_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p^{k_r}\mathbb{Z}$ where $r = r_1$. Denote $A = H_1(M)(p)$ and consider the covering

$$M_1 = N \rightarrow M$$

with

$$1 \rightarrow \pi_1(N) \rightarrow \pi_1(M) \rightarrow A \rightarrow 1$$

exact.

We will write the s.s. of the covering $N \rightarrow M$ in integer homology:

| $\mathbb{Z}$ | $A \ldots$ |
|-------------|-------------|
| 0           | 0 \ldots    |
| $(H_1(N))_A$ | $H_1(A, H_1(N))$ \ldots $H_{i+j}(M, \mathbb{Z})$ |
| $\Lambda^2_{\mathbb{Z}}A$ | $A$ |
| $\mathbb{Z}$ | $A$ |

we see immediately that $d_2 : \Lambda^2_{\mathbb{Z}}A \rightarrow (H_1(N))_A$ should be injective (since $H_2(M) = 0$). So $\text{rank } H_1(N)(p) \geq \text{rank } (H_1(N)(p))_A \geq \frac{r^2 - r}{2}$, as stated. The proof for $\tilde{r}_i$ is identical.

To prove (2) we will write the s.s. of the covering defined by the exact sequence

$$1 \rightarrow \pi_1(N) \rightarrow \pi_1(M) \rightarrow A \rightarrow 1,$$

where now $A = \mathbb{Z}/p \oplus \ldots \oplus \mathbb{Z}/p\mathbb{Z}$ in cohomology.

We have
By proposition 1.1, the exponent of $H^\ast(A, \mathbb{Z})$ is $p$. Since the $E_\infty$ should look like
\[
p^r \mathbb{Z} \\
* \\
0
\]
we conclude that the exponent of $H^2(A, H^2(N))$ is at least $p^r - 1$, hence the exponent of $H^2(N) \approx H_1(N)$ is at least $p^r - 1$, which concludes the proof.

10.2. The following result specifies the structure of pro-$p$ completion of $\pi_1(M)$. Informally speaking, a counterexample to the Thurston Conjecture would give rise to a “$p$-adic three manifold”, with “fundamental group” $\mathfrak{G}$.

**Theorem (10.2).** In conditions of 10.1, let $\mathfrak{G}$ be the pro-$p$ completion of $\pi_1(M)$. Then $\mathfrak{G}$ is a Poincaré duality pro-$p$ group in dimension 3.

**Proof.** First we notice that the subsequent quotients of class towers

\[
\pi_1(M) = \pi_1(M) \supset \pi_1(M_2) \supset \ldots \\
\text{and } \mathfrak{G} = \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \ldots
\]

are identical. It follows that $\mathfrak{G}_k = [\mathfrak{G}_{k-1}, \mathfrak{G}_{k-1}]$ is a pro-$p$ completion of $\pi_1(M_k)$. Hence $H_1(\mathfrak{G}_k)$ is finite and $H^1(\mathfrak{G}_k, \mathbb{Z}) = 0$. Let $G_k = \mathfrak{G}/\mathfrak{G}_k$; then $G_k$ is a finite $p$-group. Now, for all $k$, $H^2(G_k, \mathbb{Z}) \approx \widehat{H}_1(G_k) = \widehat{H}_1(M_k)_{(p)}$, so $H^2(\mathfrak{G}, \mathbb{Z}) \approx \widehat{H}_1(M)_{(p)}$. Consider the s.s. of the extension $1 \to \mathfrak{G}_k \to \mathfrak{G} \to G_k \to 1$ in integral cohomology. We get

\[
\begin{array}{cccccc}
W_k^{G_k} & H^1(G_k, \mathfrak{W}_k) & H^2(G_k, \mathfrak{W}_k) & H^3(G_k, \mathfrak{W}_k) & H^4(G_k, \mathfrak{W}_k) \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & H^2(G_k, \mathbb{Z}) & H^3(G_k, \mathbb{Z}) & H^4(G_k, \mathbb{Z}) \\
\end{array} \Rightarrow \begin{array}{c}
H^{i+j}(\mathfrak{G}, \mathbb{Z})
\end{array}
\]

where $\mathfrak{W}_k = H^2(\mathfrak{G}_k, \mathbb{Z})$. We wish to compare this s.s. of the covering $M_k \to M$

\[
\begin{array}{cccccc}
H^2(M_k, \mathbb{Z})^{G_k} & H^1(G_k, \mathfrak{W}_k) & \ldots & \Rightarrow & H^{i+j}(M, \mathbb{Z}) \\
0 & 0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & H^2(G_k, \mathbb{Z}) & H^3(G_k, \mathbb{Z}) & H^4(G_k, \mathbb{Z}) \\
\end{array}
\]

and $H^2(M_k, \mathbb{Z}) \approx H_1(M_k, \mathbb{Z}) \approx \mathfrak{W}_k$, so that the first three rows of the two s.s are identical. Observe that there is a natural map from the first s.s. to the second so that the differential $d_3$ from the third row to the first is the same for both.
Now, since the wedge map $H^2(G_k, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is an isomorphism dual to $H_1(M, \mathbb{Z}) \approx G_k/\langle G_k, G_k \rangle$, we see that $d_3 : H^2(M, \mathbb{Z})^G_k \to H^3(G_k, \mathbb{Z})$ is injective for the second s.s., and, in fact, an isomorphism, since the finite group $coker d_3$ should inject to $H^3(M, \mathbb{Z}) = \mathbb{Z}$, so $coker d_3 = 0$. That means in terms of the first s.s. that the wedge map $H^3(G_k, \mathbb{Z}) \to H^3(\mathfrak{g}, \mathbb{Z})$ is zero, so $H^3(\mathfrak{g}, \mathbb{Z}) = \lim H^3(G_k, \mathbb{Z}) = 0$. Next, the group $coker d_3 : H^1(G_k, \hat{W}_k) \to H^4(G_k, \mathbb{Z})$ from the second s.s. should be killed by $d_4$ from the cyclic subgroup of $H^3(M, \mathbb{Z}) = \mathbb{Z}$, so it is cyclic. In terms of the first s.s. it means that the image of $H^4(G_k, \mathbb{Z})$ in $H^4(\mathfrak{g}, \mathbb{Z})$ is cyclic, so $H^4(\mathfrak{g}, \mathbb{Z})$ is either $\mathbb{Z}/p^N \mathbb{Z}$ or $(\mathbb{Q}/\mathbb{Z})_{(p)}$. We claim that the first option is impossible.

