Gluonium contributions to the form factors of $B_c$ transitions into $\eta'$

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We calculated the form factors of $B_c$ meson transitions into $\eta'$ meson, where the $B_c$ meson is a bound state of two different heavy flavors and is treated as a nonrelativistic state, while the masses of the mesons $\eta$ and $\eta'$ are smaller compared to the transition momentum scale and are treated as light-cone objects. The mechanism of two gluon transition into $\eta'$ dominated the form factors of $B_c$ decays into $\eta'$. We considered the $\eta - \eta'$ mixing effect, especially the gluonium contributions to the $\eta'$, and then obtained their influences on the form factors. The form factors of $B_c$ transitions into $\eta'$ in the maximum momentum recoil point were obtained as follows: $f_{0+}(q^2 = 0) = 3.13^{+0.12}_{-0.09} \times 10^{-3}$ and $f_{0+}(q^2 = 0) = 1.09^{+0.15}_{-0.12} \times 10^{-2}$. Also phenomenological discussions for semileptonic $B_c \rightarrow \eta' + \ell + \nu_\ell$ and $D_s \rightarrow \eta + \ell + \nu_\ell$ decays are given.

Keywords: $B_c$ meson decays, $\eta$ and $\eta'$ mixing, perturbative calculations

I. INTRODUCTION

Hadron-hadron colliders currently provide the unique platform to investigate the production and decay properties of the $B_c$ meson as the bound state of two different heavy flavors. In pace with the running of the CERN Large Hadron Collider (LHC) with the luminosity of about $\mathcal{L} \approx 10^{33} \text{cm}^{-2}\text{s}^{-1}$, one can expect around $10^9 B_c$ events per year [1]. When a tremendous amount of $B_c$ events are reconstructed, one can systematically and precisely test the golden decay channels of the $B_c$ meson or hunting for its rare decays [2].

The $B_c^-$ meson has two different heavy flavors and its decay modes can be classified into three categories: (i) the anti-charm quark decays with $\bar{c} \rightarrow \bar{d}, \bar{s}$; (ii) the bottom quark decays with $b \rightarrow u, c$; and (iii) the weak annihilation where both the bottom and anti-charm decays. These three categories of decay modes contribute to the total decay width of the $B_c^-$ meson are around 70, 20, and 10 percent, respectively [3]. There are currently a lot of theoretical and experimental works on the singly heavy quark decays of the $B_c$ meson, some of which with the bottom quark decays can be found in Refs. [4,10]. And the studies of the rare weak annihilation decays of the $B_c$ meson are few, some of which can be found in Refs. [11,15].

In this paper, we will investigate the decay properties of the $B_c$ meson into the light pseudoscalar mesons $\eta$ and $\eta'$. The light pseudoscalar mesons are organized into two representations: singlet and octet according to flavor SU(3) symmetry. Due to Isospin symmetry, the form factors of the $B_c$ meson into Isospin triplet $\pi^0$ is trivial, where the contributions from the quark contents $u\bar{u}$ and $d\bar{d}$ in $\pi^0$ will be cancelled out. Thus the form factors of the $B_c$ meson into the light meson $\pi^0$ will only depend on a small Isospin symmetry breaking effect, which is very similar to the case where the cross sections of $e^+e^- \rightarrow J/\psi \eta(\eta')$ are around pb while there is no signal for $e^+e^- \rightarrow J/\psi \pi$ [16,17].

In the flavor SU(3) symmetry, $\eta^{(')}$ mesons exhibit some different properties from the $\pi^0$, $\eta$ belongs to the flavor-octets while $\eta'$ belongs to the flavor-singlet. Since the strange quark’s mass is very different compared with the up and down quarks, there is a large flavor SU(3) symmetry breaking effect and their masses are then different. Besides, the flavor singlet and octet contents, even then the gluonium state will be mixed with each other with the identical $J^{PC}$ and form the physical $\eta^{(')}$ states. The $\eta$ meson is viewed as the mixing state between flavor singlet and octet contents, and the gluonium content is usually suppressed. However the conventional singlet-octet basis is not enough to explain the content of $\eta'$. For example, the gluonium contribution reached few percent in $B \rightarrow \eta'$ decays. Thus the $\eta'$ is viewed as the mixing state among $q\bar{q}$, $s\bar{s}$, and $gg$.

