Atom-number squeezing and bipartite entanglement of two-component Bose–Einstein condensates: analytical results

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Abstract

We investigate spin dynamics of a two-component Bose–Einstein condensate with weak Josephson coupling. Analytical expressions of atom-number squeezing and bipartite entanglement are presented for atom–atom repulsive interactions. For attractive interactions, there is no number squeezing; however, the squeezing parameter is still useful to recognize the appearance of Schrödinger’s cat state.

Atom-number squeezing has attracted much attention due to its potential applications in quantum metrology and quantum information [1–6]. As a special case of spin squeezed states [7–15], the number squeezed state shows reduced spin fluctuation of the $J_z$ component below the standard quantum limit (SQL), which in turn suppresses the deleterious effects arising from phase diffusion [16–21]. Dynamical generation of the number squeezing has been theoretically investigated based upon a two-mode boson model with a weak Josephson coupling [22, 23, 25–28].

So far, the number squeezing has been demonstrated indirectly by detecting an increased phase fluctuation [1, 6], but not atom-number variance. However, the variances of relative number and phase have a nontrivial relation so a direct measurement of the number fluctuation is of interest and necessary [27]. In this paper, we study atom-number squeezing and bipartite entanglement of a two-component BEC with the Josephson coupling. Two analytical expressions are obtained, which provide us a more direct way to measure the number variance and entropy of entanglement through extracting phase coherence (i.e. the visibility) in atomic interference experiments.

We consider a two-component Bose–Einstein condensate with hyperfine states $|1\rangle$ and $|2\rangle$ coupled by an external microwave (or rf) field [29–32]. For the BEC confined in a deep three-dimensional harmonic potential, we adopt a single-mode approximation [33–36], which results in the following Hamiltonian ($\hbar = 1$):

$$H = \delta J_z - \Omega J_x + \chi J_z^2,$$

where the detuning $\delta$, the Josephson-like coupling constant $\Omega$ and the mean-field interaction strength $\chi$ can be controlled artificially. Angular momentum operators $J_x = \langle J_x \rangle$ and $J_z = (N_2 - N_1)/2$ are introduced, where $a_\sigma$ and $N_\sigma = \langle a_\sigma^\dagger a_\sigma \rangle$ are the annihilation and number operators respectively for the two modes $\sigma = 1, 2$. Equation (1) is equivalent with a two-site Bose–Hubbard Hamiltonian [37], where $\delta$ denotes a potential bias of a double well, $\Omega$ the hopping term and $\chi$ the on-site interaction. The total particle number $N = N_1 + N_2$ is assumed to be a fixed number, so atomic number operators $N_\sigma = \langle a_\sigma^\dagger a_\sigma \rangle$ and $N = N_1 + N_2$. Atom-number fluctuations, defined as usual $(\Delta N_\sigma)^2 = \langle N_\sigma^2 \rangle - \langle N_\sigma \rangle^2$, are the same and equal to the variance $(\Delta J_z)^2$.

For any state vector $|\Psi\rangle$, one can determine the mean spin: $\langle J \rangle = \langle J_x \rangle + i \langle J_y \rangle$, where $\langle J_x \rangle = \text{Re}(J_x)$ and $\langle J_y \rangle = \text{Im}(J_x)$. The expectation value $\langle J_x \rangle$ relates to the first-order cross-correlation function [20, 21]:

$$g^{(1)} = \frac{|\langle a_1^\dagger a_2 \rangle|}{\sqrt{\langle N_1 \rangle \langle N_2 \rangle}} = \frac{|\langle J_x \rangle|}{\sqrt{j^2 - \langle J_z \rangle^2}},$$