Indeed, if this is the case then $|coker d_3 : H^1(G_k, \hat{W}_k) \to H^4(G, \mathbb{Z})| \leq p^N$. (since $H^3(\mathfrak{g}, \mathbb{Z}) = 0$, the forth row of the first s.s. is zero). On the other hand, since the image of $H^3(M, \mathbb{Z})$ in $H^3(M, \mathbb{Z})$ has index $|G_k|$, the product of orders of groups $coker d_3 : H^1(G_k, \hat{W}_k) \to H^4(G, \mathbb{Z})$ and $Im d_2 : H^3(M, \mathbb{Z}) \to H^2(G_k, \hat{W}_k)$ is $|G_k|$, so the former group has order $\geq \frac{|G_k|}{p^N}$ since these groups are recipients of differentials from $H^3(M, \mathbb{Z})$.

So $|G_k| \leq p^2N$ which is impossible by Theorem 10.1. So $H^4(\mathfrak{g}, \mathbb{Z}) = (\mathbb{Q}/\mathbb{Z})_{(p)}$. We claim that for $i \geq 5$, $H^i(\mathfrak{g}, \mathbb{Z}) = 0$. Indeed, the first s.s. above now looks like

\[
\begin{array}{ccccccc}
\vdots \\
H^*(G_k, (\mathbb{Q}/\mathbb{Z})_{(p)}) \\
0 & H^*(G_k, \hat{W}_k) & \Rightarrow & H^{i+j}(\mathfrak{g}, \mathbb{Z}) \\
0 & H^*(G_k, \mathbb{Z})
\end{array}
\]

and $d_3 : H^*(G_k, (\mathbb{Q}/\mathbb{Z})_{(p)}) \to H^{*+3}(G_k, \hat{W}_k)$ is the same that $d_2 : H^*(G_k, \mathbb{Z}) \to H^{*+2}(G_k, \hat{W}_k)$ from the second s.s. after identification $H^*(G_k(\mathbb{Q}/\mathbb{Z})_{(p)}) \approx H^{*+1}(G_k, \mathbb{Z})$. This implies immediately that $H^i(G_k, \mathbb{Z}), i \geq 5$ is killed in the first s.s. because it is killed in the second. Summing up, we have the following table:

\[
\begin{array}{cccccccc}
H^0(\mathfrak{g}, \mathbb{Z}) & H^1(\mathfrak{g}, \mathbb{Z}) & H^2(\mathfrak{g}, \mathbb{Z}) & H^3(\mathfrak{g}, \mathbb{Z}) & H^4(\mathfrak{g}, \mathbb{Z}) & 0 & \ldots \\
\mathbb{Z} & 0 & \hat{H}_1(M)_{(p)} & 0 & (\mathbb{Q}/\mathbb{Z})_{(p)} & 0 & \ldots 
\end{array}
\]

hence in $\mathbb{F}_p$-coefficients we have

\[
\begin{array}{cccccccc}
H^0(\mathfrak{g}, \mathbb{F}_p) & H^1(\mathfrak{g}, \mathbb{F}_p) & H^2(\mathfrak{g}, \mathbb{F}_p) & H^3(\mathfrak{g}, \mathbb{F}_p) & 0 & \ldots \\
\mathbb{F}_p & V & V^* & \mathbb{F}_p & 0 & \ldots 
\end{array}
\]

where $V = Hom(H_1(M), \mathbb{Z}/p\mathbb{Z})$ and it is easy to prove that the duality $V \times V^* \to \mathbb{F}_p$ is induced by multiplication.

Q.E.D.

11. Nazarova-Royter theory and the structure of anisotropic and isotropic extensions, II

11.1. We should recall the Nazarova-Royter classification of finite modules over the cyclic group $C_p$, c.f. [NR], [Le].

**Definition.** A left elementary block is $\mathbb{Z}/p^n\mathbb{Z}$ with trivial action of $C_p$. 

20
Definition. A right elementary block is a module of the form \( \mathcal{O}/(1 - \zeta)^N \mathcal{O} \), where \( \mathcal{O} = \mathbb{Z}[C_p]/1 + \zeta + \ldots + \zeta^{p-1} \) the ring of cyclotomic integers. The prime \( 1 - \zeta \in \mathcal{O} \) is called \( \pi \).

Definition. A diagram

represents a fibered sum of a left module \( L \) and a right module \( R \) over \( L/pL \approx R/\pi R = \mathbb{F}_p \):

\[
0 \to W \to L \oplus R \to \mathbb{F}_p \to 0
\]

Definition. A diagram

represents a quotient of \( L \oplus R \) which identifies Ker(\( \times p \)) in \( L \) with Ker(\( \times \pi \)) in \( R \).

Definition. An open module is represented by a diagram

\( (p \ x_i = 0) \).

Definition. A closed module is a quotient of an open module \( W \) by a relation \( \sum a_i x_i = 0, a_i \in \mathbb{F}_p \), where \( \sum a_i t^i \) is a power of irreducible polynomial \( \neq ax \) over \( \mathbb{F}_p \).
Theorem (11.2) (Nazarova-Royter). Any f.g. indecomposable module over $C_p$ which is a $p$-torsion is either open or closed module.

Theorem (11.3) ([AL]). If $W$ is an open module. Then $H^i(C_p, W) = \mathbb{Z}/p\mathbb{Z}$ for $i \geq 1$. If $W$ is a closed module, then $H^i(C_p, W) = 0$ for $i \geq 1$.

