Congruence and Metrical Invariants of Zonotopes

Eugene Gover

The defining matrix $A$ of zonotope $Z(A) \subset \mathbb{R}^n$ determines the zonotope as both the linear image of a cube and the Minkowski sum of line segments specified by the columns of the matrix. A zonotope is also a convex polytope with centrally symmetric faces in all dimensions. When a zonotope is represented by a matrix, its volume is the sum of the absolute values of the maximal-rank minors. Sub-maximal rank minors compute the lower-dimensional volumes of facets. Maximal-rank sub-matrices determine various tilings of a zonotope, while those of submaximal rank define the angles between facets, normal vectors to facets, and can be used to demonstrate rigidity and uniqueness of a zonotope given various facet-volume and normal-vector data. Some of these properties are known. Others, are new. They will all be presented using defining matrices.

The first section focuses on the central symmetry of faces and facets of convex polytopes, and gives new proofs of theorems of Minkowski and Cauchy-Alexandrov in the case of congruences between zonotopes. The same matrix also plays an important part in the third section where new proofs of theorems of Shephard and McMullen are given in the case of submaximal rank. The second section introduces the Gram matrix $A^T A$, called the shape matrix of the zonotope, and gives it the central role in a discussion of congruences between zonotopes. The same matrix also plays an important part in the third section where new proofs of theorems of Minkowski and Cauchy-Alexandrov are given in the case of zonotopes.

1. Central Symmetry and Zonotopes

A nonempty subset $X \subset \mathbb{R}^n$ is centrally symmetric if there exists a translation $\tau$ such that $\tau(X) = -X$. The center of symmetry or reflection of the set is $c = \frac{1}{2} \tau^{-1}(0)$. In terms of $c$, $\tau(x) = -2c + x$. Equivalently, $X$ is centrally symmetric provided there exists $c \in \mathbb{R}^n$ such that $c + x \in X$ iff $c - x \in X$. This is the same as saying $x \in X$ iff $2c - x \in X$. In general, the center need not belong to the set, but it will belong if the set is convex. Given $c \in \mathbb{R}^n$, the set \{tc + (1 - t)x | x \in X, 0 \leq t \leq 2\} = cone_c(X, t) will be called the symmetric cone of $X$ centered at $c$. The subset $cone_c(X, 2) = \{2c - x | x \in X\} = X_c$ is the reflection or symmetric image of $X$ with respect to $c$. Note that $cone_c(X, 0) = X$, and $cone_c(X, 1) = \{c\}$.

The following properties are easily verified:

**Lemma 1.1.** (a) $(X_c)_c = X$; $(X_{c_1})_{c_2} = 2(c_2 - c_1) + X$; $((X_{c_1})_{c_2})_{c_3} = X_{c_3 - c_2 + c_1}$;

$(((X_{c_1})_{c_2})_{c_3})_{c_4} = 2(c_4 - c_3 + c_2 - c_1) + X$, etc.

(b) For any $c \in \mathbb{R}^n$, $cone_c(X, t)$ is centrally symmetric with center of symmetry $c$.

(c) For any $c \in \mathbb{R}^n$, $X \cup X_c$ is centrally symmetric with center of symmetry $c$.

(d) $X$ is centrally symmetric iff $X_c$ is centrally symmetric for every $c$.

(e) $X$ is centrally symmetric iff there exists $c$ such that $X = X_c$.

(f) $X$ is centrally symmetric iff for any and for every $c$ there exists $v$ such that $X_c = v + X$.

Part (a) says two successive reflections with the same center leaves a set unchanged while using different centers results in translation by twice the difference between the centers. More generally, an odd number of reflections with centers $c_1, \ldots, c_{2n+1}$ is equivalent to a single reflection with respect to the alternating sum of the centers; in particular, the image of $x \in X$ will be $2(c_{2n+1} - \cdots + c_1) - x \in X_{(c_{2n+1} - \cdots + c_1)}$. An even number of reflections is equivalent to translation by twice the alternating
sum of the centers. Parts (b) and (c) say that the symmetric cone of a set and the union of the set with any reflection are centrally symmetric. Part (d) says that a set is centrally symmetric if and only if every reflection is centrally symmetric. Part (e) says that a set is centrally symmetric if it equals some reflection of itself. The center of that reflection becomes the center of symmetry of the set. Part (f) says $X$ is centrally symmetric iff it can be translated to any and every reflection of itself.

Consider a unit cube positioned along the coordinate axes of $\mathbb{R}^k$. Its image in $\mathbb{R}^n$ under a linear transformation defined with respect to the standard bases by a real $n \times k$ matrix $A$ with columns $a_1, \ldots, a_k$ is the set $Z(A) = \{ \sum t_i a_i | 0 \leq t_i \leq 1 \}$. The set will be called the **zonotope** generated by the columns of $A$, which will in turn be called the **defining matrix** of $Z(A)$. The **rank** of the zonotope is the rank of its defining matrix. In the special case when $n \geq k$ and the columns are independent (i.e., rank $A = k$), the image is also a **parallelotope** and can be denoted as $P(A)$ or $P(a_1, \ldots, a_k)$. The parallelotopes we will consider are generated by the independent columns of tall and thin, or square matrices of rank $k$ with $n \geq k$. They are skewed, stretched, or shrunken images of cubes. The zonotopes that are not parallelotopes will be generated by the dependent columns of matrices of rank $r$ with $r < k$. They are flattened images of cubes.

The **Minkowski sum** of sets of $S_1, \ldots, S_k \subset \mathbb{R}^n$, denoted with the symbol $\oplus$, is the set $S_1 \oplus \cdots \oplus S_k = \{ s_1 + \cdots + s_k | s_i \in S_i \}$. A zonotope is the Minkowski sum of line segments: $Z(A) = l a_1 \oplus \cdots \oplus l a_k$ where the line segment $l a_i = \{ t a_i | 0 \leq t \leq 1 \}$. For a parallelotope, the generators are linearly independent and the Minkowski sum of the corresponding line segments yields a prism whose base is any Minkowski sum leaving out one of segments. (Note that the parallelotopes we will consider form a proper subset of the polytopes that fill space by translation, which are also called parallelotopes.) Cubes are convex, centrally symmetric, and the convex hulls of finite point sets. It follows that zonotopes, which are their linear images, are also convex, centrally symmetric polytopes. (As polytopes, zonotopes are also finite intersections of half-spaces.) Regarded as Minkowski sums of line segments, zonotopes are centrally symmetric for another reason: for each $\sum t_i a_i \in Z(A)$, there corresponds $\sum_i (1-t_i) a_i \in Z(A)$; these two points have center of symmetry $\sum a_i / 2$, which becomes the center of the entire zonotope.

Suppose zonotope $Z(A)$ is defined by the matrix $[a_1, \ldots, a_k] \in \mathbb{R}^{n \times k}$ of rank $r \leq k$. A subzonotope of rank $s \leq r$ of the form $Z(a_{j_1}, \ldots, a_{j_s})$ to which no further generators can be added without increasing the rank will be called a **generating face of dimension** $s$ or an **$s$-face** of $Z(A)$. The generators themselves are considered the **generating 0-faces**. A line segment $Z(a_i) = l a_i$ will be a generating 1-face or **edge** unless there is a larger, maximal collection $a_{i_1}, \ldots, a_{i_s}$ of generators containing $a_i$ with each generator a scalar multiple of the others. In that case, $Z(a_{i_1}, \ldots, a_{i_s})$ becomes an edge of $Z(A)$ containing each of the $Z(a_i)$'s. A generating $(r-1)$-face will be called a **generating facet**. A rank $r$ subzonotope with exactly $r$ generators will be called a **generating paralleloptope** of $Z(A)$. (The paralleloptope will not be a generating $r$-face unless it is the zonotope itself.)

A **bounding face** is a translation of a generating face to the boundary of the zonotope using generators not used in the definition of that face. For a generating facet $F = Z(a_{j_1}, \ldots, a_{j_r})$, the associated bounding facets can be given explicitly. Consider any $r-1$ linearly independent generators of the facet. For example, suppose $a_{j_1}, a_{j_2}, \ldots, a_{j_{r-1}}$ are independent. The cross-product of these generators is then a normal vector to the facet. (See, for example, [4].) We write this as $n_F = \times (a_{j_1}, \ldots, a_{j_{r-1}})$. Note that any two sets of $r-1$ independent generators of $F$ will give cross-products that are scalar multiples of each other. Relabel all generators $a_1, \ldots, a_k$ of the zonotope as $a_{p_1}^1, \ldots, a_{p_1}^p, a_{p_1+1}^1, \ldots, a_{q_1}^1, a_{q_1+1}^1, \ldots, a_{k_1}^1$ with superscripts designating the generators with zero, negative, and positive projections on $n_F$. It follows that $a_1^0, \ldots, a_p^0$ is a maximal set of generators of rank $r-1$ with $p = t$ and $[a_1^0, \ldots, a_p^0] = \{ a_{j_1}, \ldots, a_{j_r} \}$, and that $Z(a_1^0, \ldots, a_p^0) + a_{p+1}^- + \cdots + a_k^-$ will be one translation of $F$ to a bounding facet, while $Z(a_1^0, \ldots, a_p^0) + a_{p+1}^+ + \cdots + a_k^+$ will be the corresponding facet on the opposite side of the boundary.
We wish to revisit some results of Shephard and McMullen from [7-10] that examine how the central symmetry of the faces of a zonotope relates to the symmetry of the entire zonotope. As zonotopes in their own right, the faces of a zonotope are always centrally symmetric. For an arbitrary convex polytope, it turns out that the central symmetry of all faces of a given dimension implies the symmetry of the faces of the next higher dimension, while the central symmetry of all faces in any dimension below that of the facets implies the symmetry of the faces of the next lower dimension (McMullen, [7, 8]). Moreover, polytopes whose 2-faces are all centrally symmetric are zonotopes. Consequently, zonotopes may be characterized as the convex polytopes of dimension \( n \) whose faces of any one particular dimension \( k \) are centrally symmetric, where \( 2 \leq k \leq n - 2 \).

In order to establish these and similar results, we start by considering zones of faces of polytopes. Given a \( k \)-dimensional face \( F \) of polytope \( P \), the \( k \)-zone \( Z_k(F) \) induced by \( F \) is defined as the union of all proper faces that contain translations of \( F \). It clearly suffices to take the union only of facets, and \( Z_k(F) \) satisfies:

\[
\text{if } k < j, \text{ then } Z_k(F) = \bigcup \{ Z_j(F') \mid F \subset F' \text{ and } F' \text{ is a } j\text{-face} \}.
\]

The 1-zone \( Z_1(E) = Z(E) \) induced by an edge \( E \) is called simply a \textit{zone}. It is the traditional zone that give rise to the name zonotope.

The following is a consequence of Shephard’s Theorem 2, from [9].

