The resummed quark form factor in dimensional regularization

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The resummed expression for the quark form factor illustrates the fact that dimensional continuation provides a regularization not only for ultraviolet and infrared singularities of fixed order QCD amplitudes, but also for the Landau pole arising in resummed calculations. Explicit renormalization group invariant analytic expressions for the logarithm of the form factor are given up to two–loop order in the QCD $\beta$ function.

1. Introduction

The resummation of perturbation theory for the form factor of charged gauge particles has a long history, and has gone through several levels of refinement, starting with the seminal work of Sudakov [1]. The complete exponentiation of the form factor can be achieved by deriving and solving an evolution equation governing its energy dependence, as was done in Ref. [2], by several authors, using different techniques. An instructive introduction to some of these techniques can be found in Ref. [3]. The usefulness of dimensional regularization for the implementation and the solution of the evolution equation was first noticed in Ref. [4], where an explicit exponentiated solution for the form factor, directly comparable with diagrammatic calculations, was given, and also the ratio of the timelike to the spacelike form factor was computed. This solution expresses the logarithm of the form factor in terms of integrals over the scale $\mu^2$ of the running coupling, with an integration region extending all the way to $\mu^2 = 0$, as always in resummed expressions for QCD amplitudes and cross sections. In $d = 4$, these integrals are ill–defined because of the Landau pole singularity in the running coupling, and typically the ensuing ambiguity in the resummed expression is taken as an estimate for the size of nonperturbative, power–suppressed effects [5]. More recently [6], I have shown that dimensional continuation provides a natural and gauge invariant way to regularize the Landau singularity; the integrals over the scale of the coupling can then be explicitly evaluated in terms of analytic functions of the coupling and of the space–time dimension, which are renormalization group invariant by inspection to the desired accuracy in the QCD $\beta$ function. Here, I will briefly review the results of Ref. [6], emphasizing the special role played by dimensional regularization and the fate of the Landau pole.

2. Resumming the form factor with dimensional regularization

For definiteness, consider the timelike electromagnetic quark form factor in dimensionally regularized massless QCD, defined by

$$\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) = \langle 0| J_\mu(0)| p_1, p_2 \rangle$$

$$= -ieq_\gamma \bar{u}(p_2)\gamma_\mu u(p_1) \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right),$$

where $J_\mu$ is the electromagnetic current, $Q^2 = (p_1 + p_2)^2$, and $\epsilon = 2 - d/2 < 0$ to regulate IR divergences in the renormalized theory. As described in Ref. [6], the $Q^2$ dependence of the form factor is determined by an evolution equation of the form

$$Q^2 \frac{\partial}{\partial Q^2} \log \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right)$$

$$= \frac{1}{2} \left[ K (\epsilon, \alpha_s(\mu^2)) + G \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right].$$

The function $G$ contains the $Q^2$ dependence and is finite in the limit $\epsilon \to 0$, while the function $K$ is
a pure counterterm; furthermore, $K$ and $G$ must renormalize additively with the same anomalous dimension $\gamma_K$ to preserve the renormalization group invariance of the full form factor: thus, $dG/d\ln \mu = -dK/d\ln \mu = \gamma_K(\alpha_s)$. The functions $K$, $G$ and $\gamma_K$ are perturbatively calculable and known to two loops.

Dimensional regularization affects Eq. (2.2) in two crucial ways. First, renormalization group evolution is dictated by the $d$–dimensional $\beta$ function,

$$\beta(\epsilon, \alpha_s) = \frac{\partial \alpha_s}{\partial \mu^2} = -2\epsilon \alpha_s + \hat{\beta}(\alpha_s), \quad (2.3)$$

where $\hat{\beta}(\alpha_s)$ is the usual $\beta$ function in $d = 4$. As a consequence, for example, the one–loop running coupling takes the form

$$\frac{\alpha_s}{\mu^2} = \frac{\alpha_s}{\mu_0^2} \left[ \left( \frac{\mu^2}{\mu_0^2} \right)^\epsilon - \frac{1}{\epsilon} \left(1 - \left( \frac{\mu^2}{\mu_0^2} \right) \right) b_0 \alpha_s(\mu_0^2) \right]^{-1},$$

with $b_0 = (11C_A - 2n_f)/3$. The second effect of dimensional regularization is that one may explicitly solve the evolution equation (2.2) with the simple boundary condition

$$\Gamma\left(0, \alpha_s(\mu^2), \epsilon\right) = 1.$$  

(2.5)