A combination with Proposition 2.1 yields immediately the following result of key importance:

Theorem (11.4) (The structure of anisotropic and isotropic extensions, II). Let $1 \rightarrow \pi_1(N) \rightarrow \pi_1(M) \rightarrow C_p \rightarrow 1$ is a normal covering of rational homology spheres, represented by an element $z$ of order $p$ in $H_1(M)$, that is, the map $H_1(M) \rightarrow C_p$ is given by $(\cdot, z)_M$. Then the structure of $H_1(N)(p)$ as a $C_p$-module is as follows:

(a) if $(z, z) = 0$, then $H_1(N)(p)$ is a direct sum of open modules

(b) if $(z, z) \neq 0$, then $H_1(N)(p)$ is a direct sum of exactly one closed module and several open modules.

12. Non-existence of split anisotropic constituents

12.1. Let $G$ be a finite group acting on a manifold $N$ freely. Suppose $V \subset H_1(N)$ be a $G$-invariant subspace, and put $W = \frac{H_1(N)}{V}$. Let $Q \rightarrow N$ be a covering, defined by a map $\pi_1(N) \rightarrow H_1(N) \rightarrow W$. Then $Q \rightarrow M$ is a normal covering with a Galois group $\mathcal{P}$, which is an extension

$$0 \rightarrow W \rightarrow \mathcal{P} \rightarrow G \rightarrow 1.$$  

12.2. Now suppose $1 \rightarrow \pi_1(N) \rightarrow \pi_1(M) \rightarrow C_p \rightarrow 1$ is a covering of rational homology three-spheres, and $N$ is virtually non-Haken (in particular, has virtual first Betti number zero).

The main result of this section is as follows.

Theorem (12.2). (a) If $p \neq 2$, then $H_1(N)(p)$ does not have a split anisotropic constituent, that is, an invariant cyclic subgroup generated by an element $z$ with $\text{ord}(z, z)_N = \text{ord} z$.

(b) Suppose $p = 2$ and $H_1(N)(2)$ has such a subgroup $W$. Consider a $C_2$-invariant orthogonal splitting $H_1(N)(2) = W \oplus V$. Let $Q \rightarrow N$ be as in 12.1. Then the group $\mathcal{P}$ is a generalized quaternionic (binary dihedral) group, and $H^i(\mathcal{P}, H_1(Q)) = 0$ for $i \geq 1$. In particular, the action on $W$ is the multiplication by $(-1)$.

Proof. We will prove only the case (b), that is, $p = 2$, since the case of odd $p$ is completely similar (and easier).

Lemma (12.2). In condition of Theorem 12.2 (b) we have:

$$H^i(W, Q) = 0 \quad \text{for} \quad i \geq 1$$

$$H^0(W, Q) \approx V \quad \text{as} \quad C_2 \quad \text{module}.$$

Proof. This follows by induction from the Shrinking Theorem 7.2 and the argument of 3.1.
Lemma (12.3). $H^i(\mathcal{P}, H_1(Q))$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$ for all $i \geq 1$.

Proof. The Lyndon-Serre-Hochschild s.s. for the $\mathcal{P}$-module $H_1(Q)$ degenerates to $H^i(C_2, V)$ by the previous lemma. Since $V$ is a split component of $H_1(N)$, the statement follows from Proposition 2.1.

Now, we have exactly three possibilities for $\mathcal{P}$, except that of quaternionic group:

(i) $\mathcal{P} = C_2 \times C_{2^n}$

(ii) $\mathcal{P}$ is dihedral $D_{2^n}$

(iii) $\mathcal{P} = C_{2^n} \rtimes C_2$ with the $C_2$-action on $C_{2^n}$ given by multiplication by $\pm(2^{n-1}-1)$

In cases (i) and (ii) the cohomology $H^i(C_2, W) \approx \mathbb{Z}/2\mathbb{Z}$ and the proof of the Lemma 12.3 shows that $H^i(\mathcal{P}, H_1(Q)) = H^i(C_2, V) = 0$ for $i \geq 1$. Hence the s.s. of the group extension shows that $H^i(\mathcal{P}, \mathbb{Z})$ should be 4-periodic, which is not the case.

In case (iii) a theorem of Wall [W], see also [Th], shows that $H^{even}(\mathcal{P}, \mathbb{Z})$ is freely multiplicatively generated by two elements $\xi, \eta$ of degree 2 subject to relations $2\xi = 0$ and $2^n\eta = 0$.

Since the ranks of $H^{2k}(\mathcal{P}, \mathbb{Z})$ are not bounded, there should be a nontrivial $d_3 : (\text{coker} d_2)^{odd} \to H^{even}(\mathcal{P}, \mathbb{Z})$ in the s.s. of the covering $Q \to M$ so that its image contains some $0 \neq v \in H^{even}(\mathcal{P}, \mathbb{Z})$. But then the Proposition 1.4 implies that the ranks of $\text{coker} d_2$, hence of $H^*(\mathcal{P}, H^2(Q))$ are unbounded because $d_3(\text{coker} d_2)$ contains all $\zeta^i \cdot \eta^j \cdot v$, which is a contradiction to Lemma 12.3.

We are now ready to determine the pro-$p$ completion of $\pi_1(M)$ when $H_1(M)(p)$ is “small”.

Theorem (12.4). Let $H_1(M)(p)$ be cyclic. Then either $M$ is virtually non-Haken, or $M$ is a quotient of a 2-homology sphere by a cyclic group. The pro-$p$ completion $\mathfrak{S}$ of $\pi_1(M)$ is then finite cyclic.

Proof. Follows immediately from the Shrinking Theorem 7.2.

Theorem (12.5). Let $H_1(M)(2) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then:

(i) if the linking form in $H_1(M)(2)$ is $\langle 1 \rangle \oplus \langle 1 \rangle$ then is virtually Haken or $M$ is a quotient of a 2-homology sphere by a generalized quaternionic group $Q_{2^n}$, $n \geq 4$, and $\mathfrak{S} \approx Q_{2^n}$.

