BEYOND STRINGS, MULTIPLE TIMES AND GAUGE THEORIES OF
AREA-SCALINGS RELATIVISTIC TRANSFORMATIONS

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University
Atlanta, Georgia 30314

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ABSTRACT

Nottale’s special scale-relativity principle was proposed earlier by the author as a plausible geometrical origin to string theory and extended objects. Scale Relativity is to scales what motion Relativity is to velocities. The universal, absolute, impassible, invariant scale under dilatations in Nature is taken to be the Planck scale, which is not the same as the string scale. Starting with ordinary actions for strings and other extended objects, we show that gauge theories of volume-resolutions scale-relativistic symmetries, of the world volume measure associated with the extended “fuzzy” objects, are a natural and viable way to formulate the geometrical principle underlying the theory of all extended objects. Gauge invariance can only be implemented if the extendon actions in $D$ target dimensions are embedded in $D+1$ dimensions with an extra temporal variable corresponding to the scaling dimension of the original string coordinates. This is achieved upon viewing the extendon coordinates, from the fuzzy worldvolume point of view, as noncommuting matrices valued in the Lie algebra of Lorentz-scale relativistic transformations. Preliminary steps are taken to merge motion relativity with scale relativity by introducing the gauge field that gauges the Lorentz-scale symmetries in the same vain that the spin connection gauges ordinary Lorentz transformations and, in this fashion, one may go beyond string theory to construct the sought-after General Theory of Scale-Motion Relativity. Such theory requires the introduction of the scale-graviton (in addition to the ordinary graviton) which is the field that gauges the symmetry which converts motion dynamics into scaling-resolutions dynamics and vice versa (the analog of the gravitino that gauges supersymmetry). To go beyond the quantum string geometry most probably would require a curved fractal spacetime description (curved from both scaling and motion points of views) with a curvilinear fractal coordinate system. Non-Archimedean geometry and $p$-adic numbers are essential ingredients comprising the geometrical arena of such extensions of quantum string geometry.

I. Introduction

Despite the tremendous progress in string theory in the past two years a geometrical foundation of the theory, like Einstein’s General Theory of Relativity, is still lacking. The fact that string theory contains gravity is, per se, no satisfactory explanation of what is the underlying reason behind it nor why it should be so, especially insofar that a background independent formulation of string theory is still missing. Spacetime should emerge from the string. Tsetlyn [1] has remarked that the vanishing of the beta functions for the couplings of the non-linear $\sigma$ model associated with the string propagation in curved backgrounds must contain a clue to the non-perturbative and geometrical formulation of string theory.
Some time ago, Amati, Ciafaloni and Veneziano [4] pointed out that some sort of enlarged equivalence principle is operating in string theory in which dynamics is not only independent of coordinate transformations but also of structures occurring at distances shorter than the string length in units of $c = 1$, $\lambda_s = \sqrt{2\alpha\hbar}$, where distances smaller than $\lambda_s$ are not relevant in string theory. The string scale differs from the Planck scale $\Lambda = \sqrt{(\hbar G/c^3)}$ where $G$ is Newton’s constant in four dimensions. There is however an intermediate regime between both scales and $D$-branes probes have been suggested to explore such distances below $\lambda_s$. Yoneya and references therein [23].

In [3] we obtained modifications/corrections/extensions to the stringy-uncertainty principle within the framework of the theory of special Scale-Relativity developed by Nottale [2]. In particular, the size of the string was shown to be bounded by the minimum Planck scale and an upper, impassible, absolute scale, invariant under dilations, that incorporates the principle of scale-relativity to cosmology as well. Such principle has allowed Nottale to propose a very elegant and “simple” resolution to the cosmological constant problem. Since the Planck scale plays the same role in scale relativity that the speed of light did in special relativity, the Planck energy decouples from the Planck scale: it takes an infinite amount of energy to probe finite Planck scale resolutions! hence the Planck scale serves as a natural ultraviolet regulator in QFT. Whereas the upper scale serves as an infrared one. For the importance that the Planck scale may have as a natural regulator of matter QFT in quantum gravity, noncommutative geometry, quantum groups, .... we refer to the authors [5,14,15,25,27] and the role that loop spaces, spin networks, graphs and networks,.... may have in the discrete structures of spacetime at Planck scale we refer to [6,16,17,22].

The aim of this work is to argue that the principle of scale-relativity can be incorporated into the theory of extended objects [3]; i.e. we will show that applying this principle to the resolutions (that any physical measurement can make no smaller than the Planck scale) of the extended object’s world-volume coordinates, a fuzzy $p$-brane, [19], allows the implementation of the principle of Lorentzian-scale relativistic invariance to the world-volume measure of these extended objects, as they are dynamically embedded in a curved target spacetime background. Scale-relativity invariance of area-resolutions, volume-resolutions,.... can be maintained in all Dirac-Nambu-Goto actions of extended objects if one enlarges the $D$ background spacetime to an effective $D + 1$ spacetime where an additional temporal dimension is added. This extra time dimension stems from the scaling dimension of the original $p$-brane $X^\mu$ coordinates. The scaling dimension is determined by the $X^\mu$ coordinates transformation properties under volume-resolutions scale-relativistic transformations. For example, the 10$D$ superstring would require an 11$D$ space of signature $(9,2)$. The 11$D$ supermembrane requires a 12$D$ space of signature $(10,2)$ etc....which is what Vafa and Bars have been advocating recently concerning, F, S, theories. [31].

Once area-resolutions, volume-resolutions,....are included in the physical processes, the $X^\mu$ fuzzy coordinates are metamorphosed into matrices carrying internal indices associated with the Lie-algebra generators of scale-relativistic transformations of the world-volume resolutions. In the string case, the $X^\mu \rightarrow X^\mu_{\Delta^a \Delta^b} \Sigma^a \Delta^b$ where the matrix generators are $\Sigma^a \Delta^b$. In the past months there has been growing evidence that the $D = 11$ M theory in the infinite momentum frame bears a connection to large $N$ Matrix models;
i.e. supersymmetric gauge quantum mechanical models [26] associated with the algebra of area-preserving diffs (Hoppe, Flume, Baake et al.), and that the string coordinates can be viewed as noncommuting matrices resulting from the collective modes of the positions of D-branes (Witten, Polchinski). The membrane ground state appears as a “condensation” of an infinity of 0-branes.

