Finite-Sample Average Bid Auction*

Haitian Xie†

August, 2020

Abstract

The paper studies the problem of auction design in a setting where the auctioneer accesses the knowledge of the valuation distribution only through statistical samples. A new framework is established that combines the statistical decision theory with mechanism design. Two optimality criteria, maxmin, and equivariance, are studied along with their implications on the form of auctions. The simplest form of the equivariant auction is the average bid auction, which set individual reservation prices proportional to the average of other bids and historical samples. This form of auction can be motivated by the Gamma distribution, and it sheds new light on the estimation of the optimal price, an irregular parameter. Theoretical results show that it is often possible to use the regular parameter population mean to approximate the optimal price. An adaptive average bid estimator is developed under this idea, and it has the same asymptotic properties as the empirical Myerson estimator. The new proposed estimator has a significantly better performance in terms of value at risk and expected shortfall when the sample size is small.

JEL Classification: C44, C57, D44, D82

Keywords: Statistical Auction Design, Regret Ratio, Equivariance, Average Bid Auction, Empirical Myerson Auction, Price-mean Ratio.

*I am indebted to Graham Elliott and Yixiao Sun for their constant couragements and insightful comments. I am deeply grateful for the guidance from Ying Zhu, who leads me to the exploration of the statistical issue in auctions. I also thank Jin Xi, Wanchang Zhang, Yu-Chang Chen, and participants in the econometrics lunch seminar at UC San Diego.

†Department of Economics, University of California San Diego. Email: hax082@ucsd.edu.
1 Introduction

This paper studies the problem of auction design under the assumption that the optimal reservation price is unknown due to the lack of exact knowledge of the valuation distribution. Instead, independent samples recorded from past auctions are assumed to be available to the auctioneer.

In the classical Bayesian mechanism design literature, the auctioneer knows the distribution from which the bidders’ valuations are drawn.\(^1\) In this case, the optimal auction is derived in the seminal work of Myerson (1981), which is in the form of a second-price auction with the optimal reservation price. The optimal reserve price depends on the valuation distribution through

\[
p(F) = \arg \max_p p(1 - F(p)),
\]

where \(F\) is the valuation distribution. The job of acquiring a reasonable estimate of the optimal price is left to the statisticians and computer scientists. So the complete process of auction design is divided artificially into two steps: (i) deriving the optimal price as if the distribution were known, and then (ii) approximating this price with data.

However, as shown in the paper, the price \(p(F)\) is not a pathwise differentiable functional of the distribution \(F\); thus, it can only be estimated at rates slower than the \(n^{1/2}\)-rate. This is because the optimal reserve price depends on the density, which can only be estimated at a slower rate if no parametric form is imposed. A natural question arises that, when considering the entire problem, whether it would be favorable to sacrifice the optimality in the first step (when the distribution is known) in exchange for a faster convergence rate in the second step by targeting a suboptimal but regular parameter.

The investigation begins by framing the problem inside the statistical decision framework, where the allocation rule and transfer payments of an auction are the statistical decision rules with the bids and historical samples being the data inputs. In this framework of statistical auction design under finite sample, methods that resemble statistical estimation are integrated into the classical mechanism design theory that mainly deals with the incentive compatibility issues under asymmetric information. The maxmin and equivariance principles are studied. The framework encompasses two special cases. First, when there are no available historical samples, then the problem becomes the classical mechanism design\(^2\). Second, when there is only one bidder, then the problem becomes monopolistic pricing with samples.

As a result, a new form of auction, called the average bid auction, is proposed. It assigns the auction item to the bidder with the highest valuation. Each bidder faces an individual reservation price that is proportional to the average of other bidders’ bids and historical samples. This is because, for each bidder, the other bids plays the same role as the sample observations in constructing the estimator of the optimal price. In contrast, his own bid has to be excluded for incentive

\(^1\)The usage of “prior” and “Bayesian” in this literature is somewhat different from that in statistics. The buyers’ valuations are modeled as random variables in this context, whose distribution is referred to as the prior, and Bayesian means that the designer knows this distribution.

\(^2\)Even in this case, the paper still contributes to the literature by studying the equivariance principle.
compatibility issues.

This simple form of average is motivated using the Gamma distribution, but it has surprising insights for the estimation of \( p(F) \). A set of interesting findings shows that the population mean often serves as a useful reference point for the optimal price. Based on this idea, the paper proposes an adaptive way to conduct the average bid procedure, which uses the a pilot estimator for the coefficient in the average bid procedure. This adaptive estimator has the same asymptotic properties as the empirical Myerson estimator, and it has a significantly better performance in terms of value at risk and expected shortfall when the sample size is small.

1.1 Related Literature

In the economics literature, empirical estimates have been obtained for the independent private value auctions. Paarsch (1997) uses a parametric approach. For nonparametric estimation, Athey et al. (2002) considers solving the first-order condition of (1) after estimating the density of \( F \). Another popular approach is to directly maximize the empirical version of (1). Studies related to this approach include Segal (2003); Prasad (2008); Coey et al. (2020). This method can be referred to as the empirical Myerson auction because the price is set as if the empirical distribution is the true distribution \( F \).

Studies of similar topics can be found under the name of algorithmic mechanism design in the theoretical computer science literature, including works by Cole and Roughgarden (2014); Dhangwatnotai et al. (2015); Huang et al. (2018); Guo et al. (2019). The primary solution understudy in this literature is the empirical Myerson auction and its variants. As summarized in Babaioff et al. (2018), the two directions studied so far in this literature are two opposite directions: that of asymptotic results where only the rate of convergence is of concern, and that where only a single sample is available. The adaptive average bid procedure proposed in the paper partially fills this important gap between the large sample case and the one (or two) sample case.

There is a massive amount of literature on the structural econometrics on auction, as recently surveyed by Perrigne and Vuong (2019). The main goal of this literature is to identify and estimate the valuation distribution from the observed bids in first-price auctions. However, this paper assumes that the samples come from a truthful mechanism, where the bids are equal to the valuations themselves, so identification is no longer a problem. Instead, this paper pushes the task one step further, asking the question on how to set the allocation rule with a finite sample when all relevant quantities are identified.

The rest of the paper is organized as follows. Section 2 introduces the statistical auction design framework, where statistical methods are integrated into the classical mechanism design theory. Section 3 studies general results regarding the maxmin and equivariant principle. Section 4 motivates the average bid auction under Gamma distribution. Section 5 proposes the adaptive average bid procedure and studies its performance under general distributional assumptions both theoretically and with simulations. Section 6 concludes. Appendix A contains tables for simulation results. Proofs are collected in Appendix B.
2 Statistical Auction Design

There is a single item to sell. There are $k$ bidders, whose valuations of the item are collected in $V = (V_1, \ldots, V_k)$. Each bidder knows his own valuation. The seller doesn’t observe the valuations, instead he observes a collection of $n$ samples $W = (W_1, \ldots, W_n)$, possibly from past auctions run on similar items. The valuations $V_1, \ldots, V_k$ and samples $W_1, \ldots, W_n$ are i.i.d. draws from the same distribution $F$. Throughout the paper, $F$ is assumed to be absolute continuous, with support $[\underline{v}, \overline{v}]$ or $[\underline{v}, \infty)$ where $\underline{v} \geq 0$.

An auction is defined by the allocation rule, which determines to whom the item is sold, and the payment rule, which determines the transfer of money from the buyers to the seller. The seller makes these decisions based on the collected bids and the sample data. The formal definition is as follows.

**Definition 1.** A statistical auction consists of functions $q_i$ and $t_i$ for $i = 1, \ldots, k$, where $q_i : \mathbb{R}_+^k \times \mathbb{R}_+^n \rightarrow [0, 1]$, such that $\sum_{i=1}^n q_i \leq 1$, and $t_i : \mathbb{R}_+^k \times \mathbb{R}_+^n \rightarrow \mathbb{R}$.

The inputs of the statistical auction are bids $b \in \mathbb{R}_+^k$ and samples $w \in \mathbb{R}_+^n$. Each $q_i(b, w)$ is the probability of assigning the item to bidder $i$, and $t_i(b, w)$ is the payment from bidder $i$ to the seller. The condition $\sum_{i=1}^k q_i \leq 1$ simply means that the allocation probabilities sum up to 1, in particular, $1 - \sum_{i=1}^k q_i$ is the probability that the item is not sold. Note that a classical auction without the samples can be treated as a special case where $n = 0$. It is worthy to note that the revelation principle applies in this setting, so the restriction to the direct mechanism is without loss of generality.

The expected revenue $R(q, F)$ from implementing the rule $q$ under the distribution $F$ is defined as

$$R(q, F) = \sum_{i=1}^k \mathbb{E}[t_i(V, W)],$$

which is the sum of the expected payment from each bidder. The expectation is taken over both the valuations $V$ and samples $W$.

Since the bidders are strategic players in the auction, there are some sensible restrictions to place on the form of the statistical auction. Each bidder $i$ reports bid $b_i$ based on his own valuations. The quasilinear form of utility is assumed, which for each bidder $i$ is defined as $v_i q_i(b, w) - t_i(b, w)$. When bidder $i$ is bidding his own valuation $v_i$, he receives utility

$$U_i(v_i, b_{-i}, w) = v_i q_i((v_i, b_{-i}), w) - t_i((v_i, b_{-i}), w).$$

where $b_{-i} \in \mathbb{R}_+^{k-1}$ is the set of bids from bidders other than $i$, and $(v_i, b_{-i})$ is a complete set of bids that combines $v_i$ and $b_{-i}$. It is desired that the auction to be designed in a way that the

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3The $W$'s can also be considered as independent signals of the valuation distribution $F$. If they come from different auctions, then $F$ is the probability measure of the bigger population encompassing all these auctions. We do not need to assume that the bidders observe $W$. 

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bidders reveal their true valuations under any distribution $F$. In other words, bidding truthfully is a dominant strategy for each bidder, as defined in the following.

**Definition 2.** A statistical auction is dominant strategy incentive compatible (DSIC) if for any bidder $i$, it holds for all $v_i, b_i, b_{-i}$, and $w$ that

$$U_i(v_i, b_{-i}, w) \geq v_i q_i((b_i, b_{-i}), w) - t_i((b_i, b_{-i}), w).$$

The other important constraint is called individual rationality, which means that the bidders' utility cannot go below some lower bound so that they are willing to participate in the auction in the first place.

**Definition 3.** A statistical auction is individual rational (IR) if for any bidder $i$, it holds for all $v_i, b_{-i}$, and $w$ that

$$U_i(v_i, b_{-i}, w) \geq 0.$$

Since the bids in DSIC auctions are the valuations, henceforth the notation $v_i$ will be used to represent both the valuation and the bid for bidder $i$. A relationship between $q$ and $t$ can be derived for any auction that satisfies DSIC and IR, which is the well-known Revenue Equivalence Principle in mechanism design theory. The following lemma is the statistical version of this result that involves the sample $W$.

**Lemma 1.** *(Revenue Equivalence)* A statistical auction $(q, t)$ is DSIC if and only if, for each bidder $i$ it holds that

(i) $q_i((v_i, v_{-i}), w)$ is non-decreasing in $v_i$.

(ii) For every $v_i$ in the support of $F$, the transfer payment satisfies:

$$t_i((v_i, v_{-i}), w) = v_i q_i(v, w) - \int_0^{v_i} q_i((u, v_{-i}), w)du - U_i((v, v_{-i}), w).$$

Moreover, if $(q, t)$ maximizes the auctioneer’s expected revenue, then

$$t_i((v_i, v_{-i}), w) = v_i q_i(v, w) - \int_0^{v_i} q_i((u, v_{-i}), w)du. \tag{3}$$

This lemma means that, without loss of optimality, the allocation rule $q = \{q_i\}_{i=1}^k$ completely determines the transfer payment $t = \{t_i\}_{i=1}^k$, hence the statistical auction. Throughout the paper, statistical auctions are considered to be of the form in Equation (3), and the allocation rule is often used to denote the corresponding auction. Below are some examples of statistical auctions in view of Lemma 1.