(ii) if the linking form in $H_1(M)(2)$ is hyperbolic $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then either $M$ is virtually Haken, or $M$ is a quotient of a 2-homology sphere by the quaternionic group $Q_8$, and $\mathfrak{S} \approx Q_8$.

Proof. Let $z \in H_1(M)$ be any isotropic element and consider a $C_2$-covering $N \to M$ defined by $(\cdot, z) : H_1(M) \to \mathbb{Z}/2\mathbb{Z}$. Then $Q = H_1(N)(2)$ is a $C_2$-module and we know that it is a sum of exactly one open module and some closed modules. Since $W^{C_2} \approx \mathbb{Z}/2\mathbb{Z}$, in fact $W$ is just one open module. Now, since the linking form in $W$ is $C_2$-invariant we have $W \approx \hat{W}$, which gives, along with the main result of [NR] and [Le], that $W$ is either of a form
Now, the condition $W^C_2 \approx \mathbb{Z}/2\mathbb{Z}$ implies that $W$ is actually cyclic with either trivial action ($L$-module) or $(-1)$-action ($R$-module). By the Theorem 12.4 we have $N = Q/W$ where $Q$ is a 2-homology sphere, and by the Theorem 12.2, $\mathfrak{G}$ is a (generalized) quaternionic group $Q_{2^n}$, $n \geq 3$.

Now, if the linking form is hyperbolic, then any element is isotropic, hence whatever homomorphism $\mathfrak{G} \to \mathbb{Z}/2\mathbb{Z}$ we consider, the kernel would be cyclic. This rules out the possibility $n \geq 4$, so $\mathfrak{G} \approx Q_8$. Conversely, if $n = 3$, there is a map of odd degree from $M$ to $S^3/Q_8$, which induces an isomorphism in 2-torsion of $H_1$. Since $S^3/Q_8$ has a selfhomeomorphism $\zeta$ of order 3 (the quotient of the normalizer $N(Q_8)$ by $Q_8$ in $SL_2(\mathbb{F}_5)$ has order 3), which does not have fixed nonzero elements in $H_1(S^3/Q_8)(2)$, the linking form of $S^3/Q_8$ is hyperbolic.

Q.E.D.

13. A SPECTRAL SEQUENCE IN GROUP COHOMOLOGY

In this section, we introduce a new powerful spectral sequence, converging to $\mathbb{F}_p$-cohomology of a normal subgroup of index $p$ in a given group $G$. There are “commutative analogs” of this s.s. in algebraic geometry, which appear to be well-known [Se1]¹.

Let $1 \to K \to G \overset{s}{\to} C_p \to 1$ be an exact sequence of groups.

**Theorem (13.1).** There is a spectral sequence with $E_1$-term

$$E_1^{i,j} = H^{i+j}(G, \mathbb{F}_p) \Rightarrow H^{i+j}(K, \mathbb{F}_p), \ i + j \geq 0, \ 0 \leq i \leq p - 1$$

The differential $d_1 : H^i(G, \mathbb{F}_p) \to H^{i+1}(G, \mathbb{F}_p)$ is a multiplication by the class $s$ viewed as an element of $H^1(G, C_p)$.

**Theorem (13.2).** Let $p = 2$. There is an exact sequence

$$\ldots H^i(G, \mathbb{F}_2) \overset{r}{\to} H^i(K, \mathbb{F}_2) \overset{t}{\to} H^i(G, \mathbb{F}_2) \overset{x^s}{\to} H^{i+1}(G, \mathbb{F}_2) \to \ldots$$

¹I am grateful to David Eisenbud for this reference
where \( r \) is the restriction and \( t \) is the transfer map.

**Proof.** We begin with a lemma.

**Lemma (13.3).** Consider the regular module \( \mathbb{Z}/p^N\mathbb{Z}[C_p] \) over \( C_p \). There is a canonical filtration

\[
0 = V_0 \subset V_1 \subset \ldots V_{N+1} = \mathbb{Z}/p^N\mathbb{Z}[C_p]
\]
of \( C_p \)-modules with all successive factors \( \mathbb{F}_p \).

**Proof.** Let \( \zeta \) be a fixed generator of \( C_p \). Consider the exact sequence

\[
0 \to \mathbb{Z}/p^N\mathbb{Z}[1+\zeta+\ldots+\zeta^{p-1}] \to \mathbb{Z}/p^N\mathbb{Z}[C_p]/(1+\zeta\ldots\zeta^{p-1}) \to 0
\]

Now, the latter module may be represented as \( (\mathbb{Z}[C_p]/(1+\zeta+\ldots+\zeta^{p-1}))/p^N\mathbb{Z}[C_p]/(1+\zeta+\ldots+\zeta^{p-1}) \). Let \( \mathcal{O} = \mathbb{Z}[C_p]/(1+\zeta+\ldots+\zeta^{p-1}) \) be the ring of cyclotomic integers.

Since \( p = \text{unit} \times (1-\zeta)^{p-1} \) we see that the module above is just \( \mathcal{O}/(1-\zeta)^{N(p-1)} \) and has canonical filtration by \( (1-\zeta)^l\mathcal{O}/(1-\zeta)^{N(p-1)} \).

Q.E.D.

**Proof of Theorem 13.1.** Since \( H^i(K, \mathbb{F}_p) = H^i(G, \mathbb{F}_p[G/K]) \) by Shapiro’s lemma, we may deal with the latter group. Now, we may rewrite it as \( \text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_p[G/K]) \).

The filtration of \( G/K \) modules \( 0 = V_0 \subset V_1 \subset \ldots \subset V_p = \mathbb{F}_p[G/K] \) becomes a filtration of \( G \)-modules, so the standard s.s. of filtrated modules \([CE]\) gives a s.s. with \( E_1 \)-term \( \text{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, V_i/V_{i+1}) \Rightarrow \text{Ext}^{i+j}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{F}_p[G/K]) \), as stated.

