A FLUX TUBE SOLAR DYNAMO MODEL BASED ON THE COMPETING ROLE OF BUOYANCY AND DOWNFLOWS

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ABSTRACT

A magnetic flux tube can be considered both as a separate body and as a confined field. As a field, it is affected by both differential rotation (Ω-effect) and cyclonic convection (α-effect). As a body, the tube experiences not only a buoyant force, but also a dynamic pressure due to downflows above the tube. These two competing dynamic effects are incorporated into the α-Ω dynamo equations through the total magnetic turbulent diffusivity, leading to a flux tube dynamo operating in the convection zone. We analyze and solve the extended dynamo equations in the linear approximation by adopting the observed solar internal rotation and assuming a downflow effect derived from numerical simulations of a solar convection zone. The model reproduces the 22 yr cycle period; the extended butterfly diagram with the confinement of strong activity to low heliographic latitudes |Φ| ≤ 35°; the evidence that at low latitudes the radial field is in an approximately π phase lag compared to the toroidal field at the same latitude; the evidence that the poleward branch is in a π/2 phase lag with respect to the equatorward branch; and the evidence that most of the magnetic flux is present in an intermittent form, concentrated into strong flux tubes.

Subject headings: convection — Sun: magnetic fields

1. INTRODUCTION

The magnetic activity of the Sun and late-type stars is currently attributed to a dynamo action, namely, the interaction of conducting fluid motions and magnetic fields in a suitable spherical shell. This was first suggested by Larmor (1919), but the development of these ideas proved to be very complex, especially in view of Cowling’s theorem (1934), which indicated that a dynamo could not operate with strictly axisymmetric magnetic fields. The generation and evolution of magnetic fields in the Sun is mostly described in terms of an α-Ω dynamo operating in the convection zone or in the boundary (overshoot) layer just beneath the convection zone, as the well known recent results of helioseismology may suggest. Although the physics of the interaction of rotation and convective modes is very complex, and poorly understood in spite of ever more refined attempts, the simple (in principle) approach based on differential rotation and cyclonic turbulence has been productive in modeling solar activity.

A good dynamo model should, of course, reproduce the main characteristics of the solar cycle: equatorward migration of spots and, in general, of active regions where strong magnetic fields are observed, starting from a latitude of about 35°; poleward migration of some magnetic tracers such as high-latitude prominences and polar faculae (e.g., Makarov & Sivaraman 1989); polarity reversals of the polar fields every 11 yr; polarity rules in the two hemispheres (Hale’s law); ratio and phase relation of the poloidal field to the toroidal field (Stix 1976); 22 yr cycle period; intermittent form of most of the magnetic flux, which is mainly concentrated into flux tubes; rigid rotation of the preferential longitudes (some fixed longitudes where spots show a rigid rotation); flip-flop effect in the preferential longitudes (namely the alternate occurrence of spots at these longitudes in the two 11 yr halves of the 22 yr magnetic cycle); and possibly long-term periodicities shown by solar activity, and absence of activity at some epochs (Maunder minima).

The location of the α-Ω dynamo action is an important theoretical problem. Over the years, several locations have been proposed: mostly the whole convection zone or the stable layers at the bottom of the convection zone (this has been done in an extensive series of papers starting from the fundamental work by Parker 1955); but also the boundary (overshoot layer) just beneath the convection zone (e.g., Belvedere et al. 1991; Rüdiger & Brandenburg 1995). It is usually assumed that the Ω and α-effects are located in the same region, since a spatial separation is not plausible, as this would imply problems of upward and downward magnetic field transport, which are not easily overcome by means of the turbulent diffusion. An exception are the so-called interface models (Parker 1993; Charbonneau & MacGregor 1997; Markiel & Thomas 1999), which assume the Ω-effect to operate on the low-diffusivity side of an interface (the base of the convection zone), while the α-effect works on the high-diffusivity side, and, in some aspects, also the advective models (Choudhuri et al. 1995; Durney 1995, 1996, 1997; Nandy & Choudhuri 2001; see also the recent work by Bonanno et al. 2002). In these models, the location of the two effects may be varied, sometimes having the characteristics of an interface dynamo, sometimes of a Babcock-Leighton dynamo (with a deeply located Ω-effect and a surface-located α-effect). The advection dynamos essentially work like a conveyor belt that is provided by the meridian circulation in the form of a single or more cells in the convection zone (Wang & Sheeley 1991; Dikpati & Charbonneau 1999).

Of course, the validity of the various approaches and models is measured by their capability to reproduce as many features of the solar activity as possible. But this is not the only criterion with
which to evaluate a dynamo model: in fact, the current framework of solar dynamo theory, which appears to be in agreement with the basic features of solar activity, arose from a series of attempts that dealt separately with partial, although fundamental aspects of the problem.

Among those attempts, in the framework of this paper, we consider flux tube dynamos. It is likely that, in future work on MHD of stellar interiors and dynamo theory, flux tube dynamics will be particularly important. Earlier studies on the dynamics of thin flux tubes, intermittently erupting at the solar surface, were carried out by, e.g., Schuessler (1980), Spruit (1981), Spruit & Van Ballegooijen (1982a, 1982b), van Ballegooijen (1983), Schmitt (1987), Ferriz-Mas & Schuessler (1994,1995), Caligari et al. (1995), and Schuessler et al. (1996).

In the thin tube approximation, the tube radius must be much smaller than its radius of curvature, the local scale height, the rotational shear scale, and the size of the relevant perturbations, including the large-scale turbulence size (i.e., the size of dominant eddies). The basic point is that strong axisymmetric toroidal fields located in the subadiabatic and relatively stable layers at the bottom of the convection zone, generated from poloidal fields by rotational shear, can be represented as an ensemble of isolated flux tubes, subjected to magnetic buoyancy and other perturbations such as aerodynamical drag, downward flows, and rotationally induced effects such as Coriolis force and differential centrifugal force.

In this paper, we argue that, due to the competition between two effects, namely buoyancy force and downward motions, the rising time of the tubes may be comparable to the timescale of the solar cycle. In fact, numerical simulations of the outer solar convection zone indicate a major presence of downward-moving plumes (Chan & Sofia 1989; Kim & Chan 1998; Nordlund 1999; Robinson et al. 2003). A similar idea was already presented by Parker (1975), who considered the aerodynamical drag as the competing effect. The new approach we adopt in this paper consists in giving a particular relevance to the downward flows and in incorporating the two competing effects into the mean field dynamo equations, by deriving an expression for the total turbulent diffusivity that takes into account the action of both buoyancy and downward motions. Then, by solving the linear problem in the usual perturbative methods, we obtain, in a self-consistent way, results in close agreement with the main observational results such as those of Schuessler et al. (1995), and Schmitt (1987), Ferriz-Mas & Schuessler (1994,1995), Caligari & Van Ballegooijen (1982a, 1982b), van Ballegooijen (1983), Schmitt (1987), Ferriz-Mas & Schuessler (1994,1995), Caligari et al. (1995), and Schuessler et al. (1996).