In our case, the \( D X^\mu \) matrices are then embedded into a larger space of \( D+1 \) matrices \( Z^M \), where the extra temporal dimension is the resolution dependent scaling dimension, \( \delta(\Delta \sigma^a) \) of the original \( X^\mu \) coordinates (matrices). After the metric \( G_{\mu\nu} \) is embedded into \( G_{MN} \) we show that the Dirac-Nambu-Goto action of the fuzzy string moving in the enlarged \( D+1 \) spacetime is in fact invariant under area-resolutions scale-relativistic symmetries and, most importantly, the scale-relativistic version of the beta functions, \( \beta_M^G \neq 0, \beta^Z \neq 0 \). One can find vanishing \( \beta_M^G, \beta^Z \) but the solutions are trivial. If one wanted to find nontrivial solutions this would entail imposing unnatural constraints among the \( X^\mu \) coordinates. i.e. scale-relativistic transformations preserving the world-volumes of the extended objects are compatible with the \( \beta_M^G \neq 0, \beta^Z \neq 0 \) conditions. This should be the starting point to explore string propagation in non-conformal backgrounds. Although we must remark that the scale-relativity beta functions are not the same as the ordinary 2D conformal field theory ones.

The main point of this work is that in the same way that Lorentz invariance requires a Minkowski spacetime, volume-resolution scale-relativistic invariance requires an extra scaling temporal dimension so the effective spacetime is \( D+1 \) dimensional. Furthermore, instead of restricting ourselves to volume-scalings but instead concentrating on the most general coordinate-resolutions scale-relativistic transformations without the restriction of scaling the volumes as a whole, one will have then many scaling-temporal dimensions. This could be an explanation of the multiple-time/signatures spacetime backgrounds which are very popular to-day.

Care must be taken not to confuse transformations involving the resolutions of the fuzzy worldsheet coordinates with the string worldsheet coordinates themselves!. Extended objects can also be interpreted as a gauge theory of volume preserving diffeomorphisms [11,12]. We were able to show in [11] that \( p \)-branes can be seen as composite antisymmetric tensor field theories of the volume preserving diffs (of the type proposed by Guendelman, Nissimov and Pacheva [11] and using the local gauge theory reformulation of extended objects developed by Aurilia, Ansoldi, Smailagic and Spallucci [11]). The relevance of these composite theories is that it allowed the author to build-in, from the very beginning and without any conjectures, the analogs of \( S \) and \( T \) duality symmetries in extended objects. Furthermore, we also have shown that the \( p \)-branes worldvolumes had a natural correspondence with nonabelian composite-antisymmetric tensor fields which were not of the Yang-Mills type fields, (only for the membrane case), verifying the conjecture of Bergshoeff et al [11].

After reviewing Nottale’s essential results in section II we proceed into the implementation program of scale-relativity in III and show that strings can be viewed as gauge theories of area-resolutions scale-relativistic transformations in the enlarged \( D+1 \) spacetime including the extra temporal scaling variable. Also in III we discuss how one might achieve the goal of building the general theory of scale-motion relativity and go beyond
Finally, in the conclusion, other topics are briefly discussed, like Weyl-Finsler geometries, renormalization group flows, curved backgrounds, extensions of the quantum string geometry [33] based on curved fractal spacetimes, non-Archimedean geometry and $p$-adics numbers, among some. The supersymmetrization program should be carried out to see how far these ideas can be taken and be able to make comparisons with the current results of string-duality, moduli spaces, etc.....

II.

We shall present a brief review of Nottale's principle of special scale relativity. For a detailed account of the theory of Scale Relativity that originated with the study of fractals, we recommend the reader to study Nottale’s work that appeared in [2]. The principle of coordinate-resolutions special scale-relativity is essentially stated: “that the laws of physics should be covariant under any state of “scaling-motion” of all frames of reference associated with all systems of observers (carrying coordinate-resolutions rulers with their physical measurements apparatus). The Planck scale is taken to be the minimum-resolution scale in Nature.

Essentially, one has a collection of scalar fields, $\varphi$, that under Lorentzian-scalings (of the spacetime coordinate resolutions that our physical apparatus can resolve) behave like the ordinary spacetime coordinates under ordinary Lorentzian transformations (change of frame of reference). The analog of the speed of light, $c$, is played by the logarithm of the ratio of two resolutions: One is the resolution, $\lambda_o$, with respect to which we measure other resolutions, $\Delta x_o \leq \lambda_o$; and another is the Planck scale, $\Lambda$ in the appropriate dimension. The analog of time is played by the scaling dimension of the scalar fields, $\varphi$, which Nottale labeled by $\delta$. The origins of scale-relativity were motivated by the fractality of spacetime microphysics and, the very plausible fractality of cosmological structures as well. Ord [2], found that the relativistic quantum mechanics of particles could be reinterpreted in terms of fractal trajectories: continuous but nowhere differentiable in spacetime.

We do not intend to fall hostage of the debate of “what is quantum” and what is “classical”. Nottale’s view is that a nowhere differentiable spacetime should not be viewed any longer as “classical”. Our aims are less ambitious. We just go ahead and ask ourselves whether or not one can implement the scale-relativity principle to string theory and extended objects. We start with some definitions:

The scaling behaviour of $\varphi = \varphi(x, \Delta x)$ under scale-relativistic transformations is:

$$\Delta x_0 \rightarrow \Delta x. \quad ln(\varphi/\varphi_o) = ln[(\Delta x_0/\Delta x)^{\delta(\Delta x)}] \Rightarrow \varphi(x, \Delta x) = \varphi_o(x)(\Delta x_0/\Delta x)^{\delta(\Delta x)}.$$  \hspace{1cm} (2.1)

The scale dimension of the field $\varphi(x, \Delta x)$ is measured w.r.t the frame of reference whose resolution is $\Lambda \leq \Delta x_0 \leq \lambda_o$. Since the quantity $\lambda_0/\Lambda$ will play the role of the speed of light, $c$, and because $\Lambda$ is taken to be the invariant scale this implies also that $\lambda_o$ must be a fiducial and fixed scale w.r.t which we measure the running scales. This $\lambda_o$ scale was taken by Nottale to be the scale which signals the classical geometry-fractal spacetime transition and usually is taken to be the deBroglie wavelength of electron, for example.
However, this scale is free. One may set it equal to the string scale if one wishes. The scaling dimension is then:

$$\delta(x) = \frac{\delta_0(\Delta x_0)}{\sqrt{1 - \frac{\ln^2(\Delta x_0/\Delta x)}{\ln^2(\lambda_0/\Lambda)}}}, \quad \Lambda \leq \Delta x_0 \leq \lambda_0. \quad (2.2)$$