The $B_c$ meson is treated as a nonrelativistic bound state, where the heavy quark relative velocity is small in the rest frame of the meson. The nonrelativistic QCD (NRQCD) effective theory is employed to deal with the decays of the $B_c$ meson. Considering the light meson masses are less than the $B_c$ meson, i.e. $m_{\eta^{(')}}^2 << m_{B_c}^2$, a large momentum is transferred in the $B_c$ transitions into $\eta^{(')}$. The $\eta^{(')}$ can be treated as a light cone object in the rest frame of the $B_c$ meson. In the maximum momentum recoil point with $q^2 = 0$, the form factors of the $B_c$ transitions into $\eta^{(')}$ can be factorized as the hadron long-distance matrix elements and the corresponding perturbative short distance coefficients.

We will discuss the properties of the form factors of the $B_c$ transitions into $\eta^{(')}$, and especially on the gluonium contributions. We will employ the form factors formulae...
into the related semileptonic decays, naming \( B_c \to \eta^{(')} + e + \bar{\nu}_e \).

The paper is organized as the following. In Sec. II, we will introduce the NRQCD approach and the \( \eta - \eta' \) mixing effect. In Sec. III, we will calculate the form factors of the \( B_c \) meson into \( \eta^{(')} \). Especially, we will determine the gluonium contributions to the form factors and discuss their properties. In Sec. IV, we will study the semileptonic decays of the \( B_c \) meson into \( \eta^{(')} \). And we will tentatively analyze the processes \( D_s \to \eta + \ell + \bar{\nu}_\ell \). We summarize and conclude in the end.

II. FACTORIZATION FORMULAE

A. NRQCD effective theory

The heavy quark relative velocity is a small quantity inside the heavy quarkonium and then the heavy quark pair is nonrelativistic in the rest frame of heavy quarkonium. The quark relative velocity squared is estimated as \( v^2 \approx 0.3 \) for \( J/\psi \) and \( v^2 \approx 0.1 \) for \( \Upsilon \) [18]. The \( B_c \) meson is usually treated as a nonrelativistic state and the quark reduced velocity squared is estimated in the region \( 0.1 < v^2 < 0.3 \). The calculations of the productions and decays of the heavy quarkonium and the \( B_c \) meson with a large momentum transmitted usually refer to the NRQCD effective theory established by Bodwin, Braaten, and Lepage [18].

In the NRQCD effective theory, the Lagrangian is written as [18]

\[
\mathcal{L}_{\text{NRQCD}} = \psi^\dagger \left( iD_t + \frac{D^2}{2m} \right) \psi + \frac{c_F}{2m} \psi^\dagger \gamma_5 \cdot g_s B \psi + \frac{2}{8m^3} \psi^\dagger D^2 \psi + \frac{c_D}{8m^2} \psi^\dagger (D \cdot g_s \mathbf{E} - g_s \mathbf{E} \cdot D) \psi + \frac{ic_g}{8m^2} \psi^\dagger \gamma_5 \gamma_\lambda \cdot (D \times g_s \mathbf{E} - g_s \mathbf{E} \times D) \psi + (\psi \to i\sigma^\dagger \chi^*, A_\mu \to -A_\mu^S) + \mathcal{L}_{\text{light}}, \tag{1}
\]

where \( \psi \) and \( \chi \) represent the Pauli spinor field that annihilates a heavy quark and creates a heavy antiquark, respectively. \( \mathcal{L}_{\text{light}} \) denotes the Lagrangian for the light quarks and gluons. The short-distance coefficients \( c_D, c_F, \) and \( c_S \) can be perturbatively calculated according to the matching procedure between QCD and NRQCD calculations.

Within the framework of NRQCD, the heavy quarkonium inclusive annihilation decay width is factorized as [18]

\[
\Gamma(H) = \sum_n \frac{2m}{m_{d_n} - \frac{1}{2}} \langle H|O_n(\mu_A)|H \rangle, \tag{2}
\]

where \( \langle H|O_n(\mu_A)|H \rangle \) are NRQCD decay long-distance matrix elements (LDMEs), which involve nonperturbative information and obey the power counting rules, which are ordered by the relative velocity between the heavy quark and antiquark inside the heavy quarkonium \( H \).