In this thin tube approximation, the tube radius must be much smaller than its radius of curvature, the local scale height, the rotational shear scale, and the size of the relevant perturbations, including the large-scale turbulence size (i.e., the size of dominant eddies). The basic point is that strong axisymmetric toroidal fields located in the subadiabatic and relatively stable layers at the bottom of the convection zone, generated from poloidal fields by rotational shear, can be represented as an ensemble of isolated flux tubes, subjected to magnetic buoyancy and other perturbations such as aerodynamical drag, downward flows, and rotationally induced effects such as Coriolis force and differential centrifugal force.

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downward flow has an inward dynamic pressure \( \rho_0 \xi^2 \). If this pressure can balance the buoyant force of the flux tube, then the flux tube can be stored in the convection zone, so that it can be amplified by the \( \alpha-\Omega \) dynamo. As a body, a magnetic flux tube experiences not only a buoyancy, but also a dynamic pressure from the downflows. The total pushdown force of downward plumes is equal to the dynamic pressure \( \rho \xi^2 \) times the total cross section of the plumes \( \xi \). The resulting acceleration \( g_c \) can be expressed as

\[
g_c = 2 sv_2^2 \xi/\pi a (1 - B^2/B^2_g)
\]

(10)

where \( s = 1 \) when \( v_2 > 0 \), \( s = -1 \) when \( v_2 < 0 \), and \( \xi = S/2aL \leq 1 \) is the fractional area of the downflows. The upward-moving plumes underneath the flux tube \( (s = 1) \) accelerate the magnetic buoyant diffusion of the tube, while the downward-moving plumes above the flux tube \( (s = -1) \) go against the magnetic diffusion. These downflows act as an antidiffusion process. In general, \(-1 \leq s \leq +1\), depending on the location of the flux tube.

The condition that the dynamic pressure of the downflows balances the buoyancy determines a critical value of the magnetic field \( B_c \), which is given by \( g_c + g_r = 0 \),

\[
B_c = B_g (-2sv_2^2 \xi/\pi a)^{1/2}.
\]

(11)

When \( B < B_c \), the buoyancy is smaller than the dynamic pressure of the downflows, and the dynamo magnetic field may grow; when \( B > B_c \), the dynamo field may damp. These two ingredients, buoyancy and downflows, are shown in Figure 1. Since downflows occur in the convection zone and the overshoot region near the base of the convection zone, this dynamo mechanism may operate in both of these regions.

The magnetic field in a flux tube can be considered to be a “mean field,” since the mean-field concept is relative (e.g., Krause & Rädler 1980). In fact, it depends on the spatiotemporal range over which the average is made. For the mean field in a flux tube, the spatiotemporal range is the volume and lifetime of a typical magnetic flux tube. With this in mind, the field in a flux tube obeys the classic dynamo equation (Parker 1955),

\[
\frac{\partial B}{\partial t} = \nabla \times (U \times B + \mathcal{E}) - \nabla \times (\eta \nabla \times B).
\]

(12)

Using this equation, we cannot study the formation of flux tubes from the large-scale magnetic field, but we can determine if the flux tube fields can be amplified. We assume a pure rotation motion \( U = \Omega \times r \), where \( \Omega \) is the angular velocity. This will cause the \( \Omega \)-effect.

Both the buoyant and dynamic diffusivities are direction dependent, so they are anisotropic. Such anisotropic diffusivities must be treated as tensors in principle (Krause & Rädler 1980). However, it is hard to do so in practice. To our knowledge, almost all dynamo models (including most conveyor-belt ones) consider thermal, dynamical, and magnetic diffusivities as scalars. All recent books that deal also with solar dynamo models (e.g., Stix 1989; Mestel 1999; Ruediger & Hollerbach 2004) quote and review only papers that assume scalar diffusivities. So, a solar model that includes the anisotropy of diffusivity in tensorial form is currently beyond the state of the art.

Physically, the tensorial form of diffusivity should be used only when a strong source of asymmetry exists. This is not the stellar case, where spherical symmetry is generally assumed, and no strong axial electromagnetic force, and thus no fast rotation, occur. Practically, the dynamo theory that uses the scalar form of diffusivities has captured the essentials of the solar cycle, and is able to reproduce the butterfly diagram and other observational facts. Mathematically, the scalar form of diffusivities can be considered to be the zero-order approximation of the general tensorial form. For these reasons, we use the following approximation form of the general electromagnetic force (Krause & Rädler 1980)

\[
\mathcal{E} \approx -\alpha B - \beta \nabla \times B.
\]

(13)

where

\[
\beta \approx \eta + \beta_t + \beta_b + \beta_s
\]

(14)

Defining

\[
\alpha \approx -\alpha^0; \quad \beta \approx \beta^0;
\]

(15)

we obtain the governing equation of a magnetic flux tube,

\[
\frac{\partial B}{\partial t} = \nabla \times (U \times B + \alpha B) - \nabla \times (\beta \nabla \times B).
\]

(16)

Since \( \beta \) depends on \( B \), this equation is nonlinear. In the azimuthal symmetric case, this equation is equivalent to the following two scalar equations:

\[
\frac{\partial A}{\partial t} = \alpha B + \beta \left( \frac{1}{r} \frac{\partial A}{\partial r} + \cot \theta \frac{\partial A}{\partial \theta} + \frac{\partial^2 A}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \phi^2} \right).
\]

(17)

\[
\frac{\partial B}{\partial t} = a_1 A + a_2 \frac{\partial A}{\partial r} + a_3 \frac{\partial A}{\partial \theta} - \alpha \left( \frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} \right) + b_1 B + b_2 \frac{\partial B}{\partial r} + b_3 \frac{\partial B}{\partial \theta} + \beta \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right).
\]

(18)

See Appendix A for the detailed derivation and definitions for those coefficients \( a_i \) and \( b_i \). Here \( B \) and \( A \) are defined in equation (1).

The magnetic diffusion caused by the microscopic gas collision motion, has been included in equation (12) in terms of the classic...
magnetic diffusivity \( \eta \nabla \times \mathbf{B} \). For a homogeneous convection, \( \alpha \) can be approximately expressed as (Krause & Rädler 1980)

\[
\alpha = \frac{1}{2} \mathbf{h} \tau_{\text{cor}},
\]

(19)

where \( \mathbf{h} = \mathbf{v} \cdot (\nabla \times \mathbf{v}) \) is the mean helicity over the correlation time \( \tau_{\text{cor}} \). But in the general case, \( \alpha \) may deviate from this expression. Since the turbulent velocity \( \mathbf{v} \) is related to the background and dynamo magnetic field \( \mathbf{B} \), \( \alpha \) depends on \( \mathbf{B} \). This means that \( \alpha \)-effects must be nonlinear.