**Proof of Theorem 13.2.** This follows from the previous proof, or may be seen from the short exact sequence of \( G \)-modules

\[
0 \rightarrow \mathbb{F}_2 \xrightarrow{\times (1+\zeta)} \mathbb{F}_2[G/K] \xrightarrow{\text{aug}} \mathbb{F}_2 \rightarrow 0
\]

**Corollary 13.3.** Let \( G \) be a pro-\( p \) group and \( K \) a subgroup of index \( p \). Then there is a s.s. as in Theorem 12.1, converging to \( H^*(K, \mathbb{F}_p) \).

**Proof.** Inductive limit by the directed set of finite index subgroups.

**14. Strengthened Adem inequalities and other applications**

Recently Adem \([Ad]\) published several strong results, showing a possible range for finite \( p \)-group cohomology. All these results may be derived from our spectral sequence, as we will see soon. For \( p = 2 \) we show some simple consequences for the structure of the cohomology ring; these are related to a well-known theorem of Serre, see \([Se2]\), \([Se3]\).

**Theorem (14.1).** Let \( G \) be a finite \( p \)-group and let \( K < G \) be a subgroup of index \( p \). Then

\begin{align*}
(i) \quad b_i(K, \mathbb{F}_p) &\leq pb_i(G, \mathbb{F}_p), \quad i \geq 1 \\
(ii) \quad b_i(G, \mathbb{F}_p) &> 0 \text{ for all } i
\end{align*}
(iii) if \( p = 2 \) and \( G/[G, G] \) is elementary abelian, then \( b_1(K, \mathbb{F}_p) \leq p(b_1(G, \mathbb{F}_p) - 1) \)

(iv) let \( r_i(G) = \log_p |H^i(G, \mathbb{Z})| \). Then for \( i \) even,

\[
p \cdot r_i(G) \geq r_i(K) + 1,
\]

in particular \( r_i(G) > 0 \) for all even \( i \); for \( i \) odd,

\[
p \cdot r_i(G) \geq r_i(K) - 1.
\]

**Proof.** (i) is immediate from Theorem 13.1. (ii) follows from (i) by induction, since it is yielded for a cyclic group. For proving (iii), we first notice that \( C \neq 0 \), which kills \((p - 1)\) independent elements in \( G \). For \( p = 2 \), \( s \cdot s = s^2 = p(s) \neq 0 \), so \( d_1 : H^1 \to H^2 \) also kills \( s \) from the second \( H^1(G) \) in 13.2, hence \( b_1(K) \leq 2(b_1(G) - 1) \). To prove (iv), we will first show the idea, proving it for \( p = 2 \). Consider the truncated exact sequence

\[
C : 0 \to \mathbb{F}_p \to H^1(G) \to H^1(F) \to H^1(G) \to \ldots \to H^i(G) \to H^i(F) \xrightarrow{\psi} \text{Im} \psi \subset H^i(G) \to 0
\]

We have:

\[
0 = \chi(C) \geq 1 - (b_1(G) - b_2(G) + \ldots + (-1)^{i+1}b_i(G)) + b_1(F) + \ldots + (-1)^{i+1}b_i(F)
\]

for \( i \) even and the opposite sign for \( i \) odd.

Now, because of the short coefficient sequence \( 0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{F}_p \to 0 \) we have \( b_i = r_i + r_{i+1} \) and \( r_1 = 0 \), which gives (iv) for \( p = 2 \).

Recall the result of Serre [Se2], [Se3]: if \( G \) is not elementary abelian then there are nonzero elements \( x_1 \ldots x_n \in H^1(G) \), such that \( \beta x_1 \ldots \beta x_n = 0 \). For \( p = 2 \) we have the following

**Theorem (14.2).** Suppose \( G \) is not elementary abelian and let \( r \) be the elementary abelian rank of \( G \). Then

(i) The commutative \( \mathbb{F}_2 \)-algebra \( H^*(G, \mathbb{F}_2) \) has divisors of zero

(ii) Moreover, for \( s \in H^1(G) \) let \( k_i = \dim \text{ker}(H^i(G) \xrightarrow{x_s} H^{i+1}(G)) \). Then there exists \( s \in H^1(G) \) such that

\[
k_i + k_{i+1} \geq \text{const.} \cdot r^i, \quad \text{for} \quad i \geq 1.
\]

**Proof.** Let \( A \subset G \) be a maximal elementary abelian group. Choose \( s : G \to \mathbb{Z}/2\mathbb{Z} \) such that \( A \subset \ker s \) and put \( K = \ker s \). Then \( r(k) = r(G) = r \). From Theorem 13.2 we have

\[
b_i(K) = k_i + b_i(G) - (b_{i-1}(G) - k_{i-1}) = k_i + k_{i-1} + (b_i(G) - b_{i-1}(G))
\]

Now, from the main result of Quillen [Q] it follows that \( b_i(K) \sim \text{const.} \cdot r^i \) and \( b_i(G) - b_{i-1}(G) \sim \text{const.} \cdot r^{i-1} \), which proves (ii), hence (i).
15. Multiplication in cohomology: \( \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) case

In this chapter we start a deeper analysis of homology of coverings for manifolds with small \( H_1 \). The main result of this chapter is a next theorem, identifying the multiplication in cohomology in case \( H_1(M)(2) = (\mathbb{Z}/2)^3 \). Observe that in \((\mathbb{Z}/2)^3\) any nondegenerate linking form is diagonalizable.

**Theorem 15.4.** Let \( M \) be a non-virtually Haken manifold with \( H_1(M)(2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). Then there exists a basis \( x, y, z \) in \( H^1(M, \mathbb{F}_2) \) such that either \( xy = yz = xz = 0 \) or \( x^2 = yz, y^2 = xz, z^2 = xy \).

**Proof.** Let \( x, y, z \) be an orthonormal base for \((\cdot, \cdot)_M\). We always identify \( v \in H_1(M)(2) \) and \((\cdot, v) \in H^1(M, \mathbb{F}_2)\). Let \( w = x + y \), then \( w \) is isotropic. Let \( N \to M \) be a \( C_2 \)-covering, defined by \( w \). Let \( W = H_1(N)(2) \). Then \( W \) is a sum of one open module and several closed modules by the Theorem 11.4, and moreover \( W_{C_2} \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). That means that either we have one open module, or a direct sum of an open module and a closed module.