Under two consecutive Lorentz-scale transformations of the resolutions: $\Delta x_0 \rightarrow \Delta x$ and $\Delta x \rightarrow \Delta x'$, the logarithm of $\varphi$ and the scaling dimension $\delta$ transform like the components of a two-vector:

$$\Delta x_0 \rightarrow \Delta x \rightarrow \Delta x'. \quad \ln(\varphi'/\varphi_o) = \frac{\ln(\varphi/\varphi_o) - \delta \ln \rho}{\sqrt{1 - \frac{\ln^2 \rho}{\ln^2(\lambda_0/\Lambda)}}}. \quad (2.3)$$

$$\delta' = \delta + \frac{(\ln \rho)(\ln \lambda_o)}{\ln^2(\lambda_0/\Lambda)} \ln(\lambda_0/\Lambda). \quad (2.4)$$

where the composition of dilations (analog of addition of velocities) is:

$$\ln \rho = \frac{\ln(\Delta x/\Delta x_0) \pm \ln(\Delta x'/\Delta x)}{1 \pm \frac{\ln(\Delta x/\Delta x_0) \ln(\Delta x'/\Delta x)}{\ln^2(\lambda_0/\Lambda)}}. \quad (2.5)$$

If one chooses, $\Delta x_0 = \Delta x$ then in this particular case one recovers $\varphi(x, \Delta x_0) \equiv \varphi_o(x)$ so the above equations simplify:

$$\ln \rho = \ln(\Delta x'/\Delta x) = \ln(\Delta x'/\Delta x_0), \quad \varphi'(x, \Delta x') = \varphi_o(x)(\Delta x'/\Delta x)^{-\delta'}. \quad (2.6)$$

where:

$$\delta' = \frac{\delta(\Delta x_0)}{\sqrt{1 - \frac{\ln^2 \rho}{\ln^2(\lambda_0/\Lambda)}}} = \delta(\Delta x_0)[(1 - \beta^2)^{-1/2}]. \quad (2.7)$$

$\beta \equiv [\ln(\rho)/\ln(\lambda_0/\Lambda)]$ which is the analog of $v/c$ in motion relativity, one can recognize eq-(2.7) as the analog of time dilation in motion-relativity.

We could set $c = \ln(\lambda_0/\Lambda) = 1$. Eq-(2.6) has exactly the same form as (2.1), as it should if covariance is to be maintained. Henceforth, we shall omit the suffix $\Delta x'$. A finite Lorentz-scale transformation from the fiducial scale-frame of reference $\Lambda \leq \Delta x_0 \leq \lambda_o$, to a new scale $\Delta x$ implies:

$$\Delta x_0 \rightarrow \Delta x, \quad \varphi_o(x, \lambda_o) \rightarrow \varphi_o(x)e^{-\beta \delta} = \varphi(x, \Delta x), \quad \delta_o(\Delta x_0) \rightarrow \delta_o(\Delta x_0)[(1 - \beta^2)^{-1/2} = \delta(\Delta x), \quad \beta = \frac{\ln(\Delta x/\Delta x_0)}{\ln(\lambda_0/\Lambda)}. \quad (2.8)$$
The infinitesimal scaling transformations from one frame, whose relative velocity w.r.t the fiducial frame is \( \beta \), to another frame whose relative velocity w.r.t the fiducial frame is \( \beta + \Delta \beta \) are obtained from the relations:

\[
\ln(\varphi/\varphi_0) = -\beta \delta. \quad \ln(\varphi'/\varphi_0) = -\beta' \delta'. \quad \beta' = \beta + \Delta \beta. \quad \delta(\beta + \Delta \beta) = \delta + \Delta \beta \frac{\partial(\delta)}{\partial \beta}. \quad (2.9)
\]

so under infinitesimal scaling-relativistic transformations:

\[
\delta_{\beta}[\ln(\varphi/\varphi_0)] = \ln(\varphi'/\varphi_0) - \ln(\varphi/\varphi_0) = -[(\beta + \Delta \beta)(\delta + \Delta \delta) - \beta \delta] \sim -\gamma^2 \delta \Delta \beta. \quad (2.10)
\]

Therefore, we shall define the infinitesimal scaling-relativistic transformations:

\[
\delta_{\beta}[\ln(\varphi/\varphi_0)] = \frac{\partial \ln(\varphi/\varphi_0)}{\partial \beta} \Delta \beta = -(\gamma^2 \delta) \Delta \beta \Rightarrow \delta_{\beta}[(\varphi/\varphi_0)] = -(\gamma^2 \delta)[(\varphi/\varphi_0)] \Delta \beta. \quad (2.11)
\]

where we explicitly write \([\partial \ln(\varphi/\varphi_0)/\partial \beta]\) to emphasize the fact that one is performing a resolution-scaling transformation, a change of the scaling frame of reference, and not a differentiation w.r.t the \( \beta \) variable. In ordinary relativity we don’t have expressions like \( \partial X/\partial v \) where \( v \) is the relative velocity between two frames of reference. The latter equations show that \( \delta, \ln(\varphi/\varphi_0), \beta \) play the same role as time, space coordinates and velocity, respectively, in motion relativity. The scaling-dimension (a function of the resolutions) associated with the field \( \varphi, \delta(\varphi) \), is evaluated at two different points, \( \beta, \beta + \Delta \beta \); however the functional form of \( \delta \) does not change. Notice the subtlety in the difference upon naive differentiation w.r.t the \( \Delta x \) and performing a scaling transformation. The reason is the following.

Extreme caution must be taken in order not to confuse the scaling dimension, \( \delta \), with its transformation properties under Lorentz-scalings. For example, the quantities \( \Delta x', \Delta x, \Delta x_0 \) can all flow in such a fashion that their respective ratios remain constant. Imagine scaling the \( \Delta x', \Delta x, \Delta x_0 \) scales by a common factor (the \( \lambda_o \) and \( \Lambda \) scales remain fixed in all the formulae) so their ratio remains invariant, then the quantity \( \beta = [\ln(\rho)/\ln(\lambda_o/\Lambda)] \) still remains constant since both \( v, c \) do. This means that the gamma-dilation factor does not change either. We can notice also that the scaling velocity \( \beta = v/c \) and the gamma dilation factor is also invariant under the analog of a T duality transformation \( R \leftrightarrow (1/R) \): \( \Delta x/\Delta x' \rightarrow (\Delta x'/\Delta x) \) and \( (\lambda_o/\Lambda) \rightarrow (\Lambda/\lambda_o) \).