In order to work out the various magnetic diffusivities mentioned above, we observe that the diffusivity of a force is proportional to its impulse per unit mass. The impulse is defined as the force times its action time. The impulse per unit mass has the dimension of acceleration \( \times \) time. The action time can be taken to be the dynamic response time of a magnetic tube. Since it is assumed that the process is isothermal in calculating the buoyant acceleration, the dynamic response time is the travel time of a perturbation across the diameter of the tube at the isothermal sound speed, \( C_s = (P/\rho)^{1/2} \), so \( t_{\text{act}} = 2a/C_s \). However, diffusivity has the dimension of acceleration \( \times \) time \( \times \) length. The shortest characteristic length for a flux tube is its diameter 2\( a \). We use this length \( a \) to meet the need of dimensionality for the buoyant and the dynamic diffusivity. Consequently, we have

\[
\beta_h = g_0 \left( \frac{2a}{C_s} \right) 2a = \left( \frac{\rho}{P} \right)^{1/2} \frac{4a^2 g B^2}{B_y - B^2},
\]

(20)

\[
\beta_v = g_0 \left( \frac{2a}{C_s} \right) 2a = \frac{\rho}{P} \frac{4a^2 \eta v^2 \xi B^2}{\pi (B_y - B^2)}.
\]

(21)

The magnetic turbulent diffusivity \( \beta_t = \frac{1}{2} \nu^2 \tau_{\text{cor}} \) for an isotropic turbulence is known (Krause & Rädler 1980), where \( \nu^2 \) is the turbulent velocity.

3. LINEAR ANALYSIS

The flux tube model equations obtained in the previous section are nonlinear partial differential equations. Since such equations are difficult to solve analytically, while a simple linear analysis may reveal some important characteristics about their solution, we analyze and solve the equation in the linear approximation in this paper. Unlike numerical simulations, the linear analysis cannot take into account full nonlinearity, although it reveals some nonlinear effects, such as critical magnetic fields of a flux tube.

For the sake of simplicity, we neglect \( \eta \) in this analysis, namely, we assume

\[
\beta \approx \beta_t + \beta_h + \beta_v.
\]

(22)

The second simplifying assumption is \( B^2 \ll B_y \), which means

\[
\beta_h \approx \left( \frac{\rho}{P} \right)^{1/2} \frac{4a^2 g B^2}{B_y},
\]

(23)

\[
\beta_v \approx \left( \frac{\rho}{P} \right)^{1/2} \frac{4a^2 \eta v^2 \xi B^2}{\pi (B_y - B^2)}.
\]

(24)

3.1. Linearized Dynamo Equations

Since \( \beta_t \) and \( \beta \) in equations (17)–(18) depend upon \( \mathbf{B} \), \( \partial \mathbf{B}/\partial r \), and/or \( \partial \mathbf{B}/\partial \theta \), we need to specify \( \mathbf{B} = \mathbf{B}_0 \) and \( \mathbf{A} = \mathbf{A}_0 \) as the reference state to investigate their perturbations \( \mathbf{B}' \) and \( \mathbf{A}' \) near the reference state. See Appendix B for the detailed linearization of equations (17)–(18). Omitting all the superscript and sub-script zeros (0), and bearing in mind that all the coefficients are evaluated at the reference state, we obtain the linearized dynamo equations from equations (B3)–(B4) when the \( \alpha \)-effects are neglected:

\[
\frac{\partial \mathbf{A}'}{\partial t} = \left( \alpha + c_0 \right) \mathbf{B}'
\]

(25)

\[
\begin{align*}
\frac{\partial \mathbf{B}'}{\partial t} &= a_1 \mathbf{A}' + a_2 \frac{\partial \mathbf{A}'}{\partial r} + a_3 \frac{\partial \mathbf{A}'}{\partial \theta} \\
&+ c_1 \mathbf{B}' + c_2 \frac{\partial \mathbf{B}'}{\partial r} + c_3 \frac{\partial \mathbf{B}'}{\partial \theta} + \beta \left( \frac{\partial^2 \mathbf{B}'}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \mathbf{B}'}{\partial \theta^2} \right),
\end{align*}
\]

(26)

where

\[
\begin{align*}
a_1 &= \frac{\Omega}{r} \left( \cos \theta \tilde{\Omega}_r - \tilde{\Omega}_\theta \sin \theta \right), \\
a_2 &= -\Omega \tilde{\Omega}_\theta \sin \theta, \\
a_3 &= \frac{\Omega}{r} \tilde{\Omega}_r \sin \theta, \\
c_0 &= \frac{2 \beta_h}{B} \left( \frac{\partial \mathbf{A}}{\partial r} \cot \theta \frac{\partial \mathbf{A}}{\partial \theta} + \frac{\partial^2 \mathbf{A}}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \mathbf{A}}{\partial \theta^2} \right), \\
c_1 &= \frac{1}{r^2} (4 \hat{\mathbf{B}}_r + 2 \hat{\mathbf{B}}_\theta \cot \theta) - \frac{1}{r^2} (2 \beta_h + \beta \csc^2 \theta + \frac{\hat{\beta}}{r^2}) \\
&+ \frac{2 \beta_h}{r^2} \left( \frac{\hat{\mathbf{B}}_r^2 + \hat{\mathbf{B}}_\theta^2}{\hat{\mathbf{B}}} + 2 \beta_h \frac{\partial^2 \mathbf{B}}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \mathbf{B}}{\partial \theta^2} \right), \\
c_2 &= \frac{2 \beta_h}{r} + \frac{2 \hat{B}_r}{r} (1 + 2 \hat{B}_r) + \frac{\hat{\beta}}{r}, \\
c_3 &= \frac{2 \beta_h}{r^2} \cot \theta + \frac{2 \beta_h}{r^2} (\cot \theta + 2 \hat{B}_r).
\end{align*}
\]

We define \( \tilde{\Omega}_r, \tilde{\Omega}_\theta \), and \( \hat{\beta} \) in Appendix A. Similarly, \( \hat{\mathbf{B}}_r \equiv \partial \ln B/\partial \ln r \), and \( \hat{\mathbf{B}}_\theta \equiv \partial \ln B/\partial \theta \).