The leading order NRQCD decay operators for the decay of \( S \)-wave heavy quarkonium are

\[
\mathcal{O}(S_0^{[1]}) = \psi^\dagger \chi \gamma^\dagger \psi, \tag{3}
\]

\[
\mathcal{O}(S_0^{[1]}) = \psi^\dagger \sigma \chi \gamma^\dagger \psi. \tag{4}
\]

These operators are also valid for the \( B_c \) family with two different heavy flavors.

For a certain process, the matching coefficients multiplying decay LDMEs are determined through perturbative matching between QCD and NRQCD at the amplitude level. The covariant projection method is another equivalent but more convenient approach to extract the short-distance coefficients of the NRQCD LDMEs. The corresponding projection operators are defined as

\[
\Pi_{S=0,1}(k) = \frac{i}{\sqrt{2E_1 E_2 \omega}} \left( \alpha \hat{p}_H - \vec{k} + m_1 \right) \frac{\hat{p}_H + E_1 + E_2}{E_1 + E_2} \Gamma_S (\beta \hat{p}_H + \vec{k} - m_2) \otimes \frac{1}{\sqrt{N_c}} \sqrt{2T^*}, \tag{5}
\]

where \( \omega = \sqrt{E_1 + m_1 \sqrt{E_2 + m_2}} \) with \( E_1 = \sqrt{m^2 - \vec{k}^2} \) and \( E_2 = \sqrt{m^2 - k^2} \). We have the spin \( S = 0 \) and \( \Gamma_{S=0} = \gamma^5 \) for the spin-singlet combination. For the spin-triplet combination, we have the spin \( S = 1 \) and \( \Gamma_{S=1} = \lambda \). \( H = \varepsilon_\mu (p_H) \gamma^\mu. \left\{ \frac{1}{\sqrt{N_c}}, \sqrt{2T^*} \right\} \) denote the color-singlet and color-octet projection in the \( SU(3) \) color space.

The heavy quarkonium state is not limited to heavy quark pairs in a color singlet configuration according to NRQCD. The heavy quark pairs in a color singlet configuration is only the leading order of Fock state of the quarkonium. Other Fock states sometimes play an important role in the inclusive production of heavy quarkonium. In the form factors of the \( B_c \) meson into \( \eta^{(')} \), the
dominant contribution is from the color singlet configuration.

\[ \langle 0 | q^\gamma q | \eta_{l}(p) \rangle = i f_{q_{l}^\prime} f_{p_{l}} \mu, \quad (q' = q, s). \]  

(6)

The traditional quark-flavor bases is not enough to explain the content of \( \eta' \). \( \eta' \) may have gluonium content, and shall be treated as the mixing state among \( q\bar{q}, s\bar{s} \), and \( gg \). We have

\[ |\eta\rangle = \cos \phi |\eta_q\rangle - \sin \phi |\eta_s\rangle, \]
\[ |\eta'\rangle = \cos \phi_G |\eta_q\rangle + \sin \phi |\eta_s\rangle + \sin \phi_G |\eta_g\rangle, \]  

(7)

(8)

where the gluonium component \( |\eta_g\rangle = |gg\rangle \), \( \phi \) is the mixing angle between \( q\bar{q} \) and \( s\bar{s} \), while \( \phi_G \) is introduced to describe the mixing between the quarks and gluonium content of \( \eta' \).

\[ \Phi_{\eta'}^{(g,s)}(x, \mu) = 6x\bar{x} \left\{ 1 + \sum_{n=2,4,\ldots} a_n^{(g,s)}(\mu_0) \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\gamma_n} + \rho_n^{(g,s)}(\mu_0) \left( \frac{a_n(\mu_0)}{\alpha_s(\mu_0^2)} \right)^{-\gamma_n} \right\} C_{n}^{3/2}(x - \bar{x}), \]

\[ \Phi_{\eta'}^{(g)}(x, \mu) = x\bar{x} \sum_{n=2,4,\ldots} \left[ \rho_n^{(g,s)}(\mu_0) \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{-\gamma_n} + a_n^{(g)}(\mu_0) \left( \frac{a_n(\mu_0)}{\alpha_s(\mu_0^2)} \right)^{-\gamma_n} \right] C_{n-1}^{5/2}(x - \bar{x}), \]  

(9)

(10)

(11)

The evolution of the LCDA of the quark contents will mix with the gluonium state for \( \eta' \). In Ref. 21, the evolution equation for the LCDA of the mixing \( \eta' \) state has been calculated. The corresponding light cone distribution amplitudes are 22, 23.