However, the quantity \( \delta(\Delta x) \) does change because \( \Delta x \) has flowed. A flowing value for \( \Delta x \) is not the same as a change of a reference frame. One must not confuse the values that a coordinate, in a given frame, can take with its transformation properties under Lorentz-scale transformations. We shall take \( c = 1 \) from now on and by choosing a frame of reference we mean fixing the value of the relative velocities \( v/c = v = \beta = \ln(\rho) \) in (2.5,2.6) despite the fact that both quantities \( \Delta x, \Delta x' \) can both flow maintaining its ratio fixed. It is in this context that there is a crucial difference between taking ordinary differentiation of \( \varphi(x, \Delta x) \) w.r.t the \( \Delta x \) flowing variable and performing an infinitesimal
scaling transformation, $\delta_\beta \varphi$, given by eqs-(2.10,2.11). Therefore, by a scaling transformation one means:

$$\delta_\beta \ln(\varphi/\varphi_o) = \Delta_\beta \frac{\delta \ln(\varphi/\varphi_o)}{\delta \beta} = -\gamma^2 \delta_\beta. \tag{2.12}$$

And by the analog of “scaling-motion” (resolution-motion) of the $\varphi$ field w.r.t the fiducial reference field, $\varphi_o$, due to the flowing values of $\Delta x$ (imagine the motion of a particle whose coordinate is $\varphi$ moving with velocity $\beta$ w.r.t a particle whose frame of reference carries the coordinates $\varphi_o$):

$$\ln(\varphi^2(\Delta x)/\varphi_o) - \ln(\varphi^1(\Delta x)/\varphi_o) = \ln(\varphi(\Delta x^2)/\varphi_o) - \ln(\varphi(\Delta x^1)/\varphi_o). \tag{2.13}$$

and the scale-velocity is:

$$\frac{d[\ln(\varphi/\varphi_o)]}{d\delta}|_\beta = -\beta. \tag{2.14}$$

where we implement the resolution-motion of the “coordinates” $\varphi$ between two instants of time, $\delta^1, \delta^2$, by defining two functions, $\varphi^1(\Delta x), \varphi^2(\Delta x)$, obeying:

$$\varphi^1(\Delta x) = \varphi(\Delta x^1). \quad \varphi^2(\Delta x) = \varphi(\Delta x^2)$$

$$\delta(\varphi^1) = \delta^1(\Delta x) = \delta(\Delta x^1). \quad \delta(\varphi^2) = \delta^2(\Delta x) = \delta(\Delta x^2). \tag{2.15}$$

It is now when we can speak of a scaling-coordinates interval, $\Delta(\varphi/\varphi_o)$, versus a scaling-time interval, $\Delta \delta$, w.r.t a given fiducial frame of reference. In this fashion it is sensible to view the scaling dimension as the true analog of a time coordinate. This is not new in string theory and quantum mechanics/quantum cosmology. The Liouville mode in non-critical strings has played the role of a “temporal” direction as advocated many times by the authors [10] in connection to the origin of the arrow of time. The quantum phase space origins of a point particle from string solitons and $D$ brane scaffolding dynamics can also be studied within this framework [10]. The importance that duality and scaling in quantum mechanics has in connection with the emergence of an intrinsic fractal scaling time variable was discussed by Datta [18].

Summarizing, the analog of a time interval w.r.t a fiducial frame of reference in scale relativity is:

$$\Delta(\delta) \equiv \delta^2 - \delta^1 = \delta(\Delta x^2) - \delta(\Delta x^1); \quad \Delta(\delta') = \delta'^2 - \delta'^1 = \gamma \Delta(\delta); \quad \gamma = [1 - \beta^2]^{-1/2}. \tag{2.16}$$

and the relative “velocity” between the $\Delta x'$, $\Delta x$ frames is the one given in the r.h.s of (2.5). Similar reasoning applies to the analog of a spatial interval, $\Delta \varphi = \varphi^2 - \varphi^1$, where $\varphi^2 = \varphi(\Delta x^2); \varphi^1 = \varphi(\Delta x^1)$.

Therefore, the scale-relativistic analog of a Lorentz invariant spacetime world line interval is:
\[ d\eta^2 = [\ln^2(\lambda_0/\Lambda)](d\delta)^2 - \frac{(d(\varphi/\varphi_0))^2}{(\varphi/\varphi_0)^2} = [\ln^2(\lambda_0/\Lambda)](d\delta)^2[1 - \frac{1}{(\ln^2(\lambda_0/\Lambda))}(d\ln(\varphi/\varphi_0))^2]. \] 

(2.17)

and one gets the usual time dilation expression: \( \gamma(d\eta) = d\delta \) where we have used in the last term of (2.17) the scaling-velocity relation:

\[ \ln(\varphi/\varphi_0) = -\beta \delta \Rightarrow \frac{d\ln(\varphi/\varphi_0)}{d\delta}|_\beta = -\beta. \] 

(2.18)

Therefore, in all frames we have the scale-relativistic invariant analog of proper time:

\[ \frac{(d\delta)^2}{\gamma^2} = \frac{(d\delta')^2}{\gamma'^2} = \frac{(d\delta'')^2}{\gamma''^2} = \ldots = \frac{(d\delta_o)^2}{\gamma_o^2} \frac{1}{\gamma_o^2} = (d\eta)^2. \] 

(2.19)

For a collection of fields, \( \varphi^i \), all with the same scaling dimensions one has the generalization of flat Minkowski spacetime:

\[ d\eta^2 = [\ln^2(\lambda_0/\Lambda)](d\delta)^2 - \frac{\sum_i (d\varphi^i)^2}{\sum_i (\varphi^i)^2}. \] 

(2.20)

from now on we shall omit the \( \varphi_o \) in our formulae for convenience but should not be forgotten!

The two-dim metric in (2.11) is flat: \( dT^2 - dU^2 \) with \( U = \ln(\varphi) \). Similarly, the metric in (2.20) is also flat as one can see by performing the suitable change of coordinates:

\[ \sum_i (d\varphi^i)^2 = \sum_i (d\varphi^i)^2 = \sum_i [d(ln\zeta^i)]^2 = \sum_i (dU^i)^2. \]

(2.21)

Setting \( \ln^2(\lambda_0/\Lambda) = 1 \), the interval becomes:

\[ d\eta^2 = [\ln^2(\lambda_0/\Lambda)](d\delta)^2 - \frac{\sum_i (d\varphi^i)^2}{\sum_i (\varphi^i)^2} = (d\delta)^2 - \sum_i (dU^i)^2. \] 

(2.22)

and it is invariant under scale-relativistic transformations. The interval (2.22) is the analog of the spacetime interval in Minkowski space of signature \((+, -, -, - , \ldots)\). This “completes” the review of Nottale’s scale relativity.