3.2. Dispersion Relation

In order to investigate the linear instability, we try the wave-like solutions of the form

\[
\begin{align*}
\mathbf{B}' &= B'_0 \exp \left[ i(\omega t - \mathbf{k} \cdot \mathbf{x}) \right], \\
\mathbf{A}' &= A'_0 \exp \left[ i(\omega t - \mathbf{k} \cdot \mathbf{x}) \right]
\end{align*}
\]

(27)

(28)

where \( \omega \) is complex, while the other variables are real. Substituting equations (27)–(28) into equations (25)–(26), we obtain

\[
[i \omega + \beta (ik_r/r + ik_\theta \cot \theta/r^2 + k^2)]A'_0 = (\alpha + c_0)B'_0,
\]

(29)

\[
(i \omega - c_1 + ic_2 k_r + ic_3 k_\theta + \beta k^2)B'_0
\]

\[
= (a_1 - ia_2 k_r - ia_3 k_\theta + \alpha k^2)A'_0,
\]

(30)

where \( k^2 \equiv k_x^2 + k_z^2/r^2 \). The typical length scale in the radial direction is \( R_c \), while the typical angular scale in the latitude direction is \( \pi \); therefore we have \( k_r = 2\pi m/R_c \) and \( k_\theta = 2\pi n/\pi = 2n \), where \( m \) and \( n \) are nonzero integers. For the fundamental wave, \( m = n = 1 \). Consequently, \( k = 2[\pi^2 m^2 + (R_c/m^2)^2]^{1/2}/R_c \).
The dispersion equation can be obtained by multiplying equations (29) and (30). Defining
\[
\tilde{\omega} = i \omega + \frac{1}{2} \left[ \beta \left/ (r + c_2) \right. + \left. \beta \cot \theta + c_3 \right] k \right] - \frac{1}{2} c_1 + \beta k^2, \quad a_4 = a_2 k_r + a_3 k_\theta,
\]
\[
c_4 = \left/ (r + c_2) \right. k_r + \left. \beta \cot \theta + c_3 \right] k_\theta, \quad c_5 = \left/ (r + c_2) \right. k_r + \left. \beta \cot \theta + c_3 \right] k_\theta,
\]
\[
d_1 = c_1^2 - c_4^2 + 4c_5 + 4(a + c_3) a_1, \quad d_2 = 4c_1^2 + 4c_2 + \left/ (r + c_2) \right. \cot \theta + c_3 \right] k_\theta = 4(a + c_3) a_4 - 2c_1 c_4,
\]
we obtain
\[
\tilde{\omega}^2 = \frac{1}{4} (d_1 + id_2). \quad (31)
\]
Its solutions are
\[
\tilde{\omega}_+ = \pm \frac{1}{2} \left( \sqrt{D + d_1} + i \sqrt{D - d_1} \right), \quad (32)
\]
where
\[
D = (d_1^2 + d_2^2)^{1/2}. \quad (33)
\]
The required solution is \( \omega_+ \) because we need a positive real part of \( \omega \) to express the wave frequency:
\[
\omega = \omega_+ = \omega_0 + i \omega_1, \quad (34)
\]
where
\[
\omega_0 = \frac{1}{2} \left[ \sqrt{D - d_1} - \left/ (r + c_2) \right. k_r + \left. \beta \cot \theta + c_3 \right] k_\theta \right], \quad (35)
\]
\[
\omega_1 = - \frac{1}{2} \left[ \sqrt{D + d_1} + c_1 - 2 \beta k^2 \right]. \quad (36)
\]
The real part of \( \omega, \omega_0 > 0 \), determines the period of the dynamo wave, and the imaginary part of \( \omega, \omega_1 \), determines whether the dynamo wave grows (\( \omega_1 < 0 \)) or decays (\( \omega_1 > 0 \)).

The period \( T \) is defined by
\[
T = 2\pi / \omega_0,
\]
and the growth rate \( \Gamma \) is defined by
\[
\Gamma = - \omega_1.
\]
These are two basic parameters of a dynamo. From their expressions we can see which mechanisms play a role.

3.3. \( \alpha-\Omega \) Dynamo

When the buoyancy of the magnetic tube and the dynamic pressure of downflows are neglected, \( \beta \equiv 0 \), the dispersion relation reduces to the usual version of the dynamo wave:
\[
\omega = \sqrt{\alpha \left( \sqrt{D_0 - a_1} / 2 - i \sqrt{D_0 + a_1} / 2 \right)}, \quad (37)
\]
\[
(\beta/r + c_2) k_r + (\beta \cot \theta + c_3) k_\theta = 0. \quad (39)
\]
If we demand \((\beta/r + c_2)k_r = 0\) and \((\beta \cot \theta/r^2 + c_3)k_\theta = 0\), we find that the reference field satisfies

\[
\hat{B}_\theta = -\frac{1}{2} \left(1 + \frac{\beta}{\beta_b}\right) \cot \theta,
\]

(40)

\[
\hat{B}_r = -\frac{1}{4} \left(2 + 3 \frac{\beta}{\beta_b}\right) - \frac{\beta}{4\beta_b}.
\]

(41)

The amplitude is assumed to be proportional to the critical field \(B_c\). We can further calculate the second-order derivatives that appear in the expression of \(c_1\),

\[
\frac{\partial^2 B}{\partial \theta^2} = B \hat{B}_\theta^2 + \frac{1}{2} (1 + \beta/\beta_b) \csc^2 \theta + (\beta/\beta_b) \hat{B}_\theta \cot \theta,
\]

\[
\frac{\partial^2 B}{\partial r^2} = \frac{B}{r^2} \left(\hat{B}_\theta^2 - \hat{B}_r + \frac{3}{2} \beta \hat{B}_r + \frac{1}{4} (\hat{p} - \hat{P}) \hat{B}_r - \frac{1}{8} \frac{\beta \hat{p}}{\beta_b}(2 + \hat{P}) - \frac{3 \beta}{4 \beta_b} (2 + \hat{P}) + \frac{1}{8} \left(3 \frac{\partial \hat{p}}{\partial \ln r} + \frac{\partial \hat{P}}{\partial \ln r} - \frac{1}{\beta_b} \left( \frac{\partial \hat{p}}{\partial \ln r} - \frac{\partial \hat{P}}{\partial \ln r} \right) \right),
\]

where \(\hat{p} = \partial \ln \rho / \partial \ln r = \rho / H_p\) and \(\hat{P} = \partial \ln P / \partial \ln r = \rho / H_p\). Here \(H_p\) and \(H_p\) are density and pressure scale heights, respectively.

We assume \(A = \rho B\), where \(\rho\) is a constant factor. A negative \(\beta\) indicates that the poloidal field is in antiphase with the toroidal field. Consequently, we obtain

\[
\frac{\partial A}{\partial \theta} = \rho \hat{B}_\theta,
\]

\[
\frac{\partial A}{\partial r} = \rho \hat{B}_r,
\]

\[
\frac{\partial^2 A}{\partial \theta^2} = \rho \hat{B}_\theta^2,
\]

\[
\frac{\partial^2 A}{\partial r^2} = \frac{2 \rho B}{r} \hat{B}_r + \rho \frac{\partial^2 B}{\partial \theta^2}.
\]

These expressions are used to calculate \(c_0\).