The light cone distribution amplitudes (LCDA) of \( \eta_q \) and \( \eta_s \) components in \( \eta \) have the form

\[ \Phi_{\eta}^{(q,s)}(x, \mu) = 6x\bar{x} \left[ 1 + \sum_{n=2,4,\ldots} a_n(\mu) C_{n}^{3/2}(x - \bar{x}) \right], \]  

(9)

where \( x \) and \( \bar{x} = 1 - x \) are the momentum fractions of the light quark and antiquark inside \( \eta_{q,s} \), respectively. \( C_{n}^{3/2} \) and the following \( C_{n}^{5/2} \) are the Gegenbauer polynomials. \( a_n(\mu) \) is obtained by the scale evolution at leading-order logarithmic accuracy

\[ a_n(\mu) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{\frac{\beta_0}{\beta_0}} a_n(\mu_0), \]

(10)

and

\[ \rho_n^0 \equiv -\frac{1}{61 - P_n}, \quad \rho_n^0 \equiv \frac{6}{Q_n}, \]
\[ P_n = \frac{\gamma_n^+ + \gamma_n^-}{\gamma_n^+ + \gamma_n^-}, \quad Q_n = \frac{\gamma_n^+ - \gamma_n^-}{\gamma_n^+ - \gamma_n^-}. \]  

(18)

(19)

C. The mechanism of two gluons transition into \( \eta' \)

In the \( B_c \) meson decays into \( \eta' \), the mechanism of two gluons transition into \( \eta' \) will play an important role. Two gluons transition mechanism is blind to quark charges and light quark flavors, so the amplitude is identical to \( q\bar{q} \) and \( s\bar{s} \) except the mixing factor and the decay constant.

The amplitudes of two gluons transitions into quarks and gluonium contents of \( \eta' \) in lowest-order perturba-
transformation theory can be obtained by calculating the corresponding Feynman diagrams which are plotted in Fig. 1.

The amplitudes of two gluons to quark-antiquark content of $\eta^{(f)}$ are

$$M^{(q,s)} = -i F^{(q,s)}_{\eta^{(f)}g^*g^*}(q_1^2, q_2^2) \delta_{ab} \epsilon^{\mu\nu\rho\sigma} \epsilon^b_{\mu} \epsilon^2_{\nu} q_1 \rho q_2 \sigma,$$

where the momenta of two initial virtual gluons are denoted as $q_1$ and $q_2$, respectively. The polarization vectors of two initial gluons are denoted as $\epsilon_1(q_1)$ and $\epsilon_2(q_2)$, respectively. The form factors $F^{(q,s)}_{\eta^{(f)}g^*g^*}$ of two gluons transitions to quark-antiquark content of $\eta^{(f)}$ can be written as

$$F^{(q,s)}_{\eta^{(f)}g^*g^*}(q_1^2, q_2^2) = \frac{2\pi\alpha_s(\mu^2)}{N_c} C_{\eta^{(f)}} \int_0^1 dx \Phi^{(q,s)}(x, \mu)$$

$$\times \left[ \frac{1}{x q_1^2 + \bar{x} q_2^2 - x \bar{x} m_{\eta^{(f)}}^2} + i \epsilon + (x \leftrightarrow \bar{x}) \right].$$

(21)

For $\eta$, we have $C_\eta = \sqrt{2} f_\eta + f_\eta^*$. While for $\eta'$, we have $C_{\eta'} = \sqrt{2} f_{\eta'} + f_{\eta'}^*$. The decay constants for the corresponding components are: $f_\eta = f_g \cos \phi$, $f_{\eta'} = -f_g \sin \phi$, $f_{\eta'}^* = f_g \sin \phi$ and $f_{\eta'}^* = f_g \cos \phi$.

