Parabolic Bundles,
Products of Conjugacy Classes,
and Quantum Cohomology

C. Teleman
C. Woodward

Vienna, Preprint ESI 995 (2001)

Supported by Federal Ministry of Science and Transport, Austria
Available via anonymous ftp from FTP.ESI.AC.AT
or via WWW, URL: http://www.esi.ac.at
PARABOLIC BUNDLES, PRODUCTS OF CONJUGACY CLASSES, AND QUANTUM COHOMOLOGY

C. TELEMAN AND C. WOODWARD

Abstract. We prove a condition for the existence of flat bundles on the punctured two-sphere with prescribed holonomies around the punctures, involving Gromov-Witten invariants of generalized flag varieties. This generalizes the case of special unitary connections described by Agnihotri and the second author [1] and Belkale [5].

1. Introduction

Let $x_1, \ldots, x_b$ be distinct points on a Riemann surface $X$, and let $G$ be a connected complex reductive group. A parabolic $G$-bundle is a holomorphic principal $G$-bundle on $X$ with additional structure at each $x_i$, given by a reduction of the structure group to a parabolic subgroup, together with an element of the positive Weyl chamber, which we call a marking. Just as for bundles without parabolic structure, in order to obtain a good moduli space one has to restrict attention to bundles satisfying a semistability condition. The construction of the moduli space is carried out by Bhosle-Ramanathan [7], who also generalize the Narasimhan-Seshadri theorem [26], at least for markings sufficiently close to 0. They construct a bijection with the moduli space of flat bundles on the punctured surface $X \setminus \{x_1, \ldots, x_b\}$ with fixed holonomy around the punctures $x_i$ and structure group a maximal compact subgroup $K$ of $G$. Equivalence classes of semistable parabolic bundles may also be identified with isomorphism classes of flat bundles on surfaces with boundary, with fixed holonomy around the boundary components. In some sense, this explains why these spaces play an important role in the study of invariants of moduli spaces of bundles without parabolic structure. For invariants such as the Hilbert polynomial or the symplectic volume, one has sewing rules, suggested by quantum field theories and later proved rigorously, which express the invariants as a sum over intermediate markings of the moduli spaces of parabolic bundles.

Most of this paper is taken up with extension of the theory of parabolic vector bundles to the setting of principal bundles. First, we give an approach to the Bhosle-Ramanathan correspondence via equivariant bundles, the Atiyah-Bott canonical reduction, and results on the Yang-Mills flow of Donaldson, Daskalopolous, and Råde. It has the advantage of working for arbitrary rational markings, while the Bhosle-Ramanathan results depend

Date: Revised February 16, 2001.
MSC 53D30, 14N35.
on the existence of a representation for which the markings are mapped to parabolic weights in the sense of Seshadri.\footnote{The proof given there seems to have an error in the proof of Lemma 1.4 of \cite{7}. Namely, if $\rho : G \to \text{Gl}(V)$ is a representation, and $\tau$ the marking, then $\rho(\tau) = (\rho(\tau)_1, \ldots, \rho(\tau)_n)$ is not a marking in the sense of Mehta-Seshadri since in general $\rho(\tau)_1 - \rho(\tau)_n$ is not less than one. As a result, the proof of Proposition 2.1, etc., holds under the assumption that $\rho(\tau)_1 - \rho(\tau)_n < 1$, that is, that $\rho(\tau)$ lies in the fundamental alcove for $\text{Gl}(V)$.}

At the end of the paper we apply these results to our motivating problem, which is to determine which of the moduli spaces are non-empty in the genus zero case, generalizing the results of Agnihotri and the second author [1] and Belkale [5] for vector bundles. This is equivalent to determining the decomposition of a product of conjugacy classes in the compact group or the decomposition of a fusion product of level $nk$ (projective, positive energy) representations

$$V_{n\mu_1} \otimes_{nk} \cdots \otimes_{nk} V_{n\mu_b},$$

of the loop group in the limit $n \to \infty$. From now on, we take $K$ to be simple and simply connected. Let $T \subset K$ be a maximal torus, $t_+ \subset t$ a positive Weyl chamber, and $\alpha_0$ the highest root. The set of conjugacy classes in $K$ is parametrized by the fundamental alcove

$$\mathfrak{A} = \{ \xi \in t_+, \; \alpha_0(\xi) \leq 1 \}.$$

That is, any conjugacy class has a unique representative of the form $\exp(\mu)$; we denote the conjugacy class by $C_{\mu}$. For any integer $b > 0$, define

$$\Delta(b) = \{ (\mu_1, \ldots, \mu_b) \in \mathfrak{A}^b \mid C_{\mu_1} \cdots C_{\mu_b} \supset e \}$$

where $e$ is the group unit. By the correspondence theorem, the moduli space of semi-stable parabolic bundles on the projective line with markings $1, \ldots, b$ is non-empty if and only if $(\mu_1, \ldots, \mu_b) \in \Delta(b)$. According to the convexity theorem for Hamiltonian actions of loop groups proved by Meinrenken and the second author [24], $\Delta(b)$ is a convex polytope of maximal dimension in $\mathfrak{A}^b$.

Our description of the inequalities is in terms of quantum cohomology of the generalized flag varieties $G/P$, where $P$ is a maximal parabolic subgroup. Let $W_P \subset W$ the subgroup of $W$ generated by simple reflections of roots of the Levi subgroup of $P$. Let $B$ be a Borel subgroup. For any $w \in W/W_P$, the Schubert variety

$$Y_w = BwP \subset G/P$$

is a normal subvariety of $G/P$. The homology classes $[Y_w]$ form a basis for $H_*(G/P)$. The quantum product is an associative, commutative product on the vector space

$$QH_*(G/P, \mathbb{C}) := H_*(G/P, \mathbb{C}) \otimes \mathbb{C}[q].$$

Fix $X = \mathbb{P}^1$ and choose general points $x_1, \ldots, x_b \in \mathbb{P}^1$. Define

$$[Y_{w_1}] \ast \cdots \ast [Y_{w_{b-1}}] = \sum_{d \in \mathbb{N}, [w_b] \in W/W_P} n_d(w_1, \ldots, w_b)q^d [Y_{w_b}^\vee].$$
where \( ^\lor \) denotes Poincaré dual and the structure coefficients (Gromov-Witten invariants) \( n_d(w_1, \ldots, w_b) \) are defined as follows. The degree of a holomorphic map \( \varphi : X \to G/P \) is the homology class of its image \( [\varphi(X)] \in H_2(G/P) \cong \mathbb{Z} \). Let \( g_i Y_{w_i}, i = 1, \ldots, b \) be general translates of the Schubert varieties \( Y_{w_i} \). Define \( n_d(w_1, \ldots, w_b) \) to be the number of holomorphic maps \( \varphi : X \to G/P \) of degree \( d \) such that \( \varphi(x_i) \in g_i Y_{w_i} \), if this number is finite, and zero otherwise. The resulting product is commutative, associative, independent of the choice of general \( x_i \) and \( g_i \). It would be interesting to understand better the Gromov-Witten invariants \( n_d(w_1, \ldots, w_b) \) for \( G/P \). Some results have been announced by Peterson [27].

For any maximal parabolic subgroup \( P \), let \( \omega_P \) denote the corresponding fundamental weight. Our main result is

**Theorem 1.1.** The polytope \( \Delta(b) \) is the set of points \( (\mu_1, \ldots, \mu_b) \in \mathbb{R}^b \) satisfying

\[
\sum_{i=1}^{b} \omega_P(w_i \mu_i) \leq d
\]

for all maximal parabolic subgroups \( P \subset G \) and all \( [w_1], \ldots, [w_b] \in W/W_P \) and non-negative integers \( d \) such that the Gromov-Witten invariant \( n_d(w_1, \ldots, w_b) = 1 \).

A number of special cases of this result were already known. In the case \( K = SU(n) \), this result was proved by Agnihotri-Woodward [1] for \( n_d > 1 \); and independently in Belkale [5], with the improvement to \( n_d = 1 \). A purely symplectic approach to the problem is developed in Entov [15]. For conjugacy classes near the identity, the problem is equivalent to one involving sums of coadjoint orbits studied in Berenstein-Sjamaar [6] and Leeb-Millson [25].

An interesting outstanding question is to determine which inequalities are independent.

The organization of the paper is as follows. In Section 2 we show an equivalence between parabolic and equivariant bundles. The same strategy was used to prove the correspondence theorem for parabolic vector bundles by Furuta-Steer [17] and Boden [9]. We could have been slightly more general in this section and dealt with bundles over orbifolds, but this seemed unnecessary since all the orbifolds we need can be constructed as global quotients. In Section 3 we review the construction of the moduli space of flat \( K \)-bundles, with fixed holonomy around the marked points. In Section 4, we prove (or rather, cite results of Donaldson, Daskalopolous, and Råde which prove) a correspondence theorem between the two moduli spaces. In Section 5, we restrict to bundles over \( \mathbb{P}^1 \).

After finishing the first version of this paper, we received the preprint [3] of Balaji, Biswas, and Nagaraj which constructs the moduli space of parabolic principal bundles and proves a correspondence theorem for smooth projective varieties of arbitrary dimension. However, their definition of parabolic bundle is different from ours, and we need results on the canonical reduction not discussed there.
2. PARABOLIC $G$-BUNDLES

2.0.1. Principal bundles. Let $X$ be a complex manifold. A principal $G$-bundle over $X$ is a complex manifold $E \to X$ with a right action of $G$ that is locally trivial. That is, any point in $X$ is contained in a neighborhood $U$ such that $E|_U$ is $G$-equivariantly biholomorphic to $U \times G$. By a scheme we mean a separated scheme of finite type over $\mathbb{C}$. For $X$ a scheme, principal $G$-bundles over $X$ are required to be locally trivial in the étale topology. For $X$ a curve, local triviality holds even in the Zariski topology.

2.0.2. Parabolic structures. Let $X$ be a curve with distinct marked points $x_1, \ldots, x_b$, and $E \to X$ a principal $G$-bundle.

Definition 2.1. A parabolic structure at $x_i$ consists of the following data.

(a) a standard parabolic subgroup $P_i \subset G$;
(b) a reduction $\varphi_i \in E_{x_i}/P_i$ of the fiber to $P_i$;
(c) a marking $\mu_i \in \mathfrak{A}$ with $\alpha_0(\mu_i) < 1$, whose stabilizer $G_{\mu_i}$ under the adjoint action is a Levi subgroup of $P_i$.

We note that the Lie algebra $\mathfrak{p}_i$ of $P_i$ is related to $\mu_i$ by

$$\mathfrak{p}_i = \mathfrak{t}_C + \sum_{(\alpha, \mu_i) \geq 0} \mathfrak{g}_\alpha.$$  

A parabolic bundle on $(X, x_1, \ldots, x_b)$ is a bundle $E$ with parabolic structures at $x_1, \ldots, x_b$.

The definition of parabolic bundle given above generalizes the following definition of Mehta and Seshadri for vector bundles $V \to X$. A parabolic structure at $x_i$ is a partial flag

$$V_{x_i}^1 \subset V_{x_i}^2 \subset \cdots \subset V_{x_i}^{l_i} = V_{x_i}$$

together with a set of markings

$$\mu_{i,1} \geq \mu_{i,2} \geq \cdots \geq \mu_{i,l_i}$$

such that $\mu_{i,1} - \mu_{i,l_i} < 1$.

2.0.3. Semistability. We begin by recalling the semistability condition for parabolic vector bundles given by Mehta and Seshadri. Assume, for simplicity, that the parabolic structure are regular, so that $l_i = n$ for all $j$. The parabolic degree of $V$ is

$$\text{pardeg}(V) := \deg(V) + \sum_{i=1}^{b} \sum_{j=1}^{n} \mu_{i,j}.$$  

The parabolic slope of $V$ is $\mu(V) := \deg(V)/\text{rk}(V)$.