**Case A.** \( W \) is an open module, hence of the type

by the argument of 12.5. The first case is definitely impossible, since then \( W_{C_2} \) would have direct summand \( \mathbb{Z}/2^n \), \( n \geq 2 \). (Recall that \( W_{C_2} \approx W_{C_2} \) since \( W \approx W \)). In the second case we see immediately that the only possibility is

and as an abelian group \( W \approx \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \).

**Lemma (15.5).** \( z^2 \neq w \cdot v \) in \( H^2(M, \mathbb{F}_2) \).

**Proof.** Suppose \( z^2 = (x + y)(ax + by + cz) \). Multiplying by \( x \) and accounting the identity \( p^2 \cdot q = p \cdot q^2 = (p, q)_M \) we have \( 0 = xz^2 = a + cxyz \). Multiplying by
Let $x = W$ by $x$ is zero, or $yz$. So the only possibilities are $xy$. We can assume that $\ker H$ is one-dimensional, say $\alpha x + \beta y + \gamma z$. We have $0 = (x + y)(\alpha x + \beta y + \gamma z)$. Arguing as above, we find $0 = \alpha + \gamma xyz; 0 = \beta + \alpha xyz$, so $\alpha = \beta$. If $\gamma = 0$ that gives $\alpha = \beta = 0$, which is impossible, so $\gamma = 1$ and $\alpha = \beta = xyz$.

Case $B_1$. $xyz = \alpha = \beta = 1$, so that $x^2 + y^2 = xz + yz$. Then replacing $w$ by $x + z$ or $y + z$ and accounting that $xyz = 1$ we find also $x^2 + z^2 = xy + yz, y^2 + z^2 = xy + xz$.

Now, we look at the anisotropic covering $N \to M$, defined by $x$. Again we put $W = H_1(N)_2$; it is a sum of closed modules. What is a kernel of $H^1 \to H^2$? Let $x(ax + by + cz) = 0$, then we find as above $a = 0$. Now, $x(y + z) = x^2 + y^2$. So the only possibilities are $xy = 0$ or $xz = 0$. Let us say, the first. Replacing $x$ by $z$ and repeating the procedure, we find that either the kernel $H^1 \to \ker H^2$ is zero, or $yz = 0$, or $xz = 0$. But that contradicts the identities above. So we can assume that $\ker H^1 \to \ker H^2$ is zero and $\ker H^1 \to \ker H^2$ is zero. Now, we have $xz = \alpha x^2 + \beta y^2 + \gamma z^2$ for some $\alpha, \beta, \gamma$. Equalities $xz = \alpha x^2$ and $xz = \gamma z^2$ would give $x(z - \alpha x) = 0$, resp. $z(x - \gamma z) = 0$, which is impossible by the above. Moreover,
$xz = x^2 + z^2$ would give $xy = x^2 + y^2$ and $y^2 = y^2 + z^2$. Otherwise, $\beta = 1$, so $xz = \alpha x^2 + y^2 + \gamma z^2$. If $\alpha = \gamma = 1$ we have $xy = y^2 + z^2 - (x^2 + y^2 + z^2) = x^2$, a contradiction. Suppose $\alpha = 0, \gamma = 1$, so that $xz = y^2 + z^2$, then $xy = 0$, a contradiction. Likewise $\alpha = 1, \gamma = 0$ is impossible. Hence $\alpha = \gamma = 0$ and $xz = y^2, xy = z^2, yz = x^2$.

We claim that the case $xz = x^2 + z^2$ above is impossible. Indeed, in that case there would be a map $\pi_1(M) \to Q_8$, such that the induced map in $H^1(\cdot, \mathbb{F}_2)$ would send $H^1(Q_8, \mathbb{F}_2)$ on the subspace spanned by $x, y$ (see lemma 15.5 below). But this contradicts $x^3 = 1$.

So we conclude that $x^2 = yz, y^2 = xz, z^2 = xy$. Moreover, since $(x + y)^2 + (x + z)^2 + (x + y)(x + z) = 0$, we have a map $M \to S^2/Q_8$, inducing a homomorphism $H^1(Q_8, \mathbb{F}_2) \to H^1(M, \mathbb{F}_2)$ with $x + y$ and $x + z$ in the image. This map has odd degree, since $(x + y)^2 \cdot (x + z) = 1$. In particular, $W_1$ above is $\mathbb{Z}/4\mathbb{Z}$.

Now, the exact sequence of the Theorem 13.2 gives $b_1(N) = 2$, so $W$ as an abelian group has two generators. We have two cases again

**Case $B_{11}$.** $W$ is a sum of two closed modules, $W = W_1 \oplus W_2$ and both $W_i$ are cyclic groups necessarily of degree $\geq 8$. If their order is different, then the one with a bigger order is anisotropic, which is impossible by the Theorem 12.2 (b). So $|W_1| = |W_2| \geq 8$. Moreover, if $W_1$ is not anisotropic, then $W_1 \approx \hat{W}_2$ and the action in $W_1$ and $W_2$ is the same, that is multiplication by either $2^{n+1} + 1$, or $2^{n-1} - 1$ (where $2^n = |W_i|$). So the action in $W$ is multiplication by $2^{n-1} \pm 1$. Now, we can choose $W_i$ such that $W_1$ projects on $y$ and $W_2$ projects on $z$. By the argument of , we have a map $\pi_1(M) \to \Phi$, where $\Phi$ is a semidirect product $W_1 \rtimes \mathbb{Z}/2\mathbb{Z}$, which induces the map $H_1(M) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, sending $z$ to zero.