The amplitudes of two gluons to gluonium content of $\eta'$ are

$$M^{(g)} = -i F^{(g)}_{\eta'g^*g^*} \delta_{ab} \epsilon^{\mu\nu\rho\sigma} \epsilon^a_{\mu} \epsilon^2_{\nu} q_1 \rho q_2 \sigma,$$

where the form factors $F^{(g)}_{\eta'g^*g^*}$ of gluonium transitions to gluonium content of $\eta'$ can be written as

$$F^{(g)}_{\eta'g^*g^*}(q_1^2, q_2^2) = \frac{4\pi\alpha_s(\mu^2)}{Q^2} \frac{C_{\eta'}}{2} \int_0^1 dx \Phi^{(g)}(x, \mu)$$

$$\times \left[ \frac{x q_1^2 + \bar{x} q_2^2 - (1 + x \bar{x}) m_{\eta'}^2}{x q_1^2 + \bar{x} q_2^2 - x \bar{x} m_{\eta'}^2 + i \epsilon} - (x \leftrightarrow \bar{x}) \right].$$

(23)

where the typical scale $Q^2$ is introduced to preserve the dimensionless for the transition form factors. The choice of $Q^2$ has some freedom, and $Q^2$ is adopted to $|q_1^2|$ or $|q_2^2 + q_3^2|$ in Ref. [22, 23]. In this paper, $Q^2$ is adopted $m_{\eta'}^2$, and the LCDA of gluonium content of $\eta'$ is adopted as the form in Eq. (12).

III. FORM FACTORS OF $B_c$ INTO $\eta'$

The form factors of $B_c$ into $\eta$, i.e., $f_+^c$ and $f_{1c}$ are defined in common [24, 25],

$$\langle \eta(p)|\bar{c}g\gamma^\mu b|B_c(P)\rangle = f_+^c(q^2)(k^\mu - \frac{m_{B_c}^2 - m_\eta^2}{q^2} q^\mu)$$

$$+ f_{1c}^c(q^2) \frac{m_{B_c}^2 - m_\eta^2}{q^2} q^\mu,$$

(24)

where the momentum transfer is defined as $q = P - p$ with the $B_c$ meson momentum $P$ and the light meson momentum $p$, and the momentum $k$ is defined as $k = P + p$. And the form factors $f_+^c(q^2)$ and $f_{1c}^c(q^2)$ can be defined by the exchange of $\eta \rightarrow \eta'$.

Typical Feynman diagrams for the form factors of $B_c$ into $\eta'$ are plotted in Fig. 2. Other Feynman diagrams can be obtained by changing the gluon vertex to anticharm quark line.

Considering the hard scattering mechanism of two gluons transition into $\eta^{(f)}$, the Feynman diagrams for $B_c$ into $\eta^{(f)}$ by the vector current can be written as

$$M = \langle 0|\bar{c}^1\gamma\psi|B_c\rangle \text{Tr}[A^\mu(0)\Pi_{S=0}(k = 0)],$$

(25)

where
with the bottom quark momentum \( p_1 \) and the anti-charm quark momentum \( p_2 \).

The Mathematica software is employed with the help of the packages FeynCalc\[29\], FeynArts\[27\], and LoopTools\[28\] in the calculation of the form factors. In order to obtain the values of form factors at the maximum recoil point, we will adopt the parameter values as follows: \( m_{B_c} = 6.276 \text{GeV} \), \( m_q = 547.85 \text{MeV} \), \( m_{q'} = 957.78 \text{MeV} \) \[24\]. The heavy quark masses are adopted as \( m_c = (1.5 \pm 0.1) \text{GeV} \) and \( m_b = (4.8 \pm 0.1) \text{GeV} \). The decay constants for the quark contents of \( \eta(q') \) are: \( f_q = (1.07 \pm 0.02) f_\pi \), \( f_{q'} = (1.34 \pm 0.06) f_\pi \) with the pion decay constant \( f_\pi = 130.4 \text{MeV} \), and the mixing angle is adopted as \( \phi = 39.3^\circ \pm 1.0^\circ \) \[30\] \[31\]. The Gegenbauer momenta are adopted as \( a_2^{2+}(1 \text{GeV}) = 0.44 \pm 0.22 \) \[30\] and \( a_2^0(1 \text{GeV}) = 0.1 \) and \( \sin^2 \phi_G = 0.26 \) \[17\]. The running strong coupling constant is adopted at the bottom quark mass scale.

