Symplectic Resolutions for Symmetric Products of Surfaces

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Abstract

Let $S$ be a smooth complex connected analytic surface which admits a holomorphic symplectic structure. Let $S^{(n)}$ be its $n$th symmetric product. We prove that every projective symplectic resolution of $S^{(n)}$ is isomorphic to the Douady-Barlet resolution $S^{[n]} \rightarrow S^{(n)}$.

1 Introduction

Let $X$ be a normal complex analytic variety. Following [Bea], the variety $X$ is said to have symplectic singularities if there exists a holomorphic symplectic 2-form $\omega$ on $X_{\text{reg}}$ such that for any resolution of singularities $\pi : \tilde{X} \rightarrow X$, the 2-form $\pi^* \omega$ defined a priori on $\pi^{-1}(X_{\text{reg}})$ can be extended to a holomorphic 2-form on $\tilde{X}$. If furthermore the 2-form $\pi^* \omega$ extends to a holomorphic symplectic 2-form on the whole of $\tilde{X}$ for some resolution of $X$, then we say that $X$ admits a symplectic resolution, and the resolution $\pi$ is called symplectic.

There are two classes of examples of symplectic singularities. One consists of normalizations of closures of nilpotent orbits in a complex semi-simple Lie algebra. For these singularities, we proved in [Fu] that every symplectic resolution is isomorphic to the collapsing of the zero section of the cotangent bundle of a projective homogeneous space.

The other class of examples consists of so called quotient singularities, i.e. singularities of the form $V/G$, with $V$ a symplectic variety and $G$ a finite group of symplectic automorphisms of $V$. In this note, we are interested in the following special case. Let $S$ be a complex analytic manifolds which
admits a holomorphic symplectic structure. Then by Proposition 2.4 (Bea), the symmetric product \( S^{(n)} = S^n / \Sigma_n \) is a normal variety with symplectic singularities, where \( \Sigma_n \) is the permutation group of \( n \) letters. By Theorem 2.1 (Fub), \( S^{(n)} \) (for \( n > 1 \)) admits no symplectic resolution as soon as \( \dim(S) > 2 \). From now on, we will suppose that \( \dim(S) = 2 \), i.e. \( S \) is a smooth complex connected analytic surface admitting a symplectic structure. Examples of such surfaces include \( K3 \) surfaces, abelian surfaces, cotangent bundles of algebraic curves etc..

Recall that the Douady space \( S^{[n]} \) parametrizes zero-dimensional analytic subspaces of length \( n \) in \( S \). It is well-known that \( S^{[n]} \) is a \( 2n \)-dimensional smooth complex manifolds. If \( S \) is algebraic, it is the usual Hilbert scheme \( \text{Hilb}^n(S) \) of points on \( S \). The Douady-Barlet morphism \( S^{[n]} \to S^{(n)} \) provides a projective resolution of singularities for \( S^{(n)} \). If \( S \) admits a symplectic structure, then \( S^{[n]} \) is again symplectic (see Bea). In this case \( S^{[n]} \) gives a projective symplectic resolution for \( S^{(n)} \). Our purpose of this note is to prove the following:

**Theorem 1.** Let \( S \) be a smooth connected complex analytic surface, which admits a symplectic structure. Then every projective symplectic resolution of \( S^{(n)} \) is isomorphic to the Douady-Barlet resolution \( S^{[n]} \to S^{(n)} \).

One should bear in mind that in general there may be more than one symplectic resolutions for a variety with symplectic singularities. Such an example is given by a Richardson nilpotent orbit admitting two or more polarizations with non-conjugate Levi factors (see Fu).

A similar result for symplectic quotients of \( \mathbb{C}^{2n} \) has been proved by D. Kaledin (see Theorem 1.9 [Ka]).

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## 2 Proof of the theorem

**Lemma 2.** Let \( X \) be a normal variety with symplectic singularities and \( U \) its smooth part. Let \( \text{Sing}(X) = \bigcup_{i=1}^k F_i \) be the decomposition into irreducible components of the singular part of \( X \). Suppose that \( X \) admits a projective symplectic resolution \( \pi : Z \to X \). Furthermore we suppose that:

(i) \( \text{Pic}(U) \) is a torsion group;
(ii). for any \( i \), there exists an analytic proper sub-variety \( B_i \) in \( F_i \), such that the restricted map \( \pi' : Z_* \to X_* \) is the blow-up of \( X_* \) with center \( \bigcup_i (F_i - B_i) \), where \( X_* = X - \bigcup_i B_i \) and \( Z_* = \pi^{-1}(X_*) \).

Then every \( F_i \) is of codimension \( 2 \) in \( X \), and the morphism \( \pi : Z \to X \) is isomorphic to the blow-up \( Bl(X, \cap_i m_i^{d_i}) \) for some integers \( d_i \), where \( m_i \) is the ideal \( m_{F_i} \) defining \( F_i \).