**Example 1.** Consider the case of one bidder and one sample, i.e. $k = n = 1$. Let the allocation rule be $q_1(v_1, w_1) = 1\{v_1 > w_1\}$. Then the associated payment is

$$t_1(v_1, w_1) = v_1 1\{v_1 > w_1\} - \int_0^{v_1} 1\{u > w_1\}du = \begin{cases} w_1, & \text{if } v_1 > w_1 \\ 0, & \text{if } v_1 < w_1. \end{cases}$$
This is the single sample identity pricing rule, where the seller posts a price equal to the sample observation.

**Example 2.** When \( k = 2 \) and \( n = 1 \), there are 2 bidders and 1 sample. Let the allocation rule be

\[
q_i(v_i, v_{i'}, w_1) = 1\{v_i > v_{i'}, v_i > w_1\}, \ i \neq i' \in \{1, 2\},
\]

which means the item is allocated to the bidder with bids higher than both the sample and the other bid. The associated payment is

\[
t_i(v_i, v_{i'}, w_1) = \begin{cases} 
v_{i'}, & \text{if } v_i > v_{i'} > w_1 \\
w_1, & \text{if } v_i > w_1 > v_{i'} \\
0, & \text{if } v_i < w_1.
\end{cases}
\]

This is the second-price auction with a reservation price set equal to the sample observation.

The discussion up to this point does not involve the distribution \( F \), because we are considering dominant strategies. The next step is to characterize the optimal revenue when \( F \) is known to the seller, so as to provide a reference point for the revenue earned by any statistical auction. Two useful definitions are introduced. Let \( f \) be the pdf of \( F \).

**Definition 4.** For a absolute continuous distribution \( F \), the virtual valuation is defined as

\[
\phi_F(v) = v_i - \frac{1 - F(v_i)}{f(v_i)}.
\]

The distribution \( F \) is said to be regular if \( \phi_F \) is strictly increasing. Let \( p(F) \) be defined as the unique root of \( \phi_F(\cdot) = 0 \).

Combining Lemma 1 with Equation (2), the revenue has a more explicit representation as

\[
R(q, F) = \sum_{i=1}^{k} \mathbb{E}[q_i(V, W)\phi_F(V_i)].
\]

This is the statistical version of what is being referred to as the Myerson’s lemma in the literature, which is very useful in a lot of the derivations afterwards.

When \( F \) is regular, Myerson (1981) showed that the optimal auction \( q^F \) is the second price auction with reserve price \( p(F) \). Note that when \( F \) is known, the optimal auction no longer depends on the sample \( W \). The allocation rule for this auction is

\[
q_i^F(v) = 1\{v_i > \max_{i' \neq i} v_{i'}, v_i > p(F)\}.
\]

4This is equivalent to the definition in Equation (1)

5Throughout the paper, the case of multiple winners \( (v_i = \max_{i' \neq i} v_{i'}) \) is ignored, since the distributions under consideration are absolute continuous.
Example 3. Consider $F$ to be the exponential distribution, i.e. $f(v) = e^{-v}$. The virtual valuation function $\phi_F(v) = v - 1$ is increasing. The optimal reserve price is $p(F) = 1$. The Myerson optimal allocation rule is $q_i^F(v) = 1\{v_i > \max_{v' \neq i} v_{i'}, v_i > 1\}$.

With a little abuse of notation, denote the optimal revenue under $F$ as $R(F) = R(q_F, F)$. The utility function of the auctioneer in the entire statistical auction design problem is modeled as the ratio between the expected revenue and the optimal revenue. I call it the regret ratio, of statistical auction $q$ under a regular distribution $F$,

$$r(q, F) = \frac{R(q, F)}{R(F)}.$$ 

The name regret ratio follows from the fact that each auction is valued based not on the absolute revenue gained but on the relative revenue compared to the best achievable revenue under the same distribution.\(^6\) The regret ratio is a solid choice of the objective function from the decision-theoretic perspective, as explained in the next section, along with the maxmin criterion.

3 Optimality Criteria

3.1 Maximin Principle

The maxmin principle is used to protect against the worst possible distribution. It ranks the decision rules based on their worst-case performances. In statistics, it is often referred to as the minimax principle, for the utility function is expressed in terms of loss. Let $Q$ denote the set of all allocation rules that satisfies the first condition in Lemma 1. The formal definition of maxmin auction is as follows.

**Definition 5.** Given a class of distributions $\mathcal{F}$, a DSIC statistical auction $q$ is maxmin if

$$\inf_{F \in \mathcal{F}} r(q, F) = \sup_{q' \in Q} \inf_{F \in \mathcal{F}} r(q', F).$$

(4)

Since the objective is the regret ratio function, the principle can also be considered as maxmin-regret in this case. The maxmin-regret principle is commonly applied in the treatment assignment literature (Manski, 2004; Stoye, 2009, 2012; Tetenov, 2012).\(^7\) However, instead of using the ratio as the regret, those works use the difference between the best achievable and the achieved as the regret. To distinguish, the latter is referred to as regret difference. Arguably, the regret ratio is a better quantity in measuring the relative gains. Because the range of the regret difference varies with the magnitude of the best achievable revenue, while the regret ratio always lies within the unit interval. This property renders the regret ratios comparable across different distributions. Despite this imparity, the two types of regret play the common role in avoiding selecting the trivial rule that

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\(^6\)It is also sometimes called the competitive ratio.

\(^7\)Besides these works on finite-sample maxmin regret analysis, Kitagawa and Tetenov (2018) also studies the welfare regret from the perspective of non-asymptotic risk bound.
completely ignores information contained in the data.\textsuperscript{8} A discussion on this issue can be found in Sadler (2015). For a thorough comparison among maxmin, maxmin regret difference, and maxmin regret ratio, from an axiomatic perspective, the readers are referred to Brafman and Tennenholtz (2000).

The maxmin procedure can also be seen as a zero-sum game played by the auctioneer against nature. The auctioneer picks a statistical auction \( q \in Q \), while nature picks a distribution \( F \in \mathcal{F} \). The set \( Q \) is already convexified by the randomization as in Definition 1. Nature can also randomized. To avoid technical issues, assume \( \mathcal{F} \) is finite-dimensional. The nature’s mixed strategy is a prior distribution \( \pi \in \Delta(\mathcal{F}) \). Nature’s payoff is given by the negative of the regret ratio. A well-known result is that if the game has a Nash equilibrium \((q^*, \pi^*)\), then \( q^* \) is a maxmin auction. The prior \( \pi^* \) is called least favorable prior. This fact alleviates the difficulty in proving maxminity since now one only needs to show that \( q^* \) is the best response against \( \pi^* \), and vice versa.

The item in the auction often possesses attributes that covariate with its valuation. The attribute is modeled by a finite-valued covariate \( X \) supported on \( \mathcal{X} \). Denote \( F_x \in \mathcal{F} \) as the conditional distribution of the valuation given \( X = x \). The sample data is the set \( \{(W_j, X_j)\}_{j=1}^n \). The vector of samples \( W \) can be partitioned according to the covariate, \( W = \{W_x : x \in \mathcal{X}\} \), where each \( W_x \) contains the sample valuations whose associated attribute is \( x \). The elements in \( W_x \) are iid draws from \( F_x \). The covariate of the item in the current auction is also observed. When the attribute of the current item is observed to be \( x \), the DSIC bids contained in \( V \) are considered as equal in distribution to elements in \( W_x \).

An illustrative example is provided to enhance understanding. Consider the auction on an antique china teacup. Historical data contains transaction prices, from second-price auction conducted in the past, for both antique china teacups and plates. The attribute, in this case, is binary, representing teacup or plate. The bidders’ valuations for the teacup in this auction is assumed to share a common distribution with the historical prices of teacups in the data, while potentially distributed differently from the prices of plates. The next goal is to show that the maxmin auction only uses the data for teacups to estimate the optimal reserve price, and completely ignores the data for plates.

With more information at hand, the auctioneer can condition the allocation on the covariate. Denote this more general form of allocation rule by \( \kappa = \{\kappa_x : x \in \mathcal{X}\} \), where each \( x \) signifies the attribute of the item in the current auction. Each conditional allocation \( \kappa_x (V, W, X) \) depends on the entire set of available information. The notation \( q(V, W) \) is retained for the previously defined unconditional auction. It is important to note that, even after fixing the value \( x \), the auction \( \kappa_x \) is still different from an unconditional one in the sense that the elements in \( W \) can be treated differently accordingly to the value of \( X \). The expected revenue, for an item with attribute \( x \), is

\[
R(\kappa_x, \hat{F}) = \mathbb{E} \left[ \sum_{i=1}^k \phi_{F_x}(V_i)\kappa_x(V, W, X) \right],
\]

\textsuperscript{8}See Manski (2004) and Savage (1954) for illustrative examples on this issue.
where \( \tilde{F} = \{ F_x : x \in \mathcal{X} \} \in \mathcal{F}^X \). Dividing \( R(\kappa_x, \tilde{F}) \) by \( R^*(F_x) \) gives the conditional version of regret ratio \( r(\kappa_x, \tilde{F}) \). The maxmin problem becomes finding a \( \kappa_x \) for each \( x \) such that

\[
\inf_{F \in \mathcal{F}^X} r(\kappa_x, \tilde{F}) = \sup_{\kappa_x} \inf_{F \in \mathcal{F}^X} r(q, \tilde{F}),
\]

(5)

where \( \kappa_x \) varies in the set of all conditional auctions, and nature is allowed to vary each \( F_x \) in \( \mathcal{F} \) without any restrictions. Note that the regret ratio also implicitly depends on the marginal distribution of \( X \), but nature is assumed not to manipulate that. In fact, as shown in the proof of Proposition 1 the marginal distribution of \( X \) does not affect the maxmin auction.

For any unconditional rule \( q \) as described in Definition 1, one can define a set of associated conditional auctions \( \{ \kappa_x, x \in \mathcal{X} \} \), that uses no cross-covariate information, by letting

\[
\kappa_x(V, W, X) = q(V, W, X).
\]

(6)

Essentially, the conditional auction \( \kappa_x \) discards all samples with covariate value different than \( x \), and proceeds with the allocation \( q \).

**Proposition 1.** Assume that the unconditional rule \( q^* \) is the maxmin solution to (4), with the least favorable prior being \( \pi^* \in \Delta \mathcal{F} \). Then the associated conditional auction \( \kappa^*_x \), as defined in Equation (6), is the maxmin solution to (5).

This type of result first appears in Stoye (2009). The intuition is that nature can choose a prior that renders the data \( W_{x'} \) uninformative about items with attribute \( x \neq x' \). Then the auctioneer is best responding by discarding irrelevant information. This result provides practical guidance from the maxmin principle that only the most relevant data should be used in designing an auction on a specific item. However, this means that, in each auction, the effective sample size is going to be relatively small. To overcome this issue, a form of auction is introduced in the next section that performed particularly well under small to moderate sample size. Before that, another useful statistical decision principle is studied.

### 3.2 Equivariance Principle

The equivariance principle, in statistical estimation problems, is a group-theoretic formalization of certain intuitively appealing decisions. This intuition, in the context of statistical auction design, is that the final allocation should not be altered if the monetary unit of the bids and samples are changed. More specifically, if the auctioneer decides to allocate the item to a bidder when the input data are in dollars, then it is natural to assume that this decision remains the same when the input data are in Yen.