The parabolic structure on $V$ induces a parabolic structure on any subbundle $U$ as follows. Define a flag in $U_{x_i}$ by removing repeating terms from $U_{x_i} \cap V_{x_i}^{j}$. Define parabolic weights $\nu_{i,j}$ by $\nu_{i,j} = \mu_{i,k}$ where $k$ is the largest integer such that $U_{x_i}^{j} \subset V_{x_i}^{k}$. $V$ is semistable if and only if

$$\mu(U) \leq \mu(V).$$  

for all subbundles \( U \subset V \).

Ramanathan’s definition of semistability for \( G \)-bundles is also defined in terms of a slope condition. Let \( P \subset G \) be a parabolic subgroup. A reduction of the structure group of \( E \) to \( P \) is a map

\[
\sigma : X \to E/P.
\]

Let \( \Lambda_P^* \) denote the set of weights corresponding to characters of \( P \). For any \( \lambda \in \Lambda_P^* \), we denote by \( \sigma^*E(\lambda) \) the corresponding line bundle over \( X \). Its degree \( \deg(\sigma^*E(\lambda)) \) is an element of \( H^2(X) \cong \mathbb{Z} \). \( E \) is semistable if

\[
\deg(\sigma^*E(\lambda)) \leq 0
\]

for all \( \lambda \in \Lambda_{P,+}^* \), where \( \Lambda_{P,+}^* \subset \Lambda_P^* \) is the subset of dominant weights. It suffices to check the condition for maximal parabolics \( P \) and fundamental weights \( \lambda = \omega_P \).

The definition of semistability for parabolic \( G \)-bundles involves the relative position of the reduction \( \sigma \) and the parabolic reductions \( \varphi_i \). Given two parabolic subgroups \( P_1' = \text{Ad}(g_1)P_1, P_2' = \text{Ad}(g_2)P_2 \subset G \), where \( P_1 \) and \( P_2 \) are standard parabolics, define their relative position

\[
(P_1', P_2') \in W_{P_1} \backslash W/W_{P_2}
\]

to be the image of \((g_1, g_2)\) under the map

\[
G \times G \to G \backslash (G \times G)/P_1 \times P_2 \cong P_1 \backslash G/P_2 \cong W_{P_1} \backslash W/W_{P_2}.
\]

Note that

\[
(P_2', P_1') = (P_1', P_2')^{-1}, \quad (P', P') = [e].
\]

**Definition 2.2.** A parabolic bundle \( E \) is stable (resp. semistable) if for any maximal parabolic subgroup \( P \) and reduction \( \sigma : X \to E/P \) we have

\[
\text{deg}(\sigma^*E(\omega_P)) + \sum_{i=1}^b \omega_P(w_i \mu_i) < 0 \quad (\text{resp. } \leq 0)
\]

where \( w_i = (\varphi_i, \sigma(x_i)). \)

We call the left-hand-side of (2) the parabolic degree \( \text{pardeg}(\sigma) \) of \( \sigma \). It is straightforward to check that for \( G = \text{Gl}(n, \mathbb{C}) \), the definition reduces to that of Mehta-Seshadri.

2.1. **From equivariant bundles to parabolic bundles.** Let \( \Gamma \) denote a cyclic group of order \( N \) acting on a curve \( \tilde{X} \), with quotient \( X = \Gamma \backslash \tilde{X} \) such that the projection \( \pi : \tilde{X} \to X \) is totally ramified with ramification points \( x_1, \ldots, x_b \). It was first noted by Mehta and Seshadri that a parabolic vector bundle on \( X \) is equivalent to a \( \Gamma \)-equivariant bundle on \( \tilde{X} \). In this section we generalize this correspondence to other structure groups.

The correspondence is conveniently formulated as an isomorphism of moduli functors. Fix \( \mu_1, \ldots, \mu_b \in \mathfrak{A} \) with \( \exp(N\mu_i) \) equal to the identity for \( i = 1, \ldots, b \). Let \( \text{Bun}_\Gamma(\tilde{X}, x, \mu) \) (resp. \( \text{Bun}_{\text{hol}}(\tilde{X}, x, \mu) \)) be the functor which assigns to any scheme (resp. complex manifold) \( S \), the isomorphism classes of \( \Gamma \)-equivariant bundles \( \tilde{E} \to \tilde{X} \), such that the action of \( \gamma \) on the fiber at \( x_i \) is in the conjugacy class of \( \exp(\mu_i) \). Let \( \text{Bun}(X, x, \mu) \)
be the functor that assigns to any complex manifold $S$ the set of isomorphism classes of parabolic bundles $E$ on $(X; x_1, \ldots, x_b)$ with markings $\mu_1, \ldots, \mu_b$. Let $\text{Bun}^s(\tilde{X}, x, \mu)$ (resp. $\text{Bun}^a(\tilde{X}, x, \mu)$) denote the sub-functor assigning to any complex manifold $S$ the set of isomorphism classes of bundles semistable at every point in $S$.

**Theorem 2.3.** There exist isomorphisms of functors

$$\text{Bun}_E(\tilde{X}, x, \mu) \rightarrow \text{Bun}(X, x, \mu), \quad \text{Bun}^a(\tilde{X}, x, \mu) \rightarrow \text{Bun}^a(X, x, \mu)$$

and similarly for the versions in the holomorphic category.

That is, there is a natural bijection between isomorphism classes of $\Gamma$-equivariant bundles (resp. semistable bundles) on $S \times \tilde{X}$ with action at $x_i$ conjugate to $\mu_i$, and isomorphism classes of parabolic bundles (resp. semistable bundles) on $S \times X$ with parabolic weights $\mu_i$ at the points $x_i$.

For vector bundles, the correspondence is given by taking the sheaf of invariant sections [23, 17, 9, 8]. That is, if $V \rightarrow \tilde{X}$ is a $\Gamma$-equivariant vector bundle, then $V$ is the vector bundle whose sheaf of sections is sheaf of $\Gamma$-invariant sections of $\tilde{V}$. The parabolic structures are the filtrations at the ramification points induced by order of vanishing.

We now extend the construction to principal bundles. We begin with the discussion in the holomorphic category. Let $E \rightarrow \tilde{X}$ be a $\Gamma$-equivariant principal $G$-bundle, and consider for simplicity of notation the case of a single fixed point $x = x_j$ with marking $\mu = \mu_j$. Choose a neighborhood $\tilde{U} \rightarrow U$ with local coordinate $z$ so that the projection is given by $z \mapsto z^N$, and the action of $\Gamma$ by $z \mapsto \exp(2\pi i/N)z$. Consider the one parameter subgroup,

$$C^* \rightarrow G, \quad z \mapsto z^{N\mu/2\pi i} := \exp(\ln(z)N\mu/2\pi i).$$

Let $\tilde{\Theta}^{-N\mu}$ denote the set of $\Gamma$-invariant meromorphic sections $s : \tilde{U} \rightarrow \tilde{E}/P$ such that $s(z)z^{-N\mu/2\pi i}$ is regular. By Proposition 2.5 below, $\tilde{\Theta}^{-N\mu}$ is non-empty.

**Lemma 2.4.** There is a parabolic bundle $(E, \varphi, \mu)$ isomorphic to $\Gamma \backslash \tilde{E}$ over $\Gamma \backslash (\tilde{X}\setminus x)$ such that $\tilde{\Theta}^{-N\mu}$ is the set of sections of $E$ over $U$ which take values in the parabolic reduction $\varphi$ at the marked point $x$.

**Proof.** Let $s_0 : U \rightarrow \tilde{E}$ be a section over $U$, and $s \in \tilde{\Theta}^{-N\mu}$. Let $g_0 : U \rightarrow G$ be the map such that $s(z) = s_0(z)g_0(z)$. Since $s(z)z^{-N\mu/2\pi i}$ is regular, so is $g_0(z)z^{-N\mu/2\pi i}$. Form a bundle $\tilde{E}^{-N\mu}$ by patching together $\tilde{E}|_{\tilde{X}\setminus\{x\}}$ with $\tilde{E}|_{\tilde{U}}$, using the transition map $g_0(z)^{-1}$, that is,

$$s_0(z)g(z) \mapsto s_0(z)g_0(z)^{-1}g(z).$$

The section $s = s_0g_0$ extends to a regular section of $\tilde{E}^{-N\mu}$. Since $s$ is invariant, the action of $\Gamma$ extends to $\tilde{E}^{-N\mu}$ and is trivial in the trivialization given by $s$ near $x$. It is straightforward to check that $\tilde{E}^{-N\mu}$ is independent of the choice of $s_0$. Consider some other choice $s' = s_0g'_0$. The bundles constructed using the two choices are identical away from $x$. The identity map is given in the local trivializations near $x$ by $\tau(z) = g_0'(z)^{-1}g_0(z)$.

We claim that $\tau(z)$ is regular and $\tau(0) \in P$. We have $\tau'(z) := \text{Ad}(z^{N\mu/2\pi i})\tau(z)$ regular and $\tau(\gamma z) = \tau(z)$, since $s$ and $s'$ are invariant. Therefore, $\tau'(\gamma z) = \text{Ad}(\exp(\mu))\tau'(z)$.
section s by patching together the resulting ¡-equivariant bundle ~

Let sections of ~

one sees that the components

are regular at z = 0. Indeed, \( \tau_\alpha(0) = 0 \) for \( \alpha < 0 \) so that \( \tau(0) \in P \), and if \( \alpha > 0 \) then

By the same reasoning, for any \( \Gamma \)-invariant map \( \tau : \bar{U} \to G \) with \( \tau(0) \in P \), \( \tau'(z) := \text{Ad}(z^{N\mu/2\pi i})\tau(z) \) is regular which implies that

so that \( \tau_\alpha(z) \) is regular. By the same reasoning, for any \( \Gamma \)-invariant sections of \( \tilde{E}^{-N\mu} \), and the parabolic structure \( \hat{\varphi} := s(0)P \) is independent of the choice of \( s \). Define \( E = \Gamma \setminus \tilde{E}^{-N\mu} \), which is well-defined because \( \Gamma \) acts trivially in the fiber at the ramification points \( x_i \). Using the isomorphisms \( \iota_{x_i} : E_{x_i} \cong \tilde{E}^{-N\mu} \), set \( \varphi = \iota_{x_i}^{-1}(\hat{\varphi}) \).
The sections of \( E \) are the \( \Gamma \)-invariant sections of \( \tilde{E}^{-N\mu} \).

The existence of an invariant section \( s \) with \( sz^{-N\mu/2\pi i} \) regular follows from the equivariant version of Grauert’s Oka principle due to Heinzner-Kutzschebauch.

**Proposition 2.5.** [20, Section 11] Let \( G \) be a complex Lie group, and \( \tilde{E} \) a \( \Gamma \)-equivariant principal \( G \)-bundle. There exists a \( \Gamma \)-neighborhood \( \bar{U} \subset \tilde{X} \) of \( x \), such that \( \tilde{E} \) is \( \Gamma \)-trivial over \( \bar{U} \).

That is, there exists a \( \Gamma \)-equivariant biholomorphic map

such that the action of \( \Gamma \) is given by \( \gamma(z, g) = (\exp(2\pi i/N)z, \exp(\mu)g) \). The section given by \( s(z) = z^{N\mu/2\pi i} \) is \( \Gamma \)-invariant with \( s(z)z^{-N\mu/2\pi i} \) regular.

The construction of parabolic bundles from equivariant bundles is reversible in an obvious way. Let \( E \to X \) be a principal \( G \)-bundle with parabolic structures at \( x_1, \ldots, x_b \).

