Global Schrödinger map flows to Kähler manifolds with small data in critical Sobolev spaces: High dimensions.

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Abstract In this paper, we prove that the Schrödinger map flows from $\mathbb{R}^d$ with $d \geq 3$ to compact Kähler manifolds with small initial data in critical Sobolev spaces are global. This is a companion work of our previous paper where the energy critical case $d = 2$ was solved. In the first part of this paper, for heat flows from $\mathbb{R}^d$ ($d \geq 3$) to Riemannian manifolds with small data in critical Sobolev spaces, we prove the decay estimates of moving frame dependent quantities in the caloric gauge setting, which is of independent interest and may be applied to other problems.

Keywords: Schrödinger map flow; critical Sobolev space; global regularity; energy supercritical

1 Introduction

Let $(\mathcal{M}, g)$ be a Riemannian manifold, and $(\mathcal{N}, J, h)$ be a Kähler manifold. Given a map $u : \mathbb{R}^d \to \mathcal{M}$, the Dirichlet energy $\mathcal{E}(u)$ is defined by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^d} |du|^2 dx. \quad (1.1)$$

The heat flow is the gradient flow of energy functional $\mathcal{E}(u)$. A map $u(x, t) : \mathbb{R}^d \times [0, \infty) \to \mathcal{M}$ is called heat flow (of harmonic maps) if $u$ satisfies

$$\begin{cases} u_t = \tau(u) \\ u \mid_{t=0} = u_0(x) \end{cases} \quad (1.2)$$

Here, the tension field $\tau(u)$ is defined by

$$\tau(u) = \sum_{j=1}^{d} \nabla_j \partial_j u$$

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where \( \{\nabla_j^d\}_{j=1}^d \) denote the induced covariant derivatives on the pullback bundle \( u^*TM \).

The Hamiltonian analogy of heat flows is the so-called Schrödinger map flow. A map \( u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathcal{N} \) is called Schrödinger map flow (SL) if \( u \) satisfies

\[
\begin{align*}
  u_t &= J\tau(u) \\
  u \mid_{t=0} &= u_0(x).
\end{align*}
\] (1.3)

(1.2) is related to the liquid crystal theory (see e.g. [9]), while (1.3) plays a fundamental role in solid-state physics ([20]).

In our previous work [23] where global well-posedness of small energy 2D Schrödinger map flows into compact Kähler manifolds was proved, we noticed that to prove global well-posedness it is essential to firstly establish the parabolic decay estimates of differential fields and connection coefficients associated with heat flows in the caloric gauge setting. Hence, in the first part of this paper, we prove these decay estimates for heat flows with small data in critical Sobolev spaces with \( d \geq 3 \). In the second part, we apply decay estimates to prove global existence of SL by iteration argument, dynamic separation involved in our previous work [23] and new ingredients adapted to high dimensions.

We briefly recall the following non-exhaustive list of works on Cauchy problems and dynamic behaviors of SL. More details and references can be found in [23]. The local Cauchy theory of SL was developed by Sulem-Sulem-Bardos [36], Ding-Wang [7], McGahagan [24]. The global theory for small data Cauchy problem was pioneered by Chang-Shatah-Uhlenbeck [6], Ionescu-Kenig [14, 15], Bejenaru [1]. The small data global theory in critical Sobolev spaces for target \( S^2 \) was completed by Bejenaru-Ionescu-Kenig [2] \((d \geq 4)\) and Bejenaru-Ionescu-Kenig-Tataru [3] \((d = 2)\). In the other direction, for large data in equivariant classes, global theories on stability/instability and threshold scattering were studied by works of Gustafson, Kang, Tsai, Nakanish [11, 12] and Bejenaru, Ionescu, Kenig, Tataru [11]. The singularity formulation in equivariant class was achieved by Merle-Raphael-Rodnianski [25] and Perelman [31]. And for large data without equivariant assumptions, Rodnianski-Rubinstein-Staffilani [32] studied global regularity for \( d = 1 \) and Dodson, Smith [8, 33] studied the conditional global regularity for solutions with controlling dispersed norms for \( d = 2 \).

One of the main part of this paper is the decay estimates of moving frame dependent quantities of heat flows in the caloric gauge setting. The \( d = 2 \) case was established by Tao [37] for the \( \mathbb{H}^n \) target and by Smith [31] for general targets below threshold. Since \( d \geq 3 \) is energy supercritical, generally, these decay estimates could be expected only for small data. When \( d \geq 3 \) is even, the issue is relatively easy in the small data case. In fact, Bochner inequalities, bootstrap and Sobolev inequalities will suffice. The case when
$d$ is odd requires much more efforts. On one side, the involved quantities such as connection coefficients, curvature terms depend on both frames and the map itself, and geometric inequalities such as Bochner inequalities only provide bounds for covariant derivatives which are of integer orders. On the other side, the critical Sobolev spaces for odd dimensions are of fractional order. The conflict becomes prominent while bounding curvature dependent quantities. To solve this problem, we use parallel transport and difference characterization of Besov spaces. In fact, the difference characterization enables us to avoid directly apply fractional derivatives to geometric quantities. And the parallel transport enables us to compare geometric quantities in different points of the manifold.

The other main ingredient of this paper is the new parabolic decay estimates of the curvature terms beyond $L^\infty_t H^{d/2}_x$. In fact, in order to bound the curvature term in the $F_k(T)$ space, we observed in the 2D case that by iteration argument and dynamic separation it reduces to parabolic decays of curvature terms in the space $L^\infty_t H^1_x$. However, for $d \geq 5$, especially for $d$ large, controlling curvature terms in the space $L^\infty_t H^{d/2}_x$ is not sufficient to give an $F_k$ bound. In fact, after dynamic separation for twice, one of the most difficult curvature term reads as

$$
\int_s^\infty \phi_s(\nabla^2 R)(e_{i_0}, e_{i_1}, \ldots, e_{i_k}) ds'
$$

The Low $\times$ High interaction of $\phi_s(\nabla^2 R)(\ldots)$ in the space $F_k$ fails if we only have $L^\infty_t H^{d/2}_x$ decay for the $(\nabla^2 R)(\ldots)$ part. The remedy is to track the decay estimates of curvature part $P_k[(\nabla^2 R)(\ldots)]$ in the $L^p_x L^\infty_t$ space along the heat flow. In order to avoid the somewhat troublesome norm $L^\infty_t$ in the space $L^p_x L^\infty_t$, we turn to bound the much flexible norms $\|\partial_t[\ldots]\|_{L^p_x L^\infty_t}$, $\|\ldots\|_{L^p_x L^\infty_t}$ which control $L^p_x L^\infty_t$ by interpolation.

1.1 Main Results

Suppose that $\mathcal{M}$ is isometrically embedded into $\mathbb{R}^M$. Denote the embedding map by $\mathcal{P}$. Given a point $Q \in \mathcal{M}$, define the extrinsic Sobolev space $H^k_Q$ by

$$
H^k_Q := \{ u : \mathbb{R}^d \to \mathbb{R}^M \mid u(x) \in \mathcal{M} \text{ a.e. } x \in \mathbb{R}^d, \|u - Q\|_{H^k(\mathbb{R}^d)} < \infty \},
$$

with the metric $d_Q(f, g) = \|f - g\|_{H^k}$. Let

$$
Q(\mathbb{R}^d, \mathcal{M}) := \bigcap_{k=1}^{\infty} H^k_Q.
$$

Our main theorems are as follows:
Theorem 1.1. Let \( d \geq 3 \) and \( M \) be an \( m \)-dimensional closed Riemannian manifold. And \( P : M \to \mathbb{R}^M \) is an isometric embedding. Let \( v(s,x) \) be the solution of heat flow (1.2) with initial data \( v_0 \in Q(\mathbb{R}^d,M) \). There exists sufficiently small positive constant \( \epsilon_1 \) such that if

\[
\|v_0\|_{\dot{H}^{\frac{d}{2}}} \leq \epsilon_1, \tag{1.5}
\]

then \( v \) is global with respect to \( s \in \mathbb{R}^+ \) and converges uniformly to \( Q \) as \( s \to \infty \). And there exists a unique Tao’s caloric gauge \( \{e_l\}_{l=1}^m \) for which

\[
\nabla_s e_l = 0, \quad \lim_{s \to \infty} e_l = e_l^\infty, \quad l = 1, \ldots, m, \tag{1.6}
\]

where \( \{e_l^\infty\} \) are the given frames for \( T_Q M \). Denote the connection coefficients and differential fields under the caloric gauge condition by \( \{A_i\}_{i=1}^d \) and \( \{\psi_i\}_{i=1}^d \) respectively. Then we have

\[
\|\partial_j^s v\|_{\dot{H}^{\frac{d}{2}}} \lesssim s^{-\frac{j}{2}} \epsilon_1, \tag{1.7}
\]

\[
\|\partial_j^s (dP(e_l) - \chi_l^\infty)\|_{\dot{H}^{\frac{d}{2}}} \lesssim s^{-\frac{j}{2}} \epsilon_1 \tag{1.8}
\]

\[
\sum_{i=1}^d \|\partial_j^s A_i\|_{\dot{H}^{d-1}} \lesssim s^{-\frac{j}{2}} \epsilon_1, \tag{1.9}
\]

where \( \chi_l^\infty = \lim_{s \to \infty} dP(e_l) \) for \( l = 1, \ldots, m \).

Theorem 1.1 can be applied to prove the small data global regularity of SL in high dimensions:

Theorem 1.2. Let \( N \) be a compact Kähler manifold which is isometrically embedded into \( \mathbb{R}^N \). Let \( Q \in N \) be a fixed given point. Let \( u_0 \in Q(\mathbb{R}^d,N) \) with \( d \geq 3 \). There exists a sufficiently small constant \( \epsilon_0 > 0 \) such that if \( u_0 \) satisfies

\[
\|u_0\|_{\dot{H}^{\frac{d}{2}}} \leq \epsilon_0, \tag{1.10}
\]

then (1.3) with initial data \( u_0 \) evolves into a global unique solution \( u \in C(\mathbb{R}; Q(\mathbb{R}^d,N)) \). Moreover, for all \( \sigma \in \mathbb{Z}^+ \) there holds

\[
\|u\|_{L^\infty_t \dot{H}^\sigma_Q \cap \dot{H}^{\frac{d}{2}}} \leq C(\|u_0\|_{\dot{H}^{\frac{d}{2}}}). \tag{1.11}
\]

Remark 1.1 Tataru raised the problem of proving global well-posedness for small initial data in the critical Sobolev spaces for general Kähler targets as an open question in the survey [13]. Our previous work [23] solved the case \( d = 2 \). Here, Theorem 1.2 solves the case \( d \geq 3 \).

Remark 1.2 For small data global regularity of heat flows in energy supercritical dimensions, we recall the remarkable results obtained by Struwe
are two different styles of giving small data. Struwe \cite{35} proved in $d \geq 3$ that for any initial data $u_0$ satisfying $\|du_0\|_{L^\infty_t(L^2_{x})} \leq K$ there exists $\epsilon(K)$ such that if $\mathcal{E}(u_0) \leq \epsilon(K)$ then the solution of heat flow equation is global and converges to constant map as time goes to infinity. In our work, we impose no smallness condition on the energy but require critical Sobolev norms to be small. These two settings are two different styles of giving small data.

**Notations.** We fix two constants $\vartheta = 1 - \frac{1}{10^m}$, $\delta = \frac{1}{d10^m}$. The notation $A \lesssim B$ means there exists some $C > 0$ such that $A \leq CB$. We denote $P_k$ with $k \in \mathbb{Z}$ the Littlewood-Paley projection with Fourier multiplier supported in the frequency annual $\{2^{k-1} \leq |\eta| \leq 2^{k+1}\}$.

The connections of $TM$ and $u^*TM$ are denoted by $\overline{\nabla}$ and $\nabla$ respectively. Without confusion, we also denote connections of $TN$ and $u^*TN$ by $\overline{\nabla}$ and $\nabla$ respectively. Let $\mathbf{R}$ denote the curvature tensor of $\mathcal{M}$ or $\mathcal{N}$.

### 1.2 Road map for the proof of Theorem 1.2

Let us outline the proof for Theorem 1.2.

#### 1. Bootstrap and iteration argument.

Suppose we have solution $u \in C([-T, T]; \mathcal{Q}(\mathbb{R}^d, \mathcal{N}))$ for SL with initial data $u_0$. Let $\{c_k(\sigma)\}$, $\{c_k\}$ be frequency envelopes associated with $u_0$:

\begin{align}
2^\frac{d}{2}k \|P_k u_0\|_{L^2_t L^2_x} & \leq c_k \quad (1.12) \\
2^\frac{d}{2}k + \sigma k \|P_k u_0\|_{L^2_t L^2_x} & \leq c_k(\sigma). \quad (1.13)
\end{align}

Let $v(s, t)$ be the heat flow initiated from $u(t)$. Let $\{\phi_i\}_{i=0}^d$ and $\{A_i\}_{i=0}^d$ be differential fields and connection coefficients associated with $v$ under the caloric gauge. (The index 0 refers to $t$) Assume that $u$ satisfies

\begin{align}
2^\frac{d}{2}k \|P_k u\|_{L^\infty_t L^2_x} & \leq \epsilon^{\frac{1}{2}}c_k \quad (1.14) \\
2^{d-k} \|P_k \phi_x(\{s=0\})\|_{G_k(T)} & \leq \epsilon^{\frac{1}{2}}c_k. \quad (1.15)
\end{align}

**Step 1. Before iteration.** In Step 1, we assume $\sigma \in [0, \vartheta]$.

#### Step 1.1. Parabolic estimates along the heat direction.

Let $\{b_k(\sigma)\}$ and $\{b_k\}$ be frequency envelopes of $\{\phi_i\}_{i=0}^m$ in $G_k(T)$ norm:

\begin{equation}
 b_k(\sigma) := \sum_{i=1}^d \sup_{k' \in \mathbb{Z}} 2^{-\delta|k-k'|} 2^\frac{d}{2}k' - k' 2^{d\sigma k'} \|P_{k'} \phi_i\|_{s=0} \|_{G_{k'}(T)}, \quad (1.16)
\end{equation}

and $b_k := b_k(0)$. Then the connection coefficients $\{A_i\}_{i=0}^d$ satisfy

\begin{equation}
\sum_{i=1}^d 2^{d-k} 2^{d\sigma k} \|P_k A_i(s)\|_{F_k(T) \cap S_k(T)} \lesssim (1 + s 2^{2k})^{-4} b_{k, \vartheta}(\sigma) \quad (1.17)
\end{equation}
(1.17) is essential to derive parabolic estimates for all other differential fields, especially it implies

\[ 2\sum_{i=1}^{d} 2^{k-2}\|P_{k}A_{i} \|_{s=0} \|_{L^{2}(T)} \lesssim \epsilon \sigma_{k}^{k} \tag{1.18} \]

\[ \sum_{i=1}^{d} 2^{k-2}\|P_{k}A_{i} \|_{s=0} \|_{L^p(T)} \lesssim \epsilon \sigma_{k}^{k}. \tag{1.19} \]

**Step 1.2. Estimates along the Schrödinger direction.** By studying (6.7) and linear estimates in $G_k v.s. N_k$ spaces for linear Schrödinger equation established by [3], (1.18) and (1.19) give

\[ b_k(\sigma) \lesssim c_k(\sigma) \tag{1.20} \]

for all $\sigma \in [0, \vartheta]$.  

**Step 2. Iteration.** In Step 2, we assume $\sigma \in [0, 2\vartheta]$.

**Step 2.1. Parabolic estimates along the heat direction.** Let \( \{b_{k}(1)(\sigma)\} \) and \( \{b_{k}(1)\} \) be

\[ b_{k}(1)(\sigma) = \begin{cases} b_k(\sigma), & \text{if } \sigma \in [0, \vartheta] \\ b_k(\sigma) + c_k(\sigma - \vartheta)c_k(\vartheta), & \text{if } \sigma \in (1, 2\vartheta] \end{cases} \tag{1.21} \]

Then the connection coefficients \{A_i\}_{i=0}^{d} satisfy

\[ \sum_{i=1}^{d} 2^{k-2}\|P_{k}A_{i}(s)\|_{L^{\infty}(T)} \cap S_{k}^{\vartheta}(T) \lesssim (1 + s2^{2k})^{-4}b_{k,s}^{(1)}(\sigma), \tag{1.22} \]

and it implies

\[ 2^{k-2}\|P_{k}A_{t} \|_{s=0} \|_{L^{2}(T)} \lesssim \epsilon b_{k}(1)(\sigma) \tag{1.23} \]

\[ \sum_{i=1}^{d} 2^{k-2}\|P_{k}A_{i} \|_{s=0} \|_{L^{p}(T)} \lesssim \epsilon b_{k}(1)(\sigma). \tag{1.24} \]

**Step 2.2. Estimates along the Schrödinger direction.** By (6.7), (1.23) and (1.24) show

\[ b_k(\sigma) \lesssim c_k(\sigma) + c_k(\vartheta)c_k(\sigma - \vartheta) \tag{1.25} \]

as well for arbitrary $\sigma \in [\vartheta, 2\vartheta]$. Thus, as a corollary of embedding $G_k(T) \hookrightarrow L^{\infty}_{t}L^{2}_{x}$, we get

\[ \sum_{i=1}^{d} 2^{k-2}\|P_{k}\phi_{i} \|_{L^{\infty}_{t}L^{2}_{x}} \lesssim c_{k}^{(1)}(\sigma). \tag{1.26} \]
Step 3. Global regularity. Doing iteration for $K$ times gives

$$
\sum_{i=1}^{d} 2^{2k} 2^{2k-k} \| P_k \phi_i \|_{L_t^\infty L_x^2} \lesssim c_k^{(K)}(\sigma).
$$

(1.27)

with $\sigma \in [0, K\theta]$. And transforming (1.27) to bounds of $u$ gives

$$
\| u(t) \|_{L_t^\infty H_t^1 \cap \dot H_t^2} \lesssim C(\| u_0 \|_{H_Q^1}).
$$

(1.28)

for $L = \frac{d}{2} + K$. By the local Cauchy theory of [7, 24], we see $u$ is globally smooth provided $u_0 \in Q(\mathbb{R}^d, N)$.

2. Reduction to Decay of heat flows

Overview of first time iteration

The new difficulty in the general targets case is that the curvature term depends not only on the differential fields but also the map $u$ itself. This was overcome by synthetically using kind of dynamic separation and iteration argument. Here, the dynamic separation may be seen as the “freezing coefficient method” originally developed in the elliptic PDEs.

The curvature terms emerge in (6.8), (6.6), (6.7). The equation (6.8) is used to control connection coefficients (see (1.21)), while (6.6) and (6.7) are used to track the evolution of differential fields along heat and Schrödinger direction respectively.

Proof of (1.17). We will follow the framework of our previous work [Lemma 3.1, 23]. The curvature term $R(\phi_s, \phi_i)$ in (6.8) can be schematically written as

$$
\sum (\phi_s \circ \phi_i)(R(e_{j_0}, e_{j_1}) e_{j_2}, e_{j_3}).
$$

(1.29)

The $\{\phi_i\}_{i=0}^d$ part will be controlled by bootstrap assumption. Denote the remainder part by

$$
G(s) = \sum \langle R(e_{j_0}, e_{j_1}) e_{j_2}, e_{j_3} \rangle(s).
$$

(1.30)

By caloric gauge condition, $G$ can be expanded as

$$
\langle R(e_{j_0}, e_{j_1}) e_{j_2}, e_{j_3} \rangle(s) = \Gamma^\infty + R_1 + R_2 + R_3
$$

where we denote

$$
R_1 := \Xi_2 \int_s^\infty \sum_{i=1}^{2} (\partial_i \phi_i)^l ds' + \Xi_1 \int_s^\infty \sum_{i=1}^{2} (A_i \phi_i)^l ds'
$$

$$
R_2 := \int_s^\infty \sum_{i=1}^{2} (\partial_i \phi_i)^l \langle G' \rangle_i ds'
$$
\[
\mathcal{R}_3 := \sum \int_s^{\infty} (A_i \phi_i)(\bar{s})(G')_l \, ds
\]

\[
(G')_l := (\nabla^2 R)(e_l; e_{l_0}, e_{l_1}, e_{l_2}, e_{l_3}) - \Xi_l'^{\infty}.
\]

and \(\Gamma^\infty, \{\Xi_l^\infty\}\) are constant vectors. In the \(d = 2\) case, by bilinear estimates and the above decomposition, our previous work [23] show that for (1.21) it suffices to prove the parabolic estimates for \(P_k G'\):

\[
2^{\frac{d}{2}} \|P_k G'\|_{L^\infty L^2_x} \lesssim_L \|u\|_{L^\infty H^2_x} (1 + s2^{2k})^{-L}, \forall L \in \mathbb{N}. \quad (1.31)
\]

But for \(d \geq 5\), (1.31) is not enough. In fact, we need following new parabolic decay estimates for high dimensions:

\[
2^{\frac{d}{2}} \|P_k G'\|_{L^\infty L^2_x \cap L^{p_d}_t} \lesssim_L \|u\|_{L^\infty H^{\frac{d}{2}}_x} (1 + s2^{2k})^{-L}, \forall L \in \mathbb{N}, d \geq 3 \quad (1.32)
\]

\[
2^{\frac{d}{4}} \|P_k \partial_s G'\|_{L^{p_d}_t} \lesssim \|u\|_{L^\infty H^{\frac{d}{4}}_x}, d \geq 5 \quad (1.33)
\]

They will be proved in Section 3 and Section 4 by geodesic parallel transport and difference characterization of Besov spaces.

**Proof of (1.22).** For (1.22), inspired by our previous work [Lemma 5.1, 23], it suffices to prove more refined parabolic estimates for \(P_k G'\) than (1.32)-(1.33):

\[
2^{\frac{d}{2}} \|P_k G'\|_{L^\infty L^2_x \cap L^{p_d}_t} \lesssim_L 2^{-\sigma k} b_k(\sigma)(1 + s2^{2k})^{-L}, \forall L \in \mathbb{N}, d \geq 3 \quad (1.34)
\]

\[
2^{\frac{d}{4}} \|P_k \partial_s G'\|_{L^{p_d}_t} \lesssim 2^{-\sigma k} b_k(\sigma), d \geq 5. \quad (1.35)
\]

for all \(\sigma \in [0, \vartheta]\). By caloric gauge condition, \(G'\) can be further decomposed as

\[
(G')_l(s) = \sum \left( \int_s^{\infty} \phi^p_s(s')(\nabla^2 R)(e_l; e_{l_0}, e_{l_1}, e_{l_2}, e_{l_3}) \right) ds' \]

\[
= \sum \left( \int_s^{\infty} \phi^p_s(s') ds' \right) \Omega_{lp} + \sum \int_s^{\infty} \phi^p_s(s')((G'')_l - \Omega_{lp}) ds',
\]

where we denote

\[
(G'')_l = (\nabla^2 R)(e_l; e_{l_0}, e_{l_1}, e_{l_2}, e_{l_3}) - \Omega_{lp}^\infty
\]

Applying bilinear Littlewood-Paley decompositions reduces the proof of (1.31)-(1.33) to verify

\[
2^{\frac{d}{2}+k} \|P_k \phi_s\|_{L^\infty L^2_x \cap L^{p_d}_t} \lesssim (1 + s2^{2k})^{-M} 2^{-\sigma k} b_k(\sigma), d \geq 3 \quad (1.36)
\]

\[
2^{\frac{d}{4}+k} \|P_k (G'')\|_{L^\infty L^2_x \cap L^{p_d}_t} \lesssim (1 + s2^{2k})^{-M} \|u\|_{L^\infty H^{\frac{d}{4}}_x}, d \geq 3 \quad (1.37)
\]
2d\kappa \| P_k(\partial_t G') \|_{L^p_t} \lesssim \| u \|_{L^\infty_t H^d_x}, \quad d \geq 5 \tag{1.38} \]

**Remark.** For $d = 3, 4$ the (1.38) type estimates are not necessary.

**Proof of (1.28).** Let $K = 0$ in (1.28). To pass from the bounds for moving frame dependent quantities stated in (1.26) to the bounds of $u$ itself stated in (1.28), the key is to deduce frequency bounds for frames:

$$2d^k + \sigma k \| P_k( (\partial_s v) ) \|_{L^\infty_t L^2_x \cap L^p_t} \lesssim 2^{d_k} (1 + s2^{2k})^{-M} b_k(\sigma), \quad d \geq 3 \tag{1.39}$$

for $\sigma \in [0, \vartheta]$. Moreover, (1.36) is a corollary of (1.39) and

$$2d^k \| P_k( (D(dP)(e_p; e_l)) ) \|_{L^\infty_t L^2_x \cap L^p_t} \lesssim 2^{d_k} \| u \|_{L^\infty_t H^d_x} \tag{1.40}$$

Furthermore, using dynamic separation and bilinear Littlewood-Paley decomposition, for $\sigma \in [0, \vartheta]$, (1.36) reduces to (1.40) and

$$2d^k \| P_k( DS_{ij}^l ) \|_{L^\infty_t L^2_x \cap L^p_t} \lesssim 2^{d_k} (1 + s2^{2k})^{-M} \| u \|_{L^\infty_t H^d_x}. \tag{1.41}$$

To prove (1.40), except for using the heat flow equation, one also needs

$$2d^k \| P_k( DS_{ij}^l ) \|_{L^\infty_t L^2_x \cap L^p_t} \lesssim 2^{d_k} \| u \|_{L^\infty_t H^d_x}. \tag{1.42}$$

**Logic graph.** We summarize the above reduction process as the following graph for convenience:

**Step 1.**

| 1.31 | 1.36 | 1.37 | 1.38 |
|---|---|---|---|
| + Eq. (6.6) | + Eq. (6.6) | + Eq. (6.6) | + Eq. (6.7) |
| 1.21 | 1.18 | 1.19 | 1.20 |

**Step 2.**

| 1.31 | 1.34 | 1.35 | 1.36 | 1.37 | 1.38 |
|---|---|---|---|---|---|
| + Eq. (6.6) | + Eq. (6.6) | + Eq. (6.6) | + Eq. (6.6) | + Eq. (6.7) |
| 1.21 | 1.19 | 1.18 | 1.20 | 1.26 |

Therefore, it suffices to prove (1.31), (1.37)-(1.38), (1.39)-(1.42).