Besides this intuitive argument, there are many practical reasons for employing the equivariance principle. First, the equivariance simplifies the maximization of the regret ratio. Proposition 2 shows that the regret ratio of an equivariant auction is constant along each scale family. Second, as in
the estimation problem, the equivariance principle provides an effective way to restrict the set of statistical rules under consideration. As shown in Proposition 3, the optimal equivariant auction derived under a specific distributional assumption does not ignore the data. For this purpose, the equivariance principle is similar to unbiasedness principle, except that there is no natural analog of unbiasedness in auction design. Another reason is that the relationship between equivariance and maxmin through the generalized Hunt-Stein theorem, which states that the maxmin equivariant rule is maxmin overall. This is addressed in the next subsection after the introduction of the maxmin principle.

We first introduce the equivariance structure in the statistical auction design problem. Consider in turn the scale transformation on three types of objects: distribution of valuations, bids and samples, and the payment transfer. For any regular $F$ and $\theta > 0$, let $F_\theta$ denote the CDF of $\theta V$, i.e. $F_\theta(v) = F\left(\frac{v}{\theta}\right)$. The corresponding density is $f_\theta(v) = \frac{1}{\theta} f\left(\frac{v}{\theta}\right)$. The following lemma shows that the optimal revenue, from the Myerson auction, is scale-equivariant.

**Lemma 2.** For any regular $F$ and $\theta > 0$, we have $p(F_\theta) = \theta p(F)$, and $R(F_\theta) = \theta R(F)$.

Next, define a special class of statistical allocation rules, that are invariant to the scale transformation on bids and samples.

**Definition 6.** An allocation rule is (scale-)invariant if, for any $\theta > 0$,

$$q(\theta v, \theta w) = q(v, w), \ v \in \mathbb{R}_+^k, w \in \mathbb{R}_+^n.$$  \hspace{1cm} (7)

A statistical auction is (scale-)equivariant if its allocation is (scale-)invariant.

**Lemma 3.** For a scale-invariant allocation rule, the associated payment transfer rule satisfies, for any $\theta > 0$ and $i = 1, \cdots, n$,

$$t_i(\theta v, \theta w) = \theta t_i(v, w), \ v \in \mathbb{R}_+^k, w \in \mathbb{R}_+^n.$$  

Lemma 3 justifies the name “equivariant” by showing that for any scale-invariant allocation, the associated payment transfer is scale-equivariant.\footnote{The usage of “equivariant” and “invariant” in the text is the same as in “equivariant estimator” and “invariant test” in traditional statistics literature.} The behavior of the scale transformation is summarized by Lemma 2 and Lemma 3. There is a notable difference between these two lemmas. The revenue from the Myerson optimal auction is equivariant when the distribution scales by $\theta$. The revenue from the equivariant statistical auction is equivariant when the input valuations and samples scale by $\theta$. In particular, fix a distribution $F$, the revenue from the Myerson optimal auction is not equivariant.

In the literature, a discussion on the formalized equivariance principle can be found in Allouah and Besbes (2018) in the setting of auction design with two bidders (and no sample). Most common auctions are in the equivariant form, including the first and second-price auctions. The equivariant

\footnote{The usage of “equivariant” and “invariant” in the text is the same as in “equivariant estimator” and “invariant test” in traditional statistics literature.}
auction does not the information on the magnitudes of the valuations and samples, since it can be represented as a function of the ratios:

\[ q \left( \frac{v_1}{w_n}, \cdots, \frac{v_k}{w_n}, \frac{w_1}{w_n}, \cdots, \frac{w_{n-1}}{w_n}, 1 \right). \]

The auctions in Example 1 and 2 are both equivariant. Because the allocation in Example 1 is

\[ q(v_1, w_1) = q \left( \frac{v_1}{w_1}, 1 \right) = 1 \{ v_1/w_1 > 1 \}, \]

and the allocation in Example 2 is

\[
\begin{align*}
q_1(v_1, v_2, w_1) &= q_1 \left( \frac{v_1}{w_1}, \frac{v_2}{w_1}, 1 \right) = 1 \{ v_1/w_1 > 1, (v_1/w_1)/(v_2/w_1) > 1 \}, \\
q_2(v_1, v_2, w_1) &= q_2 \left( \frac{v_1}{w_1}, \frac{v_2}{w_1}, 1 \right) = 1 \{ v_1/w_1 < 1, (v_1/w_1)/(v_2/w_1) < 1 \}.
\end{align*}
\]

To complete the equivariance structure, we show that the regret ratio of any equivariant auction is constant with respect to the scale parameter of the valuation distribution. From Lemma 2 and Lemma 3 state that when the distribution scales by \( \theta \), both the optimal revenue and the expected revenue from an equivariant statistical auction scale by \( \theta \). Thus, we have the following proposition.

**Proposition 2.** For any regular distribution \( F \), and equivariant auction, it holds that

\[ r(q, F_\theta) = r(q, F_{\theta'}), \text{ for any } \theta, \theta' > 0. \]

The statistical auction design problem is equivariant with respect to the group of scale transformations. In the group-theoretic language, the set \( \{ F_\theta : \theta > 0 \} \) is called an orbit, within which the regret ratio is constant. The most straightforward result from the equivariance principle is to simplify the space of distributions by collapsing it into the sets of orbits.

By the generalize the Hunt Stein theorem we can expect the maxmin equivariant auction to be the overall maxmin auction. The mathematical statement is

\[
\sup_{q \in \mathcal{Q}_e} \inf_{F \in \mathcal{F}} r(q, F) = \sup_{q \in \mathcal{Q}} \inf_{F \in \mathcal{F}} r(q, F),
\]

where \( \mathcal{Q}_e \subset \mathcal{Q} \) denotes the set of all equivariant auctions. This means in finding the maxmin statistical auction, attentions can be restricted to the equivariant auctions. Bondar and Milnes (1981) shows that such result holds when the associated group transformation satisfies the amenability condition. And they have also shown that the scale transformation group (possibly one of the simplest transformation group) is amenable. See Chapter 5 of Lehmann and Casella (2006), Bondar and Milnes (1981), and Wesler (1959) for more discussions of the generalized Hunt-Stein theorem.

Next, we study the general structure of the optimal equivairant auction. The following proposition shows the common representation of the optimal equivariant auction against a given scale family of valuation distributions.
Proposition 3 (General Representation for Equivariant Auctions). For any regular distribution $F$, the optimal equivariant auction that maximizes the regret ratio for the scale family $\{F_\theta : \theta > 0\}$ has the allocation rule of the form

$$q_{F,i}(V, W) = 1 \left\{ V_i > \rho_F(V_{-i}, W), V_i > \max_{i' \neq i} V_{i'} \right\}, i = 1, \ldots, k,$$

where $\rho_F$ is symmetric and homogeneous of degree one. When $\min\{k, n\} \to \infty$, $\rho_F(V_{-i}, W)$ converges in probability to $p(F)$.

There are several features to the form of auction in (8). Notice that the denominator of the regret ratio, the optimal revenue, is a constant in this case. So the auctioneer only needs to focus on optimizing the revenue. The resulting auction is standard in the sense that a bidder wins only if his bid is the highest, which is represented by the part “$V_i > \max_{i' \neq i} V_{i'}$”. There is an individual reservation price $\rho_F(V_{-i}, W)$ for each bidder $i$. In order to be the winner, the bidder needs to have a valuation higher than this reservation price.

In setting the price for each bidder $i$, the effective sample used is in fact $(V_{-i}, W)$, where bidder $i$’s own valuation $V_i$ is excluded for the incentive compatibility issue. So the effective sample size is $k + n - 1$. Using the law of iterated expectations, the revenue from a bidder $i$ can be written as

$$\mathbb{E} \left[ q_i(V, W) \phi_F(V_i) \right] = \mathbb{E} \left[ q_i \left( \frac{V_1}{W_n}, \ldots, \frac{V_k}{W_n}, \frac{W_1}{W_n}, \ldots, \frac{W_{n-1}}{W_n} \right) \phi_F \left( \frac{V_1}{W_n}, \ldots, \frac{V_k}{W_n}, \frac{W_1}{W_n}, \ldots, \frac{W_{n-1}}{W_n} \right) \right],$$

where

$$\phi_F \left( \frac{V_1}{W_n}, \ldots, \frac{V_k}{W_n}, \frac{W_1}{W_n}, \ldots, \frac{W_{n-1}}{W_n} \right) = \mathbb{E} \left[ \phi_F(V_i) \left| \frac{V_1}{W_n}, \ldots, \frac{V_k}{W_n}, \frac{W_1}{W_n}, \ldots, \frac{W_{n-1}}{W_n} \right. \right]$$

(10)

can be considered as the equivariant version of the virtual valuation. In Myerson (1981), the optimal reserve price is derived as the root of the virtual valuation function. Similarly, here the reserve price $\rho_F(V_{-i}, W)$ is derived based on the root of the equivariant virtual valuation $\hat{\phi}_F$. From the classical estimation viewpoint, we can consider $\rho_F(V_{-i}, W)$ as an equivariant estimator of the optimal price $p(F)$ with the risk function being designed as the (negative) regret ratio.

The estimator $\rho_F(V_{-i}, W)$ consistently estimates the true optimal price when there is a large number of bidders or samples because then the auctioneer can estimate the valuation distribution accurately. The function $\rho_F$ is symmetric so other bidders’ valuations $V_{-i}$ and the samples $W$ play the exact same role in setting the individual reservation price. The fact that the samples $W$ are utilized already shows the power of the equivariance principle. Because the overall optimal auction is not equivariant and completely ignores the samples $W$ since it set the reservation price to be $p(F)$.

When the sample size $n$ is large, little is lost if we ignore the information in $V_{-i}$ and using $W$ alone to estimate the reserve price. However, in the finite sample case, when $k$ and $n$ are of
the same magnitude, it is rather important to take advantage of the information coming from the current auction.

4 Average Bid Auction

After discussion on the general representation of equivariant auctions, we study a specific form of the equivariant auction. The simplest form of $\rho_F$ one can think of is perhaps the average $\bar{S}_{-i} = \frac{1}{k+n-1} \left( \sum_{i' \neq i}^k V_i + \sum_{j=1}^n W_j \right)$ over the effective sample $(V_{-i}, W)$. Following this idea, the average bid (AB) auction $q_\beta$ is defined as:

$$q_\beta(V, W) = 1 \left\{ V_i > \beta \bar{S}_{-i}, V_i > \max_{i' \neq i} V_{i'} \right\}, i = 1, \cdots, k,$$

where $\beta$ is a positive constant. The abbreviation AB-$\beta$ is used when the coefficient $\beta$ is under discussion. We can see that the AB-$\beta$ auction sets the reserve price to be $\beta$ times the sample average.

In the case of one bidder and one sample ($k = n = 1$), the AB-$\beta$ auction sets the price to be $\beta W_1$. It is shown in Huang et al. (2018) that, for distributions with increasing hazard rate $(1 - F)/f$, the pricing rule $0.85W_1$ guarantees a regret ratio of 0.589, while the empirical Myerson pricing rule (1) only attains the maxmin regret ratio of 0.5. Notice that, in this case, the empirical Myerson pricing rule sets the price to the observed sample $W_1$, and it is also called the single sample identity pricing rule. When there are more than one bidder and sample observation, we proceed to show that the AB auction is optimal under the Gamma distribution. Then in the next section, we show that the AB auction is favorable in the finite-sample setting under general distributional assumptions.

4.1 Optimal Equivariant Auction

Statisticians studying finite-sample estimation and hypothesis testing have long made progress by imposing parametric assumptions to make the problem tractable. Similarly, we make progress in this section by restricting attention inside the Gamma family. The choice of Gamma distribution in the auction data analysis is not new (see e.g. Friedman, 1956; Hossein Bor, 1977; Skitmore, 2014; Takano et al., 2014). In a more recent paper, Ballesteros-Pérez and Skitmore (2017) shows Gamma distribution is among the distributions that provide the best fit to the empirical auction data.