**Lemma.** $b_2(\Phi) = \dim H^2(\Phi, \mathbb{F}_2) = 2$.

**Proof.** We first notice that $H^3(\Phi, \mathbb{Z}) = 0$, because $H^i(\mathbb{Z}/2\mathbb{Z}, W_1) = 0$ and hence $H^i(\mathbb{Z}/2\mathbb{Z}, H^2(W_1)) = 0$ for $i \geq 1$. Now from the exact coefficient system $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ we get $b_2(\Phi) = \text{rank } H^2(\Phi, \mathbb{Z}) = \text{rank } H_1(\Phi, \mathbb{Z}) = 2$.

By the lemma, there should be a relation between $x$ and $y$ in $H^2(M, \mathbb{F}_2)$. However, $xy = z^2$ and $x^2, y^2, z^2$ are independent, a contradiction. So the case $B_{11}$ is impossible.

**Case $B_{12}$.** $W$ is one closed module. Since $W_{C^2} \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $W$ should of a form

but this has more than two generators as an abelian group, a contradiction.

**Case $B_2$.** $xyz = 0$ and $(x + y)z = 0$. Then similarly $(xz)y = 0$ so $xz = yz = xy$. But $(xz)y = (yz)x = (xy)z = 0$, so the element $xz = yz = xy$ should be zero by Poincaré duality. This proves the theorem.
**Corollary.** There exists homomorphisms $\pi_1(M) \to \mathbb{Q}_{2^n}$, $n \geq 4$, such that the induced map $H_1(M) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is given by $((x,\cdot), (y,\cdot))$, respectively $((x,\cdot), (z,\cdot))$ and $((y,\cdot), (z,\cdot))$.

**Proof.** We know that the covering $N$, defined by $(x+y,\cdot)$ has $H_1$ as in the diagram

So, according to 12.1, we have a map $\pi_1(M) \to \mathbb{Q}_{2^n}$, where $n \geq 3$. The image of $H_1(N)_{(2)}$ in $H_1(M)_{(2)}$ is $\ker H_1(M) \xrightarrow{(x+y,\cdot)} \mathbb{Z}/2\mathbb{Z}$, which is generated by $x+y$ and $z$. By discussion of , $u$ projects to $x+y$, so $v$ and $w$ project either to $z$, or to $x+y+z$. Now, the induced map $H_1(M) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ which we want to understand is just the quotient by the subspace generated by the image of $v$ or $w$. Suppose $v$ and $w$ project to $x+y+z$, then the map $H_1(M) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is given by $((x+y,\cdot), (x+z,\cdot))$. If $n = 3$ then by the lemma below we should have $(x+y)^2 + (x+z)^2 + (x+y)(x+z) = 0$, however, this expression equals $x^2 + y^2 + z^2$ by the previous Theorem. So $n = 4$, but then we should have $(x+y)(x+z) = 0$ or $((x+y)(y+z) = 0)$. But $(x+y)(x+z) = x^2 \neq 0$, so this is also impossible. Hence $u$ and $v$ project to $z$ and the map $H_1(M) \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ indeed looks like $((x,\cdot), (y,\cdot))$. Since $x^2 + y^2 + z^2 = 0$, we see that $n \geq 4$. This proves the Corollary.

**Lemma (15.5).** (a) For any generators $x, y$ of $H^1(Q_8, \mathbb{F}_2)$ we have $x^2 + y^2 + xy = 0$. (b) There are generators $x, y$ of $H^1(Q_{2^n}, \mathbb{F}_2)$, $n \geq 4$, such that $xy = 0$.

**Proof.** Consider the extension $0 \to \mathbb{Z}/2\mathbb{Z} \to Q_8 \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0$. There is a $\mathbb{Z}/3\mathbb{Z}$ -action in this exact sequence, so the class of extensions should be $\mathbb{Z}/3\mathbb{Z}$ -invariant. But the only invariant element in $H^2(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{F}_2)$ is $x^2 + y^2 + xy$, as mentioned in , which proves (a). To prove (b) we notice that there is a map $Q_{2^n} \to D_{2^{n-1}}$, which induces isomorphisms in $H^1$. Similarly, $D_{2^{n-1}}$ maps on $D_8$. Now, the extension class of $0 \to \mathbb{Z}/2\mathbb{Z} \to D_8 \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0$ is $xy$ in some basis [B]. This proves (b).

16. **Homology of coverings for $H_1(M) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$**

16.1. In this chapter we do actual computation of the homology of all $C_2$ -coverings in the case $H_1(M)_{(2)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, the smallest nontrivial case. This computation is based heavily on the results of the previous chapter. Here is our main result.

**Theorem (16.1).** Suppose $H_1(M)_{(2)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $M$ is not virtually Haken. Then the 2-torsions in three $C_2$ -coverings of $M$ are $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n+1\mathbb{Z}$, $n \geq 1$.

**Proof.** By assumption the first homology of $M$ looks like
Now, let \( U = (2u, \cdot) \) and \( V = (v, \cdot) \) be a natural basis in \( H^1(M, \mathbb{F}_2) \). Then \( U^2 = 0 \). Since \( UV \cdot U = 0 \) and \( UV \cdot V = UV^2 = U^2V = 0 \) we see that \( U \cdot V = 0 \) by Poincaré duality. On the other hand, \( V^2 = \beta V \neq 0 \), and therefore \( V^3 \neq 0 \) again by Poincaré duality. So the ring \( H^*(M, \mathbb{F}_2) \) looks like \( U^2 = 0, \ UV = 0, \ V^3 = 1 \).