\textbf{Lemma 1.2.} Let \( P \) be a convex \( n \)-dimensional polytope in \( \mathbb{R}^n \) whose faces of dimension \((j + 1)\) are all centrally symmetric, where \((j + 1)\) is such that \( 2 \leq (j + 1) \leq n \). Consider an orthogonal projection of \( \mathbb{R}^n \) to a complement of the \( j \)-dimensional affine subspace supporting a particular \( j \)-dimensional face \( F \) of \( P \). Then the image of \( P \) under this projection is an \( (n - j) \)-dimensional convex polytope, \( \pi(P) \), and all faces of \( P \) of dimension \( j \) that are congruent to \( F \) map in one-to-one fashion to the vertices of \( \pi(P) \). For each value \( k \) with \( j \leq k \leq n \), all \( k \)-dimensional faces of \( P \) containing \( F \) are mapped in one-to-one fashion to all \((k - j)\)-dimensional faces of \( \pi(P) \).

Using this lemma, it is possible to give a new proof of an \( n \)-dimensional version of a theorem of P. Alexandrov different from the proofs given in [3] and [9].

\textbf{Proposition 1.3.} If all facets of a convex \( n \)-polytope \((n > 2)\) are centrally symmetric, then the polytope is centrally symmetric.

\textit{Proof.} Consider an \( n \)-dimensional polytope \( P \) in \( \mathbb{R}^n \). Let \( F_1 \) be a facet of \( P \) with center of symmetry \( c_1 \), and let \( F_{1,1} \) be an \((n - 2)\)-face of \( P \) that is a facet of \( F_1 \). Central symmetry ensures that the reflection \((F_{1,1})_{c_1} = F_{1,2} \) is the face of \( F_1 \) opposite to \( F_{1,1} \). This face is shared with an adjacent facet, \( F_2 \). Let \( c_2 \) be the center of \( F_2 \). The reflection \((F_{1,2})_{c_2} = F_{1,3} \) is then the face opposite \( F_{1,2} \) on the boundary of \( F_2 \). (It is also a translation of \( F_{1,1} \).) Face \( F_{1,3} \) is shared with another facet, \( F_3 \). In this way, successive \((n - 2)\)-faces \( F_{1,1}, F_{1,2}, \ldots, F_{1,m_1+1} = F_{1,1} \) are determined that are alternately reflections and translations of \( F_{1,1} \). The faces determine a corresponding chain of facets, \( F_1, F_2, \ldots, F_{m_1+1} = F_1 \), whose union, \( F_1 \cup F_2 \cup \cdots \cup F_{m_1} = Z_{(n-2)}(F_{1,1}) \), is an \((n - 2)\)-zone on the boundary of \( P \).

Choose a face \( F_{2,1} \) adjacent to \( F_{1,1} \) on the boundary of \( F_1 \). This determines another sequence of \((n - 2)\)-dimensional faces, \( F_{2,1}, F_{2,2}, \ldots, F_{2,m_2+1} = F_{2,1} \), consisting of reflected and translated copies of \( F_{2,1} \) and another sequence of facets whose union is a second \((n - 2)\)-zone, \( Z_{(n-2)}(F_{2,1}) \), on the boundary of \( P \). Project \( \mathbb{R}^n \) to the orthogonal complement of the \((n - 3)\)-dimensional affine subspace supporting the face \( F_{1,2,1} = F_{1,1} \cap F_{2,1} \). It follows from Lemma 1.2 that the projections of the two \((n - 2)\)-zones of facets become zones of \( 2 \)-faces on the boundary of \( 3 \)-dimensional \( \pi(P) \). Zones on a convex polyhedron are circumferential; any two intersect twice. As the projected zones on \( \pi(P) \) both include \( \pi(F_1) \), they must therefore intersect a second time. It follows that the \((n - 2)\)-zones of preimages must also intersect twice. In other words, if \((n - 2)\)-zones on the boundary of a convex \( n \)-dimensional polytope with centrally symmetric facets intersect at all, then they intersect twice.

The two \((n - 2)\)-zones under consideration intersect at \( F'_{1} = F_1 \). Hence they also intersect at \( F'_j = F_k \) for some \( j, k > 1 \). Facet \( F_k \) then includes congruent copies of both \( F_{1,1} \) and \( F_{2,1} \) as part
of its boundary. The same is true for $F_1$. The two facets are therefore parallel. By convexity, $F_k$ is the unique facet of $P$ parallel to $F_1$. Denote it as $F_1^{\text{op}}$. In this way, every facet of $P$ is paired with a unique parallel, opposite facet. In particular, $P$ and all zones of facets of $P$ contain even numbers of facets in parallel, opposite pairs.

We wish to show that $F_1$ and $F_1^{\text{op}}$ are symmetric images of each other. To see that this is so, let $c_1$ be the center of $F_1$ and let $c_1^{\text{op}}$ be the center of $F_1^{\text{op}}$. Set $c = \frac{1}{2}(c_1 + c_1^{\text{op}})$. Consider the symmetric image $(F_1)_c$ of $F_1$, which is a centrally symmetric $(n-1)$-polytope that must lie in the same hyperplane, $H$, as $F_1^{\text{op}}$. We will see that $(F_1)_c$ and $F_1^{\text{op}}$ are identical. Each can be defined in terms of the intersection of $H$ with half-spaces determined by the hyperplanes supporting all adjacent facets. Suppose $F_*$ is a facet of $P$ adjacent to $F_1$. Then $(F_*)_c$ will be adjacent to $F_c$, and $F_*^{\text{op}}$ will be adjacent to $F^{\text{op}}$. Denote the hyperplanes supporting $F_*$, $(F_*)_c$, and $F_*^{\text{op}}$ by $H_*$, $(H_*)_c$, and $(H_*)^{\text{op}}$ respectively. These hyperplanes are parallel, and the latter two contain the center of symmetry $(c_*)_c = c_*^{\text{op}}$ common to both $(F_*)_c$ and $F_*^{\text{op}}$. Hence $(H_*)_c = (H_*)^{\text{op}}$. This hyperplane defines two half-spaces one of which contains $F_*$ and is included among the half-spaces whose intersection with $H$ defines both $(F_*)_c$, and $F_*^{\text{op}}$. The other half-spaces defining the two facets are determined in a similar manner. As a result, $(F_*)_c$ and $F_*^{\text{op}}$ have the same definition in terms of intersections, and so $(F_*)_c = F_*^{\text{op}}$. Consequently, $F_1$ and $F_1^{\text{op}}$ are reflections of each other. $F_1 \cup F_1^{\text{op}}$ is therefore centrally symmetric with center of symmetry $c_{1,1} = c = \frac{1}{2}(c_1 + c_1^{\text{op}})$. The same can then be said for all facets of $P$.

Thus, for each facet $F_j$ of $P$, the union $F_j \cup F_j^{\text{op}}$ is centrally symmetric with center of symmetry $c_{j,j} = \frac{1}{2}(c_j + c_j^{\text{op}})$. Moreover, if $F_j$ and $F_k$ are adjacent facets sharing an $(n-2)$-face, the centers of symmetry agree on that face making those centers the same: $c_{j,j} = c_{k,k}$. By moving around the entire boundary of $P$ from facet to adjacent facet, all opposite pairs of facets share a common center of symmetry. This becomes the center of symmetry for the entire polytope, which is therefore centrally symmetric.

An immediate consequence is:

**Corollary 1.4.** If the $k$-dimensional faces of an $m$-dimensional convex polytope in $\mathbb{R}^n$ are centrally symmetric for a particular value of $k \geq 2$, then the $(k+1)$-dimensional faces are also centrally symmetric.

Proposition 1.3 and Corollary 1.4 apply to a convex polytope when $k \geq 2$. When $k = 1$ and $m = 2$, a convex polygon (the convex hull of finitely many points in a 2-dimensional affine subspace of $\mathbb{R}^n$) has 1-dimensional edges that are centrally symmetric, but the polygon itself need not be centrally symmetric. The following condition shows when an arbitrary polygon or when any closed arrangement of line segments is centrally symmetric.

**Proposition 1.5.** A 2-dimensional polygon $\subset \mathbb{R}^n$ is centrally symmetric if and only if it has pairs of equal, parallel, opposite edges. The same is true for any 1-dimensional closed configuration of line segments in $\mathbb{R}^n$.

**Proof.** One direction is clear. In the other direction, assume a polygon or closed configuration of line segments has pairs of equal, parallel opposite edges. (Such a closed configuration of line segments consists of a finite sequence of line segments $s_1, \ldots, s_{2n}$ where $s_j$ and $s_{n+j}$ are parallel of equal length for $j = 1, \ldots, n$, $s_j$ and $s_{j+1}$ share an endpoint for $j = 1, \ldots, 2n$, and $s_{2n+1} = s_1$.) Consider adjacent edges $s_1$ and $s_2$, and their equal and parallel opposites, $s_1^{\text{op}}$ and $s_2^{\text{op}}$. The pairs $s_1 \cup s_1^{\text{op}}$ and $s_2 \cup s_2^{\text{op}}$ are each centrally symmetric with centers of symmetry $c_1$ and $c_2$ respectively. The centers $c_1$ and $c_2$ must then be the same because $s_1$ and $s_2$ share a common endpoint. By moving around the entire polygon or configuration, all centers of symmetry of pairs of opposite edges must be the same, so the polygon or configuration is centrally symmetric.

**Proposition 1.6.** A convex $m$-polytope $\subset \mathbb{R}^n$ in which all 2-faces are centrally symmetric is a zonotope.
Proof. When \( m = 2 \), consider a centrally symmetric filled-in polygon in the plane. Select any edge of the polygon together with the opposite edge, which is its reflection with respect to the center of symmetry. Connecting the endpoints of the two edges produces a strip in the form of a parallelogram, which by virtue of convexity is wholly contained within the polygon. If the strip is removed, translation by a vector defined by the selected edge joins the endpoints of each removed edge to produce a smaller, centrally symmetric polygon with two fewer edges. The original polygon is the Minkowski sum of the new polygon and the selected edge. By downward induction, the original polygon becomes the Minkowski sum of edges and hence is a zonogon.

Now suppose \( m \geq 3 \). Assume as an induction hypothesis that the proposition is true for all polytopes of dimension \( < m \). Consider an \( m \)-dimensional convex polytope \( \mathcal{P} \) with centrally symmetric 2-faces. Select an edge \( \mathcal{E} \) of \( \mathcal{P} \) and let the zone \( Z(\mathcal{E}) = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_t \) be the union of all facets containing \( \mathcal{E} \). All facets belonging to \( Z(\mathcal{E}) \) have centrally symmetric 2-faces, so by the induction hypothesis, each is a zonotope containing \( \mathcal{E} \) as an edge. Each \( \mathcal{F}_j \) therefore decomposes as the Minkowski sum of \( \mathcal{E} \) with a smaller zonotope, \( \mathcal{F}_j^* \). Thus, \( \mathcal{F}_j = \mathcal{F}_j^* \oplus \mathcal{E} \). It follows that
\[
Z(\mathcal{E}) = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_t = (\mathcal{F}_1^* \oplus \mathcal{E}) \cup (\mathcal{F}_2^* \oplus \mathcal{E}) \cup \cdots \cup (\mathcal{F}_t^* \oplus \mathcal{E}) = (\mathcal{F}_1^* \cup \mathcal{F}_2^* \cup \cdots \cup \mathcal{F}_t^*) \oplus \mathcal{E}.
\]
(Note that for any subsets \( A, B, C \subset \mathbb{R}^n \), \( (A \oplus C) \cup (B \oplus C) = (A \cup B) \oplus C \).) Suppose the remaining facets of \( \mathcal{P} \) are \( \mathcal{F}_1', \mathcal{F}_2', \ldots, \mathcal{F}_s' \) so that the boundary is \( (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots \cup \mathcal{F}_t) \cup (\mathcal{F}_1' \cup \mathcal{F}_2' \cup \cdots \cup \mathcal{F}_s') \). After replacing each \( \mathcal{F}_j \) with \( \mathcal{F}_j^* \), it follows that
\[
(\mathcal{F}_1^* \cup \mathcal{F}_2^* \cup \cdots \cup \mathcal{F}_t^*) \cup (\mathcal{F}_1' \cup \mathcal{F}_2' \cup \cdots \cup \mathcal{F}_s')
\]
is the boundary of a polytope \( \mathcal{P}' \) with fewer edges than \( \mathcal{P} \), with centrally symmetric 2-faces, and with \( \mathcal{P} = \mathcal{P}' \oplus \mathcal{E} \). Once more, by downward induction on the number of non-parallel edges, \( \mathcal{P} \) will become the Minkowski sum of edges and hence a zonotope. In this way, every convex polytope of dimension \( m \) with centrally symmetric 2-faces will be a zonotope. \( \square \)

A different proof of Proposition 1.6 can be found in [3, Proposition 2.2.14].