Let \( s_i \) be a section of \( E \) defined in a neighborhood \( U_i \) of \( x_i \), such that \( s(x_i) \in \varphi \). The section \( s_i \) induces a local trivialization of \( E \) and therefore also of \( \pi^*E \). Form the bundle \( \tilde{E} \) by patching together \( \pi^*E \setminus \pi^*E_{x_i} \) and \( \pi^*E|_{\pi U_i} \), using \( z^{N\mu/2\pi i} \) as a transition map.
The action of \( \Gamma \) on \( \pi^*E \setminus \pi^*E_{x_i} \) extends to an action on \( \tilde{E}_{x_i} \), conjugate to \( \exp(\mu_i) \).
The resulting \( \Gamma \)-equivariant bundle \( \tilde{E} \) is independent up to isomorphism of the choice of \( s_i \).

To prove Theorem 2.3, one checks that the construction works in families. Let \( \tilde{E} \to S \times \tilde{X} \) be a family of equivariant bundles over \( \tilde{X} \) parametrized by a complex manifold \( S \). Let \( S_j \) be a cover of \( S \), over which there exist sections \( \Gamma \)-invariant sections with \( s_jz^{-N\mu/2\pi i} \) regular, and the bundle \( \tilde{E}|_{S_j} \) is trivial. The bundles \( E_{S_j} := \Gamma \setminus \tilde{E}^{-N\mu}_{S_j} \) are canonically isomorphic over the intersections \( S_j \cap S_k \), via the map which is the identity away from the points \( x_i \). Therefore, the gluing maps satisfy the cocycle condition and the parabolic bundles \( E_{S_j} \) patch together to a parabolic bundle \( E \to S \times X \). The reverse
construction also works in families, by essentially the same argument. The construction
is natural with respect to holomorphic maps \( f : S_1 \to S_2 \) of complex manifolds. That is,
if \( \tilde{E} \to S_2 \times \tilde{X} \) is a \( \Gamma \)-equivariant bundle and \( E \to S_2 \times X \) the corresponding parabolic
bundle, then \( f^*E \) is the parabolic bundle corresponding to \( f^*\tilde{E} \). This follows immediately
from the independence of the construction on the choice of cover and section \( s \), since one
can take these to be pull-backs from \( S_2 \).

2.1.1. The algebraic version. In this section, we briefly discuss how to make the above
constructions in the algebraic setting, by replacing the equivariant Oka principle by
a cohomology argument, and the gluing by formal gluing. For background on formal
neighborhoods, see [19, II.9]. Let \( \tilde{E} \to \tilde{X} \) be a \( \Gamma \)-equivariant \( G \)-bundle, and \( x \in \tilde{X} \) a
fixed point. Let \( z \) be a local coordinate near \( x \). Over the formal disk \( D = \text{Spec}(\mathbb{C}[[z]]) \)
at \( x \), the bundle \( \tilde{E} \) is trivial and the action of \( \Gamma \) is given by
\[
\gamma(z, \zeta) = (\gamma z, g(\gamma, z)\zeta)
\]
for some \( g : \Gamma \to G[[z]] \). Since \( \gamma^N = 1 \) we have
\[
g(\gamma, \gamma^{N-1}z)g(\gamma, \gamma^{N-2}z)\ldots g(\gamma, z) = e.
\]
In particular, \( g(\gamma, 0)^N = e \). More generally, if \( \tilde{E} \) is a \( \Gamma \)-equivariant bundle over \( D_R := \text{Spec}(R[[z]]) \), where \( R \) is any \( \mathbb{C} \)-algebra, then the action is given by an automorphism
\( g \in G(R[[z]]) \).

**Lemma 2.6.** Let \( \tilde{E} \) be a \( \Gamma \)-equivariant bundle over \( D_R \), for any \( \mathbb{C} \)-algebra \( R \). Then there
exists an automorphism \( \tau \in G(R[[z]]) \) which transforms the \( \Gamma \)-action on \( \tilde{E}|_{D_R} \cong D_R \times G \)
to the product action, that is,
\[
\tau(\gamma z)^{-1}g(\gamma, z)\tau(z) = g(\gamma, 0).
\]

**Proof.** Consider the element of \( C^1(\Gamma, G(R[[z]])) \) defined by \( \gamma \mapsto g(\gamma, z) \). Since
\[
g(\gamma_1\gamma_2, z) = g(\gamma_1, \gamma_2 z)g(\gamma_2 z) = \gamma_2^*g(\gamma_1, z)g(\gamma_2, z)
\]
g(\cdot, z) is a cocycle in the cohomology of \( \Gamma \) with values in \( G[[z]] \). Similarly \( g(\cdot, 0) \in Z^1(\Gamma, G(R)) \) which maps to \( Z^1(\Gamma, G(R[[z]])) \). Our claim is that there exists a 0-chain \( \tau \)
such that \( (\delta \tau) : g(\cdot, z) \mapsto g(\cdot, 0) \). We construct \( \tau \) order-by-order. Let \( G_l = G(R[z]/z^l = 0) \). Let \( N_l \) be the kernel of the truncation map \( G_{l+1} \to G_l \). The exact sequence of groups
\[
1 \to N_l \to G_{l+1} \to G_l \to 1
\]
induces an exact sequence of pointed sets in non-abelian cohomology ([31, p. 49])
\[
H^1(\Gamma, N_l) \to H^1(\Gamma, G_{l+1}) \to H^1(\Gamma, G_l).
\]
Since \( N_l \) is nilpotent, \( H^1(\Gamma, N_l) \) is trivial, by induction on the length of the central
series which reduces to the case that \( N_l \) is a \( \Gamma \)-module. Therefore, \( H^1(\Gamma, G_{l+1}) \) injects
into \( H^1(\Gamma, G_l) \) for all \( l \).

The complexes \( C^0(\Gamma, G_l), C^1(\Gamma, G_l) \) satisfy the Mittag-Leffler condition, that is, the
image of \( C^0(\Gamma, G_{l'}) \) (resp. \( C^1(\Gamma, G_{l'}) \)) in \( C^0(\Gamma, G_l) \) resp. \( C^1(\Gamma, G_l) \) stabilizes as \( l' \to \infty \).
Indeed, let $f_t : \text{Spec}(R_t) \to G$. Extending $f_t$ to a map $f_{t+1} : \text{Spec}(R_{t+1}) \to G$ is equivalent to extending the map $f_0^{-1}f_{t+1}$; the latter extends because $G$ is isomorphic to $\mathfrak{g}$ near the identity. Therefore, $G_{t+1} \to G_t$ is surjective, which implies the same result for the chain complexes.

The Mittag-Leffler condition implies (see [19, II.9.1] for the abelian case) that

$$H^1(\Gamma, G[[z]]) = \lim_{l \to \infty} H^1(\Gamma, G_l)$$

and therefore also injects into $H^1(\Gamma, G)$. The claim follows since $g(\cdot, z)$ and $g(\cdot, 0)$ both map to $g(\cdot, 0)$ in $H^1(\Gamma, G(R))$. \hfill \square

Let $\hat{E} \to S \times \hat{X}$ be a $\Gamma$-equivariant $G$-bundle. By the theorem of Drinfeld and Simpson [13] (another proof is given in [33]) the bundle $\hat{E}$ is trivial locally in the product of the Zariski topology for $X$ and the étale topology in $S$. By the Lemma, we can find a trivialization of the bundle $\hat{E}$ for which the action is of product form, on a neighborhood of $S \times \{x\}$ which is the product of an étale neighborhood in $S$ and formal neighborhood of $x$ in $X$. Recall the description of bundles on $\hat{X}$ by formal gluing data [4], [22, Section 3]. For any algebra $R$, let

$$\tilde{X}_R = \hat{X} \times \text{Spec}(R).$$

Let $T$ denote the functor which associates to any algebra $R$ the set of isomorphism classes of triples $(E, \rho, \sigma)$ where $E$ is a $G$-bundle over $\tilde{X}_R$, $\rho$ a trivialization over $\tilde{X}\setminus\{x\}$, and $\sigma$ a trivialization over the formal disk $D_R$. The functor $T$ is represented by $G(R((z)))$ [22, 3.8]. In particular, this implies that the set of isomorphism classes of bundles on $\tilde{X}_R$ has an action of the formal loop group $G(R((z)))$. Choose a set of trivializations of $\hat{E}$ in formal neighborhoods of the form $R[[z]]$ as described above. Let $\hat{E}_R^{-N\mu}$ denote the bundle obtained from twisting by $z^{-N\mu/2\pi i} \in G(R((z)))$. The bundles $\hat{E}_R^{-N\mu}$ are canonically isomorphic away from $x$, and the same order-of-vanishing argument given above shows that the canonical isomorphisms extend to $\tilde{X}$ and the extension preserves the parabolic structures at the ramification points. By a simple case of étale descent, the bundles $\hat{E}_R^{-N\mu}$ patch together to a parabolic bundle $\hat{E}^{-N\mu} \to S \times \tilde{X}$. Since the gluing data for $\hat{E}^{-N\mu}$ are $\Gamma$-invariant, they define a $G$-bundle $E \to S \times X$, with parabolic structure induced by the parabolic structure on $\hat{E}$.

2.1.2. The isomorphism of semistable functors. Let $\hat{E} \to \hat{X}$ be a $\Gamma$-equivariant bundle, and $E = \hat{E}^{-N\mu}/\Gamma$. Any parabolic reduction $\sigma : X \to E/P$ induces an invariant parabolic reduction $\hat{\sigma}$ of $\hat{E}$ and vice-versa, by completion [19, I.6.8]. We fix a local trivialization $U_i \times G$ near $x_i$, so that the action of $\Gamma$ is given by $\exp(\mu_i)$ on the fiber. The bundle $E$ is formed by twisting by $z^{-N\mu_i/2\pi i}$ near $x_i$, and quotienting by $\Gamma$. Using this local trivialization, the fixed point set of $\Gamma$ on $\hat{E}_{x_i}/P$ has components indexed by the double coset space of the Weyl group

$$\hat{E}_{x_i}/P) \cong (G/P)^{\exp(\mu)} = \bigcup_{w \in W_P \setminus W/W_P} LwP$$

(4)
where $L$ is the standard Levi subgroup of $P$.

**Lemma 2.7.** Let $\tilde{\sigma} : \tilde{X} \to \tilde{E}/P$ be a parabolic reduction, and $\sigma : X \to E/P$ the parabolic reduction defined by completion. Then $\tilde{\sigma}(x_i) \in LwP$ for some $w \in W_P \setminus W/W_P$, if and only if the relative position of $(\varphi_i, \sigma(x_i))$ is $[w]$.

*Proof.* Let $O_w \subset G/P$ be the open cell containing $[w]$, that is, $O_w = wB^{opp}P$, where $B^{opp}$ is the Borel opposite to $B$. Let $C_w = B^{opp}wP$. Let $\Delta_-(P)$ be the set of weights of $\mathfrak{g}/\mathfrak{p}$, i.e. roots of the negative unipotent complementary to $P$. Using the $T$-equivariant isomorphisms

$$O_w \cong \bigoplus_{\alpha \in w\Delta_-(P)} \mathfrak{g}_\alpha, \quad C_w \cong \bigoplus_{\alpha \in w\Delta_-(P) \cap \Delta_-(B)} \mathfrak{g}_\alpha,$$

it suffices to show that each component $\tilde{\sigma}_\alpha^{-N\mu}$ is regular at $z = 0$ and $\tilde{\sigma}_\alpha^{-N\mu}(0) = 0$ unless $\alpha \in \Delta_-(B)$. Since $\tilde{\sigma}$ is $\gamma$-invariant, $\tilde{\sigma}_\alpha(\gamma \cdot z) = \exp(-2\pi i (\mu, \alpha))\tilde{\sigma}_\alpha(z)$ and so

$$\tilde{\sigma}_\alpha(z) = \sum_{j \geq 0, \, j \equiv N-(N\mu, \alpha)} c_j z^j.$$

Therefore

$$\sigma_\alpha^{-N\mu}(z) = \sigma_\alpha(z)^{(N\mu, \alpha)} = \sum_{j \geq 0, \, j \equiv N-(N\mu, \alpha)} c_j z^{j+(N\mu, \alpha)}.$$

It follows that $\sigma_\alpha^{-N\mu}(z) = 0$ at $z = 0$ if $(\alpha, \mu) > 0$, and is regular in any case. \hfill $\Box$

The proof of the second statement in Theorem 2.3 is completed by the following lemma.