**Remarks on iteration.** The above is just a toy model for iteration scheme up to once time. The key point we want to address is that the key and the engine for iteration is to improve estimates of $\partial_s v$ step by step. The true scheme is a sophisticated combination of bootstrap and the above iteration.

**Remarks on frequency envelopes.** The notion of frequency envelopes introduced by Tao is very convenient in doing frequency estimates and becomes standard in the study of dispersive PDEs. Due to the iteration argument used here, we need emphasize the “order” of envelopes applied in our
previous work [23]: We say a positive $\ell^2$ summmable sequence $\{a_k\}_{k \in \mathbb{Z}}$ is a frequency envelope of $\delta$ order if
\[ a_k \leq a_j 2^{\delta|k-j|}, \forall k,j \in \mathbb{Z}. \tag{1.43} \]
Throughout the paper, all the frequency envelopes are assumed to be of $\delta$ order except $\{c_k^{(j)}, c_k, c_k(\sigma)\}_{j \in \mathbb{N}, k \in \mathbb{Z}}$. If we want to reach $\sigma = \lfloor K\theta \rfloor + 1$ in Theorem 1.1, then $\{c_k^{(j)}, c_k, c_k(\sigma)\}_{j \in [0,K+1], k \in \mathbb{Z}}$ are defined to be $\frac{1}{2^{2\pi + 1}}$ order.

2 Decay estimates of heat flows

We have seen in Section 1.2 that the whole proof reduces to prove decay estimates of heat flows such as (1.31), (1.37)-(1.38), (1.41)-(1.42). The $L^p$ part especially (1.38) requires more efforts than $L^\infty_t L^2_x$. Thus we leave them into later sections.

In this section, we prove decay estimates for quantities related to caloric gauge for small data heat flows from $\mathbb{R}^d$ with $d \geq 3$ in critical Sobolev spaces. This part is of independent interest and can be applied to other problems.

2.1 Global evolution of Heat flows

Let us fix two small constants $0 < \epsilon_1 \ll \epsilon \ll 1$.

In this subsection we prove that $v$ is global and satisfies (1.7) stated in Theorem 1.1. Suppose that the target manifold $M$ is isometrically embedded into $\mathbb{R}^m$. Let $\{S^q_k\}$ denote the second fundamental form of the embedding $M \hookrightarrow \mathbb{R}^m$. Then the heat flow equation is written as
\[ \partial_s v^q - \Delta_{\mathbb{R}^d} v^q = \sum_{i=1}^d S^q_{ij} \partial_i v^q \partial_j v^q, \tag{2.1} \]
where $\{q,l,j\}$ run over $\{1,\ldots,m\}$.

Lemma 2.1. Let $d \geq 3$ and $v_0 \in Q(\mathbb{R}^d, M)$. Denote $\{\gamma_k(\sigma)\}$ the frequency envelope of $v_0$:
\[ \gamma_j(\sigma) := \sum_{j_1 \in \mathbb{Z}} 2^{-\delta|j-j_1|} 2^{(\frac{d}{2}+\sigma)j_1} \| P_{j_1} v_0 \|_{L^2_x}. \tag{2.2} \]
There exists $0 < \epsilon_1 \ll 1$ such that if
\[ \|v_0\|_{H^{\frac{d}{2}}} \leq \epsilon_1, \tag{2.3} \]
then the heat flow initiated from $v_0$ is global and satisfies
\[ \sup_{s \in (0,\infty)} 2^{(\frac{d}{2}+\sigma)k} (1 + s2^k)^N \| P_k v \|_{L^2_x} \lesssim_N \gamma_k(\sigma), \tag{2.4} \]
provided that $N \in \mathbb{N}$, $\sigma \in [0,\theta]$, $s \geq 0$, $k \in \mathbb{Z}$. Particularly, (1.7) holds.
Proof. Let \( \bar{s} > 0 \) be the maximal time such that for all \( s \in [0, \bar{s}), \) \( j \in \mathbb{N}, \) \( j' \in \mathbb{N}_+ \) there holds

\[
\begin{align*}
  s^{\frac{d}{2}} \| \partial_{x}^{j} v \|_{\dot{H}^{\frac{d}{2}}} & \leq C_{j} \varepsilon_{\frac{1}{2}}^{\frac{1}{2}}, \\
  s^{\frac{d}{2}} \| \partial_{x}^{j'} v \|_{L_{t}^{\infty}} & \leq C_{j'} \varepsilon_{\frac{1}{2}}^{\frac{1}{2}}.
\end{align*}
\]  

(2.5) \hspace{1cm} (2.6)

**Step 1.** We first verify that

\[
\| \partial_{x}^{L} \left( S_{j}^{\alpha}(v) - S_{j}(Q) \right) \|_{\dot{H}_{x}^{\frac{d}{2}}} \leq C_{L} s^{-\frac{d}{2}} \varepsilon_{\frac{1}{2}}^{\frac{1}{2}}.
\]  

(2.7)

If \( d \) is even, (2.7) follows by chain rules, Sobolev embedding and (2.5), (2.6). The case when \( d \) is odd requires slightly more efforts. Let \( d = 2d_{0} + 1 \) with \( d_{0} \in \mathbb{N}. \) Then by chain rule we have

\[
\| \partial_{x}^{L} \left( S_{j}^{\alpha}(v) - S_{j}(Q) \right) \|_{\dot{H}_{x}^{\frac{d}{2}}} \leq \sum_{0 \leq l, l' \leq L+d_{0}, |\alpha_{1}|+\ldots+|\alpha_{l}|=L+d_{0}} \| S^{(l')}(v) \partial_{x}^{\alpha_{1}} v \ldots \partial_{x}^{\alpha_{l}} v \|_{\dot{H}_{x}^{\frac{d}{2}}}.
\]  

(2.8)

where we denote the derivatives of \( \{ S_{j}^{\alpha} \} \) by \( S^{(l')}(v) \) for simplicity. Then fractional Leibnitz formula shows

\[
\begin{align*}
  \| S^{(l')}(v) \partial_{x}^{\alpha_{1}} v \ldots \partial_{x}^{\alpha_{l}} v \|_{\dot{H}_{x}^{\frac{d}{2}}} & \lesssim \| \partial_{x}^{\alpha_{1}} v \ldots \partial_{x}^{\alpha_{l}} v \|_{L_{t}^{2}} \| S^{(l')} (v) \|_{\dot{H}_{x}^{\frac{d}{2}}} + \| \partial_{x}^{\alpha_{1}} v \ldots \partial_{x}^{\alpha_{l}} v \|_{\dot{H}_{x}^{\frac{d}{2}}} \| S^{(l')} (v) \|_{L_{t}^{2}},
\end{align*}
\]  

(2.9)

where \( r_{2} \in (2, \infty) \) is taken as

\[
\frac{1}{d} \left( \frac{d}{2} - \frac{1}{2} \right) = \frac{1}{2} - \frac{1}{r_{2}},
\]  

(2.10)

and \( r_{1} \) satisfies \( \frac{1}{r_{1}} + \frac{1}{r_{2}} = \frac{1}{2}. \) For \( 0 < |\beta| \leq \frac{d}{2}, \) by Sobolev embedding

\[
\| | \nabla_{x} |^{\beta} v \|_{L_{t}^{r_{1}}} \lesssim \| v \|_{\dot{H}_{x}^{\frac{d}{2}}},
\]  

(2.11)

with \( \frac{1}{d} \left( \frac{d}{2} - |\beta| \right) = \frac{1}{2} - \frac{1}{r_{1}}. \) Similarly, for \( d_{0}+n \leq |\alpha| < d_{0}+n+1 \) with \( n \in \mathbb{N}, \) by Sobolev embedding

\[
\| \partial_{x}^{\alpha} v \|_{L_{x}^{(\alpha(n)+1)}} \lesssim \| \partial_{x}^{\alpha} v \|_{\dot{H}_{x}^{\frac{d}{2}}},
\]  

(2.12)

with \( \frac{1}{d} \left( \frac{d}{2} + n - |\alpha| \right) = \frac{1}{2} - \frac{1}{\alpha(n)}. \) Assume that among all \( \{ \alpha_{k} \}_{k=1}^{l} \) there exists some \( |\alpha_{k}| \geq \frac{d}{2}, \) then by Hölder, (2.7), (2.11) and (2.12) we have

\[
\| \partial_{x}^{\alpha_{1}} v \ldots \partial_{x}^{\alpha_{l}} v \|_{\dot{H}_{x}^{(\alpha(n)+1)}} \lesssim (\Pi_{k \in \{1, \ldots, l\} \setminus \{k'\}} \| \partial_{x}^{\alpha_{k}} v \|_{L_{x}^{2}}) \| \partial_{x}^{\alpha_{k'}} v \|_{\dot{H}_{x}^{\alpha(n)}}.
\]
We obtain for 
\begin{align*}
\sum
\end{align*}
where we make the convention that \( n_k = 0 \) if \( |\alpha_k| \leq d_0 \) and \( n_k = n \) if \( n + d_0 \leq |\alpha_k| < d_0 + n + 1 \). By definition, we see \( |\alpha_{k'}| - n_{k'} = d_0 \) and \( \tilde{\alpha}_{(k')} = r_1 \). Hence, (2.13) gives
\begin{align*}
\| \partial_{x}^{\alpha_1} v \cdots \partial_{x}^{\alpha_l} v \|_{L^p_x} \lesssim \epsilon_1^r s^{-L}. \tag{2.14}
\end{align*}
Assume that all \( \{\alpha_k\}_{k=1}^l \) satisfy \( |\alpha_k| \leq \frac{d}{2} \). Then by Hölder, (2.7), (2.11) and interpolation we have
\begin{align*}
\| \partial_{x}^{\alpha_l} v \|_{L^p_x} \lesssim \epsilon_1^r s^{-\frac{1}{2}\alpha_k(1-\frac{2}{p})}, \tag{2.15}
\end{align*}
for all \( p \in [\alpha_k^*, \infty] \). Since by definition \( \alpha_k^* = \frac{d}{\alpha_k} \), we get a tight form of (2.15) as
\begin{align*}
\| \partial_{x}^{\alpha_k} v \|_{L^p_x} \lesssim \epsilon_1^r s^{-\frac{1}{2}\alpha_k} s^{\frac{d}{2p}}. \tag{2.16}
\end{align*}
Thus for \( \sum_{k=1}^l \frac{1}{p_k} = \frac{1}{r_1} \) (recall \( r_1 = \frac{d}{dx} \)), we conclude
\begin{align*}
\| \partial_{x}^{\alpha_1} v \cdots \partial_{x}^{\alpha_l} v \|_{L^p_x} \lesssim \Pi_{1 \leq k \leq l} \| \partial_{x}^{\alpha_k} v \|_{L^p_x}
\lesssim \epsilon_1^r s^{-\frac{1}{2} \sum_{k=1}^l \alpha_k} s^{\frac{d}{2p}}
\lesssim \epsilon_1^r s^{-\frac{1}{2}L}. \tag{2.17}
\end{align*}
Meanwhile, since \( \mathcal{S}^{(l')} \) is Lipchitz, we have
\begin{align*}
\| \mathcal{S}^{(l')} (v) \|_{B^{\frac{1}{2}+}_{x} B^{-\frac{1}{2}+}_{x}} \lesssim \| v \|_{B^{\frac{1}{2}}_{x} B^{-\frac{1}{2}}_{x}} \lesssim \| v \|_{H^L_x}, \tag{2.18}
\end{align*}
where we used Sobolev embedding and (2.10) in the last inequality. Therefore, (2.14) and (2.17) imply the first term in the RHS of (2.9) is dominated by \( \epsilon_1^r s^{-L} \) up to constants \( C_L \).

Now we turn to the second term in the RHS of (2.9). As before, we consider two subcases. Assume that all \( \{\alpha_k\}_{k=1}^l \) satisfy \( |\alpha_k| \leq \frac{d}{2} \). And especially we have \( |\alpha_k| + \frac{1}{2} \leq \frac{d}{2} \) since \( d \) is odd. Then, by (2.17) and fractional Leibnitz formula, we obtain for \( \sum_{k=1}^l \frac{1}{p_k} = \frac{1}{2} \) and \( p_k \in [\alpha_k^*, \infty] \) (Recall \( \alpha_k^* = \frac{d}{\alpha_k} \)) that
\begin{align*}
\| \partial_{x}^{\alpha_1} v \cdots \partial_{x}^{\alpha_l} v \|_{H^L_x} \lesssim \epsilon_1^r \sum_{1 \leq i \leq l} \left( s^{-\frac{1}{2}(\alpha_i + \frac{1}{2})} \right)^{\frac{d}{p_k}} \Pi_{1 \leq k \leq l, k \neq i} s^{-\frac{1}{2}\alpha_k} s^{\frac{d}{2p_k}}
\lesssim \epsilon_1^r s^{-\frac{1}{2}L}. \tag{2.19}
\end{align*}
Assume that among \( \{\alpha_k\}_1 \) there exists some \( k' \) such that \( |\alpha_{k'}| \geq \frac{d}{2} \). Then, by \( (2.5) \), \( (2.6) \), \( (2.11) \), \( (2.12) \) and fractional Leibnitz formula, we obtain
\[
\|\partial_{x_1}^{\alpha_1} v \ldots \partial_{x_i}^{\alpha_i} v\|_{L^2_x} \lesssim \|\partial_{x_1}^{\alpha_1} v\|_{L^2_x} \prod_{k \in \{1, \ldots, k]\} \|\partial_{x_i}^{\alpha_i} v\|_{L^2_x}
+ \|\partial_{x_1}^{\alpha_1} v\|_{L^2_x} \|\nabla_i^{\alpha_1+\frac{1}{2}} v\|_{L^2_x} \prod_{k \in \{1, \ldots, k\} \setminus \{k'\}} \|\partial_{x_i}^{\alpha_i} v\|_{L^2_x}
\lesssim \epsilon_1 s^{-\frac{1}{2}(\alpha_{k'} - \frac{d}{2} + \frac{1}{2})} \prod_{k \in \{1, \ldots, k\} \setminus \{k'\}} s^{-\frac{1}{2} \alpha_k}
+ \epsilon_1 s^{-\frac{1}{2}(\alpha_{k'} - \frac{d}{2})} s^{-\frac{1}{2}(\alpha_1 + \frac{1}{2})} \prod_{1 \leq k \leq \ell, k \neq k'} s^{-\frac{1}{2} \alpha_k}
\lesssim \epsilon_1 s^{-\frac{1}{2} + \frac{1}{2}}.
\] (2.20)
Therefore, \( (2.20) \) and \( (2.19) \) show the second term in the RHS of \( (2.9) \) is dominated by \( \epsilon_1 s^{-L} \) up to constants \( C_L \). And thus \( (2.5) \) follows.

**Step 2.** Applying \( (2.7) \) and following the lines of our previous paper \cite[Proposition 7.1, Step 1]{paper}, one obtains by Lemma \( 7.2 \) that \( (2.3) \) holds with \( \epsilon_1 \), i.e.
\[
\sup_{s \in [0, s]} 2^{(\frac{d}{2} + \sigma)k} (1 + s2^{2k})^N \|P_k v\|_{L^2_x} \lesssim_{N} \gamma_k(\sigma),
\] (2.21)
by bilinear Littlewood-Paley decomposition. Since this part is routine, we leave the details for readers.

**Step 3.** \( (2.21) \) shows
\[
2^{\frac{d}{2}k} (s2^{2k})^N \|P_k v\|_{L^2_x} \lesssim_{N} \gamma_k.
\]
Then by Bernstein inequality we get
\[
2^{\frac{d}{2}k} \|P_k (|\nabla|^j v)\|_{L^2_x} \lesssim_j s^{-\frac{1}{2} \gamma_k}.
\]
and thus
\[
\|\partial_{x}^{j} v\|_{L^2_x} \lesssim_j s^{-\frac{1}{2} \epsilon_1}.
\]
By Gagliardo-Nirenberg inequality and Sobolev embedding we have for \( j \in \mathbb{Z}_+ \)
\[
\|\partial_{x}^{j} (v - Q)\|_{L^\infty_x} \lesssim_j \|\partial_{x}^{j} (v - Q)\|_{L^2_x}^{\frac{2}{d+1}} \|\partial_{x}^{\frac{d}{2} + j + 1} (v - Q)\|_{L^2_x}^{\frac{1}{d+1}}
\lesssim \epsilon_1 s^{-\frac{1}{2}}, \text{ d} \in 2\mathbb{N}
\|\partial_{x}^{j} (v - Q)\|_{L^\infty_x} \lesssim_j \|v - Q\|_{L^2_x}^{\frac{2}{d+1}} \|\partial_{x}^{\frac{d}{2} + j + \frac{1}{2}} (v - Q)\|_{L^2_x}^{\frac{1}{d+1}}
\lesssim \epsilon_1 s^{-\frac{1}{2}}, \text{ d} \in 2\mathbb{N} + 1.
\]
Then by \( (2.3) \), one obtains \( (2.6) \) hold with \( \epsilon_1^{\frac{d}{2}} \) replaced by \( \epsilon_1 \). Hence \( \bar{s} = \infty \) and \( (2.21) \) yields \( (2.4) \).
Lemma 2.2. (Space-time estimates) Let $v$ be the global heat flow in Lemma 2.1 with initial data $v_0 \in Q(\mathbb{R}^d, \mathcal{M})$. Then we have

$$\|\partial_x v\|_{L^2_t H^\frac{n}{2}_x} \lesssim \epsilon_1. \quad (2.22)$$

Proof. The proof is based on energy arguments and trilinear Littlewood-Paley decomposition, see Appendix C. \hfill \square

Corollary 2.1. Let $v$ be the global heat flow in Lemma 2.1 with initial data $v_0 \in Q(\mathbb{R}^d, \mathcal{M})$. Then for all $a \in \mathbb{N}$ and $0 \leq j \leq \lfloor \frac{d}{2} - 1 \rfloor$, there holds

$$\|\nabla^a x^j \partial_x v(s)\|_{L^\infty_t L^1_x} \lesssim \epsilon_1 s^{j - a + 1} \quad (2.23)$$

$$\|\nabla^a_x \partial_s v(s)\|_{L^\infty_t L^2_x} \lesssim \epsilon_1 s^{-a + \frac{d}{2}} \quad (2.24)$$

Moreover, if $k \geq \frac{d}{2} - 1$, $p \in [2, \infty]$, we have

$$\|\nabla^k_x \partial_x v(s)\|_{L^p_t L^\frac{d}{p} x} \lesssim \epsilon_1 s^{-k + 1} + \frac{d}{p} \quad (2.25)$$

Proof. Basic theories of embedded sub-manifolds show the following inequality

$$|\nabla^a_x \partial_i v| \lesssim \sum_{j=1}^{a+1} \sum_{|\beta|=a+1, \beta \in \mathbb{Z}_+} |\partial_{x}^{\beta_i} v| ... |\partial_{x}^{3} v|. \quad (2.27)$$

Then (2.23) follows by (1.7), (2.24) follows by (2.23) and the identity $\partial_x v = \sum_{i=1}^d \nabla_i \partial_j$. And (2.25) follows by Sobolev embedding inequalities and Hölder inequalities. Lastly, we prove (2.26) by interpolating (2.23) with

$$\|\nabla^a_x \partial_i v\|_{L^2_t} \lesssim \epsilon_1 s^{-\frac{k+1}{2} + \frac{d}{4}}. \quad (2.28)$$

In order to prove (2.28), we consider two subcases: Case 1. All $\{\beta_i\}_{i=1}^d$ satisfy $\beta_i < \frac{d}{2} - 1$; Case 2. There exists some $1 \leq l_* \leq j$ such that $\beta_{l_*} \geq \frac{d}{2} - 1$. Then (2.28) follows as Step 1 of Lemma 2.1. \hfill \square

2.2 Non-critical theory for heat flows

This subsection involves some estimates which depend on both $\|\nabla^{[d/2]} v_0\|_{L^2_x}$ and $\|dv_0\|_{L^2_x}$. Thus all theses estimates are not in the critical level. But they are necessary for setting up our bootstrap in the next subsection. Most of the techniques in this subsection are classical and we present them in detail just for reader’s convenience.
Lemma 2.3. Let \( v \) be the global heat flow in Lemma 2.1 with initial data \( v_0 \in Q(\mathbb{R}^d, \mathcal{M}) \). Then the heat flow \( v \) will uniformly converge to \( Q \) as \( s \to \infty \).

Proof. The Bochner-Weitzenböck identity for \( |\partial_s v|^2 \) is

\[
(\partial_s - \Delta)|\partial_s v|^2 + 2|\nabla \partial_s v|^2 = \sum_{i=1}^d \langle R(\partial_s v, \partial_i v)\partial_s v, \partial_i v \rangle.
\] (2.29)

And we claim that

\[
\|\partial_s v\|_{L^2} \lesssim s^{-\frac{d}{2}}\|v_0\|_{L^2}.
\] (2.30)

Then, by smoothing effect of heat equations and (1.7), we get

\[
\|\partial_s v(s)\|_{L^\infty} \lesssim s^{-\frac{d}{2}}\|\partial_s v(\frac{s}{2})\|_{L^2}^2 + \int_{\frac{s}{2}}^s \|\partial_s v\|_{L^\infty}^2 \|\partial_s v\|_{L^2}^2 \tau^{-\frac{d}{2}} d\tau
\]

\[
\lesssim s^{-\frac{d}{2} - 1}\|\nabla v_0\|_{L^2}^2.
\] (2.31)

Hence, we conclude

\[
\|v(s_1, \cdot) - v(s_2, \cdot)\|_{L^\infty} \leq \int_{s_1}^{s_2} \|\partial_s v(s, \cdot)\|_{L^\infty} ds \lesssim s_1^{-\frac{d+2}{4}} + 1.
\] (2.32)

which implies \( v \) converges uniformly since \( d \geq 3 \). Denote the limit map of \( v \) by \( \Theta : \mathbb{R}^d \to \mathcal{M} \). Then by \( \|dv\|_{L^\infty} \lesssim s^{-\frac{1}{4}} \), we have \( \Theta \) is a constant map. (2.33) now reads as

\[
\sup_{x \in \mathbb{R}^d} |v(s, x) - \Theta| \lesssim s^{-\frac{2d+4}{4} + 1} \|dv_0\|_{L^2}.
\] (2.33)

Since \( v \in Q(\mathbb{R}^d, \mathcal{M}) \) implies \( \lim_{|x| \to \infty} v = Q \), (2.33) shows \( \Theta = Q \) by contradiction argument.

Therefore, it suffices to verify the claim (2.30). By Duhamel principle and smoothing effect of linear heat equation,

\[
\|\Delta v(s)\|_{L^2} \lesssim \|e^{s\Delta}v(\frac{s}{2})\|_{L^2}^2 + \int_{\frac{s}{2}}^s \|e^{(s-\tau)\Delta}(\nabla(S(v)|\nabla v|)^2(\tau))\|_{L^2}^2 d\tau
\]

\[
\lesssim s^{-\frac{1}{2}}\|\nabla v_0\|_{L^2}^2 + \int_{\frac{s}{2}}^s (s - \tau)^{-\frac{1}{2}} \left( \|\nabla v\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2} \|\nabla v\|_{L^\infty} \right)
\]

\[
\lesssim s^{-\frac{1}{2}}\|\nabla v_0\|_{L^2}^2 + \int_{\frac{s}{2}}^s (s - \tau)^{-\frac{1}{4}} \|\nabla v_0\|_{L^2}^2 (\epsilon_1^2 \tau^{-1} + \epsilon_1 \tau^{-\frac{1}{2}} \|\Delta v\|_{L^2}) d\tau
\]

\[
\lesssim s^{-\frac{1}{2}}\|\nabla v_0\|_{L^2}^2 + \epsilon_1 \int_{\frac{s}{2}}^s (s - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|\Delta v\|_{L^2} d\tau,
\]
where in the third line we applied
\[ \|\nabla v\|_{L^2_x} \lesssim \|v_0\|_{L^2_x} \]
\[ \|\nabla v\|_{L^\infty_x} \lesssim \epsilon_1 s^{-\frac{1}{2}}. \]

Let \( X(s) = \sup_{\tilde{s} \in [0,s]} \tilde{s}^\frac{1}{2} \|\Delta v(\tilde{s})\|_{L^2_x} \), thus
\[ X(s) \lesssim \|dv_0\|_{L^2_s} + \epsilon_1 X(x), \]
which shows
\[ \|\Delta v\|_{L^2_s} \lesssim s^{-\frac{1}{2}} \|dv_0\|_{L^2_s}. \]
And thus
\[ \|\partial_s v\|_{L^2_s} \lesssim \|\Delta v\|_{L^2_s} + \|\nabla v\|_{L^\infty_x} \|\nabla v\|_{L^2_x} \lesssim s^{-\frac{1}{2}} \|dv_0\|_{L^2_s}, \]
from which (2.30) follows. So the proof has been completed.