**Corollary 1.7.** A convex polytope is a zonotope if and only if it decomposes into zonotopes. Equivalently, a convex polytope is a zonotope if and only if it decomposes into parallelotopes.

**Proof.** A zonotope trivially decomposes into zonotopes. By a theorem of Shephard and McMullen (see [4] or [10]), it decomposes into parallelotopes. (Note that the decomposition, also called a tiling, means that the zonotope is the union of parallelotopes meeting each other in lower-dimensional facets.)

Conversely, suppose an \( m \)-polytope \( \mathcal{P} \subset \mathbb{R}^n \) decomposes into zonotopes, and hence into parallelotopes. It follows that in every dimension \( < m \), every face of \( \mathcal{P} \) also decomposes into parallelotopes. In particular, each 2-face decomposes into (filled-in) parallelograms and each edge decomposes into edges from those parallelograms. Consider a specific 2-face \( \mathcal{F} \), a particular edge that is part of the boundary of \( \mathcal{F} \), and a parallelogram \( \mathcal{P}^2 \) that is part of the decomposition of \( \mathcal{F} \) and shares an edge with part of the designated edge of \( \mathcal{F} \). The edge of \( \mathcal{P}^2 \) opposite to the one shared with the edge of \( \mathcal{F} \) is itself shared with another parallelogram in the decomposition of \( \mathcal{F} \). By tracking this edge from parallelogram to parallelogram, a strip of parallelograms sharing translated copies of the edge extends across \( \mathcal{F} \) to its far side. The process can be repeated with each of the parallelograms that shares an edge with part of the designated edge of \( \mathcal{F} \). Taken together, the resulting strips produce a translated copy of the designated edge of \( \mathcal{F} \) on the far side of the boundary of \( \mathcal{F} \). (A complete edge of \( \mathcal{F} \) must be obtained in this way because if part of an edge on the far side was not reached by such a strip of parallelograms, a strip formed in reverse would produce a copy of that part back on the originally designated edge of \( \mathcal{F} \).) In this way, every edge of \( \mathcal{F} \) is paired with a translated, parallel, opposite copy of that edge. It then follows from Proposition 1.5 that \( \mathcal{F} \) is centrally symmetric.

Once all 2-faces are centrally symmetric, the polytope is a zonotope by Proposition 1.6. \( \square \)

McMullen [7] demonstrated that central symmetry for faces migrates to lower as well as higher dimensions in a convex polytope provided one starts by assuming the central symmetry of faces.
in a dimension lower than that of the facets. We give McMullen’s proof rephrased in the current notation.

**Proposition 1.8.** If the \((n - 2)\)-dimensional faces of \(n\)-dimensional convex polytope \(P \subset \mathbb{R}^n\) are centrally symmetric, then the \((n - 3)\)-dimensional faces are centrally symmetric.

**Proof.** Consider an \((n - 3)\)-face, \(F_{1,1,1}\), on the boundary of \((n - 2)\)-face, \(F_{1,1}\), which is in turn on the boundary of facet, \(F_1\), of \(P\). Central symmetry implies there are \((n - 3)\)-zones of \((n - 2)\)-faces on the boundary of \(F_1\) induced by \(F_{1,1,1}\).

If an \((n - 3)\)-zone of \((n - 2)\)-faces induced by some \((n - 3)\)-face is of length four, then the \((n - 3)\)-face must be centrally symmetric. To see this, suppose \(F_{1,1} \cup F_{1,2} \cup F_{1,3} \cup F_{1,4}\) is a zone of length four induced by \(F_{1,1,1}\). From Proposition 1.3, the facet \(F_1\) is centrally symmetric, so the face opposite \(F_{1,1}\) in this zone satisfies \(F_{1,3} = (F_{1,1})c_1\) where \(c_1\) is the center of \(F_1\). In particular,

\[ F_{1,2} \cap F_{1,3} = (F_{1,1,1})c_1. \]

At the same time, central symmetry of \(F_{1,1}\) followed by central symmetry of \(F_{1,2}\) imply by Lemma 1.1(a) that

\[ F_{1,2} \cap F_{1,3} = 2(c_{1,2} - c_{1,1}) + F_{1,1,1}. \]

From these two equations, Lemma 1.1(f) implies that \(F_{1,1,1}\) is centrally symmetric.

To complete the proof, it suffices to demonstrate that any \((n - 3)\)-face such as \(F_{1,1,1}\) must be contained in some \((n - 3)\)-zone of length four on the boundary of \(F_1\). This can be done by projecting \(\mathbb{R}^n\) orthogonally to the complement of the affine \((n - 3)\)-dimensional subspace containing \(F_{1,1,1}\). The image of \(P\) under this projection is a 3-dimensional centrally symmetric polyhedron with centrally symmetric facets—a zonohedron, \(\pi(P)\). Lemma 1.2 guarantees that images of translated and reflected copies of \(F_{1,1,1}\) remain distinct on the boundary of \(\pi(P)\) and become its vertices, and that the images of the \((n - 2)\)-faces containing \(F_{1,1,1}\) become, in one-to-one fashion, the edges on the boundary of \(\pi(P)\). The existence of a parallelogram of edges bounding a face of \(\pi(P)\) would therefore demonstrate that the preimage is an \((n - 3)\)-zone of \((n - 2)\)-faces containing \(F_{1,1,1}\) of length four on the boundary of \(F_1\). But such a parallelogram must exist on the boundary of \(\pi(P)\) because any zonohedron contains at least six parallelogram faces, as can be seen using Euler’s formula.

In more detail, if \(f_j = \#\) faces with \(j\) edges and \(v_j = \#\) vertices of index \(j\), then \(f = \sum f_j, v = \sum v_j, \) and \(2e = \sum jf_j = \sum jv_j\). Using these values in Euler’s formula \(v - e + f = 2\) yields two equations,

\[ \sum (2 - j)f_j + \sum 2v_j = 4 \quad \text{and} \quad 2f_j + \sum (2 - j)v_j = 4. \]

Combining the first equation with twice the second,

\[ \sum_{j \geq 3} (6 - j)f_j + \sum_{j \geq 3} (6 - 2j)v_j = 12, \]

from which it follows that

\[ 3f_3 + 2f_4 + f_5 \geq 12 + \sum_{j \geq 7} (j - 6)f_j. \]

When all faces are centrally symmetric, \(f_3 = f_5 = 0, \) so from the preceding inequality, \(f_4 \geq 6. \)

Propositions 1.3 and 1.8 can be combined to obtain:

**Theorem 1.9.** For a polytope of dimension \(m\) in \(\mathbb{R}^n\), \(m \leq n, \) if all \(j\)-dimensional faces are centrally symmetric for a particular value, \(2 \leq j \leq (m - 2), \) then the faces in every dimension, including the polytope itself, are centrally symmetric and the polytope is a zonotope.

A point made in the proof of Proposition 1.8 is that existence of a \(k\)-zone of length \(\equiv 0 \mod 4\) implies that the \(k\)-face generating the zone is centrally symmetric. If one considers zones of a specific polytope, the possible lengths for the zones are limited by the nature of the \(k\)-faces. For example, the 24-cell \(P_{24} \subset \mathbb{R}^4\) is a regular polytope with twenty four facets, each of which is a regular octahedron. The four pairs of opposite 2-faces of a particular facet give rise to four 2-zones
of length $6 \equiv 2 \mod 4$ on the boundary. No zone can have a length that is a multiple of 4 because the generating 2-face of such a zone would have to be centrally symmetric, which is not the case for the triangular 2-faces of this polytope.

The 2-zones of $\mathcal{P}_{24}$ also provide a discrete Hopf fibration of its boundary. Starting from a particular facet $F_1$, the four zones of facets generated by opposite pairs of 2-faces of this facet are each of length 6. The zones meet at $F_1$ and again in their fourth facets, denoted $F_1^{op}$. The zones can be written as $F_1 \cup F_2^{op} \cup F_3^{op} \cup F_4^{op} \cup F_2 \cup F_3$ for $j = 1, \ldots, 4$. Together, these zones account for $(4 \cdot 4) + 2 = 18$ of the facets. The remaining six facets, labeled $F_1^*, \ldots, F_6^*$, fill the interstices and complete the boundary of $\mathcal{P}_{24}$. Consider one of the zones, say $F_1 \cup F_2^{op} \cup F_3^{op} \cup F_1^{op} \cup F_1 \cup F_2^{op}$. This zone and three new 2-zones $F_2^{op} \cup F_1^{op} \cup F_3 \cup F_2 \cup F_3^{op} \cup F_2^{op}$, $F_2^{op} \cup F_3 \cup F_1^{op} \cup F_2 \cup F_1^{op}$, and $F_1^{op} \cup F_2^{op} \cup F_3 \cup F_1^{op} \cup F_2^{op}$ are mutually disjoint. Together, they include all the facets of $\mathcal{P}_{24}$ and constitute a discrete Hopf fibration of the boundary of $\mathcal{P}_{24}$. Other examples with similar fibrations are two more regular polytopes in $\mathbb{R}^4$: the 120-cell, which has regular dodecahedral facets, and the 600-cell, which has icosahedral facets. Prisms in $\mathbb{R}^4$ also have simple discrete fibrations. This raises the question of whether there might exist sequences of polytopes in $\mathbb{R}^4$ with increasing numbers of 2-zones that allow discrete Hopf fibrations, which in the limit give the fibration of the 3-sphere.

With regard to $(n - 2)$-zones—as opposed to 1-zones—on the boundaries of zonotopes, start by considering the 4-cube. One standard projection of the 4-cube consists of inner and outer cubes whose corresponding vertices are connected by additional edges. The facets in this projection consist of two 3-cubes that can be labeled inner and outer, and six more 3-cells surrounding the inner cube that can be labeled in pairs as up/down, front/back, and left/right. Two non-intersecting $(n - 2) = 2$-zones consisting of the facets front-down-back-up and inner-left-outer-right then form a decomposition of the boundary of the 4-cube. This is the simplest discrete version of the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$ for spheres that is realizable for a polytope.