In the maximum momentum recoil point, the form factors appearing in the expression \[24\] can be perturbatively calculated reliably. We give the form factors in the maximum momentum recoil point in Tab. \[1\].

**TABLE I:** Form factors of the \( B_c \) into \( \eta(q') \) in the maximum recoil point with \( q^2 = 0 \). Here and in the following tables, the uncertainty is from the choice of the bottom and charm quark masses. Note that \( f_4(0) = f_6(0) \).

| Contributions  | \( 10^{-3} f_0^q(q^2 = 0) \) | \( 10^{-2} f_0^{q'}(q^2 = 0) \) |
|---------------|-----------------|-----------------|
| \( q\bar{q} \) with LO Gegenbauer | \( 2.77^{+0.08}_{-0.06} \) | \( 1.02^{+0.03}_{-0.00} \) |
| \( q\bar{q} \) with NLO Gegenbauer | \( 3.13^{+0.12}_{-0.09} \) | \( 1.10^{+0.03}_{-0.03} \) |
| \( gg \) | \(- \) | \( 1.05^{+0.15}_{-0.12} \) |
| Total | \( 3.13^{+0.12}_{-0.09} \) | \( 1.09^{+0.15}_{-0.12} \) |

The form factors dependent on the meson momentum fraction are sensitive to the shapes of the Gegenbauer series of the light meson. We give the form factors dependent on the meson momentum fraction in Fig. \[3\] and \[4\]. For the quarks contents contributions, the momentum fraction dependent shapes of form factors with leading order (LO) Gegenbauer momentum have only one peak, while that of form factors with next-to-leading (NLO) Gegenbauer momentum will have two peaks. For the gluonium contributions, the momentum fraction dependent shape of form factor has two peaks.

**FIG. 3:** The form factor of \( B_c \) into \( \eta(q') \) dependent on the meson momentum fraction. Note that \( f_0^q(q^2 = 0) \equiv f_0^{q'}(q^2 = 0) \).

**FIG. 4:** The form factor of \( B_c \) into \( \eta(q') \) dependent on the meson momentum fraction. Note that \( f_4^q(q^2 = 0) \equiv f_4^{q'}(q^2 = 0) \).

At the minimum momentum recoil point, the perturbative calculations for the form factors become invalid. In these region, one has to refer to Lattice QCD simulations or some certain models. In order to extrapolate the form factors to the minimum momentum recoil region, the pole model are generally adopted in many literatures \[32\] \[33\]. Thus the \( q^2 \) distribution of the form...
factors can be parametrized as
\[ f_{0,+}(q^2) = \frac{f_{0,+}(0)}{1 - \frac{q^2}{m_{B_c}^2}(1 - a \frac{q^2}{m_{B_c}^2} + b \frac{q^2}{m_{B_c}^2})}, \] (27)
where the \( a \) and \( b \) are model independent parameters and can be fitted when the data are available. Here we let \( a = b = 0 \) for simplicity. We plotted the form factors dependent on the momentum transfer squared in Fig. 5 and 6.

The semileptonic decay widths and the branching ratios can be obtained after integrating the momentum recoil squared \( q^2 \). In Tab. II we give the results for the \( B_c \to \eta + \ell + \bar{\nu}_\ell \) with \( \ell = e, \mu, \tau \). The masses of the leptons are: \( m_e = 0.50 \text{MeV}, m_\mu = 105.6 \text{MeV} \) and \( m_\tau = 1777 \text{MeV} \). The form factors with both LO and NLO Gegenbauer series are considered in the semileptonic decays. From the table, their decay widths are around \( 10^{-18} \text{GeV} \), while the branching ratios are around \( 10^{-6} \). For \( \ell = e, \mu \), their decay widths are nearly identical since their masses can be discarded in the \( B_c \) meson decays to \( \eta \). Besides, the LO and NLO Gegenbauer series have less influence in the semileptonic decay width of \( B_c \) into \( \eta \). In Tab. III we give the results for the \( B_c \to \eta' + \ell + \bar{\nu}_\ell \) with \( \ell = e, \mu, \tau \). The form factors from the quark contents with NLO Gegenbauer series and from the gluonium contribution are considered in the semileptonic decays. As given in above, we chose \( \sin^2 \varphi_G = 0.26 \), naming the gluonium content takes up 26% in \( \eta' \). From the table, their decay widths are around \( 10^{-17} \text{GeV} \), while the branching ratios are around \( 10^{-5} \).