The proof of Lemma 2.4 shows \( \|\Delta v\|_{L^\infty_x} \) indeed decays faster than that stated in Lemma 2.1 if one takes \( \|\nabla v_0\|_{L^2_s} \) into consideration. These faster rates will be useful in the set up of bootstrap. And in fact decay estimates of higher order derivatives of \( v \) can be obtained similarly by induction.

**Lemma 2.4.** Let \( v \) be the global heat flow in Lemma 2.1 with initial data \( v_0 \in Q(\mathbb{R}^d, \mathcal{M}) \). Then for all \( L \in \mathbb{N} \) one has

\[ \|\nabla^L_x \partial_s v\|_{L^2_s} \lesssim 2^{-\frac{L}{2}} \|dv_0\|_{L^2_s} \quad (2.34) \]
\[ \|\nabla^L_x \partial_s v\|_{L^\infty_x} \lesssim 2^{-\frac{2L+1}{2}} \|dv_0\|_{L^2_s} \quad (2.35) \]
\[ \|2^{\frac{1}{2}L(L-1)} \nabla^L_x \partial_s v\|_{L^2_s L^2_s} \lesssim \|dv_0\|_{L^2_s} \quad (2.36) \]
\[ \|\nabla^L_x \partial_s v\|_{L^2_s} \lesssim 2^{-\frac{L+1}{2}} \|dv_0\|_{L^2_s} \quad (2.37) \]
\[ \|\nabla^L_x \partial_s v\|_{L^\infty_x} \lesssim 2^{-\frac{L+1}{2}-\frac{d}{4}} \|dv_0\|_{L^2_s} \quad (2.38) \]

Moreover, let \( \{e_i\}_{i=1}^m \) be an orthonormal frame for the pullback bundle \( v^* T\mathcal{M} \) and \( \{\psi_i\}_{i=1}^d, \psi_s \) be the sections of trivial bundle \( [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^m \) induced by \( v \) via:

\[ \psi_i^L := \langle \partial_i v, e_i \rangle, \quad \psi_s^L := \langle \partial_s v, e_i \rangle. \]

Denote \( \{D_i, D_s\}_{i=1}^d \) the induced covariant derivatives on the bundle \( ([0, \infty) \times \mathbb{R}^d, \mathbb{R}^m) \). Then we also have

\[ \|D^L_x \psi_x\|_{L^2_s} \lesssim 2^{-\frac{L}{2}} \|dv_0\|_{L^2_s} \quad (2.39) \]
\[ \|D^L_x \psi_x\|_{L^\infty_x} \lesssim 2^{-\frac{2L+1}{2}} \|dv_0\|_{L^2_s}, \quad (2.40) \]
\[ \|2 \frac{1}{2}(L-1)^s D^s_x \psi_x \|_{L^2_x L^2_z} \lesssim \|d v_0\|_{L^2_x} \] (2.41)
\[ \|D^s_x \psi_x \|_{L^2_x} \lesssim \|d v_0\|_{L^2_x} \] (2.42)
\[ \|D^s_x \psi_x \|_{L^\infty_x} \lesssim \|d v_0\|_{L^2_z} \] (2.43)
\[ \|D^s_x \psi_x \|_{L^\infty_x} \lesssim \epsilon_s \frac{L+2}{L} \] (2.44)
\[ \|D^s_x \psi_x \|_{L^\infty_x} \lesssim \epsilon_s \frac{L+2}{L} \] (2.45)

where the simplified notations \( \psi_x, D^s_x \) refer to differential fields \( \{\psi_i\}_{i=1}^d \) and various combinations of \( \{D_i\}_{i=1}^d \) up to order \( L \).

**Proof.** The proof is based on well-known techniques. We sketch it for reader’s convenience. (2.37) - (2.38) follow from (2.34) - (2.35) by the identity

\[ \frac{\partial}{\partial x} v = \sum_{i=1}^{d} \nabla_x \partial_i v. \] (2.39)
and (2.40) follow from (2.34) - (2.35) since \( \nabla_x \partial_i v \) controls \( |D^s_x \psi_x| \) point-wisely. And by the same reason, (2.44), (2.45) follow by (2.23), (2.24) while (2.42), (2.43) follow from (2.39), (2.40).

Therefore, it suffices to prove (2.34) - (2.36). We denote

\[ X_{j,\infty}(s) := \sup_{\tilde{s} \in [0,s]} \tilde{s}^{d+2} \|\nabla^{\frac{j}{2}} \partial_x v(\tilde{s})\|_{L^\infty_x} \] (2.46)
\[ X_{j,2}(s) := \sup_{\tilde{s} \in [0,s]} \tilde{s}^{\frac{j}{2}} \|\nabla^{\frac{j}{2}} \partial_x v(\tilde{s})\|_{L^2_x} \] (2.47)
\[ Y_{j,2}(s) := \|s^{\frac{j}{2}} \nabla^{\frac{j}{2}} \partial_x v(s)\|_{L^2_x L^2_z} \] (2.48)

Recall the Bochner inequality (see e.g. [34, 37]):

\[(\partial - \Delta) |\nabla^{\frac{j}{2}} \partial_x v|^2 + 2|\nabla^{j+1} \partial_x v|^2 \leq \sum_{z=3}^{j+3} \sum_{(1+n_1)+...+(1+n_z)=j+3} |\nabla^{n_1} \partial_x v|...|\nabla^{n_z} \partial_x v| |\nabla^{\frac{j}{2}} \partial_x v|. \] (2.49)

We notice that the RHS of (2.49) can be further expanded as

\[(\partial - \Delta) |\nabla^{\frac{j}{2}} \partial_x v|^2 + 2|\nabla^{j+1} \partial_x v|^2 \]
\[ \lesssim |d v|^2 |\nabla^{\frac{j}{2}} \partial_x v|^2 + \sum_{z=3}^{j+3} \sum_{\sum_{i=1}^{z}(1+n_i)=j+3, \forall i, n_i < j} |\nabla^{n_1} \partial_x v|...|\nabla^{n_z} \partial_x v| |\nabla^{\frac{j}{2}} \partial_x v|. \] (2.50)

Then it is easy to see

\[ X_{j,2}^2(s) + 2Y_{j+1,2}^2(s) \]
\[ \lesssim jY_{j,2}(s) + c_j Y_{j,2}(s) \]
\[ + \sum_{z=3}^{j+3} \sum_{\sum_{i=1}^{z}(1+n_i)=j+3, \forall i, n_i < j} \int_0^s s^j \|\nabla^{\frac{j}{2}} \partial_x v\|_{L^2_x} \|\nabla^{n_1} \partial_x v\|_{L^2_x} \|...\|\nabla^{n_z} \partial_x v\|_{L^\infty_x} ds \]
where we used (1.7) in the first line and (2.23) in the last line. Thus we have seen

\[ Y_{l,2}(s, s) \lesssim \|dv_0\|_{L^2_x}, \forall 1 \leq l \leq j \implies X_{j,2}(s) \lesssim \|dv_0\|_{L^2_x} \]  \hspace{1cm} (2.51)

\[ X_{j,2}(s) + Y_{l,2}(s) \lesssim \|dv_0\|_{L^2_x}, \forall 1 \leq l \leq j \implies Y_{j+1,2}(s) \lesssim \|dv_0\|_{L^2_x} \]  \hspace{1cm} (2.52)

These two induction relations show for (2.34), (2.36) it suffices to verify

\[ Y_{1,2}(s) + X_{0,2}(s) \lesssim \|dv_0\|_{L^2_x}. \]  \hspace{1cm} (2.53)

Integration by parts gives

\[ \|\nabla dv\|_{L^2_x}^2 \lesssim \|\tau(v)\|_{L^2_x}^2 + \|dv\|_{L^2_x}^4. \]

By Gagliardo-Nirenberg inequality

\[ \|dv\|_{L^4_x}^4 \lesssim \|dv\|_{L^2_x}^2 \|dv\|_{H^2_x}^2. \]

Then (2.53), (2.22) yield

\[ Y_{1,2}(s) + X_{0,2}(s) \lesssim \|dv_0\|_{L^2_x}. \]

Thus (2.34), (2.36) are done.

It remains to prove (2.35). (2.50) and Kato’s inequality show

\[(\partial_s - \Delta)|\nabla_x^j \partial_x v| \lesssim \|dv\|_{L^2_x}^2 |\nabla_x^j \partial_x v| + \sum_{j=3}^{j+3} \sum_{\sum_{i=1}^r (1+n_i) = j+3, |n_i| < j} |\nabla_x^{n_1} \partial_x v| \ldots |\nabla_x^{n_r} \partial_x v|.|\]

Suppose that (2.35) hold for \( L < j \). Then by Duhamel principle and smoothing effect of heat equation one has

\[ \|\nabla_x^j \partial_x v(s)\|_{L^\infty_x} \]

\[ \lesssim s^{-\frac{2}{4}} \|\nabla_x^j \partial_x v(s/2)\|_{L^2_x} + \int_{s/2}^s \|dv\|_{L^2_x}^2 \|\nabla_x^j \partial_x v\|_{L^\infty_x} d\tau \]

\[ + \sum_{j=3}^{j+3} \sum_{\sum_{i=1}^r (1+n_i) = j+3, |n_i| < j} \int_{s/2}^s \|\nabla_x^{n_1} \partial_x v\|_{L^\infty_x} \ldots \|\nabla_x^{n_r} \partial_x v\|_{L^\infty_x} d\tau \]

\[ \lesssim s^{-\frac{2}{4}} \|\nabla_x^j \partial_x v(s/2)\|_{L^2_x} + s^{-\frac{2k+2}{4}} \int_{s/2}^s \|dv\|_{L^\infty_x}^2 d\tau \] \( X_{j,\infty}(s) \)

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Then the convergence in Step (ii). The uniqueness is standard by the boundary condition (2.55). Let \( e_j \) for arbitrary \( s \). Thus by (1.7),

\[
X_{j,\infty}(s) \lesssim X_{j,2}(s) + e_1 X_{j,\infty}(s) + \|dv_0\|_{L^2}.
\]

Hence (2.34) shows \( X_{j,\infty}(s) \lesssim \|dv_0\|_{L^2} \) and thereby our lemma follows.

\[\Box\]

**Lemma 2.5.** Let \( v \) be the global heat flow in Lemma 2.4 with initial data \( v_0 \in Q(\mathbb{R}^d, M) \). Given limit orthonormal frames \( \{e_i^\infty\}_{i=1}^m \), there exists a unique gauge \( \{\tilde{e}_i\}_{i=1}^m \) for \( v^*TM \) such that

\[
\nabla s e_l = 0 \quad (2.54)
\]

\[
\lim_{s \to \infty} e_l = e_l^\infty, \forall l = 1, \ldots, m. \quad (2.55)
\]

Frames satisfying (2.54), (2.55) are called Tao’s caloric gauge. Moreover, the connection coefficients \( A_x \) satisfy

\[
A_i = \int_s^\infty \mathcal{R}(\psi_s, \psi_i)ds' \quad (2.56)
\]

and the estimates

\[
\|\partial_s^L A_x(s)\|_{L^2} \lesssim s^{-\frac{3}{2} - \frac{d}{4} + \frac{1}{2}} \|dv_0\|_{L^2}, \quad (2.57)
\]

for \( s \geq 1 \). And the frames \( \{e_l\}_{l=1}^m \) satisfy

\[
\|\partial_s^L (D^s e_l) - D^s (e_l^\infty)\|_{L^2} \lesssim s^{\frac{2(l-1)+d}{4}} \|dv_0\|_{L^2}. \quad (2.58)
\]

**Proof.** The existence of caloric gauge follows by the standard line: (i) Take arbitrary \( \{\tilde{e}_l\}_{l=1}^m \) as the initial data of (2.54); (ii) Suppose that the solution to (2.54) with initial data \( \{\tilde{e}_l\}_{l=1}^m \) is \( \{\tilde{e}_l(s, x)\}_{l=1}^m \). Prove that \( D^s \tilde{e}_l(s, x) \) converges uniformly to some \( D^s \tilde{e}_l^\infty(x) \) as \( s \to \infty \); (iii) Apply an independent gauge transformation \( \Lambda(x) \in SO(m) \) to \( \{\tilde{e}_l^\infty(x)\} \) such that \( \Lambda(x)\tilde{e}_l^\infty(x) = e_l^\infty \).

Then \( \{\Lambda(x)\tilde{e}_l(s, x)\}_{l=1}^m \) is the desired caloric gauge satisfying (2.54), (2.55).

Therefore, to prove the existence of caloric gauge, it suffices to prove the convergence in Step (ii). The uniqueness is standard by the boundary condition (2.55).

For the simplicity of notations, we denote \( \{e_l\} \) instead of \( \{\tilde{e}_l\} \). By caloric condition \( \nabla s e_l = 0 \), one has

\[
\partial_s D^s (e_l) = (D^s D^s)(\partial_s v; e_l).
\]
where \( \mathbf{D} \) denotes the induced connection on the bundle \( \mathcal{P}^*T\mathbb{R}^m \). Thus
\[
\|d\mathcal{P}(e_1)(s_2) - d\mathcal{P}(e_1)(s_1)\|_{L_\infty^2} \lesssim \int_{s_1}^{s_2} \|\partial_x v\|_{L_\infty^2} ds \lesssim \|d\nu_0\|_{L_2^2} \int_{s_1}^{s_2} s^{-\frac{d+1}{4}} ds \\
\lesssim s^{-\frac{d}{4} + \frac{1}{4}},
\]
which shows \( d\mathcal{P}(e_1) \) converges uniformly in \( \mathbb{R}^d \) as \( s \to \infty \). Denote the limit of \( d\mathcal{P}(e_1) \) by \( \chi_1^\infty \). Hence, the convergence in Step (ii) has been verified. And
\[
\chi_1^\infty = \lim_{s \to \infty} d\mathcal{P}(e_1)(s, x) = \lim_{s \to \infty} d\mathcal{P}(e_1^\infty(Q)) \tag{2.59}
\]
is constant in \( x \). In the rest we prove \( (2.56) - (2.58) \).

Similarly one has for \( k \geq 2 \)
\[
\|\partial_x^k (d\mathcal{P}(e_1)(s_2) - d\mathcal{P}(e_1)(s_1))\|_{L_2^2} \\
\lesssim \sum_{z=0}^{k+1} \sum_{j_0+j_1+\ldots+j_z=k} \int_{s_1}^{s_2} \|D_{x}^{j_0} \psi_1 D_{x}^{j_1} \psi_2 \ldots D_{x}^{j_z} \psi_x\|_{L_2^2} ds \\
\lesssim \|d\nu_0\|_{L_2^2} \int_{s_1}^{s_2} s^{-\frac{k+1}{2}} ds \tag{2.60}
\]
\[
\lesssim s_1^{-\frac{k+1}{2}} \|d\nu_0\|_{L_2^2}. \tag{2.61}
\]
(2.61) proves (2.58) for \( d \geq 4 \) by letting \( s_2 \to \infty \) (One may apply Gagliardo-Nirenberg inequality when \( d \) is odd). The \( d = 3 \) case of (2.58) should be considered separately since (2.60) is not integrable for \( k = 1 \). For \( d = 3 \), by Gagliardo-Nirenberg inequality we have
\[
\|\partial_x^1 (d\mathcal{P}(e_1)(s_2) - d\mathcal{P}(e_1)(s_1))\|_{L_2^4} \\
\lesssim \|\partial_x^{j+2} (d\mathcal{P}(e_1)(s_2) - d\mathcal{P}(e_1)(s_1))\|_{L_2^4} \\
\lesssim \sum_{z=0}^{j+2} \sum_{j_0+j_1+\ldots+j_z=j+1} \int_{s_1}^{s_2} \|D_{x}^{j_0} \psi_1 D_{x}^{j_1} \psi_2 \ldots D_{x}^{j_z} \psi_x\|_{L_2^4} ds \\
\lesssim \sum_{z=0}^{j+2} \sum_{j_0+j_1+\ldots+j_z=j+1} \sum_{j_0'+j_1'+\ldots+j_z'=j+2} \int_{s_1}^{s_2} \|D_{x}^{j_0} \psi_1 D_{x}^{j_1} \psi_2 \ldots D_{x}^{j_z} \psi_x\|_{L_2^4} ds \\
\lesssim \|d\nu_0\|_{L_2^4} \int_{s_1}^{s_2} s^{-\frac{j+2}{2}} ds \\
\lesssim s_1^{-\frac{j+2}{4}} \|d\nu_0\|_{L_2^4}.
\]
Therefore, (2.58) has been proved for all \( d \geq 3 \).

Since \( \partial_x A_1 = \mathcal{R} (\psi, \psi_1) \), one has for \( s_2 > s_1 \geq 1 \)
\[
\|\partial_x^1 (A_x(s_2) - A_x(s_1))\|_{L_2^2}
\]
\[
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\]
\begin{align*}
\lesssim & \int_{s_1}^{s_2} \|\partial_j^2 R(\psi_s, \psi_x)\|_{L^2_x} ds \\
\lesssim & \sum_{j=1}^{j+1} \sum_{j_1+j_2+\ldots+j_s=j} \int_{s_1}^{s_2} \|D_j^0 \psi_s D_j^1 \psi_x \ldots D_j^z \psi_x\|_{L^2_x} ds \\
\lesssim & \|d\psi_0\|_{L^2_x} \sum_{j=1}^{j+1} \sum_{j_1+j_2+\ldots+j_s=j} \int_{s_1}^{s_2} s^{\frac{m+1}{4}} s^{-\frac{m-4}{4}} s^{-\frac{m-4}{4}} \ldots s^{-\frac{m-4}{4}} ds \\
\lesssim & s_1^{\frac{j}{4} - \frac{m-4}{4} + \frac{1}{2}} \|d\psi_0\|_{L^2_x}
\end{align*}

where in the forth line we applied (2.33) to (2.43) and the bounds (2.45), (2.44). Thus $A_x(s)$ converges in $H^k$ for all $k \in \mathbb{N}$ as $s \to \infty$. Denote $A_x^\infty$ the limit of $\lim_{s \to \infty} A_x(s, x)$. Then we summarize that

\begin{equation}
\|\partial_j^2 (A_x(s) - A_x^\infty)\|_{L^2_x} \lesssim s^{\frac{j}{4} - \frac{m-4}{4} + \frac{1}{2}} \|d\psi_0\|_{L^2_x} \tag{2.62}
\end{equation}

for $s \geq 1$.

To prove (2.56), it suffices to verify

\begin{equation}
A_x^\infty = 0. \tag{2.63}
\end{equation}

And (2.62) gives (2.57) if we have shown $A_x^\infty = 0$. Hence it only remains to check (2.63). By the identity

\begin{equation}
\partial_i dP(e_l) = dP(\nabla_i e_l) + (DdP)(\partial_i v; e_l), \tag{2.64}
\end{equation}

and the isometry of $dP$, we see

\begin{equation}
|\nabla_i e_l| \lesssim |\partial_i dP(e_l)| + |\partial_i v| \tag{2.65}
\end{equation}

By (2.59), $|\partial_i dP(e_l)| \to 0$ as $s \to \infty$. Meanwhile, $|\partial_i v| \to 0$ as $s \to \infty$ by Lemma 2.1. Thus (2.64) shows

\begin{equation*}
\lim_{s \to \infty} |\nabla_i e_l| = 0.
\end{equation*}

Thus $A_x^\infty = 0$ and the whole proof is completed. \qed

\section{Proof of Theorem 1.1}

We begin with a simple $L^\infty$ bound Lemma for connections and frames.

\begin{lemma}
For caloric gauge in Lemma 2.5, the connection coefficients $A_x$ and frames $\{e_l\}$ satisfy

\begin{align*}
\|\partial_x^L A_x\|_{L^\infty_x} & \lesssim_L \epsilon s^{-\frac{L+1}{4}} \tag{3.1} \\
\|\partial_x^L (dP(e) - \chi^\infty)\|_{L^\infty_x} & \lesssim \epsilon s^{-\frac{L}{4}} \tag{3.2}
\end{align*}

\end{lemma}

\begin{proof}
By (2.45)-(2.44), (3.1) follows by direct calculations as (2.57). Then (3.2) follows from (3.1) and (2.45). \qed
\end{proof}

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3.1 Setting of Bootstrap

Let $s_* \geq 0$ be the smallest time such that for all $s_* \leq s < \infty$ there holds

$$\| \partial_t^L A_s \|_{\dot{H}^{\frac{d}{2}}_s} \lesssim L \varepsilon s^{-\frac{d}{2}} \quad (3.3)$$

$$\| \partial_t^L (dPe \chi^\infty) \|_{\dot{H}^{\frac{d}{2}}_s} \lesssim L s^{-\frac{d}{2}}. \quad (3.4)$$

By (2.57), (2.58), for $s$ sufficiently large depending on $\|dv_0\|_{L^2}$, (3.3)-(3.4) hold. Our aim is to prove $s_* = 0$.

First, we improve the bounds for frames.

**Lemma 3.2.** Let $v : [0, \infty) \times \mathbb{R}^d \to \mathcal{M}$ be heat flow with initial data $v_0 \in Q(\mathbb{R}^d, \mathcal{M})$. Assume also that (2.3) holds. Let $\{e_l\}_{l=1}^m$ be the corresponding caloric gauge with given limit $\{e^\infty_l\}_{l=1}^m$. Recall that the isometric embedding $\mathcal{M} \hookrightarrow \mathbb{R}^M$ is denoted by $\mathcal{P}$ and $\lim_{s \to \infty} (d\mathcal{P})(e_p) = \chi^\infty_p$. Then if (2.3)-(3.4) hold in $s \in [s_*, \infty)$, one has the improved bound

$$\| \partial_t^L (\{dPe_l\} - \chi^\infty) \|_{\dot{H}^{\frac{d}{2}}_s} \lesssim L \varepsilon s^{-\frac{d}{2}}, \quad (3.5)$$

for all $s \in [s_*, \infty)$.