While some $(n - 2)$-zones on the boundaries of $n$-zonotopes might not intersect, an argument given in the course of the proof of Proposition 1.3 does establish:

**Proposition 1.11.** If two $(n - 2)$-zones on the boundary of a zonotope intersect, then they intersect precisely twice.

Meanwhile, for the 1-zones (traditional zones), the situation is different. For example, on the 4-cube, there are four zones each consisting of 6 facets. Any two zones intersect in 4 common facets. In general,

**Proposition 1.12.** Any two zones on the boundary of an $n$-zonotope intersect each other $2^{n-2}$ times.

**Proof.** Recall that a zone (or 1-zone) on the boundary of a zonotope is the union of all faces containing a designated edge as a Minkowski summand. This is the same as the union of all facets containing the edge. Straightforward induction on the number of edges of the zonotope then establishes the proposition. 

2. **Congruences of Zonotopes**

For the study of congruence, start with an identity that comes from the matrix $A^T A$, which will be called the **shape matrix** of $Z(A)$. If two zonotopes have the same shape matrix, they are congruent because the transformation taking generating vectors of one zonotope to corresponding vectors of the other is an isometry. The matrix formulation is the following:

**Proposition 2.1.** If $A$ and $B$ are $n \times k$ matrices ($n$ and $k$ arbitrary), then $A^T A = B^T B$ if and only if $B = QA$ where $Q$ is an $n \times n$ orthogonal matrix.
Proof. We prove only the non-trivial direction and assume $A^T A = B^T B$. Let $A = [a_1, \ldots, a_k]$ and $B = [b_1, \ldots, b_k]$. Observe first that independence of the columns of $A$ is equivalent to nonsingularity of $A^T A$. (Independence of the columns and $A^T A x = 0 \Rightarrow x^T A^T A x = 0 \Rightarrow \|A x\|^2 = 0 \Rightarrow A x = 0 \Rightarrow x = 0$. Nonsingularity of $A^T A$ and $A x = 0 \Rightarrow A^T A x = 0 \Rightarrow x = 0$.) The same can be said for $B$.

Case 1: $A$ has independent columns. By the initial observation, independence of the columns of $A \Leftrightarrow A^T A = B^T B$ nonsingular$\Leftrightarrow$independence of the columns of $B$. Hence col$(A) \perp$ and col$(B) \perp$ both have dimension $n - k$. Let $a_{k+1}, \ldots, a_n$ and $b_{k+1}, \ldots, b_n$ be orthonormal bases of col$(A) \perp$ and col$(B) \perp$ respectively. Let $Q$ be the matrix of the transformation defined by $Qa_i = b_i$ for $i = 1, \ldots, n$. Thus, in particular, $QA = B$. $Q$ is obviously orthogonal on col$(A) \perp$. It then remains to show $Q$ is also orthogonal on col$(A)$. To do so, consider vectors $x, y \in$ col$(A)$ written as $x = Ac$ and $y = Ad$. We then have

$$(Qx) \cdot (Qy) = (QAc)^T (QAd) = (Bc)^T (Bd) = c^T B^T Bd = c^T A^T Ad = (Ac)^T (Ad) = x \cdot y.$$ Hence $Q$ preserves inner products on col$(A)$ and is therefore orthogonal.

Case 2: $A$ has dependent columns. Let $A_0 = [a_{i_1}, \ldots, a_{i_l}]$ ($l < k$) where $a_{i_1}, \ldots, a_{i_l}$ is a maximal collection of independent columns of $A$, and set $B_0 = [b_{i_1}, \ldots, b_{i_l}]$. The hypothesis $A^T A = B^T B \Rightarrow A_0^T A_0 = B_0^T B_0$ so from case 1, $B_0 = QA_0$ for some orthogonal $Q$. We wish to show $B = QA$ and so that $Qa_t = b_t$ for each $t \neq i_1, \ldots, i_l$. By the observation made at the start of the proof, maximal independence of the columns of $A_0$ implies the same for the columns of $B_0$. As $a_t$ and $b_t$ are thus dependent on the columns of $A_0$ and $B_0$ respectively, we may write $a_t = \sum_{j=1}^l c_j a_{ij} = A_0 c$ and $b_t = \sum_{j=1}^l d_j b_{ij} = B_0 d$. Meanwhile, $Qa_t = QA_0 c = B_0 c$. To complete the proof, we must show $c = d$. But this follows from the series of implications:

$A^T A = B^T B \Rightarrow A_0^T A_0 = B_0^T B_0 \Rightarrow A_0^T A_0 c = B_0^T B_0 d = A_0^T A_0 d \Rightarrow A_0^T A_0 (c - d) = 0 \Rightarrow c - d = 0.$

The last implication follows from the nonsingularity of $A_0^T A_0$, which itself follows from yet another application of the initial observation of the proof. 

Applications of this proposition (in the complex case and with a different proof) were given in [6], but no geometric interpretations involving zonotopes were mentioned. Some results from that article take on added significance when the matrices (in the real case) are regarded as shape matrices of zonotopes. The proposition in the form given here together with some of its consequences represent past joint work with Nishan Krikorian. We now extend some of the results that relate to zonotopes.

One piece of information that the shape matrix does not contain is the dimension of the space in which the zonotope resides. What if two such objects have the same shape matrix but lie in different dimensional Euclidean spaces? Then Proposition 2.1 becomes:

If $A$ is $m \times k$ and $B$ is $n \times k$ $(m \leq n)$, then $A^T A = B^T B \Leftrightarrow B = QA$ where $Q$ is $n \times m$ with orthonormal columns.

This is just a slight generalization whose proof is omitted.

Another piece of geometric information about a zonotope that the shape matrix does not give is its orientation or embedding in Euclidean space. So far, two congruent zonotopes are assumed to be attached to the origin at corresponding vertices. Suppose the two are identical and have the same orientation but are attached to the origin at non-corresponding vertices. The isometry relating them is then just a translation. In this case, because a copy of each generating vector or its negative emanates from every vertex of a zonotope, placing different vertices at the origin changes only the signs attached to some of the generating vectors. For example, if $Z(a_1, \ldots, a_k)$ is translated along edge $a_1$, the resulting zonotope will be $-a_1 + Z(a_1, \ldots, a_k) = Z(-a_1, a_2, \ldots, a_k)$. A sequence of such translations can be used to reach any vertex yielding $Z(a'_1, \ldots, a'_k)$, which will therefore be related to $Z(a_1, \ldots, a_k)$ simply by $a'_i = \pm a_i$. Hence, the generating matrices will be related by $A' = AJ$ and the shape matrices by $(A')^T A' = J A^T A J$ for some $k \times k$ diagonal matrix.
$J$ with $\pm 1$'s on the principal diagonal. If, in addition, we wish to reorder the generating vectors (to match, for example, the order of generating vectors of some congruent zonotope), this can be done by pre-multiplying $A$ by a permutation matrix, $\Sigma$. The general statement about congruence is then:

**Theorem 2.2.** $Z(A)$ and $Z(B)$ are congruent, where $A$ is $m \times k$ and $B$ is $n \times k$, if and only if $(A')^T A' = B^T B$ where $A' = \Sigma J$, $\Sigma$ is a $k \times k$ permutation matrix, and $J$ is a diagonal matrix with $\pm 1$'s on the diagonal, or equivalently, if and only if there exists an $m \times n$ matrix $Q$ with orthonormal columns such that $B = QA\Sigma J$.

Now, consider a pair of generating matrices $A$ and $B$ of shape $n \times k$ with independent columns, along with the paralleotopes $\mathcal{P}(A)$ and $\mathcal{P}(B)$, and the zonotopes $Z(A^T)$ and $Z(B^T)$. We may think of $\mathcal{P}(A)$ and $\mathcal{P}(B)$ as column-paralleotopes, and refer to $Z(A^T)$ and $Z(B^T)$—which are defined using the columns of $A^T$ and $B^T$—as the corresponding row-zonotopes (of $A$ and $B$). We ask for conditions under which congruence of one pair of objects, coming from equivalence of the corresponding shape matrices, implies congruence of the other pair and how these conditions relate to the congruences. For example,

$$A_1 = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}, B_1 = \sqrt{2} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}; A_2 = \frac{1}{11} \begin{bmatrix} 3 & -12 \\ 12 & -3 \\ 4 & 4 \end{bmatrix}, B_2 = \frac{1}{11} \begin{bmatrix} 15 & -9 \\ 7 & 1 \\ 8 & 10 \end{bmatrix};$$

$$A_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B_3 = \frac{1}{\sqrt{84}} \begin{bmatrix} 46 & 48 \\ 82 & 124 \end{bmatrix}; A_4 = \begin{bmatrix} 26 & 8 \\ 24 & 2 \\ 18 & -16 \\ 32 & 26 \end{bmatrix}, B_4 = \sqrt{2} \begin{bmatrix} 17 & 9 \\ 13 & 11 \\ 1 & 17 \\ 29 & 3 \end{bmatrix},$$

are four pairs of matrices where both $(A_i)^T A_i = (B_i)^T B_i$ and $A_i(A_i)^T = B_i(B_i)^T$. In other words, $\mathcal{P}(A_i)$ is congruent to $\mathcal{P}(B_i)$ and $Z((A_i)^T)$ is congruent to $Z((B_i)^T)$.

Start by considering the case where $n = k$ and all four objects are $n$-paralleotopes in $\mathbb{R}^n$. Consider the shape matrix $A^T A$ of column-paralleotope $\mathcal{P}(A)$ and shape matrix $AA^T$ of the corresponding row-paralleotope $\mathcal{P}(A^T)$. Another matrix, $A^2 = AA = (A^T)^T A$, can now be thought of as the **comparison matrix** between the generating vectors of the row-paralleotope and the column-paralleotope. It seems plausible to conjecture that if matrices $A$ and $B$ have equal comparison matrices ($A^2 = B^2$), the shape matrices of the row-paralleotopes will be the same ($AA^T = BB^T$) if and only if the shape matrices of the column-paralleotopes are the same ($A^T A = B^T B$). This is in fact true.

**Corollary 2.3.** If $A$ and $B$ are square nonsingular matrices, and if $A^2 = B^2$, then $AA^T = BB^T$ if and only if $A^T A = B^T B$.

**Proof.** It suffices to prove one of the implications, say $\Leftarrow$. Suppose $A^T A = B^T B$. From Proposition 2.1 it then follows that $B = QA$, or $A = QT B$. Therefore, $AA = AQ^T B = BB$, which because $B$ is nonsingular implies $AQ^T = B$, and so $AA^T = AQ^T (AQ^T)^T = BB^T$. $\square$

Corollary 8.1 of [6] gives this same result in complex form but makes no reference to the geometric interpretation involving row and column-paralleotopes.

Another reasonable geometric conjecture is that if $A$ and $B$ have congruent row-paralleotopes ($AA^T = BB^T$) and congruent column-paralleotopes ($A^T A = B^T B$), then their comparison matrices are identical ($A^2 = B^2$) if and only if the two congruences are identical. This too is true.