**Proof.** That \( X \) is normal implies \( X - U \) has codimension at least \( 2 \) in \( X \), thus the Weil divisor group \( Cl(X) \) is isomorphic to the Picard group \( Pic(U) \), which is of torsion by hypothesis (i). This shows that \( X \) is \( \mathbb{Q} \)-factorial. Now the first affirmation follows from Corollary 1.3 \([Fu]\).

We will use an idea of D. Kaledin \([Ka]\) to prove the second affirmation. Since \( \pi \) is projective, \( Z = \text{Proj} \oplus_k \pi_* L^k \) for some holomorphic line bundle \( L \) on \( Z \). Notice that \( Pic(\pi^{-1}(U)) = Pic(U) \) is of torsion, replacing \( L \) by some positive power, we can suppose \( L|_{\pi^{-1}(U)} \) is trivial, thus \( \pi_* L|_U \simeq O_X|_U \).

Since \( X \) is normal and \( X - U \) has codimension \( \geq 2 \), this gives an embedding \( \pi_* L \to O_X \), thus \( \pi \) is identified with the blow-up \( Bl(X, \pi_* L) \).

Let \( i : Z_* \to Z \) and \( j : X_* \to X \) be the natural inclusions. We have the following commutative diagram:

\[
\begin{array}{ccc}
Z_* & \xrightarrow{i} & Z \\
\downarrow {\pi'} & & \downarrow \pi \\
X_* & \xrightarrow{j} & X \\
\end{array}
\]

The projection formula gives \( L = i_* i^* L \), so \( \pi_* L = \pi_* i_* i^* L = j_* \pi'_* i^* L \). By hypothesis (ii), we have \( \pi'_* : Z_* \to X_* \) is the blow-up of \( X_* \) with center \( \bigcup_i (F_i - B_i) \), thus \( \pi'_* i^* L = j^*(\cap_i m_i^{d_i}) \) for some positive integers \( d_i \). This gives that \( \pi_* L = j_* j^*(\cap_i m_i^{d_i}) = \cap_i m_i^{d_i} \), which concludes the proof.

**Lemma 3.** Let \( \Delta = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| + |z_2| < 1 \} \) be the disc in \( \mathbb{C}^2 \) and \( X = \Delta^{(k_1)} \times \cdots \times \Delta^{(k_l)} \). Then \( X \) is a normal variety with symplectic singularities, \( \text{Sing}(X) \) consists of \( N = \# \{i | k_i > 1 \} \) co-dimension \( 2 \) irreducible components \( F_i \) and any projective symplectic resolution of \( X \) is isomorphic to a blow-up \( Bl(X, \cap_i m_i^{d_i}) \) for some integers \( d_i \), where \( m_i = m_{F_i} \).

**Proof.** The first affirmation is easy. For the second affirmation, one notices that for \( k > 1 \), \( \text{Sing}(\Delta^{(k)}) \) is irreducible and of codimension \( 2 \) in \( \Delta^{(k)} \). We will apply Lemma 2 to prove the third affirmation. Denote by

\[ p : Y := \Delta^{k_1} \times \cdots \times \Delta^{k_l} \to X = \Delta^{(k_1)} \times \cdots \times \Delta^{(k_l)} \]
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the natural quotient by the product of permutation groups $\Sigma_{k_1} \times \cdots \times \Sigma_{k_l}$. Let $U$ be the smooth part of $X$ and $V = p^{-1}(U)$, then $Y - V$ is of codimension 2 in $Y$. In particular, we have $\text{Pic}(V) = \text{Pic}(Y) = 0$, which gives that $\text{Pic}(U)$ is of torsion, so condition (i) of Lemma 2 is verified.

We consider one component, say $F_1 = \text{Sing}(\Delta(k_1) \times \Delta(k_2) \times \cdots \times \Delta(k_l)) \ (k_1 > 1)$. Denote by $F_{1,*}$ the open set of $F_1$ consists of cycles of the form $2x_1 + x_2 + \cdots + x_{n-1}$, where $n = \sum_j k_j$, $x_i$ are distinct points and $2x_1 + x_2 + \cdots + x_{k_1-1}$ is in $\text{Sing}(\Delta(k_1))$. We define $B_i = F_i - F_{i,*}$. In fact, $F_{1,*}$ is the smooth part of $F_1$ and $B_1$ is its singular part. Then locally on $F_{1,*}$, the singularities looks like $\mathbb{C}^{2n-2} \times (Q, 0)$, where $Q$ is the cone over a smooth conic and 0 is its vertex. Recall that every crepant resolution of $Q$ is the blow-up of $Q$ with center 0. This gives that locally on $F_{1,*}$, any symplectic resolution $\pi : Z \to X$ is the blow-up of $X$ along the subvariety $F_{1,*}$, so the second condition of Lemma 2 is also verified.

\end{proof}

\begin{remark}
The author does not know whether the product $S^{(n_1)} \times S^{(n_2)} \times \cdots \times S^{(n_l)}$ admits other projective symplectic resolutions than the one given by $S^{[n_1]} \times S^{[n_2]} \times \cdots \times S^{[n_l]}$.
\end{remark}

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