**Proof.** As before, the case when $d$ is odd requires more efforts. From now on assume that $d = 2d_0 + 1$ with $d_0 \in \mathbb{N}_+$. Denote the connection on the bundle $\mathcal{P}^* T\mathbb{R}^m$ by $D$. Denote the induced covariant derivatives on the bundle $\nu^*(\mathcal{P}^* T\mathbb{R}^m)$ by $\{D_i\}_{i=1}^d$. Then direct calculations show

$$\partial_i (\{dPe_p\} - \chi^\infty_p) = (\{D_i (d\mathcal{P})\}(e_p) + d\mathcal{P}(\nabla_i e_p))$$

$$\partial_{x,i} (\{dPe_p\} - \chi^\infty_p) = (D^2 (d\mathcal{P})) (\partial_i v, \partial_j v; e_p) + (D(d\mathcal{P})) (\nabla_j \partial_i v; e_p)$$

$$+ (D(d\mathcal{P})) (\partial_i v; \nabla_j e_p) + (D(d\mathcal{P})) (\partial_j v; \nabla_i e_p) + d\mathcal{P}(\nabla_j \nabla_i e_p).$$

And schematically we write

$$\partial_{x}^\alpha (\{dPe_p\} - \chi^\infty) = \sum_{k=0}^{\left| \alpha \right|} \sum_{\sum_{l=1}^{k}(a_{l}+1) = \left| \alpha \right|} \left(D^k (d\mathcal{P})\right) (\nabla_x^{a_1} \partial_x v, ..., \nabla_x^{a_k} \partial_x v; \nabla_x^{\alpha} e_p) \quad (3.6)$$

In order to estimate the $\dot{H}^{\frac{d}{2}}_s$ norm, it is convenient to use the difference characterization of $\dot{H}^{\frac{d}{2}}_s$ and the geodesic parallel transport. Given $h \in \mathbb{R}^+$, for fixed $(s, x) \in [0, \infty) \times \mathbb{R}^d$, let $\gamma(\zeta)$ be the shortest geodesic connecting $v(s, x + h)$ and $v(s, x)$. There may exist more than one shortest geodesic, it suffices to pick up one of them. Suppose that $\zeta$ is normalized to be the
arclength parameter. For any given vector field $V$ on $v^*\mathcal{M}$, denote the parallel transport of $V$ along $\gamma(\zeta)$ by $\tilde{V}(\gamma(\zeta))$, i.e.

\[
\begin{align*}
\nabla_{\gamma(\zeta)} \tilde{V}(\gamma(\zeta)) &= 0, \\
\tilde{V} |_{\zeta=0} &= V(\gamma(0)),
\end{align*}
\]

for $\zeta \in [0, \text{dist}(v(s, x), v(s, x + h))]$. Since $\mathcal{P}$ is an isometric embedding, we see $\text{dist}(v(s, x), v(s, x + h)) = |v(s, x) - v(s, x + h)|$. Introduce the difference operator

\[\Delta_h f = f(x + h) - f(x).\]

Denote

\[
I_1 = \left( D^k(d\mathcal{P})(v(s, x)) \right) \left( \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v; \nabla_x^{0} e_p \right)
\]
\[
I_2 = \left( D^k(d\mathcal{P})(v(s, x + h)) \right) \left( \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v; \nabla_x^{0} e_p \right)
\]

Then (3.7) gives (Recall that $I_1, I_2$ now take values in $\mathbb{R}^M$)

\[
I_1 - I_2 = \int_0^{\Delta_h v(s)} \partial_\zeta \left[ \left( D^k(d\mathcal{P})(\gamma(\zeta)) \right) \left( \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v; \nabla_x^{0} e_p \right) \right] d\zeta
\]
\[
= \int_0^{\Delta_h v(s)} \left( D^{k+1}(d\mathcal{P})(\gamma(\zeta)) \right) \left( \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v, \gamma; \nabla_x^{0} e_p \right) d\zeta.
\]

Hence, we have point-wisely that

\[
|I_1 - I_2| = |\Delta_h v(s)| \sup_{y \in \gamma} \left| \nabla_x^{a_1} v \right| ... \left| \nabla_x^{a_k} v \right| \left| \nabla_x^{0} e_p \right| (y).
\]

By (3.7), we observe that $|\tilde{V}(\gamma(\zeta))| = |V(\gamma(0))|$. Thus we arrive at

\[
|I_1 - I_2| \leq |\Delta_h v(s)| \left| \nabla_x^{a_1} v(x) \right| ... \left| \nabla_x^{a_k} v(x) \right| \left| \nabla_x^{0} e_p(x) \right|.
\]

Then, by (2.27), (3.9) gives

\[
|I_1 - I_2| \lesssim |\Delta_h v(s)||\nabla_x^{0} e_p| \Pi_{1 \leq l \leq k} \left( \sum_{j=1}^{a_k+1} \sum_{j_1=1}^{|\gamma|} |\partial^{j_1} v|...|\partial^{j_k} v| \right).
\]

Denote

\[
I_3 = \left( D^k(d\mathcal{P})(v(s, x + h)) \right) \left( \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v; \nabla_x^{0} e_p \right)
\]

Then it is easy to see

\[
|I_3 - I_2| \lesssim \left( D^k(d\mathcal{P})(v(s, x + h)) \right) \left( \nabla_x^{a_1} v - \nabla_x^{a_1} v, ..., \nabla_x^{a_k} v, \nabla_x^{0} e_p \right) + ...
\]
\[
\begin{align*}
&\big(\nabla^a x \partial_x v, \ldots, \nabla^a x \partial_x v - \nabla^a x \partial_x v; V^a e_p\big) \\
&+ \big(\nabla^a x \partial_x v, \ldots, \nabla^a x \partial_x v; V^a e_p - \nabla^a x e_p\big)
\end{align*}
\]

To estimate this formula, we firstly estimate \(\nabla V - V\) for \(V = \nabla_x^a \partial_x v, \nabla_x^a e_p\). It is convenient to bound \(|dP(\nabla V - V)|\) instead, because the later takes value in \(\mathbb{R}^M\) and equals the former due to the isometric embedding.

**Step 2.** Before bounding \(|dP(\nabla V - V)|\) for \(V = \nabla_x^a \partial_x v, \nabla_x^a e_p\), we use the extrinsic quantities \(\partial_v v\) to express the intrinsic ones \(\nabla_x^a \partial_x v\). It is easy to check (we adopt the same notation \(v\) to denote both \(P \circ v\) and the map \(v\) itself without confusion)

\[
\begin{align*}
&dP(\partial_v v) = \partial_v v; \\
&dP(\nabla_j \partial_v v) = \partial_j [dP(\partial_v v)] - (DdP)(\partial_j v; \partial_i v) \\
&= \partial_{ij} v - (DdP)(\partial_j v; \partial_i v). \quad (3.10)
\end{align*}
\]

In Step 1, we have seen that using parallel transport

\[
\begin{align*}
&\left| (D^k(dP)(V_1, \ldots, V_k; V_0)(x + h) - (D^k dP)(V_1, \ldots, V_k; V_0)(x) \right| \\
&\leq \sum_{i=1}^k \left| (D^k dP)(\widetilde{V}_1, \ldots, \widetilde{V}_{i-1}, \widetilde{V}_i - \tilde{V}_i, \widetilde{V}_{i+1}, \ldots; V_0)(x + h) \right| \\
&+ \left| (D^k dP)(\widetilde{V}_1, \ldots, \widetilde{V}_k; V_0 - \tilde{V}_0)(x + h) \right| \\
&+ \left| (D^k dP)(V_1, \ldots, V_k; \tilde{V}_0)(x) - (D^k dP)(\widetilde{V}_1, \ldots, \widetilde{V}_k; \tilde{V}_0)(x + h) \right|. \quad (3.11)
\end{align*}
\]

Moreover, (3.12) is dominated by

\[
\begin{align*}
&\left( \sum_{i=0}^k \max_{y \in \{x,x+h\}} |V(y)| \right) \triangle_h v(s). \\
&\quad (3.13)
\end{align*}
\]

We also recall the inequality

\[
\begin{align*}
&\left| dP V(x) - dP \tilde{V}(x + h) \right| \lesssim \left( \max_{y \in \{x,x+h\}} |V(y)| \right) \triangle_h v(s). \quad (3.14)
\end{align*}
\]

Since \(P\) is isometric, (3.14) further yields

\[
\begin{align*}
&\left| V(x + h) - \tilde{V}(x + h) \right| \lesssim \left( \max_{y \in \{x,x+h\}} |V(y)| \right) \triangle_h v(s) + |\triangle_h dP V|. \quad (3.15)
\end{align*}
\]

Thus one has by (3.13), (3.15), (3.11) that

\[
\begin{align*}
&\left| (D^k dP)(V_1, \ldots, V_k; V_0)(x + h) - (D^k dP)(V_1, \ldots, V_k; V_0)(x) \right|
\end{align*}
\]
Thus by (3.14) we obtain
\[ \nabla \nabla \] 
\[ \nabla \nabla \] 
We now turn to estimate
\[ \mid \Delta_h v(s) \mid \max_{y \in \{x, x+h\}} \mid V_i(y) \mid + \mid \Delta_h dP V_i \mid 
\]
\[ + \left( \prod_{i=0}^{k} \max_{y \in \{x, x+h\}} \mid V_i(y) \mid \right) \mid \Delta_h v(s) \mid. \]

(3.16)

Therefore, applying (3.16) to (3.10) yields
\[ \mid (D dP)(\partial_j v; \partial_i v)(x + h) - (D dP)(\partial_j v; \partial_i v)(x) \mid \lesssim \mid \Delta_h v(s) \mid C_{ij}^2 + \mid \partial_j v - \partial_j v \mid \mid \partial_i v \mid (x + h) + \mid \partial_i v - \partial_i v \mid \mid \partial_j v \mid (x + h) \]
\[ \lesssim C_{ij}^2 \mid \Delta_h v \mid + (D_{ij} + C_{ij} \mid \Delta_h v \mid) C_{ij} \]

where we denote
\[ C_{ij} := \max_{y \in \{x, x+h\}} \mid \partial_j v(y) \mid + \max_{y \in \{x, x+h\}} \mid \partial_i v(y) \mid 
\]
\[ D_{ij} := \mid \Delta_h \partial_j v \mid + \mid \Delta_h \partial_i v \mid. \]

We conclude for the second order intrinsic derivatives \( \nabla_x \partial_x v \) that
\[ \mid \Delta_h dP(\nabla_j \partial_i v) \mid \lesssim \Delta_h \partial_{ij}^2 v + C_{ij}^2 \mid \Delta_h v \mid + \mid \Delta_h (\partial_x v) \mid C_{ij}. \]

By induction, we summarize that
\[ \mid \Delta_h dP(\nabla_x^k \partial_x v) \mid \lesssim \sum_{p=0}^{k} \sum_{l+p \sum_{\mu=1}^{p} j_\mu = k+1, j_\mu, l \in \mathbb{N}} C_{(i_1)}^{j_1} \ldots C_{(i_p)}^{j_p} \Delta_h \partial_{x}^l v. \] (3.17)

where we adopt the notation
\[ C_{(i)} := \sum_{i} \max_{y \in \{x, x+h\}} \mid \nabla_x^i \partial_x v(y) \mid, \text{ if } i \geq 0. \]

Thus by (3.14) we obtain
\[ \mid \nabla_x^k \partial_x v - \nabla_x^k \partial_x v \mid (x + h) \]
\[ \lesssim \sum_{p=0}^{k} \sum_{l+p \sum_{\mu=1}^{p} j_\mu = k+1, j_\mu, l \in \mathbb{N}} C_{(i_1)}^{j_1} \ldots C_{(i_p)}^{j_p} \Delta_h \partial_{x}^l v + \Delta_h (v) C_{(k+1)}. \] (3.18)

We now turn to estimate \( \Delta_h dP(\nabla_x^{k+1} \partial_x v) \). Different from the above, we express \( \nabla_x^{k+1} \partial_x v \) by connection coefficients \{ \partial_x A_i \} rather than by extrinsic quantities \{ \partial_x dP \partial_x v \}. Schematically, we write
\[ dP(\nabla_x v) = A_x dP(v); \]

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\[
    dP(\nabla_x^\alpha e) = \prod_{j=|\alpha|}^{N} (\partial^{i_j} A_x)^{j_j} dP(e). \tag{3.19}
\]

Then, we deduce that
\[
    \left| \nabla_k dP(\nabla_x^k e) \right| \lesssim \sum_{q=1}^{k} \sum_{\mu, i_q=1}^{\Omega} D_{(i_1)}^{j_1} \ldots D_{(i_q)}^{j_q} \left| \nabla_k dP(e) \right| \left| D_{(m_1)}^{k_1} \ldots D_{(m_z)}^{k_z} \right| \left| \nabla_x \partial_{x}^{m_0} A_x \right|,
\]

where \(D_{(j)} := \sum_{l=1}^{d} \sum_{\alpha=1}^{\Omega} \max_{y \in \{x,x+h\}} \left| \partial_x^{\alpha} A_i(y) \right|\), for \(j \geq 0\).

**Step 3.** We bound \(\|C_{(i)}\|_{L^p} \quad \text{and} \quad \|D_{(j)}\|_{L^p}\) in this step. \(2.23\) and \(2.25\) show for all \(k \in \mathbb{N}\), \(0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1\)
\[
    \|C_{(k)}\|_{L^p} \lesssim \epsilon_1 s^{-k+1} \tag{3.21}
\]
\[
    \|C_{(j)}\|_{L^2} \lesssim \epsilon_1. \tag{3.22}
\]

Meanwhile, \(3.1\) and \(3.3\) show for all \(k \in \mathbb{N}\), \(0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor - 1\)
\[
    \|D_{(k)}\|_{L^p} \lesssim \epsilon_1 s^{-k+1} \tag{3.23}
\]
\[
    \|D_{(j)}\|_{L^2} \lesssim \epsilon_1. \tag{3.24}
\]

**Step 4.** Recall \(d = 2d_0 + 1\). Inserting the bounds \(3.20\) and \(3.18\) to \(3.10\) with \(V_i = \nabla_x^{\alpha_i} \partial_{x}^{\nu} v, \quad i = 1, \ldots, k, \quad V_0 = \nabla_x^{\nu_0} e\), we arrive at
\[
    \left\| \nabla_k \partial_{x}^k (\left( dPE - \chi^\infty \right) \right\|_{L^2} \leq \sum_{l=1}^{k} \sum_{\Omega_1} C_{(i_1)}^{j_1} \ldots C_{(i_q)}^{j_q} D_{(q_1)}^{j_1} \ldots D_{(q_r)}^{j_r} \left| \nabla_k dP(e_p) \right| \tag{3.25}
\]
\[
    + \sum_{1 \leq k \leq \Omega_2} \sum_{m_1}^{m_z} C_{(m_1)}^{j_1} \ldots C_{(m_2)}^{j_2} D_{(m_1)}^{k_1} \ldots D_{(m_2)}^{k_2} \left| \nabla_k \partial_{x}^{m_0} A_x \right| \tag{3.26}
\]
\[
    + \sum_{1 \leq k \leq \Omega_3} \sum_{\Omega_2}^{\Omega_3} C_{(j_1)}^{j_1} \ldots C_{(j_q)}^{j_q} D_{(j_1)}^{j_1} \ldots D_{(j_r)}^{j_r} \left| \nabla_k \partial_{x}^{m_0} v \right| \tag{3.27}
\]

where the index sets \(\Omega_1, \Omega_2, \Omega_3\) are defined by
\[
    \Omega_1 : \sum_{\mu=1}^{l} j_\mu (1 + i_\mu) + \sum_{\nu=1}^{r} j_\nu (j_\nu' + 1) = k; \quad i_\mu, j_\mu, j_\nu, j_\nu' \in \mathbb{N}
\]
\[ \Omega_2 : \sum_{\mu=1}^z (1 + m_{\mu}) n_{\mu} + \sum_{\nu=1}^{z'} (1 + m'_{\nu}) n'_{\nu} = k - m_b - 1; \ n_{\mu}, m_{\mu}, n'_{\nu}, m'_{\nu} \in \mathbb{N} \]

\[ \Omega_3 : \sum_{\mu=1}^z (1 + q_{\mu}) p_{\mu} + \sum_{\nu=1}^{z'} (1 + q'_{\nu}) p'_{\nu} = k - m_b; \ p_{\mu}, q_{\mu}, p'_{\nu}, q'_{\nu} \in \mathbb{N} \]

Then the lemma follows by the difference characterization of \( \dot{H}^{\frac{1}{2}} \), Sobolev embeddings and Hölder. We present the details of this part in the following Lemma.

\[ \square \]

**Remark** One can also prove Lemma 3.2 without using parallel transport. But the following curvature bounds seem to rely heavily on parallel transport. We will deal with these geometric quantities in a unified way set up in the proof of Lemma 3.2.

**Lemma 3.3.** Let \( v \in C([0, \infty); \mathcal{Q}(\mathbb{R}^d, \mathcal{M})) \) be heat flow with initial data \( v_0 \) for which (2.3) holds for all \( s_* < s < \infty \). Let \( \{e_l\}_{l=1}^m \) be the corresponding caloric gauge with given limit \( \{e_{\infty l}\}_{l=1}^m \). Then

\[ \| \partial^L_x G' \|_{L_t^\infty \dot{H}^{\frac{d}{2}}_x} \lesssim L \epsilon s^{-\frac{d}{4}}, \ \forall L \in \mathbb{N} \]  

(3.28)

\[ \| \partial^L_x G'' \|_{L_t^\infty \dot{H}^{\frac{d}{2}}_x} \lesssim L \epsilon s^{-\frac{d}{4}}, \ \forall L \in \mathbb{N} \]  

(3.29)

**Proof.** Recall the definition of \( G', G'' \):

\[ (G')_l = (\tilde{\nabla} R)(e_l; e_{i_0}, e_{i_1}, e_{i_2}, e_{i_3}) - \Gamma^\infty_{l} \]

\[ (G'')_{pl} = (\tilde{\nabla}^2 R)(e_p, e_l; e_{i_0}, e_{i_1}, e_{i_2}, e_{i_3}) - \Omega^\infty_{pl}. \]

where we view \( R \) as a \((0, 4)\) tensor. The same arguments as Lemma 3.2 show

\[ \partial^2_x (G') = \sum_{k=0}^{[|\alpha|]} \sum_{i_0 + \sum_{l=1}^k (i_l + 1) + \sum_{\mu=0}^3 j_\mu = |\alpha|} (\tilde{\nabla}^k R)(\nabla^i_0 e, \nabla^{i_1}_x \partial_x v, ..., \nabla^{i_k}_x \partial_x v; \nabla^j_0 e_{i_0}, ..., \nabla^j_3 e_{i_3}) \]

(3.30)

\[ \partial^2_x (G'') = \sum_{k=1}^{[|\alpha|-1]} \sum_{\Omega} (\tilde{\nabla}^{k+1} R)(\nabla^i_0 e, \nabla^{i_0}_x e, \nabla^{i_1}_x \partial_x v, ..., \nabla^{i_k}_x \partial_x v; \nabla^j_0 e_{i_0}, ..., \nabla^j_3 e_{i_3}), \]

(3.31)

where the index set \( \Omega \) is defined by

\[ \Omega : i'_0 + i_0 + \sum_{l=1}^k (1 + i_l) + \sum_{\mu=0}^3 j_\mu = |\alpha|. \]  

(3.32)
Recall \( d = 2d_0 + 1 \). By difference characterization of Besov spaces and \( (3.30) \), to prove \( (3.28) \) it suffices to verify

\[
\tau^{-\frac{1}{2}} \sup_{|h| \leq \tau} \| \Delta_h (\tilde{V}^k R)(\nabla_x^i, e; \nabla_x^i \partial_x v; \nabla_x^i e_0, \ldots, \nabla_x^i e_3) \|_{L^p_x L^2}\]

\[
\lesssim L \epsilon \tau^{-\frac{1}{2}}
\]

provided that

\[
i_0 + \sum_{l=1}^{k} (i_l + 1) + \sum_{\mu=0}^{3} j_\mu = L + d_0, 0 \leq k \leq L + d_0.
\]

Using the \( (3.16) \) type estimates and \( (3.20) \), we obtain

\[
| \Delta_h (\tilde{V}^k R)(\nabla_x^i, e; \nabla_x^i \partial_x v; \nabla_x^i e_0, \ldots, \nabla_x^i e_3) |
\]

\[
\lesssim \sum_{q=1}^{L+d_0} \sum_{\Omega_1} C_{(i_1)}^{\nu_1} \cdots C_{(i_z)}^{\nu_z} D_{(i'_1)}^{\nu'_1} \cdots D_{(i'_{z'})}^{\nu'_{z'}} | \Delta_h dP(e_\mu) |
\]

\[
+ \sum_{1 \leq b \leq z', z \geq 1} \sum_{\Omega_2} C_{(m_1)}^{\nu_1} \cdots C_{(m_z)}^{\nu_z} D_{(m'_1)}^{\nu'_1} \cdots D_{(m'_{z'})}^{\nu'_{z'}} | \Delta_h \partial_x^{m_0} A_x |
\]

\[
+ \sum_{1 \leq b \leq z', z \geq 1} \sum_{\Omega_3} C_{(q_1)}^{\nu_1} \cdots C_{(q_z)}^{\nu_z} D_{(q'_1)}^{\nu'_1} \cdots D_{(q'_{z'})}^{\nu'_{z'}} | \Delta_h \partial_x^{m_0} v |
\]

where the index sets \( \Omega_1, \Omega_2, \Omega_3 \) are defined by

\[
\Omega_1 : \sum_{\nu=1}^{z} j_\nu (1 + i_\nu) + \sum_{\mu=1}^{z'} (i'_\mu + 1) j'_\mu = L + d_0, j_\nu, i'_\mu, j'_\mu, i_\nu \in \mathbb{N}
\]

\[
\Omega_2 : \sum_{\nu=1}^{z} (1 + m_\nu) n_\nu + \sum_{\mu=1}^{z'} (1 + m'_\mu) n'_\mu = L + d_0 - (m'_0 + 1), n_\nu, n'_\mu, m'_0, m_\nu \in \mathbb{N}
\]

\[
\Omega_3 : \sum_{\nu=1}^{z} (1 + q_\nu) p_\nu + \sum_{\mu=1}^{z'} (1 + q'_\mu) p'_\mu = L + d_0 - m_\nu, p_\nu, p'_\mu, q'_\mu, q_\nu \in \mathbb{N}
\]

As Before, we consider two subcases. Case 1. Assume that all \( \{i'_\mu\}, \{i_\nu\} \) in \( (3.31) \) satisfy

\[
0 \leq i_\nu \leq \frac{d}{2} - 1, 0 \leq i'_\mu \leq \frac{d}{2} - 1.
\]

Then \( (3.22), (3.24) \) show for

\[
\frac{1}{r_\nu} = \frac{1}{2} - \frac{1}{d} \frac{d}{2} - i_\nu - 1
\]

(3.38)
\[
\frac{1}{t_\mu} = \frac{1}{2} - \frac{1}{d} \left( \frac{d}{2} - i_\mu - 1 \right) \tag{3.39}
\]

there hold
\[
\| C_{\nu} \|_{L^\infty L^p_\nu} \lesssim \varepsilon \tag{3.40}
\]
\[
\| D_{i_\nu} \|_{L^\infty L^p_\nu} \lesssim \varepsilon \tag{3.41}
\]

And interpolating (3.40), (3.41) with \( L^\infty \) bounds given by (3.21)-(3.23), we have
\[
\| C_{i_\nu} \|_{L^\infty L^p_\nu} \lesssim \varepsilon^{-\frac{1}{j_1}} \tag{3.42}
\]
\[
\| D_{i_\nu} \|_{L^\infty L^p_\nu} \lesssim \varepsilon^{-\frac{1}{j'_1}} \tag{3.43}
\]
for all \( \tau \in [\tau_\nu, \infty] \) and for all \( \tau \in [\tau_\nu, \infty] \). Without loss of generality we assume \( \text{Case 1a. } j_1 > 0 \) or \( \text{Case 1b. } j'_1 > 0 \).

In the Case 1a, let \( \{ \tau_\nu \} \) be fixed exponents such that \( \tau_\nu \in [\tau_\nu, \infty] \), \( \tau_\nu' \in [\tau_\nu, \infty] \) and
\[
\sum_{\nu=1}^{z} \frac{d}{\tau_\nu} j_\nu + \sum_{\mu=1}^{z'} \frac{d}{\tau_\nu'} j'_\mu = d_0. \tag{3.44}
\]

The exponents \( \{ \tau_\nu \} \), \( \{ \tau_\nu' \} \) in (3.44) do exist. In fact, since \( j_1 > 0 \) the LHS of (3.44) is a continuous decreasing function with respect to any of \( \tau_\nu \in [\tau_\nu, \infty] \), \( \tau_\nu' \in [\tau_\nu, \infty] \). Hence the LHS of (3.44) ranges over \([0, L + d_0]\). Thus (3.44) holds with appropriate \( \{ \tau_\nu, \tau_\nu' \} \). Then, in Case 1a one obtains
\[
\left\| \frac{1}{\tau} \sup_{|h| \leq \tau} \left| C_{(i_1)}^{j_1} ... D_{(i_z)}^{j_z} D_{(i'_1)}^{j'_1} ... D_{(i'_{z'})}^{j'_{z'}} \right| \Delta h dP(e_p) \right\|_{L^\infty L^2(\tau^{-1} d\tau)L^2_\tau} \tag{3.45}
\]
\[
\leq \left\| C_{(i_1)}^{j_1} \right\|_{L^\infty L^2_\tau} \left\| C_{(i_2)}^{j_2} \right\|_{L^\infty L^2_\tau} ... \left\| C_{(i_z)}^{j_z} \right\|_{L^\infty L^2_\tau} \times \left\| D_{(i'_1)}^{j'_1} \right\|_{L^\infty L^2_\tau} ... \left\| D_{(i'_{z'})}^{j'_{z'}} \right\|_{L^\infty L^2_\tau} \left\| \frac{1}{\tau} \sup_{|h| \leq \tau} \left| \Delta h dP(e_p) \right| \right\|_{L^\infty L^2(\tau^{-1} d\tau)L^2_\tau} \tag{3.46}
\]
where in the second line to apply Hölder inequality we used the following equality
\[
\frac{1}{2} - \frac{1}{2d} = \sum_{\nu=1}^{z} \frac{j_\nu}{\tau_\nu} + \sum_{\mu=1}^{z'} \frac{j'_\mu}{\tau_\nu'}
\]
which follows from (3.44) and (3.39). Thus (3.45), (3.43) and (3.42) imply
\[
\left\| \frac{1}{\tau} \sup_{|h| \leq \tau} \left| C_{(i_1)}^{j_1} ... C_{(i_z)}^{j_z} D_{(i'_1)}^{j'_1} ... D_{(i'_{z'})}^{j'_{z'}} \right| \Delta h dP(e_p) \right\|_{L^\infty L^2_\tau(R^+L^2_\tau)}
\]

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\[ \lesssim \epsilon s^{-\frac{3}{2}} \left\| \frac{1}{\tau} \sup_{|h| \leq \tau} |\nabla_h dP(e_p)| \right\|_{L_t^\infty L_x^2(\mathbb{R}^+)} \]
\[ \lesssim \epsilon s^{-\frac{5}{2}} \left\| dP(e_p) \right\|_{L_t^\infty H_x^\frac{d}{2}}.\]

where the power $\frac{5}{2}$ in $s^{-\frac{5}{2}}$ of the second line results from (3.44). Therefore, we get by (3.4) that
\[ \sup_{|h| \leq \tau} |h| \lesssim \tau \left| \sum_{j_1}^{j_q} \cdots \sum_{j_1}^{j_q} D_{j_1}^{(i_1)} \cdots D_{j_q}^{(i_q)} |\nabla_h dP(e_p)| \right\|_{L_t^\infty L_x^\infty} \]
\[ \lesssim \epsilon s^{-\frac{3}{2}}. \]

The Case 1.b follows by the same way. Thus (3.34) has been done. The left two terms (3.35)-(3.36) can be dominated as (3.34) by (3.3) and Lemma 2.1.