The Gamma family of distributions has densities of the form

$$f(v, \alpha, \theta) = \frac{\theta^\alpha v^{\alpha-1} e^{-v/\theta}}{\Gamma(\alpha)}, \alpha \geq 1, \theta > 0,$$

where $\Gamma$ is the Gamma function, and $\alpha, \theta$ are respectively the shape and scale parameter. We first fix $\alpha$, and study the optimal equivariant auction for the scale family $\{f(\cdot, \alpha, \theta) : \theta > 0\}$. Then we can let $\theta = 1$ since the regret ratio does not depend on $\theta$ due to Proposition 2. In the rest of this
Following Proposition 3 and the discussion after it, we know the form of the equivariant reserve price \(\rho_\alpha\) depends on maximizing the revenue from one bidder (without constraining the sum of \(q_i\)'s to be less than 1). Without loss of generality, we study the revenue from bidder 1 and normalize the other valuations and samples by the valuation \(V_1\):

\[
\mathbb{E}[t_1(V, W)] = \int q_1(v, w)\phi_\alpha(v_1) \prod_{i=1}^k f(v_i, \alpha, 1) \prod_{j=1}^n f(w_j, \alpha, 1) dvdw
\]

\[
= \int q_1 \left( \frac{1}{v_1}, \frac{1}{v_1} \right) \phi_\alpha(v_1) \prod_{i=1}^k f(v_1 \cdot (v_i/v_1), \alpha, 1) \prod_{j=1}^n f(v_1 \cdot (w_j/v_1), \alpha, 1) dvdw
\]

\[
= \int q_1(\sigma) d\sigma \int \phi_\alpha(v_1) \prod_{i=1}^{k+n} f(v_1 \sigma_i, \alpha, 1) v_1^{n+k-1} dv_1,
\]

where the change of variable \(\sigma = (1, \sigma_2, \ldots, \sigma_{k+n}) = (1, v_2/v_1, \ldots, v_k/v_1, w_1/v_1, \ldots, w_n/v_1)\) is used. So \(q_1\) has been taken out from the first layer of integration. To study the first layer of integration, define a function

\[
\varphi_m(\sigma, \alpha) = \int_0^\infty \phi_\alpha(v) \prod_{i=1}^m f(v_1 \sigma_i, \alpha, 1) v^{m-1} dv
\]

\[
= \frac{\Gamma(1 + m \alpha)}{\Gamma(\alpha^m)} \left( \prod_{i=2}^m \sigma_i \right)^{\alpha-1} \times \left[ (1 + \sum_{i=2}^m \sigma_i)^{-1-m\alpha} - \frac{2F_1(1 + (m - 1)\alpha, 1 + m\alpha, 2 + (m - 1)\alpha, - (\sum_{i=2}^m \sigma_i))}{1 + (m - 1)\alpha} \right].
\]

where \(2F_1\) is the Gaussian hypergeometric function.\(^{10}\) Also define \(\varphi_m^0\) as the factor of \(\varphi_m\) that only depends on the sum \(\sum_{i=2}^m \sigma_i\),

\[
\varphi_m^0(s, \alpha) = \frac{\Gamma(1 + (m + 1)\alpha)}{\Gamma(\alpha^{m+1})} \left[ (1 + s)^{-1-(m+1)\alpha} - \frac{2F_1(1 + m\alpha, 1 + (m + 1)\alpha, 2 + m\alpha, -s)}{1 + m\alpha} \right].
\]

We can see that the sign of \(\varphi_m\) is completelt determined by the sign of \(\varphi_m^0\), that is,

\[
\text{sgn}(\varphi_m(\sigma, \alpha)) = \text{sgn} \left( \frac{\varphi_m^0 \left( \sum_{i=2}^m \sigma_i, \alpha \right)}{\varphi_m^0} \right).
\]

\(^{10}\) This equality can be verified using Mathematica. The hypergeometric function is defined as the analytic continuation of the power series

\[
2F_1(a, b, c, x) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} x^n, |x| < 1,
\]

where \((a)_n = a(a + 1) \cdots (a + n - 1) 1_{\{n \geq 1\}}\), and \((b)_n, (c)_n\) defined similarly.
Therefore, the maximizer $q_1$ of (12) is only a function of $\sum_{i=2}^{k+n-1} v_i = \left(\sum_{i=2}^{k} v_i + \sum_{j=1}^{n} w_j\right)/v_1$.\(^{11}\)

Combining this fact with the general representation of equivariant auction in Proposition 3, we know the equivariant reserve $\rho_\alpha$ must be proportional to the sample average $\bar{S}_{-i}$. Thus, we have proved the following corollary.

**Corollary 1.** The AB-$\beta$ auction is the optimal equivariant auction for the scale Gamma family $\{f(\cdot, \alpha, \theta) : \theta > 0\}$, where the constant $\beta$ depends on $k, n, \text{and} \alpha$.

Figure 1 graphs the functions $\varphi_m^0(s, \alpha)$ for $m = 3, 4$ and $\alpha = 1.5, 2.0, 2.5$. Each curve starts as a decreasing function, crosses the horizontal axis once from above, and then starts increasing. In fact, the two axes are the two asymptotes of the curve. However, these properties are not required for the proof of Corollary 1. The optimal equivariant auction is of the average bid form as long as the sign of $\varphi_m$ only depends on the sum of ratios $\sum_{i=2}^{m} \sigma_i$.

![Figure 1: The graph of $\varphi_3^0(s, \alpha)$ and $\varphi_4^0(s, \alpha)$ for different values of $\alpha$.](image)

Even though the knowledge of $\alpha$ is assumed, the resulting auction is still useful for practical purposes due to the equivariance restriction as mentioned before. The effective sample average $\bar{S}_{-i}$ is the method of moment estimator for $\alpha$ when $\theta = 1$. Without the equivariance restriction, $\bar{S}_{-i}$ would be replaced by its true value $\alpha$ in the AB auction, then the resulting auction would not be equivariant and is less practical since it requires the exact knowledge of $\alpha$. Also, the use of estimator $\bar{S}_{-i}$ in the optimal auction verifies the findings in Zaigraev and Podraza-Karakulska (2008) that, the moment based estimator performs better than the maximum likelihood estimator, in terms of both finite sample bias and variance for the estimation of $\alpha$.

If the AB-$\beta$ auction is the unique optimal equivariant auction under $\alpha$, then it is an admissible equivariant auction among all regular distributions, which means its regret ratio cannot be entirely dominated by another equivariant auction. Such uniqueness is guaranteed if $\varphi_{k+n}^0 > 0$ has a unique root.\(^{12}\)

---

\(^{11}\)The condition $v_1 > \max_{i \neq 1} v_i$ is not incorporated into the expression since $\sum_i q_i \leq 1$ is not yet imposed.

\(^{12}\)This property is shown in Figure 1 rather than being proved mathematically.
By the equivalence between $\phi_k^0 > 0$ and $v_1 > \beta S_{-1}$, we know $(k + n - 1)/\beta$ is the root of $\phi_k^0 (\cdot, \alpha)$. Figure 2 shows the value of $\beta$ computed by the numerical root of $\phi_k^0 (\cdot, \alpha)$. The $\beta$ curve increases to the “true value” $p(\alpha)/\alpha$ very rapidly as $k+n$ increases. This observation, together with the fact that $S_{-1}$ consistently estimates $\alpha$, verifies the consistency statement in Proposition 3. Figure 2 also shows another important fact that $p(\alpha)/\alpha$ is bounded in the tight $[0.7, 1]$ for all $\alpha$. This means that any choice of $\beta$ in such region is a reasonable one for the Gamma family.

### 4.2 Maxmin Auction

The previous discussion is based on the case that the shape parameter $\alpha$ is known. Next, an informal discussion is provided on how to find the maxmin auction under the Gamma distribution family with an unknown shape parameter $\alpha$. As explained earlier, the maxmin solution can be found through a game-theoretic approach. The mixed-strategy Nash equilibrium corresponds to the maxmin solution of the problem.

The shape of the regret ratio curve can be derived from Figure 2. Consider setting $\beta$ to be somewhere in $[0.8, 0.9]$, say, then as $\alpha$ increases from 1, the regret ratio first increases, then decreases, and eventually increases again. So for a careful choice of $\beta$, it is possible that the regret ratio have two minima, one at $\alpha = 1$ and the other at $\alpha^*$. When the regret ratio have two minima, any probability mixture between the two minima is a best response for nature. If for a certain
mixture $q_\beta$ is the auctioneer’s best response, then we have pinned down the maxmin auction.

Figure 3 shows the simulated regret ratios curves of $q_\beta$ for the case of two bidders and multiple samples with $\beta$ selected so that the curves approximately has two minima. The smaller minima are at 1, and the larger minima are around 10. This verifies the conjecture about the shape of the regret ratio curve. Nature’s best response is a prior over $\{1, \alpha^*\}$. Denote $\pi$ as the probability of nature choosing 1, and $1 - \pi$ is the probability of nature choosing $\alpha^*$. Given this prior, the auctioneer chooses the auction that maximizes the Bayesian regret ratio $\pi r(q, 1) + (1 - \pi) r(q, \alpha^*)$, which is a linear combination of $R(q, 1)$ and $R(q, \alpha^*)$. So the optimal form of auction is still the AB auction. And with a careful choice of $\pi$, one can get the desired value of $\beta$ as the best response of the auctioneer.

From a practical point of view, the goal of the maxmin procedure is to flatten the regret ratio curve so the corresponding statistical auction has stable performance. This goal is clearly achieved as in Figure 3, where each regret ratio curve is approximately flat. With only 5 samples, the AB-$\beta$ auction (with $\beta = 0.913$) for 2 bidders guarantees more than 94% of the optimal revenue.

5 General Performance

The main feature of the AB auction is that it uses (a fraction of) the sample average to estimate the optimal price $p(F)$. To make this feature more salient, we study the case of monopolistic pricing
where there is only one buyer. In this case, the AB-β auction amounts to setting the price to be

$$\hat{p}_β = \beta \bar{W}_n = \frac{\beta}{n} \sum_{j=1}^{n} W_j.$$  

The alternative estimator for comparison purpose is the empirical Myerson (EM) estimator commonly seen in the literature. The EM procedure estimates the reservation price by maximizing the expected revenue over the empirical distribution of the sample $W$, that is,

$$\hat{p}_{EM} = \arg \max_p \left( \frac{1}{n} \sum_{i=1}^{n} 1\{W_i \geq p\} \right).$$  (13)

This can be considered as the empirical version of Equation (1). This estimator can be made more robust against heavy tail distributions by trimming the large observations (see e.g. Dhangwatnotai et al., 2015; Huang et al., 2018).

5.1 A Two-Step Approximation

The mechanism design theory suggests that $p(F)$ is the optimal price. In other words, $p(F)$ is the target parameter. The usual way to proceed would be to find an estimator that works based on asymptotic theory. For example, such estimator is often consistent and converges to the target parameter at a certain rate. However, as shown by Prasad (2008), the convergence rate of the EM estimator $\hat{p}_{EM}$ is $n^{1/3}$, which is considerably slower than the usual $n^{1/2}$ rate.\textsuperscript{13} Moreover, the following result states that this slow convergence rate is not peculiar to the EM estimator. In fact, $p(F)$ itself is not a regular parameter in the appropriate sense.

**Proposition 4.** The parameter $p(F)$ as a functional of $F$ is not pathwise differentiable in the sense of Van Der Vaart (1991), hence it is not $\sqrt{n}$-estimable.