**Lemma 2.8.** The parabolic degree of $\sigma$ and degree of $\tilde{\sigma}$ are related by

$$(5) \quad \text{pardeg}(\sigma) = \deg(\tilde{\sigma})/N.$$

*Proof.* In the local trivialization of $\tilde{E}$ near $x_i$, the reduction $\tilde{\sigma}$ is given by $\tilde{\sigma}_i(z)P$ for some map $\tilde{\sigma}_i : \tilde{U}_i \to G$. By Lemma 2.7 we may assume $\tilde{\sigma}_i(0) = w_i^{-1}$. By equivariant Oka 2.5 applied to $\tilde{\sigma}^*\tilde{E}$, we may assume that $\tilde{\sigma}_i = w_i^{-1}$ is constant. Therefore, the bundle $(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}$ is formed by gluing together $\tilde{\sigma}_i^*\tilde{E} \setminus \bigcup \tilde{\sigma}_i^*\tilde{E}_{x_i}$ and $\bigcup \tilde{\sigma}_i^*\tilde{E}|_{\tilde{U}_i}$ using the transition maps $\text{Ad}(w_i)z^{-N\mu_i/2\pi i} = z^{-Nw_i\mu_i/2\pi i}$. This implies that the gluing maps for $(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}(\omega_P)$ are

$$\chi_P(\text{Ad}(w_i)z^{-N\mu_i/2\pi i}) = z^{-\omega_P(Nw_i\mu_i)}.$$

The degree of the line bundle is therefore

$$\deg((\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}(\omega_P)) = \deg(\tilde{\sigma}^*\tilde{E}(\omega_P)) - \sum_{i=1}^b N\omega_P(w_i\mu_i).$$

Since $\sigma = \Gamma \setminus \tilde{\sigma}^N\mu$, the degree of $\sigma^*E(\omega_P)$ is $1/N$ times the degree of $(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}(\omega_P)$.

Hence,

$$\deg(\sigma^*E(\omega_P)) + \sum_{i=1}^b \omega_P(w_i\mu_i) = \frac{1}{N} \deg(\tilde{\sigma}^*\tilde{E}(\omega_P)).$$
2.2. The canonical reduction for parabolic bundles. If a bundle is unstable, there is a canonical parabolic reduction violating the slope inequality. In the case of vector bundles, it is the Harder-Narasimhan filtration, defined as follows. Let $V$ be a parabolic vector bundle over $X$. There is a unique sub-bundle $V_{\text{max}} \subset V$ such that the slope $\mu(V_{\text{max}})$ is maximal among all sub-bundles, and the rank of $V_{\text{max}}$ is maximal among sub-bundles with that slope. The Harder-Narasimhan filtration $V_1 \subset \ldots \subset V_k = V$ is defined inductively by

$$V_{i+1}/V_i = (V/V_i)_{\text{max}}.$$ 

The quotients $V_{i+1}/V_i$ of the canonical filtration are semistable and slopes $\mu_i = \mu(V_{i+1}/V_i)$ decreasing. The Harder-Narasimhan filtration is the unique filtration with slopes $\mu_i$ and dimensions $d_i = \dim(V_{i+1})$.

The generalization of the Harder-Narasimhan filtration to arbitrary $G$ in the case of a curve $X$ without markings is due to Atiyah-Bott [2, Section 10]. They construct a canonical parabolic reduction $\sigma_E : E \rightarrow G/P$, with the property that if $L$ is a Levi subgroup of $P$ and $r : P \rightarrow L$ the projection, then $r_*\sigma^*E$ is a semistable $L$-bundle. For $G = \text{Gl}(n)$, $\sigma_E$ is corresponds to the Harder-Narasimhan filtration. For any parabolic reduction $\sigma : X \rightarrow E/P$, defines its slope $\mu(\sigma) \in \Lambda_P$ by

$$\mu(\sigma) : \lambda \mapsto \deg(\sigma^*E(\lambda))$$

for $\lambda \in \Lambda_P^*$, the weights of characters of $P$. We may consider $\mu(\sigma)$ as an element of $t$ via the embedding $\Lambda_P \rightarrow t$. The type of the bundle $E$ is the slope $\mu(\sigma_E)$ of its canonical reduction. It lies in the interior of the open face of $t_+$ corresponding to $P$. The canonical reduction $\sigma_E$ is the unique parabolic reduction with slope $\mu(\sigma_E)$. Indeed, consider an embedding $\phi : G \rightarrow \text{Gl}(V)$. Let $\sigma$ be another reduction with slope $\mu(\sigma_E)$ and parabolic $P$. By functoriality of the canonical reduction [2, 10.4], there is a parabolic subgroup $P'$ of $\text{Gl}(V)$ such that $P' \cap \phi(G) = P$ and the Harder-Narasimhan reduction of $\phi_*E$ is the reduction induced by the homomorphism $\phi|_P : P \rightarrow P'$. (The first property seems not be mentioned explicitly in [2], although it is certainly necessary for the discussion there.) Let $\phi_*\sigma$ be the parabolic reduction of $\phi_*E$ to $P'$ induced by $\sigma$. Since $\deg((\phi_*\sigma)^*\phi_*E(\lambda)) = \deg(\sigma^*E(D\phi^*\lambda))$ for any weight $\lambda \in \Lambda_P^*$, we have

$$\mu(\sigma_{\phi_*E}) = D\phi(\mu(\sigma_E)) = D\phi(\mu(\sigma)) = \mu(\phi_*\sigma).$$

By uniqueness of the Harder-Narasimhan filtration, $\phi_*\sigma = \phi_*\sigma_E$. But since $\phi(P) = P' \cap \phi(G)$, this implies that $\sigma = \sigma_E$.

The canonical reduction may be used to give a short proof of the following result of Ramanathan:

**Proposition 2.9. (Ramanathan [29, 3.17])** Let $\phi : G \rightarrow H$ be a homomorphism of Lie groups. If $E$ is semi-stable, then $\phi_*E$ is semi-stable. If $D\phi$ is injective and $\phi_*E$ is semistable then $E$ is semistable.
We extend the theory of the canonical reduction to parabolic bundles, by using the equivalence with equivariant bundles. Let \( \gamma : X \to X \) be an automorphism of the curve \( X \). Let \( E_1, E_2 \) be principal \( G \)-bundles over \( X \), and \( \gamma_E : E_1 \cong E_2 \) an isomorphism of bundles covering \( \gamma \). The associated isomorphism of adjoint bundles \( \gamma(\mathfrak{g}) : E_1(\mathfrak{g}) \cong E_2(\mathfrak{g}) \) induces an isomorphism of Harder-Narasimhan filtrations \( \sigma_1 \cong \sigma_2 \), by uniqueness. That is, \( \sigma_1 = \gamma^* \sigma_2 \). If \( E \) is an \( \gamma \)-equivariant bundle, then the canonical reduction is \( \gamma \)-invariant.

Let \( \Gamma \) be a group of automorphisms of \( E \). We will call \( \Gamma \)-stable (resp. \( \Gamma \)-semi-stable) if
\[
\deg(\sigma^*E(\lambda)) \leq 0 \quad \text{(resp. } < 0 \text{ )}
\]
for all \( \Gamma \)-invariant parabolic reductions \( \sigma \). By uniqueness of the canonical reduction, a principal \( G \)-bundle is \( \Gamma \)-semistable if and only if it is ordinary semistable. On the other hand, \( \Gamma \)-stability, or \( \Gamma \)-irreducibility of \( E \) is not in general the same as ordinary stability or irreducibility. For any parabolic bundle \( E = (E, \{(\varphi_i, \mu_i)\}) \), let \( \sigma_E \) denote its canonical reduction, defined by the one-to-one correspondence between invariant parabolic reductions of \( E \) and parabolic reductions of \( E \). Define the slope of any parabolic reduction \( \sigma \) by
\[
\mu(\sigma) : \lambda \mapsto \deg \sigma^*E(\lambda) + \sum \lambda(w_i \mu_i).
\]
The type of \( E \) is the slope of \( \sigma_E \); by the discussion above \( \sigma_E \) is the unique reduction of this slope.

2.3. Moduli spaces. In order to obtain a moduli space of semistable \( G \)-bundles, an equivalence relation has to be imposed unless there are no strictly semistable bundles. Ramanathan’s grade equivalence relation on \( G \)-bundles \( E \to X \) may be defined in terms of parabolic reductions and projections to Levi subgroups. Let \( \sigma : S \times X \to E/P \) be a parabolic reduction, \( r : P \to L \) a projection to a Levi subgroup \( L \subset P \), and \( \iota : L \to G \) the inclusion of \( L \) in \( G \). The reduction \( \sigma \) is admissible if \( \deg(\sigma^*E(\lambda))|_{s \times X} = 0 \) for all dominant weights \( \lambda \) and points \( s \in S \). The equivalence relation on semistable bundles generated by
\[
E \sim \iota_s r_\sigma^*E,
\]
as \( \sigma \) ranges over all admissible reductions, is called grade equivalence [29, 30].

By Ramanathan’s thesis, there exists a coarse moduli scheme for equivalence classes of semistable bundles. A coarse moduli scheme for \( F \) is a scheme \( M \) and a morphism \( c : F \to \text{Map}(\cdot, M) \) such that (i) \( c \) induces a bijection of points \( c(*) : F(*) \to \text{Map}(*, M) \), where \(* = \text{Spec}(\mathbb{C})\), and (ii) for any scheme \( N \) and morphism \( c' : F \to \text{Map}(\cdot, N) \), there is a unique morphism \( \phi : \text{Map}(\cdot, M) \to \text{Map}(\cdot, N) \) such that \( c' = \phi \circ c \). Usually, we omit the morphism \( c \) from the notation. Let \( \overline{\text{Bun}}^s(X) \) denote the functor which associates to any scheme \( S \) the set of equivalence classes of semi-stable algebraic principal \( G \)-bundles over \( S \times X \).

**Theorem 2.10** (Ramanathan [30], see also [16]). There is an irreducible, normal projective variety \( \mathcal{M}_G(X) \) that is a coarse moduli space for \( \overline{\text{Bun}}^s(X) \).
There are similar theorems for equivariant and for parabolic bundles. For equivariant bundles let grade\(_\Gamma\)-equivalence be the relation generated by \(E \twoheadrightarrow \mathbb{P} \quad \rho \mapsto \quad \varrho \mapsto \quad E\), where \(\varrho : X \to E/P\) is an invariant parabolic reduction of \(E\). Let \(\text{Bun}_{\text{ss}}^\mathbb{P}(X)\) (resp. \(\text{Bun}_{\text{hol},\mathbb{P}}^\mathbb{P}(X)\)) denote the functor that assigns to any scheme (resp. complex manifold) \(S\) the set of grade\(_\Gamma\)-equivalence classes of \(\Gamma\)-equivariant bundles over \(S \times X\).

