Malliavin sensitivity analysis with polynomial growth payoff functions under the Black-Scholes model

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Abstract

In this paper, we provide some numerical results for Malliavin sensitivity analysis in finance. We consider payoff functions with polynomial growth, and with and without discontinuous points and compare the results to that of the finite-difference and pathwise methods.

Keywords sensitivity analysis, Greeks, Malliavin calculus, localization

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1. Introduction

Sensitivity analysis in mathematical finance is important in risk management, especially for derivatives hedging. Fronić et al. [1] introduced Malliavin sensitivity analysis, and researchers since have studied it widely [2–4]. Simulating the first-order sensitivities, such as Delta, of options with digital type payoff functions using Malliavin calculus works well compared to the finite-difference methods (FDM) and pathwise method (PM), which we have explained below. However, considering them with the standard call or put payoff requires the use of variance reduction techniques, like localization, to obtain more stable results. Thus, the Malliavin method works well when payoff functions are discontinuous, and not when payoff functions are continuous. However, there are no studies with numerical results under general payoff functions besides standard call, put, and digital types. This study investigates the case in which the underlying asset price $X_t$ depends on parameters such as $x_0$, $\sigma$, and so on; we would then like to obtain values of its sensitivities (Greeks in finance) for hedging.

Let $\beta$ be one parameter whose sensitivity we want to obtain. Denoting $X_t = X^\beta_0$ and $C = C(\beta)$, the first-order sensitivity in $\beta$ is

$$\frac{\partial C(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta} e^{-rT} E^Q[\phi(X^\beta_T)].$$

The second-order sensitivities are similarly defined. FDM are well-known methods to simulate (3). In our numerical experiments below, we use the central difference for the first-order derivatives. The important point of FDM is that the numerical results strongly depend on the choice of approximation parameter size $h$.

2. Settings and methods

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ be a complete filtered probability space with the usual condition, where $Q$ is a risk-neutral measure. Let $r$ $(>0)$ and $\sigma$ $(>0)$ be constants denoting the interest rate of the risk-free asset and the volatility of a security price, respectively. Assume that the security price process $X_t$ under $Q$ follows the Black-Scholes (BS) model:

$$dX_t = rX_t dt + \sigma X_t dB_t, \quad X_0 = x_0,$$

where $x_0$ is a positive constant denoting the initial value of the security, and $B_t$ is a standard Brownian motion under $Q$. Let $\phi$ be a payoff function which satisfies $E^Q[|\phi(X_T)|] < \infty$, where $E^Q[\cdot]$ denotes the expectation under $Q$. The price of European-type options with $\phi$ is generally expressed as:

$$C = e^{-rT} E^Q[\phi(X_T)].$$

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2.1 Pathwise method

In this subsection, we will explain the PM. In general, payoff functions in finance are not differentiable; however, such indifferentiable points are often only one point. Thus, we can calculate its derivative formally and directly using the chain rule by ignoring the indifferentiable point:

$$\frac{\partial C(\beta)}{\partial \beta} \approx e^{-rT} E^Q \left[ \phi'(X^\beta_T) \frac{\partial X^\beta_T}{\partial \beta} \right].$$
Note that when the underlying asset follows the BS model, the formula (4) hopefully holds as the equal sign in case of continuous payoff functions. When $\beta = x_0$ (Delta) and $\beta = \sigma$ (Vega), $\partial X_T^\beta/\partial \beta$ are respectively given as
\[
\frac{\partial X_T^{x_0}}{\partial x_0} = \frac{X_T^{x_0}}{x_0}, \quad \frac{\partial X_T^{\sigma}}{\partial \sigma} = X_T^\sigma (B_T - \sigma T).
\]

When $\beta = x_0$, the second derivative (Gamma) is
\[
\frac{\partial^2 C(x_0)}{\partial x_0^2} \approx e^{-rT} \mathbb{E}^Q \left[ \phi''(X_T^{x_0}) \left( \frac{X_T^{x_0}}{x_0} \right)^2 \right].
\]