**Case 2.a** Assume that among \( \{i_{\mu}': j_{\mu} > 0, \mu = 1, \ldots, q\} \) in (3.34) there exists an \( i_{\mu}' \) such that
\[ i_{\mu}' > d - 1. \] (3.47)

Since \( \|g\|_{L_t^\infty} \lesssim \|g\|_{H_t^\frac{d}{2}} \) with
\[ \frac{1}{\rho} = \frac{1}{2} - \frac{1}{2d}, \]
we deduce by interpolation and (3.3) that
\[ \left\| D_{i_{\mu}'} \right\|_{L_t^\infty L_x^\rho} \lesssim \epsilon s^{-\frac{1}{2}(i_{\mu}'+\frac{d}{2}-\frac{d}{4})}. \] (3.48)

Without loss of generality assume that \( i_{\mu}' = 1 \). Then in Case 2a, there holds
\[ \left\| \frac{1}{\tau} \sup_{|h| \leq \tau} |D_{i_{1}^{(i_1)}}^{(i_1)} \cdots D_{i_{1}^{(i_q)}}^{(i_q)} |\nabla_h dP(e_p)| \right\|_{L_t^\infty L_x^2(\mathbb{R}^+)} \]
\[ \lesssim \left\| C_{(i_1)} \right\|_{L_t^\infty L_x^\infty} \cdots \left\| C_{(i_q)} \right\|_{L_t^\infty L_x^\infty} \left\| D_{i_{1}^{(i_1)}} \right\|_{L_t^\infty L_x^\infty} \left\| D_{i_{1}^{(i_q)}} \right\|_{L_t^\infty L_x^\infty} \left( \prod_{\mu=2}^\infty \left\| D_{(i\mu)} \right\|_{L_t^\infty L_x^\infty} \right) \]
\[ \lesssim \epsilon s^{-\frac{1}{2}(i_1+\frac{d}{2}-\frac{d}{4})} \prod_{\mu=2}^q s^{-\frac{1}{2}(i_\mu+1)} \left( \prod_{\mu=2}^q s^{-\frac{1}{2}(i_\mu+1)} \right). \] (3.49)

where we applied $L^\infty$ bounds given by (3.3) and Lemma 3.2 in the last line. It is easy to check that the RHS of (3.49) is exactly $\epsilon s^{-\frac{3}{2}}$. 

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Case 2.b Assume that among \( \{ i_\nu : j_\nu > 0, \nu = 1, \ldots, q \} \) in (3.34) there exists an \( i_\nu' \) such that \( i_\nu' > d^2 - 1 \). This follows by the same path if one has the analogy of (3.48) for \( C_{(ii)} \):

\[
\| C_{i_\nu'} \|_{L^\infty L^d_x} \lesssim \varepsilon s^{-\frac{1}{2}(i_\nu' + \frac{d^2}{2} - \frac{d}{2})}.
\]  

(3.50)

Notice that (3.50) follows by (2.26).

The left (3.35), (3.36) parts follow by the same argument in Case 2.

Therefore, as a summary, we have proved (3.28) with assuming (3.3)-(3.4). The left (3.29) is the same.

Lemma 3.4. Let \( v : [0, \infty) \times \mathbb{R}^d \to \mathcal{M} \) be heat flow with initial data \( v_0 \in Q(\mathbb{R}^d, \mathcal{M}) \). Assume also that (2.3) holds and (3.3)-(3.4) hold in \( s_* \leq s < \infty \). Let \( \{ e_i \}_{i=1}^d \) be the corresponding caloric gauge. Then for \( s_* \leq s < \infty \), the differential fields \( \{ \psi^i \}_{i=1}^d \) satisfy

\[
\| \partial^L_x \psi^i \|_{H^{d-1}_x} \lesssim L \varepsilon s^{-\frac{L+1}{d}}
\]  

(3.51)

\[
\| \partial^L_s \psi^i \|_{L^\infty_x} \lesssim L \varepsilon s^{-\frac{L+1}{2}}.
\]  

(3.52)

And the heat tension field \( \psi_s \) satisfies

\[
\| \partial^L_x \partial_s \psi \|_{H^{d-1}_x} \lesssim L \varepsilon s^{-\frac{L+1}{2}}
\]  

(3.53)

\[
\| \partial^L_s \partial_s \psi \|_{L^\infty_x} \lesssim L \varepsilon s^{-\frac{L+3}{2}}
\]  

(3.54)

\[
\| \partial^L_x \psi_s \|_{H^{d-1}_x} \lesssim L \varepsilon s^{-\frac{L+2}{2}}
\]  

(3.55)

\[
\| \partial^L_s \psi_s \|_{L^\infty_x} \lesssim L \varepsilon s^{-\frac{L+2}{2}}
\]  

(3.56)

Moreover, we have

\[
\| \psi_s \|_{L^2_x H^{d-1}_x} \lesssim L \varepsilon.
\]  

(3.57)

Proof. By the definition of differential field \( \{ \psi^i_x \} \) and the isometry of \( dP \), we obtain

\[
\psi^i = \partial_i v \cdot dP(\xi_1)
\]

(3.58)

\[
\psi^j_s = \partial_s v \cdot dP(\xi_1),
\]

(3.59)

where we write \( \partial_i v \) instead of \( \partial_i (P \circ v) \) for simplicity and \( i = 1, \ldots, d \). Generally one has

\[
\partial^L_x \psi^i = \sum_{k_1, k_2} \partial^{k_1+1}_x v \cdot \partial^{k_2}_x dP(e)
\]

(3.60)

\[
\partial^L_s \psi^i = \sum_{k_1, k_2} \partial^{k_1+1}_s v \cdot \partial^{k_2}_s dP(e).
\]

(3.61)

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and Lemma 3.2 yield the bounds for $P e$:

$$\| \partial_x^L (P(e) - \chi^\infty) \|_{\dot{H}^{\frac{1}{2}}_x} \lesssim \epsilon s^{-\frac{1}{2}} \tag{3.62}$$

$$\| \partial_x^L (P(e) - \chi^\infty) \|_{L^\infty_x} \lesssim \epsilon s^{-\frac{1}{2}}. \tag{3.63}$$

Then (3.62)-(3.63), Lemma 2.1, fractional Leibnitz rule and Sobolev inequalities imply (3.51). Meanwhile, (3.2) and (1.7) yield (3.52) by (3.60).

(3.54) directly follows from the heat flow equation and (1.7). (3.61) together with (3.54) gives (3.56). And by the heat flow equation, (3.53) reduces to prove

$$\| \partial_x^L [S(v)(\partial_x v, \partial_x v)] \|_{\dot{H}^{\frac{1}{2}} - 1} \lesssim \epsilon s^{-\frac{L^2}{4}} + \epsilon,$$

which follows by fractional Leibnitz rule, Sobolev inequalities and Lemma 2.1. Similarly, (3.61) together with (3.53) gives (3.55). Lastly, (3.57) follows by (2.22) and (3.61)-(3.63).

Lemma 3.5. Let $v : [0, \infty) \times \mathbb{R}^d \to M$ be heat flow with initial data $v_0 \in Q(\mathbb{R}^d, M)$. Assume also that (2.3) holds and (3.3)-(3.4) hold in $s \in [s^*, \infty)$. Let $\{e_l\}_{l=1}^m$ be the corresponding caloric gauge. Then the connection coefficients satisfy

$$\| \partial_x^L A_x \|_{\dot{H}^{\frac{4}{3}} - 1} \lesssim L \epsilon^2 s^{-\frac{1}{2}} \tag{3.64}$$

$$\| \partial_x^L A_x \|_{L^\infty_x} \lesssim L \epsilon^2 s^{-\frac{L+1}{2}}. \tag{3.65}$$

Proof. By (6.3), we have

$$[A_i]_p^x (s) = \int_s^\infty \langle R(v(s'))(\partial_s v(s'), \partial_i v(s')) e_p, e_q \rangle ds', \tag{3.66}$$

which can be schematically written as

$$[A_i]_p^x (s) = \int_s^\infty (\psi_s \circ \psi_x) (R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3}) ds'. \tag{3.66}$$

Following arguments of Lemma 3.3 gives

$$\| \partial_x^L \langle R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3} \rangle \|_{\dot{H}^{\frac{4}{3}}_x} \lesssim \epsilon s^{-\frac{1}{2}}$$

$$\| \partial_x^L \langle R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3} \rangle \|_{L^\infty_x} \lesssim c(L)s^{-\frac{1}{2}},$$

where $c(0) = 1$ and $c(L) = \epsilon$ if $L \geq 1$ in the second line. Then Lemma 3.4 and fractional Leibnitz rules show

$$\| \partial_x^L (\psi_x \circ \psi_s (R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3})) \|_{\dot{H}^{\frac{4}{3}} - 1} \lesssim \epsilon^2 s^{-\frac{L+2}{2}} \tag{3.67}$$

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\[
\left\| \partial_x^L (\psi_x \circ \psi_s (R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3})) \right\|_{L^\infty_x} \lesssim \epsilon^2 s^{-\frac{d+1}{2}}.
\] (3.68)

Thus integrating (3.68) in \( s' \in [s, \infty) \) gives (3.65) for all \( L \geq 0 \). And integrating (3.67) in \( s' \in [s, \infty) \) gives (3.64) for all \( L \geq 1 \). Hence it suffices to prove (3.64) with \( L = 0 \).

As before, assume that \( d = 2d_0 + 1 \) is odd. Then one has by fractional Leibnitz rule and Sobolev embedding that

\[
\left\| \partial_x^L (\psi_x \circ \psi_s (R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3})) \right\|_{L^1_x \dot{H}^{\frac{d}{2}-1}} 
\lesssim \sum_{i=1}^3 \sum_{j_i = d_0 - 1} \left\| \partial_x^{j_1} \psi_x \partial_x^{j_2} \psi_s \partial_x^{j_3} (R(e_{i_0}, e_{i_1}) e_{i_2}, e_{i_3}) \right\|_{L^1_x \dot{H}^{1}} 
\lesssim \left\| \psi_x \right\|_{L^2_x \dot{H}^{\frac{d}{2}}} \left\| \psi_s \right\|_{L^2_x \dot{H}^{\frac{d}{2}-1}} 
\lesssim \epsilon^2,
\]

where in the \( p(\beta_1), q(\beta_2), r(\beta_3) \) are defined by

\[
\frac{\beta_1}{d} = \frac{1}{p(\beta_1)}, \quad \frac{1}{d} + \frac{\beta_2}{d} = \frac{1}{q(\beta_2)}, \quad \frac{\beta_3}{d} = \frac{1}{r(\beta_3)}.
\]

Thus (3.64) for all \( L = 0 \) is done as well.

3.2 Proof of Theorem 1.1.

We want to prove Theorem 1.1 by bootstrap. First, there exists a sufficiently large \( s_0 \) such that (3.3)-(3.4) hold for \( s \in [s_0, \infty) \) because Lemma 2.5 says the left hand sides of (3.3) and (3.4) decay to 0 as \( s \to \infty \). Second, by Lemma 3.5, one can push \( s_0 \) to be 0. Therefore, all the bounds stated by Lemma 2.1-Lemma 3.5 hold for all \( s \in [0, \infty) \).

4 Decay estimates in block spaces of \( F_k \)

According to the definition of \( F_k \) space, it is natural to track the following four block spaces of \( F_k \):

\[
L^\infty_t L^2_x, \quad L^p_{t,x}, \quad L^p_{x,t} L^\infty_t, \quad L^2_{t,x} \quad (4.1)
\]
along the heat flow. We will see in Section 5 there is no need to track the $L^2_e L^\infty_x$ blocks of $F_k$ in the heat direction, which makes large convenience for us.

### 4.1 Tracking the $L^p_t \cap L^\infty_x L^2_x$ block

Let $u$ be a solution to SL. Theorem 1.1 with additional efforts yields

**Corollary 4.1.** Let $u \in \mathcal{C}([-T,T]; \mathcal{Q}(\mathbb{R}^d,\mathcal{N}))$ be a solution to SL. Let $v$ be the solution of heat flow with initial data $u$. Assume also that $v$ is global in the heat direction and $\{e_i\}_{i=1}^{2n}$ denotes the caloric gauge. Define the frequency envelopes $\{h_k(\sigma)\}$ by

$$h_k(\sigma) = \sup_{s \geq 0, k' \in \mathbb{Z}} (1 + s 2^{2k'})^4 2^{-6 |k-k'|^2} 2^{(d-1)k'} \|P_k' \psi_x\|_{L^\infty_t L^2_x}$$

$$h_k := h_k(0).$$

Suppose that $\{h_k(\sigma)\}$ satisfy

$$\sum_{k \in \mathbb{Z}} h_k^2(\sigma) < \infty, \forall \sigma \in [0,1].$$

Then $v$ satisfies in $s \in \mathbb{R}^+$ that

$$\|\partial^j_x v\|_{L^\infty_t \dot{H}^{d-j-1}_x} \lesssim s^{-\frac{j}{2}} \epsilon_1.$$  

(4.3)

And the corresponding differential fields and connection coefficients satisfy

$$\|\partial^j_x \phi_x\|_{L^\infty_t \dot{H}^{d-j-1}_x} \lesssim j s^{-\frac{j}{2}}$$

(4.4)

$$\|\partial^j_x A_x\|_{L^\infty_t \dot{H}^{d-j-1}_x} \lesssim j s^{-\frac{j}{2}}$$

(4.5)

$$\|\partial^j_x \phi_x\|_{L^\infty_t L^\infty_x} \lesssim j s^{-\frac{j+1}{2}}$$

(4.6)

$$\|\partial^j_x A_x\|_{L^\infty_t L^\infty_x} \lesssim j s^{-\frac{j+1}{2}}$$

(4.7)

and for all $j \in \mathbb{N}$, $s > 0$.

**Proof.** Compared with Theorem 1.1, we have assumption (1.2) here rather than (1.5). In order to transpose (1.2) to (1.5), we need a tricky bootstrap argument.

First, we reconstruct the subcritical theory (energy dependent) estimates presented in Lemma 2.4 with assumption (1.2). Since the energy conserves along the Schrodinger map flow we get

$$\|du\|_{L^\infty_t L^2_x} \leq \|du_0\|_{L^2_x}.$$  

(4.8)
Checking the proof of (2.57) of Lemma 2.5 one finds only the following estimates are a-priori used:
\[
\|dv\|_{L^\infty_t L^\infty_x} \lesssim \epsilon s^{-\frac{1}{2}} \tag{4.9}
\]
\[
\|\nabla L dv\|_{L^\infty_t L^\infty_x} \lesssim \epsilon s^{-\frac{1}{2}} \tag{4.10}
\]
\[
\|\nabla L dv\|_{L^\infty_t L^2_x} \lesssim \|dv_0\|_{L^2_x} s^{-\frac{1}{2}} \tag{4.11}
\]
\[
\|\nabla L dv\|_{L^\infty_t L^\infty_x} \lesssim \|dv_0\|_{L^2_x} s^{-\frac{2L+4}{4}}. \tag{4.12}
\]
Thus if (4.9)-(4.12) are obtained with assumption (4.2), then the estimate (2.57) in Lemma 2.5 holds here as well. Notice that (4.9) follows by (4.2), Gagliardo-Nirenberg inequality and Sobolev embedding:
\[
\|dv\|_{L^\infty_t L^\infty_x} \lesssim \|\phi_x\|_{L^\infty_t H^{\frac{1}{2}}_x} \tag{4.13}
\]
\[
\|\phi_x\|_{L^\infty_t H^{\frac{1}{2}}_x} \lesssim \epsilon s^{-\frac{1}{2}}. \tag{4.14}
\]
Now, we turn to prove (4.10). Denote
\[
Z_{\infty,k}(s) = \sup_{\tilde{s} \in [0,s]} \tilde{s}^{\frac{k}{2}} \|\partial^k_x v(\tilde{s})\|_{L^\infty_t L^\infty_x}.
\]
By the heat flow equation, one has
\[
\|\partial^{k+1}_x v(s)\|_{L^\infty_t L^\infty_x} \lesssim s^{-\frac{1}{2}} \|\partial^k_x v(s/2)\|_{L^\infty_t L^\infty_x} + \int_{\frac{s}{2}}^{s} (s-s')^{\frac{k}{2}} \|\partial_x^k (S(v)(\partial_x, \partial_x))\|_{L^\infty_t L^\infty_x} ds' \\
\lesssim s^{-\frac{k+1}{2}} Z_{\infty,k}(s) + \int_{\frac{s}{2}}^{s} (s-s')^{\frac{k}{2}} \|\partial^{k+1}_x v\|_{L^\infty_t L^\infty_x} ds' \\
+ \sum_{l=2}^{k+2} \sum_{\sum_{i=1}^{l-1} j_i = k+2, 1 \leq j_i \leq k} \int_{\frac{s}{2}}^{s} (s-s')^{\frac{1}{2}} \|\partial^{j_1}_x v\|_{L^\infty_t L^\infty_x} \cdots \|\partial^{j_l}_x v\|_{L^\infty_t L^\infty_x} ds'.
\]
Then we have by (4.9) that
\[
Z_{\infty,1}(s) \lesssim \epsilon \\
Z_{\infty,k'}(s) \lesssim \epsilon k' \leq k \Rightarrow Z_{\infty,k+1}(s) \lesssim \epsilon Z_{\infty,k+1}(s) + \epsilon.
\]
Therefore, using this induction relation we get
\[
\tilde{s}^{\frac{k}{2}} \|\partial^k_x v(\tilde{s})\|_{L^\infty_t L^\infty_x} \lesssim \epsilon. \tag{4.15}
\]
And transposing this extrinsic bound to the intrinsic quantities $|\nabla^j \partial_x v|$ gives (4.10). Denote
\[
Z_{2,k}(s) = \sup_{\tilde{s} \in [0,s]} \tilde{s}^{\frac{k-1}{2}} \|\partial^k_x v(\tilde{s})\|_{L^\infty_t L^2_x}.
\]

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Then similarly one has by the heat flow equation that
\[
\|\partial_s^{k+1} v(s)\|_{L_t^\infty L_x^2} \lesssim s^{-\frac{1}{2}} \|\partial_s^k v(s/2)\|_{L_t^\infty L_x^2} + \int_0^s (s-s')^\frac{1}{2} \|\partial_s^k(S(s)(\partial_x, \partial_x))\|_{L_t^\infty L_x^2} ds'
\]
\[
\lesssim s^{-\frac{k+1}{2}} Z_{2,k}(s) + \int_0^s (s-s')^\frac{1}{2} \|\partial_s^k v\|_{L_t^\infty L_x^2} \|dv\|_{L_t^\infty} ds'
\]
\[
+ \sum_{l=2}^{k+2} \sum_{j_i=k+2,1 \leq j_i \leq k} \int_0^s (s-s')^\frac{1}{2} \|\partial_s^j v\|_{L_t^\infty L_x^2} \ldots \|\partial_s^1 v\|_{L_t^\infty} ds'.
\]
Thus we obtain by (4.9), (4.15) that
\[
Z_{2,1}(s) \lesssim \|dv_0\|_{L_t^\infty L_x^2}
\]
\[
Z_{2,k}(s) \lesssim \|dv_0\|_{L_t^\infty L_x^2} \quad \forall k \leq k \Rightarrow Z_{2,k+1}(s) \lesssim \epsilon Z_{2,k+1}(s) + \|dv_0\|_{L_t^\infty L_x^2},
\]
which gives us (4.11) by transposing to intrinsic quantities. And we end the first step by pointing that (4.12) follows by (4.9)-(4.11) and applying which gives us (4.11) by transposing to intrinsic quantities. And we end
\[
Then by \partial_t v = dP(e_l)\psi_l we get for s > \hat{s}
\[
\|\partial_t v\|_{L_t^\infty H_x^{\frac{d}{2}-1}} \leq \epsilon.
\]
Then applying Theorem 1.1 with initial time \hat{s} we get for all s \geq \hat{s}
\[
\|dP(e_l) - \chi_{e_l}^\infty\|_{L_t^\infty H_x^{\frac{d}{2}}} \lesssim \epsilon.
\]
Comparing (4.16) with (4.18), we see by bootstrap that (4.16) indeed holds for all s \geq 0. And thus (4.17) holds for all s \geq 0 as well. Then our corollary follows directly by Theorem 1.1.

Our main result for this section is

**Proposition 4.1.** Let \( \sigma \in [0, \bar{\nu}] \). Assume that u is a solution to SL given in Corollary 4.1. With abuse of notations, denote \( \{h_k(\sigma)\} \) the frequency envelopes
\[
h_k(\sigma) = \sup_{s \geq 0, k \in \mathbb{Z}} (1 + s2^k)^4 2^{-\delta|k-k'|} 2^{(d-1)k'} 2^{\sigma k'} \|P_k\psi_{sz}\|_{L_t^\infty L_x^2 \cap L_t^{p_k}}
\]
(4.19)
Proof. Step 1. Proof of (4.22). Suppose that

\[ \sum_{k} h_k^2 \leq \epsilon_1. \]  

Then for all \( L \in \mathbb{N} \),

\[ \|P_k(dP(e) - \chi^\infty)\|_{L_{t,x}^{p_3} \cap L_t^{s_3} L_x^2} \lesssim (1 + s2^{2k})^{-L}2^{-\frac{d}{2}}2^{-\sigma_k}h_k(\sigma). \]  

Moreover, for all \( L \in \mathbb{N}, \mathcal{G}', \mathcal{G}'' \) satisfy

\[ \|P_k(d\mathcal{G}')\|_{L_{t,x}^{p_3} \cap L_t^{s_3} L_x^2} \lesssim (1 + s2^{2k})^{-L}2^{-\frac{d}{2}}2^{-\sigma_k}h_k(\sigma) \]  

and

\[ \|P_k(d\mathcal{G}'')\|_{L_{t,x}^{p_3} \cap L_t^{s_3} L_x^2} \lesssim (1 + s2^{2k})^{-L}2^{-\frac{d}{2}}2^{-\sigma_k}h_k(\sigma). \]  

The connection coefficients \( A_x \) satisfy

\[ \|\partial_x^k A_x\|_{L_{t,x}^{p_4} B_{p_4,\infty}^{4-1}} \lesssim C(L)\epsilon_1 s^{-\frac{d}{2}}. \]  

Proof. Step 1. Proof of (4.22). We first verify the bounds for \( \partial_x v \):

\[ 2^{\frac{d}{2}k-k}\|P_k\partial_x v(\cdot=0)\|_{L_{t,x}^{s_3} L_x^2} \lesssim 2^{-\sigma_k}h_k(\sigma) \]  

(4.26)

\[ 2^{\frac{d}{2}k-k}\|P_k\partial_x v(\cdot)\|_{L_{t,x}^{s_3} L_x^2} \lesssim 2^{-\sigma_k}h_k(\sigma)(1 + s2^{2k})^{-L}. \]  

(4.27)

Now, we turn to prove (4.23). We will frequently use the following bilinear estimates:

\[ \|P_k(f g)\|_{L_t^{p_3} L_x^{s_3} L_x^d} \lesssim \|P_{k-4+k+4} f\|_{L_t^{p_3} L_x^{s_3} L_x^d} \|P_{k-4} g\|_{L_t^{s_3} L_x^d} + \sum_{k_3 \geq k-4} 2^{\frac{d}{2}k-k}\|P_{k_3} f\|_{L_t^{p_3} L_x^{s_3} L_x^d} \|P_{k_3} g\|_{L_t^{s_3} L_x^d} \]  

(4.28)

\[ \|P_k(f g)\|_{L_t^{s_3} L_x^2} \lesssim \|P_{k-4+k+4} f\|_{L_t^{s_3} L_x^2} \|P_{k-4} g\|_{L_t^{s_3} L_x^2} + \sum_{k_3 \geq k-4} 2^{\frac{d}{2}k-k}\|P_{k_3} f\|_{L_t^{s_3} L_x^2} \|P_{k_3} g\|_{L_t^{s_3} L_x^2}. \]  

(4.29)

By definition and Corollary 4.1,

\[ 2^{\frac{d}{2}k-k}\|P_k \psi_x|_{s=0}\|_{L_{t,x}^{p_3} \cap L_t^{s_3} L_x^2} \leq 2^{-\sigma_k}h_k(\sigma) \]  

(4.30)

\[ 2^{\frac{d}{2}k}(1 + s2^{2k})^{-L}\|P_k(dP(e) - \chi^\infty)\|_{L_t^{s_3} L_x^2} \leq \epsilon. \]  

(4.31)
Then we obtain by the identity $\partial_v v = \sum_{l=1}^{2n} \psi_l^t dP(e_l)$ and bilinear estimates \((4.28)-(4.29)\) that

$$\| P_k \partial_v v \|_{L_t^1 L^\infty} \lesssim 2^{-\frac{d}{2} k + k^2} 2^{-\sigma k} h_k(\sigma) + 2^{-\frac{d}{2} k} \sum_{k_1 \geq k - 4} 2^{-d k_1 + k_1^2} 2^{-\sigma k_1} h_{k_1}(\sigma)$$

$$+ 2^{-\frac{d}{2} k} 2^{\frac{d}{2} - \frac{1}{p_d}) k} \sum_{k_2 \leq k - 4} 2^{-\sigma k_2} 2^{-\frac{d}{2} k_2} h_{k_2}(\sigma).$$

Since $d \geq 3$, by the slow variation of frequency envelopes we get for $\sigma \in [0, \vartheta]$

$$2^\frac{d}{2} k^{k-\sigma k} \| P_k \partial_v v \|_{L_t^1 L^\infty} \lesssim 2^{-\sigma k} h_k(\sigma). \quad (4.32)$$

Thus \((4.20)\) is done.