Numerical calculations using the standard solar model (Winnick et al. 2002) show that

\[
c_1 < 0,
\]

\[
d_1 \approx c_1^2 + 4\alpha' a_1 \gg |d_2|,
\]

\[
d_2 \approx 4c_1(\beta k_r/r + k_\theta \cot \theta/r^2) + 4\alpha' a_4,
\]

where \(\alpha' = \alpha + c_0\). Since \(c_0\) depends on \(B\), \(\alpha'\) depends on \(B\) even if \(\alpha\) does not depend upon \(B\). The expression of \(c_0\) given above provides us with a concrete nonlinear \(\alpha\) model. Consequently, the general dispersion relation reduces to

\[
\omega \approx -\beta(k_r/r + k_\theta \cot \theta/r^2) + \alpha' a_1/c_1 + i(\beta k^2 + \alpha' a_1/c_1 + d_2^2/16c_1^2).
\]

(42)

Letting \(\alpha' = 0\), \(\Gamma \approx -\beta k^2\), which confirms the qualitative requirements stated above.

In order to investigate the phase relationship between the poloidal and toroidal field, we set \(\alpha = 0\). Since \(a_1 < 0\) near the surface, \(c_1 < 0\), and the sign of \(c_0\) is determined by the sign of \(f\), we see that the term \(\alpha' a_1/c_1 > 0\) when \(f\) is positive and vice versa. This shows that the growth rate \(\Gamma \approx -\beta k^2 + \alpha' a_1/c_1\) is larger when the two fields are in antiphase with each other than when they are in phase. Obviously, this antiphase relationship can be attributed to the nonlinear \(\alpha\) effect. Physically, it is due to the nonlinear interaction between the poloidal and toroidal fields.

The observed equatorward and poleward branches of the solar magnetic field wave suggest

\[
k_r = \begin{cases}
2\pi(+1)/R_\odot & \text{if } \theta > 45\degree, \\
2\pi(-1)/R_\odot & \text{if } \theta < 45\degree,
\end{cases}
\]

(43)

\[
k_\theta = \begin{cases}
2\pi(+2)/\pi & \text{if } \theta > 45\degree, \\
2\pi(-2)/\pi & \text{if } \theta < 45\degree.
\end{cases}
\]

(44)

The third term in the expression of the growth rate,

\[
\Gamma = -(\beta k^2 + \alpha' a_1/c_1 + d_2^2/16c_1^2),
\]

may explain the observed branching if we use the following \(\alpha\) model:

\[
\alpha = \begin{cases}
\alpha + c_0 & \text{if } \theta < 45\degree \text{ and } k_r > 0 \text{ and } k_\theta > 0, \\
-\alpha_0 & \text{if } \theta < 45\degree \text{ and } k_r < 0 \text{ and } k_\theta > 0, \\
-c_1 \text{ if } \theta > 45\degree \text{ and } k_r > 0 \text{ and } k_\theta < 0, \\
+c_1 \text{ if } \theta > 45\degree \text{ and } k_r < 0 \text{ and } k_\theta < 0.
\end{cases}
\]

(45)

where \(\alpha_0\) and \(\alpha_1\) may slightly depend on \(\theta\), but may strongly depend on \(r\), since the important \(\theta\)-dependence of \(\alpha\) has been written down explicitly. This is another nonlinear \(\alpha\) effect, different from that specified by \(c_0\). In fact, \(\alpha\) can be represented by the perturbation magnetic field \(B'\) as (e.g., Seehafer et al. 2003)

\[
\alpha = \frac{1}{2} \tau_{cor} B' \cdot (\nabla \times B').
\]

Using the wavelike solution, equation (46), one can obtain equation (45).

We have two options to qualitatively explain why the poleward branch is weaker than the equatorward branch:

1. \(\alpha_1 < \alpha_0\).
2. The observed poleward branch is the reflection wave, which is one part of the original wave.

Because of the phase delay of the poleward branch with respect to the equatorward branch, the second option is preferable.

Solving equation (29) for \(A'_r\), we obtain

\[
A' = R_1 e^{\Phi_1} B',
\]

where

\[
R_1 \approx |\alpha'| \left[ (d_2^2/16c_1^2) + (\alpha' a_4/c_1)^2 \right]^{1/2},
\]

\[
\Phi_1 \approx \arctan \left[ \alpha' a_4/c_1 - (d_2^2/16c_1^2) \right].
\]

Using \(R_1\) and \(\Phi_1\), we can express the temporal evolution of the toroidal magnetic field \(B\) and the poloidal vector potential \(A\).
in terms of the initial field $B_0$ for each magnetic flux tube as follows:

$$B = B_0 e^{\Gamma t} \begin{cases} 
\exp \left[i \left( \frac{2\pi t}{T} - k \cdot x + \Phi_0 \right) \right] & \text{if } \theta > 45^\circ, \\
\exp \left[i \left( \frac{2\pi t}{T} + k \cdot x + \Phi_0 \right) \right] & \text{if } \theta < 45^\circ,
\end{cases}$$

$$A = B_0 R e^{\Gamma t} e^{i\Phi_0} \begin{cases} 
\exp \left[i \left( \frac{2\pi t}{T} - k \cdot x + \Phi_0 \right) \right] & \text{if } \theta > 45^\circ, \\
\exp \left[i \left( \frac{2\pi t}{T} + k \cdot x + \Phi_0 \right) \right] & \text{if } \theta < 45^\circ,
\end{cases}$$

where $\Phi_0$ is determined by the initial condition. Substituting these expressions into equation (1), we obtain

$$B = B_0 e^{\Gamma t} \begin{cases} 
\exp \left[i \left( \frac{2\pi t}{T} - k \cdot x + \Phi_0 \right) \right] \times \left[ e_\phi + (R_{1/r}) e^{i\psi} \right] & \text{if } \theta > 45^\circ, \\
\exp \left[i \left( \frac{2\pi t}{T} + k \cdot x + \Phi_0 \right) \right] \times \left[ e_\phi + (R_{1/r}) e^{i\psi} \right] & \text{if } \theta < 45^\circ,
\end{cases}$$

where

$$a^- = (\cot \theta - i k_0) e_\phi - (1 - i r k_0) e_\theta,$$

$$a^+ = (\cot \theta + i k_0) e_\phi - (1 + i r k_0) e_\theta.$$ 

For the equatorward branch, we can investigate how its peak propagates with time by setting

$$2\pi / T - k_r r - k_0 \theta + \Phi_0 = (n - 1) \pi,$$ 

where $n$ is equal to the integer part of $1 + 2t/T$. Since $k_r = \pi / R_\odot$ and $k_0 = 4$, we obtain the migration equation of the wave peak

$$\theta = 2 \pi / 9 + [2t / T - (n - 1)] \pi / 4,$$ 

where we have assumed $\theta = 40^\circ$ when $t = 0$. Obviously, the integer $n$ plays the role of cycle number.