Therefore, even though the EM estimator seems to be a natural choice from the asymptotics viewpoint, it may not have desired performance when the sample size is finite due to its slow convergence rate. Based on this observation, the estimation of $p(F)$ by using $\hat{p}_β$ can be considered as a two-step approximation. In the first step, the irregular parameter $p(F)$ is approximated by a regular parameter $\beta \mu(F)$, where $\mu(F)$ denotes the population mean of the distribution $F$. In the second step, the parameter $\beta \mu(F)$ is approximated by the estimator $\hat{p}_β$. The sample average is a regular estimator for the population mean. So if in the first step $\beta \mu(F)$ can approximate $p(F)$ well, then the AB estimator $\bar{p}_β$ can quickly converge to the neighborhood region of $p(F)$ even with a small sample size. So the two-step approximation can be seen as a regularization in the finite-sample estimation of the irregular parameter $p(F)$. While the EM estimator outperforms the AB estimator eventually with a large sample size, the AB estimator may perform better when the sample size is small, which is the harder case for conducting statistical analysis.

\textsuperscript{13}The limiting distribution of $n^{1/3}(\hat{p}_{EM} - \bar{p})$ is not the Normal distribution. The general theory of cube-root asymptotics is developed in Kim and Pollard (1990).
The question remains whether \( p(F) \) can be approximated by \( \beta \mu(F) \) for general distributions. This amounts to whether the price-mean ratio, \( p(F)/\mu(F) \), can be well-approximated by a constant \( \beta \). We already know from Figure 2 that for the Gamma distributions, the price-mean ratio is bounded in \([0.7, 1]\), a rather tight interval. The next result shows the price-mean ratio can be bounded likewise under assumptions a little stronger than regularity.

**Proposition 5.** Let \( F \) be a regular distribution.

(i) If the function \( \frac{f}{1-F} \) is concave, then \( p(F)/\mu(F) \leq 1 \).

(ii) If the function

\[
\lambda v - \frac{1 - F(v)}{f(v)}
\]

is increasing w.r.t. \( v \) for some \( \lambda \in [0, 1] \). Then the price-mean ratio can be bounded as

\[
\frac{(1 - \lambda)^\frac{1}{\lambda}}{1 - F(p(F))} \leq p(F)/\mu(F) \leq (1 - \lambda)^{-\frac{1}{\lambda}},
\]

where \( (1 - \lambda)^{\frac{1}{\lambda}} \) is taken to be \( e^{-1} \) (the limit) when \( \lambda = 0 \).

This result demonstrates the theoretical possibility to bound the price-mean ratio. The condition that (14) is increasing is called \( \lambda \)-regularity in Schweizer and Szech (2019). The lower bound in (15) contains the unknown \( p(F) \), a crude bound can be obtained by replacing the denominator \( 1 - F(p(F)) \) by 1. More discussions on the \( \lambda \)-regularity condition can be found in Cole and Roughgarden (2014); Cole and Rao (2017). In Kleinberg and Yuan (2013), the condition that \( p(F) \geq c\mu(F) \) for some \( c > 0 \) is directly assumed under the name \( c \)-boundedness. They show that this condition has attractive implications on the revenue to welfare ratio.

Apart from the theoretical result, Table 1 shows the numerical ranges of the price-ratio computed for common distributions. For these distributions, the price-mean ratio is bounded around one. For the log-normal distribution, the price-mean ratio can exceed one since the virtual valuation is not always concave in this case. Note that the price-mean ratio is scale-invariant, thus the scale parameter in parametric families can be set to 1 for the calculation.

### 5.2 Adaptive Average Bid Estimator

Consider an adaptive procedure for the AB estimator, where the coefficient \( \beta \) is estimated by a pilot estimator. Let \( \hat{\beta} = \hat{p}_{\text{EM}}/\hat{W}_n \) be the ratio between the EM estimator and the sample average. Estimator \( \hat{\beta} \) itself cannot be directly used as the pilot because that would result in the EM estimator. Instead, a coarsening operation is applied. Consider \( [\underline{\beta}, \overline{\beta}] \) as the interval containing the true price-mean ratio, which can be derived based on Proposition 5 or Table 1. Let \( \underline{\beta} = b_0 < b_1 < \cdots < b_{L_n} = \overline{\beta} \) be a set of \( L_n \) partitioning points of \([\underline{\beta}, \overline{\beta}]\), which depends on the
sample size $n$. The pilot estimator $\hat{\beta}$ is defined as the closest partitioning point to $\tilde{\beta}$, i.e.

$\hat{\beta} = b_l$, such that $|b_l - \tilde{\beta}| \leq |b_l' - \tilde{\beta}|$, for $l' = 0, \ldots, L_n$.\(^\text{14}\)

The pilot estimator $\hat{\beta}$ provides a simple way for choosing the coefficient for the AB estimator. The adapted AB estimator is defined as the plug-in estimator $\hat{p}_\beta$. If the partition gets dense in the interval $[\hat{\beta}, \tilde{\beta}]$ in a suitable rate as $n$ increases, then the pilot estimator $\hat{\beta}$ would become a $n^{1/3}$-consistent estimator of the price-mean ratio. Consequently, the adapted estimator $\hat{p}_\beta$ consistently estimates $p(F)$ with the $n^{1/3}$ rate. So with this adaptive procedure, the AB estimator has the same asymptotic properties as the EM estimator. This fact is summarized in the following proposition.

**Proposition 6.** Assume $\hat{p}_EM$ is $n^{1/3}$-consistent and $n^{1/3} (\hat{p}_EM - p(F))$ converges in distribution to a absolute continuous random variable.\(^\text{15}\) If the mesh of the partition satisfies

$$n^{1/3} \max_{1 \leq l \leq L_n} |b_l - b_{l-1}| \to 0,$$

then both the pilot estimator $\hat{\beta}$ and the adapted AB estimator $\hat{p}_\beta$ are $n^{1/3}$-consistent for $p(F)/\mu(F)$ and $p(F)$, respectively.

Simulation studies are conducted for several regular distributions, including Gamma, Generalized Normal, Student-t, Lognormal, and Generalized Pareto. Here is a description of the simulation procedure. Under each distribution, the optimal revenue is computed first. Then the AB and EM estimators are simulated, each with $10^5$ replications. The sample size $n$ is chosen to be $10, 20, 30, 40, 50, 100$, and $200$. For the EM estimator, the top $10\%$ of the sample is trimmed (not considered in the maximization in (13)) for robustness against large observations. For the AB estimator, the interval $[\hat{\beta}, \tilde{\beta}]$ is chosen to be $[0, 1.5]$, and the mesh $|b_l - b_{l-1}| = n^{-1/3}/4$. The estimates are then transformed into realized revenues using the true DGP, and regret ratios are computed.

\(^{14}\)Ties only occur with zero probability, so any tie-breaking rule can be used.

\(^{15}\)Primitive conditions for this assumption can be found in Prasad (2008). The limiting distribution is the unique maximizer of some Gaussian process (related to the Chernoff distribution), and is indeed absolute continuous.
Summary statistics of the simulated regret ratios are reported in Appendix A. The value at risk VaR\(_{\delta\%}\) is the lower \(\delta\) quantile of the regret ratio. The expected shortfall ES\(_{\delta\%}\) is the conditional expectation of the regret ratio given that it is lower than VaR\(_{\delta\%}\). Simulation results show that when the sample size is large (\(n = 100, 200\)), both the AB and EM estimators perform well. However, when the sample size is small, the AB estimator has significantly better performance, especially in terms of the risk measures VaR and ES. The improvement in ES can be as big as 30\% of regret ratio.

6 Conclusion

The AB procedure gives rise to a novel methodology that “regularizes” the estimation of a irregular parameter. Such regularization depends on the existence of a regular parameter that universally approximates the irregular parameter under reasonable assumptions, which is a rather hard problem. This method is new in the literature and further development of which is left for future works.

The AB estimator is a more competitive alternative for EM when data is sparse. There are three reasons for emphasizing small sample cases in pricing problems. First, as an application of the maxmin principle, a result was derived in the paper for cases with covariates, which showed that only the sample data with the most relevant covariate value should be used for data analysis. The practical implication is that the effective samples are going to be very small for each specific auction. Second, for emerging markets, where the design of auction is most important, the amount of data is arguable at a small level. Lastly, if the sample size is significant, then the choice of statistical method is less critical for a good performance.

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## A Simulation Results

| Parameter |  $n$  | Median | Mean | SD | VaR$_{5\%}$ | VaR$_{1\%}$ | ES$_{5\%}$ |
|-----------|------|--------|------|----|-------------|-------------|------------|
|           |      | AB     | EM   | AB | EM          | AB          | EM         |
| 1         | 10   | 0.92   | 0.85 | 0.96 | 0.93       | 0.10        | 0.19       | 0.70 | 0.43 | 0.56 | 0.21 | 0.61 | 0.30 |
|           | 20   | 0.93   | 0.89 | 0.96 | 0.95       | 0.08        | 0.15       | 0.77 | 0.54 | 0.67 | 0.31 | 0.71 | 0.40 |
|           | 30   | 0.95   | 0.91 | 0.97 | 0.96       | 0.06        | 0.13       | 0.82 | 0.62 | 0.73 | 0.39 | 0.77 | 0.48 |
|           | 40   | 0.95   | 0.93 | 0.97 | 0.97       | 0.05        | 0.11       | 0.84 | 0.69 | 0.77 | 0.47 | 0.79 | 0.56 |
|           | 50   | 0.96   | 0.94 | 0.98 | 0.97       | 0.05        | 0.09       | 0.85 | 0.74 | 0.78 | 0.54 | 0.81 | 0.62 |
|           | 100  | 0.97   | 0.96 | 0.98 | 0.98       | 0.04        | 0.06       | 0.90 | 0.85 | 0.85 | 0.72 | 0.87 | 0.77 |
|           | 200  | 0.98   | 0.98 | 0.99 | 0.99       | 0.03        | 0.03       | 0.92 | 0.91 | 0.88 | 0.84 | 0.90 | 0.87 |
| 3         | 10   | 0.90   | 0.89 | 0.95 | 0.95       | 0.14        | 0.16       | 0.59 | 0.52 | 0.41 | 0.24 | 0.48 | 0.36 |
|           | 20   | 0.93   | 0.93 | 0.97 | 0.97       | 0.10        | 0.10       | 0.72 | 0.72 | 0.54 | 0.50 | 0.61 | 0.59 |
|           | 30   | 0.95   | 0.95 | 0.98 | 0.98       | 0.07        | 0.08       | 0.80 | 0.80 | 0.64 | 0.64 | 0.70 | 0.70 |
|           | 40   | 0.96   | 0.96 | 0.98 | 0.98       | 0.06        | 0.06       | 0.83 | 0.83 | 0.70 | 0.70 | 0.75 | 0.75 |
|           | 50   | 0.97   | 0.97 | 0.99 | 0.99       | 0.05        | 0.05       | 0.86 | 0.86 | 0.75 | 0.75 | 0.79 | 0.80 |
|           | 100  | 0.98   | 0.98 | 0.99 | 0.99       | 0.03        | 0.03       | 0.92 | 0.92 | 0.85 | 0.85 | 0.88 | 0.88 |
|           | 200  | 0.99   | 0.99 | 0.99 | 0.99       | 0.02        | 0.02       | 0.95 | 0.95 | 0.91 | 0.91 | 0.93 | 0.93 |
| 5         | 10   | 0.91   | 0.91 | 0.96 | 0.96       | 0.13        | 0.13       | 0.63 | 0.63 | 0.41 | 0.37 | 0.49 | 0.48 |
|           | 20   | 0.94   | 0.95 | 0.98 | 0.98       | 0.08        | 0.08       | 0.78 | 0.78 | 0.60 | 0.60 | 0.67 | 0.67 |
|           | 30   | 0.96   | 0.96 | 0.98 | 0.98       | 0.06        | 0.06       | 0.84 | 0.84 | 0.71 | 0.71 | 0.76 | 0.76 |
|           | 40   | 0.97   | 0.97 | 0.99 | 0.99       | 0.05        | 0.05       | 0.87 | 0.87 | 0.76 | 0.77 | 0.80 | 0.81 |
|           | 50   | 0.97   | 0.97 | 0.99 | 0.99       | 0.04        | 0.04       | 0.89 | 0.89 | 0.80 | 0.80 | 0.83 | 0.84 |
|           | 100  | 0.98   | 0.98 | 0.99 | 0.99       | 0.02        | 0.02       | 0.93 | 0.94 | 0.88 | 0.89 | 0.90 | 0.90 |
|           | 200  | 0.99   | 0.99 | 1.00 | 1.00       | 0.01        | 0.01       | 0.96 | 0.96 | 0.93 | 0.93 | 0.94 | 0.94 |
| 7         | 10   | 0.92   | 0.92 | 0.96 | 0.97       | 0.12        | 0.12       | 0.67 | 0.67 | 0.44 | 0.44 | 0.53 | 0.53 |
|           | 20   | 0.95   | 0.95 | 0.98 | 0.98       | 0.07        | 0.07       | 0.80 | 0.81 | 0.65 | 0.65 | 0.71 | 0.71 |
|           | 30   | 0.96   | 0.96 | 0.98 | 0.98       | 0.05        | 0.05       | 0.85 | 0.86 | 0.74 | 0.74 | 0.78 | 0.79 |
|           | 40   | 0.97   | 0.97 | 0.99 | 0.99       | 0.04        | 0.04       | 0.88 | 0.89 | 0.79 | 0.79 | 0.83 | 0.83 |
|           | 50   | 0.97   | 0.98 | 0.99 | 0.99       | 0.04        | 0.04       | 0.90 | 0.91 | 0.83 | 0.83 | 0.85 | 0.86 |
|           | 100  | 0.98   | 0.99 | 0.99 | 0.99       | 0.02        | 0.02       | 0.94 | 0.94 | 0.90 | 0.90 | 0.91 | 0.92 |
|           | 200  | 0.99   | 0.99 | 1.00 | 1.00       | 0.01        | 0.01       | 0.97 | 0.97 | 0.94 | 0.94 | 0.95 | 0.95 |