Now, let us consider \((4.27)\). This follows by \((4.20)\) and the route of our previous work [Step 1, Lemma 7.1, [23]]. Moreover, the argument of [Step 1, Lemma 7.1, [23]] and Lemma [7.2] in fact yield the following refined bounds

$$2^\frac{d}{2} k^{k-\sigma k} \| P_k v \|_{L_t^1 L^\infty} \lesssim 2^{-\sigma k} \tilde{h}_k(\sigma)(1 + 2^k s)^{-L} \quad (4.33)$$

$$\| P_k \partial_v v \|_{L_t^1 L^\infty} \lesssim 2^{-\sigma k} (1 + 2^k s)^{-L} 2^{2k - \frac{d}{2} k} \tilde{h}_k(\sigma). \quad (4.34)$$

where $\{\tilde{h}_k(\sigma)\}$ are defined by

$$\tilde{h}_k(\sigma) = \sup_{k \in \mathbb{Z}} 2^{-\delta k - k'} 2^\frac{d}{2} k^{k-\sigma k} \| P_k v \|_{L_t^1 L^\infty}.$$

And \((4.32)\) shows for $\sigma \in [0, \vartheta]$

$$\tilde{h}_k(\sigma) \lesssim h_k(\sigma). \quad (4.35)$$

**Step 1.2.** Second, we transfer \((4.34)\) to bounds for $\psi_v$. Similar to \((4.30)\), one has

$$2^\frac{d}{2} k^{k-\sigma k} \| P_k (D^s P(e; e) - \Lambda^\infty) \|_{L_t^1 L^\infty} \leq \epsilon. \quad (4.36)$$

Then applying \((4.36), (4.34)\) and bilinear estimates \((4.28)-(4.29)\) to $\partial_v v (D^s P(e) - \chi^\infty)$ gives

$$2^\frac{d}{2} k^{k-\sigma k} \| P_k (d^s L^\infty) \|_{L_t^1 L^\infty} \leq 2^{-\sigma k} h_k(\sigma) \quad (4.37)$$

for $\sigma \in [0, \vartheta]$. Thus using bilinear estimates \((4.28)-(4.29)\) to control $\partial_v v (d^s P(e) - \chi^\infty)$, we conclude by \((4.31), (4.37)\) that

$$(1 + s 2^{2k}) \| P_k \psi_v \|_{L_t^1 L^\infty} \lesssim 2^\frac{d}{2} k^{k-\sigma k} 2^{-\sigma k} h_k(\sigma). \quad (4.38)$$

for all $\sigma \in [0, \vartheta]$.
Step 1.3. Third, we calculate \(\| P_k(dP(e) - \chi^\infty) \|_{L^p_t L^q_x} \). Recall the formula
\[
dP(e) - \chi^\infty = \int_s^\infty (d\,dP)(e; e) \psi_s ds' \tag{4.39}
\]
\[
(d\,dP)(e; e) - \Lambda^\infty = \int_s^\infty (D^2\,dP)(e; e) \psi_s ds' \tag{4.40}
\]
where \(\Lambda^\infty\) denotes the limit at \(s \to \infty\). And we have the \(L^\infty_t L^2_x\) bounds for \((D^2\,dP)(e; e; e)\)
\[
\| \partial_x (D^2\,dP(e; e; e)) \|_{L^\infty_t H^2_x} \lesssim \epsilon 2^{-\frac{4}{3}s} \tag{4.41}
\]
Then applying bilinear estimates \((4.28)-(4.29)\) to \((4.40)\) with \((4.38), (4.41)\) gives
\[
\| P_k(dP(e; e) - \Lambda^\infty) \|_{L^\infty_t L^2_x} \lesssim 2^{-\frac{4}{3}h_k} 2^{-\sigma_k} (1 + s^{2k})^{-L} \tag{4.42}
\]
Thus we obtain by applying \((4.28)-(4.29)\) to \((4.39)\) that
\[
\| P_k(dP(e) - \chi^\infty) \|_{L^\infty_t L^2_x} \lesssim (1 + s^{2k})^{-\frac{4}{3}h_k} 2^{-\sigma_k} \tag{4.43}
\]
which particularly yields \((4.22)\).

Step 2. Proof of \((4.50)-(4.51)\). The same arguments of Step 1.3 give \((4.50)\) and \((4.51)\) by using \((4.33)\) and \((4.22)\) that
\[
\| G' \|_{L^\infty_t L^2_x} \lesssim \epsilon (1 + s^{2k})^{-L} \tag{4.44}
\]

Step 3. Proof of \((4.25)\). Lastly, we prove \((4.25)\). Applying bilinear Littlewood-Paley decomposition to \(\partial_t vP(e) = \psi_t\), we obtain by \((4.33)\) and \((4.22)\) that
\[
2^{\frac{4}{3}k} \| P_k \psi_x \|_{L^p_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma) (1 + 2^{2k}s)^{-L} \tag{4.45}
\]
Recall the schematic formula \((3.66)\) of \(A_x\) and the first order decomposition of \(G\):
\[
[A_i]_q^p(s) = \int_s^\infty (\psi_s \circ \psi_x) R(e_{t_0}, e_{t_1}, e_{t_2}, e_{t_3}) ds' \tag{4.46}
\]
\[ G := R(e_0, e_1, e_2, e_3) = \Gamma^\infty + \mathcal{G} \]  
\[ = \Gamma^\infty + \Xi^\infty \int_s^\infty \psi_s ds' + \int_s^\infty \psi_s \mathcal{G}' ds' \]  
(4.45)  
(4.46)

By (4.50) and (4.38), the curvature term \( \mathcal{G} \) satisfies

\[ 2^{\frac{d}{2k}} \| P_k \mathcal{G} \|_{L^p_{t,x} \cap L^\infty_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma)(1 + 2^{2k}s)^{-L}. \]  
(4.47)

Thus by (4.38), (4.43) and (4.44), we get

\[ \| P_k A_x \|_{L^p_{t,x} \cap L^\infty_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma) \]  
(4.48)

where \( h_{k,s}(\sigma) \) is defined by

\[ h_{k,s}(\sigma) = \left\{ \begin{array}{ll}
-\sum_{-j \leq l \leq k} h_l(\sigma) h_l, & s \in [2^{2j-1}, 2^{2j+1}], k + j \leq 0 \\
2^{j} h_{-j} h_{k}(\sigma), & s \in [2^{2j-1}, 2^{2j+1}], k + j \geq 0 
\end{array} \right. \]  
(4.49)

4.2 Differential fields with respect to \( t \) variable

Proposition 4.1 has tracked the bounds of \( \mathcal{G}', \mathcal{G}'' \) in \( L^p_{t,x} \cap L^\infty_t L^2_x \) along the heat flow direction. The left unknown block spaces of \( F_k \) is \( L^p_{t,x} L^\infty_t \).

First, we reduce the estimate of \( L^p_{t,x} L^\infty_t \) norms to the space \( L^p_{t,x} L^\infty_t \) which is more flexible when handling with geometric quantities.

Lemma 4.1. If we have verified that for all \( L \in \mathbb{N} \), \( f = \mathcal{G}', \mathcal{G}'' \) there holds

\[ 2^{\frac{d}{2k}} \| P_k f \|_{L^p_{t,x} \cap L^\infty_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma)(1 + 2^{2k}s)^{-L} \]  
(4.50)

\[ 2^{\frac{d}{2k}} \| \partial_t P_k f \|_{L^p_{t,x} \cap L^\infty_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma). \]  
(4.51)

Then, consequently we have

\[ 2^{-\frac{d}{2k}} \| \partial_x^L P_k f \|_{L^p_{t,x} \cap L^\infty_t L^2_x} \lesssim 2^{-\sigma_k} h_k(\sigma)(1 + 2^{2k}s)^{-L} \]  
(4.52)

for all \( L \in \mathbb{N} \) and \( f = \mathcal{G}', \mathcal{G}'' \).

Proof. By Gagliardo-Nirenberg inequality and Hölder inequality, one has

\[ \| f \|_{L^p_{t,x} \cap L^\infty_t} \lesssim \| f \|_{L^p_{t,x}} \| \partial_t f \|_{L^p_{t,x}}, \]  
(4.53)

where \( \frac{1}{p} + \frac{1}{p_d} = 1 \). Then the Lemma follows directly. \( \Box \)

Let \( \sigma \in [0, \vartheta] \), (4.50) has been verified in Proposition 4.1. Thus Lemma 4.1 reduces the problem to prove (4.51). We recall the following result.
Lemma 4.2. Let \( d \geq 3 \). Assume that \( u \) is a solution to SL given in Proposition 4.1. Let (4.24) hold. Assume in addition that

\[
\|P_k(A_x)\|_{F_k} \lesssim 2^{-\frac{d}{2}k+2k - \sigma}h_k(\sigma)(1+2^{2k}s)^{-4} \tag{4.54}
\]

\[
\|P_k(\mathcal{G})\|_{F_k} \lesssim 2^{-\frac{d}{2}k+2 - \sigma}h_k(\sigma)(1+2^{2k}s)^{-4} \tag{4.55}
\]

Then the corresponding differential field \( \phi_t \) and connection coefficient \( A_t \) satisfy

\[
\|P_k\phi_t\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{2}k+2k - \sigma}h_k(\sigma)(1+2^{2k})^{-2}. \tag{4.56}
\]

\[
\|P_k(A_t)\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{2}k+2k - \sigma}h_k(\sigma)(1+s2^{2k})^{-1}. \tag{4.57}
\]

Proof. The proof of [Lemma 5.6, 3] and [Section 3, 23] reveal that (4.54)-(4.55) imply

\[
\|P_kA_x(\mid s=0)\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{2}k+2k - \sigma}h_k(\sigma). \tag{4.58}
\]

And the proof of [Lemma 5.6, 3] and [Section 3, 23] reveal that (4.56) is a corollary of (4.54)-(4.55) if one has obtained

\[
\|P_k\phi_t(\mid s=0)\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{2}k+2k - \sigma}h_k(\sigma). \tag{4.59}
\]

Let us verify (4.59). When \( s=0 \), \( \phi_t = \sqrt{-1}D_t\phi_t \). Using (4.58) to bound \( \|P_k(A_t\phi_t)(\mid s=0)\|_{L^p_{t,x}} \), we obtain (4.59) by bilinear Littlewood-Paley decomposition. Thus (4.56) is done.

Now let us prove (4.57). Recall the formula

\[
A_t(s) = \Gamma^\infty \int_s^\infty \phi_s \circ \phi_t ds' + \int_s^\infty (\phi_s \circ \phi_t)\mathcal{G} ds'. \tag{4.60}
\]

We have seen for \( d \geq 3 \) in (4.38)

\[
2^{-\frac{d}{2}k-2k} \|P_k\phi_s\|_{L^\infty_t L^2_x} \lesssim 2^{-\sigma}h_k(\sigma)(1+s2^{2k})^{-1}. \tag{4.61}
\]

Since \( d \geq 3 \), bilinear Littlewood-Paley decomposition gives

\[
\|P_k(\phi_s \circ \phi_t)\|_{L^p_{t,x}} \lesssim \sum_{k_1 \leq k-4, |k_2-k| \leq 4} 2^{\frac{d}{2}k_1} \|P_{k_1}\phi_s\|_{L^\infty_t L^2_x} \|P_{k_2}\phi_t\|_{L^p_{t,x}}
\]

\[
+ \sum_{k_2 \leq k-4, |k_1-k| \leq 4} 2^{d(\frac{1}{2} - \frac{1}{p_2})} 2^{\frac{d}{2}k_1} \|P_{k_1}\phi_s\|_{L^\infty_t L^2_x} 2^{\frac{d}{2}k_2} \|P_{k_2}\phi_t\|_{L^p_{t,x}}
\]

\[
+ \sum_{k_1, k_2 \leq k-4, |k_1-k_2| \leq 8} 2^{\frac{d}{2}k} \|P_{k_1}\phi_s\|_{L^\infty_t L^2_x} \|P_{k_2}\phi_t\|_{L^p_{t,x}}
\]
Proposition 4.2. Let \( s > \frac{3}{4} \) given in Proposition 4.1. Then we have for all \( d \leq 3 \),

\[
\leq 1_{k+j \geq 0} 2^{-\frac{d}{2} + 2k} (1 + s 2^{2k})^{-2 - \sigma k} \tilde{h}_k(\sigma) + 1_{k+j \leq 0} 2^{d-j} 2^{\frac{d}{2} + 2 - \sigma k} \tilde{h}_{-j}(\sigma) h_{-j}.
\]  

(4.62)

By \( d \geq 3 \), integrating (4.62) in \( s' \geq [s, \infty) \) yields

\[
\int_s^\infty \| P_k(\phi_s \circ \phi_t) \|_{L^\infty L^2_z \cap L^p_t} ds' \leq 2^{-\frac{d}{2} + 2k} (1 + s 2^{2k})^{-1 - 2 - \sigma k} \tilde{h}_k(\sigma),
\]  

(4.63)

by which the first term in the RHS of (4.60) is done. Moreover, (4.62), (4.55) and bilinear Littlewood-Paley decomposition lead to

\[
\| P_k(\phi_s \circ \phi_t) \mathcal{G} \|_{L^p_t L^\infty} \leq 1_{k+j \geq 0} \tilde{h}_k(\sigma) 2^{-\frac{d}{4} + 2k} (1 + s 2^{2k})^{-2 - \sigma k} \tilde{h}_k(\sigma) h_{-j}.
\]  

(4.64)

Since \( d \geq 3 \), integrating (4.64) in \( s' \geq [s, \infty) \) gives the second term in the RHS of (4.60). Thus (4.57) is done.

The following is the main result for this section.

**Proposition 4.2.** Let \( d \geq 3 \), \( \sigma \in [0, \theta] \). Assume that \( u \) is a solution to SL given in Proposition 4.1. Then we have for all \( s > \infty \), \( L \in [0, 10^{10} - 1] \),

\[
\| P_k(\mathcal{G}') \|_{L^\infty L^2_z \cap L^p_t} \leq 2^{-\frac{d}{4} + 2k} 2^{-\sigma k} \tilde{h}_k(\sigma) (1 + s 2^{2k})^{-L}.
\]  

(4.65)

\[
2^{-\frac{d}{4} + 2k} \| P_k(\mathcal{G}') \|_{L^p_t L^\infty} \leq 2^{-\frac{d}{4} + 2k} 2^{-\sigma k} \tilde{h}_k(\sigma) (1 + s 2^{2k})^{-L}.
\]  

(4.66)

In fact, (4.65)-(4.66) hold for all \( \{\mathcal{G}^{(j)}\}_{j=0}^\infty \), where we denote

\[
\mathcal{G}^{(j)} = (\nabla^j \mathbf{R})(e, \ldots, e; e, \ldots, e) - \text{limit.}
\]  

(4.67)

**Proof.** (4.65) has been proved in Proposition 4.1. By Lemma 4.1 it suffices to prove (4.51). We have

\[
\partial_t \mathcal{G}' = \nabla^2 \mathbf{R} (\partial_t v, e_x; e_x, \ldots, e_x) + \sum_{j_i} \nabla \nabla \left( \nabla^j e_x; \nabla^j e_x, \ldots, \nabla^j e_x \right).
\]

Schematically we write

\[
\partial_t \mathcal{G}' = \phi_t \mathcal{G}'' + A_t \mathcal{G}'.
\]  

(4.68)

Since \( d \geq 3 \), applying bilinear Littlewood-Paley decomposition to (4.68) we obtain from (4.65) and Lemma 4.2 that

\[
\| P_k(\phi_t \mathcal{G}') \|_{L^p_t L^\infty} \leq 2^{-\frac{d}{4} + 2k} 2^{-\sigma k} \tilde{h}_k(\sigma) (1 + 2^{2k} s)^{-1},
\]  

(4.69)

by which (4.51) follows. Thus the proof is completed. Lastly, we observe that (4.67) holds for all \( j \) by repeating the previous arguments.
5 Proof of Theorem 1.2 for $d \geq 3$.

5.1 Before iteration

As mentioned in Section 2, the key estimates are the $F_k \cap S^\frac{1}{2}$ norm of $A_x$ along the heat direction (see (1.21)) and the $F_k$ norm of $\tilde{G}$ along the heat direction.

Recall the expression for $\tilde{G}$:

$$\tilde{G} = G - \Gamma_\infty = \Xi_\infty \int_\infty^s (\partial_i \phi_i) ds' + \Xi_\infty \int_s^\infty (A_i \phi_i) ds' + \int_s^\infty (\partial \phi_i) G' ds' + \int_s^\infty (A_i \phi_i) G' ds'$$

$$:= U_0 + U_1 \tag{5.3}$$

where we denote $U_0$ the last RHS of (5.1) and $U_1$ the RHS of (5.2).

Lemma 5.1. Let $u$ be solution to $SL$ in $C([-T,T]; Q(\mathbb{R}^d, N))$. And let $\{h_k(\sigma)\}$ be frequency envelope such that

$$2^{d\frac{k}{2} - 1} \|\phi_x(s)\|_{F_k} \lesssim 2^{-\sigma k} h_k(\sigma)(1 + s2^{2k})^{-4}. \tag{5.4}$$

Suppose that

$$\|h_k(0)\|_{L^2} \leq \epsilon \ll 1. \tag{5.5}$$

Moreover, we assume that

$$2^{d \frac{k}{2}} \|P_k A_x\|_{F_k \cap S^\frac{1}{2}} \lesssim (1 + s2^{k})^{-4} 2^{\sigma k} h_k(\sigma). \tag{5.6}$$

Then we have

$$2^{d \frac{k}{2} - k} \|P_k \tilde{G}\|_{F_k} \lesssim (1 + s2^{2k})^{-4} 2^{-\sigma k} h_{k,s}(\sigma). \tag{5.7}$$

where $h_{k,s}(\sigma)$ is defined by (4.49).

Proof. Since (5.4) dominates (5.1), and (5.6) bounds (5.2), we get

$$2^{d \frac{k}{2}} \|P_k \tilde{G}\|_{F_k} \leq (1 + s2^{2k})^{-4} 2^{-\sigma k} h_k(\sigma). \tag{5.8}$$

Let $B \geq 1$ denote the smallest constant such that

$$(1 + s2^{2k})^4 2^{d \frac{k}{2} - k} \|P_k A_x\|_{F_k \cap S^\frac{1}{2}} \leq B 2^{-\sigma k} h_{k,s}(\sigma), \tag{5.9}$$

for all $\sigma \in [0, \vartheta], k \in \mathbb{Z}, s > \infty$. Then one can check

$$2^{d \frac{k}{2} + \sigma k} \int_s^\infty \|P_k[(\phi_x \circ \phi_x) \tilde{G}]\|_{F_k \cap S^\frac{1}{2}} ds' \lesssim (B\epsilon + 1)h_{k,s}(\sigma). \tag{5.10}$$
It is now standard to derive (5.10) from (5.8) and (5.9), see our previous paper ([23], Lemma 3.1, Step 1). Therefore, we obtain

\[ B \lesssim 1 + \epsilon B, \]

by which (5.7) follows. \( \square \)

We now prove a stronger estimate of (5.6) which by bootstrap implies (5.7) without assumption (5.6).

**Lemma 5.2.** Let \( u \) be solution of SL satisfying Lemma 5.1. Then for all \( k \in \mathbb{Z} \) we have

\[ \| P_k U_1 \|_{F_k} \lesssim \epsilon 2^{-\frac{k}{4}k^2 - \sigma k} h_k(\sigma)(1 + s2^{2k})^{-4}. \]  

(5.11)

**Proof.** Recall that \( F_k \rightarrow L^\infty_{u, t} \cap L_\infty L^2 \) in \( L^2_t \). Let \( d \geq 3 \), then Lemma 5.1 and (4.8) show that for \( u \) in Lemma 5.2 the assumptions (4.53)-(4.55) of Lemma 4.2 hold. Thus by Proposition 4.2 we obtain (4.65)-(4.66). 

**Bilinear estimates.** We will use Lemma 7.1 to do bilinear estimates. Assume that \( s \in [2^{2j_0} - 1, 2^{2j_0 + 1}] \). We prove the lemma according to \( k + j_0 \geq 0 \) or \( k + j_0 \leq 0 \). For \( s' \in [2^{2j} - 1, 2^{2j+1}] \), assumption (5.4) and Lemma 5.1 give

\[ \| A_i \phi_i \|_{F_k} \lesssim 2^{k - \frac{k}{4}k^2 - \sigma k} 2^{-\frac{k}{4}k^2 - \sigma k} h_k(\sigma) \]  

if \( k + j \leq 0 \)  

(5.12)

\[ \| A_i \phi_i \|_{F_k} \lesssim 2^{k - \frac{k}{4}k^2 - \sigma k} (1 + 2^{2k+2j})^{-4} h_k(\sigma) \]  

if \( k + j \geq 0 \).  

(5.13)

Thus (5.1) is done by integrating (5.12), (5.13) w.r.t. \( j \geq j_0 \).

For (5.2), we apply Lemma 7.1. We take the more complex term \( \int_0^\infty (A_i \phi_i) G' ds' \) of (5.2) as the candidate, the \( \partial_t \phi_i \) term is easier. Using Lemma 7.1 the High \( \times \) Low interaction of \( (A_i \phi_i) G' \) is dominated by

\[ \sum_{|k_1 - k| \leq 4} \| P_k (P_{k_1} (A_i \phi_i) P_{\leq k-4} G') \|_{F_k} \lesssim \| P_k (A_i \phi_i) \|_{F_k}. \]  

(5.14)

Thus the High \( \times \) Low part is done by (5.12), (5.13).