**Theorem 2.11.** There is a normal projective variety \(\mathcal{M}_{\Gamma}(X)\) that is a coarse moduli space for \(\text{Bun}_{\text{ss}}^\mathbb{P}(X)\).

The proof, which is analogous to the case with \(\Gamma\) trivial, is sketched in the appendix.

Finally, we consider a parabolic bundle \((E, (\varphi_i, \mu_i))\). Define parabolic reductions of \(r_*\sigma^*E\) by

\[
r(\varphi_i \cap \sigma(x_i)) \subset \sigma^*E_{x_i}/U \cong (r_*\sigma^*E)_{x_i}.
\]

These are parabolic reductions to the subgroup \(L \cap w_i P_i\) of \(L = P/U\), where \([w_i]\) is the position of \(\sigma(x_i)\) with respect to \(\varphi_i\). They induce parabolic reductions of \(\varrho \mapsto r_*\sigma^*E_{x_i}\) to the parabolic subgroups \((P \cap w_i P_i)U\). Let \(\text{Bun}_{\text{ss}}^\text{par}(X, x, \mu)\) denote the functor assigning to any scheme \(S\) the set of equivalence classes of semi-stable parabolic \(G\)-bundles over \(S \times X\) with markings \(\mu\). By Theorems 2.11,2.3 and Lemma A.4

**Theorem 2.12.** There is an irreducible, normal, projective variety \(\mathcal{M}_{G}(X, x, \mu)\) that is a coarse moduli space for the functor \(\text{Bun}_{\text{ss}}^\text{par}(X, x, \mu)\).

### 3. Flat \(K\)-bundles

On the other side of the Narasimhan-Seshadri correspondence are the flat unitary bundles on \(X\). Atiyah and Bott [2] explained how to view the moduli space of flat bundles as a symplectic quotient of the affine space of connections by gauge transformations, as follows. Let \(X\) be a compact oriented two-manifold, and \(Z\) a smooth principal \(K\)-bundle over \(X\) (trivial if \(K\) is connected and simply-connected). Let

\[
\mathcal{A}(Z) := \Omega^1(Z, \mathfrak{k})^K, \quad \mathcal{K}(Z) := \text{Aut}_K(Z)
\]

be the space of connections on \(Z\), respectively the group of gauge transformations of \(Z\). Choose an invariant inner product

\[
(\cdot, \cdot) : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}
\]
on \(\mathfrak{k}\), and let \((\wedge)\) denote the induced map \(\Omega^1(Z, \mathfrak{k}) \wedge \Omega^1(Z, \mathfrak{k}) \to \Omega^2(Z, \mathbb{R})\). The affine space \(\mathcal{A}(Z)\) has a symplectic form

\[
a_1, a_2 \mapsto \int_X (a_1 \wedge a_2).
\]
The action of \(\mathcal{K}(Z)\) on \(\mathcal{A}(Z)\) is Hamiltonian, with moment map given by the curvature

\[
\mathcal{A}(Z) \hookrightarrow \Omega^2(X, Z(\mathfrak{k})), \quad A \mapsto \text{curv}(A),
\]

where \(Z(\mathfrak{k})\) is the adjoint bundle of Lie algebras. The symplectic quotient

\[
\mathcal{R}(Z) = \mathcal{A}(Z)//\mathcal{K}(Z) = \{A \in \mathcal{A}(Z) \mid \text{curv}(A) = 0\}/\mathcal{K}(Z)
\]
is the moduli space of flat connections on $E$. The moduli space of flat $K$-bundles $\mathcal{R}_K(X)$ is the union of the components $\mathcal{R}(Z)$, as $Z$ ranges over isomorphism classes of smooth principal $K$-bundles.

In the case that $X$ is a compact, oriented two-manifold with boundary, the moment map for the action of $\mathcal{K}(Z)$ is given by the curvature plus a boundary term

$$\mathcal{A}(Z) \mapsto \Omega^2(X, \mathcal{T}) \oplus \Omega^1(\partial X, \partial Z(\mathfrak{t})), \quad A \mapsto (\text{curv}(A), \iota_{\partial X}A).$$

The symplectic quotients of $\mathcal{A}(Z)$ by $\mathcal{K}(Z)$ may be identified with moduli space of flat connections on $\mathcal{A}(Z)$, with fixed holonomy around the boundary, as follows [24]. Let $b$ denote the number of components of $\partial X$. The orbits of $\mathcal{K}(Z)$ on $\Omega^1(\partial X, \partial Z(\mathfrak{t}))$ are parametrized by $b$-tuples $\mu = (\mu_1, \ldots, \mu_b) \in \mathbb{A}^b$. Let

$$\text{Hol}_i : \Omega^1(\partial X, \partial Z(\mathfrak{t})) \to K$$

denote the holonomy around the $i$-th boundary component. Then two connections $A_1, A_2 \in \Omega^1(\partial X, \partial Z(\mathfrak{t}))$ are in the same orbit of $\mathcal{K}(Z)$ if and only if $\text{Hol}_i(A_1)$ is conjugate to $\text{Hol}_i(A_2)$, for $i = 1, \ldots, b$. The symplectic quotient

$$\mathcal{R}_K(Z, \mu) = \mathcal{A}(Z) \sslash \mu \mathcal{K}(Z) = \{A \in \mathcal{A}(Z), \text{curv}(A) = 0, \text{Hol}_i(A) \in \mathcal{C}_\mu\} / \mathcal{K}(Z)$$

is the moduli space of flat connections on $Z$, with fixed holonomy. We will see in a moment that the homeomorphism class of $\mathcal{R}_K(Z, \mu)$ does not depend on the choice of marked points $x_i$, which justifies dropping them from the notation. The union

$$\mathcal{R}_K(X, \mu) = \bigcup_{[Z]} \mathcal{R}_K(Z, \mu),$$

where $Z$ ranges over topological types $[Z]$ of $K$-bundles on $X$, is the moduli space of flat bundles of $X$. Assume $K$ is connected and simply-connected; then any bundle $Z$ is trivial.

To get back to the language used by Mehta and Seshadri in their description, one notes that the space $\mathcal{R}_K(X, \mu)$ may be identified with the space of isomorphism classes of representations of $\pi_1(X)$ in $K$. Any flat bundle $(Z, A)$ determines a representation of $\pi_1(X)$ by holonomy

$$\text{Hol}(A) : \pi_1(X) \to K.$$

The $i$-th boundary component $(\partial X)_i$ determines a conjugacy class $[(\partial X)_i] \subset \pi_1(X)$. Two flat bundles are isomorphic if and only if their holonomy representations are conjugate by the action of $K$. Therefore,

$$(7) \quad \mathcal{R}_K(X, \mu) \cong \{\rho \in \text{Hom}(\pi_1(X), K), \rho([(\partial X)_i]) \subset \mathcal{C}_\mu\} / K.$$

The holonomy description shows that there is essentially no difference between working with manifolds with boundary and manifolds with punctures. Let $X$ be a compact, oriented two-manifold without boundary, and $x_1, \ldots, x_b \in X$ distinct marked points. Then the moduli space of flat bundles on $X \setminus \{x_1, \ldots, x_b\}$ with holonomy around $x_i$ in $\mathcal{C}_i$ is $\mathcal{R}_K(X', \mu)$, where $X'$ is the manifold obtained by removing a small open disk.
containing each marked point \( x_i \). We denote this space by \( \mathcal{R}_K(X, \mu) \). In the case \( X = \mathbb{P}^1 \), the fundamental group of \( X \setminus \{x_1, \ldots, x_b\} \) is
\[
\pi_1(X \setminus \{x_1, \ldots, x_b\}) = \langle c_1, \ldots, c_b \rangle / \Pi c_i = 1.
\]
By (7), the moduli space of flat bundles is given by
\[
\mathcal{R}_K(\mathbb{P}^1; \mu_1, \ldots, \mu_b) = \{ (k_1, \ldots, k_b) \in \mathcal{C}_{\mu_1} \times \cdots \times \mathcal{C}_{\mu_b} \mid \Pi k_i = e \} / K.
\]
Corollary 3.1. The polytope \( \Delta(b) \) is the set of \( (\mu_1, \ldots, \mu_b) \) such that \( \mathcal{R}_K(\mathbb{P}^1; \mu_1, \ldots, \mu_b) \) is non-empty.

The moduli spaces of flat connections on surfaces with boundary are homeomorphic to moduli spaces of \( \Gamma \)-invariant flat connections on the ramified cover \( \tilde{X} \). Let \( \tilde{Z} \) be a \( \Gamma \)-equivariant smooth principal \( K \)-bundle on \( \tilde{X} \), and
\[
\mathcal{R}_K^\Gamma(\tilde{Z}) = \mathcal{A}(\tilde{Z})^\Gamma /\mu \mathcal{K}(\tilde{Z})^\Gamma
\]
the moduli space of \( \Gamma \)-invariant flat connections on \( \tilde{Z} \). Define, as before,
\[
\mathcal{R}_K^\Gamma(\tilde{X}) = \bigcup_{[\tilde{Z}]} \mathcal{R}_K^\Gamma(\tilde{Z})
\]
where the union is over isomorphism classes of \( \Gamma \)-equivariant principal \( K \)-bundles \( \tilde{Z} \). Let \( \tilde{Z} \) be an equivariant bundle, and \( A \) an invariant connection. Define a bundle \( Z \) on \( X \setminus \{x_i\} \) by
\[
Z = \Gamma \setminus \tilde{Z}\big|_{X \setminus \{x_i\}}.
\]
The connection \( \tilde{A} \) descends to a connection \( A \) on \( Z \). The following is well-known and left to the reader.

**Proposition 3.2.** The map \( (\tilde{Z}, \tilde{A}) \mapsto (Z, A) \) defines a homeomorphism
\[
\mathcal{R}_K^\Gamma(\tilde{X}) = \bigcup_{\mu \in \mathcal{A} \cap \Lambda / N^\mu} \mathcal{R}_K(X, \mu).
\]
Here \( \mathcal{A} \cap \Lambda / N^\mu \) is the set of \( (\mu_1, \ldots, \mu_b) \) such that \( \exp(N\mu_i) = e \) for \( i = 1, \ldots, b \). The action of \( \Gamma \) on \( \tilde{Z}_x \) is identified (up to conjugacy) with \( \exp(\mu_i) \), for some \( \mu_i \in \mathcal{A} \), if and only if \( \text{Hol}_{x_i}(A) \in \mathcal{C}_{\mu_i} \).

4. **Correspondence theorems**

Now we are ready to prove the correspondence between unitary and semistable holomorphic bundles, for arbitrary structure group and markings. The main result is the following.

**Theorem 4.1.** Let \( G \) be a connected simple, simply-connected Lie group with maximal compact subgroup \( K \), and \( X \) a curve with distinct marked points \( x_1, \ldots, x_b \). Let \( \mu_1, \ldots, \mu_b \in \mathbb{C}^+ \) be rational points with \( c_{\mu_i}(\mu_i) < 1 \). Minimizing the Yang-Mills functional on each orbit of the group of complex gauge transformations gives a homeomorphism
\[
\mathcal{M}_G(X, x, \mu) \rightarrow \mathcal{R}_K(X, \mu).
\]
Remark 4.2. (a) A slightly weaker result true for projective varieties $X$ of any dimension is proved in [3].

(b) It is straightforward to derive from this result a correspondence theorem for arbitrary complex reductive groups. However, in this case one has to allow a central holonomy around an additional puncture (corresponding to the fact that semistable bundles admit only central, rather than flat, connections.)