This can be seen by considering the perturbative expansion for $\Gamma(Q^2)$, in which each term must be proportional to a positive integer power of $(\mu^2/(-Q^2))$, or alternatively by noting that the $d$–dimensional running coupling in Eq. (2.4) vanishes, when $\mu^2 \to 0$, as $\mu^{-2\epsilon}$. One can then solve Eq. (2.3) obtaining

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \right\}$$

$$\left[ K(\epsilon, \alpha_s(\mu^2)) + G\left(-1, \frac{\xi^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right), \epsilon \right]$$

$$+ \frac{1}{2} \int \frac{d\lambda^2}{\lambda^2} \gamma_K\left( \frac{\lambda^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right\}.$$  

(2.6)

Eq. (2.6) was used in Ref. 4 to derive an explicit representation for the ratio $\Gamma(Q^2)/\Gamma(-Q^2)$, which is phenomenologically relevant because it enters directly the resummed partonic cross section of the Drell–Yan process 5.

A further illustration of the power of dimensional regularization is given by the computation of the “counterterm” function $K(\epsilon, \alpha_s)$ 6. In a minimal scheme, $K$ depends on $\mu$ only through the coupling $\alpha_s(\mu^2)$; thus, its RG equation becomes a recursion relation, expressing all higher order poles in terms of the perturbative coefficients of the simple pole, these in turn being determined by the anomalous dimension $\gamma_K$. In particular, writing

$$K(\epsilon, \alpha_s) = \sum_{m=0}^\infty K_m(\epsilon, \alpha_s), \quad (2.7)$$

$$K_m(\epsilon, \alpha_s) = \sum_{n=1}^\infty K_n^{(m+n)} \left( \frac{\alpha_s}{\pi} \right)^{n+m} \frac{1}{\epsilon^n},$$

one observes that all leading poles $(m = 0)$ are determined by one loop calculations, while in general the coefficients of the poles contributing to $K_m$ require a calculation to $m + 1$ loops. It turns out 7 that the recursion relation determining the coefficients $K_n^{(m)}$, which depends on the $\beta$ function given by Eq. (2.3), can be solved completely, including all orders in $\beta(\alpha_s)$. Furthermore, all the resulting series of poles $K_m(\epsilon, \alpha_s)$ can be summed, and yield analytic functions of $\alpha_s$ and $\epsilon$ that are regular as $\epsilon \to 0$ for $m > 0$. The only singularity in the resummed expression for $K(\epsilon, \alpha_s)$ is logarithmic and completely determined by the one–loop $\beta$ function and the one–loop coefficient in the anomalous dimension $\gamma_K$. Specifically, including only one–loop results, one finds 8

$$K_n^{(m)} = \frac{1}{2m} \left( \frac{-b_0}{4} \right)^{n-1} \gamma_K^{(m-n+1)}, \quad (2.8)$$

which is exact for $n = m$ and implies

$$K_0(\epsilon, \alpha_s) = \frac{2\gamma(1)}{b_0} \ln \left( 1 + \frac{b_0\alpha_s}{4\pi\epsilon} \right).$$

(2.9)

Notice that the function $K_0(\epsilon, \alpha_s)$ has a cut in the $\epsilon$ complex plane running from $\epsilon = -b_0\alpha_s/(4\pi)$ to $\epsilon = 0$. This cut is a direct consequence of the Landau singularity in the one–loop running coupling, as I will discuss in the next section.
3. The fate of the Landau pole

To understand why dimensional continuation allows for such explicit evaluations of resummed quantities in QCD, one may consider the one–loop β function at ϵ < 0, as given by Eq. (2.3). It is apparent that instead of the usual (double) zero at α_s = 0, the dimensionally continued β function has two distinct zeroes, one at the origin and one located at α_s = -4πϵ/b_0. For ϵ < 0, this second zero is the asymptotically free one, while at the origin in the α_s plane the β function vanishes with positive derivative, as it would in a QED–like theory. This is a different way of expressing the fact that the running coupling vanishes like a power of the scale for ϵ < 0, as pointed out in Section 2. The presence of the second, asymptotically free zero of the β function is manifest in the explicit expression, Eq. (2.4), for the one–loop running coupling. In fact, Eq. (2.4), just like the four–dimensional running coupling, has a simple pole in the µ^2 complex plane, located at