Now let us consider the High \( \times \) High part of \( (A_i \phi_i) G' \). By Lemma 7.1 and (4.63),

\[ \sum_{|k_1 - k_2| \leq 8, k_1, k_2 \geq k-4} \| P_k (P_{k_1} (A_i \phi_i) P_{k_2} G') \|_{F_k} \]  

\[ \lesssim \sum_{|k_1 - k_2| \leq 8, k_1, k_2 \geq k-4} (2^{\frac{d}{4}2(k_1 - k)} + 2^{\frac{d}{4}2(k_1 - k)}) \| P_{k_1} (A_i \phi_i) \|_{F_k} \| P_{k_2} G' \|_{L^\infty} \]  

\[ \lesssim \sum_{k_1, k_2 \geq k-4} (2^{\frac{d}{4}2(k_1 - k)} + 2^{\frac{d}{4}2(k_1 - k)}) \| P_{k_1} (A_i \phi_i) \|_{F_{k_1}} h_{k_1}(1 + 2^{2k_1+2j})^{-20}. \]  

(5.15)
Applying (5.13), for \( k + j_0 \geq 0 \), we get (5.15) is bounded by
\[
\sum_{k_1 \geq k-4} (2\frac{\pi}{2}(k_1-k) + 2\frac{d+1}{2}(k_1-k))2^{2k_1 - \frac{d}{2}k_1 - \frac{1}{2} - \sigma_1} (1 + 2^{2k_1+2j})^{-24} h_{-j}^2 h_{k_1} h_{k_1}(\sigma)
\lesssim 2^{2k - \frac{d}{2}k - \sigma k} (1 + 2^{2k+2j})^{-20} h_{k}^3 h_{k}(\sigma).
\]
Thus the High × High part for \( k + j_0 \geq 0 \) is done. If \( k + j \leq 0 \), (5.15) is dominated by
\[
\sum_{k_1 \geq -j} \left[ 2\frac{\pi}{2}(k_1-k) + 2\frac{d+1}{2}(k_1-k) \right] 2^{2k_1 - \frac{d}{2}k_1 - \frac{1}{2} - \sigma_1} (1 + 2^{2k_1+2j})^{-24} h_{-j}^2 h_{k_1} h_{k_1}(\sigma)
+ \sum_{k_1 \leq k \leq -j} \left[ 2\frac{\pi}{2}(k_1-k) + 2\frac{d+1}{2}(k_1-k) \right] 2^{-\frac{d+1}{2}k_1} 2^{2k_1 - \frac{d}{2}k_1} h_{-j}^2 h_{k} h_{k}(\sigma)
\lesssim 2^{2k - \frac{d}{2}k - \sigma k} h_{k}^2 h_{k}(\sigma) + 2^{-\sigma k} h_{-j}^2 h_{-j}(\sigma) F_{j,k}(d)
\]
where \( F_{j,k}(d) \) is defined by
\[
F_{j,k}(d) = \begin{cases} 
2\frac{\pi}{2}(-j-k)2^{-2j} + 2 \frac{d+1}{2} 2^{-\frac{d}{2}k} & d = 3, 4 \\
2^{-\frac{d+1}{2}k} 2^{2k - \frac{d}{2}k} (1 + 2^{2j+2k})^{-20} & d \geq 5
\end{cases}
\]
Then if \( k + j_0 \leq 0 \), we have
\[
\int_s^\infty \| P_k^{h_k} (A_i \phi_i) G' \| F_k \, ds
\lesssim \sum_{j \geq -k} 2^{2j} 2^{2k - \frac{d}{2}k} h_{k}^2 h_{k}(\sigma)(1 + 2^{2j+2k})^{-20} + \sum_{j_0 \leq -k} 2^{2j} 2^{-\sigma k} h_{-j}^2 h_{-j}(\sigma) 2^{-\sigma k} F_{j,k}(d)
\lesssim 2^{-\sigma k} 2^{-\frac{d}{2}k} h_{k}^2 h_{k}(\sigma).
\]
Hence the High × High part for all \( s > 0 \) is done.

Now let us consider the Low × High part of \( (A_i \phi_i) G' \). By Lemma 7.1
(4.63) - (4.66)
\[
\sum_{|k_2 - k| \leq 4, k_1 \leq -4} \| P_k (P_{k_1} (A_i \phi_i) P_{k_2} G') \| F_k
\lesssim \sum_{|k_1 - k| \leq 4, k_1 \leq -4} 2^{\frac{d+1}{2}(k_1-k)} \| P_{k_1} (A_i \phi_i) \| F_{k_1} \| P_{k_2} G' \| L_\infty + \| P_{k_1} (A_i \phi_i) \| L_\infty \| P_{k_2} G' \| L_\infty^{d+1, L_\infty}
\lesssim \sum_{|k_1 - k| \leq 4, k_1 \leq -4} 2^{-\sigma k} h_{k}(\sigma) \left( 2^{\frac{d+1}{2}(k_1-k)} \| P_{k_1} (A_i \phi_i) \| F_{k_1} + 2^{-\frac{d}{2}k_1} 2^{\frac{d}{2}k_1} \| P_{k_1} (A_i \phi_i) \| F_{k_1} \right).
\]
(5.16)
Then by (5.12), for \( k + j_0 \geq 0 \) one has (5.16) is dominated by
\[
(1 + 2^{2k+2j})^{-10} 2^{-\sigma k} h_{k}(\sigma) \sum_{-j \leq k \leq -4} \left( 2^{\frac{d+1}{2}(k_1-k)} 2^{-\frac{d}{2}k_1+2k} + 2^{-\frac{d}{2}k_1} 2^{2k_1} \right) h_{-j}^2 h_{k_1},
\]
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Hence, the Low × \text{k} (5.5), (5.4). Then for all Corollary 5.1. Let \( u \) \( u \), the connection coefficients satisfy

\[
\text{Define the function } \Phi : [0, \infty) \rightarrow \mathbb{R} \text{ by }
\Phi(T) := \sup_{T' \in [0,T]} \sup_{s > 0, k \in \mathbb{Z}} \left| 2^{\frac{s}{2} + \frac{1}{2}} (k_{1} - k) 2^{-\frac{3}{4}k_{1} + \frac{1}{4}k_{2}} + 2^{-\frac{3}{2}k_{2}} \right| \left( P_{k}(\sigma) \right) \left( 1 + s 2^{k} \right)^{-4}. \]

Moreover, the connection coefficients satisfy

\[
\left| P_{k} U_{1} \right|_{L_{x}^{2}} \leq 2^{-\frac{s}{2} + \frac{1}{2}} \left( k_{1} - k \right) 2^{-\frac{3}{4}k_{1} + \frac{1}{4}k_{2}} \left( 1 + s 2^{k} \right)^{-4}. \]

Proof. Define the function \( \Phi : [0, T_{*}] \rightarrow \mathbb{R} \) by

\[
\Phi(T) := \sup_{T' \in [0,T]} \sup_{s > 0, k \in \mathbb{Z}} \left| 2^{\frac{s}{2} + \frac{1}{2}} (k_{1} - k) 2^{-\frac{3}{4}k_{1} + \frac{1}{4}k_{2}} + 2^{-\frac{3}{2}k_{2}} \right| \left( P_{k}(\sigma) \right) \left( 1 + s 2^{k} \right)^{-4}. \]

Let \( u(s, x) \) denote the solution to the heat flow equation with initial data \( u_{0} \). By the relation \( F_{k} \downarrow L_{x}^{2} L_{t}^{\infty} \cap L^{p,t} \), we see

\[
\lim_{T \downarrow 0} \Phi(T) = \sup_{s > 0, k \in \mathbb{Z}} 2^{\frac{s}{2} + \frac{1}{2}} (k_{1} - k) 2^{-\frac{3}{4}k_{1} + \frac{1}{4}k_{2}} \left( 1 + s 2^{k} \right)^{-4}. \]
where $U_1$ is defined by

$$U_1 = \int_s^\infty \phi_s G' ds', \quad (5.21)$$

with all values taken at the point $u(s', x)$ in the above integral. We have seen in the proof of Proposition 4.1 that the RHS of (5.20) is controlled by $\epsilon_1$. Thus we get

$$\lim_{T \downarrow 0} \Phi(T) \lesssim \epsilon_1. \quad (5.22)$$

And we have seen (5.6) $\Rightarrow$ (5.11) in Lemma 5.2. Hence there holds

$$\Phi(T) \lesssim 1 \Rightarrow \Phi(T) \lesssim \epsilon_1. \quad (5.23)$$

By bootstrap, for all $T \in [0, T^*)$ we conclude

$$\Phi(T) \lesssim \epsilon_1. \quad (5.24)$$

Then the bound (5.17) for $\tilde{\mathcal{G}}$ follows by adding the $\|U_0\|_{F_k}$ part.

The connection bound (5.18) suffices to bound the evolution of $\phi_{x,t}$ along the heat direction. And for $s = 0$, (5.18) suffices to control the evolution of $\phi_x$ along the Schrödinger direction. We omit the details for this part in high dimensions since it is relatively easy to supplement them following our previous work [Section 3, 4, 5, 23]. In fact, it suffices to bound the cubic terms of the form

$$\phi_{\mu} \phi_{\nu} \tilde{\mathcal{G}} \quad (5.25)$$

in the $F_k, L^p$ and $N_k$ spaces.

Then one can get

**Proposition 5.1.** Let $\sigma \in [0, \vartheta]$ and $\epsilon_0$ be a sufficiently small constant. Let $u \in C([-T, T]; \mathcal{Q}(\mathbb{R}^d, \mathcal{N}))$ is the solution to SL with initial data $u_0$. Let $\{c_k\}$ be an $\epsilon_0$-frequency envelope of order $\frac{1}{8} \delta$ with $0 < \epsilon_0 \leq \epsilon_*$. And let $\{c_k(\sigma)\}$ be another frequency envelope of order $\frac{1}{8} \delta$ for which

$$2^{\frac{d}{8}k} \|P_k u_0\|_{L^2_x} \leq c_k \quad (5.25)$$

$$2^{\frac{d}{8}k} \|P_k u_0\|_{L^2_x} \leq c_k(\sigma) 2^{-\sigma k} \quad (5.26)$$

Denote $\{\phi_i\}_{i=1}^d$ the corresponding differential fields of the heat flow initiated from $u$. Suppose also that at the heat initial time $s = 0$,

$$\sum_{i=1}^d 2^{\frac{d}{8}k-\frac{k}{2}} \|P_k \phi_i\|_{G_k(T)} \leq \epsilon_0 \frac{1}{2} c_k. \quad (5.27)$$

Then when $s = 0$, we have for all $i = 1, \ldots, d$, $k \in \mathbb{Z}$,

$$2^{\frac{d}{8}k-\frac{k}{2}} \|P_k \phi_i\|_{G_k(T)} \lesssim c_k \quad (5.28)$$

$$2^{\frac{d}{8}k-\frac{k}{2}} \|P_k \phi_i\|_{G_k(T)} \lesssim c_k(\sigma) 2^{-\sigma k}. \quad (5.29)$$

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Proof. By bootstrap assumption (5.27) and the fact \( \{c_k\} \) is an \( \epsilon_0 \)-frequency envelope, we see the frequency envelope

\[
b_k(\sigma) = \sup_{k' \in \mathbb{Z}} 2^{-\delta|k-k'|}2^{\frac{d}{2}k'-k'}2^{\sigma k'}\|P_{k'}\phi_x(\mid s=0)\|_{G_k'}
\]

satisfies

\[
\sum_{k \in \mathbb{Z}} b_k^2 \leq \epsilon_0.
\]  

(5.31)

Thus by \( G_k \hookrightarrow F_k \), we see that the assumptions (5.5), (5.4) hold. Then applying Corollary 5.1 gives at \( s = 0 \)

\[
\|P_k A_x(\mid s=0)\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{4}k-\epsilon k} c_k(\sigma) \tag{5.32}
\]

\[
\|P_k A_t(\mid s=0)\|_{L^2_{t,x}} \lesssim 2^{-\frac{d}{4}k+2\epsilon k-\sigma k} c_k(\sigma) \tag{5.33}
\]

\[
\|P_k \phi_i(\mid s=0)\|_{L^2_{t,x}} \lesssim 2^{-\frac{d}{4}k+2\epsilon k-\sigma k} c_k(\sigma). \tag{5.34}
\]

Using the evolution equation of \( \phi_x \) along the Schrödinger direction and argument of our previous work [Section 5, [23]], one can prove

\[
b_k(\sigma) \lesssim c_k(\sigma) + \epsilon b_k(\sigma). \tag{5.35}
\]

Thus (5.28) and (5.29) are proved. \( \square \)

5.2 Iteration for once

Proposition 5.2. Given \( \sigma \in [1, 2\theta] \). Let \( \epsilon_0 \) be a sufficiently small constant. Let \( u \in C([-T,T]; Q(\mathbb{R}^d, N)) \) be the solution to SL with initial data \( u_0 \). Let \( \{c_k\} \) be an \( \epsilon_0 \)-frequency envelope of order \( \frac{1}{16}\delta \) with \( 0 < \epsilon_0 \leq \epsilon_* \). And let \( \{c_k(\sigma)\} \) be another frequency envelope of order \( \frac{1}{16}\delta \) which satisfies

\[
2^{\frac{d}{4}k} \|P_k u_0\|_{L^2_t} \leq c_k
\]

\[
2^{\frac{d}{4}k} \|P_k u_0\|_{L^2_t} \leq 2^{-\sigma k} c_k(\sigma)
\]

Denote \( \{\phi_i\}_{i=1}^d \) the corresponding differential fields of the heat flow initiated from \( u \). Suppose also that at the heat initial time \( s = 0 \),

\[
2^{\frac{d}{4}k-\epsilon k}\|P_k \phi_i(s=0)\|_{G_k(T)} \leq \epsilon_0^{\frac{1}{4}} c_k. \tag{5.36}
\]

Then when \( s = 0 \), we have for all \( i = 1, \ldots, d, k \in \mathbb{Z} \),

\[
2^{\frac{d}{4}k-\epsilon k}\|P_k \phi_i\|_{G_k(T)} \lesssim c_k \tag{5.37}
\]

\[
2^{\frac{d}{4}k-\epsilon k}\|P_k \phi_i\|_{G_k(T)} \lesssim 2^{-\sigma k}[c_k(\sigma) + c_k(\sigma - 1)c_k(1)]. \tag{5.38}
\]
Proof. The key point and the engine for our iteration is the estimates of \( \partial_t v \). Thus we begin with improving \( \partial_t v \) in Proposition 4.1 (see (4.32)).

Let \( \sigma \in [1, 2] \). Applying Proposition 5.1 with \( \sigma_0 \in [0, 1] \), we have seen

\[
2^\frac{d}{2} k^{-k} \| P_k \phi_x(s = 0) \|_{F_k} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)
\]

(5.39)

\[
2^\frac{d}{2} k^{-k} \| P_k \phi_x(s) \|_{F_k} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)(1 + 2^{2k})^{-1},
\]

(5.40)

which combined with Proposition 4.1 gives

\[
2^\frac{d}{2} k \| P_k (dP(e) - \chi^\infty) \|_{L^p_{t,x}} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)(1 + s2^{2k})^{-1} L
\]

(5.41)

\[
2^\frac{d}{2} k \| P_k v \|_{L^p_{t,x}} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)(1 + 2^{2k})^{-1} L
\]

(5.42)

\[
\| P_k (S^{(1)}(v) - S^{(1)}(\varphi)) \|_{L^p_{t,x}} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)(1 + 2^{2k})^{-1} L
\]

(5.43)

\[
2^\frac{d}{2} k - 2k \| P_k \partial_t v(s) \|_{L^p_{t,x}} \lesssim 2^{-\sigma_0 k} c_k(\sigma_0)(1 + 2^{2k})^{-1} L.
\]

(5.44)

Define the frequency envelope \( \{ b_k(\sigma) \} \) as [5.30] with \( \sigma \in [0, 2] \). Then by Proposition 5.1

\[
b_k(\sigma_0) \lesssim c_k(\sigma_0), \forall \sigma_0 \in [0, \vartheta].
\]

(5.45)

Then by definition of \( b_k(\sigma) \) and (5.41) we infer from bilinear Littlewood-Paley decomposition that

\[
2^\frac{d}{2} k - k \| P_k (\partial_t v) \|_{L^\infty_t L_x^2 \cap L^p_{t,x}} \lesssim 2^{-\sigma} b_k(\sigma) + 2^{-\sigma} b_k(1)c_k(\sigma - 1).
\]

(5.46)

Let \( J_1(s) \) be the positive continuous function defined on \([0, \infty)\) via

\[
J_1(s) = \sup_{k \in \mathbb{Z}, \varrho \in [0, s]} 2^\frac{d}{2} \varrho^k \| P_k v \|_{L^\infty_t L_x^2 \cap L^p_{t,x}} 1/b_k^{(1)}(\sigma)
\]

(5.47)

where \( b_k^{(1)}(\sigma) \) is defined by

\[
b_k^{(1)}(\sigma) = \begin{cases} b_k(\sigma), & \sigma \in [0, \vartheta] \\ b_k(\sigma) + 2^{-\sigma} c_k(1)c_k(\sigma - 1) & \sigma \in [1, 2\vartheta] \end{cases}
\]

(5.48)

Then by (5.40) and (5.45), we see

\[
\lim_{s \to 0} J_1(s) \lesssim 1.
\]

(5.49)

By Duhamel principle for the heat flow equation, we get

\[
\| P_k v \|_{L^\infty_t L_x^2 \cap L^p_{t,x}} \lesssim e^{-c(d)2^k} \| P_k v_0 \|_{L^\infty_t L_x^2 \cap L^p_{t,x}}
\]

\[
+ \int_0^s e^{-c(d)(s-t)2^k} \| P_k (S(v)(\partial_t v, \partial_x v)) \|_{L^\infty_t L_x^2 \cap L^p_{t,x}}.
\]

(5.50)
By Lemma 7.3, (5.43) and the definition of $b_k^{(1)}(\sigma)$, (5.50) is dominated by

$$2^{\frac{d}{k}}\|P_k(S(v)(\partial_x v, \partial_x v))\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} \lesssim j_1^2(s)2^{-\sigma k+2b_k^{(1)}(\sigma)}. \quad (5.51)$$

for all $\sigma \in [0,2\theta]$, $d \geq 3$. We remark that since (5.43) only reaches $\sigma_0 \in [0,1]$, one needs to gain $c_k(\theta)$ from $\partial_x v \partial_x v$ while applying Lemma 7.3. Precisely, this problem only occurs in the $I_1$ case (see Proof of Lemma 7.3), and we can estimate $I_1$ as

$$\sum_{k_1 \leq k \leq k_2} \|P_k(P_k S(v)P_k \partial_x v P_k \partial_x v)\|_{L_{t,x}^{p_k} \cap L_t^\infty L_x^2} \lesssim 2^{-\frac{d}{k}} b_k^{(1)} \left( \sum_{k_1 \leq k} 2^{k_1} \right)^2 \lesssim 2^{-\frac{d}{k}} 2^{2k-2\sigma \theta} c_k(\sigma - \theta) \left( \sum_{k_1 \leq k} 2^{k_1 - \partial k c_1(\theta)} \right)^2 c_k \lesssim 2^{-\frac{d}{k}} 2^{2k-\sigma \theta} b_k^{(1)}(\sigma). \quad (5.52)$$

Thus we arrive at

$$(1 + 2^{2k} s)^L \|P_k v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} \lesssim (1 + 2^{2k} s L e^{-c(d)2^{2k}} \|P_k v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} + (1 + 2^{2k} s)^L \int_0^s e^{-c(d)(s-\tau)2^{2k}} \tilde{J}_1^2(s)2^{-\sigma k+2b_k^{(1)}(\sigma)} d\tau,$$

which further shows

$$\tilde{J}_1(s) \lesssim 1 + \epsilon \tilde{J}_1^2(s).$$

Therefore, by (5.49), one gets for all $\sigma \in [0,2\theta]$

$$2^{\frac{d}{k}} \|P_k v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} \lesssim (1 + 2^{2k} s)^{-L} 2^{-\sigma k b_k^{(1)}(\sigma)}. \quad (5.53)$$

and using the heat flow equation, (5.53) and (5.51) yield

$$2^{\frac{d}{k}} 2^{-2k} \|P_k \partial_s v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} \lesssim 2^{-\sigma k b_k^{(1)}(\sigma)(1 + 2^{2k} s)^{-L}}. \quad (5.54)$$

Then by bilinear Littlewood-Paley decomposition, (5.54) and the frame bound (5.41), $\phi_s$ is improved to be dominated by

$$2^{\frac{d}{k} - 2k} \|P_k \phi_s\|_{L_{t,x}^{p_k} \cap L_t^\infty L_x^2} \lesssim 2^{-\sigma k b_k^{(1)}(\sigma)(1 + 2^{2k} s)^{-L}} \quad (5.55)$$

for all $\sigma \in [0,2\theta]$. Then by (5.55) the frame bound (5.41) now can be ameliorated as

$$2^{\frac{d}{k}} \|P_k (d\mathcal{P} e - \chi^\infty)\|_{L_t^\infty L_x^2 \cap L_{t,x}^{p_k}} \lesssim 2^{-\sigma k b_k^{(1)}(\sigma)(1 + 2^{2k} s)^{-L}}. \quad (5.56)$$
for all $\sigma \in [0, 2\vartheta]$. And similarly,
\[
2^{4^k} \| P_k(G') \|_{L^2_T L^2_x \cap L^4_T L^\infty_x} \lesssim 2^{-\sigma k b_k^{(1)}(\sigma)(1 + 2^{2^k})^{-L}}. \tag{5.57}
\]

Until now, we have improved all the results before Section 4.2 to $\sigma \in [0, 2\vartheta]$. Then repeating the arguments of Section 4.2, one obtains, for $d \geq 3$, $\sigma \in [0, 2\vartheta]$,
\[
2^{-\pi d/2^k} \| P_k(G') \|_{L^2_T L^2_x \cap L^4_T L^\infty_x} \lesssim 2^{-\sigma k b_k^{(1)}(\sigma)(1 + s 2^{2^k})^{-L}}. \tag{5.58}
\]

With (5.57) and (5.59), running the bootstrap programme in Section 5.1 again gives
\[
b_k(\sigma) \lesssim c_k(\sigma) + \epsilon b_k^{(1)}(\sigma) \tag{5.59}
\]
for all $\sigma \in [0, 2\vartheta]$. Then Proposition 5.2 follows.

\section*{5.3 \textit{j}–th iteration and Proof of Theorem 1.2}

Repeating the above iteration scheme for $k$ times yields

**Proposition 5.3.** Let $\vartheta \in [1 - 10^{-9}, 1 - 10^{-10}]$ be a fixed constant. Let $\delta = \frac{1}{d_0 \vartheta^m}$. Given $\sigma \in \{j \vartheta, (j + 1) \vartheta\}$, $j \in \mathbb{N}$. Let $\epsilon_0$ be a sufficiently small constant. Let $u \in C([-T,T]; Q(\mathbb{R}^d, N))$ be the solution to SL with initial data $u_0$. Let $\{c_k\}$ be an $\epsilon_0$-frequency envelope of order $\frac{1}{2^{2^k} + \delta}$ with $0 < \epsilon_0 \leq \epsilon_*$.

And let $\{c_k(\sigma)\}$ be another frequency envelope of order $\frac{1}{2^{2^k} + \delta}$ which satisfies
\[
2^{4^k} \| P_k u_0 \|_{L^2_x} \leq c_k \tag{5.60}
\]
\[
2^{4^k} \| P_k u_0 \|_{L^2_x} \leq 2^{-\sigma k c_k(\sigma)} \tag{5.61}
\]

Denote $\{\phi_i\}_{i=1}^d$ the corresponding differential fields of the heat flow initiated from $u$. Suppose also that at the heat initial time $s = 0$,
\[
2^{4^k} \| P_k \phi_i(s = 0) \|_{G_k(T)} \leq \epsilon_0^{\frac{1}{2}} c_k. \tag{5.62}
\]

Then when $s = 0$, we have for all $i = 1,...d$, $k \in \mathbb{Z}$,
\[
2^{4^k} \| P_k \phi_i(s = 0) \|_{G_k(T)} \lesssim 2^{-\sigma k c_k^{(j)}(\sigma)} \tag{5.63}
\]
\[
2^{4^k} \| P_k (d\mathbb{P}(e) - \chi^\infty) \|_{L^2_T L^\infty_x} \lesssim 2^{-\sigma k c_k^{(j)}(\sigma)} \tag{5.64}
\]

where $c_k^{(j)}(\sigma)$ is defined by induction:
\[
c_k^{(0)}(\sigma) = c_k(\sigma), \text{ if } \sigma \in [0, 1] \tag{5.65}
\]
\[
c_k^{(j+1)}(\sigma) = c_k^{(j)}(\sigma), \text{ if } \sigma \in [0, j] \tag{5.66}
\]
\[
c_k^{(j+1)}(\sigma) = c_k(\sigma) + c_k^{(j)}(\sigma - \vartheta) c_k(\vartheta), \frac{\sigma \in (j \vartheta, (j + 1) \vartheta]}{51}
Proof. To make the statement clear we introduce following notations:

\[ \lambda \subset S^a \frac{\partial L^a S(v)}{L_k^\infty L_2} \leq \varepsilon s^{-L/2}, \forall 0 \leq L \leq K_0 + N \]

\[ (S^a_{j,N}) : 2 \lambda \frac{2 k^2 L^a S(v)}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-L}, \forall 0 \leq L \leq K_0 + N \]

\[ (V_{j,N}) : 2 \lambda \frac{2 k^2 L^a S(v)}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-L}, \forall 0 \leq L \leq K_0 + N \]

These are heat flow quantities. Introduce the following notations for curvature parts:

\[ \lambda \in \mathbb{H} : ||\lambda ||_{L_k^\infty L_2} \leq \varepsilon s^{-L/2}, \forall L \in [0, K_0 + N] \]

\[ (G^a_N) : 2 \lambda \frac{2 k^2 L^a G^a}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-K}, \forall K \in [0, K_0 + N] \]

\[ (E^a_N) : 2 \lambda \frac{2 k^2 L^a D^a \lambda}{L_k^\infty L_2} \leq \varepsilon s^{-L/2}, \forall L \in [0, K_0 + N] \]

and connection parts:

\[ (AO_j) 2 \lambda \frac{2 k^2 L^a A_j}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-L} \]

\[ (B_j) 2 \lambda \frac{2 k^2 L^a A_j}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-L} \]

\[ (\Phi_j) 2 \lambda \frac{2 k^2 L^a A_j}{L_k^\infty L_2 \cap L_{1,x}^\infty} \leq 2^{-\sigma c_k(j) \lambda} (1 + 2^k s)^{-L} \]

Now, the induction relation can be written as

\[ (\lambda_1) (S^a_{j,N}) \Rightarrow (S^a_{j,N-1}) ; (S^a_{j,N-1}) \Rightarrow (S^a_{j,N-2}) \]

\[ (\lambda_2) (S^a_{j,N}) \Rightarrow (V_{j,N}) \]

\[ (\lambda_3) (S^a_{j,N}) \Rightarrow (V_{j,N}) \]

\[ (\lambda_4) (E^a_{j,N}) \Rightarrow (E^a_{j,N}) \]

\[ (\lambda_5) (E^a_{j,N}) \Rightarrow (E^a_{j,N-1}) \]

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\[(\lambda_6)(AO_j) + (G_{j,N}) \Rightarrow (AO_{j+1})\]
\[\left(\Phi^j_1\right) + \left(G^j_{b,N}ight) \Rightarrow \left(G^j_{b,N}\right)\]
\[(AO_j) + (G_{j,N}) \Rightarrow (\Phi^j_1)\]
\[(G_{j,N}) \Rightarrow (AO_j).\]

\[(\lambda_7)(V^0_j) \Rightarrow (V_{j,N})\]
\[\left(E^j_0\right) + (\Phi^j_0) \Rightarrow (V^0_j)\]
\[(\lambda_8)SL\ equation \Rightarrow (\Phi^j_0).\]

And each time estimates like \((5.52)\) give additional \(c_k(\vartheta)\) which inspires the definition of \(c^{(j)}_k(\sigma)\). Thus in order to reach \(\sigma \in [j \vartheta, (j + 1) \vartheta]\), the top derivative orders and the sufficient decay order in \((S^j_N), (G^j_N), (E^j_N)\) need are
\[
S^j_{4+2j}, \ G^j_{8+2j}, \ E^j_{4+2j}. \tag{5.68}
\]

5.4 Uniform Sobolev bounds

By \((5.3)\) and \((5.64)\) we see
\[
2^{\frac{j}{d}}\|P_k^j v\|_{L^\infty_t L^2_x} \lesssim c^{(j)}_k(\sigma) 2^{-\sigma k}, \tag{5.69}
\]
which shows
\[
\|\nabla^\beta u\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{H^{\frac{d}{2}}_x \cap H^\sigma_x}. \tag{5.70}
\]
Hence considering the conservation of energy we conclude
\[
\|\partial_x u\|_{L^\infty_t H^j_Q} \lesssim \|\partial_x u_0\|_{H^j_Q}, \tag{5.71}
\]
for all \(j \geq \left[\frac{d}{2}\right] + 1\). Applying the local well-posedness result of [7] or [24] shows \(u\) is global and the Sobolev bounds hold uniformly in \(t \in \mathbb{R}\).