III. Strings as Gauge Theories of Area Scaling-Relativistic Transformations

3.1 Area-Scale-Relativity

In analogy with the transformations given in the previous section by Nottale we can define the scalings under area-resolutions where the Planck area, \( \Lambda^2 \), is chosen to be the minimum resolution of area in nature. Now we define:

\[ \ldots \]
\[ \varphi(x^1, x^2; \Delta x^1, \Delta x^2) \equiv \varphi(x^1, x^2; \Delta x^1 \wedge \Delta x^2) = \varphi(x^1, x^2; \Delta A). \quad (3.1) \]

Mutatis mutandis

\[ \Delta A = \Delta x^1 \wedge \Delta x^2 \to \Delta A' = \Delta x'^1 \wedge \Delta x'^2 = \rho \Delta A. \quad (3.2a) \]

\[ \ln(\varphi'/\varphi_o) = \frac{\ln(\varphi/\varphi_o) - \delta(\ln \rho)}{\sqrt{1 - \frac{\ln^2 \rho}{\ln^2(\lambda^2_o/\Lambda^2)}}}. \quad (3.2b) \]

\[ \delta' = \frac{\delta + \frac{(\ln \rho)(\ln \varphi_o)}{\ln(\lambda^2_o/\Lambda^2)}}{\sqrt{1 - \frac{\ln^2 \rho}{\ln^2(\lambda^2_o/\Lambda^2)}}}. \quad (3.3) \]

where the composition of area dilatations (analog of addition of velocities) is:

\[ \ln \rho = \frac{\ln(\Delta A/\Delta A_0) \pm \ln(\Delta A'/\Delta A)}{1 \pm \frac{\ln(\Delta A/\Delta A_0)\ln(\Delta A'/\Delta A)}{\ln^2(\lambda^2_o/\Lambda^2)}}. \quad (3.4) \]

A finite area-resolution Lorentz-scale transformation implies:

\[ \Delta A_0 \to \Delta A, \varphi_o(x, \Delta A_0) \to \varphi_o(x, \Delta A_0)e^{-\beta \delta}, \delta(\Delta A_0) \to \delta(\Delta A_0)(1 - \beta^2)^{-1/2}. \quad (3.5) \]

\[ \beta = \frac{\ln(\Delta A/\Delta A_0)}{\ln(\lambda^2_o/\Lambda^2)}. \]

Identical results occur for \( p \)-branes when volume-resolutions scaling-relativistic transformations replace area-scalings.

### 3.2 Area-Scale-Relativity and Strings

We are now ready to implement the scale-relativistic transformation to string theory and extended objects; i.e. to the Nambu-Goto actions. Let’s take the string case as example. The Dirac-Nambu-Goto action:

\[ S = \int d\sigma^1 d\sigma^2 \sqrt{\text{det} \{ G_{\mu\nu} \partial_{\sigma_1} X^\mu \partial_{\sigma_2} X^\nu \}}. \quad (3.6) \]

where \( G_{\mu\nu}[X^\mu(\sigma^1, \sigma^2)] \) is the target spacetime metric. Our purpose is to embed the \( X^\mu \) coordinates into a larger space whose generalized coordinates are \( Z^\mu \) and write now: \( Z^\mu(\sigma^1, \sigma^2, \Delta \sigma^1, \Delta \sigma^2) \) to denote the resolution dependence as well. Similar arguments apply to the superspace formulation of supergravity/supersymmetry where the bosonic coordinates are part of a larger space. The target spacetime coordinates are scalar fields from the world sheet point of view. There will be two main obstacles to overcome.
The first one is the following. If the $X^\mu$ are to play similar role as the previous scalars $\varphi^i$ with common scaling dimension, $\delta$ there will be difficulties in matching the coordinates with the $\varphi, \delta$. Because now there are two independent resolutions, $\Delta\sigma^1, \Delta\sigma^2$ the analog of velocity and scaling dimension will be:

$$
\beta_1 = \frac{\ln(\Delta\sigma^1/\Delta\sigma^1_o)}{\ln(\lambda_o^1/\Lambda)}, \quad \beta_2 = \frac{\ln(\Delta\sigma^2/\Delta\sigma^2_o)}{\ln(\lambda_o^2/\Lambda)}.
$$

$$
\delta_1 = (1 - \beta_1^2)^{-1/2}\delta_1(\Delta\sigma^1_o), \quad \delta_2 = (1 - \beta_2^2)^{-1/2}\delta_2(\Delta\sigma^2_o).
$$

(3.7)

where $(\lambda_o^1, \lambda_o^2)$ are the two reference scales with respect to which we measure the resolutions $\Delta\sigma^1, \Delta\sigma^2$, respectively. As such there are two independent scaling dimensions, $\delta_1, \delta_2$ and the two dimensional version of scaling transformations under $\Delta\sigma^1_o \to \Delta\sigma^1, \Delta\sigma^2_o \to \Delta\sigma^2$, are

$$
\varphi_o(\sigma^1, \sigma^2, \Delta\sigma^1_o, \Delta\sigma^2_o) \to \varphi_o(\sigma^1, \sigma^2, \Delta\sigma^1_o, \Delta\sigma^2_o)e^{-\beta_1\delta_1 - \beta_2\delta_2} = \varphi'(\sigma^1, \sigma^2, \Delta\sigma^1, \Delta\sigma^2).
$$

(3.8)

A problem arises if one wanted to match the $X^\mu$ coordinates with the $\varphi^i, \delta_1, \delta_2$ quantities because there are now two scaling dimensions but one temporal coordinate $X^0$. We won’t delve into the possibility of choosing two temporal dimensions. Spacetimes with different signatures have appeared recently in the string literature, in Vafa’s F theory, and Bars’ S theory, where spacetimes with $D = (10, 2), (11, 3)$, .. involving the propagation of extended objects with signatures $(p, p)$ must be incorporated to implement the duality symmetries associated with nonperturbative superstring theories. Nevertheless this could be a possible avenue to pursue and we will make some comments about this below. It is for this reason that it is more natural to study the scale relativity principle applied to areas instead of lengths within the context of string theory. $p$-branes will require $p + 1$ volume scalings and gauge theories of volume-scale relativity. We should not confuse, once again, resolutions with ordinary worldvolume coordinates and gauge theories of volume preserving diffs with volume-scale relativity of resolutions. In this fashion one has one scaling dimension instead of two and then we could match the $X^0$ with $\delta(\Delta A_0)$ transforming under area scalings as:

$$
\delta(\Delta A_0) \to \delta(\Delta A_0)(1 - \beta^2)^{-1/2}, \quad \beta = \frac{\ln(\Delta A/\Delta A_0)}{\ln(\lambda_o^2/\Lambda^2)}.
$$

(3.9)

The second obstacle is that now one should incorporate motion and scaling dynamics on equal footing. At this point the scaling dynamics has been trivial ( gauge degrees of freedom). If one wished to generalize matters, one must incorporate the resolution scaling dynamics and extend the notion of a metric to the resolution “displacements” of the type $d(\Delta\sigma^1), ..$; i.e. for intervals in the world sheet like:

$$
h_{\Delta\sigma^a, \Delta\sigma^b}d(\Delta\sigma^a)d(\Delta\sigma^b), h_{\Delta\sigma^a, \sigma^b}d(\Delta\sigma^a)d(\sigma^b), ....
$$

(3.10)

and write the generalization of the Dirac Nambu Goto action accordingly.
For example, supergravity can be visualized as the extension of ordinary Riemannian geometry to a supermanifold where the metric gauges translations and the gravitino gauges supersymmetry. Fields are now quantities depending on the superspace coordinates \((x, \theta)\) where \(\theta\) are the usual Grassmannian valued coordinates. The super Poincare group acts as transformations (translations, rotations) in superspace. A superspace metric and measure exists where the supervielbein has bosonic/fermionic entries (directions). In this same fashion we must treat the displacement of resolutions and its metric. The fields \(\varphi(\sigma, \Delta \sigma)\) must be viewed in the same vain as the superfield \(\Phi(x, \theta)\) with the difference that the resolutions are also bosonic variables. The fuzzy string action will now involve a generalized area in the extended space comprising coordinates and resolutions.