\[ \mu^2 = \Lambda^2 \equiv Q^2 \left( 1 + \frac{4\pi\epsilon}{b_0\alpha_s(Q^2)} \right)^{-\frac{1}{\epsilon}}, \quad (3.10) \]

which can be used to define the coupling, just as in conventional dimensional transmutation,

\[ \alpha_s(Q^2) = \frac{4\pi\epsilon}{b_0 \left[ \left( \frac{\mu^2}{\Lambda^2} \right)^\epsilon - 1 \right]}. \quad (3.11) \]

What has changed as an effect of dimensional continuation is the fact that now the Landau pole is not necessarily located at real values of the renormalization scale. In fact, for ϵ < -b_0α_s/(4π), ϵ ≠ -1/n, Λ^2 acquires a nonvanishing imaginary part. For such (large) values of the space–time dimension, the coupling decreases smoothly to zero, starting from the boundary value α_s(Q^2), without encountering any singularity for real values of the scale. On the other hand, when the dimension of space time is sufficiently close to d = 4, the Landau pole migrates to the real axis in the µ^2 complex plane, which is also the integration contour for resummed expressions such as Eq. (2.6).

As a result, the analytic expressions obtained by evaluating the integrals will develop a cut, which might appropriately be called Landau cut.

One sees that dimensional continuation succeeds in regularizing the Landau singularity, arising from resummation, much in the same way as it regularizes ultraviolet and infrared divergences in fixed order perturbative calculations: instead of an ill–defined expression, we will now find an analytic function of the coupling and of the space–time dimension, with a specified singularity (in this case a cut) when the physical value d = 4 is approached. Of course, the singularity has not been eliminated, since it has a physical meaning and is ultimately related to confinement; rather, the singularity is now parametrized in a gauge–invariant manner, which might provide insights into nonperturbative corrections to resummed physical quantities, if the method can be applied to infrared safe observables.

4. One–loop analytic resummation

Given the discussion in the previous section, one expects that it should be possible to evaluate explicitly the integrals in Eq. (2.6). Furthermore, using for the running coupling the solution of the RG equation to a given order in 1/\Lambda, one expects the result to be RG invariant, i.e. independent of the renormalization scale µ^2, to the same accuracy. One can readily verify these claims at the one–loop level. Using Eq. (2.4), and changing variables according to

\[ \lambda^2 \rightarrow z = \left( \frac{\mu^2}{\lambda^2} \right)^\epsilon - 1, \quad (4.12) \]

the anomalous dimension integral is easily performed, yielding a logarithm. The logarithm is not integrable over the scale ξ^2 because of a singularity at ξ^2 = 0, which however is cancelled by the ξ–independent contribution of the counterterm function K, Eq. (2.5). Inserting the values of the one–loop coefficients of K, γ_K and G, and defining for simplicity

\[ a(\mu^2) = \frac{b_0}{4\pi} \alpha_s(\mu^2), \quad (4.13) \]
one finds
\[
\log \Gamma \left( \frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = -\frac{2C_F}{b_0} \log \Gamma \left( -1, \alpha_s(Q^2), \epsilon \right) 
\]
\[
\times \left\{ \frac{1}{\epsilon} \operatorname{Li}_2 \left[ \frac{(\mu^2)}{Q^2} \frac{a(\mu^2)}{a(\mu^2) + \epsilon} \right] 
- C(\epsilon) \ln \left[ 1 - \left( \frac{(\mu^2)}{Q^2} \frac{a(\mu^2)}{a(\mu^2) + \epsilon} \right) \right] \right\},
\]
where for clarity I considered the spacelike rather than the timelike form factor, and

\[
C(\epsilon) = (3 - \epsilon(\zeta(2) - 8))/2 + O(\epsilon^2)
\]

arises from the one–loop contribution to the function \( G \).