(2)
which measures phase coherence of the two-component BEC. It is observable by extracting the visibility of atomic interference fringes [1–6, 31, 37, 38]. A similar definition of the phase coherence has been proposed in [39, 40]. The degree of atom-number squeezing is quantified by a parameter [4, 22]

\[ \xi^2 = \frac{N(\Delta N_{\theta})^2}{\langle N_1 \rangle \langle N_2 \rangle} = \frac{2j(\Delta J_j)^2}{J^2(J_j)} , \]

where the total particle number \( N = 2j \) is assumed to be fixed. It has been shown that there is no squeezing (i.e. \( \xi^2 = 1 \)) for the coherent spin state (CSS) [41]:

\[ |\theta, \phi\rangle = \exp[i\theta(J_x \sin \phi - J_y \cos \phi)] |j, j\rangle , \]

which yields \( \langle J_x \rangle = j \cos \theta \) and \( \langle \Delta J_j \rangle^2 = \langle j/2 \rangle^2 \sin^2 \theta \). The condition of the number squeezing is therefore \( \xi^2 < 1 \) [22], which is consistent with the previous one [6]: \( \langle \Delta J_j \rangle^2 < j/2 \) for a symmetric BEC with the population imbalance \( (J_j) = 0 \).

Firstly, let us consider spin dynamics of the symmetric BEC (\( \delta = 0 \)) with repulsive interactions (\( \chi > 0 \)) for an initial CSS state \( |\Psi(0)\rangle = |j, 0\rangle = |j, j\rangle \), which is also an eigenvector of \( J_z \) with the eigenvalue \( j \). Experimentally, such a state has been prepared by applying a two-photon \( \pi/2 \) pulse to the condensed atoms occupied in the internal state \( |j\rangle \) [29]. The state vector at any time \( t \) can be expanded as \( |\Psi\rangle = \sum c_m |j, m\rangle \), where the probability amplitudes \( c_m \) are determined by the time-dependent Schrödinger equation with the initial condition \( c_m(0) = |j, m\rangle |j, j\rangle = \frac{1}{N} (2j)^{1/2} \). Note that the initial state shows the population imbalance \( (J_j(0)) = 0 \) and the number variance \( \langle \Delta J_j(0) \rangle^2 = J/2 \) (i.e. \( \xi^2 = 1 \)).

As the simplest case, we consider Hamiltonian (1) with \( \Omega = 0 \). This is the one-axis twisting model, proposed originally by Kitagawa and Ueda [7]. The phase coherence can be solved exactly as \( g^{(1)}(t) = \cos^{2j-1}(|\chi|t) \). In the short-time limit, it decays exponentially as \( g^{(1)}(t) \sim \exp[-(t/\tau_J)^2] \) with \( \chi J = j^{-1/2} \), denoting a characteristic time scale for phase coherence [16–19]. The damping of phase coherence, known as phase diffusion [16–21], has been observed in an experiment by extracting the visibility of the Ramsey fringe [38]. During phase diffusion, the number squeezing \( \xi^2 \) remains constant because of the conserved probability distribution \( |c_m|^2 \).

The interplay between the nonlinear interaction and the Josephson coupling leads to suppressed phase diffusion due to the appearance of number squeezing. So far, the number squeezing has been demonstrated for the BEC in optical lattices [1–4], optical trap [5] and atom chip [6]. To understand how this works, we consider unitary evolution of the symmetric BEC governed by Hamiltonian (1). In the Schrödinger picture, the amplitudes always satisfy the relation \( c_{-m} = c_m \), which in turn leads to \( \langle J_x \rangle = \langle J_y \rangle = 0 \) and \( \langle J_z \rangle = \langle J_y \rangle \neq 0 \). Moreover, equations (2) and (3) now reduce to \( g^{(1)}(t) = |J_z(t)|/j \) and \( \xi^2 = 2J^2/J_z^2 \). A relation between \( g^{(1)} \) and \( \xi^2 \) can be obtained by examining the Heisenberg equations of motion:

\[ J_x = -\chi(J_x J_y + J_y J_z) , \]
\[ J_y = \Omega J_x + \chi(J_x J_y + J_y J_z) , \]
\[ J_z = -\Omega J_y . \]