**Corollary 2.4.** If $A$ and $B$ are square nonsingular matrices such that $AA^T = BB^T$ and $A^T A = B^T B$, then $A^2 = B^2 \iff$ there exists an orthogonal matrix $Q$ such that $B = QA$ and $B^T = QA^T$.

**Proof.** Only $\Rightarrow$ is proved as $\Leftarrow$ is trivial. From Proposition 2.1, $B = Q_1 A$ and $B^T = Q_2 A^T$. We must show that $Q_1 = Q_2$. Meanwhile, $A^2 = B^2$ can be restated as $BA^{-1} = B^{-1} A$. From the first and last of these several equalities, $Q_1 = BA^{-1} = B^{-1} A$. The second equality may also be rewritten as $B = A(Q_2)^T$, implying $Q_2 = B^{-1} A$. Thus, $Q_1 = B^{-1} A = Q_2$, as was required. $\square$
When there is no orthogonal $Q$ such that $B = QA$ and $B^T = QA^T$ both hold, it is possible to have (1) $A^TA = B^TB$ and (2) $AA^T = BB^T$, but $A^2 \neq B^2$. This happens, for example, in the case of the third pair of matrices given above. The condition $A^2 = B^2$ will be replaced with a weaker comparison condition that does hold whenever (1) and (2) hold and therefore seems more closely tied to these two conditions. Indeed, whenever any two of (1), (2), and the new condition hold, it will turn out that the third holds as well. Moreover, the condition will be defined and the implications will hold in the more general setting of rectangular $n \times k$ matrices $A$ and $B$ with independent columns. In that case, the column-parallelotopes $\mathcal{P}(A)$ and $\mathcal{P}(B)$ will reside in $\mathbb{R}^n$ (with $n \geq k$) while the row-zonotopes $\mathcal{Z}(A^T)$ and $\mathcal{Z}(B^T)$ belong to $\mathbb{R}^k$. In order to obtain comparison matrices in this setting, the parallelotopes are moved to congruent copies within $\mathbb{R}^k$ by taking QR-decompositions $A = PR$ and $B = QS$ where $P$ and $Q$ are $n \times k$ with orthonormal columns, and $R$ and $S$ are $k \times k$ upper triangular of rank $k$. By Theorem 2.2, $\mathcal{P}(A)$ is congruent to $\mathcal{P}(R) \subset \mathbb{R}^k$ and $\mathcal{P}(B)$ is congruent to $\mathcal{P}(S) \subset \mathbb{R}^k$. The parallelotopes in $\mathbb{R}^k$ can now be compared to the corresponding row-zonotopes $\mathcal{Z}(A^T)$ and $\mathcal{Z}(B^T)$.

In order to make the comparison between $\mathcal{Z}(A^T)$ and $\mathcal{P}(R)$, and between $\mathcal{Z}(B^T)$ and $\mathcal{P}(S)$, it does not suffice to use $AR$ and $BS$. There are cases where (1) and (2) hold but $AR = BS$ does not. An example is the pair $A_4$ and $B_4$ given above. The problem is that the parallelotopes $\mathcal{P}(R)$ and $\mathcal{P}(S)$ are both in the same “generic” position with respect to the standard basis for $\mathbb{R}^k$. In order to make valid comparisons with the corresponding zonotopes, each parallelotope must be allowed to independently reorient itself with respect to its zonotope. For this purpose, additional orthogonal matrices $Q_1$ and $Q_2$ are introduced so that $\mathcal{Z}(A^T)$ is compared with $\mathcal{P}(Q_1 R)$ using the matrix $AQ_1 R$, and $\mathcal{Z}(B^T)$ is compared with $\mathcal{P}(Q_2 S)$ using $BQ_2 S$. Now everything works. Setting

$$ AQ_1 R = BQ_2 S, $$

(3) it turns out that when any two of (1), (2), and the new condition (3) hold, then so does the third. The following lemma will be used to establish the result.

**Lemma 2.5.** Suppose $R$ and $S$ are $k \times k$ upper triangular matrices with $R^T R = S^T S$. Then there is a diagonal matrix $J$ with $\pm 1$’s on the diagonal such that $R = JS$.

**Proof.** We compute the first two rows of $R$ and $S$. A straightforward induction (omitted) then completes the proof.

Let the columns of $R$ be $r_1, \ldots, r_k$ and those of $S$ be $s_1, \ldots, s_k$. Let $m_{ij}$ be the $ij^{th}$ entry of $R^T R = S^T S$. Then

$$ m_{11} = r_1^T r_1 = (r_{11})^2 = s_1^T s_1 = (s_{11})^2, $$

from which $r_{11} = \pm s_{11}$. In addition, for each $j = 2, \ldots, k$,

$$ m_{1j} = r_1^T r_j = r_{11} r_{1j} = s_1^T s_j = s_{11} s_{1j}. $$

If $r_{11} = s_{11}$, then $r_{1j} = s_{1j}$ for all $j = 1, \ldots, k$ making the first rows of $R$ and $S$ identical. If $r_{11} = -s_{11}$, then $r_{1j} = -s_{1j}$ for all $j = 1, \ldots, k$, so the first row of $R$ is the negative of the first row of $S$.

Next, consider

$$ m_{22} = r_2^T r_2 = (r_{12})^2 + (r_{22})^2 = s_2^T s_2 = (s_{12})^2 + (s_{22})^2. $$

We have already seen that $(r_{1j})^2 = (s_{1j})^2$ for every $j = 2$. It follows that $(r_{22})^2 = (s_{22})^2$, or $r_{22} = \pm s_{22}$. Meanwhile, for each $j = 3, \ldots, k$,

$$ m_{2j} = r_2^T r_j = r_{12} r_{1j} + r_{22} r_{2j} = s_2^T s_j = s_{12} s_{1j} + s_{22} s_{2j} = s_{12} s_{1j} $$

and so $r_{22} r_{2j} = s_{22} s_{2j}$. Consequently, either $r_{2j} = s_{2j}$ for every $j = 2, \ldots, k$, or else $r_{2j} = -s_{2j}$ for every $j = 2, \ldots, k$. Therefore, the second rows of $R$ and $S$ are either identical or negatives of
each other. Continuing with similar computations, induction shows that each row of $R$ is either the same or the negative of the corresponding row of $S$. It follows that $R = JS$ as asserted.

From the lemma, if $A = QR = Q'R'$ are two QR-decompositions of an $n \times k$ matrix with independent columns, then $R' = JR$ and $Q' = A(R')^{-1} = AR^{-1}J = QJ$. In other words, the QR-decomposition of $A$ is unique up to a diagonal $k \times k$ matrix $J$ with $\pm 1$'s on the diagonal (which changes the signs of specified columns of $Q$ and the corresponding rows of $R$). Thus, there are $2^k$ distinct QR-decompositions of $A$.

**Proposition 2.6.** Let $A = PR$ and $B = QS$ be $n \times k$ matrices with independent columns and QR-decompositions as indicated. Consider the three conditions:

1. $A^T A = B^T B$,
2. $AA^T = BB^T$, and
3. there exist orthogonal matrices $Q_1$ and $Q_2$ such that $AQ_1 R = BQ_2 S$.

If any two of the conditions hold, then so does the third. On the other hand, no one of these conditions implies either of the other two.

**Proof.** (a) Suppose (3) and $A^T A = B^T B$ hold. It follows that $R^T R = S^T S$, so by Lemma 2.5, $R = JS$ (or $S = JR$) for some diagonal matrix $J$ with $\pm 1$'s on the diagonal. Condition (3) then reduces to $B = AQ_1 Q_2^T J$, or $B = QJ_2 Q_1^T A^T = Q'A^T$ where $Q' = JQ_2 Q_1^T$ is orthogonal, from which it follows that $BB^T = AA^T$.

(b) Suppose (3) and $AA^T = BB^T$ hold. From (3), it follows that $B = AQ_1 RS^{-1} Q_2^T$. Substituting the second equality in the first and simplifying, $R^T R = S^T S$. Once again, Lemma 2.5 implies $S = JR$ for a diagonal $J$ with $\pm 1$'s on the diagonal, so

$$A^T A = R^T R = R^T J J R = S^T S = B^T B.$$  

(c) Finally, suppose conditions (1) and (2) hold. $A^T A = B^T B \Rightarrow R^T R = S^T S$, so by Proposition 2.1 there exists an orthogonal $Q_1$ such that $S = Q_1 R$. At the same time, applying Proposition 2.1 to $AA^T = BB^T$ guarantees existence of an orthogonal $Q_2$ such that $B^T = Q_2 A^T$. This last may be rewritten as $A = BQ_2$. It then follows that

$$AQ_1 R = AS = BQ_2 S.$$  

As for the last assertion of the proposition, it is clear that neither (1) nor (2) by itself implies either of the remaining two conditions. Giving an example where (3) holds but the other conditions do not will complete the proof. To that end, let $A = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix} = B^2$, but $A^T A \neq B^T B$ and $AA^T \neq BB^T$. Meanwhile, QR-decompositions for $A$ and $B$ may be taken as $A = PR = IA$ and $B = QS = IB$. Also choosing $Q_1 = Q_2 = I$ then leads to $AQ_1 R = A^2 = B^2 = BQ_2 S$, so (3) holds while (1) and (2) do not.

We make several observations concerning the proposition. First, the conclusion of the proposition may be rephrased as:

The pairs of conditions—(1) and (3), (2) and (3), (1) and (2)—are equivalent; but no one condition—(1), (2), or (3)—implies either of the other two.

Second, if $A$ and $B$ are square, and if $Q_1 = P$ and $Q_2 = Q$ where $A = PR$ and $B = QS$ are QR-decompositions, then condition (3) becomes $A^2 = B^2$. When this version of the condition holds, Proposition 2.6 includes Corollary 2.3 and so is a generalization of that corollary.

Third, as (3) must hold whenever (1) and (2) hold, simultaneous congruence of the column-parallelepipeds and congruence of the row-zonotopes ensures that $P(RQ_1 R) = Q(SQ_2 S)$. This implies that the column spaces of $P$ and $Q$ are the same and forces the column-parallelepipeds $P(A)$ and $P(B)$ to lie in the same $k$-dimensional subspace of $\mathbb{R}^n$.  

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Fourth, denoting the rows of $A$ and $B$ as $a^1, \ldots, a^n$ and $b^1, \ldots, b^n$ respectively, and writing $R$ and $S$ in terms of their columns as $R = [r_1, \ldots, r_k]$ and $S = [s_1, \ldots, s_k]$, (3) becomes $a^j \cdot Q_1(r_j) = b^j \cdot Q_2(s_j)$ for every pair $(i, j)$. With (2) and (1) also holding, $\|a^j\| = \|b^j\|$ and $\|Q_1(r_j)\| = \|Q_2(s_j)\|$. Comparison condition (3) then says that all corresponding angles between the pairs $(a^j, Q_1(r_j))$ and $(b^j, Q_2(s_j))$ are equal. (These are the angles between the respective pairs of edges from the row-zonotope $Z(A^T)$ and reordered column-parallelotope $P(Q_1(R))$ on the one hand, and $Z(B^T)$ and $P(Q_2(S))$ on the other.)