(c) If $\alpha_0(\mu_i) = 1$, then the moduli space of flat $K$-bundles with fixed holonomy is homeomorphic to a moduli space of equivalence classes of torsors (non-abelian cohomology classes) for a group sheaf which is locally a standard parabolic subgroup of $G((z))$; see [33] for definitions. Only in the case $G = SU(n)$, all parabolic subgroups of $G((z))$ are conjugated to such by outer automorphisms which is why this case does not need to be considered for moduli spaces of vector bundles. However, it seems difficult to phrase the stability condition in this language.

Let $Z \to X$ be a smooth principal $K$-bundle, and $Z_C$ the associated smooth $G$-bundle. There is a one-to-one correspondence between the affine space of connections on $Z$ and the space of holomorphic structures on $Z_C$; see Singer [32]. The infinite time limit of the contragradient of the Yang-Mills functional defines a correspondence between semi-stable $G$-bundles and flat $K$-bundles, as follows. For any connection $A \in \mathcal{A}$, let

$$d_A : \Omega^*(Z(\mathfrak{k})) \to \Omega^{*+1}(Z(\mathfrak{k}))$$

denote the corresponding covariant differentiation operator, and

$$F_A \in \Omega^2(Z(\mathfrak{k}))$$

the curvature of $A$. Let

$$* : \Omega^*(Z(\mathfrak{k})) \to \Omega^{2-*}(Z(\mathfrak{k}))$$

denote the Hodge star operator defined by the conformal structure on $X$ and metric $B$ on $\mathfrak{k}$. The Yang-Mills functional is

$$A \mapsto \|F_A\|_{L^2}^2.$$ 

Its contragradient is the Yang-Mills vector field, given by

$$\Theta(A) = -d_A * F_A.$$ 

If $\Theta(A) = 0$, the connection $A$ is a Yang-Mills connection. The following summarizes results of Donaldson, Daskalopoulos, and Råde for $\text{GL}(n)$; the extension to arbitrary groups is almost immediate.

**Theorem 4.3.** Let $Z$ be a smooth principal $K$-bundle over a complex curve $X$.

(a) The flow $\Phi_t$ of the Yang-Mills vector field exists for all times $t$.

(b) The limit $A_\infty = \lim_{t \to \infty} \Phi_t(A)$ exists for any $A$ and is a Yang-Mills connection.

(c) The Yang-Mills limit $A_\infty$ is a flat connection if and only if $A$ is semi-stable, and the map $A \mapsto A_\infty$ defines a retract of the space of semi-stable connections onto the space of flat connections.
First, let $K = U(n)$. The existence of the finite-time flow of the Yang-Mills vector field is proved in Donaldson-Kronheimer [12], for four-manifolds, and for the easier two-dimensional case in Daskalopolous [10]. The convergence at infinite time is due to Råde [28], for two or three dimensions. This proves (a) and (b). Part (c) is due to Donaldson [11] for stable bundles, and Daskalopolous [10] for semi-stable bundles in the case $K = U(n)$. For arbitrary $K$, embed $K \to U(n)$, and let the metric on $\mathfrak{g}$ be the pull-back of an invariant metric on $U(n)$. The Yang-Mills flow on $U(n)$-connections pulls back to the Yang-Mills flow on $K$-connections. Hence (b) and (c) follow from the preceding paragraph. Part (c) follows from Lemma 2.9.

We say that two connections $A_1, A_2$ are Yang-Mills equivalent if the Yang-Mills limits $\tilde{A}_1, \tilde{A}_2$ are in the same $K$-orbit. In the appendix we check that this equivalence relation is the same as Ramanathan’s equivalence on holomorphic $G$-bundles. Therefore, we have the following generalization of the Donaldson-Daskalopolous theorem to arbitrary structure groups.

**Theorem 4.4.** The map $A \mapsto A_\infty$ induces a homeomorphism $\mathcal{M}_G(X) \to \mathcal{R}_K(X)$.

**4.1. Correspondence for equivariant bundles.** Let $\tilde{E} \to \tilde{X}$ be a $\Gamma$-equivariant bundle. The underlying smooth principal $G$-bundle $\tilde{Z}_\mathbb{C}$ admits a reduction to a $\Gamma$-equivariant smooth principal $K$-bundle $\tilde{Z}$, by embedding in $\text{GL}(n)$. Let $\mathcal{R}_{K,\Gamma}(\tilde{X})$ denote the moduli space of $\Gamma$-invariant connections on $\tilde{Z}$. If $\tilde{A}$ is a $\Gamma$-invariant connection, then the tangent vector $\Theta(\tilde{A})$ is also $\Gamma$-invariant. The Yang-Mills limit $\tilde{A}_\infty$ is therefore a $\Gamma$-invariant flat connection. If $\tilde{A}$ is semi-stable, then $\tilde{A}_\infty$ is flat, by 4.3 (c).

**Theorem 4.5.** The map $\tilde{A} \mapsto \tilde{A}_\infty$ induces a homeomorphism $\mathcal{M}_{G,\Gamma}(\tilde{X}) \to \mathcal{R}_{K,\Gamma}(\tilde{X})$.

**Proof.** It only remains to show that the map is a bijection, which follows from Lemma A.3 below. \qed

In order to apply the theory of $\Gamma$-equivariant bundles to the parabolic case, we need the following lemma on existence of finite totally ramified covers.

**Lemma 4.6.** [14, 5.2] If $N$ is odd or $b$ is even, then there exists a $\Gamma$-cover $\pi : \tilde{X} \to X$ totally ramified at $x_1, \ldots, x_b$.

Therefore, we can assume that $\tilde{X}$ exists, at least after adding a marked point with marking $\mu = 0$. Theorem 4.1 now follows by combining Theorems 2.3, 4.5, 3.2, and Lemma 4.6.

5. **Parabolic bundles on the projective line**

We now turn to the question of which moduli spaces are non-empty. For genus $g > 0$, all moduli spaces are non-empty. This is a consequence of the following well-known fact:

**Lemma 5.1.** The commutator map $K \times K \to K$ is surjective.
Proof. Let \( w \in W \) be a Coxeter element. The map \( T \to T, \ t \mapsto [w,t] \) is surjective, since \( \text{Ad}(w) \) has no eigenvalue equal to 1. The claim now follows from conjugation equivariance of the commutator map. \( \square \)

Therefore, the question is interesting only in genus zero.

**Lemma 5.2.** For any curve \( X \) and markings \( \mu_1, \ldots, \mu_b \), if the moduli space \( \mathcal{M}_G(X, x, \mu) \) is non-empty then the general element of \( \mathcal{M}_G(X, x, \mu) \) has an underlying bundle that is ordinary semi-stable.

**Proof.** Since \( K \) is simply-connected, any smooth \( K \)-bundle \( Z \) on \( X \) is trivial. We identify the space of connections \( \mathcal{A}(Z) \) on \( Z \) with the space of holomorphic structures on \( Z_\mathbb{C} \). The subset \( \mathcal{A}(Z)_\text{ss} \) of ordinary semistable holomorphic structures on \( Z_\mathbb{C} \) is open and dense in the space of all holomorphic structures [2]. Let \( \mathcal{A}(Z, x) \) denote the set of holomorphic structures plus parabolic reductions at the marked points, \( \pi : \mathcal{A}(Z, x) \to \mathcal{A}(Z) \) the projection, and \( \mathcal{A}(Z, x)_\text{ss} \) the dense subset corresponding to parabolic semistable bundles with markings \( \mu \). The moduli space \( \mathcal{M}_G(X, x, \mu) \) is the quotient of \( \mathcal{A}(Z, x)_\text{ss} \) by the grade equivalence relation \( \sim \). Therefore, \( (\mathcal{A}(Z, x)_\text{ss} \cap \pi^{-1}(\mathcal{A}(Z)_\text{ss}))/\sim \) is dense in \( \mathcal{M}_G(X, x, \mu) \). \( \square \)

**Corollary 5.3.** Let \( X = \mathbb{P}^1 \). Then \( \mathcal{M}_G(X, x, \mu) \) is non-empty if and only if the trivial bundle \( E = X \times G \) with general parabolic structures at the marked points \( x_1, \ldots, x_b \) is parabolic semi-stable.

**Proof.** By the lemma, \( \mathcal{M}_G(X, x, \mu) \) is non-empty if and only if a semistable bundle \( E \) with general parabolic structures \( \varphi_i \) is parabolic semi-stable. By the Birkhoff-Grothendieck theorem [18] any principle \( G \)-bundle admits a reduction of the structure group to a complex maximal torus \( H \). A principal \( H \)-bundle of degree 0 is semi-stable if and only if it is trivial, which completes the proof. \( \square \)

The trivial bundle \( X \times G \) with parabolic structures \( (\varphi_i, \mu_i) \) is parabolic stable (resp. semi-stable) if and only if

\[
\sum_{i=1}^{b} \omega_P(w_i, \mu_i) < d \quad \text{(resp.} \quad \leq d \quad \text{)}
\]  

for every maximal parabolic \( P_i, ([w_1], \ldots, [w_b]) \in W_i \text{\textbackslash}W/P \) such that there exists a reduction \( \sigma : X \to G/P \) with \( \text{deg}(\sigma^*E(\omega_P)) = d \) with \( \sigma(x_i) \) in position \( w_i \) relative to \( \varphi_i \) for each \( i = 1, \ldots, b \).

Since \( \Delta(b) \) is a polytope of maximal dimension, it suffices to consider the case that \( \mu_1, \ldots, \mu_b \) are rational and lie in the interior of \( \mathfrak{A} \). In this case, \( P_i = B \) for all \( x_i \). Let \( \pi : G \to G/B \) denote the projection. Let \( \varphi_i = \pi(g_i) \) for some \( g_1, \ldots, g_b \in G \). The element \( \sigma(x_i) \in G/P \) lies in position \( w_i \) relative to \( \varphi_i \in G/B \) only if \( \sigma(x_i) \) lies in the Schubert cell \( g_iC_{w_i} \). Therefore,
Proposition 5.4. The polytope $\Delta(b)$ is the set of points $(\mu_1, \ldots, \mu_b) \in \mathbb{A}^b$ satisfying

$$\omega_P \left( \sum_{i=1}^b w_i \mu_i \right) \leq d$$

for all maximal parabolics $P$ and $(w_1, \ldots, w_b) \in (W/W_P)^b$ such that there exists a holomorphic curve $\sigma : \mathbb{P}^1 \to G/P$ with $\sigma(x_i) \in g_i C_{[w_i]}$, for general $g_i \in G$.

We call an inequality (8) essential if it actually defines a facet (codimension one face) of $\Delta(b)$. It remains to show that the essential inequalities are those corresponding to the structure coefficients $n_d(w_1, \ldots, w_b) = 1$. The argument is the same as that of Belkale [5] in the vector bundle case. Let $(P; w_1, \ldots, w_b; d)$ define an essential inequality, and let $(w_1, \ldots, w_b) \in \mathbb{A}^b$ be a point which violates that inequality, and no others. Let $E$ be a trivial $G$-bundle over $\mathbb{P}^1$, with generic parabolic structures $\varphi_i$ and parabolic weights $\mu_i$. Since $E$ is unstable, the canonical parabolic reduction $\sigma_E$ is non-trivial, and defines an inequality which is violated by $(\mu_1, \ldots, \mu_b)$. Since only one inequality is violated, $\sigma_E$ must be a reduction to a maximal parabolic, and the corresponding inequality must be given by the data $(\omega_P, w_1, \ldots, w_b, d)$. The type of $E$ is given by $\omega_P \mapsto -d + \sum_i \omega_P(w_i \mu_i)$. Since the canonical reduction is the unique reduction of this slope, we must have $n_d(w_1, \ldots, w_b) = 1$.