For time-independent \( \Omega \) and \( \chi \), we have \( dJ_z^2/dt = (\Omega/\chi) dJ_x/dt \), and thus

\[ \xi^2(t) = 1 - \frac{2\Omega}{\chi}[1 \mp g^{(1)}(t)] , \]

with the upper sign for the mean spin \( \langle J_x \rangle \geq 0 \) and the lower one for \( \langle J_x \rangle < 0 \). Equation (8) provides us an exact relation between the number squeezing \( \xi^2 \) and the phase coherence \( g^{(1)} \). To confirm it, we consider the two-particle (\( N = 2 \)) case. It is easy to obtain \( g^{(1)} = 1 - \frac{1}{2}(\chi \sin \omega_2 t/\omega_2)^2 \) and \( \xi^2 = 1 - \Omega^2 \sin(\omega_2 t/\omega_2)^2 \), with \( \omega_2 = (\Omega^2 + \chi^2/4)^{1/2} \). Obviously, equation (8) holds for the two-particle case. In addition, we find that the local minimum of \( \xi^2 \) (i.e. the maximal number squeezing) occurs at time \( t_{\text{min}} = \pi/(2\omega_2) \). If an optimal coupling \( \Omega/\chi = 1/2 \) is applied, the system will evolve into the maximally number-squeezed state (MSS): \( |\Psi_{\text{MSS}}\rangle = \text{e}^{-i\chi \omega_2 t/\omega_2} |1, 0\rangle \), which exhibits the perfect squeezing \( (\Delta J_z)^2 = 0 \) [8].

For a large \( N \) case, there exist no exact solutions; however, some approximated solutions are obtainable if the coupling is strong enough. In this case, the mean spin \( \langle J_x \rangle \) almost remains unchanged at \( j \). As a result, we adopt frozen-spin approximation (FSA) [8, 23, 24], i.e. replacing \( J_y \) by \( j (= N/2) \) in the Heisenberg equations, and obtain \( J_x = -\Omega J_x - \chi (J_x J_y + J_y J_z) = -\omega_2^2 J_z \), where \( \omega_N = \sqrt{\Omega(\Omega + N\chi)} \) [23]. Solving the above equation, it is easy to obtain harmonic solutions of \( J_x(t) \) and \( J_z(t) \) [24]. Inserting them into equation (5), we obtain \( J_x(t) \) and also \( g^{(1)}(t) = |J_x(t)|/j \leq 1 = \frac{1}{2}(\chi \sin \omega_2 t/\omega_2)^2 \). Now, the phase coherence \( g^{(1)}(t) \) almost exhibits sinusoidal oscillations (thin lines of figure 1), but not exponential damping as the previous case. In other words, the phase diffusion is strongly suppressed due to the Josephson coupling. From equation (8), we further obtain

\[ \xi^2(t) \simeq 1 - N\Omega \left( \frac{\sin \omega_2 t}{\omega_2} \right)^2 . \]
Clearly, the maximal squeezing appears at time $t_{\text{min}} = \pi/(2\omega \chi)$ with $\xi_{\text{min}}^2 = \Omega^2/\omega^2 < 1$ [23, 24]. Dunningham et al [25] have independently derived the time $t_{\text{min}}$ using a semiclassical analysis of Hamiltonian (1) [34]. For a large $N$ case (> $10^3$), the optimal squeezing obeys the power rule $\Omega/\chi \sim 0.58N^{0.32}$ [26, 27]. In figure 1(a), time evolution of $g^{(1)}$ and $\xi^2$ is plotted for the optimal coupling $\Omega/\chi = 1.732$ and $N = 20$. The FSA works (green dashed curves) quite well to predict $t_{\text{min}}$. For relatively large $\Omega/\chi$, say $\Omega/\chi = 10$ for $N = 20$, the FSA follows full evolution of $\xi^2$ (see figure 1(b)). As shown in figures 1(c) and (d), equation (8) remains hold for the negative $\chi$ case (see below).