![FIG. 5: The form factor of \( B_c \) into \( \eta' \) dependent on the recoil momentum squared.](image)

![FIG. 6: The form factor of \( B_c \) into \( \eta' \) dependent on the recoil momentum squared.](image)

**IV. SEMILEPTONIC DECAYS OF \( B_c \) INTO \( \eta' \)**

In this section, we will employ the form factors into the semileptonic decays of \( B_c \) into \( \eta' \). We retain the lepton masses, and the semileptonic decay width of \( B_c \) into \( \eta \) can be written as

\[
\frac{d\Gamma}{dq^2} = \frac{G_F^2 \lambda(m^2_{B_c}, m^2_\eta, q^2)^{1/2}|V_{cb}|^2}{384\pi^3 m^2_{B_c}} \left( \frac{q^2 - m^2_\eta}{q^2} \right)^2 \frac{1}{q^2} \times \left[ (m^2_{B_c} + 2q^2) \lambda(m^2_{B_c}, m^2_\eta, q^2)(f_+(q^2))^2 + 3m^2_{B_c} - m^2_\eta \right] (f_0'(q^2))^2, \] (28)

where \( \lambda(m^2_{B_c}, m^2_\eta, q^2) = (m^2_{B_c} + m^2_\eta - q^2)^2 - 4m^2_{B_c}m^2_\eta. \) And the similar formulae can be gotten for the semileptonic decay width of \( B_c \) into \( \eta' \).

![TABLE II: The semileptonic decay widths and the branching ratios of \( B_c \) into \( \eta' \). Here and in the following \( q \bar{q} \) denote the quark contents, and the life time \( \tau_{B_c} = 0.50 \text{ps} \).](table)

| \( \ell = e, \mu \) | \( q \bar{q} \) (NLO) | \( \Gamma_{\eta'}/(10^{-17} \text{GeV}) \) | \( B_{\eta'}/(10^{-6}) \) |
|----------------|-----------------|-----------------|-----------------|
| \( q \bar{q} \) (LO) | 1.58 ± 0.12 & 1.20 ± 0.09 |
| \( q \bar{q} \) (NLO) | 1.24 ± 0.07 & 0.94 ± 0.06 |
| \( \ell = \tau \) | \( q \bar{q} \) (NLO) | 1.53 ± 0.12 & 1.17 ± 0.09 |
| \( q \bar{q} \) (LO) | 1.20 ± 0.07 & 0.92 ± 0.04 |

| \( \ell = e, \mu \) | \( q \bar{q} \) (NLO) | \( \Gamma_{\eta'}/(10^{-17} \text{GeV}) \) | \( B_{\eta'}/(10^{-6}) \) |
|----------------|-----------------|-----------------|-----------------|
| \( q \bar{q} \) (LO) | 1.58 ± 0.08 & 1.20 ± 0.06 |
| \( gg \) | 1.45 ± 0.45 & 1.10 ± 0.34 |
| total | 1.55 ± 0.44 & 1.17 ± 0.25 |
| \( \ell = \tau \) | \( q \bar{q} \) (NLO) | 1.33 ± 0.07 & 1.01 ± 0.05 |
| \( gg \) | 1.22 ± 0.37 & 0.92 ± 0.29 |
| total | 1.30 ± 0.37 & 0.99 ± 0.29 |