RG invariance of Eq. (4.14) is readily verified by reexpressing the running coupling in terms of \( \alpha_s(Q^2) \), using Eq. (2.4). One finds that the \( \mu^2 \) dependence cancels, obtaining

\[
\log \Gamma \left( \frac{-Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \log \Gamma \left( -1, \alpha_s(Q^2), \epsilon \right)
\]
\[
= -\frac{2C_F}{b_0} \left\{ \frac{1}{\epsilon} \operatorname{Li}_2 \left[ \frac{a(Q^2)}{a(Q^2) + \epsilon} \right] 
+ C(\epsilon) \ln \left[ 1 + \frac{a(Q^2)}{\epsilon} \right] \right\}. 
\]

Eq. (4.16) is a striking illustration of the power of dimensional continuation; the resummed form factor has a simple analytic structure, characterized by a cut (the “Landau cut”), which can be taken to run from \( \epsilon = -a(Q^2) \) to \( \epsilon = 0^- \), as expected; one can reexpand Eq. (4.16) in powers of \( \alpha_s(Q^2) \) for fixed \( \epsilon \) to recover known perturbative results, or to generate the coefficients of all leading and next–to–leading infrared and collinear poles of the form factor; on the other hand, one can examine the behavior of the resummed expression in the vicinity of the physical point \( \epsilon = 0 \); one finds then

\[
\log \Gamma \left( -1, \alpha_s(Q^2), \epsilon \right) = \frac{2C_F}{b_0} \left[ -\frac{\zeta(2)}{\epsilon} + \frac{1}{a(Q^2)} \right] + \frac{1}{2} \log \left( \frac{a(Q^2)}{\epsilon} \right) + O(\epsilon).
\]

One observes that the resummation of all leading poles in the logarithm of the form factor gives just a single pole, with a residue independent of the coupling and of the energy, up to logarithmic corrections. As will be seen below, one may conjecture that this is in fact the only pole in the complete perturbative resummation of the logarithm of the form factor. It is also worth noticing that, in the vicinity of \( \epsilon = 0 \), the form factor contains a finite term of the form \( \exp(c/\alpha_s) \), which would correspond to a power–behaved contribution of the type \( (\Lambda^2/Q^2)^{-c} \) in the four–dimensional theory. Although in the present case this term is of no direct physical significance, being associated with an IR divergent quantity, it is encouraging to see a term of this kind arising in a gauge–invariant way from the present formalism. Finally, it should be mentioned that Eq. (4.16) can be tested by computing the ratio of the timelike to the spacelike form factor, which is found to agree with the results of [4].

5. Two–loop analytic resummation

To generalize to higher perturbative orders the calculation performed in Section 4, one can take advantage of the fact that integrations over the renormalization scale can be replaced by integrations over the coupling itself, using

\[
\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta(\epsilon, \alpha_s)},
\]

and truncating the perturbative \( \beta \) function to the desired order. It is easy to reproduce the one–loop result, Eq. (4.16), with this method. At two loops, one must change variables in both scale integrals according to

\[
\frac{d\mu^2}{\mu^2} = -\frac{d\alpha_s}{\alpha_s \epsilon + \frac{b_0}{4\pi} \alpha_s + \frac{b_1}{4\pi^2} \alpha_s^2},
\]

and of course one must include in the calculation the two–loop coefficients of the functions \( K, G \) and \( \gamma_K \). All integrations can be explicitly performed by partial fractioning, and they yield logarithms and dilogarithms, expressed in terms of the two nontrivial zeroes of the two–loop \( \beta \) function. Writing

\[
\alpha_{\pm} = -\frac{b_0}{2b_1} \left( 1 \pm \sqrt{1 - \frac{16eb_1}{b_0^2}} \right).
\]
one finds
\[
\log \Gamma \left( -1, \alpha_s(Q^2), \epsilon \right) = -\frac{2}{b_1(a_+ - a_-)} \\
\times \left\{ \left( G^{(1)}(\epsilon) + a_+ G^{(2)}(\epsilon) \right) \log \left( 1 - \frac{\alpha_s(Q^2)}{\pi a_+} \right) \\
+ 2 \left( \gamma^{(1)}_K + a_+ \gamma^{(2)}_K \right) \left[ -\frac{1}{4\epsilon} \text{Li}_2 \left( \frac{\alpha_s(Q^2)}{\pi a_+} \right) \\
+ \frac{1}{2b_1 a_+(a_+ - a_-)} \log^2 \left( 1 - \frac{\alpha_s(Q^2)}{\pi a_+} \right) \\
- \frac{1}{b_1 a_-(a_+ - a_-)} \left( \text{Li}_2 \left( \frac{\alpha_s(Q^2)}{\pi a_+} \right) - \frac{\alpha_s(Q^2) - \alpha_s(Q^2)}{\pi(a_+ - a_-)} \right) \\
\right. \\
\left. + \log \left( 1 - \frac{\alpha_s(Q^2)}{\pi a_+} \right) \log \left( \frac{\alpha_s(Q^2) - \alpha_s(Q^2)}{\pi(a_+ - a_-)} \right) \\
- \text{Li}_2 \left( \frac{a_+}{a_+ - a_-} \right) \right\} + (a_+ \leftrightarrow a_-) \right. .
\] 
(5.21)