QR-decompositions of $n \times k$ matrices with independent columns, such as $A$ and $B$ with $A = PR$ and $B = QS$, produce “generic” parallelotopes $P(R)$ and $P(S)$ in $\mathbb{R}^k$. If an additional requirement is imposed on either $R$ or $S$ that the entries on its principal diagonal be positive, then that upper-triangular matrix is uniquely determined. It represents a “template” parallelotope from which all other congruent copies of that given shape of parallelotope can be obtained by mapping using $Q$-type matrices with $k$ orthonormal columns into appropriate $k$-dimensional subspaces of Euclidean spaces of arbitrary dimension. A template parallelotope is a $k$-parallelotope in $\mathbb{R}^k$ in “standard position” meaning that the $j$-face defined by the first $j$ columns of the matrix—or by the $j$ corresponding edges of the parallelotope—always lies in the subspace spanned by the first $j$ standard basis vectors of the ambient space. The requirement that the diagonal entries of the triangular matrix be positive implies, in addition, that there exists a half-space such that the parallelotope and all of the standard basis vectors $e_1, \ldots, e_n$ are contained within that half-space.

**Proposition 2.7.** Suppose $A$ and $B$ are $n \times k$ matrices with independent columns, $Q'$ is $m \times n$ with orthonormal columns, $A' = QA$, and $B' = QB$. Then conditions (1) and (2) from Proposition 2.6 hold for $A$ and $B$ if and only if the same conditions hold for $A'$ and $B'$.

**Proof.** $\Rightarrow$: Suppose conditions (1) and (2) hold for $A$ and $B$. Then

$$A^T A' = A^T Q'T Q' A = A^T A = B^T B = B^T Q'T Q' B = B'T B'$$

and

$$A' A'^T = Q'A A'^T Q'^T = Q'B B'^T Q'^T = B'B'^T.$$  

$\Leftarrow$: Suppose conditions (1) and (2) hold for $A'$ and $B'$. Then

$$A^T A = A'^T Q'T Q' A = A'^T A' = B'^T B' = B'T Q'^T Q' B = B^T B,$$

and from which it follows that $A^T A = B^T B$. \hfill $\Box$

3. **Volumes, Normal Vectors, and Rigidity of Zonotopes**

Symmetric cones can be used to derive a well-known volume formula for zonotopes in a new way.

**Proposition 3.1.** Let $Z(A)$ be an $n$-dimensional zonotope in $\mathbb{R}^n$ defined by an $n \times k$ matrix $A = (a_1, \ldots, a_k)$ of rank $n$ where the $a_j$’s are the columns of $A$. Then

$$\operatorname{vol}_n(Z(A)) = \sum_{1 \leq j_1 < \cdots < j_n \leq k} |\operatorname{det}(A^{j_1, \ldots, j_n})|$$

where $A^{j_1, \ldots, j_n} = [a_{j_1}, \ldots, a_{j_k}]$.

**Proof.** Central symmetry and convexity ensure that the zonotope decomposes completely into symmetric cones defined by pairs of opposite facets:

$$Z(A) = \bigcup_{1 \leq j_1 < \cdots < j_{n-1} \leq k} \{\operatorname{cone}(Z(a_{j_1}, \ldots, a_{j_k}))\}$$
where \( \mathbf{c} = \frac{1}{2}(\mathbf{a}_1 + \cdots + \mathbf{a}_k) \) is the center of symmetry of \( Z(A) \) and the facets \( Z(\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_n}) \) of \( Z(A) \) are translated copies of generating facets defined by \( n \times (n-1) \) submatrices of the form \( A_{j_1 \cdots j_{n-1}} \).

In degenerate cases, several facets might lie in the same hyperplane forming actual facets of the zonotope that are larger than parallelotopes, but this has no effect on the computation of volume. For each submatrix of rank \( n-1 \), the normalized cross-product provides a unit normal vector for the corresponding non-degenerate facet:

\[
\mathbf{n}_{j_1 \cdots j_{n-1}} = \frac{\times (\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{n-1}})}{|\times (\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{n-1}})|} = \frac{\times (\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{n-1}})}{\text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}})}.
\]

(Details about volumes defined by cross-products can be found, for example, in [4].) The \( n \)-volume of each symmetric cone is \( \frac{1}{n} \) times the \( (n-1) \)-volume of the base times the height, where the height is the distance between the pair of opposite bases of the cone. That distance is simply the sum of the magnitudes of the projections of all \( \mathbf{a}_j \)'s onto \( \mathbf{n}_{j_1 \cdots j_{n-1}} \). The magnitude of each projection is of the form

\[
|\mathbf{n}_{j_1 \cdots j_{n-1}} \cdot \mathbf{a}_j| = \left| \frac{\times (\mathbf{a}_{j_1}, \ldots, \mathbf{a}_{j_{n-1}})}{\text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}})} \cdot \mathbf{a}_j \right| = \frac{|\det (A_{j_1 \cdots j_{n-1} t})|}{\text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}})}
\]

where the second equality is obtained from the Laplace expansion of the determinant in its rightmost column. The height is therefore

\[
\sum_{j \neq j_1 \cdots j_{n-1}} \frac{|\det (A_{j_1 \cdots j_{n-1} t})|}{\text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}})}
\]

and the \( n \)-volume of the symmetric cone is

\[
\text{vol}_n \left( \text{cone}_c(F_{j_1 \cdots j_{n-1}, t}) \right) = \frac{1}{n} \text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}}) \cdot \sum_{j \neq j_1 \cdots j_{n-1}} \frac{|\det (A_{j_1 \cdots j_{n-1} t})|}{\text{vol}_{n-1}(F_{j_1 \cdots j_{n-1}})}
\]

\[
= \frac{1}{n} \sum_{j \neq j_1 \cdots j_{n-1}} |\det (A_{j_1 \cdots j_{n-1} t})|.
\]

It follows that

\[
\text{vol}_n (Z(A)) = \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq k} \text{vol}_n \left( \text{cone}_c(F_{j_1 \cdots j_{n-1}, t}) \right)
\]

\[
= \frac{1}{n} \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq k} |\det (A_{j_1 \cdots j_{n-1} t})|
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq k} |\det (A_{j_1 \cdots j_{n-1} t})|.
\]

The last displayed equality holds because each term \( |\det (A_{j_1 \cdots j_{n-1} t})| \) on the next-to-last line occurs \( n \) times in equivalent forms within that sum. \( \square \)

The volume formula also follows from the (non-unique) tiling of a zonotope into translated copies of its generating parallelotopes, each used exactly once. This decomposition was cited in the proof of Corollary 1.7.

**Proposition 3.2.** Every zonotope formed from \( k \) generating vectors in \( \mathbb{R}^n \) decomposes into unique translated copies of its generating parallelotopes that intersect each other only in lower-dimensional faces. This forms a tiling of the zonotope.
Proof. A one-dimensional zonotope (line segment) decomposes into subsegments that are translations of all of its generating line segments. And in all dimensions, parallelotopes decompose trivially as themselves. Hence the proposition is true for all zonotopes of dimension 1 and for zonotopes with \( k = n \) generators in any dimension \( n \). Assume by induction that a decomposition of the required type exists for all zonotopes of every dimension \( < n \) as well as for zonotopes with fewer than \( k > n \) generators in dimension \( n \). Now consider \( Z(A) = Z(a_1, \ldots, a_k) = Z(a_1, \ldots, a_{k-1}) \oplus Z(a_k) \in \mathbb{R}^n \) where the first summand is already \( n \)-dimensional. Assume for now that none of the other generators lie in the 1-dimensional subspace spanned by \( a_k \). The visible surface of \( Z(a_1, \ldots, a_{k-1}) \) in the direction of \( a_k \) consists (by Lemma 3.2 of [4]) of unique translates of all of the generating facets of the zonotope. Each facet is defined by fewer than \( k \) generators in any dimension \( n \). Thus, in addition to the \( \binom{k-1}{n} \) parallelotopes in the decomposition of \( Z(a_1, \ldots, a_{k-1}) \), which exist by the induction assumption, there are \( \binom{k-1}{n-1} \) parallelotopes of the type just described for a total of \( \binom{k}{n} \) parallelotopes that together form a decomposition of \( Z(A) \) of the required type.

In the case where several generators \( a_{j+1}, \ldots, a_k \) all lie in a single 1-dimensional subspace, convexity of \( Z \) forces the edges defined by these generators to be contiguous. The sum of these generators then replaces \( a_k \) in the previous description. As a result, \( (k-1) \cdot \binom{n-1}{n} \) distinct \( n \)-dimensional parallelotopes are created where the bases are \( (n-1) \)-dimensional parallelotopes from the visible surface and whose remaining edge is, in every case, \( a_k \). The\( \) parallelotopes from the preceding proof that contain edge \( a_k \) are sandwiched between the visible surface of \( Z(a_1, \ldots, a_{k-1}) \) in the direction of \( a_k \) and its translation by \( a_k \). Their union defines a partial shell of parallelotopes forming an exterior wall of zonotope \( Z(A) = Z(a_1, \ldots, a_k) \), which Shephard [10] called a cup of cubes. Labeling this as \( \mathcal{C}_{1, \ldots, k-1}(a_k) \), we have the decomposition \( Z(A) = Z(a_1, \ldots, a_{k-1}) \cup \mathcal{C}_{1, \ldots, k-1}(a_k) \). A similar decomposition of the smaller zonotope yields \( Z(A) = Z(a_1, \ldots, a_{k-2}) \cup \mathcal{C}_{1, \ldots, k-2}(a_{k-1}) \cup \mathcal{C}_{1, \ldots, k-1}(a_k) \).

More generally, \( Z(A) = Z(a_{j_1}, \ldots, a_{j_n}) \cup \mathcal{C}_{j_1, \ldots, j_n}(a_{j_{n+1}}) \cup \cdots \cup \mathcal{C}_{j_1, \ldots, j_{k-1}}(a_{j_k}) \) where \( Z(a_{j_1}, \ldots, a_{j_n}) \) is a generating parallelotope of \( Z(A) \). Thus, when \( Z(A) \) is developed as the Minkowski sum of successive line segments, various decompositions of the intermediate zonotopes in the manner just shown produce different decompositions of \( Z(A) \) in terms of generating parallelotopes. These decompositions, however, do not lead to all possible tilings of the zonotope. (See Shephard [10].)