The fact that translations of an object can induced scalings in its size was formulated by Weyl himself using his field of dilatations. Nottale has made some interesting remarks in relation to the electric charge quantization and charge/mass ratios \([2]\) as results of scale-relativistic dilatation gauge invariance. Conversely, internal symmetries like strong interactions can induced spacetime diffs has been shown by Ne’eman and Sijacki \([7]\) in their version of chromo-gravity. To present a rough illustration of what is needed to merge scalings and motions into a single theory that we may label omega symmetry we will choose a four-dim “fuzzy” worldvolume whose coordinates are \(\sigma^A, \Delta \sigma^A\ A, B = 1, 2, 3, 4\). Its tangent space indices are labeled by lower case latin letters \(a, b = 1, 2, 3, 4\). The string coordinates are just scalars living on the worldvolume. The ordinary vielbein (graviton) has a correspondence with the scale-graviton:

\[
e_a^a A \partial_a \rightarrow \tilde{e}_{\Delta a}^a A \partial_{\Delta A}.
\]  

(3.12a)

The spin connection ↔ the scale-spin connection:

\[
\omega^{ab} A \rightarrow \tilde{\omega}_{\Delta a}^a \Delta b A \Sigma_{\Delta a} \Delta b.
\]  

(3.12b)

and so forth. The \(X^\mu\) string coordinates will behave like matter fields (sections of a bundle) and their partners (analogs of fermionic superpartners) will be the \(\Psi^\mu\) fields. Auxiliary fields would be needed in order to match degrees of freedom. Let’s call such symmetry that converts coordinates into resolutions, the omega symmetry. Ordinary covariant derivatives require the connections:

\[
D_A X^\mu = (\partial_A + \omega_{ab}^a + \tilde{\omega}_{\Delta A}^a \Delta b) X^\mu.....
\]  

(3.11c)

and the analogs of curvature/field strengths and torsion quantities would be:

\[
R \sim D \omega + \omega \wedge \omega. \quad \tilde{R} \sim D \tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega}. \quad T \sim D \tilde{e} + \omega \wedge e. \quad \tilde{T} \sim D \tilde{e} + \tilde{\omega} \wedge \tilde{e} +.....
\]  

(3.11d)

The actions are of the matter +geometry form:

\[
(D_A X^\mu)^2 + .... + R + \tilde{R} + R^2 + \tilde{R}^2 + torsion +.....
\]  

(3.12)

Instead of pursuing this approach at this moment we will opt to take the simplest of all scenarios below.
3.3 Gauge Theories of Area-Scale Relativistic Transformations

An alternative simpler route than the previous one of constructing the General Theory of Scale-Motion Relativity is to work in a “flat” background from the scalings point of view. We shall freeze the scaling dynamics by rendering them trivial in the sense that we will set the scaling-geometrical fields to their corresponding flat values; i.e the scale-spin connection \( \tilde{\omega} = 0 \), the scale-graviton \( \tilde{e} \) will be set to its flat value (imagine setting the gravitino to a constant multiple of the Dirac gamma matrices), the scaling-field strengths associated with the scaling-spin connection are zero etc... and we recur to the ordinary principle of gauge invariance. We shall incorporate the area-resolutions scale-relativistic transformations as an integral part of an internal space where the fuzziness of the string coordinates manifest themselves. We proceed first by working with \( D \)-dim target spacetime background and by matching the the \( X^\mu(\sigma^a) \) string variables, \( \mu = 0, 1, 2, \ldots, D-1 \) with the the original \( \varphi^i(\sigma^a, \Delta \sigma^a) \) fields. Where \( i = 0, 1, 2, \ldots, D-1 \). Since \( X^0 \) is a time coordinate one should Wick-rotate it to match the Euclidean form of the \( \varphi^0 \). It is not necessary to impose a factorization condition on the \( \varphi^i \) fields : \( \varphi^i(\sigma^a)\phi^i(\Delta \sigma^a) \) but instead one must view the \( X^\mu \) coordinates as matrix-valued :

\[
\varphi^i(\sigma^a, \Delta \sigma^a) \leftrightarrow X^\mu(\sigma^a) \equiv \Sigma^a \Delta \sigma^a X^\mu_{\Delta \sigma^a} (\sigma^a).
\]

where \( \Sigma^a \Delta \sigma^a \) are the generators of scale-relativistic transformations. These involve scaling-rotations and scaling-boosts and don’t differ from the usual Lorentz generators of the Lorentz group as we saw in section II. In the string case, these are : two resolutions scaling-boosts and one \( U(1) \)-like rotation giving a total of 3 generators. In the area-scaling case there will be one scaling-boost only and one rotation giving a total of two generators. In the four-dim worldvolume case the counting goes : 4 scaling-boosts, and 6 rotations giving a total of 10 generators, etc... In this fashion the resolution-dependence is encoded in the matrix-generators indices and the gauge transformation of the \( X^\mu \) matrices associated to the area-scale-relativistic transformations is then :

\[
X^\mu_o \rightarrow X^\mu(\sigma^a) = X^\mu_o(\sigma^a)e^{-\beta(\Delta A)\delta(\Delta A)}
\]

The matrices \( X^\mu \) simply “rotate” under resolution-scalings as matter fields in gauge theories do. They behave like field strenghts in ordinary gauge theories. In order to evaluate \((X^\mu)^2\) requires taking the trace w.r.t the matrix generators indices \( \Sigma^a \Delta^b \) (and not an integration w.r.t the internal space fiber coordinates, the \( \Delta \sigma^a \) space ). The matrix displacement \( dX^\mu \) involves taking derivatives w.r.t the \( \sigma^a \) variables and not w.r.t the \( \Delta \sigma^a \).