The relation between spin squeezing and quantum entanglement is of interest. It was shown that the obtained squeezing is useful for quantum metrology [8, 9] and entanglement [12], provided that $\xi_S = \sqrt{2J(\Delta J_y)/(\chi J_y)} = \xi/g^{(1)} < 1$ [4]. For the system considered here, however, only entanglement between the two modes (i.e. bipartite entanglement) is accessible due to the indistinguishability of identical bosons [43]. A standard measure of bipartite entanglement is the so-called entropy of entanglement $\chi$ [43, 44]:

$$E(t) = -\text{Tr}[\rho_1 \log(\rho_1)] = \sum_{m=-j}^{j} |c_m|^2 \log(|c_m|^2),$$

where $\rho_1 = \text{Tr}_2(\rho)$ is the reduced density operator for mode 1 obtained by partial trace over mode 2. The value of $E$ varies between 0, for the separable product states, to a maximum of $\log(d)$, for maximally entangled states $|\Psi_{\text{MES}}>| = d^{-1/2} \sum_{m=0}^d |m, m>| [43], where $d = (2j + 1)$ is the dimension of the Hilbert space. Utilizing $c_m = d^{-1/2}$, equation (10) gives $E_{\text{MES}} = \log(d)$, which is the maximum value of the entanglement for the system [43, 44]. As an ansatz, the spin state $|\Psi(i)>$ can be treated as a Gaussian [17]:

$$|c_m|^2 \approx \frac{1}{(2\pi(\Delta J_y)^2)^{1/2}} \exp \left[ -\frac{m^2}{2(\Delta J_y)^2} \right],$$

with its width $\Delta J_y = (j/2)^{1/2} \xi^2$, determined by the number squeezing parameter $\xi^2(t)$. Substituting equation (11) into equation (10), and replacing the discrete sum over $m$ by an integral, we arrive at

$$E(t) \approx \frac{1}{2} \log[e\pi j \xi^2(t)],$$

which provides us an analytical relation between the number squeezing and the two-mode entanglement. Considering $\xi^2 = 1$ for the initial CSS $|j, j>$, we have $E(0) = E_{\text{CSS}} \approx \frac{1}{2} \log[e\pi j]$. Numerically, Hines et al [43] have found that $E_{\text{CSS}}/E_{\text{MES}}$ remains finite for large $N$. Our result shows $E_{\text{CSS}}/E_{\text{MES}} \rightarrow \frac{1}{2}$ as $N \rightarrow \infty$. In figures 2(a) and (b), we plot time evolution of the entropy for the finite $N = 20$ case. One can find that equation (12) (red circles) agrees very well with the exact numerical solution of equation (10). In the initial stage, $E(t)$ decreases from $E_{\text{CSS}}$ to its local minimum at $t_{\text{min}}$, due to the appearance of the MSS. In comparison with the CSS, the MSS approaches to the localized twin–Fock state [45]:

$$|N/2, 0, N/2, 0>, = |j, 0>,$$

which exhibits $g^{(1)} = \xi^2 = E = 0$. To confirm it, we plot evolution of probability distribution $|c_m|^2$ in figure 3(a). For the optimal coupling $\Omega/\chi = 1.732$ and $N = 20$, the system evolves into the MSS after a duration $t_{\text{min}} = 0.284$, which shows that the probability distribution peaked at $m = 0$. The quasi-probability distribution $Q(\theta, \phi)$ of the initial state is isotropic (figure 3(c)), representing the minimal uncertainty relationship of a coherent state [7]. It takes an elliptic shape due to the squeezing along $J_y$ and the anti-squeezing along $J_x$ (figure 3(d)).