For the poleward branch, we can only see the reflected wave because it propagates into the interior of the Sun, $k_r = -2 \pi / R_\odot$. The reflection point is assumed to be located at the midpoint of the convection zone for the dynamo waves generated near the surface layers. This leads to $r_{\text{ref}} \approx 0.86 R_\odot$. The phase velocity of the wave in the radial direction approximately equals 100 cm s$^{-1}$. Therefore, it takes about $t_0 = 6$ yr for the wave to travel from the source region ($r_{\text{src}} = 0.9995 R_\odot$) to the reflection point ($r_{\text{ref}} = 0.86 R_\odot$), and then to the surface ($r = R_\odot$). As a result, the wave peak propagation equation for the poleward branch reads

$$2\pi (t + t_0) / T + k_r r + k_0 \theta + \Phi_0 = (n' - 1) \pi,$$ 

where $n'$ is equal to the integer part of $1 + 2(t + t_0)/T$. The solution of this equation is

$$\theta = 5 \pi / 18 - [2(t + t_0) / T - (n' - 1)] \pi / 4,$$ 

where we have assumed $\theta = 50^\circ$ when $t + t_0 = 0$. This implies a phase lag of $\pi/2$ with respect to the equatorward branch, as shown in Figure 3, which plots equations (48) and (50).

4. COMPARISON WITH OBSERVATIONS

In this section we try to compare the anti-$\beta$ dynamo model with relevant observations. The model parameters are as follows: (1) the downflow velocity $v_z \approx -2.1 \times 10^8$ cm s$^{-1}$ is fixed by the third numerical simulations (Robinson et al. 2003); (2) we adjust the flux tube radius $a$ to obtain the observed cycle period $T \approx 22$ yr; (3) $\xi = 1$, which means that the flux tube is totally located within the downflow regions; (4) the reflection point of the dynamo wave is assumed to be located at the midpoint of the convection zone; (5) the reference magnetic fields vary from zero to the critical field $B_c$, which means that we should integrate all the observable quantities such as the cycle period over the reference field from zero to $B_c$; (6) $\beta = \frac{1}{2} \nu_\phi (\rho \phi)^{-2} 2 \pi; (7)$ $\alpha = 0$; and (8) $f = -10^{-9}$.

4.1. Period

The $\beta$-effect dominates in the anti-$\beta$ dynamo model; the period $T$ near the surface can be approximately expressed as

$$T(a, \theta) = 2\pi / \left\{ \begin{array}{ll}
\frac{\beta}{f_0} - \beta \left( \frac{2 \pi}{R_\odot} + \frac{4 \cot \theta}{R_\odot^3} \right) dB / B_c & \text{if } \theta > 45^\circ, \\
\frac{\beta}{f_0} - \beta \left( \frac{2 \pi}{R_\odot} + \frac{4 \cot \theta}{R_\odot^3} \right) dB / B_c & \text{if } \theta < 45^\circ.
\end{array} \right.$$ 

If one fixes $T \approx 22$ yr, one finds $a = (1.15, 1.1, 1.05, \ldots, 0.65, 0.59, 0.53, 0.45, 0.37, 0.27, 0.15)$ Mm for $\theta = (5^\circ, 10^\circ, 15^\circ, \ldots, 55^\circ, \ldots, 85^\circ)$, respectively. This is comparable to 2 Mm, the typical size of flux tubes observed in the low-latitude region.

4.2. Growth Rate

Similarly, the growth rate $\Gamma$ of a flux tube near the surface can be approximately expressed as

$$\Gamma(a, \theta) = \int_0^{B_c} -\beta k^2 dB / B_c \quad \text{for } 0^\circ < \theta < 90^\circ.$$
where \( k^2 = (2\pi + 4)^2/R_0^2 \). For the parameters given above, one finds \( \Gamma^{-1} = (0.46, 0.48, 0.51, 0.53, 0.56, 0.59, 0.62, 0.66, 0.71, 0.76, 0.82, 0.94, 1, 1.18, 1.44, 1.97, 3.54) \) yr. This shows that the growth time of a typical flux tube of size \( a = 2 \) Mm is about several months.

The flux tubes should stay in the convection zone long enough (i.e., for a time longer than or equal to their growth time) for them to be amplified by the dynamo action. Since our formulation has already included the buoyant dissipation effect, a positive growth rate measured by the growth time indicates that the downflows can inhibit the buoyant rise of the flux tube on the timescale specified by the growth time. Here we assume that the flux tube is so distorted that it is totally immersed in the downflow regions by setting \( \xi = 1 \). From Figure 5 of Fan et al. (2003), we can see that only a small fraction of their flux tube is immersed in the downflow regions, which means that their \( \xi \) parameter is much smaller than the value we use. Therefore, our results are not in conflict with their three-dimensional simulation, which shows that the pumping has almost no effect in inhibiting the buoyant rise of the flux tube when the energy of the tube is larger than that of the surrounding environment. If the energy of the tube is comparable to that of the surrounding area, it still rises to the surface, but is largely distorted. If we use a small \( \xi \) parameter value, we can obtain similar results.

4.3. Active Regions

Here we have two explanations of how the strong activity may be confined to low heliographic latitudes \( \{\Phi \} \leq 35^\circ \) of the equatorward branch, as observed:

1. The growth rate at low latitude is larger than that at high latitude; or
2. what we observe at high heliographic latitudes are the reflected waves. They must be weaker than the original waves, since the reflection is not necessarily complete.

4.4. Phase Dilemmas

The phase dilemmas are twofold:

1. Equation (46) shows that the phase difference between the poloidal and toroidal field is \( \Phi_c \). Calculations show \( \Phi_c \approx 170^\circ \) for \( 0^\circ < \theta < 180^\circ \). This is in good agreement with the observation; the poloidal field is almost in antiphase with the toroidal component.
2. The \( \pi/2 \) phase lag of the poleward branch with respect to the equatorward branch can be explained by the time lag of 6 yr for the former to travel from the source region to the reflection point, and then from the reflection point to the surface.

4.5. Butterfly Diagram

Figure 3 depicts the latitude of the peak of the dynamo wave as a function of time. Only when we select the midpoint of the convection zone as the reflection point is the time delay of the poleward branch about 6 yr. This is a bit arbitrary.