Table 2: Simulation Results under Gamma Distribution.
| Parameter | n  | Median  | Mean  | SD  | VaR 5% | VaR 1% | ES 5% |
|-----------|----|---------|-------|-----|--------|--------|-------|
|           |    | AB | EM | AB | EM | AB | EM | AB | EM | AB | EM | AB | EM |
| 10        | 0  | 0.92 | 0.88 | 0.96 | 0.93 | 0.10 | 0.13 | 0.70 | 0.60 | 0.55 | 0.43 | 0.61 | 0.50 |
| 20        | 0  | 0.95 | 0.90 | 0.97 | 0.95 | 0.07 | 0.11 | 0.80 | 0.67 | 0.67 | 0.52 | 0.72 | 0.58 |
| 30        | 0  | 0.96 | 0.91 | 0.98 | 0.95 | 0.06 | 0.10 | 0.83 | 0.71 | 0.73 | 0.57 | 0.77 | 0.62 |
| 40        | 0  | 0.96 | 0.92 | 0.98 | 0.96 | 0.05 | 0.09 | 0.86 | 0.73 | 0.76 | 0.61 | 0.80 | 0.65 |
| 50        | 0  | 0.97 | 0.93 | 0.98 | 0.96 | 0.05 | 0.09 | 0.87 | 0.74 | 0.79 | 0.64 | 0.82 | 0.68 |
| 100       | 0  | 0.98 | 0.94 | 0.99 | 0.97 | 0.03 | 0.07 | 0.91 | 0.79 | 0.85 | 0.71 | 0.87 | 0.74 |
| 200       | 0  | 0.98 | 0.96 | 0.99 | 0.98 | 0.02 | 0.05 | 0.93 | 0.84 | 0.89 | 0.77 | 0.91 | 0.80 |
| 10        | 10 | 0.92 | 0.85 | 0.96 | 0.92 | 0.10 | 0.17 | 0.70 | 0.47 | 0.55 | 0.27 | 0.60 | 0.35 |
| 20        | 0  | 0.94 | 0.88 | 0.97 | 0.94 | 0.07 | 0.15 | 0.79 | 0.53 | 0.67 | 0.34 | 0.72 | 0.42 |
| 30        | 0  | 0.95 | 0.90 | 0.97 | 0.96 | 0.06 | 0.13 | 0.83 | 0.59 | 0.74 | 0.41 | 0.78 | 0.48 |
| 50        | 0  | 0.96 | 0.91 | 0.98 | 0.96 | 0.05 | 0.12 | 0.85 | 0.64 | 0.78 | 0.45 | 0.81 | 0.53 |
| 100       | 0  | 0.97 | 0.95 | 0.98 | 0.98 | 0.03 | 0.07 | 0.91 | 0.80 | 0.87 | 0.63 | 0.89 | 0.70 |
| 200       | 0  | 0.98 | 0.97 | 0.99 | 0.99 | 0.02 | 0.04 | 0.93 | 0.89 | 0.91 | 0.79 | 0.92 | 0.83 |
| 10        | 10 | 0.92 | 0.85 | 0.96 | 0.92 | 0.10 | 0.18 | 0.70 | 0.45 | 0.55 | 0.24 | 0.61 | 0.32 |
| 20        | 0  | 0.94 | 0.88 | 0.97 | 0.94 | 0.07 | 0.15 | 0.79 | 0.53 | 0.68 | 0.32 | 0.73 | 0.41 |
| 30        | 0  | 0.95 | 0.90 | 0.97 | 0.96 | 0.06 | 0.14 | 0.84 | 0.59 | 0.75 | 0.39 | 0.78 | 0.47 |
| 7         | 40 | 0.96 | 0.92 | 0.98 | 0.97 | 0.05 | 0.12 | 0.85 | 0.64 | 0.78 | 0.44 | 0.81 | 0.52 |
| 50        | 0  | 0.96 | 0.93 | 0.98 | 0.97 | 0.04 | 0.11 | 0.88 | 0.67 | 0.81 | 0.49 | 0.84 | 0.56 |
| 100       | 0  | 0.97 | 0.96 | 0.98 | 0.98 | 0.03 | 0.07 | 0.91 | 0.82 | 0.87 | 0.66 | 0.89 | 0.72 |
| 200       | 0  | 0.98 | 0.97 | 0.99 | 0.99 | 0.02 | 0.04 | 0.93 | 0.90 | 0.91 | 0.81 | 0.92 | 0.84 |
| 10        | 10 | 0.92 | 0.85 | 0.96 | 0.92 | 0.10 | 0.18 | 0.70 | 0.44 | 0.55 | 0.23 | 0.61 | 0.31 |
| 20        | 0  | 0.94 | 0.88 | 0.97 | 0.95 | 0.07 | 0.16 | 0.79 | 0.52 | 0.69 | 0.31 | 0.73 | 0.40 |
| 30        | 0  | 0.95 | 0.90 | 0.97 | 0.96 | 0.06 | 0.14 | 0.84 | 0.60 | 0.75 | 0.38 | 0.78 | 0.47 |
| 10        | 40 | 0.96 | 0.92 | 0.98 | 0.97 | 0.05 | 0.12 | 0.86 | 0.66 | 0.79 | 0.44 | 0.81 | 0.53 |
| 50        | 0  | 0.96 | 0.93 | 0.98 | 0.97 | 0.04 | 0.10 | 0.88 | 0.70 | 0.81 | 0.50 | 0.84 | 0.58 |
| 100       | 0  | 0.97 | 0.96 | 0.98 | 0.98 | 0.03 | 0.06 | 0.91 | 0.83 | 0.87 | 0.68 | 0.89 | 0.74 |
| 200       | 0  | 0.98 | 0.98 | 0.99 | 0.99 | 0.02 | 0.04 | 0.93 | 0.90 | 0.91 | 0.82 | 0.92 | 0.85 |
| 10        | 10 | 0.92 | 0.85 | 0.96 | 0.92 | 0.10 | 0.19 | 0.70 | 0.43 | 0.55 | 0.22 | 0.61 | 0.30 |
| 20        | 0  | 0.94 | 0.88 | 0.97 | 0.95 | 0.07 | 0.16 | 0.80 | 0.53 | 0.69 | 0.31 | 0.74 | 0.40 |
| 30        | 0  | 0.95 | 0.91 | 0.97 | 0.96 | 0.06 | 0.13 | 0.84 | 0.60 | 0.76 | 0.39 | 0.79 | 0.47 |
| 15        | 40 | 0.96 | 0.92 | 0.98 | 0.97 | 0.05 | 0.11 | 0.86 | 0.67 | 0.79 | 0.45 | 0.81 | 0.54 |
| 50        | 0  | 0.96 | 0.93 | 0.98 | 0.97 | 0.04 | 0.10 | 0.88 | 0.72 | 0.82 | 0.52 | 0.84 | 0.60 |
| 100       | 0  | 0.97 | 0.96 | 0.98 | 0.98 | 0.03 | 0.06 | 0.91 | 0.84 | 0.87 | 0.69 | 0.89 | 0.75 |
| 200       | 0  | 0.98 | 0.98 | 0.99 | 0.99 | 0.02 | 0.04 | 0.93 | 0.91 | 0.91 | 0.83 | 0.92 | 0.86 |

Table 3: Simulation Results under Generalized Normal Distribution.
| Parameter | n   | Median | Mean | SD  | VaR 5% | VaR 1% | ES 5% |
|-----------|-----|--------|------|-----|--------|--------|-------|
| 1.5       | 10  | 0.91   | 0.88 | 0.95| 0.93   | 0.11   | 0.14  |
|           | 20  | 0.93   | 0.89 | 0.96| 0.94   | 0.09   | 0.12  |
|           | 30  | 0.94   | 0.90 | 0.96| 0.95   | 0.08   | 0.11  |
|           | 40  | 0.94   | 0.91 | 0.96| 0.95   | 0.07   | 0.11  |
|           | 50  | 0.94   | 0.91 | 0.97| 0.96   | 0.07   | 0.10  |
|           | 100 | 0.96   | 0.94 | 0.98| 0.97   | 0.05   | 0.08  |
|           | 200 | 0.97   | 0.96 | 0.99| 0.99   | 0.04   | 0.06  |
| 2         | 10  | 0.90   | 0.85 | 0.95| 0.92   | 0.12   | 0.18  |
|           | 20  | 0.92   | 0.87 | 0.96| 0.94   | 0.10   | 0.16  |
|           | 30  | 0.93   | 0.89 | 0.96| 0.95   | 0.08   | 0.14  |
|           | 40  | 0.94   | 0.91 | 0.97| 0.96   | 0.08   | 0.13  |
|           | 50  | 0.94   | 0.92 | 0.97| 0.97   | 0.07   | 0.12  |
|           | 100 | 0.96   | 0.95 | 0.98| 0.98   | 0.05   | 0.08  |
|           | 200 | 0.98   | 0.97 | 0.99| 0.99   | 0.04   | 0.04  |
| 5         | 10  | 0.90   | 0.85 | 0.95| 0.93   | 0.12   | 0.19  |
|           | 20  | 0.93   | 0.90 | 0.96| 0.96   | 0.09   | 0.15  |
|           | 30  | 0.94   | 0.92 | 0.97| 0.97   | 0.07   | 0.12  |
|           | 40  | 0.95   | 0.94 | 0.97| 0.97   | 0.06   | 0.10  |
|           | 50  | 0.96   | 0.95 | 0.98| 0.98   | 0.05   | 0.08  |
|           | 100 | 0.97   | 0.97 | 0.99| 0.99   | 0.04   | 0.05  |
|           | 200 | 0.98   | 0.98 | 0.99| 0.99   | 0.03   | 0.03  |
| 10        | 10  | 0.91   | 0.86 | 0.95| 0.94   | 0.11   | 0.19  |
|           | 20  | 0.93   | 0.91 | 0.97| 0.96   | 0.08   | 0.14  |
|           | 30  | 0.95   | 0.93 | 0.97| 0.97   | 0.06   | 0.10  |
|           | 40  | 0.95   | 0.94 | 0.98| 0.98   | 0.06   | 0.08  |
|           | 50  | 0.96   | 0.95 | 0.98| 0.98   | 0.05   | 0.07  |
|           | 100 | 0.97   | 0.97 | 0.99| 0.99   | 0.03   | 0.04  |
|           | 200 | 0.98   | 0.98 | 0.99| 0.99   | 0.02   | 0.02  |
| 15        | 10  | 0.91   | 0.86 | 0.95| 0.94   | 0.11   | 0.18  |
|           | 20  | 0.94   | 0.91 | 0.97| 0.96   | 0.08   | 0.13  |
|           | 30  | 0.95   | 0.93 | 0.97| 0.97   | 0.06   | 0.10  |
|           | 40  | 0.95   | 0.95 | 0.98| 0.98   | 0.06   | 0.08  |
|           | 50  | 0.96   | 0.95 | 0.98| 0.98   | 0.05   | 0.07  |
|           | 100 | 0.97   | 0.97 | 0.99| 0.99   | 0.03   | 0.04  |
|           | 200 | 0.98   | 0.98 | 0.99| 0.99   | 0.02   | 0.02  |