Since the scaling dimension \( \delta(\Delta A) \) still remains we must add an additional scaling time variable denoted by \( T(\Delta A) \) which solely depends on the area-resolutions. The new time variable can be thought of as a multiple of the constant unit matrix where the proportionality factor is a function of \( \Delta \sigma^a; \Delta A \) variables only. This is the analog of the spinorial representation of the spacetime coordinates in twistor methods using Pauli spin matrices : \( X^\mu \leftrightarrow X^0 1 + X^i \sigma_i \). It is no surprising to find similarities between Penrose’s description of twistors because in twistor space a point in complexified and compactified spacetime is smeared out (fuzzy) into a complex line in projective space. It is now when one embeds the
\( D \) dimensional space into a \( D + 1 \) space, \( Z^M \), where the common scaling dimension of the \( \varphi^i \) fields plays the role of the additional time coordinate. The scale-relativistic invariant world interval is then equated to (we have omitted the \( \varphi_0 \)):

\[
(d\eta)^2 = G_{MN}dZ^MdZ^N = dT^2 + G_{\mu\nu}dX^\mu dX^\nu \leftrightarrow (d\delta)^2 - \sum_i (d\varphi^i)^2 \sum_i (\varphi^i)^2.
\] (3.15)

\[
dX^\mu dX^\nu = tr[\Sigma^a \Sigma^b dX^\mu_{\Delta^a \Delta^b}(\sigma^a) \Sigma^c \Sigma^d dX^\nu_{\Delta^c \Delta^d}(\sigma^a)].
\]

\[
dX^\mu_{\Delta^a \Delta^b}(\sigma^a) = \frac{\partial X^\mu_{\Delta^a \Delta^b}(\sigma^a)}{\partial \sigma^a}d\sigma^a.
\]

\[
G_{\mu\nu}dX^\mu dX^\nu \leftrightarrow -\sum_i (d\varphi^i)^2 \sum_i (\varphi^i)^2. \quad X^\mu \leftrightarrow \int \frac{(dZ^m)}{\sqrt{\sum_n (Z^n)^2}} \leftrightarrow \int \frac{(d\varphi^i)}{\sqrt{\sum_i (\varphi^i)^2}}.
\] (3.16)

where \( iX^0 \leftrightarrow \varphi^0 \leftrightarrow Z^0 \).

The ordinary string action is:

\[
S = \int d^2\sigma \sqrt{\text{det}[h_{ab}]} \quad h_{ab} = G_{\mu\nu}\partial_aX^\mu\partial_bX^\nu.
\] (3.17)

with \( h_{ab} \) being the induced worldsheet metric due to the string’s embedding in the ordinary spacetime. Adding the scaling dimension as the extra time dimension yields the extended action:

\[
S = \int d^2\sigma \sqrt{\text{det}[H_{ab}]} \quad H_{ab} = G_{MN}\partial_aZ^M\partial_bZ^N.
\] (3.18)

and now \( H_{ab} \) is the induced worldsheet metric due to the string’s embedding into the extended space \( Z^M = T, X^\mu \) space of dimension \( D + 1 \). Imposing invariance of the extended action \( S \) under area-scaling-relativistic transformations:

\[
\delta_\beta S = 0 \Rightarrow \frac{1}{2} \sqrt{\text{det}[H_{ab}]}H^{ab}\delta_\beta H_{ab} = 0.
\] (3.19)

Eq-(3.19) is trivially satisfied as a result of the definition of the induced worldsheet metric. It is fairly clear that if one had started with a Lorentz-scale invariant metric (an invariant proper-time interval in the extended target spacetime), as a result of the embedding, the induced worldsheet metric, \( H_{ab} \) will automatically be scale invariant because under scalings of resolutions the coordinates, \( \sigma^a \) are inert. Since \( H_{ab}d\sigma^ad\sigma^b = G_{MN}dZ^MdZ^N \), and the latter interval is scale-relativistic invariant by construction, it follows that \( \delta_\beta(H_{ab}d\sigma^ad\sigma^b) = [\delta_\beta H_{ab}]d\sigma^ad\sigma^b = 0 \). Since this is true for all displacements \( d\sigma^a \) then \( \delta_\beta H_{ab} = 0 \). Therefore, the worldsheet area element must be invariant as well because each component of the two-dim metric \( H_{ab} \) is invariant under scale-relativistic
transformations. Let check that this is true. The metric $G_{MN}(Z^M)$ and the $Z^M$ obey the following:

$$
\delta_\beta H_{ab} = [\delta_\beta G_{MN}] \partial_a Z^M \partial_b Z^N + G_{MN} \partial_a [\delta_\beta Z^M] \partial_b Z^N + G_{MN} \partial_a Z^M \partial_b [\delta_\beta Z^N] = 0. \tag{3.20}
$$

if:

$$
\delta_\beta G_{MN} = (\partial_{Z^M} G_{MN}) \delta_\beta Z^M = (\partial_{Z^M} G_{MN}) A(\beta, \delta) Z^M = -2A G_{MN} = M(\beta, \delta) G_{MN} \Rightarrow -2A = M. \tag{3.21}
$$

and

$$
\delta_\beta G_{MN} = M(\beta, \delta) G_{MN}. \quad \delta_\beta Z^M = A(\beta, \delta) Z^M. \tag{3.22}
$$

From eqs-(2.2, 2.11) we learnt that:

$$
\delta_\beta (\delta_\beta) = (\Delta_\beta) \beta \gamma^2 \delta \Rightarrow \delta_\beta Z^m = -(\Delta_\beta) (\gamma^2 \delta) Z^m. \tag{3.23}
$$

since:

$$
G_{Z^DZ^D} = 1 \Rightarrow \partial_{Z^M} G_{Z^DZ^D} = 0 \Rightarrow \delta_\beta G_{Z^DZ^D} = 0. \quad \partial_a Z^D = 0.
$$

$$
\delta_\beta Z^m = -(\Delta_\beta) (\gamma^2 \delta) Z^m. \quad \partial_a (\delta_\beta Z^m) = -(\Delta_\beta) (\gamma^2 \delta) \partial_a Z^m.
$$

because the scaling dimension only depends on the resolutions $\Delta A$ and:

$$
G_{mn} = -\frac{\delta_{mn}}{\sum(Z^i)^2} \Rightarrow \partial_{Z^m} G_{mn} = \frac{2Z^m}{\sum(Z^i)^2} \Rightarrow \sum Z^m \partial_{Z^m} G_{mn} = -2G_{mn}. \tag{3.23}
$$

with $m = i = 0, 1, 2....D - 1$. Using the equations above it is straightforward to show that $\delta_\beta H_{AB} = 0$ due to relationship $2A + M = 0$ and $\delta_\beta G_{mn} = -2A G_{mn}$, without introducing any constraints whatsoever on the $Z^M$ variables because the $(\partial_a Z^M)(\partial_a Z^N)$ terms decouple (can be factored out). This problem was raised earlier in [3]. And conversely, if $\delta_\beta H_{AB} = 0$ one can show that the metric $G_{mn}, G_{Z^DZ^D}$ components have the required form as in eq-(3.15) if one does not wish to constrain the variables $Z^M$.