4.6. Cyclic Variations of Solar Global Parameters and Helioseismic Frequencies

We construct models of the structure and evolution of the Sun that include variable magnetic fields and turbulence. The magnetic effects are (1) magnetic pressure, (2) magnetic energy, and (3) magnetic modulation to turbulence. The effects of turbulence are (1) turbulent pressure, (2) turbulent kinetic energy, (3) turbulent inhibition of the radiative energy loss of a convective eddy, and (4) turbulent generation of magnetic fields. Using these ingredients we construct five types of solar variability models (including the standard solar model) with magnetic effects. These models are in part based on three-dimensional numerical simulations of the superadiabatic layers near the surface of the Sun. The models are tested with several sets of observational data, namely, the changes of (1) the total solar irradiance, (2) the photospheric temperature, (3) radius, (4) the position of the convection zone base, and (5) low- and medium-degree solar oscillation frequencies. We find that turbulence plays a major role in solar variability, and only a model that includes a magnetically modulated turbulent mechanism can agree with all the currently available observational data. We find that because of the somewhat poor quality of all observations (other than the helioseismological ones), we need all data sets in order to restrict the range of models. These studies, presented in Li et al. (2003), strongly suggest a near-surface dynamo model. The anti-\( \beta \) dynamo, which operates near the surface, may meet this need.

5. CONCLUSIONS

Helioseismic observations of the cyclic variations of low- and medium-degree solar oscillation frequencies are very precise. The cyclic variation of the position of the convection zone base, obtained by helioseismic inversions (Basu & Antia 2000; Basu 2002), is also very accurate. These high-precision observations seem to exclude the location of strong magnetic fields at the base of the convection zone or in the overshoot layer, as is generally assumed on the basis of stability against magnetic buoyancy and equipartition arguments. This is one of the main motivations to reconsider dynamos located in the convection zone. The second motivation arises from trying to reproduce the observed cyclic variations of the total solar irradiance (Fröhlich & Lean 1998), the solar effective temperature (Gray & Livingston 1997), and the solar radius (see § 2.3 of Li et al. 2003 for a brief review). Extensive numerical experiments (Li & Sofia 2001) show that only magnetic fields located in the convection zone can simultaneously produce the observed cyclic variations of these three global solar parameters. The third motivation originates from numerical simulations of the solar convection zone (Chan & Sofia 1989; Nordlund 1999; Robinson et al. 2003), because they indicate a major presence of downward-moving plumes of high velocity.

However, convection zone dynamos have to face the magnetic buoyancy instability problem. The best way to solve this problem is to incorporate the buoyancy into the dynamo equations. This forces us to treat the mean field in a flux tube. To include the dynamic pressure of a flow, the flux tube should be regarded as a body. As a field, both the cyclonic convection (\( \alpha \)-effect) and differential rotation (\( \Omega \)-effect) play a role. As a body, the tube experiences not only a buoyant force, but also a dynamic pressure of downflows above the tube. We show that these two dynamic effects can be incorporated into the dynamo equations by converting them into two diffusion terms.

We analyzed and solved the extended dynamo equations in the linear approximation by adopting the observed solar internal rotation rate, and by assuming a downflow suggested by numerical simulations of the solar convection zone. The results are in agreement with the basic properties of the solar cycle, and give a theoretical basis for understanding the variations of solar irradiance, temperature, radius, and helioseismic frequencies in the course of the magnetic cycle.
Finally, the model results give a theoretical basis for interpreting the observed cyclic variations of the total solar irradiance, the solar effective temperature, the solar radius, and the helioseismic frequencies (Li et al. 2003, 2002; Li & Sofia 2001; Sofia & Li 2001).

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APPENDIX A

DYNAMO EQUATIONS IN SPHERICAL COORDINATES

In this paper we assume azimuthal symmetry, which means that all quantities do not depend on the azimuthal coordinate \( \phi \).

Using equation (1) to expand equation (16), we obtain

\[
\frac{\partial B}{\partial t} \phi + \nabla \times \left[ \left( \frac{\partial A}{\partial t} - \alpha B - \beta \nabla^2 A \right) \right] = \nabla \times \left\{ U e_\phi \times \left[ \nabla \times (A e_\phi) \right] + \alpha \nabla \times (A e_\phi) - \beta \nabla \times (B e_\phi) \right\},
\]

(A1)

where we have used

\[
U = \Omega \times r = r \Omega \sin \theta e_\phi = U e_\phi.
\]

(A2)

We have also used the mathematical equality

\[
\nabla \times \nabla \times (A e_\phi) = \nabla (\nabla \cdot (A e_\phi)) - \nabla^2 (A e_\phi) = -e_\phi \nabla^2 A.
\]

(A3)

The second equality holds well when \( A \) does not depend on \( \phi \).

We use the following mathematical equality to calculate the curl of a vector field \( F \):

\[
\nabla \times F = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta F_\theta \right) - \frac{\partial F_\phi}{\partial \phi} \right] e_r + \frac{1}{r} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta F_r \right) - \frac{\partial}{\partial \phi} (r F_\phi) \right] e_\theta + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r F_\phi) - \frac{\partial F_r}{\partial \theta} \right] e_\phi.
\]

(A4)

Since we assume that the system is in azimuthal symmetry, we know

\[
\nabla \times (F e_\phi) = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta F_\theta \right) e_r - \frac{1}{r} \frac{\partial}{\partial \phi} (r F_\phi) e_\theta
\]

(A5)

for \( F = F(r, \theta) e_\phi \). Using these formulae in equation (A1), we obtain

\[
\left( \frac{\partial B}{\partial t} - \frac{M}{r} \right) e_\phi + \nabla \times \left[ \left( \frac{\partial A}{\partial t} - \alpha B - \beta \nabla^2 A \right) e_\phi \right] = 0,
\]

(A6)

where

\[
M = r \nabla \times \left\{ \left[ \frac{U \partial}{\partial r} \left( r A \right) + \frac{\alpha}{r \sin \theta} \partial (\sin \theta) - \frac{\beta}{r \sin \theta} \partial (\sin \theta) \right] e_r + \left[ \frac{U}{r \sin \theta} \partial \left( \frac{\alpha}{r} \right) + \frac{\beta}{r} \partial \left( \frac{\beta}{r} \right) \right] e_\theta \right\} e_\phi.
\]

(A7)

Equation (A5) shows that the second term of equation (A6) on the left-hand side is perpendicular to the first term. Consequently, equation (A6) is equivalent to the following two scalar equations:

\[
\frac{\partial A}{\partial t} = \alpha B + \beta \left( \frac{1}{r} \frac{\partial A}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial A}{\partial \theta} + \frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} \right),
\]

(A8)

\[
\frac{\partial B}{\partial t} = a_1 A + a_2 \frac{\partial A}{\partial r} + a_3 \frac{\partial A}{\partial \theta} - \alpha \left( \frac{\partial^2 A}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} \right) + b_1 B + b_2 \frac{\partial B}{\partial r} + b_3 \frac{\partial B}{\partial \theta} + \beta \left( \frac{\partial^2 B}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 B}{\partial \theta^2} \right),
\]