2.2 Malliavin calculus method (MCM)

One method to calculate Greeks using integration by parts formula in Malliavin calculus was introduced by Fourni et al. [1]. The formula in [1] is
\[
\frac{\partial C(x)}{\partial \beta} = e^{-rT} \mathbb{E}^Q \left[ \phi (X_T^\beta) \mathbb{H}_\beta \right],
\]
where $H_\beta$ is a certain random variable. When $\beta = x_0$, $H_{x_0} = B_T(x_0)T$, and when $\beta = \sigma$, $H_\sigma = B_T^\sigma/(\sigma T) - B_T - (1/\sigma)$. Note that the derivative of $\phi$ does not appear in the expression (5). [1] similarly provides a formula for the second-order sensitivity:
\[
\frac{\partial^2 C(x)}{\partial x^2} = e^{-rT} \mathbb{E}^Q \left[ \phi'(X_T^x) \mathbb{H}_{x,\beta} \right],
\]
where $H_{x,\beta}$ is a certain random variable. When $\beta = x_0$, $H_{x_0} = (B_T^2 - \sigma TB_T - T)/(x_0^2\sigma^2T^2)$. By combining this with PM, we also have an alternative expression of Gamma:
\[
\frac{\partial^2 C(x_0)}{\partial x_0^2} = e^{-rT} \mathbb{E}^Q \left[ \phi'(X_T^{x_0})H_{x_0} \right],
\]
where $H_{x_0} = (B_TX_T^{x_0} - \sigma TX_T^{x_0})/(x_0^2\sigma T)$. For convenience, we call (6) MCM(Gamma1) and (7) as MCM(Gamma2) in the graphs below.

3. Payoff functions and Greeks

3.1 Payoff functions

In this subsection, we give the two types of payoff functions considered in this study. Let $K > 0$ be a constant denoting a strike price, and $1_D$ be the indicator function for a set $D$. For $i = 1, 2$, define
\[
\phi^i_p(x) = x^p 1_{\{x \geq K\}} - K^p 1_{\{x > K\}} 1_{\{i = 1\}}.
\]

When $i = 1$, $\phi^1_p(x)$ is a continuous function with polynomial growth on the order of $p$ for $x > K$, but not differentiable at $x = K$. In addition, for $p = 1$, $\phi^1_1(x)$ is the standard call payoff function. When $i = 2$, $\phi^2_p(x)$ is a discontinuous function at $x = K$ and has polynomial growth in the order of $p$ for $x > K$. For $p = 0$, $\phi^2_0(x)$ is the digital call payoff function.

$\phi^i_p(x)$ and $\phi^i_q(x)$ have the following relationship: for $x \leq K$, $\phi^i_p(x) = \phi^i_q(x) \equiv 0$ holds and for $x > K$, $\phi^i_p(x) = \phi^i_q(x) + 1$ holds when $K = 1$. Through these two types of functions, we attempt to find the impacts of growth orders and the existence of discontinuous points on the accuracies of simulations.

3.2 Option price

In this subsection, we give an analytical representation of option prices $\psi^i_p = e^{-rT}\mathbb{E}^Q[\phi^i_p(X_T)]$ under the BS model (1). Let $f(x)$ be the density function of the standard normal distribution and $F(x)$ be the distribution function of $f(x)$. Furthermore, set $A = x_0^p(C^{(p-1)}(rT + \sigma^2 T^2)/2).

Theorem 1 Option price $\psi^0_0$ is
\[
\psi^0_0 = AF(d_1) - e^{-rT}K^pF(d_2)1_{\{i = 1\}}, \quad (8)
\]

where
\[
d_2 = \frac{\log x_0/K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.
\]

We can derive (8) through standard calculations.

3.3 Sensitivity analysis

Here, we give explicit representations of Delta, Vega, and Gamma for (8) given in Theorem 1.

Theorem 2 The Delta, Vega, and Gamma for (8) are as follows.

(i) When $i = 1$, we have
\[
\frac{\partial \psi^0_1(x)}{\partial x_0} = \frac{A}{x_0} F(d_1), \quad \frac{\partial \psi^0_1(x)}{\partial \sigma} = pAV/\sqrt{T} \left\{ (p - 1)\sigma V F(d_1) + f(d_1) \right\},
\]

(ii) When $i = 2$, we have
\[
\frac{\partial \psi^0_2(x)}{\partial x_0} = \frac{A}{x_0\sigma\sqrt{T}} f(d_1), \quad \frac{\partial \psi^0_2(x)}{\partial x_0^2} = \frac{A}{x_0\sigma} \left( \sqrt{T} + d_2 \right) f(d_1),
\]

\[
\frac{\partial^2 \psi^0_2(x)}{\partial x_0^2} - \frac{A d_2 + \sigma^2 T}{x_0^2 \sigma^2 T^2} f(d_1).
\]