To illustrate one possibility of what might happen to the facets and tiling of a zonotope, consider

\[
A_0 = \begin{bmatrix}
1 & 0 & 1 & 0 & -1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\quad \text{and} \quad
A_\epsilon = \begin{bmatrix}
1 & 0 & 1 & 0 & -1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & \epsilon & 1 & 1
\end{bmatrix}.
\]

The first three columns of \( A_0 \) are dependent while the other triples of columns are independent. In \( A_\epsilon \) for \( \epsilon \) small, all ten triples of columns are independent. Consequently, there are nine out of ten possible generating parallelotopes of \( Z(A_0) \) while for \( Z(A_\epsilon) \), all ten are parallelotopes. In each case, translations of the generating parallelotopes can be arranged in various ways to form a tiling of the
The generating subzonotope defined by the first three columns of $A_0$ is two-dimensional and translates to a pair of symmetrically opposite hexagonal facets of the zonotope. Each of these facets coincides with a union of translations of the three generating facets $Z(a_1, a_2)$, $Z(a_1, a_3)$, and $Z(a_2, a_3)$, where $a_1, \ldots, a_5$ are the columns of $A_0$. Zonotope $Z(A_0)$ has $2\binom{5}{2} = 20$ generating facets but only 14 geometric facets; translates of three generating facets make up each hexagonal facet. In the case of $Z(A_\epsilon)$, denote the columns of $A_\epsilon$ by $a_1', \ldots, a_5'$. Letting $\epsilon \to 0$ demonstrates how a translate of the generating parallelootope $Z(a_1', a_2', a_3')$ flattens out and approaches one of the two hexagonal facets on the boundary of $Z(A_\epsilon)$. Which facet is approached depends on the choice of tiling. At the same time, two different sets of translates of the generating facets $Z(a_1', a_2')$, $Z(a_1', a_3')$, and $Z(a_2', a_3')$ approach co-planarity on opposite sides of the boundary of the zonotope to form copies of that same hexagonal facet.

Angles between edges $la_i$ and $la_j$ of an $n$-zonotope $Z(A) = Z(a_1, \ldots, a_k)$ can be computed as $\theta = \arccos(\langle a_i \cdot a_j / |a_i||a_j| \rangle)$. Dihedral angles between facets can be computed similarly using normal vectors to the facets. For facet $Z(A_{i_1, \ldots, i_{n-1}}) = Z(a_{i_1, \ldots, i_{n-1}})$, the unit normal vector is

$$n_{i_1, \ldots, i_{n-1}} = \frac{\times (a_{i_1, \ldots, i_{n-1}})}{|\times (a_{i_1, \ldots, i_{n-1}})|}.$$

The cross-product used to find this normal vector can also be recovered from $\wedge^{n-1}(A)$, the map representing the map $\wedge^{n-1}f : \wedge^{n-1}\mathbb{R}^k \to \wedge^{n-1}\mathbb{R}^k$ with respect to reverse lexicographically ordered bases of the exterior powers $\wedge^{n-1}\mathbb{R}^k$ and $\wedge^{n-1}\mathbb{R}^n$. The entries of $\wedge^{n-1}(A)$ are the $(n-1) \times (n-1)$ minors of $A$ with the minor in row $i$ and column $j$ the one defined by omitting those row and column indices. If $\wedge^{n-1}(A)$ is modified so that its rows and columns alternate in sign with the $(i, j)$-th entry multiplied by $(-1)^{n+i+j}$, then the $j$-th column of the new matrix $\wedge^{n-1}(A)$ will be the cross product (or its negative) of the columns of $A$ with the complementary column indices. That is,

$$\left( \wedge^{n-1}_{\pm}(A) \right)^j = (-1)^j \left[ \times (a_1, \ldots, \hat{a}_j, \ldots, a_n) \right].$$

The norm of the cross product then gives the $(n-1)$-volume of the parallelootope defined by those columns. (See, for example, Corollary 1.3 of [4]. If the columns are not independent, they define a zonotope of rank less than $n-1$ whose $(n-1)$-volume is 0.) Thus, the columns of $\wedge^{n-1}_{\pm}(A)$ are normal vectors to the parallelotopes that comprise the generating facets of $Z(A)$, and the norms of the vectors give the $(n-1)$-volumes of those parallelotopes. (Note that from Corollary 1.8, each facet decomposes into such paralleloptes.)

We wish to examine from the perspective of zonotopes two classic results in the theory of convex polytopes. The first is due to Minkowski. (See, for example, [5]):

**Theorem 3.3.** Given distinct unit vectors $u_1, \ldots, u_t$ that span $\mathbb{R}^n$ and corresponding arbitrary positive real numbers $a_1, \ldots, a_t$, then up to translation, there exists a unique convex polytope $P \in \mathbb{R}^n$ for which the vectors are the outward-pointing normals to the facets and the numbers are the $(n-1)$-volumes of the facets, if and only if $\sum a_iu_i = 0$.

The second, due to Cauchy in $\mathbb{R}^3$, extended to arbitrary $\mathbb{R}^n$ by Alexandrov (see, for instance, [1]), and a basic part of geometric rigidity theory, is:

**Theorem 3.4.** If combinatorially equivalent convex polytopes in $\mathbb{R}^n$, $n \geq 3$, have congruent corresponding facets, then the polytopes are congruent.

Considering first Minkowski’s theorem, observe that for polytopes whose facets come in pairs with equal $(n-1)$-volumes and unit normal vectors that are negatives of each other, the condition $\sum a_iu_i = 0$ is automatically satisfied. Indeed, it will turn out that given any distinct set of unit
Proposition 3.5. Given distinct unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_t$ spanning $\mathbb{R}^n$, no two of which are negatives of each other, and corresponding arbitrary positive real numbers $a_1, \ldots, a_t$, there exists a unique centrally-symmetric polytope $\mathcal{P}$ with $2t$ facets such that $\mathbf{u}_i$ and $-\mathbf{u}_i$ are outward-pointing normals to facets $\mathcal{F}_i$ and $\mathcal{F}_{i}^{op}$, and $\text{vol}_{n-1}\mathcal{F}_i = \text{vol}_{n-1}\mathcal{F}_{i}^{op} = a_i$.

Proof. The hypotheses ensure that the sum $\sum a_i \mathbf{u}_i + \sum a_i (-\mathbf{u}_i) = \mathbf{0}$ is taken over the $(n-1)$-volumes of all facets times their corresponding normal vectors. Theorem 3.3 therefore applies and guarantees existence of a unique convex polytope $\mathcal{P}$ with the given vectors and their negatives as normal vectors and the given numbers as $(n-1)$-volumes of $t$ pairs of corresponding facets. The normal vectors come in opposite pairs, so opposite pairs of facets lie in parallel hyperplanes. The polytope is thus the intersection of the slabs that lie between the pairs of parallel hyperplanes. In order to bound a closed polytope, there must be at least $n$ such slabs.

If there are exactly $n$ given vectors, they form a basis for $\mathbb{R}^n$. There is then a family of parallelopetopes $\{\mathcal{P}\}$ with $\mathcal{P} \subset \mathbb{R}^n$, each of whose facet normals agree with those of $\mathcal{P}$. By adjusting the distance between each pair of corresponding parallel hyperplanes to match the distances for $\mathcal{P}$, a parallelopetope from the family is determined that is the same intersection of slabs as $\mathcal{P}$. Hence the two are the same, and so $\mathcal{P}$ is a parallelopetope. It is then also centrally symmetric, which establishes the proposition when $t = n$.

Assume, by induction, that the proposition holds for all sets of $< m$ normal vectors and corresponding facet-volumes for some fixed value $m$. Now, consider $m$ unit normal vectors and $m$ corresponding facet volumes. Again by Theorem 3.3, there exists a unique convex polytope $\mathcal{P}$ whose facets satisfy these conditions. Removing the $m$-th vector and volume value, there exists a unique polytope $\mathcal{P}'$ defined by the remaining vectors, their negatives, and the remaining volume values, which by the inductive assumption is centrally symmetric. $\mathcal{P}$ will equal the intersection of $\mathcal{P}'$ with a slab bounded by hyperplanes $\mathcal{H}_m$ and $\mathcal{H}_{-m}$ whose outward-pointing normal vectors are $\mathbf{u}_m$ and $\mathbf{u}_{-m}$. In order to satisfy the condition that the $(n-1)$-volumes of the last pair of opposite facets have the same $(n-1)$-volume, $a_m$, the two hyperplanes must intersect $\mathcal{P}'$ in cross sections of equal $(n-1)$-volumes of that same value. Given that $\mathcal{P}'$ is centrally symmetric, the width and location of the slab will be constrained by this equal-cross-section condition.

Let $\mathcal{H}$ be the supporting hyperplane of $\mathcal{P}'$ for which $\mathbf{u}_m$ is the outward-pointing normal, and let $\mathcal{H}'$ be the parallel supporting hyperplane on the opposite side of $\mathcal{P}'$. The distance $w$ between the hyperplanes is the width of $\mathcal{P}'$ in the direction of $\mathbf{u}_m$. Let $\mathcal{H}_{tw}$ be the hyperplane parallel to and between $\mathcal{H}$ and $\mathcal{H}'$ whose distance from $\mathcal{H}$ is $tw$, where $0 \leq t \leq 1$. Define a non-negative real-valued function $C : [0, 1] \rightarrow \mathbb{R}$ by $C(t) = \text{the } (n-1)\text{-dimensional cross-sectional volume of } \mathcal{P}' \cap \mathcal{H}_{tw}$. As $\mathcal{P}'$ is centrally symmetric, this function is unimodal and symmetric about the value $t = \frac{1}{2}$, where it attains its maximum. Moreover, the cross section at $t = \frac{1}{2}$ contains the center $c'$ of $\mathcal{P}'$. (These are all consequences of Corollary 2.2 of [2].) There are now only two possibilities for the slab whose intersection with $\mathcal{P}'$ produces $\mathcal{P}$. One is that the hyperplanes producing equal cross-sectional volumes $a_m$ are $\mathcal{H}_{\frac{1}{2}-t_0} = \mathcal{H}_m$ and $\mathcal{H}_{\frac{1}{2}+t_0} = \mathcal{H}_{-m}$ for some specific value $0 \leq t_0 \leq \frac{1}{2}$.

In this case, the central hyperplane of the resulting slab contains the center $c'$ of $\mathcal{P}'$. It then follows that after the slab intersects $\mathcal{P}'$, the resulting polytope, $\mathcal{P}$, retains $c'$ as its center of symmetry and so is itself centrally symmetric. The only other possibility is that the maximal value of $C$ is $a_m$, which remains constant over some subinterval of $[0, 1]$ centered at $\frac{1}{2}$. In this case, the cross-sections for that subinterval are constant, ensuring that the intersection of $\mathcal{P}'$ with the slab between any two hyperplanes $\mathcal{H}_{t_1w}$ and $\mathcal{H}_{t_2w}$ from that subinterval is a prism with a centrally symmetric base. Such a prism is itself centrally symmetric.
Therefore in all cases, $\mathcal{P}$, which is the intersection of $\mathcal{P}'$ with a slab producing new, opposite facets of $(n-1)$-volume $a_n$, is centrally symmetric. \hfill \Box

**Corollary 3.6.** If $\{u_1, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$ and $a_1, \ldots, a_n$ are arbitrary positive real numbers, then there exists a unique parallelepiped with pairs of opposite facets having the $u_i$'s as normal vectors and the $a_i$'s as the $(n-1)$-volumes of the facets.

**Proof.** The first step in the proof of the proposition demonstrates existence of a unique convex polytope $\mathcal{P}$ satisfying the given conditions. At the same time, independence of the $u_i$ guarantees existence of a family of parallelepipeds with facets parallel to the facets of $\mathcal{P}$. Adjusting the distances between pairs of opposite facets of these parallelepipeds to match those of $\mathcal{P}$ determines a unique parallelepiped that coincides with $\mathcal{P}$ and consequently satisfies all of the given conditions. \hfill \Box

As a result, parallelepipeds can also be found with arbitrary dihedral angles $0 < \theta < 2\pi$ and facet volumes.