(A9)
where we have used
\[ \nabla^2 A = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial A}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial A}{\partial \theta} \right) = \frac{2}{r} \frac{\partial A}{\partial r} + \cot \theta \frac{\partial A}{\partial \theta} + \frac{\partial^2 A}{r^2} + \frac{1}{r^2} \frac{\partial^2 A}{\partial \theta^2} \] (A10)
and defined
\[ a_1 = \frac{1}{r} \left[ \frac{\partial}{\partial r} \left( U \cot \theta - \alpha \right) - \frac{1}{r} \frac{\partial}{\partial \theta} (U + \alpha \cot \theta) \right] = \frac{1}{r} \left( \Omega \cos \hat{\Omega} r - \Omega \hat{\Omega} \sin \theta + \frac{\alpha}{r} \csc^2 \theta - \frac{\partial \alpha}{\partial \theta} - \frac{1}{r} \cot \theta \frac{\partial \alpha}{\partial \theta} \right), \]
\[ a_2 = \frac{1}{r} \left\{ \left( U \cot \theta - \alpha \right) - \frac{\partial}{\partial r} (\alpha r) \right\} = -\Omega \hat{\Omega} \sin \theta - \frac{2\alpha}{r} \frac{\partial \alpha}{\partial r}, \]
\[ a_3 = -\frac{1}{r} \left\{ \frac{U + \alpha \cot \theta}{r} + \frac{\partial}{\partial \theta} \left( \alpha/r \right) \right\} = \frac{1}{r} \left( \Omega \hat{\Omega} \left( \sin \theta - \frac{\alpha \cot \theta}{r} - \frac{1}{r} \frac{\partial \alpha}{\partial \theta} \right) \right), \]
\[ b_1 = \frac{1}{r} \left[ \frac{\partial \beta}{\partial r} + \frac{\partial}{\partial \theta} \left( \frac{\beta \cot \theta}{r} \right) \right] = \frac{2\beta_b}{r^2} \left( \frac{\partial B}{\partial r} + \cot \theta \frac{\partial B}{\partial \theta} - \frac{\partial \beta \csc \theta}{r^2} + \frac{\hat{\beta}}{r^2} \right), \]
\[ b_2 = \frac{1}{r} \left[ \beta + \frac{\partial}{\partial r}(\beta r) \right] = \frac{2}{r^2} \left( \frac{\partial \beta}{\partial \theta} + \frac{\hat{\beta}}{r} \right), \]
\[ b_3 = \frac{1}{r} \left[ \beta \frac{\partial \cot \theta}{\partial r} + \frac{\partial}{\partial \theta} (\beta/r) \right] = \frac{1}{r^2} \left( \beta \cot \theta + \frac{2\beta_b \partial B}{B} \right). \] (A11)

The valid condition for the second equality sign is \( B \ll \sqrt{8\pi P} \). We define \( \hat{\Omega} = \partial \ln \Omega / \partial \ln r \) and \( \hat{\Omega} = \partial \Omega / \partial \theta \). We also define
\[ \hat{\beta} \equiv \left( \frac{\partial \beta}{\partial \ln r} \right) = \frac{\beta_b}{2} \left( \frac{\partial \ln \mu}{\partial \ln r} - \frac{\partial \ln P}{\partial \ln r} \right) - \beta_b \left( \frac{2}{2} \frac{\partial \ln P}{\partial \ln r} - \frac{1}{2} \frac{\partial \ln \mu}{\partial \ln r} - \frac{1}{2} \frac{\partial \ln P}{\partial \ln r} \right) + \beta_b \left( \frac{2}{2} \frac{\partial \ln P}{\partial \ln r} - \frac{1}{2} \frac{\partial \ln \mu}{\partial \ln r} \right). \]

**APPENDIX B**

**LINEARIZATION OF DYNAMO EQUATIONS**

In order to linearize equations (17)–(18), we decompose \( B \) and \( A \) into two parts:
\[ B = B_0 + B', \]
\[ A = A_0 + A', \] (B1)
(B2)
where \( B_0 \) and \( A_0 \) represent the reference state of the system, while \( B' \) and \( A' \) represents the perturbations of \( B \) and \( A \) near \( B_0 \) and \( A_0 \). Substituting equations (B1)–(B2) into equations (17)–(18), we obtain two linear equations that govern \( B' \) and \( A' \),
\[ \frac{\partial A'}{\partial t} = \left( \alpha + c_0 \right) A' + \beta_b \left( \frac{1}{r} \frac{\partial A'}{\partial r} + \cot \theta \frac{\partial A'}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 A'}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 A'}{\partial \theta^2} \right), \] (B3)
\[ \frac{\partial B'}{\partial t} = a_1 A' + a_2 \frac{\partial A'}{\partial r} + a_3 \frac{\partial A'}{\partial \theta} - \alpha \left( \frac{\partial^2 A'}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 A'}{\partial \theta^2} \right) + c_1 B' + c_2 \frac{\partial B'}{\partial r} + c_3 \frac{\partial B'}{\partial \theta} + \beta_0 \left( \frac{\partial^2 B'}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 B'}{\partial \theta^2} \right), \] (B4)
where superscript or subscript 0 means that the quantities are evaluated at the reference state, for example, \( a_1 = a_1 |_{B=B_0,A=A_0} \), and
\[ c_0 = \frac{2\beta_b}{B_0} \left( \frac{1}{r} \frac{\partial A_0}{\partial r} + \cot \theta \frac{\partial A_0}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 A_0}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 A_0}{\partial \theta^2} \right), \]
\[ c_1 = b_1 \frac{\partial B_0}{\partial B_0} + \frac{\partial b_1}{\partial B_0} B_0 + \frac{\partial b_2}{\partial B_0} B_0 + \frac{\partial b_3}{\partial B_0} B_0 + \frac{2\beta_b}{B_0} \left( \frac{\partial^2 B_0}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 B_0}{\partial \theta^2} \right), \]
\[ c_2 = b_2 \frac{\partial B_0}{\partial B_0} + \frac{\partial b_2}{\partial B_0} B_0 + \frac{\partial b_3}{\partial B_0} B_0 + \frac{\partial b_4}{\partial B_0} B_0 + \frac{\partial b_5}{\partial B_0} B_0, \]
\[ c_3 = b_3 \frac{\partial B_0}{\partial B_0} + \frac{\partial b_3}{\partial B_0} B_0 + \frac{\partial b_4}{\partial B_0} B_0 + \frac{\partial b_5}{\partial B_0} B_0. \]
Here \( B'_0 = \partial B_0 / \partial r \) and \( B''_0 = \partial B_0 / \partial \theta \). Using the expressions for \( b \) given in Appendix A, we can calculate their derivatives with respect to \( B_0 \), \( B'_0 \), and \( B''_0 \):