In Ref. [11], the semileptonic branching ratios of \( B_c \) into \( \eta' \) have been already predicted in perturbative QCD, where the \( Br(B_c \to \eta + \ell + \bar{\nu}_\ell) = 3.98 \times 10^{-6} \) and \( Br(B_c \to \eta' + \ell + \bar{\nu}_\ell) = 5.24 \times 10^{-5} \) with \( \ell = e, \mu \) and \( m_{B_c} = 2 \text{MeV}, m_d = 4 \text{MeV} \) and \( m_s = 80 \text{MeV} \). Compared with these predictions in Perturbative QCD, our results are smaller due to the choice of the decay constant of \( \eta' \). Currently, there is no report on semileptonic decays of \( B_c \) into \( \eta' \). However, the hunting for the signals of \( B_c \) into \( \eta' \) is accessible in future LHCb experiments when considering the large cross section of \( B_c \) meson.
In the end it is very interesting to find out whether the formulæ in above can guide the studies of the processes $D_s \to \eta + \ell + \bar{\nu}_\ell$. The BESIII Collaboration have measured these channels and given $Br(D_s \to \eta + \ell + \bar{\nu}_\ell) = (2.42 \pm 0.46 \pm 0.11)\%$ and $Br(D_s \to \eta' + \ell + \bar{\nu}_\ell) = (1.06\pm 0.54 \pm 0.07)\%$ [34]. For $D_s \to \eta^{(*)} + \ell + \bar{\nu}_\ell$, the $c \to s$ transition with another spectator strange quark will be present in the $D_s \to \eta^{(*)}$ form factors. Considering the transferred momentum is near 0.9 GeV in $D_s \to \eta' + \ell + \bar{\nu}_\ell$, the perturbative calculation may be invalid, so we only consider the channel $D_s \to \eta + \ell + \bar{\nu}_\ell$. As the tentatively analysis, it is interesting to find out how large the mechanism of two gluon transitions contributes to processes $D_s \to \eta + \ell + \bar{\nu}_\ell$. Employing the above formulæ, and taking the replacement of $b \to c$, $c \to s$ and $B_c \to D_s$, we may tentatively give the order of magnitude of their decay widths since the $D_s$ meson is not a really non-relativistic bound state and the transferred momentum is near $m_c$. We found that the mechanism of two gluon transitions gives $Br(D_s \to \eta + \ell + \bar{\nu}_\ell) \sim 10^{-4}$ and only contributes to 0.5% in the channel $D_s \to \eta + \ell + \bar{\nu}_\ell$. The $c \to s$ transition thus dominates the form factor of $D_s \to \eta$. To extrapolate the form factors of $D_s \to \eta$ to the minimum momentum recoil region, the pole model is still useful [33]. Combining the experimental data, the $c \to s$ transition leads to the $f_{D_s}^{\eta}(q^2 = 0) = 0.50 \pm 0.05$.

V. CONCLUSION

We investigated the form factors of $B_c$ into $\eta^{(*)}$ and employed the form factors into their semileptonic decays. Unlike the decay of $D_s$ into $\eta^{(*)}$, the mechanism of two-gluon transition dominated the contribution for the form factors of $B_c$ into $\eta^{(*)}$. For $\eta'$, the gluonium content is included and its contribution is calculated. At the maximum momentum recoil point, the form factors of $B_c$ into $\eta^{(*)}$ is factored as the LDMEs along with the corresponding short-distance perturbatively calculable coefficients. The results of form factors in the maximum momentum recoil point were obtained as: $f_{B_c}^{\eta}(q^2 = 0) = 3.13^{+0.12}_{-0.09} \times 10^{-3}$ and $f_{B_c}^{\eta'}(q^2 = 0) = 1.09^{+0.13}_{-0.12} \times 10^{-2}$. Using the pole model, the form factors of $B_c$ into $\eta^{(*)}$ is extrapolated into the minimum momentum recoil region.

We calculated the semileptonic decay widths and the branching ratios. The branching ratio of $B_c \to \eta + \ell + \bar{\nu}_\ell$ is around $10^{-6}$, while the branching ratio of $B_c \to \eta' + \ell + \bar{\nu}_\ell$ is around $10^{-5}$. Future LHCb experimental shall test these predictions, which is helpful to understand the $\eta - \eta'$ mixing and the decay properties of $B_c$ meson.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grant No. 11705092, 11775117 and 11235005, and by Natural Science Foundation of Jiangsu under Grant No. BK20171471.
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