APPENDIX A. EQUIVALENCE OF SEMI-STABLE BUNDLES

As discussed above, in order to construct a moduli space for semi-stable bundles one must impose an equivalence relation stronger than isomorphism. There are several ways of defining this equivalence, and we must show the different definitions lead to the same result. First, we consider equivalence relations on bundles without additional structure. According to Ramanathan [29, Proposition 3.12], for any semi-stable bundle $E \to X$, there is a semi-stable bundle $\text{Gr}(E)$, unique up to isomorphism, defined by the condition that there is an admissible reduction $\sigma : X \to E/P$ such that $r_* \sigma^* E$ is stable and $\text{Gr}(E) \cong \mathcal{L}_* r_* \sigma^* E$. The set of isomorphism classes of semi-stable $G$-bundles $E$ such that $E \cong \text{Gr}(E)$ form a set of representatives for the equivalence classes of semi-stable $G$-bundles over $X$. That is, two bundles $E_1, E_2 \to X$ are grade equivalent, if and only if their grade bundles $\text{Gr}(E_1), \text{Gr}(E_2)$ are isomorphic.

There are three other ways of defining equivalence on semi-stable bundles, besides grade equivalence. Let $E \to S \times X$ be a family of principal $G$-bundles, and $E_0$ a bundle such that $E_s \cong E_0$, for $s$ varying in a dense open subset of $S$. Then $E_s$ is $S$-equivalent to $E_0$, for any $s \in S$. Proposition 3.24 in [29] states that semi-stable bundles are grade equivalent, if and only if they are $S$-equivalent. If $E_s$ is a continuous family of holomorphic structures on a smooth principal $G$-bundle, such that $E_s$ lie in the orbit of a bundle $E_0$ for $s$ varying in a dense open subset of $S$, we say that $E_s$ are topologically $S$-equivalent. Finally, we say that two semi-stable bundles are Yang-Mills equivalent, if the corresponding flat $K$-bundles are isomorphic.
Lemma A.1. The grade, holomorphic $S$, topological $S$, and Yang-Mills equivalence relations are identical.

The identity of grade and holomorphic $S$-equivalence is proved in Ramanathan [29]. By [29, 4.15.2], there exists a (non-singular, projective) universal space for semistable $G$-bundles on $X$, which we call $\mathcal{U}_G(X)$ (Ramanathan’s $R_3$). What we want to check is that $\mathcal{U}_G(X)$ has the universal property for topological families, at least locally. That is, a continuous family (say, in the Sobolev topology) $E_s$ of semistable $G$-bundles defines a continuous family $x_s$ in $\mathcal{U}_G(X)$, in a neighborhood of any $s_0 \in S$. Indeed, the space $\mathcal{U}_G(X)$ is constructed from an embedding $\iota : G \to \text{Gl}(V)$, and a line bundle $L \to X$, such that any bundle $\iota_s(E) \otimes L$ is generated by globally sections, and the higher cohomology of $\iota_s(E) \otimes L$ vanishes. A point in $\mathcal{U}_G(X)$ is essentially a choice of generating sections for $\iota_s(E)$, together with a $G$-structure on $\iota_s(E)$. Since higher cohomology vanishes, the global sections of $\iota_s(E_s) \otimes L$ form a topological vector bundle over the parameter space $S$. Therefore, we can choose a continuous family of generating sections $f_1(s), \ldots, f_N(s)$, which generate $\iota_s(E_s)$ for any $s \in S_0$, for a neighborhood $S_0$ of $s_0$. Together with the $G$-structure on $\iota_s(E_s)$ these give the family $x_s$. Since $\mathcal{M}_G(X)$ is a good quotient of $\mathcal{U}_G(X)$, the family $[E_s]$ is a continuous path in $\mathcal{M}$. This shows that $E_0$ and $E_s$ are $S$-equivalent, for any $s \in S$.

Remark A.2. The discussion also shows that $\mathcal{M}_G(X)$ is a coarse moduli space in the topological category. That is, it represents the functor from topological spaces to sets that assigns to any topological space $S$ the set of continuous families $E_s, s \in S$ of equivalence classes of semistable holomorphic $G$-bundles over $X$.

It remains to show that Yang-Mills equivalence is the same as topological $S$-equivalence. This is a consequence of Ramanathan’s correspondence theorem, but one can also prove it directly as follows. For simplicity of notation we identify the space of connections $\mathcal{A}(Z)$ on a smooth $K$-bundle $Z$ with the space of holomorphic structures on $Z_C$. If $A_1, A_2 \in \mathcal{A}^a(Z)$ are Yang-Mills equivalent, then they are obviously topologically $S$-equivalent. Conversely, suppose $A_1, A_2$ are semistable and grade equivalent. Since $A_{i, \infty}$ is flat, $r_s \sigma^* A_{i, \infty} \cong \sigma^* A_{i, \infty}$ for any parabolic reduction $\sigma$. This is well-known for vector bundles [11]; the general case follows from embedding [29, 3.15]. Therefore, $A_{i, \infty} \cong \text{Gr}_\Gamma (A_{i, \infty})$. The connection $A_{i, \infty}$ is topologically $S$-equivalent to $A_i$ and therefore grade-equivalent, so $A_{i, \infty} \cong \text{Gr}_\Gamma (A_i)$. That is, the Yang-Mills limit $A_{i, \infty}$ is the grade bundle of $A_i$, so $A_{1, \infty} \cong A_{2, \infty}$ as holomorphic bundles. Finally, flat connections isomorphic by a complex gauge transformation are related by a unitary gauge transformation [11, p.276], so $A_{1, \infty} \cong A_{2, \infty}$ are isomorphic as connections on $Z$.

A.1. The equivariant case. Let $\Gamma$ be a finite group acting on $X$, and $E \to S \times X$ be a family (resp. topological family) of $\Gamma$-equivariant principal $G$-bundles, and $E_0$ an equivariant bundle such that $E_s \cong E_0$, for $s$ varying in a dense open subset of $S$. Then $E_s$ is called $S^\Gamma$-equivalent (resp. topologically $S^\Gamma$-equivalent) to $E_0$, for any $s \in S$. 
Lemma A.3. The grade\(_T\), holomorphic \(S\), topological \(S\), and equivariant Yang-Mills equivalence relations are identical.

In order to prove the lemma, we sketch a construction of the moduli space of equivariant \(G\)-bundles, as a subquotient of the moduli space of bundles with level structure. A level structure on \(E\) at a point \(x \in X\) is a point \(e_x\) in the fiber \(E_x\). Bundles with level structure have no automorphisms, since the map \(\text{Aut}(E) \to \text{Aut}(E_x)\) is injective, for any bundle \(E\). A morphism of bundles with level structure \((E_1, e_{1,x}), (E_2, e_{2,x})\) is a morphism \(\varphi: E_1 \cong E_2\) such that \(\varphi(e_{1,x}) = e_{2,x}\). Let \(\text{Bun}(X; y_1, \ldots, y_N)\) denote the functor which associates to any scheme \(S\) the set of isomorphism classes of \(G\)-bundles over \(S \times X\) with level structures at points \(y_1, \ldots, y_N \in X\). Let \(\text{Bun}^{ss}(X; y_1, \ldots, y_N)\) denote the open subfunctor defined by the condition that the underlying bundle is semi-stable. Gieseker’s construction in the case \(G = \text{Gl}(n)\), (or the more general construction given in [21]) implies that \(\text{Bun}^{ss}(X; y_1, \ldots, y_N)\) is represented by a smooth quasi-projective moduli space \(\mathcal{M}_G(y_1, \ldots, y_N)\).

For other connected complex reductive groups, it follows from the embedding argument of [29, 4.8.1]. The right action of \(G\) on the fiber at each marked point induces an action of \(G^N\) on \(\mathcal{M}_G(y_1, \ldots, y_N)\). The map \(\mathcal{M}_G(y_1, \ldots, y_N) \to \mathcal{M}_G\) forgetting the level structures, is a good quotient for the action. Indeed, via the theory of Hilbert schemes as in [30, Section 5] one may construct a universal space (see e.g. [30, 4.6] for the definition)) \(U_G(y_1, \ldots, y_N)\) for \(G\)-bundles with level structure at \(y_1, \ldots, y_N\), such that \(U_G(y_1, \ldots, y_N) \to \mathcal{M}_G(y_1, \ldots, y_N)\) is a good quotient. The map \(U_G(y_1, \ldots, y_N) \to U_G\) is a principal \(G^N\)-bundle, and therefore a good quotient. Therefore, \(\mathcal{M}_G(y_1, \ldots, y_N) \to \mathcal{M}_G\) is also a good quotient.

Now we turn to equivariant level structures. Suppose that the set \(\{y_1, \ldots, y_N\}\) is invariant under \(\Gamma\), and the stabilizers \(\Gamma_{y_i}\) are trivial. An equivariant bundle with level structure is a bundle with level structure \((E, e_{y_1}, \ldots, e_{y_N})\) together with a lift of the \(\Gamma\)-action such that \(\gamma(e_{y_i}) = e_{\gamma(y_i)}\). Let \(\mathcal{M}_{G,\Gamma}(y_1, \ldots, y_N)\) denote the set of equivariant bundles with level structure, up to isomorphism, whose underlying bundle is semistable. Any equivariant bundle with level structure defines a bundle with level structure whose isomorphism class is fixed by the action by pullback of \(\Gamma\). Since bundles with level structure have no automorphisms, forgetting the equivariant structure defines an injection \(\mathcal{M}_{G,\Gamma}(y_1, \ldots, y_N)\) into \(\mathcal{M}_G(y_1, \ldots, y_N)\). Its image is the fixed point set of \(\Gamma\), which is a non-singular quasi-projective variety. Let \(G^N_\Gamma\) denote the subgroup of \(G^N\) invariant under the action of \(\Gamma\) on \(G^N\) induced by the action of \(\Gamma\) on the set \(\{y_1, \ldots, y_N\}\). Since the action of \(G(y_1, \ldots, y_N)\) on \(\mathcal{M}_G(y_1, \ldots, y_N)\) has a good quotient, the action of the subgroup \(G^N_\Gamma\) also has a good quotient. Therefore, the action of \(G^N_\Gamma\) on the subvariety \(\mathcal{M}_\Gamma(y_1, \ldots, y_N)\) also has a good quotient, which we denote \(\mathcal{M}_{G,\Gamma}\). Since \(\mathcal{M}_\Gamma(y_1, \ldots, y_N)\) is normal and quasi-projective, so is \(\mathcal{M}_{G,\Gamma}\).