Table 4: Simulation Results under Student-t Distribution.
| Parameter n | Median | Mean | SD | VaR5\% | VaR1\% | ES5\% |
|------------|--------|------|----|--------|--------|-------|
| 1.5 10     | 0.90   | 0.89 | 0.95 | 0.94   | 0.12   | 0.13  |
|           | 0.65   | 0.62 | 0.49 | 0.46   | 0.55   | 0.52  |
|           | 0.94   | 0.92 | 0.97 | 0.96   | 0.08   | 0.09  |
|           | 0.77   | 0.72 | 0.63 | 0.59   | 0.69   | 0.64  |
|           | 0.95   | 0.94 | 0.98 | 0.97   | 0.06   | 0.08  |
|           | 0.82   | 0.77 | 0.70 | 0.66   | 0.75   | 0.71  |
|           | 0.97   | 0.95 | 0.99 | 0.97   | 0.05   | 0.06  |
|           | 0.84   | 0.80 | 0.74 | 0.70   | 0.78   | 0.74  |
|           | 0.98   | 0.96 | 0.99 | 0.98   | 0.03   | 0.04  |
|           | 0.91   | 0.87 | 0.84 | 0.81   | 0.87   | 0.84  |
|           | 0.99   | 0.97 | 1.00 | 0.98   | 0.02   | 0.03  |
|           | 0.94   | 0.91 | 0.89 | 0.87   | 0.91   | 0.88  |
| 1.2 10     | 0.92   | 0.88 | 0.96 | 0.93   | 0.10   | 0.14  |
|           | 0.72   | 0.57 | 0.57 | 0.37   | 0.62   | 0.45  |
|           | 0.94   | 0.89 | 0.97 | 0.94   | 0.07   | 0.13  |
|           | 0.79   | 0.63 | 0.67 | 0.46   | 0.72   | 0.52  |
|           | 0.95   | 0.91 | 0.97 | 0.95   | 0.06   | 0.11  |
|           | 0.83   | 0.66 | 0.73 | 0.52   | 0.77   | 0.57  |
|           | 0.96   | 0.91 | 0.97 | 0.96   | 0.05   | 0.10  |
|           | 0.85   | 0.69 | 0.77 | 0.56   | 0.80   | 0.61  |
|           | 0.97   | 0.96 | 0.99 | 0.98   | 0.03   | 0.08  |
|           | 0.90   | 0.77 | 0.85 | 0.67   | 0.87   | 0.71  |
|           | 0.98   | 0.96 | 0.98 | 0.98   | 0.03   | 0.06  |
|           | 0.92   | 0.84 | 0.89 | 0.74   | 0.91   | 0.78  |
| 1.0 10     | 0.91   | 0.85 | 0.95 | 0.92   | 0.11   | 0.17  |
|           | 0.67   | 0.47 | 0.52 | 0.28   | 0.58   | 0.35  |
|           | 0.92   | 0.88 | 0.95 | 0.94   | 0.09   | 0.16  |
|           | 0.73   | 0.52 | 0.61 | 0.34   | 0.66   | 0.41  |
|           | 0.93   | 0.89 | 0.96 | 0.95   | 0.08   | 0.14  |
|           | 0.77   | 0.57 | 0.68 | 0.40   | 0.71   | 0.47  |
|           | 0.94   | 0.91 | 0.96 | 0.96   | 0.07   | 0.13  |
|           | 0.79   | 0.61 | 0.71 | 0.45   | 0.74   | 0.51  |
|           | 0.96   | 0.92 | 0.97 | 0.97   | 0.07   | 0.11  |
|           | 0.80   | 0.65 | 0.72 | 0.48   | 0.75   | 0.55  |
|           | 0.97   | 0.96 | 0.98 | 0.98   | 0.05   | 0.08  |
|           | 0.85   | 0.79 | 0.80 | 0.61   | 0.82   | 0.69  |
|           | 0.97   | 0.97 | 0.99 | 0.99   | 0.04   | 0.04  |
|           | 0.89   | 0.89 | 0.84 | 0.78   | 0.86   | 0.82  |
| 0.7 10     | 0.88   | 0.85 | 0.94 | 0.94   | 0.15   | 0.20  |
|           | 0.56   | 0.37 | 0.39 | 0.16   | 0.45   | 0.24  |
|           | 0.91   | 0.90 | 0.96 | 0.96   | 0.12   | 0.15  |
|           | 0.64   | 0.58 | 0.48 | 0.28   | 0.54   | 0.41  |
|           | 0.93   | 0.93 | 0.97 | 0.97   | 0.09   | 0.11  |
|           | 0.72   | 0.70 | 0.57 | 0.45   | 0.63   | 0.55  |
|           | 0.95   | 0.95 | 0.98 | 0.98   | 0.08   | 0.09  |
|           | 0.77   | 0.77 | 0.62 | 0.56   | 0.68   | 0.65  |
|           | 0.97   | 0.97 | 0.99 | 0.99   | 0.04   | 0.04  |
|           | 0.89   | 0.90 | 0.80 | 0.80   | 0.84   | 0.84  |
|           | 0.98   | 0.98 | 0.99 | 0.99   | 0.02   | 0.02  |
|           | 0.94   | 0.94 | 0.89 | 0.89   | 0.91   | 0.91  |
| 0.5 10     | 0.90   | 0.89 | 0.96 | 0.96   | 0.14   | 0.16  |
|           | 0.56   | 0.55 | 0.35 | 0.25   | 0.43   | 0.37  |
|           | 0.94   | 0.94 | 0.97 | 0.97   | 0.09   | 0.10  |
|           | 0.74   | 0.74 | 0.53 | 0.53   | 0.62   | 0.62  |
|           | 0.95   | 0.96 | 0.98 | 0.98   | 0.07   | 0.07  |
|           | 0.82   | 0.82 | 0.66 | 0.66   | 0.72   | 0.72  |
|           | 0.97   | 0.97 | 0.99 | 0.99   | 0.05   | 0.05  |
|           | 0.88   | 0.88 | 0.78 | 0.78   | 0.81   | 0.82  |
|           | 0.98   | 0.98 | 0.99 | 0.99   | 0.03   | 0.03  |
|           | 0.93   | 0.93 | 0.87 | 0.87   | 0.89   | 0.90  |
|           | 0.99   | 0.99 | 1.00 | 1.00   | 0.02   | 0.02  |
|           | 0.96   | 0.96 | 0.92 | 0.93   | 0.94   | 0.94  |

Table 5: Simulation Results under Lognormal Distribution.
| Parameter | n  | Median AB | Median EM | Mean AB | Mean EM | SD AB | SD EM | VaR5% AB | VaR5% EM | VaR1% AB | VaR1% EM | ES5% AB | ES5% EM |
|-----------|----|-----------|-----------|--------|--------|-------|-------|-----------|----------|-----------|----------|--------|--------|
| 2         | 10 | 0.92      | 0.88      | 0.96   | 0.93   | 0.10  | 0.13  | 0.70      | 0.60     | 0.55      | 0.43     | 0.61   | 0.50   |
|           | 20 | 0.95      | 0.90      | 0.97   | 0.95   | 0.07  | 0.11  | 0.80      | 0.67     | 0.67      | 0.52     | 0.72   | 0.58   |
|           | 30 | 0.96      | 0.91      | 0.98   | 0.95   | 0.06  | 0.10  | 0.83      | 0.70     | 0.73      | 0.57     | 0.77   | 0.62   |
| 100       | 200| 0.98      | 0.96      | 0.99   | 0.98   | 0.02  | 0.05  | 0.93      | 0.84     | 0.89      | 0.77     | 0.91   | 0.80   |
| 5         | 10 | 0.92      | 0.85      | 0.96   | 0.92   | 0.10  | 0.17  | 0.70      | 0.47     | 0.55      | 0.27     | 0.60   | 0.35   |
|           | 20 | 0.94      | 0.88      | 0.97   | 0.94   | 0.07  | 0.15  | 0.79      | 0.53     | 0.67      | 0.34     | 0.72   | 0.42   |
|           | 30 | 0.95      | 0.90      | 0.97   | 0.96   | 0.06  | 0.13  | 0.83      | 0.59     | 0.74      | 0.41     | 0.78   | 0.48   |
| 100       | 200| 0.98      | 0.97      | 0.99   | 0.99   | 0.02  | 0.04  | 0.93      | 0.89     | 0.91      | 0.79     | 0.92   | 0.83   |
| 7         | 10 | 0.92      | 0.85      | 0.96   | 0.92   | 0.10  | 0.18  | 0.70      | 0.45     | 0.55      | 0.24     | 0.61   | 0.32   |
|           | 20 | 0.94      | 0.88      | 0.97   | 0.94   | 0.07  | 0.15  | 0.79      | 0.53     | 0.68      | 0.32     | 0.73   | 0.41   |
|           | 30 | 0.95      | 0.90      | 0.97   | 0.96   | 0.06  | 0.14  | 0.84      | 0.59     | 0.75      | 0.39     | 0.78   | 0.47   |
| 100       | 200| 0.98      | 0.97      | 0.99   | 0.99   | 0.02  | 0.04  | 0.93      | 0.90     | 0.91      | 0.81     | 0.92   | 0.84   |
| 10        | 10 | 0.92      | 0.85      | 0.96   | 0.92   | 0.10  | 0.18  | 0.70      | 0.44     | 0.55      | 0.23     | 0.61   | 0.31   |
|           | 20 | 0.94      | 0.88      | 0.97   | 0.95   | 0.07  | 0.16  | 0.79      | 0.52     | 0.69      | 0.31     | 0.73   | 0.40   |
|           | 30 | 0.95      | 0.90      | 0.97   | 0.96   | 0.06  | 0.14  | 0.84      | 0.60     | 0.75      | 0.38     | 0.78   | 0.47   |
| 100       | 200| 0.98      | 0.98      | 0.99   | 0.99   | 0.02  | 0.04  | 0.93      | 0.90     | 0.91      | 0.82     | 0.92   | 0.85   |
| 15        | 10 | 0.92      | 0.85      | 0.96   | 0.92   | 0.10  | 0.19  | 0.70      | 0.43     | 0.55      | 0.22     | 0.61   | 0.30   |
|           | 20 | 0.94      | 0.88      | 0.97   | 0.95   | 0.07  | 0.16  | 0.80      | 0.53     | 0.69      | 0.31     | 0.74   | 0.40   |
|           | 30 | 0.95      | 0.91      | 0.97   | 0.96   | 0.06  | 0.13  | 0.84      | 0.60     | 0.76      | 0.39     | 0.79   | 0.47   |
| 100       | 200| 0.98      | 0.98      | 0.99   | 0.99   | 0.02  | 0.04  | 0.93      | 0.91     | 0.91      | 0.83     | 0.92   | 0.86   |

Table 6: Simulation Results under Generalized Pareto Distribution.
B Proofs

Proof of Lemma 1. This result follows from standard arguments for DSIC auction design. See, for example, Chapter 4 in Börgers (2015).