Let \( N \to M \) be a \( C_2 \)-covering, defined by \( (2u, \cdot) = U \). By the Shrinking Theorem, \( W = H_1(N)_{(2)} \) looks like \( \mathbb{Z}/2\mathbb{Z} \oplus S \). Moreover, \( S \) should be a closed module with \( S^{C_2} \approx \mathbb{Z}/2\mathbb{Z} \). Hence \( S \) looks like

Now if \( n > 2 \), then \( S \) is a cyclic group \( \mathbb{Z}/2^n\mathbb{Z} \) with the action by multiplication by \( \pm(2^{n-1} - 1) \), which is impossible by the Theorem 12.2 (6). So \( n = 2 \) and \( W \) looks like

where \( x \) is anisotropic, but \( p \) and \( q \) may be isotropic. However, since \( \zeta p = q \) we claim that \( p+q \) is isotropic, because \( (p+q, p+q) = (p, p) + (q, q) = 2(p, p) = 0 \). We now consider the covering \( Q \to N \) defined by \( (p+q) \). The action corresponding to this covering in \( R = H_1(Q)_{(2)} \) will be called \( \eta \), and we keep the notation \( \zeta \) for the induced action in \( R \). So, as a \( \eta \)-module, \( R \) looks like

by the proof of Theorem 15.4. We wish now to understand the possibilities for \( \zeta \)-action.

*Case A.* \( t \) may be chosen so that the cyclic group \( T \) generated by it is \( \zeta \)-invariant. By the Theorem 12.2 (b) the action of \( \zeta \) in \( T \) must be multiplication by \( (-1) \). But
then the free involution $\zeta \eta$ will act trivially on $T$ which is impossible by the same
Theorem. So the case $A$ is impossible.

Case $B$. Since $t$ projects (to $W$) to $p+q$, which is $\zeta$-invariant, we must have $\zeta t = (2m+1)t+$ (something that projects to zero). In other words, $\zeta t = (2l+1)t+(r+s)$. Now, for $\zeta r$ we have four possibilities:

(i) $\zeta r = r$

(ii) $\zeta r = r + 2^{n-1}t$

(iii) $\zeta r = s$

(iv) $\zeta r = s + 2^{n-1}t$

Since $\zeta$ and $\eta$ commute, we have $\zeta s = s$ in case (i). But then the group generated by $r$ and $s$ is $\zeta$-invariant. Since the action of $\zeta$ is orthogonal, its complement, that is $T$, is also $\zeta$-invariant and we are brought back to the case A. This shows (i) is impossible. (iii) is reduced to $i$ by changing $\zeta$ to $\zeta \eta$. Likewise (ii) is equivalent to (iv), so we will only study (iv). In this case we have $\zeta r = s + 2^{n-1}t$, $\zeta s = r + 2^{n-1}t$, $\zeta t = (2l+1)t+(r+s)$. It follows that $\text{Ker} (1-\zeta)$ is generated by $r+s$ and $2t$ in case $2l+1 = 1$, or $r+s$ and $2^{n-1}t$ in case $2l+1 = -1$, or $r+s$ and $t+r$ in case $2l+1 = 2^{n-1}+1$ or $r+s$ and $2^{n-1}t$ in case $2l+1 = 2^{n-1}-1$. In all cases $(1+\zeta)r = r+s + 2^{n-1}t$ and $(1+\zeta)t = (2l+2)t+(r+s)$. If $2l+2 \neq 0$, $2^{n-1}$, then $(1+\zeta)r = r+s + 2^{n-1}t$ and $(1+\zeta)t = (2l+2)t+(r+s)$. If $2l+2 = 0$, then $(1+\zeta)r = r+s + 2^{n-1}t$ and $(1+\zeta)t = (2l+2)t+(r+s)$. If $2l+2 = 1$, then $(1+\zeta)r = r+s + 2^{n-1}t$ and $(1+\zeta)t = (2l+2)t+(r+s)$.

The final case $2l+1 = 2^{n-1} \pm 1$ will be studied later.

So in all cases we had so far, $H^1(\mathbb{Z}/2\mathbb{Z}, H_1(Q)) = 0$ for the $\zeta$-action. We can replace the chain of coverings $Q \xrightarrow{\eta} N \xrightarrow{\zeta} M$ by $Q \xrightarrow{\zeta} G \xrightarrow{\eta} M$ and then see that we have the situation of case $B$ of 3.1 which has been shown to be impossible in 3.5.

Now consider the possibility

$$\zeta t = (2^{n-1} - 1)t + (r + s)$$
$$\zeta r = r + 2^{n-1}t$$
$$\zeta s = r + 2^{n-1}t$$

Relabeling $\zeta \eta$ by $\zeta$ we have an action

$$\zeta t = (2^{n-1} + 1)t + (r + s)$$
$$\zeta r = r + 2^{n-1}t$$
$$\zeta s = s + 2^{n-1}t$$

Now, the kernel $\text{Ker} (1-\zeta)$ is generated by $r+2^{n-2}t$ and $s+2^{n-2}t$ subject the relation $2((r+2^{n-2}t) + (s+2^{n-2}t)) = 0$, so $\text{Ker} (1-\zeta) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Moreover,
$Im(1 + \zeta)$ is generated by $2^{n-1}t$ and $(r + s)$, so $Im(1 + \zeta) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so $H^1(\mathbb{Z}/2\mathbb{Z}, H_1(Q)) = \mathbb{Z}/2\mathbb{Z}$, which means that $Q \to G$ is isotropic. Moreover, since $H_1(Q)C_2 \approx H_1(Q)^C_2 \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, we see that $H_1(G)(2)$ contains $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ as an index 2 subgroup. Hence there are three possibilities for $H_1(G)(2)$:

Case I . $H_1(G)(2) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

Case II . $H_1(G)(2) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

Case III . $H_1(G)(2) \approx \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

On the other hand, $G \eta \rightarrow M$ is given by anisotropic element $v$, so $H_1(G)(2)$ under the $\eta$ -action should be a direct sum of closed modules and $(H_1(G)(2))C_2$ should be $\mathbb{Z}/4\mathbb{Z}$, so we actually have just one closed module, which must look like

It follows that only case I can be realized, and $m = 3$, so our module is

which is just $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ with permutation action. The remaining case

$$\zeta t = (2^{n-1} + 1)t + (r + s)$$
$$\zeta r = s + 2^{n-1}t$$
$$\zeta s = r + 2^{n-1}t$$

is dealt in the same manner. In fact, we determined homology of all $C_2$ -coverings of $M$, these are: $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for the unique isotropic extension and $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^{n+1}\mathbb{Z}$ for two anisotropic extensions.

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