3.4 Localization method

We now introduce a variance reduction method for MCM, or localization, which Fourni et al. [1] also introduced. Let $g^i_p$ and $h^i_p$ be functions satisfying $g^i_p(x) = g^i_p(x) + h^i_p(x)$, where $g^i_p$ is a differentiable function and $h^i_p$ is a function with compact support. Note that once we obtain $g^i_p$, then $h^i_p = g^i_p - g^i_p$. For Delta and Vega, we use $g^1_p$ as $g^1_p(x) = a^1_p x^3 + b^1_p x^2 + c^1_p x + d^1_p$ on $[L_i, U_i]$ for $i = 1, 2$, and for Gamma, we use $g^2_p(x) = a^2_p x^3 + b^2_p x^2 + c^2_p x^2 + d^2_p x + e^2_p$ on $[L_i, U_i]$. To determine the coefficients and satisfy the differentiability of $g^i_p$, we use the following conditions:

Take $L_i = K - \varepsilon_{loc} 1_{\{i = 2\}}$ and $U_i = K + \varepsilon_{loc}$ for $\varepsilon_{loc} > 0$. Then, $g^i_p(x) = \phi^i_p(x)$ for $x \notin [L_i, U_i], g^i_p(L_i) = 0$, $g^i_p(U_i) = \phi^i_p(U_i)$, and $(g^i_p)'(U_i) = (\phi^i_p)'(U_i)$. For Gamma, add $(g^i_p)'(L_i) = 0$ and $(g^i_p)''(U_i) = (\phi^i_p)''(U_i)$. Note that in this variance reduction method, we need to introduce an approximation parameter $\varepsilon_{loc}$, such as FDM.
Then, we rewrite our target (3) as
\[
\frac{\partial C(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta} e^{-rT} \mathbb{E}^Q \left[ g^p_i(X_T^0) \right] + \frac{\partial}{\partial \beta} e^{-rT} \mathbb{E}^Q \left[ h^p_i(X_T^0) \right].
\]
The first term on the right hand side has differentiability, so we can apply PM. On the other hand, the second term does not have differentiability, so we apply MCM. However, from our construction \( h^p_i = \phi^p_i - g^p_i \) and \( g^p_i = \phi^p_i \) for \( [L_i, U_i]^c \), where \( c \) denotes the complement, \( h^p_i \) has compact support. Thanks to this property, we might have a small variance in the simulation and better numerical results in some cases.

We have some possibilities for expressing Gamma:
\[
\frac{\partial^2 C(x_0)}{\partial x_0^2} \approx \mathbb{E}^Q \left[ (g^p_i)'\left( X_T^{x_0} \right) \left( \frac{X_T^{x_0}}{x_0} \right)^2 \right] + \mathbb{E}^Q \left[ h^p_i \left( X_T^{x_0} \right) H^1_{x_0} \right],
\]
\[
\frac{\partial^2 C(x_0)}{\partial x_0^2} \approx \mathbb{E}^Q \left[ (g^p_i)'\left( X_T^{x_0} \right) \left( \frac{X_T^{x_0}}{x_0} \right)^2 \right] + \mathbb{E}^Q \left[ (h^p_i)' \left( X_T^{x_0} \right) H^2_{x_0} \right].
\]

For convenience, we call (9) Loc(Gamma1) and (10) as Loc(Gamma2) in the graphs below.

4. Numerical experiments

We now show results from some numerical experiments related to sensitivity analysis using Malliavin calculus under the BS Model (1). We use the following parameters for the parameters in our simulation: \( r = 0.05, \sigma = 0.3, T = 1, K = 1 \) and \( x_0 = 1 \) (at-the-money). We use the Euler-Maruyama scheme to simulate sample paths with 100 steps until maturity \( T \), and obtain an approximation of their expectations, using the \( 10^6 \) times Monte-Carlo method. In the following figures, the \( x \)-axis denotes the power \( p \) and the \( y \)-axis denotes the relative error rates from the true values, which we obtain from Theorem 2. For \( i = 1 \), we simulate each sensitivity with \( p \) from 0.25 to 3 with 0.25 step-size. For \( i = 2 \), we simulate each sensitivity with \( p \) from 0 to 3 with 0.25 step-size.

4.1 Delta

First, we show the numerical results for Delta \( \partial C^p_i/\partial x_0 \) for the continuous payoff functions (\( i = 1 \)) in Fig. 1.

Fig. 1 shows that FDM with \( h = 0.01 \), PM, localization with \( \varepsilon_{\text{loc}} = 0.01 \), and localization with \( \varepsilon_{\text{loc}} = 0.1 \) give rather high-accuracy results for all \( p \). MCM also gives high-accuracy results, such that the relative error rates are less than 1% for all \( p \). MCM also gives high-accuracy results, such that the relative error rates are less than 1% for all \( p \). On the other hand, FDM with \( h = 0.1 \) does not work in this situation; thus, we see that FDM depends on the choice of the approximation parameter size \( h \) in the Delta calculation for a continuous payoff function. Note that for a Delta with continuous payoff functions, the choice of the approximation parameter size \( \varepsilon_{\text{loc}} \) in localization does not have much impact on the results.