Proof of Lemma 2. By definition,
\[
\phi_{F \theta}(\theta p(F)) = \theta p - \frac{1 - F_\theta(\theta p(F))}{f_\theta(\theta p(F))} = \theta \left( p(F) - \frac{1 - F(p(F))}{f(p(F))} \right) = 0.
\]
Thus \( p(F_\theta) = \theta p(F) \). The Myerson auction under \( F_\theta \) is
\[
t_i^{F_\theta}(v) = 1\{v_i > \max_{i' \neq i} v_{i'}, v_i > \theta p(F)\}.
\]
The optimal revenue is
\[
R(F_\theta) = \sum_{i=1}^{k} \int_{\mathbb{R}^k_+} 1\{v_i > \max_{i' \neq i} v_{i'}, v_i > \theta p(F)\} \left( v_i - \frac{1 - F_\theta(v_i)}{f_\theta(v_i)} \right) \prod_{i=1}^{k} f_\theta(v_i) dv
= \sum_{i=1}^{k} \int_{\mathbb{R}^k_+} \left\{ v_i > \max_{i' \neq i} v_{i'}, v_i > p(F) \right\} \theta \left( v_i - \frac{1 - F(v_i)}{f(v_i)} \right) \prod_{i=1}^{k} f(v_i) d\left( \frac{1}{\theta} v \right)
= \theta R(F).
\]

Proof of Lemma 3. Using the revenue equivalence result in Lemma 1, we have
\[
t_i(\theta v, \theta w) = \theta v_i q_i(\theta v_i, \theta w) - \int_0^{v_i} q_i((u, \theta v_{-i}), \theta w) du
= \theta q_i(\theta v, \theta w) - \theta \int_0^{v_i} q_i((u, v_{-i}), w) du
= \theta t_i(v, w).
\]

Proof of Proposition 1. The proof follows closely the proof of Proposition 3 in Stoye (2009). Define the prior \( \tilde{\pi}^* \in \Delta(F^X) \) by \( \tilde{\pi}^* = \prod_{x \in X} \pi^* \). By construction, the marginals of \( \tilde{\pi}^* \) are all identical to \( \pi^* \), while the states \( \{F_x\} \) are mutually independent. We want to show that \( (\kappa^*, \tilde{\pi}^*) \) forms a Nash equilibrium. For \( x \neq x' \), the independence between \( F_x \) and \( F_{x'} \) (under prior \( \tilde{\pi}^* \)) implies that \( W_{x'} \) is uninformative about \( F_x \). Thus \( \kappa^*_x \) is a best response against \( \tilde{\pi}^* \). Next, given that the auctioneer picks \( \kappa^*_x \), the conditional regret ratio is \( r(\kappa^*_x, F) \). By the definition of \( \kappa^*_x \), this regret ratio equals to the one from applying \( q^* \) to the tuple \( (V, W_x) \). Thus nature is best responding by using any prior that has marginal distributions equal to \( \pi^* \).
Proof of Proposition 3. Due to Proposition 2, the regret ratio is the same for all \( \theta \). So we can let \( \theta = 1 \) and consider the distribution \( F \). Thus the denominator of the regret ratio can be omitted. The problem is to find the allocation \( q \) that maximizes the revenue:

\[
\max_{q \in \mathcal{Q}} R(q, F) = \sum_{i=1}^{k} \mathbb{E} [q_i(V, W)\phi_F(V_i)]
\]

s.t. \( q_i \in [0,1], \sum_i q_i \leq 1 \), and \( q_i \) scale-invariant.

We can restrict the \( q_i \)'s to be either 1 or 0 because a \( q_i \) taking values in \( (0,1) \) can never be optimal unless there are equal bids, which happens with probability zero. The maximization can be achieved in two steps. In the first step, we solve the maximization of (9) for each \( q_i \) without imposing the restriction that \( \sum_i q_i \leq 1 \). By monotonicity of \( q_i \) in \( V_i \) and scale-invariance, the solution is of the form \( 1 \{ V_i \geq \rho_F(V_{-i}, W) \} \), where \( \rho_F \) is homogeneous of degree 1. \( \rho_F \) is symmetric and does not depend on \( i \) due to the symmetry in the maximization problem. Then in the second step we can pick among the non-zero \( q_i \)'s a unique one to be 1. Since \( \phi_F \) is increasing, we should choose the \( i \) with the largest \( V_i \), which leads to the expression \( 1 \{ V_i > \max_{i \neq i} V_i \} \). Notice that this expression satisfies the monotonicity and scale-invariance properties. Combining the two steps, we get the expression (8).

For the last statement, consider the average bid auction defined by Equation (11). When \( \beta = p(F)/\mu(F) \) is the true price-mean ratio, \( \beta \bar{S}_{-i} \) is consistent for \( p(F) \) as \( \min\{k,n\} \to \infty \). In this case, the regret ratio of the AB-\( \beta \) auction converges to 1. Now since the auction with \( \rho_F \) maximizes the regret ratio among all equivariant auctions, its regret ratio must also converge to 1. This means \( \rho_F \) must converges in probability to the true optimal price \( p(F) \) since \( \rho_F(V_{-i}, W) \) is independent of \( V_i \).

Proof of Proposition 4. Consider a one-dimensional regular parametric submodel with parameter \( \beta \). The PDF and CDP are parametrized as \( f(\cdot, \eta) \) and \( F(\cdot, \eta) \) respectively. The true probability distribution is indexed by \( \eta_0 \). The parameter \( p(F) \) is implicitly defined through

\[
p(\eta)f(p(\eta), \eta) - (1 - F(p(\eta), \eta)) = 0.
\]

Denote the true optimal price by \( p_0 = p(\eta_0) \). By the implicit function theorem, the derivative of \( p \) with respect to \( \eta \) is

\[
\frac{d}{d\eta} p(\eta)|_{\eta=\eta_0} = -\frac{F_\eta(p_0, \eta_0) + p_0 f_\eta(p_0, \eta_0)}{2f(p_0, \eta_0) + p_0 f_\eta(p_0, \eta_0)}.
\]

(16)

The goal is to turn the above expression into a linear function of the score function \( s(\cdot, \eta) = \frac{f_\eta(\cdot, \eta)}{f(\cdot, \eta_0)} \). The denominator in Equation (16) is constant when the true distribution \( f(\cdot, \eta_0) \) is fixed, thus can
be ignored. The first term in the numerator, $F_{\eta}$, can be written as

$$F_{\eta}(p_0, \eta_0) = \frac{\partial}{\partial \eta} \int 1_{[\xi, \rho_0]}(u) f(u, \eta) du \bigg|_{\eta = \eta_0}$$

$$= \int 1_{[\xi, \rho_0]}(u) f_{\eta}(u, \eta) du$$

$$= \mathbb{E}_{\eta_0} \left[ \int 1_{[\xi, \rho_0]}(V) s(V, \eta_0) \right],$$

which is a continuous, linear functional of $s(\cdot, \eta_0)$. The second term in the numerator is $p_0 f_{\eta}(p_0, \eta_0)$, where $p_0$ is a constant. The term

$$f_{\eta}(p_0, \eta_0) = s(p_0, \eta_0) f(p_0, \eta),$$

where $f(p_0, \eta)$ is again a constant. It boils down to the term $s(p_0, \eta_0)$, which can be thought of as the Dirac delta $\delta_{p_0}$ applied to the score $s(\cdot, \eta_0)$, where $\delta_{p_0}(s)$ evaluates a function $s$ at $p_0$. The functional $\delta_{p_0}$ is indeed linear. However, it is not bounded in the Hilbert space $L_2(F_{\eta}, \eta_0)$, thus not continuous. Hence the RHS of Equation (16) is not a continuous, linear operator of the score function. Thus the parameter $p(F)$ is not pathwise differentiable. \[\square\]

**Proof of Proposition 5.** When the hazard rate is concave, so is the virtual valuation function $\phi$. By Jensen’s inequality, we have $\phi(p) = 0 = \mathbb{E} [ \phi(V) ] \leq \phi(\mu)$. Thus $p \leq \mu$. For the second result, the lower bound follows directly from Proposition 7 in Schweizer and Szech (2019). For the upper bound, we have

$$\mu(F) = \int_0^\infty 1 - F(v) dv \geq p(F)(1 - F(p(F))) \geq p(F)(1 - \lambda)^{1/\lambda},$$

where the last inequality follows from Lemma 3 in Schweizer and Szech (2019). \[\square\]

**Proof of Proposition 6.** By the assumption on $p_{\text{EM}}$, we know $\tilde{\beta}$ is $n^{1/3}$-consistent for $p/\mu$ and $n^{1/3}(\tilde{\beta} - p/\mu)$ converges in distribution to a absolute continous random variable. For $\tilde{\beta}$, consider any $\epsilon > 0$, we have

$$P \left( n^{1/3} \left| \tilde{\beta} - p/\mu \right| \geq \epsilon \right) = P \left( \left| \tilde{\beta} - p/\mu \right| \geq n^{-1/3} \epsilon \right)$$

$$\leq P \left( \left| \tilde{\beta} - p/\mu \right| \geq n^{-1/3} \epsilon \right) + P \left( \left| \tilde{\beta} - p/\mu \right| < \max_{1 \leq l \leq L_n} |b_l - b_{l-1}| \right),$$

where the last inequality follows from the fact that

$$\left| \tilde{\beta} - p/\mu \right| \leq \left| \tilde{\beta} - p/\mu \right| \Rightarrow \left| \tilde{\beta} - p/\mu \right| < \max_{1 \leq l \leq L_n} |b_l - b_{l-1}|.$$
By the condition \( n^{1/3} \max_{1 \leq t \leq L_n} |b_t - b_{t-1}| \to 0 \), we have

\[
P \left( \left| \hat{\beta} - p/\mu \right| < \max_{1 \leq t \leq L_n} |b_t - b_{t-1}| \right) = P \left( n^{1/3} |\hat{\beta} - p/\mu| < n^{1/3} \max_{1 \leq t \leq L_n} |b_t - b_{t-1}| \right) \to 0.
\]

So \( \hat{\beta} \) is \( n^{1/3} \)-consistent. Then \( \hat{p}_\beta \) is also \( n^{1/3} \)-consistent since

\[
n^{1/3}(\hat{p}_\beta - p) = n^{1/3}(\hat{\beta}\hat{W}_n - (p/\mu)\hat{W}_n + (p/\mu)\hat{W}_n - p)
\]

\[
= n^{1/3}(\hat{\beta} - p/\mu)\hat{W}_n + n^{1/3}(p/\mu)(\hat{W}_n - \mu) = O_p(1).
\]

\[\Box\]