It is fairly straightforward to verify that when the two–loop coefficients \( \{b_1, \gamma^{(2)}_K, G^{(2)}(\epsilon)\} \) are taken to zero Eq. (5.21) reduces to Eq. (4.16). One can also study the behavior of Eq. (5.21) in the vicinity of \( \epsilon = 0 \); one finds that, as announced, the simple pole in the logarithm of the form factor, given in Eq. (4.17), is not affected by the inclusion of two–loop effects, while the logarithmic singularities are enhanced to \( \log^2 \) strength. This result, together with the all–loop evaluation of the counterterm function \( K \), strongly suggests that the residue of the simple pole in Eq. (4.17) receives no higher order perturbative corrections.

It is apparent that the calculation leading to Eq. (5.21) can be generalized to any number of loops: with the change of variables in Eq. (5.18), one must deal at most with a double integral of a rational function, which is in general computable in terms of polylogarithms by means of partial fractioning. While this generalization would be purely formal at this stage, since the three–loop perturbative coefficients of the functions \( G \) and \( \gamma_K \) are not known at present, one may rely on the fact that the present method and results can be systematically improved upon when higher order perturbative calculations become available.

6. Outlook

I have shown that, in the case of the resummed quark form factor, dimensional conti-

uation provides a gauge– and RG–invariant regular-

ization of the Landau singularity. A general-

ization of the present formalism to more compli-

cated QCD amplitudes and cross sections, and in par-

ticular to the resummation of real soft gluon emis-

sion, would be of great theoretical and phenom-

enological interest, since it would in principle lead towards the construction of resummed, RG–invariant partonic cross section, and might provide useful insights in the nature of nonper-

turbative corrections to factorization theorems. With this in mind, it is encouraging to note that the form factor plays a key role also in the re-

summation of real gluon emission \( [9] \), and that “Sudakov” resummation techniques are available also for more complicated QCD processes, involving nonsiglet color exchanges \( [10] \).

REFERENCES

1. V.V. Sudakov, Sov. Phys. JETP 3 (1956) p. 65.
2. A.H. Mueller, Phys. Rev. D 20 (1979) p. 2037; J.C. Collins, Phys. Rev. D 22 (1980) p. 1478; A. Sen, Phys. Rev. D 24 (1981) p. 3281; G.P. Korchemsky and A.V. Radyushkin Nucl. Phys. B 283 (1987) p. 342.
3. J.C. Collins, in Perturbative Quantum Chromodynamics, ed. A.H. Mueller, World Scientific, Singapore, 1989, p. 573.
4. L. Magnea and G. Sterman, Phys. Rev. D 42 (1990) p. 4222.
5. M. Beneke, Phys. Rep. 317 (1999) p. 1, hep-ph/9807443.
6. L. Magnea, hep-ph/0006255.
7. G. Sterman, Nucl. Phys. B 281 (1987) p. 310; S. Catani and L. Trentadue, Nucl. Phys. B 327 (1989) p. 323; L. Magnea, Nucl. Phys. B 349 (1991) p. 703.
8. See, e.g., G. Parisi, Statistical field theory, Addison Wesley (1988).
9. A. Bassetto, M. Ciafaloni and G. Marchesini, Phys. Rep. 100 (1983) p. 201.
10. N. Kidonakis and G. Sterman, Nucl. Phys. B 505 (1997) p. 321, hep-ph/9705234; R. Bonciani, S. Catani, M.L. Mangano and P. Nason, Nucl. Phys. B 529 (1998) p. 424, hep-ph/9801373.