We claim that \(\mathcal{M}_{G,\Gamma}\) is a coarse moduli space for the functor of \(S_\Gamma\)-equivalence classes of \(\Gamma\)-equivariant bundles. Let \(E\) be a \(\Gamma\)-equivariant semi-stable bundle over \(S \times X\), and \(s\) any point in \(S\). In a neighborhood \(S_1\) of \(s\), \(E\) admits equivariant level structures at \(y_1, \ldots, y_N\). Therefore \(E|_{S_1}\) is induced from a map \(S \to \mathcal{M}_\Gamma(y_1, \ldots, y_N)\). If \(E_s\) are equivariantly

PARABOLIC BUNDLES

21
isomorphic for \( s \) in an open subset \( S_0 \subset S \), then the image of \( S_0 \cap S_1 \) in \( \mathcal{M}^\Gamma(y_1, \ldots, y_N) \) is contained in the closure of a single orbit. Conversely, if \( E_0 \in \mathcal{M}^\Gamma(y_1, \ldots, y_N) \) lies in the closure \( \overline{E} \) of the orbit of \( E_1 \in \mathcal{M}^\Gamma(y_1, \ldots, y_N) \), then forgetting the level structure on \( \overline{E} \) shows that \( E_0 \) and \( E_1 \) are \( S_\Gamma \)-equivalent. This shows that the points of \( \mathcal{M}_{G,\Gamma} \) are \( S_\Gamma \)-equivalence classes of semistable bundles. For any family \( E \to S \times X \) of equivariant semi-stable \( G \)-bundles which admits equivariant level structure over \( S_1 \subset S \), let

\[
\varphi_{E,S_1} : S_1 \to \mathcal{M}_{G,\Gamma}
\]
denote the map induced by adding some level structure, \( S_1 \to \mathcal{M}^\Gamma(y_1, \ldots, y_N) \), and then composing with the projection. It is clear that \( \varphi_E \) does not depend on the choice of level structure, so that \( \varphi_{E,S_1} \) patches together to a map \( \varphi_E \), and \( E \mapsto \varphi_E \) defines a morphism of functors \( c_\Gamma : \text{Bun}_\Gamma(X, x, \mu) \to \mathcal{M}_{G,\Gamma} \). By a similar argument, for any morphism of functors \( c' : \text{Bun}_\Gamma(X, x, \mu) \to \mathcal{M}' \), there is a unique morphism \( \varphi : \mathcal{M}_{G,\Gamma} \to \mathcal{M}' \) such that \( c' = c \circ \varphi \). This shows that \( (\mathcal{M}_{G,\Gamma}, c_\Gamma) \) is a coarse moduli space for the functor \( \text{Bun}_\Gamma(X, x, \mu) \). Projectivity of the moduli space follows from the correspondence theorem in Section 4, since the moduli space of flat bundles with fixed holonomy is compact.

Finally we check the identity of the four equivalence relations. The \( \Gamma \)-equivariant version of the argument in Ramanathan [29, Section 3] shows that \( S_\Gamma \)-equivalence, is the same as grade\( \Gamma \)-equivalence. Topological \( S \)-equivalence is also the same, since as explained above a continuous path \( E_s \) of holomorphic structures on a fixed smooth principal \( G \)-bundle induces a continuous path in \( \mathcal{M}_G(y_1, \ldots, y_N) \). If \( E_s \) are \( \Gamma \)-equivariant, \( E_s \) gives a continuous path in \( \mathcal{M}_{G,\Gamma} \). If \( E_s \) are isomorphic to \( E_0 \), for a dense open subset of \( s \in S \), then the image of \( E_s \) in \( \mathcal{M}_{G,\Gamma} \) is a single point. Therefore, \( E_s \) is \( S_\Gamma \)-equivalent to \( E_0 \) for any \( s \in S \). This shows that topological \( S_\Gamma \)-equivalence implies \( S_\Gamma \)-equivalence. The proof that the equivariant Yang-Mills equivalence relation on semistable \( \Gamma \)-equivariant holomorphic bundles is the same as grade\( \Gamma \) equivalence and \( S_\Gamma \)-equivalence, is the same as in the non-equivariant case.

A.2. The parabolic case. Let \( \Gamma \) be a cyclic group, acting on \( \hat{X} \) so that \( r : \hat{X} \to X \) is totally ramified at \( x_1, \ldots, x_b \). The \( S \) equivalence relations on equivariant bundles on \( \hat{X} \), resp. parabolic bundles on \( X \) are identical by the isomorphism of functors Theorem 2.3.

Lemma A.4. Two \( \Gamma \)-equivariant bundles are grade\( \Gamma \)-equivalent, if and only if the corresponding parabolic bundles are parabolic grade equivalent.

Proof. Let \( \hat{E} \to \hat{X} \) be a \( \Gamma \)-equivariant bundle, and \((E, \varphi, \mu)\) the corresponding parabolic bundle. Let \( \hat{\sigma} \) be a \( \Gamma \)-invariant parabolic reduction to a parabolic subgroup \( P \) and \( \sigma \) the corresponding parabolic reduction of \( E \). It suffices to show that \( \iota_*, r_* \sigma^* E \) is the parabolic bundle corresponding to \( \iota_* r_* \hat{\sigma}^* \hat{E} \).

Let \( \hat{U}_i \times G \) be a local trivialization near \( x_i \), so that the action of \( \Gamma \) is \((u, g) \mapsto (\exp(2\pi i/N)z, \exp(\mu)g)\). The parabolic reduction \( \hat{\sigma} \) is given near \( x_i \) by \( \hat{\sigma}_i(z)P \), for some map \( \hat{\sigma}_i : \hat{U}_i \to G \) with \( \hat{\sigma}_i(0) = w_i^{-1} \). By equivariant Oka 2.5 applied to \( \hat{\sigma}_i^* \hat{E} \), we may take \( \hat{\sigma}(z) = w_i^{-1} \) constant. The local trivialization of \( \hat{\sigma}_i \hat{E} \) induces a local trivialization
of $r_*\tilde{\sigma}^*\tilde{E}$ near $x_i$. The action of $\Gamma$ is given in this local trivialization by 
$$(z, l) \mapsto (\exp(2\pi i/N)z, \exp(w_i\mu_i)l).$$

Let $\mu_{L,i}$ be the markings for $L$ conjugate to the elements $w_i\mu_i$. The parabolic bundle corresponding to $r_*\tilde{\sigma}^*\tilde{E}$ is $(E_L, \varphi_L, \mu_L)$ where $E_L = (r_*\tilde{\sigma}^*\tilde{E})^{-N\mu_L}/\Gamma$.

The bundle $E_L$ is isomorphic to $r_*\sigma^*E$. Indeed, define $w_{L,i}$ by 
$$w_{L,i}\mu_{L,i} = w_i\mu_i.$$

An invariant section $s_{L,i}$ of $r_*\tilde{\sigma}^*\tilde{E}$ with $s_{L,i}z^{-N\mu_{L,i}/2\pi i}$ regular is given by 
$$s_{L,i}(z) = w_{L,i}z^{N\mu_{L,i}/2\pi i}.$$

The gluing maps for $(r_*\tilde{\sigma}^*\tilde{E})^{-N\mu_L}$ are given by $s_{L,i}(z)^{-1}$. The gluing map for $r_*(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}$ is $z^{-Nw_i\mu_i/2\pi i}$. Since 
$$z^{-Nw_i\mu_i/2\pi i}w_{L,i}z^{N\mu_{L,i}/2\pi i} = w_{L,i}$$
is regular at $z = 0$, the bundles $(r_*\tilde{\sigma}^*\tilde{E})^{-N\mu_L}$, $r_*(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}$ are isomorphic. Therefore, their quotients by $\Gamma$ are isomorphic, which proves the claim. The parabolic structure for $(r_*\tilde{\sigma}^*\tilde{E})^{-N\mu_L}$ at $x_i$ is given by the standard parabolic $w_{L,i}^{-1}(L \cap w_iP_i)$ in the trivialization near $x_i$. Under the isomorphism with $r_*(\tilde{\sigma}^{-N\mu})^*\tilde{E}^{-N\mu}$ this maps to the parabolic structure $L \cap w_iP_i$. In trivialization-free notation, we have $\varphi_{L,i} = r(\tilde{\sigma}(x_i) \cap \varphi_i)$ which proves the lemma.

References

[1] S. Agnihotri and C. Woodward. Eigenvalues of products of unitary matrices and quantum Schubert calculus. Math. Res. Lett., 5(6):817–836, 1998.
[2] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. Phil. Trans. Roy. Soc. London Ser. A, 308:523–615, 1982.
[3] V. Balaji, I. Biswas, and D.S. Nagaraj. On the principal bundles over projective manifolds with parabolic structure over a divisor, 2000. preprint, to appear in Tohuku Math. J.
[4] A. Beauville and Y. Laszlo. Un lemme de descente. C. R. Acad. Sci. Paris Sér. I Math., 320(3):335–340, 1995.
[5] P. Belkale. Local systems on $P^1 - S$ for $S$ a finite set, 1998. University of Chicago preprint.
[6] A. Berenstein and R. Sjamaar. Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion. J. Amer. Math. Soc., 13(2):433–466 (electronic), 2000.
[7] U. Bhosle and A. Ramanathan. Moduli of parabolic $G$-bundles on curves. Math. Z., 202(2):161–180, 1989.
[8] I. Biswas. Parabolic bundles as orbifold bundles. Duke Math. J., 88(2):305–325, 1997.
[9] H. Boden. Representations of orbifold groups and parabolic bundles. Comment. Math. Helvetici, 66:389–447, 1991.
[10] G. D. Daskalopoulos. The topology of the space of stable bundles on a compact Riemann surface. J. Differential Geom., 36(3):699–746, 1992.
[11] S. K. Donaldson. A new proof of a theorem of Narasimhan and Seshadri. J. Differential Geom., 18(2):269–277, 1983.
[12] S. K. Donaldson and P. Kronheimer. The geometry of four-manifolds. Oxford Mathematical Monographs. Oxford University Press, New York, 1990.
[13] V. G. Drinfeld and C. Simpson. B-structures on G-bundles and local triviality. *Math. Res. Lett.*, 2(6):823–829, 1995.

[14] A. L. Edmonds, R. S. Kulkarni, and R. E. Stong. Realizability of branched coverings of surfaces. *Trans. Amer. Math. Soc.*, 282(2):773–790, 1984.

[15] M. Entov. K-area, hofer metric and geometry of conjugacy classes in Lie groups. math.SG/0009111.

[16] G. Faltings. Stable G-bundles and projective connections. *J. Algebraic Geom.*, 2(3):507–568, 1993.

[17] M. Furuta and B. Steer. Seifert fibred homology 3-spheres and the Yang-Mills equations on Riemann surfaces with marked points. *Adv. Math.*, 96(1):38–102, 1992.

[18] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[19] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[20] P. Heinzner and F. Kutzschebauch. An equivariant version of Grauert’s Oka principle. *Invent. Math.*, 119(2):317–346, 1995.

[21] D. Huybrechts and M. Lehn. Stable pairs on curves and surfaces. *J. Algebraic Geom.*, 4(1):67–104, 1995.

[22] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic G-bundles over curves and their sections. *Ann. Sci. École Norm. Sup. (4)*, 30(4):499–525, 1997.

[23] V.B. Mehta and C.S. Seshadri. Moduli of vector bundles on curves with parabolic structure. *Math. Ann.*, 248:205–239, 1980.

[24] E. Meinrenken and C. Woodward. Hamiltonian loop group actions and Verlinde factorization. *Journal of Differential Geometry*, 50:417–470, 1999.

[25] J. Millson and B. Leeb. Convex functions on symmetric spaces and geometric invariant theory for spaces of weighted configurations on flag manifolds, 2000. preprint.

[26] M. S. Narasimhan and C. S. Seshadri. Holomorphic vector bundles on a compact Riemann surface. *Math. Ann.*, 155:69–80, 1964.

[27] D. Peterson. Lectures on quantum cohomology of G/P. Montreal, 1997.

[28] J. Råde. On the Yang-Mills heat equation in two and three dimensions. *J. Reine Angew. Math.*, 431:123–163, 1992.

[29] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. *Proc. Indian Acad. Sci. Math. Sci.*, 106(3):301–328, 1996.

[30] A. Ramanathan. Moduli for principal bundles over algebraic curves. II. *Proc. Indian Acad. Sci. Math. Sci.*, 106(4):421–449, 1996.

[31] J.-P. Serre. *Cohomologie galoisienne*. Springer-Verlag, Berlin, fifth edition, 1994.

[32] I. M. Singer. The geometric interpretation of a special connection. *Pacific J. Math.*, 9:585–590, 1959.

[33] C. Teleman. Borel-Weil-Bott theory on the moduli stack of G-bundles over a curve. *Invent. Math.*, 134(1):1–57, 1998.