Next, we show a numerical result of Delta \( \partial C^p_2/\partial x_0 \) for the discontinuous payoff functions (\( i = 2 \)) in Fig. 2.

First, note that in Fig. 2, the numerical results from the PM do not appear because the errors are huge compared to the other errors. This is because the PM ignores the discontinuous point, and thus ignores the derivative of this point; that is, the PM ignores the Dirac-delta function at this point, which leads to relatively large error rates. This is the same reason as in other cases related to derivatives of discontinuous payoff functions. From Fig. 2, we can see that FDM with \( h = 0.01 \), and MCM and localization with \( \varepsilon_{\text{loc}} = 0.01 \) give highly accurate results. Namely, the relative error rates are below 1%. However FDM with \( h = 0.1 \) and localization with \( \varepsilon_{\text{loc}} = 0.1 \) give large relative error rates, so compared to Delta with continuous payoff functions, Delta with discontinuous payoff functions have numerical results that depend more on the approximation parameters.

4.2 Vega

We now comment on the numerical results related to Vega \( \partial C^p / \partial \sigma \) without figures due to space limitation.

For Vega with continuous payoff functions, the numerical results are quite similar to those of Delta with continuous payoff functions. However, for Vega with discontinuous payoff functions, the numerical results are much different from those of Delta with discontinuous payoff functions. Each method has low accuracy for Vega with discontinuous payoff functions, and only FDM with \( h = 0.01 \) and MCM work with about 5% relative error rates, except when \( p = 1 \). When \( p = 1 \), a spike appears and the results with both methods have about 26 – 27%
relative error rates.

4.3 Gamma

In this subsection, we will illustrate the numerical results for Gamma \( \frac{\partial^2 C_i}{\partial x_i^2} \).

First, we show the results for Gamma with continuous payoff function in Fig. 3.

Compared to the results for Delta and Vega with continuous payoff functions, the accuracy for Gamma in Fig. 3 is rather low. However, as [4] points out, for the standard call payoff function, MCM (Gamma2) (7), which is the mixed method of the pathwise derivative and Malliavin calculus, also works well for continuous payoff functions with polynomial growth. The other choices, FDM with \( h = 0.01, 0.1 \), and localization with \( \varepsilon_{loc} = 0.1 \) has about 1%, a high-accuracy for this situation. However, note that they depend on their approximation parameter sizes. The other methods do not work well in this situation.

Next, we show the results for Gamma with discontinuous payoff functions in Fig. 4.

From Fig. 4, we can easily see that it is difficult to simulate Gamma for a discontinuous payoff function with polynomial growth. However, MCM (Gamma1) performs better in that the relative error rates are less than 3%, except around \( p = 1 \). Second, FDM (\( h = 0.1 \)) also gives better results, but again note that FDM depends on the choice of the approximation parameter size. Unfortunately, other methods do not work well in this situation.

4.4 Other results

We also numerically tested for \( p = 4, 5, \ldots, 10 \), in which MCM has large standard errors compared to the other methods. Hence, MCM without any variance reduction does not work well for large \( p \), even if the payoff functions are discontinuous. We can guess that adding variance reduction techniques for Malliavin weights, such as Kohatsu-Higa et al. [3], will improve the results. In addition, we take \( K = 1 \) in this study, and from the payoff function \( \phi^p \), we normalize the jump size of the payoff functions to 1. Therefore, the impact of jump size becomes relatively small for large \( p \) because in this case, the slope of the payoff functions becomes large and such sensitivities become more important as a whole. Consequently, we gradually do not need to study how to cope with jumps. In this sense, PM gives better results for discontinuous payoff functions when \( p \) is large. However, put another way, even if \( p \) is large, the accuracy of PM might still not be good and MCM might work well for large jump sizes.

The cases of Delta and Vega for localization with continuous payoff functions, the choice of parameter size does not affect the results so much. However, it has some effect on the results in other cases. Another point of localization is that we adopt a polynomial function on the order of 3 in this study for first derivatives, and a polynomial function on the order of 5 for second derivatives; however, there are other choices for localization functions that require consideration.

5. Conclusion

We numerically confirm that our results related to the characteristics of sensitivities for functions with polynomial growth, and with and without discontinuous points are essentially similar to those obtained for the standard call, put, and digital payoff functions in many other studies (e.g., see [4]). This is the first study attempting to determine the characteristics of MCM under general payoff functions, and represents the first step in this direction.

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