6 Appendix A. Gauges

6.1 Moving frame

In this subsection, we make the convention that Roman indices range in \(\{1, ..., m\}\) or \(\{1, ..., d\}\) according to the context. Let \(\mathcal{M}\) be a m-dimensional compact Riemannian manifold and \(v : \mathbb{R} \times \mathbb{R}^d \to \mathcal{M}\) be a smooth map. Let \(\{e_i(t,x)\}_{i=1}^m\) be global orthonormal frames for \(v^*(\mathcal{T}\mathcal{M})\). \(u\) induces scaler fields \(\{\psi^j_l\}_{j=0}^m\) which are defined on \(\mathbb{R} \times \mathbb{R}^d\) and take values in \(\mathbb{R}^m\):
\[
\psi^j_l = \langle \partial_j u, e_l \rangle, \ l = 1, ..., m. \tag{6.1}
\]
where and in the following we make the convention that \( j = 0 \) refers to \( t \) and \( j = 1, \ldots, d \) refers to \( x_j \) respectively. Conversely, sections of the trivial vector bundle \( ([0, T] \times \mathbb{R}^d; \mathbb{R}^m) \) yield sections of \( v^*TM \):

\[
\varphi \in \Gamma(([0, T] \times \mathbb{R}^d; \mathbb{R}^m)) \quad \longrightarrow \quad \varphi^l e_l \in \Gamma(u^*TM)
\]

\[
\varphi^l = \langle X, e_l \rangle, \quad l = 1, \ldots, m \in \Gamma(([0, T] \times \mathbb{R}^d; \mathbb{R}^m)) \quad \longrightarrow \quad X \in \Gamma(u^*TM).
\]

And similarly there exists a correspondence between \( m \times m \) matrices valued functions in \( ([0, T] \times \mathbb{R}^d; \mathbb{R}^m) \) and linear transformations in \( v^*(T\mathcal{M}) \):

\[
A \quad \longrightarrow \quad \mathfrak{A} : \quad X^l e_l \mapsto \quad (A^p_q X^q)e_p, \quad \forall X = X^l e_l \in u^*TM
\]

\[
A^p_q = \langle \mathfrak{A} e_p, e_q \rangle, \quad A \in \text{gl}(\mathbb{R}^m) \quad \longrightarrow \quad \mathfrak{A}.
\]

\( v \) induces a covariant derivative on \( ([0, T] \times \mathbb{R}^d; \mathbb{R}^m) \) by

\[
D_i \psi^j = \partial_i \psi^j + \sum_{q=1}^m \left( [A^j_i]_q \right) \psi^q,
\]

where the induced connection coefficient matrices are defined by \( [A^j_i]_q = \langle \nabla_i e_p, e_q \rangle \). The following identities are very often used throughout the paper:

\[
\text{torsion free identity} \quad D_{\mu} \psi_{\nu} = D_{\nu} \psi_{\mu} \quad (6.2)
\]

\[
\text{commutator identity} \quad \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\nu}, A_{\mu}] \quad \longrightarrow \quad \mathcal{R}(\partial_{\mu} v, \partial_{\nu} v). \quad (6.3)
\]

(6.3) will be schematically written as

\[
[D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j] = \mathcal{R}(\psi_i, \psi_j).
\]

The choices for gauges are very important for geometric dispersive PDEs. Nowadays, four gauges are often used: Coulomb gauge (see e.g. \([27, 19]\) for wave maps), caloristic gauge (see \([37, 34, 21]\) for wave maps and \([28, 29, 30]\) for hyperbolic Yang-Mills), microlocal gauge (see \([37, 41]\) for wave maps), Lorentz gauge.

### 6.2 Gauge for Schrödinger flows

If the target manifold is Kähler say 2\(n\)-dimensional manifold \( \mathcal{N} \) with complex structure \( J \) and metric \( h \), it is convenient to work with trivial vector bundle over \([0, T] \times \mathbb{R}^d \) with fiber \( \mathbb{C}^n \) instead of \( \mathbb{R}^{2n} \). In this case, the moving frame is chosen to be \( \{ e_i, Je_i \}_{i=1}^n \) and the induced scaled valued fields on \( ([0, T] \times \mathbb{R}^d; \mathbb{C}^n) \) are

\[
\phi^\gamma_i = \psi^\gamma_i + \sqrt{-1} \psi^{\gamma+n}_i, \quad i = 0, \ldots, d; \quad \gamma = 1, \ldots, n.
\]

Let \( \mathbb{C}^n \) valued function \( \varphi \) be a section of \( ([0, T] \times \mathbb{R}^d; \mathbb{C}^n) \), then it induces a section of \( u^*T\mathcal{N} \) via

\[
\varphi e := \varphi^\gamma e_\gamma + \varphi^{\gamma+n} Je_\gamma.
\]
The induced derivative on the complex vector bundle \([0, T] \times \mathbb{R}^d, \mathbb{C}^n\) is
\[ D_i = \partial_i + A_i, \]
where \(\{A_i\}\) denote the induced connection coefficient matrices which are defined by
\[ A_i^\gamma_\beta = [A_i]^\gamma_\beta + \sqrt{-1}[A_i]^\gamma_\beta + n, \quad i = 0, \ldots, d; \quad \gamma, \beta = 1, \ldots, n. \]

The torsion free identity and the commutator identity are recalled as follows:
\[ D_i \phi_j = D_j \phi_i \] (6.4)
\[ ([D_i, D_j] \varphi) e = ([\partial_i A_j - \partial_j A_i + [A_i, A_j]] \varphi) e = R(\partial_i u, \partial_j u) \varphi. \] (6.5)

The heat flow equation shows the heat tension filed \(\phi_s\) satisfies
\[ \sum_{d} D_i \phi_i. \] (6.6)

At the heat initial time \(s = 0\), \(\{\phi_j\}_{j=1}^d\) satisfy under the Schrödinger flow evolution
\[ -\sqrt{-1} D_i \phi_j = \sum_{i=1}^d D_i D_i \phi_j + \sum_{i=1}^d R(\phi_j, \phi_i) \phi_i. \] (6.7)

And for \(i = 0, 1, \ldots, d, \ s > 0\), the connection coefficients can be written as
\[ A_i^\gamma_\beta(s, t, x) = \int_{s}^{\infty} \langle R(v(\kappa)) (\partial_\kappa v(\kappa), \partial_\kappa v(\kappa)) e_p, e_q \rangle d\kappa. \] (6.8)

Let \(E\) be a manifold. Let \(D\) be the equipped connection. Suppose \(T\) is a type \((0, r)\) tensor on \(E\). The \(k\)-th covariant derivative of \(T\) is of \((0, r + k)\) type. And we denote
\[ (D^1 T)(X_1; Y_1, \ldots, Y_r) := (D_{X_1} T)(Y_1, \ldots, Y_r) \]
\[ (D^k T)(X_1, \ldots, X_k; Y_1, \ldots, Y_r) := \left[ D_{X_k} (D^{k-1} T) \right](X_1, \ldots, X_{k-1}; Y_1, \ldots, Y_r) \]
where \(\{X_j\}, \{Y_i\}_{i=1}^r\) are vector fields on \(E\).

### 7 Appendix B. Function Spaces

We recall the spaces developed by [3, 22, 15]. Given a unite vector \(e \in S^{d-1}\) we denote its orthogonal complement of \(\mathbb{R}^d\) by \(e^\perp\). The lateral space \(L^p_q\) is defined by
\[ \|g\|_{L^p_q} = \left( \int_{\mathbb{R}} \left( \int_{e^\perp \times \mathbb{R}} |g(t, x_1 e + x')|^q dx' dx \right)^{\frac{p}{q}} dx_1 \right)^{\frac{1}{p}}, \] (7.1)
Lemma 7.1. If \( |k_1 - k| \leq 4 \), then
\[
\| P_k (P_{k_1} f) \|_{F_k (T)} \lesssim \| P_{k_1} f \|_{F_k (T)}.
\]
(7.5)

If \( |k_2 - k| \leq 4 \), \( k_1, k_2 \geq 4 \), then
\[
\| P_k (P_{k_1} f P_{k_2} g) \|_{F_k (T)} \lesssim \left( 2^\frac{1}{2} (k_1 - k) + 2^\frac{d}{4} (k_1 - k) \right) \| P_{k_1} f \|_{L^\infty} \| P_{k_2} g \|_{F_k (T)}.
\]
(7.6)

Let \( S_k^* (T) \), \( F_k (T) \) denote the normed space of functions in \( L_k^2 (T) \) for which the corresponding norm
\[
\| g \|_{S_k^* (T)} := 2^{k_1} \left( \| g \|_{L_{k_1}^\infty} \| g \|_{L_{k_1}^1} + \| g \|_{L_{k_1}^1} \right).
\]
(7.3)

is finite.

The following bilinear estimates are used to control the curvature terms.

Theorem 7.1. If \( |k_1 - k| \leq 4 \), then
\[
\| P_k (P_{k_1} f) \|_{F_k (T)} \lesssim \| P_{k_1} f \|_{F_k (T)}.
\]
(7.5)

If \( |k_2 - k_1| \leq 4 \), \( k_1, k_2 \geq 4 \), then
\[
\| P_k (P_{k_1} f P_{k_2} g) \|_{F_k (T)} \lesssim \left( 2^\frac{1}{2} (k_1 - k) + 2^\frac{d}{4} (k_1 - k) \right) \| P_{k_1} f \|_{L^\infty} \| P_{k_2} g \|_{F_k (T)}.
\]
(7.6)

Let \( S_k^* (T) \), \( F_k (T) \) denote the normed space of functions in \( L_k^2 (T) \) for which the corresponding norm
\[
\| g \|_{S_k^* (T)} := 2^{k_1} \left( \| g \|_{L_{k_1}^\infty} \| g \|_{L_{k_1}^1} + \| g \|_{L_{k_1}^1} \right).
\]
(7.3)

is finite.

The following bilinear estimates are used to control the curvature terms.

Theorem 7.1. If \( |k_1 - k| \leq 4 \), then
\[
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\]
(7.5)

If \( |k_2 - k_1| \leq 4 \), \( k_1, k_2 \geq 4 \), then
\[
\| P_k (P_{k_1} f P_{k_2} g) \|_{F_k (T)} \lesssim \left( 2^\frac{1}{2} (k_1 - k) + 2^\frac{d}{4} (k_1 - k) \right) \| P_{k_1} f \|_{L^\infty} \| P_{k_2} g \|_{F_k (T)}.
\]
(7.6)

Let \( S_k^* (T) \), \( F_k (T) \) denote the normed space of functions in \( L_k^2 (T) \) for which the corresponding norm
\[
\| g \|_{S_k^* (T)} := 2^{k_1} \left( \| g \|_{L_{k_1}^\infty} \| g \|_{L_{k_1}^1} + \| g \|_{L_{k_1}^1} \right).
\]
(7.3)

is finite.

The following bilinear estimates are used to control the curvature terms.
Lemma 7.2. \([3, 23]\) Let \(F : \mathbb{R}^M \to \mathbb{R}\) be a smooth function and \(v : \mathbb{R}^d \times [-T, T] \to \mathbb{R}^M\) be a smooth map. Define

\[
\beta_k = \sum_{|k' - k| \leq 20} 2^{3k'} \|P_{k'} v\|_{L^\infty L^2_t},
\]

\[
\alpha_k = \sum_{|k' - k| \leq 20} 2^{4k'} \|P_{k'} (F(v))\|_{L^\infty L^2_t}.
\]

Assume that \(\|v\|_{L^\infty} \lesssim 1\) and \(\sup_{k \in \mathbb{Z}} \beta_k \leq 1\). Then

\[
2^{3k} \|P_k F(v)(\partial_x v, \partial_x v)\|_{L^\infty L^2_t} \lesssim 2^k \beta_k \sum_{k_1 \leq k} \beta_{k_1} 2^{k_1} + \sum_{k_2 \geq k} 2^{-d|k-k_2|} 2^{2k_2} \beta_{k_2}^2
\]

\[
+ \alpha_k (\sum_{k_1 \leq k} 2^{k_1} \beta_{k_1})^2 + \sum_{k_2 \geq k} 2^{d(k-k_2)} 2^{k_2} \alpha_{k_2} \beta_{k_2} \sum_{k_1 \leq k_2} 2^{k_1} \beta_{k_1}.
\]

(7.8)

Lemma 7.3. Let \(F : \mathbb{R}^M \to \mathbb{R}\) be a smooth function and \(v : \mathbb{R}^d \times [-T, T] \to \mathbb{R}^M\) be a smooth map. Define

\[
\tilde{\beta}_k = \sum_{|k' - k| \leq 30} 2^{3k'} \|P_{k'} v\|_{L^\infty L^2_t \cap L^{p_d}_t},
\]

\[
\tilde{\alpha}_k = \sum_{|k' - k| \leq 30} 2^{4k'} \|P_{k'} (F(v))\|_{L^\infty L^2_t \cap L^{p_d}_t}.
\]

Assume that \(\|v\|_{L^\infty} \lesssim 1\). Then

\[
2^{3k} \|P_k F(v)(\partial_x v, \partial_x v)\|_{L^\infty L^2_t \cap L^{p_d}_t} \lesssim 2^k \tilde{\beta}_k \sum_{k_1 \leq k} \tilde{\beta}_{k_1} 2^{k_1} + \sum_{k_2 \geq k} 2^{-d|k-k_2|} 2^{2k_2} \tilde{\beta}_{k_2}^2
\]

\[
+ \tilde{\alpha}_k (\sum_{k_1 \leq k} 2^{k_1} \tilde{\beta}_{k_1})^2 + \sum_{k_2 \geq k} 2^{d(k-k_2)} 2^{k_2} \tilde{\alpha}_{k_2} \tilde{\beta}_{k_2} \sum_{k_1 \leq k_2} 2^{k_1} \tilde{\beta}_{k_1}.
\]

(7.9)

Proof. The proof is an adaptation of \([3, \text{Lemma 8.2}]\). By Lemma 7.2 and symmetry, it suffices to bound the \(\|P_k (F(v)(P_{k_1} \partial_x v P_{k_2} \partial_x v))\|_{L_{t,x}^{p_d}}\) part with \(k_1 \leq k_2\). Then we consider three subcases:

\[
\sum_{k_1 \leq k_2} \|P_k (F(v)(P_{k_1} \partial_x v P_{k_2} \partial_x v))\|_{L_{t,x}^{p_d}}
\]

\[
\lesssim \sum_{k_1 \leq k_2, k_3 \geq k_2 + 5} \|P_k (P_{k_3} F(v)(P_{k_1} \partial_x v P_{k_2} \partial_x v))\|_{L_{t,x}^{p_d}}
\]

\[+ \sum_{k_1 \leq k_2 |k_2 - k_3| \leq 4} \|P_k (P_{k_3} F(v)(P_{k_1} \partial_x v P_{k_2} \partial_x v))\|_{L_{t,x}^{p_d}}
\]

\[+ \sum_{k_1 \leq k_2, k_3 \geq k_2 + 4} \|P_k (P_{k_3} F(v)(P_{k_1} \partial_x v P_{k_2} \partial_x v))\|_{L_{t,x}^{p_d}}
\]

:= I_1 + I_2 + I_3.
$I_1$ is dominated by
\[
\sum_{k_1 \leq k_2, k_3 \geq k_2+5} \|P_k(F(v)P_k \partial_x v P_k \partial_x v)\|_{L^p_{t,x}} \lesssim \sum_{|k-k_3| \leq 4} \|P_{k_3}F(v)\|_{L^p_{t,x}} \sum_{k_1 \leq k_2 \leq k-4} \|P_{k_1}2^{\frac{k_1}{2}} \partial_k v\|_{L^p_{t,x}} 2^{\frac{k_2}{2}} \|P_k \partial_x v\|_{L^\infty_{t,x}} \leq 2^{-\frac{d}{2}k} \tilde{\alpha}_k \left( \sum_{k_1 \leq k} 2^{k_1} \tilde{\beta}_{k_1} \right)^2 .
\]

$I_2$ is bounded by
\[
\sum_{k_1 \leq k_2-4, |k_2-k_3| \leq 4, k_2, k_3 \geq k-5} \|P_k(F(v)P_k \partial_x v P_k \partial_x v)\|_{L^p_{t,x}} + \sum_{|k_1-k_2| \leq 4, |k_2-k_3| \leq 4, k_1, k_2, k_3 \geq k-10} \|P_k(F(v)P_k \partial_x v P_k \partial_x v)\|_{L^p_{t,x}} \lesssim 2^{\frac{k}{4}} \sum_{k_2 \geq k-5} \|P_{k_2}F(v)\|_{L^p_{t,x}} \|P_k \partial_x v\|_{L^p_{t,x}} \left( \sum_{k_1 \leq k_2} 2^{\frac{k_1}{2}} \|P_{k_1} \partial_x v P_k \partial_x v\|_{L^\infty_{t,x}} \right) \lesssim 2^{\frac{k}{4}} \sum_{k_2 \geq k-5} 2^{-dk_2} \tilde{\alpha}_{k_2} \tilde{\beta}_{k_2} \left( \sum_{k_1 \leq k_2} 2^{k_1} \tilde{\beta}_{k_1} \right).
\]

$I_3$ is dominated by
\[
\sum_{k_1 \leq k_2-4, |k_2-k_1| \leq 4} \|F(v)\|_{L^p_{t,x}} 2^{\frac{k}{4}} \|P_{k_1} \partial_x v\|_{L^p_{t,x}} \|P_k \partial_x v\|_{L^p_{t,x}} \|P_{k_2} \partial_x v\|_{L^p_{t,x}} + \sum_{k_1 \leq k_2, |k_2-k_1| \leq 8, k_1, k_2 \geq k-9} 2^{\frac{k}{4}} \|F(v)\|_{L^p_{t,x}} \|P_{k_1} \partial_x v\|_{L^p_{t,x}} \|P_k \partial_x v\|_{L^p_{t,x}} \|P_{k_2} \partial_x v\|_{L^p_{t,x}} \lesssim 2^{-\frac{d}{4}k} \tilde{\beta}_k \left( \sum_{k_1 \leq k} 2^{k_1} \tilde{\beta}_{k_1} \right) + 2^{\frac{k}{4}} \sum_{k_1 \geq k-4} 2^{2k_1} \tilde{\beta}_{k_1} .
\]

We also recall the general form of fractional Leibnitz rule (Kato-Ponce inequality), see [10] and the reference therein.

**Lemma 7.4.** Let $\frac{4}{5} < r < \infty$, and $1 < p_1, q_1, p_2, q_2 \leq \infty$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $s > \max(0, \frac{4}{5} - d)$ or $s \in 2\mathbb{N}$, for $f, g \in \mathcal{S}(\mathbb{R}^d)$ we have
\[
\| |\nabla|^s (fg)\|_{L^r_x} \lesssim \|g\|_{L^{p_1}_x} \||\nabla|^s f\|_{L^{q_1}_x} + \|f\|_{L^{p_2}_x} \||\nabla|^s g\|_{L^{q_2}_x} \tag{7.10}
\]

**8 Appendix C. Proof of Lemma 2.2.**

**Proof.** By the heat flow equation,
\[
\frac{d}{ds} \|v\|^2_{H^s_{x,t}} = -\|\partial_x v\|^2_{H^s_{x,t}} + \langle |\nabla|^s v, |\nabla|^s [S(v)(\partial_x v, \partial_x v)]\rangle_{L^2_x} \tag{8.1}
\]
We have seen in Lemma 2.1 that
\[
\|S(v)\|^2_{H^d_x} \lesssim \epsilon_1. \tag{8.2}
\]
Let \(\{\alpha_k\}, \{\beta_k\}\) be defined in Lemma 7.2, and define
\[
\zeta_k(\sigma) = \sup_{k_1 \in \mathbb{Z}} 2^{-d[k-k_1]} \sum_{k_1 \leq k_2 \leq 20} 2^{-d(k-k_2)} \|P_{k_2}v\|_{L^\infty_t L^2_x}
\]
\[
\alpha_k = \sup_{k' \in \mathbb{Z}} 2^{-d|k-k'|} \alpha_{k'}
\]
Since frequency envelopes are of slow variation, Lemma 7.2 shows
\[
\|\nabla [\frac{d}{dt} P_k[S(v)(\partial_x v, \partial_x v)]]\|_{L^\infty_t L^2_x} \lesssim \zeta_k(1) \left( \sum_{k_1 \leq k} \zeta_{k_1}(0) 2^{k_1} \right) + \sum_{k_2 \geq k} 2^{-d[k-k_2]} \zeta_{k_2}(1) \zeta_{k_2}(0)
\]
\[
+ \alpha_k \left( \sum_{k_1 \leq k} 2^{k_1} \zeta_{k_1}(\frac{1}{2}) \right)^2 + \sum_{k_2 \geq k} 2^{d(k-k_2)} 2^{k_2} \alpha_{k_2} \zeta_{k_2} \left( \frac{1}{2} \right) \left[ \sum_{k_1 \leq k_2} \zeta_{k_1} \left( \frac{1}{2} \right) 2^{\frac{k_2}{2}} \right]
\]
\[
\lesssim 2^k \zeta_k(1) \zeta_k(0) + \alpha_k 2^k \zeta_k(\frac{1}{2})^2.
\]
Therefore, by (8.2) and Lemma 2.1 we get from \(\zeta_k(1/2) \leq \sqrt{\zeta_k(0) \zeta_k(1)}\) that
\[
\sum_{k \in \mathbb{Z}} \|\nabla [\frac{d}{dt} P_k[S(v)(\partial_x v, \partial_x v)]]\|_{L^\infty_t L^2_x} \lesssim \sum_{k \in \mathbb{Z}} \epsilon \|\zeta_k(1)\|^2 + \epsilon \|\zeta_k(\frac{1}{2})\|^4
\]
\[
\lesssim \epsilon \|\partial_x v\|^2_{L^\infty_t H^d_x}.
\]
Thus (8.1) reduces to
\[
\frac{d}{ds} \|v\|^2_{H^d_x} + (1-\epsilon) \|\partial_x v\|^2_{H^d_x} \leq 0 \tag{8.3}
\]
Integrating (8.3) in \(s \in [0, \infty)\) yields (2.22) since \(\|v\|_{H^d_x} \lesssim \epsilon_1\) by Lemma 2.1.

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