Hence one arrives at:

$$
Z^D(\Delta A) = T(\Delta A) \leftrightarrow \delta(\Delta A). \quad X^\mu \leftrightarrow \int \frac{d\varphi^i(\sigma^a, \Delta A)}{\sqrt{\sum_i(\varphi^i(\sigma^a, \Delta A))^2}}. \tag{3.25}
$$

and conclude that Dirac-Nambu-Goto actions are scale-relativistic invariant if, only if, one embeds the string in $D+1$ dimensions. The same argument applies to all $p$-branes/extendons, gauge invariance of actions under volume-resolutions scaling-relativistic transformations.
associated with the fuzzy world-volume in a $D+1$ spacetime with the extra scaling-temporal dimension.

**IV Concluding Remarks**

Summarizing, we have shown that area-resolutions scale-relativistic invariance is a symmetry of string theory that requires embedding the $D$ coordinates $X^\mu$ into $D+1$ dimensions with the extra temporal variable being precisely the common scaling dimension of all the string coordinates w.r.t scale-relativistic transformations. In one scoop we have achieved:

1) Why strings cannot probe distances below the Planck scale.
2) Why the string coordinates behave like matrices from the “fuzzy” world sheet point of view.
3) Why extra temporal dimensions appear in strings.
4) All extended objects admit similar symmetries when areas are replaced by volumes. Therefore, string theory, membranes, etc....are all unified in this manner.

It remains to study the string propagation in curved backgrounds from the *scalings* point of view and to write the scale relativity analog of the the Callan-Symanzik Renormalization Group Equation associated with a whole family of actions that respect scale-relativistic invariance. In eqs-(3.20-3.22) we saw that the form of the metric ( up to diffeomorphisms) is tightly constrained as a result of scale-relativistic invariance/covariance. “Scaled-curved” metrics must deviate from the scale-flat form in (3.15); i.e the scale-relativity version of Einstein’s equations obey equations of the Callan-Symanzik Renormalization Group type. This was also noticed by Nottale [2].

It is unknown why the string quantum effective actions give the classical background Einstein, Yang-Mills, antisymmetric tensor, dilaton,.....equations of motion. Strings can consistently propagate in those ( conformal) backgrounds if, and only if, the $2D$ CFT beta functions associated with the string couplings to the background fields vanishes. A sort of quantum/classical duality seems to be operating from the world-sheet/spacetime view. The idea that a quantum/classical duality might exist in Nature has been previously discussed by Nottale pertaining small/large scales: Quantum like structures emerge at the very large and the very small scales. The classical physics regime lives in between. This is another manifestation of the analog of the $T$ duality symmetry in string theory operating in scale-relativity.

The fact that scale-relativity invariance, at the classical level, already constrains the form of the background metric is a very promising fact that we believe may answer why there is a connection between the quantum string effective action and the classical background field equations. A very important work concerning universality and integrability has been provided by Fairlie et al [32]. An infinity of Lagrangians furnished the same universal equations of motion. Is scale-relativistic invariance tantamount of universality? In the sense that nonperturbative string physics exhibits duality symmetries among different Lagrangians that describe the same theory in different corners of the moduli space? In the same fashion that ordinary relativistic covariance is tantamount of the independence on the coordinate systems to describe physical phenomena, scale-relativity might signal the
independence of nonperturbative string physics on the redundant Lagrangian descriptions.

Area-momentum uncertainty relations of a string based on their propagation in loop spaces have been recently analyzed by [13, 21] where they studied the Hausdorff dimension and fractal like behaviour of the string’s world sheet. The area played the role of temporal evolution parameter. The scale relativity modifications of such area-momentum uncertainty are straightforward following our results in [3] based on [2]. Pavsic [20] also has discussed the propagation of strings and p-branes from a loop space point of view and wrote wave-functional equations of motion of the Wheeler-De Witt type (Schrodinger like).

The fact that points really do not exist as such due to their smearing and fuzzy-like behaviour into string-bits, area-bits, volume-bits, might bear important connection with the work of [24] and with Quantum Groups [25] and Connes Noncommutative Geometry. The latter made its first appearance in strings with Witten open string field theory formulation using BRST and path integral techniques which culminated with Zweibach closed string field theory based on Batalin-Vilkovisky Antibracket algebraic (operads) methods. Is the circle finally closed?

The extensions of ordinary 2D conformal field theories, $W_\infty$ CFT and $W_\infty$ geometry, deserves further study than the performed so far. What is $W_\infty$ geometry? does scale-relativity provide clues to find an answer? The connection among $W_\infty$ noncritical strings, affine Toda solitons, integrable models, continuous Toda theories, self-dual membrane, $SU(\infty)$ self dual Yang Mills, Plebanski’s heavenly equations, quantum Lie algebras, Moyal deformation quantization, etc was provided by the author in [29] based on earlier work by Chapline and Yamagashi, Nissimov and Pacheva and many others. We refer to [29] for an extensive list of references.

Physical applications of Finsler geometries in connection to the maximal proper four-accelerations in string theory (minimal scale) have been discussed by Brandt [8]. Conformal Weyl-Finsler structures has been studied by [9]. These Finsler metrics are no mathematical curiosities; these metrics are imposed by Stringy-Physics: “maximal” proper four accelerations. Weyl-Finsler Geometry is thus another natural and plausible geometrical setting to start (and attempt) to build the geometrical foundations of string theory. Weyl-Finsler geometries allow for the introduction of Torsion as well. Riemannian geometry is recovered in a certain limit. In particular, Einstein’s equations appear in the limit of infinite maximal proper acceleration by taking the $\Lambda \rightarrow 0$ limit, the analog of the $c \rightarrow \infty$ limit: the Galilean limit.

A challenging question would be if one can maintain scale-relativity invariance at the quantum level. The construction of General Scale-Motion Relativity remains open. We offered some clues at the end of 3.2. At the quantum level fractals should have a pivotal role. To go beyond the Quantum String Geometry [33] may require generalizations of Riemannian geometry that include curvilinear fractal coordinate systems [2]. A notion of fractal derivative, fractal integration, fractal measure, ....appearing in fractal geometries has already been built. What remains is to construct the version of metrics, connections, curvatures....that would enable us to define inertial and accelerated systems of reference in fractal spacetimes. We are unaware if such mathematical tools are available to-day and
for this reason we have followed the simplest route that has been paved over the years and that originated with Weyl: gauge invariance. Hints that p-adics numbers and non Archimedean geometries might very relevant to tackle this very difficult challenge at the Planck scale have been given among many others in [28].

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