Until now, we consider spin dynamics of Hamiltonian (1) with $\chi > 0$ and the initial state $|j, j>$. This state is a stable fixed point in the phase space [42], so the phase coherence $g^{(1)}$, the number squeezing $\xi^2$ and the two-mode entanglement $E$ oscillate regularly with the same period. Using equations (8) and (12), it is possible to measure both $\xi^2$ and $E$ by extracting $g^{(1)}$ (i.e. the visibility) in an atomic interference experiment [1–6, 31, 38]. There are two alternative experimental setups. One possibility is to trap the condensed $^{23}$Na atoms in a symmetric double-well potential formed by atom chip [6], and the other is the two-component BEC in an optical dipole trap [30]. In both cases, positive and large enough $\chi \sim (\Delta_{11} + \Delta_{22} - 2\Delta_{12})$ is required, which speeds up dynamics of the system such that the deleterious effects such as atom losses can be neglected [25].

To proceed, let us consider another scenario: spin dynamics of Hamiltonian (1) with the negative $\chi$ case [1]. Now the initial state $|j, j>$ corresponds to an unstable point at the separatrix [42], which leads to a more complex dynamics with a quite different characteristic time scale. From equation (8), we find that there is no number squeezing for the negative $\chi$ case (see also figures 1(c) and (d)). Instead of preparing the MSS, the latter model can be used to generate a Schrödinger’s cat state like $|\Psi_{\text{CAT}}> = \frac{1}{\sqrt{2}}(|j, -j> + |j, j>)$ [46]. In figures 2(c) and (d), we plot time evolution of $E(t)$ for the $N = 20$ case. Our results show that equation (12) follows

[1] Phase separation of the two-component BEC occurs for the negative $\chi$ case, which can be avoided by considering positive $\chi$ and the initial state $|j, -j>$, (cf [43]). Our results still hold.
the exact results of $E$ in the early stages of the evolution, and then diverges as $|\Psi\rangle \rightarrow |\Psi\rangle_{\text{MES}}$. This is because the population distribution of $|\Psi\rangle_{\text{MES}}$ is no longer a Gaussian, and equation (12) cannot simulate the entropy. However, the right-hand side of equation (12), a monotonic function of $\xi^2$, is still useful to recognize the appearances of the cat state, which exhibits the largest number variance $(\Delta J_z)^2_{\max} = j^2$ and $\xi^2_{\max} = 2j$. In real evolution, an approximate cat state with $\xi = \pm j$ (figure 1(d)) can be obtained at time $|\chi|t_{\max} \sim \ln(8N)/N$ for the optimal coupling $\Omega /|\chi| = N/2$. [46]. From the right panel of figure 3, we also find that this state shows the probability distribution $|c_m|^2$ peaked at $m = \pm j$ and maximal values of $Q(\theta, \phi)$ pointed to the north and the south poles of the Bloch sphere.

In summary, we have investigated spin dynamics of a symmetric BEC with repulsive interactions ($\chi > 0$) evolved from a coherent spin state $|j, j\rangle$. As main results of our work, we find analytical expressions of the number-squeezing parameter $\xi^2$ and the entropy of entanglement $E$ as equations (8) and (12) respectively. Both of them can be, in principle, at least, measured by extracting the phase coherence $g^{(1)}(t)$ (i.e., the visibility) in atomic interference experiments. For the case of attractive interactions ($\chi < 0$), though there exists no number squeezing, the squeezing parameter $\xi^2$ or equation (12) is still useful to recognize the appearance of Schrödinger’s cat state.

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Figure 3. Probability distribution $|c_m|^2$ (a) and (b), and quasi-probability distribution $Q(\theta, \phi) = |\langle \theta, \phi | \Psi(t) \rangle|^2$ (c)–(f), where the CSS $|\theta, \phi\rangle$ is given by equation (4). The red lines in (a) and (b) represent $|c_m|^2$ for the MSS at $t_{\max} = 0.284 \chi^{-1}$ and the cat state at $t_{\max} = 0.275 \chi^{-1}$, respectively. The parameters in the left and the right panels are the same as in figures 1(a) and (d). (Colour online.)
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