In the simple case of boxes, one might give high school students taking elementary algebra the problem of finding the dimensions of a box whose opposite pairs of faces have areas $1$, $2$, and $3$, or, for that matter, any three arbitrarily picked positive areas. We also note the following algebraic consequence of the preceding geometric corollary:

**Corollary 3.7.** Every $n \times n$ non-singular real matrix, $B$, is the $(n-1)$-st exterior power of a unique $n \times n$ matrix. That is, $B = \wedge^{n-1}(A)$ for some $n \times n$ matrix $A$, which may be regarded as the $(n-1)$-st exterior root of $B$.

**Proof.** Let $B$ be an arbitrary non-singular $n \times n$ matrix. Regard the columns of $B$ and their negatives as the normal vectors to pairs of facets of a parallelepiped where the norms of these vectors are the corresponding facet-volumes. Recall from the discussion preceding Theorem 3.3 that for an $n \times n$ matrix $A$, the $j$-th column of $\wedge^{n-1}_\pm(A)$ is $(-1)^j \times (a_1, \ldots, \hat{a}_j, \ldots, a_n)$. This column vector is the outward normal for one of the pair of facets of the parallelepiped $\mathcal{P}(A)$ whose corresponding generating facet is $\mathcal{P}(a_1, \ldots, \hat{a}_j, \ldots, a_n)$ and whose norm is the $(n-1)$-volume of this facet. The previous corollary guarantees existence of a unique parallelepiped $\mathcal{P}(A)$ with specified outward-pointing normals and facet-volumes. Its defining matrix $A$ with perhaps the columns permuted and some of their signs changed then satisfies $\wedge^{n-1}_\pm(A) = B$. If $B$ is first altered to $B'$ where the $(i, j)$-th entry of $B'$ is $(-1)^{n+i+j}$ times the corresponding entry of $B$, then $\wedge^{n-1}(A) = B$. \hfill \Box

Conditions on matrices of various shapes that guarantee the existence of $k$-th exterior roots for different values of $k$ are less known.

We now consider the Cauchy-Alexandrov Theorem for zonotopes, where a direct proof is possible.

**Proposition 3.8.** If combinatorially equivalent zonotopes in $\mathbb{R}^n$, $n \geq 3$, have congruent corresponding facets, then the zonotopes are congruent.

**Proof.** Consider zonotopes $\mathcal{Z}(A)$ and $\mathcal{Z}(A')$ of rank $n$ defined by $n \times k$ matrices $A = [a_1, \ldots, a_k]$ and $A' = [a'_1, \ldots, a'_k]$ where $k \geq n$. By permuting and changing the signs of columns as necessary, we may assume that all corresponding pairs of facets determined by the matrices are congruent.

Consider the case where $k = n$. The columns of both matrices are then independent and the zonotopes they define are parallelepipeds. Corresponding generating facets of the parallelepipeds are of the form $\mathcal{P}(a_1, \ldots, \hat{a}_j, \ldots, a_n)$ and $\mathcal{P}(a'_1, \ldots, \hat{a}'_j, \ldots, a'_n)$, which can be denoted briefly as $\mathcal{P}(A(j))$ and $\mathcal{P}(A'(j))$. As the facets in each pair are congruent, Theorem 2.2 implies that $(A(j))^T (A(j)) = (A'(j))^T (A'(j))$. Taken over all values $j = 1, \ldots, n$, all columns of the matrices are covered by equivalent comparisons, and so $A^T A = (A')^T A'$. By Theorem 2.2, the parallelepipeds are thus congruent to each other. (Indeed, it suffices to consider just three congruent pairs of corresponding
facets in order to guarantee that all corresponding pairs of edges of the parallelotopes have been compared.

More generally, for an arbitrary matrix of rank \( n \) with \( k \geq n \), consider any \( n \times n \) submatrix of rank \( n \) and three \( n \times (n - 1) \) submatrices of that matrix. Make a corresponding selection of such submatrices from \( A \) and \( A' \). The \( n \times n \) submatrices now represent corresponding generating parallelotopes of \( Z(A) \) and \( Z(A') \). The argument given in the preceding paragraph guarantees congruence for that pair of parallelotopes. By Corollary 1.7, unique translated copies of all such parallelotopes from each zonotope can be assembled in various ways to form a tiling of that zonotope. Choose a specific tiling of one zonotope and then assemble a similar tiling for the other zonotope. All corresponding parallelotopes of the two tilings will be congruent thereby implying congruence of the zonotopes themselves. \( \square \)

It was observed in the first paragraph of the proof that parallelotopes of dimension at least 3 are congruent to each other if they have three pairs of congruent facets. It becomes less obvious and perhaps more surprising that congruence of three pairs of facets still suffices in dimensions much greater than 3. It should also be noted that requiring the dimension to be at least 3 is necessary in order to guarantee that comparison of shape matrices for three corresponding pairs of facets,

\[
(A(j_i))^T(A(j_i)) = (A'(j_i))^T(A'(j_i))
\]

with \( i = 1, 2, 3 \) is sufficient to imply \( A^TA = (A')^TA' \). When \( n = 2 \),

\[
(a_1)^T(a_1) = (a_1')^T(a_1') \quad \text{and} \quad (a_2)^T(a_2) = (a_2')^T(a_2')
\]

do not force

\[
(a_1, a_2)^T(a_1, a_2) = (a_1', a_2')^T(a_1', a_2')
\]

because they lack the comparison \( (a_1)^T(a_2) = (a_1')^T(a_2') \). Indeed, zonogons in \( \mathbb{R}^2 \) with congruent corresponding edges need not be congruent.

For combinatorially equivalent zonotopes, the uniqueness part of Minkowski’s Theorem can be proven directly. Theorem 3.3 and its consequences will not be used, but Corollary 3.7, which has an independent algebraic proof, will be.

**Proposition 3.9.** If combinatorially equivalent zonotopes \( Z(A) \) and \( Z(A') \) have corresponding equal unit facet normals and facet volumes, then they are congruent.

**Proof.** Let \( A = [a_1, \ldots, a_k] \) and \( A' = [a_1', \ldots, a_k'] \) be matrices of rank \( n \) with \( k \geq n \). Let \( u_1, \ldots, u_k \) together with their negatives be the outward-pointing unit normal vectors to the corresponding pairs of facets of the zonotopes that are defined by the matrices, and suppose \( a_1, \ldots, a_k \) determine the corresponding volumes of the facets.

The proof is by induction on \( k \). When \( k = n \), the matrices are non-singular and define parallelotopes \( P(A) \) and \( P(A') \). Corresponding facets of the parallelotopes have the same normal vectors, so the dihedral angles between pairs of facets are also equal. Corresponding facets also have the same volumes. Taken together, these comparisons ensure that the defining matrices satisfy

\[
(\wedge_{\pm}^{n-1}A)^T(\wedge_{\pm}^{n-1}A) = (\wedge_{\pm}^{n-1}A')^T(\wedge_{\pm}^{n-1}A'),
\]

and therefore also

\[
(\wedge^{n-1}A)^T(\wedge^{n-1}A) = (\wedge^{n-1}A')^T(\wedge^{n-1}A').
\]

Moreover, \( (\wedge^{n-1}A)^T(\wedge^{n-1}A) = \wedge^{n-1}(A^TA) \), from which

\[
\wedge^{n-1}(A^TA) = \wedge^{n-1}(A'^TA').
\]
Corollary 3.7 then implies
\[ A^T A = A'^T A, \]
so by Theorem 2.2, \( Z(A) \) and \( Z(A') \) are congruent.

Now assume the proposition holds for zonotopes with fewer than \( k \) generators, for some fixed value \( k > n \). Suppose \( Z(A) \) and \( Z(A') \) satisfy the normal-vector and facet-volume conditions and are defined by \( n \times k \) matrices. Let \( A(\hat{k}) \) and \( A'(\hat{k}) \) be the corresponding matrices with \( k \)-th columns omitted. It follows that \( Z(A) = Z(A(\hat{k})) \oplus l\mathbf{a}_k \) and \( Z(A') = Z(A'(\hat{k})) \oplus l\mathbf{a}'_k \). The normal vector to each facet of \( Z(A) \) belonging to the zone (that is, 1-zone) of facets containing \( \mathbf{a}_k \) is orthogonal to \( \mathbf{a}_k \). All of these vectors span a hyperplane orthogonal to \( \mathbf{a}_k \). A similar relationship holds in \( Z(A') \). As the normal vectors and hyperplanes are the same, it follows that \( \mathbf{a}_k \) and \( \mathbf{a}'_k \) are parallel. Moreover, corresponding facets in the zones for \( \mathbf{a}_k \) and \( \mathbf{a}'_k \) have the same volumes. The facets in these zones are Minkowski sums of faces from either \( A(\hat{k}) \) with \( l\mathbf{a}_k \) or from \( A'(\hat{k}) \) with \( l\mathbf{a}'_k \). The faces are either \((n - 2)\) or \((n - 1)\)-dimensional and the resulting facets, after forming the sums, are then either prisms in the first case, or convex hulls of translated facets, when \( \mathbf{a}_k \) or \( \mathbf{a}'_k \) lies in the hyperplane containing the facet, in the second case. In either case, congruence of the corresponding base faces, the fact that \( \mathbf{a}_k \) and \( \mathbf{a}'_k \) are parallel, and equality of volumes of the resulting facets, forces \( \mathbf{a}_k \) and \( \mathbf{a}'_k \) to have the same length. Once the vectors are parallel and of the same length, the corresponding facets formed as Minkowski sums using these vectors, are congruent. Thus, all corresponding pairs of facets from \( Z(A) \) and \( Z(A') \) are congruent, and the two zonotopes are themselves congruent by Proposition 3.8. \[ \square \]

REFERENCES

[1] A. D. Alexandrov, *Convex Polyhedra*, Springer-Verlag, New York, 2005.
[2] D. Avis, et. al., *On the sectional area of convex polytopes*, Proceedings of the XII Annual Symposium on Computational Geometry, New York, 1996.
[3] A. Björner, et. al., *Oriented Matroids* (2nd ed.), Cambridge U. Press, New York, 1999.
[4] E. Gover and N. Krikorian, *Determinants and the Volumes of Parallelotopes and Zonotopes*, Linear Algebra and its Applications, 433 (2010), 28–40.
[5] B. Grünbaum, *Convex Polytopes*, Springer-Verlag, New York, 2003.
[6] R. Horn and I. Olkin, *When does \( A^*A = B^*B \) and why does one want to know?*, MAA Monthly, 103 (1996), 470–482.
[7] P. McMullen, *Polytopes with centrally symmetric faces*, Israel J. Math. 8 (1970), 194–196.
[8] P. McMullen, *Polytopes with centrally symmetric facets*, Israel J. Math. 23 (1976), 337–338.
[9] G. Shephard, *Polytopes with centrally symmetric faces*, Canadian J. Math. 19 (1967), 1206–1213.
[10] G. Shephard, *Combinatorial Properties of Associated Zonotopes*, Canadian J. Math. 26 (1974), 302–321.

Eugene Gover
Department of Mathematics
Northeastern University
Boston, MA 02115, U.S.A.
e.gover@neu.edu